Degenerate Dynamical Systems

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I. INTRODUCTION

A number of dynamical systems of physical interest possess field-dependent symplectic forms which degenerate, becoming noninvertible for some particular configurations. Systems as diverse as vortex interactions in fluids, and gravitation theories in dimensions $d > 4$ containing higher powers of curvature in the Lagrangian exhibit this feature (see e.g. [3]). Models of this kind naturally arise in different contexts of current high energy physics, ranging from cosmology and brane worlds to strings an M-theory.

The problem is how to describe the evolution of the system near a degenerate configuration and, if it could reach such state, how it would evolve afterwards. The standard hypotheses in the treatment of dynamical systems, however, exclude the possibility that the symplectic form degenerates on surfaces which are generically domain walls. This kind of surfaces cannot be understood as dense sets of Poincaré singularities. Roughly speaking, a symplectic degeneracy is the counterpart of a Poincaré singularity in that, in the latter the gradient of the Hamiltonian vanishes, whereas the former can be interpreted as an infinite gradient.

The previous point can be made more explicit, by considering the simplest example of a degenerate system, whose phase flow satisfies

\[
\left( \begin{array}{cc} 0 & x_2 \\ -x_2 & 0 \end{array} \right) \left( \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right) = \left( \begin{array}{c} E_1 \\ E_2 \end{array} \right),
\]

with $E_1 E_2 \neq 0$, which degenerates at $x_2 = 0$. An equivalent formulation in the $x_2 \neq 0$ region is

\[
\left( \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right) = \frac{1}{x_2} \left( \begin{array}{c} -E_2 \\ E_1 \end{array} \right),
\]

which can be viewed as a phase flow where the gradient of the Hamiltonian diverges as $x_2 \to 0$. The required symplectomorphism (canonical transformation) to obtain Eq. (3) from Eq. (1) is noninvertible throughout phase space, however.

II. FIRST-ORDER LAGRANGIANS AND THEIR SYMPLECTIC FORMS

Let us consider a system whose action is a one-form $A$, integrated over a $(0 + 1)$-dimensional worldline embedded in a $(2n + 1)$-dimensional spacetime of signature $(-, +, \ldots +)$,

\[
S[z; 1, 2] = \int_{1}^{2} A_{\mu} \dot{z}^{\mu} d\tau,
\]

The field $A_{\mu}$ is a prescribed set of $2n + 1$ functions of the embedding coordinates $z^{\mu}$, which are the dynamical variables. This action is manifestly invariant under reparametrizations of the worldline $\tau \to \tau'(\tau)$, and diffeomorphisms $z^{\mu} \to z^{\mu}(z)$. Identifying the affine parameter with the timelike embedding coordinate $z^0 := t$, so that $z^i = z^i(t)$, the action reads

\[
S[z; 1, 2] = \int_{1}^{2} A_{\mu} \dot{z}^{\mu} d\tau.
\]
The equations of motion are given by
\[ F_{ij} \dot{z}^j + E_i = 0 , \]
where we have defined \( E_i \equiv \partial_t A_0 - \partial_i A_i \) and \( F_{ij} \equiv \partial_i A_j - \partial_j A_i \). In the following, we assume \( A_i \) and \( A_0 \) to be time-independent.

These dynamical systems are naturally classified according to the rank \( \rho \) of the symplectic form \( F_{ij} \). Thus, three cases are distinguished: (A) Regular Hamiltonian systems, for which the symplectic form has constant maximal rank, \( \rho(F_{ij}) = 2n \) throughout phase space \( \Gamma \). (B) Singular or constrained Hamiltonian systems, which have a constant nonmaximal rank, \( \rho(F_{ij}) = 2m < 2n \) throughout \( \Gamma \). And, (C) Degenerate systems, which have nonconstant rank \( \rho(F_{ij}) \) throughout \( \Gamma \).

III. DEGENERATE SYSTEMS

We will focus our discussion in the degenerate case (C), which has been traditionally left aside in the literature. We will assume that the zero-measure subset of \( \Gamma \) given by
\[ \Sigma = \{ z \in \Gamma / F = 0 \} , \]
where \( F \equiv \det(F_{ij}) \), is not dense. Thus, outside \( \Sigma \), the symplectic form \( F_{ij} \) has a constant rank \( 2n \), and the dynamical structure there is described through cases (A) above.

Under these conditions, nothing prevents the system, starting from a generic state for which \( F \neq 0 \), from reaching a point on \( \Sigma \) after some finite time. Having this scenario in mind, we address the following points:

- The description of the locus of \( \Sigma \).
- Classification of the phase flow near \( \Sigma \).
- Whether \( \Sigma \) can be reached and, in that case, the fate of the system thereafter.

A. Degeneracy Surfaces \( \Sigma \)

As is well known, a skew-symmetric \( 2n \times 2n \) matrix \( F_{ij}(z) \) can be brought into the block-diagonal form by an orthogonal transformation. Thus the two-form \( \mathcal{F} = \frac{1}{2} F_{ij} dz^i \wedge dz^j \) can be block diagonalized in an open set, under a local \( O(2n) \) coordinate transformation \( z^i \to x^i(z) \),
\[ \mathcal{F} = \sum_{r=1}^{n} f_r(z) dx^{2r-1} \wedge dx^{2r} . \]

However, in open sets containing points of the degeneracy surfaces, the Darboux-like coordinates \( x^i \) cannot be brought into the standard canonical form, because at least one of the \( f_r \)’s in \( \mathcal{F} \) vanishes at \( \Sigma \). Hence, further (finite) rescalings cannot normalize the \( f_r \)’s to 1. As a consequence, the set \( \Sigma \) is the union of the \((2n-1)\)-dimensional surfaces
\[ \Sigma_r = \{ z \in \Gamma / f_r(z) = 0 \} , \]
that is, \( \Sigma = \cup_{r=1}^{n} \Sigma_r \).

Moreover, by virtue of the Bianchi identity (\( d\mathcal{F} = 0 \)), it can be shown that \( f_r(x) \) depends only on the pair of conjugate coordinates \((x^{2r-1}, x^{2r})\). This means that the degeneracy surfaces are constant along the remaining coordinates.

We assume that the \( f_r \)’s are smooth Morse functions on the corresponding \((x^{2r-1} - x^{2r})\) planes, which ensures that they possess only simple zeros except at isolated points; the cases where \( f_r \) has zeros of higher order can be thought of as the merging of simple zeros. Hence, the level curves \( f_r(x^{2r-1}, x^{2r}) = 0 \) divide the \((x^{2r-1} - x^{2r})\)-plane into nonoverlapping sets and therefore,

**Lemma 1:** The locus of the degeneracy surfaces \( \Sigma \) corresponds to a collection of domain walls, splitting the phase space \( \Gamma \) into a number of nonoverlapping regions.

B. Characterization of the Phase Flow near \( \Sigma \)

Generically, at a surface \( \Sigma_r \), the rank \( \rho(F_{ij}) \) is lowered by \( 2 \), and at points where \( k \) of these surfaces intersect, \( \rho \) is lowered by \( 2k \). In a sufficiently small neighborhood of the surface \( \Sigma_r \), the behavior of the system is dominated by the dynamical variables \( x^\alpha = (x^{2r-1} - x^{2r}) \), whose corresponding equations of motion can be read from Eq. (8) as
\[ \epsilon_{\alpha\beta} f(x) z^\beta = -E_\alpha , \]
where for simplicity, we have set \( r = 1 \), so that \( \alpha \) and \( \beta = 1, 2 \) and \( f := f_1 \). Near a degeneracy surface \( \Sigma_r \), the remaining dynamical variables \( z^\alpha , ( \alpha = 3, ..., 2n ) \), behave like the phase space coordinates of a regular system.

Here it is assumed that \( E_\alpha \) remains finite and does not vanish on \( \Sigma_1 \) (i.e., Poincaré singularities are assumed to be located outside \( \Sigma \)), therefore, Eq. (8) implies that the velocity becomes tangent to the \((x^3 - x^7)\) plane, because the components \( z^\alpha \) become unbounded as the orbit
approaches \( \Sigma_1 \), while the other components \( \dot{z}^\alpha \) remain finite.

Due to the fact that \( f \) has a simple zero at \( \Sigma_1 \), \( \dot{x}^\alpha \) reverses its sign across the degeneracy surface. Consequently, the phase flow evolves in opposite directions on each side of \( \Sigma \). Thus, in a local neighborhood of \( \Sigma \), one of the following three situations occur: (a) Orbits flow towards \( \Sigma \) and end there, (b) the orbits originate at the degeneracy surface and flow away from it, or (c) the orbits run parallel to \( \Sigma \), but in opposite directions on each side.

Hence, the surfaces act as sinks or sources for the orbits in cases (a) and (b) respectively, which naturally suggests a classification of the local nature of \( \Sigma \) into \( \Sigma^{(-)} \), \( \Sigma^{(+)} \), and \( \Sigma^{(0)} \) for the cases (a), (b) and (c), respectively (see Fig. 1).

In all three cases there is no flux across the degeneracy surface, and therefore,

**Lemma 2**: The regions on either side of \( \Sigma \) are causally disconnected and dynamically independent from each other.

An immediate consequence of this, is the violation of Liouville’s theorem at the surfaces of degeneracy. In fact, outside the degeneracy surfaces, the Liouville current

\[
j^i = \sqrt{F} \dot{z}^i ,
\]

is divergence-free \( (\partial_i j^i = 0) \) by virtue of the equations of motion and the identity \( \partial_i (\sqrt{F} F^{ij} E_j) = 0 \), with \( F^{ij} F_{jk} = \delta^i_k \). This means that Liouville’s theorem holds outside \( \Sigma \), where the dynamical behavior is regular. Moreover, \( j^i \) has a finite limit as the system approaches a degeneracy surface, whose only nonvanishing components on each side of \( \Sigma \) are

\[
j^\alpha = |f| \dot{x}^\alpha = \text{sgn}(f) \epsilon^{\alpha\beta} E_\beta .
\]

The local character of the degeneracy surfaces \( \Sigma \), can be inferred from the flux of \( j^i \) across a pill box enclosing a portion of \( \Sigma \). The flux density \( \Phi = j^i n_i \) across the lids of the pill box is given by the projection of \( j^i \) along the normal to the surface \( n_i = \partial_i F^{1/2} \), whose only nonvanishing components are \( n_\alpha = \partial_\alpha |f| \), that is,

\[
\Phi = -F^{1/2} F^{ij} E_j \partial_i F^{1/2} = \partial_\alpha f \epsilon^{\alpha\beta} E_\beta .
\]

Note that \( \Phi \) is not only finite, but continuous on \( \Sigma \). Therefore,

**Lemma 3**: The local character of the degeneracy surfaces is given by \( \Sigma^{(\eta)} \) with \( \eta = \text{sgn}(\Phi) \). Furthermore, in general, \( \Sigma \) is globally piecewise attractive \( (\Sigma^{(-)}) \) or repulsive \( (\Sigma^{(+)}) \), and is of type \( \Sigma^{(0)} \) at the intersections with the surfaces \( \Pi = \{ z \in \Gamma / \Phi(z) = 0 \} \) (see Fig. 1.d).

Hence, \( \Sigma^{(0)} \) generically corresponds to the boundaries between \( \Sigma^{(-)} \) and \( \Sigma^{(+)} \) (that is, \( \Sigma^{(0)} = \partial \Sigma^{(-)} \)) which is a subset of codimension 2 in phase space.

In the particular case, when both surfaces \( \Sigma \) and \( \Pi \) coincide on an open set, \( \Sigma \) is globally of type \( \Sigma^{(0)} \). This occurs for example, if \( E_i |_{\Sigma^{(0)}} = \partial_1 (h(z^1)F^{1/2}) \), whose only nonvanishing components are of the form \( E_\alpha = \tilde{h}(z^\alpha) \partial_\alpha f \) for some functions \( h \) and \( \tilde{h} \neq 0 \).

**C. Evolution towards \( \Sigma^{(-)} \)**

The degeneracy surfaces \( \Sigma^{(+)} \) and \( \Sigma^{(-)} \) represent sets of initial and final states of the system, respectively. Configurations at a surface \( \Sigma^{(+)} \) are unstable against small perturbations, and it seems unlikely that a system could be prepared there. On the other hand, if one considers the system at \( \Sigma^{(-)} \), a small perturbation to move it away from the surface would require an infinite acceleration. In this sense, the surfaces \( \Sigma^{(-)} \) represent stable final states for the evolution of the system, and any initial configuration sufficiently near the degeneracy surface is doomed to fall on it. Then, the question whether the system can be consistently defined on \( \Sigma^{(-)} \) naturally arises.

For simplicity, let us consider a system possessing a single surface of degeneracy which is globally of type \( \Sigma^{(-)} \). We will now show that when the system reaches \( \Sigma^{(-)} \), two coordinates become non dynamical; the system acquires a new gauge symmetry on the degeneracy surface which corresponds to displacements along \( \Sigma^{(-)} \), and one degree of freedom is lost.
whose Poisson brackets are \( \{ \phi_i, \phi_j \} = F_{ij} \). Outside \( \Sigma^(-) \), the invertibility of \( F_{ij} \) implies that the constraints \( \phi_i \) are second class. However, at the degeneracy surface, the rank of \( F_{ij} \) is reduced by two, thus, two of the \( \phi_i \)'s have vanishing Poisson brackets with the whole set of constraints.

Although the constraint structure changes abruptly at \( \Sigma^(-) \), after the system reaches this surface, its evolution can be described by a standard constrained system, as can be seen through a suitable change of basis for the constraints \( \phi_i \).

Linear combinations of the form \( \varphi^{(\alpha)} = e^{(\alpha)}_i \phi_i \), become first class provided \( e^{(\alpha)}_i \) are null vectors of \( F_{ij} \). This can only happen at the degeneracy surface, where there are two of such vectors. They can be chosen so that one is tangent and the other is normal to the surfaces \( F = \text{constant} \), namely, \( e^{(1)}_{i}F_{ij} = \frac{1}{2}\partial_j F \) and \( e^{(2)}_{i}F_{ij} = F_{ij}\sqrt{F} \). In Darboux-like coordinates, their only nonvanishing components are \( e^{(1)}_{\alpha} = \epsilon^{\alpha\beta}\partial_\beta f \) and \( e^{(2)}_{\alpha} = \delta^{\alpha\beta}\partial_\beta f \), with \( \alpha = 1, 2 \).

In the basis \( \phi_i = \{ \varphi^{(\alpha)}; \phi_a \} \), with \( a = 3, ..., 2n \), the constraint algebra reads,

\[
\{ \varphi^{(\alpha)}; \varphi^{(\beta)} \} \approx \frac{1}{4} e^{(\alpha)}_i e^{(\beta)}_j F^{-\frac{1}{2}}(\partial_k F)^2 = f \epsilon^{(\alpha)(\beta)}|\partial f|^2 , \\
\{ \varphi^{(\alpha)}; \phi_a \} \approx e^{(\alpha)}_i F_{ib} = 0 , \\
\{ \phi_a; \phi_b \} = F_{ab} .
\]

From this it is apparent that, on the surface \( \Sigma^(-) \), the constraints \( \varphi^{(\alpha)} \) have vanishing Poisson brackets, and are therefore candidates for first class constraints.

In order to examine whether \( \varphi^{(\alpha)} \) are first or second class at the degeneracy surface \( (f = 0) \), it is necessary to compute their Poisson brackets with \( f \). The only non vanishing bracket involving \( f \) is

\[
\{ f; \varphi^{(2)} \} = e^{(2)}_i \partial_\alpha f = |\partial_\alpha f|^2 ,
\]

which cannot vanish on \( \Sigma \) because, by hypothesis, \( f \) has a simple zeros there. This shows that \( \varphi^{(1)} \) is first class, while \( (f, \varphi^{(2)}) \) form a conjugate pair of second class constraints.

The transformations generated by \( \varphi^{(\alpha)} \) correspond to \( \delta z^\alpha = 0 \), and

\[
\delta z^\alpha = \{ \xi^\beta; \varphi^{(\beta)} \} = \epsilon^{(\beta)}_\beta e^{(\alpha)}_\beta = \xi^\alpha .
\]

Thus, the constraints \( \varphi^{(1)} \) and \( \varphi^{(2)} \) generate tangent and normal displacements to \( \Sigma^(-) \) respectively, as expected. Hence, \( f \approx 0 \) can be viewed as the gauge fixing condition associated with the “gauge generator” \( \varphi^{(2)} \). This is summarized in the following

**Lemma 4:** On the degeneracy surface \( \Sigma^(-) \), the system acquires a new gauge invariance, because the second class constraint \( \varphi^{(1)} \) becomes first class, while the number of second class constraints \( (f, \varphi^{(2)}; \phi_a) \) remains the same \( (2n) \). Since each first class constraint eliminates one degree of freedom, we conclude that one degree of freedom is dynamically frozen on the degeneracy surface.

We illustrate these results in the following examples.

### IV. EXAMPLES

#### A. Simplest Degenerate System

The simplest case of a degenerate dynamical system is provided by the Lagrangian

\[
L_D = A_0 \dot{x}^2 + A_0 ,
\]

with \( A_1 = 0, A_2 = x_1 x_2, A_0 = -\nu x_1 \). The symplectic form, \( F_{\alpha\beta} = \epsilon_{\alpha\beta} x_2 \), degenerates at the surface \( x_2 = 0 \), which is of type \( \Sigma^{(0)} \), with \( \eta = \text{sgn}(\nu) \). The orbits run perpendicular to \( \Sigma^{(0)} \) and take a finite time to connect a point on the surface with a point outside.

This example captures the essence of the behavior of any degenerate system in a neighborhood of a degeneracy surface of type \( \Sigma^{(+)} \) or \( \Sigma^{(-)} \). In particular, the shock-wave solutions of Burgers’ equation,

\[
\partial_t u + u \partial_x u = \nu \partial_x^2 u ,
\]

which is relevant in the context of turbulence, exhibit this behavior. These solutions are of the form

\[
u(x, t) = -2\nu \sum_{k=1}^{2n} (x - z_k(t))^{-1} ,
\]

where \( z_k(t) \) are complex coordinates which come in conjugate pairs and satisfy a vortex-like equation [14]. The corresponding equations of motion for \( z_k(t) \) can be obtained from an action of the form [8], which for \( n = 1 \) and \( z = x_1 + i x_2 \) reads

\[
\left( \begin{array}{c} 0 & x_2 \\ -x_2 & 0 \end{array} \right) \left( \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right) = \left( \begin{array}{c} \nu \\ 0 \end{array} \right) ,
\]

whose associated Lagrangian, is precisely given by [11]. This solution describes a one dimensional shock wave centered at \( x = x_1 \), with peaks at \( x = x_1 \pm x_2(t) \) of height \( \mp 2\nu/x_2(t) \), travelling outwards from \( x_1 \).

#### B. Coupling with a regular system

The next example examines explicitly the fate of a degenerate system when it reaches a surface of type \( \Sigma^{(-)} \). A simple Lagrangian for which this occurs is of the form

\[
\phi_i(z, p) \equiv p_i - A_i(z, t) \approx 0 ,
\]

with \( A_1 = 0, A_2 = x_1 x_2, A_0 = -\nu x_1 \). The symplectic form, \( F_{\alpha\beta} = \epsilon_{\alpha\beta} x_2 \), degenerates at the surface \( x_2 = 0 \), which is of type \( \Sigma^{(0)} \), with \( \eta = \text{sgn}(\nu) \). The orbits run perpendicular to \( \Sigma^{(0)} \) and take a finite time to connect a point on the surface with a point outside.

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Here,
\[ L = L_D(x) + L_R(z) - V_\lambda(x, z) \, , \]
with $\alpha = 1, 2$, is some two-dimensional degenerate system possessing a global surface of type $\Sigma^{(-)}$ at $f(x^\alpha) = 0$; $L_R(z^\alpha)$ is a Regular system with Hamiltonian $H_R(z^\alpha)$, and $V_\lambda(x^\alpha, z^\alpha)$ is an interaction term of the form
\[ V_\lambda = \lambda f(x^\alpha)H_R(z^\alpha) \, . \]
This coupling is chosen so that it vanishes on $\Sigma^{(-)}$ and does not change the flux density $\Phi$ there, so that the character of the degeneracy surface does not depend on the coupling constant $\lambda$. Note that this coupling would be trivial in case of nondegenerate systems. Furthermore, the presence of $H_R$ in the coupling implies that, besides the conservation of the total Hamiltonian $H = H_D + H_R + V_\lambda$, the equations of motion
\[ z^\alpha = (1 + \lambda f(x))F^{ab}\partial_bH_R \, , \]
give rise to a separate conservation law for $H_R$, because $H_R = z^a\partial_aH_R = 0$. In turn, this implies that the remaining equations of motion
\[ \epsilon_{\alpha\beta}f(x)\dot{x}^\beta = \partial_\alpha(H_D + \lambda f(x)H_R) \, , \]
can be integrated as an autonomous two-dimensional subsystem. Once these equations have been solved, and their solutions substituted in (23), it is apparent that, the solutions of Eqs. (23) describe the same orbits as in the decoupled case ($\lambda = 0$) but with a reparametrized time,
\[ z^\alpha(t) = z^\alpha(\tau) \, , \]
with
\[ \frac{d\tau}{dt} = 1 + \lambda f(x(t)) \, . \]
Note that as the orbits approach the surface $\Sigma^{(-)}$, this time reparametrization remains finite.

Once the system reaches the degeneracy surface ($f(x) = 0$), both time coordinates become identical and, on $\Sigma^{(-)}$, all traces of the degenerate subsystem disappear, including the information about its initial conditions $x^\alpha(t_0)$.

Thus from the moment the degeneracy surface is reached, the system becomes a regular one, described by $L_R(z^\alpha)$, and the degrees of freedom of the degenerate system are forever lost.

In order to illustrate this point, consider the degenerate Lagrangian given by Eq. (14) with $\nu < 0$, coupled with a one dimensional harmonic oscillator in the form (22). In that case, the total energy is $E = \mathcal{E}_R(1 + \lambda x_2) + \nu x_1$, where $\mathcal{E}_R$ is the energy of the harmonic oscillator, which is separately conserved. Eq. (24) is readily integrated as
\[ x_2(t) = \pm \sqrt{2\nu t + (x_2(t_0))^2} \, , \]
for $t < \frac{(x_2(t_0))^2}{2\nu}$, and $x_2(t) = 0$ afterwards.

Hence, the harmonic oscillator coordinates $Z = z^1 + iz^2$ evolve according to
\[ Z(t) = Z_0 \exp(\lambda t) \, , \]
with $|Z_0|^2 = 2\mathcal{E}_R$, where the reparametrized time is given by
\[ \tau = t + \frac{\lambda}{3\nu}[2\nu t + (x_2(t_0))^2]^{3/2} \, , \]
for $t < \frac{(x_2(t_0))^2}{2\nu}$, and $\tau = t$ afterwards.

V. DISCUSSION & OVERVIEW

The degeneracy of the symplectic form opens up the possibility of a violation of Liouville’s theorem. In fact, the divergence of the current $j^i = \sqrt{\mathcal{F}}\dot{z}^i$ reads
\[ \partial_i j^i = -\partial_i[\sqrt{\mathcal{F}}]\partial_\alpha A_0 - \sqrt{\mathcal{F}}\partial_j\partial_\alpha A_0 \, . \]
If $A_0 = -H$ is continuous and differentiable, the second term in the r.h.s. vanishes identically. However, the first term can give rise to a non-zero contribution, responsible for the jump in the flow across $\Sigma$. In this sense, the problem we address here is the counterpart of Poincaré classical study of singularities in the phase flow. Both cases correspond to different classes of possible singularities in the phase flow, and hence, the degeneracy surfaces cannot be understood as a dense set of Poincaré’s singularities.

It is reasonable to expect that the extension of our analysis to field theory would lead to the possibility that the symplectic form degenerates for field configurations where some local degrees of freedom should freeze out and some field components become nondynamical. In the case of higher dimensional gravity, this means that as the system reaches a degeneracy surface, some dynamical components of the metric become redundant, which would correspond to a sort of dynamical dimensional reduction mechanism.

The quantum mechanical analysis of this kind of degenerate systems, shows that there is no tunneling across a surface of degeneracy $\Sigma$, but there is a nonvanishing propagation amplitude between states in the bulk and on $\Sigma$ [13]. These results would be relevant for the quantum Hall effect [16] and also for strings propagating in a background possessing a nonconstant B-field [17].

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