On the Phase Space in Double Field Theory

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Abstract

We present a model of (double) kinetic theory which paves the way to describe matter in a Double Field Theory background. Generalized diffeomorphisms acting on double phase space tensors are introduced. The generalized covariant derivative is replaced by a generalized Liouville operator as happens in relativistic kinetic theory. The section condition is consistently extended and the closure of the generalized transformations is still given by the C-bracket. In this context we propose a generalized Boltzmann equation and compute the moments of the latter, obtaining an expression for the generalized energy-momentum tensor and its conservation law.
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1 Introduction

The Einstein’s field equations,

\[ G_{\mu\nu} = T_{\mu\nu} \]  

are a fundamental intermediate step in order to describe the dynamics of matter coupled to gravity in a Riemannian d-dimensional background \((\mu = 0, \ldots, d - 1)\). The left hand side of the equation is given by the Einstein tensor, which is a divergenceless and symmetric tensor that depends on the geometric properties of the d-dimensional space-time and the right hand side of the equation is related to the matter and energy content of the system.

Kinetic theory is one usual way to describe matter from first principles. In this scheme the energy-momentum tensor is the second moment of the one-particle distribution function \( f = f(x, p) \) \([1, 2, 3]\),

\[ T^{\mu\nu} = \int \sqrt{g} d^d p \ p^\mu p^\nu f \]  

with \( p^\mu \) the d-momentum and \( g \) the determinant of the metric tensor. The evolution of \( f \) is given by the relativistic Boltzmann equation

\[ p^\mu D_\mu f = C[f] \]  

where \( C[f] \) is the collision term and \( D_\mu \) is the Liouville operator defined as

\[ D_\mu = \nabla_\mu - \Gamma^\sigma_{\mu\nu} p^\nu \frac{\partial}{\partial p^\sigma}. \]  

In the previous expression \( \nabla_\mu \) is the covariant derivative of the space-time using the Levi-Civita connection \( \Gamma^\sigma_{\mu\nu} \). For applications and extensions of the formalism see \([4]\). In appendix \([A]\) we include the conventions used throughout this work.

The Boltzmann equation \([1.3]\) describes the evolution of the number of particles in a given volume of the 2d-dimensional phase space, which in this context is defined as follows: for each position \( x \) on the d-manifold \( M \), a momentum space \( \mathbb{P}_x \) attached to \( M \) is introduced. Then the phase space is a collection \((x, \mathbb{P}_x)\) which defines a fiber bundle. On account of this the right hand side of the Einstein equation also holds a rich geometric structure that is part of the relativistic kinetic theory. For instance, the Liouville operator
$D_\mu$ can be understood as the covariant derivative of a phase-space since infinitesimal
diffeomorphisms for a generic phase-space tensor $v_\mu^\nu(x,p)$ receive an extra contribution
of the form \cite{5},
\begin{equation}
\delta_\xi v_\mu^\nu = L_\xi v_\mu^\nu + p^\rho \frac{\partial \xi^\sigma(x)}{\partial x^\rho} \frac{\partial v_\mu^\nu}{\partial p^\sigma},
\end{equation}
where $L_\xi$ is the Lie derivative acting on tensors defined as
\begin{equation}
L_\xi v_\mu^\nu = \xi^\sigma \frac{\partial v_\mu^\nu}{\partial x^\sigma} + \frac{\partial \xi^\rho}{\partial x^\mu} v_\nu^\rho - \frac{\partial \xi^\nu}{\partial x^\rho} v_\mu^\rho + \omega \frac{\partial \xi^\sigma}{\partial x^\sigma} v_\mu^\nu,
\end{equation}
$\omega$ is a weight constant and $\xi_\mu = \xi_\mu(x)$ is an infinitesimal parameter. The closure of (1.5)
is still given by the Lie bracket,
\begin{equation}
\xi^\mu_{12}(x) = \xi^\mu_1 \frac{\partial \xi^\nu_2}{\partial x^\rho} - (1 \leftrightarrow 2).
\end{equation}
On the other hand the energy-momentum tensor (1.2) is a symmetric and divergence-
less tensor related with the variation of the matter action $S_m$ with respect to the (inverse)
metric tensor as
\begin{equation}
T_\mu^\nu = \frac{-2}{\sqrt{g}} \delta S_m \delta g_\mu^\nu.
\end{equation}
Then, (1.1) is the equation of motion of the metric tensor when we couple matter to a
Riemannian background.

In this work we present a model of kinetic theory which paves the way to describe
matter in a Double Field Theory (DFT) background. DFT \cite{6,7,8} is a generalization of
Riemannian geometry which is manifestly invariant under the group $G = O(d,d)$. The
previous group is closely related with an exact symmetry of String Theory \cite{9}. However
since the dimension of the fundamental representation of $G$ is 2$d$, the ordinary space-time
must be doubled to accomplish $O(d,d)$ as a global symmetry of the theory \footnote{Check \cite{10} for reviews about DFT.}. The general-
ized coordinates of the double space $X^M = (x^\mu, \tilde{x}_\mu)$ are in the fundamental representation
of $G$, where $\tilde{x}_\mu$ is the extra set of coordinates and $M = 0, \ldots, 2d - 1$. Derivatives in the
double space are constrained by the section condition (or strong constraint),
\begin{equation}
\partial_M (\partial^M \ast) = (\partial_M \ast)(\partial^M \ast) = 0,
\end{equation}
where $\star$ means a product of arbitrary generalized fields. This constraint effectively removes the dependence on $\tilde{x}_\mu$.

The invariant metric of $G$ is

$$
\eta_{MN} = \begin{pmatrix}
0 & \delta^\nu_{\mu} \\
\delta^\mu_\nu & 0
\end{pmatrix}.
$$

(1.10)

This metric raises and lowers the indices $M, N, \ldots$ and is left invariant under generalized diffeomorphisms, generated infinitesimally by $\xi^M$ through the generalized Lie derivative, defined by

$$
\mathcal{L}_\xi V_M(X) = \xi^N \partial_N V_M(X) + (\partial_M \xi^N - \partial^N \xi_M) V_N(X) + \omega(\partial_N \xi^N) V_M(X),
$$

(1.11)

where $V_M(X)$ is an arbitrary (double) vector and $\omega$ is a weight constant. The closure of the generalized diffeomorphisms,

$$
[\delta_{\xi_1}, \delta_{\xi_2}] V^M(X) = \delta_{\xi_{21}} V^M(X)
$$

(1.12)

is given by the C-bracket,

$$
\xi^M_{12}(X) = \xi^P_1 \frac{\partial \xi^M_2}{\partial X^P} - \frac{1}{2} \xi^P_1 \frac{\partial \xi^P_2}{\partial X^M} - (1 \leftrightarrow 2).
$$

(1.13)

The field content of DFT consist of a dynamical metric $\mathcal{H}_{MN}(X) = \mathcal{H}_{MN}$, called the generalized metric and a scalar $d(X) = d$, called the generalized dilaton. The partial derivative of a vector in the double space does not transform like a vector. The generalized covariant derivative is defined as

$$
\nabla_M V_N = \partial_M V_N + \Gamma_{MN}^P V_P,
$$

(1.14)

and demanding compatibility with the invariant group, with the generalized metric and demanding that the generalized torsion vanishes is not enough to completely determine the generalized affine connection. However, the generalized Ricci tensor $\mathcal{R}_{MN}$ and the generalized Ricci scalar $\mathcal{R}$ are fully determined, and therefore an action principle can be defined in the following way,

$$
\int d^{2d}X e^{-2d} \left( \mathcal{R}(\mathcal{H}, d) + \mathcal{L}_m \right)
$$

(1.15)

where $\mathcal{L}_m$ is the matter Lagrangian.
1.1 Main results

The main goal of this work is to elaborate on the (double) phase space of DFT and to define an \( O(d, d) \) invariant kinetic theory which paves the way to describe matter in the double space.

We start defining a generalized momentum space \( \mathbb{P}_X \) attached to the double space. The double phase space is the collection

\[
\left\{ X^M, \mathcal{P}^M \right\}, \quad (1.16)
\]

where \( \mathcal{P}^M \) is an extra coordinate that is also an \( O(d, d) \) vector. The infinitesimal generalized diffeomorphisms acting on a double phase space vector \( V^Q = V^Q(X, \mathcal{P}) \) are defined as

\[
\delta_\xi V^Q = \mathcal{L}_\xi V^Q + \mathcal{P}^N \frac{\partial \xi^M}{\partial X^N} \frac{\partial V^Q}{\partial \mathcal{P}^M} - \mathcal{P}^N \frac{\partial \xi_N}{\partial X^M} \frac{\partial V^Q}{\partial \mathcal{P}^M}, \quad (1.17)
\]

where \( \mathcal{L}_\xi \) is the generalized Lie derivative and \( \xi = \xi(X) \) is an infinitesimal parameter. We demand a section condition for the momentum derivatives,

\[
\left( \frac{\partial}{\partial \mathcal{P}^M} \right)^* \left( \frac{\partial}{\partial \mathcal{P}^M} \right)^* = 0 \quad (1.18)
\]

and for the mixed derivatives,

\[
\left( \frac{\partial}{\partial X^M} \right)^* \left( \frac{\partial}{\partial \mathcal{P}^M} \right)^* = \left( \frac{\partial}{\partial X^M} \right)^* \left( \frac{\partial}{\partial \mathcal{P}^M} \right)^* = 0. \quad (1.19)
\]

Imposing the previous constraints, the last term of (1.17) vanishes. We solved the strong constraint of the double phase space with the following solution

\[
\frac{\partial}{\partial \mathcal{P}_\mu} = 0, \quad (1.20)
\]

which is enough to recover the usual phase space diffeomorphisms from the generalized ones and then \( \mathcal{P}^\mu = p^\mu \) is the d-momentum. The closure of (1.17),

\[
\left[ \delta_{\xi_1}, \delta_{\xi_2} \right] V^M(X, \mathcal{P}) = \delta_{\xi_{21}} V^M(X, \mathcal{P}) \quad (1.21)
\]

is given by the C-bracket, as it must be for DFT-diffeomorphisms,

\[
\xi^M_{12}(X) = \xi_1^P \frac{\partial \xi_2^M}{\partial X^P} - \frac{1}{2} \xi_1^P \frac{\partial \xi_2^P}{\partial X^M} - (1 \leftrightarrow 2). \quad (1.22)
\]
The covariant derivative of the double phase space is given by the generalized Liouville operator $\mathcal{D}_M$ defined as

$$\mathcal{D}_M = \nabla_M - \Gamma_{MN}^Q \mathcal{P}_N^Q \frac{\partial}{\partial \mathcal{P}_Q}.$$  \hfill (1.23)

We define a generalized distribution function $F$ as a function $F = F(X, \mathcal{P})$ that satisfies,

$$\mathcal{P}^M \mathcal{D}_M F = \mathcal{C}[F],$$  \hfill (1.24)

where $\mathcal{C}[F]$ is a generalized collision term. We assume the existence of an equilibrium state such that

$$\mathcal{C}[F_{eq}] = 0.$$  \hfill (1.25)

The integration of the product of (1.24) and a generic phase space object $\Psi^M$ leads to

$$\nabla_N \left[ \int \Psi^M \mathcal{P}^N F e^{-2d} d^2\mathcal{P} \right] - \int F \mathcal{P}^N \mathcal{D}_N \Psi^M e^{-2d} d^2\mathcal{P} - \int F \mathcal{P}^M \mathcal{P}^N \mathcal{U}_N e^{-2d} d^2\mathcal{P} = 0,$$

where

$$\mathcal{U}_N = \mathcal{U}_N(X) = -6 \frac{\partial d}{\partial X^N}.$$  \hfill (1.26)

If we set $\Psi^M$ to a constant, we find

$$\int \mathcal{P}^M F e^{-2d} d^2\mathcal{P} = \mathcal{N}^M$$  \hfill (1.27)

with the following conservation law for the generalized d-current $\mathcal{N}^M$,

$$\nabla_M \mathcal{N}^M = \mathcal{U}_M \mathcal{N}^M.$$  \hfill (1.28)

The previous law shows that the generalized dilaton acts as an effective force on an element of volume of the double phase space. On the other hand considering $\Psi^M = \mathcal{P}^M$ in (1.26) leads to the generalized energy-momentum tensor

$$\mathcal{T}^{MN}(X) = \int \mathcal{P}^M \mathcal{P}^N F e^{-2d} d^2\mathcal{P},$$  \hfill (1.29)

and its conservation law

$$\nabla_N \mathcal{T}^{MN} = -2\Gamma_{NQ}^M \mathcal{T}^{NQ} + \mathcal{U}_N \mathcal{T}^{MN},$$  \hfill (1.30)
which is fully determined in terms of the field content of DFT imposing

\[
\mathcal{T}^{MN} = \int (P^M P^N + \overline{P}^M \overline{P}^N) F e^{-2d} d^d P,
\]

(1.31)

where we use the standard notation for projected indices,

\[
V_M = P^N V_N \quad \text{and} \quad \overline{V}_M = \overline{P}^N V_N,
\]

(1.32)

with \( P_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}_{MN}) \) and \( \overline{P} = \frac{1}{2}(\eta_{MN} + \mathcal{H}_{MN}) \).

Similar to the case of the generalized d-current, the divergence of the generalized energy-momentum tensor does not vanish. Interestingly enough, since \( \delta \mathcal{R}/\delta \mathcal{H}^{MN} \) does not satisfy the properties of an Einstein tensor \([11]\), it is possible to consider an Einstein-type equation in DFT of the form

\[
\mathcal{G}_{MN} = \mathcal{T}_{M\overline{N}} + \mathcal{T}_{\overline{M}N}
\]

(1.33)

from a variational principle of the total action, namely

\[
\frac{\delta (S_{\text{vacuum, DFT}} + S_m)}{\delta \mathcal{H}^{MN}} = 0
\]

(1.34)

with

\[
\mathcal{G}_{MN} = \frac{1}{e^{-2d}} \frac{\delta S_{\text{vacuum, DFT}}}{\delta \mathcal{H}^{MN}} = \frac{\delta \mathcal{R}}{\delta \mathcal{H}^{MN}} = \mathcal{R}_{M\overline{N}} + \mathcal{R}_{\overline{M}N}
\]

(1.35)

and

\[
\mathcal{T}_{MN} = -\frac{1}{e^{-2d}} \frac{\delta S_m}{\delta \mathcal{H}^{MN}} = -\frac{\delta \mathcal{L}_m}{\delta \mathcal{H}^{MN}}.
\]

(1.36)

The mixed index structure of the generalized Ricci tensor and the generalized energy-momentum tensor in (1.33) and (1.35) comes from the fact that the generalized metric is an element of \( O(d, d) \) and therefore its variation is constrained. Finally, we understand (1.30) as a constraint on \( \mathcal{L}_m \) which might be a first step to describe an \( O(d,d) \) invariant formulation of matter in the context of DFT.
1.2 Outline

This work is organized as follows: In section 2 we review the main aspects of general relativistic kinetic theory. We focus on the construction of the double phase space and its symmetries. Special attention is put on the Liouville operator, that is defined as the natural extension of the covariant derivative. The latter allows to construct the relativistic Boltzmann equation and to inspect the properties of the current of particles and the energy-momentum tensor. Section 3 is dedicated to review the basic aspects of the geometry of DFT. We start defining generalized diffeomorphisms through a generalized Lie derivative and then we introduce a generalized affine connection that unlike general relativity is undetermined. After that we review the differential Bianchi identities and discuss about the tension between the generalized Einstein tensor and the generalized energy-momentum tensor. In section 4 we elaborate on kinetic theory on the double space. We start giving a consistent deformation of the generalized diffeomorphisms with a section condition for the generalized d-momentum derivatives. The bracket of the previous transformations is the C-bracket, but the covariant derivative takes the form of the generalized Liouville operator. After that we present a generalized Boltzmann equation and a generalized transfer equation, and we obtain the conservation laws for the generalized d-current and the generalized energy-momentum tensor. Finally in 5 we conclude the work elaborating on open questions and some future directions.

2 Relativistic Kinetic Theory

2.1 Basics

We start with a d-manifold $M$ with coordinates $x^\mu$, $\mu = 0, \ldots, d$, equipped with a metric tensor $g_{\mu\nu}$ and a Levi-Civita connection $\Gamma^\rho_{\mu\nu}$. For each point $x$ on $M$ with coordinate $x^\mu$, we introduce its tangent space $\mathbb{P}_x$ whose vectors are the d-momenta $p^\mu$. In consequence the phase space is a collection $(x, \mathbb{P}_x)$ which defines a tangent bundle \cite{5, 12}. From this point of view, the d-momentum can be considered independent of the position and thus

$$\frac{\partial p^\nu}{\partial x^\mu} = 0.$$ (2.1)
This condition holds in an off-shell formulation of the general relativistic kinetic theory which is the scenario that we will deal with. Conversely the on-shell condition $p^\mu p_\mu = m^2$ spoils the independence between momenta itself and with position coordinates.

Let us observe that coordinate transformations on $M$, induce transformations in the fiber and therefore the infinitesimal diffeomorphisms of a phase space scalar $v$ with constant weight $\omega$ can be written as

$$
\delta_\xi v = L_\xi v + \rho^\rho \frac{\partial \xi^\sigma(x)}{\partial x^\rho} \frac{\partial v}{\partial p^\sigma}, \quad (2.2)
$$

where $L_\xi$ is the usual Lie derivative defined as

$$
L_\xi v = \xi^\sigma \frac{\partial v^\nu}{\partial x^\sigma} + \omega \frac{\partial \xi^\sigma}{\partial x^\sigma} v, \quad (2.3)
$$

with $\xi^\mu = \xi^\mu(x)$ an infinitesimal parameter that characterizes the transformation. We may extend (2.2) to tensors by taking the usual Lie derivative acting on different tensor structures, e.g. for a $(1,1)$ tensor we have

$$
L_\xi v_{\mu}^{\nu} = \xi^\sigma \frac{\partial v_{\mu}^{\nu}}{\partial x^\sigma} + \frac{\partial \xi^\rho}{\partial x^\mu} v_{\rho}^{\nu} - \frac{\partial \xi^\nu}{\partial x^\rho} v_{\mu}^{\rho} + \omega \frac{\partial \xi^\sigma}{\partial x^\sigma} v_{\mu}^{\nu}. \quad (2.4)
$$

It is straightforward to check the closure of the transformation (2.4),

$$
\left[\delta_{\xi_1}, \delta_{\xi_2}\right] v_{\mu}^{\nu} = \delta_{\xi_{21}} v_{\mu}^{\nu} \quad (2.5)
$$

and show that the bracket is given by the Lie Bracket,

$$
\xi_{12}^\mu(x) = \xi_1^\rho \frac{\partial \xi_2^\mu}{\partial x^\rho} - (1 \leftrightarrow 2). \quad (2.6)
$$

Since we have tensors acting on the phase space we need to define a natural extension of the covariant derivative in the phase space, namely the Liouville operator $D_\mu$ (see e.g. Appendix A of [13]). Regarding that we have taken the collection $(x^\mu, p^\mu)$ to be the basis of the phase space, the Liouville operator for an arbitrary tensor reads

$$
D_\mu A^{\rho\lambda}(x,p) = \nabla_\mu A^{\rho\lambda}(x,p) - \Gamma^\sigma_{\mu\nu} p^\nu \frac{\partial A^{\rho\lambda}(x,p)}{\partial p^\sigma}, \quad (2.7)
$$

where $\nabla_\mu$ is the well-known covariant derivative. In particular it satisfies

$$
D_\mu p^\nu = D_\mu p_\nu = 0. \quad (2.8)
$$
Finally the diffeomorphism invariant volume element of the phase space is the product of the coordinate and momentum invariant volume elements, namely

$$
\sqrt{gd^d p} \sqrt{gd^d x} = gd^d p d^d x ,
$$

with $g$ the determinant of the metric tensor.

### 2.2 The relativistic Boltzmann equation

The relativistic Boltzmann equation rules the evolution of the one-particle distribution function (1pdf) $f = f(x,p)$, which is a phase space scalar. In its simplest form this equation is

$$
p^\mu D_\mu f = C[f] .
$$

The right hand side of (2.10) is the collision term which takes into account the non-gravitational interactions between particles. If an equilibrium state is achieved the 1pdf takes its equilibrium form $f = f_{eq}$ and $C[f_{eq}] = 0$.

In this context we want to extract the geometric properties of the first and second momentum of the Boltzmann equation in order to present the so-called transfer equations for the particle current and the energy-momentum tensor. We start by integrating the product of the relativistic Boltzmann equation and an arbitrary 1-index object of the phase space $\Psi^\nu(x,p)$, over the phase space, i.e.,

$$
\int \Psi^\nu (p^\mu \frac{\partial f}{\partial x^\mu} - \frac{\partial f}{\partial p^\sigma} \Gamma^\sigma_{\mu\rho} p^\mu p^\rho ) gd^d p d^d x = \int \Psi^\nu C(f,f) gd^d p d^d x .
$$

When considering an equilibrium state the RHS of (2.11) is vanishing as we have mentioned above. The LHS of (2.11) can be simplified using the Leibniz rule, the equation

$$
p^\mu \frac{\partial g}{\partial x^\mu} - \frac{\partial}{\partial p^\rho} (g \Gamma^\sigma_{\mu\rho} p^\mu p^\nu) = 0 ,
$$

which follows from (A.4), and the divergence theorem

$$
\int \frac{\partial}{\partial p^\rho} (f \Psi^\nu \Gamma^\sigma_{\mu\rho} p^\rho ) gd^d p d^d x = 0 .
$$
Thus we obtain the following conservation laws, independently of the integration on $\sqrt{g} d^d x$,
\[
\nabla_\mu \left[ \int \Psi^\nu p^\mu f \sqrt{g} d^d p \right] - \int f p^\mu \nabla_\mu \Psi^\nu \sqrt{g} d^d p + \int \frac{\partial \Psi^\nu}{\partial p^\sigma} \Gamma^\rho_{\mu\sigma} p^\mu p^\sigma f \sqrt{g} d^d p = 0. \quad (2.14)
\]
If we set the arbitrary function to a constant scalar, we have
\[
\int p^\mu f(x, p) \sqrt{g} d^d p = N^\mu(x) \quad (2.15)
\]
with the usual conservation law or transfer function for the particle current,
\[
\nabla_\mu N^\mu = 0. \quad (2.16)
\]
If instead we take $\Psi^\nu = p^\nu$, since $p^\nu$ is in the kernel of $D_\mu$, the remaining terms in (2.14) cancel out and we finally get the expression of the energy-momentum tensor
\[
\int p^\mu p^\nu f(x, p) \sqrt{g} d^d p = T^\mu_\nu(x), \quad (2.17)
\]
and its conservation law
\[
\nabla_\mu T^\mu_\nu = 0. \quad (2.18)
\]
Interestingly enough, if we left the equilibrium state, the conservation laws (2.16) and (2.18) still holds since the zeroth and first order moment of the collision term are vanishing for any $f$. This is so due to $p^\nu$ is a collisional conserved quantity or, in other words, a summational invariant [3].

3 Double Field Theory

3.1 Double space and generalized fields

The geometry of DFT is based on a double space equipped with two metrics. On the one hand, we have the invariant metric of $O(d, d)$, $\eta_{MN}$, where $2d$ is the amount of dimensions of the theory. The indices $M, N, \ldots$ are in the fundamental representation of $O(d, d)$ and are raised and lowered with $\eta^{MN}$ and $\eta_{MN}$ respectively. On the other hand we have the generalized metric $H_{MN}$ that encodes the field content of the universal NS-NS sector of
the low energy effective superstring theory, namely, a metric tensor $g_{\mu \nu}$, a Kalb Ramond field $b_{\mu \nu} = -b_{\nu \mu}$ and the dilaton $\phi$. The generalized metric is an element of $O(d, d)$ and therefore satisfies

$$H_{MP}\eta^{PQ}H_{QN} = \eta_{MN}.$$  \hspace{1cm} (3.1)

The main purpose of DFT is to define a theory manifestly invariant under $O(d, d)$, which is closely related with a symmetry of String Theory [9]. Because of that, all the DFT fields and parameters are $O(d, d)$ multiplets or group-invariant objects. Since the dimension of the fundamental representation of $O(d, d)$ is $2d$, the coordinates of DFT are $X^M = (x^\mu, \tilde{x}_\mu)$. The coordinates $\tilde{x}_\mu$ are known as the dual coordinates and are taken away imposing the strong constraint,

$$\partial_M(\partial^M \star) = (\partial_M \star)(\partial^M \star) = 0,$$  \hspace{1cm} (3.2)

where $\star$ means a product of arbitrary generalized fields. In this section the notation for the derivatives is not ambiguous and $\partial_M = \frac{\partial}{\partial X^M}$. Using the previous constraint, the components of the fields of DFT depend only on $x^\mu$. The parametrization of the invariant metric is,

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^\mu_{\nu} \\ \delta^\nu_{\mu} & 0 \end{pmatrix},$$  \hspace{1cm} (3.3)

while the parametrization of the generalized metric is

$$H_{MN} = \begin{pmatrix} g^{\mu \nu} & -g^{\mu \sigma}b_{\sigma \nu} \\ b_{\mu \sigma}g^{\sigma \nu} & g_{\mu \nu} - b_{\mu \sigma}g^{\sigma \rho}b_{\rho \nu} \end{pmatrix}.$$  \hspace{1cm} (3.4)

It is straightforward to check that the previous parametrization satisfies (3.1).

In addition to the global $O(d, d)$ symmetry, DFT is invariant under generalized diffeomorphisms, generated infinitesimally by $\xi^M$ through the generalized Lie derivative, defined by

$$L_\xi V_M = \xi^N \partial_N V_M + (\partial_M \xi^N - \partial^N \xi_M)V_N + \omega(\partial_N \xi^N)V_M,$$  \hspace{1cm} (3.5)

where $V_M$ is an arbitrary vector and $\omega$ is a weight constant. This expression is trivially extended to other tensors with different index structure. For example, the generalized metric is a generalized tensor with vanishing weight and $\eta_{MN}$ is trivially invariant.
The field content of DFT consists in a generalized dilaton $d$ in addition to the generalized metric. The former transforms as a tensor with weight $\omega = 1$ and thus

$$\delta_{\xi}(e^{-2d}) = \partial_P(\xi^P e^{-2d}), \quad (3.6)$$

which means that $e^{-2d}$ transforms as a density and $e^{-2d}dX$ defines the invariant volume of DFT.

Similarly to general relativity, the partial derivative of a tensor does not transform as a tensor and therefore a covariant derivative must be included. We develop this issue in the next part of the work.

### 3.2 Generalized affine connection and torsion

Having defined a generalized Lie derivative, it is natural to seek a covariant derivative. The later is defined as

$$\nabla_M V_N = \partial_M V_N + \Gamma_{MN}^P V_P, \quad (3.7)$$

with trivial extension to tensors with more indices. Here we have introduced a generalized affine connection $\Gamma_{MN}^P$ whose transformation properties must compensate the failure of the partial derivative of a tensor to transform covariantly under generalized diffeomorphisms.

We can now demand some properties on the connection, namely:

- **Compatibility with $\eta_{MN}$:**

  $$\nabla_M \eta_{NP} = 0, \quad (3.8)$$

  and then the generalized affine connection is antisymmetric in its last two indices, i.e.

  $$\Gamma_{MN}^Q \eta_{QP} = \Gamma_{MNP} = -\Gamma_{MPN}. \quad (3.9)$$

- **Compatibility with $H_{MN}$:**

  $$\nabla_M H_{NP} = 0. \quad (3.10)$$
In order to discuss this item is convenient to define $O(d,d)$ projectors,

$$P_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}_{MN}) \quad \text{and} \quad \overline{P}_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{H}_{MN}) , \quad (3.11)$$

which satisfy the following properties

$$\overline{P}_{MQ} P_{QN} = P_{MN}, \quad P_{MN} P_{QN} = P_{MN}, \quad P_{MQ} \overline{P}^Q_N = 0, \quad \overline{P}_{MN} + P_{MN} = \eta_{MN}. \quad (3.12)$$

Using the previous projectors, an arbitrary vector $V_M$ can be decomposed as

$$V_M = V_M + V_{\overline{M}} = P_M V_N + \overline{P}_M V_N . \quad (3.13)$$

Then, the projections $\Gamma_{MNP}$ and $\Gamma_{MNP}$ remains undetermined after imposing $\nabla_M P_{NP} = 0$ and $\nabla_M \overline{P}_{NP} = 0$.

- Partial integration in the presence of the generalized density $e^{-2d}$:

$$\int e^{-2d} V \nabla_M V^M d^2X = - \int e^{-2d} V^M \nabla_M V d^2X \quad (3.14)$$

for arbitrary $V$ and $V^M$. The previous item forces

$$\Gamma_{MN}^M = 2\partial_N d . \quad (3.15)$$

- Vanishing torsion:

$$\Gamma_{[MNP]} = T_{MNP} = 0 . \quad (3.16)$$

Let us observe that the generalized torsion $T_{MNP}$ is antisymmetric in all its indices and transforms as a tensor (unlike $\Gamma_{[MN]P}$).

As we have showed, while in general relativity demanding metric compatibility and vanishing torsion determines the connection completely (the affine connection turns to the Levi-Civita connection), in this approach of DFT these requirements turn out to leave undetermined components of the generalized version of the affine connection [14]. At this point it is important to mention that there exists an equivalent approach to DFT known as semi-covariant formalism in which the generalized connection is fully determined [8].
In the present approach, the generalized Riemann tensor of the theory is undetermined but it is possible to take traces on it and obtain a generalized Ricci scalar \( R \) that is fully determined as a function of the generalized metric and the generalized dilaton,

\[
R = \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} + 4 \mathcal{H}^{MN} \partial_M \partial_N d + 4 \mathcal{H}^{MN} \partial_N d - 4 \mathcal{H}^{MN} \partial_M d \partial_N d - \partial_M \partial_N H^{MN}.
\]  

(3.17)

In the next part of the work we discuss the differential Bianchi identities of DFT and its relation with the generalized energy-momentum tensor.

### 3.3 Generalized energy-momentum tensor and differential Bianchi identities

The action principle of DFT is

\[
S = \int d^2 X e^{-2d} \left( R(\mathcal{H}, d) + \mathcal{L}_m \right),
\]  

(3.18)

where \( \mathcal{L}_m \) represents matter coupled to the vacuum solution of the double space. Using \( \mathcal{L}_m \), parametrizing the generalized metric as showed in \( \mathcal{H} \) and the generalized dilaton as

\[
e^{-2d} = \sqrt{g} e^{-2\phi},
\]  

(3.19)

and imposing the strong constraint, the DFT action reduces to the following action,

\[
S = \int d^d x \sqrt{g} e^{-2\phi} \left( R + 4(\partial \phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + L_m \right),
\]  

(3.20)

where \( H_{\mu\nu\rho} = 3 \partial_{[\mu} b_{\nu\rho]} \) is the curvature of the Kalb-Ramond field and \( L_m \) is the parametrization of the matter terms in \( \mathcal{L}_m \). The variation of the previous action with respect to \( g^{\mu\nu} \) leads to

\[
\frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{g} e^{-2\phi} \left( G_{\mu\nu} + 4 \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4} H_{\mu\sigma\rho} H^{\nu\sigma\rho} + \frac{\delta L_m}{\delta g^{\mu\nu}} \right)
\]

\[
- \frac{1}{2} g_{\mu\nu} (4(\partial \phi)^2 - \frac{1}{12} H_{\epsilon\lambda\rho} H^{\epsilon\lambda\rho} + L_m) = 0,
\]  

(3.21)

where

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R
\]  

(3.22)
is the Einstein tensor. The action (3.20) has an ambiguity with respect to the terms $4(\partial \phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}$. These terms can be consider as part of the matter Lagrangian or as a vacuum solution. In this work we consider the latter and we define

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} + 4\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{4}H_{\mu\sigma\rho}H^{\nu\sigma\rho} - \frac{1}{2}g_{\mu\nu}(4(\partial \phi)^2 - \frac{1}{12}H_{\epsilon\lambda\rho}H^{\epsilon\lambda\rho}) \quad (3.23)$$

as an effective Einstein tensor and a generalized Einstein equation

$$\tilde{G}_{\mu\nu} = T_{\mu\nu} \quad (3.24)$$

with

$$T_{\mu\nu} = \frac{-1}{\sqrt{g}e^{-2\phi}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{1}{2}g_{\mu\nu}L_m - \frac{\delta L_m}{\delta g^{\mu\nu}} \quad (3.25)$$

and $S_m$ the matter action. Disregarding the other EOMs coming from the variation of the action (3.20) with respect to the rest of the field content, namely $\phi$ or $H_{\mu\nu\rho}$, we find that

$$\nabla_\mu \tilde{G}^{\mu\nu} \neq 0. \quad (3.26)$$

Analogously to the previous case, we consider a set of EOMs from the variational principle of (3.18) with respect to the generalized metric tensor as

$$\frac{\delta S}{\delta H^{MN}} = 0 \quad (3.27)$$

which implies an Einstein-type equation

$$G_{MN} = T_{MN} \quad (3.28)$$

with

$$G_{MN} = \frac{\delta R}{\delta H^{MN}} = R_{MN} + R_{M\overline{N}} \quad (3.29)$$

and the definition of the generalized energy-momentum tensor,

$$T_{MN} = \frac{-1}{e^{-2\phi}} \frac{\delta S_m}{\delta H^{MN}} = -\frac{\delta L_m}{\delta H^{MN}} \quad (3.30)$$

Due to the normalization of the vacuum action, in GR it is usual to define the energy-momentum tensor as (1.8) which is twice (3.25). Since we do not actually parametrize $L_m$, in the eventual case of a comparison we may double it.
Since $\mathcal{H}_{MN}$ is a constrained field, the generalized energy-momentum tensor satisfies

$$\mathcal{T}^{MN} = \mathcal{T}^{MN} + \mathcal{T}^{\overline{M}N}.$$  \hfill (3.31)

On the other hand, the differential Bianchi identities of DFT reads $[14]$,\footnote{14}

$$\nabla_\mathcal{L} \mathcal{R} - 4\nabla^{\mathcal{M}} \mathcal{R} \mathcal{L}_M = 0$$
$$\nabla_\mathcal{T} \mathcal{R} + 4\nabla^{\mathcal{M}} \mathcal{R} \mathcal{T}_M = 0.$$  \hfill (3.32)

We notice that

$$\nabla_M \mathcal{G}^{MN} = -\frac{1}{4} \mathcal{H}^{NP} \nabla_P \mathcal{R},$$  \hfill (3.33)

and we do not necessarily expect a conservation law of the form

$$\nabla_M \mathcal{T}^{MN} = 0,$$  \hfill (3.34)

from the (double) kinetic side according to (3.26).

Although it is possible to define a generalized tensor $\mathcal{G}^{MN}$, by adding the Bianchi identities $[3.32]$, such that

$$\nabla_M \mathcal{G}^{MN} = 0$$  \hfill (3.35)

as was shown in $[11]$. In this work, we consider equation $[3.35]$ as a constraint on $\mathcal{L}_m$ and we propose the following relation (cf. 3.28),

$$\mathcal{G}_{MN} - \mathcal{T}_{MN} - \mathcal{T}_{\overline{M}N} = 0$$  \hfill (3.36)

and consequently

$$\nabla_M \mathcal{G}^{MN} = \nabla_M \mathcal{T}^{MN}.$$  \hfill (3.37)

The main goal of this work is to compute the RHS of (3.37) from a (double) kinetic theory.

4 Double Kinetic Theory

4.1 The double phase space

Similarly to the general relativistic kinetic formalism summarized in the Section 2, we define the notion of double phase space as an extension of the double space,

$$\{X^M, \mathcal{P}^M\},$$  \hfill (4.1)
where for each point of the double space we consider its double tangent space whose vectors are the generalized momenta $\mathcal{P}^M$. Further the momenta $\mathcal{P}^M$ are $O(d, d)$ vectors. It is well known that the double space is not a (double) manifold and in consequence we need to demand

$$\frac{\partial \mathcal{P}^M}{\partial X^N} = 0. \quad (4.2)$$

We set a particular deformation of the generalized diffeomorphisms,

$$\delta \xi V^Q(X, \mathcal{P}) = \mathcal{L}_\xi V^Q(X, \mathcal{P}) + \mathcal{P}^N \frac{\partial \xi^M}{\partial X^N} \frac{\partial V^Q(X, \mathcal{P})}{\partial \mathcal{P}^M} - \mathcal{P}^N \frac{\partial \xi^N}{\partial X^M} \frac{\partial V^Q(X, \mathcal{P})}{\partial \mathcal{P}^M}, \quad (4.3)$$

where $\xi^M = \xi^M(X)$ and $V^Q(X, \mathcal{P})$ is a generic vector of the double phase space. The expression (4.3) is trivially extended to tensors in the double phase space with different index structure. We demand a section condition for the momentum derivatives,

$$\left(\frac{\partial}{\partial \mathcal{P}^M}\right) \left(\frac{\partial}{\partial \mathcal{P}^M}\right) = \frac{\partial}{\partial \mathcal{P}^M} \left(\frac{\partial}{\partial \mathcal{P}^M}\right) = 0 \quad (4.4)$$

and for the mixed derivatives,

$$\left(\frac{\partial}{\partial X^M}\right) \left(\frac{\partial}{\partial \mathcal{P}^M}\right) = \frac{\partial}{\partial X^M} \left(\frac{\partial}{\partial \mathcal{P}^M}\right) = 0. \quad (4.5)$$

Imposing the previous constraint and its simplest solution

$$\frac{\partial}{\partial \mathcal{P}^\mu} = 0 \quad (4.6)$$

is enough to recover the usual phase space diffeomorphisms (2.2) from (4.3) and then $\mathcal{P}^\mu = p^\mu$ is the ordinary d-momentum.

The closure of (4.3),

$$\left[\delta_{\xi_1}, \delta_{\xi_2}\right] V^M(X, \mathcal{P}) = \delta_{\xi_{21}} V^M(X, \mathcal{P}) \quad (4.7)$$

is given by the C-bracket,

$$\xi_{12}^M(X) = \xi_1^P \frac{\partial \xi_2^M}{\partial X^P} - \frac{1}{2} \xi_1^P \frac{\partial \xi_2^P}{\partial X^M} - (1 \leftrightarrow 2), \quad (4.8)$$

as in ordinary DFT (See appendix B).
4.2 The generalized Liouville operator

The new terms that we add to the standard generalized diffeomorphisms now force us to define the natural extension of the covariant derivative of the double phase space, namely the generalized Liouville operator. Inspecting the form of (4.3) and the transformation rule of $\Gamma_{MNP}$ it is possible to conclude that this operator must be

$$D_M = \nabla_M - \Gamma_{MN}^Q \partial_P \frac{\partial}{\partial P^Q},$$

(4.9)

if we want that the covariant derivative of a phase space scalar transforms as a phase space scalar. In (4.9) $\nabla_M$ is the DFT covariant derivative defined in (3.7).

Further we define the generalized distribution function $F$ as a function $F = F(X,P)$ and, analogously to (2.10), we propose that its evolution equation given by the generalized Boltzmann equation that reads

$$\mathcal{P}^M D_M F = \mathcal{C}[F],$$

(4.10)

with $\mathcal{C}[F]$ a generalized collision term. Let us observe that in this case $\mathcal{P}_M$ is not in the kernel of $D_M$. On the other hand, the invariant volume of the double phase space must be

$$e^{-2d} d^{2d} X e^{-2d} d^{2d} P = e^{-4d} d^{2d} X d^{2d} P.$$

(4.11)

As we did in the Section 2, we shall compute the transfer equations by an analogous procedure but in this generalized DFT scheme. Let us integrate the product of a generic 1-index object $\Psi^M$ with the equation presented in (4.10) assuming the existence of an equilibrium state such that $F = F_{eq}$ and $\mathcal{C}[F_{eq}] = 0$, so

$$\int \Psi^M \mathcal{P}^N \mathcal{D}_N F_{eq} e^{-4d} d^{2d} X d^{2d} P = \int \Psi^M \mathcal{C}[F_{eq}] e^{-4d} d^{2d} P d^{2d} X = 0.$$

(4.12)

Again the LHS of (4.12) can be simplified using the Leibniz rule, the equation

$$e^{-4d} \mathcal{P}^N \left( 4 \frac{\partial d}{\partial X^N} + \Gamma_{MN}^Q \delta_M^Q \right) = -e^{-4d} \mathcal{P}^N \mathcal{U}_N,$$

(4.13)

where

$$\mathcal{U}_N = -6 \frac{\partial d}{\partial X^N}.$$

(4.14)
and the divergence theorem. The final expression is independent of the integration over the coordinate volume $e^{-2d}d^{2d}X$, and it reads

$$\nabla_N \left[ \int \Psi^M \mathcal{P}^N \mathcal{F}_{eq} e^{-2d} d^{2d} \mathcal{P} \right] - \int \mathcal{F}_{eq} \mathcal{P}^N \mathcal{D}_N \Psi^M e^{-2d} d^{2d} \mathcal{P} - \int \mathcal{F}_{eq} \Psi^M \mathcal{P}^N \mathcal{U}_N e^{-2d} d^{2d} \mathcal{P} = 0.$$ (4.15)

If we set the arbitrary function $\Psi^M$ to a constant scalar, we may define, again by analogy with the Section 2, the generalized particle current as

$$\mathcal{N}^M = \int \mathcal{P}^M \mathcal{F}_{eq} e^{-2d} d^{2d} \mathcal{P}$$ (4.16)

with the following conservation law

$$\nabla_M \mathcal{N}^M = \mathcal{U}_M \mathcal{N}^M.$$ (4.17)

Considering $\Psi^M = \mathcal{P}^M$ in (4.15) leads to the generalized energy-momentum tensor

$$\mathcal{T}^{MN}(X) = \int \mathcal{P}^M \mathcal{P}^N \mathcal{F}_{eq} e^{-2d} d^{2d} \mathcal{P}$$ (4.18)

and its conservation law

$$\nabla_N \mathcal{T}^{MN} = -2 \Gamma^M_{NQ} \mathcal{T}^{NQ} + \mathcal{U}_N \mathcal{T}^{MN},$$ (4.19)

which is completely determined in terms of the field content of DFT. As was previously discussed, the energy-momentum tensor can be projected in the following way,

$$\mathcal{T}^{MN} = \int (\mathcal{P}^M \mathcal{P}^N + \mathcal{P}^N \mathcal{P}^M) \mathcal{F}_{eq} e^{-2d} d^{2d} \mathcal{P},$$ (4.20)

since the generalized projectors do not depend on the generalized momentum.

The previous results show that the role of a non-vanishing generalized dilaton in DFT introduce an effective force in an element of volume of the double phase space and then the divergence of the generalized current of particles is not vanishing as well as the divergence of the energy-momentum tensor. On the other hand, the relations (4.17) and (4.19) must be treated as constraints on the Lagrangian matter of DFT considering an Einstein-type equation in DFT of the form (3.28)

$$\mathcal{G}_{MN} = \mathcal{T}_{MN},$$ (4.21)

from a variational principle of the total action (3.18).
5 Outlook

We present a model of kinetic theory in the context of DFT. We define a double phase space where tensors depend both on the generalized coordinates $X^M$ and the generalized d-momentum $P^M$. Generalized diffeomorphisms on the phase space are consistently deformed and the covariant derivative is replaced by a generalized Liouville operator, as happens in ordinary general relativistic kinetic theory. The closure of the transformations is still given by the C-bracket. The previous formalism allows us to introduce the analogue of the Boltzmann equation for a generalized distribution function which describes the evolution of the number of particles in a volume element of the double phase-space. From the previous equation we extracted the conservation laws of the generalized d-current and energy-momentum tensor, which strongly constrains the matter Lagrangian of DFT.

The results of this work open the door to a large number of questions and future directions. We elaborate on some important points:

(i) **Collisions in double space**

The conservation laws of the ordinary d-current $N^\mu$ and energy-momentum tensor $T^{\mu\nu}$ in GR still holds out of the equilibrium state. The key point of this statement is that these conservation laws come from the transfer equation considering a collisional invariant quantity, the momentum $p^\mu$ for the procedure of taking moments. It is possible that a T-duality invariant treatment of collisions in the double space could be captured using the framework presented here and we expect a generalized notion of collision-invariants related to the generalized d-momentum.

(ii) **H-theorem and thermodynamics**

The transfer equation,

$$\int \Psi^\nu (p^\mu \frac{\partial f}{\partial x^\mu} - \frac{\partial f}{\partial p^\sigma} \Gamma^\nu_{\mu\rho} p^\rho)gd^d pd^d x = \int \Psi^\nu C[f]gd^d pd^d x, \quad (5.1)$$

allows to define an entropy current $S^\mu$ considering

$$\Psi^\mu \propto - \ln \left( \frac{f^{1/3}}{g_s} \right) \quad (5.2)$$
where $h$ is the Planck constant and $g_s$ the degeneracy factor. The previous statement is known as the H-theorem [3]. In this context the conservation law of the entropy current,

$$\nabla_\mu S^\mu \geq 0$$  \hspace{1cm} (5.3)

is understood as the second law of thermodynamics. Finding a T-duality generalization of the previous results using the generalized transfer equation (4.12) could be an interesting direction to continue the present work. We expect that the previous treatment allows to establish equivalences between our model and [15].

(iii) T-duality invariant hydrodynamic formalism

The energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (e + p)u^\mu u^\nu + pg^{\mu\nu},$$  \hspace{1cm} (5.4)

where $e$ is the internal energy density and $p$ is the pressure. The previous expression is easily obtained considering a variational principle for an Einstein-Hilbert action coupled to a matter Lagrangian of the form [2],

$$L_m \propto p.$$  \hspace{1cm} (5.5)

Finding the explicit form of this Lagrangian from a DFT point of view is an open program that we present in order to obtain a T-duality invariant hydrodynamic formalism.

(iv) Double Cosmology

The construction of a low energy effective string cosmology based on DFT is a promising area of work [16]. One interesting aspect of this kind of approaches is that the inclusion of the dual coordinates $\hat{x}$ provides that the cosmological singularities of a homogeneous and isotropic universe may disappear thanks to the T-duality symmetry. An apparent big bang singularity in the ordinary supergravity framework is (T-)dual to an expanding universe in the dual dimensions when the section condition is suitably applied. The present work could be an intermediate step in finding a manifestly T-duality invariant energy-momentum tensor for a perfect fluid in the double-space which could give a description of geometry and matter in the double cosmology through a variational principle (see also [17]).
(v) **Generalized distribution functions and equilibrium**

The equilibrium states and their properties are very well-known in GR [3], particularly the equilibrium 1pdf as a function of the momenta and different lagrange multipliers. The generalization of the latter to the double space could be a great step in the description of matter in DFT and it would allow us to explicitly evaluate the generalized particle current, the generalized energy-momentum tensor and its conservation laws.

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### A Conventions

In this appendix we introduce the notation used throughout the paper.

The Christoffel connection is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \quad (A.1)$$

while its transformation rule under infinitesimal diffeomorphisms is

$$\delta_\xi \Gamma^\rho_{\mu\nu} = \delta^\text{cov}_\xi \Gamma^\rho_{\mu\nu} + \partial_\mu \xi^\rho \quad (A.2)$$

where $\delta^\text{cov}_\xi$ is the transformation rule of a (2,1) tensor.

The covariant derivative of a generic (1,1) tensor is given by

$$\nabla_\rho v^\mu_\nu = \partial_\rho v^\mu_\nu - \Gamma^\sigma_{\rho\mu} v_\sigma^\nu + \Gamma^\nu_{\rho\sigma} v_\mu^\sigma \quad (A.3)$$

The trace of the connection is

$$\Gamma^\nu_{\mu\nu} = \partial_\mu (\ln(\sqrt{g})) \quad (A.4)$$
The d-momentum coordinates satisfies,

$$\frac{\partial p^\mu}{\partial p^\nu} = \delta^\mu_\nu.$$  \hfill (A.5)

We work in units such that $c=1$.

## B  Closure

We show the closure for a generic vector of the double phase space,

$$\left[\delta_{\xi_1}, \delta_{\xi_2}\right] V_M = \delta_{\xi_1} \left( \xi_2^N \partial_N V_M + (\partial_M \xi_2^N - \partial^N \xi_2 M) V_N \right. + \omega(\partial_N \xi_2^N) V_M + \delta_{\xi_2}^{(p)} V_M \left. \right) - (1 \leftrightarrow 2).$$  \hfill (B.1)

The extension to tensors of the double phase space is straightforward. In (B.1) we introduce the following notation,

$$\delta^{(p)}_{\xi_2} V_M = \mathcal{P}^Q (\partial_Q \xi_2^R) \frac{\partial V_M}{\partial \mathcal{P}^R}$$  \hfill (B.2)

where $\partial_M = \frac{\partial}{\partial X^M}$. We want to show that the previous expression is equivalent to

$$\delta_{\xi_2} V_M,$$  \hfill (B.3)

with

$$\xi_1^M (X) = \xi_1^P \partial_P \xi_2^M - \frac{1}{2} \xi_1^P \partial^M \xi_2 P - (1 \leftrightarrow 2).$$  \hfill (B.4)

Since the standard generalized diffeomorphisms close with the C-bracket, we just need to show that the extra terms in (B.1) are

$$\mathcal{P}^N \partial_N (\xi_2^P \partial_P \xi_1 Q - \frac{1}{2} \xi_2^P \partial_Q \xi_1 P) \frac{\partial V_M}{\partial \mathcal{P}^Q} - (1 \leftrightarrow 2).$$  \hfill (B.5)

Let us note that the last term of the previous expression is trivially null using the strong constraint, and therefore we need to recover only the first one.

The extra terms in (B.1) are

$$\left( \xi_2^N \partial_N (\delta_{\xi_1}^{(p)} V_M) + (\partial_M \xi_2^N - \partial^N \xi_2 M) (\delta_{\xi_1}^{(p)} V_N) + \omega(\partial_N \xi_2^N) (\delta_{\xi_1}^{(p)} V_M) \right. + (\partial_{\xi_2}^{(p)} \partial_N \xi_2 Q \frac{\partial V_M}{\partial \mathcal{P}^Q} + \mathcal{P}^N \partial_N \xi_2 Q \frac{\partial}{\partial \mathcal{P}^Q} (\delta_{\xi_1} V_M) \left. \right) - (1 \leftrightarrow 2).$$  \hfill (B.6)
The first term of the second line of the previous expression is trivially null using the transformation of \( P \) given by (4.3) and the strong constraint. The remaining terms are

\[
\begin{align*}
&= \left( P^Q \xi_2^N \partial_N (\partial_Q \xi_1^R) \frac{\partial V_M}{\partial P_R} + P^Q \xi_2^N \partial_Q \xi_1^R \partial_N \left( \frac{\partial V_M}{\partial P_R} \right) \right) \\
&\quad + (\partial_M \xi_2^N - \partial_N \xi_2^M) (P^Q \partial_Q \xi_1^R \frac{\partial V_N}{\partial P_R}) + \omega (\partial_N \xi_1^R) (P^Q \partial_Q \xi_1^R \frac{\partial V_M}{\partial P_R}) \\
&\quad + P^N \partial_N \xi_2^Q \frac{\partial}{\partial P_Q} (\xi_1^R \partial_R V_M + (\partial_M \xi_1^R - \partial_R \xi_1^M) V_R + \omega (\partial_R \xi_1^R) V_M) \\
&\quad + P^N \partial_N \xi_2^Q \partial_Q \xi_1^R \frac{\partial V_M}{\partial P_R} - (1 \leftrightarrow 2),
\end{align*}
\]

where we have used that \( \frac{\partial P_M}{\partial P_N} = \delta_M^N \). Up to this point is easy to note that the closure does not depend on the weight factor \( \omega \). The terms with two derivatives acting on \( V_M \) also simplify and we have,

\[
\begin{align*}
&= \left( P^N \xi_2^Q \partial_Q (\partial_N \xi_1^R) \frac{\partial V_M}{\partial P_R} + P^N \partial_N \xi_2^Q \partial_Q \xi_1^R \frac{\partial V_M}{\partial P_R} \right) - (1 \leftrightarrow 2) \\
&= \left( P^N \partial_N (\xi_2^Q \partial_Q \xi_1^R) \frac{\partial V_M}{\partial P_R} \right) - (1 \leftrightarrow 2),
\end{align*}
\]

which matches with (B.7).

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