CONTRACTIONS WITH RANK ONE DEFECT OPERATORS AND TRUNCATED CMV MATRICES

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Abstract. The main issue we address in the present paper are the new models for completely nonunitary contractions with rank one defect operators acting on some Hilbert space of dimension $N \leq \infty$. These models complement nicely the well-known models of Livšic and Sz.-Nagy–Foias. We show that each such operator acting on some finite-dimensional (respectively separable infinite-dimensional Hilbert space) is unitarily equivalent to some finite (respectively semi-infinite) truncated CMV matrix obtained from the “full” CMV matrix by deleting the first row and the first column, and acting in $\mathbb{C}^N$ (respectively $\ell^2(N)$). This result can be viewed as a nonunitary version of the famous characterization of unitary operators with a simple spectrum due to Cantero, Moral and Velázquez, as well as an analog for contraction operators of the result from [4] concerning dissipative non-self-adjoint operators with a rank one imaginary part. It is shown that another functional model for contractions with rank one defect operators takes the form of the compression $f(\zeta) \to P_K (\zeta f(\zeta))$ on the Hilbert space $L^2(\mathbb{T}, d\mu)$ with a probability measure $\mu$ onto the subspace $K = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$. The relationship between characteristic functions of sub-matrices of the truncated CMV matrix with rank one defect operators and the corresponding Schur iterates is established. We develop direct and inverse spectral analysis for finite and semi-infinite truncated CMV matrices. In particular, we study the problem of reconstruction of such matrices from their spectrum or the mixed spectral data involving Schur parameters. It is pointed out that if the mixed spectral data contains zero eigenvalue, then no solution, unique solution or infinitely many solutions may occur in the inverse problem for truncated CMV matrices. The uniqueness theorem for recovered truncated CMV matrix from the given mixed spectral data is established. In this part the paper is closely related to the results of Hochstadt and Gesztesy–Simon obtained for finite self-adjoint Jacobi matrices.

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1. Introduction

It is well known [2] that every self-adjoint or unitary operator with a simple spectrum acting on some separable Hilbert space is unitarily equivalent to the operator of multiplication by the independent variable on the Hilbert space $L^2(\mathbb{R}, d\mu)$ or $L^2(\mathbb{T}, d\mu)$, respectively, where $d\mu$ is a probability measure on the real line $\mathbb{R}$ or on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The matrix representation of self-adjoint operators with simple spectrum was established for the first time by Stone [1]. He proved that every self-adjoint operator with a simple spectrum is unitarily equivalent to a certain Jacobi (tri-diagonal) matrix of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $a_k > 0$, and $b_k$ are real numbers for all $k \in \mathbb{N}$. The non-self-adjoint version of the Stone theorem has been recently obtained in [4] for dissipative non-self-adjoint operators with rank one imaginary part. It turned out that the matrix representation of such operators is a non-self-adjoint Jacobi matrix of the form (1.1) with only nonreal first entry $b_1$ satisfying $\text{Im} b_1 > 0$.

The problem of the canonical matrix representation of a unitary operator with a simple spectrum has been recently solved by M. Cantero, L. Moral and L. Velázquez in [11]. They introduced and studied five-diagonal unitary matrices of the form

$$C = C(\{\alpha_n\}) = \begin{pmatrix} \bar{\alpha}_0 & -\bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \rho_0 & -\bar{\alpha}_1\rho_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ 0 & -\bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\rho_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Such matrix appears as a matrix representation of the unitary operator $(Uf)(\zeta) = \zeta f(\zeta)$ in $L^2(\mathbb{T}, d\mu)$ with respect to the orthonormal system $\{\chi_n\}$ obtained by orthonormalization of the sequence $\{1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \ldots\}$. The so called Schur parameters or Verblunsky coefficients $\{\alpha_n\}$, $|\alpha_n| < 1$, arise in the Szegő recurrence formula

$$\zeta \Phi_n(\zeta) = \Phi_{n+1}(\zeta) + \bar{\alpha}_n \zeta^n \Phi_n(1/\zeta), \quad n = 0, 1, \ldots.$$
for monic orthogonal with respect to $d\mu$ polynomials $\{\Phi_n\}$, and $\rho_n := \sqrt{1 - |\alpha_n|^2}$. The matrices $\mathcal{C}(\{\alpha_n\})$ are called the CMV matrices. The spectral analysis of CMV matrices has recently attracted much attention, and we refer on this matter to the papers [11, 12, 20, 36, 37, 38].

The spectral theory of non-self-adjoint and nonunitary operators and their models is based on the concept of characteristic function of the corresponding operator or the operator coligations [6, 9, 10, 26, 27, 28, 29, 30, 31, 32, 33, 39].

In this paper we employ the Sz.-Nagy–Foias theory [39] and the unitary colligations approach [10] to the spectral analysis of contractions acting on Hilbert spaces. The corresponding characteristic function belongs to the Schur class of operator-valued functions holomorphic in the open unit disk $\mathbb{D}$. By Sz.-Nagy–Foias theorem [39, Proposition VI.2.1] each completely nonunitary contraction $T$ with rank one defect operators $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ (shortly, with rank one defects) is unitarily equivalent to the operator (functional model) of the form

$$\mathcal{H}_\Theta = (H^2 \oplus \text{clos} \Delta L^2(\mathbb{T})) \ominus \left\{ \Theta u \oplus \Delta u : u \in H^2 \right\} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in H^2, \ g \in \text{clos} \Delta L^2(\mathbb{T}), \ P_{H^2}(\Theta f + \Delta g) = 0 \right\},$$

$$\Xi_\Theta \left( \begin{pmatrix} f \\ g \end{pmatrix} \right) = P_{\mathcal{H}_\Theta} \zeta \left( \begin{pmatrix} f \\ g \end{pmatrix} \right), \quad \Xi^*_\Theta \left( \begin{pmatrix} f \\ g \end{pmatrix} \right) = \left( \begin{pmatrix} \zeta f - f(0) \\ \bar{\zeta} g \end{pmatrix} \right) \left( \begin{pmatrix} f \\ g \end{pmatrix} \right) \in \mathcal{H}_\Theta,$$

where $H^2$ is the Hardy space,

$$\Theta = \Theta_T(z) = \left( -T + zD_{T^*}(I - zT^*)^{-1}D_T \right) \mid \mathcal{D}_T$$

is the characteristic function of $T$, $\Delta^2 = 1 - |\Theta|^2$, $P_{H^2}$ is the orthogonal projection onto $H^2$ in $L^2(\mathbb{T})$, and $P_{\mathcal{H}_\Theta}$ is the orthogonal projection onto the model space $\mathcal{H}_\Theta$.

We obtain a new functional model that complements the above mentioned Sz.-Nagy–Foias functional model, and show that every completely nonunitary contraction $T$ with rank one defects is unitarily equivalent to the compression $f(\zeta) \to P_K(\zeta f(\zeta))$ on the Hilbert space $L^2(\mathbb{T}, d\mu)$ with a probability measure $\mu$ onto subspace $K = L^2(\mathbb{T}, d\mu) \oplus \mathbb{C}$.

We studied the so called truncated CMV matrix $\mathcal{T}$ obtained from the “full” CMV matrix $\mathcal{C} = \mathcal{C}(\{\alpha_n\})$ (1.2) by deleting the first row and the first column:

$$\mathcal{T} = \mathcal{T}(\{\alpha_n\}) = \begin{pmatrix} -\bar{\alpha}_1 \alpha_0 & -\rho_1 \alpha_0 & 0 & 0 & \ldots \\ \bar{\alpha}_2 \rho_1 & -\bar{\alpha}_2 \alpha_1 & \bar{\alpha}_3 \rho_2 & \rho_3 \rho_2 & \ldots \\ \rho_2 \rho_1 & -\rho_2 \alpha_1 & -\bar{\alpha}_3 \alpha_2 & -\rho_3 \alpha_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$  

In the semi-infinite case $\mathcal{T}$ takes on the block-matrix form (see Section 4.3)

$$\mathcal{T} = \begin{pmatrix} B_1 & C_1 & 0 & 0 & 0 & \cdots \\ A_1 & B_2 & C_2 & 0 & 0 & \cdots \\ 0 & A_2 & B_3 & C_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$  

It turned out that the truncated CMV matrix $\mathcal{T}(\{\alpha_n\})$ is a contraction with rank one defects, and the Sz.-Nagy–Foias characteristic function that agrees with the Schur function $f$ associated with the measure $\mu$ [37]. Moreover, we show that the sub-matrix $\mathcal{T}^{(k)}(\{\alpha_n\})$ obtained
from \( T(\{\alpha_n\}) \) by deleting the first \( k \) rows and columns is also a contraction with rank one defects, and its characteristic function agrees with the well known \( k^{\text{th}} \) Schur iterate
\[
f_k(z) = \frac{f_{k-1}(z) - \alpha_{k-1}}{z(1 - \overline{\alpha}_{k-1}f_{k-1}(z))}, \quad f_0(z) = f(z).
\]
This relation is an analog of the corresponding relation between the \( m \)-function of a Jacobi matrix and the \( m \)-function of its sub-matrix (cf. [21]).

Our main result states that each completely nonunitary contraction \( T \) with rank one defects is unitarily equivalent to the one-parameter family \( T(\{e^{it}\alpha_n\}) \), where \( \{\alpha_n\} \) are the Schur parameters of the Sz.-Nagy–Foias characteristic function of \( T \). We develop direct and inverse spectral analysis for finite and semi-infinite truncated CMV matrices. It is shown that given an arbitrary set of \( N \) not necessarily distinct numbers from \( \mathbb{D} \) there is a one-parameter family of unitarily equivalent \( N \times N \) truncated CMV matrices having those numbers as the eigenvalues counting algebraic multiplicity. We prove the uniqueness of \( N \times N \) truncated CMV matrix \( \mathcal{T} \) with given not necessarily distinct eigenvalues \( z_1, \ldots, z_r \) and given first \( N - r + 1 \) Schur parameters \( \alpha_0(\mathcal{T}), \ldots, \alpha_{N-r}(\mathcal{T}) \). This result on inverse spectral analysis of finite truncated CMV matrices is an analog of the Hochstadt [23] and Gesztesy-Simon [21] uniqueness theorems for finite self-adjoint Jacobi matrices as well as for established in [4] uniqueness theorem for finite non-self-adjoint Jacobi matrices with rank one imaginary part. We obtain the existence of \( N \times N \) truncated CMV matrix \( \mathcal{T} \) when its eigenvalues \( z_1, \ldots, z_m \) and the last Schur parameters \( \alpha_m(\mathcal{T}), \ldots, \alpha_N(\mathcal{T}) \) are known.

Here is a summary of the rest of the paper. In Sections 2 and 3 we discuss some basics from the Sz.-Nagy–Foias theory and the unitary colligations with the focus upon the characteristic function and its properties. Section 4 provides a brief overview of the theory of orthogonal polynomials on the unit circle and CMV matrices. The main results concerning truncated CMV matrices and the models of completely nonunitary contractions with rank one defects are presented in Section 5 and 6. The final section 7 deals with the inverse spectral analysis for truncated CMV matrices.

2. Contractions, unitary colligations, and their characteristic functions

2.1. Contractions and the Sz.-Nagy–Foias characteristic functions. Let \( H \) be a separable Hilbert space with the inner product \((\cdot, \cdot)\). A bounded linear operator \( T \) in \( H \) is called a contraction if \( \|T\| \leq 1 \) (for the basic properties of contractions see [39 Chapter I]). If \( T \) is a contraction then the operators
\[
D_T := (I - T^*T)^{1/2}, \quad D_{T^*} := (I - TT^*)^{1/2}
\]
are called the defect operators of \( T \) or shortly defects, and the subspaces \( \mathcal{D}_T = \text{ran} \ D_T \), \( \mathcal{D}_{T^*} = \text{ran} \ D_{T^*} \) the defect subspaces of \( T \). The dimensions \( \dim \mathcal{D}_T \), \( \dim \mathcal{D}_{T^*} \) are known as the defect numbers of \( T \). Given a pair of numbers \( n, n^* = 0, 1, \ldots, \infty \) it is easy to construct a contraction with \( n = \dim \mathcal{D}_T \), \( n^* = \dim \mathcal{D}_{T^*} \). Each contraction \( T \) acting on a finite dimensional Hilbert space has equal defect numbers: \( n = n^* \).

The defect operators satisfy the following intertwining relations
\[
TD_T = D_{T^*}T, \quad T^*D_{T^*} = D_T T^*.
\]
and the block-operators
\[
\begin{pmatrix}
-T^* & D_T \\
D_T^* & T
\end{pmatrix} : \mathcal{D}_T \to \mathcal{D}_T, \quad \begin{pmatrix}
-T & D_T \\
D_T^* & T^*
\end{pmatrix} : \mathcal{D}_T \to \mathcal{D}_T.
\]
are unitary operators in the corresponding orthogonal sums of the spaces. It follows from [2.1] that \( T \mathcal{D}_T \subset \mathcal{D}_T^*, \ T^* \mathcal{D}_T \subset \mathcal{D}_T, \) and \( T(\ker D_T) = \ker D_T^*, \ T^*(\ker D_T^*) = \ker D_T. \) Moreover, \( T \upharpoonright \ker D_T \) and \( T^* \upharpoonright \ker D_T^* \) are isometric operators. It follows that \( T \) is a quasi-unitary extension [26] of the isometric operator \( V = T \upharpoonright \ker D_T \) (for the definition see Section 6.2).

A contraction \( T \) is called completely nonunitary if there is no nontrivial reducing subspace of \( T, \) on which \( T \) generates a unitary operator. One of the fundamental results of the contractions theory [39, Theorem I.3.2] reads that, given a contraction \( T \) in \( H, \) there is a canonical orthogonal decomposition
\[
H = H_0 \oplus H_1, \quad T = T_0 \oplus T_1, \quad T_j = T \upharpoonright H_j, \quad j = 0, 1,
\]
where \( H_0 \) and \( H_1 \) reduce \( T, \) \( T_0 \) is a completely nonunitary contraction, and \( T_1 \) is a unitary operator. Moreover,
\[
H_1 = \left( \bigcap_{n \geq 1} \ker D_{T^n} \right) \bigcap \left( \bigcap_{n \geq 1} \ker D_{T^{* n}} \right),
\]
so,
\[
T \text{ is completely nonunitary} \iff \left( \bigcap_{n \geq 1} \ker D_{T^n} \right) \bigcap \left( \bigcap_{n \geq 1} \ker D_{T^{* n}} \right) = \{0\}. \tag{2.2}
\]
Clearly,
\[
\bigcap_{n \geq 1} \ker D_{T^n} = H \ominus \operatorname{span} \left\{ T^{* n} D_T H, n = 0, 1, \ldots \right\}, \tag{2.3}
\]
\[
\bigcap_{n \geq 1} \ker D_{T^{* n}} = H \ominus \operatorname{span} \left\{ T^n D_T^* H, n = 0, 1, \ldots \right\}.
\]
Let \( V \) be an isometry in \( H. \) A subspace \( \Omega \) in \( H \) is called wandering for \( V \) if \( V^p \Omega \perp V^q \Omega \) for all \( p, q \in \mathbb{Z}_+, p \neq q. \) Since \( V \) is an isometry, the latter is equivalent to \( V^n \Omega \perp \Omega \) for all \( n \in \mathbb{N}. \) If \( H = \sum_{n=0}^{\infty} V^n \Omega, \) then \( V \) is called a unilateral shift and \( \Omega \) is called the generating subspace. The dimension of \( \Omega \) is called the multiplicity of the unilateral shift \( V. \) It is well known [39, Theorem I.1.1] that \( V \) is a unilateral shift if and only if \( \bigcap_{n=0}^{\infty} V^n H = \{0\}. \) Clearly, if an isometry \( V \) is the unilateral shift in \( H, \) then \( \Omega = H \ominus VH \) is the generating subspace for \( V. \)

Given a contraction \( T \) in \( H \) and a subspace \( \mathcal{H} \subset H, \) the unilateral shift \( V : \mathcal{H} \to \mathcal{H} \) is said to be contained in \( T, \) if \( \mathcal{H} \) is invariant for \( T, \) and \( T \upharpoonright \mathcal{H} = V \) [13]. The subspaces \( \bigcap_{n \geq 1} \ker D_{T^n} \) and \( \bigcap_{n \geq 1} \ker D_{T^{* n}} \) are invariant for \( T \) and \( T^*, \) respectively, and the operators \( V_T := T \upharpoonright \bigcap_{n \geq 1} \ker D_{T^n} \) and \( V_{T^*} := \uparrow \bigcap_{n \geq 1} \ker D_{T^{* n}} \) are unilateral shifts. Moreover, \( V_T \) and \( V_{T^*} \) are the maximal unilateral shifts contained in \( T \) and \( T^*. \) The multiplicities of the shifts \( V_T \) and \( V_{T^*} \) do not exceed the defect numbers \( \dim \mathcal{D}_{T^n} \) and \( \dim \mathcal{D}_{T^*}, \) respectively [15]. If \( T \) is a
completely nonunitary contraction with rank one defects, then (see [13, 15 Theorem 1.7])

\[ T \text{ does not contain the unilateral shift } \iff T^* \text{ does not contain the unilateral shift} \]

\[ \bigcap_{n \geq 1} \ker D_{T^n} = \{0\} \iff \bigcap_{n \geq 1} \ker D_{T^{*n}} = \{0\}. \]

The function (see [39 Chapter VI])

\[ \Theta_T(z) = (-T + zD_{T^*}(I - zT^{*-1})D_T) | \mathcal{D}_T \]

is known as the characteristic function of the Sz.-Nagy – Foias type of a contraction \( T \). This function belongs to the Schur class \( S(\mathcal{D}_T, \mathcal{D}_T^*) \) of \( \mathcal{L}(\mathcal{D}_T, \mathcal{D}_T^*) \)-valued holomorphic in the unit disk \( \mathbb{D} \) operator-functions, i.e., \( \|\Theta_T(z)\| \leq 1 \) for \( z \in \mathbb{D} \). Moreover, the function \( \Theta_T \) satisfies the condition \( \|\Theta_T(0)f\| < \|f\| \) for all \( f \in \mathcal{D}_T \setminus \{0\} \). The characteristic functions of \( T \) and \( T^* \) are connected by the relation

\[ \Theta_{T^*}(z) = \Theta_T^*(\bar{z}), \quad z \in \mathbb{D}. \]

Two operator-valued functions \( \Theta_1 \in S(\mathcal{M}_1, \mathcal{N}_1) \) and \( \Theta_2 \in S(\mathcal{M}_2, \mathcal{N}_2) \) are said to agree if there are two unitary operators \( V : \mathcal{N}_1 \rightarrow \mathcal{N}_2 \) and \( W : \mathcal{M}_2 \rightarrow \mathcal{M}_1 \) such that

\[ V\Theta_1(z)W = \Theta_2(z), \quad z \in \mathbb{D}. \]

It is well known [39 Theorem VI.3.4], that two completely nonunitary contractions \( T_1 \) and \( T_2 \) are unitary equivalent if and only if their characteristic functions \( \Theta_{T_1} \) and \( \Theta_{T_2} \) agree.

Every operator-valued function \( \Theta \) from the Schur class \( S(\mathcal{M}, \mathcal{N}) \) has almost everywhere nontangential strong limit values \( \Theta(\zeta), \zeta \in \mathbb{T} \). A function \( \Theta \in S(\mathcal{M}, \mathcal{N}) \) is called inner if \( \Theta^*(\zeta)\Theta(\zeta) = I_{\mathcal{M}} \) for a.e. \( \zeta \in \mathbb{T} \), and co-inner if \( \Theta(\zeta)\Theta^*(\zeta) = I_{\mathcal{N}} \) for a.e. \( \zeta \in \mathbb{T} \). A function \( \Theta \in S(\mathcal{M}, \mathcal{N}) \) is called bi-inner, if it is both inner and co-inner. A contraction \( T \) on a Hilbert space \( \mathcal{H} \) belongs to the classes \( C_0, (C_0), \) if

\[ s - \lim_{n \to \infty} T^n = 0 \quad (s - \lim_{n \to \infty} T^{*n} = 0), \]

respectively. By definition \( C_{00} := C_0 \cap C_0 \). The completely nonunitary part of a contraction \( T \) belongs to the class \( C_0, C_0, \) or \( C_{00} \) if and only if its characteristic function \( \Theta_T(z) \) is inner, co-inner, or bi-inner, respectively (cf. [39 Section VI.2]).

In the following statement [39 Theorem VI.4.1] the spectrum of completely nonunitary contractions is described.

**Theorem 2.1.** Let \( T \) be a completely nonunitary contraction on \( H \). Denote by \( S_T \) the set of points \( z \in \mathbb{D} \) for which the operator \( \Theta_T(z) \) is not boundedly invertible, together with those \( z \in \mathbb{T} \) not lying on any of the open arcs of \( \mathbb{T} \) on which \( \Theta_T \) is a unitary operator valued analytic function. Furthermore, denote by \( S^0_T \) the set of points \( z \in \mathbb{D} \) for which \( \Theta_T(z) \) is not invertible at all. Then the spectrum \( \sigma(T) \) of \( T \) agrees with \( S_T \), and the point spectrum \( \sigma_p(T) \) with \( S^0_T \).

If \( T \) is a completely nonunitary contraction with rank one defects, and if \( z_0 \) is an eigenvalue of \( T \), then the geometric multiplicity of \( z_0 \) is one, the algebraic multiplicity is finite, and the characteristic function \( \Theta_T \) admits the following factorization

\[
\Theta_T(z) = c \prod_{k} \frac{\bar{z}_k z_k - z}{1 - \bar{z}_k z} \exp \left( - \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right) \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} \ln k(t) dt \right),
\]
where $|c| = 1$, $k(t) \geq 0$, $\ln k(t) \in L_1[0,2\pi]$, $\mu$ is a finite nonnegative measure singular with respect to the Lebesgue measure, and $\{z_k\}$ are the eigenvalues of $T$. In addition, if $\dim H = N < \infty$, and $T$ is a completely nonunitary contraction in $H$ with rank one defects, then its characteristic function is the finite Blaschke product of order $N$ of the form

$$b(z) = e^{i\varphi} \prod_{k=1}^{m} \left( \frac{z - z_k}{1 - \bar{z}_k z} \right)^{l_k},$$

where $z_1, \ldots, z_m$ are distinct eigenvalues of $T$ with the algebraic multiplicities $l_1, \ldots, l_m$, respectively, $l_1 + \ldots + l_m = N$, and $\varphi \in [0,2\pi)$. Hence, a finite-dimensional completely nonunitary contraction $T$ with rank one defects belongs to the class $C_{00}$, and $\lim_{n \to \infty} ||T^n|| = 0$.

It is easily seen from Theorem 2.1 that the point spectrum of a contraction $T$ with rank one defects agrees with $D$ if and only if $\Theta_T \equiv 0$.

2.2. Unitary colligations and their characteristic functions. Every contraction $T$ acting on Hilbert space $H$ can be included into the unitary operator colligation

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathcal{M}, \mathcal{N}, H \right\},$$

where $\mathcal{M}$ and $\mathcal{N}$ are separable Hilbert spaces, and

$$U = \begin{pmatrix} S \\ F \end{pmatrix} : \mathcal{M} \to \mathcal{N}, \quad \begin{pmatrix} G \\ T \end{pmatrix} : \mathcal{N} \to H$$

is a unitary operator. $T$ is called the basic operator of the unitary colligation $\Delta$. The spaces $\mathcal{M}$ and $\mathcal{N}$ are called the left outer space and right outer space, respectively. The unitarity of $U$ means

$$U^*U = \begin{pmatrix} I_{\mathcal{M}} & 0 \\ 0 & I_H \end{pmatrix}, \quad UU^* = \begin{pmatrix} I_{\mathcal{N}} & 0 \\ 0 & I_H \end{pmatrix}.$$

or equivalently,

$$T + G^*G = I_H, \quad F + S^*S = I_{\mathcal{M}}, \quad T^*F + G^*S = 0, \quad TT^* + FF^* = I_H, \quad GG^* + SS^* = I_{\mathcal{N}}, \quad TG^* + FS^* = 0.$$ (2.5)

The colligation

$$\Delta_0 = \left\{ \begin{pmatrix} -T^* \\ D_T \end{pmatrix} \begin{pmatrix} D_T^* \\ T \end{pmatrix} ; \mathcal{D}_T, \mathcal{D}_T^*, H \right\}$$

provides an example of the unitary colligation with given basic operator $T$.

Let $\Delta = \left\{ \begin{pmatrix} S \\ F \end{pmatrix}; \mathcal{M}, \mathcal{N}, H \right\}$ be a unitary colligation. Define the following subspaces in $H$

$$H^{(c)} = \text{span} \{ T^n F \mathcal{M}, \ n = 0, 1, \ldots \}, \quad H^{(o)} = \text{span} \{ T^n G^* \mathcal{N}, \ n = 0, 1, \ldots \}.$$ (2.7)

The subspaces $H^{(c)}$ and $H^{(o)}$ are called the controllable and the observable subspaces, respectively. Let

$$(H^{(c)})^\perp := H \ominus H^{(c)}, \quad (H^{(o)})^\perp := H \ominus H^{(o)}.$$ (2.8)

\footnote{also known as the \textit{conservative system} [5]}
A unitary colligation $\Delta$ is called prime if

$$\left( H^{(c)} \right)^\perp \cap \left( H^{(o)} \right)^\perp = \{0\},$$

Clearly, the latter condition is equivalent to

$$H^{(c)} + H^{(o)} = H.$$  

(2.5) and (2.8) we get

$$\left( H^{(c)} \right)^\perp = \bigcap_{n\geq 0} \ker(F^*T^n) = \bigcap_{n\geq 0} \ker(D_T^*T^n) = \bigcap_{n\geq 1} \ker(D_{T^n}).$$

$$\left( H^{(o)} \right)^\perp = \bigcap_{n\geq 0} \ker(GT^n) = \bigcap_{n\geq 0} \ker(D_T^n) = \bigcap_{n\geq 1} \ker(D_{T^n}).$$

(2.9)

It follows now from (2.2) that the unitary colligation $\Delta = \left\{ \left( \begin{array}{cc} S & G \\ F & T \end{array} \right); M, N, H \right\}$ is prime if and only if $T$ is a completely nonunitary operator.

Given a unitary colligation $\Delta = \left\{ \left( \begin{array}{cc} S & G \\ F & T \end{array} \right); M, N, H \right\}$, its characteristic function is defined by

$$\Theta_\Delta(z) = S + zG(I_H - zT)^{-1}F, \quad z \in \mathbb{D}.$$  

This function belongs to the Schur class $S(M, N)$ of $L(M, N)$-valued holomorphic in the unit disk $\mathbb{D}$ operator-functions. In particular, the characteristic function of the unitary colligation $\Delta_0$ is in fact the Sz.-Nagy – Foias characteristic function of the operator $T^*$.

Two prime unitary colligations

$$\Delta_1 = \left\{ \left( \begin{array}{cc} S & G_1 \\ F_1 & T_1 \end{array} \right); M, N, H_1 \right\} \quad \text{and} \quad \Delta_2 = \left\{ \left( \begin{array}{cc} S & G_2 \\ F_2 & T_2 \end{array} \right); M, N, H_2 \right\}$$

which have equal characteristic functions are unitarily equivalent in the following sense [10, Theorem 3.2]: there exists a unitary operator $V : H_1 \to H_2$ such that

$$VT_1 = T_2V, \quad VF_1 = F_2, \quad G_2V = G_1 \iff \left( \begin{array}{cc} I_M & 0 \\ 0 & V \end{array} \right) \left( \begin{array}{cc} S & G_1 \\ F_1 & T_1 \end{array} \right) \left( \begin{array}{cc} I_M & 0 \\ 0 & V \end{array} \right) = \left( \begin{array}{cc} S & G_2 \\ F_2 & T_2 \end{array} \right) \left( \begin{array}{cc} I_M & 0 \\ 0 & V \end{array} \right).$$

Besides, given $\Theta \in S(M, N)$, there exists a prime unitary colligation

$$\Delta = \left\{ \left( \begin{array}{cc} S & G \\ F & T \end{array} \right); M, N, H \right\}$$

such that $\Theta_\Delta = \Theta$ in $\mathbb{D}$ [10, Theorem 5.1].

Later on in Section 3 we will need the following result.

**Theorem 2.2.** Let $T$ be a contraction acting on Hilbert space $H$ with finite defect numbers. Suppose that $M$ and $N$ are two given Hilbert spaces such that $\dim N = \dim \mathcal{D}_T$ and $\dim M = \dim \mathcal{D}_{T^*}$. Then all unitary colligations with the basic operator $T$ and outer subspaces $M$ and $N$ take the form

$$\Delta = \left\{ \left( \begin{array}{cc} -KT^*M & KD_T^* \\ DT^*M & T \end{array} \right); M, N, H \right\},$$

(2.10)

2*the transfer function* of the system [5]
where \( K : \mathcal{D}_T \to \mathcal{M} \) and \( M : \mathcal{M} \to \mathcal{D}_{T^*} \) are unitary operators. The characteristic function of \( \Delta \) is
\[
\Theta_\Delta(z) = K\Theta_{T^*}(z)M, \quad z \in \mathbb{D},
\]
i.e., \( \Theta_\Delta \) agrees with the characteristic function \( \Theta_{T^*} \) of \( T^* \).

**Proof.** Let \( \Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathcal{M}, \mathcal{N}, H \right\} \) be a unitary colligation. From the relation \( G^*G + T^*T = I_H \) it follows that
\[
||Gf||^2 = ||D_Tf||^2, \quad f \in H.
\]
Hence, the operator \( K : \mathcal{D}_T \to \mathcal{M} \) defined by
\[
KD_Tf = Gf, \quad f \in H,
\]
is isometric, and \( \text{ran} \, K = \mathcal{M} \). Similarly, the relation \( FF^* + TT^* = I_H \) yields that the operator \( N : \mathcal{D}_{T^*} \to \mathcal{M} \) given by the relation
\[
ND_{T^*}f = F^*f, \quad f \in H
\]
is isometric, and \( \text{ran} \, N = \mathcal{M} \). So \( M = N^* : \mathcal{M} \to \mathcal{D}_{T^*} \) is unitary, and \( F = D_{T^*}M \).

From the relation \( T^*F + G^*S = 0 \) we get \( T^*D_{T^*}M + D_TK^*S = 0 \). Hence by \((2.1)\)
\( T^*M + K^*S = 0 \). As \( \text{ran} \, M = \mathcal{D}_{T^*} \), ran \( K^* = \mathcal{D}_T \), and \( T \mathcal{D}_{T^*} \subset \mathcal{D}_T \), we have
\[
S = -KT^*M.
\]
Observe also that
\[
TG^* + FS^* = TD_TK^* - D_TMM^*TK^* = 0
\]
\[
SS^* + GG^* = KT^*MM^*TK^* + KD_T^2K^* = K(T^*T + I - T^*T)K^* = I_{\mathcal{M}},
\]
\[
S^*S + F^*F = M^*TK^*KT^*M + M^*D_{T^*}M = M^*(TT^* + I - TT^*)M = I_{\mathcal{M}}.
\]
Thus, all conditions \( (2.5) \) are satisfied, i.e., the colligation \( \Delta \) is of the form \( (2.10) \).

Conversely, if \( \text{dim} \, \mathcal{N} = \text{dim} \, \mathcal{D}_T < \infty \), \( \text{dim} \, \mathcal{M} = \text{dim} \, \mathcal{D}_{T^*} < \infty \), and \( K : \mathcal{D}_T \to \mathcal{N} \) and \( M : \mathcal{M} \to \mathcal{D}_{T^*} \) are unitary operators, then one can easily see that \( U = \begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : \)
\[
\begin{pmatrix} \mathcal{M} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathcal{N} \\ \mathcal{H} \end{pmatrix}
\]
is a unitary operator, i.e., the relations \( (2.5) \) are satisfied. It follows that
\[
\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathcal{M}, \mathcal{N}, H \right\}
\]
is a unitary colligation, where \( G = KD_T \), \( F = D_{T^*}M \), \( S = -KT^*M \).

For the characteristic function \( \Theta_\Delta \) we obtain for all \( z \in \mathbb{D} \)
\[
\Theta_\Delta(z) = S + zG(I - zT)^{-1}F = -KT^*M + zKD_T(I - zT)^{-1}D_{T^*}M = K\Theta_{T^*}(z)M.
\]

**Corollary 2.3.** Let \( T \) be a contraction with finite defect numbers, \( \text{dim} \, \mathcal{N} = \text{dim} \, \mathcal{D}_T \), \( \text{dim} \, \mathcal{M} = \text{dim} \, \mathcal{D}_{T^*} \), and let \( \Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathcal{M}, \mathcal{N}, H \right\} \) be a unitary colligation. Then all other unitary colligations with the basic operator \( T \) and outer subspaces \( \mathcal{M} \) and \( \mathcal{N} \) take the form
\[
\tilde{\Delta} = \left\{ \begin{pmatrix} C_1SC_2 & C_1G \\ FC_2 & C_2T \end{pmatrix} ; \mathcal{M}, \mathcal{N}, H \right\}.
\]
where \( C_1 \) and \( C_2 \) are unitary operators in \( \mathcal{N} \) and \( \mathcal{M} \), respectively.

**Proof.** By Theorem 2.2 we have
\[
G = KD_T, \quad F = D_{T^*}M, \quad S = -KT^*M,
\]
where \( K : \mathcal{D}_T \to \mathcal{M} \) and \( M : \mathcal{N} \to \mathcal{D}_{T^*} \) are unitary operators. If \( \tilde{\Delta} = \left\{ \left( \begin{array}{c} \tilde{S} \\ \tilde{F} \\ \tilde{T} \end{array} \right) ; \mathcal{M}, \mathcal{N}, H \right\} \) is some other unitary colligation then \( \tilde{G} = \tilde{K}D_T, \quad \tilde{F} = D_{T^*}\tilde{M}, \quad \tilde{S} = -\tilde{K}T^*\tilde{M} \), where \( \tilde{K} : \mathcal{D}_T \to \mathcal{M} \) and \( \tilde{M} : \mathcal{N} \to \mathcal{D}_{T^*} \) are unitary operators. Let \( C_1 := \tilde{K}K^{-1}, \quad C_2 := M^{-1}\tilde{M} \). Then \( C_1 \) and \( C_2 \) are unitary operators in \( \mathcal{N} \) and \( \mathcal{M} \), respectively, and
\[
\tilde{G} = C_1G, \quad \tilde{F} = FC_2, \quad \tilde{S} = C_1SC_2,
\]
as needed. \( \square \)

### 3. Completely nonunitary contractions with rank one defects and the corresponding unitary colligations

**Theorem 3.1.** Each contraction \( T \) with rank one defects on the Hilbert space \( H \) can be included into the unitary colligation
\[
\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathcal{C}, \mathcal{C}, H \right\}.
\]

Let \( \vec{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C} \oplus H \), and let the subspaces \( (H^{(c)})^\perp \) and \( (H^{(o)})^\perp \) in \( H \) be defined by (2.8). Then
\[
(H^{(c)})^\perp = (\mathbb{C} \oplus H) \oplus \overline{\text{span}} \{ U^n\vec{1} ; \, n = 0, 1, \ldots \}, \\
(H^{(o)})^\perp = (\mathbb{C} \oplus H) \oplus \overline{\text{span}} \{ U^*n\vec{1} ; \, n = 0, 1, \ldots \},
\]
and so the following conditions are equivalent:

(i) the unitary colligation \( \Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathcal{C}, \mathcal{C}, H \right\} \) is prime;

(ii) \( T \) is a completely nonunitary contraction;

(iii) \( \vec{1} \) is the cyclic vector for \( U : \overline{\text{span}} \{ U^n\vec{1} ; \, n \in \mathbb{Z} \} = \mathbb{C} \oplus H \).

All other unitary colligations with the basic operator \( T \) and the outer spaces \( \mathbb{C} \) are of the form
\[
\tilde{\Delta} = \left\{ \begin{pmatrix} c_1c_2S & c_1G \\ c_2F & T \end{pmatrix} ; \mathcal{C}, \mathcal{C}, H \right\},
\]
where \( |c_1| = |c_2| = 1 \).

**Proof.** Since \( \dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = 1 \), by Theorem 2.2 we can choose the unitary colligation \( \Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathcal{C}, \mathcal{C}, H \right\} \) of the form (2.10), i.e., \( S = -KT^*M, \quad G = KD_T, \quad F = D_{T^*}M, \quad K : \text{ran } D_T \to \mathbb{C}, \quad M : \mathbb{C} \to \text{ran } D_{T^*} \) are isometric operators. So \( U = \begin{pmatrix} S & G \\ F & T \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \) is the unitary operator.
To prove (3.1), suppose that the vector \( \vec{h} = \left( \begin{matrix} z \\ h \end{matrix} \right) \in \mathbb{C} \oplus H \) is orthogonal to the subspace \( \text{span} \{ U^n \bar{1}, \ n = 0, 1, \ldots \} \). Then \( U^{*n} \vec{h} \perp \bar{1}, \ n = 0, 1, \ldots \) so \( z = 0 \) and \( \vec{h} = \left( \begin{matrix} 0 \\ h \end{matrix} \right) \). By using \( U^* = \left( \begin{array}{cc} S^* & F^* \\ G^* & T^* \end{array} \right) \), we get consequently

\[
F^*h = 0, \quad F^*T^*h = 0, \quad F^*T^zh = 0, \quad \ldots, \quad F^*T^k h = 0, \ldots
\]

It follows from (2.9) that \( h \in (H^{(c)})^\perp \). Conversely, if \( h \in (H^{(c)})^\perp \) then \( h \perp \text{span} \{ U^n \bar{1}, \ n = 0, 1, \ldots \} \). Similarly, \( (H^{(o)})^\perp = (\mathbb{C} \oplus H) \oplus \left( \text{span} \{ U^n \bar{1}, \ n = 0, 1, \ldots \} \right) \), as needed.

We arrive at the following conclusion:

\( \bar{1} \) is a cyclic vector for \( U \iff (H^{(c)})^\perp \cap (H^{(o)})^\perp = \{ 0 \} \iff \)

the unitary colligation \( \Delta = \left\{ \left( \begin{array}{cc} S & G \\ F & T \end{array} \right); \mathbb{C}, \mathbb{C}, H \right\} \) is prime \iff

the operator \( T \) is completely nonunitary.

By Corollary 2.3 all other unitary colligations with the basic operator \( T \) and the outer subspace \( \mathbb{C} \) are given by (3.2) with \( |c_1| = |c_2| = 1 \). \( \square \)

**Remark 3.2.** In terms of the Naimark dilations of a probability operator-valued measure on the unit circle, the main result of Theorem 3.1 is proved in [14, Theorem 1.20].

Let us give more precise expressions for the operators \( F, G, \) and \( S \). Let \( \hat{\varphi}_1 \in \mathcal{D}_T, \hat{\varphi}_2 \in \mathcal{D}_{T^*} \).

Put

\[
\varphi_1 = \frac{\hat{\varphi}_1}{\| \hat{\varphi}_1 \|}, \quad \varphi_2 = \frac{\hat{\varphi}_2}{\| \hat{\varphi}_2 \|}.
\]

Then

\[
Kh = b_1(h, \varphi_1), \quad h \in \text{ran} \ D_T,
\]

\[
M^*g = b_2(g, \varphi_2), \quad g \in \text{ran} \ D_{T^*},
\]

where \( |b_1| = |b_2| = 1 \). Observe that \( T\varphi_1 = -\alpha_0 \varphi_2 \) and \( T^*\varphi_2 = -\bar{\alpha}_0 \varphi_1 \), where \( \alpha_0 \) is a complex number from \( \mathbb{D} \). It follows that

\[
D^2_T\varphi_1 = (1 - |\alpha_0|^2)\varphi_1, \quad D^2_{T^*}\varphi_2 = (1 - |\alpha_0|^2)\varphi_2.
\]

Let \( \rho_0 = \sqrt{1 - |\alpha_0|^2} \). Since \( \dim(\text{ran} \ D^2_T) = \dim(\text{ran} \ D^2_{T^*}) = 1 \), the number \( \rho_0 \) is a unique positive eigenvalue of \( D_T(D_{T^*}) \). Next,

\[
Gh = b_1(D_T h, \varphi_1) = b_1(h, D_T \varphi_1) = b_1 \rho_0(h, \varphi_1),
\]

\[
F^*h = b_2(D_T^* h, \varphi_2) = b_2(h, D_{T^*} \varphi_2) = b_2 \rho_0(h, \varphi_2), \quad h \in H.
\]

Hence \( F^1 = \rho_0 \bar{b}_2\varphi_2 \). Since \( S = -KT^*M \), we get

\[
S1 = -b_1 \bar{b}_2(T^*\varphi_2, \varphi_1) = b_1 \bar{b}_2 \bar{\alpha}_0.
\]

In the case \( \dim H = N < \infty \) the operator \( T \) can be given by the \( N \times N \) matrix with respect to some orthonormal basis and we can choose \( \hat{\varphi}_1 \) (respectively, \( \hat{\varphi}_2 \)), as one of the nonzero columns of the matrix \( I - T^*T \) \( (I - TT^*) \). In addition,

\[
\text{Trace}(I - T^*T) = \text{Trace}(I - TT^*) = \rho_0^2.
\]
Thus, if \( \varphi_2 = \begin{pmatrix} \varphi_2^{(1)} \\ \varphi_2^{(2)} \\ \vdots \\ \varphi_2^{(N)} \end{pmatrix} \), then the column \( F \) takes the form \( F = \bar{b}_2 \rho_0 \begin{pmatrix} \varphi_2^{(1)} \\ \varphi_2^{(2)} \\ \vdots \\ \varphi_2^{(N)} \end{pmatrix} \).

If \( \varphi_1 = \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_1^{(2)} \\ \vdots \\ \varphi_1^{(N)} \end{pmatrix} \), then the row \( G \) takes the form \( G = b_1 \rho_0 \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_1^{(2)} \\ \vdots \\ \varphi_1^{(N)} \end{pmatrix} \). Finally, the number \( S \) is given by \( -b_1 \bar{b}_2 (T^* \varphi_2, \varphi_1) \).

If \( \dim H = N \) and \( T \) is a completely nonunitary contraction with rank one defects, then \( \Theta_\Delta \) is a finite Blaschke product

\[
\Theta_\Delta(z) = e^{i\varphi} \prod_{k=1}^{N} \frac{z - \bar{z}_k}{1 - z_k z},
\]

where the numbers \( z_1, \ldots, z_N \) are the eigenvalues of \( T \). Since all other unitary colligations are of the form \( (3.2) \), for the characteristic function \( \Theta_\Delta(z) \) we get

\[
\Theta_\Delta(z) = c_1 c_2 \Theta_\Delta(z), \quad z \in \mathbb{D}, \quad t \in [0, 2\pi).
\]

Let \( U \) be a unitary operator with a cyclic vector \( e \), acting on the Hilbert space \( H \). The spectral measure \( \mu \) associated with \( U \) and \( e \) provides the relation

\[
(F(U)e, e) = \int_T F(\zeta)d\mu(\zeta),
\]

which is the Spectral Theorem for unitaries. For instance,

\[
F(z) = ((U + zI)(U - zI)^{-1}e, e) = \int_T \frac{\zeta + z}{\zeta - z} d\mu(\zeta), \quad z \in \mathbb{D}
\]

is the Carathéodory function \( (1.11) \), i.e., \( F \) is holomorphic in the unit disc \( \mathbb{D} \), \( \text{Re} F > 0 \) in \( \mathbb{D} \), and \( F(0) = 1 \).

**Theorem 3.3.** Let \( T \) be a completely nonunitary contraction with rank one defects, \( \Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix} ; \mathbb{C}, \mathbb{C}, H \right\} \) be the prime unitary colligation, and \( \Theta_\Delta \) be its characteristic function. Put

\[
F(z) = \begin{pmatrix} S & G \\ F & T \end{pmatrix} \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix}, \quad z \in \mathbb{D},
\]

where \( U = \begin{pmatrix} S & G \\ F & T \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \). Then

\[
\Theta_\Delta(\bar{z}) = \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}, \quad F(z) = \frac{1 + z \Theta_\Delta(\bar{z})}{1 - z \Theta_\Delta(\bar{z})}, \quad z \in \mathbb{D}.
\]

**Proof.** We use the well known Schur–Frobenius formula for the inverse of block operators (see, e.g., \[16\ Section 0.2\], \[17\ p. 57\]). Let \( \mathfrak{H}_1 \) and \( \mathfrak{H}_2 \) be two Hilbert spaces, and \( \Phi \) an operator in \( \mathfrak{H}_1 \oplus \mathfrak{H}_2 \) given by the block operator matrix

\[
\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}.
\]
Suppose that $D^{-1} \in \mathcal{L}(\mathcal{H}_2)$ and $(A - BD^{-1}C)^{-1} \in \mathcal{L}(\mathcal{H}_1)$. Then $\Phi^{-1} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H}_2)$ and

$$\Phi^{-1} = \begin{pmatrix} K^{-1} & -K^{-1}BD^{-1} \\ -D^{-1}CK^{-1} & D^{-1} + D^{-1}CK^{-1}BD^{-1} \end{pmatrix},$$

where $K = A - BD^{-1}C$.

Applying this formula for $\Phi = I - zU = 1 - zS - z^2G(I - zT)^{-1}F = 1 - z\Theta_\Delta(z)$. Therefore

$$\left((I - zU)^{-1}\bar{1}, \bar{1}\right) = \frac{1}{1 - z\Theta_\Delta(z)}, \quad z \in \mathbb{D}.$$

Let

$$\Psi(z) = \left((I + zU)(I - zU)^{-1}\bar{1}, \bar{1}\right), \quad z \in \mathbb{D}.$$ 

Clearly, the equality $F(z) = \overline{\Psi(\bar{z})}$ holds, which yields (3.5). □

**Remark 3.4.** Relations (3.5) is proved in [14, Theorem 1.20, Comments 2.8]. Our proof is different.

### 4. OPUC and CMV matrices

#### 4.1. Basics of OPUC

It is well recognized now that the theory of orthogonal polynomials on the real line plays an important role in the spectral theory of self-adjoint operators (and close to such operators) acting on Hilbert spaces. Likewise, the theory of orthogonal polynomials on the unit circle (OPUC) appears in the same fashion in the study of unitary operators and close to such operators. Here we recall some rudiments and advances of the OPUC theory.

If $\mu$ is a nontrivial probability measure on $\mathbb{T}$ (that is, not supported on a finite set), the monic orthogonal polynomials $\Phi_n(z, \mu)$ (or $\Phi_n$ if $\mu$ is understood) are uniquely determined by

$$\Phi_n(z) = \prod_{j=1}^{n}(z - z_{n,j}), \quad \int_{\mathbb{T}} \zeta^{-j}\Phi_n(\zeta) \, d\mu = 0, \quad j = 0, 1, \ldots, n - 1,$$

so on the Hilbert space $L^2(\mathbb{T}, d\mu)$, $\langle \Phi_n, \Phi_m \rangle = 0, n \neq m$. We also consider the orthonormal polynomials $\phi_n$ of the form $\phi_n = \Phi_n/\|\Phi_n\|$.

In case when $\mu$ is supported on a finite set, that is,

$$\mu = \sum_{k=1}^{N} \mu_k \delta(\zeta_k), \quad \zeta_k \in \mathbb{T},$$

a finite number of orthogonal polynomials $\{\Phi_k\}_{k=0}^{N-1}$ can be defined in the same manner.

Clearly, (4.1) and the fact that the space of polynomials of degree at most $n$ has dimension $n + 1$ imply

$$\deg(P) = n, \quad P \perp \zeta^j, \quad j = 0, 1, \ldots, n - 1 \Rightarrow P = c\Phi_n.$$
On $L^2(\mathbb{T}, d\mu)$ the anti-unitary map $f^*(\zeta) := \zeta^n \overline{f(\zeta)}$ (which depends on $n$) is naturally defined. The set of polynomials of degree at most $n$ is left invariant:

$$P(z) = \sum_{j=0}^{n} p_j z^j \Rightarrow P^*(z) = \sum_{j=0}^{n} \bar{p}_{n-j} z^j.$$  \hfill (4.4)

(4.3) now implies

$$\text{deg}(P) \leq n, \quad P \perp \zeta^j, \quad j = 1, \ldots, n \Rightarrow P = c \Phi_n^*.$$  \hfill (4.5)

A key feature of the unit circle is that the multiplication $Uf = zf$ in $L^2(\mathbb{T}, d\mu)$ is a unitary operator. So the difference $\Phi_n(z) - \zeta \Phi_n(z)$ is of degree $n$ and orthogonal to $z^j$ for $j = 1, 2, \ldots, n$, and by (4.5)

$$\Phi_{n+1}(z) = z \Phi_n(z) - \bar{\alpha}_n(\mu) \Phi_n^*(z)$$

with some complex numbers $\alpha_n(\mu)$, called the Verblunsky coefficients [37]. (4.6) is known as the Szegő recurrences after its first occurrence in the celebrated book [40] of G. Szegő. (4.6) at $z = 0$ imply

$$\alpha_n(\mu) = \alpha_n = -\Phi_{n+1}(0).$$  \hfill (4.7)

It is known that for nontrivial measures $|\alpha_n| < 1$ for all $n = 0, 1, 2, \ldots$, and for trivial measures (4.2) one has a finite set of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{N-1}$ with $|\alpha_n| < 1$, $n = 0, 1, \ldots, N - 2$, and $|\alpha_{N-1}| = 1$. Since it arises often, define

$$\rho_j := \sqrt{1 - |\alpha_j|^2}, \quad 0 \leq \rho_j \leq 1, \quad |\alpha_j|^2 + \rho_j^2 = 1.$$  \hfill (4.8)

The inverse Szegő recurrences are also of interest (cf. [37] Theorem 1.5.4]):

$$z \Phi_n(z) = \rho_n^{-2} (\Phi_{n+1}(z) + \bar{\alpha}_n \Phi_n^*(z)).$$  \hfill (4.9)

The norm of the polynomials $\Phi_n$ in $L^2(\mathbb{T}, d\mu)$ can be computed by:

$$||\Phi_n|| = \prod_{j=0}^{n-1} \rho_j, \quad n = 1, 2, \ldots.$$  \hfill (4.10)

Let $\mathbb{D}^\infty$ be the set of complex sequences $\{\alpha_j\}_{j=0}^\infty$ with $|\alpha_j| < 1$. The map $S$, from $\mu \to \{\alpha_j(\mu)\}_{j=0}^\infty$, is a well defined map from the set $\mathcal{P}$ of nontrivial probability measures on $\mathbb{T}$ to $\mathbb{D}^\infty$. It was S. Verblunsky who proved that $S$ is a bijection. As a matter of fact, $S$ is a homeomorphism, provided $\mathcal{P}$ is equipped with the weak*-topology, and $\mathbb{D}^\infty$ with the topology of component convergence. Moreover, it follows directly from (4.6) that for two measures $\mu_1$ and $\mu_2$

$$\alpha_j(\mu_1) = \alpha_j(\mu_2), \quad j = 0, 1, \ldots, n - 1 \Rightarrow \Phi_j(z, \mu_1) = \Phi_j(z, \mu_2), \quad j = 0, 1, \ldots, n.$$  \hfill (4.11)

Conversely, by (4.9)

$$\Phi_n(z, \mu_1) = \Phi_n(z, \mu_2) \Rightarrow \alpha_j(\mu_1) = \alpha_j(\mu_2), \quad j = 0, 1, \ldots, n - 1.$$  \hfill (4.12)

The orthonormal set $\{\phi_n\}_{n \geq 0}$ does not necessarily form a basis in $L^2(\mathbb{T}, d\mu)$ (e.g., if $d\mu = dm$ is the normalized Lebesgue measure on $\mathbb{T}$, then $\phi_n = \zeta^n$ and $\zeta^{-1}$ is orthogonal to all $\phi_n$). A celebrated result of Szegő – Kolmogorov – Krein reads that $\{\phi_n\}$ is a basis in $L^2(\mathbb{T}, d\mu)$ if and only if $\log \mu' \not\in L^1(\mathbb{T})$, where $\mu'$ is the Radon – Nikodym derivative of $\mu$ with respect to $dm$. In addition, the following result holds true (cf. [37] Theorem 1.5.7)].
Theorem 4.1. For any nontrivial probability measure \( \mu \) on the unit circle, the following are equivalent

(i) \( \lim_{n \to \infty} \| \Phi_n \| = 0 \);

(ii) \( \sum_{n=0}^{\infty} |\alpha_n|^2 = \infty \);

(iii) the system \( \{ \phi_n \}_{n=0}^{\infty} \) is the orthonormal basis in \( L^2(\mathbb{T}, d\mu) \).

Note that if \( \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \) and \( P \) is the orthogonal projection in \( L^2(\mathbb{T}, d\mu) \) onto \( \text{span} \{ \zeta^n, n = 0, 1, \ldots \} \), then (see [38])

\[
\| (I - P) \zeta \| = \prod_{n=0}^{\infty} \rho_n.
\]

Let us now turn to the basic properties of zeros \( \{ z_{n,j} \}_{j=1}^{n} \) of OPUC. It is well known (cf., e.g., [37, Theorem 1.7.1]) that \( |z_{n,j}| < 1 \) for all \( n \) and \( j \). Moreover, a result of Geronimus [37, Theorem 1.7.5] reads that given a monic polynomial \( P_n \) of degree \( n \) with all its zeros inside \( \mathbb{D} \), there is a (nontrivial) probability measure \( \mu \) on \( \mathbb{T} \) such that \( P_n = \Phi_n(\mu) \). Actually, there are infinitely many such measures, all of them have the same Verblunsky coefficients up to the order \( n - 1 \), and the same moments up to the order \( n \). Given a monic polynomial \( P_n \) with all its zeros inside the disk, let us call a monic polynomial \( Q_{n+m} + m \) an extension of \( P_n \), if there is a measure \( \mu \) such that \( P_n = \Phi_n(\mu), Q_{n+m} = \Phi_{n+m}(\mu) \).

To obtain all such extensions one just has to extend a sequence of Verblunsky coefficients \( \alpha_0, \ldots, \alpha_{n-1} \), which are completely determined by \( P_n \), by a sequence \( \beta_0, \ldots, \beta_{m-1} \) with arbitrary \( \beta_j \in \mathbb{D} \) and then apply (4.6).

One of the most recent advances in the study of zeros of OPUC is the theorem of Simon and Totik [37, Theorem 1.7.15], which claims that given a polynomial \( P_n \) as above, and an arbitrary set of points \( z_1, \ldots, z_m \) in the unit disk, not necessarily distinct, there is an extension \( Q_{n+m} \) of \( P_n \) such that \( Q_{n+m}(z_j) = 0, j = 1, 2, \ldots, m \), counting the multiplicity. The latter as usual means that

\[
z_k = z_{k+1} = \ldots = z_{k+p} \Rightarrow Q_{n+m}(z_k) = Q'_{n+m}(z_k) = \ldots = Q^{(p)}_{n+m}(z_k) = 0.
\]

The uniqueness of such extension is an open problem. A particular case \( m = 1 \) appeared earlier in [3]. Now \( \beta_0 = \alpha_n \) is defined uniquely from (4.6) by

\[
0 = Q_{n+1}(z_1) = z_1 P_n(z_1) - \bar{\alpha}_n P_n^*(z_1).
\]

This result will play a key role in the inverse problems with mixed data in Section 7.

4.2. Geronimus theory. There is an important analytic aspect of the OPUC theory which was developed by Geronimus [18, 19] in 1940’s.

Given a probability measure \( \mu \) on \( \mathbb{T} \), define the Carathéodory function by

\[
F(z) = F(z, \mu) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) = 1 + 2 \sum_{n=1}^{\infty} \beta_n z^n, \quad \beta_n = \int_{\mathbb{T}} \zeta^{-n} d\mu
\]
the moments of $\mu$. $F$ is an analytic function in $\mathbb{D}$ which obeys $\text{Re} F > 0$, $F(0) = 1$. The Schur function is then defined by

\begin{equation}
(4.12) \quad f(z) = f(z, \mu) := \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}, \quad F(z) = \frac{1 + zf(z)}{1 - zf(z)},
\end{equation}

so it is an analytic function in $\mathbb{D}$ with $\sup_{\mathbb{D}} |f(z)| \leq 1$. A one-to-one correspondence can be easily set up between the three classes (probability measures, Carathéodory and Schur functions). Under this correspondence $\mu$ is trivial, that is, supported on a finite set, if and only if the associate Schur function is a finite Blaschke product. Moreover, this Blaschke product has the order $N - 1$ for measures (4.12).

We proceed with the Schur algorithm. Given a Schur function $f = f_0$, which is not a finite Blaschke product, define inductively

\begin{equation}
(4.13) \quad f_{n+1}(z) = \frac{f_n(z) - \gamma_n}{z(1 - \gamma_n f_n(z))}, \quad \gamma_n = f_n(0).
\end{equation}

It is clear that the sequence $\{f_n\}$ is an infinite sequence of Schur functions (called the $n^{th}$ Schur iterates) and neither of its terms is a finite Blaschke product. The numbers $\{\gamma_n\}$ are called the Schur parameters:

$$
Sf = \{\gamma_0, \gamma_1, \ldots\}.
$$

In case when

$$
f(z) = e^{i\varphi} \prod_{k=1}^{N} \frac{z - z_k}{1 - \bar{z}_k z}
$$

is a finite Blaschke product of order $N$, the Schur algorithm terminates at the N-th step. The sequence of Schur parameters $\{\gamma_k\}_{k=0}^{N}$ is finite, $|\gamma_k| < 1$ for $k = 0, 1, \ldots, N - 1$, and $|\gamma_N| = 1$.

If a Schur function $f$ is not a finite Blaschke product, the connection between the non-tangential limit values $f(\zeta)$ and its Schur parameters $\{\gamma_n\}$ is given by the formula

\begin{equation}
(4.14) \quad \prod_{n=0}^{\infty} (1 - |\gamma_n|^2) = \exp \left\{ \int_{\mathbb{T}} \ln(1 - |f(\zeta)|^2) dm \right\}
\end{equation}

(see [8]). It follows that

$$
\sum_{n=0}^{\infty} |\gamma_n|^2 = \infty \iff \ln(1 - |f(\zeta)|^2) \notin L^1(\mathbb{T}).
$$

In addition, if one of the conditions

1. $\limsup_{n \to \infty} |\gamma_n| = 1$,
2. $\lim_{n \to \infty} \gamma_n \gamma_{n+m} = 0$ for each $m = 1, 2, \ldots$, but $\limsup_{n \to \infty} |\gamma_n| > 0$,

is fulfilled, then $f$ is the inner function (see [34, 24]).

Later in Section 7 we will make use of the following fundamental result of Schur [35]: the set of all Schur functions $f$ with prescribed first Schur parameters $\gamma_0, \ldots, \gamma_n$ is given by the linear fractional transformation

\begin{equation}
(4.15) \quad f(z) = \frac{A(z) + zB^*(z)s(z)}{B(z) + zA^*(z)s(z)},
\end{equation}

where
where \( s \) is an arbitrary Schur function, and \( A, B \) are polynomials of degree at most \( n \). Moreover,

\[
Sf = \{\gamma_0, \ldots, \gamma_n, \gamma_0(s), \gamma_1(s), \ldots\}
\]

The pair \((A, B)\), known as the Wall pair, is completely determined by \(\{\gamma_j\}_{j=0}^n\). Specifically,

\[
W(z) := \begin{pmatrix} zB^*(z) & A(z) \\ zA^*(z) & B(z) \end{pmatrix} = Q_{\gamma_0}(z) Q_{\gamma_1}(z) \cdots Q_{\gamma_n}(z),
\]

where

\[
Q_\omega(z) = \frac{1}{\sqrt{1 - |\omega|^2}} \begin{pmatrix} z & \omega \\ z\bar{\omega} & 1 \end{pmatrix}, \quad \omega \in \mathbb{D}.
\]

By computing determinants, we see that

\[
B^*(z)B(z) - A^*(z)A(z) = z^n \prod_{j=0}^n (1 - |\gamma_j|^2)^{1/2},
\]

so \( A \) and \( B \) have no common zeros in \( \mathbb{C} \setminus \{0\} \). In fact they have no common zeros at all since \( B(0) = 1 \). It is known also that \( B \neq 0 \) in \( \mathbb{D} \), and both \( AB^{-1} \) and \( A^*B^{-1} \) are Schur functions.

A straightforward computation shows that \( Q_\omega \) (and hence \( W \)) are \( j \)-inner matrix functions:

\[
W^*(z)jW(z) \geq j \quad \text{for} \quad z \in \mathbb{D},
\]

\[
W^*(z)jW(z) = j \quad \text{for} \quad z \in \mathbb{T}
\]

with the signature matrix

\[
j = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

For further properties of the Wall pairs see [24, Section 4], [37, Chapter 1.3.8].

A curious situation when the Schur parameters for a finite Blaschke product can be computed explicitly was found by Khrushchev [25, formula (1.12)]. Let \( \mu \) be a nontrivial probability measure (or measure of the form (4.2) with big enough \( N \)) with Verblunsky coefficients \( \{\alpha_k\} \), and \( \Phi_n \) be its \( n \)th monic orthogonal polynomial. Consider the following Blaschke product of order \( n \)

\[
b_0(z) := \Phi_n(z) = \prod_{j=1}^n \frac{z - z_{n,j}}{1 - \bar{z}_{n,j}z}, \quad b_0(0) = -\bar{\alpha}_{n-1}.
\]

It is a matter of a simple computation based on (4.19) to make sure that

\[
b_1(z) = \frac{b_0(z) - b_0(0)}{z(1 - b_0(0)b_0(z))} = \frac{\Phi_{n-1}(z)}{\Phi^*_{n-1}(z)}.
\]

Hence the Schur parameters of \( b_0 \) are of the form

\[
Sb_0 = \{-\bar{\alpha}_{n-1}, -\bar{\alpha}_{n-2}, \ldots, -\bar{\alpha}_0, 1\}.
\]

The fundamental paper of Schur [35] had appeared a few years before Szegö introduced the notion of orthogonal polynomials on the unit circle. Amazingly, neither of them benefited from the ideas of the other. Only 20 years later Geronimus put them together and came up with the following fundamental result (see [18, Theorem IX, p. 111])
Theorem 4.2. Let $\mu$ be a nontrivial probability measure on $\mathbb{T}$ and $f$ its Schur function with the Schur parameters $\gamma_n(f)$. Then $\gamma_n(f) = \alpha_n(\mu)$. For measures $\mu$ the latter equality holds for $n = 0, 1, \ldots, N - 1$.

It is clear now why a minus and conjugate is taken in (4.6).

We complete with the result which will be used later on in Section 7.

Theorem 4.3. Given two sets $\alpha_0, \ldots, \alpha_{n-1}$ and $z_1, \ldots, z_m$ of complex numbers in $\mathbb{D}$, and $\gamma \in \mathbb{T}$, there exists a finite Blaschke product $b$ of order $n + m$ such that

(i) $Sb = \{\omega_0, \ldots, \omega_{m-1}, \alpha_0, \ldots, \alpha_{n-1}, \gamma\}$,

(ii) $b(z_j) = 0$, $j = 1, \ldots, m$, counting multiplicity.

Proof. Denote $\beta_k := -\gamma \alpha_{n-k-1}$, $k = 0, 1, \ldots, n - 1$ and construct a system of monic orthogonal polynomials $\{\Phi_k(z, \beta)\}_{k=0}^n$ by (4.6). The theorem of Simon-Totik claims that there is a measure $\mu$ with

$$
\Phi_n(z, \mu) = \Phi_n(z, \beta), \quad \Phi_{n+m}(z_j, \mu) = 0, \quad j = 1, \ldots, m,
$$
counting the multiplicity. The first equality means that $\alpha_k(\mu) = \beta_k$, $k = 1, \ldots, n - 1$. Finally, put

$$
b(z) := \gamma \frac{\Phi_{n+m}(z, \mu)}{\Phi_{n+m}(z, \mu)}.
$$

The result now follows from Khrushchev’s formula (4.16). \qed

Note that for $m = 1$ the Blaschke product is uniquely determined.

4.3. CMV matrices. One of the most interesting developments in the OPUC theory in recent years is the discovery by Cantero, Moral, and Velázquez [11, 12] of a matrix realization for the operator of multiplication by $\zeta$ on $L^2(\mathbb{T}, d\mu)$ which is a unitary matrix of finite band size (i.e., $|\langle \chi_m, \chi_n \rangle| = 0$ if $|m - n| > k$ for some $k$); in this case, $k = 2$ to be compared with $k = 1$ for the Jacobi matrices, which correspond to the real line case. The CMV basis (complete, orthonormal system) $\{\chi_n\}$ is obtained by orthonormalizing the sequence $1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \ldots$, and the matrix, called the CMV matrix,

$$
\mathcal{C} = \mathcal{C}(\mu) = \|e_{n,m}\|_{m,n=0}^\infty = \|\langle \chi_m, \chi_n \rangle\|, \quad m, n \in \mathbb{Z}_+
$$
is five-diagonal. Remarkably, the $\chi$’s can be expressed in terms of $\phi$’s and $\phi^*$’s:

$$
\chi_{2n}(z) = z^{-n}\phi^*_{2n}(z), \quad \chi_{2n+1}(z) = z^{-n}\phi_{2n+1}(z), \quad n \in \mathbb{Z}_+,
$$
and the matrix elements in terms of $\alpha$’s and $\rho$’s:

$$
\mathcal{C} = \mathcal{C}(\{\alpha_n\}) = 
\begin{pmatrix}
\bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\
\rho_0 & -\bar{\alpha}_1\rho_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\
0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \ldots \\
0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\alpha_3\alpha_2 & \rho_3\alpha_2 & \ldots \\
0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\alpha_4\alpha_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix},
$$

$\alpha$’s are the Verblunsky coefficients and $\rho$’s are given in (4.8).

It is not hard to write down a general formula for the matrix entries $c_{ij}$ (see [22]). Let $2\epsilon_m := 1 - (-1)^m$, $m \in \mathbb{Z}_+$, and $\epsilon_{-1} = 1$, so $\{\epsilon_m\}_{m \ge 0} = \{0, 1, 0, 1, \ldots\}$,

$$
\epsilon_m + \epsilon_{m+1} = 1, \quad \epsilon_m \epsilon_{m+1} = 0, \quad \epsilon_m - \epsilon_{m+1} = (-1)^{m+1}.
$$
Then
\begin{align}
    c_{mm} &= -\bar{\alpha}_m \alpha_{m-1}, \\
    c_{m+2,m} &= \rho_m \rho_{m+1} \epsilon_m, \\
    c_{m,m+2} &= \rho_m \rho_{m+1} \epsilon_{m+1},
\end{align}
and
\begin{align}
    c_{m+1,m} &= \bar{\alpha}_{m+1} \rho_m \epsilon_m - \alpha_{m-1} \rho_m \epsilon_{m+1}, \\
    c_{m,m+1} &= \bar{\alpha}_{m+1} \rho_m \epsilon_{m+1} - \alpha_{m-1} \rho_m \epsilon_m.
\end{align}

It is clear (cf. [7, Theorem 1]), that any semi-infinite CMV matrix $C$ \((4.17)\) can be written in the three-diagonal block-matrix form
\begin{align}
    C &= \begin{pmatrix}
        B_0 & C_0 & 0 & 0 & 0 & \cdots \\
        A_0 & B_1 & C_1 & 0 & 0 & \cdots \\
        0 & A_1 & B_2 & C_2 & 0 & \cdots \\
        \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
    \end{pmatrix}
\end{align}
with
\begin{align}
    B_0 &= (\bar{\alpha}_0), \\
    C_0 &= (\bar{\alpha}_1 \rho_0 \rho_1 \rho_0), \\
    A_0 &= \begin{pmatrix}
        \rho_0 \\
        0
    \end{pmatrix}, \\
    A_n &= \begin{pmatrix}
        \rho_{2n} \rho_{2n-1} & -\rho_{2n} \alpha_{2n-1} \\
        0 & 0
    \end{pmatrix}, \\
    B_n &= \begin{pmatrix}
        -\bar{\alpha}_{2n-1} \alpha_{2n-2} & -\rho_{2n-1} \alpha_{2n-2} \\
        \bar{\alpha}_{2n} \rho_{2n-1} & -\bar{\alpha}_{2n} \alpha_{2n-1}
    \end{pmatrix}, \\
    C_n &= \begin{pmatrix}
        0 & 0 \cdots \\
        \bar{\alpha}_{2n+1} \rho_{2n} & \rho_{2n+1} \rho_{2n}
    \end{pmatrix}, \\
    n &= 1, 2, \ldots.
\end{align}

There is a nice multiplicative structure of the CMV matrices. In the semi-infinite case $C$ is the product of two matrices: $C = L \mathcal{M}$, where
\begin{align}
    L &= \Psi(\alpha_0) \oplus \Psi(\alpha_2) \oplus \cdots \Psi(\alpha_{2m}) \oplus \cdots, \\
    \mathcal{M} &= 1_{1 \times 1} \oplus \Psi(\alpha_1) \oplus \Psi(\alpha_3) \oplus \cdots \oplus \Psi(\alpha_{2m+1}) \oplus \cdots,
\end{align}
and $
\Psi(\alpha) = \begin{pmatrix}
    \bar{\alpha} \\
    \rho \\
    -\alpha
\end{pmatrix}.
$ The finite $(N+1) \times (N+1)$ CMV matrix $C$ obeys $\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in \mathbb{D}$ and $|\alpha_N| = 1$ is also the product $C = L \mathcal{M}$, where in this case $\Psi(\alpha_N) = (\bar{\alpha}_N)$.

It is just natural to take the ordered set $1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \ldots$ instead of $1, \zeta, \zeta^{-1}, \zeta_1, \zeta_2, \ldots$, that leads to the alternate CMV basis \{X_n\} and the alternate CMV matrix
\begin{align}
    \tilde{C} = \|\zeta X_m, X_n\| &= \begin{pmatrix}
        \bar{\alpha}_0 & \rho_0 & 0 & 0 & 0 & \cdots \\
        \bar{\alpha}_1 \rho_0 & -\bar{\alpha}_1 \alpha_0 & \bar{\alpha}_2 \rho_1 & \rho_2 \rho_1 & 0 & \cdots \\
        \rho_1 \rho_0 & -\rho_1 \alpha_0 & -\bar{\alpha}_2 \alpha_1 & -\rho_2 \alpha_1 & 0 & \cdots \\
        0 & 0 & \rho_3 \rho_2 & -\rho_3 \alpha_2 & -\bar{\alpha}_4 \rho_3 & \cdots \\
        0 & 0 & 0 & \rho_3 \alpha_2 & -\rho_3 \alpha_2 & \bar{\alpha}_4 \alpha_3 & \cdots \\
        \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
    \end{pmatrix},
\end{align}

which turns out to be the transpose of $C$ (see [37, Corollary 4.2.6]). Furthermore, $L = L^t$ and $\mathcal{M} = \mathcal{M}^t$ imply $\tilde{C} = C^t = \mathcal{M} L$.

An important relation between CMV matrices and monic orthogonal polynomials similar to the well-known property of orthogonal polynomials on the real line
$$
\Phi_n(z) = \det(zI_n - C^{(n)}),
$$
holds, where \( C^{(n)} \) is the principal \( n \times n \) block of \( C \).

One of the most important results of Cantero, Moral, and Velázquez \cite{CMV05} states that each unitary operator \( U \) with the simple spectrum (i.e. having a cyclic vector \( e_1 \)) acting on some infinite-dimensional separable Hilbert space (respectively, finite-dimensional Hilbert space) is unitarily equivalent to a certain CMV matrix in \( l_2(\mathbb{Z}_+) \) (respectively, in \( \mathbb{C}^n \)). The corresponding \( \alpha \)'s come up as the Verblunsky coefficients of the spectral measure \( d\mu \) of \( U \) associated with \( e_1 \). This is the analog of Stone’s self-adjoint cyclic model theorem. To be more precise, let us, following \cite{JA20}, call a cyclic unitary model a unitary operator \( U \) acting on a separable Hilbert space \( \mathcal{H} \) with the distinguished cyclic unit vector \( v_0 \). Two cyclic unitary models, \((\mathcal{H}, U, v_0)\) and \((\tilde{\mathcal{H}}, \tilde{U}, \tilde{v}_0)\) are called equivalent if there is a unitary operator \( W \) from \( \mathcal{H} \) onto \( \tilde{\mathcal{H}} \) such that \( W v_0 = \tilde{v}_0 \) and \( WUW^{-1} = \tilde{U} \). It is clear that \( \delta_0 = (1, 0, 0, \ldots) \) is cyclic for any CMV matrix \( C \). Moreover, every class of equivalent unitary models contains exactly one CMV model \((\ell^2, C, \delta_0)\).

5. A model in the space \( L^2(\mathbb{T}, d\mu) \) of a completely nonunitary contraction with rank one defects

**Theorem 5.1.** Let \( T \) be a completely nonunitary contraction with rank one defects. Then there exists a probability measure \( \mu \) on \( \mathbb{T} \) such that \( T \) is unitarily equivalent to the following operator

\[
T h(\zeta) = P_{\delta}(\zeta h(\zeta)), \quad h \in \mathcal{H} := L^2(\mathbb{T}, d\mu) \ominus \mathbb{C},
\]

where \( P_{\delta} \) is the orthogonal projection in \( L^2(\mathbb{T}, d\mu) \) onto \( \mathcal{H} \). The Schur function associated with \( \mu \) is exactly the characteristic function of \( T \).

**Proof.** Include \( T \) into a prime unitary colligation \( \Delta = \{(S \ G \ F \ T); \mathbb{C}, \mathbb{C}, \mathcal{H}\} \). The characteristic function \( \Theta_{\Delta} \) agrees with the characteristic function of \( T^* \). By Theorem \ref{thm:PrimeColligation} the vector \( \vec{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is cyclic for the unitary operator \( U = \begin{pmatrix} S & G \\ F & T \end{pmatrix} \).

Let \( E_U(\zeta) \) be the resolution of identity for \( U \). Define \( d\mu(\zeta) := (dE_U(\zeta)\vec{1}, \vec{1}) \) and put

\[
U f(\zeta) = \zeta f(\zeta)
\]

the unitary multiplication operator in \( L^2(\mathbb{T}, d\mu) \). By the spectral theorem for unitaries with cyclic vectors (cf. \cite{Sar78} Section 1.4.5) there exists a unitary operator \( W : \mathbb{C} \oplus \mathcal{H} \to L^2(\mathbb{T}, d\mu) \) such that

\[
U = W^{-1}UW \quad \text{and} \quad W\vec{1} = 1.
\]

It follows that \( W \) takes the block-operator form

\[
W = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathcal{H} \end{pmatrix},
\]

where \( \mathcal{H} = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C} \), \( V : \mathcal{H} \to L^2(\mathbb{T}, d\mu) \ominus \mathbb{C} \) is a unitary operator. If \( \mathcal{U} \) is given by \( \mathcal{U} := P_{\delta}U|_{\mathcal{H}} \), then

\[
\mathcal{U} := P_{\delta}U|_{\mathcal{H}} = VTV^{-1},
\]

i.e., \( T \) is unitarily equivalent to \( \mathcal{U} \). Clearly, \( U \) has the block form

\[
U = \begin{pmatrix} P_{\delta}U|_{\mathcal{H}} & P_{\delta}U|_{\mathcal{H}} \\ P_{\delta}U|_{\mathcal{H}} & \mathcal{U} \end{pmatrix}.
\]
where $P_C$ is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto the subspace $\mathbb{C}$ of the constant functions in $L^2(\mathbb{T}, d\mu)$. The unitary colligation $\Delta$ is unitarily equivalent to the unitary colligation

\begin{equation}
(5.2) \quad \left\{ \begin{pmatrix} P_C U & 1 \\ P_H U & 1 \end{pmatrix}, \mathbb{C}, \mathbb{C}, \mathbb{H} \right\}.
\end{equation}

Note that

\[ P_C(U1) = \int_{\mathbb{T}} \zeta \, d\mu, \quad P_H(U1) = \zeta - \int_{\mathbb{T}} \zeta \, d\mu, \quad P_C(U^* 1) = \overline{\zeta} - \int_{\mathbb{T}} \overline{\zeta} \, d\mu. \]

Let $F(z) = \left( (U + zI)(U - zI)^{-1}1, 1 \right)$. Then

\[ F(z) = \left( (U + zI)(U - zI)^{-1}1, 1 \right) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta), \]

i.e., $F$ is the Carathéodory function associated with $\mu$. From Theorem 3.3 we conclude

\[ \Theta_{\Delta}(\bar{z}) = \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}, \]

and so by (4.12) $\Theta_{\Delta}(\bar{z})$ agrees with the Schur function associated with $\mu$. \hfill \Box

Let $\{\Phi_n\}$ be the system of monic polynomials orthogonal with respect to $\mu$, and let $\{\alpha_n\}$ be the corresponding Verblunsky coefficients. By Geronimus’ theorem $\{\alpha_n\}$ are the Schur parameters of $f$. Let $\mathcal{H}^{(c)}$ be the controllable subspace of the unitary colligation (5.2). From (3.1) it follows that

\[ (\mathcal{H}^{(c)})^\perp = L^2(\mathbb{T}, d\mu) \ominus \overline{\text{span}} \{\zeta^n, \ n = 0, 1, \ldots\}. \]

If $\mu$ is a nontrivial measure, then in view of (4.10) we obtain

\[ \|P_{(\mathcal{H}^{(c)})^\perp}\zeta\| = \prod_{n=0}^{\infty} (1 - |\alpha_n|^2)^{1/2}. \]

The latter is equivalent to

\[ \|P_{(\mathcal{H}^{(c)})^\perp}P_C(U^* 1)\| = \prod_{n=0}^{\infty} (1 - |\alpha_n|^2)^{1/2}. \]

Hence, from (2.10) and (2.7) we have the equivalence

\begin{equation}
(5.3) \quad \text{span} \{ \mathcal{T}^n \mathcal{D}_{\mathcal{H}^*}, \ n = 0, 1, \ldots\} = \mathcal{H} \iff \sum_{n=0}^{\infty} |\alpha_n|^2 = \infty.
\end{equation}

Remark 5.2. By the construction of Theorem 5.1 the Schur function $f$ associated with $\mu$ is exactly $\Theta_{\Delta}(\bar{z})$. Another (unitary equivalent) models of $T$ are connected with the operators $U_\lambda = \begin{pmatrix} \overline{\lambda S} & G \\ \overline{\lambda F} & T \end{pmatrix}$, where $|\lambda| = 1$. The characteristic function of the unitary colligation $\Delta_\lambda = \left\{ \begin{pmatrix} \overline{\lambda S} & G \\ \overline{\lambda F} & T \end{pmatrix}, \mathbb{C}, \mathbb{C}, \mathbb{H} \right\}$. 

as \( \bar{\lambda} \Theta_{\Delta} \). The model operator \( \mathcal{H}_\lambda \) takes the form
\[
\mathcal{H}_\lambda = L^2(\mathbb{T}, d\mu_\lambda) \oplus \mathbb{C}, \quad \mathcal{H}_\lambda h(\zeta) = P_{\delta_\lambda}(\zeta h(\zeta)), \quad h(\zeta) \in \mathcal{H}_\lambda.
\]
The Schur function \( f_\lambda \) associated with \( \mu_\lambda \) is \( f_\lambda = \lambda f \). The connection between the Carathéodory functions \( F_\lambda(z) = \left((U_\lambda + zI)(U_\lambda - zI)^{-1}I_1, I_1\right) \) and \( F \) is given by
\[
F_\lambda(z) = \frac{(1 - \lambda) + (1 + \lambda)F(z)}{(1 + \lambda) + (1 - \lambda)F(z)}.
\]
The measures \( \mu_\lambda \) are known as the Aleksandrov measures associated with \( \mu \) \cite[Section 1.3.9]{37}.

6. Truncated CMV matrices

6.1. Truncated CMV matrix as a model for contractions with rank one defects.

Let \( \mathcal{C} = \mathcal{C}(\{\alpha_n\}) \) be the CMV matrix given by \( (4.17) \). Recall that \( \mathcal{C}(\{\alpha_n\}) \) is the matrix representation of the unitary operator \( \mathcal{U} \) of multiplication by \( \zeta \) in \( L^2(\mathbb{T}, d\mu) \), where \( \mu \) is the probability measure with Verblunsky coefficients \( \{\alpha_n\} \). By the Geronimus theorem the Schur parameters of the Schur function \( (4.12) \) associated with \( \mu \) are \( \{\alpha_n\} \).

The matrix \( \mathcal{C} \) determines the unitary operator in the space \( l_2(\mathbb{Z}_+) \) (respectively in \( \mathbb{C}^{N+1} \) in the case of \( (N + 1) \times (N + 1) \) matrix). The vector \( \delta_0 = (1, 0, 0, \ldots)' \) is cyclic for \( \mathcal{C} \). Consider the matrix
\[
(6.1) \quad \mathcal{T} = \mathcal{T}(\{\alpha_n\}) = \begin{pmatrix}
-\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\
\bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \ldots \\
\rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \ldots \\
0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

obtained from \( \mathcal{C} \) by deleting the first row and the first column. It is clear from \( (4.20) \) that a semi-infinite \( \mathcal{T} \) takes on the three-diagonal \( 2 \times 2 \) block-matrix form
\[
\mathcal{T} = \begin{pmatrix}
\mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & 0 & \ldots \\
\mathcal{A}_1 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & 0 & \ldots \\
0 & \mathcal{A}_2 & \mathcal{B}_3 & \mathcal{C}_3 & 0 & \ldots \\
\mathcal{A}_\infty & \mathcal{B}_\infty & \mathcal{C}_\infty & \mathcal{A}_\infty & \mathcal{B}_\infty & \mathcal{C}_\infty & \mathcal{A}_\infty & \mathcal{B}_\infty & \mathcal{C}_\infty & \ldots
\end{pmatrix},
\]

where \( \mathcal{A}_n, \mathcal{B}_n, \) and \( \mathcal{C}_n \) are defined in \( (4.21) \). Henceforth \( \mathcal{T} \) is called a truncated CMV matrix. \( \mathcal{T} \) is the matrix of the operator \( \mathcal{H} = \mathcal{P}_3 \mathcal{U}(\delta) \), where \( \mathcal{P}_3 \) is the orthogonal projection in \( L^2(\mathbb{T}, d\mu) \) onto the subspace \( \delta = L^2(\mathbb{T}, d\mu) \). \( \mathcal{C} \).

It is easy to see that given \( \mathcal{T} \) \( (6.1) \), the values \( \alpha_n \) are uniquely determined. Indeed, from \( (2, 2) \) and \( (3, 2) \) entries we have by \( (4.8) \) \( |\alpha_1|^2 = |\bar{\alpha}_2\alpha_1|^2 + |\rho_3\alpha_2|^2 \), so \( |\alpha_1| \) and \( \rho_1 > 0 \) are known, and we find \( \alpha_0, \alpha_2 \) from \( (1, 2) \) and \( (2, 1) \) entries of \( (6.1) \). From \( (2, 1) \) and \( (2, 2) \) entries we get \( \rho_2 > 0 \), then \( \alpha_1, \alpha_3, \) etc. We call \( \alpha_n = \alpha_n(\mathcal{T}) \) the parameters of \( \mathcal{T} \) \( (6.1) \).

As it was mentioned in Section 4.3, \( \mathcal{C} = \mathcal{L} \mathcal{M} \), \( \mathcal{L} \) and \( \mathcal{M} \) are defined in \( (4.22) \). Given a matrix \( A \), we denote by \( A_r \) \( (A_c) \) the matrix obtained from \( A \) by deleting the first row (column). Clearly, \( A_{rc} = (A_r)_c \). So we have \( \mathcal{T} = \mathcal{C}_{rc} = \mathcal{L}_r \mathcal{M}_c \). \( \mathcal{M}_c \) is isometric with \( \text{dim ran } (I - \mathcal{M}_c \mathcal{M}_c^*) = 1 \), whereas \( \mathcal{L}_r \) is coisometric with \( \text{dim ran } (I - \mathcal{L}_r^* \mathcal{L}_r) = 1 \).
Let \( P_{\delta_0^+} \) be the orthogonal projection in \( l_2(\mathbb{Z}_+) \) (\( \mathbb{C}^{N+1} \)) onto the subspace \( \delta_0^+ \cong l_2(\mathbb{N}) \) (\( \mathbb{C}^N \)). Then the matrix \( \mathcal{T} \) determines on the Hilbert space \( \delta_0^+ \) the operator \( \mathcal{T} = P_{\delta_0^+} \mathcal{C} \mid \delta_0^+ \).

Let the operators (matrices) \( \mathcal{S} : \mathbb{C} \to \mathbb{C} \), \( \mathcal{F} : \mathbb{C} \to \delta_0^+ \) and \( \mathcal{G} : \delta_0^+ \to \mathbb{C} \) be given by

\[
\mathcal{S}1 = \bar{\alpha}, \quad \mathcal{F}1 = \begin{pmatrix} \rho_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathcal{G} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \bar{\alpha}_1 \rho_0 h_1 + \rho_1 \rho_0 h_2.
\]

Hence, the matrix \( \mathcal{C} \) takes the block form

\[
\mathcal{C} = \begin{pmatrix} \mathcal{S} & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix}.
\]

From [2,10] it follows that

\[
\left\| \mathcal{G} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = \rho_0^2 |\bar{\alpha}_1 h_1 + \rho_1 h_2|^2, \quad \mathcal{D}_\mathcal{T} = \{ \lambda (\alpha_1 \delta_1 + \rho_1 \delta_2), \ \lambda \in \mathbb{C} \},
\]

\[
\left\| \mathcal{F}^* \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = \rho_0^2 |h_1|^2, \quad \mathcal{D}_{\mathcal{F}^*} = \{ \lambda \delta_1, \ \lambda \in \mathbb{C} \},
\]

\[
D_{\mathcal{T}} h = \rho_0 (h, \alpha_1 \delta_1 + \rho_1 \delta_2)(\alpha_1 \delta_1 + \rho_1 \delta_2), \quad D_{\mathcal{F}^*} h = \rho_0 (h, \delta_1) \delta_1, \quad h \in \ell^2(\mathbb{N}) (\mathbb{C}^N),
\]

\[
\mathcal{T}(\alpha_1 \delta_1 + \rho_1 \delta_2) = -\alpha_0 \delta_1.
\]

Since \( \delta_0 \) is the cyclic vector for \( \mathcal{C} \), then by Theorem [3,1] the unitary colligation

\[
\Delta_{\mathcal{C}} = \left\{ \begin{pmatrix} \mathcal{S} & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix} : \mathbb{C}, \mathbb{C}, \delta_0^+ \right\}
\]

is prime, and \( \mathcal{T} \) is a completely nonunitary operator with rank one defects on the Hilbert spaces \( l_2(\mathbb{N}) \) or \( \mathbb{C}^N \).

Let

\[
F(z) = ((\mathcal{C} + zI)(\mathcal{C} - zI)^{-1}\delta_0, \delta_0), \quad f(z) = \frac{1}{2} \frac{F(z) - 1}{F(z) + 1}
\]

be the Carathéodory and the Schur functions associated with \( \mathcal{C} \). By Theorems 2.2 and 3.3 \( f \) agrees with the characteristic function of \( \mathcal{T} \).

**Proposition 6.1.**

(1) For a semi-infinite truncated CMV matrix \( \mathcal{T} = \mathcal{T}(\{\alpha_n\}) \) the following statements are equivalent:

(a) the matrix \( \mathcal{T} \) does not contain a unilateral shift;

(b) the matrix \( \mathcal{T}^* \) does not contain a unilateral shift;

(c) \( \text{span} \{\mathcal{T}^n \delta_1, \ n = 0, 1, \ldots \} = \ell^2(\mathbb{N}) \);

(d) \( \text{span} \{\mathcal{T}^{+n}(\alpha_1 \delta_1 + \rho_1 \delta_2), \ n = 0, 1, \ldots \} = \ell^2(\mathbb{N}) \);
\( \sum_{n=0}^{\infty} |\alpha_n|^2 = \infty; \)

(f) \( \ln(1 - |f(e^{it})|^2) \notin L^1[-\pi, \pi]. \)

(2) If \( T \) is a semi-finite truncated CMV matrix, and one of the conditions
(a) \( \limsup_{n \to \infty} |\alpha_n| = 1, \)
(b) \( \lim_{n \to \infty} \alpha_n \alpha_{n+m} = 0 \) for \( m = 1, 2, \ldots, \) but \( \limsup_{n \to \infty} |\alpha_n| > 0 \)

is fulfilled, then
\[
\lim_{n \to \infty} T^n = s - \lim_{n \to \infty} T^{*n} = 0.
\]

(3) If \( T \) is a finite truncated CMV matrix, then \( \lim_{n \to \infty} \|T^n\| = 0. \)

\textbf{Proof.} (1) Since \( \{\alpha_n\} \) are the Schur parameters of the Schur function \( f \) associated with the full CMV matrix \( C(\{\alpha_n\}) \), and \( f \) agrees with the characteristic function of \( T(\{\alpha_n\}) \), the equivalence of the statements (a)–(f) follows from (2.3), (2.4), (2.7), (2.9), (4.14), (6.2), (6.3), and Theorems 3.1 and 4.1.

(2) Each condition (a) or (b) implies \( f \) is inner (see subsection 4.2). Hence \( T \) belongs to the class \( C_{00}, \) i.e., \( s - \lim_{n \to \infty} T^n = s - \lim_{n \to \infty} T^{*n} = 0. \)

(3) The function \( f \) is a finite Blaschke product and so inner. Since \( T \) is finite-dimensional, we get \( \lim_{n \to \infty} \|T^n\| = 0. \) \hfill \Box

\textbf{Proposition 6.2.} Let \( T(\{\alpha_n\}) \) and \( T(\{\beta_n\}) \) be truncated CMV matrices. Then \( T(\{\alpha_n\}) \) and \( T(\{\beta_n\}) \) are unitarily equivalent if and only if \( \beta_n = e^{it}\alpha_n \) for all \( n \) and \( t \in [0, 2\pi) \).

Moreover, if \( V \) is the diagonal unitary matrix of the form
\[
V = \text{diag}(e^{it}, 1, e^{it}, 1, \ldots),
\]

then
\[
V T(\{\alpha_n\}) V^{-1} = T(\{e^{it}\alpha_n\}).
\]

\textbf{Proof.} Consider two CMV matrices \( C(\{\alpha_n\}) \) and \( C(\{\beta_n\}) \), and associated with them Schur functions \( f_{\alpha} \) and \( f_{\beta} \). Since these functions agree with the characteristic functions of \( T(\{\alpha_n\}) \) and \( T(\{\beta_n\}) \), respectively, the operators \( T(\{\alpha_n\}) \) and \( T(\{\beta_n\}) \) are unitarily equivalent if and only if \( f_{\alpha} \) and \( f_{\beta} \) differ by a scalar unimodular factor, which in turn yields \( \beta_n = e^{it}\alpha_n \) for all \( n \) and \( t \in [0, 2\pi) \).

Equality (6.6) with \( V \) (6.5) can be verified by the direct calculation based on (4.18), (4.19). So \( T(\{\alpha_n\}) \) and \( T(\{e^{it}\alpha_n\}) \) are unitarily equivalent. \hfill \Box

\textbf{Remark 6.3.} The similar problem for “full” CMV matrices can be considered as well. Let two CMV matrices \( C(\{\alpha_n\}) \) and \( C(\{\beta_n\}) \) be unitarily equivalent by a unitary preserving \( \delta_0 \). Then they are identical (see [38, Theorem 2.3]). In general, two unitaries with simple spectra are unitarily equivalent if and only if their spectral measures are in the same measure class. This is a standard issue in what is called multiplicity theory. So, two CMV matrices are unitarily equivalent if and only if their measures are mutually absolutely continuous. For instance, a CMV matrix is unitarily equivalent to the free one \( (\alpha_n \equiv 0) \) if and only if the associated measure \( \mu \) has the property \( \mu' > 0 \) a.e. and does not have a singular part.

From (6.6) it follows that
\[
T(\{e^{it}\alpha_n\}) = e^{itA} T(\{\alpha_n\}) e^{-itA}, \quad t \in \mathbb{R},
\]
where \( \mathcal{A} \) is a self-adjoint diagonal matrix \( \mathcal{A} = \text{diag}(1, 0, 1, 0, \ldots) \). Hence the matrix \( \mathcal{T}(\{e^{i\alpha_n}\}) \) satisfies the differential equation
\[
\frac{dT(t)}{dt} = i(\mathcal{A}T(t) - T(t)\mathcal{A})
\]
and \( \mathcal{T}(0) = \mathcal{T}(\{\alpha_n\}) \).

The next theorem states that truncated CMV matrices are models of completely nonunitary contractions with rank one defects.

**Theorem 6.4.** Let \( T \) be a completely nonunitary contraction with rank one defects acting on infinite-dimensional separable Hilbert space \( H \) (respectively, finite-dimensional Hilbert space). Then \( T \) is unitarily equivalent to the operator acting on \( l_2(\mathbb{N}) \) (respectively, on \( \mathbb{C}^N \) in the case \( \dim H = N \)) determined by the truncated CMV matrix \( \mathcal{T} = \mathcal{T}(\{\alpha_n\}) \), where \( \{\alpha_n\} \) are the Schur parameters of the characteristic function of \( T \). In particular, every completely nonunitary contraction with rank one defects is a product of co-isometric and isometric operators with rank one defects.

**Proof.** Include \( T \) into a prime unitary colligation \( \Delta = \{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathbb{C}, \mathbb{C}, H\} \). By Theorem 3.1 the vector \( \vec{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is a cyclic for the unitary operator \( U = \begin{pmatrix} S & G \\ F & T \end{pmatrix} \). From the results of [11, 12] (see also [36, 37]) there exists a unique CMV matrix \( \mathcal{C} \) such that
\[
U = W^{-1}\mathcal{C}W, \quad \delta_0 = W\vec{1},
\]
where \( W \) is a unitary operator from \( \mathbb{C} \oplus H \) onto \( l_2(\mathbb{Z}_+) \) (\( \mathbb{C}^{N+1} \)), and \( \delta_0 = (1, 0, 0, \ldots)' \). It follows that the operator \( W \) takes the block-operator form
\[
W = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{X} \end{pmatrix}; \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{C} \\ \delta_0^\perp \end{pmatrix},
\]
where \( \mathcal{X} : H \rightarrow \delta_0^\perp \) is a unitary operator. Hence \( \mathcal{T} = \mathcal{X}T\mathcal{X}^{-1} \), i.e., the operator \( T \) is unitarily equivalent to the operator in \( l_2(\mathbb{N}) \) (\( \mathbb{C}^N \)) given by the truncated CMV matrix \( \mathcal{T} = \mathcal{T}(\{\alpha_n\}) \). From representation (4.11) of \( F(z) = \begin{pmatrix} (U + zI)(U - zI)^{-1}\vec{1}, \vec{1} \end{pmatrix} \) and Theorem 3.3 it follows that \( \{\alpha_n\} \) are the Schur parameters of the function \( \Theta_\Delta(z) \) that agrees with the characteristic function of \( T \).

Let \( \mathcal{Q} \) be an arbitrary unitary operator in \( \delta_0^\perp \). Since \( \mathcal{T} = \mathcal{L}_r\mathcal{M}_c \), we get
\[
T = \mathcal{X}^{-1}\mathcal{T}\mathcal{X} = \mathcal{X}^{-1}\mathcal{L}_r\mathcal{M}_c\mathcal{X} = \mathcal{X}^{-1}\mathcal{L}_r\mathcal{Q}\mathcal{Q}^{-1}\mathcal{M}_c\mathcal{X} = L M,
\]
where \( M = \mathcal{Q}^{-1}\mathcal{M}_c\mathcal{X} \) is an isometric operator with rank one defect, and \( L = \mathcal{X}^{-1}\mathcal{L}_r\mathcal{Q} \) is a co-isometric operator with rank one defect. \( \square \)

Note that the unitary colligation (6.3) is unitary equivalent to the unitary colligation (5.2).

6.2. The Livšic theorem for quasi-unitary contractive extensions and the corresponding truncated CMV matrix. Let \( V \) be an isometric operator acting on some Hilbert space \( H \) with the domain \( \text{dom} V \) and the range \( \text{ran} V \). The numbers \( \dim(H \oplus \text{dom} V) \) and \( \dim(H \oplus \text{ran} V) \) are called the defect indices of \( V \). The isometric operator \( V \) is called prime if there is no nontrivial subspace on which \( V \) is unitary. In [26, 27] M. Livšic developed the spectral theory of isometric operators with equal defect indices, and their quasi-unitary
extensions. A nonunitary operator $S$ on $H$ is called a quasi-unitary extension of the isometric operator $V$ with the defect indices $(n, n)$, if $S$ agrees with $V$ on $\mathrm{dom} \ V$ and maps $H \oplus \mathrm{dom} \ V$ into $H \ominus \mathrm{ran} \ V$.

Let $\mathcal{U}$ be the bilateral shift in $\ell^2(\mathbb{Z})$, i.e., $\mathcal{U}\delta_k = \delta_{k-1}$, $k \in \mathbb{Z}$, where $\{\delta_k, \ k \in \mathbb{Z}\}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z})$. Define $\mathcal{V}_0$ by

$$\mathrm{dom} \mathcal{V}_0 = \delta_o^\perp, \quad \mathcal{V}_0 = \mathcal{U}\upharpoonright \mathrm{dom} \mathcal{V}_0,$$

Then $\mathrm{ran} \mathcal{V}_0 = \delta_o^\perp$. Let the quasi-unitary extension $\mathcal{S}_0$ of $\mathcal{V}_0$ be given by $\mathcal{S}_0\delta_0 = 0$, $\mathcal{S}_0\upharpoonright \mathrm{dom} \mathcal{V}_0 = \mathcal{V}_0$. Then each point of $\mathbb{D}$ is the eigenvalue of $\mathcal{S}_0$. So the spectrum of $\mathcal{S}_0$ agrees with $\mathbb{D}$. The following result is essentially due to M. Livšic [26].

**Theorem 6.5.** Let $S$ be a quasi-unitary contractive extension of a prime isometric operator $V$ with the defect indices $(1, 1)$. If the whole open disk $\mathbb{D}$ consists of the point spectrum of $S$, then $V$ and $S$ are unitarily equivalent to $\mathcal{V}_0$ and $\mathcal{S}_0$, respectively.

Clearly, the rank of the defect operators $(I - \mathcal{S}_0^*\mathcal{S}_0)^{1/2}$ and $(I - \mathcal{S}_0\mathcal{S}_0^*)^{1/2}$ is equal to one. Since the point spectrum of $\mathcal{S}_0$ is $\mathbb{D}$, the Sz.-Nagy–Foias characteristic function $\Theta$ of $\mathcal{S}_0$ is identically equal to zero. On the other hand, one can easily show (and it is well known) that a completely nonunitary contraction with rank one defects and zero characteristic function is unitarily equivalent to the operator $S \oplus S^*$, where $S$ is the unilateral shift in $\ell^2(\mathbb{N})$. So the operators $\mathcal{S}_0$ and $S \oplus S^*$ are unitarily equivalent. Since all Schur parameters of the function $\Theta = 0$ are zeros, the corresponding truncated CMV matrix $\mathcal{T}_0 = \|t_0(i, j)\|$ takes the form

$$\mathcal{T}_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix},$$

i.e., $t_0(2k, 2k+2) = t_0(2k+1, 2k-1) = 1$, $k \geq 1$, and the rest $t_0(i, j) = 0$. The matrix $\mathcal{T}_0$ is a submatrix of the free CMV matrix $\mathcal{C}_0$ corresponding to zero Schur parameters. Each point $z$ of $\mathbb{D}$ is the eigenvalue of $\mathcal{T}_0$. The corresponding eigensubspace is

$$\mathcal{M}_z = \{\lambda (0, 1, 0, z, 0, z^2, 0, z^3, \ldots)^t, \quad \lambda \in \mathbb{C}\}.$$

Hence, the spectrum of $\mathcal{T}_0$ is the closed unit disk $\overline{\mathbb{D}}$.

Let $\mathcal{V}_0$ be the operator in $\ell^2(\mathbb{N})$

$$\mathrm{dom} \mathcal{V}_0 = \ell^2(\mathbb{N}) \ominus \{c\delta_2\} = \ker D_{\mathcal{T}_0}, \quad \mathcal{V}_0 = \mathcal{T}_0\upharpoonright \mathrm{dom} \mathcal{V}_0.$$

Then $\mathrm{ran} \mathcal{V}_0 = \ell^2(\mathbb{N}) \ominus \{c\delta_1\} = \ker D_{\mathcal{T}_0}$, and $\mathcal{V}_0$ is isometric with the defect indices $(1, 1)$. The contraction $\mathcal{T}_0$ is the quasi-unitary extension of $\mathcal{V}_0$ with the zero characteristic function. Therefore, the truncated CMV matrix $\mathcal{T}_0$ is unitarily equivalent to the operator $\mathcal{S}_0$, and by Livšic theorem [26] the isometric operator $\mathcal{V}_0$ is unitarily equivalent to $\mathcal{V}_0$. 

All other quasi-unitary contractive extensions of $\mathcal{V}_0$ are given by the truncated CMV matrices $\mathcal{T} = \|t(i, j)\|

\begin{align*}
\mathcal{T} &= \begin{pmatrix}
0 & -re^{i\varphi} & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
& \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix},
\end{align*}

(6.8)

i.e., $t(2k, 2k + 2) = t(2k + 1, 2k - 1) = 1$, $k \geq 1$, $t(1, 2) = -re^{i\varphi}$, $r \in (0, 1)$, $\varphi$ is an arbitrary number from the interval $[0, 2\pi)$, and the rest $t(i, j) = 0$. The characteristic function of $\mathcal{T}$ is the constant function $\Theta = re^{i\varphi}$. The spectrum of each such matrix is the unit circle $\mathbb{T}$. Because $|\Theta^{-1}| = r^{-1}$, each of such matrix is similar to unitary matrix [39, Theorem IX.1.2].

The matrices $\mathcal{T}_0$ and $\mathcal{T}$ contain the shift

$$\text{dom } \mathcal{W} = \overline{\text{span}} \{\delta_1, \delta_3, \ldots, \delta_{2n-1}, \ldots\}, \quad \mathcal{W} \left( \sum_{n=1}^{\infty} h_n \delta_{2n-1} \right) = \sum_{n=1}^{\infty} h_n \delta_{2n+1}.$$  

The matrices $\mathcal{T}_0^*$ and $\mathcal{T}^*$ contain the shift

$$\text{dom } \mathcal{W}_* = \overline{\text{span}} \{\delta_2, \delta_4, \ldots, \delta_{2n}, \ldots\}, \quad \mathcal{W}_* \left( \sum_{n=1}^{\infty} h_n \delta_{2n} \right) = \sum_{n=1}^{\infty} h_n \delta_{2n+2}.$$  

Let $T$ be a completely nonunitary contraction with rank one defects and the constant characteristic function $\Theta$, $0 < |\Theta(z)| = r < 1$. Then by Theorem 6.4 $T$ is unitarily equivalent to the truncated CMV matrices (6.8).

6.3. Sub-matrices of truncated CMV matrices and iterates of their Schur functions. Along with truncated CMV matrices $\mathcal{T}(\{\alpha_n\})$ (6.1), we consider here truncated CMV matrices $\tilde{\mathcal{T}}(\{\alpha_n\})$ obtained from the alternate CMV matrix $\tilde{\mathcal{C}}(\{\alpha_n\})$ (4.23) by the same procedure. The matrix $\tilde{\mathcal{T}}(\{\alpha_n\})$ is the transpose of $\mathcal{T}(\{\alpha_n\})$

$$\tilde{\mathcal{T}} = \begin{pmatrix}
-\bar{\alpha}_1 \alpha_0 & \bar{\alpha}_2 \rho_1 & \rho_2 \rho_1 & 0 & \ldots \\
-\rho_1 \alpha_0 & -\bar{\alpha}_2 \bar{\alpha}_1 & -\rho_2 \bar{\alpha}_1 & 0 & \ldots \\
0 & \bar{\alpha}_3 \rho_2 & -\bar{\alpha}_3 \alpha_2 & \bar{\alpha}_4 \rho_3 & \ldots \\
0 & \bar{\rho}_3 \rho_2 & -\rho_3 \alpha_2 & -\bar{\alpha}_4 \alpha_3 & \ldots \\
& \ldots & \ldots & \ldots & \ldots
\end{pmatrix},$$

(6.9)

and

$$\tilde{\mathcal{T}}(\{\alpha_n\}) = \mathcal{T}^t(\{\alpha_n\}) = (\mathcal{M}_c)^t(\mathcal{L}_r)^t = \mathcal{M}_r \mathcal{L}_c.$$  

As in Section 6.1, it is not hard to show that $\tilde{\mathcal{T}}(\{\alpha_n\})$ is a completely nonunitary contraction with rank one defects, and its characteristic function $\tilde{f}$ agrees with the Schur function associated with Verblunsky coefficients (Schur parameters) $\{\alpha_n\}$. Indeed (cf. (6.4))

$$\tilde{(\mathcal{C} + z I)}(\mathcal{C} - z I)^{-1} = (\mathcal{C}^t + z I)(\mathcal{C}^t - z I)^{-1} = ((\mathcal{C} + z I)(\mathcal{C} - z I)^{-1})^t,$$

and so $\tilde{F}(z) := ((\mathcal{C} + z I)(\mathcal{C} - z I)^{-1} \delta_0, \delta_0) = F(z)$, $\tilde{f} = f$, as claimed. So, the matrices $\mathcal{T}(\{\alpha_n\})$ and $\tilde{\mathcal{T}}(\{\alpha_n\})$ are unitarily equivalent.
Denote by $\mathcal{T}^{(k)}$ ($\widetilde{\mathcal{T}}^{(k)}$) the matrix obtained from $\mathcal{T}$ ($\widetilde{\mathcal{T}}$) by deleting the first $k$ rows and columns. The following result provides the characteristic function of $\mathcal{T}^{(k)}$.

**Theorem 6.6.** Let $\mu$ be a probability measure on $\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^N$, $N \leq \infty$, and let $f$, $\mathcal{C}(\{\alpha_n\})$, $\widetilde{\mathcal{C}}(\{\alpha_n\})$, $\mathcal{T}(\{\alpha_n\})$, $\widetilde{\mathcal{T}}(\{\alpha_n\})$ be the corresponding Schur function, CMV and truncated CMV matrices, respectively. Then $\mathcal{T}^{(k)}$, $\widetilde{\mathcal{T}}^{(k)}$ are completely nonunitary contractions with rank one defects, and the following relations hold:

$$
\mathcal{T}^{(2m-1)}(\{\alpha_n\}_{n=0}^N) = \widetilde{\mathcal{T}}(\{\alpha_n\}_{n=2m-1}^N), \quad \mathcal{T}^{(2m)}(\{\alpha_n\}_{n=0}^N) = \mathcal{T}(\{\alpha_n\}_{n=2m}^N), \quad m = 1, 2, \ldots
$$

So, the characteristic function of $\mathcal{T}^{(k)}$ agrees with the $k$th Schur iterate of $f$.

**Proof.** The relations

$$
\mathcal{T}^{(1)}(\{\alpha_n\}_{n=0}^N) = \widetilde{\mathcal{T}}(\{\alpha_n\}_{n=1}^N), \quad \mathcal{T}^{(1)}(\{\alpha_n\}_{n=1}^N) = \mathcal{T}(\{\alpha_n\}_{n=2}^N)
$$

follows directly from (6.1) and (6.9). The rest is a matter of simple induction and the definition of the $k$th Schur iterates.

The relation between characteristic functions of the sub-matrices $\mathcal{T}^{(k)}(\{\alpha_n\}_{n=0}^N)$ and the $k$th Schur iterates established in the above mentioned theorem is a complete analog of the result concerning the connections between m-functions of a Jacobi matrix and its sub-matrices [21].

Let us now go back to the model of Section 5.

**Theorem 6.7.** Let $\mu$ be a probability measure on $\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^N$, $N \leq \infty$. Consider three subspaces in $L^2(\mathbb{T}, \mu)$

$$
\mathcal{H}_{2m} := \text{span}\{1, \zeta, \zeta^2, \ldots, \zeta^m\}, \quad \mathcal{H}_{2m-1} := \text{span}\{1, \zeta, \zeta^2, \ldots, \zeta^{m-1}\}, \quad \widetilde{\mathcal{H}}_{2m-1} := \text{span}\{1, \zeta, \zeta^2, \ldots, \zeta^{m-1}, \zeta^m\}.
$$

Denote by $\mathcal{H}_{2m}$ ($\mathcal{H}_{2m-1}$, $\widetilde{\mathcal{H}}_{2m-1}$) their orthogonal complements in $L^2(\mathbb{T}, \mu)$, and by $P_{2m}$ ($P_{2m-1}$, $\widetilde{P}_{2m-1}$) the orthogonal projections onto $\mathcal{H}_{2m}$ ($\mathcal{H}_{2m-1}$, $\widetilde{\mathcal{H}}_{2m-1}$), respectively. Then the operators

$$
\Sigma_k h(\zeta) = P_k (\zeta h(\zeta)), \quad h(\zeta) \in \mathcal{H}_k, \quad \Sigma_{2m-1} h(\zeta) = \widetilde{P}_m (\zeta h(\zeta)), \quad h(\zeta) \in \widetilde{\mathcal{H}}_{2m-1},
$$

are completely nonunitary contractions with rank one defects. The characteristic function of $\Sigma_k$ agrees with the $k$th Schur iterate of the Schur function $f(\mu)$, the characteristic function of $\Sigma_{2m-1}$ with $(2m - 1)$th Schur iterate of $f(\mu)$. So, the operator $\Sigma_k$ is unitarily equivalent to the operator

$$
h(\zeta) \rightarrow P_0^{(k)} (\zeta h(\zeta)), \quad h(\zeta) \in L^2(\mathbb{T}, d\mu(\{\alpha_n\}_{n=0}^N)) \ominus \mathbb{C},
$$

where $P_0^{(k)}$ is the orthogonal projection onto $L^2(\mathbb{T}, d\mu(\{\alpha_n\}_{n=0}^N)) \ominus \mathbb{C}$. In addition, $\Sigma_{2m-1}$ is unitarily equivalent to $\Sigma_{2m-1}$.

**Proof.** Recall that CMV matrices $\mathcal{C}(\{\alpha_n\})$ and $\widetilde{\mathcal{C}}(\{\alpha_n\})$ represent the unitary operator $U h(\zeta) = \zeta h(\zeta)$ in $L^2(\mathbb{T}, d\mu(\{\alpha_n\}))$ with respect to the complete orthonormal systems $\{\chi_n\}$ and $\{x_n\}$, respectively. Moreover

$$
\mathcal{H}_{2m} = \text{span}\{\chi_0, \chi_1, \ldots, \chi_{2m}\} = \text{span}\{x_0, x_1, \ldots, x_{2m}\},
$$

$$
\mathcal{H}_{2m-1} = \text{span}\{\chi_0, \chi_1, \ldots, \chi_{2m-1}\},
$$

$$
\widetilde{\mathcal{H}}_{2m-1} = \text{span}\{x_0, x_1, \ldots, x_{2m-1}\}.
$$
Since $\mathcal{T}(\{\alpha_n\}_{n=0}^N)$ is the matrix of $\mathcal{F}(\{\alpha_n\}_{n=0}^N)$ with respect to the basis $\{x_n\}_{n=1}^N$, the operators $\mathcal{T}_{2m}, \mathcal{T}_{2m-1}$, and $\tilde{\mathcal{T}}_{2m-1}$ have the matrices $\mathcal{T}^{(2m)}, \mathcal{T}^{(2m-1)}$, and $\tilde{\mathcal{T}}^{(2m-1)}$, respectively. From Theorem 6.6 it follows that $\mathcal{T}_k$ are completely nonunitary contractions with rank one defects for all $k$, and their characteristic functions agree with the $k^{th}$ Schur iterates of $f$. By Theorems 6.6 and 6.7 the operator $\mathcal{T}_k$ is unitarily equivalent to the operator given by (6.11). We also have

$$\tilde{\mathcal{T}}^{(2m-1)}(\{\alpha_n\}_{n=0}^N) = \mathcal{T}(\{\alpha_n\}_{n=2m-1}^N).$$

Therefore, the characteristic function of $\tilde{\mathcal{T}}^{(2m-1)}(\{\alpha_n\}_{n=0}^N)$ agrees with the $(2m - 1)^{th}$ iterate $f_{2m-1}$ of $f$, and hence the operators $\mathcal{T}^{(2m-1)}(\{\alpha_n\}_{n=0}^N)$ and $\tilde{\mathcal{T}}^{(2m-1)}(\{\alpha_n\}_{n=0}^N)$ are unitarily equivalent.

We complete the section with the general result from the contractions theory which is proved with the help of the truncated CMV model.

**Proposition 6.8.** Let $T$ be a completely nonunitary contraction with rank one defects in a separable Hilbert space $H$, $\dim H \geq 2$, and let $P_{\ker D_T^*}, P_{\ker D_T}$ be the orthogonal projections onto $\ker D_T^*$ and $\ker D_T$ in $H$, respectively. Then the operators

$$T_1 := P_{\ker D_T}, T_1 := P_{\ker D_T^*}$$

are unitarily equivalent completely nonunitary contractions with rank one defects, and their characteristic functions agree with the function

$$h(z) := \frac{1}{z} \frac{h(z) - h(0)}{1 - h(0)h(z)},$$

where $h$ is the characteristic function of $T$.

**Proof.** By Theorem 6.4 the operator $T$ is unitarily equivalent to the truncated CMV matrices $\mathcal{T} = \mathcal{T}(\{\alpha_n\}_{n=0}^N)$ and $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}(\{\alpha_n\}_{n=0}^N)$, where $\{\alpha_n\}_{n=0}^N$ are the Schur parameters of $h$, $N \leq \infty$. So, there exists a unitary operators $V, \tilde{V} : \delta_0^+ \to H$ such that

$$V \mathcal{T} V^{-1} = \tilde{V} \tilde{\mathcal{T}} \tilde{V}^{-1} = T.$$

It follows that

$$VD_T^*V^{-1} = D_T^*, \quad VD_T \tilde{V}^{-1} = D_T,$$

and hence $V \ker D_T^* = \ker D_T^*, \tilde{V} \ker D_T = \ker D_T$. Due to (6.2) we have

$$\mathcal{D}_T^* = \mathcal{D}_T = \text{span} \{\delta_1\}$$

and

$$\mathcal{T}^{(1)} = P_{\ker D_T^*} \mathcal{T} \downarrow \ker D_T^*, \quad \tilde{\mathcal{T}}^{(1)} = P_{\ker D_T^*} \tilde{\mathcal{T}} \downarrow \ker D_T.$$

Hence

$$V \mathcal{T}^{(1)} V^{-1} = T_1, \quad \tilde{V} \mathcal{T}^{(1)} \tilde{V}^{-1} = \tilde{T}_1.$$

Now from Theorem 6.6 it follows that $T_1$ and $\tilde{T}_1$ are completely nonunitary contractions with rank one defects, and their characteristic functions agree with the first Schur iterate $h_1$ of $h$. Hence $T_1$ and $\tilde{T}_1$ are unitarily equivalent.

$\square$
7. INVERSE SPECTRAL PROBLEMS FOR FINITE AND SEMI-INFINITE TRUNCATED CMV MATRICES

Consider a $N \times N$ truncated CMV matrix

$$
\mathcal{T} = \mathcal{T}(\{\alpha_n\}) = \begin{pmatrix}
-\alpha_1 & -\rho_1 & 0 & \ldots & 0 \\
\bar{\alpha}_2 & -\alpha_2 & \bar{\alpha}_3 & \rho_2 & \ldots & 0 \\
\rho_2 & -\rho_2 & -\alpha_3 & \bar{\alpha}_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\bar{\alpha}_N & \rho_{N-1} & \bar{\alpha}_N & \rho_{N-2} & \ldots & -\alpha_N \alpha_{N-1}
\end{pmatrix}
$$

(for even $N$ it looks a bit different). The problem under investigation in the present section is the reconstruction of the matrix $\mathcal{T}$ (7.1) from either the complete set of its eigenvalues or from the mixed spectral data: the part of the spectrum and the part of the parameters $\alpha_n(\mathcal{T})$.

7.1. Existence of a finite truncated CMV matrix with the given spectrum.

**Theorem 7.1.** Let $z_1, z_2, \ldots, z_N$ be not necessarily distinct numbers from the open unit disk. Then there exists a truncated $N \times N$ CMV matrix $\mathcal{T}$ (7.1) which has eigenvalues $z_1, z_2, \ldots, z_N$, counting their algebraic multiplicities. Such matrix is determined uniquely up to multiplication of its parameters $\alpha_n(\mathcal{T})$ by the same unimodular factor.

**Proof.** Let

$$
b(z) = e^{i\varphi} \prod_{k=1}^{N} \frac{z - z_k}{1 - \bar{z}_k z}, \quad z \in \mathbb{D}, \quad \varphi \in [0, 2\pi).
$$

We want to show that $b$ is the characteristic function of a truncated CMV matrix $\mathcal{T}$ (7.1). Put

$$
F(z) = \frac{1 + zb(z)}{1 - zb(z)},
$$

which is a rational function with $N + 1$ distinct simple poles lying on $\mathbb{T}$, $\operatorname{Re} F(z) > 0$, $z \in \mathbb{D}$, and $F(0) = 1$. It follows that there exists a probability measure $d\mu$ on the unit circle supported at those poles, so that

$$
F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).
$$

Let $\{\alpha_0, \ldots, \alpha_{N-1}, \alpha_N\}$ be the Schur parameters of $b$, that is the same as the Verblunsky coefficients of $\mu$. Construct the $(N + 1) \times (N + 1)$ unitary CMV matrix $\mathcal{C}$ of the form (4.17). Then

$$
F(z) = ((\mathcal{C} + zI)(\mathcal{C} - zI)^{-1}\delta_0, \delta_0), \quad |z| < 1,
$$

where $\delta_0 = (1, 0, \ldots, 0)^t \in \mathbb{C}^{N+1}$. Let $\mathcal{T}$ be $N \times N$ be truncated CMV matrix of the form (7.1). $\mathcal{C}$ has the block form

$$
\mathcal{C} = \begin{pmatrix}
\mathcal{S} & \mathcal{G} \\
\mathcal{F} & \mathcal{T}
\end{pmatrix}.
$$
where $S = \tilde{\alpha}_0$, $\mathcal{G} = (\tilde{\alpha}_1 \rho_0, \rho_1 \rho_0, 0, \ldots, 0)$, and $\mathcal{F} = \begin{pmatrix} \rho_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Since $\delta_0$ is a cyclic vector for $\mathcal{C}$, the unitary colligation $\Delta = \left\{ \begin{pmatrix} S \\ \mathcal{G} \\ \mathcal{F} \\ \mathcal{T} \end{pmatrix}, \mathbb{C}, \mathbb{C}, \mathbb{C}^N \right\}$ is prime. Hence $\mathcal{T}$ is a completely nonunitary contraction with rank one defect operators. Let $\Theta_{\Delta}(z)$ be the transfer function of $\Delta$. By Theorem 3.3 we have
\[
\Theta_{\Delta}(\bar{z}) = \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}, \quad \Theta_{\Delta}(z) = b(z).
\]
So $b(z)$ agrees with the characteristic function of $\mathcal{T}$. Therefore $\mathcal{T}$ has eigenvalues $z_1, \ldots, z_N$, counting their algebraic multiplicities [39].

Finally, let $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$ be two such matrices. Each of them is a completely nonunitary matrix with rank one defects, and their characteristic functions agree with $b$ (7.2). Hence they are unitarily equivalent, and Proposition 6.2 completes the proof. \hfill \Box

**Example 7.2.** Let $T$ be a completely nonunitary contraction with rank one defects on $N$-dimensional Hilbert space, and let $T$ have just one eigenvalue $z = 0$ of the algebraic multiplicity $N$. Then its characteristic function agrees with $f(z) = e^{i\varphi} z^N$. The corresponding Schur parameters are $\{0, \ldots, 0, e^{i\varphi}\}$. It follows that $\rho_n = 1$ for $n = 0, \ldots, N - 1$. Hence $T$ is unitarily equivalent to the $N \times N$ truncated CMV matrix $\mathcal{T}_N$ (see the expressions for $\mathcal{T}_5$ and $\mathcal{T}_6$):

\[
\mathcal{T}_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{i\varphi} \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad \mathcal{T}_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{i\varphi} & 0
\end{pmatrix}.
\]

### 7.2. Uniqueness and reconstruction of a finite truncated CMV matrix from mixed spectral data

It is easily seen from (7.1) that a truncated $N \times N$ CMV matrix $\mathcal{T}$ is completely determined by $N + 1$ independent parameters $\alpha_j(\mathcal{T})$, $j = 0, 1, \ldots, N$. The problem we discuss here is whether $\mathcal{T}$ can be restored from the part of its spectrum (the eigenvalues $z_1, \ldots, z_m$, of the algebraic multiplicity $l_k$, $k = 1, \ldots, m$, with $l_1 + \ldots + l_m = r$), and the first $N - r + 1$ parameters $\alpha_0(\mathcal{T}), \ldots, \alpha_{N-r}(\mathcal{T})$. As we will see later on, the solution of this problem is unique (if it exists).

We begin with a simple result from complex analysis. We don’t know where exactly it appears in the literature, but by all means it is known to experts.

**Lemma 7.3.** Let $z_1, \ldots, z_m$ be distinct points in $\mathbb{D}$, $l_1, \ldots, l_m$ positive integers, and $r = l_1 + \ldots + l_m$. Suppose that the Nevanlinna-Pick interpolation problem with multiple nodes
\[
(7.3) \quad b^{(j)}(z_k) = w^{(j)}_k, \quad j = 0, 1, \ldots, l_k - 1, \quad k = 1, 2, \ldots, m
\]
has two solutions $b_1$ and $b_2$, both the Blaschke products of order $\leq r - 1$. Then $b_1 = b_2$. 

Proof. Assume first that \( z_k \neq 0, w_k^{(0)} \neq 0, k = 1, \ldots, m \). Given a Blaschke product \( s \), we see by differentiating the equality \( s(1/z) = s^{-1}(z) \) that
\[
\frac{1}{s^{(j)}(1/z)} = \frac{P_j(s(z), s'(z), \ldots, s^{(j)}(z))}{s^{2j}(z)},
\]
where \( P_j \) is a polynomial of its variables. Hence
\[
\frac{1}{s^{(j)}(1/z_k)} = \frac{P_j(s(z_k), \ldots, s^{(j)}(z_k))}{s^{2j}(z_k)}, \quad k = 1, 2, \ldots, m
\]
so we have
\[
b_1^{(j)}(z_k) = b_2^{(j)}(z_k), \quad b_1^{(j)}(\frac{1}{z_k}) = b_2^{(j)}(\frac{1}{z_k}), \quad j = 0, 1, \ldots, l_k - 1, \quad k = 1, 2, \ldots, m.
\]
Then for the difference \( u = b_1 - b_2 \) the relations
\[
u^{(j)}(z_k) = u^{(j)}(\frac{1}{z_k}) = 0, \quad j = 0, 1, \ldots, l_k - 1, \quad k = 1, 2, \ldots, m.
\]
hold. Let now
\[
b_l(z) = \frac{p_l(z)}{q_l(z)}, \quad l = 1, 2, \quad u(z) = \frac{p_1(z)q_2(z) - p_2(z)q_1(z)}{q_1(z)q_2(z)} = \frac{p(z)}{q(z)},
\]
where \( p, q \) are polynomials of degree \( \leq 2r - 2 \). The Leibniz formula
\[
u^{(n)}(z) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^{(k)}(z) \left( \frac{1}{q} \right)^{(n-k)}(z)
\]
shows by induction that (7.4) imply
\[
p^{(j)}(z_k) = p^{(j)}(\frac{1}{z_k}) = 0, \quad j = 0, 1, \ldots, l_k - 1, \quad k = 1, 2, \ldots, m.
\]
But \( \deg p \leq 2r - 2 \), and there are \( 2r \) conditions in (7.5), so \( p \equiv 0 \), as needed.

Assume next that \( z_k \neq 0, k = 1, \ldots, m \) and some of \( w_k^{(0)} \) are zero. Take \( \varepsilon \in \mathbb{D}, \varepsilon \neq w_k^{(0)} \) and put
\[
s_0 := \frac{z - \varepsilon}{1 - \varepsilon z}, \quad \tilde{b}_l(z) := s_0(b_l(z)), \quad l = 1, 2.
\]
Then both \( \tilde{b}_1 \) and \( \tilde{b}_2 \) are Blaschke products of order \( \leq r - 1 \) which solve the interpolation problem
\[
\tilde{b}_l^{(j)}(z_k) = \tilde{w}_k^{(j)}, \quad j = 0, 1, \ldots, l_k - 1, \quad k = 1, 2, \ldots, m, \quad l = 1, 2,
\]
where \( \tilde{w}_k^{(0)} = s_0(w_k^{(0)}) \neq 0 \) and \( \tilde{w}_k^{(j)} = (s_0(b_l(z)))^{(j)} \big|_{z=z_k} \). The above argument applied to \( \tilde{b}_l \) gives \( \tilde{b}_1 = \tilde{b}_2 \Rightarrow b_1 = b_2 \), as needed.

Finally, assume that \( z_1 = 0 \). Let \( \varepsilon \neq -z_k \) for all \( k \), and put
\[
\tilde{b}_l(z) := b_l(s_0(z)), \quad l = 1, 2.
\]
Then the Blaschke products \( \tilde{b}_1, \tilde{b}_2 \) of order \( \leq r - 1 \) satisfy
\[
\tilde{b}_l^{(j)}(z_k) = \tilde{w}_k^{(j)}, \quad j = 0, 1, \ldots, l_k - 1, \quad k = 1, 2, \ldots, m, \quad l = 1, 2
\]
and \( \tilde{z}_k = (z_k + \varepsilon)(1 + \bar{z}_k z)^{-1} \neq 0 \). Hence \( \tilde{b}_1 = \tilde{b}_2 \), and so \( b_1 = b_2 \). The proof is complete.

\[ \Box \]

**Theorem 7.4.** Let \( z_1, \ldots, z_m \) be distinct nonzero points in \( \mathbb{D} \), \( l_1, \ldots, l_m \) be positive integers, and \( r = l_1 + \ldots + l_m \leq N \). Let \( \alpha_0, \ldots, \alpha_{N-r} \in \mathbb{D} \). If there exists a \( N \times N \) truncated CMV matrix \( T \) such that \( z_1, \ldots, z_m \) are eigenvalues of \( T \) with the algebraic multiplicities \( l_1, \ldots, l_m \), and \( \alpha_j(T) = \alpha_j \), \( j = 0, \ldots, N - r \), then this matrix is unique.

**Proof.** If the required \( T \) exists then its characteristic function \( \Theta_T(z) \) is the Blaschke product of order \( N \) and of the form

\begin{equation}
(7.6)
\begin{align*}
b(z) = e^{it} \prod_{k=1}^{m} \left( \frac{z - z_k}{1 - \bar{z}_k z} \right)^{l_k} \prod_{j=1}^{N-r} \frac{z - v_j}{1 - \bar{v}_j z},
\end{align*}
\end{equation}

with the given first \( N - r + 1 \) Schur parameters \( \alpha_0(b), \ldots, \alpha_{N-r}(b) \). Our goal is to prove the uniqueness of such function \( b \).

According to the result of Schur [35] (see Section 4.2) the set of all Schur functions \( b \) with given first \( N - r + 1 \) Schur parameters is parametrized by

\begin{equation}
(7.7)
b(z) = \frac{A(z) + zB^*(z)s(z)}{B(z) + zA^*(z)s(z)},
\end{equation}

where \( s(z) \) is an arbitrary Schur function, and \( A, B \) are polynomials of degree at most \( N - r \). Since \( b \) is the Blaschke product of order \( N \), it is clear that so is \( s(z) \), \( \deg s(z) = r - 1 \), and

\[ Sb = \{ \alpha_0, \ldots, \alpha_{N-r}, \alpha_0(s), \ldots, \alpha_{r-1}(s) \}. \]

Let us solve (7.7) for \( s \):

\[ s(z) = \frac{A(z) - B(z)b(z)}{-zB^*(z) + zA^*(z)b(z)}, \]

so \( s(z) \) satisfies the Nevanlinna-Pick interpolation problem (7.3), where \( w_k^{(j)} \) are completely determined from the given nonzero \( z_k \)'s and \( \alpha_j \)'s. By Lemma [7.3] there is at most one such \( s(z) \), and the uniqueness of \( b \) is proved. \( \Box \)

**Remark 7.5.** Suppose that \( z_1, \ldots, z_m \) are distinct nonzero points in \( \mathbb{D} \), and \( l_1 + \ldots + l_m = N \), so the only \( \alpha_0 \) is prescribed. It is clear that \( \alpha_0 \) is completely determined by the choice of \( z_j \) and their multiplicities \( l_j \):

\[ b(z) = e^{it} \prod_{k=1}^{m} \left( \frac{z - z_k}{1 - \bar{z}_k z} \right)^{l_k}, \quad \alpha_0 = b(0) = e^{it} \prod_{j=1}^{m} (-z_j^{l_j}). \]

So for all other \( \alpha_0 \) the inverse problem has no solution.

In the case when one of the eigenvalues is zero, all three possibilities (no solution, unique solution, and infinitely many solutions) may occur for the inverse problem in question. For instance, there is no solution at all as long as \( z_1 = 0, \alpha_0 \neq 0 \). Assume next, that \( r = l_1 = 1, z_1 = 0 \), and the points \( \alpha_0, \alpha_1, \ldots, \alpha_{N-1} \) are taken in \( \mathbb{D} \), with the only restriction \( \alpha_0 = 0, \alpha_1 \neq 0 \). The Blaschke products \( b_\gamma \) with the Schur parameters \( \{ \alpha_0, \alpha_1, \ldots, \alpha_{N-1}; \gamma \} \) and arbitrary \( \gamma \in \mathbb{T} \) are of the form

\[ b_\gamma(z) = e^{it} \prod_{j=1}^{N-1} \frac{z - v_j}{1 - \bar{v}_j z}, \]
and the corresponding $N \times N$ truncated CMV matrices $\mathcal{T}_g$ solve the problem.

Finally, assume that except for the zero eigenvalue of multiplicity $k$ ($z_1 = z_2 = \ldots = z_k = 0$), a few more nonzero (and not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_r$ are given, as well as the points $\alpha_0 = \ldots = \alpha_{k-1} = 0$, $\alpha_k \neq 0, \ldots, \alpha_{N-r}$ in $\mathbb{D}$. If the solution of the corresponding mixed inverse problem $\mathcal{T}$ exists, its characteristic function takes the form

$$b(z) = e^{it} z^k \prod_{j=1}^r \frac{z - \lambda_j}{1 - \lambda_j z} g(z),$$

where $g$ is the Blaschke product of order $N - k - r$, $g(0) \neq 0$, and the first $N - k - r + 1$ Schur parameters of $h = z^{-k} b$ are given numbers $\alpha_k, \ldots, \alpha_{N-r}$. Clearly, $h$ is exactly the $k$th Schur iterate of $b$. If the required truncated CMV matrix $\mathcal{T}$ exists, then by Theorem 6.6 the characteristic function of $\mathcal{T}^{(k)}$ agrees with $h$. It follows now from Theorem 7.4 that $\mathcal{T}^{(k)}$ is unique, and since $\alpha_j(T) = 0$, $j = 0, \ldots, k - 1$, the matrix $\mathcal{T}$ is unique as well.

The situation changes dramatically if we assume that the last parameters of $\mathcal{T}$ are known. In this case we can prove the existence, but not the uniqueness of the solution.

**Theorem 7.6.** Let $z_1, \ldots, z_m$ and $\alpha_m, \ldots, \alpha_{N-1}$ be two collections of arbitrary complex number from the open unit disk, and let $\alpha_N \in \mathbb{T}$. Then there exists a $N \times N$ truncated CMV matrix $\mathcal{T}$ of the form (7.1) such that

(i) $z_1, \ldots, z_m$ are eigenvalues of $\mathcal{T}$, counting the algebraic multiplicity,

(ii) $\alpha_n(T) = \alpha_n$, $n = m, m+1, \ldots, N$.

**Proof.** By Theorem 4.3 there exists a Blaschke product $b(z)$ of order $N$ such that $b(z_k) = 0$, $k = 1, \ldots, m$, with the Schur parameters

$$\alpha_n(b) = \alpha_n, \quad n = m, m + 1, \ldots, N.$$

Take now the matrix $\mathcal{T}$ with $\alpha_n(T) = \alpha_n$, $n = 0, 1, \ldots, N$. By Theorem 3.3 the characteristic function of $\mathcal{T}$ agrees with $b(z)$, that completes the proof. □

Theorem 7.6 thereby says that a $N \times N$ truncated CMV matrix $\mathcal{T}$ can be reconstructed from its $m$ eigenvalues and the lower principal block of order $N - m$. The latter is either the truncated CMV matrix $\mathcal{T}^{(\{\alpha_n\}_{n=m})}$ or its transpose $\tilde{\mathcal{T}}$.

### 7.3. Inverse problem for semi-infinite truncated CMV matrix.**

In this subsection we consider the criterion when given complex numbers $z_n$, $n = 1, 2, \ldots$ from $\mathbb{D}$ are the eigenvalues counting algebraic multiplicity of some semi-infinite truncated CMV matrix.

**Proposition 7.7.** Given complex numbers $z_n$, $n = 1, 2, \ldots$ are eigenvalues counting algebraic multiplicity of some semi-infinite truncated CMV matrix if and only if

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

**Proof.** The convergence of the sum is equivalent to the convergence of the Blaschke product

$$b(z) = \prod_{k=1}^{\infty} \frac{z_k - z}{z_k - \bar{z_k} z}.$$
Let \( \{\alpha_n\} \) be the Schur parameters of \( b \). The characteristic function of the truncated CMV matrix \( \mathcal{T}(\{\alpha_n\}) \) agrees with \( b \). Hence the eigenvalues of \( \mathcal{T}(\{\alpha_n\}) \) are precisely the complex numbers \( \{z_n\} \).

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