Warped and compact extra dimensions: 
5D branes in 6D models

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Abstract

We consider six dimensional brane world models with a compact and a warped extra dimension with five dimensional branes. We find that such scenarios have many interesting features arising from both ADD and Randall-Sundrum -models. In particular we study a class of models with a single 5D brane and a finite warped extra dimension, where one of the brane dimensions is compact. In these models the hierarchy problem can be solved on a single positive tension brane.

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Introduction

Extra dimensions, both compact (ADD) and warped (RS), have received an immense amount of attention in recent years. The theoretical motivation coming from string theory and the realizations that gravity is not experimentally well known at smaller than millimeter scales have inspired much of the research effort. Furthermore the properties of warped extra dimensions that help to alleviate the hierarchy problem with an experimentally reachable gravity scale are of great interest.

The majority of the papers on the subject of warped extra dimensions have concentrated on the Randall-Sundrum -scenarios, with one or more branes, in five dimensions. More recently, a number of six dimensional constructions have been considered as well (see e.g. [4]-[7]). In the case of compact surplus spaces, there is a large number of studies from one up to several dimensions. To solve the hierarchy problem, the ADD-case requires, however, at least two compact dimensions.

In this paper we are interested in the properties of 6D models that combine both the compact and warped geometries. In general, however, we are considering extra dimensions with a non-trivial topology, i.e. the topology can be more complicated than the simple direct product, \( S^1 \times \mathbb{R} \). This allows us to include new properties to the model. Indeed, our scenario has no 3-brane but instead a (5-dimensional) 4-brane surrounded by 6-dimensional space. This setting puts then physical constraints on the extra brane dimension which has to be compact.

The plan of the paper is as follows: in Section 2 we write down the metric and the corresponding Einstein’s equations. Static solutions of the Einstein’s equations are then studied in empty space. Branes are added to the picture in Section 3. In Section 4 we consider gravitons to determine the condition for the zero-mode localization and write the corrections to Newton’s law. In Section 5 we study more carefully a particular single brane model with finite extra dimensions. Cosmological aspects of 6D scenarios are considered in Section 6. The conclusions have been drawn in Section 7.

Einstein’s Equations

The metric that includes a compact and a warped dimension, can be written in the form

\[
d s^2 = \eta(\tau, z)^2 d\tau^2 - R(\tau, z)^2 \delta_{ij} dx^i dx^j - a(\tau, z)^2 dz^2 - b(\tau, z)^2 d\theta^2.
\]

The \( z \)-coordinate corresponds to the warped direction and the \( \theta \)-coordinate to the compact dimension. Einstein’s equations read

\[
G_{AB} \equiv R_{AB} - \frac{1}{2} g_{AB} R = -8\pi G_6 T_{AB} - \Lambda_{AB},
\]

where \( G_6 \) is the 6D Newton’s constant, \( T_{AB} \) the energy-impulse tensor and the components of the inhomogeneous 6D cosmological constant \( \Lambda \), \( \Lambda_z \) and \( \Lambda_\theta \) are included in

\[
\Lambda^R_A = \text{diag}(\Lambda, \Lambda, \Lambda, \Lambda, \Lambda_z, \Lambda_\theta).
\]

Note that either or both \( \Lambda_z \) and \( \Lambda_\theta \) may be unequal to \( \Lambda \) because there are more degrees of freedom in the metric [6, 8].

The non-zero components of the Einstein’s tensor in this general case can be computed in a straightforward manner:

\[
G_{00} = -\frac{\eta^2 a' b'}{a^3 b} - \frac{3 \eta^2 a' R'}{a^3} + \frac{3 \eta^2 b' R'}{a^2 b} + \frac{3 \eta^2 R'^2}{a^2} + \frac{\eta^2 b''}{a^2 b} + 
\]
where a dot denotes a derivative with respect to the conformal time $\tau$ and a prime denotes a derivative with respect to the $z$-coordinate. Note that the $(2,2)$ and $(3,3)$ components are equal to the $(1,1)$ component and are therefore omitted.

## 2.1 Static solutions in empty space

Before considering brane-world scenarios or cosmological evolution, we first study the possible static space-time configurations allowed by the Einstein’s equations. In the static 4D Poincare invariant -case ($R(z) = \eta(z)$) the metric can be written in the form

$$ds^2 = a(z)^2 \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 - b(z)^2 d\theta^2.$$  

The Einstein’s equations in empty space ($T_{AB} = 0$) are then simplified to

$$\frac{3 b' a'}{b a} + \frac{3 a'^2}{a^2} + \frac{b''}{b} + \frac{3 a''}{a} = -\Lambda$$  

$$\frac{4 b' a'}{b a} + \frac{6 a'^2}{a^2} = -\Lambda_z$$  

$$\frac{6 a'^2}{a^2} + \frac{4 a''}{a} = -\Lambda_\theta,$$

where the prime again indicates a derivative with respect to $z$. Note that the dynamical solutions, where $\eta(\tau) \neq R(\tau)$ are possible, and expected, in cosmological situations where the extra dimensions undergo evolution. These will be discussed in a later Section.

For static case, depending on the values of $\Lambda$, there is a number of possibilities for the solutions of $a$ and $b$. In addition to the trivial solution with vanishing cosmological constants, a number of other solutions can also be found. From (17), we find the usual exponential solution,

$$a(z) = a_0 e^{-k(z-z_0)},$$
which requires that $k^2 = -\Lambda_\theta/10$. The exponential solution for $a(z)$ implies that

$$b(z) = b_0 e^{-l(z-z_1)},$$

where

$$l = \frac{1}{4k}(-\Lambda_z + \frac{3}{5}\Lambda_\theta).$$

Eq. (15) then gives a constraint for the different $\Lambda$ parameters

$$-\frac{3}{2}\Lambda_\theta + 4\Lambda - \frac{5}{2}\Lambda_z^2 = 0.$$ (21)

From (20) we see that $k$ and $l$ can have different signs if $-\Lambda_z + \frac{3}{5}\Lambda_\theta < 0$.

As special cases we have two simple possibilities. If $b(z)$ is a constant, the model is simply the RS-model with a compact dimension. The values of the cosmological constants are related then by $\Lambda = \Lambda_z = \frac{3}{5}\Lambda_\theta$ and $a(z) = a_0 \exp(-kz)$. If, on the other hand, $a(z)$ is constant, we must set $\Lambda_z = \Lambda_\theta = 0$, $l^2 = -\Lambda$, with $b(z) = b_0 \exp(-lz)$.

We see that different values of $\Lambda_i$ allows us to have different types of space-time configurations. The most commonly considered possibility (e.g. see [4]) is that both $k$ and $l$ and are positive and hence $a$ and $b$ decrease with increasing $z$. One can also have a situation where $k > 0$ and $l < 0$ which implies that the radius of the compact dimension grows exponentially with $z$. An interesting possibility is also the $k < 0$, $l > 0$ case.

3 Branes

We now add branes to the picture and consider the possible space-time configurations. The energy momentum tensor of a 5-brane located at $z = z_0$, can be of the form [8]

$$T^A_B = \delta(z - z_0) \begin{pmatrix} \sigma \delta^\nu_\mu & 0 \\ 0 & \sigma_\theta \end{pmatrix}. $$ (22)

Note that by assuming that matter branes are described by Eq. (22), the SM fields, as well as gravity of course, are free to propagate in the compact dimension. Experiments hence give constraints on the possible parameters of the model. This will be discussed in more detail in Section 5.2.

From the Einstein’s equations (15)-(17) we see that the jump conditions at the brane at $z = z_0$ are:

$$\frac{[b']}{b} \bigg|_{z_0} + 3 \frac{[a']}{a} \bigg|_{z_0} = -8\pi G_6 \sigma$$

$$4 \frac{[a']}{a} \bigg|_{z_0} = -8\pi G_6 \sigma_\theta$$ (23)

These jump conditions then lead to the fine-tuning between the bulk cosmological constant(s) and brane tension, e.g. in the case where $\Lambda = \Lambda_z = \Lambda_\theta$, $k = l$, and if the brane is set at the origin, we need to set, $\sigma = \sigma_\theta = -\Lambda/(10\pi G_6 k)$.

With branes in the picture, we can construct a number of different models with 4-branes, i.e. compact branes located at $z > 0$. Some of the possibilities are studied in the following.
3.1 Brane setups

In the general case, we now have the two jump conditions (23) determined by the brane tensions $\sigma$ and $\sigma_\theta$. One can device a number of different possible space-time configurations. With $k > 0$, the scale factor $a$ shrinks with increasing $z$, similar to RS-models and one can do the obvious generalizations of the 5D RS-models to 6D. Fig. 1a. is a configuration where the four-brane is actually a string located at $z = 0$. This model has been considered in detail in [4]. We can also move the brane from the origin, Fig. 1b., and get another 6D version of the RSII-setup with 5D branes. Obviously, one can always add another brane along the radial dimension which leads to a setup similar to the RSI-model. In this case the warped direction is no longer infinite and one can hope to alleviate the hierarchy problem with such a setup. Similar constructions have been considered in [7]. We can also have similar setups with $k < 0$, which may be acceptable or not, depending on the localization of the zero-mode. We can also combine the two solutions, so that $e.g. a = a_0 \exp(-k|z - z_0|)$, and we have a volcano universe. Note that in order to account for the jump of $a'$ at the origin, we must also add a brane there.

So far we have assumed that the values of $\Lambda_i$ are same constant in all regions. If we allow for different values of $\Lambda_i$, we see that in addition to the exponential solution there exists also the trivial solution, $a = \text{const}$, when $\Lambda_\theta = \Lambda_z = 0$. One can then consider configurations like those depicted in Fig 1c.

4 Graviton spectrum

An important aspect of the brane world constructions is the behaviour of gravitons. Consider a perturbation of the static 4D-Poincare invariant metric (14),

$$ds^2 = a(z)^2(\eta_{\mu\nu} + h_{\mu\nu}(x, z, \theta))dx^\mu dx^\nu - dz^2 - b(z)^2d\theta^2.$$  \hfill (24)

Using the results derived in [3], the equation for linear fluctuations of the metric is easily calculable:

$$h'' + \left(\frac{b'}{b} + 4\frac{a'}{a}\right)h' + \frac{1}{b^2}\partial_\theta^2 h - \frac{1}{a^2}\Box h = 0,$$  \hfill (25)
where $\Box$ is the 4D flat space d’Alembertian operator and prime denotes differentiation with respect to the $z$-coordinate. Note that in deriving (25), the usual fine tuning between the bulk cosmological constant, $\Lambda$, and brane tension, $\sigma$, has been assumed. Hence, when considering space-times with a different cosmological constant on different sides of the brane, one must careful in using (25) to calculate the graviton spectrum.

Decomposing the fluctuations, $h(z, x, \theta) = \psi(z)\varphi(x)e^{i\theta}$, and assuming that a mode has a 4D KK-mass of $m$, (25) can be written in the form

$$\psi'' + \left(\frac{b'}{b} + 4\frac{a'}{a}\right)\psi' + \left(\frac{m^2}{a^2} - \frac{n^2}{b^2}\right)\psi = 0.$$  
(26)

From (26) it is clear that the zero-mode, $m = 0$, $n = 0$, always has a solution $\psi = \text{const}$. Normalizability of the zero-mode, $\int dz \sqrt{-g}g^{00} < \infty$, hence requires that

$$\int dz \, a^2 b < \infty.$$  
(27)

### 4.1 Localization of the graviton zero-mode

With Eq. (27) we can study the different scenarios presented in the previous Section. Clearly, we do not have to worry about the normalization, and hence localization, of the graviton zero-mode unless we have an infinite $z$-dimension. In the RSII-type setup with an infinite $z$-direction with the exponential solutions of the scale factors, (27) dictates that we must require that $2k + l > 0$. In addition to the obvious case where both $a$ and $b$ decrease with growing $z$, it is also possible to have a scenario where one of the scale factor shrinks, i.e. $kl < 0$. The requirement that the zero-mode localizes translates in this case ($kl < 0$) into a condition for the cosmological constants:

$$\frac{3}{5}\Lambda_\theta < \Lambda_z < -\frac{1}{5}\Lambda_\theta,$$  
(28)

where the first inequality comes from $kl < 0$ whereas the second inequality comes from $2k + l > 0$. This means that we can have an exponentially growing infinite $z$-dimension as long as the $\theta$-dimension decreases rapidly enough. The requirement for the localization of the zero-mode also constrains the allowed values of brane tension, via (23), $4\sigma > \sigma_\theta$.

### 4.2 Massive RS-modes

In the 6D scenario with 5D branes the bulk particles may propagate towards two perpendicular directions. They can propagate in the compact, locally flat dimension labelled by $\theta$ or towards the radial direction labelled by $z$. Thus we have pure KK-excitation modes in both directions together with mixed modes. Because the excitation along the compact dimension resembles clearly the ones in ADD-models, we call them ADD-modes. In the very same spirit we call the perpendicular excitations as RS-modes, because there share common properties with excitation of RS-models. Again, note that the SM particles also have the ADD-modes but not the RS-ones. We study next the massive RS-modes and after that the ADD-modes. Similar considerations have been carried out in [4, 6].

Assuming the exponential forms (18)-(19) for the scale factors $a$ and $b$ (with $z_0 = z_1$), (24) takes the form

$$\psi'' - (l + 4k)\psi' + \left(\frac{m^2}{a_0^2} - \frac{n^2}{b_0^2}\right)e^{2(l-k)(z-z_0)}\psi = 0.$$  
(29)

We can transform this into the form of a 2D Schrödinger equation by writing

$$\rho = \frac{1}{k}\left(e^{k(z-z_0)} - 1\right), \quad \psi = (k\rho + 1)^{\frac{k}{2}}\Psi,$$  
(30)
in which case (23) transforms into
\[
-\frac{1}{2b} \partial_\rho (b \partial_\rho \Psi) + \left( \frac{3}{8} \frac{2l + 5k}{4l + k} \right) \frac{n^2}{2b} (k \rho + 1) 2^{(l/k-1)} \Psi = \frac{m^2}{2a^2} \Psi. \tag{31}
\]

From (31) we see that the angular modes \( n \neq 0 \) are separated by a mass gap \( a_0/b_0 \) from the zero-mode on the brane. The \( s \)-waves can be solved from (24):
\[
\psi_m(z) = e^{\frac{i}{2k} (z-z_0)} C_m [Y_{1+\frac{l}{2k}}(m \rho)J_{2+\frac{l}{2k}}(m \rho e^{k(z-z_0)}) - J_{1+\frac{l}{2k}}(m \rho)Y_{2+\frac{l}{2k}}(m \rho e^{k(z-z_0)})], \tag{32}
\]
where \( C_m \) is determined by the normalization condition
\[
\int dz \ a^2 \ b \ \psi_m \psi_n = \delta_{mn}. \tag{33}
\]

In the case of an infinite extra-dimensions, the mass spectrum is continuous, starting from \( m = 0 \) \cite{6,9}. The correction to the Newton’s law due to the \( s \)-modes is \cite{6,9}
\[
\Delta V(r) \sim r^{-3-\frac{1}{k}} \equiv r^{-\alpha}. \tag{34}
\]

The exponent is bounded due to the requirement of zero-mode normalizability, \( 2k + l > 0 \). Hence, we see that the power of the correction \cite{[1]} gives us information of the internal structure of the extra dimensions: if \( 1 < \alpha < 3 \), one of the scale factors is growing while the other is shrinking. Note that this also indicates the presence of more than one extra dimension since in the RS-scenario \( \alpha = 3 \). If on the other hand, \( \alpha \geq 3 \), both scale factor are decreasing with \( z \).

Putting another brane at \( z = z_1 \), Fig. 1c, modifies the normalization of the graviton wavefunctions, \( \psi_m \), as well as discretizes the mass spectrum of the gravitons. The \( s \)-waves (32) are such that \( \psi'(z_0) = 0 \). When we now set another brane at \( z = z_1 \), we must again require that \( \psi'(z_1) = 0 \), which obviously then leads to a discretization of the allowed graviton masses, \( m \). The requirement \( \psi'(z_1) = 0 \) leads to a condition
\[
J_{1+\frac{l}{2k}}(m \rho e^{k(z_1-z_0)}) = J_{1+\frac{l}{2k}}(m \rho), \tag{35}
\]
which needs to be solved numerically to obtain the exact mass spectrum. However, we can estimate the mass gap between the states easily for large \( e^{k(z_1-z_0)} \), \( \Delta m \approx \pi k a_0 e^{-k(z_1-z_0)} \). Note that the expression for the mass gap is valid only at large \( m \).

### 4.3 ADD-modes

Since in this scenario we also have a compact dimension, ADD-gravitons propagating along the brane are present. The radius of the compact dimension is given by the ratio of the scale factor at the brane at \( z = z_0 \),
\[
R(z_0) = \frac{b(z_0)}{a(z_0)}. \tag{36}
\]

The spectrum of ADD-masses can be read from (26),
\[
m_n = \frac{a_0}{b_0} n. \tag{37}
\]

It is crucial to note, that this is the mass formula also for particles localized at the 4-brane. In particular the standard model particles have excitation given by (37). This put limits on the mass of the lightest massive excitation \( m_{ADD} = \frac{a_0}{b_0} \). Using the expressions (18), (19) we get
\[
R(z) = \frac{b_0}{a_0} e^{(k-l)(z-z_0)}, \ m_n(z) = \frac{a_0}{b_0} e^{(l-k)(z-z_0)} n, \tag{38}
\]
from which we see that the ADD-spectrum on different branes can vary greatly. Moreover, the correction to the Newton’s law due to ADD-spectrum is well known \[10\]. It can be shown that the leading correction to the Newton’s law is \( \Delta V(r) \sim r^{-2} \) because the compact dimension is 1-dimensional.

### 4.4 4D gravitational constant and corrections to Newtonian potential

Since the gravitational action is of the form

\[
S_{\text{grav}} = \frac{1}{16\pi G_6} \int d^4x \int d\theta dz \sqrt{g_6} R_6 + \ldots,
\]

we see that the 4D Newton’s constant in a scenario with two 5D branes, is given by

\[
G_4 = \frac{2k + l}{2\pi a_0^2 b_0 [1 - e^{-(2k+l)(z_1 - z_0)}]} G_6.
\]

If we normalize the scale factors so that on the other brane \( a^2(z_1)b(z_1) = 1 \), we get instead

\[
G_4 = \frac{2k + l}{2\pi [e^{(2k+l)(z_1 - z_0)} - 1]} G_6.
\]

In the six dimensional scenario, unlike in the RS-scenarios, one can accommodate a setup where gravity on the positive tension brane is suppressed. As an example, consider a two brane setup with branes at \( z = 0 \) and \( z = z_1 \). From \[23\] we see that the brane at the origin has a positive brane tension if \( l + 3k > 0 \). On the other hand, from \[40\] it is clear that if we wish to alleviate the hierarchy problem on the same brane, \( 2k + l < 0 \). Hence, \( k \) and \( l \) must satisfy the conditions

\[
-\frac{l}{3} < k < -\frac{l}{2}, \quad k > 0, \quad l < 0.
\]

Condition \[42\] implies that \( b(z) \) grows while \( a(z) \) shrinks with increasing \( z \). Such a setup is not possible in the normal RS-scenarios and becomes possible only in models with more than one extra dimension \[11\].

We can now also compare the corrections to the Newtonian potential arising from the RS- and ADD-excitations. We have already noted that the sign of the product \( kl \) reflects to the power of the leading RS-correction. If we compare the leading ADD-correction \( \sim r^{-2} \) and the RS-correction \( \sim r^{-3+\frac{1}{l}} \), we find that if \( l/k < -1 \), the RS-correction is the leading one. In the opposite case ADD-correction is the leading one. However, one must keep in mind that in realistic models the ADD-correction is only significant at distances smaller than the radius of the compact extra dimension, \( i.e. \) \( r \lesssim 1/\text{TeV} \), and cannot be measured directly. One can then expect that measurements on the short range gravitational potential can reveal structure about extra dimensions due to the RS-excitations. Hence, we can gain information from the very small compact dimensions, even though they are too small to be studied directly, by studying the RS-excitations.

### 5 Bowl-universe

We have already seen that there exists at least two types of solutions to the Einstein’s equations in 6D, the trivial constant solution as well as the exponential solutions. With branes added to the picture, we can construct a large number of different types of scenarios, some of which were shown
in Section 3. In the case of exponentially growing or shrinking scale factors, it is obvious that if we wish to include the origin, \( z = 0 \), in our space, we must place a string-like brane at the origin (unless the 6D space around the origin is flat). It is therefore interesting to ask whether this is always necessary or is there a solution which does not require any matter at the origin.

If there is no matter at the origin, the second derivative of the scale factors must be continuous. Hence, we must require that \( a'(0) = b'(0) = 0 \). From the second Einstein’s equation, Eq. (1), we immediately see that \( \Lambda_z = 0 \). From the set of static Einstein’s equations we find that there are three different solutions which smoothly include the origin:

\[
\begin{align*}
I : & \quad a(z) = a_0, \quad b(z) = b_0, \quad \Lambda = \Lambda_\theta = 0 \\
II : & \quad a(z) = a_0, \quad b(z) = b_0 \frac{\cosh(kz)}{\cosh(kz_0)}, \quad k^2 = -\Lambda, \quad \Lambda_\theta = 0 \\
III : & \quad a(z) = a_0 \frac{\cosh^{2/5}(\kappa z)}{\cosh^{4/5}(\kappa z_0)}, \quad b(z) = b_0 \frac{\text{sech}^{3/5}(\kappa z)}{\text{sech}^{1/5}(\kappa z_0)}, \quad k^2 = -\Lambda, \quad \Lambda_\theta = \frac{8}{5} \Lambda,
\end{align*}
\]

where \( \kappa = k/\sqrt{10} \). Note that the sign of \( \Lambda \) is very significant in the constructions: if \( \Lambda > 0 \), \( z \) is bounded by the requirement \( a(z) > 0 \), \( b(z) > 0 \), and hence

\[
\begin{align*}
II : & \quad z < \frac{\pi}{2\sqrt{\Lambda}} \\
III : & \quad z < \pi \sqrt{\frac{2}{5\Lambda}}
\end{align*}
\]

Note also, that this type single brane solution has no 5-D brane analogy! However, there is a singularity at finite distance from the brane.

We now add a brane at \( z = z_0 \) to the setup and consider a case where the brane has been wrapped around the origin so that the SM fields can propagate along the compact extra dimension. This opens an interesting possibility that there is no space outside the brane, i.e. gravitons can only propagate inwards as well as along the brane. In the cases II and III, the volume element \( a^2 b \) is a bowl-shaped function and therefore we refer to this particular model as the bowl-model. Since SM fields and ADD-gravitons are present in the particle spectrum, the size of the compact extra dimension must be small enough so that the lightest ADD-excitations are out of reach of the collider experiments, i.e. \( m_{\text{ADD}} \gtrsim 1 \text{ TeV} \).

The tension of the single 4-brane in the different scenarios is easily calculable from (23). In the trivial first case we see that there can be no matter on the brane unless we allow for a non-trivial space-time structure outside the space bounded by the brane. The second and the third case are more interesting: In the second case we see that \( \sigma \) is always positive while \( \sigma_\theta \) vanishes. In the third case, \( \sigma \) is positive for \( \cosh(\kappa z_0) > 2/3 \) and \( \sigma_\theta > 0 \) for all \( z_0 \).

The value of the gravitational constant on the brane is easily calculable from (39):

\[
\frac{1}{2\pi a_0^2 b_0} \frac{G_6}{G_4} = \begin{cases} 
\cosh^{-1}(kz_0) \left( \frac{z_0}{2} + \frac{1}{4k} \sinh(2kz_0) \right) & (I) \\
\cosh^{-1/5}(\kappa z_0) \int_0^{z_0} \cosh^{1/5}(\kappa z) \, dz & (II) \\
\cosh^{-1/5}(\kappa z_0) \int_0^{z_0} \cosh^{1/5}(\kappa z) \, dz & (III)
\end{cases}
\]

where the integral in the third case can be approximated by

\[
\int_0^{z_0} \cosh^{1/5}(\kappa z) \approx \begin{cases} 
\frac{311}{15} \frac{1}{\kappa} + \frac{5}{14} \frac{\kappa^2 z_0^3}{z_0^2}, & \kappa z_0 < 1 \\
\frac{245}{14} \frac{\kappa}{\kappa^2} (e^{\kappa z_0/2} - e^{1/5}), & \kappa z_0 > 1.
\end{cases}
\]

### 5.1 Gravitons

The graviton spectrum is again given by Eq. (26). In all of the cases, a massless zero-mode \( \psi = \psi_0 \) is present. The existence of massive modes is dependent on the behaviour of the scale factors.
5.1.1 Case I

The trivial solution, \( a = a_0 \) and \( b = b_0 \), is just a scenario with two compact flat dimensions. In our setup, the SM fields feel one of the extra dimensions while gravitons can propagate along both of them. Clearly such a setup is insufficient in alleviating the hierarchy problem.

5.1.2 Case II

Consider the graviton equation in case II. A zero-mode, \( \psi = \psi_0 \), is present as always. The massive graviton modes are described by

\[
\psi'' + k\tanh(kz)\psi' + \left(\frac{m^2}{a_0^2} - \frac{n^2 \cosh^2(kz_0)}{b_0^2 \cosh^2(kz)}\right)\psi = 0, \tag{46}
\]

which has a solution with \( \psi'(0) = 0 \), of the form \( m = 0 \)

\[
\psi(z) = C_m \left( P_\mu(i \sinh(kz))Q_{\mu-1}(0) - Q_\mu(i \sinh(kz))P_{\mu-1}(0) \right), \tag{47}
\]

where

\[
\mu = \frac{1}{2} \left( -1 + \sqrt{1 - 4\frac{m^2}{k^2a_0^2}} \right) \tag{48}
\]

and \( P \) and \( Q \) are the Legendre functions. The overall constant factor is determined by the normalization condition, \( (33) \). The spectrum of the graviton masses is strongly dependent on the sign on \( \Lambda \), just like the behaviour of the scale factors. If \( \Lambda > 0 \), there are no massive s-waves with \( \psi'(z_0) = 0 \). On the other hand, higher waves, with \( n > 0 \), do exist.

If, however \( \Lambda < 0 \), s-waves are present in the graviton spectrum. To find the graviton masses, we require that \( \psi'(z_0) = 0 \), i.e. the derivative vanishes on the brane. This is in general a numerical problem. From the numerical work it is clear that as \( z_0 \) grows, the mass spectrum becomes more dense and the mass of the lightest massive mode decreases.

5.1.3 Case III

In the third case, the graviton spectrum is calculable from

\[
\psi'' + k\tanh(kz)\psi' + \left(\frac{m^2 \cosh^{4/5}(kz_0)}{a_0^2 \cosh^{4/5}(kz)} - \frac{n^2 \cosh^{6/5}(kz_0)}{b_0^2 \cosh^{6/5}(kz_0)}\right)\psi = 0. \tag{49}
\]

Eq. (49) needs to be solved numerically. Again, \( \Delta m \) and the mass of the lightest mode decrease with increasing \( z_0 \).

In the large \( \sinh(kz) \) limit, s-waves can be approximated by

\[
\psi(z) = \sinh^{-1/2}(\kappa z)[A_mJ_{5/4}\left(\frac{5m}{2k\bar{a}_0} \sinh^{-2/5}(\kappa z)\right)] + B_mY_{5/4}\left(\frac{5m}{2k\bar{a}_0} \sinh^{-2/5}(\kappa z)\right)], \tag{50}
\]

where \( \bar{a}_0 = a_0 / \cosh^{2/5}(\kappa z_0) \).

5.2 Hierarchy problem

We can now look for parameters which alleviate the hierarchy problem while keeping the KK-excitations along the brane heavy. Assume that the fundamental 6D gravity scale is \( M_* \). The
non-observation of KK-excitations combined with \([13]\) then gives the following constraints for the value of the parameters:

\[
\begin{align*}
\frac{1}{2\pi a_0^2} m_{ADD} z_0^{-1} M_{Pl}^2 &\lesssim M_\ast^4 &\lesssim \frac{1}{2\pi a_0^2} z_0^{-1} M_{Pl}^2 \frac{M_\ast^4}{m_{ADD}^4} & (I) \\
\frac{2k}{\pi a_0^2} e^{-kz_0} m_{ADD} M_{Pl}^2 &\lesssim M_\ast^4 &\lesssim \frac{2k}{\pi a_0^2} e^{-kz_0} M_{Pl}^2 \frac{M_\ast^4}{m_{ADD}^4} & (II) \\
\frac{\kappa}{4\pi a_0^2} e^{-3\kappa z_0/10} m_{ADD} M_{Pl}^2 &\lesssim M_\ast^4 &\lesssim \frac{\kappa}{4\pi a_0^2} e^{-3\kappa z_0/10} M_{Pl}^2 \frac{M_\ast^4}{m_{ADD}^4} & (III)
\end{align*}
\]

where the experimental limit for the mass of the lightest ADD-mode is denoted by \(m_{ADD}\). We are interested in the lower limit for \(M_\ast\). Assuming that \(\tilde{m} = 1\) TeV and using \(a_0 = 1\), we get

\[
\begin{align*}
(z_0/\text{GeV}^{-1})^{-1/4} \times 10^{10} \text{GeV} &\lesssim M_\ast & (I) \\
(k/\text{GeV})^{1/4} e^{-kz_0/4} \times 10^{10} \text{GeV} &\lesssim M_\ast & (II) \\
(\kappa/\text{GeV})^{1/4} e^{-3\kappa z_0/40} \times 10^{10} \text{GeV} &\lesssim M_\ast & (III)
\end{align*}
\]

If we wish that the fundamental 6D gravity scale is \(\sim 1\) TeV, we must in the I case require that \(z_0 \sim 10^{28}\) GeV\(^{-1}\), making the hierarchy problem worse by introducing a new, large scale to the theory. In case II, we see that if \(k \sim M_\ast, k \sim 90\), where as in case III, \(\kappa z_0 \sim 300\). A setup with the fundamental scale around 1 TeV is hence feasible in this scenario. The graviton mass spectrum in this case is well approximated by a continuous spectrum, starting at \(m = 0\).

### 6 Cosmology

In addition to the interesting properties that static brane world configurations possess, the dynamical evolution of space-time also offer novel properties that have an effect on cosmology. Cosmological evolution in the brane world scenarios have been shown to possess interesting properties \([12, 13]\). In order to study cosmology on a brane in the 6D scenario, we adopt a metric:

\[
ds^2 = \eta(\tau, z)^2 d\tau^2 - R(\tau, z)^2 \delta_{ij} dx^i dx^j - a(\tau, z)^2 dz^2 - b(\tau, z)^2 d\theta^2.
\]

The components of the Einstein’s tensor \(G_{\mu\nu}\) are:

\[
G_{00} = -\frac{\eta^2 a' b'}{a^3 b} + \frac{3 \eta^2 a' R'}{a^2 b R} + \frac{3 \eta^2 b R^2}{a^2 R^2} + \frac{\eta^2 b''}{a^2 b} + \frac{3 \eta^2 R'}{a^2 R} - \frac{a b}{a R} - \frac{b R}{a R} + \frac{3 \bar{R}^2}{R^2} \quad (54)
\]

\[
G_{11} = \frac{R^2 a' b'}{a^2 b} + \frac{R^2 a' \eta'}{a^2 \eta} - \frac{R^2 b' \eta'}{a^2 \eta} + \frac{2 R a' R'}{a^3} - \frac{2 R b' R'}{a^2 b} - \frac{2 R \eta' R'}{a^2\eta} - \frac{R^2}{a^2} \quad (55)
\]

\[
G_{44} = \frac{3 a^2 \eta R}{\eta^3 R} + \frac{3 a^2 R}{\eta R} - \frac{3 a^2 \eta'}{\eta^2 R^2} + \frac{3 a^2 \eta'}{\eta R^2} + \frac{3 a^2 \eta'}{\eta^2 R} - \frac{3 a^2 \eta'}{\eta R} + \frac{3 a^2 \eta'}{\eta R} + \frac{3 a^2 \eta'}{\eta R} \quad (58)
\]

\[
G_{55} = \frac{b^2 a' \eta'}{a^3 \eta} + \frac{3 b^2 a' R'}{a^3 R} - \frac{a^2 \eta}{a^2 \eta R} + \frac{3 b^2 R^2}{a^2 R^2} - \frac{b^2 \eta'}{a^2 \eta} + \frac{3 b^2 R''}{a^2 R} - \frac{b^2 \eta'}{a^2 \eta} + \frac{3 b^2 R''}{a^2 R} \quad (59)
\]
Matter on brane has the form

\[ T^R_A = f(\tau, z) \text{diag}(\rho(\tau), -P(\tau), -P(\tau), -P(\tau), 0, -P_v(\tau)), \]

with an arbitrary prefactor \( f(\tau, z) \), whose significance becomes clear later on. Jumps are required for components \((0,0), (1,1), (5,5)\):

\[
\begin{align*}
\frac{1}{a_0^2} \left[ \left. \frac{b'}{b} \right|_{\tau = \tau_0} \right] + 3 \frac{1}{a_0^2} \left[ \left. \frac{R'}{R} \right|_{\tau = \tau_0} \right] &= 8\pi G_6 f_{0}\rho \\
\frac{1}{a_0^2} \left[ \left. \frac{\eta'}{\eta} \right|_{\tau = \tau_0} \right] + 2 \frac{1}{a_0^2} \left[ \left. \frac{R'}{R} \right|_{\tau = \tau_0} \right] &= -8\pi G_6 f_{0}P \\
\frac{1}{a_0^2} \left[ \left. \frac{\eta'}{\eta} \right|_{\tau = \tau_0} \right] + 3 \frac{1}{a_0^2} \left[ \left. \frac{R'}{R} \right|_{\tau = \tau_0} \right] &= -8\pi G_6 f_{0}P_v,
\end{align*}
\]

where the index 0 refers to values on the brane. The jumps are in the direction perpendicular to the brane and can easily be calculated from the Gauss-Codacci equations with the unit vector field normal to the 4-brane chosen as \( n^A = (0, 0, 0, 0, 1/a, 0) \).

The non-trivial continuity equations \( T^B_A;_B = 0 \) are:

\[
\begin{align*}
T^B_0;_B &= \dot{\rho} + 3(P + \rho) \frac{\dot{R}}{R} + \rho(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{f}}{f}) + P_v \frac{\dot{b}}{b} = 0 \quad (66) \\
T^B_4;_B &= P_v \frac{b'}{b} - \rho \frac{\eta'}{\eta} + 3P \frac{R'}{R} = 0. \quad (67)
\end{align*}
\]

From the \( T^B_0;_B = 0 \) continuity equation we see that in order to recover the usual continuity equation on the brane we have a number of different choices:

\[
\begin{align*}
(i) & \quad \frac{\dot{b}}{b} = 0 \quad f = 1/a \\
(ii) & \quad P_v = 0 \quad f = 1/(ab) \\
(iii) & \quad P_v = -\rho \quad f = 1/a.
\end{align*}
\]

In each case we can then solve for the jump factors from \( (65) \). From the first continuity equation we see that the dynamics of the extra dimensions can also lead to the non-conservation of energy density on the brane.

The jump of \((4,4)\) gives a constraint relating the mean values of the scale factors,

\[
\rho \frac{\eta}{\eta_0} - 3P \frac{R'}{R_0} - P_v \frac{\dot{b}}{b_0} = 0, \quad (69)
\]

where \( \frac{\eta}{\eta_0} \equiv \frac{1}{2}(f(z_0^+) + f(z_0^-)) \) is the mean value across the brane. Note that is also apparent from the \((4)\) component of the continuity equations.

Taking the mean value of \( G_{44} \) and assuming for simplicity that space on the brane is locally invariant under space inversions about the brane, \textit{i.e.} \( \frac{\eta}{\eta_0} = 0 \), and choosing \( \eta_0 = 1 \), we get the Friedmann-type evolution equation of the scale factor \( R \) on the brane:

\[
\frac{\dot{R}}{R} + \frac{\ddot{R}}{R^2} + \frac{\dot{b}}{b} \frac{\dot{R}}{R} = -\frac{8\pi G_6 f_0}{32} ((\rho + P - P_v)^2 + 3\rho P_v) + \frac{1}{3} \frac{\eta_2}{ab} \frac{P_v}{\rho} - \frac{\dot{b}}{3b} + \frac{1}{3} \Lambda z. \quad (70)
\]
We see that the evolution of the scale factor is, like in the RS-scenario \[12\], fundamentally different from the Friedmann equation since here \( H = \dot{R}/R \sim \rho \), instead of \( H^2 \sim \rho \). This is in fact a general property of extra dimensional models with a single warped extra dimension, as one can see from considering a \( D \) dimensional metric,

\[
ds^2 = \eta(\tau, z)^2 d\tau^2 - R(\tau, z)^2 \delta_{ij} dx^i dx^j - a(\tau, z)^2 dz^2 - \sum_{i=5}^{D-5} b_i(\tau, z)^2 d\theta_i^2,
\]

along with matter on the brane, \( T_{\mathcal{B}}^{\mu\nu} = f(\tau, z) \text{diag}(\rho, -P, -P, -P, 0, -P_1, \ldots, -P_{D-5}) \). From the Gauss-Codacci -equations one can easily show that in the case of a (locally) spatially symmetric brane, the four dimensional curvature scalar (and hence \( H^2 \)) is proportional to \( T^2/(D-2) - T_{AB}T^{AB} \) and one cannot choose \( P_i \in \mathbb{R} \) in such a way that the \( \rho^2 \) term vanishes. One can recover the standard Friedmann equation on the brane by adding a energy density on the brane, like in the RS-scenario \[13, 14\].

From (70) we also see that in the 6D brane world the evolution of the compact dimension can significantly affect the evolution on the brane. For example, if \( \dot{b}/b \) is large compared to \( H \), we have a situation where \( H \sim \rho^2 \). The evolution of the scale factor can hence be much more complicated than in standard cosmology. Furthermore, a changing \( b \) also would indicate a varying tower of KK-masses as well as have an effect on the gravitational constant on the brane.

7 Conclusions

In the present paper we have studied the possibility of having a six dimensional theory with a 5-brane such that one dimension of the brane forms a small compact space. In this scenario we thus have one ADD-like spatial dimension together with one Randall-Sundrum -like dimension. The size of the ADD-space is naturally constrained by the requirement that the Kaluza-Klein excitations are heavy enough compared to the experimental limits. Besides our brane where standard model particles live, these models may have an additional brane which can be 4-dimensional but may also be 5-dimensional. As a special case, there is the natural model with only a single 3-brane \[4\]. Obviously, one can also extend the considerations presented here to scenarios with several compact dimensions along with a single warped dimension.

Six dimensional constructions make possible aspects that one cannot have in 5D models. An interesting property is the possibility that one can have a positive tension brane while at the same time alleviating the hierarchy problem. This is made possible by the extra degree of freedom introduced by the scale factor of the compact dimension.

An interesting case among numerous other possible constructions, is a simple model with only one 5-dimensional brane. In this single brane model the parameters are chosen so that there is no extra brane at the origin, something one cannot do in 5D construction. This bowl-model has thus no mirror world and it includes the interesting property that the hierarchy problem is solved with a single positive tension brane. As we have seen, one can construct both one and two brane models where ordinary matter lies on a positive tension brane while solving the hierarchy problem. Phenomenologically these models differ from each other; e.g. there can be both gravitational and SM neutrino interactions between two branes, which are obviously absent in the single brane scenario. These pleasant properties of the single brane construction make it interesting among 6D models. The bowl-model, and other 6D constructions, can also in principle be tested experimentally by observing the type of the leading correction to the Newtonian gravitational potential. The power of the correction can reveal information of the structure of the extra dimensions and possibly distinguish between different types of models, even if the radius of the compact dimension is much too small to be detected directly.
Cosmologically, these 6-dimensional models, thus including the bowl-model, share the difficulties of all extra dimensional models on restoring consistently the ordinary Friedmann evolution on the brane \[13, 14\]. Thus, a more detailed study of the cosmological models is clearly needed.

It would be also interesting to study more some of the other features of the 6D models, and in particular the simple bowl-model. Issues, like creation of the matter by dynamics certainly would have interest of its own. Also the possibility to have a changing gravitational constant and ADD-excitation masses, \textit{i.e.} graviton and standard model excitation masses, would certainly have interesting consequences. The bowl-model also allows a construction where with a positive cosmological constant together with 6-dimensional brane there is at a finite coordinate distance a singularity like in some Randall-Sundrum -models. The effect of these possible features to cosmology remains to be studied. Stabilization of the extra dimensions is also an important issue that needs further consideration \[16, 17\].

Six dimensional models with a compact and a warped dimension open novel perspectives to the brane world models. It seems that one can combine nice features of both ADD- and RS-models in such a way that many of the central problems are alleviated.

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