Majority vote model with ancillary noise in complex networks

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We analyze the properties of the majority-vote (MV) model, one of the simplest nonequilibrium systems exhibiting an order-disorder phase transition, in the presence of an additional noise in which a local spin can be changed independently of its neighborhood. In the standard MV, spins are aligned with their local majority with probability 1−f and with complementary probability f, the majority rule is not followed. In the noisy MV (NMV), a random spin flip is succeeded with probability p (with complementary 1−p the usual rule is accomplished). Such extra ingredient was considered by Vieira and Crokidakis [Physica A 450, 30 (2016)] for the square lattice. Here, we generalize the NMV for an arbitrary network, including homogeneous [random regular (RR) and Erdős Renyi (ER)] and heterogeneous [Barabasi-Albert (BA)] structures, through mean-field calculations and numerical simulations. Results coming from both approaches are in excellent agreement with each other. Our results show that the presence of additional noise does not affect the classification of phase transition, which remains continuous irrespective of the network degree and its distribution. The critical point and the threshold probability p∗ marking the disappearance of the ordered phase depend on the node distribution and increase with the connectivity k. The critical behavior, investigated numerically, exhibits a common set of critical exponents for RR and ER topologies, but different from BA and regular lattices. In addition, a previous proposition of a first-order transition in the NMV for large k’s is dropped.

I. INTRODUCTION

Phase transitions and spontaneous breaking symmetry appear in myriad systems in the scope of physics \cite{1,2,3,4,5,6,7,8}, biology \cite{6}, chemistry, social dynamics \cite{7,8} and others. Among the several microscopic models proposed to study such systems, the Majority Vote (MV) model is probably one of the simplest nonequilibrium down Z\textsubscript{2} symmetry examples \cite{4}. Its dynamics mimics the existence/formation of distinct opinions (±1 in such case) in a community and contemplates the role of individuals that do not adopt the local prevailing (majority) opinion, commonly referred to as contrarians or nonconformist characters. Thence it has attracted considerable attention recently, not only for theoretical purposes (including the investigation of critical behavior, universality classes and phase coexistence) but also for the description (at least under a simplified level) of the mechanisms leading to the opinion formation \cite{7,8}. It is known that the phase transition exhibited by the MV model is continuous, signed by a spontaneous symmetry-breaking \cite{4,9,10}, although the critical behavior depends on lattice topology \cite{4,10}.

An entirely different behavior was recently uncovered \cite{11,12} with the inclusion of a term proportional to the local spin (an inertial term), in which the phase transition becomes discontinuous. The effects of other ingredients, such as partial inertia \cite{13}, more states per site (instead of “up” or “down” as in the MV) \cite{11,14} and diffusion \cite{15} have also been investigated.

The effect of an ancillary noise in the MV model was proposed recently by Vieira and Crokidakis \cite{16}. In such noisy MV model (NMV), besides the intrinsic noise f, ruling the majority interaction to be or not to be accepted, one includes an independent noise allowing a spin be flipped irrespective of its neighborhood. In the social language jargon, this ingredient corresponds to another kind of nonconformism, usually referred to as independence. Results for the square lattice indicated that the phase transition and critical behavior are identical to the standard MV model, although the presence of such auxiliary noise provides an additional route for the emergence of the phase transition, even in the absence of misalignment term f\textsuperscript{*}.

A relevant issue concerns in comparing the NMV with alike models in which the effect of independent noise has been undertaken. Recent studies for the q-voter model \cite{3}, in which a spin is flipped to the value of its q nearest neighbors with a certain rate, reveal that the transition becomes discontinuous for large values of q whenever the independence ingredient is included. Due to the similarities between the q–voter model and the NMV, a natural inquiry is if that shift in the order of the phase transition is also verified in the latter case for large connectivities \cite{16}.

Aimed at tackling such above queries, here we give a
further step by analyzing the NMV in distinct topologies. By means of mean-field calculations (MFT), we derive expressions for the values of the critical points as a function of the extra noise for an arbitrary network topology. A different (MFT) approach than Refs. [3, 17], based on an extension of the ideas from Ref. [4] for an arbitrary network structure, is performed. The critical behavior is also investigated numerically by employing a finite size scaling analysis. Although MFT provide an approximated description of the phase transition, we observe an excellent agreement with numerical results in the regime of large connectivities. Our results reveal the phase transition is continuous for all topologies and connectivities. Thus, contrasting to the $q$–voter model, a first-order transition is not observed in the NMV. Results indicate the critical behavior in scale-free is different from homogeneous networks.

This paper is organized as follows: In Sec. II, we derive the mean field theory for the model. Next, numerical results are shown in Sec. III. Conclusions are drawn in Sec. IV.

II. MODEL AND MEAN FIELD ANALYSIS

The MV model is defined in an arbitrary lattice topology, in which each node $i$ of degree $k$ is attached to a binary spin variable, $\sigma_i$, that can take the values $\sigma_i = \pm 1$. In the original case, with probability $1 - f$ each node $i$ tends to align itself with its local neighborhood majority, and with complementary probability $f$, the majority rule is not followed. The increase of the misalignment quantity $f$ gives rise to an order-disorder (continuous) phase transition [4, 9, 10]. The NMV differs from the original MV for the inclusion of an additional rule for spin inversion independently of the neighborhood.

\[
p_+ - \frac{1}{2} = \frac{(1 - p)(1 - 2f)}{2\langle k \rangle} \sum_{k=1}^{\infty} kP(k) \sum_{n=[k/2]}^{k} C_n^k [p_+^{k-n} - p_-^{k-n}].
\]

It is possible to derive a simpler expression for $\langle S[X]\rangle$ in the regime of large $k$’s, in which each term from the binomial distribution approaches to a gaussian one with mean $kp_{\pm 1}$ and variance $\sigma^2 = kp_+p_-$. So that

\[
\langle S[X]\rangle = \frac{1}{2} \left[ 2\text{erf} \left( \sqrt{\frac{2ky}{1 - 4y^2}} \right) + \frac{\sqrt{2k(\frac{1}{2} - y)}}{\sqrt{1 - 4y^2}} \left[ y \sqrt{\frac{1}{1 - 4y^2}} \right] \right],
\]

where $\text{erf}(x)$ denotes the error function and, for simplicity, we have rewritten $p_{\pm}$ in terms of the variable $y$ through formula $p_{\pm} = \frac{1}{2} \pm y$. Since the terms in the numerator dominate over the denominator, $\langle S[X]\rangle$ reduces to the simpler form $\langle S[X]\rangle = \text{erf}(y\sqrt{2k})$, from which we

Since we are dealing with markovian systems described by a master equation, our analysis starts by deriving the time evolution of the local magnetization is given by

\[
\frac{d}{dt} m_k = -m_k + (1-p)(1-2f)\langle S[X]\rangle,
\]

where $S[X]$ denotes the signal function, evaluated over the spins of the $k$ nearest neighbors of the site $i$ and reading $S[X] = \text{sign}(X)$ if $X \neq 0$ and $S[0] = 0$. Note that for $p = 0$ one recovers the original MV model. Thus, the steady state satisfies the relation

\[
m_k = (1 - p)(1 - 2f)\langle S[X]\rangle.
\]
arrive at the following expression for the steady state
\[ m_k = (1-p)(1-2f)\text{erf}(y\sqrt{2}k), \]  
(5)
whose steady \( y_0 \)'s are given by
\[ y = \frac{1}{2}\left(1-p\right)\left(1-2f\right)\sum_k\left|\text{erf}(y\sqrt{2}k)\right|P(k). \]  
(6)
Having the steady \( y_0 \)'s [from Eqs. 4 and 8], the corresponding \( m_k \)'s are obtained (for fixed \( f \) and \( p \)) from Eq. 3, whose mean magnetization \( m \) is finally evaluated through \( m = \sum_{k=1}^{\infty} m_k P(k) \).

In order to derive a closed expression for the critical point, we should note that Eq. 4 presents two solutions \((y = \pm y_0 \neq 0)\) for \( f < f_c \) (besides the trivial \( y = 0 \)) and only the trivial solution \( y = 0 \) for \( f > f_c \). Since \( y_0 \) is expected to be small close to the critical point, the right side of Eq. 4 can be expanded in Taylor series whose dependence on \( f \) and \( p \) (for arbitrary lattice topology) reads
\[ y_0 = \frac{1}{A(f,p)}\left(-1 + (1-p)(1-2f)\sqrt{\frac{2}{\pi}}\frac{(k^{3/2})}{\langle k \rangle}\right)^{1/2}, \]
where \( \langle k^{3/2} \rangle = \sum_k k^{3/2}P(k) \) and \( A(f,p) = \sqrt{\frac{8}{\pi}}\frac{(k^{3/2})}{\langle k \rangle}(1-p)(1-2f) \). The critical \( f_c \) is finally given by
\[ f_c = \frac{1}{2}\left(1 - \sqrt{\frac{\pi}{2}}\frac{1}{1-p}\frac{\langle k \rangle}{\langle k^{3/2} \rangle}\right), \]
with \( A(p, f_c) = 2\frac{(k^{5/2})}{(k^{3/2})} \). In particular, for \( p = 0 \) one recovers the expression \( f_c = \frac{1}{2}\left(1 - \sqrt{\frac{\pi}{2}}\frac{\langle k \rangle}{\langle k^{3/2} \rangle}\right) \), in consistency with the results from Ref. 9. Complementary, an order-disorder phase transition is also obtained in the absence of misalignment \( f = 0 \) by increasing the ancillary noise \( p \), whose critical rate \( p_c \) satisfies the relation
\[ p_c = \left\{1 - \sqrt{\frac{\pi}{2}}\frac{\langle k \rangle}{\langle k^{3/2} \rangle}\right\}. \]
Therefore, the former conclusion predicted from the MFT concerns that above expression extends, for an arbitrary network distribution, the conjecture \( p_c(f = 0) = 2f_c(p = 0) \) obtained for the square lattice 17. A second MFT upshot to be drawn is that, contrasting to the generalized q-voter model 5, the inclusion of an independent noise does not alter the classification of phase transition, irrespective of the lattice topology and the neighborhood. The critical point, on the other hand, depends on the network topology.

The first structure to be considered is a random regular network in which nodes follow the distribution \( P(k) = \delta(k - k_0) \), \( k_0 \) being the degree and thus Eq. 5 becomes
\[ f_c = \frac{1}{2}\left(1 - \sqrt{\frac{\pi}{2k_0}}\frac{1}{1-p}\right). \]
(9)
The second topology is the Erdős-Rényi (ER), an iconic example of a homogeneous random network, in which the degree distribution is given by \( P(k) = \langle k \rangle^k e^{-\langle k \rangle}/k! \).

Finally, a heterogeneous network is considered, in which nodes are distributed according to a power-law distribution \( P(k) \sim k^{-\gamma} \). For avoiding divergences when \( k \to 0 \) in the PL, we have imposed a minimum degree \( k_0 \) and the averages \( \langle k \rangle \) and \( \langle k^{3/2} \rangle \) become \( \langle k \rangle = \frac{1}{1-\gamma/k_0}k_0 \) and \( \langle k^{3/2} \rangle = \frac{1}{\gamma-5/2}k_0^{3/2} \), respectively. Thus, the critical point \( f_c \) reads
\[ f_c = \frac{1}{2}\left(1 - \sqrt{\frac{\pi}{2k_0}}\frac{\gamma - 5/2}{\gamma - 2}\frac{1}{1-p}\right). \]
(10)
We shall focus on the analysis for \( \gamma = 3 \), which is a hallmark of scale-free structures 19. In the next section, we are going to confirm above findings by performing numerical simulations.

### III. NUMERICAL RESULTS AND PHASE DIAGRAMS

We have performed extensive numerical simulations for networks with sizes \( N \) ranging from \( N = 1000 \) to \( N = 20000 \). We generated the RR networks through the classical configuration model, introduced by Bollobás 18. The ER networks are constructed by connecting each pair of nodes with probability \( \langle k \rangle/N \). When the size of the graph \( N \to \infty \), the degree distribution is poissonian type, with mean \( \langle k \rangle \). The Barabasi-Albert (BA) scale-free network is a typical representation of heterogeneous structures 19, in which the degree distribution follows a power-law \( P(k) \sim k^{-\gamma} \) with scaling exponent \( \gamma = 3 \).

Besides the order parameter vanishment, continuous phase transitions are signed by an algebraic divergence of the variance \( \chi = N\langle (m^2) - \langle m \rangle^2 \rangle \) at the critical point \( f_c \) for \( N \to \infty \). Since only finite systems can be simulated, the above quantities become rounded at the vicinity of criticality due to finite size effects. For calculating the critical point and the critical exponents, we resort to the finite size scaling theory, in which \( (m) \) and \( \chi \) are rewritten as \( \langle m \rangle = N^{-\beta/\nu}f(N^{1/\nu}\epsilon) \) and \( \chi = N^{\gamma/\nu}g(N^{1/\nu}\epsilon) \), with \( f \) and \( g \) being scaling functions and \( \epsilon = f_c/f \). For \( \epsilon = 0 \), the above relations acquire the dependence on the system size reading \( \langle m \rangle = N^{-\beta/\nu}f(0) \) and \( \chi = N^{\gamma/\nu}g(0) \), in which a log-log plot of \( (m) \) and \( \chi \) versus \( N \) furnish the exponents \( \beta/\nu \) and \( \gamma/\nu \), respectively. The critical point can be properly located through the reduced cumulant \( U_4 = 1 - \frac{(m^4)}{3(m^2)^2} \), since curves for distinct \( N \)'s cross at \( f = f_c \) (\( \epsilon = 0 \)) and \( U_4 \) becomes constant \( U_4 = U_4^0 \). Off the critical point, \( U_4 \to 2/3 \) and 0 for the ordered and disordered phases, respectively when \( N \to \infty \).

Figs. 1 and 2 exemplify the behavior of above quantities for the RR and ER cases for \( p = 0.1 \) \( (k_0 = 10) \), \( p = 0.5 \) \( (k_0 = 40) \) and \( p = 0.4, 0.6 \) \( (k_0 = 30) \), respectively. Panels (a) and (b) reproduce the typical trademarks of critical transitions: \( (m) \) decreases smoothly by
raising \( f \) (or \( p \)) and \( \chi \) presents a maximum whose peak becomes more pronounced as \( N \) increases. The nature of phase transitions is reinforced by examining the crossing among the curves of \( U_4 \) for distinct system sizes. In all cases, the crossing is characterized by an apparent universal value \( U_0' = 0.28(1) \).

Analysis of the critical exponents furnish results consistent with \( \beta/\nu = 1/4 \), \( \gamma/\nu = 1/2 \) and \( 1/\nu = 1/2 \). Also, they satisfy the relation \( 2\beta/\nu + \gamma/\nu = D_{eff} \), with \( D_{eff} = 1 \), in agreement with Ref. \[10\]. Hence, above analysis reveals a universal behavior for homogeneous topologies. On the other hand, all of them are very different from the values \( \beta = 1/8 \), \( \gamma = 7/4 \) and \( \nu = 1 \) for bidimensional (regular) lattices \[4,10\].

A slightly critical different behavior is obtained for BA case, exemplified in Fig. 3 for \( p = 0.4 \) \( (k_0 = 20) \) and \( p = 0.6 \) \( (k_0 = 10) \). In both cases, the crossing value \( U_0' \) and set of critical exponents are different from those both homogeneous structures ones, reading \( U_0' = 0.16(1) \) and critical exponents \( \beta/\nu = 0.34(1) \) and \( \gamma/\nu = 0.32(1) \), respectively. Although distinct from the RR and ER, they also fulfill the relation \( 2\beta/\nu + \gamma/\nu = 1 \).

Now we examine the phase diagrams in turn. A comparison among results for distinct topologies is shown in Figs. 4–6 for distinct \( p \)'s, \( k_0 \)'s (RR and BA) and \( \langle k \rangle \)'s (ER). Panels (a) show that all estimates agree very well for large connectivities, but some discrepancies arise for lower system degrees. These trends reveal not only the reliability of one-site MFT but also its accuracy for the
FIG. 4. Panel (a) shows, for the RR networks, the transition rates \( f_c \) versus the degree node \( k_0 \) for distinct values of \( p \). Circles and lines correspond to estimates obtained from Eqs. (3) and (9), respectively. The symbol × correspond to the numerical values obtained from the crossing among \( U_4 \) for distinct \( N \)’s. In (b), the phase diagram \( p_c \) versus \( f_c \) for distinct \( k_0 \)’s.

FIG. 5. Panel (a) shows, for the ER case, the transition rates \( f_c \) versus the mean degree \( \langle k \rangle \) for distinct values of \( p \). Circles and lines correspond to estimates obtained from Eqs. (3) and (8), respectively. The symbol × correspond to the numerical values obtained from the crossing among \( U_4 \) for distinct \( N \)’s. In (b), the phase diagram \( p_c \) versus \( f_c \) for distinct \( k_0 \)’s.

location of the critical point. As in the MFT, the phase transitions are critical, irrespective of the lattice topology and the system degree. Thus, contrasting to the \( q \)–voter model, the inclusion of “independence” does not shift the phase transition to a discontinuous one. Also, increasing the ancillary \( p \) does not alter the discrepancies among methods. Finally we observe that [panels (b)] the inclusion of additional noise shortens the ordered phase and thus the disordered region enlarges.

FIG. 6. Panel (a) shows, for the BA case, the transition rates \( f_c \) versus the minimum degree \( k_0 \) for distinct values of \( p \). Circles and lines correspond to estimates obtained from Eqs. (3) and (10), respectively. The symbol × correspond to the numerical values obtained from the crossing among fourth-order reduced cumulant for distinct network sizes. In (b), the phase diagram \( p_c \) versus \( f_c \) for distinct \( k_0 \)’s.

IV. CONCLUSIONS

The majority vote model in the presence of two distinct kinds of noise has been investigated in the framework of numerical simulations and mean-field theory for complex networks. Homogeneous and scale-free structures have been considered. We have derived approximated expressions (through a different mean-field approach than Refs. [9, 17]) for the critical point in terms of the lattice topology and noise parameter \( p \), which work very well in the regime of large connectivities. The critical behavior and the set of critical exponents have been detailed investigated. Numerical results strongly suggest a common set of critical exponents for random regular and Erdős-Rényi cases, but different for different heterogeneous structures, hence suggesting a novel universality class for the MV in scale-free structures. Due to the scarcity of results [20], we believe that our findings constitute an important step for typifying the critical behavior in scale-free networks. However, we remark that further studies are still required. In particular, a comparison between the critical behavior coming from other heterogeneous structures, such as the uncorrelated configuration model (UCM) [21], can be very interesting. Previous proposal of a discontinuous phase transition in the regime of large connectivities is discarded, irrespective of the network topology.

[1] J. Marro and R. Dickman, Nonequilibrium Phase Transitions in Lattice Models (Cambridge University Press, Cambridge, 1999).
[2] G. Ódor, *Universality In Nonequilibrium Lattice Systems: Theoretical Foundations* (World Scientific, Singapore, 2007).

[3] M. Henkel, H. Hinrichsen and S. Lubeck, *Non-Equilibrium Phase Transitions Volume I: Absorbing Phase Transitions* (Springer-Verlag, The Netherlands, 2008).

[4] M. J. de Oliveira, J. Stat. Phys. 66, 273 (1992).

[5] P. Nyczka, K. Sznajd-Weron and J. Cislo, Phys. Rev. E, 86, 011105 (2012).

[6] T. Vicsek and A. Zafeiris, Physics Reports 517, 71 (2012).

[7] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. 81, 591 (2009).

[8] C. Castellano, M. Marsili, and A. Vespignani, Phys. Rev. Lett. 85, 3536 (2000).

[9] H. Chen, C. Shen, G. He, H. Zhang and Z. Hou, Phys. Rev. E 91, 022816 (2015).

[10] L. F. C. Pereira and F. G. B. Moreira, Phys. Rev. E 71, 016123 (2005).

[11] H. Chen, C. Shen, H. Zhang, G. Li, Z. Hou and J. Kurths, Phys Rev. E 95, 042304 (2017).

[12] J. M. Encinas, P. E. Harunari, M. M. de Oliveira and C. E. Fiore, Sci. Rep. 8, 9338 (2018).

[13] P. E. Harunari, M. M. de Oliveira and C. E. Fiore, Phys. Rev. E 96, 042305 (2017).

[14] D. F. F. Melo, L. F. Pereira and F. G. B. Moreira, J. Stat. Mech. P11032 (2010); F. W. S. Lima, Physica A 391, 1752 (2012)

[15] N. Crokidakis and P. M. C. de Oliveira, Phys. Rev. E 85, 041147 (2012).

[16] A. R. Vieira and N. Crokidakis, Physica A 450, 30 (2016).

[17] C. Castellano and R. Pastor-Satorras, J. Stat. Mech. p. P05001 (2006).

[18] B. Bollobás, Europ. J. Combinatorics. 1, 311 (1980).

[19] A.-L. Barabási and R. Albert, Science 286, 509 (1999).

[20] F. W. S. Lima, Int. J. Mod. Phys. C, 17, 1257 (2006).

[21] M. Catanzaro, M. Boguna and R. Pastor-Satorras, Phys. Rev. E 71, 027103 (2005).