GraphFlow: A New Graph Convolutional Network Based on Parallel Flows

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Abstract

In view of the huge success of convolution neural networks (CNN) for image classification and object recognition, there have been attempts to generalize the method to general graph-structured data. One major direction is based on spectral graph theory and graph signal processing. In this paper, we study the problem from a completely different perspective, by introducing parallel flow decomposition of graphs. The essential idea is to decompose a graph into families of non-intersecting one dimensional (1D) paths, after which, we may apply a 1D CNN along each family of paths. We demonstrate that the our method, which we call GraphFlow, is able to transfer CNN architectures to general graphs. To show the effectiveness of our approach, we test our method on the classical MNIST dataset, synthetic datasets on network information propagation and a news article classification dataset.

Index Terms

Graph convolutional network, graph-structured data, parallel flow, graph decomposition, convolutional neural network

I. INTRODUCTION

Suppose $G$ is a graph, which can be weighted and directed. A graph signal on $G$ is a function that assigns a number to each node of $G$. The graph either describes relative physical positioning of the nodes or captures correlations between the signals at each vertex. It is an important task to retrieve graph information by processing and analyzing graph signals. An important example of a graph signal is “images”. In this case, the graph is just a 2D lattice, while the graph signals are either RGB values and grey scale. In recent years, convolution neural network (CNN) has

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been used extensively in a large array of applications (e.g., [1]–[4]) with notable success in image processing.

There are attempts to extend the CNN architecture to general graphs (e.g., [5]–[14]). Most of these approaches are based on graph signal processing (see [15] for an overview) using spectral graph theory and graph Fourier transforms. In this paper, we propose a completely different approach that exploits the geometric structure of a graph. To motivate the proposed work, we briefly review some key ingredients of CNN. For convenience, we regard each pixel of an image as a node in the graph (shown in Figure 1). Each node is connected to the neighboring pixels (including the neighbors in the diagonal positions). A convolution filter (Figure 1(b)) can be viewed as a signal on a smaller lattice. The convolution operation is performed by taking dot product of the filter placed at various positions of the image lattice. The filter is used to examine local properties of the image signal, such as existence of edges in different directions.

An important feature of this approach is that the filter is shared at different places of the image, as the 2D lattice (with diagonal connections) is homogeneous and most of the nodes (except those at the image boundary) have the same local structure. However, an attempt to generalize this approach to general graph-structured data faces the immediate difficulty that different nodes may have different number of neighbors and local neighborhoods of each node may differ greatly. Such inhomogeneity renders filter sharing difficult to achieve.

![Fig. 1. In this example, an image is represented by a 2D lattice with diagonal connections as shown in (a). The entire graph has four directions, highlighted in different colors. In (b), we show a typical 3 × 3 filter. It can be decomposed as four 1D filters, each corresponds to a direction of the lattice in (a).](image)

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In the proposed work, we take the point of view that a small filter, such as a 3 × 3 filter, can be decomposed as four 1D filters, each of size 3 × 1. (Figure 1(b)). Each represents a direction of the
2D lattice. On the other hand, each direction of the 2D lattice consists of non-crossing paths, or parallel paths. Moreover, each edge of the lattice belongs to one among the four directions. This filtering scheme with shared 1D filters is readily generalizable to general graphs, as long as we find the correct notion of “directions” (called parallel flows below) with the properties discussed above. In Figure 2, we show some examples, none of which is a lattice. We demonstrate how we can decompose each graph into different directions. Therefore, to design filters, we only need to design a single 1D filter for each direction.

![Figure 2](image)

Fig. 2. In (a), we have two directions for the tree, highlighted in blue and red. In (b), we can also cover the tree by two directions, with purple edges common to both directions. In (c), we have a more general graph, and we use three directions. Of course, we may have different ways to give directions to the same graph, as shown in the simple example (d).

While we have explained the motivation and some basic ideas, we provide more details in the rest of the paper organized as follows. In Section II, we introduce the notion of parallel flows and establish theoretical foundations for the paper. In Section III, we explain the building blocks of a graph CNN model using parallel flows and discuss how to build such a model. We present experimental results on different applications in Section IV and conclude in Section V.
II. PARALLEL FLOWS

In this section, we assume that $G = (V, E)$ is a connected, undirected, unweighted simple graph. We shall describe how to remove such restrictions in building up a graph CNN model in the next section.

We assume that a subset of nodes $V' \subset V$ is fixed. An intuition is that $V'$ serves as a choice of boundary. Moreover, $V'$ can be the empty set.

Example 1. (a) A typical example is when $G$ is a tree and $V'$ consists of the degree 1 nodes of $G$.
(b) If $G$ is a finite lattice, then there is a natural choice of the boundary $V'$ as the nodes with degree $\leq 3$.

Definition 1. A path $P$ of $G$ is a connected subgraph of $G$ such that each node of $P$ has degree $\leq 2$ in $P$. In particular, a single node or a closed loop is considered as a path with this definition.

A path $P$ is not extendable if either $P$ is a single node, a loop or the ends of $P$ belong to the set $V'$. We allow the interior of $P$ to contain nodes of $V'$.

Two paths $P_1$ and $P_2$ are said to be parallel to each other if they do not intersect in $G$.

The following are the key concepts we want to introduce, and they allow us to generalize the notion of "directions" in 2D lattice (as discussed in the introduction).

Definition 2. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a union of paths. Then $\mathcal{P}$ is a parallel flow if the following two conditions hold:
(a) $P_1, \ldots, P_n$ are not extendable.
(b) $P_1, \ldots, P_n$ are pairwise parallel.

Definition 3. Given a $0 < \epsilon \leq 1$, a set of parallel flows $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is an $\epsilon$-cover of $G$ if the union of all the paths in all the involved parallel flows and the boundary $V'$ contains $\epsilon$ fraction of all the edges of $G$. If $\epsilon = 1$, we simply call any 1-cover a cover. The smallest $m$ such that there is an $\epsilon$-cover consisting of $m$ parallel flows deserves special attention, and we denote it by $\mu(G, V', \epsilon)$, and we abbreviate $\mu(G, V', 1)$ as $\mu(G, V')$.

It is sometimes of practical use to have $\epsilon < 1$. For example, in Figure 2 (d), if we allow $\epsilon = 0.9$, then we only need to use one parallel flow to obtain (the two red horizontal paths) a
We first study some general properties.

**Lemma 1.** (a) If $V' \subset V''$, then $\mu(G, V'', \epsilon) \leq \mu(G, V', \epsilon)$.

(b) Suppose $G_1 = (V_1, E_1)$ is a subtree of a tree $G_2 = (V_2, E_2)$ and $V_i' \subset V_i$, $i = 1, 2$. Moreover, for each $v \in V_2'$, the node of $V_i$ closest to $v$ belongs to $V_i'$. Then $\mu(G_1, V_1') \leq \mu(G_2, V_2')$.

(c) Suppose $G = G_1 \cup G_2$ and $V_1' \cup V_2' \subset V'$. Then $\mu(G, V') \leq \mu(G_1, V_1') + \mu(G_2, V_2')$.

**Proof.**

(a) We notice that as $V' \subset V''$, any non-extendable path w.r.t. $V'$ is also not extendable w.r.t. $V''$. Hence any $\epsilon$-cover by parallel flows of $G$ w.r.t. $V'$ is also an $\epsilon$-cover by parallel flows w.r.t. $V''$, whence the claim.

(b) As both $G_1$ and $G_2$ are trees, there are no loops in them. Let $P$ be a non-extendable path of $G_2$. By the assumptions, $P \cap G_1$ is a union of non-extendable paths of $G_1$. It is clear that being parallel is preserved by taking intersection with $G_1$. Therefore, by taking intersection, a cover of $G_2$ by parallel flows yields a cover of $G_1$ by parallel flows. The inequality follows.

(c) Any union of covers of $G_i$, $i = 1, 2$ by parallel flows is also a cover of $G$ by parallel flows, under the assumption that $V_1' \cup V_2' \subset V'$. Hence $\mu(G, V') \leq \mu(G_1, V_1') + \mu(G_2, V_2')$. $\square$

**Theorem 1.** Suppose $G = (V, E)$ is a tree. Let $d_{\text{max}}$ be the maximal degree of $G$ and $V'$ consists of nodes with degree strictly smaller than $d_{\text{max}}$. Then $\mu(G, V') = \lfloor (d_{\text{max}} + 1)/2 \rfloor$.

**Proof.** The inequality $\mu(G, V') \geq \lfloor (d_{\text{max}} + 1)/2 \rfloor$ is clear. This is because that at any $v$ with maximal degree, at least $\lfloor (d_{\text{max}} + 1)/2 \rfloor$ paths are required to cover all the neighboring edges. Moreover, none of them are parallel, and hence $\mu(G, V')$ is at least $\lfloor (d_{\text{max}} + 1)/2 \rfloor$.

We prove $\mu(G, V') \leq \lfloor (d_{\text{max}} + 1)/2 \rfloor$ by starting with some special cases.

**Case 1:** $d_{\text{max}}$ is even and all the nodes of $V'$ are of degree 1.

The case $d_{\text{max}} = 2$ is trivial. We assume $d_{\text{max}} \geq 4$. Starting from a fixed node $v_0$ with degree $d_{\text{max}}$, we label its neighbors in pairs using $1, \ldots, d_{\text{max}}/2$. If $v$ is a neighbor of $v_0$ of degree $d_{\text{max}}$, let $k$ be the label of the edge connecting $v$ and $v_0$. We can again label the remaining adjacent edges of $v$ in pairs except $k$. This procedure can be continued for all the nodes without any conflict as $G$ is a tree. For each $k$, the union $\mathcal{P}_k$ of edges with the same label forms a parallel flow (see Figure 3), with the following reasons:
Fig. 3. In this example \( d_{\text{max}} = 6 \). We label the tree using three colors and obtain a cover with three parallel flows.

(a) For each node, there are at most two adjacent edges with the same label. Hence any two

distinct paths of \( P_k \) do not intersect.

(b) Moreover, if \( v \) has degree \( d_{\text{max}} \), then it is of degree 2 in a path in \( P_k \). Hence each path of

\( P_k \) is non-extendable w.r.t. \( V' \).

The union of \( P_k, 1 \leq k \leq d_{\text{max}}/2 \) clearly covers \( G \). This proves that \( \mu(G, V') \leq d_{\text{max}}/2 \) as

we have presented one such cover.

**Case 2:** \( d_{\text{max}} = 3 \) and all the nodes of \( V' \) are of degree 1. Therefore, we want to show that

\( \mu(G, V') = 2 \).

We prove this by induction on \( |V| \). The base case \( |V| = 4 \) is trivial. For \( |V| > 4 \), we are able

to find a subtree \( G_1 \) (see Figure 4) of \( G \) such that \( G_1 = (V_1, E_1) \) such that the following holds:

(a) \( G_1 \) also belongs to Case 2 with \( |V_1| = |V| - 2 \).

(b) There is a node \( v \) of \( G_1 \) of degree 1 such that \( G \) is obtained from \( G_1 \) by attaching two

edges \( e_1, e_2 \).

By the induction hypothesis, two parallel flows \( P_1, P_2 \) cover \( G_1 \). Without loss of generality,
we assume that there is a nontrivial path \( P_1 \in P_1 \) that contains \( v \) as an end point.

Suppose \( P_2 \) does not contain any nontrivial path that ends at \( v \). Then we can first extend \( P_1 \)
by attaching \( e_1 \) to form \( P_1' \) and add the path \( e_1 \cup e_2 \) to \( P_2 \) to form \( P_2' \). Clearly, both \( P_1' \) and \( P_2' \)
are parallel flows and they cover \( G \).
On the other hand, if $P_2$ contains a path $P_2$ also ends at $v$, then we extend $P_2$ by attaching $e_2$ to form $P'_2$. Again, both $P'_1$ and $P'_2$ are parallel flows and they cover $G$.

**Case 3:** $d_{\text{max}}$ is odd and all the nodes of $V'$ are of degree 1.

In this case, we apply induction to $d_{\text{max}}$. The base case $d_{\text{max}} = 3$ is shown and we assume that $d_{\text{max}} \geq 5$. We first claim that there is a parallel flow $P$ such that: for each $v \in V$, there is a path $P \in \mathcal{P}$ containing $v$. To construct such a parallel flow, one can first include any non-extendable path $P$. Then for each neighbor $v$ of any path already in $\mathcal{P}$ with degree $d_{\text{max}}$, we include in $\mathcal{P}$ a non-extendable path parallel to $P$. This procedure can be repeated until each $v$ with degree $d_{\text{max}}$ is contained in a path in $\mathcal{P}$.

We construct a (possibly disconnected) new graph $G_1$ by removing the edges in $\mathcal{P}$ from $G$ (see Figure 5). Each component of $G_1$ satisfy the same condition of Case 3 with maximal degree the same as $d_{\text{max}} - 2$. Moreover, as we assume $d_{\text{max}} \geq 5$, the nodes with degree 1 in $G_1$ are of degree 1 in $G$ as well. Therefore, by the induction hypothesis, we are able to find a cover of $G_1$ with $\lfloor (d_{\text{max}} - 1)/2 \rfloor$ parallel flows. Taking union with $\mathcal{P}$, we obtain a cover of $G$ with $\lfloor (d_{\text{max}} + 1)/2 \rfloor$ parallel flows.

Finally, we prove the general case. Given $G$, we construct a tree $G_1$ containing $G$. For each $v$ with degree $d_G(v)$ satisfying $1 < d_G(v) < d_{\text{max}}$, we add $d_{\text{max}} - d_G(v)$ edges to $v$ so that the degree...
Fig. 5. This is an example with $d_{\text{max}} = 5$ of Case 3. The dashed red paths form the parallel flow $\mathcal{P}$. When we remove $\mathcal{P}$, we obtain $G_1$, which is a union of 3 trees each with maximal degree 3.

of $v$ in $G_1$ is $d_{\text{max}}$. Let $V'_1$ be the nodes of $G_1$ with degree 1 and the condition of Lemma 1(b) holds true for $V_1$ and $V'_1$. Then by the special cases considered above, $\mu(G_1, V'_1) \leq \lfloor (d_{\text{max}} + 1)/2 \rfloor$. Now the inequality for $\mu(G, V') \leq \lfloor (d_{\text{max}} + 1)/2 \rfloor$ holds by Lemma 1(b).

As an immediate consequence, we have that if $G = (V, E)$ is a tree, then $\mu(G, V) = \lfloor (d_{\text{max}} + 1)/2 \rfloor$.

Now for a general graph $G$, we can always find a spanning subtree $G_1$ of $G$ (for example a breadth-first-search spanning tree from any non-isolated node). The maximal degree of $G_1$ is clearly no greater than $d_{\text{max}}$. Moreover, if we let $G_2$ be the (possibly disconnected) subgraph of $G$ by removing those edges contained in $G_1$, then the maximal degree of $G_2$ is strictly smaller than $d_{\text{max}}$. Therefore, a simple induction yields the following estimation for a general graph.

**Corollary 1.** For any graph $G = (V, E)$ (which can be disconnected), let $d_{\text{max}}$ be the maximal degree of $G$. Then $\mu(G, V) \leq (\lfloor (d_{\text{max}} + 1)/2 \rfloor + 1)\lfloor (d_{\text{max}} + 1)/2 \rfloor$.

### III. GraphFlow: A Parallel Flow Based Graph Convolutional Network

For a graph $G = (V, E)$, recall that a graph signal or a graph label is a function that assigns a (real) number to each node $v \in V$. On each node $v$ of a path in a parallel flow, it retains the signal of the corresponding node in the original graph.

In this section, we propose a graph convolutional network framework based on the concept of parallel flow decomposition of a graph introduced in the previous section.
A. Architectural components

For a fixed $0 < \epsilon \leq 1$, let $\{P_1, \ldots, P_m\}$ be a fixed $\epsilon$-cover of the graph $G = (V, E)$ by parallel flows. To build up a graph convolution neural network (GCN), we have the following components:

(a) convolution layers
(b) pooling layers
(c) fusion layers
(d) fully connected layers.

**Convolution layers:** As each parallel flow $P_i$ is the union of linear paths. We may apply common set of standard 1-dimensional filters. We may briefly recall that they are specified by the parameters: $p$ as the padding size, $n$ as the filter size, $s$ as the stride number and $c$ as the number of channels. It is preferable that $n$ is an odd number such that there is a unique center.

**Pooling layers:** Similar to the convolution layers, the pooling layer is the standard 1-dimensional pooling specified: by $n$ (preferably being odd) the pooling size, and $s$ the stride number. We usually use max pooling, though average pooling is also a viable option.

**Fusion layers:** This is a new type of architectural component to establish communications among different parallel flows. We fix an ordering of the nodes of $G$, and each node has a unique index. For each node of the linear paths in the parallel flows, we record the index of the node in the $G$. A node in $G$ might appear in different parallel flows, and all of them have the same index. Each convolution or pooling operation centered at node $v$ makes $v$ retains its index. In the fusion layer, we apply a fusion function $f$ across all the nodes with the same index in all the parallel flows (see Figure 6 for an example). Preferred choices of $f$ include the max function and the average function.

We would like to remark that the the fusion layer is designed so that we are able to formally recover a 2D pooling layer as a concatenation of a 1D pooling layer and a fusion layer. However, some experiments suggest that the usage of 1D pooling layers alone might be sufficient, and the additional involvement of fusion layers does not add much to the performance.

**Fully connected layers:** These layers are again standard. They are the last few layers of a architectural pipeline when the dimension of the feature vector is sufficiently small.
Fig. 6. In (a), both the central node of the horizontal and vertical paths are indexed by $i$ with signals $x$ and $y$ respectively. After the fusion layer with $\max$ fusion function $f$, they both have the common signal $\max(x, y)$.

B. Graph convolution neural network framework

In this section, we describe how various architectural components might be brought up together to form a complete picture.

Given a graph $G = (V, E)$ and $0 < \epsilon \leq 1$, the initial step is to construct an $\epsilon$-cover $\{P_1, \ldots, P_m\}$ by parallel flows of $G$.

For certain graphs, $\epsilon$-covers can be obtained by inspection. For example, for the 2D lattice, we may use the canonical cover $\{P_1, P_2\}$, where $P_1$ and $P_2$ consist of horizontal and vertical paths respectively. If $G$ is obtained from the 2D lattice by connecting nodes that are diagonally adjacent, we may further include parallel flows $P_3$ and $P_4$ consists of the paths in the two diagonal directions respectively.

There are certainly numerous graphs such that a parallel flow decomposition cannot be obtained by inspection. In such cases, discussions made in Section II (for example, the proof of Corollary [1]) tell us how we may proceed. More precisely, let $S$ be an operation that take $G$ and $v \in V$ as inputs and a spanning tree $T$ of $G$ as an output. Examples of $S$ include the (breadth-first-search) BFS and the (depth-first-search) DFS algorithms. Therefore, we may choose a nodes $v \in V$ and generate a spanning tree $T = S(G, v)$. By removing the edges belonging to $T$ from $G$, we obtain a (possibly disconnected) subgraph $G'$ of $G$ with strictly smaller number of edges.
On the other hand, the proof of Theorem 1 Case 1 can be used to decompose a tree into parallel flows. This procedure can be repeated until sufficiently amount of edges are included in the parallel flows.

Once an $\epsilon$-cover of $G$ by parallel flows is obtained, there is an option to regularize the size of the parallel flows by removing short paths and sub-dividing long paths into shorter ones.

The rest of the steps can be derived by modifying any CNN model. For the applications of this paper, we mainly use variations of the following simple model: input layer $\rightarrow$ convolution layer $\rightarrow$ pooling layer $\rightarrow$ fusion layer $\rightarrow$ convolution layer $\rightarrow$ pooling layer $\rightarrow$ fusion layer $\rightarrow$ fully connected layers $\rightarrow$ output layer. We call such a model a GraphFlow.

As long as the operation $S$ in the parallel flow decomposition can be applied to directed graphs, so does GraphFlow. Moreover, it works for weighted graphs as well, as we only need to form the weighted dot product with edge weights in the convolution layers.

C. Further discussions

We first make a comparison between GraphFlow and CNN. To emphasize the similarities between them, we compare them side-by-side in the following table, which also serves as a dictionary between these two frameworks.

|                  | CNN            | GraphFlow       |
|------------------|----------------|-----------------|
| Convolution      | 2D filters     | Multiple 1D filters |
| Down sampling    | 2D pooling     | 1D pooling & fusion |
| Striding         | 2D striding    | 1D striding     |
| Dense layers     | Fully conn.    | Fully conn.     |

From Table I, we see that GraphFlow bears much resemblance to CNN, while GraphFlow has the advantage of being generalizable to arbitrary graphs. Moreover, we expect that more sophisticated CNN models can be “translated” into GraphFlow models using Table I.

One of the most popular graph CNN architectures is based on spectral graph signal processing (e.g., [5]–[7], [9]–[15]), abbreviated by GSP-CNN for convenience. Both GSP-CNN and GraphFlow can be useful in different applications. However, we would like to point out a difference underlying the main principles behind these methods. For GraphFlow and CNN, a filter assigns
different weights to the neighbors of the central node in the convolution operation. On the other
hand, if the graph adjacency of Laplacian matrix is used as the graph shift operator for GSP-
CNN, then a filter assigns a common weight to all the neighbors of the central node and nodes
receive different weights only when their distances to the central node are different. In this
respect, GraphFlow is a closer relative to CNN.

IV. NUMERICAL EXPERIMENTS

The problems considered in this paper are of the following type: given a signal on a graph
G, we want to determine in which category does such a signal belong to.

A. MNIST dataset revisit

In this paper, we introduce GraphFlow not as a substitute of CNN on 2D lattices, but as an
alternative interpretation of CNN, which can be readily generalized to general graphs. Therefore,
we would like to apply GraphFlow to MNIST dataset\textsuperscript{1} to demonstrate that GraphFlow has
comparable performance to CNN on its realm of applications.

Now, we describe the specifications of GraphFlow in this application. As we have pointed out
earlier, the parallel flow decomposition can be obtained by inspection, namely, the parallel flows
consist of horizontal paths, vertical paths and paths in the two diagonal directions. We use two
layers of convolution layers with 16 channels of size 3 1D filters, with stride 1. Immediately
after each convolution layer, there is a 1D pooling layer of size 3, with stride 2. Following the
second pooling layer, there is an optional fusion layer with $f = \max$ (with or without the fusion
layer has the same performance). The last two layers are fully connected layers.

GraphFlow is able to achieve a 98.7\% accuracy, which is comparable with accuracy achievable
by CNN architectures such as 98.3\% by LeNet-1 and 98.9\% by LeNet-4 (\textsuperscript{2}).

B. Information propagation on social networks

In this section, we study two problems on information propagation over social networks using
synthetic datasets.

There are quite a few models proposed for information propagation over networks. For
example, under the independent cascade framework, there are SI and SIRI models, depending

\textsuperscript{1}http://yann.lecun.com/exdb/mnist/
on whether a node has acquired the information, i.e., called an infected node, can recover and being re-infected subsequently. For the SI model, any infected node have a positive probability to infect its neighbors, and any infected node remains in that status. On the other hand, in the SIRI model, any infected node can recover and become infected again. The difference in the mechanisms causes different dynamical behaviors. For the SI model, all the nodes becomes infected almost surely eventually; while for the SIRI model, it can happen that all the nodes are recovered if the recover rate is high and infection rate is low. There are numerous research works on this theme (e.g., [16]–[19]), which mainly use classical signal processing techniques. We showcase that GraphFlow can also be used for specific problems.

1) Number of information sources: If information starts the spreading from a few sources of the network, it is of interest to determine the number of sources. We start the spreading from 1, 2 or 3 randomly chosen sources on the Enron email network $G$, and generate a cascade using the SI model. A snapshot observation of the infected nodes (about 20% of the total population) is made. We would like to determine the exact number of sources.

If we assign 1 to each infected node and 0 to the remaining nodes, we obtain a graph signal from a snapshot observation. Therefore, the problem is essentially a classification into 3 categories from a graph signal. We can apply GraphFlow.

The network $G$ contains 500 nodes with average degree 12.57 and maximum degree 141. To obtain a cover of $G$ by parallel flows, we use the BFS algorithm for the spanning tree generation operation $S$ (with details given in Section III-B). The resulting cover contains 80 flows. Therefore, for the initial convolution layer, we need 80 distinct 1D filters for each channel.

We synthetically generate 8000 samples for training and 2000 samples for testing. GraphFlow is able to achieve a 93.5% accuracy in determining the number of sources.

2) Information propagation type: As we have mentioned earlier, apart from the SI model, we also have the SIRI model. For the second problem, we want to determine the information propagation type from snapshot observations. The information spreadings are generated either using the SI or the SIRI model such that

(a) About 20% of the nodes are infected when the observation is made for either model.

(b) The spreading parameters are chosen such that: in the SI model, all the nodes are eventually infected; while in the SIRI model, only a small fraction of nodes are infected eventually.

2https://snap.stanford.edu/data/email-Enron.html
This is a classification problem into 2 categories using graph signals. We may apply GraphFlow. We use the exact same architecture as the previous problem. Again, we generate 8000 samples for training and 2000 samples for testing. GraphFlow achieves a 96.1% accuracy.

C. News article classification

We apply GraphFlow to news article classification. The dataset we use is the publicly available 20news dataset (20). The graph signals are constructed as in (6): each document $x$ is represented using a normalized bag-of-words model and the underlying graph $G$ (of size 1000) is constructed using a 16-NN graph on the word2vec embedding (21) using the 1000 most common words (instead of 10000 keywords as in (6)). There are 16617 texts with each label from one among 20 categories.

The decomposition of $G$ into parallel flows is more subtle in this case. We apply the BFS algorithm based decomposition three times. Recall that the BFS algorithm requires an initial node. Therefore, for each of the three instances, we use different initial node selection schemes. Similarly, we apply the DFS algorithm based decomposition three times. The union of the six decompositions of $G$ by parallel flows contains a lot of short paths. We remove all paths with length smaller than 6. There are 43 parallel flows that remain, and they form a 0.91-cover of $G$.

The model is changed slightly here: for the pooling layer, size 2 max pooling is used with stride 1.

Along with the test accuracy of GraphFlow, we also include the performance of a few classification methods from (6), including GC32 (which is a GSP-CNN), FC2500, FC2500-FC500 and Softmax (we refer the reader to (6) for the detailed model architectures) for comparison in Table II. We see that GraphFlow outperforms all the other methods.

After examining the dataset, we notice that many errors occur for those texts with sparse graph signals, i.e., most of the components are 0. On the other hand, some of the class labels are similar and hard to distinguish for texts with sparse graphs signals, such as “soc.religion.Christian” and “talk.religion.misc”. Therefore, we perform another test on the sub-dataset with 12000 texts by removing those whose graph signals has < 13 nonzero components. The results are summarized in Table III. Again, we see that GraphFlow performs the best.

\[^3\text{We thank the authors of (6) for providing their source codes.}\]
| Method       | Accuracy |
|-------------|----------|
| **GraphFlow** | 64.3%    |
| GC32        | 55.9%    |
| FC2500      | 57.1%    |
| FC2500-FC500| 56.3%    |
| Softmax     | 54.6%    |

**TABLE II**

Performance of GraphFlow and other models on 20news dataset.

| Method       | Accuracy |
|-------------|----------|
| **GraphFlow** | 70.0%    |
| GC32        | 67.1%    |
| FC2500      | 66.5%    |
| FC2500-FC500| 67.7%    |
| Softmax     | 67.3%    |

**TABLE III**

Performance of GraphFlow and other models on 20news dataset with non-sparse graph signals.

V. CONCLUSION

In this paper, we introduced a new convolution neural network framework (GraphFlow) on general graphs, based on the idea of decomposing a graph into parallel flows. This approach allows us to mimic CNN architectures already developed for 2D lattices. We presented a few applications to demonstrate the effectiveness of the approach.

Currently, the parallel flow decomposition scheme is problem specific. It would be interesting to have a more systematic general procedure for a large family of graphs. We shall explore such a possibility in the future.

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