Duality for normal lattice expansions and sorted residuated frames with relations

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Abstract. We revisit the problem of Stone duality for lattices with quasioperators, presenting a fresh duality result. The new result is an improvement over that of our previous work in two important respects. First, the axiomatization of frames is now simplified, partly by incorporating Gehrke’s proposal of section stability for relations. Second, morphisms are redefined so as to preserve Galois stable (and co-stable) sets and we rely for this, partly again, on Goldblatt’s recently proposed definition of bounded morphisms for polarities. In studying the dual algebraic structures associated to polarities with relations we demonstrate that stable/co-stable set operators result as the Galois closure of the restriction of classical (though sorted) image operators generated by the frame relations to Galois stable/co-stable sets. This provides a proof, at the representation level, that non-distributive logics can be regarded as fragments of sorted residuated (poly)modal logics, a research direction recently initiated by this author.

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1. Introduction

In this article we address and resolve the problem of Stone duality for the category of lattices with quasioperators. The work presented here is a significant improvement over our own [21] (with Dunn) and [15]. The axiomatization of frames and frame relations in [15] was rather cumbersome and it is now simplified, partly by incorporating Gehrke’s proposal [9] of section stability for relations. Morphisms in the category of frames (polarities with relations) dual to lattices with quasioperators are defined so that not only clopens in the
dual complex algebra of the frame be preserved (as in [21,15]) but so that the
inverses of frame morphisms are morphisms of the complete lattices of sta-
ble sets dual to frames (their full complex algebras), preserving arbitrary joins
and meets. Polarity morphisms, as we define them, are the same as Goldblatt’s
bounded morphisms for polarities [11], but we diverge from [11] in the exten-
sion to morphisms for polarities with relations. We also diverge from [9,11] in
the definition of canonical relations and in the way operators are defined from
frame relations. We do this with the express objective of demonstrating that
quasioperators in the complex algebra of stable sets of a frame can be obtained
as the Galois closure of the restriction to Galois sets of the classical though
sorted image operators generated from the relations, in the Jónsson–Tarski
style [24]. The logical significance of this is that it demonstrates at the repre-
sentation level that non-distributive logics are fragments of sorted residuated
polymodal logics (their modal companions).

The structure of this article is as follows. Section 2 is an introductory
section, defining the category \( \text{NLE}_\tau \) of normal lattice expansions of some sim-
ilarity type \( \tau \) (a sequence of distribution types of lattice operators).

In Section 3 we present basic definitions and results for polarities (equiva-
antly, sorted residuated frames). In Remark 3.2 we carefully list all our no-
tational conventions, to ensure the reader has an effortless and seamless un-
derstanding of our notation.

With Section 3.1 we address the first issue of significance for our purposes,
which is to define operators from relations, properly axiomatized, so as to
ensure complete distribution properties of the defined operators. In the same
section we list the first four axioms for the objects of the category \( \text{SRF}_\tau \) of
sorted residuated frames (polarities) with relations.

Section 3.2 turns to studying morphisms, first for frames in the absence
of additional relations (Section 3.2.1) and then for frames with additional
relations of sort types in some similarity type \( \tau \) (Section 3.2.2). Further axioms
for frame morphisms are stated and the category \( \text{SRF}_\tau \) defined (Definition
3.26).

Section 4 defines a contravariant functor from the category \( \text{NLE}_\tau \) to
the category \( \text{SRF}_\tau \). For the lattice representation, we rely on [21] and for
the representation of normal lattice operators we draw on [14] and [15] and
we review the canonical frame construction in Section 4.1. Sections 4.2 and
4.3 are devoted to proving that the frame axioms hold for the dual frame
of a normal lattice expansion and for the duals of normal lattice expansion
homomorphisms.

Stone duality is addressed in Section 5. To ensure a Stone duality theorem
can be proven we extend the axiomatization for frames (defining a smaller
category \( \text{SRF}_\tau^* \) and topologizing the frames), drawing on [21,15], we verify
that all additional axioms hold for the canonical frame construction and we
conclude with a Stone duality theorem (Theorem 5.8).

In the Conclusions in Section 6 we summarize the results obtained and
sketch a potential area for further research.
2. Normal Lattice Expansions (NLEs)

Let \(\{1, \partial\}\) be a 2-element set, \(L^1 = L\) and \(L^\partial = L^{\text{op}}\) (the opposite lattice). Extending established terminology [24], a function \(f : L_1 \times \cdots \times L_n \rightarrow L_{n+1}\) will be called additive and normal, or a normal operator, if it distributes over finite joins of the lattice \(L_i\), for each \(i = 1, \ldots, n\), delivering a join in \(L_{n+1}\).

**Definition 2.1.** An \(n\)-ary operation \(f\) on a bounded lattice \(L\) is a normal lattice operator of distribution type \(\delta(f) = (i_1, \ldots, i_n; i_{n+1})\in\{1, \partial\}^{n+1}\) if it is a normal additive function \(f : L_{i_1} \times \cdots \times L_{i_n} \rightarrow L_{i_{n+1}}\) (distributing over finite joins in each argument place), where each \(i_j\), for \(j = 1, \ldots, n+1\), is in the set \(\{1, \partial\}\), hence \(L_{i_j}\) is either \(L\), or \(L^{\partial}\).

If \(\tau\) is a tuple (sequence) of distribution types, a normal lattice expansion of (similarity) type \(\tau\) is a lattice with a normal lattice operator of distribution type \(\delta\) for each \(\delta\) in \(\tau\).

Normal lattice operators in the above sense are sometimes referred to in the literature as quasioperators, see for example Moshier and Jipsen \[27, 28\]. Remarks 4.2 and 4.8 provide some clarifications on the connections between \[27, 28\] and related work by this author \[14, 21, 15\], including the present article.

**Definition 2.2.** The category \(\text{NLE}_\tau\), for a fixed similarity type \(\tau\), has normal lattice expansions of type \(\tau\) as objects. Its morphisms are the usual algebraic homomorphisms.

**Example 2.3.** A modal normal diamond operator \(\Diamond\) is a normal lattice operator of distribution type \(\delta(\Diamond) = (1; 1)\), i.e., \(\Diamond : L \rightarrow L\), distributing over finite joins of \(L\). A normal box operator \(\Box\) is also a normal lattice operator in the sense of Definition 2.1, of distribution type \(\delta(\Box) = (\partial; \partial)\), i.e., \(\Box : L^{\partial} \rightarrow L^{\partial}\) distributes over finite joins of \(L^{\partial}\), which are then just meets of \(L\).

An \(\text{FL}_{ew}\)-algebra (also referred to as a full \(\text{BCK}\)-algebra, or a commutative integral residuated lattice) \(A = (L, \wedge, \vee, 0, 1, \circ, \rightarrow)\) is a normal lattice expansion, where \(\delta(\circ) = (1, 1; 1)\), \(\delta(\rightarrow) = (1, \partial; 1)\), i.e., \(\circ : L \times L \rightarrow L\) and \(\rightarrow : L \times L^{\partial} \rightarrow L^{\partial}\) are both normal lattice operators with the familiar distributive properties.

The Grishin operators \[13\] \(\leftarrow, *\), \(\rightarrow\), satisfying the familiar co-residuation conditions \(a \geq c \leftarrow b\) iff \(a * b \geq c\) iff \(b \geq a \rightarrow c\) have the respective distribution properties, which are exactly captured by assigning to them the distribution types \(\delta(*) = (\partial, \partial; \partial)\) (* behaves like a binary box operator), \(\delta(\leftarrow) = (1, \partial; 1)\) and \(\delta(\rightarrow) = (\partial, 1; 1)\).

Distributive normal lattice expansions are the special case where the underlying lattice is distributive. BAOs (Boolean algebras with operators) \[24, 25\] are the special case where the underlying lattice is a Boolean algebra and all normal operators distribute over finite joins of the Boolean algebra, i.e., they are all of distribution types of the form \((1, \ldots, 1; 1)\). For BAOs, operators of other distribution types can be obtained by composition of operators with Boolean complementation. For example, in studying residuated Boolean algebras \[26\], Jönsson and Tsinakis make use of a notion of conjugate operators,
introduced in [24, 25], and they show that intensional implications (division operations) \(\setminus, \div\) (the residuals of the product operator \(\odot\)) are interdefinable with the conjugates (at each argument place) \(\angle, \rangle\) of \(\odot\), i.e., \(a \setminus b = (a \rangle b^\angle)^\neg\) and \(a \div b = (a \angle b^\rangle)^\neg\) (and similarly for \(/, \langle\) and \(\langle, \rangle\), see [26] for details). Note that \(\setminus, \div\) are not operators, whereas \(\angle, \rangle\) are.

The relational representation of BAOs in [24], extending Stone’s representation [35] of Boolean algebras using the space of ultrafilters of the algebra, forms the technical basis of the subsequently introduced by Kripke possible worlds semantics, which has had a well-known impact on the development of normal modal logics. This has been extended to the case of the logics of distributive lattices with various quasioperators, see [7] and [32, 33], for example, now based on the Priestley representation [31] of distributive lattices in ordered Stone spaces (simplifying Stone’s original representation [34] of distributive lattices), using the space of prime filters.

For non-distributive lattices, a number of different representation results have been published [37, 23, 21, 14, 30]. Following the definition of canonical extensions for general lattices [10], interest in the subject was renewed [3, 4, 6, 5], see also [12, 27, 28]. A brief review of the area was given in [16], slightly expanded in [22], to which we refer the reader for more details. As noted in [22], there appear to be two settings in which both a canonical extension construction and a full categorical duality can be carried through for bounded lattices. The first is obtained by combining as in [4, 6, 3] results from Urquhart’s representation [37] and its variants [30] (Ploščica) and [1] (Allwein and Hartonas). The second is based on the Hartonas and Dunn representation and duality [21], or on the Moshier and Jipsen [27] simplification of the representation and duality of [14], combining with the proofs in [10, 27] that [21, 27] deliver a canonical lattice extension. In [5], Craig and Haviar have in fact established a connection between the first and the second approach.

A Stone type duality for normal lattice expansions however has only first been presented in [15], extending [21]. Part of the difficulty was in defining an appropriate notion of morphism and Goldblatt [11, page 1021] reviews the related attempts to this issue. In [15] this problem was somewhat sidestepped, by restricting the definition of frame morphisms to such that their inverses are homomorphisms of the sublattices of clopen elements of the full complex algebras of the frames, as in the lattice duality of [21]. Another source of difficulty has been to define operators from suitably axiomatized relations on frames, so that the framework can serve the semantics of logics without distribution as Jónsson and Tarski’s BAOs [24] have served the semantics of modal logics. In [15] we proposed an axiomatization of frames and relations, though the axiomatization appears to be somewhat forced and we provide a significant improvement in the present article.

3. Sorted Residuated Frames (SRFs)

Regard \(\{1, \partial\}\) as a set of sorts and let \(Z = (Z_1, Z_\partial)\) be a sorted set. Sorted residuated frames \(\mathfrak{F} = (Z_1, I, Z_\partial)\) are triples consisting of nonempty sets \(Z_1 =\)
$X, Z_0 = Y$ and a binary relation $I \subseteq X \times Y$. The relation $I$ generates residuated operators $\Diamond : \mathcal{P}(X) \rhd \mathcal{P}(Y) : \blacksquare$ $(U \subseteq V \iff \Diamond U \subseteq V)$, defined by

$$\Diamond U = \{ y \in Y | \exists x \in X \ (x I y \land x \in U) \}$$

$$\blacksquare V = \{ x \in X | \forall y \in Y \ (x I y \rightarrow y \in V) \}.$$

The dual sorted residuated modal algebra of the (sorted residuated) frame $(X, I, Y)$ is the algebra $\Diamond : \mathcal{P}(X) \rhd \mathcal{P}(Y) : \blacksquare$. By residuation, the compositions $\blacksquare \Diamond$ and $\Diamond \blacksquare$ are closure operators on $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, respectively.

For a sorted frame $(X, I, Y)$, the complement of the frame relation $I$ will be consistently designated by $\bot$ and referred to as the Galois relation of the frame. It generates a Galois connection $(\ )^\bot : \mathcal{P}(X) \rhd \mathcal{P}(Y)^\bot : \bot(\ )$ $(V \subseteq U^\bot \iff U \subseteq \bot V)$

$$U^\bot = \{ y \in Y | \forall x \in U \ x \bot y \} \{ y \in Y | U \bot y \}$$

$$\bot V = \{ x \in X | \forall y \in V \ x \bot y \} = \{ x \in X | x \bot V \}.$$

Triples $(X, R, Y), R \subseteq X \times Y$, where $R$ is treated as the Galois relation of the frame, are variously referred to in the literature as polarities, after Birkhoff [2], as formal contexts, in the Formal Concept Analysis (FCA) tradition [8], or as object-attribute (categorization, classification, or information) systems [29,38], or as generalized Kripke frames [9], or as polarity frames in the bi-approximation semantics of [36].

Note that the residuated and Galois connected operators generate the same closure operators, on $\mathcal{P}(X)$, $\blacksquare \Diamond U = \bot(U^\bot)$ and on $\mathcal{P}(Y)$, $\Diamond \blacksquare V = (\bot V)^\bot$. This follows from the fact that $\bot V = \blacksquare(-V)$ and $U^\bot = \Diamond(-U)$.

**Proposition 3.1.** The discrete categories of polarities and sorted residuated frames are equivalent. $\square$

The equivalence allows us to move in our arguments and definitions back-and-forth between sorted residuated frames and polarities. Indeed, for our purposes, both the residuated pairs $\blacksquare, \Diamond$ and $\Diamond, \blacksquare$, as well as the Galois connection will be involved in definitions and arguments and we do not differentiate between polarities $(X, \bot, Y)$ and their associated sorted residuated frames $(X, I, Y)$, with $I$ being the complement of $\bot$.

A subset $A \subseteq X$ will be called stable if $A = \blacksquare \Diamond A = \bot(A^\bot)$. Similarly, a subset $B \subseteq Y$ will be called co-stable if $B = \Diamond \blacksquare B = (\bot B)^\bot$. Stable and co-stable sets will be referred to as Galois sets, disambiguating to Galois stable or Galois co-stable when needed and as appropriate.

By $G(X), G(Y)$ we designate the complete lattices of stable and co-stable sets, respectively. Note that the Galois connection restricts to a dual isomorphism $(\ )^\bot : G(X) \rhd G(Y)^\bot : \bot(\ )$.

Preorders, relations are induced on each of the sorts, by setting for $x, z \in X$, $x \leq z$ iff $\{ x \}^\bot \subseteq \{ z \}^\bot$ and, similarly, for $y, v \in Y$, $y \leq v$ iff $\{ y \}^\bot \subseteq \{ v \}^\bot$. A (sorted) frame is called separated if the preorders $\leq$ (on $X$ and on $Y$) are in fact partial orders $\leq$.

**Remark 3.2 (Notational conventions).** For a sorted relation $R \subseteq \prod_{j=1}^{j=n+1} Z_{i_j}$, where $i_j \in \{ 1, \partial \}$ for each $j$ (and thus $Z_{i_j} = X$ if $i_j = 1$ and $Z_{i_j} = Y$ when $i_j = \partial$), we make the convention to regard it as a relation $R \subseteq Z_{i_n+1} \times \prod_{j=1}^{j=n} Z_{i_j}$,
we agree to write its sort type as \( \sigma(R) = (i_{n+1}; i_1 \ldots i_n) \) and for a tuple of points of suitable sort we write \( uRu_1 \ldots u_n \) for \((u, u_1, \ldots, u_n) \in R \). We often display the sort type as a superscript, as in \( R^\sigma \). Thus, for example, \( R^{\partial_i} \) is a subset of \( Y \times (X \times Y) \). In writing then \( yR^{\partial_i} xv \) it is understood that \( x \in X = Z_1 \) and \( y, v \in Y = Z_\beta \). The sort superscript is understood as part of the name designation of the relation, so that, for example, \( R^{\partial_1} \), \( R^{\partial_1} \) name two different relations.

We use \( \Gamma \) to designate upper closure \( \Gamma U = \{ z \in X|\exists x \in U \ x \preceq z \} \), for \( U \subseteq X \), and similarly for \( U \subseteq Y \). The set \( U \) is increasing (an upset) iff \( U = \Gamma U \). For a singleton set \( \{x\} \subseteq X \) we write \( \Gamma x \), rather than \( \Gamma(\{x\}) \) and similarly for \( \{y\} \subseteq Y \).

We typically use the standard FCA [8] priming notation for each of the two Galois maps \( \partial(\ ),(\ )^\partial \). This allows for stating and proving results for each of \( G(X), G(Y) \) without either repeating definitions and proofs, or making constant appeals to duality. Thus for a Galois set \( G, G' = G^\partial \), if \( G \in G(X) \) (\( G \) is a Galois stable set), and otherwise \( G' = G^\partial \), if \( G \in G(Y) \) (\( G \) is a Galois co-stable set).

For an element \( u \) in either \( X \) or \( Y \) and a subset \( W \), respectively of \( Y \) or \( X \), we write \( u\mid W \), under a well-sorting assumption, to stand for either \( u \perp W \) (which stands for \( u \perp w \), for all \( w \in W \)), or \( W \perp u \) (which stands for \( w \perp u \), for all \( w \in W \)), where well-sorting means that either \( u \in X, W \subseteq Y \), or \( W \subseteq X \) and \( u \in Y \), respectively. Similarly for the notation \( u\mid v \), where \( u, v \) are elements of different sort.

We designate \( n \)-tuples (of sets, or elements) using a vectorial notation, setting \( (G_1, \ldots, G_n) = \hat{G} \in \prod_{j=1}^{j=n} G(Z_{i_j}), \hat{U} \in \prod_{j=1}^{j=n} \wp(Z_{i_j}), \hat{\bar{u}} \in \prod_{j=1}^{j=n} Z_{i_j} \) (where \( i_j \in \{1, \partial\} \)). Most of the time we are interested in some particular argument place \( 1 \leq k \leq n \) and we write \( \hat{G}[F]_k \) for the tuple \( \hat{G} \) where \( G_k = F \) (or \( G_k \) is replaced by \( F \)). Similarly \( \hat{\bar{u}}[x]_k \) is \( \{u_1, \ldots, u_{k-1}, x, u_{k+1}, \ldots, u_n\} \).

For brevity, we write \( \bar{u} \leq \bar{v} \) for the pointwise ordering statements \( u_1 \leq v_1, \ldots, u_n \leq v_n \). We also let \( \bar{u} \in \hat{W} \) stand for the conjunction of componentwise membership \( u_j \in W_j \), for all \( j = 1, \ldots, n \).

To simplify notation, we write \( \Gamma \bar{u} \) for the \( n \)-tuple \( (\Gamma u_1, \ldots, \Gamma u_n) \). For a unary map \( f \) and a tuple \( \bar{u} \) we write \( f[\bar{u}] \) for the tuple \( (f(u_1), \ldots, f(u_n)) \).

Note that the same notation is used for the image \( f[S] = \{ f(x) | x \in S \} \) of a set under a function \( f \), but context will make it clear what the intended meaning is. The convention can be nested, so that if \( S \) is a set (or sequence) of tuples \( \bar{u}_i \), then \( f[S] \) is the set (or sequence) consisting of the elements \( f[\bar{u}_i] \).

To refer to sections of relations (the sets obtained by leaving one argument place unfilled) we make use of the notation \( \bar{u}_[.]_k \) which stands for the \((n - 1)\)-tuple \( (u_1, \ldots, u_{k-1}, [\ ], u_{k+1}, \ldots, u_n) \) and similarly for tuples of sets, extending the membership convention for tuples to cases such as \( \bar{u}[.]_k \in \hat{F}[.]_k \) and similarly for ordering relations \( \bar{u}[.]_k \leq \bar{v}[.]_k \). We also quantify over tuples (with, or without a hole in them), instead of resorting to an iterated quantification over the elements of the tuple, as for example in \( \exists \bar{u}[.]_k \in \hat{F}[.]_k \exists v, w \in G \ wR\bar{u}[v]_k \).
We extend the vectorial notation to distribution types, summarily writing \( \sigma = (i_j; i_{n+1}) \) for \((i_1, \ldots, i_n; i_{n+1})\). Then, for example, \(i_j[\partial]_k\) is the tuple with \(i_k = \partial\). Furthermore, we let \(i_j = \partial\), if \(i_j = 1\) and \(i_j = 1\), when \(i_j = \partial\).

**Lemma 3.3.** Let \(\mathfrak{F} = (X, \perp, Y)\) be a polarity and \(u \in Z = X \cup Y\).

1. \(\perp\) is increasing in each argument place (and thereby its complement \(I\) is decreasing in each argument place).
2. \((\Gamma u)\)' = \(\{u\}'\) and \(\Gamma u = \{u\}''\) is a Galois set.
3. Galois sets are increasing, i.e., \(u \in G\) implies \(\Gamma u \subseteq G\).
4. For a Galois set \(G\), \(G = \bigcup_{u \in G} \Gamma u\).
5. For a Galois set \(G\), \(G = \bigvee_{u \in G} \Gamma u = \bigcap_{v \in G} \{v\}'\).
6. For a Galois set \(G\) and any set \(W\), \(W'' \subseteq G\) iff \(W \subseteq G\).

**Proof.** By simple calculation. Proof details are included in [15, Lemma 2.2]. For claim 4, \(\bigcup_{u \in G} \Gamma u \subseteq G\) by claim 3 (Galois sets are upsets). In claim 5, given our notational conventions, the claim is that if \(G \in \mathcal{G}(X)\), then \(G = \bigcap_{G \perp y} \{y\}\) and if \(G \in \mathcal{G}(Y)\), then \(G = \bigcap_{x \perp G} \{x\}^\perp\).

**Definition 3.4 (Closed and open elements).** The principal upper sets of the form \(\Gamma_x\), with \(x \in X\), will be called closed, or filter elements of \(\mathcal{G}(X)\), while sets of the form \(\{y\}^\perp\), with \(y \in Y\), will be referred to as open, or ideal elements of \(\mathcal{G}(X)\). Similarly for \(\mathcal{G}(Y)\). A closed element \(\Gamma u\) is clopen iff there exists an element \(v\), with \(u | v\), such that \(\Gamma u = \{v\}'\).

By Lemma 3.3, the closed elements of \(\mathcal{G}(X)\) join-generate \(\mathcal{G}(X)\), while the open elements meet-generate \(\mathcal{G}(X)\) (similarly for \(\mathcal{G}(Y)\)).

**Definition 3.5 (Galois dual relation).** For a relation \(R\), of sort type \(\sigma\), its Galois dual relation \(R'\) is the relation defined by \(uR'\bar{v}\) iff \(\forall w (wR\bar{v} \longrightarrow w | u)\). In other words, \(R'\bar{v} = (R\bar{v})'\).

For example, given a relation \(R^{111}\) its Galois dual is the relation \(R^{\partial11}\) where for any \(x, z \in X\), \(R^{\partial11}xz = \{y \in Y | \forall u \in X (uR^{111}xz \longrightarrow u | y)\} = (R^{111}xz)^\perp\) and, similarly, for a relation \(S^{\partial1\partial}\) its Galois dual is the relation \(S^{11\partial}\) where for any \(z \in X, v \in Y\) we have \(S^{11\partial}zv = \{y | S^{\partial1\partial}zv\}\), i.e., \(xS^{11\partial}zv\) holds iff for all \(y \in Y\), if \(yS^{\partial1\partial}zv\), then \(x \perp y\).

**Definition 3.6 (Sections of relations).** For an \((n + 1)\)-ary relation \(R^\sigma\) and an \(n\)-tuple \(\bar{u}\), \(R^\sigma \bar{u} = \{w | wR^\sigma \bar{u}\}\) is the section of \(R^\sigma\) determined by \(\bar{u}\). To designate a section of the relation at the \(k\)-th argument place we let \(\bar{u}[-]_k\) be the tuple with a hole at the \(k\)-th argument place. Then \(wR^\sigma \bar{u}[-]_k = \{v | wR^\sigma \bar{u}[v]_k\} \subseteq Z_{i_k}\) is the \(k\)-th section of \(R^\sigma\).

We defer the definition of the category \(\text{SRF}_\tau\) of sorted residuated frames of type \(\tau\) for later (see Definition 3.26), after establishing the necessary facts.

### 3.1. Frame relations and operators

If \(R^\sigma\) is a relation on a sorted residuated frame \(\mathfrak{F} = (X, I, Y)\), of some sort type \(\sigma = (i_{n+1}; i_1 \ldots i_n)\), then as in the unsorted case, \(R^\sigma\) (but we shall drop the displayed sort type when clear from context) generates a (sorted) image.
operator $\alpha_R$, defined by \((3.1)\), of sort $\sigma(\alpha_R) = (i_1, \ldots, i_n; i_{n+1})$, defined by the obvious generalization of the Jónsson–Tarski image operators \([24]\),

$$\alpha_R(\bar{W}) = \left\{ w \in Z_{i_{n+1}} \mid 3 \bar{w} (wR\bar{w} \land \bigwedge_{j=1}^{j=n} (w_j \in W_j)) \right\} \cup R\bar{w}, \quad (3.1)$$

where for each $j$, $W_j \subseteq Z_{i_j}$ (and recall that $Z_{i_j} = X$ when $i_j = 1$ and $Z_{i_j} = Y$, if $i_j = \partial$).

Thus $\alpha_R : \prod_{j=1}^{j=n} \wp(Z_{i_j}) \longrightarrow \wp(Z_{i_{n+1}})$ is a sorted normal and completely additive function in each argument place, therefore it is residuated, i.e., for each $k$ there is a set-operator $\beta^k_R$ satisfying the condition

$$\alpha_R(\bar{W}[V]_k) \subseteq U \text{ iff } V \subseteq \beta^k_R(\bar{W}[U]_k). \quad (3.2)$$

Hence $\beta^k_R(\bar{W}[U]_k)$ is the largest set $V$ s.t. $\alpha_R(\bar{W}[V]_k) \subseteq U$ and it is thereby definable by

$$\beta^k_R(\bar{W}[U]_k) = \bigcup \{V \mid \alpha_R(\bar{W}[V]_k) \subseteq U\}. \quad (3.3)$$

**Definition 3.7.** $\tilde{\sigma}_R$ is the closure of the restriction of $\alpha_R$ to Galois sets $\bar{F}$,

$$\tilde{\sigma}_R(\bar{F}) = (\alpha_R(\bar{F}))'' = \left( \bigcup_{j=1, \ldots, n} R\bar{w} \right)' = \bigvee (R\bar{w})'', \quad (3.4)$$

where $F_j \in G(Z_{i_j})$, for each $j \in \{1, \ldots, n\}$.

In Theorem 3.12 we establish conditions under which the sorted operation $\tilde{\sigma}_R$ on Galois sets is completely distributive, in each argument place.

The operator $\tilde{\sigma}_R$ is sorted and its sorting is inherited from the sort type of $R$. For example, if $\sigma(R) = (\partial; 11)$, $\alpha_R : \wp(X) \times \wp(X) \longrightarrow \wp(Y)$, hence $\tilde{\sigma}_R : G(X) \times G(X) \longrightarrow G(Y)$. Single-sorted operations

$$\tilde{\sigma}^R : G(X) \longrightarrow G(X) \quad \text{and} \quad \tilde{\sigma}_R^\partial : G(Y) \times G(Y) \longrightarrow G(Y)$$

can then be extracted by composing appropriately with the Galois connection: $\tilde{\sigma}^R(\bar{F}_1, \bar{F}_2, \bar{F}_3) = (\tilde{\sigma}^R(A, C))'$ (where $A, C \in G(X)$) and, similarly, $\tilde{\sigma}_R^\partial(\bar{F}_1, \bar{F}_2) = \tilde{\sigma}_R(\bar{F}_1', \bar{F}_2')$ (where $A, C \in G(Y)$). Similarly for the $n$-ary case and for an arbitrary distribution type.

**Definition 3.8** (Full complex algebra). Let $\bar{F} = (X, \partial, Y, R)$ be a polarity with a relation $R$ of some sort $\sigma(R) = (i_{n+1}; i_1 \ldots i_n)$. The full complex algebra of $\bar{F}$ is the structure $\bar{F}^\bar{\partial} = (G(X), \tilde{\sigma}_R^\partial)$ and its dual full complex algebra is the structure $\bar{F}^\partial = (G(Y), \tilde{\sigma}_R^\partial)$. Subalgebras of full complex algebras will be referred to as complex algebras of a frame.

Most of the time we work with the dual sorted algebra of Galois sets

$$\left\{ ( )^\pm : G(X) \preceq G(Y)^{\partial} : ( ), \quad \tilde{\sigma}_R : \prod_{j=1}^{j=n} G(Z_{i_j}) \longrightarrow G(Z_{i_{n+1}}) \right\},$$

$$\left( Z_1 = X, \quad Z_\partial = Y \right)$$

as it allows for considering sorted operations that distribute over joins in each argument place (which are either joins of $G(X)$, or of $G(Y)$, depending on the
sort type of the operation). Single-sorted normal operators are then extracted in the complex algebra by composition with the Galois maps, as indicated above.

**Definition 3.9 (Conjugates).** Let $\alpha$ be an image operator (generated by some relation $R$) of sort type $\sigma(\alpha) = (i, j; i_{n+1})$ and $\bar{\alpha}$ the closure of its restriction to Galois sets in each argument place, as defined above. A function $\overline{\gamma}^k$ on Galois sets, of sort type $\sigma(\overline{\gamma}^k) = (i, j; i_{n+1})$ (where $i = \partial$ if $i = 1$ and $i = \bar{\partial}$ is a conjugate of $\overline{\alpha}$ at the $k$-th argument place (or a $k$-conjugate) iff the following condition

$$\overline{\alpha}(\bar{F}) \subseteq G \iff \overline{\gamma}^k(\bar{F}[G'])_k \subseteq F'_k$$

holds for all Galois sets $F_j \in G(Z_{ij})$ and $G \in G(G(Z_{in})).$

It follows from the definition of a conjugate function that $\overline{\gamma}$ is a $k$-conjugate of $\overline{\alpha}$ iff $\bar{\alpha}$ is one of $\overline{\gamma}$ and we thus call $\overline{\alpha}, \overline{\gamma}$ $k$-conjugates. Note that the priming notation for both maps of the duality $(\cdot)\overline{\gamma} : G(X) \subseteq G(Y)_\partial : \overline{\gamma}(\cdot)$ packs together, in one form, four distinct (due to sorting) cases of conjugacy.

**Example 3.10.** In the case of a ternary relation $R^{111}$ of the indicated sort type, an image operator $\alpha_R = \bigcirc : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is generated. Designate the closure of its restriction to Galois stable sets by $\bigcirc : G(X) \times G(X) \rightarrow G(Y)$ Then $\overline{\alpha} = \bigcirc$ is of sort type $\sigma(\bigcirc) = (1, 1, 1)$ If $\overline{\gamma}_R^k = : G(X) \times G(Y) \rightarrow G(Y)$, with $\sigma(\cdot) = (1, \partial, 1)$, then $\bigcirc, \overline{\gamma}$ are conjugates $\iff$ for any Galois sets $A, F, C \in G(X)$ it holds that $A \bigcirc F \subseteq C$ $\iff$ $A \overline{\gamma} C' \subseteq F'$.

Note that, given an operator $\overline{\sigma} : G(X) \times G(X) \rightarrow G(Y)$, if we now define $\Rightarrow : G(X) \times G(Y) \rightarrow G(Y)$ by $A \Rightarrow C = (A \overline{\gamma} C')' = (A \overline{\gamma} C')$, it is immediate that $\bigcirc, \overline{\gamma}$ are conjugates $\iff$ $\bigcirc, \Rightarrow$ are residuated. In other words

$$A \bigcirc F \subseteq C \iff A \overline{\gamma} C' \subseteq F' \iff F \subseteq A \Rightarrow C.$$ 

**Lemma 3.11.** The following are equivalent.

1. $\overline{\alpha}_R$ distributes over any joins of Galois sets at the $k$-th argument place
2. $\overline{\alpha}_R$ has a $k$-conjugate $\overline{\gamma}^k_R$ defined on Galois sets by

$$\overline{\gamma}^k_R(\bar{F}) = \bigcap \{G[\overline{\alpha}_R(\bar{F}[G']_k)] \subseteq F'_k\}$$

3. $\overline{\alpha}_R$ has a $k$-residual $\overline{\beta}^k_R$ defined on Galois sets by

$$\overline{\beta}^k_R(\bar{F}[G')_k] = (\overline{\gamma}_R^k(\bar{F}[G']_k))' = \bigcup \{G'[\overline{\alpha}_R(\bar{F}[G']_k) \subseteq F'_k\}.$$

**Proof.** Existence of a $k$-residual is equivalent to distribution over arbitrary joins and the residual is defined by

$$\overline{\beta}^k_R(\ldots, F_{k-1}, H, F_{k+1}, \ldots) = \bigcup \{G[\overline{\alpha}_R(\ldots, F_{k-1}, G, F_{k+1}, \ldots) \subseteq H\}.$$ 

We show that the distributivity assumption 1) implies that 2) and 3) are equivalent, i.e., that

$$\overline{\alpha}_R(\bar{F}[G)_k] \subseteq H \iff \overline{\gamma}^k_R(\bar{F}[H'])_k \subseteq G' \iff G \subseteq \overline{\beta}^k_R(\bar{F}[H)_k].$$

We illustrate the proof for the unary case only, as the other parameters remain idle in the argument.
Assume $\alpha_R(G) \subseteq H$ and let $\gamma_R(H') = \bigcap \{ E|\alpha_R(E') \subseteq H \}$, a Galois set by definition, given that $G,H,E$ are assumed to be Galois sets. Then $G'$ is in the set whose intersection is taken. Hence $\gamma_R(H') \subseteq G'$ follows from the definition of $\gamma_R$. It also follows by definition that $G \subseteq \overline{\beta}_R(H) = (\gamma_R(H'))'$.

Assuming $G \subseteq \overline{\beta}_R(H)$ we obtain by definition that $G \subseteq (\gamma_R(H'))'$, hence $G \subseteq \bigvee \{ E'|\alpha_R(E') \subseteq H \}$, using the definition of $\gamma_R$ and duality. Hence by the distributivity assumption $\alpha_R(G) \subseteq \bigvee \{ \alpha_R(E')|\alpha_R(E') \subseteq H \} \subseteq H$. This establishes that $\alpha_R(G) \subseteq H$ iff $\gamma_R(H') \subseteq G'$ iff $G \subseteq \overline{\beta}_R(H)$.

**Theorem 3.12.** Let $\mathfrak{F} = (X,\leq,Y,R)$ be a frame with an $(n+1)$-ary sorted relation, of some sort $\sigma(R) = (i_{n+1};i_j)$ and assume that for any $w \in Z_{i_{n+1}}$ and any $(n-1)$-tuple $\bar{p} \{\_\}_k$ with $p_j \in Z_{i_j}$, for each $j \in \{1,\ldots,n\} \setminus \{k\}$, the sections $wR'\bar{p} \{\_\}_k$ of the Galois dual relation $R'$ of $R$ are Galois sets. Then $\alpha_R$ distributes at the $k$th argument place over arbitrary joins in $\mathcal{G}(Z_{i_k})$.

**Proof.** Define the relation $T$ from $R$ by setting,

$$vT\bar{p}[w]_k \iff w \in (vR'\bar{p} \{\_\}_k)'$$

then use equation (3.6) below, to define a relation $S$

$$\forall v \in Z_{i_{n+1}} \forall \bar{p} \{\_\}_k \in \mathcal{Z}_{i_j} \forall w \in Z_{i_k} (vT\bar{p}[w]_k \leftrightarrow wS\bar{p}[v]_k) \tag{3.6}$$

Note that the sort type of $S$, as defined, is $\sigma(S) = (\bar{i_k};i_j[i_{n+1}]_k)$. Let $\gamma_S$ be the closure of the restriction of the image operator $\eta_S$ to Galois sets, according to the sort type of $S$. We show that $\alpha_R$ and $\gamma_S$ are $k$-conjugates. To establish the conjugacy condition $\alpha_R(\mathcal{F}) \subseteq G$ iff $\gamma_S(\mathcal{F}[G']_k) \subseteq F'_k$ it suffices by Lemma 3.3 to verify that $\alpha_R(\mathcal{F}) \subseteq G$ iff $\eta_S(\mathcal{F}[G']_k) \subseteq F'_k$. We have

$$\alpha_R(\mathcal{F}) \subseteq G \iff \bigcup_{\bar{p} \in \mathcal{F}} R\bar{p} \subseteq G \iff \forall \bar{p} (\bar{p} \in \mathcal{F} \rightarrow (R\bar{p} \subseteq G))$$

if $\forall \bar{p} (\bar{p} \in \mathcal{F} \rightarrow (G' \subseteq R\bar{p}))$

if $\forall \bar{p} (\bar{p} \in \mathcal{F} \rightarrow \forall v \in Z_{i_{n+1}} (G'v \rightarrow vR'\bar{p}))$

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (\bar{p} \{\_\}_k \in \mathcal{F}_i \wedge \bar{p} \in \mathcal{F}_k \wedge G'v \rightarrow vR'\bar{p}[p_k])$

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (\bar{p} \{\_\}_k \in \mathcal{F}_i \wedge G'v \rightarrow (p_k \in F_k \rightarrow vR'\bar{p}[p_k]))$

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (\bar{p} \{\_\}_k \in \mathcal{F}_i \wedge G'v \rightarrow (F_k \subseteq vR'\bar{p}[p_k]))$

(using the hypothesis that the $k$-th sections of $R'$ are Galois sets)

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (\bar{p} \{\_\}_k \in \mathcal{F}_i \wedge G'v \rightarrow ((vR'\bar{p} \{\_\}_k)' \subseteq F'_k))$

(using the definition of $T$)

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (\bar{p} \{\_\}_k \in \mathcal{F}_i \wedge G'v \rightarrow \forall w \in Z_{i_k} (vT\bar{p}[w]_k \rightarrow F_k[w]))$

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (vT\bar{p}[w]_k \wedge \bar{p} \{\_\}_k \in \mathcal{F}_i \wedge G'v \rightarrow F_k[w])$

(using the definition of $S$)

if $\forall \bar{p} \forall v \in Z_{i_{n+1}} (wS\bar{p}[v]_k \wedge \bar{p} \{\_\}_k \in \mathcal{F}_i \wedge G'v \rightarrow F_k[w])$

if $\bigcup_{\bar{p}[v]_k \in \mathcal{F}[G']_k} wS\bar{p}[v]_k \subseteq F'_k$

if $\eta_S(\mathcal{F}[G']_k) \subseteq F'_k$.

Hence $\alpha_R$ and $\gamma_S$ are $k$-conjugates. Consequently, by Lemma 3.11, $\alpha_R$ distributes at the $k$-th argument place over arbitrary joins in $\mathcal{G}(Z_{i_k})$. \hfill \Box
Definition 3.13. We let $\beta_{R_i}^k$ be the restriction of $\beta^k_R$ of equation (3.3) to Galois sets, according to its sort type, explicitly defined by (3.7)
\[
\beta_{R_i}^k(\bar{E}[G]_k) = \bigcup \{ F \in G(Z_{i_k}) | \alpha_R(\bar{E}[F]_k) \subseteq G \}. \tag{3.7}
\]

Theorem 3.14. If $\bar{\alpha}_R$ is residuated in the $k$-th argument place, then $\beta_{R_i}^k$ is its residual and $\beta_{R_i}^k(\bar{E}[G]_k)$ is a Galois set, i.e., the union in equation (3.7) is actually a join in $G(Z_{i_k})$.

Proof. We illustrate the proof for the unary case only, since the other parameters that may exist remain idle in the argument. In the unary case, $\beta_{R_i}(G) = \bigcup \{ F | \alpha_R(F) \subseteq G \}$, for Galois sets $F,G$.

Note first that $\bar{\alpha}_R(\bar{F}) \subseteq \bar{G}$ iff $\bar{F} \subseteq \beta_{R_i}(\bar{G})$. Left-to-right is obvious by definition and by the fact that for a Galois set $G$ and any set $U$, $U'' \subseteq G$ iff $U \subseteq G$. If $F \subseteq \beta_{R_i}(\bar{G}) \subseteq \beta_{R_i}(\bar{F})$, then by residuation $\alpha_R(F) \subseteq G$. Given that $G$ is a Galois set, it follows $\bar{\alpha}_R(F) \subseteq G$.

If indeed $\bar{\alpha}_R$ is residuated on Galois sets with a map $\bar{\beta}_R$, then the residual is defined by $\bar{\beta}_R(G) = \bigvee \{ F | \bar{\alpha}_R(F) \subseteq G \} = \bigvee \{ F | \alpha_R(F) \subseteq G \}$ and this is precisely the closure of $\beta_{R_i}(G) = \bigcup \{ F | \alpha_R(F) \subseteq G \}$.

But in that case we obtain $F \subseteq \bar{\beta}_R(G)$ iff $\bar{\alpha}_R(F) \subseteq G$ iff $\alpha_R(F) \subseteq G$ if $F \subseteq \beta_{R_i}(\bar{G})$ and setting $F = \bar{\beta}_R(G)$ it follows that $\bar{\beta}_R(G) \subseteq \beta_{R_i}(\bar{G}) \subseteq \bar{\beta}_R(G)$.

Lemma 3.15. $\beta_{R_i}^k$ is equivalently defined by (3.8) and by (3.9)
\[
\begin{align*}
\beta_{R_i}^k(\bar{E}[G]_k) &= \bigcup \{ \Gamma u \in G(Z_{i_k}) | \alpha_R(\bar{E}[\Gamma u]_k) \subseteq G \}, \tag{3.8} \\
\beta_{R_i}^k(\bar{E}[G]_k) &= \{ u \in Z_{i_k} | \alpha_R(\bar{E}[\Gamma u]_k) \subseteq G \}. \tag{3.9}
\end{align*}
\]

Proof. $\beta_{R_i}^k$ is defined by equation (3.7), so if $u \in \beta_{R_i}^k(\bar{E}[G]_k)$, let $F \in G(Z_{i_k})$ be such that $u \in F$ and $\alpha_R(\bar{E}[F]_k) \subseteq G$. Then $\Gamma u \subseteq F$ and by monotonicity of $\alpha_R$ we have $\alpha_R(\bar{E}[\Gamma u]_k) \subseteq \alpha_R(\bar{E}[F]_k) \subseteq G$ and this establishes the left-to-right inclusion for the first identity of the lemma. The converse inclusion is obvious since $\Gamma u$ is a Galois set.

For the second identity, the inclusion right-to-left is obvious. Now if $u$ is such that $\alpha_R(\bar{E}[\Gamma u]_k) \subseteq G$ and $u \leq w$, then $\Gamma w \subseteq \Gamma u$ and then by monotonicity of $\alpha_R$ it follows that $\alpha_R(\bar{E}[\Gamma w]_k) \subseteq \alpha_R(\bar{E}[\Gamma u]_k) \subseteq G$.

This shows that $\bigcup \{ \Gamma u \in G(Z_{i_k}) | \alpha_R(\bar{E}[\Gamma u]_k) \subseteq G \}$ is contained in the set $\{ u \in Z_{i_k} | \alpha_R(\bar{E}[\Gamma u]_k) \subseteq G \}$, and given the first part of the lemma, the second identity holds as well.

We summarize our results obtained thus far with the following observations.

Let $C_\tau$ be the class of sorted residuated frames (equivalently, polarities) with relations $R_\sigma$ of sort type $\sigma$, for each $\sigma = (i_{n+1}; i_j) \in \{ 1, \partial \}^{n+1}$ in the similarity type $\tau$. Assume the stability axiom below (F4 in Table 1) for $C_\tau$.

- For each relation $R$ of type $\sigma = (i_{n+1}; i_j)$ and each $w \in Z_{i_{n+1}}$ and $\bar{u}[w]_k$ with $u_j \in Z_{i_j}$ for each $j \in \{ 1, \ldots, n \} \setminus \{ k \}$, the $k$-th section $wR'\bar{u}[w]_k$ of the Galois dual relation $R'$ of $R$ is a Galois set, for each $k = 1, \ldots, n$. 

Let \( \alpha_R \) be the classical sorted image operator generated by \( R \), as in equation (3.1), and \( \beta^k_R \) its \( k \)-residual for any \( k = 1, \ldots, n \), defined as usual by equation (3.3). Then

1. the closure \( \overline{\alpha}_R \) (Definition 3.7) of the restriction of \( \alpha_R \) to Galois sets is residuated at the \( k \)-th argument place with the restriction \( \beta^k_R \) (Definition 3.13) of \( \beta^k_R \) to Galois sets (Lemma 3.11, Theorem 3.12, Theorem 3.14).

2. a completely normal operator \( \overline{\alpha}^1_R : G(X)^n \rightarrow G(X) \) of distribution type \( \delta = (\vec{i}; i_{n+1}) \) is obtained by composition with the Galois connection

\[
\overline{\alpha}^1_R(A_1, \ldots, A_n) = \begin{cases} 
\overline{\alpha}_R \left( \prod_{i=1}^{n} A_j \right) & \text{if } i_{n+1} = 1 \\
\left( \overline{\alpha}_R \left( \prod_{i=1}^{n} A_j \right) \right)' & \text{if } i_{n+1} = \partial
\end{cases}
\]

3. similarly for its dual operator \( \overline{\alpha}^\partial_R : G(Y)^n \rightarrow G(Y) \).

We list in Table 1 the frame axioms we shall assume in the sequel, for a sorted residuated frame with relations \( \mathfrak{F} = (X, I, Y, (R_\sigma)_{\sigma \in \tau}) \).

Note that axioms F1 and F2 imply that there is a (sorted) function \( \vec{f}_R \) on the points of the frame such that \( \vec{f}_R(\vec{u}) = w \iff R\vec{u} = \Gamma w \). The following immediate observation will be useful in the sequel.

**Lemma 3.16.** Let \( \mathfrak{F} \) be a frame of similarity type \( \tau \) and assume that axioms F1–F3 in Table 1 hold. Then for a frame relation \( R \) of type \( \sigma \) in \( \tau \), \( \overline{\alpha}_R(\Gamma \vec{u}) = R\vec{u} = \alpha_R(\Gamma \vec{u}) = \Gamma(\vec{f}_R(\vec{u})) \).

**Proof.** By definition (3.1), \( \alpha_R(\Gamma \vec{u}) = \bigcup_{\vec{u} \leq \vec{w}} R\vec{w} \). By axiom F3, \( \bigcup_{\vec{u} \leq \vec{w}} R\vec{w} = R\vec{u} \), which is a closed element by axiom F2, generated by a unique point \( w = \vec{f}_R(\vec{u}) \), by axiom F1, so that \( \alpha_R(\Gamma \vec{u}) = R\vec{u} = \Gamma w = (\alpha_R(\Gamma \vec{u}))'' = \overline{\alpha}_R(\Gamma \vec{u}) \), where \( \Gamma w = \Gamma(\vec{f}_R(\vec{u})) \). \( \Box \)

The axiomatization will be strengthened in Section 5, imposing among others a topology on each of \( X \) and \( Y \), in order to be able to carry out a Stone duality proof.
3.2. Weak bounded morphisms

Recall that a bounded morphism \( p : (W_1, R_1) \to (W_2, R_2) \), for classical Kripke frames, is defined as a map that preserves the frame relation, i.e., \( uR_1v \) implies that \( p(u)R_2p(v) \) and so that its inverse \( p^{-1} \) is a homomorphism of the dual modal algebras \( p^{-1} : (\bar{\mathcal{P}}(W_2), \Diamond_2) \to (\bar{\mathcal{P}}(W_1), \Diamond_1) \), i.e., such that \( p^{-1}(\Diamond_2V) = \Diamond_1(p^{-1}V) \). This can be re-written as the familiar first-order condition typically used to define bounded morphisms.

For sorted frames, when their dual sorted residuated modal algebras are of interest, morphisms of sorted frames can be taken to be the natural generalization of bounded morphisms to the sorted case, to wit a pair of maps \( (p,q) : (X_2, I_2, Y_2) \to (X_1, I_1, Y_1) \), such that their inverses commute with the residuated set-operators \( \Diamond_1, \blacksquare_1 \) and \( \Diamond_2, \blacksquare_2 \) (equivalently, with the Galois connections). One direction of the required inclusions is ensured by requiring preservation of the frame relation, as in the unsorted case. Since inverse maps preserve unions and every set can be written as the union of the singletons of its elements, the reverse inclusion \( \Diamond_2p^{-1}(U) \supseteq q^{-1}(\Diamond_1U) \) will hold if it holds for singletons \( \Diamond_2p^{-1}(\{x\}) \supseteq q^{-1}(\Diamond_1\{x\}) \). Rephrasing and expressing it as a first-order condition we obtain condition (3.10). Similarly for the other reverse inclusion, after replacing boxes with diamonds, working with co-atoms \( \{-x\} \) and contraposing a number of times we obtain the equivalent first-order condition (3.11).

\[
\forall x \in X_1 \forall y' \in Y_2 (xI_1q(y') \to \exists x' \in X_2 (x = p(x') \land x'I_2y')) \quad \text{(3.10)}
\]

\[
\forall x' \in X_2 \forall y \in Y_1 (p(x')I_1y \to \exists y' \in Y_2 (y = q(y') \land x'I_2y')) \quad \text{(3.11)}
\]

We then arrive at the natural generalization and a sorted bounded morphism \( (p,q) : (X_2, I_2, Y_2) \to (X_1, I_1, Y_1) \) is defined as a pair of maps \( p : X_2 \to X_1, q : Y_2 \to Y_1 \) such that the relation preservation condition (3.12) below,

\[
\forall x' \in X_2 \forall y' \in Y_2 (x'I_2y' \to p(x')I_1q(y')) \quad \text{(3.12)}
\]

as well as conditions (3.10) and (3.11) hold. Note that sorted bounded morphisms preserve the closure operators

\[
p^{-1}(\blacksquare_1\Diamond_1U) = \blacksquare_2\Diamond_2p^{-1}(U) \quad \text{and} \quad q^{-1}(\square_1\blacksquare_1V) = \square_2\blacksquare_2q^{-1}(V),
\]

therefore they preserve arbitrary joins, since these are closures of unions and as inverse maps preserve both unions and intersections, sorted bounded morphisms are homomorphisms of the complete lattices of Galois stable and co-stable sets.

3.2.1. Morphisms for sorted residuated frames. Singleton sets are atoms of the powerset Boolean algebras, they join-generate any subset, i.e., \( U = \bigcup_{u \in U} \{u\} \), and this was used in computing the first-order conditions (3.10), (3.11) for sorted bounded morphisms. For stable and co-stable sets, join generators are the closed elements \( \Gamma x (x \in X) \) and \( \Gamma y (y \in Y) \) so that we have, respectively, \( A = \bigvee_{x \in A} \Gamma x = \bigcup_{x \in A} \Gamma x \), using Lemma 3.3. We have, for any \( x \in X \),

\[
q^{-1}(\square_1\Diamond_1x) \subseteq \square_2\Diamond_2p^{-1}(\Gamma x) \quad \text{iff} \quad \forall y' \in Y_2 (y' \in q^{-1}(\square_1\Diamond_1x) \Rightarrow y' \in \square_2\Diamond_2p^{-1}(\Gamma x))
\]
If \( y' \in Y_2(q(y')) \in \bigtriangleup_1 \Gamma x \Rightarrow y' \in \bigtriangleup_2 p^{-1}(\Gamma x) \)

iff \( y' \in Y_2(\exists z \in X_1(zI_1 q(y') \land x \leq z) \Rightarrow y' \in \bigtriangleup_2 p^{-1}(\Gamma x) \)

iff \( y' \in Y_2(xI_1 q(y') \Rightarrow y' \in \bigtriangleup_2 p^{-1}(\Gamma x) \)

iff \( y' \in Y_2(xI_1 q(y') \Rightarrow \exists x' \in X_2(x \leq p(x') \land x'I_2 y')) \)

and this is the weakened version of (3.10) we shall need. We point out that the proof used the fact that the frame relation \( I \) is decreasing in both argument places (Lemma 3.3), hence \( \exists z \in X_1(zI_1 q(y') \land x \leq z) \) iff \( xI_1 q(y') \).

**Definition 3.17.** If \( (p, q) : (X_2, I_2, Y_2) \rightarrow (X_1, I_1, Y_1) \), with \( p : X_2 \rightarrow X_1 \) and \( q : Y_2 \rightarrow Y_1 \), then we let \( \pi = (p, q) \) and we define \( \pi^{-1} \) by setting

\[
\pi^{-1}(W) = \begin{cases} 
  p^{-1}(W) \in \wp(X_2) & \text{if } W \subseteq X_1 \\
  q^{-1}(W) \in \wp(Y_2) & \text{if } W \subseteq Y_1.
\end{cases}
\]

Similarly, we let

\[
\pi(w) = \begin{cases} 
  p(w) \in X_1 & \text{if } w \in X_2 \\
  q(w) \in Y_1 & \text{if } w \in Y_2.
\end{cases}
\]

**Lemma 3.18.** If \( \pi = (p, q) : (X_2, I_2, Y_2) \rightarrow (X_1, I_1, Y_1) \) is a pair of maps \( p : X_2 \rightarrow X_1, \ q : Y_2 \rightarrow Y_1 \), then the following are equivalent:

1. for any increasing subset \( A \subseteq X_1, \pi^{-1}(\bigtriangleup_1 A) \not\subseteq \bigtriangleup_2 \pi^{-1}(A) \)
2. for any \( x \in X_1, \pi^{-1}(\bigtriangleup_1 \Gamma x) \not\subseteq \bigtriangleup_2 \pi^{-1}(\Gamma x) \)
3. \( \forall x \in X_1 \forall y' \in Y_2(xI_1 \pi(y') \Rightarrow \exists x' \in X_2(x \leq \pi(x') \land x'I_2 y')) \).

**Proof.** (1)\(\Rightarrow\)(2) Immediate, since \( \Gamma x = \{ z \in X_1 \mid x \leq z \} \subseteq X_1 \) is increasing.
(2)\(\iff\)(3) This was shown above. (3)\(\Rightarrow\)(1) Let \( y' \in q^{-1}(\bigtriangleup_1 A) \), i.e., \( q(y') \in \bigtriangleup_1 A \) and let then \( x \in X_1 \) be such that \( xI_1 q(y') \) and \( x \in A \). From \( xI_1 q(y') \) and condition (3) we obtain that there exists \( x' \in X_2 \) such that \( x'I_2 y' \) and \( x \leq p(x') \).

Given the assumption that \( A \) is an increasing subset and since \( x \in A \) it follows that \( p(x') \in A \), as well. This shows that \( y' \in \bigtriangleup_2 p^{-1}(A) \).

Therefore, (1)\(\iff\)(2)\(\iff\)(3). \(\square\)

Similarly, we obtain the following lemma.

**Lemma 3.19.** If \( \pi = (p, q) : (X_2, I_2, Y_2) \rightarrow (X_1, I_1, Y_1) \) is a pair of maps \( p : X_2 \rightarrow X_1, \ q : Y_2 \rightarrow Y_1 \), then the following are equivalent:

1. for any decreasing subset \( B \subseteq Y_1, \bigtriangledown_2 \pi^{-1}(B) \not\subseteq \pi^{-1}(\bigtriangledown_1 B) \)
2. for any \( y \in Y_1, \bigtriangledown_2 \pi^{-1}(-\Gamma y) \not\subseteq \pi^{-1}(\bigtriangledown_1 (-\Gamma y)) \)
3. \( \forall x' \in X_2 \forall y \in Y_1(\pi(x')I_1 y' \Rightarrow \exists y' \in Y_2(y \leq \pi(y') \land x'I_2 y')) \).

Note that case (3) of Lemma 3.19 is a weakened analogue of (3.11).

**Definition 3.20.** If \( \pi = (p, q) : (X_2, I_2, Y_2) \rightarrow (X_1, I_1, Y_1) \) is a pair of maps \( p : X_2 \rightarrow X_1, \ q : Y_2 \rightarrow Y_1 \), then \( \pi \) will be called a (sorted) weak bounded morphism iff

1. \( \forall x' \in X_2 \forall y' \in Y_2(\pi(x')I_1 \pi(y')) \)
2. \( \forall x \in X_1 \forall y' \in Y_2(xI_1 \pi(y') \Rightarrow \exists x' \in X_2(x \leq \pi(x') \land x'I_2 y')) \)
3. \( \forall x' \in X_2 \forall y \in Y_1(\pi(x')I_1 y \Rightarrow \exists y' \in Y_2(y \leq \pi(y') \land x'I_2 y')) \).
Corollary 3.21. The inverse $\pi^{-1} = (p, q)^{-1}$ of a weak bounded morphism is a complete lattice homomorphism of the lattices of Galois stable sets of sorted residuated frames.

Proof. The proof is an immediate consequence of Lemmas 3.18 and 3.19.

Summarizing, we have shown that all squares in the diagrams in the middle and right below commute

$$
\begin{array}{c}
(X_1, \perp_1, Y_1) & \xrightarrow{\Phi^1(X_1)} & \Phi^1(Y_1) \\
\downarrow p & & \downarrow q \\
(X_2, \perp_2, Y_2) & \xrightarrow{\Phi^1(X_2)} & \Phi^1(Y_2)
\end{array}
$$

where $\Phi^1$, $\Phi^\downarrow$ denote, respectively, the set of increasing and decreasing subsets.

Remark 3.22. Goldblatt [11] was first to propose bounded morphisms for polarities and our definition is equivalent to his. For morphisms of frames with relations (restricted in [11] to relations of sort type that generate either join-preserving or meet-preserving operators only) we will diverge from his definition, as the relations Goldblatt considers on frames, expanding on Gehrke’s [9], do not coincide with ours and they can be in fact construed as the Galois duals of the frame relations we consider. In [19] we discussed the connections between our approach and Gehrke’s generalized Kripke frames approach.

For later use, we make an observation in the next lemma.

Lemma 3.23. If $\pi^{-1}$ preserves closed elements, then it preserves clopen elements, as well.

Proof. Letting $\pi^{-1}(\Gamma u) = \Gamma w_u$ and $\pi^{-1}(\Gamma v) = \Gamma w_v$, we have $(\pi^{-1}(\Gamma v))' = (\Gamma w_v)' = \{w_v\}$ and $\Gamma w_u = \pi^{-1}(\Gamma u) = \pi^{-1}((\Gamma v)') = \pi^{-1}((\Gamma v)') = (\pi^{-1}(\Gamma v))' = \{w_u\}$, assuming $\Gamma u = \{v\}'$ is clopen, using Lemma 3.3 and using also the fact that weak bounded morphisms commute with the Galois connection (by the results of Section 3.2.1, specifically, Corollary 3.21).

3.2.2. Morphisms for frames with relations. Let $\pi$ be a weak bounded morphism, $\pi = (p, q) : (X_2, I_2, Y_2, (S_\sigma)_{\sigma \in \mathbb{F}}) \rightarrow (X_1, I_1, Y_1, (R_\sigma)_{\sigma \in \mathbb{F}})$, and let $R_\sigma, S_\sigma$ be corresponding relations in the two frames, of the same sort type. For simplicity, we omit the subscript $\sigma$ in the sequel.

Proposition 3.24. If for any $\bar{u}$ it holds that $\pi^{-1}\overline{\alpha}_R(\Gamma \bar{u}) = \overline{\alpha}_S(\pi^{-1}[\Gamma \bar{u}])$, then for any tuple $\bar{F}$ of Galois sets of the required sort $\pi^{-1}\overline{\alpha}_R(\bar{F}) = \overline{\alpha}_S(\pi^{-1}[\bar{F}])$.

Proof. We have

$$
\pi^{-1}\overline{\alpha}_R(\bar{F}) = \pi^{-1} \bigvee_{\bar{u} \in \bar{F}} \overline{\alpha}_R(\Gamma \bar{u}) \\
= \bigvee_{\bar{u} \in \bar{F}} \pi^{-1}\overline{\alpha}_R(\Gamma \bar{u}) \\
= \bigvee_{\bar{u} \in \bar{F}} \overline{\alpha}_S(\pi^{-1}[\Gamma \bar{u}]) \\
= \bigvee_{\bar{u} \in \bar{F}} \bigvee_{\bar{w} \in \pi^{-1}[\bar{F}]} \overline{\alpha}_S(\Gamma \bar{w}) \\
= \bigvee_{\bar{w} \in \pi^{-1}[\bar{F}]} \overline{\alpha}_S(\Gamma \bar{w}) = \overline{\alpha}_S(\pi^{-1}[\bar{F}]).
$$
For the last line, note that if \( \bar{u} \in \check{F} \) and \( \bar{u} \leq \pi[\bar{w}] \), then \( \pi[\bar{w}] \in \check{F} \), since Galois sets are increasing. Hence \( \bigvee_{\bar{u} \leq \pi[\bar{w}]} \overline{\alpha}_{\check{F}}(\Gamma \bar{u}) \leq \bigvee_{\bar{w} \in \pi^{-1}[\check{F}]} \overline{\alpha}_{\check{S}}(\Gamma \bar{w}) \). Conversely, if \( \bar{w} \in \pi^{-1}[\check{F}] \), let \( \bar{u} = \pi[\bar{w}] \), so that \( \bar{u} \in \check{F} \) and \( \bar{u} \leq \pi[\bar{w}] \), which shows that the converse inclusion also holds. \( \Box \)

**Lemma 3.25.** Assuming the frame axioms of Table 1, the condition in the statement of Proposition 3.24 can be replaced by the requirement in equation (3.14), which is equivalent to condition (3.13)

\[
\pi^{-1}\alpha_R(\Gamma \bar{u}) = \alpha_S(\pi^{-1}[\Gamma \bar{u}]),
\]

(3.13)

\[
\pi(\nu)R\bar{u} \iff \exists \bar{w}(\bar{u} \leq \pi[\bar{w}] \land \nu S \bar{w}).
\]

(3.14)

**Proof.** By Lemma 3.16 for any relation \( R \) in the frame it holds that \( \overline{\alpha}_R(\Gamma \bar{u}) = R\bar{u} = \alpha_R(\Gamma \bar{u}) \). It follows that if equation (3.14) holds, then \( \alpha_S(\pi^{-1}[\Gamma \bar{u}]) \) is a Galois set, hence \( \alpha_S(\pi^{-1}[\Gamma \bar{u}]) = \overline{\alpha}_S(\pi^{-1}[\Gamma \bar{u}]) \). Hence the identity in (3.14) implies the hypothesis in the statement of Proposition 3.24. Since

\[
\alpha_S(\pi^{-1}[\Gamma \bar{u}]) = \bigcup_{\bar{w} \in \pi^{-1}[\Gamma \bar{u}]} \alpha_S(\Gamma \bar{w}) = \bigcup_{\bar{u} \leq \pi[\bar{w}]} \alpha_S(\Gamma \bar{w}) = \bigcup_{\bar{u} \leq \pi[\bar{w}]} S \bar{w}
\]

and \( \nu \in \pi^{-1}\alpha_R(\Gamma \bar{u}) \) iff \( \pi(\nu)R\bar{u} \) it follows that the two conditions in (3.14) and (3.13) are equivalent. \( \Box \)

We conclude with the definition of the category of \( \tau \)-frames, for a similarity type (sequence of distribution types) \( \tau \).

**Definition 3.26.** The objects \( \mathfrak{F} = (X, I, Y, (R_\sigma)_{\sigma \in \tau}) \) of the category \( \text{SRF}_\tau \) of \( \tau \)-frames are sorted residuated frames (equivalently, polarities) with a relation of sort type \( \sigma \), for each \( \sigma \) in \( \tau \), subject to axioms F1–F4 of Table 1. Its morphisms \( \pi = (p, q) : (X_2, I_2, Y_2, (S_\sigma)_{\sigma \in \tau}) \rightarrow (X_1, I_1, Y_1, (R_\sigma)_{\sigma \in \tau}) \) are the weak bounded morphisms specified in Definition 3.20 that, in addition, satisfy condition (3.13) (axiom M4). Table 2 collects together all axioms.

Hereafter, by a weak bounded morphism we shall always mean that condition (3.13) (axiom M4) is also satisfied. Note that axiom M4 assumes proper sorting of \( \bar{u}, \nu \) and \( \bar{w} \) as necessitated by the sorting of the relations. We also point out that the axiomatization of the category \( \text{SRF}_\tau \) will be strengthened in Section 5, including an axiom that both \( X, Y \) are carriers of a Stone topology, for the purpose of deriving a Stone duality result.

### 4. Dual sorted residuated frames of NLEs

A bounded lattice expansion is a structure \( L = (L, \leq, \land, \lor, 0, 1, \mathcal{F}_1, \mathcal{F}_\partial) \), where \( \mathcal{F}_1 \) consists of normal lattice operators \( f \) of distribution type \( \delta(f) = (\check{i}_j; 1) \) (i.e., of output type 1), while \( \mathcal{F}_\partial \) consists of normal lattice operators \( h \) of distribution type \( \delta(h) = (\check{i}_j; \partial) \) (i.e., of output type \( \partial \)). For representation purposes, nothing depends on the size of the operator families \( \mathcal{F}_1 \) and \( \mathcal{F}_\partial \) and we may as well assume that they contain a single member, say \( \mathcal{F}_1 = \{f\} \) and \( \mathcal{F}_\partial = \{h\} \). In addition, nothing depends on the arity of the operators, so we may assume they are both \( n \)-ary.
Table 2. Axioms for the frame Category $\text{SRF}_\tau$

| (F1) | The frame is separated |
| (F2) | For each $\sigma = (i_j;i_{n+1})$ in the similarity type $\tau$, each $\bar{u} \in \prod_{j=1}^{j=n} Z_{i_j}$, $R_\sigma \bar{u}$ is a closed element of $\mathcal{G}(Z_{i_{n+1}})$ |
| (F3) | For each $\sigma = (i_j;i_{n+1})$ in the similarity type $\tau$, each $w \in Z_{i_{n+1}}$, the n-ary relation $wR_\sigma$ is decreasing in every argument place |
| (F4) | All sections of the Galois dual relations $R'_\sigma$ of $R_\sigma$, for each $\sigma$ in $\tau$, are Galois sets |

For $\pi = (p,q): (X_2,I_2,Y_2,(S_\sigma)_{\sigma \varepsilon \tau}) \rightarrow (X_1,I_1,Y_1,(R_\sigma)_{\sigma \varepsilon \tau})$

(M1) \forall x' \in X_2 \forall y' \in Y_2 (x'I_2 y' \rightarrow p(x')I_1 p(y'))

(M2) \forall x \in X_1 \forall y' \in Y_2 (xI_1 q(y') \rightarrow \exists x' \in X_2 (x \leq p(x') \wedge x'I_2 y'))

(M3) \forall x' \in X_2 \forall y \in Y_1 (p(x')I_1 y \rightarrow \exists y' \in Y_2 (y \leq q(y') \wedge x'I_2 y'))

(M4) for all $\bar{u}$ and $v$, $\bar{u} R_\sigma v$ iff there exists $\bar{v}$ such that $\bar{u} \leq \pi[\bar{v}]$ and $vS_\sigma \bar{v}$

4.1. Canonical lattice frame construction

The canonical frame is constructed as follows, based on [20, 21, 14, 15].

First, the base polarity $\mathcal{Z} = (\text{Filt}(\mathcal{L}), \bot, \text{Idl}(\mathcal{L}))$ consists of the sets $X = \text{Filt}(\mathcal{L})$ of filters and $Y = \text{Idl}(\mathcal{L})$ of ideals of the lattice and the relation $\bot \subseteq \text{Filt}(\mathcal{L}) \times \text{Idl}(\mathcal{L})$ is defined by $x \bot y$ if $x \cap y = \emptyset$, while the representation map $\zeta_1$ sends a lattice element $a \in L$ to the set of filters that contain it, $\zeta_1(a) = \{x \in X | a \in x\} = \{x \in X | x_a \subseteq x\} = \Gamma x_a$. Similarly, a co-representation map $\zeta_0$ is defined by $\zeta_0(a) = \{y \in Y | y \subseteq a\} = \{y \in Y | y_a \subseteq y\} = \Gamma y_a$. It is easily seen that $(\zeta_1(a))' = \zeta_0(a)$ and, similarly, $(\zeta_0(a))' = \zeta_1(a)$. The images of $\zeta_1, \zeta_0$ are precisely the families (sublattices of $\mathcal{G}(X), \mathcal{G}(Y)$, respectively) of clopen elements of $\mathcal{G}(X), \mathcal{G}(Y)$, since clearly $\Gamma x_a = \downset{y_a}$ and $\Gamma y_a = \upset{x_a}$.

Second, for each normal lattice operator a relation is defined, such that if $\delta = (i_1, \ldots, i_n; i_{n+1})$ is the distribution type of the operator, then $\sigma = (i_{n+1}; i_1 \ldots i_n)$ is the sort type of the relation. Without loss of generality, we have restricted to the families of operators $\mathcal{F}_1 = \{f\}$ and $\mathcal{F}_0 = \{h\}$, so that we shall define two corresponding relations $R, S$ of respective sort types $\sigma(R) = (1; i_1 \ldots i_n)$ and $\sigma(S) = (\partial; t_1 \ldots t_n)$, where for each $j$, $i_j$ and $t_j$ are in $\{1, \partial\}$. In other words $R \subseteq X \times \prod_{j=1}^{j=n} Z_{i_j}$ and $S \subseteq Y \times \prod_{j=1}^{j=n} Z_{t_j}$. To define the relations, we use the point operators introduced in [14] (see also [15]). In the generic case we examine, we need to define two sorted operators

$$\bar{f}: \prod_{j=1}^{j=n} Z_{i_j} \rightarrow Z_1 \quad \bar{h}: \prod_{j=1}^{j=n} Z_{t_j} \rightarrow Z_\partial$$

(recall that $Z_1 = X, Z_\partial = Y$).

Assuming for the moment that the point operators have been defined, the canonical relations $R, S$ are defined by

$$xR\bar{u} \text{ iff } \bar{f}(\bar{u}) \subseteq x \left(\text{for } x \in X \text{ and } \bar{u} \in \prod_{j=1}^{j=n} Z_{i_j}\right),$$

$$xS\bar{v} \text{ iff } \bar{h}(\bar{v}) \subseteq x \left(\text{for } x \in Y \text{ and } \bar{v} \in \prod_{j=1}^{j=n} Z_{t_j}\right).$$
Returning to the point operators and letting $x_e, y_e$ be the principal filter and principal ideal, respectively, generated by a lattice element $e$, these are uniformly defined as follows, for $\hat{u} \in \prod_{j=1}^{j=n} Z_{t_j}$ and $\hat{v} \in \prod_{j=1}^{j=n} Z_{t_j}$

$$\hat{f}(\hat{u}) = \bigvee \{ x_{f(a)} | a \in \hat{u} \} \quad \hat{h}(\hat{v}) = \bigvee \{ y_{h(a)} | a \in \hat{v} \}. \quad (4.2)$$

In other words, $\hat{f}(\hat{u})$ is the filter generated by the set $\{ f(a) | a \in \hat{u} \}$. Similarly $\hat{h}(\hat{v})$ is the ideal generated by the set $\{ h(a) | a \in \hat{v} \}$.

**Example 4.1.** $(\text{FL}_{ew})$ We consider as an example the case of associative, commutative, integral residuated lattices $\mathcal{L} = (L, \leq, \land, \lor, 0, 1, \circ, \to)$, the algebraic models of $\text{FL}_{ew}$ (the associative full Lambek calculus with exchange and weakening), also referred to in the literature as full $\text{BCK}$. By residuation of $\circ, \to$, the distribution types of the operators are $\delta(\circ) = (1, 1; 1)$ and $\delta(\to) = (1, \partial, \partial)$. Let $(\text{Filt}(\mathcal{L}), \bot, \text{Idl}(\mathcal{L}))$ be the canonical frame of the bounded lattice $(L, \leq, \land, \lor, 0, 1)$. Designate the corresponding canonical point operators by $\circ$ and $\to$, respectively. They are defined by (4.2)

$$x \circ z = \bigvee \{ x_{a \circ c} | a \in x \land c \in z \} \in \text{Filt}(\mathcal{L}) \quad (x, z \in \text{Filt}(\mathcal{L}))$$

$$x \to v = \bigvee \{ y_{a \to v} | a \in x \land c \in v \} \in \text{Idl}(\mathcal{L}) \quad (x \in \text{Filt}(\mathcal{L}), v \in \text{Idl}(\mathcal{L}))$$

where recall that we write $x_e, y_e$ for the principal filter and ideal, respectively, generated by the lattice element $e$, so that $x \circ z \in \text{Filt}(\mathcal{L})$, while $(x \to v) \in \text{Idl}(\mathcal{L})$.

The relations $R^{111}, S^{\partial L \partial}$ are then defined by

$$u R^{111} x z \text{ iff } x \circ z \subseteq u \quad y S^{\partial \partial} x v \text{ iff } (x \to v) \subseteq y$$

of sort types $\sigma(R) = (1; 11)$ and $\sigma(S) = (\partial, 10 \partial)$. The canonical $\text{FL}_{ew}$-frame is therefore the structure $\mathfrak{F} = (\text{Filt}(\mathcal{L}), \bot, \text{Idl}(\mathcal{L}), R^{111}, S^{\partial \partial})$.

**Remark 4.2.** In [27] a lattice duality is presented, $\text{Filt} : \text{Lat} \cong \text{BL}^{\partial} : \text{KOF}$, where for a given lattice $\mathcal{L}$ we have $\mathcal{L} \cong \text{KOF}(\text{Filt}(\mathcal{L})) \subseteq \text{FSat}(\text{Filt}(\mathcal{L}))$ where the latter is shown in [27] to be a concrete realization of a canonical extension of $\mathcal{L}$. In their sequel article [28], Moshier and Jipsen extend their work to lattice expansions $(L, j)$, proving that normal lattice operators $j$ (which they refer to as quasioperators) are in bijective correspondence with functions $f_j$ on points of the corresponding product of their dual $\text{BL}$-spaces that are succinctly characterized as (1) strongly continuous in the product topology and (2) preserving finite meets in each argument place. It follows by simple calculation that the definition of the point operators $f_j$ given by Moshier and Jipsen in [28] is a rephrasing of the definition of the point operators introduced in [14] (and thereafter used in [15], as well as in the current article). Indeed, examining for simplicity only the case of a quasioperator that is additive (join-preserving), $j : L_1 \times \cdots \times L_n \longrightarrow L_{n+1}$, Moshier and Jipsen define $f_j : \text{Filt}(L_1) \times \cdots \times \text{Filt}(L_n) \longrightarrow \text{Filt}(L_{n+1})$ by setting

$$f_j(u_1, \ldots, u_n) = \bigcap \{ x \in \text{Filt}(L_{n+1}) | u_1 \times \cdots \times u_n \subseteq j^{-1}(x) \}. \quad (4.3)$$
Designate by \( x_e \) the principal filter generated by a lattice element \( e \). Letting
\[
\hat{j}(u_1, \ldots, u_n) = \bigvee \left\{ x_{j(a_1, \ldots, a_n)} \mid \bigwedge_{k=1}^{k=n} (a_k \in u_k) \right\} \in \text{Filt}(L_{n+1}) \tag{4.4}
\]
be the point operator introduced in [14, page 420], also used above in equation (4.2), it is readily seen that
\[
f_j(u_1, \ldots, u_n) = \bigcap \{ x \in \text{Filt}(L_{n+1}) \mid \hat{j}(u_1, \ldots, u_n) \subseteq x \} = \hat{j}(u_1, \ldots, u_n)
\]
i.e., the function \( f_j \) defined by Moshier and Jipsen in [28] coincides with the function \( \hat{j} \) of [14], since \( u_1 \times \cdots \times u_n \subseteq j^{-1}(x) \iff \bigwedge_{k=1}^{k=n} (a_k \in u_k) \iff j(a_1, \ldots, a_n) \in x \iff \bigvee \{ x_{j(a_1, \ldots, a_n)} \mid \bigwedge_{k=1}^{k=n} (a_k \in u_k) \} \subseteq x \iff \hat{j}(u_1, \ldots, u_n) \subseteq x \). In [14, Lemma 6.8] the distribution properties of the point operators were established, but no topological characterization as given in [28] was sought for in [14].

### 4.2. Properties of the canonical frame

We first verify that axioms F1–F3 of Table 2 hold for the canonical sorted residuated frame (polarity).

**Lemma 4.3.** The following hold for the canonical frame.

1. The frame is separated.
2. For \( \tilde{u} \in \prod_{j=1}^{j=n} Z_{t_j} \) and \( \tilde{v} \in \prod_{j=1}^{j=n} Z_{t_j} \), the sections \( R\tilde{u} \) and \( S\tilde{v} \) are closed elements of \( G(X) \) and \( G(Y) \), respectively.
3. For \( x \in X, y \in Y \), the n-ary relations \( xR, yS \) are decreasing in every argument place.

**Proof.** For (1), just note that the ordering \( \preceq \) is set-theoretic inclusion (of filters, and of ideals, respectively), hence separation of the frame is immediate.

For (2), by the definition of the relations, \( R\tilde{u} = \{ x \mid \hat{f}(\tilde{u}) \subseteq x \} = \Gamma(\hat{f}(\tilde{u})) \) is a closed element of \( G(X) \) and similarly for \( S\tilde{v} \).

For (3), if \( w \subseteq u_k \), then \( \{ x_{f(a_1, \ldots, a_n)} \mid a_k \in w \land \bigwedge_{j=1}^{j=n} (a_j \in u_j) \} \) is a subset of the set \( \{ x_{f(a_1, \ldots, a_n)} \mid \bigwedge_{j=1}^{j=n} (a_j \in u_j) \} \), hence taking joins it follows that \( \hat{f}(\tilde{u}[w]_k) \subseteq \hat{f}(\tilde{u}) \). By definition, if \( xR\tilde{u} \) holds, then we obtain \( \hat{f}(\tilde{u}[w]_k) \subseteq \hat{f}(\tilde{u}) \subseteq x \), hence \( xR\tilde{u} \) holds as well. Similarly for the relation \( S \). \( \square \)

**Lemma 4.4.** In the canonical frame, \( xR\tilde{u} \) holds iff \( \forall \tilde{a} \in L^n (\tilde{a} \in \tilde{u} \iff f(\tilde{a}) \in x) \). Similarly, \( yS\tilde{v} \) holds iff \( \forall \tilde{a} \in L^n (\tilde{a} \in \tilde{v} \iff h(\tilde{a}) \in y) \).

**Proof.** By definition \( xR\tilde{u} \) holds iff \( \hat{f}(\tilde{u}) \subseteq x \), where \( \hat{f}(\tilde{u}) \), by its definition (4.2) is the filter generated by the elements \( f(\tilde{a}) \), for \( \tilde{a} \in \tilde{u} \), hence clearly \( \tilde{a} \in \tilde{u} \) implies \( f(\tilde{a}) \in x \). Similarly for the relation \( S \). \( \square \)

**Lemma 4.5.** Where \( R', S' \) are the Galois dual relations of the canonical relations \( R, S \), \( yR'\tilde{u} \) holds iff \( \hat{f}(\tilde{u}) \perp y \iff \exists \tilde{b} (\tilde{b} \in \tilde{u} \land f(\tilde{b}) \in y) \). Similarly, \( xS'\tilde{v} \) holds iff \( x \perp \hat{h}(\tilde{v}) \iff \exists \tilde{e} (\tilde{e} \in \tilde{v} \land h(\tilde{e}) \in x) \).

**Proof.** The proof is given in [19, Lemma 24], to which we refer the reader. \( \square \)

We can now prove that the frame axiom F4 of Table 2 also holds in the canonical frame.
Lemma 4.6. In the canonical frame, all sections of the Galois dual relations \( R', S' \) of the canonical relations \( R, S \) are Galois sets.

Proof. There are two cases to handle, one for each of the relations \( R', S' \), with two subcases for each one, depending on whether \( i_k \) is 1, or \( \partial \). The cases of the two relations are similar, we have presented the proof for the relation \( R' \) in [19, Lemma 25] and we only detail here the other case.

Case of the relation \( S' \):

The section \( S'\tilde{v} = (S\tilde{v})' \) is a Galois (stable) set, by its definition. Recall that the sort type of \( S \) is \( \sigma(S) = (\partial; t_1 \ldots t_n) \), where \( t_k \in \{1, \partial\} \) for each \( 1 \leq k \leq n \), and that \( S \) was defined given the lattice normal operator \( h \), of distribution type \( \delta(h) = (t_1, \ldots, t_n; \partial) \).

Let now \( x \in X \) and consider any section \( xS'\tilde{v}[\_]_k \). We distinguish the subcases \( i_k = 1 \), or \( i_k = \partial \). When \( i_k = \partial \) (same as the output type of \( h \)), then \( h \) is monotone at the \( k \)-th argument place and it distributes over finite lattice meets, whereas when \( i_k = 1 \), then \( h \) is antitone at the \( k \)-th argument place and it co-distributes over finite lattice joins, turning them to meets. Furthermore, by Lemma 4.5, \( xS'\tilde{y} \) holds iff \( x \perp \overline{\tilde{h}}(\tilde{y}) \) iff \( \exists \tilde{e}(\tilde{e} \in \tilde{y} \land h(\tilde{e}) \in x) \).

Subcase \( i_k = \partial \): Then \( xS'\tilde{y}[\_]_k \subseteq Y = \text{Idl}(\mathcal{L}) \) and note that the output type of \( h \) is also \( t_{n+1} = \partial \).

Let \( W = \{ b \in L | \exists \tilde{e}[\_]_k \in \tilde{y}[\_]_k \, h(\tilde{e}[b]_k) \in x \} \) and \( w \) be the filter generated by \( W \). If \( v \) is an ideal such that \( xS'\tilde{y}[\_]_k \) holds, then by Lemma 4.5 \( x \perp \overline{\tilde{h}}(\tilde{y}[v]_k) \), equivalently, for some tuple of lattice elements \( \tilde{e}[b]_k \in \tilde{y}[v]_k \) we have \( h(\tilde{e}[b]_k) \in x \). Then \( b \in w \cap v \), i.e., \( w \perp v \) and then \( w \perp xS'\tilde{y}[\_]_k \).

Let now \( q \) be an ideal \( q \in (xS'\tilde{y}[\_]_k)'' \). We show that \( xS'\tilde{y}[q]_k \) holds.

By the assumption on \( q \) and the fact that \( w \perp xS'\tilde{y}[\_]_k \) we obtain \( w \perp q \), i.e., there is some element \( b \in w \cap q \neq \emptyset \). By definition of \( w \), there exist lattice elements \( b_1, \ldots, b_s \in W \), for some positive integer \( s \), such that \( b_1 \land \cdots \land b_s \leq b \).

Since \( b_r \in W \), for \( 1 \leq r \leq s \), there exist tuples of lattice elements \( \tilde{c}[\_]_k = (c_1^r, \ldots, c_{k-1}^r, c_{k+1}^r, \ldots, c_n^r) \), for \( 1 \leq r \leq s \), such that \( h(\tilde{c}[b_r]_k) \in x \). Define

\[
e_j = \begin{cases} c_j^1 \land \cdots \land c_j^s & \text{if } t_j = 1 \\ c_j^1 \lor \cdots \lor c_j^s & \text{if } t_j = \partial \end{cases}
\]

Considering the monotonicity properties of \( h \), observe that for each \( 1 \leq r \leq s \) we have

\[
h(\tilde{e}[b_r]_k) = h(\tilde{c}[b_r]_k[e_j]_k^{t_j} = \partial [e_j']_k^{t_j'} = 1}) \geq h(\tilde{c}[b_r]_k) \in x \in \text{Filt}(\mathcal{L}) \tag{4.5}
\]

and so \( \bigwedge_{r=1}^{s} h(\tilde{e}[b_r]_k) \in x \). In (4.5), \( \tilde{c}[b_r]_k[e_j]_k^{t_j} = \partial [e_j']_k^{t_j'} = 1 \) designates the result of replacing \( c_j^r \) by \( e_j \) in \( \tilde{c}[b_r]_k \), for every \( j \neq k \) from 1 to \( n \) such that \( t_j = \partial \) in the distribution type of \( h \), and also replacing \( c_j^r \) by \( e_j' \), for every \( j' \neq k \) from 1 to \( n \) such that \( t_{j'} = 1 \).

By the case assumption, \( h \) is monotone and it distributes over finite meets at the \( k \)-th argument place, hence we obtain

\[
h(\tilde{c}[b]_k) \geq h(\tilde{e}[b_1 \land \cdots \land b_s]_k) = \bigwedge_{r=1}^{s} h(\tilde{e}[b_r]_k) \in x.
\]
By Lemma 4.5, given \( \bar{e}[b]_k \in \bar{v}[q]_k \) and \( h(\bar{e}[b]_k) \in x \) we conclude that \( xS'\bar{v}[q]_k \) holds and this proves that the section \( xS'\bar{v}[z]_k \) is a Galois (co-stable) set.

Subcase \( i_k = 1 \):

Then \( xS'\bar{y}[z]_k \subseteq X = \text{Filt}(\mathcal{L}) \).

Let \( W = \{ b \in L | \exists \bar{e}[z]_k \in \bar{y}[z]_k \text{ and } h(\bar{e}[b]_k) \in x \} \) and \( v \) be the ideal generated by \( W \). If \( z \) is any filter such that \( xS'\bar{y}[z]_k \), then by Lemma 4.5 there is a tuple of lattice elements \( \bar{e}[b]_k \in \bar{y}[z]_k \) such that \( h(\bar{e}[b]_k) \in x \). Thus \( b \in z \cap v \) and this shows that \( xS'\bar{y}[z]_k \perp \perp v \).

We now assume that \( z \in (xS'\bar{y}[z]_k)'' \) and show that \( xS'\bar{y}[z]_k \) holds.

By \( x \in (xS'\bar{y}[z]_k)'' \) and \( xS'\bar{y}[z]_k \perp \perp v \) it follows that \( z \perp \perp v \), so for some lattice element \( b \) we have \( b \in z \cap v \neq \emptyset \).

By definition of the ideal \( v \), there exist elements \( b_1, \ldots, b_s \in W \), for some positive integer \( s \), such that \( b \leq b_1 \lor \cdots \lor b_s \). By definition of \( W \) there exist tuples of lattice elements \( \bar{e}^r[\cdot]_k \), with \( 1 \leq r \leq s \), such that \( h(\bar{e}^r[b_r]_k) \in x \) for each \( 1 \leq r \leq s \). Define

\[
e_j = \begin{cases} e^1_j \land \cdots \land e^s_j & \text{if } t_j = 1 \\ e^1_j \lor \cdots \lor e^s_j & \text{if } t_j = \partial. \end{cases}
\]

Considering the monotonicity properties of \( h \) and using the notation introduced in the previous case, observe that, for each \( 1 \leq r \leq s \) we have

\[
h(\bar{e}[b_r]_k) = h(\bar{e}^r[b_r]_k[e^r_j]_{t_j = \partial} [e^r_{j'}]_{t_j = 1}) \geq h(\bar{e}^r[b_r]_k) \in x \in \text{Filt}(\mathcal{L})
\]

and so \( \bigwedge_{r=1}^{r=s} h(\bar{e}[b_r]_k) \in x \). By the case assumption, \( h \) is antitone and it co-distributes over finite joins at the \( k \)-th argument place, turning them to meets, hence we obtain that

\[
h(\bar{e}[b]_k) \geq h(\bar{e}[b_1 \lor \cdots \lor b_s]_k) = \bigwedge_{r=1}^{r=s} h(\bar{e}[b_r]_k) \in x.
\]

By Lemma 4.5, given \( \bar{e}[b]_k \in \bar{y}[z]_k \) and \( h(\bar{e}[b]_k) \in x \), we conclude that \( xS'\bar{y}[z]_k \) holds and this shows that the section \( xS'\bar{y}[z]_k \) is a Galois (stable) set. \( \Box \)

The canonical frame for a lattice expansion \( \mathcal{L} = (L, \leq, \land, \lor, 0, 1, f, h) \), where \( \delta(f) = (i_1, \ldots, i_n; 1) \) and \( \delta(h) = (t_1, \ldots, t_n; \partial) \) \( (i_j, t_j \in (1, \partial)) \) is the structure \( \mathcal{L}_+ = \mathfrak{F}(\mathcal{L}), \mathfrak{X}, \mathfrak{Y}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{I}, \mathfrak{J}, \mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{O}, \mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, \mathfrak{U}, \mathfrak{V}, \mathfrak{W}, \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}) \) and by Lemma 4.6, the canonical relations \( R, S \) are compatible with the Galois connection generated by \( \perp \perp \subseteq X \times Y \), in the sense that all sections of their Galois dual relations are Galois sets. Set operators \( \alpha_R, \eta_S \) are defined as in Section 3 and we let \( \alpha_R, \eta_S \) be the closures of their restrictions to Galois sets (according to their distribution types). Note that \( \alpha_R(\mathcal{F}) \in \mathcal{G}(X) \), while \( \eta_S(\mathcal{G}) \in \mathcal{G}(Y) \), given the output types of \( f, h \) (alternatively, given the sort types of \( R, S \)).

It follows from Theorem 3.12 and Lemma 4.6, that the sorted operators \( \alpha_R, \eta_S \) on Galois sets distribute over arbitrary joins of Galois sets (stable or co-stable, according to the sort types of \( R, S \)) in each argument place.

Note that \( \alpha_R, \eta_S \) are sorted maps, taking their values in \( \mathcal{G}(X) \) and \( \mathcal{G}(Y) \), respectively. We define single-sorted maps on \( \mathcal{G}(X) \) (analogously for \( \mathcal{G}(Y) \)) by
composition with the Galois connection

$$\overline{\alpha}_f(A_1, \ldots, A_n) = \overline{\alpha}_R \left( \ldots, A_j, \ldots, A'_r, \ldots \right) \quad (A_1, \ldots, A_n \in G(X)), \quad (4.6)$$

$$\overline{\eta}_h(B_1, \ldots, B_n) = \overline{\eta}_S \left( \ldots, B_r, \ldots, B'_j, \ldots \right) \quad (B_1, \ldots, B_n \in G(Y)). \quad (4.7)$$

Given that the Galois connection is a duality of Galois stable and Galois co-stable sets, it follows that the distribution type of $\overline{\alpha}_f$ is that of $f$ and that $\overline{\alpha}_f$ distributes, or co-distributes, over arbitrary joins and meets in each argument place, according to its distribution type, returning joins in $G(X)$. Similarly, for $\overline{\eta}_h$.

**Definition 4.7.** The lattice representation maps $\zeta_1 : (L, \preceq, \land, \lor, 0, 1) \rightarrow G(X)$ and $\zeta_\partial : (L, \preceq, \land, \lor, 0, 1) \rightarrow G(Y)$ are extended to maps $\zeta_1 : L \rightarrow G(X)$ and $\zeta_\partial : L \rightarrow G(Y)$ by setting

$$\zeta_1(f(a_1, \ldots, a_n)) = \overline{\alpha}_f(\zeta_1(a_1), \ldots, \zeta_1(a_n))$$

$$= \overline{\alpha}_R \left( \ldots, \zeta_1(a_j), \ldots, \zeta_\partial(a_r), \ldots \right)$$

$$\zeta_\partial(f(a_1, \ldots, a_n)) = (\overline{\alpha}_f(\zeta_1(a_1), \ldots, \zeta_1(a_n)))' \quad (4.8)$$

$$\zeta_1(h(a_1, \ldots, a_n)) = (\overline{\eta}_h(\zeta_\partial(a_1), \ldots, \zeta_\partial(a_n)))' \quad (4.9)$$

**Remark 4.8.** Recall from Remark 4.2 that Moshier and Jipsen [28] show that normal lattice operators $j$ are in bijective correspondence with meet-preserving strongly continuous maps $\overline{j}$ on the dual lattice spaces (BL-spaces, in the terminology of [27, 28]) and that these maps, as defined in [28], coincide with the maps introduced in [14] and subsequently used in [16, 15, 22, 19] and in this article, defined by equation (4.2). Furthermore, given a meet-preserving strongly continuous $n$-ary map $f$ on BL-spaces $X_i$, a join-preserving map $j_f$ is defined in [28] on $\prod_i \text{FSat}(X_i)$ with values in $\text{FSat}(X_n)$ by

$$j_F(F_0, \ldots, F_{n-1}) = \bigcap \{ G \in \text{FP}(X_n) \mid F_0 \times \cdots \times F_{n-1} \subseteq f^{-1}(G) \}$$

and this provides for a representation of a normal lattice operator as a map on the canonical extension of the lattice, but the issue is not discussed in any detail in [28]. There is however a clear similarity with the approach we have taken, as witnessed by Lemma 3.16 and its preceding remark.

In [10], Gehrke and Harding introduced the so-called $\sigma$ and $\pi$-extensions of lattice maps. The reader may recall from [10] that if $(\alpha, C)$ is a canonical extension of a bounded lattice $L$, and $K, 0$ are its sets of closed and open elements, the $\sigma$ and $\pi$-extensions $f_\sigma, f_\pi : L_\sigma \rightarrow L_\sigma$ (where, following the notation of
eralized Kripke frames approach [9]. In [19, Remark 29] we have detailed the representation of operators is sought for and this is also the case in the gen-

4.3. Duals of lattice expansion homomorphisms

With the next proposition, we verify that the frame axioms M1–M4 also hold in the canonical frame construction.

Proposition 4.9. Let h : L → L* be a homomorphism of normal lattice expansions of similarity type τ, with corresponding normal operators fσ, fσ*, for each σ ∈ τ, and \( \mathfrak{F} = (X, I, Y, (R_\sigma)_{\sigma \in \tau}) \), \( \mathfrak{F}^* = (X^*, I^*, Y^*, (S_\sigma)_{\sigma \in \tau}) \) their canonical dual frames, as defined in Section 4.1. For \( x^* \in X^* \) and \( y^* \in Y^* \) define \( p(x^*) = h^{-1}[x^*] = \{a \in L | h(a) \in x^*\} \), \( q(y^*) = h^{-1}[y^*] = \{a \in L | h(a) \in y^*\} \). Then \( \pi = (p, q) : (X^*, I^*, Y^*, (S_\sigma)_{\sigma \in \tau}) \rightarrow (X, I, Y, (R_\sigma)_{\sigma \in \tau}) \) is a weak bounded mor-

Proof. For the first condition (relation preservation, axiom M1 of Table 2), assume \( x^* I^* y^* \), i.e., \( x^* \cap y^* = \emptyset \). If \( p(x^*) \cap q(y^*) = \emptyset \) and \( a \in h^{-1}[x^*] \cap h^{-1}[y^*] \), then \( h(a) \in x^* \cap y^* \neq \emptyset \), contradicting the hypothesis.

For the second condition (axiom M2 of Table 2), let \( x \in X, y^* \in Y^* \) be arbitrary and assume \( xlq(y^*) \), which means \( x \cap h^{-1}[y^*] = \emptyset \), i.e., \( \forall a \in L (a \in x \rightarrow h(a) \notin y^*) \). Let \( x^* \) be the filter generated by the set \( \{h(a) | a \in x\} = h[x] \). Then \( x \subseteq h^{-1}[x^*] = p(x^*) \). Suppose now that \( e \in x^* \cap y^* \). Let \( a_1, \ldots, a_n \in x \) such that \( h(a_1) \land \cdots \land h(a_n) \leq e \). Since \( h \) is a lattice homomorphism and \( x \) is a filter, if \( a = a_1 \land \cdots \land a_n \), then \( a \in x \) and \( h(a_1) \land \cdots \land h(a_n) = h(a) \leq e \in y^* \), but \( y^* \) is an ideal, hence \( h(a) \notin y^* \), contradiction. Hence it also holds for this \( x^* \) that \( x^* I^* y^* \).

The proof of the third condition (axiom M3 of Table 2) is similar, but we detail it anyway. So let \( x^* \in X^*, y \in Y \), arbitrary, and assume that \( p(x^*) I y \), i.e., \( h^{-1}[x^*] \cap y = \emptyset \). In other words, \( \forall a \in L (a \in y \rightarrow h(a) \notin x^*) \). Let \( y^* \) be the ideal generated by the set \( h[y] = \{h(e) | e \in y\} \). Then \( y \subseteq h^{-1}[y^*] = q(y^*) \). If
$e \in x^* \cap y^*$, let $a_1, \ldots, a_n \in y$ such that $e \leq h(a_1) \vee \cdots \vee h(a_n) = h(a_1 \vee \cdots \vee a_n)$, since $h$ is a homomorphism. Put $a = a_1 \vee \cdots \vee a_n \in y$. Then $e \in x^*$, a filter, hence $h(a) \in x^*$, while $a \in y$, contradiction. Hence we obtain $x^* I^* y^*$ for the ideal $y^*$ defined above.

Let $R_\sigma, S_\sigma$ be the canonical relations in the two frames, of the same sort type $\sigma = (i_{n+1}; i_j)$, corresponding to the normal lattice operators $f_\sigma$ and $f_\sigma^*$. Let $\alpha_R, \alpha_S$ (we drop the subscript $\sigma$, for simplicity) be the induced image operators.

We show that $\pi^{-1}$ is a homomorphism, i.e., that axiom M4 of Table 2 holds, or, equivalently by Lemma 3.25, that $\pi^{-1}\alpha_R(\Gamma \bar{u}) = \alpha_S(\pi^{-1}[\Gamma \bar{u}])$.

We calculate that

$v \in \pi^{-1}\alpha_R(\Gamma \bar{u})$ iff $\pi(v) \in \alpha_R(\Gamma \bar{u})$

iff $h^{-1}[v] \in \alpha_R(\Gamma \bar{u})$ definition of $\pi$ ($v$ a filter, or an ideal)

iff $h^{-1}[v] \in R \bar{u}$ Lemma 3.16

iff $\widetilde{f} \bar{u} \leq h^{-1}[v]$ the order is inclusion

iff $\widetilde{f} \bar{u} \subseteq \gamma h^{-1}[v]$ in the canonical frame

iff $\forall \bar{a} \bar{a} \in \bar{u} \longrightarrow f(\bar{a}) \in h^{-1}[v]$ definition of $\widetilde{f}$

iff $\forall \bar{a} \bar{a} \in \bar{u} \longrightarrow h(f(\bar{a})) \in v$ $h$ is a homomorphism.

Note that $\bar{w} \in \pi^{-1}[\Gamma \bar{u}]$ iff $\bar{w} \subseteq \pi[\bar{w}]$ and since $\pi^{-1}[\Gamma \bar{u}]$ is a Galois set, by Lemma 3.3 we obtain $\pi^{-1}[\Gamma \bar{u}] = \bigcup_{\bar{w} \subseteq \pi[\bar{w}]} \Gamma \bar{w}$, hence we compute

$\alpha_S \pi^{-1}[\Gamma \bar{u}] = \alpha_S \bigcup_{\bar{w} \subseteq \pi[\bar{w}]} \Gamma \bar{w} = \bigcup_{\bar{w} \subseteq \pi[\bar{w}]} \alpha_S \Gamma \bar{w}$

$= \bigcup_{\bar{w} \subseteq \pi[\bar{w}]} \bar{w} \bar{S} = \bigcup_{\bar{w} \subseteq \pi[\bar{w}]} \Gamma(\bar{f} \bar{w})$

$= \bigcup_{h(\bar{u}) \leq \bar{w}} \Gamma(\bar{f} \bar{w}).$

We first prove that $\bigcup_{h(\bar{u}) \leq \bar{w}} \Gamma(\bar{f} \bar{w}) \subseteq \pi^{-1}\alpha_R(\Gamma \bar{u})$. To clarify notation, $h(\bar{u})$ is the tuple $(h(u_1), \ldots, h(u_n))$, where $h(u_j) = \{h(e) \mid e \in u_j\}$. Then by $h(\bar{u}) \leq \bar{w}$ we mean the (conjunction of the) pointwise inclusions $h(u_j) \subseteq w_j$.

Let then $\bar{w}$ be such that $h(\bar{u}) \subseteq \bar{w}$ and assume that $\bar{f} \bar{w} \subseteq v$. To show that $v \in \pi^{-1}\alpha_R(\Gamma \bar{u})$ we assume that $\bar{a} \in \bar{w}$ and prove that $f^*(h(\bar{a})) \in v$.

By $\bar{a} \in \bar{w}$ and $h(\bar{a}) \subseteq \bar{w}$ we obtain $h(\bar{a}) \in \bar{w}$ (i.e., $h(a_j) \in w_j$, for all $j = 1, \ldots, n$). By definition in the canonical frame $\bar{f} \bar{w}$ is (the filter, or ideal, depending on the distribution type $\sigma$) generated by the set $\{f^*(\bar{e}) \mid \bar{e} \in \bar{w}\}$. By the hypothesis that $\bar{f} \bar{w} \subseteq v$ we obtain that $\{f^*(\bar{e}) \mid \bar{e} \in \bar{w}\} \subseteq v$. Since $h(\bar{a}) \in \bar{w}$, we have in particular that $f^*(h(\bar{a})) \in v$.

Conversely, we show that $\pi^{-1}\alpha_R(\Gamma \bar{u}) \subseteq \bigcup_{h(\bar{u}) \leq \bar{w}} \Gamma(\bar{f} \bar{w})$.

Let $v \in \pi^{-1}\alpha_R(\Gamma \bar{u})$ which means, by the calculation above, that $\pi(v) R \bar{u}$, equivalently in the canonical frame it means that $\{f^*(h(\bar{a})) \mid \bar{a} \in \bar{w}\} \subseteq v$. The claim is that there is a tuple $\bar{w} = (w_1, \ldots, w_n)$, where $w_j$ is a filter if $i_j = 1$ in the distribution type of $f$, $f^*$ and $w_j$ is an ideal when $i_j = \partial$, such that $h(\bar{u}) \subseteq \bar{w}$ and $\bar{f} \bar{w} \subseteq v$. 
Define \( w_j = \{ h[u_j] \} \) to be the filter of \( \mathcal{L}^* \) generated by the set \( h[u_j] \), if \( i_j = 1 \) and otherwise let it be the ideal generated by the same set. Thus trivially \( h[\bar{u}] \subseteq \bar{w} \) holds. By definition of \( \bar{f}^* \), it suffices to get its set of generators \( \{ f^*(\bar{e}) | \bar{e} \in \bar{w} \} \) to be contained in \( v \).

Let \( \bar{e} \in \bar{w} \) so that for \( 1 \leq j \leq n, e_j \in w_j \).

If \( i_j = 1 \), then \( w_j \) is the filter generated by the set \( h[u_j] \) (where \( u_j \) is also a filter) and there are elements \( a_1^j, \ldots, a_s^j \in u_j \) such that \( h(a_1^j) \wedge \cdots \wedge h(a_s^j) \leq e_j \). Since \( h \) is a homomorphism and \( u_j \) is a filter, letting \( a^j = a_1^j \wedge \cdots \wedge a_s^j \in u_j \) we obtain \( h(a^j) \leq e_j \).

If \( i_j = \partial \), then \( w_j \) is the ideal generated by the set \( h[u_j] \) (where \( u_j \) is also an ideal) and there are elements \( a_1^j, \ldots, a_t^j \in u_j \) such that \( e_j \leq h(a_1^j) \vee \cdots \vee h(a_t^j) \). Letting \( a^j \) be the disjunction \( a^j = a_1^j \vee \cdots \vee a_t^j \) we similarly obtain \( e_j \leq h(a^j) \).

If the output type \( i_{n+1} = 1 \), then \( v \) is a filter, \( f, f^* \) are monotone at the \( j \)-th position, whenever \( i_j = 1 \) and they are antitone at any position \( i' \) with \( i_j' = \partial \). Therefore, \( f^*(h(a^j)) \leq f^*(e_{1}, \ldots, e_n) \), which we may compactly write as \( f^*(h[\bar{a}]) \leq f^*(\bar{e}) \). From the hypothesis on \( v \) we have that \( f^*(h[\bar{a}]) \in v \), which is a filter when \( i_{n+1} = 1 \), hence also \( f^*(\bar{e}) \in v \).

If the output type \( i_{n+1} = \partial \), then \( v \) is an ideal, \( f, f^* \) are monotone at the \( j \)-th position, whenever \( i_j = \partial \) and they are antitone at any position \( i_j' \) with \( i_j' = 1 \). Thereby, \( f^*(-(e_1, \ldots, e_n)) \leq f^*(-(a^1, \ldots, a^n)) \), i.e., \( f^*(\bar{e}) \leq f^*(h[\bar{a}]) \) in \( v \), now an ideal, hence again \( f^*(\bar{e}) \in v \).

\( \Box \)

5. Stone duality

The results we have presented can be extended to a Stone duality, by combining them with our results in [21, 15]. The functor \( F : \text{NLE}_\tau \rightarrow \text{SRF}^\text{op}_\tau \) sends a normal lattice expansion \( \mathcal{L} \) in \( \text{NLE}_\tau \) to its dual frame \( \mathcal{L}_+ = \mathfrak{F} \) in \( \text{SRF}_\tau \) (detailed in Section 4.1) and a lattice expansion homomorphism \( h \) to a weak bounded morphism \( \pi \) (detailed in Section 4.3). Conversely, we have constructed a functor \( L : \text{SRF}^\text{op}_\tau \rightarrow \text{NLE}_\tau \), sending a frame \( \mathfrak{F} \) in \( \text{SRF}_\tau \) to its full complex algebra \( \mathfrak{F}^+ \) (Definition 3.8), which is a complete normal lattice expansion, and a weak bounded morphism \( \pi \) to a complete normal lattice expansion homomorphism \( \pi^{-1} \) (detailed in Section 3.2.2).

For a full Stone duality, a subcategory \( \text{SRF}^*_\tau \) will be identified, by strengthening the axiomatization of sorted residuated frames with relations, imposing in particular that the sort sets of a sorted frame are carriers of a Stone topology. In addition, we replace axiom F2 with a stronger version, we add axioms F5–F7 from [21, 15] as well as axioms M5, M6. The full axiomatization of the category \( \text{SRF}^*_\tau \) is presented in Table 3. Call a point \( u \in X \cup Y \) clopen if \( \Gamma u \) is clopen, i.e., if there exists a point \( v \) of the dual sort such that \( \Gamma u = \{ v \}' \).

Remark 5.1. Axioms F1 and F5–F7 are an equivalent axiomatization of the L-frames of [21]. Lattice expansions were studied in [15] and axioms R7–R10 were postulated in order to derive that the defined operators from relations have the complete distribution properties corresponding to the distribution type of the operator represented. We have followed in this article an alternative approach,
Table 3. Axioms for the Subcategory $\text{SRF}^*_\tau$ of $\text{SRF}_\tau$

| (F1) | The frame is separated |
| (F2) | For each $\sigma = (\tilde{i}_j; i_{n+1})$ in the similarity type $\tau$, each $\tilde{u} \in \prod_{j=1}^{n} Z_i$, $R_{\sigma} \tilde{u}$ is a closed element of $G(Z_{i_{n+1}})$ and if all points $u_j$ are clopen, then $R_{\sigma} \tilde{u}$ is a clopen element of $G(Z_{i_{n+1}})$ |
| (F3) | For each $\sigma = (\tilde{i}_j; i_{n+1})$ in the similarity type $\tau$, each $w \in Z_{i_{n+1}}$, the $n$-ary relation $wR_{\sigma}$ is decreasing in every argument place |
| (F4) | All sections of the Galois dual relations $R'_{\sigma}$ of $R_{\sigma}$, for each $\sigma$ in $\tau$, are Galois sets |
| (F5) | Clopen sets are closed under finite intersections in each of $G(X), G(Y)$ |
| (F6) | The family of closed sets, for each of $G(X), G(Y)$, is the intersection closure of the respective set of clopens |
| (F7) | Each of $X, Y$ carries a Stone topology generated by the sub-basis of their respective families of clopen sets and their complements |

For a sorted map $\pi : (X, I, Y, (S_{\sigma})_{\sigma \in \tau}) \rightarrow (X, I_1, Y_1, (R_{\sigma})_{\sigma \in \tau})$, where $\pi = (p, q), p : X_2 \rightarrow X_1$ and $q : Y_2 \rightarrow Y_1$,

- (M1) $\forall x' \in X_2 \forall y' \in Y_2 (x'I_2y' \rightarrow \pi(x')I_1\pi(y'))$
- (M2) $\forall x \in X_1 \forall y' \in Y_2 (y_1\pi(y') \rightarrow \exists x' \in X_2 (x \leq \pi(x') \land x'I_2y'))$
- (M3) $\forall x' \in X_2 \forall y \in Y_1 (\pi(x')I_1y \rightarrow \exists y' \in Y_2 (y \leq \pi(y') \land x'I_2y'))$
- (M4) for all $\tilde{u}$ and $v$, $\pi(v)R_{\sigma}\tilde{u}$ iff there exists $\tilde{w}$ s.t. $\tilde{u} \leq \pi[\tilde{w}]$ and $vS_{\tilde{u}}\tilde{w}$
- (M5) for all points $u$, $\pi^{-1}(\Gamma u) = \Gamma v$, for some (unique, by separation) $v$
- (M6) $\pi$ is continuous in the topological sense

presenting a simpler axiomatization of frames, relying only on axiom F4 in order to obtain proof of the required distribution properties. Though differences in the presentation exist, the representation of normal lattice operators is the same in this article and in [15], see in particular [15, Remark 3.2]. Frame morphisms in this article are the weak bounded morphisms axiomatized by M1–M6, whereas in [15] a weaker notion of morphism was employed, requiring essentially only preservation of clopen sets.

For a frame $\mathfrak{F}$ of $\text{SRF}^*_\tau$, the following result ensures that its lattice of clopen elements is an object of $\text{NLE}_\tau$.

**Proposition 5.2.** Let $\mathfrak{F} = (X, I, Y, (R_{\sigma})_{\sigma \in \tau})$ be a sorted residuated frame in the category $\text{SRF}^*_\tau$, with a relation $R_{\sigma}$ of sort $\sigma = (i_{n+1}; \tilde{i}_j)$ for each $\sigma$ in $\tau$. Then the clopen elements in $G(X)$ form a lattice and $\alpha^I_{R_{\sigma}}$ restricts to a normal operator of distribution type $\delta = (\tilde{i}_j; i_{n+1})$ on the lattice of clopens.

**Proof.** By the fact that the Galois connection restricts to a duality on clopens, the clopen elements of $G(X)$ and $G(Y)$ are lattices, given axiom F5. By definition and axiom F2 (Table 3), $\alpha^I_{R_{\sigma}}(\tilde{F}) = \bigvee_{\tilde{u} \in \tilde{F}} (R_{\sigma}\tilde{u})'''' = \bigvee_{\tilde{u} \in \tilde{F}} R_{\sigma}\tilde{u} = \bigvee_{\tilde{u} \in \tilde{F}} \Gamma w,$
for some $w$ depending on $\bar{u}$ (which is unique, by the separation axiom F1).
In particular, $\overline{\alpha_{R_\sigma}}(\Gamma w_1, \ldots, \Gamma w_n) = \bigvee_{\bar{u} \leq \bar{q}} R_\sigma \bar{u}$. By axiom F3, if $\bar{w} \leq \bar{u}$, then $R_\sigma \bar{u} \subseteq R_\sigma \bar{w}$, hence $\overline{\alpha_{R_\sigma}}(\Gamma w_1, \ldots, \Gamma w_n) = R_\sigma \bar{w}$, a closed element, by axiom F2.
Again by axiom F2 (Table 3), if all sets $\Gamma w_j$ are clopen elements, then also $R_\sigma \bar{w}$ is a clopen element. Hence $\overline{\alpha_{R_\sigma}}$ restricts to an operator on clopen elements.

By Theorem 3.12, using axiom F4, $\overline{\alpha_{R_\sigma}}$ (which is a sorted operator, of sort $(\vec{i_j}; i_{n+1})$), distributes over arbitrary joins in each argument place. It follows that the single-sorted operator $\alpha_{R_\sigma}^1$ obtained by appropriate composition with the Galois connection is a normal operator of distribution type $\delta = (\vec{i_j}; i_{n+1})$ on the lattice of clopen elements in $\mathcal{G}(X)$.

Propositions 5.2 and the next proposition verify that $L^*$ is a well defined functor, $L^* : \text{SRF}_\tau^* \rightarrow \text{NLE}_\tau$.

**Proposition 5.3.** Let $L^*$ be defined so that for a sorted residuated frame $\mathfrak{F}$ with relations in $\text{SRF}_\tau^*$, $L^*(\mathfrak{F})$ is the normal lattice expansion of its stable clopen elements (the clopens of $\mathcal{G}(X)$). If $\pi = (p, q) : \mathfrak{F}_2 \rightarrow \mathfrak{F}_1$ is a morphism in $\text{SRF}_\tau^*$, then $L^*(\pi) = \pi^{-1}$ is a homomorphism of normal lattice expansions from clopens in $\mathcal{G}(X_1)$ to clopens of $\mathcal{G}(X_2)$.

**Proof.** Axiom M5 and Lemma 3.23 ensure that $L^*(\pi) = \pi^{-1}$ maps clopens to clopens, hence it restricts to a homomorphism of the normal lattice expansions of clopens. $\Box$

We next verify that the functor $F : \text{NLE}_\tau \rightarrow \text{SRF}_\tau^*$ is in fact a functor $F : \text{NLE}_\tau \rightarrow \text{SRF}_\tau^*$.

**Proposition 5.4.** The canonical frame of a normal lattice expansion is a sorted residuated frame in the category $\text{SRF}_\tau^*$.

**Proof.** Axioms F1–F3 were verified in Lemma 4.3. For the strengthened axiom F2 (Table 3), the proof follows from the fact, proven in [14, Lemma 6.7], that the point operators $\vec{f}$ map principal filters (ideals) to principal filters (ideals). Lemma 4.6 verified axiom F4. Clopens in the canonical frame are the sets $\Gamma x_a$, for a principal filter $x_a$ (similarly for ideals) and $\Gamma x_a \cap \Gamma x_b = \Gamma x_{a \lor b}$, so axiom F5 holds. By join-density of principal filters (similarly for ideals) $x = \bigvee_{a \subseteq x} x_a$ and then $\Gamma x = \Gamma(\bigvee_{a \subseteq x} x_a) = \bigcap_{a \subseteq x} \Gamma x_a$, hence axiom F6 is true in the canonical frame. Clopen sets $\Gamma x_a$ are precisely the sets in the image of the representation map $\zeta_1(a) = \{ x \in \text{Filt}(\mathcal{L}) | a \in x \}$ and it is by a standard argument in Stone duality that the topology generated by the subbasis $\mathcal{S} = \{ \zeta_1(a) | a \in \mathcal{L} \} \cup \{-\zeta_1(a) | a \in \mathcal{L} \}$ is compact and totally separated and the compact-open sets (clopens, since the space is totally separated, hence Hausdorff) are precisely the sets $\zeta_1(a)$, for a lattice element $a$. Proof details can be found in [21, Lemma 2.5], and thereby axiom F7 holds for the canonical frame as well, which is then an object of the category $\text{SRF}_\tau^*$, as claimed. $\Box$

**Theorem 5.5.** The representation map $\zeta_1$ (Definition 4.7) is a homomorphism of normal lattice expansions, indeed an isomorphism of the normal lattice expansion $\mathcal{L}$ and its second dual $(\mathcal{L}_+)^+$, i.e., the normal lattice expansion of the...
clopen sets of filters. Similarly, $\zeta$ is a dual isomorphism of $\mathcal{L}$ and the normal lattice expansion of clopen sets of ideals.

Proof. The fact that $\zeta_1$ is a lattice isomorphism was shown in [21, Theorem 2.4]. By the axiomatization of the category $\mathsf{SRF}^*_\tau$ and Proposition 5.4, the operations $\overline{\sigma}_R$, for a frame relation $R$, restrict to additive operators on clopen sets (in $\mathcal{G}(X)$, or in $\mathcal{G}(Y)$, in accordance to the sort of the relation $R$), hence the operators derived by composition with the Galois connection as in Definition 4.7 are normal lattice operators of the same distribution type as that of the lattice operator they represent. The representation of normal lattice operators in this article is no different from that of [15] (see, in particular, [15, Remark 3.2]), the difference of approach between this article and [15] having to do only with the axiomatization of frame relations. Hence we may appeal to the results of [15], in particular, [15, Theorem 4.5], concluding the proof of the present theorem.

Proposition 5.6. Let $h : \mathcal{L} \to \mathcal{L}^*$ be a morphism in the category $\mathsf{NLE}_\tau$ and $\pi = (p,q) : \mathcal{L}^*_+ \to \mathcal{L}^*_+$ be the canonical $\mathsf{SRF}^*_\tau$ morphism, $p = h^{-1} : X^* \to X$ and $q = h^{-1} : Y^* \to Y$. Then for a filter $u \in X$, $\pi^{-1}(\Gamma u)$ is a closed element in $G(X^*)$. Similarly for ideals.

Proof. Let $w_u$ be the filter generated by the set $h[u] = \{ h(a) | a \in u \}$. By calculating $\pi^{-1}(\Gamma u)$, it is easily seen that $\pi^{-1}(\Gamma u) = \Gamma w_u$. Similarly for ideals.

Proposition 5.7. The canonical map $\pi = (p,q) : \mathcal{L}^*_+ \to \mathcal{L}^*_+$ is continuous, in the topological sense.

Proof. The proof is part of the argument in the proof of [21, Lemma 2.5].

By the above arguments, $\mathcal{L}^*$ and $F$ are contravariant functors on our categories of interest $F : \mathsf{NLE}_\tau \cong (\mathsf{SRF}^*_\tau)^{op} : \mathcal{L}^*$.

Theorem 5.8. (Stone duality) For any objects $\mathcal{L}, \mathfrak{F}$ of the categories $\mathsf{NLE}_\tau$ and $\mathsf{SRF}^*_\tau$, respectively, $\mathcal{L} \cong L^*F(\mathcal{L})$ and $\mathfrak{F} \cong FL^*(\mathfrak{F})$.

Proof. The isomorphism $\mathcal{L} \cong L^*F(\mathcal{L})$ was handled in Theorem 5.5, referring for proof details to [21, Theorem 2.4], for the lattice isomorphism, and to [15, Theorem 4.5], for the case of lattice expansions of similarity type $\tau$.

For the second isomorphism, $\mathfrak{F} \cong FL^*(\mathfrak{F})$, we may base the argument either on [21], or on [16]. In [21], we argued that any sorted frame $\mathfrak{F}$ ($L$-frame in [21]) subject to the axioms F1, F5–F7 is the frame dual to a lattice. More specifically, if $S = (C_s)_{s \in L}$ is the family of compact-open subsets of $X$ ($S$ is in fact a normal lattice expansion of similarity type $\tau$, by the frame axioms and by Proposition 5.2), indexed in some set $L$, then $L$ inherits the structure of the family $S$. Hence, it is a normal lattice expansion $\mathcal{L}$ of type $\tau$ isomorphic to $S$ and $\mathfrak{F}$ is, up to isomorphism, its dual frame. In other words, $L^*(\mathfrak{F}) = S \cong \mathcal{L}$ and $\mathfrak{F} \cong F(\mathcal{L})$. Therefore, we have $FL^*(\mathfrak{F}) \cong FL^*F(\mathcal{L}) \cong F(\mathcal{L}) \cong \mathfrak{F}$. 

\qed
6. Concluding remarks and further research

We revisited in this article the question of Stone duality for lattices with quasioperators (normal lattice expansions, in our terminology), first addressed in [15]. We improved on the results of [15] in the following sense.

First, the category of frames is specified with a simpler axiomatization on relations, to ensure that the induced operators are completely normal lattice operators. Gehrke’s notion of stability of sections [9] was used and, by introducing a notion of sorted conjugate operators we argued that the induced sorted operators, for any sort type $\sigma = (i_{n+1}; \vec{r}_j)$ of the relations, distribute over arbitrary joins of Galois sets (stable, or co-stable, according to the sort type of the relation). By composition with the Galois connection, completely normal single-sorted operators of distribution type $\delta = (\vec{r}_j; i_{n+1})$ are obtained. The section stability requirement implies complete distribution, but as far as we can see the two are not equivalent.

Second, frame morphisms in [15] were tailored to the need to prove a Stone duality and were thus keyed only to the requirement that their inverses preserve clopens. Based on Goldblatt’s recent notion of bounded morphisms for polarities [11] we defined weak bounded morphisms for polarities (equivalently, sorted residuated frames) and extended to the case of frames with relations. The extended definition for morphisms is different and simpler than Goldblatt’s. The extended notion of a frame morphism was shown to satisfy the requirement that its inverse is a homomorphism of the full complex algebras of the frames. To ensure that a Stone duality result is provable, we strengthened the axiomatization of frame relations and morphisms in Section 5.

Third, we expanded on our results in [18] by showing that completely normal operators on the lattice of stable sets of a frame are obtained by taking the closure of the restriction of classical sorted image operators to Galois sets. This provides a proof, at the representation level, that the logics of normal lattice expansions are fragments of corresponding sorted residuated polymodal logics (their modal companions).

This latter development opens up some new problems to investigate, given the results established in this article. Essentially, the research direction opened is one of reducing problems on non-distributive logics (via translation to their modal companions) to problems on sorted residuated polymodal logics. We leave these issues for further research (initiated in [17,18]).

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