AN EXTENSION OF THE LÖWNER–HEINZ INEQUALITY

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Abstract. We extend the celebrated Löwner–Heinz inequality by showing that if $A, B$ are Hilbert space operators such that $A > B \geq 0$, then

$$A^r - B^r \geq ||A||^r - \left( ||A|| - \frac{1}{||(A - B)^{-1}||} \right)^r > 0$$

for each $0 < r \leq 1$. As an application we prove that

$$\log A - \log B \geq \log ||A|| - \log \left( ||A|| - \frac{1}{||(A - B)^{-1}||} \right) > 0.$$  

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ equipped with the operator norm $\| \cdot \|$. There are three types of ordering on the real space of all self-adjoint operators as follows. Let $A, B \in \mathbb{B}(\mathcal{H})$ be self-adjoint. Then

1. $A \geq B$ if $\langle Ax, x \rangle \geq \langle Bx, x \rangle$.
2. $A \succ B$ if $\langle Ax, x \rangle > \langle Bx, x \rangle$ holds for all non-zero elements $x \in \mathcal{H}$.
3. $A > B$ if $A \geq B$ and $A - B$ is invertible.

Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1) but the reverse implications are not valid in general. For instance, if $A$ is the diagonal operator $(1, 1/2, 1/3, \cdots)$ on $\ell^2$, then $A > 0$ but $A \not\succeq 0$.

Of course, in the case where $H$ is of finite dimension, (2) and (3) are equivalent. A continuous real valued function $f$ defined on an interval $J$ is called operator monotone if $A \geq B$ implies that $f(A) \geq f(B)$ for all self-adjoint operators $A, B$ with spectra in $J$. The Löwner–Heinz inequality says that, $f(x) = x^r$ $(0 < r \leq 1)$ is operator monotone on $[0, \infty)$. Löwner [10] proved the inequality for matrices. Heinz [8] proved it for positive operators acting on a Hilbert space of arbitrary dimension. Based on the $C^*$-algebra theory, Pedersen [11] gave a shorter proof of the inequality.

There exist several operator norm inequalities each of which is equivalent to the Löwner–Heinz inequality, see [7]. One of them is $\|A^r B^r\| \leq \|AB\|^r$, called the Cördes inequality in the literature, in which $A$ and $B$ are positive operators and $0 < r \leq 1$. A generalization of the Cördes inequality for operator monotone functions is given in

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It is shown in [1] that this norm inequality is related to the Finsler structure of the space of positive invertible elements.

Kwong [9] showed that if \( A > B \) (\( A \succ B \), resp.), then \( A^r > B^r \) (\( A^r \succ B^r \), resp.) for \( 0 < r \leq 1 \). Uchiyama [12] showed that for every non-constant operator monotone function \( f \) on an interval \( J \), \( A \succ B \) implies \( f(A) \succ f(B) \) for all self-adjoint operators \( A, B \) with spectra in \( J \).

There are several extensions of the Löwner–Heinz inequality. The Furuta inequality [6], which states that if \( A \geq B \geq 0 \), then for \( r \geq 0 \),
\[
(A^{r/2} A^p A^{r/2})^{1/q} \geq (A^{r/2} B^p A^{r/2})^{1/q}
\]
holds for \( p \geq 0 \) and \( q \geq 1 \) with \( (1 + r)q \geq p + r \), is known as an exquisite extension of the Löwner–Heinz inequality; see the survey article [5] and references therein.

If \( f \) is an operator monotone function on \((-1, 1)\), then \( f \) can be represented as
\[
f(t) = f(0) + f'(0) \int_{-1}^{1} \frac{t}{1 - \lambda t} d\mu(\lambda)
\]
where \( \mu \) is a positive measure on \((-1, 1)\). It is known that
\[
t^r = \frac{\sin(r\pi)}{\pi} \int_{0}^{\infty} \frac{t}{\lambda + t} \lambda^{-1} d\lambda,
\]
in which \( 0 < r < 1 \), and
\[
A^r = \frac{\sin(r\pi)}{\pi} \int_{0}^{\infty} \frac{A}{\lambda + A} \lambda^{-1} d\lambda,
\]
where \( A \) is positive and \( 0 < r < 1 \); see e.g. [3, Chapter V].

In this paper we extend the Löwner–Heinz inequality by showing that if \( A, B \in \mathbb{B}(\mathcal{H}) \) such that \( A > B \geq 0 \), then
\[
A^r - B^r \geq \|A\|^r - \left(\|A\| - \frac{1}{\|(A - B)^{-1}\|}\right)^r > 0
\]
for each \( 0 < r \leq 1 \). As an application we prove that
\[
\log A - \log B \geq \log \|A\| - \log \left(\|A\| - \frac{1}{\|(A - B)^{-1}\|}\right) > 0.
\]

2. The results

We start our work with the following useful lemma.

**Lemma 2.1.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be invertible positive operators such that \( A - B \geq m > 0 \). Then
\[
B^{-1} - A^{-1} \geq \frac{m}{(\|A\| - m) \|A\|},
\]
Proof. Since \( f(t) = \frac{1}{t} \) is a decreasing operator monotone function on \([0, \infty)\) we have \( B^{-1} \geq (A - m)^{-1} \). On the other hand

\[
(A - m)^{-1} \geq A^{-1} + \frac{m}{||A|| - m||A||}
\]

\[
\iff (A^{-1} + \frac{m}{||A|| - m||A||})(A - m) \leq 1
\]

\[
\iff \frac{A^2}{||A|| - m||A||} - \frac{mA}{||A|| - m||A||} \leq 1
\]

\[
\iff A^2 - mA \leq (||A|| - m)||A||
\]

\[
\iff ||A^2 - mA|| \leq (||A|| - m)||A||.
\]

There exists \( \lambda_0 \in \text{sp}(A) \) such that \( ||A|| = \lambda_0 \). Since \( A \geq m > 0 \), we have

\[
||A^2 - mA|| = \max\{\lambda : \lambda \in \text{sp}(A^2 - mA)\}
\]

\[
= \max\{\lambda^2 - m\lambda : \lambda \in \text{sp}(A)\}
\]

\[
= \lambda_0^2 - m\lambda_0
\]

\[
= (||A|| - m)||A||.
\]

So \( B^{-1} \geq (A - m)^{-1} \geq A^{-1} + \frac{m}{||A|| - m||A||}. \)

Now we use Lemma 2.1 to prove an analogous but different result to the main theorem of Uchiyama [12] in an easy fashion as an offshoot of our work.

**Proposition 2.2.** Let \( f \) be a non-constant operator monotone function on an interval \( J \) and \( A, B \) be self-adjoint operators with spectra in \( J \) such that \( A > B \). Then \( f(A) > f(B) \).

Proof. Without loss of generality we assume that \( J = (-1, 1) \). Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be self-adjoint operators with spectra in \((-1, 1)\) and \( A - B \) is positive and invertible. So there exists \( m > 0 \) such that \( A - B \geq m > 0 \). Put \( f_\lambda(t) = \frac{t}{1-t\lambda} \) for each \( \lambda \) with \( |\lambda| < 1 \).

We shall show that \( f_\lambda(A) - f_\lambda(B) \) is bounded blow and so invertible. It is clear that the claim is true for \( \lambda = 0 \). If \( 0 < \lambda < 1 \), then \( (1 - \lambda B) - (1 - \lambda A) = \lambda(A - B) > \lambda m > 0 \) as well as \( 1 - \lambda B \) and \( 1 - \lambda A \) are positive invertible operators. Since

\[
\frac{t}{1-\lambda t} = \frac{-1}{\lambda} + \frac{1}{\lambda} \left( \frac{1}{1-\lambda t} \right),
\]
by Lemma 2.1, we have
\[ f(\lambda(A)) - f(\lambda(B)) = \frac{1}{\lambda} \left( \frac{1}{1-\lambda A} - \frac{1}{1-\lambda B} \right) \]
\[ \geq \frac{1}{\lambda} \left( \frac{\lambda m}{(||1-\lambda B|| - \lambda m) ||1-\lambda B||} \right) \quad \text{(by (2.1))} \]
\[ = \frac{m}{(||1-\lambda B|| - \lambda m) ||1-\lambda B||} > 0 \]
A similar argument shows that
\[ f(\lambda(A)) - f(\lambda(B)) \geq \frac{m}{(||1-\lambda A|| + \lambda m) ||1-\lambda A||} > 0 \]
for each \(-1 < \lambda < 0\). Since \(f\) is operator monotone on \((-1, 1)\), it can be represented as
\[ f(t) = f(0) + f'(0) \int_{-1}^{1} f(\lambda(t)) d\mu(\lambda), \]
where \(\mu\) is a nonzero positive measure on \((-1, 1)\). Since \(f\) is nonconstant, \(f'(0) > 0\), [2, Lemma 2.3]. Hence
\[ f(A) - f(B) = f'(0) \int_{-1}^{1} \left( \frac{A}{1-\lambda A} - \frac{B}{1-\lambda B} \right) d\mu(\lambda) \]
\[ = f'(0) \int_{-1}^{1} (f(\lambda(A)) - f(\lambda(B))) d\mu(\lambda) \]
\[ \geq f'(0) \int_{-1}^{1} m_\lambda d\mu(\lambda), \]
where
\[ m_\lambda = \frac{m}{(||1-\lambda B|| - \lambda m) ||1-\lambda B||} \]
if \(0 \leq \lambda < 1\), and
\[ m_\lambda = \frac{m}{(||1-\lambda A|| + \lambda m) ||1-\lambda A||} \]
if \(-1 < \lambda < 0\). Since \(\mu\) is a nonzero positive measure and \(m_\lambda > 0\), we have
\[ f(A) - f(B) \geq f'(0) \int_{-1}^{1} m_\lambda d\mu(\lambda) > 0. \]
Therefore \(f(A) > f(B)\). \( \square \)

Our main result reads as follows.

\textbf{Theorem 2.3.} Let \(A, B \in \mathbb{B}(\mathcal{H})\) be positive operators such that \(A - B \geq m > 0\) and \(0 < r \leq 1\). Then
\[ A^r - B^r \geq ||A||^r - (||A|| - m)^r. \]
Proof. Let $0 < r < 1$. First note that,

$$\frac{A}{\lambda + A} - \frac{B}{\lambda + B} = \lambda \left( \frac{1}{\lambda + B} - \frac{1}{\lambda + A} \right) \geq \frac{\lambda m}{(||A + \lambda|| - m)||A + \lambda||} \quad \text{by (2.1)}$$

for each $\lambda > 0$. By using (1.3) we have

$$A^r - B^r = \sin \left( \frac{r\pi}{\pi} \right) \int_0^\infty \lambda^{r - 1} \left( \frac{A}{\lambda + A} - \frac{B}{\lambda + B} \right) d\lambda \geq \frac{\sin(r\pi)}{\pi} \int_0^\infty \left( \frac{m \lambda^r}{(||A|| + \lambda - m)(||A|| + \lambda)} \right) d\lambda.$$

We need to compute

$$I = \int_0^\infty \frac{\lambda^r}{(\lambda + ||A||)(\lambda + (||A|| - m))} d\lambda$$

where $0 < r < 1$. We will need the branch cut for $z^r = \rho^r e^{ir\theta}$, in which $z = \rho e^{i\theta}$ and $0 \leq \theta \leq 2\pi$. Consider

$$\int_C \frac{z^r}{(z + ||A||)(z + (||A|| - m))} dz,$$

where the keyhole contour $C$ consists of a large circle $C_R$ of radius $R$, a small circle $C_\epsilon$ of radius $\epsilon$ and two lines just above and below the branch cuts $\theta = 0$; see Figure 1. The contribution from $C_R$ is $O(R^{r-2})2\pi R = O(R^{r-1}) = 0$ as $R \to \infty$. Similarly the contribution from $C_\epsilon$ is zero as $\epsilon \to 0$. The contribution from just above the branch cut and from just below the branch cut is $I$ and $-e^{2r\pi i}I$, respectively, as $\epsilon \to 0$ and $R \to \infty$. Hence, taking the limits as $\epsilon \to 0$ and $R \to \infty$,

$$(1 - e^{2r\pi i})I = \int_C \frac{z^r}{(z + ||A||)(z + (||A|| - m))} dz = -2\pi i e^{r\pi i} \left( \frac{||A||^r - (||A|| - m)^r}{||A|| - (||A|| - m)} \right)$$

by the Cauchy residue theorem. So

$$I = \frac{\pi}{m \sin(r\pi)} \left( ||A||^r - (||A|| - m)^r \right).$$
Therefore

\[ A^r - B^r \geq \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{m\lambda^r}{(||A|| + \lambda - m)(||A|| + \lambda)} d\lambda = ||A||^r - (||A|| - m)^r. \]

\[ \square \]

**Corollary 2.4.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be positive operators such that \( A - B \geq m > 0 \). Then

\[ \log A - \log B \geq \log ||A|| - \log(||A|| - m) . \]

**Proof.** Put \( f_n(t) = n(t^{\frac{1}{n}} - 1) \) on \([0, \infty)\). Then the sequence \( \{f_n\} \) uniformly converges to \( \log t \) on any compact subset of \((0, \infty)\). Hence

\[ \log A - \log B = \lim_{n \to \infty} f_n(A) - f_n(B) \geq \lim_{n \to \infty} n(||A||^{\frac{1}{n}} - (||A|| - m)^{\frac{1}{n}}) = \log ||A|| - \log(||A|| - m). \]

\[ \square \]

**Corollary 2.5.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \) such that \( A > B \geq 0 \). Then

\[ (i) A^r - B^r \geq ||A||^r - \left(||A|| - \frac{1}{||A - B||^{-1}}\right)^r \]
for all $0 < r \leq 1$

$$(ii) \log A - \log B \geq \log ||A|| - \log \left( \frac{1}{||A-B||^{-1}} \right).$$

Proof. It follows from $A > B \geq 0$ that $A - B \geq \frac{1}{||A-B||^{-1}} > 0$. Now the assertions are deduced from Theorem 2.3 and Corollary 2.4. □

Remark 2.6. The inequality in Corollary 2.5 is sharp. Indeed for positive scalars $a, b$, if $a > b$, then

$$a^r - b^r = a^r - \left(a - \frac{1}{(a-b)^{-1}}\right)^r$$

and

$$\log a - \log b = \log a - \log \left(a - \frac{1}{(a-b)^{-1}}\right).$$

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