Generalized Fibonacci Primitive Roots
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Abstract: This note generalizes the Fibonacci primitive roots to the set of integers. An asymptotic formula for counting the number of integers with such primitive root is introduced here.

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1 Introduction
Let $p \geq 5$ be a prime. A primitive root $r \neq \pm 1, s^2$ modulo $p$ is a Fibonacci primitive root if $r^2 \equiv r + (1 \text{ mod } p)$. Several basic properties of Fibonacci primitive roots were introduced in [38]. A few other plausible generalizations are available in the literature. Fibonacci primitive roots have applications in finite field calculations, and other area of discrete mathematics, [14], [44]. The complete statement appears in Theorem 8.

The subset of primes with Fibonacci primitive roots is defined by
$$P_F = \{ p \in \mathbb{P} : \text{ord}_p(r) = \varphi(p) \text{ and } r^2 \equiv r + 1 \text{ mod } p \} \subset \mathbb{P},$$
(1)

The asymptotic formula for the counting function has the form
$$P_F(x) = \# \{ p \leq x : \text{ord}_p(r) = \varphi(p) \text{ and } r^2 \equiv r + 1 \text{ mod } p \}$$
(2)
conditional on the generalized Riemann hypothesis, was established in [25], and [39]. This note introduces a generalization to the subset of integers
$$N_F = \{ n \in \mathbb{N} : \text{ord}_n(r) = \lambda(n) \text{ and } r^2 \equiv r + 1 \text{ mod } n \} \subset \mathbb{N},$$
(3)
and provides the corresponding asymptotic counting function
$$N_F(x) = \# \{ n \leq x : \text{ord}_n(r) = \lambda(n) \text{ and } r^2 \equiv r + 1 \text{ mod } n \}.$$  
(4)

Theorem 1. Assume the generalized Riemann hypothesis. Then, are infinitely many composite integers $n \geq 1$ which have Fibonacci primitive roots. Moreover, the number of such integers $n \leq x$ has the asymptotic formula
$$N_F(x) = \left( e^{\gamma_F - \gamma} + o(1) \right) \frac{x}{(\log x)^{1-\alpha_F}} \prod_{p \in \mathcal{A}} \left( 1 - \frac{1}{p^2} \right),$$
(5)
where $\mathcal{A} = \{ p \in \mathbb{P} : \text{ord}_p(r) = p - 1, \text{ ord}_{p^2}(r) \neq p(p - 1) \text{ or } r^2 \neq r + 1 \text{ mod } p^2 \}$, and $\alpha_F = 0.265705...$ is a constant.
The constant $\alpha_F = (27/38) \prod_{p \geq 2} (1 - 1/p(p-1))$ is the average density of the densities of the primes in the residue classes mod 20 that have Fibonacci primitive roots, confer [38] for the calculations.

Sections 2 to 7 provide some essential background and the required basic results. The proof of Theorem 1 is settled in Section 8. And Section 9 has a result for the harmonic sum constrained by Fibonacci primitive roots.

### 2 Some Arithmetic Functions

The Euler totient function counts the number of relatively prime integers $\varphi(n) = \#\{k : \gcd(k, n) = 1\}$. This counting function is compactly expressed by the analytic formula $\varphi(n) = n \prod_{p|n} (1 - 1/p)$, $n \in \mathbb{N}$.

**Lemma 2.** (Fermat-Euler) If $a \in \mathbb{Z}$ is an integer such that $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \mod n$.

The Carmichael function is basically a refinement of the Euler totient function to the finite ring $\mathbb{Z}/n\mathbb{Z}$. Given an integer $n = p_1^{v_1} p_2^{v_2} \cdots p_t^{v_t}$, the Carmichael function is defined by

$$\lambda(n) = \gcd(n, n - 1) = \gcd(n, n - 2) = \cdots = \gcd(n, 1) = \prod_{p^n||n} \frac{p^v - 1}{p - 1},$$

where the symbol $p^v||n, v \geq 0$, denotes the maximal prime power divisor of $n \geq 1$, and

$$\lambda(p^n) = \begin{cases} \varphi(p^n) & \text{if } p \geq 3 \text{ or } v \leq 2, \\
2^{v-2} & \text{if } p = 2 \text{ and } v \geq 3. \end{cases}$$

The two functions coincide, that is, $\varphi(n) = \lambda(n)$ if $n = 2, 4, p^m$, or $2p^m, m \geq 1$. And $\varphi(2^m) = 2 \lambda(2^m)$. In a few other cases, there are some simple relationships between $\varphi(n)$ and $\lambda(n)$. In fact, it seamlessly improves the Fermat-Euler Theorem: The improvement provides the least exponent $\lambda(n)|\varphi(n)$ such that $a^{\lambda(n)} \equiv 1 \mod n$.

**Lemma 3.** ([3 p. 233]) (i) For any given integer $n \in \mathbb{N}$ the congruence $a^{\lambda(n)} \equiv 1 \mod n$ is satisfied by every integer $a \geq 1$ relatively prime to $n$, that is $\gcd(a, n) = 1$.

(ii) In every congruence $x^{\lambda(n)} \equiv 1 \mod n$, a solution $u$ exists which is a primitive root mod $n$, and for any such solution $u$, there are $\varphi(\lambda(n))$ primitive roots congruent to powers of $u$.

**Proof:** (i) The number $\lambda(n)$ is a multiple of every $\lambda(p^n)$ such that $p^n|n$. Ergo, for any relatively prime integer $a \geq 2$, the system of congruences

$$a^{\lambda(n)} \equiv 1 \mod p_1^{v_1}, \quad a^{\lambda(n)} \equiv 1 \mod p_2^{v_2}, \quad \ldots, \quad a^{\lambda(n)} \equiv 1 \mod p_t^{v_t},$$

where $t = \omega(n)$ is the number of prime divisors in $n$, is valid. ■

An integer $u \in \mathbb{Z}$ is called a primitive root mod $n$ if $\min\{m \in \mathbb{N} : u^m \equiv 1 \mod n\} = \lambda(n)$.
3 Characteristic Function For Fibonacci Primitive Roots

The symbol \( \text{ord}_{p^k}(u) \) denotes the order of an element \( r \in (\mathbb{Z}/p^k\mathbb{Z})^\times \) in the multiplicative group of the integers modulo \( p^k \). The order satisfies the divisibility condition \( \text{ord}_{p^k}(r)|\lambda(n) \), and primitive roots have maximal orders \( \text{ord}_{p^k}(r) = \lambda(n) \). The basic properties of primitive root are explicated in [2], [35], et cetera. The characteristic function and primitive roots have maximal orders \( \text{ord}_{p^k}(r) = \lambda(n) \). Let 

**Lemma 4.** Let \( p^k, k \geq 1 \), be a prime power, and let \( r \in \mathbb{Z} \) be an integer such that \( \gcd (r, p^k) = 1 \). Then

(i) The characteristic \( f \) function of the primitive root \((r \mod p^k)\) is given by

\[
 f(p^k) = \begin{cases} 
 0 & \text{if } p \leq 3, k \leq 1, \\
 1 & \text{if } \text{ord}_{p^k}(r) = p^{k-1}(p-1) \text{ and } r^2 \equiv r + 1 \mod p^k, \text{ for any } p \geq 5, k \geq 1, \\
 0 & \text{if } \text{ord}_{p^k}(r) \neq p^{k-1}(p-1), \text{ and } p \geq 5, k \geq 2. 
\end{cases}
\]

(ii) The function \( f \) is multiplicative, but not completely multiplicative since

(iii) \( f(pq) = f(p)f(q), \text{ gcd}(p, q) = 1 \),

(iv) \( f(p^2) \neq f(p)f(p), \text{ if } \text{ord}_{p^2}(r) \neq p(p-1) \).

**Proof:** The congruences \( x^2 - x - 1 \equiv 0 \mod 2 \) and \( x^2 - x - 1 \equiv 0 \mod 3 \) have no roots, so \( f(2^k) = 0 \) and \( f(3^k) = 0 \) for all \( k \geq 1 \). Moreover, for primes \( p \geq 5 \) and \( k \geq 1 \), function has the value \( f(p^k) = 1 \) if and only if the element \( r \in (\mathbb{Z}/p^k\mathbb{Z})^\times \) is a Fibonacci primitive root modulo \( p^k \). Otherwise, it vanishes: \( f(p^k) = 0 \). The completely multiplicative property fails for \( p = 5 \). Specifically, \( 0 = f(5^2) \neq f(5)f(5) = 1 \).

Observe that the conditions \( \text{ord}_r(r) = p - 1 \) and \( \text{ord}_{p^2}(r) \neq p(p-1) \) imply that the integer \( r \neq \pm 1, s^2 \) cannot be extended to a primitive root mod \( p^k, k \geq 2 \). But that the condition \( \text{ord}_{p^2}(r) = p(p-1) \) implies that the integer \( r \) can be extended to a primitive root mod \( p^k, k \geq 2 \).

4 Wirsing Formula

This formula provides decompositions of some summatory multiplicative functions as products over the primes supports of the functions. This technique works well with certain multiplicative functions, which have supports on subsets of primes numbers of nonzero densities.

**Lemma 5.** ([47] p. 71]) Suppose that \( f: \mathbb{N} \rightarrow \mathbb{C} \) is a multiplicative function with the following properties.

(i) \( f(n) \geq 0 \) for all integers \( n \in \mathbb{N} \).

(ii) \( f(p^k) \leq c^k \) for all integers \( k \in \mathbb{N} \), and \( c < 2 \) constant.

(iii) \( \sum_{p \leq x} f(p) = (\tau + o(1))x/\log x, \text{ where } \tau > 0 \text{ is a constant as } x \rightarrow \infty. \)

Then
\[
\sum_{n \leq x} f(n) = \left( \frac{1}{e^{\gamma s} \Gamma(s)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots \right) \tag{10}
\]

The gamma function appearing in the above formula is defined by \( \Gamma(s) = \int_0^\infty t^{s-1}e^{-st}dt, \ s \in \mathbb{C} \). The intricate proof of Wirsing formula appears in [47]. It is also assembled in various papers, such as [19], [33, p. 195], and discussed in [30, p. 70], [43, p. 308]. Various applications are provided in [27], [32], [48], et alii.

5 Harmonic Sums And Products Over Fibonacci Primes

The subset of primes \( \mathcal{P}_F = \{ p \in \mathbb{P} : \text{ord}(r) = \varphi(p) \text{and} r^2 \equiv r + 1 \text{ mod } p \} \subset \mathbb{P} \) consists of all the primes with Fibonacci primitive roots \( r \in \mathbb{Z} \). By Theorem 8, it has nonzero density \( \alpha_F = \delta(\mathcal{P}_F) > 0 \). The real number \( \alpha_F > 0 \) is a rational multiple of the well known Artin constant, see [38], and [25] for the derivation.

The proof of the next result is based on standard analytic number theory methods in the literature, refer to [32, Lemma 4].

**Lemma 6.** Assume the generalized Riemann hypothesis, and let \( x \geq 1 \) be a large number. Then, there exists a pair of constants \( \beta_F > 0 \), and \( \gamma_u > 0 \) such that

(i) \[ \sum_{p \leq x, p \in \mathcal{P}_F} \frac{1}{p} = \alpha_F \log \log x + \beta_F + O \left( \frac{\log \log x}{\log^2 x} \right), \]

(ii) \[ \sum_{p \leq x, p \in \mathcal{P}_F} \frac{\log p}{p-1} = \alpha_F \log x - \gamma_F + O \left( \frac{\log \log x}{\log^2 x} \right). \]

**Proof:** (i) Let \( \pi_F(x) = \# \{ p \leq x : \text{ord}(r) = \varphi(p) \text{ and } r^2 \equiv r + 1 \text{ mod } p \} \) be the counting measure of the corresponding subset of primes \( \mathcal{P}_F \). To estimate the asymptotic order of the prime harmonic sum, use the Stieltjes integral representation:

\[
\sum_{p \leq x, p \in \mathcal{P}_F} \frac{1}{p} = \int_{x_0}^x \frac{1}{t} d\pi_F(t) = \frac{\pi_F(x)}{x} + c(x_0) + \int_{x_0}^x \frac{\pi_u(t)}{t^2} dt, \tag{11}
\]

where \( x_0 > 0 \) is a constant. By the GRH, \( \pi_F(x) = \alpha_F \pi(x) = \alpha_F x / \log x + O \left( \log \log x / \log^2 x \right) \), see Theorem 8. This yields

\[
\int_{x_0}^x \frac{1}{t} d\pi_F(t) = \frac{\alpha_F}{\log x} + O \left( \frac{\log \log x}{\log^2(x)} \right) + c_0(x_0) + \alpha_F \int_{x_0}^x \left( \frac{1}{t \log t} + O \left( \frac{\log \log t}{t \log^2(t)} \right) \right) dt \tag{12}
\]

\[
= \alpha_F \log \log x - \log \log x_0 + c_0(x_0) + O \left( \frac{\log \log x}{\log^2 x} \right),
\]

where \( \beta_F = - \log \log x_0 + c_0(x_0) \) is a generalized Mertens constant. The statement (ii) follows
A generalized Mertens constant $\beta_F$ and a generalized Euler constant $\gamma_F$ have other equivalent definitions such as

$$\beta_F = \lim_{x \to \infty} \left( \sum_{p \leq x, p \in P_F} \frac{1}{p} - \alpha_F \log \log x \right) \quad \text{and} \quad \beta_u = \gamma_u - \sum_{p \in P_F} \sum_{k \geq 2} \frac{1}{kp^k},$$

respectively. These constants satisfy $\beta_F = \beta_1 \alpha_F$ and $\gamma_F = \gamma \alpha_F$. If the density $\alpha_F = 1$, these definitions reduce to the usual Euler constant and the Mertens constant, which are defined by

$$\gamma = \lim_{x \to \infty} \left( \sum_{p \leq x} \frac{\log p}{p-1} - \log x \right) \quad \text{and} \quad \beta_1 = \lim_{x \to \infty} \left( \sum_{p \leq x} \frac{1}{p} - \log \log x \right),$$

or some other equivalent definitions, respectively. Moreover, the linear independence relation in (13) becomes $\beta = \gamma - \sum_{p \geq 2} \sum_{k \geq 2} (kp^k)^{-1}$, see [18, Theorem 427].

A numerical experiment utilizing a small subset of primes with Fibonacci primitive roots gives the following approximate values:

(i) $\alpha_F = \frac{27}{38} \prod_{p \geq 2} \left( 1 - \frac{1}{p(p-1)} \right) = 0.26570548288843681890137\ldots,$

(ii) $\beta_F \approx \sum_{p \leq 1301, p \in P_F} \frac{1}{p} - \alpha_F \log \log x = 0.05020530308647012230491\ldots,$

(iii) $\gamma_F \approx \sum_{p \leq 1301, p \in P_F} \frac{\log p}{p-1} - \alpha_F \log x = 0.0221594862523476326826286\ldots.$

**Lemma 7.** Assume the generalized Riemann hypothesis, and let $x \geq 1$ be a large number. Then, there exists a pair of constants $\gamma_F > 0$ and $\nu_F > 0$ such that

(i) $\prod_{p \leq x, p \in P_F} \left( 1 - \frac{1}{p} \right)^{-1} = e^{\gamma_F} \log(x)^{\alpha_F} + O \left( \frac{\log \log x}{\log^2 x} \right),$

(ii) $\prod_{p \leq x, p \in P_F} \left( 1 + \frac{1}{p} \right) = e^{\gamma_F} \prod_{p \in P_F} \left( 1 - p^{-2} \right) \log(x)^{\alpha_F} + O \left( \frac{\log \log x}{\log^2 x} \right),$

(iii) $\prod_{p \leq x, p \in P_F} \left( 1 - \frac{\log p}{p-1} \right)^{-1} = e^{\nu_F} x^{-\gamma_F} x^{\alpha_F} + O \left( \frac{x^{\alpha_F} \log \log x}{\log^2 x} \right).$

**Proof:** (i) Express the logarithm of the product as

$$\sum_{p \leq x, p \in P_F} \log \left( 1 - \frac{1}{p} \right)^{-1} = \sum_{p \leq x, p \in P_F} \sum_{k \geq 1} \frac{1}{kp^k} = \sum_{p \leq x, p \in P_F} \frac{1}{p} + \sum_{p \leq x, p \in P_F} \sum_{k \geq 2} \frac{1}{kp^k}.$$

Apply Lemma 6 and relation (13) to complete the verification. For (ii) and (iii), use similar methods as in the first one. $\blacksquare$
The constant $\nu_F > 0$ is defined by the double power series (an approximate numerical value for $p \leq x = 1301$ is shown):

$$\nu_2 = \sum_{p \in \mathbb{P}, \ k \geq 2} \frac{1}{k} \left( \frac{\log p}{p-1} \right)^k \approx 0.188622600886988493134287... .$$

\(16\)

### 6 Density Correction Factor

The subsets of primes $\mathcal{P}_F = \{ p \in \mathbb{P} : \text{ord}(r) = p-1 \}$ has an important partition as a disjoint union $\mathcal{P}_F = \mathcal{A} \cup \mathcal{B}$, where

(i) $\mathcal{A} = \{ p \in \mathbb{P} : \text{ord}(r) = p-1, \ \text{ord}_p r \neq p^{k-1}(p-1), \text{ or } r^2 \not\equiv r + 1 \text{ mod } p^2 \}$ and

(ii) $\mathcal{B} = \{ p \in \mathbb{P} : \text{ord}(r) = p-1, \text{ord}_p r = p^{k-1}(p-1), \text{ and } r^2 \equiv r + 1 \text{ mod } p^2 \}$.

This partition has a role in the determination of the density of the subset of integers $\mathcal{N}_F = \{ n \in \mathbb{N} : \text{ord}(r) = \lambda(n) \}$ with Fibonacci primitive roots $r \in \mathbb{Z}$. The finite prime product occurring in the proof of Theorem 1 splits into two finite products over these subsets of primes. These are used to reformulate an equivalent expression suitable for this analysis:

$$P(x) = \prod_{p^k \leq x, \ \text{ord}_p(r) = p-1, \ \text{ord}_p^2 r \neq p^{k-1}(p-1) \text{ or } r^2 \not\equiv r + 1 \text{ mod } p^2} \left( 1 + \frac{1}{p} \right) \prod_{p^k \leq x, \ \text{ord}_p^2 r = p^{k-1}(p-1) \text{ and } r^2 \equiv r + 1 \text{ mod } p^2} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right)$$

$$= \prod_{p^k \leq x, \ \text{ord}_p(r) = p-1, \ \text{ord}_p^2 r \neq p^{k-1}(p-1) \text{ or } r^2 \not\equiv r + 1 \text{ mod } p^2} \left( 1 - \frac{1}{p^2} \right) \prod_{p^k \leq x, \ \text{ord}_p^2 r = p^{k-1}(p-1) \text{ and } r^2 \equiv r + 1 \text{ mod } p^2} \left( 1 - \frac{1}{p} \right)^{-1}$$

$$= \prod_{p \leq x, p \in \mathcal{A}} \left( 1 - \frac{1}{p^2} \right) \prod_{p \leq x, p \in \mathcal{P}_F} \left( 1 - \frac{1}{p} \right)^{-1}$$

$$= \prod_{p \in \mathcal{A}} \left( 1 - \frac{1}{p^2} \right) \prod_{p \leq x, p \in \mathcal{P}_F} \left( 1 - \frac{1}{p} \right)^{-1} + O \left( \frac{\log x}{x} \right),$$

\(17\)

where the convergent partial product is replaced with the approximation

$$\prod_{p \leq x, p \in \mathcal{A}} \left( 1 - \frac{1}{p^2} \right) = \prod_{p \in \mathcal{A}} \left( 1 - \frac{1}{p^2} \right) + O \left( \frac{1}{x} \right).$$

\(18\)

The product $\prod_{p \in \mathcal{A}} \left( 1 - p^{-2} \right)$ reduces the density to compensate for those primes for which the Fibonacci primitive root $r \mod p$ that cannot be extended to a Fibonacci primitive root $r \mod p^2$. This seems to be a density correction factor similar to case for primitive roots over the prime numbers. The correction required for certain densities of primes with respect to fixed primitive roots over the primes was discovered by the Lehmers, see [41].
7 The Proof Of The Theorem

The algebraic constraint \( r^2 = r + 1 \) restricts the Fibonacci primitive roots \( r \) modulo a prime \( p \geq 5 \) to the algebraic integers \( (1 \pm \sqrt{5})/2 \) in the finite field \( \mathbb{F}_p \). This in turns forces the corresponding primes to the residue classes \( p \equiv \pm 1 \mod 10 \), the analysis appears in \[38\]. A conditional proof for the conjecture of Shank was achieved in \[25\] and \[39\].

**Theorem 8.** (\[25\]) If the extended Riemann hypothesis hold for the Dedekind zeta function over Galois fields of the type \( \mathbb{Q}(\sqrt[r]{r}, \sqrt[q]{r}) \), where \( r \) is a squarefree integer, and \( d \mid n \). Then, the density of primes which have Fibonacci primitive roots is \( \alpha_F = (27/38) \prod_{p \equiv 2}(1 - 1/p(p - 1)) \), and the number of such primes is

\[
P_F(x) = \alpha_F \frac{x}{\log x} + O \left( \frac{x \log \log x}{\log^2 x} \right) \quad \text{as} \quad x \rightarrow \infty.
\]

**Proof of Theorem 1:** By Theorem 8, the density \( \alpha_F = \delta(P_F) > 0 \) of the subset of primes \( P_F \) is nonzero. Put \( \tau = \alpha_F \) in Wirsing formula, Lemma 6, and replace the characteristic function \( f(n) \) of Fibonacci primitive roots in the finite ring \( \mathbb{Z}/p^k \mathbb{Z}, k \geq 1 \), see Lemma 4, to produce

\[
\sum_{n \leq x} f(n) = \left( \frac{1}{e^{\tau F(\tau)}} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{f(p) + f(p^2)}{p^2} + \cdots \right)
\]

\[
= \left( \frac{1}{e^{\tau \alpha F}} + o(1) \right) \frac{x}{\log x} \prod_{\text{ord}_p(r) = 1, \text{ord}_{p^2}(r) \neq 1 \text{ or } r^2 \equiv r + 1 \mod p^2} \left( 1 + \frac{1}{p} \right)
\]

\[
\times \prod_{\text{ord}_p(r) = p(p-1) \text{ and } r^2 \equiv r + 1 \mod p^2} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right).
\]

**Note:** In terms of the subsets \( \mathcal{A} \) and \( \mathcal{B} \), see Section 6, the characteristic function \( f(n) \) of Fibonacci primitive roots has the simpler description

\[
f(p^k) = \begin{cases} 1 & \text{if } p \in \mathcal{A}, \text{ and } k = 1, \\ 0 & \text{if } p \in \mathcal{A}, \text{ and } k \geq 2, \end{cases} \quad \text{and} \quad f(p^k) = \begin{cases} 1 & \text{if } p \in \mathcal{B}, \\ 0 & \text{otherwise}. \end{cases}
\]

Replacing the equivalent product, see \[17\] in Section 6, and using Lemma 7, yield

\[
\sum_{n \leq x} f(n) = \left( \frac{1}{e^{\tau \alpha F}} + o(1) \right) \frac{x}{\log x} = \prod_{p \leq x} \left( 1 - \frac{1}{p^2} \right) \prod_{p \leq x, p \in P_F} \left( 1 - \frac{1}{p} \right)^{-1}
\]

\[
= \left( \frac{e^{\tau \gamma_F - \tau \alpha F}}{\Gamma(\alpha F)} + o(1) \right) \frac{x}{(\log x)^{1 - \alpha F}} \prod_{p \in \mathcal{A}} \left( 1 - \frac{1}{p^2} \right)
\]

where \( \alpha_F > 0 \) is a constant, and \( \gamma_F \) is a generalized Euler constant, see Lemma 6 for detail. \(\blacksquare\)

The constant \( \alpha_F = (27/38) \prod_{p \equiv 2}(1 - 1/p(p - 1)) = 0.265705... \) is the average density of the densities of the primes in the residue classes mod 20 that have Fibonacci primitive roots, confer \[38\] for the calculations.
8 Harmonic Sum For Fibonacci Primitive Roots

The first few primes in the subset of primes \( \mathcal{P}_F = \{ p \leq x : \text{ord}(r) = \varphi(p) \text{ and } r^2 \equiv r + 1 \text{ mod } p \} \), which have Fibonacci primitive roots, are

\[
\mathcal{P}_F = \{ 5, 11, 19, 31, 41, 59, 61, 71, 79, 109, 131, 149, 179, 191, 239, 241, 251, 269, 271, 311, \\
359, 379, 389, 409, 419, 431, 439, 449, 479, 491, 499, 569, 571, 599, 601, 631, 641, 659, \\
701, 719, 739, 751, 821, 839, 929, 971, 1019, 1039, 1051, 1091, 1129, 1171, 1181, \\
1201, 1259, 1301, \ldots \}.
\] (23)

The subset of primes \( \mathcal{P}_F \subset \mathbb{P} \) of density \( \alpha_F = \delta(\mathcal{P}_F) > 0 \) generates a subset of composite integers \( \mathcal{N}_F = \{ n \in \mathbb{N} : \text{ord}(r) = \lambda(n) \text{ and } r^2 \equiv r + 1 \text{ mod } n \} \), which have Fibonacci primitive roots. The first few are listed as

\[
\mathcal{N}_F = \{ 5, 11, 19, 31, 41, 55, 59, 61, 71, 79, 5 \cdot 19, 109, 11^2, 5 \cdot 19, 149, 5 \cdot 31, \ldots \}.
\] (24)

The first such composite number \( n = 55 \) has two Fibonacci primitive roots \( r = 8 \) and 48 of order \( \text{ord}_r = \lambda(55) = 20 \). Let

\[
\mathcal{N}_F = \{ n \in \mathbb{N} : \text{ord}(r) = \lambda(n) \text{ and } r^2 \equiv r + 1 \text{ mod } n \},
\] (25)

and let

\[
N_F(x) = \# \{ n \leq x : \text{ord}(r) = \lambda(n) \text{ and } r^2 \equiv r + 1 \text{ mod } n \}
\] (26)

be the corresponding discrete counting measure. This was determined in Theorem 1.

The subset \( \mathcal{N}_F \subset \mathcal{R} \) is a proper subset \( \mathcal{R} = \{ n \in \mathbb{N} : p|n \Rightarrow p \in \mathcal{P}_F \} \). This subset of integers, which contains all the primes powers, is generated by the subset of primes \( \mathcal{P}_F = \{ p \in \mathbb{P} : \text{ord}(r) = \lambda(p) \text{ and } r^2 \equiv r + 1 \text{ mod } p \} \). Since there are some primitive root \((r \text{ mod } p)\) which are not primitive root modulo prime powers \( p^m, m \geq 2 \), the subset \( \mathcal{R} \) is slightly larger than the subset \( \mathcal{N}_F \). For example, \( r = 3 \) is a Fibonacci primitive root \( \text{mod } p = 5 \), but \((3 + 5k)^2 \not\equiv 3 + 5k \text{ mod } 5^2 \) for \( k \in \mathbb{F}_p \). Thus it cannot be extended to a Fibonacci primitive root modulo \( 5^m, m \geq 2 \). More precisely, the prime powers \( 5^m \in \mathcal{R}, m \geq 1 \), but \( 5^2 \not\in \mathcal{N}_F, m \geq 2 \).

An asymptotic formula for the harmonic sum over the subset of integers \( \mathcal{N}_F \) is determined here.

**Lemma 9.** Let \( x \geq 1 \) be a large number, let \( \alpha_F = \delta(\mathcal{P}_F) > 0 \) be the density of the subset of primes \( \mathcal{P}_F \), and let \( \mathcal{N}_F \subset \mathbb{N} \) be a subset of integers generated by \( \mathcal{P}_F \). Then

\[
\sum_{n \leq x, n \in \mathcal{N}_F} \frac{1}{n} = \kappa_F (\log x)^{\alpha_F} + \gamma_F + O\left(\frac{1}{(\log x)^{1-\alpha_F}}\right).
\] (27)

The number \( \alpha_F = 0.265705484288 \ldots \) is a generalized Artin constant, and \( \gamma_F > 0 \) is a generalized Euler constant, see (13) for the definition. The other constant is

\[
\kappa_F = \frac{e^{\gamma_F - \alpha_F}}{\alpha_F \Gamma(\alpha_F)} \prod_{p \in \mathcal{A}} \left(1 - \frac{1}{p^2}\right),
\] (28)

where the index of the product ranges over the subset primes \( \mathcal{A} \), see Section 6.
Proof: Use the discrete counting measure \( N_F(x) = (\alpha_F K_F + o(1)) x (\log x)^{\alpha_F - 1} \), Theorem 1, to write the finite sum as an integral, and evaluate it:

\[
\sum_{n \leq x, n \in \mathcal{N}_F} \frac{1}{n} = \int_{x_0}^{x} \frac{1}{t} dN_F(t) = \frac{N_F(t)}{t} \bigg|_{x_0}^{x} + \int_{x_0}^{x} \frac{N_F(t)}{t^2} dt, \tag{29}
\]

where \( x_0 > 0 \) is a constant. Continuing the evaluation yields

\[
\int_{x_0}^{x} \frac{1}{t} dN_2(t) = \frac{(\alpha_F K_F + o(1))}{(\log x)^{1-\alpha_F}} + c_0(x_0) + \int_{x_0}^{x} \frac{(\alpha_F K_F + o(1))}{t (\log t)^{1-\alpha_F}} dt \tag{30}
\]

\[
= \kappa_F (\log x)^{\alpha_F} + \gamma_F + O \left( \frac{1}{(\log x)^{1-\alpha_F}} \right),
\]

where \( c_0(x_0) \) is a constant. Moreover,

\[
\gamma_F = \lim_{x \to \infty} \left( \sum_{n \leq x, n \in \mathcal{N}_F} \frac{1}{n} - \kappa_F \log^{\alpha_F} x \right) = c_0(x_0) + \int_{x_0}^{\infty} \frac{(\alpha_F K_F + o(1))}{t (\log t)^{1-\alpha_F}} dt \tag{31}
\]

is a second definition of this constant. \( \blacksquare \)

The integral lower limit \( x_0 = 4 \) appears to be correct one since the subset of integers is \( \mathcal{N}_2 = \{5, 11, \ldots\} \).

The Fibonacci primitive root problem is a special case of a more general problem that investigates the existence of infinitely many primes such that the polynomial congruence \( f(n) \equiv 0 \mod p \) has primitive root solutions. The algebraic and analytic analysis for these more general polynomials are expected to be interesting.

9 Problems

Problem 8.1 Let \( \mathcal{A} = \{ p \in \mathbb{P} : \text{ord}(r) = p - 1, \text{ord}(r) \neq p(p - 1) \text{ or } r^2 \neq r + 1 \mod p^2 \} \). Determine whether or not the subset of primes \( \mathcal{A} = \{ 5, \ldots \} \) is finite. There is an algebraic characterization, see (16). Sketch of the proof: Use a) \( r^2 \equiv r + 1 \mod p \), b) \( 1 < r < p \) to show that \( (r + mp)^2 \equiv r + mp + \mod p^2 \), \( m \in \mathbb{F}_p \) has no solutions for all primes \( p > 5 \).

Problem 8.2 Let \( \varphi(n)/\lambda(n) \) be the index of the finite group \( \mathbb{Z}/n\mathbb{Z} \), and suppose it contains a Fibonacci primitive root. Explain which maximal subgroup \( G \subset (\mathbb{Z}/p^k \mathbb{Z})^\times \) of cardinality \#\( G = \lambda(n) \) contains the Fibonacci primitive root. Some theoretical/numerical data is available in [7].

Problem 8.3 Show that the constants satisfy the equation \( \gamma_F - \gamma \alpha_F = 0 \), confer equation (13).

Problem 8.4 Determine the least primitive fibonacci root modulo \( n \).

Problem 8.5 Show that \( L(s, \chi_F) = \sum_{n \geq 1} \chi_F(n)/n^s \) is analytic on the half plane \( \Re(s) > 1 \), and has a pole at \( s = 1 \). Here the character is defined by \( \chi_F(n) = 1 \) if \( n \) is a fibonacci integer, else it vanishes.
**Problem 8.6** Show that the logarithm derivative $-L'(s, \chi_F)/L(s, \chi_F)^2 = \sum_{n \geq 1} \Lambda_F(n)/n^s$ is analytic on the half plane $\Re(s) > 1$, and has a pole at $s = 1$. Here the prime characteristic function is defined by $\Lambda_F(n) = \log n$ if $n$ is a fibonacci prime, else it vanishes.

**Problem 8.7** Generalize the analysis for fibonacci primitive root solutions of the quadratic equation $x^2 - x - 1 \equiv 0 \mod n$ for infinitely many integers $n \geq 1$ to the existence of primitive root solutions for a general quadratic equation $ax^2 + bx + c \equiv 0 \mod n$ for infinitely many integers $n \geq 1$.

**References**

[1] Ambrose, Christopher. Artin primitive root conjecture and a problem of Rohrlich. Math. Proc. Cambridge Philos. Soc. 157(2014), no. 1, 79-99.

[2] Apostol, Tom M. Introduction to analytic number theory. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[3] Balog, Antal; Cojocaru, Alina-Carmen; David, Chantal. Average twin prime conjecture for elliptic curves. Amer. J. Math. 133,(2011), no. 5, 1179-1229.

[4] Roger C. Baker, Paul Pollack, Bounded gaps between primes with a given primitive root, II, [arXiv:1407.7186].

[5] Carmichael, R. D.; On Euler’s $\varphi$-function. Bull. Amer. Math. Soc. 13 (1907), no. 5, 241-243.

[6] Cristian Cobeli, On a Problem of Mordell with Primitive Roots, [arXiv:0911.2832].

[7] Peter J. Cameron and D. A. Preece, Notes on primitive lambda-roots, [http://www.maths.qmul.ac.uk/~pjc/csgnotes/lamda.pdf].

[8] Cobeli, Cristian; Zaharescu, Alexandru. On the distribution of primitive roots mod p . Acta Arith. 83, (1998), no. 2, 143-153.

[9] Joseph Cohen, Primitive roots in quadratic fields, II, Journal of Number Theory 124 (2007) 429-441.

[10] H. Davenport, On Primitive Roots in Finite Fields, Quarterly J. Math. 1937, 308-312.

[11] Rainer Dietmann, Christian Elsholtz, Igor E. Shparlinski, On Gaps Between Primitive Roots in the Hamming Metric, [arXiv:1207.0842].

[12] Paul Erdos, Harold N. Shapiro, On The Least Primitive Root Of A Prime, 1957, euclid-project.org.

[13] Adam Tyler Felix, Variations on Artin Primitive Root Conjecture, PhD Thesis, Queen University, Canada, August 2011.

[14] S. W. Golomb, Algebraic constructions for Costas arrays, Journal of Combinatorial Theory A, vol. 37, no. 1, pp. 13—21, 1984.

[15] Goldfeld, Morris. Artin conjecture on the average. Mathematika 15, 1968, 223-226.
[16] Gupta, Rajiv; Murty, M. Ram. A remark on Artin’s conjecture. Invent. Math. 78 (1984), no. 1, 127-130.

[17] D. R. Heath-Brown, Artins conjecture for primitive roots, Quart. J. Math. Oxford Ser. (2) 37, No. 145, 27-38, 1986.

[18] Hardy, G. H.; Wright, E. M. An introduction to the theory of numbers. Sixth edition. Revised by D. R. Heath-Brown and J. H. Silverman. With a foreword by Andrew Wiles. Oxford University Press, 2008.

[19] Hildebrand, Adolf. Quantitative mean value theorems for nonnegative multiplicative functions. II. Acta Arith. 48 (1987), no. 3, 209-260.

[20] C. Hooley, On Artins conjecture, J. Reine Angew. Math. 225, 209-220, 1967.

[21] Li, Shuguang; Pomerance, Carl. On generalizing Artin’s conjecture on primitive roots to composite moduli. J. Reine Angew. Math. 556 (2003), 205-224.

[22] Iwaniec, Henryk; Kowalski, Emmanuel. Analytic number theory. AMS Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004.

[23] Konyagin, Sergei V.; Shparlinski, Igor E. On the consecutive powers of a primitive root: gaps and exponential sums. Mathematika 58 (2012), no. 1, 11-20.

[24] H. W. Lenstra Jr, P. Moree, P. Stevenhagen, Character sums for primitive root densities, arXiv:1112.4816

[25] Lenstra, H. W., Jr. On Artin conjecture and Euclid algorithm in global fields. Invent. Math. 42, (1977), 201-224.

[26] Lidl, Rudolf; Niederreiter, Harald. Finite fields. With a foreword by P. M. Cohn. Second edition. Encyclopedia of Mathematics and its Applications, 20. Cambridge University Press, Cambridge, 1997.

[27] Moree, Pieter. Counting numbers in multiplicative sets: Landau versus Ramanujan. Math. Newsl. 21 (2011), no. 3, 73-81.

[28] Pieter Moree. Artin’s primitive root conjecture -a survey. arXiv:math/0412262.

[29] Moree, P. Artin prime producing quadratics. Abh. Math. Sem. Univ. Hamburg 77 (2007), 109-127.

[30] Montgomery, Hugh L.; Vaughan, Robert C. Multiplicative number theory. I. Classical theory. Cambridge University Press, Cambridge, 2007.

[31] Narkiewicz, W. The development of prime number theory. From Euclid to Hardy and Littlewood. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[32] Pappalardi, Francesco; Saidak, Filip; Shparlinski, Igor E. Square-free values of the Carmichael function. J. Number Theory 103 (2003), no. 1, 122-131.

[33] A. G. Postnikov, Introduction to analytic number theory, Translations of Mathematical Monographs, vol. 68, American Mathematical Society, Providence, RI, 1988.

[34] Roskam, Hans. Artin primitive root conjecture for quadratic fields. J. Theory Nombres Bordeaux, 14, (2002), no. 1, 287-324.
[35] Rose, H. E. A course in number theory. Second edition. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

[36] Ribenboim, Paulo, The new book of prime number records, Berlin, New York: Springer-Verlag, 1996.

[37] Shanks, Daniel; Taylor, Larry. An observation on Fibonacci primitive roots. Fibonacci Quart. 11, (1973), no. 2, 159-160.

[38] Shanks, Daniel Fibonacci primitive roots. Fibonacci Quart. 10, (1972), no. 2, 163-168, 181.

[39] Sander, J. W. On Fibonacci primitive roots. Fibonacci Quart. 28, (1990), no. 1, 79-80.

[40] Stevenhagen, Peter. The correction factor in Artin’s primitive root conjecture. Les XXIIemes Journees Arithmetiques (Lille, 2001). J. Theor. Nombres Bordeaux 15 (2003), no. 1, 383-391.

[41] Stevenhagen, Peter. The correction factor in Artin’s primitive root conjecture. Les XXIIemes Journees Arithmetiques (Lille, 2001). J. Theor. Nombres Bordeaux 15 (2003), no. 1, 383-391.

[42] Stephens, P. J. An average result for Artin conjecture. Mathematika 16, (1969), 178-188.

[43] G. Tenenbaum, Introduction to analytic and probabilistic number theory, Cambridge Studies in Advanced Mathematics 46, Cambridge University Press, Cambridge, 1995.

[44] Tim Trudgian, Qiang Wang, The $T_{-4}$ and $G_{-4}$ constructions of Costas arrays, arXiv:1409.6827

[45] Vaughan, R. C. Some applications of Montgomery’s sieve. J. Number Theory 5 (1973), 64-79.

[46] Winterhof, Arne. Character sums, primitive elements, and powers in finite fields. J. Number Theory 91, 2001, no. 1, 153-163.

[47] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen, Math. Ann. 143 (1961) 75-102.

[48] Williams, Kenneth S. Note on integers representable by binary quadratic forms. Canad. Math. Bull. 18 (1975), no. 1, 123-125.