Domain Wall in MQCD and
Supersymmetric Cycles in
Exceptional Holonomy Manifolds

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Abstract

It was conjectured by Witten that a BPS-saturated domain wall exists in the
M-theory fivebrane version of QCD (MQCD) and can be represented as a supersymmetric three-cycle in the sense of Becker et al with an appropriate asymptotic behavior. We derive the differential equation which defines an associative cycle in $G_2$ holonomy seven-manifold corresponding to the supersymmetric three-cycle and show that it contains a sum of the Poisson brackets. We study solutions of the differential equation with prescribed asymptotic behavior.

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1 Introduction

Recently, many interesting results about supersymmetric gauge theories in different dimensions have been obtained by embedding gauge theories in string theory and using properties of the latter to study the former. D-branes has provided an arena to exchange results of these theories and to advance both of them. Three and four dimensional theories have been studied by using configurations of branes in M-theory [1].

Witten used this method to explore the minimal $N = 1$ model with an $SU(n)$ vector multiplet in four dimensions [2]. He showed how for this model some of the outstanding properties of the ordinary QCD such as confinement, a mass gap and spontaneous breaking of a discrete chiral symmetry can be approached from M-theory point of view. A configuration arising from $n$ Dirichlet fourbranes suspended between two NS fivebranes was used to study this theory. It was shown that the obtained theory (MQCD) is confining, it has strings or flux tubes and undergoes spontaneous breaking of the $Z_n$ chiral symmetry.

A consequence of the spontaneously broken chiral symmetry is that there can be a domain wall separating different vacua. Witten has suggested that a BPS-saturated domain wall exists in MQCD and that it can be represented as a supersymmetric three-cycle in the sense of Becker et al [3] with a prescribed asymptotic behavior. BPS-saturated domain walls in four dimensional supersymmetric field theories have been considered in [4].

The aim of this note is to study the Witten conjecture on domain walls in MQCD. First we will derive a differential equation for the supersymmetric three-cycle. It turns out that this equation contains a sum of the Poisson brackets. Then we investigate solutions of this equation.

2 Domain Wall in MQCD

Domain walls separate spatial domains containing different vacua. In [2] it was suggested that domain walls exist in supersymmetric gauge theory obtained by using a brane configuration in M-theory. The domain wall is described [2] as an M-theory fivebrane of the form $R^3 \times S$ where $R^3$ is parameterized by $x^0, x^1, x^2$ and $S$ is a three-surface in the seven manifold $\tilde{Y} = R \times Y$. Here $R$ is the copy of $x^3$ direction and $Y = R^5 \times S^1$. Near $x^3 = -\infty$, $S$ should look like $R \times \Sigma$, where $\Sigma$ is the Riemann surface defined by $w = \zeta v^{-1}, t = v^n$. Here $w, v, t$ are complex coordinates in $R^5 \times S^1$ with coordinates $x^4, x^5, x^7, x^8, x^6, x^{10}: v = x^4 + ix^5, w = x^7 + ix^8, s = x^6 + ix^{10}, t = e^{-s}, 0 \leq x^{10} \leq 2\pi, \zeta$ is a constant. Near $x^3 = +\infty$, $S$ should look like $R \times \Sigma'$, where $\Sigma'$ is the Riemann surface of an ”adjacent” vacuum, defined by $w = exp(2\pi i/n)\zeta v^{-1}, t = v^n$. We have to find such $S$ which is a supersymmetric three-cycle with the described asymptotic behavior. First we will derive a differential equation for the supersymmetric three-cycle. It turns out that this equation contains a sum of the Poisson brackets. Then we study solutions of this equation.
3 Supersymmetric Cycle in $G_2$ Holonomy Manifolds

A supersymmetric cycle is defined by the property that the world-volume theory of a brane wrapping around it is supersymmetric. The three-cycle is supersymmetric if the global supersymmetry transformation can be undergone by $\kappa$-transformation, which implies that

$$P_2\Psi = \frac{1}{2}(1 - \frac{i}{3!}e^{\alpha\beta\gamma}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \Gamma_{MNP})\Psi = 0,$$

where $\Psi$ is covariantly constant ten-dimensional spinor, $\Gamma_{MNP}$ are ten-dimensional $\Gamma$ matrices.

The conditions for the supersymmetric cycles in Calabi-Yau 3-folds have been analyzed on [5]. It was shown that a supersymmetric three-cycle is one for which the pullback of Kähler form $J$ vanishes and the pullback of the holomorphic 3-form $\Omega$ is a constant multiple of volume element, namely $\ast X(J) = 0$, $\ast X(\Omega) \sim 1$, where $\ast$ denotes the pullback and $\ast$ is a Hodge dual on membrane world-volume. In [3] it was mentioned that supersymmetric cycles may be complex or special Lagrangian submanifolds and also of exceptional type which exist in $Spin(7)$, $SU(4)$ and $G_2$ holonomy manifolds. To study supersymmetric cycles one uses the concept of calibration [3]. A calibration is a closed $p$-form $\Phi$ on a Riemannian manifold of dimension $n$ such that its restriction to each tangent $p$-plane of $M$ is less or equal to the volume of the plane. Submanifolds for which there is equality are said to be calibrated by $\Phi$.

In the case of domain walls in MQCD [4] one deals with a seven dimensional flat manifold of $G_2$ holonomy and with the associative calibration. $G_2$ is a subgroup of $O(7)$ which leaves the constant spinor invariant. Just as in Calabi-Yau case in the case of $G_2$ holonomy manifold there is a canonical 3 form $\Phi$ and its Hodge dual $\ast \Phi$ which are covariantly constant [3]. If we choose the local veilbein so that the metric is $\sum_{i=1}^{n} e_i \otimes e_i$, locally the $G_2$ invariant 3-form $\Phi$ can be written as [7] with

$$\Phi = e_1 \wedge e_2 \wedge e_7 + e_1 \wedge e_3 \wedge e_6 + e_1 \wedge e_4 \wedge e_5 - e_2 \wedge e_4 \wedge e_6 + e_3 \wedge e_4 \wedge e_7 + e_5 \wedge e_6 \wedge e_7. \quad (1)$$

A supersymmetric three-cycle $S$ in $\tilde{Y}$ is one for which the pullback $\Phi|_S$ of this threeform is a constant multiple of the volume element. This condition is written below as a partial differential equation in terms of Poisson brackets.

4 Differential equation for a supersymmetric cycle

Let us derive a differential equation defining a supersymmetric cycle $S$.

Introducing notations.

Let us choose $v, \bar{v}, x^3$ as coordinates on $S$. The 3-cycle $S$ is embedded in $R^7$ as

$$w = w(x^3, v, \bar{v}), \quad s = s(x^3, v, \bar{v}), \quad (2)$$

$$v = x^4 + ix^5, \quad w = x^7 + ix^8, \quad s = x^6 + ix^{10}.$$

Here functions $s$ and $w$ have asymptotic behavior of the form
\[
\begin{align*}
\text{and} & \\
& w(\pm\infty, \nu, \bar{\nu}) = e^{\pm \frac{2\pi i}{n} \zeta \nu^{-1}}, \\
& s(\pm\infty, \nu, \bar{\nu}) = -n \ln \nu,
\end{align*}
\]

and
\[
\begin{align*}
& w(-\infty, \nu, \bar{\nu}) = \zeta \nu^{-1}, \\
& s(-\infty, \nu, \bar{\nu}) = -n \ln \nu,
\end{align*}
\]

We denote \( x^6 = A, \ x^7 = C, \ x^8 = D, \ x^{10} = B \) with \( A, B, C \) and \( D \) being functions of \( x^3, x^4, x^5 \).

Let us set
\[
e_1 = dx^{10}, \ e_2 = dx^5, \ e_3 = dx^3, \ e_4 = dx^7, \ e_5 = dx^4, \ e_6 = dx^6, \ e_7 = dx^8,
\]

and denoted \( x^4 = y_1, \ x^5 = y_2, \ x^3 = y_3 \) then the form (3) will take the form
\[
\Phi = dB \wedge dy_2 \wedge dD + dB \wedge dy_3 \wedge dA + dB \wedge dC \wedge dy_1 - dy_2 \wedge dC \wedge dA
\]
\[
+ dy_3 \wedge dC \wedge dD + dy_1 \wedge dA \wedge dD + dy_2 \wedge dC \wedge dy_1.
\]

**Differential equation and boundary conditions.**

**Lemma A.** The surface \( S \) embedded in \( \mathbb{R}^7 \) as \( (4) \) is a supersymmetric three-cycle if \( A, B, C, D \) as functions of \( x^3, x^4, x^5 \) satisfy the following non-linear partial differential equation
\[
1 + \{ A, B \}_{(1,2)} + \{ C, D \}_{(1,2)} + \{ B, D \}_{(1,3)} +
\]
\[
\{ B, C \}_{(2,3)} + \{ C, A \}_{(1,3)} + \{ A, D \}_{(2,3)} = \sqrt{g},
\]

where we denoted \( x^4 = y_1, \ x^5 = y_2, \ x^3 = y_3 \) and defined the Poisson brackets as
\[
\{ f, g \}_{(i,j)} = \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j} - \frac{\partial g}{\partial y_i} \frac{\partial f}{\partial y_j}.
\]

Here \( f = f(y_1, y_2, y_3), \ g = g(y_1, y_2, y_3), \ i, j = 1, 2, 3 \) and \( g \) is the determinant of the induced metric \( g_{ij} = \delta_{ij} + \partial_i A \partial_j A + \partial_i B \partial_j B + \partial_i C \partial_j C + \partial_i D \partial_j D \)

We will consider here the case \( n = 2 \) and \( \zeta \) being a real number. Then the boundary conditions \( (3), (4) \) will take the form:
\[
C(y_1, y_2, \pm \infty) = \pm c(y_1, y_2), \quad c(y_1, y_2) = -\frac{y_1 \zeta}{y_1^2 + y_2^2},
\]
\[
D(y_1, y_2, \pm \infty) = \pm d(y_1, y_2), \quad d(y_1, y_2) = \frac{y_2 \zeta}{y_1^2 + y_2^2},
\]
\[
A(y_1, y_2, \pm \infty) = \pm a(y_1, y_2), \quad a(y_1, y_2) = -\ln (y_1^2 + y_2^2),
\]

\[\text{In the previous version of this paper the determinant was set equal to 1. This lead to an extra condition for the solution. I am grateful to J. Maldacena for pointing out this fact.}\]
\[ B(y_1, y_2, \pm\infty) = b(y_1, y_2), \quad b(y_1, y_2) = -2 \arctan \frac{y_2}{y_1} + \pi. \] (9)

**Proof.** The equation defining a supersymmetric three-cycle \( S \) is

\[ \Phi|_S = \sqrt{g} dx^3 \wedge dx^4 \wedge dx^5. \] (10)

Since we set \( \Phi \) the form \( \Phi \) takes the form of a sum of the Poisson brackets

\[ \Phi = (1 + \{A, B\}_{(1,2)} + \{C, D\}_{(1,2)} + \{B, D\}_{(1,3)} + \{B, C\}_{(2,3)} + \{A, D\}_{(2,3)}) dy_1 \wedge dy_2 \wedge dy_3. \] (11)

The condition for the surface to be a supersymmetric three-cycle is reduced to \( 7 \).

**Vacuum solutions.**

Let us check that the vacuum configurations satisfy equation \( 7 \).

**Lemma B.** The surface \( S_0 \) embedded in \( \mathbb{R}^7 \) as \( 2 \) with no dependence on \( x_3 \) and analytical functions \( w \) and \( s \), i.e. \( w = w(v) \), \( s = s(v) \) is a supersymmetric three-cycle. **Proof.** We denote \( A, B \) and \( C, D \) defining \( S_0 \) as \( A = a, B = b, C = c \) and \( D = d \). They satisfy the Cauchy-Riemann conditions

\[ \partial_1 a = \partial_2 b, \quad \partial_2 a = -\partial_1 b. \] (12)

For \( S_0 \) we have

\[ \Phi|_{S_0} = (1 + \{a, b\}_{(1,2)} + \{c, d\}_{(1,2)}) dy_1 \wedge dy_2 \wedge dy_3. \] (13)

Due to the Cauchy-Riemann conditions \( 12 \) we have

\[ 1 + \{a, b\}_{(1,2)} + \{c, d\}_{(1,2)} = 1 + (\partial_1 a)^2 + (\partial_2 a)^2 + (\partial_1 c)^2 + (\partial_2 c)^2. \] (14)

The induced metric on \( S_0 \) is

\[ (g_{ik})|_{S_0} = \begin{pmatrix} g^{(0)}_{11} & 0 & 0 \\ 0 & g^{(0)}_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(0)}_{11} = 1 + (\partial_1 a)^2 + (\partial_2 a)^2 + (\partial_1 c)^2 + (\partial_2 c)^2. \] (15)

The square root of the determinant of the induced metric on \( S_0 \) is

\[ \sqrt{g}|_{S_0} = 1 + (\partial_1 a)^2 + (\partial_2 a)^2 + (\partial_1 c)^2 + (\partial_2 c)^2, \]

and it coincides with \( 14 \), that proves the Lemma B.

**Ansatz.**

To solve equation \( 7 \) with boundary conditions \( 3 \) we will use an expansion into series over the powers of the function

\[ \gamma(y_3) = \frac{1}{e^{y_3} + e^{-y_3}}. \]
which vanishes in the limit $y_3 = x^3 \to \pm \infty$. This function satisfies $|\gamma(x)| < 1/2$.

The following ansatz will be used

\begin{align}
A &= a(y_1, y_2) + \sum_{k=1}^{\infty} \gamma^{2k}(y_3)a_{2k}(y_1, y_2), \\
B &= b(y_1, y_2) + \sum_{k=1}^{\infty} \gamma^{2k}(y_3)b_{2k}(y_1, y_2), \\
C &= \beta(y_3)c(y_1, y_2), \\
D &= \beta(y_3)d(y_1, y_2),
\end{align}

(16)-(19)

here

\[
\beta(y_3) = \frac{e^{y_3} - e^{-y_3}}{e^{y_3} + e^{-y_3}}.
\]

The boundary conditions (9) are satisfied for the above ansatz. In fact, the ansatz (16)-(19) can be used for arbitrary functions $a, b, c, d$ satisfying the Cauchy-Riemann conditions. In order to solve equations (7) we have to find functions $a_{2k}(y_1, y_2)$, $b_{2k}(y_1, y_2)$. Let us mention that in principle one can add to (18) and (19) series over $\gamma(y_3)$ as we did in (16) and (17). We fix here this arbitrariness to simplify calculations.

Calculations.

Due to a simple differential algebra

\[
\partial_3 \beta = 4\gamma^2, \quad \partial_3 \gamma^k = -k\beta \gamma^k, \quad \beta^2 = 1 - 4\gamma^2,
\]

(20)

we have the following expressions for combinations of Poisson brackets for the ansatz (16)-(19):

\[
\{C, A\}_{(1,3)} + \{A, D\}_{(2,3)} = 4\gamma^2(-c\partial_1 A + d\partial_2 A),
\]

(21)

\[
\{B, D\}_{(1,3)} + \{B, C\}_{(2,3)} = 4\gamma^2(c\partial_2 B + d\partial_1 B).
\]

(22)

Therefore, representing

\[
A = a + A, \quad B = b + B
\]

one can write

\[
\{C, A\}_{(1,3)} + \{A, D\}_{(2,3)} + \{B, D\}_{(1,3)} + \{B, C\}_{(2,3)} = 4\gamma^2(-c(\partial_1 A - \partial_2 B) + d(\partial_2 A + \partial_1 B)),
\]

(23)

i.e. this combination of the Poisson brackets starts from the 4-th order on $\gamma$, 

\[
\{C, A\}_{(1,3)} + \{A, D\}_{(2,3)} + \{B, D\}_{(1,3)} + \{B, C\}_{(2,3)} =
\]
One also has

\[
4\gamma^2 \sum_{k=1}^{\infty} \gamma^{2k}[-c(\partial_1 a_{2k} - \partial_2 b_{2k}) + d(\partial_2 a_{2k} + \partial_1 b_{2k})]
\]

We see that only the powers of \(\gamma^2\) enter the expression

\[
P \equiv 1 + \{C, D\}_{(1,2)} + \{A, B\}_{(1,2)} + \{C, A\}_{(1,3)} + \{A, D\}_{(2,3)} + \{B, D\}_{(1,3)} + \{B, C\}_{(2,3)}
\]

and in the lowermost orders on \(\gamma^2\) we have

\[
P = g_{11}^{(0)} - 4\gamma^2((\partial_1 c)^2 + (\partial_2 c)^2) + \gamma^2[\partial_1 a(\partial_2 b_2 + \partial_1 a_2) - \partial_2 a(\partial_1 b_2 - \partial_2 a_2)] + \\
+ \gamma^4[\partial_1 a(\partial_2 b_4 + \partial_1 a_4) - \partial_2 a(\partial_1 b_4 - \partial_2 a_4) + 4c(-\partial_1 a_2 + \partial_2 b_2) + 4d(\partial_2 a_2 + \partial_1 b_2) + \\
+ \partial_1 a_2 \partial_2 b_2 - \partial_2 a_2 \partial_1 b_2]
\]

Let us now examine the form of the determinant. Under ansatz (13)-(19) we have

\[
\begin{pmatrix}
1 + \Delta_{11}^{(0)} + \gamma^2 \Delta_{11}^{(1)} + \gamma^4 \Delta_{11}^{(2)} + (...) & \gamma^2 \Delta_{12}^{(1)} + \gamma^4 \Delta_{12}^{(2)} + (...) & \beta \gamma^2 (\Delta_{13}^{(1)} + \gamma^2 \Delta_{13}^{(2)} + ...) \\
\gamma^2 \Delta_{12}^{(1)} + \gamma^4 \Delta_{12}^{(2)} + (...) & 1 + \Delta_{12}^{(0)} + \gamma^2 \Delta_{22}^{(1)} + \gamma^4 \Delta_{22}^{(2)} + (...) & \beta \gamma^2 (\Delta_{13}^{(1)} + \gamma^2 \Delta_{13}^{(2)} + ...) \\
\beta \gamma^2 (\Delta_{23}^{(1)} + \gamma^2 \Delta_{23}^{(2)} + ...) & \beta \gamma^2 (\Delta_{23}^{(1)} + \gamma^2 \Delta_{23}^{(2)} + ...) & 1 + \gamma^4 \Delta_{33}^{(2)} + ...
\end{pmatrix}
\]

\[
\Delta_{11}^{(1)} = 2\partial_1 a \partial_1 a_2 + 2\partial_1 b \partial_1 b_2 - 4(\partial_1 c)^2 - 4(\partial_2 c)^2,
\]

\[
\Delta_{22}^{(1)} = 2\partial_2 a \partial_2 a_2 + 2\partial_2 b \partial_2 b_2 - 4(\partial_1 c)^2 - 4(\partial_2 c)^2,
\]

\[
\Delta_{12}^{(1)} = \partial_1 a \partial_2 a_2 + \partial_1 a \partial_2 a_2 + \partial_1 b \partial_1 a - \partial_2 a \partial_2 b_2,
\]

\[
\Delta_{13}^{(1)} = 4(c \partial_1 c + d \partial_1 d) - 2(a_2 \partial_1 a - b_2 \partial_1 a),
\]

\[
\Delta_{23}^{(1)} = 4(c \partial_2 c + d \partial_2 d) - 2(a_2 \partial_2 a + b_2 \partial_1 a),
\]

\[
\Delta_{33}^{(2)} = 16(c^2 + d^2) + 4(a_2^2 + b_2^2),
\]

\[
\Delta_{11}^{(2)} = 2\partial_1 a \partial_1 a_4 - (\partial_1 a_2)^2 + 2\partial_1 b_1 \partial_1 b_4 + (\partial_1 b_2)^2,
\]

\[
\Delta_{22}^{(2)} = 2\partial_2 a \partial_2 a_4 - (\partial_2 a_2)^2 + 2\partial_2 b_1 \partial_2 b_4 + (\partial_2 b_2)^2.
\]

Performing calculation of the determinant and expanding the result over powers of \(\gamma\) we see that it contains only the powers of \(\gamma^2\) and does not contain the terms proportional to \(\beta\). We have
\[ g = (g_{11}^{(0)})^2 + \gamma^2 g_{11}^{(0)} \rho_2 + \gamma^4 \rho_4 + \ldots \]  

(33)

where

\[ \rho_2 = \Delta_{11}^{(1)} + \Delta_{22}^{(1)}, \]

\[ \rho_4 = \Delta_{11}^{(1)} \Delta_{22}^{(1)} - \Delta_{12}^{(1)} \Delta_{12}^{(1)} - g_{11}^{(0)} (\Delta_{13}^{(1)} \Delta_{13}^{(1)} + \Delta_{23}^{(1)} \Delta_{23}^{(1)}) + g_{11}^{(0)} (\Delta_{11}^{(2)} + \Delta_{22}^{(2)}) + \Delta_{33}^{(2)} (g_{11}^{(0)})^2. \]

The condition for the surface to be a supersymmetric three-cycle will lead us to the following equation

\[ P^2 = g. \]  

(34)

Let us require vanishing of the coefficients in front of \( \gamma^k \) for all \( k \). This will lead us to the system of recursive differential equations for \( a_{2k}, b_{2k} \). The coefficients in front of \( \gamma^2 \) in the right and left hand sides of (34) vanish identically. In the fourth order we get an equation in which the terms with \( a_4 \) and \( b_4 \) are canceled and in the fourth order we left with nonlinear equation for \( a_2 \) and \( b_2 \). We can take one of them equal to zero or assume some relation between them.

Let us assume the Cauchy-Riemann conditions on \( A \) and \( B \) as functions of \( y_1, y_2 \). From equation (23) we see that if one assumes the Cauchy-Riemann conditions for \( A \) and \( B \) we will get that

\[ P\big|_{CR} = 1 + (\partial_1 A)^2 + (\partial_2 A)^2 + \beta^2 ((\partial_1 C)^2 + (\partial_2 C)^2). \]

(35)

The determinant also simplifies. We have

\[ g_{11} = g_{22} = 1 + (\partial_1 A)^2 + (\partial_2 A)^2 + (\partial_1 C)^2 + (\partial_2 C)^2, \]

(36)

\[ g_{12} = g_{21} = \partial_1 A \partial_2 A + \partial_1 B \partial_2 B + \partial_1 C \partial_2 C + \partial_1 D \partial_2 D = 0, \]

(37)

\[ g_{13} = g_{31} = \partial_1 A \partial_3 A + \partial_1 B \partial_3 B + \partial_1 C \partial_3 C + \partial_1 D \partial_3 D, \]

\[ g_{23} = g_{32} = \partial_2 A \partial_3 A + \partial_2 B \partial_3 B + \partial_2 C \partial_3 C + \partial_2 D \partial_3 D. \]

We notice that

\[ P = g_{11} = g_{22}, \]  

(38)

and the metric has the form

\[ (g_{ik})\big|_{CR} = \begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & g_{11} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}. \]

(39)

Therefore

\[ \text{det}(g_{ik})\big|_{CR} = g_{11} (g_{11} g_{33} - (g_{13}^2 + g_{23}^2)). \]

(40)

Hence the equation (4) under the assumption of analyticity has the form

\[ g_{11} (g_{33} - 1) = g_{13}^2 + g_{23}^2. \]

(41)
or explicitly,
\[
[\partial_1 A \partial_3 A - \partial_2 A \partial_3 B + \partial_1 C \partial_3 C - \partial_2 C \partial_3 D]^2 + [\partial_2 A \partial_3 A + \partial_1 A \partial_3 B + \partial_2 C \partial_3 C + \partial_1 C \partial_3 D]^2 =
\]
\[
[1 + (\partial_1 A)^2 + (\partial_2 A)^2 + (\partial_1 C)^2 + (\partial_2 C)^2] \cdot [(\partial_1 A)^2 + (\partial_3 B)^2 + (\partial_3 C)^2 + (\partial_3 D)^2] \tag{42}
\]

The form of equation (42) depends on the embedding of the surface \( S \) and also on the choice of the basis (33). We will consider now another nonlinear equation which is similar to (42) and for which we prove the existence of a solution with the prescribed asymptotic behavior as a series over \( \gamma \).

5 Solution of a nonlinear equation

Now instead of (10) we set
\[
e_1 = dx^6, e_2 = dx^4, e_3 = dx^5, e_4 = dx^7, e_5 = dx^3, e_6 = dx^8, e_7 = dx^{10}, \tag{43}
\]
then the form (11) will take the form
\[
\Phi = dA \wedge dx^4 \wedge dB + dA \wedge dx^5 \wedge dD + dA \wedge dC \wedge dx^3 + dx^4 \wedge dx^5 \wedge dx^3
\]
\[
- dx^4 \wedge dC \wedge dD + dx^5 \wedge dC \wedge dB + dx^3 \wedge dD \wedge dB. \tag{44}
\]
We consider now the condition \( \Phi|_S = dx^3 \wedge dx^4 \wedge dx^5 \). It can be written as
\[
\{A, B\}_{(3,2)} + \{A, D\}_{(1,3)} + \{A, C\}_{(1,2)} + \\
\{C, D\}_{(3,2)} + \{C, B\}_{(3,1)} + \{D, B\}_{(1,2)} = 0, \tag{45}
\]

To solve equation (44) with boundary conditions (40) we will use the same ansatz as below, see (46)-(49). We have
\[
\{A, D\}_{(3,1)} = -4d\partial_1 a \cdot \gamma^2 - 4\gamma^2 d \cdot \sum_{k=1}^{\infty} \gamma^{2k} \partial_1 a_{2k} - 2(1 - 4\gamma^2)\partial_1 d \cdot \sum_{k=1}^{\infty} k\gamma^{2k} a_{2k}, \tag{46}
\]
\[
\{A, B\}_{(3,2)} = -2\beta(\sum_{k=1}^{\infty} k\gamma^{2k} a_{2k})(\partial_2 b + \sum_{m=1}^{\infty} \gamma^{2m} \partial_2 b_{2m}) + \\
2\beta(\sum_{k=1}^{\infty} k\gamma^{2k} b_{2k})(\partial_2 a + \sum_{m=1}^{\infty} \gamma^{2m} \partial_2 a_{2m}), \tag{47}
\]
\[
\{A, C\}_{(1,2)} = \beta(\partial_1 a \partial_2 c - \partial_2 a \partial_1 c) + \beta \cdot \sum_{k=1}^{\infty} \gamma^{2k}(\partial_2 c \partial_1 a_{2k} - \partial_1 c \partial_2 a_{2k}), \tag{48}
\]
\[
\{C, D\}_{(3,2)} = 4\beta \gamma^2 (c \partial_2 d - d \partial_2 c), \tag{49}
\]
where \( \partial_i \) denotes \( \frac{\partial}{\partial y_i} \), \( i = 1, 2 \). One gets \( \{B, C\}_{(3,1)} \) from (46) after the substitution \( A \rightarrow B, \ a_k \rightarrow b_k, \ D \rightarrow C \) and \( \{B, D\}_{(1,2)} \) from (48) after the substitution \( A \rightarrow B, \ a_k \rightarrow b_k, \ C \rightarrow D \).
Only terms $\beta \gamma^{2k}$ and $\gamma^{2k}$ contain the dependence on $y_3$. Let us require vanishing of the coefficients in front of $\beta \gamma^{2k}$ and $\gamma^{2k}$ for all $k$. This will lead us to the following recursive system of linear equations for $a_{2k}(y_1, y_2)$, $b_{2k}(y_1, y_2)$

\begin{equation}
- \partial_2 c \cdot a_{2k} + \partial_1 c \cdot b_{2k} = \frac{1}{2k} \cdot P_k;
\end{equation}

\begin{equation}
- \partial_1 a \cdot a_{2k} + \partial_2 a \cdot b_{2k} - \frac{1}{2k}(D_1 a_{2k} + D_2 b_{2k}) = \frac{1}{2k} \cdot R_k,
\end{equation}

where $D_i, i = 1, 2$ are given by

\begin{equation}
D_1 = \partial_1 c \cdot \partial_2 - \partial_2 c \cdot \partial_1, \quad D_2 = \partial_1 c \cdot \partial_1 + \partial_2 c \cdot \partial_2,
\end{equation}

$P_k$ and $R_k$ depend only on $a_i, b_i, c_i, d_i$ with $i \leq 2k - 2$, namely

\begin{equation}
P_k = \delta_{k,1} P_1 - 4(c \partial_1 b_{2k-2} + d \partial_1 a_{2k-2}) + 8(k - 1)(\partial_1 c \cdot b_{2k-2} - \partial_2 c \cdot a_{2k-2}),
\end{equation}

\begin{equation}
P_1 = -4(c \partial_1 b + d \partial_1 a)
\end{equation}

and

\begin{equation}
R_k = \delta_{k,1} R_1 + 2 \sum_{s=1}^{k-1} s \cdot [a_{2s} \partial_2 b_{2k-2s} - b_{2s} \partial_2 a_{2k-2s}],
\end{equation}

$R_1 = -4(c \partial_2 d - d \partial_2 c)$.

Note that in our notations $a_0 = b_0 = 0$. If one finds $a_{2k}$ from (51) and substitutes to (51) then due to the Cauchy-Riemann conditions for $(a, b)$ and $(c, d)$ one gets the equation containing only derivative over $y_2$:

\begin{equation}
(\partial_2 + V_{2k})b_{2k} = L_{2k},
\end{equation}

where

\begin{equation}
V_{2k} = kW_1 + W_0;
\end{equation}

$W_1$ and $W_0$ do not depend on $k$:

\begin{equation}
W_1 = \frac{2(\partial_1 a \partial_1 c - \partial_2 a \partial_2 c)}{(\partial_1 c)^2 + (\partial_2 c)^2};
\end{equation}

\begin{equation}
W_0 = \frac{\partial_2(\partial_1 c/\partial_2 c) \cdot \partial_1 c - \partial_1(\partial_1 c/\partial_2 c) \cdot \partial_2 c}{(\partial_1 c)^2 + (\partial_2 c)^2} \cdot \partial_2 c,
\end{equation}

\begin{equation}
L_{2k} = \frac{\partial_2 c}{(\partial_1 c)^2 + (\partial_2 c)^2} \cdot [-R_k + P_k \partial_1 a \partial_2 c + \frac{1}{2k} D_1 P_k \partial_2 c - \frac{1}{2k} P_k \cdot D_1(\partial_2 c)].
\end{equation}

By substituting $a$ and $c$ from (9) to these formulae we’ll get the following recursive differential equation

\begin{equation}
\partial_2 b_{2k} - \left(\frac{4ky_1(y_1^2 - 3y_2^2)}{\zeta(y_1^2 + y_2^2)} + \frac{1}{y_2}\right)b_{2k} = L_{2k},
\end{equation}

where

\begin{equation}
L_{2k} = -\frac{2y_1 y_2}{\zeta} \cdot \left[R_k + y_1^2 + y_2^2 P_k - \frac{(y_1^2 - y_2^2)}{2y_1 y_2} \cdot \partial_2 P_k \frac{1}{2k} - \partial_1 P_k \frac{1}{2k} + \frac{y_1^2 + y_2^2}{y_1 y_2} \cdot P_k \right],
\end{equation}

9
\[ P_k = \frac{4 \zeta}{y_1^2 + y_2^2} \cdot (y_1 \partial_1 b_{2k-2} - y_2 \partial_1 a_{2k-2}) + \frac{8(k-1)\zeta}{(y_1^2 + y_2^2)^2} \cdot [(y_1^2 - y_2^2) \cdot b_{2k-2} - 2y_1y_2 \cdot a_{2k-2}], \quad (62) \]

\( k \neq 1 \) and \( P_k = 16y_1y_2\zeta/(y_1^2 + y_2^2)^2 \) and \( R_{2k} \) is given by (54) and \( R_1 = 4y_1\zeta^2/(y_1^2 + y_2^2)^2 \).

Let us demonstrate how this works, in the first (nontrivial) order, when the equations (50), (51) take the form:

\[ 2\partial_2 b \cdot a_2 - 2\partial_2 a \cdot b_2 + \{c, a_2\}_{(1,2)} + \{d, b_2\}_{(2,1)} = 4c\partial_1 c - 4d\partial_2 c, \quad (64) \]

\[ -\partial_2 c \cdot a_2 + \partial_1 c \cdot b_2 = -2c\partial_1 b - 2d\partial_1 a. \quad (65) \]

In accordance with (60), (61) and (62) we have

\[ \partial_2 b_2 - \left(\frac{4y_1(y_1^2 - 3y_2^2)}{\zeta(y_1^2 + y_2^2)} + \frac{1}{y_2}\right)b_2 = -\zeta \cdot \frac{8y_1^2y_2}{(y_1^2 + y_2^2)^2} - \frac{1}{\zeta} \cdot \frac{32y_1^2y_2}{y_1^2 + y_2^2}, \quad (66) \]

This is an ordinary differential equation that can be solved in quadratures.

We get the following

**Theorem.** Let \( \mathcal{S} \) be a three-surface:

\[ w(x^3, v) = \frac{\zeta}{v} \tanh x^3, \quad s(x^3, v, \bar{v}) = -2 \ln v + \sum_{k=1}^{\infty} 2^{2k}(\cosh x^3)^{-2k} g_{2k}(v, \bar{v}), \quad (67) \]

where \( g_{2k}(v, \bar{v}) = a_{2k}(y_1, y_2) + ib_{2k}(y_1, y_2), v = y_1 + iy_2, b_{2k} \) is a solution of recursive differential equations (61) and \( a_{2n} \) is given by (63) then \( \mathcal{S} \) satisfies \( \Phi_{\mathcal{S}} = dx^3 \wedge dx^4 \wedge dx^5 \) with the following boundary conditions: near \( x^3 = -\infty \), \( \mathcal{S} \) looks like \( \mathbb{R} \times \Sigma \), with \( \Sigma \) being the Riemann surface defined by \( w = \zeta v^{-1}, s = -2 \ln v \); near \( x^3 = +\infty \), \( \mathcal{S} \) looks like \( \mathbb{R} \times \Sigma' \) with \( \Sigma' \) being the Riemann surface defined by \( w = -\zeta v^{-1}, s = -2 \ln v \).

In this theorem we don’t discuss the convergence of the series, i.e. we have constructed the surface as a formal series. The problem of convergence will be discussed below.

**Remark 1. Large \( \zeta \)**

Equation (66) crucially simplifies if one sends \( \zeta \to \infty \). In fact sending \( \zeta \to \infty \), one gets the following equation for \( b_2 \):

\[ \partial_2 b_2 - \frac{b_2}{y_2} = -\zeta \cdot \frac{8y_1^2y_2}{(y_1^2 + y_2^2)^2}. \quad (68) \]
From this ordinary differential equation one can easily get the solution for $b_2$:

$$b_2 = -\zeta \cdot \frac{4y_2(y_1y_2 + (y_1^2 + y_2^2) \arctan y_2/y_1)}{y_1(y_1^2 + y_2^2)} + \zeta \cdot f(y_1)y_2,$$

(69)

here $f(y_1)$ is an arbitrary function. Therefore in this regime $b_2 \sim \zeta$. According to (63) $a_2 \sim \zeta$ and

$$a_2 = \frac{y_1^2 - y_2^2}{2y_1y_2} b_2.$$

(70)

Equation (60) for arbitrary $k > 2$ is also simplified and has the form

$$\partial_2 b_{2k} - \frac{1}{y_2} b_{2k} = L_{2k},$$

(71)

Let us show that $b_{2k}$ for large $\zeta$ is proportional to $\zeta$. For $k = 1$ we have just proved it. Under assumption that $b_{2r} \sim \zeta$ for $1 < r < k$, we get that $P_k \sim \zeta^2$, $R_k \sim \zeta^2$, $k > 1$, so we neglect the second term in the square brackets in (61) and we left with

$$L_{2k} = -\frac{2y_1y_2}{\zeta} \left[ R_k - \frac{(y_1^2 - y_2^2)}{2y_1y_2} \frac{\partial_2 P_k}{4k} - \frac{\partial_1 P_k}{2k} + \frac{y_1^2 - y_2^2}{y_1y_2^2} \frac{P_k}{k} \right],$$

(72)

i.e. $L_{2k} \sim \zeta$ and we get $b_{2k} \sim \zeta$ and $a_{2k} \sim \zeta$, $k > 0$.

Remark 2. Small $\zeta$

Let us consider the case when $\zeta \to 0$. First let us examine the behavior of $b_2$ for small $\zeta$. For small $\zeta$ we leave $1/\zeta$ terms in the (66) and we get an algebraic relation

$$b_2 = \frac{8y_1y_2}{y_1^2 - 3y_2^2},$$

(73)

here we assume that $y_1^2 \neq 3y_2^2$. Therefore we have

$$s(y_3, v, \dot{v}) = -2\ln v + \frac{4}{(e^{y_3} + e^{-y_3})^2} \cdot \frac{v\dot{v} - \dot{v}^2}{v^2 + \dot{v}^2 - v\dot{v}} + \ldots$$

(74)

For arbitrary $k$ in the left hand side of (51) one also leaves only $1/\zeta$ term

$$b_{2k} = \frac{y_2(y_1^2 + y_2^2)}{2k(y_1^2 - 3y_2^2)} \left[ R_k + \frac{y_1^2 + y_2^2}{\zeta y_2} P_k \right].$$

(75)

Since $P_k$ is proportional to $\zeta$ we have to keep the second term in the right hand side of (73) and we have that $b_{2k}$ and $a_{2k}$ do not depend on $\zeta$ for small $\zeta$.

Remark 3. On the convergence of the series

Let us make few comments about the convergence. For this purpose let us examine the behavior of the system (41) and (51) for the large $k$. For the large $k$ we can drop out the last two terms in the left hand side of (71) and get an algebraic system

$$- \partial_2 c \cdot a_{2k} + \partial_1 c \cdot b_{2k} = \frac{1}{2k} \cdot P_k;$$

(76)
\[- \partial_1 a \cdot a_{2k} + \partial_2 a \cdot b_{2k} = \frac{1}{2k} R_k. \quad (77)\]

For $k > r$ with $r$ being large enough we have from (55)

\[k W_1 b_{2k} = L_{2k}^{as}, \quad (78)\]

where

\[L_{2k}^{as} = \frac{\partial_2 c}{(\partial_1 c)^2 + (\partial_2 c)^2} \cdot \left[ - R_k + \frac{\partial_1 a}{\partial_2 c} \right]. \quad (79)\]

More explicitly this can be written as

\[- \frac{4ky_1(y_1^2 - 3y_2^2)}{\zeta(y_1^2 + y_2^2)} b_{2k} = L_{2k}^{as}, \quad (80)\]

where

\[L_{2k}^{as} = \frac{2y_1y_2}{\zeta} \cdot \left[ R_k + \frac{y_1^2 + y_2^2}{\zeta y_2} P_k \right]. \quad (81)\]

Note that for the large $k$ in the left hand side of (53) we can leave only the terms proportional to $k$ and this gives

\[P_k^{as} = 8k(\partial_1 c \cdot b_{2k-2} - \partial_2 c \cdot a_{2k-2}), \quad (82)\]

or from (62)

\[P_k^{as} = \frac{8k\zeta}{(y_1^2 + y_2^2)^2} \cdot \left[ (y_1^2 - y_2^2) \cdot b_{2k-2} - 2y_1y_2 \cdot a_{2k-2} \right], \quad (83)\]

Taking into account (51) we conclude that

\[- \partial_2 c \cdot a_{2k} + \partial_1 c \cdot b_{2k} = 4(\partial_2 c \cdot a_{2k-2} + \partial_1 c \cdot b_{2k-2}), \quad (84)\]

i.e.

\[- \partial_2 c \cdot a_{2k} + \partial_1 c \cdot b_{2k} = 4^k P, \quad (85)\]

where

\[P_r = \frac{1}{4r} \left( - \partial_2 c \cdot a_{2r} + \partial_1 c \cdot b_{2r} \right), \quad (86)\]

Therefore we have

\[P_k^{as} = 2k \cdot 4^k P. \quad (87)\]

Using (59), we have

\[a_{2k} = \frac{\partial_1 c}{\partial_2 c} b_{2k} - \frac{1}{\partial_2 c} \cdot 4^k P, \quad (88)\]

Using (54) one can represent the sum over $s$ in (54) as

\[R_k = -2\partial_2(\partial_1 c/\partial_2 c) \sum_{s=1}^{k-1} s b_{2s} b_{2k-2s} - \sum_{s=1}^{k-1} P_s \partial_2 b_{2k-2s} + \sum_{s=1}^{k-1} \frac{s}{k-s} b_{2s} \partial_2 P_{k-2s}. \quad (89)\]

In the second and the third sum we can use the asymptotic relation (87) for $s > r$ and $k - s > r$, respectively. So, schematically we can represent $R_k$ for the large $k$ as

\[R_k \sim \sum_{s=1}^{k-1} \left[ s\gamma b_{2s} b_{2k-2s} + \lambda_s \partial_2 b_{2k-2s} + \omega_s b_{2k-2s} \right], \quad (90)\]
\( \lambda_s \) and \( \omega_s \) depend only on the first \( r \) terms \( b_{2k} \) and \( a_{2k} \); \( \gamma = -2\partial_2 (\partial_1 c/\partial_2 c) \).

So we have to solve the following problem: to prove the convergence of series \( \sum_{k>r} b_{2k} \tau^k \) in some circle, \( |\tau| < \rho \) if we have the following recursive relations

\[
kW_1 b_{2k} = \frac{-2\partial_2 c}{(\partial_1 c)^2 + (\partial_2 c)^2} \left[ \sum_{s=r}^{k-1} s(\lambda_s b_{2s-2} + \omega_s b_{2s-2}) + \gamma s b_{2s} b_{2s-2} + \theta_k \right].
\]

(91)

with uniformly bounded functions \( \lambda_s(y_1, y_2) \), \( \omega(y_1, y_2) \) and \( \theta_s(y_1, y_2) \),

\[
|\lambda_s| < \lambda^*, \quad |\omega_s| < \omega^*, \quad |\theta_s| < \theta^*.
\]

Since we deal with series (16) and (17), in our case \( \tau = \gamma(y^3) \) and \( |\gamma| \leq 1/2 \).

To solve this problem it seems suitable to introduce a generating function

\[
\Psi((y_1, y_2, \tau)) = \sum_{k>1} b_{2k}(y_1, y_2) \tau^k.
\]

(92)

This generating function satisfies the following partial differential equation

\[
\partial_\tau \Psi(y_1, y_2, \tau) + \Gamma(y_1, y_2, \tau) \Psi(y_1, y_2, \tau) \partial_\tau \Psi(y_1, y_2, \tau) + \Lambda(y_1, y_2, \tau) \partial_{y_2} \Psi(y_1, y_2, \tau) + \Omega(y_1, y_2, \tau) \partial_{y_2} \Psi(y_1, y_2, \tau) = \Theta(y_1, y_2, \tau).
\]

(93)

The explicit formulae for \( \Gamma(y_1, y_2, \tau) \), \( \Omega_{y_2}(y_1, y_2, \tau) \), \( \Lambda(y_1, y_2, \tau) \) and \( \Theta(y_1, y_2, \tau) \) follow from equation (50), (88) and (91). We see that the generating function satisfies the nonlinear partial differential equation over \( y_2 \) and \( \tau \) of the first order. The variable \( y_1 \) enters here as a parameter. It is well known that under suitable conditions on \( \Omega, \Lambda, \Gamma \) and \( \Phi \) the Cauchy problem for equation (93) has an unique solution and the solution can be obtained by finding the first integrals of equations for characteristics.

6 Conclusion

In this note the equation for the domain wall in MQCD was derived and some properties of solution of this equation have been discussed. The problem of the rigorous proof of the existence of the domain wall solution of the equation requires a further consideration.

Recently a different approach was suggested in [8] where an explicit intersecting fivebrane configuration was found and it was interpreted as a domain wall in MQCD. Another approach to the problem of domain walls in MQCD has been explored in [9].

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References

[1] A. Hanany and E. Witten, "Type IIB Superstrings, BPS Monopoles, and Three-Dimensional Gauge Dynamics," Nucl. Phys. B492 (1997) 152.

[2] E. Witten, "Branes and the dynamics of QCD," hep-th/9706109.

[3] K. Becker, M. Becker, D. R. Morrison, H. Ooguri, Z. Yin, "Supersymmetric Cycles in Exceptional Holonomy Manifolds and Calabi-Yau Fourfolds," Nucl. Phys B480(1996)225.

[4] G. Dvali and M. Shifman, "Domain Walls In Strongly Coupled Theories", hep-th/9706089. A. Kovner, M. Shifman, and A. Smilga, "Domain Walls in Supersymmetric Yang-Mills Theories," hep-th/9706089.

[5] K. Becker, M. Becker and A. Strominger, "Fivebranes, membranes and nonperturbative string theory," Nucl. Phys. B456 (1995) 130.

[6] D. Joyce, "Compact Riemannian 7-Manifolds with Holonomy G_2, I;II", J. Diff. Geom. 43(1996)291;329.

[7] S. L. Shatashvili and C. Vafa, "Superstrings and Manifolds of Exceptional Holonomy", hep-th/9407025.

[8] A.Fayyazuddin and M.Spalinski, "Extended Objects in MQCD", hep-th/9711083

[9] J.Sonnenschein, S.Theisen, private communications.