Spectra for semiclassical operators with periodic bicharacteristics in dimension two

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Abstract

We study the distribution of eigenvalues for selfadjoint $h$–pseudodifferential operators in dimension two, arising as perturbations of selfadjoint operators with a periodic classical flow. When the strength $\varepsilon$ of the perturbation is $\ll h$, the spectrum displays a cluster structure, and assuming that $\varepsilon \gg h^2$ (or sometimes $\gg h^{N_0}$, for $N_0 > 1$ large), we obtain a complete asymptotic description of the individual eigenvalues inside subclusters, corresponding to the regular values of the leading symbol of the perturbation, averaged along the flow.

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The spectral theory of selfadjoint partial differential operators, whose associated classical flow is periodic, has a long and distinguished tradition, starting with the classical works of J. J. Duistermaat-V. Guillemin [9] and A. Weinstein [27], in the high energy limit, in the case of compact manifolds. Subsequently, many important contributions to the theory were given, [5], [12], [1], [28], and the case of semiclassical pseudodifferential operators was treated in [13], [8], [21]. In particular, assuming that the Hamilton flow is periodic in some energy shell, the cluster structure of the spectrum has been established in [15]. That work also contains some precise results concerning the semiclassical asymptotics for the counting function of eigenvalues in the clusters, with the celebrated Bohr-Sommerfeld quantization rule obtained as a special case in dimension one, see also [7]. Let us also remark that apart from their intrinsic interest in spectral theory, starting from the work [9], operators with periodic classical flow have frequently served as a source of examples of situations where various spectral estimates become optimal — see [2] for a recent manifestation of this in the context of uniform $L^p$ resolvent estimates for the Laplacian on a compact Zoll manifold.

The purpose of this paper is to show how the microlocal techniques of [16], [17], [18], developed in the context of analytic non-selfadjoint perturbations of selfadjoint operators with periodic classical flow in dimension two, apply to a class of selfadjoint operators of the form $P_\varepsilon = P(x, hD_x) + \varepsilon Q(x, hD_x)$. Here $P = P(x, hD_x)$ and $Q = Q(x, hD_x)$ are selfadjoint, with $P$ elliptic at infinity and with the classical flow of $P$ periodic in a band of energies around 0. It is then well-known, and will be recalled below, that the spectrum of $P$ near 0 exhibits a cluster structure, each cluster being of size $\leq O(h^2)$, and with the separation between the adjacent clusters
of size $h$. The parameter $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, measures the strength of the selfadjoint perturbation, and in order to have the clustering for the spectrum of $P_\varepsilon$, one should have $\varepsilon \ll h$. The general problem is then to understand the internal structure of the spectral clusters of the perturbed operator $P_\varepsilon$ in some detail, in the semiclassical limit $h \to 0$. In this work, assuming that $\varepsilon \gg h^{N_0}$, where $N_0 \geq 2$ depends on the size of the spectral clusters of $P$, we obtain semiclassical complete asymptotic expansions for the individual eigenvalues of $P_\varepsilon$ in subclusters, corresponding to regular values of the leading symbol of the perturbation $Q$, averaged along the classical flow. We remark that contrary to [16], [17], [18], no analyticity assumptions are needed here, and the spectral analysis is carried out within the framework of the standard $L^2$–spaces.

Let $M$ stand for $\mathbb{R}^2$ or a smooth compact 2–dimensional Riemannian manifold. When $M = \mathbb{R}^2$, we let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h), \quad 0 < h \ll 1,$$

(1.1)

be the $h$–Weyl quantization on $\mathbb{R}^2$ of a symbol $P(x, \xi, \varepsilon; h)$ depending smoothly on $\varepsilon \in \text{neigh}(0, \mathbb{R})$ taking values in the symbol class $S(m)$. Here $m$ is assumed to be an order function on $\mathbb{R}^4$ in the sense that $m > 0$ and

$$\exists C_0 \geq 1, N_0 > 0, \ m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y), \ X, Y \in \mathbb{R}^4.$$

(1.2)

The symbol class $S(m)$ is given by

$$S(m) = \{ a \in C^\infty(\mathbb{R}^4); \forall \alpha \in \mathbb{N}^4, \exists C_\alpha > 0, \forall X \in \mathbb{R}^4, \ |\partial_\alpha^a a(X)| \leq C_\alpha m(X) \}.$$

We shall assume throughout that

$$m \geq 1.$$

(1.3)

Assume furthermore that

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi, \varepsilon), \quad h \to 0,$$

(1.4)

in the space $S(m)$. We make the ellipticity assumption,

$$|p_0(x, \xi, \varepsilon)| \geq \frac{1}{C} m(x, \xi), \quad |(x, \xi)| \geq C,$$

(1.5)

for some $C > 0$. 

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When $M$ is a compact manifold, we first recall the standard class of semiclassical symbols on $T^*M$,

$$S^m(T^*M) = \left\{ a(x, \xi; h) \in C^\infty(T^*M \times (0,1]) : \left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi; h) \right| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|} \right\}.$$  

Associated to $S^m(T^*M)$ is the corresponding class of semiclassical pseudodifferential operators denoted by $L^m(M)$. Let $P_\varepsilon$ be a $C^\infty$ function of $\varepsilon \in \text{neigh}(0, \mathbb{R})$ with values in $L^m(M)$, $m > 0$. Let $\tilde{M} \subset M$ be a coordinate chart identified with a convex bounded domain in $\mathbb{R}^n$ in such a way that the Riemannian volume element $\mu(dx)$ reduces to the Lebesgue measure in $\tilde{M}$. We then have on $\tilde{M}$, for every $u \in C^\infty_0(\tilde{M})$,

$$P_\varepsilon u(x) = \frac{1}{(2\pi h)^2} \int \int e^{i \frac{(x-y) \cdot \xi}{h}} p \left( \frac{x+y}{2}, \xi, \varepsilon; h \right) u(y) dy d\xi + Ru(x). \quad (1.6)$$

Here $p(x, \xi, \varepsilon; h)$ is a smooth function of $\varepsilon$ with values in $S^m_{\text{loc}}(\tilde{M} \times \mathbb{R}^2)$, and $R$ is negligible in the sense that its Schwartz kernel $R(x,y)$ satisfies $\partial_\xi^\alpha \partial_y^\beta R(x,y) = O(h^\infty)$, for all $\alpha, \beta$. We further assume that the symbol $p(x, \xi, \varepsilon; h)$ has an asymptotic expansion in $S^m_{\text{loc}}(\tilde{M} \times \mathbb{R}^2)$, as $h \to 0$,

$$p(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi, \varepsilon), \quad p_j \in S^m_{\text{loc}}.$$

(1.7)

The semiclassical principal symbol of $P_\varepsilon$ in this case is given by $p_0(x, \xi, \varepsilon)$, and we make the ellipticity assumption,

$$|p_0(x, \xi, \varepsilon)| \geq \frac{1}{C} \langle \xi \rangle^{m}, \quad (x, \xi) \in T^*M, \quad |\xi| \geq C,$$

for some $C > 0$. Here we recall that since $M$ has been equipped with some Riemannian metric, $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are well defined. Let us also recall from [24] that while the complete symbol $p$ in (1.7) depends on the choice of local coordinates, the principal symbol $p_0$ and the subprincipal symbol $p_1$ are invariantly defined, provided that we use local coordinates in (1.6) for which the Riemannian volume density becomes equal to the Lebesgue measure.

In what follows, we shall write $p_\varepsilon$ for the principal symbol $p_0(x, \xi, \varepsilon)$ of $P_\varepsilon$, and simply $p$ for $p_0(x, \xi, \varepsilon = 0)$. We shall assume that for $\varepsilon \in \text{neigh}(0, \mathbb{R})$,

$$P_\varepsilon \text{ is formally selfadjoint.} \quad (1.8)$$

In the case when $M$ is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$.  

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For $h > 0$ small enough and when equipped with the domain $H(m)$, the naturally defined Sobolev space associated with the order function $m$ (so that in the compact case, $H(m)$ is the standard semiclassical Sobolev space $H^m(M)$), $P_\varepsilon$ becomes a well-defined selfadjoint operator on $L^2(M)$. Moreover, the assumptions above imply that the spectrum of $P_\varepsilon$ in a fixed neighborhood of 0 is discrete, for $h > 0$ and $\varepsilon \geq 0$ small enough.

We shall assume that the energy surface $p^{-1}(0) \subset T^*M$ is connected and that $dp \neq 0$ along $p^{-1}(0)$. Let $H_p = p_\xi \cdot \frac{\partial}{\partial \xi} - p_x \cdot \frac{\partial}{\partial x}$ be the Hamilton vector field of $p$. We introduce the following basic assumption, assumed to hold throughout this work: for $E \in \text{neigh}(0, \mathbb{R})$,

\begin{equation}
\text{The } H_{p_{\xi}}\text{-flow is periodic on } p^{-1}(E) \text{ with minimal period } T(E) > 0 \text{ depending smoothly on } E.
\end{equation}

Let $g : \text{neigh}(0, \mathbb{R}) \rightarrow \mathbb{R}$ be the smooth function defined by

\begin{equation}
g'(E) = \frac{T(E)}{2\pi}, \quad g(0) = 0,
\end{equation}

so that $g \circ p$ has a $2\pi$-periodic Hamilton flow. Set $f = g^{-1}$. We then have the following well known result, due to [13], following the earlier works [27], [5].

**Theorem 1.1** Assume that the subprincipal symbol of $P_0$ vanishes. Then the spectrum of $P_0$ near 0 is contained in the union of the intervals of the form

\begin{equation}
I_k = f(h(k - \theta)) + [-\mathcal{O}(h^2), \mathcal{O}(h^2)], \quad k \in \mathbb{Z},
\end{equation}

pairwise disjoint for $h > 0$ small enough. Here $\theta = \alpha_1/4 + S_1/2\pi h$, where $\alpha_1 \in \mathbb{Z}$ and $S_1 \in \mathbb{R}$ are the Maslov index and the classical action, respectively, computed along a closed $H_{p_{\xi}}$-trajectory $\subset p^{-1}(0)$, of period $T(0)$.

**Remark.** We refer to [22] for a self-contained discussion of Maslov indices of loops of Lagrangian subspaces and closed Hamiltonian trajectories.

**Remark.** Let us observe that up to a constant, the function $g$ in (1.10) is equal to $1/2\pi$ times the classical action along a closed $H_{p_{\xi}}$-orbit in $p^{-1}(E)$, $E \in \text{neigh}(0, \mathbb{R})$, see [7].

Let us write

\begin{equation}
p_\varepsilon = p + \varepsilon q + \mathcal{O}(\varepsilon^2 m),
\end{equation}

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in the case $M = \mathbb{R}^2$, and $p_\varepsilon = p + \varepsilon q + \mathcal{O}(\varepsilon^2 \langle \xi \rangle^m)$ in the compact case. Here $q$ is smooth and real-valued on $T^* M$. Let

$$
\langle q \rangle = \frac{1}{T(E)} \int_{-T(E)/2}^{T(E)/2} q \circ \exp (tH_p) \, dt \quad \text{on } p^{-1}(E),
$$

and notice that $H_p \langle q \rangle = 0$. In view of (1.9), the space of closed $H_p$–orbits in $p^{-1}(0)$,

$$
\Sigma = p^{-1}(0)/\exp (R H_p),
$$

is a 2-dimensional symplectic manifold, and $\langle q \rangle$ can naturally be viewed as a function on $\Sigma$.

The following is the main result of this work.

**Theorem 1.2** Let us assume that (1.9) holds and that the subprincipal symbol of $P_0$ vanishes. Let $F_0$ be a regular value of $\langle q \rangle$, considered as a function on $\Sigma$. Assume that $\langle q \rangle^{-1}(F_0) \subset \Sigma$ is a connected closed curve, and let us introduce the corresponding Lagrangian torus,

$$
\Lambda_{0,F_0} = \{ \rho \in T^* M; p(\rho) = 0, \langle q \rangle(\rho) = F_0 \}.
$$

When $\gamma_1$ and $\gamma_2$ are the fundamental cycles in $\Lambda_{0,F_0}$ with $\gamma_1$ being given by a closed $H_p$–trajectory of minimal period, we write $S = (S_1, S_2)$ and $\alpha = (\alpha_1, \alpha_2)$ for the actions and the Maslov indices of the cycles, respectively. Assume next that the spectrum of $P_0$ near 0 clusters into bands of size $\mathcal{O}(1)h^{N_0}$, for some $N_0 \geq 2$. Let us assume that

$$
h^{N_0} \ll \varepsilon \ll h.
$$

Let $C > 0$ be large enough. There exists a smooth function

$$
f(\xi_1; h) = f(\xi_1) + \sum_{j=2}^{N_0-1} h^j f_j(\xi_1), \quad \xi_1 \in \text{neigh}(0, \mathbb{R}),
$$

Figure 1: Spectral clusters for the unperturbed operator $P_0$. The size of each cluster is $\mathcal{O}(h^2)$, with the separation between adjacent clusters being of order $h$. 

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such that for each $k \in \mathbb{Z}$, with $h(k - \alpha_1/4) - S_1/2\pi$ small enough, the eigenvalues of $P_\varepsilon$ in the set
\[
|z - f\left( h(k - \alpha_1/4) - S_1/2\pi, h \right) - \varepsilon F_0| < \frac{\varepsilon}{C}
\] (1.15)
are given by
\[
\hat{P}\left( h(k - \alpha_1/4) - S_1/2\pi, h(\ell - \alpha_2/4) - S_2/2\pi, \varepsilon, \frac{h^{N_0}}{\varepsilon}; h \right) + \mathcal{O}(h^\infty), \quad \ell \in \mathbb{Z}. \tag{1.16}
\]
Here $\hat{P}(\xi,\varepsilon,h^{N_0}/\varepsilon;h)$ is smooth in $\xi \in \text{neigh}(0,\mathbb{R}^2)$, smooth in $\varepsilon, \frac{h^{N_0}}{\varepsilon} \in \text{neigh}(0,\mathbb{R})$, and has a complete asymptotic expansion in the space of such functions, as $h \to 0$,
\[
\hat{P}(\xi,\varepsilon,\frac{h^{N_0}}{\varepsilon};h) \sim f(\xi_1;h) + \varepsilon \left( r_0(\xi,\varepsilon,\frac{h^{N_0}}{\varepsilon}) + hr_1(\xi,\varepsilon,\frac{h^{N_0}}{\varepsilon}) + \ldots \right).
\]
We have
\[
r_0(\xi) = \langle q \rangle(\xi) + \mathcal{O}\left( \varepsilon + \frac{h^{N_0}}{\varepsilon} \right), \quad r_j = \mathcal{O}(1), \quad j \geq 1,
\]
corresponding to the action-angle coordinates near the Lagrangian torus $\Lambda_{0,F_0}$.

We have
\[
f(h(k - \theta)) \quad f(h(k - \theta)) + \varepsilon F_0
\]
\[
\sim \varepsilon h
\]
\[
\mathcal{O}(\varepsilon)
\]

Figure 2: Spectral asymptotics in a subcluster of the $k$th spectral cluster of the operator $P_\varepsilon$, corresponding to the regular value $F_0$ of the leading symbol of the perturbation, averaged along the classical flow. Here $h^2 \ll \varepsilon \ll h$. The red crosses represent the eigenvalues of $P_\varepsilon$ in (1.15), given by (1.16).

Remark. In the case when the compact manifold $\Lambda_{0,F_0}$ has several connected components, the result of Theorem 1.2 may be extended by showing that the set of eigenvalues $z$ in (1.15) agrees with the union of the spectral contributions coming from each of the connected components, modulo $\mathcal{O}(h^\infty)$, each contribution having the description as in the theorem. See also the discussion in Section 3 below.

Example. Let $M$ be a compact symmetric surface of rank one, and let $P_0 = -h^2 \Delta - 1$ on $M$. The assumption (1.9) then holds and from [12] we know that the spectrum
of $P_0$ clusters into bands of diameter 0 and separation of order $h$. Furthermore, the eigenvalues of $P_0 + 1$ depend quadratically on $h$, and we may conclude that the functions $f_j$ in (1.14) vanish, for $j > 2$, while $f_2$ is a constant. Taking $2 < N_0 \in \mathbb{N}$ to be any fixed integer, we see that the result of Theorem 1.2 applies to the Schrödinger operator $P_\epsilon = P_0 + \epsilon q = \hbar^2 (-\Delta + q) - 1$, where $\epsilon = \hbar^2$ and $q \in C^\infty(M)$ is real-valued, cf. with [27], [6], [10]. Let now $A$ be a smooth real-valued 1-form on $M$ and consider the magnetic Schrödinger operator on $M$ given by

$$P_\epsilon = \hbar^2 d_A^\ast d_A + \hbar^2 q - 1.$$  \hspace{1cm} (1.17)

Here $d_A u = du + iA \wedge u$, and $d_A^\ast$ is the Riemannian adjoint of $d_A$. Theorem 1.2 applies to the operator $P_\epsilon$ in (1.17) when $\epsilon = \lambda \hbar$, provided that $0 < \lambda \ll 1$ is sufficiently small but fixed. In this case, $q(x, \xi) = q_A(x, \xi) = 2\langle \xi, A^\sharp \rangle$, where $A^\sharp$ is the vector field associated to $A$ by means of the Riemannian metric. We may also remark if $B = dA$ is the magnetic field and $dA = d\tilde{A}$, then, since $H^1(M) = 0$, we have $\tilde{A} = A + d\varphi$, where $\varphi \in C^\infty(M)$ is real-valued. It follows that $\langle q_{\tilde{A}} - q_A \rangle = H_0^p \varphi$, where $p = \xi^2$ is the leading symbol of $P_0$, and therefore the flow average $\langle q_{\tilde{A}} \rangle = \langle q_A \rangle$ depends on the magnetic 2-form $B = dA$ only. See also Section 6 below.

**Remark.** It seems quite likely that the result of Theorem 1.2 can be extended to the case when $F_0$ is a non-degenerate critical value of $\langle q \rangle$, cf. with [6], [10]. We would also like to mention that the result of Theorem 1.2 can be viewed as a Bohr-Sommerfeld quantization condition in the spectral clusters, corresponding to the regular values for a reduced one-dimensional operator, and here there are some direct links with [4], [3], and the theory of Toeplitz operators on reduced compact symplectic spaces such as $\Sigma$. See also [14].

The plan of the paper is as follows. In Section 2, after re-deriving the clustering of the spectrum of $P_0$, we carry out an averaging reduction of $P_\epsilon$, microlocally in an energy shell. In Section 3, we microlocalize further to a suitable Lagrangian torus and construct a quantum Birkhoff normal form for $P_\epsilon$ near the torus, very much following the approach of [16]. In Section 4 we solve a suitable global Grushin problem for $P_\epsilon$ and identify the spectrum in the subclusters precisely, thereby completing the main part of the proof of Theorem 1.2. In Section 5, we complete the discussion by addressing the case when the spectral clusters of $P_0$ are of size $O(\hbar^{N_0})$, $N > 2$, and show how to reach smaller values of $\epsilon$ in Theorem 1.2 in this case. In Section 6, we finally give an application to the magnetic Schrödinger operator on $\mathbb{R}^2$ in the resonant case.

**Acknowledgements.** The first author was supported by the UCLA Dissertation Year Fellowship. The third author was supported by the grant NOSEVOL ANR 2011 BS 01019 01.
2 Clustering of eigenvalues and averaging reduction

For future reference, it will be convenient and natural for us to start by recalling a proof of Theorem 1.1 — see also Proposition 2.1 of [17]. When \( z \in \text{neigh}(0, \mathbb{R}) \), let us consider the equation

\[
(P_0 - z)u = v, \quad u \in H(m). \tag{2.1}
\]

Let \( \chi \in C_0^\infty(T^*M; [0,1]) \) be such that \( \chi = 1 \) near \( p^{-1}(0) \). Semiclassical elliptic regularity gives, with the \( L^2 \) norms throughout, that

\[
|| (1 - \chi) u || \leq \mathcal{O}(1)|| v || + \mathcal{O}(h^\infty)|| u ||, \tag{2.2}
\]

where \( \chi = \text{Op}_h^w(\chi) \) is the corresponding quantization. Here and in what follows, when \( M = \mathbb{R}^2 \), we use the \( h \)-Weyl quantization, while when \( M \) is compact, we fix the choice of the quantization map \( \text{Op}_h^w : S^m(T^*M) \to L^m(M) \), given by the Weyl quantization in special local coordinates as recalled in the introduction, with the associated symbol map: \( L^m(M) \to S^m(T^*M)/h^2S^{m-2}(T^*M) \).

Turning the attention to a neighborhood of \( p^{-1}(0) \), let \( \gamma \subset p^{-1}(0) \) be a closed \( H_{p^*} \) orbit, where we know that \( T(0) \) is the minimal period of \( \gamma \). From Section 3 of [16], we recall the following result.

**Proposition 2.1** There exists a smooth real-valued canonical transformation

\[
\kappa : \text{neigh}(\gamma, T^*M) \to \text{neigh}(\tau = x = \xi = 0, T^*(S^1 \times \mathbb{R}), \tag{2.3}
\]

mapping \( \gamma \) onto \( \{ \tau = x = \xi = 0 \} \), such that \( p \circ \kappa^{-1} = f(\tau) \). Here \( f \) has been defined in (1.10).

Following [16], we recall that the canonical transformation \( \kappa \) can be implemented by a multi-valued microlocally unitary \( h \)-Fourier integral operator \( U = \mathcal{O}(1) : L^2(M) \to L^2_f(S^1 \times \mathbb{R}) \), so that the improved Egorov property holds — see the discussion in Section 2 of [16]. Here \( L^2_f(S^1 \times \mathbb{R}) \) is the space of functions defined microlocally near \( \tau = x = \xi = 0 \) in \( T^*(S^1 \times \mathbb{R}) \), which satisfy the Floquet-Bloch periodicity condition,

\[
u(t - 2\pi, x) = e^{2\pi i \theta} u(t, x), \quad \theta = \frac{S_1}{2\pi h} + \frac{\alpha_1}{4}. \tag{2.4} \]
As explained in [16], the multi-valuedness of $U$ is a reflection of the fact that the domain of definition of the canonical transformation $\kappa$ is not simply connected, the corresponding first homotopy group being generated by the closed trajectory $\gamma$.

It follows that there exists a selfadjoint operator $\tilde{P}$ with the leading symbol $f(\tau)$ near $\{\tau = x = \xi = 0\}$ and with vanishing subprincipal symbol, so that $\tilde{P}U = UP_0$ microlocally near $\gamma$, so that

$$(\tilde{P}U - UP_0) \text{Op}_h^w(\chi_1) = \mathcal{O}(\hbar^\infty) : L^2(M) \to L^2(M),$$

(2.5)

and

$$\chi_2^w(x, hD_x)(\tilde{P}U - UP_0) = \mathcal{O}(\hbar^\infty) : L^2_f(S^1 \times \mathbb{R}) \to L^2_f(S^1 \times \mathbb{R}),$$

for every $\chi_1 \in C_0^\infty(\text{neigh}(\gamma, T^*M))$ and for every $\chi_2 \in C_0^\infty(T^*(S^1 \times \mathbb{R}))$ supported near $\tau = x = \xi = 0$. The operator $\tilde{P}$ acts on the space of functions satisfying the Floquet-Bloch condition (2.4), defined microlocally near $\tau = x = \xi = 0$ in $T^*(S^1 \times \mathbb{R})$.

Let us remark next that an orthonormal basis for the space $L^2_f(S^1)$ of functions $u \in L^2_{\text{loc}}(\mathbb{R})$ satisfying a Floquet-Bloch periodicity condition analogous to (2.4), with the $x$-variable suppressed, consists of the functions

$$e_k(t) = \frac{1}{\sqrt{2\pi}} \exp(i(k - \theta)t), \quad \theta = \frac{S_1}{2\pi \hbar} + \frac{\alpha_1}{4}, \quad k \in \mathbb{Z},$$

(2.6)

which satisfy $f(hD_t)e_k(t) = f(h(k - \theta))e_k(t)$. It follows that if $z \in \text{neigh}(0, \mathbb{R})$ is such that

$$|z - f(h(k - \theta))| \geq C\hbar^2, \quad k \in \mathbb{Z},$$

for $C > 1$ sufficiently large but fixed, then the operator

$$\tilde{P} - z = f(hD_t) + \hbar^2 R - z, \quad R = \mathcal{O}(1) : L^2_f(S^1 \times \mathbb{R}) \to L^2_f(S^1 \times \mathbb{R}),$$

is invertible, microlocally near $\tau = x = \xi = 0$, with the norm of the inverse being $\mathcal{O}(\hbar^{-2})$.

Let us now take finitely many closed $H_\rho$-trajectories $\gamma_1, \ldots, \gamma_N \subset p^{-1}(0)$ and small open neighborhoods $\Omega_j$ of $\gamma_j$, with $\Omega_j$ invariant under the $H_\rho$-flow, $1 \leq j \leq N$, such that $p^{-1}(0) \subseteq \bigcup \Omega_j$. Associated to this open cover, we take cutoff functions $0 \leq \chi_j \in C_0^\infty(\Omega_j)$ such that $H_\rho \chi_j = 0$ and $\sum \chi_j = 1$ near $p^{-1}(0)$. Let $U_j$ denote a multi-valued microlocally unitary $h$-Fourier integral operator associated to the canonical transformation near $\gamma_j$, as in Proposition 2.1.
For each \( j, 1 \leq j \leq N \), using (2.1), we see that
\[
(P_0 - z)\chi_j u + [\chi_j, P_0]u = \chi_j v, \tag{2.7}
\]
and therefore,
\[
U_j (P_0 - z)\chi_j u = (\bar{P} - z)U_j \chi_j u + \mathcal{O}(h^\infty)u = U_j \chi_j v + U_j [P_0, \chi_j]u. \tag{2.8}
\]
When \( z \in \text{neigh}(0, \mathbb{R}) \) avoids the intervals \( I_k \) in \((1.11)\), we just saw that the operator
\[
\bar{P} - z = f(hD_t) + h^2R_j - z,
\]
possesses a microlocal inverse of norm \( \mathcal{O}(1/h^2) \). Using (2.8), we conclude that
\[
\| \chi_j u \| \leq \mathcal{O}\left(\frac{1}{h^2}\right) \| v \| + \mathcal{O}\left(\frac{1}{h^2}\right) \| [P, \chi_j]u \| + \mathcal{O}(h^\infty)\| u \|. \tag{2.9}
\]
Since the subprincipal symbols of \( P_0 \) and \( \chi_j \) both vanish, we have in the operator sense, \([P_0, \chi_j] = \mathcal{O}(h^3)\) — see also \([11]\) for composition rules for the subprincipal symbols. Using (2.9) and summing over \( j \) we get
\[
\| \sum_{j=1}^N \chi_j u \| \leq \mathcal{O}\left(\frac{1}{h^2}\right) \| v \| + \mathcal{O}(h)\| u \|. \tag{2.10}
\]
Combining (2.2) and (2.10), we obtain
\[
\| u \| \leq \mathcal{O}\left(\frac{1}{h^2}\right) \| v \| + \mathcal{O}(h)\| u \|. \tag{2.11}
\]
Taking \( h \) small enough, we conclude that \((P_0 - z)^{-1}\) exists and satisfies \((P_0 - z)^{-1} = \mathcal{O}(1/h^2) : L^2 \to L^2\), when \( z \in \text{neigh}(0, \mathbb{R}) \) avoids the intervals \( I_k, k \in \mathbb{Z} \).

Having recalled a proof of Theorem 1.1, let us now proceed to carry out an averaging reduction for the selfadjoint operator \( P_\varepsilon \), replacing the leading symbol \( q \) of the perturbation by its average along closed orbits of the \( H_p \)-flow. We notice that such a reduction has a very long tradition \([27], [13]\), and the following discussion will be therefore somewhat brief.

Let \( G_0 \in C^\infty \) in a neighborhood of \( p^{-1}(0) \) be real-valued and such that
\[
H_p G_0 = q - \langle q \rangle, \tag{2.12}
\]
where \( \langle q \rangle \) is the flow average, defined in (1.13). As recalled in [16], we may take
\[
G_0 = \frac{1}{T(E)} \int_{-T(E)/2}^{T(E)/2} \left[ 1_{R_-}(t) \left( t + \frac{1}{2} T(E) \right) + 1_{R_+}(t) \left( t - \frac{1}{2} T(E) \right) \right] q \circ \exp(t H_p) \, dt,
\]
on \( p^{-1}(E) \). By a Taylor expansion, we then get
\[
p_\varepsilon \circ \exp(\varepsilon H_{G_0}) = p + \varepsilon \langle q - H_p G_0 \rangle + \mathcal{O}(\varepsilon^2) = p + \varepsilon \langle q \rangle + \mathcal{O}(\varepsilon^2).
\]
Similarly, with \( G_1, G_2, \ldots \) denoting a sequence of smooth real-valued functions to be determined, and \( G \sim \sum_{j=0}^{\infty} \varepsilon^j G_j \), if we expand \( p_\varepsilon \circ \exp(\varepsilon H_G) \) asymptotically, we claim that we can iteratively solve for \( G_j \) so that
\[
p_\varepsilon \circ \exp(\varepsilon H_G) = p + \varepsilon \langle q \rangle + \mathcal{O}(\varepsilon^2),
\]
where the \( \mathcal{O}(\varepsilon^2) \) error term is real-valued and Poisson commutes with \( p \), modulo \( \mathcal{O}(\varepsilon^\infty) \). Explicitly, if \( G_{\leq N} = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots + \varepsilon^N G_N \) satisfies
\[
p_\varepsilon \circ \exp(\varepsilon H_{G_{\leq N}}) = p + \varepsilon \langle q \rangle + \varepsilon^2 q_2 + \ldots + \varepsilon^{N+1} q_{N+1} + \varepsilon^{N+2} r_{N+2} + \mathcal{O}(\varepsilon^{N+3}),
\]
where \( q_j \) are real-valued and \( H_p q_j = 0, 2 \leq j \leq N + 1 \), then for \( G_{\leq N+1} = G_{\leq N} + \varepsilon^{N+1} G_{N+1} \), with \( G_{N+1} \in C^\infty \) to be determined, we have by a variation on the Campbell-Hausdorff formula [20],
\[
\exp(\varepsilon H_{G_{\leq N+1}}) = \exp(\varepsilon H_{G_{\leq N}} + \varepsilon^{N+2} H_{G_{N+1}}) = \exp(\varepsilon H_{G_{\leq N}}) \exp(\varepsilon^{N+2} H_{G_{N+1}})(1 + \mathcal{O}(\varepsilon^{N+3})), \quad (2.13)
\]
where the \( \mathcal{O}(\varepsilon^{N+3}) \)-bound is in the \( C^\infty \)-sense. This implies that
\[
p_\varepsilon \circ \exp(\varepsilon H_{G_{\leq N+1}}) = p + \varepsilon \langle q \rangle + \varepsilon^2 q_2 + \ldots + \varepsilon^{N+1} q_{N+1} + (r_{N+2} - H_p G_{N+1}) \varepsilon^{N+2} + \mathcal{O}(\varepsilon^{N+3}). \quad (2.14)
\]
As above, we may find a smooth real-valued solution of \( H_p G_{N+1} = r_{N+2} - \langle r_{N+2} \rangle \), defined near \( p^{-1}(0) \).

The functions \( G_j, j \geq 0 \), are defined in a fixed neighborhood of \( p^{-1}(0) \), and by Borel’s lemma we may choose \( G(x, \xi, \varepsilon) \in C^\infty \) near \( p^{-1}(0) \), smooth in \( \varepsilon \in \text{neigh}(0, \mathbb{R}) \), which is given by
\[
G \sim \sum_{j=0}^{\infty} \varepsilon^j G_j, \quad (2.15)
\]

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asymptotically in the $C^\infty$-sense. We have then achieved that $p_\varepsilon \circ \exp (\varepsilon H_G)$ is in involution with $p$ modulo $O(\varepsilon^\infty)$, in a fixed neighborhood of $p^{-1}(0)$, as desired.

Now an application of Cartan’s formula shows that the canonical transformation $\exp (\varepsilon H_G)$ is exact in the sense that the 1-form $(\exp (\varepsilon H_G))^* \lambda - \lambda$ is exact, where $\lambda$ is the fundamental 1-form on $T^*M$. By Egorov’s theorem, we may therefore quantize the real-valued smooth canonical transformation $\exp (\varepsilon H_G)$ by a (single-valued) $h$-Fourier integral operator $U_\varepsilon = O(1) : L^2(M) \to L^2(M)$, which is microlocally unitary near $p^{-1}(0)$. Then we have that the selfadjoint operator $\tilde{P}_\varepsilon := U_\varepsilon^{-1} P_\varepsilon U_\varepsilon$, defined microlocally near $p^{-1}(0)$, is such that its leading symbol is of the form $p + \varepsilon \langle q \rangle + O(\varepsilon^2)$, where the $O(\varepsilon^2)$ term is in involution with $p$, modulo $O(\varepsilon^\infty)$.

Furthermore, by the results of Section 2 of [16], we know that if we choose the principal symbol of the microlocally unitary Fourier integral operator $U_\varepsilon$ to be of constant argument, then $U_\varepsilon$ enjoys the improved Egorov property, so that on the level of symbols we have $\tilde{P}_\varepsilon = P_\varepsilon \circ \exp (\varepsilon H_G) + O(h^2)$. A natural choice of $U_\varepsilon$ is therefore given by $U_\varepsilon = e^{-i\varepsilon G/h}$, since then the principal symbol of $U_\varepsilon$ solves a real transport equation, using also that the subprincipal symbol of $G$ vanishes.

We summarize the discussion above in the following result.

**Proposition 2.2** There exists $G(x, \xi, \varepsilon) \in C^\infty(\text{neigh}(p^{-1}(0), T^*M))$ real-valued, depending smoothly on $\varepsilon \in \text{neigh}(0, \mathbb{R})$, with the asymptotic expansion (2.15) in the space of real-valued smooth functions in a fixed neighborhood of $p^{-1}(0)$, such that the microlocally defined selfadjoint operator

$$\tilde{P}_\varepsilon = e^{i\varepsilon G/h} P_\varepsilon e^{-i\varepsilon G/h}$$

depends on $\varepsilon$ in a $C^\infty$-fashion and has the leading symbol of the form $p + \varepsilon \langle q \rangle + O(\varepsilon^2)$, where the $O(\varepsilon^2)$-term Poisson commutes with $p$ modulo $O(\varepsilon^\infty)$. The subprincipal symbol of $\tilde{P}_\varepsilon$ is $O(\varepsilon)$. Assume furthermore that $\varepsilon \ll h$, so that the spectrum of $P_\varepsilon$ near 0 retains a cluster structure, being confined to the union of intervals

$$I_k(\varepsilon) = f(h(k-\theta)) + [-O(h^2 + \varepsilon), O(h^2 + \varepsilon)], \quad k \in \mathbb{Z}.$$  

If $z \in \text{Spec}(P_\varepsilon) \cap \text{neigh}(0, \mathbb{R})$ is such that $z \in I_k(\varepsilon)$, for some $k$, then we have

$$\varepsilon \min_{\text{neigh}(p^{-1}(0))} \langle q \rangle - O(\varepsilon^2 + h^2) \leq z - f(h(k-\theta)) \leq \varepsilon \max_{\text{neigh}(p^{-1}(0))} \langle q \rangle + O(\varepsilon^2 + h^2).$$

Here the last estimate follows by an application of sharp Gårding’s inequality.
3 Normal form near a Lagrangian torus

In Proposition 2.2, we have reduced ourselves to a microlocally defined selfadjoint operator $\tilde{P}_\varepsilon$, acting on $L^2(M)$, with the leading symbol of the form

$$p + \varepsilon \langle q \rangle + O(\varepsilon^2),$$

where the $O(\varepsilon^2)$-term Poisson commutes with $p$, modulo $O(\varepsilon^\infty)$. The subprincipal symbol of $\tilde{P}_\varepsilon$ is $O(\varepsilon)$. In what follows, when working with the operator $\tilde{P}_\varepsilon$, to simplify the notation, we shall drop the tilde and write $P_\varepsilon$ instead.

Let $F_0 \in \mathbb{R}$ be such that $\min_{p^{-1}(0)} \langle q \rangle < F_0 < \max_{p^{-1}(0)} \langle q \rangle$ and assume that $F_0$ is a regular value of $\langle q \rangle$, viewed as a function on the space of closed orbits $\Sigma$. After replacing $q$ by $q - F_0$ we may assume that $F_0 = 0$, and let us consider the $H_p$-flow invariant set

$$\Lambda_{0,0} = \{ \rho \in T^*M; p(\rho) = 0, \langle q \rangle (\rho) = 0 \}.$$

We know that $T(0)$ is the minimal period of all closed $H_p$-trajectories in $\Lambda_{0,0}$ and since $dp, d\langle q \rangle$ are linearly independent at each point of $\Lambda_{0,0}$, we see that $\Lambda_{0,0}$ is a Lagrangian manifold which is a union of finitely many tori. Assume for simplicity that $\Lambda_{0,0}$ is connected so that it is equal to a single Lagrangian torus. Since the functions $p, \langle q \rangle$ are in involution, they form a completely integrable system in a neighborhood of $\Lambda_{0,0}$. We have action-angle coordinates near $\Lambda_{0,0}$ [19], given by a smooth real-valued canonical transformation $\kappa: \text{neigh}(\xi = 0, T^*T^2) \to \text{neigh}(\Lambda_{0,0}, T^*M), \ T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2,$

(3.1)

mapping the zero section in $T^*T^2$ onto $\Lambda_{0,0}$, and such that $p \circ \kappa = p(\xi), \langle q \rangle \circ \kappa = \langle q \rangle (\xi)$. Here we make the identification $T^*T^2 \cong (\mathbb{R}/2\pi\mathbb{Z})^2 \times \mathbb{R}^2$. Since the classical flow of $p$ is periodic with minimal period $T(0)$ in $\Lambda_{0,0}$, we may and will choose $\kappa$ so that in fact $p \circ \kappa = p(\xi_1)$, by letting $\xi_1$ be the normalized action of a closed $H_p$-trajectory of minimal period — see the discussion in Section 4 of [16]. The linear independence of the differentials of $p$ and $\langle q \rangle$ implies that $p'(0) \neq 0, \partial_{\xi_2} \langle q \rangle (0) \neq 0$. We may also remark that when expressed in terms of the action coordinate $\xi_1$, the function $p$ becomes $p(\xi_1) = f(\xi_1)$, where the smooth function $f$ has been introduced after (1.10).

Implementing the real canonical transformation $\kappa$ in (3.1) by means of a multi-valued microlocally unitary $h$-Fourier integral operator $U: L^2(T^2) \to L^2(M)$, which also has the improved Egorov property [16], we get a new selfadjoint operator $U^{-1}P_\varepsilon U$, which for simplicity, will still be denoted by $P_\varepsilon$,

$$P_\varepsilon : L^2(T^2) \to L^2(T^2).$$

(3.2)
Here the operator $P_\varepsilon$ is defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$, with the full (Weyl) symbol of the form

$$P_\varepsilon \sim \sum_{j=0}^\infty \hbar^j p_j(x,\xi,\varepsilon),$$

(3.3)

the principal symbol being

$$p_0(x,\xi,\varepsilon) = p(\xi_1) + \varepsilon \langle q \rangle (\xi) + O(\varepsilon^2)$$

(3.4)

with the $O(\varepsilon^2)$ error term independent of $x_1$ modulo $O(\varepsilon^\infty)$. Furthermore,

$$p_1(x,\xi,\varepsilon) = O(\varepsilon),$$

and all the terms in the expansion (3.3) are smooth and real-valued. The dependence on $\varepsilon \in [0,\varepsilon_0]$ in (3.3) is still $C^\infty$. The space $L_2^2(\mathbb{T}^2)$ here stands for the subspace of $L^2_{\text{loc}}(\mathbb{R}^2)$ consisting of Floquet periodic functions $u(x)$, satisfying

$$u(x - \nu) = e^{i\nu \cdot \Theta} u(x), \quad \nu \in (2\pi \mathbb{Z})^2, \quad \Theta = \frac{S}{2\pi} + \frac{\alpha}{4}.$$

Here $S = (S_1, S_2)$ with $S_j$ being the action of the generator $\gamma_j$ of the first homotopy group of $\Lambda_{0,0}$, with $\gamma_1$ being given by a closed $H_p$-trajectory of minimal period, and $\alpha = (\alpha_1, \alpha_2)$ is the corresponding Maslov index.

### 3.1 Removing the $x_1$ dependence

Our first goal is to eliminate the $x_1$-dependence in $p_j$, $j \geq 1$, in (3.3). Let $A = A(x,\xi,\varepsilon) \in C^\infty$ be real-valued, $x \in \mathbb{T}^2$, $\xi \in \text{neigh}(0,\mathbb{R}^2)$, smooth in $\varepsilon \geq 0$, and let us consider the conjugation of the selfadjoint operator $P_\varepsilon$ by the unitary $h$-pseudodifferential operator $e^{iAw}$. We have

$$e^{-iA_w}P_\varepsilon e^{iA_w} = e^{-i\text{ad}A_w}P_\varepsilon = \sum_{k=0}^\infty \frac{(-i\text{ad}A_w)^k}{k!}P_\varepsilon, \quad (\text{ad}A_w)P_\varepsilon = [A_w, P_\varepsilon].$$

Identifying the symbols with the corresponding $h$-Weyl quantizations, we obtain that

$$e^{-iA_w}P_\varepsilon e^{iA_w} = P_\varepsilon + e^{-iA_w}[P_\varepsilon, e^{iA_w}]$$

$$= p_0(x,\xi,\varepsilon) + h(p_1(x,\xi,\varepsilon) + H_{p_0}A(x,\xi,\varepsilon)) + O(h^2).$$
We shall now show that $A$ can be chosen real-valued smooth, so that $p_1 + H_{p_0}A$ becomes independent of $x_1$, modulo $O(\varepsilon^\infty)$. In doing so, we shall construct the $C^\infty$-symbol $A$ as a formal power series in $\varepsilon$. Introducing the Taylor expansions,

$$p_0(x, \xi, \varepsilon) \sim \sum_{\ell=0}^{\infty} \varepsilon^\ell p_{0,\ell}(x, \xi), \quad p_1(x, \xi, \varepsilon) \sim \sum_{\ell=1}^{\infty} \varepsilon^\ell p_{1,\ell}(x, \xi),$$

and writing

$$A \sim \sum_{\ell=1}^{\infty} \varepsilon^\ell a_\ell(x, \xi),$$

we compute the power series expansion of the Poisson bracket,

$$H_{p_0}A \sim \sum_{k \geq 0, \ell \geq 1} \varepsilon^{k+\ell} \{p_{0,k}, a_\ell\} = \sum_{m=1}^{\infty} \varepsilon^m f_m,$$

where

$$f_m = \sum_{k + \ell = m, k \geq 0, \ell \geq 1} \{p_{0,k}, a_\ell\}.$$

We would like to choose the coefficients $a_\ell$, $\ell \geq 1$, so that $p_1, + f_\ell$ is independent of $x_1$, for all $\ell \geq 1$. When $\ell = 1$, we have $p_{1,1} + f_1 = p_{1,1} + \partial_\xi p \partial_{x_1} a_1$, and since $\partial_\xi p(0) \neq 0$, we can determine $a_1$ real-valued by solving the transport equation,

$$p_{1,1} + \partial_\xi p \partial_{x_1} a_1 = \langle p_{1,1}\rangle_{x_1},$$

the right hand side standing for the average of $p_{1,1}$ with respect to $x_1$. Arguing inductively, assume that the smooth real-valued functions $a_1, \ldots, a_m$ have already been determined. The term $p_{1,m+1} + f_{m+1}$ is of the form

$$p_{1,m+1} + \partial_\xi p \partial_{x_1} a_{m+1} + \sum_{k + \ell = m+1, \ell < m+1} \{p_{0,k}, a_\ell\},$$

and it is therefore clear that we can choose $a_{m+1}$ real, so that this expression becomes independent of $x_1$. Arguing in this fashion, we obtain a sequence $a_j \in C^\infty(\text{neigh}(\xi = 0, T^*T^2))$, $a_j$ real-valued, so that if $A \in C^\infty$ in all variables and real-valued, is such that

$$A(x, \xi, \varepsilon) \sim \sum_{j=1}^{\infty} \varepsilon^j a_j,$$
then the subprincipal symbol of the selfadjoint operator $e^{-iA^w}P_\varepsilon e^{iA^w}$ is $O(\varepsilon)$ and independent of $x_1$, modulo $O(\varepsilon^\infty)$.

Assume inductively that we have found $A_0 = A, \ldots, A_{N-1}$ real-valued so that the selfadjoint operator

$$P_\varepsilon^{(N)} := e^{-i\text{ad}(h^{N-1}A_{N-1})} \circ \ldots \circ e^{-i\text{ad}(hA_1)} \circ e^{-i\text{ad}(A)} P_\varepsilon$$

is of the form $\sim \sum_{j=0}^\infty h^j p_j$, where $p_j$ are independent of $x_1$ modulo $O(\varepsilon^\infty)$, for $j \leq N$. We then look for a conjugation by a unitary $h$-pseudodifferential operator of the form $e^{ih^N A^w_N}$, and we see as before that the leading symbol of $e^{-ih^N A^w_N} [P_\varepsilon^{(N)}, e^{ih^N A^w_N}]$ is $h^{N+1} H p_0 A_N$. Therefore, $e^{-ih^N A^w_N} P_\varepsilon^{(N)} e^{ih^N A^w_N}$ is of the form $\sim \sum_{j=0}^\infty h^j \tilde{p}_j$, where $\tilde{p}_j = p_j$ for $j \leq N$, and $\tilde{p}_{N+1} = p_{N+1} + H p_0 A_N$. It is therefore clear that we can determine $A_N$, real-valued and smooth, as a formal power series in $\varepsilon$, so that $\tilde{p}_{N+1}$ becomes independent of $x_1$, modulo $O(\varepsilon^\infty)$. Using the Campbell-Hausdorff formula and Borel’s lemma, we see that there exists a selfadjoint $h$-pseudodifferential operator $A$ with a real-valued symbol $\sim \sum_{\nu=0}^\infty h^{\nu} a_{\nu}(x, \xi, \varepsilon) \in \mathcal{S}(1)$, smooth in $\varepsilon \in [0, \varepsilon_0)$, with $a_0 = O(\varepsilon)$, such that

$$e^{-i\text{ad}A^w} \sim \ldots e^{-i\text{ad}A^w_N} \circ e^{-i\text{ad}(A^w_1)} \circ e^{-i\text{ad}(A^w_0)},$$

and we conclude that the selfadjoint operator $\tilde{P}_\varepsilon = e^{-i\text{ad}A^w} P_\varepsilon$ is of the form

$$\tilde{P}_\varepsilon = \sum_{j=0}^\infty h^j \tilde{p}_j(x_2, \xi, \varepsilon),$$

modulo $O((h, \varepsilon)^\infty)$. Here $\tilde{p}_j$ are real-valued, smooth in $\varepsilon \in [0, \varepsilon_0)$ and independent of $x_1$, with $\tilde{p}_0(x_2, \xi, \varepsilon) = p(\xi_1) + \varepsilon \langle q \rangle (\xi) + O(\varepsilon^2)$, and $\tilde{p}_1(x_2, \xi, \varepsilon) = O(\varepsilon)$.

### 3.2 Removing the $x_2$ dependence

In the previous subsection, using only the fact that $\partial_{\xi_1} p(0) \neq 0$, we have carried out repeated averagings along the $H_p$-flow, thus eliminating the $x_1$-dependence in the full symbol of $P_\varepsilon$ by means of a conjugation by an elliptic unitary $h$-pseudodifferential operator. We have reduced ourselves to a selfadjoint operator of the form,

$$\tilde{P}_\varepsilon \sim \sum_{j=0}^\infty h^j \tilde{p}_j(x_2, \xi, \varepsilon) \quad \text{on } L^2_j(T^2), \quad (3.5)$$

modulo $O((\varepsilon, h)^\infty)$, where $\tilde{p}_j$ in (3.5) are real-valued, smooth in $\varepsilon \in [0, \varepsilon_0)$, with

$$\tilde{p}_0(x_2, \xi, \varepsilon) = p(\xi_1) + \varepsilon \langle q \rangle (\xi) + O(\varepsilon^2)$$
with the $O(\varepsilon^2)$ error term independent of $x_1$. Also, $\tilde{p}_1(x_2, \xi, 0) = 0$.

Continuing to argue in the spirit of [16], [17], we shall now look for an additional conjugation by means of unitary $h$–Fourier integral operators, which eliminates the $x_2$-dependence in the full symbol in (3.5). Following [16], to that end it will be convenient to construct the conjugating operator by viewing $\varepsilon$ and $h^2/\varepsilon$ as two independent small parameters, provided that $\varepsilon$ is not too small.

On the level of symbols, we write, using that $\tilde{p}_1(x_2, \xi, \varepsilon) = \varepsilon q_1(x_2, \xi, \varepsilon)$, where $q_1$ is real-valued and $C^\infty$ in all variables,

$$\tilde{P}_\varepsilon = p(\xi_1) + \varepsilon \left( \langle q \rangle (\xi) + O(\varepsilon) + h q_1(x_2, \xi, \varepsilon) + \frac{h^2}{\varepsilon} \tilde{p}_2 + \frac{h^2}{\varepsilon} \tilde{p}_3 + \ldots \right)$$

$$= p(\xi_1) + \varepsilon \left( r_0(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) + h r_1(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) + h^2 r_2 + \ldots \right), \quad (3.6)$$

with

$$r_0(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) = \langle q \rangle (\xi) + O(\varepsilon) + \frac{h^2}{\varepsilon} \tilde{p}_2, \quad (3.7)$$

$$r_1(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) = q_1(x_2, \xi, \varepsilon) + \frac{h^2}{\varepsilon} \tilde{p}_3,$$

$$r_j(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) = \frac{h^2}{\varepsilon} \tilde{p}_{j+2}, \quad j \geq 2.$$

When eliminating the variable $x_2$, let us introduce the basic assumption that

$$\varepsilon = O(h^\delta), \quad \delta > 0, \quad (3.8)$$

and also, assume that

$$\frac{h^2}{\varepsilon} \leq \delta_0, \quad (3.9)$$

for some $\delta_0 > 0$ sufficiently small but independent of $h$. Replacing first (3.9) by the strengthened hypothesis,

$$\frac{h^2}{\varepsilon} \leq O(h^{\delta_1}), \quad \delta_1 > 0, \quad (3.10)$$

let us describe the construction of a unitary conjugation eliminating the $x_2$–dependence in $\tilde{P}_\varepsilon$.

When $b_0 = b_0(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon})$ is real-valued and smooth for $\xi \in \text{neigh}(0, \mathbb{R}^2)$, $\varepsilon$, $h^2/\varepsilon \in [0, \varepsilon_0)$, and is such that $b_0 = O(\varepsilon + h^2/\varepsilon)$ in the $C^\infty$–sense, we consider the selfadjoint operator

$$e^{\frac{i}{\varepsilon} B_0} \tilde{P}_\varepsilon e^{-\frac{i}{\varepsilon} B_0}, \quad B_0 = b_0^* (x_2, hD_x, \varepsilon, h^2/\varepsilon). \quad (3.11)$$
Since the commutator $[B_0, p(hD_{x_1})] = 0$, we see that the full symbol of the conjugated operator (3.11) is real-valued and of the form

$$p(\xi_1) + \varepsilon (\hat{r}_0 + h\hat{r}_1 + \ldots),$$

where by Egorov’s theorem,

$$\hat{r}_0 = r_0 \circ \exp (H_{b_0}) = \sum_{k=0}^{\infty} \frac{1}{k!} H_{b_0}^k r_0,$$

while $\hat{r}_j = O(1)$ for $j \geq 1$. Since the canonical transformation $\exp (H_{b_0})$ is exact, we see that the conjugated operator still acts on the space $L^2_T(T^2)$ of Floquet periodic functions.

It follows from (3.7) that

$$\hat{r}_0 = \langle q \rangle (\xi) + O\left(\varepsilon + \frac{h^2}{\varepsilon}\right) - \partial_{\xi_2} \langle q \rangle \partial_{x_2} b_0 + O\left(\varepsilon^2 + \frac{h^2}{\varepsilon}\right),$$

and using that $\partial_{\xi_2} \langle q \rangle \neq 0$, it becomes clear how to construct a real-valued smooth symbol $b_0 = O(\varepsilon + h^2/\varepsilon)$, defined near $\xi = 0$ in $T^*T^2$, smooth in $\varepsilon, h^2/\varepsilon \in \text{neigh}(0, \mathbb{R})$, as a formal Taylor series in $\varepsilon, h^2/\varepsilon$, so that $\hat{r}_0 = \langle q \rangle + O(\varepsilon + h^2/\varepsilon)$ is independent of $x$, modulo $O(h^\infty)$, in view of (3.8), (3.10).

Dropping the assumption (3.10), we now come to discuss the construction of a conjugating Fourier integral operator when only (3.9) is valid. Following [16], we consider the eikonal equation for $\varphi = \varphi(x_2, \xi, \varepsilon, h^2/\varepsilon)$,

$$r_0 \left( x_2, \xi_1, \xi_2 + \partial_{x_2} \varphi, \varepsilon, \frac{h^2}{\varepsilon} \right) = \langle r_0(\cdot, \xi, \varepsilon, \frac{h^2}{\varepsilon}) \rangle,$$

(3.12)

where $\langle \cdot \rangle$ in the right hand side stands for the average with respect to $x_2$. Since $\partial_{\xi_2} \langle q \rangle \neq 0$, by Hamilton-Jacobi theory, (3.12) has a smooth real-valued solution with $\partial_{x_2} \varphi$ single-valued and

$$\partial_{x_2} \varphi = O\left(\varepsilon + \frac{h^2}{\varepsilon}\right).$$

Taylor expanding (3.12) and using that

$$\partial_{\xi_2} r_0 \left( x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon} \right) = \partial_{\xi_2} \langle q \rangle (\xi) + O\left(\varepsilon + \frac{h^2}{\varepsilon}\right),$$

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we get
\[ \varphi = \varphi_{\text{per}} + x_2 \zeta_2, \]
where \( \varphi_{\text{per}} = O(\varepsilon + h^2/\varepsilon) \) is periodic in \( x_2 \) and \( \zeta_2(\xi, \varepsilon, h^2/\varepsilon) = O((\varepsilon, h^2/\varepsilon)^2). \)

Let us set
\[ \eta = \eta\left( \xi, \varepsilon, \frac{h^2}{\varepsilon} \right) = (\xi_1, \xi_2 + \zeta_2), \]
and
\[ \psi\left( x_2, \eta, \varepsilon, \frac{h^2}{\varepsilon} \right) = \varphi_{\text{per}} + x \cdot \eta, \]
where \( \varphi_{\text{per}} \) is viewed as a function of \( \eta \) rather than \( \xi \). Associated to the function \( \psi \) is the real-valued smooth canonical transformation
\[ \kappa : (\psi', \eta) \to (x, \psi'_x), \tag{3.13} \]
which is an \( O(\varepsilon + h^2/\varepsilon) \)–perturbation of the identity in the \( C^\infty \)–sense, and such that if \( (x, \xi) = \kappa(y, \eta) \), then \( \xi_1 = \eta_1 \). We have by construction,
\[ (r_0 \circ \kappa)(y, \eta, \varepsilon, h^2\varepsilon) = r_0(x, \psi'_x, \varepsilon, h^2/\varepsilon) = \langle r_0(\cdot, \xi, \varepsilon, \frac{h^2}{\varepsilon}) \rangle \]
\[ = \langle r_0(\cdot, \eta, \varepsilon, \frac{h^2}{\varepsilon}) \rangle + O\left( (\varepsilon, h^2/\varepsilon)^2 \right), \]
which is a function of \( (y, \eta) \), independent of \( y \).

We can quantize the canonical transformation \( \kappa \) in (3.13) by a microlocally unitary Fourier integral operator, and after conjugation by this operator, we obtain a new operator, still denoted by \( \tilde{P}_\varepsilon \), which is of the form (3.6), where
\[ r_0 = \langle q \rangle(\xi) + O\left( \varepsilon + \frac{h^2}{\varepsilon} \right) \]
is independent of \( x \), and \( r_j = O(1) \) in the \( C^\infty \)–sense, for \( j \geq 1 \). Furthermore, as explained in Section 4 of [16], the conjugated operator \( \tilde{P}_\varepsilon \) still acts on the space \( L^2_f(\mathbb{T}^2) \) of Floquet periodic functions.

Let us consider therefore an operator of the form
\[ \tilde{P}_\varepsilon = p(\xi_1) + \varepsilon \left( r_0(\xi, \varepsilon, \frac{h^2}{\varepsilon}) + hr_1(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) + \ldots \right), \]
where $r_0 = \langle q \rangle(\xi) + \mathcal{O}(\varepsilon + h^2/\varepsilon)$ is independent of $x$, and $r_j = \mathcal{O}(1)$, $j \geq 1$. Furthermore, all the terms $r_j$ are real-valued, smooth, and depend smoothly on $\varepsilon$, $h^2/\varepsilon \in \text{neigh}(0, R)$. To eliminate the $x_2$-dependence in the lower order terms $r_j$, $j \geq 1$, we could argue as in the previous step, making the terms $r_j$ independent of $x_2$ one at a time, but here we would like to describe a slightly different method, which has the merit of being more direct. Let us look for a conjugation by an elliptic unitary pseudodifferential operator of the form $e^{iB/h}$, where

$$B(x_2, \xi, \varepsilon, h) = \sum_{\nu=1}^{\infty} h^\nu b_\nu(x_2, \xi, \varepsilon, h^2/\varepsilon).$$

Here $b_\nu$ are real-valued smooth and depend smoothly on $\varepsilon$, $h^2/\varepsilon \in \text{neigh}(0, R)$. To eliminate the $x_2$-dependence in the lower order terms $r_j$, $j \geq 1$, we could argue as in the previous step, making the terms $r_j$ independent of $x_2$ one at a time, but here we would like to describe a slightly different method, which has the merit of being more direct. Let us look for a conjugation by an elliptic unitary pseudodifferential operator of the form $e^{iB/h}$, where

$$B(x_2, \xi, \varepsilon, h) = \sum_{\nu=1}^{\infty} h^\nu b_\nu(x_2, \xi, \varepsilon, h^2/\varepsilon).$$

Here $b_\nu$ are real-valued smooth and depend smoothly on $\varepsilon$, $h^2/\varepsilon \in \text{neigh}(0, R)$. The conjugated operator

$$e^{iB} \tilde{P}_\varepsilon e^{-iB} = e^{i\text{ad} B} \tilde{P}_\varepsilon = \sum_{k=0}^{\infty} \frac{(i\text{ad} B)^k}{h^k k!} \tilde{P}_\varepsilon$$

is selfadjoint and can be expanded as follows,

$$p(\xi_1) + \varepsilon \sum_{k=0}^{\infty} \sum_{j_1=1}^{\infty} \ldots \sum_{j_k=1}^{\infty} \sum_{\ell=0}^{k} h^{\ell+j_1+\ldots+j_k} \frac{1}{k!} \left( \frac{i}{h} \text{ad} b_{j_1} \right) \ldots \left( \frac{i}{h} \text{ad} b_{j_k} \right) r_{\ell}$$

$$= p(\xi_1) + \varepsilon \sum_{n=0}^{\infty} h^n \tilde{r}_n. \quad (3.14)$$

Here $\tilde{r}_n$ is equal to the sum of all the coefficients for $h^n$ coming from the expressions

$$h^{\ell+j_1+\ldots+j_k} \frac{1}{k!} \left( \frac{i}{h} \text{ad} b_{j_1} \right) \ldots \left( \frac{i}{h} \text{ad} b_{j_k} \right) r_{\ell}, \quad (3.15)$$

with $\ell + j_1 + \ldots + j_k \leq n$ and $j_\nu \geq 1$. In particular, we see that $\tilde{r}_n$ are all real-valued, thanks to the observation that if $A, B$ are selfadjoint, then so is the operator $i[A, B] = (i\text{ad} A)B$. Then $\tilde{r}_0 = r_0, \tilde{r}_1 = r_1 + H b_1 r_0 = r_1 - H b_1, \ldots, \tilde{r}_n = r_n - H b_n + s_n$, where $s_n$ only depends on $b_1, \ldots, b_{n-1}$ and is the sum of all coefficients of $h^n$ arising in the expressions (3.15) with $\ell + j_1 + \ldots + j_k \leq n, j_1, \ldots, j_k, \ell < n, j_\nu \geq 1$.

It is therefore clear how to find $b_1, b_2, \ldots$ real-valued smooth, successively, with $b_j = \mathcal{O}(1)$, such that all the coefficients $\tilde{r}_j$ in (3.14) are independent of $x$ and $= \mathcal{O}(1)$.

The discussion in this section may be summarized in the following theorem.
Theorem 3.1 Let us make all the general assumptions of Section 1 and let $F_0 \in \mathbb{R}$ be a regular value of $\langle q \rangle$, viewed as a function on the space of closed orbits $\Sigma$. Assume that the Lagrangian manifold

$$\Lambda_{0,F_0} : p = 0, \quad \langle q \rangle = F_0$$

is connected. When $\gamma_1$ and $\gamma_2$ are the fundamental cycles in $\Lambda_{0,F_0}$ with $\gamma_1$ corresponding to a closed $H_p$-trajectory of minimal period, we write $S = (S_1, S_2)$ and $\alpha = (\alpha_1, \alpha_2)$ for the actions and the Maslov indices of the cycles, respectively. Assume furthermore that $\varepsilon = O(h^\delta)$, $\delta > 0$, is such that $h^2/\varepsilon \leq \delta_0$, for some $\delta_0 > 0$ sufficiently small but fixed. There exists a smooth Lagrangian torus $\tilde{\Lambda}_{0,F_0} \subset T^*M$, which is an $O(\varepsilon + h^2/\varepsilon)$–perturbation of $\Lambda_{0,F_0}$ in the $C^\infty$–sense, such that when $\rho \in T^*M$ is away from a small neighborhood of $\tilde{\Lambda}_{0,F_0}$ and $|p(\rho)| \leq 1/C$, for $C > 0$ sufficiently large, we have

$$|\langle q \rangle(\rho) - F_0| \geq \frac{1}{O(1)}.$$

There exists a $C^\infty$ real-valued canonical transformation

$$\kappa : \text{neigh}(\tilde{\Lambda}_{0,F_0}, T^*M) \to \text{neigh}(\xi = 0, T^*T^2),$$

mapping to $\tilde{\Lambda}_{0,F_0}$ to $\xi = 0$, and a corresponding uniformly bounded $h$-Fourier integral operator

$$U = O(1) : L^2(M) \to L^2_f(T^2),$$

which has the following properties:

1. The operator $U$ is microlocally unitary near $\tilde{\Lambda}_{0,F_0}$: if $U^* = O(1) : L^2_f(T^2) \to L^2(M)$ is the complex adjoint, then for every $\chi_1 \in C^\infty_0(\text{neigh}(\tilde{\Lambda}_{0,F_0}, T^*M))$, we have

$$\left(U^*U - 1\right) \text{Op}_h^w(\chi_1) = O(h^\infty) : L^2(M) \to L^2(M). \quad (3.16)$$

For every $\chi_2 \in C^\infty_0(\text{neigh}(\xi = 0, T^*T^2))$, we have

$$\left(UU^* - 1\right) \chi_2^w(x, hD_x) = O(h^\infty) : L^2_f(T^2) \to L^2_f(T^2).$$

2. We have a normal form for $P_\varepsilon$: Acting on $L^2_f(T^2)$, there exists a selfadjoint operator $\hat{P}(hD_x, \varepsilon, \frac{h^2}{\varepsilon}; h)$ with the symbol

$$\hat{P}(\xi, \varepsilon, \frac{h^2}{\varepsilon}; h) \sim p(\xi) + \varepsilon \sum_{j=0}^{\infty} h^j r_j \left(\xi, \varepsilon, \frac{h^2}{\varepsilon}\right), \quad |\xi| \leq \frac{1}{O(1)},$$

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smooth in $\xi \in \text{neigh}(0, \mathbb{R}^2)$, and smooth in $\varepsilon$, $h^2/\varepsilon \in \text{neigh}(0, \mathbb{R})$, such that

$$r_0 = \langle q \rangle(\xi) + \mathcal{O}\left(\varepsilon + \frac{h^2}{\varepsilon}\right),$$

and

$$r_j = \mathcal{O}(1), \quad j \geq 1,$$

and such that $\hat{P}U = UP_\varepsilon$ microlocally near $\hat{\Lambda}_{0,F_0}$, i.e.

$$\left(\hat{P}U - UP_\varepsilon\right) \text{Op}_\hbar^w(\chi_1) = \mathcal{O}(h^\infty), \quad \chi_2^w(x, hD_x) \left(\hat{P}U - UP_\varepsilon\right) = \mathcal{O}(h^\infty),$$

in the operator sense, for every $\chi_1, \chi_2$ as in $1$).

## 4 Eigenvalue asymptotics in subclusters

Throughout this section, we shall assume that $\varepsilon \ll h$ and that the lower bound $h^2/\varepsilon \leq \delta_0 \ll 1$ is valid. We then know that Theorem 3.1 applies and that the spectrum of $P_\varepsilon$ near 0 is confined to the union of intervals,

$$I_k(\varepsilon) = f(h(k - \theta)) + [-\mathcal{O}(\varepsilon), \mathcal{O}(\varepsilon)], \quad k \in \mathbb{Z}, \quad \theta = \frac{S_1}{2\pi h} + \frac{\alpha_1}{4},$$

disjoint for all $h > 0$ small enough.

When proving Theorem 1.2, following [17], let us first check that if $z \in \text{neigh}(0, \mathbb{R})$ is such that

$$|z - f(h(k - \theta)) - \varepsilon F_0| \leq \frac{\varepsilon}{C}, \quad C \gg 1,$$

for some $k \in \mathbb{Z}$, and $z$ avoids the union of the pairwise disjoint open intervals $J_\ell(h)$ of length $\varepsilon h/\mathcal{O}(1)$, that are centered at the quasi–eigenvalues

$$\hat{P}\left(h(k - \frac{\alpha_1}{4}) - \frac{S_1}{2\pi}, h(\ell - \frac{\alpha_2}{4}) - \frac{S_2}{2\pi}, \varepsilon, \frac{h^2}{\varepsilon}; h\right),$$

for $\ell \in \mathbb{Z}$, then the operator

$$P_\varepsilon - z : H(m) \to L^2(M)$$

is bijective.
To that end, consider a partition of unity on $T^*M$, 
\[ 1 = \chi + \psi_{1,+} + \psi_{1,-} + \psi_{2,+} + \psi_{2,-}. \] (4.3)

Here $\chi \in C^\infty_0(T^*M)$ is supported in a small flow invariant neighborhood of $\hat{\Lambda}_{0,F_0}$ where the operator $U$ of Theorem 3.1 is defined and unitary, and where $P_\varepsilon$ is intertwined with $\hat{P}$, and $\chi = 1$ near $\hat{\Lambda}_{0,F_0}$. Thanks to Theorem 3.1, we also assume, as we may, that on the operator level,
\[ [P_\varepsilon, \chi] = \mathcal{O}(h^\infty) : L^2 \to L^2. \] (4.4)

Furthermore, the functions $\psi_{1,\pm} \in C^\infty_0(T^*M)$ are supported in flow invariant regions $\Omega_{\pm}$, such that $\pm(\langle q \rangle - F_0) \geq 1/\mathcal{O}(1)$ in $\Omega_{\pm}$, respectively. Moreover, we can arrange so that $\psi_{1,\pm}$ are in involution with $p$, the principal symbol of $P_{\varepsilon=0}$. Finally, $\psi_{2,\pm} \in C^\infty_0(T^*M)$ are such that $\pm p > 1/\mathcal{O}(1)$ in the support of $\psi_{2,\pm}$.

Let us consider the equation,
\[ (P_\varepsilon - z)u = v, \quad u \in H(m), \]
when $z \in \text{neigh}(0, \mathbb{R})$ satisfies (4.1) for some $k \in \mathbb{Z}$. We then claim that, with the norms taken in $L^2$,
\[ ||(1 - \chi)u|| \leq \mathcal{O}\left(\frac{1}{\varepsilon}\right) ||v|| + \mathcal{O}(h^\infty)||u||. \] (4.5)

When establishing (4.5), we only have to prove this bound with $\psi_{1,\pm}$ in place of $1 - \chi$, as the estimate involving $\psi_{2,\pm}$ follows from the semiclassical elliptic regularity.

Let $\gamma \subset p^{-1}(0)$ be a closed $H_p$-orbit away from $\hat{\Lambda}_{0,F_0}$, and assume, to fix the ideas, that $\langle q \rangle \geq F_0 + 1/C$ near $\gamma$. Let $\psi, \tilde{\psi} \in C^\infty_0$ be supported in a small flow-invariant neighborhood of $\gamma$ and assume that $H_p\psi = H_p\tilde{\psi} = 0$ and that $\tilde{\psi} = 1$ near supp $\psi$. In view of a standard iteration argument [16], it suffices to prove that
\[ ||\psi u|| \leq \mathcal{O}\left(\frac{1}{\varepsilon}\right) ||v|| + \mathcal{O}(h)||\tilde{\psi} u|| + \mathcal{O}(h^\infty)||u||. \] (4.6)

In doing so, we shall use the normal form for $P_\varepsilon$ near $\gamma$, recalled in the proof of Theorem 1.1 in Section 2. We have
\[ (P_\varepsilon - z)\psi u = \psi v + [P_\varepsilon, \psi]u. \]
Here \([P_\varepsilon, \psi] = \mathcal{O}(h^3 + \varepsilon h) = \mathcal{O}(\varepsilon h)\) as an operator on \(L^2\), since \(h^2 \leq \varepsilon\) and the subprincipal symbols of \(P_0\) and \(\psi\) vanish. Applying the Fourier integral operator \(U\) introduced in the proof of Theorem 1.1 and using Egorov’s theorem, we obtain, modulo an error term of norm \(\mathcal{O}(h^\infty)||u||\),

\[
(f(hD_t) + \varepsilon \langle q \rangle (hD_t, x, hD_x) + \mathcal{O}(\varepsilon^2 + h^2) - z) U\psi u = U (\psi v + [P_\varepsilon, \psi] u). \quad (4.7)
\]

Let us now check that the operator \(f(hD_t) + \varepsilon \langle q \rangle (hD_t, x, hD_x) - z\), acting on \(L^2(S^1 \times \mathbb{R})\), is invertible, microlocally near \(\tau = x = \xi = 0\), with the norm of the inverse being \(\mathcal{O}(1/\varepsilon)\), provided that \(z \in \text{neigh}(0, \mathbb{R})\) is such that (4.1) holds. To that end, we consider a direct sum orthogonal decomposition,

\[
f(hD_t) + \varepsilon \langle q \rangle (hD_t, x, hD_x) - z = \bigoplus_{k' \in \mathbb{Z}} (f(h(k' - \theta)) + \varepsilon \langle q \rangle (h(k' - \theta), x, hD_x) - z), \quad (4.8)
\]

where it is understood that we only consider the values of \(k' \in \mathbb{Z}\) for which \(h(k' - \theta)\) is small enough. Using that \(\varepsilon \ll h\), we see that for each \(k' \neq k\), with \(k\) given in (4.1), the corresponding direct summand in (4.8) is invertible, microlocally near \(x = \xi = 0\) with a norm of the inverse being \(\mathcal{O}(h^{-1})\). When verifying the microlocal invertibility in the case \(k' = k\), we write \(z = f(h(k - \theta)) + \varepsilon w\), where \(|w - F_0| \leq 1/C\), \(C \gg 1\).

We have \(\langle q \rangle(\tau, x, \xi) - F_0 \sim 1\), for \(\tau, x, \xi \approx 0\), and the operator

\[
f(h(k - \theta)) + \varepsilon \langle q \rangle (h(k - \theta), x, hD_x) - z = \varepsilon (\langle q \rangle (h(k - \theta), x, hD_x) - w)
\]

is therefore invertible, microlocally near \(x = \xi = 0\), with the \(\mathcal{O}(\varepsilon^{-1})\) bound for the norm of the inverse.

From (4.7) we therefore infer that

\[
||\psi u|| \leq \mathcal{O}\left(\frac{1}{\varepsilon}\right) (||v|| + ||[P_\varepsilon, \psi] u||) + \mathcal{O}(h^\infty)||u||, \quad (4.9)
\]

and using also that

\[
||[P_\varepsilon, \psi] u|| \leq \mathcal{O}(\varepsilon h)||\tilde{\psi} u|| + \mathcal{O}(h^\infty)||u||,
\]

we obtain the bounds (4.6) and then (4.5).

Relying upon (4.5), we shall now complete the proof of the fact that the spectrum of \(P_\varepsilon\) in the region (4.1) is contained in the union of the intervals \(J_\ell(h)\) centered at the quasi-eigenvalues (4.2). Let us write

\[
(P_\varepsilon - z)\chi u = \chi v + [P_\varepsilon, \chi] u,
\]
where from (4.4) we know that the norm of the commutator term does not exceed $O(h^{\infty})\|u\|$. Applying the unitary Fourier integral operator $U$ of Theorem 3.1, we get, modulo an error term of norm $O(h^{\infty})\|u\|$, 

$$
(\hat{P} - z)U\chi u = U(\chi v + [P_\varepsilon, \chi]u).
$$

Now an expansion in a Fourier series shows that the operator $\hat{P} - z$ is invertible, microlocally near $\xi = 0$, with a microlocal inverse of the norm $O((\varepsilon h)^{-1})$, provided that $z$ in the set (4.1) avoids the intervals $J_\ell(h)$. We get 

$$
\|\chi u\| \leq O\left(\frac{1}{\varepsilon h}\right)\|v\| + O(h^{\infty})\|u\|,
$$

and combining this estimate together with (4.5) we infer that the operator $P_\varepsilon - z : H(m) \to L^2(M)$ is injective, hence bijective, since it is a Fredholm operator of index zero by general arguments, for $h > 0$ small enough.

When $z$ in (4.1) varies in an interval $J_\ell(h)$ centered around the quasi-eigenvalue in (4.2), contained in the set in (4.1), for some $\ell \in \mathbb{Z}$, we may follow Section 6 of [16] and set up a globally well posed Grushin problem for the operator $P_\varepsilon - z$. Since the corresponding discussion here is even simpler than that of [16], we shall only recall the main steps. Let us define the rank one operators 

$$
R_+ : L^2(M) \to \mathbb{C}, \quad R_- : \mathbb{C} \to L^2(M),
$$

given by 

$$
R_+ u = (U\chi u, e_{k\ell}), \quad R_- u_- = u_- U^* e_{k\ell}.
$$

Here 

$$
e_{k\ell}(x) = \frac{1}{2\pi} e^{\frac{h}{2}(k-\theta_1)x_1 + h(\ell-\theta_2)x_2}, \quad \theta_j = \frac{\alpha_j}{4} + \frac{S_j}{2\pi h}, \quad j = 1, 2,
$$

the scalar product in the definition of $R_+$ is taken in the space $L^2_j(\mathbb{T}^2)$, and $U^*$ is the complex adjoint of $U$. The arguments of Section 6 of [16] can now be applied as they stand to show that for every $(v, v_+) \in L^2(M) \times \mathbb{C}$, the Grushin problem 

$$
(P_\varepsilon - z)u + R_- u_- = v, \quad R_+ u = v_+,
$$

has a unique solution $(u, u_-) \in H(m) \times \mathbb{C}$. We have the corresponding estimate 

$$
\varepsilon h\|u\| + |u_-| \leq O(1) (\|v\| + \varepsilon h |v_+|),
$$

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and if we write the solution in the form

\[ u = Ev + E_+ v_+, \quad u_- = E_- v + E_- v_+, \]

then repeating the arguments of [16], we find that

\[ E_+(z) = z - \hat{P} \left( h(k - \theta_1), h(\ell - \theta_2), \varepsilon, \frac{h^2}{\varepsilon}; h \right) + O(h^\infty). \]

Since the eigenvalues of \( P_\varepsilon \) in the interval \( J_k(h) \) are precisely the values of \( z \) for which \( E_+(z) \) vanishes \[25\], we see that we have established Theorem 1.2, in the general case when the clusters of \( P_0 \) are of size \( O(h^2) \), and when \( \varepsilon \) is in the range \( h^2 \ll \varepsilon \ll h \).

Remark. The number of the eigenvalues of \( P_\varepsilon \) in the subcluster (4.1) is \( \sim h^{-1} \), which is of the same order of magnitude as the total number of eigenvalues of \( P_\varepsilon \) in the \( k \)th spectral cluster \( f(h(k - \theta)) + [-O(\varepsilon), O(\varepsilon)] \). See also Chapter 15 of [7].

5 Improving parameter range for thin clusters

In this section, following [18], we shall extend the range of \( \varepsilon \) in Theorem 1.2, in the case when the spectrum of \( P_0 \) near 0 clusters into bands of size \( O(h^{N_0}) \), for some integer \( N_0 > 2 \).

Let \( P_\varepsilon, \varepsilon \in \text{neigh}(0, \mathbb{R}) \), be a smooth family of selfadjoint operators, such the assumptions of the introduction are satisfied. As we saw in Section 3, microlocally near the Lagrangian torus \( \Lambda_{0,P_0} \), the operator \( P_0 \) can be reduced by successive averaging procedures to an operator of the form

\[ P_0 \sim \sum_{j=0}^\infty h^j p_j(x_2, \xi), \quad (5.1) \]

defined near \( \xi = 0 \) in \( T^*T^2 \), and such that \( p_0 = p(\xi_1), p_1 = 0 \). We then have the following result.

**Proposition 5.1** Assume that the subprincipal symbol of \( P_0 \) vanishes and that the spectrum of \( P_0 \) clusters into intervals of size \( \leq O(h^{N_0}) \), for some integer \( N_0 > 2 \). Then the terms \( p_j(x_2, \xi) = p_j(\xi_1) \) in (5.1) are independent of \( (x_2, \xi_2) \) when \( 1 \leq j \leq N_0 - 1 \).
Proposition 5.1 is an analog of Proposition 12.1 of [18], where a microlocal model for the selfadjoint operator $P_0$ near a closed $H_p$-trajectory was considered. This minor difference does not affect the validity of the result, and the proof of Proposition 5.1 is essentially the same as that of Proposition 12.1 in [18], making use of a suitable family of $O(h^{1/2})$-Gaussian quasimodes on the one-dimensional torus.

An application of the discussion in Section 3 together with Proposition 5.1 allows us to conclude that when $0 \neq \varepsilon \in \text{neigh}(0, \mathbb{R})$, microlocally near the torus $\Lambda_{0,R_0}$, the operator $P_\varepsilon$ can be reduced to the following form,

$$P_\varepsilon = \sum_{j=0}^{\infty} h^j p_j(x_2, \xi, \varepsilon), \quad (x, \xi) \in T^* T^2,$$

where

$$p_0(x_2, \xi, \varepsilon) = p(\xi_1) + \varepsilon \langle q \rangle + O(\varepsilon^2)$$

is independent of $x_1$, and and

$$p_1(x_2, \xi, \varepsilon) = \varepsilon q_1(x_2, \xi, \varepsilon),$$

$$p_j(x_2, \xi, \varepsilon) = p_j(\xi_1) + \varepsilon q_j(x_2, \xi, \varepsilon), \quad 2 \leq j \leq N_0 - 1.$$

It follows that we can write,

$$P_\varepsilon = p(\xi_1; h) + \varepsilon \left( r_0 \left( x_2, \xi, \varepsilon, \frac{h^{N_0}}{\varepsilon} \right) + h r_1 \left( x_2, \xi, \varepsilon, \frac{h^{N_0}}{\varepsilon} \right) + h^2 r_2 \ldots \right), \quad (5.2)$$

where

$$p(\xi_1; h) = p(\xi_1) + \sum_{j=2}^{N_0-1} h^j p_j(\xi_1),$$

$$r_0 \left( x_2, \xi, \varepsilon, \frac{h^{N_0}}{\varepsilon} \right) = \langle q \rangle(\xi) + O(\varepsilon) + \frac{h^{N_0}}{\varepsilon} p_{N_0}(x_2, \xi, \varepsilon),$$

$$r_1 \left( x_2, \xi, \varepsilon, \frac{h^{N_0}}{\varepsilon} \right) = q_1(x_2, \xi, \varepsilon) + \frac{h^{N_0}}{\varepsilon} p_{N_0+1}(x_2, \xi, \varepsilon),$$

and more generally,

$$r_j \left( x_2, \xi, \varepsilon, \frac{h^{N_0}}{\varepsilon} \right) = q_j(x_2, \xi, \varepsilon) + \frac{h^{N_0}}{\varepsilon} p_{N_0+j}(x_2, \xi, \varepsilon), \quad 1 \leq j \leq N_0 - 1,$$

$$r_j \left( x_2, \xi, \varepsilon, \frac{h^{N_0}}{\varepsilon} \right) = \frac{h^{N_0}}{\varepsilon} p_{j+N_0}(x_2, \xi, \varepsilon), \quad j \geq N_0.$$
The analysis of Subsection 3.2 can then be applied to the operator in (5.2), provided that
\[ \frac{h^{N_0}}{\varepsilon} \leq \delta_0 \ll 1, \]
and we see that a natural analog of Theorem 3.1 is valid, with the small parameter \( h^2/\varepsilon \) replaced by \( h^{N_0}/\varepsilon \). The arguments of Section 4 can therefore also be applied, with minor modifications, and we obtain the full statement of Theorem 1.2, for \( \varepsilon \) in the range \( h^{N_0} \ll \varepsilon \ll h \).

6 Magnetic Schrödinger operators in the resonant case

Let us consider the magnetic Schrödinger operator on \( \mathbb{R}^2 \),

\[ P = \sum_{j=1}^{2} (hD_{x_j} + A_j(x))^2 + V(x). \quad (6.1) \]

Here the magnetic and electric potentials \( A = (A_1, A_2) \) and \( V \) are assumed to be smooth and real-valued, with \( \partial^\alpha A, \partial^\alpha V \in L^\infty(\mathbb{R}^2) \), for all \( \alpha \in \mathbb{N}^2 \). It is then well known that \( P \) is essentially selfadjoint on \( L^2(\mathbb{R}^2) \), starting from \( C^\infty_0(\mathbb{R}^2) \).

Let us assume that \( V \geq 0 \) with equality at 0 only and that \( V''(0) > 0 \). We further assume that

\[ \liminf_{|x| \to \infty} V(x) > 0. \]

The spectrum of the selfadjoint nonnegative operator \( P \) is then discrete in a neighborhood of 0.

Associated to \( P \) in (6.1) is the Weyl symbol given by

\[ p(x, \xi) = \sum_{j=1}^{2} (\xi_j + A_j(x))^2 + V(x), \quad x, \xi \in \mathbb{R}^2. \quad (6.2) \]

Assume that near 0, for \( j = 1, 2 \), we have

\[ A_j(x) = \mathcal{O}(x^{m-1}), \quad (6.3) \]

for some \( m \geq 3 \), and that

\[ V(x) = \frac{1}{2} V''(0)x \cdot x + \mathcal{O}(x^m). \quad (6.4) \]
After a linear symplectic change of coordinates, we obtain, as \((x, \xi) \to 0\),

\[
p(x, \xi) = p_2(x, \xi) + \sum_{j=1}^{2} A_{j,m-1}(x) \xi_j + p_m(x) + O((x, \xi)^{m+1}). \tag{6.5}
\]

Here

\[
p_2(x, \xi) = \sum_{j=1}^{2} \frac{\lambda_j}{2}(x_j^2 + \xi_j^2), \quad \lambda_j > 0,
\]

\(A_{j,m-1}\) is a homogeneous polynomial of degree \(m - 1\), and \(p_m(x)\) is a homogeneous polynomial of degree \(m\). In what follows, in order to fix the ideas, we shall consider the case \(m = 4\). Assume also, for simplicity, that the electrical potential \(V\) satisfies \(V(-x) = V(x)\) and the magnetic potential \(A\) satisfies \(A(-x) = -A(x)\). We can then rewrite (6.5) as follows,

\[
p(x, \xi) = p_2(x, \xi) + \sum_{j=1}^{2} A_{j,3}(x) \xi_j + p_4(x) + O((x, \xi)^{6}). \tag{6.6}
\]

We assume that \(\lambda = (\lambda_1, \lambda_2)\) fulfills the resonant condition,

\[
\lambda \cdot k = 0, \tag{6.7}
\]

for some \(0 \neq k \in \mathbb{Z}^2\). We shall then be interested in eigenvalues \(E\) of \(P\) with \(E \sim \varepsilon\), where \(h^{2\delta} < \varepsilon \ll 1\), \(0 < \delta < 1/2\). The general arguments of [23] imply that the corresponding eigenfunctions are microlocally concentrated in the region where \((x, \xi) = O(\varepsilon^{1/2})\), and we introduce therefore the change of variables \(x = \varepsilon^{1/2} y\). Then

\[
\frac{1}{\varepsilon} P(x, hD_x) = \frac{1}{\varepsilon} P(\varepsilon^{1/2} y, \tilde{h}D_y), \quad \tilde{h} = \frac{h}{\varepsilon} \ll 1.
\]

It follows from (6.6) that the symbol of the corresponding \(\tilde{h}\)-pseudodifferential operator is

\[
\frac{1}{\varepsilon} p(\varepsilon^{1/2} y, \eta) = p_2(y, \eta) + \varepsilon q(y, \eta) + O(\varepsilon^2),
\]

to be considered in the region where \(|(y, \eta)| = O(1)\). Here

\[
q(y, \eta) = \sum_{j=1}^{2} A_{j,3}(y) \eta_j + p_4(y). \tag{6.8}
\]
The resonant assumption (6.7) implies that the $H_{p_2}$-flow is periodic on $p_2^{-1}(E)$, for $E \in \text{neigh}(1, R)$, with period $T > 0$ which does not depend on $E$, and we shall assume that $T$ is the minimal period for the $H_{p_2}$-flow. We may therefore apply Theorem 1.2 to discuss the invertibility of
\[ P(x, hD_x) - \varepsilon(1 + z) = \varepsilon \left( \frac{1}{\varepsilon} P(x, hD_x) - 1 - z \right), \quad z \in \text{neigh}(0, R), \]
in the range of energies $E = \varepsilon(1 + z)$, given by
\[ h^{N_0/(N_0+1)} \ll E \ll h^{1/2}, \]
for all $N_0 = 2, 3, \ldots$. Notice also that since the eigenvalues of $p_2^w(x, \tilde{h}D_x)$ depend linearly on $\tilde{h}$, the functions $f_j$, $j \geq 2$, occurring in Theorem 1.2, all vanish. We obtain the following result.

**Proposition 6.1** Assume that (6.7) holds and that the $H_{p_2}$-flow has a minimal period $T > 0$ on $p_2^{-1}(1)$. Let $\langle q \rangle$ stand for the average of the homogeneous function $q$ in (6.8) along the trajectories of the Hamilton vector field of $p_2$, and assume that $\langle q \rangle$ is not identically zero. Let $F_0 \in \mathbb{R}$ be a regular value of $\langle q \rangle$ restricted to $p_2^{-1}(1)$. Let $\varepsilon$ satisfy
\[ h^{N_0/(N_0+1)} \ll \varepsilon \ll h^{1/2}, \]
for some $N_0 \geq 2$ fixed. Then for $z \in \text{neigh}(0, R)$ in the set
\[ \left| z - f \left( \tilde{h}(k - \frac{\alpha_1}{4}) - \frac{S_1}{2\pi} \right) - \varepsilon F_0 \right| < \frac{\varepsilon}{\mathcal{O}(1)}, \quad f(E) = \frac{2\pi}{T} E, \quad \tilde{h} = \frac{h}{\varepsilon}, \]
the eigenvalues of $P$ of the form $\varepsilon(1 + z)$ are given by
\[ z = \tilde{P} \left( \tilde{h}(k - \frac{\alpha_1}{4}) - \frac{S_1}{2\pi}, \tilde{h}(\ell - \frac{\alpha_2}{4}) - \frac{S_2}{2\pi}, \frac{\tilde{h}^{N_0}}{\varepsilon}, \frac{\tilde{h}}{\varepsilon} \right) + \mathcal{O}(h^{\infty}), \quad \ell \in \mathbb{Z}. \]

Here $\tilde{P}(\xi, \varepsilon, \tilde{h}^{N_0}/\varepsilon; \tilde{h})$ has an expansion, as $\tilde{h} \to 0$,
\[ \tilde{P} \left( \xi, \varepsilon, \frac{\tilde{h}^{N_0}}{\varepsilon}; \tilde{h} \right) \sim f(\xi_1) + \varepsilon \sum_{n=0}^{\infty} \tilde{h}^n r_n \left( \xi, \varepsilon, \frac{\tilde{h}^{N_0}}{\varepsilon} \right), \]
where
\[ r_0(\xi) = \langle q \rangle(\xi) + \mathcal{O} \left( \varepsilon + \frac{\tilde{h}^{N_0}}{\varepsilon} \right), \quad r_j = \mathcal{O}(1), \quad j \geq 1. \]
The coordinates $\xi_1 = \xi_1(E)$ and $\xi_2 = \xi_2(E, F)$ are the normalized actions of the Lagrangian tori

$$\Lambda_{E,F} : p_2 = E, \quad \langle q \rangle = F,$$

for $E \in \text{neigh}(1, \mathbb{R})$, $F \in \text{neigh}(F_0, \mathbb{R})$, given by

$$\xi_j = \frac{1}{2\pi} \left( \int_{\gamma_j(E,F)} \eta \, dy - \int_{\gamma_j(1,F_0)} \eta \, dy \right), \quad j = 1, 2,$$

with $\gamma_j(E, F)$ being fundamental cycles in $\Lambda_{E,F}$, such that $\gamma_1(E, F)$ corresponds to a closed $H_{p_2}$-trajectory of minimal period $T$. Furthermore,

$$S_j = \int_{\gamma_j(1,F_0)} \eta \, dy,$$

and $\alpha_j \in \mathbb{Z}$ is fixed, $j = 1, 2$.

We shall finish this section by providing an explicit example, illustrating Proposition 6.1 in the case when $\lambda = (1, 1)$. Then $T = 2\pi$ is the minimal period for the $H_{p_2}$-flow, and our task becomes computing the flow average $\langle q \rangle$ and determining its critical values, viewed as a function on the compact symplectic manifold $\Sigma$. In this case, as we saw in [18], the manifold $\Sigma$ can naturally be identified with the complex projective line $\mathbb{C}P^1 \cong S^2$.

Continuing to follow [18], let us recall first how to compute the trajectory average of a monomial $x^\alpha \xi^\beta$ with $|\alpha| + |\beta| = m$, for some $m \in \{3, 4, 5, ...\}$. To this end, it is convenient to introduce

$$z_j = x_j + i\xi_j \in \mathbb{C}, \quad j = 1, 2, \quad (6.9)$$

and we then notice that along a $H_{p_2}$-trajectory we get in the $z_1, z_2$ coordinates:

$$z_j(t) = e^{-i\lambda_j t} z_j(0). \quad (6.10)$$

Then we write $x_j(t) = \text{Re} \, z_j(t)$, $\xi_j(t) = \text{Im} \, z_j(t)$, so that

$$x(t)^\alpha \xi(t)^\beta = \prod_{j=1}^2 (\text{Re} \, z_j(t))^{\alpha_j} (\text{Im} \, z_j(t))^{\beta_j}$$

$$= \frac{1}{2^{|\alpha|+|\beta|}} \prod_{j=1}^2 \left( (z_j(0)e^{-i\lambda_j t} + \overline{z_j(0)}e^{i\lambda_j t})^{\alpha_j} (z_j(0)e^{-i\lambda_j t} - \overline{z_j(0)}e^{i\lambda_j t})^{\beta_j} \right). \quad (6.11)$$
Expanding the product by means of the binomial theorem, we see that the time average is equal to the time-independent term, and since this average is constant along each trajectory we shall replace the symbols $z_j(0)$ simply by $z_j$.

For simplicity, we shall assume that $p_4 = 0$ in (6.8), and then we write

$$A_{j,3}(x) = \sum_{k=0}^{3} a_{j,k} x_1^k x_2^{3-k}, \quad j = 1, 2.$$  \hfill (6.12)

The associated magnetic field

$$B(x) = \frac{\partial A_{2,3}}{\partial x_1} - \frac{\partial A_{1,3}}{\partial x_2}$$

is given by

$$B(x) = b_2 x_1^2 + b_1 x_1 x_2 + b_0 x_2^2,$$ \hfill (6.13)

where

$$b_2 = 3a_{2,3} - a_{1,2}, \quad b_1 = 2(a_{2,3} - a_{1,1}), \quad b_0 = a_{2,1} - 3a_{1,0}.$$ \hfill (6.14)

Using (6.11), we get

$$\langle \xi_{1} x_1^3 \rangle = 0,$$

$$\langle \xi_{1} x_1^2 x_2 \rangle = \frac{1}{16i} \left( z_1^2 \overline{z}_2 - \overline{z}_1^2 z_2 \right) = -\frac{1}{2} \rho_1^{3/2} \rho_2^{1/2} \sin(\theta_1 - \theta_2),$$

$$\langle \xi_{1} x_1 x_2^2 \rangle = \frac{1}{16i} \left( z_1^2 \overline{z}_2^2 - \overline{z}_1^2 z_2^2 \right) = -\frac{1}{2} \rho_1 \rho_2 \sin 2(\theta_1 - \theta_2),$$

$$\langle \xi_{1} x_2^3 \rangle = \frac{3}{16i} \left( z_1 z_2 \overline{z}_2 - \overline{z}_1 z_2 \overline{z}_2 \right) = -\frac{3}{2} \rho_1^{1/2} \rho_2^{3/2} \sin(\theta_1 - \theta_2),$$

$$\langle \xi_{2} x_1^3 \rangle = \frac{3}{16i} \left( z_1 \overline{z}_2 z_2^2 - \overline{z}_1 \overline{z}_2 z_2^2 \right) = \frac{3}{2} \rho_1^{3/2} \rho_2^{1/2} \sin(\theta_1 - \theta_2),$$

$$\langle \xi_{2} x_1^2 x_2 \rangle = \frac{1}{16i} \left( z_1^2 \overline{z}_2 - \overline{z}_1^2 z_2 \right) = \frac{1}{2} \rho_1 \rho_2 \sin 2(\theta_1 - \theta_2),$$

$$\langle \xi_{2} x_1 x_2^2 \rangle = \frac{1}{16i} \left( z_1^2 \overline{z}_2^2 - \overline{z}_1^2 z_2^2 \right) = \frac{1}{2} \rho_1^{1/2} \rho_2^{3/2} \sin(\theta_1 - \theta_2),$$

$$\langle \xi_{2} x_2^3 \rangle = 0.$$  

Here $(\rho_j, \theta_j)$ are the action–angle variables given by

$$z_j = \sqrt{2\rho_j e^{-i\theta_j}}.$$  

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Recalling the expressions for $q$ in (6.8) and $B$ in (6.13), (6.14), we get

$$2\langle q \rangle = b_2 \rho_1^{3/2} \rho_2^{1/2} \sin(\theta_1 - \theta_2) + \frac{b_1}{2} \rho_1 \rho_2 \sin 2(\theta_1 - \theta_2) + b_0 \rho_1^{1/2} \rho_2^{3/2} \sin(\theta_1 - \theta_2).$$  \hspace{1cm} (6.15)

In particular, as was already observed in the example in the introduction, the flow average $\langle q \rangle$ depends on the magnetic field $B$ only. Let us consider the special case when $b_2 = b_0 = 0$ while $b_1 \neq 0$. In this case, a straightforward computation shows that $\langle q \rangle$, viewed as a function on the space $\Sigma$, has exactly three critical values, given by $\pm b_1/16$ and 0. When $F_0 \in \mathbb{R}$ is in the range of $\langle q \rangle$, $F_0$ away from $\pm b_1/8$ and 0, Proposition 6.1 applies.

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