Deformation Expression for Elements of Algebras (II) –(Weyl algebra of 2m-generators)–

Hideki Omori* Yoshiaki Maeda† Naoya Miyazaki‡
Tokyo University of Science Keio University Keio University

Akira Yoshioka§
Tokyo University of Science

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*Department of Mathematics, Faculty of Sciences and Technology, Tokyo University of Science, 2641, Noda, Chiba, 278-8510, Japan, email: omori@ma.noda.tus.ac.jp
†Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Yokohama, 223-8522, Japan, email: maeda@math.keio.ac.jp
‡Department of Mathematics, Faculty of Economics, Keio University, 4-1-1, Hiyoshi, Yokohama, 223-8521, Japan, email: miyazaki@hc.cc.keio.ac.jp
§Department of Mathematics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Tokyo, 102-8601, Japan, email: yoshioka@rs.kagu.tus.ac.jp
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This is a noncommutative version of the previous work \([14]\) in the same title with numbering \((I)\). In general in a noncommutative algebra, there is no canonical way to express elements in univalent way, which is often called “ordering problem”. In this note we discuss this problem in the case of the Weyl algebra of \(2m\)-generators. By fixing an expression, we extends Weyl algebra transcendentally. We treat \(*\)-exponential functions of linear forms, and quadratic forms of crossed symbol under generic expression parameters. In the extended algebra, we find a group which is isomorphic to \(SO(m, \mathbb{C}) \times \mathbb{Z}_2\) in the normal ordered expression, and several strange behaviour of \(*\)-exponential functions of quadratic forms. After some investigation for \(*\)-exponential functions and intertwiners we find there is an expression parameter \(K_s\) under which a certain system of \(*\)-exponential functions of quadratic forms generates a Clifford algebra. Hence, the transcendentally extended Weyl algebra has the property which may be called the Weyl-Clifford algebra.

1 Weyl algebra in the normal ordered expression

Throughout this paper, we use notations

\[
\hat{u} = (\tilde{u}_1, \cdots, \tilde{u}_m), \quad \hat{v} = (\tilde{v}_1, \cdots, \tilde{v}_m), \quad u = (u_1, u_2, \cdots, u_{2m}) = (\tilde{u}, \tilde{v}).
\]

The Weyl algebra \(W_\hbar(2m)\) is the algebra generated by \(\hat{u} = (\tilde{u}_1, \cdots, \tilde{u}_m), \quad \hat{v} = (\tilde{v}_1, \cdots, \tilde{v}_m)\) with the commutation relations

\[
[\tilde{u}_i, \tilde{v}_j] = -\hbar \delta_{ij}, \quad [\tilde{u}_i, \tilde{u}_j] = 0 = [\tilde{v}_i, \tilde{v}_j],
\]

where \([a, b] = ab - ba\) and \(\hbar\) is a positive constant. When we consider the case \(m = 1\), \((\hat{u}, \hat{v})\) stands for \((\tilde{u}_1, \tilde{v}_1)\) (or \((u_1, u_2)\)). One of the way for univalent expression of elements of \(W_\hbar(2m)\) is to write \(\hat{u}\) in the l.h.s in each monomial. For instance, we write \(\hat{v} \ast \hat{u}, \hat{v}^2 \ast \hat{u}\) as \(\tilde{u} \hat{v} + \hbar \hat{v} \tilde{u}, \tilde{u} \hat{v} + 2i\hbar \tilde{u}\hat{v}\). This way is called normal ordering.

Furthermore, we identify a normally expressed element with a usual polynomial and we denote, for instance

\[
:\hat{v} \ast \hat{u} \ast \hat{v} + \hat{v} \ast \hat{u};_{K_0} = \tilde{u} \tilde{v}^2 + \tilde{u} \tilde{v} + i\hbar \tilde{v} + i\hbar, \quad \hat{v} \ast \hat{u};_{K_0} = \tilde{u}, \quad \hat{v};_{K_0} = \tilde{v},
\]

where the suffix \(\ast_{K_0}\) indicate the normal ordered expression. The sign \(\ast\) is omitted in normally ordered expressions as they are ordinary polynomials.

For general \(m\), the \(*\)-product (denoted by \(*_{K_0}\)) of two normally ordered elements is given by the abbreviate formula

\[
(1.1) \quad f(u)*_{K_0} g(u) = f(\exp \hbar i \{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}) g, \quad (\PsiDO-product formula)
\]
where \( \frac{\partial}{\partial \bar{u}} \frac{\partial}{\partial u} = \sum_i \overline{\frac{\partial}{\partial v_i}} \frac{\partial}{\partial u_i} \). More precisely it is

\[
f(u)^{*_{K_0}} g(u) = \sum_k \frac{(i\hbar)^k}{k!} \partial_{\bar{v}_1} \cdots \partial_{\bar{v}_k} f(u) \partial_{u_1} \cdots \partial_{u_k} g(u).
\]

We give several formulas which will be used later. For \( m = 1 \), we note first that the associativity and the commutation relation \( \bar{u}^{*} \bar{v}^{*} - \bar{v}^{*} \bar{u}^{*} = -i\hbar \) gives for every polynomial \( p(\bar{u}^{*} \bar{v}^{*}) \) of \( \bar{u}^{*} \bar{v}^{*} \) that

\[
p(\bar{u}^{*} \bar{v}^{*}) \bar{u}^{*} = \bar{u}^{*} p(\bar{v}^{*} \bar{u}^{*}) = \bar{u}^{*} p(\bar{u}^{*} \bar{v}^{*} + i\hbar), \quad \text{(bumping identity)}
\]

Let \( \bar{u} \bar{v} = \frac{1}{2}(\bar{u} \bar{v} + \bar{v} \bar{u}) \); the symmetric product. The bumping identity gives

\[
\bar{u}^{*} (\bar{u}^{*} \bar{v}^{*}) \bar{v}^{*} = \bar{u}^{*} (\bar{u} \bar{v} + \frac{1}{2}i\hbar) \bar{v}^{*} = (\bar{u} \bar{v} + \frac{1}{2}i\hbar) \bar{v}^{*} (\bar{u} \bar{v} + \frac{3}{2}i\hbar).
\]

### 1.1 Star-exponential functions of crossed symbols

For every \( C = (C_{ij}) \in M_\mathbb{C}(m) \) \((m \times m\)-complex matrix\), we denote \( C(\bar{u}, \bar{v}) = \sum C_{ij} \bar{u}_i \bar{v}_j \). This special class of quadratic forms is very convenient for calculation in the normal ordered \((K_0\text{-ordered})\) expression. In this section, normal ordered expressions are mainly used except otherwise stated.

By a direct calculation via \( \Psi \text{DO}-\text{product formula} \) \( (1.1) \) we have

\[
e^{\frac{2 \pi i}{\hbar} \sum A_{kl} \bar{u}_k \bar{v}_l} \big|_{K_0} = e^{\frac{2 \pi i}{\hbar} \sum B_{kl} \bar{u}_k \bar{v}_l} = e^{\frac{2 \pi i}{\hbar} \sum C_{ij} \bar{u}_i \bar{v}_j},
\]

\[
C = A + B + 2AB.
\]

If we denote \( e^{\frac{2 \pi i}{\hbar} \sum A_{kl} \bar{u}_k \bar{v}_l} \) by \( [A] \), then the product formula is read as

\[
[A] \big|_{K_0} [B] = [A + B + 2AB], \quad A, B \in M_\mathbb{C}(m)
\]

This is viewed as

\[
(I + 2A)(I + 2B) = I + 2(A + B + 2AB).
\]

By the correspondence \( A \leftrightarrow I + 2A \), the multiplicative structure of usual matrix algebra \( M_\mathbb{C}(m) \) is translated into the space \( \{ e^{\frac{2 \pi i}{\hbar} C} : C \in M_\mathbb{C}(m) \} \). Let \( O'_{K_0} = \{ X \in M_\mathbb{C}(m) : \det(I_m + 2X) \neq 0 \} \).

**Proposition 1.1** \( O'_{K_0} \) forms a group under the \( \big|_{K_0} \)-product, isomorphic to \( GL(m, \mathbb{C}) \). \( 0 \) is the identity, and the inverse \( X_{\big|_{K_0}}^{-1} \) is given by \( -I + 2X \).\]

But, note here that the additive unit \( 0 \in M_\mathbb{C}(m) \) is shifted to \(-\frac{1}{2}I_m(u,v)\), that is,

\[
-\frac{1}{2}I + C + 2(-\frac{1}{2}I)C = -\frac{1}{2}I, \quad \text{i.e.}
\]

\[
e^{\frac{2 \pi i}{\hbar} C(\bar{u}, \bar{v})} \big|_{K_0} e^{-\frac{1}{2}I_m(\bar{u}, \bar{v})} = e^{-\frac{1}{2}I_m(\bar{u}, \bar{v})} \big|_{K_0} e^{\frac{2 \pi i}{\hbar} C(\bar{u}, \bar{v})} = e^{-\frac{1}{2}I_m(\bar{u}, \bar{v})}.
\]

On the other hand, we see \(-I + 2(\bar{I})(\bar{I}) = 0\). In general \( (1.3) \) gives the following:
Proposition 1.2 $e^{-\frac{1}{\hbar}C(u,\bar{u})_{K_0}} e^{-\frac{1}{\hbar}C(\bar{u},\bar{v})} = e^{\frac{2}{\hbar}i} = 1$ if and only if $C^2 = C$. In particular, let

$$I_k(u, \bar{v}) = \bar{u}_{i_1} \bar{v}_{i_1} + \cdots + \bar{u}_{i_k} \bar{v}_{i_k}$$

for mutually distinct arbitrary $i_1, \ldots, i_k$. Then, $e^{-\frac{1}{\hbar}I_k(u, \bar{v})_{K_0}} e^{-\frac{1}{\hbar}I_k(\bar{u}, \bar{v})} = 1$.

By (1.3) we have the exponential law

$$e^{\frac{1}{\hbar}(e^{iC} - I_m)(\bar{u}, \bar{v})_{K_0}} e^{\frac{1}{\hbar}(e^{iC} - I_m)(\bar{u}, \bar{v})} = e^{\frac{1}{\hbar}(e^{iC} - I_m)(\bar{u}, \bar{v})}.$$

Differentiate this exponential law (1.4) to obtain the *-exponential function

$$e^{\frac{1}{\hbar}\sum C_k \bar{u}_k \bar{v}_k_{K_0} e} = e^{\frac{1}{\hbar}\sum (C_k - I_m) \bar{u}_k \bar{v}_k}$$

where $\bar{u}_k \bar{v}_k$ stands for $\bar{u}_k \bar{v}_k_{K_0}$, but this is $\bar{u}_k \bar{v}_k$ in the normal ordered expression. Here, we used the notation such as $e^{\frac{1}{\hbar}\sum C_k \bar{u}_k \bar{v}_k_{K_0}}$ to avoid possible confusion. It is remarkable that the amplitudes are kept to be 1 in this calculation.

In what follows of this section, we use $\bar{u}_i \bar{v}_j$ often instead of $\bar{u}_i \bar{v}_j$, where

$$\bar{u}_i \bar{v}_j = \frac{1}{2} (\bar{u}_i \bar{v}_j + \bar{v}_j \bar{u}_i), \quad \bar{u}_i \bar{v}_j_{K_0} = \bar{u}_i \bar{v}_j + \frac{1}{2} i \hbar [\bar{u}_i, \bar{v}_j].$$

The reason of this is that the expression by using $\bar{u}_i \bar{v}_j$ has a rich symmetric properties, just like Stratonovich formula in stochastic integrals.

By using this notation, (1.3) is changed into

$$\sum C_k \bar{u}_k \bar{v}_k_{K_0} = e^{\frac{1}{\hbar}\sum (C_k - I_m) \bar{u}_k \bar{v}_k}$$

with a nontrivial amplitude part, but note here that the amplitude part is determined by its phase part. If $C^2 = C$, then $e^{iC} = I + (e^i - 1)C$, hence

$$e^{\frac{1}{\hbar}\sum C_k \bar{u}_k \bar{v}_k_{K_0}} = e^{\frac{1}{\hbar}\sum (C_k \bar{u}_k \bar{v}_k - \frac{1}{2} C_k \bar{u}_k \bar{v}_k)}, \quad e^{\frac{1}{\hbar}\sum C_k \bar{u}_k \bar{v}_k_{K_0}} = e^{\frac{1}{\hbar}\sum (C_k \bar{u}_k \bar{v}_k)}.$$

A typical example is

$$C(\theta) = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}, \quad e^{\frac{1}{\hbar}\sum C_k (\theta) \bar{u}_k \bar{v}_k_{K_0}} = (\pm i) e^{\frac{i}{\hbar}\sum C_k (\theta) \bar{u}_k \bar{v}_k}.$$

Consider now the group $\mathfrak{G}$ generated by $\sum A_k \bar{u}_k \bar{v}_k_{K_0}$, $A \in \mathcal{O}_{K_0}$ and two homomorphisms

$$\pi : \mathfrak{G} \to [\mathcal{O}'_{K_0}], \quad \hat{\alpha} : \mathfrak{G} \to \mathbb{C}_*$$

defined by

$$\pi(\sum A_k \bar{u}_k \bar{v}_k)_{K_0} = e^{\frac{1}{\hbar}\sum A_k \bar{u}_k \bar{v}_k}, \quad \hat{\alpha}(\sum A_k \bar{u}_k \bar{v}_k)_{K_0} = e^{\frac{i}{\hbar}\sum A_k \bar{u}_k \bar{v}_k}.$$ 

As it is shown in Proposition 1.2 and in (1.6) even though $\pi(X) = 1$, it may occur $\hat{\alpha}(X) = -1$. 
Proposition 1.3 In the normal ordered expression, \( e^{\frac{i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 \), \( A \in \mathcal{O}' \), generates a group isomorphic to a connected double covering of \( GL(m, \mathbb{C}) \) embedded naturally in \( \mathbb{C}_* \times GL(m, \mathbb{C}) \) in the form \( (\hat{a}(X), \pi(X)) \), \( X \in \mathfrak{g} \).

Our main concern in this section is elements \( \epsilon_{\alpha} = e^{\frac{i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 \), \( k = 1, 2, \ldots, m \).

Although, \( e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \) are defined only for the normal ordered expression at this stage, we denote it by 

\[
\epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 = e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 = e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 = e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 \tag{1.8}
\]

for we will define in the later section an abstract \( e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \) in generic ordered expressions. We call \( \epsilon_{\alpha}(k) \) partial polar elements and \( \epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 \) its normal ordered expression. We see 

\[
\epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 = ie^{-\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 = 0
\]

(In the later section, we see Weyl ordered expression of \( \epsilon_{\alpha}(k) \) diverges.)

By the product formula under the normal ordered expression \( K_0 \), the natural commutativity 

\[
\epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 \cdot K_0 \cdot \pi_i \cdot K_0 = e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 \cdot K_0 \cdot \pi_i \cdot K_0 = e^{\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0 \cdot K_0 \cdot K_0 \cdot \pi_i \cdot K_0 \cdot K_0
\]

holds. Summarizing these we have

**Proposition 1.4** \( \epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 \cdot \pi_i \cdot K_0 = \epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 \cdot \pi_i \cdot K_0 \cdot \pi_i \cdot K_0 = -1 \)

However, this does not imply that \( \epsilon_{\alpha}(k) \) are commuting each other in general nor 

\( \epsilon_{\alpha}(k)^2 = -1 \) in general. Indeed, it will be shown in §3.2 that there is an expression parameter \( k \) such that \( \epsilon_{\alpha}(k) \) are mutually anti-commuting under \( K \)-expression. Moreover, in §3.2 we see there is a class of expression parameters \( K \) such that 

\( \epsilon_{\alpha}(k)^2 = 1 \).

We set \( \epsilon_{\alpha}(k) \) by 

\[
\epsilon_{\alpha}(k) = e^{\frac{i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k} \cdot \pi_i \cdot K_0
\]

and call it the total polar element. Its \( K_0 \)-expression is 

\[
\epsilon_{\alpha}(k) \cdot \pi_i \cdot K_0 = i^m e^{-\frac{\pm i}{\hbar} \sum a_k \hat{u}_k \hat{v}_k}
\]

If \( \langle \hat{a}, \hat{a} \rangle = \hat{a}^\dagger \hat{a} = 1 \), then by noting \( (\hat{a} \hat{a})^n = \hat{a} \hat{a} \hat{a} \hat{a} \), we see 

\[
e^{\hat{a} \hat{a}} = I + (e^\hat{a} - 1) \hat{a} \hat{a}
\]

For \( \hat{a}, \hat{b} \in \mathfrak{c} \), we set 

\[
\langle \hat{a}, \hat{b} \rangle = \sum_{i=1}^m a_i \hat{u}_i, \langle \hat{a}, \hat{v} \rangle = \sum_{i=1}^m a_i \hat{v}_i etc.
\]

Then, it is easy to see \( \{\langle \hat{a}, \hat{u} \rangle, \langle \hat{a}, \hat{v} \rangle \} = -hi\langle \hat{a}, \hat{b} \rangle \). Hence, if \( \langle \hat{a}, \hat{a} \rangle = 1 \), then \( \langle \hat{a}, \hat{u} \rangle \) and \( \langle \hat{a}, \hat{v} \rangle \) form a canonical conjugate pair.

Let \( S_{\mathbb{C}}^m = \{a \in \mathbb{C}^m; \langle \hat{a}, \hat{a} \rangle = 1\} \), and \( S_{\mathbb{R}}^m = \{a \in \mathbb{R}^m; \langle \hat{a}, \hat{a} \rangle = 1\} \). Then, for every \( \hat{a} \in S_{\mathbb{C}}^m \), the quadratic form 

\[
\alpha\langle \hat{a}, \hat{u} \rangle^2 + \beta\langle \hat{a}, \hat{v} \rangle^2 + 2\gamma\langle \hat{a}, \hat{a} \rangle\langle \hat{a}, \hat{v} \rangle
\]

can be considered as if it were only 2 variables, where 

\( \langle \hat{a}, \hat{u} \rangle \cdot \langle \hat{a}, \hat{v} \rangle = \frac{1}{2} (\langle \hat{a}, \hat{u} \rangle \cdot \langle \hat{a}, \hat{v} \rangle + \langle \hat{a}, \hat{v} \rangle \cdot \langle \hat{a}, \hat{u} \rangle) \) 

and \( \langle \hat{a}, \hat{u} \rangle \cdot \langle \hat{a}, \hat{v} \rangle \) etc means the product by *. Let \( D = \gamma^2 - \alpha \beta \) be its discriminant.
Hence, we have

\[ (1.11) \]

\[ :e^\frac{i}{\hbar}\langle \hat{a}, \hat{u} \rangle \langle \hat{a}, \hat{v} \rangle \in K_0 = e^{i\beta} e^{\frac{i}{\hbar}(1-\epsilon)(\hat{a}, \hat{u} \rangle \langle \hat{a}, \hat{v} \rangle} \]

We define

\[ (1.12) \]

\[ \epsilon_0(\hat{a}) = e^{\frac{i}{\hbar} \langle \hat{a}, \hat{u} \rangle \langle \hat{a}, \hat{v} \rangle} \]

We call \( \epsilon_0(\hat{a}) \) also a partial polar element. Since \( \{\hat{a}, \hat{u}, \hat{b}, \hat{v}\} = -\hbar i \langle \hat{a}, \hat{b} \rangle \), the product formula \( (1.3) \) gives

\[ :\epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) :_{K_0} = -e^{-\frac{i}{\hbar} \langle (\hat{a}, \hat{u} \rangle \langle \hat{a}, \hat{v} \rangle + (\hat{b}, \hat{u} \rangle \langle \hat{b}, \hat{v} \rangle - 2(\hat{a}, \hat{b} \rangle \langle \hat{a}, \hat{v} \rangle} \]

In particular, if \( \langle \hat{a}, \hat{b} \rangle = 0 \), then

\[ \epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) = \langle \hat{b}, \hat{u} \rangle \ast \epsilon_0(\hat{a}), \quad \epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) = \langle \hat{b}, \hat{v} \rangle \ast \epsilon_0(\hat{a}) \]

and

\[ :\epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) :_{K_0} = :\epsilon_0(\hat{b}) \ast \epsilon_0(\hat{a}) :_{K_0} \]

By this observation, we have the following:

**Proposition 1.5** If \( \langle \hat{a}, \hat{b} \rangle = 0 \), then

\[ \epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) = \langle \hat{b}, \hat{u} \rangle \ast \epsilon_0(\hat{a}), \quad \epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) = \langle \hat{b}, \hat{v} \rangle \ast \epsilon_0(\hat{a}) \]

and

\[ :\epsilon_0(\hat{a}) \ast \epsilon_0(\hat{b}) :_{K_0} = :\epsilon_0(\hat{b}) \ast \epsilon_0(\hat{a}) :_{K_0} \]

Let \( \mathfrak{P}_{K_0}^{(\ell)}, \ell = 1, 2, 4 \), be the set consisting of all elements written by

\[ \epsilon_0(\hat{a}_1) \ast \epsilon_0(\hat{a}_2) \ast \cdots \ast \epsilon_0(\hat{a}_k); \quad k \in \mathbb{N}, \hat{a}_j \in S_{C}^{m-1} \]

Since \( \epsilon_0(\hat{a}_j)^4 = 1 \), we have a series of subgroups \( \mathfrak{P}_{K_0}^{(1)} \supset \mathfrak{P}_{K_0}^{(2)} \supset \mathfrak{P}_{K_0}^{(4)} \) of \( \mathfrak{G} \).

**Lemma 1.1** \( \mathfrak{P}_{K_0}^{(4)} \) is the group generated by \{\( \epsilon_0(\hat{a}) \ast \epsilon_0^{-1}(\hat{b}); \hat{a}, \hat{b} \in S_{C}^{m-1} \)}, which is a connected subgroup of \( \mathfrak{G} \). However, \(-1\) is not contained in \( \mathfrak{P}_{K_0}^{(4)} \). Moreover, \( \mathfrak{P}_{K_0}^{(2)} = \mathfrak{P}_{K_0}^{(4)} \cup (-\mathfrak{P}_{K_0}^{(4)}) \), and

\[ \mathfrak{P}_{K_0}^{(1)} = \bigcup_{k=0}^{3} \langle j \rangle \mathfrak{P}_{K_0}^{(4)} \]

**Proof** It is easy to see that \( \bigcup_{\ell} \mathfrak{P}_{K_0}^{(4\ell)} \) is a group. Write its element as \( \epsilon_0(\hat{a}_1) \ast \epsilon_0(\hat{a}_2) \ast \cdots \ast \epsilon_0(\hat{a}_4) \). Since \( S_{C}^{m-1} \) is arcwise connected, this is arcwise connected to \( 1^4 = 1 \). For the second assertion, we have only to note that the product formula \( (1.3) \) allows that the phase part and the amplitude part are computed independently. \( \square \)
1.1.1 Bumping identity

For a while, we use notations \((u,v)\) for \((\bar{u}, \bar{v})\) for simplicity. Using the uniqueness of the real analytic solution, we have the following useful

**Lemma 1.2** (bumping identity) \(v * e^{it_u v} = e^{it_v u} * v, \; e^{it_u v} * u = u * e^{it_v u}\) holds.

**Proof** This is given by the bumping identity \(u*(v*u) = (u*v)*u\), if the polynomial approximation theorem holds. Here, this is proved by the uniqueness of the real analytic solution, hence the proof can be applied for other expressions. The continuity of \(v\) and the associativity give

\[
\frac{d}{dt} v * e^{it_u v} = v * ((iu * v) * e^{it_u v}) = iv * u * (v * e^{it_u v}).
\]

On the other hand, the continuity of \(*v\) and the associativity give

\[
\frac{d}{dt} e^{it_u v} * v = ((iv * u) * e^{it_u v}) * v = iv * u * (e^{it_u v} * v).
\]

Both satisfy the differential equation \(\frac{d}{dt} f_t = iv * u * f_t\) with the initial condition \(v\). The second one is shown by the same proof.

Note here that we used only the real analyticity of \(e^{it_u v}\) in the normal ordered expression. Note that the bumping identity gives in particular

\[
v * e^{it_u v} = e^{it_u (u v + i)} * v = -e^{it_u v} * v, \quad u * e^{it_u v} = -e^{it_u v} * u.
\]

The next Proposition is stated under \(K_0\)-expression by using the original notations, but we see the bumping identity holds in every ordered expression defined in the next section §[2]

**Proposition 1.6**: \(\langle \tilde{a}, \tilde{v} \rangle * \varepsilon_{00} (\tilde{a}) :_{K_0} = -\varepsilon_{00} (\tilde{a}) * \langle \tilde{a}, \tilde{v} \rangle :_{K_0} \) and \(\langle \tilde{a}, \tilde{v} \rangle * \varepsilon_{00} (\tilde{a}) :_{K_0} = -\varepsilon_{00} (\tilde{a}) * \langle \tilde{a}, \tilde{v} \rangle :_{K_0} \).

1.2 Double covering group of \(SO(m, \mathbb{C})\) in \(\mathbb{C} \times GL(m, \mathbb{C})\)

Our main concern in this section is the mutual relations between \(*\)-exponential functions of degenerate quadratic forms of small rank.

By Propositions 1.5 and 1.6 we have easily that

\[
e^{\frac{i(u_1 \ast \tilde{u}_1)}{\pi}} * (\sum_{i=1}^{m} b_i \tilde{u}_i) * e^{\frac{i(-u_1 \ast \tilde{u}_1)}{\pi}} = -b_1 \tilde{u}_1 + \sum_{i=2}^{m} b_i \tilde{u}_i.
\]

In general, for every \(\langle \tilde{a}, \tilde{v} \rangle, \tilde{a} \in S_{\mathbb{C}}^{m-1} \) and \(\tilde{b} \in \mathbb{C}^m\), we set \(\tilde{b} = \langle \tilde{a}, \tilde{b} \rangle \tilde{a} + \tilde{c}, \; \langle \tilde{a}, \tilde{c} \rangle = 0\). By computing the same way as above, we have the reflection with respect to \(\tilde{a}\):

\[
\begin{align*}
e^{\frac{i(u \ast \tilde{u})}{\pi}} * (\langle \tilde{b}, \tilde{u} \rangle * e^{\frac{i(-u \ast \tilde{u})}{\pi}}) + (\langle \tilde{b}, \tilde{u} \rangle * e^{\frac{i(-u \ast \tilde{u})}{\pi}}) &= (\tilde{b} - 2 \langle \tilde{a}, \tilde{b} \rangle \tilde{a}, \tilde{u}) \\
e^{\frac{i(u \ast \tilde{u})}{\pi}} * (\langle \tilde{b}, \tilde{v} \rangle * e^{\frac{i(-u \ast \tilde{u})}{\pi}}) + (\langle \tilde{b}, \tilde{v} \rangle * e^{\frac{i(-u \ast \tilde{u})}{\pi}}) &= (\tilde{b} - 2 \langle \tilde{a}, \tilde{b} \rangle \tilde{a}, \tilde{v})
\end{align*}
\]

For simplicity we use notations in the group theory

\[
Ad(\varepsilon_{00} (\tilde{a})) \ast X = \varepsilon_{00} (\tilde{a}) \ast X \ast \varepsilon_{00}^{-1} (\tilde{a}), \quad Ad(\varepsilon_{00} (\tilde{a}) * \varepsilon_{00} (\tilde{b})) \ast X = \varepsilon_{00} (\tilde{a}) * \varepsilon_{00} (\tilde{b}) * X \ast \varepsilon_{00}^{-1} (\tilde{b}) * \varepsilon_{00}^{-1} (\tilde{a}).
\]

Note that \(Ad(\varepsilon_{00} (\tilde{a})) = Ad(\varepsilon_{00}^{-1} (\tilde{a}))\).

Now, Proposition 1.5 gives the following
**Proposition 1.7** \( \text{Ad}(\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b})) \) generate \( SO(m, \mathbb{C}) \). Since \( \Psi_{K_0}^{(4)} \) does not contain \(-1, \)
\[
\{\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b}); \bar{a}, \bar{b} \in S_{C}^{m-1}\}
\]
generates a group isomorphic to \( SO(m, \mathbb{C}) \). If \( \bar{a}, \bar{b} \) are restricted in real vectors, then \( \text{Ad}(\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b})) \) generate \( SO(m) \), hence \( \{\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b})\} \) generate a group isomorphic to \( SO(m) \).

Since
\[
\text{Ad}(\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b})) = \text{Ad}(\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b}))
\]
we conclude the following:

**Theorem 1.1** In the normal ordered expression, \( \{\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b}); \bar{a}, \bar{b} \in S_{C}^{m-1}\} \) forms a group \( \widetilde{SO}(m, \mathbb{C}) \) which is a subgroup of \( \mathfrak{so} \), and \( \text{Ad}(\widetilde{SO}(m, \mathbb{C})) = SO(m, \mathbb{C}) \).

\[
\text{Ad} : \widetilde{SO}(m, \mathbb{C}) \rightarrow SO(m, \mathbb{C})
\]
is a 2-to-1 surjective homomorphism.

Moreover, the mapping \( \pi \) given by forgetting the amplitude part gives also a 2-to-1 homomorphism onto a subgroup of \( \mathcal{O}_{K_0}' \), which will be denoted by \( SO_{*}(m, \mathbb{C}) \).

It is not hard to consider the adjoint action of the group \( SO_{*}(m, \mathbb{C}) \). This is denoted by \( \text{Ad}' \). This is an isomorphism. It is remarkable that Theorem 1.1 is obtained without using Clifford algebra. However, if the obtained group \( \widetilde{SO}(m) = \text{Spin}(m) \) is constructed via Clifford algebra, we must have \( (\varepsilon_{00}(\bar{a})*\varepsilon_{00}^{-1}(\bar{b}))_{x}^{2} = -1 \), but such an anti-commutative property is not given by the normal ordered expression.

Hence, we have in normal ordered expression that
\[
\widetilde{SO}(m, \mathbb{C}) \cong SO(m, \mathbb{C}) \times \mathbb{Z}_{2}, \quad \widetilde{SO}(m) \cong SO(m) \times \mathbb{Z}_{2}.
\]

In the next subsection, we prove some strange nature of partial polar elements in the normal ordered expression.

### 1.3 Partial polar elements are double-valued

For every \( \bar{a} \in S_{C}^{m-1} \), we consider
\[
\varepsilon_{00}(\bar{a}) = e_{*}^{rac{\overline{\langle \bar{a}, \bar{u} \rangle}}{\varepsilon_{00}^{-1}(\bar{a})}} = e_{*}^{-\frac{\overline{\langle \bar{a}, \bar{u} \rangle}}{\varepsilon_{00}^{-1}(\bar{a})}}.
\]

If \( \langle \bar{a}, \bar{a} \rangle = 1 \), then \( \langle \bar{a}, \bar{u} \rangle \) and \( \langle \bar{a}, \bar{v} \rangle \) form a canonical conjugate pair. In this subsection, we set \( u = \langle \bar{a}, \bar{u} \rangle \) and \( v = \langle \bar{a}, \bar{v} \rangle \) for simplicity.

In the normal ordered expression \( K_0 \), we set
\[
\frac{\varepsilon_{00}(au^2 + bv^2 + 2cu^1v)}{\varepsilon_{00}^{-1}(\bar{a})} = \psi(t)e^{X(t)u^2 + Y(t)v^2 + 2Z(t)uv}.
\]
then we have a system of ordinary differential equations with initial conditions $X(0) = Y(0) = Z(0) = 0$ and $\psi(0) = 1$

\[
\begin{align*}
X'(t) &= \frac{1}{\hbar} a + 4icX(t) - 4hbX(t)^2 \\
y'(t) &= \frac{1}{\hbar} b + 4ibZ(t) - 4hbZ(t)^2 \\
z'(t) &= \frac{1}{\hbar} c + 2icZ(t) + 2ibX(t) - 4hbX(t)Z(t) \\
\psi'(t) &= -2hbX(t)\psi(t)
\end{align*}
\]

Solving the evolution equation (1.14), we have the solution (1.15) for $D = c^2 - ab = 1$.

\[
\begin{align*}
x(t) &= \frac{a}{2}\frac{\sin(2t)}{\cos(2t) - ic\sin(2t)} \\
y(t) &= \frac{b}{2}\frac{\sin(2t)}{\cos(2t) - ic\sin(2t)} \\
z(t) &= \frac{1}{2}\left(1 - \frac{1}{\cos(2t) - ic\sin(2t)}\right) \\
\psi(t) &= \frac{e^{-itc}}{\sqrt{\cos(2t) - ic\sin(2t)}}
\end{align*}
\]

Readers have only to check this is a real analytic solution, and need not to care about how this form is obtained.

Although $e^{\frac{t}{\hbar}(iau+ibv+c(\omega_1+\omega_2))}$ diverges in the Weyl ordered expression, recall that its normal ordered expression has been given in the previous section.

**Strange behaviour of solutions**

Here, we mention an anomalous behaviour of the solution in the normal ordered expression. We think these have never emphasized in physics literatures, but these are crucial from our viewpoint.

1. **(a)** It is remarkable that if $c \neq 0$, e.g. $c = \pm 1$, then $\sqrt{\cos(2t) - ic\sin(2t)}$ must change sign on $[0, \pi]$, since the curve turning around the origin. Thus, one has to set $\sqrt{\cos(2\pi) - ic\sin(2\pi)} = -1$, whenever $\sqrt{\cos(0) - ic\sin(0)} = 1$ is needed. Thus, we have

\[
\begin{align*}
: e_{+}^{\frac{t}{\hbar}(au^2+bu^2+2cu^v)} &:_{K_0} = -e^{-\pi itc}
\end{align*}
\]

depending only on $c$ whenever $c^2 - ab = 1$.

2. **(b)** Branched singular points are distributed $\pi$-periodically along two lines parallel to the real line lying in upper or lower half-plane depending on $c$. On the other hand, if $c = 0$, $\sqrt{\cos(2t)}$ has two singular points at $t = \frac{\pi}{4} + \frac{2\pi}{4} + \frac{3\pi}{4}$.

3. **(c)** Along the pure-imaginary direction, if $|\text{Re} c| < 1$, then $e_{+}^{\frac{t}{\hbar}(au^2+bu^2+2cu^v)}{:_{K_0}}$ is rapidly decreasing in both sides. Hence, the integral $\int_{\mathbb{R}} : e_{+}^{\frac{t}{\hbar}(au^2+bu^2+2cu^v)}{:_{K_0}}$ converges. If $\text{Re} c = 1$, then we have nonvanishing limits

\[
\lim_{t \to \infty} : e_{+}^{\frac{t}{\hbar}(au^2+bu^2+2cu^v)}{:_{K_0}} = \lim_{t \to -\infty} : e_{+}^{\frac{t}{\hbar}(au^2+bu^2+2cu^v)}{:_{K_0}}.
\]
The exponential function, we must set:
\[ e^{(1.17)} \]
connected in the set \( \varepsilon \).
These limits play the role of ground states (vacuums) in matrix representations (cf. (2.17)).

The most strange phenomenon will be seen below:
(d) Set \( u+v=\frac{1}{2}(u+v+u*) \). By noting that \( 2u+v=2u+v+i\hbar \), \( (1.15) \) shows that the term \( e^{-it\varepsilon} \) disappear in the normal ordered expression of \( e^{\frac{it}{\hbar}((u^2+b^2+2cu+iv)^*)} \), and hence at \( t=\pi \), we have
\[ :e^{\frac{\pi}{\hbar}((u^2+b^2+2cu+iv)^*)}:_{K_0} = -1 \]
independent of \( a, b, c \) whenever \( c^2-\text{ab}=1 \). Moreover, at \( t=\frac{\pi}{2} \) we have
\[ :e^{\frac{\pi}{\hbar}((u^2+b^2+2cu+iv)^*)}:_{K_0}=\sqrt{-1}e^{-\frac{\pi}{\hbar}uv} \]
independent of \( a, b, c \) whenever \( c^2-\text{ab}=1 \). Since the manifold \( \{c^2-\text{ab}=1\} \) is arcwise connected, the sign ambiguity of \( \sqrt{-1} \) must be eliminated, and
\[ \{ :e^{\frac{\pi}{\hbar}((u^2+b^2+2cu+iv)^*)}:_{K_0}; a, b, c\in\mathbb{C}, \ c^2-\text{ab}=1 \} \]
must be viewed as a single element. In particular, since \( (a, b, c) = (0, 0, 1) \) and \( (0, 0, -1) \) are arcwise connected in the set \( c^2-\text{ab}=1 \), we have
\[ (1.17) :e^{\frac{\pi}{\hbar}2uv}:_{K_0} = \sqrt{-1}e^{-\frac{\pi}{\hbar}uv} = :e^{-\frac{\pi}{\hbar}2uv}:_{K_0} \]
Recall that the \(*\)-exponential function \( :e^{\frac{\pi}{\hbar}2uv}:_{K_0} \) is holomorphic for \( t \in \mathbb{C} \) and by the definition of the exponential function, we must set \( :e^{\frac{\pi}{\hbar}2uv}:_{K_0} = -1 \) and \( :e^{\frac{\pi}{\hbar}2uv}:_{K_0} = ie^{-\frac{\pi}{\hbar}uv} \) by fixing 1 at \( t=0 \). The exponential law gives
\[ (1.18) :e^{\frac{\pi}{\hbar}2uv} * :e^{\frac{\pi}{\hbar}2uv}:_{K_0} = :e^{\frac{\pi}{\hbar}2uv}:_{K_0} = -1. \]
Hence, we have the anomalous identity
\[ :e^{\frac{\pi}{\hbar}2uv}:_{K_0} = -ie^{-\frac{\pi}{\hbar}2uv}:_{K_0}, \]
which appears to contradict to (1.17). Note that the exponential law is established by the uniqueness of the real analytic solution of the evolution equation.

Indeed, this is not a contradiction, but \( :e^{\frac{\pi}{\hbar}2uv}:_{K_0} \) is a double-valued single element caused since the ambiguity of \( \sqrt{-1} \) cannot be removed. We called it the polar element and denote it by \( \varepsilon_{00} \). \( \varepsilon_{00} \) is an element something like \( \sqrt{-1} \), or an operator whose eigenvalues are \( \pm i \).

In fact, this pathological phenomenon is caused by singular points of \( :e^{\frac{\pi}{\hbar}((u^2+b^2+2cu+iv)^*)}:_{K_0} \) lying in the domain \( \{(t, a, b, c); 0<t<\frac{\pi}{2}, c^2-\text{ab}=1\} \) (cf. [11], [12]). This is just like \( \frac{1}{\sqrt{\cos(2t)}} \) in (b) above.

To explain this, we first note that \( \frac{1}{\sqrt{2}}(u^2+v^2), u]=2v, \frac{1}{\sqrt{2}}(u^2+v^2), v]=-2u. \) By this we have for each \( \theta \), one parameter groups \( \text{Ad}(e^{\frac{\pi}{\hbar}((u^2+v^2)^*)})e^{2uv} \) with respect to \( t \):
\[ \text{Ad}(e^{\frac{\theta}{\hbar}((u^2+v^2)^*)})e^{2uv} = e^{\frac{\theta}{\hbar}((u^2+v^2)^*)}e^{2uv} = e^{2uv} \]
In particular, setting \( t = \frac{\pi}{2\hbar} \) we have
\[
\text{Ad}(e^{i\theta(u^2+v^2)} e^{\pi u^2v}) = e^{-\frac{s}{4} u^2v}.
\]

On the other hand, if we fix \( t = \frac{\pi}{2} \) first, then the element \( \text{Ad}(e^{i\theta(u^2+v^2)} e^{00} : K_0) \) is independent of \( \theta \) by the formula mentioned in (d), since the discriminant of the quadratic form of the r.h.s. is identically 1. The normal ordered expression of the r.h.s. is identically:
\[
\text{Ad}(e^{i\theta(u^2+v^2)} e^{\pi u^2v}) = e^{\frac{s}{4} u^2v}.
\]

In fact, the r.h.s. \( e^{\frac{s}{4} u^2v} \) is not on the exponential function but on another “exponential function” in the opposite sheet which is \(-1\) at \( t = 0 \). Exchanging sheet is caused by the branched singular point at \( 2\theta = \frac{\pi}{2} \).

**Remark** Although examples in this section are stated in the normal ordered expression for various quadratic forms, \([2,23]\) in the next section shows that same phenomena must appear for a single quadratic form \( \frac{1}{4\hbar} u^2v \) under various expression parameters.

## 2 General product formulas and intertwiners

To understand strange phenomena mentioned in the previous section, we are requested to make a wider view about expression for elements of transcendentally extended Weyl algebra.

For that, we start in a general setting as follows: Let \( \mathcal{G}(n) \) and \( \mathfrak{A}(n) \) be the spaces of complex symmetric matrices and skew-symmetric matrices respectively, and \( \mathcal{M}(n) = \mathcal{G}(n) \oplus \mathfrak{A}(n) \). For an arbitrary fixed \( n \times n \)-complex matrix \( \Lambda \in \mathcal{M}(n) \), we define a product \( \ast_{\Lambda} \) on the space of polynomials \( \mathbb{C}[u] \) by the formula
\[
f \ast_{\Lambda} g = f e^{\frac{\pi i}{2} \sum \delta_{ii}\Lambda_{ij} \partial_{ij}} g = \sum \frac{(i\hbar)^k}{k!} \Lambda_{11} \cdots \Lambda_{kk} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g.
\]

It is known and not hard to prove that \((\mathbb{C}[u], \ast_{\Lambda})\) is an associative algebra. Clearly, if \( \Lambda \) is symmetric, then the obtained algebra is commutative and it is isomorphic to the standard polynomial algebra with \( \hbar \).

For every \( \Lambda \), \( \partial_{u_i} \) acts as a derivation of the algebra \((\mathbb{C}[u], \ast_{\Lambda})\). Noting this, for any other constant symmetric matrix \( K \), define a new product \( \ast_{\Lambda,K} \) by the formula
\[
f \ast_{\Lambda,K} g = f e^{\frac{\pi i}{2} \sum \delta_{ii} \Lambda_{ij} \partial_{ij} + \delta_{ii} \partial_{ij}} g
\]
\[
= \sum \frac{(i\hbar)^k}{k!} K_{11} \cdots K_{kk} (\partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f) \ast_{\Lambda} (\partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g).
\]

This is also an associative algebra \((\mathbb{C}[u], \ast_{\Lambda,K})\). Since \( \Lambda, K \) are constant matrices and the non-commutativity of matrix algebra is not used in calculation of product formulas, the new product formula can be rewritten as
\[
f \ast_{\Lambda,K} g = \sum \frac{(i\hbar)^k}{k!} ((\Lambda+K)_{11} \cdots (\Lambda+K)_{kk} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g.
\]
by noting that exchanging indices of $\partial_{u_1 \cdots u_k}$ is permitted. That is, $\star_{\Lambda, K} = \star_{\Lambda + K}$.

This formula may be written as

$$fe^{\frac{ih}{2}(\sum \tilde{b}_{ij}(\Lambda + K)_{ij} \tilde{\alpha}_{ij})} g = fe^{\frac{ih}{2}(\sum \tilde{b}_{ij}K_{ij}K_{ij}^{\dagger}(\sum \tilde{b}_{ij}^{\dagger} \Lambda_{ij} \tilde{\alpha}_{ij}) \tilde{\alpha}_{ij})} g. \tag{2.2}$$

Using a symmetric matrix $K$, we compute

$$\frac{1}{(2\pi)^d} \sum K_{ij} \partial_{u_i} \partial_{u_j}^* (g \star f^*)^k = \left( \left( \frac{1}{2}\sum K_{ij} \partial_{u_i} \partial_{u_j} \right)^k \right)^* g.$$ 

Using this formula, we have the following formula:

$$e^{\frac{ih}{2} \sum K_{ij} \partial_{u_i} \partial_{u_j}} \left( \left( e^{-\frac{ih}{2} \sum K_{ij} \partial_{u_i} \partial_{u_j}} f \right)^* \left( e^{-\frac{ih}{2} \sum K_{ij} \partial_{u_i} \partial_{u_j}} g \right) \right) = fe^{\frac{ih}{2}(\sum \tilde{b}_{ij}K_{ij}K_{ij}^{\dagger} \tilde{\alpha}_{ij})} g = f \star_{\Lambda + K} g. \tag{2.3}$$

The next one is proved directly by formula (2.3):

**Corollary 2.1** Let $I_0^K(f) = e^{\frac{ih}{2} \sum K_{ij} \partial_{u_i} \partial_{u_j}} f$ and $I_0^J(f) = e^{-\frac{ih}{2} \sum K_{ij} \partial_{u_i} \partial_{u_j}} f$. Then $I_0^K$ is an isomorphism of $(\mathbb{C}[u] ; \star_{\Lambda})$ onto $(\mathbb{C}[u] ; \star_{\Lambda + K})$.

Set $\Lambda = K + J$ where $K$, $J$ are the symmetric part and the skew part of $\Lambda$, respectively. Since the commutator $[u_i, u_j] = ihJ_{ij}$ is given by the skew part of $\Lambda$, the algebraic structure of $(\mathbb{C}[u] ; \star_{\Lambda})$ depends only on $J$, whose isomorphism class may be denoted by $(\mathbb{C}[u] ; \star_J)$ or simply by $(\mathbb{C}[u] ; \star)$ by noticing this class consists of a single algebra.

It is clear that the product $f \star_{\Lambda} g$ is defined if one of $f, g$ is a polynomial and another is a smooth function.

Let $\text{Hol}(\mathbb{C}^n)$ be the space of all holomorphic functions on the complex $n$-plane $\mathbb{C}^n$ with the uniform convergence topology on each compact domain. The following two propositions are useful:

**Proposition 2.1** $\text{Hol}(\mathbb{C}^n)$ with the topology above is a Fréchet space defined by a countable family of seminorms.

**Proposition 2.2** For every polynomial $p(u) \in \mathbb{C}[u]$, the left-multiplication $f \rightarrow p(u) \star_{\Lambda} f$ and the right-multiplication $f \rightarrow f \star_{\Lambda} p(u)$ are both continuous linear mappings of $\text{Hol}(\mathbb{C}^n)$ into itself.

If two of $f, g, h$ are polynomials, then associativity $(f \star_{\Lambda} g \star_{\Lambda} h) = f \star_{\Lambda} (g \star_{\Lambda} h)$ holds.

### 2.1 Expression parameters and intertwiners

As used in the previous section, we recall notations

$$u = (u_1, u_2, \cdots, u_{2m}) = (\tilde{u}, \tilde{v}), \quad \tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_m), \quad \tilde{v} = (\tilde{v}_1, \cdots, \tilde{v}_m). \tag{2.4}$$

The skew part $J$ is fixed to be the standard skew-symmetric matrix $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. The algebra is called the Weyl algebra and the isomorphism class is denoted by $W_h(2m)$. 
In the standard theory of algebraic system, we are only concerned with isomorphism classes. For the case of a universal enveloping algebra of a Lie algebra, Poincaré-Birkhoff-Witt theorem ensures that this is realized on the space of ordinary polynomials by giving a new associative product. However, there is no standard way of unique expression of elements for algebras.

Note that if the generator system is fixed, then the product formula (2.1) also gives the unique expression of elements of this algebra by the ordinary polynomials.

For instance, computing $u_i u_j da u_k$ using (2.1) gives the expression of $u_i u_j u_k$ as a polynomial.

Thus, the product formula (2.1) will be referred as a $K$-ordered expression, i.e. if generators are fixed, giving an ordered expression is nothing but giving a product formula on the space of polynomials which defines the Weyl algebra $W_\hbar(2m)$.

By this formulation of orderings, the intertwiner between $K$-ordered expressions and $K'$-ordered expressions is explicitly given as follows:

**Proposition 2.3** For every $K, K' \in \mathfrak{S}(n)$, the intertwiner is defined by

$$I_K^{K'}(f) = \exp \left( \frac{i \hbar}{4} \sum_{i,j} (K'_{ij} - K_{ij}) \partial_{u_i} \partial_{u_j} \right) f = \left( I_0^{K'} \right)^{-1}(i_0^K f),$$

and by (2.3), it gives an isomorphism $I_K^{K'} : (\mathbb{C}[u]; *_{K,J}) \to (\mathbb{C}[u]; *_{K',J})$. Namely, the following identity holds for any $f, g \in \mathbb{C}[u]$:  

$$I_K^{K'}(f *_{\Lambda} g) = I_K^{K'}(f) *_{\Lambda'} I_K^{K'}(g),$$

where $\Lambda = K + J$, $\Lambda' = K' + J$.

Intertwiners do not change the algebraic structure $*$, but do change the expression of elements by the ordinary commutative structure.

If the skew part $J$ is fixed, we often use notation $*_{K}$ instead of $*_{\Lambda}$.

As in the case of one variable, infinitesimal intertwiner

$$dI_K(K') = \frac{d}{dt}|_{t=0} I_K^{K+tk'} = \frac{1}{4i\hbar} K'_{ij} \partial_{u_i} \partial_{u_j}$$

is viewed as a flat connection on the trivial bundle $\prod_{K \in \mathfrak{S}(n)} \text{Hol} (\mathbb{C}^n)$. The equation of parallel translation along a curve $K(t)$ is given by

$$\frac{d}{dt} f_t = dI_K(\dot{K}(t)) f_t, \quad \dot{K}(t) = \frac{d}{dt} K(t),$$

but this may not have a solution for some initial function.
2.1.1 Several remarks on product formulas and notations

In what follows, we use the notation $*_K$ instead of $*_{\Lambda}$, since the skew part $J$ is fixed as the standard skew-matrix. We use notations

$$
\mathbf{u} = (u_1, u_2, \ldots, u_{2m}) = (\mathbf{u}, \mathbf{v}), \quad \tilde{\mathbf{u}} = (\tilde{u}_1, \ldots, \tilde{u}_m), \quad \mathbf{v} = (\tilde{v}_1, \ldots, \tilde{v}_m).
$$

All results in [11] hold for functions $f(\langle \mathbf{a}, \mathbf{u} \rangle)$ by setting $\tau = i\hbar \langle \mathbf{a}K, \mathbf{a} \rangle$.

Note that according to the choice of $K = 0, K_0, -K_0, I$, where

$$
(0, K_0, -K_0, I) = \begin{pmatrix}
0 & 0 & I & 0 \\
0 & I & 0 & -I \\
I & -I & I & 0 \\
0 & I & 0 & I
\end{pmatrix},
$$

the product formulas (2.1) give the Weyl ordered expression, the normal ordered expression, the antinormal ordered expression respectively, but the last one, called the unit ordered expression is not so familiar in physics.

For each ordered expression, the product formulas are given respectively by the following formulas:

\[ f(\mathbf{u})*_{\mathbf{g}} f(\mathbf{u}) = f \exp \frac{i\hbar}{2} \{ \tilde{\partial}_v \wedge \tilde{\partial}_u \} g, \quad (\text{Moyal product formula}) \]

\[ f(\mathbf{u})*_{K_0} f(\mathbf{u}) = f \exp i\hbar \{ \tilde{\partial}_v \partial_u \} g, \quad (\PsiDO-product formula) \]

\[ f(\mathbf{u})*_{-K_0} f(\mathbf{u}) = f \exp -i\hbar \{ \tilde{\partial}_u \partial_v \} g, \quad (\overline{\PsiDO}-product formula) \]

where $\tilde{\partial}_v \wedge \tilde{\partial}_u = \sum_i (\tilde{\partial}_{\tilde{v}_i} \partial_{\tilde{a}_i} - \tilde{\partial}_{\tilde{a}_i} \partial_{\tilde{v}_i})$ and $\tilde{\partial}_v \partial_u = \sum_i \tilde{\partial}_{\tilde{v}_i} \partial_{\tilde{a}_i}$.

The product formula for the unit ordered expression is a bit complicated to write down, but it is easy to obtain. For instance,

$$
u_i *_{\mathbf{g}} \nu_i = \nu_i^2 + \frac{i\hbar}{2}, \quad \nu_i *_{\mathbf{g}} e^{-\frac{\hbar}{2m} \nu_i^2/2} = \left( e^{-\frac{\hbar}{2m} \nu_i^2} *_{\mathbf{g}} \right) \nu_i \quad \text{etc}
$$

while the Weyl ordered expression gives

$$
\tilde{v}_i *_{\mathbf{g}} e^{-\frac{\hbar}{2} \tilde{v}_i \tilde{v}_i} = 0 = e^{-\frac{\hbar}{2m} \tilde{v}_i \tilde{v}_i} *_{\mathbf{g}} \tilde{v}_i.
$$

Formulas of unit ordered expression mainly obtained via the intertwiners mentioned above.

For $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{2m}$, we set $\langle \mathbf{a} \Gamma, \mathbf{b} \rangle = \sum_{ij=1}^{2m} \Gamma_{ij} a_i b_j$, $\langle \mathbf{a}, \mathbf{u} \rangle = \sum_{i=1}^{2m} a_i u_i$. These will be denoted also by $\mathbf{a} \Gamma \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{u} \rangle = \mathbf{a} \cdot \mathbf{u}$.

For $f(\mathbf{u}) \in Hol(\mathbb{C}^{2m})$, the direct calculation via the product formula (2.1) by using Taylor expansion gives the following:
\[ e^{\frac{i}{\hbar}(a,u)} f(u) = e^{\frac{i}{\hbar}(a,u)} f(u + \frac{s}{2}a(K+J)), \]
\[ f(u)^s e^{-\frac{i}{\hbar}(a,u)} = f(u + \frac{s}{2}a(-K+J)) e^{-\frac{i}{\hbar}(a,u)} \]

as natural extension of the product formula. This gives also the associativity of computations involving two functions of exponential growth and a holomorphic function.

Throughout this series, we use notation \( \cdot_k \) to indicate the expression parameter for elements of \( W_k(2m) \). For instance, we write
\[ :u_i* u_j;_k = u_i u_j + \frac{i}{2}(K+J)_{ij}, \, \text{etc.} \]

**2.1.2 For the case \( m = 1 \)**

In the case \( m = 1 \), it is convenient to use notations \((\tilde{u}, \tilde{v})\) for \((u_1, u_2)\). A remarkable feature of the first three formulas of (2.8) is seen as follows:

\[
\begin{align*}
\sum a_{kl}u^k v^l,_{\cdot K_0} &= \sum a_{kl}u^k v^l, \quad \text{(normal ordering)}, \\
\sum a_{kl}u^k v^l,_{\cdot -K_0} &= \sum a_{kl}u^k v^l, \quad \text{(anti-normal ordering)}, \\
\frac{1}{2}(\tilde{u}^2 + \tilde{v}^2;_0) &= \tilde{u}^2, \quad \frac{1}{3}(\tilde{u}^2 + \tilde{v}^2 + \tilde{u}^2)_0 = \tilde{u}^2, \\
\frac{1}{6}(\tilde{u}^2 + \tilde{v}^2 + \tilde{u}^2 + \tilde{v}^2 + \tilde{u}^2)_0 &= \tilde{u}^2 \tilde{v}^2, \, \text{etc.}.
\end{align*}
\]

In general, define \( W_* (\tilde{u}^k \tilde{v}^l) \) by \( \frac{1}{(k+l)!} \sum X_1 X_2 \cdots X_{k+l} \), where \( X_i \) is \( \tilde{u} \) or \( \tilde{v} \) and the summation runs through all possible rearrangement of \( \tilde{u}^k \tilde{v}^l \).

\[ (\tilde{u} + \tilde{v})_*^n = \sum_k n C_k W_* (\tilde{u}^k \tilde{v}^{n-k}). \]

It is easy to see
\[ \sum a_{kl}W_* (\tilde{u}^k \tilde{v}^l)_0 = \sum a_{kl}\tilde{u}^k \tilde{v}^l. \]

The next result is trivial, but important.

**Proposition 2.4** If \( K \) is fixed, then every entire function \( f(\tilde{u}, \tilde{v}) = \sum a_{kl} \tilde{u}^k \tilde{v}^l \) can be viewed as a \( K \)-ordered expression of an element.

However it is not easy to write down the relations between elements written by different expression parameters.

Set \( H_* = a \tilde{u}^2 + b \tilde{v}^2 + 2c \tilde{u} \tilde{v}, \quad \tilde{u} \tilde{v} = \frac{1}{2}(\tilde{u}^2 + \tilde{v}^2 + \tilde{u} \tilde{v}), \) and \( c^2-ab = D \). It is easy to see that \( H_*;_0 = a \tilde{u}^2 + b \tilde{v}^2 + 2c \tilde{u} \tilde{v}. \) The normal ordered expression of the \(*\)-exponential function \( e_*^{t(a \tilde{u}^2 + b \tilde{v}^2 + 2c \tilde{u} \tilde{v})} \) is given by (1.15) by removing \( e^{-itc} \) in the amplitude part. In this section, its Weyl ordered expression \( e_*^{t(a \tilde{u}^2 + b \tilde{v}^2 + 2c \tilde{u} \tilde{v})}_0 \) will be given.
For that purpose we set \( e^{t(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v})} : 0 = F(t, \dot{u}, \dot{v}) \) and consider the real analytic solution of the evolution equation

\[
\frac{\partial}{\partial t} F(t, \dot{u}, \dot{v}) = (a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}) \ast_0 F(t, \dot{u}, \dot{v}), \quad F(0, \dot{u}, \dot{v}) = 1
\]

(2.11)

By the Moyal product formula in \( \S 2.1 \) we have

\[
(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}) \ast_0 F(t, \dot{u}, \dot{v}) = (a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}) F + \hbar \{(b\dddot{v} + c\dot{u})\partial_t F - (a\dddot{u} + c\dot{v})\partial_t F\}
\]

\[
- \frac{\hbar^2}{4} \{b\partial_u^2 F - 2c\partial_v \partial_u F + a\partial_v^2 F\}
\]

Keeping the uniqueness of the real analytic solution in mind, we set by using a function \( f(x) \) of one variable

\[
e^{t(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v})} : 0 = f_1(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v})
\]

to obtain a simplified form

\[
(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}) \ast_0 f_1(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}) = (a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}) f_1(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v})
\]

\[
- \hbar^2 (ab - c^2) (f_1'(a\ddot{u}^2 + b\ddot{v}^2 - 2c\dot{u}\dot{v}) + f_1''(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v})(a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v}))
\]

Setting \( x = a\ddot{u}^2 + b\ddot{v}^2 + 2c\dot{u}\dot{v} \), we obtain the equation

\[
\frac{d}{dt} f_1(x) = xf_1'(x) + \hbar^2 D(f_1'(x) + xf_1''(x))
\]

(2.12)

where \( D = c^2 - ab \) is the discriminant of \( H_* \).

**Lemma 2.1** The solution of the differential equation (2.12) with the initial function 1 is

\[
e^{tH_*} : 0 = \frac{1}{\cos(\hbar\sqrt{D} t)} \exp \{ - \frac{1}{\hbar\sqrt{D}} \tan(\hbar\sqrt{D} t) \}.
\]

**Proof** Set \( f_1(x) = g(t)e^{h(t)x} \). Plugging this to obtain

\[
\{g'(t) - Dh^2 g(t)h(t) + xg(t)\{h'(t) - 1 - Dh^2 h(t)^2\}\}e^{h(t)x} = 0.
\]

Hence, \( h'(t) - 1 - Dh^2 h(t)^2 = 0 \). By this \( h(t) \) is obtained as

\[
h(t) = \frac{1}{\hbar\sqrt{D}} \tan(\hbar(\sqrt{D})t).
\]

The sign ambiguity of \( \sqrt{D} \) does not suffer the result.

Next, solving

\[
g'(t) - g(t)Dh^2 \frac{1}{\hbar\sqrt{D}} \tan(\hbar(\sqrt{D})t) = 0
\]

we have \( g(t) = \frac{1}{\cos(\hbar\sqrt{D} t)} \). The sign ambiguity of \( \pm \sqrt{D} \) does not suffer the result, and \( t \) is allowed to be a complex number. \( \Box \)

Consequently, we have the following:
Theorem 2.1 The Weyl ordered expression of the *-exponential function $e_*^{t(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}$ is given by

$$e_*^{t(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}_0 = \frac{1}{\cos(h\sqrt{D} t)} \exp \left( \frac{1}{h\sqrt{D}} \tan(h\sqrt{D} t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v}) \right)$$

where $\frac{1}{h\sqrt{D}} \tan(h\sqrt{D} t) = t$ in the case $D = 0$.

Remark. A partial polar element $\varepsilon_{00}(a)$ cannot be expressed by the Weyl ordered expression.

If $D \neq 0$, then this case is represented by the case $D = 1$ where $H_*$ is viewed as

$$H_* = \frac{1}{2} \left( (\alpha\bar{u} + \beta\bar{v})^* (\gamma\bar{u} + \delta\bar{v}) + (\gamma\bar{u} + \delta\bar{v})^* (\alpha\bar{u} + \beta\bar{v}) \right), \quad [(\alpha\bar{u} + \beta\bar{v}), (\gamma\bar{u} + \delta\bar{v})] = -i\hbar,$$

the canonical conjugate pair.

If $D = 1$, then $e_*^{t(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}_0$ is singular at $t = \pm \frac{\pi}{2\hbar}$ in the Weyl ordered expression. However, this does not imply that $e_*^{t(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}$ is singular at $t = \pm \frac{\pi}{2\hbar}$, since this singular point disappears in the normal ordered expression as it will be seen in §1.3. Singular points depend on expression parameters.

By noting $\cosh(is) = \cos s$, $\tanh(is) = i \tan s$, we see also

$$e_*^{t(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}_0 = \frac{1}{\cos(h\sqrt{-D} t)} e^{(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})/\sqrt{-D}} \tan(h\sqrt{-D} t)$$

Replacing $t$ by $t/\hbar$ we see that

$$e_*^{\frac{t}{\hbar}H_*}_0 = \frac{1}{\cos \sqrt{D} t} e^{\frac{1}{\hbar} \tan(\sqrt{D} t)H_*}_0$$

It may be better to rewrite this as

$$e_*^{\frac{t}{\hbar}H_*}_0 = \frac{1}{\cosh \sqrt{D} t} e^{\frac{1}{\hbar} \tan(\sqrt{D} t)H_*}_0.$$  

This is rapidly decreasing on $\mathbb{R}$.

Although $e_*^{\frac{t}{\hbar}H_*}_0$ has singularities, the exponential law holds by the uniqueness of real analytic solution of the defining equation:

$$e_*^{(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})} * e_*^{(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})} = e_*^{(s+t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}$$

$$e_*^{(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v} + \alpha)} = e^{\alpha t} e_*^{(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}, \quad \alpha \in \mathbb{C}.$$  

Recalling $2\bar{u}^2 \bar{v} = 2\bar{u}\bar{v} - i\hbar$, we have by (2.16)

$$\lim_{t \to \infty} e_*^{\frac{t}{\hbar}(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}_0 = \lim_{t \to \infty} e^{\frac{t}{\hbar}(\tan(t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v}) + e^{\frac{1}{\hbar} \tan(t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})} = e^{\frac{1}{\hbar} \tan(t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})},$$

$$\lim_{t \to -\infty} e_*^{\frac{t}{\hbar}(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}_0 = \lim_{t \to -\infty} e^{\frac{-t}{\hbar}(\tan(t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v}) + e^{\frac{-1}{\hbar} \tan(t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})} = e^{\frac{-1}{\hbar} \tan(t)(a\bar{u}^2 + b\bar{v}^2 + 2c\bar{u}\bar{v})}.}$$
These have idempotent property by the exponential law. These elements are called vacuums in what follows. Note that such elements are not classical elements for they are not defined for $h = 0$. On the other hand, we easily see

$$
\lim_{t \to -\infty} e^\frac{t}{\hbar} (a\hat{u}^2 + b\hat{v}^2 + 2c\hat{u}\hat{v}) = 0, \quad \lim_{t \to \infty} e^\frac{t}{\hbar} (a\hat{u}^2 + b\hat{v}^2 + 2c\hat{u}\hat{v}) = 0.
$$

Therefor an operation such as

$$
\lim_{t \to \infty} e^\frac{t}{\hbar} (a\hat{u}^2 + b\hat{v}^2 + 2c\hat{u}\hat{v}) \star p(\hat{u}, \hat{v}) \star e^\frac{-t}{\hbar} (a\hat{u}^2 + b\hat{v}^2 + 2c\hat{u}\hat{v})
$$

is not defined.

### 2.2 Star-exponential functions of linear functions

By a direct calculation of intertwiners, we see that

$$
I_K (e^{\frac{t}{\hbar}}(a, u)) = e^{\frac{t}{\hbar}}(a(e^{t}K) - K, a) e^{\frac{t}{\hbar}}(a, u).
$$

Hence, $\{e^{\frac{t}{\hbar}}(aK, a) e^{\frac{t}{\hbar}}(a, u) ; K \in \mathcal{S}(2m)\}$ is a parallel section of $\prod_{K \in \mathcal{S}(2m)} \text{Hol}(\mathbb{C}^{2m})$.

We shall denote this element symbolically by $e^{\frac{t}{\hbar}}(a, u)$. Namely, we define

$$
e^{\frac{t}{\hbar}}(a, u) : = e^{\frac{t}{\hbar}}(aK, a) e^{\frac{t}{\hbar}}(a, u) = e^{\frac{t}{\hbar}}(aK, a) e^{\frac{t}{\hbar}}(a, u).
$$

It is remarkable that if $K = 0$, then $e^{\frac{t}{\hbar}}(a, u) : = e^{\frac{t}{\hbar}}(a, u)$, that is, $\star$-exponential functions of linear functions are ordinary exponential functions.

By using the product formula for $K$-ordered expression, we have easily the exponential law

$$
e^{\frac{t}{\hbar}}(a, u) \star e^{\frac{t}{\hbar}}(a, u) \star e^{\frac{t}{\hbar}}(a, u) \star e^{(s+t)\frac{1}{\hbar}}(a, u) = e^{t} e^{\frac{t}{\hbar}}(a, u) = e^{\frac{t}{\hbar}}(a, u)
$$

The exponential law may be written by omitting the suffix $K$ as

$$
e^{\frac{t}{\hbar}}(a, u) \star e^{\frac{t}{\hbar}}(a, u) \star e^{\frac{t}{\hbar}}(a, u) \star e^{(s+t)\frac{1}{\hbar}}(a, u) = e^{t} e^{\frac{t}{\hbar}}(a, u) = e^{\frac{t}{\hbar}}(a, u)
$$

together with the exponential law with the ordinary exponential functions.

Furthermore, for every $K$, $e^{\frac{t}{\hbar}}(a, u)$ is the solution of the evolution equation

$$
\frac{d}{dt} e^{\frac{t}{\hbar}}(a, u) = \frac{1}{i\hbar} (a, u) \cdot e^{\frac{t}{\hbar}}(a, u) \cdot e^{\frac{t}{\hbar}}(a, u) \cdot e^{(s+t)\frac{1}{\hbar}}(a, u) \text{ with initial data } 1: K = 1.
$$

Note that $e^{(a, u)} : = e^{(a, u)}$. $e^{\frac{t}{\hbar}}(a, u) = \{e^{\frac{t}{\hbar}}(aK, a) e^{\frac{t}{\hbar}}(a, u) ; K \in \mathcal{S}(2m)\}$ forms a one parameter group of parallel sections.

By applying (2.19) to $e^{\pm \frac{t}{\hbar}}(a, u)$ carefully, we have for every $f \in \text{Hol}(\mathbb{C}^{n})$ that

$$
: e^{\frac{t}{\hbar}}(a, u) \star f(a, u) \star e^{\frac{-t}{\hbar}}(a, u) : = f(a + sa)J : = e^{\frac{t}{\hbar}}(a, u) \star (f(a, u) \star e^{\frac{-t}{\hbar}}(a, u)) : K.
$$
This gives also the associativity and the real analyticity of $e_{s^1}(a'u) * f_s(u) * e_{s^{-1}}(a'u)$ in $s$. However if we know the associativity in advance by using Theorem 2.2 for instance, then it is better to compute as follows:

Differentiating $F_s(s) = e_{s^1}(a'u) * f_s(u) * e_{s^{-1}}(a'u)$ in $s$, we have

$$\frac{d}{ds} F_s(s) = \left[ \frac{1}{i\hbar} \langle a, u \rangle, F_s(s) \right], \quad F_s(0) = f_s(u).$$

On the other hand, $f_s(u + saJ)$ satisfies the same equation

$$\frac{d}{ds} f_s(u + saJ) = \left[ \frac{1}{i\hbar} \langle a, u \rangle, f_s(u + saJ) \right].$$

Thus, the uniqueness of real analytic solution gives (2.20).

The product formula gives

(2.21) $$e_{s^1}(a'u) * e_{s^1}(b'u) = e_{s^1}(a'b') * e_{s^1}(a+b'u).$$

This is equivalent with $e_{s^1}(a'u) * e_{s^1}(b'u) = e_{s^1}(a'b', a+b'u).$. This forms a noncommutative group isomorphic to the group $\mathbb{C}^{2m} \times \mathbb{C}$ with the group structure

(2.22) $$(a, \lambda) * (b, \mu) = (a+b, \lambda+\mu+\frac{1}{2}\langle aJ, b \rangle).$$

This is viewed as a central extension of the abelian group $\mathbb{C}^{2m}$, called sometimes Heisenberg group. The algebra generated by $\{e_{s^1}(a'u) : a \in \mathbb{C}^n\}$ is called the noncommutative torus.

Remark for notations of $*$-products. Since $*$-commutators are independent of expression parameters, we often omit the suffix : $\mathcal{K}$ or $\mathcal{K}$ in computations involving only commutation relations.

It is obvious that the correspondence

$$x \rightarrow e_{s^1}(x'u), \quad c \rightarrow e^c$$

gives an isomorphism of Heisenberg group onto the noncommutative torus.

| $e_{s^1}(a'u)$ | $e_{s^1}(b'u)$ | $e_{s^1}(a+b'u)$ |
|----------------|----------------|------------------|
| $e_{s^1}(a'u)$ | $e_{s^1}(b'u)$ | $e_{s^1}(a+b'u)$ |

2.2.1 Linear change of generators

Next, we consider the effect of a linear change of generators

$$u_i' = \sum u_k S^k_i, \quad S \in GL(n, \mathbb{C}), \quad (u' = uS).$$

By the help that

$$\partial u_i = \sum S^k_i \partial u_k'$$

the product formula is rewritten by using new generators as

(2.23) $$f *_\Lambda g = f e^{\frac{i}{\hbar} \langle x S \partial_i (S^k_i) \partial_j \rangle} g.$$
Thus the notation $\star$ is better to be replaced by $\star^\prime$, where $\Lambda' = \Sigma S AS$. Therefor the algebraic structure of $(\mathbb{C}[u]; \star)$ depends only on the conjugacy class of the skew part $J$.

If $\Sigma S JS = J$, that is, $S$ is a symplectic linear change of generators

$$u_i' = \sum u_k S_i^k, \quad S \in Sp(m, \mathbb{C}),$$

then the mapping $u \rightarrow u'$ does not change the algebraic structure. Thus, a symplectic change of generators is recovered by the intertwiner $I_{K'}^{SKS'}$. Change of generators are viewed often as coordinate transformations, but note here that $I_{K'}^{SKS'}$ is something like the “square root” of symplectic coordinate transformations.

Since $\det S = 1$ for $S \in Sp(m, \mathbb{C})$, we see $\det \Sigma S KS = \det K$, hence the isomorphic change by the intertwiner $I_{K'}^K$ cannot be recovered by a coordinate transformation if $\det K \neq \det K'$.

Even if $S \in GL(n, \mathbb{C})$ is not in $Sp(m, \mathbb{C})$, setting $u_i' = \sum u_k S_i^k$ and $J' = \Sigma S JS$ gives an isomorphism

$$\Phi_S : (\mathbb{C}[u]; \star_J) \rightarrow (\mathbb{C}[u']; \star_{J'}),$$

of Weyl algebras.

Keeping that $u$ are complex variables, we have the following formula: Let $u = u'S + b$, $S \in Sp(m, \mathbb{C})$. Then,

$$e^{\frac{1}{1!}(a, u'S + b)} = e^{\frac{1}{1!}(a, b)} e^{\frac{1}{1!}(a'S, u')}$$

and

$$e^{\frac{1}{1!}(a, u'S + b)}^\prime_{\Sigma S KS, \Sigma K'} = e^{\frac{1}{1!}(a' SKS, a) + \frac{1}{1!}(a, u'S + b)} = e^{\frac{1}{1!}(a' SKS, a)} e^{\frac{1}{1!}(a, u)}$$

where $:\Sigma K'_{\Sigma K'}$ means the $K$-ordered expression with respect to the generator system $u'$. The above formula may be written as the formula

$$\Phi_S^J : e_{a, u}^J\Sigma S K S, \Sigma K'_{\Sigma K'} = e_{a, u}^J\Sigma S K S, \Sigma K'_{\Sigma K'}.$$  

By this formula, we see that the change of expression parameters can be traced by the change of generator systems.

### 2.3 Remarks on real analyticity and associativity

A mapping $f : U \rightarrow F$ from an open subset $U$ of $\mathbb{R}$ into a Fréchet space $F$ is called real analytic, if for every $a \in U$ there is an $\varepsilon(a) > 0$ such that $f$ is written in the form

$$f(a + s) = \sum_k \frac{1}{k!} a_k s^k, \quad a_k \in F, \quad |s| < \varepsilon(a),$$

where $a_k$ is given by $a_k = \partial^k_s f|_{s=0}$.

If $F$ is a Banach space and $\sum_k \frac{1}{k!} ||a_k|| ||s||^k$ converges, then the power series $\sum_k a_k s^k$ is said to converge absolutely under the norm.

If a Fréchet space $F$ is defined by a countable family of seminorms $\{\|f\|_\ell; \ell = 1, 2, 3, \cdots\}$, then replace this part by the absolute convergence of $\sum_k \frac{1}{k!} ||a_k||_\ell ||s||^k$ w.r.t. seminorms $\| \cdot \|_\ell$. A power series $\sum_k a_k s^k$ converges if this converges absolutely under every seminorms.
Lemma 2.2 For a power series \( \sum_k a_k s^k \), \( a_k \in F \), there exists a unique real number \( R \) (\( 0 \leq R \leq \infty \)) satisfying (1) and (2) below:

1. If \( |s| < R \), then the power series \( \sum_k a_k s^k \) converges absolutely under every seminorm \( \| \cdot \|_\ell \).
2. If \( |s| > R \), then \( \sum_k a_k s^k \) does not converge.

Proof Suppose \( \sum_k a_k s^k \) converges at \( s_0 \). Then \( a_k s_0^k \) is bounded under every seminorm \( \| \cdot \|_\ell \). Set \( \sup_k \| a_k s_0^k \|_\ell \leq M_\ell \). Then for every \( s \) such that \( |s| < |s_0| \) we see

\[
\sum_k \| a_k s^k \|_\ell \leq \sum_k M_\ell |s/s_0|^k = M_\ell \frac{1}{1-|s/s_0|} < \infty.
\]

Then the convergence of \( \sum_k a_k s^k \) follows. \( \square \)

Lemma 2.3 \( \sum_{k\geq0} a_k s^k \) and \( \sum_{k\geq1} k a_k s^{k-1} \) have same radius of convergence.

Real analyticity is left invariant under every continuous linear transformation.

Lemma 2.4 Let \( F,G \) be Fréchet spaces and \( \varphi : F \rightarrow G \) be a continuous linear mapping. If \( f : U \rightarrow F \) is real analytic, then \( \varphi f : U \rightarrow G \) is also real analytic.

Remarks for the associativity Products of exponential functions of quadratic forms may not be defined, and even if the product is defined associativity may not hold. In general, we do not have associativity even for a polynomial \( p(u) \)

\[
(e^{H(u)}*p(u))*e^{K(u)}, \quad e^{H(u)}*(p(u)*e^{K(u)}),
\]

since \( p(u) \) has two different \(*\)-inverses in general.

However, if we can treat elements in \( (\mathbb{C}[u[[h]],*_{\lambda}) \), the space of formal power series of \( h \), then \(*_{\lambda}\)-product is always defined by the product formula (2.1) and the associativity holds.

Elements of \( Hol(\mathbb{C}^n) \) are often given as real analytic functions of \( h \) defined on certain interval containing \( h = 0 \). The following is easy to see:

Theorem 2.2 Suppose \( f(h,u), g(h,u) \) and \( h(h,u) \) are given as real analytic function of \( h \) in some interval \( [0,H] \). If all of these

\[
f(h,u)*_{\lambda} g(h,u), \quad (f(h,u)*_{\lambda} g(h,u))*_{\lambda} h(h,u), \quad g(h,u)*_{\lambda} h(h,u), \quad f(h,u)*_{\lambda} (g(h,u)*_{\lambda} h(h,u))
\]

are defined as real analytic functions on \( h \in [0,H] \), then the associativity holds: i.e.

\[
(f(h,u)*_{\lambda} g(h,u))*_{\lambda} h(h,u) = f(h,u)*_{\lambda} (g(h,u)*_{\lambda} h(h,u)).
\]
We refer to this theorem as the **formal associativity theorem**.

**Remark 1.** In what follows, elements are often given in the form \( f(\frac{1}{\hbar}\varphi(t), u) \) by using a real analytic function \( f(t, u), t \in [0, T] \), where \( \varphi(t) \) is a real analytic function such that \( \varphi(0) = 0 \). (Cf. [2.19], [2.13], [1.5]). In such a case, replacing \( t \) by \( \hbar \) gives a real analytic function of \( \hbar \), and such an element is embedded in \( (\mathbb{C}[u][[h]], \ast \kappa) \). Thus, we can apply the above theorem. We call such elements **classical elements**. However, there are many elements in \( \text{Hol}(\mathbb{C}^n) \) written in the form \( f(\frac{1}{\hbar}\varphi(t), u) \) such that \( \varphi(0) \neq 0 \).

Using Lemma [2.1] we have the following:

**Lemma 2.5** Let \( U \) be an connected open neighborhood of 0 of \( \mathbb{R}^l \) Suppose \( \psi : U \to \text{Hol}(\mathbb{C}^n) \) be a real analytic mapping. Then \( x \to p(u) + \psi(x) + q(u) \) is also a real analytic on \( U \) for every polynomial \( p(u), q(u) \).

**Proof** It is easy by using that \( X \to p(u) + X + q(u) \) is a continuous linear mapping. \( \square \)

In the noncommutative torus multiplicative commutators play the same role as commutators:

**Lemma 2.6** The multiplicative commutator gives

\[
: e_x^{-\frac{1}{\hbar}}(b, u) * e_x^{-\frac{1}{\hbar}}(a, u) * e_x^{-\frac{1}{\hbar}}(b, u) * e_x^{-\frac{1}{\hbar}}(a, u) : \kappa = e_x^{-\frac{1}{\hbar}}(b, a)
\]

which belongs to the center independent of expression parameters. For the case \( m = 1 \), the multiplicative commutator gives the area \( (b, J, a) \) of the rectangular domain spanned by \( b = (b_1, b_2) \) and \( a = (a_1, a_2) \).

On the other hand, regarding \( h \) as a member of generators, the Lie algebra generated by \( h \) and \( \{ \langle a, u \rangle, a \in \mathbb{C}^n \} \) with relations \( [\bar{u}_i, \bar{v}_j] = -\sqrt{-1}h \delta_{ij} \) is called also the (complex) **Heisenberg Lie algebra**. Its universal enveloping algebra is called the **Heisenberg algebra**. We denote this algebra by \( \mathcal{H}(2m) \). In contrast with the Weyl algebra \( W_h(2m) \) in previous sections, \( h \) is not treated as a scalar, but a member of generators, hence \( \frac{1}{m} \) is not an element of \( \mathcal{H}(2m) \).

### 2.3.1 Subalgebras and their two-sided ideals

\( (\text{Hol}(\mathbb{C}^{2m}), \ast \kappa) \) contains various systems which closed under the \( \ast \kappa \)-product, which will be called **subalgebras**. Weyl algebra \( (W_h(2m), \ast \kappa) \) is a dense subalgebras.

**Lemma 2.7** There is no nontrivial two-sided ideal of the Weyl algebra \( (W_h(2m), \ast \kappa) \). On the other hand the Heisenberg algebra \( (\mathcal{H}(2m), \ast \kappa) \) has two-sided ideals corresponding to points of \( \mathbb{C}^{2m} \).

**Proof** is easy by observing the following: Suppose \( \psi \) is a homomorphism of an algebra into \( \mathbb{C} \), and suppose \( [x, y]_s = z \), then \( \psi(z) = 0 \). It follows that there is no nontrivial two-sided ideal of the Weyl algebra \( (W_h(2m), \ast \kappa) \).

On the other hand \( h \ast \mathcal{H}(2m) \) is a two-sided ideal of \( \mathcal{H}(2m) \) such that quotient algebra is the usual commutative polynomial ring \( \mathbb{C}[u] \). It is easy to see that for every \( a \in \mathbb{C}^{2m} \), the two-sided ideal of \( \mathbb{C}[u] \) generated by \( u - a \) is pull back to give an nontrivial two-sided ideal of \( \mathcal{H}(2m) \). \( \square \)

Hence, to treat the Heisenberg algebra as a topological algebra, it is better to write the generators as \( \{ i\hbar, (a, u), a \in \mathbb{C}^n \} \) without using \( \frac{1}{\hbar} \). Then, the Heisenberg algebra may be treated in \( (\text{Hol}(\mathbb{C}^{2m+1}), \ast) \). These are seen in [S], pp195-200, pp300-305.
3 Intertwiners for exponential functions of quadratic forms

In this section we investigate intertwiners on the space of exponential functions of quadratic forms $\mathbb{C}e^{\mathbb{S}(2m)}$. This will be used also to obtain $K$-ordered expressions of star-exponential functions of quadratic forms. In the argument in this section, the skew part $J$ of $\Lambda = K + J$ need not be nondegenerate. So the arguments can be applied for the case $J = 0$.

3.1 Restrictions to the space of exponential functions

If the generator system/fundamental coordinate system is fixed, infinitesimal intertwiners are viewed naturally a flat connection defined on the trivial bundle over the space of expression parameters. Let $\mathbb{C}e^{\mathbb{S}(2m)}$ be the multiplicative space of all exponential functions of quadratic forms. We consider the product bundle

$$\bigotimes_{K \in \mathbb{S}(2m)} \mathbb{C}e^{\mathbb{S}(2m)} \subset \bigotimes_{K \in \mathbb{S}(2m)} \text{Hol}(\mathbb{C}^{2m})$$

(parallel subbundle).

We restrict the connection (infinitesimal intertwiner) to the subbundle

$$\bigotimes_{K \in \mathbb{S}(2m)} (e^{\mathbb{S}(2m)}; *_{K})$$

A horizontal distribution $H_{K}(ge^{(u_{\frac{1}{2m}} A,u)})$ at $ge^{(u_{\frac{1}{2m}} A,u)}$ defined on $\bigotimes_{K \in \mathbb{S}(2m)} \mathbb{C}e^{\mathbb{S}(2m)}$ is given by applying the infinitesimal intertwiner $dI_{K}(K')$ as follows:

$$H_{K}(ge^{(u_{\frac{1}{2m}} A,u)}) = \{ (K'; g(\frac{1}{2} \text{Tr} K'A + (u_{\frac{1}{2m}} A,u))e^{(u_{\frac{1}{2m}} A,u)}); K' \in \mathbb{S}(2m) \}.$$

The infinitesimal intertwiner/horizontal distribution is viewed as a flat connection on these bundles. The intertwiners can be viewed as parallel translations, though a parallel displacement is not defined on the whole space in general. However, since functions are restricted to the space of exponential functions of quadratic forms, the equation of parallel displacement can be solved locally.

Let $u = (u_{1}, \ldots, u_{2m})$. The exact formula to parallel translation is obtained by solving the evolution equation

$$\frac{d}{dt}g(t)e^{\frac{1}{\hbar}(uQ(t),u)} = \sum_{ij} K^{ij}_{} \partial_{u_{i}} \partial_{u_{j}}(g(t)e^{\frac{1}{\hbar}(uQ(t),u)}), \quad Q(0) = A, \quad g(0) = g$$

by setting

$$\sum_{ij} K^{ij}_{} \partial_{u_{i}} \partial_{u_{j}}(g(t)e^{\frac{1}{\hbar}(uA,u)}) = g(t)(2\text{Tr} K \frac{1}{i\hbar} Q(t) + 4 \frac{1}{(i\hbar)^{2}} (QKQ)_{ij} u_{i} u_{j}) e^{\frac{1}{\hbar}(uQ(t),u)}.$$

A direct calculation gives

$$\sum_{i} K^{ij}_{\cdot} \partial_{u_{i}} \partial_{u_{j}}(g(t)e^{\frac{1}{\hbar}(uQ(t),u)}) = g(t)\left(2\text{Tr} K \frac{1}{i\hbar} Q(t) + 4 \frac{1}{(i\hbar)^{2}} (QKQ)_{ij} u_{i} u_{j}\right) e^{\frac{1}{\hbar}(uQ(t),u)}.$$
By uniqueness of the real analytic solution, we only have to solve a system of ordinary differential equations:

\[
\begin{cases}
\frac{d}{dt}Q(t) = \frac{4}{ith}Q(t)KQ(t) \\
\frac{d}{dt}g(t) = g(t)\left(\frac{2}{ith}\Tr KQ(t)\right)
\end{cases} \quad Q(0) = A, \quad g(0) = g.
\]

Hence, we have \(Q(t) = \frac{1}{I-\frac{4}{i\hbar}AK}A, \quad g(t) = g(\det(I - \frac{4i}{\hbar}AK))^{-1/2}.\)

Here, the inverse matrix of \(X\) is denoted by \(\frac{1}{X}\). Note that \(\frac{1}{\frac{1}{X}} = \frac{1}{X}\). It is easy to check that \(\frac{1}{I-\frac{4}{i\hbar}AK}A\) is a symmetric matrix by the bumping identity

\[
(3.2) \quad \frac{1}{I-\frac{4}{i\hbar}AK}A = A\frac{1}{I-KA}.
\]

Setting \(t = \frac{\hbar}{4}\), we have the intertwiner \(I_0^K\):

\[
(3.3) \quad Q\left(\frac{\hbar i}{4}\right) = \frac{1}{I-\frac{4}{i\hbar}AK}A, \quad g\left(\frac{\hbar i}{4}\right) = g(\det(I - AK))^{-1/2}.
\]

For simplicity, we denote \(ge^{\frac{1}{i\hbar}(u,A,u)}\) by \((g; A)\), and we call \(g\) and \(A\) the amplitude and the phase part of \(ge^{\frac{1}{i\hbar}(u,A,u)}\). In this notation, we see that

\[
I_0^K(g; A) = \left(g \det(I - AK)^{-1/2}; T_K(A)\right),
\]

where \(T_K : \mathcal{G}(2m) \to \mathcal{G}(2m), \quad T_K(A) = \frac{1}{I-\frac{4}{i\hbar}AK}A\) is viewed as the phase part of the intertwiner \(I_0^K\).

Computing the inverse \(I_0^K = (I_0^K)^{-1}\), and the composition \(I_0^K I_0^{K'}\), we easily see

\[
(3.4) \quad I_0^{K'}(g; A) = \left(g \det(I - A(K' - K))^{-1/2}; \frac{1}{I-A(K'-K)}A\right).
\]

This mapping is singular at \(A\) such that \(\det(I-A(K'-K))=0\), and the sign ambiguity cannot be removed. \(T_K^{K'}(A) = \frac{1}{I-A(K'-K)}A\) is viewed as the phase part of the intertwiner.

Note that the identities

\[
T_K^{K'} \sim T_K, (T_K)^{-1}, \quad I_0^{K'} \sim I_0^{K} I_0^K
\]

hold. Here \(\sim\) means the equality in algebraic calculations such as \(x/x = 1, \sqrt{1+x}/\sqrt{1+x} = 1\). Singularities are moved by this algebraic trick. Moving branched singularities are the remarkable feature of this calculus.

By the concrete form of intertwiners for exponential functions of quadratic forms, we see the following

**Theorem 3.1** There is no globally defined parallel section of \(\coprod_{K \in \mathcal{G}(2m)} Ce^{\mathcal{G}(2m)}\) except constant scalar sections (trivial sections), and every nontrivial parallel section is two-valued.

Since setting \(\tilde{a}a = (a_i a_j) = A\) we see

\[
\langle \tilde{a}, \tilde{u} \rangle \star \langle \tilde{a}, \tilde{v} \rangle_{\cdot K_0} = \langle \tilde{a}, \tilde{u} \rangle \langle \tilde{a}, \tilde{v} \rangle = A(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v})\left(\frac{1}{2} \begin{bmatrix} 0 & \tilde{a}a \\ \tilde{a}a & 0 \end{bmatrix}\right) \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}
\]
and the eigenvalue of this rank 2 matrix \[
\begin{bmatrix}
0 & \bar{\alpha}\alpha \\
\bar{\alpha}\alpha & 0
\end{bmatrix}
\]
is \(\pm\langle \alpha, \bar{\alpha} \rangle\) and 2\((m-1)\) zeros. Hence, \(\varepsilon_{00}(\alpha)\) has a nontrivial \(K\)-ordered expression for \(K\) such that \(\det \left( I - \begin{bmatrix}
0 & \bar{\alpha}\alpha \\
\bar{\alpha}\alpha & 0
\end{bmatrix} \right) \neq 0\). In the previous section, we have seen that polar elements behaves delicately depending on expression parameters. But, we first recall the reason why the double-valued nature of \(\varepsilon_{00}(\alpha)\) appears.

We explain the reason by using the notaions \(u = \langle \alpha, \bar{\alpha} \rangle\) and \(v = \langle \alpha, \bar{\alpha} \rangle\). There is an adjoint rotation of one parameter subgroups such that

\[
\text{Ad}(b(s))(e^{\frac{\pi}{\hbar}wuv}) = e^{\frac{\pi}{\hbar}wuv},
\]
where \(1 \text{ at } t = 0\) is required by definition of one parameter subgroups. Here, we note that the polar element \(\varepsilon_{00} = e^{\frac{\pi}{\hbar}wuv}\) is a member of one parameter subgroup \(e^{\frac{\pi}{\hbar}wuv}\) of crossed symbol. In spite that, if one fixes \(t = \pi i\) first, then the normal ordered expression :\(\text{Ad}(b(s))(e^{\frac{\pi}{\hbar}wuv})\): \(K_0\) is independent of \(s\). It follows

\[
\varepsilon_{00}(\alpha)^{\dagger} = :\text{Ad}(b(\pi))(e^{\frac{\pi}{\hbar}wuv})::K_0 = :e^{\frac{\pi}{\hbar}wuv}::K_0.
\]

By the same observation as above, we see that \(\varepsilon_{00}(\alpha)^{-1} = :\varepsilon_{00}(\alpha)^{-1}::K_0\). On the other hand by the exponential law, we see \(\varepsilon_{00}(\alpha)\) satisfies

\[
\varepsilon_{00}(\alpha)^2 = (\varepsilon_{00}^{-1}(\alpha))^2 = -1, \quad \varepsilon_{00}(\alpha) \ast \varepsilon_{00}^{-1}(\alpha) = 1
\]
in the normal ordered expression. This was the reason why \(\varepsilon_{00}(\alpha)\) should be regarded as a two valued element.

In what follows, we show that such double-valued nature is not violated by intertwiners.

### 3.1.1 Intertwiners are 2-to-2 mappings

Recalling (3.4) may be rewritten as

\[
I_K^{K'} \left( \frac{g}{\sqrt{\det(I-\bar{A}K')}}: \frac{1}{I-\bar{A}K} A \right) = \left( \frac{g}{\sqrt{\det(I-\bar{A}K')}}: \frac{1}{I-\bar{A}K} A \right)
\]

if \(I-\bar{A}K, I-\bar{A}K'\) are invertible.

Let \(D_K = \{ A \in \mathbb{G}(2m): \det(I-\bar{A}K) \neq 0 \}\), and let \(D_A = \{ K \in \mathbb{G}(2m): \det(I-\bar{A}K) \neq 0 \}\).

First, we consider the case where \(A\) is fixed, then

\[
\left\{ \left( \frac{1}{\sqrt{\det(I-\bar{A}K')}}: \frac{1}{I-\bar{A}K} A \right): K \in D_A \right\}
\]
is a double-valued parallel section defined on \(D_A\). If \(A\) is nonsingular, then

\[
\left( \frac{\det A}{\sqrt{\det(I-\bar{A}K')}}: \frac{1}{I-\bar{A}K} A \right) = \left( \frac{1}{\sqrt{\det(A^{-1}K')}}: \frac{1}{A^{-1}K} \right)
\]
is also a parallel section on on \(D_A\). Taking the limit \(A^{-1} \to 0\), we have a little strange double-valued parallel section

\[
\left( \frac{1}{\sqrt{\det(-K')}}: \frac{1}{K} \right), \quad K \in D_\infty,
\]
where \( \mathcal{D}_\infty = \{ K; \det K \neq 0 \} \). If \( K = K_0 \), then this is \( \frac{1}{\sqrt{(-1)^m}} e^{-\frac{i}{\hbar} \sum_k u_k v_k} \). Hence, (3.6) may be regarded as the \( K \)-ordered expression of the total polar element (11).

Next, we consider the case where \( K \) is fixed and \( A \) is moving, then the space
\[
\tilde{D}_K = \left\{ \left( \frac{1}{\sqrt{\det(I-AK)}}, \frac{1}{I-AK} A \right); A \in \mathcal{D}_K \right\}
\]
for \( K \neq 0 \) is viewed as a nontrivial double cover of the space \( \mathcal{D}_K \). \( \tilde{D}_0 \) for \( K = 0 \) is viewed as \( \{(\pm 1; A); A \in \mathcal{D}_0 \} \). Let \( \pi_K \) be the natural projection, and let \( \mathcal{D}_{K,K'} = \mathcal{D}_K \cap \mathcal{D}_{K'} \).

Proposition 3.1 The intertwiner \( I_K^{K'} \) is then a 2-to-2 mapping from \( \tilde{D}_K \) to \( \tilde{D}_{K'} \). Hence, the intertwiner keeps the double-valued nature of the \( \ast \)-exponential functions of quadratic forms.

Precisely speaking, the intertwiner \( I_K^{K'} \) is defined as a mapping of \( \pi_K^{-1} \mathcal{D}_{K,K'} \) onto \( \pi_K^{-1} \mathcal{D}_{K,K'} \). Since the transformation \( T_K: \frac{1}{1-AK} A \rightarrow \frac{1}{1-AK'} A \) changes the homotopical nature of closed curves via the movement of singularities, the notion of “lift” of closed curves by parallel displacement along closed curves is not stable. Recall again that these arguments have nothing to do with the Weyl algebra.

Note Intertwiners fails the cocycle condition as one-to-one mappings, i.e. \( I_K^{K''} I_K^{K'} I_K^{K''} \) may not equal 1, but it is \( \pm 1 \) for exponential functions of quadratic forms. This is similar to \( \mathbb{Z}_2 \)-gerbes.

If \( g \) is fixed \( \mathcal{D}_K \) is a double covering space of \( \mathcal{D}_K \). As in the case of one variable, take the following diagram in mind
\[
\begin{array}{ccc}
\tilde{D}_K & \supset \pi^{-1}(\mathcal{D}_K \cap \mathcal{D}_{K'}) & I_K^{K'} \\
\downarrow \pi & \downarrow \pi & \downarrow \pi \\
\mathcal{D}_K & \supset \mathcal{D}_K \cap \mathcal{D}_{K'} & \subset \mathcal{D}_{K'} \\
\end{array}
\]

Turning around the circle in the picture of the l.h.s., the sign does not change as \( \circ \) is not a singular point. However, turning around the circle in the r.h.s., the sign changes as \( \bullet \) is a branched singular point. Similarly, as the \( \bullet \) in the l.h.s. picture is a branched singular point, the sign changes around this point. Hence, provided \( c \) is a constant, \( \pi^{-1}(p) \) must be two points, which one cannot distinguish, for these two points exchange each other when one goes around the point \( \bullet \). On the other hand, since \( \circ \) is not a singular point in the r.h.s., these two point can be distinguished.

Consequently, one cannot trace how points of \( \pi^{-1}(\mathcal{D}_K \cap \mathcal{D}_{K'}) \) map onto points of \( \pi^{-1}(\mathcal{D}_{K} \cap \mathcal{D}_{K'}) \) with one-to-one correspondence. In spite of this difficulty, one can trace this mapping as a 2-to-2 mapping. If one views these two points as a singleton, then this produces nothing but the identification of \( \mathcal{D}_K \) with \( \mathcal{D}_{K'} \), and the mapping is nothing but the identity mapping of \( \mathcal{D}_K \) onto \( \mathcal{D}_{K'} \).
Since this procedure loses much information, we prefer to regard such a mapping as a 2-to-2 mapping, for these two points can be distinguished locally.

Moreover, one can define a kind of group operation via the definition of 2-to-2 mappings. Even in such a situation, some partial area of object one may fix a unit, inverse and product in a univalent way to obtain a genuine group. Hence, local differential geometry can be done without any difficulty.

To treat elements with double-valued nature, we have to discuss the intertwiners to generic ordered expressions. Setting \( u=(\tilde{u}_1, \ldots, \tilde{u}_m, \tilde{v}_1, \ldots, \tilde{v}_m) \) and recalling \( K_0=\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \), the formula \((1.6)\) is rewritten as

\[
(3.7) \quad \exp_{e^s} \frac{s}{2i\hbar} \langle 0 \mid C \mid 0 \rangle, u \rangle_{K_0} = e^{\frac{s}{2i\hbar}C} \exp \frac{1}{2i\hbar} \langle 0 \mid e^{sC-I} \mid 0 \rangle, u \rangle_{K_0}.
\]

In particular, we see

\[
: e^s_{\pi} (\tilde{u}_1 \oplus \tilde{v}_1 \oplus \cdots \oplus \tilde{u}_m \oplus \tilde{v}_m) ;_{K_0} = e^{\frac{s}{2i\hbar}} \exp \frac{1}{2i\hbar} \langle 0 \mid e^{sI-I} \mid 0 \rangle, u \rangle_{K_0}.
\]

The sign ambiguity of \( \sqrt{\cdot} \) does not appear on this expression, but precisely speaking we should write the l.h.s.

\[
: e^s_{\pi} (\tilde{u}_1 \oplus \tilde{v}_1 \oplus \cdots \oplus \tilde{u}_m \oplus \tilde{v}_m) ;_{K_0} = (-1)^m, \quad : e^s_{\pi} (\tilde{u}_1 \oplus \tilde{v}_1 \oplus \cdots \oplus \tilde{u}_m \oplus \tilde{v}_m) ;_{K_0} = i^m \exp \frac{1}{i\hbar} \langle 0 \mid I \mid 0 \rangle, u \rangle_{K_0},
\]

where \([0 \rightarrow a]\) implies the path given by the straight line segment.

It is very natural to expect that there is \( K \) such that

\[
\hat{\varepsilon}_{00}(\tilde{a}) \ast_K \hat{\varepsilon}_{00}(\tilde{b}) = -\hat{\varepsilon}_{00}(\tilde{b}) \ast_K \hat{\varepsilon}_{00}(\tilde{a}), \quad \text{(if } \langle \tilde{a}, \tilde{b} \rangle = 0).\]

### 3.2 Clifford algebras in \((\text{Hol}(\mathbb{C}^{2m}), \ast_K)\)

Since the group \( \text{Spin}(m) \) is usually constructed as a Clifford algebra, it is natural to think that the group ring of the group \( \tilde{SO}(m) \) under some other expression parameter \( K \) considered in \((\text{Hol}(\mathbb{C}^n), \ast_K)\) has the structure of Clifford algebra.

At this moment, this is supported only by the following strange phrase:

Since the \( \varepsilon_{00}(k) \)'s are defined as double-valued elements, the identities

\[
\varepsilon_{00}(k) = -\varepsilon_{00}(k), \quad \varepsilon_{00}(k) \ast \varepsilon_{00}(\ell) = -\varepsilon_{00}(\ell) \ast \varepsilon_{00}(k)
\]

do not contradictory.

The goal of this section is the following.

**Theorem 3.2** There is an expression parameter \( K_s \) having the following properties: Let \( V_s \) be 1 or any partial polar element without involving \( \varepsilon_{00}(k), \varepsilon_{00}(\ell) \) \((k, \ell)\) such that \( k \neq \ell \). Then,

\[
: \varepsilon_{00}(k) \ast V_s ;_{K_s} = -: V_s ;_{K_s}
\]

\[
: (\varepsilon_{00}(k) \ast \varepsilon_{00}(\ell)) \ast V_s ;_{K_s} = -: V_s ;_{K_s}.
\]
Since the identity above gives
\[ :\varepsilon_{00}(k)\ast\varepsilon_{00}(l):_{K_s} = - :\varepsilon_{00}(l)^{-1}\ast\varepsilon_{00}(k)^{-1}:_{K_s},\]
by noting that \(\varepsilon_{00}(k)^{-1} = \pm\varepsilon_{00}(k)\) by the double-valued nature, but the \(\pm\)-sign can be controlled to be independent of \(k\) (cf [12]), we see that
\[ :\varepsilon_{00}(k)\ast\varepsilon_{00}(l):_{K_s} = - :\varepsilon_{00}(l)\ast\varepsilon_{00}(k):_{K_s}.\]

Under such an ordered expression \(K = K_s\), the system
\[ p(u, v)^* K :\varepsilon_{00}(1)^{\varepsilon_1} \ast \cdots \ast K \ast \varepsilon_{00}(m)^{\varepsilon_m} \]
naturally forms an algebra under the \(*_K\)-product, which may be called the Weyl-Clifford algebra. This means the super-theoretic expressions are already built in the extended Weyl algebra. That is, we have no need to construct a new mathematical theory to absorb the super manifold theory.

The next three steps are essential for the proof of Theorem 3.2:
1. Note first that in the normal ordered expression, partial polar elements form a commutative algebra. Moreover, we already see that
\[ \varepsilon_{00}(l)^{\varepsilon_1} \ast \cdots \ast \varepsilon_{00}(m)^{\varepsilon_m} :_{K_0} = :\varepsilon_{00}(l)^{\varepsilon_1} \ast \cdots \ast \varepsilon_{00}(m)^{\varepsilon_m} :_{K_0}. \]

2. Let \(V_s\) be any partial polar element or \(\pm 1\). Applying the intertwiner \(I_{K_0}^{K_s}\) to this system, we show the following:
   2.1 \(I_{K_0}^{K_s}(\varepsilon_{00}(l)^{\varepsilon_1} \ast \cdots \ast \varepsilon_{00}(m)^{\varepsilon_m} :_{K_0})\) has no singular point on the interval \([0, \pi]\).
   2.2 In spite of this, if \(V_s\) does not contain \(\varepsilon_{00}(l)^{\varepsilon_1} \ast \varepsilon_{00}(l)^{\varepsilon_2} \ast \varepsilon_{00}(m)^{\varepsilon_m}\), then \(I_{K_0}^{K_s}(\varepsilon_{00}(l)^{\varepsilon_1} \ast \cdots \ast \varepsilon_{00}(m)^{\varepsilon_m} :_{K_0})\) has singular points on the open intervals \((0, \pi)\) and \((\pi, 2\pi)\).
   3. If \(V_s\) does not contain \(\varepsilon_{00}(l)^{\varepsilon_1} \ast \varepsilon_{00}(l)^{\varepsilon_2} \ast \varepsilon_{00}(m)^{\varepsilon_m}\), then \(\varepsilon_{00}(l)^{\varepsilon_1} \ast \cdots \ast \varepsilon_{00}(m)^{\varepsilon_m} :_{K_0}\) has no singular point in \([0, \pi] \times [0, \pi]\) except on the diagonal set \(s = t\).

The proof of Theorem 3.2 is given as follows: Suppose \(V_s\) does not contain \(\varepsilon_{00}(k), \varepsilon_{00}(l)\). Note that \(\varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s = \varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s\) has no singular point on the lower triangular domain \(\{(s, t); 0 \leq t < s \leq \pi\}\). There is one singular point at \((\mu, \mu)\). Therefore, we see
\[ :\varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s:_{K_s} \ast \varepsilon_{00}(s-t) :_{K_s} \ast V_s:_{K_s} \]
which is defined by taking the clockwise path avoiding the singular point.

Note here that products such as \(\varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s:_{K_s}\), \(\varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s:_{K_s}\), and \(\varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s:_{K_s}\) are defined by solving the evolution equation with initial data \(V_s\). Such a procedure for constructing products will be called the path connecting product.

**Singularity makes change of sign.** Here, we show that the singularity makes the change of sign, and hence

\[ :\varepsilon_{00}(k)\ast \varepsilon_{00}(l)\ast V_s:_{K_s} = - :\varepsilon_{00}(l)\ast \varepsilon_{00}(k)\ast V_s:_{K_s}, \]

(3.8)
where it is assumed that \( V_\epsilon \) does not contain \( \epsilon_{00}(k) , \epsilon_{00}(\ell) \).

Consider the product \( e^{sH_s} \cdot e^{tK_t} \) for two quadratic forms \( H_s , K_t \) in \( (s,t) \in \mathbb{C}^2 \) such that \( [H_s , K_t] = 0 \). In our situation it may be assumed \( e^{sH_s} \cdot e^{tK_t} = \pm e^{tK_t} \cdot e^{sH_s} \) with the sign ambiguity, that is, the phase parts of both sides coincides and the sign ambiguity appears only in the amplitude parts.

In general, \( e^{sH_s} \cdot e^{tK_t} \) has a singular set \( S \) of complex codimension 1. We see that the origin \((0,0)\) is not contained in \( S \). Since \( S \) is a branched singularity, we have to prepare two sheets \( \mathbb{C}^2_+ \), \( \mathbb{C}^2_- \) and “slit” \( \Sigma \) of real codimension 1 to connect these two sheets. \( \Sigma \) is set so that \( \mathbb{C}^2 \setminus \Sigma \) is simply connected and there is no singular point.

Now, restrict the parameter \((s,t) \in \mathbb{R}^2 \) in \( e^{sH_s} \cdot e^{tK_t} \). One may assume that \( \mathbb{R}^2 \) is transversal to \( S \) in generic ordered expression. Hence, if \( S \cap \mathbb{R}^2 \neq \emptyset \), then this is a discrete set and \( \Sigma \cap \mathbb{R}^2 \) is a collection of (real one dimensional) curves starting at a singular point ending another singular point or \( \infty \).

We indicate this by the notation \( e^{[0 \rightarrow s_1]} H_s \cdot e^{[t]} K_t \). This is the clockwise chasing from the origin. On the contrary, \( e^{[s \rightarrow t]} K_t \cdot e^{s_1 H_s} \) means the anti-clockwise chasing from the origin. Now suppose there is a singular point \((s_0, t_0)\) and a slit as it is seen in the left figure, then \( e^{[0 \rightarrow t]} K_t \cdot e^{s_1 H_s} \) is lying in the opposite sheet. By this way, the sign changes around a singular point.

### 3.2.1 Intertwiners to generic ordered expressions

In the first part of discussions, we recall (3.4). Via the intertwiner \( I^K_{K_0} \) from (3.7), the \( K \)-expression of the \( * \)-exponential function \( e^{\frac{1}{2}\Sigma C_k u_k \langle v, v \rangle} \) is easily obtained. Recall \( e^{\frac{1}{2}(t_1 u_1 + \cdots + t_m u_m) \langle v, v \rangle} \) is an entire function of \((t_1, \cdots , t_m)\), \( t_i \in \mathbb{C}\), in the normal ordered expression which is written as \( e^{\frac{1}{2}(t_1 + \cdots + t_m) \langle u, A, u \rangle} \), where

\[
A = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}, \quad C = \text{diag}(\tau_1, \cdots , \tau_m), \quad \tau_k = \frac{1}{2}(\epsilon^{it_k} - 1), \quad \text{cf.} \ (1.6).
\]

In this section, we use special ordered expressions \( K_s \), where \( K_s \) is given step by step by (3.9), (3.11), (3.12) and (3.13) as follows: We set

\[
K_s = \begin{bmatrix} S & T \\ T & S \end{bmatrix}, \quad tS = S, \quad tT = T, \quad S, T \in M(m, \mathbb{C}).
\]

By (3.4), we need to know \( \sqrt{\det(I - A(K - K_0))} \). \( \det(I - A(K - K_0)) \) is given by elementary transformation as follows:

\[
\begin{vmatrix}
I - C(T - I) & -CS \\
-C S & I - C(T - I)
\end{vmatrix} = \det(I - C(T + S - I)) \det(I - C(T - S - I))
\]
Note that $S+T=U, S-T=V$ are arbitrary symmetric matrices.

As we want to use a $K$-ordered expression such that

$$\text{sgn}(e_{m}^{\pm i\tilde{u}_{i}\mp\tilde{v}_{j}};K) = \text{sgn}(e_{m}^{\pm i\tilde{u}_{j}\mp\tilde{v}_{i}};K)$$

for every $i,j$, we restrict $K$ to symmetric matrices such that

(3.11) $$K_s = \begin{bmatrix} i\rho I & cI \\ cI & i\rho I \end{bmatrix} + \begin{bmatrix} S' & T' \\ T' & S' \end{bmatrix}, \quad \rho, c \in \mathbb{R},$$

where the diagonal components of $S', T'$ are zero, and all other entries are the same complex constant. It is easy to see that the formula for $\rho_{\ast}$ is written by replacing $i$ by $j$ in the formula for $\rho_{\ast}(\tilde{u}_{i}, \tilde{v}_{i})_{K}$, since we use only $i\rho$ and $c$ in the computation of $\rho_{\ast}(\tilde{u}_{i}, \tilde{v}_{i})_{K}$.

In what follows we set that

(3.12) $$T' = \begin{bmatrix} 0 & a & a & \cdots & a \\ a & 0 & a & \cdots & a \\ a & a & 0 & \cdots & a \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a & a & a & \cdots & 0 \end{bmatrix}, \quad S' = \begin{bmatrix} 0 & ib & ib & \cdots & ib \\ ib & 0 & ib & \cdots & ib \\ ib & ib & 0 & \cdots & ib \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ ib & ib & ib & \cdots & 0 \end{bmatrix}, \quad a, b \in \mathbb{R},$$

so that $T-S = \overline{T+S}$. Further, in (3.13) below we put the additional condition that $c = a > 0$ and $\rho > b$. Hence, $T$ in (3.3) is a matrix such that $T_{ij} = c > 0$.

We refer to $K_s$ as the special ordered expression, or $K_s$-expression.

### 3.2.2 Vertexes and 2-dimensional nets

To establish the product formula for polar elements in general $K$-ordered expressions, we have to prepare several tools used in the definition of products.

Denote by $(t_1, t_2, \cdots, t_m)$ a point of $\mathbb{R}^m$. The lattice point is the subset $(\pi \mathbb{Z})^m$ of $\mathbb{R}^m$, and the 1-dimensional lattice is the subset of $\mathbb{R}^m$ such that only one of $(t_1, t_2, \cdots, t_m)$ is in $\mathbb{R}$ and others are $k\pi$, $k \in \mathbb{Z}$. A vertex is a point $(\delta_1, \cdots, \delta_m)$ where $\delta_i = 0$ or $\pi$. The number of $\pi$’s is called the index of the vertex.

The 2-dimensional lattice is the subset of $\mathbb{R}^m$ with only two of $(t_1, t_2, \cdots, t_m)$ in $\mathbb{R}$ and all others $k\pi$, $k \in \mathbb{Z}$. Denote the 2-dimensional lattice, the 1-dimensional lattice and the set of lattice points by $L_m(2), L_m(1), L_m(0)$ respectively.

We often use $(t_1, t_2, \cdots, t_m) \in \mathbb{R}^m$ to indicate the $*$-exponential function

$$e_{\ast}^*(t_1 u_1 v_1 + \cdots + t_m u_m v_m)_{K}. $$

Hence, lattice points in $K_0$-ordered expression are

$$L_{\ast}^*_{K_0} = \{e_{\ast}^*(\delta_1 u_1 v_1 + \cdots + \delta_m u_m v_m)_{K_0}, \quad \delta_i = 0 \text{ or } \pi \ell. $$

The next proposition is a basic result proved by the uniqueness of the real analytic solutions of evolution equations.
Proposition 3.2 For a lattice point $L_s$, if $\varepsilon_{\frac{t}{m}}^{\pm} L_{s, k}$ is not singular on $t \in [0, \pi]$, then $\varepsilon_{00}(k) * L_s : K$ is defined as a single element. This gives the $K$-ordered expression of vertex $\varepsilon_{00}(k) * L_s$.

We often use the variable $\tau = \frac{1}{2} (e^{it} - 1)$ or $\tau^{-1}$ instead of $t$, when the variables $e^{it}$ is restricted in the unit circle.

Note that

$$\tau = \frac{e^{i0} - 1}{2} = 0, \text{ hence } \tau^{-1} + 1 = \infty, \quad \text{and } \tau = \frac{e^{i\pm i\pi} - 1}{2} = -1, \text{ hence } \tau^{-1} + 1 = 0.$$ 

To obtain a single-valued product formula between partial polar elements, we have to consider elements $\varepsilon_{\frac{t}{m}}^{\pm}(t_1 u_1, \cdots + t_m u_m : K)$ where some of $(t_1, \cdots, t_m)$ are 0 or $\pm \pi$. Say $t_{i_1}, \cdots, t_{i_k}$ are 0, and $t_{j_1}, \cdots, t_{j_\ell}$ are $\pm \pi$. By a suitable change of rows and columns, we can assume $\tau_{i_1} = \cdots = \tau_{i_k} = 0$ without loss of generality, and the computation of the determinant is reduced to the case $(m-k) \times (m-k)$-matrices, where $C = diag(\tau_{i_1}, \cdots, \tau_{m-k-\ell}, \pm \pi, \cdots, \pm \pi)$ with $\sharp (\pm \pi) = \ell$ (the index of the vertex).

Since $\tau_i \neq 0$ in the reduced matrices, we have, by setting $\sigma = m - k - \ell$,

$$\det (I - C (T + S - I)) \det (I - C (T - S - I))$$

$$= (\det C)^2 \det (C^{-1} + I - (T + S)) \det (C^{-1} + I - (T - S))$$

$$= (\det C)^2 \det \left((\text{diag}(\tau^{-1}_{i_1}, \cdots, \tau^{-1}_{\sigma}, 0, \cdots, 0)) - (T + S)\right)$$

$$\times \det \left((\text{diag}(\tau^{-1}_{i_1}, \cdots, \tau^{-1}_{\sigma}, 0, \cdots, 0)) - (T - S)\right),$$

where the number of 0’s is $\sharp 0 = \ell$.

For simplicity, we set $\alpha = e + i\rho$, $\beta = a + i b$. Hence, the determinant is decomposed into two factors $F \times \overline{F}$, and one of them is given by the formula as follows:

$$\begin{vmatrix}
\tau^{-1} + 1 + \alpha & \beta & \cdots & \beta \\
\beta & \alpha & \cdots & \beta \\
\vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \cdots & \alpha
\end{vmatrix} - (\tau^{-1} + 1)$$

$$= \begin{vmatrix}
1 & 0 & \cdots & 0 \\
0 & \alpha & \cdots & \beta \\
\vdots & \vdots & \ddots & \vdots \\
0 & \beta & \cdots & \alpha
\end{vmatrix}$$

$$= \begin{vmatrix}
\alpha & \beta & \cdots & \beta \\
\beta & \alpha & \cdots & \beta \\
\vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \cdots & \alpha
\end{vmatrix}$$

$$= \begin{vmatrix}
\alpha & \beta & \cdots & \beta \\
\beta & \alpha & \cdots & \beta \\
\vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \cdots & \alpha
\end{vmatrix}$$
For $\ell=0$, this is given by $
abla^{-1}+\alpha$, and for $\ell\geq 1$, it is
\[
(\tau^{-1}+1)(\alpha+(\ell-1)\beta)(\alpha-\beta)^{\ell-1}+(\alpha+\ell\beta)(\alpha-\beta)^\ell
= \left((\tau^{-1}+1)(\alpha+(\ell-1)\beta)+(\alpha+\ell\beta)(\alpha-\beta)\right)(\alpha-\beta)^{\ell-1}.
\]

We now assume the following
\begin{equation}
(\alpha+\ell \beta)(\alpha-\beta) \neq 0
\end{equation}
In particular, this implies $c=a>0$. Note that we have three dimensions of freedom for $(a, b, \rho)$.

\begin{lemma}
Under the assumption (3.13), we see that $\text{Re}(\alpha)>0$ for $\ell=0$, and for $\ell \geq 1$,
\[
\text{Re}\left((\alpha-\beta)(1+\frac{\beta}{\alpha+(\ell-1)\beta})\right)>0.
\]
Thus, there is no singular point on the domain $\text{Re}(\tau^{-1}+1) \geq 0$.
\end{lemma}

\textbf{Proof}. The case $\ell=0$ is trivial. Since $\alpha-\beta$ is positive pure imaginary under (3.13), we easily see $\text{Re}\left((\alpha-\beta)(1+\frac{\beta}{\alpha+(\ell-1)\beta})\right)<0$ is equivalent to $\text{Im}(\frac{\beta}{\alpha+(\ell-1)\beta})<0$, and this is equivalent to
\[
\text{Im}(\frac{\alpha+(\ell-1)\beta}{\beta})=\text{Im}(\frac{\alpha-\beta}{\beta})>0.
\]
Hence, this is equivalent to $\text{Re}(\beta)>0$. \qed

Recall that the information for the signs of the imaginary parts give information about the sign change of the square roots.

Recall another factor of the determinant is given by the complex conjugate. Keeping these in mind, we have the following.

\begin{proposition}
Under the $K_s$-ordered expression the assumption together with (3.13), there is no singular point on lines of the 1-dimensional lattice of any index. Moreover, $\varepsilon_{\eta^2\eta^2}^* V_{s:K}$ is alternating $2\pi$-periodic w.r.t the variable $t \in \mathbb{R}$. In particular $\varepsilon_{\eta^2\eta^2}^* V_{s:K}^* = -V_{s:K}$.
\end{proposition}

Hence, by Proposition 3.2 applied to the vertex of index 1 gives that the product via the connecting paths is defined to give
\[
\varepsilon_{00}(i_{\ell-1}) \varepsilon_{00}(i_{\ell}) =: \varepsilon_{\eta^2\eta^2}^*(u_{i_{\ell-1}} v_{i_{\ell-1}} + u_{i_{\ell}} v_{i_{\ell}})_{s:K}, \quad \text{or} \quad -\varepsilon_{\eta^2\eta^2}^*(u_{i_{\ell-1}} v_{i_{\ell-1}} + u_{i_{\ell}} v_{i_{\ell}})_{s:K}.
\]
The reason of the ambiguity of ± sign is that the expression parameter $K$ will be so chosen that
\[ \varepsilon_{s}^{(u_{i_{t-1}} \circ v_{i_{t-1}} + u_{i_{t}} \circ v_{i_{t}})} : K \]
has a singular point on $t \in [0, \pi]$, and the ± sign is determined by the path avoiding the singular point. On the other hand, since the left hand side is defined without ambiguity, we have the equality
\[ :\varepsilon_{00}(i_{t-1}) \ast \varepsilon_{00}(i_{t}) : = \gamma \varepsilon_{s}^{(u_{i_{t-1}} \circ v_{i_{t-1}} + u_{i_{t}} \circ v_{i_{t}})} : K, \]
where $\gamma = 1$, or $-1$
depending on the path. By Proposition 3.2 again, we have
\[ :\varepsilon_{s}^{(u_{i_{t-2}} \circ v_{i_{t-2}})} \ast (\varepsilon_{00}(i_{t-1}) \ast \varepsilon_{00}(i_{t})) : K = :\varepsilon_{s}^{(u_{i_{t-2}} \circ v_{i_{t-2}} + u_{i_{t-1}} \circ v_{i_{t-1}} + u_{i_{t}} \circ v_{i_{t}})} : K, \]
for every $s \in [0, \pi]$. Hence, at $s=\pi$, we have
\[ :\varepsilon_{00}(i_{t-2}) \ast (\varepsilon_{00}(i_{t-1}) \ast \varepsilon_{00}(i_{t})) : K = :\varepsilon_{s}^{(u_{i_{t-2}} \circ v_{i_{t-2}} + u_{i_{t-1}} \circ v_{i_{t-1}} + u_{i_{t}} \circ v_{i_{t}})} : K, \]
Repeating this procedure, we see that
\[ :e_{s}^{(u_{1} \circ v_{1})} \ast :\varepsilon_{00}(i_{1}) \ast \cdots \ast \varepsilon_{00}(i_{t}) : K = :\varepsilon_{s}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2} + \cdots + u_{t} \circ v_{t})} : K. \]
and that $:e_{s}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2} + \cdots + u_{t} \circ v_{t})} : K$ is defined and this gives at $s=\pi$
\[ \gamma \varepsilon_{s}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2} + \cdots + u_{t} \circ v_{t})} : K. \]
Hence, inductive use of Proposition 3.2 together with Proposition 2.1 gives

**Proposition 3.4** Products $\varepsilon_{00}(k_{1}) \ast \varepsilon_{00}(k_{2}) \ast \cdots \ast \varepsilon_{00}(k_{p})$ are welldefined in the special ordered expression by the path connecting products.

### 3.2.3 Several properties of the $K$-ordered expression of $e_{s}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2})} \ast V_{*}$

First, we consider the case of $V_{*} = 1$. The $K$-ordered expression $\varepsilon_{s}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2})} : K$ is given by computing the intertwiner
\[ I_{K}^{K}(e_{*}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2})} A, u) = \frac{e_{*}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2})}}{\sqrt{\det I-A(K-K_{0})}} e_{*}^{(u_{1} \circ v_{1} + u_{2} \circ v_{2}, A, u)}, \]
Set $\tau = \frac{1}{2}(e^{\iota}-1), C=\text{diag}(\tau, \tau), A = \begin{bmatrix} 0 & \tau I_{2} \\ \tau I_{2} & 0 \end{bmatrix}$, $K = \begin{bmatrix} S & T \\ T & S \end{bmatrix}$, $\iota S=S, \iota T=T$ for simplicity.

Concerning only the amplitude by (3.4), we only have to know $\sqrt{\det(I-A(K-K_{0}))}$. The determinant is given by elementary transformation as follows:

\[ \begin{vmatrix} I-C(T-I) & -CS \\ -CS & I-C(T-I) \end{vmatrix} = \det(I-C(T+S-I)) \det(I-C(T-S-I)) \]
\[ = \tau^{4} \det(C^{-1}+I-(T+S)) \det(C^{-1}+I-(T-S)) \]
\[ = \tau^{4} \det((\text{diag}(\tau^{-1}+1, \tau^{-1}+1))-(T+S)) \det((\text{diag}(\tau^{-1}+1, \tau^{-1}+1))-(T-S)) \].
Recall we have assumed that all entries of $S+T$ other than diagonal are constant $\beta$. Note at first by such a condition, we are considering all pairs $k,l$, possibly $k = l$, at the same time. Since only $(k,l)$-submatrices of $K$ are used in the computation when $k,l$ are fixed, the computation is reduced to the case $m=2$.

For the case that the index of the vertex is 0, that is, $\ell=0$, that is the case $V_*=1$, we set $\tau = \frac{1}{2} (e^{it} - 1)$, and we may assume $m=2$, $A = \begin{bmatrix} 0 & \tau I_2 \\ \tau I_2 & 0 \end{bmatrix}$ without loss of generality. The intertwiner is written as

$$I_{K_0}^{K} (e^{it} e^{i\frac{1}{2} (u A, u)}) = \frac{e^{it}}{\sqrt{\det I_2 - A (K - K_0)}} e^{i\frac{1}{2} (u A - A (K - K_0) A, u)}$$

where

$$K = \begin{bmatrix} i \rho I + S' & c I + T' \\ c I + T' & i \rho I + S' \end{bmatrix}, \quad K_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

and

$$I - A (K - K_0) = \begin{bmatrix} I - \tau (c - 1) I + T' & -\tau (i \rho I + S') \\ -\tau (i \rho I + S') & I - \tau (c - 1) I + T' \end{bmatrix}.$$  \hfill (3.15)

What we want to obtain is that in the $K_*$-ordered expression $\mathcal{E}_{\mathcal{K}}^{\mathcal{K}} (u k, v k) * V_* * K$ has singularities of order 1 on the open intervals $(0, \pi)$, and $(\pi, 2 \pi)$ for every $k,l$. The determinant of (3.15) is written by the elementary transformations as

$$\det (I - \tau ((c - 1 + i \rho) I + T' + S')) \det (I - \tau ((c - 1 + i \rho) I + T' + S')).$$  \hfill (3.16)

Setting $S' + T' = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$, this vanishes when

$$\tau^{-1} + 1 + \alpha)^2 - \beta^2 = 0 \quad \text{or} \quad (\tau^{-1} + 1 + \alpha)^2 - \beta^2 = 0.$$

That is

$$- (\tau^{-1} + 1) = \alpha \pm \beta, \quad \text{or} \quad - (\tau^{-1} + 1) = \alpha \pm \beta.$$

By the condition (3.13), we see $\Re(\alpha + \beta) > 0$. Hence,

$$- (\tau^{-1} + 1) = \alpha - \beta, \quad - (\tau^{-1} + 1) = \alpha - \beta$$

are the singular points. Hence, we have the desired result for the case $V_* = 1$.

We next consider $\mathcal{E}_{\mathcal{K}}^{\mathcal{K}} (u k, v k) * V_* * K$ by taking $K$ in (3.11) with (3.13). For the case $V_* \neq 1$, i.e. $\ell \geq 1$, we have only to use the diagonal matrix $\text{diag}(\tau, -1, \ldots, -1)$ instead of $\tau$ in the previous case of $V_* = 1$, where the number of $-1$ is the index $\ell$ of the vertex $V_*$. We assume that $V_*$ does not contain $\varepsilon_{00}(k)$ and $\varepsilon_{00}(\ell)$. What we want to show is that

$$\mathcal{E}_{\mathcal{K}}^{\mathcal{K}} (u k, v k) * V_* * K$$

is $2\pi$-periodic on $\mathbb{R}$ and it has singular points $\mu$, $0 < \mu < \pi$, and $\nu$, $\pi < \nu < 2\pi$. 

The first factor of the determinant is written as
\[
\begin{vmatrix}
\tau^{-1}+1+\alpha & \beta & \beta & \ldots & \beta \\
\beta & \tau^{-1}+1+\alpha & \beta & \ldots & \beta \\
\beta & \beta & \alpha & \ldots & \beta \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \beta & \ldots & \alpha
\end{vmatrix} = (\tau^{-1}+1)^2(\alpha+(\ell-1)\beta)(\alpha-\beta)\ell^1
\]
\[
+2(\tau^{-1}+1)(\alpha+\ell\beta)(\alpha-\beta)^\ell
\]
\[
+(\alpha+(\ell+1)\beta)(\alpha-\beta)^{\ell+1}.
\]
It follows that the determinant vanishes if and only if
\[
-\tau^{-1}+1 = \alpha-\beta, \quad \text{or} \quad \frac{(\alpha+(\ell+1)\beta)(\alpha-\beta)}{\alpha+(\ell-1)\beta} = \left(1 + \frac{2\beta}{\alpha+(\ell-1)\beta}\right)(\alpha-\beta).
\]
Since \(\alpha-\beta = id_+\), \(d_+ > 0\), the same argument as in the proof of Lemma 3.1 gives that
\[
\text{Re}\left(1 + \frac{2\beta}{\alpha+(\ell-1)\beta}\right)(\alpha-\beta) > 0.
\]
Hence, we have only to assume (3.13).

For the second factor, the determinant is given by the complex conjugate. Hence, the requested conditions are satisfied by (3.13). The singular points together with the second factor are given by
\[
e^{is} = \frac{id_+ + 1}{id_+ - 1}, \quad \frac{id_+ - 1}{id_+ + 1}.
\]

**Proposition 3.5** Under the condition (3.13) for \(K\), \(e^{\frac{2\pi}{2\pi}(u_k,v_l)} * v_{k,l}\), \(k\neq l\), has singular points at \(t_0 \in (0, \pi)\) and \(2\pi - t_0\) for every vertex \(V_s\), and
\[
e^{\frac{2\pi}{2\pi}(u_k,v_l)} * v_{k,l} = v_{k,l}
\]
for every \(1 \leq k, l \leq m\), if we take the anti-clockwise half-circle path avoiding the singular point.

The \(\pm\) sign of \(e^{\frac{2\pi}{2\pi}(u_k,v_l)} * v_{k,l}\) is determined by the path avoiding the singularity.

### 3.2.4 Determinant equation of two variables

Here, we show that \(e^{\frac{2\pi}{2\pi}(t_1u_k,v_l) + t_2u_l,v_l)} * v_{k,l}\), \(k\neq l\), has no singular point other than the singular point lying in the diagonal \((t_0, t_0) \in [0, \pi] \times [0, \pi]\). Here, we assume that \(V_s\) does not contain \(\varepsilon_{00}(k)\) and \(\varepsilon_{00}(l)\).

As singular points are given by zeros of (3.10), in this section we consider the equation
\[
\begin{vmatrix}
\tau^{-1}+1+\alpha & \beta & \beta & \ldots & \beta \\
\beta & \tau^{-1}+1+\alpha & \beta & \ldots & \beta \\
\tau^{-1}+1+\alpha & \beta & \beta & \ldots & \beta \\
\beta & \beta & \alpha & \ldots & \beta \\
\beta & \beta & \beta & \ldots & \alpha
\end{vmatrix} = 0
\]
of two variables under the restriction \( \tau_i \in i\mathbb{R} \). For simplicity, we set \( ix=\tau_1^{-1}+1, iy=\tau_2^{-1}+1 \). The first factor is written as

\[
-xy(\alpha+(\ell-1)\beta)(\alpha-\beta)^{\ell+1}+i(x+y)(\alpha+\ell\beta)(\alpha-\beta)^{\ell}+(\alpha+(\ell+1)\beta)(\alpha-\beta)^{\ell+1}=0, \quad (\ell \geq 0).
\]

i.e.

\[
-xy+i(x+y)(\alpha+\ell\beta)(\alpha-\beta)^{\ell}+(\alpha+(\ell+1)\beta)(\alpha-\beta)^{\ell+1}(\alpha+(\ell-1)\beta) =0.
\]

If we set \( A=\frac{(\alpha+\ell\beta)(\alpha-\beta)}{\alpha+(\ell-1)\beta} \) and

\[
B^2=\left(\frac{\alpha+\ell\beta}{\alpha+(\ell-1)\beta}\right)^2 - \left(\frac{(\alpha+\ell\beta)(\alpha-\beta)}{\alpha+(\ell-1)\beta}\right) = \left(\frac{(\alpha-\beta)}{\alpha+(\ell-1)\beta}\right)^2,
\]

then the first factor is rewritten as \((ix+A)(iy+A)-B^2\). Similarly, the second factor is written as \((ix+\bar{A})(iy+\bar{A})-\bar{B}^2\).

We put in (3.13) the condition that \( \alpha-\beta \) is a positive pure imaginary number, and set \( \alpha-\beta=\iota d_+ \).

Setting \( A=A_0+iA_1, \ B=B_0+iB_1 \), we have

\[
A_0=-d_+\text{Im}\frac{\beta}{\alpha+(\ell-1)\beta}, \quad A_1=d_+\left(1+\text{Re}\frac{\beta}{\alpha+(\ell-1)\beta}\right),
\]

\[
B_0 = -d_+\text{Im}\frac{\beta}{\alpha+(\ell-1)\beta}, \quad B_1=d_+\text{Re}\frac{\beta}{\alpha+(\ell-1)\beta}.
\]

Hence, we see \( A_0=B_0, A_1-B_1=d_+ \). Let \( \alpha=a+i\rho, \beta=a+ib \).

Note now that

\[
\text{Im}\frac{\beta}{\alpha+(\ell-1)\beta}<0 \iff \text{Im}\frac{\alpha}{\beta}>0 \iff a(\rho-b)>0,
\]

\[
\text{Re}\frac{\beta}{\alpha+(\ell-1)\beta}>0 \iff \text{Re}(\ell-1+\frac{\alpha}{\beta})>0 \iff \ell-1+\frac{a^2+\rho b}{a^2+b^2}>0.
\]

Since \( \rho b>0 \) is assumed, if \( \ell \geq 1 \) then \( \ell-1+\frac{a^2+\rho b}{a^2+b^2}>0 \). For \( \ell=0 \), if \( \rho-b>0 \) then \( -1+\frac{a^2+\rho b}{a^2+b^2}>0 \).

In addition, suppose in addition that \( \alpha=a+i\rho = c+i\rho, \beta=a+ib \) satisfy that \( a>b>0, \ \rho>b>0 \). Then we easily see that \( A_0=B_0>0, A_1^2>B_1^2 \) for every \( \ell \geq 0 \). Note that all additional conditions other than the condition \( c=a \) are open conditions, hence we have three real dimensions of freedom.

Suppose we have an equation of pure imaginary variables \( ix, iy \)

\[(3.18) \quad (ix+A)(iy+A)-B^2=0, \quad A, B \in \mathbb{C}, \quad x, y \in \mathbb{R}.
\]

Set \( A=A_0+iA_1, \ A=B_0+iB_1 \)

**Lemma 3.2** If \( A_0=B_0 \neq 0 \), then taking the imaginary part and the real part of the equation (3.18), we have

\[
x+y=2(B_1-A_1), \quad xy=(B_1-A_1)^2.
\]

This shows that the solution of (3.18) is degenerate.

It follows that \( e_k \frac{1}{(t_1 \mu k \phi_k + t_2 \mu k \sqrt{\tau})} \ast V_{\ast k}, k \neq l \), has no other singular point than those sitting in the diagonal set \( (t_0, t_0) \in [0, \pi] \times [0, \pi] \).
3.2.5 Emergence of Clifford algebra

In the case $V_*$ does not contain $\varepsilon_{00}(k)$ but it contains $\varepsilon_{00}(\ell)$, we see by Proposition 3.3 that $\varepsilon_{00}(\ell)^*V_* = -V'_*$. Hence, $V_* = \varepsilon_{00}(\ell)^*V'_*$ where $V'_*$ does not contain $\varepsilon_{00}(k)$, $\varepsilon_{00}(\ell)$. Thus, by Proposition 3.3 and (3.8) we see

$$\varepsilon_{00}(\ell)^*\varepsilon_{00}(\ell)^*V_* = \varepsilon_{00}(\ell)^*\varepsilon_{00}(\ell)^*V'_* = -\varepsilon_{00}(\ell)^*V'_*.$$ 

On the other hand

$$\varepsilon_{00}(\ell)^*\varepsilon_{00}(\ell)^*V_* = \varepsilon_{00}(\ell)^*\varepsilon_{00}(\ell)^*V'_* = -\varepsilon_{00}(\ell)^*\varepsilon_{00}(\ell)^*V'_* = \varepsilon_{00}(\ell)^*V'_*.$$ 

In the case $V_*$ contains $\varepsilon_{00}(k)$, $\varepsilon_{00}(\ell)$, we set $\varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*V_* = V'_*$ where $V'_*$ does not contain $\varepsilon_{00}(k)$, $\varepsilon_{00}(\ell)$. In this situation, we have

$$\varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*V_* = -\varepsilon_{00}(\ell)^*\varepsilon_{00}(k)^*V'_*.$$ 

Suppose $V_* = \varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*V'_*$, then

$$\varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*V_* = \varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*\varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*V'_* = -\varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*\varepsilon_{00}(k)^*V'_* = \varepsilon_{00}(k)^*V'_* = V'_*.$$ 

On the other hand,

$$\varepsilon_{00}(\ell)^*\varepsilon_{00}(k)^*V_* = \varepsilon_{00}(\ell)^*\varepsilon_{00}(k)^*\varepsilon_{00}(k)^*\varepsilon_{00}(\ell)^*V'_* = -\varepsilon_{00}(\ell)^*\varepsilon_{00}(k)^*V'_* = V'_*.$$ 

Thus, Theorem 3.2 is proved by these observation. Namely, we see

**Theorem 3.3** In a $K_\kappa$-ordered expression, $(Hol(\mathbb{C}^{2m}), \ast_{K_\kappa})$ contains the Clifford algebra $Cliff(m)$. 

Since $(\varepsilon_{00}(k)^*\varepsilon_{00}(\ell))^2 = -1 = \varepsilon_{00}(k)^2$ in $K_\kappa$-ordered expression, we see in particular

**Corollary 3.1** In the $K_\kappa$-ordered expression, the group $\mathfrak{P}^{(4)}_{K_\kappa}$ coincides with $\mathfrak{P}^{(2)}_{K_\kappa}$. It follows that $\mathfrak{P}^{(2)}_{K_\kappa}$ is a connected double cover of $SO(m, \mathbb{C})$, hence $\mathfrak{P}^{(2)}_{K_\kappa} \cong Spin(m) \otimes \mathbb{C}$.

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