1. Introduction

Let $X$, $Y$ and $Z$ be topological spaces. For a mapping $f : X \times Y \to Z$ and a point $(x, y) \in X \times Y$ we write $f^x(y) = f(x, y)$. A mapping $f : X \times Y \to Z$ is said to be separately continuous, if mappings $f^x : Y \to Z$ and $f_y : X \to Z$ are continuous for all $x \in X$ and $y \in Y$. If $f : X \to Y$ is a pointwise limit of a sequence of separately continuous mappings $f_n : X \to Y$, then $f$ is a Baire-one mapping or $f$ belongs to the first Baire class.

In 1898 H. Lebesgue \cite{1} proved that if $X = Y = \mathbb{R}$ then every separately continuous function $f : X \times Y \to Z$ belongs to the first Baire class. A collection of topological spaces $(X, Y, Z)$ with this property we call a Lebesgue triple.

The result of Lebesgue was generalized by many mathematicians (see \cite{2, 3, 4, 5, 6} and the references given there). In particular, A. Kalancha and V. Maslyuchenko \cite{4} showed that $(\mathbb{R}, \mathbb{R}, Z)$ is a Lebesgue triple if $Z$ is a topological vector space. T. Banakh \cite{5} proved that $(\mathbb{R}, \mathbb{R}, Z)$ is a Lebesgue triple in the case when $Z$ is an equiconnected space. It follows from \cite{6} Theorem 3 that for every metrizable arcwise connected and locally arcwise connected space $Z$ a collection $(\mathbb{R}, \mathbb{R}, Z)$ is a Lebesgue triple.

In the connection with the above-mentioned results V. Maslyuchenko put the following question.

**Question 1.1.** Does there exist a topological space $Z$ such that $(\mathbb{R}, \mathbb{R}, Z)$ is not a Lebesgue triple?

Here we give the positive answer to this question. Moreover, we prove that $(X, Y, Z)$ is not a Lebesgue triple for topological spaces $X$ and $Y$ of a wide class of spaces, which includes, in particular, all spaces $\mathbb{R}^n$, and for a space $Z = X \times Y$ endowed with the cross-topology (see definitions in Section 2). In Sections 2 and 3 we give some auxiliary properties of this topology. Section 4 contains a proof of the main result. In the last section we show that connectedness-like conditions on spaces $X$ and $Y$ in the main result are essential. We prove that $(X, Y, Z)$ is a Lebesgue triple when $X$ is a strongly zero-dimensional metrizable space, $Y$ and $Z$ are arbitrary topological spaces.

2. Compact sets in the cross-topology

Let $X$ and $Y$ be topological spaces. We denote by $\gamma$ the collection of all subsets $A$ of $X \times Y$ such that for every point $(x, y)$ of $A$ there exist such neighborhoods $U$ and $V$ of $x$ and $y$ in $X$ and $Y$, respectively, that $(\{x\} \times V) \cup (U \times \{y\}) \subseteq A$. The system $\gamma$ forms a topology on $X \times Y$, which is called the cross-topology. A product $X \times Y$ with the cross-topology we denote by $(X \times Y, \gamma)$.

For a point $p = (x, y) \in X \times Y$ by cross($p$) we denote the set $(\{x\} \times Y) \cup (X \times \{y\})$. For an arbitrary $A \subseteq X \times Y$ let cross($A$) = $\bigcup_{p \in A}$ cross($p$).

**Proposition 2.1.** Let $X$ and $Y$ be $T_1$-spaces and $(p_n)_{n=1}^\infty$ a sequence of points $p_n = (x_n, y_n) \in X \times Y$ such that $x_n \neq x_m$ and $y_n \neq y_m$ if $n \neq m$. Then $P = \{p_n : n \in \mathbb{N}\}$ is a $\gamma$-discrete set.

**Proof.** Since every one-point subset of $X$ or of $Y$ is closed, $P$ is $\gamma$-closed. Similarly, every subset $Q \subseteq P$ is also $\gamma$-closed. Hence, $P$ is closed discrete subspace of $(X \times Y, \gamma)$. \qed
Proposition 2.2. Let $X$ and $Y$ be $T_1$-spaces and let $K \subseteq X \times Y$ be a $\gamma$-compact set. Then there exists a countable set $A \subseteq X \times Y$ such that $K \subseteq \text{cross}(A)$.

Proof. Assume that $K \not\subseteq \text{cross}(A)$ for any finite set $A \subseteq X \times Y$. We choose an arbitrary point $p_i \in K$ and by the induction on $n \in \mathbb{N}$ we construct a sequence of points $p_n \in K$ such that $p_{n+1} \in K \setminus \text{cross}(P_n)$, where $P_n = \{p_k : 1 \leq k \leq n\}$ for every $n \in \mathbb{N}$. According to Proposition 2.1 the set $P = \{p_n : n \in \mathbb{N}\}$ is infinite $\gamma$-discrete in $K$ which contradicts the fact that $K$ is $\gamma$-compact. □

Proposition 2.3. Let $X$ and $Y$ be $T_1$-spaces and let $A$ and $B$ be discrete sets in $X$ and $Y$, respectively. Then the topology of product $\tau$ and the cross-topology $\gamma$ coincide on the set $C = \text{cross}(A \times B)$.

Proof. Fix $p = (x, y) \in C$. Using the discreteness of $A$ and $B$, we choose such neighborhoods $U$ and $V$ of $x$ and $y$ in $X$ and $Y$, respectively, that $|U \cap A| \leq 1$ and $|V \cap B| \leq 1$. Then $C \cap (U \times V) = C \cap \text{cross}(c)$ for some point $c \in C$. Then $\tau = \gamma$ on the set $C \cap (U \times V)$. □

Propositions 2.2 and 2.3 immediately imply the following characterization of $\gamma$-compact sets.

Proposition 2.4. Let $X$ and $Y$ be $T_1$-spaces and $K \subseteq X \times Y$. Then $K$ is $\gamma$-compact if and only if when

1. $K$ is compact;
2. $K \subseteq \text{cross}(C)$ for a finite set $C \subseteq X \times Y$.

3. Connected sets and cross-mappings

Proposition 3.1. Let $X$ and $Y$ be connected spaces, $A$ a dense subset of $X$, let $\emptyset \neq B \subseteq Y$ and $C \subseteq X \times Y$ be such sets that $\text{cross}(A \times B) \subseteq C$. Then $C$ is connected.

Proof. Let $U$ and $V$ be open subsets of $C$ such that $C = U \cup V$. Since $X$ and $Y$ are connected, for every $p \in A \times B$ either $\text{cross}(p) \subseteq U$, or $\text{cross}(p) \subseteq V$. Since $\text{cross}(p) \cap \text{cross}(q) \neq \emptyset$ for distinct points $p, q \in X \times Y$, $\text{cross}(A \times B) \subseteq U$ or $\text{cross}(A \times B) \subseteq V$. Taking into account that $\text{cross}(A \times B)$ is dense in $X \times Y$, and consequently, in $C$, we obtain that $C \subseteq U$ or $C \subseteq V$. Therefore, $U = \emptyset$ or $V = \emptyset$. Hence, $C$ is connected. □

Corollary 3.2. Let $X$ and $Y$ be infinite connected $T_1$-spaces. Then the complement to any finite subset of $X \times Y$ is connected.

Proof. Let $C \subseteq X \times Y$ be a finite set. We choose finite sets $A \subseteq X$ and $B \subseteq Y$ such that $C \subseteq A \times B$. Remark that $A_1 = X \setminus A$ and $B_1 = Y \setminus B$ are dense in $X$ and $Y$, respectively, and $\text{cross}(A_1 \times B_1) \subseteq (X \times Y) \setminus C$. It remains to apply Proposition 3.1. □

Definition 3.3. A topological space $X$ is said to be a $C_1$-space (or a space with the property $C_1$), if the complement to any finite subset has finite many components.

Let us observe that the real line $\mathbb{R}$ has the property $C_1$. Moreover, a finite product of $C_1$-spaces is a $C_1$-space.

Let $X$ and $Y$ be topological spaces and $P \subseteq X \times Y$. A mapping $f : P \to X \times Y$ is called a cross-mapping, if $f(p) \subseteq \text{cross}(p)$ for every $p \in P$.

Lemma 3.4. Let $X$ and $Y$ be Hausdorff spaces, $U \subseteq X$, $V \subseteq Y$, let $f : U \times V \to X \times Y$ be a continuous cross-mapping, let $A \subseteq X$ and $B \subseteq Y$ be finite sets and the following conditions hold:

1. $U$, $V$ be connected $C_1$-spaces;
2. $f(U \times V) \subseteq \text{cross}(A \times B)$.
Then either \( f(U \times V) \subseteq \{a\} \times Y \) for some \( a \in A \), or \( f(U \times V) \subseteq X \times \{b\} \) for some \( b \in B \).

**Proof.** If both \( U \) and \( V \) are finite, then (1) imply that \( U \) and \( V \) are one-point sets and the lemma follows from (2). If \( U \) is finite (one-point) and \( V \) is infinite, then \( F = \{ z \in U \times V : f(z) \in \text{cross}(A \times B) \setminus (A \times Y) \} \) is finite clopen subset of \( U \times V \). The connectedness of \( U \times V \) implies \( F = \emptyset \). Hence, \( f(U \times V) \subseteq A \times Y \). Since \( f \) is continuous, \( f(U \times V) \subseteq \{a\} \times Y \) for some \( a \in A \).

Now let \( U \) and \( V \) be infinite. Then it follows from (1) that \( U \) and \( V \) have no isolated points. Since \( A_1 = A \cap U \) and \( B_1 = B \cap V \) are closed and nowhere dense in \( U \) and \( V \), respectively, the set

\[
C = (U \times V) \cap \text{cross}(A \times B) = (U \times V) \cap \text{cross}(A_1 \times B_1)
\]

is closed and nowhere dense in \( Z = U \times V \).

Let \( \alpha : U \times V \to X \) and \( \beta : U \times V \to Y \) be continuous functions such that \( f(x, y) = (\alpha(x, y), \beta(x, y)) \) for all \( (x, y) \in Z \). Put

\[
Z_\alpha = \{(x, y) \in Z : \alpha(x, y) = x\}, \quad Z_\beta = \{(x, y) \in Z : \beta(x, y) = y\}.
\]

Notice that

\[
P_\alpha = \{z \in Z_\alpha : \alpha(z) \in A\} = Z_\alpha \cap (A \times Y) = Z_\alpha \cap (A_1 \times Y)
\]

is nowhere dense in \( Z \). Therefore, \( Q_\alpha = \{z \in Z_\alpha : \alpha(z) \notin A\} \) is dense in \( \text{int}_Z(Z_\alpha) \), where by \( \text{int}_Z(D) \) we denote the interior of \( D \subseteq Z \) in \( Z \), and by \( \overline{D} \) we denote the closure of \( D \) in \( Z \). Condition (2) implies that \( Q_\alpha \) is contained in the closed set \( \{z \in Z : \beta(z) \in B\} \). Hence,

\[
\text{int}_Z(Z_\alpha) \subseteq \overline{Q_\alpha} \subseteq \{z \in Z : \beta(z) \in B\},
\]

i.e. \( f(\text{int}_Z(Z_\alpha)) \subseteq X \times B \). Similarly, \( f(\text{int}_Z(Z_\beta)) \subseteq A \times Y \).

Since \( f \) is a cross-mapping, \( Z = Z_\alpha \cup Z_\beta \). Remark that \( Z_\alpha \) and \( Z_\beta \) are closed in \( Z \). Let

\[
G = Z \setminus C.
\]

Taking into account that \( C \) is closed and nowhere dense in \( Z \), we have that \( G \) is open and dense in \( Z \). According to (1) the sets \( U \setminus A \) and \( V \setminus B \) have finite many components, therefore, the set \( G = (U \setminus A) \times (V \setminus B) \) has finite many components \( G_1, \ldots, G_k \). Then \( G = \bigcup_{i=1}^k G_i \), where all \( G_i \) are closed in \( G \). Hence, all the sets \( G_i \) are clopen in \( G \), in particular, open in \( Z \). Notice that \( Z_\alpha \cap Z_\beta = \{z \in Z : f(z) = z\} \subseteq f(Z) \subseteq \text{cross}(A \times B) \). Thus, \( G \cap Z_\alpha \cap Z_\beta = \emptyset \). Then \( G_i \subseteq (Z_\alpha \cap G_i) \cup (Z_\beta \cap G_i) \), consequently \( G_i \subseteq Z_\alpha \) or \( G_i \subseteq Z_\beta \) for every \( 1 \leq i \leq k \). Let

\[
I_\alpha = \{1 \leq i \leq k : G_i \subseteq Z_\alpha\}, \quad I_\beta = \{1 \leq i \leq k : G_i \subseteq Z_\beta\},
\]

\[
U_\alpha = \bigcup_{i \in I_\alpha} G_i, \quad U_\beta = \bigcup_{i \in I_\beta} G_i.
\]

Remark that

\[
f(U_\alpha) \subseteq f(\text{int}_Z(Z_\alpha)) \subseteq X \times B, \quad f(U_\beta) \subseteq f(\text{int}_Z(Z_\beta)) \subseteq A \times Y.
\]

Hence, for any \( z = (x, y) \in U_\alpha \) we have \( \alpha(x, y) = x \) and \( \beta(x, y) \in B \). Similarly, \( \alpha(x, y) \in A \) and \( \beta(x, y) = y \) for every \( z = (x, y) \in U_\beta \). Therefore, \( z = f(z) \in A \times B \) for any \( z \in U_\alpha \cup U_\beta \). Consequently, \( Z_0 = U_\alpha \cap U_\beta \) is finite.

Denote \( \bar{E} = \overline{U_\alpha \setminus Z_0} \) and \( \bar{D} = \overline{U_\beta \setminus Z_0} \). Since by Proposition 1.3 the set \( Z \setminus Z_0 \) is connected, nonempty and \( Z \setminus Z_0 = E \cup D \), taking into account that \( \overline{E \cap D} = \emptyset \) and \( E \cap \overline{D} = \emptyset \), we obtain \( E = \emptyset \) or \( D = \emptyset \). Assume that \( E = \emptyset \). Then \( U_\beta \) is dense in \( Z \) and

\[
f(Z) \subseteq f(U_\beta) \subseteq A \times Y.
\]
Since \( U \times V \) is connected, \( f(U \times V) \) is connected too, therefore, there is such \( a \in A \) that \( f(U \times V) \subseteq \{a\} \times Y \).

\[ \square \]

4. The main result

**Proposition 4.1.** Let \( X \) and \( Y \) be \( T_1 \)-spaces, \( z_0 \in X \times Y \) and let \((z_n)_{n=1}^{\infty}\) be \( \gamma \)-convergent to \( z_0 \) sequence of points \( z_n = (x_n, y_n) \in X \times Y \). Then there exists \( m \in \mathbb{N} \) such that \( z_n \in \text{cross}(z_0) \) for all \( n \geq m \).

**Proof.** Assume the contrary. Then by the induction on \( k \in \mathbb{N} \) it is easy to construct a strictly increasing sequence of numbers \( n_k \in \mathbb{N} \) such that \( x_{n_k} \neq x_{n_j} \) and \( y_{n_k} \neq y_{n_j} \) for distinct \( i, j \in \mathbb{N} \), and \( z_{n_k} \notin \text{cross}(z_0) \) for all \( k \in \mathbb{N} \). Now the sequence \((p_k)_{k=1}^{\infty}\) of points \( p_k = z_{n_k} \) converges to \( z_0 \), and from the other side the set \( G = (X \times Y) \setminus \{p_k : k \in \mathbb{N}\} \) is a neighborhood of \( z_0 \), a contradiction. \( \square \)

A system \( \mathcal{A} \) of subsets of a topological space \( X \) is called a \( \pi \)-pseudobase \( [8] \), if for every nonempty open set \( U \subseteq X \) there exists a set \( A \in \mathcal{A} \) such that \( \text{int}(A) \neq \emptyset \) and \( A \subseteq U \).

**Theorem 4.2.** Let \( X \) and \( Y \) be Hausdorff spaces without isolated points and let \( X \) and \( Y \) have \( \pi \)-pseudobases which consist of connected compact \( C_1 \)-sets, and let \( f : X \times Y \to X \times Y \) be the identical mapping. Then \( f \notin B_1(X \times Y, (X \times Y, \gamma)) \).

**Proof.** Assuming the contrary, we choose a sequence of continuous functions \( f_n : X \times Y \to (X \times Y, \gamma) \) such that \( f_n(x, y) \to (x, y) \) in \((X \times Y, \gamma)\) for all \((x, y) \in X \times Y \).

Remark that every \( f_n \) : \( X \times Y \to X \times Y \) is continuous. Then for every \( n \in \mathbb{N} \) the set \( P_n = \{p \in X \times Y : f_n(p) \in \text{cross}(p)\} \) is closed. Hence, for every \( n \in \mathbb{N} \) the set

\[ F_n = \bigcap_{m \geq n} P_m = \{p \in X \times Y : \forall m \geq n \ f_m(p) \in \text{cross}(p)\} \]

is closed too. Moreover, by Proposition 4.1

\[ X \times Y = \bigcup_{n=1}^{\infty} F_n. \]

The conditions of the theorem imply that \( Z = X \times Y \) has a \( \pi \)-pseudobase of compact sets. Then the product contains an open everywhere dense locally compact subspace, in particular, the product \( X \times Y \) is Baire. We choose a number \( n_0 \in \mathbb{N} \) and compact connected \( C_1 \)-sets \( U \subseteq X \) and \( V \subseteq Y \) such that \( U \times V \subseteq F_{n_0} \), \( U_0 = \text{int}(U) \neq \emptyset \) and \( V_0 = \text{int}(V) \neq \emptyset \).

Let \( W = U \times V \). According to Proposition 2.4 there exist such sequences of finite sets \( A_n \subseteq X \) and \( B_n \subseteq Y \) that \( f_n(W) \subseteq (A_n \times Y) \cup (X \times B_n) \) for every \( n \in \mathbb{N} \). Since \( X \) and \( Y \) have no isolated points, the sets \( U_0 \) and \( V_0 \) are infinite. Take such points \( p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in U_0 \times V_0 \) that \( p_1 \notin \text{cross}(p_2) \). Since \( X \) and \( Y \) are Hausdorff, there exist neighborhoods \( U_1 \) and \( U_2 \) of \( x_1 \) and \( x_2 \) in \( U_0 \) and neighborhoods \( V_1 \) and \( V_2 \) of \( y_1 \) and \( y_2 \) in \( V_0 \), respectively, such that \( U_1 \cap U_2 = V_1 \cap V_2 = \emptyset \). Now we choose a number \( N \geq n_0 \) such that \( f_N(p_1) \in U_1 \times V_1 \) and \( f_N(p_2) \in U_2 \times V_2 \).

Since \( f_N|_W \) is a cross-mapping, Lemma 3.4 implies that \( f_N(W) \subseteq \{a\} \times Y \) for some \( a \in A \) or \( f_N(W) \subseteq X \times \{b\} \) for some \( b \in B \). Assume that \( f_N(W) \subseteq \{a\} \times Y \) for some \( a \in X \). Then \( (U_1 \times V_1) \cap (\{a\} \times Y) \neq \emptyset \) and \( (U_2 \times V_2) \cap (\{a\} \times Y) \neq \emptyset \). Therefore, \( a \in U_1 \cap U_2 \), which is impossible. \( \square \)

**Corollary 4.3.** Let \( n, m \geq 1 \) and let \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m \) be the identical mapping. Then \( f \notin B_1(\mathbb{R}^n \times \mathbb{R}^m, (\mathbb{R}^n \times \mathbb{R}^m, \gamma)) \).

**Corollary 4.4.** The collection \((\mathbb{R}^n, \mathbb{R}^m, (\mathbb{R}^n \times \mathbb{R}^m, \gamma))\) is not a Lebesgue triple for all \( n, m \geq 1 \).
5. Separately continuous mappings on zero-dimensional spaces

Recall that a nonempty topological space $X$ is strongly zero-dimensional, if it is completely regular and every finite functionally open cover of $X$ has a finite disjoint open refinement [9, p. 529].

Theorem 5.1. Let $X$ be a strongly zero-dimensional metrizable space, let $Y$ and $Z$ be topological spaces. Then $(X, Y, Z)$ is a Lebesgue triple.

Proof. Let $d$ be a metric on $X$, which generates its topology. For every $n \in \mathbb{N}$ we consider an open cover $B_n$ of $X$ by balls of the diameter $\leq \frac{1}{n}$. It follow from [7] that every $B_n$ has locally finite clopen refinement $\forall \alpha < \beta$ if $\alpha > 0$. Then $V_n = (V_{\alpha,n} : 0 \leq \alpha < \beta_n)$ is a locally finite disjoint cover of $X$ by clopen sets $V_{\alpha,n}$ which refines $B_n$.

Let $f : X \times Y \to Z$ be a separately continuous function. For all $n \in \mathbb{N}$ and $0 \leq \alpha < \beta_n$ we choose a point $x_{\alpha,n} \in V_{\alpha,n}$. Let us consider functions $f_n : X \times Y \to Z$ defined as the following:

$$f_n(x, y) = f(x_{\alpha,n}, y),$$

if $x \in V_{\alpha,n}$ and $y \in Y$. Clearly, for every $n \in \mathbb{N}$ the function $f_n$ is jointly continuous, provided $f$ is continuous with respect to the second variable. We show that $f_n(x, y) \to f(x, y)$ on $X \times Y$. Fix $(x, y) \in X \times Y$ and choose a sequence $(\alpha_n)_{n=1}^\infty$ such that $x \in V_{\alpha_n,n}$. Since $\text{diam} V_{\alpha_n,n} \to 0$, $x_{\alpha_n,n} \to x$. Taking into account that $f$ is continuous with respect to the first variable, we obtain that

$$f_n(x, y) = f(x_{\alpha_n,n}, y) \to f(x, y).$$

Hence, $f \in B_1(X \times Y, Z)$. \qed

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