COMBINATORICS OF NORMAL SEQUENCES OF BRAIDS

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Abstract. Many natural counting problems arise in connection with the normal form of braids—and seem to have not been much considered so far. Here we solve some of them by analysing the normality condition in terms of the associated permutations, their descents and the corresponding partitions. A number of different induction schemes appear in that framework.

Ubiquitous and connected with a number of domains, Artin’s braid groups have received much attention in the recent years. However, not so many works are devoted to a purely combinatorial study of braids, presumably because counting arguments did not prove so far to be much helpful for investigating braids. Nevertheless, although braid groups are infinite, they admit several filtrations leading to finite sets and, therefore, to natural enumeration problems.

For each presentation of braid groups, (at least) two natural counting problems arise, namely, on the one hand, counting how many braids admit an expression of a given length, in particular evaluating the associated growth rate—we shall refer to this as Question 1 in the sequel—and, on the other hand, counting, for a given braid, how many words represent that specific braid, a relevant question when the number is finite, typically when we discard the inverses of the generators and only consider positive expressions, i.e., when we restrict to some submonoid of $B_n$—we shall refer to that as Question 2.

In the case of the Artin generators $\sigma_i$, both types of questions have been addressed, and at least partially solved: Question 1, actually not for $B_n$ but for the submonoid $B_n^+$ of $B_n$ generated by the $\sigma_i$’s, was investigated in [25], and completely solved in [5]. As for Question 2, it is natural in this context to address it for the particular elements $\Delta_n$, where $\Delta_n$ is Garside’s fundamental braid [18]. It was investigated and solved for $n = 3$ in [9].

In this paper, we address Question 1 for another natural generating set, namely the so-called simple braids, also called the Garside generators below [18]. These generators, which are the divisors of $\Delta_n$ in the monoid $B_n^+$, are in one-to-one correspondence with permutations of $n$ objects, and they give rise to a remarkable unique decomposition for each braid, usually called its normal form [14] [11] [16] [15]. Because of its uniqueness and of its many nice properties, expressed in particular in the existence of a bi-automatic structure, the normal form of braids is the preferred way of specifying braids in many recent developments, in particular those of algorithmic or cryptographical nature [19] [21].

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In the case of the Garside generators and the associated normal form(s), what we called Question 2 above is not relevant, as each braid has one unique distinguished decomposition in terms of the generators, and counting the non-normal expressions would appear artificial. But Question 1, namely counting the number of braids with a normal form of a given length, is quite natural. The question was briefly considered by R. Charney in [8]: she observed that, because normal words can be recognized by a finite state automaton, the number of braids with length $d$ obeys a linear induction rule, and the associated generating function is rational. Explicit values are given in the case of 3 and 4 strand braids.

The aim of this paper is to go further in the investigation of counting problems connected with the normal form of braids. We consider the case of the monoid $B_n^+$, and mainly study the number of positive $n$ strand braids with a normal form of length (at most) $d$, i.e., we consider for Garside generators the problem solved in [5] in the case of Artin generators. The main difference is that, in the case of Garside generators, the length is no longer an additive parameter: for instance, multiplying two simple braids may result in a simple braid, i.e., multiplying two braids of length 1 may result in a braid of length 1. This makes the current study much more uneasy.

Let $b_{n,d}$ denote the number of positive $n$ strand braids with a normal form of length at most $d$, i.e., the number of divisors of $\Delta^d_n$ in $B_n^+$. We establish various results about the numbers $b_{n,d}$, and about the connected numbers $b_{n,d}(x)$ that count, for $x$ a simple braid, the positive $n$ strand braid with a normal form of length at most $d$ whose $d$th factor is precisely $x$. Two types of results are established, namely results for fixed braid index $n$, and results for fixed degree $d$. When $n$ is fixed and $d$ varies, as was recalled above, the numbers $b_{n,d}$ and $b_{n,d}(x)$ obey a linear induction rule associated with a certain $n! \times n!$ incidence matrix $M_n$. Here we show that $M_n$ can be replaced with a smaller matrix of size $p(n) \times p(n)$, where $p(n)$ is the number of partitions of $n$. The result relies on analysing the descents of the permutations associated with simple braids, and it is connected with a classical result by Solomon [24]. It is then easy to deduce the numerical value of $b_{n,d}$ for small $n, d$, as well as explicit formulas, at least for $n \leq 4$. We are also led to several conjectures about the eigenvalues of the matrix $M_n$ that seem to have never been considered so far. The most puzzling one claims that the characteristic polynomial of $M_{n-1}$ divides that of $M_n$. It holds at least for $n \leq 10$.

When $d$ is fixed and $n$ varies, quite different induction rules appear. Everything is trivial for $d = 1$, and an explicit formula for $b_{n,2}$ can be deduced from the results of [6, 7]. It seems difficult to go further in general, but new results (and new induction schemes) appear when we consider the numbers $b_{n,d}(\Delta_{n-r})$ with $1 \leq r \leq n$, typically in the (non-trivial) case $r = 1$, and, more generally, when $r$ is fixed. In particular, we obtain explicit values for $b_{n,3}(\Delta_{n-1})$, $b_{n,3}(\Delta_{n-2})$, and $b_{n,4}(\Delta_{n-1})$.

The specific questions investigated in this paper, in particular that of the value of $b_{n,d}(\Delta_{n-r})$, arose in [12]. There exists a distinguished linear ordering of braids $<$ relying on the notion of a $\sigma$-positive braid word [11], and the aim of [12] is to develop a new approach to that ordering based on the study of its connection with the Garside structure. It turns out that certain parameters describing the restriction of $<$ to positive $n$-braids of degree at most $d$ can be expressed in terms of the numbers $b_{n,d}(\Delta_r)$, an initial motivation for our current study of these numbers. However, we think that the formulas and methods developed in the current paper go beyond the above specific
applications. In particular, the great diversity of the induction schemes appearing in connection with various specializations of the general problem is remarkable. At the least, the current study should demonstrate the richness of the combinatorics underlying the normal form of braids.

Still other presentations of the braid groups are known, in particular the one involving the so-called dual monoid \([2, 4]\), which gives rise to an alternative Garside structure, and, therefore, to an alternative normal form analogous to that considered here, where the role of simple braids is played by elements that are in one-to-one correspondence with non-crossing partitions. All questions considered in the current paper could be similarly addressed for the dual structure, and, more generally, for the many presentations of \(B_n\) known to date. Similarly, Artin’s braid groups \(B_n\) belong to larger families of groups, typically Artin-Tits groups of spherical type and, more generally, Garside groups \([13, 10, 22]\). Once again, all questions considered here extend to such frameworks naturally. However, mainly because of the specific applications mentioned above, we find it interesting to consider here the specific framework of braids and permutations, and we leave the extensions for further investigation.

The paper is organized as follows. Section 1 sets the framework and the basic definitions. In Section 2 we introduce the incidence matrix \(M_n\) that controls the sequences \(b_{n,d}\) for fixed \(n\) and show how to reduce their size from \(n!\) to \(2^{n-1}\). In Section 3 we show how to further reduce the size to \(p(n)\), and solve the induction for small values of \(n\). Finally, in Section 4 we turn to the cases when the degree is fixed and the braid index varies.

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1. BACKGROUND AND PRELIMINARY RESULTS

Our notation is standard, and we refer to textbooks like [3] or [16] for basic results about braid groups. We recall that the \(n\) strand braid group \(B_n\) is defined for \(n \geq 1\) by the presentation

\[
B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1}; \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2 \right\rangle.
\]

So, \(B_1\) is a trivial group \(\{1\}\), while \(B_2\) is the free group generated by \(\sigma_1\). The elements of \(B_n\) are called \(n\) strand braids, or simply \(n\)-braids. We use \(B_\infty\) for the group generated by an infinite sequence of \(\sigma_i\)’s subject to the relations of (1.1), i.e., the direct limit of all \(B_n\)’s under the inclusion of \(B_n\) into \(B_{n+1}\).

By definition, every \(n\)-braid \(x\) admits (infinitely many) expressions in terms of the generators \(\sigma_i\), \(1 \leq i < n\). Such a expression is called an \(n\) strand braid word. Two braid words \(w, w'\) representing the same braid are said to be equivalent; the braid represented by a braid word \(w\) is denoted \([w]\).

It is standard to associate with every \(n\) strand braid word \(w\) an \(n\) strand braid diagram by stacking elementary diagrams associated with the successive letters according to the rules.
Then two braid words are equivalent if and only if the diagrams they encode are the projections of ambient isotopic figures in $\mathbb{R}^3$, i.e., one can deform one diagram into the other without allowing the strands to cross or moving the endpoints.

1.1. The monoid $B^+_n$ and the braids $\Delta_n$. Let $B^+_n$ be the monoid admitting the presentation \[(1.1)\] The elements of $B^+_n$ are called positive $n$-braids.

Definition 1.1. For $x, y$ in $B^+_n$, we say that $x$ is a left divisor of $y$, denoted $x \preceq y$, or, equivalently, that $y$ is a right multiple of $x$, if $y = xz$ holds for some $z$ in $B^+_n$. We denote by $\text{Div}(y)$ the (finite) set of all left divisors of $y$ in $B^+_n$.

As $B^+_n$ is not commutative for $n \geq 3$, there are the symmetric notions of a right divisor and a left multiple—but we shall mostly use left divisors here. Note that $x$ is a (left) divisor of $y$ in the sense of $B^+_n$ if and only if it is a (left) divisor in the sense of $B^+_\infty$, so there is no need to always specify the index $n$.

With respect to left divisibility, $B^+_n$ has the structure of a lattice \cite{18}: any two positive $n$-braids $x, y$ admit a greatest common left divisor, denoted $\gcd(x, y)$, and a least common right multiple. A special role is played by the lcm of the elements $\sigma_1, \ldots, \sigma_{n-1}$, traditionally denoted $\Delta_n$, which we recall is inductively defined by \[(1.2)\]

It is well known that $\Delta_n^2$ belongs to the centre of $B_n$ (and even generates it for $n \geq 3$), and that the inner automorphism $\phi_n$ of $B_n$ corresponding to conjugation by $\Delta_n$ exchanges $\sigma_i$ and $\sigma_{n-i}$ for $1 \leq i \leq n-1$.

1.2. The normal form. In $B^+_n$, the left and the right divisors of $\Delta_n$ coincide, and they make a finite sublattice of $(B^+_n, \preceq)$ with $n!$ elements. These braids will be called simple in the sequel. Geometrically, simple braids are those positive braids that can be represented by a braid diagram in which any two strands cross at most once.

For each positive $n$-braid $x$ distinct of 1, the simple braid $\gcd(x, \Delta_n)$ is the maximal simple left divisor of $x$, and we obtain a distinguished expression $x = x_1 x'$ with $x_1$ simple. By decomposing $x'$ in the same way and iterating, we obtains the so-called normal expression \cite{15,16}.

Definition 1.2. A sequence $(x_1, \ldots, x_d)$ of simple $n$-braids is said to be normal if, for each $k$, one has $x_k = \gcd(\Delta_n, x_k \ldots x_d)$.

Clearly, each positive braid admits a unique normal expression. It will be convenient here to consider the normal expression as unbounded on the right by completing it with as many trivial factors 1 we need. In this way, we can speak of the $d$th factor (in the normal form) of $x$ for each positive braid $x$. We say that a positive braid has degree $d$ if $d$ is the largest integer such that the $d$th factor of $x$ is not 1. It is well known that the positive $n$-braids of degree at most $d$ coincide with the (left or right) divisors of $\Delta_n^d$.

The only properties of the normal form we shall use here are as follows:
Lemma 1.3. \[15\] Assume that \((x_1, \ldots, x_d)\) is a sequence of simple \(n\)-braids. Then the following are equivalent:

(i) The sequence \((x_1, \ldots, x_r)\) is normal;
(ii) For \(1 \leq k < d\), the sequence \((x_k, x_{k+1})\) is normal;
(iii) For \(1 \leq k < d\), every \(\sigma_i\) dividing \(x_{k+1}\) on the left divides \(x_k\) on the right.

Definition 1.4. For \(x\) a simple \(n\)-braid, we define \(D_L(x)\) (resp. \(D_R(x)\)) to be the set of all \(i\)'s such that \(\sigma_i\) is a left (resp. right) divisor of \(x\).

The example of \(\sigma_2\) and \(\sigma_2\sigma_1\sigma_2\sigma_2\), for which both \(D_L\) and \(D_R\) is \(\{2\}\), shows that these sets do not determine a simple braid. However, as far as normal sequences are concerned, they contain all needed information, as Lemma 1.3 can be restated as:

Lemma 1.5. A sequence of simple \(n\)-braids \((x_1, \ldots, x_d)\) is normal if and only if, for each \(k < d\), we have \(D_R(x_k) \supseteq D_L(x_{k+1})\).

1.3. Connection with permutations. Everywhere in the sequel, we write \([1, n]\) for \(\{1, \ldots, n\}\). By mapping \(\sigma_i\) to the transposition \((i, i+1)\), one defines a surjective homomorphism \(\pi\) of \(B_n\) onto the symmetric group \(\mathfrak{S}_n\). The restriction of \(\pi\) to simple braids is a bijection: for every permutation \(f\) of \([1, n]\), there exists exactly one simple braid \(x\) satisfying \(\pi(x) = f\).

The Exchange Lemma for Coxeter groups connects the sets \(D_L(x)\) and \(D_R(x)\) with the permutation associated with \(x\) and their descents. For \(f\) a permutation, use \(\ell(f)\) for the minimal number of factors occurring in a decomposition of \(f\) as a product of transpositions. The precise statement is

Lemma 1.6. Let \(x\) be a simple \(n\)-braid \(x\). For \(1 \leq i < n\), the following are equivalent:

(i) The braid \(\sigma_i\) is a left divisor of \(x\) in \(B_n^+\), i.e., \(i\) belongs to \(D_L(x)\);
(ii) The strands starting at positions \(i\) and \(i+1\) cross in any (positive) diagram for \(x\);
(iii) We have \(\pi(x)^{-1}(i) > \pi(x)^{-1}(i+1)\);
(iv) We have \(\ell(\pi(\sigma_i x)) < \ell(\pi(x))\), i.e., \(i\) is a descent of \(\pi(x)^{-1}\).

Symmetrically, the following are equivalent:

(i') The braid \(\sigma_i\) is a right divisor of \(x\) in \(B_n^+\), i.e., \(i\) belongs to \(D_R(x)\);
(ii') The strands finishing at positions \(i\) and \(i+1\) cross in any (positive) diagram for \(x\);
(iii') We have \(\pi(x)(i) > \pi(x)(i+1)\);
(iv') We have \(\ell(\pi(\sigma_i x)) < \ell(\pi(x))\), i.e., \(i\) is a descent of \(\pi(x)\).

So, for \(x\) a simple braid, the indices \(i\) such that \(\sigma_i\) is a right divisor of \(x\) are the descents of the associated permutation \(\pi(x)\), while those such that \(\sigma_i\) is a left divisor of \(x\) are the descents of \(\pi(x)^{-1}\).

1.4. The numbers \(b_{n,d}\) and \(b_{n,d}(x)\). Our aim in this paper is to solve various counting problems involving the normal form of positive braids. The main numbers we investigate are as follows:

Definition 1.7. For \(n, d \geq 1\), we denote by \(b_{n,d}\) the number of positive \(n\) strand braids of degree at most \(d\), i.e., the number of divisors of \(\Delta_n^d\) in the braid monoid.

By Lemma 1.3 \(b_{n,d}\) is the number of normal sequences of length \(d\), i.e., the number of sequences \((x_1, \ldots, x_d)\) where all \(x_k\) are simple braids and \(D_L(x_k) \supseteq D_R(x_{k+1})\) holds for
By Lemma 1.6, it is also the number of sequences of permutations $(f_1, \ldots, f_d)$ such that, for each $k < d$, the descents of $f_{k+1}$ are included in those of $f_k$.

For $d = 1$, the bijection between simple $n$ strand braids and permutations of $[1, n]$ immediately gives

$$b_{n, 1} = n!,$$

which implies for all $n, d$

$$b_{n, d} \leq (n!)^d.$$

In the sequel, we shall have to count normal sequences satisfying some constraints. So we introduce one more notation.

Definition 1.8. For $n, d \geq 1$ and $x$ a simple $n$-braid, we denote by $b_{n, d}(x)$ the number of positive $n$ strand braids of degree at most $d$ with $d$th factor equal to $x$.

In other words, $b_{n, d}(x)$ is the number of normal sequences of the form $(x_1, \ldots, x_{d-1}, x)$. Some connections are obvious:

Proposition 1.9. For all $n, d$, we have

$$b_{n, d} = \sum_{x \text{ simple}} b_{n, d}(x) = b_{n, d+1}(1).$$

Proof. The first equality is obvious. The second one follows from the fact that $(x_1, \ldots, x_d)$ is normal if and only if $(x_1, \ldots, x_d, 1)$ is: indeed, 1 has no left divisor but itself, so, by Lemma 1.3, every sequence $(x, 1)$ is normal.

□

2. Adjacence matrices

In this section, we study the numbers $b_{n, d}$ and $b_{n, d}(x)$ when $n$ is fixed and $d$ varies. By Lemma 1.3, normal sequence of simple braids is characterized by a purely local criterion that only involves adjacent entries. It follows that the set of all normal sequences can be recognized by a finite state automaton [16], and, as a consequence, the associated counting numbers obey a linear induction rule specified by a certain incidence matrix [17]. In this section, we define the matrix involved in the current situation, and show how its size, which is originally $n!$, can be lowered to $2^{n-1}$.

2.1. Enumeration of simple braids. Below we consider matrices whose entries are indexed by simple braids (or, equivalently, permutations). Fixing an enumeration of simple braids is not important at a conceptual level, but this is necessary when the objects are to be specified explicitly. We shall use the restriction of the canonical linear ordering of braids denoted $<^c$ in [14]—which gives for each $n$ a well-ordering of ordinal type $\omega^{n-2}$ on $B_n^+$. The corresponding increasing enumeration of simple $n$-braids can be constructed directly using induction on $n$. We start from the following easy remark:

Lemma 2.1. For $1 \leq i \leq n$, write $\sigma_{i, n}$ for $\sigma_i\sigma_{i+1} \ldots \sigma_{n-1}$ (so that $\sigma_{n, n}$ is 1). Then every simple $n$-braid $x$ admits a unique decomposition $x = \sigma_{i, n} y$ with $1 \leq i \leq n$ and $y$ a simple $(n - 1)$-braid.

Proof. (Figure 1) Let $i = \pi(x)(n)$. Then we can realize $x$ by a diagram in which the $i$th strand is first sent to the rightmost position, and it remains a simple $(n - 1)$-braid. Conversely, we have $i = \pi(\sigma_{i, n} y)(n)$, so the decomposition is unique. □
Definition 2.2. We inductively define an enumeration $S_n$ of simple $n$-braids by
\begin{equation}
S_1 := (1), \quad S_n := S_{n-1} \circ \sigma_{n-1,n}S_{n-1} \circ \ldots \circ \sigma_{1,n}S_{n-1},
\end{equation}
where $\circ$ stands for list concatenation, and $xS$ is the list obtained from $S$ by multiplying all entries by $x$ on the left. The $k$th element in $\bigcup_n S_n$ is denoted $\tau_k$.

The first $\tau_k$’s are, in increasing order,
\begin{align*}
\tau_1 &= 1, \\
\tau_2 &= \sigma_1, \\
\tau_3 &= \sigma_2, \\
\tau_4 &= \sigma_2\sigma_1, \\
\tau_5 &= \sigma_1\sigma_2, \\
\tau_6 &= \sigma_1\sigma_2\sigma_1, \\
\tau_7 &= \sigma_3, \ldots.
\end{align*}

Lemma 2.1 guarantees that all simple braids occur in the above enumeration. Note that, for every $n$, we have $\Delta_n = \tau_n!$.

It is easy to check that the ordering of simple braids we use corresponds to a reversed antilexicographic ordering of the inverses of the associated permutations: $x$ occurs before $y$ if and only if we have $\pi(x)^{-1} < \pi(y)^{-1}$, where $f < g$ is said to hold if we have $f(i) > g(i)$ for the largest $i$ for which $f$ and $g$ do not agree.

2.2. The matrix $M_n$. Everywhere in the sequel, we write $(M)_{x,y}$ for the $(x, y)$-entry of a matrix $M$.

Definition 2.3. For $n \geq 1$, we define $M_n$ to be the $n! \times n!$ matrix satisfying
\begin{equation}
(M_n)_{k,\ell} = \begin{cases} 
1 & \text{if } (\tau_k, \tau_\ell) \text{ is normal}, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Instead of referring to integer entries, it will be often convenient to think of the entries of $M_n$ as directly indexed by simple braids; for $x, y$ simple braids, we simply write $(M_n)_{x,y}$ for the corresponding entry.

Example 2.4. The first 3 matrices $M_n$ are
\begin{align*}
M_1 &= 1, \\
M_2 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\
M_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.
\end{align*}

The construction of the matrix $M_n$ immediately implies the following results:

Lemma 2.5. (i) The first column and the last row of $M_n$ contain only 1’s; the first row, its first entry excepted, and the last column, its last entry excepted, contain only 0’s.

(ii) The first $(n-1)!$ columns of $M_n$ consist of $n$ stacked copies of $M_{n-1}$.

(iii) If $D_R(\tau_k) = D_R(\tau_k')$ holds, then the $k$th and the $k'$th rows in $M_n$ coincide. Similarly, if $D_L(\tau_\ell) = D_L(\tau_\ell')$ holds, then the $\ell$th and the $\ell'$th columns in $M_n$ coincide.
Proof. (i) By construction, we have \( \tau_1 = 1 \) and \( \tau_{n!} = \Delta_n \). Now \((x,1)\) is always normal, and so is \((\Delta_n, x)\). On the other hand, \((1,x)\) is normal only for \(x = 1\), and \((x, \Delta_n)\) is normal only for \(x = \Delta_n\).

(ii) Assume \(k \leq (n-1)!\) and \(k' = k + (n - i) \cdot (n - 1)!\). Our enumeration of simple braids implies \(\tau_{k'} = \sigma_i \tau_k\). Then Figure 1 makes the equality \(D_R(\tau_{k'}) \cap [1, n - 1] = D_R(\tau_k)\) clear. For every simple \((n-1)\)-braid \(y\), the set \(D_L(y)\) is included in \([1, n - 1]\), and it follows that \((\tau_{k'}, y)\) is normal if and only if \((\tau_k, y)\) is. In other words, we have \((M_n)_{k', \ell} = (M_n)_{k, \ell}\) for \(\ell \leq (n-1)!\).

(iii) By Lemma 2.3, the value of \((M_n)_{k, \ell}\) only depends on \(D_R(\tau_k)\) and on \(D_L(\tau_\ell)\).

The connection between the numbers \(b_{n,d}(x)\) and the matrix \(M_n\) is straightforward:

Lemma 2.6. For every simple \(y\) and every \(d \geq 1\), we have

\[
(2.2) \quad b_{n,d}(y) = \left((1, 1, \ldots, 1) M_n^{d-1}\right) y. 
\]

Proof. Induction on \(d\). For \(d = 1\), and for each simple \(n\)-braid \(x\), there is exactly one braid of degree at most 1 whose first factor is \(y\), namely \(y\) itself, and we have \(b_{n,1}(y) = 1\). Assume \(d \geq 2\). By Lemma 2.3, \((x_1, \ldots, x_{d-1}, y)\) is normal if and only if \((x_1, \ldots, x_{d-1})\) and \((x_{d-1}, y)\) are normal, so we get

\[
b_{n,d}(y) = \sum_{(x,y) \text{ normal}} b_{n,d-1}(x) = \sum_x b_{n,d-1}(x) (M_n)_{x,y},
\]

and (2.2) follows inductively.

Remark 2.7. As last row of \(M_n\) is \((1, \ldots, 1)\), we have \((1, \ldots, 1) = (0, \ldots, 0, 1)M_n\), and we can replace (2.2) with

\[
(2.3) \quad b_{n,d}(y) = \left((0, \ldots, 0, 1) M_n^d\right) y.
\]

Example 2.8. Using the value of \(M_2\), we immediately find \(b_{2,d}(1) = d\), \(b_{2,d}(\sigma_1) = 1\), as could be expected: there are \(d+1\) braids of degree at most \(d\), namely the braids \(\sigma_1^k\) with \(k < d\), whose \(d\)th factor is 1, and \(\sigma_1^d\), whose \(d\)th factor is \(\Delta_2\), i.e., \(\sigma_1\).

The computation for \(n \geq 3\) is more complicated, and we postpone it. For the moment, we just point that, as the numbers \(b_{n,d}(x)\) obey the linear recurrence (2.2), standard arguments imply that they can be expressed in terms of the eigenvalues of \(M_n\):

Proposition 2.9. Let \(\rho_1, \ldots, \rho_k\) be the non-zero eigenvalues of \(M_n\). Then, for each simple \(n\)-braid \(x\), there exist polynomials \(P_1, \ldots, P_k\) with \(\deg(P_i)\) at most the multiplicity of \(\rho_i\) for \(M_n\) such that, for each \(d \geq 0\), we have

\[
(2.4) \quad b_{n,d}(x) = P_1(d)\rho_1^d + \cdots + P_k(d)\rho_k^d.
\]

Corollary 2.10. For all \(n, x\), the generating function of the numbers \(b_{n,d}(x)\)'s with respect to \(d\) is rational.

2.3. Reducing the size. The size \(n!\) of the incidence \(M_n\) is uselessly large, and we shall see now how to lower it. This will be done in two steps. The first one relies on the fact, pointed out in Lemma 2.3(iii), that many columns in \(M_n\) are equal. For subsequent use, it will be useful to introduce a new sequence of numbers:
Definition 2.11. For $I, J \subseteq [1, n - 1]$, we denote by $a_{n,I,J}$ (resp. $\tilde{a}_{n,I,J}$) the number of simple $n$-braids satisfying $D_L(x) = I$ (resp. $D_L(x) \supseteq I$) and $D_R(x) \supseteq J$.

Lemma 2.12. For $n \geq 1$, let $M'_n$ be the $2^{n-1} \times 2^{n-1}$ matrix with entries indexed by subsets of $[1, n - 1]$ defined by $(M'_n)_{I,J} = a_{n,I,J}$. Then the characteristic polynomials of $M'_n$ and $M_n$ coincide up to a power of $x$, and, for every simple $y$ with $D_L(y) = J$ and every $d \geq 1$, we have

$$b_{n,d}(y) = ((1,1,\ldots,1)M'_n)^{d-1})J.$$ 

Proof. Gathering the columns corresponding to simples with the same $D_L$ set and summing the corresponding lines amounts to replacing $M_n$ with a similar matrix of the form $\begin{pmatrix} M'_{n-1} & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$, so the result about the characteristic polynomial is clear.

As for the value of $b_{n,d}(y)$, the argument is similar to that for Lemma 2.6. The induction starts as $a_{n,I,[1,n-1]} = 1$ holds for each $I$. For the general step, we find

$$b_{n,d}(y) = \sum_{(x,y) \text{ normal}} b_{n,d-1}(x) = \sum_{D_R(x) \supseteq J} b_{n,d-1}(x) = \sum_{D_R(x) \supseteq J} \sum_{D_L(x) = I} b_{n,d-1}(x)$$

$$= \sum_{I} b'_{n,d-1}(I)a_{n,I,J} = \sum_{I} ((1,\ldots,1)M'_n)^{d-2})I(M'_n)^{d-1})J = ((1,\ldots,1)M'_n)^{d-1})J,$$

where $b'_{n,d}(I)$ denotes the common value of $b_{n,d}(x)$ for $x$ with $D_L(x) = I$. \hfill \Box

For $n = 3$, and using the enumeration $\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$ that is induced by our enumeration of simple braids, we obtain $M'_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Observe that the second and third columns in $M'_3$ coincide, which suggests a further reduction step.

3. Partitions associated with a simple braid

We can indeed reduce the size of the matrices once more: we can replace the incidence matrix $M'_n$ with a new matrix $\overline{M}_n$, whose size is $p(n)$, the number of partitions of $n$. Here the result is deduced from elementary remarks about simple braids (or, equivalently, about permutations); it can also be deduced from classical results about Solomon’s algebra of descents—and therefore extends to all Artin–Tits groups of spherical type.

3.1. Computation of $\tilde{a}_{n,I,J}$. We shall start from an explicit determination of the value of the numbers $\tilde{a}_{n,I,J}$ in terms of the block compositions of $I$ and $J$. We first recall the notions of composition and partition.

Definition 3.1. For $I \subseteq [1, n - 1]$, the $n$-composition $[I]_n$ of $I$ is defined to be the sequence $(p_1, p_2 - p_1, \ldots, p_k - p_{k-1})$, where $p_1, \ldots, p_k$ is the increasing enumeration of $[1, n] \setminus I$. The $n$-partition $\{I\}_n$ of $I$ is the non-increasing rearrangement of $[I]_n$. 

Example 3.2. The composition of $I$ consists of the sizes of the blocks of adjacent elements in $I$, augmented by 1: for instance, the 10-composition of $\{1,2,4,5,6,9\}$ is $(3,4,1,2)$, as the blocks are of size $2,3,0,1$. The 10-partition of $I$ is therefore $(4,3,2,1)$. Note that the $n$-composition of $I$ determines $I$, but its $n$-partition does not.

The geometric observation is the following one:

Lemma 3.3. Assume $J \subseteq [1,n-1]$, and let $(q_1, \ldots, q_\ell)$ be the $n$-composition of $J$. For $x$ a simple $n$-braid, define $f_x : [1,n] \to [1,\ell]$ by $f_x(i) = j$ if the $i$th strand of $x$ finishes in the $j$th block of $J$. Then $x \mapsto f_x$ establishes a bijection between the simple $n$-braids $x$ that satisfy $D_R(x) \supseteq J$ and the functions from $[1,n]$ to $[1,\ell]$. Moreover, for $D_R(x) \supseteq J$, we have, for every $i$,

$$i \in D_L(x) \iff f_x(i) \geq f_x(i+1). \quad (3.1)$$

Proof. Assume that $x$ is a simple $n$-braid satisfying $D_R(x) \supseteq J$, i.e., that is right divisible by $\sigma_j$ for each $j$ in $J$. By hypothesis, the first block of consecutive elements of $J$ is $1, \ldots, q_1 - 1$. Then $x$ being right divisible by $\sigma_1, \ldots, \sigma_{q_1-1}$ is equivalent to its being right divisible by the left lcm of these elements, which is $\Delta_{q_1}$. Similarly, the second block in $J$ is $\sigma_{q_1+1}, \ldots, \sigma_{q_1+q_2-1}$, and being right divisible by these elements amounts to being right divisible by their left lcm, which is $\text{sh}^{q_1}(\Delta_{q_2})$, where $\text{sh}$ denotes the shift endomorphism of $B_\infty$ that maps each $\sigma_1$ to $\sigma_{i+1}$. Now, by construction, the blocks $\{1, \ldots, q_1 - 1\}$ and $\{q_1 + 1, \ldots, q_1 + q_2 - 1\}$ are separated by $q_1$, and, therefore, the corresponding $\sigma$’s commute. In particular, $\Delta_{q_1}$ and $\text{sh}^{q_1}(\Delta_{q_2})$ commute, and their left lcm is their product. Finally, a simple braid $x$ satisfies $D_R(x) \supseteq J$ if and only if it is a right multiple of the element

$$\Delta_J = \Delta_{q_1} \text{sh}^{q_1}(\Delta_{q_2}) \cdots \text{sh}^{q_1+\cdots+q_{\ell-1}}(\Delta_{q_\ell}),$$

i.e., we have $x = x'\Delta_J$ for some $x'$.

We claim that $f_x$ determines $x'$, hence $x$. Indeed, in a simple braid, any two strands cross at most once. Now, in a $\Delta$-diagram, any two strands cross. So, if the $i$th and the $i'$th strands go to the same block of $J$, i.e., if we have $f_x(i) = f_x(i')$, then these strands cross in the final $\Delta$-part, and therefore they cannot cross in (any diagram representing) $x'$. So, when $f_x$ is given, there is only one way to construct $x'$, namely taking the strands to the entrance of the specified $\Delta$-block in increasing order (Figure 2).

Consider now $i, 1 \leq i < n$. We wonder whether $\sigma_i$ is a left divisor of $x$, i.e., if the $i$th and the $i+1$ strands cross in the diagram of $x$. If we have $f_x(i) = f_x(i+1)$, the $i$th and $i+1$ strand go to the same $J$-block, where they certainly cross. If we have $f_x(i) > f_x(i+1)$, then the $i$th strand goes to a $J$-block on the right of the $J$-block to which the $i+1$ strand goes, so they must cross in the $x'$ part. On the contrary, for $f_x(i) < f_x(i+1)$, the strands cannot cross in the $x'$ part—if they crossed once, they would have to cross a second time before exiting, and this is forbidden—and they do not cross in the $\Delta$ part either. So $\Box$ holds.

We deduce the following characterization of $\hat{a}_{n,1,J}$:

Proposition 3.4. Assume that $I, J$ are subsets of $[1,n-1]$ with respective $n$-compositions $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_\ell)$. Then $\hat{a}_{n,1,J}$ is the number of $k \times \ell$ matrices with non-negative integer entries such that, for all $i, j$, the $i$th row has sum $p_i$ and the $j$th column
The strands going to the same block cannot cross in the $x'$ part, as they cross inside the block; so the strands starting from $i$ and $i'$ with $i < i'$ cross if and only if they go to different blocks and $i$ goes to a block on the right of the block $i'$ goes to.

Figure 2. A simple braid divisible by all $\sigma_j$'s with $j$ in $J$ can be represented by a diagram finishing with $\Delta$'s corresponding to the blocks of $J$: the strands going to the same block cannot cross in the $x'$ part, as they cross inside the block; so the strands starting from $i$ and $i'$ with $i < i'$ cross if and only if they go to different blocks and $i$ goes to a block on the right of the block $i'$ goes to.

has sum $q_j$. In particular, we have

$$\hat{a}_{n,\emptyset} = \frac{n!}{p_1! \ldots p_k!} \quad \text{and} \quad \hat{a}_{n,\emptyset,J} = \frac{n!}{q_1! \ldots q_\ell!}.\tag{3.2}$$

Proof. Lemma 3.3 immediately implies

$$a_{n,I,J} = \# \{ f : [1, n] \to [1, \ell] ; (\forall j)(#f^{-1}(j) = q_j) \& (i \in I \iff f(i) \geq f(i + 1)) \}. \tag{3.3}$$

Assume that $f$ is a function of $[1, n]$ to $[1, \ell]$ satisfying the constraints of (3.4). Let $A_f$ be the $k \times \ell$-matrix whose $(i, j)$-entry is the number of $k$'s in the $i$th block of $I$ satisfying $f(k) = j$, i.e., it is $q_j$. We claim that $A_f$ determines $f$. Indeed, (3.4) requires that $f$ be non-increasing on each block of $I$, so there is only one possibility once the number of $k$'s going to the various $j$ is fixed.

The first equality in (3.2) follows: for $\ell = n$, there is exactly one nonzero entry in each column, so choosing a convenient matrix amounts to choosing among $n$ elements the $q_1$ columns with a 1 in the first row, the $q_2$ columns with a 1 in the second row, etc. The second equality is similar with rows and columns exchanged. □

Corollary 3.5. (i) The number $\hat{a}_{n,I,J}$ only depends on the partitions $\{I\}_n$ and $\{J\}_n$.

(ii) For each $n$ and $I$, the number $a_{n,I,J}$ only depends on the partition $\{J\}_n$.

Proof. Point (i) directly follows from the characterization of Proposition 3.4 as the latter clearly involves the sizes of the blocks of $I$ and $J$ only. As for (ii), the usual inclusion-exclusion formula gives

$$a_{n,I,J} = \sum_{K \cap I = \emptyset} (-1)^K \hat{a}_{n,I \cup K,J}.\tag{3.4}$$

By (i), each term in the sum only depends on $\{J\}_n$, and so does the sum. □
It is easy to check that the value of $a_{n,I,J}$ does not only depend on $\{I\}_n$ in general: when we apply the inclusion-exclusion formula, the sizes of the blocks in $I \cup K$ do not only depend on the sizes of the block in $I$.

**Remark 3.6.** Corollary 3.5 can also be deduced from the classical results by Solomon about the descent algebra—and, therefore, it extends to all Artin–Tits groups of spherical type. The argument is as follows. For $f$ a permutation, let $D(f)$ denote the sets of descents of $f$. In the group algebra $\mathbb{Q}[\mathfrak{S}_n]$, let $d_I := \sum \{ f; D(f) = I \}$ and $e_J = \sum \{ f; D(f) \supseteq J \}$. Using $w_0$ for the flip permutation, we have $w_0 e_J = \sum \{ f; D(f) \supseteq J \}$, and therefore $a_{n,I,J} = \langle d_I, w_0 e_J \rangle$, where $\langle .., .. \rangle$ is the inner product defined by $\langle f, g \rangle = 1$ for $g = f^{-1}$, and $= 0$ otherwise. Using the isometry result of [20], we deduce $a_{n,I,J} = \langle \theta(d_I), \theta(w_0 e_J) \rangle$, where $\theta(e_K)$ denotes the character of $\mathfrak{S}_n$ induced by the trivial character of the standard parabolic subgroup generated by (the transpositions $s_i$ with $i$ in) $K$. By [24], the subspace of $\mathbb{Q}[\mathfrak{S}_n]$ generated by the $e_K$ is a subalgebra, and the kernel of $\theta$ is generated by the elements $d_I - d_J$ with $I, J$ associated with the same partition, and it follows from the above expression that $a_{n,I,J}$ only depends on the partition associated with $J$.

### 3.2. The matrix $\overline{M}_n$

We can now come back to the matrix $M'_n$, and replace it for the computation of the numbers $b_{n,d}$ with a new matrix of smaller size. Indeed, a direct application of Corollary 3.5 is

**Lemma 3.7.** Assume that $J, J'$ are subsets of $[1, n-1]$ with the same $n$-partition. Then the $J$th and $J'$th columns of $M'_n$ are equal.

Thus the process used to replace $M_n$ with $M'_n$ can be applied again, i.e., the form a new matrix by gathering the equal columns and summing the corresponding rows.

**Definition 3.8.**

(i) For $\lambda$ a partition (or a composition) of $n$, we denote by $\overline{\lambda}$ the unique subset $I$ of $[1, n-1]$ satisfying $|I|_n = \lambda$.

(ii) For $\lambda, \mu \vdash n$ (i.e., partitions of $n$), we put

$$\overline{\pi}_{\lambda, \mu} = \sum_{\{ I \}_n = \lambda} a_{n,I,\overline{\mu}} = \# \{ x; \{ D_L(x) \}_n = \lambda \text{ & } D_R(x) \supseteq \overline{\mu} \},$$

and we let $\overline{M}_n$ be the matrix with rows and columns indexed by partitions of $n$ and whose $(\lambda, \mu)$-entry is $\overline{\pi}_{\lambda, \mu}$.

In this way the size of the matrix has been reduced from $n!$ to $p(n)$, the number of partitions of $n$. For instance, enumerating partitions in the order induced by the previous order on $\mathfrak{P}([1,n])$, we obtain

$$\overline{M}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \overline{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 4 & 1 & 0 & 0 \\ 5 & 3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \overline{M}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 8 & 0 & 2 & 0 & 0 & 0 \\ 23 & 12 & 4 & 5 & 0 & 1 & 0 \\ 43 & 21 & 5 & 10 & 0 & 2 & 0 \\ 8 & 6 & 4 & 4 & 2 & 2 & 0 \\ 18 & 12 & 6 & 8 & 2 & 4 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$
Proposition 3.9. For \( n \geq 1 \), the characteristic polynomials of \( \overline{M}_n \) and \( M_n \) coincide up to a power of \( x \), and, for every simple \( n \)-braid \( x \), we have for each \( d \)

\[
b_{n,d}(x) = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})_{\overline{M}_n}^{d-1}\lambda,
\]

where \( \lambda \) is the \( n \)-partition of \( D_L(x) \). In particular we have

\[
b_{n,d} = ((1, 1, \ldots, 1)_{\overline{M}_n}^{d})_{(1,1,\ldots,1)}.
\]

Table 1 gives the first few values deduced from the above formulas.

| \( d \) | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|
| \( b_{2,1}(1) \) | 1 | 2 | 3 | 4 | 5 | 6 |
| \( b_{3,1}(1) \) | 1 | 6 | 19 | 48 | 109 | 234 |
| \( b_{3,1}(\Delta_2) \) | 1 | 3 | 7 | 15 | 31 | 63 |
| \( b_{4,1}(1) \) | 1 | 24 | 211 | 1380 | 8077 | 45252 |
| \( b_{4,1}(\Delta_2) \) | 1 | 12 | 83 | 492 | 2765 | 15240 |
| \( b_{4,1}(\Delta_3) \) | 1 | 4 | 15 | 64 | 309 | 1600 |
| \( b_{5,1}(1) \) | 1 | 120 | 3651 | 79140 | 1548701 | 2937546 |
| \( b_{5,1}(\Delta_2) \) | 1 | 60 | 1501 | 30540 | 585811 | 11044080 |
| \( b_{5,1}(\Delta_3) \) | 1 | 20 | 311 | 5260 | 94881 | 1755360 |
| \( b_{5,1}(\Delta_4) \) | 1 | 5 | 31 | 325 | 4931 | 86565 |
| \( b_{6,1}(1) \) | 1 | 720 | 90921 | 7952040 | 634472921 | 49477263360 |
| \( b_{6,1}(\Delta_2) \) | 1 | 360 | 38559 | 3228300 | 254718389 | 19808530620 |
| \( b_{6,1}(\Delta_3) \) | 1 | 120 | 8727 | 649260 | 49654757 | 383162580 |
| \( b_{6,1}(\Delta_4) \) | 1 | 30 | 1075 | 61620 | 4387195 | 332578230 |
| \( b_{6,1}(\Delta_5) \) | 1 | 6 | 63 | 1955 | 116423 | 8448606 |

Table 1. First values of \( b_{n,d}(\Delta_r) \) for \( 1 \leq r < n \); the values of \( b_{n,d} \) can be also read, as Proposition 3.9 gives \( b_{n,d} = b_{n,d+1}(1) \), so, for instance, we find \( b_{3,3} = 48 \), and \( b_{4,5} = 45252 \).

3.3. Small values of \( n \). For small values of \( n \), it is easy to complete the computations and to obtain an explicit form for the expansion of \( b_{n,d}(x) \) announced in Proposition 2.9.

Example 3.10. Assume \( n = 3 \). The matrix \( \overline{M}_3 \) is invertible with eigenvalues 1 (double) and 2. By solving the recurrences, we find

\[
b_{3,1}(x) = \begin{cases} 
4 \cdot 2^d - 3d - 4 & \text{for } x \text{ with partition } (1,1,1), \text{i.e., } x = 1 \\
2^d - 1 & \text{for } x \text{ with partition } (2,1), \text{i.e., } x = \sigma_1, \sigma_2, \sigma_2 \sigma_1, \text{ or } \sigma_1 \sigma_2, \\
1 & \text{for } x \text{ with partition } (3), \text{i.e., } x = \Delta_3, 
\end{cases}
\]

and we deduce \( b_{3,1} = 8 \cdot 2^d - 3d - 7 \).
Assume now with associated partition as indicated $p$ of $i.e.$ and 1 for associated partition (4), those of $M$.

For each $n$, the characteristic polynomial of $M_n$—hence of $M_n(x)$ is displayed in Table 2.

Table 2. Characteristic polynomial of $M_n$ for $n \leq 8$, and the corresponding largest eigenvalue—which is to be compared with $n!$, the growth rate for the number of $n$-braids of degree at most $d$ if all sequences were normal.

| $n$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\rho_{\text{max}}(M_n)$ | $1$ | $1$ | $2$ | $5.449$ | $18.717$ | $77.405$ | $373.990$ | $2066.575$ |
| $\rho_{\text{max}}(M_n) \cdot n$ | $0.5$ | $0.667$ | $0.681$ | $0.687$ | $0.689$ | $0.690$ | $0.691$ |

Example 3.11. Assume now $n = 4$. The matrix $\overline{M}_4$ admits 4 eigenvalues, namely those of $\overline{M}_3$, plus $\rho_1 = 3 + \sqrt{6}$ and $\rho_2 = 3 - \sqrt{6}$. Solving the recurrences yields for $b_{n,d}(x)$ with associated partition as indicated

\[(1, 1, 1, 1) : \quad \frac{1}{20}(18 + 7\sqrt{6}) \rho_1^d + \frac{1}{20}(18 - 7\sqrt{6}) \rho_2^d - \frac{256}{20} \cdot 2^d + 6k + 11,\]
\[(2, 1, 1) : \quad \frac{1}{20}(18 + 7\sqrt{6}) \rho_1^d + \frac{1}{20}(18 - 7\sqrt{6}) \rho_2^d - \frac{2}{5} \cdot 2^d + 1,\]
\[(2, 2) : \quad \frac{1}{2} (\sqrt{6}) \rho_1^d - \frac{1}{2} (\sqrt{6}) \rho_2^d,\]
\[(3, 1) : \quad \frac{1}{60} (6 + \sqrt{6}) \rho_1^d + \frac{1}{60} (6 - \sqrt{6}) \rho_2^d + \frac{4}{5} \cdot 2^d - 1,\]

and 1 for associated partition (4), i.e., for $x = \Delta_4$. As the characteristic polynomial of $\overline{M}_4$ is $(x^2 - 6x + 3)(x - 2)(x - 1)^2$, we can equivalently determine $b_{4,d}(x)$ and $b_{4,d}$ by inductions on $d$ of the form

\[(3.7) \quad u_d = 6u_{d-1} - 3u_{d-2} + \alpha 2^d + \beta d + \gamma\]

where $\alpha, \beta, \gamma$ are determined using special values of $u_d$. For instance, $b_{4,d}$ is determined by [3.7] with $\alpha = 32, \beta = -12, \gamma = -34$ and the values $u_{-1} = 0, u_0 = 1$. Generating functions can be deduced easily.

3.4. Eigenvalues of $M_n$. By Proposition 3.9 the value of $b_{n,d}$ and $b_{n,d}(x)$, and in particular its asymptotic behaviour when $d$ grows to infinity, are connected with the non-zero eigenvalues of $M_n$, which, by Proposition 3.9 coincide with those of $\overline{M}_n$. The characteristic polynomial of $\overline{M}_n$—hence of $M_n$ up to an $x^d$ factor—for small values of $n$ is displayed in Table 2.

These values support the following

Conjecture 3.12. For each $n$, the characteristic polynomial of $M_{n-1}$ divides that of $M_n$. More precisely, the spectrum of $\overline{M}_n$ is the spectrum of $\overline{M}_{n-1}$, plus $p(n) - p(n-1)$ simple non-zero eigenvalues.
It is not hard to check the above statement for \( n \leq 10 \). Specifically, let \( \hat{M}_n \) be the size \( p(n) - 2 \) matrix obtained from \( M_n \) by deleting the first and the last rows and columns. For each small value of \( n \), one can directly check that \( \hat{M}_n \) is similar to a matrix of the form \( \begin{pmatrix} \hat{M}_{n-1} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{M}_1 \end{pmatrix} \), and deduce the properties asserted in Conjecture 3.12. But no generic argument is known so far.

The growth rate of the numbers \( b_{n,d}(x) \) is connected with the largest eigenvalue \( \rho_{\text{max}}(M_n) \) of \( M_n \). For \( n \leq 6 \), all \( b_{n,d}(x) \) except \( b_{n,d}(\Delta_n) \), which is 1, and therefore all \( b_{n,d} \) as well, grow like \( \rho_{\text{max}}(M_n)^d \).

**Question 3.13.** Do all \( b_{n,d}(x) \) except \( b_{n,d}(\Delta_n) \) grow like \( \rho_{\text{max}}(M_n)^d \)?

**Question 3.14.** What is the asymptotic behaviour of \( \rho_{\text{max}}(M_n) \) with \( n \)?

The trivial upper bound of (1.4) suggests to compare \( \rho_{\text{max}}(M_n) \) with \( n! \), or, rather, \( \rho_{\text{max}}(M_n) \) with \( n \cdot \rho_{\text{max}}(M_{n-1}) \).

4. **LETTERING THE DEGREE VARY**

So far, we kept the braid index \( n \) fixed, and studied how the numbers \( b_{n,d} \) or \( b_{n,d}(x) \) vary with \( d \), thus letting linear inductions appear. Quite different induction schemes appear when we fix the degree and let the braid index vary. No systematic method is known so far, and we only mention a few partial results motivated by the approach of [12].

4.1. **The numbers \( b_{n,2} \).** Very little is known about \( b_{n,d} \) in general. The case \( d = 1 \) is trivial, as we already observed the equality

\[ b_{n,1} = n! \]

For \( d = 2 \), the value can be deduced from earlier results of [6, 7] about permutations. We shall use the following very general observation about duality in Garside groups:

**Lemma 4.1.** For \( x \in \Delta_n \), let \( *x \) and \( x^* \) be defined by \( *x \ x = x \ x^* = \Delta_n \). Then \( x \mapsto *x \) and \( x \mapsto x^* \) are permutations of \( \Delta_n \), and, for each simple \( x \), we have

\[ D_R(*x) = [1, n] \setminus D_L(x) \quad \text{and} \quad D_L(x^*) = [1, n] \setminus D_R(x). \]

**Proof.** Assume \( x \in \Delta_n \). Then, by hypothesis, \( x \) is a left and a right divisor of \( \Delta_n \), hence \( *x \) and \( x^* \) are positive braids, and they are divisors of \( \Delta_n \) in \( B_n^+ \), so they are simple. That the mappings \( x \mapsto *x \) and \( x \mapsto x^* \) are injective is clear, and the surjectivity follows from the finiteness of \( \Delta_n \).

Now, \( \sigma_i \) being a right divisor of \( *x \) is equivalent to \( x \sigma_i \) not being simple, hence to the non-existence of \( y \) satisfying \( *x \sigma_i y = \Delta_n \), and finally to the non-existence of \( y \) satisfying \( x = \sigma_i y \). This implies the first equality in (4.1). The second equality follows from a symmetric argument. \( \square \)

**Proposition 4.2.** The numbers \( b_{n,2} \) are determined by the induction

\[ b_{0,2} = 1, \quad b_{n,2} = \sum_{i=0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 b_{i,2}. \]
Their double exponential generating function is

\begin{equation}
\sum_{n=0}^{\infty} \frac{b_{n,2} x^n}{n!^2} = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!^2} \right)^{-1} = \frac{1}{J_0(\sqrt{x})},
\end{equation}

where \(J_0(x)\) is the Bessel function.

**Proof.** By definition, \(b_{n,2}\) is the number of pairs of simple \(n\)-braids \((x_1, x_2)\) satisfying \(D_R(x_1) \supseteq D_L(x_2)\), i.e., by Lemma 1.1, \(D_R(x_1) \cap D_R(x_2) = \emptyset\). By Lemma 1.6, this number is also the number of pairs of permutations \((f, g)\) in \(S_n\) with no descent in common, i.e., such that there exists no \(i\) satisfying both \(f(i) > f(i+1)\) and \(g(i) > g(i+1)\). Such pairs of permutations have been counted in [6,23] (see also [23]), with the result indicated above. □

4.2. The numbers \(b_{n,2}(\Delta_{n-r})\). Specific results appear when we consider the numbers \(b_{n,d}(x)\) with \(x\) of the form \(\Delta_{n-r}\) with \(1 \leq r \leq n\). In particular, we can complete the computation when \(r\) is fixed and \(d\) is small. We obviously have \(b_{n,1}(\Delta_{n-r}) = 1\) for \(1 \leq r \leq n\), so the first case to consider is \(d = 2\). The general principle that makes the computation of \(b_{n,d}(\Delta_{n-r})\) relatively easy is the following observation:

**Lemma 4.3.** For all \(n, d, r\), we have

\begin{equation}
b_{n,d}(\Delta_{n-r}) = \sum_{x \text{ right divisible by } \Delta_{n-r}} b_{n,d-1}(x).
\end{equation}

**Proof.** The argument is similar to that for Proposition 1.9. A sequence \((x_1, \ldots, x_{d-1}, \Delta_{n-r})\) is normal if and only if both \((x_1, \ldots, x_{d-1})\) and \((x_{d-1}, \Delta_{n-r})\) are normal. Now \((x_{d-1}, \Delta_{n-r})\) is normal if and only if every \(\sigma_i\) dividing \(\Delta_{n-r}\) on the left divides \(x_{d-1}\) on the right. The \(\sigma_i\)'s dividing \(\Delta_{n-r}\) on the left are \(\sigma_1, \ldots, \sigma_{n-r-1}\). The simple braids that are right divisible by \(\sigma_1, \ldots, \sigma_{n-r-1}\) are those right divisible by \(\Delta_{n-r}\). Then (4.4) follows. □

**Proposition 4.4.** For \(1 \leq r \leq n\), we have

\begin{equation}
b_{n,2}(\Delta_{n-r}) = \frac{n!}{(n-r)!}.
\end{equation}

**Proof.** By Lemma 1.3, \(b_{n,2}(\Delta_{n-r})\) is the number of simple \(n\)-braids \(x\) that are right divisible by \(\Delta_{n-r}\), i.e., that satisfy \(D_R(x) \supseteq [1, n-r]\). The block composition of \([1, n-r]\) in \(n\) is \((n-r, 1, \ldots, 1)\), so (3.2) directly gives (4.5). □

4.3. The numbers \(b_{n,3}(\Delta_{n-r})\). Things become more interesting for \(d = 3\).

**Proposition 4.5.** For \(1 \leq r \leq n\), there exist polynomials \(P_1, \ldots, P_{n-r}\) with integer coefficients and \(P_i\) of degree at most \(n-r-i+1\) such that, for every \(n\), we have

\begin{equation}
b_{n,3}(\Delta_{n-r}) = (n-r)! \left( n-r+1 \right)^n + \sum_{i=1}^{n-r} P_i(n) x^{i+r-i-1}.
\end{equation}

The explicit values for \(r = 1, 2\) are

\begin{equation}
b_{n,3}(\Delta_{n-1}) = 2^{n-1},
\end{equation}

\begin{equation}
b_{n,3}(\Delta_{n-2}) = 2 \cdot 3^n - (n+6) \cdot 2^{n-1} + 1.
\end{equation}
Proof. We begin with (4.7). By Lemma 4.3, \( b_{n,3}(\Delta_{n-1}) \) is the sum of all \( b_{n,2}(x) \) with \( x \) right divisible by \( \Delta_{n-1} \), i.e., it is the number of normal sequences \((x_1, x_2)\) such that \( x_2 \) is right divisible by \( \Delta_{n-1} \). Let \( S \) be the set of all such normal sequences. We partition \( S \) according to the value of \( D_L(x_2) \), i.e., for each subset \( I \) of \([1, n-1]\), we count how many pairs \((x_1, x_2)\) satisfy \( D_L(x_2) = I \). So assume that \( x_2 \) is right divisible by \( \Delta_{n-1} \). Two cases are possible. Either \( x_2 \) is right divisible by (hence equal to) \( \Delta_n \), and then we have \( D_L(x_2) = [1, n-1] \). Or \( x_2 \) is not divisible by \( \Delta_{n-1} \), and then Lemma 3.3 shows that \( x_2 \) must be \( \sigma_i, n \Delta_{n-1} \) for some \( i \) with \( 2 \leq i \leq n \), so that the block composition of \( D_L(x_2) \) is \((i-1, n-i+1)\). So the possible compositions for the set \( D_L(x_2) \) are \((n)\), and \((p, n-p)\) with \( 1 \leq p \leq n-1 \). Conversely, the previous analysis shows that, for each \( I \) of the previous form, there exists exactly one possible \( x_2 \). Now, Proposition 3.4 says that there is one choice for \( x_1 \) in the case of \((n)\)—namely \( x_1 = \Delta_n \)—and \( \binom{n}{p} \) choices for \( x_1 \) in the case of \((p, n-p)\). We deduce

\[
b_{n,3}(\Delta_{n-1}) = 1 + \sum_{p=1}^{n-1} \binom{n}{p} = 2^{n-1}.
\]

The method is similar for computing \( b_{n,3}(\Delta_{n-2}) \) in (4.8). Assume that \((x_1, x_2)\) is a normal sequence with \( x_2 \) right divisible by \( \Delta_{n-2} \). The hypothesis is \( D_L(x_2) \supseteq [1, n-3] \), so three cases may occur, namely \( D_L(x_2) \supseteq [1, n-2] \), \( D_L(x_2) = [1, n-3] \), and \( D_L(x_2) = [1, n-3] \cup \{n-1\} \). The first case was analysed above. In the second case, \( D_L(x_2) \) has three blocks, and, conversely, each set \( I \) with three blocks gives exactly one eligible \( x_2 \). In the third case, \( D_L(x_2) \) has either two blocks, or it has three blocks with the middle one at least 2; conversely, each set \( I \) of the previous form gives one eligible \( x_2 \). Using as above Proposition 3.4 to count the eligible \( x_1 \)'s for each possible \( I \), we obtain that \( b_{n,3}(\Delta_{n-2}) \) is

\[
(4.9) \quad b_{n,3}(\Delta_{n-1}) + \sum_{p_1+p_2+p_3=n} \frac{n!}{p_1!p_2!p_3!} + \sum_{p_1+p_2=n} \frac{n!}{p_1!p_2!} + \sum_{p_1+p_2=p_3=n} \frac{n!}{p_1!p_2!p_3!}.
\]

Using the fact that \( 3^n \) is the sum of all \( \frac{n!}{p_1!p_2!p_3!} \) with \( p_1+p_2+p_3 = n \), one deduces (4.8) by bookkeeping.

Applying the same method in the general case leads to (4.10). Indeed, always by Lemma 4.3, specifying a simple \( n \)-braid \( x_2 \) satisfying \( D_R(x_2) \supseteq [1, r-1] \) amounts to choosing a permutation of the \( n-r \) last strands and the \( n-r \) positions \((i_1, \ldots, i_{n-r})\) where these strands start from. In the generic case, the resulting set \( D_L(x_2) \) is \((i_1 - 1, \ldots, i_{n-r} - 1)\), whose composition consists of \( n-r \) blocks. The special cases are when at least two adjacent strands among the last \( n-r \) ones start from adjacent positions; according to whether these strands cross or not in the final part, one then obtains either a composition with a block of size 2 at least, or a composition with less than \( n-r \) blocks. Conversely, for every subset \( I \) of \([1, n-1]\) with \( n-r \) blocks, there exists in general \((n-r)!\) eligible \( x_2 \)'s, one for each choice of the final permutation of the last \( n-r \) strands. There may be less than \((n-r)!\) choices for \( x_2 \) when 1 occurs in the composition of \( I \). Also, subsets of \([1, n-1]\) with fewer than \( n-r \) blocks may lead to eligible \( x_2 \)'s. Multiplying by the number of eligibles \( x_1 \)'s for each \( I \) and summing up yields an expression similar to (4.9), involving \((n-r)!\) sums of the form...
\[ \sum_{p_1+\ldots+p_{n-r+1}=n} \frac{n!}{p_1!\ldots p_{n-r+1}!} \] with possible order constraints on \( p_1, \ldots, p_{n-r+1} \). Each of them leads to a factor \((n-r+1)^n\), plus additional factors corresponding to specializing arguments to 0 or 1 or to grouping them.

\[ \text{4.4. The numbers } b_{n,d}(\Delta_{n-1}). \] For \( d = 4 \), it seems hopeless to complete the computation of \( b_{n,d}(\Delta_{n-r}) \). However, this can be done for \( r = 1 \). The remarkable point is that still another induction scheme appears.

**Proposition 4.6.** For \( n \geq 1 \), we have

\[ b_{n,4}(\Delta_{n-1}) = \sum_{i=0}^{n-1} \frac{n!}{i!}. \]

**Proof.** According to Lemma 3.3 again, we have now to count the normal sequences \((x_1, x_2, x_3)\) with \( x_3 \) of the form \( \sigma_{i,n}\Delta_{n-1} \), \( 2 \leq i \leq n \). We partition the family according to the value \( I \) of \( D_{L}(x_2) \), and count how many sequences may correspond to a given \( I \).

Let \((p_1, \ldots, p_k)\) denote the block composition of \( I \).

Let us first consider the case \( I = [1, n-1] \). Then we must have \( x_2 = \Delta_n \), hence \( x_1 = \Delta_n \) as well. There are \( n \) possible choices for \( x_3 \), and the total number of corresponding sequences \((x_1, x_2, x_3)\) is \( n \).

We assume now \( I \neq [1, n-1] \), i.e., \( k \geq 2 \). As for \( x_1 \), Proposition 4.4 directly gives the number of choices, namely \( \frac{n!}{p_1!\ldots p_k!} \). So we are left with counting how many pairs \((x_2, x_3)\) are eligible. The case \( x_3 = \Delta_n \) is excluded since it implies \( x_2 = \Delta_n \) hence \( I = [1, n-1] \). As in the case of \( b_{n,3}(\Delta_{n-1}) \), the hypothesis that \( x_3 \) is \( \sigma_{i,n}\Delta_{n-1} \) for some \( i \) with \( 2 \leq i \leq n \) implies that the block composition of \( D_{L}(x_3) \) consists of two nonempty blocks, and, conversely, each partition of \([1, n]\) into two nonempty blocks gives a unique \( x_3 \) of the convenient form. So the number of pairs \((x_2, x_3)\) associated with \( I \) is the number of \( x_3 \)’s satisfying \( D_{L}(x_2) = I \) and such that \( D_{R}(x_2) \) has two blocks.

By 3.3, this number is the number of functions \( f \) of \([1, n]\) to \([1, 2]\) such that \( f(i) < f(i+1) \) holds exactly for \( i \notin I \). As only two values are possible, this condition means that we have \( f(i) = 1 \) and \( f(i+1) = 2 \) for \( i \notin I \), and \( f(i+1) \leq f(i) \) for \( i \in I \).

Consider the blocks of \( I \). In each block, except possibly the first and the last ones, the value of \( f \) has to be 2 on the first element, and to be 1 on the last element. Inbetween, \( f \) is non-increasing. So the values consist of a series of 2’s, followed by a series of 1’s. The only parameter to specify is the position where \( f \) switches from 2 to 1, so, for a block of size \( p \), there are \( p-1 \) possible choices (see Figure 3). The cases of the first and the last blocks are special, because there is no constraint on the left for the first block, and on the right for the last block. So, in these special cases, there are \( p \) choices instead of \( p-1 \). The conclusion is that, for \( I \) of block composition \((p_1, \ldots, p_k)\), there are \( p_1(p_2-1)\ldots(p_k-1-1)p_k \) choices for the pairs \((x_2, x_3)\) associated with \( I \). Merging the result for \( x_1 \) and for \((x_2, x_3)\) and summing up over \( I \) gives

\[ b_{n,4}(\Delta_{n-1}) = \sum_{p_1+\ldots+p_{n-r+1}=n} \frac{n!}{p_1!\ldots p_k!} p_1(p_2-1)\ldots(p_k-1-1)p_k, \]

the sum being taken over all finite compositions \((p_1, \ldots, p_k)\) of \( n \): indeed, the value for \( I = [1, n-1] \), namely \( n \), corresponds to the missing term \( \frac{n!}{n!} \) of the sum.
We can now simplify the right hand term in (4.11). To this end, we observe that

\begin{equation}
\sum_{p_1 + \cdots + p_k = i + 1} \frac{p_1 - 1}{p_1!} \cdots \frac{p_r - 1}{p_r!} \frac{p_k}{p_k!} = 1
\end{equation}

holds for \( i \geq 0 \). Indeed, let \( F(i) \) be the left hand side of (4.12). We prove (4.12) using induction on \( i \). For \( i = 0 \), we get 1 = 1. Assume \( i \geq 1 \) and consider the sequences \((p_1, \ldots, p_k)\) satisfying \( p_1 + \cdots + p_k = i + 1 \). On the one hand, we have \((i+1)\), whose contribution to \( F(i) \) is \( \frac{1}{i!} \). On the other hand, we have the sequences of length at least 2. Now, for each \( p \) with \( 1 \leq p \leq i \), the contribution of \((p, p_2, \ldots, p_k)\) to \( F(i) \) is \( \frac{p-1}{p!} \) times the contribution of \((p_2, \ldots, p_k)\) to \( F(i-p) \). Hence the total contribution of the sequences beginning with \( p \) to \( F(i) \) is \( \frac{p-1}{p!} \) \( F(i-p) \), so, by induction hypothesis, it is \( \frac{(p-1)}{p!} \) \( F(i-p) \). We deduce \( F(i) = \frac{0}{1!} + \frac{1}{2!} + \cdots + \frac{i-1}{(i-1)!} + \frac{1}{i!} \), which is clearly 1.

Consider now the right hand side in (4.11). For \( 0 \leq i < n - 1 \), the contribution of \((i+1, p_2, \ldots, p_k)\) to the sum is \( \frac{n!}{i!} \) times the quantity \( \frac{p_1 - 1}{p_1!} \frac{p_2 - 1}{p_2!} \cdots \frac{p_k}{p_k!} \) involved in (4.12). Using the latter equality, we deduce that the total contribution of the sequences beginning with \( i+1 \) is \( \frac{n!}{i!} \). As for \( i = n - 1 \), the contribution of \((n)\) to the right hand side in (4.11) is \( n \), which is \( \frac{n!}{(n-1)!} \), so the general formula remains valid. By summing over \( i \), we obtain (4.10). \( \square \)

**Corollary 4.7.** The numbers \( b_{n,4}(\Delta_{n-1}) \) are determined by the induction

\[ u_1 = 1, \quad u_n = nu_{n-1} + 2n - 1. \]

Another consequence of (4.10) is the equality

\[ b_{n,4}(\Delta_{n-1}) = \left[ n!e \right] - 1, \]

with \( e = \exp(1) \).

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