Deflection angle of light for an observer and source at finite distance from a rotating global monopole

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Abstract

By using a method improved with a generalized optical metric, the deflection of light for an observer and source at finite distance from a lens object in a stationary, axisymmetric and asymptotically flat spacetime has been recently discussed [Ono, Ishihara, Asada, Phys. Rev. D 96, 104037 (2017)]. In this paper, we study a possible extension of this method to an asymptotically nonflat spacetime. We discuss a rotating global monopole. Our result of the deflection angle of light is compared with a recent work on the same spacetime but limited within the asymptotic source and observer [Jusufi et al., Phys. Rev. D 95, 104012 (2017)], in which they employ another approach proposed by Werner with using the Nazim’s osculating Riemannian construction method via the Randers-Finsler metric. We show that the two different methods give the same result in the asymptotically far limit. We obtain also the corrections to the deflection angle due to the finite distance from the rotating global monopole.

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I. INTRODUCTION

Since the experimental confirmation of the theory of general relativity [1] succeeded in 1919 [2], a lot of calculations of the gravitational bending of light have been done not only for black holes [3] but also for other objects such as wormholes and gravitational monopoles [4]. Gibbons and Werner (2008) proposed an alternative way of deriving the deflection angle of light [5]. They assumed that the source and receiver are located at an asymptotic Minkowskian region and they used the Gauss-Bonnet theorem to a spatial domain described by the optical metric, for which a light ray is described as a spatial curve. Ishihara et al. have recently extended Gibbons and Werner’s idea in order to investigate finite-distance corrections in the small deflection case (corresponding to a large impact parameter case) [6] and also in the strong deflection limit for which the photon orbits may have the winding number larger than unity [7]. In particular, the asymptotic receiver and source have not been assumed. Our method and Werner’s one are limited within asymptotically flat spacetimes.

In this paper, we discuss an extension of our method to a rotating global monopole that is one of asymptotically nonflat spacetimes. A static solution of a global monopole was found in a paper by Barriola and Vilenkin [8]. According to their model, global monopoles are configurations whose energy density decreases with the distance as $r^{-2}$ and whose spacetimes exhibit a solid deficit angle given by $\delta = 8\pi^2 \eta^2$, where $\eta$ is the scale of gauge-symmetry breaking. Recently, global monopoles have been discussed as spacetimes with a cosmological constant, e.g. in [9]. Static spherically symmetric composite global-local monopoles have also been studied [10]. Gravitational lensing in spacetimes with a non-rotating global monopole has been intensively investigated, for instance by Cheng and Man [11] who studied strong gravitational lensing of a Schwarzschild black hole with a solid deficit angle owing to a global monopole. More recently, it has also been proposed that gravitational microlensing by global monopole may even be used to test Verlinde’s emergent gravity theory [12]. As mentioned above, we investigate a possible extension of our method to stationary, axisymmetric spacetimes with a solid deficit angle, especially in order to examine finite-distance corrections to the deflection angle of light. The geometrical setups in the present paper are not those in the optical geometry, in the sense that the photon orbit has a non-vanishing geodesic curvature, though the light ray in the four-dimensional spacetime obeys a null geodesic.
This paper is organized as follows. Section II discusses a generalized optical metric for a rotating global monopole. Section III discusses how to define the deflection angle of light in a stationary, axisymmetric spacetime with the deficit angle. In particular, it is shown that the proposed definition of the deflection angle is also coordinate-invariant by using the Gauss-Bonnet theorem. We discuss also how to compute the gravitational deflection angle of light by the proposed method. Section IV is devoted to the conclusion. Throughout this paper, we use the unit of $G = c = 1$, and the observer may be called the receiver in order to avoid a confusion between $r_O$ and $r_0$ by using $r_R$.

II. GENERALIZED OPTICAL METRIC FOR ROTATING GLOBAL MONOPOLE

A. Rotating global monopole

By applying the method of complex coordinate transformation, an extension of the static global monopole solution to a rotating global monopole spacetime was described by R. M. Teixeira Filho and V. B. Bezerra in Ref. [13].

Its spacetime metric reads

\[
\begin{align*}
\frac{ds^2}{g_{\mu\nu}} &= dx^\mu dx^\nu \\
&= - \left(1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right) dt^2 \\
&\quad + \left[\frac{r^2 - a^2 \{ (1 - \beta^2) \sin^2 \theta - \cos^2 \theta \}}{r^2 - 2Mr + a^2} - (1 - \beta^2) \frac{a^2 \sin^2 \theta \{2Mr - a^2 (1 - \sin^4 \theta)\}}{(r^2 - 2Mr + a^2)^2}\right] dr^2 \\
&\quad + \beta^2 (r^2 + a^2 \cos^2 \theta) d\theta^2 \\
&\quad + \sin^2 \theta \left[\frac{\beta^2 r^4 + \{1 - (1 - 2\beta^2) \cos^2 \theta\} a^2 r^2 + 2Ma^2 r \sin^2 \theta + a^4 \cos^2 \theta (\beta^2 \cos^2 \theta + \sin^2 \theta)}{r^2 + a^2 \cos^2 \theta}\right] d\phi^2 \\
&\quad - \frac{4aMr \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dt d\phi + 2(1 - \beta^2) \frac{a \{r^2 \sin^2 \theta - a^2 \cos^2 \theta (1 + \cos^2 \theta)\}}{r^2 - 2Mr + a^2} dr d\phi, \\
\end{align*}
\]

where the coordinates are $-\infty < t < +\infty$, $2M \leq r < +\infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. We denote

\[
\beta^2 = 1 - 8\pi \eta^2, \tag{2}
\]
where $\eta$ is the scale of a gauge-symmetry breaking.

The rotating global monopole by Eq. (1) is a rotating generalization of the global monopole black hole in Ref. [14]. Here, $M$ denotes the global monopole core mass. The parameter $a$ is the total angular momentum of the global monopole, which gives rise to the Lense-Thirring effect in general relativity, and the parameter $\beta$ is the deficit angle of the spacetime where $\beta$ satisfies $0 < \beta \leq 1$.

**B. Generalized optical metric**

By following Ref. [15], we define the generalized optical metric $\gamma_{ij}$ ($i, j = 1, 2, 3$) by a relation as

$$dt = \sqrt{\gamma_{ij}dx^i dx^j + \beta_i dx^i},$$

which is directly obtained by solving the null condition ($ds^2 = 0$) for $dt$. Note that $\gamma_{ij}$ is not the induced metric in the Arnowitt-Deser-Misner (ADM) formalism. We define a three-dimensional space $(\mathcal{M})$ by the generalized optical metric $\gamma_{ij}dx^i dx^j$.

For the rotating global monopole by Eq. (1), we find the components of the generalized optical metric as

$$\gamma_{ij}dx^i dx^j = \left(\frac{a^2 \cos^2 \theta + r^2}{a^2 + r(r - 2M)}\right)^2 \left[\frac{a^2 \cos^2 \theta + r(r - 2M)}{a^2 \cos^2 \theta + r(r - 2M)}\right] \times \left[\frac{a^4 (\beta^2 - 1) \sin^6 \theta + a^2 \{a^2 + r(r - 2M)\} \cos^2 \theta + a^2 r^2 (\beta^2 - 1) \sin^2 \theta + (a^2 - 2Mr + r^2) r^2}{a^2 \cos^2 \theta + r(r - 2M)}\right] dr^2$$

$$+ \frac{\beta^2 (a^2 \cos^2 \theta + r^2)^2}{a^2 \cos^2 \theta + r(r - 2M)} d\theta^2 + \frac{2a (1 - \beta^2) \left[r^2 \sin^2 \theta - a^2 \cos^2 \theta \left(\cos^2 \theta + 1\right)\right]}{a^2 + r(r - 2M) \left(1 - \frac{2Mr}{a^2 \cos^2 \theta + r^2}\right)} drd\phi$$

$$+ \frac{\sin^2 \theta \left(a^2 \cos(2\theta) + a^2 + 2r^2\right)}{8 \left[r(r - 2M) + a^2 \cos^2 \theta\right]^2} d\phi^2.$$

We obtain the components of $\beta_i$ as

$$\beta_i dx^i = -\frac{2aMr \sin^2 \theta}{a^2 \cos^2 \theta + r(r - 2M)} d\phi.$$
In the rest of the paper, we focus on the light rays in the equatorial plane, namely \( \theta = \pi/2 \). Note that the generalized optical metric \( \gamma_{ij} \) doesn’t mean an asymptotically flat space, because there is the deficit angle of spacetime (if \( \beta \neq 1 \)).

III. DEFLECTION ANGLE OF LIGHT BY A ROTATING GLOBAL MONOPOLE

A. Deflection angle of light in asymptotically flat spacetimes

Let us begin this section with briefly summarizing the generalized optical metric method that enables us to calculate the deflection angle of light for non-asymptotic receiver (denoted as \( R \)) and source (denoted as \( S \)) [15].

We define the deflection angle of light as

\[
\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}.
\]

(6)

Here, \( \Psi_R \) and \( \Psi_S \) are angles between the light ray tangent and the radial direction from the lens object, defined in a covariant manner using the generalized optical metric, at the receiver location and the source, respectively. On the other hand, \( \phi_{RS} \) is the coordinate angle between the receiver and source, where the coordinate angle is associated with the rotational Killing vector in the spacetime. If the space under study is Euclidean, this \( \alpha \) becomes the deflection angle of the curve. This is consistent with the thin lens approximation in the standard theory of gravitational lensing.

By using the Gauss-Bonnet theorem as [16]

\[
\int \int_{\infty \square S} K dS + \oint_{\partial T} \kappa_g d\ell + \sum_{i=1}^n \Theta_i = 2\pi.
\]

(7)

Eq. (6) can be recast into [15]

\[
\alpha = -\int \int_{\infty \square S} K dS + \int_{S}^{R} \kappa_g d\ell,
\]

(8)

where \( K \) is defined as the Gaussian curvature at some point on the two-dimensional surface, \( dS \) denotes the infinitesimal surface element defined with \( \gamma_{ij} \), \( \infty \square S \) denotes a quadrilateral embedded in a curved space with \( \gamma_{ij}, \kappa_g \) denotes the geodesic curvature of the light ray in this space and \( d\ell \) is an arc length defined with the generalized optical metric (See Fig. 2 in Ref. [15]). It is shown by Asada and Kasai that this \( d\ell \) for the light ray is an affine parameter [17].
B. Deflection angle of light in spacetimes with deficit angle

When we consider the deflection angle of light in a spacetime with the deficit angle, we should modify the definition of deflection angle of light as

$$\alpha \equiv \Psi_R - \Psi_S + \beta \phi_{RS},$$

in order to match the fact that the total circumference angle with deficit angle is $2\pi \beta$ in stead of $2\pi$, where we note that the coordinate $\phi$ means the angular coordinate $0 \leq \phi \leq 2\pi$. The last term of Eq.(9) reflects that the deflection angle of light is affected through the global angle coordinate by the deficit angle of the spacetime.

Note that the surface integral and path integral terms appear in the right hand side of Eq. (8) if $\beta_i = 0$ (See [6]). However, in the rotating global monopole, Eq.(8) is modified by the deficit angle

$$\frac{1}{2} \int_{\mathcal{R} \times S} K dS + \int_{r_{\infty}}^{R} \bar{k}_g dl + \int_{r_{\infty}}^{r_{\infty}} \bar{k}_g dl - \int_{C_r} ^{C_r} \bar{k}_g dl + \int_{C_{\infty}} ^{C_{\infty}} \bar{k}_g dl + \Psi_R + (\pi - \Psi_S) + \pi = 2\pi \beta \int_0 ^{\Psi_S - \Psi_R + \alpha} d\phi + \Psi_R - \Psi_S = 0$$

$$\frac{1}{2} \int_{\mathcal{R} \times S} K dS + \int_{r_{\infty}}^{R} \bar{k}_g dl + \int_{S} ^{r_{\infty}} \bar{k}_g dl - \int_{C_r} ^{C_r} \bar{k}_g dl + \bar{\beta} \int_0 ^{\Psi_S - \Psi_R + \alpha} d\phi + \Psi_R - \Psi_S = 0$$

where $\bar{k}_g$ is a geodesic curvature along the radial line from the infinity to the receiver, $\bar{k}_g$ is a geodesic curvature along the radial line from the source to the infinity, $\bar{k}_g$ is a geodesic curvature along the light ray from the source to the receiver and $\bar{k}_g$ is a geodesic curvature along the path $C_{\infty}$. The path $C_r$ is light ray from the receiver to the source in generalized optical metric, $C_{\infty}$ is circular arcsegment of radius $R >> r_R, r_S$ and we use $dl = \sqrt{1 + \frac{4\nu}{r}} dr = \{1 + \mathcal{O}(M/r)\} dr$. We shall explain in more detail this calculation in Sec.III-D-3. Therefore, the deflection angle of light by the rotating global monopole is

$$\alpha \equiv - \int_{\mathcal{R} \times S} K dS + \int_{R} ^{r_{\infty}} \bar{k}_g dl - \int_{S} ^{r_{\infty}} \bar{k}_g dl + \int_{C_r} ^{C_r} \bar{k}_g dl.$$  

The deflection angle is also a coordinate-invariant in the spacetimes with deficit angle, because $\Psi_R$ and $\Psi_S$ are obtained by the inner product at a receiver and a source respectively.

We have two ways in order to calculate the deflection angle of light. We shall make detailed calculations of the right-hand side of Eq. (11) and the right-hand side of $\alpha \equiv \Psi_R - \Psi_S + \phi_{RS}$ below.
C. Gaussian curvature

For the equatorial case of a rotating global monopole, the Gaussian curvature in the weak field approximation is calculated as

\[ K = \frac{R_{r\phi r\phi}}{\det \gamma_{ij}^{(2)}} \]

\[ = \frac{1}{\sqrt{\det \gamma_{ij}^{(2)}}} \left[ \frac{\partial}{\partial \phi} \left( \sqrt{\det \gamma_{ij}^{(2)}} \Gamma_{rr}^{\phi} \right) - \frac{\partial}{\partial r} \left( \sqrt{\det \gamma_{ij}^{(2)}} \Gamma_{r\phi}^{\phi} \right) \right] \]

\[ = \left[ -\frac{2}{r^3} - \frac{6}{r^5} \beta^2 a^2 \right] M + \frac{3}{r^4} M^2 \]

\[ + \mathcal{O}(M^3/r^5, a^4 M/r^7, a^2 M^3/r^7), \] (12)

where \( \gamma_{ij}^{(2)} \) denotes the two-dimensional generalized optical metric in the equatorial plane \( \theta = \pi/2 \). The \( a \) and \( M \) are book-keeping parameters in the weak field approximation. As for the first line of Eq. (12), please see e.g. the page 263 in Reference [20]. We note that the first term in the second line of Eq. (12) does not contribute because \( \Gamma_{rr}^{\phi} = 0 \). It is not surprising that this Gaussian curvature does not agree with Eq. (26) in Jusufi and Övgün [18], because their Gaussian curvature describes another surface that is associated with the Randers-Finsler metric different from our optical metric, though the same four-dimensional spacetime is considered.

In order to perform the surface integral of the Gaussian curvature in Eq. (8), we have to know the boundary shape of the integration domain. In other words, we need to describe the light ray as a function of \( r(\phi) \). For the later convenience, we introduce the inverse of \( r \) as \( u \equiv r^{-1} \). The orbit equation in this case becomes

\[ \left[ 1 + 2Mu \right] \left( \frac{du}{d\phi} \right)^2 - \left[ \frac{2a (1 - \beta^2) (b^2 u^2 - \beta^2)}{b^2} - \frac{4aMu (\beta^2 - 1) (b^2 u^2 - 2\beta^2)}{b^2} \right] \frac{du}{d\phi} \]

\[ + \left[ \beta^2 u^2 - \frac{\beta^4}{b^2} \right] + \left\{ - \frac{2\beta^4 u}{b^2} + \frac{4a\beta^4 u}{b^2} \right\} M \] \[ + \mathcal{O}(a^2 u^2, M^2 u^2, aM^2 u^3, M^3 u^3) = 0, \] (13)

where \( b \) is the impact parameter of the photon. See e.g. Reference [15] on how to obtain the photon orbit equation in the axisymmetric and stationary spacetime. The orbit equation is
iteratively solved as
\[
\begin{align*}
\frac{\partial u}{\partial \phi} &= \frac{\beta}{b} \sin \{\beta \phi + \phi_0(1 - \beta)\} + \frac{\beta^2 + \beta^2 \cos^2 \{\beta \phi + \phi_0(1 - \beta)\}}{b^2} \frac{M}{b} \\
&\quad + \frac{\beta(\beta^2 - 1) \sin[2 \{\beta \phi + \phi_0(1 - \beta)\}]}{2b^2} aM \\
&\quad + \frac{\beta^2[-4 + (-1 + \beta^2) \cos \{\beta \phi + \phi_0(1 - \beta)\} + (-1 + \beta^2) \cos \{3 \beta \phi + 3 \phi_0(1 - \beta)\}]}{2b^2} aM \\
&\quad + \mathcal{O}(M^2/b^3).
\end{align*}
\]

The area element of the equatorial plane \(dS\) is
\[
dS = \sqrt{\det \gamma_{ij}^{(2)}} \, drd\phi = \sqrt{\beta^2 r^2} + \mathcal{O}(Mr) drd\phi = \{\beta r + \mathcal{O}(M, M^2/r)\} drd\phi.
\]

By using Eq. (14) as the iterative solution for the photon orbit, the surface integral of the Gaussian curvature in Eq. (8) is calculated as
\[
\begin{align*}
- \int \int_{S_R} K dS &= \int_{\phi_S}^{\phi_R} \int_{r_S}^{r_R} d\phi \int_\beta^\infty \left( -\frac{2M}{r^3} \right) r\beta + \mathcal{O}(M^3/b^3, a^2 M^3/b^5, a^4 M^2/b^6) \\
&= \int_0^{u(\phi)} du \int_{\phi_S}^{\phi_R} d\phi (2M\beta) + \mathcal{O}(M^3/b^3, a^2 M^3/b^5, a^4 M^2/b^6) \\
&= 2M\beta \int_{\phi_S}^{\phi_R} d\phi \left( \frac{\beta}{b} \sin \{\beta \phi + \phi_0(1 - \beta)\} + \frac{\beta(\beta^2 - 1) \sin[2 \{\beta \phi + \phi_0(1 - \beta)\}]}{2b^2} aM \right) \\
&= 2M\beta b \left[ \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} \right] + \frac{aM(1 - \beta^2)}{\beta} [u_R^2 - u_S^2] \\
&\quad + \mathcal{O}(M^3/b^3, a^2 M^3/b^5, a^4 M^2/b^6),
\end{align*}
\]

where we used \(\sin \{\beta \phi + \phi_0(1 - \beta)\} = \frac{bu_S}{\beta} + \frac{(1-\beta^2)\sqrt{1-\frac{b^2 u_S^2}{\beta^2}}}{\beta} u_S a - \frac{\beta(2-\frac{b^2 u_S^2}{\beta^2})}{b} M + \mathcal{O}(aM/b^2)\)

and \(\sin \{\beta \phi + \phi_0(1 - \beta)\} = \frac{bu_R}{\beta} - \frac{(1-\beta^2)\sqrt{1-\frac{b^2 u_R^2}{\beta^2}}}{\beta} u_R a - \frac{\beta(2-\frac{b^2 u_R^2}{\beta^2})}{b} M + \mathcal{O}(aM/b^2)\) by Eq. (14) in the last line.

**D. Geodesic curvature**

1. **Light ray in optical metric**

The geodesic curvature plays an important role in our calculations of the light deflection, though it is not usually described in standard textbooks on general relativity. Hence, we
follow Reference [15] to briefly explain the geodesic curvature here. The geodesic curvature can be defined in the vector form as (e.g. [19])

$$\kappa_g \equiv \vec{T}' \cdot \left( \vec{T} \times \vec{N} \right), \quad (17)$$

where we assume a parameterized curve with a parameter, $\vec{T}$ is the unit tangent vector for the curve by reparameterizing the curve using its arc length, $\vec{T}'$ is its derivative with respect to the parameter, and $\vec{N}$ is the unit normal vector for the surface. Eq. (17) can be rewritten in the tensor form as

$$\kappa_g = \epsilon^{ijk} N^i a^j e^k, \quad (18)$$

where $\vec{T}$ and $\vec{T}'$ correspond to $e^k$ and $a^j$, respectively. Here, the Levi-Civita tensor $\epsilon^{ijk}$ is defined by $\epsilon^{ijk} \equiv \sqrt{\gamma} \epsilon_{ijk}$, where $\gamma \equiv \text{det} (\gamma_{ij})$, and $\epsilon_{ijk}$ is the Levi-Civita symbol ($\epsilon_{123} = 1$).

In the present paper, we use $\gamma_{ij}$ in the above definitions but not $g_{0i}$ [15]. Note that $a^i \neq 0$ in the three-dimensional optical metric by nonvanishing $g_{0i}$ [15], even though the light signal follows a geodesic in the four-dimensional spacetime. On the other hand, we emphasize that $a^i = 0$ and thus $\kappa_g = 0$ for the geodesics in the optical metric, because $\beta_i = 0$.

As shown first in Reference [15], Eq. (18) is rewritten in a convenient form as

$$\kappa_g = -\epsilon^{ijk} N_i \beta_{j|k}, \quad (19)$$

where we use $\gamma_{ij} e^i e^j = 1$.

Henceforth, we focus on the equatorial plane ($\theta = \pi/2$). Then, let us denote the unit normal vector as $N_p$. This vector is normal to the $\theta$-constant surface. Therefore, it satisfies $N_p \propto \nabla_p \theta = \delta_p^\theta$, where $\nabla_p$ is the covariant derivative associated with $\gamma_{ij}$. Hence, $N_p$ is written in a form as $N_p = N_\theta \delta_p^\theta$. By noting that $N_p$ is a unit vector ($N_p N_q \gamma^{pq} = 1$), we obtain $N_\theta = \pm 1/\sqrt{\gamma^{\theta\theta}}$. Therefore, $N_p$ can be expressed as

$$N_p = \frac{1}{\sqrt{\gamma^{\theta\theta}}} \delta_p^\theta, \quad (20)$$

where we choose the upward direction without loss of generality.

For the equatorial case, one can show

$$\epsilon^{\theta pq} \beta_{q|p} = -\frac{1}{\sqrt{\gamma}} \beta_{\phi,r}, \quad (21)$$
where the comma denotes the partial derivative, we use $\epsilon^{\theta r\phi} = -1/\sqrt{\gamma}$ and we note $\beta_{r,\phi} = 0$ owing to the axisymmetry. By using Eqs. (20) and (21), the geodesic curvature of the light ray with the generalized optical metric becomes

$$\kappa_g = -\sqrt{\frac{1}{\gamma}}\beta_{\phi,r}.$$  

(22)

For the global monopole case, this is obtained as

$$\kappa_g = -\frac{2}{\beta r^3}aM - \frac{2}{\beta r^4}aM^2 + \mathcal{O}(aM^3/r^5, aM^4/r^6, aM^5/r^7, a^5 M/r^7).$$  

(23)

We examine the contribution from the geodesic curvature. This contribution is the path integral along the light ray (from the source to the receiver), which is computed as

$$\int_{C_R} \kappa_g d\ell = -\int_{S}^{R} \frac{2}{\beta r^3}aMd\ell + \mathcal{O}(a^3 M/b^4)$$

$$= -\int_{S}^{\phi} \frac{2}{\beta r^3}aM \frac{b}{\cos{\{\beta \phi + \phi_0(1 - \beta)\}}} d\phi + \mathcal{O}(aM^2/b^4)$$

$$= -\frac{2}{\beta} aM \int_{S}^{\phi} \left( \frac{\beta \cos{\{\beta \phi + \phi_0(1 - \beta)\}}}{b} \right)^3 \frac{b}{\cos^2{\{\beta \phi + \phi_0(1 - \beta)\}}} d\phi + \mathcal{O}(aM^2/b^4)$$

$$= -\frac{2}{b^2} aM \int_{S}^{\phi} \cos{\{\beta \phi + \phi_0(1 - \beta)\}} d\phi + \mathcal{O}(aM^2/b^4)$$

$$= -\frac{2aM\beta}{b^2} [\sin{\{\beta \phi_R + \phi_0(1 - \beta)\}} - \sin{\{\beta \phi_S + \phi_0(1 - \beta)\}}] + \mathcal{O}(aM^2/b^4)$$

$$= -\frac{2aM\beta}{b^2} \left[ \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} \right] + \mathcal{O}(aM^2/b^4),$$  

(24)

where we use $dl = r d\phi$, $r = \frac{\beta \cos{\{3\phi + \phi_0(1 - \beta)\}}}{b} + \mathcal{O}(a/b, M/b)$. In the last line, we used $\sin{\{\beta \phi_R + \phi_0(1 - \beta)\}} = \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} + \mathcal{O}(a u_R, M u_R)$ and $\sin{\{\beta \phi_S + \phi_0(1 - \beta)\}} = -\sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} + \mathcal{O}(a u_S, M u_S)$ by Eq. (14). The sign of the right hand side of Eq. (24) changes, if the photon orbit is retrograde.

2. **Radial lines in the generalized optical metric**

The unit tangent vector along a radius line in $^{(3)}M$ is $R^i = (R^r, 0, 0)$. On the equatorial plane, from

$$\gamma_{ij} R^i R^j = \gamma_{rr}(R^r)^2 = 1,$$  

(25)
we obtain

\[ R^r = \frac{1}{\sqrt{\gamma_{rr}}}. \]  

(26)

The acceleration vector \( a^i \) of this line is

\[ a^i = R^i_{\ j} R^j. \]  

(27)

Its explicit form is

\[
a^i = \left( \frac{1}{2} \left( \frac{\partial}{\partial r} \gamma_{rr} \right) + \frac{\gamma^{rr}}{2 \gamma_{rr}} \frac{\partial \gamma_{rr}}{\partial r} + \frac{\gamma^{r\phi} \gamma_{r\phi} \partial \gamma_{r\phi}}{\gamma_{rr}} \partial_r, 0, \frac{\gamma^{r\phi} \partial \gamma_{rr}}{2 \partial r} \right).
\]  

(28)

Here, the vector \( a^i \ (i = r, \theta, \phi) \) becomes

\[
a^r = \frac{2(\beta^2 - 1)^2}{\beta^2 r^4} a^2 M + O(a^4/r^5),
\]

\[ a^\theta = 0, \]

\[ a^\phi = \frac{2(\beta^2 - 1)}{\beta^2 r^4} a M + O(a^3/r^5, a M^2/r^5). \]  

(29)

This means \( a^i \) is zero vector in Kerr or Schwarzschild cases \((\beta = 1)\). The unit normal vector for the equatorial plane satisfies

\[
N_i \propto \nabla_i \theta (= \pi/2) = \delta_i^\theta,
\]

\[
\gamma_{ij} N^i N^j = 1 \quad \text{(normalization)}.
\]  

(30)

Therefore, it becomes

\[
N^i = \left( 0, \frac{1}{\sqrt{\gamma_{\theta\theta}}}, 0 \right).
\]  

(31)

By using Eqs. (26), (28) and (31), an explicit form of \( \kappa_g \) is obtained as

\[
\kappa_g = \epsilon_{ijk} N^i a^j R^k = \sqrt{\gamma} \left( \gamma^{r\phi} \frac{\partial \gamma_{r\phi}}{\partial r} + \frac{\gamma^{r\phi} \partial \gamma_{rr}}{2 \partial r} \right),
\]  

(32)

where we defined \( \gamma = \det(\gamma_{ij}) \) and \( \epsilon_{ijk} \) is Levi-Civita tensor.
Moreover, by substituting functions of metric $\gamma_{ij}$ for this equation, we obtain $\kappa_g$ as

$$\kappa_g = -\sqrt{(r - 2M)^2 \left\{ a^2 + r(r - 2M) \right\} \left\{ a^2 (\beta^2 - 1) (2Mr + 1) + r^2 \right\}} \times \left[ a \left( \beta^2 - 1 \right) r^2 \left\{ a^4 (\beta^2 - 1) \left\{ -8M^2 r + M (3r^2 - 5) + 2r \right\} + a^2 r \left\{ 12 (\beta^2 - 1) M^3 r \\
- 8 (\beta^2 - 1) M^2 (r^2 - 1) + Mr \left\{ -6 \beta^2 + (\beta^2 - 1) r^2 + 3 \right\} + \beta^2 r^2 \right\} + 2Mr^3 (2M - r) \right\} \right] \bigg/ \left[ (r - 2M)^2 \left\{ a^2 + r(r - 2M) \right\} \right]^2 \left\{ a^2 (\beta^2 - 1) (2Mr + 1) + r^2 \right\} \times \left\{ r^2 \left( a^6 (\beta^2 - 1) (2Mr + 1) + a^4 r \left\{ -4 (\beta^4 - 1) M^2 r + 2 (\beta^4 - 1) M (r^2 - 1) + \beta^4 r \right\} + 2a^2 \beta^2 r^2 (2M - r) \right\} - (\beta^2 - 1) M (r^2 - 1) - r \right\} + \beta^2 r^4 (r - 2M)^2 \right\}^{1/2} \right] . \tag{33}$$

This is approximated as

$$\kappa_g = \frac{2 (\beta^2 - 1)}{\beta r^3} aM - \frac{\beta (\beta^2 - 1)}{r^4} a^3 + \frac{10 (\beta^2 - 1)}{\beta r^4} aM^2 + O(a^3 M/r^5, aM^3/r^5), \tag{34}$$

where this $\kappa_g$ vanishes in Kerr or Schwarzschild spacetime ($\beta = 1$), since the acceleration vector $a^i$ becomes 0.

Let us integrate the leading term of $\kappa_g$ from the source to the infinity

$$\int_S^{r_\infty} \frac{2 (\beta^2 - 1)}{\beta r^3} aM dl = \int_{r_s}^{\infty} \frac{2 (\beta^2 - 1)}{\beta r^3} aM dr \left[ \frac{1}{r^2} \right]_{r_s}^{\infty} = \frac{(\beta^2 - 1) aM}{\beta r_s^2} + O(aM^2/r_s^3). \tag{35}$$

Similarly, the integral of $\kappa_g$ from the receiver to the infinity is computed as

$$\int_R^{r_\infty} \frac{2 (\beta^2 - 1)}{\beta r^3} aM dl = -\frac{(1 - \beta^2) aM}{\beta r R^2} + O(aM^2/r R^3), \tag{36}$$

where we use $dl = \sqrt{1 + \frac{4M}{r}} dr = \{1 + O(M/r)\} dr$.
3. Geodesic curvature of circular arcsegment in optical metric

The orbital equation Eq. (13) can be solved as

\[
\frac{du}{d\phi} = \frac{1}{F_\pm(u)}.
\]

\[
F_+(u) = \frac{1}{\beta\sqrt{u_0^2 - u^2}} - \left(1 - \frac{1}{\beta^2}\right) a + \frac{u_0^3 - u^3}{\beta(u_0^2 - u^2)^{3/2}} M
\]

\[
- 2uaM \left(1 - \frac{1}{\beta^2} + \frac{u_0^3(u_0 - u)}{u\beta^2(u_0^2 - u^2)^{3/2}}\right)
\]

\[+ O(a^2u, M^2u, aM^2u^2, M^3u^2), \tag{37}\]

\[
F_-(u) = -\frac{1}{\beta\sqrt{u_0^2 - u^2}} - \left(1 - \frac{1}{\beta^2}\right) a + \frac{u_0^3 - u^3}{\beta(u_0^2 - u^2)^{3/2}} M
\]

\[
- 2uaM \left(1 - \frac{1}{\beta^2} - \frac{u_0^3(u_0 - u)}{u\beta^2(u_0^2 - u^2)^{3/2}}\right)
\]

\[+ O(a^2u, M^2u, aM^2u^2, M^3u^2), \tag{38}\]

where we use

\[
b = \frac{\beta}{u_0} + \beta M - 2u_0aM + O(a^2u_0, M^2u_0, aM^2u_0^2, M^3u_0^2). \tag{39}\]

\(u_0\) is the inverse of the closest approach.

At \(r = r_\infty\) (\(r_\infty\) is an infinite constant radius of the circular arc segment), we obtain

\[dl^2 = r_\infty^2 \beta^2 d\phi^2,\] geodesic curvature \(\kappa_g = \frac{1}{r_\infty} + O(M/r_\infty^2).\) Let us integrate as

\[
\beta \phi_{RS} = \int_S^R \kappa_g dl = \int_S^R \beta d\phi = \beta \int_{\phi_S}^{\phi_R} d\phi = \beta \int_{u_S}^{u_R} F_+(u)du + \beta \int_{u_0}^{u_R} F_-(u)du. \tag{40}\]

\[
\int F_+(u)du = \int \left\{ \frac{1}{\beta\sqrt{u_0^2 - u^2}} - \left(1 - \frac{1}{\beta^2}\right) a + \frac{u_0^3 - u^3}{\beta(u_0^2 - u^2)^{3/2}} M
\]

\[
- 2uaM \left(1 - \frac{1}{\beta^2} \pm \frac{u_0^3(u_0 - u)}{u\beta^2(u_0^2 - u^2)^{3/2}}\right) \right\} du
\]

\[= \pm \frac{1}{\beta} \arcsin \left(\frac{u}{u_0}\right) - \left(1 - \frac{1}{\beta^2}\right) au + \frac{2u_0^2 + u\sqrt{u_0^2 - u^2}}{\beta(u_0 + u)} M
\]

\[+ \left\{ -(1 + \frac{1}{\beta^2}) u^2 \pm \frac{2u_0^2\sqrt{u_0^2 - u^2}}{\beta^2(u_0 + u)} \right\} aM
\]

\[+ O(M^2/u_0^2). \tag{41}\]
\[
\phi_{RS} = \frac{\pi}{\beta} - \frac{1}{\beta} \left\{ \arcsin \left( \frac{bu_S}{\beta} \right) + \arcsin \left( \frac{bu_R}{\beta} \right) \right\} - \left( 1 - \frac{1}{\beta^2} \right) (u_R - u_S)a \\
+ \left\{ \frac{(2 - \frac{b^2 u_R^2}{\beta^2})}{b \sqrt{1 - \frac{b^2 a^2}{\beta^2}}} + \frac{(2 - \frac{b^2 u_S^2}{\beta^2})}{b \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}}} \right\} M \\
+ \left\{ - \left( 1 - \frac{1}{\beta^2} \right) (u_R^2 - u_S^2) - \frac{2}{b^2 \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}}} - \frac{2}{b^2 \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}}} \right\} aM + O(M^2/b^2),
\]

(42)

where we use \(u_0 = \frac{\alpha}{b} + \frac{b^2 M}{b^2} - \frac{2b^2 aM}{b^r} \). This \( \phi_{RS} \) becomes that for the Kerr case, only if one takes the limit \( \beta \to 1 \).

E. Jump angles

In the previous section, the unit tangent vector along the radius line in \(^{(3)}M\) is obtained as

\[
R^t = \left( \frac{1}{\sqrt{\gamma_{rr}}}, 0, 0 \right),
\]

(43)
the unit tangential vector along the spatial curve is also obtained as

\[ e^i = \xi \left( \frac{dr}{d\phi}, 0, 1 \right), \]

\[ \xi_{+R} = \frac{b}{r_R^2 \beta^2} - \frac{2b}{r_R^3 \beta^2} M + \frac{(-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}}{r_R^2 \beta^2} a \]
\[ + \frac{2r_R^2 \beta^2 (1 - \frac{b^2}{r_R^2 \beta^2}) + b^2 (-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}}{r_R^5 \beta^4 (1 - \frac{b^2}{r_R^2 \beta^2})} aM + \mathcal{O}(M^2/r_R^3), \]

\[ \xi_{-R} = -\frac{b}{r_R^2 \beta^2} + \frac{2b}{r_R^3 \beta^2} M - \frac{(-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}}{r_R^2 \beta^2} a \]
\[ - \frac{2r_R^2 \beta^2 (1 - \frac{b^2}{r_R^2 \beta^2}) + b^2 (-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}}{r_R^5 \beta^4 (1 - \frac{b^2}{r_R^2 \beta^2})} aM + \mathcal{O}(M^2/r_R^3), \]

\[ \xi_{+S} = \frac{b}{r_S^2 \beta^2} - \frac{2b}{r_S^3 \beta^2} M - \frac{(-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}}{r_S^2 \beta^2} a \]
\[ + \frac{2r_S^2 \beta^2 (1 - \frac{b^2}{r_S^2 \beta^2}) + b^2 (-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}}{r_S^5 \beta^4 (1 - \frac{b^2}{r_S^2 \beta^2})} aM + \mathcal{O}(M^2/r_S^3), \]

\[ \xi_{-S} = -\frac{b}{r_S^2 \beta^2} + \frac{2b}{r_S^3 \beta^2} M + \frac{(-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}}{r_S^2 \beta^2} a \]
\[ - \frac{2r_S^2 \beta^2 (1 - \frac{b^2}{r_S^2 \beta^2}) + b^2 (-1 + \beta^2) \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}}{r_S^5 \beta^4 (1 - \frac{b^2}{r_S^2 \beta^2})} aM + \mathcal{O}(M^2/r_S^3), \]

where \( \xi_+ \) means that \( e^i \) is the tangent vector of the prograde photon orbit and \( \xi_- \) means that \( e^i \) is the tangent vector of the retrograde photon orbit. In addition, S and R for \( \xi_{\pm} \) mean respectively from the source to the closest approach and from the receiver to the closest approach. Therefore, we can define the angle measured from the outgoing radial direction
by

\[
\cos \Psi_R \equiv \gamma_{ij} e^i R^j \\
= \gamma_{rr} e^r R^r + \gamma_{\phi r} e^\phi R^r \\
= \sqrt{\gamma_{rr}} \frac{dr}{d\phi} \bigg|_+ + \frac{\gamma_{\phi r} \xi + R}{\sqrt{\gamma_{rr}}} \\
= \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}} + \frac{b^2 M}{r_R^3 \beta^2 \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}} + \frac{b(1 - \beta^2)}{r_R^2 \beta^2} a \\
- \frac{2b}{r_R^3 \beta^2 \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}} aM + \mathcal{O}(M^2 / r_R^2), (44)
\]

\[
- \cos(\pi - \Psi_S) \equiv \gamma_{ij} e^i R^j \\
= \gamma_{rr} e^r R^r + \gamma_{\phi r} e^\phi R^r \\
= \sqrt{\gamma_{rr}} \frac{dr}{d\phi} \bigg|_+ + \frac{\gamma_{\phi r} \xi + S}{\sqrt{\gamma_{rr}}} \\
= - \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}} - \frac{b^2 M}{r_S^3 \beta^2 \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}} + \frac{b(1 - \beta^2)}{r_S^2 \beta^2} a \\
+ \frac{2b}{r_S^3 \beta^2 \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}} aM + \mathcal{O}(M^2 / r_S^2), (45)
\]

where Eq. (44) is at the receiver position and Eq. (45) is at the source. Therefore, \( \Psi_R \) and \( \Psi_S \) are obtained as

\[
\Psi_R = \arcsin \left( \frac{b}{r_R \beta} \right) - \frac{bM}{r_R^2 \beta \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}} + \frac{(\beta^2 - 1)a}{r_R \beta} \\
+ \frac{2 + (\beta^2 - 1)}{r_R^2 \beta \sqrt{1 - \frac{b^2}{r_R^2 \beta^2}}} aM + \mathcal{O}(M^2 / r_R^2), (46)
\]

\[
\pi - \Psi_S = \arcsin \left( \frac{b}{r_S \beta} \right) - \frac{bM}{r_S^2 \beta \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}} - \frac{(\beta^2 - 1)a}{r_S \beta} \\
+ \frac{2 - (\beta^2 - 1)}{r_S^2 \beta \sqrt{1 - \frac{b^2}{r_S^2 \beta^2}}} aM + \mathcal{O}(M^2 / r_S^2). (47)
\]
F. Deflection angle

By bringing together Eqs. (16), (24), (35), (36), (46) and (47), the deflection angle of light for the prograde case is obtained as

$$\alpha_{\text{prog}} = \frac{2M\beta}{b} \left[ \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} \right] + \frac{aM(1 - \beta^2)}{\beta} \left[ u_R^2 - u_S^2 \right]$$

$$- \frac{2aM\beta}{b^2} \left[ \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} \right] + \frac{(1 - \beta^2)aM}{\beta r_S^2} - \frac{(1 - \beta^2)aM}{\beta r_R^2} + O(M^2/b^2)$$

$$= \frac{2M\beta}{b} \left[ \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} \right] - \frac{2aM\beta}{b^2} \left[ \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} \right]$$

$$+ O(M^2/b^2).$$

The deflection angle for the retrograde case is

$$\alpha_{\text{retro}} = \frac{2M\beta}{b} \left[ \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} \right] + \frac{2aM\beta}{b^2} \left[ \sqrt{1 - \frac{b^2 u_R^2}{\beta^2}} + \sqrt{1 - \frac{b^2 u_S^2}{\beta^2}} \right]$$

$$+ O(M^2/b^2).$$

For both cases, the source and receiver may be located at finite distance from the monopole. As a matter of course, these results are also obtained by substituting Eqs. (42), (46) and (47) to Eq. (5). Eqs. (48) and (49) show that the light deflection is affected by deficit angle.

One can see that, in the limit as $r_R \to \infty$ and $r_S \to \infty$, Eqs. (48) and (49) become

$$\alpha_{\text{prog}} \to \frac{4M\beta}{b} - \frac{4aM\beta}{b^2} + O\left(\frac{M^2}{b^2}\right),$$

$$\alpha_{\text{retro}} \to \frac{4M\beta}{b} + \frac{4aM\beta}{b^2} + O\left(\frac{M^2}{b^2}\right).$$

These equations coincide with Eq. (53) in Jusufi and Övgün [18], in which they are restricted within the asymptotic source and receiver ($r_R \to \infty$ and $r_S \to \infty$).

IV. POSSIBLE ASTRONOMICAL APPLICATIONS

In this section, we discuss possible astronomical applications. The above calculations discuss the deflection angle of light. In particular, we do not assume that the receiver and the source are located at the infinity. The finite-distance correction to the deflection angle
of light, denoted as $\delta \alpha$, is the difference between the asymptotic deflection angle $\alpha_\infty$ and the deflection angle for the finite distance case. It is expressed as

$$\delta \alpha \equiv \alpha_\infty - \alpha.$$  \hspace{1cm} (50)

The finite-distance correction to the deflection angle of light is roughly estimated as

$$\delta \alpha \sim \frac{Mbu_S^2}{\beta} + \frac{Mbu_R^2}{\beta} + O(aM/b^2, M^2/b^2).$$  \hspace{1cm} (51)

We consider the mass of the rotating global monopole equals to $SgrA^*$ ($M_{Sgr} \simeq 4 \times 10^6 M_\odot$, $M_\odot$ is Solar mass), the spin angular momentum of the rotating global monopole is $a = 2/3 M_{Sgr}$ and the parameter $\beta = 0, 0.999, 359/360$. We assume $r_R$ is the distance from Earth to $SgrA^*$ ($r_R \simeq 2.6 \times 10^{17}$[km]). The deflection angle of light for prograde orbit is $\alpha_{\text{prog}}$, for retrograde case is $\alpha_{\text{retro}}$. Under these assumptions, Eq. (51) implies

$$\delta \alpha \sim \frac{Mb}{\beta r_S^2} \sim 8.6 \times 10^{-5} \text{arcsec.} \times \left( \frac{M}{4 \times 10^6 M_\odot} \right) \left( \frac{b}{4 \times 10^8 M_\odot} \right) \left( \frac{0.1\text{pc}}{r_S} \right)^2 \left( \frac{0.999}{\beta} \right).$$  \hspace{1cm} (52)

FIG. 1: $\alpha_{\text{prog}}$, where we assume the $SgrA^*$. The vertical axis denotes the deflection angle of light with the finite-distance correction and the horizontal axis denotes the source distance $r_S$. The red solid curve, blue dash curve and green dot curve correspond to $\beta = 0$ (Kerr spacetime), $\beta = 0.999$ and $\beta = 359/360$, respectively. The impact parameter is assumed to be $b = 10^2 M_{Sgr}$. 

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FIG. 2: $\alpha_{\text{retro}}$, where we assume the SgrA$^*$. The vertical axis denotes the deflection angle of light with the finite-distance correction and the horizontal axis denotes the source distance $r_S$. The red solid curve, blue dash curve and green dot curve correspond to $\beta = 0$ (Kerr spacetime), $\beta = 0.999$ and $\beta = 359/360$, respectively. The impact parameter is assumed to be $b = 10^2 M_{\text{Sgr}}$. 
V. CONCLUSION

In the weak field approximation, we have discussed a possible modified deflection angle of light for an observer and source at finite distance from a rotating global monopole with deficit angle. We have shown that both of the Werner’s method and the generalized optical metric method give the same deflection angle at the leading order of the weak field approximation, if the receiver and source are at the null infinity. Therefore, our result is a possible extension to asymptotically nonflat spacetimes. We have also found corrections for the deflection angle due to the finite distance from the global monopole. It is left for future to study higher order terms in the weak field approximation of a rotating global monopole and to examine also the strong deflection limit.

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