On a Family of Laurent Polynomials Generated by $2 \times 2$ Matrices

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Abstract To $2 \times 2$ matrix $G$ with complex entries, the sequence of Laurent polynomial $L_n(z, G) = \text{tr} \left( G \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} G^* \right)^n$ is related. It turns out that for each $n$, the family $\{L_n(z, G)\}_G$, where $G$ runs over the set of all $2 \times 2$ matrices, is a three-parametric family. A natural parametrization of this family is found. The polynomial $L_n(z, G)$ is expressed in terms of these parameters and the Chebyshev polynomial $T_n$. The zero set of the polynomial $L_n(z, G)$ is described.

Keywords $2 \times 2$-matrices · Laurent polynomials · Chebyshev polynomials · Entire functions with real $\pm 1$-points

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- $\mathbb{R}$ stands for the set of all real numbers.
- $\mathbb{C}$ stands for the set of all complex numbers.
- If $z \in \mathbb{C}$, $z = x + iy$, $x, y \in \mathbb{R}$, then $\overline{z} = x - iy$ is the complex conjugate number.
- If $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is a matrix, then $M^* = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is the Hermitian conjugate matrix.
- For a matrix $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, $\text{tr} \ M$ stands for the trace of $M$: $\text{tr} \ M = m_{11} + m_{22}$, $\det \ M$ stands for the determinant of $M$.

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1 Laurent Polynomials Generated by $2 \times 2$ Matrices

Let

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

(1.1)

be a $2 \times 2$ matrix with complex entries. For $z \in \mathbb{C}$, let us define

$$S(z, G) = G \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} G^*,$$

(1.2a)

$$L_n(z, G) = \text{tr}(S(z, G))^n.$$  

(1.2b)

Lemma 1.1 Considered as a function of $z$, $L_n(z, G)$ is a Laurent polynomial:

$$L_n(z, G) = \sum_{-n \leq k \leq n} c_{k,n}(G)z^k.$$  

(1.3)

The “leading” coefficients $c_{\pm n,n}(G)$ are:

$$c_{-n,n}(G) = (|g_{12}|^2 + |g_{22}|^2)^n, \quad c_{n,n}(G) = (|g_{11}|^2 + |g_{21}|^2)^n.$$  

(1.4)

Proof The $2 \times 2$ matrix function $S(z, G)$ can be presented as a linear combination:

$$S(z, G) = P_1(G)z + P_{-1}(G)z^{-1},$$

(1.5)

where

$$P_1(G) = \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} \end{bmatrix}, \quad P_{-1}(G) = \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} \begin{bmatrix} g_{12} & g_{22} \end{bmatrix}.$$  

(1.6)

Since $(S(z, G))^n = (P_1(G)z + P_{-1}(G)z^{-1})^n$, it is clear that

$$(S(z, G))^n = (P_1(G))^n z^n + \cdots + (P_{-1}(G))^n z^{-n}.$$  

(1.7)

The matrices $(P_1(G))^n$, $(P_{-1}(G))^n$ and their traces can be calculated easily:

$$(P_1(G))^n = \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} (|g_{11}|^2 + |g_{21}|^2)^{n-1} \begin{bmatrix} g_{11} & g_{21} \end{bmatrix},$$

$$(P_{-1}(G))^n = \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} (|g_{12}|^2 + |g_{22}|^2)^{n-1} \begin{bmatrix} g_{12} & g_{22} \end{bmatrix},$$

$$\text{tr}(P_1(G))^n = (|g_{11}|^2 + |g_{21}|^2)^n, \quad \text{tr}(P_{-1}(G))^n = (|g_{12}|^2 + |g_{22}|^2)^n.$$  

(1.8)

Since the value $\text{tr} M$ is linear with respect to $2 \times 2$ matrix $M$, the equality (1.4) follows from the definition (1.2b) and from (1.8).

Let us “normalize” the polynomial $L_n(z, G)$.
Definition 1.2 We say that the matrix \( G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \) is generic if the condition

\[
(|g_{11}|^2 + |g_{21}|^2)(|g_{12}|^2 + |g_{22}|^2) \neq 0
\]

holds.

For a generic matrix \( G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \), let us define

\[
R_1(G) = (|g_{11}|^2 + |g_{21}|^2)^{\frac{1}{2}}, \quad R_2(G) = (|g_{12}|^2 + |g_{22}|^2)^{\frac{1}{2}},
\]

\[
H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} (R_1(G))^{-1} & 0 \\ 0 & (R_2(G))^{-1} \end{bmatrix}
\]

The matrix \( H \) satisfies the normalizing condition

\[
|h_{11}|^2 + |h_{21}|^2 = 1, \quad |h_{12}|^2 + |h_{21}|^2 = 1.
\]

Lemma 1.3 The Laurent polynomials \( L_n(., H) \) and \( L_n(., G) \) are related by the equality

\[
L_n(z, G) = R^n L_n(\rho z, H),
\]

where

\[
R = R_1(G)R_2(G), \quad \rho = R_1(G)/R_2(G),
\]

\( R_1(G), R_2(G) \) are defined by (1.10).

Lemma 1.4 Let \( H \) be an arbitrary \( 2 \times 2 \) matrix with complex entries and \( F \) be the nonnegative square root of the matrix \( H^*H \):

\[
F^2 = H^*H, \quad F \geq 0.
\]

Then the Laurent polynomials \( L_n(z, H) \) and \( L_n(z, F) \) coincides:

\[
L_n(z, H) \equiv L_n(z, F).
\]

Proof According to the definitions (1.2a) and (1.2b),

\[
L_n(z, H) = \text{tr} \left( H \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} H^* \cdot H \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} H^* \cdot \cdots \cdot H \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} H^* \cdot H \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} H^* \right)
\]

(1.17)
Permuting the matrices $H$ and 
\[
\begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix}
\]
$H^* \cdot H \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} H^* \cdots H \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} H^* .
\]
$H \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} H^*$, we obtain
\[
L_n(z, H) = \text{tr}\left( \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] H^* H \cdot \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] H^* H \cdots \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] H^* H \cdot \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] H^* H \right).
\]
Taking into account (1.15), we obtain
\[
L_n(z, H) = \text{tr}\left( \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F^2 \cdot \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F^2 \cdots \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F^2 \cdot \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F \cdot F \right).
\]
Permuting the matrices $\begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} F^2 \cdot \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F^2 \cdots \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F^2 \cdot \left[ \begin{array}{cc}
  z & 0 \\
  0 & z^{-1}
\end{array} \right] F$ and $F$, we obtain
\[
L_n(z, H) = \text{tr}\left( F \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} F \cdot F \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} F \cdots F \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} F \cdot F \begin{bmatrix}
  z & 0 \\
  0 & z^{-1}
\end{bmatrix} F \right)
\]
(1.18)
According to the definitions (1.2a) and (1.2b), the function in the right hand side of (1.18) is the polynomial $L_n(z, F)$.

We apply Lemma 1.4 to the normalized matrix $H$ of the form (1.10)-(1.11). In view of (1.12), the matrix $H^* H$ is of the form
\[
H^* H = \begin{bmatrix}
  1 & \gamma \\
  \bar{\gamma} & 1
\end{bmatrix}, \quad \text{where} \quad \gamma \in \mathbb{C}, \quad |\gamma| \leq 1.
\]
(1.19)
Let $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ be the non-negative square root of the matrix $H^* H$. Since $F \geq 0$, the conditions
\[
f_{11} \geq 0, \quad f_{22} \geq 0, \quad f_{12} = \overline{f_{21}}
\]
(1.20)
hold. In particular,
\[
|f_{12}| = |f_{21}|
\]
(1.21)
From (1.19) and from the equality $F^2 = H^* H$ it follows that
\[
(f_{11})^2 + |f_{12}|^2 = 1, \quad |f_{21}|^2 + (f_{22})^2 = 1.
\]
(1.22)
Since $f_{11} \geq 0, \ f_{22} \geq 0$, from (1.21) and (1.22) it follows that
\[
f_{11} = f_{22}.
\]
(1.23)
From (1.22), (1.23) and $f_{12} = \overline{f_{21}}$ it follows that there exist $\theta \in [0, \pi/2]$ and $a \in \mathbb{C}, |a| = 1$ such that $f_{11} = f_{22} = \cos \theta, f_{12} = a \sin \theta, f_{21} = \sin \theta \overline{a}$. Since $F \geq 0$, 
the inequality \( \det F \geq 0 \) holds. Therefore actually \( \theta \in [0, \pi/4] \). It is evident that such \( \theta \) and \( a \) are unique.

Thus the following result is obtained:

**Lemma 1.5** Let \( H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \) be an arbitrary \( 2 \times 2 \) matrix with complex entries which satisfy the normalizing condition (1.12). Let \( F \) be the non-negative square root of the matrix \( H^*H \).

Then \( F \) is of the form \( F = F_{\theta,a} \), where

\[
F_{\theta,a} = \begin{bmatrix} \cos \theta & a \sin \theta \\ \sin \theta \overline{a} & \cos \theta \end{bmatrix},
\]

whith \( \theta \in [0, \pi/4] \), \( a \in \mathbb{C} \), \( |a| = 1 \).

According to Lemma 1.4, the Laurent polynomials \( L_n(z, H) \) and \( L_n(z, F_{\theta,a}) \) coincide:

\[
L_n(z, H) = L_n(z, F_{\theta,a}), \quad n = 1, 2, 3, \ldots
\]

Let us relate the matrix \( U_a \) to the number \( a \in \mathbb{C} \):

\[
U_a = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.
\]

If \( |a| = 1 \), then the matrix \( U_a \) is unitary: \( U_a U_a^* = U_a^* U_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). For \( |a| = 1 \), the equalities

\[
F_{\theta,a} = U_a F_{\theta} U_a^*, \quad U_a \begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix} U_a^* = \begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix}.
\]

hold, where

\[
F_{\theta} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.
\]

Therefore

\[
S(z, F_{\theta,a}) = U_a S(z, F_{\theta}) U_a^*, \quad \forall a \in \mathbb{C} : |a| = 1,
\]

and for any \( n = 1, 2, 3, \ldots \),

\[
(S(z, F_{\theta,a}))^n = U_a (S(z, F_{\theta}))^n U_a^*, \quad \forall a \in \mathbb{C} : |a| = 1,
\]

If \( M \) is an arbitrary matrix and \( U \) is an unitary matrix, then \( \text{tr } U M U^* = \text{tr } M \). In particular, \( \text{tr } U_a (S(z, F_{\theta}))^n U_a^* = \text{tr}(S(z, F_{\theta}))^n. \) Thus

\[
L_n(z, F_{\theta,a}) = L_n(z, F_{\theta}), \quad \forall a \in \mathbb{C} : |a| = 1.
\]

Comparing (1.25) and (1.30), we obtain the following result:
Theorem 1.6 Let \( H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \) be an arbitrary \( 2 \times 2 \) matrix with complex entries which satisfies the normalizing condition (1.12). Then there exists an unique \( \theta \in [0, \pi/4] \) such that

\[
L_n(z, H) = L_n(z, F_\theta), \quad n = 1, 2, 3, \ldots,
\]

where the matrix \( F_\theta \) is defined by (1.28).

Let us summarize the above consideration.

Theorem 1.7 Let \( G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \) be a \( 2 \times 2 \) matrix with complex entries. We assume that \( G \) is generic, that in no-one of two columns \( \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} \) and \( \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} \) vanishes. Let the Laurent polynomial \( L_n(z, G) \) is defined by (1.2).

Then

1. There exists the number \( \theta \in [0, \pi/4] \) such that the Laurent polynomial \( L_n(., G) \) generated by the matrix \( G \) can be expressed in terms of the Laurent polynomial \( L_n(., F_\theta) \) generated by the matrix \( F_\theta \):

\[
L_n(z, G) \equiv R^n L_n(\rho z, F_\theta), \quad z \in \mathbb{C},
\]

for every \( n = 1, 2, 3, \ldots \), where the matrix \( F_\theta \) is defined by (1.28), the numbers \( R \) and \( \rho \) are the same that appears in (1.14).

2. The parameters \( \theta \) is determined by the matrix \( G \) uniquely. In particular \( \theta \) does not depend on \( n \).

3. The parameter \( \theta \) takes the value \( \theta = 0 \) if and only if the columns \( \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} \) and \( \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} \) of the matrix \( G \) are orthogonal, that is \( g_{11} \overline{g_{12}} + g_{21} \overline{g_{22}} = 0 \).

The parameter \( \theta \) takes the value \( \theta = \pi/4 \) if and only if the columns \( \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} \) and \( \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} \) of the matrix \( G \) are proportional, that is the matrix \( G \) is of rank one.

2 Properties of the Polynomials \( L_n(z, F_\theta) \).

Theorem 2.1 Let the Laurent polynomial \( L_n(z, F_\theta), n = 1, 2, 3, \ldots \), be defined as

\[
L_n(z, F_\theta) = \text{tr} \left( S(z, F_\theta) \right)^n.
\]

where

\[
S(z, F_\theta) = F_\theta \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} F_\theta,
\]

the matrix \( F_\theta \) is defined by (1.28), and \( \theta \in [0, \pi/4] \).
Then

1. The Laurent polynomial $L_n(z, F_\theta)$ is of the form

$$L_n(z, F_\theta) = z^n + z^{-n} + \sum_{-(n-1) \leq k \leq (n-1)} p_{k,n}(\theta) z^k,$$

$$p_{k,n}(\theta) = p_{-k,n}(\theta).$$  \hfill (2.2)

2. For $\theta \in (0, \pi/4)$, the coefficients $p_{k,n}(\theta)$ vanish if $k \neq n \mod 2$ and are strictly positive if $k = n \mod 2$:

$$p_{k,n}(\theta) = 0, \quad -(n - 1) \leq k \leq n - 1, \quad k \neq n \mod 2,$$

$$p_{k,n}(\theta) > 0, \quad -(n - 1) \leq k \leq n - 1, \quad k = n \mod 2.$$  \hfill (2.3a, 2.3b)

3. For $\theta \in [0, \pi/4]$, the Laurent polynomial $L_n(z, F_\theta)$ can be expressed in terms of the Chebyshev polynomial $T_n, T_n(\zeta) = \cos(n \arccos \zeta)$:

$$L_n(z, F_\theta) = 2(\cos 2\theta)^n \cdot T_n\left(\frac{z + z^{-1}}{2\cos 2\theta}\right).$$ \hfill (2.4)

**Remark 2.2** For $\theta = 0$,

$$L_n(z, F_0) = z^n + z^{-n}.$$ \hfill (2.5)

So, all coefficients $p_{k,n}(0)$ of the Laurent polynomial $L_n(z, F_0)$ vanish:

$$p_{k,n}(0) = 0, \quad -(n - 1) \leq k \leq n - 1.$$ \hfill (2.6)

For $\theta = \pi/4$,

$$L_n(z, \pi/4) = \left(z + z^{-1}\right)^n,$$ \hfill (2.7)

so

$$p_{k,n}\left(\pi/4\right) = 0 \quad \text{if} \quad k \neq n \mod 2, \quad p_{k,n}\left(\pi/4\right) = \left(\frac{n}{2\cos \pi/2}\right) \quad \text{if} \quad k = n \mod 2.$$ \hfill (2.8)

**Proof of Theorem 2.1.** 1. The equalities (2.1) are the equalities (1.2) for the matrix $G = F_\theta$.

2. It is clear that $S(z, F_\theta) = z P_1(\theta) + P_{-1}(\theta) z^{-1}$, where

$$P_1(\theta) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \end{bmatrix}, \quad P_{-1}(\theta) = \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} \begin{bmatrix} \sin \theta & \cos \theta \end{bmatrix}.$$ \hfill (2.9)

Thus

$$(S(z, F_\theta))^n = \sum_{\epsilon} z^{\epsilon(\theta)} P_{\epsilon_1}(\theta) \cdot P_{\epsilon_2}(\theta) \cdot \cdots \cdot P_{\epsilon_n}(\theta),$$ \hfill (2.10)
the sum in (2.10) runs over all combinations of subscripts with either \( \varepsilon_k = 1 \) or \( \varepsilon_k = -1 \), \( v(\varepsilon) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n \).

It is clear that

\[
v(\varepsilon) = n - 2v_-(\varepsilon) = 2v_+ (\varepsilon) - n,
\]

where

\[
v_+ (\varepsilon) = \#\{ k : \varepsilon_k = +1 \}, \quad v_- (\varepsilon) = \#\{ k : \varepsilon_k = -1 \}.
\] (2.11)

Therefore

\[
v(\varepsilon) = n \pmod{2} \quad \forall \varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n).
\] (2.12)

Regrouping summands in (2.10), we obtain

\[
(S(z, F_\theta))^n = \sum_{-n \leq k \leq n} z^k \left( \sum_{\varepsilon : v(\varepsilon) = k} P_{\varepsilon_1}(\theta) \cdot P_{\varepsilon_2}(\theta) \cdot \cdots \cdot P_{\varepsilon_n}(\theta) \right).
\] (2.13)

Thus the coefficients \( p_{k,n}(\theta) \) of the polynomial \( L_n(z, F_\theta) \), (2.2), are:

\[
p_{k,n}(\theta) = \sum_{\varepsilon : v(\varepsilon) = k} \text{tr} \left( P_{\varepsilon_1}(\theta) \cdot P_{\varepsilon_2}(\theta) \cdot \cdots \cdot P_{\varepsilon_n}(\theta) \right), \quad -(n - 1) \leq k \leq n - 1,
\] (2.14)

the sum in (2.14) runs over the set \( \{ \varepsilon : v(\varepsilon) = k \} \).

According to (2.12), if \( k \neq n \pmod{2} \), then the set \( \{ \varepsilon : v(\varepsilon) = k \} \) is empty. Thus the sum in (2.14) vanishes if \( k \neq n \pmod{2} \). In other words, the condition (2.3a) holds.

If an integer \( k \) satisfies the conditions

\[
k = n \pmod{2}, \quad -(n - 1) \leq k \leq (n - 1),
\] (2.15)

then the set \( \{ \varepsilon : v(\varepsilon) = k \} \) is not empty. The equality \( v(\varepsilon) = k \) means that

\[
v_+ (\varepsilon) = \frac{n+k}{2}, \quad v_- (\varepsilon) = \frac{n-k}{2}.
\]

Moreover if an integer \( k \) satisfies the condition (2.15), then

\[
\#\{ \varepsilon : v(\varepsilon) = k \} = \binom{n}{\frac{n+k}{2}} = \binom{n}{\frac{n-k}{2}}.
\] (2.16)

For \( \theta \in (0, \pi/2) \), all entries each of the matrices \( P_{\varepsilon_j}(\theta) \) are strictly positive. Hence all the entries each of the matrices \( P_{\varepsilon_1}(\theta) \cdot P_{\varepsilon_2}(\theta) \cdot \cdots \cdot P_{\varepsilon_n}(\theta) \) are strictly positive. All the more \( \text{tr} \left( P_{\varepsilon_1}(\theta) \cdot P_{\varepsilon_2}(\theta) \cdot \cdots \cdot P_{\varepsilon_n}(\theta) \right) > 0 \). Therefore the condition (2.3b) holds.

\[1\] There are \( 2^n \) such combinations.
3. For a $2 \times 2$ matrix $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, let $\lambda_1(M)$ and $\lambda_2(M)$ be the eigenvalues of $M$, that is the roots of the characteristic equation $\det \left( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - M \right) = 0$. For any power $M^n$ of the matrix $M$, $n = 1, 2, 3, \ldots$, the equalities

$$\lambda_1(M^n) = (\lambda_1(M))^n, \quad \lambda_2(M^n) = (\lambda_2(M))^n$$

hold. In particular, the trace $\operatorname{tr} M^n$ of the matrix $M^n$ can be expressed in terms of the eigenvalues of the matrix $M$:

$$\operatorname{tr} M^n = (\lambda_1(M))^n + (\lambda_2(M))^n, \quad n = 1, 2, 3, \ldots$$  \hspace{1cm} (2.17)

We apply (2.17) to the matrix $M = S(z, F_\theta)$. Taking into account (2.1a), we come to the equality

$$L_n(z, F_\theta) = (\lambda_1(S(z, F_\theta)))^n + (\lambda_2(S(z, F_\theta)))^n, \quad z \in \mathbb{C}. \hspace{1cm} (2.18a)$$

The eigenvalues of the matrix $S(z, F_\theta)$ can be found explicitly:

$$\lambda_1(S(z, F_\theta)) = \frac{z + z^{-1}}{2} + \sqrt{\left(\frac{z + z^{-1}}{2}\right)^2 - \cos^2 2\theta}, \hspace{1cm} (2.18b)$$

$$\lambda_2(S(z, F_\theta)) = \frac{z + z^{-1}}{2} - \sqrt{\left(\frac{z + z^{-1}}{2}\right)^2 - \cos^2 2\theta}. \hspace{1cm} (2.19b)$$

The Chebyshev polynomial $T_n(\zeta)$ can be represented as

$$T_n(\zeta) = \frac{1}{2} \left( (\mu_1(\zeta))^n + (\mu_2(\zeta))^n \right), \hspace{1cm} (2.19a)$$

where

$$\mu_1(\zeta) = \zeta + \sqrt{\zeta^2 - 1}, \quad \mu_2(\zeta) = \zeta - \sqrt{\zeta^2 - 1}. \hspace{1cm} (2.19b)$$

Comparing (2.18b) with (2.19b), we conclude that

$$\lambda_1(S(z, F_\theta)) = \cos 2\theta \mu_1 \left( \frac{z + z^{-1}}{2\cos 2\theta} \right), \quad \lambda_2(S(z, F_\theta)) = \cos 2\theta \mu_2 \left( \frac{z + z^{-1}}{2\cos 2\theta} \right),$$

$$\theta \in [0, \pi/4), \quad z \in \mathbb{C}\backslash\{0\}. \hspace{1cm} (2.20)$$

Comparing (2.18a), (2.19a) and (2.20), we obtain (2.4). \hfill \Box

Remark 2.3 Since $2T_n \left( \frac{z + z^{-1}}{2} \right) = z^n + z^{-n}$, the equality (2.5) is the special case of the equality (2.4) corresponding to the value $\theta = 0$.

Since $T_n(\zeta) = 2^{n-1} \zeta^n + o(|\zeta|^n)$ as $|\zeta| \to \infty$, the equality (2.7) is the limiting case of the equality (2.4) corresponding to the “value” $\theta = \frac{\pi}{4} - 0$. 

Remark 2.4 Applying the binomial formula, we derive from (2.19) that

\[ T_n(\zeta) = \sum_{0 \leq j \leq n} \binom{n}{j} \zeta^{n-2j} (\zeta^2 - 1)^j. \]  

(2.21)

Substituting \( \zeta = \frac{z + z^{-1}}{2 \cos 2\theta} \) into (2.21) and taking into account (2.4), we obtain the equality

\[ L_n(z, F_\theta) = 2^{-(n-1)} \sum_{0 \leq j \leq n} \binom{n}{j} (z + z^{-1})^{n-2j} \left( z^2 + z^{-2} + 2(1 - \cos 2\theta) \right)^j. \]

(2.22)

From (2.22) it is evident that

\( (-1)^n L_n(-z, F_\theta) = L_n(z, F_\theta). \)

Hence the condition (2.3a) holds. The condition (2.3b) can be derived from (2.22). The coefficients \( p_{k,n}(\theta) \) of the Laurent polynomial \( L_n(z, F_\theta) \) majorize the coefficients of the Laurent polynomial \( 2^{-(n-1)}(z + z^{-1})^n. \)

Notation Let \( f : \mathbb{C} \to \mathbb{C} \) be a mapping and \( E \subset \mathbb{C} \) be a subset of \( \mathbb{C}. \) By \( f^{-1}(E) \) we denote the preimage of the set \( E \) with respect to the mapping \( f: \)

\[ f^{-1}(E) = \{ \zeta \in \mathbb{C} : f(\zeta) \in E \}. \]

Lemma 2.5 Let \( T_n \) be the Chebyshev polynomial of degree \( n, n = 1, 2, 3, \ldots. \) Then

\[ T_n^{-1}([-1, 1]) = [-1, 1]. \]

(2.23)

Proof a. If \( \zeta \in [-1, 1], \) then \( T_n(\zeta) \in [-1, 1]. \) Thus \( T_n^{-1}([-1, 1]) \supseteq [-1, 1]. \)

b. If \( s \in [-1, 1], \) then the equation \( T_n(\zeta) = s \) has \( n \) roots \( \zeta_1(s), \ldots, \zeta_n(s) \) located within the interval \([-1, 1]. \) (If \( s \in (-1, 1) \) these roots are even different.) Since the polynomial \( T_n \) is of degree \( n, \) the equation \( T_n(t) = s \) has no other roots. Thus, if \( s \in [-1, 1], \) then \( T_n^{-1}([s]) \subset [-1, 1]. \) Thus \( T_n^{-1}([-1, 1]) \subseteq [-1, 1]. \) \( \Box \)

Let us introduce the mapping \( \Psi_\theta : \mathbb{C}\backslash 0 \to \mathbb{C}: \)

\[ \Psi_\theta(z) = \frac{z + \frac{1}{z}}{2 \cos 2\theta}, \]

(2.24)

\( \theta \in [0, \frac{\pi}{4}) \) is considered as a parameter. The mapping \( \Psi_\theta \) is related to the Joukowski mapping \( Jo : \mathbb{C}\backslash 0 \to \mathbb{C}: \)

\[ Jo(z) = \frac{z + \frac{1}{z}}{2}. \]
Concerning the Joukowski mapping, see for example [2, Section I.5, pp. 67–68.]

From properties of the Joukowski mapping we derive the following

Properties of the mapping $\Psi_\theta$:

1. $\Psi_\theta^{-1}([-1, 1]) = T^\circ_\theta \cup T^-_\theta$, $\Psi_\theta^{-1}((-1, 1)) = T^\circ_\theta \cup T^-_\theta$, (2.25)

where

$$
T^+_\theta = \{ z \in \mathbb{C} : |z| = 1, 2\theta \leq \arg z \leq \pi - 2\theta \},
$$

$$
T^-_\theta = \{ z \in \mathbb{C} : |z| = 1, 2\theta \leq -\arg z \leq \pi - 2\theta \},
$$

$$
T^\circ_\theta = \{ z \in \mathbb{C} : |z| = 1, 2\theta < \arg z < \pi - 2\theta \},
$$

$$
T^-_\theta = \{ z \in \mathbb{C} : |z| = 1, 2\theta < -\arg z < \pi - 2\theta \}.
$$

2. $\Psi_\theta$ maps $T^+_\theta$ onto $[-1, 1]$ homeomorphically, and $\Psi_\theta'(z) \neq 0$ for $z \in T^+_\theta$.

3. $\Psi_\theta$ maps $T^-_\theta$ onto $[-1, 1]$ homeomorphically, and $\Psi_\theta'(z) \neq 0$ for $z \in T^-_\theta$.

**Theorem 2.6** For each $\theta \in [0, \frac{\pi}{4})$ and for each $n = 1, 2, 3, \ldots$, all roots of the Laurent polynomial $L_n(z, F_\theta)$ are located within the set $T^\circ_\theta \cup T^-_\theta$ and are simple (i.e. of multiplicity one).

**Proof** We consider the function $T_n \left( \frac{z+z^{-1}}{2\cos 2\theta} \right)$ which appears in (2.4) as a composition $T_n \circ \Psi_\theta$ of the Chebyshev polynomial $T_n$ and the function $\Psi_\theta$ defined by (2.24). The roots of the polynomial $T_n$ form the set $T_n^{-1}(\{0\})$. Since $\{0\} \in (-1, 1)$ and $\Psi_\theta^{-1}((-1, 1)) = T^\circ_\theta \cup T^-_\theta$, all roots of the function $T_n \left( \frac{z+z^{-1}}{2\cos 2\theta} \right)$ lie within the set $T^\circ_\theta \cup T^-_\theta$. Since all roots of $T_n$ are simple and the derivative $\Psi_\theta'(z)$ does not vanish for $z \in T^\circ_\theta \cup T^-_\theta$, all roots of the function $T_n \left( \frac{z+z^{-1}}{2\cos 2\theta} \right)$ are simple. Now Theorem 2.6 is a consequence of the statement 3 of Theorem 2.1. (See the equality (2.4).) $\Box$

**Remark 2.7** Since

$$
L_n(z, F_\theta) = (-1)^n L_n(-z, F_\theta), \quad L_n(\bar{z}, F_\theta) = \overline{L_n(z, F_\theta)} \quad \forall z \in \mathbb{C} \setminus 0,
$$

the set of all roots of the Laurent polynomial $L_n(z, F_\theta)$ is symmetric both with respect to the real axis and with respect to the imaginary axis.

3 The Parametrization of the Set $\{L_n(z, G)\}_G$ of Laurent Polynomials by Free Parameters

**Theorem 3.1** For each $n = 2, 3, \ldots$, the family of Laurent polynomials $\{L_n(z, G)\}_G$, where $G$ runs over the set of all generic $2 \times 2$ matrices with complex entries, is a three-parametric family. The representation (1.32) is a parametrization of this family by free parameters $R$, $\rho$, $\theta$:
1. Given a generic $2 \times 2$ matrix $G$ with complex entries, then for every $n = 1, 2, 3, \ldots$ the Laurent polynomial $L_n(z, G)$ is representable in the form

$$L_n(z, G) = R^n L_n(\rho z, F_\theta), \quad z \in \mathbb{C}, \quad (3.1)$$

with some $R \in (0, \infty)$, $\rho \in (0, \infty)$, $\theta \in [0, \pi/4]$.

2. Given a triple $(R, \rho, \theta)$ of numbers which satisfy the condition

$$R \in (0, \infty), \quad \rho \in (0, \infty), \quad \theta \in [0, \pi/4], \quad (3.2)$$

then there exists the generic matrix $G_{R, \rho, \theta}$ such that the equalities

$$L_n(z, G_{R, \rho, \theta}) = R^n L_n(\rho z, F_\theta) \quad (3.3)$$

hold for every $n = 1, 2, 3, \ldots$

3. If the triples $(R_1, \rho_1, \theta_1)$ and $(R_2, \rho_2, \theta_2)$ satisfy the condition (3.2) and the functions $R^n_1 L_n(\rho_1 z, F_{\theta_1})$ and $R^n_2 L_n(\rho_2 z, F_{\theta_2})$ of variable $z$ coincide for some $n \geq 2$, then $R_1 = R_2$, $\rho_1 = \rho_2$, and $\theta_1 = \theta_2$. If the functions $R^n_1 L_1(\rho_1 z, F_{\theta_1})$ and $R^n_2 L_1(\rho_2 z, F_{\theta_2})$ coincide, then $R_1 = R_2$, $\rho_1 = \rho_2$, but $\theta_1, \theta_2$ can be arbitrary.

Proof 1. The statement 1 of Theorem 3.1 coincides with the statement 1 of Theorem 1.7.

2. Given a triple $(R, \rho, \theta)$, we define

$$r_1 = \sqrt{R \cdot \rho}, \quad r_2 = \sqrt{R/\rho}, \quad G_{R, \rho, \theta} = F_\theta \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}. \quad (3.4)$$

Then

$$G_{R, \rho, \theta} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} G^*_{R, \rho, \theta} = R F_\theta \begin{bmatrix} \rho z & 0 \\ 0 & (\rho z)^{-1} \end{bmatrix} F_\theta.$$

In other words,

$$S(z, G_{R, \rho, \theta}) = R \cdot S(\rho z, F_\theta).$$

Finally

$$\text{tr} \left( S(z, G_{R, \rho, \theta})^n \right) = R^n \cdot \text{tr} \left( S(\rho z, F_\theta)^n \right).$$

Thus the equality (3.3) holds.

3. We assume that

$$R^n_1 L_n(\rho_1 z, F_{\theta_1}) = R^n_2 L_n(\rho_2 z, F_{\theta_2}) \quad \forall z \in \mathbb{C} \setminus 0 \quad (3.5)$$
by some $n$. According to (2.2), the equality (3.5) implies that

$$R_1^n \left( (\rho_1 z)^n + (\rho_1 z)^{-n} + \sum_{-(n-1) \leq k \leq -(n-1)} p_k(\theta_1)(\rho_1 z)^k \right) = R_2^n \left( (\rho_2 z)^n + (\rho_2 z)^{-n} + \sum_{-(n-1) \leq k \leq -(n-1)} p_k(\theta_2)(\rho_2 z)^k \right).$$

Comparing the coefficients by the leading terms $z^n$ and $z^{-n}$ we see that

$$R_1^n \rho_1^n = R_2^n \rho_2^n, \quad R_1^n \rho_1^{-n} = R_2^n \rho_2^{-n}.$$ 

From these equalities it follows that $R_1 = R_2$ and $\rho_1 = \rho_2$. Now the equality (3.5) is reduced to the equality

$$L_n(z, F_{\theta_1}) = L_n(z, F_{\theta_2}) \quad \forall z \in \mathbb{C}\setminus\{0\}.$$ 

In particular,

$$L_n(1, F_{\theta_1}) = L_n(1, F_{\theta_2}).$$

According to the statement 2 of Theorem 2.1, the value $L_n(1, F_\theta)$ increases strictly monotonically in the interval $\theta \in [0, \pi/4]$ if $n \geq 2$. Therefore $\theta_1 = \theta_2$. The Laurent polynomial $L_1(z, F_\theta) = z + z^{-1}$ does not depend on $\theta$. \qed

4 Trigonometric Polynomials Generated by $2 \times 2$ matrices

The formula (2.4) suggests to relate the family of trigonometric polynomials $\tau_{n,\theta}(t)$ to the family of Laurent polynomials $L_n(z, \theta)$:

**Definition 4.1** For $\theta \in [0, \pi/4)$ and $n = 1, 2, 3, \ldots$, we define the function $\tau_{n,\theta}(t)$ of variable $t \in \mathbb{C}$:

$$\tau_{n,\theta}(t) \overset{\text{def}}{=} \frac{1}{2(\cos 2\theta)^n} \cdot L_n(e^{it}, F_\theta),$$

(4.1)

where the function $L_n(z, F_\theta)$ was defined by (2.1).

**Lemma 4.2** The function $\tau_{n,\theta}(t)$ is an even trigonometric polynomial of degree $n$:

$$\tau_{n,\theta}(t) = \frac{1}{(\cos 2\theta)^n} \cos nt + \sum_{0 \leq k \leq (n-1)} \tau_{k, n}(\theta) \cos kt,$$

(4.2)

where the coefficients $\tau_{k, n}(\theta)$ are related to the coefficients $p_{k, n}(\theta)$ of the Laurent polynomial $L_n(z, F_\theta)$, (2.2), by the equalities

$$\tau_{0, n}(\theta) = \frac{p_{0, n}(\theta)}{2(\cos 2\theta)^n}, \quad \tau_{k, n}(\theta) = \frac{p_{k, n}(\theta)}{(\cos 2\theta)^n}, \quad 1 \leq k \leq n - 1.$$
In particular,
\[
\tau_{k,n}(\theta) = 0, \quad -(n-1) \leq k \leq n-1, \quad k \neq n \pmod{2},
\]
\[\tau_{k,n}(\theta) > 0, \quad -(n-1) \leq k \leq n-1, \quad k = n \pmod{2}.
\]

The following result is an immediate consequence of Theorem 2.1, statement 3:

**Theorem 4.3** The trigonometric polynomial \(\tau_n(t, \theta)\) and the Chebyshev polynomial \(T_n(\zeta)\) are related by the equality
\[
\tau_{n,\theta}(t) = T_n\left(\cos t \cos 2\theta\right), \quad t \in \mathbb{C}, \quad \theta \in [0, \frac{\pi}{4}).
\]

Let \(\Phi_\theta : \mathbb{C} \to \mathbb{C}\) be the mapping defined as
\[
\Phi_\theta(t) = \frac{\cos t}{\cos 2\theta},
\]
where \(\theta \in [0, \frac{\pi}{4})\) is considered as a parameter.

**Lemma 4.4** For \(\theta \in \left[0, \frac{\pi}{4}\right]\), the function \(\Phi_\theta\) possesses the following properties:

1. \(\Phi_\theta^{-1}([-1, 1]) = \mathcal{P}\), \(\Phi_\theta^{-1}((-1, 1)) = \mathcal{P}\),

where
\[
\mathcal{P} = \bigcup_{-\infty < p < \infty} [p\pi + 2\theta, (p+1)\pi - 2\theta],
\]
\[
\mathcal{P} = \bigcup_{-\infty < p < \infty} (p\pi + 2\theta, (p+1)\pi - 2\theta),
\]
are periodic systems of closed or open intervals respectively.

2. For each \(p\), the function \(\Phi_\theta\) maps the interval \([p\pi + 2\theta, (p+1)\pi - 2\theta]\) onto the interval \([-1, 1]\) homeomorphically.

3. For each \(p\),
\[
\Phi_\theta'(t) \neq 0 \quad \forall t \in (p\pi + 2\theta, (p+1)\pi - 2\theta).
\]

**Proof** 1. Let \(\mathcal{S} = \{t \in \mathbb{C} : \cos t \in \mathbb{R}\}\). Then \(\mathcal{S}\) is the union of the real axis and the countable set of vertical lines:
\[
\mathcal{S} = \mathbb{R} \cup \left( \bigcup_{-\infty < q < \infty} t_q + i\mathbb{R} \right), \quad t_q = \frac{\pi}{2} (1 + 2q).
\]

If \(s \in \mathbb{R}\setminus0\), then \(|\cos(t_q + is)| > 1\). Therefore, \(t \in \mathbb{R}\) if \(\Phi_\theta(t) \in [-1, 1]\). Thus (4.6) holds.

2. On each interval \([p\pi + 2\theta, (p+1)\pi - 2\theta]\), the function \(\Phi_\theta(t)\) behaves strictly monotonically. It decreases if \(p\) is even and increases if \(\pi\) is odd.
3. 

\[(−1)^{p−1} \Phi_0'(t) = \frac{|\sin t|}{\cos 2\theta} > 0 \quad \forall \ t \in (p\pi + 2\theta, (p+1)\pi - 2\theta).\]

\(\square\)

According to Theorem 4.3, the mapping \(\tau_{n,\theta}\) is a composition of the mappings \(\Phi_\theta\) and \(T_n\): \(\tau_{n,\theta} = T_n \circ \Phi_\theta\). Therefore the following result holds:

Lemma 4.5 For each \(n = 1, 2, 3, \ldots\) and \(\theta \in \left[0, \frac{\pi}{4}\right]\), the preimage \(\tau_{n,\theta}^{-1}([-1, 1])\) of the interval \([-1, 1]\) with respect to the mapping \(\tau_{n,\theta}\) is the system \(\mathcal{P}\) of intervals that appears in (4.6b):

\[
\tau_{n,\theta}^{-1}([-1, 1]) = \mathcal{P}. \tag{4.8}
\]

Theorem 4.6 For each \(n = 1, 2, 3, \ldots\) and \(\theta \in \left[0, \frac{\pi}{4}\right]\):

1. All roots of the equation

\[
\tau_{n,\theta}(t) = 0 \tag{4.9}
\]

are real and simple. Moreover the roots of the equation (4.9) are located within the set \(\mathcal{P}\).

2. All roots of the equation

\[
(\tau_{n,\theta}(t))^2 = 1 \tag{4.10}
\]

are real.

Proof 1. For each \(n = 1, 2, 3, \ldots\),

\[
T_n^{-1}(\{0\}) \subset (-1, 1).
\]

Since \(\tau_{n,\theta} = T_n \circ \Phi_\theta\),

\[
\tau_{n,\theta}^{-1}(\{0\}) \subset \Phi_\theta^{-1}((-1, 1)).
\]

In view of (4.6a),

\[
\tau_{n,\theta}(\{0\}) \subset \mathcal{P}.
\]

In particular,

\[
\tau_{n,\theta}^{-1}(\{0\}) \subset \mathbb{R}.
\]

All roots of the Chebyshev polynomial are simple. Since \(\Phi_\theta'(t) \neq 0 \ \forall \ t \in \mathcal{P}\), all roots of the trigonometric polynomial \(\tau_{n,\theta} = T_n \circ \Phi_\theta\) are simple as well.
2. The set of roots of the equation (4.10) is the set
\[ \tau_{n,\theta}^{[-1]}((-1) \cup (+1)) \subset \tau_{n,\theta}^{[-1]}((-1, 1)) = \mathcal{P} \subset \mathbb{R}. \]

We denote by \( \mathfrak{F} \) the class of all real entire functions \( f(z) \) having the property (F): all roots of the equation \( f^2(z) - 1 = 0 \) are real.

Functions of the class \( \mathfrak{F} \) arise in matters:

1. Stability theory of linear differential equations with periodic coefficients, [3];
2. Spectral theory of linear differential equations with periodic coefficients, [4].
3. Approximation theory, [5,6].

Functions belonging to the class \( \mathfrak{F} \) admit a description in terms of comb functions. A comb function is a function which effect a conformal mapping of the open upper half-plane onto a comb region. See [1].

The comb domain related to the function \( \tau_{n,\theta}(t) \) is shown in Fig. 1, where
\[ \cosh h = \frac{1}{\cos 2\theta}. \]

The function\(^2\)
\[ u_\theta(t) = i \ln \left( \frac{\cos t}{\cos 2\theta} + \sqrt{\frac{\cos t}{\cos 2\theta}}^2 - 1 \right) \]
effects the conformal mapping of the upper half-plane \( \{ t \in \mathbb{C} : \text{Im} \ t > 0 \} \) onto the comb domain shown in Fig. 1. The normalizing conditions are:
\[ u_\theta(0) = ih, \quad \lim_{t \to i\infty} t^{-1} u_\theta(t) = 1. \]

\(^2\) Here \( \ln z = 0 \) for \( z = 1 \) and \( \sqrt{z^2 - 1} > 0 \) for \( z \in (1, +\infty) \).
Figure 2 illustrates the boundary correspondence by the mapping \( t \rightarrow u_\theta(t) \).

The function \( \tau_{n,\theta}(t) \) is representable in the form

\[
\tau_{n,\theta}(t) = \cos n u_\theta(t).
\]

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