Loop Calculus and Belief Propagation for q-ary Alphabet: Loop Tower

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Outline

• Forney-style graphical model formulation for statistical inference problem
• Traces and graphical traces
• Gauge-invariant formulation of loop calculus
• Loop towers for q-ary alphabet
• Relation to the Bethe free energy approach
• Continuous and supersymmetric cases
• Homotopy approach to loop decomposition
**Statistical Inference**

| \( \sigma_{\text{orig}} \) | \( \Rightarrow \) | \( \chi \) | \( \Rightarrow \) | \( \sigma \) |
|--------------------------|-----------------|-------------|-----------------|--------|
| original data            |                 | noisy channel | corrupted data: | statistical inference |
| \( \sigma_{\text{orig}} \in \mathcal{C} \) |                 | \( \mathcal{P}(\chi|\sigma) \) | log-likelihood magnetic field | possible preimage \( \sigma \in \mathcal{C} \) |

\[
\sigma = (\sigma_1, \ldots, \sigma_N), \quad N \text{ finite, } \quad \sigma_i = \pm 1 (\text{example})
\]

**Maximum Likelihood**

| symbol Maximum-a-Posteriori |
|--------------------------------|

\[
\text{ML} = \arg \max_{\sigma} \mathcal{P}(\chi|\sigma) \quad \text{MAP}_{i} = \arg \max_{\sigma_i} \sum_{\sigma \backslash \sigma_i} \mathcal{P}(\chi|\sigma)
\]

Exhaustive search is generally expensive: complexity \( \sim 2^N \)
Forney-style graphical model formulation

\[ C_0 = (\mathcal{V}_0, \mathcal{E}_0) \]

\[ \mathcal{V}_0 = \{a\} \]

\[ \mathcal{E}_0 = \{(ab)\} \]

q-ary variables reside on edges

\[ \sigma_{ab} = \sigma_{ba} = 0, \ldots, (q - 1) \]

Forney '01; Loeliger '01

Probability of a configuration

\[ p(\sigma) = Z_{C_0}^{-1} \prod_a f_a(\sigma_a), \quad Z_{C_0} = \sum_\sigma \prod_a f_a(\sigma_a) \]

Partition function

Marginal probabilities

\[ p_a(\sigma_a) \equiv \sum_{\sigma \setminus \sigma_a} p(\sigma), \quad p_{ab}(\sigma_{ab}) \equiv \sum_{\sigma \setminus \sigma_{ab}} p(\sigma), \quad \sigma_a \equiv \{\sigma_{ab}|(ab) \in \mathcal{E}_0\} \]

Reduced variables

can be expressed in terms of the derivatives of the free energy with respect to factor-functions

\[ \mathcal{F}_{C_0} = -\ln Z_{C_0} \]
Loop calculus (binary alphabet)

Belief Propagation (BP) is exact on a tree

**Loop Series:**

**Exact (II) expression in terms of BP**

\[
Z = \sum_{\sigma} \prod_{a} f_a(\sigma_a) = Z_0 \left( 1 + \sum_{C} r(C) \right)
\]

\[
r(C) = \frac{\prod_{a \in C} \mu_a}{\prod_{(ab) \in C} (1 - m_{ab})} = \prod_{a \in C} \tilde{\mu}_a
\]

\(C \in \text{Generalized Loops} = \text{Loops without loose ends}\)

\[
m_{ab} = \int d\sigma_a b_a^{(bp)}(\sigma_a) \sigma_{ab}
\]

\[
\mu_a = \int d\sigma_a b_a^{(bp)}(\sigma_a) \prod_{b \in a, C} (\sigma_{ab} - m_{ab})
\]

- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.
Equivalent models: gauge fixing and transformations

**Replace the model with an equivalent more convenient model**

**Invariant approach**

(i) Introduce an invariant object that describes partition function $Z$
(ii) Different equivalent models correspond to different coordinate choices (gauge fixing)
(iii) Gauge transformations are changing the basis sets

**Coordinate approach**

(i) Introduce a set of gauge transformations that do not change $Z$
(ii) Gauge transformations build new equivalent models

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**General strategy (based on linear algebra)**

(i) Replace q-ary alphabet with a q-dimensional vector space
(ii) (letters are basis vectors)
(iii) Represent $Z$ by an invariant object **graphical trace**
(iii) Gauge fixing is a basis set choice
(iv) Gauge transformations are linear transformation of basis sets
Gauge invariance: matrix formulation

Gauge transformations of factor-functions

\[ f_a(\sigma_a = (\sigma_{ab}, \cdots)) \rightarrow \sum_{\sigma'_{ab}} G_{ab}(\sigma_{ab}, \sigma'_{ab}) f_a(\sigma'_{ab}, \cdots), \]

with orthogonality conditions

\[ \sum_{\sigma_{ab}} G_{ab}(\sigma_{ab}, \sigma') G_{ba}(\sigma_{ab}, \sigma'') = \delta(\sigma', \sigma''), \]

do not change the partition function

\[ Z_{C_0} = \sum_{\sigma} \prod_a \left( \sum_{\sigma'_a} f_a(\sigma'_a) \prod_{b \in a} G_{ab}(\sigma_{ab}, \sigma'_{ab}) \right) \]

\[ \equiv \sum_{\sigma} \bar{\rho}\{\hat{G}|\sigma\} \equiv \text{Tr} \left( \bar{\rho}\{\hat{G}|\sigma\} \right), \]
BP equations: matrix (coordinate) formulation

No-loose-ends condition

\[ \sigma_{ac_1} = 0 \]
\[ \sigma_{ac_{n-1}} = 0 \]

Results in BP equations

\[ \sum_{\sigma'_{a}} f_{a}(\sigma') G_{ab}^{(bp)}(\sigma_{ab} \neq 0, \sigma'_{ab}) \prod_{c \in a, c \neq b} G_{ac}^{(bp)}(0, \sigma'_{ac}) = 0, \]

with

\[ \rho_{a} = \sum_{\sigma'_{a}} f_{a}(\sigma') \prod_{c \in a} G_{ac}^{(bp)}(0, \sigma'_{ac}). \]
BP equations: standard form

A standard form of BP equations

\[
\frac{\exp \left( \eta^{(bp)}_{ab}(\sigma_{ab}) \right)}{\sum_{\sigma_{ab}} \exp \left( \eta^{(bp)}_{ab}(\sigma_{ab}) + \eta^{(bp)}_{ba}(\sigma_{ab}) \right)} = \frac{\sum_{\sigma_a} f_{a}(\sigma_{a}) \exp \left( \sum_{b \in a} \eta^{(bp)}_{ab}(\sigma_{ab}) \right)}{\sum_{\sigma_a} f_{a}(\sigma_{a}) \exp \left( \sum_{b \in a} \eta^{(bp)}_{ab}(\sigma_{ab}) \right)}
\]

is reproduced using the following representation for the ground state

\[
\epsilon_{ab} = G_{ab}(0, \sigma) = \frac{\exp (\eta_{ab}(\sigma))}{\sum_{\sigma} \exp (\eta_{ab}(\sigma) + \eta_{ba}(\sigma))}
\]

Side remark: relation to iterative BP
“Reduced Bethe free energy” (variational approach)

Reduced Bethe free energy \[ F_0(\varepsilon) = -\ln(Z_0(\varepsilon)) \]

\[ Z_0(\hat{\varepsilon}) \equiv \bar{p}\{G|0\} = \prod_a \rho_a(\varepsilon_a), \quad \text{with} \quad \varepsilon_{ab}(\sigma_{ab}) \equiv G_{ab}(0, \sigma_{ab}) \]

\[ \varepsilon_{ab} \cdot \varepsilon_{ba} = 1 \]

is an attempt to approximate the partition function \( Z \) in terms of the ground-state contribution in a proper gauge.

BP equations are recovered by the stationary point conditions \[ \frac{\partial F_0(\varepsilon)}{\partial \varepsilon_{ab}} = 0 \]

Not a standard variational scheme: corrections can be of either sign

What is the relation of the introduced functional to the Bethe free energy (Yedidia, Freeman, Weiss ‘01)?
Graphical representation of trace and cyclic trace

**Trace**

\[ Tr(f) = \sum f_j = \sum f_{ij} g^{ji} \]

**Cyclic trace**

\[ Tr(f^n) = \sum f_{j_1} f_{j_2} f_{j_3} \ldots f_{j_n} = \sum f_{i_1j_1} g^{ji_2} f_{i_2j_2} g^{ji_3} \ldots f_{i_nj_n} g^{ji_1} \]
Graphic trace and partition function

Collection of tensors (poly-vectors)

\[ f = \{ f^a \}_{a=1,\ldots,N} = \{ f_{a_1 \ldots a_n} \}_{a=1,\ldots,N} \]
\[ g = \{ g_{ab} \}_{a \in b} = \{ g^{i_a b} \}_{a \in b} \]

Trigonometric trace

\[ \text{Tr}(f) = \text{Tr}_g \left( \prod_a f^a \right) = \sum \left( \ldots f^a_{a_1 \ldots a_k \ldots a_n} \ldots f^b_{b_1 \ldots b_j \ldots b_{nb}} \ldots \right) \]

Scalar products

\[ u \cdot w = g_{ab} (u \otimes w) \]
\[ u \in W_{ab} \quad w \in W_{ba} \]

Orthogonality condition

\[ g^{ij} = g_{ab} (e^i_{ab} \otimes e^j_{ba}) = e^i_{ab} \cdot e^j_{ba} = \delta^{ij} \]

Tensors and factor-functions

\[ f^a = \sum_{\sigma_1 \ldots \sigma_n} f_a (\sigma_1, \ldots, \sigma_n) e^{\sigma_1}_{ab_1} \otimes \ldots \otimes e^{\sigma_n}_{ab_n} \]

\[ Z = \text{Tr}(f) \]
Partition function and graphic trace: gauge invariance

Dual basis set of co-vectors
(elements of the dual space)

$$\xi_{ab} \in W^*_{ab} \quad \xi_{ab,i}(e^j_{ab}) = \delta^j_i$$

Orthogonality condition (two equivalent forms)

$$\xi_{ab,i} \cdot \xi_{ba,j} = \delta_{ij} \quad g_{ab} = \sum_j \xi_{ab,j} \otimes \xi_{ba,j}$$

Graphic trace: Evaluate scalar products (reside on edges) on tensors (reside vertices)

$$Tr(f) = \sum_{\sigma \sigma'} f_a(\sigma_{ab}, \sigma_{ac}, \sigma_{ad}) f_b(\sigma_{ba}, \sigma_{bs}) ...$$

Gauge invariance: graphic trace is an invariant object, factor-functions are basis-set dependent

$$f_a(\sigma_1, \ldots, \sigma_n) = \xi_{ab_1, \sigma_1} \otimes \ldots \otimes \xi_{ab_n, \sigma_n} (f^a)$$

“Gauge fixing” is a choice of an orthogonal basis set
Belief propagation gauge and BP equations

Introduce local ground states $\epsilon_{ab} \in W^*_{ab}$ and excited (painted) states $u_{ab} \in W^*_{ab}$

$$\epsilon_{ab} \cdot \epsilon_{ba} = 1$$
$$u_{ab} \cdot \epsilon_{ba} = 0$$

BP gauge: painted structures with loose ends should be forbidden (in particular, no allowed painted structures in a tree case)

$$u_{ab} \otimes \epsilon_{ab_1} \otimes \cdots \otimes \epsilon_{ab_{n-1}} (f^a) = u_{ab} (\epsilon_{ab_1} \otimes \cdots \otimes \epsilon_{ab_{n-1}} (f^a)) = 0$$

or, stated differently, results in BP equations in invariant form:

$$g_{ab} (\epsilon_{ab_1} \otimes \cdots \otimes \epsilon_{ab_{n-1}} (f^a)) = \rho_{ab} \epsilon_{ba}$$

$$g_{ab} (\epsilon_{ab_1} \otimes \cdots \otimes \epsilon_{ab_{n-1}} (f^a)) \in W^*_{ba}$$
Loop decomposition: binary case

\[ Z_{C_0} = Z_{0; C_0} (1 + \sum_{C_1} r(C_1)), \quad r(C_1) = Z_{0; C_0}^{-1} \bar{p}(G | \sigma_{C_1}). \]

\[ Z_{0; C_0} \equiv \bar{p}(G | \sigma_0), \quad \sigma_0 \equiv \{ \sigma_{ab} = 0 \mid (ab) \in C_0 \}, \]

\[ \sigma_{C_1} \equiv \left\{ \begin{array}{ll} \sigma_{ab} = 1 & (ab) \in C_1 \\ \sigma_{ab} = 0 & (ab) \in C_0 \setminus C_1. \end{array} \right\}. \]

Beliefs (marginal probabilities)

\[ b_{ab}^{(bp)} (\sigma_{ab}) = G_{ab}^{(bp)} (0, \sigma_{ab}). \]

\[ b_a^{(bp)} (\sigma_a) = \frac{f_a (\sigma_a) \prod_{b \in a} G_{ab}^{(bp)} (0, \sigma_{ab})}{\sum_{\sigma_a} f_a (\sigma_a) \prod_{b \in a} G_{ab}^{(bp)} (0, \sigma_{ab})}. \]

\[ r(C_1) = \frac{\prod_{a \in C_1} \mu_a}{\prod_{(ab) \in C_1} (1 - m_{ab}^2)}, \quad m_{ab} \equiv \sum_{\sigma_{ab}} \sigma_{ab} b_{ab}^{(bp)} (\sigma_{ab}), \]

\[ \mu_a \equiv \sum_{\sigma_a} \left( \prod_{b \in a, C_1} (\sigma_{ab} - m_{ab}) \right) b_a^{(bp)} (\sigma_a). \]

A generalized loop visualizes a single-configuration contribution to the partition function in BP gauge.
Loop towers for q-ary alphabet: first step

A generalized loop defines a vertex model on the corresponding subgraph with \((q-1)\)-ary alphabet (first store above the ground store)

\[
Z_{C_0} = Z_{0;C_0} + \sum_{C_1 \in \Omega(C_0)} Z_{C_1}, \quad Z_{C_1} = \sum_{\sigma_{C_1}} \tilde{p}(G^{(bp)}_{C_1}|\sigma_{C_1})
\]

\(q>2\) (non-binary case): more than one local excited state

Partition function for the subgraph model

\[
Z_{C_1} = \sum_{\sigma_{C_1}} \prod_{a \in C_1} f_{1;a}(\sigma_{a;C_1}), \quad f_{1;a}(\sigma_{a;C_1}) = \\
= \sum_{\sigma'_a} f_a(\sigma'_a) \prod_{b \in a, C_0} G^{(bp)}_{ab;C_0}(\sigma_{ab}, \sigma'_{ab}) \prod_{b \in a, C_0} \delta(\sigma_{ab}, 0)
\]
Loop-tower expansion for q-ary alphabet

\[ j = 1, \ldots, q - 2 : \quad Z_{C_j} = Z_{C_0; C_j} + \sum_{C_{j+1} \in \Omega(C_j)} Z_{C_{j+1}}. \]

Building the next level (store)

\[ Z_{C_j} = \sum_{\sigma_{C_j}} \prod_{a \in C_j} f_{j;a}(\sigma_a; C_j), \]

\[ f_{j;a}(\sigma_a; C_j) = \sum_{\sigma'_{a; C_{j-1}}} f_{j-1;a}(\sigma'_{a; C_{j-1}}) \]

\[ \times \prod_{b \in a, C_{j-1}} G_{ab; C_{j-1}}^{(bp)}(\sigma_{ab}, \sigma'_{ab}) \prod_{b \notin C_j} \delta(\sigma_{ab}, j - 1). \]
Bethe free energy for q-ary alphabet

BP equations can be obtained as stationary points of the Bethe free energy functional of beliefs

\[ \Phi_{\text{Bethe}} = \sum_a \sum_{\sigma_a} b_a(\sigma_a) \ln \left( \frac{b_a(\sigma_a)}{f_a(\sigma_a)} \right) \]

\[ - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab}). \]

with natural constraints

\[ 0 \leq b_a(\sigma_a), b_{ac}(\sigma_{ac}) \leq 1, \]

\[ \sum_{\sigma_a} b_a(\sigma_a) = 1, \quad \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) = 1, \]

\[ b_{ac}(\sigma_{ac}) = \sum_{\sigma_a \setminus \sigma_{ac}} b_a(\sigma_a), \quad b_{ac}(\sigma_{ca}) = \sum_{\sigma_c \setminus \sigma_{ca}} b_c(\sigma_c). \]
Bethe effective Lagrangian

$$\mathcal{L}_{\text{Bethe}} = \sum_a \sum_{\sigma_a} b_a(\sigma_a) \ln \left( \frac{b_a(\sigma_a)}{f_a(\sigma_a)} \right) - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab})$$

$$+ \sum_{(ab)} \left( \sum_{\sigma_{ab}} \ln(\varepsilon_{ab}(\sigma_{ab})) \right) \left( b_{ab}(\sigma_{ab}) - \sum_{\sigma_a \setminus \sigma_{ab}} b_a(\sigma_a) \right)$$

$$+ \sum_{\sigma_{ba}} \ln(\varepsilon_{ba}(\sigma_{ba})) \left( b_{ab}(\sigma_{ba}) - \sum_{\sigma_b \setminus \sigma_{ba}} b_b(\sigma_b) \right)$$

Values of beliefs

$$b_a^{(*)}(\sigma_a) = (\varrho_a(\varepsilon_a))^{-1} f_a(\sigma_a) \prod_{b \in a} \varepsilon_{ab}(\sigma_{ab})$$

$$b_{ab}^{(*)}(\sigma_{ab}) = \varrho_{ab}^{-1}(\varepsilon_{ab}, \varepsilon_{ba}) \varepsilon_{ab}(\sigma_{ab}) \varepsilon_{ba}(\sigma_{ab}),$$

$$\varrho_a(\varepsilon_a) = \sum_{\sigma_a} f_a(\varepsilon_a) \prod_{c \in a} \varepsilon_{ac}(\sigma_{ac}),$$

$$\varrho_{ab}(\varepsilon_{ab}, \varepsilon_{ba}) = \sum_{\sigma_{ab}} \varepsilon_{ab}(\sigma_{ab}) \varepsilon_{ba}(\sigma_{ab}),$$

Variation of beliefs

$$\mathcal{F}_B(\hat{\varepsilon}) = - \sum_a \ln \varrho_a(\varepsilon_a) + \sum_{(ab)} \ln (\varrho_{ab}(\varepsilon_{ab}, \varepsilon_{ba}))$$
Relation to Bethe free energy

\( \mathcal{L}_{\text{Bethe}} = \sum_a \sum_{\sigma_a} b_a(\sigma_a) \ln \left( \frac{b_a(\sigma_a)}{f_a(\sigma_a)} \right) - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab}) \)

\[ + \sum_{(ab)} \left( \sum_{\sigma_{ab}} \ln(\varepsilon_{ab}(\sigma_{ab})) \left( b_{ab}(\sigma_{ab}) - \sum_{\sigma_n \setminus \sigma_{ab}} b_a(\sigma_a) \right) \right) \]

\[ + \sum_{\sigma_{ba}} \ln(\varepsilon_{ba}(\sigma_{ba})) \left( b_{ab}(\sigma_{ba}) - \sum_{\sigma_b \setminus \sigma_{ba}} b_b(\sigma_b) \right) \]

Variation of the ground state

\( \mathcal{F}_B(\hat{\varepsilon}) = - \sum_a \ln \varrho_a(\varepsilon_a) + \sum_{(ab)} \ln (\varrho_{ab}(\varepsilon_{ab}, \varepsilon_{ba})) \)

Variation of beliefs

\( \Phi_{\text{Bethe}} = \sum_a \sum_{\sigma_a} b_a(\sigma_a) \ln \left( \frac{b_a(\sigma_a)}{f_a(\sigma_a)} \right) \)

\[ - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab}(\sigma_{ab}) \ln b_{ab}(\sigma_{ab}). \]

Gauge fixing

\[ \sum_{\sigma_{ab}} \varepsilon_{ab}(\sigma_{ab}) \varepsilon_{ba}(\sigma_{ab}) = 1. \]

\( \mathcal{F}_0(\hat{\varepsilon}) \) Reduced Bethe free energy
Summary

- We have extended the loop expansion for general statistical inference problem to the case of general q-ary alphabet
- In the general case the loop decomposition goes over the loop towers
- We have formulated the statistical inference problem in terms of a graphical trace, which leads to the invariance of the partition function under a set of gauge transformations
- BP equations have been interpreted as a special choice of gauge
- The introduced Bethe effective Lagrangian establishes a connection between the gauge-invariant and Bethe free-energy approaches
- Generalization to the continuous and supersymmetric cases
Path forward: interplay of topological and geometrical equivalence

Topological structure: the graph

Use topologically equivalent models

e.g. Weitz '06

Geometrical structure: factor-functions

Use geometrically equivalent models

Combine

+ • improving BP
   • quantum version
   • etc
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Continuous and supersymmetric case: graphical sigma-models

Scalar product: the space of states and its dual are equivalent

\[ \varphi(\psi) = \int_M d\sigma \varphi(\sigma) \psi(\sigma) \]

No-loose-end requirement

\[ \psi_{ba}^{(0)} = \lambda_{ab} \int d\sigma_{ac} f_a(\sigma_a) \prod_{c \in a \atop c \neq b} \psi_{ac}^{(0)}(\sigma_{ac}) \]

Continuous version of BP equations

\[ \psi_{ba}^{(j-1)} = \lambda_{ab} P_{C,ab} \int \prod_{c \in a \atop c \neq b} d\sigma_{ac} f_a(\sigma_a) \prod_{c \in a \atop c \neq b} \psi_{ac}^{(j-1)}(\sigma_{ac}) \quad \text{and} \quad P_{C,ab} \psi_{ba}^{(j-1)} = 0 \]
Supersymmetric sigma-models: supermanifolds

$\mathbb{Z}_2$-graded manifolds (supermanifolds)

substrate (usual) manifold $\tilde{M} \subset M$

additional Grassman (anticommuting variables)

Functions on a supermanifold

$$\psi(\sigma) = \psi(x, \theta) = \psi^{(0)}(x) + \sum_{i_1} \psi^{(1)}_{i_1}(x) \theta_{i_1} + \sum_{i_1 < i_2} \psi^{(2)}_{i_1 i_2}(x) \theta_{i_1} \theta_{i_2} + \ldots$$

Berezin integral (measure in a supermanifold)

$$\int d\theta_i = 0; \quad \int \theta_i d\theta_i = 1; \quad d\theta_i d\theta_j = -d\theta_j d\theta_i; \quad d\theta_i \theta_j = -\theta_j d\theta_i$$

Any function on a supermanifold can be represented as a sum of its even and odd components
Supersymmetric sigma models: graphic supertrace I

Natural assumption: factor-functions are even functions on

\[ M_a = \prod_{b \in a} M_{ab} \]

Introduce parities of the beliefs

\[ p_{ab} = 0, 1 \]

BP equations for parities

\[ (\psi^{(0)}_{ab}, \psi^{(0)}_{ba}) \neq 0 \]

Follows from the first two

\[ p_{ba} = \sum_{c \neq b} p_{ac}; \quad p_{ba} = p_{ab}; \quad \sum_{c \in a} p_{ac} = 0 \]

Edge parity is well-defined

\[ p \in H_1(X; \mathbb{Z}_2) \]

\[ 2^{B_1} \] elements

\[ H_0(X; \mathbb{Z}_2) = \mathbb{Z}_2 \]

\[ B_0 = 1 \]

(number of connected components)

Euler characteristic

\[ B_1 - B_0 = \text{card}(E) - \text{card}(V) \]
Supersymmetric sigma models: graphic supertrace II

Decompose the vector spaces

\[ \mathcal{W}_{ab} = \mathcal{W}_{ab}^{(+)} \oplus \mathcal{W}_{ab}^{(-)} \]

into reduced vector spaces

\[ \mathcal{W}_{ab}^{(\alpha_{ab})} \text{ with } \alpha_{ab} = (-1)^{p_{ab}} \]

Graphic supertrace decomposition (generalizes the supertrace)

\[ Z = \sum_{p \in H_1(X; \mathbb{Z}_2)} Z_p; \quad \text{Tr}\mathcal{F} = \sum_{p \in H_1(X; \mathbb{Z}_2)} \text{Tr}\mathcal{F}_p \]

results in a multi-reference loop expansion

\[ \mathcal{F}_p \] is the graphic trace (partition function) of a reduced model
Homotopy approach to loop decomposition

Both models are equivalent

Loop calculus for the bouquet model (independent loops) constitutes a resummation for the original model (generalized loops)