THE GORDIAN DISTANCE OF HANDLEBODY-KNOTS AND ALEXANDER BIQUANDLE COLORINGS

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ABSTRACT. We give lower bounds for the Gordian distance and the unknotting number of handlebody-knots by using Alexander biquandle colorings. We construct handlebody-knots with Gordian distance $n$ and unknotting number $n$ for any positive integer $n$.

1. Introduction

The Gordian distance of two classical knots is the minimal number of crossing changes needed to be deformed each other. In particular, we call the Gordian distance of a classical knot and the trivial one the unknotting number of the classical knot. Clark, Elhamdadi, Saito and Yeatman [2] gave a lower bound for the Nakanishi index [18], which induced a lower bound for the unknotting number of classical knots. This is an generalization of the Przytycki’s result [19]. In this paper, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots, which is a generalization of a classical knot with respect to a genus.

Ishii [5] introduced an enhanced constituent link of a spatial trivalent graph, and Ishii and Iwakiri [8] introduced an $A$-flow of a spatial graph, where $A$ is an abelian group, to define colorings and invariants of handlebody-knots. Iwakiri [14] gave a lower bound for the unknotting number of handlebody-knots by using Alexander quandle colorings of its $Z_2$ or $Z_3$-flowed diagram. Ishii, Iwakiri, Jang and Oshiro [9] introduced a $G$-family of quandles, which is an extension of the above structures. Recently, Ishii and Nelson [13] introduced a $G$-family of biquandles, which is a biquandle version of a $G$-family of quandles.

In this paper, we extend the result in [14] in three directions. First, we extend from $Z_2$, $Z_3$-flows to any $Z_m$-flow. Second, we extend from quandles to biquandles. Finally, we extend from unknotting numbers to Gordian distances. Thus we can determine the Gordian distance and the unknotting number of handlebody-knots more efficiently. We construct handlebody-knots with Gordian distance $n$ and unknotting number $n$ for any $n \in Z_{>0}$ and note that one of them cannot be obtained by using Alexander quandle colorings introduced in [14].

This paper is organized into seven sections. In Section 2, we recall the definition of a handlebody-knot and introduce the Gordian distance and the unknotting number of handlebody-knots. In Section 3, we recall the definition of a (bi)quandle and a $G$-family of (bi)quandles. In Section 4, we introduce a coloring of a diagram of a handlebody-knot by using a $G$-family of biquandles. In Section 5, we show that there are linear relationships for Alexander biquandle colorings of a diagram of a handlebody-knot. In Section 6, we give lower bounds for the Gordian distance and the unknotting number of handlebody-knots by using $Z_m$-family of Alexander biquandles colorings. In section 7, we construct handlebody-knots with Gordian distance $n$ and

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unknotting number \( n \) for any \( n \in \mathbb{Z}_{>0} \). Moreover, we note that one of them can not be obtained by using Alexander quandle colorings with \( \mathbb{Z}_2, \mathbb{Z}_3 \)-flows introduced in [14].

2. The Gordian distance of handlebody-knots

A **handlebody-link**, which is introduced in [5], is the disjoint union of handlebodies embedded in the 3-sphere \( S^3 \). A **handlebody-knot** is a handlebody-link with one component. In this paper, we assume that every component of a handlebody-link is of genus at least 1. An \( S^1 \)-**orientation** of a handlebody-link is an orientation of all genus 1 components of the handlebody-link, where an orientation of a solid torus is an orientation of its core \( S^1 \). Two \( S^1 \)-oriented handlebody-links are **equivalent** if there exists an orientation-preserving self-homeomorphism of \( S^3 \) sending one to the other preserving the \( S^1 \)-orientation.

A **spatial trivalent graph** is a graph whose vertices are valency 3 embedded in \( S^3 \). In this paper, a trivalent graph may have a circle component, which has no vertices. A **Y-orientation** of a spatial trivalent graph is a direction of all edges of the graph satisfying that every vertex of the graph is both the initial vertex of a directed edge and the terminal vertex of a directed edge (Figure 1). A vertex of a Y-oriented spatial trivalent graph can be allocated a sign; the vertex is said to be positive or negative, or to have sign \(+1\) or \(-1\). The standard convention is shown in Figure 1. For a Y-oriented spatial trivalent graph \( K \) and an \( S^1 \)-oriented handlebody-link \( H \), we say that \( K \) represents \( H \) if \( H \) is a regular neighborhood of \( K \) and the \( S^1 \)-orientation of \( H \) agrees with the Y-orientation. Then any \( S^1 \)-oriented handlebody-link can be represented by some Y-oriented spatial trivalent graph. We define a **diagram** of an \( S^1 \)-oriented handlebody-link by a diagram of a Y-oriented spatial trivalent graph representing the handlebody-link. An \( S^1 \)-oriented handlebody-link is **trivial** if it has a diagram with no crossings. Then the following theorem holds.

![Figure 1. Y-orientations and signs.](image)

**Theorem 2.1** ([7]). For a diagram \( D_i \) of an \( S^1 \)-oriented handlebody-link \( H_i \) \((i = 1, 2)\), \( H_1 \) and \( H_2 \) are equivalent if and only if \( D_1 \) and \( D_2 \) are related by a finite sequence of R1–R6 moves depicted in Figure 2 preserving Y-orientations.

In this paper, for a diagram \( D \) of an \( S^1 \)-oriented handlebody-link, we denote by \( \mathcal{A}(D) \) and \( \mathcal{S} \mathcal{A}(D) \) the set of all arcs of \( D \) and the one of all semi-arcs of \( D \) respectively, where a semi-arc is a piece of a curve each of whose endpoints is a crossing or a vertex. An orientation of a (semi-)arc of \( D \) is also represented by the normal orientation obtained by rotating the usual orientation counterclockwise by \( \pi/2 \) on the diagram. For any \( m \in \mathbb{Z}_{\geq 0} \), we put \( \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} \).

A **crossing change** of an \( S^1 \)-oriented handlebody-link \( H \) is that of a spatial trivalent graph representing \( H \). This deformation can be realized by switching two handles depicted in Figure 3. It is easy to see that any two \( S^1 \)-oriented handlebody-knots of the same genus can be related by a finite sequence of crossing changes. For any two \( S^1 \)-oriented handlebody-knots \( H_1 \) and \( H_2 \) of
Figure 2. The Reidemeister moves for handlebody-links.

the same genus, we define their Gordian distance $d(H_1, H_2)$ by the minimal number of crossing changes needed to be deformed each other. In particular, for any $S^1$-oriented handlebody-knot $H$ and the $S^1$-oriented trivial handlebody-knot $O$ of the same genus, we define $u(H) := d(H, O)$, which is called the unknotting number of $H$.

Figure 3. A crossing change of an $S^1$-oriented handlebody-link.

3. A Biquandle and a $G$-Family of Biquandles

We recall the definitions of a quandle and a biquandle.

**Definition 3.1 (15, 16).** A quandle is a non-empty set $X$ with a binary operation $*: X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x * x = x$.
- For any $x \in X$, the map $S_x : X \to X$ defined by $S_x(y) = y * x$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

**Definition 3.2 (3).** A biquandle is a non-empty set $X$ with binary operations $\bar{\star}, \bar{\circ}, : X \times X \to X$ satisfying the following axioms.

- For any $x \in X$, $x \bar{\star} x = x \bar{\circ} x$.
- For any $x \in X$, the map $\bar{S}_x : X \to X$ defined by $\bar{S}_x(y) = y \bar{\star} x$ is a bijection.
- For any $x \in X$, the map $\bar{S}_x : X \to X$ defined by $\bar{S}_x(y) = y \bar{\circ} x$ is a bijection.
- The map $S : X \times X \to X \times X$ defined by $S(x,y) = (y \bar{\star} x, x \bar{\circ} y)$ is a bijection.
- For any $x, y, z \in X$,
  - $(x \bar{\star} y) \bar{\star} (z \bar{\star} y) = (x \bar{\star} z) \bar{\star} (y \bar{\star} z)$,
  - $(x \bar{\circ} y) \bar{\circ} (z \bar{\circ} y) = (x \bar{\circ} z) \bar{\circ} (y \bar{\circ} z)$,
  - $(x \bar{\star} y) \bar{\circ} (z \bar{\circ} y) = (x \bar{\circ} z) \bar{\star} (y \bar{\circ} z)$.
We define $x^n := \mathbb{Z}_x^n$ and $\bar{x}^n := \mathbb{Z}_x^{-n}$ for any $n \in \mathbb{Z}$. We note that $(X, \ast)$ is a quandle if and only if $(X, \ast, \bar{\ast})$ is a biquandle with $x \bar{\ast} y = x$. For any $m \in \mathbb{Z}_{\geq 0}$, a $\mathbb{Z}_m[s^\pm 1, t^\pm 1]$-module $X$ is a biquandle with $a \ast b = ta + (s-t)b$ and $a \bar{\ast} b = sa$, which we call an Alexander biquandle. When $s = 1$, an Alexander biquandle coincides with an Alexander quandle.

**Definition 3.3** ([13]). Let $X$ be a biquandle. We define two families of binary operations $\ast^n, \bar{\ast}^n : X \times X \rightarrow X(n \in \mathbb{Z})$ by the equalities

\[
\begin{align*}
a \ast^n b &= a, \quad a \ast^1 b = a \ast b, \quad a \ast^{i+j} b = (a \ast^i b) \ast^j (b \ast^i b), \\
\bar{x}^n b &= a, \quad a \bar{\ast}^1 b = a \bar{\ast} b, \quad a \bar{\ast}^{i+j} b = (a \bar{\ast}^i b) \bar{\ast}^j (b \bar{\ast}^i b)
\end{align*}
\]

for any $i, j \in \mathbb{Z}$.

Since $a = a \ast^{[0]} b = (a \ast^{-1} b) \ast^{[1]} (b \ast^{-1} b) = (a \ast^{-1} b) \ast^{-1} (b \ast^{-1} b)$, we have $a \ast^{-1} b = a \ast^{-1} (b \ast^{-1} b)$ and $(b \ast^{-1} b) \ast (b \ast^{-1} b) = b$. Then for an Alexander biquandle $X$, we have $a \ast^n b = t^na + (s^n - t^n)b$ and $a \bar{\ast}^n b = s^n a$ for any $a, b \in X$.

We define the type of a biquandle $X$ by

\[\text{type} X = \min \{ n > 0 \mid a \ast^n b = a = a \bar{\ast}^n b \ (\forall a, b \in X) \}.\]

Any finite biquandle is of finite type [13].

We also recall the definitions of a $G$-family of quandles and a $G$-family of biquandles.

**Definition 3.4** ([9]). Let $G$ be a group with the identity element $e$. A $G$-family of quandles is a non-empty set $X$ with a family of binary operations $\ast^g : X \times X \rightarrow X (g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$, $x \ast^g x = x$.
- For any $x, y \in X$ and $g, h \in G$, $x \ast^g y = (x \ast^g y) \ast^h y$ and $x \ast^e y = x$.
- For any $x, y, z \in X$ and $g, h \in G$, $(x \ast^g y) \ast^h z = (x \ast^g y) \ast^h z \ast^g (y \ast^h z)$.

**Definition 3.5** ([10] [13]). Let $G$ be a group with the identity element $e$. A $G$-family of biquandles is a non-empty set $X$ with two families of binary operations $\ast^g, \bar{\ast}^g : X \times X \rightarrow X (g \in G)$ satisfying the following axioms.

- For any $x \in X$ and $g \in G$,
  \[x \ast^g x = x \ast^g x.\]

- For any $x, y \in X$ and $g, h \in G$,
  \[x \ast^g y = (x \ast^g y) \ast^h (y \ast^g y), \quad x \ast^e y = x, \quad x \bar{\ast}^g y = (x \bar{\ast}^g y) \bar{\ast}^h (y \bar{\ast}^g y), \quad x \bar{\ast}^e y = x.\]

- For any $x, y, z \in X$ and $g, h \in G$,
  \[(x \ast^g y) \ast^h (z \bar{\ast}^g y) = (x \ast^h z) \ast^{-1} g (y \ast^h z), \quad (x \bar{\ast}^g y) \bar{\ast}^h (z \bar{\ast}^g y) = (x \bar{\ast}^h z) \bar{\ast}^{-1} g (y \bar{\ast}^h z), \quad (x \bar{\ast}^g y) \bar{\ast}^h (z \bar{\ast}^g y) = (x \bar{\ast}^h z) \bar{\ast}^{-1} g (y \bar{\ast}^h z).\]

For a biquandle $(X, \ast, \bar{\ast})$ with type $X < \infty$, $(X, \{\ast^n\}_{n \in \mathbb{Z}_{\text{type} X}}, \{\bar{\ast}^n\}_{n \in \mathbb{Z}_{\text{type} X}})$ is a $\mathbb{Z}_{\text{type} X}$-family of biquandles [13]. In particular, when $X$ is an Alexander biquandle, $(X, \{\ast^n\}_{n \in \mathbb{Z}_{\text{type} X}}, \{\bar{\ast}^n\}_{n \in \mathbb{Z}_{\text{type} X}})$ is called a $\mathbb{Z}_{\text{type} X}$-family of Alexander biquandles.
4. Colorings

In this section, we introduce a coloring of a diagram of an $S^1$-oriented handlebody-link by a $G$-family of biquandles. Let $G$ be a group and let $D$ be a diagram of an $S^1$-oriented handlebody-link $H$. A $G$-flow of $D$ is a map $\phi: A(D) \to G$ satisfying

\[
\begin{array}{c}
a \quad b \\
\downarrow & \downarrow \\
ab & b^{-1}ab
\end{array}
\]

at each crossing and each vertex. In this paper, to avoid confusion, we often represent an element of $G$ with an underline. We denote by $(D, \phi)$, which is called a $G$-flowed diagram of $H$, a diagram $D$ given a $G$-flow $\phi$ and put $\text{Flow}(D; G) := \{ \phi \mid \phi : G$-flow of $D \}$. We can identify a $G$-flow $\phi$ with a homomorphism from the fundamental group $\pi_1(S^3 - H)$ to $G$.

Let $G$ be a group and let $D$ be a diagram of an $S^1$-oriented handlebody-link $H$. Let $D'$ be a diagram obtained by applying one of Reidemeister moves to the diagram $D$ once. For any $G$-flow $\phi$ of $D$, there is an unique $G$-flow $\phi'$ of $D'$ which coincides with $\phi$ except near the point where the move applied. Therefore the number of $G$-flow of $D$, denoted by $\#\text{Flow}(D; G)$, is an invariant of $H$. We call the $G$-flow $\phi'$ the associated $G$-flow of $\phi$ and the $G$-flowed diagram $(D', \phi')$ the associated $G$-flowed diagram of $(D, \phi)$.

For any $m \in \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_m$-flow $\phi$ of a diagram $D$ of an $S^1$-oriented handlebody-link $H$, we define $\gcd \phi := \gcd \{ \phi(a), m \mid a \in A(D) \}$. Then we have the following lemma in the same way as in [11].

**Lemma 4.1.** For any $m \in \mathbb{Z}_{\geq 0}$, let $(D, \phi)$ be a $\mathbb{Z}_m$-flowed diagram of an $S^1$-oriented handlebody-link $H$ and let $(D', \phi')$ be the associated $\mathbb{Z}_m$-flowed diagram of $(D, \phi)$. Then it follows that $\gcd \phi = \gcd \phi'$.

Let $G$ be a group, $X$ be a $G$-family of biquandles and let $(D, \phi)$ be a $G$-flowed diagram of an $S^1$-oriented handlebody-link $H$. An $X$-coloring of $(D, \phi)$ is a map $C: SA(D, \phi) \to X$ satisfying

\[
\begin{array}{c}
x \quad \bar{y}^2 \quad x \\
y \quad \bar{x} \quad x \\
x \quad \bar{y} \quad x
\end{array}
\]

at each crossing and each vertex, where $SA(D, \phi)$ is the set of all semi-arcs of $(D, \phi)$. We denote by $\text{Col}_X(D, \phi)$ the set of all $X$-colorings of $(D, \phi)$. We note that $\text{Col}_X(D, \phi)$ is a vector space over $X$ when $X$ is a field.

**Proposition 4.2** ([13]). Let $X$ be a $G$-family of biquandles and let $(D, \phi)$ be a $G$-flowed diagram of an $S^1$-oriented handlebody-link $H$. Let $(D', \phi')$ be the associated $G$-flowed diagram of $(D, \phi)$. For any $X$-coloring $C$ of $(D, \phi)$, there is an unique $X$-coloring $C'$ of $(D', \phi')$ which coincides with $C$ except near the point where the move applied.

We call the $X$-coloring $C'$ the associated $X$-coloring of $C$. By this proposition, we have $\#\text{Col}_X(D, \phi) = \#\text{Col}_X(D', \phi')$. 

Proposition 4.3. Let $G$ be a group and let $X$ be a $G$-family of biquandles. Then the following hold.

(1) Let $(D, \phi)$ be a $G$-flowed diagram of an $S^1$-oriented handlebody-link. Then it follows that $\# \text{Col}_X(D, \phi) \geq \#X$.

(2) Let $(O, \psi)$ be a $G$-flowed diagram of an $S^1$-oriented $m$-component trivial handlebody-link. Then it follows that $\# \text{Col}_X(O, \psi) = (\#X)^m$.

Proof. (1) By Theorem 2.1 and [17], we can deform $(D, \phi)$ into the $G$-flowed diagram $(D', \phi')$ depicted in Figure 4 by a finite sequence of Reidemeister moves preserving Y-orientations, where $b$ is a classical $l$-braid, and $a_{i,1}, a_{i,m_1}, b_{i,1}, \ldots, b_{i,n_i} \in G$ for any $i = 1, \ldots, s$. We note that $\prod_{j=1}^{m_i} a_{i,j} = \prod_{j=1}^{n_i} b_{i,j}$ for any $i = 1, \ldots, s$, and $x \text{ or } x \text{ } \overline{\text{or}} x \text{ for any } x \in X$ and $g \in G$. By Proposition 4.2, it is sufficient to prove that $\# \text{Col}_X(D', \phi') \geq \#X$. Here for any $x \in X$ and $g \in G$, we write $x \text{ or } x \text{ } \overline{\text{or}} x \text{ simply. Then for any } x \in X$, the assignment of elements of $X$ to each semi-arc of $(D', \phi')$ as shown in Figures 4 and 5 is an $X$-coloring, where each $g_i$ represents an element of $G$ in Figure 5. Therefore we have $\# \text{Col}_X(D', \phi') \geq \#X$.

Figure 4. A $G$-flowed diagram $(D', \phi')$ and its X-coloring.

(2) It is sufficient to prove that $\# \text{Col}_X(O, \psi) = \#X$ when $m = 1$. Let $(O_g, \psi_g)$ be a $G$-flowed diagram of an $S^1$-oriented trivial handlebody-knot of genus $g$. By Theorem 2.1, we can deform $(O_g, \psi_g)$ into the $G$-flowed diagram $(O'_g, \psi'_g)$ depicted in Figure 6 by a finite sequence of Reidemeister moves preserving Y-orientations, where $a_i \in G$ for any $i = 1, \ldots, g$, and $e$ is the identity of $G$. By Proposition 4.2, it is sufficient to prove that $\# \text{Col}_X(O'_g, \psi'_g) = \#X$. For any $x \in X$, the assignment of $x$ to each semi-arc of $(O'_g, \psi'_g)$ as shown in Figure 6 is an $X$-coloring. On the other hand, since any $X$-coloring of $(O'_g, \psi'_g)$ is given by Figure 6 for some $x \in X$, we have $\# \text{Col}_X(O'_g, \psi'_g) = \#X$.

□

5. Linear relationships for Alexander biquandle colorings

For any $Z_m$-flowed diagram $(D, \phi)$ of an $S^1$-oriented handlebody-link, we define the Alexander numbering of $(D, \phi)$ by assigning elements of $Z_m$ to each region of $(D, \phi)$ as shown in Figure
where the unbounded region is labeled 0. It is an extension of the Alexander numbering of a classical knot diagram \([1]\). It is easy to see that for any \(\mathbb{Z}_m\)-flowed diagram \((D, \phi)\) of an \(S^1\)-oriented handlebody-link, there uniquely exists the Alexander numbering of \((D, \phi)\). For example, a \(\mathbb{Z}_m\)-flowed diagram of the handlebody-knot \(5_2\) \([12]\) with the Alexander numbering is depicted in Figure 8. For any semi-arc \(\alpha\) of \((D, \phi)\), we denote by \(\rho(\alpha)\) the Alexander number of the region which the normal orientation of \(\alpha\) points to.

In the following, every component of a diagram of any \(S^1\)-oriented handlebody-link may have a crossing at least 1. Let \((D, \phi)\) be a \(\mathbb{Z}_m\)-flowed diagram of an \(S^1\)-oriented handlebody-link with the Alexander numbering and let \(X\) be a \(\mathbb{Z}_m\)-family of Alexander biquandles. We put \(C(D, \phi) = \{c_1, \ldots, c_n\}\) and \(V(D, \phi) = \{\tau_1, \ldots, \tau_{2k}\}\), where \(C(D, \phi)\) and \(V(D, \phi)\) are the set of all crossings of \((D, \phi)\) and the one of all vertices of \((D, \phi)\) respectively, where the sign of \(\tau_i\) is 1.
for any $i = 1, \ldots, k$ and $-1$ for any $i = k + 1, \ldots, 2k$. Then we denote by $x_i$ each semi-arc of $(D, \phi)$ as shown in Figure 9 which implies $SA(D, \phi) = \{x_1, \ldots, x_{2n+3k}\}$.

We denote by $u_i$, $v_i$, $v'_i$, $w_i$, $\alpha_i$, $\beta_i$ and $\gamma_i$ the semi-arcs incident to a crossing $c_i$ or a vertex $\tau_i$ as shown in Figure 10. We put $\phi_i := \phi(u_i) = \phi(w_i)$, $\psi_i := \phi(v_i) = \phi(v'_i)$, $\eta_i := \phi(\alpha_i)$ and $\theta_i := \phi(\beta_i)$. We denote by $\epsilon_{c_i} \in \{\pm 1\}$ and $\epsilon_{\tau_i} \in \{\pm 1\}$ the signs of a crossing $c_i$ and a vertex $\tau_i$ respectively (see Figure 10).

For any semi-arcs $y, y' \in SA(D, \phi)$, we put

$$\delta(y, y') := \begin{cases} 1 & (y = y'), \\ 0 & (y \neq y'). \end{cases}$$
Then we define a matrix

\[ A(D, \phi; X) = (a_{i, j}) \in M(2n + 4k, 2n + 3k; X) \]

by

\[
\begin{align*}
& a_{i, j} =
\begin{cases}
\delta(u_i, x_j)t^{\psi_i} + \delta(v_i, x_j)(s^{\psi_i} - t^{\psi_i}) - \delta(w_i, x_j) & (1 \leq i \leq n), \\
-\delta(v_{i-n}, x_j)s^{\phi_{i-n}} + \delta(v'_{i-n}, x_j) & (n + 1 \leq i \leq 2n), \\
\delta(\alpha_{i-2n}, x_j) - \delta(\gamma_{i-2n}, x_j) & (2n + 1 \leq i \leq 2n + 2k), \\
\delta(\beta_{i-2n-2k}, x_j) - \delta(\gamma_{i-2n-2k}, x_j)s^{n-2n-2k} & (2n + 2k + 1 \leq i \leq 2n + 4k).
\end{cases}
\end{align*}
\]

We note that \( A(D, \phi; X) \) is determined up to permuting of rows and columns of the matrix, and it follows that

\[ \text{Col}_X(D, \phi) = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n+3k} \end{bmatrix} \in X^{2n+3k} \mid A(D, \phi; X) \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n+3k} \end{bmatrix} = 0 \right\}. \]

For example, let \((E, \psi)\) be the \(\mathbb{Z}_m\)-flowed diagram of the handlebody-knot depicted in Figure 11. Then we have

\[ A(E, \psi; X) = \begin{bmatrix}
-1 & 0 & s^a - t^a & t^a & 0 & 0 & 0 \\
0 & -1 & 0 & s^b - t^b & 0 & t^b & 0 \\
0 & 1 & -s^b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -s^a & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & -s^a & 0 & 0 \\
0 & 0 & 0 & 0 & -s^a & 0 & 1
\end{bmatrix}. \]

\[ (E, \psi) \]

\[ \text{Figure 11. A } \mathbb{Z}_m\text{-flowed diagram } (E, \psi). \]

Then we have the following proposition.

**Proposition 5.1.** Let \((D, \phi)\) be a \(\mathbb{Z}_m\)-flowed diagram of an \(S^1\)-oriented handlebody-link with the Alexander numbering and let \(X\) be a \(\mathbb{Z}_m\)-family of Alexander biquandles. Let \(a_i\) be the \(i\)-th...
row of \(A(D, \phi; X)\), that is,
\[
A(D, \phi; X) = (a_{i,j}) = \begin{pmatrix}
a_1 \\
a_2 \\ \\
\vdots \\
a_{2n+4k}
\end{pmatrix}.
\]

Then it follows that
\[
\sum_{i=1}^{n} \epsilon_i t^{-\rho(u_i)}(s^{\phi_i} - t^{\phi_i})a_i + \sum_{i=1}^{n} \epsilon_i t^{-\rho(v_i)}(s^{\psi_i} - t^{\psi_i})a_{n+i} \\
+ \sum_{i=1}^{2k} \epsilon_{c_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})a_{2n+i} + \sum_{i=1}^{2k} \epsilon_{\tau_i} t^{-\rho(\lambda_i)}(s^{\theta_i} - t^{\theta_i})a_{2n+2k+i} = 0.
\]

**Proof.** For any semi-arc \(y\) incident to a crossing or a vertex \(\sigma\), we put
\[
\epsilon(y; \sigma) := \begin{cases} 
1 & \text{if the orientation of } y \text{ points to } \sigma, \\
-1 & \text{otherwise}.
\end{cases}
\]

It is sufficient to prove that for any \(j = 1, 2, \ldots, 2n + 3k,\)
\[
\sum_{i=1}^{n} \epsilon_i t^{-\rho(u_i)}(s^{\phi_i} - t^{\phi_i})a_{i,j} + \sum_{i=1}^{n} \epsilon_i t^{-\rho(v_i)}(s^{\psi_i} - t^{\psi_i})a_{n+i,j} \\
+ \sum_{i=1}^{2k} \epsilon_{c_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})a_{2n+i,j} + \sum_{i=1}^{2k} \epsilon_{\tau_i} t^{-\rho(\lambda_i)}(s^{\theta_i} - t^{\theta_i})a_{2n+2k+i,j} = 0.
\]

For the first term, we have
\[
\epsilon_{c_i} t^{-\rho(u_i)}(s^{\phi_i} - t^{\phi_i})\delta(u_i, x_j)t^{\psi_i} = \delta(u_i, x_j)\epsilon(u_i; c_i)t^{-\rho(u_i)}(s^{\phi(u_i)} - t^{\phi(u_i)}),
\]
\[
= \epsilon_{c_i} t^{-\rho(u_i)}(s^{\phi_i} - t^{\phi_i})\delta(v_i, x_j)(s^{\psi_i} - t^{\psi_i}) = \epsilon_{c_i} t^{-\rho(u_i)}(s^{\phi_i} - t^{\phi_i})\delta(v_i, x_j)(s^{\phi(u_i)} - t^{\phi(u_i)}).
\]

For the second term, we have
\[
\epsilon_{c_i} t^{-\rho(v_i)}(s^{\psi_i} - t^{\psi_i})\delta(v_i, x_j)t^{\phi_i} = \delta(v_i, x_j)\epsilon(v_i; c_i)t^{-\rho(v_i)}(s^{\psi(v_i)} - t^{\psi(v_i)}),
\]
\[
= \epsilon_{c_i} t^{-\rho(v_i)}(s^{\psi_i} - t^{\psi_i})\delta(v_i, x_j)(s^{\phi_i} - t^{\phi_i}) = \delta(v_i, x_j)\epsilon(v_i; c_i)t^{-\rho(v_i)}(s^{\phi(v_i)} - t^{\phi(v_i)}).
\]

For the third term, we have
\[
\epsilon_{\tau_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})\delta(\alpha_i, x_j) = \delta(\alpha_i, x_j)\epsilon(\alpha_i; \tau_i)t^{-\rho(\alpha_i)}(s^{\phi(\alpha_i)} - t^{\phi(\alpha_i)}),
\]
\[
= \epsilon_{\tau_i} t^{-\rho(\alpha_i)}(s^{\eta_i} - t^{\eta_i})\delta(\gamma_i, x_j) = \delta(\gamma_i, x_j)\epsilon(\gamma_i; \tau_i)t^{-\rho(\gamma_i)}(s^{\phi(\gamma_i)} - t^{\phi(\gamma_i)}).
\]

For the last term, we have
\[
\epsilon_{\tau_i} t^{-\rho(\lambda_i)}(s^{\theta_i} - t^{\theta_i})\delta(\lambda_i, x_j) = \delta(\lambda_i, x_j)\epsilon(\lambda_i; \tau_i)t^{-\rho(\lambda_i)}(s^{\phi(\lambda_i)} - t^{\phi(\lambda_i)}),
\]
\[
= \epsilon_{\tau_i} t^{-\rho(\lambda_i)}(s^{\theta_i} - t^{\theta_i})\delta(\lambda_i, x_j) = \delta(\lambda_i, x_j)\epsilon(\lambda_i; \tau_i)t^{-\rho(\lambda_i)}(s^{\phi(\lambda_i)} - t^{\phi(\lambda_i)}).
\]
We note that

\[(1) + (2) = \delta(v_i, x_j) \epsilon(v_i; c_i) t^{-\rho(v_i)} (s^{\phi_i}(v_i) - t^{\phi_i}(v_i)),\]

\[(3) + (4) = \delta(\gamma_i, x_j) \epsilon(\gamma_i; \tau_i) t^{-\rho(\gamma_i)} (s^{\phi_i}(\gamma_i) - t^{\phi_i}(\gamma_i)).\]

Therefore for any \(j = 1, 2, \ldots, 2n + 3k\), it follows that

\[
\sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w_i)} (s^{\phi_i} - t^{\phi_i}) a_{i,j} + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(v_i)} (s^{\phi_i} - t^{\phi_i}) a_{n+i,j} + \sum_{i=1}^{2k} \epsilon_{c_i} t^{-\rho(\alpha_i)} (s^{\phi_i} - t^{\phi_i}) a_{2n+i,j} + \sum_{i=1}^{2k} \epsilon_{c_i} t^{-\rho(\beta_i)} (s^{\phi_i} - t^{\phi_i}) a_{2n+2k+i,j}
\]

\[
= \sum_{i=1}^{n} \delta(u_i, x_j) \epsilon(u_i; c_i) t^{-\rho(u_i)} (s^{\phi_i}(u_i) - t^{\phi_i}(u_i))
+ \delta(v_i, x_j) \epsilon(v_i; c_i) t^{-\rho(v_i)} (s^{\phi_i}(v_i) - t^{\phi_i}(v_i))
+ \delta(v_i', x_j) \epsilon(v_i'; c_i) t^{-\rho(v_i')} (s^{\phi_i}(v_i') - t^{\phi_i}(v_i'))
+ \delta(w_i, x_j) \epsilon(w_i; c_i) t^{-\rho(w_i)} (s^{\phi_i}(w_i) - t^{\phi_i}(w_i))
+ \sum_{i=1}^{2k} \delta(\alpha_i, x_j) \epsilon(\alpha_i; \tau_i) t^{-\rho(\alpha_i)} (s^{\phi_i}(\alpha_i) - t^{\phi_i}(\alpha_i))
+ \delta(\beta_i, x_j) \epsilon(\beta_i; \tau_i) t^{-\rho(\beta_i)} (s^{\phi_i}(\beta_i) - t^{\phi_i}(\beta_i))
+ \delta(\gamma_i, x_j) \epsilon(\gamma_i; \tau_i) t^{-\rho(\gamma_i)} (s^{\phi_i}(\gamma_i) - t^{\phi_i}(\gamma_i))
= t^{-\rho(x_j)} (s^{\phi(x_j)} - t^{\phi(x_j)}) - t^{-\rho(x_j)} (s^{\phi(x_j)} - t^{\phi(x_j)})
= 0.
\]

\(\square\)

Let \(X\) be an Alexander biquandle and let \(m = \text{type} X\). Then \(X\) is also a \(Z_m\)-family of Alexander biquandles. Let \(D\) be an oriented classical link diagram. We can regard \(D\) as a \(Z_m\)-flowed diagram \((D, \phi_{(1)})\) of an \(S^1\)-oriented handlebody-link whose components are of genus 1, where \(\phi_{(1)}\) is the constant map to 1. Hence we can regard an \(X\)-coloring of \(D\) as an \(X\)-coloring of \((D, \phi_{(1)})\). We define a matrix \(A(D; X) \in M(2n, 2n; X)\) by \(A(D; X) = A(D, \phi_{(1)}; X)\), where \(n\) is the number of crossings of \(D\). Then the set of all \(X\)-colorings of \(D\), denoted by \(\text{Col}_X(D)\), is given by

\[
\text{Col}_X(D) = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \end{pmatrix} \in X^{2n} \mid A(D; X) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \end{pmatrix} = 0 \right\}.
\]

Therefore we obtain the following corollary.
**Corollary 5.2.** Let $D$ be a diagram of an oriented classical link with the Alexander numbering and let $X$ be an Alexander biquandle. Let $a_i$ be the $i$-th row of $A(D; X)$, that is,

$$A(D; X) = (a_{i,j}) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{pmatrix}.$$ 

Then it follows that

$$\sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w_i)}(s-t)a_i + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(v_i')}(s-t)a_{n+i} = 0.$$ 

6. Main theorem

In this section, we give lower bounds for the Gordian distance and the unknotting number of $S^1$-oriented handlebody-knots.

**Theorem 6.1.** Let $H_i$ be an $S^1$-oriented handlebody-knot of genus $g$ and let $D_i$ be a diagram of $H_i$ ($i = 1, 2$). Let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander biquandles, where $p$ is a prime number, $s \in \mathbb{Z}_p[t^{\pm 1}]$, and $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$ is an irreducible polynomial. Then it follows that

$$\max_{\phi_1 \in \text{Flow}(D_1; \mathbb{Z}_m)} \min_{\phi_2 \in \text{Flow}(D_2; \mathbb{Z}_m)} |\dim \text{Col}_X(D_1, \phi_1) - \dim \text{Col}_X(D_2, \phi_2)| \leq d(H_1, H_2).$$

**Proof.** Let $(D, \phi)$ be a $\mathbb{Z}_m$-flowed diagram of an $S^1$-oriented handlebody-knot and let $C(D, \phi) = \{c_1, \ldots, c_n\}$ and $V(D, \phi) = \{\tau_1, \ldots, \tau_{2k}\}$. Let $(\overline{D}, \overline{\phi})$ be the $\mathbb{Z}_m$-flowed diagram of an $S^1$-oriented handlebody-knot which is obtained from $(D, \phi)$ by the crossing change at $c_1$ and let $C(\overline{D}, \overline{\phi}) = \{\overline{\tau}_1, \ldots, \overline{\tau}_n\}$ and $V(\overline{D}, \overline{\phi}) = \{\overline{\tau}_1, \ldots, \overline{\tau}_{2k}\}$, where $\overline{\phi}, \overline{\tau}_i$ and $\overline{\tau}_i$ originate from $\phi, c_i$ and $\tau_i$ naturally and respectively (see Figure 12). In the following, we show that

$$|\dim \text{Col}_X(D, \phi) - \dim \text{Col}_X(\overline{D}, \overline{\phi})| \leq 1,$$

that is,

$$|\text{rank } A(D, \phi; X) - \text{rank } A(\overline{D}, \overline{\phi}; X)| \leq 1.$$ 

**Figure 12.** The crossing change at $c_1$.

We may assume that $c_1$ is a positive crossing and $\overline{\tau}_i$ is a negative crossing. We denote by $\overline{x}_i$ each semi-arc of $(\overline{D}, \overline{\phi})$ in the same way as in Figure 9 with respect to $\overline{\tau}_i$ or $\overline{\tau}_i$ and so are $\overline{v}_i$,
\[ w_i, \bar{\alpha}_i, \bar{\beta}_i, \bar{\phi}_i, \bar{\psi}_i, \bar{\eta}_i, \bar{\tau}_i, \bar{\tau}_r (\text{see Figure 10}). \text{ We denote by } x_{j_1} \text{ and } x_{j_2} \text{ the semi-arcs which point to the crossing } c_1 \text{ of } (D, \phi) \text{ as shown in Figure 12, and we put } a := \phi_1 = \bar{\psi}_1 \text{ and } b := \psi_1 = \bar{\phi}_1. \text { We note that } \text{Col}_X(D, \phi) \text{ and } \text{Col}_X(D, \bar{\phi}) \text{ are vector spaces over } X \text{ since } X \text{ is a field.} \]

Let

\[
A(D, \phi; X) = (a_{i,j}) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n+4k} \end{pmatrix}, \quad A(D, \bar{\phi}; X) = (\bar{a}_{i,j}) = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_{2n+4k} \end{pmatrix}
\]

and let

\[
\hat{A}(\bar{D}, \bar{\phi}; X) = (\hat{a}_{i,j}) = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_{2n+4k} \end{pmatrix},
\]

where \( \hat{a}_i \) is a vector obtained by permuting the first entry and the \((n+1)\)-th entry of \( \bar{a}_i \). Then we have \( a_i = \hat{a}_i \) when \( i \neq 1, n+1 \). We note that \( \text{rank } A(D, \bar{\phi}; X) = \text{rank } \hat{A}(\bar{D}, \bar{\phi}; X) \) and

\[
\begin{align*}
a_1 &= (-1, 0, \ldots, 0, t^a_1, 0, \ldots, 0, s^b_1 - t^b_1, 0, \ldots, 0), \\
a_n+1 &= (0, \ldots, 0, 1, 0, \ldots, 0, -s^a_1, 0, \ldots, 0), \\
\bar{a}_1 &= (t^a, 0, \ldots, 0, s^a - t^a, 0, \ldots, 0, -1, 0, \ldots, 0), \\
\bar{a}_{n+1} &= (0, \ldots, 0, -s^b_1, 0, \ldots, 0, 1, 0, \ldots, 0), \\
\hat{a}_1 &= (0, \ldots, 0, s^a - t^a, 0, \ldots, 0, -1, 0, \ldots, 0, t^a_1, 0, \ldots, 0), \\
\hat{a}_{n+1} &= (1, 0, \ldots, 0, -s^b_1, 0, \ldots, 0).
\end{align*}
\]

By Proposition 5.1 we obtain

\[
\sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w_i)} (s^{\phi_i} - t^{\phi_i}) a_i + \sum_{i=1}^{n} \epsilon_{c_i} t^{-\rho(w'_i)} (s^{\psi_i} - t^{\psi_i}) a_{n+i} + \sum_{i=1}^{2k} \epsilon_{c_i} t^{-\rho(c_i)} (s^{\eta_i} - t^{\eta_i}) a_{2n+i} + \sum_{i=1}^{2k} \epsilon_{r_i} t^{-\rho(\beta_i)} (s^{\theta_i} - t^{\theta_i}) a_{2n+2k+i} = 0
\]
and
\[\sum_{i=1}^{n} \varepsilon_c t^{-\rho(\pi_i)} (s_{\pi_i} - t_{\pi_i}) \alpha_i + \sum_{i=1}^{n} \varepsilon_c t^{-\rho(\pi_i)} (s_{\pi_i} - t_{\pi_i}) \alpha_{n+i} + 2k \sum_{i=1}^{2k} \varepsilon_c t^{-\rho(\pi_i)} (s_{\pi_i} - t_{\pi_i}) \alpha_{2n+i} + 2k \sum_{i=1}^{2k} \varepsilon_c t^{-\rho(\pi_i)} (s_{\pi_i} - t_{\pi_i}) \alpha_{2n+2k+i} = 0.\]

If \(\varepsilon_c t^{-\rho(\pi_1)} (s_{\phi_1} - t_{\phi_1}) = 0\), we have \(s_{\phi_1} - t_{\phi_1} = s^a - t^a = 0\), which implies that \(a_{n+1} = -\hat{a}_1\). Hence it follows that
\[| \text{rank } A(D, \phi; X) - \text{rank } A(\mathcal{D}, \bar{\phi}; X) | = | \text{rank } A(D, \phi; X) - \text{rank } \hat{A}(\mathcal{D}, \bar{\phi}; X) | \leq 1.\]

If \(\varepsilon_c t^{-\rho(\pi_1)} (s_{\phi_1} - t_{\phi_1}) = 0\), we have \(s_{\phi_1} - t_{\phi_1} = s^b - t^b = 0\), which implies that \(a_1 = -\hat{a}_{n+1}\). Hence it follows that
\[| \text{rank } A(D, \phi; X) - \text{rank } A(\mathcal{D}, \bar{\phi}; X) | = | \text{rank } A(D, \phi; X) - \text{rank } \hat{A}(\mathcal{D}, \bar{\phi}; X) | \leq 1.\]

If \(\varepsilon_c t^{-\rho(\pi_1)} (s_{\phi_1} - t_{\phi_1}) \neq 0\) and \(\varepsilon_c t^{-\rho(\pi_1)} (s_{\phi_1} - t_{\phi_1}) \neq 0\), we can represent \(a_1\) and \(\bar{a}_1\) as linear combinations of \(a_2, \ldots, a_{2n+4k}\) and \(\bar{a}_2, \ldots, \bar{a}_{2n+4k}\) respectively. Hence it follows that
\[\text{rank } A(D, \phi; X) = \text{rank } \left( \begin{array}{c} a_2 \\ \vdots \\ a_{2n+4k} \end{array} \right), \quad \text{rank } A(\mathcal{D}, \bar{\phi}; X) = \text{rank } \left( \begin{array}{c} \bar{a}_2 \\ \vdots \\ \bar{a}_{2n+4k} \end{array} \right),\]
which implies that
\[| \text{rank } A(D, \phi; X) - \text{rank } A(\mathcal{D}, \bar{\phi}; X) | = \left| \begin{array}{c} a_2 \\ \vdots \\ a_{2n+4k} \end{array} \right| - \left| \begin{array}{c} \bar{a}_2 \\ \vdots \\ \bar{a}_{2n+4k} \end{array} \right| = \left| \begin{array}{c} a_2 \\ \vdots \\ a_{2n+4k} \end{array} \right| - \left| \begin{array}{c} \hat{a}_2 \\ \vdots \\ \hat{a}_{2n+4k} \end{array} \right| \leq 1.\]

Consequently, if we can deform \(H_1\) into \(H_2\) by crossing changes at \(l\) crossings, then for any \(\mathbb{Z}_m\)-flowed diagram \((D_1, \phi_1)\) of \(H_1\), there exists a \(\mathbb{Z}_m\)-flowed diagram \((D_2, \phi_2)\) of \(H_2\) satisfying \(\gcd \phi_1 = \gcd \phi_2\) and
\[| \dim \text{Col}_X(D_1, \phi_1) - \dim \text{Col}_X(D_2, \phi_2) | \leq l\]
by Lemma [1]. Therefore it follows that
\[\max_{\phi_1 \in \text{Flow}(D_1; \mathbb{Z}_m)} \min_{\phi_2 \in \text{Flow}(D_2; \mathbb{Z}_m)} | \dim \text{Col}_X(D_1, \phi_1) - \dim \text{Col}_X(D_2, \phi_2) | \leq d(H_1, H_2).\]
By Proposition 4.3 and Theorem 6.1, the following corollary holds immediately.

**Corollary 6.2.** Let $H$ be an $S^1$-oriented handlebody-knot and let $D$ be a diagram of $H$. Let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander biquandles, where $p$ is a prime number, $s \in \mathbb{Z}_p[t^{\pm 1}]$ and $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$ is an irreducible polynomial. Then it follows that

$$\max_{\phi \in \text{Flow}(D; \mathbb{Z}_m)} \dim \text{Col}_X(D, \phi) - 1 \leq u(H).$$

7. Examples

In this section, we give some examples. In Example 7.1, we give a handlebody-knot with unknotting number 2, and in Remark 7.2, we note that it can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2$, $\mathbb{Z}_3$-flows introduced in [14]. In Example 7.3, we give three handlebody-knots with unknotting number $n$ for any $n \in \mathbb{Z}_{>0}$. In Example 7.4, we give two handlebody-knots with their Gordian distance $n$ for any $n \in \mathbb{Z}_{>0}$.

**Example 7.1.** Let $H$ be the handlebody-knot represented by the $\mathbb{Z}_{10}$-flowed diagram $(D, \phi)$ depicted in Figure 13. Then we show that $u(H) = 2$.

Let $s = 1 \in \mathbb{Z}_3[t^{\pm 1}]$ and let $f(t) = t^4 + 2t^3 + t^2 + 2t + 1 \in \mathbb{Z}_3[t^{\pm 1}]$, which is an irreducible polynomial. Then $X := \mathbb{Z}_3[t^{\pm 1}]/(f(t))$ is a $\mathbb{Z}_{10}$-family of Alexander biquandles. Then for any $x, y, z \in X$, the assignment of them to each semi-arc of $(D, \phi)$ as shown in Figure 13 is an $X$-coloring of $(D, \phi)$, which implies $\dim \text{Col}_X(D, \phi) \geq 3$. By Corollary 6.2, we obtain $2 \leq u(H)$. On the other hand, we can deform $H$ into a trivial handlebody-knot by the crossing changes at two crossings surrounded by dotted circles depicted in Figure 13. Therefore it follows that $u(H) = 2$.

![Figure 13. A $\mathbb{Z}_{10}$-flowed diagram $(D, \phi)$ of $H$.](image)

**Remark 7.2.** We show that the result in Example 7.1 can not be obtained by using Alexander quandle colorings with $\mathbb{Z}_2, \mathbb{Z}_3$-flows introduced in [14].

Let $H$ be the handlebody-knot represented by the $\mathbb{Z}_m$-flowed diagram $(D, \phi(a, b))$ depicted in Figure 14 for any $m = 2, 3$ and $a, b \in \mathbb{Z}_m$. Let $p$ be a prime number, $s = 1 \in \mathbb{Z}_p[t^{\pm 1}]$, $f(t)$ be an irreducible polynomial in $\mathbb{Z}_p[t^{\pm 1}]$ and let $X = \mathbb{Z}_p[t^{\pm 1}]/(f(t))$ which is a $\mathbb{Z}_m$-family of Alexander (bi)quandles. We note that $\text{Col}_X(D, \phi(a, b))$ is generated by $x, y, z \in X$ as shown in Figure 14 for any $m = 2, 3$ and $a, b \in \mathbb{Z}_m$. If $(a, b) = (1, 0)$, $x, y$ and $z$ need to satisfy the
following relations:
\[(t^2 - t + 1)x - (t^2 - t + 1)y = 0,\]
\[-t(t^2 - t + 1)x + t^{-1}(t + 1)(t - 1)(t^2 - t + 1)y + t^{-1}(t^2 - t + 1)z = 0,\]
\[-t^{-1}(t - 1)(t^2 - t + 1)x + t^{-2}(t^2 - t - 1)(t^2 - t + 1)y + t^{-2}(t^2 - t + 1)z = 0,\]
\[((t^3 + t^2 - 1)(t^2 - t + 1) - t)x - ((t^3 + t^2 - 1)(t^2 - t + 1) - t)z = 0,\]
that is,
\[M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\]
where
\[M = \begin{pmatrix} t^2 - t + 1 & -t(t^2 - t + 1) & 0 \\ -t(t^2 - t + 1) & t^{-1}(t + 1)(t^2 - t + 1) & t^{-1}(t^2 - t + 1) \\ -(t^3 + t^2 - 1)(t^2 - t + 1) - t & t^{-2}(t^2 - t - 1)(t^2 - t + 1) & t^{-2}(t^2 - t + 1) + t \end{pmatrix}.\]
These relations are obtained from crossings \(c_1, c_2, c_3\) and \(c_4\) as shown in Figure 14. When \(t^2 - t + 1 \neq 0\) in \(X\), it is clearly that \(\text{rank } M \geq 1\). When \(t^2 - t + 1 = 0\) in \(X\), we have
\[M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -t & 0 & t \end{pmatrix},\]
which implies that \(\text{rank } M = 1\). Hence we have \(\text{dim Col}_X(D, \phi(1, 0)) = 3 - \text{rank } M \leq 2\). Therefore we can not obtain \(2 \leq u(H)\).

We can prove the remaining cases in the same way.

**Example 7.3.** Let \(A_n, B_n\) and \(C_n\) be the handlebody-knots represented by the \(\mathbb{Z}_8\)-flowed diagram \((D_{A_n}, \phi_{A_n})\), the \(\mathbb{Z}_{24}\)-flowed diagram \((D_{B_n}, \phi_{B_n})\) and the \(\mathbb{Z}_8\)-flowed diagram \((D_{C_n}, \phi_{C_n})\) depicted in Figure 15, 16 and 17 respectively for any \(n \in \mathbb{Z}_{>0}\). Then we show that \(u(A_n) = u(B_n) = u(C_n) = n\).
(1) Let \( s = t + 1 \in \mathbb{Z}_3[t^{\pm 1}] \) and let \( f(t) = t^2 + t + 2 \in \mathbb{Z}_3[t^{\pm 1}] \), which is an irreducible polynomial. Then \( X := \mathbb{Z}_3[t^{\pm 1}]/(f(t)) \) is a \( \mathbb{Z}_8 \)-family of Alexander biquandles. Then for any \( x_0, x_1, \ldots, x_n \in X \), the assignment of them to each semi-arc of \((D_{A_n}, \phi_{A_n})\) as shown in Figure 15 is an \( X \)-coloring of \((D_{A_n}, \phi_{A_n})\), which implies \( \dim \text{Col}_X(D_{A_n}, \phi_{A_n}) \geq n + 1 \). By Corollary 6.2 we obtain \( n \leq u(A_n) \). On the other hand, we can deform \( A_n \) into a trivial handlebody-knot by the crossing changes at \( n \) crossings surrounded by dotted circles depicted in Figure 15. Therefore it follows that \( u(A_n) = n \).

![Figure 15. A \( \mathbb{Z}_8 \)-flowed diagram \((D_{A_n}, \phi_{A_n})\) of \( A_n \).](image)

(2) Let \( s = t^2 + 1 \in \mathbb{Z}_5[t^{\pm 1}] \) and let \( f(t) = t^2 + 2t + 4 \in \mathbb{Z}_5[t^{\pm 1}] \), which is an irreducible polynomial. Then \( X := \mathbb{Z}_5[t^{\pm 1}]/(f(t)) \) is a \( \mathbb{Z}_{24} \)-family of Alexander biquandles. Then for any \( x_0, x_1, \ldots, x_n \in X \), the assignment of them to each semi-arc of \((D_{B_n}, \phi_{B_n})\) as shown in Figure 16 is an \( X \)-coloring of \((D_{B_n}, \phi_{B_n})\), which implies \( \dim \text{Col}_X(D_{B_n}, \phi_{B_n}) \geq n + 1 \). By Corollary 6.2 we obtain \( n \leq u(B_n) \). On the other hand, we can deform \( B_n \) into a trivial handlebody-knot by the crossing changes at \( n \) crossings surrounded by dotted circles depicted in Figure 16. Therefore it follows that \( u(B_n) = n \).

![Figure 16. A \( \mathbb{Z}_{24} \)-flowed diagram \((D_{B_n}, \phi_{B_n})\) of \( B_n \).](image)

(3) Let \( s = 2t - 1 \in \mathbb{Z}_3[t^{\pm 1}] \) and let \( f(t) = t^2 + t + 2 \in \mathbb{Z}_3[t^{\pm 1}] \), which is an irreducible polynomial. Then \( X := \mathbb{Z}_3[t^{\pm 1}]/(f(t)) \) is a \( \mathbb{Z}_8 \)-family of Alexander biquandles. Then for any \( x_0, x_1, \ldots, x_n \in X \), the assignment of them to each semi-arc of \((D_{C_n}, \phi_{C_n})\) as shown in Figure 17 is an \( X \)-coloring of \((D_{C_n}, \phi_{C_n})\), which implies \( \dim \text{Col}_X(D_{C_n}, \phi_{C_n}) \geq n + 1 \). By Corollary 6.2 we obtain \( n \leq u(C_n) \). On the other hand, we can deform \( C_n \) into
a trivial handlebody-knot by the crossing changes at \( n \) crossings surrounded by dotted circles depicted in Figure 17. Therefore it follows that \( u(C_n) = n \).

![Figure 17. A \( \mathbb{Z}_8 \)-flowed diagram \((D_{C_n}, \phi_{C_n})\) of \( C_n \).](image)

**Example 7.4.** Let \( H_n \) and \( H'_n \) be the handlebody-knots represented by the \( \mathbb{Z}_3 \)-flowed diagrams \((D_n, \phi_n)\) and \((D'_n, \phi'_n(a, b))\) respectively depicted in Figure 18 for any \( n \in \mathbb{Z}_{>0} \) and \( a, b \in \mathbb{Z}_3 \). Then we show that \( d(H_n, H'_n) = n \).

Let \( s = 1 \in \mathbb{Z}_2[t^{\pm 1}] \) and let \( f(t) = t^2 + t + 1 \in \mathbb{Z}_2[t^{\pm 1}] \), which is an irreducible polynomial. Then \( X := \mathbb{Z}_2[t^{\pm 1}]/(f(t)) \) is a \( \mathbb{Z}_3 \)-family of Alexander (bi)quandles. Then for any \( x_0, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \), the assignment of them to each semi-arc of \((D_n, \phi_n)\) as shown in Figure 18 is an \( X \)-coloring of \((D_n, \phi_n)\), which implies \( \dim \text{Col}_X(D_n, \phi_n) \geq 2n + 1 \).

On the other hand, we note that \( \text{Col}_X(D'_n, \phi'_n(a, b)) \) is generated by \( x_0, x_1, x'_1, \ldots, x_n, y_1, y'_1, \ldots, y_n, y'_n \in X \) as shown in Figure 18 for any \( a, b \in \mathbb{Z}_3 \). If \((a, b) = (0, 0)\), it is easy to see that \( \dim \text{Col}_X(D'_n, \phi'_n(a, b)) = 1 \). If \((a, b) = (1, 1), (1, 2), (2, 1), (2, 2)\), we obtain that \( x_i = x'_i = y_i = y'_i \) for any \( i = 1, 2, \ldots, n \), which implies \( \dim \text{Col}_X(D'_n, \phi'_n(a, b)) \leq n + 1 \). If \((a, b) = (0, 1), (0, 2)\), we have

\[
\begin{align*}
x_0 &= x_1 = x_2, \\
x_{i+2} &= x'_i (i = 1, 2, \ldots, n - 2), \\
x'_i &= \begin{cases} 
  x_i \pm b y'_i & (i \text{: odd}), \\
  x_i \pm b y'_i & (i \text{: even}),
\end{cases} \\
x_n &= x'_{n-1}, \\
y_i &= y'_i (i = 1, 2, \ldots, n).
\end{align*}
\]

Hence \( \text{Col}_X(D'_n, \phi'_n(a, b)) \) is generated by \( x_0, y_1, \ldots, y_n \in X \), which implies \( \dim \text{Col}_X(D'_n, \phi'_n(a, b)) \leq n + 1 \). If \((a, b) = (1, 0), (2, 0)\), in the same way as when \((a, b) = (0, 1), (0, 2)\), \( \text{Col}_X(D'_n, \phi'_n(a, b)) \)
is generated by $x_0, x_1, \ldots, x_n \in X$, which implies $\dim \text{Col}_X(D'_n, \phi'_n(a, b)) \leq n + 1$. Hence for any $a, b \in \mathbb{Z}_3$, $\dim \text{Col}_X(D'_n, \phi'_n(a, b)) \leq n + 1$, which implies that

$$\dim \text{Col}_X(D_n, \phi_n) - \dim \text{Col}_X(D'_n, \phi'_n(a, b)) \geq n.$$ 

By Theorem 6.1, it follows that $n \leq d(H_n, H'_n)$.

Finally, we can deform $H'_n$ into $H_n$ by the crossing changes at $n$ crossings surrounded by dotted circles depicted in Figure 18. Therefore it follows that $d(H_n, H'_n) = n$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure18.png}
\caption{\textit{Z}_3\text{-flowed diagrams $D_n, \phi_n$ and $D'_n, \phi'_n$ of $H_n$ and $H'_n$.}}
\end{figure}

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