On the non-perturbative graviton propagator

V.M. Khatsymovsky

Budker Institute of Nuclear Physics
of Siberian Branch Russian Academy of Sciences
Novosibirsk, 630090, Russia
E-mail address: khatsym@gmail.com

Abstract

To reduce general relativity (GR) to the canonical Hamiltonian formalism and construct the path (functional) integral in a simpler and, especially in the discrete case, less singular way, one extends the configuration superspace, as in the connection representation. The result of the functional integration over connection can be written in the form of expansions: the phase of the result arises in the leading order of the stationary phase method where it is equal to the Regge action, the module of the result arises in the leading order of the expansion over a scale of discrete lapse-shift functions and has maxima at finite (Planck scale) areas/lengths and rapidly decreases at large areas/lengths. This means the appearance of a kind of lattice with spacings dynamically fixed (loosely). We consider a typical propagator for this system using an example of a simplified pseudo-mini-superspace structure - a cubic lattice.

keywords: general relativity; piecewise flat spacetime; Regge calculus; discrete connection; functional integral; graviton propagator

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1 Introduction

Description of general relativity (GR) on a certain class of Riemannian manifolds, namely piecewise flat manifolds, proposed by Regge [1], can be quite self-sufficient, because
one can approximate any smooth Riemannian manifold by piecewise flat ones with an arbitrary accuracy [2, 3]. This attracts attention, in particular, in view of the formal non-renormalizability of continuous GR on the one hand, and the countability of the number of degrees of freedom of a piecewise flat manifold and the related possibility to apply for quantizing the system the tools used to quantize on the lattice [4], on the other hand. The most convenient and universal for the analysis of quantum Regge gravity is the functional integral approach where a certain freedom is in defining the functional measure. Using reasonable physical arguments it turns possible to fix the measure and use it to obtain physical quantities such as the Newtonian potential [5, 6]. In [7], a review of the Regge theory and some other approaches is given. Recently the Causal Dynamical Triangulations approach related to the Regge theory has allowed to get interesting results in quantum gravity [8].

One of the approaches to fixing the functional measure might be the canonical Hamiltonian approach, which gives it in the form symbolically as \(dpdq\) for some set of canonical coordinates \(p, q\), which can be expressed in terms of a Jacobian of the Poisson brackets of the constraints in the original coordinates. In GR, the most convenient is using the tetrad-connection (Cartan-Weyl) form of the action for constructing the Hamiltonian formalism. Especially this refers to the discrete GR or Regge gravity where the Regge action in terms of the original edge length variables is quite singular from the viewpoint of passing to the canonical variables. If we succeeded in performing a canonical analysis of the tetrad-connection gravity, then we can make the functional integration over the connection variables and arrive at the functional integral expression in terms of the purely tetrad or lengths variables. In [9], we have performed the functional integration over connection for the exact selfdual plus anti-selfdual connection representation of the Regge action of the type which we have suggested in [10].

In the connection representation, the Hamiltonian formalism with area tensors and connection matrices being conjugate variables leads to the Jacobian of the Poisson brackets of the constraints which is singular at the flat metric. This is typical for discrete gravity due to the lack of diffeomorphism invariance, as is reviewed, eg, in [11]. More convenient is using an analogue of the considered in the literature "area Regge calculus" [12] where the areas of the triangles are independent variables. The equations of motion for this system mean vanishing the defect angles, which, however, because of the lack of the usual geometric interpretation does not mean flat spacetime.
In our case, the natural modification consists in considering area tensors as independent variables [13]. In the configuration superspace of independent area tensors, the points of the physical hypersurface run through sets of area tensors corresponding to all possible sets of edge vectors. Simple equations of motion ensure commuting constraints and simple form of the functional measure, and we should project the measure onto the physical hypersurface. This is achieved by introducing an appropriate delta-function factor with support on the physical hypersurface.

In this paper we consider the functional integral and, in particular, the gravitational propagator, aiming at a certain diagram technique. In the next Section, the mentioned selfdual plus anti-selfdual form of the action and the associated notation are given. In Section 3, we consider the procedure of the functional integration over connection and the possibility to take proper different (but compatible) expansions for the phase and module of the result. In [9], we have calculated the module in the leading order of the expansion over the discrete lapse-shift functions, which gives (a part of) the measure. The phase in the leading order of the stationary phase expansion is the connection form of the action on the classical solution for connection, that is, the Regge action. In Section 4, we touch upon the construction of a perturbation series if the resulting measure in the functional integral has maxima. In particular, the optimal background values of variables in such an expansion are determined in part by the maxima of the measure. In Section 5 we write out the functional measure in the system with independent area tensors. In Section 6 we project (the area tensor part of) this measure onto the physical hypersurface, functional integration over the connection part of this measure is done earlier. We estimate the optimal background values of variables (areas or lengths). In Section 7, this is used to write out the propagator on a simplified pseudo-mini-superspace (cubic) lattice.

2 Selfdual plus anti-selfdual form of the action

The piecewise flat spacetime is considered as the simplicial complex consisting of the 4-dimensional tetrahedra or 4-simplices $\sigma^4$, their 3-dimensional faces $\sigma^3$ (usual tetrahedrons), 2-dimensional faces or triangles $\sigma^2$, edges $\sigma^1$ and vertices $\sigma^0$. On these objects, different variables can be given. To each 4-simplex we assign a local pseudo-Euclidean frame. There are $\text{SO}(3,1)$ connection matrices $\Omega_{\sigma^3}$ on the 3-faces $\sigma^3$ and curvature ma-
trices $R_{\sigma^2}$ on the triangles $\sigma^2$. The discrete analogs of connection and curvature were introduced by Fröhlich [14]. The matrix $R_{\sigma^2}$ is the holonomy of $\Omega_{\sigma^3}$, a path ordered product of $\Omega_{\sigma^3}, \sigma^3 \supset \sigma^2$, taken along the loop enclosing $\sigma^2$. There are vectors $l_{\sigma^1}$ of the edges $\sigma^1$ and area tensors $v_{\sigma^2}^{ab}$ of the triangles $\sigma^2$. The matrices $\Omega_{\sigma^3}$ can be decomposed multiplicatively into the selfdual $+\Omega_{\sigma^3}$ and anti-selfdual $-\Omega_{\sigma^3}$ parts, elements of SO(3,C), and accordingly the matrices $R_{\sigma^2}$ decompose into $+R_{\sigma^2}$ and $-R_{\sigma^2}$ in the same way. The matrices $v_{\sigma^2}^{ab}$ decompose into selfdual $+v_{\sigma^2}^{ab}$ and anti-selfdual $-v_{\sigma^2}^{ab}$ parts additively.

The action $\frac{1}{2} \int R \sqrt{-g} d^4x$ on the piecewise flat manifold in the representation which we use in the paper [9] takes the form

$$S = \frac{1}{2} \sum_{\sigma^2} \left[ \left( 1 + \frac{i}{\gamma} \right) \sqrt{+v_{\sigma^2}^{2}} \arcsin \frac{+v_{\sigma^2} \ast +R_{\sigma^2}(\Omega)}{\sqrt{+v_{\sigma^2}^{2}}} ight]$$

$$+ \left( 1 - \frac{i}{\gamma} \right) \sqrt{-v_{\sigma^2}^{2}} \arcsin \frac{-v_{\sigma^2} \ast -R_{\sigma^2}(\Omega)}{\sqrt{-v_{\sigma^2}^{2}}} \right]. \quad (1)$$

Here, $v \ast R \equiv \frac{1}{2} v^a R^{bc} \epsilon_{abc}$, $\epsilon_{123} = +1$, and $\pm v$ (or $\pm v_k$ in components) are 3-vectors parameterizing via $\pm v_{ab} = \pm v^k \pm \Sigma_{kab}/2$ selfdual $+v_{ab}$ and antiselfdual $-v_{ab}$ parts of the antisymmetric tensor $v_{ab}$ expanded over triple of (anti-)selfdual basis matrices $\pm \Sigma_k$ obeying algebra of the Pauli matrices times $-i$,

$$v_{ab} = +v_{ab} - v_{ab}, \quad \pm v_{ab} = \frac{1}{2} v_{ab} \pm \frac{i}{4} \epsilon_{abcd} v_{cd}, \quad 2 \pm v_k = -\epsilon_{klm} v_{lm} \pm i(v_{k0} - v_{0k}). \quad (2)$$

Here $\epsilon^{0123} = +1$, the metric $g_{ab} = \text{diag}(-1,1,1,1)$. In particular, if $v_{ab}$ is the area bivector built on some edge vectors $l_1^a$ and $l_2^b$, $v_{ab} = \frac{1}{2} \epsilon_{abcd} l_1^c l_2^d$, then

$$2 \pm v = \pm il_1 \times l_2 - l_1 l_2^0 + l_2 l_1^0. \quad (3)$$

The quantity $\gamma$ is a discrete analog of the Barbero-Immirzi parameter [15, 16], which parameterizes the parity odd Holst term [17, 18] in the continuum theory.

We adopt a partly regular structure of the 4-dimensional simplicial complex. There are 3-dimensional leaves of the foliation, simplicial complexes of the same structure numbered by an integer coordinate $t$. The 4-dimensional geometry is constructed by connecting analogous vertices in the neighboring leaves by edges. We call these edges $t$-like. Besides that, these leaves are connected by diagonal edges. The diagonal edge connects a vertex $\sigma^0_1$ in one leaf and a vertex $\sigma^0_2$ in the neighboring leaf analogous to a vertex in the former leaf neighboring to $\sigma^0_1$. We call edges completely contained in a leaf by leaf edges. A simplex containing a $t$-like edge will be called a $t$-like simplex, a simplex
completely contained in a leaf will be called a leaf simplex, and any other simplex will be called a diagonal one.

The region between any two neighboring leaves gets divided into 4-dimensional prisms whose lateral 3-dimensional surface is formed by \( t \)-like tetrahedra \( \sigma^3 \); the bases of each prism are analogous 3-simplices in these leaves, and each prism is divided into four 4-simplices. If the situation is compared with the usual continuum theory, the terms "\( t \)-like" and "leaf/diagonal" are characteristics of the world vector indices while "timelike" and "spacelike" refer to the local frame indices. The vectors of the \( t \)-like edges are discrete analogs of the Arnowitt-Deser-Misner [19] lapse-shift functions if \( t \) is considered as a time coordinate. The lapse-shift functions in the continuum theory can be considered as gauge parameters, fixation of which means fixing four degrees of freedom in the metric tensor associated with diffeomorphisms.

In this construction, any given 4-simplex \( \sigma^4 \) contains a \( t \)-like edge with a 4-vector \( l_0^a \), discrete lapse-shift functions, and three else edges with 4-vectors \( l_\alpha^a \), \( \alpha = 1, 2, 3 \), with a common vertex \( O \). The tetrad \( l_\lambda^a \), \( \lambda = 0, 1, 2, 3 \), forms six bivectors

\[
v_{ab}^{\lambda \mu} \equiv \frac{1}{2} \varepsilon^{ab}_{\ cd} l_\lambda^c l_\mu^d.
\]  

Three of them \( v_{12}^{ab} \equiv v_{23}^{ab}, \ldots \), are bivectors of some leaf/diagonal triangles, and three \( \tau_{\alpha}^{ab} \equiv v_{0\alpha}^{ab} \) are bivectors of some \( t \)-like ones.

### 3 Functional integration over connection

The functional integration of \( \exp(iS) \) over connection in the continuum theory is Gaussian and gives \( \exp \left( i \frac{1}{2} \int R \sqrt{-g} d^4x \right) \) in the tetrad/metric variables. In the considered discrete formulation, the integration over \( D\Omega \) (the product of the invariant or Haar measures on the instances of SO(3,1) group) is not Gaussian, but we can use the stationary phase method and expand the action as a function of the connection around the classical solution. This classical solution for connection just leads to the Regge action by construction of the connection representation. Thus, the Regge action constitutes the phase of the result of the considered integration in the leading order.

As for the module of this result, it arises in the leading order of another expansion, that one in powers of the lapse-shift functions. The contribution of the \( t \)-like triangles to the action is proportional to the scale of the lapse-shift functions. It can be considered as
a correction to the rest of action, the contribution of the leaf and diagonal triangles, which is of zero order in the scale of the lapse-shift functions. This is successfully combined with the fact that the holonomies just on the leaf and diagonal triangles can be taken as independent. This simplifies the considered procedure of integration reducing the result in the leading order to the product of independent integrals over $\mathcal{D}R_{\sigma^2}$ for the leaf and diagonal triangles $\sigma^2$. In the next orders in the scale of the lapse-shift functions, the integration can be performed using (multiplicative) expressions of $R_{\sigma^2}$s for the $t$-like $\sigma^2$s in terms of $R_{\sigma^2}$s for the leaf/diagonal $\sigma^2$s (these just solve some algebraical identities, the Bianchi identities, [1]). (Some preliminary discussion of the strategy of integration over connection considered here is made in our paper [20].)

The phase of this expansion appears beginning with the correction of the first order in the lapse-shift functions, and calculating it even in this order is a difficult task.

Important is the question of the simultaneous validity of both the expansions. If we base our experience on the continuum theory, the Palatini action is there a local second order polynomial of the connection, and non-Gaussian corrections to it would generate a series of the corrections to the effective action over negative powers of $\det \| e_{\lambda}^a \| = \epsilon_{abcd} e_{\lambda}^a e_{\lambda}^b e_{\lambda}^c e_{\lambda}^d \propto N$ where $e_{\lambda}^a$ is the continuum tetrad, $N$ is just the scale of the lapse-shift functions $e_{\lambda}^0$. This is simply because the considered corrections to the effective action diverge just in the case of the degeneracy of the quadratic form of the connection, and the determinant of this form is some power of $\det \| e_{\lambda}^a \|$. The inequality of $\det \| e_{\lambda}^a \|$ to zero makes integrations over all the connection components $\omega_{\lambda}^{ab}$ convergent in the given point.

However, in the discrete theory, the dependence of the action on the connection is not local, including at the level of the second order in (the variations of) the connection. This is in the sense that the matrix of the quadratic form of the connection is large and close to the diagonal, but can not be considered block-diagonal: this form consists of pairwise products of generators of the connection matrices on the pairs of 3-simplices having a common 2-face. The dependence on the connection becomes effectively local if the independent curvatures (ie, those on the leaf/diagonal triangles) are taken as (a part of) the connection variables and the expansion in powers of the scale of the lapse-shift functions is considered. The objects, on which this dependence is localized, are not the vertices or 4-simplices as one would expect from the naive analogy with the continuum theory, but the triangles.
Now we integrate not over infinitesimal, but over finite rotations, and there is a difference in what these rotations are, Euclidean (compact) or Lorentzian (noncompact). The rotation around a timelike area is Euclidean, and the integration over it is finite in any case; the rotation around a spacelike area is Lorentzian, and the value of this area being nonzero is important for that the integration over this rotation to converge. An example is the following integral,

\[ \int_{-\infty}^{+\infty} \exp(i|v| \sinh \psi) d\psi = \int_{-\infty}^{+\infty} \exp(-|v| \cosh \psi) d\psi, \]

(5)

where |v| stands for the module of the (spacelike) area, \( v = \sqrt{v^2} \), and \( \psi \) stands for the Lorentz angle parameter. In the direct SO(3,1) (not expanded into self- and anti-selfdual) formulation, the integrations over the independent rotations around the leaf/diagonal triangles with spacelike areas are already convergent; the contributions of the \( t \)-like triangles with timelike areas to the action can only contribute to the convergence of the integrations over the (Euclidean) rotations around them (if any) that are already finite, and add nothing to the convergence of the integrations over the independent rotations around the leaf/diagonal triangles. That is, the dependence on the discrete lapse-shift \( l_0^a \) and on the areas of the \( t \)-like triangles is not singular in the neighborhood of zero, and both the expansions can exist simultaneously for such a system.

In the present formulation, the selfdual and anti-selfdual rotations on a triangle \( \sigma^2 \) are combinations of those around \( \sigma^2 \) and in the plane of \( \sigma^2 \), the convergence of the integrations is improved, and, together with "arcsin" functions in the exact representation, this leads to the integration results that are even less singular with respect to the leaf/diagonal triangle areas than (5) at small \( v \) (in fact, nonsingular). Besides that, we have calculated [21] the typical expressions arising in the expansion of the result of the functional integration over connection in powers of the scale of the lapse-shift. These expressions also turn out to be regular at small \( v \) and exponentially suppressed at large \( v \).

4 Starting point for the perturbation expansion

We have the result of the functional integration over connection of the form

\[ \int F(l) d^n l \exp(iS(l)). \]

(6)
Here \( l = (l_1, \ldots, l_n) \) is the set of the edge lengths, \( S(l) \) is the Regge action, \( F(l) \) has maxima for some finite values of \( l \) and rapidly decreases at large \( l \) (exponentially).

It is intuitively clear that the configuration of the system will be fixed (to some extent loosely) in the neighborhood of these points. We have interplay of the two factors, monotonic and oscillating, each with its own preferred points, to force the system to be in the vicinity of them. This is illustrated by the following simple example,

\[
\int \exp \left[-\frac{(x - c)^2}{b^2}\right] \exp \left[i \left(2xq + \frac{x^2}{L^2}\right)\right] dx \propto \exp \frac{i(2qc + c^2L^{-2}) - b^2q^2}{1 - ib^2L^{-2}}.
\]

(7)

Here, \( b \) models a typical width of the probability distribution \( F(x) \). This example shows that if the classical equations of motion cease to control \( x \) (at \( L^{-1} = 0 \)) or cause \( x \) to go to infinity (at \( L \to \infty \)), then \( x \) is controlled by the measure \( F(x)dx \) at \( b/L \ll 1 \), and in the leading approximation we replace \( x \) in \( S(x) \) by some optimal \( x_0 \) of the type of the maximum of \( F(x) \). Although, usually \( x_0 \) which replaces \( x \) in \( S(x) \) when developing the perturbative series is a solution to the equations of motion.

In particular, if the flat background is given, the equations of motion (the Regge equations) are satisfied identically and do not constrain any edge length (this corresponds to \( L^{-1} = 0 \) in the example). On the curved background, geometry is changed when changing edge lengths, and it is natural to expect that the effect is defined by the angle defects \( \alpha_{\sigma^2} \); this corresponds to \( b^2/L^2 \) of the order of the typical angle defect \( \alpha \). Given the typical curvature \( R \), \( \alpha \) is of the order of \( Ra^2 \), which is extremely small for \( R \) encountered in practice if the typical edge length \( a \) is of the order of the Planck scale: \( b/L \ll 1 \). The optimal \( x_0 \) around which the perturbative series should be made is defined by \( F(x) \) in these cases.

Consider this point from the viewpoint of constructing a formal perturbation theory. We pass from \( l \) to a new \( n \)-vector variable \( u = (u_1, \ldots, u_n) \) which makes the measure to be the Lebesgue one. We make the Taylor expansion of \( S(l) \) around some point \( l_0 = l(u_0) \) \((u_0 = (u_{01}, \ldots, u_{0n}), \ l_0 = (l_{01}, \ldots, l_{0n})) \) over \( \Delta u = u - u_0 \),

\[
F(l) d^n l = d^n u, \quad S(l(u)) = S(l_0) + \sum_{j,k} \frac{\partial S(l_0)}{\partial l_j} \frac{\partial l_j(u_0)}{\partial u_k} \Delta u_k
+ \frac{1}{2} \sum_{j,k,l} \frac{\partial }{\partial u_l} \left( \frac{\partial l_j}{\partial u_k} \frac{\partial S}{\partial l_j} \right) \bigg|_{u=u_0} \Delta u_k \Delta u_l + \ldots.
\]

(8)

The requirement that there be no term linear in \( \Delta u \) in the latter means the classical
equations of motion (the Regge equations),

\[ \frac{\partial S(l_0)}{\partial l} = 0, \quad S(l) = \frac{1}{2} \sum_{j,k,l,m} \frac{\partial^2 S(l_0)}{\partial l_j \partial l_l} \frac{\partial l_j(u_0)}{\partial u_k} \frac{\partial l_l(u_0)}{\partial u_m} \Delta u_k \Delta u_m + \ldots \]  \hspace{1cm} (9)

Consider the flat background spacetime. Any skeleton \((l_1, \ldots, l_n)\) that is realizable in this spacetime satisfies the Regge equations. One can imagine the 4-dimensional spacetime as a hypersurface embedded in a flat spacetime of a sufficiently large dimensionality \([22]\) \(D\) with the pseudoEuclidean metric \(\eta_{AB} = \text{diag}(-1, +1, +1, \ldots, +1)\). Let the flat 4-dimensional spacetime be the set of points with the coordinates \(x^A\) different from zero only at \(A = 0, 1, 2, 3\) (a hyperplane). The coordinates of the vertices \(x^{A}_{\sigma^0}\) for any edge \(\sigma^1\) with the ending vertices \(\sigma^0_1, \sigma^0_2\) define its vector \(l^A_{\sigma^1} = x^A_{\sigma^0_1} - x^A_{\sigma^0_2}\) and, in particular, the length \(l_{\sigma^1}\). The motion of a vertex by \(\delta x^A_{\sigma^0}\) can be decomposed into its translation in the hyperplane \(\delta_\parallel x^A_{\sigma^0} = (\ast, \ast, \ast, 0, \ldots, 0)\) (\(\delta_\parallel x^A_{\sigma^0} = 0\) at \(A \geq 4\)) and physical fluctuation \(\delta_\perp x^A_{\sigma^0} = (0, 0, 0, 0, \ast, \ast, \ldots, \ast)\) (\(\delta_\perp x^A_{\sigma^0} = 0\) at \(A = 0, 1, 2, 3\)), which makes the flat spacetime curved. The translations \(\delta_\parallel x^A_{\sigma^0}\) or \(\delta_\parallel l_{\sigma^1}\) leave the spacetime flat and generate gauge transformations on it \([23, 24, 25]\). In particular, \(\delta_\parallel^2 S = 0\) on the flat spacetime. At the same time, \(\delta_\parallel \delta_\perp S\) is generally nonzero even on the flat spacetime (since \(\delta_\parallel S\) is nonzero on the curved spacetime), as well as \(\delta_\perp^2 S\). In overall, (the matrix of) the quadratic form \(\delta^2 S\) on the flat spacetime where \(\delta\) is \(\delta_\parallel\) or \(\delta_\perp\) has zero block \(\delta_\parallel^2 S\), but its dimension (4 parameters per vertex) is less than half of the dimension of \(\delta_\parallel^2 S\) itself (\(D - 4\) parameters per vertex with \(D \geq 10\)). Therefore \(\det \|\partial^2 S/\partial l_i \partial l_k\| \neq 0\) for a random skeleton (with proper initial/boundary conditions imposed).

If the integral \(\int d^n u = \int F(l) d^n l\) is finite (which is indeed the case), the boundary of the range of the variable \(u\) is located mainly at finite \(u\); when approaching this boundary, then \(D(l_1, \ldots, l_n)/D(u_1, \ldots, u_n) = 1/F(l) \to \infty\). That is, this shows up as an infinite potential wall in \(S(l(u))\). Since, as it turns out, \(F(l)\) is zero if any \(l_i\) is zero or infinity, the system does not go to these points, but on the contrary, the dominant contribution to the path integral comes from the neighborhood of the point \(l = l_0\) which minimizes the determinant of the considered form \(\delta^2 S\) in the action (9) or maximizes the inverse of it,

\[ F(l_0)^2 \det \|\partial^2 S(l_0)\|^{-1} \]  \hspace{1cm} (10)

Although, an accidental additional symmetry of the skeleton is also possible leading to \(\det \|\partial^2 S(l_0)/\partial l_i \partial l_k\| = 0\) \([23, 24, 25]\); let the rank of the \(n \times n\) dimensional matrix \(\partial^2 S/\partial l_i \partial l_k\) be, say, \(n - 1\). We can consider (combinations of) the lengths \((l_1, \ldots, l_n)\)
which are chosen so that $\delta^2 S$ at $l = l_0$ does not depend on $\delta l_n$. We can fix the gauge by fixing $l_n$ [23] and omitting the integration over $dl_n$. Especially since, most likely, the measure $F(l)$ also does not fix $l_n$, since, being obtained from the exact representation of the Regge action, it can have its symmetries. As a result, we also have (10), where now $\partial^2 S/\partial l_i \partial l_k$ is a $(n - 1) \times (n - 1)$ dimensional matrix: $i, k = 1, 2, ..., n - 1$.

Thus, in order to find the initial point of the perturbative expansion, we, as usual, solve the equations of motion. Here we take the flat spacetime as such a solution. The variables which are not defined by the equations of motion, the edge lengths of the (flat) skeleton, are defined, roughly speaking, by maximizing $F(l_0)$.

For the general solution, the curved background spacetime, situation is more complex. Some edge lengths are fixed by the equations of motion, others taken as independent ones are not fixed by them. At least one variable, the overall scale of the lengths, is not fixed by the equations of motion. It is quite expectable that if the curved spacetime tends to the flat one (the angle defects $\alpha$ tend to zero), the edge length fluctuations that become gauge translations in the flat spacetime case go to infinity according to the equations of motion. In reality, as mentioned in the above example (7), this means that these fluctuations are governed by the measure, as if they were gauge, as in the case of the flat spacetime. Thus, passing from the curved background spacetime to the flat one is continuous in the functional integral itself, and the task is to develop an appropriate technical procedure which in the particular case of the flat background reduces to the one considered here.

Important is invariance of the definition of the physical point in the configuration superspace at $l = l_0$ by maximizing (10) with respect to an arbitrary nondegenerate redefinition of the variables $l \rightarrow q = q(l)$. An important example is considered in the present paper estimate of the scale of the lengths $l$ or areas $A = l^2$. If the measure for the scale of areas takes the form $F(A)dA$, then the measure for the lengths is proportional to $F(l^2)dl$. The locations of the maxima of the functions $F(A)$ and $F(l^2)$ are different (not related by $A = l^2$). Taking into account that the action depends on $l$ as $l^2$, we get the scale of $\partial^2 S/\partial l_i \partial l_k$ equal to 1, and the problem is that of maximizing $F(l^2)l$. In area variables, $\partial^2 S/\partial A_i \partial A_k$ has the scale $1/A$, and the problem reduces to that of maximizing $F(A)\sqrt{A}$, that is, the same. Thus, the calculation of $l_0$ must be carried out with taking into account the dependence of the action on the variables $l$. 
5 Independent area tensors

Consider an extended configuration superspace where the area bivectors are generalized to arbitrary antisymmetric tensors that do not necessarily correspond to any edge vectors. An extension of Regge calculus to this superspace can be called "area tensor Regge calculus", in analogy with area Regge calculus [12], in which the triangle areas are independent freely chosen variables.

Now in area tensor Regge calculus we can develop the canonical Hamiltonian formalism described by the first class constraints and construct the path integral measure [26]. For that, we first pass to the continuous time limit by tending $\Delta t$ between the leaves, $v_{\sigma^2}$ for the $t$-like $\sigma^2$s and $\Omega_{\sigma^3} - 1$ for the leaf/diagonal $\sigma^3$s as $\Delta t$ to zero. The canonically conjugate variables are $v_{\sigma^2}$ for the leaf/diagonal triangles $\sigma^2$ and $\Omega_{\sigma^3}$ for the $t$-like tetrahedra $\sigma^3$. We can write out the canonical path integral measure. Then we choose the measure in the full discrete theory which would result in the canonical measure in the continuous time limit whatever direction is chosen as time one. This measure is $\prod_{\sigma^2 \in \mathcal{F}} d^6 v_{\sigma^2} \prod_{\sigma^3} D\Omega_{\sigma^3}$, where $\mathcal{F}$ is a set of triangles, which can be taken to be $t$-like, and throwing away the integration over the tensors of these is a kind of gauge fixing analogous to fixing the lapse-shift functions. The area tensor part of this measure can be rewritten as $\prod_{\sigma^2 \in \mathcal{F}} \delta^6 (v_{\sigma^2} - v_{\sigma^2}^{(0)}) \prod_{\sigma^2} d^6 v_{\sigma^2}$, which clearly shows the fixation of tensors $v_{\sigma^2}$ as constants $v_{\sigma^2}^{(0)}$.

Any $v_{\sigma^2}$ is defined in the local frame of a certain 4-simplex $\sigma^4$ which contains $\sigma^2$. This fact can be reflected by denoting $v_{\sigma^2}$ as $v_{\sigma^2 | \sigma^4}$. In principle, the measure can be made maximally symmetric by inserting the factors $d^6 v_{\sigma^2 | \sigma^4}$ for all other $\sigma^4$s that contain this $v_{\sigma^2}$. As long as nothing depends on the newly introduced variables $v_{\sigma^2 | \sigma^4}$, and also some intermediate regularization limiting the limits of integration to large but finite areas is implied, the new integrations in the measure lead simply to a normalization factor.

Another thing is that not all the area tensors in the 4-simplex are independent - only six out of ten. Therefore, there can be integrations for six triangles in the 4-simplex. As such, we can take the six triangles that contain the above common vertex $O$ for the tetrad. Thus, the product of the differentials of area tensors in the measure takes the form $\prod_{\sigma^4} \prod_{\sigma^2 \subset \sigma^4} d^6 v_{\sigma^2 | \sigma^4}$, the prime on the product over triangles in the 4-simplex means the restriction to the independent six triangles.

Once we accept the above described partly regular structure, where each 4-simplex
contains a \( t \)-like edge, we take the common vertex \( O \) as the initial (past) end point of the \( t \)-like edge. There are three \( t \)-like and three leaf/diagonal triangles for which the integrations are introduced. In the physical case, these six bivectors are parameterized in terms of the discrete tetrad 4-vectors \( (4) \).

The solution of the equations of motion of area tensor Regge calculus is trivial, but as long as there is no well-defined metric geometry, we can not say that the spacetime is flat. Projecting the functional measure in this theory in the configuration superspace onto the physical hypersurface whose points denote the sets of area tensors to which there correspond some sets of edge vectors, we get the measure for Regge calculus. When passing to the latter, the conditions originally introduced in area tensor Regge calculus as gauge ones will completely fix the dynamics, and therefore they should be relaxed. There are four components of the \( t \)-like vector at each vertex, the analogs of the lapse-shift functions, and the same number of values related to the \( t \)-like area tensors should be fixed. For example, considering the tetrad and bivectors \( (4) \) chosen at the given vertex \( \sigma^0 \) and defined in some 4-simplex as functions of this vertex, we can take the following four conditions,

\[
\tau_\alpha(\sigma^0) \circ \tau_\alpha(\sigma^0) = \varepsilon^2, \quad \alpha = 1, 2, 3, \quad \tau_1(\sigma^0) \circ \tau_2(\sigma^0) = 0. \tag{11}
\]

Here \( A \circ B \equiv A_{ab}B^{ab}/2 \) for matrices. A certain feature of these conditions is that if \( l^0_\alpha(\sigma^0) = 0, \alpha = 1, 2, 3 \) (Schwinger time gauge \( [27] \)), and \( l_\alpha(\sigma^0) \cdot l_\beta(\sigma^0) \propto \delta_{\alpha\beta} \) is given, then \( (11) \) fixes lapse \( l^0_0(\sigma^0) \propto \varepsilon \) and shift \( l_0(\sigma^0) = 0. \) Thus, the measure to be further projected onto the physical hypersurface is

\[
\prod_{\sigma^0} \left( \delta(\tau_1(\sigma^0) \circ \tau_2(\sigma^0)) \prod_{\alpha=1}^3 \delta(\tau_\alpha(\sigma^0) \circ \tau_\alpha(\sigma^0) - \varepsilon^2) \right) \prod_{\sigma^4} \prod'_{t-\text{like}} d^6\tau_{t^2} d^6\sigma^4 \cdot \prod_{\sigma^4} \prod'_{n_{t-\text{like}}} d^6\nu_{t^2} \prod_{\sigma^3} D\Omega_{\sigma^3}. \tag{12}
\]

6 Projecting path integral measure onto the physical hypersurface

To project the measure onto the physical hypersurface, we introduce the \( \delta \)-function factors required to single out area tensors such that, first, they correspond to certain edge vectors in each 4-simplex and, second, the resulting edge lengths of any two neighboring
4-simplices coincide on their common 3-face (the induced on the 3-face metric is continuous). We call these factors $\delta_{\text{metric}}$ and $\delta_{\text{cont}}$. The factor $\delta_{\text{metric}}$ enforces the conditions

$$\epsilon_{abcd}v_{\lambda \mu}^{ab}v_{\nu \rho}^{cd} \sim \epsilon_{\lambda \mu \nu \rho}$$

in each 4-simplex. These conditions are covariant in terms of the world indices. Therefore, the delta-function factor enforcing them is a scalar density. This agrees with the fact that the measures on the metric in the continuum GR are generally defined up to a power of $-g$ [28, 29]. We can write [26] the general form of this factor which is a scalar density symmetrically as

$$\int V^\eta \delta^{21} (\epsilon_{abcd}v_{\lambda \mu}^{ab}v_{\nu \rho}^{cd} - V\epsilon_{\lambda \mu \nu \rho}) dV$$

in the 4-simplex under consideration, and $\delta_{\text{metric}}$ is the product of such expressions for the 4-simplices. We consider further the value of the parameter $\eta = 20$, which is singled out by that then $\delta_{\text{metric}}$ is a scalar, that is, it is invariant under an arbitrary deformation of the 4-simplex and thus can be considered to express itself some local property of the metric, not of the 4-simplex.

It is convenient to divide $\delta_{\text{metric}}$ into two factors,

$$\delta_{\text{metric}} = \delta_{\text{metric}}(\tau, v)\delta_{\text{metric}}(v)$$

where $\delta_{\text{metric}}(v)$ is the part of this expression establishing conditions ($\epsilon_{abcd}v_{\alpha \beta}^{ab}v_{\gamma \delta}^{cd} = 0$) on the purely leaf/diagonal area tensors $v_{\alpha \beta}^{ab}$ ($v_{1}^{ab} \equiv v_{23}^{ab}, 2$ cycle perm) and invariant under an arbitrary linear transformation over the 3-dimensional world indices 1, 2, 3; that is, it is invariant under an arbitrary deformation of the 3-face so that it expresses some local property of the 3-metric, not of this 3-simplex.

It is convenient to parameterize area tensors by selfdual and antiselfdual vectors so that $\delta_{\text{metric}}(v)$ is the product of the expressions of the type

$$[v_1 \times v_2 \cdot v_3]^{4}\delta^6(v_{\alpha} \cdot v_{\beta} - v_{\alpha}^{\ast} \cdot v_{\beta}^{\ast})$$

over the 4-simplices. Here $v \equiv v^{\ast}$, $v^{\ast} = -v$. There are six conditions imposed by (16), which restrict 18 components of the three leaf/diagonal area tensors $v_{\alpha}^{ab}$ to correspond to 12 degrees of freedom of the 4-dimensional triad $l_{\alpha}^{a}$. The rest 14 conditions in $\delta_{\text{metric}}(\tau, v)$ restrict 18 components of the three $t$-like area tensors $\tau_{\alpha}^{ab}$ to four degrees of freedom, which just correspond to four components of the $t$-like vector $l_{0}^{a}$, as required.
In terms of the selfdual and antiselfdual vectors, we can write for the measure
\[ \prod \alpha \, d^6v^{ab}_\alpha \propto \prod \alpha \, d^3v_\alpha d^3v^*_\alpha, \]
where \( d^3v d^3v^* \equiv 2^3 d^3\text{Re} v d^3\text{Im} v \). The factor (16) implies that the triples of area vectors are connected according to
\[ v^*_\alpha = O v_\alpha, \quad O \in O(3, \mathbb{C}). \]
In fact, this \( O \) is a combination of inversion and rotation by an imaginary angle,
\[ O = -\exp(-i \psi \times \cdot), \quad \psi = 2 \xi |\xi| \text{arth} |\xi|, \quad \xi = l^0_\alpha l^\alpha, \quad l^\alpha = \frac{\varepsilon^{\alpha\beta\gamma} l^\beta \times l^\gamma / 2}{|l_1 \times l_2 \cdot l_3|}. \]  
(17)

The three degrees of freedom in \( O \) can be fixed by using Schwinger time gauge \( l^0_\alpha = 0, \alpha = 1, 2, 3 \). With the factor (16) applied, the measure becomes
\[ \prod \alpha \, d^3v_\alpha d^3v^*_\alpha \rightarrow [v_1 \times v_2 \cdot v_3]^3 \prod \alpha \, d^3v_\alpha. \]  
(18)

A feature of the factor (16) is its invariance with respect to the scaling (by means of a real number) of any involved area vector \( v_\alpha \). Accordingly, in (18) the order of a part of the measure in any vector \( v_\alpha \) goes to the same power of its scale \( v_\alpha = \sqrt{v^2_\alpha} \).

The factor \( \delta_{\text{cont}} \) ensures that the resulting edge lengths of any two neighboring 4-simplices coincide on their common 3-face. The situation when the two neighboring 4-simplices do not coincide on their common 3-face can be interpreted not necessarily as an ambiguity of the coordinates of the vertices of the common 3-face, but also as only a discontinuity on this 3-face of the metric induced from within each of these two 4-simplices. This allows to construct the \( \delta \)-function factor [30]; in particular, for the given 3-face it follows from the requirement of the invariance with respect to an arbitrary deformation of the 3-face leaving it in the same 3-plane. (This \( \delta \)-function factor can be also found from the properly regularized formally infinite terms in the Einstein action in the path integral arising when substituting the discontinuous metric there [31].) Let \( s_{\sigma^1|\sigma^4} \) be the length squared of the edge \( \sigma^1 \) in the 4-simplex \( \sigma^4 \). For a given 3-face \( \sigma^3 \) shared by 4-simplices \( \sigma^4 \) and \( \sigma^4' \), the considered factor takes the form
\[ V_{\sigma^3}^4 \prod_{\sigma^1 \subset \sigma^3} \delta(s_{\sigma^1|\sigma^4} - s_{\sigma^1|\sigma^4'}), \]  
(19)

where \( V_{\sigma^3} \) is the volume of \( \sigma^3 \). (The product of these factors for all \( \sigma^3 \)'s meeting at a given \( \sigma^2 \) is poorly defined due to superfluous delta-functions expressing the independence of the lengths of the edges of \( \sigma^2 \) in all the 4-simplices meeting at \( \sigma^2 \), and these deltas are properly canceled in a symmetrical way in \( \delta_{\text{cont}} \).)

The factor (19) can be reformulated in terms of the area vectors of the triangles constituting the 3-face. Namely, taking the area vectors \( v_{\sigma^2_i}, i = 1, 2, 3 \) of any three
with the help of this factor as

$$[v_{\sigma_2^3} \times v_{\sigma_2^2} \cdot v_{\sigma_2^3}^\prime] \delta^6(v_{\sigma_2^3} - v_{\sigma_2^2} \cdot v_{\sigma_2^3}).$$  \hspace{1cm} (20)$$

If there are integrations in the measure over the sets of $v_{\sigma_2^3}$ and $v_{\sigma_2^3}^\prime$, they are ”glued” with the help of this factor as

$$\prod_i d^3v_{\sigma_2^3} d^3v_{\sigma_2^3}^\prime \longrightarrow [v_{\sigma_2^3} \times v_{\sigma_2^2} \cdot v_{\sigma_2^3}^\prime]^3 \mathcal{O} \prod_i d^3v_{\sigma_2^3}^\prime. $$  \hspace{1cm} (21)$$

Here $\mathcal{O}$ ($\in O(3)$ or $\in O(2,1)$ depending on the signature of the subspace spanned by this triad) presents the remaining rotational degrees of freedom in $v_{\sigma_2^3}^\prime$.

Figure 1: 4-simplices surrounding a given triangle $\sigma^2$. Fat lines in the 4-simplices show sets of triangles with tensors defined in these 4-simplices integration over which is present in the measure.

In particular, consider the dependence on $v_{\sigma^2}$ for some given leaf/diagonal triangle $\sigma^2$, Fig.1. There are two $t$-like 3-simplices having $\sigma^2$ as their 2-face - past (in $t$) and future $\sigma^3$. There are two 4-simplices, $\sigma^4$ and $\sigma^4^\prime$, having $\sigma^3$ as their 3-face. It is the integrations over $v_{\sigma^2}$ and $v_{\sigma^2}^\prime$ that are exactly contained in the measure, the sets of the related area vectors being denoted as $v, \tau, v', \tau'$. Suppose the area vectors on this future 3-face are denoted as $\tau_1, \tau_2, v_3$ and $\tau_1', \tau_2', v_3'$ in $\sigma^4$ and $\sigma^4^\prime$, respectively, where $v_3$ is just the vector of the considered $\sigma^2$. The integrations on this 3-face from both the sides of $\sigma^4$ and $\sigma^4^\prime$ are ”glued” with the help of the factor (20) as

$$d^3\tau_1 d^3\tau_2 d^3v_3 d^3\tau_1' d^3\tau_2' d^3v_3' \longrightarrow [\tau_1 \times \tau_2 : v_3]^3 \mathcal{O} d^3\tau_1 d^3\tau_2 d^3v_3, $$  \hspace{1cm} (22)$$

$\mathcal{O} \in O(2,1)$. Considering this procedure with respect to the dependence on the scale $v_3$ of $v_3$, we can say that the measure $d^3v_3'$ goes to $v_3^3$. Before this we have already obtained $v_3^3 d^3v_3$ from $d^6v_3$ by means of the factor $\delta_{\text{metric}}$ in (18) and analogously $v_3^3 d^3v_3'$ (where $v_3' = v_3$). Finally, we can pass to spherical coordinates of $v_3$. In overall, as far as the
dependence on the area scale $v$ of the leaf/diagonal triangle is considered, we get $v^{11}dv$ for the measure on the physical hypersurface.

We note some conventionality of determining orders in individual area vector scales or areas. Typically, we have a determinant type factor, in which the areas can be formally multiplicatively separated like $[v_1 \times v_2 \cdot v_3] = v_1v_2v_3[n_1 \times n_2 \cdot n_3]$ where $n_\alpha = v_\alpha/v_\alpha$.

Using $v_4 = v_1 + v_2 + v_3$ for the fourth area vector of the tetrahedron surface, we can rewrite this factor as $[v_1 \times v_2 \cdot v_4]$ having another dependence on the areas proportional to $v_1v_2v_4$. The reason for such an ambiguity is that the area tensors are a dependent set of variables on the physical hypersurface and, strictly speaking, when changing an area, some other variables should be changed too. We use the result on the order in the individual area for the rough estimate of the extremum point of the corresponding factor in the full measure. It takes more precise meaning for the definition of the overall order in the common scale of the leaf/diagonal areas; vice versa, having estimated the overall order and adopting equivalence of the different triangles in this respect, we get the above twelve as the best choice for the order in the individual area if we try to approximate (a part of) the measure by the product over separate areas.

The estimated phase factor can be substituted into the path integral measure. In [9] we have found (the module of) the result of the functional integration over connection in the factorization approximation (the leading order in the expansion over the discrete lapse-shift functions), which is the product of some functions $\mathcal{N}_0(v_{\sigma^2}, v_{\sigma^3})$ over the leaf/diagonal triangles $\sigma^2$. Note that this calculation indirectly corresponds to the considered independent area tensor Regge calculus in that the (conditionally convergent) integral over the connection is defined from the requirement that it can be analytically continued as a function of area tensors to the independent such tensors. The resulting estimate for the measure referred to any such triangle with the scale of the area tensor $v$ takes the form

$$\mathcal{N}_0v^{11}dv = \left| \frac{1}{\frac{1}{2} \left( \frac{1}{2} - i \right)^2 v^2 + 1 \sh \left( \frac{\pi}{2} \left( \frac{1}{2} - i \right) v \right) } \right|^2 v^{11}dv. \quad (23)$$

In the physical spacelike region, $v^2 = -|v|^2$. The product of the measures of the type (23) over some number $T$ of the triangles in a certain domain contains a measure on their common area scale $v$ (expectedly, $v$ is close to the maxima of separate expressions of the form (23)) of the type $(\mathcal{N}_0v^{12})^Tv^{-1}dv$. According to the end of Section 4, since the action depends linearly on $v$, the optimal point $v = v_0$ is defined by the maximum
of \((N_0v^{12})^T v^{-1/2}\) or, at large \(T\), the maximum of \(N_0v^{12}\). There are two mechanisms for arising a maximum of this function in different regions of the parameter \(\gamma\).

1) Closeness to a pole (in the unphysical region \(\text{Im } v^2 \neq 0\)). At \(\gamma \ll 1\), there is a principal maximum at \(v^2 = -4\gamma^2\) (and some other maxima at \(v^2 = -4\gamma^2n^2, n \ll \gamma^{-1}\)).

2) Interplay between a power function and a decreasing exponent. For \(\gamma\) not small, including the case when the parity violation term is absent at \(\gamma = \infty\), the function behaves like \(v^{12}\) at small \(v\) and \(\exp(-\pi|v|)\) at large \(v\). The maximum of the function \(|v|^{12}\exp(-\pi|v|)\) is reached at \(|v| = 12/\pi\).

Passing to some length scale \(a\), \(2v_0 = ia^2\), we find that \(a\) varies from approximately \(2\sqrt{\gamma}\) at \(\gamma \ll 1\) to approximately \(\sqrt{24/\pi}\) at large \(\gamma\) (or \(\sqrt{2(\eta - 8)}/\pi\) if \(\eta\) in (14) is not predefined). In the usual units, \(a^2\) also includes the factor \(8\pi G\).

7 Propagator on a simple pseudo-mini-superspace system

Thus, we have a Regge lattice, which fluctuates around certain optimal areas, and we can consider its dynamics. In [23, 24, 25], the authors consider small fluctuations of the edge lengths of the simplest periodic Regge lattice and show correspondence with the continuum theory in the long wavelength limit.

To illustrate application of the above consideration, we use a simplified pseudo-mini-superspace system with the fields given at the nodes of the hypercubic lattice. One can not built a curved manifold of the flat cubes, therefore it is simply a finite difference approximation of the metric. There is the above freely chosen triad \(l_\alpha\) or \(v_\alpha\) at each vertex and practically one-to-one correspondence between these triads. There are three leaf/diagonal areas at each vertex, for which we assume existence of the optimal values according to the above consideration of the piecewise flat manifold. Besides that, from reflection symmetry, it is natural to take \(l_\alpha \cdot l_\beta = 0\) at \(\alpha \neq \beta\) as optimal. The optimal values of these three areas determine the optimal values of the three lengths \(|l_\alpha|\), for which we take the above \(a\). The discrete lapse-shift vector \(l^0_\alpha\) with the three \(l_\alpha\) forms three \(t\)-like areas with the tensors \(\tau_\alpha\). It can be chosen freely, but then fixed by, eg, (11) as an analog of the gauge condition of the continuum theory. In fact, we get some finite difference GR action. For the metric \(ds^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta - dt^2\) (synchronous frame), i. e.
the lapse-shift \( l_0^0 = 1, l_0 = 0 \), the continuum action has the form
\[
\frac{1}{2} \int R \sqrt{-g} \, d^4x = \frac{1}{4} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left( \gamma^{\alpha\gamma} \gamma^{\beta\delta} - \gamma^{\alpha\beta} \gamma^{\gamma\delta} \right) \gamma_{\alpha\beta} \gamma_{\gamma\delta} 
+ \left[ \gamma^{\alpha\zeta} \left( \gamma^{\beta\gamma} \gamma^{\epsilon\delta} - \gamma^{\beta\gamma} \gamma^{\epsilon\delta} \right) + \frac{1}{2} \gamma^{\alpha\zeta} \left( \gamma^{\alpha\beta} \gamma^{\epsilon\delta} - \gamma^{\alpha\beta} \gamma^{\epsilon\delta} \right) \right] \gamma_{\alpha\beta,\gamma\delta} \gamma_{\epsilon\zeta} \right\}. \tag{24}
\]

The propagator for the perturbations \( \Delta \gamma_{\alpha\beta} \) of the background metric \( \gamma_{\alpha\beta} \) in accordance with the standard definition using a source term in the action,
\[
S_J = \frac{1}{2} \int R \sqrt{-g} \, d^4x + \int J^{\alpha\beta} \Delta \gamma_{\alpha\beta} \sqrt{-g} \, d^4x,
\]
\[
\Delta \gamma_{\alpha\beta}(x) = - \int G_{\alpha\beta} \gamma^\delta(x,y) J_{\gamma\delta}(y) \sqrt{-g} \, d^4y,
\]  
(25)
takes the form in the momentum representation (if the background \( g_{\lambda\mu} = \text{const} \))
\[
G_{\alpha\beta} \gamma^\delta \sqrt{-g} = \frac{2}{p_0^2 - p^2} \left\{ \delta_0^\alpha \delta_0^\beta + \delta_0^\alpha \delta_0^\gamma - \gamma_{\alpha\beta} \gamma^\gamma + \frac{1}{p_0^2} \left[ \gamma_{\alpha\beta} p_\gamma p_\zeta \gamma^{\epsilon\gamma} \gamma^{\zeta\delta} + p_\alpha p_\beta \gamma^\delta 
- \left( p_\alpha \delta_0^\gamma + p_\beta \delta_0^\gamma \right) p_\epsilon \gamma^{\epsilon\delta} - \left( p_\alpha \delta_0^\delta + p_\beta \delta_0^\delta \right) p_\epsilon \gamma^{\epsilon\gamma} \right] + \frac{1}{p_0^2} p_\alpha p_\beta p_\epsilon p_\zeta \gamma^{\epsilon\gamma} \gamma^{\zeta\delta} \right\}. \tag{26}
\]
Here \( p^2 = p_\alpha p_\beta \gamma_{\alpha\beta} \). At \( l_0^0 \neq 1, l_0 = 0 \), the replacement \( p_0 \to p_0(l_0^0)^{-1} \) should be made.

The whole perturbation can be decomposed into the traceless transverse one \( \Delta \gamma_{\alpha\beta}^\perp \) (actually graviton degrees of freedom), longitudinal 3-vector, and scalar components,
\[
\Delta \gamma_{\alpha\beta} = \Delta \gamma_{\alpha\beta}^\perp + C \gamma_{\alpha\beta} + p_\alpha \chi_\beta + p_\beta \chi_\alpha.
\]  
(27)
Accordingly, the propagator (26) can also be decomposed into the proper graviton one \( G_{\alpha\beta}^{\perp} \gamma^\delta \), scalar \( G_{\alpha\beta}^{(0)} \gamma^\delta \), and vector \( G_{\alpha\beta}^{(1)} \gamma^\delta \) contributions,
\[
G_{\alpha\beta} \gamma^\delta = G_{\alpha\beta}^{\perp} \gamma^\delta + G_{\alpha\beta}^{(0)} \gamma^\delta + G_{\alpha\beta}^{(1)} \gamma^\delta,
\]
\[
G_{\alpha\beta}^{\perp} \gamma^\delta \sqrt{-g} = \frac{2}{p_0^2 - p^2} \left\{ \delta_0^\alpha \delta_0^\delta + \delta_0^\alpha \delta_0^\gamma - \gamma_{\alpha\beta} \gamma^\gamma + \frac{1}{p_0^2} \left[ \gamma_{\alpha\beta} p_\gamma p_\zeta \gamma^{\epsilon\gamma} \gamma^{\zeta\delta} + p_\alpha p_\beta \gamma^\delta 
- \left( p_\alpha \delta_0^\gamma + p_\beta \delta_0^\gamma \right) p_\epsilon \gamma^{\epsilon\delta} - \left( p_\alpha \delta_0^\delta + p_\beta \delta_0^\delta \right) p_\epsilon \gamma^{\epsilon\gamma} \right] + \frac{1}{p_0^2} p_\alpha p_\beta p_\epsilon p_\zeta \gamma^{\epsilon\gamma} \gamma^{\zeta\delta} \right\},
\]
\[
G_{\alpha\beta}^{(0)} \gamma^\delta \sqrt{-g} = - \frac{2}{p_0^2} \left\{ \gamma_{\alpha\beta} p_\gamma p_\zeta \gamma^{\epsilon\gamma} \gamma^{\zeta\delta} \right\},
\]
\[
G_{\alpha\beta}^{(1)} \gamma^\delta \sqrt{-g} = \frac{2}{p_0^2} \left\{ \frac{1}{p^2} \left[ -p_\alpha p_\beta \gamma^\delta + \left( p_\alpha \delta_0^\gamma + p_\beta \delta_0^\gamma \right) p_\epsilon \gamma^{\epsilon\delta} \right] \right\} + \left( p_\alpha \delta_0^\delta + p_\beta \delta_0^\delta \right) p_\epsilon \gamma^{\epsilon\gamma} \right\} - \left( \frac{1}{p_0^2} \right) p_\alpha p_\beta p_\epsilon p_\zeta \gamma^{\epsilon\gamma} \gamma^{\zeta\delta}. \tag{28}
\]

Above we consider the implicitly introduced variables \( u \) reducing the measure to the Lebesgue one (in Section 4), but to write out the propagator it is more convenient to use the original length or metric variables. This means that instead of \( \langle \Delta u_j \Delta u_k \rangle \) we consider
this correlator, linearly transformed to the variables of the squared lengths $s_j = l_j^2$ or metric, $\sum_{i,m}(\partial s_j(u_0)/\partial u_l)(\Delta u_l\Delta u_m)\partial s_k(u_0)/\partial u_m$. Of course, calculating the correlator of the exact length variables, $\langle \Delta l_j \Delta l_k \rangle$, $\Delta l_j = l_j(u_0 + \Delta u) - l_j(u_0)$, requires the knowledge of the exact measure $F(l)$ and, eg, the summation of the corresponding diagrams with $u$-lines.

Now in the considered pseudo-mini-superspace hypercubic system, the derivatives in the action $\partial \lambda$ should be replaced by finite difference operators. Such an operator for the difference between the points $x^\lambda + 1$ and $x^\lambda$ is $\delta \lambda = \exp(ip\lambda) - 1$ in the momentum representation, where $p_\lambda$ is now the quasimomentum. But it is most convenient to take a symmetrical finite difference analogue $(\delta \lambda - \bar{\delta} \lambda)/2$ for $\partial \lambda$. Then the discrete propagator follows simply by the replacement of $p_\lambda$ by $(\delta \lambda - \bar{\delta} \lambda)/2i = \sin p_\lambda$ in the continuum version. For the metric, we should take $\gamma_{\alpha\beta} = a^2 \delta_{\alpha\beta}$. Since we fix the scale of the $t$-like area tensors at the level $\epsilon$ (11), this fixes $l_0^a$: $l_0^a = 2\varepsilon a^{-1}, l_0^0 = 0$. One might take $\epsilon$ arbitrarily small to pass to a continuous time coordinate, but the Regge manifold shrunk in some direction is, to some extent, singular, and we approach from the side of relatively small values of $\epsilon$ to the symmetric variant $\epsilon = a^2/2, l_0^a = a$. The propagator becomes

$$a^2 G_{\alpha\beta}^\gamma^\delta = \frac{2}{\sin^2 p_0 - \sum_{\alpha} \sin^2 p_\alpha} \left\{ \delta_{\alpha\beta}^\gamma^\delta + \delta_{\alpha\gamma}^\delta^\beta - \gamma_{\alpha\beta}^\gamma^\delta \right\}
+ \frac{1}{\sin p_0} \left[ \gamma_{\alpha\beta} \sin p_\varsigma \sin p_\varsigma^\gamma p_\varsigma^\gamma \gamma_{\varsigma\delta}^\gamma^\beta + \sin p_\alpha \sin p_\delta \gamma^\gamma^\delta - \left( \sin p_\alpha \delta_{\beta}^\gamma + \sin p_\beta \delta_{\alpha}^\gamma \right) \sin p_\varsigma \gamma^\varsigma^\delta \right.
- \left( \sin p_\alpha^\lambda \delta_{\beta}^\delta + \sin p_\beta^\lambda \delta_{\alpha}^\delta \right) \sin p_\varsigma \gamma^\varsigma^\gamma \right\} + \frac{1}{\sin p_0} \sin p_\alpha \sin p_\beta \sin p_\gamma^\gamma^\delta.
$$

Since $a^{-1} \Delta \gamma_{\alpha\beta}$ and $a^3 \gamma^\gamma^\alpha \gamma^\delta^\beta \Delta \gamma_{\alpha\beta}$ have the meaning of length variations $\Delta l_j$, the LHS is a correlator of the type of $\langle \Delta l_j \Delta l_k \rangle$ (with $\Delta l_j \approx \sum_k (\partial l_j(u_0)/\partial u_k)\Delta u_k$). In the usual units, the RHS also includes the factor $8\pi G$.

The scalar and vector contributions to this propagator, as the continuum $G^{(0)}_{\alpha\beta} \gamma^\delta$ and $G^{(1)}_{\alpha\beta} \gamma^\delta$ (28), show that the perturbations of these fields are not fully propagated, but have the ability to grow monotonically (quadratically) in time. As these perturbations grow, they inevitably cease to be small as compared to the background metric. The considered cut off properties of the path integral measure prevent the metric from going far from the established optimal values. This can serve as a circumstance that limits the growth of perturbations and, possibly, leads to a modification of the propagator at long times (small $p_0$).
8 Conclusion

The modern canonical quantum GR, called Loop Quantum Gravity (LQG) [32], is based on the formulation of the classical GR as a theory of connections, rather than metrics. We just issue from an exact connection representation and the canonical formalism of the discrete mini-superspace theory (Regge calculus). Further, LQG aims at constructing a theory in which the gravitational field acts as a natural UV cut-off. Again, our approach has in common with this in that the elementary length scale $a$ is set dynamically.

The "non-perturbativity" refers to the procedure of excluding the connection type variables. This shows up in the existence of an optimal elementary area/length or background metric around which the required perturbative expansion can already be performed. Roughly speaking, the series of the diagrams with the connection field loops corresponding to the deviation of the dependence of the action on the connection from the bilinear one is summed up exactly in the suggestions of smallness of some tetrad variables (discrete lapse-shift functions) that can be fixed like gauge parameters in the continuum GR. As a result, the path integral measure obtained shows that the elementary area or length is located mainly in the Planck scale region (separated from zero and infinity). This means the appearance of a certain background metric for the vertices coordinatized in a given way (this is just the possibility that we choose) or, equivalently, stabilizing the coordinate lattice once a background metric is given.

Note that using Palatini (Christoffel connection) form of the Regge action we have obtained [20] even a more pronounced, delta-functional type of area fixation by the measure in the leading order of the lapse-shift for limiting cases of $\gamma$; although an expression in closed form for the measure for a finite nonzero $\gamma$, as that which we have in the case of the orthogonal connection $N_0$ (23), is not yet available.

The original theory has space-time symmetry, and a priori we are free to choose the direction in which the edge vectors are fixed manually, like a gauge, then in the orthogonal 3-dimensional subspace the edges will stabilize dynamically. Probably, what is happening looks similar to the one that takes place in the theory of Hořava [33], but in its origin has more similarity with spontaneous symmetry breaking.

The strategy used to transform the path integral assumes an analog of the fixation of the gauge consisting in fixing the four lapse-shift functions. Then we obtain a contribution to the propagator that is singular at small $p_0$ or at long times. This means that
some perturbations increase with time and should be stabilized by the measure. This point is to be studied to lead to a calculational prescription.

The background spacetime as a solution to the equations of motion (Regge equations) is taken to be flat, and the edge lengths in this flat spacetime are governed by the functional measure as being near some Planck scale values. The curved background spacetime as a solution to the equations of motion also can be considered. At small defects \( \alpha \ll 1 \), the edge length fluctuations that become gauge translations in the flat spacetime limit \( (\alpha \to 0) \) are governed by the functional measure. In this respect, in the first approximation, the spacetime can be taken flat; again, a calculational prescription should be developed to take into account the effect of the measure and determine the corrections.

The analysis is based on obtaining the result of the functional integration over connection (Section 3), both its phase (with the Regge action in the leading order) and module (the measure). For the phase, the stationary phase expansion is in increasing powers of the coupling constant or a dimensionless parameter - the ratio of the Planck length squared to the typical length \( l \) squared or simply \( l^{-2} \) for \( l \) in the Planck units. Later \( l \) is replaced by \( a \), and we have found that \( a^{-2} \simeq \pi/[2(\eta - 8)] \) is possible (the end of Section 6), and if we descend from the side of large \( \eta \), this indeed can be considered as a small parameter.

For the measure, we study corrections in powers of the scale of the lapse-shift functions. Also the area tensor part of the measure should be analyzed in more detail. In the action expanded in the Taylor series over (the variations of) the lengths, the interaction terms (cubic and higher order) come with additional negative powers of the scale \( a \). Therefore, the perturbative series for any correlator is, roughly, a power series in the parameter \( a^{-2} \). A complication is that we need to take into account the measure \( F(l) \) in the calculations in each order. Or, in terms of the above variables \( u \) in which the measure is a Lebesgue one, we need to substitute \( l = l(u) \) and sum up the diagrams with \( u \)-lines in each order. Of course, simplifying assumptions can be studied like linearizing the dependence \( l = l(u) \) at \( u = u_0 \) around which there is the perturbative expansion, \( \Delta l_j = \sum_k (\partial l_j(u_0)/\partial u_k) \Delta u_k \), which can be implied using the propagator written out here.

An immediate task might be to repeat the analysis for some real Regge lattice. The expression for the propagator is expected to be much more bulky than (29).

In what follows, we must average the result over possible Regge lattices. In this case,
it is not the propagator itself that needs to be averaged, but the final diagrams and amplitudes calculated on its basis.

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