Abstract: Dawson’s integral and related functions in mathematical physics that include the complex error function (Faddeeva’s integral), Fried–Conte (plasma dispersion) function, Jackson function, Fresnel function and Gordeyev’s integral are analytically evaluated in terms of the confluent hypergeometric function. And hence, the asymptotic expansions of these functions on the complex plane \( \mathbb{C} \) are derived by using the asymptotic expansion of the confluent hypergeometric function.

Keywords: Dawson’s integral, complex error function, plasma dispersion function, Fresnel functions, Gordeyev’s integral, confluent hypergeometric function, asymptotic evaluation

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1 Introduction

Let us consider the first-order initial value problem,

\[
D' + 2zD = 1, \quad D(0) = 0.
\]

Its solution given by the definite integral

\[
daw z = D(z) = e^{-z^2} \int_{0}^{z} e^{\eta^2} d\eta
\]

is known as Dawson’s integral \([1, 15, 22, 25]\). Dawson’s integral is related to several important functions (in integral form) in mathematical physics that include Faddeeva’s integral (also known as the complex error function or Kramp function, see \([8, 9, 17, 20, 25]\))

\[
w(z) = e^{-z^2}\left[1 + \frac{2i}{\sqrt{\pi}} e^{z^2} \text{daw} z\right] = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_{0}^{z} e^{\eta^2} d\eta\right),
\]

the Fried–Conte function (or plasma dispersion function, see \([4, 10]\))

\[
Z(z) = i \sqrt{\pi} w(z) = i \sqrt{\pi} e^{-z^2}\left[1 + \frac{2i}{\sqrt{\pi}} e^{z^2} \text{daw} z\right] = i \sqrt{\pi} e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_{0}^{z} e^{\eta^2} d\eta\right),
\]

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the Jackson function (see [13])

\[ G(z) = 1 + zZ(z) = 1 + i \sqrt{\pi} \omega z e^{-z^2} \left[ 1 + \frac{2i}{\sqrt{\pi}} e^{z^2} \right] \]

and the Fresnel functions \( C(z) \) and \( S(z) \) (see [1]) defined by the relation

\[ \frac{e^{i\pi z}}{\sqrt{1/z}} \ daw(\sqrt{1/z}) = \int_0^z e^{i\pi \eta^2} \ d\eta = C(z) + iS(z), \]

where

\[ C(z) = \int_0^z \cos(\pi \eta^2) \ d\eta \quad \text{and} \quad S(z) = \int_0^z \sin(\pi \eta^2) \ d\eta. \]

There is also Gordeyev’s integral [12] which is related to Dawson’s integral via the plasma dispersion (Fried–Conte) function \( Z \), and is given by

\[ G_v(\omega, \lambda) = \omega \int_0^\infty e^{i(\omega t - \lambda(1 - \cos t) - \nu t^2)/2} \ dt = \frac{-i\omega}{\sqrt{2\nu}} e^{-\lambda} \sum_{n=-\infty}^\infty I_n(\lambda)Z\left(\frac{\omega - n}{\sqrt{2\nu}}\right), \ \text{Re}(\nu) > 0, \]

where \( I_n \) is the modified Bessel function of the first kind [1], the real part of \( \omega \) is the wave frequency of an electrostatic wave propagating in a hot magnetized plasma, and \( \lambda \) and \( \nu \) are respectively the squares of the perpendicular and parallel components of the wave vector [19].

Another function related to Dawson’s integral was defined by Sitenko [23], and it only takes real arguments. It is given by

\[ \varphi(x) = 2x \ daw x = 2xe^{-x^2} \int_0^x e^{\eta^2} \ d\eta. \]

In that case, for a real argument \( x \), the Jackson function \( G(x) \), given by (1.5), can then take the form

\[ G(x) = 1 - \varphi(x) + i\sqrt{\pi}xe^{-x^2}. \]

These functions find their applications in astronomy, celestial mechanics, optical physics, plasma physics, planetary atmosphere, radiophysics, spectroscopy and so on [4, 5, 10, 12, 13, 17, 23, 24]. Therefore, it is important to adequately evaluate them. Having computed Dawson’s integral or Faddeeva’s integral, it is then straightforward to compute the other related integrals or functions such as the Fried–Conte function, Jackson function, Fresnel integral and Gordeyev function.

Moreover, these integrals are non-elementary. Being non-elementary means that they can neither be expressed in terms of elementary functions such as polynomials of finite degree, exponentials and logarithms, nor in terms of mathematical expressions obtained by performing finite algebraic combinations involving elementary functions [14, 18, 21]. For this reason, it is not possible to evaluate analytically these integrals in closed form or, in other words, in terms of elementary functions [3, 14, 18, 21]. To this end, intensive works have mainly focused on numerical approximations [2, 3, 6–9, 11, 15, 16, 20, 25, 26]. But numerical integrations do have drawbacks, they become very expensive and inaccurate very quickly as \( z \) becomes large or for some values of \( z \); see for example the recent work by Abrarov and Quine [3].

None of the above integrals can be evaluated analytically in close form, or in terms of elementary functions, as pointed out by Abrarov and Quine [3] of course, but one can express them in terms of a special function, the confluent hypergeometric function \( _1F_1 \) (see [1, 27]). Noting that Nijimbere [18, Theorem 1] has evaluated the non-elementary integral \( I_a^b e^{\lambda x} dx, \ a \geq 2 \) for any constant \( \lambda \) in terms of the confluent hypergeometric function \( _1F_1 \), as a first objective of this paper, we use the results in [18] to obtain an analytical
expression for Dawson’s integral in terms of the confluent hypergeometric function $1F_1$. And hence, we write Faddeeva’s integral and the other related integrals in terms of $1F_1$.

On the other hand, the confluent hypergeometric function is an entire function on the whole complex plane $\mathbb{C}$, and its properties are well known [1, 27]. For instance, its asymptotic expansion is given in [1, 27]. Therefore, as a second objective of this work, the asymptotic expansion of the confluent hypergeometric function is used to obtain the asymptotic expansions of the above functions (integrals) on the complex plane $\mathbb{C}$.

It is worth clarifying that the asymptotic expansion for Dawson’s integral and the related functions above would for instance be derived by using the asymptotic expansion of the complementary error function $\text{erfc} z$; see for example [27, (7.12.1)]. However dominant terms such as $|z| \gg 1$ are missing in the asymptotic expansion [27, (7.12.1)]. In other words, using a formula such as [27, (7.12.1)] would give incomplete asymptotic expansions for Dawson’s integral and the other related functions, and we would not like to see this in the applications. One should follow Nijimbere [18] in order to correctly derive the asymptotic expansion of $\text{erfc} z$.

2 Evaluation of Dawson’s integral and related functions in terms of $1F_1$

In this section, Dawson’s integral is evaluated in terms of the confluent hypergeometric $1F_1$, and relations (1.3)–(1.6) and (1.7) are used to express the Faddeeva, Fried–Conte, Jackson, Fresnel and Gordeyev integrals, respectively, in terms of $1F_1$.

2.1 Evaluation of Dawson’s integral in terms of $1F_1$

We use [18, Theorem 1] to express (1.2) in terms of the confluent hypergeometric functions $1F_1$, and then obtain

$$\text{daw} z = D(z) = e^{-z^2} \int_0^z e^{\eta^2} d\eta = ze^{-z^2} 1F_1 \left( \frac{1}{2} ; \frac{3}{2} ; z^2 \right). \quad (2.1)$$

One may also solve (1.1) and obtain the general solution (see [18])

$$D(z) = e^{-z^2} \left( \int e^{\eta^2} d\eta + C \right) = ze^{-z^2} 1F_1 \left( \frac{1}{2} ; \frac{3}{2} ; z^2 \right) + Ce^{-z^2}. \quad (2.2)$$

Therefore, applying the initial condition $D(0) = 0$ gives (2.1).

2.2 Evaluation of related functions in terms of $1F_1$

We now can evaluate the other functions related to Dawson’s integral (see Section 1) in terms of the confluent hypergeometric function. By using (1.3), Faddeeva’s integral is now given by

$$w(z) = e^{-z^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{\eta^2} d\eta \right) = e^{-z^2} 1F_1 \left( \frac{1}{2} ; \frac{3}{2} ; z^2 \right) \left[ 1 + \frac{2iz}{\sqrt{\pi}} 1F_1 \left( \frac{1}{2} ; \frac{3}{2} ; z^2 \right) \right]. \quad (2.3)$$

By using (1.4), the plasma dispersion function $Z(z)$, also called Fried–Conte function, is given by

$$Z(z) = i\sqrt{\pi} w(z) = i\sqrt{\pi} e^{-z^2} \left[ 1 + \frac{2iz}{\sqrt{\pi}} 1F_1 \left( \frac{1}{2} ; \frac{3}{2} ; z^2 \right) \right]. \quad (2.4)$$

By using (1.5), the Jackson function, denoted by $G(z)$, is given by

$$G(z) = 1 + zZ(z) = 1 + i\sqrt{\pi} w(z) = 1 + i\sqrt{\pi} e^{-z^2} \left[ 1 + \frac{2iz}{\sqrt{\pi}} 1F_1 \left( \frac{1}{2} ; \frac{3}{2} ; z^2 \right) \right]. \quad (2.5)$$
Using (1.6) (one may also use [18, Proposition 1]) gives
\[ z \int_{0}^{\infty} e^{i \pi \eta^2} \, d\eta = e^{i \pi z^2 / 4} \text{daw}(\sqrt{i \pi z}) = z \, _1F_1\left( \frac{1}{2} ; 1 ; i \pi z^2 \right), \] (2.6)
which is equivalent to [1, (7.3.25)]. Now using [18, (53) in Theorem 2] and [18, (57) in Theorem 3] (or using
[18, (50) and (51)]), we obtain
\[ C(z) = z \int_{0}^{\infty} \cos(\pi \eta^2) \, d\eta = z \, _1F_2\left( \frac{1}{4} ; \frac{1}{2} , \frac{5}{4} ; -\pi z^2 / 4 \right), \] (2.7)
and
\[ S(z) = z \int_{0}^{\infty} \sin(\pi \eta^2) \, d\eta = \frac{\pi z^3}{3} \, _1F_2\left( \frac{3}{4} ; \frac{3}{2} , \frac{7}{4} ; -\pi z^2 / 4 \right). \] (2.8)

By setting 
\[ z = \frac{\omega - n}{\sqrt{2} \nu} \]
in the Fried–Conte function and substituting in (1.7), Gordeyev’s integral can now be written in terms of \( _1F_1 \) as
\[ G_\nu(\omega, \lambda) = -i \omega e^{-\lambda} \sum_{n=\infty}^{\infty} I_n(\lambda) 2^\nu \Gamma\left( \frac{\nu + 1}{2} \right) + e^{-z^2} \left[ 1 + \frac{2i(\omega - n)}{\sqrt{2} \pi \nu} \right] \frac{\omega - n}{\sqrt{2} \nu} \, _1F_1\left( \frac{1}{2} ; 1 ; \frac{\omega - n}{\sqrt{2} \nu} \right). \] (2.9)

3 Asymptotic evaluation

In this section, we derive the asymptotic expansion of Dawson’s integral and related functions that include the complex error function, Fried–Conte function, Jackson function, Fresnel functions and Gordeyev’s function, using the asymptotic expansion of the confluent hypergeometric \( _1F_1 \). The results are summarized in
Theorems 3.2–3.4.

3.1 Asymptotic evaluation of Dawson’s integral

**Lemma 3.1.** Let \(|z| \gg 1\) and let \(a\) be any constant. If \(a\) is even, then
\[ z \, _1F_1\left( \frac{1}{a} ; 1 ; z^a \right) \sim \Gamma\left( 1 + \frac{1}{a} \right) e^{i \pi z^2 / 2|z|} \left[ 1 + \frac{\Gamma\left( \frac{1}{a} + 1 \right)}{z^a} + O\left( \frac{1}{z^{2a}} \right) \right] + \frac{e^{z^2}}{az^{a-1}} \left[ 1 + \frac{\Gamma\left( \frac{2}{a} - \frac{1}{a} \right)}{z^a} + O\left( \frac{1}{z^{2a}} \right) \right]. \] (3.1)

If \(a\) is odd on the other hand, then
\[ z \, _1F_1\left( \frac{1}{a} ; 1 ; z^a \right) \sim \Gamma\left( 1 + \frac{1}{a} \right) e^{i \pi z^2 / 2|z|} \left[ 1 + \frac{\Gamma\left( \frac{1}{a} + 1 \right)}{z^a} + O\left( \frac{1}{z^{2a}} \right) \right] + \frac{e^{z^2}}{az^{a-1}} \left[ 1 + \frac{\Gamma\left( \frac{2}{a} - \frac{1}{a} \right)}{z^a} + O\left( \frac{1}{z^{2a}} \right) \right]. \] (3.2)

And the positive sign in (3.1) and (3.2) is taken if
\[ -\frac{\pi}{2a} + \frac{2k\pi}{a} < \arg(z) < \frac{3\pi}{2a} + \frac{2k\pi}{a}, \] (3.3)
while the negative sign is taken if
\[- \frac{3\pi}{2a} + \frac{2k\pi}{a} < \arg(z) < -\frac{\pi}{2a} + \frac{2k\pi}{a}, \]
for \(k = 0, 1, 2, \ldots\).

**Proof.** To prove (3.1) and (3.2), we use the asymptotic expansion of the confluent hypergeometric function valid for \(|\xi| \gg 1\) (see [1, (13.5.1)]):

\[
\frac{1}{\Gamma(b)} \frac{\Gamma(a; \xi)}{\Gamma(b-a)} = e^{\frac{\xi}{b-a}} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (-\xi)^{-n} + O(|\xi|^{-R}) \right\}
\]

\[
+ e^{\frac{\xi}{b-a}} \left\{ \sum_{n=0}^{S-1} \frac{(b-a)_n(1-a)_n}{n!} (-\xi)^{-n} + O(|\xi|^{-S}) \right\},
\]

where \(a \) and \(b \) are constants, and the positive sign being taken if
\[- \frac{\pi}{2} < \arg(\xi) < \frac{3\pi}{2}, \]
and the negative sign being taken if
\[- \frac{3\pi}{2} < \arg(\xi) \leq -\frac{\pi}{2}. \]

We now set \(\xi = z^a, a = \frac{1}{a} + 1 \) in (3.4), and obtain

\[
\frac{1}{\Gamma(\frac{1}{a} + 1)} \frac{1}{\Gamma(a; z)} = e^{\frac{z}{2a}} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-R} \right\}
\]

\[
+ \frac{1}{\Gamma(\frac{1}{a})} \frac{1}{z^a} \left\{ \sum_{n=0}^{S-1} \frac{(1)_n(1-\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-S} \right\}.
\]

Then for \(|z| \gg 1\),

\[
e^{\frac{z^a}{2a}} \left\{ \sum_{n=0}^{\infty} \frac{\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-R} \right\} \sim \left\{ \frac{e^{\frac{z^a}{2a}}}{|z|} \left[ 1 + \frac{\Gamma(\frac{1}{a} + 1)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right]\right. \text{ if } a \text{ is even,}
\]

\[
\left. + \frac{e^{\frac{z^a}{2a}}}{z} \left[ 1 + \frac{\Gamma\left(\frac{1}{a} + 1\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right]\right. \text{ if } a \text{ is odd,}
\]

while

\[
\frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{1}{z^a} \left\{ \sum_{n=0}^{S-1} \frac{(1)_n(1-\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-S} \right\} \sim \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(2 - \frac{1}{a}\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right].
\]

Therefore, for \(|z| \gg 1\),

\[
1 \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{1}{z^a} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-R} \right\} \sim \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(\frac{1}{a} + 1\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right]
\]

for any even constant \(a\). And for any odd constant \(a\),

\[
1 \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{1}{z^a} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-R} \right\} \sim \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(\frac{1}{a} + 1\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right].
\]

Hence for \(|z| \gg 1\), and for any even \(a\),

\[
z \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{1}{z^a} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-R} \right\} \sim \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(\frac{1}{a} + 1\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right] + \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(2 - \frac{1}{a}\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right],
\]

while for \(|z| \gg 1\), and for any odd \(a\),

\[
z \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{1}{z^a} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{a})_n}{n!} (z^a)^{-n} + O(z^a)^{-R} \right\} \sim \frac{1}{\Gamma\left(\frac{1}{a}\right)} \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(\frac{1}{a} + 1\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right] + \frac{e^{\frac{z^a}{2a}}}{z^a} \left[ 1 + \frac{\Gamma\left(2 - \frac{1}{a}\right)}{z^a} + O\left(\frac{1}{z^{2a}}\right) \right].
\]
On the other hand, we observe that 

\[ \xi = z^a \] implies \[ z = \xi^{\frac{1}{a}} = |\xi|^{\frac{1}{a}} e^{i \text{arg}(\xi)} \frac{1}{a} = |\xi|^{\frac{1}{a}} e^{i \text{arg}(\xi) \frac{1}{a}}. \]

Therefore, (3.6) gives 

\[- \frac{\pi}{2a} + \frac{2k\pi}{a} < \text{arg}(z) < \frac{3\pi}{2a} + \frac{2k\pi}{a},\]

which is exactly (3.3), while (3.7) gives 

\[- \frac{3\pi}{2a} + \frac{2k\pi}{a} < \text{arg}(z) < - \frac{\pi}{2a} + \frac{2k\pi}{a},\]

which is exactly (3.4), and where \( k = 0, 1, 2, \ldots. \)

**Theorem 3.2.** Let \( |z| \gg 1. \) If \(- \frac{\pi}{4} + k\pi < \text{arg}(z) < \frac{3\pi}{4} + k\pi, k = 0, 1, 2, \ldots, \) then

\[
daw z \sim i \frac{\sqrt{\pi}}{2} e^{-z^2} \left[ 1 + \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right) \right] + \frac{1}{2z} + \frac{\sqrt{\pi}}{4z^3} + O\left( \frac{1}{z^5} \right).
\]

(3.8)

If, on the other hand, \(- \frac{3\pi}{4} + k\pi < \text{arg}(z) < - \frac{\pi}{4} + k\pi, \) then

\[
daw z \sim -i \frac{\sqrt{\pi}}{2} e^{-z^2} \left[ 1 + \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right) \right] - \frac{1}{2z} - \frac{\sqrt{\pi}}{4z^3} + O\left( \frac{1}{z^5} \right).
\]

(3.9)

Therefore, for \( |z| > 1, \) if \(- \frac{\pi}{4} + k\pi < \text{arg}(z) < \frac{3\pi}{4} + k\pi, k = 0, 1, 2, \ldots, \) then the following assertions hold:

(i) Feddeeva’s integral \( w(z) \) is approximated by

\[
w(z) \sim e^{-z^2} \left[ - \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right) \right] + \frac{i}{\sqrt{\pi}} \left[ \frac{1}{z} + \frac{\sqrt{\pi}}{2z^3} + O\left( \frac{1}{z^5} \right) \right].
\]

(3.10)

(ii) The Fried–Conte (plasma dispersion) function \( Z(z) \) is approximated by

\[
Z(z) \sim i e^{-z^2} \left[ - \frac{\pi}{2z^2} + O\left( \frac{1}{z^4} \right) \right] - \frac{1}{2z} - \frac{\sqrt{\pi}}{4z^3} + O\left( \frac{1}{z^5} \right).
\]

(3.11)

(iii) The Jackson function \( G(z) \) is approximated by

\[
G(z) \sim i e^{-z^2} \left[ - \frac{\pi}{2z^2} + O\left( \frac{1}{z^4} \right) \right] - \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right).
\]

(3.12)

While on the other hand, if \(- \frac{3\pi}{4} + k\pi < \text{arg}(z) < - \frac{\pi}{4} + k\pi, \) then the following assertions hold:

(iv) Feddeeva’s integral \( w(z) \) is approximated by

\[
w(z) \sim e^{-z^2} \left[ 2 + \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right) \right] + \frac{i}{\sqrt{\pi}} \left[ \frac{1}{z} + \frac{\sqrt{\pi}}{2z^3} + O\left( \frac{1}{z^5} \right) \right].
\]

(3.13)

(v) The Fried–Conte (plasma dispersion) function \( Z(z) \) is approximated by

\[
Z(z) \sim i \sqrt{\pi} e^{-z^2} \left[ 2 + \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right) \right] - \frac{1}{2z} - \frac{\sqrt{\pi}}{2z^3} + O\left( \frac{1}{z^5} \right).
\]

(3.14)

(vi) The Jackson function \( G(z) \) is approximated by

\[
G(z) \sim i \sqrt{\pi} e^{-z^2} \left[ 2 + \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right) \right] - \frac{\sqrt{\pi}}{2z^2} + O\left( \frac{1}{z^4} \right).
\]

(3.15)

**Proof.** For \( \alpha = 2, \) and having in mind that \( \alpha = 2 \) is even, (3.1) becomes

\[
z_1 F_1 \left( \frac{1}{2}; \frac{3}{2}; z^2 \right) \sim \Gamma\left( \frac{3}{2} \right) \frac{z^{1/2} e^{z/2}}{|z|} \left[ 1 + \frac{\Gamma\left( \frac{3}{2} \right)}{2z^2} + O\left( \frac{1}{z^4} \right) \right] + \frac{e^{z/2}}{2z} \left[ 1 + \frac{\Gamma\left( \frac{3}{2} \right)}{2z^2} + O\left( \frac{1}{z^4} \right) \right].
\]

(3.16)
where the positive sign is taken if
\[-\frac{\pi}{4} + k\pi < \arg(z) < \frac{3\pi}{4} + k\pi,\]
while the negative sign is taken if
\[-\frac{3\pi}{4} + k\pi < \arg(z) < -\frac{\pi}{4} + k\pi,\]
for \(k = 0, 1, 2, \ldots\).

When substituting (3.16) in (2.2), the resulting equation together with (3.17) and (3.18) respectively gives (3.8) if \(-\frac{\pi}{4} + k\pi < \arg(z) < \frac{3\pi}{4} + k\pi\) and (3.9) if \(-\frac{3\pi}{4} + k\pi < \arg(z) < -\frac{\pi}{4} + k\pi\). Hence, substituting (3.8) and (3.9) in (2.3), (2.4) and (2.5) respectively gives (3.10), (3.11) and (3.12) if \(-\frac{\pi}{4} + k\pi < \arg(z) < \frac{3\pi}{4} + k\pi\), and (3.13), (3.14) and (3.15) if \(-\frac{3\pi}{4} + k\pi < \arg(z) < -\frac{\pi}{4} + k\pi\).

### 3.2 Asymptotic evaluation of Fresnel functions

In this section, we use the asymptotic expansion of the confluent hypergeometric function \(_1F_1\) to derive the asymptotic expansion of Fresnel's functions \(C(z)\) and \(S(z)\). And the results are described in Theorem 3.3.

**Theorem 3.3.** For \(|z| \gg 1\), Fresnel's functions \(C(z)\) and \(S(z)\) are asymptotically given by

\[
C(z) \sim \left\{ \begin{array}{ll}
\frac{\sqrt{2}}{4} - \frac{1}{\pi} \frac{\sin(\pi z)}{2 \pi z} - \frac{\cos(\pi z)}{4(\pi)^{3/2} z^3} + O\left(\frac{1}{z^4}\right) & \text{if } -\frac{\pi}{2} + k\pi < \arg(z) < -\frac{\pi}{2} + k\pi,
\end{array} \right.
\]

\[
S(z) \sim \left\{ \begin{array}{ll}
\frac{\sqrt{2}}{4} + \frac{1}{\pi} \frac{\sin(\pi z)}{2 \pi z} + \frac{\cos(\pi z)}{4(\pi)^{3/2} z^3} + O\left(\frac{1}{z^4}\right) & \text{if } -\frac{\pi}{2} + k\pi < \arg(z) < -\frac{\pi}{2} + k\pi,
\end{array} \right.
\]

where \(k = 0, 1, 2, \ldots\).

**Proof.** Let us first assume that, for \(|z| \gg 1\), Fresnel's functions are approximated by \(C(z) \sim \tilde{C}(z)\) and \(S(z) \sim \tilde{S}(z)\). Then using (2.6)–(2.8) yields

\[
\int_0^z e^{in\eta^2} d\eta = e^{inz^2}/\sqrt{i\pi} \text{daw}(\sqrt{i\pi}z) = z \, _1F_1\left(\frac{1}{2}; \frac{3}{2}; i\pi z^2\right)
\]

\[
= \int_0^z \cos(n\eta^2) \, d\eta + i \int_0^z \sin(n\eta^2) \, d\eta
\]

\[
= \pi \, _1F_2\left(\frac{1}{4}; \frac{1}{2}; \frac{\pi z^2}{4}\right) + \pi \, _1F_2\left(\frac{3}{4}; \frac{5}{2}; \frac{\pi z^2}{4}\right)
\]

\[
= C(z) + iS(z) \sim \tilde{C}(z) + i\tilde{S}(z), \quad |z| \gg 1.
\]

And we observe that the asymptotic expansion of (3.21) is a sum of two parts \(\tilde{C}(z)\) and \(\tilde{S}(z)\).

Now, setting \(\xi = iz^2\), \(a = \frac{1}{4} = \frac{1}{4}\) and \(b = \frac{3}{2} = \frac{3}{2}\) in (3.5) and taking into account that \(a = 2\) is even yields

\[
\frac{1}{\Gamma\left(\frac{3}{2}\right)} e^{iz^2} (inz^2)^{\frac{1}{2}} \sum_{n=0}^{R-1} \frac{(-iz^2)^{2n}}{n!} + O(|z|^{-R})
\]

\[
= \frac{1}{\Gamma\left(\frac{3}{2}\right)} iinz^2 \sum_{n=0}^{R-1} \frac{(1/2)_n}{n!} (\xi)^{-n} + O(|\xi|^{-S}).
\]

Moreover,

\[
\xi = iz^2 \quad \text{implies} \quad z = \left(\frac{\xi}{i\pi}\right)^{\frac{1}{2}} = \left(\frac{\xi}{i\pi}\right)^{\frac{1}{2}} e^{\frac{\text{arg}(\xi)}{2}},
\]
Then for any fixed $i$ and the confluent hypergeometric function

This gives

$$\arg(z) = \frac{\arg(\xi)}{2} - \frac{\pi}{4} + k\pi, \quad k = 0, 1, 2, \ldots .$$

Rearranging terms on one hand, while neglecting higher-order terms on another hand, yields

$$\int_0^z e^{i\eta^2} \, d\eta = z \, I_1\left(\frac{1}{2} \left( \frac{3}{2} + i\pi z^2 \right) \right) = \pm \frac{\sqrt{z}}{4} + \sqrt{\frac{z}{\pi}} - \frac{\sin(\pi z^2)}{2\pi z} - \cos(\pi z^2) 4(\pi z^2)^{1/2} + O\left( \frac{1}{|z|^2} \right)$$

$$+ \left[ \frac{\sqrt{z}}{4} + \sqrt{\frac{z}{\pi}} - \frac{\sin(\pi z^2)}{2\pi z} + \cos(\pi z^2) 4(\pi z^2)^{1/2} + O\left( \frac{1}{|z|^2} \right) \right],$$

(3.23)

where the positive sign is now taken if

$$-\frac{\pi}{2} + k\pi < \arg(z) < -\frac{\pi}{2} + k\pi,$$

and the negative sign being taken if

$$-\pi + k\pi < \arg(z) < -\pi + k\pi.$$

Hence, comparing (3.22) with (3.23) gives (3.19) and (3.20).

3.3 Asymptotic evaluation of Gordeyev's integral

In this section, we derive the asymptotic expansion of Gordeyev's integral using the asymptotic expansion of the confluent hypergeometric function $I_1$ and present the results in Theorem 3.4.

We first observe that for complex $\omega = \omega_r + i\omega_i$ and complex $v = v_r + iv_i$, if we set $\sqrt{v_r + iv_i} = \tilde{v}_r + i\tilde{v}_i$, where the subscripts $r$ and $i$ stand for the real and imaginary parts respectively, then

$$\frac{\omega - n}{\sqrt{2}v} = \frac{(\omega_r + n)v_r + \omega_i v_i}{\sqrt{2}(v_r^2 + v_i^2)} + i\frac{\omega iv_i - (\omega_r + n)v_i}{\sqrt{2}(v_r^2 + v_i^2)}.$$

And so

$$\arg\left( \frac{\omega - n}{\sqrt{2}v} \right) = \arctan \left[ \frac{\omega iv_i - (\omega_r + n)v_i}{(\omega_r + n)v_i + \omega iv_i} \right].$$

**Theorem 3.4.** Let $\lambda = \lambda_r + i\lambda_i$, $\omega = \omega_r + i\omega_i$, $v = v_r + iv_i$ and $\sqrt{v} = \sqrt{v_r + iv_i} = \tilde{v}_r + i\tilde{v}_i$, where the subscripts $r$ and $i$ stand for the real and imaginary parts respectively, and let

$$\theta = \arg\left( \frac{\omega - n}{\sqrt{2}v} \right) = \arctan \left[ \frac{\omega iv_i - (\omega_r + n)v_i}{(\omega_r + n)v_i + \omega iv_i} \right].$$

(i) Then for any fixed $v$ and any fixed $\omega$, Gordeyev's integral $G_v(\omega, \lambda)$ is asymptotically given by

$$G_v(\omega, \lambda) \sim -\frac{i\omega}{2\sqrt{\pi v\lambda}} \sum_{n=\infty}^{\infty} \left[ 1 - \frac{4n^2 - 1}{8\lambda} + O\left( \frac{1}{\lambda^2} \right) \right] Z\left( \frac{\omega - n}{\sqrt{2}v} \right), \quad |\lambda| \gg 1,$$

(3.24)

where $Z$ is the plasma dispersion (Fried–Conte) function, see (1.4), and $-\frac{\pi}{4} < \arg(\lambda) < \frac{\pi}{4}$.

(ii) For any fixed $\lambda$, if $|\omega| \gg 1$ and $v$ is fixed, or if $|v| \sim 0$ and $\omega$ is fixed, then

$$G_v(\omega, \lambda) \sim -\frac{i\omega}{2\sqrt{\pi v}} e^{-\lambda} \sum_{n=\infty}^{\infty} I_n(\lambda) \left[ i e^{-\frac{\pi}{8\lambda}} \left( \frac{\sqrt{2v}}{\omega - n} \right)^2 + O\left( \frac{\sqrt{2v}}{\omega - n} \right)^2 \right]$$

$$- \frac{1}{2} \left( \frac{\sqrt{2v}}{\omega - n} \right) - \frac{\sqrt{\pi}}{4} \left( \frac{\sqrt{2v}}{\omega - n} \right)^3 + O\left( \frac{\sqrt{2v}}{\omega - n} \right)^5$$

(3.25)

if $-\frac{\pi}{4} + k\pi < \theta < \frac{3\pi}{4} + k\pi$, $k = 0, 1, 2, \ldots$ And

$$G_v(\omega, \lambda) \sim -\frac{i\omega}{2\sqrt{v}} e^{-\lambda} \sum_{n=\infty}^{\infty} I_n(\lambda) \left[ i\sqrt{\pi} e^{-\frac{\pi^2}{8\lambda}} \left( 2 + \frac{\sqrt{\pi}}{2} \left( \frac{\sqrt{2v}}{\omega - n} \right)^2 + O\left( \frac{\sqrt{2v}}{\omega - n} \right)^2 \right) \right]$$

$$- \frac{1}{2} \left( \frac{\sqrt{2v}}{\omega - n} \right) - \frac{\sqrt{\pi}}{4} \left( \frac{\sqrt{2v}}{\omega - n} \right)^3 + O\left( \frac{\sqrt{2v}}{\omega - n} \right)^5$$

(3.26)

if $-\frac{3\pi}{4} + k\pi < \theta < -\frac{\pi}{4} + k\pi$. 

yields respectively
\[ G_\nu(\omega, \lambda) \sim \frac{-i\omega}{2\sqrt{\pi n}} \sum_{n=-\infty}^{\infty} \left\{ i\sqrt{\pi} e^{-\frac{(\omega-n)^2}{4\lambda}} \left[ 1 - \frac{\pi}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 + O\left( \frac{\sqrt{2\nu}}{\lambda(\omega - n)} \right)^2 \right] \right. 
\[ - \frac{1}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 + \frac{\pi}{4} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^3 + O\left( \frac{\sqrt{2\nu}}{\lambda^2(\omega - n)} \right) \right\} \] 
(3.27)

if \( -\frac{n}{4} + k\pi \theta < \frac{3\pi}{4} + k\pi, k = 0, 1, 2, \ldots \) And

\[ G_\nu(\omega, \lambda) \sim \frac{-i\omega}{2\sqrt{\pi n}} \sum_{n=-\infty}^{\infty} \left\{ i\sqrt{\pi} e^{-\frac{(\omega-n)^2}{4\lambda}} \left[ 1 - \frac{\pi}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 + \frac{\pi}{4} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^3 + O\left( \frac{\sqrt{2\nu}}{\lambda^2(\omega - n)} \right) \right] \right\} \] 
(3.28)

if \( -\frac{3n}{4} + k\pi \theta < \frac{\pi}{4} + k\pi, k = 0, 1, 2, \ldots \)

\[ \text{Proof.} \ (i) \text{ For } |\lambda| \gg 1, \text{ we have, using [1, (9.7.1)], that} 
\] 
\[ I_n(\lambda) = e^{\lambda^2/2\lambda} \left[ 1 - \frac{4n^2 - 1}{8\lambda} + O\left( \frac{1}{\lambda^2} \right) \right], \quad -\frac{n}{2} < \arg(\lambda) < \frac{n}{2}. \]

Substituting in (2.9) gives (3.24).

(ii) Setting \( z = \frac{\omega-n}{\sqrt{2\nu}} \) in (3.11) and (3.14) and letting \( |\nu| \gg 1 \) while \( \nu \) is fixed, or \( |\nu| \to 0 \) while \( \omega \) is fixed, yields respectively

\[ Z(\omega - n) \sim ie^{-\frac{(\omega-n)^2}{2\nu}} \left[ -\frac{\pi}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 + O\left( \frac{\sqrt{2\nu}}{\lambda(\omega - n)} \right)^4 \right] \]
\[ - \frac{1}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 + \frac{\pi}{4} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^3 + O\left( \frac{\sqrt{2\nu}}{\lambda^2(\omega - n)} \right)^5 \] 
(3.29)

if \( -\frac{n}{4} + k\pi \theta < \arg(\omega - n/\sqrt{2\nu}) < \frac{3n}{4} + k\pi, k = 0, 1, 2, \ldots \), and

\[ Z(\omega - n) \sim i\sqrt{\pi} e^{-\frac{(\omega-n)^2}{2\nu}} \left[ 2 + \frac{\pi}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 + O\left( \frac{\sqrt{2\nu}}{\lambda^2(\omega - n)} \right)^4 \right] \]
\[ - \frac{1}{2} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^2 - \frac{\pi}{4} \left( \frac{\sqrt{2\nu}}{\omega - n} \right)^3 + O\left( \frac{\sqrt{2\nu}}{\lambda^2(\omega - n)} \right)^5 \] 
(3.30)

if \( -\frac{3n}{4} + k\pi \theta < \arg(\omega - n/\sqrt{2\nu}) < -\frac{n}{4} + k\pi, k = 0, 1, 2, \ldots \). Hence, substituting (3.29) and (3.30) in (2.10) respectively gives (3.25) and (3.26).

(iii) On the other hand, combining (3.25) and (3.26) with (3.24) respectively gives (3.27) and (3.28).

\[ \Box \]

\section*{4 Discussion and conclusions}

Having evaluated Dawson’s integral in terms of the confluent hypergeometric function, the other related functions including the complex error function (Faddeeva’s integral), Fried–Conte (plasma dispersion) function, Jackson function, Fresnel functions and Gorbeyev’s integral were also evaluated in terms of the confluent hypergeometric function.

By using the asymptotic expansions of the confluent hypergeometric function, the asymptotic expansion for \(|z| \gg 1\) of Dawson’s integral were derived and consequently the asymptotic expansions of the complex error function (Faddeeva’s integral), Fried–Conte (plasma dispersion) function, Jackson function, Fresnel
functions and Gordeyev’s integral were evaluated (Theorem 3.2, Theorem 3.3 and Theorem 3.4). To obtain, on the other hand, the asymptotic expressions of these functions for small arguments $|z| \ll 1$, one should keep the first few terms in the series representing the confluent hypergeometric function.

It is also important to point out that asymptotic expansions of Gordeyev’s integral that take into account the properties of an electromagnetic wave propagating in a hot plasma, which are the wave frequency, the perpendicular and parallel components of the wave vector, were also carefully derived (Theorem 3.4).

Moreover, writing these functions in terms of the confluent hypergeometric function confirms once again that these functions are entire on the whole complex plane $\mathbb{C}$ since the confluent hypergeometric function is entire on the whole complex plane $\mathbb{C}$.

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