High Energy Hadron Production
as Self-Organized Criticality

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Abstract

In high energy nuclear collisions, production rates of light nuclei as well as those of hadrons and hadronic resonances agree with the predictions of an ideal gas at a temperature \( T = 155 \pm 10 \) MeV. In an equilibrium hadronic medium of this temperature, light nuclei cannot survive. We propose that the observed behavior is due to an evolution in global non-equilibrium, leading to self-organized criticality. At the confinement point, the initial quark-gluon medium becomes quenched by the vacuum, breaking up into all allowed free hadronic and nuclear mass states, without formation of any subsequent thermal hadronic medium.

The production rates of hadrons and hadronic resonances in high energy collisions, from \( e^+e^- \) to \( A-A \), have for some years been somewhat enigmatic. The relative yields are found to agree with those obtained from an ideal resonance gas \cite{1} at the (pseudo)critical confinement temperature \( T_c = 155 \pm 10 \) MeV \cite{2,3}; with an additional parameter specifying an overall production volume \( V \) \cite{4}, so are the absolute yields\cite{4}. The conventional interpretation assumes that the collision leads to a hot quark-gluon plasma, which evolves, cools and at \( T_c \) undergoes a transition to a hadron gas of that temperature. This interacting hadron gas then expands, cools further, and eventually freezes out into free hadrons. The curious feature is that the hadron abundances are already specified once and for all at \( T_c \) and are not subsequently modified in the evolution of the hadron gas. In general, the interactions are taken to cease completely only at a kinematic freeze-out at a somewhat lower temperature. The ideal gas description at \( T_c \) is found to be valid even for resonances such as the \( \rho \) or the \( \Delta \), which are in principle easy to break up in a hadron gas of 155 MeV temperature. To maintain for hadron production in nuclear collisions both the ideal hadron gas abundances as given at \( T_c \) and the existence of a thermal hadronic medium below the critical point, one has taken recourse to various features: a very rapid decrease of the hadron gas density, very weak interactions through hadronic collisions, a very short

\footnote{At lower energies, additional parameters for baryon density and strangeness suppression may be needed.}
life-time of the hadron gas, and more (see e.g. [5]). In elementary collisions \((e^+e^- \text{ or } p-p)\), hadron production has been accounted for by quantum mechanisms leading to behavior of a thermal form, without implying a subsequent thermal hadronic medium [6–9].

The mentioned enigma was further enhanced by recent LHC data of \(Pb-Pb\) collisions at \(\sqrt{s} = 2.76\) TeV, taken by the ALICE collaboration [4]. It is observed that even the yields for light nuclei, deuteron, \(3\)Helium, hyper-triton, \(4\)Helium and their antiparticles, are in accord with a formation temperature of 155 MeV. Since these states have binding energies of a few MeV and are generally of larger than hadronic size, their survival in the assumed hot hadron gas poses an even more striking puzzle, often characterized as “snowball in hell” [10]. In a situation of approximate global equilibrium, with much slower evolution, as presumably existed in the nucleosynthesis after the Big Bang, the rate of deuteron and helium production has in fact nothing to do with the hadronization temperature: it is determined by the binding energies of the nuclei. Even if in the early universe the great abundance of photons can enhance the break-up of possible nuclei, an interacting hadron gas with many pions would also do this. Our basic question thus is: why are the yields for the production of light nuclei determined by the rates as specified at the critical hadronization temperature, although in hot hadron gas they would immediately be destroyed?

In the present note, we want to show that a solution to this puzzle can be obtained by abandoning the idea of a thermal hadron medium existing below the confinement point. Instead, we propose that the hot quark-gluon system, when it cools down to the hadronization temperature, is effectively quenched by the cold physical vacuum. The relevant basic mechanism for this is self-organized criticality, leading to universal scale-free behavior, as in fact also obtained in many other cases. We first recall the corresponding scenario.

Self-organized criticality (SOC) [11] is the evolution of a non-equilibrium system to a critical attractor, driven by the individual interactions, without any tuning of external parameters (for surveys, see [12–14]). At the critical point, the system becomes scale-free, so that components of all sizes are accounted for by the same law. While equilibrium systems require the tuning of thermal parameters (temperature, density) to reach critical behavior, non-equilibrium systems subject to SOC reach the critical point through interactions within the system itself.

The appearance of scale-free behavior at criticality is readily seen by considering the correlation function \(\Gamma(r)\) of a many-body system, generally taken to have the form

\[
\Gamma(r, T) \sim \frac{\exp[-r/\xi(T)]}{r^p},
\]  

where \(r\) denotes the separation distance of two constituents and \(\xi(T)\) the correlation length in the system at temperature \(T\). The exponent \(p\) is equal to unity in the conventional Ornstein-Zernike formulation in three space dimensions; the general form in \(d \geq 3\) space dimension and with the fractal extension \(\eta\) [15] is \(p = d - 2 + \eta\). In most cases, \(\eta\) is small or vanishes. The correlation length \(\xi(T)\) defines a temperature-dependent scale specifying at what separation constituents still are in touch. At the critical point \(T = T_c\) of a continuous phase transition, \(\xi \rightarrow \infty\), so that there is no longer a scale-dependent
parameter measuring the role of different separations. The correlation function now shows universal power-law behavior,

$$\Gamma(r,T_c) \sim r^{-p}$$  \hspace{1cm} (2)

for all separations $r$, and self-organized criticality means that the system is governed by such a scale-free form. As a corollary, the corresponding susceptibility in the three-dimensional case

$$\chi(T) = \frac{1}{kT} \int d^3r \Gamma(r,T) \rightarrow \frac{4\pi}{kT} \int dr r^{1-\eta}$$  \hspace{1cm} (3)

will diverge at the critical point.

The typical illustration of SOC proposed in the pioneering work [11] is the behavior of sandpiles. Pouring sand onto a flat surface leads to a pile increasing in size up to a point where the pile has a certain critical slope. The addition of further sand now results in avalanches of various sizes, preventing an overall increase of the slope. The number $N(s)$ of different avalanches of size $s$ observed over a long period is found to vary as a power of $s$. Such power-law behavior

$$N(s) = \alpha s^{-p}$$  \hspace{1cm} (4)

implies

$$\log N(s) = A - p \log s, A = \log \alpha$$  \hspace{1cm} (5)

as shown in Fig. 1. Large avalanches are thus governed by the same law as small ones - the phenomenon is scale-free. This breaks down only at the ends, with single grains of sand and the entire pile.

![Figure 1: Distributions of the number $N(s)$ of avalanches vs. their size $s$](image)

Such behavior is in fact found in a variety of situations, from sandpiles to earthquakes to partitioning of integers. Much of the complexity of our world may well result from self-organized criticality. In high energy physics, it is immediately reminiscent of the statistical bootstrap model of Hagedorn [16], who had “fireballs composed of fireballs, which in turn are composed of fireballs, and so on”. Such scale-free composition cascades of massive hadronic states arise also in the dual-resonance model [17–19]. The general pattern has been shown to be due to an underlying structure analogous to the partitioning of an integer into integers [20], the well-known *partitio numerorum* problem of number
theory \cite{21}, and in fact sand piles and integers were shown to result in similar structural patterns \cite{22}. This in turn has immediate consequences on hadron production observed in high energy collisions, as we shall show.

The perhaps simplest form of self-organized criticality is provided by partitioning integers. Consider the *ordered* partitioning of an integer \( n \) into integers. The number \( q(n) \) of such partitionings is for \( n = 3 \) equal to four: \( 3, 2+1, 1+2, 1+1+1 \), i.e., \( p(3) = 4 \). It is easily shown \cite{20} that in general

\[
q(n) = 2^{n-1} = \frac{1}{2} \exp\{n \ln 2\}. \tag{6}
\]

The problem of unordered partitions is more difficult and only solved asymptotically \cite{21}. In the ordered case considered above, one thus finds that the number of partitions increases exponentially with the size of the integer. Given an initial integer \( n \), we would now like to know the number \( N(k,n) \) specifying how often a given integer \( k \) occurs in the set of all partitionings of \( n \). To illustrate, in the above case of \( n = 3 \), we have \( N(3,3) = 1, N(3,2) = 2 \) and \( N(3,1) = 5 \). To apply the formalism of self-organized criticality, we have to attribute a strength \( s(k) \) to each integer. It seems natural use the number of partitions for this, i.e., set

\[
s(k) = q(k) = \frac{1}{2} \exp\{k \ln 2\}. \tag{7}
\]

The desired number \( N(k,n) \) in a scale-free scenario is then given by

\[
N(k,n) = \alpha(n)[q(k)]^{-p}, \tag{8}
\]

leading to

\[
\log N(k,n) = -[p \log e \ln 2] k + p \log 2 + \log \alpha(n) \tag{9}
\]

as counterpart of eq. (5). For small values of \( n \), \( N(k,n) \) is readily obtained explicitly. In Fig. 2 we thus see that relation (9) is in fact well satisfied already for \( n = 4, 5 \) and 6, except for slight deviations in the limit \( k = n \). In particular, we find \( [p \log e \ln 2] \simeq 0.38 \), so that the critical exponent becomes \( p \simeq 1.26 \). The appearence of a specific integer \( k \) in the set of all partitions of \( n \) thus corresponds to the appearence of an avalanche of a given size in the average over a long time period of the sandpile case.

![Figure 2: Distributions of the number \( N(k,n) \) of integers \( k \) for \( n = 4, 5 \) and 6](image-url)
Hagedorn’s bootstrap approach [16] proposes that a hadronic state of overall mass m can be partitioned into structurally similar states, and so on. If these states were at rest, the situation would be identical to the above partitioning problem. Since the constituent fireballs have an intrinsic motion, the number of states \( \rho(m) \) corresponding to a given mass \( m \) is determined by the bootstrap equation

\[
\rho(m) = \delta(m-m_0) + \sum_N \frac{1}{N!} \left[ \frac{4\pi}{3(2\pi m_0)^2} \right]^{N-1} \int \prod_{i=1}^N [dm_i \rho(m_i) d^3p_i] \delta^4(\Sigma_ip_i - p),
\]

with \( m_0 \) denoting the lowest possible mass (the single grain of sand). The equation can be solved analytically [23], giving

\[
\rho(m) \sim m^{-a}e^{m/T_H} \rightarrow \ln \rho \sim \frac{m}{T_H} - a \ln m,
\]

and \( T_H \) as solution of

\[
\left( \frac{2}{3\pi} \right) \left( \frac{T_H}{m_0} \right) K_2(m_0/T_H) = 2 \ln 2 - 1,
\]

where \( K_2(x) \) is a Hankel function of pure imaginary argument. For \( m_0 = m_\pi \approx 130 \text{ MeV} \), this leads to the Hagedorn temperature \( T_H \approx 150 \text{ MeV} \), i.e., to approximately the critical hadronization temperature found in statistical QCD. The cited solution gave \( a = 3 \), but other exponents have also been discussed [24–26].

The form of eq. (11) is an asymptotic solution of the bootstrap equation; it evidently diverges for \( m \to 0 \) and must be modified for small masses. Using a similar result for \( \rho(m) \) obtained in the dual resonance model [27], Hagedorn proposed [28]

\[
\rho(m) = \text{const.}(1 + (m/\mu_0))^{-a} \exp(m/T_H)
\]

where \( \mu_0 \approx 1 - 2 \text{ GeV} \) is a normalization constant.

At this point we should emphasize that the forms (5), (9) and (11) are entirely due to the self-organized nature of the components, with an integer consisting of integers, a fireball of fireballs. They are in no way a result of thermal behavior. We have expressed the slope coefficient of \( m \) in eq. (11) in terms of the Hagedorn “temperature” only in reference to subsequent applications. In itself, it is totally of combinatorial origin. It is of course possible to construct a thermodynamics of integer partitions [20], with an entropy \( S(n) = \ln q(n) = n \ln 2 - \ln 2 \), leading to a temperature \( \Theta = dS(n)/dn = \ln 2 \). In that sense, the integers then form a gas of partitions at the critical temperature \( \Theta \).

We now want to apply the formalism of self-organized criticality to strong interaction physics. There exists some early work in that direction, in which it was argued that Reggeon field theory [29] in fact shows such behavior, with critical exponents in the same universality class as a specific avalanche model [30, 31]. In the framework of QCD as basic theory of strong interactions, numerical lattice studies have shown that the deconfined/confinement transition is in fact a rapid cross-over rather than a genuine thermodynamic phase transition [32]. In the chiral limit of two flavor QCD with vanishing quark masses \( m_q \), one does recover critical behavior [33, 34]; there is a continuous transition at \( T = T_c \), with the chiral condensate \( M = \langle \psi\bar{\psi} \rangle \) as order parameter,

\[
M(T) \sim (T_c - T)^\beta, \quad T \leq T_c
\]
in terms of the critical exponent $\beta$. Thus $T_c$ is defined as the temperature point at which for $m_q = 0$ the chiral condensate $M(T)$ vanishes, $M(T = T_c, m_q = 0) = 0$. For a system at $T = T_c$ with $m_q \to 0$, one finds singular behavior,

$$M(T_c) \sim m_q^{1/\delta},$$

(15)

with the critical exponent $\delta$. For $m_q = 0$, the correlation length characterizing fluctuations of the chiral condensate diverges as

$$\xi(T, m_q = 0) \sim |T - T_c|^{-\nu},$$

(16)

in terms of the critical exponent $\nu$.

The small but finite $u$ and $d$ quark masses $m_q$ in physical QCD act like a weak external field in spin system (see eq. (15)), preventing genuine singular behavior [30]. The behavior of the system for the actual small quark mass values is nevertheless thought to be strongly influenced by the near-by singularity. As a result, specific thermodynamic variables are sharply peaked, defining a pseudo-critical point $T_{pc}(m_q)$ close to $T_c(m_q = 0)$. The fluctuation-dissipation theorem relates the correlation length to the chiral susceptibility $\chi_M \sim \partial M(T)/\partial m_q$, and in Fig. 3 this is seen to show a pronounced peak in temperature [35], which is taken as the pseudocritical temperature of QCD.

Figure 3: The temperature dependence of the chiral susceptibility for three different light quark masses [35].

This is now to be applied to high energy nuclear collisions. The picture we have in mind assumes a sudden quench of the partonic medium produced in the collision. The initial hot system of deconfined quarks and gluons (also interpreted as the molten color glass [37]) rapidly expands and cools; while this system is presumably in local thermal equilibrium, the difference between transverse and longitudinal motion implies a global non-equilibrium behavior. The longitudinal expansion quickly drives the system to the hadronisation point, and it is now suddenly thrown into the cold physical vacuum. The
process is not unlike that of a molten metal being dumped into cold water. In this quenching process, the system freezes out into the degrees of freedom presented by the system at the transition point and subsequently remains as such, apart from possible hadron or resonance decays. There never is an evolving warm metal. In other words, in our case there is no hot interacting hadron gas. To obtain that, we would have to adiabatically lower the temperature of a closed quark-gluon system; it is the sudden immersion into the vacuum that causes the quench. The snowball is one of the allowed states of the system, and so it can appear at the quenching point. Subsequently, however, it is not in hell, but together with all other fragments, it finds itself freestreaming in the cold physical vacuum. Whatever thermal features are observed, such as radial or elliptic hydrodynamic flow, must then have originated from local equilibrium in the earlier deconfined stage \[38,39\]. The mechanism driving the system rapidly to the critical point is the global non-equilibrium due to the longitudinal motion provided by the collision.

In such a scenario, high energy nuclear collisions lead to a system which at the critical point breaks up into components of different masses \( m \), subject to self-similar composition and hence of a strength \( \rho(m) \) as given by the above eq. \[13\]. In the self-organized criticality formalism, this implies that the interaction will produce

\[
N(m) = \alpha [\rho(m)]^{-p} \tag{17}
\]

hadrons of mass \( m \). With \( \rho(m) \) given by eq. \[13\], the resulting powerlaw form

\[
\log N(m) = -m \left( \frac{p \log e}{T_H} \right) \left[ 1 - \left( \frac{a T_H}{m} \right) \ln(1 + \frac{m}{\mu_0}) \right] + \text{const.} \tag{18}
\]

is found to show a behavior similar to that obtained from an ideal resonance gas in equilibrium. We emphasize that it is here obtained assuming only scale-free behavior (self-organized criticality) and a mass weight determined by the number of partitions. No equilibrium thermal system of any kind is assumed.

We now consider the mentioned ALICE data \[4\]. In Fig. 4 the production yields for the different mass states in central \( Pb-Pb \) collisions at \( \sqrt{s} = 2.76 \text{ GeV} \) are shown; in each case, the yield is divided by the relevant spin degeneracy. We see that the yields show essentially powerlike behavior, and the light nuclei follow the same law as the elementary hadrons. The solid line in Fig. 4 shows the behavior obtained from eqs. \[18\], ignoring for the moment the second term in the square brackets,

\[
\log[(dN/dy)/(2s + 1)] \simeq -m \left( \frac{0.43 p}{T_H} \right) + A, \tag{19}
\]

with \( T_H = 0.155 \text{MeV} \) and fit values \( p = 0.9, A = 3.4 \). The form is evidently in good agreement with the data.

Including the correction term to linear behavior that we had omitted above, we have

\[
\log[(dN/dy)/(2s + 1)] \simeq -m \left( \frac{0.43 p}{T_H} \right) + + p a \log[1 + (m/\mu)] + A, \tag{20}
\]

The additional term is, as indicated, rather model dependent. It will effectively turn the yield curve down for decreasing masses. This is in fact necessary, since the decay
of heavier resonances will enhance the direct low mass meson yields. To illustrate the
effect of the term, we choose $a = 3$, corresponding to the mentioned solution (11) of the
bootstrap equation [23], and $\mu = 2$ GeV for the normalization. The result is included in
Fig. 4.

Our main result, eq. (20), is quite similar to the result of the yield calculation in the
resonance gas model [1]. We have here obtained it, however, without the assumption
that the confinement transition produces a hot interacting resonance gas. Instead, the
yield values arise in the sudden self-organized critical quench which takes place when the
hot partonic system hits the cold vacuum. We thus conclude that the agreement of high
energy hadron production yields with an ideal resonance gas model does not establish that
such collisions produce a thermal hadronic medium for $T < T_c$. The observation that the
yields of light nuclei also agree with the same pattern in fact throws serious doubt on the
existence of such a medium. In the global non-equilibrium scenario proposed here, any
state, also still heavier nuclei, can arise in the quench.

As a caveat, we note, however, that the argumentation based on the statistical bootstrap
model result (13) is inherently of a qualitative nature. The form (13) itself holds in
the limit of large $m$; moreover, to obtain it, the discrete hadron spectrum was replaced
by a continuum. Furthermore, the isospin, strangeness and baryon number structure of
the spectrum are not taken into account. We can therefore expect at best qualitative

Figure 4: Yield rates of species at central rapidity vs. their mass $m$ [4]. The solid line
corresponds to eq. (19), the dashed line to eq. (20).
agreement with the data - in detail, our simple model cannot be expected to reproduce the results of a full ideal resonance gas analysis and is meant mainly to illustrate the SOC approach. Further work on a more detailed SOC analysis is in progress.

As final comment, we recall the behavior of proton-nucleus or nucleus-nucleus collisions at low energy, leading to what is denoted as nuclear multifragmentation [40, 41]. The result of such collisions will be nuclear fragments of size $A$, and the distribution of these fragments is found to obey the so-called Fisher law [42]

$$P(A) = \text{const.} A^{-\tau},$$

(21)

corresponding to the critical point of droplet condensation, with $\tau = 2.33$. It thus constitutes another instance of self-organized criticality, with all fragments governed by the same law. The difference between this form and the one for high energy collisions is that at low energy, the the break-up is simply into mass fragments, whereas at high energy it produces all possible excitation states.

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