On a diophantine inequality with prime numbers of a special type

D. I. Tolev

Abstract

We consider the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E},$$

where $1 < c < \frac{15}{14}$, $N$ is a sufficiently large real number and $E > 0$ is an arbitrarily large constant. We prove that the above inequality has a solution in primes $p_1, p_2, p_3$ such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ has at most \left\lfloor \frac{369}{180 - 168c} \right\rfloor$ prime factors, counted with the multiplicity.

1 Introduction and statement of the result

We consider the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \Delta,$$

where $c > 1$ is a constant, $N$ is a sufficiently large real number and $\Delta = \Delta(N)$ is a function such that $\Delta(N) \to 0$ as $N \to \infty$. Having in mind I. M. Vinogradov’s famous theorem about Goldbach’s ternary problem (see [21]), one may expect that if $c$ is not much greater than 1 and $\Delta(N)$ is a suitable function, then inequality (1) has a solution in prime numbers $p_1, p_2, p_3$. A result of this type with $1 < c < \frac{15}{14}$ and with $\Delta = N^{-\kappa}$ for certain $\kappa = \kappa(c) > 0$ was established in 1992 by the author [16]. Several improvements were made since then and the strongest of them is due to Baker and Weingartner [11]. In 2014 they established that (1) is solvable in primes, provided that $N$ is large enough, $1 < c < \frac{14}{9}$ and $\Delta = N^{-\kappa}$ for certain $\kappa = \kappa(c) > 0$.

Suppose that $r$ is a natural number and let $\mathcal{P}_r$ be the set of positive integers having at most $r$ prime factors, counted with the multiplicity. (We say that the numbers from $\mathcal{P}_r$ are almost primes of order $r$.) In 1973 Chen [3], improving results of other mathematicians, established that there exist infinitely many primes $p$ such that $p + 2 \in \mathcal{P}_2$. Bearing in mind Chen’s result, one may try to study the arithmetical properties of the
set of primes $p$ such that $p + 2 \in \mathcal{P}_r$ for a fixed $r \geq 2$ and, in particular, to establish
the solvability of diophantine equations or inequalities in such primes. For example, Matomäki and Shao \[11\], improving author’s results from \[17\] and \[18\] as well as a result
of Matomäki \[10\], proved that every sufficiently large odd integer $N$ can be represented
as a sum of three primes $p_1, p_2, p_3$ such that $p_i + 2 \in \mathcal{P}_r$, $i = 1, 2, 3$. Other results of this
kind were found by the author \[19\], and by Dimitrov and Todorova \[6\].

One may expect that if the constant $c > 1$ is close to 1 then inequality (1), with
a suitable $\Delta$ satisfying $\Delta \to 0$ as $N \to \infty$, is solvable in primes $p_i$ such that
$p_i + 2$ are almost primes of certain fixed order. An attempt to establish a result of this type was
made by Dimitrov \[5\], but he considers the inequality (1) only when $c < \frac{15}{14}$, whilst the
case $c > 1$ is more interesting. Dimitrov recently announced that he is able to prove the
solvability of (1) in the case $1 < c < \frac{121}{120}$ but such a result has not been published.

In the present paper we assume that $c$ is a constant such that

$$1 < c < \frac{15}{14}.$$  \hspace{1cm} (2)

We consider the inequality (1) with

$$\Delta = (\log N)^{-E},$$  \hspace{1cm} (3)

where $E > 0$ is an arbitrarily large constant and we prove the following

**Theorem 1.** Let $c$ be a constant satisfying (2) and let $N$ be a sufficiently large real
number. Then inequality (1), with $\Delta$ specified by (3), has a solution in primes $p_i$ such that each of the numbers $p_i + 2$, $i = 1, 2, 3$ has at most \(\left\lfloor \frac{360}{180-168c} \right\rfloor\) prime factors,

counted with the multiplicity.

It follows from Theorem 1 that if $c > 1$ is close to 1, then inequality (1), with $\Delta$ given
by (3), has a solution in primes $p_i$ such that $p_i + 2 \in \mathcal{P}_{30}$.

We can establish a similar result for the inequality (1) with $\Delta = N^{-\kappa}$ for certain
$\kappa > 0$, but then we would have $p_i + 2 \in \mathcal{P}_m$, $i = 1, 2, 3$, where $m$ depends on $c$ and $\kappa$.

Notations in the paper shall be as follows. By $\varepsilon$ and $A$ we denote an arbitrarily small positive number and respectively, an arbitrarily large constant which may not be the
same in different formulae. The letter $p$ always denotes a prime number. By $\tau(n)$, $\mu(n)$,
$\varphi(n)$ and $\Lambda(n)$ we denote the number of divisors of $n$, Möbius’ function, Euler’s function
and Von Mangoldt’s function respectively. We shall use $(m, n)$ and $[m, n]$ for the greatest
common divisor and the least common multiple of the integers $m, n$. (We denote in this
way also open and closed intervals from the real line, but the meaning will be clear from
the context). Let $[t]$ be the integer part of the real number $t$ and $e(t) = e^{2\pi it}$. With $\chi$
we denote a Dirichlet’s character. As usual, $\sum_{\chi (\text{mod } q)}$ means that the summation is
taken over all Dirichlet’s characters modulo $q$. Respectively, $\sum_{\chi (\text{mod } q)}^*$ means that the
summation is taken over the primitive Dirichlet’s characters modulo $q$.

Suppose that $\chi$ is a Dirichlet’s character and $L(s, \chi)$ is the corresponding $L$-function.
If $T \geq 2$ and $0 \leq \sigma \leq 1$ we denote by $N(T, \sigma, \chi)$ the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$
such that $|\gamma| \leq T$ and $\sigma \leq \beta \leq 1$. We also write $N(T, \chi) = N(T, 0, \chi)$.
By

\[ \psi(y) = \sum_{n \leq y} \Lambda(n), \quad \psi(y, k, l) = \sum_{n \leq y, n \equiv l \pmod{k}} \Lambda(n), \]

we denote Chebyshev’s functions and we define \( \Delta(y, k, l) \) by

\[ \Delta(y, k, l) = \psi(y, k, l) - \frac{y}{\varphi(k)}. \]

For a given Dirichlet’s character \( \chi \) we write

\[ \psi(y, \chi) = \sum_{n \leq y} \Lambda(n) \chi(n). \]

Finally, by \( \Box \) we mark an end of a proof or its absence.

2 Beginning of the proof

Let \( \eta, \delta, \xi, \mu \) be positive real numbers depending on \( c \). We shall specify them later but for now only assume that they satisfy the conditions

\[ \xi + 3\delta < \frac{12}{25}, \quad 2 < \frac{\delta}{\eta} < 3, \quad \mu < 1. \]

We define

\[ X = N^\frac{1}{2}, \quad z = X^\eta, \quad D = X^\delta, \quad \tau = X^{\xi-c}, \]

\[ r = \lfloor \log X \rfloor, \quad \Xi = (\log X)^{E+3} \]

and

\[ P(z) = \prod_{2 < p < z} p, \]

where the product is taken over prime numbers.

Consider the sum

\[ \Gamma = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3), \]

where the summation is taken over the primes \( p_1, p_2, p_3 \) from the interval \( (\mu X, X] \) which satisfy \( (\Pi) \) (with \( \Delta \) given by \( (\Xi) \)), as well as the conditions

\[ (p_1 + 2, P(z)) = (p_2 + 2, P(z)) = (p_2 + 2, P(z)) = 1. \]

If we prove the inequality

\[ \Gamma > 0, \]
then the equation (11) would have a solution in primes \( p_1, p_2, p_3 \) satisfying (12). If the number \( p_i + 2 \) has \( l \) prime factors counted with multiplicity, then from (8), (10), (12) and from the condition \( \mu X < p_i \leq X \) we easily find that
\[
l \leq \frac{1}{\eta}.
\] (14)

This means that \( p_i + 2 \) would be an almost-prime of order \( \eta^{-1} \). Therefore, to prove the theorem we have to establish (13) for a suitable choice of \( \eta \).

Firstly, we use the following

**Lemma 2.** Let \( a, \delta \) be real numbers, \( 0 < \delta < \frac{a}{4} \), and let \( r \) be a positive integer. There exists a function \( \theta(y) \) which is \( r \) times continuously differentiable and such that
\[
\theta(y) = 1 \quad \text{for } |y| \leq a - \delta,
\]
\[
0 < \theta(y) < 1 \quad \text{for } a - \delta < |y| \leq a + \delta,
\]
\[
\theta(y) = 0 \quad \text{for } |y| \geq a + \delta,
\]
and its Fourier transform
\[
\Theta(x) = \int_{-\infty}^{\infty} \theta(y)e(-xy) \, dy
\] (15)
satisfies the inequality
\[
|\Theta(x)| \leq \min \left( \frac{2a}{|x|}, \frac{1}{|x|} \left( \frac{r}{|x|\delta} \right)^r \right).
\] (16)

**Proof.** This is an old result which goes back to Segal [13]. \( \square \)

We apply this lemma with \( r \) given by (9) and with \( a = \frac{7\Delta}{8}, \delta = \frac{\Delta}{8} \), where \( \Delta \) is specified by (3). Hence we have
\[
\theta(y) = 0 \quad \text{for } |y| \geq \Delta,
\]
\[
0 < \theta(y) \leq 1 \quad \text{for } |y| < \Delta
\] (17)
and from (11) and (17) we find that
\[
\Gamma \geq \Gamma' := \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \theta(p_1^c + p_2^c + p_3^c - N)
\] (18)

For \( i = 1, 2, 3 \) we consider the quantities
\[
\Lambda_i = \sum_{d|(p_i + 2, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (p_i + 2, P(z)) = 1, \\ 0 & \text{otherwise}. \end{cases}
\] (19)

Bearing in mind (12) and (18) we find that
\[
\Gamma' = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1 \Lambda_2 \Lambda_3 \theta(p_1^c + p_2^c + p_3^c - N).
\] (20)

Now we apply the following fundamental result from sieve theory
Lemma 3. Suppose that $D > 4$ is a real number. There exist arithmetical functions $\lambda^\pm(d)$ (called Rosser's functions of level $D$) with the following properties.

1) For any positive integer $d$ we have

$$|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \text{ if } d > D \text{ or } \mu(d) = 0. \quad (21)$$

2) If $n$ is a positive integer then

$$\sum_{d|n} |\lambda^-(d)| \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^+(d).$$

3) If $z$ is a real number such that $z^2 \leq D \leq z^3$ and if

$$P(z) = \prod_{2 < p < z} p, \quad \Phi = \prod_{2 < p < z} \left(1 - \frac{1}{p-1}\right), \quad \mathfrak{N}^\pm = \sum_{d|P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}, \quad s_0 = \frac{\log D}{\log z}, \quad (22)$$

then we have

$$\Phi \leq \mathfrak{N}^+ \leq \Phi \left(F(s_0) + O \left(\left(\log D\right)^{-\frac{1}{3}}\right)\right), \quad (23)$$

$$\Phi \geq \mathfrak{N}^- \geq \Phi \left(f(s_0) + O \left(\left(\log D\right)^{-\frac{1}{3}}\right)\right), \quad (24)$$

where $F(s)$ and $f(s)$ (the functions of the linear sieve) satisfy

$$f(s) = 2e^s s^{-1} \log(s - 1), \quad F(s) = 2e^s s^{-1} \text{ for } 2 \leq s \leq 3. \quad (25)$$

Here $\gamma$ stands for the Euler constant.

Proof. This is a special case of a more general result — see Greaves [7, Ch. 4].

Next we use the following

Lemma 4. Suppose that $\Lambda_i, \Lambda_i^\pm, i = 1, 2, 3$ are real numbers satisfying (27). Then we have

$$\Lambda_1 \Lambda_2 \Lambda_3 \geq \Lambda_1^\pm \Lambda_2^\pm \Lambda_3^\pm + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ + \Lambda_1^- \Lambda_2^- \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+. \quad (28)$$
Proof. The proof is elementary and similar to the proof of [2, Lemma 13]).

We apply (20), (27) and (28) and then we substitute the quantity from the right side of (28) for \( \Lambda_1 \Lambda_2 \Lambda_3 \) in (20). We find that

\[
\Gamma' \geq \Gamma_1 + \Gamma_2 + \Gamma_3 - 2\Gamma_4,
\]

where \( \Gamma_1, \ldots, \Gamma_4 \) are the contributions coming from the consecutive terms of the right side of (28). It is clear that

\[
\Gamma_1 = \Gamma_2 = \Gamma_3 = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \theta (p_1^c + p_2^c + p_3^c - N),
\]

(29)

\[
\Gamma_4 = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \theta (p_1^c + p_2^c + p_3^c - N).
\]

(30)

Hence, we get

\[
\Gamma' \geq 3\Gamma_1 - 2\Gamma_4.
\]

(31)

Consider \( \Gamma_1 \). (The study of \( \Gamma_4 \) is likewise). We apply Fourier’s inversion formula

\[
\theta(t) = \int_{-\infty}^{\infty} \Theta(x) e(xt) \, dx
\]

as well as (20) to find that

\[
\Gamma_1 = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \int_{-\infty}^{\infty} \Theta(x) e (x (p_1^c + p_2^c + p_3^c - N)) \, dx
\]

\[
= \int_{-\infty}^{\infty} \Theta(x) L^{-}(x) L^{+}(x)^2 e(-Nx) \, dx,
\]

(32)

where

\[
L^{\pm}(x) = \sum_{\mu X < p \leq X} (\log p) e (xp^c) \sum_{d_{(p+2,P(z))}} \lambda^{\pm}(d).
\]

Changing the order of summation, we get

\[
L^{\pm}(x) = \sum_{d_{|P(z)}} \lambda^{\pm}(d) \sum_{\mu X < p \leq X} (\log p) e (xp^c).
\]

(33)
We divide the integral from (32) into three parts as follows:

\[ \Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} + \Gamma_1^{(3)}, \]

where

\[ \Gamma_1^{(1)} = \int_{|x|<\tau} \Theta(x) L^-(x) (L^+(x))^2 e(-Nx) \, dx, \]

\[ \Gamma_1^{(2)} = \int_{\tau<|x|<\Xi} \Theta(x) L^-(x) (L^+(x))^2 e(-Nx) \, dx, \]

\[ \Gamma_1^{(3)} = \int_{|x|>\Xi} \Theta(x) L^-(x) (L^+(x))^2 e(-Nx) \, dx. \]

Similarly, for the quantity \( \Gamma_4 \) defined by (30) we find

\[ \Gamma_4 = \Gamma_4^{(1)} + \Gamma_4^{(2)} + \Gamma_4^{(3)}, \]

where

\[ \Gamma_4^{(1)} = \int_{|x|<\tau} \Theta(x) L^+(x)^3 e(-Nx) \, dx, \]

\[ \Gamma_4^{(2)} = \int_{\tau<|x|<\Xi} \Theta(x) L^+(x)^3 e(-Nx) \, dx, \]

\[ \Gamma_4^{(3)} = \int_{|x|>\Xi} \Theta(x) L^+(x)^3 e(-Nx) \, dx. \]

It is easy to estimate \( \Gamma_1^{(3)} \) and \( \Gamma_4^{(3)} \). It is clear from (33) that \( L^\pm(x) \ll X^{1+\varepsilon} \). We also use (3), (9) and (16) to find that

\[ \Gamma_1^{(3)}, \Gamma_4^{(3)} \ll X^{3+\varepsilon} \left( \frac{r}{\Delta} \right)^r \ll X^{3+\varepsilon} \left( \frac{r}{\Delta \Xi} \right)^r \ll 1. \]

From (18), (31), (34) and (38) it follows that

\[ \Gamma \geq 3\Gamma_1^{(1)} - 2\Gamma_4^{(1)} - c_0 \left( |\Gamma_1^{(2)}| + |\Gamma_4^{(2)}| + 1 \right), \]

where \( c_0 > 0 \) is an absolute constant.
3 The integrals $\Gamma_1^{(1)}$ and $\Gamma_4^{(1)}$

In this section we find asymptotic formulae for $L^\pm(x)$, provided that $|x| < \tau$. The arithmetic structure of the Rosser weights $\lambda^\pm(d)$ is not important here, so we consider a sum of the form

$$L(x) = \sum_{d \leq D} \lambda(d) \sum_{\mu X < p \leq X} \frac{1}{n} \log p \ e(xp^\epsilon),$$

and we assume that $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2 \nmid d \quad \text{or} \quad \mu(d) = 0. \quad (45)$$

We also define

$$I(x) = \int_{\mu X}^{X} e(x^t) \ dt, \quad (46)$$

To study $I(x)$, we need the following

**Lemma 5.** Consider the integral

$$I = \int_{a}^{b} G(x) e(F(x)) dx$$

where $G(x), F(x)$ are real functions with continuous second derivatives. Assume that the function $G(x)/F'(x)$ is monotonous and suppose that $|G(x)| \leq H$ for all $x \in [a, b]$.

If $|F'(x)| \geq h > 0$ for all $x \in [a, b]$ then $I \ll Hh^{-1}$.

If $|F''(x)| \geq h > 0$ for all $x \in [a, b]$ then $I \ll Hh^{-\frac{3}{2}}$.

**Proof.** See [14, p. 71].

We also need Bombieri–Vinogradov’s theorem:

**Lemma 6.** Suppose that $x > 2, \ Q > 2$ and consider the sum

$$\Sigma = \sum_{d \leq Q} \max_{y \leq x} \max_{\chi} |\Delta(y, d, l)|,$$

where $\Delta(y, d, l)$ is defined by (3). For any constant $A > 0$ there exists $B = B(A) > 0$ such that if $Q \leq \sqrt{x} (\log x)^{-B}$, then $\Sigma \ll x(\log x)^{-A}$.

**Proof.** See [4, Ch. 28].
Lemma 7. Suppose that $2 \leq T \leq y$.

We have

$$
\psi(y) = y - \sum_{|\gamma| < T} \frac{y^\rho}{\rho} + O \left( \frac{y \log y}{T} \right),
$$

(47)

where the summation runs over the non-trivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function such that $|\gamma| < T$.

If $r > 1$ and $\chi$ is a primitive character (mod $r$), then we have

$$
\psi(y, \chi) = -\sum_{|\gamma| < T} \frac{y^\rho}{\rho} + \sum_{|\gamma| < 1} \frac{1}{\rho} + O \left( \frac{y \log(ry)}{T} \right),
$$

(48)

where the summation is taken over the non-trivial zeros $\rho = \beta + i\gamma$ of Dirichlet’s $L$-function $L(s, \chi)$ such that $|\gamma| < T$ and $|\gamma| < 1$, respectively.

Proof. See [4, Ch. 17] and [4, Ch. 19]

The next lemma provides an information about the density of the zeroes of Dirichlet’s $L$-functions.

Lemma 8. If $\chi$ is a primitive Dirichlet’s character modulo $d$ and if $T \geq 2$, then we have

$$
N(T, \chi) \ll T \log(dt).
$$

(49)

If $Q \geq 1$ and $T \geq 2$, then for the sum

$$
\Sigma(T, \sigma, Q) = \sum_{d \leq Q} \sum_{\chi \bmod d^*} N(T, \sigma, \chi)
$$

(50)

we have

$$
\Sigma(T, \sigma, Q) \ll \begin{cases} 
(Q^2 T)^{\frac{11 - \sigma}{\sigma}} (\log(QT))^{9} & \text{if } \frac{1}{3} \leq \sigma \leq \frac{4}{5}, \\
(Q^2 T)^{\frac{11 - \sigma}{\sigma}} (\log(QT))^{14} & \text{if } \frac{4}{5} \leq \sigma \leq 1.
\end{cases}
$$

(51)

Proof. The proof of (49) can be found in [4, Ch. 16] and for the proof of (51) see [12, Theorem 12.2].

In the next lemma we present an analog of the estimate from [16] Lemma 5].

Lemma 9. Consider the sum

$$
\mathcal{L}(T, Q, X) = \sum_{d \leq Q} \sum_{\chi \bmod d^*} \sum_{|\gamma| \leq T} X^\beta,
$$

(52)

where $X \geq 2$, $Q \geq 1$, $T \geq 2$ and where the summation in the inner sum is taken over the non-trivial zeros $\rho = \beta + i\gamma$ of Dirichlet’s $L$-function $L(s, \chi)$ such that $|\gamma| \leq T$. 9
Suppose that
\[ Q^2 T \leq X^{\frac{2}{3}}. \]  
(53)

Then we have
\[ \mathfrak{L}(T, Q, X) \ll (\log X)^{20} \left( X + X^{\frac{1}{2}} Q T^{\frac{1}{2}} \right). \]  
(54)

**Proof.** The proof of (54) is standard but for reader’s convenience we present the arguments. Suppose that \( \chi \) is a primitive character modulo \( d \) and \( \rho = \beta + i\gamma \) is a non-trivial zero of \( L(s, \chi) \). We start with the identity
\[ X^\beta = 1 + (\log X) \int_0^1 X^\sigma \Phi(\sigma, \beta) d\sigma, \]  
(55)

where \( \Phi(\sigma, \beta) = 0 \) for \( \sigma > \beta \) and \( \Phi(\sigma, \beta) = 1 \) for \( \sigma \leq \beta \). It is clear that
\[ \sum_{|\gamma| \leq T} \Phi(\sigma, \beta) = N(T, \sigma, \chi), \]

hence applying (55) and the estimate (52) from Lemma 8 we find that
\[ \sum_{|\gamma| \leq T} X^\beta = N(T, \chi) + (\log X) \int_0^1 X^\sigma N(T, \sigma, \chi) d\sigma \]
\[ \ll X^{\frac{1}{2}} T (\log X)^2 + (\log X) \int_\frac{1}{2}^1 X^\sigma N(T, \sigma, \chi) d\sigma. \]

From the above formula, (50) and (52) we find that
\[ \mathfrak{L}(T, Q, X) \ll X^{\frac{1}{2}} Q^2 T (\log X)^2 + (\log X) \int_\frac{1}{2}^1 X^\sigma \Sigma(T, \sigma, Q) d\sigma. \]

We apply the estimate (51) from Lemma 8 and find that
\[ \mathfrak{L}(T, Q, X) \ll (\log X)^{15} \left( X^{\frac{1}{2}} Q^2 T + I_1 + I_2 \right), \]  
(56)

where
\[ I_1 = \int_\frac{1}{2}^1 X^\sigma (Q^2 T)^\frac{3(1-\sigma)}{2-\sigma} d\sigma, \quad I_2 = \int_\frac{1}{2}^1 X^\sigma (Q^2 T)^\frac{2(1-\sigma)}{\sigma} d\sigma. \]

To estimate the integral \( I_1 \), we write it in the form
\[ I_1 = \int_\frac{1}{2} e^{h_1(\sigma)} d\sigma, \quad h_1(\sigma) = \sigma(\log X) + \frac{3(1-\sigma)}{2-\sigma} \log(Q^2 T). \]
Using the condition (53), it is easy to verify that \( h'_1(\sigma) \geq 0 \) for \( \frac{1}{2} \leq \sigma \leq \frac{4}{5} \). This means that \( \max_{\frac{1}{2} \leq \sigma \leq \frac{4}{5}} h_1(\sigma) = h_1\left(\frac{4}{5}\right) \) and therefore

\[
I_1 \ll e^{h_1\left(\frac{4}{5}\right)} = X^{\frac{4}{5}}QT^{\frac{1}{2}}. \tag{57}
\]

Consider \( I_2 \). We have

\[
I_2 = \int_{\frac{4}{5}}^1 e^{h_2(\sigma)} d\sigma, \quad h_2(\sigma) = \sigma(\log X) + \frac{2(1 - \sigma)}{\sigma} \log(Q^2T).
\]

We have \( h''_2(\sigma) > 0 \) for \( \frac{4}{5} \leq \sigma \leq 1 \), hence \( \max_{\frac{4}{5} \leq \sigma \leq 1} h_2(\sigma) = \max\left(h_2\left(\frac{4}{5}\right), h_2(1)\right) \) and therefore

\[
I_2 \ll e^{h_2\left(\frac{4}{5}\right)} + e^{h_2(1)} = X^{\frac{4}{5}}QT^{\frac{1}{2}} + X. \tag{58}
\]

The estimate (54) is a consequence of (53) and (56) – (58). □

We shall prove the following

**Lemma 10.** Suppose that \( D \) and \( \tau \) are defined by (8) and that \( \xi \) and \( \delta \) satisfy (7). If \( L(x) \) and \( I(x) \) are defined by (44) and (46) and if \( |x| < \tau \) then we have

\[
L(x) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(x) + O\left( X(\log X)^{-A} \right), \tag{59}
\]

where \( A > 0 \) is an arbitrarily large constant.

**Proof.** One may easily see that

\[
L(x) = L_1(x) + O\left( X^{\frac{1}{2} + \varepsilon} \right), \tag{60}
\]

where

\[
L_1(x) = \sum_{d \leq D} \lambda(d) S(x, d), \quad S(x, d) = \sum_{\mu X < n \leq X \atop d \mid n + 2} \Lambda(n) e(xn^\varepsilon). \tag{61}
\]

To find an asymptotic formula for \( L_1(x) \), we shall proceed in different ways according to the size of \( |x| \).

Firstly, consider the case

\[
|x| \leq X^{-c}(\log X)^B, \tag{62}
\]
where $B > 0$ is a constant which we shall specify later. For the sum $S(x, d)$, defined by (61), we apply Abel’s formula and (4) to find that

$$S(x, d) = e(x X^c) \sum_{\mu X < n \leq X} \Lambda(n) - \int_\muX^{X} \sum_{\mu X < n \leq X} \Lambda(n) \frac{d}{dt} e(x t^c) \, dt$$

$$= e(x X^c) \left( \psi(X, d, -2) - \psi(\mu X, d, -2) \right)$$

$$- \int_\muX^{X} \left( \psi(t, d, -2) - \psi(\mu X, d, -2) \right) \frac{d}{dt} e(x t^c) \, dt.$$

Now we use (5) to get

$$S(x, d) = e(x X^c) \left( \frac{X - \mu X}{\varphi(d)} + \Delta(X, d, -2) - \Delta(\mu X, d, -2) \right)$$

$$- \int_\muX^{X} \left( \frac{t - \mu X}{\varphi(d)} + \Delta(t, d, -2) - \Delta(\mu X, d, -2) \right) \frac{d}{dt} e(x t^c) \, dt.$$

Therefore, using (46) and the assumption (62) we find that

$$S(x, d) = \frac{1}{\varphi(d)} \left( e(x X^c) (X - \mu X) - \int_\muX^{X} (t - \mu X) \frac{d}{dt} e(x t^c) \right)$$

$$+ O \left( (1 + |x| X^c) \max_{y \leq X} \max_{(l, d) = 1} |\Delta(y, d, l)| \right)$$

$$= \frac{I(x)}{\varphi(d)} + O \left( (\log X)^B \max_{y \leq X} \max_{(l, d) = 1} |\Delta(y, d, l)| \right).$$

The contribution from the remainder term in (63) is $\ll X (\log X)^{-A}$. We also take into account (60) to conclude that in the case (62) the asymptotic formula (59) is true.

Consider now the case

$$X^{-c}(\log X)^B < |x| \leq \tau.$$

We proceed with the sum $S(x, d)$, specified by (61), in a different way. Using the prop-
erties of Dirichlet’s characters we get

\[ S(x, d) = \sum_{\mu X < n \leq X} \Lambda(n) e(xn^c) \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \chi(n) \overline{\chi(-2)} \]

\[ = \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \overline{\chi(-2)} \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) e(xn^c) . \]

We separate the contribution from the principal character to get

\[ S(x, d) = \frac{1}{\varphi(d)} Y(X) + O \left( \frac{\log X^2}{\varphi(d)} \right) + \mathfrak{M}, \quad (65) \]

where

\[ Y(X) = \sum_{\mu X < n \leq X} \Lambda(n) e(xn^c) \quad (66) \]

and where \( \mathfrak{M} \) comes from the non-principal characters. We express \( \mathfrak{M} \) as a sum over primitive characters and find that

\[ \mathfrak{M} = \frac{1}{\varphi(d)} \sum_{1 < r \mid d, r > 1} \sum_{\chi \pmod{r}} \overline{\chi(-2)} \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) e(xn^c) . \]

If we omit the condition \((n, d) = 1\) imposed in the sum over \(n\), then the resulting error will be \(O((\log X)^2)\). (We leave the easy verification to the reader). We use (64) and substitute the expression for \(S(x, d)\) in the sum \(L_1(x)\) from (61). Taking into account (60) we find that

\[ L(x) = \sum_{d \leq D} \lambda(d) \frac{Y(X)}{\varphi(d)} + O \left( X^{\frac{1}{2} + \epsilon} \right) + \mathfrak{M}, \quad (67) \]

where

\[ \mathfrak{M} = \sum_{d \leq D} \lambda(d) \frac{\sum_{1 < r \mid d, r > 1} \sum_{\chi \pmod{r}} \overline{\chi(-2)} Y(X, \chi)}{\varphi(d)} , \]

and

\[ Y(X, \chi) = \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) e(xn^c) . \quad (68) \]

We change the order of summation to get

\[ \mathfrak{M} = \sum_{1 < r \leq D} \left( \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} \right) \sum_{\chi \pmod{r}} \overline{\chi(-2)} Y(X, \chi) . \]

Using (45), we easily find that

\[ \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} \ll \frac{\log X}{\varphi(r)} , \]
hence
\[ \mathfrak{M} \ll (\log X) \sum_{1 < r \leq D} \frac{1}{\varphi(r)} \sum_{\chi \pmod{r}} |Y(X, \chi)|. \quad (69) \]

Consider the sum \(Y(X, \chi)\). We apply Abel’s formula and use (6) to find that
\[
Y(X, \chi) = e(xX^c) \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) - \int_{\mu X}^{X} \left( \sum_{\mu X < n \leq t} \Lambda(n) \chi(n) \right) \frac{d}{dt} e(xt^c) \, dt
\]
\[= e(xX^c) (\psi(X, \chi) - \psi(\mu X, \chi)) - \int_{\mu X}^{X} (\psi(t, \chi) - \psi(\mu X, \chi)) \frac{d}{dt} e(xt^c) \, dt. \]

We choose
\[T = |x|X^c D(\log X)^B. \quad (70)\]
From (7), (8) and (64) we easily see that \(2 \leq T \leq \mu X\). Now we apply formula (48) from Lemma 7 and find that
\[
Y(X, \chi) = e(xX^c) \left( - \sum_{|\gamma| < T} \frac{X^\rho - (\mu X)^\rho}{\rho} + O \left( \frac{X(\log X)^2}{T} \right) \right)
\]
\[+ \int_{\mu X}^{X} \left( \sum_{|\gamma| < T} \frac{t^\rho - (\mu X)^\rho}{\rho} + O \left( \frac{X(\log X)^2}{T} \right) \right) \frac{d}{dt} e(xt^c) \, dt. \]

We estimate the contributions from the error terms, then we change the order of summation and integration and finally we integrate by parts to get
\[
Y(X, \chi) = - \sum_{|\gamma| < T} I_{\rho}(x) + O \left( \frac{X}{T}(\log X)^2 \left( 1 + |x|X^c \right) \right),
\]
where
\[I_{\rho}(x) = \int_{\mu X}^{X} t^{\rho-1} e(xt^c) \, dt. \quad (71)\]

Now, we substitute the last expression for \(Y(X, \chi)\) in (69) and use (64) to obtain
\[
\mathfrak{M} \ll (\log X) \sum_{1 < r \leq D} \frac{1}{\varphi(r)} \sum_{\chi \pmod{r}} |I_{\rho}(x)| + T^{-1} |x| X^{1+c} D (\log X)^3. \quad (72)\]

We study the sum \(Y(X)\), defined by (66), in the same manner but now we apply formula (17) from Lemma 7. We take the parameter \(T\), defined by (70), and after some calculations we find that
\[
Y(X) = I(x) - \sum_{|\gamma| < T} I_{\rho}(x) + O \left( T^{-1} |x| X^{1+c} (\log X)^3 \right),
\]

14
where \( I(x) \) and \( I_\rho(x) \) are defined respectively by (66) and (71) and \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \) such that \( |\gamma| < T \).

Substituting the last expression for \( Y(X) \) in (67), together with (70) and (72) gives

\[
L(x) - \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(x) \ll X (\log X)^{3-B} + (\log X)^3 \mathfrak{R},
\]

where

\[
\mathfrak{R} = \sum_{d \leq D} \frac{1}{d} \sum_{\chi \pmod{d}} \sum_{|\gamma| \leq T} |I_\rho(x)|.
\]

Consider the integral \( I_\rho(x) \) defined by (71) and let \( \rho = \beta + i\gamma \). With a change of variables we can write the integral in the form

\[
I_\rho(x) = \frac{1}{c} \int_{(\mu X)^c} X^{\frac{1}{c}} \frac{1}{x} e(xu) du = \int_{(\mu X)^c} X^{\frac{1}{c}} \frac{1}{x} e(h(u)) du,
\]

where

\[
h(u) = \frac{\gamma}{2\pi c} \log u + xu.
\]

We have

\[
h'(u) = \frac{\gamma}{2\pi c u} + x, \quad h''(u) = \frac{-\gamma}{2\pi c u^2}.
\]

Suppose that \( |\gamma| \geq 4\pi c|x|X^c \). Then we have \( |h'(u)| \asymp \frac{|\gamma|}{X^c} \) and we estimate the integral using Lemma 3. We find that

\[
I_\rho(x) \ll \frac{X^\beta}{|\gamma|} \quad \text{if} \quad 4\pi c|x|X^c \leq |\gamma| \leq T.
\]

If \( \pi c\mu^c|x|X^c < |\gamma| < 4\pi c|x|X^c \), then we have \( |h''(u)| \asymp \frac{|\gamma|}{X^{2c}} \asymp \frac{|\gamma|}{X^c} \) and using Lemma 5 we find that

\[
I_\rho(x) \ll \frac{X^{\beta-c}}{\sqrt{|x|X^{-c}}} = \frac{X^\beta}{\sqrt{|x|X^c}}.
\]

Finally, if \( |\gamma| \leq \pi c\mu^c|x|X^c \), then we have \( |h'(u)| \asymp |x| \) and applying again Lemma 5 we get

\[
I_\rho(x) \ll \frac{X^\beta}{|x|X^c}.
\]

However, from (63) it follows that \( |x|X^c \geq \sqrt{|x|X^c} \), hence from the last two estimates we obtain

\[
I_\rho(x) \ll \frac{X^\beta}{\sqrt{|x|X^c}} \quad \text{if} \quad |\gamma| < 4\pi c|x|X^c.
\]
From (74) – (76) we find that
\[ \mathcal{R} \ll \mathcal{R}' + \mathcal{R}'', \tag{77} \]
where \( \mathcal{R}' \) is the contribution coming from the terms for which the condition (75) for \( \gamma \) is satisfied and respectively, \( \mathcal{R}'' \) comes from the terms for which \( \gamma \) satisfies the condition for \( \gamma \) given in (76).

Consider \( \mathcal{R}' \). We have
\[
\mathcal{R}' = \sum_{d \leq D} \frac{1}{d} \sum_{x \pmod{d}} \sum_{4\pi c(x|X^c|) = |\gamma|} X^\beta / |\gamma|.
\]
We divide the sum over \( d \) into \( O(\log X) \) sums in which \( d \) runs over an interval of the form \( (Q, Q') \), where \( Q' \leq \min(2Q, D) \). We proceed with the sum over \( \gamma \) in the same way. Hence, we obtain
\[
\mathcal{R}' \ll (\log X)^2 \max_{Q \in [1, D]} \max_{L \in [4\pi c(x|X^c|, T]} \left( (QL)^{-1} \mathcal{L}(L, Q, X) \right), \tag{78}
\]
where \( \mathcal{L}(L, Q, X) \) is the sum defined by (52).

Having in mind (7), (8), (64), (70) and since for the above sums we have \( Q \leq D \) and \( L \leq T \), we easily verify the condition \( Q^2 L \leq X^{12/5} \) imposed in Lemma 9. Then using (54), (64) and (78), we find that
\[
\mathcal{R}' \ll X(\log X)^{22 - \frac{B}{2}}. \tag{79}
\]

Consider now the quantity \( \mathcal{R}'' \) for which we have
\[
\mathcal{R}'' \ll \frac{1}{\sqrt{|x|X^c}} \sum_{d \leq D} \frac{1}{d} \sum_{x \pmod{d}} \sum_{|\gamma| \leq 4\pi c(x|X^c|} X^\beta.
\]
Using (52), the estimate (54) from Lemma 9 and (64) we find that
\[
\mathcal{R}'' \ll \frac{\log X}{\sqrt{|x|X^c}} \max_{Q \in [1, D]} \left( Q^{-1} \mathcal{L} (4\pi c(x|X^c, Q, X) \right) \ll X(\log X)^{22 - \frac{B}{2}}. \tag{80}
\]
From (77), (79) and (80) we obtain
\[
\mathcal{R} \ll X(\log X)^{22 - \frac{B}{2}}.
\]
We substitute this estimate for \( \mathcal{R} \) in (73) and, as the constant \( B \) can be taken arbitrarily large, we see that the asymptotic formula (59) is correct also in the case (64). This proves the lemma.

\[\square\]

The next lemma is an analog of [16, Lemma 7].
Lemma 11. Let \( \tau \) and \( \Xi \) be defined by (8) and (9). Then for the sum \( L(x) \) and for the integral \( I(x) \), defined respectively by (44) and (46), we have

\[
\int_{|x|<\tau} |L(x)|^2 \, dx \ll X^{2-c}(\log X)^6, \tag{81}
\]

\[
\int_{|x|<\tau} |I(x)|^2 \, dx \ll X^{2-c}(\log X)^4, \tag{82}
\]

\[
\int_{|x|<\Xi} |L(x)|^2 \, dx \ll X^{\Xi}(\log X)^6. \tag{83}
\]

Proof: We shall only prove (81), the other inequalities can be proved likewise. Denote by \( J \) the integral on the left side of (81). We have

\[
J = \int_{|x|<\tau} L(x)L(-x) \, dx
\]

\[
= \int_{|x|<\tau} \sum_{d_1 \leq D} \lambda(d_1) \sum_{\mu X<p_1 \leq X \atop d_1|p_1+2} (\log p_1)e^{(xp_1^c)} \sum_{d_2 \leq D} \lambda(d_2) \sum_{\mu X<p_2 \leq X \atop d_2|p_2+2} (\log p_2)e^{(-xp_2^c)} \, dx
\]

\[
= \sum_{d_1,d_2 \leq D} \lambda(d_1)\lambda(d_2) \sum_{\mu X<n_1,n_2 \leq X \atop d_1|n_1+2 \atop d_2|n_2+2, \ i=1,2} (\log p_1)(\log p_2) \int_{|x|<\tau} e^{(x(p_1^c-p_2^c))} \, dx
\]

\[
\ll (\log X)^2 \sum_{d_1,d_2 \leq D} \sum_{\mu X<n_1,n_2 \leq X \atop d_1|n_1+2 \atop d_2|n_2+2, \ i=1,2} \min\left(\tau, \frac{1}{|n_1^c-n_2^c|}\right).
\]

We change the order of summation and use the obvious inequality \( uv \leq u^2 + v^2 \) to get

\[
J \ll (\log X)^2 \sum_{\mu X<n_1,n_2 \leq X} \tau(n_1+2) \tau(n_2+2) \min\left(\tau, \frac{1}{|n_1^c-n_2^c|}\right)
\]

\[
\ll (\log X)^2 \sum_{\mu X<n_1,n_2 \leq X} \tau^2(n_1+2) \min\left(\tau, \frac{1}{|n_1^c-n_2^c|}\right).
\]

Now we proceed as in the proof of [16, Lemma 7] and we also use the well-known inequality \( \sum_{n \leq y} \tau^2(n) \ll y(\log y)^3. \) In this way we prove (81) — we leave the details to the reader. \( \square \)
We shall find asymptotic formulae for the integrals \( \Gamma_1^{(1)} \) and \( \Gamma_4^{(1)} \) defined respectively by (85) and (89). We consider only \( \Gamma_1^{(1)} \) because the study of \( \Gamma_4^{(1)} \) is similar.

From (22), (46) and Lemma 10 we know that if \( |x| < \tau \), then we have
\[
L^\pm(x) = \mathcal{N}^\pm I(x) + O\left(X (\log X)^{-A}\right),
\]
where \( A > 0 \) is arbitrarily large. Let
\[
\mathcal{J}_0 = \int_{|x| < \tau} \Theta(x) e (-Nx) I(x)^3 \, dx.
\]
(84)

We use the identity
\[
L^- (x) \left( L^+ (x) \right)^2 = \mathcal{N}^- (\mathcal{N}^+)^2 I(x)^3 + (L^- (x) - \mathcal{N}^- I(x)) (\mathcal{N}^+)^2 I(x)^2
\]
\[
+ L^- (x) (L^+ (x) - \mathcal{N}^+ I(x)) \mathcal{N}^+ I(x) + L^- (x) L^+ (x) (L^+ (x) - \mathcal{N}^+ I(x))
\]
and the obvious estimate
\[
\mathcal{N}^\pm \ll \log X
\]
(85)
to find that
\[
\left| L^- (x) \left( L^+ (x) \right)^2 - \mathcal{N}^- (\mathcal{N}^+)^2 I(x)^3 \right| \ll X (\log X)^{3-A} \left(|I(x)|^2 + |L^- (x)|^2 + |L^+ (x)|^2\right).
\]
From the above inequality, the estimate (16) with \( a = \frac{7}{8} \Delta \), from (85), (84) and the estimates (81), (82) from Lemma 11 it follows that
\[
\Gamma_1^{(1)} - \mathcal{N}^- (\mathcal{N}^+)^2 \mathcal{J}_0 \ll \Delta X^{3-c} (\log X)^{9-A}.
\]
(86)

Consider now the integral
\[
\mathcal{J} = \int_{-\infty}^{\infty} \Theta(x) e (-Nx) I(x)^3 \, dx.
\]
(87)

Following the proof of (16) Lemma 6] we see that for a suitable \( \mu \in (0, 1) \) depending on \( c \) we have
\[
\mathcal{J} \gg \Delta X^{3-c}.
\]
(88)

We leave the easy verification to the reader.

Further, we have \( \left| \frac{\pi}{\mathcal{N}} (xt^c) \right| \gg |x| X^{-1} \) for \( t \in [\mu X, X] \). Therefore, applying again Lemma 5 we find that \( I(x) \ll |x|^{-1} X^{1-c} \). From this estimate, (8), (16) with \( a = \frac{7}{8} \Delta \), (84) and (87) we find that
\[
|\mathcal{J} - \mathcal{J}_0| \leq \int_{|x| > \tau} |\Theta(x)| |I(x)|^3 \, dx \ll \Delta X^{3-3c} \int_{x > \tau} \frac{dx}{x^3} \ll \Delta X^{3-3c} \tau^{-2} \ll \Delta X^{3-c-2\varepsilon}.
\]
(89)

We apply (86) (with \( A = 12 \)) as well as (85) and (89) to find
\[
\Gamma_1^{(1)} = \mathcal{N}^- (\mathcal{N}^+)^2 \mathcal{J} + O\left(\Delta X^{3-c} (\log X)^{-4}\right).
\]
(90)

We proceed with \( \Gamma_4^{(1)} \) is the same way and prove that
\[
\Gamma_4^{(1)} = (\mathcal{N}^+)^3 \mathcal{J} + O\left(\Delta X^{3-c} (\log X)^{-4}\right).
\]
(91)
4 The estimation of $\Gamma_1^{(2)}$ and $\Gamma_4^{(2)}$, and the end of the proof

From (43) we see that in order to find a non-trivial lower bound for $\Gamma$ we have to prove that the integrals $\Gamma_1^{(2)}$ and $\Gamma_4^{(2)}$ are small enough. To establish this we need estimates for the sums $L^\pm(x)$ provided that $\tau \leq |x| \leq \Xi$.

We apply the next lemma, which is a special case of Vaughan’s identity.

**Lemma 12.** Let $f(n)$ be a complex valued function defined for integers $n \in (\mu X, X]$. Then we have

$$
\sum_{\mu X < n \leq X} \Lambda(n) f(n) = S_1 - S_2 - S_3,
$$

where

$$
S_1 = \sum_{k \leq X^{1/3}} \mu(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} (\log l) f(kl),
$$

$$
S_2 = \sum_{k \leq X^{1/2}} c(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} f(kl),
$$

$$
S_3 = \sum_{X^{1/3} < k \leq X^{1/2}} a(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} \Lambda(l) f(kl)
$$

and where $a(k), c(k)$ are real numbers satisfying

$$
|a(k)| \leq \tau(k), \quad |c(k)| \leq \log k.
$$

**Proof.** Can be found in [20].

Follows Van der Corput’s inequality.

**Lemma 13.** Let $\alpha, \beta$ be real numbers with $\beta - \alpha \geq 1$ and let $H$ be a positive integer. Suppose that for any integer $l \in (\alpha, \beta]$ there is a complex number $\Upsilon(l)$. Then we have

$$
\left| \sum_{\alpha < l \leq \beta} \Upsilon(l) \right|^2 \leq \frac{\beta - \alpha + H}{H} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H}\right) \sum_{\alpha < l, l+h \leq \beta} \Upsilon(l+h) \Upsilon(l).
$$

**Proof.** Can be found in [8, Lemma 8.17].

The next lemma presents Van der Corput’s estimate for exponential sums.
Lemma 14. Suppose that $\alpha, \beta$ are real numbers with $\beta - \alpha \geq 1$ and let $f(y)$ be two times continuously differentiable function in the interval $[\alpha, \beta]$. Assume also that for some $\lambda > 0$ we have $|f''(y)| \asymp \lambda$ uniformly for $y \in [\alpha, \beta]$. Then we have

$$\left| \sum_{\alpha < n \leq \beta} e(f(n)) \right| \ll (\beta - \alpha)\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}.$$

Proof. See [9, Chapter 1, Theorem 5]. 

From this point onwards we assume that

$$\xi = \frac{459c - 435}{125}, \quad \delta = \frac{180 - 168c}{125}. \quad (97)$$

(It is easy to verify that (97) implies the first inequality from (7)). We prove the following

Lemma 15. Suppose that $D, \tau$ and $\Xi$ are defined by (8) and (9) and $\xi, \delta$ are specified by (97). Suppose also that the real numbers $\lambda(d)$ satisfy (45) and $L(x)$ is defined by (44).

Then there exists $\varkappa(c) > 0$ such that

$$\sup_{\tau \leq |x| \leq \Xi} |L(x)| \ll X^{2-c-\varkappa(c)}. \quad (98)$$

Proof. Instead of $L(x)$ we consider the sum $L_1(x)$ given by (61) and we take into account (60). We write $L_1(x)$ in the form

$$L_1(x) = \sum_{\mu X < n \leq X} \Lambda(n) f(n), \quad f(n) = \sum_{d \leq D \atop d | n+2} \lambda(d) e(xn^c) \quad (99)$$

and we apply Lemma 12 to find that

$$L_1(x) = S_1 - S_2 - S_3, \quad (100)$$

where $S_1, S_2$ and $S_3$ are the sums defined respectively by (92) – (94) with the function $f(n)$ given by (99).

From (93) we find that

$$S_2 = S'_2 + S''_2, \quad (101)$$

where

$$S'_2 = \sum_{k \leq X^{\frac{1}{2}}} c(k) \sum_{\frac{\mu X}{X} < l \leq \frac{\Xi}{X}} f(kl), \quad (102)$$

$$S''_2 = \sum_{X^{\frac{1}{2}} < k \leq X^{\frac{1}{2}}} c(k) \sum_{\frac{\mu X}{X} < l \leq \frac{\Xi}{X}} f(kl). \quad (103)$$

Therefore we have

$$L_1(x) \ll |S_1| + |S'_2| + |S''_2| + |S_3|. \quad (104)$$
Consider the sum $S_2'$. We use (99) and (102) and change the order of summation to write it in the form

$$S_2' = \sum_{d \leq D} \lambda(d) \sum_{k \leq X^{\frac{1}{2}}} c(k) \sum_{\frac{X}{d} < l \leq \frac{X}{k}} e(x(kl)^c).$$

Since $\lambda(d) = 0$ for $2 \mid d$, then from $d \mid kl + 2$ it follows that $(k, d) = 1$. Hence there exists an integer $l_0$ such that $d \mid kl + 2$ is equivalent to $l \equiv l_0 \pmod{d}$, which means that $l = l_0 + md$ for some integer $m$. Therefore we get

$$S_2' = \sum_{d \leq D} \lambda(d) \sum_{k \leq X^{\frac{1}{2}}} c(k) \sum_{\frac{X}{d} < l \leq \frac{X}{k}} e(h(m)), \quad h(m) = xk^c(l_0 + md)^c. \quad (105)$$

We have $h''(m) = c(c - 1)xk^c d^2 (l_0 + md)^{c-2}$, hence $|h''(m)| \asymp |x|^2d^2X^{c-2}$ and using Lemma 14 we find that the sum over $m$ in (105) is

$$\ll \frac{X}{kd} \left( |x|^2d^2X^{c-2} \right)^{\frac{1}{2}} + \left( |x|^2d^2X^{c-2} \right)^{-\frac{1}{2}} \ll |x|^{\frac{1}{2}}X^{\frac{c}{2}} + |x|^{-\frac{1}{2}}(kd)^{-1}X^{1-\frac{c}{2}}.$$

Then using (45), (95) and (105) we find that

$$S_2' \ll X^\varepsilon \left( D|x|^{\frac{1}{2}}X^{\frac{c}{2}} + |x|^{-\frac{1}{2}}X^{1-\frac{c}{2}} \right). \quad (106)$$

To estimate $S_1$ we apply Abel’s summation formula to get rid of the factor $(\log l)$ in the sum from (92) and then proceed as above. In this way we find that

$$S_1 \ll X^\varepsilon \left( D|x|^{\frac{1}{2}}X^{\frac{c}{2}} + |x|^{-\frac{1}{2}}X^{1-\frac{c}{2}} \right). \quad (107)$$

We leave the simple calculations to the reader.

Consider now the sum $S_3$. We divide it into $O(\log X)$ sums of type

$$W(K) = \sum_{K < k \leq K_1} a(k) \sum_{\frac{X}{k} < l \leq X^{\frac{1}{2}}} \Lambda(l) \sum_{d \leq D} \lambda(d) e(x(kl)^c), \quad (108)$$

where

$$K_1 \leq 2K, \quad X^{\frac{1}{2}} \leq K < K_1 \leq X^{\frac{3}{4}}. \quad (109)$$

Consider the case

$$X^{\frac{1}{2}} \leq K. \quad (110)$$

It is clear that

$$W(K) \ll X^\varepsilon \sum_{K < k \leq K_1} \left| \sum_{\frac{X}{k} < l \leq X^{\frac{1}{2}}} \Psi(l) \right|, \quad (111)$$

21
where
\[ \Upsilon(l) = \Lambda(l) \sum_{d \leq D, d | kl+2} \lambda(d) \, e \left( x(kl)^c \right). \] (112)

Applying Cauchy’s inequality, we find that
\[ |W(K)|^2 \ll X^e K \sum_{K < k \leq K_1} \left| \sum_{\frac{\mu X}{K} < l \leq \frac{X}{K}} \Upsilon(l) \right|^2. \] (113)

Assume that \( H \) is an integer satisfying
\[ 1 \leq H \ll \frac{X}{K} \] (114)
and let \( \Upsilon(l) \) be defined by (112). For the inner sum in (113) we apply the inequility (96) from Lemma [13] and we find that
\[ |W(K)|^2 \ll \frac{X^{1+e}}{H} \sum_{K < k \leq K_1} \sum_{|h| < H} \left( 1 - \frac{|h|}{H} \right) \sum_{\frac{\mu X}{K} < l, l+h \leq \frac{X}{K}} \Lambda(l) \Lambda(l-h) \mathcal{F}, \] (115)
where
\[ \mathcal{F} = \sum_{\tilde{K}_1 < k \leq \tilde{K}_1} \sum_{d_1 | k | l+h+2} e \left( xk^c \left( (l+h)^c - l^c \right) \right) \]
and
\[ \tilde{K}_1 = \min \left( K_1, \frac{X}{l}, \frac{X}{l+h} \right). \]

We change the order of summation and find
\[ |W(K)|^2 \ll \frac{X^{1+e}}{H} \sum_{d_1, d_2 \leq D} \lambda(d_1) \lambda(d_2) \sum_{|h| < H} \left( 1 - \frac{|h|}{H} \right) \sum_{\frac{\mu X}{K} < l, l+h \leq \frac{X}{K}} \Lambda(l) \Lambda(l+h) \mathcal{F}, \] (115)
where
\[ \mathcal{F} = \sum_{\tilde{K}_1 < k \leq \tilde{K}_1} \sum_{d_1 | k | l+h+2} e \left( xk^c \left( (l+h)^c - l^c \right) \right) \]
and
\[ \tilde{K}_1 = \max \left( K_1, \frac{X}{l}, \frac{X}{l+h} \right). \]

Since \( \lambda(d) = 0 \) for \( 2 \nmid d \), we may assume that \( 2 \nmid d_1 d_2 \). Hence \( (d_1, l) = (d_2, l+h) = 1 \) because otherwise the sum \( \mathcal{F} \) would be empty. Therefore, there exists an integer \( k_0 \) depending on \( l, h, d_1, d_2 \) such that the pair of conditions \( d_1 | kl+2 \) and \( d_2 | k(l+h)+2 \) is equivalent to the congruence \( k \equiv k_0 \pmod{[d_1, d_2]} \). Hence we may write the sum \( \mathcal{F} \) as
\[ \mathcal{F} = \sum_{\frac{k-k_0}{[d_1, d_2]} < m \leq \frac{k_1-k_0}{[d_1, d_2]}} e(F(m)), \]
where
\[ F(m) = xk^c \left( (l+h)^c - l^c \right). \]
where
\[ F(m) = x ((l + h)^c - l^c) (k_0 + m[d_1, d_2])^c. \]

Obviously, we have
\[ \mathfrak{F} \ll \frac{K}{[d_1, d_2]} \quad \text{if} \quad h = 0. \quad (116) \]

Consider now the case \( h \neq 0 \). We have
\[ F''(m) = c(c - 1) x ((l + h)^c - l^c) (k_0 + m[d_1, d_2])^{c-2} [d_1, d_2]^2, \]

hence
\[ |F''(m)| \asymp |x| ((l + h)^c - l^c) K^{c-2}[d_1, d_2]^2. \]

We apply Lemma 14 and find that
\[ \mathfrak{F} \ll \frac{K}{[d_1, d_2]} \left( |x| |(l + h)^c - l^c| K^{c-2}[d_1, d_2]^2 \right)^{\frac{1}{2}} + \left( |x| |(l + h)^c - l^c| K^{c-2}[d_1, d_2]^2 \right)^{-\frac{1}{2}} \]
\[ \ll |x|^{\frac{1}{2}} K^{\frac{c}{2}} |l + h|^c - l^c|^{\frac{1}{2}} + |x|^{-\frac{1}{2}} K^{1 - \frac{c}{2}} |l + h|^c - l^c|^{-\frac{1}{2}} [d_1, d_2]^{-1}. \]

We use the above estimate, (115), (116) as well as the estimate
\[ \sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \ll (\log X)^3 \]
(we leave the easy verification to the reader) to obtain
\[ |W(K)|^2 \ll X^{2+\varepsilon} H^{-1} + X^{1+\varepsilon} D^2 |x|^{\frac{1}{2}} K^{\frac{c}{2}} H^{-1} \Sigma_1 + X^{1+\varepsilon} |x|^{-\frac{1}{2}} K^{1-\frac{c}{2}} H^{-1} \Sigma_1, \]

where
\[ \Sigma_1 = \sum_{0 < |h| \leq H} \sum_{\frac{X}{h^c} < l + h \leq \frac{X}{K}} |(l + h)^c - l^c|^{\frac{1}{2}}, \]
\[ \Sigma_2 = \sum_{0 < |h| \leq H} \sum_{\frac{X}{h^c} < l + h \leq \frac{X}{K}} |(l + h)^c - l^c|^{-\frac{1}{2}}. \]

By a straightforward calculation, which we leave to the reader, one obtains
\[ \Sigma_1 \ll H^{\frac{3}{2}} X^{\frac{1}{2} - \frac{c}{2}} K^{-\frac{1}{2} - \frac{c}{2}}, \quad \Sigma_2 \ll H^{\frac{1}{2}} X^{\frac{3}{2} - \frac{c}{2}} K^{-\frac{1}{2} - \frac{c}{2}}. \]

Therefore, we find
\[ |W(K)|^2 \ll X^{\varepsilon} \left( X^2 H^{-1} + X^{\frac{4\varepsilon}{3}} D^2 |x|^{\frac{1}{2}} K^{1 - \frac{c}{2}} H^{\frac{1}{2}} + X^{\frac{2\varepsilon}{3}} |x|^{-\frac{1}{2}} K^{1 - \frac{c}{2}} H^{-\frac{1}{2}} \right). \quad (117) \]

We choose
\[ H = \left[ \min(H_0, X K^{-1}) \right], \quad \text{where} \quad H_0 = X^{\frac{1-\varepsilon}{3}} K^{\frac{1}{2}} D^{-\frac{1}{2}} |x|^{-\frac{1}{2}}. \quad (118) \]

23
It is easy to verify that (114) holds. We note that
\[ H^{-1} \approx H_0^{-1} + KX^{-1}. \] (119)

Using (109), (110) and (117) – (119) we obtain
\[ W(K) \ll X^\varepsilon \left( X^{\frac{1}{2}+\delta} D^\frac{2}{3}|x|^{\frac{1}{3}} + X^{\frac{1}{2}+\delta} D^\frac{2}{3}|x|^{\frac{1}{3}} + X^{1-\frac{1}{2}} D^\frac{2}{3}|x|^{-\frac{1}{2}} + X^{1-\frac{1}{2}} |x|^{-\frac{1}{2}} \right). \] (120)

Consider now the sum \( W(K) \) defined by (108) in the case \( K < X^{\frac{1}{2}} \). (121)

We write it in the form
\[ W(K) = \sum_{\frac{K_1}{X} < l \leq X} \Lambda(l) \sum_{\max(K,\frac{X}{2}) < k \leq \min(K_1,\frac{X}{2})} a(k) \sum_{d \leq D} \lambda(d) e \left( x( kl)^c \right). \]

Now we have \( \frac{X}{K} \gg X^{\frac{1}{2}} \) and we may proceed as above but with roles of \( k \) and \( l \) reversed.

Finally, we establish again the estimate (120).

Since the sum \( S_3 \) consists of \( O \left( \log X \right) \) sums of type \( W(K) \), it can be estimated by the expression from the right side or (120), too. We study the sum \( S_2'' \) in the same manner and we obtain
\[ S_2'', S_3 \ll X^\varepsilon \left( X^{\frac{1}{2}+\delta} D^{\frac{2}{3}} |x|^{\frac{1}{3}} + X^{\frac{1}{2}+\delta} D^{\frac{2}{3}} |x|^{\frac{1}{3}} + X^{1-\frac{1}{2}} D^{\frac{2}{3}} |x|^{\frac{1}{2}} + X^{1-\frac{1}{2}} |x|^{-\frac{1}{2}} \right). \] (122)

From (60), (104), (106), (107) and (122) we find that
\[ L(x) \ll X^\varepsilon \left( X^{\frac{1}{2}+\delta} D |x|^{\frac{1}{3}} + X^{1-\frac{1}{2}} |x|^{-\frac{1}{2}} + X^{\frac{1}{2}+\delta} D^{\frac{2}{3}} |x|^{\frac{1}{3}} + X^{\frac{1}{2}+\delta} D^{\frac{2}{3}} |x|^{\frac{1}{3}} + X^{1-\frac{1}{2}} D^{\frac{2}{3}} |x|^{\frac{1}{2}} + X^{1-\frac{1}{2}} |x|^{-\frac{1}{2}} \right). \]

Now we use (6) to find that if \( \tau \leq |x| \leq \Xi \), then we have \( X^{\varepsilon-\tau} \leq |x| \ll X^{\varepsilon} \). Hence
\[ L(x) \ll X^\varepsilon \left( X^{\frac{1}{2}+\delta+\delta} + X^{\frac{1}{2}+\delta+\delta} + X^{1-\frac{1}{2}+\delta} + X^{1-\frac{1}{2}} \right). \]

It remains to use (2) and (97) and after a simple calculation we obtain (98).

We are now in position to estimate the quantities \( \Gamma_1^{(2)} \) and \( \Gamma_4^{(2)} \) defined respectively by (36) and (40). We apply Lemma 15 and use (16) with \( a = \frac{7}{8} \Delta \) to find that
\[ \Gamma_1^{(2)}, \Gamma_4^{(2)} \ll X^{2-c-(c)} \Delta \int_{|x| \leq \Xi} |L^+ (x)|^2 dx. \]

From the above formula, the definitions of \( \Delta \) and \( \Xi \) (see (3) and (9)) and formula (83) of Lemma 11 we conclude that
\[ \Gamma_1^{(2)}, \Gamma_4^{(2)} \ll \Delta X^{3-c} (\log X)^{-4}. \]
From the last formula and (43), (88), (90), (91) we conclude that

\[ \Gamma \geq |3 \mathfrak{N} - 2 \mathfrak{N}^+| (\mathfrak{N}^+)^2 \overline{3} + O \left( \Delta X^{3-c} (\log X)^{-4} \right). \]  

(123)

Now we shall find a lower bound for the difference $3 \mathfrak{N} - 2 \mathfrak{N}^+$. It is clear that for the quantity $\mathfrak{P}$ defined by (22) we have

\[ \mathfrak{P} \asymp (\log X)^{-1}. \]  

(124)

From (23) and (24) we see that

\[ 3 \mathfrak{N} - 2 \mathfrak{N}^+ \geq \mathfrak{P} \left( 3f(s_0) - 2F(s_0) \right) + O \left( (\log X)^{-\frac{4}{3}} \right), \]  

(125)

where $s_0$ is defined by (22) and $F(s), f(s)$ are the functions specified by (25). We take $s_0 = 2,95$ which means that

\[ \eta = \frac{\delta}{2,95} = \frac{180 - 168c}{368, 75}. \]

Hence, using (25) we see that $3f(s_0) - 2F(s_0) > 0$.

It remains to take into account the lower bound for $\mathfrak{N}^+$ in (23) as well as (88), (124), (125) to obtain

\[ \Gamma \gg \Delta X^{3-c} (\log X)^{-3}. \]

Therefore $\Gamma > 0$ and the equation (1) with $\Delta$ specified by (3) has a solution in primes $p_1, p_2, p_3$ such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ has at most \[ \left\lfloor \frac{369}{180 - 168c} \right\rfloor \] prime factors. This completes the proof of Theorem 1.

$\square$

References

[1] Baker R., Weingartner A., A ternary diophantine inequality over primes, Acta Arith., 162, (2014), 159-196.

[2] Brüdern J., Fouvry E., Lagrange’s Four Squares Theorem with almost prime variables, J. Reine Angew. Math., 454 (1994), 59–96.

[3] Chen J. R. On the representation of a lagre even integer as the sum of a prime and the product of at most two primes, Sci. Sinica, 16, (1973), 157-167.

[4] Davenport H., Multiplicative Number Theory, Sec. ed., Springer, 1980.

[5] Dimitrov S. I., Studying diophantine inequalities and arithmetical progressions using number theory methods, Thesis, Technical University - Sofia, 2016, (in Bulgarian).

[6] Dimitrov S. I., Todorova T. L., Diophantine approximation by prime numbers of a special form, Annuaire Univ. Sofia, Fac. Math. Inform., vol.102, (2015), 71-90.
[7] Greaves G. *Sieves in number theory*, Springer, 2001.

[8] Iwaniec H., Kowalski E., *Analytic Number Theory*, American Mathematical Society, 2004.

[9] Karatsuba A. A., *Basic analytic number theory*, Springer, 1993.

[10] Matomäki K., *A Bombieri-Vinogradov type exponential sum result with applications*, J. Number Theory, 129 (2009), 2214–2225.

[11] Matomäki K., Shao H., Vinogradov’s three primes theorems with almost twin primes, arXiv:1512.03213v1 [math.NT]

[12] Montgomery H. L., *Topics in multiplicative number theory*, Springer, 1971.

[13] Segal B. I., *On a theorem analogous to Waring’s theorem*, Dokl. Akad. Nauk SSSR, (N. S.) 2, (1933), 47-49 (in Russian).

[14] Titchmarsh E. G., *The Theory of the Riemann Zeta-function* (revised by D. R. Heath-Brown), Clarendon Press, Oxford 1986.

[15] Todorova T.L., Tolev D.I., *On the distribution of αp modulo one for primes p of a special form*, Math. Slovaca 60, (2010), 771–786.

[16] Tolev D. I., *On a diophantine inequality involving prime numbers*, Acta Arith., 61,(3), (1992), 289-306.

[17] Tolev D. I., *Arithmetic progressions of prime-almost-prime twins*, Acta Arith., 88, (1999), 67-98.

[18] Tolev D. I., *Representations of large integers as sums of two primes of special type*, in “Algebraic Number Theory and Diophantine Analysis”, Walter de Gruyter, 2000, 485-495.

[19] Tolev D. I., *Additive problems with prime numbers of special type*, Acta Arith. 96, 11 (2000), 53–88. Corrigendum: Acta Arith. 105, 2, (2002), 205.

[20] Vaughan R. C., *An elementary method in prime number theory*, Acta Arith. 37 (1980), 111–115.

[21] Vinogradov I. M., *Representation of an odd number as a sum of three primes*, Dokl. Akad. Nauk SSSR 15 (1937), 169-172 (in Russian).

Faculty of Mathematics and Informatics
Sofia University “St. Kl. Ohridsky”
5 J. Bourchier, 1164 Sofia, Bulgaria

dtolev@fmi.uni-sofia.bg