The Controlled-$U$ and Unitary Transformation in Two-Qudit

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We concretely construct an extension of the controlled-$U$ gate in qudit from some elementary gates. We also construct unitary transformation in two-qudit by means of the extended controlled-$U$ gate and show the universality of it.

I. INTRODUCTION

In study on networks of quantum computer, qubit is mainly used for unit of circuits. A qubit has two states, $|0\rangle$ and $|1\rangle$, and preserves their superposition. It is a unit of computation. A network of quantum computer consists of a bundle of them and quantum computation is a sequence of quantum gates, which are unitary transformations, on them. Although we do not know what kind of unitary transformation is required, it is shown that any unitary transformation can be constructed in qubit\cite{1,2}.

In qubit number of steps of computation tends to be large because algorithm is given in two-value. When network is constructed in actual physical system, there arises decoherence which is caused by interaction with environment\cite{3,4}. Also in this point of view, it is better that number of steps is small. Further physical systems are not always two-level. For example, atom has infinite energy levels. There is no need to restrict to only two of them.

It is considered that unit of computation has three or more states. This unit is called by “qudit”. Quantum computation in qudit is a sequence of unitary transformations on a bundle of qudits as well as in qubit case.

Computation in qudit has some advantage to that in qubit. Use of multi-valued unit may decrease number of steps. This is favorable for the decoherence problem. Waste of high excited states in physical systems may fall off.

As stated above, quantum gates in qudit are unitary transformations on a bundle of qudits. Such gates have been devised by some researchers\cite{5–12}. However, as far as we know, there is no work on construction of unitary transformation from elementary gates concretely and proof of universality of it. It seems that the universality is approved on the analogy of qubit, however, it should be proved strictly. So in this paper we construct unitary transformation from elementary gates concretely and show the universality of it.

The constitution of the paper is as follows. In Sec.II we give some elementary gates. We build any network by sequence of them. We extend the controlled-unitary gate to qudit, which was introduced in \cite{13} to construct unitary transformation in qubit in Sec. III. In Sec. IV we show that we can construct any unitary transformation in two-qudit using the controlled-unitary gates constructed in the previous section. The last section is devoted to the discussion.

II. ELEMENTARY GATES FOR QUDIT

There are some ways to construct networks in qudit and there may be different elementary gates in each way. So first we introduce elementary gates in our method.

We assume the following. First we can perform any unitary transformation in a single qudit. In spite of this fact there exist some important gates. We introduce them in Sec.II A. Second there are at least one gate to connect two (or more) qudits for entanglement. We introduce such a gate in Sec. II B.

We consider a $d$-level system, thus there are $d$ states in a qudit. We label them as $|k\rangle$ ($k = 0, 1, 2, \ldots, d-1$).

When we express states and operators in matrix form we identify it with $d$-dimensional vector

$$|k\rangle = t(0, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where $t$ means transpose.

A. Elementary gates in a single qudit

First we introduce $P_{ab}$ which exchange two states whose operation is

$$P_{ab}|c\rangle = \begin{cases} |b\rangle, & \text{if } |c\rangle = |a\rangle \\ |a\rangle, & \text{if } |c\rangle = |b\rangle \\ |c\rangle, & \text{otherwise} \end{cases} \ (a, b, c = 0, 1, 2, \ldots, d-1).$$

At this point $P_{ba}$ is the same with $P_{ab}$, so we restrict to $a \leq b$ unessentially. The diagram is
one gate is fundamental. For example, we choose
d−
which is one of elementary matrices.
where
ab
satisfies
ab
= δij + δia(−δja + δjb) + δib(δja − δjb)
which is one of elementary matrices.
Pab satisfies
Pab = †Pab = P†
ab,

P2
ab = 1
d,
where † means hermitian conjugate.
Number of Pab’s is 2C2 = d(d − 1)/2, however, only
d − 1 of them are more fundamental. In fact if we put
Pab = P0a (a = 1, 2, . . . , d − 1),
others are expressed by
Pab = PabPbPa.

Nevertheless in actual physical systems all gates may be handled equally.
Second we introduce another important gate which is an extension of the Walsh-Hadamard gate. The operation is

\( H_{ab}|c\rangle = \begin{cases} 
\frac{1}{\sqrt{2}}(|a\rangle + |b\rangle), & \text{if } |c\rangle = |a\rangle \\
\frac{1}{\sqrt{2}}(|a\rangle - |b\rangle), & \text{if } |c\rangle = |b\rangle \\
|c\rangle, & \text{otherwise}
\end{cases} \)

and its matrix form is

\[
(H_{ab})_{ij} = \delta_{ij} + \delta_{ia}\left((-1 + \frac{1}{\sqrt{2}})\delta_{ja} + \frac{1}{\sqrt{2}}\delta_{jb}\right) + \delta_{ib}\left(\frac{1}{\sqrt{2}}\delta_{ja} - (1 + \frac{1}{\sqrt{2}})\delta_{jb}\right).
\]

\( H_{ab} \) satisfies

\( H = H^\dagger = H_{01}, \quad H^2 = 1_d. \)

Similar to \( P_{ab} \), number of \( H_{ab} \) is 2C2, however, only one gate is fundamental. For example, we choose

\( H = H_{01}, \)

others are expressed by

\( H_{ab} = P_{ba}P_{1b}HP_{1b}P_{ba}, \quad (1 \leq a \leq b). \)

B. Controlled gate

Now we introduce a symbol \( \tilde{C}^a(U) \) which means the controlled gate. \( C \) means the controlled operation and exponent \( a \) indicates the state of the control bit in which the unitary transformation \( U \) is applied. The tilde over \( C \) means two qudit operation. In equation the operation of \( \tilde{C}^a(U) \) is

\[ \tilde{C}^a(U)|c\rangle|d\rangle = \begin{cases} 
|a\rangle(U|d\rangle), & \text{if } |c\rangle = |a\rangle \\
|c\rangle|d\rangle, & \text{otherwise}
\end{cases} . \]

Also we introduce the diagram of \( \tilde{C}^a(U) \) as follows:

As the controlled gate, the controlled-NOT or the controlled-\( \sigma_z \) is usually used in qubit. In this paper we extend the controlled-\( \sigma_z \) in qudit. There are some extensions. For example, one of them is that if the control bit is some state then all of the states in the target bit are shifted at once. However our extension is the following.

If the control bit is \( |a\rangle \), then the sign of \( |b\rangle \) is reversed, otherwise the target bit is left alone.

In equation

\[ \tilde{C}^a(M_b)|c\rangle|d\rangle = \begin{cases} 
-|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\
|c\rangle|d\rangle, & \text{otherwise}
\end{cases} , \quad (a, b, c, d = 0, 1, 2, . . . , d − 1), \quad (1) \]

where \( M_b \) is a single qudit operation which reverses the sign of \( |b\rangle \):

\[ M_b|c\rangle = \begin{cases} 
-|b\rangle, & \text{if } |c\rangle = |b\rangle \\
|c\rangle, & \text{otherwise}
\end{cases} , \quad (b, c = 0, 1, . . . , d − 1). \]

The matrix form of \( \tilde{C}^a(M_b) \) is

\[ (\tilde{C}^a(M_b))_{ij} = \delta_{ij}(1 - 2\delta_{i, ad+b}), \quad (i, j = 0, 1, . . . , d^2 − 1), \quad (2) \]

where

\[ M_b = \text{diag}(1, 1, . . . , -1, . . . , 1), \quad (i, j = 0, 1, . . . , d − 1), \]

and the diagram is
From (1) or (2), $\tilde{C}^a(M_b)$ apparently satisfies

$$\tilde{C}^a(M_b)(\tilde{C}^a(M_b))^\dagger = I_{d^2}$$

and also

$$M_b^a = b M_a$$

holds.

Number of $\tilde{C}^a(M_b)$’s is $d^2$, however, only one is fundamental. For example, if we choose $\tilde{C}^a(M_b)$, others are obtained by

$$\tilde{C}^a(M_b) = (\mathbb{1} \otimes P_{a\bar{a}})(\mathbb{1} \otimes P_{b\bar{b}})\tilde{C}^a(M_b)(\mathbb{1} \otimes P_{b\bar{b}})(\mathbb{1} \otimes P_{a\bar{a}}).$$

Nevertheless, similar to $P_{ab}$, all $\tilde{C}^a(M_b)$’s may be realized equally in actual physical systems.

### III. CONSTRUCTION OF THE CONTROLLED-U GATE

#### A. Preparation

In this section we construct the controlled-U gate in qudit by connecting the elementary gates given in the previous section. Although we can consider some extensions, we adopt the following.

If the control bit is $|a\rangle$, then a unitary transformation $U$ is applied to the target bit, otherwise the target bit is left alone.

In equation

$$\tilde{C}^a(U)|b\rangle = \begin{cases} |a\rangle(U|b\rangle), & \text{if } |c\rangle = |a\rangle \\ |c\rangle|b\rangle, & \text{otherwise} \end{cases},$$

and the diagram is

\[
\begin{array}{c}
|a\rangle \\
\hspace{1cm} \text{U} \\
|b\rangle \\
\end{array}
\]

First we construct the controlled-exchange gate $\tilde{C}^a(P_{bc})$ whose operation is

if the control bit is $|a\rangle$, then exchange the states $|b\rangle$ and $|c\rangle$ in the target bit.

In equation

$$\tilde{C}^a(P_{bc})|c\rangle = \begin{cases} |a\rangle|c\rangle, & \text{if } |d\rangle = |a\rangle|b\rangle \\ |a\rangle|b\rangle, & \text{if } |d\rangle = |a\rangle|c\rangle \\ |d\rangle|c\rangle, & \text{otherwise} \end{cases}.$$  

This gate is built by

$$\tilde{C}^a(P_{bc}) = (\mathbb{1} \otimes H_{bc})\tilde{C}^a(M_c)(\mathbb{1} \otimes H_{bc})$$

whose diagram is

\[
\begin{array}{c}
|a\rangle \\
\hspace{1cm} \text{P}_{bc} \\
|b\rangle \\
\end{array}
\]

Indeed, if the control bit is $|a\rangle$

\[
\tilde{C}^a(P_{bc})(|a\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle) = |a\rangle(\alpha_c|b\rangle + \alpha_b|c\rangle + \sum_{k=0}^{d-1} \alpha_k|k\rangle)
\]

and otherwise

$$\tilde{C}^a(P_{bc})(|l\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle) = (\mathbb{1} \otimes \mathbb{1})|l\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle = |l\rangle \sum_{k=0}^{d-1} \alpha_k|k\rangle,$$

where $\sum^{\lor b,c}$ means a sum except for $b, c$.  

Second by means of $\tilde{C}^a(P_b)$ we construct the gate whose operation is

if the control bit is $|a\rangle$, then perform the phase shift $e^{-i\theta}$ to $|0\rangle$ and $e^{i\theta}$ to $|b\rangle$.

In equation

$$\tilde{C}^a(\Theta_b(\theta))|c\rangle|d\rangle = \begin{cases} e^{-i\theta}|a\rangle|0\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|0\rangle \\ e^{i\theta}|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\ |c\rangle|d\rangle, & \text{otherwise} \end{cases}$$

where $\Theta_b(\theta)$ is a single qudit operation whose matrix form is

$$\Theta_b(\theta) \equiv \text{diag}(e^{-i\theta}, 1, \ldots, e^{i\theta}, \ldots, 1).$$

This gate is realized by

$$\tilde{C}^a(\Theta_b(\theta)) \equiv (\mathbb{1} \otimes \Theta_b(\theta/4))\tilde{C}^a(P_{0b})(\mathbb{1} \otimes \Theta_b(-\theta/2)) \otimes \tilde{C}^a(P_{0b})(\mathbb{1} \otimes \Theta_b(\theta/4)) \otimes \tilde{C}^a(P_{0b})(\mathbb{1} \otimes \Theta_b(\theta/4))$$

Next we consider a single qudit operation:

$$\Theta(\theta)|b\rangle = \begin{cases} e^{-i(\theta_1+\theta_2+\ldots+\theta_{d-1})}|0\rangle, & \text{if } |b\rangle = |0\rangle \\ e^{i\theta_b}|b\rangle, & \text{otherwise} \end{cases},$$

where we abbreviate

$$\theta = (\theta_1, \theta_2, \ldots, \theta_{d-1}).$$

The matrix form is

$$\Theta(\theta) = \text{diag}(e^{-i(\theta_1+\theta_2+\ldots+\theta_{d-1})}, e^{i\theta_1}, \ldots, e^{i\theta_{d-1}}).$$

Then the controlled operation

$$\tilde{C}^a(\Theta(\theta))|c\rangle|b\rangle = \begin{cases} |a\rangle\Theta(\theta)|b\rangle, & \text{if } |c\rangle = |a\rangle \\ |c\rangle|b\rangle, & \text{otherwise} \end{cases}$$

is realized by

$$\tilde{C}^a(\Theta(\theta)) = \prod_{b=1}^{d-1} \tilde{C}^a(\Theta_b(\theta_b))$$

and the diagram is

\[\text{Diagram for } \tilde{C}^a(\Theta(\theta))\]

B. Construction of the controlled-$U$

First we construct the controlled-$U$ for not any unitary transformation but special unitary transformation $W \in \text{SU}(d)$.

For any $W \in \text{SU}(d)$, there exists $V \in \text{SU}(d)$ which
satisfies
\[ W = V^\dagger \Theta(\theta)V, \]
\[ \Theta(\theta) \equiv \text{diag}(e^{-i(\theta_1+\theta_2+\cdots+\theta_{d-1})}, e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{d-1}}), \]
where \( \Theta(\theta) \) is an appropriate diagonal matrix. By this fact we obtain the diagram of the controlled-\( U \) for \( SU(d) \) as follows:

![Diagram]

\[ a \]
\[ W \]
\[ = \]
\[ V \]
\[ \Theta(\theta) \]
\[ V^\dagger \]

To extend the above result to \( U(d) \), we introduce the phase gate \( \tilde{C}^a(S) \):
\[ \tilde{C}^a(S)|c\rangle|b\rangle = \begin{cases} e^{i\delta}|a\rangle|b\rangle, & \text{if } |c\rangle = |a\rangle \\ |c\rangle|b\rangle, & \text{otherwise} \end{cases} \]
following [13]. In the similar way to qubit case, the diagram is given by

![Diagram]

\[ a \]
\[ S \]
\[ = \]
\[ E_a \]
\[ W \]

where
\[ E_a \equiv \text{diag}(1,1,\ldots,1,e^{i\delta},1,\ldots,1) \in SU(d). \]
Indeed, in two-qudit representation
\[ E_a \otimes \mathbb{1} = \text{diag}(1,1,\ldots,1,e^{i\delta},1,\ldots,1) = \tilde{C}^a(S). \]

Making use of the phase gate, we can construct the controlled-\( U \) for \( U(d) \). Any \( U \in U(d) \) is decomposed to
\[ U = e^{i\delta}W \quad (W \in SU(d)). \]
By this fact the controlled-\( U \) for \( U(d) \) is given by
\[ \tilde{C}^a(U) = \tilde{C}^a(W)\tilde{C}^a(S). \]
and the diagram is

![Diagram]

\[ a \]
\[ U \]
\[ = \]
\[ a \]
\[ E_a \]
\[ W \]

IV. CONSTRUCTION OF UNITARY TRANSFORMATION IN TWO-QUDIT

In this section we construct unitary transformation \( \tilde{U} \in U(d^2) \) in two-qudit making use of the controlled-\( U \) in the previous section. We follow the method by Deutch[1]. In Sec. IV A we show that any state is transformed to an arbitrary basis vector in a single qudit. The similar result in two-qudit is shown in Sec. IV B. Then making use of this fact we construct unitary transformation in two-qudit in Sec. IV C.

A. Transformation to a basis vector in a single qudit

A state in a single qudit is written by
\[ |x\rangle = \sum_{k=0}^{d-1} c_k|k\rangle, \quad (c_k \in \mathbb{C}, \ k = 0, 1, 2, \ldots, d-1) \]
or, in the matrix form
\[ x = (c_0, c_1, \ldots, c_{d-1}). \]

For any state \(|x\rangle\), there exists a unitary transformation \( U \) which satisfies
\[ N_0|0\rangle = U|x\rangle \]
where
\[ N_n \equiv (\sum_{i=n}^{d-1} |c_i|^2)^{1/2} \]
or in the matrix form
\[ ^t(N_0, 0, \ldots, 0) = Ux. \]
Such $U$ is concretely (but not necessarily efficiently) constructed as follows. We put

$$h_k = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \frac{c_k}{N_k} & \cdots & \cdots & \frac{c_{k+1}}{N_{k+1}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \frac{c_{k+1}}{N_{k+1}} & \cdots & \cdots & \frac{c_{k+2}}{N_{k+2}} \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix},$$

then we obtain

$$(h_{d-2}h_{d-3} \cdots h_1 h_0) \mathbf{x} = \mathcal{T}(N_0, 0, \ldots, 0).$$

We also obtain an arbitrary basis vector as follows:

$$\mathcal{T}(0, 0, \ldots, N_0, 0, \ldots, 0) = U \mathbf{x}.$$  \hfill (3)

Indeed, if we multiply the exchange operator after operating $h$'s, we obtain

$$P_{0k}(h_{d-2}h_{d-3} \cdots h_1 h_0) \mathbf{x} = \mathcal{T}(N_0, 0, \ldots, 0) = \mathcal{T}(0, 0, \ldots, N_0, 0, \ldots, 0).$$

We temporarily call this operator as $T_k(\mathbf{x})$. We note the operation again:

$$T_k(\mathbf{x}) \mathbf{x} = \mathcal{T}(0, 0, \ldots, N_0, 0, \ldots, 0) (k = 0, 1, \ldots, d - 1).$$

### B. Transformation to a basis vector in two-qudit

Making use of $T_k(\mathbf{x})$, we construct the unitary operator $\hat{S}_k^0(\mathbf{x})$, which transforms a state $\hat{\mathbf{x}}$ in two-qudit to a basis vector.

A state in two-qudit is written by

$$| \mathbf{x} \rangle = \sum_{i,j=0}^{d-1} c_{ij} | i \rangle | j \rangle, \quad \left( \sum_{i,j=0}^{d-1} | c_{ij} |^2 = 1, \right)$$

or, in the matrix form

$$\hat{\mathbf{x}} = \mathcal{T}(c_{00}, c_{01}, \ldots, c_{0,d-1}, c_{10}, c_{11}, \ldots, c_{1,d-1}, \ldots, c_{d-1,0}, \ldots, c_{d-1,d-1})$$

where

$$\mathbf{x}_i \equiv (c_{i0}, c_{i1}, \ldots, c_{i,d-1}) \quad (i = 0, 1, \ldots, d - 1).$$

Then the diagram is

\[ \text{Diagram} \]

where we put

$$\mathbf{y} \equiv (| \mathbf{x}_0 |, | \mathbf{x}_1 |, \ldots, | \mathbf{x}_{d-1} |) \in V_d(\mathbb{C}).$$

In this circuit we find

$$\hat{S}_k^0(\mathbf{x}) | \mathbf{y} \rangle = | a \rangle | b \rangle.$$
C. Construction of unitary transformation in two-qudit

In this subsection, finally, we construct unitary transformation in two-qudit. Let \( \tilde{U} \in U(d^2) \) be the unitary transformation whose eigenvalues are \( e^{i\sigma(a,b)} \) (\( a, b = 0, 1, 2, \ldots, d - 1 \)) and the corresponding eigenstates \( |\sigma(a,b)\rangle \).

We introduce \( \tilde{X}(a, b) \) whose operation is

\[
\tilde{X}(a, b)|c\rangle|d\rangle = \begin{cases} 
  e^{i\sigma(a,b)}|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\
  |c\rangle|d\rangle, & \text{otherwise}
\end{cases}
\]

\( \tilde{X}(a, b) \) is given by

\[ \tilde{X}(a, b) = \tilde{C}^a(X(a, b)) \]

where \( X(a, b) \) is a single qudit gate:

\[
X(a, b)|c\rangle|d\rangle = \begin{cases} 
  e^{i\sigma(a,b)}|a\rangle|b\rangle, & \text{if } |c\rangle|d\rangle = |a\rangle|b\rangle \\
  |c\rangle|d\rangle, & \text{otherwise}
\end{cases}
\]

The diagram is

\[ \tilde{X}(a, b) \quad \equiv \quad X(a, b) \]

In the matrix form

\[ X(a, b) \equiv \text{diag}(1, 1, \ldots, 1, e^{i\sigma(a,b)}, 1, \ldots, 1) \]

with

\[
\begin{pmatrix}
  1 \\
  \vdots \\
  1 \\
  e^{i\sigma(a,b)} \\
  1 \\
  \vdots \\
  1
\end{pmatrix}
\]

or, in component,

\[
(X(a, b))_{ij} = \delta_{ij} \{ 1 + \delta_i a + b (1 + e^{i\sigma(a,b)}) \};
\]

\[
(\tilde{X}(a, b))_{ij} = \delta_{ij} \{ 1 + \delta_i b (1 + e^{i\sigma(a,b)}) \}.
\]

By the result of Sec. IVB, there exists \( \tilde{S}^a_b(\sigma(a,b)) \) which satisfies

\[ \tilde{S}^a_b(\sigma(a,b))|a\rangle|b\rangle = |a\rangle|b\rangle, \quad (a, b = 0, 1, \ldots, d - 1). \]

Then we introduce the operator

\[ \tilde{Z}(a, b) \equiv \tilde{S}^{-1}_b(\sigma(a,b))\tilde{X}(a, b)\tilde{S}(\sigma(a,b)). \]

The diagram is
They satisfy

\[ \tilde{Z}(a, b) |\sigma(c, d)\rangle = \begin{cases} 
  e^{i\sigma(a, b)} |\sigma(a, b)\rangle, & |\sigma(c, d)\rangle = |\sigma(a, b)\rangle \\
  |\sigma(c, d)\rangle, & |\sigma(c, d)\rangle \neq |\sigma(a, b)\rangle 
\end{cases} . \]

Finally we construct \( \tilde{U} \) by

\[ \tilde{U} = \prod_{a, b=0}^{d-1} \tilde{Z}(a, b). \]

The diagram is

\[ \tilde{U} = \tilde{Z}(0, 0) \quad \tilde{Z}(0, 1) \quad \tilde{Z}(d-1, d-1) \]

Indeed, this operator satisfies

\[ \tilde{U} |\sigma(a, b)\rangle = e^{i\sigma(a, b)} |\sigma(a, b)\rangle, \quad (a, b = 0, 1, 2, \ldots, d - 1), \]

to be diagonal in the eigenstate of \( \tilde{U} \). Thus we find that this is the circuit which perform \( \tilde{U} \).

V. DISCUSSIONS

We have constructed the controlled-\( U \) gate in qudit and unitary transformation in two-qudit. In the similar way to extend the controlled-\( U \) to the controlled\(^n \)-\( U \) in qubit[13], the controlled-\( U \) will be extended in qudit.

However, in this method as well as in the qubit case, the larger \( n \) is the larger steps exponentially and the more difficult calculation is. To avoid the problem some quite new ideas may be required.

When we construct the controlled-\( U \), we do not use the Euler decomposition but use the diagonalization. The former may fit the property of laser and have advantage to construct the circuit with laser operation. However, when number of states in one qudit is large, the decomposition seems to be complicated [14, 15]. Our method may not fit the property of laser, however, the method of diagonalization is well-known and even when number of states is large, network is relatively easy built. Laser is not necessarily needed to construct networks and there may exist physical systems fit to the diagonalization.

We adopt the controlled-\( M_b \) gate as the elementary gate of the controlled operation. This choice stems from the notion that in physical systems manipulation between only two states is allowed. However, there may exist physical systems in which more than two states are manipulated at once. In such cases, we can choose other gate to decrease steps of calculation.

As stated in the introduction, there are the problem of decoherence through interaction with environment in construction of networks. Taking this fact into consideration, qudit has advantage to qubit. However, to realize qudit in physical systems, for example, if we make use of energy levels of electron in an atom, the energy differences between high excited states are very small, thus in actual not so many levels are used in construction.

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