SIX WAYS TO QUANTIZE
(2+1)-DIMENSIONAL GRAVITY

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1. Introduction

There is an old joke—feel free to adapt it to your local politics—that if you put three socialists together in a room, you’ll end up with four political parties. The same can be said of physicists working on quantum gravity. Quantum gravity is hard, and in the absence of compelling experimental or mathematical guidance, the past forty years of research has given us a remarkable variety of approaches to the subject.

For realistic quantum gravity in four spacetime dimensions, the best that can be said is that not all of these approaches have been shown to fail. There may be cause for optimism about some lines of research—notably loop variables and string theory—but we are still far from having a satisfactory theory. In this situation, a valuable tactic is to look for simpler models that share the basic conceptual problems of quantum gravity, while at the same time being computationally tractable.

In the past few years, it has become increasingly clear that general relativity in 2+1 dimensions can serve as such a model. Gravity in 2+1 dimensions has a finite number of physical degrees of freedom, and quantum field theory is effectively reduced to quantum mechanics. At the same time, most of the underlying problems of quantum gravity—the need to find diffeomorphism-invariant observables, the “problem of time,” issues of topology and topology change, the basic question of what it means to quantize geometry—are still present. And in contrast to the (3+1)-dimensional case, in 2+1 dimensions we suffer an embarrassment of riches: many

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of the existing approaches can be made to work, but they do not all give the same answers.

The goal of this talk is to discuss six of these approaches and compare the resulting quantum theories. My hope is that such a comparison can provide some insight into the problems of realistic (3+1)-dimensional quantum gravity. Each of these methods is worthy of a talk in its own right, and my treatment will necessarily be cursory, but I hope it will serve as a useful introduction.

2. Three Approaches to Classical Gravity

In order to investigate quantum gravity in three spacetime dimensions, it is helpful to first understand the classical theory. This is particularly true because different points of view on classical gravity suggest very different approaches to quantization. In this section, I will briefly summarize three techniques for solving the empty space Einstein field equations in 2+1 dimensions. For simplicity, I will largely restrict my attention to the most elementary nontrivial spacetime topology, \( M = \mathbb{R} \times T^2 \), where the spatial topology \( T^2 \) is that of a two-dimensional torus.

Let us begin with the question of why general relativity is so much simpler in three dimensions than in four. The easiest starting point is a straightforward counting argument. In four dimensions, the spatial metric at a fixed time has six independent components, giving six configuration space degrees of freedom per spacetime point. Four of these are “gauge” degrees of freedom, however, which can be eliminated by a choice of four coordinates. Two physical degrees of freedom per point remain, corresponding roughly to the two possible polarizations of a gravitational wave.

In three spacetime dimensions, on the other hand, the spatial metric has three independent components, and we can choose three coordinates, leaving no degrees of freedom per point. This counting argument reappears mathematically in the statement that the curvature tensor \( R_{\mu\nu\rho\sigma} \) is completely determined by the Ricci tensor:

\[
R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \tag{2.1}
\]

The field equations of general relativity thus tell us that the curvature tensor depends algebraically on the stress-energy tensor, and in particular vanishes in the vacuum.

At first sight, this seems to make three-dimensional gravity a completely trivial theory, hardly likely to be a useful model. In a topologically trivial spacetime, this is indeed the case: the vanishing of the curvature tensor completely determines the geometry, and no gravitational dynamics remains. If a spacetime contains noncontractible curves, however, this is no longer the case.

Recall that the curvature tensor is defined by parallel transport around closed curves. In particular, if a curve \( \gamma \) encloses an area in which \( R_{\mu\nu\rho\sigma} \) vanishes, then parallel transport around \( \gamma \) is trivial. In a simply connected spacetime, flatness thus guarantees that parallel transport is path-independent, allowing a global definition
of parallelism. In a multiply connected spacetime, on the other hand, not all closed 
curves enclose areas—think of a circumference of a torus—and this argument breaks 
down. Instead, the geometry of a flat spacetime is characterized by the results of 
parallel transport around such noncontractible curves, that is, by their holonomies.

This picture suggests a natural way to construct solutions of the (2+1)-dimensional 
vacuum field equations, Thurston’s method of geometric structures. Any flat 
spacetime can be covered by contractible coordinate patches, and in each patch, the 
vanishing of the curvature ensures that the geometry is simply that of Minkowski 
space. All of the geometric information is now hidden in the transition functions 
describing the overlaps between patches. Moreover, since the metric in each patch 
can be chosen to be the standard Minkowski metric, these transition functions must 
be isometries of $\eta_{\mu\nu}$, i.e., elements of the three-dimensional Poincaré group ISO(2,1).

Such a geometry, in which a manifold $M$ is built out of patches of a “model space” $X$ 
 glued together by isomorphisms of some structure on $X$, is known as a geometric 
structure, or $G$ structure, on $M$. We can rephrase the (2+1)-dimensional field equa-
tions as a statement that empty spacetime has an ISO(2,1), or Lorentzian, structure. 
The general properties of geometric structures have been widely studied by mathe-
maticians, and some powerful results are available. In particular, it can be shown 
that only a small number of transition functions are needed, one for each generator 
of the fundamental group $\pi_1 M$. In fact, a geometric structure determines a group 
homomorphism $\Gamma : \pi_1 M \to G$, where $G$ is the appropriate group of isomorphisms 
(for us, ISO(2,1)). The image $\Gamma(\pi_1 M)$, which tells us the transition function around 
each closed curve, is known as the holonomy group of the geometric structure. Any 
two conjugate holonomy groups represent the same geometric structure, since overall 
conjugation is merely a simultaneous isometry in all of the coordinate patches. For a 
Lorentzian structure, the space of possible holonomy groups is thus

$$\mathcal{M} = \text{Hom}_0(\pi_1 M, \text{ISO}(2,1)) / \sim,$$

$$\rho_1 \sim \rho_2 \text{ if } \rho_2 = h \cdot \rho_1 \cdot h^{-1}, \quad h \in \text{ISO}(2,1). \quad (2.2)$$

Relativists are familiar with simple examples of such constructions. A flat torus, 
for instance, can be built from Euclidean space by gluing together the edges of a 
parallelogram; the holonomies are simply the pair of translations that identify the 
edges. Similarly, the general two-dimensional version of this procedure is familiar 
to string theorists: it is a form of the uniformization theorem, which allows one to 
characterize a surface of genus $g > 1$ by means of a 2$g$-generator subgroup of SL(2, $\mathbb{R}$), 
the isometry group of the two-dimensional metric of constant negative curvature.

For a compact manifold, the specification of a homomorphism $\rho \in \mathcal{M}$ completely 
determines the geometric structure, and Eq. (2.2) classifies the solutions of the vacuum

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1 I am glossing over a topological subtlety. The space of homomorphisms $\text{Hom}(\pi_1 M, \text{ISO}(2,1))$ has 
more than one component, and only one—that for which the SO(2,1) projection is Fuchsian—gives 
rise to Lorentzian spacetimes. This is the meaning of the subscript 0 in Eq. (2.2).
Einstein field equations. For a more physically realistic noncompact topology, this is not quite the case. But Mess has shown that for topologies of the form $\mathbb{R} \times \Sigma$ with a compact spatial section $\Sigma$, Eq. (2.2) is good enough: a homomorphism $\rho$ determines a unique maximal domain of dependence.

In principle, the specification of a holonomy group thus determines the geometry of a (2+1)-dimensional spacetime. In practice, such a set of data is rather difficult to work with. In a sense, this is a classical version of the problem of observables in quantum gravity: the holonomies of an ISO(2,1) structure provide a complete set of diffeomorphism-invariant observables, but it is not easy to translate their values into more conventional information about the metric or other familiar geometric objects. This approach is not entirely alien to classical general relativity, however, since it resembles the technique of Regge calculus, in which a spacetime is built by gluing together flat simplices. We might therefore expect quantum theories based on Regge calculus to have some contact with the theory of geometric structures.

The two-dimensional analog offers one possible solution to these problems of interpretation. It is a standard result that if a compact Riemann surface $\Sigma$ is characterized by a geometric structure with holonomy group $\Gamma \subset \text{SL}(2, \mathbb{R})$, then it can be represented as a quotient space $\mathbb{H}^2/\Gamma$, where $\mathbb{H}^2$ is the upper half-plane with its standard constant negative curvature metric. This result gives us a constructive procedure for determining $\Sigma$, and allows a practical computation of many of its geometric properties. The generalization of this result to three-dimensional spacetimes is nontrivial, but Mess has again provided the relevant theorems, showing that a similar quotient construction is possible for topologies $\mathbb{R} \times \Sigma$.

For concreteness, let us apply this rather abstract argument to the case of a spacetime with the topology $M = \mathbb{R} \times T^2$. The fundamental group of the torus—and thus of $M$—is the abelian group $\mathbb{Z} \oplus \mathbb{Z}$, with one generator for each of the two independent circumferences. The holonomy group must therefore be generated by two commuting Poincaré transformations, say $(\Lambda_1, a_1)$ and $(\Lambda_2, a_2)$.

We first consider the SO(2,1) transformations $\Lambda_1$ and $\Lambda_2$. Any Lorentz transformation in 2+1 dimensions fixes a vector $n$, and for $\Lambda_1$ and $\Lambda_2$ to commute, they must fix the same vector. The space of holonomies thus splits into three topological components, according to whether $n$ is spacelike, null, or timelike, and it is not hard to show that a well-behaved geometric structure can occur only when $n$ is spacelike. We therefore demand that either $\Lambda_1$ and $\Lambda_2$ fix a spacelike vector, say $(0,0,1)$, or else that they both be the identity.

If $\Lambda_1 = \Lambda_2 = I$, the holonomy group is generated by a pair of arbitrary spacelike translations $a_1$ and $a_2$, and the spacetime is simply a static flat torus. We can choose coordinates such that $a_1 = (0, 0, a)$ and $a_2 = (0, a\tau_1, a\tau_2)$; a fundamental region for the holonomy group on a spatial slice is then a parallelogram with vertices at $(0, 0), (a, 0), (a\tau_1, a\tau_2)$, and $(a(\tau_1 + 1), a\tau_2)$. The torus formed by identifying the opposite sides of such a parallelogram is said to have modulus $\tau = \tau_1 + i\tau_2$; its area is $a^2\tau_2$.

If instead $\Lambda_1$ and $\Lambda_2$ stabilize a spacelike vector, both must be boosts. We can
use our remaining freedom of overall conjugation to transform the two generators to the form

\[ H(\gamma_1) : (t, x, y) \rightarrow (t \cosh \lambda + x \sinh \lambda, x \cosh \lambda + t \sinh \lambda, y + a) \]

\[ H(\gamma_2) : (t, x, y) \rightarrow (t \cosh \mu + x \sinh \mu, x \cosh \mu + t \sinh \mu, y + b), \quad (2.3) \]

where \( \lambda \) and \( \mu \) are now unique up to the single remaining identification \( (\lambda, \mu) \sim (-\lambda, -\mu) \). To find the desired quotient space, it is convenient to choose new Minkowski space coordinates \( t = T^{-1} \cosh u, \ x = T^{-1} \sinh u \). The holonomies (2.3) then reduce to translations of \( u \) and \( y \), and the quotient space is simply a torus on each surface of constant \( T \). The modulus and area of this torus are not hard to compute:

\[ \tau = \tau_1 + i\tau_2 = \left( a + \frac{i\lambda}{T} \right)^{-1} \left( b + \frac{i\mu}{T} \right) \quad (2.4) \]

and

\[ A(T) = \frac{a\mu - \lambda b}{T} \quad (2.5) \]

Furthermore, the coordinate \( T \) has been chosen so that any surface of constant \( T \) has constant mean extrinsic curvature \( \text{Tr} K = T \). We have thus obtained an explicit expression for the spacetime geometry as a spatial torus with a time-dependent modulus and area, with \( \text{Tr} K \) serving as time.

In contrast to the case of purely translational holonomies, such a spacetime clearly has nontrivial dynamics. It is easy to check that

\[ \left[ \tau_1 - \frac{1}{2} \left( \frac{\mu}{\lambda} - \frac{b}{a} \right) \right]^2 + \tau_2^2 = \frac{1}{4} \left[ \frac{\mu}{\lambda} - \frac{b}{a} \right]^2, \quad (2.6) \]

so the modulus describes a circular motion in the upper half plane. This is not quite the whole story, however, since not all different values of \( \tau \) represent distinct geometries. This is because of the presence of “large diffeomorphisms,” diffeomorphisms that cannot be smoothly deformed to the identity. The simplest such diffeomorphism is a Dehn twist, a transformation in which the torus is cut open along a circumference and reglued with a \( 2\pi \) twist of one edge. The full group of large diffeomorphisms of the torus is well understood; it is generated by a pair of transformations \( S \) and \( T \), which act on the modulus as

\[ S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1, \quad (2.7) \]

or equivalently,

\[ S: (a, \lambda) \rightarrow (b, \mu), \quad (b, \mu) \rightarrow (-a, -\lambda) \quad (2.8) \]

\[ T: (a, \lambda) \rightarrow (a, \lambda), \quad (b, \mu) \rightarrow (b + a, \mu + \lambda). \quad (2.9) \]
These transformations are known to mathematicians as modular transformations, or as elements of the mapping class group. A fundamental region for the group acting on \( \tau \) is the famous “keyhole” region \(-1/2 \leq \tau_1 \leq 1/2, |\tau| \geq 1\).

We now have a complete description of the solutions of the vacuum Einstein equations for the topology \( \mathbb{R} \times T^2 \). But other techniques may offer additional insight, and it is useful to discuss them. One alternative, of course, is to try to actually solve the field equations, in either first or second order form. The first order form is considerably easier, since as Witten has pointed out, the action can then be treated as a Chern-Simons action for an ordinary gauge theory. In particular, if we take the local frame \( e_a^\mu \) and the spin connection \( \omega_{a\mu} = \frac{1}{2} \epsilon_{abc} \omega^b_{\mu} \) as independent variables, the standard Einstein action is

\[
S = \int d^3 x \epsilon^{\rho\mu\nu}e^a_\rho (\partial_\mu \omega_{a\nu} - \partial_\nu \omega_{a\mu} + \epsilon_{abc} \omega^b_{\mu} \omega^c_{\nu}) \quad (2.10)
\]

\[
= 2 \int dt \int_\Sigma d^2 x (-\epsilon^{ij} e_{ai} \dot{\omega}_{aj} + e^a_0 \tilde{\Theta}_a + \omega_{ab} \Theta^a) \quad (2.11)
\]

with constraints

\[
\Theta^a = \frac{1}{2} \epsilon^{ij} (\partial_i e_{aj} - \partial_j e_{ai} + \epsilon^{abc} (\omega_{bi} e_{cj} - \omega_{ci} e_{bj})) \quad (2.12)
\]

\[
\tilde{\Theta}^a = \frac{1}{2} \epsilon^{ij} (\partial_i \omega^a_{j} - \partial_j \omega^a_{i} + \epsilon^{abc} \omega_{hi} \omega_{cj}). \quad (2.13)
\]

These constraints generate the Lie algebra ISO(2,1); moreover, as Witten observed, the frame \( e_{ai} \) and the spin connection \( \omega_{ai} \) together constitute an ISO(2,1) connection on \( \Sigma \). The conditions \( \tilde{\Theta}^a = \Theta^a = 0 \) then force this connection to be flat, while at the same time generating local gauge transformations, requiring us to identify gauge-equivalent connections. A solution of the field equations is thus described by an equivalence class of flat ISO(2,1) connections on \( \Sigma \).

But any flat connection is determined by its holonomies—the word is now used in its fiber bundle sense—which in turn can be described by a homomorphism \( \Gamma \) from \( \pi_1 \Sigma \) to the gauge group ISO(2,1). Moreover, gauge transformations have the effect of conjugating \( \Gamma \) by an arbitrary group element. We thus recover the description of the space of solutions, with the holonomy group \( \Gamma \) of the geometric structure now replaced by the holonomy group \( \tilde{\Gamma} \) of a flat ISO(2,1) connection. This new approach gives us a more direct relationship between the holonomy group and more conventional geometric quantities, however: given a homomorphism \( \rho \in \mathcal{M} \) with image \( \tilde{\Gamma} \), we are instructed to find a flat connection with \( \tilde{\Gamma} \) as its holonomy group, perform a gauge transformation to make the translation component \( e^a_\mu \) nonsingular, and then interpret \((\omega, e)\) as the spin connection and triad field of the geometry. Unruh and Newbury have recently shown how to write the action (2.10) explicitly in terms of such holonomies; the result is closely related to the Wess-Zumino-Witten action for the group ISO(2,1).
For the torus, for instance, the simplest connection with the holonomies \((2.3)\) is
\[
e^{(2)} = a \, dx + b \, dy, \quad \omega^{(2)} = \lambda \, dx + \mu \, dy.
\]
(2.14)
The candidate triad is singular, but this can be easily fixed by an appropriate ISO(2,1) gauge transformation; we obtain
\[
e^{(0)} = \frac{dT}{T^2}, \quad \omega^{(0)} = 0,
\]
e^{(1)} = \frac{1}{T}(\lambda \, dx + \mu \, dy), \quad \omega^{(1)} = 0,
\]
e^{(2)} = a \, dx + b \, dy, \quad \omega^{(2)} = \lambda \, dx + \mu \, dy.
\]
(2.15)
It is straightforward to show that this triad gives a spatial metric with modulus \((2.4)\) and area \((2.5)\), confirming the classical equivalence of these approaches.

Finally, we can ignore the special features of three dimensions and try to solve the Einstein field equations in their standard metric form. This program is most easily carried out in ADM variables, and has been studied carefully by Moncrief\(^{16}\) and Hosoya and Nakao.\(^{17}\)
The Einstein action is now
\[
S = \int d^3x (-^{(3)}g)^{1/2} \, ^{(3)}R = \int dt \int \Sigma d^2x (\pi^{ij} \dot{g}_{ij} - N^i \mathcal{H}_i - N \mathcal{H}),
\]
(2.16)
where the momentum conjugate to \(g_{ij}\) is \(\pi^{ij} = \sqrt{g} (K^{ij} - g^{ij} K)\), with \(K^{ij}\) the extrinsic curvature of the surface \(t = \text{const.}\), and
\[
\mathcal{H}_i = -2 \nabla_j \pi^j_i, \quad \mathcal{H} = \frac{1}{\sqrt{g}} g_{ij} g_{kl} (\pi^{ik} \pi^{jl} - \pi^{ij} \pi^{kl}) - \sqrt{g} R
\]
(2.17)
are the supermomentum and super-Hamiltonian constraints. Let us choose York’s time slicing\(^{13}\) \(K = \pi/\sqrt{g} = T\), foliating the spacetime by surfaces of constant \(\text{Tr}K\). Moncrief shows that such a foliation always exists for solutions of the field equations. By the two-dimensional uniformization theorem, we can write the spatial metric as
\[
g_{ij} = e^{2\lambda} \tilde{g}_{ij},
\]
(2.18)
where \(\tilde{g}\) is a metric of constant curvature \(k\) on \(\Sigma\); \(k = 1\) for the sphere, \(0\) for the torus, and \(-1\) for surfaces of genus greater than one. The conjugate momenta have a corresponding decomposition into a trace \(\pi\) and a traceless part \(\tilde{\pi}^{ij}\), and the Hamiltonian constraint reduces to an elliptic differential equation for \(\lambda\),
\[
\Delta_{\tilde{g}} \lambda - \frac{1}{4} T^2 e^{2\lambda} + \frac{1}{2} \left[ \tilde{g}^{-1} \tilde{g}_{ij} \tilde{g}_{kl} \tilde{\pi}^{ik} \tilde{\pi}^{jl} \right] e^{-2\lambda} - \frac{k}{2} = 0.
\]
(2.19)
This equation has no solution when \(k = 1\), and a unique solution otherwise. For the torus, in particular, one finds that
\[
e^{4\lambda} = \frac{2}{T^2} \tilde{g}^{-1} \tilde{g}_{ij} \tilde{g}_{kl} \tilde{\pi}^{ik} \tilde{\pi}^{jl},
\]
(2.20)
eliminating $\lambda$ as an independent degree of freedom.

As always, the flat spatial metric $\tilde{g}_{ij}$ is determined by a complex modulus $\tau = \tau_1 + i\tau_2$, and we can define its conjugate momentum by

$$p^\alpha = \int_{\Sigma} d^2x e^{2\lambda} \left[ \tilde{\pi}^{ij} \frac{\partial}{\partial \tau_\alpha} \tilde{g}_{ij} \right].$$

(2.21)

Inserting Eq. (2.20) back into Eq. (2.16), we obtain a reduced phase space action

$$S = \int dT \left[ p^\alpha \frac{d\tau_\alpha}{dT} - H(p, \tau) \right]$$

(2.22)

with an effective Hamiltonian

$$H(p, \tau) = \frac{1}{T} \left[ \tau_2^{-2} \left( (p_1)^2 + (p_2)^2 \right) \right]^{1/2}$$

(2.23)

that describes the remaining physical degrees of freedom.

The expression in brackets in Eq. (2.23) is the square of the momentum with respect to the constant negative curvature (Poincaré) metric

$$d\sigma^2 = \tau_2^{-2} d\tau d\bar{\tau}$$

(2.24)

on the torus moduli space. Except for the square root and the rather trivial time dependence, (2.23) is the Hamiltonian for a free particle moving in a curved background described by this metric. One can easily verify that the classical solutions are simply the geodesics in such a background, i.e., semicircles centered on the real axis, precisely as expected from Eq. (2.6). The ADM approach thus gives the same set of solutions we already obtained, and confirms the identification (2.4) between moduli and holonomies. Moreover, using Eq. (2.4) we can write the Hamiltonian as

$$H = \frac{a\mu - \lambda b}{T},$$

(2.25)

which we recognize as the area of the surface of constant $\text{Tr}K = T$.

3. Quantum Theory I: Reduced ADM Phase Space Quantization

Having understood the classical solutions for our model, we can finally turn to the problem of quantization. The most obvious approach is suggested by the reduced phase space action (2.22), the classical action for the physical degrees of freedom in the York time slicing. The number of degrees of freedom is now finite—for the torus, two positions and two canonical momenta—and we are left with a straightforward
problem of quantum mechanics. We can now simply follow our noses: we make
\[ p = p^1 + ip^2 \]
and \( \tau \) into operators, with the standard commutation relations
\[ [\hat{\tau}_\alpha, \hat{p}^\beta] = i\delta_\alpha^\beta, \tag{3.1} \]
acting on the Hilbert space of square integrable functions of \( \tau \). The Schrödinger equation for this system can be read off from the action; it is
\[ i\frac{\partial \psi}{\partial T} = T^{-1} \Delta_0^{1/2} \psi, \tag{3.2} \]
where
\[ \Delta_0 = -\tau^2 \left( \frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) \tag{3.3} \]
is the scalar Laplacian on the upper half plane with respect to the Poincaré metric. \( \Delta_0 \) has no negative eigenvalues, and the (positive) square root can be defined by a spectral decomposition; our solutions automatically have positive “energy.” We have seen that the Hamiltonian on a surface of fixed \( T \) is essentially the area, so this positive frequency condition is a restriction to wave functions describing an expanding universe. Such a picture is classically consistent, since no exact solution in 2+1 dimensions recollapses, but the generalization to 3+1 dimensions is problematic.

The one real subtlety in this formulation comes from the question of how to treat the large diffeomorphisms (2.7). Quantum gravity should presumably be invariant under the entire group of diffeomorphisms, and it may be argued that we should therefore restrict ourselves to wave functions that are invariant under these transformations. This point of view is not undisputed—one can argue instead that the only true invariances of the quantum theory are those generated by the constraints—but there seems to be good evidence that the classical limit of point particle scattering can be recovered only if one requires full mapping class group invariance. Modular invariant eigenfunctions of the Laplacian (3.3) are called weight zero Maass forms, and have been studied extensively by mathematicians; the exact functional forms are not known, but lowest nonzero eigenvalues have been investigated numerically.

There appears to be nothing inconsistent with this approach to quantizing (2+1)-dimensional gravity. The dynamics is straightforward, and the physical significance of the fundamental operators \( \hat{\tau} \) and \( \hat{p} \) is relatively clear. The method depends heavily on a classical choice of time slicing, however, in our case the choice of a foliation by surfaces of constant \( \text{Tr}K \). As a result, certain apparently reasonable questions—for instance, questions about the geometry of different time slices—seem difficult or impossible to ask. It is not even known whether different choices of time slicing lead to equivalent theories.
4. Quantum Theory II: Chern-Simons Theory/Connection Representation

A second straightforward approach to quantizing our system starts with the first order Chern-Simons form of the action. While this technique does not have any exact analog in 3+1 dimensions, it is closely related to Ashtekar’s connection dynamics. We now begin with the action (2.10), which is already in the first order form \( p\dot{q} + \ldots \), and impose the equal time commutation relations

\[
[\omega_{ai}(x), e^b_j(x')] = -\frac{i}{2} \delta^b_a \epsilon_{ij} \delta^2(x - x').
\] (4.1)

It is natural to consider the \( \omega_{ai} \) as “positions” and the \( e^b_j \) as “momenta” in this representation (hence the name “connection dynamics”), in part because the constraint \( \Theta^a \) of Eq. (2.12) is the “derivative” of the constraint \( \tilde{\Theta}^a \): if we perform an infinitesimal variation of \( \omega \) in \( \tilde{\Theta}^a \), the resulting equation for \( \delta\omega \) will be precisely the constraint \( \Theta^a \) evaluated at \( e = \delta\omega \). The space of solutions \( (\omega, e) \) of the constraints is thus the cotangent bundle of the space of flat spin connections \( \omega \). We shall see in the next section that this is a consequence of the structure of the group ISO(2,1).

With this choice of variables, the \( \tilde{\Theta}^a \) constraint tells us that the SO(2,1) connection \( \omega \) must be flat, while \( \Theta^a \) generates SO(2,1) gauge transformations, requiring us to identify gauge-equivalent connections. Wave functions are thus functions of equivalence classes of flat SO(2,1) connections. For the torus topology, in particular, the SO(2,1) piece of the connection is parametrized by the holonomies \( \lambda \) and \( \mu \), and wave functions are thus square integrable functions \( \chi(\lambda, \mu) \). Again, we should presumably demand modular invariance, in the form of invariance under the transformations (2.8). The remaining holonomies \( a \) and \( b \) now become operators, whose commutators can be determined, for instance, by comparing Eq. (2.15) and Eq. (4.1); we find that

\[
[\hat{a}, \hat{\mu}] = [\hat{\lambda}, \hat{b}] = \frac{i}{2},
\] (4.2)

with all other commutators vanishing.

In this first order formalism, the notion of time has rather mysteriously disappeared. The Hamiltonian of a Chern-Simons theory is identically zero, and there seems to be no dynamics—we have obtained what is commonly known as a “frozen time” formalism. This should not be surprising, however, since we have already encountered the same “problem of time” classically. Our wave functions \( \chi \) depend only on the holonomies \( \lambda \) and \( \mu \) of a geometric structure, which describe the entire spacetime, not a particular time slice; it is already nontrivial to extract dynamical information at the classical level.

Classically, though, we know how to solve this problem. A geometric structure determines an entire spacetime, but once we have constructed that spacetime, we can simply choose our favorite time slicing and look at the corresponding dynamics.
Eq. (2.4), for instance, gives the moduli of a surface of constant $\text{Tr}K = T$ in terms of the corresponding holonomies. It is natural to carry this equation over into the quantum theory, defining a one-parameter family of operators

$$\hat{\tau}(T) = \left( \hat{a} + \frac{i \hat{\lambda}}{T} \right)^{-1} \left( \hat{b} + \frac{i \hat{\mu}}{T} \right). \quad (4.3)$$

The Hamiltonian becomes

$$\hat{H} = \frac{\hat{a} \hat{\mu} - \hat{\lambda} \hat{b}}{T}, \quad (4.4)$$

and it is not hard to check that the operators $\hat{a}$, $\hat{b}$, $\hat{\lambda}$, and $\hat{\mu}$ obey the correct Heisenberg equations of motion.

From this point of view, Chern-Simons quantization should be understood as a Heisenberg picture. Each choice of time slicing will determine a family of “time”-dependent operators analogous to the moduli (4.3); the corresponding Hamiltonian, in turn, provides us with a quantum description of dynamics in that time slicing. Operators such as (4.3) are manifestly diffeomorphism-invariant, and are examples of what Rovelli calls “evolving constants of motion.”

Kuchař has raised the important question of whether the operators coming from different time slicings can be simultaneously made self-adjoint; this issue is not yet resolved.

It should be stressed that explicit constructions such as (4.3) depend on our ability to solve the field equations in the corresponding time slicing. In 3+1 dimensions, this will no longer be possible, and we will presumably have to develop a suitable perturbation theory for “time”-dependent operators.

It is natural to ask how the two quantum theories discussed so far fit together. In particular, we might hope that the reduced phase space ADM theory is a Schrödinger picture corresponding to the connection representation Heisenberg picture. In other words, we can ask whether the wave functions $\psi(T)$ of the last section are simply the eigenfunctions of the operators $\hat{\tau}(T)$ of this section.

As it turns out, they are not quite. For this simple model, it is possible to explicitly diagonalize the operators $\hat{\tau}$ and $\hat{\tau}^\dagger$. The resulting wave functions satisfy a Schrödinger equation of the form

$$i \frac{\partial \chi}{\partial T} = T^{-1} \Delta_1^{1/2} \chi, \quad (4.5)$$

where

$$\Delta_1 = \Delta_0 + i \tau_2 \frac{\partial}{\partial \tau_1} - \frac{1}{4} \quad (4.6)$$

is the Laplacian for Maass forms of weight one-half, essentially one-component spinors on moduli space. If the ADM Schrödinger equation is an ordinary square root of a

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\[\footnote{I am neglecting subtleties involving operator ordering; suffice it to say that compatibility of the modular transformations \((2.7)\) and \((2.8)\) places strong restrictions on possible orderings.} \]
Klein-Gordon equation on moduli space, the connection representation Schrödinger equation is a kind of Dirac square root.

Let me mention in passing that this analysis again involves subtleties concerning operator ordering. By rather unnatural changes of ordering, we can actually obtain Laplacians acting on Maass forms of arbitrary weight, i.e., arbitrary tensors on moduli space. Such changes implicitly involve new inner products, however, and therefore alter the definition of the adjoint; in particular, we can reproduce the ADM results only at the cost of losing hermiticity of the holonomy operators.

Finally, we must confront a set of problems arising from the topology of the space of connections. So far, I have only discussed connections whose holonomies correspond to geometric structures—for the torus, holonomies that fix a spacelike vector. In the Chern-Simons formulation, there seems to be no compelling reason to omit the “nongeometrical” connections, whose presence could be taken as a sign of additional phases of the theory.

If we insist on mapping class group invariance, however, these extra sectors will not appear. Consider, for instance, the “timelike sector” for our $\mathbb{R} \times T^2$ topology. The hyperbolic sines and cosines in Eq. (2.3) will be replaced by ordinary sines and cosines, and the variables analogous to $\lambda$ and $\mu$ will be angles, say $\theta$ and $\phi$. Modular transformations parallel to (2.8) then tell us to identify, for instance, $(\theta, \phi)$ and $(\theta, \phi + n\theta)$. But for generic values of $\theta$, $\phi + n\theta$ can be made arbitrarily close to any other angle. This is a symptom of the fact that a typical orbit of the mapping class group is dense, which in turn means that the only continuous invariant wave functions $\psi(\theta, \phi)$ in the “timelike sector” are constants.

5. Quantum Theory III: Covariant Canonical Quantization

The methods of the previous section are closely related to a somewhat different approach known as covariant canonical quantization. This approach grew out of the traditional “covariant vs. canonical” debate in quantum gravity. We understand quantum theories most clearly in a canonical framework. But ordinary canonical quantization requires us to make a choice of time from the start, violating at least the spirit of general covariance. On the other hand, the only standard covariant approach we have available, the path integral background field method, is fundamentally perturbative, a feature widely believed to be disastrous for quantum gravity.

Covariant canonical quantization attempts to avoid this dilemma by finding a covariant setting in which to do canonical quantization. That setting is the space of solutions of the field equations. For a theory with a well-posed initial value problem, it is easy to see that this space of solutions is isomorphic to the phase space—any point in phase space determines a set of initial values that can be used to construct a solution, and any solution determines a set of initial data on an (arbitrary but fixed) initial hypersurface.

To apply this program to (2+1)-dimensional gravity, let us return to the descrip-
tion (2.2) of the space of classical solutions. To define a symplectic structure on this space—a necessary first step for quantization—we observe that the group manifold of \( \text{ISO}(2,1) \) is already a cotangent bundle, with base space \( \text{SO}(2,1) \). Indeed, given two curves \( \Lambda_1(t) \) and \( \Lambda_2(t) \) through a common point \( \Lambda(0) \) of the \( \text{SO}(2,1) \) manifold, the “product cotangent vector” is

\[
\left. \frac{d}{dt} (\Lambda_1 \Lambda_2)(t) \right|_{t=0} \cdot (\Lambda_1 \Lambda_2)^{-1}(t)
\]

\[
= \left. \frac{d}{dt} \Lambda_1(t) \right|_{t=0} \Lambda_1^{-1}(t) + \Lambda_1(0) \left[ \left. \frac{d}{dt} \Lambda_2(t) \right|_{t=0} \Lambda_2^{-1}(t) \right] \Lambda_1^{-1}(0),
\]

which can be recognized as the usual semidirect product structure for \( \text{ISO}(2,1) \). It is not hard to see that the space of solutions \( \mathcal{M} \) of Eq. (2.2) is therefore itself a cotangent bundle (ignoring difficult questions of smoothness), whose base space

\[
\mathcal{N} = \text{Hom}_0(\pi_1 M, \text{SO}(2,1)) / \sim
\]

is the space of equivalence classes of \( \text{SO}(2,1) \) holonomies.

Such a cotangent bundle automatically carries a symplectic structure, with points in the base space serving as “positions” and cotangent vectors as “momenta.” For our simple model, we can again quantize by follow our noses, replacing the corresponding Poisson brackets by commutators. For the torus, in particular, the base space is parametrized by the \( \text{SO}(2,1) \) holonomies \( \lambda \) and \( \mu \), and the Poisson brackets of the cotangent bundle structure lead to commutators identical to those of Eq. (1.2). The analysis of the previous section therefore goes through unchanged, but now in a framework that can be generalized to higher dimensions.

6. Quantum Theory IV: The Loop Representation

In 3+1 dimensional gravity, recent work on Ashtekar’s variables and the connection representation has led to an interesting “dual” approach to quantization, known as the loop representation.\(^{22,23,29}\) This method starts with the observation that a connection on a manifold \( M \) with gauge group \( G \) is essentially a map from closed loops on \( M \) into \( G \). Given any loop \( \gamma \) based at a point \( p \in M \), the connection determines an element of \( G \) describing the result of parallel transport around \( \gamma \). Conversely, any set of parallel transport matrices obeying certain algebraic constraints determines a connection. Connections and loops are thus dual to each other, and a wave function \( \Psi[A] \) of a connection determines a dual wave function

\[
\tilde{\Psi}[\gamma] = \int [dA] \Psi[A] T^0[\gamma][A],
\]

where

\[
T^0[\gamma][A] = \text{Tr} P \exp \left\{ \int_\gamma A \right\}
\]

\[
(6.1)
\]

\[
(6.2)
\]
is the holonomy of $\gamma$ (the trace removes any dependence on the base point).

In 3+1 dimensions, the functional integral in Eq. (6.1) requires some careful definition, and it is more useful to look instead for representations of the algebra of the loop operators $T(0)$ and their cotangent vectors $T(1)$. In 2+1 dimensions, on the other hand, the picture is simpler; the relevant connections are now the flat SO(2,1) connections $\omega$, and instead of loops we need only consider homotopy classes of loops. The loop transformation is then well-defined, and can be investigated in detail.

For the $\mathbb{R} \times T^2$ topology, any homotopy class $[\gamma]$ is labeled by two winding numbers $m$ and $n$, and loop representation wave functions are functions $\tilde{\psi}(m,n)$. It is not hard to check that

$$T^0[(m,n)](\lambda,\mu) = 1 + 2 \cosh(m\lambda + n\mu)$$
$$T^1[(m,n)](\lambda,\mu, a, b) = 2 \sinh(m\lambda + n\mu)(m\hat{a} + n\hat{b}), \tag{6.3}$$

and that the standard loop operator commutators give back Eq. (4.2). Moreover, using the symmetry $\chi(\lambda,\mu) = \chi(-\lambda, -\mu)$, we can recognize Eq. (6.1) as an ordinary two-sided Laplace transform from coordinates $(\lambda,\mu)$ to $(m,n)$,

$$\tilde{\chi}(m,n) = 2 \int d\lambda \int d\mu e^{2(m\lambda + n\mu)} \chi(\lambda,\mu), \tag{6.4}$$

up to an overall constant that disappears if one uses SU(1,1) rather than SO(2,1) traces.

It is now natural to ask whether the transformation (6.4) is an isomorphism, that is, whether the connection and loop representations are equivalent. As Marolf has recently shown, it is not.\textsuperscript{30} The basic problem is that the Laplace transform (6.4) is only evaluated at integral values of $m$ and $n$, and information at these points is not enough to invert the transformation. In fact, the loop transform has a kernel that is dense in the connection representation Hilbert space—any wave function $\chi(\lambda,\mu)$ is the limit of a sequence of wave functions that each have a transform $\tilde{\chi} = 0$.

There is a technical way to avoid this problem, essentially by defining the transform (6.4) only on a subset of connection representation wave functions, but the procedure seems rather contrived. Alternatively, one can define loop representation wave functions on the “generalized loops” of Di Bartolo et al.\textsuperscript{31} A generalized loop is a distribution on $\Sigma$ that behaves roughly like a loop with fractional winding number. If we allow such distributions, the integers $m$ and $n$ can be replaced by continuous “winding numbers,” and we recover a quantization equivalent to that of the connection representation. But again, there seems to be no strong justification for such a choice, and the physical significance of generalized loops is far from clear.

One possible loophole remains. We have not yet imposed the condition of modular invariance, which for loop representation wave functions requires that

$$\tilde{\chi}(m,n) = \tilde{\chi}(n,-m) = \tilde{\chi}(m-n,n). \tag{6.5}$$

These expressions are slightly different from those of Ref.\textsuperscript{30} because I am taking SO(2,1) rather than SU(1,1) traces.
I do not know whether the transformation between such loop states and the corresponding connection states is an isomorphism; this question deserves further study.

7. Quantum Theory V: The Wheeler-DeWitt Equation

Yet another common approach to quantum gravity is that of the Wheeler-DeWitt equation. The methods discussed so far have required us to solve the supermomentum and super-Hamiltonian constraints before quantizing; that is, we have quantized only the physical degrees of freedom of the classical theory. Following Dirac, we could instead allow wave functions to be arbitrary functionals on the full configuration space, and impose the constraints as operator equations to define physical wave functions.

In the first order formalism of section 4, this procedure leads to nothing new, essentially because the constraints are first order in the canonical momenta. If we start with a wave function \( \Phi[\omega] \), the constraint \( \Theta^a \) of Eq. (2.12) acts multiplicatively, again requiring \( \Phi[\omega] \) to have its support on the flat spin connections. \( \Theta^a \) depends on \( e^b j \) as well as \( \omega \), and we must substitute

\[
e^b_j = -\frac{i}{2} \epsilon^{jk} \frac{\delta}{\delta \omega^b_k}.
\]

But the resulting functional differential equation is first order, and can be integrated exactly; it tells us merely that the wave function must be invariant under SO(2,1) transformations of \( \omega \), reproducing our previous results.

In the metric formalism of section 3, on the other hand, the super-Hamiltonian constraint (2.17) is second order in the momenta, and the results are rather different. For the \( \mathbb{R} \times T^2 \) topology, our initial wave functions will now be functionals \( \Psi[\lambda, \tau] \) of the scale factor \( \lambda \) and the modulus \( \tau \). The supermomentum constraints are again first order, and merely require that \( \Psi \) be invariant under spatial diffeomorphisms. But the \( \mathcal{H} = 0 \) constraint must now be imposed as a functional differential equation, the Wheeler-DeWitt equation, which for this topology takes the form

\[
\left\{ \frac{1}{8} e^{2\lambda} \frac{\delta}{\delta \lambda} e^{-2\lambda} \frac{\delta}{\delta \lambda} + \frac{1}{2} \Delta_0 + 2e^{2\lambda}(\Delta_\tilde{g}\lambda) \right\} \Psi[\lambda, \tau] = 0,
\]

where \( \Delta_\tilde{g} \) is the Laplacian on \( \Sigma \) with respect to the flat metric \( \tilde{g} \), and \( \Delta_0 \) is again the Laplacian on the torus moduli space.

At first sight, it is hard to see how the solutions of the Wheeler-DeWitt equation relate to the wave functions \( \psi(\tau, T) \) of section 3. Wave functions satisfying Eq. (7.2) are functions of a “many-fingered time” \( \lambda \), and it is not easy to extract a single parameter \( T \) to describe their evolution. Part of the problem is now hidden in the choice of inner product on the space of solutions of the Wheeler-DeWitt equation. Eq. (7.2) is a functional Klein-Gordon equation, whose inner product should presumably be
something like a Klein-Gordon inner product, say
\[ \langle \Psi_1 | \Psi_2 \rangle = \frac{1}{2\mathcal{I}} \prod_{x \in \Sigma} \int \frac{d^2\tau}{\tau_2^2} \left( \Psi_1^*[\lambda, \tau] \frac{\delta}{\delta \lambda} \Psi_2[\lambda, \tau] \right) \bigg|_{\lambda = \lambda_0}. \tag{7.3} \]

Woodard has argued that such an inner product should be understood as a consequence of gauge-fixing. According to this interpretation, we should start with a standard inner product
\[ \langle \Psi_1 | \Psi_2 \rangle = \int [d\lambda] \int \frac{d^2\tau}{\tau_2^2} \Psi_1^*[\lambda, \tau] \Psi_2[\lambda, \tau]. \tag{7.4} \]

Since we have not fixed a time slicing, this functional integral is divergent, and must be gauge-fixed in the standard Faddeev-Popov manner. If we choose a gauge \( \lambda = \lambda_0 \), we recover Eq. (7.3), complete with the functional derivative, which arises as a Faddeev-Popov determinant. If we choose instead a constant mean curvature gauge \( \pi/\sqrt{g} = T \), it can be shown that the inner product takes the form
\[ \langle \Psi_1 | \Psi_2 \rangle = \int [d\lambda] \int \frac{d^2\tau}{\tau_2^2} \tilde{\Psi}_1(T, \tau) \tilde{\Psi}_2(T, \tau), \tag{7.5} \]
where
\[ \tilde{\Psi}(T, \tau) = \int [d\lambda] \exp \left\{ i T \int_{\Sigma} e^{2\lambda} \right\} \nu[\lambda, \tau] \Psi[\lambda, \tau] \tag{7.6} \]
and \( \nu[\lambda, \tau] \) is a measure factor coming from the Faddeev-Popov determinant,
\[ \nu[\lambda, \tau] = \det^{1/2} |\Delta_{\tilde{g}} - 2\Delta_{\tilde{g}} \lambda - T^2 e^{2\lambda}|. \tag{7.7} \]

The gauge-fixed inner product is thus determined by a set of wave functions that depend only in \( T \) and \( \tau \). The obvious question is whether they are the same functions we found in section 3. While this problem has not been completely analyzed, the answer is probably that they are not. If we ignore the Faddeev-Popov factor \( \nu[\lambda, \tau] \) in Eq. (7.3), and insert Eq. (7.2), we find that
\[ \left( T^2 \frac{\partial^2}{\partial T^2} + \Delta_0 \right) \tilde{\Psi}(T, \tau) \tag{7.8} \]
\[ = \int [d\lambda] \exp \left\{ i T \int_{\Sigma} e^{2\lambda} \right\} \left\{ T^2 \left( e^{4\lambda} - \left( \int_{\Sigma} e^{2\lambda} \right)^2 \right) - 4e^{2\lambda} \Delta_{\tilde{g}} \lambda \right\} \Psi[\lambda, \tau]. \]

If the functional integral were limited to configurations with spatially constant \( \lambda \), the right-hand side of Eq. (7.8) would vanish, and we would recover the square of the Schrödinger equation (3.2) of reduced phase space quantization. This would be the case if \( \lambda \) were required to satisfy the classical constraint (2.19). Instead, however, all values of \( \lambda \) contribute to the functional integral, and there is no reason to expect the
right-hand side of Eq. (7.8) to disappear. Based on preliminary calculations, it seems unlikely that the inclusion of the measure $\nu[\lambda, \tau]$ will change this conclusion. For small values of $T$, however—recall that these correspond to late values of time—it seems that the factor $\nu[\lambda, \tau]$ strongly damps any contributions with $\Delta g \lambda \neq 0$, so the Schrödinger equation (3.2) may still be a good approximation.

8. Quantum Theory VI: Lattice Approaches

Let me conclude with a brief mention of some of the lattice and combinatorial approaches to quantum gravity in 2+1 dimensions. This is a topic worthy of a talk of its own, and I will only be able to touch on a few highlights.

A crucial observation, due to Waelbroeck and ’t Hooft, is that in classical (2+1)-dimensional gravity, a Regge calculus lattice picture is exact. This is a consequence of the fact that classical solutions are flat; in contrast to the more familiar (3+1)-dimensional case, a representation by flat simplices is thus a complete description. This, in turn, means that extra edges may be added at will, or conversely that only a few edge lengths are needed to completely specify the geometry.

Waelbroeck uses this fact to express the lattice action in terms of precisely the number of parameters needed to determine a geometric structure. For the $\mathbb{R} \times T^2$ topology, these are just the four holonomies $a, b, \lambda,$ and $\mu$. While the resulting quantum theory has not yet been fully developed, it seems very likely that the result will be equivalent to that of covariant canonical quantization.

An alternative lattice approach starts with the old observation of Ponzano and Regge that the Regge calculus action in three dimensions can be approximated as a sum over angular momenta of 6-j symbols. The picture is one of attaching an angular momentum $j$ to each edge of a tetrahedral lattice; a tetrahedron’s contribution to the action depends on the corresponding 6-j symbol. The result is invariant under subdivision, again reflecting the exactness of the lattice approximation in three dimensions. By using the quantum group analog of angular momentum, Turaev and Viro have recently showed how to evaluate the sums over $j$; in an appropriate limit, the result is equivalent to the Ponzano-Regge action on the one hand and to the Euclideanized version of an ISO(2,1) Chern-Simons theory on the other. This lattice approach thus provides a bridge between the first and second order formalisms. While the implications for quantum gravity have not been fully developed, the connections to topological field theory and three-manifold invariants provide interesting directions.

A very recent article by Rovelli ties the Turaev-Viro approach to the loop representation. Rovelli argues that the natural basis of lattice states discussed by Ooguri, based on “colorings” or values of $j$ on the boundary of a three-manifold, is identical to the loop representation basis of section 6, and that values of $j$ are simply edge lengths in the loop representation. This might cause some concern, since we saw above that the loop and connection representations are inequivalent. But the Turaev-Viro lattice picture probably corresponds to Euclideanized (2+1)-dimensional gravity, for which
the loop transform (6.4) is a Fourier transformation rather than a Laplace transform, making the two representations equivalent. Perhaps the lesson here is that one must be very careful about analytically continuing to Riemannian metrics.

9. Conclusion and Prospects

In ordinary quantum field theory, we are used to the idea that many different approaches to quantization lead to the same results. In quantum gravity, as we have seen, this is no longer the case. There seem to be at least four inequivalent quantum theories of (2+1)-dimensional gravity—the reduced phase space ADM theory, the theory arising from Chern-Simons or covariant canonical quantization, the loop representation, and the Wheeler-DeWitt formalism—and more undoubtedly await discovery. The basic problem, I believe, is that the equivalence proofs in quantum field theory rely on renormalizability and locality. General relativity is not renormalizable in 3+1 dimensions, and more to the point, the fundamental physical variables are not local. As a result, a candidate for a quantum theory of gravity is likely to depend sensitively on choices of variables and techniques of quantization.

In one sense, this is discouraging news. For now, we have no criteria for picking out the “right” quantization, and it is frustrating to realize that we may be working on formulations that are ultimately irrelevant to the real universe. On the other hand, this variability is also a cause for hope, since it means that a failure of one approach need not doom others. And some general results—the importance of the mapping class group, the possibility of describing geometry concretely in terms of nonlocal observables, the key role of the space of classical solutions—seem not to depend on a particular choice of quantization. It is clear, however, that we need to find better physical criteria to determine which approaches are most likely to be correct.

In this talk, I have only touched on a few of the avenues that have been used to explore (2+1)-dimensional quantum gravity. For instance, I have not mentioned the work of Nelson and Regge on the algebra of observables or the many uses of path integral methods. In restricting myself to one simple topology, I have also ignored the interesting issues of topology change. But I hope this talk has provided a starting point for what can be a truly fascinating subject.

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