ON CONCRETE SPECTRAL PROPERTIES OF A TWISTED-LAPLACIAN ASSOCIATED TO A CENTRAL EXTENSION OF THE REAL HEISENBERG GROUP

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ABSTRACT. We consider the magnetic Laplacian $\Delta_{\nu,\mu}$ on $\mathbb{R}^{2n} = \mathbb{C}^n$ given by

$$\Delta_{\nu,\mu} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2i\nu \left( E + \bar{E} + n \right) + 2\mu \left( E - \bar{E} \right) - \left( \nu^2 + \mu^2 \right) |z|^2.$$

We show that $\Delta_{\nu,\mu}$ is connected to the sub-Laplacian of a group of Heisenberg type given by $\mathbb{C} \times_\omega \mathbb{C}^n$ realized as a central extension of the real Heisenberg group $H_{2n+1}$. We also discuss invariance properties of $\Delta_{\nu,\mu}$ and give some of their explicit spectral properties.

1. INTRODUCTION

In the present paper we study the spectral properties of the second order differential operator

$$\Delta_{\nu,\mu} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2(\mu + i\nu)E - 2(\mu - i\nu)\bar{E} - (\nu^2 + \mu^2)|z|^2 + 2i\nu n,$$

acting on the free Hilbert space $L^2(\mathbb{C}^n, dm)$, where $E$ is the Euler operator and $\bar{E}$ is its complex conjugate. The parameters $\nu$ and $\mu$ are assumed to be real and $\mu > 0$. The particular case of $\nu = 0$ and $\mu = 2b$ with $n = 1$ leads to minus four times the special Hermite operator $L_b$.

$$-4L_b = 4 \left\{ \frac{\partial^2}{\partial z \partial \bar{z}} + b \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) - b^2 |z|^2 \right\}.$$

Such operator is the Hamiltonian describing the quantum behavior of a charged particle on the configuration space $\mathbb{C}^n$ under the influence of a constant magnetic field. Geometrically, $L_b$ represents a Bochner Laplacians $\nabla^* \nabla$ on the smooth sections of a Hermitian line bundle with connection $\nabla$ over the manifold $M = \mathbb{C}^n$.

The main results to which is aimed this paper concern the realization of $\Delta_{\nu,\mu}$ as a magnetic Schrödinger operator associated to a specific potential vector (Section 4). The connection to the sub-Laplacian of a group of Heisenberg type given by $\mathbb{C} \times_\omega \mathbb{C}^n$ is also established (see Section 3). The group $\mathbb{C} \times_\omega \mathbb{C}^n$ is realized as a central extension $N_\omega = (\mathbb{C} \times \mathbb{C}^n, \cdot_\omega)$ of the standard Heisenberg group $H_{2n+1} = (\mathbb{R} \times \mathbb{C}^n, \cdot_{3n+1})$. In this new group, the symplectic form is extended and replaced by an Hermitian product (details in Section 2). Invariance properties of $\Delta_{\nu,\mu}$ are discussed in Section 3 and concrete description of its $L^2$-spectral analysis is presented in Section 5. In Section 6, we use the factorization method to generate eigenfunctions of $\Delta_{\nu,\mu}$ in terms of multivariate version.

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of complex Hermite polynomials. For the case of the twisted Laplacian of the standard Heisenberg group, one can refer to \[10,19\].

2. THE GROUP $N_\omega = \mathbb{C} \times \omega \mathbb{C}^n$ AS A CENTRAL EXTENSION OF THE HEISENBERG GROUP $H_{2n+1} = \mathbb{R} \times \mathrm{Im} \omega \mathbb{C}^n$

We realize $N_\omega := \mathbb{C} \times \omega \mathbb{C}^n$ as a central extension of the Heisenberg group $H_{2n+1} := \mathbb{R} \times \mathrm{Im} \omega \mathbb{C}^n$, where $\omega(z,w)$ denotes the standard Hermitian form on $\mathbb{C}^n$. To this end, we follow the exposition given in \[13\]. Being indeed, if $(K, \cdot)$ and $(G, \circledast)$ are two abelian groups and $\psi : K \times K \rightarrow G$ a given mapping. On $G \times K$ we define the $\psi$-law by

$$(z_0, z) \cdot \psi (w_0, w) = (z_0 \circ w_0 \circ \psi(z, w); z \cdot w).$$

We say that $G \times \psi K$ is a central extension of $(K, \cdot)$ by $(G, \circledast)$ associated to $\psi$ if the short sequence

$$0 \rightarrow K \rightarrow G \times \psi K \rightarrow G \rightarrow 0$$

is exact, and such that $K$ is in $Z(G)$, the center of the group $E$. This holds if one of the following two equivalent assertions is satisfied, to wit

i) $\psi$ preserves the neutral element $\psi(0_K, 0_K) = 0_G$ and verifies the cocycle relation

$$\psi(x, y) \circ \psi(x \cdot y, z) = \psi(x, y \cdot z) \circ \psi(y, z)$$

for every $x, y, z \in K$.

ii) $G \times \psi K := (G \times K, \cdot \psi)$ is a group.

Now, let $\mathbb{R}^2 = \mathbb{R}_s \times \mathbb{R}_t$ be the real $(s, t)$-plane identified with the complex plane $\mathbb{C} = \{z_0 = s + it; s, t \in \mathbb{R}\}$ and $\mathbb{C}^n$ denotes the complex $n$-space endowed with its standard Hermitian form

$$\omega(z,w) := \langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$$

for $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$. Thus, we define $N_\omega = \mathbb{C} \times \omega \mathbb{C}^n$ to be the set $\mathbb{C} \times \mathbb{C}^n$ endowed with the $\omega$-law given by

$$\omega(z,w) := \langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j.$$

Under (2.1), $N_\omega := \mathbb{C} \times \omega \mathbb{C}^n$ is a non-commutative nilpotent group of step two with center $Z(N_\omega) = \mathbb{C} \times \omega \{0\} = (\mathbb{R} \times \mathbb{R}) \times \omega \{0\}$. The identity element is $(0;0)$ and the symmetric of an element $(z_0; z)$ is $(-z_0 - \langle z, z \rangle; -z)$. Notice for instance that the $\omega$-law given by (2.1) can be rewritten in the coordinates $z_0 = (s, t), w_0 = (s', t')$ and $z \in \mathbb{C}^n$ and $z, w \in \mathbb{C}^n$ as follows

$$\omega(z,w) := \langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j.$$

Hence, endowing the set $\mathbb{R}_t \times \mathbb{C}^n$ with the $\cdot_\mathrm{Im} \omega$-law given by

$$(t; z) \cdot_\mathrm{Im} \omega (t'; w) = (t + t' + \mathrm{Im} \langle z, w \rangle; z + w),$$

makes $\mathbb{R} \times \mathrm{Im} \omega \mathbb{C}^n$ a group, which is nothing else than the classical real Heisenberg group of dimension $2n + 1$. One can notice easily that $(\mathbb{C} \times \mathbb{C}^n, \cdot_\omega)$, in addition of being the central extension of $\mathbb{C}^n$ by $\mathbb{C}$ associated to the map $\psi = \omega$, can also be viewed, due to (2.2), as the central extension of $(\mathbb{R}_t \times \mathbb{C}^n, \cdot_\mathrm{Im} \omega)$ by $\mathbb{R}_s$ associated to $\psi = \Re \omega$. This can be stated otherwise using directly the definition; if we denote by $q$ the projection mapping from $(\mathbb{R}_s \times \mathbb{R}_t) \times \omega \mathbb{C}^n$ onto $\mathbb{R}_t \times \mathrm{Im} \omega \mathbb{C}^n$ given by $q(s, t; z) = (t; z)$, one check that the
mapping $q$ is a homomorphism from the group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ onto the Heisenberg group $H_{2n+1} = \mathbb{R} \times_{1,\omega} \mathbb{C}^n$ and that the kernel of $q$ is given by

$$\ker q = \{(s, 0; 0), s \in \mathbb{R}\} = (\mathbb{R}_s \times \{0\}) \times_\omega \{0\}.$$ 

Since $\ker q$ is contained in the center $Z(N_\omega)$ of $N_\omega$, we may say that the group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ is a central extension of the Heisenberg group $H_{2n+1} = \mathbb{R} \times_{1,\omega} \mathbb{C}^n$ by $(\mathbb{R}_s, +)$; i.e. we have $\mathbb{C} \times_\omega \mathbb{C}^n / \ker q = \mathbb{C} \times_\omega \mathbb{C}^n / \mathbb{R}_s = H_{2n+1}$. Accordingly, harmonic analysis on our group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ will have many links to that on the classical Heisenberg group.

3. Explicit Formula for the Sub-Laplacian $\mathcal{L}_\omega$ on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$

The group $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ with the $\omega$-law given in (2.1) is a real Lie group of dimension $2n + 2$, and its tangent space at its neutral element $e = (0; 0) \in \mathbb{C} \times \mathbb{C}^n$ is given by $T_{(0,0)}N_\omega = (\mathbb{C}, +) \times (\mathbb{C}^n, +)$ as a real vector space of dimension $2n + 2$. In fact, $N_\omega$ is naturally equipped with the standard differentiable structure on euclidean spaces generated by the coordinates system $\{(\mathbb{C} \times \mathbb{C}^n, x)\}$, where $x$ is the coordinates map

$$x : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{R}^{2n+2}; \quad (z_0, z) \longmapsto (s, t, x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

The group action and the group symmetric maps are smooth under this differentiable structure. Let denote by $n_\omega$ its associated Lie algebra composed of all left invariant vector fields on $N_\omega$ and endowed with the standard bracket on vector fields. It is a well known fact that $n_\omega \cong T_{(0,0)}N_\omega$. For the sake of giving the explicit formula for the sub-Laplacian $\mathcal{L}_\omega$ on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$, we need to build a basis of $n_\omega$ which will be constructed as first order differential operators on functions of $N_\omega$. Define the left action by a fixed element $(z_0; z) \in N_\omega$ by

$$\ell_{(z_0; z)} : N_\omega \longrightarrow N_\omega; \quad (w_0; w) \longmapsto (z_0; z) \cdot_\omega (w_0; w).$$

This map is a diffeomorphism with respect to the Lie group structure. Hence, it is possible to extend its push-forward to act on vector fields. Furthermore, its action on a vector field $X$ is given explicitly by

$$\ell_{g*}X_\phi = X_\phi (f \circ \ell_\phi),$$

for test data $p$ and $f$ such that $p \in N_\omega$ and $f$ is a smooth function of $N_\omega$. By definition, a vector field $X$ is said to be left invariant if the equality $\ell_{g*}X = X$ holds.

In order to construct a left invariant vector field basis, we take a basis of the tangent vectors at the identity and generate from each vector of the tangent basis, a left invariant vector field by pushing it forward using $\ell_{(z_0; z)}$. Recall that a basis of the tangent vector space $T_{(0,0)}N_\omega$ acting on smooth functions $f$ is given by

$$\left(\frac{\partial}{\partial x^i}\right)_{(0,0)}f := \partial_i(f \circ x^{-1})\bigg|_{x(0,0)}; \quad i = 1, 2, \ldots, 2n + 2, \quad (3.1)$$

where $\partial_i$ is the ordinary partial derivative with respect to the $i$-th variable. We can now carry out the following computation in order to find generators for $n_\omega$:

$$\ell_{(z_0; z)*}\left(\frac{\partial}{\partial x^i}\right)_{(0,0)}f = \left(\frac{\partial}{\partial x^i}\right)_{(0,0)}(f \circ \ell_{(z_0; z)}) = \partial_i((f \circ \ell_{(z_0; z)}) \circ x^{-1})\bigg|_{x(0,0)}.$$
We plug in $x^{-1} \circ x$ in the middle of the last equation and we use the multivariable chain rule to get

$$\ell_{(z_0,z)} \left( \frac{\partial}{\partial x^i} \right)_{(0,0)} f = \partial_i((f \circ x^{-1}) \circ (x \circ \ell_{(z_0,z)} \circ x^{-1})) \bigg|_{x(0,0)}$$

$$= \sum_{m=1}^{2n+2} \partial_m(f \circ x^{-1}) \bigg|_{x \circ \ell_{(z_0,z)} \circ x^{-1} \circ x(0,0)} \times \partial_i(x^m \circ \ell_{(z_0,z)} \circ x^{-1}) \bigg|_{x(0,0)}$$

$$= \sum_{m=1}^{2n+2} J_{m,i} \times \left( \frac{\partial}{\partial x^m} \right)_{(z_0,z)},$$

where $J_{m,i} := \frac{\partial}{\partial x^m}(x^m \circ \ell_{(z_0,z)} \circ x^{-1}) \bigg|_{x(0,0)}$ and $x^m \circ \ell_{(z_0,z)} \circ x^{-1}$ is the $m$-th coordinate map of $x \circ \ell_{(z_0,z)} \circ x^{-1}$. Explicitly, we have

$$x \circ \ell_{(z_0,z)} \circ x^{-1}(s', t', x'_1, y'_1, \ldots, x'_n, y'_n) =$$

$$\left( s + s' + \sum_{j=1}^{n} (x_j x'_j + y_j y'_j), t + t' + \sum_{j=1}^{n} (y_j x'_j - x_j y'_j), x_1 + x'_1, y_1 + y'_1, \ldots, y_n + y'_n \right)$$

Therefore, it follows that $J_{m,i}$ can be viewed as the components of the following Jacobian matrix

$$J := J_{(0,\ldots,0)} = \begin{pmatrix}
1 & 0 & x_1 & y_1 & \cdots & x_n & y_n \\
0 & 1 & -x_1 & y_1 & \cdots & y_n & -x_n \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 1
\end{pmatrix}$$

Reading vertically, column by column, we find the following basis

$$(3.2) \qquad S = \left( \frac{\partial}{\partial s} \right) \qquad T = \left( \frac{\partial}{\partial t} \right) \qquad X_j = x_j \left( \frac{\partial}{\partial s} \right) + y_j \left( \frac{\partial}{\partial t} \right) + \left( \frac{\partial}{\partial x_j} \right) \qquad Y_j = y_j \left( \frac{\partial}{\partial s} \right) - x_j \left( \frac{\partial}{\partial t} \right) + \left( \frac{\partial}{\partial y_j} \right).$$

Note that we are using the coordinates $z_0 = s + it$ and $z_j = x_j + iy_j$ with

$$(3.3) \quad \frac{\partial}{\partial x^1} = \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial x^2} = \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x^{2j+1}} = \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial x^{2j+2}} = \frac{\partial}{\partial y_j},$$

for $j = 1, \ldots, n$. We summarize the above discussion on $N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n$ and its associated Lie algebra $\mathfrak{n}_\omega$ with some additional remarks by making the following statement.
Proposition 3.1. The real vector fields \( S = \frac{\partial}{\partial s}, T = \frac{\partial}{\partial t} \) together with \( X_j, Y_j; j = 1, \ldots, n \) given by

\[
X_j = x_j \left( \frac{\partial}{\partial s} \right) + y_j \left( \frac{\partial}{\partial t} \right) + \left( \frac{\partial}{\partial x_j} \right) \quad \text{and} \quad Y_j = y_j \left( \frac{\partial}{\partial s} \right) - x_j \left( \frac{\partial}{\partial t} \right) + \left( \frac{\partial}{\partial y_j} \right)
\]

form a basis for \( n_\omega \). Moreover, they satisfy the following commutation relations of Heisenberg type

\[
\begin{align*}
[S, X_j] &= [S, Y_j] = 0 \\
[T, X_j] &= [T, Y_j] = 0 \\
[S, T] &= 0 \\
[X_j, X_k] &= [Y_j, Y_k] = 0 \\
[X_j, Y_k] &= -2\delta_{jk} T 
\end{align*}
\]

for all \( j, k = 1, \ldots, n \).

Remark 3.2. As expected we see, in view of the above proposition, that the Lie algebra \( n_\omega \) of \( N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n \) with \( \omega(z, w) = \langle z, w \rangle \) is also a central extension of the classical Heisenberg algebra \( H_{2n+1} = \mathbb{R} \times_{1\omega} \mathbb{C}^n \) generated by the vector fields

\[
\begin{align*}
T = \frac{\partial}{\partial t}, \quad \dot{X}_j &= -y_j \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}, \quad \dot{Y}_j &= x_j \frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}
\end{align*}
\]

with the nontrivial commutation relation \([\dot{X}_j, \dot{Y}_k] = -2T\), where \( (x_j, y_j); j = 1, \ldots, n\) are coordinates of \( \mathbb{C}^n \).

Remark 3.3. To build such left invariant vector fields, one can also look for a one parameter group of \( N_\omega \), i.e., a group homomorphism \( \gamma : (\mathbb{R}, +) \longrightarrow N_\omega \) (curves \( \gamma(\epsilon) \in N_\omega; \epsilon \in \mathbb{R} \)) satisfying

\[
\dot{\gamma}(0) = \left. \frac{d\gamma}{d\epsilon}(\epsilon) \right|_{\epsilon=0} = (v_0; v) \in T_{(0,0)}N_\omega = \mathbb{C} \times \mathbb{C}^n.
\]

Next, we define in below the sub-Laplacian by setting

Definition 3.4. Let \( X_j, Y_j; j = 1, \ldots, n \), be the vector fields given in Proposition 3.1. Then, the operator

\[
\mathcal{L}_\omega = \sum_{j=1}^{n} X_j^2 + Y_j^2
\]

is called here the sub-Laplacian of \( N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n \).

The following proposition gives the explicit differential expression of \( \mathcal{L}_\omega \) in terms of the Laplace-Beltrami \( \Delta_{\mathbb{R}^{2n}} \) of \( \mathbb{C}^n = \mathbb{R}^{2n} \) and the first order differential operators \( E_{x,y} \) and \( F_{x,y} \) defined by

\[
\Delta_{\mathbb{R}^{2n}} := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}; \quad E_{x,y} := \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}; \quad F_{x,y} := \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}.
\]

Namely, we have

Proposition 3.5. The sub-Laplacian \( \mathcal{L}_\omega \) as defined in the above definition is given explicitly in the coordinates \( t, s, x_j, y_j; j = 1, \ldots, n \), of \( N_\omega = \mathbb{C} \times_\omega \mathbb{C}^n \) as follows

\[
(3.4) \quad \mathcal{L}_\omega = \Delta_{\mathbb{R}^{2n}} + 2(E_{x,y} + n) \frac{\partial}{\partial s} - 2F_{x,y} \frac{\partial}{\partial t} + (|x|^2 + |y|^2)(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}),
\]
where \(|x|^2 = \sum_{j=1}^{n} x_j^2\) and \(|y|^2 = \sum_{j=1}^{n} y_j^2\).

The explicit expression of \(\mathcal{L}_\omega\) given in Proposition 3.5 can be handled by straightforward computations.

**Remark 3.6.** If we consider the coordinates \((s, t) \in \mathbb{R}^2 = \mathbb{C}\) and \(z = (z_1, \ldots, z_n) \in \mathbb{C}^n\) with \(z_j = x_j + iy_j\), then the sub-Laplacian \(\mathcal{L}_\omega\) in (3.4) can be rewritten as

\[
\mathcal{L}_\omega = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2(E + \bar{E} + n) \frac{\partial}{\partial s} - 2i(E - \bar{E}) \frac{\partial}{\partial t} + |z|^2 \left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right),
\]

where \(E = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}\) is the complex Euler operator and \(\bar{E} = \sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\) is its complex conjugate.

**Remark 3.7.** The action of \(\mathcal{L}_\omega\) on functions \(F(t; z)\) on \(N_{\omega} = \mathbb{C} \times_{\omega} \mathbb{C}^n\) that are independent of the argument \(s\), reduces to that of the sub-Laplacian

\[
\tilde{\mathcal{L}}_{1_{\omega \omega}} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} - 2i(E - \bar{E}) \frac{\partial}{\partial t} + |z|^2 \frac{\partial^2}{\partial t^2}
\]

of the classical Heisenberg group \(\mathbb{R} \times_{1_{\omega \omega}} \mathbb{C}^n = \mathbb{H}_{2n+1}\).

We conclude this section by mentioning that both operators \(\mathcal{L}_\omega\) and \(\tilde{\mathcal{L}}_{1_{\omega \omega}}\) are not elliptic. But they are not far from being such in many aspects of their spectral theory. We will make this precise by discussing in a concrete manner the spectral eigenfunction problem on \(\mathbb{C}^n\) of the associated elliptic differential operator

\[
\Delta_{\nu, \mu} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2i\nu(E + \bar{E} + n) + 2\mu(E - \bar{E}) - (\nu^2 + \mu^2)|z|^2
\]

\[
= 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2(\nu + \mu)E - 2(\mu - \nu)\bar{E} - \left( \nu^2 + \mu^2 \right) |z|^2 + 2ivn.
\]

Formally, \(\Delta_{\nu, \mu}\) is related to \(\mathcal{L}_\omega\) using partial Fourier transform in \((s, t)\) with \((iv, i\mu)\) as dual arguments.

In the next section, we see that the operator \(\Delta_{\nu, \mu}\) can also be regarded as Schrödinger operator on \(\mathbb{C}^n = \mathbb{R}^{2n}\) in the presence of a uniform magnetic field \(\mathbf{B}_\mu = i\theta \nu, \mu \) on \(\mathbb{C}^n = \mathbb{R}^{2n}\) associated to a specific differential 1-form \(\theta_{\nu, \mu}\).

4. **Realization of \(\Delta_{\nu, \mu}\) as a magnetic Schrödinger operator and invariance property**

Magnetic Schrödinger operator on a complete oriented Riemannian manifold \((M, g)\) is defined to be

\[
H_\theta = (d + \text{ext } \theta)^*(d + \text{ext } \theta),
\]

where \(\theta\) is a given \(C^1\) real differential 1-form on \(M\) (potential vector). Here \(d\) stands for the usual exterior derivative acting on the space of differential \(p\)-forms \(\Omega^p(M)\), \(\text{ext } \theta\) is the
operator of exterior left multiplication by \( \theta \), i.e., \((\text{ext} \theta)\omega = \theta \wedge \omega \) and \((d + \text{ext} \theta)^* \) is the formal adjoint of \( d + \text{ext} \theta \) with respect to the Hermitian product

\[
\langle \alpha, \beta \rangle_{\Omega^p} = \int_M \alpha \wedge \ast \beta
\]

induced by the metric \( g \) on \( \Omega^p = \Omega^p(M) \), where \( \ast \) denotes the Hodge star operator associated to the volume form. From general theory of Schrödinger operators on non-compact manifolds, it is known that the operator \( H_\theta \), viewed as an unbounded operator in \( L^2(M, dm) \), is essentially self-adjoint for any smooth measure \( dm \).

In our framework \( M \) is the complex \( n \)-space \( \mathbb{C}^n \) equipped with its Kähler metric

\[
g = ds^2 = -i \sum_{j=1}^{n} dz_j \otimes d\bar{z}_j = \sum_{j=1}^{n} dx_j \otimes dy_j
\]

and the corresponding volume form is \( \text{Vol} = dx_1 dy_1 \cdots dx_n dy_n \). Associated to the parameters \( \nu \) and \( \mu \), we consider the potential vector

\[
\theta_{\nu, \mu}(z) := -\frac{\mu - i\nu}{2} \sum_{j=1}^{n} \bar{z}_j dz_j + \frac{\mu + i\nu}{2} \sum_{j=1}^{n} z_j d\bar{z}_j.
\]

Thus, we prove the following result concerning the twisted Laplacian defined by \((3.7)\).

**Proposition 4.1.** For every complex-valued \( C^\infty \) function \( f \) on \( \mathbb{C}^n \), we have

\[
\Delta_{\nu, \mu} f = -H_{\theta_{\nu, \mu}} f = -(d + \text{ext} \theta_{\nu, \mu})^* (d + \text{ext} \theta_{\nu, \mu}) f.
\]

**Sketched proof.** We start by writing \( H_{\theta_{\nu, \mu}} := (d + \text{ext} \theta_{\nu, \mu})^* (d + \text{ext} \theta_{\nu, \mu}) \) as

\[
H_{\theta_{\nu, \mu}} = d^* df + d^* \text{ext} \theta_{\nu, \mu} f + (\text{ext} \theta_{\nu, \mu})^* df + (\text{ext} \theta_{\nu, \mu})^* \text{ext} \theta_{\nu, \mu} f.
\]

Next, using the well-known facts \( d^* = -\ast d \ast \) and \( (\text{ext} \theta)^* = \ast \text{ext} \theta \ast \), we establish the following

\[
d^* d = -4 \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j},
\]

\[
d^* \text{ext} \theta_{\nu, \mu} f + (\text{ext} \theta_{\nu, \mu})^* d = \sum_{j=1}^{n} \left( -2(\mu + i\nu) z_j \frac{\partial}{\partial z_j} + 2(\mu - i\nu) \bar{z}_j \frac{\partial}{\partial \bar{z}_j} - 2i \nu \right),
\]

\[(\text{ext} \theta_{\nu, \mu})^* \text{ext} \theta_{\nu, \mu} = (\mu^2 + \nu^2) |z|^2.
\]

One of the advantages of the formula for \( \Delta_{\nu, \mu} \) as given by \((4.3)\) with the differential 1-form \( \theta_{\nu, \mu} \) in \((4.2)\) is that we can derive easily some invariance properties of the Laplacian \( \Delta_{\nu, \mu} \) with respect to the group of rigid motions of the complex Hermitian space \((\mathbb{C}^n, ds^2)\); \( ds^2 = \sum_{j=1}^{n} dz_j \otimes d\bar{z}_j \). Let \( G \) denote the group of biholomorphic mapping of \( \mathbb{C}^n \) that preserve the Hermitian metric \( ds^2 \). Then, \( G = U(n) \ltimes \mathbb{C}^n \) is the group of semi-direct product of the unitary group \( U(n) \) of \( \mathbb{C}^n \) with the additive group \((\mathbb{C}^n, +)\). It can be represented as

\[
G := U(n) \ltimes \mathbb{C}^n = \left\{ g = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} =: [A, b]; \ A \in U(n), b \in \mathbb{C} \right\}
\]
and acts transitively on $\mathbb{C}^n$ via the mappings $g.z = Az + b$. The pull-back $g^*\theta_{v,\mu}$ of the differential 1-form $\theta_{v,\mu}$ by the above mapping $z \mapsto g.z$ is related to $\theta_{v,\mu}$ by the following identity for every $g \in G = U(n) \ltimes \mathbb{C}^n$.

**Proposition 4.2.** Let $\theta_{v,\mu}$ be as in (4.2) and $g \in G = U(n) \times \mathbb{C}^n$. Then, for every $g \in G = U(n) \times \mathbb{C}^n$ we have

$$g^*\theta_{v,\mu} = \theta_{v,\mu} + \frac{d}{2} \left[ \langle z, g^{-1}.0 \rangle + \langle z, g^{-1}.0 \rangle \right] + \frac{\mu}{2} \left[ \langle z, g^{-1}.0 \rangle - \langle z, g^{-1}.0 \rangle \right]$$

where

$$j^\nu,\mu(g, z) = \exp(i\phi_{v,\mu}(g, z)).$$

The phase function $\phi_{v,\mu}(g, z)$ is given by

$$\phi_{v,\mu}(g, z) = -v \Re \left( \langle z, g^{-1}.0 \rangle \right) + \mu \Im \left( \langle z, g^{-1}.0 \rangle \right).$$

**Proof.** The identity (4.5) holds by component-wise straightforward computations. Indeed, direct computation yields

$$g^*\theta_{v,\mu}(z) = \theta_{v,\mu}(z) - \frac{iv}{2} d \left[ \langle z, g^{-1}.0 \rangle + \langle z, g^{-1}.0 \rangle \right] + \frac{\mu}{2} \left[ \langle z, g^{-1}.0 \rangle - \langle z, g^{-1}.0 \rangle \right]$$

where $g^{-1}$ is the inverse mapping of $z \mapsto g.z$ and $g^{-1}.0 = -A^{-1}b = -A^*b$ for $g = [A, b] \in U(n) \ltimes \mathbb{C}^n$. We conclude since $d j^\nu,\mu(g, z) = id(\phi_{v,\mu}(g, z)) j^\nu,\mu(g, z)$. □

Notice that the relation (4.5) reads also as $g^*\theta_{v,\mu} = \theta_{v,\mu} + d \log(j^\nu,\mu(\gamma, z))$ and shows that the differential 1-form $\theta_{v,\mu}$ is not $G$-invariant. But $g^*\theta_{v,\mu}$ and $\theta_{v,\mu}$ are in the same class of the de Rham cohomology group. Also it gives insight how to make, in view of the expression (4.3), the Laplacian $\Delta_{v,\mu}$ invariant with respect to a $G$-action on functions built with the help of the following automorphic factor $j^\nu,\mu(g, z)$ defined through (4.6) and satisfying the chain rule

$$j^\nu,\mu(gg', z) = j^\nu,\mu(g, z)j^\nu,\mu(g', z)$$

for every $g \in G = U(n) \ltimes \mathbb{C}^n$ and $z \in \mathbb{C}^n$. Associated to $j^\nu,\mu$, we define $T_g^{\nu,\mu}$ to be the operator acting on differential $p$-forms $\omega$ of $\mathbb{C}^n$ through the formula

$$T_g^{\nu,\mu} \omega = j^\nu,\mu(g, z)g^*\omega.$$ 

On $C^\infty$ complex-valued functions $f$ on $\mathbb{C}^n$, it reduces further to

$$[T_g^{\nu,\mu} f](z) = j^\nu,\mu(g, z)g^* f(z) = j^\nu,\mu(g, z)f(g.z).$$

Thus, the following invariance property for $\Delta_{v,\mu}$ holds.

**Proposition 4.3.** For every $g \in U(n) \ltimes \mathbb{C}^n$, we have

$$\Delta_{v,\mu} T_g^{\nu,\mu} = T_g^{\nu,\mu} \Delta_{v,\mu}.$$ 

**Proof.** Using the well-known facts $g^*d = dg^*$ and $g^*(\alpha \wedge \beta) = g^*\alpha \wedge g^*\beta$, we get

$$T_g^{\nu,\mu}((d + \ext \theta_{v,\mu}) f) = j^\nu,\mu(\gamma, z) \left( d[g^* f] + [g^* \theta_{v,\mu}] \wedge [g^* f] \right).$$
Now, by means of the identity (4.5), it follows
\[ T^{v,\mu}_g ((d + \text{ext } \theta_{v,\mu}) f) = j^{v,\mu}(\gamma, z)[g^* f] + \theta_{v,\mu}[g^* f] = (d + \text{ext } \theta_{v,\mu})(T^{v,\mu}_g f). \]
Moreover, \( T^{v,\mu}_g \) commutes also with \((d + \text{ext } \theta_{v,\mu})^*\) for \( T^{v,\mu}_g \) being a unitary transformation. Therefore, by means of the expression of \( \Delta_{v,\mu} = -(d + \text{ext } \theta_{v,\mu})^*(d + \text{ext } \theta_{v,\mu}) \) as a magnetic Schrödinger operator \( H_{\theta_{v,\mu}} \), we deduce easily that \( \Delta_{v,\mu} \) and \( T^{v,\mu}_g \) commute. This ends the proof.

**Remark 4.4.** For \( g \in \{1\} \times \mathbb{C}^n = (\mathbb{C}^n, +) \), the unitary operators \( T^{v,\mu}_g \) given in (4.9) define projective representation of \( G \) on the space of \( \mathcal{C}^\infty \) functions on \( \mathbb{C}^n \). In fact, they are the so-called magnetic translation operators that arise in the study of Schrödinger operators in the presence of uniform magnetic field.

5. **Spectral properties of \( \Delta_{v,\mu} \) acting on \( \mathcal{C}^\infty(\mathbb{C}^n) \) and on \( \mathcal{H} = L^2(\mathbb{C}^n, dm) \)**

We denote by \( \mathbb{C}^n \) the Frechet space of complex-valued functions on \( \mathcal{C}^\infty(\mathbb{C}^n) \) endowed with the compact-open topology, while \( L^2(\mathbb{C}^n, dm) \) denotes the usual Hilbert space of square integrable complex-valued functions \( F(z) \) on \( \mathbb{C}^n \) with respect to the usual Lebesgue measure \( dm(z) \). In the sequel, we will give a concrete description of the eigenspaces of \( \Delta_{v,\mu} \) in both \( \mathcal{C}^\infty(\mathbb{C}^n) \) and \( L^2(\mathbb{C}^n, dm) \). To this end, let \( \lambda \) be any complex number in \( \mathbb{C} \) and \( E_\lambda(\Delta_{v,\mu}) \) be the eigenspace of \( \Delta_{v,\mu} \) corresponding to the eigenvalue \(-2\mu(2\lambda + n)\) in \( \mathcal{C}^\infty(\mathbb{C}^n) \), i.e.,
\[
E_\lambda(\Delta_{v,\mu}) = \{ F \in \mathcal{C}^\infty(\mathbb{C}^n); \Delta_{v,\mu}F = -2\mu(2\lambda + n)F \}.
\]
Also, by \( A_\lambda^2(\Delta_{v,\mu}) \) we denote the subspace of \( L^2(\mathbb{C}^n, dm) \) whose elements \( F(z) \) satisfy \( \Delta_{v,\mu}F = -2\mu(2\lambda + n)F \). Namely, by elliptic regularity of \( \Delta_{v,\mu} \), we have
\[
A_\lambda^2(\Delta_{v,\mu}) = L^2(\mathbb{C}^n, dm) \cap E_\lambda(\Delta_{v,\mu}).
\]

The first result related to \( E_\lambda(\Delta_{v,\mu}) \) and \( A_\lambda^2(\Delta_{v,\mu}) \) is the following.

**Proposition 5.1.** The eigenspaces \( E_\lambda(\Delta_{v,\mu}) \) and \( A_\lambda^2(\Delta_{v,\mu}) \) are invariants under the \( T^{v,\mu}_g \)-action given by (4.9).

**Proof.** This can be handled easily making use the invariance property (4.10) of \( \Delta_{v,\mu} \) by the unitary transformations \( T^{v,\mu}_g \). \( \square \)

**Proposition 5.2.** The set of spherical eigenfunctions of \( \Delta_{v,\mu} \) with \(-2\mu(2\lambda + n)\) as eigenvalue is a one dimensional vector subspace of \( E_\lambda(\Delta_{v,\mu}) \) generated by
\[
\phi_{\lambda}^{v,\mu}(z) = e^{-\frac{2\mu(2\lambda + n)}{n} |z|^2} \, _1F_1(-\lambda; n; \mu |z|^2)
\]
where \( _1F_1(a; c; x) \) is denoting here the usual confluent hypergeometric function,
\[
_1F_1(a; c; x) = 1 + \frac{a \, x}{c} + \frac{a(a + 1) \, x^2}{c(c + 1) \, 2!} + \cdots; \quad x \in \mathbb{C}.
\]

**Remark 5.3.** By a “spherical” (or radial here) eigenfunction of \( \Delta_{v,\mu} \), we mean \( U(n) \)-invariant function \( f \) satisfying \( f(hz) = f(z) \) for all \( h \in U(n) \) and \( z \in \mathbb{C}^n \).
Sketched proof. To prove the statement, we write $\Delta_{\nu,\mu}$ in polar coordinates $z = r\theta$ with $r \geq 0$ and $\theta \in S^{n-1}$ as

$$
\Delta_{\nu,\mu} = \frac{\partial^2}{\partial r^2} + \left(\frac{2n-1}{r} + 2iv\right) \frac{\partial}{\partial r} - (v^2 + \mu^2)r^2 + 2ivn + L^\theta_{\nu,\mu},
$$

where $L^\theta_{\nu,\mu}$ stands for the tangential component of $\Delta_{\nu,\mu}$. The eigenvalue problem $\Delta_{\nu,\mu}f = -2\mu(2\lambda + n)f$ for radial functions $f(z) = \psi(x)$, with $x = r^2$, reduces to the differential equation

$$
\left\{ x \frac{\partial^2}{\partial x^2} + (n + ivx) \frac{\partial}{\partial x} - \left[\frac{v^2 + \mu^2}{4}x + \frac{i\nu - \mu - \lambda}{2}\right] \right\} \psi = 0.
$$

Next, making use of the appropriate change of function $\psi(x) = e^{\frac{i\nu - \mu}{2}x}y(x)$, we see that the previous equation leads to the confluent hypergeometric differential equation [15, page 193]

$$
xy'' + (n - \mu x)y' + \mu \lambda y = 0
$$

whose regular solution at $x = 0$ is the confluent hypergeometric function $_1F_1(-\lambda; n; \mu x)$.

**Remark 5.4.** According to the proof of the previous result, we make the following key observation that can deserve as outline of the proofs of Proposition 5.2 and the assertions below. Indeed, the operators $\Delta_{\nu,\mu}$ and $\Delta_0,\mu$ are unitary equivalent in $L^2(\mathbb{C}^n, dm)$. More precisely, we have

$$
\Delta_{\nu,\mu} = e^{-\frac{\mu}{2}|\cdot|^2} \Delta_0,\mu e^{\frac{\mu}{2}|\cdot|^2}.
$$

Accordingly, we claim the following

**Proposition 5.5.** Let $(\nu, \mu) \in \mathbb{R}^2$ with $\mu > 0$ and $\lambda \in \mathbb{C}$. Then, the eigenspace $A^2_\lambda(\Delta_{\nu,\mu})$ as defined (5.2) is non-zero (Hilbert) space if and only if $\lambda = l$ with $l = 0, 1, 2, \ldots$, is a positive integer number. Moreover, the spaces $A^2_l(\Delta_{\nu,\mu})$, $l = 0, 1, 2, \ldots$, are pairwise orthogonal in $L^2(\mathbb{C}^n, dm)$ and we have the following orthogonal decomposition in Hilbertian subspaces

$$
L^2(\mathbb{C}^n, dm) = \bigoplus_{l=0}^{\infty} A^2_l(\Delta_{\nu,\mu}).
$$

**Remark 5.6.** The claim 5.5 asserts that the spectrum of $\Delta_{\nu,\mu}$ in $L^2(\mathbb{C}^n, dm)$ is purely discrete and each of its eigenvalue $-2\mu(2l + n)$, $l \in \mathbb{Z}^+$, is a positive integer number. Moreover, the eigenspace $A^2_l(\Delta_{\nu,\mu})$ in (5.2) is of infinite dimension.

**Proposition 5.7.** Let $(\nu, \mu) \in \mathbb{R}^2$ with $\mu > 0$. For fixed $l = 0, 1, 2, \ldots$, let $P_l$ be the orthogonal eigenprojector operator from $L^2(\mathbb{C}^n, dm)$ onto the eigenspace $A^2_l(\Delta_{\nu,\mu})$ with $-2\mu(2l + n)$ as eigenvalue. Then the Schwartz kernel $P^\nu,\mu_l(z, w)$ of the operator $P_l$ is given by the following explicit formula

$$
P^\nu,\mu_l(z, w) = \left(\frac{\mu}{\pi}\right)^l \frac{(n - l + 1)!}{(n - 1)!} j^\nu,\mu_l(z, w) e^{-\frac{\mu}{2}|z - w|^2} _1F_1(-l; n; \mu|z - w|^2),
$$

where the factor $j^\nu,\mu_l(z, w)$; $z, w \in \mathbb{C}^n$ is given by

$$
j^\nu,\mu_l(z, w) = e^{-\frac{\mu}{2}(|z|^2 - |w|^2)} + \frac{\mu}{2}(z, w) - (z, w).
$$
Sketched proof. The proof for \( \nu = 0 \) is contained in \([3,2,6]\). For arbitrary \( \nu \), the proof can be handled in a similar way or making use of the key observation that in \( L^2(\mathbb{C}^n, dm) \), the operators \( \Delta_{v,\mu} \) and \( \Delta_{0,\mu} \) are unitary equivalents and we have

\[
\Delta_{v,\mu} = e^{-\frac{\mu}{2}|z|^2} \Delta_{0,\mu} e^{\frac{\mu}{2}|z|^2}.
\]

as in [].

Remark 5.8. A direct proof of Proposition \([5,5]\) can be handled using Proposition \([5,2]\) and the asymptotic behavior of the confluent hypergeometric function given by \([15\, \text{page} \, 332]\).

\[
(6.2)
\]

\[
\text{as } x \rightarrow +\infty. \text{ This asymptotic behaviour can also be used to show that the radial function } q^{\nu,\mu}_{\lambda}
\]
given by \((5.5)\) is bounded if and only if \( \lambda = l; l = 0,1,2, \ldots \).

6. Factorisation of \( \Delta_{v,\mu} \) and associated Hermite polynomials

In this section we study the spectral theory of \( \Delta_{v,\mu} \) on \( L^2(\mathbb{C}^n, dl) \) using the factorisation method. This method finds its origin in the works of Dirac \([4]\) and Schrödinger \([16]\), then developed by Infeld and Hull \([8]\) in order to solve eigenvalue problems appearing in quantum theory. Notice for instance that the operator \( \Delta_{0,\mu} = L_{\mu} \) is refereed in physiological literature as the Landau operator on \( \mathbb{R}^{2n} = \mathbb{C}^n \) (or Schrödinger operator on \( \mathbb{R}^{2n} \) in the presence of a uniform magnetic field \( B_{\mu} = \sqrt{-1}d\theta_{v,\mu} \)) and for which many of their spectral properties that we are considering go back to Landau’s work in 1930 on the Hamiltonian in \( \mathbb{R}^2 = \mathbb{C} \) given by

\[
L_{\mu} = -4 \frac{\partial^2}{\partial z \bar{z}} - 2\mu \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) + \mu^2 |z|^2 = 4 \left( -\frac{\partial}{\partial z} + \frac{\mu}{2} \bar{z} \right) \left( \frac{\partial}{\partial \bar{z}} + \frac{\mu}{2} z \right) - 2\mu I
\]

More generally the Laplacian \( \Delta_{v,\mu} \) can be rewritten as

\[
\Delta_{v,\mu} = -4 \sum_{j=1}^{n} \left( -\frac{\partial}{\partial z_j} + \frac{\mu - iv}{2} \bar{z}_j \right) \left( \frac{\partial}{\partial \bar{z}_j} + \frac{\mu + iv}{2} z_j \right) - 2\mu n I.
\]

Hence in view of the above remarks, the spectral properties of \( \Delta_{v,\mu} \) on \( L^2(\mathbb{C}^n, dm) \) or on \( C^\infty(\mathbb{C}^n) \) can be derived from Landau’s work \([12]\). To this end, It will be helpful to define

\[
\tilde{\Delta}_{v,\mu} := -\frac{1}{4} \Delta_{v,\mu}.
\]

We also need to define, for \( j = 1,2, \ldots, n \), the following first order differential operators

\[
(6.1)
\]

\[
a_j^+ = -\frac{\partial}{\partial z} + \frac{\mu - iv}{2} \bar{z}_j = -e^{-\frac{\mu - iv}{2} |z|^2} \frac{\partial}{\partial z_j} e^{-\frac{\mu - iv}{2} |z|^2},
\]

and

\[
(6.2)
\]

\[
a_j^- = \frac{\partial}{\partial \bar{z}} + \frac{\mu + iv}{2} z_j = e^{-\frac{\mu + iv}{2} |z|^2} \frac{\partial}{\partial \bar{z}_j} e^{\frac{\mu + iv}{2} |z|^2}.
\]
These operators satisfy the commutation relationships 
\[ [a_j^-, a_k^+] = n \mu \delta_{jk}, \]
where \( \delta_{jk} \) is the Krönecker symbol. They are linked to the Laplacian \( \tilde{\Delta}_{v, \mu} \) through
\[
\sum_{j=1}^{n} a_j^+ a_j^- = \tilde{\Delta}_{v, \mu} - \frac{n}{2} \mu \quad \text{and} \quad \sum_{j=1}^{n} a_j^- a_j^+ = \tilde{\Delta}_{v, \mu} + \frac{n}{2} \mu.
\]

Moreover, we have the following creation and annihilation equalities
\[
\tilde{\Delta}_{v, \mu} a_j^+ = a_j^+ (\tilde{\Delta}_{v, \mu} + \mu) \quad \text{and} \quad \tilde{\Delta}_{v, \mu} a_j^- = a_j^- (\tilde{\Delta}_{v, \mu} - \mu)
\]
and allow the determination of the eigenvalues and eigenvectors of \( \tilde{\Delta}_{v, \mu} \). Indeed, if \( \psi \) is an eigenvector of \( \tilde{\Delta}_{v, \mu} \) associated to the eigenvalue \( \lambda \) we have the following
\[
\tilde{\Delta}_{v, \mu}(a_j^+ \psi) = a_j^+(\tilde{\Delta}_{v, \mu} + \mu)\psi = (\lambda + \mu)a_j^+\psi,
\]
\[
\tilde{\Delta}_{v, \mu}(a_j^- \psi) = a_j^-(\tilde{\Delta}_{v, \mu} - \mu)\psi = (\lambda - \mu)a_j^-\psi.
\]

Thus, we need only to know those associated to the lowest eigenvalue. In fact, since \( \tilde{\Delta}_{v, \mu} \) is positive semi-definite, all the eigenvalues are real and nonnegative. Moreover, from symmetry and ellipticity of \( \tilde{\Delta}_{v, \mu} \) we know that \( \tilde{\Delta}_{v, \mu} \) has an infinite sequence of nonnegative eigenvalues (see for example [11]):
\[ 0 \leq \lambda_0 < \lambda_1 < \cdots \uparrow \infty. \]

Therefore, if \( \psi_0 \) is an eigensolution associated to \( \lambda_0 \), we have necessary \( a_j^- \psi_0 = 0 \) for every \( j = 1, 2, \cdots, n \), thanks to (6.5). This implies, by using the second expression in (6.2), that
\[ \psi_0(z) = e^{-\frac{\mu}{2} - \frac{i\nu}{2} |z|^2} f(z), \]
where \( f \) is any arbitrary holomorphic function. Consequently,
\[ A_0 = \text{Ker}(a_j^-) = \text{span} \{ z^m e^{-\frac{\mu}{2} - \frac{i\nu}{2} |z|^2}; \ m \in (\mathbb{Z}^+)^n \}. \]

Here we have used the multivariate notation \( z^m \) for given multi-index \( m := (m_1, m_2, \cdots, m_n) \) to mean \( z^m := z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \). Making use of the creation operators leads to the following family of multi-indexed functions \( h_{r,s}^{v, \mu}; r = (r_1, \cdots, r_n); s = (s_1, \cdots, s_n) \),
\[
h_{r,s}^{v, \mu} = (a_1^{+r_1}) \cdots (a_n^{+r_n}) (z^s e^{-\frac{\mu}{2} + \frac{i\nu}{2} |z|^2})
\]
\[
= (-1)^{|r|} e^{\frac{\mu}{2} - \frac{i\nu}{2} |z|^2} D_z^r e^{-\frac{\mu}{2} + \frac{i\nu}{2} |z|^2} (z^s e^{-\frac{\mu}{2} - \frac{i\nu}{2} |z|^2})
\]
\[
= (-1)^{|r|+|s|} e^{-\frac{\mu}{2} - \frac{i\nu}{2} |z|^2} D_z^r D_z^s e^{-\frac{\mu}{2} - \frac{i\nu}{2} |z|^2},
\]
where \(|m|\) and \(|m|!\) stand for \(|m| := m_1 + \cdots + m_n\) and \(|m|! := m_1! \cdots m_n!\) respectively, and \( D_z^m \) and \( D_z^m \) are defined by
\[ D_z^m := \frac{\partial^{ |m|}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \quad \text{and} \quad D_z^m := \frac{\partial^{ |m|}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}. \]

According to the above discussion, \( h_{r,s}^{v, \mu} \) are eigensolutions associated to the eigenvalue \( \frac{n}{2} \mu + |r| \mu \) and \( \lambda_l := \mu(\frac{n}{2} + l); l = 0, 1, 2, \cdots \), are, indeed, eigenvalues of \( \tilde{\Delta}_{v, \mu} \). The following proposition shows that \( \{ \lambda_l := \mu(\frac{n}{2} + l); l = 0, 1, 2, \cdots \} \) are the only eigenvalues of \( \tilde{\Delta}_{v, \mu} \).
Proposition 6.1. The $h_{r,s}^{v,\mu}$ form a complete orthogonal system in the Hilbert space $L^2 (\mathbb{C}^n, d\lambda)$. Moreover, we have the following decomposition

$$L^2 (\mathbb{C}^n, d\lambda) = \bigoplus_{l=1}^{\infty} A_l,$$

where

$$A_l := \text{span} \{ h_{r,s}^{v,\mu}; \ r,s \in (\mathbb{Z}^+)^n, \ |r| = l \}.$$ 

Proof. The identity (6.6) shows that $h_{r,s}^{v,\mu}$, up to $e^{iv|z|^2/2}$, are essentially the high-dimensional analogue of the univariate complex Hermite functions

$$h_{m,n}^{r} (\xi, \xi) := (-1)^{m+n} e^{\sigma|\xi|^2} \frac{\partial^{m+n}}{\partial \xi^m \partial \bar{\xi}^n} e^{-\sigma|\xi|^2}; \ \xi \in \mathbb{C}, \sigma > 0,$$

considered in [9, 5]. The main idea of the proof is then to separate the variable $z$ in the expression (6.6) into its components $z_j$ to get

$$h_{r,s}^{v,\mu} (z, \bar{z}) = \mu^{-|s|} e^{-\frac{\beta}{2} |z|^2} \prod_{j=1}^{n} (-1)^{r_j + s_j} e^{\mu |z_j|^2} \frac{\partial^{r_j + s_j}}{\partial z_j \partial \bar{z}_j} e^{-\mu |z_j|^2}$$

$$= \mu^{-|s|} e^{-\frac{\beta}{2} |z|^2} \prod_{j=1}^{n} h_{r_j, s_j} (\sqrt{\mu} z_j, \sqrt{\mu} \bar{z}_j).$$

This implies that the eigenvalues of $\Delta_{v,\mu} = -4\Delta_{v,\mu}$ are

$$\{-2\mu (n + 2l); \ l = 0, 1, 2, \cdots \},$$

which coincide with the results in Section 5. Notice as well that $A_l^2 (\Delta_{v,\mu})$ and $A_l$ refer to the same set.

Remark 6.2. The Hermite functions $h_{r,s}^{v,\mu}$ are given explicitly by

$$h_{r,s}^{v,\mu} = \mu^{-|s|} e^{-\frac{\beta}{2} |z|^2} \sum_{|k|=0}^{\min(r,s)} \frac{(\sqrt{\mu} |r| + |s| - 2|k|)!}{k!(r - k)!(s - k)!} z^{s-k} z^{r-k},$$

where $r = (r_1, \cdots, r_n)$, $s = (s_1, \cdots, s_n)$ and $\min(r, s) := (\min(r_1, s_1), \cdots, \min(r_n, s_n))$. 

7. CONCLUDING REMARKS

The consideration of the unitary transformations $T_{v,\mu}^g ; g \in G = U(n) \ltimes \mathbb{C}^n$, and the $T_{v,\mu}^g$-invariance property satisfied by the magnetic Laplacian $\Delta_{v,\mu}$ give rise to new class of automorphic functions associated to the automorphic factor $f_{v,\mu}^g (g, z)$ when we restrict $g$ to belong in a full-rank discrete subgroup $\Gamma$ of $G$. We call them automorphic functions of bi-weight $(v, \mu)$. The considered $\Delta_{v,\mu}$ leaves invariant this space and therefore the corresponding eigenvalue problem is well defined. Thus, a detailed description of the spectral properties of $\Delta_{v,\mu}$ when acting on bi-weighted automorphic functions with respect to any discrete subgroup of $(\mathbb{C}^n, +)$ (not necessary of full-rank) is of great interest. We hope to focus on this in a near future. We conclude, noting that the particular case $v = 0$ and $\Gamma = \{1\} \ltimes \mathbb{C}^n$, these functions reduce further the classical one studied in [7].


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