A quantum secret sharing scheme among three parties utilizing four-qubit Smolin bound entangled state

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Four-qubit Smolin bound entangled state \cite{1} has a distinct feature: the state is not distillable when every qubit is separated from each other; but it makes two separated qubit entangled if the other qubits group together. Here the feature is applied to quantum secret sharing, a QSS protocol similar to Ekert 91 protocol of QKD is proposed. The security problem, disadvantage and advantage of this protocol are discussed.

A. Introduction

In the past years, with the development of quantum key distribution \cite{2,3}, quantum secret sharing (QSS) \cite{6} attracts much attention in both the theoretical and experimental aspects of quantum communication. QSS is a protocol to split a message to several parts so that no subset of parts is sufficient to read the message but the entire set is. For example, suppose that Alice wants to send a secret to two distant parties, Bob and Charlie. One of them, Bob or Charlie, is not entirely trusted by Alice, but she knows if they who can come together and carry it out for her, but she doesn’t know whether they are honest or whether there is an eavesdropper in the channel. She cannot simply send a message to both by a classical channel, because if the two of them coexist, the honest one will keep the dishonest one from doing any damage. Instead of giving entire secret messages to either of them, it may be desirable for Alice to split the secret messages into two encrypted parts and send each one a part so that neither individual is able to obtain all of the original information unless they collaborate. To achieve this end, classical cryptography can use a technique called secret sharing \cite{4}. QSS is the generation of this concept to quantum scenario.

Up to now, there are many kinds quantum secret sharing protocols with and without entanglement. The first QSS protocol has been proposed by Hillery et al \cite{6}, in which three-qubit GHZ (Greenberger-Horne-Zeilinger) entangled states is employed to allow information splitting and eavesdropper protection simultaneously. Moreover, Koashi and Imoto considered the correlation of the two-qubit Bell state in their quantum secret sharing scheme \cite{7}. Then Bagherinezhad and Karimipour introduced a work for quantum secret sharing which utilizes the reusable GHZ states as secure carriers \cite{13}. Entanglement swapping is another method used to realize QSS. Karimipour et al. \cite{5} proposed d-level secret sharing via entanglement swapping between a generalized cat states for d-level systems and a generalized Bell states. Other improved versions of QSS based on entanglement swapping were also presented \cite{8–10}. Product states are alternative resources for realizing QSS. Li-Yi Hsu et al \cite{14} suggested QSS schemes with a particular set of orthogonal product states in which an unknown quantum state cannot be determined only by LOCC(local operation and classical communication) if the order of the local measurements is private. A BB84-like QSS Scheme was given by the Group Guang-can Guo \cite{15}, which security is based on the quantum no-cloning theory. Recently, the experimental demonstrations of QSS by GHZ states \cite{11} and by a single qubit \cite{12} were reported, respectively.

In this work, we use a bound entangled state as quantum resource to accomplish QSS task. Bound entanglement (BE) \cite{16} is a kind of entanglement in multi-parties system which cannot be distilled to pure entangled form only by local operations and classical communication (LOCC). But for some bound entangled states, collective operation between some subparties induces distillable entanglement shared between the others. Four-party Smolin entangled state \cite{1} is such state: when two parties come together, they can by LOCC enable the other two parties to have some pure entanglement. The Smolin state can be expressed as:

\[
\rho = \frac{1}{4}(|\Phi^+\rangle_{12}\langle \Phi^+| \otimes |\Phi^+\rangle_{34}\langle \Phi^+| + |\Phi^−\rangle_{12}\langle \Phi^−| \otimes |\Phi^−\rangle_{34}\langle \Phi^−| \\
+ |\Psi^+\rangle_{12}\langle \Psi^+| \otimes |\Psi^+\rangle_{34}\langle \Psi^+| + |\Psi^−\rangle_{12}\langle \Psi^−| \otimes |\Psi^−\rangle_{34}\langle \Psi^−| )
\]  

(1)

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where we use the usual notation for the maximally entangled states of two qubit (Bell state)

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

(2)

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

(3)

In other word, if 1 and 2 particle come together and do the nonlocal Bell measurement on their systems, they can determine reliably which Bell state they have since the four Bell states are orthogonal, and 3, 4 have the same one with 1,2. The feature of Smolin state is helpful for QSS scheme [1,17,18].

Our work is organized as follows. In Sec. II we present a QSS scheme utilizing four party bound entangled states (Smolin states for short), and analyze the security in the sense of the violation to Bell inequality. In Sec.III the scheme is compared with others QSS protocol, and the conclusions are given.

B. Quantum Secret Sharing With Smolin States

To achieve the purpose of sharing secret, the scheme is divided into three stages. The first stage is for preparation and distribution of Smolin states. Alice produces a series of Smolin states and sends two qubit in every event to Bob and Charlie. Secondly, for checking the security of quantum channel, Alice chooses randomly some Smolin states from the series. The three man determine whether the correlation in the Smolin states violate the Bell inequality or not. Then if the channel is secure, after Alice operates qubits in hand, Bob and Charlie share Alices’ private classical information which can be revealed only by Bob and Charlie’s collaboration.

In the first phase, Alice prepares N copies of four-qubit bound entangled Smolin states each of which has a corresponding record number. For each Smolin state qubits 1, 2 are in the possession of Alice, and qubits 3, 4 are send to Bob and Charlie, respectively. Once Bob and Charlie receive one qubit, they publicly announce the facts. As a result, Alice has qubits 1, 2 of each Smolin state, Bob has qubit 3 and Charlie qubit 4. And the shared Smolin states between the three parties will serve as carriers of private classical information.

In the phase of examining the security of quantum channel, Alice chooses randomly M copies from the Smolin states, and inform Bob and Charlie which their record numbers are. Then Alice projects qubit 1 into the basis $|x\rangle, |y\rangle$ and qubit 2 into $|x\rangle, |y\rangle$. Bob and Charlie measure their qubits 3, 4 in the base of $|x\rangle, |y\rangle$ and send their results back to Alice, respectively. Collecting all results of measurements on four qubits, Alice is able to calculate the value of correlation function $E$ of four qubits in Smolin state.

For four qubits, a two-setting Bell-inequality similar to standard CHSH [20] inequality for two particles is given as [17]:

$$|E(1, 1, 1, 1) + E(1, 1, 1, 2) + E(2, 2, 2, 1) − E(2, 2, 2, 2)| \leq 2.$$  

(4)

But for four-qubit Smolin entangled state, the correlation function $E_{QM}$ satisfies

$$|E_{QM}(1, 1, 1, 1) + E_{QM}(1, 1, 1, 2) + E_{QM}(2, 2, 2, 1) − E_{QM}(2, 2, 2, 2)| = 2\sqrt{2}.$$  

(5)

which gives violation, and it is proved that the above violation to two-setting Bell inequality is maximal [17]. Therefore, if the qubits are not directly or indirectly disturbed, Alice gets the Equation (5), and the quantum channel is secure and can be used for private communication.

In the phase of transferring information, Alice first measures the qubits 1,2 of the rest (N-M) copies of Smolin states in Bell basis $\{\Phi^\pm, |\Psi^\pm\rangle\}$, and encode the results obtained, for example as $|\Phi^+\rangle = 00, |\Phi^-\rangle = 01, |\Psi^+\rangle = 10, |\Psi^-\rangle = 11$. Hence, after the measurement, Alice creates a random and private bit string which, simultaneously, is send to and shared between Bob and Charlie when finishing the measurement is announced. To reveal the bit string, Bob and Charlie have to come together and determine that in which one of four Bell states their qubits are, then read out Alice’s secure bit-string information. Actually and very importantly, in this phase, the key is created, send and shared just at the same time. As a result, the task of QSS is achieved.

Then let’s go back to consider how to detect possible eavesdropping attacks. If there is Eve in the channel who wants to extract out Alice’s private information. As discussed in Ref. [21], Eve cannot elicit any information from the qubits while in transit from Alice to Bob and Charlie, because the private information is created only after Alice’s measurements on qubits 1,2 and announcement for finishing it. So Eve has to intercept and clone the qubits 3,4 send...
to Bob and Charlie, and distribute one copy between them. However the intervention of Eve will introduce noise to the original Smolin state. Now the modified Smolin state is expressed in a general form:

$$\rho^{\text{noisy}} = \frac{1-p}{16} I^\otimes 4 + p \rho$$

(6)

where I stands for identity on one-qubit space, $p$ is scaling parameter which $p \leq \frac{2}{3}$ because of the intervention of Eve. The corresponding correlation function $E_{QM}$ is amended as follows:

$$E_{QM}(1, 1, 1, 1)(\rho^{\text{noisy}}(p)) + E_{QM}(1, 1, 1, 2)(\rho^{\text{noisy}}(p))$$

$$+ E_{QM}(2, 2, 1, 1)(\rho^{\text{noisy}}(p)) - E_{QM}(2, 2, 2, 2)(\rho^{\text{noisy}}(p)) = 2\sqrt{2}p$$

(7)

For $p \leq \frac{2}{3}$, the value of the Eq.[7] do not violate two-setting Bell-inequality of four qubits. Therefore, the existence of Eve will be detected while Alice tests the violation of the correlation function $E$ to Bell-inequality.

There may be another method for Eve to choose. Eve makes Bell measurement on qubits 3,4, prepares two copies same as the result, and dispenses one to Bob and Charlie. It makes that both of Eve and Alice have the same Bell states as shared between Bob and Charlie. Eve’s trick can also be detected if Alice simply tests whether the correlation function between qubit 1,2 violate Bell-inequality of two qubits or not. Then the security for the present quantum secret sharing is guaranteed.

C. Conclusion

This investigation introduces a quantum secret sharing scheme using four-qubit Smolin bound entangled state as private channel. The idea is very similar to Ekert’s 91 protocol about quantum key distribution [21]. By testing the violation of the four-qubit correlation function to two-setting Bell-inequality, the existence of eavesdropper can be detected.

Comparing with the QSS protocol via GHZ state in which a half of qubits must be discarded because of incorrect direction for measurement, the efficiency of the present scheme can reach 100% if the quantum channel is secure. Another QSS scheme with the help of bound entangled states is discussed in Ref. [18]. However, its efficiency can only approach 50% in principle. On other hand, because each Bell state carries two bits hidden classical information, for every copy of Smolin state Alice can send 2 bits shared by Bob and Charlie.

But the generalization of this quantum secret sharing scheme to multi-parties is not good as expected. In each Smolin state, the number of qubits in hand of Alice is equal to the number of parties sharing secret information. With the growing of the parties, the dimension of collective measurement is increasing, while collective measurement on multi-qubit system is more difficult to accomplish.

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E. References

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