On the finiteness of uniform sinks

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Abstract

We study the finiteness of uniform sinks for flow. Precisely, we prove that, for \( \alpha > 0 \) \( T > 0 \), if a vector field \( X \) has only hyperbolic singularities or sectionally dissipative singularities, then \( X \) can have only finitely many \((\alpha, T)\)-uniform sinks. This is a generalized version of a theorem of Liao [3].

1 Introduction

In this work, we give a generalized version of a theorem of Liao [3]. It could be seen as an extension of the remarkable Pliss’ theorem [4] in the setting of singular flows.

Let \( M \) be a compact smooth Riemannian manifold and \( X \) be a smooth vector field on \( M \). We know that \( X \) will generate a smooth flow \( \phi_t \). If \( X(\sigma) = 0 \), \( \sigma \) is called a singularity of \( X \). If \( \phi_t(p) = p \) for some \( t > 0 \) and \( X(p) \neq 0 \), \( p \) is called a periodic point. We use \( \text{Sing}(X) \) and \( \text{Per}(X) \) to denote the sets of singularities and periodic points.

The flow \( \Phi_t = d\phi_t : TM \rightarrow TM \) is called the tangent flow. Note that every periodic orbit has at least one zero Lyapunov exponent w.r.t. \( \Phi_t \). To understand the dynamics in a small neighborhood of a periodic orbit, Poincaré used the Poincaré return map: for any point in the periodic orbit, one takes a cross section at that point, then the flow defines a local diffeomorphism in a small neighborhood of the cross section. The dynamics of the flow in a small neighborhood of the periodic orbit can be understood by the dynamics of the diffeomorphism.

By extending this idea to the general non-periodic case, for any regular point \( x \) and any \( t \in \mathbb{R} \), one considers local normal cross sections at \( x \) and \( \phi_t(x) \), then the flow gives a local diffeomorphism between these two cross sections. Its linearization is the linear Poincaré flow \( \psi_t \), which is defined as the following: given a regular point \( x \in M \), consider a vector \( v \) in the orthogonal complement of \( X(x) \), one defines

\[
\psi_t(v) = \Phi_t(v) - \frac{<\Phi_t(v), X(\phi_t(x))>}{|X(\phi_t(x))|^2}X(\phi_t(x)).
\]

Note that \( \psi_t \) cannot be defined on the singularities.

*2000 Mathematics Subject Classification. 37D30
†D. Yang was partially supported by NSFC 11271152, Ministry of Education of P. R. China 20100061120098
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Given $\alpha > 0$ and $T > 0$, a periodic orbit $\Gamma$ is called an $(\alpha, T)$-uniform sink if there are $m \in \mathbb{N}$ and times $0 = t_0 < t_1 < t_2 \cdots < t_n = m\pi(\Gamma)$ ($\pi(\Gamma)$ is the period of $\Gamma$) satisfying $t_i - t_{i-1} \leq T$ for any $1 \leq i \leq n$ such that for any $x \in \Gamma$, one has

$$\prod_{i=1}^{n} \|\psi_{t_i-t_{i-1}}(\phi_{t_{i-1}}(x))\| \leq e^{-\alpha m\pi(\Gamma)}.$$ 

A singularity $\sigma$ is called sectionally dissipative if the following is true: when we list all the eigenvalues of $DX(\sigma)$ as $\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$, we have $\text{Re}(\lambda_i) + \text{Re}(\lambda_j) \leq 0$ for any $1 \leq i < j \leq d$. Here $d$ is the dimension of the manifold $M$.

Notice that for a sectionally dissipative non-hyperbolic singularity, the maximal of the real parts of its eigenvalues should be zero.

**Theorem A.** Let $\alpha > 0$ $T > 0$. If a vector field $X$ has only hyperbolic singularities or sectionally dissipative singularities, then $X$ can have only finitely many $(\alpha, T)$-uniform sinks.

Similar results for diffeomorphisms or non-singular flows were got by Pliss [4]. If $X$ has no singularities and $X$ has infinitely many $(\alpha, T)$-uniform sinks $\{\gamma_n\}$, then by using Pliss Lemma, for any $\gamma_n$, there is a point $x_n \in \gamma_n$ such that $x_n$ has its stable manifold of uniform size which is independent with $n$. From the fact that $M$ has finite volume, we can get a contradiction.

Liao [3] proved Theorem A with an additional assumption: $X$ is a star vector field. As $X$ is star, every singularity of $X$ is hyperbolic. If every singularity of $X$ is hyperbolic and $X$ has infinitely many $(\alpha, T)$-uniform sinks $\gamma_n$, then by his estimation, for each $n$, there is a point $x_n \in \gamma_n$ such that $x_n$ has its stable manifold with some uniformity (after the rescaling of the flow). So if $x_n$ is far away from singularities, we can get a contradiction. Thus we can assume that $\lim_{n \to \infty} x_n = \sigma$ for some singularity $\sigma$. Then Liao proved that $\sigma$ has a strong unstable one-dimensional manifold and for $n$ large enough, the basin of $x_n$ intersects the strong unstable manifold. Since a one-dimensional strong unstable manifold contains only two orbits, we can get a contradiction.

But in Liao’s argument, if $\sigma$ is not hyperbolic, we will encounter two difficulties:

- $\sigma$ may not have a strong unstable one-dimensional manifold. To solve this difficulty, we need to analysis the dynamics on the central manifold of $\sigma$.
- When $\sigma$ is hyperbolic, then we know that for $x$ close to $\sigma$,

$$\frac{|X(x)|}{d(x, \sigma)}$$

is uniformly bounded. Thus, Liao needed to consider a cone along the unstable direction. When $\sigma$ is non-hyperbolic, we need to consider some general cone-like region along the central manifold of $\sigma$. 

2
2 Preliminaries

2.1 The rescaled linear Poincaré flow and the stable manifold theorem

Let $X$ be a $C^1$ vector field. For each regular point $x$, one defines its normal space $N_x$ to be $N_x = \{ v \in T_xM : \langle v, X(x) \rangle = 0 \}$. Denote by

$$N = \bigcup_{x \in M \setminus \text{Sing}(X)} N_x.$$ 

$N$ is called the normal bundle of $X$. Note that $M \setminus \text{Sing}(X)$ may be not compact. Thus, $N$ may be defined on some non-compact set. We notice that $\psi_t$ is defined on $N$. Given any regular point $x \in M$, $t \in \mathbb{R}$ and any $v \in N_x$, we define

$$\psi^*_t(v) = \frac{\psi_t(v)}{\| \Phi_t(x) \|} = \frac{|X(x)|}{|X(\phi_t(x))|} \psi_t(v).$$

$\psi^*_t$ is called the rescaled linear Poincaré flow w.r.t. $X$.

Definition 2.1. Let $C > 0$, $\eta > 0$ and $T > 0$. A regular point $x \in M$ is called $(C, \eta, T)$-$\psi^*$-contracted, if there is a sequence of times $0 = t_0 < t_1 < \cdots < t_n < \cdots$ such that

- $t_i - t_{i-1} \leq T$ and $\lim_{n \to \infty} t_n = \infty$.
- $\prod_{i=1}^n \| \psi^*_{s_{i-1}}(\phi_{t_{i-1}}(x)) \| \leq C e^{-\eta n}$ for any $n \geq 1$.

One says that a regular point $x \in M$ is $(C, \eta, T)$-expanded if it is $(C, \eta, T)$-contracted for $-X$.

For a normed vector space $V$ and $r > 0$, denote by

$$V(r) = \{ v \in V : |v| \leq r \}.$$ 

For any regular point $x \in M$, we define the local normal manifold $N_x(\beta) = \exp_x(N_x(\beta))$ $(\beta > 0)$. The flow $\phi_t$ defines a local diffeomorphism from a small neighborhood of $N_x(\beta)$ to $N_{\phi_t(x)}(\beta)$, which is denoted by $P_{x,\phi_t(x)}$, and which is called the sectional Poincaré map.

Liao [3] had the following estimations on the size of stable manifolds. One can see [1, Section 2] for a geometric proof.

Lemma 2.2. Let $X$ be a $C^1$ vector field on $M$. Given $C > 0$, $\eta > 0$ and $T > 0$, there is $\delta = \delta(C, \eta, T) > 0$ such that for any $(C, \eta, T)$-$\psi^*$-contracted point $x$, one has $N_x(\delta|X(x)|)$ is in the domain of the sectional Poincaré map $P_{x,\phi_t(x)}$ for any $t \geq 0$, and

$$\lim_{t \to \infty} \text{diam}(P_{x,\phi_t(x)}(N_x(\delta|X(x)|))) = 0.$$ 

It follows that $N_x(\delta|X(x)|)$ is in the stable set of $x$ after a reparametrization. Although we don’t want to give the proof of Lemma 2.2 again, we would like to give some idea about the proof. First we can consider the fibered map $P_{t,x}(x) : N_x(\beta) \to N_{\phi_t(x)}(\beta)$, which is defined by $P_{t,x}(x) = \exp_{\phi_t(x)} \circ P_{x,\phi_t(x)} \circ \exp_x$, whose dynamics are conjugate to $P_{t,\phi_t(x)}$. The linearization of $P_{t,\phi_t(x)}$ is the linear Poincaré flow $\psi_t$. But since $X$ may contain
singularities, the linearized neighborhood is not uniform. Then we define the rescaling of $P_{x,\phi_t(x)}$ by

$$P_{x,\phi_t(x)}^*(v) = P_{x,\phi_t(x)}(X(x)|v) |X(\phi_t(x))|.$$

The linearization of $P_{x,\phi_t(x)}^*$ is $\psi_t^*$ and the linearized neighborhood is uniform; by a careful calculation, $DP_{x,\phi_t(x)}^*$ also have some uniform continuity properties (See [II]). By our assumption, if $x$ is $(C, \eta, T)$- contraction, then $x$ has its stable manifold of uniform size w.r.t. $P^*$: the proof follows from the classical case of diffeomorphisms (see [5, Corollary 3.3] for instance).

2.2 A lemma of Pliss type

**Lemma 2.3.** Given $C > 0$ and $0 < \lambda_1 < \lambda_2 < 1$, there is $N = N(C, \lambda_1, \lambda_2) \in \mathbb{N}$ such that: for any sequence of numbers $\{a_n\}$ satisfying the following properties:

$$\sum_{i=1}^{n} a_i \leq C + n\lambda_1, \quad \forall n \in \mathbb{N},$$

then there is $L \leq N$ such that

$$\sum_{i=1}^{n} a_{L+i} \leq n\lambda_2, \quad \forall n \in \mathbb{N}.$$

**Proof.** We choose $N$ such that $C + N\lambda_1 < N\lambda_2$. Given any sequence of numbers $\{a_n\}$ satisfying $\sum_{i=1}^{n} a_i \leq C + n\lambda_1$, $\forall n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, one has

$$\sum_{i=1}^{n} a_{m+i} \leq n\lambda_2.$$

This is because otherwise, for each $j$, there is $n_j$ such that $\sum_{i=1}^{n_j} a_{j+i} \geq n_j\lambda_2$. Thus $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} a_i \geq \lambda_2$. This fact contradicts the assumption.

Now we will show the existence of $L$, which is required to be less than $N$. If not, there is a sequence of numbers $\{a_n\}$ satisfying

$$\sum_{i=1}^{n} a_i \leq C + n\lambda_1, \quad \forall n \in \mathbb{N},$$

but for any $m$ satisfying

$$\sum_{i=1}^{n} a_{m+i} \leq n\lambda_2, \quad \forall n \in \mathbb{N}$$

one has $m > N$. We take a minimal $m$ with the above property. This implies $a_{m-1} > \lambda_2$. Inductively, we can have that

**Claim.** For any $j \leq m - 1$, $\sum_{k=0}^{m-1-j} a_{j+k} > (m-j)\lambda_2$. 

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Proof. We know that it is true for \( j = m - 1 \). If it is already true for \( j, j + 1, \ldots, m - 1 \), and it is not true for \( j - 1 \), then we will have

\[
\sum_{k=0}^{m-1-(j-1)} a_{j-1+k} \leq (m - j + 1)\lambda_2,
\]

\[
\sum_{k=0}^{m-1-\ell} a_{\ell+k} > (m - \ell)\lambda_2, \quad \forall \ell \in \{j, j + 1, \ldots, m - 1\}.
\]

This will imply that for any \( 0 \leq \ell \leq m - j - 1 \), one has

\[
\sum_{i=0}^{m-j} a_{j-1+i} \leq (m - j + 1)\lambda_2.
\]

By the definition of \( m \), we know that \( j - 1 \) also have the same property of \( m \). This will contradict to the minimality of \( m \).

As a consequence of the above claim, one has \( \sum_{i=1}^{m-1} a_i > (m - 1)\lambda_2 \geq N\lambda_2 \geq C + N\lambda_1 \). This gives a contradiction to the assumption of \( \{a_n\} \).

\[\square\]

3 The splitting of the singularity

Recall the sphere bundle \( S^1(M) \) which consists of all unit tangent vectors of the tangent bundle \( TM \):

\[S^1(M) = \{ v \in TM, |v| = 1 \}.\]

Thus, \( \Phi_t \) induces a continuous flow \( \Phi_t^1 \) on \( S^1(M) \), where

\[\Phi_t^1(v) = \frac{\Phi_t(v)}{|\Phi_t(v)|} \quad (v \in S^1(M)).\]

Define the following frame by

\[\mathcal{F}_2(M) = \bigcup_{x \in M} \{(u, v) : u, v \in T_xM, u \neq 0, u \bot v\}.\]

and the normalized frame by

\[\mathcal{F}^\#_2(M) = \bigcup_{x \in M} \{(u, v) : u, v \in T_xM, |u| = 1, u \bot v\}.\]

Define the flows \( \chi_t \) on \( \mathcal{F}_2(M) \) and \( \chi_t^\# \) on \( \mathcal{F}^\#_2(M) \) by the following way:

\[\chi_t(u, v) = \{ \Phi_t(u), \Phi_t(v) - \frac{<\Phi_t(u), \Phi_t(v)>}{|\Phi_t(v)|^2}\Phi_t(u) \}\]

\[\chi_t^\#(u, v) = \{ \frac{\Phi_t(u)}{|\Phi_t(u)|}, \Phi_t(v) - \frac{<\Phi_t(u), \Phi_t(v)>}{|\Phi_t(v)|^2}\Phi_t(u) \}\]

Note that \( \mathcal{F}^\#_2(M) \) is a complete metric space and \( \chi_t^\# \) is a continuous flow.
Lemma 3.2. If $\Lambda \subset S^1(M)$ is dominated w.r.t. $\chi_t$, then its closure $\overline{\Lambda}$ is also dominated w.r.t. $\chi_t$.

Proof. By taking a limit, we know that this lemma is true. \hfill $\square$

Lemma 3.3. For every singularity $\sigma$, given $t \in \mathbb{R}$, $\Phi_t(u)$ is $C^\infty$ w.r.t. $u \in T_\sigma M$ (although $X$ is only $C^1$).

Proof. This is true because that $\Phi_t$ is linear w.r.t. $u \in T_\sigma M$. \hfill $\square$

Lemma 3.4. For any $u \in S^1_{\sigma}M$ and $t \in \mathbb{R}$, one has

$$D_u \Phi_t^1 = \frac{\text{proj}_2 \chi_t(u, \cdot)}{|\Phi_t(u)|}.$$

Proof. For each $u \in S^1_{\sigma}M$, $T_u S^1_{\sigma}M$ can be identical with $N_u$, where $N_u = \{ v \in T_\sigma M : v \perp u \}$. $D_u \Phi_t^1$ is a map from $N_u$ to $N_{\Phi_t^1(u)}$, which can be got by the following way: for each $v \in N_u$, we need to project $\Phi_t(v)$ to $N_{\Phi_t^1(u)}$. This process gives $\text{proj}_2 \chi_t(u, \cdot)$. Since $\Phi_t(v)$ may not be unit, we need to do a scaling. This ends the proof. \hfill $\square$

Proposition 3.5. Let $C > 0$, $\eta > 0$ and $T > 0$. For a singularity $\sigma$, if there is a sequence of $(C, \eta, T)$-psi*-contracted points $\{ x_n \}$ satisfying $\lim_{n \to \infty} x_n = \sigma$, then $\sigma$ admits a dominated splitting $E \oplus F$ w.r.t. the tangent flow $\Phi_t$ and $\dim F = 1$. Moreover, any accumulation point of $\{ X(x_n)/|X(x_n)| \}$ in $S^1M$ is not in $E$.

Proof. By our assumptions, $\{ X(x_n)/|X(x_n)| \}$ is dominated w.r.t. $\chi_t$. So any accumulation point $u$ of $X(x_n)/|X(x_n)|$ is dominated w.r.t. $\chi_t$. Given $T > 0$, let $f = \Phi_t^1$. By the assumptions, there are $C > 0$ and $\lambda \in (0, 1)$ such that for any $n \in \mathbb{N}$, one has

$$\prod_{i=0}^{n-1} \| Df(f^i(u)) \| \leq C \lambda^n.$$

Fix some $\lambda_1 \in (\lambda, 1)$. By Lemma 2.3, there exists an infinite sequence $\{ n_i \}_{i \in \mathbb{N}}$ such that

$$\prod_{j=0}^{n-1} \| Df(f^j(f^{n_i}(u))) \| \leq \lambda_1^n, \forall \ n \in \mathbb{N}.$$

Choose $n_i, n_j$ such that $n_j > n_i$, $f^{n_i}(u)$ and $f^{n_j}(u)$ are close enough. Thus, there is $\delta > 0$ such that $f^{n_j-n_i}$ is a contracting map on $B(f^{n_i}(u), \delta)$. This implies that $f$ has a periodic point. Thus, there is $T' > 0$ and $u' \in S^1_{\sigma}M$ such that $\Phi_{T'}(u') = u'$ and $u'$ is dominated w.r.t. $\chi_t$.

Now we have that $\Phi_{T'}$ has the following form w.r.t. $\langle u' \rangle \oplus N_{u'}$

$$\begin{pmatrix} \Phi_{T'}(u') & 0 \\ A & \text{proj}_2 \chi_{T'}(u', \cdot) \end{pmatrix}.$$
This implies that $\Phi_T$ has a unique largest eigenvalue. From these facts, one can get the dominated splitting on $T_x M$.

Once we know that we have the dominated splitting $T_\sigma M = E \oplus F$ with $\dim F = 1$, we know that any point which is dominated w.r.t. $\chi_t$ cannot be in $E$. Thus, every accumulation point of $X(x_n)/|X(x_n)|$ is not in $E$. □

4 The intersection of local invariant manifolds

4.1 The central manifold at $\sigma$

In this subsection, we assume that $\sigma$ is a singularity and $T_\sigma M$ admits a dominated splitting $E \oplus F$ w.r.t. the tangent flow $\Phi_t$ and $\dim F = 1$. We will talk about the local dynamics around $\sigma$.

Since $T_\sigma M = E \oplus F$ with $\dim F = 1$ is a dominated splitting of $\Phi_t$, by the plaque family theorem of [2, Theorem 5.5], there is a local embedded one-dimensional manifold $\chi^+ \subset \chi^0$ in a small neighborhood of $\sigma$. If $E$ is uniformly contracting, then $\chi^+ \subset \chi^0$ in the following sense: there is a $C^1$ map $g : F \to E$ in $T_\sigma M$ satisfying $g(0) = 0$ and $Dg(0) = 0$, if we denote by $W^F_\sigma(\sigma) = \exp_x(g(r, r))$, we have that for any $\varepsilon > 0$, there is $\delta > 0$ such that $\chi^+ \subset \chi^0$.

We notice that $W^F(\sigma)$ may not be unique in general. $W^F(\sigma)$ has two separetrix, which are denoted by $W^{F, +}(\sigma)$ and $W^{F, -}(\sigma)$. We discuss $W^{F, +}$ only since they are in a symmetric position. If $E$ is uniformly contracting, then $W^{s, +}(\sigma)$ separates a small neighborhood of $\sigma$ into two parts: the upper part which contains $W^{F, +}(\sigma)$, and the lower part which contains $W^{F, -}(\sigma)$. If $W^{F, +}(\sigma)$ is Lyapunov stable in the following sense: for any $\varepsilon > 0$, there is $\delta > 0$ such that $\varphi_t(W^{F, +}_{\delta}(\sigma)) \subset W^{F, +}_{\varepsilon}(\sigma)$ for any $t > 0$, then the upper part will be foliated by locally strong stable foliations.

Lemma 4.1. If $W^{F, +}(\sigma)$ is contained in the unstable manifold of $\sigma$, then it is uniquely defined.

Proof. Under the assumptions, $F$ cannot be a contracting bundle. If $F$ is an expanding bundle, then the conclusion follows from the classical theorem about the existence of unstable manifolds.

So we can assume that $F$ is non-hyperbolic. In this case, since we have the dominated splitting $T_\sigma M = E \oplus F$, we have that $E = E^s$ is contracting. We can extend the bundle $E$ and $F$ in a small neighborhood $U$ of $\sigma$, which are still denoted by $E$ and $F$. For $\alpha > 0$, for any point $x \in U$, we define the cone field $C^E_\alpha(x) \subset T_x M$

$$C^E_\alpha(x) = \{v \in T_x M : v = v_E + v_F, \ v_E \in E(x), \ v_F \in F(x), \ |v_F| \leq \alpha|v_E|\}.$$

By reducing $U$ if necessary, one has

Claim. There are $\alpha > 0$, $T > 0$ and $\varepsilon > 0$ such that

1. For any $x \in U$, if $\phi_{[-T, 0]}(x) \in U$, then $\Phi_{-T}(x)(C^E_\alpha(x)) \subset C^E_\alpha(\phi_{-T}(x))$.

2. For any $x, y \in U$ such that $d(x, y) < \varepsilon$, if $\phi_{[-T, 0]}(x) \in U$ and $\phi_{[-T, 0]}(y) \in U$, and if $\exp_{\phi_{-T}(x)}^{-1}(y) \in C^E_\alpha(x)$, then $\exp_{\phi_{-T}(x)}^{-1}(y) \in C^E_\alpha(\phi_{-T}(x))$, and moreover $d(\phi_{-T}(x), \phi_{-T}(y)) > 2d(x, y)$.
Proof of the Claim. First at the singularity \( \sigma \), there is \( T > 0 \) and \( \alpha > 0 \) such that

- \( \Phi_{-T}(C_\alpha^E(\sigma)) \subset C_\alpha^E(\phi_{-T}(\sigma)) \),
- for any unit vector \( v \in C_\alpha^E(\sigma) \), \( |\Phi_{-T}(v)| > 2 \).

So by the continuity of \( E \), by reducing \( U \) if necessary, we have it is also true for any \( x \in U \). Thus Item 1 of the claim is true. Next, since the linearization of \( \phi_{-T} \) is uniformly continuous, we have that the existence of \( \varepsilon \) such that Item 2 of the claim is true.

4.2 The cone-like region

For \( \alpha > 0 \), one considers the cone

\[
C_\alpha^F = \{ v \in T_\sigma M : v = v_E + v_F, \, v_E \in E, \, v_F \in F, \, \text{and} \, |v_E| \leq \alpha|v_F| \}.
\]

and considers \( C_\alpha^F = \exp_\sigma(C_\alpha^F \cap T_\sigma M(\beta)) \).

We extend \( E \) and \( F \) continuously to a small neighborhood \( U \) of \( \sigma \). Let \( \alpha > 0 \) and \( \beta > 0 \) be given. Now we define a cone-like region \( D_\alpha^F(\beta) \),

\[
D_\alpha^F(\beta) = \{ x \in M : d(x, \sigma) < \beta, X(x) = X_F(x) + X_E(x), X_F(x) \in F(x), \, X_E(x) \in E(x), \, |X_E(x)| < \alpha|X_F(x)| \}.
\]

Lemma 4.2. Assume that \( DX(\sigma)|_E \) is non-singular. For any \( \delta > 0 \), there are \( \alpha > 0 \) and \( \beta > 0 \) such that for any \( y \in D_\alpha^F(\beta) \), one has \( \exp_y(\mathcal{N}_y(\delta|X(y)|)) \cap W^F(\sigma) \neq \emptyset \).

Proof. Without loss of generality, we can identify a small neighborhood \( U \) to be an open set in \( \mathbb{R}^d \), and \( \sigma \) to be the origin point 0. \( E \) and \( F \) are two subspaces perpendicular to each other. Thus \( W^F(\sigma) \) is a \( C^1 \) curve tangent to \( F \) at the origin. Denote by \( x = (x_E, x_F) \) for each point \( x \) in the small neighborhood. We assume that the flow generated by the vector field in a small neighborhood of \( \sigma \) is the solution of the following differential equation:

\[
\frac{dx_E}{dt} = A_E x_E + f_E(x), \quad \frac{dx_F}{dt} = A_F x_F + f_F(x),
\]

where \( f_E \) and \( f_F \) are higher order terms of \( |x| \), and the matrix
\[
\begin{pmatrix}
A_E & 0 \\
0 & A_F
\end{pmatrix}
\]

can be regarded as \(DX(\sigma)\). Under our assumptions, we always know that \(A_E\) is non-singular since \(E\) is dominated by \(F\).

Thus we can find a constant \(c > 0\) which depends only on \(A\) such that for small enough \(\beta > 0\), if \(d(x, \sigma) < \beta\) then \(|x_E| \leq c|X_E(x)|\).

Through an argument of basic geometry, if \(|X_E(x)| \leq \alpha|X_F(x)|\), then the lengths of \(E\) component of \(N_\alpha(tX(x))\) is greater than \(c_0t|X(x)|\) provided \(x\) close to \(\sigma\) enough for some uniform constant \(c_0 = c_0(\alpha) > 0\).

Now let \(\delta > 0\) be given. Choose \(\alpha > 0\) satisfying \(\alpha c/c_0 \leq \delta\). We also choose \(\beta > 0\) satisfying the conditions above. Let \(y \in D^E_\alpha(\beta)\). To prove \(\exp_y(N_y(\delta|X(y)|)) \cap W^F(\sigma) \neq \emptyset\), it is enough to show that the length of \(E\) component of \(\exp_y(N_y(\delta|X(y)|))\) is greater than \(|y_E|\). From the first part, we have

\[|y_E| \leq c|X_E(y)| \leq c_0|X_F(y)| \leq c_0|X(y)|.\]

From the second part, the size of \(E\) component of \(N_y(\delta X(y))\) is greater that \(c_0\delta|X(y)|\), which is greater than \(c\alpha|X(y)|\) and hence greater than \(|E_y|\).

**Lemma 4.3.** Let \(\eta > 0\) and \(T > 0\). Assume that \(\sigma\) is a singularity and there is a sequence of \((1, \eta, T)\)-\(\psi^*\)-contracted points \(\{x_n\}\) such that \(\lim_{n \to \infty} x_n = \sigma\). For any \(\alpha > 0\), there is \(L = L(\alpha) > 0\) such that for any \(L' > L\) and any \(\beta > 0\), there exists an integer \(N = N(\alpha, \beta, L')\) such that for any \(n \geq N\), \(\phi_{L,L'}(x_n) \in D^F_\alpha(\beta)\).

**Proof.** By taking a subsequence if necessary, we have that \(\lim_{n \to \infty} X(x_n)/|X(x_n)| = v \in T_\alpha M\). First by Proposition 3.5, \(\sigma\) admits a dominated splitting \(E \oplus F\) with respect to the tangent flow and \(\dim F = 1\). Given \(\alpha > 0\), by the domination and \(v \notin E\), there exists \(L = L(\alpha)\) such that for any \(L > L_0\), \(\Phi_t(v) \in D^F_\alpha\).

Let \(\beta > 0\) and \(L' > L\) be given. By the fact that \(x_n \to \sigma\), \(\frac{X(x_n)}{|X(x_n)|} \to v\) and the continuity of \(\Phi_t\), there is some positive integer \(N = N(\alpha, \beta, L')\) such that for any \(n \geq N\), we have \(d(x_n, \sigma) < \beta\), \(\Phi_{L,L'}(x_n)/|X(x_n)|) \subset C^F_\alpha\). The last relation implies that \(|X_E(x)| < \alpha|X_F(x)|\) for \(x \in \phi_{L,L'}(x_n)\) and implies that \(\phi_{L,L'}(x_n) \subset D^F_\alpha(\beta)\).

\[\square\]

5 The final proof of Theorem A

We will prove Theorem A in this section. Assume that we are under the assumptions of Theorem A. We will prove it by contradiction. We assume that the conclusion of Theorem A is not true. That is, there are infinitely many distinct \((\alpha, T)\)-sinks \(\{\gamma_n\}\) of \(X\). If the period of \(\gamma_n\) is bounded, then the limit point of \(\{\gamma_n\}\) in the Hausdorff topology should be a non-hyperbolic periodic orbit \(\gamma\). Then each periodic orbit close to \(\gamma\) cannot be a \((\alpha, T)\)-sink. This gives a contradiction.

Hence, we can assume that the period of \(\gamma_n\) tends to infinity. By Pliss Lemma [4], there is \(x_n \in \gamma_n\) which is \((1, \eta, T)\)-\(\psi^*\)-contracted for some \(\eta > 0\). Then by Lemma 2.2 there is \(\delta > 0\) such that \(N_{x_n}(\delta|X(x_n)|)\) is contained in the basin of \(\gamma_n\). As a corollary, the ball \(B(x_n, \delta|X(x_n)|)\) is also contained in the basin of \(\gamma_n\) by reducing \(\delta\) if necessary.
Without loss of generality, we can assume that \( \{x_n\} \) converges. If the limit point is a regular point, then the basin of \( \gamma_n \) will contains a uniform ball for each \( n \). This will contradict to the infiniteness. Thus, we can assume that \( \lim_{n \to \infty} x_n = \sigma \), where \( \sigma \) is a singularity.

By the assumptions, \( \sigma \) will be hyperbolic or sectionally dissipative. By Proposition 3.5, there is a dominated splitting \( T_\sigma M = E \oplus F \) w.r.t. the tangent flow, where \( \dim F = 1 \).

In any case we will have \( DX(\sigma)|_E \) is non-singular.

By the theory of invariant manifolds, we know the existence of \( W^F_\varepsilon(\sigma) \).

**We consider the case that \( F \) is expanded.** Let \( \delta = \delta(1, \eta/2, T) \) given by Lemma 2.2. By Lemma 4.2, for this \( \delta \), there exist \( \alpha > 0 \) and \( \beta > 0 \) such that for any \( y \in \mathcal{D}_\alpha^F(\beta) \), one has \( \exp_y(\mathcal{N}_y(\delta|x(y)|)) \cap W^F(\sigma) \neq \emptyset \). Lemma 1.3 gives the number \( L = L(\alpha, \beta) > 0 \). \( \phi_{|L,L'|}(x_n) \in \mathcal{D}_\alpha^F(\beta) \) for any given \( L' > L \) provided that \( n \) is large. So there exists \( C > 0 \) which depends on \( L \), such that \( y_n = \phi_L(x_n) \in (C, \eta, T) \)-\( \psi^* \)-contracted. By Lemma 2.3, there exists \( L' > L \), such that some points \( z_n = \phi_{L}(x_n) \) \( (L < \bar{L} < L') \) is \( (1, \frac{3}{4}, T) \)-\( \psi^* \)-contracted. By Lemma 2.2 and our choice of \( \delta \) above, whenever \( n \) is large enough, \( \exp_{x_n} \mathcal{N}_{x_n}(\delta|x(z_n)|) \) is in the stable domain of the sectional Poincaré map \( P_t \) for any \( t \geq 0 \). By Lemma 4.2 we have \( \exp_{x_n} \mathcal{N}_{x_n}(\delta|x(z_n)|) \cap W^F(\sigma) \neq \emptyset \). In other words, in \( n \) is large enough, the basin of \( \gamma_n \) intersects the \( W^F(\sigma) \). But \( W^F(\sigma) \) contains only two orbits. The two orbits can only go to at most two sinks by forward iterations. This gives a contradiction.

**Now we will the consider the case that \( F \) is not expanding.** By the domination, \( E = E^s \) is uniformly contracted. Thus the stable manifold \( W^s(\sigma) \) separates a small neighborhood of \( \sigma \) into two parts: the upper part which contains \( W^{F,+}(\sigma) \) and the lower part which contains \( W^{F,-}(\sigma) \). By taking a subsequence if necessary, we assume that \( \{x_n\} \) accumulates \( \sigma \) in the upper part.

As in [6], there are two cases for \( W^{F,+}(\sigma) \):

1. \( W^{F,+}(\sigma) \) is Lyapunov stable in the following sense: for any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( \varphi_t(W^F_{\delta,+}(\sigma)) \subset W^F_{\varepsilon,+}(\sigma) \) for any \( t > 0 \).

2. \( W^{F,+}(\sigma) \) is contained in the unstable manifold of \( \sigma \).

In Case 1, the upper half part of the small neighborhood will be foliated by a strong stable foliation. This means that some point in \( W^{F,+} \) is contained in the stable manifold of some \( \gamma_n \). This contradicts to the local invariance of \( W^{F,+}(\sigma) \).

For Case 2, by Lemma 4.1 \( W^{F,+} \) is uniquely defined. The argument will be similar to the case when \( F \) is expanding. For completeness, we repeat it again.

Let \( \delta = \delta(1, \eta/2, T) \) be the number given by Lemma 2.2. By Lemma 4.2, for this \( \delta \), there exist \( \alpha > 0 \) and \( \beta > 0 \) such that for any \( y \in \mathcal{D}_\alpha^F(\beta) \), one has \( \exp_y(\mathcal{N}_y(\delta|x(y)|)) \cap W^F(\sigma) \neq \emptyset \). Lemma 1.3 gives the number \( L = L(\alpha, \beta) > 0 \). \( \phi_{|L,L'|}(x_n) \in \mathcal{D}_\alpha^F(\beta) \) for any given \( L' > L \) provided that \( n \) is large. So there exists \( C > 0 \) which depends on \( L \), such that \( y_n = \phi_L(x_n) \in (C, \eta, T) \)-\( \psi^* \)-contracted. By Lemma 2.3, there exists \( L' > L \), such that some points \( z_n = \phi_{L}(x_n) \) \( (L < \bar{L} < L') \) is \( (1, \frac{3}{4}, T) \)-\( \psi^* \)-contracted. By Lemma 2.2 and our choice of \( \delta \) above, whenever \( n \) is large enough, \( \exp_{x_n} \mathcal{N}_{x_n}(\delta|x(z_n)|) \) is in the stable domain of the sectional Poincaré map \( P_t \) for any \( t \geq 0 \). By Lemma 4.2 we have...
exp_{z_n}(N_{z_n}(\delta|X(z_n)|)) \cap W^{F,+}(\sigma) \neq \emptyset. In other words, if n is large enough, the basin of \gamma_n intersects \ W^{F,+}(\sigma). W^{F,+}(\sigma) is just one orbit and uniquely defined. Thus it can only go to at most one sink by forward iterations. This gives a contradiction. The proof of Theorem A is complete.

Acknowledgements. We would like to thank Shaobo Gan for the knowledge from him about singular flows, especially on Liao’s work.

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