Recommendation Algorithms that Increase Access to Influencers in a Network

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Abstract

This paper introduces and studies the novel problem of improving fairness in networks defined as access to influencers. In particular, the paper makes several contributions:

a) It studies a novel problem of network fairness that considers equitability of access to influencers, nodes that influence access to opportunity for others in the graph;
b) A new fairness index based on minimum path length to the influencer set is proposed to quantify this notion of network fairness and is appropriately normalized through a Barabasi-Albert random graph; and
c) Modifications to existing node recommendation algorithms are proposed to increase the new fairness index.

Our approach recommends nodes using standard node recommendation algorithms that are based on the number of triangles between the source and target node with probability $P$. With probability $1-P$, it introduces weak ties and diversity in the network by recommending nodes using an importance sampling algorithm. This sampling algorithm is based on a polynomial function of degree of the target node and its distance from the influencer set. Through extensive simulations on real-world network data, we show that it is feasible to measure fairness and achieve more fairness through the new recommendation algorithm. We also study and show the robustness of the algorithm to different parameter choices and provide insights on when to use the different importance sampling methods based on the structure of the network.
1. Introduction

Networks are ubiquitous and influence our lives in significant ways. Often these networks have a small set of nodes that are highly influential and having access to these nodes is a significant advantage. For instance, on a social media site like LinkedIn, assuming everything else in terms of skills and other characteristics to be the same, access to decision makers at large corporations can increase the likelihood of obtaining jobs and other opportunities like sales. On Twitter and Facebook, access to influencers can help businesses market messages more easily and extensively. In academic networks, access to influential researchers can increase the likelihood of receiving a favorable recommendation. When deciding to build a new road, easy access to the more central roads can lead to better commerce and trade for the local township.

To ensure a level playing field for all nodes in a network, we must ensure nodes in a graph have similar access to influencers as much as possible. However, in many real world applications, new edges are formed via recommendation systems that are based on network attributes like triangle closings (number of mutual friends) and node characteristics (ex. demographics, industry, school, interests, etc). While these systems are great at optimizing the number of edges that are formed in the network over time, they create unintentional biases in the system where some nodes have an unfair advantage of gaining more access to opportunities and utilities through influencers. The algorithms that power these recommendation systems further exacerbate the situation due to the attributes they rely on. For instance, on LinkedIn, we ideally want a system where professionals with the same skills and qualifications have equal access to opportunities. However, due to the network bias that is created, this is hardly the case. Professionals who have large personal networks get connected to influential people more easily than those who are new to LinkedIn or do not have a strong network. Such network effects eventually create bias in who has access to opportunities in the future.

There is a need to change the algorithms powering these recommendations to reduce this network bias such that everyone is allowed equal access to opportunities based on what they know and not who they know. We show that it is indeed possible to do so by introducing some randomness in a controlled fashion to existing node recommendation algorithms. One way to view the randomness is by thinking of it as the introduction of some weak ties in the graph that increases access to influential nodes for more nodes in the network. For instance, seminal research by Mark Granovetter (Granovetter, 1973) showed that introducing weak ties in a network of professionals increases the likelihood of a node finding a job. To the best of our knowledge, a systematic study of introducing this diversification into node
recommendation algorithms in the context of networks to increase fairness in terms of access to influencers has not been studied before. Since we don’t have access to real recommendation systems, this paper uses simulation on real network datasets from various applications to study the problem and illustrate our methodology.

It is important to study the problem of fairness due to the societal conditions that prevail in the world today. There are people who are otherwise qualified but are not necessarily obtaining the opportunities they deserve because they may not always have the right connections. Social media plays a significant role in increasing unfairness and a large part of that is due to the algorithms that power the recommendation systems. These systems often aim at optimizing these algorithms to increase engagement and eventually revenue via advertising which happens through increasing the volume of connections. This unintentionally ends up increasing unfairness for nodes, especially those who do not know influencers and who are not part of social circles that include them. This project investigates one way of mitigating this bias in recommendation algorithms. By doing so, we hope people will not be penalized based on who they know and will get rewarded purely based on what they know. This will make our society much more equitable even as the influence of social media and digitization keeps increasing.
2. Related Work

Social networks are not always fair. Whether it be Facebook, Instagram, LinkedIn, a sales network, or anything else, there are always some users that have an advantage over others simply because of who they are connected to. In fact, there has been a recent study done by the MIT Technology Review on Facebook and how their ad algorithm discriminates by race by recommending low-paying jobs more frequently to Mexicans and Hispanics than Caucasians (Hao, 2019). There have also been studies on the bias in LinkedIn machine learning models and how the recommendation systems that LinkedIn uses are inherently biased (Saint-Jacques, G., Sepehri, A., Perisic, I., & Li, N, 2020).

The above are only some documented examples of unfairness in networks; there are countless others. Although it is hard to completely eradicate all kinds of unfairness in networks, we can certainly reduce them, especially those that arise due to algorithms that are used to recommend new edges. Not much research has been done in this area, but this is an important problem. This paper attempts to find ways of reducing unfairness in a network by modifying the underlying recommendation algorithms.

The mathematical model used to modify the recommendation algorithm in this paper is related to the model studied by Watts and Strogatz (W-S). In their paper (Watts & Strogatz, 1998), W-S present a network where there are many links that form due to homophily and triadic closure, while also presenting weak long-range ties to properly simulate the small-world phenomenon. It was the first model that showed how a small amount of randomness can lead to more connections in a social network and reduce the diameter of the network. However, the main motivation of W-S was to perform efficient decentralized search - finding the best way to efficiently transmit a message to a target node from a source node using the existing network. Our focus in this paper is to recommend new nodes to existing nodes and in doing so ensure equal access to influencers (set of target nodes). Our mathematical model of introducing controlled diversity is also different from W-S as we use a function of both degree of a node and its distance to the influencer set. We show that this function leads to significant improvement in fairness.

There has also been research on mechanisms that can reduce conflict by increasing social welfare (Judd, Kearns & Vorobeychik, 2011). Researchers posit a model where kings provide “side-payments” to pawns which reduces conflict and increases social welfare. In our case, we can think of influencers as kings and all other nodes as pawns. As a node gets connected to a new node that reduces its distance from influencers, it receives a side-payment.

Additionally, there has been research done on how increasing a network’s structural power can increase collective fairness (Santos FP, Pacheco JM, Paiva A & Santos FC, 2017). However, while collective fairness may have increased, individual fairness did not, as this method made individuals meet
the same partner in multiple groups and gave people very few opportunities for influencers to be in an user’s network. Our paper aims to not only increase collective fairness, but individual fairness as well.

Another seminal paper that studied the problem of increasing weak ties in networks was *The Theory of Weak Ties* (Granovetter, 1973). Granovetter did a survey in Boston and empirically showed that adding weak ties increased the likelihood of nodes getting connected to influential people (job posters) and finding a job. However, this paper did not study a method to improve recommendations to increase such fairness; it only showed that the existence of such ties was effective.

Before we can increase fairness, it is important to have a principled method to measure it. There have been many solutions proposed to measure fairness in networks. These include the Atkinson Index (Saint-Jacques, G., Sepehri, A., Perisic, I., & Li, N, 2020) and using various other algorithms in networks (Mehrabi, Morstatter, Saxena, Lerman & Galstyan, 2019; Mehrabi, N., Morstatter, F., Saxena, N., Lerman, K., & Galstyan, A., 2019). However, the fairness measure proposed in this paper is different from previous ones; it measures the shortest distance to an influencer set in a network and finds the fairness of a graph by taking the average of the top \( t \% \) of node fairesses. While there has been research done on determining influencers in a network (Aman Ullah, Bin wang, Jinfang Sheng, Jun Long, Nasrullah Khan, 2021; J. Dai et al. 2019; B. Rozemberczki, R. Davies, R. Sarkar and C. Sutton, 2018), our approach goes a lot further. It calculates a scale free fairness measure using the influencer set, and improves the fairness in a network via novel node recommendation algorithms.
3. Background

We will use graphs and networks interchangeably in this paper. Graphs are used to represent many different scenarios, including social networks, road networks, power grid networks, among others.

To measure the influence of a node on the network, we use betweenness centrality as our measure of centrality. It measures the proportion of times a node acts as a bridge along the shortest path between two other nodes (Brandes, 2001). The equation for betweenness centrality for a node \( v \) is as follows:

\[
 c(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}
\]

where \( \sigma_{st} \) is the total number of shortest paths from a source node \( s \) to a target node \( t \) and \( \sigma_{st}(v) \) is the number of those paths that pass through \( v \) (Freeman, 1977). In other words, it is the proportion of shortest paths in the network that involve \( v \).

We now describe our novel fairness measure introduced in this paper. Fairness is defined to be the minimum shortest path to the influencer set. Using Dijkstra’s algorithm (Dijkstra, 1959), the fairness of a node can be measured to be:

\[
\text{fairness}(v) = \min\{s \in S\} \text{dist}(v, s)
\]

where \( S \) is the influencer set and \( v \) is a node in the graph.

The fairness of a graph is defined to be the average of the top \( t\% \) of node fairness values. This can be modeled by the equation:

\[
\text{fairness}(G) = \text{top } t\% \{v \in V \text{ and fairness}(v) \neq \infty\} \text{ fairness}(v)
\]

where \( G \) is the graph in question and \( V \) is the set of vertices in the graph not including influencers. We assume the graph \( G \) is connected, otherwise the fairness would have been returned as infinity. For disconnected graphs, we can still study fairness using this notion but we describe this case separately in the experimental section.

To normalize the fairness measure above to make it independent of the number of nodes and edges in the observed graph (scale free), we compute the fairness of a random graph having the same number of nodes and edges as the observed graph. The two methods we used in this paper to produce random graphs are the Erdos-Renyi graph and the Barabasi-Albert graph.

The Erdos-Renyi graph connects nodes at random with every edge being included in the graph with probability \( p \). The probability for generating a graph with \( n \) nodes and \( m \) edges can be modeled by:
where as probability $p$ increases, it becomes more likely to add more edges to the given graph (Erdos & Renyi, 1959).

The Barabasi-Albert graph begins with an initial connected network of $n$ nodes, where new nodes are added to existing nodes with probability $p_i$ where $p_i$ is the probability that a new node is connected to node $i$. This is modeled through the equation:

$$p_i = \frac{k_i}{\sum_j k_j}$$

where $k_i$ is the degree of node $i$ and the baseline sum is made over all pre-existing nodes $j$ (Barabasi & Albert, 1999).
4. Technical Approach

We will assume the given graph has nodes with the same attributes and the edges represent the connections among these nodes. In other words, the likelihood of any pair of nodes connecting is exactly the same in terms of attributes but they differ only because of graph structure (e.g., due to differences in the number of mutual edges).

There are two main parts to our technical approach:

1) finding an appropriate measure for fairness; and
2) finding an approach to modify the node recommendation algorithms that are used in practice such that the modification introduces more fairness as measured by the index above 1).

4.1 - Finding an Appropriate Measure for Fairness

This part has three main methods:

1) finding the influencers in the graph using graph theoretic measures
2) finding the fairness measure of each node and of the overall graph; and
3) normalizing the graph fairness measure using a random graph to obtain a fairness index that makes it possible to compare fairness after adjusting for the varying number of edges and nodes in the graph.

4.1.1 - Finding Influencers

We find influencers by calculating the betweenness centrality of every node in the graph. We take the top $k$ percent of nodes as our influencers.

4.1.2 - Finding The Fairness Measure

We first find the fairness of a single node and then aggregate it to compute the fairness of the entire network. If a path exists from the node to the influencer set, we define node fairness as the shortest path of the node to the influencer set (computed using the Djikstra’s algorithm). If no path exists from a node to the influencer set, individual node fairness is not defined. This only happens when the graph is disconnected. excluding the nodes that don’t have any paths to any influencers for a connected graph.
4.1.2.2- Getting Fairness of Total Graph

For a connected graph, network fairness is defined as the average of the top \( t \) percent of node fairesses. If the graph is disconnected, some nodes may not have any path to the influencer set. In this case we define fairness as the proportion of nodes that cannot reach the influencer path and we will use this as a measure of fairness: the higher the proportion, the more unfair the graph is.

4.1.3- Calculating the Fairness Index

Since we are working with graphs that have different numbers of nodes and edges, we have to normalize our fairness values to ensure it is scale free for connected graphs (the fairness measure for disconnected graphs is already scale free). We do so by using two random graphs: Erdos-Renyi and Barabasi-Albert.

4.2 Recommendation Algorithms to Increase the Fairness Index in a Social Network

The main idea of our proposed recommendation algorithm is motivated by a similar idea in the Watts-Strogatz model in the context of the small-world experiment and decentralized search. The high level mathematical model behind our approach is as follows. For a given node \( n \), with some probability \( P \), we recommend a node based on the number of mutual friends between \( n \) and the candidate node (the larger the mutual friends, the more likely we are to recommend the node). With probability \((1-P)\), we add some randomness and create a weak tie by forming a new edge with a randomly selected target node. We also introduce a modification where the weak tie is selected via an importance sampling algorithm that is proportional to some function based on degree and distance to the influencer set of the candidate node. More details are provided below.

4.2.1 - Details

For a given node \( n \), consider the array \( M[n] = \{(m, i(m), w(i(m)))\} \). Here, \( m \) denotes the target candidate nodes that \( n \) is not yet connected to. For each such node \( m \), \( i(m) \) denotes the number of mutual edges between \( n \) and \( m \) and \( w(i(m)) \) denotes the weight we have for \( m \). We do not include the influencer set in \( M[n] \) to avoid trivial solutions.

Calculate \( w(i(m)) \) values

We want \( w(i(m)) \) to be an increasing function of \( i(m) \) because the more mutual friends there are between \( n \) and \( m \), the more likely they are to connect and hence we should choose to recommend it with a
higher chance. One popular w function that is used is the sigmoid function $q(i) = 1/(1 + e^{a-i})$ where $a$ is a constant. Since $q(0) = 1/(1 + e^{a})$ is a constant, we can choose

$$w(i) = q(i)/q(0) = (e^{-a} + 1)/(e^{-a} + e^{-i}) \approx e^i.$$ We make this approximation assuming $a$ is a very large number. Hence, $w(i) = e^i$.

Now consider another array for a given node $n$ as $R[n] = \{ m, D(m), d(m), wr(a, b) \}$. Here, $D(m)$ is the degree of $m$, $d(m)$ is the minimum distance of $m$ to the influencer set, $wr(a, b)$ is the weight function written as $wr(a, b) = D(m)^a/d(m)^b$; $a$ and $b$ are non-negative constants. We will explain the reasoning behind choosing this weight function later below.

**Algorithm Details**

With probability $P$, we recommend a node for a given visit node $n$ by sampling a node from $M$ with weights proportional to $w(i)$’s and with probability $(1-P)$ we recommend a node by sampling from $R$ with weights proportional to $wr(a, b)$. Note that $wr(0,0) = 1$. Just like in the Watts-Strogatz model, with probability $P$, we recommend a node that is close to $n$’s neighborhood and with probability $(1-P)$, we select a node at random (assuming $a=b=0$). These random nodes shorten the path to the influencer set and create more fairness. Also for simplicity and efficiency in our simulations, we always connect to the recommended node. In practice, this does not happen and recommendations have different connection probabilities that are not constant. For instance, the connection probability to a node with zero mutual friends is typically lower than connecting to a node with ten mutual friends. Since our goal in this paper is only to study the impact of adding diversity to fairness, we can make this assumption without loss of generality. For instance, we could have assumed connections when sampling a node at random with probability $(1-P)$ happens with some probability less than 1, but this is equivalent to choosing some other value of $P$ and setting the connection probability to 1 throughout. In the end, what really matters is the budget we have in terms of adding nodes at random and the fairness index it produces.

Like in the Watts-Strogatz model, diversity can be introduced in a controlled manner. We use two intuitions for this:

1) Sampling nodes with higher degrees with higher weight is helpful since they are likely to create more paths to the influencer set; and

2) Given two nodes that have the same high degree, the one that is closer to the influencer set is more likely to increase fairness.

However, it is not clear what is better: a very high degree node far away from the influencer set (e.g., a node with a degree of 100 that is a distance of 3 from an influencer) versus a less high degree node closer to the influencer set (e.g., a node with a degree of 50 that is a distance 1 from an influencer). In our
simulations, we test multiple $a$ and $b$ values, which will be described in further detail in the experiments section.

This algorithm is illustrated below in pseudocode:

```
Parameters: $N$ (number of visits), $a$, $b$, $P$, $inf$ (influencer array), $G$ (graph)

1 $g_{new} = G$
2 nvisits = 0
3 while(nvisits < $N$):
4     select random node $n$
5     generate $M(n)$
6     generate $R(n)$
7     with probability $P$:
8         sample an edge from $M$
9     else with probability 1-$P$:
10        sample an edge from $R$
11    $g_{new} = g_{new} \cup$ new edge
12    nvisits+=1

function generateM($n$):
1     get nodes that $n$ is not connected to (excluding influencers)
2     compute mutual friends with $n$ for each node
3     compute weights for each node using $e^i$ where $i$ is the number of mutual friends

function generateR($n$):
1     get nodes that $n$ is not connected to (excluding influencers)
2     compute degree and the minimum distance to an influencer of each node
3     sample with weights equal to $D^a/d^b$
```
5. Experiments

We conducted simulations on multiple datasets to show the efficacy of our methods: a Facebook social circles graph (McAuley & Leskovec, 2012), a LastFM graph (Rozemberczki & Sarkar, 2020), and a road graph (Rossi & Ahmed, 2015). The first two datasets are connected graphs while the last dataset is a disconnected graph. Each of these data-sets has a unique network structure and provides complementary insights to our methodology. We also conduct robustness study for choice of influencer set.

Figure 1. Visualization of Facebook network using NetworkX draw feature

Figure 2. Visualization of LastFM network using NetworkX draw feature
From Figure 1, it is apparent that in the Facebook data there are several dense communities loosely connected by some weak ties. From Figure 2, we observe that for the LastFM data there is one central community with some nodes on the periphery. Figure 3 shows the road network is a sparse and disconnected graph.

5.1- Experimental Setup

All random graphs and visualizations were generated using python module NetworkX v2.5, and all other graphs were created with Python module Matplotlib v3.4.2.

For each data set, we ran the $D^a/d^b$ algorithm for the following pairs of $a$ and $b$:

- $a = 0, b = 0$ (completely random)
- $a = 1, b = 0$ (proportional to degree)
- $a = 1, b = 1$ (proportional to degree/distance(D/d))
- $a = \frac{1}{2}, b = \frac{1}{2}$ (proportional to $\sqrt{D/d}$)
- $a = 0, b = 2$ (proportional to $1/d^2$)
- $a = 2, b = 2$ (proportional to $(D/d)^2$)
- $a = 1, b = 2$ (proportional to $D/d^2$)

We also calculated the fairness index of nodes by using the Barabasi-Albert random graph since it converges faster than the Erdos-Renyi random graph.
As can be seen through Figure 4, the Barabasi-Albert random graph converges faster than the Erdos-Renyi graph, and therefore provides a better normalizing value for calculating the fairness index.
5.2 - Facebook Data

In each of these graphs, the lower the fairness index, the better the fairness in the graph since there are shorter paths to the influencer relative to the Barabasi-Albert graph. Therefore, in Figure 5, we observe that degree is the best algorithm for the Facebook graph as it is the consistently the lowest line on the graph, and \(D/d^2\) and \((D/d)^2\) are close seconds.

We also observe that each of the algorithms have the same value for \(P = 1\). This is because the algorithms at that point are completely based on mutual friends and not randomness. When decreasing \(P\), meaning there is a higher chance of randomness in the network, the fairness index is decreasing as expected.

To investigate why degree was the best algorithm for the Facebook graph, we made a scatterplot comparing degree and distance to the influencer set.
From Figure 6, we can see how very high degree nodes are not close to the influencers and hence $D/d$ selects them with very low probability, hurting the fairness performance of $D/d$. On the other hand however, there are a significant number of nodes with high-degrees and hence selecting based on degree leads to creation of more paths to the influencers.
5.3 - LastFM Data

![Fairness Index Graph](image)

**Figure 7.** Changes in Fairness with Different Recommendation Algorithms for Last FM (Lower Fairness Index means More Fairness)

From Figure 7, we observe that $(D/d)^2$ is clearly the best algorithm for the LastFM data. To investigate why, we made a scatterplot comparing degree and distance.

![Degree vs 1/Distance Scatterplot](image)

**Figure 8.** Scatterplot of LastFM Data with Degree on X-axis and 1/Distance on y-axis
In Figure 8, we can see how there are some high degree nodes that are farther away from the influencers while others are closer. Adding randomness just proportional to degree would select both nodes with that degree with the same probability regardless of their distance to the influencer. This is why selecting randomness proportional to \( D/d \) is the most accurate since it penalizes nodes that are far away from influencers despite having a high-degree.

The reason why \((D/d)^2\) seems to be the best algorithm is because it looks like the higher the power of \( D/d \), the more the function penalizes the nodes that are far away from influencers, and the better the fairness value it returns.

### 5.4 - Studying the Impact of Changing Influencers on the Fairness Index

We manually changed the influencers of the LastFM graph to show the robustness of this algorithm. These influencers had very low betweenness centralities. This is representative of situations where influencers who don’t have a big social media presence are still influential in the real world.

![Comparing Degree vs 1/Distance(Last FM)](image)

**Figure 9.** Scatterplot of LastFM Data with Manually Changed Influencers

From Figure 9, we can see how even with a change in influencers and a drop in fairness, the same pattern emerges where there are high-degree nodes both close to influencers and far away.
5.5 - Road Network Data

Unlike previous data sets examined, it is apparent from Figure 3 that the road network has many disconnected components, which drastically affects the results of our different recommendation systems. Therefore instead of measuring fairness as the minimum distance to an influencer, we looked at the proportion of nodes that didn’t have a path to the influencer as a function of different $P$-values.
Figure 11. Looking at Different Recommendation Algorithms for a Road Network

From Figure 11, we can see how increasing the amount of randomness in the network overall decreased the percentage of nodes that didn’t have a path to the influencer, and hence increased fairness. This data set shows how the algorithm can work with not just connected graphs, but disconnected ones as well.

5.6 - Summary of Experimental Results

Overall, our main findings were:

1) As we increase the amount of randomness in the network (decreasing $P$-value), the overall fairness in the network in terms of access to opportunity increases.

2) Overall, it seems like $(D/d)^2$ is the best recommendation algorithm to increase fairness in the network, although it is specific to the data-set at hand.

3) There is robustness in the algorithm with the number of influencers and whether the graph is connected or not; although the overall fairness deteriorates, it still follows the same pattern.
6. Conclusion

In this paper, a new way of measuring fairness was introduced. Fairness was defined to be the shortest distance from a node to any of the influencers in the graph, and a fairness index was calculated in respect to the Barabasi-Albert random graph. Through different recommendation algorithms, this paper showed how increasing the amount of randomness in a network increases fairness and gives users opportunities to connect with influencers. It also showed that overall \((D/d)^2\) is the best algorithm for increasing fairness and that this algorithm is robust with changes in influencers and whether a graph is connected or not.

In terms of future directions of this paper, we will work to scale the method discussed in this paper to larger graphs that are more representative of current social networks. We will work on other measures of centrality for identifying the influencers in a network and see how that impacts the results.

We also want to simulate the activeness of a user and take that into account when running simulations. We assume that all nodes are equally likely to visit the network in our simulations; however, in reality, it depends on how active an user is in the network that determines their frequency in visits. In general, nodes with higher degrees tend to visit more often. Additionally, in this paper we assumed that the number of nodes in a graph is constant. However in reality, users are constantly joining and leaving social networks, which will affect the total fairness; we want to study this in future research. We also want to look at other importance sampling functions and see if any of those increase the fairness more than the \(D^\alpha/d^\beta\) algorithm.

Lastly, we want to run our experiments on real recommendation systems by working at a social media company. Such experiments would also help us understand the opportunity cost in terms of engagement loss and revenue as we introduce different amounts of fairness.
References

Aman Ullah, Bin Wang, Jinfang Sheng, Jun Long, Nasrullah Khan. (2021). "Identification of Influential Nodes via Effective Distance-based Centrality Mechanism in Complex Networks", *Complexity*, vol. 2021, Article ID 8403738, 16 pages.

B. Rozemberczki and R. Sarkar. (2020). Characteristic Functions on Graphs: Birds of a Feather, from Statistical Descriptors to Parametric Models. *Proceedings of the 29th ACM International Conference on Information and Knowledge Management (CIKM '20)*. pp. 1325-1334.

B. Rozemberczki, R. Davies, R. Sarkar and C. Sutton. (2018). GEMSEC: Graph Embedding with Self Clustering.

Barabási, A.-L., & Albert, R. (1999). Emergence of scaling in random networks. *Science*, 286(5439), 509-512.

Brandes, U. (2001). A Faster Algorithm for Betweenness Centrality. *Journal of Mathematical Sociology*, 25(2), 163-177.

Dijkstra, E. W. (1959) “A Note on Two Problems in Connexion with Graphs.” *Numerische Mathematik*, vol. 1, no. 1, pp. 269–271.

Erdős, P.; Rényi, A. (1959). "On Random Graphs. I". *Publicationes Mathematicae*. 6: 290–297.

Freeman, L. (1977). A Set of Measures of Centrality Based on Betweenness. *Sociometry*, 40(1), 35-41.

Granovetter, M. S. (1973). The strength of weak ties. *American Journal of Sociology*, 78(6), 1360-1380.

Hao, K. (2019, April 5). Facebook's ad-serving algorithm discriminates by gender and race. *MIT Technology Review*. https://www.technologyreview.com/2019/04/05/1175/facebook-algorithm-discriminates-ai-bias/
J. Dai et al. (2019). "Identifying Influential Nodes in Complex Networks Based on Local Neighbor Contribution," in IEEE Access, vol. 7, pp. 131719-131731.

J. McAuley and J. Leskovec. (2012). Learning to Discover Social Circles in Ego Networks. NIPS, pp. 1-9.

Jian Kang, Jingrui He, Ross Maciejewski, and Hanghang Tong. (2020). InFoRM: Individual Fairness on Graph Mining. Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining. pp. 379–389.

Judd S., Kearns M., Vorobeychik Y. (2011). Behavioral Conflict and Fairness in Social Networks. In: Chen N., Elkind E., Koutsoupias E. (eds) Internet and Network Economics. WINE 2011. Lecture Notes in Computer Science, vol 7090. Springer, Berlin, Heidelberg.

Mehrabi, N., Morstatter, F., Saxena, N., Lerman, K., & Galstyan, A. (2019, September 17). A Survey on Bias and Fairness in Machine Learning. arXiv: 1908.09635.

Saint-Jacques, G., Sepehri, A., Perisic, I., & Li, N. (2020, February 14). Fairness through Experimentation: Inequality in A/B Testing as an approach to responsible design. arXiv: 2002.05819.

Santos FP, Pacheco JM, Paiva A, Santos FC. (2017). Structural power and the evolution of collective fairness in social networks. PLOS ONE 12(4): e0175687.

Watts, D., Strogatz, S. (1998). Collective dynamics of 'small-world' networks. Nature 393, 440–442.