GEOMETRIC CHARACTERIZATIONS OF BÄCKLUND TRANSFORMATIONS OF SINE-GORDON TYPE

JEANNE N. CLELLAND AND THOMAS A. IVEY

Abstract. We consider several properties commonly (but not universally) possessed by Bäcklund transformations between hyperbolic Monge-Ampère equations: wavelike nature of the underlying equations, preservation of independent variables, quasilinearity of the transformation, and autonomy of the transformation. The goal of this paper is to show that, while these properties all appear to depend on the formulation of both the underlying PDEs and the Bäcklund transformation in a particular coordinate system, in fact they all have intrinsic geometric meaning, independent of any particular choice of local coordinates. The problem of classifying Bäcklund transformations with these properties will be considered in a future paper.

1. Introduction

The classical Bäcklund transformation for the sine-Gordon equation

\( \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \sin z \)  (1.1)

is a system of two partial differential equations for unknown functions \( u(x, y) \) and \( v(x, y) \):

\[
\begin{align*}
    u_x - v_x &= 2\lambda \sin \left( \frac{u + v}{2} \right), \\
    u_y + v_y &= \frac{2}{\lambda} \sin \left( \frac{u - v}{2} \right).
\end{align*}
\] (1.2)

The system (1.2) has the property that if \( z = u(x, y) \) is a given solution of (1.1), then solving the system for \( v(x, y) \) with \( \lambda \) fixed gives a 1-parameter family of new solutions \( z = v(x, y) \) of (1.1). (For example, starting with the trivial solution \( u = 0 \) gives the 1-soliton solutions of sine-Gordon, with initial position depending on a constant of integration and velocity depending on the choice of the nonzero constant \( \lambda \).) Note that, given \( u \), the partial derivatives of \( v \) are completely determined; the compatibility condition of the two resulting equations for \( v \) is precisely that \( u \) satisfies the sine-Gordon equation (1.1).

In general, Bäcklund transformations provide a way to obtain new solutions of a partial differential equation (or system of PDEs) by starting with a given solution of the same (or a different) PDE and solving an auxiliary system of ordinary differential equations. In this context, some special properties of the transformation (1.2) are:

1. the transformation (1.2) is an auto-Bäcklund transformation, i.e., it links two solutions of the same PDE—namely, the sine-Gordon equation (1.1);
the underlying PDE (1.1) is a wavelike equation—i.e., a hyperbolic PDE of the form 
\[ z_{xy} = f(x, y, z, z_x, z_y); \]
(3) the transformation (1.2) preserves the independent variables \( x \) and \( y \);
(4) the relations between the partial derivatives of \( u \) and \( v \) defined by the transformation (1.2) are linear;
(5) the transformation (1.2) has no explicit dependence on \( x \) or \( y \); thus we say that the transformation (1.2) is autonomous;
(6) the transformation (1.2) is actually a one-parameter family of transformations, depending on \( \lambda \).

Parametric Bäcklund transformations were the subject of one of our earlier papers [2]. In the present paper, we will be less concerned with the parametric property (6) and will concentrate on the other properties on this list. The goal of this paper is to show that, while these properties all appear to depend on the formulation of both the PDE (1.1) and the Bäcklund transformation (1.2) in a particular coordinate system, in fact they all have intrinsic geometric meaning, independent of any particular choice of local coordinates. The problem of classifying certain types of Bäcklund transformations will be considered in the sequel [3].

This paper is organized as follows: in §2 we review the geometric formulation given in [1] for Bäcklund transformations of hyperbolic Monge-Ampère systems in terms of exterior differential systems. In §3 we show how properties (2)-(5) of the transformation (1.2) listed above are encoded in certain invariants of the associated exterior differential system, independent of any choice of local coordinate system. In §4 we outline some of the techniques that will be used in the sequel [3] to give a partial classification of quasilinear, wavelike Bäcklund transformations, both in the autonomous and non-autonomous cases, and discuss some of the limitations of our approach.

Perhaps surprisingly, a geometric characterization of property (1)—being an auto-Bäcklund transformation—in terms of invariants for the associated exterior differential system remains elusive; we hope to consider this issue in a future paper.

2. Geometric formulation of Bäcklund transformations

In this section we will show how to formulate hyperbolic Monge-Ampère PDEs, as well as Bäcklund transformations between them, as exterior differential systems. We will use the sine-Gordon equation (1.1) and its Bäcklund transformation (1.2) as examples to illustrate the general constructions.

2.1. Hyperbolic Monge-Ampère systems. The existence of a Bäcklund transformation between two partial differential equations is a property that is independent of changes of coordinates. The geometric viewpoint we adopt for studying such properties is that of exterior differential systems, in which a PDE or system of PDEs is described by a differentially closed ideal \( \mathcal{I} \) of differential forms on a manifold, and solutions to the PDE are in one-to-one correspondence with submanifolds to which the forms in the ideal pull back to be zero. (Such submanifolds are called integral submanifolds or integrals of the system.)

For example, if we let \( \mathcal{I} \) be the differential ideal generated by the differential forms
\[ \theta = du - pdx - q dy, \quad \Omega = (dp - (\sin u) dy) \wedge dx \] (2.1)
on the manifold \( \mathbb{R}^5 \) with coordinates \((x, y, u, p, q)\), then solutions of the sine-Gordon equation (1.1) are in one-to-one correspondence with surfaces in \( \mathbb{R}^5 \) on which \( \theta, \Omega \) and their exterior
derivatives vanish, and on which the 2-form $dx \wedge dy$ is never zero. This can be seen as follows: the condition that $dx \wedge dy$ is nonvanishing on a surface $\Sigma \subset \mathbb{R}^5$ is equivalent to the condition that $\Sigma$ is a graph over the the $xy$ plane. Thus $\Sigma$ is defined by equations of the form

$$u = u(x, y), \quad p = p(x, y), \quad q = q(x, y).$$

Then the vanishing of $\theta$ implies that $p = u_x$ and $q = u_y$, while the vanishing of $\Omega$ implies that $u_{xy} = p_y = \sin u$.

This is an example of a Monge-Ampère exterior differential system on a 5-dimensional manifold. Any such exterior differential system $I$ is generated locally by a contact 1-form $\theta$ and a 2-form $\Omega$, with the property that at each point $\Omega$ is linearly independent from $d\theta$ and wedge products with $\theta$. As an ideal, $I$ is generated algebraically by $\theta$, $d\theta$ and $\Omega$. Given such a system, the Pfaff theorem implies that there always exist local coordinates $(x, y, u, p, q)$ such that (up to a nonzero multiple)

$$\theta = du - p\, dx - q\, dy.$$ 

Then by subtracting off suitable multiples of $\theta$ and $d\theta$, we can assume that

$$\Omega = A\, dp \wedge dy + B\, (dx \wedge dp + dq \wedge dy) + C\, dx \wedge dq + D\, dp \wedge dq + E\, dx \wedge dy$$

for some functions $A, B, C, D, E$. Thus, by the same argument as above, integral surfaces of $I$ on which $dx \wedge dy$ is never zero are in one-to-one correspondence with solutions of a Monge-Ampère PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + D(u_{xx}u_{yy} - u_{xy}^2) + E = 0,$$  

(2.2)

where $A, B, C, D, E$ are functions of the variables $(x, y, u, u_x, u_y)$.

Monge-Ampère equations are the smallest class of second-order PDEs for one function of two variables that is invariant under contact transformations and contains the quasilinear equations (see, e.g., Chapter 2 in [5]). So, if one is interested in studying equations like the sine-Gordon equation (1.1) from a geometric viewpoint, it is natural to focus on Monge-Ampère systems. Note that in the example (2.1), the 2-form generator $\Omega$ is decomposable, i.e., a wedge-product of two 1-forms. In fact, we can choose algebraic generators for $I$ consisting of $\theta$ and two 2-forms $\Omega_1, \Omega_2$ that are both decomposable. Specifically, if we take

$$\Omega_1 = (dp - (\sin u)\, dy) \wedge dx, \quad \Omega_2 = (dq - (\sin u)\, dx) \wedge dy,$$

then we have

$$\Omega = \Omega_1, \quad d\theta = - (\Omega_1 + \Omega_2),$$

and so

$$I = \{\theta, d\theta, \Omega\} = \{\theta, \Omega_1, \Omega_2\}.$$

Monge-Ampère systems with this property are called hyperbolic, because this decomposability condition is equivalent to the condition that the corresponding PDE (2.2) is hyperbolic in the usual sense.

2.2. Bäcklund transformations of hyperbolic Monge-Ampère systems. The following geometric definition of a Bäcklund transformation between two hyperbolic Monge-Ampère systems is based on that given in [4] and [6].

Let $(\mathcal{M}, I)$, $(\mathcal{M}, \mathcal{I})$ be hyperbolic Monge-Ampère systems on 5-dimensional manifolds $\mathcal{M}, \mathcal{M}$ respectively, algebraically generated by differential forms

$$I = \{\theta, \Omega_1, \Omega_2\}, \quad \mathcal{I} = \{\theta, \Omega_1, \Omega_2\}.$$
A Bäcklund transformation between \((M, \mathcal{I})\) and \((\bar{M}, \bar{\mathcal{I}})\) is a 6-dimensional manifold \(B\), equipped with submersions \(\pi : B \to M\) and \(\bar{\pi} : B \to \bar{M}\) and a Pfaffian exterior differential system \(J\) of rank 2 on \(B\), with the property that \(J\) is an integrable extension of both \(\mathcal{I}\) and \(\bar{\mathcal{I}}\).

\[
\begin{array}{c}
\pi \\
\downarrow \\
M \\
\downarrow \\
\bar{\pi} \\
\bar{M}
\end{array}
\]

Specifically, \(J\) is the differential ideal on \(B\) generated by the 1-forms \(\pi^*\theta, \pi^*\bar{\theta}\) and their exterior derivatives. The condition that \(J\) be an integrable extension of both \(\mathcal{I}\) and \(\bar{\mathcal{I}}\) means

\[
\begin{align*}
\pi^*d\theta &\equiv 0 \mod \pi^*\theta, \pi^*\bar{\theta}, \pi^*\Omega_1, \pi^*\Omega_2, \\
\pi^*d\bar{\theta} &\equiv 0 \mod \pi^*\theta, \pi^*\bar{\theta}, \pi^*\Omega_1, \pi^*\Omega_2.
\end{align*}
\]

Consequently, when \(J\) is restricted to the inverse image (under \(\pi\)) of an integral submanifold of \(\mathcal{I}\), it satisfies the Frobenius integrability condition, and similarly for the inverse image of an integral of \(\bar{\mathcal{I}}\). We will also assume that the Bäcklund transformation satisfies the technical condition that \(\pi^*d\theta\) and \(\pi^*d\bar{\theta}\) are linearly independent modulo \(\pi^*\theta\) and \(\pi^*\bar{\theta}\); we call such transformations normal.

For example, in the sine-Gordon example above, the systems \((M, \mathcal{I}), (\bar{M}, \bar{\mathcal{I}})\) are each taken to be copies of the exterior differential system described in the previous section: \(M = \mathbb{R}^5\) with coordinates \((x, y, u, p, q)\), \(\bar{M} = \mathbb{R}^5\) with coordinates \((x, y, u, \bar{p}, \bar{q})\), and

\[
\begin{align*}
\mathcal{I} &= \{\theta = du - p \, dx - q \, dy, \quad \Omega_1 = (dp - (\sin u) \, dy) \wedge dx, \quad \Omega_2 = (dq - (\sin u) \, dx) \wedge dy\}, \\
\bar{\mathcal{I}} &= \{\bar{\theta} = d\bar{u} - \bar{p} \, d\bar{x} - \bar{q} \, d\bar{y}, \quad \bar{\Omega}_1 = (d\bar{p} - (\sin \bar{u}) \, d\bar{y}) \wedge d\bar{x}, \quad \bar{\Omega}_2 = (d\bar{q} - (\sin \bar{u}) \, d\bar{x}) \wedge d\bar{y}\}.
\end{align*}
\]

\(B\) is the 6-dimensional submanifold of \(M \times \bar{M}\) defined by the equations \(x = x, y = y\) (because the transformation preserves the independent variables \(x, y\)) and the two equations

\[
p - \bar{p} = 2\lambda \sin \left(\frac{u + \bar{u}}{2}\right), \quad q + \bar{q} = 2\lambda \sin \left(\frac{u - \bar{u}}{2}\right), \tag{2.4}
\]

which are equivalent to equations (1.2). (We will see later that it is advantageous to regard these equations as defining \(p, q\) as functions of the independent variables \((x, y, u, \bar{p}, \bar{q})\) on \(B\).) It is straightforward to check that the pullbacks of \(\mathcal{I}, \bar{\mathcal{I}}\) to \(B\) satisfy the integrability conditions (2.3), and that \((B, J)\) is a normal Bäcklund transformation between \((M, \mathcal{I})\) and \((\bar{M}, \bar{\mathcal{I}})\). The fact that the restriction of \(J\) to the inverse image of an integral submanifold of \(\mathcal{I}\) satisfies the Frobenius condition is equivalent to the statement that whenever the function \(u(x, y)\) is a solution of the sine-Gordon equation (1.1), the system (1.2) is a compatible system for the unknown function \(v(x, y)\) whose solutions can be constructed by solving ODEs.

In [1] it is shown that a normal Bäcklund transformation between two hyperbolic Monge-Ampère systems determines, and is determined by, a \(G\)-structure \(\mathcal{P}\) on the 6-dimensional manifold \(B\), where \(G \subset GL(6, \mathbb{R})\) consists of matrices of the form

\[
\begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & B & 0 \\
0 & 0 & 0 & A
\end{pmatrix}, \quad A, B \in GL(2, \mathbb{R}), \quad a = \det A, b = \det B. \tag{2.5}
\]
(In other words, \( \mathcal{P} \) is a principal bundle on \( \mathcal{B} \) with fiber group \( G \), and with a right \( G \)-action induced by the inclusion of \( G \) into \( GL(6, \mathbb{R}) \).) We will use the notation \((\hat{\theta}, \hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3, \hat{\omega}^4)\) for the components of the canonical \( \mathbb{R}^6 \)-valued 1-form on \( \mathcal{P} \), which satisfy structure equations

\[
\begin{bmatrix}
\hat{\theta} \\
\hat{\omega}^1 \\
\hat{\omega}^2 \\
\hat{\omega}^3 \\
\hat{\omega}^4
\end{bmatrix} = \Upsilon \wedge
\begin{bmatrix}
\hat{\theta} \\
\hat{\omega}^1 \\
\hat{\omega}^2 \\
\hat{\omega}^3 \\
\hat{\omega}^4
\end{bmatrix} +
\begin{bmatrix}
A_1(\hat{\omega}^1 - C_1\hat{\theta}) \wedge (\hat{\omega}^2 - C_2\hat{\theta}) + \hat{\omega}^3 \wedge \hat{\omega}^4 \\
\hat{\omega}^1 \wedge \hat{\omega}^2 + A_2(\hat{\omega}^3 - C_3\hat{\theta}) \wedge (\hat{\omega}^4 - C_4\hat{\theta}) \\
B_1\hat{\theta} \wedge \hat{\theta} + C_1\hat{\omega}^3 \wedge \hat{\omega}^4 \\
B_2\hat{\theta} \wedge \hat{\theta} + C_2\hat{\omega}^3 \wedge \hat{\omega}^4 \\
B_3\hat{\theta} \wedge \hat{\theta} + C_3\hat{\omega}^1 \wedge \hat{\omega}^2 \\
B_4\hat{\theta} \wedge \hat{\theta} + C_4\hat{\omega}^1 \wedge \hat{\omega}^2
\end{bmatrix}.
\] (2.6)

Here \( \Upsilon \) is a connection form which takes values in the Lie algebra \( \mathfrak{g} \) of \( G \), and \( A_i, B_i, C_i \) are well-defined torsion functions on \( \mathcal{P} \) (see [1] for more details). The relationship between the \( G \)-structure and the Pfaffian systems involved in the Bäcklund transformation is that, if \((\theta, \omega^1, \omega^2, \omega^3, \omega^4)\) is a local section of \( \mathcal{P} \), then on its domain \( \pi^*\mathfrak{L} \) is generated algebraically by \( \{\theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\} \), and \( \pi^*\mathfrak{T} \) by \( \{\theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4\} \). Because \( \theta \) and \( \hat{\theta} \) must each have Pfaff rank 5, \( A_1 \) and \( A_2 \) must be nonzero at every point; furthermore, the condition of normality implies that \( A_1A_2 - 1 \) is also nonzero everywhere.

The \( G \)-structure endows \( T^*\mathcal{B} \) with a well-defined splitting

\[
T^*\mathcal{B} = L \oplus \underline{L} \oplus W_1 \oplus W_2
\] (2.7)

such that, given any local section of \( \mathcal{P} \), \( L \) is spanned by \( \theta, \underline{L} \) by \( \theta \), \( W_1 \) by \( \{\omega^1, \omega^2\} \), and \( W_2 \) by \( \{\omega^3, \omega^4\} \). In [1] it is shown that the torsion functions are components of well-defined tensors on \( \mathcal{B} \) which are maps between bundles associated to terms in this splitting. One way to see this is to study how these functions vary along the fibers. For example, if \( g \in G \) is the group element given by (2.5), then

\[
R_g^*A_1 = a^{-1}bA_1, \quad R_g^*A_2 = b^{-1}aA_2.
\]

Notice that this implies that the product \( A_1A_2 \) is a well-defined function on \( \mathcal{B} \). It also follows that \( A_1 \) and \( A_2 \) are components of well-defined tensors in \( L \otimes \Lambda^2W_2^* \) and \( \underline{L} \otimes \Lambda^2W_1^* \), respectively. Similarly, the vectors \([C_1, C_2]\) and \([C_3, C_4]\) are components of well-defined tensors \( \tau_1 \in \Gamma(W_1^* \otimes \Lambda^2W_2) \) and \( \tau_2 \in \Gamma(W_2^* \otimes \Lambda^2W_1) \). In fact, these tensors are just the exterior derivative followed by an appropriate quotient map; for example \( \tau_1 \) is simply the exterior derivative applied to sections of \( W_1 \), modulo 1-forms in \( L, \underline{L} \) and \( W_1 \).
For the sine-Gordon example above, one can take the following local section of $\mathcal{P}$ (recall that $\mathcal{B} \subset \mathbb{R}^5 \times \mathbb{R}^5$ is defined by $\underline{x} = x, \underline{y} = y$, and equations (2.4)):

$$
\theta = du - \frac{p}{2} dx - q dy = du - \left(p + 2\lambda \sin\left(\frac{u + u}{2}\right)\right) dx - q dy,
$$

$$
\omega^1 = dx,
$$

$$
\omega^2 = dp - (\sin u) dy + \lambda \cos\left(\frac{u + u}{2}\right) \theta,
$$

$$
\omega^3 = dy,
$$

$$
\omega^4 = dq - (\sin u) dx - \frac{1}{\lambda} \cos\left(\frac{u - u}{2}\right) \theta.
$$

The specific multiples of $\theta, \overline{\theta}$ appearing in $\omega^2, \omega^4$ are uniquely determined by the conditions

$$
d\theta \equiv 0 \mod \theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4,
$$

$$
d\overline{\theta} \equiv 0 \mod \theta, \omega^1 \wedge \omega^2, \omega^3 \wedge \omega^4,
$$

which are implied by the structure equations (2.6). The torsion functions associated to this local section are:

$$
A_1 = 1, \quad A_2 = -1, \quad B_1 = B_3 = C_1 = C_3 = 0,
$$

$$
B_2 = -\frac{\lambda}{2} \sin\left(\frac{u + u}{2}\right), \quad B_4 = \frac{1}{2\lambda} \sin\left(\frac{u - u}{2}\right),
$$

$$
C_2 = -\lambda \cos\left(\frac{u + u}{2}\right), \quad C_4 = -\frac{1}{\lambda} \cos\left(\frac{u - u}{2}\right).
$$

3. Wavelike, quasilinear, autonomous Bäcklund transformations

In this section, we show how properties (2)-(5) of the Bäcklund transformation (1.2) may be characterized geometrically, in terms of the invariants $A_i, B_i, C_i$ associated to the $G$-structure $\mathcal{P}$ determined by the exterior differential system $(\mathcal{B}, \mathcal{J})$.

3.1. Wavelike Bäcklund transformations. A hyperbolic Monge-Ampère PDE is called wavelike if it may be expressed in local coordinates as

$$
u_{xy} = f(x, y, u, u_x, u_y); \quad (3.1)
$$

this is equivalent to the condition that the characteristics are tangent to the coordinate directions at each point. In terms of these local coordinates, a Bäcklund transformation between two wavelike Monge-Ampère PDEs is called wavelike if it preserves the characteristic directions; this is equivalent to the condition that it preserves the characteristic independent variables $x, y$ up to a transformation of the form

$$
\underline{x} = \phi(x), \quad \underline{y} = \psi(y).
$$

By making an analogous change of independent variables for one PDE or the other, we may assume without loss of generality that a wavelike Bäcklund transformation satisfies the condition $\underline{x} = x, \underline{y} = y$. 


A wavelike Bäcklund transformation between two wavelike PDEs

\[ u_{xy} = f(x, y, u, u_x, u_y), \quad v_{xy} = g(x, y, v, v_x, v_y) \]

is generally described by equations of the form

\[ u_x = F(x, y, u, v, u_x, v_x), \quad v_y = G(x, y, u, v, u_y, v_x). \]

(This will be made more precise in the proof of Proposition 3.3.) In this case the following coframing of \( \mathcal{B} \) is a local section of \( \mathcal{P} \):

\[
\begin{align*}
\theta &= du - F dx - q dy, \\
\vartheta &= dv - p dx - G dy, \\
\omega^1 &= dx, \\
\omega^2 &= dp - g dy + r_1 \theta, \\
\omega^3 &= dy, \\
\omega^4 &= dq - f dx - r_2 \theta,
\end{align*}
\]

where \( r_1, r_2 \) are functions on \( \mathcal{B} \) uniquely determined by the conditions (2.8).

Remark 3.2. Bäcklund transformations where one (or both) of the tensors \( \tau_1 \) or \( \tau_2 \) corresponding to the vectors \([C_1, C_2], [C_3, C_4] \) is identically zero are highly degenerate, and are classified in [1]. Here, we will assume that both are nonzero at each point; in particular, this implies that neither of the Pfaffian systems \( W_1, W_2 \) are completely integrable.

Proposition 3.3. Let \( \mathcal{P} \searrow \mathcal{B} \) define a wavelike normal Bäcklund transformation for which both tensors \( \tau_1, \tau_2 \) are nonvanishing on \( \mathcal{B} \). Then near any point on \( \mathcal{B} \) there exist local coordinates \( x, y, u, v, p, q \) and functions \( f, g, F, G \) such that the following is a section of \( \mathcal{P} \):

\[
\begin{align*}
\theta &= du - F dx - q dy, \\
\vartheta &= dv - p dx - G dy, \\
\omega^1 &= dx, \\
\omega^2 &= dp - g dy - (F_v/F_p) \theta, \\
\omega^3 &= dy, \\
\omega^4 &= dq - f dx - (G_u/G_q) \theta,
\end{align*}
\]

and \( f, g, F, G \) satisfy the following partial differential equations:

\[
\begin{align*}
F_q &= G_p = 0, \\
f - gF_p &= F_y + qF_u + GF_v, \\
g - fG_q &= G_x + FG_u + pG_v, \\
0 &= f_vF_p - f_pF_v = g_uG_q - g_qG_u.
\end{align*}
\]

Moreover, the quantities \( F_p, G_q, \) and \( F_pG_q - 1 \) are nonzero at every point of \( \mathcal{B} \); in particular, equations (3.5), (3.6) can be solved for \( f \) and \( g \). In these coordinates, \( \mathcal{B} \) represents a Bäcklund transformation.
transformation between the wavelike partial differential equations

\[ u_{xy} = f(x, y, u, u_x, u_y), \quad v_{xy} = g(x, y, v, v_x, v_y), \quad (3.8) \]

with the transformation given by the equations

\[ u_x = F(x, y, u, v, v_x), \quad v_y = G(x, y, u, v, u_y). \quad (3.9) \]

**Proof.** Take a local section of \( \mathcal{P} \) such that \( \omega^1 \) and \( \omega^3 \) span the integrable subsystems of \( W_1 \) and \( W_2 \), respectively. Using the \( G \)-action, we can modify the section (specifically, by multiplying \( \omega^1, \omega^3 \) by suitable scaling functions) so that \( \omega^1 = dx \) and \( \omega^3 = dy \) for some locally defined functions \( x, y \) on \( \mathcal{B} \). The structure equations (2.6) imply that \( \{\theta, dx, dy\} \) is a Frobenius system; therefore, there are a locally defined functions \( u, p_1, q_1 \) on \( \mathcal{B} \) such that

\[ \theta = \mu(du - p_1 dx - q_1 dy) \]

for some nonvanishing multiple \( \mu \). Moreover, because \( \theta \) has Pfaff rank 5, the functions \( u, x, y, p_1, q_1 \) must have linearly independent differentials. Similarly, there must exist locally defined functions \( v, p_2, q_2 \) such that

\[ \theta = \mu(dv - p_2 dx - q_2 dy) \]

for some nonvanishing multiple \( \mu \). Using the \( G \)-action, we can modify the section by scaling so that \( \mu = \mu = 1 \).

The structure equations imply that \( d\theta \equiv A_1 dx \wedge \omega^2 + dy \wedge \omega^4 \) modulo \( \theta \). Substituting \( \theta = du - p_1 dx - q_1 dy \) into this equation shows that we must have

\[ A_1 \omega^2 = dp_1 - s_1 dx - f dy - r_3 \theta, \quad (3.10) \]

\[ \omega^4 = dq_1 - f dx - s_2 dy - r_2 \theta \quad (3.11) \]

for some functions \( f, r_2, r_3, s_1, s_2 \). Similarly, substituting \( \theta = dv - p_2 dx - q_2 dy \) into the equation \( d\theta \equiv dx \wedge \omega^2 + A_2 dy \wedge \omega^4 \) mod \( \theta \) shows that

\[ A_2 \omega^2 = dp_2 - s_3 dx - g dy - r_1 \theta, \quad (3.12) \]

\[ A_2 \omega^4 = dq_2 - g dx - s_4 dy - r_4 \theta \quad (3.13) \]

for some functions \( g, r_1, r_4, s_2, s_4 \). We may use the remaining freedom in the \( G \)-action to modify the section, by adding multiples of \( dx \) and \( dy \) to \( \omega^2 \) and \( \omega^4 \), respectively, to arrange that \( s_2 = 0 \) and \( s_4 = 0 \).

Comparing (3.10) and (3.12) shows that the functions \( p_1, p_2, x, y, u, v \) are functionally dependent, and comparing (3.13) and (3.11) shows that the functions \( q_1, q_2, x, y, u, v \) are functionally dependent; on the other hand, linear independence of the forms \( \theta, \theta, dx, dy, \omega^2 \) and \( \omega^4 \) shows that \( x, y, u, v, p_2 \) and \( q_1 \) are a local coordinate system on \( \mathcal{B} \). Thus, locally there exist functions \( F \) and \( G \) such that

\[ p_1 = F(x, y, u, v, p_2) \quad q_2 = G(x, y, u, v, q_1). \]

Substituting these equations, together with equations (3.11), (3.12), and the expressions for \( \theta \) and \( \theta \) into equations (3.10) and (3.13) yields

\[ A_1(dp_2 - g dy - r_1(dv - p_2 dx - G dy)) = dF - s_1 dx - f dy - r_3(du - F dx - q_1 dy) \]

\[ A_2(dq_1 - f dx - r_2(du - F dx - q_1 dy)) = dG - g dx - s_4 dy - r_4(dv - p_2 dx - G dy). \quad (3.14) \]
For convenience, set \( p = p_2 \) and \( q = q_1 \). Expanding \( dF, dG \) and equating the coefficients of the differentials of the coordinates in equations (3.14) yields
\[
A_1 = F_p, \quad r_1 = -F_u/F_p, \quad r_3 = F_u, \quad f - F_p g = F_y + F_u q + F_v G, \\
A_2 = G_q, \quad r_2 = -G_u/G_q, \quad r_4 = G_v, \quad g - G_q f = G_x + G_u F + G_v p.
\]
This establishes equations (3.5) and (3.6) in the statement of the proposition, as well as the form of the coefficients of \( dx \) in \( \omega^2 \) and \( dy \) in \( \omega^4 \) in (3.3).

The structure equations (2.6) imply that
\[
d\hat{\omega}^4 \wedge \hat{\omega}^3 \wedge \hat{\omega}^2 \wedge \hat{\theta} = \frac{C_4}{2A_1} d\hat{\theta} \wedge d\hat{\theta} \wedge \hat{\theta}.
\]
Pulling this condition back to our particular section gives
\[
-df \wedge dx \wedge dy \wedge dq \wedge du = \frac{C_4}{A_1} dF \wedge dx \wedge dq \wedge dy \wedge du.
\]
The local invariants for this section satisfy \( C_1 = C_3 = 0 \); therefore, our assumption that \( W_1, W_2 \) are not completely integrable implies that the functions \( C_2, C_4 \) are nonzero. Thus it follows from this equation that \( df \wedge dF \equiv 0 \) modulo \( dx, dy, du \) and \( dq \). This establishes the first equation on the last line (3.7) in the proposition. Moreover, it implies that \( f \) may be regarded as a function of the variables \( x, y, u, v, p_1, q_1 \); hence \( f \) is locally a well-defined function on \( M \). Similarly, the second equation in (3.7) may be derived from the condition
\[
d\hat{\omega}^2 \wedge \hat{\omega}^1 \wedge \hat{\omega}^0 \wedge \hat{\theta} = \frac{C_2}{2A_2} d\hat{\theta} \wedge d\hat{\theta} \wedge \hat{\theta},
\]
and \( g \) may be regarded as a function of the variables \( x, y, v, p_2, q_2 \); hence \( g \) is locally a well-defined function on \( M \).

Since we have \( A_1 = F_p, A_2 = G_q \), the condition that \( B \) is a normal Bäcklund transformation implies that \( F_p, G_q \neq 0 \) and \( F_p G_q - 1 \neq 0 \) at every point of \( B \). Any integral surface of \( \mathcal{J} \) on which \( dx \wedge dy \neq 0 \) is given by specifying \( u, v, p, q \) as functions of \( x \) and \( y \) that satisfy the conditions \( p = u_x, q = u_y \), and
\[
u_x = F(x, y, u, v, p), \quad u_y = G(x, y, u, v, q).
\]
By taking total derivatives of these equations with respect to \( y \) and \( x \) respectively, it follows that \( u \) and \( v \) satisfy the wavelike PDEs (3.8), where \( f, g \) are the functions determined by equations (3.5), (3.6).

\[\square\]

3.2. Quasilinear wavelike Bäcklund transformations. The Bäcklund transformation (1.2) for the sine-Gordon equation is defined by quasilinear PDEs—i.e., PDEs that are linear in the partial derivatives \( u_x, u_y, v_x, v_y \). For the general case of a wavelike Bäcklund transformation between two wavelike PDEs, this condition may be expressed in terms of the local coordinates given by Proposition 3.3 as the condition that the functions \( F, G \) are linear in the variables \( p, q \), respectively. In this case, the invariants associated to the section (3.3) of \( \mathcal{P} \) have the property that the functions \( A_1 = F_p \) and \( A_2 = G_q \) are functions of the variables \( x, y, u, v \) alone. It follows that the product \( A_1 A_2 \), which is well-defined on \( B \) independent of the choice of section of \( \mathcal{P} \), is also a function of the variables \( x, y, u, v \) alone.

This condition can be expressed geometrically as follows: any wavelike Bäcklund transformation \( B \) is endowed with a well-defined rank 4 Pfaffian system \( K \) that is the direct sum of
and the integrable subsystems of $W_1$ and $W_2$; moreover, this system is Frobenius. (In terms of the local coordinates given by Proposition 3.3, $K$ is spanned by $\{dx, dy, du, dv\}$.)

**Definition 3.4.** A wavelike Bäcklund transformation between two wavelike hyperbolic Monge-Ampère equations will be called quasilinear if the function $A_1A_2$ is constant along the integral surfaces of $K$ (or equivalently, if $d(A_1A_2) \in K$).

In terms of the local normal form (3.15) given by Proposition 3.3, this is equivalent to the condition that the functions $F$ and $G$ are linear in the variables $p$ and $q$, respectively. Thus, in the quasilinear case we may set

$$F = F^0 + F^1p, \quad G = G^0 + G^1q$$

where $F^i$ and $G^i$ are functions of the variables $x, y, u, v$, and the quantities $F^1, G^1$ and $F^1G^1 - 1$ are nonzero at each point. Then the equations (3.15) defining the Bäcklund transformation become:

$$u_x = F^1v_x + F^0, \quad v_y = G^1u_y + G^0.$$  \hfill (3.17)

The following proposition shows that in the quasilinear case, the local coordinates of Proposition 3.3 may be refined in such a way that the functions $f, g$ become linear with respect to the variables $p, q$. Consequently, the PDEs (3.18) underlying a quasilinear Bäcklund transformation (3.16) may both be assumed to be quasilinear as well.

**Proposition 3.5.** Let $\mathcal{P} \backslash \mathcal{B}$ define a quasilinear wavelike normal Bäcklund transformation. Near any point on $\mathcal{B}$ there exist local coordinates $x, y, u, v, p, q$ and functions $f, g, F, G$ satisfying the conditions of Proposition 3.3 together with the additional conditions that $F$ is linear with respect to $p$, $G$ is linear with respect to $q$, and the functions $f, g$ are linear with respect to the variables $p, q$. (In particular, $f$ and $g$ contain no terms involving the product $pq$.)

**Proof.** Let $x, y, u, v, p, q$ be the local coordinates provided by Proposition 3.3. As noted above, the assumption that $\mathcal{B}$ is quasilinear implies that $F, G$ have the form (3.16). Substituting these expressions into equations (3.5), (3.6) yields the following system of equations for $f, g$:

$$\begin{bmatrix} \frac{1}{G^1} & -F^1 \\ -G^1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} F_y + G^0F_v & F_y + G^0F_v \\ G^0 + F^0G_u & G^0 + F^0G_u \end{bmatrix} \begin{bmatrix} F^0 + G^1F^1 & F^0 + G^1F^1 \\ G^1 + F^1G^1 & G^1 + F^1G^1 \end{bmatrix} \begin{bmatrix} 1 \\ p \\ q \\ pq \end{bmatrix}.$$  \hfill (3.18)

Therefore, $f$ and $g$ are given by:

$$\begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 1 \\ F^1 \\ G^1 \end{bmatrix} \begin{bmatrix} F_y + G^0F_v & F_y + G^0F_v \\ G^0 + F^0G_u & G^0 + F^0G_u \end{bmatrix} \begin{bmatrix} F^0 + G^1F^1 & F^0 + G^1F^1 \\ G^1 + F^1G^1 & G^1 + F^1G^1 \end{bmatrix} \begin{bmatrix} 1 \\ p \\ q \\ pq \end{bmatrix},$$ \hfill (3.19)

where $\Delta = 1 - F^1G^1$.

As is evident from (3.18) and the linearity relations (3.16), such transformations link solutions of hyperbolic equations of the form

$$u_{xy} = Au_xu_y + Bu_x + Cu_y + D$$  \hfill (3.19)
where \( A, B, C, D \) are functions of \( x, y, u \). Moreover, the form of such equations, along with the form (3.17) for the transformation, is invariant under point transformations defined by \( U = \varphi(u, x, y) \). In order to complete the proof, it remains to show that we can use such changes of variable to eliminate the first-order nonlinearity in the right-hand side of (3.19).

Under a change of variable \( U = \varphi(x, y, u) \), the PDE (3.19) is transformed to the PDE

\[
U_{xy} = \tilde{A} U_x U_y + \tilde{B} U_x + \tilde{C} U_y + \tilde{D}
\]

for the function \( U(x, y) \), where, in particular,

\[
\tilde{A} = \frac{\varphi_{uu} + A \varphi_u}{\varphi_u^2}.
\]

By choosing \( \varphi(x, y, u) \) so that it satisfies the first-order PDE

\[
\varphi_u = \int e^{-A(x,y,u)} du,
\]

we can arrange that \( \varphi_{uu} + A \varphi_u = 0 \); thus, the PDE satisfied by \( U \) has no nonlinear first-order term. Similarly, we can simultaneously make a change of variable \( V = \psi(x, y, v) \), to arrange that the PDE satisfied by \( V \) has no nonlinear first-order term. When we do so, the quasilinear wavelike form (3.17) of the Bäcklund transformation is unchanged. (However, the coefficients in (3.17) are altered; for example, \( F^1 \) is replaced by \( \frac{\psi \psi V}{\varphi_u} \) and \( G^1 \) is replaced by \( \frac{\psi \psi G^1}{\varphi_u} \).) Now re-labeling \( U \) as \( u \) and \( V \) as \( v \) gives the desired local coordinates. \( \square \)

### 3.3. Autonomous wavelike Bäcklund transformations.

A PDE is called **autonomous** if it contains no explicit dependence on the independent variables \( x, y \). Thus, the wavelike Bäcklund transformation (3.15) is autonomous if \( F \) and \( G \) are functions of the variables \( u, v, p, q \) alone, with no dependence on the variables \( x, y \). In this case, we can see from equations (3.5), (3.6) that the functions \( f, g \) are also independent of the variables \( x, y \), and hence the PDEs underlying the Bäcklund transformation have the form

\[
u_{xy} = f(u, u_x, u_y), \quad \psi_{xy} = g(v, v_x, v_y).
\]

Geometrically, the condition that the system is autonomous is represented by the presence of a 2-dimensional symmetry group of the exterior differential system \( \mathcal{J} \) on \( \mathcal{B} \)—namely, the group of simultaneous translations in the independent variables \( x, y \). Specifically, if \( \mathcal{J} \) represents an autonomous Bäcklund transformation, then for any real numbers \( a, b \), the diffeomorphism \( \phi : \mathcal{B} \to \mathcal{B} \) defined by

\[
\phi(x, y, u, v, p, q) = (x + a, y + b, u, v, p, q)
\]

has the property that \( \phi^* \mathcal{J} = \mathcal{J} \).

It is generally more convenient to work with the **infinitesimal symmetries** of \( \mathcal{J} \). These are the vector fields on \( \mathcal{B} \) that generate the symmetry group; in the case of the translation symmetry group above, the infinitesimal symmetries are generated by the two commuting vector fields

\[
X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}.
\]

In general, we have the following definition:
Definition 3.6. A vector field $X$ on a manifold $B$ is an infinitesimal symmetry of the exterior differential system $J$ on $B$ if for every differential form $\Phi \in J$, the Lie derivative $\mathcal{L}_X \Phi$ is contained in $J$.

Since the $G$-structure $P \to B$ is canonically associated to the exterior differential system $J$ on $B$, any symmetry $\phi$ of $(B, J)$ must also be a symmetry of $P$, and vice-versa. In other words, for any local section $\omega$ of $P$, $\phi^* \omega$ must also be a local section of $P$. In particular, if $X$ is an infinitesimal symmetry then the Lie derivative $\mathcal{L}_X$ must preserve the splitting (2.7) of the cotangent bundle $T^*B$. Furthermore, if the Bäcklund transformation is wavelike, then $\mathcal{L}_X$ must also preserve the 1-dimensional integrable subsystems of $W_1$ and $W_2$.

The following proposition shows that in the wavelike case, the presence of a pair of commuting infinitesimal symmetries characterizes the autonomous examples; more precisely, if a wavelike Bäcklund transformation has two linearly independent, commuting infinitesimal symmetries (subject to a transversality condition which will be made precise below), then there exist local coordinates with respect to which the PDEs defining the Bäcklund transformation are autonomous.

Proposition 3.7. Let $P \to B$ define a normal wavelike Bäcklund transformation. Let $X$ and $Y$ be pointwise linearly independent, commuting vector fields on $B$ which are infinitesimal symmetries of $P$ and are transverse to the integrable subsystems of $W_1$ and $W_2$; i.e., if $\omega^1$ and $\omega^3$ are local sections of these integrable subsystems, then

$$\det \begin{bmatrix} \omega^1(X) & \omega^1(Y) \\ \omega^3(X) & \omega^3(Y) \end{bmatrix} \neq 0 \quad (3.20)$$

at each point of $B$. Then near any point in $B$ there exist local coordinates $x, y, u, v, p, q$ and functions $F, G, f, g$ satisfying the conditions of Proposition 3.3, with the additional property that $F, G, f, g$ are independent of $x$ and $y$.

Proof. Because the vector fields $X$ and $Y$ commute, they are tangent to a local foliation of $B$ with two-dimensional leaves. Thus, any point in $B$ has a neighborhood on which there exist functions $u, v, p, q$ whose differentials are linearly independent and annihilate $X$ and $Y$; i.e.,

$$\{du, dv, dp, dq\} \perp \{X, Y\}.$$

Moreover, the span of $\{du, dv, dp, dq\} \subset T^*B$ is uniquely determined by this condition.

Let $x, y$ be locally defined functions, possibly defined on a smaller neighborhood of the given point, such that $dx$ and $dy$ span the integrable subsystems of $W_1$ and $W_2$, respectively. Then by our hypothesis (3.20) the functions $x, y, u, v, p, q$ form a local coordinate system, and

$$\text{span} \{X, Y\} = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$$

at each point.

First, we will show that we can replace $X, Y$ by constant coefficient linear combinations

$$\tilde{X} = aX + bY, \quad \tilde{Y} = cX + dY \quad (3.21)$$

(so that $\tilde{X}, \tilde{Y}$ commute as well) and $x, y$ by functions $\tilde{x}(x), \tilde{y}(y)$ to arrange that

$$\tilde{X} = \frac{\partial}{\partial \tilde{x}}, \quad \tilde{Y} = \frac{\partial}{\partial \tilde{y}}.$$
To this end, note that if \( Z \) is any symmetry vector field of \((\mathcal{B}, J)\), then
\[
\mathcal{L}_Z dx = d(Z \cdot dx) \equiv 0 \mod dx,
\]
and similarly, \( d(Z \cdot dy) \equiv 0 \mod dy \). In other words, the function \( Z \cdot dx \) is a function of \( x \) alone, and \( Z \cdot dy \) is a function of \( y \) alone. Thus, if we let \( M \) be the matrix
\[
M = \begin{bmatrix} X \cdot dx & Y \cdot dx \\ X \cdot dy & Y \cdot dy \end{bmatrix},
\]
then the top row entries \( M_{11}, M_{12} \) are functions of \( x \) alone, while the bottom row entries \( M_{21}, M_{22} \) are functions of \( y \) alone. Moreover, by replacing \( X, Y \) by constant-coefficient linear combinations \( \tilde{X}, \tilde{Y} \) as in (3.21), we can assume that \( M_{12} \) and \( M_{21} \) vanish at the given point of \( \mathcal{B} \). It follows that \( M_{12} \) also vanishes on the hypersurface through the given point where \( x \) is constant, while \( M_{21} \) vanishes on the hypersurface through the given point where \( y \) is constant. But we also have
\[
\tilde{X}(M_{12}) = \tilde{X}(\tilde{Y} \cdot dx)
= \tilde{X}(\tilde{Y}(x)) \quad \text{(by definition)}
= \tilde{Y}(\tilde{X}(x)) \quad \text{(because \( \tilde{X}, \tilde{Y} \) commute)}
= \tilde{Y}(M_{11})
= M'_{11}(x)\tilde{Y}(x) \quad \text{(by the chain rule)}
= M'_{11}(x)M_{12}.
\]
So, by the local uniqueness theorem for ODEs, \( M_{12} \) vanishes identically in a neighborhood of the given point, and a similar argument shows that the same is true for \( M_{21} \). Now let \( \tilde{x} \) be a function of \( x \) satisfying
\[
\frac{d\tilde{x}}{dx} = \frac{1}{M_{11}(x)};
\]
then we have \( \tilde{X}(\tilde{x}) = 1 \). Similarly, let \( \tilde{y}(y) \) be a function of \( y \) such that \( \tilde{Y}(\tilde{y}) = 1 \). Dropping the tildes, the yields a local coordinate system \( x, y, u, v, p, q \) such that
\[
X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y},
\]
as claimed.

Next, we will show that we can choose a nonvanishing local section \( \theta_1 \) of \( L \) of the form
\[
\theta_1 = A du + B dv + C dp + D dq + R dx + S dy,
\]
where \( A, B, C, D, R, S \) are locally defined functions on \( \mathcal{B} \) that are independent of the variables \( x, y \). Start by choosing any nonvanishing local section \( \theta_0 \) of \( L \). (Here and in what follows we will shrink our local coordinate neighborhood around the given point as necessary.) Because \( \{\theta_0, dx, dy\} \) is a Frobenius system, there exist locally defined functions \( U, P, Q \) such that, up to a scalar multiple,
\[
\theta_0 = dU - P dx - Q dy
= U_u du + U_v dv + U_p dp + U_q dq + (U_x - P) dx + (U_y - Q) dy.
\]
Because \( X \) and \( Y \) are symmetries of the system, we must have
\[
\mathcal{L}_X \theta_0 \equiv \mathcal{L}_Y \theta_0 \equiv 0 \mod \theta_0.
\]
Direct computation using equation (3.22) shows that
\[
\mathcal{L}_X \theta_0 = U_{xu} du + U_{xv} dv + U_{xp} dp + U_{xq} dq + (U_{xx} - P_x) dx + (U_{xy} - Q_x) dy,
\]
\[
\mathcal{L}_Y \theta_0 = U_{yu} du + U_{yv} dv + U_{yp} dp + U_{yq} dq + (U_{xy} - P_y) dx + (U_{yy} - Q_y) dy.
\]
In order that both of these expressions be scalar multiples of \( \theta_0 \), it must be true that the ratio of any pair of the functions
\[
U_u, U_v, U_p, U_q, (U_x - P), (U_y - Q)
\]
is independent of the variables \( x, y \). Then if, say, \( U_u \neq 0 \), we can write
\[
\theta_0 = U_u \left( du + \frac{U_v}{U_u} dv + \frac{U_p}{U_u} dp + \frac{U_q}{U_u} dq + \left( \frac{U_x - P}{U_u} \right) dx + \left( \frac{U_y - Q}{U_u} \right) dy \right),
\]
and it follows that \( \theta_0 = e^\lambda \theta_1 \), where \( e^\lambda = U_u \) and
\[
\theta_1 = A \ du + B \ dv + C \ dp + D \ dq + R \ dx + S \ dy
\]
for functions some \( A, B, C, D, R, S \) that are independent of the variables \( x, y \), as claimed. A similar argument shows that there exists a nonvanishing section \( \theta_1 \) of \( L \) of the form
\[
\theta_1 = A \ du + B \ dv + C \ dp + D \ dq + R \ dx + S \ dy,
\]
where the functions \( A, B, C, D, R, S \) are independent of the variables \( x, y \).

Finally, we will show that we can modify our local coordinates and rescale the sections \( \theta_1, \theta_1 \) to arrive at sections \( \theta, \bar{\theta} \) of \( L, \bar{L} \) respectively, of the form
\[
\theta = du - F(u, v, p) \ dx - q \ dy,
\]
\[
\bar{\theta} = dv - p \ dx - G(u, v, q) \ dy
\]
for some functions \( F, G \) that are independent of the variables \( x, y \). It will follow that the Bäcklund transformation is given in terms of these local coordinates the the equations
\[
u_x = F(u, v, v_x), \quad v_y = G(u, v, u_y),
\]
which will complete the proof of the Proposition. To simplify the exposition, we introduce the following notations: let \( I_1 \subset T^*B \) and \( I_2 \subset T^*B \) be the complementary local sub-bundles spanned by \( \{ dx, dy \} \) and \( \{ du, dv, dp, dq \} \), respectively. Let \( \pi_1, \pi_2 \) denote projections onto these sub-bundles. There is an induced splitting of the local 2-forms on \( B \); namely,
\[
\Lambda^2(T^*B) = \Lambda^2 I_1 \oplus (I_1 \otimes I_2) \oplus \Lambda^2 I_2.
\]
(3.24)
By abuse of notation, we also let \( \pi_1 \) and \( \pi_2 \) denote projections onto the first and last summands in equation (3.24), respectively.

By equation (3.22), we have
\[
\pi_2(d\theta_0) = \pi_2(-dP \wedge dx - dQ \wedge dy) = 0.
\]
Substituting \( \theta_0 = e^\lambda \theta_1 \) into this equation yields (after cancelling a factor of \( e^\lambda \))
\[
\pi_2 \left( d\lambda \wedge \theta_1 + d\theta_1 \right) = 0.
\]
(3.25)
Let
\[
\phi = \pi_1(\theta_1) = A \ du + B \ dv + C \ dp + D \ dq, \quad \psi = \pi_2(\theta_1) = R \ dx + S \ dy.
\]
Substituting $\theta_1 = \phi + \psi$ into equation (3.25) yields

$$0 = \pi_2 (d\lambda \wedge \phi + d\lambda \wedge \psi + d\phi + d\psi)$$

$$= \pi_2(d\lambda) \wedge \phi + d\phi,$$

where in the second line we have used the facts that $\pi_2(\psi) = 0$ (and hence $\pi_2(d\lambda \wedge \psi) = 0)$, $\pi_2(d\psi) = 0$, and $\pi_2(d\phi) = d\phi$ (which follows from the fact that $A, B, C, D$ are independent of $x, y$). But this implies that the 1-form $\phi$ is integrable, and by the Pfaff Theorem, locally there must exist functions $\tilde{U}, \mu$ such that

$$\phi = e^\mu d\tilde{U}.$$

Moreover, since $\phi$ has no dependence on the variables $x, y$, we can assume that $\tilde{U}, \mu$ have no dependence on $x, y$ as well. Thus, we can define a nonvanishing local section $\theta$ of $L$ by

$$\theta = e^{-\mu}\theta_1 = d\tilde{U} - \tilde{P}_1 dx - \tilde{Q}_1 dy,$$

where $\tilde{P}_1 = e^{-\mu}R$ and $\tilde{Q}_1 = e^{-\mu}S$ are functions of the variables $u, v, p, q$, with no dependence on the variables $x, y$. By a similar argument, there exist functions $\tilde{V}, \tilde{P}_2$ and $\tilde{Q}_2$ of the variables $u, v, p, q$ such that

$$\theta = d\tilde{V} - \tilde{P}_2 dx - \tilde{Q}_2 dy,$$

is a nonvanishing local section of $\mathcal{L}$.

Now we re-label the coordinates $\tilde{U}, \tilde{V}, \tilde{P}_2, \tilde{Q}_1$ as $u, v, p, q$ respectively. Then we have

$$\theta = du - \tilde{P}_1(u,v,p,q) dx - q dy,$$

$$\theta = dv - p dx - \tilde{Q}_2(u,v,p,q) dy.$$

Set $F = \tilde{P}_1$ and $G = \tilde{Q}_2$. As was shown in the proof of Proposition 3.3, $F$ is in fact independent of $q$, and $G$ is independent of $p$. Thus, the Bäcklund transformation is given by

$$u_x = F(u,v,u_x),$$

$$v_y = G(u,v,u_y),$$

where there is no explicit dependence on $x$ or $y$ in the right-hand sides, as claimed. As noted previously, it follows that the right-hand sides $f, g$ of the Monge-Ampère equations (3.8) are also independent of $x$ and $y$.

In the quasilinear case, we have the following corollary:

**Corollary 3.8.** Let $\mathcal{P} \owns \mathcal{B}$ define a quasilinear wavelike Bäcklund transformation, and assume that there exist vector fields $X, Y$ on $\mathcal{B}$ that satisfy the conditions of Proposition 3.7. Then near any point of $\mathcal{B}$ there are coordinates $x, y, u, v, p, q$ and functions $F^0, F^1, G^0, G^1$ satisfying the conclusions of Proposition 3.5 and which are independent of $x$ and $y$. □

4. Discussion

In order to classify wavelike normal Bäcklund transformations for which both tensors $\tau_1, \tau_2$ are nonvanishing, as in Proposition 3.3, it would be necessary to find all solutions of the PDE system (3.4)-(3.7) satisfying the appropriate nondegeneracy conditions. After using equations (3.5), (3.6) to eliminate $f, g$, the remaining equations are an overdetermined system of four PDEs for the two functions $F, G$. The Cartan theory of exterior differential systems [6] provides a powerful technique for determining the solution space for such
overdetermined systems; in the sequel [3], we will apply these methods to give a partial clas-
sification of quasilinear, wavelike Bäcklund transformations, both in the autonomous and
non-autonomous cases.

We close by noting that not all interesting Bäcklund transformations involving integrable
PDEs fit the definition given at the beginning of §2.2. In particular, a more general definition
of Bäcklund transformation allows for a total space $\mathcal{B}$ where the fibers of $\pi$ and $\pi'$ have
arbitrary dimension, and $\mathcal{B}$ carries a Pfaffian system $\mathcal{J}$ which is an integrable extension
of the Monge-Ampère systems on either side (see, e.g., Defn. 6.5.10 in [6]). For example,
consider the Tzitzeica equation

$$(\ln h)_{xy} = h - h^{-2},$$

(4.1)

which arises in connection with the construction of affine spheres [7]. (Note that this PDE can
be put into the wavelike form (3.1) by setting $u = \ln h$.) This equation has an auto-Bäcklund
transformation which is defined by a compatible system of total differential equations

$$\alpha_x = \frac{h_x \alpha + \lambda \beta}{h} - \alpha^2, \quad \alpha_y = h - \alpha \beta,$$

$$\beta_x = h - \alpha \beta, \quad \beta_y = \frac{h_y \beta + \lambda^{-1} \alpha}{h},$$

where $\lambda$ is an arbitrary nonzero constant. Given a solution $h$ of (4.1), one solves for $\alpha$ and
$\beta$, and then the new solution of (4.1) is given by $h' = 2\alpha \beta - h$. (Note that the system is
symmetric under the interchanging of $h$ and $h'$.) The system for $\alpha$ and $\beta$ is equivalent to
a Pfaffian system of rank 2 defined on a 7-dimensional total space $\mathcal{B}$, with submersions to
5-manifolds $\mathcal{M}, \mathcal{M}'$, each carrying a copy of the Monge-Ampère system encoding (4.1).

To give another example, in our previous paper [4] we proved that (almost) all hyperbolic
Monge-Ampère equations that are Darboux-integrable at second order are linked by
a Bäcklund transformation to the wave equation $z_{xy} = 0$, and this transformation is of the
type described in §2.2. These transformations may be composed, in an obvious way, to yield
a more general Bäcklund transformation between any two of these equations (for example,
$u_{xy} = 2\sqrt{u_x u_y}/(x + y)$ and $v_{xy} = 2v/(x + y)^2$) where, again, the total space has dimension 7.
However, as we will see in the sequel, certain pairs of these equations are linked to each other
by quasilinear wavelike Bäcklund transformations of the type discussed in this paper, where
the total space is 6-dimensional, are which do not appear to involve the wave equation.

References

[1] J. Clelland, Homogeneous Bäcklund transformations of hyperbolic Monge-Ampère systems, Asian J. Math. 6 (2002), 433–480.
[2] J. Clelland, T. Ivey, Parametric Bäcklund transformations I: Phenomenology, Trans. Amer. Math. Soc. 357 (2005), 1061–1093.
[3] –, Classifying Backlund transformations of sine-Gordon type, in preparation.
[4] –, Bäcklund Transformations and Darboux Integrability for Nonlinear Wave Equations, Asian J. Math. 13 (2009), 13–64.
[5] E. Goursat, Leçons sur l’intégration des équations aux dérivées partielles du second ordre à
deux variables indépendantes, Tome I, Hermann, 1896.
[6] T. Ivey, J.M. Landsberg, Cartan for Beginners: Differential geometry via moving frames and
exterior differential systems, Graduate Studies in Mathematics vol. 61, American Mathematical So-
ciety, 2003.
[7] C. Rogers, W. Schief, Bäcklund and Darboux Transformations: Geometry and Modern Ap-
plications in Soliton Theory, Cambridge, 2002.
