Dinamic of Atlas (SXV)

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Abstract: The current orbit of Atlas was analyzed using frequency phase space mapping. Finding that the Corotation and Lindblad resonances are separated by about 4 kilometers, the latter is related to Atlas’ eccentricity greater than 0.0095. Extending the Dynamic Maps concept we find, in addition to the 53:52 resonance (Cooper et al. 2015), the 55:54 resonance. Finally, we demonstrate how gravitational perturbations by Pandora contribute to additional oscillations of the critical angle for the 54:53 resonance.

Keywords: Celestial mechanics, dynamics of natural satellites, resonances, Atlas

1. Introduction

The visit of Voyager I and II space probes Saturn system in 1980s contributed to the discovery of Atlas, Prometheus and Pandora satellites (Greenberg 1984). However, their physical and dynamic characteristics only began to be understood in 2000s with data obtained by the Cassini space probe, whose images revealed that Atlas has an ellipsoidal shape with an average radius of ~ 15 km and an extensive equatorial mountain range orbiting the A ring outer edge, while Prometheus and Pandora are ellipsoidal in shape with mean radii on the order of 40 and 42 km, respectively, orbiting the F ring inner and outer edges, respectively (Thomas and Helfenstein 2020).

Due to the gravitational interactions between satellites, the orbits of Saturn’s many satellites suffer from quasi-periodic perturbations that are not negligible. Such perturbations cause many satellite pairs to have their orbits near mean-motion commensurabilities resulting in resonant interactions (Callegari and Yokoyama 2020). The effect of mean-motion resonance perturbations can range depending on satellite size and mass (assuming same density).

Using a combination of data produced by the Voyager missions, data obtained by the Hubble Space Telescope, Earth-based telescopes, and by recent data from the Cassini spacecraft Spitale et al. (2006) conducted the first studies to characterize the orbital dynamics of Atlas. In particular, Spitale et al. (2006) was first to suggest the proximity of Atlas to the 54:53 mean-motion resonances with Prometheus and 70:67 with Pandora given by the angular combination $\phi_1 = 54\lambda_{Pr} - 53\lambda_{At} - \sigma_{Pr}$ and $\psi_1 = 70\lambda_{Pa} - 67\lambda_{At} - 3\sigma_{Pa}$, to explain the oscillatory perturbations in their orbits where $\lambda$ refers to mean longitude and $\sigma$ the longitude of the pericenter, $Pr$, $At$, $Pa$ refer to Prometheus, Atlas and Pandora, respectively. However, Spitale et al. (2006) did not demonstrated the temporal behavior of these angles. As Prometheus and Pandora have chaotic orbits (Goldreich and Rappaport, 2003) Spitale et al. (2006) suggested that there are reasons to suspect that Atlas’ orbit would exhibit chaotic dynamics as well.

Subsequently, in a follow-up study, Cooper et al. (2015) followed the same methodology as described in Spitale et al. (2006) and investigated the time-series of the arguments $\phi_1$ and $\phi_2 = 54\lambda_{Pr} - 53\lambda_{At} - \sigma_{At}$ with the latter being associated with the 54:53 mean-motion resonance between Atlas and Prometheus over a 20 year time period. They found that these critical angles displayed alternations...
between circulations and librations. Furthermore, Cooper et al. (2015), from applying the Fast Lyapunov Indicator (FLI) method, determined the orbit of Atlas to be chaotic caused by a coupling between the eccentricity-corotation (CER) and eccentricity-Lindblad (LER) 54:53 resonances with Prometheus. They numerically determined a Lyapunov exponent of order 10 years confirming the conjecture put forward by Spitale et al. (2006).

Based on the CoraLin analytical model (El Moutamid et al. 2014), Renner et al. (2016) investigated the dynamical cause for the origin of the chaotic motion of the time evolution of Atlas. Following the work by Renner et al. (2016) the alternation between episodes of librations and circulations originates due to resonance overlap between the CER and LER resonances. According to Renner et al. (2016) the chaotic dynamics is lifted for a Prometheus-like satellite for an orbital eccentricity smaller than $10^{-5}$.

Furthermore, there is no discussion in the works by Spitale et al. (2006) and Cooper et al. (2015) related to the 70:67 mean-motion resonance in the Atlas-Pandora pair. This was subsequently investigated by Renner et al. (2016). The authors concluded that the overlap for the four arguments related to the 70:67 resonance (Renner et al. 2016, their table 3) does not contribute to the emergence of chaos and as such does not contribute significantly on the dynamical time evolution of the orbit of Atlas. However, we note, that Renner et al. (2016) is lacking to provide any further analysis details that could contribute to this statement.

Both previous works, Spitale et al. (2006), Cooper et al. (2015) and Renner et al. (2016), claim that perturbation by Pandora do not cause significant disturbances in the orbital evolution of Atlas.

The main purpose of this work is to extend the work presented in Cooper et al. (2015) and Renner et al. (2016) in an attempt to solve the conundrum related to the observed intermittent behaviour in time variations for the phi1 and phi2 critical angles related to the 54:53 mean-motion resonances between the Atlas-Prometheus pair. In Section 2, we will provide an extensive investigation into the current orbit of Atlas as we deem this approach to be necessary. To this end we considered several systems formed by the Atlas-Prometheus and Atlas-Pandora pairs and the tender (do you mean fragile?) Atlas-Prometheus-Pandora and investigate the dynamical properties using different numerical models as discussed in Section 3. In Section 4, we generalize the properties of the 54:53 Atlas-Prometheus mean-motion resonance and explore this particular resonance in some more detail, where a large set of Atlas-like test satellites were numerically integrated and the time evolution of their orbital elements analyzed with a spectral analysis method. This approach allowed us to create dynamic maps of relevant portions of phase space to obtain a detailed mapping of the structures associated with Atlas-like resonant dynamics (Callegari and Yokoyama, 2010, 2020; Callegari et al. 2021).

2. Methodology

In this work, we follow the same methodology described in Callegari and Yokoyama (2010, 2020) and Callegari et al. (2021). Results were obtained from numerically integrating the equations of motion for $N$ satellites orbiting Saturn. We consider full mutual gravitational interactions between the satellites and the disturbing effects due the oblateness (rotational deformation with Saturn as an oblate spheroid) of Saturn as a central body. The latter effect is quantified through the expansion of non-symmetric potential. Our simulations consider a maximum of three satellites, namely Atlas, Prometheus and Pandora.

The simulations can contain up to 3 satellites, named Atlas, Prometheus and Pandora. Two different models were adopted and compared to each other. i) the exact equation model for orbital
motion given in Callegari and Yokoyama (2010), which considers all the satellites listed above plus \( J_2 \) and \( J_4 \) (the dimensionless coefficients present in the expansion of Saturn's potential up to 4th order); ii) direct Mercury package application (Chambers 1999) with term \( J_6 \) addition (dimensionless coefficient present in the Saturn's potential of expansion to the 6th order). In both cases i) and ii) we apply the code "RA15" (Everhart 1985) for the numerical solution of the system of ordinary differential equations with local error of 15th order. Initial conditions and numerical parameters are listed in Table 2. Details of specific numerical simulations are indicated in figures corresponding legends (see Figure 1, as an example).

Figure 1: The argument of the pericenter (left) and the orbital eccentricity (right) of Atlas. The black curve represents the Atlas osculating pericenter (\( \omega \)) obtained from http://ssd.jpl.nasa.gov/horizons.cgi with a starting date of January 1\(^{st} \), 2000. The data were obtained for 3 days with an interval of 5 minutes. The yellow curve was generated from scheme ii) where only the perturbation due to the \( J_2 \) term was considered. The green curves represent the geometric pericenter (left) and the geometric eccentricity (right) calculated from the scheme ii) considering the perturbations caused by the terms \( J_2, J_4 \) and \( J_6 \).

Two additional efforts were always be applied to analyze our numerical dataset: I) in some cases, the orbits were compared with the osculating orbital elements provided by the Horizons\(^4\) ephemeris system (Giorgini et al. 1996) ensuring validity of our numerical data; II) from the so-called state vector, that is, the vector that contains the position coordinates and the respective instantaneous velocities for the satellite obtained from the osculating elements of the numerical simulation, we calculate the so-called geometric elements by the algorithm direct application described in Renner and Sicardy (2006).

For Callegari et al. (2021) the geometric elements are calculated to obtain a better orbit fit when a satellite is very close to a strong non-central field (such as Saturn) and the orbital eccentricity is very small in relation to a threshold value, very close to zero. In this case, the pericenter argument (\( \omega \)) periodically advances with the same frequency as the mean-movement. This fast component observed in \( \omega \) is caused by \( J_2 \) (Callegari et al. 2021).

Fig. 1 illustrates our method of calculation and comparison with the different numerical schemes. On the left in Fig. 1 we have the short-time advance of \( \omega \) in the case of Atlas. The black curve

\(^4\) http://ssd.jpl.nasa.gov/horizons.cgi
was generated with the data obtained from the *Horizons* system. The yellow curve was obtained with a numerical simulation performed with scheme ii) where an Atlas-like satellite on the effects of $J_2$ was included in the simulation along with Prometheus and Pandora. The good alignment between the black and yellow curves shows that the fast circulation of the Atlas pericenter is governed by the term $J_2$. The green curve represents the Atlas geometric pericenter calculated with scheme II) from the state vector generated in simulation ii) considering the perturbations caused by the terms $J_2$, $J_4$ and $J_6$ along with Prometheus and Pandora.

Finally, another method applied throughout this work was that the orbits obtained numerically were analyzed in the frequency domain, so that variables such as the osculating semi-major axis, the osculating eccentricity and osculating orbital inclination were analyzed via Fast Fourier Transform (FFT) with the algorithm given in Press et al. (1996). A large number of significant peaks appear in the distinct variables and are identified according to the different disturbing forces that will be described in detail later. A systematic analysis of the over a wide set of initial conditions allowed us to follow the dependence of the main frequencies of the main amplitudes and the corresponding frequencies. The results are presented in two different forms: the Individual Power Spectra given in Fig. 6 in section 4.1 and the Dynamic Maps (MD) as in Fig. 8, section 4.2. Both approaches constitute the “resonance mapping” in the frequency domain: with IPS the dependence of the main system frequencies with a free parameter is investigated; with MD a new dimension is added, named the orbital eccentricity, in that case the amplitudes was analyzed. Miczenko and Ferraz-Mello (2001) introduced this technique in planetary dynamics and Callegari and Yokoyama (2010, 2020) followed by Callegari et al. (2021) who successfully introduced it to analyze the resonant dynamics of Saturn’s satellite pairs, namely: Mimas-Tethys, Dione-Enceladus, Rhea-Titan, Mimas-Anthe and Mimas-Methone.

### 3. The current orbit of Atlas

The main goal of this section is to identify the short-and long-period and resonant components in the time variation of the Atlas elements due to several perturbations. For this reason, we used several numerical simulations that provided us with output data similar to the actual orbital Atlas elements. The left and right columns in Fig. 2 show osculating elements and their geometric counterparts: from top to bottom, we can see the time variation for the major semi-axis, the eccentricity, and orbital inclination with respect to Saturn’s equatorial plane. All these variables were used to interpret Atlas’ orbit, as will be seen later.

#### 3.1 The $J_2$ effects on the pericenter argument ($\omega$) and the length of the ascending node ($\Omega$) of Atlas and Prometheus

Callegari et al. (2021) define “long mutual secular module” as those dictated by the secular perturbation of $J_2$ with the non-resonant, long-period interactions between Atlas-Prometheus. The perturbation caused by $J_2$ contributes significantly to the secular variation of the pericenter argument ($\omega$) and to the longitude of the ascending node ($\Omega$) are given respectively by (e.g. Danby 1988):

\[
\frac{d\Omega}{dt} = -\frac{3}{2} \frac{J_2 n R_s^2}{a^2 (1-e^2)^2} \cos i, \quad (1)
\]

\[
\frac{d\omega}{dt} = \frac{3}{2} \frac{J_2 n R_s^2}{a^2 (1-e^2)^2} \left(2 - \frac{5}{2} \sin^2 i\right), \quad (2)
\]

where $n$, $a$, $e$, $i$, $R_s$ are the mean motion, the major semi-axis, the eccentricity and orbital inclination with respect to the equator of the main body, and the equatorial radius of the planet. When the orbit is close
to the equatorial plane, \( \cos i \sim 1 \) and \( \sin^2 i \sim 0 \). Considering the initial date January 1st, 2000, we can apply Eq. (1) for Atlas and Prometheus, obtaining the periods for the retrograde variation of the ascending node of Atlas and Prometheus which are 130.95 and 136.67 days, similarly, by Eq. (2) we have that \( \omega \) of Atlas and Prometheus circulates in a prograde direction with period 65.47 and 68.34 days respectively, similar values to what we can observe in Fig. 3(a-b). The reason for rapid variation of \( \omega_{At} \) and \( \omega_{Pr} \), where the subscripts \( At \) and \( Pr \) refer to Prometheus and Atlas, respectively, is associated with the perturbations caused by \( J_2 \), a phenomenon well known by classical works (e.g. Greenberg, 1981).
**Figure 2**: Comparison of the left (black) and right (green) geometric osculating elements for the time variation of the major semi-axis, eccentricity and orbital inclination (degree) with respect to Saturn’s equator. We can observe the long-period oscillations for the geometric eccentricity and inclination caused by the long period secular mode. This simulation used the initial date January 1\textsuperscript{st}, 2000 with a 1 day interval; satellites included Atlas, Prometheus and Pandora, along with the terms $J_2$, $J_4$ and $J_6$.

As in Callegari et al. (2021), an important variable used in several analyses is the relative variation for osculating longitude ascending node of Prometheus and Atlas $\Delta \Omega_{Pr-At} \equiv \Omega_{Pr} - \Omega_{At}$. From the rate of variation $\Omega_{At}$ and $\Omega_{Pr}$ obtained above, we have that $\Delta \Omega_{Pr-At}$ circulates in the prograde direction with a period of approximate $T \sim 3123.6$ days, where $T = \frac{1}{130.95} - \frac{1}{136.68}$. Similarly, we obtained the relative variation for the pericenter osculator argument $\Delta \varpi_{Pr-At} \equiv \varpi_{Pr} - \varpi_{At}$, in the retrograde sense with period approximately $3133.15$ days. Both angular quantities show a slow and smooth circulation. This behavior is responsible for oscillations in the eccentricity and inclination of Atlas (see Fig. 2), which will be discussed below.

**Figure 3**: The short period dynamics for the pericenter argument ($\omega$) and the ascending node longitude ($\Omega$) can be seen at the top. The dotted curves in black represent the osculator and in green the geometric $\omega$ and in blue the $\Omega$. Initial conditions on January 1\textsuperscript{st}, 2000 calculated by 30,000-day interval with 1-day step.
We can see in Fig. 3 how the osculator $\omega$ has oscillatory components that make visual analysis difficult. We can notice that $\Delta \sigma_{Pr-At}$ is of difficult physical interpretation (a diffuse group of points), since it $\Delta \sigma_{Pr-At}$ contains the pericenter of both satellites, being strongly influenced by the fast component due to the $J_2$ term. For the others, we can observe the period related to the long-term secular module (~3123.6 days).

In the upper part of Fig. 3(a-b) we can observe the short period dynamics for $\omega$ and $\Omega$ for Atlas and Prometheus. At first sight, if we look at the osculator argument for the pericenter (black dotted curve) we see that it becomes difficult to interpret due to the term $J_2$, so we analyze the geometric one (green dotted line). We note that the geometric presents its behavior circling in the prograde direction with a period of approximately 60 days for Atlas and Prometheus (a more accurate value would be ~62.5 days for Atlas and ~65.5 days for Prometheus obtained by FFT). On the other hand, the osculating longitude of the ascending node (blue dotted line) exhibits its retrograde circulation with period ~130 days (more exact ~125.1 day for Atlas and 131.1 days for Prometheus obtained by FFT). The behavior for $\Delta \sigma_{Pr-At}$ and $\Delta \Omega_{Pr-At}$ can be observed in Fig. 3(c-f). We see on the left the osculator $\Delta \sigma_{Pr-At}$, at first sight, the physical interpretation becomes complicated due to the noise caused by the term $J_2$, but when we consider the geometric one, we see a slow prograde circulation with period $P_{\Delta \sigma} \sim 3123.6$ days. In contrast, both elements, osculator and geometric for $\Delta \Omega_{(Pr-At)}$ can be observed in a slow prograde circulation with period $P_{\Delta \Omega} \sim 3123.6$ days. For Callegari et al. (2021) these periods were defined as “long period mutual secular module” and can be seen in Fig. 3(d-f). This long-period component appears as a regular oscillation in eccentricity and inclination. In Fig. 2 we identify the oscillations related to the long-period secular mode.

3.2 The proximity of Atlas-Prometheus with the 54:53 mean-motion resonance

Spitale et al. (2006) proposed the Atlas-Prometheus proximity with a 54:53 mean movements resonance and Cooper et al. (2015) present in their Fig. 12 the time variation for the arguments $\phi_1$ and $\phi_2$ for a period of 20 years (~7305 days) suggesting that both arguments are librating in an irregular behavior, i.e., alternating between oscillations and “quasi-periodic” circulations. For Cooper et al. (2015) this behavior is not related to other disturbances arising from the Prometheus-Pandora pair, showing that only Prometheus has a strong influence on Atlas.

3.2.1 The corotation angle: $\phi_1$

In Fig. 4 we see the time variation for the geometric arguments $\phi_1$ and $\phi_2$ for a period of 30,000 days. If we compare the 15:14 Mimas-Methone (Callegari et al. 2021) and 11:10 Mimas-Anthe (Callegari and Yokoyama 2020) mean-motion resonances with the 54:53 resonance between Atlas-Prometheus, we see that the behavior for the geometric argument $\phi_1$ (left) presents periods of alternation between oscillation around 180° and circulation. This behavior will be easily explained in section 4.2.
Figure 4: Time variation for the geometric arguments $\phi_1$ and $\phi_2$ for a period of 30,000 days. Physically the libration around 180° means that the line of conjunctions between the Atlas-Prometheus pair oscillates around the apocenter of Prometheus, but as there is an irregularity in the time variation of $\phi_1$, alternation between oscillation and circulation, the line of conjunctions is not well defined. As with $\phi_1$, the argument $\phi_2$ presents its irregular temporal variation: alternation between circulation and oscillation. Initial conditions like Fig. 2, except the Pandora satellite.

We can observe that for the initial date January 1st, 2000, the geometric argument $\phi_1$ is oscillating around 180° for an approximate main period 5600 days, alternating its regime for a brief period of circulation. Physically the libration around 180° means that the Atlas-Prometheus pair conjunctions oscillate around a line that passes through the apocenter of Prometheus and this irregular behavior shows that the line between the Atlas-Prometheus pair conjunction is not well defined. For Callegari et al. (2021) a resonant system can only be determined by the true libration of a certain critical angle, but as seen in Fig. 4, we cannot consider that the orbit of Atlas is stuck in the 54:53 resonance of mean movements with Prometheus, but as we will see later, it can be considered close to this resonance.

3.2.2 The Lindblad angle: $\phi_2$

The behavior for the argument $\phi_2$ is also irregular, as can be seen in Fig. 4 (right). The observed retrograde circulation can be explained by the rewriting of $\phi_2$, as in Callegari and Yokoyama (2020), were we can rewrite the Lindblad angle as an angular combination between the geometrics $\phi_1$ and $\Delta\sigma_{Pr-At}$ with a simple addition and subtraction of the argument $\sigma_{Pr}$ and rearranging the terms as follows: $\phi_2 = 54\lambda_{Pr} - 53\lambda_{At} - \sigma_{At} + \sigma_{Pr} - \sigma_{Pr} = \phi_1 - \Delta\sigma_{Pr-At}$ Since $\Delta\sigma_{Pr-At}$ is circulating in the prograde direction, the minus sign makes $\phi_2$ retrograde. Thus, the dominant component in the Lindblad angle is the long-period mutual secular component, present in Atlas’ eccentricity and orbital inclination (Fig. 2). Therefore, Atlas is not trapped in the Lindblad resonance.

3.3 The secular variation for the eccentricity and inclination of Atlas

The forced and proper components for a given orbit due to secular interactions in general are well represented in polar coordinates. The mutual secular components $\Delta\sigma_{Pr-At}$ and $\Delta\Omega_{Pr-At}$ are mainly linked to the eccentricity and slope, respectively. The left column in Fig. 5(a-b) shows the projection for the geometric elements (green curve) obtained with schemes ii) and II) represented in a period of 5,000 days and the classical secular solution (red curve) for an Atlas-like satellite in the plane $e_{At}\cos(\Delta\sigma_{Pr-At}) \times e_{At}\sin(\Delta\sigma_{Pr-At}) \times i_{At}\cos(\Omega_{Pr-At}) \times i_{At}\sin(\Omega_{Pr-At})$.

As we can see in Fig. 5(a), the trajectory related to the Atlas eccentricity is very close to the origin; on the other hand, in Fig. 5(b) we can see that the trajectory related to the Atlas inclination is far away in relation to origin.
In Fig. 5(a) the green curve represents the own component for the geometric eccentricity of Atlas, $e_{pg} \sim 0.002$, while the black curve represents the orbit obtained by Horizons and the red curve represents the classical secular solution for a satellite that has the initial conditions similar to Atlas. We can estimate that the own component for this eccentricity is $e_p \sim 0.006$, i.e., almost triple the geometric component. The force inclination of Atlas is $i_f \sim 0.0032^\circ$, a good estimate measured from the average radius of the black trajectory depicted in Fig. 5(b). On the other hand, unlike the geometric own component for the eccentricity, the own component for the geometric inclination has a value similar to that obtained with classical secular theory, that is, $-0.004^\circ$.

The geometric elements are weakly affected by these short period variations; however, the geometric eccentricity of Atlas suffers from long period oscillations caused by the long period secular mutual component. The column on the right represents the time variation for the eccentricity and inclination of the simulations present in the column on the left. In order to compare the geometric elements and the classical solution, we add the eccentricity and inclination osculating elements (black curves). We can isolate the long-period components for the proper eccentricity and inclination by solving Classical Secular Theory, by the system of differential equations given in section 7.7 of Murray and Demont (1999) for a system formed by Saturn with the coefficients $J_2$ and $J_4$, Prometheus and a particle that has the same initial conditions as the osculating orbit of Atlas, representing the second order approximation in eccentricity and inclination. The result is the red lines.

In Fig. 5(c) we can see that the simultaneous comparison of these three elements is complicated by scaling issues. The geometric eccentricity is much smaller than the osculating one, making it difficult to visually identify long-period components. On the other hand, in Fig. 5(d) we can identify the long-period secular mutual component for the inclination. We can note that the period obtained with classical secular theory is similar to the present value $\sim$3100 days.

Differently from what Callegari et al. (2021) found in their Fig. 6, we cannot identify resonant components since the current orbit of Atlas is not locked into the 54:53 mean movements resonance with Prometheus.
Figure 5: Black curves represent the osculating elements obtained with scheme ii); green curves geometric elements obtained with scheme II), red curves obtained with the classical secular theory including Saturn with coefficients $J_2$ and $J_4$ and Prometheus. (a) projection of trajectories into the plane $e_{Al}\cos(\Delta \varpi_{Pr-Al}) \times e_{Al}\sin(\Delta \varpi_{Pr-Al})$, (b) projections of trajectories into the plane $l_{Al}\cos(\Omega_{Pr-Al}) \times l_{Al}\sin(\Delta \Omega_{Pr-Al})$, where $l_{Al}$ is in degree, (c) time variation for the eccentricity corresponding to (a), (d) variation of the orbital inclinations corresponding to (b). We can see in (d) the long period secular mutual component in the inclination.

4. The 54:53 Atlas-Prometheus resonance mapping in frequency domain and phase space

In the previous section, we provided a detailed description for the dynamics of Atlas considering only the current orbital configuration (relative to the epoch January 1st, 2000) and of the other bodies. As seen in Section 3.2.1, Atlas is not locked into the 54:53 mean-motion resonance with Prometheus, but rather we can consider that it lies close to this resonance. Considering this hypothesis, our starting point will now be to explore numerically the phase space domain of this mean-motion resonance in the neighborhood of Atlas real orbit. For this, we build a dense set of thousands of orbits for satellites that have their initial conditions close to the actual Atlas orbit, and numerically analyze the Fourier spectrum for these orbits. Our map was constructed by considering the spectral peaks obtained for each of the individual spectra.

As in Callegari and Yokoyama (2020) and Callegari et al. (2021), this mapping will be done in two steps: first we performed a one-dimensional mapping of the resonance adopting in the numerical simulation the major semi-axis of an Atlas-like satellite, $a_0$, as a free parameter. The results were called
the "Individual Power Spectrum", denoted by IPS. Then, a new dimension, named orbital eccentricity, \( e_0 \), was added, resulting in a two-dimensional map, denoted by Dynamic Map (MD).

While the simulations are calculated for a dense set of initial conditions of \( a_0 \) (IPS) or \((a_0, e_0)\) (MD), the other orbital elements for the test satellite were assumed to be considered the same as Atlas with respect to the date January 1st, 2000 (see Table 2). For each initial condition the osculating elements major semi-axis, eccentricity and orbital inclination were analyzed with the Fourier spectrum. The y-axis in the IPS displays for each initial condition the period associated with the peak in the spectrum with amplitude greater than the present value. We denote this value as the "reference amplitude", which usually represents 1% or 5% of the largest peak recorded in each of the individual spectra. The dynamic maps are based on a specific number \( N \) defined as the number of significant peaks in the individual spectra, i.e. those that are larger than the reference amplitude. A color scale in the mapping was related to the value of \( N \), from white (\( N=0 \)) to dark blue (\( N>100 \)). For intermediate values of \( N \), the representative color was white/blue and for larger values the color tended toward dark blue.

### 4.1 The Individual Power Spectrum

In Fig. 6 we can see the IPS obtained from 1000 orbits Atlas-like satellites with initial conditions given in the range \( 138315 \leq a_0 \leq 138335 \) km. The variable used in the Fourier analysis for Fig. 5(a) was the osculating semi-major axis, in Fig. 5(b) was the osculating eccentricity, and in Fig. 5(c) the osculating orbital inclination. For Callegari and Yokoyama (2020) when interpreting an IPS we must consider that the spectrum of solutions obtained with the equations of movement contains many frequencies, their harmonics and linear combinations between them, so the main objective was the identification of these frequencies and the dependence of these values with the initial parameter and their relevant physical meaning. The fundamental location periods were distributed in the vertical direction and in Fig. 5(a) two main regions was identified:

- a) Region A, where the periods associated with the long-period disturbances caused by \( J_2 \) are distributed, the periods associated with proximity to the 54:53 Atlas-Prometheus resonance, all the corresponding harmonics and linear combinations between them.

- b) Region B, where we find the amplitudes related to the short period. But this region has no relevant physical significance for dynamic characterization, being maintained just for question of IPS completeness (Callegari and Yokoyama 2020). We can see that the constant line in Fig. 6(a) represents the Atlas orbital period, ~0.6 day.

In Callegari and Yokoyama (2020) Fig. 5 and Callegari et al. (2021) Fig. 6, we can notice that the site for the period of the resonant argument has a small number of significant periods in its spectrum along the osculator semi-major axis interval and this feature contributes to the confinement of the orbit in the respective mean movement resonance. However, Fig. 6(a) shows this region, \( 138324 \leq a_0 \leq 138325.5 \) km, with a dense concentration of significant peaks that are not well spaced, impairing the identification of fundamental resonant periods. This high number of peaks suggests the proximity of the orbits of these test satellites to the 54:53 mean-motion resonance with Prometheus. In Fig 6(b-c) we can observe the period \( \approx 2859.75 \) days related to the circulation of the secular component mutual in the eccentricity, \( P_{\Delta e} \), and inclination \( P_{\Delta \Omega} \), a different value from that obtained in section 3.1.
Figure 6: IPS for 1000 Atlas-like satellites in the interval of initial major semi-axis, $a_0$, given on the x-axis. The IPS was constructed from the spectrum for (a) the osculating major semi-axis, (b) the osculating eccentricity, and (c) the osculating orbital inclination. The y-axis is given in logarithmic scale (base 10). The integration time for each initial condition is 172.25 years, sampled with an interval of 0.06 days. The numerical simulation was performed with scheme ii) including gravitational perturbations $J_2$, $J_4$ and $J_6$. Atlas and Prometheus without the presence of Pandora. With the exception of the major semi-axis, the other initial conditions were held constant with respect to epoch January 1st, 2000. The location for the fundamental periods is distributed in the vertical direction and two main regions can be identified: A (long periods, resonant and their harmonics) and B (short periods and their harmonics). We can observe in (a) the orbital period of Atlas (~ 0.6 day) and the periods related to the long-period secular mutual component eccentricity (b) and inclination (c). The letter C represents the corotation region associated with the 54:53 Atlas-Prometheus resonance.

When the spectrum related to a given orbit contains a high number of peaks, the orbital motion can be characterized as “quasi-periodic”, irregular or chaotic. As opposed to the regular orbital motion of Anthe and Methone discussed in Callegari and Yokoyama (2020) and Callegari et al. (2021), the spectra for the major semi-axis and Atlas eccentricity osculating elements given in Fig. 7 exhibit this behavior. For Callegari and Yokoyama (2020), this represents a spectrum of an orbit that is at the edge of the corotation resonance. This fact will be discussed in more detail in the next section.

Figure 7: (a) Fourier spectrum for the osculating major semi-axis and (b) osculating eccentricity of an Atlas-like satellite. We only consider periods longer than 100 days, corresponding to long-period dynamics.
We can observe in Fig. 7 the high number of peaks. The total integration time for this orbit with initial conditions relative to the epoch January 1st, 2000 was 1720 years, with an interval of 0.06 day. In (b) note is the period related to the mutual secular component for the eccentricity ($P\Delta\omega$).

4.2 The dynamic mapping for the 54:53 mean motions resonance between Atlas and Prometheus in the frequency phase space

We explore the phase space domain of the 54:53 mean-motion resonance. The mapping was performed considering the frequencies domain obtained with Fourier spectrum for a set of numerically integrated orbits, whose initial conditions are close to the real orbit of Atlas (Callegari and Yokoyama, 2010; 2020).

Callegari and Yokoyama (2010, 2020) analyze the phase space around the orbital neighborhood of a satellite according to the value of the spectral number $N$. To construct this $N$ we consider a pre-set value, usually 5% of the reference amplitude for the Fourier spectrum for each of the individual orbits of these test satellites. Therefore $N$ represents the number of peaks in the spectrum that are greater than or equal to the pre-set reference value.

In Fig. 8 we can observe the dynamic maps constructed from the spectra of a dense grid of initial conditions for $(a_0, e_0)$. The red star represents the initial condition $(a_0, e_0) = (138325.32$ km, 0.0059) for Atlas.

The map on the left Fig. 8(a) represents the phase space of the spectral domain for the semi-axis and the map on the right Fig. 8(b) represents the spectral domain for the orbital eccentricity. We can notice three distinct regions for the map on the left:

a) C: presents a small region ~4 km long isolated by dark blue regions starting at $e_0 \sim 0.003$ and ending at $e_0 \sim 0.006$ between that containing the domain in phase space for the 54:53 corotation resonance between Atlas and Prometheus. In the corotation region, the orbital movement was regular and the value of the spectral number $N$ was low. For orbits that have initial conditions $(a_0, e_0)$ interior to this region, we had the libration of the geometric argument $\phi_1$ (the co-rotation angle) around 180°. The dark blue regions near the corotation region showed a high value for $N$ and geometric argument $\phi_1$ alternated its time variation between oscillations around 180° and circulations, a typical feature for regions that represent the edges of the resonance (Callegari and Yokoyama, 2020);

b) A: region characterized by the low value of $N$ limited between the left edge of region C extending to values of $e_0$ between 0 to 0.11. The orbits inside this region with low value for $e_0$ presented the geometric argument $\phi_1$ circling in the retrograde direction, and those with a higher value for $e_0$ had the geometric argument $\phi_1$ circling in the prograde direction;

c) B: region that has similar characteristics to region A, but for orbits that present a small value for $e_0$ the argument $\phi_1$ circulates in the prograde direction and circulating in the retrograde direction when $e_0$ have a high value;

d) L: for $e_0 \geq 0.009$ we had a region isolated by dark blue regions ~6 km long in which the value for the spectral number $N$ was low. The geometric argument $\phi_2$ started to present its time variation alternating between oscillations and circulations and when we considered values for $e_0 \geq 0.01$ $\phi_2$ began to librate around 0, implying that the line of conjunctions between the Atlas-like satellite and Prometheus passed through the pericenter of the Atlas-like satellite characterizing this region as the domain for the 54:53 Lindblad resonance.
Figure 8: (a) Dynamic map constructed from the spectrum of the osculating major semi-axis of 15,000 Atlas-like satellites. The colors stand for: orbits whose conditions \((a_0, e_0)\) are interior to region C neon red symbols; orbits whose initial conditions \((a_0, e_0)\) are interior to region L orange symbols; orbits with initial conditions \((a_0, e_0)\) interior to region A and B yellow symbols. (b) Map constructed similarly to (a), but we use the spectrum of the orbital osculating eccentricity.

In Fig. 8 N represents spectral number defined in section 4.2. C and L represent the regions for the 54:53 Corotation and Lindblad resonances associated with Atlas-Prometheus. The equations system and initial conditions are the same as those used in Fig. 1, except that Pandora is not present in the numerical simulations. For each of the initial condition the total integration time in the numerical simulation was \(~188,743.68\) days (~516.75 years) sampled in 0.18 day intervals. The different symbols present in this map correspond to initial conditions for the orbits represented in Fig. 9 and the red star corresponds to the orbit of Atlas with respect to 01/01/2000 epoch. The green horizontal lines indicate cross-sections in the map that will be used in the discussion of Fig. 10. Colored symbols and the other orbital osculating elements are fixed with respect to Table 2.

A simple visualization in Fig. 8(a) reveals that the orbit of Atlas relative to the epoch January 1\(^{st}\), 2020 was at the edge of the resonance and this location justifies the alternation between oscillation and circulation observed in the geometric argument \(\phi_1\) by Spitale et al. (2006). This feature and the others above can be verified by analyzing the individual numerical simulations for each Atlas-like satellites represented in Fig.8.

4.2.1 The region for the Corotation resonance

The time variation for 4 geometric arguments \(\phi_1 = 54\lambda_{pr} - 53\lambda_S - \omega_{pr},\) \(e_2 = 54\lambda_{pr} - 53\lambda_S - \omega_S,\) where \(S\) represents an Atlas-like test satellite that has its initial condition \((a_0, e_0)\) inside or close the edge of region C were represented by the distinct neon red symbols. For the initial conditions \((138322\) km, 0.004), \((138323\) km, 0.005) and \((138323.25, 0.0047)\), we see that the argument \(\phi_1\) pounds around 180° and \(\phi_2\) circulates in a prograde movement, Fig. 9(a-c). On the other hand, the initial condition given by \((138324.25\) km, 0.005) near the edge of region C presented the time variation of the argument \(\phi_1\) oscillating around 180° followed by a brief period of circulation returning to oscillate,
changing to a short period of circulation to then oscillate again around 180° and \( \phi_2 \) circulating in prograde movement.

The red star in Fig. 8(a) represents the initial condition (138325.32 km, 0.0059), that is, the values for the osculating semi-major axis and for the osculating eccentricity of Atlas with respect to epoch 01/01/2000. Thus, we can observe that Atlas lies on the edge between region C and the dark blue region. Fig. 4(a) represents the time variation for the geometric argument \( \phi_1 \) which has a variation between oscillation around 180° followed by circulation periods. This behavior is characteristic of satellites that have their orbits at the edge of the resonance, therefore, with the help of dynamic map we conclude that Atlas is not stuck in the 54:53 mean movements resonance with Prometheus, but rather, lies at the edge of this region (Callegari Jr and Yokoyama 2020, Callegari et al. 2021).

Renner et al. (2016) mapped for the first time the 54:53 resonance domain of Atlas-Prometheus mean-motion by applying the analytical model with two degrees of freedom called CoraLin, described in El Moutamid et al. (2014). For comparison between results, we calculated the actions \( J_C \) and \( J_L \) described in Table 1 of El Moutamid et al. (2014), using the values for the geometric elements semi-major axis and eccentricity, obtained from the state vector generated by the Horizons system for Atlas and Prometheus. We adopted the geometric value 137665.519 km for the parameter \( a_0 \) which represents the center for the resonance according to the CoraLin model (El Moutamid et al. 2014). El Moutamid et al. (2014) and Renner et al. (2016) plotted \( \chi \equiv (J_C - J_L) \times \phi_1 \) showing that the location and extent for chaotic regions depends on the value of the resonant argument with respect to the epoch (Callegari and Yokoyama 2020). The CoraLin model has the fundamental parameter \( D \), described in Table 1 of El Moutamid et al. (2014), which represents the distance, in the frequency domain, between the corotation and Lindblad regions, i.e., the \( D \) value is an indicative to estimate the coupling between the two resonances. For \( |D| = 0 \) the resonances are overlapping and for \( |D| \) large the resonances are separated (Renner et al. 2016). Using the geometric orbital elements obtained from the Atlas and Prometheus state vector obtained via Horizons, we find the value \( |D| \approx 0.49 \), indicating that the corotation and Lindblad resonances are separated, confirming the result observed in Fig. 8(a).
Figure 9: Geometric arguments $\phi_1$ and $\phi_2$ obtained from the numerical simulations corresponding to the initial conditions represented by the different symbols in region C of Fig. 8(a).

In Fig. 9(a) we have an orbit with initial condition near the lower edge of region C, in Fig. 9(b) we have the orbit near the center of region C, in Fig. 9(c) the orbit is near a right upper edge of C and in Fig. 9(d) we have the orbit at the edge of region C with the dark blue region. We see that Fig. 9(a-c) the
argument $\phi_1$ is librating around 180° and $\phi_2$ is circulating in retrograde movement, on the other hand, in Fig. 9(d) the argument $\phi_1$ shows alternation between oscillation and circulation of its movements and $\phi_2$ is also circling in retrograde movement. The simulation was performed in the same way as Fig. 1, without Pandora, but for a period of 30,000 days with an interval of 1 day.

In Fig. 10 we have the plots $\chi \times \phi_1$ for the orbits with the initial conditions represented in Fig. 9. Each of these orbits is represented with its respective plot in different colors. In Fig. 10, each of the orbits has the value for $|\chi| < 2$ characteristic for regular orbits (Callegari et al 2021). The plot in bold represents the geometric orbit obtained from the Horizons system, for the time interval between 01/01/2000 - 01/01/2050, totaling 18264 days, and is represented for comparison effect with the plot in red, which represents the geometric orbit for Atlas obtained from the initial conditions 01/01/2000 integrated for a period of 30,000 days.

![Figure 10: Plot for the orbits represented in Fig. 9.](image)

The plots in bold and red represent the comparison between the geometric orbit obtained directly from Horizons and the geometric orbit obtained through numerical integration in Fig. 10. We see the limitation of Horizons in this comparison, as we obtained only the period 01/01/2000-01/01/2050.

4.2.2 Regions A and B

The regions represented in the dynamic map for which the value of the spectral number $N$ is small are usually associated with the frequency domain for regular orbital movement, associated with high-order resonances, or other regions that have no physical significance (Callegari et al 2021).

In Fig. 8(a) we identify regions A and B, bounded between the left and right edges, of those with high value for $N$. A brief analysis for two of these orbits with initial conditions inside to these regions revealed the following features: these orbits are represented by the yellow symbols in Fig. 8(a) and the time variation for the geometric arguments $\phi_1$ and $\phi_2$ as described in section 4.2.1, reveals their rapid circulation. In Fig. 11(a) we have the orbit represented by the initial condition (138318 km, 0.004), both arguments, $\phi_1$ e $\phi_2$ are circulating with retrograde movement. On the other hand, Fig 11(b)
represents the orbit with initial condition (138325 km, 0.004) and the geometric arguments $\phi_1$ e $\phi_2$ are both circling with prograde movement. This behavior suggests the regularity of the orbits represented by these two regions.

a) $a_0=138318$ km, $e_0=0.004$ full circle

\[ \begin{align*}
\phi_1 & \text{ [degree]} \\
\phi_2 & \text{ [degree]}
\end{align*} \]

Time (day) 0 2000 4000 6000 8000 10000

b) $a_0=138325$ km, $e_0=0.004$ full square

\[ \begin{align*}
\phi_1 & \text{ [degree]} \\
\phi_2 & \text{ [degree]}
\end{align*} \]

Time (day) 0 2000 4000 6000 8000 10000

Figure 11. Time variation for the geometric arguments $\phi_1$ and $\phi_2$ represented by the magenta symbols in Fig. 8(a).

In Fig. 11(a) we have retrograde movement and, in Fig. 11(b), prograde movement. Such behavior suggests the periodicity of orbits represented by these regions. The simulation is similar to Fig. 9, but we present the variation of the arguments for a period of 10,000 days due to their fast circulation period.

The plot $\chi \times \phi_1$ for the orbits represented in Fig. 11 is in Fig. 12. These orbits have $|\chi| < 2$ characteristic of periodic orbits.
4.2.3 The region for the Lindblad resonance

Considering the values of the major semi-axis and eccentricity osculating elements with respect to epoch 01/01/2000, Lindblad’s geometric argument $\phi_2$ circulates in the retrograde direction (Fig 4(b)). However, with the mapping in phase space of the 54:53 Atlas-Prometheus resonance we determine the region to which the argument $\phi_2$ libre, defining the region of the Lindblad resonance.

The Lindblad region is located in the upper right of Figs. 8(a,b) delimited by dark blue regions with high value for $N$. Fig. 13 presents the geometric arguments $\phi_1$ and $\phi_2$ defined in section 4.2.1 with the time variation for 2 orbits with initial conditions interior to the Lindblad region.

In Fig. 13(a) we have the initial condition (138329 km, 0.0095) near the lower edge of the Lindblad region. The geometric argument $\phi_2$ presents its time variation oscillating around 0 and alternating to a prograde circulation, characteristic for orbits that are close to the resonance edge, and the argument $\phi_1$ presents its time variation circulating in the prograde direction. On the other hand, Fig 13(b) presents the variation for the arguments $\phi_1$ and $\phi_2$ from an orbit inside the Lindblad region, with initial condition given by (138333 km, 0.012). We see the libration for the argument $\phi_2$ around 0 and circulation in the prograde direction of $\phi_1$. Physically, the libration around 0 means that the conjunctions between the “Satellite”-Prometheus pair occur close to a line passing through the pericenter of “Satellite” (here the term “Satellite” refers to the Atlas-like test satellite used in the numerical simulations).
Geometric arguments $\phi_1$ and $\phi_2$ obtained from numerical simulations corresponding to the initial conditions represented by the different symbols in region C of Fig. 8(a) and considering the initial conditions in the Lindblad region.

In Fig. 13(a) we have an orbit near the lower edge of this region, the time variation of its geometric argument $\phi_2$ presents oscillation around 0 and circulation, characteristic of an orbit at the edge of the resonance. In Fig. 13(b) we have the libration of the geometric argument $\phi_2$ around 0 defining that the line of conjunctions with Prometheus passes through the pericenter of Atlas. In both initial conditions, the argument $\phi_1$ is circulating with a prograde movement.

In the comparison between Fig. 8(a) and Fig. 8(b) we can see that the Lindblad resonance excites the osculating eccentricity of the test satellites much more than the major semi-axis, showing a good approximation between the mapping and the theoretical prediction given by El Moutmaid et al. (2014). Since the mapping in Fig. 8(b) considered the spectrum of the orbital eccentricity at all edges of the upper right part in Fig. 8(b) the spectral number $N$ is greater than or equal to the spectral number in Fig. 8(a), where the spectrum of the major semi-axis was used, with similar result to that found by Callegari and Yokoyama (2020) in the phase space mapping for the 11:10 Mimas-Anthe resonance.

However, in the Lindblad region the test satellites have their initial eccentricity much larger than the initial eccentricity of Prometheus $e_{Pr} \approx 0.0025$, although, if we consider planar orbits and the interval $138315 \text{ km} \leq a_0 \leq 138335 \text{ km}$, the orbital crossings occur for $0.097 < e_0 < 0.098$. This result is easily obtained with the classical relation given by: $a_{Pr}(1 - e_{Pr}) = a_0(1 + e_0)$, where the symbol $Pr$ refers to Prometheus. But it is important to point out the fact that the Atlas inclination ($i_{At} \approx 0.004$) is twice the inclination of Prometheus ($i_{Pr} \approx 0.008$), where the trajectory of these Atlas-like satellites will not cross the orbit of Prometheus (Callegari and Yokoyama 2020), however, the libration of the argument $\phi_1$ prevents the conjunctions between the “Satellite”-Prometheus pair from occurring at the pericenter of the “Satellite” (Callegari and Yokoyama 2020).
When we consider the plot $\chi \times \phi_1$ for the orbits represented in Fig. 13, we see a fuzzy set of points, indicating another good approximation of the numerical simulations with the analytical CoraLin model (El Moutamid et al. 2014). The result can be observed in Fig. 14.

![Plot $\chi \times \phi_1$ for the orbits represented in Fig. 13.](image)

$\phi_1$ [rad]

We can compare in a qualitatively way IPS and dynamic mapping. In fact, with IPS it is possible to check in a more accurate way, the structure and length of the resonance shown in the map, and see the distribution of frequencies (or periods) as a function of the major semi-axis. As an example, the initial value for the eccentricity of Atlas is $e_0 = 0.0059$ in Fig. 6(a). But if we consider the intersection of the horizontal green line defined by $e_0 = 0.0059$ in Fig. 8(a), we can make a brief comparison between the two. We can observe the correspondence between the regular (low value for N in the map), non-regular (high value for N in the map) and the resonance edge regions in both as well.

Performing the same comparison for other initial eccentricity values that represented by the horizontal green lines in Fig. 8(a), namely: $e_0 = 0.00025$, $e_0 = 0.004$, $e_0 = 0.005$, $e_0 = 0.009$, $e_0 = 0.01$ and $e_0 = 0.012$, we get results corresponding to the IPS's represented in Fig. 15.0
Figure 15. IPS for 1000 Atlas-like satellites in the interval of initial range major semi-axis, $a_0$, given on the x-axis for various values of osculating eccentricity indicated at the top of each plot. C and L represent the corotation and Lindblad regions, respectively, associated with the 54:53 Atlas-Prometheus resonance. For each initial condition represented in the IPS’s the total integration time is ~172.25 years, sampled every 0.06 days.

The combined analysis between the IPS and the map revealed a small region for which the Atlas-like test satellite exhibits the $\phi_2$ librating argument, a result that expands the domain for the Lindblad resonance, seen in Fig. 15(a). For the orbits represented in Fig. 15(b-c) we see the region for corroding, but for the orbit represented in Fig. 15(d) the region in which we expect to find the Lindblad resonance showed a dense set of periods, making it difficult to identify this resonance. As for the orbits represented in Fig. 15(d-e) we identified the region related to the Lindblad resonance. The combined analysis between the IPS and the map revealed a small region for which the Atlas-like test satellite presents the $\phi_2$ librating argument, a result that expands the domain for the Lindblad resonance, observed in Fig. 15(a). For the orbits represented in Fig. 15(b-c) we see the region for corroding, but for the orbit represented in Fig. 15(d) the region in which we expect to find the Lindblad resonance showed a dense set of periods, making it difficult to identify this resonance. As for the orbits represented in Fig. 15(d-e) we identified the region related to the Lindblad resonance.

The IPS represented by Fig. 6(a) revealed that around the edges of the resonance, other complex structures are present and that these structures are related to other commensurabilities of mean movements and higher order resonances (Callegari and Yokoyama 2010, 2020).
Extending the interval of the osculating range major semi-axis represented in Fig. 8(a) to $138280 \text{ km} \leq a_0 \leq 138380 \text{ km}$ revealed two first order resonances namely 53:52 and 55:54 between the Atlas-Prometheus pair observed in Fig. 16 which represents the mapping to the semi-major axis for 30,000 Atlas-like satellites. The first description of the 53:52 resonance between Atlas-Prometheus was given in Cooper et al. (2015), but this resonance is about 40 km to the left of the 54:53 resonance.

Figure 16: Dynamic map constructed from the interval $138280 \text{ km} \leq a_0 \leq 138380 \text{ km}$ for the osculating major semi-axis for 30,000 Atlas-like satellites. The colors stand for: orbits whose conditions $(a_0, e_0)$ are interior to region C neon red symbols; orbits whose initial conditions $(a_0, e_0)$ are interior to region L orange symbols; orbits with initial conditions $(a_0, e_0)$ interior to region A and B yellow symbols, but considering 30,000 satellites similar to Atlas.

(b) Map constructed similarly to (a), but we use the spectrum of the orbital osculating eccentricity.

This mapping in Fig.16 revealed the location of two 1$^{st}$ order resonances: 53:52 and 55:54 between the Atlas-Prometheus pair. The different black symbols present in this map correspond to initial conditions for the orbits represented in Fig. 17.

The time variation for the geometric arguments $\phi_1 = 53\lambda_{Pr} - 52\lambda_S - \omega_{Pr}$ and $\phi_2 = 53\lambda_{Pr} - 52\lambda_S - \omega_S$, where $S$ is an Atlas-like satellite, is represented in fig. 17(a). In it, we observe the circulation of $\phi_1$ and the libration of $\phi_2$ around 0, indicating the Lindblad region associated with 53:52 resonance. On the other hand, we have in Fig. 17(b) the time variation for the geometric arguments $\phi_1 = 55\lambda_{Pr} - 54\lambda_S - \omega_{Pr}$ and $\phi_2 = 55\lambda_{Pr} - 524 - \omega_S$, where $\phi_1$ is librating around 180°, thus defining the region.
for the corotation associated with the resonance 55:54 between the pair “Satellite”-Prometheus and $\phi_2$ is circulating.

**Figure 17**: Time variation for the geometric arguments associated with the resonances (a) 53:52 and (b) 55:54 between the “Satellite”-Prometheus pair, where satellite refers to the Atlas-like “Satellite” present in the numerical simulations.

4.3 The influence of Prometheus eccentricity on the 54:53 resonance of mean movements

The maps presented in Section 4.2 considered the value for the osculating eccentricity of Prometheus equal to $e_p \approx 0.00252$, however, Renner et al. (2016) suggest that a condition for the argument libration to occur $\phi_1$ would be that the geometric eccentricity of Prometheus be smaller than 0.00008. The main idea of this section is to investigate this condition.

To do this, we built the dynamic map similar to the one represented in Fig. 8, but considering a fictitious satellite to Prometheus with osculating eccentricity equal to 0.00008.
Figure 18: Dynamic map constructed from a fictitious Prometheus with eccentricity equal to 0.00008 for 15,000 Atlas-like satellites. The colors stand for: orbits whose conditions \((a_0, e_0)\) are interior to region C neon red symbols; orbits whose initial conditions \((a_0, e_0)\) are interior to region L orange symbols; orbits with initial conditions \((a_0, e_0)\) interior to region A and B yellow symbols, but considering 30,000 satellites similar to Atlas. Colored circles represent the orbits given in Fig. 20.

A brief comparison between Fig.8(a) and Fig.18(a) we see that the domain of the corotation region in the 54:53 resonance phase space between Atlas-“fictitious Prometheus” has undergone an increase, extending for initial eccentricity values up to \(e_0 \sim 0.011\). We can see that the Lindblad resonance related region has undergone a contraction to a bluish white “small spot” in the upper right corner. The dark blue edges surrounding the region of corotation in Fig. 8(a) have narrowed to the low eccentricity of fictional Prometheus.

The analysis done in Section 4.2.1 was repeated here. The colored filled circles in Fig. 18(a) represent orbits for Atlas-like satellites and was analyzed the variation of their geometric arguments \(\phi_1 = 54\lambda_{PS} - 53\lambda_S - \alpha_{PS}\) and \(\phi_2 = 54\lambda_{PS} - 53\lambda_S - \alpha_S\), where \(PS\) refers to a fictitious Prometheus and \(S\) to the Atlas-like satellite. The time variation for these arguments can be observed in Fig. 19.

In Fig. 19 (a) we see an initial condition inside the dark region, high value for \(N\) and we do not have the libration of the geometric argument \(\phi_1\). For Fig. 19(b-f) the libration for the geometric argument \(\phi_1\) can be observed for all initial conditions inside the corotation region.

Orbits internal to regions with a high value for the spectral number \(N\) have irregular time variation for the geometric arguments. An example is seen in Fig. 19(a), where we see the irregular time variation for the geometric arguments \(\phi_1\) and \(\phi_2\). For the other arguments Fig.19(b-f) we see the libration of \(\phi_1\) around 180° and the circulation of \(\phi_2\).
Figure 19. Dynamic map consider a fictitious Prometheus with osculating orbital eccentricity equal to 0.00008. The colors stand for: orbits whose conditions \((a_0, e_0)\) are interior to region C neon red symbols; orbits whose initial conditions \((a_0, e_0)\) are interior to region L orange symbols; orbits with initial conditions \((a_0, e_0)\) interior to region A and B yellow symbols. (b) Map constructed similarly to (a), but we use the spectrum of the orbital osculating eccentricity.

4.4 Adrian

5. Pandora’s perturbation in the orbit of Atlas

In previous works, Pandora’s effects on the orbital dynamics of Atlas were not considered, as these effects were found to be of small magnitude (Spitale et al. 2006, Cooper et al. 2015, Renner et al. 2016), a fact that can be observed in Fig. 23, which shows the mapping obtained with the spectrum of the osculating semi-major axis for the 70:67 resonance of mean-motion between Atlas and Pandora. Unlike Fig. 8(a), the orbits for the Atlas-like satellites represented in that map have low value for the spectral number N.
Figure 23. Phase space mapping for the 70:67 resonance between the Atlas-Pandora pair. We observe the Dynamic Map constructed for 15,000 Atlas-like satellites orbits integrated using the system of equations in Fig. 1, except for the exchange of Prometheus for Pandora. The green horizontal lines indicate cross-sections in the map that will be used in the discussion of Fig. 24 and Fig. 25. The other osculating orbital elements are fixed with respect to Table 2.

A comparison between the dynamic map represented in Fig. 23 and the IPS given by Fig. 24(a) reveals an overlap of structures. For Renner et al. (2016), in the frequency domain, the four geometric arguments
\[
\psi_1 = 70\lambda_P - 67\lambda_{At} - 3\sigma_P, \quad \psi_2 = 70\lambda_P - 67\lambda_{At} - 2\sigma_P - \sigma_{At}, \quad \psi_3 = 70\lambda_P - 67\lambda_{At} - \sigma_P - 2\sigma_{At} \quad \text{and} \quad \psi_4 = 70\lambda_P - 67\lambda_{At} - 3\sigma_{At},
\]
related to resonance 70:67 of mean-motions between Atlas-Pandora, are overlapped. With the help of the IPS depicted in Fig. 24(b), which represents a "zoom" of the central region depicted in Fig. 24(a), we can observe the structures related to each of the \(\psi_i, i = 1, 2, 3, \) and 4 mentioned by Renner et al. (2016) and these are found very close together. The red dashed vertical line in Fig. 24(b) that represents the larger osculating semimajor axis of Atlas with respect to epoch 01/01/2000. We note that, like the 54:53 resonance between Atlas-Prometheus, Atlas lies close to the edge of this region, revealed by analysis of the IPS given in Fig. 23(b).
Figure 24. IPS for the major osculating major semi-axis of Atlas obtained by solving the system of equations given by Fig. 1, but with Pandora without the Prometheus presence. (a) We consider 1,000 orbits of Atlas-like satellite. (b) An approximation of the region [138323 km, 138326 km] to observe finer structures, considering 3,000 orbits of Atlas-like satellite obtained as Fig. 6.

A qualitative comparison between IPS and resonance mapping reveals 70:67 resonance domain. Fig. 24(a) reveals the IPS for an Atlas-like satellite with initial eccentricity $e_0 = 0.001$. We see that the structures are overlapping, on the other hand, in Fig. 24(b) with an eccentricity $e_0 = 0.005$ we cannot distinguish the structures as in Fig. 23(a), but Fig. 24(c-d) we have these structures.

In contrast, when we consider the presence of Pandora in the 54:53 resonance mapping between Atlas-Prometheus we see a sharp change in the structures that were present when considering only Atlas and Prometheus. Fig. 5.24 presents this mapping. We noticed that regions A, B, C, and L started to show a high value for the spectral number $N$. 
To check whether, in fact, the addition of Pandora contributed to the increase in the spectral number $N$, we constructed a series of maps considering in addition to Prometheus, a fictitious satellite to Pandora, which we varied its mass to a percentage of Pandora’s total mass.
Figure 26. Dynamic map consider Pandora. The colors stand for: orbits whose conditions \((a_0, e_0)\) are interior to region C neon red symbols; orbits whose initial conditions \((a_0, e_0)\) are interior to region L orange symbols; orbits with initial conditions \((a_0, e_0)\) interior to region A and B yellow symbols. (b) Map constructed similarly to (a), but we use the spectrum of the orbital osculating eccentricity.

In Fig. 27 we have a sequence of 5 mappings, constructed as follows: (a) Pandora with total mass, (b) fictitious Pandora with 50% of its mass, (c) fictitious Pandora with 25% of its mass, (d) fictitious Pandora with 10% of its mass and (e) without Pandora. We see that the gradual decrease in Pandora's mass contributed to the reestablishment of the regions of regular motion, and with the 25% mass the region for the corrodin and Lindblad resonances is already resumed.

Figure 27. Maps obtained for different fictitious Pandoras with their mass varying. (a) 100% of Pandora mass, (b) 50% of Pandora mass, (c) 25% of Pandora mass, (d) 10% of Pandora mass, and (e) no Pandora. We see that the gradual decrease of Pandora's mass.

Since dynamic mapping considers the number of significant peaks above a pre-established percentage, here 5% of the largest peak recorded in the spectrum of the individual orbit of the Atlas-like satellites, Fig. 5.27(a) suggests that the presence of Pandora has contributed to increase this number of peaks, and to investigate this fact we used IPS, since we are analyzing the orbits in the frequency domain.
Figure 28. (a) IPS obtained with Atlas on the effects of Prometheus and Pandora. (b) In blue, IPS for Atlas with Prometheus and in red the IPS for Atlas and Pandora. The overlap of both revealed the overlap between the two mean movements resonances, responsible for the irregular movement of Atlas.

Fig. 28(a) shows the IPS constructed for the Atlas osculating semi-major axis considering in addition to Prometheus the Pandora effects. We see that in the region in which the periods associated with corotation resonance occur, there was an increase in the number of peaks as observed in Fig. 5.25(a) and to investigate this fact we analyzed the IPS for Atlas-Prometheus, Fig. 6(a) and Atlas-Pandora Fig. 23(a) superimposed. The result of this overlap is represented in Fig. 5.28(b).

In Fig. 5.28(b) in blue we have the IPS for Atlas-Prometheus and in red the IPS for Atlas-Pandora, revealing that both mean movements resonances, 54:53 between Atlas-Prometheus and 70:67 between Atlas-Pandora, are overlapping.

We note that for the regions of the 54:53 resonance between Atlas-Prometheus that did not contain a dense number of peaks are occupied by peaks originating from the 70:67 resonance between Atlas-Pandora and this overlap between the resonances has strong implications for orbital dynamics as observed by Spitale et al. (2006), Cooper et al. (2015) and Renner et al. (2016)

6. Final Considerations

We show an important result revealed by the phase space mapping of the 54:53 resonance between Atlas-Prometheus is that Atlas is located at the edge of the mean movements resonance implying the irregular behavior of the geometric argument $\phi_1$ observer by Spitale et al. (2006), Cooper et al. (2015) and Renner et al. (2016).

With the mapping we determine the regions for the corotation and Lindblad resonances, showing that the regions in phase space containing these resonances are well determined and isolated by regions that have a high value for the spectral number $N$, related to irregular orbital movement.
The corotation region is limited for values of orbital eccentricity between $0.003 < e_0 < 0.006$ and occupies a very narrow band in phase space $\sim 4$ km, within the range for the major semi-axis $138321 \text{ km} < a_0 < 138325 \text{ km}$. For orbits that are interior to this region, its geometric argument $\phi_1 = 54\lambda_{Pr} - 53\lambda_S - \sigma_{Pr}$, where $S$ represents an Atlas-like test satellite exhibits libration around $180^\circ$, physically this means that the conjunction between the Atlas-Prometheus pair oscillates around a line that passes through the apocenter of Prometheus.

For initial eccentricity values $e_0 > 0.01$ and with initial semi-major axis $a_0 > 13827 \text{ km}$ begins the domain for the Lindblad resonance. For the orbits inside this region, the geometric argument $\phi_2 = 54\lambda_{Pr} - 53\lambda_S - \sigma_S$ presents its libration around 0. However, for initial eccentricity values near zero and with an initial major semi-axis near $138316 \text{ km}$, the test satellite also presented the libration of the geometric argument $\phi_2$. This is revealed by the qualitative analysis between the mapping and the IPS.

A condition proposed by Renner et al. (2016) to occur the libration of the geometric argument $\phi_1$ would be that the orbital eccentricity of Prometheus be close to zero. To illustrate this fact, our mapping with a fictitious Prometheus and with orbital eccentricity $0.00008$ revealed the increase in the corotation region and the narrowing into the Lindblad region.

The addition of Pandora to the phase space mapping of the 54:53 Atlas-Prometheus resonance implied the loss of the corotation and Lindblad region. A series of maps obtained with the Pandora mass variation revealed that this satellite causes significant perturbations ito the Atlas orbit, a fact that had not been considered in previous works.

The IPS and mapping of the 70:67 resonance between Atlas-Pandora revealed that Atlas is close to this resonance and the overlap of this resonance with the 54:53 resonance implies the irregular orbital dynamics observed in Atlas, a conclusion obtained with the overlap of both IPS.

Unlike the small satellites Aegaeon, Anthe, and Methone, Atlas is not locked into the 54:53 medium movements resonance with Prometheus.

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Appendix A: Parameters and Initial Conditions

Table 1 provides the Saturn physical parameters and Table 2 provides the masses and orbital osculating elements provided by the Ephemeris system at the time January 1st, 2000.

**Table 1**: Physical constants for Saturn $^a$ *Horizon*, $^b$ Jacobson et al. (2006)

| Constant | Value       | Unit   |
|----------|-------------|--------|
| Mass$^{(a)}$ | $568,317 \times 10^{24}$ | kg     |
| $r_y^{(a)}$ | 60268       | km     |
| $J_2^{(b)}$ | $16290.6 \times 10^{-6}$ |        |
| $J_4^{(b)}$ | $-935.1 \times 10^{-6}$ |        |
| $J_6^{(b)}$ | $86.8 \times 10^{-6}$ |        |

**Table 2**: Osculating elements for Atlas, Prometheus and Pandora with respect to epoch January 1st, 2000 obtained from Ephemerides *Horizons* (Accessed 05/29/2022). $^b$Mass obtained from Thomas and Helfenstein (2020)
|                  | Atlas                | Prometheus           | Pandora              |
|------------------|----------------------|----------------------|----------------------|
| Mass\(^{(a)}\) (kg) | 0.575 \times 10^{16} | 0.15 \times 10^{18}  | 0.137 \times 10^{18} |
| a (km)           | 138325.37            | 140024.64            | 142346.13            |
| e                | 0.00591              | 0.00252              | 0.00145              |
| i (degree)       | 0.00419              | 0.00807              | 0.05024              |
| \(\omega\) (degree) | 200.77               | 201.36               | 332.15               |
| \(\Omega\) (degree) | 235.44               | 309.14               | 149.98               |
| \(n\) (degree/day) | 592.97               | 581.87               | 567.69               |