Abstract

In a RAC drawing of a graph, vertices are represented by points in the plane, adjacent vertices are connected by line segments, and crossings must form right angles. Graphs that admit such drawings are RAC graphs. RAC graphs are beyond-planar graphs and have been studied extensively. In particular, it is known that a RAC graph with \( n \) vertices has at most \( 4n - 10 \) edges.

We introduce a superclass of RAC graphs, which we call arc-RAC graphs. In an arc-RAC drawing, edges are drawn as circular arcs whereas crossings must still form right angles. We provide a Turán-type result showing that an arc-RAC graph with \( n \) vertices has at most \( 14n - 12 \) edges and that there are \( n \)-vertex arc-RAC graphs with \( 4.5n - O(\sqrt{n}) \) edges.

1 Introduction

A *drawing* of a graph in the plane is a mapping of its vertices to distinct points and each edge \( uv \) to a curve whose endpoints are \( u \) and \( v \). Planar graphs, which admit crossing-free drawings, have been studied extensively. They have many nice properties and several algorithms for drawing them are known, see, e.g., [20,21]. However, in practice we must also draw non-planar graphs and crossings make it difficult to understand a drawing. For this reason, graph classes with restrictions on crossings are studied, e.g., graphs that can be drawn with at most \( k \) crossings per edge (known as \( k \)-planar graphs) or where the angle formed by each crossing is “large”. These classes are categorized as beyond-planar graphs and have experienced increasing interest in recent years [14].

As introduced by Didimo et al. [13], a prominent beyond-planar graph class that concerns the crossing angles is the class of \( k \)-bend right-angle-crossing graphs, or RAC\(_ k \) graphs for short, that admit a drawing where all crossings form 90° angles and each edge is a poly-line with at most \( k \) bends. Using right-angle crossings and few bends is motivated by several cognitive studies suggesting a positive correlation between large crossing angles or small curve complexity and the readability of a graph drawing [17,19]. Didimo et al. [13] studied the edge density of RAC\(_ k \) graphs. They showed that RAC\(_ 0 \) graphs with \( n \) vertices have at most \( 4n - 10 \) edges (which is tight), that RAC\(_ 1 \) graphs have at most \( O(n^{2.5}) \) edges, that RAC\(_ 2 \)}
graphs have at most $O(n^{4.7})$ edges and that all graphs are $RAC_3$. Dujmović et al. [15] gave an alternative simple proof of the $4n - 10$ bound for $RAC_0$ graphs using charging arguments similar to those of Ackerman and Tardos [2] and Ackerman [1]. Arikushi et al. [6] improved the upper bounds to $6.5n$ for $RAC_1$ graphs and to $74.2n$ for $RAC_2$ graphs. The bound of $6.5n - 13$ for $RAC_1$ graphs was also obtained by charging arguments. They also provided a $RAC_1$ graph with $4.5n - O(\sqrt{n})$ edges. The best known lower and upper bound for the edge density of $RAC_1$ graphs of $5n - 10$ and $5.5n - 11$, respectively, are due to Angelini et al. [4].

We extend the class of $RAC_0$ graphs by allowing edges to be drawn as circular arcs but still requiring $90^\circ$ crossings. An angle at which two circles intersect is the angle between the two tangents to each of the circles at an intersection point. Two circles intersecting at a right angle are called orthogonal. For any circle $\gamma$, let $C(\gamma)$ be its center and let $r(\gamma)$ be its radius.

**Observation 1.1.** Let $\alpha$ and $\beta$ be two circles. Then $\alpha$ and $\beta$ are orthogonal if and only if $r(\alpha)^2 + r(\beta)^2 = |C(\alpha)C(\beta)|^2$; see Figure 1.

Similarly, two circular arcs $\alpha$ and $\beta$ are orthogonal if they intersect properly (that is, ignoring intersections at endpoints) and the underlying circles (that contain the arcs) are orthogonal. For the remainder of this paper, all arcs will be circular arcs. We consider any straight-line segment an arc with infinite radius. We call a drawing of a graph where the edges are drawn as arcs such that any pair of intersecting arcs is orthogonal an arc-$RAC$ drawing; see Figure 2. A graph that admits an arc-$RAC$ drawings is called an arc-$RAC$ graph.

The idea of drawing graphs with arcs dates back to at least the work of the artist Mark Lombardi who drew social networks, featuring players from the political and financial sector [23]. Indeed, user studies [26, 28] state that users prefer edges drawn with curves of small curvature; not necessarily for performance but for aesthetics. Drawing graphs with arcs can help to improve certain quality measures of a drawing such as angular resolution [3, 12] or visual complexity [22, 27].
An immediate restriction on the edge density of arc-RAC graphs is imposed by the following known result.

Lemma 1.2 (\cite{24}). In an arc-RAC drawing, there cannot be four pairwise orthogonal arcs.

It follows from Lemma 1.2 that arc-RAC graphs are 4-quasi-planar; that is, an arc-RAC drawing cannot have four edges that pairwise cross. This implies that an arc-RAC graph with $n$ vertices can have at most $72(n - 2)$ edges \cite{1}.

Our main contribution is that we reduce this bound to $14n - 12$ using charging arguments similar to those of Ackerman \cite{1} and Dujmović et al. \cite{15}; see Section 2. For us, the main challenge was to apply these charging arguments to a modification of an arc-RAC drawing and to exploit, at the same time, geometric properties of the original arc-RAC drawing to derive the bound. We also provide a lower bound of $4.5n - O(\sqrt{n})$ on the edge density of arc-RAC graphs based on the construction of Arikushi et al. \cite{6}, see Section 3. We conclude with some open problems in Section 4. Throughout the paper our notation won’t distinguish between the entities (vertices and edges) of an abstract graph and the geometric objects (points and curves) representing them in a drawing.

2 Edge Density Upper Bound

Let $G$ be a 4-quasi-planar graph, and let $D$ be a 4-quasi-planar drawing of $G$. In his proof of the upper bound on the edge density of 4-quasi-planar graphs, Ackerman \cite{1} first modified the given drawing as to remove faces of small degree. We use a similar modification that we now describe.

Consider two edges $e_1$ and $e_2$ in $D$ that intersect multiple times. A region in $D$ bounded by pieces of $e_1$ and $e_2$ that connect two consecutive intersection points is called a lens. If one of the intersection points is a vertex of $G$, then we call such a lens a 1-lens, otherwise a 0-lens. (From now on, by intersection point we always mean a proper intersection point.) A lens that does not contain a vertex of $G$ is empty. Every drawing with 0-lenses has a smallest empty 0-lens, that is, an empty 0-lens that does not contain any other empty 0-lenses in its interior. We can swap the two curves that bound a smallest empty 0-lens; Figure 3. We do it repeatedly until there are no empty 0-lenses left. We call the new empty-0-lens-free drawing $D'$ of $G$ a simplified drawing and the empty-0-lens-removal process simplification. As mentioned above, Ackerman \cite{1} used a similar modification to prepare a 4-quasi-planar drawing for his charging arguments; note, that unlike Ackerman, we do not resolve 1-lenses. We look at simplification in more detail, in particular, we consider how it changes the order in which edges share points. Note that during simplification, no new pairs of crossing edges are introduced, because each iteration only resolves a smallest empty lens. Therefore, the drawing remains 4-quasi-planar. Because the drawing remains 4-quasi-planar and because we always resolve a smallest empty 0-lens we have the following observation.

Figure 3 A simplification step resolves a smallest empty 0-lens; if two edges $e_1$ and $e_2$ change the order in which they share points with the edge $e$, they form an empty 0-lens intersecting $e$ before the step, and thus, in the original 4-quasi-planar drawing.
Observation 2.1. If two edges $e_1$ and $e_2$ change the order in which they share points with another edge $e$ after a simplification step, then they form an empty 0-lens intersecting $e$ in the original 4-quasi-planar drawing; see Figure 3.

We now focus on the special type of 4-quasi-planar drawings we are interested in. Suppose that $G$ is an arc-RAC graph, $D$ is an arc-RAC drawing of $G$, and $D'$ is a simplified drawing of $D$. If two edges $e_1$ and $e_2$ share an intersection point in $D'$, then they do not form a 0-empty lens in $D$, as otherwise simplification would remove both intersection points of these two edges. If $e_1$ and $e_2$ share a vertex of $G$ then they also do not form a 0-empty lens in $D$, as otherwise they would share three points in $D$ (the two intersection points of the lens and the common vertex of $G$). Thus, we have the following observation.

Observation 2.2. If two edges $e_1$ and $e_2$ share a point in the simplified drawing, then they do not form an empty 0-lens in the original arc-RAC drawing.

In the following, we first state the main theorem of this section and provide the structure of its proof (deferring one small lemma and the main technical lemma until later). Then, we carefully prove the remaining technical details in Lemmas 2.4 to 2.9 to establish the result.

Theorem 2.3. An arc-RAC graph with $n$ vertices can have at most $14n - 12$ edges.

Proof. Let $G = (V, E)$ be the given arc-RAC graph with an arc-RAC drawing $D$, let $D'$ be a simplified drawing of $D$, and let $G' = (V', E')$ be the planarization of $D'$. Our charging argument consists of three steps.

First, each face $f$ of $G'$ is assigned an initial charge $ch(f) = |f| + v(f) - 4$, where $|f|$ is the degree of $f$ in the planarization and $v(f)$ is the number of vertices of $G$ on the boundary of $f$. Applying Euler’s formula several times, Ackerman and Tardos showed that $\sum_{f \in G'} ch(f) = 4n - 8$, where $n$ is the number of vertices of $G$. In addition, we set the charge $ch(v)$ of a vertex $v$ of $G$ to $16/3$. Hence the total charge of the system is $4n - 8 + 16n/3 = 28n/3 - 8$.

In the next two steps (described below), similarly to Dujmović et al. [15], we redistribute the charges among faces of $G'$ and vertices of $G$ so that, for every face $f$, the final charge $ch_{\text{fin}}(f)$ is at least $v(f)/3$ and the final charge of each vertex is non-negative. Observing that $28n/3 - 8 \geq \sum_{f \in G'} ch_{\text{fin}}(f) \geq \sum_{f \in G'} v(f)/3 = \sum_{v \in G} \deg(v)/3 = 2|E|/3$ yields that the number of edges of $G$ is at most $14n - 12$ as claimed.

After the first charging step above it is easy to see that $ch(f) \geq v(f)/3$ holds if $|f| \geq 4$. We call a face $f$ of $G'$ a $k$-triangle, $k$-quadrilateral, or $k$-pentagon if $f$ has the corresponding shape and $v(f) = k$. Similarly, we call a face of degree two a $1$-digon. Note that any digon is a 1-digon, because it must be incident to a vertex of $G$.

In the second charging step, we need to find 4/3 units of charge for each digon, one unit of charge for each 0-triangle, and 1/3 unit of charge for each 1-triangle.

To charge a digon $d$ incident to a vertex $v$ of $G$ we decrease $ch(v)$ by 4/3 and increase $ch(d)$ by 4/3; see Figure 1a. We say that $v$ contributes charge to $d$.

To charge triangles, we proceed similarly to Ackerman [1] and Dujmović et al. [15]. Theorem 7.

Consider a 1-triangle $t_1$. Let $v$ be the unique vertex incident to $t_1$, and let $s_1 \in E'$ be the edge of $t_1$ opposite of $v$; see Figure 4a. Note that the endpoints of $s_1$ are intersection points in $D'$. Let $f_1$ be the face on the other side of $s_1$. If $f_1$ is a 0-quadrilateral, then we
consider its edge $s_2 \in E'$ opposite to $s_1$ and the face $f_2$ on the other side of $s_2$. We continue iteratively until we meet a face $f_k$ that is not a 0-quadrilateral. If $f_k$ is a triangle, then all the faces $t_1, f_1, f_2, \ldots, f_k$ belong to the same empty 1-lens $l$ (recall that $D'$ does not contain empty 0-lenses) incident to the vertex $v$ of $t_1$. In this case, we decrease $\text{ch}(v)$ by $1/3$ and increase $\text{ch}(t_1)$ by $1/3$; see Figure 4a. Otherwise, $f_k$ is not a triangle and $|f_k| + v(f_k) - 4 \geq 1$ (see Figure 4b). In this case, we decrease $\text{ch}(f_k)$ by $1/3$ and increase $\text{ch}(t_1)$ by $1/3$. We say that the face $f_k$ contributes charge to the triangle $t_1$ over its side $s_k$.

For a 0-triangle $t_0$, we repeat the above charging over each side. If the last face on our path is a triangle $t'$, then $t_0$ and $t'$ are contained in an empty 1-lens and $t'$ is a 1-triangle incident to a vertex $v$ of $G$. In this case, we decrease $\text{ch}(v)$ by $1/3$ and increase $\text{ch}(t_0)$ by $1/3$; see Figure 4c.

Thus, at the end of the second step, each digon or triangle $f$ has charge of at least $v(f)/3$. Note that the charge for $f$ comes either from a higher-degree face or from a vertex $v$ incident to an empty 1-lens containing $f$.

In the third step, we do not modify the charging any more, but we need to ensure that

(i) $\text{ch}(f) \geq v(f)/3$ still holds for each face $f$ of $G'$ with $|f| \geq 4$ and

(ii) $\text{ch}(v) \geq 0$ for each $v$ of $G$.

We first show statement (i). Ackerman [1] noted that a face $f$ can contribute charges over each of its edges at most once. Therefore, if $|f| + v(f) \geq 6$, then $f$ still has a charge of at least $v(f)/3$. It remains to verify that 1-quadrilaterals and 0-pentagons, which initially had only one unit of charge, have a charge of at least $1/3$ or zero, respectively, at the end of the second step.

A 1-quadrilateral $q$ can contribute charge to at most two triangles since the endpoints of any edge of $G'$ over which a face contributes charge must be intersection points in $D'$; see Figure 4d and recall that $q$ now plays the role of $f_k$ in Figure 4b.

A 0-pentagon cannot contribute charge to more than three triangles; see Lemma 2.9.

Now we show statement (ii). Recall that a vertex $v$ can contribute charge to a digon incident to $v$ or to at most two triangles contained in an empty 1-lens incident to $v$. Observe that two empty 1-lenses with either triangles or a digon taking charge from $v$ cannot overlap; see Figure 4a. We show in Lemma 2.4 that $v$ cannot be incident to more than four such empty 1-lenses. In the worst case, $v$ contributes 4/3 units of charge to each of the at most four incident digons representing these empty 1-lenses. Thus, $v$ has non-negative charge at the end of the second step.

Lemma 2.4. In any simplified arc-RAC drawing, each vertex is incident to at most four non-overlapping empty 1-lenses.
6 Drawing Graphs with Circular Arcs and Right Angle Crossings

**Figure 5** The edges of an empty 1-lens form a $\pi/2$ angle at the vertex of the lens.

**Figure 6** The operator $\Pi(\cdot; \cdot)$ is not meant to describe all intersection points along an edge; here $\Pi(e; e_1, e_2, e_3, e_4, e_5, e_6)$, $\Pi(e; e_1, e_3, e_4, e_5)$, and $\Pi(e; e_2, e_4, e_3, e_5)$ all hold at the same time.

**Proof.** Let $v$ be a vertex incident to some 1-lenses. Let $l$ be one of these 1-lenses. Then $l$ forms an angle of $90^\circ$ between the two edges incident to $v$ that form $l$; see Figure [5]. This is due to the fact that the other “endpoint” of $l$ is a proper intersection point where the two edges must meet at $90^\circ$. Thus $v$ is incident to at most four empty 1-lenses.

We now set the stage for proving Lemma [2.9], which shows that a 0-pentagon in a simplified drawing does not contribute charge to more than three triangles. The proof goes by contradiction. First we consider the edges of the graph forming a 0-pentagon contributing charge to at least four triangles in the simplified drawing. Then we describe the order in which these edges share points in the simplified drawing and show that this order is preserved in the original arc-RAC drawing. Finally, we use some geometric arguments to show that such an arc-RAC drawing of the edges does not exist; see Lemma [2.8].

Let $D$ be an arc-RAC drawing of some arc-RAC graph $G = (V, E)$, let $D'$ be its simplified drawing, and let $p$ be a 0-pentagon that contributes charge to at least four triangles. Let $s_0, s_1, \ldots, s_4$ be the sides of $p$ in clockwise order and denote the edges of $G$ that contain these sides as $e_0, e_1, \ldots, e_4$ so that edge $e_0$ contains side $s_0$ etc. Since $p$ contributes charge over at least four sides, these sides are consecutive around $p$. Without loss of generality, we assume that $s_4$ is the side over which $p$ does not necessarily contribute charge.

For $i \in \{0, 1, 2, 3\}$, let $t_i$ be the triangle that gets charge from $p$ over the side $s_i$. The triangle $t_i$ is bounded by the edges $e_{i-1}$ and $e_{i+1}$. (Indices are taken modulo 5.) Note that all faces bounded by $e_{i-1}$ and $e_{i+1}$ that are between $t_i$ and $p$ must be 0-quadrilaterals. If $t_i$ is a 1-triangle, then $e_{i-1}$ and $e_{i+1}$ share a vertex of the graph at a vertex of the triangle, otherwise $t_i$ is a 0-triangle and $e_{i-1}$ and $e_{i+1}$ intersect at a vertex of the triangle. Let $A'_{i-1,i+1}$ denote this common point of $e_{i-1}$ and $e_{i+1}$, and let $E_p = \{e_0, \ldots, e_4\}$; see Figure [7a].

We now describe the order in which the edges in $E_p$ share points in $D'$. To this end, we orient the edges in $E_p$ so that this orientation conforms with the orientation of a clockwise walk around the boundary of $p$ in $D'$. In addition, we write $\Pi(e_k; e_i, e_{i_1}, \ldots, e_{i_t})$ if the edge $e_k$ shares points (either intersection points or vertices of the graph) with the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_t}$ in this order with respect to the orientation of $e_k$; see Figure [8]. (Note that we can have $\Pi(e_k; e_i, e_{i_1}, e_{i_2})$ as edges may intersect twice. We will not consider more than two edges sharing the same endpoint.) Due to the order in which we numbered the edges in $E_p$, it holds in $D'$ that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$, and, for $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2, i-1, i+1, i+2})$; see Figure [7a]. Now we show that this order is preserved in $D$. Obviously every pair of edges $(e_{i-1}, e_{i+1})$ that shares a vertex of $G$ in $D'$ also shares a vertex of $G$ in $D$. Furthermore, every
We will show that the edges \( e_i \) and \( e_j \) share points with \( D \); thus, the order in which the edges \( e_i \) and \( e_j \) share points with \( D \) is preserved in \( D' \), that is, \( \Pi(e_k; e_i, e_j) \) in \( D' \).

If \( (i, j) \in \{(0, 3), (3, 0)\} \), then, according to Lemma 2.5, the edges \( e_i \) and \( e_j \) do not cross in \( D \), so they do not form an empty 0-lens in \( D \), and thus, by Observation 2.1, \( e_i \) and \( e_j \) share points with \( e_k \) in the same order in \( D \) as in \( D' \), that is, \( \Pi(e_k; e_i, e_j) \) in \( D' \).

Otherwise the edges \( e_i \) and \( e_j \) share a point in \( D' \); see Figure 7a. Therefore, according to Observation 2.2, \( e_i \) and \( e_j \) do not form an empty 0-lens in \( D \), and thus, according to Observation 2.1, \( e_i \) and \( e_j \) share points with \( e_k \) in the same order in \( D \) as in \( D' \), that is, \( \Pi(e_k; e_i, e_j) \) in \( D' \).
Assume that the edges in $E_p$ share points in $D'$ is preserved in $D$; see Figure 7b. We show now that such an arc-RAC drawing does not exist; see Lemma 2.8. This is the main ingredient to prove Lemma 2.9 which says that a 0-pentagon in a simplified arc-RAC drawing contributes charge to at most three triangles.

For simplicity of presentation and without loss of generality, we assume that the points $A_{i-1,i+1}'$ are vertices of $G$, which we denote by $v_{i-1,i+1}$. To prove Lemma 2.8 we need the following simple observation.

**Observation 2.7.** For two orthogonal circles the tangent to one circle at one of the intersection points goes through the center of the other circle; see Figure 7. In particular, a line is orthogonal to a circle if the line goes through the center of the circle.

**Lemma 2.8.** The edges in $E_p$ do not admit an arc-RAC drawing where it holds that $\Pi(e_0;e_4,e_1,e_2)$, $\Pi(e_3;e_1,e_2,e_4)$, and, for $i \in \{1,2,4\}$, $\Pi(e_i;e_{i-2},e_{i-1},e_{i+1},e_{i+2})$. 

**Proof.** Assume that the edges in $E_p$ admit an arc-RAC drawing where they share points in the order indicated above. For $i \in \{0,\ldots,4\}$, let $P_{i+1}$ be the intersection point of $e_i$ and $e_{i+1}$; see Figure 7b. Note that on $e_i$, the point $P_{i-1,i}$ is before the point $P_{i,i+1}$ (due to $\Pi(e_i;e_{i-1},e_{i+1})$).

Recall that an inversion [24] with respect to a circle $\alpha$, the inversion circle, is a mapping that takes any point $P \neq C(\alpha)$ to a point $P'$ on the ray $C(\alpha)P$ so that $|C(\alpha)P'| = r(\alpha) = r(\alpha)^2$. Inversion maps each circle not passing through $C(\alpha)$ to another circle and each circle passing through $C(\alpha)$ to a line. The center of the inversion circle is mapped to the “point at infinity”. It is known that inversion preserves angles and the topology of the original drawing. Therefore, the image of an inversion of an arc-RAC drawing is also an arc-RAC drawing and the order in which the edges in $E_p$ share points is preserved.

We invert the drawing of the edges in $E_p$ with respect to a small inversion circle centered at $v_{24}$. Let $e_i'$ be the image of $e_i$, $v_{i-1,i+1}'$ be the image of $v_{i-1,i+1}$ ($v_{24}'$ is the point at infinity), and $P_{i,i+1}'$ be the image of $P_{i,i+1}$. We consider two cases regarding whether the edges $e_2$ and $e_4$ belong to two different circles or not.

**Case I:** $e_2$ and $e_4$ belong to two different circles.

One of the intersection points of their circles is $v_{24}$, and we let $X$ denote the other intersection point. Here we have that $e_2'$ and $e_4'$ are two straight line rays whose lines intersect at the point $X'$ which is the image of $X$; see Figure 8.

We now assume for a contradiction that the arc $e_2'$ forms a concave side of the triangle $\Delta_1 = P_{12}v_{41}'X';$ see Figure 8a where the triangle is filled gray. (Symmetrically, we can...
show that the arc \( e_0^0 \) cannot form a concave side of the triangle \( \Delta_0 = P_{40}^0v_{02}^0X^0. \) By Observation 2.7, \( C(e_1^4) \) must lie on the ray \( e_2^2. \) Since we assume that the arc \( e_1^2 \) forms a concave side of the triangle \( \Delta_1, \) \( C(e_1^4) \) and \( v_{02}^0 \) are separated by \( P_{12}^0 \) on \( e_2^2. \) Consider the tangent \( l_0 \) to \( e_0^0 \) at \( P_{01}^0. \) Again in light of Observation 2.7, \( l_0 \) has to go through \( C(e_1^4) \) because \( e_0^0 \) and \( e_1^4 \) are orthogonal. On the one hand, \( v_{02}^0 \) is to the same side of \( l_0 \) as \( P_{12}^0; \) see Figure 8a. On the other hand, \( l_0 \) separates \( P_{12}^0 \) and \( v_{11}^1, \) due to \( \Pi(e_1; e_4, e_0, e_2). \) Moreover, \( l_0 \) does not separate \( v_{11}^1 \) and \( P_{40}^0 \) since it intersects the line of \( e_2^3 \) when leaving the gray triangle \( \Delta_1. \) So the two points \( v_{02}^0 \) and \( P_{40}^0 \) of the same arc \( e_0^0 \) are separated by \( l_0, \) which is a tangent of this arc; contradiction.

Thus, the arc \( e_1^2 \) forms a convex side of the triangle \( \Delta_1, \) and \( e_0^0 \) forms a convex side of \( \Delta_0; \) see Figure 9. Now, due to Observation 2.7, \( C(e_0^0) \) is between \( v_{11}^1 \) and \( P_{40}^0, \) and \( C(e_1^4) \) is between \( v_{02}^0 \) and \( P_{12}^0, \) because that is where the tangents \( t_1 \) of \( e_1^3 \) and \( t_0 \) of \( e_0^0 \) in \( P_{01}^0 \) intersect the lines of \( e_3^2 \) and \( e_2^2, \) respectively. Taking into account that \( C(e_1^4) = X^0, \) because \( e_3^2 \) is orthogonal to both \( e_3^2 \) and \( e_1^3, \) we obtain that the points \( C(e_0^0), C(e_1^4), P_{12}^0, P_{23}^0 \) appear on the line of \( e_2^3 \) in this order. Thus, the circle of \( e_1^3 \) is contained within the circle of \( e_3^2. \) This is a contradiction because \( e_3^2 \) and \( e_1^3 \) must share a point; namely \( v_{13}^0. \)

**Case II:** \( e_2 \) and \( e_4 \) belong to the same circle.

Here \( e_2^2 \) and \( e_4^2 \) are two disjoint straight-line rays on the same line \( l \) (meeting at infinity at \( v_{24}^2; \) see Figure 9). We direct \( l \) as \( e_2^2 \) and \( e_4^2 \) (from right to left in Figure 9). Because \( e_0^0, e_1^3, \) and \( e_2^2 \) are orthogonal to \( l, \) their centers have to be on \( l. \) Due to our initial assumption, we have \( \Pi(e_4; e_2, e_3, e_0, e_1) \) and \( \Pi(e_2; e_0, e_1, e_3, e_4). \) Hence, along \( l, \) we have \( P_{34}^0, P_{40}^0, v_{31}^1, \) (on \( e_4^2 \)) and then \( v_{02}^0, P_{12}^0, P_{23}^0 \) (on \( e_2^2 \)). Therefore, the circle of \( e_1^3 \) is contained in that of \( e_3^2. \) Hence, \( e_1^3 \) does not share a point with \( e_3^2; \) a contradiction. ▶

**Lemma 2.9.** A 0-pentagon in a simplified arc-RAC drawing contributes charge to at most three triangles.

**Proof.** As discussed above, if a 0-pentagon formed by edges \( e_0, e_1, \ldots, e_4 \) contributes charge to more than three triangles in a simplified drawing (see Figure 7a), then this implies the existence of an arc-RAC drawing where it holds that \( \Pi(e_0; e_4, e_1, e_2), \Pi(e_3; e_1, e_2, e_4) \) and, for \( i \in \{1, 2, 4\}, \Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2}); \) see Figure 8b. This, however, contradicts Lemma 2.8. ▶

With the proofs of Lemmas 2.4 and 2.9 now in place, the proof of Theorem 2.3 is complete.

### 3 Edge Density Lower Bound

We give a lower bound on the number of edges by using the construction of Arikushi et al. 5 that they used to give the lower bound on edge density for RAC1 graphs. Let \( G \) be
an embedded graph where the vertices of $G$ are points of the hexagonal lattice clipped in a rectangle; see Figure 10a. The edges of $G$ are the edges of the lattice and, inside each hexagon that is bounded by cycle $(P_0, \ldots, P_3)$, six additional edges $(P_i, P_{i+2})$. We refer to a part of the drawing made up of a single hexagon and its diagonals as a *tile*. In Theorem 3.3 we show that each hexagon can be drawn as a regular hexagon and its diagonals can be drawn as sets of arcs $A = \{\alpha_0, \alpha_1, \alpha_2\}$ and $B = \{\beta_0, \beta_1, \beta_2\}$, so that the arcs in $A$ are pairwise orthogonal and for each arc in $B$ intersecting another arc in $A$ the two arcs are orthogonal; we use this construction to establish Theorem 3.3. In particular, the arcs in $A$ form the triangle $(P_0, P_2, P_4)$ while the arcs in $B$ form the triangle $(P_1, P_3, P_5)$.

We first define the radii and centers of the arcs in a tile and show that they form only orthogonal crossings. We use the geometric center of the tile as the origin of our coordinate system in the following analysis. We now discuss the arcs in $A$; then we turn to the arcs in $B$. For each $j \in \{0, 1, 2\}$, the arc $\alpha_j$ has radius $r_A = 1$ and center $C(\alpha_j) = (x_A \cos(\pi/6 + j^{2\pi}/3), x_A \sin(\pi/6 + j^{2\pi}/3))$ where $x_A = \sqrt{2}/3$; see Figure 11a.

**Lemma 3.1.** The arcs in $A$ are pairwise orthogonal.

**Proof.** Consider the equilateral triangle $\triangle C(\alpha_0)C(\alpha_1)C(\alpha_2)$ on the centers of the three arcs. Because the origin is in the center of the triangle it is easy to see that the length of the side of the triangle is $2x_A \cos \pi/6 = \sqrt{2}$, and so the distance between the centers of any two arcs is $\sqrt{2}$. Because, in addition, the radius of each arc is 1, by Observation 1.1 every two arcs are orthogonal. □

As in Figure 11b for each $j \in \{0, 1, 2\}$, the arc $\beta_j$ has radius $r_B = \sqrt{70 + 40\sqrt{3}}/6$ and center $C(\beta_j) = (x_B \cos(\pi/2 + (j+1)\pi/3), x_B \sin(\pi/2 + (j+1)\pi/3))$, where $x_B = \sqrt{3} + \sqrt{70 + 40\sqrt{3}}/6$.

**Lemma 3.2.** If an arc in $B$ intersects an arc in $A$, then the two arcs are orthogonal.

**Proof.** Let $i, j \in \{0, 1, 2\}$. If $j = i$, then $\|C(\alpha_i) - C(\beta_i)\| = \sqrt{112 + 64\sqrt{3}}/6 > 1 + \sqrt{70 + 40\sqrt{3}} = r_A + r_B$, so the arcs $\alpha_i$ and $\beta_i$ do not intersect, otherwise $\|C(\alpha_i) - C(\beta_j)\| = 1 + \sqrt{70 + 40\sqrt{3}} = (r_A^2 + r_B^2)$, so by Observation 1.1 the arcs $\alpha_i$ and $\beta_j$ are orthogonal. □

**Theorem 3.3.** There exist arc-RAC graphs with $4.5n - O(\sqrt{n})$ edges.
Proof. We start by constructing a tile and then show that its drawing is indeed a valid arc-RAC drawing. After that it is easy to construct the drawing of the embedded graph $G$ with the claimed edge-density.

Consider two intersecting circles $\alpha$ and $\beta$ so that one of their intersection points is closer to the origin than the other, we denote by $X^-_{\alpha\beta}$ the intersection point which is closer to the origin and by $X^+_{\alpha\beta}$ the intersection point which is further from the origin.

Let the vertices of the hexagon in a tile be $P_0 = X^+_{\alpha_0\alpha_1}$, $P_1 = X^-_{\beta_0\beta_1}$, $P_2 = X^+_{\alpha_1\alpha_2}$, $P_3 = X^-_{\beta_1\beta_2}$, $P_4 = X^+_{\alpha_2\alpha_0}$, and $P_5 = X^-_{\beta_2\beta_0}$. Due to the symmetric definitions of the arcs the angle between two consecutive vertices of the hexagon is $\pi/3$. Moreover, by a simple computation, we see that for each $j \in \{0, 1, 2\}$ and with $b = \sqrt{2}/(\sqrt{3} + 1)$ we have:

$$X^+_{\alpha_j\alpha_{j+1}} = (b \cos(\pi/3 + j \pi/3), b \sin(\pi/3 + j \pi/3))$$
$$X^-_{\beta_j\beta_{j+1}} = (b \cos(\pi/3 + (j + 2) \pi/3), b \sin(\pi/3 + (j + 2) \pi/3)).$$

Thus, all the vertices of the hexagon are equidistant from its center, so the hexagon is regular. Now let the arcs of $A$ and $B$ be exactly the arcs with radii and centers specified above, such that they are completely contained in the regular hexagon. According to Lemmas 3.1 and 3.2 all the crossings of arcs belonging to the same tile are orthogonal. Moreover, the arcs do not intersect the interior of the edges of the hexagon. To see this take, for example, the arc $\alpha_2$. Its center $C(\alpha_2)$ is below the segment connecting $P_2$ and $P_4$. This segment makes an orthogonal angle in the hexagon with the side of the hexagon $P_2P_1$, therefore, the tangent of $\alpha_2$ at $P_2$ intersects the interior of the hexagon. Thus, $\alpha_2$ cannot intersect an edge of the hexagon. Similarly we can show that the arcs in $B$ do not intersect the interior of the edges of the hexagon. Therefore, the drawing of a tile, and hence of $G$, is valid.

As mentioned by Arikushi et al. [5] almost all vertices of the lattice with the exception of at most $O(\sqrt{n})$ vertices at the lattice’s boundary have degree 9. Hence the number of edges is $4.5n - O(\sqrt{n})$.

▶ Corollary 3.4. The class of arc-RAC graphs is a proper superclass of RAC$_0$ graphs.

4 Open Problems and Conjectures

An obvious open problem is to tighten the bounds on the edge density of arc-RAC graphs in Theorems 2.3 and 3.3.
Another immediate question is the relation to RAC\(_1\) graphs, which also extend the class of RAC\(_0\) graphs. This is especially intriguing as the best known lower bound for the edge density of RAC\(_1\) graphs is indeed larger than our lower bound for arc-RAC graphs whereas potentially there might be arc-RAC graphs that are denser than the densest RAC\(_1\) graphs.

The relation between RAC\(_k\) graphs and 1-planar graphs is well understood \[8\,9\,11\,16\]. What about the relation between arc-RAC graphs and 1-planar graphs? In particular, is there a 1-planar graph which is not arc-RAC?

We are also interested in the area required by arc-RAC drawings. Are there arc-RAC graphs that need exponential area to admit an arc-RAC drawing? (A way to measure this off the grid is to consider the ratio between the longest and the shortest edge in a drawing.)

Finally, the complexity of recognizing arc-RAC graphs is open, but likely NP-hard.

References

1. Eyal Ackerman. On the maximum number of edges in topological graphs with no four pairwise crossing edges. *Discrete Comput. Geom.*, 41(3):365–375, 2009. doi:10.1007/s00454-009-9143-9
2. Eyal Ackerman and Gábor Tardos. On the maximum number of edges in quasi-planar graphs. *J. Combin. Theory, Ser. A*, 114(3):563–571, 2007. doi:10.1016/j.jcta.2006.08.002
3. Oswin Aichholzer, Wolfgang Aigner, Franz Aurenhammer, Kateřina Čech Dobšáková, Bert Jüttler, and Günter Rote. Triangulations with circular arcs. In Marc van Kreveld and Bettina Speckmann, editors, *Proc. Graph Drawing (GD’11)*, volume 7034 of LNCS, pages 296–307. Springer, 2012. doi:10.1007/978-3-642-25878-7_29
4. Patrizio Angelini, Michael A. Bekos, Henry Förster, and Michael Kaufmann. On RAC drawings of graphs with one bend per edge. In Therese Biedl and Andreas Kerren, editors, *Proc. Graph Drawing & Network Vis. (GD’18)*, volume 11282 of LNCS, pages 123–136. Springer, 2018. doi:10.1007/978-3-030-04414-5_9
5. Karin Arikushi, Radoslav Fulek, Baláazs Keszegh, Filip Morić, and Csaba D. Tóth. Graphs that admit right angle crossing drawings. In Dimitrios M. Thilikos, editor, *Proc. Graph Theoretic Concepts in Comput. Sci. (WG’10)*, volume 6410 of LNCS, pages 135–146. Springer, 2010. doi:10.1007/978-3-642-16926-7_14
6. Karin Arikushi, Radoslav Fulek, Baláazs Keszegh, Filip Morić, and Csaba D. Tóth. Graphs that admit right angle crossing drawings. *Comput. Geom.*, 45(4):169–177, 2012. doi:10.1016/j.comgeo.2011.11.008
7. Christian Bachmaier, Franz J. Brandenburg, Kathrin Hanauer, Daniel Neuwirth, and Josef Reislhuber. NIC-planar graphs. *Discrete Appl. Math.*, 232:23–40, 2017. doi:10.1016/j.dam.2017.08.015
8. Michael A. Bekos, Walter Didimo, Giuseppe Liotta, Saeed Mehrabi, and Fabrizio Montecchiani. On RAC drawings of 1-planar graphs. *Theoretical Comput. Sci.*, 689:48–57, 2017. doi:10.1016/j.tcs.2017.05.039
9. Franz J. Brandenburg, Walter Didimo, William S. Evans, Philipp Kindermann, Giuseppe Liotta, and Fabrizio Montecchiani. Recognizing and drawing IC-planar graphs. *Theoret. Comput. Sci.*, 636:1–16, 2016. URL: https://arxiv.org/abs/1509.00388 doi:10.1016/j.tcs.2016.04.026
10. Steven Chaplick, Henry Förster, Myroslav Kryven, and Alexander Wolff. On arrangements of orthogonal circles. In Daniel Archambault and Csaba D. Tóth, editors, *Proc. Graph Drawing and Network Visualization (GD’19)*, volume 11904 of LNCS, pages 216–229. Springer, 2019. URL: https://arxiv.org/abs/1907.08121 doi:10.1007/978-3-030-35802-0_17
11. Steven Chaplick, Fabian Lipp, Alexander Wolff, and Johannes Zink. Compact drawings of 1-planar graphs with right-angle crossings and few bends. *Comput. Geom.*, 84:50–68, 2019. Special issue on EuroCG 2018. doi:10.1016/j.comgeo.2019.07.006
12 C. C. Cheng, Christian A. Duncan, Michael T. Goodrich, and Stephen G. Kobourov. Drawing planar graphs with circular arcs. *Discrete Comput. Geom.*, 25:405–418, 2001. doi:10.1007/s004540010080

13 Walter Didimo, Peter Eades, and Giuseppe Liotta. Drawing graphs with right angle crossings. *Theoret. Comput. Sci.*, 412(39):5156–5166, 2011. doi:10.1016/j.tcs.2011.05.025

14 Walter Didimo, Giuseppe Liotta, and Fabrizio Montecchiani. A survey on graph drawing beyond planarity. *ACM Comput. Surv.*, 52(1):4:1–4:37, 2019. doi:10.1145/3301281

15 Vida Dujmović, Joachim Gudmundsson, Pat Morin, and Thomas Wolle. Notes on large angle crossing graphs. In A. Potanin and A. Viglas, editors, *Proc. Comput. Australasian Theory Symp. (CATS’10)*, volume 109 of CRPIT, pages 19–24. Australian Computer Society, 2010. URL: http://dl.acm.org/citation.cfm?id=1862317.1862320

16 Peter Eades and Giuseppe Liotta. Right angle crossing graphs and 1-planarity. *Discrete Appl. Math.*, 161(7):961–969, 2013. doi:10.1016/j.dam.2012.11.019

17 Weidong Huang. Using eye tracking to investigate graph layout effects. In Seok-Hee Hong and Kwan-Liu Ma, editors, *Proc. Asia-Pacific Symp. Visual. (APVIS’07)*, pages 97–100. IEEE, 2007. doi:10.1109/APVIS.2007.329282

18 Weidong Huang, Peter Eades, and Seok-Hee Hong. Larger crossing angles make graphs easier to read. *J. Vis. Lang. Comput.*, 25(4):452–465, 2014. doi:10.1016/j.jvlc.2014.03.001

19 Weidong Huang, Seok-Hee Hong, and Peter Eades. Effects of crossing angles. In *Proc. IEEE VGTC Pacific Visualization (PacificVis’08)*, pages 41–46, 2008. doi:10.1109/PACIFICVIS.2008.4475457

20 Michael Jünger and Petra Mutzel, editors. *Graph Drawing Software*. Springer, Berlin, Heidelberg, 2004. doi:10.1007/978-3-642-18638-7

21 Michael Kaufmann and Dorothea Wagner, editors. *Drawing Graphs: Methods and Models*. Springer, Berlin, Heidelberg, 2001. doi:10.1007/3-540-44969-8

22 Myroslav Kryven, Alexander Ravsky, and Alexander Wolff. Drawing graphs on few circles and few spheres. *J. Graph Algorithms Appl.*, 23(2):371–391, 2019. doi:10.7155/jgaa.00495

23 Mark Lombardi and Robert Hobbs, editors. *Mark Lombardi: Global Networks*. Independent Curators, 2003.

24 C. Stanley Ogilvy. *Excursions in Geometry*. Oxford Univ. Press, New York, 1969.

25 János Pach, Radoš Radoičić, and Géza Tóth. Relaxing planarity for topological graphs. In Ervin Györi, Gyula O. H. Katona, László Lovász, and Tamás Fleiner, editors, *More Sets, Graphs and Numbers: A Salute to Vera Sós and András Hajnal*, pages 285–300. Springer Berlin Heidelberg, 2006. doi:10.1007/978-3-540-32439-3_12

26 Helen C. Purchase, John Hamer, Martin Nöllenburg, and Stephen G. Kobourov. On the usability of Lombardi graph drawings. In Walter Didimo and Maurizio Patrignani, editors, *Proc. Graph Drawing (GD’12)*, volume 7704 of LNCS, pages 451–462. Springer, 2013. doi:10.1007/978-3-642-36763-2_40

27 André Schulz. Drawing graphs with few arcs. *J. Graph Algorithms Appl.*, 19(1):393–412, 2015. doi:10.7155/jgaa.00366

28 Kai Xu, Chris Rooney, Peter Passmore, and Dong-Han Ham. A user study on curved edges in graph visualisation. In Philip Cox, Beryl Plimmer, and Peter Rodgers, editors, *Proc. Theory Appl. Diagrams (DIAGRAMS’10)*, volume 7352 of LNCS, pages 306–308. Springer, 2012. doi:10.1007/978-3-642-31223-6_34