MODULI STACKS OF SEMISTABLE SHEAVES AND REPRESENTATIONS OF EXT-QUIVERS

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Abstract. We show that the moduli stacks of semistable sheaves on smooth projective varieties are analytic locally on their coarse moduli spaces described in terms of representations of the associated Ext-quivers with convergent relations. When the underlying variety is a Calabi-Yau 3-fold, our result describes the above moduli stacks as critical locus analytic locally on the coarse moduli spaces. The results in this paper will be applied to the wall-crossing formula of Gopakumar-Vafa invariants defined by Maulik and the author.

1. Introduction

1.1. Motivation. The purpose of this paper is to give descriptions of moduli stacks of semistable sheaves on smooth projective varieties in terms of quivers with (formal but convergent) relations, analytic locally on their coarse moduli spaces. The relevant quiver is the Ext-quiver associated to the simple collection of coherent sheaves, determined by a polystable sheaf corresponding to a point of the coarse moduli space. Probably the main results have been folklore for experts of moduli of sheaves (at least on formal neighborhoods at closed points of the coarse moduli space), but we cannot find any reference and our purpose is to give precise statements and details of the proofs. The main results in this paper will be used in the companion paper [Tod] in the proof of wall-crossing formula of Gopakumar-Vafa invariants introduced by Maulik and the author [MT].

1.2. Results. Let $X$ be a smooth projective variety over $\mathbb{C}$ and $\omega$ an ample divisor on it. Let $\mathcal{M}_\omega$ be the moduli stack of $\omega$-Gieseker semistable sheaves on $X$, and $M_\omega$ the coarse moduli space of $S$-equivalence classes of them. There is a natural morphism

$$p_M : \mathcal{M}_\omega \to M_\omega$$

sending a semistable sheaf to its $S$-equivalence class. A closed point of $M_\omega$ corresponds to a polystable sheaf, i.e. a direct sum

$$E = \bigoplus_{i=1}^k V_i \otimes E_i$$

(1.1)

where each $E_1, \ldots, E_k$ are mutually non-isomorphic $\omega$-Gieseker stable sheaves with the same reduced Hilbert polynomials.
The Ext-quiver $Q$ associated to the collection $(E_1, \ldots, E_k)$ is defined by the quiver whose vertex is $\{1, \ldots, k\}$ and the number of arrows from $i$ to $j$ is the dimension of $\text{Ext}^1(E_i, E_j)$. We denote by $\mathcal{M}_Q$ the moduli stack of finite dimensional $Q$-representations with dimension vector $(\dim V_i)_{1 \leq i \leq k}$, and $M_Q$ the coarse moduli space of semi-simple $Q$-representations with dimension vector as above. We have the natural morphism

$$p_Q : \mathcal{M}_Q \to M_Q$$

sending a $Q$-representation to its semi-simplification. There is a point $0 \in M_Q$ represented by the semi-simple $Q$-representation $\bigoplus_{i=1}^k V_i \otimes S_i$, where $S_i$ is a simple $Q$-representation corresponding to the vertex $i$. The following is the main result in this paper.

**Theorem 1.1.** (Theorem 3.2) For \(p \in M_\omega\) represented by a polystable sheaf (1.1), let $Q$ be the Ext-quiver associated to $(E_1, \ldots, E_k)$. Then there exist analytic open neighborhoods $p \in U \subset M_\omega$, $0 \in V \subset M_Q$, closed analytic substack $Z \subset p_Q^{-1}(V)$ with the natural morphism to its coarse moduli space $p_Q : Z \to U$.

Indeed we can define the (formal but convergent) relation $I$ of the Ext-quiver $Q$, using the minimal $A_\infty$-structure of the dg-category generated by $(E_1, \ldots, E_k)$. The convergence of $I$ will be proved by generalizing the gauge theory arguments of [Fuk03, Tu] for deformations of vector bundles to the case of resolutions of coherent sheaves by complexes of vector bundles. The substack $Z \subset p_Q^{-1}(V)$ is then defined to be the stack of $Q$-representations satisfying the relation $I$.

When $X$ is a smooth projective Calabi-Yau (CY) 3-fold, we can take the relation $I$ to be the derivation of a convergent super-potential of the quiver $Q$. So we have the following corollary of Theorem 1.1.

**Corollary 1.2.** (Corollary 5.7) In the situation of Theorem 1.1, suppose that $X$ is a smooth projective CY 3-fold. Then there is a morphism of complex analytic stacks $W : p_Q^{-1}(V) \to \mathbb{C}$ such that

$$Z = \{dW = 0\} \xrightarrow{\sim} p_M^{-1}(U).$$

A result similar to (5.7) was already proved in [JS12, BBBBJ15], where the stack $M_\omega$ is described as a critical locus locally on $M_\omega$. Our description is more global, as we describe the stack $M_\omega$ as a critical locus on the preimage of an open subset of the coarse moduli space $M_\omega$. The result of Corollary 5.7 is also compatible with the $d$-critical structure introduced by Joyce [Joy15]. By [PTVV13], the stack $M_\omega$ is a truncation of a derived scheme with a $(-1)$-shifted symplectic structure [PTVV13]. Using
this fact, it is proved in [BBBJ15] that the stack $\mathcal{M}_\omega$ has a canonical $d$-critical structure. From the construction of $W$ in Corollary 1.2, the data $(p_M^{-1}(U), p_Q^{-1}(V), W)$ is shown to give a $d$-critical chart of the $d$-critical stack $\mathcal{M}_\omega$ (see [Tod] Appendix A).

In the case of moduli spaces of one dimensional sheaves, we also investigate the wall-crossing phenomena of these moduli spaces with respect to the twisted stability. Let $A(X)_\mathbb{C}$ be the complexified ample cone of $X$ and take an element

$$\sigma = B + i\omega \in A(X)_\mathbb{C}.$$ 

Let $M_\sigma$ be the coarse moduli space of one dimensional $B$-twisted $\omega$-semistable sheaves on $X$. We will see that the result of Theorem 1.1 is also applied for the moduli space $M_\sigma$ of twisted semistable sheaves. If we take $\sigma^+ \in A(X)_\mathbb{C}$ to be sufficiently close to $\sigma$, we have the natural projective morphism

$$q_M: M_{\sigma^+} \to M_{\sigma}. \quad (1.2)$$

**Theorem 1.3.** (Theorem 7.7) For $p \in M_\sigma$, let an open subset $p \in U \subset M_\sigma$, a quiver $Q$, and an analytic space $Z$ be as in Theorem 1.1. Then there is a stability condition $\xi$ on the category of $Q$-representations such that we have the commutative diagram of isomorphisms

$$Z_\xi \xrightarrow{\sim} q^{-1}_M(U) \xrightarrow{q_M} \mathbb{C}.$$ 

Here $Z_\xi$ is the coarse moduli space of $\xi$-semistable $Q$-representations satisfying the relation $I$.

When $X$ is a K3 surface, the morphism (1.2) was studied by Arbarello-Saccà [AS]. In this case, they showed that the morphism (1.2) is analytic locally on $M_\sigma$ described as a symplectic resolution of singularities of Nakajima quiver varieties via variation of stability conditions of representations of quivers. One can check that the result of Theorem 1.3 gives the same description of the morphism (1.2) as in [AS], if we know the formality of the dg-algebra $R\text{Hom}(E, E)$ for a polystable sheaf $[E] \in M_\sigma$.

The results of Corollary 1.2 and Theorem 1.3 will be used in [Tod] to show the wall-crossing formula of (generalization of) Gopakumar-Vafa (GV) invariants introduced by Maulik and the author [MT]. The idea is roughly speaking as follows. In [Tod], we construct some perverse sheaves $\phi_{M_{\sigma^+}}$, $\phi_{M_{\sigma}}$ on the moduli spaces $M_{\sigma^+}$, $M_{\sigma}$ in (1.2) respectively, following the analogy of BPS sheaves introduced by Davison-Meinhardt [DM]. It turns out that there is a natural morphism

$$\phi_{M_{\sigma}} \to Rq_M^*\phi_{M_{\sigma^+}} \quad (1.4)$$

and we want to show that the above morphism is an isomorphism. The results of Corollary 1.2 and Theorem 1.3 enable us to reduce to the case
of quivers with convergent super-potentials. In the case of quivers with super-potentials, the similar question was addressed and solved in [DM], and we can use the results and arguments in loc.cit. to show that (1.4) is an isomorphism.

In a similar way, using the result of Corollary 1.2 it should be possible to reduce several problems in Donaldson-Thomas (DT) theory on CY 3-folds to the case of representations of quivers with convergent super-potentials, which is easier in many cases. For example it is recently announced by Davison-Meinhardt that the integrality conjecture of generalized DT invariants [JS12, KS] on CY 3-folds can be proved using the result of Corollary 1.2.

1.3. Plan of the paper. The organization of this paper is as follows. In Section 2 we introduce the notion of quivers with convergent relations and construct the moduli spaces of their representations. In Section 3 we fix some notation on the moduli spaces of semistable sheaves and state the precise form of Theorem 1.1. In Section 4 we describe deformation theory of coherent sheaves in terms of minimal $A_\infty$-structures. In Section 5 we complete the proof of Theorem 1.1. In Section 6 we recall NC deformation theory and relate it with the result of Theorem 1.1. In Section 7 we prove Theorem 1.3.

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2. Quivers with convergent relations

In this section, we recall some basic notions on quivers, their representations and moduli spaces. We also introduce the concept of convergent relations of quivers, and moduli spaces of quiver representations satisfying such relations.

2.1. Representations of quivers. Recall that a quiver $Q$ consists data

$$Q = (V(Q), E(Q), s, t)$$

where $V(Q), E(Q)$ are finite sets and $s, t$ are maps

$$s, t : E(Q) \to V(Q).$$

The set $V(Q)$ is the set of vertices and $E(Q)$ is the set of edges. For $e \in E(Q)$, $s(e)$ is the source of $e$ and $t(e)$ is the target of $e$. For $i, j \in V(Q)$, we use the following notation

$$E_{i,j} := \{ e \in E(Q) : s(e) = i, t(e) = j \}$$

i.e. $E_{i,j}$ is the set of edges from $i$ to $j$. 

A $Q$-representation consists of data
\[ V = \{(V_i, u_e) : i \in V(Q), e \in E(Q), u_e : V_{s(e)} \to V_{t(e)}\} \]
where $V_i$ is a finite dimensional $\mathbb{C}$-vector space and $u_e$ is a linear map. For a $Q$-representation (2.2), the vector
\[ \vec{m} = (m_i)_{i \in V(Q)}, m_i = \dim V_i \]
is called the dimension vector.

Given a dimension vector (2.3), let $V_i$ be a $\mathbb{C}$-vector space with dimension $m_i$. Let us set
\[ G := \prod_{i \in Q(V)} \text{GL}(V_i), \quad \text{Rep}_Q(\vec{m}) := \prod_{e \in E(V)} \text{Hom}(V_{s(e)}, V_{t(e)}). \]
The algebraic group $G$ acts on $\text{Rep}_Q(\vec{m})$ by
\[ g \cdot u = \{g_{t(e)}^{-1} \circ u_e \circ g_{s(e)}\}_{e \in E(Q)} \]
for $g = (g_i)_{i \in V(Q)} \in G$ and $u = (u_e)_{e \in E(Q)}$. A $Q$-representation with dimension vector $\vec{m}$ is determined by a point in $\text{Rep}_Q(\vec{m})$ up to $G$-action. The moduli stack of $Q$-representations with dimension vector $\vec{m}$ is given by the quotient stack
\[ \mathcal{M}_Q(\vec{m}) := [\text{Rep}_Q(\vec{m})/G]. \]
It has the coarse moduli space, given by
\[ p_Q : \mathcal{M}_Q(\vec{m}) \to M_Q(\vec{m}) := \text{Rep}_Q(\vec{m})//G. \]
Here in general, if a reductive algebraic group $G$ acts on an affine scheme $Y = \text{Spec } R$, then its affine GIT quotient is given by
\[ Y//G := \text{Spec } R^G. \]
For two points in $x_1, x_2 \in Y$, they are mapped to the same point in $Y//G$ iff their $G$-orbit closures intersect, i.e.
\[ G \cdot x_1 \cap G \cdot x_2 \neq \emptyset. \]
In the case of $G$-action on $\text{Rep}_Q(\vec{m})$, the above condition is also equivalent to that the corresponding $Q$-representations have the isomorphic semi-simplifications. The quotient space $M_Q(\vec{m})$ parametrizes semi-simple $Q$-representations with dimension vector $\vec{m}$, and the map (2.5) sends a $Q$-representation to its semi-simplification (see [Muk03, Section 5], [Kin94, Section 3] for details).

For $i \in V(Q)$, let $S_i$ be the simple $Q$-representation corresponding to the vertex $i$, i.e. it is the unique $Q$-representation with dimension vector $m_i = 1$ and $m_j = 0$ for $j \neq i$. The point $0 \in \text{Rep}_Q(\vec{m})$ and its image $0 \in M_Q(\vec{m})$ by the map (2.5) correspond to semi-simple $Q$-representation $\oplus_{i \in V(Q)} V_i \otimes S_i$. A $Q$-representation (2.2) is called nilpotent if any sufficiently large number of compositions of the linear maps $u_e$ becomes zero. It is easy to see that a
Q-representation is nilpotent iff it is an iterated extensions of simple objects \( \{ S_i \}_{i \in V(Q)} \). In particular, the fiber
\[
p_Q^{-1}(0) \subset \mathcal{M}_Q(\vec{m})
\]
for the morphism \( \text{(2.5)} \) consists of nilpotent \( Q \)-representations with dimension vector \( \vec{m} \).

2.2. Quivers with convergent relations. Recall that a path of a quiver \( Q \) is a composition of edges in \( Q \)
\[ e_1 e_2 \ldots e_n, \, e_i \in E(Q), \, t(e_i) = s(e_{i+1}). \]
The number \( n \) above is called the length of the path. The path algebra of a quiver \( Q \) is a \( \mathbb{C} \)-vector space spanned by paths in \( Q \):
\[
\mathbb{C}[Q] := \bigoplus_{n \geq 0} \bigoplus_{e_1, \ldots, e_n \in E(Q), t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n.
\]
Here a path of length zero is a trivial path at each vertex of \( Q \), and the product on \( \mathbb{C}[Q] \) is defined by composition of paths. By taking the completion of \( \mathbb{C}[Q] \) with respect to the length of the path, we obtain the formal path algebra:
\[
\mathbb{C}[\![Q\!]] := \prod_{n \geq 0} \bigoplus_{e_1, \ldots, e_n \in E(Q), t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n.
\]
Note that an element \( f \in \mathbb{C}[\![Q\!]] \) is written as
\[
(2.6) \quad f = \sum_{n \geq 0} \sum_{\psi: \{1, \ldots, n+1\} \rightarrow V(Q)} a_{\psi, e_{\bullet}} \cdot e_1 e_2 \ldots e_n.
\]
Here \( a_{\psi, e_{\bullet}} \in \mathbb{C}, \, e_{\bullet} = (e_1, \ldots, e_n) \) and \( E_{\psi(i), \psi(i+1)} \) is defined as in \( \text{(2.1)} \). The above element \( f \) lies in \( \mathbb{C}[Q] \) iff \( a_{\psi, e_{\bullet}} = 0 \) for \( n \gg 0 \).

Definition 2.1. We define the subalgebra
\[
\mathbb{C}\{Q\} \subset \mathbb{C}[\![Q\!]]
\]
to be elements \( \text{(2.6)} \) such that \( |a_{\psi, e_{\bullet}}| < C^n \) for some constant \( C > 0 \) which is independent of \( n \).

Note that \( \mathbb{C}\{Q\} \) contains \( \mathbb{C}[Q] \) as a subalgebra. For an element \( f \in \mathbb{C}\{Q\} \), we write it as \( \text{(2.6)} \) and consider the following \( \text{Hom}(V_a, V_b) \)-valued formal function of \( u = (u_e)_{e \in E(Q)} \in \text{Rep}_Q(\vec{m}) \)
\[
(2.7) \quad f(a, b, \vec{m}) := \sum_{n \geq 0, \psi: \{1, \ldots, n+1\} \rightarrow V(Q)} \sum_{\psi(1) = a, \psi(n+1) = b} a_{\psi, e_{\bullet}} \cdot u_{e_n} \circ \cdots \circ u_{e_2} \circ u_{e_1}.
\]
By the definition of $\mathbb{C}\{Q\}$, the above $\text{Hom}(V_a, V_b)$-valued formal function on $\text{Rep}_Q(\vec{m})$ has a convergent radius. So there is an analytic open neighborhood
\begin{equation}
0 \in \mathcal{U} \subset \text{Rep}_Q(\vec{m})
\end{equation}
such that the function (2.7) absolutely converges on it and determines the complex analytic map
\[f(a, b, \vec{m}) : \mathcal{U} \to \text{Hom}(V_a, V_b).\]
In particular, the equations $f(a, b, \vec{m}) = 0$ for all $a, b \in V(Q)$ determines the closed complex analytic subspace of $\mathcal{U}$.

2.3. **Saturated open subsets.** We will extend the arguments in the previous subsection to a preimage of an open subset in $\text{Rep}_Q(\vec{m})/\!/G$. Before doing this, we prepare some general definitions and lemmas for the action of a reductive algebraic group on affine schemes or analytic spaces.

**Definition 2.2.** Let $G$ be a reductive group acting on an affine algebraic $\mathbb{C}$-scheme $Y$. Then an analytic open set $U \subset Y$ is called saturated if for any $x \in U$, the orbit closure $G \cdot x \subset Y$ is contained in $U$.

Note that a saturated open subset is in particular $G$-invariant. Let
\begin{equation}
\pi_Y : Y \to Y/\!/G
\end{equation}
be the quotient map and $V \subset Y/\!/G$ be an analytic open subset. Then $\pi_Y^{-1}(V)$ is obviously saturated. Indeed, the converse is also true. In order to see this, we recall the following fact on the topology of affine GIT quotient $Y/\!/G$.

**Theorem 2.3.** ([Nee85, Sch89]) In the situation of Definition 2.2, let $K \subset G$ be a maximal compact subgroup of $G$. Then there is a $K$-invariant closed subset $S \subset Y$ in analytic topology, called Kempf-Ness set, satisfying the following: for any $x \in S$ the $G$-orbit $G \cdot x \subset Y$ is closed in $Y$ and the inclusion $S \subset Y$ induces the homeomorphism
\begin{equation}
\iota : S/K \xrightarrow{\cong} Y/\!/G.
\end{equation}
Here the topology of $S/K$ is a quotient topology induced from the analytic topology of $S$, and that of $Y/\!/G$ is the analytic topology. In particular, the analytic topology of $Y/\!/G$ is the quotient topology induced from the analytic topology of $Y$.

The following lemma follows from the above theorem:

**Lemma 2.4.** In the situation of Definition 2.2, an analytic open subset $U \subset Y$ is saturated iff there is an analytic open set $V \subset Y/\!/G$ such that $U = \pi_Y^{-1}(V)$ where $\pi_Y : Y \to Y/\!/G$ is the quotient morphism.

**Proof.** For $x \in U$ and $y \in Y$, suppose that $\pi_Y(x) = \pi_Y(y)$, i.e. $G \cdot x$ and $G \cdot y$ intersect. Since $U$ is saturated, we have $G \cdot x \subset U$. Then we have $G \cdot y \cap U \neq \emptyset$, and since $U$ is open there is $g \in G$ such that $g \cdot y \in U$. 

Therefore we have \( y \in U \). This implies that there is a subset \( V \subset Y//G \) such that \( U = \pi_Y^{-1}(V) \). By Theorem 2.3, the subset \( V \) is analytic open, hence the lemma holds.

We also have the following lemma.

**Lemma 2.5.** In the situation of Definition 2.2, let \( y \in Y \) be a \( G \)-fixed point and \( U \subset Y \) a \( G \)-invariant analytic open subset with \( y \in U \). Then there is an analytic open subset \( U' \subset Y \), which is saturated and satisfies \( 0 \in y \in U' \subset U \).

**Proof.** Let \( S \subset Y \) be the Kempf-Ness set as in Theorem 2.3. Since \( y \in Y \) is \( G \)-fixed, we have \( y \in S \) by the homeomorphism (2.10). Then we have \( y \in S \cap U \), and \( S \cap U \) is a \( K \)-invariant open subset in \( S \). Therefore we have \( S \cap U = \pi_S^{-1}(V) \) for some open subset \( V \subset S/K \), where \( \pi_S: S \to S/K \) is the quotient map. Since the map \( \tau \) in (2.10) is a homeomorphism, the subset \( \iota(V) \subset Y//G \) is open. We set a saturated open subset \( U' \subset Y \) to be \( U' = \pi_Y^{-1}(\iota(V)) \) for the quotient map (2.9). Since \( \pi_S(y) \in V \), we have \( y \in U' \). It is enough to check that \( U' \subset U \). By the construction of \( U' \), for \( x \in U' \) there is \( z \in S \cap U \) such that \( \pi_Y(x) = \pi_Y(z) \), i.e. the closures of \( G \cdot x \) and \( G \cdot z \) intersect. Since \( G \cdot z \) is closed, we have \( z \in G \cdot x \). Therefore there is \( g \in G \) such that \( g \cdot x \in U \). Since \( U \) is \( G \)-invariant, we have \( x \in U \), hence the lemma is proved.

2.4. **Analytic Hilbert quotients.** Later we will take GIT-type quotients for non-algebraic complex analytic spaces. Here we recall the basic notions for such quotients. The following definition appears in [HMP98, Gre15] for reduced complex analytic spaces.

**Definition 2.6.** Let \( G \) be a reductive algebraic group acting on a complex analytic space \( Z \). Then a complex analytic space \( Z//G \) together with a morphism

\[
\pi_Z: Z \to Z//G
\]

is called an analytic Hilbert quotient if the following conditions hold:

1. \( \pi_Z \) is a locally Stein map, i.e. there is an open cover \( Z//G = \bigcup_\lambda U_\lambda \) by Stein open subsets \( U_\lambda \) such that \( \pi_Z^{-1}(U_\lambda) \) is Stein.
2. We have \( (\pi_Z^*\mathcal{O}_Z)^G = \mathcal{O}_{Z//G} \).

An analytic Hilbert quotient is known to exist when \( Z \) is a reduced Stein space, which is unique up to isomorphism [Hei91]. In [HMP98, Gre15], analytic Hilbert quotients are discussed under the assumption that \( Z \) is reduced. It seems that such quotients for non-reduced analytic spaces are not available in literatures. We don’t develop generality of such quotients for non-reduced analytic spaces, but show the existence of such quotients in some special cases discussed below, and their universality.
We show the following lemma on the existence of analytic Hilbert quotients, which may be well-known, but we include it here as we cannot find a reference.

**Lemma 2.7.** Let $Y$ be an affine algebraic $C$-scheme with $G$-action. Then for the affine GIT quotient $\pi_Y : Y \to Y//G$, its analytification

$$\pi^{an}_Y : Y^{an} \to (Y//G)^{an}$$

is an analytic Hilbert quotient.

**Proof.** The condition (1) in Definition 2.6 is obvious as $Y^{an}$ and $(Y//G)^{an}$ are Stein, so we only prove (2). First suppose that $Y = C^n$ and the $G$-action on it is linear. In this case, the condition (2) in Definition 2.6 is proved in \cite{Lun76}. In general, there is a $G$-invariant closed embedding $Y \subset C^n$ where $G$ acts on $C^n$ linearly, and the commutative diagram

(2.12)

Here since $G$ is reductive, the functor $(-)^G$ sending a $G$-representation to its $G$-invariant part is exact. So the natural map $\Gamma(O_{C^n})^G \to \Gamma(O_Y)^G$ is surjective, so the bottom arrow of (2.12) is a closed embedding.

By taking the analytification of (2.12), we obtain the commutative diagram of analytic sheaves on $(C^n//G)^{an}$

(2.13)

Since $\pi^{an}_{C^n}$ is locally Stein, and the functor $(-)^G$ is exact, the vertical arrows of (2.13) are surjections. Therefore the bottom arrow of (2.13) is surjective. Also as $O_{Y//G} = (\pi_Y)_*O_{Y//G}$ for Zariski sheaves, we have an injection $O_{Y//G} \leftrightarrow \pi_Y_*O_Y$, which is also injective after taking completions at each closed point of $O_{Y//G}$. Hence the bottom arrow of (2.13) is also injective, so it is an isomorphism, i.e. $\pi^{an}_Y$ satisfies the condition (2) in Definition 2.6 \[\Box\]

By Lemma 2.7 for an analytic open subset $U \subset Y//G$ the map

(2.14)

$$\pi_Y : \pi^{-1}_Y(U) \to U$$

is an analytic Hilbert quotient of $\pi^{-1}_Y(U)$. We also have the following lemma:

**Lemma 2.8.** Let $Z \subset \pi^{-1}_Y(U)$ be a $G$-invariant closed analytic subspace. Then there is a closed analytic subspace $Z//G \to U$ and an analytic Hilbert quotient $\pi_Z : Z \to Z//G$. 

Proof. Since (2.14) is an analytic Hilbert quotient and the functor \((-)^G\) is exact, we have the surjection
\[
\mathcal{O}_U = (\pi_Y^* \mathcal{O}_{\pi_Y^{-1}(U)})^G \rightarrow (\pi_Y^* \mathcal{O}_Z)^G.
\]
Therefore by setting \(Z//G\) to be the complex analytic subspace of \(U\) defined by the ideal of the above kernel, we obtain the analytic Hilbert quotient \(\pi_Z = \pi_Y|_Z : Z \rightarrow Z//G\).

By gluing the above construction, we have the following lemma:

**Lemma 2.9.** Let \(Y\) be an algebraic \(\mathbb{C}\)-scheme with \(G\)-action and \(\pi_Y : Y \rightarrow Y'\) a \(G\)-invariant morphism of algebraic \(\mathbb{C}\)-schemes where \(G\) acts on \(Y'\) trivially. Suppose that \(Y' = \bigcup_{i \in I} V'_i\) is an affine open cover such that \(V_i = \pi_Y^{-1}(V'_i)\) is affine and \(\pi|_{V_i} : V_i \rightarrow V'_i\) is isomorphic to \(V_i \rightarrow V_i//G\). Then for an analytic open subset \(U \subset Y'\) and a \(G\)-invariant closed analytic subspace \(Z \subset \pi_Y^{-1}(U)\), the analytic Hilbert quotient \(Z//G\) exists as a closed analytic subspace of \(U\).

Proof. Let \(U_i = U \cap V'_i\) and \(Z_i = Z \cap V_i\). By applying Lemma 2.8 for \(Z_i \subset \pi_Y^{-1}(U_i) \subset V_i\), we obtain the analytic Hilbert quotient \(Z_i//G \subset U_i\). By the construction, they glue to give a desired analytic Hilbert quotient \(Z//G \subset U\).

**Remark 2.10.** The situation of Lemma 2.9 happens for a GIT quotient of semistable locus w.r.t. a \(G\)-linearization on a quasi-projective scheme.

We next discuss the universality of analytic Hilbert quotients:

**Definition 2.11.** An analytic Hilbert quotient (2.11) satisfies the universality if for any \(G\)-invariant analytic map \(h : Z \rightarrow Z'\) to a complex analytic space \(Z'\), there is a unique factorization
\[
h : Z \xrightarrow{\pi_Z} Z//G \rightarrow Z'.
\]

The above universality is proved in [Hei91, Corollary 4] when \(Z\) is a reduced Stein space and \(Z' = \mathbb{C}^n\). Below show the universality for the analytic Hilbert quotients given in Lemma 2.9. We prepare the following lemma:

**Lemma 2.12.** Let \(\pi_Z : Z \rightarrow Z//G\) be the analytic Hilbert quotient given in Lemma 2.9. Then for any family of \(G\)-invariant closed (not necessary analytic) subsets \(\{W_\lambda\}_{\lambda \in \Lambda}\) in \(Z\), the image \(\pi_Z(W_\lambda)\) is closed in \(Z//G\) and we have the identity
\[
\pi_Z \left( \bigcap_{\lambda \in \Lambda} W_\lambda \right) = \bigcap_{\lambda \in \Lambda} \pi_Z(W_\lambda).
\]

Proof. The question is local on \(Z//G\), so we may assume that \(Y\) is affine and \(Y' = Y//G\). Since \(Z, Z//G\) are closed in \(\pi_Y^{-1}(U)\), \(U\), we may also assume that \(Z = \pi_Y^{-1}(U), Z//G = U\). Let \(S \subset Y\) be a Kempf-Ness set as in...
Then for $S' := \pi_y^{-1}(U) \cap S$, we have the homeomorphism $S'/K \cong U$. Therefore for $W'_\lambda := S' \cap W_\lambda$, we have $\pi_Z(W'_\lambda) = \pi_Z(W_\lambda)$. Since each $W'_\lambda$ is a $K$-invariant closed subset of $S'$, its image $\pi_Z(W'_\lambda)$ is a closed subset of $U$ and the identity (2.16) holds.

The desired universality is proved in the following lemma:

**Lemma 2.13.** The analytic Hilbert quotient $\pi_Z: Z \to Z//G$ in Lemma 2.9 satisfies the universality in Definition 2.11.

**Proof.** Let $h: Z \to Z'$ be a $G$-invariant analytic map to a complex analytic space $Z'$. We take an open cover $Z' = \bigcup_{\lambda \in \Lambda} U'_\lambda$ such that $U'_\lambda$ is a closed analytic subspace of an open subset in $\mathbb{C}^n$. Let $W'_\lambda := Z' \setminus U'_\lambda$ and $W_\lambda := h^{-1}(W'_\lambda)$. Then each $W_\lambda$ is a $G$-invariant closed subset of $Z$. By Lemma 2.12 the image $\pi_Z(W_\lambda) \subset Z//G$ is closed and

$$\bigcap_{\lambda \in \Lambda} \pi_Z(W_\lambda) = \pi_Z \left( \bigcap_{\lambda \in \Lambda} W_\lambda \right) = \pi_Z \circ h^{-1} \left( \bigcap_{\lambda \in \Lambda} W'_\lambda \right) = \emptyset.$$ 

Here the last identity follows because $\{U'_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $Z'$. It follows that by setting $U_\lambda := (Z//G) \setminus \pi_Z(W_\lambda)$, we have an open cover $Z//G = \bigcup_{\lambda \in \Lambda} U_\lambda$ and the diagram

$$\begin{array}{ccc}
\pi_Z^{-1}(U_\lambda) & \longrightarrow & h^{-1}(U'_\lambda) \\
\pi_Z \downarrow & & \downarrow h \\
U_\lambda & \longrightarrow & U'_\lambda \longrightarrow \mathbb{C}^n.
\end{array}$$

Here the top horizontal arrow is an open immersion, and the right horizontal arrow is a locally closed embedding. By the property (2) in Definition 2.6 there is a unique analytic map $U_\lambda \to U'_\lambda$ which makes the above diagram commute. By the uniqueness, they glue to give a desired factorization (2.15).

2.5. **Moduli spaces of representations of quivers with convergent relations.** We return to the situation of Section 2.2.

**Definition 2.14.** A convergent relation $I$ of a quiver $Q$ is a collection of finite number of elements

$$I = (f_1, \ldots, f_l), \ f_i \in \mathbb{C}(Q).$$

Using the lemmas in the previous subsection, we have the following:

**Lemma 2.15.** Given a convergent relation $I = (f_1, \ldots, f_l)$ of a quiver $Q$ and its dimension vector $\vec{m}$, there is an analytic open neighborhood of $0$

$$0 \in V \subset M_Q(\vec{m})$$
such that each $\text{Hom}(V_a, V_b)$-valued formal function $f_i(a, b, \vec{m})$ defined by (2.7) for $f = f_i$ absolutely converges on $\pi_Q^{-1}(V)$. Here $\pi_Q$ is the quotient map

$$\pi_Q : \text{Rep}_Q(\vec{m}) \to M_Q(\vec{m}).$$

Proof. Let $U$ be an open neighborhood of $0 \in \text{Rep}_Q(\vec{m})$ as in (2.8), where each $f_i(a, b, \vec{m})$ absolutely converges on $U$. Since for $g = (g_i)_{i \in V(Q)} \in G$ and $u = (u_e)_{e \in E(Q)}$ we have

$$f_i(a, b, \vec{m})(g \cdot u) = g_0^{-1} \circ f_i(a, b, \vec{m})(u) \circ g_a$$

the $\text{Hom}(V_a, V_b)$-valued function $f_i(a, b, \vec{m})$ absolutely converges on $G \cdot U$. By Lemma 2.5 there is a saturated open subset $0 \in V \subset G \cdot U$. Then by Lemma 2.4, $V = \pi_Q^{-1}(V)$ for an open subset $0 \in V \subset M_Q(\vec{m})$. $\square$

For a quiver $Q$ with a convergent relation $I = (f_1, \ldots, f_l)$, let $\vec{m}$ be its dimension vector and take an open subset $V \subset M_Q(\vec{m})$ as in Lemma 2.15. By Lemma 2.15, we have the $G$-invariant closed analytic subspace of $\pi_Q^{-1}(V)$

$$\text{Rep}_{(Q,I)}(\vec{m})|_V \subset \pi_Q^{-1}(V)$$

(2.17)

whose structure sheaf is given by

$$\mathcal{O}_{\text{Rep}_{(Q,I)}(\vec{m})|_V} = \mathcal{O}_{\pi_Q^{-1}(V)}/(f_i(a, b, \vec{m})_{jk}, a, b \in V(Q)).$$

Here $f_i(a, b, \vec{m})_{jk}$ is the matrix component of the analytic map

$$f_i(a, b, \vec{m}) : \pi_Q^{-1}(V) \to \text{Hom}(V_a, V_b).$$

By taking the quotient by $G$, we have the following definition:

**Definition 2.16.** Let $Q$ be a quiver with a convergent relation $I$, and $\vec{m}$ its dimension vector. Then for a sufficiently small analytic open neighborhood $0 \in V \subset M_Q(\vec{m})$, we define the complex analytic stack $M_{(Q,I)}(\vec{m})|_V$ and complex analytic space $M_{(Q,I)}(\vec{m})|_V$ by

$$M_{(Q,I)}(\vec{m})|_V := [\text{Rep}_{(Q,I)}(\vec{m})|_V/G],$$

$$M_{(Q,I)}(\vec{m})|_V := \text{Rep}_{(Q,I)}(\vec{m})|_V//G.$$

Here $\text{Rep}_{(Q,I)}(\vec{m})|_V//G$ is the analytic Hilbert quotient of $\text{Rep}_{(Q,I)}(\vec{m})|_V$, given in Lemma 2.5.

2.6. **Convergent super-potential.** For a quiver $Q$, its convergent super-potential is defined as follows.

**Definition 2.17.** A convergent super-potential of a quiver $Q$ is an element

$$W \in \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}].$$

A convergent super-potential $W$ of $Q$ is represented by a formal sum

$$W = \sum_{n \geq 1} \sum_{\{1, \ldots, n+1\} \subseteq V(Q), e_i \in E_{\psi(i), \psi(i+1)}} \sum_{\psi(n+1) = \psi(1)} a_{\psi,e} \cdot e_1 e_2 \ldots e_n$$
with \(|a_{\psi,e}| < C^n\) for a constant \(C > 0\).

For \(i,j \in V(Q)\), let \(E_{i,j}\) be the \(C\)-vector space spanned by \(E_{i,j}\). We set

\[
E_{i,j}^\vee := \{ e^\vee : e \in E_{i,j} \} \subset E_{i,j}^\vee.
\]

Here for \(e \in E_{i,j}\), the element \(e^\vee \in E_{i,j}^\vee\) is defined by the condition \(e^\vee(e) = 1\) and \(e^\vee(e') = 0\) for any \(e \neq e' \in E_{i,j}\), i.e. \(E_{i,j}^\vee\) is the dual basis of \(E_{i,j}\).

For a map \(\psi : \{1, \ldots, n+1\} \to V(Q)\) with \(\psi(1) = \psi(n+1)\) and elements \(e_i \in E_{\psi(i),\psi(i+1)}, e \in E(Q)\), we set

\[
\partial e^\vee (e_1 \ldots e_n) = \sum_{a=1}^n e^\vee(e_a)e_{a+1} \ldots e_ne_1 \ldots e_{a-1}.
\]

Here \(e^\vee(e_a) = 0\) if \((s(e_a), t(e_a)) \neq (s(e), t(e))\). The above partial differential extends to a linear map

\[
\partial e^\vee : \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}] \to \mathbb{C}\{Q\}.
\]

For a convergent super-potential \(W\), the set of elements in \(\mathbb{C}\{Q\}\)

\[
\partial W := \{ \partial e^\vee W : e \in E(Q) \}
\]

is a convergent relation of \(Q\).

For a dimension vector \(\vec{m}\) of \(Q\), let \(tr W\) be the formal function of \(u = (u_e)_{e \in E(Q)} \in \text{Rep}_Q(\vec{m})\) defined by

\[
tr W(u) := \sum_{n \geq 1} \sum_{\{1, \ldots, n+1\} \to V(Q)} \sum_{\psi(1) = \psi(n+1) = \psi(i)} a_{\psi,e} \cdot \text{tr}(u_{e_n} \circ u_{e_{n-1}} \circ \cdots \circ u_{e_1}).
\]

The above formal function on \(\text{Rep}_Q(\vec{m})\) is \(G\)-invariant. By the argument of Lemma 2.15, there is an analytic open neighborhood \(0 \in V \subset M_Q(\vec{m})\) such that the formal function \(tr W\) absolutely converges on \(\pi^{-1}_Q(V)\) to give a \(G\)-invariant holomorphic function

\[
tr W : \pi^{-1}_Q(V) \to \mathbb{C}.
\]

Then for the relation \(I = \partial W\), it is easy to see (and well-known when \(W\) is a usual super-potential of \(Q\)) that the analytic subspace \((2.17)\) equals to the critical locus of \(tr W\) in \(\pi^{-1}_Q(V)\):

\[
\text{Rep}_{Q,\partial W}(\vec{m})|_V = \{ d(tr W) = 0 \}.
\]

In particular, we have

\[
\mathcal{M}_{Q,\partial W}(\vec{m})|_V = [\{ d(tr W) = 0 \}/G].
\]

3. Moduli stacks of semistable sheaves

In this section, we recall some basic notions and facts on moduli spaces of semistable sheaves, whose details are available in [HL97]. Then we state the precise form of Theorem 1.1 in Theorem 3.2. In what follows, we always assume that the varieties or schemes are defined over \(\mathbb{C}\).
3.1. Gieseker semistable sheaves. Let 

$$(X, \mathcal{O}_X(1))$$

be a polarized smooth projective variety with $\omega = c_1(\mathcal{O}_X(1))$. For a coherent sheaf $E$ on $X$, its Hilbert polynomial is defined by

$$\chi(E \otimes \mathcal{O}_X(m)) = a_d m^d + a_{d-1} m^{d-1} + \cdots$$

where $d = \dim \text{Supp}(E)$ and $a_d$ is a positive rational number. The reduced Hilbert polynomial is defined by

$$\overline{\chi}(E, m) := \frac{\chi(E \otimes \mathcal{O}_X(m))}{a_d} \in \mathbb{Q}[m].$$

For polynomials $p_i(m) \in \mathbb{Q}[m]$ with $i = 1, 2$, we write $p_1(m) \asymp p_2(m)$ if $\deg p_1 < \deg p_2$ or $\deg p_1 = \deg p_2$, $p_1(m) > p_2(m)$ for $m \gg 0$. Then $(\mathbb{Q}[m], \asymp)$ is an ordered set.

By definition, a coherent sheaf $E$ on $X$ is said to be $\omega$-Gieseker (semi)stable if for any non-zero subsheaf $E' \subseteq E$, we have the inequality

$$\overline{\chi}(E', m) \asymp (\leq) \overline{\chi}(E, m).$$

For any Gieseker semistable sheaf $E$ on $X$, it has a filtration (called J"{o}rdar-H"{o}lder (JH) filtration)

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = E$$

such that each $F_i/F_{i-1}$ is $\omega$-Gieseker stable whose reduced Hilbert polynomial coincides with $\overline{\chi}(E, m)$. The JH filtration is not necessary unique, but its subquotient

$$\text{gr}(E) := \bigoplus_{i=1}^{k} F_i/F_{i-1}$$

is uniquely determined up to isomorphism. For two $\omega$-Gieseker semistable sheaves $E, E'$ on $X$, they are called $S$-equivalent if $\text{gr}(E)$ and $\text{gr}(E')$ are isomorphic.

3.2. Moduli spaces of semistable sheaves. Let $\mathcal{M}$ be the 2-functor

$$(3.1) \quad \mathcal{M}: \text{Sch}/\mathbb{C} \to \text{Groupoid}$$

which sends a $\mathbb{C}$-scheme $S$ to the groupoid of $S$-flat coherent sheaves on $X \times S$. The stack $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$. Let $\Gamma$ be the image of the Chern character map

$$\Gamma := \text{Im}(\text{ch}: K(X) \to H^*(X, \mathbb{Q})).$$

For each $v \in \Gamma$, we have an open substack of finite type

$$\mathcal{M}_\omega(v) \subset \mathcal{M}$$

consisting of flat families of $\omega$-Gieseker semistable sheaves with Chern character $v$. 
The stack $\mathcal{M}_\omega(v)$ is constructed as a global quotient stack of a quasi-projective scheme. For $[E] \in \mathcal{M}_\omega(v)$, we take $m \gg 0$ and a vector space $V$ satisfying

$$\dim V = \chi(E(m)) = \dim H^0(E(m)).$$

The above condition depends only on $v$, and independent of $E$ for $m \gg 0$. Let $\text{Quot}(V, v)$ be the Grothendieck Quot scheme parameterizing quotients

$$s : V \otimes \mathcal{O}_X(-m) \to E$$

in $\text{Coh}(X)$ with $\text{ch}(E) = v$. Then there is an open subscheme

$$\text{Quot}_c(V, v) \subset \text{Quot}(V, v)$$

parameterizing quotients (3.2) such that $E$ is $\omega$-Gieseker semistable and the induced linear map $V \to H^0(E(m))$ is an isomorphism. The algebraic group $\text{GL}(V)$ acts on $\text{Quot}_c(V, v)$ by

$$g \cdot (V \otimes \mathcal{O}_X(-m) \xrightarrow{s} E) = (V \otimes \mathcal{O}_X(-m) \xrightarrow{s \circ g} E)$$

and the stack $\mathcal{M}_\omega(v)$ is described as

$$\mathcal{M}_\omega(v) = \left[\text{Quot}_c(V, v)/\text{GL}(V)\right].$$

The above construction is compatible with the Geometric Invariant Theory (GIT). If we take the closure of $\text{Quot}_c(V, v)$,

$$\text{Quot}^c(V, v) \subset \text{Quot}(V, v)$$

then there is a $\text{GL}(V)$-linearized polarization on $\text{Quot}^c(V, v)$ such that its open locus $\text{Quot}^c(V, v)$ is the GIT semistable locus with respect to the above $\text{GL}(V)$-linearized polarization. In particular, we have the good quotient morphism (which is in particular a good moduli space in the sense of [Alp13])

$$p_M : \mathcal{M}_\omega(v) \to \mathcal{M}_\omega(v):= \text{Quot}^c(V, v)/\text{GL}(V).$$

Namely, there is a $\text{GL}(V)$-invariant affine open cover

$$\text{Quot}^c(V, v) = \bigcup_i U_i, \ U_i = \text{Spec} R_i$$

such that $\mathcal{M}_\omega(v)$ has the following affine open cover

$$\mathcal{M}_\omega(v) = \bigcup_i U_i/\text{GL}(V), \ U_i/\text{GL}(V) = \text{Spec} R_i^{\text{GL}(V)}.$$

By the GIT construction of $\mathcal{M}_\omega(v)$, two points $x_1, x_2 \in \text{Quot}^c(V)$ are mapped to the same point by $p_M$ if and only if their orbit closures intersect, i.e.

$$\text{GL}(V) \cdot x_1 \cap \text{GL}(V) \cdot x_2 \neq \emptyset.$$ 

It is also known that the above condition is equivalent to that, if $x_i$ corresponds to a $\omega$-Gieseker semistable sheaf $E_i$, then $E_1$ and $E_2$ are $S$-equivalent. In fact, the projective scheme $\mathcal{M}_\omega(v)$ is the coarse moduli space of $S$-equivalence classes of $\omega$-Gieseker semistable sheaves with Chern character.
v. So every point \( p \in M_\omega(v) \) is represented by a direct sum of \( \omega \)-Gieseker stable sheaves \( E \) (called a polystable sheaf), written as

\[
E = \bigoplus_{i=1}^{k} V_i \otimes E_i.
\]

Here each \( V_i \) is a finite dimensional vector space, \( E_i \) is a \( \omega \)-Gieseker stable sheaf with \( \chi(E_i, m) = \chi(E, m) \) for all \( i \).

3.3. Ext-quiver. Suppose that \( E \in \text{Coh}(X) \) is of the form (3.3). Then the collection of the sheaves \((E_1, \ldots, E_k)\) forms a simple collection, defined below:

**Definition 3.1.** A collection of coherent sheaves \((E_1, \ldots, E_k)\) is called a simple collection if \( \text{Hom}(E_i, E_j) = \mathbb{C} \cdot \delta_{ij} \).

Let \( E_\bullet = (E_1, \ldots, E_k) \) be a simple collection of coherent sheaves on \( X \). For each \( 1 \leq i, j \leq k \), we fix a finite subset

\[
E_{i,j} \subset \text{Ext}^1(E_i, E_j)^\vee
\]

(3.4)

giving a basis of \( \text{Ext}^1(E_i, E_j)^\vee \). We define the quiver \( Q_{E_\bullet} \) as follows. The set of vertices and edges are given by

\[
V(Q_{E_\bullet}) = \{1, 2, \ldots, k\}, \quad E(Q_{E_\bullet}) = \coprod_{1 \leq i, j \leq k} E_{i,j}.
\]

The maps \( s, t : E(Q_{E_\bullet}) \to V(Q_{E_\bullet}) \) are given by

\[
s|_{E_{i,j}} = i, \quad t|_{E_{i,j}} = j.
\]

The resulting quiver \( Q_{E_\bullet} \) is called the Ext-quiver of \( E_\bullet \).

We can now state the precise statement of Theorem 1.1:

**Theorem 3.2.** Let \( X \) be a smooth projective variety, and let \( M_\omega(v) \) be the moduli stack of \( \omega \)-Gieseker semistable sheaves on \( X \) with Chern character \( v \). We have the natural morphism to its coarse moduli space

\[
p_M : M_\omega(v) \to M_\omega(v).
\]

For \( p \in M_\omega(v) \), it is represented by a sheaf \( E \) of the form

\[
E = \bigoplus_{i=1}^{k} V_i \otimes E_i
\]

where \( E_\bullet = (E_1, \ldots, E_k) \) is a simple collection. Let \( Q_{E_\bullet} \) be the corresponding Ext-quiver and \( \vec{m} \) its dimension vector given by \( \vec{m} = (m_1, \ldots, m_k) \), where \( m_i = \dim V_i \). Then there is a convergent relation \( I_{E_\bullet} \) of \( Q_{E_\bullet} \), analytic...
open neighborhoods \( p \in U \subset M_\omega(v), \ 0 \in V \subset M_{QE_\bullet}(\tilde{m}) \) and commutative isomorphisms

\[
\begin{array}{c}
M_{(QE_\bullet, IE_\bullet)}(\tilde{m})|_V \xrightarrow{\cong} p_M^{-1}(U) \\
p_Q \\ M_{(Q_E_\bullet, I_E_\bullet)}(\tilde{m})|_V \xrightarrow{\cong} U.
\end{array}
\]

(3.5)

Here the bottom arrow sends 0 to \( p \).

The proof of Theorem 3.2 will be completed in Proposition 5.4 below.

4. DEFORMATIONS OF COHESIVE SHEAVES

In this section, we describe deformation theory of coherent sheaves via dg-algebras and their minimal \( A_\infty \)-models. The arguments are already known for vector bundles \( [Fuk03] \) and we apply similar arguments for resolutions of coherent sheaves by vector bundles.

The above description will give local atlas of the moduli stack \( \mathcal{M} \) in Subsection 3.2 via finite dimensional \( A_\infty \)-algebras. More precisely for a given coherent sheaf \( E \) on a smooth projective variety \( X \), we compare the following three descriptions of the deformation space of \( E \):

1. An open neighborhood of the algebraic stack \( \mathcal{M} \) given in Subsection 3.2 at the point \( [E] \in \mathcal{M} \).
2. The Mauer-Cartan locus associated with the infinite dimensional dg-algebra \( \mathbf{R}\text{Hom}(E,E) \).
3. The Mauer-Cartan locus associated with the finite dimensional minimal \( A_\infty \)-algebra \( \text{Ext}^*(E,E) \).

We will compare the above descriptions by first constructing the map \( (3) \Rightarrow (2) \) in Lemma 4.2. Then we will construct a map \( (2) \Rightarrow (1) \), and then composing we get a desired atlas \( (3) \Rightarrow (1) \) in Proposition 4.3.

4.1. DEFORMATIONS OF VECTOR BUNDLES. We recall some basic facts on the deformation theory of vector bundles via gauge theory, and fix some notation (see \( [Fuk03] \) for details). For a holomorphic vector bundle \( \mathcal{E} \to X \) on a smooth projective variety \( X \), we denote by \( \mathcal{A}_{p,q}(\mathcal{E}) \) the sheaf of \( \mathcal{E} \)-valued \((p,q)\)-forms on \( X \), and set

\[ \mathcal{A}_{p,q}(\mathcal{E}) := \Gamma(X, \mathcal{A}_{p,q}(\mathcal{E})). \]

The holomorphic structure on \( \mathcal{E} \) is given by the Dolbeaut connection

\[ \overline{\partial}_\mathcal{E} : \mathcal{A}^{0,0}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{E}). \]

The Dolbeaut connection extends to the Dolbeaut complex

\[ 0 \to \mathcal{A}^{0,0}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{E}) \to \cdots \to \mathcal{A}^{0,i}(\mathcal{E}) \to \mathcal{A}^{0,i+1}(\mathcal{E}) \to \cdots \]
giving a resolution of $E$. The complex $A^{0,*}(E)$ is an elliptic complex (see [Wel73 Chapter IV, Section 5]), whose global section computes $H^*(X, E)$, i.e.

$$H^k(X, E) = \mathcal{H}^k(A^{0,*}(E)).$$

Any other holomorphic structure on $E$ is given by the Dolbeaut connection of the form

$$\overline{\partial}_E + A : A^{0,0}(E) \to A^{0,1}(E)$$

for some $A \in A^{0,1}(\text{End}(E))$. Conversely given $A \in A^{0,1}(\text{End}(E))$, the connection $\overline{\partial}_E + A$ gives a holomorphic structure on $E$ if and only if its square is zero, i.e.

$$\text{ad}(\overline{\partial}_E)(A) + A \circ A = 0.$$

The above equation is the Mauer-Cartan (MC) equation of the dg-algebra

$$g^* := A^{0,*}(\text{End}(E)).$$

The quotient of the solution space of the MC equation of $g^*$ by the gauge group of $C^\infty$-automorphisms of $E$ describes the deformation space of $E$ as holomorphic vector bundles.

### 4.2. Deformations of complexes.

We have a similar deformation theory for complexes of vector bundles. Let

$$E^\bullet = (\cdots \to 0 \to E^i \xrightarrow{d^i} E^{i+1} \to \cdots \to E^j \to 0 \to \cdots)$$

be a bounded complex of holomorphic vector bundles on $X$. By taking the Dolbeaut complex $A^{0,*}(E^i)$ for each $E^i$, we obtain the double complex $A^{0,*}(E^\bullet)$. Let $\text{Tot}(-)$ means the total complex of the double complex. We set

$$A^{0,*}(E^\bullet) := \text{Tot}(\Gamma(X, A^{0,*}(E^\bullet))).$$

Similarly to the vector bundle case, the complex $\text{Tot}(A^{0,*}(E^\bullet))$ is elliptic, and its global section computes the hyper cohomology of $E^\bullet$

$$\mathcal{H}^k(\mathbb{R}\Gamma(X, E^\bullet)) = \mathcal{H}^k(A^{0,*}(E^\bullet)).$$

Applying the construction (4.3) to the inner $\text{Hom}$ complex $\text{Hom}^*(E^\bullet, E^\bullet)$, we obtain the complex

$$g^\bullet := A^{0,*}(\text{Hom}^*(E^\bullet, E^\bullet)).$$

Its degree $k$ part is given by

$$g^k_\bullet = \bigoplus_{p+q=k} \prod_i A^{0,q}(\text{Hom}(E^i, E^{i+p})).$$

and the differential $d_q$ is induced by the Dolbeaut connections $\overline{\partial}_E$ on each $E^i$ together with the differentials $d^*$ in (4.2). Also the composition

$$A^{0,q}(\text{Hom}(E^i, E^{i+p})) \times A^{0,q'}(\text{Hom}(E^{i+p}, E^{i+p+p'})) \to A^{0,q+q'}(\text{Hom}(E^i, E^{i+p+p'}))$$
defines the product structure \( \cdot \) on \( \mathfrak{g}^{\bullet}_{\mathcal{E}} \). Then it is straightforward to check that the data

\[(4.6) \quad (\mathfrak{g}^{\bullet}_{\mathcal{E}}, d_{\mathfrak{g}}, \cdot)\]

is a dg-algebra.

Let \( \mathfrak{mc} \) be the map defined by

\[
\mathfrak{mc}: \mathfrak{g}^{1}_{\mathcal{E}} \to \mathfrak{g}^{2}_{\mathcal{E}}, \quad \alpha \mapsto d_{\mathfrak{g}}(\alpha) + \alpha \cdot \alpha.
\]

Its zero set

\[(4.7) \quad \text{MC}(\mathfrak{g}^{\bullet}_{\mathcal{E}}) = \{ \alpha \in \mathfrak{g}^{1}_{\mathcal{E}} : \mathfrak{mc}(\alpha) = 0 \}\]

is the solution of the Mauer-Cartan equation of the dg-algebra \( \mathfrak{g}^{\bullet}_{\mathcal{E}} \). Note that an element \( \alpha \in \mathfrak{g}^{1}_{\mathcal{E}} \) satisfies the MC equation iff

\[
(d_{\mathfrak{A}^{0,\bullet}}(\mathcal{E}^{\bullet}) + \alpha)^2 = 0
\]
on \( \mathfrak{A}^{0,\bullet}(\mathcal{E}^{\bullet}) \). In this case, the data

\[(4.8) \quad (\mathfrak{A}^{0,\bullet}(\mathcal{E}^{\bullet}), d_{\mathfrak{A}^{0,\bullet}}(\mathcal{E}^{\bullet}) + \alpha)\]
determines a dg-\( \mathfrak{A}^{0,\bullet}(\mathcal{O}_{X}) \)-module. Then (4.8) is a bounded complex of \( \mathcal{O}_{X} \)-modules whose cohomologies are coherent (see [Blo10, Lemma 4.1.5]), giving a deformation of the complex (4.2) in the derived category.

More explicitly, by (4.5) an element \( \alpha \in \mathfrak{g}^{1}_{\mathcal{E}} \) consists of data

\[(4.9) \quad \alpha = (\alpha^{0,0}_0, \alpha^{0,1}_1, \alpha^{0,2}_2, \ldots), \quad \alpha^{i}_j \in \mathfrak{A}^{0,j}(\mathfrak{H}om(\mathcal{E}^{i}, \mathcal{E}^{i-j+1}))\]

Suppose that the above \( \alpha \) satisfies the MC equation \( \mathfrak{mc}(\alpha) = 0 \). Then the diagram

\[
\cdots \to \mathfrak{A}^{0,0}(\mathcal{E}^{i-1}) \to \mathfrak{A}^{0,0}(\mathcal{E}^{i}) \to \mathfrak{A}^{0,0}(\mathcal{E}^{i+1}) \to \cdots
\]

\[
\downarrow \quad \overline{\partial}_{\mathcal{E}^{i}} + \alpha^{i}_1
\]

\[
\cdots \to \mathfrak{A}^{0,1}(\mathcal{E}^{i-1}) \to \mathfrak{A}^{0,1}(\mathcal{E}^{i}) \to \mathfrak{A}^{0,1}(\mathcal{E}^{i+1}) \to \cdots
\]

\[
\downarrow \quad \overline{\partial}_{\mathcal{E}^{i}} + \alpha^{i}_1
\]

\[
\cdots \to \mathfrak{A}^{0,2}(\mathcal{E}^{i-1}) \to \mathfrak{A}^{0,2}(\mathcal{E}^{i}) \to \mathfrak{A}^{0,2}(\mathcal{E}^{i+1}) \to \cdots
\]

satisfies the following: it is a complex in the horizontal direction, each square is commutative, and the compositions of vertical arrows are homotopic to zero with homotopy given by \( \alpha^{i}_2 \).

In particular if \( \alpha^{i}_j = 0 \) for \( j \geq 2 \), then the above diagram extends to a double complex. In this case

\[
\mathcal{E}^{i}_{\alpha} = (\mathfrak{A}^{0,0}(\mathcal{E}^{i}), \overline{\partial}_{\mathcal{E}^{i}} + \alpha^{i}_1)
\]
is a holomorphic structure on \( \mathcal{E}^{i} \). By setting

\[
d'_{\alpha} = d^{i} + \alpha^{i}_0: \mathfrak{A}^{0,0}(\mathcal{E}^{i}) \to \mathfrak{A}^{0,0}(\mathcal{E}^{i+1})
\]
we have the bounded complex of holomorphic vector bundles on $X$

$$
\cdots \to 0 \to \mathcal{E}_{-n} \xrightarrow{d_{-n}} \cdots \to \mathcal{E}_{-1} \xrightarrow{d_{-1}} \mathcal{E}_{0} \to 0 \to \cdots \tag{4.10}
$$
giving a deformation of $\mathcal{E}^\bullet$ as complexes. Conversely given a deformation of $\mathcal{E}^\bullet$ as a complex, then it gives rise to the solution of MC equation of the form $\alpha = (\alpha_0^0, \alpha_1^1, 0, \ldots)$.

For $\alpha, \alpha' \in \text{MC}(g^\bullet)$, $\alpha$ and $\alpha'$ are called \textit{gauge equivalent} if there exist

$$\gamma = \{(\gamma_0^i, \gamma_1^i, \gamma_2^i, \ldots)\}_i \in g^\bullet, \gamma_j^i \in A^0,j(\text{Hom}(\mathcal{E}^i, \mathcal{E}^{i-j}))$$

where $\gamma_0^i$ gives an isomorphism $\mathcal{E}^i \xrightarrow{\cong} \mathcal{E}^i$ as $C^\infty$-vector bundles, such that we have

$$\gamma \circ (d_{A^0,*}(\mathcal{E}^\bullet) + \alpha) \circ \gamma^{-1} = d_{A^0,*}(\mathcal{E}^\bullet) + \alpha'. \tag{4.11}$$

In this case, we have the isomorphism of the dg-$A^0,*(\mathcal{O}_X)$-modules

$$\gamma: (A^0,*(\mathcal{E}^\bullet), d_{A^0,*}(\mathcal{E}^\bullet) + \alpha) \xrightarrow{\cong} (A^0,*(\mathcal{E}^\bullet), d_{A^0,*}(\mathcal{E}^\bullet) + \alpha')$$

giving isomorphic deformations of (4.2) in the derived category.

Suppose that the complex (4.2) is quasi-isomorphic to a coherent sheaf $E$. Let $\text{Def}_E$ be the deformation functor

$$\text{Def}_E: Art \to \text{Set}$$

sending a finite dimensional commutative local $\mathbb{C}$-algebra $(R, \mathfrak{m})$ to the set of isomorphism classes of $R$-flat deformation of $E$ to $X \times \text{Spec} R$. Then it is shown in [FIM12, Section 8] that we have the functorial isomorphism

$$\text{MC}(g^\bullet \otimes \mathfrak{m})/(\text{gauge equivalence}) \xrightarrow{\cong} \text{Def}_E(R)$$

by sending a solution of the MC equation to the cohomology of the corresponding deformation (4.8).

4.3. \textbf{Resolutions of coherent sheaves.} For a smooth projective variety $X$, we consider deformation theory of a sheaf

$$E \in \text{Coh}(X)$$

in terms of dg-algebra. As we recalled in Section 4.1 when $E$ is a vector bundle its deformation theory is described in terms of the dg-algebra (4.11). In general, we take a resolution of $E$ by vector bundles and consider the associated dg-algebra (4.6).

We first fix a resolution of $E$ by vector bundles in the following way. Let $\mathcal{O}_X(1)$ be an ample line bundle on $X$. Then for $m_0 \gg 0$ we have the surjection

$$H^0(E(m_0)) \otimes \mathcal{O}_X(-m_0) \to E.$$
Applying this construction to the kernel of the above morphism and repeating, we obtain the resolution of $E$ of the form

$$
\cdots \to W^i \otimes \mathcal{O}_X(-m_i) \xrightarrow{d^i} W^{i+1} \otimes \mathcal{O}_X(-m_{i+1}) \to \cdots \\
\cdots \to W^0 \otimes \mathcal{O}_X(-m_0) \to E \to 0
$$

for finite dimensional vector spaces $W^i$. Since $X$ is smooth, the kernel of $d^i$ for $i = -N$ with $N \gg 0$ is a vector bundle on $X$. Therefore we obtain the bounded resolution of $E$

$$
0 \to \mathcal{E}^{-N} \xrightarrow{d^{-N}} \cdots \to \mathcal{E}^{-1} \xrightarrow{d^{-1}} \mathcal{E}^0 \to E \to 0
$$

where $\mathcal{E}^{-N} = \text{Ker}(d^{-N})$ and $\mathcal{E}^i = W^i \otimes \mathcal{O}_X(-m_i)$ for $-N < i \leq 0$.

By replacing $m_i$ and $n_i$ if necessary, the above construction can be extended to local universal family of deformations of $E$. Let $\mathcal{M}$ be the stack (3.1), and take its local atlas

$$
(A,p) \rightarrow (\mathcal{M},[E])
$$

at $[E] \in \mathcal{M}$, such that $A$ is a finite type affine scheme and a point $p \in A$ is sent to $[E]$. Let

$$
E_A \in \text{Coh}(X \times A)
$$

be the universal family. Let $\mathcal{O}_{X \times A}(1)$ be the pull-back of $\mathcal{O}_X(1)$ to $X \times A$. For $m_0 \gg 0$, the $\mathcal{O}_A$-module $H^0(E_A(-m_0))$ is locally free of finite rank and we have the surjection

$$
H^0(E_A(m_0)) \otimes \mathcal{O}_A \mathcal{O}_{X \times A}(-m_0) \rightarrow E_A.
$$

Similarly as above, we obtain the resolution of $E_A$ of the form

$$
\cdots \to \mathcal{W}^i \otimes \mathcal{O}_A \mathcal{O}_{X \times A}(-m_i) \rightarrow \mathcal{W}^{i+1} \otimes \mathcal{O}_A \mathcal{O}_{X \times A}(-m_{i+1}) \to \cdots \\
\cdots \to W^0 \otimes \mathcal{O}_A \mathcal{O}_{X \times A}(-m_0) \to E_A \to 0
$$

for locally free $\mathcal{O}_A$-modules $\mathcal{W}^i$ of finite rank. By taking the kernel at $i = -N$ for $N \gg 0$, we obtain the resolution of $E_A$

$$
0 \to \mathcal{E}^{-N}_A \rightarrow \cdots \rightarrow \mathcal{E}^{-1}_A \rightarrow \mathcal{E}^0_A \to E_A \to 0.
$$

For $N \gg 0$, each $\mathcal{E}^i_A$ is a vector bundle on $X \times A$, since $E_A$ is a $A$-flat perfect object. By restricting it to $X \times \{p\}$, we obtain the resolution (4.12).

4.4. **Minimal $A_\infty$-algebras.** For a coherent sheaf $E$ on $X$, we fix a resolution $\mathcal{E}^\bullet$ as in (4.12) and consider the dg-algebra (4.6)

$$
\mathfrak{g}_E^\bullet := \mathfrak{g}_E^\bullet.
$$

When $E$ is a vector bundle, we just take the dg-algebra (4.1) in the argument below. By (4.4) we have

$$
\text{Ext}^k(E,E) = \mathcal{H}^k(\mathfrak{g}_E^\bullet).
$$
By the homological transfer theorem, there exists a minimal $A_\infty$-algebra structure $\{m_n\}_{n \geq 2}$ on $\text{Ext}^*(E, E)$, and a quasi-isomorphism

$$I: (\text{Ext}^*(E, E), \{m_n\}_{n \geq 2}) \to (g^*_E, d_g, \cdot)$$

as $A_\infty$-algebras. Here the $A_\infty$-structure on $\text{Ext}^*(E, E)$ consists of linear maps

$$m_n: \text{Ext}^*(E, E) \to \text{Ext}^{*+2-n}(E, E), \ n \geq 2$$

and the quasi-isomorphism (4.16) is a collection of linear maps

$$I_n: \text{Ext}^*(E, E) \otimes^n \to g^{*+1-n}_E.$$

Both of $m_n$ and $I_n$ satisfy the $A_\infty$-constraints. The maps $m_n$ and $I_n$ are explicitly described in terms of Kontsevich-Soibelman’s tree formula [KS01] given as follows.

Let us choose a Kähler metric on $X$, Hermitian metrics on vector bundles $E_i$, and fix them. A standard argument in Hodge theory for elliptic complexes (for example, see [Wel73]) yields linear embedding

$$i: \text{Ext}^*(E, E) \hookrightarrow g^*_E$$

which identifies $\text{Ext}^*(E, E)$ with $\Delta = 0$ where $\Delta$ is the Laplacian operator

$$\Delta = d_g d^*_g + d^*_g d_g: g^*_E \rightarrow g^*_E.$$

Here $d^*_g$ is the adjoint map of $d_g$ with respect to the above chosen Kähler metric on $X$ and Hermitian metrics on $E_i$. Moreover we have linear operators

$$p: g^*_E \rightarrow \text{Ext}^*(E, E), \ h: g^*_E \rightarrow g^{*+1}_E$$

satisfying the following relations

$$p \circ i = \text{id}, \ i \circ p = \text{id} + d_g \circ h + h \circ d_g.$$

The homotopy operator $h$ is given by

$$h = -d^*_g \circ G$$

where $G$ is the Green’s operator, which is an operator of order $-2$ (see [Wel73 Chapter IV]), hence $h$ is of order $-1$.

The $A_\infty$-product (4.17) is described by Kontsevich-Soibelman’s tree formula as

$$m_n = \sum_{T \in \mathcal{O}(n)} \pm m_{n, T}$$

where $\mathcal{O}(n)$ is the set of isomorphism classes of binary rooted trees with $n$-leaves. Here $m_{n, T}$ is the operation given by the composition associated to $T$, by putting $i$ on leaves, the product map $\cdot$ of $g^*_E$ on internal vertices, the homotopy $h$ on internal edges, and the projection $p$ on the root of $T$. For example, $m_3$ is given by

$$m_3(x_1, x_2, x_3) = \pm p(h(i(x_1) \cdot i(x_2)) \cdot i(x_3)) \pm p(i(x_1) \cdot h(i(x_1) \cdot i(x_2))).$$
The operation $I_n$ is similarly given by
\begin{equation}
I_n = \sum_{T \in \mathcal{O}(n)} \pm I_{n,T}
\end{equation}
where $I_{n,T}$ is defined by replacing $p$ by $h$ in the construction of $m_{n,T}$. For example, $I_3$ is given by
\[I_3(x_1, x_2, x_3) = \pm h(h(i(x_1) \cdot i(x_2)) \cdot i(x_3)) \pm h(i(x_1) \cdot h(i(x_1) \cdot i(x_2))).\]

By [Tu, Appendix A], there exists another $A_\infty$-homomorphism
\begin{equation}
P: (g^* \hat{\otimes} d_g, \cdot) \to (\operatorname{Ext}^*(E, E), \{m_n\}_{n \geq 2})
\end{equation}
which is a homotopy inverse of $I$, i.e.
\[P \circ I = \text{id}, \quad I \circ P \text{ homotopic } \sim \text{id.}\]

Here two $A_\infty$-morphisms $f_1, f_2: A_1 \to A_2$ between $A_\infty$-algebras $A_1, A_2$ are called homotopic if there is an $A_\infty$-homomorphism
\[H: A_1 \to A_2 \otimes \Omega^{*}_{[0,1]}\]
such that $H(0) = f_1$ and $H(1) = f_2$, where $\Omega^{*}_{[0,1]}$ is the dg-algebra of $C^\infty$-differential forms on the interval $[0, 1]$. The $A_\infty$-homomorphism $P$ consists of linear maps
\[P_n: (g^* \hat{\otimes})^n \to \operatorname{Ext}^{*+1-n}(E, E)\]
which are also described in terms of tree formula, whose details we omit (see [Tu, Appendix A] for details).

Later we will use some boundedness properties of linear maps $m_n, I_n$ and $P_n$. Let us take an even number $l \gg 0$, e.g. $l > 2 \dim X$, and consider the Sobolev $(l, 2)$-norm $\|.-\|_l$ on $g^*_E$. It also induces a norm $\|.-\|_l$ on $\operatorname{Ext}^*(E, E)$ by the embedding $i$ in (4.18). We denote by
\[g^*_E \subset \hat{g}^*_E, l\]
the completion of $g^*_E$ with respect to the Sobolev norm $\|.-\|_l$.

**Lemma 4.1.** There is a constant $C > 0$ independent of $n$ such that
\[\|m_n\|_l < C^n, \quad \|I_n\|_l < C^n, \quad \|P_n\|_l < C^n.\]
Here $\|.-\|_l$ for linear maps mean the operator norm with respect to the norm $\|.-\|_l$ on $g^*_E$ or $\operatorname{Ext}^*(E, E)$.

**Proof.** When $E$ is a vector bundle, the lemma is proved in [Fuk03, Proposition 2.3.2] and [Tu, Lemma A.1.1, Lemma A.1.2, Lemma A.1.5]. The key ingredient of the proof is that the maps $m_n, I_n, P_n$ are constructed as in (1.21) using rooted trees, whose cardinality is bounded as
\[\#\mathcal{O}(n) = \frac{(2n - 2)!}{(n - 1)!n!} < 4^{n-1},\]
and the fact that the homotopy operator $h$, the product map on $\mathfrak{g}_E^*$ are extended to bounded operators

$$\hat{\mathfrak{g}}_{E,l}^* \stackrel{h}{\rightarrow} \hat{\mathfrak{g}}_{E,l}^*, \quad \hat{\mathfrak{g}}_{E,l}^* \times \hat{\mathfrak{g}}_{E,l}^* \rightarrow \hat{\mathfrak{g}}_{E,l}^*.$$  

When $E$ is a coherent sheaf which is not necessary a vector bundle, the above property still hold for the complex (4.12) without any modification: the boundedness of $h$ is a general fact for elliptic complexes (see [Wel73, Theorem 4.12]), as it is an operator of degree $-1$ given by (4.20), and that of the product · follows from our choice of $l \gg 0$ and a standard result of Sobolev spaces (for example see [Won, Theorem 25]). Therefore the same argument for the vector bundle case proves the lemma. □

4.5. **Deformations by $A_\infty$-algebras.** For $x \in \text{Ext}^1(E, E)$, we consider the infinite series

$$\kappa(x) := \sum_{n \geq 2} m_n(x, \ldots, x) \tag{4.24}$$

where each term $m_n(x, \ldots, x)$ is an element of $\text{Ext}^2(E, E)$. By Lemma 4.1, there is an analytic open neighborhood

$$0 \in U \subset \text{Ext}^1(E, E) \tag{4.25}$$

such that the series (4.24) absolutely converges on $U$ to give a complex analytic morphism

$$\kappa : U \rightarrow \text{Ext}^2(E, E). \tag{4.26}$$

The equation $\kappa(x) = 0$ is the Maurer-Cartan equation for the $A_\infty$-algebra (4.17). We set $T$ to be

$$T := \kappa^{-1}(0) \subset U \tag{4.27}$$

i.e. $T$ is the closed complex analytic subspace defined by the ideal of zero of the map (4.26).

On the other hand, for $x \in \text{Ext}^1(E, E)$ we also consider the infinite series

$$I_*(x) := \sum_{n \geq 1} I_n(x, \ldots, x) \tag{4.28}$$

where each term $I_n(x, \ldots, x)$ is an element of $\mathfrak{g}_E^1$. By Lemma 4.1, for sufficiently small open subset (4.25) the series (4.28) absolutely converges on $U$ to give a morphism of Banach analytic spaces

$$I_* : U \rightarrow \hat{\mathfrak{g}}_E^1. \tag{4.29}$$

**Lemma 4.2.** The morphism (4.29) restricts to the morphism of Banach analytic spaces

$$I_* : T \rightarrow \text{MC}(\mathfrak{g}_E^*). \tag{4.30}$$

Here $\text{MC}(\mathfrak{g}_E^*)$ is the solution of the Maurer-Cartan equation (4.7) of the dg-algebra $\mathfrak{g}_E^*$. 
Proof. The result is proved in [Tu, Section 2.2, Lemma A.1.3] when $E$ is a vector bundle, and the same argument applies for the complex (4.12). Since $I_*$ is an $A_{\infty}$-homomorphism, it preserves the MC locus, so it sends $T$ to $\text{MC}(\hat{\mathfrak{g}}_{E,T})$. For $x \in T$, the smoothness of $I_*(x)$ follows along with the argument of [Tu, Lemma A.1.3], by replacing $\mathfrak{g}$ in loc.cit. by the differential $d_{\hat{g}}$ of $\hat{\mathfrak{g}}_{E,T}$. Therefore we obtain the morphism (4.30). □

Let $M$ be the moduli stack of coherent sheaves on $X$, and we regard it as a complex analytic stack. The above lemma implies the following proposition:

**Proposition 4.3.** By shrinking $U$ if necessary, the morphism (4.29) induces the morphism of complex analytic stacks

\[ I_*: T \to M. \]

Proof. The map in Lemma 4.2 corresponds to the element

\[ \alpha \in \mathfrak{g}_E^* \otimes \Gamma(\mathcal{O}_T) \]

satisfying the MC equation of the dg-algebra $\mathfrak{g}_E^* \otimes \Gamma(\mathcal{O}_T)$. Then we obtain the dg-$A_0^0(\mathcal{O}_X) \otimes \mathcal{O}_T$-module

\[ (A_0^0(\mathcal{E}^*) \otimes \mathcal{O}_T, d_{A^0_0(\mathcal{E}^*)} + \alpha). \]

Here $\mathcal{E}^*$ is the complex (4.12).

The dg-module (4.32) is a bounded complex of $\mathcal{O}_{X \times T}$-modules. We can show that each cohomology of (4.32) is a coherent $\mathcal{O}_{X \times T}$-module as in [Blo10, Lemma 4.1.5], which essentially follows the argument in [DK90, p51-52]. Indeed for each $t \in T$ and $x \in X$, by the proof of [Blo10, Lemma 4.1.5] there is an open neighborhood $x \in U$ such that there is a degree zero $C^\infty$-isomorphism

\[ \phi_t: A_0^0(\mathcal{E}^*)|_U \cong A_0^0(\mathcal{E}^*)|_U \]

satisfying that

\[ \phi_t^{-1} \circ (d_{A^0_0(\mathcal{E}^*)} + \alpha_t) \circ \phi_t = d_{A^0_0(\mathcal{E}^*)} + \beta_t. \]

Here in the notation (4.33), $\beta_t$ is of the form

\[ \beta_t = ((\beta_0^i)_t, 0, 0, \ldots), \ (\beta_0^i)_t \in \text{Hom}(\mathcal{E}^i|_U, \mathcal{E}^{i+1}|_U). \]

This implies that the dg-module (4.32) restricted to $U \times \{t\}$ is gauge equivalent to a complex which is quasi-isomorphic to a bounded complex of holomorphic vector bundles on $U$. The isomorphism (4.33) can be found by solving a certain differential equation, as in [DK90, p51-52]. As remarked in [DK90, p52], the solution $\phi_t$ is analytic in $t \in T$ as $\alpha_t$ is. Therefore by shrinking $U, T$ if necessary we see that (4.32) restricted to $U \times T$ is gauge equivalent to a complex which is quasi-isomorphic to a bounded complex of analytic vector bundles on $U \times T$. In particular, each cohomology of (4.32) is coherent.
Therefore (4.32) determines an object
\[ \mathcal{E}_T^\bullet \in \mathcal{D}^b_{\text{Coh}(X \times T)}(\text{Mod} \, \mathcal{O}_{X \times T}). \]

We show that by shrinking \( U \) if necessary, the object \( \mathcal{E}_T^\bullet \) is quasi-isomorphic to a \( T \)-flat sheaf
\[ E_T := \mathcal{H}^0(\mathcal{E}_T^\bullet) \in \text{Coh}(X \times T). \]

By the construction of \( \mathcal{E}_T^\bullet \), at \( t = 0 \) we have \( E_T^\bullet \otimes_{\mathcal{O}_T} \mathcal{O}_{\{0\}} \cong \mathcal{E}_T^\bullet \). We have the spectral sequence
\[ E_2^{p,q} = \text{Tor}_{X \times T}(\mathcal{H}^q(\mathcal{E}_T^\bullet), \mathcal{O}_{X \times \{0\}}) \Rightarrow \mathcal{H}^{p+q}(E). \]

Let \( q_0 \) be the maximal \( q \in \mathbb{Z} \) such that \( \mathcal{H}^q(\mathcal{E}_T^\bullet) \neq 0 \). If \( q_0 > 0 \), then by the above spectral sequence we have \( \mathcal{H}^q(\mathcal{E}_T^\bullet)|_{t=0} = 0 \). Therefore by shrinking \( \mathcal{U} \), we have \( q_0 \leq 0 \), and as \( E \neq 0 \) it follows that \( q_0 = 0 \) by the above spectral sequence. Moreover we have \( E_2^{-1,0} = 0 \), which implies that \( E_T \) is flat at \( t = 0 \), hence \( E_2^{p,0} = 0 \) for any \( p < 0 \). Then by the above spectral sequence again, we have \( E_2^{0,-1} = 0 \), hence we may assume \( \mathcal{H}^{-1}(\mathcal{E}_T^\bullet) = 0 \). Inductively, by shrinking \( \mathcal{U} \) we see that \( \mathcal{H}^{q}(\mathcal{E}_T^\bullet) = 0 \) for any \( q < 0 \). Therefore the above claim holds.

By the universal property of \( \mathcal{M} \), the sheaf \( E_T \) defines the morphism (4.31). □

**Proposition 4.4.** The morphism of complex analytic stacks \( I_* : T \to \mathcal{M} \) in (4.31) is smooth of relative dimension \( \dim \text{Aut}(E) \).

**Proof.** We first show that \( I_* : T \to \mathcal{M} \) is smooth. Let \( (S, s) \) be a complex analytic space and \( (S, s) \to (\mathcal{M}, [E]) \) a morphism of complex analytic stacks which sends \( s \) to \([E]\). It is enough to show that, after replacing \( S \) by its open neighborhood at \( s \in S \) if necessary, we have the factorization
\[ (S, s) \to (T, 0) \xrightarrow{f_1} (\mathcal{M}, [E]). \]

By shrinking \( S \) if necessary, we may assume that \( S \to \mathcal{M} \) factors through
\[ (S, s) \xrightarrow{f_1} (A, p) \to (\mathcal{M}, [E]) \]

where the right morphism is the local atlas in (4.13). Let \( \mathcal{E}_A^\bullet \) be the complex on \( X \times A \) constructed in (4.14). By pulling \( \mathcal{E}_A^\bullet \) back by \( f_1^* \), we obtain the complex
\[ \mathcal{E}_S^\bullet = f_1^* \mathcal{E}_A^\bullet. \]

Then as described in Section 4.1, the complex structures of each term of \( \mathcal{E}_S^\bullet \) and their differentials give rise to the solution of the MC equation of the dg-algebra \( g_E^* \otimes \mathcal{O}_S(S) \). Thus we obtain a map of Banach analytic spaces
\[ f_2 : (S, s) \to (\text{MC}(g_E^*), 0). \]
We are left to prove the existence of the morphism $f_3: (S, s) \to (T, 0)$ such that the composition

$$(S, s) \xrightarrow{f_3} (T, 0) \xrightarrow{I_*} (\text{MC}(g_E^*), 0)$$

differs from $f_2$ only up to gauge equivalence. The existence of such $f_3$ is proved in [Tu, Theorem 2.2.2] when $E$ is a vector bundle, and the same argument applies for the complex of vector bundles (4.12). Below we give an outline of the proof.

For $y \in g_E^1$, consider the series

$$P_*(y) := \sum_{n \geq 1} P_n(y, \ldots, y)$$

where $P$ is the homotopy inverse of $I$ in (4.23). By Lemma 4.1, there is an open neighborhood $0 \in U' \subset g_E^1$ in $\|\cdot\|_1$-norm such that $P_*$ gives the analytic map

$$P_*: U' \to \text{Ext}^1(E, E).$$

Since $P$ is an $A_\infty$-homomorphism, after shrinking $U'$ if necessary the above map induces the morphism of Banach analytic spaces

$$P_*: \text{MC}(g_E^*) \cap U' \to T.$$

Therefore by shrinking $S$ if necessary so that $f_2(S) \subset U'$, we have the analytic map

$$f_3 = P_* \circ f_2: (S, s) \to (T, 0).$$

It remains to show that two maps

$$I_* \circ f_3 = I_* \circ P_* \circ f_2, \quad f_2: (S, s) \to (\text{MC}(g_E^*), 0)$$

are gauge equivalent. Since $P$ is a homotopy inverse of $I$, there is an $A_\infty$-homomorphism

$$H: g_E^* \to g_E^* \otimes \Omega^*_{[0,1]}$$

such that $H(0) = \text{id}$ and $H(1) = I_* \circ P$. Then $H$ also satisfies the boundedness property as in Lemma 4.1 (see [Tu, Corollary A.2.7]), so that after shrinking $U'$ if necessary the $A_\infty$-homomorphism $H$ induces the analytic map

$$H_*: \text{MC}(g_E^*) \cap U' \to \text{MC}(g_E^* \otimes \Omega^*_{[0,1]}).$$

Then the analytic map

$$H_* \circ f_2: S \to \text{MC}(g_E^* \otimes \Omega^*_{[0,1]})$$

satisfies

$$H_* \circ f_2(0) = f_2, \quad H_* \circ f_2(1) = I_* \circ P_* \circ f_2.$$

This implies that $f_2$ and $I_* \circ P_* \circ f_2$ are gauge equivalent in the sense of [Fuk03, Definition 2.2.2]. As proved in [Fuk03, Lemma 2.2.2], this notion of gauge equivalence coincides with the gauge equivalence in (4.11). Therefore the smoothness of $I_*$ follows.
Finally, the relative dimension of $I_* : T \to \mathcal{M}$ is $\dim \text{Aut}(E)$ since the dimension of the tangent space of $T$ at $0$ is $\dim \text{Ext}^1(E, E)$, and that of $\mathcal{M}$ at $[E]$ is $\dim \text{Ext}^1(E, E) - \dim \text{Aut}(E)$. □

5. Local descriptions of moduli stacks of semistable sheaves

In this section, we use the results in the previous sections to prove Theorem 3.2. By applying the arguments to the CY 3-fold case, we also obtain Corollary 5.7.

5.1. Convergent relation of the Ext-quiver. For a smooth projective variety $X$, let $E_* = (E_1, \ldots, E_k)$ be a simple collection of coherent sheaves on $X$, and $Q_{E_*}$ the associated Ext-quiver (see Subsection 3.3). Here we construct a convergent relation of $Q_{E_*}$ from the minimal $A_\infty$-structure on the derived category of coherent sheaves on $X$.

Let us consider the sheaf on $X$ of the form
\begin{equation}
E = \bigoplus_{i=1}^k V_i \otimes E_i
\end{equation}
for vector spaces $V_i$, and set $m_i = \dim V_i$. Note that we have the decomposition
\begin{equation}
\text{Ext}^*(E, E) = \bigoplus_{1 \leq a, b \leq k} \text{Hom}(V_a, V_b) \otimes \text{Ext}^*(E_a, E_b).
\end{equation}
Let us take a resolution $E^\bullet \to E$ as in (4.12). From its construction, it naturally decomposes into the direct sum of resolutions of $E_i$. Namely, let
\begin{equation}
0 \to \mathcal{E}_i^{-N} \xrightarrow{d_i^{-N}} \cdots \to \mathcal{E}_i^{-1} \xrightarrow{d_i^{-1}} \mathcal{E}_i^0 \to E_i \to 0
\end{equation}
be the resolution (4.12) applied for $E_i$. By taking $N \gg 0$, we may assume that $N$ is independent of $i$. Then the complex $\mathcal{E}^\bullet$ in (4.12) is
\begin{equation}
\mathcal{E}^\bullet = \bigoplus_{i=1}^k V_i \otimes \mathcal{E}_i^\bullet.
\end{equation}
Therefore we have the decompositions
\begin{equation}
\mathfrak{g}_E^* = \bigoplus_{1 \leq a, b \leq k} \text{Hom}(V_a, V_b) \otimes A_0^0(\text{Hom}^*(\mathcal{E}_a^\bullet, \mathcal{E}_b^\bullet)).
\end{equation}
Here $\mathfrak{g}_E^*$ is the dg-algebra (4.15), defined via the above complex $\mathcal{E}^\bullet$. The decomposition of $\mathfrak{g}_E^*$ is compatible with the Laplacian operator $\Delta$. Indeed each
complex $A^{0,*}(\text{Hom}^*(E_a^\bullet, E_b^\bullet))$ is elliptic and hence we have linear operators

\[ i_{a,b}: \text{Ext}^*(E_a, E_b) \to A^{0,*}(\text{Hom}^*(E_a^\bullet, E_b^\bullet)) \]
\[ p_{a,b}: A^{0,*}(\text{Hom}^*(E_a^\bullet, E_b^\bullet)) \to \text{Ext}^*(E_a, E_b) \]
\[ h_{a,b}: A^{0,*}(\text{Hom}^*(E_a^\bullet, E_b^\bullet)) \to A^{0,*-1}(\text{Hom}^*(E_a^\bullet, E_b^\bullet)) \]

satisfying the same relations as (4.19) and

\[ \star = \bigoplus_{1 \leq a, b \leq k} \text{id}_{\text{Hom}(V_a, V_b)} \otimes \star_{a,b} \]

where $\star$ is either $i$ or $p$ or $h$ given in Subsection 4.4.

Let $E$ be the coherent sheaf on $X$ defined by

\[ E := \bigoplus_{i=1}^k E_i \]

and consider the $A_\infty$-product

\[ m_n: \text{Ext}^1(E, E)^\otimes n \to \text{Ext}^2(E, E). \]

By the relation (5.4) and the explicit formula (4.17) of the $A_\infty$-product, the map (5.6) only consists of the direct sum factors of the form

\[ m_n: \text{Ext}^1(E_{\psi(1)}, E_{\psi(2)}) \otimes \text{Ext}^1(E_{\psi(2)}, E_{\psi(3)}) \otimes \cdots \]
\[ \cdots \otimes \text{Ext}^1(E_{\psi(n)}, E_{\psi(n+1)}) \to \text{Ext}^2(E_{\psi(1)}, E_{\psi(n+1)}) \]

for maps $\psi: \{1, \ldots, n+1\} \to \{1, \ldots, k\}$, which give a minimal $A_\infty$-category structure on the dg-category generated by $(E_1, \ldots, E_k)$. By taking the dual and the products of (5.7) for all $n \geq 2$, we obtain the linear map

\[ m^\vee := \prod_{n \geq 2} m_n^\vee: \text{Ext}^2(E, E)^\vee \to \prod_{n \geq 2} \bigoplus_{\psi: \{1, \ldots, n+1\} \to \{1, \ldots, k\}} \text{Ext}^1(E_{\psi(1)}, E_{\psi(2)})^\vee \otimes \cdots \]
\[ \cdots \otimes \text{Ext}^1(E_{\psi(n)}, E_{\psi(n+1)})^\vee. \]

Note that an element of the RHS is an element of $\mathbb{C}[Q_{E_\bullet}]$ by (2.6). Let \( \{o_1, \ldots, o_l\} \) be a basis of $\text{Ext}^2(E, E)^\vee$ and set

\[ f_i = m^\vee(o_i) \in \mathbb{C}[Q_{E_\bullet}]. \]

Then by Lemma 4.11 we have $f_i \in \mathbb{C}\{Q_{E_\bullet}\}$. We obtain the convergent relation of $Q_{E_\bullet}$

\[ I_{E_\bullet} := (f_1, \ldots, f_l). \]
5.2. Deformations of direct sums of simple collections. We consider the deformations of sheaves of the form \((5.1)\). By the decomposition \((5.2)\), the space \(\text{Ext}^1(E, E)\) is identified with the space of \(Q_{E_*}\)-representations
\[
\text{Ext}^1(E, E) = \text{Rep}_{Q_{E_*}}(\vec{m}).
\]
Here \(\vec{m}\) is the dimension vector of \(Q_{E_*}\) given by \(m_i = \dim V_i\). We also have
\[
G = \text{Aut}(E) = \prod_{i=1}^k \text{GL}(V_i)
\]
and the adjoint action of \(\text{Aut}(E)\) on \(\text{Ext}^1(E, E)\) coincides with the action \((2.1)\) under the identification \((5.9)\). Recall that in \((4.26)\) and \((4.29)\), we constructed analytic maps
\[
\kappa: \mathcal{U} \to \text{Ext}^2(E, E), \quad I_*: \mathcal{U} \to \hat{g}_{E,l}^*
\]
for a sufficiently small analytic open subset \(0 \in \mathcal{U} \subset \text{Ext}^1(E, E)\). Explicitly under the identification \((5.9)\), for a \(Q_{E_*}\)-representation
\[
u = (u_e)_{e \in E(Q_{E_*})} \in \mathcal{U}, \quad u_e: V_{s(e)} \to V_{t(e)},
\]
we have the following identities by the decompositions \((5.2), (5.3), (5.4)\).
\[
k(\nu) = \sum_{n \geq 2} \sum_{\{e_i \in E^{(i)}, \psi_{(i+1)}\} \subseteq \{1, \ldots, k\}} m_n(e_1^\vee, \ldots, e_n^\vee) \cdot u_{e_n} \circ \cdots \circ u_{e_2} \circ u_{e_1},
\]
\[
I_*(\nu) = \sum_{n \geq 2} \sum_{\{e_i \in E^{(i)}, \psi_{(i+1)}\} \subseteq \{1, \ldots, k\}} I_n(e_1^\vee, \ldots, e_n^\vee) \cdot u_{e_n} \circ \cdots \circ u_{e_2} \circ u_{e_1}.
\]
Here for \(e \in E_{i,j}\), the element \(e_i^\vee \in \text{Ext}^1(E_i, E_j)\) is defined as in \((2.18)\).

**Lemma 5.1.** There is a saturated open subset \(\mathcal{V}\) in \(\text{Ext}^1(E, E)\) w.r.t. the \(G\)-action on \(\text{Ext}^1(E, E)\), satisfying
\[
0 \in \mathcal{V} \subset G \cdot \mathcal{U} \subset \text{Ext}^1(E, E)
\]
such that the maps in \((5.11)\) induce \(G\)-equivariant analytic maps
\[
k: \mathcal{V} \to \text{Ext}^2(E, E), \quad I_*: \mathcal{V} \to \hat{g}_{E,l}^*
\]
Here \(G\) acts on \(\text{Ext}^2(E, E)\) and \(\hat{g}_{E,l}^*\) by adjoint.

**Proof.** The formal series \(\kappa\) and \(I_*\) in \((5.12)\) are obviously \(G\)-equivariant. Therefore for a choice of \(\mathcal{U}\) in \((4.26), (4.29)\), the maps \(\kappa, I_*\) can be extended to analytic maps
\[
k: G \cdot \mathcal{U} \to \text{Ext}^2(E, E), \quad I_*: G \cdot \mathcal{U} \to \hat{g}_{E,l}^*
\]
By Lemma \((2.5)\) there is a saturated analytic open subset \(\mathcal{V} \subset G \cdot \mathcal{U}\) which contains \(0 \in \text{Ext}^1(E, E)\), so the lemma follows. \(\blacksquare\)
Let $V \subset \text{Ext}^1(E, E)$ be as in Lemma 5.1. By Lemma 2.4, it is written as

$$V = \pi_{Q, E^*}^{-1}(V)$$

for some analytic open subset $0 \in V \subset M_{Q, E^*}(\bar{m})$, where $\pi_{Q, E^*}$ is the quotient map

$$\pi_{Q, E^*} : \text{Rep}_{Q, E^*}(\bar{m}) \to M_{Q, E^*}(\bar{m}).$$

Let $R \subset V$ be the closed analytic subspace given by

$$R := \kappa^{-1}(0) \subset V \subset \text{Ext}^1(E, E).$$

By the definition of $I_{E^*}$ in (5.8), under the identification (5.9) we have

$$R = \text{Rep}_{(Q, E^*), I_{E^*}}(\bar{m})|_V.$$ 

Here we have used the notation (2.17) for the RHS. Therefore in the notation of Definition 2.16, we have

$$\mathcal{M}_{(Q, E^*), I_{E^*}}(\bar{m})|_V = [R/G].$$

Lemma 5.2. By shrinking $V$ if necessary, the map $I_*$ in Lemma 5.1 induces the smooth morphism of relative dimension zero

$$I_* : \mathcal{M}_{Q, E^*}((\bar{m}))|_V \to \mathcal{M}.$$ 

Here $\mathcal{M}$ is the moduli stack of coherent sheaves on $X$.

Proof. By Lemma 4.2 and Proposition 4.4, the map $I_*$ in Lemma 5.1 gives the analytic maps

$$I_* : R \cap U \to \text{MC}(g_{E^*}), \quad I_* : R \cap U \to \mathcal{M}.$$ 

Then by the $G$-equivalence of $I_*$ and the property $V \subset G \cdot U$ in Lemma 5.1, the above maps extend to the $G$-equivariant analytic maps

$$I_* : R \to \text{MC}(g_{E^*}), \quad I_* : R \to \mathcal{M}.$$ 

Here the right map is induced by the left map as in the proof of Proposition 4.4. By the $G$-equivariance of $I_*$, the right map of (5.14) descends to the quotient by $G$ to induce (5.13), which is of relative dimension zero by Lemma 5.1. □

5.3. Functoriality of $I_*$. In this subsection, by the explicit description (5.12) of the map $I_*$ in Proposition 5.4, we see that it has some functorial property. In particular, it implies that $I_*$ sends subsheaves to subrepresentations of Ext-quivers. This fact will not be used in the rest of this section, but will be used in the proof of Theorem 6.8 which will be used in Theorem 7.7 to compare stability conditions of sheaves and quiver representations.

For each $i \in V(Q, E^*) = \{1, 2, \ldots, k\}$, let $V_i, V'_i$ be vector spaces with dimensions $m_i, m'_i$, and set

$$E = \bigoplus_{i=1}^{k} V_i \otimes E_i, \quad E' = \bigoplus_{i=1}^{k} V'_i \otimes E_i.$$
Let us take
\[ u = (u_e)_{e \in E(Q_{E_*})}, \quad u' = (u'_e)_{e \in E(Q_{E_*})} \]
where \( u_e, u'_e \) are linear maps
\[ u_e : V_{s(e)} \to V_{t(e)}, \quad u'_e : V'_{s(e)} \to V'_{t(e)}, \]
whose operator norms are sufficiently small so that they give \( Q_{E_*} \)-representations satisfying the relation \( I_{E_*}. \) Let \( \phi_i : V_i \to V'_i \) be linear maps for \( 1 \leq i \leq k \) such that the following diagram commutes for each \( e \in E(Q_{E_*}) \)

\[
\begin{array}{ccc}
V_{s(e)} & \xrightarrow{u_e} & V_{t(e)} \\
\phi_{s(e)} & \downarrow & \phi_{t(e)} \\
V'_{s(e)} & \xrightarrow{u'_e} & V'_{t(e)}
\end{array}
\]

Then each term of
\[ I_\ast(u) \in \text{MC}(\underline{\mathfrak{g}}^*_{E}), \quad I_\ast(u') \in \text{MC}(\underline{\mathfrak{g}}^*_{E'}) \]
in (5.12) satisfy
\[ I_n(e_1^\vee, \ldots, e_n^\vee) \cdot \phi_{t(e_n)} \circ u_{e_n} \circ \cdots \circ u_{e_1} = I_n(e_1^\vee, \ldots, e_n^\vee) \cdot u'_e \circ \cdots \circ u'_e \circ \phi_{s(e_1)}. \]

This implies that the map
\[
\bigoplus_{i=1}^k \phi_i \otimes \text{id}: \left( A^{0,*} \left( \bigoplus_{i=1}^k V_i \otimes \mathcal{E}_i^* \right), d_{A^{0,*}} \left( \bigoplus_{i=1}^k V_i \otimes \mathcal{E}_i^* \right) + I_\ast(u) \right)
\]
\[ \to \left( A^{0,*} \left( \bigoplus_{i=1}^k V'_i \otimes \mathcal{E}_i^* \right), d_{A^{0,*}} \left( \bigoplus_{i=1}^k V'_i \otimes \mathcal{E}_i^* \right) + I_\ast(u') \right) \]
is a map of dg-\( A^{0,*}(\mathcal{O}_X) \)-modules. By taking the cohomology of the above map, we obtain the morphism of coherent sheaves
\[ H^0 \left( \bigoplus_{i=1}^k \phi_i \otimes \text{id} \right) : E_u \to E_{u'}. \]

Here \( E_u, E_{u'} \) are coherent sheaves corresponding to \( u, u' \) under the map in Proposition 4.3 respectively.

**Remark 5.3.** In the above argument, we assumed that the operator norms of \( u, u' \) are enough small so that \( I_\ast \) is defined. We can relax this condition in the following cases. First suppose that each \( \phi_i \) is injective or surjective. Then the operator norm of \( u \) is bounded by that of \( u' \), so if the operator norm of \( u' \) is enough small then so is \( u \) and \( I_\ast(u) \) is defined. Next if \( u, u' \) correspond to nilpotent \( Q_{E_*} \)-representations, then whatever the operator norms of \( u, u' \) the infinite sums \( I_\ast(u), I_\ast(u') \) in (5.12) are finite sums. So in the above cases, \( E_u, E_{u'} \) and the morphism (5.17) are well-defined.
5.4. Étale slice. Below we use the notation in Subsection 3.2. Let $\mathcal{M}_\omega(v)$ be the moduli stack of $\omega$-Gieseker semistable sheaves on $X$ with Chern character $v$, $M_\omega(v)$ its coarse moduli space. Let $E$ be a polystable sheaf of the form (3.3), and take closed points $p = [E] \in M_\omega(v)$, $p' = [E] \in M_\omega(v)$.

For $m \gg 0$, let $V$ be the vector space given by

$$V = H^0(E(m)) = \bigoplus_{i=1}^k V_i \otimes H^0(E_i(m)).$$

Let $q \in \text{Quot}^\circ(V, v)$ be a point which is mapped to $p'$ under the quotient morphism $\text{Quot}^\circ(V, v) \to \mathcal{M}_\omega(v)$. Then we have

$$\text{Stab}_{\text{GL}(V)}(q) = G \subset \text{GL}(V)$$

where $G$ is given as in (5.10). By Luna’s étale slice theorem [Lun73], there is an affine locally closed $G$-invariant subscheme

$$q \in Z \subset \text{Quot}^\circ(V, v)$$

such that the natural $\text{GL}(V)$-equivariant morphism

$$\text{GL}(V) \times_G Z \to \text{Quot}^\circ(V, v)$$

is étale. Moreover by taking the quotients by $\text{GL}(V)$, we obtain the Cartesian diagram

$$\begin{array}{ccc}
Z/G & \longrightarrow & \mathcal{M}_\omega(v) \\
\downarrow \quad p_M & & \downarrow \\
\quad Z\times G & \longrightarrow & M_\omega(v)
\end{array}$$

such that each horizontal arrows are étale. Therefore there is a saturated analytic open subset $W \subset Z$ (w.r.t. the $G$-action on $Z$) which contains $q$ and the Cartesian diagram of complex analytic stacks

$$\begin{array}{ccc}
[W/G] & \longrightarrow & \mathcal{M}_\omega(v) \\
\downarrow \quad p_M & & \downarrow \\
W\times G & \longrightarrow & M_\omega(v)
\end{array}$$

such that each horizontal arrows are analytic open immersions.

On the other hand, let us consider the morphism $I_*$ in Lemma 5.2 applied for the above polystable sheaf $p' = [E] \in M_\omega(v)$. By the openness of stability, by shrinking $U$ in Lemma 5.1 if necessary, the map $I_*$ in Lemma 5.2 factors through the open substack $M_\omega(v) \subset \mathcal{M}$:

$$I_* : \mathcal{M}(\text{Quo}^\circ(V, v))(\bar{m})|_V \to \mathcal{M}(\omega(v)).$$

Now the following proposition completes the proof of Theorem 3.2.
Proposition 5.4. By shrinking $V$ in Lemma 5.1 and $W$ if necessary (while keeping the condition to be saturated in $\text{Ext}^1(E, E)$, $Z$ respectively) the map \((5.19)\) induces the commutative isomorphisms

\[(5.20) \quad \begin{array}{ccc}
\mathbb{R}/G & \xrightarrow{I_*} & \mathbb{W}/G \\
p_Q & & p_W \\
\mathbb{R}/G & \mathbb{W}/G.
\end{array}
\]

Proof. The map \((5.19)\) induces the analytic map $\mathbb{R}/G \to M_{\omega}(v)$. So by shrinking $0 \in V \subseteq M_{Q_\bullet}(\bar{m})$ if necessary, we may assume that the above map factors through $\mathbb{R}/G \to \mathbb{W}/G$. Then we have the commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}/G & \xrightarrow{I_*} & \mathbb{W}/G \\
p_Q & & p_W \\
\mathbb{R}/G & \mathbb{W}/G.
\end{array}
\]

Let $K \subset G$ be a maximal compact subgroup, and take a sufficiently small $K$-invariant analytic open subset $q \in \mathbb{W}_1 \subset \mathbb{W}$. Then as in the proof of Proposition 4.4, the composition

\[
\mathbb{W}_1 \to \mathbb{W} \to [\mathbb{W}/G] \subset M_{\omega}(v)
\]

admits a lift $\phi: \mathbb{W}_1 \to \mathbb{R}$ using the homotopy inverse $P$ of $I$. Moreover the proof in \textit{loc.cit.} immediately implies that $\phi$ can taken to be $K$-equivariant. (Indeed if the map $f_2$ in \textit{loc.cit.} is $K$-equivariant, then so is $f_3$ as $P$, is $K$-equivariant.) So we have the commutative diagram

\[(5.21) \quad \begin{array}{ccc}
\mathbb{R} & \xrightarrow{\phi} & \mathbb{W}_1 \\
\mathbb{R}/G & \xrightarrow{I_*} & [\mathbb{W}/G].
\end{array}
\]

Note that the bottom arrow is a smooth morphism of relative dimension zero by Lemma 5.2. Let $0 \in R_1 \subset \mathbb{R}$ be a sufficiently small $K$-invariant analytic open neighborhood. Since both of $R_1$ and $\mathbb{W}_1$ are the bases of versal families of flat deformations of $E$ with tangent space $\text{Ext}^1(E, E)$, and $\phi$ is isomorphism at the tangent by the diagram \((5.21)\), the $K$-equivariant map $\phi$ gives an isomorphism $\psi: \mathbb{W}_1 \overset{\cong}{\to} R_1$ for some suitable choices of $\mathbb{W}_1$. 
By setting $\psi = \phi^{-1}$, we obtain the commutative diagram

$$
\begin{array}{ccc}
R_1 & \xrightarrow{\psi} & W_1 \\
\downarrow & & \downarrow \\
[R/G] & \xrightarrow{I_*} & [W/G].
\end{array}
$$

By Lemma 5.5 below, after shrinking $R_1$ if necessary we can extend the $K$-equivariant isomorphism $\psi: R_1 \xrightarrow{\sim} W_1$ to a $G$-equivariant isomorphism between $G$-invariant open subsets in $R$ and $W$.

$$
\tilde{\psi}: R_2 := G \cdot R_1 \xrightarrow{\sim} W_2 := G \cdot W_1
$$

by sending $g \cdot x$ to $g \cdot \psi(x)$ for $g \in G$ and $x \in R_1$. Then by Lemma 5.6 below, the isomorphism (5.23) restricts to the isomorphism of saturated open subsets. By taking the quotients of $G$-actions, we obtain the desired isomorphisms (5.20).

In the proof of the above proposition, we postponed the following two lemmas:

**Lemma 5.5.** The map (5.23) is well-defined and an isomorphism.

**Proof.** The lemma is essentially proved in the proof of [JS12, Theorem 5.5]. In order to show that (5.23) is well-defined, it is enough to show that if $g_1 R_1 \cap g_2 R_1 \neq \emptyset$ for $g_1, g_2 \in G$, then we have the identity $g_1 \psi g_1^{-1} = g_2 \psi g_2^{-1}$ on $g_1 R_1 \cap g_2 R_1$. By applying $g_2^{-1}$, we may assume that $g_2 = 1$. Let $G' \subset G$ be the open subset given by

$$
G' := \{ g \in G : g R_1 \cap R_1 \neq \emptyset \}.
$$

If we define $G''$ to be

$$
G'' := \{ g \in G' : g \psi g^{-1} = \psi \text{ on } g R_1 \cap R_1 \}
$$

then $G''$ is a closed analytic subset of $G'$ which contains $K$. Therefore if $(G')^o, (G'')^o$ are the connected components of $G', G''$ which contain $K$, then we have $(G')^o = (G'')^o$. Then we take a sufficiently small $K$-invariant open subset $0 \in R'_1 \subset R_1$ satisfying the following: for any $x_1, x_2 \in R'_1$ with $G \cdot x_1 = G \cdot x_2$, the connected component of $(G \cdot x_1) \cap R_1$ containing $x_1$ should contain $x_2$. The above choice of $R'_1$ implies that

$$
G''' := \{ g \in G : g R'_1 \cap R'_1 \neq \emptyset \} \subset (G')^o.
$$

Therefore as $(G')^o = (G'')^o$, for $g \in G'''$ we have $g \psi g^{-1} = \psi$ on $g R'_1 \cap R'_1 \neq \emptyset$. By replacing $R_1$ with $R'_1$, we see that (5.23) is well-defined. Applying the above argument for the inverse of $\psi: R_1 \xrightarrow{\sim} W_1$, we have the inverse of (5.23), showing that (5.23) is an isomorphism. □
Lemma 5.6. There exist saturated open subsets $\tilde{V} \subset \text{Ext}^1(E, E)$, $\tilde{W} \subset Z$ satisfying $0 \in R \cap \tilde{V} \subset R_2$, $q \in \tilde{W} \subset W_2$ such that the isomorphism (5.23) restricts to the isomorphism

$$\tilde{\psi}: R \cap \tilde{V} \cong \tilde{W}.$$ 

Proof. Let $W_3 \subset Z$ be a saturated open subset in $Z$ satisfying $q \in W_3 \subset W_2$, which exists by Lemma 2.5 and set $R_3 := \tilde{\psi}^{-1}(W_3) \subset R_2$. Then $R_3$ is written as $R_3 = R \cap V'$ for some $G$-invariant open subset $0 \in V' \subset V$. Let $V'' \subset \text{Ext}^1(E, E)$ be a saturated open subset satisfying $0 \in V'' \subset V'$, which again exists by Lemma 2.5 and set $R_4 := R \cap V'' \subset R_3$. Let $W_4 := \tilde{\psi}(R_4)$. We show that $W_4$ is a saturated open subset in $Z$. Indeed for $x \in W_4$, the orbit closure $G \cdot x$ in $Z$ is contained in $W_3$ since $W_3$ is saturated. Take $y \in G \cdot x$ and consider $\tilde{\psi}^{-1}(y) \in R_3$. Then since $V''$ is saturated, we have $\tilde{\psi}^{-1}(y) \in R_4$, hence $y \in W_4$ as desired. Now $V'', W_4$ are saturated in $\text{Ext}^1(E, E)$, $Z$. By setting $V = V'', \tilde{W} = W_4$, we obtain the lemma. \hfill \Box

5.5. Calabi-Yau 3-fold case. We keep the situation in the previous subsections. Suppose furthermore that $X$ is a smooth projective CY 3-fold, i.e.

$$\dim X = 3, \ O_X(K_X) \cong O_X.$$ 

In this case, the $A_\infty$-structure (5.6) is cyclic (see [Pol01]), i.e. for a map

$$\psi: \{1, \ldots, n+1\} \to \{1, \ldots, k\}, \ \psi(1) = \psi(n+1)$$

and elements

$$a_i \in \text{Ext}^1(E_{\psi(i)}, E_{\psi(i+1)}), \ 1 \leq i \leq n,$$

we have the relation

(5.24) \quad $(m_{n-1}(a_1, \ldots, a_{n-1}), a_n) = (m_{n-1}(a_2, \ldots, a_n), a_1)$.

Here $m_n$ is the $A_\infty$-product (5.7), $\langle -, - \rangle$ is the Serre duality pairing

(5.25) \quad $\langle -, - \rangle: \text{Ext}^j(E_a, E_b) \times \text{Ext}^{3-j}(E_b, E_a) \to \text{Ext}^3(E_a, E_a) \overset{f_X\text{tr}}{\to} \mathbb{C}$.

Let $W_{E_*} \subset \mathbb{C}[Q_{E_*}]$ be defined by

$$W_{E_*} := \sum_{n \geq 3} \sum_{\{1, \ldots, n+1\} \in \psi(1) = \psi(n+1)} \sum_{\psi(1) = \psi(n+1)} a_{\psi, E_*} \cdot e_1 e_2 \ldots e_n.$$

Here the coefficient $a_{\psi, E_*}$ is given by

(5.26) \quad $a_{\psi, E_*} = \frac{1}{n}(m_{n-1}(e_1^\vee, e_2^\vee, \ldots, e_{n-1}^\vee), e_n^\vee)$.

Then by Lemma 4.1, we have

$$W_{E_*} \subset \mathbb{C}[Q_{E_*}] \subset \mathbb{C}[Q_{E_*}].$$
Therefore $W_{E_*}$ determines a convergent super-potential of $Q_{E_*}$ (see Definition 2.17).

Let $E$ be the object given by (5.5). By the Serre duality, $\text{Ext}^2(E, E^\vee)$ is identified with $\text{Ext}^1(E, E)$. Thus

\[(5.27) \quad \{e^\vee : e \in E(Q_{E_*})\} \subset \text{Ext}^1(E, E)\]

gives a basis of $\text{Ext}^2(E, E^\vee)$. Using this basis, the relation $I_{E_*}$ defined in (5.8) satisfies

\[I_{E_*} = \{m^\vee(e^\vee) : e \in E(Q_{E_*})\} = \partial W_{E_*}.\]

Here the first identity is due to the definition of $I_{E_*}$ via the basis (5.27), and the second identity follows from the construction of $W_{E_*}$ and the cyclic condition (5.24). As a corollary of Theorem 3.2, we obtain the following:

**Corollary 5.7.** In the situation of Theorem 3.2, suppose furthermore that $X$ is a smooth projective CY 3-fold. Then there is a convergent super-potential $W_{E_*}$ of $Q_{E_*}$, analytic open neighborhoods $p \in U \subset M_\omega(v)$, $0 \in V \subset M_{Q_{E_*}}(\vec{m})$ and commutative isomorphisms

\[(5.28) \quad p_M^{-1}(U) \xrightarrow[I_*]{} M_{(Q_{E_*}, \omega_{E_*})}(\vec{m})|_V \xrightarrow{\pi_Q^{-1}(V)/G} \Rightarrow \{d(\text{tr} W_{E_*}) = 0\}/G \xrightarrow{\pi_Q^{-1}(V)/G} \mathbb{C}.\]

Here the bottom arrow sends 0 to $p$, $\pi_Q : \text{Rep}_{Q_{E_*}}(\vec{m}) \to M_{Q_{E_*}}(\vec{m})$ is the quotient morphism, and $\text{tr} W_{E_*}$ is the $G$-invariant analytic function on the smooth analytic space $\pi_Q^{-1}(V)$ (see Subsection 2.6).

### 6. Non-commutative deformation theory

Note that the diagram (3.5) in Theorem 3.2 in particular implies the isomorphism

\[(6.1) \quad I_* : p_Q^{-1}(0) \xrightarrow{\sim} p_M^{-1}(p).\]

In this section, we recall the NC deformation theory associated to a simple collection of sheaves, and explain its relationship to the isomorphism (6.1).

More precisely in Theorem 6.8, using NC deformation theory we show that the map $I_*$ gives an equivalence of categories between the category of nilpotent representations of the Ext-quiver and the subcategory of coherent sheaves on $X$ generated by the given simple collection. The result of Theorem 6.8 immediately implies the isomorphism (6.1), so giving an interpretation of (6.1) via NC deformation theory. The result of Theorem 6.8 will be only used in the proof of Lemma 7.8 in the next section, but seems to
be an interesting result in its own right as it gives intrinsic understanding of the isomorphism \((6.1)\).

6.1. NC deformation functors. Let \(X\) be a smooth projective variety, and take a simple collection of coherent sheaves on it

\[
E_\bullet = (E_1, E_2, \ldots, E_k).
\]

The NC deformation theory associated to the simple collection \((6.2)\) is formulated for such a collection \([Laup02, Eri10, Kaw, BR]\). The following convention is due to Kawamata \([Kaw]\).

By definition, a \(k\)-pointed \(\mathbb{C}\)-algebra is an associative ring \(R\) with \(\mathbb{C}\)-algebra homomorphisms

\[
\mathbb{C}^k \xrightarrow{p} R \xrightarrow{q} \mathbb{C}^k
\]

whose composition is the identity. Then \(R\) decomposes as

\[
R = \mathbb{C}^k \oplus m, \; m := \text{Ker } q.
\]

For \(1 \leq i \leq k\), let \(m_i\) be the kernel of the composition

\[
R \xrightarrow{q} \mathbb{C}^k \to \mathbb{C}
\]

where the second map is the \(i\)-th projection. Note that \(m = \cap_{i=1}^k m_i\). We define \(Art_k\) to be the category of finite dimensional \(k\)-pointed \(\mathbb{C}\)-algebras

\[
R = \mathbb{C}^k \oplus m\text{ such that } m \text{ is nilpotent.}
\]

For a simple collection \((6.2)\), we have the NC deformation functor

\[
(6.3) \quad \text{Def}_{E_\bullet}^{\text{nc}} : Art_k \to \text{Set}.
\]

The above functor is defined by sending \(R = \mathbb{C}^k \oplus m\) to the set of isomorphism classes of pairs

\[
(\mathcal{E}, \psi), \; \mathcal{E} \in \text{Coh}(R \otimes_{\mathbb{C}} \mathcal{O}_X)
\]

where \(\mathcal{E}\) is a coherent left \(R \otimes_{\mathbb{C}} \mathcal{O}_X\)-module which is flat over \(R\), and \(\psi\) is an isomorphism \(R/m \otimes_R \mathcal{E} \cong_{\oplus} \oplus_i E_i\) which induces isomorphisms

\[
R/m_i \otimes_R \mathcal{E} \cong E_i, \; 1 \leq i \leq k.
\]

6.2. Pro-representable hull. Let \(\widehat{Art}_k\) be the category whose objects consist of \(\mathbb{C}^k\)-algebras given by inverse limits of objects in \(Art_k\). An object \(A \in \widehat{Art}_k\) is called a pro-representable hull of the functor \(\text{Def}_{E_\bullet}^{\text{nc}}\) if there is a formally smooth morphism

\[
\text{Hom}_{\widehat{Art}_k}(A, -) \to \text{Def}_{E_\bullet}^{\text{nc}}(-)
\]

which are isomorphisms in first orders. A pro-representable hull is, if it exists, unique up to non-canonical isomorphisms (see \([Sch68]\)).

A pro-representable hull of the functor \(\text{Def}_{E_\bullet}^{\text{nc}}\) is known to exist by \([Laup02, Eri10]\). By \([Kaw]\), it is explicitly constructed by taking the iterated universal extensions of sheaves \(E_i\), which we review here. We first set \(E^{(0)}_i = E_i\) for
1 ≤ i ≤ k. Suppose that \( E^{(n)}_i \) is constructed for some \( n ≥ 0 \) and all \( 1 ≤ i ≤ k \). Then \( E^{(n+1)}_i \) is constructed as the universal extension

\[
0 \to \bigoplus_{j=1}^{k} \text{Ext}^1(E^{(n)}_i, E_j)^\vee \otimes E_j \to E^{(n+1)}_i \to E^{(n)}_i \to 0.
\]

Let us set

\[
E^{(n)} := \bigoplus_{i=1}^{n} E^{(n)}_i, \quad R^{(n)} := \text{Hom}(E^{(n)}, E^{(n)}).
\]

Then \( R^{(n)} \) is an object of \( \text{Art}_k \), and \( E^{(n)} \) is an element of \( \text{Def}_{E^{(n)}}^{\text{nc}}(R^{(n)}) \) by [Kaw, Theorem 4.8]. Moreover by [Kaw, Lemma 4.3, Corollary 4.6, Theorem 4.8], there exist natural surjections \( R^{(n+1)} \to R^{(n)} \) such that the inverse limit

\[
\text{mod}_{E^{(n)}}^{\text{nc}} := \lim_{\leftarrow} R^{(n)} \in \text{Art}_k
\]

is a pro-representable hull of \( E^{(n+1)} \). Moreover the surjection \( E^{(n+1)} \to E^{(n)} \) induces the isomorphism

\[
R^{(n)} \otimes_{R^{(n+1)}} E^{(n+1)} \cong E^{(n)}.
\]

By the surjection \( R^{(n+1)} \to R^{(n)} \), we have the fully-faithful embedding

\[
\text{mod} R^{(n)} \hookrightarrow \text{mod} R^{(n+1)}.
\]

Then the category \( \text{mod}_{E^{(n)}}^{\text{nc}} \) is defined by

\[
\text{mod}_{E^{(n)}}^{\text{nc}} := \lim_{\rightarrow} \left( \text{mod} R^{(n)} \right).
\]

The above category is identified with the abelian category of nilpotent finite dimensional right \( R^{\text{nc}}_{E^{(n)}} \)-modules.

### 6.3. Equivalence of categories via NC deformations

In what follows, we show that the category \( \text{mod}_{E^{(n)}}^{\text{nc}} \) is equivalent to the subcategory of \( \text{Coh}(X) \)

\[
\langle E_1, E_2, \ldots, E_k \rangle \subset \text{Coh}(X)
\]

given by the extension closure of \( E_1, \ldots, E_k \), i.e. the smallest extension closed subcategory of \( \text{Coh}(X) \) which contains \( E_1, \ldots, E_k \).

**Lemma 6.1.** For \( T \in \text{mod} R^{(n)} \), we have

\[
\Phi(T) := T \otimes_{R^{(n)}} E^{(n)} \in \langle E_1, \ldots, E_k \rangle.
\]

**Proof.** Since \( R^{(n)} \in \text{Art}_k \), it decomposes as

\[
R^{(n)} = \mathbb{C}^k \oplus m^{(n)}.
\]

We take the following filtration in \( \text{mod} R^{(n)} \)

\[
\cdots \subset T(m^{(n)})^j \subset T(m^{(n)})^{j-1} \subset \cdots \subset Tm^{(n)} \subset T.
\]

Then the subquotient

\[
T^{(j)} := T(m^{(n)})^j / T(m^{(n)})^{j+1}
\]
is a \( \mathbb{C}^k \)-module, which is zero for \( j \gg 0 \). Since \( E^{(n)} \) is an NC deformation of \( E_* \) to \( R^{(n)} \), it follows that \( T^{(j)} \otimes_{R^{(n)}} E^{(n)} \) is a direct sum of objects in \( (E_1, \ldots, E_k) \). Since \( T \) is given by iterated extensions of \( T^{(j)} \), the lemma follows.

The functor
\[
\Phi: \text{mod } R^{(n)} \to \langle E_1, \ldots, E_k \rangle
\]
given by Lemma 6.1 commutes with the embedding (6.6) by the isomorphism (6.5). Hence we obtain the functor
\[
(6.9) \quad \Phi: \text{mod}_{nil} R_{E_*}^{nc} \to \langle E_1, \ldots, E_k \rangle.
\]
Below we show that the functor (6.9) is an equivalence of categories. We prepare some lemmas.

**Lemma 6.2.** We have \( \text{Hom}(E^{(n)}_i, E_j) = \mathbb{C}^{\delta_{ij}} \) and the natural map
\[
\text{Ext}^1(E^{(n)}_i, E_j) \to \text{Ext}^1(E^{(n+1)}_i, E_j)
\]
is a zero map.

**Proof.** The lemma follows from the exact sequence
\[
0 \to \text{Hom}(E^{(n)}_i, E_j) \to \text{Hom}(E^{(n+1)}_i, E_j) \to \text{Ext}^1(E^{(n)}_i, E_j)
\]
\[
\text{id} \to \text{Ext}^1(E^{(n)}_i, E_j) \to \text{Ext}^1(E^{(n+1)}_i, E_j)
\]
obtained by applying \( \text{Hom}(-, E_j) \) to the exact sequence (6.4). \( \square \)

**Lemma 6.3.** For any \( U \in \langle E_1, \ldots, E_k \rangle \) and \( n \geq 0 \), the natural map
\[
(6.10) \quad \text{Ext}^1(E^{(n)}_i, U) \to \text{Ext}^1(E^{(n+l)}_i, U)
\]
is a zero map for \( l \gg 0 \).

**Proof.** If \( U = E_j \) for some \( 1 \leq j \leq k \), the lemma follows from Lemma 6.2. Otherwise there is an exact sequence
\[
0 \to U' \to U \to U'' \to 0, \quad U', U'' \in \langle E_1, \ldots, E_k \rangle \setminus \{0\}.
\]
Suppose that the lemma holds for \( U' \) and \( U'' \). For \( l' \gg 0 \) and \( l'' \gg 0 \), We have the commutative diagram
\[
\begin{array}{cccc}
\text{Ext}^1(E^{(n)}, U') & \to & \text{Ext}^1(E^{(n)}, U) & \to & \text{Ext}^1(E^{(n)}, U'') \\
\downarrow & & \downarrow & & \downarrow 0 \\
\text{Ext}^1(E^{(n+l')}, U') & \to & \text{Ext}^1(E^{(n+l')}, U) & \to & \text{Ext}^1(E^{(n+l')}, U'') \\
\downarrow 0 & & \downarrow & & \downarrow \\
\text{Ext}^1(E^{(n+l'+l'')}, U') & \to & \text{Ext}^1(E^{(n+l'+l'')}, U) & \to & \text{Ext}^1(E^{(n+l'+l'')}, U'').
\end{array}
\]
Here the horizontal arrows are exact sequences. The map (6.10) for \( l = l' + l'' \) is the composition of middle vertical arrows, which is zero by a diagram.
Lemma 6.4. For any \( U \in \langle E_1, \ldots, E_k \rangle \), the sequence
\[
\text{Hom}(E^{(0)}, U) \subset \text{Hom}(E^{(1)}, U) \subset \cdots \subset \text{Hom}(E^{(n)}, U) \subset \cdots
\] (6.11)
terminates for \( n \gg 0 \).

**Proof.** The lemma can be proved by the induction on the number of iterated extensions of \( U \) by \( E_1, \ldots, E_k \). If \( U = E_i \) for some \( i \), then the sequence (6.11) terminates by Lemma 6.2. Otherwise there is an exact sequence
\[
0 \to E_i \to U \to U' \to 0
\]
for some \( 1 \leq i \leq k \) and \( U' \in \langle E_1, \ldots, E_k \rangle \). By applying \( \text{Hom}(E^{(n)}, -) \), we obtain the exact sequence
\[
0 \to \text{Hom}(E^{(n)}, E_i) \to \text{Hom}(E^{(n)}, U) \to \text{Hom}(E^{(n)}, U').
\]
By Lemma 6.2 it follows that
\[
\text{hom}(E^{(n)}, U) \leq \text{hom}(E^{(n)}, U') + 1.
\]
By the induction hypothesis, \( \text{hom}(E^{(n)}, U') \) is bounded above by a number which is independent of \( n \). Therefore \( \text{hom}(E^{(n)}, U) \) is also bounded above. \( \square \)

By Lemma 6.4 we have the functor
\[
\Psi: \langle E_1, \ldots, E_k \rangle \to \mod\text{nil} R^\text{nc}_{E_\bullet}
\] (6.12)
sending \( U \) to \( \text{Hom}(E^{(n)}, U) \) for \( n \gg 0 \).

**Lemma 6.5.** The functor (6.12) is exact.

**Proof.** It is enough to show that (6.12) is right exact. Let \( 0 \to U' \to U \to U'' \to 0 \) be an exact sequence in \( \langle E_1, \ldots, E_k \rangle \). For \( n \gg 0 \) and \( l \gg 0 \), we have the commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(E^{(n)}, U) & \longrightarrow & \text{Hom}(E^{(n)}, U'') \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}(E^{(n+l)}, U) & \longrightarrow & \text{Hom}(E^{(n+l)}, U'')
\end{array}
\]
Ext\(^1\)(\( E^{(n)}, U' \)) Ext\(^1\)(\( E^{(n+l)}, U' \)).

Here the isomorphisms of the left and middle vertical arrows follow from Lemma 6.4 and the right vertical arrow is a zero map by Lemma 6.3. Therefore the right bottom horizontal arrow is a zero map, which shows that \( \text{Hom}(E^{(n)}, U) \to \text{Hom}(E^{(n)}, U'') \) is surjective for \( n \gg 0 \). Therefore the functor (6.12) is exact. \( \square \)

We then show the following proposition:

**Proposition 6.6.** The functor (6.4) is an equivalence of categories.
Proof. The functor (6.12) is a right adjoint functor of $\Phi$, so there exist canonical natural transformations

$$\text{id} \to \Psi \circ \Phi(-), \Phi \circ \Psi(-) \to \text{id}.$$ 

It is enough to show that both of them are isomorphisms of functors.

As $E^{(n)}$ is flat over $R^{(n)}$, the functor $\Phi$ is exact. The functor $\Psi$ is also exact by Lemma 6.5, so the compositions $\Psi \circ \Phi, \Phi \circ \Psi$ are also exact. Therefore by the induction on the number of iterated extensions by simple objects and the five lemma, it is enough to check the isomorphisms

$$S_i \xrightarrow{\cong} \Psi \circ \Phi(S_i), \Phi \circ \Psi(E_i) \xrightarrow{\cong} E_i.$$ 

Here $S_1, \ldots, S_k$ are simple $R(0) = C^k$-modules. Since $\Phi(S_i) = E_i$ and $\Psi(E_i) = S_i$, the above isomorphisms are obvious. □

6.4. Mauer-Cartan formalism of NC deformations. We can interpret the NC deformation functor (6.3) in terms of Mauer-Cartan formalism. The argument below is also available in [Seg08].

For $R \in \text{Art}_k$ with the decomposition $R = \mathbb{C}^k \oplus m$, an argument similar to Subsection 4.2 shows that

$$\text{Def}_{E^\bullet}(R) \cong \text{MC} \left( A^{0,*} \left( \mathcal{H}om^* \left( \bigoplus_{i,j=1}^k E_i^\bullet \oplus \bigoplus_{i=1}^k E_i^\bullet \right) \otimes m \right) \right) / \sim$$

$$= \text{MC} \left( \bigoplus_{i,j} A^{0,*} \left( \mathcal{H}om^* (E_i^\bullet, E_j^\bullet) \right) \otimes_{\mathbb{C}} m_{ij} \right) / \sim.$$ 

(6.13)

Here $\sim$ means gauge equivalence, $\otimes$ is the tensor product of $k$-pointed $\mathbb{C}$-algebras (see [Seg08 Section 1.3]), and $m_{ij} = e_i \cdot m \cdot e_j$ for the idempotents $\{e_1, \ldots, e_k\}$ of $R$. Then using the $A_\infty$-operation $\{I_n\}_{n \geq 1}$ in Subsection 4.4, we have the map

$$I_\ast : \text{MC} \left( \bigoplus_{i,j} \text{Ext}^*(E_i, E_j) \otimes_{\mathbb{C}} m_{ij} \right)$$

$$\to \text{MC} \left( \bigoplus_{i,j} A^{0,*} \left( \mathcal{H}om(\mathcal{E}_i^\bullet, \mathcal{E}_j^\bullet) \right) \otimes_{\mathbb{C}} m_{ij} \right)$$

(6.14)

which is an isomorphism after taking the quotients by gauge equivalence. Here the LHS is the solution of the MC equation of the $A_\infty$-algebra

$$\bigoplus_{i,j} \text{Ext}^*(E_i, E_j) \otimes_{\mathbb{C}} m_{ij}$$

whose $A_\infty$-product is given by (5.7), and the map $I_\ast$ is constructed as in (4.28).
Let $A$ be the $\mathbb{C}^k$-algebra defined by
\begin{equation}
A := \mathbb{C}[Q_{E\bullet}]/(f_1, \ldots, f_l)
\end{equation}
where $(f_1, \ldots, f_l)$ is the convergent relation of $Q_{E\bullet}$ given in (5.8). We have the tautological identification
\begin{equation}
\text{MC} \left( \bigoplus_{i,j} \text{Ext}^*(E_i, E_j) \otimes C \, m_{ij} \right) = \text{Hom}_{\text{Art}_k}(A, R).
\end{equation}
Here $(e_{i,j} \otimes r_{ij})$ in the LHS corresponds to $A \rightarrow R$ given by
\[
\text{Ext}^1(E_i, E_j)^* \supset E_{i,j} \ni z \mapsto e_{i,j}(z) \cdot r_{ij}.
\]
As proved in [Seg08, Proposition 2.13], under the above identification the gauge equivalence in the LHS corresponds to the conjugation by an element in $1 + \oplus_i m_{ii}$ in the RHS.

Thus we see that $A$ is a pro-representable hull of $\text{Def}^{nc}_{E\bullet}$. By the uniqueness of pro-representable hull, we have an isomorphism
\[
R^{nc}_{E\bullet} \cong A
\]
which commute with maps to $\text{Def}^{nc}_{E\bullet}$. Combined with Proposition 6.6 we have the following corollary:

**Corollary 6.7.** We have an equivalence of categories
\begin{equation}
\Phi: \text{mod}^{nil}(A) \xrightarrow{\sim} (E_1, E_2, \ldots, E_k).
\end{equation}
Here $A$ is the $\mathbb{C}^k$-algebra (6.15).

### 6.5. Equivalence of categories via $I_\bullet$

Let us take a nilpotent $Q_{E\bullet}$-representation
\begin{equation}
u = (u_e)_{e \in E(Q_{E\bullet})}, \quad u_e: V_{s(e)} \to V_{t(e)}.
\end{equation}
By the argument in Subsection 5.3 and Remark 5.3 the correspondence $u \mapsto I_\bullet(u)$ forms a functor
\begin{equation}
I_\bullet: \text{mod}^{nil}(A) \to \text{Coh}(X).
\end{equation}
We compare the above functor with the equivalence (6.17) in the following proposition:

**Theorem 6.8.** The functor (6.19) is isomorphic to the functor $\Phi$ in (6.9). In particular, the functor $I_\bullet$ in (6.19) is an equivalence of categories
\[I_\bullet: \text{mod}^{nil}(A) \xrightarrow{\sim} (E_1, E_2, \ldots, E_k) \subset \text{Coh}(X).
\]

**Proof.** Let $A = \mathbb{C}^k \oplus m$ be the decomposition, $\{e_1, \ldots, e_k\}$ the idempotents of $A$, and set $A^{(n)} := A/m^{n+1}$, $m^{(n)} := m/m^{n+1}$. Then for an element $u$ as in (6.18), the compositions of $u_e$ for $e \in E(Q_{E\bullet})$ along with the path in $Q_{E\bullet}$ defines the linear map
\[
u: m^{(n)}_{ij} \to \text{Hom}(V_i, V_j), \quad m^{(n)}_{ij} := e_i \cdot m^{(n)} \cdot e_j.
\]
On the other hand, let 
\[ c^{(n)} \in \text{MC} \left( \bigoplus_{i,j} \text{Ext}^*(E_i, E_j) \otimes_C m_{ij}^{(n)} \right) \]
be the canonical element corresponding to the surjection \( A \to A^{(n)} \) under the tautological identity (6.16). Applying the map (6.14), we obtain
\[ I_*(c^{(n)}) \in \text{MC} \left( \bigoplus_{i,j} A^{0,*}(\text{Hom}^*(E^*_i, E^*_j)) \otimes_C m_{ij}^{(n)} \right). \]
(6.20)

Then for \( n \gg 0 \), we have the identity
\[ I_*(u) = u \circ I_*(c^{(n)}) \in \text{MC}(g^*) \). \]
(6.21)

Let \( F^{(n)} \in \text{Def}^{nc}(A^{(n)}) \) the NC deformation of \( E_\bullet \) over \( A^{(n)} \) corresponding to (6.20) under the isomorphism (6.13). Note that \( F^{(n)} \) is the universal NC deformation over \( A \) pulled back by the surjection \( A \to A^{(n)} \). Let \( T \in \text{mod}_{nil}(A) \) be the object given by the \( Q_{E_\bullet} \)-representation \( u \). Then the identity (6.21) implies that
\[ I_*(T) \cong T \otimes_{A^{(n)}} F^{(n)} \).

By the construction of \( \Phi \) in (6.17), which goes back to the construction in Lemma 6.1 and the universality of \( F^{(n)} \), we have \( \Phi(T) = T \otimes_{A^{(n)}} F^{(n)} \). Therefore the proposition holds.

In the diagram (3.5), note that \( p^{-1}_Q(0) \) consists of nilpotent \( A \)-modules and \( p^{-1}_M(p) \) consists of objects in the extension closure \( \langle E_1, \ldots, E_k \rangle \). The above proposition implies that the isomorphism (6.1) is induced by the universal family over NC deformations.

7. Moduli spaces of one dimensional semistable sheaves

In this section, we focus on the case of moduli spaces of one dimensional semistable sheaves, and prove Theorem 1.3.

7.1. Twisted semistable sheaves. Let \( X \) be a smooth projective variety, and \( A(X)_C \) its complexified ample cone
\[ A(X)_C := \{ B + i\omega \in \text{NS}(X)_C : \omega \text{ is ample} \}. \]

Let
\[ \text{Coh}_{\leq 1}(X) \subset \text{Coh}(X) \]
be the abelian subcategory of coherent sheaves whose supports have dimensions less than or equal to one. For an object \( E \in \text{Coh}_{\leq 1}(X) \) and \( B + i\omega \in A(X)_C \), the \( B \)-twisted \( \omega \)-slope \( \mu_{B,\omega}(E) \) is defined by
\[ \mu_{B,\omega}(E) := \frac{\chi(E) - B \cdot \text{ch}_{d-1}(E)}{\omega \cdot \text{ch}_{d-1}(E)} \in \mathbb{R} \cup \{ \infty \}. \]
Here $d = \dim X$, and we set $\mu_{B,\omega}(E) = \infty$ if $\omega \cdot \text{ch}_{d-1}(E) = 0$, i.e. if $E$ is a zero dimensional sheaf.

**Definition 7.1.** An object $E \in \text{Coh}_{\leq 1}(X)$ is $(B,\omega)$-(semi)stable if for any non-zero subsheaf $F \subsetneq E$, we have the inequality

$$\mu_{B,\omega}(F) < (\leq) \mu_{B,\omega}(E).$$

**Remark 7.2.** If $B = 0$, then $E \in \text{Coh}_{\leq 1}(X)$ is $(0,\omega)$-(semi)stable iff it is $\omega$-Gieseker (semi)stable sheaf.

**Remark 7.3.** For any integer $k \geq 1$ and a line bundle $L$ on $X$, we have

$$\mu_{B,\omega}(E) = \mu_{kB,k\omega}(E) = \mu_{kB+c_1(L),k\omega}(E \otimes L).$$

In particular if $B,\omega$ are elements of $\text{NS}(X)_\mathbb{Q}$ so that $kB,k\omega$ are integral, then for a line bundle $L$ with $c_1(L) = -kB$ a sheaf $E \in \text{Coh}_{\leq 1}(X)$ is $(B,\omega)$-semistable iff $E \otimes L$ is a $\omega$-Gieseker semistable sheaf.

The $(B,\omega)$-stability condition is interpreted in terms of Bridgeland stability conditions [Bri07] as follows. Let $N_1(X) \subset H^2(X,\mathbb{Z})$ be the group of numerical classes of algebraic one cycles on $X$ and set

$$\Gamma_X := N_1(X) \oplus \mathbb{Z}.$$ 

Let $\text{cl}$ be the group homomorphism defined by

$$\text{cl}: K(\text{Coh}_{\leq 1}(X)) \to \Gamma_X, \ E \mapsto ([E],\chi(E))$$

where $[E]$ is the fundamental one cycle associated to $E$. By definition, a Bridgeland stability condition on $D^b(\text{Coh}_{\leq 1}(X))$ w.r.t. the group homomorphism map (7.1) consists of data

$$\sigma = (Z,A), \ Z: \Gamma_X \to \mathbb{C}, \ A \subset D^b(\text{Coh}_{\leq 1}(X))$$

where $Z$ is a group homomorphism, $A$ is the heart of a bounded t-structure satisfying some axioms (see [Bri07, KS] for details). It determines the sets of $\sigma$-(semi)stable objects: $E \in D^b(\text{Coh}_{\leq 1}(X))$ is $\sigma$-(semi)stable if $E[k] \in \mathcal{A}$ for some $k \in \mathbb{Z}$, and for any non-zero subobject $0 \neq F \subsetneq E[k]$ in $\mathcal{A}$, we have the inequality in $(0, \pi)$:

$$\arg Z(\text{cl}(F)) < (\leq) \arg Z(\text{cl}(E[k])).$$

The set of Bridgeland stability conditions (7.2) forms a complex manifold, which we denote by $\text{Stab}_{\leq 1}(X)$. The forgetting map $(Z,A) \mapsto Z$ gives a local homeomorphism

$$\text{Stab}_{\leq 1}(X) \to (\Gamma_X)_{\mathbb{C}}.$$ 

For a given element $B + i\omega \in A(X)_\mathbb{C}$, let $Z_{B,\omega}$ be the group homomorphism $\Gamma_X \to \mathbb{C}$ defined by

$$Z_{B,\omega}(\beta, m) := -m + (B + i\omega)\beta.$$ 

Then the pair

$$\sigma_{B,\omega} := (Z_{B,\omega}, \text{Coh}_{\leq 1}(X))$$

(7.4)
determines a point in Stab\(_{\leq 1}(X)\).

It is obvious that an object in Coh\(_{\leq 1}(X)\) is \((B, \omega)-(semi)stable\) if it is Bridgeland \(\sigma_{B,\omega}-(semi)stable\). We also call \((B, \omega)-(semi)stable\) sheaves as \(\sigma_{B,\omega}-(semi)stable\) objects. Moreover the map
\[
A(X)_{\mathbb{C}} \to \text{Stab}_{\leq 1}(X), \quad (B, \omega) \mapsto \sigma_{B,\omega}
\]
is a continuous injective map, whose image is denoted by
\[
U(X) \subset \text{Stab}_{\leq 1}(X).
\]

7.2. Moduli stacks of twisted semistable sheaves. For \(\sigma = \sigma_{B,\omega} \in U(X)\), and \(v \in \Gamma_X\), let
\[
\mathcal{M}_\sigma(v) \subset \mathcal{M}
\]
be the moduli stack of \(\sigma\)-semistable \(E \in \text{Coh}_{\leq 1}(X)\) with \(\text{cl}(E) = v\). As in the case of Gieseker stability, we have the following:

**Lemma 7.4.** The stack \(\mathcal{M}_\sigma(v)\) is an algebraic stack of finite type with a projective coarse moduli space \(M_\sigma(v)\). So we have the natural morphism
\[
p_\mathcal{M} : \mathcal{M}_\sigma(v) \to M_\sigma(v).
\]
Moreover for each closed point \(p \in M_\sigma(v)\), the same conclusion of Theorem 3.2 holds.

**Proof.** If \(B\) and \(\omega\) are rational, then we can reduce the lemma in the case of \(B = 0\) and \(\omega\) is integral by Remark 7.3. In that case, the lemma follows from Theorem 3.2. In general by wall-chamber structure on the space of Bridgeland stability conditions, there is a collection of real codimension one submanifolds \(\{W_j\}_{j \in J}\) in \(A(X)_{\mathbb{C}}\) called walls such that \(\mathcal{M}_\sigma(v)\) is constant if \(\sigma\) is contained in a strata
\[
\bigcap_{j \in J'} W_j \setminus \bigcup_{j \notin J'} W_j
\]
for some subset \(J' \subset J\). Each wall is given by \(\mu_{B,\omega}(\beta, n) = \mu_{B,\omega}(\beta', n')\) for other \((\beta', n') \in \Gamma_X\) which is not proportional to \((\beta, n)\), i.e.
\[
(n' \beta - n \beta')\omega = B \beta' \cdot \omega - B \beta \cdot \omega \beta'.
\]
The above equation determines a hypersurface in \(A(X)_{\mathbb{C}}\) which contains dense rational points. Therefore if \((B, \omega)\) is not rational, then we can perturb it in the strata (7.5) and can assume that \((B, \omega)\) is rational. \(\square\)

7.3. Moduli stacks of semistable Ext-quiver representations. For \(v \in \Gamma_X\) and \(\sigma = \sigma_{B,\omega} \in U(X)\), take a point \(p \in M_\sigma(v)\). Suppose that \(p\) is represented by a \((B, \omega)\)-polystable sheaf \(E\) of the form
\[
E = \bigoplus_{i=1}^k V_i \otimes E_i
\]
\[
E = \bigoplus_{i=1}^k V_i \otimes E_i
\]
where \(E_i \in \text{Coh}_{<1}(X)\) is \((B, \omega)\)-stable with \(\mu_{B, \omega}(E_i) = \mu_{B, \omega}(E)\). Then we have the Ext-quiver \(Q_{E^*}\) associated to the simple collection

\[E^* = (E_1, \ldots, E_k),\]

together with a convergent relation \(I_{E^*}\) as in (5.3). For \(i \in V(Q_{E^*}) = \{1, 2, \ldots, k\}\), let \(S_i\) be the one dimensional \(Q_{E^*}\)-representation corresponding to the vertex \(i\). We denote by \(K(Q_{E^*})\) the Grothendieck group of finite dimensional \(Q_{E^*}\)-representations, and take the group homomorphism

\[\dim: K(Q_{E^*}) \to \Gamma_Q := \bigoplus_{i=1}^k \mathbb{Z} \cdot \dim(S_i)\]

by taking the dimension vectors.

Let us take another stability condition

\[\sigma^+ = \sigma_{B^+, \omega^+} = (Z_{B^+, \omega^+}, \text{Coh}_{<1}(X)) \in U(X).\]

Then we have the group homomorphism

\[Z^+_Q: K(Q_{E^*}) \to \Gamma_Q \to \mathbb{C}, \ [S_i] \mapsto Z_{B^+, \omega^+}(E_i).\]

The above group homomorphism determines a Bridgeland stability condition on the category of \(Q_{E^*}\)-representations, and the associated (semi)stable representations. They are described in terms of slope stability condition as in Definition 7.1. Let \(\mu^+_Q\) be the slope function on the category of \(Q_{E^*}\)-representations defined by

\[\mu^+_Q(-) := -\frac{\text{Re} \ Z^+_Q(-)}{\text{Im} \ Z^+_Q(-)}.\]

Note that if \(\mathbb{V}\) is a \(Q_{E^*}\)-representation with dimension vector

\[(7.8) \quad \bar{m} = (m_i)_{1 \leq i \leq k}, \ m_i = \dim V_i\]

then we have the identity

\[(7.9) \quad \mu^+_Q(\mathbb{V}) = \mu_{B^+, \omega^+}(E)\]

where \(E\) is given by (7.6). We have the following definition:

**Definition 7.5.** A \(Q_{E^*}\)-representation \(\mathbb{V}\) is \(\mu^+_Q\)-(semi)stable if for any sub \(Q_{E^*}\)-representation \(\mathbb{V}' \subset \mathbb{V}\), we have the inequality

\[\mu^+_Q(\mathbb{V'}) < (\leq) \mu^+_Q(\mathbb{V}).\]

For the dimension vector \((7.8)\), let

\[\text{Rep}^+_Q(E^*) \subset \text{Rep}_{Q_{E^*}}(\bar{m})\]

be the (Zariski) open subset consisting of \(\mu^+_Q\)-semistable \(Q_{E^*}\)-representations. The above open subset is a GIT semistable locus with respect to a certain character of \(G\) (see [Kin94, Section 3]). The quotients by \(G\)

\[M^+_Q(E^*) = [\text{Rep}^+_Q(E^*)/G], \ M^+_Q(E^*) = \text{Rep}^+_Q(E^*)//G\]
are the moduli stack of $\mu^+_Q$-semistable $Q_{E^*}$-representations with dimension vector $\vec{m}$, and its coarse moduli space, respectively. We have the commutative diagram

$$
\begin{array}{c}
M^+_{Q_{E^*}}(\vec{m}) \quad \xrightarrow{\rho^+_Q} \quad M_{Q_{E^*}}(\vec{m}) \\
\downarrow \rho^+_Q \\
M^+_{Q_{E^*}}(\vec{m}) \quad \xrightarrow{q_Q} \quad M_{Q_{E^*}}(\vec{m}).
\end{array}
$$

Here the vertical arrows are natural morphisms to the coarse moduli spaces, the top horizontal arrow is an open immersion and the bottom horizontal arrow $q_Q$ is induced by the universality of the GIT quotients. Note that $q_Q$ is projective due to a general argument of affine GIT quotients (see [Muk03, Section 6]).

Let $0 \in V \subset M_{Q_{E^*}}(\vec{m})$ be a sufficiently small analytic open subset as in Definition 2.16. Let

$$\text{Rep}^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V \subset \text{Rep}_{Q_{E^*},I_{E^*}}(\vec{m})|_V$$

be the open locus consisting of $\mu^+_Q$-semistable representations, where the RHS is defined as in (2.17). Then we set

$$M^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V := \text{Rep}^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V / G,$$

$$M^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V := \text{Rep}^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V \# G.$$

Here $M^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V$ is the analytic Hilbert quotient given in Lemma 2.9, which is a closed analytic subspace of $V^+ = q^{-1}_Q(V)$. We have the commutative diagram

$$
\begin{array}{c}
M^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V \quad \xrightarrow{\rho^+_Q} \quad M_{Q_{E^*},I_{E^*}}(\vec{m})|_V \\
\downarrow \rho^+_Q \\
M^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V \quad \xrightarrow{q_Q} \quad M_{Q_{E^*},I_{E^*}}(\vec{m})|_V.
\end{array}
$$

Here the vertical arrows are natural morphisms to the coarse moduli spaces, the top horizontal arrow is an open immersion and the bottom horizontal arrow $q_{(Q,I)}$ is induced by the universality of analytic Hilbert quotients (see Lemma 2.13).

**Lemma 7.6.** The morphism $q_{(Q,I)}$ in the diagram (7.10) is projective.

**Proof.** We have the following commutative diagram

$$
\begin{array}{c}
M^+_{Q_{E^*},I_{E^*}}(\vec{m})|_V \quad \xrightarrow{\rho^+_Q} \quad V^+ \quad \xrightarrow{\square} \quad M^+_{Q_{E^*}}(\vec{m}) \\
\downarrow q_{(Q,I)} \\
M_{Q_{E^*},I_{E^*}}(\vec{m})|_V \quad \xrightarrow{q_Q} \quad M_{Q_{E^*}}(\vec{m}).
\end{array}
$$
Here the right diagram is a Cartesian square whose horizontal arrows are open immersions, and the horizontal arrows in the left diagram are closed immersions. Since \( q_Q \) is projective, the morphism \( q_{(Q,I)} \) is projective by the above diagram.

7.4. Moduli stacks of semistable sheaves under the change of stability. Let us take \( \sigma^+ \) in (7.7) sufficiently close to \( \sigma \). Then by wall-chamber structure on \( U(X) \), any \( \sigma^+ \)-semistable object \( E \) with \( \cl(E) = v \) is \( \sigma \)-semistable. Then we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_{\sigma^+}(v) & \xrightarrow{r_M} & \mathcal{M}_{\sigma}(v) \\
p_M^+ & \downarrow & q_M \\
\mathcal{M}_{\sigma^+}(v) & \xrightarrow{q_M} & \mathcal{M}_{\sigma}(v).
\end{array}
\]

Here the vertical arrows are natural morphisms to the coarse moduli spaces, the top arrow is an open immersion and the bottom arrow is induced by the universality of coarse moduli spaces. The following is the main result in this section.

**Theorem 7.7.** For a closed point \( p \in \mathcal{M}_{\sigma}(v) \) represented by a polystable sheaf (7.6), there is an analytic open neighborhoods \( p \in U \subset \mathcal{M}_{\sigma}(v) \) and \( 0 \in V \subset M_{Q_{E^*}}(\bar{m}) \), where \( Q_{E^*} \) is the Ext-quiver associated to \( p \) with convergent relation \( I_{E^*} \) and the dimension vector \( \bar{m} \) is given by (7.8), such that the diagram (7.11) pulled back to \( U \)

\[
\begin{array}{ccc}
r_{M}^{-1}(U) & \xrightarrow{p_M} & p_{M}^{-1}(U) \\
p_M^+ & \downarrow & q_M \\
r_{M}^{-1}(U) & \xrightarrow{q_M} & U \end{array}
\]

is isomorphic to the diagram (7.10).

**Proof.** We take \( U = \mathcal{W}/\!/G, V \subset M_{Q_{E^*}}(\bar{m}) \) and the isomorphism

\[
I_* : \mathcal{M}_{(Q_{E^*}, I_{E^*})}(\bar{m})|_V \xrightarrow{\simeq} p_M^{-1}(U)
\]

as in Proposition 5.4. It is enough to show that the isomorphism (7.12) restricts to the isomorphism

\[
I_* : \mathcal{M}_{(Q_{E^*}, I_{E^*})}^{+}(\bar{m})|_V \xrightarrow{\simeq} r_M^{-1}(U).
\]

For a \( \mathbb{C} \)-valued point \( x \in \mathcal{M}_{(Q_{E^*}, I_{E^*})}(\bar{m})|_V \), let \( \forall_x \) be the corresponding \( Q_{E^*} \)-representation, and \( E_x \in \text{Coh}_{\leq 1}(X) \) the \((B, \omega)\)-semistable sheaf corresponding to \( I_*(x) \in p_M^{-1}(U) \). Let \( Z \subset \mathcal{M}_{(Q_{E^*}, I_{E^*})}^{+}(\bar{m})|_V \) be the closed substack given by

\[
Z := \{ x \in \mathcal{M}_{(Q_{E^*}, I_{E^*})}^{+}(\bar{m})|_V : I_*(x) \notin r_M^{-1}(U) \}.
\]
Namely $x \in \mathcal{M}_{(Q_{E^*}, I_{E^*})}(\tilde{m})|_\nu$ is a $\mathbb{C}$-valued point of $\mathcal{Z}$ iff $\forall_x$ is $\mu^+_Q$-semistable but $E_x$ is not $(B^+, \omega^+)$-semistable. Below we use the notation in the diagram (7.10). By Lemma 7.8 below, we have

\[(7.14) \quad \mathcal{Z} \cap (r(Q, I))^{-1}(0) = \emptyset.\]

On the other hand, by Lemma 2.12 the subset

\[p^+_I(Q, I)(\mathcal{Z}) \subset \mathcal{M}^+_{(Q_{E^*}, I_{E^*})}(\tilde{m})|_\nu\]

is closed. Together with Lemma 7.6 we see that

\[r(Q, I)(\mathcal{Z}) = q(Q, I) \circ p^+_I(Q, I)(\mathcal{Z}) \subset \mathcal{M}^+_{(Q_{E^*}, I_{E^*})}(\tilde{m})|_\nu\]

is a closed subset. By (7.14), the above closed subset does not contain 0. Therefore by shrinking $V$ if necessary, we may assume that $\mathcal{Z} = \emptyset$, i.e. (7.12) takes $\mathcal{M}^+_{(Q_{E^*}, I_{E^*})}(\tilde{m})|_\nu$ to $r^{-1}_M(U)$.

Next for $x \in \mathcal{M}_{(Q_{E^*}, I_{E^*})}(\tilde{m})|_\nu$, suppose that $E_x$ is $(B^+, \omega^+)$-semistable, i.e. $I_*(x) \in r^{-1}_M(U)$. Note that by (7.9), we have

\[\mu^+_Q(\nu_x) = \mu_{B^+, \omega^+}(E_x).\]

By the functoriality of $I_*$ in Subsection 5.3 and the above equality, if a sub $Q_{E^*}$-representation $\nu' \subset \nu_x$ destabilizes $\nu_x$ in $\mu^+_Q$-stability, then by applying $I_*$ and noting Remark 5.3 we obtain the subsheaf $E' \subset E_x$ which destabilizes $E_x$ in $(B^+, \omega^+)$-stability. This is a contradiction, so $\nu_x$ is $\mu^+_Q$-semistable, i.e. $x \in \mathcal{M}^+_{(Q_{E^*}, I_{E^*})}(\tilde{m})|_\nu$. Therefore we obtain the desired isomorphism (7.13).

We have used the following lemma:

**Lemma 7.8.** Under the equivalence $I_*$ in Theorem 6.8 an object $\nu \in \text{mod-nil}(A)$ with $\dim \nu = \tilde{m}$ is $\mu^+_Q$-semistable iff $F = I_*(\nu)$ is $(B^+, \omega^+)$-semistable in $\text{Coh}_{\leq 1}(X)$.

**Proof.** The if direction is proved in the first part of the proof of Theorem 7.7, so we only prove the only if direction. Suppose by contradiction that $\forall$ is $\mu^+_Q$-semistable but $F$ is not $(B^+, \omega^+)$-semistable. Then there is a non-zero subsheaf $F' \subset F$ such that $\mu_{B^+, \omega^+}(F') > \mu_{B^+, \omega^+}(F)$. On the other hand, as $\sigma^+$ is sufficiently close to $\sigma$ we may assume that there is no wall between $\sigma$ and $\sigma^+$ w.r.t. the numerical class $\text{cl}(F)$. So we have $\mu_{B, \omega}(F') \geq \mu_{B, \omega}(F)$. Since $F \in (E_1, \ldots, E_k)$ and each $E_i$ is $(B, \omega)$-stable with the same slope, the sheaf $F$ is $(B, \omega)$-semistable. Therefore we have $\mu_{B, \omega}(F') \leq \mu_{B, \omega}(F)$, thus $\mu_{B, \omega}(F') = \mu_{B, \omega}(F)$ and $F'$ is also $(B, \omega)$-semistable. By the uniqueness of JH factors of $(B, \omega)$-semistable sheaves, we have $F' \in (E_1, \ldots, E_k)$. Then by the equivalence $I_*$ in Theorem 6.8 we find a subobject $\nu' \subset \nu$ in $\text{mod-nil}(A)$ with $I_*(\nu') \cong F'$. By the identity (7.9), the subobject $\nu'$ destabilizes $\nu$, hence a contradiction. \qed


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