Remarks on the Physical Meaning of Diffraction-Free Solutions of Maxwell Equations

E. Comay

School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
Tel Aviv 69978
Israel

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Abstract:

It is proved that a source of electromagnetic radiation cannot emit a diffraction-free beam at the wave zone. A Bessel $J_0$ $\varphi$-invariant beam does not hold even at the intermediate zone. These results negate claims published recently in the literature.
An idea of creating a diffraction-free beam has been published[1]. The beam’s amplitude is cylindrically symmetric ($\varphi$-invariant) where the $r$-dependence is proportional to the Bessel function of the first kind $J_0(ar)$ and $a$ is a constant having the dimension of $L^{-1}$. Reference [1] has arisen a great interest in utilization of $J_0$ beams and has been cited more than 360 times[2]. An application of [1] shows the central peak of the assumed $J_0$ beam[3] and another one refers to its peculiar $z$-component wavelength[4]. Another publication related to [1] claims that a superluminal propagation of light in air has been detected[5]. Objections to [5] have been published[6]. The purpose of this work is to show that one cannot construct a diffraction-free electromagnetic beam at the wave zone and that the Bessel function $J_0(ar)$ is unsuitable for describing diffraction free $\varphi$-invariant wave at the intermediate zone too. This outcome proves that results of papers discussing this topic, in general, and those ascribing superluminal velocity to beams that take the form of Bessel function $J_0$, in particular, should be reevaluated. Units where the speed of light $c = 1$ are used. The metric $g_{\alpha\beta}$ is diagonal and its entries are $(1,-1,-1,-1)$. $\mathbf{u}_r$, $\mathbf{u}_\varphi$ and $\mathbf{u}_z$ denote unit vectors in cylindrical coordinates and $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ are unit vectors in Cartesian coordinates.

A general analysis of diffraction-free solutions of Maxwell equations has been published[7]. Here the fields solving the problem are derived from a vector potential $\mathbf{A}$ that satisfies the wave equation together with the Lorentz-gauge requirement[8]. It turns out that this work is relevant to [1] and some of its results are analyzed here in detail. (Another work[9] is closely related to [1] and [7].) Let us start with the solution obtained in Example 1 (on p. 1557 of [7]). Using cylindrical coordinates and removing constant factors, the time dependent monochromatic electric field of this solution is obtained
from the vector potential \( \mathbf{E} = -\partial \mathbf{A}/\partial t \)

\[
\mathbf{E} = \omega J_1(ar)e^{i(bz-\omega t)}\mathbf{u}_\varphi
\]  

(1)

where \( J_1 \) is the Bessel function of the first kind of order 1. The magnetic field is \( \mathbf{B} = \text{curl}\mathbf{A} \)

\[
\mathbf{B} = -bJ_1(ar)e^{i(bz-\omega t)}\mathbf{u}_r - iaJ_0(ar)e^{i(bz-\omega t)}\mathbf{u}_z. \tag{2}
\]

Ignoring constant factors, one finds that the magnetic field (2) is dual to the electric field of Example 2 of [7]. (The factor 2 in \( U_r \) of example 2 is a misprint.) This outcome indicates that Examples 1 and 2 of [7] represent dual electromagnetic solutions where \( \mathbf{E} \to \mathbf{B}, \ \mathbf{B} \to -\mathbf{E} \) (see [8], p. 252).

Having the solution, let us examine the problem of a cylindrically shaped wave guide whose walls are made of a perfect conductor (see [8], p. 335). The length of the cylinder is much greater than both its diameter \( 2R \) and the wavelength \( \lambda = 1/\omega \) (see fig. 1). The boundary conditions along the wave guide’s walls are (see [8], p. 335)

\[
E_\parallel = 0, \ B_\perp = 0. \tag{3}
\]

Thus, the solution (1) and (2) satisfies the boundary conditions provided

\[
J_1(aR) = 0. \tag{4}
\]

Dynamical properties of the solution (1) and (2) are obtained from the energy-momentum tensor of the electromagnetic fields (see [10], p. 81 or [8], p. 605))

\[
T^{\mu\nu}_E = \frac{1}{4\pi}(F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}) \tag{5}
\]

where \( F^{\mu\nu} \) denotes the tensor of the electromagnetic fields. Expression (5) is quadratic in the fields. Hence, one should use the real part of (1) and
in an evaluation of quantities belonging to it. Let us first examine the
momentum density of the fields. This is the Poynting vector
\[ S = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B}. \] (6)

The \( z \)-component of the momentum density and energy flux are obtained
from the substitution of the appropriate real part of (1) and (2)
\[ S_z = \frac{b\omega}{4\pi} J_1'(ar)\cos^2(bz - \omega t). \] (7)

Expression (7) is non-negative at all points, a property which is consistent
with the beam’s expected flux of energy that travels away from a localized
source.

The radial component of the momentum density is obtained analogously
\[ S_r = -\frac{a\omega}{8\pi} J_1(ar)J_0(ar)\sin[2(kz - \omega t)]. \] (8)

Here one sees that, unlike the case of (7), the sign of (8) alternates periodically
in the time and \( z \)-coordinates. Moreover, for any fixed value of \( t \) and \( z \), it
changes sign along the \( r \)-axis, because zeroes of the Bessel functions \( J_0 \) and
\( J_1 \) do not coincide[11]. It follows that although the radial motion does not
vanish locally, its mean value is null. This property indicates that the radial
motion takes the type of a standing wave.

Now let us examine the interaction of the fields with the walls of the wave
guide. Point \( P \) at \( x = R, y = z = 0 \) is used as a representation of the general
case and cartesian coordinates are used. The \( x \)-component of the momentum
current at \( P \) is (see [10], p. 82 or [8], p. 605))
\[ T_{xx} = \frac{1}{8\pi} (E_y^2 + E_z^2 - E_x^2 + B_y^2 + B_z^2 - B_x^2). \] (9)

Examining the fields (1) and (2) and the boundary value (4), one finds that
only the \( z \)-component of the magnetic field makes a nonvanishing contribu-
tion. Thus, the momentum current at $P$ is

$$T_{xx} = \frac{a^2}{8\pi} J_0^2(aR) \sin^2(bz - \omega t).$$

(10)

This momentum current is absorbed by the walls, because the fields vanish in all space outside the inner part of the wave guide.

Another effect of the magnetic field (2) on the wave guide’s walls is the electric current induced in the $\varphi$-direction. Indeed, let us evaluate the line integral along the infinitesimal rectangular closed path of fig. 1. Using vector analysis, Maxwell equations and the boundary condition (4), one finds

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int \text{curl} \mathbf{B} \cdot ds = \int 4\pi j \cdot ds.$$  \hspace{1cm} (11)

Thus, a nonzero current $j$ is induced on the walls, because only $B_z$ at the inner part of the wave guide makes a nonvanishing contribution to the line integral. This outcome proves that a time-dependent (and $z$-dependent) electric current flows along the $\varphi$-direction of the wave guide’s walls and that fields of this current are part of the solution (1) and (2). This electric current sustains the $B_z$ related standing wave in the radial direction. The walls also counteract against local electromagnetic pressure.

The dual solution of example 2 of [7] behaves analogously. Using the same global factor of (1) and (2), one finds for this case

$$\mathbf{B} = \omega J_1(ar)e^{i(bz-\omega t)}u_\varphi$$

(12)

$$\mathbf{E} = bJ_1(ar)e^{i(bz-\omega t)}u_r + iaJ_0(ar)e^{i(bz-\omega t)}u_z.$$  \hspace{1cm} (13)

Hence, the boundary conditions (3) yield

$$J_0(aR) = 0.$$  \hspace{1cm} (14)

Since $J_0(ar)$ and $J_1(ar)$ have no common root[11], a nonvanishing radial electric field exists at the wave guide’s walls. It follows from Maxwell equation
$\text{div}\mathbf{E} = 4\pi \rho$ that a time dependent and $z$-dependent charge density is built on the inner part of the wave guide’s walls. Thus, we have also in Example 2 a current that flows on the walls and affects the fields inside the wave guide.

Let us examine an analogous experimental setup. Here the source of the radiation at $z = -L$ is the same as that of the first experiment but the wave guide is removed. This situation is different from the wave guide case. Indeed, the fields of a closed electromagnetic system depend on charges and currents at the retarded space-time points (see [10], pp. 158-160 or [8], p. 225). Therefore, the wave guide’s solutions clearly do not hold for this case because here the current along the wave guide walls is missing.

Since in the second experiment the region at $z = 0$ satisfies the wave zone requirements (see [10] p. 170 or [8], p. 392)

$$L \gg \lambda, \quad L \gg 2R,$$

one can use the wave zone solution. Let $\mathbf{A}$ denote the retarded vector potential at the wave zone. Thus, one finds the fields (see [10] p. 171)

$$\mathbf{B} = \dot{\mathbf{A}} \times \mathbf{n},$$

$$\mathbf{E} = (\dot{\mathbf{A}} \times \mathbf{n}) \times \mathbf{n}$$

where $\mathbf{n}$ is a unit vector in the radial direction.

It turns out that the solution for the free space experiment is inherently different from the one which fits the wave guide’s inner space. In particular, in the case of free space, fields at the wave zone are perpendicular to the radius vector from the source to the field point. On the other hand, the wave guide solution contains a $z$-component ($B_z$ or $E_z$) which is an inherent part of the solution. As shown above, the $B_z$ (or $E_z$) field is associated with the electric current induced on the wave guide’s walls. This conclusion obviously
holds for any pattern of source elements put at the same spatial region as the one used here, because the analysis does not refer to the source’s details. Thus, the results disagree with the claim of [9].

One can use general arguments for proving that a diffraction-free electromagnetic beam that has a nonvanishing $z$-component for at least one of the fields, contains transverse standing wave. Indeed, the beam carries energy and therefore $S$ of (1) does not vanish. Hence, $E$ is not parallel to $B$ and, due to the $z$-component of the fields, $S$ has a nonvanishing transverse component. Now, the diffraction-free property of the beam prevents energy from flowing transversally. Hence, the transverse component of $S$ is a standing wave.

It can also be proved that all solutions of [7] have a nonvanishing $z$-component of at least one of the fields. Indeed, the vector potential $A$ takes the form (see p. 1556 therein)

$$A = \sum_n (\alpha_n M_n + \beta_n N_n),$$

(18)

where $\alpha_n$ and $\beta_n$ are numerical coefficients of the expansion. Here

$$M_n = \text{curl}[J_n(ar)e^{i(bz+n\varphi-\omega t)}u_z]$$

(19)

and

$$N_n = \frac{1}{k}\text{curl}M_n$$

(20)

where $k$ is the wave number. Now $N_n$ contains a $z$-component (see p. 1557 therein). Hence, if $\beta_n \neq 0$ then $E = -\partial A/\partial t = i\omega A$ has a $z$-component too. In other cases all $\beta_n = 0$, which mean that for at least one $n$, $\alpha_n \neq 0$. Here the magnetic field $B = \text{curl}A = \alpha_n \text{curl}M = k\alpha_n N$, which means that $B_z \neq 0$ and the proof is completed.

It follows that the family of solutions of [7] involves standing waves associated with the $z$-components of the solutions. This diffraction-free family
of solutions may fit cylindrical wave guides but are unsuitable for the case of a free space.

Example 4 of [7] (see p. 1558) is the last one which is analyzed here in detail. This example contains one component which is proportional to \( J_0 (ar) \) and is \( \varphi \)-invariant. Although it has a \( \varphi \)-dependent \( z \)-component term which is associated with a standing wave, it looks simpler to show another problem of this solution. The vector potential of this example is given in Cartesian coordinates

\[
A = -i\alpha [aJ_0(\varphi)\mathbf{u}_x - i\frac{a^2}{b}J_1(\varphi)\mathbf{u}_z]e^{i(bz - \omega t)}. \tag{21}
\]

Using \( E = -\partial A/\partial t \), one finds

\[
E = \alpha \omega [aJ_0(\varphi)\mathbf{u}_x - i\frac{a^2}{b}J_1(\varphi)\mathbf{u}_z]e^{i(bz - \omega t)}. \tag{22}
\]

Let us examine the \( z \)-component of the Poynting vector which represents energy current flowing along the beam’s direction, namely, the quantity which is analogous to (7) of Example 1. Examining (22), one finds that only \( B_y \) is needed for this purpose. Thus, \( (\text{curl} \, A)_y \) of (21) is

\[
B_y = \alpha [(ab - \frac{a^3}{2b})J_0(\varphi) + \frac{a^3}{2b}\cos2\varphi J_2(\varphi)]e^{i(bz - \omega t)}. \tag{23}
\]

Hence, the required \( z \)-component of the Poynting vector is obtained as the product of the real parts of \( E_x \) of (22) and \( B_y \) of (23)

\[
S_z = \alpha^2 \omega [(a^2b - \frac{a^4}{2b})J_0^2(\varphi) + \frac{a^4}{2b}\cos2\varphi J_0(\varphi)J_2(\varphi)]\cos^2(bz - \omega t). \tag{24}
\]

Let us examine the \( z \)-component of the energy current near a point whose radial coordinate is \( \bar{R} \) and \( J_0(a\bar{R}) = 0 \). In this neighbourhood \( J_2 \) is dominant[12] and the contribution of the \( J_0^2(\varphi) \) term of (24) can be ignored. The rest of (24) is proportional to \( J_0(\varphi)J_2(\varphi)\cos2\varphi \). Now, let us examine the value of \( S_z \) on a circle whose radius is \( \bar{R} + \varepsilon \), where \( \varepsilon \) is an appropriate small
quantity. Due to the factor $\cos 2\varphi$, one realizes that $S_z$ takes different signs on this circle. Hence, in the solution of Example 4 of [7], energy flows in opposite $z$-directions in certain regions of space. This property of Example 4 is inconsistent with the notion of a beam, where electromagnetic energy flows away from a localized source.

It is clear from the analysis carried out above that, in free space, one cannot build a diffraction free beam from the family of Bessel functions of [7], because these functions are unsuitable at the wave zone.

Some conclusions can be drawn for the intermediate zone too. The diffraction free $\varphi$-invariant $J_0(ar)$ function proposed in [1] does not belong to the solutions of [7]. Indeed, in [7], there are only two truly $\varphi$-invariant solutions. They are the dual solutions of Examples 1 and 2 which are discussed above. As proved in this work, the $z$-component of the energy current is proportional to $J_1^2(ar)$. Hence, the flow of energy vanishes along the $z$-axis. It is also proved above that Example 4 of [7], where there is one $J_0$ term which is $\varphi$-invariant, does not describe a beam of electromagnetic radiation and its $z$-component is not $\varphi$-invariant. It follows that experiments using a $\varphi$-invariant setup and showing a strong peak at the center (like [1,3,4]) should not be interpreted by means of diffraction free solutions.
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* Email: eli@tauphy.tau.ac.il

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[12] Due to [11], the roots \( r > 0 \) of \( J_n(r) \) and \( J_{n+1}(r) \) are simple, do not coincide and interlace. Hence, the recurrence formula \( 2J_1(r)/r = J_0(r) + J_2(r) \) proves that positive roots of \( J_0(r) \) and \( J_2(r) \) do not coincide.
Figure captions:

Fig. 1:

Electromagnetic radiation is emitted from a source into a cylindrical wave guide whose radius is $R$. The source is at $z = -L$ and $L \gg 2R$. $O$ denotes the origin of coordinates and the rectangle at point $P$ denotes a closed integration path (see text).
Fig. 1

\[ z = -L \]