Algebraic vector bundles and $p$-local $\mathbb{A}^1$-homotopy theory

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Abstract

We construct many “low rank” algebraic vector bundles on “simple” smooth affine varieties of high dimension. In a related direction, we study the existence of polynomial representatives of elements in the classical (unstable) homotopy groups of spheres. Using techniques of $\mathbb{A}^1$-homotopy theory, we are able to produce “motivic” lifts of elements in classical homotopy groups of spheres; these lifts provide interesting polynomial maps of spheres and algebraic vector bundles.

1 Introduction

Fix a base field $k \subset \mathbb{C}$ and suppose $X$ is a smooth algebraic $k$-variety. There is a forgetful map $\mathcal{V}_r(X) \to \mathcal{V}_r^\text{top}(X)$ from the set of isomorphism classes of rank $r$ algebraic vector bundles on $X$ to the set of isomorphism classes of rank $r$ complex topological vector bundles on the complex manifold $X(\mathbb{C})$. A complex topological vector bundle lying in the image of this map will be called algebraizable. In general, the forgetful map is neither injective nor surjective. A necessary condition for algebraizability of a vector bundle is that the topological Chern classes should be algebraic, i.e., should lie in the image of the cycle class map $\text{CH}^i(X) \to H^{2i}(X, \mathbb{Z})$.

The forgetful map from the previous paragraph factors as:

$$\mathcal{V}_n(X) \longrightarrow [X, Gr_n]_{\mathbb{A}^1} \longrightarrow \mathcal{V}_n^\text{top}(X),$$

where $[X, Gr_n]_{\mathbb{A}^1}$ is an “algebraic” homotopy invariant mirroring classical homotopy invariance of topological vector bundles. In more detail, $Gr_n$ is an infinite Grassmannian; it may be realized as the ind-scheme $\text{colim}_N Gr_{n,N}$ (see [MV99, §4 Proposition 3.7] for further discussion). The set $[X, Gr_n]_{\mathbb{A}^1}$ is the set of maps between $X$ and $Gr_n$ in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category [MV99].

The set $[X, Gr_n]_{\mathbb{A}^1}$ has a concrete description that we now give. F. Morel proved that if $X$ is furthermore smooth and affine, then $[X, Gr_n]_{\mathbb{A}^1}$ coincides with the set of isomorphism classes of rank $n$ vector bundles on $X$ and also the quotient of the set of morphisms $X \to Gr_n$ by the equivalence relation generated by $\mathbb{A}^1$-homotopies (see [AHW17, Theorem 1]). For a smooth variety $X$, a result of Jouanolou–Thomason [Wei89, Proposition 4.4] guarantees that we may find a smooth affine variety $\tilde{X}$ and a torsor under a vector bundle $\pi : \tilde{X} \to X$; by construction $\pi$ is an isomorphism in the Morel–Voevodsky $\mathbb{A}^1$-homotopy category (any pair $(\tilde{X}, \pi)$ will be called a Jouanolou device for $X$). Thus, any element of the set $[X, Gr_n]_{\mathbb{A}^1}$ may be represented by an equivalence class of morphisms $\tilde{X} \to Gr_n$, i.e., by an actual vector bundle of rank $n$ on $\tilde{X}$; we refer to such equivalence classes as motivic vector bundles of rank $n$.

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In [AFH19], we used the above factorization to demonstrate the existence of additional cohomological restrictions to algebraizability of a bundle beyond algebraicity of Chern classes. The obstructions we described relied on failure of injectivity of the cycle class map. In this paper, we analyze the opposite situation, i.e., cases where the cycle class map is injective. A large class of such varieties is given by “cellular” varieties (we leave the precise definition of “cellular” vague, but one may use, e.g., the stably cellular varieties of [DI05, Definition 2.10]). In this setting, we formulate the following question to focus the discussion.

**Question 1.** If $X$ is a smooth complex variety that is “cellular” (e.g., $\mathbb{P}^n$), then is every complex topological vector bundle motivic? In other words, for such an $X$ is the map

$$[X, Gr_n]_{A^1} \to \mathcal{V}^{top}_n(X)$$

surjective for every integer $n \geq 1$?

Our interest in this question has two sources. First, it is a difficult problem to construct indecomposable algebraic vector bundles of rank $r$ on $\mathbb{P}^n$ when $1 < r < n$; ranks in this range will be called “small.” The problem of constructing vector bundles of small rank was explicitly stated by Schwarzenberger in the 1960s and Mumford called the special case of rank 2 bundles on $\mathbb{P}^n$, $n \geq 5$, “the most interesting unsolved problem in projective geometry that [he knew] of” [MFK94, p. 227]. In a slightly broader context, Evans and Griffith write [EG85, p. 113] that small rank bundles “seem to be rare in nature.” For sufficiently complicated smooth complex varieties one can use auxiliary techniques (e.g., small rank representations of fundamental groups, Hartshorne–Serre correspondence, unimodular rows) to describe vector bundles of small rank, but for varieties that are sufficiently simple, it seems reasonable to broaden the scope of the question still further: can one construct indecomposable small rank vector bundles on “simple” smooth complex varieties? While the definition of simple is left vague here, “cellular” varieties in the sense above should certainly be considered “simple”.

Second, one may always analyze the problem of deciding whether a given rank $r$ complex topological vector bundle on a smooth complex algebraic variety is algebraizable in two steps: decide if the given bundle is motivic, and, if it is, decide whether it lies in the image of the map $\mathcal{V}_r(X) \to [X, Gr_r]_{A^1}$. The latter question may be phrased more concretely using a Jouanolou device of $X$. Indeed, a motivic vector bundle on $X$ corresponds to an algebraic vector bundle on a Jouanolou device $(\tilde{X}, \pi)$ for $X$, and asking whether a bundle lies in the image of the map $\mathcal{V}_r(X) \to [X, Gr_r]_{A^1}$ is equivalent to asking whether the given bundle descends along $\pi$.

**Remark 2.** The problem of algebraizability of vector bundles on $\mathbb{P}^n$ has been studied for $n \leq 3$ by various authors. The case $n = 1$ being immediate, Schwarzenberger resolved the case $n = 2$. In this case, topological vector bundles are classified by their Chern classes, which may be identified with pairs of integers, and Schwarzenberger [Sch61a, Sch61b] constructed algebraic vector bundles with prescribed Chern classes. The algebraizability of vector bundles on $\mathbb{P}^3$ was more subtle. Schwarzenberger showed that rank 2 topological bundles on $\mathbb{P}^3$ had additional congruence conditions on Chern classes (stemming from the Riemann–Roch theorem). Horrocks [Hor68] showed that there exist rank 2 vector bundles on $\mathbb{P}^3$ with given Chern classes satisfying this condition. Atiyah and Rees completed the topological classification of rank 2 vector bundles on $\mathbb{P}^3$ showing that there is an additional mod 2 invariant (the $\alpha$-invariant) for rank 2 vector bundles with even first Chern class. They showed that Horrocks’ bundles actually provide algebraic representatives for each class isomorphism class of rank 2 topological vector bundles. Various other topological classification results exist and we refer the reader to [OSS11] for more details.

**Remark 3.** Suppose $k$ is a field. At the moment, we do not know a single example of a smooth $k$-variety $X$ such that the map $\mathcal{V}_n(X) \to [X, Gr_n]_{A^1}$ is not surjective. If $X$ is a smooth projective curve over
k, then it is straightforward to show that \( \mathcal{V}_n(X) \rightarrow [X, Gr_n]_{A^1} \) is surjective for any integer \( n \). We will show in [AFH] that surjectivity can be guaranteed for smooth projective surfaces over an infinite field or smooth projective 3-folds over an algebraically closed field. For a general smooth \( k \)-variety \( X \), note that if \(( \tilde{X}, \pi)\) is a Jouanolou device for \( X \), then \( X \times_X \tilde{X} \) is again a smooth affine variety, and either projection \( X \times_X \tilde{X} \rightarrow \tilde{X} \) is a torsor under a vector bundle, and therefore an isomorphism in the Morel–Voevodsky \( A^1 \)-homotopy category. It follows that \( \mathcal{V}_n(\tilde{X}) \rightarrow \mathcal{V}_n(\tilde{X} \times_X \tilde{X}) \) is always a bijection. Therefore, given any vector bundle \( \mathcal{E} \) on \( X \), there exists an isomorphism \( \theta : p_1^* \mathcal{E} \sim p_2^* \mathcal{E} \). The descent question is tantamount to asking whether \( \theta \) may be chosen to satisfy the cocycle condition.

The precise goal of this paper is to analyze the algebraizability question (more precisely, Question 1) for a class of “interesting” topological vector bundles on \( \mathbb{P}^n \) introduced by E. Rees and L. Smith. Let us recall the construction of these topological vector bundles here; we will refer to them as Rees bundles in the sequel. By a classical result of Serre [Ser53, Proposition 11], we know that if \( p \) is a prime, then the \( p \)-primary component of \( \pi_{4p-3}(S^3) \) is isomorphic to \( \mathbb{Z}/p \), generated by the composite of a generator \( \alpha_1 \) of the \( p \)-primary component of \( \pi_{2p}(S^3) \) and the \((2p-3)\)rd suspension of itself; we will write \( \alpha_2^2 \) for this class. In fact, \( \pi_{4p-3}(S^3) \) is the first odd degree homotopy group of \( S^3 \) with non-trivial \( p \)-primary torsion. The map \( \mathbb{P}^n \rightarrow S^{2n} \) that collapses \( \mathbb{P}^{n-1} \) to a point determines a function

\[
[S^{2n-1}, S^3] \cong [S^{2n}, BSU(2)] \rightarrow [\mathbb{P}^n, BSU(2)]
\]

Using the fact that \( \pi_{4p-3}(S^3) \) is the first non-trivial \( p \)-torsion in an odd degree homotopy group of \( S^3 \), Rees showed that the class \( \alpha_2^2 \) determines a non-trivial rank 2 vector bundle \( \xi_p \in [\mathbb{P}^{2p-1}, BSU(2)] \); we will refer to this bundle as a Rees bundle [Rec78]. By construction, \( \xi_p \) is a non-trivial rank 2 bundle with trivial Chern classes.

We will show here that the bundles \( \xi_p \) are motivic. Moreover, we deduce the existence of many non-trivial rank 2 algebraic vector bundles on “simple” smooth affine varieties of large dimension. To describe the result precisely, let us write \( X_n \) for any Jouanolou device for \( \mathbb{P}^n \). For example, one may use the following model for \( X_n \); let \( V \) be an \( n+1 \)-dimensional \( \mathbb{C} \)-vector space, and define \( X_n \) to be the open subvariety of \( \mathbb{P}(V) \times \mathbb{P}(V^c) \) with closed complement the incidence divisor. The projection onto a factor induces a morphism \( X_n \rightarrow \mathbb{P}^n \) that is a torsor under the tangent bundle of \( \mathbb{P}^n \).

**Theorem 4** (See Theorem 2.2.16). For every prime number \( p \), the bundle \( \xi_p \) lifts to a rank 2 algebraic vector bundle on \( X_{2p-1} \).

The proof of Theorem 4 relies on two ingredients. First, we use an extension of \( p \)-local homotopy theory [Sul05, BK72] to the setting of the Morel–Voevodsky \( A^1 \)-homotopy category developed in [AFH22]. In fact, we only need a few specific facts from that paper, most important of which is a \( p \)-local splitting result for the special linear group \( SL_n \) [AFH22, Theorem 5.2.1]. Using this \( p \)-local splitting, we give a new construction of the class \( \alpha_2^2 \) described above that lifts to algebraic geometry. Second, the algebra-geometric version of \( \alpha_2^2 \) has “weights” that prevent it from being the \( A^1 \)-homotopy class corresponding to an algebraic vector bundle. Voevodsky introduced a class \( \tau \) (see, e.g., [Voe03, Theorem 6.16]), multiplication by which shifts weights on suitably defined torsion classes. The class \( \tau \alpha_2^2 \) then provides a lift with correct weights.

**Remark 5.** The observation that multiplication by \( \tau \) can be used to eliminate problems involving weights together with information about construction of known elements of the homotopy groups of spheres of small dimension suggests the possibility that there is never an obstruction to producing motivic lifts of a given topological vector bundle on \( \mathbb{P}^n \) for small \( n \). A more systematic way of constructing topological vector bundles on \( \mathbb{P}^n \) is via the unstable Adams–Novikov spectral sequence. Motivated in part by the preceding observations, the third author was led to conjecture the Wilson space hypothesis: the \( \mathbb{P}^1 \)-infinite
loop space $\Omega_p^\infty \text{MGL}$ classifying Voevodsky’s algebraic cobordism is an even space. In notation that will be introduced later in the paper, this entails (a) $\pi_{n-1}^A(\Omega_p^\infty \text{MGL})$ vanishes for all $n \geq 0$, and (b) $\Omega_p^\infty \text{MGL}$ admits a filtration whose associated graded space is a wedge sum of motivic spheres $S^{m,n}$. We will discuss this conjecture and its connection to Question 1 elsewhere.

**Remark 6.** The motivation for Rees’ construction originated from results of Grauert–Schneider [GS77]. If the bundles $\xi_p$ were algebraizable, then the fact that they have trivial Chern classes would imply they were necessarily unstable by Barth’s results on Chern classes of stable vector bundles [Bar77, Corollary 1 p. 127] (here, stability means slope stability in the sense of Mumford). Grauert and Schneider analyzed unstable rank 2 vector bundles on projective space and they aimed to prove that such vector bundles were necessarily direct sums of line bundles; this assertion is now sometimes known as the Grauert–Schneider conjecture. In view of the Grauert–Schneider conjecture, the bundles $\xi_p$ should not be algebraizable. In this light, Theorem 4 presents an enticing dichotomy. Either $\xi_p$ descends to $\mathbb{P}^{2p-1}$ and thus produces a non-split rank 2 vector bundle on projective space, or it does not, in which case it provides a topological vector bundle that admits no algebraic structure.

The techniques used to establish Theorem 4 can also be used to answer another long-standing problem. Indeed, it is a classical problem to determine which elements of the homotopy groups of spheres admit “polynomial” representatives; this problem can be viewed as an incarnation of the philosophy behind the original Atiyah–Bott proof of Bott periodicity [AB64, p. 231]. Write $S^n \subset \mathbb{R}^{n+1}$ for the standard $n$-sphere defined by $\sum_{i=0}^{n} x_i^2 = 1$. By a real algebraic representative of a homotopy class $\alpha \in \pi_n(S^m)$ we will mean a morphism of real algebraic varieties $S^n \to S^m$ such that the continuous map obtained by taking real points is in the homotopy class $\alpha$.

The existence of real algebraic representatives for elements of homotopy groups of spheres was studied in the stable setting by [Bau67] and, independently, by R. Wood. Indeed, all elements in the stable homotopy groups of spheres admit polynomial representatives (see Baum [Bau67, Corollary 2.11; Wood’s results remain unpublished]). The situation regarding existence of real algebraic representatives of unstable homotopy classes could not be more different. Indeed, Wood [Woo68] showed that there are no non-constant polynomial maps $S^n \to S^m$ when $n \geq 2m$. In particular, this means that as soon as $p$ is a prime number $\geq 3$, the classes $\alpha_1$ and $\alpha_1^2$ cannot be represented by morphisms of real algebraic spheres. In fact, even when $p = 2$, Wood shows that $\alpha_1$ (in this case a generator of $\pi_2(S^3) \cong \mathbb{Z}/2$) admits no real algebraic representative [Woo68, Theorem 2].

Given the paucity of real algebraic representatives of homotopy classes, Wood later broadened the scope of the representation problem [Woo93] by studying “complex” polynomial representatives. Indeed, if we consider $S^n$ as a complex algebraic variety, then the set of complex points of $S^n$ is a complex manifold diffeomorphic to the tangent bundle of the standard (smooth) $n$-sphere (in particular, homotopy equivalent to the standard (smooth) $n$-sphere). By a complex algebraic representative of a homotopy class $\alpha \in \pi_n(S^m)$, we will mean a morphism of complex algebraic varieties $S^n \to S^m$ such that the associated continuous map obtained by taking complex points is in the homotopy class $\alpha$. Wood writes [Woo93, Question 2]: “[w]hat is the first element in the homotopy groups of spheres which cannot be represented by a polynomial map of quadrics?” and then remarks that one cannot rule out the possibility that all elements of the unstable homotopy groups of spheres admits complex polynomial representatives. We present evidence in support of this latter possibility, so that the situation regarding existence of complex algebraic representatives of homotopy classes appears to be essentially the opposite of that for real algebraic representatives.

We write $Q_{2n-1}$ for the smooth affine variety in $\mathbb{A}^{2n}$ defined by the equation $x_0 \cdots x_n y_0 \cdots y_n = 1$ and $Q_{2n}$ for the smooth affine variety in $\mathbb{A}^{2n+1}$ defined by $x_0 \cdots x_n y_0 \cdots y_n = z(1-z)$; these are the “split” smooth affine quadrics. Over the complex numbers, the quadrics defining the standard $n$-sphere become isomorphic to “split” quadrics of dimension $n$. From the standpoint of $\mathbb{A}^1$-homotopy theory, the varieties $Q_{2n-1}$ and $Q_{2n}$
have the \( \mathbb{A}^1 \)-homotopy types of motivic spheres [ADF17, Theorem 2]. With this notation, the next result shows that we may construct non-constant morphisms from quadrics of arbitrary high dimension to \( Q_3 \).

**Theorem 7** (See Proposition 2.2.12 and Theorem 2.2.13). *Over the field of complex numbers, for every prime number \( p \), the homotopy classes \( \alpha_1 \) and \( \alpha_2 \) admit complex polynomial representatives. In particular, there exist non-constant morphisms \( Q_{2p} \to Q_3 \) and \( Q_{4p-3} \to Q_3 \).*

Granted the construction of suitable algebro-geometric homotopy classes realizing the topological classes \( \alpha_1 \) and \( \alpha_2 \), Theorem 7 is deduced in a way similar to Theorem 4 using [AHW18] to guarantee the existence of actual morphisms of quadrics representing given \( \mathbb{A}^1 \)-homotopy classes.

**Remark 8.** On the way to establishing Theorems 4 and 7 we will answer some other questions posed by Wood (see Proposition 2.2.1) and construct a number of other non-constant morphisms between quadrics (see Proposition 2.2.14).

**Conventions**

Fix a base field \( k \). For later use, we remind the reader that a field \( k \) is called *formally real* if \(-1\) is not a sum of squares in \( k \), and not formally real otherwise. Write \( \text{Sm}_k \) for the category of schemes that are separated, smooth and have finite type over \( \text{Spec} \, k \). Write \( \text{Sp} \) for the category of simplicial presheaves on \( \text{Sm}_k \); we identify \( \text{Sm}_k \) as the full-subcategory of \( \text{Sp} \) consisting of simplicially constant representable presheaves. We write \( \ast \) for the final object of \( \text{Sp} \), and by a pointed space, we will mean a pair \((\mathcal{X}, \ast)\) consisting of a space \( \mathcal{X} \) and a morphism \( x : \ast \to \mathcal{X} \). If \( \mathcal{X} \in \text{Sp} \), we write \( \mathcal{X}_+ := \mathcal{X} \sqcup \ast \) and refer to \( \mathcal{X}_+ \) as \( \mathcal{X} \) with a disjoint base-point.

We write \( \mathcal{H}(k) \) for the Morel–Voevodsky \( \mathbb{A}^1 \)-homotopy category [MV99]; if \( \mathcal{X} \) and \( \mathcal{Y} \) are spaces, then we set \( [\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1} := \text{Hom}_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y}) \). Similar notation will be employed for the pointed homotopy category and based \( \mathbb{A}^1 \)-homotopy classes of maps. We write \( \mathbb{G}_m \) for the usual multiplicative group scheme, which is a pointed space via the identity section. For any integers \( a, b \in \mathbb{N} \), we write \( S^{a,b} \) for the motivic sphere \( S^a \wedge \mathbb{G}_m^b \) (note that this is not consistent with Voevodsky’s notation). If \( \mathcal{X} \) is a pointed space, we denote by \( \pi_{a,b}(\mathcal{X}) \) the Nisnevich sheaf (of sets, groups or abelian groups) associated with the presheaf on \( \text{Sm} \) assigning \( [S^{a,b} \wedge U_+, \mathcal{X}]_{\mathbb{A}^1} \) to \( U \in \text{Sm} \).

## 2 Proofs of the main results

Section 2.1 reviews properties of multiplication by \( \tau \) in unstable motivic homotopy theory, collected in the omnibus Theorem 2.1.13. Along the way we establish that the complex or étale realization of multiplication by \( \tau \) is the identity. Section 2.2 then reviews some results on \( p \)-local \( \mathbb{A}^1 \)-homotopy theory and constructs an \( \mathbb{A}^1 \)-homotopy class \( \alpha_1 \) using algebro-geometric techniques (see Proposition 2.2.5). We then proceed to establish the results laid out in the introduction.

### 2.1 Mod \( n \) Moore spaces and weight shifting

Suppose \( \mathcal{X} \) is a pointed space, and we are given a pointed map \( \xi : S^{a,b} \to \mathcal{X} \) representing an element of \( \pi_{a,b}(\mathcal{X})(k) \). Our first goal in this section will be to describe a procedure that, under the assumption that \( \xi \) is a torsion class in a suitable sense, allows us to produce a new morphism from \( S^{a-1,b+1} \). We introduce and study a motivic analog of the usual mod \( n \) Moore space; the main difference between our story and the classical story is that “multiplication by \( n \)” in \( \mathbb{A}^1 \)-homotopy sheaves is more subtle. More precisely, under suitable hypotheses on \( a \) and \( b \), the \( \mathbb{A}^1 \)-homotopy sheaves of \( \mathcal{X} \) inherit an action of Milnor–Witt...
K-theory sheaves. After reviewing some properties of this action in the unstable setting, we then introduce appropriate analogs of the classical multiplication by \( n \) map.

**Composition and actions of Milnor–Witt sheaves**

We begin by reviewing results of F. Morel on the structure of endomorphisms of motivic spheres. The computation is phrased in terms of the Milnor–Witt K-theory sheaves \( K_i^{MW} \) described in [Mor12, Chapter 3]. If \( GW \) is the unramified Grothendieck–Witt sheaf (i.e., the unramified sheaf whose sections over extensions \( L \) of the base-field are the given by \( GW(L) \), the Grothendieck–Witt group of symmetric bilinear forms over \( L \) and \( W \) is the unramified Witt sheaf (i.e., the unramified sheaf whose sections over \( L \) are given by the Witt group \( W(L) \)), then Morel showed [Mor12, Lemma 3.10] that \( K_0^{MW} \cong GW \) and \( K_i^{MW} \cong W \) for \( i < 0 \). Milnor–Witt K-theory sheaves form a sheaf of \( \mathbb{Z} \)-graded rings \( K_*^{MW} \).

**Theorem 2.1.1** ([Mor12, Corollary 6.43]). Suppose \( a \) and \( b \) are integers with \( a \geq 2 \) and \( b \geq 1 \). For any integer \( i \geq -b \), there is an isomorphism

\[
\pi_{a,b+i}^h(S^{a,b}) \xrightarrow{\sim} K_i^{MW}.
\]

We now discuss how Theorem 2.1.1 interacts with the graded ring structure on Milnor–Witt K-theory. Suppose \((\mathcal{X}, x)\) is a pointed space and we are given a morphism \( \varphi : S^{a,b+i}_+ \to S^{a,b}_+ \) where \( i \geq -b \). In that case, precomposition with \( \varphi \) defines a morphism of sheaves

\[
\varphi^* : \pi_{a,b}(\mathcal{X}) \to \pi_{a,b+i}(\mathcal{X}).
\]

More generally, for \( i \geq -b \), define a morphism of presheaves

\[
[S^{a,b+i}_+ \wedge U_+, S^{a,b}_+][a_1] \times [S^{a,b}_+ \wedge U_+, \mathcal{X}][a_1] \longrightarrow [S^{a,b+i}_+ \wedge U_+, S^{a,b}_+][a_1]
\]

by means of the following formula: if \( U \in \text{Sm}_k \), and \((\varphi, f)\) is a pair with \( \varphi : S^{a,b+i}_+ U_+ \to S^{a,b}_+ \) and \( f : S^{a,b}_+ \to \mathcal{X} \), then we send \((\varphi, f)\) to the composite:

\[
S^{a,b+i}_+ \wedge U_+ \xrightarrow{\varphi^* \Delta U_+} S^{a,b+i}_+ \wedge U_+ \xrightarrow{f} \mathcal{X}.
\]

Sheafifying yields a morphism

\[
\pi_{a,b+i}^h(S^{a,b}_+) \times \pi_{a,b}(\mathcal{X}) \to \pi_{a,b+i}^h(\mathcal{X}).
\]

If \( a \geq 2 \), or \( a \geq 1 \) and \( \mathcal{X} \) is an \( \mathbb{A}^1 \)-h-space, then \( \pi_{a,b}(\mathcal{X}) \) is a sheaf of abelian groups. Standard results on composition operations in homotopy groups show that the above morphism yields an action of the sheaf \( \pi_{a,b+i}^h(S^{a,b}_+) \) on \( \pi_{a,b}(\mathcal{X}) \).

In conjunction with Theorem 2.1.1, whenever \( a \geq 2 \) and \( b \geq 1 \), for any integer \( i \geq -b \), the discussion above provides an action morphism

\[
(2.1) \quad K_i^{MW} \times \pi_{a,b}(\mathcal{X}) \to \pi_{a,b+i}(\mathcal{X}).
\]

Note that this action is covariantly functorial in the pointed space \( \mathcal{X} \). In the special case \( i = 0 \), we conclude that there is a (functorial) \( GW = K_0^{MW} \)-module structure on \( \pi_{a,b}(\mathcal{X}) \). The next result, whose proof is immediate from the construction, summarizes the facts about the above action morphism that we will use in the sequel.
Lemma 2.1.2. Assume \( a \geq 2, b \geq 1 \) and \( i \geq -b \) are integers and suppose \( \varphi : S^{a,b+i} \to S^{a,b} \) is a morphism. Write \( [\varphi] \) for the class in \( K^\text{MW}_{-i}(k) \) corresponding to \( \varphi \). If \( (\mathcal{X}, x) \) is a pointed space, then the homomorphism \( \varphi^* : \pi^\text{MW}_{a,b}(\mathcal{X}) \to \pi^\text{MW}_{a,b+i}(\mathcal{X}) \) corresponds to the action of \([\varphi]\) under the morphism in (2.1).

Example 2.1.3. Assume \( b, b' > 0 \) are integers, \( a \geq 2 \) and take \( \mathcal{X} = S^{a,b'} \). In that case, \( \pi^\text{MW}_{a,b}(S^{a,b'}) = K^\text{MW}_{-i} \) and we obtain a morphisms \( K^\text{MW}_{-i} \times K^\text{MW}_{-b} \to K^\text{MW}_{-b+i} \) for \( i \geq -b \). By construction, this operation is given by the multiplication operation in Milnor–Witt K-theory.

**Degree \( n \) self-maps of the Tate circle**

Now, we introduce the analog of the self-map of the sphere of degree \( n \) that we will use in the sequel; we begin by analyzing self-maps of the Tate circle.

**Lemma 2.1.4.** If \( z \) is a coordinate on \( G_m \), then the map sending the integer \( n \) to the \( \mathbb{A}^1 \)-homotopy class of the base-point preserving morphism \( z \mapsto z^n \) defines a bijection \( \mathbb{Z} \to [G_m, G_m]_{\mathbb{A}^1} \) where the latter is the set of pointed \( \mathbb{A}^1 \)-homotopy endomorphisms of \( G_m \).

**Proof.** As a scheme, \( G_m \) is fibrant in the injective Nisnevich local model structure on simplicial presheaves and it is \( \mathbb{A}^1 \)-local since for any smooth scheme \( U \) the map \( \text{Hom}_{\text{Sm}}(U, G_m) \to \text{Hom}_{\text{Sm}}(U \times \mathbb{A}^1, G_m) \) is a bijection. It follows that the pointed \( \mathbb{A}^1 \)-homotopy endomorphisms of \( G_m \) are identified with the pointed endomorphisms of \( G_m \) as a scheme, i.e.,

\[
[G_m, G_m]_{\mathbb{A}^1} = \text{Hom}_{\text{Sm}}(G_m, G_m).
\]

In that case, there is a canonical bijection \( G_m(G_m) = \mathbb{Z} \) sending a coordinate \( z \) on \( G_m \) to \( z^n \).

We now define analogs of mod \( n \) Moore spaces following the usual procedure.

**Definition 2.1.5.** Set \( [n] : G_m \to G_m \) to be the morphism defined by \( z \mapsto z^n \). Then, set

\[
M_n := \text{hocolim}( * \leftarrow G_m \xrightarrow{[n]} G_m );
\]

here \( \text{hocolim} \) means the homotopy colimit computed in the \( \mathbb{A}^1 \)-homotopy category.

**Remark 2.1.6.** Explicitly, \( M_n \) may be realized as the ordinary pushout of the diagram \( \mathbb{A}^1 \leftarrow G_m \to G_m \) where the leftward map is the standard inclusion.

Next, we introduce an explicit homotopy class on \( M_n \). If \( \mu_n \) is the sheaf of \( n \)-th roots of unity (pointed by the identity), then there is a fiber sequence of the form \( \mu_n \to G_m \to G_m \), where the second map is \([n]\). The induced map between homotopy pushouts of the horizontal maps in the diagram

\[
\begin{array}{ccc}
\mu_n & \to & * \\
\downarrow & & \downarrow \\
G_m & \xrightarrow{[n]} & G_m \\
\end{array}
\]

is a morphism \( \Sigma \mu_n \to M_n \). A choice of section of \( \mu_n \), i.e., a choice of an \( n \)-th root of unity, may be viewed as a morphism \( S^1 \to \mu_n \). Composing the simplicial suspension of such a choice with the morphism just described, we obtain a morphism \( S^1 \to \Sigma \mu_n \to M_n \). Taking \( \tau \in \mu_n(k) \) to be a *primitive* \( n \)-th root of unity, we abuse notation and write

\[
(2.2) \quad \tau : S^1 \to M_n
\]
for the induced composite. We write \( C(\tau) \) for the homotopy cofiber of the above map so there is a cofiber sequence of the form

\[
S^1 \rightarrow M_n \rightarrow C(\tau);
\]

we proceed to analyze this cofiber sequence, its suspensions and its dependence on \( \tau \in \mu_n(k) \).

**Construction 2.1.7.** For later use, we would like to have an explicit morphism of spaces representing \( \tau \). In Remark 2.1.6, we constructed an explicit model for \( M_n \) as a pushout. Likewise, identify \( S^0_k = \text{Spec} k[t]/t(1-t) \) and consider the morphism \( S^0_k \rightarrow A^1_k \) given by the projection \( k[t] \rightarrow k[t]/t(1-t) \). With these identifications, the \( A^1 \)-homotopy type of \( S^1 \) is represented by the actual pushout of the diagram

\[
\ast \leftarrow S^0_k \rightarrow A^1_k;
\]

we write \( S^1_k = A^1_k/\{0,1\} \) for this model of \( S^1 \). Our choice of primitive \( n \)-th root of unity defines a morphism \( S^0_k \rightarrow G_m \) (send the base-point to \( 1 \in G_m(k) \)) and the non-base-point to \( \tau \in G_m(k) \). If we consider the inclusion \( G_m \hookrightarrow A^1_k \) as given by the ring homomorphism \( k[z] \subset k[z, z^{-1}] \), then we may define a map \( A^1_k \rightarrow A^1_k \) by means of the ring homomorphism \( k[z] \rightarrow k[t] \) sending \( z \) to \( (1-t) + t\tau \). By definition, these morphisms fit into a commutative diagram of the form:

\[
\begin{array}{ccc}
\ast & \leftarrow & S^0_k \\
& & \downarrow \\
G_m & \leftarrow & A^1_k
\end{array}
\]

\[
\begin{array}{ccc}
& & \downarrow \\
& & A^1_k
\end{array}
\]

\[
\begin{array}{ccc}
\ast & \leftarrow & S^0_k \\
& & \downarrow \\
G_m & \rightarrow & A^1_k
\end{array}
\]

The induced morphism of pushouts \( S^1_k \rightarrow M_n \) is a model for the morphism in (2.2).

**Remark 2.1.8.** Above, we have constructed \( \tau \) for a single \( n \), but there are compatibilities for the relevant morphisms for different choices of \( n \). Indeed, if \( n|n' \), then there are induced morphisms \( \mu_n \rightarrow \mu_{n'} \) and \( M_n \rightarrow M_{n'} \). If we fix a primitive \( n' \)-th root of unity, then it yields a primitive \( n \)-th root of unity thus given a morphism \( S^1 \rightarrow M_{n'} \), we obtain a morphism \( S^1 \rightarrow M_n \).

**Suspensions of \([n]\)**

Suspending \( [n] : G_m \rightarrow G_m \), we obtain morphisms \( S^a,b \rightarrow S^a,b \) for any \( a \geq 1 \) and \( b \geq 1 \). As soon as \( a \geq 1 \), the \( [S^a,b, S^a,b]_{A^1} \) is no longer \( \mathbb{Z} \). Indeed, Morel showed [Mor12, Theorem 7.36] that \( [S^1,1, S^1,1]_{A^1} \) is an extension of \( GW(k) \) by \( k^\times/(\pm1) \). If \( a \geq 2 \), then Theorem 2.1.1 shows \( [S^a,b, S^a,b]_{A^1} = GW(k) \).

Thus, for \( b \geq 2 \) suspension defines a map \( \mathbb{Z} = [G_m, G_m]_{A^1} \rightarrow [S^a,b, S^a,b]_{A^1} = GW(k) \).

Following F. Morel, let \( \epsilon = -(-1) \); this is the \( A^1 \)-homotopy class of the map \( \mathbb{P}^{1} \wedge \mathbb{P}^{1} \rightarrow \mathbb{P}^{1} \wedge \mathbb{P}^{1} \) corresponding to switching the two factors [Mor04, §6.1]. Define:

\[
n_{\epsilon} := \sum_{i=1}^{n}((-1)^{i-1}).
\]

We now identify the image of \([n]\) under suspension.

**Proposition 2.1.9.** For any integers \( i, j \geq 1 \), the \((i,j)\)-fold suspension \( \Sigma^{i,j}[n] \) of \([n] : G_m \rightarrow G_m \) coincides with \( n_{\epsilon} \).
Proof. We first treat the case \( i = j = 1 \). Recall from [Mor12, §7.3] that if \((X,x)\) and \((Y,y)\) are two pointed spaces, then the join \(X * Y\) of \(X\) and \(Y\) is the \(A^1\)-homotopy push-out of the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{p_1} & X 
\times Y \xrightarrow{p_2} Y,
\end{array}
\]
where \(p_i\) is the projection onto the \(i\)-th factor. It is straightforward to show that there is a (pointed) \(A^1\)-weak equivalence \(X * Y \cong \Sigma X \wedge Y\). By functoriality of homotopy pushouts, this construction is functorial in both input spaces. In particular, given any pointed endomorphism \(f : Y \rightarrow Y\), there is an induced pointed morphism \(Id * f : X * Y \rightarrow X * Y\).

Now, take \(X = Y = G_m\). In that case, the projections \(G_m \times G_m \rightarrow G_m\) are \(A^1\)-weakly equivalent to the inclusions \(G_m \times G_m \hookrightarrow G_m \times \A^1\) and \(G_m \times G_m \hookrightarrow \A^1 \times G_m\). Since these latter morphisms are cofibrations, it follows that the actual push-out of the diagram
\[
\begin{array}{ccc}
G_m \times A^1 & \longrightarrow & G_m \times G_m \longrightarrow A^1 \times G_m
\end{array}
\]
is a model for the homotopy push-out. We may view the above diagram as the standard Zariski cover of \(A^2 \setminus 0\), and thus we conclude that \(G_m * G_m\) is \(A^1\)-weakly equivalent to \(A^2 \setminus 0\).

Pick coordinates \(x\) and \(y\) on \(A^2 \setminus 0\). Using functoriality of the join for \([n] : G_m \rightarrow G_m\), and the above presentation, we conclude that \(Id * [n] : A^2 \setminus 0 \rightarrow A^2 \setminus 0\) is the map sending \((x,y) \mapsto (x,y^n)\). On the other hand, \(Id * [n] = \Sigma p_1[n]\) via the \(A^1\)-weak equivalence \(G_m * G_m \cong \Sigma G_m \wedge G_m\) and the identification \(\Sigma G_m \cong \P^1\).

To finish the proof, it suffices to identify the class in \([A^2 \setminus 0, A^2 \setminus 0]_{A^1}\) of the morphism \((a,b) \mapsto (a, b^n)\). We may use [AF14a, Corollary 4.4] (the argument given there works for \(n = 2\) as well using the Postnikov tower described, e.g., in [AF14b, Theorem 6.1]) and [Du22, Remark 6.12] to conclude that this class is \(n\), times the class of the identity morphism, which is precisely what we wanted to show.

To conclude, it remains to observe that further suspension defines an isomorphism \([S^{1,2}, S^{1,2}]_{A^1} \rightarrow [S^{1+a,2+b}, S^{1+a,2+b}]\) for any integers \(a, b \geq 0\); this is a consequence of [Mor12, Theorem 7.20] and Theorem 2.1.1 together with suitable pushout conventions for suspension.

\[\blacksquare\]

Remark 2.1.10. We may also compute \(\Sigma[n]\) as an element of \([S^{1,1}, S^{1,1}]_{A^1}\). Indeed, observe that \(z \mapsto z^n\) extends to a morphism \(A^1 \rightarrow A^1\) given by the same formula. Thus, the map in question in homogeneous coordinates sends \([x, y]\) to \([x^n, y^n]\). This map fixes 0 and \(\infty\). We may then appeal to the formulas of Cazanave [Caz12] for the degrees of pointed rational functions.

Remark 2.1.11. By construction, the map \(\Z \rightarrow GW(k)\) we have discussed above is a homomorphism for the monoid structure on each object induced by composition. This map is not, in general, a group homomorphism for the additive structure on \(GW(k)\). However, if \(-1\) is a square in \(k\), then \((-1) = 1\) and our computations show that \(\Z \rightarrow GW(k)\) is a group homomorphism.

Analyzing a connecting homomorphism

Henceforth, fix an integer \(n\) and assume our base-field \(k\) contains a primitive \(n\)-th root of unity \(\tau\). Then, smashing the morphism \(\tau\) of (2.2) with the identity map \(\iota : S^{a-1,b} \rightarrow S^{a-1,b}\) and taking homotopy cofibers yields a cofiber sequence of the form
\[
S^{a,b} \longrightarrow S^{a-1,b} \wedge M_n \longrightarrow \Sigma^{a-1,b}C(\tau).
\]
Likewise, smashing the cofiber sequence defining \(M_n\) (see Definition 2.1.5) with \(S^{a-1,b}\) and shifting, one obtains a cofiber sequence of the form
\[
S^{a-1,b} \wedge M_n \longrightarrow S^{a,b+1} \longrightarrow S^{a,b+1}.
\]
where the second map is $\Sigma^{a,b}[n]$.

The composite of the first morphism in the cofiber sequences (2.3) and (2.4) yields a morphism (2.5)

$$S^{a,b} \rightarrow S^{a,b+1}.$$  

If $a \geq 2$ and $b \geq 0$, then Theorem 2.1.1 tells us that $\pi^A_1(S^{a,b+1}) = K_1^{MW}(k)$, so the preceding morphism defines an element of $K_1^{MW}(k)$. Since $K_1^{MW}(k)$ is generated by symbols $[a]$ with $a \in k^\times$ [Mor12, Lemma 3.6(1)], the element $\tau \in \mu_n(k) \subset k^\times$ yields a symbol $[\tau] \in K_1^{MW}(k)$. The next result identifies the $\mathbb{A}^1$-homotopy class of (2.5) in terms of $[\tau]$.

**Proposition 2.1.12.** Suppose $n$ is a fixed integer $\geq 2$, $k$ is a field containing a primitive $n$-th root of unity and $a$ and $b$ are integers with $a \geq 2$ and $b \geq 0$. The class in $K_1^{MW}(k)$-corresponding to the morphism $S^{a,b} \rightarrow S^{a,b+1}$ of (2.5) is the symbol $[\tau]$.

**Proof.** Tracing through the definitions, we see that the above morphism is induced by the $S^{a,b-1}$-suspension of the composite

$$S^1 \rightarrow S^1 \wedge \mu_n \rightarrow M_n \rightarrow S^1 \wedge G_m$$

where the first map is the choice of $\tau$, the second map is the canonical map from the suspension of the fiber to the cofiber and the third map is the map from the cofiber to the suspension of the first space. By basic compatibility between fiber and cofiber sequences, the composite of the last two maps coincides with the suspension of the inclusion $\mu_n \rightarrow G_m$ (see, e.g., [May99, Chapter 8.7 Lemma p. 63]). Therefore, the map in question is the simplicial suspension of the map sending $\tau \in \mu_n(k)$ to $\tau \in G_m(k)$. Under the identification of $[S^{a+1,b-1}, S^{a+1,b}]_{\mathbb{A}^1} \cong K_1^{MW}(k)$ of [Mor12, Corollary 6.43], this corresponds precisely to $[\tau] \in K_1^{MW}(k)$.

**Weight shifting**

We now show that, at least for classes that are torsion in a suitable sense, one may use the setup above to modify weights of unstable $\mathbb{A}^1$-homotopy classes. In fact, we will establish a more refined sheaf-theoretic weight-shifting device. Given a pointed space $(\mathcal{X}, x)$, we write $\pi^A_1(S^{a,b}(\mathcal{X}; n_\epsilon))$ for the Nisnevich sheaf on $\text{Sm}_k$ associated with the presheaf $U \mapsto [S^{a,b} \wedge M_n \wedge U_+, \mathcal{X}]_{\mathbb{A}^1}$; the notation is chosen to follow that for homotopy groups with coefficients. If $a \geq 2$, $\pi^A_1(S^{a,b}(\mathcal{X}; n_\epsilon))$ is a sheaf of groups, and if $a \geq 3$, this sheaf of groups is abelian.

Smashing the cofiber sequence in (2.4) with $U_+$, mapping into $\mathcal{X}$ and sheafifying for the Nisnevich topology yields an exact sequence of $\mathbb{A}^1$-homotopy sheaves of the form

$$\cdots \rightarrow \pi^A_1(S^{a,b+1}(\mathcal{X})) \xrightarrow{(1)} \pi^A_1(S^{a,b}(\mathcal{X})) \rightarrow \pi^A_1(\mathcal{X}; n_\epsilon) \xrightarrow{(2)} \pi^A_1(\mathcal{X}^{a-1,b+1}(\mathcal{X})) \rightarrow \cdots.$$  

The arrows labelled (1) and (2) in this diagram are, by definition, given by $\Sigma^{a,b+1}[n]$ and $\Sigma^{a-1,b+1}[n]$ respectively. By Lemma 2.1.2, these homomorphisms are given by multiplication by the classes in the Grothendieck–Witt ring corresponding to these elements. Proposition 2.1.9 then guarantees that the endomorphism of $\pi^A_1(\mathcal{X})$ (and $\pi^A_1(\mathcal{X}^{a-1,b+1}(\mathcal{X}))$) in the above diagram corresponds to multiplication by $n_\epsilon$ for the $\text{GW}$-module structures on these sheaves. Write $n_\epsilon \pi^A_1(\mathcal{X})$ for the $n_\epsilon$-torsion subsheaf of $\pi^A_1(\mathcal{X})$.

Repeating the procedure described at the beginning of the preceding paragraph, the morphism $\Sigma^{a-1,b}\tau$ of (2.3) yields a morphism of sheaves of the form $\pi^{A_1}_{a-1,b}(\mathcal{X}; n_\epsilon) \rightarrow \pi^{A_1}_{a,b}(\mathcal{X})$. Putting this together with
the exact sequence of the previous paragraph, we obtain a commutative diagram of the form:

\[(2.6) \quad \pi^{\mathbb{A}_1}_{a,b+1}(\mathcal{X}) \quad \xrightarrow{n_e} \quad \pi^{\mathbb{A}_1}_{a-1,b}(\mathcal{X}; n) \quad \xrightarrow{n_e} \quad \pi^{\mathbb{A}_1}_{a-1,b+1}(\mathcal{X}) \quad \xrightarrow{0}, \]

Putting everything we have learned so far together, we obtain our claimed weight-shifting result.

**Theorem 2.1.13.** Assume \( k \) is a field containing a primitive \( n \)-th root of unity \( \tau \) and let \( a, b \) be integers with \( a \geq 2 \) and \( b \geq 1 \). The morphism \( \pi^{\mathbb{A}_1}_{a-1,b}(\mathcal{X}; n) \to \pi^{\mathbb{A}_1}_{a,b}(\mathcal{X}) \) of Diagram (2.6) factors through a morphism of sheaves

\[(2.7) \quad \tau : n_e \pi^{\mathbb{A}_1}_{a-1,b+1}(\mathcal{X}) \to \pi^{\mathbb{A}_1}_{a,b}(\mathcal{X}) /[\tau] \cdot \pi^{\mathbb{A}_1}_{a,b+1}(\mathcal{X}), \]

where \([\tau] \in K^M(k)\) is the symbol corresponding to the primitive \( n \)-th root of unity \( \tau \) and \([\tau]\) denotes the action of \([\tau]\) under (2.1). If \( k \) is quadratically closed, then the morphism \( \tau \) above induces a homomorphism

\[\tau : n \pi^{\mathbb{A}_1}_{a-1,b+1}(\mathcal{X})(k) \to \pi^{\mathbb{A}_1}_{a,b}(\mathcal{X})(k) / [\tau] \cdot \pi^{\mathbb{A}_1}_{a,b+1}(\mathcal{X})(k).\]

**Proof.** In diagram (2.6), the composite morphism \( \pi^{\mathbb{A}_1}_{a,b+1}(\mathcal{X}) \to \pi^{\mathbb{A}_1}_{a,b}(\mathcal{X}) \) is, by construction, induced by composition with the morphism from (2.5). The \( \mathbb{A}_1 \)-homotopy class of this composite is \([\tau]\) by appeal to Proposition 2.1.12. The first result then follows immediately by appeal to Lemma 2.1.2 and a diagram chase. If \( k \) is quadratically closed, then \( GW(k) = \mathbb{Z} \) via the rank map and \( n_e = n \) and the result follows by taking stalks. \( \square \)

**Complex realization and \( \tau \)**

We now analyze the behavior of \( \tau \) under complex realization. To this end, recall that if \( \iota : \mathbb{C} \to X \in Sm_k \) is a fixed embedding, then the functor sending \( X \in Sm_k \) to \( X(\mathbb{C}) \) equipped with its usual structure of a complex analytic space extends to a functor

\[\mathcal{R}_\iota : \mathcal{H}(k) \to \mathcal{H},\]

where \( \mathcal{H} \) is the usual homotopy category. The functor at the level of homotopy categories is described in [MV99, §4], but complex realization can be realized as the left Quillen functor of a Quillen pair for a suitable model for \( \mathcal{H}(k) \) [DI04, Theorem 1.4] (in particular, it preserves homotopy colimits).

**Lemma 2.1.14.** If \( \iota : \mathbb{C} \to \mathcal{X} \) is a fixed embedding, then the following diagram commutes:

\[n_e \pi^{\mathbb{A}_1}_{a,b}(\mathcal{X})(k) \quad \xrightarrow{n_e \iota_e} \quad n_e (\pi^{\mathbb{A}_1}_{a+1,b-1}(\mathcal{X})(k) / ([\tau] \cdot \pi^{\mathbb{A}_1}_{a+1,b}(\mathcal{X})(k))) \quad \xrightarrow{id} \quad n \pi_{a+b}(\mathcal{R}_\iota \mathcal{X}).\]

In particular, if \( \mathcal{X} \) is a smooth \( \mathbb{C} \)-scheme, then \( \mathcal{R}_\iota \mathcal{X} = \mathcal{X}(\mathbb{C}) \).

**Proof.** We unwind the definition of the weight-shifting map, which arises Diagram (2.6). Under complex realization, the map of degree \( n_e \) is sent to the standard degree \( n \) map. It follows that under complex realization \( n_e \pi^{\mathbb{A}_1}_{a,b}(\mathcal{X})(k) \) is sent to the \( n \)-torsion subgroup of \( \pi_{a+b}(\mathcal{R}_\iota \mathcal{X}) \). The morphism \([\tau] : S^{n+1,b-1} \to\]
$S^{a+1,b}$ becomes a morphism $S^{a+b} \to S^{a+b+1}$ under complex realization, and such a map becomes homotopically trivial. It follows that the map $n_{e} \pi_{a+1,b-1}^{k}(\mathcal{X})(k) \to \pi_{a+b}(\mathfrak{R},\mathcal{X})$ factors through a morphism $n_{e} \pi_{a+1,b-1}^{k}(\mathcal{X})(k)/([\tau]\pi_{a+1,b-1}^{k}(\mathcal{X})(k)))$ to $n_{e}^{\ell} \pi_{a+b}(\mathcal{X}(\mathbb{C}))$ as in the statement; this guarantees commutativity of the diagram for a suitable morphism in the bottom horizontal position.

To see that the bottom horizontal morphism in the diagram is the identity map, we appeal to Construction 2.1.7 and the observation that the morphism of homotopy groups in the statement is induced by composition with an explicit morphism. Since complex realization commutes with formation of homotopy colimits, we conclude that the complex realization of $M_{n}$ is a usual mod $n$ Moore space, say presented as $S^{1} \sqcup e^{2}$ where $e^{2}$ is a 2-cell, while the complex realization of $S^{1}_{\mathbb{C}}$ is the usual circle. The result then follows from the observation that our explicit model for $\tau$ corresponds to the inclusion $S^{1} \hookrightarrow S^{1} \sqcup e^{2}$. 

Étale realization and $\tau$

For a discussion of étale realization of the motivic homotopy category we refer the reader to [Isa04]. Briefly, if $\ell$ is prime, then we may define the $\ell$-complete étale realization functor on the category of schemes. Given a scheme $X$, its étale realization is an $\ell$-complete pro-simplicial set that we will denote by $Et(X)$.

The functor $Et(-)$ has the property that a morphism of schemes $f: X \to Y$ induces a weak equivalence $Et(X) \to Et(Y)$ if and only if $f^{*}: H_{\mathfrak{q}}^{*}(Y,\mathbb{Z}/\ell) \to H_{\mathfrak{q}}^{*}(X,\mathbb{Z}/\ell)$ is an isomorphism. By [Isa04, Theorem 2], if $k$ is a field and $\ell$ is different from the characteristic of $k$, the assignment $X \mapsto Et(X)$ on smooth $k$-schemes extends to a left Quillen functor of a Quillen pair for a suitable model of $\mathcal{H}_{k}(k)$; we write $Et$ for the corresponding functor on homotopy categories as well. If $k$ is furthermore separably closed, it follows from the Künneth isomorphism in étale cohomology with $\mathbb{Z}/\ell\mathbb{Z}$-coefficients that the functor $Et$ preserves finite products (and smash products of pointed spaces).

Assume furthermore that $k$ is an algebraically closed field, and let $R$ be the ring of Witt vectors of $k$. Choose an algebraically closed field $K$ and embeddings $R \hookrightarrow K$ and $\mathbb{C} \hookrightarrow K$. Suppose $G$ is a split reductive $\mathbb{Z}$-group scheme, and write $G_{S}$ for the base-change of $G$ along a morphism $\mathbb{Z}t_{0}S$. We obtain a diagram of morphisms of the form:

$$G_{k} \longrightarrow G_{R} \hookrightarrow G_{K} \longrightarrow G_{\mathbb{C}},$$

which may be used to compare the étale realization of $G_{k}$ and the corresponding complex points. In particular, for $G = G_{m}$, one knows that $Et(G_{m}) = (S^{1})_{\ell}^{\wedge}$.

Lemma 2.1.15. Suppose $k$ is a finite field with algebraic closure $k^{alg}$ and $\ell$ is a prime different from the characteristic of the base field. The following diagram commutes:

$$
\begin{array}{ccc}
\ell_{e} \pi_{a,b}^{k}(\mathcal{X})(k) & \longrightarrow & \ell_{e}(\pi_{a+1,b-1}^{k}(\mathcal{X})(k)/([\tau]\pi_{a+1,b-1}^{k}(\mathcal{X})(k))) \\
\downarrow Et & & \downarrow Et \\
\ell \pi_{a+b}(Et,\mathcal{X}) & \longrightarrow & \ell \pi_{a+b}(Et,\mathcal{X}).
\end{array}
$$

Proof. The existence of a commutative diagram with some morphism in the bottom horizontal position follows along exactly the same lines as in Lemma 2.1.14, in this case observing that the map $\ell_{e}$ is sent to the degree $\ell$ map. It then remains to conclude that the bottom horizontal arrow is the identity map, for which we appeal to explicit construction of $\tau$ from Construction 2.1.7. In this case, since étale realization commutes with homotopy colimits, we conclude that $Et(M_{e})$ is the $\ell$-completion of the usual Moore space $M_{e}$. Likewise, $Et(G_{m}) = (S^{1})_{\ell}^{\wedge}$ by lifting to characteristic zero as described before the statement. To conclude, it suffices to observe that the morphism $\tau$ is actually defined over $\mathbb{Z}[\tau]$ (here thinking of $\tau$ as the root of unity). In that case, our result follows from Lemma 2.1.14 by taking $\ell$-completions.
2.2 Building vector bundles and maps

In this section, we will prove the existence of various morphisms of quadrics. By adjunction arguments, these morphisms of quadrics give rise to corresponding vector bundles. As a warm-up we answer some old questions of R. Wood on existence of algebraic representatives of elements in classical homotopy groups of spheres. To this end, recall that the quadric $Q_{2n-1}$ has the $A^1$-homotopy type of $\Sigma^{n-1}G_m^n$ and $Q_{2n}$ has the $A^1$-homotopy type of $\Sigma^nG_m^n$ by [ADF17, Theorem 2] (both results hold over $\text{Spec } \mathbb{Z}$). Then, by [AHW18, Theorem 4.2.1], we know that if $R$ is any smooth $k$-algebra, then any element of $[\text{Spec } R, Q_{2n-1}]_{A^1}$ may be represented by an explicit morphism of schemes $\text{Spec } R \to Q_{2n-1}$. Likewise, by [Aso22, Theorem 2], any element of $[\text{Spec } R, Q_{2n}]_{A^1}$ may be represented by a morphism $\text{Spec } R \to Q_{2n}$. Combining these two observations, we see that to establish the existence of a morphism of quadrics, it suffices to show that the $A^1$-homotopy class of the resulting morphism is non-trivial, which can be checked in many ways.

Morphisms of quadrics of very low degree

The next result answers [Woo93, Question 3] and also establishes the case of Theorem 7 corresponding to $p = 2$.

**Proposition 2.2.1.** If $k = \mathbb{C}$, then there exist morphisms of quadrics $Q_4 \to Q_3$ and $Q_5 \to Q_3$ providing polynomial representatives of the suspension and double suspension of the Hopf map.

**Proof.** Consider the map $\eta : A^2 \setminus 0 \to \mathbb{P}^1$. The $G_m$-suspension of $\eta$ yields a map $\Sigma G_m \eta : S^1 \wedge G_m^3 \to A^2 \setminus 0$. Combining [Mor12, Theorems 6.13 and 7.20], we know that $\pi_{1,0}^{A^1}(A^2 \setminus 0)(k) \cong W(k)$. In particular, if $k$ is algebraically closed, then $W(k) \cong \mathbb{Z}/2$ and we conclude that $\Sigma G_m \eta$ is a 2-torsion class. Taking $k = \mathbb{C}$ and appealing to Theorem 2.1.13 there is an element $\tau \Sigma G_m \eta : \mathbb{P}^1 \wedge 2 \to A^2 \setminus 0$. Appeal to Lemma 2.1.14 shows that this element is mapped under complex realization to the suspension of the Hopf map $S^1 \to A^2$.

Now, by [ADF17, Proposition 2.1.2] we know that $Q_4 \cong \mathbb{P}^1 \wedge 2$ in $\mathcal{H}(k)$. Likewise, $Q_3 \cong A^2 \setminus 0$ in $\mathcal{H}(k)$. Then, by [AHW18, Theorem 4.2.1], since the map $\pi_0(\text{Sing} A^1Q_3)(R) \to [\text{Spec } R, Q_3]_{A^1}$ is a bijection for any smooth affine $k$-algebra $R$, we conclude that $\tau \Sigma G_m \eta$ is represented by a morphism $Q_4 \to Q_3$ as required.

For the second statement, observe that composing $\Sigma \mathbb{P}^1 \eta$ and $\tau \Sigma G_m \eta$ yields an element of $[A^3 \setminus 0, A^2 \setminus 0]_{A^1}$ that is mapped by complex realization to the suspension of the Hopf map. Again, by [AHW18, Theorem 4.2.1], this element is represented by a morphism of quadrics $Q_5 \to Q_3$. \qed

**Remark 2.2.2.** In [AF14b, Theorem 7.4], we studied $\pi_{A^1}^{A^1}(SL_2)$ under complex realization. In particular, we claimed that $\pi_{A^1}^{A^1}(SL_2)(\mathbb{C}) \cong \mathbb{Z}/2$ is sent isomorphically to $\mathbb{Z}/2 = \pi_3(S^3)$ under complex realization. The proof of this assertion in [AF14b, Theorem 7.4] is mistaken and Proposition 2.2.1 supplies a correct proof.

A quick review of localization in $A^1$-homotopy theory

In order to establish higher degree analogs of Proposition 2.2.1, we will use techniques of localization in $A^1$-homotopy theory as developed in [AFH22]. Suppose $R \subset \mathbb{Q}$ is a subring of the rational numbers (in the sequel, $R$ will be $\mathbb{Z}[\frac{1}{n}]$ for a suitable integer $n$). In [AFH22], we constructed an $R$-localization on the unstable $A^1$-homotopy category. In more detail, by [AFH22, Theorem 4.3.8] if $(\mathscr{X}, x)$ is any pointed, weakly $A^1$-nilpotent space (see [AFH22, Definition 3.3.1]), then there is a space $L_R \mathscr{X}$ together with a morphism $\mathscr{X} \to L_R \mathscr{X}$ such that $L_R \mathscr{X}$ is again weakly $A^1$-nilpotent and the induced map $\pi_{A^1}^*(\mathscr{X}) \to \pi_{A^1}^*(L_R \mathscr{X})$ is simply tensoring with $R$ for $i \geq 1$. For later use, it suffices to observe that $SL_m$, pointed by
the identity element, satisfies these hypotheses [AFH22, Example 3.4.1] since it is an $\mathbb{A}^1$-h-space. Using these localization techniques, we established the following splitting result, which is an analog of a classical result of Serre.

**Theorem 2.2.3** ([AFH22, Theorem 5.2.1]). If $k$ is a field that is not formally real, then there is an $\mathbb{A}^1$-weak equivalence

$$Q_3 \times Q_5 \times \cdots \times Q_{2n-1} \xrightarrow{\sim} SL_n$$

after inverting $(n-1)!$.

In [AFH22, Theorem 5.3.3], we also studied the rational $\mathbb{A}^1$-homotopy type of $\mathbb{A}^n \setminus 0$. In particular, we showed that if $k$ is a field that is not formally real, then the natural map $\mathbb{A}^n \setminus 0 \to K(\mathbb{Z}(m),2m-1)$ is a $Q$-$\mathbb{A}^1$-weak equivalence. The homotopy sheaves of $K(\mathbb{Z}(m),2m-1)$ are given by sheafifying Voevodsky’s motivic cohomology groups and are, in general, rather non-trivial. For example, vanishing of $\pi^1_i(K(\mathbb{Z}(m),2m-1))$ for $i \geq 2m-1$ is the Beilinson–Soulé vanishing conjecture. Nevertheless, the motivic cohomology of quadrics is rather simple and may be used to produce a large potential class of maps to which we may apply Theorem 2.1.13.

**Proposition 2.2.4.** If $k$ is a field that is not formally real, then the sheaf $\pi^1_{i,j}(\mathbb{A}^n \setminus 0)\mathbb{Q}$ vanishes for $j > n$.

**Proof.** Since $\mathbb{A}^m \setminus 0 \to K(\mathbb{Z}(m),2m-1)$ is a $Q$-$\mathbb{A}^1$-weak equivalence, it follows that $\pi^1_i(\mathbb{A}^n \setminus 0)\mathbb{Q} \cong \pi^1_i(K(\mathbb{Z}(n),2n-1))\mathbb{Q} \cong H^{2n-1-i,n}_Q$. On the other hand,

$$\pi^1_{i,j}(K(\mathbb{Z}(n),2n-1))\mathbb{Q} \cong (H^{2n-1-i,n}_Q)_{-j} \cong H^{2n-1-i-j,n-j}_Q,$$

where the latter isomorphism follows from the Tate suspension isomorphism in motivic cohomology. Since motivic cohomology in negative weight vanishes by definition, the result follows.

Unstable motivic $\alpha_1$ and related homotopy classes

We are now in a position to give the algebro-geometric construction of the class $\alpha_1$ mentioned in the introduction. The geometric motivation for the construction is provided in Remark 2.2.6, which also gives an explicit model for $\alpha_1$.

**Proposition 2.2.5.** Suppose $k$ is a field having characteristic not equal to 2. If $p$ is an odd prime number, then there exists a $p$-torsion class $\alpha_1 \in \pi^1_{p-1,p+1}(S^{1,2})$ having the following properties:

1. if $k$ has characteristic 0, then $\alpha_1$ is defined over $\mathbb{Q}[i]$; if $k$ has positive characteristic, and $p$ is different from the characteristic of $k$, then $\alpha_1$ is defined over the prime field;
2. if $k$ has characteristic 0, then the complex realization of $\alpha_1$ is a class $\alpha_1^{top} \in \pi_{2p}(S^3)$ generating the $p$-torsion subgroup (isomorphic to $\mathbb{Z}/p$) of the latter; if $k$ has characteristic different from $p$, then the étale realization of the base-extension of $\alpha_1$ to $\overline{k}$ generates the $p$-torsion subgroup of $\pi^1_{2p}(Q_3)^\wedge$; in either case $\alpha_1$ is $\mathbb{P}^1$-stably non-trivial;
3. if $k$ satisfies either of the hypotheses in Point (2) and furthermore contains a primitive $p$-th root of unity, then there exists a non-trivial class $\tau \alpha_1 \in \pi^1_{p,p}(S^{1,2})$; again this class is $\mathbb{P}^1$-stably non-trivial.

**Proof.** Since $p$ is odd, [AF14a, Theorem 3.14 and Proposition 3.15] in conjunction with [AF14a, Lemma 2.7, Remark 2.8 and Corollary 3.11] imply that there is an isomorphism of sheaves of the form:

$$\pi^1_{p-1,p+1}(SL_p) \cong \mathbb{Z}/p!\mathbb{Z}$$
We claim that the projection of $\alpha_1$ onto $\pi^A_0 SL_p(S^2 \setminus 0)$ under the splitting of Theorem 2.2.3 yields the existence statement. Indeed, Theorem 2.2.3 holds over any field of positive characteristic or over $\mathbb{Q}[i]$ (such fields are not formally real). We now establish points (1) and (2) simultaneously. Since the sheaf $\mathbb{Z}/(p!)\mathbb{Z}$ is constant, to check that $\alpha_1$ lies in the necessary summand, it suffices to check after passage to an algebraic closure of the base field. After passing to an algebraic closure, to check non-triviality, we may appeal to realization.

We first treat the case where $k = \mathbb{C}$. In that case, [AF14a, Theorem 5.5] shows that the map given by complex realization determines an isomorphism $\pi^A_0 SL_p(S^2) \cong \pi_2 p(S^2)$. Serre showed that, after inverting $(p−1)!$, the group $SU_p$ splits as a product of odd-dimensional spheres [Ser53, V.3 Corollaire 1]. In fact, the argument proving Theorem 2.2.3 realizes Serre’s splitting explicitly. On the other hand, the $p$-primary subgroup of $\pi_2 p(S^{2n−1})$, $n \leq p$ is trivial (the statement of [Ser53, Proposition 11] holds for $n = 2$ as well since $p$ is assumed an odd prime). Combining these observations, the $p$-primary subgroup of $\pi_2 p(S^2)$ is isomorphic to $\pi_2 p(S^3)$, so a generator for the former realizes the topological class $\alpha_1^{\text{top}}$.

The case where $k$ has positive characteristic is treated similarly, but one appeals to étale realization instead of complex realization; we refer the reader to the discussion around Lemma 2.1.15 for the notation. The étale homotopy type of $SL_p$ can be described by [FP81, Theorem 1]. Indeed, $Et(SL_p) \cong SL_p(S^2) \cong \pi_2 p(S^2)$ via the comparison maps described before Lemma 2.1.15, while the latter homotopy type is identified with $SU_p(S^2)$ since the inclusion $SU_p \hookrightarrow SL_p$ is a homotopy equivalence. In that case, $\pi^A_0 SL_p(k) \cong \mathbb{Z}/p$ after $p$-completion and the argument of [AF14a, Theorem 5.5] provides an explicit generator of the former group, which is compatible with “lifting to characteristic zero” and is thus non-trivial after étale realization.

Similarly, [AHW19, Lemma 3.3.3] guarantees that $Et(Q_{2n−1}) \cong (S^{2n−1})^\wedge$, which allows us to appeal to Serre’s vanishing statement from the preceding paragraph to conclude that $\alpha_1$ projects non-trivially to the $p$-primary component of $\pi_2 p(S^3)^\wedge$ and is thus non-trivial. The statements about $\mathbb{P}^1$-stable non-triviality follow from the fact that the classes $\alpha_1^{\text{top}}$ are stably non-trivial.

Finally, for Point (3), we proceed as follows. Since $\pi^A_0 SL_p SL_p(S^2)$ is a constant sheaf of abelian groups, the $GW$-module structure factors through the rank homomorphism $GW \to \mathbb{Z}$. It follows that $\alpha_1$ is a $p_t$-torsion class. We may then appeal to Theorem 2.1.13. Therefore, $\tau_\alpha$ determines a class in $\pi^A_0 SL_p(SL_2)/[\tau] \cdot \pi^A_0 SL_p(SL_2)$. We abuse terminology and write $\tau_\alpha$ for any lift of the image of $\alpha_1$ under $\tau$ to $\pi^A_0 SL_p(SL_2)$. To conclude non-triviality of $\tau_\alpha$, we simply appeal to Point (2) and Lemma 2.1.14 or 2.1.15. The stable non-triviality of $\tau_\alpha$ follows in a similar fashion.

Remark 2.2.6. For an explicit representative of the class $\alpha_1$ constructed in Proposition 2.2.5, one simply unwinds the references to [AF14a]. Indeed, consider the $A^1$-fiber sequence

$$SL_p \rightarrow SL_{p+1} \rightarrow Q_{2p+1}.$$ 

The $SL_p$-torsor $SL_{p+1} \rightarrow Q_{2p+1}$ is classified by a morphism $Q_{2p+1} \rightarrow BSL_p$. This classifying morphism is adjoint to a “clutching function”, i.e., an $A^1$-homotopy class of maps $S^{p−1} \rightarrow SL_p$. Using Suslin matrices (see [AFH22, Lemma 5.1.4]), one sees that this clutching function is killed by multiplication by $p!$, which provides the required generator. In fact, this observation provides the key geometric input to Theorem 2.2.3.

Example 2.2.7. For $p = 2$, $\pi^A_0 SL_2(SL_2) \cong (K_2^{MW})_{−3} \cong W$, which is not a constant sheaf in contrast to the assertion of Proposition 2.2.5. The Tate suspension $\Sigma G_m \eta : S^{1,3} \rightarrow S^{1,2}$ represents a generator of this
group as a $K_0^{MW}$-module. If the base field is not formally real, then $W$ is a strictly $\mathbb{A}^1$-invariant sheaf that is $2\tau$-torsion as a $K_0^{MW}$-module, and so in that case, we can build $\tau \Sigma_{G_m} \eta$, which yields a class in $\pi_{2,2}^1(SL_2)$; this yields a slight refinement of Proposition 2.2.1, and we may use $\Sigma_{G_m} \eta$ as a model of $\alpha_1$ at the prime 2.

**Example 2.2.8.** For $p = 3$, an explicit description of $\pi_{2,4}^1(SL_2)$ follows by combining [AF14b, Theorem 3.3 and Lemma 7.2]. Indeed, those results yield an exact sequence of the form

$$I \longrightarrow \pi_{2,4}^1(SL_2) \longrightarrow \mathbb{Z}/12 \longrightarrow 0.$$

If $-1$ is a sum of squares in the base field, then a classic result of Pfister implies that $I$ is 2-primary torsion sheaf [EKM08, Proposition 31.4] and thus killed by inverting 2. The 3-primary part of $\pi_{2,4}^1(SL_2)$ is thus exactly $\mathbb{Z}/3$. An explicit generator of the factor of $\mathbb{Z}/12$ is the morphism adjoint to the map $Sp_4/Sp_2 \rightarrow BSp_4$ classifying the $Sp_2$-torsor $Sp_4 \rightarrow Sp_4/Sp_2$ (under the exceptional isomorphism $Sp_2 \cong SL_2$). If the base field $k$ contains a primitive 3rd root of unity, then the class $\tau \alpha_1$ is defined and yields an explicit morphism $S^{3,3} \rightarrow SL_2$.

**Remark 2.2.9.** The existence of the class $\alpha_1$ at the prime 5 follows from [AF17, Theorem 3.2.1]. More generally, note that $\pi_{p-1,p+1}^1(SL_2) \cong \pi_{p,p+1}^1(BSL_2)$ by adjunction and that the group $\pi_{p,p+1}^1(BSL_2)(k)$ may be identified with $[Q_{2p+1}, BSL_2]_{\mathbb{A}^1}$ since $BSL_2$ is $\mathbb{A}^1$-connected [AF14a, Lemma 2.1]. Therefore, the class $\alpha_1$, of Proposition 2.2.5 yields a non-trivial rank 2 vector bundle on $Q_{2p+1}$, extending the constructions of vector bundles from [AF17].

**Lemma 2.2.10.** Fix a field $k$ and assume $p$ is a prime different from the characteristic of $k$. If $k$ is not formally real, and has characteristic different from 2, there are non-trivial $p$-torsion classes $\alpha_2^2 \in \pi_{2p-3,2p}^1(S^{1,2})(k)$. If furthermore, $k$ contains a primitive $p$th root of unity, then there are non-trivial $p$-torsion classes $\tau \alpha_1^2 \in \pi_{2p-2,2p-1}^1(S^{1,2})(k)$. If $k \subset \mathbb{C}$, then the complex realization of $\alpha_1^2$ and $\tau \alpha_1^2$ coincide with $\alpha_1^2,\text{top}$.

**Proof.** As in the proof of Proposition 2.2.5 it suffices to prove the result over $\mathbb{Q}[i]$ if $k$ has characteristic 0 and over the prime field if $k$ has positive characteristic (or a suitable cyclotomic extension of these fields where an appropriate root of unity is adjoined), so we assume this in what follows. Consider the composite map

$$S^{2p-3,2p} \xrightarrow{\Sigma^{p-2,2p-1}} S^{p-1,p+1} \xrightarrow{\alpha_1} \mathbb{A}^2 \setminus 0.$$

Proposition 2.2.5 shows that $\alpha_1$ is a $p$-torsion class, and it follows that $\alpha_2^2$ is also a $p$-torsion class. Theorem 2.1.13 then applies to show that $\tau \alpha_1^2$ is also a $p$-torsion class (a priori, this is defined in the quotient $\pi_{2p-2,2p-1}^1(S^{1,2})(k)/[\tau] \cdot \pi_{2p-2,2p}^1(S^{1,2})(k)$, but we may abuse notation and write $\tau \alpha_1^2$ for the choice of any lift to $\pi_{2p-2,2p-1}^1(S^{1,2})(k)$). The non-triviality assertion follows by appeal to complex or étale realization. Indeed, one knows that $\pi_{4p-3}(S^3) \cong \mathbb{Z}/p$ by [Ser53, Proposition 11] and appeal to [AHW19, Lemma 3.3.3] allows us to conclude that $\pi_{4p-3}(Q_3) \cong \mathbb{Z}/p$ as well. The assertion about $\alpha_1^2$ follows immediately by functoriality of realization, while the argument about $\tau \alpha_1^2$ follows by an additional appeal to Lemma 2.1.14 or Lemma 2.1.15.

**Example 2.2.11.** In Example 2.2.7, we observed that $\tau \Sigma_{G_m} \eta$ could be taken as a model for $\alpha_1$ at $p = 2$. Similarly, we take $\Sigma_{G_m} \circ \Sigma_{G_m}^\Lambda 2 \eta$ as a model for $\alpha_1^2$ at $p = 2$. If we take a base field that has characteristic not equal to 2, then $-1$ is a primitive square root of unity. If the base field is furthermore not formally real, then appealing to Theorem 2.1.13 we see that $\tau \alpha_1^2$ exists for $p = 2$ as well.
Non-constant morphisms of quadrics

Granted Lemma 2.2.10, we can now establish Theorems 4 and 7 from the introduction. In light of Example 2.2.11, we first treat the case \( p = 2 \), where the results are slightly stronger than the case of odd primes.

**Proposition 2.2.12.** If \( k \) is a field that is not formally real and has characteristic not equal to 2, the elements \( \tau \alpha_1 \) and \( \tau \alpha_1^2 \) from Examples 2.2.7 and 2.2.11 correspond to non-constant morphisms \( Q_4 \to Q_3 \) and \( Q_5 \to Q_3 \). For every \( i > 0 \), the \( \mathbb{P}^1 \)-suspensions \( \Sigma_{\mathbb{P}^1} \tau \alpha_1^2 \) yield non-constant morphisms \( Q_{5+2i} \to Q_{3+2i} \).

**Proof.** Mimicking the proof of Lemma 2.2.10, it suffices to analyze what happens under complex realization: the positive characteristic case follows by lifting to characteristic 0. The complex realization of \( \Sigma G_m \eta \) is the suspension of \( \eta_{\text{top}} \). Therefore, we conclude that \( \tau \alpha_1^2 \) has complex realization a suitable suspension of \( \eta^2 \) and likewise has non-trivial étale realization. Since \( \eta^2 \) is stably non-trivial, we conclude that \( \tau \alpha_1^2 \) is \( \mathbb{P}^1 \)-stably non-trivial as well.

**Theorem 2.2.13.** Fix a prime number \( p > 2 \). Suppose \( k \) is a field that is not formally real, has characteristic different from \( p \), and contains a primitive \( p \)-th root of unity.

- The element \( \tau \alpha_1 \) of Proposition 2.2.5 and the element \( \tau \alpha_1^2 \) of Lemma 2.2.10 correspond to non-constant morphisms \( Q_{2p} \to Q_3 \) and \( Q_{4p-3} \to Q_3 \).
- For every integer \( i \geq 0 \), the \( \mathbb{P}^1 \)-suspension \( \tau \Sigma_{\mathbb{P}^1} \alpha_1 \) corresponds to a non-constant morphism \( Q_{2(p+i)} \to Q_{3+2i} \).

**Proof.** By [ADF17, Theorem 2], we know that \( Q_{2n} \cong S^n \wedge G_m^n \) and \( Q_{2n-1} \cong S^{n-1} \wedge G_m^n \). Since \( Q_{2n} \) and \( Q_{2n-1} \) are smooth affine schemes, the result then follows immediately from [AHW18, Theorem 4.2.1] combined with Proposition 2.2.5 or Lemma 2.2.10: for any smooth affine \( k \)-scheme \( X = \text{Spec} \, R \), the map \( \pi_0(\text{Sing}^{A^1} Q_{2m-1})(X) \to [X, Q_{2m-1}]_{A^1} \) is a bijection.

We saw above that a composite of suspensions of morphisms of quadrics need not have the correct weights to yield a morphism of quadrics. If \( -1 \) is a sum of squares in the base field, then the \( A^1 \)-homotopy class of any morphism \( Q_{2n+\epsilon} \to Q_{2m-1} \) with \( \epsilon = 0, 1 \) and \( n > m \) is a torsion class by appeal to Proposition 2.2.4. Therefore, in this range, we may compose suspensions of morphisms of quadrics and \( \tau \) to yield new non-constant morphisms of quadrics.

**Proposition 2.2.14.** If \( k \) is an algebraically closed field having characteristic not equal to 2, then for every \( i > 0 \) there are non-constant morphisms \( Q_{11+2i} \to Q_{5+2i} \) and \( Q_{23+2i} \to Q_{9+2i} \).

**Proof.** Let \( \nu : S^{4,4} \to S^{2,2} \) and \( \sigma : S^{7,8} \to S^{4,4} \) be the corresponding motivic Hopf maps: \( \nu \) is obtained by applying the Hopf construction on the multiplication on \( SL_2 \) and \( \sigma \) is obtained similarly from the multiplication on the unit norm elements in the split octonion algebra. The Tate suspensions \( \Sigma G_m \nu \) and \( \Sigma G_m \sigma \) thus yield morphisms \( S^{5,5} \to S^{2,3} \) and \( S^{7,9} \to S^{4,5} \). Composing with a suitable suspension, we obtain morphisms \( \nu^2 : S^{4,7} \to S^{2,3} \) and \( \sigma^2 : S^{10,13} \to S^{4,5} \). By appeal to Proposition 2.2.4, these classes are torsion and thus it makes sense to speak of \( \tau \nu^2 \) and \( \tau \sigma^2 \). These classes have complex realization \( \nu^2_{\text{top}} \) and \( \sigma^2_{\text{top}} \), by their very definition, and therefore, are stably non-trivial. One concludes that these classes yield the relevant non-constant morphisms as in the Proof of Theorem 2.2.13.

**Remark 2.2.15.** If \( X = \text{Spec} \, R \) is a smooth affine scheme, then a morphism \( X \to A^n \setminus 0 \) corresponds to a unimodular row of length \( n \), i.e., projective \( R \)-module \( P \) of rank \( n-1 \) over \( X \) together with an isomorphism \( P \oplus R \cong R^n \). Given a morphism \( X \to Q_{2n-1} \), composing with the \( Q_{2n-1} \to A^n \setminus 0 \) thus yields a unimodular row of length \( n \). Therefore, the constructions of Proposition 2.2.12, Theorem 2.2.13 and Proposition 2.2.14 all give rise to unimodular rows of small length.
Rees bundles

Finally, we establish Theorem 4 from the introduction in a slightly more general form.

**Theorem 2.2.16.** Fix a prime number \( p \) and assume \( k \) is a field having characteristic different from \( p \). Suppose \( k \) is not formally real, and contains a primitive \( p \)-th root of unity. The image of \( \tau \alpha^2_1 \) under the map \([Q_{4p-2}, BSL_2]_{A^1} \rightarrow [X_{2p-1}, BSL_2]_{A^1}\) is a class \( \xi_p \); this class corresponds to a (non-trivial) rank 2 algebraic vector bundle on \( X_{2p-1} \). If \( k \) admits a complex embedding, then \( \xi_p \) is mapped to the class \( \xi_p^\text{top} \) of Rees.

**Proof.** First we treat the case \( p = 2 \). In that case, the result follows from either Proposition 2.2.1 or Proposition 2.2.12 and [ADF17, Theorem 4.3.6]. Indeed, for \( p = 2 \), \( \Sigma_{G_m} \eta \) is a model for \( \alpha_1 \) by the discussion of Example 2.2.7. Composing \( \Sigma_{p^3} \eta \) and \( \tau \Sigma_{G_m} \eta \) gives a class that is mapped to \( \eta_\text{top}^2 \) under complex or étale realization. This composite is a morphism \( Q_5 \rightarrow Q_3 \) and the clutching construction of [ADF17, Theorem 4.3.6] yields a class in \([Q_6, BSL_2]_{A^1}\). The fact that the class in question remains non-trivial in \( X_3 \) follows immediately from compatibility with complex realization and the relevant vector bundle exists by [AHW17, Theorem 5.2.3].

The case of odd primes follows in an analogous fashion by replacing appeal to Proposition 2.2.1 with references to Lemma 2.2.10. More precisely, \( \tau \alpha^2_1 \) gives a \( p \)-torsion class in \( \pi^1_{2p-2,2p-1}(SL_2) \). By Lemma 2.2.10, \( \tau \alpha^2_1 \) can be viewed as a non-trivial class in \( \pi^1_{2p-2,2p-1}(SL_2) \). This element thus corresponds to an actual morphism \( Q_{4p-3} \rightarrow Q_3 \) by Theorem 2.2.13 and, by means of the clutching construction of [ADF17, Theorem 4.3.6] yields a rank 2 vector bundle on \( Q_{4p-2} \). The fact that the class in question remains non-trivial under the map \([Q_{4p-2}, BSL_2]_{A^1} \rightarrow [X_{2p-1}, BSL_2]_{A^1}\) follows immediately from compatibility with complex realization. Again, the relevant vector bundle on \( X_{2p-1} \) exists by [AHW17, Theorem 5.2.3].

**Remark 2.2.17.** By the Hartshorne–Serre correspondence (see, e.g., [Arr07, Theorem 1.1]), the rank 2 bundles constructed in Theorem 2.2.16 correspond to codimension 2 subvarieties of \( Q_{4p-2} \) that are local complete intersections but not complete intersections since the Rees bundles are not direct sums of line bundles.

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