On the effect of irrelevant boundary scaling operators

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We investigate consequences of adding irrelevant (or less relevant) boundary operators to a \((1+1)\)-dimensional field theory, using the Ising and the boundary sine-Gordon model as examples. In the integrable case, irrelevant perturbations are shown to multiply reflection matrices by CDD factors: the low-energy behavior is possible, while various high-energy behaviors are possible, including “roaming” RG trajectories. In the non-integrable case, a Monte Carlo study shows that behaviors are possible, including “roaming” RG trajectories.

In this paper we study the effect of adding irrelevant or less relevant scaling operators to a given theory, focusing on boundary perturbations in a \((1+1)\)-dimensional field theory. This is a problem of important physical interest. Consider for instance tunneling in the fractional quantum Hall effect, which has been intensely studied recently in the context of Luttinger liquids, shot noise and fractional charge measurements. For filling factor \(\nu = 1/3\), there is a single relevant tunneling operator, but there are two of them for \(\nu = 1/5\). Most analytical calculations, in particular the exact solutions, do hold only when the second, less relevant operator has been scaled away. In practice, however, this operator will always be there, and it is important to be able to evaluate its role.

The standard expectation is simply that one can neglect all less relevant and irrelevant operators when computing the properties close to the IR fixed point. This is, however, entirely based on weak coupling expansions, which can be quite misleading. For instance, the added perturbations could become relevant in strong coupling, entirely changing the physics in a non-perturbative way.

Our study supports the standard expectation in a variety of cases related with the tunneling problem. On the conceptual side, one of our main results is that, in the integrable case, additional irrelevant perturbations contribute so-called CDD (Castillejo-Dalitz-Dyson) factors to the boundary reflection matrices \(R\). This is not so surprising, as discussed in Ref.\textsuperscript{3}. \(R\) is fully determined up to such CDD factors by the boundary analogues of the Yang-Baxter equation, unitarity and crossing symmetry, and our results thus provide a natural explanation for the CDD ambiguity. Nevertheless, unexpected features are found close to the UV fixed point which can be altered by the added perturbations and is approached along a “roaming” renormalization group (RG) trajectory that oscillates between different boundary conditions. The relation between irrelevant perturbations and CDD factors has been discussed independently for the bulk case in Ref.\textsuperscript{3}.

To start, we consider the simplest possible situation, which is the scaling limit of the Ising model with zero external field but with a boundary magnetic field \(h\). This integrable model is described by a free Majorana fermion field theory. Since the boundary spin operator coincides with the free fermion operator (in sharp contrast to the bulk case), the problem can be handled by solving the boundary equation of motion, which in turn determines the boundary reflection matrices. It is then easy to include additional irrelevant operators: all perturbations that are quadratic in the fermions will still lead to a solvable model.

We illustrate this for a boundary perturbation made up of two terms, the spin operator and the stress energy tensor \(T = \pi : \psi \bar{\psi} \psi \bar{\psi} :\), where \(\bar{\partial} \equiv \partial_z\) and \(z = x + iy\). The Euclidean action is

\[
A = \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \left( \bar{\psi} \partial_y \psi - \bar{\psi} \partial \psi + m \bar{\psi} \psi \right) + \frac{1}{2} \int_{-\infty}^{\infty} dy \left( \bar{\psi} \psi (x = 0, y) + \frac{1}{i} \bar{\psi} \partial_y (x = 0, y) \right) + \frac{h}{2i} \int_{-\infty}^{\infty} dy (\bar{\psi} + \psi) (x = 0, y) + \frac{\lambda}{i} \int_{-\infty}^{\infty} dy (\bar{\psi} \partial_y \psi + \bar{\psi} \partial_y \bar{\psi}) (x = 0, y)
\]

where \(a\) is a fermionic boundary degree of freedom. The case \(\lambda = 0\) was studied in Ref.\textsuperscript{3}. When \(h = \lambda = 0\), the model is at the fixed point corresponding to free boundary conditions, or in the fermion language, \(\psi = \bar{\psi} = 0\). It follows that on the boundary, \(T = -i\pi : \psi \bar{\psi} \psi \bar{\psi} :\), and the last term in Eq.\textsuperscript{3} reads \((\lambda/\pi) \int dy (T + \bar{T})\).

Let us now determine \(R\). The equations of motion at the boundary are

\[
\frac{h^2}{2i} (\psi + \bar{\psi}) = \frac{d}{dy} (\psi - \bar{\psi}) - 2\lambda \frac{d^2}{dxdy} (\psi - \bar{\psi})
\]

The bulk theory in Eq.\textsuperscript{3} contains one type of particle, namely the free fermion \(A\) with mass \(m\). The fermion fields can be expressed in terms of the particle-creation operator \(A^\dagger (\theta)\).

\[
\psi(x, y) = \int_{-\infty}^{\infty} d\theta \left[ \omega e^{\theta/2} A(\theta) e^{imx \sinh \theta - my \cosh \theta} + \bar{\omega} e^{\theta/2} A^\dagger (\theta) e^{-imx \sinh \theta + my \cosh \theta} \right].
\]
\[ \psi(x, y) = \int_{-\infty}^{\infty} d\theta \left[ \omega e^{-\theta/2} A(\theta) e^{i\pi x \sinh \theta - my \cosh \theta} + \omega e^{-\theta/2} A(\theta) e^{-i\pi x \sinh \theta + my \cosh \theta} \right], \]

where \( \omega = \exp(i\pi/4), \bar{\omega} = \exp(-i\pi/4) \). Inserting this expansion into Eq. (2) and using the defining relation \( A(\theta) = R(\theta) A(\theta)^{-1} \), it is straightforward to obtain the reflection matrix. For simplicity, we focus on the massless limit, \( m \to 0 \). In that limit, we boost the rapides \( \theta \) such that the energy of the particles \( m \sinh \theta \to \exp(\beta) \), while the momenta \( m \sinh \theta \to \pm \exp(\beta) \) for right- or left-movers, respectively. Physical rapides are characterized by \( \exp(\beta) \approx T \), where \( T \) denotes the temperature.

The reflection matrix for rapides \( \beta \) then reads

\[ R(\beta) = -i \tanh \left( \frac{\beta - \beta_B}{2} - \frac{i\pi}{4} \right) \coth \left( \frac{\beta - \beta_B}{2} - \frac{i\pi}{4} \right), \]

(4)

where (the + sign corresponds to \( \beta_B^2 \))

\[ \exp(\beta_{B}^{1,2}) = \sqrt{1 + 4M^2} \pm \frac{1}{4\lambda}. \]

(5)

Seding the additional irrelevant coupling to zero, \( \lambda \to 0 \), we obtain \( \exp(\beta_B^1) \approx h^2/2 \), while \( \exp(\beta_B^2) \approx 1/2\lambda \) diverges. Hence only the first term of the \( R \) matrix (4) matters, while the second factor saturates at minus one: the boundary magnetic field \( h \) induces a flow from free to fixed boundary conditions, or, in the fermion language, from \( \psi = \bar{\psi} \) to \( \psi = -\bar{\psi} \). Close to the UV fixed point \( \beta \to \infty \), the reflection matrix is \( R_{\text{free}} = i \). Close to the IR fixed point \( \beta \to -\infty \), we instead get \( R_{\text{fixed}} = -i \).

Let us now discuss the meaning of Eq. (4). In the limit where both bare couplings are small, we have \( \exp(\beta_B^1) \ll \exp(\beta_B^2) \). The vicinity of the free fixed point now corresponds to temperatures \( \exp(\beta_B^1) \ll T \ll \exp(\beta_B^2) \). In that regime, the coth-factor in Eq. (4) is close to minus one, and Eq. (4) describes an Ising model with free boundary conditions perturbed by a relevant (\( \propto h \)) and an irrelevant (\( \propto \lambda \)) term. There are now two possibilities. If we look at higher energies, i.e., run the RG backwards, the contribution of the irrelevant term is seen to grow while the relevant term disappears. Equation (4) for the \( R \) matrix shows that in this limit the model flows to fixed boundary conditions, \( R \to -i \). Therefore the UV fixed point is not characterized by free boundary conditions anymore, as one might have expected. If instead we look at lower energies, the irrelevant term is scaled away while the relevant term grows. From Eq. (4) we see that the model again flows to fixed boundary conditions, \( R \to -i \). The result of running the RG forward (to lower energies) is very natural and confirms the general idea that irrelevant couplings should scale away. The result of running the RG backwards (to higher energies) is less natural, and its simplicity is probably connected to the fact that we are dealing with a free theory. In more complex cases, we do not expect that such a backward RG trajectory ever converges.

The structure of Eq. (4) generalizes to more complex mixtures of \( N \) perturbations that preserve integrability. Up to a factor \(-i\), the \( R \) matrix should thereby result as a product of \( N \) (minus) hyperbolic tangents with associated characteristic “Kondo” temperatures, \( \exp(\beta_B^1) \ll \exp(\beta_B^2) \ll \ldots \ll \exp(\beta_B^{N}) \). If one considers some physical property as a function of temperature, it will exhibit “non-universal” properties for \( T \geq \exp(\beta_B^{N}) \). For \( T \ll \exp(\beta_B^{N}) \), however, all the hyperbolic tangents but the first one are saturated, and then the reflection matrix becomes \( R \approx i \tanh \left( \frac{\beta - \beta_B}{2} - \frac{i\pi}{4} \right) \), identical to the case of a single relevant perturbation. Note that in these calculations, no cut-off has ever appeared. This is because of the implicit dimensional (normal-order) regularization used in integrable systems. In practice, the theory must often be regularized using a UV cut-off, whence a different renormalization of \( \beta_B \) by the irrelevant couplings will occur.

Running the RG backwards from physical temperatures in the range \( \exp(\beta_B^1) \ll T \ll \exp(\beta_B^2) \) gives intriguing results. If \( N \) is even, the model starts at free boundary conditions and ends up at fixed ones. The UV fixed point is approached by “roaming”, where the RG trajectory comes again \( N/2 - 1 \) times close to fixed boundary conditions. For instance, if \( N = 4 \), the backward RG trajectory reads: fixed \( \leftarrow \) free \( \leftarrow \) fixed \( \leftarrow \) free. If \( N \) is odd, the model ends up at free boundary conditions, and comes close to fixed boundary conditions \( (N-1)/2 \) times. For instance, if \( N = 3 \), the backward trajectory reads: free \( \leftarrow \) fixed \( \leftarrow \) free. Of course, in a given realistic situation, we expect the equivalent of \( N \) to be infinite, so the backwards trajectory will keep roaming forever as energies get higher and higher. In a cut-off regularized theory, such a behavior will only be meaningful up to energy scales given by the UV cut-off. In contrast, the forward trajectory always describes the same flow from free to fixed boundary conditions.

The above considerations for the Ising model can now be applied to the boundary sine-Gordon (BSG) model, which in turn is related to the problem of a quantum impurity in a Luttinger liquid. Indeed,

\[ A = \frac{1}{2} \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ (\partial_x \Phi)^2 + (\partial_y \Phi)^2 \right] + h \int_{-\infty}^{\infty} dy \cos[2\pi g \Phi(x = 0, y)]. \]

Here \( \Phi(x, y) \) is a folded bosonic field. Remarkably, the \( g = 1/2 \) BSG model can be mapped on a pair of Ising models. The first of these Ising models has fixed boundary conditions \( (h = \infty) \), while the other sees an ef-
in the previous paragraph, Eq. (4) yields terms of the underlying two Ising models. By means of the operators are irrelevant with scaling dimension 2. A general combination of this sort can now be rewritten in terms of the underlying two Ising models. Clearly, the reflection matrices are consistent with the Ising R matrix.

Suppose now that we add a general combination of the two irrelevant operators \( \cos[2\sqrt{2\pi} g \Phi(x = 0)] \) and \( \exp(\beta_B x) \). At \( g = 1/2 \) and \( h \to 0 \), both operators are irrelevant with scaling dimension 2. A general combination of this sort can now be rewritten in terms of the two Ising models. By means of bosonization, we obtain the equivalent combination \( \lambda : \psi_0 \psi : + \mu : \xi_0 \xi : \). By using the same reasoning as in the previous paragraph, Eq. (5) yields

\[
P + Q = i \coth \left( \frac{\beta - \beta_B}{2} - \frac{i \pi}{4} \right),
\]

\[
P - Q = -i \tanh \left( \frac{\beta - \beta_B}{2} - \frac{i \pi}{4} \right) \coth \left( \frac{\beta - \beta_B}{2} - \frac{i \pi}{4} \right),
\]

In the limit of small bare couplings, \( \exp(\beta_B) \propto h^2 \), \( \exp(\beta_B^2) \propto 1/\lambda \), \( \exp(\beta_B^3) \propto 1/\mu \), and therefore \( \beta_B \ll \beta_B^2, \beta_B^3 \). For temperatures \( T \ll \exp(\beta_B^2), \exp(\beta_B^3) \), the coth-factors are saturated, and we recover Eq. (6). For high temperatures, in contrast, one gets \( P + Q \to i \), \( P - Q \to -i \), corresponding to Neumann (free) boundary conditions. The situation (8) is thus similar to the case \( N = 3 \) for the Ising model. Notice however that in the special case \( \mu = 0 \), we always get \( P + Q = -i \), and the situation now looks like the case \( N = 2 \) for the Ising model, with identical Dirichlet (fixed) boundary conditions in the UV and IR. The Neumann boundary condition is generally found only in the temperature regime \( \exp(\beta_B^2) \ll T \ll \exp(\beta_B^3) \).

Following Ref. 10, the exact transport properties can be computed for arbitrary boundary interactions once the reflection matrices are known. The \( g = 1/2 \) linear conductance in units of \( ge^2/h \) is

\[
G = \int_{-\infty}^{\infty} d\beta \frac{1}{1 + e^\beta} d\beta |Q(\beta + \ln T)|^2,
\]

where \( Q \) is given by Eq. (8). This formula results from Ref. 10 after an integration by parts, and holds provided \( |Q|^2 \) is not a constant, so there is a flow indeed. Figure 1 shows plots for Eq. (9). In the absence of the irrelevant couplings, the conductance goes from the perfect value \( ge^2/h \) at high \( T \) (Neumann boundary conditions) down to \( G = 0 \) as \( T \to 0 \) (Dirichlet). With the irrelevant couplings, the situation is more complex. For \( \mu = 0 \), in the UV limit Dirichlet boundary conditions \( (G = 0) \) are approached again. However, for finite \( \mu \), the UV fixed point corresponds to Neumann boundary conditions, see Fig. 1.

Let us now discuss the effect of irrelevant operators in a cut-off regularized theory. We first consider the case of Ising models (8) and only add the first of the above two operators, \( \cos[2\sqrt{2\pi} g \Phi(x = 0)] \), with a coupling constant \( \lambda \). Its effect is studied by Monte Carlo simulation for the pinning function \( F(x) \) of the Friedel oscillation. The 2\( k_F \)-oscillatory charge screening cloud \( \rho(x) \) of a spinless Luttinger liquid around the impurity at position \( x = 0 \) is

\[
\rho(x) - \rho_0 \propto \cos(2k_F x + \eta_F(x/a)^\alpha F(x), \tag{10}
\]

where \( \rho_0 = k_F/\pi \) and \( \eta_F \) is a phase shift. In the absence of the irrelevant coupling, \( F \) depends on \( x/k_F \) only, where \( x = a(ha)^{-1/(1-\gamma)} \) is a lengthscale corresponding to \( \exp(\beta_B) \). For \( T = 0 \), energy scales are related to the distance \( x \) with \( x \gg x_B \) meaning low energies. In Eq. (10) we have introduced the lattice spacing \( a \) which serves as a high-energy cut-off. General arguments show that \( F(x) \to 1 \) as \( x \to \infty \). Figure 2 shows numerical results for the \( g = 1/2 \) pinning function. From the \( \lambda \neq 0 \) data, one sees that in the low-energy portion \( x \gg x_B \), the additional irrelevant operator can simply be incorporated by a rescaling of the lengthscale \( x_B \) or the associated rapidity \( \beta_B \) (such a rescaling could also have been used in the fits of figure 1). A good estimate for the observed size of the rescaling can be obtained from a self-consistent harmonic approximation. At high energy scales, \( x \ll x_B \), however, the curves do not scale on
The inclusion of additional irrelevant or less relevant terms is not restricted to $g = 1/2$. First, recall that a general consequence of integrability is the existence of a whole infinity of quantities in involution. A model perturbed by a combination of all these quantities is then also integrable, and the associated $R$ matrices should differ from the standard ones only by the inclusion of CDD factors, giving rise to results similar to the Ising ones. As for adding operators that destroy integrability, we have to resort to numerical simulations. We have studied the combination of the usual BSG term $\cos \left[ \sqrt{2\pi g} \Phi(x = 0) \right]$ and of $\cos \left[ 2\sqrt{2\pi g} \Phi(x = 0) \right]$, which is not integrable away from $g = 1/2$. In Fig. 3 we show data for $g = 1/5$, where the second operator is relevant (with scaling dimension 4/5), albeit of course less relevant than the first. We observe that at low energy scales ($x \gg x_B$), the coupling $\lambda$ has again no effect besides a renormalization of $x_B$. The scaling form of $F$ is altered only for $x \ll x_B$, giving rise to a picture that appears to be quite similar to the integrable one.

Our results should not lead to the belief that irrelevant operators are always of little importance. The integrable situation is quite special of course, since it does not even require the introduction of a cut-off. In non-integrable cases, the necessary cut-off might give rise to more complex behaviors. More importantly, the “roaming” RG trajectories we observed when approaching the UV fixed point in the Ising model should serve as a warning that there are situations where irrelevant operators lead to counter-intuitive behavior, sometimes even changing the nature of the IR fixed point. In the boundary sine-Gordon model, however, this seems never to be the case.

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