On the non-existence of simple congruences for quotients of Eisenstein series

by

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1. Introduction. Define $p(n)$ to be the number of ways of writing $n$ as a sum of non-increasing positive integers. Ramanujan famously established the congruences

$$p(5n + 4) \equiv 0 \mod 5,$$
$$p(7n + 5) \equiv 0 \mod 7,$$
$$p(11n + 6) \equiv 0 \mod 11,$$

and noted that there does not appear to be any other prime for which the partition function has equally simple congruences. Ahlgren and Boylan \[1\] build on the work of Kiming and Olsson \[5\] to prove that there truly are no other such primes. For large enough primes $l$, Sinick \[7\] and the author \[3\] prove the non-existence of simple congruences

$$a(ln + c) \equiv 0 \mod l$$

for wide classes of functions $a(n)$ related to the coefficients of modular forms. However, all of the modular forms studied in \[1, 7, 3\] are non-vanishing on the upper half-plane. Here we prove the non-existence of simple congruences (when $l$ is large enough) for ratios of Eisenstein series.

Let $\sigma_m(n) := \sum_{d|n} d^m$ and define the Bernoulli numbers $B_k$ by $t/(e^t - 1) = \sum_{k=0}^{\infty} B_k t^k/k!$. For even $k \geq 2$, set

$$E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Note that $E_2 \equiv E_4 \equiv E_6 \equiv 1$ modulo 2 and 3. Berndt and Yee \[2\] prove congruences for the quotients of Eisenstein series in Table \[1\] where $F(q) := \sum a(n) q^n$. An obviously necessary requirement for the congruences in the $n \equiv 2 \mod 3$ column of Table \[1\] is that there are simple congruences of the

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Table 1. Congruences of Berndt and Yee [2]

| \( F(q) \) | \( n \equiv 2 \text{ mod } 3 \) | \( n \equiv 4 \text{ mod } 8 \) |
|------------|-----------------|-----------------|
| \( 1/E_2 \) | \( a(n) \equiv 0 \text{ mod } 3^4 \) |                |
| \( 1/E_4 \) | \( a(n) \equiv 0 \text{ mod } 3^2 \) |                |
| \( 1/E_6 \) | \( a(n) \equiv 0 \text{ mod } 3^3 \) | \( a(n) \equiv 0 \text{ mod } 7^2 \) |
| \( E_2/E_4 \) | \( a(n) \equiv 0 \text{ mod } 3^3 \) |                |
| \( E_2/E_6 \) | \( a(n) \equiv 0 \text{ mod } 3^2 \) | \( a(n) \equiv 0 \text{ mod } 7^2 \) |
| \( E_4/E_6 \) | \( a(n) \equiv 0 \text{ mod } 3^3 \) |                |
| \( E_2^2/E_6 \) | \( a(n) \equiv 0 \text{ mod } 3^5 \) |                |

form \( a(3n + 2) \equiv 0 \text{ mod } 3 \). All but the first form in Table 1 are covered by the following theorem.

**Theorem 1.1.** Let \( r \geq 0 \) and \( s, t \in \mathbb{Z} \). If \( E_r^2 E_4^s E_6^t = \sum a(n)q^n \) has a simple congruence \( a(ln + c) \equiv 0 \text{ mod } l \) for the prime \( l \), then either \( l \leq 2r + 8|s| + 12|t| + 21 \) or \( r = s = t = 0 \).

This theorem gives an explicit upper bound on primes \( l \) for which there can be congruences of the form \( a(ln + c) \equiv 0 \text{ mod } l^k \) as in the middle column of Table 1.

**Remark 1.2.** See Remark 4.1 for a slight improvement of Theorem 1.1 in some cases.

**Example 1.3.** The form \( E_6/E_4^{12} \) can only have simple congruences for \( l \leq 129 \). Of these, the primes \( l = 2 \) and \( 3 \) are trivial with \( E_4 \equiv E_6 \equiv 1 \text{ mod } l \). For the remaining primes, the only congruences are

\[
\begin{align*}
a(ln + c) &\equiv 0 \text{ mod } 17, \\
\left( \frac{c}{17} \right) &\equiv -1.
\end{align*}
\]

Mahlburg [6] shows that for each of the forms in Table 1 except \( 1/E_2 \), there are infinitely many primes \( l \) such that for any \( i \geq 1 \), the set of \( n \) with \( a(n) \equiv 0 \text{ mod } l^i \) has arithmetic density 1. On the other hand, our result shows that (for large enough \( l \)) every arithmetic progression modulo \( l \) has at least one non-vanishing coefficient modulo \( l \).

Section 2 recalls certain definitions and tools from the theory of modular forms. Simple congruences are reinterpreted in terms of Tate cycles, which are reviewed in Section 3. Section 4 proves Theorem 1.1.

**2. Preliminaries.** A modular form of weight \( k \in \mathbb{Z} \) on \( \text{SL}_2(\mathbb{Z}) \) is a holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) which satisfies

\[
f\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau)
\]
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for every \((\begin{array}{cc} a & b \\ c & d \end{array}) \in \text{SL}_2(\mathbb{Z})\), and which is holomorphic at infinity. Modular forms have Fourier expansions in powers of \(q = e^{2\pi i \tau}\). For any prime \(l \geq 5\), let \(\mathbb{Z}_{(l)} = \{a/b \in \mathbb{Q} : l \nmid b\}\). We denote by \(M_k\) the set of all weight \(k\) modular forms on \(\text{SL}_2(\mathbb{Z})\) with \(l\)-integral Fourier coefficients. Although \(E_k\) is a modular form of weight \(k\) whenever \(k \geq 4\), \(E_2\) is called a quasi-modular form since it satisfies the slightly different transformation rule
\[
E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6ic}{\pi}(c\tau + d).
\]

**Definition 2.1.** If \(l\) is a prime, then a Laurent series \(f = \sum_{n \geq N} a(n)q^n \in \mathbb{Z}_{(l)}((q))\) has a simple congruence at \(c\) mod \(l\) if \(a(n + c) \equiv 0 \text{ mod } l\) for all \(n\).

**Lemma 2.2.** Suppose that \(l\) is prime and that \(f = \sum a(n)q^n\) and \(g = \sum b(n)q^n \in \mathbb{Z}_{(l)}((q))\) with \(g \not\equiv 0 \text{ mod } l\). The series \(f\) has a simple congruence at \(c\) mod \(l\) if and only if the series \(fg^l\) has a simple congruence at \(c\) mod \(l\).

**Proof.** It suffices to consider the reductions mod \(l\) of the series
\[
\left(\sum a(n)q^n\right)\left(\sum b(n)q^n\right) \equiv \sum_n \left(\sum_m b(m)a(n - lm)\right)q^n \text{ mod } l.
\]
If \(a(n)\) vanishes when \(n \equiv c\) mod \(l\), then the inner sum on the right hand side will also vanish for \(n \equiv c\) mod \(l\). The converse follows via multiplication by \((\sum b(n)q^n)^{-l}\) and repetition of this argument. ■

Our main tool is Ramanujan’s \(\Theta\) operator
\[
\Theta := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}.
\]
For any prime \(l\) and any Laurent series \(f = \sum a(n)q^n \in \mathbb{Z}_{(l)}((q))\), by Fermat’s Little Theorem
\[
\Theta^l f = \sum a(n)nlq^n \equiv \sum a(n)nlq^n = \Theta f \text{ mod } l.
\]
We call the sequence \(\Theta f, \ldots, \Theta^l f \text{ mod } l\) the Tate cycle of \(f\). Note that \(\Theta^{l-1} f \equiv f \text{ mod } l\) is equivalent to \(f\) having a simple congruence at 0 mod \(l\).

We now recall some facts about the reductions of modular forms mod \(l\). See Swinnerton-Dyer [8, Section 3] for the details on this paragraph. There are polynomials \(A(Q, R), B(Q, R) \in \mathbb{Z}_{(l)}[Q, R]\) such that
\[
A(E_4, E_6) = E_{l-1}, \quad B(E_4, E_6) = E_{l+1}.
\]
Reduce the coefficients of these polynomials modulo \(l\) to get \(\tilde{A}, \tilde{B} \in \mathbb{F}_l[Q, R]\). Then the polynomial \(\tilde{A}\) has no repeated factor and is prime to \(\tilde{B}\). Furthermore, the \(\mathbb{F}_l\)-algebra of reduced modular forms is naturally isomorphic to
\[
\frac{\mathbb{F}_l[Q, R]}{\tilde{A} - 1}
\]
via $Q \to E_4$ and $R \to E_6$. Whenever a power series $f$ is congruent to a modular form, define the filtration of $f$ by

$$\omega(f) := \inf \{k : f \equiv g \mod l \}.$$ 

If $f \in M_k$, then for some $g \in M_{k+l+1}$, $\Theta f \equiv g \mod l$. The next lemma also follows from [8, Section 3].

**Lemma 2.3.** Let $l \geq 5$ be prime, $f \in M_{k_1}$, $f \not\equiv 0 \mod l$ and $g \in M_{k_2}$.

1. If $f \equiv g \mod l$, then $k_1 \equiv k_2 \mod l - 1$.
2. $\omega(\Theta f) \leq \omega(f) + l + 1$ with equality if and only if $\omega(f) \not\equiv 0 \mod l$.
3. If $\omega(f) \equiv 0 \mod l$, then for some $s \geq 1$, $\omega(\Theta f) = \omega(f) + (l + 1) - s(l - 1)$.
4. $\omega(f^i) = i\omega(f)$.

The natural grading induced by (2.1) provides a key step in the following lemma which is taken from the proof of [5, Proposition 2].

**Lemma 2.4.** A form $f \in M_k$ with $\Theta f \not\equiv 0 \mod l$ has a simple congruence at $c \not\equiv 0 \mod l$ if and only if $\Theta^{(l+1)/2} f \equiv -\left(\frac{c}{l}\right) \Theta f \mod l$.

**Proof.** Since $\Theta$ satisfies the product rule, we have

$$\Theta^{l-1} (q^{-c} f) \equiv \sum_{i=0}^{l-1} \binom{l-1}{i} (-c)^{l-1-i} q^{-c} \Theta^i f \mod l \equiv \sum_{i=0}^{l-1} c^{l-1-i} q^{-c} \Theta^i f \mod l$$

$$\equiv c^{l-1} q^{-c} f + \sum_{i=1}^{l-1} c^{l-1-i} q^{-c} \Theta^i f \mod l.$$ 

A simple congruence for $f$ at $c \not\equiv 0 \mod l$ is equivalent to a simple congruence for $q^{-c} f$ at $0 \mod l$, which in turn is equivalent to $\Theta^{l-1} (q^{-c} f) \equiv q^{-c} f \mod l$. This is equivalent to $0 \equiv \sum_{i=1}^{l-1} c^{l-1-i} q^{-c} \Theta^i f \mod l$, by the computation above, and hence to $0 \equiv \sum_{i=1}^{l-1} c^{l-1-i} \Theta^i f \mod l$. By Lemma 2.3(2) and (3), for $1 \leq i \leq (l-1)/2$ we have

$$\omega(\Theta^i f) \equiv \omega(\Theta^{i+(l-1)/2} f) \equiv \omega(f) + 2i \mod l - 1.$$ 

By Lemma 2.3(1) and the natural grading (filtration modulo $l-1$), the only way for the given sum to be zero is if for all $1 \leq i \leq (l-1)/2$ we have

$$c^{l-1-i} \Theta^i f + c^{l-1-(i+(l-1)/2)} \Theta^{i+(l-1)/2} f \equiv 0 \mod l,$$ 

which happens if and only if

$$\Theta^{i+(l-1)/2} f \equiv -c^{(l-1)/2} \Theta^i f \equiv -\left(\frac{c}{l}\right) \Theta^i f \mod l,$$ 

which happens if and only if

$$\Theta^{(l+1)/2} f \equiv -\left(\frac{c}{l}\right) \Theta f \mod l.$$
Lemma 2.5. Let $a, b, c \geq 0$ be integers and let $l > 11$ be prime. Then
\[ \omega(E_{l+1}^a E_4^b E_6^c) = al + a + 4b + 6c. \]

Proof. Since $E_{l+1}^a E_4^b E_6^c \in M_{al+a+4b+6c}$, it suffices to show that $\tilde{A}(Q, R)$ does
not divide $\tilde{B}(Q, R)^a Q^b R^c$. However $\tilde{A}$ has no repeated factors and is prime to $\tilde{B}$ and so it suffices to show
that $\tilde{A}$ does not divide $QR$. But $QR$ has weight 10 and $E_{l-1}$ has weight $l - 1 > 10$ so this is impossible. □

3. The structure of Tate cycles. The framework we use below follows Jochnowitz [4]. Let $f \in M_k$ be such that $\Theta f \not\equiv 0 \mod l$. Recall from Section 2 that the Tate cycle of $f$ is the sequence $\Theta f, \ldots, \Theta^{l-1} f \mod l$. With $s \geq 1$ as in (3) of Lemma 2.3, we have
\[ \omega(\Theta^{i+1} f) \equiv \begin{cases} \omega(\Theta^i f) + 1 \mod l & \text{if } \omega(\Theta^i f) \not\equiv 0 \mod l, \\ s + 1 \mod l & \text{if } \omega(\Theta^i f) \equiv 0 \mod l. \end{cases} \]

In particular, when $\omega(\Theta^i f) \equiv 0 \mod l$, the quantity $s$ which determines the change in filtration also controls the time until the next occurrence of $\omega(\Theta^i f) \equiv 0 \mod l$. We say that $\Theta^i f$ is a high point of the Tate cycle and $\Theta^{i+1} f$ is a low point of the Tate cycle whenever $\omega(\Theta^i f) \equiv 0 \mod l$. Elementary considerations (see, for example, [4, Section 7] or [3, Section 3]) yield

Lemma 3.1. Let $f \in M_k$ with $\Theta f \not\equiv 0 \mod l$.

(1) If the Tate cycle has only one low point, then the low point has filtration 2 mod $l$.
(2) The Tate cycle has one or two low points.

Lemma 3.2. Suppose $f \in M_k$ has a simple congruence at $c \not\equiv 0 \mod l$, where $l \geq 5$ is prime, and $\Theta f \not\equiv 0 \mod l$. Then the Tate cycle of $f$ has two low points. Furthermore, if $\Theta^i f$ is a high point, then
\[ \omega(\Theta^{i+1} f) = \omega(\Theta^i f) + (l + 1) - \left(\frac{l+1}{2}\right)(l-1) \equiv \frac{l+3}{2} \mod l. \]

Proof. By Lemma 2.4, $\omega(\Theta f) = \omega(\Theta^{(l+1)/2} f)$. Hence, the filtration is not monotonically increasing between $\Theta f$ and $\Theta^{(l+1)/2} f$, so there must be a fall in filtration (and hence a low point) somewhere in the first half of the Tate cycle. We also have $\omega(\Theta^{(l+1)/2} f) = \omega(\Theta f) = \omega(\Theta^l f)$ and so by the same reasoning there must be a low point somewhere in the second half of the Tate cycle. By Lemma 3.1, there are exactly two low points in the Tate cycle. Lemma 2.3(2) and (3) give
\[ \omega(\Theta f) = \omega(\Theta^{(l+1)/2} f) = \omega(\Theta f) + \left(\frac{l-1}{2}\right)(l+1) - s(l-1) \]
for some $s \geq 1$. Hence $s = (l + 1)/2$. By the same reasoning, the fall in filtration for the second half of the Tate cycle must also have $s = (l + 1)/2$. The lemma follows.

The proof of Theorem 1.1 uses the above lemma to determine how far the filtration falls, and the bounds of the next lemma to show a corresponding restriction on $l$.

**Lemma 3.3.** Let $l \geq 5$ be prime and suppose $f \in M_k$ has a simple congruence at $c \not\equiv 0 \mod l$. If $\omega(f) = Al + B$ where $1 \leq B \leq l - 1$, then

$$\frac{l + 1}{2} \leq B \leq A + \frac{l + 3}{2}.$$

**Proof.** Since $B \neq 0$, $\omega(\Theta f) = (A + 1)l + (B + 1)$. From the proof of Lemma 3.2, the Tate cycle has a high point before $\Theta^{(l+1)/2} f$. By Lemma 3.2, the high point is $\Theta^i f$ with $1 \leq i \leq (l - 1)/2$. Hence we have

$$\omega(\Theta^i f) = Al + B + i(l + 1) \equiv B + i \equiv 0 \mod l.$$

Together with the restrictions on $B$ and $i$, this congruence implies that $B + i = l$ and $B \geq (l + 1)/2$. Also, by Lemma 2.3 the high point has filtration

$$\omega(\Theta^{l-B} f) = \omega(f) + (l - B)(l + 1) = (A + l - B + 1)l.$$

Lemma 3.2 implies that the corresponding low point has filtration

$$\omega(\Theta^{l-B+1} f) = \left(A - B + \frac{l + 3}{2}\right)l + \left(\frac{l + 3}{2}\right).$$

The fact that $\omega(\Theta^{l-B+1} f) \geq 0$ implies the second inequality.

If $\Theta f \equiv 0 \mod l$, then the Tate cycle is trivial and the above lemmas are not applicable. We dispense with this case now.

**Lemma 3.4.** Let $f = E_2^r E_4^s E_6^t$ where $r \geq 0$ and $s, t \in \mathbb{Z}$. If $l$ is a prime such that $\Theta f \equiv 0 \mod l$, then either $l \leq 13$ or $r \equiv s \equiv t \equiv 0 \mod l$.

**Example 3.5.** We have $\Theta(E_4 E_6) \equiv 0 \mod l$ for $l = 2, 3, 11$.

**Example 3.6.** We have $\Theta(E_2^{14} E_4^{-15} E_6^{-14}) \equiv 0 \mod l$ for $l = 2, 3, 5, 7, 13$.

Note that $\Theta f \equiv 0 \mod l$ is equivalent to $f$ having simple congruences at all $c \not\equiv 0 \mod l$.

**Proof of Lemma 3.4.** Assume $l \geq 17$ and expand $f$ as a power series to get

$$f = 1 + (-24r + 240s - 504t)q$$
$$+ (288r^2 - 5760rs + 12096rt - 360r + 28800s^2$$
$$- 120960st - 26640s + 127008t^2 - 143640t)s^2 + \cdots.$$
If \( \Theta f \equiv 0 \mod l \), then the coefficients of \( q \) and \( q^2 \) vanish modulo \( l \). That is,

\[
-24r + 240s - 504t \equiv 0 \mod l,
\]
and

\[
288r^2 - 5760rs + 12096rt - 360r + 28800s^2
\]

\[
-120960st - 26640s + 127008t^2 - 143640t \equiv 0 \mod l.
\]

Furthermore, by Lemmas 2.3(2) and 2.5 and the fact that \( E_2 \equiv E_{l+1} \mod l \), we have

\[
\omega(E_{l+1}^rE_4^sE_6^t) \equiv r + 4s + 6t \equiv 0 \mod l.
\]

Solving the system of congruences given by (3.3) and (3.1) yields

\[
7r \equiv -72t \mod l,
\]
and

\[
14s \equiv 15t \mod l.
\]
Substituting (3.4) and (3.5) into 49 times (3.2) yields

\[
-8255520t \equiv 0 \mod l.
\]

Since \( 8255520 = 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13 \), the lemma follows.

4. Proof of Theorem 1.1
We begin with the trivial observation that

\[
E_2^rE_4^sE_6^t = 1 + \cdots
\]
does not have a simple congruence at \( 0 \mod l \). Hence, we assume that \( E_2^rE_4^sE_6^t \) has a simple congruence at \( c \not\equiv 0 \mod l \), where \( l \geq 5 \). Since \( E_2 \equiv E_{l+1} \mod l \), \( E_{l+1}^rE_4^sE_6^t \) has a simple congruence at \( c \mod l \).

Recall that our goal is to show \( l \leq 2r + 8|s| + 12|t| + 21 \). Hence, if \( l < |s| \) or \( l < |t| \) then we are done. Thus we assume \( l + s \geq 0 \) and \( l + t \geq 0 \). We also assume \( l > 11 \). Lemma 3.4 allows us to take \( \Theta(E_2^rE_4^sE_6^t) \not\equiv 0 \mod l \) (otherwise we are done). By Lemma 2.2 we see that

\[
E_{l+1}^rE_4^sE_6^{l+t} \in M(r+10)_{l+(r+4s+6t)}
\]

has a simple congruence at \( c \mod l \). We work with this multiplied form \( E_{l+1}^rE_4^sE_6^{l+t} \) because it is holomorphic (with positive weight) and so our filtration apparatus is applicable. By Lemma 2.5

\[
\omega(E_{l+1}^rE_4^sE_6^{l+t}) = (r + 10)l + (r + 4s + 6t).
\]

We break into four cases depending on the size of \( r + 4s + 6t \):

1. If \( l \leq |r + 4s + 6t| \) then we are done.
2. If \( 0 < r + 4s + 6t < l \) then by equation (4.1) and the first inequality of Lemma 3.3 \( (l + 1)/2 \leq r + 4s + 6t \) and we are done.
3. If \( r + 4s + 6t = 0 \), then by Lemma 2.3

\[
\omega(\Theta E_{l+1}^rE_4^sE_6^{l+t}) = (r + 11)l + 1 - s'(l - 1)
\]
for some \( s' \geq 1 \). If \( l \leq r + 13 \) then we are done, so it suffices to consider \( l > r + 13 \). Now in order for the filtration above to be non-negative, we must
have $s' \leq r + 11$. Now $\omega(\Theta E_{l+1}^r E_4^{l+s} E_6^{l+t}) \equiv s' + 1 \mod l$. By Lemma 2.4, there must be a high point of the Tate cycle before $\Theta^{(l+1)/2} E_{l+1}^r E_4^{l+s} E_6^{l+t}$. Let $i$ be the index of the first high point, so $1 \leq i \leq (l - 1)/2$. Then

$$\omega(\Theta^i E_{l+1}^r E_4^{l+s} E_6^{l+t}) \equiv s' + i \equiv 0 \mod l.$$  

Together with the restrictions on $i$ and $s'$ (namely $s' \leq r + 11 < r + 13 < l$), this congruence implies that

$$s' \geq \frac{l + 1}{2}.$$  

That is, $l \leq 2s' - 1 \leq 2r + 21$ and we are done.

(4) If $-l < r + 4s + 6t < 0$, then take $B = l + r + 4s + 6t$ and $A = r + 9$. Equation (4.1) and the second inequality of Lemma 3.3 give

$$l + r + 4s + 6t \leq r + 9 + \frac{l + 3}{2},$$

which is equivalent to $l \leq 21 - 8s - 12t$ and we are done. ■

**Remark 4.1.** Combining these four cases and recalling that the proof assumed $l + s \geq 0$, $l + t \geq 0$ and $l > 11$, we see that if $r + 4s + 6t > 0$, then

$$l \leq \max\{|s| - 1, |t| - 1, 11, 2r + 8s + 12t - 1\},$$

and if $r + 4s + 6t \leq 0$, then

$$l \leq \max\{|s| - 1, |t| - 1, 11, 21 - 8s - 12t\}.$$

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