Global $L^2$-boundedness of a New Class of Rough Fourier Integral Operators

Jiawei Dai and Qiang Huang*

Abstract. In this paper, we investigate the $L^2$ boundedness of Fourier integral operator $T_{\phi,a}$ with rough symbol $a \in L^{\infty} S^m$ and rough phase $\phi \in L^{\infty} \Phi^2$ which satisfies $|\{x : |\nabla_x \phi(x, \xi) - y| \leq r\}| \leq C(r^{n-1} + r^n)$ for any $\xi, y \in \mathbb{R}^n$ and $r > 0$. We obtain that $T_{\phi,a}$ is bounded on $L^2$ if $m < \rho (n - 1)/2 - n/2$ when $0 \leq \rho \leq 1/2$ or $m < -(n + 1)/4$ when $1/2 \leq \rho \leq 1$. When $\rho = 0$ or $n = 1$, the condition of $m$ is sharp. Moreover, the maximal wave operator is a special class of $T_{\phi,a}$ which is studied in this paper. Thus, our main theorem substantially extends and improves some known results about the maximal wave operator.

1. Introduction and main results

A Fourier integral operator (FIO) is defined as

$$T_{\phi,a} f(x) = \int_{\mathbb{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{f}(\xi) d\xi,$$

where $a$ is the symbol and $\phi$ is the phase function, and $\hat{f}$ denotes the Fourier transform of $f$. As we can see, all pseudo-differential operators are of this form with $\phi(x, \xi) = x \cdot \xi$.

In the study of FIOs, one usually assume the symbol $a(x, \xi)$ belongs to Hörmander class $S^m_{\rho,\delta}$ and the phase function $\phi$ is in the class $\Phi^2$ satisfying the strong non-degeneracy condition.

Definition 1.1. Let $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$. A function $a \in S^m_{\rho,\delta}$, if $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies

$$\sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{-m-\rho|\alpha|+\delta|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty$$

for all multi-indices $\alpha, \beta$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Received December 9, 2021; Accepted April 10, 2022.

Communicated by Sanghyuk Lee.

2020 Mathematics Subject Classification. 35S30, 42B37.

Key words and phrases. rough Fourier integral operator, $L^2$ boundedness, maximal wave operator.

This work was supported by Scientific Research Fund of Zhejiang Provincial Education Department (No. Y201738640) and the National Natural Science Foundation of China (No. 11801518).

*Corresponding author.
Definition 1.2. A real-valued function \( \phi(x, \xi) \in \Phi^2 \), if \( \phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \), is homogeneous of order 1 in the frequency variable \( \xi \) and

\[
\sup_{(x,\xi)\in\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}} |\xi|^{-1+|\alpha|} |\partial^\alpha_x \partial_\xi^\beta \phi(x, \xi)| < \infty
\]

for any \(|\alpha| + |\beta| \geq 2\).

Definition 1.3 (Strong non-degeneracy condition). A real-valued function \( \phi \in C^2(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \) satisfies the strong non-degeneracy condition, if there exists a constant \( c > 0 \) such that

\[
\left| \det \left( \frac{\partial^2 \phi(x, \xi)}{\partial x_j \partial \xi_k} \right) \right| \geq c \quad \text{for all} \ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.
\]

Obviously, when \( x \) has compact support, if \( \phi \in \Phi^2 \) and the mixed Hessian matrix \( \det \left( \frac{\partial^2 \phi}{\partial x_j \partial \xi_k} \right) \neq 0 \), then \( \phi \) satisfies the strong non-degeneracy condition.

The local \( L^2 \) boundedness of FIOs with \( \phi \in \Phi^2 \) and satisfying the determinant of the mixed Hessian matrix is non-zero on the support of the symbol was firstly investigated by Eskin [9] for \( a \in S^0_{1,0} \) and by Hörmander [13] for \( a \in S^0_{\rho,1 - \rho} \), \( 1/2 < \rho \leq 1 \). Later on, Beals [2] and Greenleaf–Uhlmann [11] extended Hörmander’s result to the case of \( a \in S^0_{1/2,1/2} \). Meanwhile, there were many studies on the global \( L^2 \) boundedness of FIOs, such as Fujiwara [10] and Asada–Fujiwara [1]. Recently, Dos Santos Ferreira and Staubach [8] established the global \( L^2 \) boundedness with \( a \in S^m_{\rho,\delta} \), \( 0 \leq \rho \leq 1 \), \( 0 \leq \delta < 1 \) and \( m \leq \min\{0, n(\rho - \delta)/2\} \).

For the \( L^p \) boundedness of FIOs, Seeger–Sogge–Stein [18] proved the local \( H^1 - L^1 \) boundedness when \( a \in S^{(1-n)/2}_{1,0} \) by using the well-known “dyadic-parabolic” decomposition. Moreover, they got the local \( L^p \)-boundedness when \( a \in S^m_{1,0} \), \( m = (1-n)|1/p - 1/2| \) and the condition of \( m \) is sharp. Later on, Ruzhansky and Sugimoto [17] proved the global \( L^p \) boundedness of FIOs with \( a \in S^m_{\rho,\delta} \), \( m = (1-n)|1/p - 1/2| \). In [3], Castro, Israelsson and Staubach established the global \( L^p \) boundedness of FIOs with \( a \in S^m_{\rho,\delta} \), \( 0 \leq \rho \leq 1 \), \( 0 \leq \delta < 1 \), \( m = -(n - \rho)|1/p - 1/2| - n \max\{0, (\delta - \rho)/2\} \) or \( a \in S^m_{\rho,1} \), \( 0 \leq \rho \leq 1 \), \( m < -(n(1 - \rho) \max(1/p, 1/2) - (n - 1)|1/p - 1/2| \). Besides, there are many results about local and global \( L^p \) boundedness of FIOs, such as [4, 6, 8, 15].

In [14], Kenig and Staubach introduced a class of pseudo-differential operators with the symbol belongs to rough Hörmander class was denoted by \( L^\infty S^m_{\rho} \), and proved the sharp \( L^2 \)-boundedness of this class of pseudo-differential operators. The specific definition of \( L^\infty S^m_{\rho} \) and the result are as follows.

Definition 1.4. Let \( m \in \mathbb{R} \) and \( 0 \leq \rho \leq 1 \). A function \( a \) belongs to the rough Hörmander class \( L^\infty S^m_{\rho} \), if it satisfies

\[
\sup_{\xi \in \mathbb{R}^n} (\xi)^{-m+\rho|\alpha|} \| \partial_\xi^\alpha a(\cdot, \xi) \|_{L^\infty} < \infty \quad \text{for all multi index} \ \alpha.
\]
Theorem 1.5. [14, Proposition 2.3] When \( a \in L^\infty S^m_\rho \), \( 0 \leq \rho \leq 1 \), then the pseudodifferential operator \( T_a \) is bounded on \( L^2 \) if and only if \( m < \frac{n}{2} (\rho - 1) \).

Inspired by the work of Kenig and Staubach [14], Dos Santos Ferreira and Staubach [8] defined the rough phase class \( L^\infty \Phi^2 \) which behaves like an \( L^\infty \) function in the spatial variable \( x \) and the rough non-degeneracy condition. The specific definitions are as follows.

Definition 1.6. A real-valued function \( \phi \) belongs to the rough phase class \( L^\infty \Phi^2 \), if \( \phi \) is homogeneous of degree 1 in the frequency variable \( \xi \) and satisfies

\[
\sup_{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}} |\xi|^{k-1} \| \nabla^k_\xi \phi(\cdot, \xi) \|_{L^\infty} < \infty \quad \text{for all } k \geq 2.
\]

Definition 1.7 (Rough non-degeneracy condition). A real valued phase \( \phi \) satisfies the rough non-degeneracy condition, if there exists a constant \( c > 0 \) such that

\[
|\nabla_\xi \phi(x, \xi) - \nabla_\xi \phi(y, \xi)| \geq c|x - y|
\]

for any \( x, y \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^n \setminus \{0\} \).

In [8], Dos Santos Ferreira and Staubach established various \( L^p \) boundedness of FIOs with \( a \in L^\infty S^m_\rho \) and \( \phi \in L^\infty \Phi^2 \) satisfying the rough non-degeneracy condition. Here, we would like to mention the \( L^2 \) boundedness of rough FIOs.

Theorem 1.8. [8, Theorem 2.8] When \( a \in L^\infty S^m_\rho \) and \( \phi \in L^\infty \Phi^2 \) satisfying the rough non-degeneracy condition, \( T_{\phi, a} \) is bounded on \( L^2 \) if

\[
m < n \left( \frac{\rho - 1}{2} - \frac{n - 1}{4} \right).
\]

On the other hand, the wave operator defined as

\[
e^{it\sqrt{-\Delta}} f(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|)} \hat{f}(\xi) d\xi
\]

which is a special class of FIO with \( a(x, \xi) = 1 \). It is well-known that for all \( f \in H^s \), if \( s > 1/2 \), \( e^{it\sqrt{-\Delta}} f \) converges to \( f \) almost everywhere as \( t \to 0 \) (see [7]) and if \( s \leq 1/2 \) the convergence fails (see [12]). The convergence is due to the following estimate of the maximal wave operator

\[
\left\| \sup_{0 < t < 1} |e^{it\sqrt{-\Delta}} f| \right\|_{L^2} \leq C \| f \|_{H^s}
\]

for \( s > 1/2 \).

By the definition of Sobolev space, we can see that (1.1) is equivalent to \( \| T g \|_{L^2} \leq C \| g \|_2 \), where

\[
T_g(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|)} (1 + |\xi|^2)^{-s/2} \hat{g}(\xi) d\xi
\]
and $t(x) \in L^\infty$, $\hat{g}(\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)$. Moreover, it is easy to prove that $(1 + |.|^2)^{-s/2} \in L^\infty S^1_{\rho_1} \subseteq L^\infty S^1_{\rho_2}$ and $x \cdot \xi + t(x)|\xi| \in L^\infty \Phi^2$ but does not satisfy the rough non-degeneracy condition. Motivated by these, we consider the $L^2$ boundedness of a class of FIOs which is generalized of (1.2). The following theorem is our main result in this paper.

**Theorem 1.9.** Let $a \in L^\infty S^m_{\rho}$ and $\phi \in L^\infty \Phi$ satisfying
\[(1.3) \quad \{x : |\nabla_\xi \phi(x, \xi) - y| \leq r\} \leq C (r^{n-1} + r^n)\]
for any $\xi, y \in \mathbb{R}^n$ and $r > 0$. Then $T_{\phi,a}$ is bounded on $L^2$ if $m < \rho (n - 1)/2 - n/2$ when $0 \leq \rho \leq 1/2$ or $m < -(n + 1)/4$ when $1/2 \leq \rho \leq 1$.

**Remark 1.10.** The reason why we replace the rough non-degeneracy condition by the condition (1.3) is that for all $t(x) \in L^\infty$, by some direct computations, we can get that $\phi(x, \xi) = x \cdot \xi + t(x)|\xi|$ does not satisfy rough non-degeneracy condition but satisfies (1.3). Moreover, we can prove that the strong non-degeneracy condition or rough non-degeneracy condition implies (1.3). So, our result extends the existing results substantially. Now, We show the proof of this conclusion below.

**Proof.** Since the rough non-degeneracy condition implies the strong non-degeneracy condition (see [8, Proposition 1.11]), we only need to prove the strong non-degeneracy condition implies (1.3). For this purpose, we consider the map $F_\xi : x \mapsto \nabla_\xi \phi(x, \xi)$. Since $\phi$ satisfies the strong non-degeneracy condition, setting $z = \nabla_\xi \phi(x, \xi)$ and by the inverse theorem, we have
\[
|x : |\nabla_\xi \phi(x, \xi) - y| \leq r\} = \int_{\{x : |\nabla_\xi \phi(x, \xi) - y| \leq r\}} dx = \int_{\{z : |z - y| \leq r\}} d(F_\xi^{-1}(z)) \\
\leq \int_{\{z : |z - y| \leq r\}} \left| \frac{\partial^2 \phi(x, \xi)}{\partial x_j \partial \xi_k} \right|^{-1} dz \leq C \int_{\{z : |z - y| \leq r\}} dz \\
\leq C r^n \leq C (r^n + r^{n-1}).
\]

**Remark 1.11.** According to [12,16], when $\rho = 0$ or $n = 1$, the bound on $m$ is sharp.

Throughout the paper, we use $C$, $c$ to denote some positive constants that are independent of $x$, $\xi$, $f$ and may vary from line to line. We denote by $B_r$ the ball in $\mathbb{R}^n$ with center 0 and radius $r$.

2. Proof of Theorem 1.9

Before proving the main theorem, we need the following two lemmas for the low frequency of $T_{\phi,a}$.
Lemma 2.1. [8, Lemma 1.17] Suppose that \( u \in C_\infty_c(\mathbb{B}_1) \) and satisfies that
\[
|\nabla^k u(x)| \leq C_k |x|^{1-k} \quad \text{for all } k \in \mathbb{N}^+,
\]
then for any \( 0 \leq \mu < 1 \), we have
\[
\left| \int_{\mathbb{R}^n} e^{ix \cdot y} u(x) \, dx \right| \leq C \langle y \rangle^{-n-\mu}.
\]

Lemma 2.2. Suppose \( a \) and \( \phi \) satisfy the assumptions of Theorem 1.9 then for any \( \eta \in C_\infty_c(\mathbb{B}_1) \), the following operator
\[
S_{0,\phi,a} f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) \eta(\xi) f(\xi) \, d\xi
\]
is bounded on \( L^2 \).

Proof. By standard dual argument, we have \( \|S_{0,\phi,a} \|^2_{L^2 \rightarrow L^2} = \|S_{0,\phi,a} S_{0,\phi,a}^* \|_{L^2 \rightarrow L^2} \), where \( S_{0,\phi,a} S_{0,\phi,a}^* f(x) = \int_{\mathbb{R}^n} k_0(x,y) f(y) \, dy \) and
\[
k_0(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\xi) - \phi(y,\xi))} a(x,\xi) a(y,\xi) \eta^2(\xi) \, d\xi.
\]
By Schur’s theorem, to prove the \( L^2 \) boundedness of \( S_{0,\phi,a} S_{0,\phi,a}^* \), it suffices to show that
\[
\sup_y \int_{\mathbb{R}^n} |k_0(x,y)| \, dx < \infty \quad \text{and} \quad \sup_x \int_{\mathbb{R}^n} |k_0(x,y)| \, dy < \infty.
\]

By choosing some \( \xi_0 \in S^{n-1} \) and setting \( h_x(\xi) = \phi(x,\xi) - \nabla_\xi \phi(x,\xi_0) \cdot \xi \), \( h_y(\xi) = \phi(y,\xi) - \nabla_\xi \phi(y,\xi_0) \cdot \xi \), we have
\[
k_0(x,y) = \int_{\mathbb{R}^n} e^{i(\nabla_\xi \phi(x,\xi_0) - \nabla_\xi \phi(y,\xi_0) \cdot \xi)} e^{i(h_x(\xi) - h_y(\xi))} a(x,\xi) a(y,\xi) \eta^2(\xi) \, d\xi.
\]

We claim that \( h_x \) satisfies the following estimate
\[
(2.1) \quad \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{-1+k} |\nabla_\xi^k h_x(\xi)| < \infty \quad \text{for all } k \geq 1.
\]
Indeed, since \( \phi \in L^\infty \Phi^2 \), using the mean value theorem, we have
\[
|\nabla_\xi h_x(\xi)| = |\nabla_\xi \phi(x,\xi) - \nabla_\xi \phi(x,\xi_0)| = |\nabla_\xi \phi(x,\xi/|\xi|) - \nabla_\xi \phi(x,\xi_0)| < \infty.
\]
When \( k \geq 2 \), we have
\[
|\nabla_\xi^k h_x(\xi)| = |\nabla_\xi^k \phi(x,\xi)| \leq C |\xi|^{1-k}
\]
as desired. Similarly, \( h_y(\xi) \) has the same estimate.
Applying (2.1) and the fact $a \in L^\infty S^m_\rho$, we can get

\[
\left| \nabla^k \left( e^{i(h_x(\xi) - h_y(\xi))} a(x, \xi) \bar{\sigma}(y, \xi) \eta^2(\xi) \right) \right| \\
\leq C_k \sum_{k_1 + k_2 + k_3 = k} \left| \nabla^{k_1} e^{i(h_x(\xi) - h_y(\xi))} \left| \nabla^{k_2} (a(x, \xi) \bar{\sigma}(y, \xi)) \right| \nabla^{k_3} \eta^2(\xi) \right| \\
\leq C_k \sum_{k_1 \leq k} \left| \nabla^{k_1} e^{i(h_x(\xi) - h_y(\xi))} \right| \\
\leq C_k \sum_{k_1 \leq k} \sum_{s=1}^{k_1} \sum_{t_1 + \cdots + t_s = k_1} \left| \nabla^{t_1} (h_x(\xi) - h_y(\xi)) \cdots \nabla^{t_s} (h_x(\xi) - h_y(\xi)) \right| \\
\leq C_k \sum_{k_1 \leq k} \sum_{s=1}^{k_1} \sum_{t_1 + \cdots + t_s = k_1} |\xi|^{1-t_1} \cdots |\xi|^{1-t_s} \\
\leq C_k \sum_{k_1 \leq k} \sum_{s=1}^{k_1} \sum_{t_1 + \cdots + t_s = k_1} |\xi|^{s-t_1-\cdots-t_s} \\
\leq C_k |\xi|^{1-k}.
\]

Then by Lemma 2.1, for any $y \in \mathbb{R}^n$ and $0 \leq \mu < 1$, we have

\[
\int_{\mathbb{R}^n} |k_0(x, y)| \, dx \leq C \int (1 + |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)|^2)^{-(n+\mu)/2} \\
\leq C \left( \int_{\{x : |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)| < 1\}} + \int_{\{x : |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)| \geq 1\}} \right) \\
(1 + |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)|^2)^{-(n+\mu)/2} \\
= I + II.
\]

For $I$, by (1.3), we have

\[
I \leq \{x : |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)| < 1\} < \infty.
\]

For $II$, we have

\[
II = \sum_{s=1}^{\infty} \int_{\{x : 2^{s-1} \leq |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)| < 2^s\}} (1 + |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)|^2)^{-(n+\mu)/2} \\
\leq C \sum_{s=1}^{\infty} 2^{-(s-1)(n+\mu)} \left\{x : |\nabla^\infty \phi(x, \xi_0) - \nabla^\infty \phi(y, \xi_0)| < 2^s\right\} \\
\leq C \sum_{s=1}^{\infty} 2^{-(s-1)(n+\mu)} (2^{s(n-1)} + 2^{sn}) < \infty.
\]

By the same method, we can also prove $\sup_y \int_{\mathbb{R}^n} |k_0(x, y)| \, dy < \infty$. Then it follows that $S_{0, \phi, a}$ is bounded on $L^2$. \qed
Now we turn to prove Theorem 1.9.

**Proof of Theorem 1.9.** First, we write $T_{\phi,a}$ as $T_{\phi,a} = S_{\phi,a} F$, where

$$S_{\phi,a} f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) f(\xi) \, d\xi$$

and $F(f) = \hat{f}$. By Plancherel’s theorem, it is enough to prove the $L^2$ boundedness of $S_{\phi,a}$.

Decomposing $S_{\phi,a}$ as

$$S_{\phi,a} f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) \chi_0(\xi) f(\xi) \, d\xi + \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) (1 - \chi_0(\xi)) f(\xi) \, d\xi$$

where $\chi_0 \in C_0^\infty(B_2)$ and $\chi_0 = 1$ in $B_1$.

We can get the $L^2$ boundedness of $S_{0,\phi,a} f$ directly from Lemma 2.2. So, it remains to prove the $L^2$ boundedness of $S_{1,\phi,a} f$. By standard dual argument, we only need to prove $L^2$ boundedness of $S_{1,\phi,a} S_{1,\phi,a}^*$, where $S_{1,\phi,a} S_{1,\phi,a}^* f(x) = \int_{\mathbb{R}^n} k_1(x,y) f(y) \, dy$ and

$$k_1(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\xi) - \phi(y,\xi))} a(x,\xi) \overline{a}(x,\xi) (1 - \chi_0(\xi))^2 \, d\xi.$$

By the well-known Littlewood–Paley decomposition, we can obtain that $(1 - \chi_0(\xi))^2 = \sum_{j=1}^{\infty} \chi_j(\xi)$, where

$$\chi_j \in C_0^\infty(B_{2^{j+1}} \setminus B_{2^{j-1}}), \quad |\nabla^k_x \chi_j(\xi)| \leq C_k 2^{-jk} \quad \text{for all } k \in \mathbb{N}.$$

Then $k_1(x,y)$ can be decomposed as

$$k_1(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\xi) - \phi(y,\xi))} a(x,\xi) \overline{a}(x,\xi) (1 - \chi_0(\xi))^2 \, d\xi$$

$$= \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} e^{i(\phi(x,\xi) - \phi(y,\xi))} a(x,\xi) \overline{a}(x,\xi) \chi_j(\xi) \, d\xi$$

$$= \sum_{j=1}^{\infty} k_{1,j}(x,y).$$

Next, we will show that

$$\sup_y \int_{\mathbb{R}^n} |k_1(x,y)| \, dx < \infty, \quad \sup_x \int_{\mathbb{R}^n} |k_1(x,y)| \, dy < \infty.$$

Then by (2.2) and Schur’s theorem, we have

$$\|T_{1,\phi,a}\|_{L^2 \to L^2} = \|S_{1,\phi,a}\|_{L^2 \to L^2} = \|S_{1,\phi,a} S_{1,\phi,a}^*\|_{L^2 \to L^2}^{1/2}.$$
Case 1: $0 \leq \rho \leq 1/2$. For any $j \in \mathbb{N}$, $B_j^\nu$ denote a ball $B(\xi^\nu_j, 2j^{1-\rho})$ with $2^{j-1} \leq |\xi^\nu_j| < 2^{j+1}$. We can observe that there are no more than $J = C2^{j+n}$ points $\xi^\nu_j \in B_{2^{j+1}} \setminus B_{2^{j-1}}$ and cut-off functions $\psi^\nu_j \in C_c^\infty(B_j^\nu)$ such that

$$\sum_{\nu=1}^J \psi^\nu_j(\xi) = 1, \quad |\nabla_\xi^k \psi^\nu_j(\xi)| \leq C_k 2^{-jk(1-\rho)} \quad \text{for all} \ k \in \mathbb{N}. \quad (2.3)$$

Then $k_{1,j}$ can be decomposed as

$$k_{1,j}(x, y) = \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - \phi(y, \xi))} a(x, \xi) \overline{\alpha}(x, \xi) \chi_j(\xi) \, d\xi$$

$$= \sum_{\nu=1}^J \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - \phi(y, \xi))} a(x, \xi) \overline{\alpha}(x, \xi) \chi_j(\xi) \psi^\nu_j(\xi) \, d\xi$$

$$= \sum_{\nu=1}^J k_{1,j,\nu}(x, y).$$

By setting $h(x, \xi) = \phi(x, \xi) - \nabla_\xi \phi(x, \xi^\nu), \ h_y(\xi) = \phi(y, \xi) - \nabla_\xi \phi(y, \xi^\nu), \ b^\nu_j(x, y, \xi) = e^{i(h_x(\xi) - h_y(\xi))} a(x, \xi) \overline{\alpha}(x, \xi) \chi_j(\xi) \psi^\nu_j(\xi)$, we can rewrite $k_{1,j,\nu}(x, y)$ as

$$k_{1,j,\nu}(x, y) = \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - \phi(y, \xi))} a(x, \xi) \overline{\alpha}(x, \xi) \chi_j(\xi) \psi^\nu_j(\xi) \, d\xi$$

$$= \int_{\mathbb{R}^n} e^{i(\nabla_\xi h(x, \xi^\nu_j) - \nabla_\xi h(y, \xi^\nu_j))} b^\nu_j(x, \xi^\nu_j) \, d\xi.$$

Since $\phi \in L^\infty \Phi^2, \ \xi \in B_j^\nu$, using the mean value theorem, we have

$$|\nabla_\xi h_x(\xi)| \leq C|\xi - \xi^\nu_j| \sup_{\xi \in B_j^\nu} |\nabla_\xi^2 \phi(x, \xi)| \leq C2^{j(1-\rho)}2^{-j} = C2^{-j\rho}. \quad (2.4)$$

For $k \geq 2$, since $0 \leq \rho \leq 1/2$, we get

$$|\nabla_\xi^k h_x(\xi)| = |\nabla_\xi^k \phi(x, \xi)| \leq C 2^{j(1-k)} \leq C 2^{-j k}. \quad (2.5)$$

Obviously, $h_y$ has the same estimates as (2.4) and (2.5).

In addition, since $a \in L^\infty S^m_\rho$ and $\psi^\nu_j$ satisfies (2.3), we obtain that

$$|\nabla_\xi^k a(x, \xi) \overline{\alpha}(x, \xi) \chi_j(\xi) \psi^\nu_j(\xi)|$$

$$\leq \sum_{k_1 + \ldots + k_4 = k} |\nabla_\xi^{k_1} a(x, \xi)||\nabla_\xi^{k_2} \overline{\alpha}(x, \xi)||\nabla_\xi^{k_3} \chi_j(\xi)||\nabla_\xi^{k_4} \psi^\nu_j(\xi)|$$

$$\leq C \sum_{k_1 + \ldots + k_4 = k} 2^{j(2m - \rho(k_1 + k_2))} 2^{-jk_3} 2^{-j(1-\rho)k_4}$$

$$\leq C 2^{j(2m - \rho k)}.$$ 

In the last inequality, we use the fact $1 - \rho \geq \rho$ when $0 \leq \rho \leq 1/2$. 

Next, we define a self adjoint operator $L$ as

$$L = I - 2^{2j\rho}\nabla^2 \xi.$$ 

For any $N \in \mathbb{N}$, by (2.4), (2.5) and (2.6), we have

$$|L^N b^\nu_j(x, y, \xi)|$$

$$\leq C \sum_{N_1+N_2\leq2N} 2^{j\rho(N_1+N_2)} \left|\nabla_N^N \left[a(x, \xi)\overline{a(x, \xi)} \psi_j^\nu(x, \xi) \right] \right| \left|\nabla_N^N e^{i(h_x(\xi)-h_y(\xi))} \right|$$

$$\leq C \sum_{N_1+N_2\leq2N} 2^{j\rho(N_1+N_2)} 2^{j(2m-\rho N_1)}$$

$$\times \sum_{t=1}^{N_2} \sum_{k_1+\ldots+k_t=N_2} \left|\nabla^{k_1} (h_x(\xi)-h_y(\xi)) \cdot \nabla^{k_t} (h_x(\xi)-h_y(\xi)) \right|$$

$$\leq C \sum_{N_1+N_2\leq2N} 2^{j(2m+\rho N_2)} \sum_{t=1}^{N_2} \sum_{k_1+\ldots+k_t=N_2} 2^{-jk_1}\ldots2^{-jk_t\rho}$$

$$\leq C 2^{2jm}.$$ 

Since $L$ is self adjoint and $\xi$-support of $b^\nu_j$ is contained in $B^\nu_j$, for any $N \in \mathbb{N}$, we have

$$|k_{1,j,\nu}(x, y)| = \left|\int_{\mathbb{R}^n} e^{i(\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi'))} b^\nu_j(x, y, \xi) d\xi \right|$$

$$= (1 + 2^{2j\rho} |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')|^2)^{-N}$$

$$\times \left|\int_{\mathbb{R}^n} e^{i(\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi'))} L^N b^\nu_j(x, y, \xi) d\xi \right|$$

$$\leq C (1 + 2^{2j\rho} |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')|^2)^{-N} 2^{j(2m+n(1-\rho))}.$$ 

Now, we estimate $\int_{\mathbb{R}^n} |k_{1,j,\nu}(x, y)| dx$. For any $y \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \left(1 + 2^{2j\rho} |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')|^2 \right)^{-N} dx$$

$$= \left(\int_{\{x: |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')| < 2^{-j\rho}\}} + \int_{\{x: |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')| \geq 2^{-j\rho}\}} \right)$$

$$\left(1 + 2^{2j\rho} |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')|^2 \right)^{-N} dx$$

$$= I + II.$$ 

For $I$, by (1.3), we have

$$I \leq \left|\{x : |\nabla_\xi \phi (x, \xi') - \nabla_\xi \phi (y, \xi')| < 2^{-j\rho}\} \right| \leq C (2^{-j\rho(n-1)} + 2^{-j\rho m}) \leq C 2^{-j\rho(n-1)}.$$
For II, choosing $N > n/2$, we can get

$$II = \int_{\{x : |\nabla_\xi \phi(x, \xi, y, \xi^\nu) - \nabla_\xi \phi(y, \xi)| \geq 2^{j\rho}\}} \left(1 + 2^{2j\rho} |\nabla_\xi \phi(x, \xi, y, \xi^\nu) - \nabla_\xi \phi(y, \xi)|^2\right)^{-N} dx$$

$$= \sum_{s=1}^{\infty} \int_{\{x : 2^{s-1} - 2^{j\rho} \leq |\nabla_\xi \phi(x, \xi, y, \xi^\nu) - \nabla_\xi \phi(y, \xi)| < 2^{s-2-j\rho}\}} \left(1 + 2^{2j\rho} |\nabla_\xi \phi(x, \xi, y, \xi^\nu) - \nabla_\xi \phi(y, \xi)|^2\right)^{-N} dx$$

$$\leq \sum_{s=1}^{\infty} 2^{-2N(s-1)} \sum_{n=s}^{\infty} 2^{-Nn} \left((-2k)2^{-j\rho}n^{-1} + (2^s2^{-j\rho})n\right)$$

So if $\int_{\mathbb{R}^n} |k_{1,v}(x, y)| dx \leq C2^{j(2m+n(1-\rho))2^{-j\rho}(n-1)}$. Therefore, for any $y \in \mathbb{R}^n$, since $J = C2^{j\rho n}$, when $m < \rho(n-1)/2 - n/2$, we obtain

$$\int_{\mathbb{R}^n} |k_1(x, y)| dx \leq \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} \int_{\mathbb{R}^n} |k_{1,j,\nu}(x, y)| dx \leq C \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} 2^{j(2m+n(1-\rho))2^{-j\rho(n-1)}}$$

$$\leq C \sum_{j=1}^{\infty} 2^{j(2m+n(1-\rho))} < \infty.$$

By the same method, we also can get $\sup_x \int_{\mathbb{R}^n} |k_1(x, y)| dy < \infty$ when $m < \rho(n-1)/2 - n/2$. So if $m < \rho(n-1)/2 - n/2$ when $0 \leq \rho \leq 1/2$, $T_{\phi,a}$ is bounded on $L^2$.

**Case 2:** $1/2 \leq \rho \leq 1$. First, let us recall the well-known “dyadic-parabolic” decomposition [18]. For $j \in \mathbb{N}$, fix a collection $\{\xi^\nu_j\}$ of unit vectors, that satisfy

(i) $|\xi^v_j - \xi^\nu_j| \geq 2^{-j/2}$, $v_1 \neq v_2$;

(ii) if $\xi \in S^{n-1}$, then there exists an $\xi^\nu_j$ so that $|\xi - \xi^\nu_j| < 2^{-j/2}$.

For each $j \in \mathbb{N}$, set $\Gamma^\nu_j = \{\xi : |\xi/|\xi| - \xi^\nu_j| \leq 2^{-j/2}\}$. Then we can construct an associated partition of unity given by $\psi^\nu_j$, such that each $\psi^\nu_j$ is homogeneous of degree 0, supported in $\Gamma^\nu_j$ and satisfies that

$$\sum_{\nu=1}^{J} \psi^\nu_j(\xi) \equiv 1 \quad \text{for all } \xi \neq 0 \quad \text{and} \quad |\nabla_\xi \psi^\nu_j(\xi)| \leq C_k |\xi|^{-k} 2^{j/2}, \quad k \in \mathbb{N}.$$

Then we can decompose $k_1(x, y)$ as

$$k_1(x, y) = \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - \phi(y, \xi))} a(x, \xi)\pi(x, \xi)(1 - \chi_0(\xi))^2 d\xi$$

$$= \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} \int_{\mathbb{R}^n} e^{i(\phi(x, \xi) - \phi(y, \xi))} a(x, \xi)\pi(x, \xi)\chi_j(\xi)\psi^\nu_j(\xi) d\xi$$
Since a
(1)

Now we define an operator as

\[ L = I - 2^{2j\rho} \partial_{\xi_1}^2 - 2^j \nabla^2_{\xi_j}. \]

where \( b_j^\nu(x, y, \xi) = e^{i(h_x(\xi) - h_y(\xi))} a(x, \xi) \overline{a}(x, \xi) \chi_j(\xi) \psi_j^\nu(\xi) \) and \( h_x(\xi) = \phi(x, \xi) - \nabla_x \phi(x, \xi^\nu) \cdot \xi, h_y(\xi) = \phi(y, \xi) - \nabla_x \phi(y, \xi^\nu) \cdot \xi. \)

Without loss of generality, by rotating coordinate axes, we can assume that \( \xi_j^\nu = (1, 0, \ldots, 0) \). If we denote \( \xi' = (0, \xi_2, \ldots, \xi_n) \), then we have

**Lemma 2.3.** [19, pp. 406–407] For \( \phi \in L^\infty \Phi^2 \) and \( \xi \in \Gamma_j^\nu \cap \{ \xi : 2^{j-1} < |\xi| \leq 2^{j+1} \} \), we have

\[ |\partial_{\xi_1}^N \nabla_{\xi_j}^M (h_x(\xi) - h_y(\xi))| \leq C 2^{-j(N + M/2)} \quad \text{if } N, M \geq 0 \text{ with } N + M \geq 1, \]

\[ |\partial_{\xi_1}^N \nabla_{\xi_j}^M \psi_j^\nu(\xi)| \leq C 2^{-j(N + M/2)} \quad \text{if } N, M \geq 0. \]

By Lemma 2.3 for any \( k, l \geq 0 \), we have

\[
|\partial_{\xi_1}^k \nabla_{\xi_j}^l e^{i(h_x(\xi) - h_y(\xi))} |
\leq C \sum_{t=1}^{k+l} \sum_{k_1 + \cdots + k_t = k \atop l_1 + \cdots + l_t = l} |\partial_{\xi_1}^{k_1} \nabla_{\xi_j}^{l_1} (h_x(\xi) - h_y(\xi)) \cdots \partial_{\xi_1}^{k_t} \nabla_{\xi_j}^{l_t} (h_x(\xi) - h_y(\xi))|
\leq C \sum_{t=1}^{k+l} 2^{-j(k_1 + l_1/2)} \cdots 2^{-j(k_t + l_t/2)}
\leq C \sum_{t=1}^{k+l} 2^{-j(k+l/2)} \leq C 2^{-j(k+l/2)}.
\]

Now we define an operator as

\[
L = I - 2^{2j\rho} \partial_{\xi_1}^2 - 2^j \nabla^2_{\xi_j}.
\]

Since \( a \in L^\infty S_{m}^m \), \( 1/2 \leq \rho \leq 1 \), applying (2.7) and (2.8), we have

\[
|L^N b_j^\nu(x, y, \xi)|
\leq C \sum_{N_1 + N_2 \leq N} 2^{2j\rho N_1} 2^{jN_2} |\partial_{\xi_1}^{2N_1} \nabla_{\xi_j}^{2N_2} (e^{i(h_x(\xi) - h_y(\xi))} a(x, \xi) \overline{a}(y, \xi) \chi_j(\xi) \psi_j^\nu(\xi))|
\leq C \sum_{N_1 + N_2 \leq N} 2^{2j\rho N_1} 2^{jN_2}
\times \sum_{k_1 + k_2 + k_3 = 2N_1 \atop l_1 + l_2 + l_3 = 2N_2} |\partial_{\xi_1}^{k_1} \nabla_{\xi_j}^{l_1} e^{i(h_x(\xi) - h_y(\xi))} | \partial_{\xi_1}^{k_2} \nabla_{\xi_j}^{l_2} (a(x, \xi) \overline{a}(y, \xi) \chi_j(\xi)) | \partial_{\xi_1}^{k_3} \nabla_{\xi_j}^{l_3} \psi_j^\nu(\xi)|
\[ \leq C \sum_{N_1+N_2 \leq N} 2^{2j\rho N_1} 2^{jN_2} \sum_{k_1+k_2+k_3=2N_1} 2^{j(2m-\rho(k_1+k_2)+2\delta)} \leq C 2^{jm}. \]

In the second to last line of above estimate, we use the condition $1/2 \leq \rho \leq 1$ to estimate the power term. It is easy to see that $|\{\xi \mid (x, y, \xi) \in \text{supp} b_j^\nu \text{ for some } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n\}| \leq C 2^{n(n+1)/2}$. Then for any $N \in \mathbb{N}$, we have

\begin{align*}
|k_{1,j,\nu}(x, y)| & = (1 + 2^{2j\rho} |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)|^2 + 2^j |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)|)^{-N} \\
& \times \left| \int_{\mathbb{R}^n} e^{i(\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu))} L^N b_j^\nu(x, y, \xi) \, d\xi \right| \\
& \leq C (1 + 2^{2j\rho} |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)|^2 + 2^j |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)|^2)^{-N} \\
& \times \left| \int_{\mathbb{R}^n} |L^N b_j^\nu(x, y, \xi)| \, d\xi \right| \\
& \leq C (1 + 2^{2j\rho} |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)|^2 + 2^j |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)|^2)^{-N} \\
& \times 2^{j(2m+n(n+1)/2)}.
\end{align*}

Now we begin to estimate $\int_{\mathbb{R}^n} |k_{1,j,\nu}(x, y)| \, dx$. First, we denote

\[ E_1 = \{x : |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)| \leq 2^{-j\rho}, |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)| \leq 2^{-j/2}\}, \]
\[ E_2 = \{x : |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)| \geq 2^{-j\rho}, |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)| \leq 2^{-j/2}\}, \]
\[ E_3 = \{x : |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)| \leq 2^{-j\rho}, |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)| \geq 2^{-j/2}\}, \]
\[ E_4 = \{x : |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)| \geq 2^{-j\rho}, |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)| \geq 2^{-j/2}\}. \]

Then

\begin{align*}
\int_{\mathbb{R}^n} (1 + 2^{2j\rho} |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)|^2 + 2^j |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)|^2)^{-N} \, dx \\
& = \left( \int_{E_1} + \int_{E_2} + \int_{E_3} + \int_{E_4} \right) \\
& \left( 1 + 2^{2j\rho} |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)|^2 + 2^j |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)|^2 \right)^{-N} \, dx \\
& = I_1 + I_2 + I_3 + I_4.
\end{align*}

Fix $y \in \mathbb{R}^n$, $r > 0$, we observe that the rectangle $\{z : |z_1 - \partial_{\xi_1} \phi(x, \xi_j^\nu)| \leq 2^{-j\rho} r, |z' - \nabla \phi(x, \xi_j^\nu)| \leq 2^{-j/2} r\}$ can be covered by no more than $C 2^{j(n-1)(\rho-1/2)}$ balls with radius $2^{-j\rho} r$. Then, since $\phi$ satisfies \(1.3\), we obtain that

\begin{align*}
\{x : |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)| \leq 2^{-j\rho} r, |\nabla \phi(x, \xi_j^\nu) - \nabla \phi(y, \xi_j^\nu)| \leq 2^{-j/2} r\} & \leq C 2^{j(n-1)(\rho-1/2)} [(2^{-j\rho} r)^{n-1} + (2^{-j\rho} r)^n].
\end{align*}
By (2.10), for $I_1$, we have
\[
I_1 \leq \left\{ x : |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| \leq 2^{-j\rho}, |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| \leq 2^{-j/2} \right\}
\leq C 2^{j(n-1)(\rho-1/2)} (2^{-j\rho(n-1)} + 2^{-jm}) \leq C 2^{-j(n-1)/2}.
\]

For $II_2$, since $E_2 \subseteq \bigcup_{s=1}^{\infty} E_s$, where
\[
E_s = \left\{ x : 2^{s-1} 2^{-j\rho} \leq |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| < 2^s 2^{-j\rho}, |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| \leq 2^{-j/2} \right\},
\]
then when $N > n/2$, we have
\[
I_2 \leq \sum_{s=1}^{\infty} \int_{E_s} \left( 1 + 2^2 2^{-N(s-1)} \left| \partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j') \right|^2 \right)^{-N}.
\]
\[
\leq \sum_{s=1}^{\infty} 2^{-2N(s-1)} \times \left\{ x : |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| < 2^s 2^{-j\rho}, |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| \leq 2^{-j/2} \right\}
\leq C \sum_{s=1}^{\infty} 2^{-2(N(s-1))} 2^{j(n-1)(\rho-1/2)} [(2^s 2^{-j\rho})^{n-1} + (2^s 2^{-j\rho})^n]
\leq C \sum_{s=1}^{\infty} 2^{-s(2N-n)} 2^{-j(n-1)/2} \leq C 2^{-j(n-1)/2}.
\]

The estimate for $I_3$ is similar to $I_2$, we omit the details here.

For $I_4$, since $E_t \subseteq \bigcup_{t=1}^{\infty} \bigcup_{s=1}^{\infty} E_{t,s}$, where
\[
E_{t,s} = \left\{ x : 2^{s-1} 2^{-j\rho} \leq |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| < 2^s 2^{-j\rho}, 2^{t-1} 2^{-j/2} \leq |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| < 2^t 2^{-j/2} \right\},
\]
Then when $N > n$, we get that
\[
I_4 \leq \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} 2^{-N(s-1)} 2^{-N(t-1)} \times \left\{ x : |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| < 2^s 2^{-j\rho}, |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| < 2^t 2^{-j/2} \right\}
\leq \sum_{s=1}^{\infty} \sum_{t \leq s} 2^{-N(s-1)} 2^{-N(t-1)} \times \left\{ x : |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| < 2^s 2^{-j\rho}, |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| < 2^s 2^{-j/2} \right\}
+ \sum_{s=1}^{\infty} \sum_{t \geq s} 2^{-N(s-1)} 2^{-N(t-1)} \times \left\{ x : |\partial_{\xi_1} \phi(x, \xi_j') - \partial_{\xi_1} \phi(y, \xi_j')| < 2^t 2^{-j\rho}, |\nabla_{\xi'} \phi(x, \xi_j') - \nabla_{\xi'} \phi(y, \xi_j')| < 2^t 2^{-j/2} \right\}.
\[
\leq C \sum_{s=1}^{\infty} \sum_{t \leq s} 2^{-N(s-1)} 2^{-N(t-1)} 2^{j(n-1)(\rho - 1/2)} [(2^s 2^{-j \rho})^{n-1} + (2^s 2^{-j \rho})^{n-1}]
+ \sum_{s=1}^{\infty} \sum_{t \geq s} 2^{-N(s-1)} 2^{-N(t-1)} 2^{j(n-1)(\rho - 1/2)} [(2^t 2^{-j \rho})^{n-1} + (2^t 2^{-j \rho})^{n-1}]
\leq C 2^{-j(n-1)/2}.
\]

Hence

\[
\int_{\mathbb{R}^n} (1 + 2^{2j \rho} |\partial_{\xi_1} \phi(x, \xi_j^\nu) - \partial_{\xi_1} \phi(y, \xi_j^\nu)|^2 + 2^j |\nabla_{\xi'} \phi(x, \xi_j^\nu) - \nabla_{\xi'} \phi(y, \xi_j^\nu)|^2)^{-N} \, dx
\leq C 2^{-j(n-1)/2},
\]

\[
\int_{\mathbb{R}^n} |k_{1,j,\nu}(x, y)| \, dx \leq C 2^{j(2m+1)}.
\]

For any \( y \in \mathbb{R}^n \), by (2.9), (2.11) and with the fact \( J = C 2^{(n-1)/2} \), then when \( m < -(n+1)/4 \), we can get

\[
\int_{\mathbb{R}^n} |k_1(x, y)| \, dx \leq \sum_{j=1}^{\infty} \sum_{\nu=1}^{J} \int_{\mathbb{R}^n} |k_{1,j,\nu}(x, y)| \, dx \leq C \sum_{j=1}^{\infty} 2^{j(2m+(n+1)/2)} < \infty.
\]

By symmetry, it is easy to get that for any \( x \in \mathbb{R}^n \), when \( m < -(n+1)/4 \), \( \int_{\mathbb{R}^n} |k_1(x, y)| \, dy < \infty \). So when \( 1/2 \leq \rho \leq 1 \), if \( m < -(n+1)/4 \), then \( T_{\phi,a} \) is bounded on \( L^2 \).

References

[1] K. Asada and D. Fujiwara, \textit{On some oscillatory integral transformations in} \( L^2(\mathbb{R}^n) \), Japan. J. Math. (N.S.) \textbf{4} (1978), no. 2, 299–361.

[2] R. Beals, \textit{Spatially inhomogeneous pseudodifferential operators II}, Comm. Pure Appl. Math. \textbf{27} (1974), 161–205.

[3] A. J. Castro, A. Israelsson and W. Staubach, \textit{Regularity of Fourier integral operators with amplitudes in general H"{o}rmander classes}, Anal. Math. Phys. \textbf{11} (2021), no. 3, Paper No. 121, 54 pp.

[4] E. Cordero, F. Nicola and L. Rodino, \textit{Boundedness of Fourier integral operators on} \( \mathcal{F}L^p \) \textit{spaces}, Trans. Amer. Math. Soc. \textbf{361} (2009), no. 11, 6049–6071.

[5] \textit{On the global boundedness of Fourier integral operators}, Ann. Global Anal. Geom. \textbf{38} (2010), no. 4, 373–398.

[6] S. Coriasco and M. Ruzhansky, \textit{On the boundedness of Fourier integral operators on} \( L^p(\mathbb{R}^n) \), C. R. Math. Acad. Sci. Paris \textbf{348} (2010), no. 15-16, 847–851.
[7] M. G. Cowling, *Pointwise behavior of solutions to Schrödinger equations*, in: Harmonic Analysis (Cortona, 1982), 83–90, Lecture Notes in Math. 992, Springer, Berlin, 1983.

[8] D. Dos Santos Ferreira and W. Staubach, *Global and local regularity of Fourier integral operators on weighted and unweighted spaces*, Mem. Amer. Math. Soc. 229 (2014), no. 1074, 65 pp.

[9] G. I. Éskin, *Degenerate elliptic pseudodifferential equations of principal type*, Mat. Sb. (N.S.) 82(124) (1970), no. 4, 585–628.

[10] D. Fujiwara, *A global version of Eskin's theorem*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977), no. 2, 327–339.

[11] A. Greenleaf and G. Uhlmann, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, J. Funct. Anal. 89 (1990), no. 1, 202–232.

[12] S. Ham, H. Ko and S. Lee, *Dimension of divergence set of the wave equation*, Nonlinear Anal. 215 (2022), Paper No. 112631, 10 pp.

[13] L. Hörmander, *Fourier integral operators I*, Acta Math. 127 (1971), no. 1-2, 79–183.

[14] C. E. Kenig and W. Staubach, *Ψ-pseudodifferential operators and estimates for maximal oscillatory integrals*, Studia Math. 183 (2007), no. 3, 249–258.

[15] F. Nicola, *Boundedness of Fourier integral operators on Fourier Lebesgue spaces and affine fibrations*, Studia Math. 198 (2010), no. 3, 207–219.

[16] L. Rodino, *On the boundedness of pseudo differential operators in the class $L^m_{\rho,1}$*, Proc. Amer. Math. Soc. 58 (1976), 211–215.

[17] M. Ruzhansky and M. Sugimoto, *A local-to-global boundedness argument and Fourier integral operators*, J. Math. Anal Appl. 473 (2019), no. 2, 892–904.

[18] A. Seeger, C. D. Sogge and E. M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. (2) 134 (1991), no. 2, 231–251.

[19] E. M. Stein, *Harmonic Analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series 43, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.

Jiawei Dai and Qiang Huang

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

E-mail addresses: jwdai123@163.com, huangqiang0704@163.com