Localization of massive fermions on the baby-skyrmion branes in 6-dimensions

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We construct brane solutions in 6-dimensional Einstein-Skyrme systems. A class of baby skyrmion solutions realizes warped compactification of the extra dimensions and gravity localization on the brane for negative bulk cosmological constant. Coupling of the fermions with the brane skyrmions lead to the brane localized fermions. In terms of the level crossing picture, emergence of the massive localized modes are observed. Nonlinear nature of the skyrmions brings richer information for the fermions level structure. It comprises doubly degenerate lowest plus single excited modes. The three generation of the fundamental fermions is associated with this distinctive structure. The mass hierarchy of quarks or leptons is appeared in terms of a slightly deformed baby-skyrmions with topological charge three.

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I. INTRODUCTION

Theories with extradimensions have been expected to solve the hierarchy problem and cosmological constant problem. Experimentally unobserved extradimensions indicate that the standard model particles and forces are confined to a 3-brane. Intensive study has been performed for the Randall-Sundram (RS) brane model in 5 space-time dimensions. In this framework, the exponential warp factor in the metric can generate a large hierarchy of scales. This model, however, requires unstable negative tension branes and the fine-tuning between brane tensions and bulk cosmological constant.

There is hope that higher dimensional brane models more than five could evade those problems appeared in 5-dimensions. In fact brane theories in 6-dimensions show a very distinct feature towards the fine-tuning and negative tension brane problems. In Refs. 3-6, it was shown that the brane tension merely produces deficit angles in the bulk and hence it can take an arbitrary value without affecting the brane geometry. The model is based on the spontaneous compactification by the bulk magnetic flux. If the compactification manifold is a sphere, two branes have to be introduced with equal tensions. If it is a disk, no second 3-brane is needed. But still the fine-tuning between magnetic flux and the bulk cosmological constant cannot be avoided although non-static solutions could be free of any fine-tuning.

Alternatively to the flux compactification in 6-dimensions, the nonlinear sigma model has been used for compactifications of the extra space dimensions. As in the flux compactification, no second 3-brane is needed if the parameters in the sigma model and bulk space-time are tuned.

Warped compactifications are also possible in 6 space-time dimensions in the model of topological objects such as defects and solitons. In this context strings were investigated, showing that they can realize localization of gravity. Interestingly, if the brane is modeled in such a field theory language, the fine-tuning between bulk and brane parameters required in the case of delta-like branes turns to a tuning of the model parameters.

The Skyrme model is known to possess soliton solutions called baby skyrmions in 2-dimensional space. In this paper we therefore consider the warped compactification of the 2-dimensional extra space by the baby skyrmions. We find that in the 6-dimensional Einstein-Skyrme systems, static solutions which realize warped compactification exist for negative bulk cosmological constant. Since the solution is regular except at the conical singularity, it has only single 3-brane. Thus no fine-tuning between brane tensions is required. The Skyrme model possess a rich class of stable multi soliton solutions. We find various brane solutions by such multi-solitons.

It should be noted that general considerations in the 6-dimensional brane model with bulk scalar fields suggest that the mechanism of regular warped compactification with single positive tension brane is not possible. However, the model under consideration is restricted to the bulk scalar field depending only on the radial coordinate in the extra space. The scalar field in the Skyrme model depend not only the radial coordinate but also the angular coordinate to exhibit nontrivial topological structure, which makes possible to realize regular warped compactification.

Study of localization of fermions and gauge fields on topological defects have been extensively studied with co-dimension one and two. Many years ago, particle localization on a domain wall in higher dimensional space time was already addressed. The authors suggested the possibility of localized massless fermions on the 1-dimensional kink background in 4+1 space-time with Yukawa-type coupling manner. Later, localization of chiral fermions on RS scenario was discussed in Ref. 20. Analysis for the massive fermionic modes was done by Ringeval et.al., in Ref. 21. For co-dimension
two, the localization on higher dimensional generalizations of the RS model was studied within the coupling of real scalar fields \[23\]. Many studies have been followed and most of them are based on the Abelian Higgs, or Higgs mediated models with the chiral fermions.

Problem of fermion mass hierarchy has been discussed in many articles \[24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 37\] within the different mechanisms. They are based on the Yukawa coupling of the fermions and the Higgs (scalar) field. In Ref.\[30\], the authors set up multiple branes and considered localization of fermions on different branes in terms of Yukawa couplings to the Higgs field. In Ref.\[27\], the fermions have quantum numbers of the rotational momenta which are origin of the generation of fermions. The authors of Ref.\[21\] deal with this problem with somewhat different approach. Conical singularity of the background branes and the orbital angular momentum of the fermions around the branes are the key role for the generation. In Ref.\[26\] hierarchy between the fermionic generations are explained in terms of multi-winding number solutions of the complex scalar fields. They observed chiral fermionic zero modes on a topological defect with winding number three and finite masses appear the mixing of these zero modes. Although any brane localization mechanism is absent in their discussion, the idea is promising. In Ref.\[37\], the authors have taken into account more realistic standard model charges. Ref.\[52\] is about discussion of the fermion families from two layer warped 6-dimensions. The authors obtained Kaluza-Klein particles in 5 dimensions at first; those are finally regarded as light, standard model fermions in 4 dimensions.

Starting point of our approach is conventional and has somewhat similarity to Ref.\[26\]. We shall consider the localization of the fermions on the baby skyrmion branes with topological charge three. The localized modes of fermions are confirmed through the analysis of spectral flow of the one particle state \[38\]. According to the Index theorem a nonzero topological charge implies the zero modes of the Dirac operator \[39\]. The zero crossing modes are found to be the localized fermions on the brane. So the generation of the fermions is defined in terms of the topological charge of the skyrmions with a special quantum number called grandspin \(K_3\). There are different profiles of the zero crossing behavior for different \(K_3\), and it is the origin of the finite mass in our point of view. Nonlinear nature of the skyrmion fields has richer information than the case of Abelian string; the level comprises lowest doubly degenerate as well as single excited modes, which can partially explains generation puzzle of the fermions. In order to manifest more realistic mass structure, breaking of the rotational symmetry and the shape deformation of the background skyrmions is taken into account.

The crucial difference of our approach from the other attempts is the representation of the fermions. It is based on our knowledge that even in the first generation of the fermions, they have small but finite masses. It means that the fermions are not pure chiral eigenstates. Therefore, in this article we employ the standard representation of the higher dimensional gamma matrices instead of the chiral one.

This paper is organized as follows. In the next section we describe the Einstein-Skyrme system in 6-dimensions and derives the coupled equations for the Skyrme and gravitational fields. We derive a class of multi-winding number solutions. Some typical numerical brane solutions are shown. Formulation of the fermions in higher dimensional curved space-time is discussed in Sec.III. Coupling of the fermions and the skyrmions is introduced in this section. Conclusion and discussion are given in Sec.IV.

\section{Construction of the Baby-Skyrmion Branes}

\subsection{Model}

We introduce a model of the 6-dimensional Einstein-Skyrme system with a bulk cosmological constant coupled to fermions \[41\]. The action comprises

\begin{equation}
S = S_{\text{gravity}} + S_{\text{brane}} + S_{\text{fermion}}.
\end{equation}

Here \(S_{\text{gravity}}\) is the 6-dimensional Einstein-Hilbert action

\begin{equation}
S_{\text{gravity}} = \int d^6x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \Lambda_b \right].
\end{equation}

In the parameter \(\kappa^2 = 1/M_6^4\), \(M_6\) is the 6-dimensional Planck mass, denoted the fundamental gravity scale, and \(\Lambda_b\) is the bulk cosmological constant.

For \(S_{\text{brane}}\) we use the action of baby-Skyrme model \[17, 18\]

\begin{equation}
S_{\text{brane}} = \int d^6x L_{\text{brane}}
\end{equation}

with

\begin{equation}
L_{\text{brane}} = \sqrt{-g} \left[ \frac{F^2}{2} \partial_M \vec{\phi} \cdot \partial^M \vec{\phi} + \frac{1}{4e^2} \left( \partial_M \vec{\phi} \times \partial_N \vec{\phi} \right)^2 + \mu^2 (1 + \vec{n} \cdot \vec{\phi}) \right],
\end{equation}

where \(M, N\) run over 0, \cdots, 5 and \(\vec{n} = (0, 0, 1)\). \(\vec{\phi} = (\phi^1, \phi^2, \phi^3)\) denotes a triplet of scalar real fields with the constraint \(\vec{\phi} \cdot \vec{\phi} = 1\). The constants \(F, e, \mu\) are the Skyrme model parameters with the dimension of \((\text{energy})^2, (\text{energy})^{-1}, (\text{energy})^0\), respectively. The first term in Eq.(1) is nothing but a nonlinear \(\sigma\)-model. The second term is the analogue of the Skyrme fourth order term in the standard Skyrme model (the Skyrme model in 3+1 dimensions) which works as a stabilizer for obtaining the soliton solution. The last term is referred to as a potential term which guarantee the stability of a baby-skyrmion.
The solutions of the model would be characterized by following topological charge in curved space-time

\[ Q = \frac{1}{4\pi} \int d^2x \hat{\phi} \cdot (\nabla_1 \phi \times \nabla_2 \phi) \]  

where \( \nabla_\mu \) means the space-time covariant derivative. Let us assume that the matter Skyrme fields depend only on the extra coordinates and impose the hedgehog ansatz

\[ \phi = (\sin f(r) \cos n\theta, \sin f(r) \sin n\theta, \cos f(r)) . \]  

The function \( f(r) \) which is often called as the profile function, has following boundary condition

\[ f(0) = -(m - 1)\pi, \quad \lim_{r \to \infty} f(r) = \pi \]  

where \((m, n)\) is arbitrary integer. This ansatz ensures the topological charge

\[ Q = n(1 - (-1)^m)/2. \]  

We consider the maximally symmetric metric with vanishing 4D cosmological constant,

\[ ds^2 = B^2(r)\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + C^2(r)d\theta^2 \]  

where \( \eta_{\mu\nu} \) is the Minkowski metric with the signature \((-+++)\) in our convention and \( 0 \leq r < \infty \) and \( 0 \leq \theta < 2\pi \). This ansatz has been proved to realize warped compactification of the extra dimension in models where branes are represented by global defects [41].

\( S_{\text{fermion}} \) is the action for fermions coupled with the skyrmions and the gravity; we shall describe it in Sec. III.

The general forms of the coupled system of Einstein equations and the equation of motion of the Skyrme model are

\[ G_{MN} = \kappa^2 (-\Lambda g_{MN} + T_{MN}) , \]  

\[ \frac{1}{\sqrt{-g}} \partial_N \left( \sqrt{-g} F^2 \phi \times \partial^N \phi \right) \]  

\[ + \sqrt{-g} \frac{1}{e^2} \partial_M \phi \cdot (\partial^M \phi + \partial^N \phi) + \mu^2 \phi \times \bar{n} = 0, \]  

where the stress-energy tensor \( T_{MN} \) is given by

\[ T_{MN} = -2 \frac{\delta L_{\text{brane}}}{\delta g_{MN}} + g_{MN} L_{\text{brane}} \]  

\[ = F^2 \partial_M \phi \cdot \partial_N \phi + \frac{1}{e^2} F^2 (\partial_M \phi \times \partial_N \phi) \cdot (\partial_B \phi \times \partial_N \phi) \]  

\[ + g_{MN} L_{\text{brane}} . \]  

Inserting Eq. (6) into Eq. (11), one obtains the Lagrangian

\[ L_{\text{brane}} = -B^4 C^4 e^2 \left[ uf^2 + \frac{n^2 \sin^2 f}{C^2} + 2\tilde{\mu}(1 + \cos f) \right] \]  

where we have introduced the dimensionless quantities

\[ \tilde{x}_\mu = eF x_\mu, \quad y = eFr, \quad \tilde{C} = eFC, \quad \tilde{\mu} = \frac{1}{eF^2} \mu \]  

and

\[ u = 1 + \frac{n^2 \sin^2 f}{C^2} . \]  

The prime denotes derivative with respect to the radial component \( y \) of the two extra space. The Skyrme field equation is thus

\[ f'' + \left( \frac{4B'}{B} + \frac{\tilde{C}'}{C} \right) f' - \frac{1}{2u} \left[ \frac{n^2 \sin^2 f}{C^2} (1 + f'^2) + 2\tilde{\mu}^2 \sin f \right] = 0 \]  

where

\[ \frac{u'}{u} = \frac{n^2}{C^2 + n^2 \sin^2 f} \left[ f' \sin 2f - 2 \tilde{C}' \sin^2 f \right] . \]  

Within this ansatz, the components of the stress-energy tensor [12] becomes

\[ T_{\mu\nu} = -F^4 e^2 B^2 \eta_{\mu\nu} \tau_0 (y) , \]  

\[ \tau_0 (y) = \frac{u}{2} f'^2 + \frac{n^2 \sin^2 f}{2C^2} + \tilde{\mu}^2 (1 + \cos f) \]  

\[ T_{rr} = -F^4 e^2 \tau_r (y) , \]  

\[ \tau_r (y) = -\frac{u}{2} f'^2 + \frac{n^2 \sin^2 f}{2C^2} + \tilde{\mu}^2 (1 + \cos f) \]  

\[ T_{\theta\theta} = -F^4 \tilde{C}^2 \tau_\theta (y) , \]  

\[ \tau_\theta (y) = \frac{u}{2} f'^2 - \frac{n^2 \sin^2 f}{2C^2} + \tilde{\mu}^2 (1 + \cos f) \]  

FIG. 1: Typical results of the profile functions \( f \) (straight line), the warp metrics \( B, C \) (dashed,dotted line,respectively), as a function of \( y \).
\[ q \] (dashed, dotted, and dot-dashed line, respectively) and the topological charge density \( q \) (straight line).

where

\[ \dot{u} = 1 - \frac{n^2 \sin^2 f}{C^2}. \]  

(21)

The Einstein equations with bulk cosmological constant are written down in the following form

\[ 3\dot{b}' + 6\dot{b}^2 + 3\dot{b}\dot{c} + \ddot{c} + \dot{c}' = -\alpha(\tilde{\Lambda}_b + \tau_0(y)) \]  

(22)

\[ 6\dot{b}^2 + 4\dot{b}\dot{c} = -\alpha(\tilde{\Lambda}_b + \tau_0(y)) \]  

(23)

\[ 4\dot{b}' + 10\dot{b}^2 = -\alpha(\tilde{\Lambda}_b + \tau_0(y)) \]  

(24)

where \( \alpha = \kappa_2 F^2 \) is a dimensionless coupling constant and \( \Lambda_b = \Lambda_b/e^2 F^4 \) is a dimensionless bulk cosmological constant. Also, we introduce \( \dot{b} \equiv B'/B, \dot{c} \equiv C'/C \) for convenience.

### B. Boundary conditions

At infinity, all components of the energy-momentum tensor vanishes and the Einstein equations (22), (24) are then reduced to

\[ 3\dot{b}' + 6\dot{b}^2 + 3\dot{b}\dot{c} + \ddot{c} + \dot{c}' = -\alpha(\tilde{\Lambda}_b) \]  

(25)

\[ 6\dot{b}^2 + 4\dot{b}\dot{c} = -\alpha(\tilde{\Lambda}_b) \]  

(26)

\[ 4\dot{b}' + 10\dot{b}^2 = -\alpha(\tilde{\Lambda}_b). \]  

(27)

The general solution has been obtained in Refs. [13, 16] which is given by

\[ \dot{b} = p \frac{Ae^{\frac{b}{2} + y} - e^{-\frac{b}{2} + y}}{Ae^{\frac{b}{2} + y} + e^{-\frac{b}{2} + y}}, \quad \dot{c} = \frac{5p^2}{2b} - \frac{3}{2}b \]  

(28)

where \( A \) is an arbitrary constant and

\[ p = \sqrt{-\frac{\alpha\Lambda_b}{10}}. \]  

(29)

Since we are interested in regular solutions with warped compactification of the extra-space, the functions \( B \) and \( C \) must converge at infinity. This is achieved only when \( \Lambda_b < 0 \) and \( A = 0 \) with the solution

\[ B \rightarrow \epsilon_1 e^{-py}, \quad C \rightarrow \epsilon_2 e^{-py} \]  

(30)

where \( \epsilon_1 \) and \( \epsilon_2 \) are arbitrary constants. Then, the asymptotic form of the metric which realizes warped compactification is given by

\[ ds^2_{50} = \epsilon_1 e^{-2\sqrt{-\frac{\alpha\Lambda_b}{10}}y} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 + \epsilon_2 e^{-2\sqrt{-\frac{\alpha\Lambda_b}{10}}y} d\theta^2. \]  

(31)

The 4-dimensional reduced Planck mass \( M_{pl} \) is derived by the coefficient of the 4 dimensional Ricci scalar, which can be calculated inserting the metric (9) into the action (2),

\[ \frac{M_{pl}^2}{2} \int d^4 x \sqrt{-g^{(4)}} R^{(4)} = \frac{M_6^4}{2} \int d^6 x \sqrt{-g} B^{-2}(r) R^{(4)} \]

\[ = \frac{M_6^4}{2} \int d^4 x \sqrt{-g^{(4)}} R^{(4)} \int dr d\theta B^2(r) C(r) \]

\[ = \frac{2\pi M^4_6}{2} \int dr B^2(r) C(r) \int d^4 x \sqrt{-g^{(4)}} R^{(4)} \]

where the superscript (4) represents a tensor defined on the 4 dimensional submanifold. Thus, we find the relation between \( M_{pl} \) and \( M_6 \) as

\[ M_{pl}^2 = 2\pi M_6^4 \int_0^{\infty} dr B^2(r) C(r). \]  

(32)
The requirement of gravity localization is equivalent to the finiteness of the 4-dimensional Planck mass. For the asymptotic solution (31), the localization is attained.

Let us consider the asymptotic solutions for skyrmions. we can write
\[ f(y) = \bar{f} + \delta f(y), \]  
where for \( y \gg 1, \bar{f} \sim 0 \). The linearized field equations are given by
\[ \delta f'' - 5p\delta f' - \mu \delta f = 0. \]  
Assuming that \( f \) falls off exponentially, one obtains for \( y \gg 1 \)
\[ \delta f(y) \rightarrow f_c e^{-qy} \quad \text{with} \quad q = \sqrt{25p^2 + 4\mu - 5p}. \]  
where \( f_c \) is an arbitrary constant.

Following regularity of the geometry at the center of the defect is imposed
\[ B'(0) = 0, \quad C(0) = 0, \quad C''(0) = 1 \]  
and we can arbitrarily fix \( B(0) = 1 \). Boundary conditions for the warp factors and the profile function at the origin are determined by expanding them around the origin. For the different topological sectors, the first few terms are schematically written down as
\[ f(y) = -(m - 1)\pi + f^{(n)}(0)y^n + O(y^{n+1}) \]  
\[ b(y) = B y + O(y^3) \]  
\[ \tilde{C}(y) = y + C y^3 + O(y^5) \]  
where \((m,n) = (1,1)\)
\[ b = \frac{\alpha}{4}(\tilde{\Lambda}_b + 2\tilde{\mu} - \frac{1}{2}f'(0)^4), \]  
\[ c = \frac{\alpha}{12}(\tilde{\Lambda}_b + 2\tilde{\mu} - 2f'(0)^2 - \frac{5}{2}f''(0)^4) \]  
\((m,n) = (1,2)\)
\[ b = \frac{\alpha}{4}(\tilde{\Lambda}_b + 2\tilde{\mu}), \quad c = \frac{\alpha}{12}(\tilde{\Lambda}_b + 2\tilde{\mu}) \]  
\((m,n) = (2,1)\)
\[ b = \frac{\alpha}{4}(\tilde{\Lambda}_b - \frac{1}{2}f'(0)^4), \]  
\[ c = \frac{\alpha}{12}(\tilde{\Lambda}_b - 2f'(0)^2 - \frac{5}{2}f''(0)^4) \]  
\((m,n) = (2,2)\)
\[ b = -\frac{\alpha}{4}\tilde{\Lambda}_b, \quad c = \frac{\alpha}{12}\tilde{\Lambda}_b. \]  
Thus one finds that the only \( f'(0) \) or \( f''(0) \) is the free parameter vicinity of the origin.

Consider linear combinations of Eqs. (22)-(24), we obtain
\[ \dot{b}' + 4\dot{b}^2 + \dot{b}\dot{c} = -\frac{1}{2}\alpha\tilde{\Lambda}_b + \frac{\alpha}{4}(\tau_r + \tau_\theta), \]  
\[ 4\dot{b}\dot{c} + \dot{c}' + \dot{c}^2 = -\frac{1}{2}\alpha\tilde{\Lambda}_b + \frac{\alpha}{4}(4\tau_0 + \tau_r - 3\tau_\theta). \]  
Integrating Eqs. (11), (15) from zero to \( y_c \), we get
\[ B^2(y_c)B'(y_c)\tilde{C}(y_c) \]  
\[ = -\frac{\alpha}{2}\tilde{\Lambda}_b \int_0^{y_c} B^4\tilde{C}dy - \frac{\alpha}{4}(\mu_r + \mu_\theta), \]  
\[ B^4(y_c)\tilde{C}'(y_c) \]  
\[ = 1 - \frac{\alpha}{2}\tilde{\Lambda}_b \int_0^{y_c} B^4\tilde{C}dy - \frac{\alpha}{4}(4\mu_0 + \mu_r - \mu_\theta). \]  
(46) is the 6-dimensional analogue of the relation determining the Tolman mass whereas Eq. (47) is the generalization of the relation giving the angular deficit. Combining these the following relations are obtained in the \( y_c \rightarrow \infty \)
\[ \alpha \int_0^{\infty} B^4\frac{n^2\sin^2 f}{C}(1 + f'^2)dy = 1 \]  
or
\[ \alpha \int_0^{\infty} B^4\left[ \frac{n^2\sin^2 f}{C} + 2\tilde{\Lambda}_b\tilde{C} + 2\tilde{\mu}\tilde{C}(1 - \cos f) \right]dy = 1. \]  
(49)

These conditions are used for checking the numerical accuracy of our calculations.

In order to study the singularity structure of the bulk solutions, the several curvature invariants are computed [13]. The explicit form for the metric (9) are given in Ref. [42] and they are
\[ R := 20\dot{b}^2 + 8\dot{b}' + 2\dot{c}' + 2\dot{c}^2 + \dot{8}\tilde{c} \]  
\[ R_{AB}R^{AB} := \]  
\[ 80\dot{b}^4 + 20\dot{b}^2 + 2\dot{c}^2 + 64\dot{b}\dot{\tilde{b}} + 4\dot{c}\dot{\tilde{c}} + 2\dot{8}\tilde{c}^2 \]  
\[ + 32\dot{\tilde{c}}\dot{\tilde{c}} + 8\dot{b}\dot{\tilde{c}} + 8\dot{\tilde{b}}\dot{\tilde{c}} + 8\dot{\tilde{c}}^2 + 8\tilde{c}\tilde{c}' + 8\tilde{b}\tilde{c}' \]  
\[ R_{ABCD}R^{ABCD} := \]  
\[ 4\dot{c}^4 + 4\dot{b}\dot{\tilde{b}} + 16\dot{b}^2\dot{c}^2 + 8\dot{c}\dot{\tilde{c}} + 4\dot{c}^2 + 32\dot{\tilde{b}}\dot{\tilde{b}} + 16\dot{b}^2 \]  
\[ C_{ABCD}C^{ABCD} := \frac{12}{5}(\dot{b}' - \dot{c}' + \dot{\tilde{b}}\dot{\tilde{c}} - \dot{\tilde{c}}^2)\dot{\tilde{c}}. \]  

C. Numerical analysis

The equations (10), (22)-(24) should be solved numerically since they are highly nonlinear. The simple technique to solve the Einstein-Skyrme equations is the shooting method combined with the 4th order RungeKutta forward integration [14]. However, a unique set of boundary conditions at \( y = 0 \) produces 2 distinct solutions, one of which grows exponentially and another decays exponentially as \( y \rightarrow \infty \). This causes instability of solutions when the forward integration is performed. Instead, we employ a backward integration which is used in Ref. [15] where the 6-dimensional vortex-like regular brane solutions were constructed. The backward integration method requires a set of boundary conditions at infinity. We, however, truncate and take the distance \( y_{max} \)
FIG. 4: Parameter space for typical solutions which exhibit gravity localization around the skyrmions with $Q = 3$.

FIG. 5: The effective potential $V_{11}$ defined by Eq. (A2) with the parameter, $\tilde{\omega} = 1.1, l = 0$ for $(m, n) = (1, 1)$ is shown. The figure clearly exhibits Volcano shape.

FIG. 6: Fermion number density for the background skyrmion with $(m, n) = (1, 1)$ with the model parameters: $\alpha = 0.5, \tilde{\Lambda}_b = -0.1, \tilde{\mu} = 0.220525915$ and $f'(0) = 0.7292181213078184$. The results for the coupling constant $\tilde{M} = 0$ (decoupled) and $\tilde{M} = 0.86$ are plotted. If the fermions couple to the skyrmions, only the state $K_3 = 0$ is localized on the brane core. The other states are not observed because they are strongly delocalized.

III. FERMIONS

A. Basic formalism

The action of the fermions coupled with the Skyrme field in a Yukawa coupling manner can be written as

$$S_{\text{fermion}} = \int d^6x L_{\text{fermion}}$$

with

$$L_{\text{fermion}} = \sqrt{-g} \left[ \bar{\Psi} (i \Gamma^A D_A - M \vec{\tau} \cdot \vec{\phi}) \Psi \right].$$

The 6-dimensional gamma matrices $\Gamma^A$ are defined with the help the vielbein $e^A_a$ and those of the flat-space $\gamma^a$, i.e., $\Gamma^A := e^A_a \gamma^a$. The covariant derivative is defined as

$$D_A := \frac{1}{2} \overrightarrow{\partial} A + \frac{1}{2} \omega^A_{\hat{a} \hat{b}} \sigma_{\hat{a} \hat{b}}$$

where $\omega^A_{\hat{a} \hat{b}} := \frac{1}{2} e^{AB} \nabla_A e^B_B$ are the spin connection with generators $\sigma_{\hat{a} \hat{b}} := \frac{1}{4} [\gamma_{\hat{a}}, \gamma_{\hat{b}}]$. The symbol $\overrightarrow{\partial}$ implies that

far enough from the origin so that the Skyrme profile would fall off before it reaches $y_{\text{max}}$. The set of boundary conditions at $y = y_{\text{max}}$ produces a unique solution which satisfies the boundary conditions at $y = 0$ and hence it is numerically stable. We present our typical numerical results in Figs. 1-2. Fig. 3 shows the fine-tuning surface in the model parameter space $(\alpha, \tilde{\mu}, \tilde{\Lambda}_b)$ corresponding to the gravity localization condition. We should stress that once the desired solutions are obtained, the parameters are no longer arbitrary, i.e., a constraint $h(\alpha, \tilde{\mu}, \tilde{\Lambda}_b) = 0$ is emerged.
ψ(∂ψ − (∂ψ) φ). Here A, B = 0, · · · , 5 are the 6-dimensional space-time index and a, b = 0, · · · , 5 correspond to the flat tangent 6-dimensional Minkowski space. The vielbein is defined through g_{AB} = e_A^a e_a B = \eta_{ab} e_A^a e_B^b. We introduce the form of vielbein which was used, e.g., at Ref. 31

\begin{align*}
\varepsilon_\mu^a &= B(r) \delta_\mu^a, \quad \mu = 0, \cdots, 3, \\
\varepsilon_4^a &= \cos \theta, \quad \varepsilon_5^a = \sin \theta, \\
\varepsilon_6^a &= C(r) \sin \theta, \quad \varepsilon_7^a = C(r) \cos \theta.
\end{align*}

(53)

Straightforward calculation shows that the nonvanishing components of the corresponding spin connections are

\begin{align*}
\omega_\mu^4 &= \delta_\mu^a B'(r) \cos \theta, \quad \omega_\mu^5 = \delta_\mu^a B'(r) \sin \theta, \\
\omega_\theta^4 &= 1 - C'(r).
\end{align*}

(54)

The Dirac equation is

\begin{align*}
\left[ i \frac{1}{B} \delta_\mu^a \gamma_\mu \partial_\mu + i (\cos \theta \gamma_4 + \sin \theta \gamma_5) (\partial_r + \frac{2B'}{B} - \frac{1 - C'}{2C}) - i (\sin \theta \gamma_4 - \cos \theta \gamma_5) \frac{1}{C} \partial_\theta - M \gamma_7 \right] \psi &= 0.
\end{align*}

(55)

The Dirac gamma matrices should satisfy the anti-commutation relations \( \{ \gamma^A, \gamma^B \} = 2 e_i^{AB} \) and there are the possible candidates preserving such Clifford algebra. In most of previous studies in the 6-dimensions, they are based on the localization of the chiral fermions on the Abelian vortex. In the chiral representation, the spinors are expanded into the right- and the left-handed components and the zero mode appears as a eigenstate of the right or the left. The massive modes emerge as their mixing states. Our approach, however, is somewhat different for treatment of the masses of the fermions. Actually, even in the first generation the fermions have intrinsic, finite masses. So, treating the massive fermionic modes directly, we employ the standard representation of the higher dimensional gamma matrices instead of the chiral one. The eigenvalues of the Dirac hamiltonian are estimated for background brane solutions with large parameter space, and the zero modes appear as the zero crossing points. The standard representation of the gamma matrices in 6-dimensional can be defined as

\begin{align*}
\gamma^\mu &= \begin{pmatrix} \gamma_1 & 0 \\ 0 & -\gamma_1 \end{pmatrix}, \quad \gamma_i^\mu = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \left( \begin{array}{cc} 0 & \bar{\sigma} \\ -\bar{\sigma} & 0 \end{array} \right) \\
\gamma^3 &= \begin{pmatrix} 0 & -iI_4 \\ -iI_4 & 0 \end{pmatrix}, \quad \gamma^5 := \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}
\end{align*}

(56)

where \( I_n \) means the identity matrix of the dimension \( n \). The 6-dimensional spinor \( \Psi \) can be decomposed into the 4 dimensional and the extra space-time components \( \Psi(x^\mu, r, \theta) = \psi(x^\mu) \otimes U(r, \theta) \). Here the 4-dimensional part \( \psi(x^\mu) \) is the solution of the corresponding Dirac equation on the brane

\begin{align*}
iv \gamma^\mu \partial_\mu \psi &= w \psi
\end{align*}

(57)

in which the eigenvalues \( w \) correspond to the 4-dimensional masses of the fermions. The Dirac equation in 6-dimensions is transformed as the 2-dimensional eigenproblem with the eigenvalue \( w \). The following replacement of the eigenfunctions greatly simplifies the equation of motion 28

\begin{align*}
u(y, \theta) := \exp \left[ 2 \ln B(y) + \frac{1}{2} \ln \tilde{C}(y) - \frac{1}{2} \int^y \frac{dy'}{C(y')} \right] U(y, \theta).
\end{align*}

The eigenproblem becomes

\begin{align*}
H_2 u &= \tilde{w} u
\end{align*}

(58)

where the hamiltonian is

\begin{align*}
H_2 := B \begin{pmatrix} M \gamma_7 \cdot \phi & -e^{-i\theta} (\partial_\theta - \frac{\partial_6}{C}) \\ e^{i\theta} (\partial_\theta + \frac{\partial_6}{C}) & -M \gamma_7 \cdot \phi \end{pmatrix}.
\end{align*}

(59)

Here we have introduced the dimensionless coupling constant and the eigenvalue \( M := M/eF \) and \( \tilde{w} := \omega/eF \). One easily confirms that \( H_2 \) commutes with “grandspin”

\begin{align*}
K_3 := l_3 + \frac{\gamma_6}{2} + \frac{\gamma_3^3}{2}
\end{align*}

(60)

where \( l_3 := -i \frac{\partial \phi}{\partial \theta} \) is the orbital angular momentum in the extra space and for convenience, we introduce \( \gamma^6 := I \otimes \sigma_3^i \). Thus the eigenstates are specified by the magnitude of the grandspin, i.e.,

\begin{align*}
K_3 = 0, \pm 1, \pm 2, \pm 3, \cdots, & \text{ for odd } n \\
K_3 = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \cdots, & \text{ for even } n.
\end{align*}

(61)
If we consider “time dependent” Dirac equation
\[ i\partial_\tau \bar{u}(\tau, r, \theta) = H_2 \bar{u}(\tau, r, \theta), \]
the equation is invariant under “time-reversal” transformation \( T \), i.e., \( \tau \rightarrow -\tau \).
\[
T H_2 T^{-1} = H_2, \quad T := i\gamma^5 \otimes \tau^2 C,
\]
Here \( C \) is the charge conjugation operator. Since the Hamiltonian is invariant under time reverse, the states of \( \pm K_3 \) are degenerate in energy.

Once the desired eigenfunctions are obtained, angular averaged fermion densities on the brane can be estimated as follows
\[
\langle \rho \rangle := \int \hat{C} d^3U^1 U = N(y)\rho(y) \tag{64}
\]
\[
\rho(y) := \int d\theta d^2U(y, \theta)u(y, \theta) \tag{65}
\]
where
\[
N(y) = \exp \left[ -4\ln B(y) + \oint dy' \frac{dy'}{C(y')} \right]. \tag{66}
\]
By using the asymptotics of the warp factors [28], we easily find out the behaviour \( N(y) \) at zero and infinity [28]
\[
N(y \rightarrow 0) \rightarrow y
\]
\[
N(y \rightarrow \infty) \rightarrow \frac{1}{\epsilon_1^2} \exp(4py + \frac{1}{p\epsilon_2} e^{py}). \tag{67}
\]

At far from the core, the densities of the excited modes are more enhanced than the localized mode.

The eigenequation (58) can be recast into a (Schrödinger like) second order differential equation. We assume a form of eigenfunction
\[
u(y, \theta) := \begin{pmatrix}
\eta_1(y)e^{il\theta} \\
\eta_2(y)e^{i(l+n+1)\theta} \\
\xi_1(y)e^{i(l+1)\theta} \\
\xi_2(y)e^{i(l+n+1)\theta}
\end{pmatrix} \tag{68}
\]
where \( l \) is an arbitrary integer. In order to eliminate \( \xi_1 \) and \( \xi_2 \) from Eq. (58), we write
\[
\xi_1 = \beta(G_- D^l \eta_1 - \ddot{M} \sin f D'^{l+n} \eta_2) \tag{69}
\]
\[
\xi_2 = -\beta(\ddot{M} \sin f D'_- \eta_1 - G_+ D'^{l+n} \eta_2) \tag{70}
\]
where
\[
\beta := \left( \frac{\ddot{\omega}^2}{B^2} - \ddot{M}^2 \right)^{-1} \tag{71}
\]
\[
G_\pm := \frac{\ddot{\omega} \pm \ddot{M} \cos f}{B} \tag{72}
\]
\[
D'_\pm := \partial_y \pm \frac{1}{C} \tag{73}
\]
and then, we obtain...
plane-wave spinor basis. For $N$ can be obtained from the secular equation; 

\begin{equation}
\{ G_{-} - \delta^{l+1} G_{-} \frac{-G'}{C} - \beta G' \frac{G_{-}}{C^2} \} \eta_1 + \{ -\beta \hat{M} \sin f \partial_y^2 \} \eta_2 = 0
\end{equation}

where $\delta^{l} := \beta + \frac{1}{C} \beta$. Furthermore the following replacement simplifies Eqs. (74), (75)

\begin{align*}
\zeta(y) &:= \exp \left[ - \int^y \frac{R - A_1}{2} \, d\tilde{y} \right] \eta_1(y), \\
\chi(y) &:= \exp \left[ - \int^y \frac{R - A_2}{2} \, d\tilde{y} \right] \eta_2(y),
\end{align*}

where $R := \frac{4 \beta^2}{C} + \eta \big( \big)$. We finally obtain the equation of the form

\begin{equation}
\begin{pmatrix}
-\partial_y^2 - R \partial_y + V_{11} & -A_2 \partial_y + V_{12} \\
-A_1 \partial_y + V_{21} & -\partial_y^2 - R \partial_y + V_{22}
\end{pmatrix}
\begin{pmatrix}
\zeta \\
\chi
\end{pmatrix} = 0. 
\end{equation}

The explicit forms of $A_i, \tilde{A}_i, V_{ij}$ $(i, j = 1, 2)$ are given in appendix A. Here we show the typical example of the effective potential $V_{11}$ in Fig. 5. The shape of the potential is like famous Volcano type. So we expect the existence of localized mode of the fermions.

**B. The numerical method**

Instead of solving the second order differential equation [78], we study the eigenproblem [68] directly. We employ the method which was originally proposed by Kahana-Ripka [35] for solving the Dirac equation with non-linear chiral background. According to the Rayleigh-Ritz variational method [44], the upper bound of the spectrum can be obtained from the secular equation;

\begin{equation}
\text{det} \left( A - \epsilon B \right) = 0
\end{equation}

with

\begin{equation}
A_{ij} = \int d^3 x \varphi^\dagger_i H \varphi_j, \quad B_{ij} = \int d^3 x \varphi^\dagger_i \varphi_j
\end{equation}

where $\{ \varphi_i \}$ $(i = 1, \ldots, N)$ is some complete set of the plane-wave spinor basis. For $N \to \infty$, the spectrum $\epsilon$ becomes exact. Eq. (79) can be solved numerically. For simplicity, we construct a plane-wave basis in large circular box with radius $D$ as a set of eigenstates of the flat, unperturbed $(B = 1, B' = 0, C = r_f = \pi)$ Hamiltonian

\begin{equation}
H_0 = -\gamma^0 \gamma^4 \partial_4 - \gamma^0 \gamma^5 \partial_5 - \gamma^0 \hat{M} r^3.
\end{equation}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig9}
\caption{Fermion number density for the background $(m, n) = (1, 3)$ skyrmion with the model parameters: $\alpha = 0.11$, $\tilde{A}_b = -0.35$, $\tilde{\mu} = 0.957648$. The case of the coupling constant $\tilde{M} = 1.0$. The states $K_3 = 0, \pm 1$ are localized on the brane core. Doubly degenerate $K_3 = \pm 1$ states are more localized (they exhibit the non-zero value at the origin).}
\end{figure}
1, \ldots, m_{\text{max}}) are discretized by the boundary conditions
\begin{equation}
J_p(k_iD) = 0, \quad J_q(l_iD) = 0
\end{equation}
where \( p := K_3 + \frac{1}{2} - \frac{n}{2}, \quad q := K_3 - \frac{1}{2} + \frac{n}{2} \). The orthogonality of the basis is then satisfied by
\begin{equation}
\int_0^D drr J_p(k_ir)J_p(k_hr) = \int_0^D drr J_p(k_iD)J_p(k_hr)
= \delta_{ij} \frac{D^2}{2} [J_{\nu \pm 1}(k_iD)]^2, \quad \nu := K_3 \pm \frac{1}{2} + \frac{n}{2}.
\end{equation}
Expanding the eigenstates of Eq.\( \text{(55)} \) in terms of the plane-wave basis, the eigenproblem reduces to the symmetric matrix diagonalization problem. A special care is taken for the estimation of the matrix element of the hamiltonian \( \text{(55)} \). In order to hold the Hermiticity of the matrix, the following differential rule is imposed
\begin{equation}
\langle \psi \mid y \phi \rangle = \int dyd\theta \mathcal{C}(y) \frac{1}{2} \left[ \psi^\dagger \partial_y \phi - (\partial_y \psi^\dagger)\phi \right].
\end{equation}
Size of the radius \( D \) is chosen so as to wrap the whole branes. Apparently the typical value \( D = 10.0 \) is sufficient because one easily observe that the brane profile functions, the stress energy tensors, and the curvature invariants are flat at \( y > 8 \) (see Figs. \ref{fig:1} and \ref{fig:2}).

Fig.\ref{fig:3} shows the densities \( \rho(y) \) for the background skyrmion of \((m,n) = (1,1)\). We display a tower of the massive modes together with the ground state. As one easily observe that only the lowest mode is peaked on the brane while all other modes escape from the core; therefore the massive modes cannot be observed on the brane core. In Fig.\ref{fig:4} we plot the fermion spectra as varying the coupling constant \( \bar{M} \). Only the lowest isolated mode decays from positive continuum to the negative. We will confirm that the spectrum corresponds to the brane localized mode.

Libanov, Troitsky have discussed relation between the topological charge and the fermionic generation \( \bar{\alpha} \). Four-dimensional fermions appear as zero-modes trapped in the core of the global vortex with winding number three. We shall investigate this speculation with our solution for \( (m,n) = (1,3) \). The results of the expansion at the origin \( \bar{\alpha}_m = \bar{\alpha}_n \) are
\begin{equation}
B = -\frac{\alpha}{4}(\bar{\Lambda}_b + 2\bar{\mu}), \quad C = \frac{\alpha}{12}(\bar{\Lambda}_b + 2\bar{\mu})
\end{equation}
and \( f^{(3)}(0) \) is the shooting parameter. The typical example of solution of the brane is shown in Fig.\ref{fig:5}. As is expected, we observe three localized solutions (Fig.\ref{fig:6}) which can be regarded as the generations of the fermions. A main difference from the previous studies is that we use non-linear type of soliton solutions for constructing the branes. They have richer information of the topology and the spectra exhibit doubly degenerate ground states and a higher excited state. On the other hand, the linear type of solitons like Abelian vortex have the spectra with double degeneracy only. The mass spectra of the fermions in our universe comprises doubly degenerate plus single level with very high energy; it suggests the underlying topology that we employ.

C. Fermion level crossing picture

In Fig.\ref{fig:7} we show that only one isolated mode dives from positive energy to negative. This behavior is called as spectral flow or level crossing picture \( \text{(55)} \). Spectral flow is defined as the number of eigenvalues of Dirac Hamiltonian that cross zero from below minus the
number of eigenvalues that cross zero from above for varying the properties of the background fields. According to the Index theorem a nonzero topological charge implies the zero modes of the Dirac operator \( \tilde{\Lambda} \). The number of flow coincides with the topological charge and the zero modes are emerged when they cross the zero. The level crossing picture extensively studied in the Dirac equation with non-linear chiral background \( \tilde{\Lambda} \), with the Higgs field of the Abelian-Higgs model \( \phi \) and with the non-trivial gauge fields \( (e.g., \text{instanton, meron}) \). In a electroweak theory, one level crossing of the fermion in the background of the sphaleron barrier is observed \([51]\). Sometimes the mechanism can be thought of as a quantum mechanical description of fermion creation/annihilation. Interestingly, we can observe the mixing of some levels, \( i.e., \) the energy levels cannot cross and the excited particle changes with the light one (see Fig.7). This behavior indicates that these fermions interact with each other via some potential. This has been thoroughly discussed in Ref.\([49]\) with somewhat different models. In this subsection, we shall examine the level crossing behavior of the fermion in the brane-skyrmion background.

Studying the spectral flow argument, the authors always investigate evolution of the spectrum by one parameter, \( e.g., \) time (which characterize the size or the strength of the background field). Our model, however, has many parameters which define the basic property of the branes. Therefore, we explore the fermion level behavior for the brane solutions with large parameter space. Fig.[10] shows the result of the spectral flow of the solutions with \((m, n) = (1, 3)\) as a function of the coupling constant \( \tilde{M} \). Fig.[11] is the corresponding fermion densities. One easily observes that the density exhibits various localization behaviors in every domain. In order to examine more thorough parameter dependence of the spectral flow, first we prepare several varieties of the brane solutions with large parameter space (see Fig.[12]). Once the brane solutions are obtained, the model parameters \( (\alpha, \tilde{\mu}, \tilde{\Lambda}_b) \) are no longer arbitrary, \( i.e., \) a constraint \( h(\alpha, \tilde{\mu}, \tilde{\Lambda}_b) = 0 \) exists. So if we fix \( \tilde{\Lambda}_b, \tilde{M} \), either \( \alpha \) or \( \tilde{\mu} \) is identified as an evolution parameter. We display in Figs.[13][14], the spectral flow corresponding to these parameters. Fig.[15] shows more general spectral flow “cascade”, \( i.e., \) the flow as functions of \((\tilde{\mu}, \tilde{M})\).

**D. Mass splitting of the generations**

Because of the time-reversal symmetry, the spectra contain doubly degenerate states and, the first two generations of the fundamental fermions should be observed as the degenerate states within our framework. Some effects can split the degenerate states. For example, in order to manifest the symmetry breaking, we introduce
the explicit mass term for the fermions of the form

\[ H_m := B\gamma^6 \hat{m}, \quad \hat{m} := \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \]  

(86)

\( H_m \) commutes with the grandspin operator and then the eigenstate of the Hamiltonian with this term still are labeled by \( K_3 \). On the other hand, it breaks the time-reversal symmetry, thus, this additional term successfully splits the degeneracy. Without loss of generality, we can set \( m_1 = 0, m_2 = \Delta m \), which is treated as a free parameter.

Another trial is based on a fine structure of the background solitons. In Ref. [52], the authors extensively studied the structure of the multisolitons with charge \( 1 \leq Q \leq 5 \) in the baby Skyrme model. They use one parameter family of the potentials \( U = \mu^2 (1 - \phi^2)^s \). For the \( Q = 3 \), the soliton has spherical symmetry below some critical value \( s \) (our calculation is the case of \( s = 1 \)). The symmetry is broken above the critical value, and the solution exhibits only \( \mathbb{Z}_2 \) symmetry. So, we expect that our branes slightly deform and the degenerate spectra split.

For the \( 3+1 \) Skyrme model, construction of the ansatz for the multisolitons with spherical coordinates \((r, \theta, \phi)\) called Rational map ansatz has been proposed by Houghton, Manton and Sutcliffe [53]. Although the solitons exhibit the complex platonic symmetries, they become tractable by using this ansatz. Coupling of the fermions to the Rational map skyrmions was already studied in Refs. [45, 54]. In the baby Skyrme model, however, no analytical form of ansatz for such non-continuous symmetry exists, then we treat the effects as a small perturbation.

In general, for the solution with non spherical symmetry, the profile function is modified \( f(r) \to F(r, \theta) \). In the case of \( \mathbb{Z}_2 \), if the deformation is small, \( F(r, \theta) \) can be expanded

\[ F(r, \theta) \sim f(r) + \frac{\epsilon}{2} f_1(r) (e^{2i\theta} + e^{-2i\theta}) \]  

(87)

where the first term of the right hand side \( f(r) \) corresponds to the profile function with spherical symmetry. One can easily confirm that the solution has \( \mathbb{Z}_2 \) symmetry, i.e., \( F(r, \theta + \pi) = F(r, \theta) \). Substituting into Eq. (88), the term \( \hat{T} \cdot \hat{\phi} \to \hat{T} \cdot \hat{\phi} + V_\epsilon \) thus Hamiltonian (59) can be written as

\[ H_2 \to \bar{H}_2(\epsilon) \equiv H_2 + B\gamma^6 \hat{M} \hat{V}_\epsilon \]  

(88)

where the potential \( \hat{V}_\epsilon \) is defined as

\[ V_\epsilon(r, \theta; \epsilon) := \epsilon \begin{pmatrix} -\sin f(r)(e^{2i\theta} + e^{-2i\theta}) & \cos f(r)(e^{-i\theta} + e^{-5i\theta}) \\ \cos f(r)(e^{i\theta} + e^{5i\theta}) & \sin f(r)(e^{2i\theta} + e^{-2i\theta}) \end{pmatrix} \]  

(89)

for the charge three \((n = 3)\). Since we have no detailed information about the deformation, we suppose \( \epsilon := \frac{\epsilon}{2} f_1(r) \) as a constant parameter describing measure of the deformation. The potential breaks both the time-reversal and the grand spin symmetry, then the coupling between states with different grandspin \( K_3 \) occur. Mixing of states \((K_3, K'_3)\) with large \( \Delta K_3 := |K_3 - K'_3| \) can be negligible for small deformation. Here we compute the eigenproblem \( \bar{H}_2(\epsilon)u = \hat{\omega}u \), taking into account only the coupling with \( \Delta K_3 \leq 2 \).

At present status of our model, we admit that it is a toy model for understanding the level structure of the realistic standard model fermions because we have no explicit realistic charges. We are able to fit just one quark/lepton sector in terms of a set of model parameters. First we
evaluate the mass of the “down sector” of quarks ($d, s, b$). Fig.10 shows the spectra. We employ a brane solution with the parameters $\alpha = 0.08, \bar{\Lambda}_b = -0.21, \bar{\mu} = 1.4029$, and the coupling constant $M = 0.94$. For obtaining the dimensionful values, we tentatively choose the parameters of the Skyrme model, $Fe = 27800$ MeV. We obtain masses of the down, the strange, and the bottom quark $m_d = 5.1$ MeV, $m_s = 98$ MeV, $m_b = 4200$ MeV, for $\epsilon = 0.012$. For the lepton sector, we have to employ a different solution which is characterized by the set of parameters $\alpha = 0.07, \bar{\Lambda}_b = -0.21, \bar{\mu} = 1.8838$ and $M = 1.03$. The Skyrme model parameter $Fe = 10506$ MeV and when the deformation parameter is $\epsilon = 0.03383$, we obtain the masses $m_e = 0.5$ MeV, $m_\mu = 107$ MeV, $m_\tau = 1779$ MeV (Fig.17). These are in quite good agreement with the corresponding experimental observations.

Here, it is worth mentioning the physical implications of our parameter choice. In Sec.11 we have computed the brane solutions with wide range of the parameters $(\alpha, \bar{\Lambda}_b, \bar{\mu})$, which are physically equivalent. The coupling constant $M$ has been chosen in order that the lowest energy levels cross zero. The splitting of the first two generations and the third can be adjusted in terms of the choice of $(\alpha, \bar{\Lambda}_b, \bar{\mu})$. Essentially, the deformation parameter $\epsilon$ is not a free parameter; it should be determined uniquely in terms of a consistent calculation of the Einstein-Skyrme systems. Thus, one can say that the splitting of the first two generations is also controlled by the brane parameter choice. Finally, the skyrmion parameter $Fe \sim 10^4$ MeV guarantees the size of the branes as $r \sim 10^{-1}$ fm, which is consistent with our observation; it is sufficiently small so as not to observe any evidence of the extra dimension.

IV. CONCLUSION

In this article, we have proposed new brane solutions in 6-dimensional space-time and have discussed about localization of the fermions on them. The brane have constructed by a baby-skyrmion, a generalization of a $O(3)$ non-linear $\sigma$ model, and have realized the regular warped compactification in 6-dimensional anti-de Sitter space-time. The metric is non-factorizable with the warp factors either exponentially diverging or decaying in static solutions. But only exponentially decaying warp factors can allow gravity localization near the brane in the sense that the 6-dimensional Planck mass takes a finite value. Such solutions could be obtained numerically if we impose suitable boundary conditions at the distance far from the brane and integrate in backward. On the other hand, the forward integration method is unstable for any boundary conditions at the origin.

After the solutions of the skyrmion branes were successfully found, we have studied the fermion localization on them. The Dirac equation in 6-dimensional curved space-time has been constructed in terms of introducing the vielbein and 6-dimensional generalization of the gamma matrices. To treat the massive modes, we have used the standard representation of the gamma matrices. In order to solve the eigenproblem, we have introduced the plane wave basis in a large circular box with radius $D$. Studying identification of the charge of the background skyrmions and the generation number of the standard model fermions in our universe, we have investigated the skyrmions of the topological charge $Q = 3$. This conjecture has been confirmed through the discussion of the fermion spectral flow. It embodies the localized modes where the number equals to the background topological charge. As a result, we have found localized fermion modes on the skyrmion branes corresponding to their topological charge. For $(m, n) = (1, 1)$, we have observed the solutions with some parameter ranges, which means the existence of the massive modes as well as the massless one. For $(m, n) = (1, 3)$, three solutions localized on the brane have been found. They comprise doubly degenerate lowest modes of plus single excited state. This level structure well describes the experimental measurements of fermion masses. On the other hand, in the Higgs mediated models, the spectra exhibit only double degeneracy.

As was found in Ref.[52], the minimal energy configuration of the baby skyrmion with charge three has no spherical symmetry, rather, it exhibits $Z_2$ symmetry. The small shape deformation has an effect on the lowest degenerate state and splits them. We have successfully obtained the tower of the “down” sector of quark generation within one parameter family of the deformation. We have treated the deformation of the skyrmion in a perturbative way and have neglected the effect for the gravity. In order to address the deformation of the brane solutions from spherical geometry, we need to examine full simula-
tion of the Einstein-Skyrme system without imposing any symmetrical ansatz for the geometry. We believe that the solution will clarify the origin of the level structure of the standard model fermions.

In Ref.\[21\], the authors had an elaborate analysis for the massive modes in a 5-dimensional anti-de Sitter space-time. They set a domain that is defined by the deformation of the background baby skyrmions as function of the parameter $\epsilon$ which determines the strength of the deformation (in MeV). The experimental values are also from Ref.\[55\].

We observed generation of masses of the down sector $(d, s, b)$ or the lepton sector $(e, \mu, \tau)$ with good agreement of experimental observations. Perhaps most serious drawback of our approach is that we have much number of free parameters and need the independent parameter sets to fit each of the quark/lepton sectors. However, this may be justified if one considers that the quarks and leptons are localized on different branes but share the 4-dimensional spacetime. Another defect of our model is of course lack of the explicit spin and charge of the fermions. Taking into account the effects of the breaking of the flavor symmetry and assigning the realistic charges for all quark/lepton sectors to our solutions will be a great advance toward full understanding of the generation mechanism of our universe.

Our choice of the representation in this article might need more thorough consideration. Although there are several advantages to treat the massive modes, mechanism of localization of left-chiral fermions on the brane is not clear. Estimation of the asymmetry of the chirality on the brane is undoubtedly important subject, at least, about the lowest zero crossing level. The result implementing this will be reported in the forthcoming paper.

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APPENDIX A: THE EFFECTIVE POTENTIAL

The explicit form of $V_{ij}$ In Eq.(78) are given by

\begin{align}
V_{11} &= -\beta^{-1} + \left( \delta^{i+1} \beta^{-1} - \beta \frac{n}{C} \tilde{M}^2 \sin^2 f + \beta G_+ G'_+ - \frac{1}{2} \beta \tilde{M}^2 \sin 2ff' \right) \frac{l + n}{C} - \frac{l \tilde{C}'}{C^2} \\
V_{12} &= -\beta \tilde{M} \frac{l + n}{C} \sin f \left( G_+ \cot ff' - G'_+ - \frac{n}{C} G_+ \right) - A_2 \frac{R - A_2}{2} \\
V_{21} &= -\beta \tilde{M} \frac{n}{C} \sin f \left( G_- \cot ff' - G'_- + \frac{n}{C} G_- \right) - A_1 \frac{R - A_1}{2} \\
V_{22} &= -\beta^{-1} + \left( \delta^{i+1} \beta^{-1} + \beta \frac{n}{C} G_+ G_- + \beta G_- G'_+ - \frac{1}{2} \beta \tilde{M}^2 \sin 2ff' \right) \frac{l + n}{C} - \frac{(l + n) \tilde{C}'}{C^2}
\end{align}

(A1) (A2) (A3) (A4)
where

\[ A_1 := \delta^1 \beta^{-1} + \beta \left( G_+ G'_- - \tilde{M}^2 \sin^2 f \, \frac{f'}{C} - \frac{1}{2} \tilde{M}^2 \sin 2f \, f' \right) \]  
(A5)

\[ A_2 := \beta \tilde{M} \sin f \left( \frac{n}{C} G_+ - G_+ \cot f \, f' + G'_+ \right) \]  
(A6)

\[ \tilde{A}_1 := \beta \tilde{M} \sin f \left( \frac{n}{C} G_- - G_- \cot f \, f' + G'_- \right) \]  
(A7)

\[ \tilde{A}_2 := \delta^1 \beta^{-1} + \beta \left( G_- G'_+ + \tilde{M}^2 \sin^2 f \, \frac{n}{C} - \frac{1}{2} \tilde{M}^2 \sin 2f \, f' \right) \]  
(A8)
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