Explicit Non-Abelian Monopoles and Instantons in SU(N) Pure Yang-Mills Theory

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Abstract

It is well known that there are no static non-Abelian monopole solutions in pure Yang-Mills theory on Minkowski space $\mathbb{R}^{3,1}$. We show that such solutions exist in SU(N) gauge theory on the spaces $\mathbb{R}^2 \times S^2$ and $\mathbb{R} \times S^1 \times S^2$ with Minkowski signature $(-+++)$. In the temporal gauge they are solutions of pure Yang-Mills theory on $T \times S^2$, where $T$ is $\mathbb{R}$ or $S^1$. Namely, imposing SO(3)-invariance and some reality conditions, we consistently reduce the Yang-Mills model on the above spaces to a non-Abelian analog of the $\phi^4$ kink model whose static solutions give SU(N) monopole (-antimonopole) configurations on the space $\mathbb{R}^{1,1} \times S^2$ via the above-mentioned correspondence. These solutions can also be considered as instanton configurations of Yang-Mills theory in $2+1$ dimensions. The kink model on $\mathbb{R} \times S^1$ admits also periodic sphaleron-type solutions describing chains of $n$ kink-antikink pairs spaced around the circle $S^1$ with arbitrary $n > 0$. They correspond to chains of $n$ static monopole-antimonopole pairs on the space $\mathbb{R} \times S^1 \times S^2$ which can also be interpreted as instanton configurations in 2+1 dimensional pure Yang-Mills theory at finite temperature (thermal time circle). We also describe similar solutions in Euclidean SU(N) gauge theory on $S^1 \times S^3$ interpreted as chains of $n$ instanton-antiinstanton pairs.

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1 Introduction and summary

Magnetic monopoles [1] are playing an important role in the nonperturbative physics in 3+1 dimensional Yang-Mills-Higgs theory [2]-[6]. In particular, quark confinement is believed to be explained by the condensation of monopoles and the dual Meissner effect [7]. Confinement should also be a property of the pure Yang-Mills theory without any matter. However, there are no non-Abelian monopoles in pure gauge theory on \( \mathbb{R}^3 \) and that is why the dual Meissner effect is discussed in terms of the Dirac monopoles (‘Abelian dominance’), whose embedding into SU(2) Yang-Mills are point (singular) Wu-Yang monopoles [8].

In this paper, we construct explicit smooth static monopole and monopole-antimonopole solutions in SU(\( N \)) pure Yang-Mills theory on the space \( \mathbb{R} \times T \times S^2 \) with Minkowski signature (\(-+++\)), where \( T = \mathbb{R} \) or \( S^1 \). These configurations can be considered as finite action solutions in pure Yang-Mills theory on the Euclidean space \( T \times S^2 \) (temporal gauge), i.e. as instanton configurations. We also describe explicit instantons and chains of instanton-antistanton pairs on the Euclidean spaces \( \mathbb{R} \times S^3 \) and \( S^1 \times S^3 \), respectively. Note that instanton-antiinstanton chains on \( \mathbb{R}^4 \) describe certain multiparticle scattering events leading to the transitions between topologically distinct vacua [10].

On the other hand, it was recently shown that SU(\( N \)) gauge theories on \( S^1 \times S^{d-1} \) demonstrate for large \( N \) and weak coupling a confinement-deconfinement transition at temperatures proportional to the inverse scale of the spheres [12, 13]. All this may justify the study of the above-mentioned solutions to the Yang-Mills equations on the spaces \( T \times S^{d-1} \) with \( d = 3, 4 \).

The outline of this paper is as follows. We start our discussion in Section 2 with SU(2) Yang-Mills theory on the space \( \Sigma^{1,1} \times S^2 \) with Minkowski signature (\(-+++\)). Here \( \Sigma^{1,1} \) is the space \( \mathbb{R}^{1,1} \) or \( \mathbb{R} \times S^1 \) and \( S^2 = \text{SO}(3)/\text{SO}(2) \) is the standard two-sphere. We consider SO(3)-invariant gauge field with the SO(3)-invariance defined up to a gauge transformation (cf. [14]). The imposition of this symmetry condition reduces SU(2) Yang-Mills theory on \( \Sigma^{1,1} \times S^2 \) to the Abelian Higgs model in 1+1 dimensions and, in the case of vanishing gauge field on \( \Sigma^{1,1} \), to the standard \( \phi^4 \) kink model on the space \( \Sigma^{1,1} \), which are discussed in Section 3. Static kink and antikink configurations on \( \mathbb{R}^{1,1} \) give static monopole and antimonopole solutions of the SU(2) Yang-Mills equations on \( \mathbb{R}^{1,1} \times S^2 \), which can also be interpreted as instanton and antiinstanton configurations in Yang-Mills theory in 2+1 dimensions. In the \( \phi^4 \) kink model on \( \mathbb{R} \times S^1 \) there exist static sphaleron configurations (chains of alternating kinks and antikinks equally spaced around the circle \( S^1 \) [15]) corresponding in our construction to chains of monopole-antimonopole pairs on the space \( S^1 \times S^2 \). The above-mentioned relation between kinks and sphalerons in 1+1 dimensions and SU(2) monopoles and chains of monopole-antimonopole pairs in 3+1 dimensions is discussed in Section 4. In Section 5 we generalize our results to the case of SU(\( N \)) pure Yang-Mills theory on the space \( \Sigma^{1,1} \times S^2 \). Namely, for any \( N > 2 \) the Yang-Mills equations are reduced to a non-Abelian analog of \( \phi^4 \) kink equation which can be interpreted as describing \( N - 1 \) interacting kinks and antikinks. It has multikink and kink-antikink static solutions. These solutions correspond to SU(\( N \)) multi-monopole and monopole-antimonopole configurations on the spaces \( \mathbb{R} \times S^2 \) and \( S^1 \times S^2 \). Finally, in Section 6 we consider SU(\( N \)) Yang-Mills equations on the Euclidean space \( T \times S^3 \) and their SO(4)-symmetric reduction to a matrix model on \( T \), where \( T = \mathbb{R} \) or \( S^1 \). We discuss further algebraic reduction of

\[ \text{See [9] for a multi-monopole generalization of the SU(2) Wu-Yang solution with monopoles located at arbitrary points in } \mathbb{R}^3. \]

\[ \text{For further discussion of the role of instantons in field theory and references see e.g. [11].} \]

\[ \text{However, Lorentz rotations in } \mathbb{R}^{1,1} \text{ yield moving dyons from the viewpoint of 3+1 dimensions.} \]
the matrix field equations to Toda-like equations and describe some of their solutions. They are instantons on the space $\mathbb{R} \times S^3$ and chains of $n$ instantons and $n$ antiinstantons on the space $S^1 \times S^3$. Possible applications of these solutions and their feasible generalizations are briefly discussed.

2 Yang-Mills theory on $\Sigma^{1,1} \times S^2$

**Manifold $\Sigma^{1,1} \times S^2$.** Let $\Sigma^{1,1}$ be the space $\mathbb{R}^2$ or $\mathbb{R} \times S^1$ with (local) real coordinates $x^i$ and Minkowski metric $\eta = (\eta_{ij}) = \text{diag}(1, -1, +1)$, where $i, j = 0, 1$. We use this unusual numbering since it is more convenient from the viewpoint of $3+1$ dimensions where axial symmetry is usually related with the $x^3$-axis. Consider the product $\Sigma^{1,1} \times S^2$ with local real coordinates $x^m$, where indices $\mu, \nu, ...$ run through $0, 1, 2, 3$ so that $x^1$ and $x^2$ are local coordinates on the two-sphere $S^2$.

We also introduce on $S^2 \cong \mathbb{C}P^1$ a local complex coordinate $y = \frac{1}{2}(x^1 + ix^2)$ and coordinates $0 \leq \theta < \pi$, $0 \leq \varphi \leq 2\pi$ via

$$y = R \tan\left(\frac{\theta}{2}\right)\exp(-i\varphi), \quad \bar{y} = R \tan\left(\frac{\theta}{2}\right)\exp(i\varphi),$$

where $R$ is the constant radius of $S^2$. In these coordinates the metric on $\Sigma^{1,1} \times S^2$ has the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = ds^2_\Sigma + ds^2_\mathbb{C}P^1 = \eta_{ij} dx^i dx^j + 2 g_{\bar{y}y} dy d\bar{y} =$$

$$= (dx^0)^2 - (dx^3)^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - (dx^0)^2 - (dx^3)^2 + \frac{4R^4}{(R^2 + y\bar{y})^2} dy d\bar{y},$$

Note that the volume two-form on $S^2$ reads

$$\omega = \omega_{\theta\varphi} d\theta \wedge d\varphi = R^2 \sin \theta d\theta \wedge d\varphi = - \frac{2i R^4}{(R^2 + y\bar{y})^2} dy \wedge d\bar{y}. \quad \text{(2.3)}$$

Here, the bar denotes complex conjugation.

**Yang-Mills equations.** We consider a rank 2 Hermitian vector bundle $\mathcal{E}$ over $M := \Sigma^{1,1} \times S^2$ with a gauge potential $A$ on $\mathcal{E}$ and the gauge field $\mathcal{F} = dA + A \wedge A$. In local coordinates, $A = A_\mu dx^\mu$ and $\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu$ with $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $\partial_\mu := \partial / \partial x^\mu$. Both $A_\mu$ and $\mathcal{F}_{\mu\nu}$ take values in the Lie algebra $su(2)$.

For the standard Yang-Mills action functional

$$S = -\frac{1}{4\pi} \int_M \text{tr} (\mathcal{F} \wedge \ast \mathcal{F}) = -\frac{1}{4\pi} \int_M d^4x \sqrt{g} \text{tr} (\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu})$$

the equations of motion are

$$\frac{1}{\sqrt{g}} \partial_\mu(\sqrt{g} \mathcal{F}^{\mu\nu}) + [A_\mu, \mathcal{F}^{\mu\nu}] = 0.$$

Here, $\ast$ is the Hodge operator and $g = |\det(g_{\mu\nu})|$. In particular, for the metric (2.2) we have $\sqrt{g} = R^2 \sin \theta$.

**SO(3)-invariant gauge potential.** We want to consider the gauge fields on the bundle $\mathcal{E} \rightarrow \Sigma^{1,1} \times \mathbb{C}P^1$ which are invariant under the SU(2) isometry group of $\mathbb{C}P^1 \cong SU(2)/U(1)$. It is natural to allow for gauge transformations to accompany the space-time SU(2) action [14]. The $\mathbb{C}P^1$ dependence in this case is uniquely determined by the rank of the bundle $\mathcal{E}$ and the degree $m$.
of the monopole bundle \( \mathcal{L}^m \) over \( \mathbb{C}P^1 \) (see e.g. [14, 16, 17]). For rank \( \mathcal{E} = 2 \) and \( m = 1 \) the gauge potential has the form

\[
A = \left( \frac{1}{2} A + a \right) \sigma_3 + \frac{1}{2} \phi \beta \sigma_+ - \frac{1}{2} \bar{\phi} \beta \sigma_- \quad \text{with} \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_1^+ \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(2.6)

where \( A \) is an Abelian gauge potential on the complex line bundle \( \tilde{L} \) over \( \Sigma^{1,1} \), \( \phi \in H^0(\Sigma^{1,1}, \tilde{L}) \) is a complex scalar, \( a \) is a one-monopole gauge potential on the bundle \( L \to \mathbb{C}P^1 \) given (locally) by

\[
a = \frac{1}{2(R^2 + y\bar{y})} (\bar{y} \, dy - y \, d\bar{y})
\]

(2.7)

and

\[
\beta = \frac{\sqrt{2} R^2}{R^2 + y\bar{y}} \, dy.
\]

(2.8)

Note that fields \( A \) and \( \phi \) depend only on coordinates on \( \Sigma^{1,1} \).

### Symmetric gauge fields.

In local complex coordinates on \( \Sigma^{1,1} \times S^2 \) the curvature \( \mathcal{F} = dA + A \wedge A \) for \( A \) of the form (2.6) has the field strength components

\[
\begin{align*}
\mathcal{F}_{ij} &= \frac{1}{2} F_{ij} \sigma_3, \\
\mathcal{F}_{y\bar{y}} &= -\frac{1}{4} g_{y\bar{y}} \left( \frac{2}{R^2} - \phi \bar{\phi} \right) \sigma_3, \\
\mathcal{F}_{iy} &= \frac{1}{2} \rho (D_i \phi) \sigma_+ \quad \text{and} \quad \mathcal{F}_{i\bar{y}} = -\frac{1}{2} \rho (\bar{D}_i \bar{\phi}) \sigma_-,
\end{align*}
\]

(2.9a)

where

\[
\begin{align*}
D_i \phi &:= \partial_i \phi + A_i \phi, \\
\bar{D}_i \bar{\phi} &:= \partial_i \bar{\phi} - A_i \bar{\phi} \quad \text{and} \quad \rho := (g_{y\bar{y}})^{1/2}, \\
F_{ij} &:= \partial_i A_j - \partial_j A_i =: -i f_{ij} \quad \text{and} \quad A_j := -i a_j \quad \text{with} \quad a_j \in \mathbb{R}.
\end{align*}
\]

(2.10)

### 3 Abelian Higgs model on \( \Sigma^{1,1} \), kinks and sphalerons

#### Abelian Higgs model.

Substituting (2.9) into (2.4) and performing the integral over \( \mathbb{C}P^1 \), we arrive at the action

\[
S = -\frac{1}{4\pi} \int_M \text{tr} \left( \mathcal{F} \wedge * \mathcal{F} \right) = R^2 \int_{\Sigma^{1,1}} d^2x \left\{ \frac{1}{2} \bar{F}^{ij} F_{ij} + \bar{D}_i \bar{\phi} D^i \phi + \frac{1}{4} \left( \frac{2}{R^2} - \phi \bar{\phi} \right)^2 \right\},
\]

(3.1)

which coincides with the action functional of the Abelian Higgs model at critical coupling \( \lambda = 1 \) (see e.g. [2, 4]). In other words, there is a one-to-one correspondence between gauge equivalence classes of solutions \((A, \phi)\) to the Abelian Higgs model (3.1) and gauge equivalence classes of \( \text{SO}(3) \)-invariant solutions \( A \) of the Yang-Mills equations on \( \Sigma^{1,1} \times S^2 \).

#### Remark.

In the Euclidean regime, the second integral in (3.1) is the Ginzburg-Landau free energy of a superconductor. Recall that in the general case of this functional the last term under the integral in (3.1) has the form

\[
V(\phi, \bar{\phi}) = \frac{\lambda}{4} \left( \frac{2}{R^2} - \phi \bar{\phi} \right)^2
\]

(3.2)

and the critical value \( \lambda = 1 \) of the coupling constant separates Type I \((0 < \lambda < 1)\) and Type II \((\lambda > 1)\) superconductivity. The normal conducting phase (Coulomb phase) corresponds to the limit
$R \to \infty$. Note that if instead of the spherically symmetric ansatz (2.6) one chooses more general ansatz [18] of the form

$$A = \frac{1}{2} (iAQ - QdQ) + \frac{i\phi R}{2\sqrt{2}} (1_2 + iQ)dQ - \frac{i\bar{\phi} R}{2\sqrt{2}} (1_2 - iQ)dQ ,$$  \hfill (3.3)

where

$$Q = i(\sin \theta \cos(m\varphi)\sigma_1 + \sin \theta \sin(m\varphi)\sigma_2 + \cos \theta \sigma_3) \quad \text{with} \quad m \in \mathbb{Z} ,$$  \hfill (3.4)

then in (3.1) one obtains the potential term (3.2) with

$$\lambda = \frac{2m^2}{m^2 + 1} ,$$  \hfill (3.5)

and $\lambda > 1$ for $m > 1$. For $m = 1$ we have $\lambda = 1$ and the potentials $A$ from (2.6) and (3.3) are related by a gauge transformation. Note that the Abelian Higgs model (3.1) and its non-Abelian generalization are integrable on compact Riemann surfaces of genus $g > 1$ [19].

**Equations of motion.** The field equations for $(A, \phi)$ can be obtained either by variation of (3.1) or by substitution of (2.6)-(2.11) into the Yang-Mills equations (2.5). In both ways we obtain the equations

$$\partial_i F^{ij} = \frac{1}{2} \left( \bar{\phi} D^j \phi - \phi D^j \bar{\phi} \right) ,$$  \hfill (3.6)

$$D_i D^j \phi + \frac{1}{2} \left( \frac{2}{R^2} - \phi^2 \right) \phi = 0 .$$  \hfill (3.7)

These equations have interesting sphaleron solutions [20, 21] (i.e. saddle-point static finite energy solutions [22] of the field equations). For their study the authors of [20, 21] consider numerical solving of classical equations of motion for the starting configuration

$$\phi = \phi_{kink} \exp(i\alpha(x^3)) \quad \text{and} \quad A_3 = -i\partial_3 \alpha(x^3) ,$$  \hfill (3.8)

where $\phi_{kink}$ will be discussed in a moment and $\alpha(+\infty) - \alpha(-\infty) = \pi$. For more details see [20, 21]. Here, we want to concentrate on explicit solutions.

For finding explicit solutions we restrict $\phi$ to be a real scalar field, $\bar{\phi} = \phi$. Then the choice $A_i = 0$ solves (3.6) and reduces (3.7) to the standard $\phi^4$ kink model equation

$$\partial_i \partial^i \phi + \frac{1}{R^2} \phi - \frac{1}{2} \phi^3 = 0 ,$$  \hfill (3.9)

which is the Euler-Lagrange equation for the consistently reduced action (3.1),

$$S_{\text{red}} = R^2 \int_{\Sigma_{1,1}} d^2x \left\{ \partial_i \phi \partial^i \phi + \frac{1}{4} \left( \frac{2}{R^2} - \phi^2 \right)^2 \right\} .$$  \hfill (3.10)

The energy of static configurations is

$$E = R^2 \int_{b_1}^{b_2} dx^3 \left\{ (\partial_3 \phi)^2 + \frac{1}{4} \left( \frac{2}{R^2} - \phi^2 \right)^2 \right\} =$$

$$= R^2 \int_{b_1}^{b_2} dx^3 \left( \partial_3 \phi \mp \frac{i}{2} \left( \frac{2}{R^2} - \phi^2 \right) \right)^2 \pm R^2 \int_{b_1}^{b_2} d\phi \left( \frac{2}{R^2} - \phi^2 \right) \geq \frac{8\sqrt{2}}{3R^2} |q| ,$$  \hfill (3.11)
where
\[
q = \frac{R}{2\sqrt{2}} \int_{b_1}^{b_2} dx^3 \partial_3 \phi = \frac{R}{2\sqrt{2}} (\phi(b_2) - \phi(b_1)) \in \{1, 0, -1\} \tag{3.12}
\]
is the topological charge. Here, \(b_1 = -\infty, b_2 = +\infty\) for \(\Sigma^{1,1} = \mathbb{R}^{1,1}, b_1 = 0, b_2 = L < \infty\) for \(\Sigma^{1,1} = \mathbb{R} \times S^1\) and \(\phi(b_1) = \pm \sqrt{2} \frac{\phi}{R}, \phi(b_2) = \pm \sqrt{2} \frac{\phi}{R}\) are the vacua of the model (3.10). Note that (3.11) is the reduction of the Yang-Mills energy functional for static configurations. The equality in (3.11) is attained on the BPS equations
\[
\partial_3 \phi = \pm \frac{1}{2} \left( \frac{2}{R^2} - \phi^2 \right), \tag{3.13}
\]
where the choice of the \(\pm\) sign corresponds to the sign of \(q\) for \(q \neq 0\).

**Kinks.** Static solution of the BPS equation (3.13) with the + sign is known as \(\phi^4\) kink,
\[
\phi_{kink} := \phi(x^3) = \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2} R} x^3 \right), \tag{3.14}
\]
which is a solution interpolating between the vacua \(-\sqrt{2} \frac{\phi}{R}\) and \(+\sqrt{2} \frac{\phi}{R}\) and having the topological charge \(q = 1\). For the antikink with \(q = -1\), interpolating between \(+\sqrt{2} \frac{\phi}{R}\) and \(-\sqrt{2} \frac{\phi}{R}\), we have
\[
\phi_{antikink} = -\frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2} R} x^3 \right). \tag{3.15}
\]
The energy density is maximal at the point \(x^3 = 0\) for both the kink and the antikink. This point is interpreted as their position which can be shifted by \(x^3\)-translations. The energy of the static kink is
\[
E[\phi] = \frac{8}{3} \frac{\sqrt{2}}{R}, \tag{3.16}
\]
and the same for the antikink.

By a Lorentz rotation of \(\mathbb{R}^{1,1}\) one can obtain a kink
\[
\phi(x^0, x^3) = \frac{\sqrt{2}}{R} \tanh \left( \frac{\gamma}{\sqrt{2} R} (x^3 - vx^0) \right) \tag{3.17}
\]
moving with the velocity \(-1 < v < 1\) and \(\gamma := (1-v^2)^{-1/2}\) is the Lorentz factor [4]. For the moving kink (3.17) the energy (3.16) is multiplied by \(\gamma\).

**Sphalerons.** For obtaining sphalerons in the model (3.9)-(3.11) one should consider \(\Sigma^{1,1} = \mathbb{R} \times S^1\) and static solutions of (3.9) defined on \(S^1\) with a circumference \(L\) such that
\[
\phi(x^3 + L) = \phi(x^3). \tag{3.18}
\]
Then sphalerons are given by [15]
\[
\phi_n(x^3; k) = 2k b(k) \text{sn}[b(k) x^3; k] \quad \text{with} \quad b(k) = \frac{1}{R(1 + k^2)^{1/2}} \quad \text{and} \quad 0 \leq k \leq 1. \tag{3.19}
\]
Here \( sn[x;k] \) is the Jacobi elliptic function with the period \( 4K(k) \). The periodicity condition (3.18) is satisfied if

\[
L = \frac{4K(k) n}{b(k)}
\]

(3.20)

for some integer \( n \). Solutions (3.19) exist if \( L \geq L_n := 2\pi R n \) [15].

By virtue of the periodic boundary condition (3.18), the topological charge of the sphaleron (3.19) is zero. In fact, the configuration (3.19) is interpreted as a chain of \( n \) kinks and \( n \) antikinks alternatively and equally spaced around the circle \( S^1 \) [4, 15]. Energy (3.11) of the sphaleron (3.19) is

\[
E[\phi_n] = \frac{4n}{3R} \left[ 8(1 + k^2)E(k) - (1 - k^2)(5 + 3k^2)K(k) \right],
\]

(3.21)

where \( E(k) \) is the complete elliptic integral of the second kind [15]. In the limit \( L \to \infty \) the energy (3.21) becomes \( 2n E[\phi] \), where \( E[\phi] \) is the energy (3.16) of the single kink (3.14).

4 Explicit SU(2) multi-monopole configurations

Non-Abelian monopoles. To obtain explicit smooth SU(2) monopole configurations in pure Yang-Mills theory we should substitute the kink, antikink and kink-antikink chain solutions (3.14), (3.15) and (3.19) into (2.6) and (2.9). In particular, the monopole gauge potential \( A \) reads

\[
A = a\sigma_3 + \frac{1}{2}\phi(\bar{\beta}\sigma_+ - \beta\sigma_-),
\]

(4.1)

where \( a \) and \( \beta \) are given in (2.7), (2.8). We see that \( A_0 = 0 = A_3 \) and \( A_y, A_{\bar{y}} \) can be easily extracted from (4.1).

For the two-form \( F \) of the one-monopole gauge field we have

\[
F = \frac{1}{2} F_{ij} dx^i \wedge dx^j + F_{i\bar{y}} dx^i \wedge d\bar{y} + F_{iy} dx^i \wedge dy + F_{\bar{y}y} dy \wedge d\bar{y} =
\]

\[
= -\frac{1}{4} \left( \frac{2}{R^2} - \phi^2 \right) \left\{ \frac{\sigma_1}{2i} \rho dx^2 \wedge dx^3 + \frac{\sigma_2}{2i} \rho dx^3 \wedge dx^1 + \frac{\sigma_3}{2i} \rho^2 dx^1 \wedge dx^2 \right\} =
\]

\[
= -\frac{1}{8} \left( \frac{2}{R^2} - \phi^2 \right) \frac{\sigma_a}{2i} \epsilon_{bc} \beta^b \wedge \beta^c,
\]

(4.2)

where

\[
\rho = \frac{4\sqrt{2} R^2}{4R^2 + (x^1)^2 + (x^2)^2}, \quad \beta^1 := \rho dx^1, \quad \beta^2 := \rho dx^2 \quad \text{and} \quad \beta^3 := dx^3.
\]

(4.3)

Here, \( \{\beta^a\} \) forms the nonholonomic basis of one-forms on \( \mathbb{T} \times S^2 \) with \( \mathbb{T} = \mathbb{R} \) or \( \mathbb{T} = S^1 \). In derivation of (4.2) we used the fact that the kink solutions (3.14) satisfy the BPS equation (3.13) with the + sign. Thus, \( F \) from (4.2) is a smooth \( su(2) \) gauge field on \( \mathbb{R} \times S^2 \) with finite energy given by (3.16) for \( \phi \) from (3.14). Similarly, substituting (3.15) into (2.9), we obtain antimonopole gauge field which coincides in form with (4.2) for \( x^3 \to -x^3 \).

Topological charges. To understand the topological nature of the solution (4.2), we consider the asymptotic behaviour of the gauge potential (4.1) which follows from the explicit form (3.14) of \( \phi \). Note that for the Euclidean space \( \mathbb{R} \times S^2 \) ‘infinity’ is the disconnected space

\[
S^2 \times Z_2 = S^2_+ \cup S^2_+,
\]

(4.4)
where \( S^2_\pm = S^2 \times \{x^3 = \pm \infty\} \) are two-spheres at \( x^3 = \pm \infty \). Introducing notation \( A_\pm := A(x^3 = \pm \infty) \), from (4.1) we obtain

\[
A_- = h \, dh^{-1} \quad \text{and} \quad A_+ = h^{-1} \, dh \quad \Rightarrow \quad F_\pm = 0 \, \text{ (vacuum)},
\]

where

\[
h = \frac{1}{\sqrt{R^2 + y \tilde{y}}} \begin{pmatrix} R & \tilde{y} \\ -y & R \end{pmatrix} \in SU(2) .
\]

For further clarification, we use \( h \) for the gauge transformation of \( A_- \) to zero, i.e. consider

\[
\tilde{A} := h^{-1} \, A \, h + h^{-1} \, dh \quad \Rightarrow \quad \tilde{A}_- = 0 \quad \text{and} \quad \tilde{A}_+ = g^{-1} \, dg ,
\]

where

\[
g := h^2 = \frac{1}{(R^2 + y \tilde{y})} \begin{pmatrix} R^2 - y \tilde{y} & 2R \tilde{y} \\ -2Ry & R^2 - y \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \exp(i \varphi) \\ -\sin \theta \exp(-i \varphi) & \cos \theta \end{pmatrix} \in SU(2) .
\]

In (4.8) we recognize the map \( g : S^2_+ \to S^2 \subset SU(2) \) of degree 1 in full agreement with the fact that the topological charge of the kink \( \phi \) is \( q = 1 \). Thus, if we consider \( x^3 \) as Euclidean time of Yang-Mills theory in \( 2 + 1 \) dimensions, then our solution (4.1), (4.2) is interpreted as the one-instanton configuration describing transition between the vacua \( \tilde{A}_- = 0 \) and \( \tilde{A}_+ = g^{-1} \, dg \). The (finite) action for this ‘instanton’ configuration is given by formula (3.16). On the other hand, from the viewpoint of Yang-Mills theory in \( 3 + 1 \) dimensions, the solution (4.1), (4.2) describes a finite-energy static one-monopole configuration with the magnetic field strength (4.2). In fact, this is the standard equivalence of instanton and monopole interpretations of finite-action Yang-Mills (-Higgs) topological solitons in \( 3 + 0 \) dimensions. It is especially obvious in the case of Yang-Mills theory on the space \( \mathbb{R} \times S^2 \).

For antimonopole, related with antikink, one can repeat the above calculations and obtain after the gauge transformation \( A \mapsto A' := h \, A \, h^{-1} + h \, dh^{-1} \) that

\[
A'_- = 0 \quad \text{and} \quad A'_+ = g \, dg^{-1} = g \, dg^\dagger.
\]

Hence, for the antimonopole we have the map \( g^\dagger : S^2_+ \to S^2 \subset SU(2) \) of degree \( q = -1 \), as expected. Note again that the topological charges of monopole and antimonopole on the space \( \mathbb{R} \times S^2 \) coincide with the topological charges of kink and antikink, respectively.

**Dyons.** The moving kink (3.17) corresponds to a non-Abelian dyon. It has the same form (4.1) of the gauge potential but for \( F \) we obtain

\[
F = \frac{1}{8} \left( \frac{2}{R^2} - \phi^2 \right) \{ \gamma \nu \sigma_1 \rho \, dx^0 \wedge dx^2 - \gamma \nu \sigma_2 \rho \, dx^0 \wedge dx^1 + \gamma i \sigma_1 \rho \, dx^2 \wedge dx^3 + \gamma i \sigma_2 \rho \, dx^1 \wedge dx^3 + i \sigma_3 \rho \, dx^1 \wedge dx^2 \}. \tag{4.10}
\]

Thus, for the dyon solution (4.10) the electric components \( F_{01} \) and \( F_{02} \) are nonvanishing if \( v \neq 0 \). Its energy is the same as of moving kink.

**Monopole-antimonopole chains.** For the solution (3.19) the equations (3.13) are not valid and we have

\[
F = \frac{1}{4} (\partial_3 \phi) i \sigma_1 \beta^2 \wedge \beta^3 + \frac{1}{4} (\partial_3 \phi) i \sigma_2 \beta^3 \wedge \beta^1 + \frac{1}{8} \left( \frac{2}{R^2} - \phi^2 \right) i \sigma_3 \beta^1 \wedge \beta^2 \tag{4.11}
\]

with \( \beta^a \) given in (4.3). This field describes a chain of \( n \) monopole-antimonopole pairs on \( S^1 \times S^2 \) with the energy (3.21).
5 Explicit monopole solutions in SU($N$) gauge theory

Here we generalize the results of the previous Sections to the case of SU($N$) Yang-Mills theory. So, we consider now a rank $N$ Hermitian vector bundle $E$ over $M := \Sigma^{1,1} \times S^2$ with fields $A_\mu$ and $F_{\mu\nu}$ in (2.4), (2.5) taking values in the Lie algebra $su(N)$.

SO(3)-invariance. For imposing SO(3)-invariance on our $A$ and $F$, we choose a particular case from those ones considered in [17] with $k_0 = \ldots = k_m = 1$ and therefore $N = m + 1$, where $m$ is the maximal first Chern number of the Dirac monopole bundles $L^{m-2\ell} := L^{\otimes (m-2\ell)} \to \mathbb{C}P^1$ with $0 \leq \ell \leq m$. On each such line bundle $L^{m-2\ell}$ we have the unitary connection

$$a^{m-2\ell} = (m - 2\ell)a,$$

where $a$ is given in (2.7), $\ell = 0, \ldots, m$ and $m := N - 1$. The simplest choice generalizing (2.6) is

$$A = \frac{1}{2} A^{(m)} + a^{(m)} + \frac{1}{2} \Phi_m \bar{\beta} - \frac{1}{2} \bar{\Phi}_m \beta,$$

where

$$A^{(m)} = \text{diag}(A^0, \ldots, A^\ell, \ldots, A^m) \quad \text{and} \quad a^{(m)} = \text{diag}(a^m, \ldots, a^{m-2\ell}, \ldots, a^{-m})$$

are diagonal traceless (i.e. $\sum_{\ell=0}^{m} A^{\ell} = 0$) $N \times N$ matrices and

$$\Phi_m = \begin{pmatrix} 0 & \phi_1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \phi_m \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

For more general ansätze see [17].

For $A$ of the form (5.2) the field strength components are

$$F_{ij} = \frac{1}{2} F^{(m)}_{ij} = \frac{1}{2} (\partial_i A^{(m)}_j - \partial_j A^{(m)}_i), \quad F_{y\bar{y}} = -\frac{1}{4} g_{y\bar{y}} \left( \frac{2}{R^2} \Upsilon_m - [\Phi_m, \Phi_m^\dagger] \right),$$

$$F_{i\bar{y}} = \frac{1}{2} \rho (D_i \Phi_m) \quad \text{and} \quad F_{iy} = -\frac{1}{2} \rho (D_i \Phi_m)^\dagger,$$

where

$$D_i \Phi_m := \partial_i \Phi_m + [A^{(m)}, \Phi_m], \quad \Upsilon_m = \text{diag}(m, \ldots, m - 2\ell, \ldots, -m)$$

and $\rho$ is given in (2.2), (2.10).

Matrix $\Phi^4$ kink model. Substituting (5.2)-(5.5) into (2.4) and integrating over $\mathbb{C}P^1$, we obtain the non-Abelian Higgs model

$$S = -\frac{1}{4\pi} \int_M \text{tr}(F \wedge * F) =$$

$$= R^2 \int_{\Sigma^{1,1}} d^2x \text{tr} \left\{ \frac{1}{4} (F^{(m)}_{ij})^\dagger F^{(m)}_{ij} + (D_i \Phi_m)^\dagger D^i \Phi_m + \frac{1}{8} (\frac{2}{R^2} \Upsilon_m - [\Phi_m, \Phi_m^\dagger])^2 \right\},$$

which is equivalent to a model of $m$ interacting complex scalar fields and $m$ Abelian gauge fields. Similarly to the SU(2) case, we can consistently impose the reality condition $\bar{\phi}_\ell = \phi_\ell$ for $\ell = 1, \ldots, m$.
and then put \( A^{(m)} = 0 \). This choice reduces (5.7) to the action functional of the matrix \( \Phi^4 \) kink model

\[
S_{red} = R^2 \int_{\Sigma^{1,1}} d^2 x \, \text{tr} \left\{ \partial_t \Phi_m \, \partial^j \Phi_m^\dagger + \frac{1}{R^2} \left( \frac{2}{R^2} \chi_m - [\Phi_m, \Phi_m^\dagger] \right)^2 \right\},
\]

(5.8)
describing interacting \( \phi^4 \)-type kinks.

From (5.8) we obtain the matrix field equation

\[
\partial_t \partial^j \Phi_m + \frac{1}{R^2} \Phi_m - \frac{1}{4} [\Phi_m, \Phi_m^\dagger], \Phi_m = 0,
\]

(5.9)
which is equivalent to the linked equations

\[
\partial_t \partial^j \phi_\ell + \frac{1}{R^2} \phi_\ell + \frac{1}{4} (\phi_{\ell-1}^2 - 2\phi_\ell^2 + \phi_{\ell+1}^2) \phi_\ell = 0
\]

(5.10)
with \( \ell = 1, \ldots, m \) and \( \phi_0 := 0 =: \phi_{m+1} \).

**Explicit kink-type solutions.** From (5.8) and (5.9) one can see that the vacua are given by

\[
\frac{\sqrt{2}}{R} \Phi^0_m \quad \text{with} \quad \Phi^0_m : \quad \phi^0_\ell = \pm \sqrt{\ell(m - \ell + 1)} \quad \text{for} \quad \ell = 1, \ldots, m
\]

(5.11)
and therefore we have \( 2^m \) distinct vacua for different combinations of signs in (5.11).

Simplest static solution of eq. (5.9) is

\[
\Phi_m = \Phi^0_m \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} x^3 \right)
\]

(5.12)
where \( \Phi^0_m \) is anyone from \( 2^m \) vacua. We are not able to describe the general solution of eqs. (5.10) but can give many explicit examples. In particular, nontrivial smooth solution can be obtained by taking \( \phi_\ell \) vanishing for \( \ell \) from a subset \( I_0 \) of indices \( I = \{1, \ldots, m\} \) and as a kink or an antikink (with the topological charge \( q_\ell = \pm 1 \)) for other values of \( \ell \in I \setminus I_0 \). For instance, we can take

\[
m=2 \quad : \quad \phi_1 = q_1 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_1) \right) \quad \text{and} \quad \phi_2 = 0 (5.13)
\]

\[
m=3 \quad : \quad \phi_1 = q_1 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_1) \right), \quad \phi_2 = 0, \quad \phi_3 = q_3 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_3) \right) (5.14)
\]

\[
m=2r \quad : \quad \phi_1 = q_1 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_1) \right), \quad \phi_2 = 0, \quad \phi_3 = q_3 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_3) \right),
\]

\[
\ldots \quad \phi_{m-1} = q_{m-1} \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_{m-1}) \right), \quad \phi_m = 0 (5.15)
\]

\[
m=2r+1 \quad : \quad \phi_1 = q_1 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_1) \right), \quad \phi_2 = 0, \quad \phi_3 = q_3 \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_3) \right),
\]

\[
\ldots \quad \phi_{m-1} = 0, \quad \phi_m = q_m \frac{\sqrt{2}}{R} \tanh \left( \frac{1}{\sqrt{2R}} (x^3 - a_m) \right) (5.16)
\]

Here, each \( \phi_\ell \neq 0 \) describes a kink (for \( q_\ell = 1 \)) or antikink (for \( q_\ell = -1 \)) located at the point \( a_\ell \in \mathbb{R} \). In (5.15), (5.16), one can consider any \( N = m + 1 > 1 \), including \( N \gg 1 \). The energy of
any such configuration is infinite due to the contribution of saddle points $\phi_\ell = 0$. However, it can be ‘regularized’ by subtracting a constant from the potential energy density.

**Sphaleron-type solutions.** This kind of static periodic solutions appear on $\Sigma^{1,1} = \mathbb{R} \times S^1$. The simplest one is

$$\Phi_m = \Phi^0_m \phi_n (x^3; k),$$

(5.17)

where the sphaleron $\phi_n$ is given by formulae (3.19)-(3.21). Furthermore, suppose we were able to choose $0 \leq k_\ell \leq 1$ and $n_\ell$ such that $\phi_\ell$ have the same period (3.20) for different pairs $(n_\ell, k_\ell)$. Then we can introduce more general than (5.17) sphaleron-type solutions

$$m=2r : \phi_1=\phi_{n_1}(x^3; k_1), \phi_2=0, \ldots, \phi_{m-1}=\phi_{n_{m-1}}(x^3; k_{m-1}), \phi_m = 0$$

(5.18a)

$$m=2r+1 : \phi_1=\phi_{n_1}(x^3; k_1), \phi_2=0, \ldots, \phi_{m-1} = 0, \phi_m = \phi_{n_m}(x^3; k_m),$$

(5.18b)

whose substitution into $\Phi_m$ does not reproduce the factorized form (5.17).

**Explicit SU(N) monopole configurations.** To obtain static monopole configuration in SU(N) pure Yang-Mills theory on $\mathbb{R} \times T \times S^2$, we should substitute into (5.2)-(5.5) with $A^{(m)}=0$ any solution $\Phi_m$ of equation (5.9) smooth on $T = \mathbb{R}$ or $T = S^1$.

For the simplest solutions (5.12) and (5.17), the explicit form of $\mathcal{F}$ can be obtained from (4.2) or (4.11) by substituting

$$\sigma_+ \to \Phi^0_m, \quad \sigma_- \to (\Phi^0_m)^\dagger \quad \text{and} \quad \sigma_3 \to \Upsilon_m,$$

(5.19)

i.e. via embedding the Lie algebra $su(2)$ into $su(N)$ since $i(\Phi^0_m + (\Phi^0_m)^\dagger)$, $\Phi^0_m - (\Phi^0_m)^\dagger$ and $i\Upsilon_m$ form the generators of the $N$-dimensional representation of SU(2). Therefore, the monopole configurations on $T \times S^2$ obtained via (5.19) are $su(2)$ solutions in disguise.

More interesting solutions, not reduced to the $su(2)$-case, will be obtained if we substitute into (5.2) and (5.5) the sandwich-type solutions (5.13)-(5.16) on $\mathbb{R}$ or (5.18) on $S^1$. We will not write down the explicit form of these smooth configurations since they are obtainable simply by the substitution of (5.13)-(5.16) or (5.18) into (5.5). Their topological charges can be obtained by summation of charges of kink/antikink ‘entering’ e.g. in (5.15), (5.16) by virtue of the kink/monopole correspondence discussed in Section 4. These solutions describe non-Abelian magnetic flux tubes extended along the $x^3$-axis since their total energy is infinite as we noted before.

**Towards supersymmetric monopoles.** It is of interest to generalize our monopole solutions to the supersymmetric case. For this, one should consider $\mathcal{N}$-extended supersymmetric Yang-Mills (SYM) theory on $\Sigma^{1,1} \times S^2$ and impose an SO(3)-invariance condition on fermionic and scalar fields from a proper supermultiplet. For instance, one can consider $\mathcal{N} = 4$ SYM model (maximal supersymmetry) and reduce it to a fermionic matrix $\Phi^4$-type kink model in $1+1$ dimensions. Also one can consider the dimensional reduction of $\mathcal{N} = 3$ SYM theory (equivalent to $\mathcal{N} = 4$) which is integrable by twistor methods [23] (see [24] for recent reviews and references). The case $\mathcal{N} = 1$ and $\mathcal{N} = 2$ can also be considered.

Recall that there exist sphaleron-type solutions not only in $\phi^4$ kink model but also in the Abelian Higgs model (3.1) on $\Sigma^{1,1}$ with $A \neq 0$ [20, 21]. Therefore, the results of Refs. [20, 21] on creation of kink-antikink pairs, nonperturbative nonconservation of fermion quantum numbers etc. can be uplifted to $\mathcal{N}$-extended SYM theory on $\Sigma^{1,1} \times S^2$. For instance, the creation of kink-antikink pairs will be equivalent to a creation of monopole-antimonopole pairs. Similar results for the case $\mathcal{N} > 2$ are expectable.
6 Instantons on $\mathbb{R} \times S^3$ and instanton-antiinstanton chains on $S^1 \times S^3$

**Manifold** $\mathbb{T} \times S^3$. Let us consider the Euclidean space $\mathbb{T} \times S^3$, where $\mathbb{T}$ is $\mathbb{R}$ or $S^1$. The three-dimensional sphere can be described via the embedding $S^3 \subset \mathbb{R}^4$ by the equation

$$\delta_{\mu,\nu} x'^\mu x''^\nu = R^2,$$

where $\mu', \nu', ... = 1, ..., 4$. On $S^3$ one can introduce left-invariant one-forms $\{e^a\}$ as (cf. [25])

$$e^a := \sqrt{2} \bar{\eta}^a_{\mu,\nu'} x'^\mu dx'^\nu,$$

where $\bar{\eta}^a_{\mu,\nu'}$ are the anti-self-dual 't Hooft tensors and $a, b, ... = 1, 2, 3$. These one-forms satisfy the Maurer-Cartan equations

$$\frac{1}{\sqrt{2} R} \epsilon^{a}_{bc} e^b \wedge e^c = 0 .$$

Introducing $e^4 := dx^4 = d\tau$, we can write the metric on $\mathbb{T} \times S^3$ in the form

$$ds^2 = \delta_{ab} e^a e^b + e^4 e^4 .$$

**SO(4)-invariant gauge fields.** Let us consider a rank $N$ Hermitian vector bundle $E$ over $\mathbb{T} \times S^3$ with a gauge potential $A$ on $E$ and the gauge field $\mathcal{F} = dA + A \wedge A$, both with values in the Lie algebra $su(N)$. Since $S^3 = SO(4)/SO(3)$ is a homogeneous $SO(4)$-space, we can introduce $SO(4)$-invariant connection $A$ and its curvature $\mathcal{F}$. In the ‘temporal gauge’ $A_\tau = 0$ this connection has the form (cf. [26])

$$A = \frac{1}{2} X_a e^a ,$$

where $X_a = X_a(\tau)$ for $a = 1, 2, 3$ are arbitrary $su(N)$-valued functions of $\tau$.

For the $SO(4)$-invariant curvature $\mathcal{F}$ of $A$ we have

$$\mathcal{F} = dA + A \wedge A = -\frac{1}{2} \dot{X}_a e^a \wedge d\tau + \frac{1}{2} \left( \frac{1}{\sqrt{2} R} \epsilon^{c}_{ab} X_c + \frac{1}{4} [X_a, X_b] \right) e^a \wedge e^b$$

and therefore

$$\mathcal{F}_{ab} = \frac{1}{\sqrt{2} R} \epsilon_{abc} X_c + \frac{1}{4} [X_a, X_b] \quad \text{and} \quad \mathcal{F}_{4a} = \frac{1}{2} \dot{X}_a := \frac{1}{2} \frac{dX_a}{d\tau} .$$

Note that we have chosen normalization coefficients in (6.2) and (6.5) so that our reduced field equations be in conformity with those from Section 3 and Section 4.

**Matrix equations.** On the Euclidean space $\mathbb{T} \times S^3$ one can consider self-dual Yang-Mills (SDYM) and anti-self-dual Yang-Mills (ASDYM) equations,

$$\text{SDYM} : \quad \mathcal{F}_{a4} = \frac{1}{2} \epsilon_{abc} \mathcal{F}_{bc}$$

$$\text{ASDYM} : \quad \mathcal{F}_{a4} = -\frac{1}{2} \epsilon_{abc} \mathcal{F}_{bc}$$

solutions of which are instantons and antiinstantons, respectively.
Substitution of (6.7) into equations (6.8) and (6.9) reduce them to the first order equations:

\[ \dot{X}_a = -\frac{\sqrt{2}}{R} X_a - \frac{1}{4} \epsilon_{abc} [X_b, X_c] \quad \text{for SDYM} , \]  
(6.10)

\[ \dot{X}_a = \frac{\sqrt{2}}{R} X_a + \frac{1}{4} \epsilon_{abc} [X_b, X_c] \quad \text{for ASDYM} . \]  
(6.11)

On the other hand, the full Yang-Mills equations (2.5) reduce to the second order equations

\[ \ddot{X}_a = \frac{2}{R^2} X_a + \frac{3}{2\sqrt{2}R} \epsilon_{abc} [X_b, X_c] + \frac{1}{4} [X_b, [X_a, X_b]] , \]  
(6.12)

which obviously follow from both equations (6.10) and (6.11) but not vice versa. These equations describe a simple matrix model which is the Euclidean version of the truncation \( \mathcal{N} = 4 \rightarrow \mathcal{N} = 0 \) of the well-known plane wave matrix model. For its relation with \( \mathcal{N} = 4 \) SYM theory on \( \mathbb{R} \times S^3 \) see e.g. [27] and references therein. For gravity dual description see e.g. [28].

**Toda-like equations.** First order equations (6.10) and (6.11) are integrable due to integrability of initial SDYM and ASDYM equations (6.8) and (6.9). They can be reduced further to equations generalizing Toda chain equations as we describe below.

Let \( \{H_\alpha, E_\alpha, E_{-\alpha}\} \) be the Chevalley basis for the Lie algebra \( su(N) \) with the commutation relations

\[ [H_\alpha, H_\beta] = 0 , \quad [E_\alpha, E_{-\beta}] = \delta_{\alpha\beta} H_\beta , \quad [H_\alpha, E_\beta] = K_{\alpha\beta} E_\beta \quad \text{and} \quad [H_\alpha, E_{-\beta}] = -K_{\alpha\beta} E_{-\beta} , \]  
(6.13)

where \( K_{\alpha\beta} \) are components of the Cartan matrix and \( \alpha, \beta, ... = 1, ..., m = N - 1 \). We choose for \( \{X_a\} \) in (6.10) the (algebraic) ansatz (cf. [29])

\[ X_1 = \sum_{\alpha=1}^{m} \rho_\alpha (E_\alpha - E_{-\alpha}) , \quad X_2 = -i \sum_{\alpha=1}^{m} \rho_\alpha (E_\alpha + E_{-\alpha}) \quad \text{and} \quad X_3 = i \sum_{\alpha=1}^{m} f_\alpha H_\alpha , \]  
(6.14)

where \( \rho_\alpha \) and \( f_\alpha \) are arbitrary real-valued functions of \( \tau \). It is not difficult to see that after substituting (6.14) into (6.10) and using (6.13) we obtain the equations

\[ \ddot{\phi}_\alpha + \frac{\sqrt{2}}{R} \dot{\phi}_\alpha = \sum_{\beta=1}^{m} K_{\alpha\beta} \exp \phi_\beta - \frac{4}{R^2} , \]  
(6.15)

where

\[ \phi_\alpha := 2 \log \rho_\alpha \quad \text{and} \quad f_\alpha = \sum_{\beta=1}^{m} K_{\alpha\beta}^{-1} \left( \dot{\phi}_\beta + \frac{2\sqrt{2}}{R} \right) \quad \text{with} \quad \sum_{\gamma=1}^{m} K_{\alpha\gamma}^{-1} K_{\gamma\beta} = \delta_{\alpha\beta} . \]  
(6.16)

The standard \( A_m \) Toda chain equations follow from (6.15) in the limit \( R \rightarrow \infty \). Note that in (6.15) one can consider the limit \( N = m + 1 \rightarrow \infty \) and obtain Toda-like lattice equations.

**Some explicit solutions.** Although (6.15) are integrable equations, we will not try to construct their general solutions here. Instead, we restrict ourselves to some simple solutions of eqs. (6.12) and (6.15) related with the \( \phi^4 \) kink equation. Namely, we consider the ansatz

\[ X_a = (\phi - c)T_a , \]  
(6.17)
where \( \phi = \phi(\tau) \) is a function of \( \tau \), \( c \) is a constant and \( T_a \)'s are generators of \( N \)-dimensional representation of SU(2). Substituting (6.17) into (6.12), we obtain the equation

\[
\ddot{\phi} + \left( \frac{2c}{R^2} - \frac{3c^2}{\sqrt{2}R} + \frac{c^3}{2} \right) - \left( \frac{2}{R^2} - \frac{6c}{\sqrt{2}R} + \frac{3c^2}{2} \right) \phi - \left( \frac{3}{\sqrt{2}R} - \frac{3c}{2} \right) \phi^2 - \frac{1}{2} \phi^3 = 0 ,
\]

which for \( c = \sqrt{2}/R \) reduces to the equation

\[
\ddot{\phi} + \frac{1}{R^2} \phi - \frac{1}{2} \phi^3 = 0 ,
\]

coinciding with the static form of \( \phi^4 \)-kink equation (3.9) after substituting \( x^3 \rightarrow \tau \). Thus, solutions (3.14) and (3.15) with \( x^3 \rightarrow \tau \) produce respectively the instanton and the anti-instanton (via (6.5) and (6.17)) solutions on \( \mathbb{R} \times S^3 \), while (3.19) gives a chain of \( n \) instantons and \( n \) anti-instantons on the space \( S^1 \times S^3 \). Recall that the circumference of \( S^1 \) is \( L \) and the radius of \( S^3 \) is \( R \).

In particular, for the one-instanton solution on \( \mathbb{R} \times S^3 \) we obtain

\[
A = \frac{1}{2} \left( \phi - \frac{\sqrt{2}}{R} \right) T_a e^a \quad \text{and} \quad F = \frac{1}{2} \left( \frac{2}{R^2} - \phi^2 \right) \left( d\tau \wedge e^a - \frac{1}{4} \epsilon_{abc} e^b \wedge e^c \right) T_a ,
\]

where

\[
\phi = \frac{\sqrt{2}}{R} \tanh \left( \frac{\tau}{\sqrt{2}R} \right) .
\]

Note that (6.17) corresponds to the ansatz

\[
\rho_\alpha^2 = \exp \phi_\alpha = \frac{1}{2} \alpha(m - \alpha + 1) \left( \phi - \frac{\sqrt{2}}{R} \right) \quad \text{and} \quad f_\alpha = -\alpha(m - \alpha + 1) \left( \phi - \frac{\sqrt{2}}{R} \right)
\]

which reduces (6.10), (6.14) and (6.15) to the equation (3.13) with \( x^3 \rightarrow \tau \), and (6.21) is a solution of this first order equation.

Similarly, substituting into (6.5)-(6.7) and (6.17) the solution

\[
\phi_n(\tau; k) = 2k b(k) \text{sn}[b(k)\tau; k]
\]

of eq. (6.19) on \( S^1 \) with all parameters given in (3.19)-(3.21), we obtain a finite-action configuration which is interpreted as a chain of \( n \) instanton-anti-instanton pairs sitting on \( S^1 \times S^3 \). Note that applications of such kind configurations on the Euclidean space \( \mathbb{R}^4 \) were considered e.g. in [10, 11]. It would be interesting to construct more general solutions of eqs. (6.12) and (6.15). Another possible generalization is to consider noncommutative multi-instantons (see e.g. [30] for ’t Hooft type instantons on the noncommutative \( \mathbb{R}^4 \) on a proper deformation of the space \( \mathbb{T} \times S^3 \). Note that this space is a Lie group both for \( \mathbb{T} = \mathbb{R} \) and \( \mathbb{T} = S^1 \). Therefore, one can consider not only various deformations of spheres (see e.g. [31]) but also a quantum group type deformation of the space \( \mathbb{T} \times S^3 \).
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