$\mathcal{C}$ is an $\left(\infty\right)$-category.
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**Example 1**

$\mathcal{C} = \text{Fin}, \mathcal{C}^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \bigsqcup_{n} B\Sigma_{n}$
$C$ is an ($\infty$-)category.

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**Example 1**

$C = \text{Fin}, C^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \bigsqcup_n B\Sigma_n$

$C^{\text{iso}}$ inherits extra structure from $C$. 
$\mathcal{C}$ is an $(\infty\text{-})$category.
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**Example 1**

$\mathcal{C} = \text{Fin}, \mathcal{C}^{\text{iso}} = \text{Fin}^{\text{iso}} \cong \coprod_n B\Sigma_n$

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**Example 2**

If $\mathcal{C}^{\oplus}$ is symmetric monoidal, $\mathcal{C}^{\text{iso}}$ inherits $\mathbb{E}_\infty$-space structure.
K Theory of categories

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If $\mathcal{C}^\oplus$ is symmetric monoidal, $\mathcal{C}^{\text{iso}}$ inherits $\mathbb{E}_\infty$-space structure.

**Example 3**

$\coprod_n B\Sigma_n$ inherits two $\mathbb{E}_\infty$-space structures from $\amalg, \times$. 
An $\mathbb{E}_\infty$-space $X$ is *grouplike* if the commutative monoid $\pi_0(X)$ is an abelian group.
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**Theorem 4**

$\Omega^\infty : Sp \rightarrow E_\infty Top$ determines an equivalence

$$E_\infty \text{Top}_{gp} \cong Sp^{\geq 0}.$$
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**Theorem 4**

$\Omega^\infty : Sp \to \mathbb{E}_\infty \text{Top}$ determines an equivalence

$$\mathbb{E}_\infty \text{Top}_{gp} \cong Sp^{\geq 0}.$$

$K(C^\oplus)$ = ‘group completion’ of the $\mathbb{E}_\infty$-space $C^{iso}$ (a spectrum).
Example 5

Perfect modules over a ring spectrum: \( C^\oplus = \text{Mod}^{\text{perf}, \oplus}_R \)
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Perfect modules over a ring spectrum: $C^\oplus = \text{Mod}^\text{perf,\oplus}_R$

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Example 6

Finite sets: $C^\oplus = \text{Fin}^{II}$
Examples

Example 5

Perfect modules over a ring spectrum: \( \mathcal{C}^\oplus = \text{Mod}^\text{perf,\oplus}_R \)
\( K(\mathcal{C}^\oplus) = K(R) \) (definition of higher algebraic K-theory)

Example 6

Finite sets: \( \mathcal{C}^\oplus = \text{Fin}^{\Pi} \)
\( K(\mathcal{C}^\oplus) \cong \mathbb{S} \) (Barratt-Priddy-Quillen theorem)
In each case, $C^\oplus$ is a ‘commutative semiring ($\infty$-)category’:
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Obstacles to making this precise:
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Alternative: categorify ordinary semirings and group completion!
Given abelian groups (commutative monoids) $A, B$, there is an abelian group (commutative monoid) $A \bigotimes B$. 

$\binom{1}{\text{Commutative Algebra of Categories}}$

$\textbf{Ab} \bigotimes \text{ComMon}$ is a symmetric monoidal category.

Monoids in $\textbf{Ab} \bigotimes \text{ComMon}$ are rings (semirings).

$\mathbb{Z}(\mathbb{N})$ is the free abelian group (commutative monoid) on one generator.

A commutative monoid is an abelian group if and only if it is a $\mathbb{Z}$-module.

$\mathbb{Z} \bigotimes \mathbb{N}$ is group completion.
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(Gepner-Groth-Nikolaus)

1. Given symmetric monoidal $\infty$-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal $\infty$-category $\mathcal{C} \otimes \mathcal{D}$. 
Semiring categories

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1. Given symmetric monoidal $\infty$-categories $\mathcal{C}, \mathcal{D}$, there is a symmetric monoidal $\infty$-category $\mathcal{C} \boxtimes \mathcal{D}$.
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Examples of commutative semiring $\infty$-categories:
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Examples of commutative semiring $\infty$-categories:

- closed monoidal categories ($\text{Set}$, $\text{Top}$, $\text{Vect}$, $\text{Set}_G$)
- categories built via some constructions ($\text{Set}^{\text{op}}$, $\text{Set}^{\text{iso}}$)
- connective commutative ring spectra ($\mathcal{S}$, $KU$, $HR$)
There is a full subcategory inclusion $\text{Sp}^{\geq 0} \subseteq \text{SymMon}_{\infty}$. 
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**Theorem 7 (B.)**

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**Theorem 7 (B.)**

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Group completion

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John D. Berman  
Commutative Algebra of Categories
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If $\mathcal{C}^\oplus, \otimes$ not a groupoid but semiadditive ($\text{Mod}_R$),

$$K(\mathcal{C}^\oplus) \cong S \otimes \text{Fun}^\oplus, \otimes (\text{Burn}[\text{Cob}_1^{\text{fr}}], \mathcal{C}).$$
$\mathcal{C}^{\oplus}$ a symmetric monoidal category, or $\mathcal{C}^{\oplus,\otimes}$ a semiring category.
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**Definition 8**

- $\mathcal{C}$ is cartesian monoidal if $\oplus = \times$. 

Example 9

$\text{Set}$ is cocartesian monoidal.

$\text{Set}^{\text{op}}$ is cartesian monoidal.

$\text{Ab}$ (or $\text{ComMon}$) is semiadditive.
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**Definition 8**

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- $C$ is cartesian monoidal if $\oplus = \times$.
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\[\]
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\[ \text{Mod}_{\text{Fin}} \cong \text{CocartMonCat} \quad (\mathcal{C}^\oplus \text{ is a Fin-module iff } \oplus = \amalg) \]
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Results are true for categories or \( \infty \)-categories.

Question: What is \( \text{Fin}^b \text{Fin} \otimes \text{Fin}^\text{op} \)?

Theorem 11 (Glasman)
The Burnside category is the free semiadditive category on one object.
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$\text{Fin} \otimes \text{Fin}^{\text{op}} \cong \text{Burn}$
| Semiring $\infty$-category $\mathcal{R}$ | $\mathcal{R}$-modules |
|----------------------------------------|------------------------|
| $\mathcal{S}$                          | Spectra                |
| $\text{Fin}^{\text{iso}}$             | Symmetric monoidal     |
| $\text{Fin}$                           | Cocartesian monoidal   |
| $\text{Fin}^{\text{op}}$              | Cartesian monoidal     |
| $\text{Fin}^{\text{inj}}$             | Symmetric monoidal with initial unit |
| $\text{Fin}^{\text{inj,op}}$          | Symmetric monoidal with terminal unit |
| $\text{Fin}^*$                         | Cocartesian monoidal with $0 = 1$ |
| $\text{Fin}^{\text{op}}_*$            | Cartesian monoidal with $0 = 1$ |
| $\text{Burn}$                          | Semiadditive           |
| $\text{Burn}_{\text{gp}}$             | Additive               |
Definition 12

A PROP (PROduct and Permutation category) is a symmetric monoidal category $\mathcal{P}^\oplus$ generated by one object under $\oplus$. 
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Think: objects labeled by finite sets, $\oplus = \sqcup$. 
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A PROP (PROduct and Permutation category) is a symmetric monoidal category $\mathcal{P}^\oplus$ generated by one object under $\oplus$.

Think: objects labeled by finite sets, $\oplus = \amalg$.

Definition 13

A $\mathcal{P}^\oplus$-algebra in $\mathcal{C}^\otimes$ is a symmetric monoidal functor

$$\text{Alg}_\mathcal{P}(\mathcal{C}^\otimes) = \text{Hom}(\mathcal{P}^\oplus, \mathcal{C}^\otimes).$$
Example 14

- Fin is the PROP for commutative monoids;
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- $\text{Fin}$ is the PROP for commutative monoids;
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If \( \mathcal{P}^{\oplus} \) is cartesian monoidal, \( \mathcal{P}^{\text{op}} \subseteq \text{Alg}_{\mathcal{P}}(\text{Set}^\times) \):
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If \(\mathcal{P}^{\oplus}\) is cartesian monoidal, \(\mathcal{P}^{\text{op}} \subseteq \text{Alg}_{\mathcal{P}}(\text{Top}^\times)\): Subcategory of finitely generated free objects.
Example 14
- Fin is the PROP for commutative monoids;
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If $\mathcal{P}^{+}$ is cartesian monoidal, $\mathcal{P}^{op} \subseteq \text{Alg}_{\mathcal{P}}(\text{Top}^{\times})$: Subcategory of finitely generated free objects.

Definition 15
A Lawvere theory is a cartesian monoidal PROP $\mathcal{L}^{\times}$. Algebras are taken in $\text{Set}^{\times}$ (1-categories) or $\text{Top}^{\times}$ ($\infty$-categories):

$$\text{Alg}_{\mathcal{L}} = \text{Alg}_{\mathcal{L}}(\text{Top}^{\times}) \cong \text{Hom}(\mathcal{L}^{\times}, \text{Top}^{\times})$$
| Lawvere theory | Set-algebras | Top-algebras |
|---------------|-------------|--------------|
| Fin^op        | Set         | Top          |
Lawvere theories

Example 16

| Lawvere theory    | Set-algebras | Top-algebras |
|-------------------|--------------|--------------|
| $\text{Fin}^{\text{op}}$ | $\text{Set}$    | $\text{Top}$   |
| $\text{Burn} = \text{Span}(\text{Fin})$ | $\text{Ab}$ | $\text{Sp}^{\geq 0}$ |
### Example 16

| Lawvere theory       | Set-algebras | Top-algebras      |
|----------------------|--------------|-------------------|
| \( \text{Fin}^{\text{op}} \) | Set          | Top               |
| \( \text{Burn} = \text{Span}(\text{Fin}) \) | Ab           | \( \text{Sp}^{\geq 0} \) |
| \( \text{Poly} = \text{Bispan}(\text{Fin}) \) | \( \text{ComRing} \) | ? (\( \text{ComRingSp}^{\geq 0} \)) |
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Theorem 17 (B.)

- A PROP is a cyclic $\text{Fin}^{\text{iso}}$-module.
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| Burn = Span(Fin) | Ab           | Sp<sup>≥0</sup> |
| Poly = Bispan(Fin) | ComRing      | ? (ComRingSp<sup>≥0</sup>) |

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- A Lawvere theory is a cyclic \( \text{Fin}^{\text{op}} \)-module.
- If \( \mathcal{P}, \mathcal{P}' \) are PROPs/Lawvere theories, so is \( \mathcal{P} \otimes \mathcal{P}' \).
- If \( \mathcal{P}^{\oplus} \) is a PROP, the associated Lawvere theory is \( \mathcal{P}^{\oplus} \otimes \text{Fin}^{\text{op}} \):

\[
\text{Alg}_{\mathcal{P}}(\text{Top}^{\times}) \cong \text{Alg}_{\mathcal{P} \otimes \text{Fin}^{\text{op}}}(\text{Top}^{\times}).
\]
Definition 18 (B.)

An equivariant Lawvere theory is a cyclic $\text{Fin}_G^{\text{op}}$-module $\mathcal{L}^\times$.

$$\text{Alg}_\mathcal{L} = \text{Hom}(\mathcal{L}^\times, \text{Top}^\times).$$
Definition 18 (B.)

An equivariant Lawvere theory is a cyclic $\text{Fin}^\text{op}_G$-module $\mathcal{L}^\times$.

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Theorem 19 (Elmendorf)

$\text{Fin}^\text{op}_G$ is the equivariant Lawvere theory for $\text{Top}_G$.
Definition 18 (B.)

An equivariant Lawvere theory is a cyclic $\text{Fin}_G^{\text{op}}$-module $\mathcal{L}^\times$.

$$\text{Alg}_{\mathcal{L}} = \text{Hom}(\mathcal{L}^\times, \text{Top}^\times).$$

Theorem 19 (Elmendorf)

$\text{Fin}_G^{\text{op}}$ is the equivariant Lawvere theory for $\text{Top}_G$.

Theorem 20 (Guillou-May)

$\text{Burn}_G = \text{Span}(\text{Fin}_G)$ is the equivariant Lawvere theory for $\text{Sp}_G^{\geq 0}$.
Definition 18 (B.)

An equivariant Lawvere theory is a cyclic $\text{Fin}_G^{\text{op}}$-module $\mathcal{L}^\times$.

$$\text{Alg}_{\mathcal{L}} = \text{Hom}(\mathcal{L}^\times, \text{Top}^\times).$$

Theorem 19 (Elmendorf)

$\text{Fin}_G^{\text{op}}$ is the equivariant Lawvere theory for $\text{Top}_G$.

Theorem 20 (Guillou-May)

Burn$_G = \text{Span}(\text{Fin}_G)$ is the equivariant Lawvere theory for $\text{Sp}^{\geq 0}_G$.

Conjecture

Poly$_G = \text{Bispan}(\text{Fin}_G)$ is the equivariant Lawvere theory for $\text{CRingSp}^{\geq 0}_G$. 
Operad $\mathcal{O}$:
- given a finite set $X$, set $\mathcal{O}(X)$ of ways to multiply objects of $X$
Operad $\mathcal{O}$:
- given a finite set $X$, set $\mathcal{O}(X)$ of ways to multiply objects of $X$
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- given a finite set $X$, set $\mathcal{O}(X)$ of ways to multiply objects of $X$
- composition maps
- associative

Example: Commutative operad $\text{Comm}$.
Operad $\mathcal{O}$:
- given a finite set $X$, set $\mathcal{O}(X)$ of ways to multiply objects of $X$
- composition maps
- associative

Example 21
Commutative operad $\text{Comm}(X) = \ast$. 
Application: operads

Symmetric monoidal category $\text{Env}(\mathcal{O})^H$:

- Objects are finite sets.
Symmetric monoidal category $\text{Env}(\mathcal{O})^H$:
- Objects are finite sets.
- Morphism $X \rightarrow Y$ is a way to turn $X$ into $Y$ using operations in $\mathcal{O}$. 
Symmetric monoidal category $\text{Env}(\mathcal{O})^\Pi$:

- Objects are finite sets.
- Morphism $X \to Y$ is a way to turn $X$ into $Y$ using operations in $\mathcal{O}$.
- Symmetric monoidal operation is $\Pi$. 

Example 22: $\text{Env}(\mathcal{O})^{\text{Comm}} \to \text{Fin}$. 

John D. Berman

Commutative Algebra of Categories
Symmetric monoidal category Env(\mathcal{O})^H:

- Objects are finite sets.
- Morphism \( X \rightarrow Y \) is a way to turn \( X \) into \( Y \) using operations in \( \mathcal{O} \).
- Symmetric monoidal operation is \( \Pi \).

Env(\mathcal{O})^H is a PROP; algebras are \( \mathcal{O} \)-algebras

\[
\text{Hom}(\text{Env}(\mathcal{O})^H, \mathcal{C}^\otimes) \cong \text{Alg}_{\mathcal{O}}(\mathcal{C}^\otimes).
\]
Symmetric monoidal category $\text{Env}(\mathcal{O})^\Pi$:

- Objects are finite sets.
- Morphism $X \to Y$ is a way to turn $X$ into $Y$ using operations in $\mathcal{O}$.
- Symmetric monoidal operation is $\Pi$.

$\text{Env}(\mathcal{O})^\Pi$ is a PROP; algebras are $\mathcal{O}$-algebras

$$\text{Hom}(\text{Env}(\mathcal{O})^\Pi, \mathcal{C}^\otimes) \cong \text{Alg}_\mathcal{O}(\mathcal{C}^\otimes).$$

**Example 22**

$\text{Env}((\text{Comm})^\Pi = \text{Fin}^\Pi$. 
Applications: operads

\[
\begin{pmatrix}
\text{Operads} \\
\mathcal{O}
\end{pmatrix} \rightarrow \begin{pmatrix}
\text{PROP}s \\
\text{Env}(\mathcal{O})
\end{pmatrix} \rightarrow \begin{pmatrix}
\text{Lawvere Theories} \\
\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}
\end{pmatrix}
\]
Applications: operads

\[(\text{Operads}) \xrightarrow{O} (\text{PROP}s) \xrightarrow{\text{Env}(O)} (\text{Lawvere Theories}) \xrightarrow{\text{Env}(O) \otimes \text{Fin}^{\text{op}}} \]

**Theorem 23**

*Given an operad $\mathcal{O}$, $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$ is:*

- the Lawvere theory associated to $\mathcal{O}$;
Applications: operads

\[
\begin{align*}
\text{Operads} \quad \mathcal{O} & \quad \rightarrow \quad \text{PROP}s \quad \text{Env}(\mathcal{O}) \quad \rightarrow \quad \text{Lawvere Theories} \quad \text{Env}(\mathcal{O}) \otimes \text{Fin}^{op}
\end{align*}
\]

**Theorem 23**

*Given an operad \( \mathcal{O} \), \( \text{Env}(\mathcal{O}) \otimes \text{Fin}^{op} \) is:*

- *the Lawvere theory associated to \( \mathcal{O} \);*
- *the PROP for \( \mathcal{O} \) – Comm–bialgebras;*
Applications: operads

\[ \left( \text{Operads} \right) \rightarrow \left( \text{PROP}s \right) \rightarrow \left( \text{Lawvere Theories} \right) \]

**Theorem 23**

*Given an operad \( \mathcal{O} \), \( \text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \) is:*

- *the Lawvere theory associated to \( \mathcal{O} \);*
- *the PROP for \( \mathcal{O} \) – Comm–bialgebras;*
- *an explicit span construction.*
Applications: operads

\[
\begin{align*}
\text{(Operads)} & \rightarrow \text{(PROP}s) & \rightarrow \text{(Lawvere Theories)} \\
\mathcal{O} & \rightarrow \text{Env} (\mathcal{O}) & \rightarrow \text{Env} (\mathcal{O}) \otimes \text{Fin}^{op}
\end{align*}
\]

**Theorem 23**

*Given an operad \( \mathcal{O} \), \( \text{Env}(\mathcal{O}) \otimes \text{Fin}^{op} \) is:*

- *the Lawvere theory associated to \( \mathcal{O} \);*
- *the PROP for \( \mathcal{O} \) – Comm–bialgebras;*
- *an explicit span construction.*

**Conjecture**

*The PROP for \( \mathcal{O} \) – \( \mathcal{O}' \)–bialgebras can be computed via a span construction.*
Push/pull square of rings:

\[
\begin{array}{ccc}
\text{Fin}^{\text{iso}} & \rightarrow & \text{Fin} \\
\downarrow & & \downarrow \\
\text{Fin}^{\text{op}} & \rightarrow & \text{Burn}
\end{array}
\]
Push/pull square of rings:

\[
\begin{array}{c}
\text{SymMon}_\infty \xrightarrow{\otimes \text{Fin}} \text{CocartMon}_\infty \\
\downarrow \otimes \text{Fin}^\text{op} \quad \downarrow \otimes \text{Burn} \\
\text{CartMon}_\infty \xrightarrow{\otimes \text{Burn}} \text{SemiaddCat}_\infty
\end{array}
\]

Descent: Can $C_b \text{SymMon}_\infty$ be reconstructed from $C_b \text{Fin}$ and $C_b \text{Fin}^{\text{op}}$?

Answer: Not always!
Future work

Push/pull square of rings:

\[
\begin{array}{ccc}
\text{Fin}^{\text{iso}} & \longrightarrow & \text{Fin} \\
\downarrow & & \downarrow \\
\text{Fin}^{\text{op}} & \longrightarrow & \text{Burn}
\end{array}
\]

**Descent:** Can \( \mathcal{C} \otimes \in \text{SymMon}_\infty \) be reconstructed from \( \mathcal{C} \otimes \text{Fin} \) and \( \mathcal{C} \otimes \text{Fin}^{\text{op}} \)?
Push/pull square of rings:

\[
\begin{array}{ccc}
\text{Fin}^{\text{iso}} & \rightarrow & \text{Fin} \\
\downarrow & & \downarrow \\
\text{Fin}^{\text{op}} & \rightarrow & \text{Burn}
\end{array}
\]

**Descent:** Can \( C \otimes \in \text{SymMon}_\infty \) be reconstructed from \( C \otimes \text{Fin} \) and \( C \otimes \text{Fin}^{\text{op}} \)?

**Answer:** Not always!
Future work

Push/pull square of rings:

\[
\begin{array}{ccc}
\text{Fin}^{\text{iso}} & \longrightarrow & \text{Fin} \\
\downarrow & & \downarrow \\
\text{Fin}^{\text{op}} & \longrightarrow & \text{Burn}
\end{array}
\]

**Descent**: Can \( C \otimes \in \text{SymMon}_{\infty} \) be reconstructed from \( C \otimes \text{Fin} \) and \( C \otimes \text{Fin}^{\text{op}} \)?

**Answer**: Not always!

**Example 24**

\( S \otimes \text{Fin} \cong S \otimes \text{Fin}^{\text{op}} \cong 0 \), but \( S \not\cong 0 \).
Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env}(\mathcal{O}) \otimes \text{Fin}$ and $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?
Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env}(\mathcal{O}) \otimes \text{Fin}$ and $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?

- $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \cong \mathcal{L}_\mathcal{O}$ (Lawvere theory)
Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env}(\mathcal{O}) \otimes \text{Fin}$ and $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?

- $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \cong \mathcal{L}_\mathcal{O}$ (Lawvere theory)
- $\text{Env}(\mathcal{O}) \otimes \text{Fin} \cong \text{Fin}$
Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env} (\mathcal{O}) \otimes \text{Fin}$ and $\text{Env} (\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?

- $\text{Env} (\mathcal{O}) \otimes \text{Fin}^{\text{op}} \cong \mathcal{L}_\mathcal{O}$ (Lawvere theory)
- $\text{Env} (\mathcal{O}) \otimes \text{Fin} \cong \text{Fin}$
- $\mathcal{L}_\mathcal{O} \otimes \text{Burn} \cong \text{Burn}$
Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env}(\mathcal{O}) \otimes \text{Fin}$ and $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?

- $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \cong \mathcal{L}_{\mathcal{O}}$ (Lawvere theory)
- $\text{Env}(\mathcal{O}) \otimes \text{Fin} \cong \text{Fin}$
- $\mathcal{L}_{\mathcal{O}} \otimes \text{Burn} \cong \text{Burn}$

Conjecture

There is an equivalence of ($\infty$-)categories between unital ($\infty$-)operads and cyclic $\text{Fin}^{\text{op}}$-modules with trivialization over $\text{Burn}$. 
Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env}(\mathcal{O}) \otimes \text{Fin}$ and $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?

- $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \cong \mathcal{L}_\mathcal{O}$ (Lawvere theory)
- $\text{Env}(\mathcal{O}) \otimes \text{Fin} \cong \text{Fin}$
- $\mathcal{L}_\mathcal{O} \otimes \text{Burn} \cong \text{Burn}$

Conjecture

There is an equivalence of ($\infty$-)categories between unital ($\infty$-)operads and cyclic $\text{Fin}^{\text{op}}$-modules with trivialization over Burn.

Applications:
- earlier conjecture on operadic bialgebras
Future work

Example 25

Can operad $\mathcal{O}$ be recovered from $\text{Env}(\mathcal{O}) \otimes \text{Fin}$ and $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}}$?

- $\text{Env}(\mathcal{O}) \otimes \text{Fin}^{\text{op}} \cong \mathcal{L}_\mathcal{O}$ (Lawvere theory)
- $\text{Env}(\mathcal{O}) \otimes \text{Fin} \cong \text{Fin}$
- $\mathcal{L}_\mathcal{O} \otimes \text{Burn} \cong \text{Burn}$

Conjecture

There is an equivalence of ($\infty$-)categories between unital ($\infty$-)operads and cyclic $\text{Fin}^{\text{op}}$-modules with trivialization over $\text{Burn}$.

Applications:

- earlier conjecture on operadic bialgebras
- equivariant $\infty$-operads