Design principle of hinge structures with duality-induced hidden symmetry

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Recently, a new type of duality was found in some deformable mechanical networks, which induces a hidden symmetry when the structures take a critical configuration at the self-dual point. However, such duality relies on meticulous structures which are usually found accidentally. In order to discover more self-dual structures with novel topological properties, a design principle of self-dual structures based on a deeper understanding of this duality is needed. In this work, we show that this duality originates from the partial center inversion (PCI) symmetry of the hinges in the structure, which gives each hinge an extra freedom degree without modifying the system dynamics. This property results in dynamic isomers for the hinge chain, i.e., dissimilar chain configurations with identical dynamic modes, which can be utilized to build a new type of flexible wave-guides. Based on this mechanism, we proposed simple rules to identify and design 1D and 2D periodic self-dual structures with arbitrary complexity. This design principle can also guide the experimental realization of the mechanical duality. At last, by taking magnon in 2D hinge lattice as an example, we show that the duality and the associated hidden symmetry is a generic property of hinge structures, independent of specific dynamics of the systems.

INTRODUCTION

Symmetry plays a central role in modern physics. continuous symmetries related with space and time give rise to the fundamental conservation laws, while space groups symmetries, like translation and rotation symmetry, can be utilized to classify the eigenstates of the Hamiltonians subject to these symmetries. There are also other fundamental symmetries related with the local/internal freedom degrees of systems, like gauge symmetry, particle-hole symmetry, time-reversal symmetry etc. These symmetries have provided a powerful tool in searching and analyzing topological materials in large scale owing to ever-developed irreducible representation tables for space groups. However, there are also many hidden symmetries beyond crystallographic symmetries, e.g., composite antiunitary symmetries based on translation, rotation, sublattice exchange, complex conjugation and local gauge transformation. These hidden symmetries usually induce isolated degeneracy at some specific points in Brillouin zone (BZ) and also have deep connection with accident degeneracy.

Duality is another fundamental concept in mathematics and physics. Recently, a new kind of hidden symmetry induced by self-duality was discovered in mechanical isostatic networks and deformable chains. These structures have a low coordination number, which creates many local hinges with unconstrained freedom degrees. By tuning the open angle of hinges \( \vartheta \) from small to large, one can change these structures continuously from an open to a folded state. Interestingly, the structure with small \( \vartheta \) are found to be dual to the structure with large \( \vartheta \), and there exist a critical \( \vartheta^* \), at which the dual counterpart of the structure is itself. At this self-dual point, hidden symmetry emerges, resulting in Kramers-like double degeneracy in the full BZ of the phononic spectrum. It also generates other interesting phenomena, like mechanical non-abelian spintronics, the degeneracy of elastic modulus, the critically-tilted Dirac cone, topological corner states and the symmetric boundary effect etc. Nevertheless, so far, only a few meticulous structures are found to be self-dual, and the origin of the duality and the hidden symmetry remain mysterious.

In this work, we illuminate the origin of this duality and propose a design principle of self-dual structures. We find that this duality originates from a special partial center inversion (PCI) symmetry of the hinge, which gives the system an extra freedom degree without changing the Hamiltonian. When multiple hinges are connected into a hinge chain, the combination of local PCI generates dynamic isomerism of the chain, whose dynamic modes are exactly the same. Based on the mechanism of duality, we also propose simple rules to design 1D and 2D periodic self-dual structures with arbitrary complexity. At last, by further studying a non-mechanical magnon system, we show that this special duality is a generic property of hinge structures, independent of specific system dynamics.

DUALITY IN A SINGLE HINGE

We first consider a simple hinge composed by two structurally identical arms \( a \) and \( b \), which can freely rotate around the hinge point in dimension \( d = 2 \). Each arm is modelled as a spring network with \( n + 1 \) nodes. The total freedom degrees in the system thus is \( (2n + 1)d \), with the hinge node shared by two arms. Under the harmonic approximation, the nodes only do small vibrations around their equilibrated positions, the time-dependent
position of node $i$ in arm $a$, $b$ can be written as

$$r_i^a(t) = R_i^a + u_i^a(t), \quad (1)$$

$$r_i^b(t) = R_i^b + u_i^b(t), \quad (2)$$

respectively, where $R_i^a$ and $R_i^b$ are the equilibrated positions, $u_i^a(t)$ and $u_i^b(t)$ are the time-dependent vibration displacement for the node $i$ in arms $a$ and $b$, respectively. The position of the hinge node is $r_0 = r_0^a - r_0^b = R_0 + u_0$. Thus $R_0 = R_0^a - R_0^b$ and $u_0 = u_0^a - u_0^b$. Since the two arms are structurally identical, they are connected by the rotational operation $\hat{R}(\theta)$ with respect to the hinge point $R_0$:

$$R_i^a = \hat{R}(\theta)R_i^b \quad (3)$$

with $\theta$ the open angle between two arms. The Hamiltonian of the system is

$$H = \frac{1}{2} m_0 v_0^2 + \sum_{i=1}^{n} m_i (v_i^2 + v_i^* v_i) + \sum_{i,k} \lambda \left\{ \left| (u_i^a - u_i^b) \cdot e_i^{a^*} \right|^2 + \left| (u_i^b - u_i^a) \cdot e_i^{b^*} \right|^2 \right\} \quad (4)$$

where $v_0$ is the velocity of the hinge node, $v_{a,i}$ and $v_{b,i}$ are the velocities of the node $i$ in arm $a$ and $b$, respectively. $e_i^{a^*} = R_k^a - R_i^a$ and $e_i^{b^*} = R_k^b - R_i^b$ with $l, k$ running over all spring connections in each arm. The vibration displacement vector $X = \{u_0, u_0^a, u_0^b, u_0^a, \cdots, u_n, u_n^a, u_n^b\}$ satisfies the linear dynamic equation

$$m \cdot \partial_t^2 X = D \cdot X \quad (5)$$

with $D$ the dynamic matrix obtained from Hamiltonian and $m$ the mass of each node. From Eq. (5), we can obtain the phononic spectrum of system, which contains the full dynamic information of the system.

For an isolated mechanical system, the Hamiltonian is invariant under the central inversion with respect to arbitrary fixed point. However, this is usually not true for a part of the system. Nevertheless, for a single hinge, one can prove that the Hamiltonian Eq. (4) is invariant when the center inversion is conducted on a single arm with respect to the hinge point, e.g.,

$$R_i^a = 2R_0 - R_i^b, \quad i \in [0,n+1] \quad (6)$$

We call this transformation partial center inversion (PCI), which is depicted in Fig. 1b,c. Furthermore, rotation symmetry guarantees that the system is invariant after a 90° counter-clockwise rotation around the hinge point (Fig. 1d). Therefore, by defining these two consecutive transformations as $\hat{V}_0$, we have the commutation relationship $[\hat{V}_0, H] = 0$ or $[\hat{V}_0, D] = 0$. It should be noticed that, although PCI leaves the vibrational freedom degree $u_i^a$ intact, the directions of $u_i^a$ are rotated in the second step.

$\hat{V}_0$ can also be interpreted as the combination of operators $\hat{K}$ and $\hat{U}_0$. Here, operator $\hat{K}$ changes the open angle of the hinge from $\theta$ to $\theta^* = \pi - \theta$, i.e., $\hat{K}D(\theta) = D(\theta^*)$, while operator $\hat{U}_0$ switches the corresponding nodes in two arms, i.e., $r_i^a = r_i^b$, and rotates all vibrational freedom degrees by 90°. Thus, we have $\hat{V}_0 = \hat{K}\hat{U}_0$. In the case of $n=2$, which is shown in Fig. 1b, $\hat{U}_0$ can be written as

$$\hat{U}_0 = \begin{pmatrix} \hat{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{r} \end{pmatrix} \quad (7)$$

where $\hat{r} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the four-fold rotation operator.

Here, the first to fifth rows in Eq. (7) correspond to $u_0^a$, $u_{a,1}$, $u_{b,1}$, $u_{a,2}$, $u_{b,2}$, respectively. As can be seen, $\hat{U}$ performs the node switching $(a,1) \leftrightarrow (b,1)$ and $(a,2) \leftrightarrow (b,2)$, while leaves the position of hinge node unchanged. Since $\hat{K}$ commutes with $\hat{U}_0$, from $[\hat{V}_0, D] = 0$ we have

$$\hat{U}_0 \cdot D(\theta) \cdot \hat{U}_0^{-1} = D(\theta^*). \quad (8)$$

Eq. (9) expresses the dual relationship between two hinge configurations at open angles $\theta$ and $\theta^*$. We call this dual relationship as "hinge duality". Especially, $\theta^* = \theta$ corresponds to the self-dual point at which the hinge
FIG. 2: Dynamic isomersm of a hinge chain. (a) A reference period hinge chain with the hinge sequence 000000···. (b) When PCI is conducted on the first and second hinge points consecutively, the hinge chain changes into the 100000··· configuration followed by the 110000··· configuration. These configurations are dynamic isomers of the configuration (a). (c, d) Dynamic isomers with the hinge sequences 111111··· and 1100011110001100···.

remains intact under \( \hat{K} \), while the dynamic modes are transformed by \( \hat{U}_0 \). One can easily prove that the vibrational modes before and after this transformation are orthogonal to each other, i.e.,

\[
\mathbf{X} \cdot (\hat{U}_0 \cdot \mathbf{X}) = 0.
\]  (9)

Therefore, for a given vibration energy or frequency, a hinge structure is double degenerated. Moreover, this hinge duality is independent of the numbers of freedom degree in the arm. This means that even in the continuous limit of \( N \to \infty \), the hinge duality is still preserved.

**DYNAMIC ISOMERISM OF HINGE CHAINS**

From the above analysis, one can see that the PCI gives the hinge an extra freedom degree which preserves the dynamic eigenmodes of the system. When multiple hinges are inter-connected to form a hinge chain, we find these extra freedom degrees are additive, i.e., a hinge chain with \( N \) hinges has \( 2^N \) dissimilar configurations whose vibration dynamics are exactly the same. We call these configurations dynamic isomers, which can be labeled by a binary sequence "000000..." of length \( N \). Here, 1 and 0 indicates two different dual states with open angle \( \theta \) and \( \theta^* \) for a single hinge. In Fig. 2, we show several configurations of dynamic isomers with different sequences, where sequence "000000..." or "111111..." represents two simplest periodic hinge chains (Fig. 2a,b,c). We also show an intermediate configuration between these two states in Fig. 2b, and a more disordered chain configuration in Fig. 2d. It’s very surprising that such a disordered chain has the same eigenmodes as that of the periodic chains. It should be noticed that mode preogation in "111111..." chain configuration is rotated by 90° compared with the original configuration with sequence “000000...”. In fact, one can also construct other kinds of periodic hinge chain, e.g., that with sequence “101010...”, in which the vibration mode propagates in a different direction. These intriguing properties can be used to build a new type of flexible wave-guides without loss. Moreover, in chemistry and biology, many small molecules and macro-molecules are either single hinge or hinge chain structure. It is very interesting to explore whether the hinges duality also exists at this molecule scale [36].

**DUALITY IN PERIODIC HINGE CHAINS**

From the above analysis, one can see that the two minimal periodic hinge chain with sequence "000000..." and "111111..." are connected by combination of PCI transformations on every hinges and a global 90° rotation of the whole chain. These two steps define the dual transformation for the periodic hinge chain, which is identical to direct applying the single dual transformation \( \hat{V}_0 \) in Fig. 1 to the hinge in the unit cells of the periodic structure. To see this more clearly, we use \( l \) to mark different
FIG. 3: Hinge duality in the periodic hinge chain. (a) Directly applying the dual transformation in Fig. 1 to the unit cell of the periodic hinge chain inverts the wave propagation direction or wave vector $\mathbf{q}$. (b) Phononic spectrums of the periodic hinge chain at three different open angles $\vartheta = 60^\circ, 90^\circ, 120^\circ$, where $\vartheta = 90^\circ$ is the self-dual point at which the hidden symmetry induces the double degeneracy.

cells and use $(l,i)$ representing the $i$ node in cell $l$. As shown in Fig. 3, nodes $(l,1)$ and $(l,2)$ is on arm $a$, while nodes $(l,3)$ and $(l+1,1)$ is on arm $b$ of cell $l$. Note that node $(l+1,1)$ also shared by arm $a$ in cell $l+1$. When the operator $\hat{V}_0$ acts on the unit cell, one has the node switching: $(l,2) \leftrightarrow (l,3)$ and $(l,1) \leftrightarrow (l+1,1)$. The switching between $(l,1)$ and $(l+1,1)$ reverses the propagation direction of vibrational modes. Thus, this transformation for periodic hinge chain should be written as $\hat{V}_1 = K\hat{U}_1$ with

$$\hat{U}_1 = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & T_{a,b} \end{pmatrix}, \hat{I} \tag{10}$$

Here, the switching between $(l,1)$ and $(l+1,1)$ is expressed as the combination of operator $T_{a,b}$, which shifts node $(l,1)$ one period, and complex conjugation $\hat{I}$ which reverses the sign of wave vector $\mathbf{q}$. Therefore, $\hat{U}_1$ is an anti-unitary matrix satisfying $\hat{U}_1^2 = -1$. Moreover, the transformation in Fig. 3 suggests that $\hat{V}_1$ commutes with $\hat{D}$, i.e., $[\hat{V}_1, \hat{D}] = 0$. A proof of this commutation relationship is also provided in Supporting Information based on the dynamic matrix in $q$ space. Therefore, we have the dual transformation relationship for the periodic chain

$$\hat{U}_1 \cdot \hat{D}(\vartheta, \mathbf{q}) \cdot \hat{U}_1^{-1} = \hat{D}(\vartheta^*, \mathbf{q}). \tag{11}$$

Similar to that in single hinge, the above dual relationship is also independent of the freedom degree in the hinge arms. In Fig. 3b, we show the phononic spectrum for the periodic hinge chains shown in Fig. 3a. We find the identical spectrum for system at $\vartheta = 60^\circ$ and $\vartheta^* = 120^\circ$, as well as the double degeneracy in the whole BZ zone.

2D PERIODIC HINGE NETWORKS

The simple design rule of 1D periodic hinge dual structure can also be extended to 2D periodic one. Nevertheless, one first needs to construct the unit cell of 2D periodic structure, which contains a hinge. The procedures are in the following: i) constructing a single hinge in which each arm of the hinge should has more than three nodes; ii) making the corresponding nodes in each arms as a pair; iii) choosing two pairs of corresponding nodes and construct two vectors that connects the nodes in each pair. These two vectors define the two lattice vectors $\mathbf{a}_1$.
and $\mathbf{a}_2$. Following the above procedure, one in principle can generate infinite number of 2D periodic structures with hinge duality by choosing different hinge arm in the unit cell. The dual transformation of 2D periodic structure is similar to 1D, which also relies the operator $\hat{V}_0$ to do the node switching between the corresponding nodes of the hinge in the unit cell. For examples, Fig.4 a,b,c show three cases of 2D periodic structures, in which the arm in the hinge is composed by three nodes and different bond connections. It should be noticed that the structure in Fig. 4c with equilateral triangle hinge arm is the twisted Kagome lattice with p31m space group symmetry, which is the first structure found to have dual properties[17, 28]. Since the relative position of the nodes are the same, these three structures have the same dual transformation

$$\hat{U}_2 \cdot D(\theta, \mathbf{q}) \cdot \hat{U}_2^{-1} = D(\theta^*, \mathbf{q}).$$

Here, the nodes switchings $(l_1, l_2, 1) \rightarrow (l_1 + 1, l_2, 1)$ and $(l_1, l_2, 2) \rightarrow (l_1, l_2 + 1, 2)$ in unit cell $(l_1, l_2)$ are related with the operator $\hat{T}_{a_1}$ and $\hat{T}_{a_2}$ along the lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively. If the hinge arm is a hourglass made up by two equilateral triangles (Fig. 4d), the corresponding 2D structure is a lattice with p2g symmetry, which can also be obtained by twisting the standard Kagome lattice. When the hinge arm is a perfect square, one obtains a 2D lattice with p4g symmetry, which is also called snub square crystal. One can prove all these structures constructed by this method have the same self-dual point at $\theta = 90^\circ$.

The above design principle can also have more complicated variation. In Fig. 3f, we show the lattice with pmg symmetry first reported in [18], which is the second isostatic structure found to have hinge duality. The unit cell of this structure is a hinge made up by two rhombus arm with a ‘dangling’ bond on each arm. However, the two ‘dangling’ bonds are arranged in a opposite direction, making two arms unable to map to each other by a single rotation. In fact, the hinge duality in this structure involves additional PCI transformation for the dangling bond (see the dashed blue bond in Fig. 4f). This example suggests the existence of other complicated rules based on multiple PCIs to build self-dual structures with different symmetries. One can see that Fig. 4a-f are all deformable networks. In fact, the extra free degrees of the hinge in the unit cell and the energy conservation during dual transformation guarantee that the structure

FIG. 4: 2D periodic lattices with hinge duality. (a-f) By constructing different hinges in 2D unit cells, one can obtain various 2D periodic lattices with hinge duality. Note that (a, b) are same lattice under different representation; (c) is the p31m twisted kagome lattice found in [17], (f) is the pmg lattice discovered in [18]. The corresponding bond pairs are drawn using the same color.
must be deformable. Furthermore, one can also see that changing the mass of the corresponding node pair, or the elastic constant for corresponding bond pair in the hinge, does not affect the hinge duality. This further increases the number of 2D hinge-dual structures. At last, the complexity of our design rules does not scale up with the number of freedom degree in the unit cell. This makes it possible to construct bulk self-dual metamaterials with desired functions.

**GENERALIZATION TO NON-MECHANICAL SYSTEMS**

The hinge duality also exists in non-mechanical dynamic systems. To see this, let us consider a 2D magnon system. The magnon is the collective magnetic excitation associated with the precession of the spin moments [40]. In the model of Heisenberg ferromagnet with exchange interaction, the Hamiltonian of the system can be written as

$$ H = -2J \sum_{i,j} s_i \cdot s_j $$

(13)

where the \((i,j)\) indicate the nearest spin pairs. The dynamic equation of the system can be obtained based on the Heisenberg picture of quantum dynamics,

$$ i\hbar \frac{d}{dt} \langle s_i(t) \rangle = \langle [s_i, H] \rangle. $$

(14)

By assuming a small perturbation of spins along the z axis, i.e., \(s_x, s_y \ll s_z\). The dynamic equation of magnon can be written as

$$ \frac{d}{dt} \tilde{S}_\perp(t) = \frac{2JS}{\hbar} D_m \cdot S_\perp $$

(15)

where \(S_\perp = (s_{1x}, s_{1y}, s_{2x}, s_{2y}, \cdots)\) and \(\tilde{S}_\perp = (s_{1y} - s_{1x}, s_{2y} - s_{2x}, \cdots)\). Here \(D_m\) is the dynamic matrix of magnon. Eq. (15) is a first-order dynamic equation, which is different from the second-order one for phonons. In Fig. 5, we show the magnon spectrums for the p31m twisted Kagome lattice under three different open angles. Similar to the phonon systems [17], we observe an identical magnon spectrum for systems at dual open angles \(\vartheta = 60^\circ\) and \(\vartheta = 120^\circ\), as well as the Kramers-like degeneracy at the self-dual point \(\vartheta = 90^\circ\). These results suggest that the hinge duality is a generic property of hinge structures, independent of the type of system dynamics.

**DISCUSS AND CONCLUSION**

In conclusion, we unveil the mechanism of hinge duality and the corresponding duality-induced hidden symmetry in mechanical and non-mechanical hinge systems. We find the hinge structure has a unique partial center inversion (PCI) symmetry, based on which the system can have a Hamiltonian-invariant dual transformation between two configurations with two different open angles. This PCI symmetry gives each hinge an extra freedom degree while leaving the dynamics of the system intact. This intriguing property leads to the dynamic isomerism of hinge chains, which can be either periodic or non-periodic. Furthermore, we prove that PCI symmetry also exists in 1D and 2D periodic structures, and propose a simple rule to design self-dual 1D and 2D periodic structures with arbitrary complexity. At last, we show that the hinge duality is a generic property of the structure that also exists in non-mechanical magnon system. During the preparation of this work, we also noticed the work from [19], which discussed the generality of duality-induced hidden symmetry in other Hamiltonian systems from a more fundamental point of view. We expect further breaking through in discovery of various kinds of hidden symmetry related with duality, especially in ape-
periodic 2D [41] and periodic 3D isostatic systems [42].

[1] Dresselhaus, M. S., Dresselhaus, G. & Jorio, A. Group theory: application to the physics of condensed matter (Springer Science & Business Media, 2007).
[2] Zhang, T. et al. Catalogue of topological electronic materials. Nature 566, 475–479 (2019).
[3] Vergniory, M. et al. A complete catalogue of high-quality topological materials. Nature 566, 480–485 (2019).
[4] Fruchart, M., Yao, C. & Vitelli, V. Systematic generation of Hamiltonian families with dualities. arXiv preprint arXiv:2107.07924 (2021).
[5] Liu, J.-M. & Semperlotti, F. Synthetic Kramers pair crossing. Proc Natl Acad Sci U S A 117, 30252–30259 (2020).
[6] Mao, X. & Lubensky, T. C. Elasticity and response in nearly isostatic periodic lattices. Phys. Rev. Lett. 103, 205503 (2009).
[7] Sun, K., Souslov, A., Mao, X. & Lubensky, T. C. Surface phonons, elastic response, and conformal invariance in twisted kagome lattices. Proc Natl Acad Sci U S A 109, 12369–74 (2012).
[8] Klein, M. & Jung, J. On a degeneracy theorem of Kramers. Am. J. Phys. 20, 65–71 (1952).
[9] Fruchart, M. & Vitelli, V. Symmetries and Dualities in the Theory of Elasticity. Phys. Rev. Lett. 124, 248001 (2020).
[10] Danawe, H., Li, H., Ba'ba'a, H. A. & Tol, S. Existence of Corner Modes in Elastic Twisted Kagome Lattices. arXiv preprint arXiv:2107.07924 (2021).
[11] Gonella, S. Symmetry of the phononic landscape of Corner Modes in Elastic Twisted Kagome Lattices. arXiv preprint arXiv:2107.07924 (2021).
[12] Karch, A. & Tong, D. Particle-vertex duality from 3d bosonization. Phys. Rev. X 6, 031043 (2016).
[13] Arndt, M. et al. Wave–particle duality of C 60 molecules. Nature 401, 680–682 (1999).
[14] Nakata, Y., Urase, Y. & Nakashima, T. Geometric Structure behind Duality and Manifestation of Self-Duality from Electrical Circuits to Metamaterials. Symmetry 11, 1336 (2019).
[15] Fruchart, M., Zhou, Y. & Vitelli, V. Dualities and non-Abelian mechanics. Nature 577, 636–640 (2020).
[16] Lei, Q.-L. et al. Self-Assembly of Isostatic Self-Dual Colloidal Crystals. Phys. Rev. Lett. 127, 018001 (2021).
[17] Fruchart, M., Yao, C. & Vitelli, V. Systematic generation of Hamiltonian families with dualities. arXiv preprint arXiv:2108.11138 (2021).
[18] Lubensky, T., Kane, C., Mao, X., Souslov, A. & Sun, K. Phonons and elasticity in critically coordinated lattices. Rep. Prog. Phys. 78, 073901 (2015).