A DIFFUSION PROBLEM OF KIRCHHOFF TYPE INVOLVING THE NONLOCAL FRACTIONAL $p$–LAPLACIAN

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Abstract. In this paper, we study an anomalous diffusion model of Kirchhoff type driven by a nonlocal integro–differential operator. As a particular case, we are concerned with the following initial–boundary value problem involving the fractional $p$–Laplacian
\[
\begin{cases}
\partial_t u + M \left( \|u\|_{s,p}^p \right) (\Delta)_p u = f(x, t) & \text{in } \Omega \times \mathbb{R}^+,
\partial_t u = \partial u / \partial t, & \text{in } \Omega,
\partial u / \partial n = 0 & \text{in } \partial \Omega,
\end{cases}
\]
where $\|u\|_{s,p}$ is the Gagliardo $p$–seminorm of $u$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial \Omega$ and $1 < p < N/s$, with $0 < s < 1$, the main Kirchhoff function $M : \mathbb{R}_+^+ \to \mathbb{R}_+^+$ is a continuous and nondecreasing function, $(-\Delta)_p$ is the fractional $p$–Laplacian, $u_0$ is in $L^2(\Omega)$ and $f \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega))$. Under some appropriate conditions, the well–posedness of solutions for the problem above is studied by employing the sub–differential approach. Finally, the large–time behavior and extinction of solutions are also investigated.

1. Introduction. In this paper, we study the fractional Kirchhoff type parabolic problem
\[
\begin{cases}
\partial_t u + M \left( \|u\|_{X^{s,p}_0(\Omega)}^p \right) \mathcal{L}_K u = f(x, t) & \text{in } \Omega \times \mathbb{R}^+,
\partial u / \partial t = u_0(x) & \text{in } \Omega,
\partial u / \partial n = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where $\|u\|_{X^{s,p}_0(\Omega)}$ is the Gagliardo $p$–seminorm of $u$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial \Omega$ and
\[
\| \varphi \|_{X^{s,p}_0(\Omega)}^p = \left( \int_{\Omega} \int_{\Omega} |\varphi(x) - \varphi(y)|^p K(x - y) dxdy \right)^{1/p},
\]
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is the norm on the main fractional Sobolev space $X_0^{s,p}(\Omega)$, which we introduce in Section 2 and which is well-defined along any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Here and in the follows

$$Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega) \quad \text{and} \quad C\Omega = \mathbb{R}^N \setminus \Omega.$$  \hspace{1cm} (1.3)

The elliptic part of problem (1.1) is represented by $\mathcal{L}_K$, which is a nonlocal integro-differential operator, defined pointwise for each $x \in \mathbb{R}^N$ by

$$\mathcal{L}_K \varphi(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} |(\varphi(x) - \varphi(y))|^{p-2}(\varphi(x) - \varphi(y))K(x-y) \, dy$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_{\varepsilon}(x)$ denotes the ball in $\mathbb{R}^N$ with radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^N$. The kernel $K: \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfies

$$\begin{cases} 
  mK \in L^1(\mathbb{R}^N), & \text{where } m(x) = \min\{|x|^p, 1\}; \\
  \text{there exists } K_0 > 0 \text{ such that } K(x) \geq K_0 |x|^{-(N+ps)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}. 
\end{cases} \hspace{1cm} (1.4)$$

A typical example for $K$ is given by $K(x) = |x|^{-N-ps}$. In this case,

$$[\varphi]_{s,p} = \left( \int_{\mathbb{R}^{2N}} |\varphi(x) - \varphi(y)|^p |x-y|^{-N-ps} \, dx \, dy \right)^{1/p}, \hspace{1cm} (1.5)$$

that is the norm in (1.2) becomes the celebrated Gagliardo semi-norm. Similarly, the operator $\mathcal{L}_K$, when $K(x) = |x|^{-N-ps}$, reduces to the fractional $p$–Laplacian $(-\Delta)_p^s$, which (up to normalization factors) may be defined for any $x \in \mathbb{R}^N$ as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}[\varphi(x) - \varphi(y)]}{|x-y|^{N+ps}} \, dy$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$. We refer to [8, 16, 17, 23, 24, 28, 32, 34, 35, 38, 39] and the references therein for further details on the fractional Laplacian operator, and to [18] for further details on the fractional Laplacian and on the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$. Throughout the paper, without further mentioning, we always assume for simplicity that $N > ps$ and that $K$ satisfies (1.4). It is worth to note that the case $N = 2s$, that is when $p = 2$, $N = 1$ and $s = 1/2$, it was recently studied in [19]. As far as we know even the case $N = 2s$ is completely new for (1.1).

Throughout the paper $M: \mathbb{R}^N_+ \to \mathbb{R}^+$ denotes the main Kirchhoff function, assumed to be continuous and nondecreasing, and $u_0$ the initial value of class $L^2(\Omega)$. The interest in studying problems like (1.1) relies not only on mathematical purposes, but also on their significance in real models, as explained by Caffarelli in [11, 12], Laskin in [26] and Vázquez in [37]. It is worthy pointing out that Applebaum in [7] states that the fractional Laplacian operators of the form $(-\Delta)^s$, are the infinitesimal generators of stable radially symmetric Lévy processes. Laskin in [27] formulated a fractional Schrödinger equation as a result of expanding the Feynman path integral, from the Brownian–like to the Lévy–like quantum mechanical paths.

Recently, Fiscella and Valdinoci in [23] first proposed a stationary Kirchhoff variational equation, involving the fractional Laplacian, which models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. Indeed, the stationary problem (1.1) is a fractional version of a model, the so–called stationary Kirchhoff equation, introduced by Kirchhoff for certain evolution phenomena. We refer e.g. to [9, 21, 33] for evolution equations of Kirchhoff type, to [32, 34, 35] for non–degenerate stationary Kirchhoff problems, to [8, 14, 30, 35, 38] for the degenerate case and the references therein.
To explain the motivation of problem (1.1), let us shortly introduce a prototype of nonlocal problem like (1.1) in $\mathbb{R}^+ \times \mathbb{R}^N$. Indeed, nonlocal evolution equations of the form
\[
\partial_t u(x,t) = \int_{\mathbb{R}^N} [u(y,t) - u(x,t)] K(x - y) dy,
\]
and its variants, have been recently widely used to model diffusion processes. More precisely, as stated by Fife in [20], if $u = u(x,t)$ is thought of as a density of population at the point $x$ and time $t$ and $K(x - y)$ is thought of as the probability distribution of jumping from location $y$ to location $x$, then $\int_{\mathbb{R}^N} u(y,t) K(x - y) dy$ is the rate at which individuals are arriving at position $x$ from all other places and $-\int_{\mathbb{R}^N} u(x,t) K(x - y) dy$ is the rate at which they are leaving location $x$ to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density $u$ satisfies problem (1.6). For recent references on nonlocal diffusions, see for example [2, 3, 4, 13, 15, 31].

If we consider the effects of total population, then problem (1.6) becomes
\[
\partial_t u(x,t) = M \left( \int_{\mathbb{R}^N} |u(x,t) - u(y,t)|^2 K(x - y) dx dy \right) \\
\times \int_{\mathbb{R}^N} [u(y,t) - u(x,t)] K(x - y) dy,
\]
where the coefficient $M : \mathbb{R}^+_0 \rightarrow \mathbb{R}^+$ denotes the possible changes of total population in $\mathbb{R}^N$. This means that the behavior of individuals is subject to total population, such as the diffusion process of bacteria. Problem (1.7) is also meaningful, since the way of measurements are usually taken in average sense. In particular, if $K(x) = |x|^{-N-2s}$ and $s \nearrow 1^-$, then equation (1.7) reduces to
\[
\partial_t u - M \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u = 0,
\]
which has been investigated in recent papers. See for instance [25] and the references therein. Moreover, if $M \equiv 1$, $K(x) = |x|^{-N-ps}$ and $s \nearrow 1^-$, then problem (1.7) reduces to
\[
\partial_t u - \Delta_p u = f(x,t),
\]
where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ and $f$ is a source term not necessarily zero. By using sub–differential calculus, Akagi and Matsuura in [2] exploited the well–posedness and asymptotic behaviors of solutions for equation (1.8) in variable exponent Sobolev spaces, see [3] for further results. Actually, the existence, uniqueness, extinction in finite time, decay and blow–up of solutions for equation (1.9) have been studied extensively in recent years, for example, see [5, 6] and the references therein.

Motivated by the above works, we focus on the well–posedness and large–time behavior of solutions of (1.1). Main difficulties arise, when dealing with this problem, because of the presence of the Kirchhoff function and of the nonlocal nature of the $p$–fractional Laplacian. To the best of our knowledge, there are no results on the diffusion problem of Kirchhoff type involving the fractional $p$–Laplacian. Clearly, the extension from the linear case $p = 2$ to the semi–linear general case $1 < p < \infty$ is also not trivial.

Problems of the above form (1.1) are mathematical models occurring in nonlocal reaction–diffusion theory, non–Newtonian fluid theory, non–Newtonian filtration and turbulent flows of a gas in a porous medium. In the non–Newtonian fluid
theory, the quantity $p$ is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

The rest of the paper is organized as follows. In Section 2, we recall some necessary definitions and properties of the fractional Sobolev spaces. In Section 3, we obtain the well-posedness of solutions of problem (1.1) for all $p > 1$ by employing the subdifferential approach. In Section 4, the asymptotic behavior of solutions of problem (1.1) is studied under some appropriate natural assumptions. Section 5 is devoted to the study of the extinction property of solutions of problem (1.1), without source term (i.e. when $f \equiv 0$) in the singular case $2N/(N + 2s) \leq p < 2$.

2. Preliminaries. In this section, we first recall some necessary properties of fractional Sobolev spaces and related notations, see also [18] for further details.

Firstly, let us recall that $X_{s,p}(\Omega)$ denotes the linear space of Lebesgue measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that the quantity

$$
\|u\|_{X_{s,p}(\Omega)} = \left(\int_{\Omega} |u|^p dx + \iint_{Q} |u(x) - u(y)|^p K(x - y) dxdy\right)^{1/p}
$$

is finite, where $Q$ is defined in (1.3). It is easy to see that bounded and Lipschitz functions belong to $X_{s,p}(\Omega)$, thus $X_{s,p}(\Omega)$ is not trivial. We refer to [22, 38] for further details on the space $X_{s,p}(\Omega)$. Moreover, $X_{0,s,p}(\mathbb{R}^N)$ denotes the space of functions $u \in X_{s,p}(\mathbb{R}^N)$ that vanish a.e. in $C^\Omega = \mathbb{R}^N \setminus \Omega$, endowed with the norm (1.2).

It is easy to see that (1.2) is equivalent to (2.1) and that $X_{0,s,p}(\Omega)$ is a uniformly convex Banach space, being $p > 1$. See [34, 35, 38] for more details. Indeed, $X_{0,s,p}(\Omega)$ can also be identified with the closure of $C_0^\infty(\Omega)$ in $X_{s,p}(\Omega)$, as shown by Fiscella, Servadei and Valdinoci in [22]. It is worthy mentioning that the space $X_{0,s,p}(\Omega)$ is similar to, but different from, the standard fractional Sobolev space $W_0^{s,p}(\Omega)$. The interested reader can refer to [22] for further comments.

Note that in both (2.1) and (1.2) the integrals can be extended to the whole spaces $\mathbb{R}^N$ and $\mathbb{R}^{2s}$, since $u = 0$ a.e. in $C^\Omega$. By Lemma 2.3 of [38], the Banach space $X_{0,s,p}(\Omega) = (X_{0,s,p}(\Omega), \| \cdot \|_{X_{0,s,p}(\Omega)})$ is continuously embedded in $L^r(\mathbb{R}^N)$ for any $r \in [1, p^*_s]$, where the critical fractional Sobolev exponent is defined by

$$
p^*_s = \begin{cases} 
Np & \text{if } N > sp, \\
N - sp & \text{if } N \leq sp.
\end{cases}
$$

Hence, for all $r \in [1, p^*_s]$ there exists $C_* > 0$ such that

$$
\|u\|_{L^r(\mathbb{R}^N)} \leq C_* \|u\|_{X_{0,s,p}(\Omega)} \quad \text{for all } u \in X_{0,s,p}(\Omega). \quad (2.2)
$$

Throughout the paper, the letters $C, C_i, i = 1, 2, \ldots$, denote positive constants which vary from line to line, but are independent of the terms which take part in any limit process.

For the reader’s convenience, we recall some related useful definitions and notations. The Banach space $L^p(0, T; X)$ consists of all strongly measurable functions $u : [0, T] \to X$, endowed with the norm

$$
\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(\cdot, t)\|_X^p dt\right)^{1/p},
$$
where $T \in \mathbb{R}^+$ is a given number and $X$ is a reflexive Banach space. Obviously, $L^p(0, T; X)$ is a reflexive Banach space. It follows from [3] Theorem 1.5] that the dual space of $L^p(0, T; X)$ can be identified with $L^{p'}(0, T; X')$, where $p' = p/(p - 1)$ and $X'$ is the dual space of $X$. The other spaces can be understood similarly.

3. Well–posedness of solutions. In this section, we prove the well-posedness of (weak) solutions of problem (1.1). To this aim, we require the following assumption

$$(M) \ M : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \text{ is a continuous and nondecreasing function and there exists a constant } m_0 > 0 \text{ such that }$$

$$M(t) \geq m_0 \quad \text{for all } t \geq 0.$$ 
A prototype for Kirchhoff function $M$ is given by

$$M(t) = b_1 t^{\gamma_1 - 1} + b_2 t^{\gamma_2 - 1} + b_3, \quad b_1, b_2 \geq 0, b_3 > 0, \ 1 \leq \gamma_1 \leq \gamma_2 < \infty,$$

for all $t \geq 0$. If $M$ in problem (1.1) is of the special type above, then Kirchhoff problem (1.1) is called non–degenerate; if $b_3 = 0$, but $b_1 + b_2 > 0$, then (1.1) is said to be degenerate. Hence the problems we treat in this paper are non–degenerate.

Without further mentioning, throughout the paper we always assume that $(M)$ holds and put

$$\mathcal{M}(t) = \int_0^t M(\tau)d\tau$$

for all $t \in \mathbb{R}^+_0$.

Here and henceforth, $C_w([0, T]; X)$ denotes the space of weakly continuous functions on $[0, T]$ into a normed space $X$ and $f \in L^2_{\text{loc}}(\mathbb{R}^+_0; L^2(\Omega))$.

**Definition 3.1.** A function $u \in C(\mathbb{R}^+_0; L^2(\Omega))$ is said to be a solution for problem (1.1), if the following conditions are satisfied

(i) $u \in W^{1, 2}_{\text{loc}}(\mathbb{R}^+_0; L^2(\Omega)) \cap C_w(\mathbb{R}^+_0; X^{s,p}(\Omega))$ and $\mathcal{L}_K u \in L^2_{\text{loc}}(\mathbb{R}^+_0; L^2(\Omega))$,

(ii) $u(\cdot, 0) = u_0$ a.e. in $\Omega$,

(iii) For all $\varphi \in X^{s,p}_0(\Omega)$, the following equality holds

$$\int_{\Omega} \partial_t u(x,t)\varphi dx + M\left(\|u(\cdot, t)\|_{X^{s,p}_0(\Omega)}\right) \langle u(\cdot, t), \varphi \rangle_{X^{s,p}_0(\Omega)} = \int_{\Omega} f(x,t)\varphi dx$$

for a.e. $t \in \mathbb{R}^+$, where

$$\langle u(\cdot, t), \varphi \rangle_{X^{s,p}_0(\Omega)} = \int_{\Omega} \int_{\Omega} |u(x,t) - u(y,t)|^{p-2}[u(x,t) - u(y,t)] \times [\varphi(x) - \varphi(y)]K(x-y)dxdy.$$ 

Now we reduce problem (1.1) to the Cauchy problem of an abstract evolution equation. Let

$$H_0 = \{u \in L^2(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

and define the functional $I : H_0 \to [0, \infty]$ by

$$I(v) = \begin{cases} \frac{1}{p} \mathcal{M}\left(\|v\|_{X^{s,p}_0(\Omega)}\right) & \text{if } v \in X^{s,p}_0(\Omega), \\ \infty & \text{otherwise,} \end{cases}$$

and by $\mathcal{D}(I) = \{v \in H_0 : I(v) < \infty\}$ the effective domain of $I$.

To prove that problem (1.1) is well–posed, we need check the lower semi–continuity of the functional $I$ in $H_0$. 

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Lemma 3.1. Let $1 < p < \infty$. Then $\mathcal{I}$ is proper, convex and lower semi–continuous in $H_0$.

Proof. It follows from \cite{22} Theorem 6] that $C_0^\infty(\Omega)$ is dense in $X_0^{s,p}(\Omega)$, so that $\mathcal{D}(\mathcal{I})$ is nonempty. Thus $\mathcal{I}$ is proper in $H_0$.

Now we show that $\mathcal{I}$ is convex in $H_0$. Let $u,v \in H_0$ and $\lambda \in [0,1]$. Clearly

$$\mathcal{I}(\lambda u + (1 - \lambda)v) \leq \lambda \mathcal{I}(u) + (1 - \lambda)\mathcal{I}(v),$$

whenever either $u$ or $v$ is in $H_0 \setminus X_0^{s,p}(\Omega)$, since then either $\mathcal{I}(u) = \infty$ or $\mathcal{I}(v) = \infty$.

It remains to consider the case in which $u,v \in X_0^{s,p}(\Omega)$. Since $M$ is nondecreasing on $\mathbb{R}_0^+$, then $\mathcal{M}$ is convex on $\mathbb{R}_0^+$. Indeed, fix $t_1, t_2 \in \mathbb{R}_0^+$, and assume, without loss of generality, that $t_1 < t_2$. Then setting $t = \lambda t_1 + (1 - \lambda)t_2$, in order to prove that

$$\mathcal{M}(t) = \mathcal{M}(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda \mathcal{M}(t_1) + (1 - \lambda)\mathcal{M}(t_2),$$

it is enough to verify that

$$\lambda(\mathcal{M}(t) - \mathcal{M}(t_1)) + (1 - \lambda)(\mathcal{M}(t) - \mathcal{M}(t_2)) \leq 0. \tag{3.3}$$

By the Lagrange mean theorem, there exist $\xi, \eta \in \mathbb{R}^+$, with $t_1 < \xi < t < \eta < t_2$, such that

$$\lambda(\mathcal{M}(t) - \mathcal{M}(t_1)) + (1 - \lambda)(\mathcal{M}(t) - \mathcal{M}(t_2)) = \lambda M(\xi)(t - t_1) + (1 - \lambda)M(\eta)(t_2 - t) .$$

Note that $t - t_1 = (1 - \lambda)(t_2 - t_1)$ and $t - t_2 = \lambda(t_1 - t_2)$. It follows from these facts that

$$\lambda(\mathcal{M}(t) - \mathcal{M}(t_1)) + (1 - \lambda)(\mathcal{M}(t) - \mathcal{M}(t_2)) = \lambda(1 - \lambda)M(\xi)(t_2 - t_1) + \lambda(1 - \lambda)M(\eta)(t_1 - t_2) = \lambda(1 - \lambda)(t_2 - t_1)[M(\xi) - M(\eta)].$$

Then claim \eqref{3.3} follows at once, since $M$ is nondecreasing in $\mathbb{R}_0^+$ and $\xi < \eta$.

Hence, being $\mathcal{M}$ nondecreasing in $\mathbb{R}_0^+$, then \eqref{3.3} implies that

$$\mathcal{I}(\lambda u + (1 - \lambda)v) = \frac{1}{p} \mathcal{M} \left( \|\lambda u + (1 - \lambda)v\|_{X_0^{s,p}(\Omega)}^p \right) \leq \frac{1}{p} \mathcal{M} \left( \lambda \|u\|_{X_0^{s,p}(\Omega)}^p + (1 - \lambda)\|v\|_{X_0^{s,p}(\Omega)}^p \right) \leq \frac{\lambda}{p} \mathcal{M}\left(\|u\|_{X_0^{s,p}(\Omega)}^p\right) + \frac{1 - \lambda}{p} \mathcal{M}\left(\|v\|_{X_0^{s,p}(\Omega)}^p\right) = \lambda \mathcal{I}(u) + (1 - \lambda)\mathcal{I}(v).$$

Therefore, $\mathcal{I}$ is convex in $H_0$, as stated.

Next, we show that $\mathcal{I}$ is lower semi–continuous in $H_0$. Let $\mu \in \mathbb{R}_0^+$ be fixed and set

$$[\mathcal{I} \leq \mu] = \{u \in H_0 : \mathcal{I}(u) \leq \mu\}.$$

Let $(u_n)_n$ be a sequence in $[\mathcal{I} \leq \mu]$ such that $u_n \to u$ strongly in $H_0$. Clearly,$\mathcal{I}(u_n) \leq \mu$ for all $n$ yields

$$\|u_n\|_{X_0^{s,p}(\Omega)}^p = \int_\Omega |u_n(x) - u_n(y)|^p K(x - y) dx dy \leq \mu p.$$

Hence $(\|u_n\|_{X_0^{s,p}(\Omega)})_n$ is bounded and, up to a subsequence, still denoted by $(u_n)_n$, then $u_n \to u$ weakly in $X_0^{s,p}(\Omega)$, since $X_0^{s,p}(\Omega)$ is reflexive.

Let $\mathcal{I}_0$ denote the restriction of $\mathcal{I}$ to $X_0^{s,p}(\Omega)$. Then $\mathcal{I}_0$ is of class $C^1(X_0^{s,p}(\Omega))$ and weakly lower semi–continuous in $X_0^{s,p}(\Omega)$, see \cite{38} for similar arguments. Moreover,
Let \( \mathcal{I}_0 \) be convex. Therefore we have \( \liminf_{n \to \infty} \mathcal{I}_0(u_n) \geq \mathcal{I}_0(u) \), which, together with \( \mathcal{I}_0(u_n) = \mathcal{I}(u_n) \leq \mu \), implies that \( u \in [\mathcal{I} \leq \mu] \). Hence \([\mathcal{I} \leq \mu]\) is closed in \( H_0 \), and so \( \mathcal{I} \) is lower semi-continuous in \( H_0 \).

Moreover, one can show the following lemma, see [38] for similar arguments.

**Lemma 3.2.** The restriction \( \mathcal{I}_0 \) of \( \mathcal{I} \) to \( X^{s,p}_0(\Omega) \) is of class \( C^1(X^{s,p}_0(\Omega)) \), and the Fréchet derivative \( d\mathcal{I}_0(u) \) of \( \mathcal{I}_0 \) at \( u \in X^{s,p}_0(\Omega) \) coincides with \( M \left( \|u\|^p_{X^{s,p}_0(\Omega)} \right) \mathcal{L}Ku \) valued at \( u \in \mathbb{R}^n \setminus \Omega \) in the sense of distribution, that is,

\[
(d\mathcal{I}_0(u), v)_{(X^{s,p}_0(\Omega))', X^{s,p}_0(\Omega)} = M \left( \|u\|^p_{X^{s,p}_0(\Omega)} \right) (u, v)_{X^{s,p}_0(\Omega)}
\]

for all \( v \in X^{s,p}_0(\Omega) \).

The sub-differential operator \( \partial \mathcal{I} : H_0 \to H_0 \) of \( \mathcal{I} \) is given by

\[
\partial \mathcal{I}(u) = \{ \xi \in H_0 : \mathcal{I}(v) - \mathcal{I}(u) \geq \langle \xi, v - u \rangle_{H_0} \text{ for all } v \in \mathcal{D}(\mathcal{I}) \}
\]

for any \( u \in \mathcal{D}(\mathcal{I}) \), where naturally \( (\cdot, \cdot)_{H_0} = (\cdot, \cdot)_{L^2(\Omega)} \) and the effective domain \( \mathcal{D}(\partial \mathcal{I}) \) of \( \partial \mathcal{I} \) is

\[
\mathcal{D}(\partial \mathcal{I}) = \{ u \in \mathcal{D}(\mathcal{I}) : \partial \mathcal{I}(u) \neq \emptyset \}.
\]

Since \( \partial \mathcal{I}(u) \subset d\mathcal{I}_0(u) \) for all \( u \in \mathcal{D}(\partial \mathcal{I}) \), we have \( \partial \mathcal{I}(u) = M(\|u\|^p_{X^{s,p}_0(\Omega)}) \mathcal{L}Ku \) in \( H_0 \).

Hence problem (1.1) is reduced to the following Cauchy abstract problem

\[
\begin{cases}
\frac{d}{dt} u(t) + \partial \mathcal{I}(u(t)) = f(t) & \text{in } H_0, \\
u(0) = u_0.
\end{cases}
\] (3.4)

The abstract evolution equation (3.4) was well studied, mainly by Brézis in [10] Chap. III, see also [30]. Thus Lemma 3.1 yields at once the next result.

**Theorem 3.1.** For all \( u_0 \in L^2(\Omega) \) and \( f \in L^2_{loc}(\mathbb{R}^n_+; L^2(\Omega)) \), there exists a unique solution \( u \) of problem (1.1) such that the function \( t \mapsto \mathcal{I}(u(\cdot,t)) \) is locally absolutely continuous in \( \mathbb{R}_0^+ \).

In particular, if \( u_0 \in X^{s,p}_0(\Omega) \), then

\[
u \in W^{1,2}_{loc}(\mathbb{R}^n_+; L^2(\Omega)) \cap C_\infty(\mathbb{R}^n_+; X^{s,p}_0(\Omega))
\]

and the function \( t \mapsto \mathcal{I}(u(\cdot,t)) \) is locally absolutely continuous in \( \mathbb{R}_0^+ \).

Furthermore, the unique solution \( u \) of problem (1.1) continuously depends on the initial datum \( u_0 \) and on \( f \) in the following sense. If \( u_i, i = 1, 2 \), denote the unique solutions of problem (1.1), with \( u_{0,i} \in L^2(\Omega) \) and \( f_i \in L^2_{loc}(\mathbb{R}^n_+; L^2(\Omega)) \), then

\[
\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^2(\Omega)} \leq \|u_{0,1} - u_{0,2}\|_{L^2(\Omega)} + \int_0^t \|f_1(\cdot,\tau) - f_2(\cdot,\tau)\|_{L^2(\Omega)} d\tau
\]

for all \( t \in \mathbb{R}_0^+ \).

**Corollary 3.2.** Let \( p \geq 2 \). For all \( u_0 \in L^2(\Omega) \) (resp. \( u_0 \in X^{s,p}_0(\Omega) \)) and all \( f \in L^2_{loc}(\mathbb{R}^n_+; L^2(\Omega)) \), the unique solution \( u \) of problem (1.1) is strongly continuous from \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_0^+ \)) into \( X^{s,p}_0(\Omega) \).

**Proof.** Let \( 0 < t_0 < t < \infty \). Then \( \mathcal{I}(u(\cdot,t)) \to \mathcal{I}(u(\cdot,t_0)) \) and \( u(\cdot,t) \to u(\cdot,t_0) \) weakly in \( L^2(\Omega) \) as \( t \to t_0 \) by Theorem 3.1. As \( p \geq 2 \), inequality (3.5) of [10] Lemma 2.27, that is

\[
\left| \frac{a + b}{2} \right|^p + \left| \frac{a - b}{2} \right|^p \leq \frac{1}{2} \left( |a|^p + |b|^p \right)
\]

for all \( a, b \in \mathbb{R} \),
yields that
\[
\frac{1}{2} \left| \frac{u(x,t) - u(y,t) + u(x,t_0) - u(y,t_0)}{2} \right|^p + \frac{1}{2} \left| \frac{u(x,t) - u(y,t) - u(x,t_0) + u(y,t_0)}{2} \right|^p
\leq \frac{1}{2} \left( \left| u(x,t) - u(y,t) \right|^p + \left| u(x,t_0) - u(y,t_0) \right|^p \right),
\]
for a.a. \((x,y) \in Q\). Hence,
\[
I\left( \frac{u(\cdot,t) + u(\cdot,t_0)}{2} \right) + I\left( \frac{u(\cdot,t) - u(\cdot,t_0)}{2} \right) \leq \frac{1}{2} [I(u(\cdot,t)) + I(u(\cdot,t_0))].
\]
On the other hand, since \(u(\cdot,t) \rightharpoonup u(\cdot,t_0)\) weakly in \(L^2(\Omega)\) and \(I\) is weakly lower semi-continuous in \(H_0\), we have
\[
I(u(\cdot,t_0)) \leq \liminf_{t \to t_0} I\left( \frac{u(\cdot,t) + u(\cdot,t_0)}{2} \right).
\]
Combining these facts with the assumption that \(I(u(\cdot,t)) \rightharpoonup I(u(\cdot,t_0))\) as \(t \to t_0\), we obtain that
\[
\limsup_{t \to t_0} I\left( \frac{u(\cdot,t) - u(\cdot,t_0)}{2} \right) = 0.
\]
Hence \(u(\cdot,t) \rightharpoonup u(\cdot,t_0)\) strongly in \(X^s_{0,p}(\Omega)\) as \(t \to t_0\).

The other cases can be proved similarly. This completes the proof. \(\square\)

Next let us prove the existence of periodic solutions for problem [3.4]. In the argument it is required that \(2 < p^*_s\), that is that \(p > 2N/(N+2s)\), and in turn \(p > \max\{1, 2N/(N+2s)\}\).

**Corollary 3.3.** Let \(p > \max\{1, 2N/(N+2s)\}\). Then for any \(T > 0\) and all \(f \in L^2_{\text{loc}}(\mathbb{R}_0^+; L^2(\Omega))\), there exists a unique solution \(u\) of problem (1.1) such that \(u(\cdot,0) = u(\cdot,T)\).

**Proof.** By Corollary 3.4 of [10], it suffices to check the coercivity of \(I\) in \(H_0\). Since the embedding \(X^s_{0,p}(\Omega) \hookrightarrow L^2(\mathbb{R}^N)\) is continuous, being \(p > 2N/(N+2s)\), we obtain by (M)
\[
I(u) \geq \frac{1}{p} \mathcal{M} \left( \|u\|_{X^s_{0,p}(\Omega)}^p \right) \geq \frac{m_0}{p} \|u\|_{X^s_{0,p}(\Omega)}^p \geq \frac{m_0 C_s^p}{p} \|u\|_{H_0}^p,
\]
where \(C_s > 0\) comes from (2.2). Hence the functional \(I\) is coercive in \(H_0\), since
\[
\lim_{\|u\|_{H_0} \to \infty} \frac{I(u)}{\|u\|_{H_0}} = \infty,
\]
being \(p > 1\). \(\square\)

4. **Large–time behaviors of solutions.** This section is concerned with the large–time behavior of solutions.

**Theorem 4.1.** Suppose that \(p > \max\{1, 2N/(N+2s)\}\). Let \(f^* \in L^2(\Omega)\) and \(f \in L^2_{\text{loc}}(\mathbb{R}_0^+; L^2(\Omega))\) be such that
\[
f - f^* \in L^2(\mathbb{R}_0^+; L^2(\Omega))
\]
and
\[
f(\cdot,t) \rightharpoonup f^* \quad \text{weakly in } L^2(\Omega) \quad \text{as } t \to \infty.
\]
Let $u$ be the unique solution of problem (1.1), with initial datum $u_0 \in X_0^{s,p}(\Omega)$. Then there exists $u^* \in X_0^{s,p}(\Omega)$ such that

$$u(\cdot, t) \to u^* \text{ strongly in } L^2(\Omega) \text{ as } t \to \infty,$$

where $\mathcal{I}$ is defined in (3.2). Moreover, $u^*$ is the unique weak solution of the problem

$$\begin{cases}
M(\|u^*\|^p_{X_0^{s,p}(\Omega)})\mathcal{L}_K u^* = f^* & \text{ in } \Omega, \\
u^* = 0 & \text{ in } \mathbb{R}^N \setminus \Omega.
\end{cases} \tag{4.3}$$

To prove Theorem 4.1 we need the following crucial result.

**Lemma 4.1.** Under the assumptions of Theorem 4.1, if $u = u(x, t)$ is the unique solution of problem (1.1), then $u(\cdot, t)$ is bounded in $X_0^{s,p}(\Omega)$ uniformly in $t$, that is

$$\sup_{t \in \mathbb{R}_0^+} \|u(\cdot, t)\|_{X_0^{s,p}(\Omega)} = \bar{C} < \infty.$$  

**Proof.** Let us first recall that (1.1) can be reduced to (3.4) and fix $t > 0$. Multiplying (3.4) by $\partial_t u(\cdot, t)$ which is in $L^2(\Omega)$, since $u(\cdot, t)$ is in $W^{1,2}_{\text{loc}}(\mathbb{R}_0^+; L^2(\Omega))$, and using the chain–rule for sub–differentials, we have

$$\begin{aligned}
\|\partial_t u(\cdot, t)\|^2_{L^2(\Omega)} + \partial_t \mathcal{I}(u(\cdot, t)) &= (f(\cdot, t) - f^*, \partial_t u(\cdot, t))_{L^2(\Omega)} + (f^*, \partial_t u(\cdot, t))_{L^2(\Omega)} \\
&\leq \frac{1}{2}\|f(\cdot, t) - f^*\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t u(\cdot, t)\|^2_{L^2(\Omega)} + \frac{d}{dt}(f^*, u(\cdot, t))_{L^2(\Omega)}.
\end{aligned}$$

This yields

$$\frac{1}{2}\|\partial_t u(\cdot, t)\|^2_{L^2(\Omega)} + \frac{d}{dt} E u(t) \leq 0 \text{ for a.e. } t \geq 0,$$

where

$$Ev = \mathcal{I}(v) - (f^*, v)_{L^2(\Omega)} - \frac{1}{2} \int_0^t \|f(\cdot, \tau) - f^*\|^2_{L^2(\Omega)} d\tau$$

along any $v \in H_0$. Hence the function $t \mapsto Ev(t)$ is nonincreasing in $\mathbb{R}_0^+$. Thus, $Ev(t) \leq Ev(0)$ in $\mathbb{R}_0^+$. Moreover, since $f - f^* \in L^2(\mathbb{R}_0^+; L^2(\Omega))$ and $\mathcal{I}$ is coercive in $H_0$, being $2N/(N + 2s) < p$, there exists a constant $C_2 > 0$ such that $|Ev(t)| \leq C_2$ for all $t \geq 0$. Thus, we obtain that

$$\int_0^\infty \|\partial_t u(\cdot, t)\|^2_{L^2(\Omega)} dt < \infty. \tag{4.4}$$

Now, let $(t_n)_n$ be an arbitrary sequence in $\mathbb{R}_0^+$ such that $t_n \to \infty$. Then by (4.4), we can take $\theta_n \in [t_n, t_n + 1]$ such that $\partial_t u(\cdot, \theta_n) \to 0$ strongly in $L^2(\Omega)$. Hence, we deduce from (4.2) that

$$\partial \mathcal{I}(u(\cdot, \theta_n)) = f(\cdot, \theta_n) - \partial_t u(\cdot, \theta_n) \rightharpoonup f^* \text{ weakly in } L^2(\Omega).$$

Noting that

$$Eu(t) \leq Eu(0) = \mathcal{I}(u_0) - (f^*, u_0)_{L^2(\Omega)},$$

and the fact that $f - f^* \in L^2(\mathbb{R}_0^+; L^2(\Omega))$, we obtain

$$\sup_{t \geq 0} \mathcal{I}(u(\cdot, t)) < \infty,$$
which, together with the coercivity of $\mathcal{I}$ in $H_0$, means that
\[
\sup_{t \geq 0} \| u(\cdot, t) \|_{X_0^{s,p}(\Omega)} = C < \infty.
\]
The proof is thus complete. \hfill \Box

**Proof of Theorem 4.1.** Let $(t_n)_n$ be any sequence in $\mathbb{R}_0^+$ such that $t_n \to \infty$. As shown in the proof of Lemma 4.1, there exists a subsequence $(\theta_n)_n$, still denoted by $(\theta_n)_n$, such that
\[
u(\cdot, \theta_n) \to \nu^* \text{ weakly in } X_0^{s,p}(\Omega),
\]
which, together with $p > 2N/(N + 2s)$, yields that
\[
u(\cdot, \theta_n) \to \nu^* \text{ strongly in } L^2(\Omega).
\]
Combining these facts with (4.5), we conclude from the demiclosedness of $\partial \mathcal{I}$ that
\[
\partial \mathcal{I}(\nu^*) = f^*,
\]
which is equivalent to an $L^2$-formulation of (4.3).

On the other hand, from the definition of sub-differential, we have
\[
\mathcal{I}(\nu(\cdot, \theta_n)) \leq \mathcal{I}(\nu^*) + (\partial \mathcal{I}(\nu(\cdot, \theta_n)), \nu(\cdot, \theta_n) - \nu^*)_{H_0', H_0} \to \mathcal{I}(\nu^*)
\]
as $n \to \infty$. Note also that the weak lower semicontinuity of $\mathcal{I}$ in $H_0$ implies that
\[
\liminf_{n \to \infty} \mathcal{I}(\nu(\cdot, \theta_n)) \geq \mathcal{I}(\nu^*).
\]

Hence, $\mathcal{I}(\nu(\cdot, \theta_n)) \to \mathcal{I}(\nu^*)$ as $n \to \infty$.

Next, we show that the sequence $(\nu(\cdot, t_n))_n$ converges in $L^2(\Omega)$, along the prescribed sequence $t_n \to \infty$. By (4.4), we get
\[
\| \nu(\cdot, t_n) - \nu(\cdot, \theta_n) \|_{L^2(\Omega)} \leq \left( \int_{t_n}^{\theta_n} \| \partial_t \nu(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau \right)^{1/2} (\theta_n - t_n)^{1/2}
\]
\[
\leq \left( \int_{t_n}^{\infty} \| \partial_t \nu(\cdot, \tau) \|_{L^2(\Omega)}^2 d\tau \right)^{1/2} \to 0
\]
as $n \to \infty$, since $\theta_n - t_n \in [0, 1]$ for all $n$. Thus,
\[
\nu(\cdot, t_n) \to \nu^* \text{ strongly in } L^2(\Omega) \text{ as } n \to \infty.
\]

On the other hand, (4.6) implies that as $n \to \infty$
\[
Eu(\theta_n) \to \mathcal{I}(\nu^*) - (f^*, \nu^*)_{L^2(\Omega)} - \frac{1}{2} \int_0^\infty \| f(\cdot, \tau) - f^* \|_{L^2(\Omega)}^2 d\tau.
\]

By the monotonicity of $Eu$, we know that $Eu(t_n)$ converges to the same limit as $t_n \to \infty$. Consequently, we conclude that as $n \to \infty$
\[
\mathcal{I}(\nu(\cdot, t_n)) = Eu(t_n) + (f^*, \nu(\cdot, t_n))_{L^2(\Omega)} + \frac{1}{2} \int_0^{t_n} \| f(\cdot, \tau) - f^* \|_{L^2(\Omega)}^2 d\tau \to \mathcal{I}(\nu^*).
\]

This completes the proof. \hfill \Box

We note in passing that the restriction $p > \max\{1, 2N/(N + 2s)\}$ is automatic whenever $p \geq 2$, being $s > 0$.

**Corollary 4.2.** Suppose that $p \geq 2$. Under the assumptions of Theorem 4.1
\[
\nu(\cdot, t) \to \nu^* \text{ strongly in } X_0^{s,p}(\Omega) \text{ as } t \to \infty.
\]
Proof. The proof is similar to that of Corollary [3.2] so we leave it to the interested reader.

In order to study the large–time behavior, let us introduce some notation and let us assume from now on that $p \geq 2$. Fix $\alpha \in [p-1, \infty)$ and take $L_M > 0$ so small that

$$m_0 C_p - L_M \alpha 2^{2 \alpha p - \alpha - p + 1} \bar{C} \alpha p > 0,$$

(4.8)

where $\bar{C} > 0$ is the number given in Lemma [4.1] $m_0$ in $(\mathcal{M})$ and $C_p$ is a positive constant for which the celebrated inequality

$$\langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \rangle \cdot (\xi - \eta) \geq C_p |\xi - \eta|^p$$

for all $\xi, \eta \in \mathbb{R}^N$ (4.9) holds, being $p \geq 2$.

On the Kirchhoff function $M$ let us also assume

$$(\tilde{\mathcal{M}}) \ |M(t_1) - M(t_2)| \leq L_M |t_1 - t_2|^\alpha \ 	ext{for all} \ t_1, t_2 \in [0, \bar{C}^p].$$

Condition $(\tilde{\mathcal{M}})$, when $p = 2$ and $\alpha = 1$, has been used to get the uniqueness of solutions of stationary Kirchhoff problems by Ma in [29]. Here, we use condition $(\tilde{\mathcal{M}})$ to get some useful asymptotic estimates.

**Theorem 4.3.** Suppose that $p \geq 2$, that $M$ satisfies $(\mathcal{M})$ and $(\tilde{\mathcal{M}})$, with $L_M > 0$ verifying (4.8), and that $f (\cdot, t) \equiv f^* \in L^2(\Omega)$ in $\mathbb{R}^N_+$. Let $u = u(x, t)$ be the unique solution of problem (1.1), with initial datum $u_0 \in L^2(\Omega)$, and let $u^*$ be the corresponding weak solution of problem (4.3).

(i) If $p > 2$, then there exists a constant $C > 0$ such that for all $t \geq 0$

$$\|u(\cdot, t) - u^*\|_{L^2(\Omega)} \leq \|u_0 - u^*\|_{L^2(\Omega)} \left( 1 + \|u_0 - u^*\|_{L^2(\Omega)}^2 C t \right)^{-1/(p-2)}.$$

(ii) If $p = 2$, then there exists a constant $C > 0$ such that for all $t \geq 0$

$$\|u(\cdot, t) - u^*\|_{L^2(\Omega)} \leq \|u_0 - u^*\|_{L^2(\Omega)} e^{-C t}.$$

Proof. Using the notation of the statement, we have

$$\partial_t u + M \left( \|u(\cdot, t)\|_{X_0^{p, p}(\Omega)}^{p-2} \right) \mathcal{L}_K u = f^* = M \left( \|u^*\|_{X_0^{p, p}(\Omega)}^{p-2} \right) \mathcal{L}_K u^*.$$  (4.10)

In the following, for convenience, we set

$$\langle (v, w) \rangle_{X_0^{p, p}(\Omega)}$$



$$= \int_\Omega \left[ |v(x) - v(y)|^{p-2} (v(x) - v(y)) - |w(x) - w(y)|^{p-2} (w(x) - w(y)) \right]$$

$$\times [v(x) - w(x) - v(y) + w(y)] K(x-y) dx dy,$$

and recall by (4.1) that

$$\langle v, w - u \rangle_{X_0^{p, p}(\Omega)}$$

$$= \int_\Omega |v(x) - v(y)|^{p-2} [v(x) - v(y)] \cdot [(v(x) - w(x) - v(y)] K(x-y) dx dy$$

for all $v, w \in X_0^{p, p}(\Omega)$. Fix $t > 0$. Multiplying (4.10) by $u(\cdot, t) - u^*$, which is in $L^2(\Omega)$, as shown above, we get

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t) - u^*\|^2_{L^2(\Omega)} + M \left( \|u(\cdot, t)\|_{X_0^{p, p}(\Omega)}^{p-2} \right) \langle (u(\cdot, t), u^*) \rangle_{X_0^{p, p}(\Omega)}$$

$$= [M \left( \|u^*\|_{X_0^{p, p}(\Omega)}^{p-2} \right) - M \left( \|u(\cdot, t)\|_{X_0^{p, p}(\Omega)}^{p-2} \right)] \langle u^*, u - u^* \rangle_{X_0^{p, p}(\Omega)}.$$  (4.11)
Now it follows from (M) that
\[
M \left( \|u^*\|_{X^0_p(\Omega)}^p \right) - M \left( \|u(\cdot, t)\|_{X^0_p(\Omega)}^p \right) \\
\leq L_M \left| \|u(\cdot, t)\|_{X^0_p(\Omega)}^p - \|u^*\|_{X^0_p(\Omega)}^p \right| \alpha \\
\leq L_M p^\alpha \left[ \int_Q \left( |u(x, t) - u(y, t)| + |u^*(x) - u^*(y)| \right)^{p-1} \right. \\
\times |u(x, t) - u^*(x) - u(y, t) + u^*(y)| K(x - y) \, dx \, dy \left. \right] ^\alpha ,
\]
since \(|a^p - b^p| \leq (a + b)^{p-1}|a - b|\) for all \(a \geq 0\) and \(b \geq 0\) by the Lagrange mean theorem. Then by the Hölder inequality we obtain
\[
M \left( \|u^*\|_{X^0_p(\Omega)}^p \right) - M \left( \|u(\cdot, t)\|_{X^0_p(\Omega)}^p \right) \\
\leq L_M p^\alpha \|u(\cdot, t) - u(\cdot, t)\|_{X^0_p(\Omega)} \|u^*(x) - u^*(y)\|_{X^0_p(\Omega)} \|u(\cdot, t) - u^*\|_{X^0_p(\Omega)} ,
\]
and
\[
|\langle u^*, u(\cdot, t) - u^* \rangle_{X^0_p(\Omega)}| \leq \|u^*\|_{X^0_p(\Omega)} \|u - u^*\|_{X^0_p(\Omega)} .
\]
Combining these two inequalities with (4.11), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 + M \left( \|u(\cdot, t)\|_{X^0_p(\Omega)}^p \right) \|u(\cdot, t), u^*\|_{X^0_p(\Omega)} \\
\leq L_M p^\alpha \|u(\cdot, t) - u^*\|_{X^0_p(\Omega)} \|u^*(x) - u^*(y)\|_{X^0_p(\Omega)} \|u(\cdot, t) - u^*\|_{X^0_p(\Omega)} \\
\leq L_M p^\alpha \|u(\cdot, t) - u^*\|_{X^0_p(\Omega)} \|u^*(x) - u^*(y)\|_{X^0_p(\Omega)} ,
\]
being \(\|u(\cdot, t) - u^*\|_{X^0_p(\Omega)} \leq 2C\) for all \(t \geq 0\) by Lemma 4.1. From (M), (4.9) and (4.12) we get
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 + C_3 \|u(\cdot, t) - u^*\|_{X^0_p(\Omega)}^p \leq 0 ,
\]
where \(C_3 = m_0 C_p - L_M p^\alpha \|2ap - ap + 1\| \geq 0\) by (4.8). It follows from (4.13) and the continuous embedding \(X^0_p(\Omega) \to L^2(\Omega)\) that
\[
\frac{d}{dt} \|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 + 2C_4 \|u(\cdot, t) - u^*\|_{L^2(\Omega)}^p \leq 0 ,
\]
where \(C_4 > 0\) is a constant. Therefore, we conclude from (4.14) that
\[
\|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 \leq \begin{cases} \\
\|u_0 - u^*\|_{L^2(\Omega)}^2 \left( 1 + (p - 2)C_4 \|u_0 - u^*\|_{L^2(\Omega)}^{p-2} \right)^{-2/(p-2)} & \text{if } p > 2 , \\
\|u_0 - u^*\|_{L^2(\Omega)}^2 e^{-2C_4 t} & \text{if } p = 2 ,
\end{cases}
\]
for all \(t \geq 0\), as desired. \(\square\)

If \(p > 2\), a weaker assumption on \(f\) and \(f^*\) than that of Theorem 4.3 can be given by assuming that for all \(t \geq 0\)
\[
\|f(\cdot, t) - f^*\|_{L^2(\Omega)}^p \leq C_0 (1 + t)^{-p/(p-2)} ,
\]
(4.15)
where \( p' = p/(p - 1) \) and \( C_0 > 0 \) is a given constant. We have the next result.

**Theorem 4.4.** Suppose that \( p > 2 \), that \( M \) satisfies (\( M \)) and (\( \tilde{M} \)), with \( L_M > 0 \) verifying instead of (\( 4.8 \)) the stronger requirement

\[
m_0 C_p - L_M p \alpha 2^{2 \alpha P - \alpha - P + 2} \tilde{C}^{\alpha P} > 0.
\]

(4.16)

Assume that \( f^* \in L^2(\Omega) \) and \( f \in L^2(\mathbb{R}_0^+; L^2(\Omega)) \) satisfy (\( 4.1 \)) and (\( 4.15 \)). Let \( u = u(x,t) \) be the unique solution of problem (\( 1.1 \)), with initial data \( u_0 \in L^2(\Omega) \), and let \( u^* \) be the corresponding weak solution of (\( 4.3 \)). Then there exists \( C > 0 \) such that

\[
\|u(\cdot,t) - u^*\|_{L^2(\Omega)} \leq C(1 + t)^{-1/(p-2)}
\]

for all \( t \geq 0 \).

**Proof.** A similar argument as that of the proof of Theorem 4.3 gives that

\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot,t) - u^*\|_{L^2(\Omega)}^2 + M \left( \|u(\cdot,t)\|_{X_0^{s,p}(\Omega)}^{p} \right) \left( \|u(\cdot,t), u^*\|_{X_0^{s,p}(\Omega)} \right) \leq L_M p \alpha 2^{2 \alpha p - \alpha - p + 1} \tilde{C}^{\alpha p} \|u(\cdot,t) - u^*\|_{X_0^{s,p}(\Omega)}^p
\]

\[
+ \|f(\cdot,t) - f^*\|_{L^2(\Omega)} \|u(\cdot,t) - u^*\|_{L^2(\Omega)}
\]

(4.17)

\[
\leq L_M p \alpha 2^{2 \alpha p - \alpha - p + 1} \tilde{C}^{\alpha p} \|u(\cdot,t) - u^*\|_{X_0^{s,p}(\Omega)}^p
\]

\[
+ C_s \|f(\cdot,t) - f^*\|_{L^2(\Omega)} \|u(\cdot,t) - u^*\|_{X_0^{s,p}(\Omega)}
\]

where \( C_s > 0 \) is the embedding constant given in (\( 2.2 \)). Furthermore, by the Young inequality, (\( 4.9 \)) and (\( M \)), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot,t) - u^*\|_{L^2(\Omega)}^2 + \frac{m_0 C_p}{2} \|u(\cdot,t) - u^*\|_{X_0^{s,p}(\Omega)}^p
\]

\[
\leq L_M p \alpha 2^{2 \alpha p - \alpha - p + 1} \tilde{C}^{\alpha p} \|u(\cdot,t) - u^*\|_{X_0^{s,p}(\Omega)}^p
\]

(4.18)

\[
+ \left( \frac{2}{m_0 C_p} \right)^{1/(p-1)} C_p' \|f(\cdot,t) - f^*\|_{L^2(\Omega)}^p
\]

Then from (\( 4.15 \), (\( 4.18 \)) and the fractional Sobolev embedding \( X_0^{s,p}(\Omega) \hookrightarrow L^2(\Omega) \) it follows that for all \( t \geq 0 \)

\[
\frac{d}{dt} \|u(\cdot,t) - u^*\|_{L^2(\Omega)}^2 + C_5 C_s^{-p} \|u(\cdot,t) - u^*\|_{L^2(\Omega)}^p \leq C_6 \|f(\cdot,t) - f^*\|_{L^2(\Omega)}^p
\]

\[
\leq C_6 C_0 (1 + t)^{-p/(p-2)}
\]

where by (\( 4.16 \))

\[
C_5 = m_0 C_p - L_M p \alpha 2^{2 \alpha p - \alpha - p + 2} \tilde{C}^{\alpha p} > 0 \quad \text{and} \quad C_6 = 2 (2/m_0 C_p)^{1/(p-1)} C_p'
\]

Let \( X = X(t) \) be a solution of the nonlinear ordinary differential equation, with source term,

\[
X' + C_5 C_s^{-p} X^{p/2} = \lambda_s (1 + t)^{-p/(p-2)},
\]

(4.19)

where the constant \( \lambda_s > 0 \) has to be determined later. A solution of (\( 4.19 \)) is given by the explicit formula

\[
X(t) = \alpha (1 + t)^{-2/(p-2)}, \quad t \in \mathbb{R}_0^+
\]
where $\alpha$ is subject to the restriction
\[
H(\alpha) = - \frac{2\alpha}{p - 2} + C_5 C_{*}^{-p} \alpha^{p/2} - \lambda_* = 0.
\]
This equation always has a solution $\alpha^* > 0$, since $H(0) = -\lambda_* < 0$ and $H(\infty) = \infty$.
Moreover, the solution $\alpha^*$ is estimated from below by
\[
\alpha^* = \left[ \frac{C_5^p}{C_5} \left( \frac{2\alpha^*}{p - 2} + \lambda_* \right) \right]^{2/p} \geq \left( \frac{C_5^p \lambda_*}{C_5} \right)^{2/p}.
\]
Now we choose $\lambda^* > 0$ so that $\lambda_* \geq \max\{C_0 C_0, C_5 C_{*}^{-p}\|u_0 - u^*\|_{L^2(\Omega)}\}$, then
\[
\|u_0 - u^*\|_{L^2(\Omega)}^2 \leq X(0) = \alpha^* \text{ and } \lambda_* \geq C_0 C_0.
\]
Set $Y(t) = \|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 - X(t)$. Then $Y(t)$ satisfies the following linear differential inequality
\[
\begin{cases}
Y'(t) + \beta Y(t) \leq (C_0 C_0 - \lambda_*) (1 + t)^{-p/(p-2)} \leq 0 & \text{for } t > 0, \\
Y(0) \leq 0,
\end{cases}
\]
where
\[
\beta = C_5 C_{*}^{-p} \frac{\|u(\cdot, t) - u^*\|_{L^2(\Omega)}^p - X(t)^{p/2}}{\|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 - X(t)}
\]
\[
= \frac{p}{2} C_5 C_{*}^{-p} \int_0^1 \left( (1 - \tau) X(t) \right)^{p/2 - 1} d\tau \geq 0.
\]
It follows that $Y(t) \leq 0$ for all $t \geq 0$, that is,
\[
0 \leq \|u(\cdot, t) - u^*\|_{L^2(\Omega)}^2 \leq X(t) = \alpha^* (1 + t)^{-2/(p-2)}.
\]
This completes the proof. \(\Box\)

5. Extinction of solutions. For the singular case $1 < p < 2$, we prove below that when $f \equiv 0$ the solution $u$ of \([1.1]\) vanishes at a finite time $T^*$, called extinction time of $u$.

**Theorem 5.1.** (Extinction property). Let $u_0 \in X_{0}^{p,N}(\Omega) \setminus \{0\}$, and let $u(\cdot, t)$ be the unique solution of problem \([1.1]\), with $f \equiv 0$. If $2N/(N + 2s) \leq p < 2$, then the following upper estimate holds
\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq \left[ \|u_0\|_{L^2(\Omega)}^{2-p} - m_0 C_{*}^{-p} (2 - p) t \right]^{1/(2-p)}
\]
for all $t \in \mathbb{R}^+$. Thus the solution $u$ extinguishes at a finite time $T^*$.

**Proof.** Fix $t > 0$. Choose $\varphi = u(\cdot, t)$ in Definition \([3.1]\) \((iii)\). Then,
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + M \left( \|u(\cdot, t)\|_{X_{0}^{p,N}(\Omega)}^p \right) \|u(\cdot, t)\|_{X_{0}^{p,N}(\Omega)}^p = 0.
\]
(5.1)
By \((M)\) and \([5.1]\), we arrive at the inequality
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + m_0 \|u(\cdot, t)\|_{X_{0}^{p,N}(\Omega)}^p \leq 0.
\]
Furthermore, \([2.2]\) implies that
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + m_0 C_{*}^{-p} \|u(\cdot, t)\|_{L^2(\Omega)}^p \leq 0.
\]
(5.2)
Thus, since $t > 0$ is arbitrary,

$$
\|u(\cdot, t)\|_{L^2(\Omega)}^2 \leq \left[ \|u_0\|_{L^2(\Omega)}^{2-p} - m_0 C_*^{-p}(2-p)t \right]^{2/(2-p)} \quad \text{for all } t \in \mathbb{R}^+_0,
$$

where

$$
\left[ \|u_0\|_{L^2(\Omega)}^{2-p} - m_0 C_*^{-p}(2-p)t \right]_+^2 = \max \left\{ \|u_0\|_{L^2(\Omega)}^{2-p} - m_0 C_*^{-p}(2-p)t, 0 \right\}.
$$

This means that the solution $u$ vanishes at a finite time $T^*$. The proof is thus complete.

**Remark 5.1.** Theorem 5.1 leaves open some interesting questions, as the determination of above and below estimates for the extinction time $T^*$ in view of initial data and the extension of Theorem 5.1 to the case where $f \not\equiv 0$ in problem (1.1). We are planning to investigate these two open problems in a forthcoming paper.

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