TOPOLOGICAL MATTER, MIRROR SYMMETRY
AND NON-CRITICAL (SUPER)STRINGS

A.V. Ramallo⋆ and J.M. Sanchez de Santos†

Departamento de Física de Partículas,
Universidad de Santiago,
E-15706 Santiago de Compostela, Spain.

ABSTRACT

We study the realization of the (super) conformal topological symmetry in two-
dimensional field theories. The mirror automorphism of the topological algebra is
represented as a reflection in the space of fields. As a consequence, a double
BRST structure for topological matter theories is found. It is shown that the
implementation of the topological symmetry in non-critical (super)string theories
depends on the matter content of the two realizations connected by the mirror
transformation.

⋆ E-mail: ALFONSO@GAES.USC.ES
† E-mail: SANTOS@GAES.USC.ES
1. Introduction

Two-dimensional topological field theories have been a subject extensively studied in the last few years [1, 2, 3]. These models are endowed with a topological BRST symmetry that allows to eliminate all their local excitations. If, in addition to the topological symmetry, the system is conformally invariant, we say that the model is a topological conformal field theory (TCFT) [4].

As any other conformal field theory, the TCFT’s are characterized by their chiral algebra, which must include the Virasoro algebra as a subalgebra. The topological nature of TCFT’s is ensured if all the generators of the chiral algebra (including the energy-momentum tensor) are exact in the cohomology defined by the topological BRST charge.

The compatibility between the topological and the (extended) conformal symmetries is encoded in the so-called topological algebra [4]. This is the operator algebra closed by the BRST current and the generators of the chiral algebra. The realization of the topological algebra in different topological matter systems is the main subject studied in this paper.

The standard procedure to obtain the local form of the topological symmetries corresponding to some chiral algebras consists in performing a redefinition (a twist) of different superconformal theories. For example, the local BRST algebra for the (unextended) Virasoro symmetry can be obtained from the $N = 2$ superconformal models [5, 6], whereas if the topological matter system is $(N = 1)$ supersymmetric, its local BRST symmetry is governed by a twisted $N = 3$ superconformal algebra [7]. In general the twisting procedure gives rise to a Virasoro algebra with a vanishing central charge. Nevertheless, the twisted algebra contains c-number anomalies, parametrized by a number (the so-called dimension) which characterizes the TCFT.

One of the features of the topological algebra that we shall study is the fact that it possesses an automorphism, the mirror transformation [8], that relates two
different realizations. Under this mirror transformation the abelian currents needed to implement locally the topological symmetry change their signs and the BRST current is exchanged with the BRST ancestor of the energy-momentum tensor. The origin of the mirror symmetry can be traced back to the twisting procedure since the two representations of the topological algebra connected by a mirror transformation are obtained when two different twisting prescriptions are applied to the same superconformal field theory. Within our general approach, the mirror transformation will be represented as a transformation of fields. In fact, when we are in a convenient basis of fields, we shall see that the mirror symmetry can be understood as a reflection in field space.

The topological conformal symmetry is realized in systems where conformal matter is coupled to 2d (super)gravity [9, 10, 11], as happens in non-critical (super)string theories. Indeed, as was first shown in ref. [12], one can improve the usual BRST current that fixes the reparametrization invariance in such a way that it realizes the topological algebra. Moreover, this result has been extended in [13] to systems where matter is coupled to supergravity and $W$-gravity (see also ref. [14]).

In these realizations of the topological symmetry, the field content is given by matter (and Liuoville) fields together with (super)diffeomorphism ghosts. The ghost sector in these matter+gravity models always contains a pair $(b, c)$ of anticommuting ghosts with spins $(2, -1)$. By means of the mirror transformation, the ghosts $b$ and $c$ acquire spins 1 and 0 respectively, whereas the matter central charge is changed. For these systems we shall find that in order to have a representation of the topological algebra with a real c-number extension (i.e. with real dimension) the central charges of the matter sectors of the two realizations of the topological symmetry connected by the mirror transformation must be restricted to a particular range of values. Actually we shall obtain a “barrier” formula which, surprisingly, involves the two realizations of the topological symmetry related by the mirror reflection.
This paper is organized as follows. In section 2 we study the topological symmetry in a system of scalar fields with vanishing Virasoro central charge. After introducing the topological algebra, we study its possible vertex operator representations. By means of a convenient bosonization, the results obtained are put in terms of a pair of fermionic fields which, after a mirror transformation, are identified with the diffeomorphism ghosts. This section concludes with a study of the reflections in field space that are equivalent to the mirror automorphism. The realization of the topological symmetry in the bosonic string theory is analysed in section 3. In this section we explore the role played by the mirror invariance in the implementation of the topological symmetry in the models of matter coupled to two-dimensional gravity.

In section 4 we analyze a supersymmetric model of scalar bosons and Majorana fermions. The supersymmetric extension of the topological algebra is reviewed at the beginning of this section. In order to get a representation of this algebra in our model, a supersymmetric ghost system is introduced. In terms of these ghosts fields the topological supersymmetric algebra is easily realized. In one of these realizations one can identify the ghost fields with the standard superdiffeomorphism ghosts of two-dimensional supergravity. Many of the results obtained in sections 2 and 3 are generalized to the supersymmetric case. In particular, the mirror transformation is represented as a simultaneous reflection of the scalar and Majorana fields. Moreover, the range of values of the central charge of the matter sector required to implement the topological symmetry with a real dimension can be determined as in the non-supersymmetric case.

We summarize our results and discuss the possible extensions of our work in section 5. Finally in the Appendix we give some details of our calculations.
2. The BRST symmetry of topological matter

Let us consider a system of conformal topological matter in two spacetime dimensions. Such a system is characterized by an energy-momentum tensor $T$ that closes the Virasoro algebra with vanishing central charge. Therefore the holomorphic component of $T$ satisfies the operator product expansion (OPE):

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \quad (2.1)$$

We shall consider a realization of the algebra (2.1) in terms of a multicomponent scalar field $\vec{\phi}(z) = (\phi_1(z), \ldots, \phi_N(z))$, $N$ being the number of scalar fields. The $\phi_i$’s are free fields and therefore they obey the basic OPE’s

$$\phi_i(z) \phi_j(w) = -\delta_{ij} \log(z-w). \quad (2.2)$$

In terms of $\vec{\phi}$, the energy-momentum tensor $T$ can be written as:

$$T = -\frac{1}{2} \overline{\partial \phi} \cdot \partial \phi + \vec{A} \cdot \partial^2 \phi, \quad (2.3)$$

where we have included some background charges parametrized by the constant vector $\vec{A}$ ($\vec{A} \in \mathbb{C}^N$). For a general $\vec{A}$, $T(z)$ closes the Virasoro algebra with central charge

$$c = N + 12 A^2. \quad (2.4)$$

Therefore, in order to satisfy eq. (2.1), we must require $c$ to vanish and, as a consequence, this fixes the value of $A^2$ to be:

$$A^2 = -\frac{N}{12}. \quad (2.5)$$

We would like this model to be a topological theory. This would be the case if we were able to find a nilpotent BRST current $Q$ such that it could be considered
as the generator of a topological symmetry. The nilpotency of $Q$ is guaranteed if the OPE of $Q$ with itself is regular:

$$Q(z)Q(w) = 0, \quad (2.6)$$

whereas the requirement of being the generator of a symmetry is fulfilled if we impose that $Q$ be a primary dimension-one operator with respect to the energy-momentum tensor $T$:

$$T(z)Q(w) = \frac{Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{z - w}. \quad (2.7)$$

What makes $Q(z)$ the generator of a topological symmetry is the fact that the energy-momentum tensor $T$ is $Q$-exact, i.e. that there exists an operator $G$ such that

$$T(z) = \{ \oint Q, G(z) \}. \quad (2.8)$$

$G$ is thus a dimension-two operator which is the BRST partner of $T$. On dimensional grounds, the local version of eq. (2.8) has the form:

$$Q(z)G(w) = \frac{d}{(z-w)^3} + \frac{R(w)}{(z-w)^2} + \frac{T(w)}{z-w}, \quad (2.9)$$

where $d$ is a c-number constant and $R$ is an abelian dimension-one current. We shall call the algebra closed by $Q$, $T$, $G$ and $R$, the topological algebra (TA). This algebra characterizes the topological symmetry of the model. In its minimal version the algebra closes without introducing extra generators. Consistency with eqs. (2.6), (2.7) and (2.9) is achieved if one requires the other OPE's of the algebra.
to be:

\[
R(z) R(w) = \frac{d}{(z - w)^2} \\
R(z) Q(w) = \frac{Q(w)}{z - w} \\
T(z) R(w) = -\frac{d}{(z - w)^3} + \frac{R(w)}{(z - w)^2} + \frac{\partial R(w)}{z - w} \\
R(z) G(w) = -\frac{G(w)}{z - w} \\
T(z) G(w) = \frac{2G(w)}{(z - w)^2} + \frac{\partial G(w)}{z - w} \\
G(z) G(w) = 0.
\]

The central extension \(d\) of the algebra will be called the dimension of the TA [4]. Notice that \(R\) is an anomalous abelian current whose anomaly is precisely given by \(d\). In fact the TA displayed in eqs. (2.1), (2.6), (2.9) and (2.10) can be obtained by twisting the \(N = 2\) superconformal algebra. The relation between the parameter \(d\) and the Virasoro central charge of the \(N = 2\) theory is simply \(c_{N=2} = 3d\).

An interesting aspect of the TA is that for any realization of the algebra one can generate another with the same value of \(d\) by means of a redefinition that exchanges the role of \(Q\) and \(G\) and changes the sign of \(R\):

\[
T \rightarrow T^* = T - \partial R \\
Q \rightarrow Q^* = G \\
G \rightarrow G^* = Q \\
R \rightarrow R^* = -R.
\]

The transformation in (2.11) will be called the mirror transformation. It has its origin in the two possible twistings \(T^{N=2} \pm 12\partial R\) that one can perform to generate a topological theory from a \(N = 2\) superconformal model.

Let us now investigate the possible realizations of the BRST current in our scalar field theory. We want \(Q\) to be a primary dimension-one operator local in
the field $\phi$. Accordingly we shall adopt the following ansatz for $Q$:

$$Q = Q_n(\partial^i \phi) \ e^{\bar{\alpha} \cdot \phi},$$  

(2.12)

where $Q_n$ is a polynomial in the derivatives of $\phi$ with dimension $n$. This non-negative integer $n$ will be called the depth of the operator $Q$. In eq. (2.12) $\bar{\alpha}$ is a constant vector. The nilpotency condition of $Q$ (eq.(2.6)) implies that:

$$\bar{\alpha}^2 \leq 0.$$  

(2.13)

On the other hand, as the conformal weight with respect to the energy-momentum tensor $T$ in eq. (2.3) of the vertex operator $e^{\bar{\alpha} \cdot \phi}$ is

$$\Delta(e^{\bar{\alpha} \cdot \phi}) = \bar{\alpha} \cdot \bar{\alpha} - \frac{1}{2} \bar{\alpha}^2,$$  

(2.14)

it follows from eq. (2.7) that

$$\Delta_Q = n + \bar{A} \cdot \bar{\alpha} - \frac{1}{2} \bar{\alpha}^2 = 1.$$  

(2.15)

Once $Q$ is fixed, one must determine an operator $G$ such that eq. (2.8) holds. In view of the ansatz we have taken for $Q$, it is natural to search for an operator $G$ of the same form, i.e. a polynomial $G_m$ of depth $m$ times a vertex operator:

$$G = G_m(\partial^i \phi) \ e^{-\bar{\alpha} \cdot \phi}.$$  

(2.16)

Notice that we are forced to have the exponential factor $e^{-\bar{\alpha} \cdot \phi}$ in $G$ if we want to fulfill eq. (2.8) for $T$ as in eq. (2.3). The dimension-two character of $G$ constrains $m$, $\bar{A}$, and $\bar{\alpha}$ to satisfy:
\[ \Delta_G = m - \vec{A} \cdot \vec{\alpha} - \frac{1}{2} \vec{\alpha}^2 = 2. \quad (2.17) \]

Adding eqs. (2.15) and (2.17), one gets a simple equation relating the depths of \( Q \) and \( G \) with the length of the vector \( \vec{\alpha} \):

\[ n + m = \vec{\alpha}^2 + 3. \quad (2.18) \]

As \( \vec{\alpha}^2 \) cannot be positive (see eq. (2.13)), we get from eq.(2.18) the following restriction on the possible values of the depths:

\[ n + m \leq 3. \quad (2.19) \]

Moreover, by subtracting eqs. (2.15) and (2.17), the scalar product \( \vec{A} \cdot \vec{\alpha} \) is determined as a function of \( n \) and \( m \):

\[ \vec{A} \cdot \vec{\alpha} = \frac{m - n - 1}{2}. \quad (2.20) \]

The rule (2.20) greatly restricts the number of possible choices for \( n \) and \( m \). Actually the mirror symmetry of the algebra pairs the \( (n,m) \) and \( (m,n) \) cases since by applying a mirror transformation to a \( (n,m) \) realization one generates a \( (m,n) \) one:

\[ (n,m)^* \approx (m,n). \quad (2.21) \]

Up to now we have imposed very few conditions extracted from the TA. In order to implement the topological symmetry one must check the closure of the full algebra. This has to be done case by case for the different values of the depths \( n \) and \( m \). It turns out that only for \( (n,m) = (0,2) \) and its mirror image \( (m,n) = (2,0) \) the algebra can be closed in general for \( N \geq 2 \). For other values of \( n \) and \( m \) the constraints generated by the algebra are either inconsistent or they can only
be solved for particular values of the number $N$ of scalar fields. The analysis of
the cases allowed by the rule (2.19) is presented in the Appendix. In the present
section we shall limit ourselves to studying the $(0, 2)$ and $(2, 0)$ cases. Notice that
according to eq. (2.20) when $n = 0$ and $m = 2$ one has:

$$\vec{\alpha}^2 = -1 \quad \vec{A} \cdot \vec{\alpha} = \frac{1}{2}. \quad (2.22)$$

Let us parametrize $Q$ and $g$ in this case as follows:

$$Q = e^{\vec{\alpha} \cdot \vec{\phi}}$$
$$G = (\vec{g} \cdot \partial^2 \vec{\phi} + \partial \vec{\phi} \cdot H \cdot \partial \vec{\phi}) e^{-\vec{\alpha} \cdot \vec{\phi}}, \quad (2.23)$$

where $\vec{g}$ is a constant vector and $H$ is a numerical $N \times N$ symmetric matrix. In
the following we shall denote by $X \cdot H \cdot Y$ to the contraction of the matrix $H$ with
any two vectors $X$ and $Y$. In order to identify $d$ and $R$ one has to compute the
OPE of $Q$ and $G$ (see eq. (2.9)). A straightforward calculation leads to the result:

$$d = \vec{g} \cdot \vec{\alpha} + \vec{\alpha} \cdot H \cdot \vec{\alpha} \quad (2.24)$$
$$R = (2\vec{\alpha} \cdot H + d \vec{\alpha}) \cdot \partial \vec{\phi} \equiv \vec{R} \cdot \partial \vec{\phi}.$$  

On the other hand, by comparing the background charge and kinetic terms of $T$
with those appearing in the simple pole singularity of the product $Q(z)G(w)$, one
gets the constraints:

$$\vec{A} = \vec{g} + \frac{d}{2} \vec{\alpha}$$
$$\frac{d}{2} \vec{\alpha} \otimes \vec{\alpha} + (H \cdot \vec{\alpha}) \otimes \vec{\alpha} + \vec{\alpha} \otimes (H \cdot \vec{\alpha}) + H = -I_2, \quad (2.25)$$

where $I$ is the $N \times N$ identity matrix. The anomalous character of the $R$-current
is reflected in the double and triple pole singularities appearing in the OPE’s
$R(z)R(w)$ and $T(z)R(w)$. Parametrizing $R = \vec{R} \cdot \partial \vec{\phi}$, with $\vec{R}$ as given in the
second equation in (2.24), one obtains two new conditions

\[
\vec{A} \cdot \vec{R} = -\frac{d}{2}, \quad \vec{R} \cdot \vec{R} = -d.
\]  (2.26)

Other OPE’s of the TA follow from the conditions we have obtained so far and therefore one can consider eqs. (2.24), (2.25) and (2.26) as a complete set of consistency conditions that ensure the fulfillment of the TA. In fact one can use eqs. (2.24)-(2.26) to get new relations between \(H, \vec{\alpha}\) and \(\vec{g}\). For instance, if we contract both sides of the second equation in (2.25) with two vectors \(\vec{\alpha}\), we get:

\[
\vec{\alpha} \cdot H \cdot \vec{\alpha} = d - \frac{1}{2}.
\]  (2.27)

Plugging this result in eqs. (2.24) and (2.26), one can readily obtain:

\[
\vec{g} \cdot \vec{\alpha} = \frac{d + 1}{2}, \quad \vec{\alpha} \cdot H \cdot \vec{g} = -\frac{d(d + 1)}{2}.
\]  (2.28)

Let us check that the constraints found for \(H, \vec{\alpha}\) and \(\vec{g}\) are enough to ensure that \(Q\) and \(G\) have good quantum numbers with respect to the \(R\) current. A direct calculation using eqs. (2.23) and (2.24) yields the result:

\[
R(z) Q(w) = -\frac{\vec{R} \cdot \vec{\alpha}}{z - w} e^{\vec{\alpha} \cdot \vec{\phi}(w)}.
\]  (2.29)

Using eqs. (2.22) and (2.27) one immediately finds for the scalar product \(\vec{R} \cdot \vec{\alpha}\) the value

\[
\vec{R} \cdot \vec{\alpha} = -1,
\]  (2.30)

from which the OPE \(R(z)Q(w)\) of eq. (2.10) follows. Similarly, after a simple calculation, one gets:

\[
R(z) G(w) = -\frac{2\vec{R} \cdot \vec{g}}{(z - w)^3} e^{-\vec{\alpha} \cdot \vec{\phi}(w)} - \frac{2\vec{R} \cdot H \cdot \partial \vec{\phi}(w)}{(z - w)^2} e^{-\vec{\alpha} \cdot \vec{\phi}(w)} + \vec{R} \cdot \vec{\alpha} \frac{G(w)}{z - w}.
\]  (2.31)

It is easy to prove that the residues of the triple and double poles in eq. (2.31)
vanish. Indeed, making use of eqs. (2.25), (2.27) and (2.28), one easily arrives at:

\[ \vec{R} \cdot \vec{g} = 0 \quad H \cdot \vec{R} = 0, \quad (2.32) \]

which, together with eq. (2.30), imply that \( R(z) \) and \( G(w) \) have the OPE displayed in eq. (2.10). Proceeding in the same way, all the remaining OPE’s in eq. (2.10) can be checked.

The realization of the TA we have obtained can be put in much simpler terms if one introduces a pair \((b,c)\) of two anticommuting fields. In terms of the bosonic field \( \vec{\phi} \), these new fields are given by standard bosonization formulas that involve the component of \( \vec{\phi} \) along the direction of \( \vec{\alpha} \):

\[ b = e^{\vec{\alpha} \cdot \vec{\phi}} \quad c = e^{-\vec{\alpha} \cdot \vec{\phi}}. \quad (2.33) \]

Notice that from eq. (2.22) the \( b \) and \( c \) fields have conformal dimensions one and zero respectively and they satisfy the OPE:

\[ b(z) c(w) = \frac{1}{z - w}. \quad (2.34) \]

When trying to translate all expressions in terms of this \((b,c)\) system, one has to compute the normal ordered products of \( b, c \) and their derivatives in terms of the field \( \vec{\phi} \). Some of these products that will be needed below are:

\[ :bc: = \vec{\alpha} \cdot \partial \vec{\phi} \]
\[ :b\partial c: = \frac{1}{2}(\vec{\alpha} \cdot \partial^2 \vec{\phi} - (\vec{\alpha} \cdot \partial \vec{\phi})^2) \]
\[ :cb\partial c: = \frac{1}{2}(\vec{\alpha} \cdot \partial^2 \vec{\phi} + (\vec{\alpha} \cdot \partial \vec{\phi})^2) e^{-\vec{\alpha} \cdot \vec{\phi}}. \quad (2.35) \]

In (2.35) by \( :cb\partial c: \) we mean \( c(\partial \partial c) \). In order to extract the contribution of the \((b,c)\) system to the generators of the TA, we shall decompose all vectors in
components parallel and orthogonal to $\vec{\alpha}$. Thus, if $\vec{X}$ is an arbitrary vector, we shall write:

$$\vec{X} = \vec{X}_\parallel + \vec{X}_\perp,$$

(2.36)

where

$$\vec{X}_\parallel = \frac{(\vec{X} \cdot \vec{\alpha})}{\vec{\alpha}^2} \vec{\alpha} = -(\vec{X} \cdot \vec{\alpha}) \vec{\alpha}$$

$$\vec{X}_\perp = \vec{X} - \vec{X}_\parallel = \vec{X} + (\vec{X} \cdot \vec{\alpha}) \vec{\alpha}.$$  

(2.37)

Using eqs. (2.22) and (2.28) one easily gets the components of $\vec{A}$, $\vec{R}$ and $\vec{g}$ parallel to $\vec{\alpha}$:

$$\vec{A}_\parallel = -\vec{\alpha}$$

$$\vec{R}_\parallel = \vec{\alpha}$$

$$\vec{g}_\parallel = -\frac{d + 1}{2} \vec{\alpha}.$$  

(2.38)

Notice that the BRST charge $Q$ is simply the $b$ field:

$$Q(z) = b(z).$$

(2.39)

To find the expression of $T$ and $R$ in the new variables, we first split them as:

$$T = T_\parallel + T_\perp$$

$$R = R_\parallel + R_\perp.$$  

(2.40)

Making use of eq. (2.38), the parallel components $T_\parallel$ and $R_\parallel$ are easily obtained:

$$T_\parallel = \frac{1}{2}(\vec{\alpha} \cdot \partial \vec{\phi})^2 - \frac{1}{2}\vec{\alpha} \cdot \partial^2 \vec{\phi} = -b\partial c$$

$$R_\parallel = \vec{\alpha} \cdot \partial \vec{\phi} = b\vec{c}.$$  

(2.41)

where we have taken into account the bosonization formulas of eq. (2.35). Let us denote the components of $\vec{\phi}$, $\vec{R}$ and $\vec{A}$ orthogonal to $\vec{\alpha}$ as:

$$\vec{\phi}_\perp = \vec{\phi}_\perp$$

$$\vec{R}_\perp = \vec{R}_\perp$$

$$\vec{a}_\perp = \vec{A}_\perp.$$  

(2.42)

Notice that $\vec{\phi}_\perp$, $\vec{R}_\perp$ and $\vec{a}_\perp$ live in an $(N - 1)$-dimensional vector space (i.e. in the
hyperplane orthogonal to \( \vec{a} \)). With these notations, \( T \) and \( R \) can be written as:

\[
T = T_\varphi + T_{(b,c)} = -\frac{1}{2}(\partial \vec{\varphi})^2 + \vec{a} \cdot \partial^2 \vec{\varphi} - b \partial c
\]

\[
R = R_\varphi + bc = \vec{r} \cdot \partial \vec{\varphi} + bc.
\]

(2.43)

It remains to express \( G \) in terms of \( b, c \) and \( \vec{\varphi} \). This is easily accomplished if one takes into account that the orthogonal decomposition of the terms \( \vec{g} \cdot \partial^2 \vec{\varphi} \) and \( \partial \vec{\varphi} \cdot H \cdot \partial \vec{\varphi} \) appearing in \( G \) is:

\[
\vec{g} \cdot \partial^2 \vec{\varphi} = \vec{a} \cdot \partial^2 \vec{\varphi} - \frac{d+1}{2} \vec{\alpha} \cdot \partial^2 \vec{\varphi} - \frac{1}{2} \partial \vec{\varphi} \cdot H \cdot \partial \vec{\varphi} = -\frac{1}{2}(\partial \vec{\varphi})^2 + \frac{d-1}{2} (\vec{\alpha} \cdot \partial \vec{\varphi})^2 - R_\varphi \vec{\alpha} \cdot \partial \vec{\varphi},
\]

where eqs. (2.27) and (2.24) have been used. We have also used to obtain (2.44) the fact that \( H \) acting on any two vectors orthogonal to \( \vec{\alpha} \) equals the matrix \(-\frac{I}{2}\):

\[
\vec{X}_\perp \cdot H \cdot \vec{Y}_\perp = -\frac{\vec{X}_\perp \cdot \vec{Y}_\perp}{2}.
\]

(2.45)

Eq. (2.45) follows easily from eq. (2.25). Substituting eq. (2.44) in the second equation of (2.23) and using the bosonization dictionary of eq. (2.35), we obtain:

\[
G = cT_\varphi - cb \partial c + R_\varphi \partial c + \frac{d}{2} \partial^2 c,
\]

where \( T_\varphi \) and \( R_\varphi \) are defined in eq. (2.43). In terms of \( \vec{\varphi} \) and \( (b,c) \) the realization of the TA of eqs. (2.39), (2.43) and (2.46) depends on two \((N-1)\)-dimensional vectors \( \vec{a} \) and \( \vec{r} \). The constraints needed to ensure the closure of the TA can be easily rephrased in terms of the scalar products of the vectors \( \vec{a} \) and \( \vec{r} \). Putting \( n = N - 1 \), we obtain from eqs. (2.5), (2.26) and (2.38):

\[
\vec{a}^2 = \frac{2-n}{12} \quad \vec{r}^2 = 1 - d \quad \vec{a} \cdot \vec{r} = -\frac{1+d}{2}.
\]

(2.47)

For \( n \geq 1 \) it is possible to solve the constraints of eq. (2.47). In fact when \( n = 1 \) the value of \( d \) is fixed ( \( d \) can only be \(-2\) or \(-\frac{1}{3}\) ), whereas for \( n > 1 \) \( d \) remains arbitrary.
Let us consider now the realization of the TA that is obtained by performing a
mirror transformation. After a redefinition as in eq. (2.11), the energy-momentum
tensor in bosonic language takes the form

\[ T^\ast = -\frac{1}{2}(\partial \vec{\phi})^2 + (\vec{A} - \vec{R}) \cdot \partial^2 \vec{\phi}. \tag{2.48} \]

Notice that the mirror transformation changes the vector of the background charges
from \( \vec{A} \) to \( \vec{A} - \vec{R} \). More illuminating is however the expression of \( T^\ast \) in terms of
the \((b,c)\) fields. Taking the expressions of \( T \) and \( R \) in eq. (2.43) into account, one
gets

\[ T^\ast = T^\ast_\varphi + T^\ast_{(b,c)} = -\frac{1}{2}(\partial \vec{\varphi})^2 + \vec{a}^\ast \cdot \partial^2 \vec{\varphi} - 2b\partial c + c\partial b, \tag{2.49} \]

where

\[ \vec{a}^\ast = \vec{a} - \vec{r}. \tag{2.50} \]

In \( T^\ast \) the fields \( b \) and \( c \) have weights 2 and \(-1\) respectively. Therefore this means
that \( T^\ast \) can be regarded as the energy-momentum tensor of a bosonic string, being
\((b,c)\) the standard reparametrization ghosts. Other generators of the TA can be
equally obtained for this mirror realization. For example \( G^\ast \) and \( R^\ast \) are given by

\[ \begin{align*}
G^\ast &= b \\
R^\ast &= R^\ast_\varphi + R^\ast_{(b,c)} = \vec{r}^\ast \cdot \partial \vec{\varphi} - bc,
\end{align*} \tag{2.51} \]

with \( \vec{r}^\ast = -\vec{r} \). The new BRST current is

\[ Q^\ast = c \left[ T^\ast_\varphi + \frac{1}{2} T^\ast_{(b,c)} \right] - \partial \left[ c \left( R^\ast_\varphi + \frac{1}{2} R^\ast_{(b,c)} \right) \right] + \frac{d}{2} \partial^2 c. \tag{2.52} \]

The first term in eq. (2.52) coincides with the standard BRST current in the
bosonic string theory. The second and third terms in this equation are total
derivatives and therefore they do not contribute to the integrated BRST charge. Therefore the topological symmetry we are dealing with in this mirror realization is just an improved version of the BRST symmetry of the bosonic string [12]. Notice that the improving term in (2.52) depends on $d$ and on the direction of the abelian current $R^*$ in the space of fields. This mirror realization is given in terms of two vectors $\vec{a}^*$ and $\vec{r}^*$, which determine respectively the background charges in the energy-momentum tensor and the $R$-charges of the bosonic fields. It is straightforward to obtain from eq. (2.47) the constraints that $\vec{a}^*$ and $\vec{r}^*$ must satisfy:

$$
(\vec{a}^*)^2 = \frac{26 - n}{12}, \quad \vec{a}^* \cdot \vec{r}^* = \frac{3 - d}{2}, \quad (\vec{r}^*)^2 = 1 - d.
$$

(2.53)

It is important to point out that for $d \neq 0$ the mirror transformation can be implemented as a transformation of the fields. Let us come back to the formalism in which only the bosonic fields appear (i.e. before the introduction of the $(b,c)$ pair). Consider the hyperplane orthogonal to the vector $\vec{R}$. For any vector $\vec{X}$, a reflection with respect to this hyperplane is a transformation that changes the sign of the component of $\vec{X}$ parallel to $\vec{R}$. Calling the reflected vector $\vec{X}^*$, we have:

$$
\vec{X} \rightarrow \vec{X}^* = \vec{X} - \frac{2(\vec{X} \cdot \vec{R})}{\vec{R}^2} \vec{R}.
$$

(2.54)

As $\vec{R}^2 = -d$, eq. (2.54) reduces to:

$$
\vec{X}^* = \vec{X} + \frac{2}{d} (\vec{X} \cdot \vec{R}) \vec{R},
$$

(2.55)

which is a well-defined orthogonal transformation for $d \neq 0$. Let us apply this transformation to the field $\vec{\phi}$, i.e. let us define a new field $\vec{\phi}^*$ as:

$$
\vec{\phi}^* = \vec{\phi} + \frac{2}{d} (\vec{\phi} \cdot \vec{R}) \vec{R}.
$$

(2.56)

In terms of $\vec{\phi}^*$, the energy-momentum tensor $T$ in (2.3) can be written as:

$$
T = -\frac{1}{2} (\partial \vec{\phi}^*)^2 + \vec{A}^* \cdot \partial^2 \vec{\phi}^*,
$$

(2.57)

where we have used the fact that the scalar product of two vectors is invariant.
under orthogonal transformations. From the general formula (2.55) and the scalar product $\vec{A} \cdot \vec{R}$ given in eq. (2.26), we get

$$\vec{A}^* = \vec{A} - \vec{R},$$

which means that $T$ is given by:

$$T = -\frac{1}{2}(\partial \vec{\phi}^*)^2 + (\vec{A} - \vec{R}) \cdot \partial^2 \vec{\phi}^*.$$  \hspace{1cm} (2.58)

By comparing eqs. (2.59) and (2.48) we conclude that the transformation $T \rightarrow T - \partial R$ is equivalent to the change of variables $\vec{\phi} \rightarrow \vec{\phi}^*$. Notice that with this interpretation of the mirror transformation we do not change the energy-momentum tensor (we just reexpress it in terms of new fields) and therefore the system and its mirror copy can be regarded as the same model. As $\vec{R}^* = -\vec{R}$ (see eq. (2.54)), the $R$-current can be written as:

$$R = -\vec{R} \cdot \partial \vec{\phi}^*,$$ \hspace{1cm} (2.60)

which means that the components of $R$ along the new field variables $\vec{\phi}^*$ are minus the ones with respect to the initial fields $\vec{\phi}$. Thus we see that changing variables as $\vec{\phi} \rightarrow \vec{\phi}^*$ without transforming $R$ is equivalent to the transformation $R \rightarrow -R$ in eq. (2.11). In order to get the full set of generators of the TA in terms of the $\vec{\phi}^*$ field, let us consider the operators obtained by replacing $\vec{\phi}$ by $\vec{\phi}^*$ in $Q$ and $G$. The conformal dimensions of the vertex operators $e^{\pm \vec{\alpha} \cdot \vec{\phi}^*}$ with respect to $T$ are:

$$\Delta(e^{\vec{\alpha} \cdot \vec{\phi}^*}) = 2 \quad \Delta(e^{-\vec{\alpha} \cdot \vec{\phi}^*}) = -1,$$  \hspace{1cm} (2.61)

as can be easily proved from eqs. (2.59), (2.22) and (2.26). It is thus natural to consider the operators

$$Q^* = (\vec{g} \cdot \partial^2 \vec{\phi}^* + \partial \vec{\phi}^* \cdot H \cdot \partial \vec{\phi}^*) e^{-\vec{\alpha} \cdot \vec{\phi}^*}$$

$$G^* = e^{\vec{\alpha} \cdot \vec{\phi}^*}.$$  \hspace{1cm} (2.62)

It is easy to convince oneself that $T$, $R$, $Q^*$ and $G^*$ close the TA. Notice that the new BRST current $Q^*$ is formally obtained by replacing $\vec{\phi}$ by $\vec{\phi}^*$ in $G$. However
$Q^* \neq G$, as can be seen by writing $Q^*$ in terms of $\vec{\phi}$. Taking into account that $\vec{A} \cdot \vec{B}^* = \vec{A}^* \cdot \vec{B}$ for any two vectors $\vec{A}$ and $\vec{B}$, we get

$$Q^* = (\vec{g}^* \cdot \partial^2 \vec{\phi} + \partial \vec{\phi} \cdot H \cdot \partial \vec{\phi})e^{-\vec{\alpha}^* \cdot \vec{\phi}},$$

(2.63)

where we have taken into account eq. (2.56) and the fact that $H \cdot \vec{R} = 0$ (see eq. (2.32)). On the other hand from the general equation (2.55) and from eqs. (2.30) and (2.32) we have

$$\vec{g}^* = \vec{g}, \quad \vec{\alpha}^* = \vec{\alpha} - \frac{2}{d} \vec{R}.$$  

(2.64)

This implies that :

$$Q^* = (\vec{g} \cdot \partial^2 \vec{\phi} + \partial \vec{\phi} \cdot H \cdot \partial \vec{\phi})e^{-(\vec{\alpha} - \frac{2}{d} \vec{R}) \cdot \vec{\phi}},$$

(2.65)

and therefore $Q^* \neq G$ as stated above. In the same way $G^*$, obtained by making the replacement of $\vec{\phi}$ by $\vec{\phi}^*$ in $Q$, differs from it and it is given by :

$$G^* = e^{(\vec{\alpha} - \frac{2}{d} \vec{R}) \cdot \vec{\phi}}.$$  

(2.66)

It is easy to see how the reparametrization ghosts with conformal weights $(2, -1)$ appear in the new variables. In complete analogy with eq. (2.33), let us introduce two new fields $b^*$ and $c^*$ defined as

$$b^* = e^{\vec{\alpha} \cdot \vec{\phi}^*}, \quad c^* = e^{-\vec{\alpha} \cdot \vec{\phi}^*}.$$  

(2.67)

From eq. (2.61) it follows that $(b^*, c^*)$ have indeed conformal dimensions $(2, -1)$ and thus we expect them to correspond to the reparametrization ghosts that are needed to interpret our system as a string theory. To verify this fact we proceed as we did previously with the $(b, c)$ fields. Let us separate everywhere the component
of $\vec{\phi}^*$ parallel to $\vec{\alpha}$ from those that are orthogonal. If we denote the latter by $\vec{\varphi}^*$, i.e. if we define

$$\vec{\varphi}^* = \vec{\phi}^*_\perp,$$

(2.68)

it is straightforward to write down the expression of $T$ and $R$ in terms of $\vec{\varphi}^*$, $b^*$, and $c^*$:

$$T = T_{\vec{\varphi}^*} + T_{(b^*,c^*)} = -\frac{1}{2}(\partial \vec{\varphi}^*)^2 + \vec{a}^* \cdot \partial^2 \vec{\varphi}^* - 2b^* \partial c^* + c^* \partial b^*$$

$$R = R_{\vec{\varphi}^*} + R_{(b^*,c^*)} = \vec{r}^* \cdot \partial \vec{\varphi}^* - b^* c^*,$$

(2.69)

where the vectors $\vec{a}^*$ and $\vec{r}^*$ are the same that appear in eqs. (2.49) and (2.51), and therefore they satisfy the constraints of eq. (2.53). In the same way one can readily prove that $Q^*$ and $G^*$ as given in eq. (2.62) can be written as

$$Q^* = c^* [T_{\vec{\varphi}^*} + \frac{1}{2}T_{(b^*,c^*)}] - \partial [c^* ( R_{\vec{\varphi}^*} + \frac{1}{2}R_{(b^*,c^*)})] + \frac{d}{2} \partial^2 c^*$$

$$G^* = b^*.$$

(2.70)

The comparison of eqs. (2.69) and (2.70) with eqs. (2.49), (2.51) and (2.42) shows that, as expected, the change of variables $\vec{\varphi} \rightarrow \vec{\varphi}^*$, $(b,c) \rightarrow (b^*,c^*)$ is equivalent to the mirror redefinition of $T$, $R$, $Q$ and $G$. This fact will be exploited in the next section, where we study the implications of the topological symmetry in critical and non-critical bosonic string theory.
3. The topological symmetry of the bosonic string theory

In this section we analyse the string theory aspects of the realization of the TA found in the previous section. In order to be coherent with the notations used so far, all quantities related to the string theory realization of the TA will be labelled by an asterisk, whereas those variables and vectors obtained after a mirror redefinition of the string fields will not carry such a label. First of all, let us suppose that we have a critical string theory. This means that the number \( n \) of components of the field \( \vec{\phi}^* \) is 26 and that the background charge of this field is zero (i.e. that \( \vec{a}^* = 0 \)). As in this case \( \vec{a}^* \cdot \vec{r}^* \) obviously vanishes, from the constraints generated by the TA (eq.(2.53)) one gets that the value of the dimension \( d \) is

\[
d (\text{critical string}) = 3. \tag{3.1}
\]

Notice that \( d = 3 \) is obtained no matter the choice of \( \vec{r}^* \). Of course, the value of \( (\vec{r}^*)^2 \) is fixed by eq. (2.53). Taking eq. (3.1) into account, we see that \( (\vec{r}^*)^2 = -2 \), which in particular means that \( \vec{r}^* \) cannot be zero.

Suppose now that our string is a non-critical string, i.e. that we identify one of the \( n \) components of \( \vec{\varphi}^* \) with the Liouville field. This identification can be achieved if we introduce an unitary vector \( \vec{u}_L \) such that its direction corresponds to the Liouville degree of freedom, whereas the components of \( \vec{\varphi}^* \) in the orthogonal complement of \( \vec{u}_L \) in the \( n \)-dimensional space represent the matter fields. Accordingly, we decompose the vectors \( \vec{a}^* \), \( \vec{r}^* \) and the field \( \vec{\varphi}^* \) as

\[
\vec{a}^* = \vec{a}_m^* + \vec{a}_L^* \quad \vec{r}^* = \vec{r}_m^* + \vec{r}_L^* \quad \vec{\varphi}^* = \vec{\varphi}_m^* + \vec{\varphi}_L^* \quad \tag{3.2}
\]

Expressed in these new variables, the energy-momentum tensor \( T \) can be written as

\[
T = -\frac{1}{2} (\partial \vec{\varphi}_m^*)^2 + \vec{a}_m^* \cdot \partial^2 \vec{\varphi}_m^* - \frac{1}{2} (\partial \vec{\varphi}_L^*)^2 + a_L^* \partial^2 \vec{\varphi}_L^* - 2b^* \partial c^* + c^* \partial b^*, \quad \tag{3.3}
\]

where we have taken into account that \( \vec{a}_L^* \) and \( \vec{\varphi}_L^* \) live in a one-dimensional vector.
subspace. The central charge of the matter can be obtained as a function of $\vec{a}_m^*$. Indeed from eq. (3.3), one has

$$c_m^* = n - 1 + 12(\vec{a}_m^*)^2.$$  \hspace{1cm} \text{(3.4)}

Of course the Liouville field $\vec{\varphi}_L^*$ has central charge $c_L^* = 26 - c_m^*$. As we have mentioned in the previous section, when one analyzes the topological symmetry of the non-critical string, it is convenient to introduce the field variables obtained after a mirror redefinition. If we decompose the mirror field $\vec{\varphi}$ into its components along the matter and Liouville directions (denoted by $\vec{\varphi}_m$ and $\vec{\varphi}_L$ respectively), we can write the energy-momentum tensor $T$ of the string given in eq. (3.3) as:

$$T = -\frac{1}{2}(\partial \vec{\varphi}_m)^2 + \vec{a}_m \cdot \partial^2 \vec{\varphi}_m - \frac{1}{2}(\partial \varphi_L)^2 + a_L \partial^2 \varphi_L - b \partial c,$$ \hspace{1cm} \text{(3.5)}

where $(b, c)$ are $(1, 0)$ ghosts obtained from the string fields as in eq. (2.67) (see below). In complete analogy with eq. (3.4), the central charge of the mirror matter fields is given by:

$$c_m = n - 1 + 12(\vec{a}_m)^2.$$ \hspace{1cm} \text{(3.6)}

The precise relation between the string variables and their mirror counterparts depends on the vector $\vec{r}^*$, which parametrizes the different topological symmetries of the string theory. Recall (see eq. (2.69)) that the components of $\vec{r}^*$ determine the topological $U(1)$-charges of the field $\vec{\varphi}$, i.e. the topological quantum numbers of $\vec{\varphi}_m^*$ and $\vec{\varphi}_L^*$. In fact, the projection of $\vec{r}^*$ into the matter hyperplane (denoted by $\vec{r}_m^*$ in eq. (3.2)) enters in the relation between $c_m^*$ and $c_m$. Taking into account that $\vec{a}_m = \vec{a}_m^* - \vec{r}_m^*$ in eq. (3.6), one immediately gets

$$c_m = c_m^* + 12(\vec{r}_m^*)^2 - 24\vec{a}_m^* \cdot \vec{r}_m^*.$$ \hspace{1cm} \text{(3.7)}

Let us now extract some information from the constraints (2.53) generated by the TA. Splitting $\vec{a}^*$ and $\vec{r}^*$ into their matter and Liouville parts, the last two
equations in (2.53) can be written as:

\[
(r_m^*)^2 + (r_L^*)^2 = 1 - d \quad \bar{a}_m^* \cdot \bar{r}_m^* + a_L^* r_L^* = \frac{3 - d}{2}.
\] (3.8)

Eliminating \( r_L^* \) in these equations, we get:

\[
\bar{a}_m^* \cdot \bar{r}_m^* + a_L^* \sqrt{1 - d - (r_m^*)^2} = \frac{3 - d}{2},
\] (3.9)

which can be converted in a quadratic equation in the variable \( d \). By solving this equation, we obtain after a short calculation

\[
d = \frac{c_m + c_m^* - 7 - 6 (r_m^*)^2 \pm \sqrt{(1 - c_m)(25 - c_m^*)}}{6}.
\] (3.10)

Eq. (3.10) gives the central extension \( d \) of the TA as a function of the matter central charges \( c_m \) and \( c_m^* \) and of the vector \( \bar{r}_m^* \) of the topological \( U(1) \) charges of the matter fields. Notice the remarkable fact that the term under the square root in eq. (3.10) depends on the differences \( 1 - c_m \) and \( 25 - c_m^* \). This means that the implementation of the topological symmetry in these matter-plus-gravity models is determined by the values of the matter central charge \( c_m^* \) and of its mirror counterpart. As a matter of fact, one can regard eq. (3.10) as a “barrier” formula. Indeed, if we choose \((r_m^*)^2\) to be real, the domains of values of \( c_m \) and \( c_m^* \) for which \( d \) is also real are:

\[
c_m \leq 1 \quad c_m^* \leq 25 \quad \text{and} \quad c_m \geq 1 \quad c_m^* \geq 25.
\] (3.11)

Notice that the central charges \( c_m \) and \( c_m^* \) appearing in (3.11) are related by (3.7). Actually if the matter degrees of freedom are neutral under the topological symmetry (i.e. if \( \bar{r}_m^* = 0 \)), \( c_m \) and \( c_m^* \) are equal and (3.10) reduces to [12]

\[
d = \frac{c_m^* - 7 \pm \sqrt{(1 - c_m^*)(25 - c_m^*)}}{6}.
\] (3.12)

Requiring in this case \( d \) to be real, we obtain the standard range for the matter central charge of the bosonic non-critical strings, namely \( \{c_m^* \leq 1\} \cup \{c_m^* \geq 25\} \).
Notice that, taking the minus sign in eq. (3.12), \( d = 0 \) corresponds to the trivial matter system of ref. [15] in which \( c^*_m = -2 \).

Let us now study the form of the mirror change of variables for the non-critical string theory. First of all we must bosonize the ghost fields \((b, c)\) and \((b^*, c^*)\). For that purpose we introduce two scalar fields \(\chi\) and \(\chi^*\), whose relation with the ghost fields is given by the standard bosonization formulas, i.e. by:

\[
\begin{align*}
    b &= e^{\chi} \\
    c &= e^{-\chi} \\
    b^* &= e^{\chi^*} \\
    c^* &= e^{-\chi^*}.
\end{align*}
\]

The fields \(\chi\) and \(\chi^*\) have the OPE’s

\[
\chi(z) \chi(w) = \chi^*(z) \chi^*(w) = \log(z - w).
\]

The contributions of the fields \(\chi\) and \(\chi^*\) to the energy-momentum tensor can be computed by using standard methods from the expressions of \(T_{(b,c)}\) and \(T_{(b^*,c^*)}\). The result is:

\[
\begin{align*}
    T_{(b,c)} &= \frac{1}{2} (\partial \chi)^2 - \frac{1}{2} \partial^2 \chi \\
    T_{(b^*,c^*)} &= \frac{1}{2} (\partial \chi^*)^2 - \frac{3}{2} \partial^2 \chi^*.
\end{align*}
\]

The change of variables corresponding to the mirror transformation was given in eq. (2.56) for the fields \(\vec{\phi}\) and \(\vec{\phi}^*\). The decomposition of these fields along the ghost, Liouville and matter directions is:

\[
\vec{\phi} = \vec{\varphi}_m + \vec{\varphi}_L - \chi \vec{\alpha} \quad \quad \vec{\phi}^* = \vec{\varphi}^*_m + \vec{\varphi}^*_L - \chi^* \vec{\alpha},
\]

where we have taken into account that \(\chi\) (\(\chi^*\)) parametrizes the component of \(\vec{\phi}\) (\(\vec{\phi}^*\)) parallel to \(\vec{\alpha}\) (see eqs. (2.33) and (2.67)). Substituting eq. (3.16) in eq. (2.56) and using the fact that \(\vec{R} \cdot \vec{\alpha} = -1\) (see eq. (2.30)) one immediately obtains the
relation of $\varphi^*_m$, $\varphi^*_L$ and $\chi^*$ with their mirror partners:

$$\begin{align*}
\varphi^*_m &= (1 + \frac{2}{d} \tilde{r}^*_m \otimes \tilde{r}^*_m) \varphi_m + \frac{2 r^*_L \tilde{r}^*_m}{d} \varphi_L + \frac{2}{d} \tilde{r}^*_m \chi \\
\varphi^*_L &= \frac{2 r^*_L}{d} \tilde{r}^*_m \cdot \varphi_m + (1 + \frac{2 (r^*_L)^2}{d}) \varphi_L + \frac{2 r^*_L}{d} \chi \\
\chi^* &= - \frac{2}{d} \tilde{r}^*_m \cdot \varphi_m - \frac{2}{d} r^*_L \varphi_L + (1 - \frac{2}{d}) \chi.
\end{align*}$$

(3.17)

By a direct calculation using eq. (3.17) one can check that the energy-momentum tensor when expressed in terms of the mirror fields takes the form displayed in eq. (3.5).

Let us now restrict ourselves for the remainder of this section to the case $n = 2$, i.e. when there is only one matter field. The minimal models coupled to two-dimensional gravity and the $c = 1$ string are particular cases of these $n = 2$ theories. The constraints (2.53) now take the form:

$$\begin{align*}
(a^*_m)^2 + (a^*_L)^2 &= 2 \\
(a^*_m r^*_m + a^*_L r^*_L) &= \frac{3 - d}{2} \\
(r^*_m)^2 + (r^*_L)^2 &= 1 - d,
\end{align*}$$

(3.18)

where the background charges $a^*_m$ and $a^*_L$ are determined by the matter central charge $c^*_m$:

$$a^*_m = i \sqrt{\frac{1 - c^*_m}{12}} \quad a^*_L = \sqrt{\frac{25 - c^*_m}{12}}.$$  

(3.19)

It is easy in this case to find the general solution of the constraints dictated by the TA. First of all, let us parametrize the matter and Liouville background charges as:

$$a^*_m = \frac{i \lambda}{2} - \frac{i}{\lambda} \quad a^*_L = \frac{\lambda}{2} + \frac{1}{\lambda},$$

(3.20)

where $\lambda$ is a function of $c^*_m$ that can be obtained by comparing eq. (3.20) with eq. (3.21).
Indeed after a short calculation one arrives at the result

\[ \lambda = \sqrt{\frac{25 - c_m^*}{12}} \pm \sqrt{\frac{1 - c_m^*}{12}}. \]  

(3.21)

The quantity \( \lambda \) is well-known in non-critical string theory. Indeed, the values of \( \lambda \) in (3.21) are such that the vertex operator \( e^{\lambda \varphi L} \) has conformal weight one and thus it can be considered as the gravitational dressing of the unit operator. Usually in the Liouville theory the minus sign in (3.21) is chosen in order to make contact with the classical limit. Here we will not specify any particular choice for this sign. Using the representation (3.20) it is straightforward to find the values of \( r_m^* \) and \( r_L^* \) that satisfy eq. (3.18)

\[
\begin{align*}
    r_m^* &= \frac{i}{4} (1 - d) \lambda - \frac{i}{\lambda} \\
    r_L^* &= \frac{1 - d}{4} \lambda + \frac{1}{\lambda},
\end{align*}
\]

(3.22)

As the background charges \( a_m \) and \( a_L \) of the mirror fields are given by \( a_m = a_m^* - r_m^* \) and \( a_m = a_m^* - r_m^* \), we find

\[
\begin{align*}
    a_m &= \frac{i}{4} (1 + d) \lambda \\
    a_L &= \frac{1}{4} (1 + d) \lambda,
\end{align*}
\]

(3.23)

from which the central charge \( c_m \) of the mirror fields in the matter direction can be easily computed. Indeed, as in this case \( c_m = 1 + 12(a_m)^2 \), it follows that

\[
c_m = 1 - \frac{(d + 1)^2}{8} \left[ 13 - c_m^* \pm \sqrt{(1 - c_m^*)(25 - c_m^*)} \right].
\]

(3.24)

The double sign in eq. (3.24) corresponds to the two possible choices in (3.21). Notice that for \( d = -1 \) one has \( c_m = 1 \) independently of the value of \( c_m^* \). In this case \( a_m = a_L = 0 \) as it can be verified by inspecting eq. (3.23). For \( d \neq -1 \) one can invert eq. (3.24) and obtain the expression of \( c_m^* \) in terms of \( c_m \). One gets:

\[
c_m^* = 13 - 4 \frac{1 - c_m}{(d + 1)^2} - 9 \frac{(d + 1)^2}{1 - c_m}.
\]

(3.25)

The topological symmetry of the string theory we have been analyzing depends on the parameter \( d \). For each value of \( d \) we have a BRST current \( Q^*(z) \). Although
$Q^*(z)$’s for different $d$’s are different, the corresponding integrated BRST charges are equal due to the fact that $d$ only enters in the total derivative terms of $Q^*(z)$. Thus $d$ parametrizes the freedom one has to choose an improving term in the BRST current of string theory compatible with the TA. As we have seen, associated to each BRST current, we have a set of fields that naturally implement the mirror version of the string topological symmetry. It could be that, when the string theory is formulated in terms of the mirror variables, new symmetries that are hidden or realized in a non-local way in the original fields, become apparent. Let us see some examples of how this can occur.

First of all, let us ask ourselves if there is any particular value of $d$ for which the spin one ghosts $(b, c)$ can be written locally in terms of the reparametrization ghosts $(b^*, c^*)$. A glance at eq. (3.17) reveals that this only occurs for $d = 1$. Indeed using eqs. (3.13) and (3.17) we obtain in this case:

$$b = e^{2\vec{r}^* \cdot \vec{\phi}^*} c^* \quad \quad c = e^{-2\vec{r}^* \cdot \vec{\phi}^*} b^*,$$

(3.26)

Spin one ghosts are characteristic of topological sigma models [2]. Thus it is natural to think that in the mirror variables one could formulate string theory as a topological sigma model. Moreover $d = 1$ is the natural value for the c-number anomaly of the BRST symmetry of a topological sigma model whose target space has complex dimension one. However, a difficulty arises when one tries to identify the $(b, c)$ fields with the ghosts of the topological sigma model. The problem is that, with respect to the $R$-current of the string, the fields $b$ and $c$ have $R$-charge $+1$ and $-1$ respectively. These values of the $R$-charge are opposite to the ones expected for the ghosts of a topological sigma model, where the antighost must have conformal weight one. In order to overcome this difficulty one must redefine $b$ and $c$. A simple exchange does not work since $b$ and $c$ have different conformal dimensions. Actually, one can redefine $b$ and $c$ in such a way that their conformal weights are interchanged and their $R$-charges are not altered. The key observation to find out the form of this redefinition is the fact that $(\vec{r}^*)^2 = 0$ for $d = 1$ (see
eq. (3.18)). This means that we can use the exponentials of $\vec{r}^* \cdot \vec{\phi}^*$ to dress the fields $(b, c)$ in such a way that their $U(1)$ charges are unaffected. Let us define new ghost fields $B$ and $C$ as:

$$B = e^{\vec{r}^* \cdot \vec{\phi}^*} b \quad C = e^{-\vec{r}^* \cdot \vec{\phi}^*} c. \quad (3.27)$$

The conformal weights of $B$ and $C$ are $\Delta(B) = 1$ and $\Delta(C) = 0$. In terms of the diffeomorphism ghosts $(b^*, c^*)$ $B$ and $C$ are given by:

$$B = e^{-\vec{r}^* \cdot \vec{\phi}^*} b^* \quad C = e^{\vec{r}^* \cdot \vec{\phi}^*} c^*. \quad (3.28)$$

Using the value of $\vec{r}^*$ for $d = 1$ (see eq. (3.22)), $B$ and $C$ can be written as a function of the matter and Liouville fields of string theory:

$$B = e^{i \frac{\lambda}{2} (\phi^*_m + i \phi^*_L)} b^* \quad C = e^{-i \frac{\lambda}{2} (\phi^*_m + i \phi^*_L)} c^*. \quad (3.29)$$

Notice that for a minimal $(p,q)$ model with $c^*_m = 1 - 6 \frac{(p-q)^2}{pq}$ and $q \geq p$, $\lambda$ is equal to $\sqrt{\frac{2p}{q}}$ (we are taking the minus sign in eq. (3.21)).

In order to verify that we are mapping string theory to a topological sigma model we need to introduce new fields that could represent coordinates in a target manifold. With this purpose in mind, let us define two new fields $x$ and $\bar{p}$ as follows:

$$x = [b^* c^* - i \frac{\lambda}{2} (\partial \phi^*_m - i \partial \phi^*_L)] e^{i \frac{\lambda}{2} (\phi^*_m + i \phi^*_L)}$$

$$\bar{p} = e^{-i \frac{\lambda}{2} (\phi^*_m + i \phi^*_L)}. \quad (3.30)$$

The energy-momentum tensor $T$ and the $U(1)$ current $R$ of eq. (2.69) have the following expressions in terms of $x$ and $\bar{p}$:

$$T = - \partial x \bar{p} - B \partial C$$
$$R = - BC. \quad (3.31)$$

After identifying $\bar{p} \equiv \partial \bar{x}$, where $\bar{x}$ is the holomorphic component of the complex conjugate of the coordinate $x$, the operator $T$ in eq. (3.31) represents the energy-
momentum tensor of a topological sigma model in one complex dimension \[2\]. The
fields \(B, C, x\) and \(\bar{p}\) satisfy the OPE’s:

\[
B(z)C(w) = x(z)\bar{p}(w) = \frac{1}{z-w}.
\] (3.32)

On the other hand, the operators \(Q^*\) and \(G^*\) of eq. (2.70) for \(d = 1\) can be written
in the remarkably simple form:

\[
Q^* = -C \partial x \quad G^* = B\bar{p}.
\] (3.33)

In the form of eq. (3.33) \(Q^*\) is the standard topological current for a topological
sigma model. We have thus succeeded in finding a set of fields in terms of which
the BRST symmetry of non-critical string theories is equivalent to the topological
symmetry of a \(d = 1\) topological sigma model. For \(c_m^* = 1\) the above mapping
was obtained in ref. [16]. This analysis was extended to \(c_m^* < 1\) string theories in
ref. [17]. In these papers it is shown how the BRST cohomology of the non-critical
string theory can be recovered from the physical states of the topological sigma
model. Notice that the field \(x\) is nothing but one of the ground ring generators of
\(c_m^* \leq 1\) string theory [18, 19].

As a second example of a non-trivial mapping generated by the mirror transfor-
mation, let us suppose next that we choose in eqs. (3.22) and (3.23) the particular
value of \(d\) given by

\[
d = -1 - \frac{2}{\lambda}.
\] (3.34)

For this value of \(d\), the background charges of the mirror fields are:

\[
a_m = -\frac{i}{2} \quad a_L = -\frac{1}{2},
\] (3.35)

which imply that our mirror system has central charge \(c_m = -2\) in the matter
sector. It turns out that when eq. (3.35) holds, there exist three dimension-one
operators that close a $SL(2)$ current algebra. In terms of the mirror fields these three currents are:

\[ H = i\partial \varphi_m + \partial \varphi_L \]
\[ J_+ = \frac{i\partial \chi}{\sqrt{2}} e^{-\chi - \varphi_L} \]
\[ J_- = \left[ \sqrt{2}(\partial \varphi_m - i\partial \varphi_L) - \frac{i\partial \chi}{\sqrt{2}} \right] e^{\chi + \varphi_L} \tag{3.36} \]

Using the expression of $T$ given in eqs. (3.5) and (3.15), one can easily check that $H$ and $J_\pm$ are primary operators with conformal dimension one. Moreover they close a $SL(2)$ current algebra without central extension:

\[ H(z) J_\pm(w) = \pm \frac{J_\pm(w)}{z - w} \]
\[ J_+(z) J_-(w) = \frac{H(w)}{z - w} \]
\[ H(z) H(w) = J_+(z) J_+(w) = J_-(z) J_-(w) = 0 \tag{3.37} \]

Actually eq. (3.36) is nothing but the standard free field realization of a zero level $SL(2)$ current algebra [20].

The energy-momentum tensor of the string theory can be written as a Sugawara bilinear in $H$ and $J_\pm$:

\[ T = \frac{1}{2} \left[ J_+ J_- + J_- J_+ + H^2 + \right] \tag{3.38} \]

where the global coefficient $\frac{1}{2}$ is the value corresponding to a $SL(2)$ current algebra of vanishing level. Although eqs. (3.37) and (3.38) are easy to check when the mirror fields are used, they are not so evident when the currents are given in terms of the original fields of the string theory. As a matter of fact, the expressions of $H$ and $J_\pm$ in terms of $\varphi_m^*, \varphi_L^*$ and $\chi^*$ can be worked out if one uses the values of $d,$
The result is:

\[
H = \frac{1}{\lambda(\lambda + 2)} \left[ i(4 - \lambda^2) \partial \varphi^*_m - (4 + \lambda^2) \partial \varphi^*_L + 4\lambda \partial \chi^* \right]
\]

\[
J_{\pm} = J_{\pm} e^{\pm\Lambda},
\]

where \(J_{\pm}\) and \(\Lambda\) are given by:

\[
J_{\pm} = (\lambda - 1) \partial \varphi^*_m + i(1 - \lambda - \frac{4}{\lambda + 2}) \partial \varphi^*_L + i(1 + \frac{2\lambda}{\lambda + 2}) \partial \chi^*
\]

\[
J_{-} = (\lambda - 1) \partial \varphi^*_m + i(1 + \lambda + \frac{4}{\lambda} - \frac{4}{\lambda + 2}) \partial \varphi^*_L - i(3 + \frac{4}{\lambda + 2}) \partial \chi^*
\]

\[
\Lambda = \frac{\lambda + 2}{2\lambda} [i(\lambda^2 - 1) \varphi^*_m + (\lambda^2 + 1)\varphi^*_L - 2\lambda \chi^*].
\]

From eqs. (3.39) and (3.40) one can easily check that, for any value of \(\lambda\), the currents \(J_{\pm}\) and \(H\) satisfy the centerless algebra of eq. (3.37). Moreover, using eqs. (3.39) and (3.40), it is straightforward to prove that, indeed, the Sugawara expression for \(T\) given in eq. (3.38) reduces to the standard form of the energy-momentum tensor of the non-critical strings, where the background charges of the matter and Liouville fields are given in eq. (3.20). Notice that \(H\) and \(J_{\pm}\), as displayed in eqs. (3.39) and (3.40), are in general non-local in the string ghost fields \((b^*, c^*)\). They are however local in the spin one ghosts \((b, c)\). In fact the ghost \(b\) (i.e. the mirror BRST charge \(Q\), see eq. (2.39)) is one of the screening operators of the \(SL(2)\) current algebra of eq. (3.37). It would be interesting to investigate the consequences of the \(SL(2)\) symmetry of the non-critical bosonic string that we have just uncovered. We will not attempt this here. Instead we shall generalize in the next section our results to the case of superconformal matter.
4. Topological superconformal matter and superstrings

The purpose of this section is to extend the results obtained so far to conformal topological matter systems that possess a superconformal symmetry. We shall realize this supersymmetry in terms of a multicomponent scalar field $\vec{\phi}$, as in the previous section, and an $N$-component Majorana fermion $\vec{\Psi}(z) = (\Psi_1(z), \ldots, \Psi_N(z))$. The OPE’s among the components of $\vec{\Psi}$ are given by:

$$\Psi_i(z) \Psi_j(w) = -\frac{\delta_{ij}}{z - w}. \quad (4.1)$$

The holomorphic component of the energy-momentum tensor $T$ is given by:

$$T = -\frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} + \vec{A} \cdot \partial^2 \vec{\phi} + \frac{1}{2} \vec{\Psi} \cdot \partial \vec{\Psi}. \quad (4.2)$$

The vanishing of the conformal anomaly $c$ of the operator $T$ in eq. (4.2) yields the following condition for the modulus of the background charge $\vec{A}$:

$$\vec{A}^2 = -\frac{N}{8}. \quad (4.3)$$

The superconformal algebra in this model is generated by a fermionic dimension-$\frac{3}{2}$ operator that we shall denote by $T_F$. The primary character of $T_F$ fixes the OPE of $T$ with $T_F$:

$$T(z) \ T_F(w) = \frac{3}{2} \frac{T_F(w)}{(z - w)^2} + \frac{\partial T_F(w)}{z - w}. \quad (4.4)$$

Moreover, the OPE of the supersymmetry generator $T_F$ with itself contains a single pole singularity whose residue is proportional to the energy-momentum tensor $T$:

$$T_F(z) \ T_F(w) = \frac{1}{2} \frac{T(w)}{z - w}. \quad (4.5)$$

Notice that, due to the vanishing of the central charge, there are no higher order poles in the right-hand side of eq.(4.5). In order to represent $T_F$ in terms of the
fields $\vec{\phi}$ and $\vec{\Psi}$, we shall adopt the following ansatz:

$$T_F = \frac{1}{2} \partial \vec{\phi} \cdot L \cdot \vec{\Psi} + \vec{M} \cdot \partial \vec{\Psi} + K_{lmn} \Psi_l \Psi_m \Psi_n,$$  \hfill (4.6)

where $\vec{M}$, $L$ and $K$ are numerical tensors of ranks one, two and three respectively. In (4.6) we have included all possible terms compatible with the dimension and statistics of $T_F$. By requiring the fulfillment of eqs. (4.4) and (4.5) one arrives at the following conditions on $\vec{M}$, $L$ and $K$:

$$L L^T = L^T L = 1$$

$$\vec{M} = - L \cdot \vec{A}$$

$$K_{lmn} = 0.$$  \hfill (4.7)

Substituting the values given by eq. (4.7) in eq. (4.6), one gets $T_F$ in terms of the orthogonal matrix $L$:

$$T_F = \frac{1}{2} \partial \vec{\phi} \cdot L \cdot \vec{\Psi} - \vec{A} \cdot L \cdot \partial \vec{\Psi}.$$  \hfill (4.8)

Actually we can absorb the unknown matrix $L$ by redefining the fermionic field $\vec{\Psi}$ as $\vec{\Psi} \rightarrow L \vec{\Psi}$. After this redefinition $T_F$ takes the form:

$$T_F = \frac{1}{2} \partial \vec{\phi} \cdot \vec{\Psi} - \vec{A} \cdot \partial \vec{\Psi}.$$  \hfill (4.9)

The presence of a superconformal symmetry in the chiral algebra of our system suggests that all fields of the model can be arranged in supersymmetry doublets. In general one of those doublets $(X, Y)$ is composed by two primary fields $X$ and $Y$ whose conformal weights $\Delta_X$ and $\Delta_Y$ differ by $\frac{1}{2}$ ($\Delta_Y = \Delta_X + \frac{1}{2}$). The action of $T_F$ on $X$ and $Y$ is given by:

$$T_F(z) X(w) = \frac{1}{2} \frac{Y(w)}{z-w}$$

$$T_F(z) Y(w) = \frac{\Delta_X X(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial X(w)}{z-w}.$$  \hfill (4.10)

Let us now consider the algebra closed by $T_F$ and the generators of the topo-
logical symmetry. In the type of models we are dealing with, it is natural to think that the operators $T$, $G$, $Q$ and $R$ will have a supersymmetric partner. As can be checked by comparing eqs. (4.5) and (4.4) with the general equation (4.10), $T_F$ is the partner of $T$. Let us denote by $Q_B$, $G_B$ and $R_F$ the partners of $Q$, $G$ and $R$ respectively. With these fields we form four supersymmetry doublets $(T_F, T)$, $(Q_B, Q)$, $(G_B, G)$ and $(R_F, R)$. The conformal weights of the lower components of these doublets are:

$$\Delta(T_F) = \frac{3}{2} \quad \Delta(G_B) = \frac{3}{2} \quad \Delta(Q_B) = \frac{1}{2} \quad \Delta(R_F) = \frac{1}{2}.$$  \hspace{1cm} (4.11)

An algebra with the field content described above can be obtained by twisting the $N = 3$ superconformal algebra [7]. Let us describe this algebra with our notations. First of all, $T$, $G$, $Q$ and $R$ close the same algebra as in the non-supersymmetric case. On the other hand, the new fields $T_F$, $G_B$, $Q_B$ and $R_F$ are primary with respect to $T$ with the conformal weights displayed in eq. (4.11). The action of $T_F$ on the generators is determined by the arrangement of these fields in supersymmetry doublets described above (see eq. (4.10)). There is however an exception. The action of $T_F$ on $(R_F, R)$ is anomalous. Actually one has:

$$T_F(z) \ R_F(w) = - \frac{\partial}{2} \left( \frac{R_F(w)}{(z - w)^2} + \frac{1}{2} \frac{R(w)}{z - w} \right)$$

$$T_F(z) \ R(w) = \frac{1}{2} \left( \frac{R_F(w)}{(z - w)^2} + \frac{1}{2} \frac{\partial R_F(w)}{z - w} \right).$$  \hspace{1cm} (4.12)

The anomaly term in eq. (4.12) is related to the anomalous behaviour of $R$ under the action of $T$. Two of the multiplets ($(T_F, T)$ and $(Q_B, Q)$) are connected to the other two ($(G_B, G)$ and $(R_F, R)$) by the action of the BRST charge $Q$: the two latter doublets are the BRST ancestors of the former ones. In fact, the supersymmetry generator $T_F$ is $Q$-exact. Its $Q$-ancestor is $G_B$ and when one tries to relate $G_B$
and $T_F$ by the action of $Q$, one gets two new fields ($R_F$ and $Q_B$):

\[
Q(z) \ G_B(w) = -\frac{1}{2} \frac{R_F(w)}{(z-w)^2} - \frac{T_F(w)}{z-w}.
\]  

(4.13)

\[
Q(z) \ T_F(w) = -\frac{1}{2} \frac{Q_B(w)}{(z-w)^2}.
\]

Notice that $R_F$ and $R$ play the same role with respect to $T_F$ and $T$ respectively (compare eq. (4.13) with eq. (2.9)). These two new fields $R_F$ and $Q_B$ are connected by $Q$:

\[
Q(z) \ R_F(w) = \frac{Q_B(w)}{z-w}.
\]  

(4.14)

\[
Q(z) \ Q_B(w) = 0.
\]

The non-vanishing singularities of the products of $Q_B$, $G_B$ and $R_F$ with themselves are given by:

\[
Q_B(z) \ G_B(w) = -\frac{d}{2} \frac{G_B(w)}{(z-w)^2} - \frac{1}{2} \frac{R(w)}{z-w}.
\]  

(4.15)

\[
R_F(z) \ R_F(w) = \frac{d}{z-w}.
\]

Finally, in order to complete the non-zero OPE’s of the algebra, let us display those of $G$ with the new generators:

\[
G(z) \ T_F(w) = -\frac{3}{2} \frac{G_B(w)}{(z-w)^2} - \frac{\partial G_B(w)}{z-w}.
\]

\[
G(z) \ Q_B(w) = \frac{1}{2} \frac{R_F(w)}{(z-w)^2} - \frac{T_F(w) - \partial R_F(w)}{z-w}.
\]  

(4.16)

\[
G(z) \ G_B(w) = 0
\]

\[
G(z) \ R_F(w) = -\frac{G_B(w)}{z-w}.
\]
and those of $R$ with $Q_B, G_B$ and $R_F$:

$$R(z) Q_B(w) = \frac{Q_B(w)}{z-w}$$

$$R(z) G_B(w) = -\frac{G_B(w)}{z-w}$$

$$R(z) R_F(w) = 0.$$ (4.17)

From eqs. (4.15) and (4.17) one notices that the operators $Q_B, G_B$ and $R$ close an $SL(2)$ current algebra with level $k = 2d$, which is a remnant of the $SL(2)$ Kac-Moody subalgebra present in the $N = 3$ superconformal algebra (before performing the topological twist $Q_B, G_B$ and $R$ have conformal dimension one). In what follows the algebra just described will be called the supersymmetric topological algebra (STA). Notice that the STA is nothing but the BRST algebra of the $N = 1$ superconformal symmetry.

As happened in the non-supersymmetric case, the STA enjoys a mirror symmetry whose origin comes from the two possible topological twists of the $N = 3$ superconformal algebra. Under this symmetry, from a given realization of the STA, we generate a new one by changing $T, Q, G$ and $R$ into $T^*, Q^*, G^*$ and $R^*$ as in eq. (2.11). The other four generators of the STA must be changed as follows:

$$T_F \rightarrow T_F^* = T_F - \partial R_F$$

$$Q_B \rightarrow Q_B^* = G_B$$

$$G_B \rightarrow G_B^* = Q_B$$

$$R_F \rightarrow R_F^* = -R_F.$$ (4.18)

This mirror symmetry will play an important role in our analysis, specially in connection with superstring theories.

We are now going to consider the realization of the STA in our supersymmetric field theory. We could proceed as in section 2 and study the possible vertex operator
representations for the fields appearing in the STA. Instead of doing that, we shall follow a shorter path, which takes into account the lessons learnt in the analysis of section 2. As a direct generalization of eq. (2.39), we represent the doublet $(Q_B, Q)$ by a doublet of fields $(\beta, b)$:

$$Q_B = \beta \quad Q = b,$$

(4.19)

where $\beta$ ($b$) is a commuting (anticommuting) field. From eq. (4.11) we see that the conformal weights of $\beta$ and $b$ must be $\Delta(\beta) = \frac{1}{2}$ and $\Delta(b) = 1$ respectively. At this point the question arises of how one can represent $\beta$ and $b$ in terms of the fields $\vec{\phi}$ and $\bar{\Psi}$. Following the spirit of the manifest supersymmetric bosonization of refs. [21, 22], let us write:

$$\beta = e^{\vec{\mu} \cdot \vec{\phi}},$$

(4.20)

$$b = - \vec{\mu} \cdot \bar{\Psi} e^{\vec{\mu} \cdot \vec{\phi}},$$

where $\vec{\mu}$ is a numerical vector. It can be easily checked that, indeed, the fields $(\beta, b)$ in eq. (4.20) form a supersymmetry doublet. Together with $(\beta, b)$ we must introduce a conjugate doublet $(c, \gamma)$, where $c$ ($\gamma$) is an anticommuting (commuting) field of dimension 0 ($\frac{1}{2}$) which is conjugate to $b(\beta)$. From now on we will refer to $b, c, \beta$ and $\gamma$ as the ghost fields. These fields can be represented as:

$$c = - \vec{\lambda} \cdot \bar{\Psi} e^{-\vec{\mu} \cdot \vec{\phi}},$$

(4.21)

$$\gamma = [\vec{\lambda} \cdot \partial \vec{\phi} - (\vec{\mu} \cdot \bar{\Psi}) (\vec{\lambda} \cdot \bar{\Psi})] e^{-\vec{\mu} \cdot \vec{\phi}},$$

where $\vec{\lambda}$ is a new numerical vector. In fact if we require that $(c, \gamma)$ satisfy the first equation in (4.10), we obtain the following condition that must be satisfied by $\vec{\lambda}$:

$$\vec{A} \cdot \vec{\lambda} = \frac{1}{2},$$

(4.22)

which can be obtained by imposing the absence of the double pole singularity in
the product $T_F(z)c(w)$. The fact that the doublets $(\beta, b)$ and $(c, \gamma)$ are conjugate is reflected in the fact that the non-vanishing OPE’s among these fields are:

$$b(z)c(w) = \frac{1}{z-w}, \quad \beta(z)\gamma(w) = \frac{-1}{z-w}. \quad (4.23)$$

After a simple calculation, one can verify that the two ghost doublets are conjugates if the scalar products of $\vec{\mu}$ and $\vec{\lambda}$ are given by:

$$\vec{\lambda} \cdot \vec{\lambda} = \vec{\mu} \cdot \vec{\mu} = 0, \quad \vec{\lambda} \cdot \vec{\mu} = -1. \quad (4.24)$$

On the other hand, $\beta$ has conformal weight $\frac{1}{2}$ if $\vec{A} \cdot \vec{\mu}$ is given by:

$$\vec{A} \cdot \vec{\mu} = \frac{1}{2}. \quad (4.25)$$

In order to separate the contributions of the ghost fields from the remaining degrees of freedom of our system, it is convenient to introduce two vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$, which are linearly related to $\vec{\mu}$ and $\vec{\lambda}$ as:

$$\vec{\alpha}_1 = \frac{1}{\sqrt{2}} (\vec{\mu} + \vec{\lambda}), \quad \vec{\alpha}_2 = \frac{i}{\sqrt{2}} (\vec{\mu} - \vec{\lambda}), \quad (4.26)$$

where the imaginary unit $i$ has been included in the definition of $\vec{\alpha}_2$ for convenience. These vectors satisfy:

$$\vec{\alpha}_i \cdot \vec{\alpha}_j = -\delta_{ij}. \quad (4.27)$$

It is now straightforward to split any vector $\vec{X}$ in its components parallel and orthogonal to the ghost plane:

$$\vec{X} = \vec{X}_\parallel + \vec{X}_\perp, \quad (4.28)$$

where

$$\vec{X}_\parallel = \frac{(\vec{X} \cdot \vec{\alpha}_1)}{\vec{\alpha}_1^2} \vec{\alpha}_1 + \frac{(\vec{X} \cdot \vec{\alpha}_2)}{\vec{\alpha}_2^2} \vec{\alpha}_2 = -(\vec{X} \cdot \vec{\alpha}_1) \vec{\alpha}_1 - (\vec{X} \cdot \vec{\alpha}_2) \vec{\alpha}_2. \quad (4.29)$$

Using this last equation, together with the values of $\vec{A} \cdot \vec{\lambda}$ and $\vec{A} \cdot \vec{\mu}$ given in eqs. (4.22) and (4.25), it is straightforward to obtain the component of the background
charge $\vec{A}$ parallel to the ghost plane:

$$\vec{A}_|| = -\frac{\vec{a}_1}{\sqrt{2}} = -\frac{1}{2}(\vec{\mu} + \vec{\lambda}).$$  \hfill (4.30)

From this result we can split $T$ and $T_F$ as:

$$T = T_|| + T_\perp \quad T_F = T_F^\parallel + T_F^\perp,$$  \hfill (4.31)

with

$$T_|| = \frac{1}{2}(\vec{a}_1 \cdot \partial \vec{\phi})^2 + \frac{1}{2}(\vec{a}_2 \cdot \partial \vec{\phi})^2 - \frac{1}{\sqrt{2}} \vec{a}_1 \cdot \partial^2 \vec{\phi}$$

$$-\frac{1}{2}(\vec{a}_1 \cdot \vec{\Psi})(\vec{a}_1 \cdot \partial \vec{\Psi}) - \frac{1}{2}(\vec{a}_2 \cdot \vec{\Psi})(\vec{a}_2 \cdot \partial \vec{\Psi})$$

$$T_F^\parallel = -\frac{1}{2}(\vec{a}_1 \cdot \partial \vec{\phi})(\vec{a}_1 \cdot \vec{\Psi}) - \frac{1}{2}(\vec{a}_2 \cdot \partial \vec{\phi})(\vec{a}_2 \cdot \vec{\Psi}) + \frac{1}{\sqrt{2}} \vec{a}_1 \cdot \partial \vec{\Psi}. \hfill (4.32)$$

These parallel components can be expressed locally in terms of the ghost fields. In fact, using eqs. (4.20) and (4.21) it is easy to check that

$$T_|| = -b\partial c - \frac{1}{2}(\beta \partial \gamma - \gamma \partial \beta)$$

$$T_F^\parallel = \frac{1}{2} b\gamma - \frac{1}{2} \beta \partial c,$$  \hfill (4.33)

where normal ordering is understood. Denoting the components of $\vec{\phi}$, $\vec{\Psi}$ and $\vec{A}$ in the orthogonal complement of the ghost plane as:

$$\vec{\varphi} = \vec{\phi}_\perp \quad \vec{\psi} = \vec{\Psi}_\perp \quad \vec{a} = \vec{A}_\perp,$$  \hfill (4.34)

we can write the total energy-momentum tensor $T$ and supersymmetry generator $T_F$ in the form:

$$T = -\frac{1}{2} \partial \vec{\varphi} \cdot \partial \vec{\varphi} + \vec{a} \cdot \partial^2 \vec{\varphi} + \frac{1}{2} \vec{\psi} \cdot \partial \vec{\psi} - b\partial c - \frac{1}{2}(\beta \partial \gamma - \gamma \partial \beta)$$

$$T_F = \frac{1}{2} \partial \vec{\varphi} \cdot \vec{\psi} - \vec{a} \cdot \partial \vec{\varphi} + \frac{1}{2} b\gamma - \frac{1}{2} \beta \partial c.$$  \hfill (4.35)

Notice that $\vec{\varphi}$, $\vec{\psi}$ and $\vec{a}$ live in an $n$-dimensional space, with $n = N - 2$. The fields $\vec{\varphi}$ and $\vec{\psi}$ will be collectively referred to as “matter” fields.
Let us now find the expression of the other operators appearing in the STA. Consider, first of all, the $R_F$ field. This operator is the generator of the STA with the lowest dimension and therefore one expects that the number of possible terms appearing in its expression will be lower that in any other generator. Recall that $R_F$ is a fermionic operator whose conformal dimension is $\frac{1}{2}$. With the fields we have at hand, the only contributions that could appear in $R_F$ are a linear combination of the components of $\bar{\psi}$ and the products $\beta c$ and $\gamma c$ of the ghost fields. The fact that the product $Q_B(z) R_F(w)$ is non-singular implies that $R_F$ cannot contain a $\gamma c$ term. Moreover, the coefficient of the $\beta c$ term can be determined by requiring the action of $Q(z)$ on $R_F(w)$ to be given by the first equation in (4.14). Thus one is led to:

$$ R_F = \beta c - \vec{r} \cdot \bar{\psi}, \quad (4.36) $$

where $\vec{r}$ is a numerical vector. From the expression of $R_F$ given in eq. (4.36) one can determine the form of $R$ by applying $T_F$ to $R_F$ and comparing the result with eq. (4.12):

$$ R = bc + \beta \gamma + \vec{r} \cdot \partial \bar{\phi}. \quad (4.37) $$

In order to match the anomalous behaviour of $R_F$ under the supersymmetry transformation (see the first equation in (4.12)), the product $\vec{a} \cdot \vec{r}$ must be fixed to a particular value that depends on $d$. Moreover $\vec{r}^2$ can be determined, for example, from the second equation in (4.15). Taking eqs. (4.3) and (4.30) into account, one gets the following set of conditions for the scalar products of the vectors $\vec{a}$ and $\vec{r}$:

$$ \vec{a}^2 = \frac{2 - n}{8} \quad \vec{a} \cdot \vec{r} = -\frac{d + 1}{2} \quad \vec{r}^2 = -d. \quad (4.38) $$

One can follow the same procedure to determine the form of $G_B$ from its behaviour under the action of $Q$ and $Q_B$. After some trial and error, it is rather
easy to arrive at the expression:

\[ G_B = -cT_{F,M} + \frac{1}{2} [bc\gamma + \frac{1}{2} \beta \gamma^2 + \beta c\partial c + (\vec{r} \cdot \partial \vec{\phi}) \gamma - (\vec{r} \cdot \vec{\psi}) \partial c + d\partial \gamma], \quad (4.39) \]

where \( T_{F,M} \) denotes the matter contribution to \( T_F \). It remains to obtain the form of \( G \). As in the case of \( R \), once \( G_B \) is known the form of \( G \) is fixed by the supersymmetry. Indeed, from the residue of the single pole singularity in the product \( T_F(z)G_B(w) \), one gets:

\[ G = c[T_M + T_{\beta,\gamma}] - cb\partial c - \gamma T_{F,M} + \gamma[\beta \partial c - \frac{1}{4} b\gamma] \]
\[ + (\vec{r} \cdot \vec{\partial} \vec{\phi}) \partial c + \frac{1}{2} (\vec{r} \cdot \vec{\psi}) \partial \gamma - \frac{1}{2} (\vec{r} \cdot \vec{\partial} \vec{\psi}) \gamma + \frac{d}{2} \partial^2 c. \quad (4.40) \]

In eq. (4.40) \( T_M \) and \( T_{\beta,\gamma} \) are the contributions to the energy-momentum tensor of the matter fields and the \((\beta, \gamma)\) system respectively. It is now straightforward (although in some cases quite tedious) to check that all the OPE’s of the STA are satisfied if \( \vec{a} \) and \( \vec{r} \) verify eq. (4.38).

Let us now apply the mirror transformation to the realization of the STA that we have just obtained. Taking into account the form of \( T, T_F, R \) and \( R_F \) (eqs. (4.35)-(4.37)) we get:

\[ T^* = -\frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} + \vec{a}^* \cdot \partial^2 \vec{\phi} + \frac{1}{2} \vec{\psi} \cdot \partial \vec{\psi} - 2b\partial c + c\partial b - \frac{3}{2} \beta \partial \gamma - \frac{1}{2} \gamma \partial \beta \]
\[ T^*_F = \frac{1}{2} \partial \vec{\phi} \cdot \vec{\psi} - \vec{a}^* \cdot \partial \vec{\psi} + \frac{1}{2} b\gamma - \frac{3}{2} \beta \partial c - \partial \beta c, \quad (4.41) \]

where, as in (2.50), \( \vec{a}^* = \vec{a} - \vec{r} \). By inspecting eq. (4.41) we conclude that in this mirror realization the field \( b, c, \beta \) and \( \gamma \) have conformal dimensions \( 2, -1, \frac{3}{2} \) and \( -\frac{1}{2} \) respectively. Therefore they can be regarded as the ghost fields of the Ramond-Neveu-Schwarz (RNS) superstring. Notice that \( G^* = b \) and \( G^*_B = \beta, \)
while
\[ R_F^* = \vec{a}^* \cdot \vec{\psi} - \beta c \quad \quad R^* = \vec{r}^* \cdot \partial \vec{\phi} - bc - \beta \gamma, \quad (4.42) \]

with \( \vec{r}^* = -\vec{r} \). The BRST charge \( Q^* \) can be written in the suggestive form:
\[ Q^* = c[T_M^* + \frac{1}{2} T_{gh}^*] - \gamma[T_F^* M + \frac{1}{2} T_{F,gh}^*] - \]
\[ -\partial[c(R_M^* + \frac{1}{2} R_{gh}^*)] + \frac{1}{2} \partial|\gamma(R_F^* M + \frac{1}{2} R_{F,gh}^*)] + \frac{d}{2} \partial^2 c, \quad (4.43) \]

where the subscripts \( M \) and \( gh \) denote respectively the matter and ghost contributions to the operators appearing in eq. (4.43). The first two terms in eq. (4.43) correspond to the usual BRST charge of the RNS superstring, whereas the last three contributions to \( Q^* \) are total derivatives that do not contribute to the integrated BRST charge. The bosonic partner of the BRST charge can be written in a way similar to eq. (4.43). Indeed, after some calculation, one may verify that \( Q_B^* \) is given by:
\[ Q_B^* = -c[T_F^* M + \frac{1}{2} T_{F,gh}^*] - \gamma[R_M^* + \frac{1}{2} R_{gh}^*] - \]
\[ -\frac{\partial c}{2} [R_F^* M + \frac{1}{2} R_{F,gh}^*] + \partial[c(R_F^* M + \frac{1}{2} R_{F,gh}^*)] + \frac{d}{2} \partial \gamma. \quad (4.44) \]

We have thus proved that the BRST operator of the RNS superstring closes the STA. In complete parallel with what occurs in the bosonic string, one has to improve the standard BRST superstring current in order to have a closed operator algebra. Notice the close relationship between the improving terms in eqs. (2.52) and (4.43). These improving terms depend on the dimension \( d \) and on the \( R^* \) and \( R_F^* \) currents. To ensure the fulfillment of the STA, the vectors \( \vec{a}^* \) and \( \vec{r}^* \) must satisfy:
\[ (\vec{a}^*)^2 = \frac{10 - n}{8} \quad \quad \vec{a}^* \cdot \vec{r}^* = \frac{1 - d}{2} \quad \quad (\vec{r}^*)^2 = -d. \quad (4.45) \]

As announced in section 1, the mirror transformation can be regarded as a transformation of the fields. In order to see how this can be achieved, let us come
back to the fields $\vec{\phi}$ and $\vec{\Psi}$ from which we extracted our ghost system. In terms of these fields, the $R$ and $R_F$ currents can be written as:

$$
R = \vec{R} \cdot \partial \vec{\phi} \quad \quad R_F = \vec{R}_F \cdot \vec{\Psi},
$$

where the numerical vectors $\vec{R}$ and $\vec{R}_F$ can be obtained by evaluating the normal-ordered products appearing in eq. (4.42). A short calculation shows that:

$$
\vec{R} = -\vec{R}_F = \vec{r} + \vec{\lambda}.
$$

(4.47)

Notice that the vectors $\vec{R}$ and $\vec{R}_F$ are collinear. Let us denote by $\vec{\phi}^*$ and $\vec{\Psi}^*$ the fields obtained after a reflection with respect to a hyperplane orthogonal to $\vec{R}$. This reflection is well-defined for $d \neq 0$ (see eq. (2.55)). The relation between $\vec{\phi}$ and $\vec{\phi}^*$ is given in eq. (2.56), whereas $\vec{\Psi}^*$ is given by a similar expression:

$$
\vec{\Psi}^* = \vec{\Psi} + \frac{2}{d} (\vec{\Psi} \cdot \vec{R}) \vec{R}.
$$

(4.48)

In terms of these reflected fields, $T$ and $T_F$ can be written as:

$$
T = -\frac{1}{2} \partial \vec{\phi}^* \cdot \partial \vec{\phi}^* + \vec{A}^* \cdot \partial^2 \vec{\phi}^* + \frac{1}{2} \vec{\Psi}^* \cdot \partial \vec{\Psi}^* \\
T_F = \frac{1}{2} \partial \vec{\phi}^* \cdot \vec{\Psi}^* - \vec{A}^* \cdot \partial \vec{\Psi}^*;
$$

(4.49)

where, as in eq. (2.58), $\vec{A}^* = \vec{A} - \vec{R}$. On the other hand, using eqs. (4.24), (4.25) and (4.47), one can immediately prove that:

$$
\vec{A}^* \cdot \vec{\mu} = \frac{3}{2},
$$

(4.50)

which implies that the conformal dimensions of the vertex operators $e^{\pm \vec{\mu} \cdot \vec{\phi}^*}$ with respect to $T$ are:

$$
\Delta(e^{\pm \vec{\mu} \cdot \vec{\phi}^*}) = \pm \frac{3}{2}.
$$

(4.51)

Let us now substitute in our bosonization formulas of eqs. (4.20) and (4.21) the fields $\vec{\phi}$ and $\vec{\Psi}$ by their reflected counterparts $\vec{\phi}^*$ and $\vec{\Psi}^*$. The resulting ghost
fields will be denoted by $b^*$, $c^*$, $\beta^*$ and $\gamma^*$. One has:

$$
\begin{align*}
    b^* &= -\vec{\mu} \cdot \vec{\Psi}^* \: e^{\vec{\mu} \cdot \vec{\phi}^*} \\
    c^* &= -\vec{\lambda} \cdot \vec{\Psi}^* \: e^{-\vec{\mu} \cdot \vec{\phi}^*} \\
    \beta^* &= e^{\vec{\mu} \cdot \vec{\phi}^*} \\
    \gamma^* &= [\vec{\lambda} \cdot \partial \vec{\phi}^* - (\vec{\mu} \cdot \vec{\Psi}^*) (\vec{\lambda} \cdot \vec{\Psi}^*)] \: e^{-\vec{\mu} \cdot \vec{\phi}^*}.
\end{align*}
$$

(4.52)

After a short calculation one can verify that, if $\vec{\lambda}$ and $\vec{\mu}$ satisfy eq. (4.24), these ghost fields have the same OPE’s than those of $b$, $c$, $\beta$ and $\gamma$ (see eq. (4.23)). Notice that, using the result (4.51), the conformal weights of the ghost fields are:

$$
\begin{align*}
    \Delta(b^*) &= 2 & \Delta(c^*) &= -1 & \Delta(\beta^*) &= \frac{3}{2} & \Delta(\gamma^*) &= -\frac{1}{2}.
\end{align*}
$$

(4.53)

Moreover, denoting by $\vec{\varphi}^*$ and $\vec{\psi}^*$ the components of $\vec{\phi}^*$ and $\vec{\Psi}^*$ in the orthogonal complement of the plane spanned by $\vec{\mu}$ and $\vec{\lambda}$, it is straightforward to prove that $T$ and $T_F$ can be written as:

$$
\begin{align*}
    T &= -\frac{1}{2} \partial \vec{\varphi}^* \cdot \partial \vec{\varphi}^* + \vec{a}^* \cdot \partial^2 \vec{\varphi}^* + \frac{1}{2} \vec{\psi}^* \cdot \partial \vec{\psi}^* \\
    &\quad - 2b^* \partial c^* + c^* \partial b^* - \frac{3}{2} \beta^* \partial \gamma^* - \frac{1}{2} \gamma^* \partial \beta^* \\
    T_F &= \frac{1}{2} \partial \vec{\varphi}^* \cdot \vec{\psi}^* - \vec{a}^* \cdot \partial \vec{\psi}^* + \frac{1}{2} b^* \gamma^* - \frac{3}{2} \beta^* \partial c^* - \partial \beta^* c^*.
\end{align*}
$$

(4.54)

Similarly, one can reexpress $R_F$ and $R$ in terms of the new fields. One has:

$$
\begin{align*}
    R_F &= -\beta^* c^* - \vec{r}^* \cdot \vec{\psi}^* & R &= -b^* c^* - \beta^* \gamma^* + \vec{r}^* \cdot \partial \vec{\varphi}^*.
\end{align*}
$$

(4.55)

It is important to point out the close relationship between eq. (4.55) and the second equation in (2.69). In both cases the ghost contributions to the topological currents change their signs when they are written in terms of the mirror variables.
whereas in the matter part $\vec{r}$ must be substituted by $\vec{r}^* = -\vec{r}$. The BRST charge $Q^*$ and its supersymmetric partner $Q_B^*$ that close the STA with the currents given in eq. (4.55) are obtained by substituting in eqs. (4.43) and (4.44) $T^*, T_F^*, R^*$ and $R_F^*$ by $T$, $T_F$, $R$ and $R_F$ respectively. The expressions for the latter in terms of the new variables are given in eqs. (4.54) and (4.55). One must also replace in eqs. (4.43) and (4.44) the ghost fields $b, c, \beta, \gamma$ by their mirror counterparts $b^*, c^*, \beta^*, \gamma^*$. On the other hand, the BRST ancestors of $T$ and $T_F$ in this realization are simply $G = b^*$ and $G_B = \beta^*$. Notice the complete analogy in the behaviour under the mirror reflection between the supersymmetric and non-supersymmetric cases (see eqs. (2.69) and (2.70)).

As we have already mentioned, in the variables labelled by an asterisk, our system can be regarded as a RNS superstring. For a critical RNS superstring the number $n$ of components of $\vec{\varphi}^*$ is 10 and $\vec{a}^* = 0$. A glance at eq. (4.45) shows that there is only one value of $d$ consistent with the closure of the STA:

$$d \text{ (critical RNS superstring)} = 1. \quad (4.56)$$

For this value of $d$, one gets from eq. (4.45) that $\vec{r}^*$ cannot be zero since $(\vec{r}^*)^2 = -1$.

If we were dealing with a non-critical string, one of the directions of the $n$-dimensional field space must be identified with the Liouville mode. Accordingly we decompose in this case $\vec{a}^*, \vec{r}^*$ and $\vec{\varphi}^*$ as in eq. (3.2), while the Majorana field $\vec{\psi}^*$ is split as:

$$\vec{\psi}^* = \vec{\psi}^*_m + \vec{\psi}^*_L. \quad (4.57)$$

The central charges $c^*_m$ and $c^*_L$ of the matter and Liouville degrees of freedom can be computed from their background charges $a^*_m$ and $a^*_L$ as in section 3. They are now constrained to satisfy $c^*_m + c^*_L = 15$. As is customary in the study of superconformal field theories, let us introduce the quantities $\hat{c}_m = \frac{2}{3} c_m$ and $\hat{c}_m^* = \frac{2}{3} c^*_m$. The dimension $d$ of the STA can be written as a function of $\hat{c}_m$. 

43
\(c_m^*\) and the component of \(\vec{r}\) parallel to the matter subspace. A short calculation, similar to the one performed in section 3 for the bosonic string, gives the result:

\[
d = \frac{\hat{c}_m + \hat{c}_m^*}{2} - 5 - 4 (\hat{r}_m^*)^2 \pm \sqrt{(1 - \hat{c}_m)(9 - \hat{c}_m^*)}.
\]

Remarkably enough, the quantity under the square root in eq. (4.58) involves the central charges of the matter coupled to supergravity \((\hat{c}_m^*)\) and of its mirror image \((\hat{c}_m)\). If we require that \((\hat{r}_m^*)^2\) and \(d\) be real, the range of the allowed central charges is \(\{\hat{c}_m \leq 1 \quad \hat{c}_m^* \leq 9\}\) and \(\{\hat{c}_m \geq 1 \quad \hat{c}_m^* \geq 9\}\). If \(\hat{r}_m^* = 0\), eq. (4.58) simplifies since in this case \(c_m = c_m^*\) and, therefore, the range of values of \(c_m^*\) that correspond to real values of \(d\) is \(\{\hat{c}_m^* \leq 1\} \cup \{\hat{c}_m^* \geq 9\}\).

When the number \(n\) of components of the fields \(\vec{\varphi}^*\) and \(\vec{\psi}^*\) is equal to two, one can solve explicitly the constraints imposed by the topological symmetry (eq. (4.45)). Notice that the \(n = 2\) theories include as particular cases the minimal models coupled to two-dimensional supergravity and the \(\hat{c} = 1\) RNS superstring. From eq. (4.45) we get that the background charges along the matter and Liouville directions satisfy \((a_m^*)^2 + (a_L^*)^2 = 1\). It is convenient to parametrize \(a_m^*\) and \(a_L^*\) as follows:

\[
a_m^* = \frac{i}{2} \left( \hat{\lambda} - \frac{1}{\hat{\lambda}} \right) \quad a_L^* = \frac{1}{2} \left( \frac{1}{\hat{\lambda}} + \hat{\lambda} \right).
\]

The quantity \(\hat{\lambda}\) can be determined from the central charge of the matter sector. After a straightforward calculation one gets:

\[
\hat{\lambda} = \sqrt{\frac{9 - \hat{c}_m^*}{8}} \pm \sqrt{\frac{1 - \hat{c}_m^*}{8}}.
\]

In terms of \(\hat{\lambda}\), one can easily get the form of \(\hat{r}_m^*\) as a function of \(d\):

\[
r_m^* = -\frac{i}{2} \left( \frac{1}{\hat{\lambda}} + d \hat{\lambda} \right) \quad r_L^* = \frac{1}{2} \left( \frac{1}{\hat{\lambda}} - d \hat{\lambda} \right).
\]

A simple calculation shows that the vectors \(\vec{a}^*\) and \(\vec{r}^*\) given in eqs. (4.59) and (4.61) solve the constraints (4.45). Notice that eq. (4.61) determines the direction
of the topological $U(1)$ symmetry in the space of fields. It is also possible to obtain the central charge $\hat{c}_m$ of the matter sector of the mirror theory as a function of $d$ and $\hat{c}_m^*$. A short calculation shows that:

$$\hat{c}_m = 1 - \frac{(d + 1)^2}{2} \left[ 5 - \hat{c}_m^* \pm \sqrt{(1 - \hat{c}_m^*)(9 - \hat{c}_m^*)} \right]. \quad (4.62)$$

For $d = -1$ the central charge $\hat{c}_m$ is equal to one for all values of $\hat{c}_m^*$, in complete analogy to what happened in the bosonic case. Indeed, for this value of $d$, $a_m^* = r_m^*$ and $a_L^* = r_L^*$, which means that the background charges $a_m$ and $a_L$ are zero. For $d \neq -1$ $\hat{c}_m$ depends on $\hat{c}_m^*$ and it is possible to invert eq. (4.62) and get $\hat{c}_m^*$ as a function of $d$ and $\hat{c}_m$.

### 5. Summary and Concluding Remarks

In this paper we have analyzed the possible realizations of the topological conformal symmetry of CFT’s with vanishing Virasoro anomaly. We have investigated the consequences of the existence of the mirror symmetry, which appears quite naturally in our approach as a transformation that relates two realizations of the topological algebra. We have seen that, once a convenient basis of fields is chosen, the mirror automorphism can be recast as a field transformation.

When the system under study is a model of matter coupled to two-dimensional (super)gravity, the mirror symmetry seems to play a relevant role. The BRST current for these theories is just an improved version of the standard BRST current that fixes the two-dimensional (super)diffeomorphism invariance. By means of the mirror transformation one relates the original model of matter plus two-dimensional (super)gravity to a system with spin-one ghosts and, what is more important, the implementation of the topological symmetry depends on the matter content of the two theories connected by the symmetry. This fact appears neatly in the expressions that give the dimension $d$ in terms of the central charges of the matter sector (eqs. (3.10) and (4.58)).
Our results imply that for these (super)gravity theories there exists an additional BRST symmetry that, apart from a sign, shares the $U(1)$ currents with the original (super)diffeomorphism invariance. In general the generator of this new BRST transformation cannot be expressed locally in terms of the original fields of the model. Only after the (super)diffeomorphism ghosts are conveniently bosonized, can a local expression for the new BRST current be given. Therefore one can consider the mirror BRST current as a hidden symmetry of the matter-plus-gravity models in two dimensions.

It is interesting to connect our analysis with the standard BRST approach to string theory. In string theory the states are obtained from the cohomology classes of the BRST charge that satisfy an additional equivariance condition that involves the zero mode of the $b^*$ ghost [23, 24]. This equivariance condition can be understood as a requirement of covariance of the physical states under general coordinate transformations of the world sheet [25]. Notice that the $b^*$ field is nothing but the $G$ operator in our realization of the topological algebra.

The role of the $G$ operator is also crucial in the topological string theory approach [4, 26, 27]. In this formalism one couples topological matter to topological gravity [28]. As has been studied in ref. [29], the coupling of matter to topological gravity can be generated by means of a gauge principle. The generator of this new symmetry is precisely $G$ and the corresponding transformations are odd analogues of the world sheet reparametrizations (recall that $G$ is the BRST partner of the energy-momentum tensor). In order to have a theory invariant under the local $G$-transformations, one must introduce an odd gauge field $\Psi_{\mu\nu}$, which is related to the world sheet metric $g_{\mu\nu}$ by means of the $Q$ symmetry. After this odd symmetry is fixed, the standard description of topological gravity is obtained. It can be shown [30] that (at least for minimal models) this topological approach gives rise to the same physical states as those obtained from the equivariant cohomology of the standard BRST charge.

To summarize the situation one can say that in both approaches to string the-
ory one has to require additional conditions to the BRST Virasoro cohomology. In the standard approach to string theory one considers the equivariant cohomology whereas in the topological string approach one enlarges the BRST symmetry in such a way that the equivariance condition is already incorporated in the topological symmetry. From this point of view the extra conditions are generated when the topological symmetry is implemented locally since only in this case the field $G$ responsible for these conditions is uniquely determined. Notice that this is consistent with what we have obtained since our basic requirement is precisely the fulfillement of the conditions generated by the topological algebra. Moreover we have obtained an extra BRST charge in (super)string theory, different from the standard (super) Virasoro one but closely related to it, which is obtained from the $G$ operator by substituting in $G$ the bosonized fields by their reflected counterparts.

Our results can be generalized in several directions. Let us only mention that it would be very interesting to study the general form of the BRST algebra for the $N = 2$ superstring [31]. In this case, by a simple counting of the central charges of the ghost sector, we expect that the lower and upper limits of the barrier will collapse and, at least for some choice of the improving terms, there will be no barrier at all. An interesting aspect to elucidate is the role played by the mirror symmetry in the implementation of the topological algebra, which in this case is a twisted $N = 4$ superconformal algebra as has been checked in refs. [32, 33]. Another interesting question is the relation (if any) of this symmetry with the target space-world sheet duality that these $N = 2$ strings have [34]. Work in these directions is in progress and we expect to report on it in a near future.

Acknowledgements:

We would like to thank O. Alvarez, J. M. F. Labastida, M. Mariño, J. Mas and G. Sierra for valuable discussions at different stages of this work. We are grateful to J. M. Isidro, P.M. Llatas and S. Roy for a critical reading of the manuscript. This work was supported in part by DGICYT under grant PB 93-0344 and by CICYT under grant AEN 94-0928.
APPENDIX A

In this appendix we complete the analysis of the realizations of the topological symmetry. Consider first of all the non-supersymmetric case of section 2. As we stated in that section, the values \( n \) and \( m \) of the depths of \( Q \) and \( G \) are restricted by the condition \( n + m \leq 3 \) (see eq. (2.19)). Apart from the cases \( n = 0, m = 2 \) and \( n = 2, m = 0 \) studied in section 2, only when \( n = 0, m = 0 \) and \( n = 1, m = 1 \) is it possible to get a consistent representation of the TA. For these depths the number of fields is fixed to some particular values. For example if \( (n, m) = (0, 0) \) one is forced to have only one scalar field. From eq. (2.18) we obtain \( \tilde{\alpha}^2 = -3 \) for this case. The explicit form of the generators can be easily worked out. One gets [35]:

\[
T = -\frac{1}{2} (\partial \phi)^2 + \frac{i}{2\sqrt{3}} \partial^2 \phi \\
Q = \frac{1}{\sqrt{3}} e^{i\sqrt{3} \phi} \\
G = \frac{1}{\sqrt{3}} e^{-i\sqrt{3} \phi} \\
R = \frac{i}{\sqrt{3}} \partial \phi.
\]

Moreover the value of \( d \) for the generators of eq. (A.1) is fixed \( (d = \frac{1}{3}) \). It is important to point out that for the representation (A.1) the mirror transformation is simply realized by the field reflection \( \phi \rightarrow -\phi \).

For \( (n, m) = (1, 1) \) one can get a representation of the TA only when the number of fields is \( N = 3 \), whereas the value of \( d \) is arbitrary. One could proceed as in section 2 and study the most general operators of depth one that can represent \( Q \) and \( G \). It is however easier to introduce a spin one anticommuting ghost system \( (b, c) \) which, in terms of the scalar field \( \tilde{\phi} \), is given by:

\[
b = e^{-\tilde{\alpha} \cdot \tilde{\phi}} \\
c = e^{\tilde{\alpha} \cdot \tilde{\phi}}.
\]

Notice the difference in the signs of the exponents in eqs. (A.2) and (2.33). As now \( \tilde{A} \cdot \tilde{\alpha} = -\frac{1}{2} \) and \( \tilde{\alpha}^2 = -1 \) (see eqs. (2.20) and (2.18)), the fields \( b \) and
c have conformal weights 1 and 0 respectively. As in section 2, one can separate in the energy-momentum tensor the contributions of the \((b, c)\) system and of the components of \(\vec{\phi}\) orthogonal to \(\vec{a}\). The latter constitute a two-component scalar field that we shall denote by \(\vec{\varphi}\). The expression of \(T\) in terms of \(b, c\) and \(\vec{\varphi}\) is the same as in eq. (2.43). The background charge \(\vec{a}\) of the field \(\vec{\varphi}\) must satisfy \(\vec{a}^2 = 0\).

Let us write down the expressions of the generators for \(d \neq 1\) and \(\vec{a} \neq 0\) (for \(d = 1\) and/or \(\vec{a} = 0\) similar expressions are obtained). The form of \(Q\) is:

\[ Q = \partial c + \frac{2}{1 - \vec{a} \cdot (\vec{a} \cdot \partial \vec{\varphi})}, \]  

(A.3)

while \(R\) and \(G\) are given by:

\[
R = -bc + \vec{r} \cdot \partial \vec{\varphi}
\]

\[
G = \frac{1 - d}{2} \partial b + b\vec{a} \cdot \partial \vec{\varphi} - b (R_M + \frac{1}{2}R_{gh}).
\]

(A.4)

This representation of the TA corresponds to the standard Coulomb gas construction of the \(N = 2\) superconformal algebra (see, for example, ref. [36]). In eq. (A.4) \(R_M\) and \(R_{gh}\) are the contributions to \(R\) of the \(\vec{\varphi}\) scalar and of the \((b, c)\) system respectively. The sign of this last term in \(R\) imply that the field \(c\) has \(R\)-charge +1. The two component vector \(\vec{r}\) satisfies \(\vec{r}^2 = 1 - d\) (as in eq. (2.47)) together with the condition:

\[ \vec{a} \cdot \vec{r} = \frac{1 - d}{2}. \]  

(A.5)

It is easy to solve these constraints and find a solution for \(\vec{r}\). If we parametrize the background charge \(\vec{a}\) as \(\vec{a} = \lambda (1, \pm i)\) (recall that \(\vec{a}^2 = 0\)), one gets:

\[
\vec{r} = (\lambda + \frac{1 - d}{\lambda}, \pm i(\lambda - \frac{1 - d}{\lambda})).
\]

(A.6)

After performing a mirror transformation to this \((1,1)\) realization, the fields \(b\) and \(c\) acquire conformal weights 0 and 1 respectively. Thus we generate a realization of the TA similar to the one in eqs. (A.3) and (A.4) with the roles of \(b\) and \(c\) exchanged.
Let us turn now to the supersymmetric algebra. It is useful in this case to label the different representations of the STA by the depths of $Q_B$ and $G_B$. In general we can represent these operators as:

$$Q_B = Q_{B,r} (\partial^i \phi, \partial^j \Psi) e^{i\tilde{\mu} \cdot \vec{\phi}}, \quad G_B = G_{B,s} (\partial^i \phi, \partial^j \Psi) e^{-i\tilde{\mu} \cdot \vec{\phi}}, \quad (A.7)$$

where $Q_{B,r}$ and $G_{B,s}$ are polynomials in the derivatives of $\vec{\phi}$ and $\vec{\Psi}$ with conformal weights $r$ and $s$ respectively. An analysis similar to the one performed in the bosonic case shows that now the depths $r$ and $s$ are restricted by the condition $r + s \leq 2$ (notice that $r$ and $s$ can be half integers). The realizations of the STA studied in section 4 (i.e. the RNS superstring and its mirror) correspond to $(r,s) = (0,2)$ and $(r,s) = (2,0)$. The study of the possible values of $(r,s)$ allowed by the depth rule yields a result which is very similar to the bosonic case. In fact only for $(r,s)$ equal to $(0,0)$ and $(1,1)$ does one arrive at a consistent realization of the topological algebra. For $(r,s) = (0,0)$ the theory must contain a scalar field and a Majorana fermion. The dimension is fixed to the value $d = \frac{1}{2}$ and the generators of the STA take the form [37]:

$$T = -\frac{1}{2} (\partial \phi)^2 + \frac{i}{2\sqrt{2}} \partial^2 \phi + \frac{1}{2} \Psi \partial \Psi \quad T_F = \frac{1}{2} \partial \phi \Psi - \frac{i}{2\sqrt{2}} \partial \Psi$$

$$Q = \frac{1}{\sqrt{2}} \Psi e^{i\sqrt{2} \phi} \quad Q_B = \frac{i}{2} e^{i\sqrt{2} \phi}$$

$$G = -\frac{1}{\sqrt{2}} \Psi e^{-i\sqrt{2} \phi} \quad G_B = \frac{i}{2} e^{-i\sqrt{2} \phi}$$

$$R = \frac{i}{\sqrt{2}} \partial \phi \quad R_F = -\frac{i}{\sqrt{2}} \Psi. \quad (A.8)$$

Notice that for the representation of eq. (A.8) the mirror transformation is realized as $\phi \rightarrow -\phi$ and $\Psi \rightarrow -\Psi$, in complete analogy with what happens for the representation of the TA given in eq. (A.1).

For $(r,s) = (1,1)$ there exists a representation of the STA with arbitrary $d$ when the number of bosonic and fermionic fields is four. In this case it is easier to
introduce a supersymmetric ghost system whose relation with the original fields \( \tilde{\phi} \) and \( \tilde{\Psi} \) is as in eqs. (4.20) and (4.21). The conformal weights of \( \beta, \gamma, b \) and \( c \) are \( \frac{1}{2}, \frac{1}{2}, 1 \) and 0 respectively and therefore \( T \) and \( T_F \) are given by eq. (4.5), where \( \tilde{\psi} \) and \( \tilde{\varphi} \) are two-component vectors. For \( d \neq 1 \) and \( \tilde{a} \neq 0 \) the expressions of \( Q_B \) and \( Q \) are:

\[
Q_B = \gamma + \frac{2}{1-d} c \tilde{a} \cdot \tilde{\psi}
\]

\[
Q = \partial c + \frac{2}{1-d} [c \tilde{a} \cdot \partial \tilde{\varphi} + \gamma \tilde{a} \cdot \tilde{\psi}].
\]

The two abelian currents \( R_F \) and \( R \) take the form:

\[
R_F = - \beta c - \vec{r} \cdot \tilde{\psi}
\]

\[
R = - bc - \beta \gamma + \vec{r} \cdot \partial \tilde{\varphi},
\]

where \( \vec{r} \) is a two-component vector such that \( \vec{r}^2 = -d \) and whose scalar product with the background charge \( \tilde{a} \) is given by eq. (A.5). Denoting by \( R_M, R_{M,gh}, R_{F,M} \) and \( R_{F,gh} \) to the \( \tilde{\varphi} \) and ghost contributions to \( R \) and \( R_F \), we can write \( G_B \) and \( G \) as:

\[
G_B = \frac{1-d}{2} \partial \beta + \frac{b}{2} (R_{F,M} + \frac{1}{2} R_{F,gh}) - \frac{\beta}{2} (R_{M} + \frac{1}{2} R_{gh}) + \frac{\tilde{a} \cdot \tilde{\psi}}{1-d} [db + \beta (R_{F,M} + \frac{1}{2} R_{F,gh})]
\]

\[
G = \frac{1-d}{2} \partial b - b (R_{M} + \frac{1}{2} R_{gh}) + \frac{\partial \beta}{2} (R_{F,M} + \frac{1}{2} R_{F,gh}) - \frac{\beta}{2} (\partial R_{F,M} + \frac{1}{2} \partial R_{F,gh}) - \frac{\tilde{a} \cdot \tilde{\varphi}}{1-d} [db + \beta (R_{F,M} + \frac{1}{2} R_{F,gh})] - \frac{\tilde{a} \cdot \tilde{\psi}}{1-d} [d\partial \beta + \beta (R_{M} + \frac{1}{2} R_{gh}) + b (R_{F,M} + \frac{1}{2} R_{F,gh})].
\]

The realization of the STA displayed in eqs. (A.9)-(A.11) was found previously (although with different notations from ours) in ref. [38]. Due to the signs of the
ghost contributions to $R$, the mirror transformation applied to this realization gives rise to a system of ghosts with spins $1$ and $\frac{1}{2}$ and with the ghost number reversed. Actually $b$, $c$, $\beta$, and $\gamma$ acquire conformal weights $1$, $0$, $\frac{1}{2}$, and $\frac{1}{2}$ respectively. The mirror realization is thus of the same type as the original one with the roles of $b$ and $\beta$ exchanged with those of $c$ and $\gamma$ respectively. As in the bosonic case one can solve the constraints for $\vec{r}$. If we put again $\vec{a} = \lambda (1, \pm i)$, we get:

$$\vec{r} = (\frac{\lambda d}{d-1} + \frac{1 - d}{4\lambda}, \pm i \left( \frac{\lambda d}{d-1} - \frac{1 - d}{4\lambda} \right) ).$$

(A.12)
REFERENCES

1. E. Witten, Comm. Math. Phys. 117(1988), 353.

2. E. Witten, Comm. Math. Phys. 118(1988), 411.

3. For a review see D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209(1991), 129.

4. R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352(1991), 59.; “Notes on topological string theory and 2d quantum gravity”, Proceedings of the Trieste spring school 1990, edited by M. Green et al. (World Scientific, Singapore, 1991).

5. W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324(1989), 427.

6. T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A4(1990), 1653; T. Eguchi, S. Hosono and S.-K. Yang, Comm. Math. Phys. 140(1991), 159.

7. H. Yoshii, Phys. Lett. B259(1991), 279.

8. For a review of the applications of mirror symmetry to the study of complex manifolds see “Essays on Mirror Manifolds”, edited by S. T. Yau (International Press, Hong Kong, 1992).

9. A.M. Polyakov, Mod. Phys. Lett. A2(1987), 893; V. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A3(1988), 819.

10. F. David, Mod. Phys. Lett. A3(1988), 1651; J. Distler and H. Kawai, Nucl. Phys. B321(1989), 509.

11. For a review see P. Ginsparg and G. Moore, “Lectures on 2D Gravity and 2D String Theory” [hep-th/9304011], in “Recent directions in particle Theory”, Proceedings of the 1992 TASI summer school, edited by J. Harvey and J. Polchinski (World Scientific, Singapore, 1993).

12. B. Gato-Rivera and A.M. Semikhatov, Phys. Lett. B293(1992), 72, Nucl. Phys. B408(1993), 133.
13. M. Bershadsky, W. Lerche, D. Nemechansky and N. Warner, *Nucl. Phys.* **B401**(1993), 304.

14. S. Mukhi and C. Vafa, *Nucl. Phys.* **B407**(1993), 667.

15. J. Distler, *Nucl. Phys.* **B342**(1990), 523.

16. H. Ishikawa and M. Kato, *Int. J. Mod. Phys.* **A**(1994), 5796.

17. P.M. Llatas and S. Roy, *Phys. Lett.* **B345**(1995), 6.

18. E. Witten, *Nucl. Phys.* **B373**(1992), 187.

19. D. Kutasov, E. Martinec and N. Seiberg, *Phys. Lett.* **B276**(1992), 437.

20. A. Gerasimov et al., *Int. J. Mod. Phys.* **A5**(1990), 2495.

21. E. Martinec and G. Sotkov, *Phys. Lett.* **B208**(1988), 240.

22. M. Takama, *Phys. Lett.* **B210**(1988), 153.

23. L. Alvarez-Gaume, C. Gomez, G. Moore and C. Vafa, *Nucl. Phys.* **B303**(1988), 455; C. Vafa, *Phys. Lett.* **B190**(1987), 47.

24. P. Nelson, *Phys. Rev. Lett.* **62**(1989), 993; J. Distler and P. Nelson, *Comm. Math. Phys.* **138**(1991), 255.

25. C.M. Becchi, R. Collina and C. Imbimbo, *Phys. Lett.* **B322**(1994), 79.

26. E. Witten, *Nucl. Phys.* **B340**(1990), 281; E. Verlinde and H. Verlinde, *Nucl. Phys.* **B348**(1991), 547.

27. K. Li, *Nucl. Phys.* **B354**(1991), 711, *Nucl. Phys.* **b354**(1991), 725.

28. J.M.F. Labastida, M. Pernici and E. Witten, *Nucl. Phys.* **B310**(1988), 611.

29. J.M.F. Labastida and P.M. Llatas, *Nucl. Phys.* **379**(1992), 220.

30. T. E. Eguchi, H. Kanno, Y. Yamada and S.-K. Yang, *Phys. Lett.* **B305**(1993), 235.

31. M. Ademollo et al., *Phys. Lett.* **B62**(1976), 105, *Nucl. Phys.* **B111**(1976), 77.
32. J. Gomis and H. Suzuki, *Phys. Lett.* **B278**(1992), 266; A. Giveon and M. Rocek, *Nucl. Phys.* **B400**(1993), 145.

33. A. Boresch, K. Landsteiner, W. Lerche and A. Sevrin, *Nucl. Phys.* **B436**(1995), 609.

34. H. Ooguri and C. Vafa, *Nucl. Phys.* **B361**(1991), 469, *Nucl. Phys.* **B367**(1991), 83, *Mod. Phys. Lett.* **A5**(1990), 1389.

35. G. Waterson, *Phys. Lett.* **B171**(1980), 77.

36. K. Ito, *Nucl. Phys.* **B332**(1990), 566.

37. A. Schwimmer and N. Seiberg, *Phys. Lett.* **B184**(1987), 191.

38. A. Fujitsu, *Phys. Lett.* **B299**(1993), 49.