THE TREE PROPERTY AT FIRST AND DOUBLE SUCCESSORS OF SINGULAR CARDINALS OF UNCOUNTABLE COFINALITY WITH AN ARBITRARY GAP

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Abstract. Let $\mu$ be a regular cardinal and assume that there is a continuous sequence of cardinals $\langle \kappa_\xi : \xi < \mu \rangle$ such that $\kappa_0 > \mu$ is a supercompact cardinal and for all $\xi < \mu$, $\kappa_{\xi+1}$ are also supercompact cardinals. Furthermore, assume that there is a weakly compact cardinal $\lambda$ above all of them. Set $\kappa = \kappa_0$ and let $\gamma \geq \lambda$ be a cardinal with $\text{cof}(\gamma) > \mu$. Assuming the GCH we construct a generic extension where $\kappa$ is strong limit, $\text{cof}(\kappa) = \mu$, $2^\kappa = \gamma$, and $\text{TP}(\kappa^+)$ and $\text{TP}(\kappa^{++})$ hold. This generalizes the main results of [FHS18] and [Sin16a].

1. Introduction

Infinite trees has a central role in infinite combinatorics. Recall that a $\kappa$-tree is called $\kappa$-Aronszajn if it has no cofinal branches. Given a regular cardinal $\kappa$ it is said that the tree property holds at $\kappa$, denoted by $\text{TP}(\kappa)$, if every $\kappa$-tree has a cofinal branch. In 1972 Mitchell proved that assuming the existence of a weakly compact cardinal $\kappa$ there is a generic extension where $\kappa = \aleph_2$, $2^{\aleph_0} = \aleph_2$ and $\text{TP}(\aleph_2)$ holds.

Thereby the consistency of a weakly compact cardinal is an upper bound for the consistency of $\text{TP}(\aleph_2)$. It is worth mentioning that the failure of CH in Mitchell’s model is necessary, for otherwise, by virtue of Specker’s theorem, there is a special $\aleph_2$-Aronszajn trees. The converse implication is also true on the basis of a theorem of Silver (see e.g. [Tho78]) who proved that if $\text{TP}(\aleph_2)$ holds then $\omega_2$ is a weakly compact cardinal in $L$. Combining both theorems, it follows that $\text{TP}(\aleph_2)$ is equiconsistent with the existence of a weakly compact cardinal.

In this paper we are interested in the forcing devised by Mitchell in [Mit72], as well as in some other similar constructions developed by several authors over the years [Abr83] [CF98] [Sin16b] [Ung13] [FH11] [FHS18].

Given a cardinal $\kappa$ we define the Mitchell forcing $\mathbb{M}(\kappa)$ with respect to $\kappa$ as the set of pairs $(p, q)$ such that $p \in \text{Add}(\omega, \kappa)$ and $q$ is a function with $\text{dom}(q) \in [\kappa]^{<\omega_1}$ such that

$$\forall \alpha \in \text{dom}(q), \models_{\text{Add}(\omega, \kappa)} "q(\alpha) \in \text{Add}(\omega_1, 1)".$$
Furthermore, it is not hard to show that there is a projection
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Aronszajn tree. Without loss of generality we may assume that
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as its collapsing component has domain \([\gamma, \kappa]\)

Moreover, Mitchell forcing \(\mathbb{M}\) can be regarded as the amalgam of two components: the first one intended to blow up the power set of \(\aleph_0\) (\(\text{Cohen component}\)) and the second one devised to collapse the interval \((\omega_1, \kappa)\) while preserving \(\omega_1\) (\(\text{Collapsing component}\)). These two roles of \(\mathbb{M}\) yield a generic extension where \(\kappa = \aleph_2\) and \(2^{\aleph_0} \geq \aleph_2\) \(\square\) so that it simply remains to prove that \(\text{TP}(\aleph_2)\) holds in this model. To this aim, one assumes that \(\kappa\) is a weakly compact cardinal and carefully analyzes the quotient forcing \(\mathbb{M}/\mathbb{M}_\gamma\), where \(\mathbb{M}_\gamma\) is the resulting forcing of restricting \(\mathbb{M}\) to the \(\gamma\) first coordinates. More precisely, given \(\gamma < \kappa\), \(\mathbb{M}_\gamma\) is the Mitchell forcing whose Cohen component is \(\text{Add}(\kappa, \gamma)\) and its collapsing component has domain \([\gamma]^{<\omega_1}\).

As it will be useful for future arguments, here we give a brief sketch of the proof of \(\text{TP}(\aleph_2)\) in Mitchell’s model: Let \(\kappa\) be weakly compact and, aiming for a contradiction, let \(\hat{T}\) be a \(\mathbb{M}\)-name for a \(\kappa\)-Aronszajn tree. Without loss of generality we may assume that \(\hat{T}\) is a \(\mathbb{M}\)-name for a subset of \(\kappa\). In the first place, let us notice that “\(\text{Add}(\kappa, \gamma)\) is a \(\kappa\)-Aronszajn tree” is a \(\Pi^1_1\) true sentence in the structure \(\langle V, \in, \hat{T}, \mathbb{M}, \kappa \rangle\) hence, by weak compactness of \(\kappa\), there is an inaccessible cardinal \(\gamma < \kappa\) where the quoted formula reflects. Namely,
\[
\langle V_\gamma, \in, \hat{T} \cap \gamma, \mathbb{M} \cap V_\gamma, \gamma \rangle \models \text{“Add}(\kappa, \gamma)\) is a \(\gamma\)-Aronszajn tree”.
\]

Since \(\gamma\) is an inaccessible cardinal, it is not difficult to see that \(\mathbb{M} \cap V_\gamma = \mathbb{M}_\gamma\), and thus that \(\mathbb{M}_\gamma\) forces \(\hat{T} \cap \gamma\) to be a \(\gamma\)-Aronszajn tree. Furthermore, it is not hard to show that there is a projection\(^2\) between \(\mathbb{M}\) and \(\mathbb{M}_\gamma\) so that we can regard Mitchell’s forcing as the two-step iteration \(\mathbb{M}_\gamma \ast \mathbb{M}/\mathbb{M}_\gamma\). In order to reach a contradiction, since we are assuming that \(\mathbb{M}\) forces “\(\hat{T}\) is a \(\kappa\)-tree”, it will be enough to prove that \(\mathbb{M}/\mathbb{M}_\gamma\) does not add new \(\gamma\)-branches to \(\hat{T} \cap \gamma\). To this aim (working in \(V^{\mathbb{M}_\gamma}\)) we first show that there is a projection between \(\mathbb{M}/\mathbb{M}_\gamma\) and the product \(\mathbb{P}_\gamma \times \mathbb{Q}_\gamma\) of two forcings which are, respectively, \(\gamma\)-Knaster and \(\omega_1\)-closed. Recall that for a regular cardinal \(\theta\) a forcing notion \(\mathbb{P}\)

\(^1\)If we further assume that \(\kappa^{\aleph_0} = \kappa\) in the ground model then the equality is already true.

\(^2\)Recall that a map \(\pi : \mathbb{P} \to \mathbb{Q}\) between two posets is a projection if \(\pi(1_\mathbb{P}) = 1_\mathbb{Q}\) and further for every \(p \in \mathbb{P}\) and every \(q \in \mathbb{Q}\) if \(q \leq_q \pi(p)\) then there is \(\bar{p} \leq_p p\) such that \(\pi(\bar{p}) \leq_q q\).
is called $\theta$-Knaster if every subset of $\mathcal{P}$ of size $\theta$ has two compatible conditions.

Notice that by the very definition of $M_{\gamma}$, $\gamma = \aleph_2$ and $2^{\aleph_0} = \gamma$ hold in the corresponding generic extension so that, since $Q_\gamma$ is $\omega_1$-closed, Silver’s theorem (see theorem 1.15) ensures that $Q_\gamma$ does not add any new branch to $T \cap \gamma$. Also, $Q_\gamma$ is a $\gamma$-cc forcing so that after forcing with it over $V^{M_{\lambda}}$, $\gamma$ remains a cardinal. Working within $V^{M_{\lambda}} \times Q_\gamma$, notice that $\mathbb{P}_\gamma$ remains $\gamma$-Knaster, as $Q_\gamma$ is $\omega_1$-closed, and thus this forcing does not introduce new $\gamma$-branches to $T \cap \gamma$, either. Summing up, this argument shows a contradiction between the fact that $T \cap \gamma$ is a $\gamma$-Aronszajn tree in $V^{M_{\lambda}}$, and the fact that $T$ is a $\kappa$-tree in $V^{M}$, hence proving that after forcing with $M$ there are no $\kappa$-Aronszajn trees.

In the light of Mitchell’s result it is natural to ask whether it is consistent to have the tree property at two consecutive cardinals. The first result in this direction was due to Abraham, who proved in 1983 that from the existence of a supercompact cardinal with a weakly compact cardinal above it, it possible to force $\text{TP}(\aleph_2)$ and $\text{TP}(\aleph_3)$ [Abr83]. It may seem surprising at first glance that for getting the consistency of $\text{TP}(\aleph_2) + \text{TP}(\aleph_3)$ one needs much stronger hypotheses than those assumed by Mitchell, specially considering that the consistency of $\text{TP}(\aleph_2) + \text{TP}(\aleph_4)$ follows from a straightforward application of Mitchell’s ideas to two weakly compact cardinals. But, as Magidor observed, to get the consistency of the tree property at two consecutive cardinals one needs to transcend the level of $0^\sharp$ (see [Abr83 Theorem 1.1]).

Some years later, and building on Abraham’s ideas, Cummings and Foreman designed a forcing that, starting with infinitely many supercompact cardinals yields a generic extension where the tree property holds at $\aleph_n$, for each $2 \leq n < \omega$ [CF98]. In that paper the authors combined Mitchell’s construction with the Prikry-type forcing technology to get a model where $\text{TP}(\kappa^{++})$ holds and $\kappa$ is a strong limit cardinal with $\text{cof}(\kappa) = \omega$, and where SCH fails [CF98]. Specifically, starting with a supercompact cardinal $\kappa$ joint a weakly compact cardinal $\lambda > \kappa$, they produce a generic extension where $2^\kappa = \lambda$, $\lambda = \kappa^{++}$, $\text{TP}(\kappa^{++})$ holds and $\kappa$ is a strong limit with $\text{cof}(\kappa) = \omega$. As the Cummings-Foreman ideas will be extensively used in the paper it should be convenient to sketch some of the main ideas of their proof.

Starting with a Laver indestructible supercompact cardinal $\kappa$ and a weakly compact cardinal $\lambda > \kappa$ choose a normal measure $\mathcal{U}$ on $\kappa$ in the generic extension given by $\text{Add}(\kappa, \lambda)$. In this generic extension one has to prove that there set of ordinals $A \subseteq \lambda$, big in the sense of the indescribable filter over $\lambda$, such that for each $\alpha \in A$ the measure $\mathcal{U}$ projects onto a normal measure $\mathcal{U}_\alpha$ over $\kappa$ in the intermediate generic extension $V^{\text{Add}(\kappa, \alpha)}$. For each such $\alpha \in A$ let, respectively, $Q$ and $Q_\alpha$
be the Prikry forcings with respect to the measures $\mathcal{U}$ and $\mathcal{U}_\alpha$. This yields a coherent filtration $\langle \mathcal{U}_\alpha : \alpha \in \mathcal{A} \cup \{\lambda\} \rangle$ of the original measure $\mathcal{U}_\lambda$. Hence, by virtue of Mathias criterion \cite{Gi10}, any generic filter $H \subseteq Q_\beta$ over $V^{\text{Add}(\kappa, \beta)}$ is also generic for $Q_\alpha$ over $V^{\text{Add}(\kappa, \alpha)}$, provided $\alpha < \beta$. Thereby, for each $\alpha < \beta$ in $\mathcal{A}$ there are projections $\pi^{\beta}_\alpha$ between the forcings $\text{Add}(\kappa, \beta) \ast Q_\beta$ and $\text{Add}(\kappa, \alpha) \ast Q_\alpha$.

Now in $V$ let us consider the Mitchell-like forcing $R$ whose Cohen component is $\text{Add}(\kappa, \lambda) \ast Q$ and the collapsing component is given by functions $q$ with domain $[\mathcal{A}]^{\leq \kappa}$ such that

$\forall \alpha \in \text{dom} (q) \models ^{\text{Add}(\kappa, \alpha) \ast Q_\alpha} q(\alpha) \in \text{Add}(\kappa, 1)$.

The order of $R$ is defined in a similar way as $\leq_M$ but using the system of projections $\langle \pi^{\beta}_\alpha : \alpha < \beta \in \mathcal{A} \cup \{\lambda\} \rangle$. Specifically, $(p, q) \leq_R (p^*, q^*)$ if and only if $p \leq_{\text{Add}(\kappa, \lambda)} p^*$, $\text{dom}(q^*) \subseteq \text{dom}(q)$ and

$\forall \alpha \in \text{dom}(q^*) \pi^{\lambda}_\alpha (p, q) \models ^{\text{Add}(\kappa, \alpha) \ast Q_\alpha} q(\alpha) \leq q^*(\alpha)$.

The rest of $\pi^{\beta}_\alpha$’s are used to make sure that there are projections between the intermediate forcings $R \upharpoonright \beta$ and $R \upharpoonright \alpha$, $\alpha < \beta$ (see \cite{CF98}). Finally, a similar analysis of the quotient forcing as the one sketched above yields the proof of TP$(\kappa^{++})$ in $V^R$.

Building on these ideas, as well as on \cite{Ung13}, Friedman, Honzik and Stejskalová \cite{FHS18} exhibited an argument to get arbitrary values of $2^\kappa$ in Cummings-Foreman’s model. In particular this shows that TP$(\kappa^{++})$ is consistent with an arbitrary failure of SCH at strong limit cardinals \cite{FHS18}. Building on \cite{FHS18} this result was subsequently generalized in \cite{GP18} to the setting of uncountable cofinalities as well as pushed down to the level of $\aleph_\mu$ under some restrictions on the power set of $\aleph_\mu$.

A related discussion is about the existence of Aronszajn trees at first successors of singular strong limit cardinals $\kappa$ where SCH fails. The origin of this problem can be traced back to the first proof of the consistency of $\neg \text{SCH}_\kappa$, due to Silver and Prikry (see \cite{Tho78}): start with $\kappa$ a measurable cardinal with $2^\kappa = \kappa^{++}$ and force with the Prikry forcing $P_\kappa$ with respect to some normal measure on $\kappa$. It is the case that $P_\kappa$ not only forces that $\kappa$ is a singular strong limit cardinal, hence SCH fails, but also that TP$(\kappa^+)$ fails. Indeed, since $\kappa$ is a measurable cardinal, in particular $\kappa^{<\kappa} = \kappa$, then $\square^*_\kappa$ holds (see e.g. \cite{CFM01}). Since Prikry forcing preserves the successor of $\kappa$, $\square^*_\kappa$ also holds in the generic extension by $P_\kappa$. On the other hand, by virtue of Jensen’s theorem, it is known that the existence of a $\square^*_\kappa$-sequence is

\footnote{In fact in \cite{GP18} it is shown that the hypothesis needed for all these results can be weakened to a strong cardinal $\kappa$ with $o(\kappa) = \mu$.}

\footnote{The consistency of this statement is exactly the existence of a measurable cardinal $\kappa$ with $o(\kappa) = \kappa^{++}$ as proved by Gitik \cite{Tho78}.}
equivalent to the existence of a special $\kappa^+$-Aronszajn tree, hence $P_\mu$ forces both the failure of $\text{SCH}_\kappa$ and the failure of $\text{TP}(\kappa^+)$. A natural question is thus if this is essentially the only way to produce a model where $\text{SCH}_\kappa$ fails or whether there are others. More precisely, given a singular strong limit cardinal $\kappa$ with $\text{cof}(\kappa) = \omega$ does $\text{TP}(\kappa^+)$ (and in particular, $\neg \Box^*_\kappa$) imply $\text{SCH}_\kappa$? This question was originally posed in 1989 by Woodin joint with other authors (see e.g. [Por05]) and remained unanswered for a long term. Possibly the most decided attempt towards settling this problem was carried out by Gitik and Sharon, who proved the consistency of $\neg \text{SCH}_\kappa + \neg \Box^*_\kappa$ from the existence of a supercompact cardinal $\kappa$ [GS89]. In that paper, Gitik and Sharon introduced the Diagonal Supercompact Prikry -which is a generalization of the Supercompact Prikry forcing used by Magidor in [Mag78], and starting with a $\kappa^+\omega^1$-supercompact cardinal $\kappa$ produced a generic extension where there is a very good scale at $\kappa$ (see definition 2.6) and both $\text{SCH}_\kappa$ and $\Box^*_\kappa$ fail.

Little time after Gitik-Sharon’s proof, Cummings and Foremann observed that the failure of $\Box^*_\kappa$ in the Gitik-Sharon model was due to the existence of a bad scale at $\kappa$. Altogether, in Gitik-Sharon’s model there is a strong limit cardinal $\kappa$ with $\text{cof}(\kappa) = \omega$ that carries both a very good scale and a bad scale, and $\text{SCH}_\kappa$ fails. The construction of a model of $\neg \text{SCH}_\kappa + \text{TP}(\kappa^+)$ finally came from Neeman, who in [Nee99] showed that starting with $\omega$-many supercompact cardinals and combining the ideas of [GS89] with the analysis of narrow systems of [MS96], one obtains the desired model.

Following up on Neeman’s ideas, Sinapova proved in [Sin16b] that the Mitchell-like forcing of [Ung13] already yields a generic extension where $\text{TP}(\kappa^+)$ and $\text{TP}(\kappa^{++})$ both hold and $\text{SCH}_\kappa$ fails. Subsequent work of Sinapova and Unger revealed that this is also true for $\kappa = \aleph_{\omega^2}$ [SU18].

In this paper we aim to combine Sinapova’s main result from [Sin16b] with Theorem 1 of [GP18], aiming for a generic extension with a singular strong limit cardinal $\kappa$ of arbitrary cofinality $\mu$ where $\text{TP}(\kappa^+)$ and $\text{TP}(\kappa^{++})$ both hold and there is an arbitrary failure of $\text{SCH}_\kappa$. More precisely we will prove the following result.

**Theorem 1.1 (Main Theorem).** Assume the GCH. Let $\mu$ be a regular cardinal and let $\langle \kappa_\xi : \xi < \mu \rangle$ be a continuous sequence of cardinals such that each $\kappa_{\xi+1}$ is supercompact. Set $\kappa = \kappa_0$, $\delta = (\sup_{\xi < \mu} \kappa_\xi)^+$ and assume there is a weakly compact cardinal $\lambda > \delta$. Then for each cardinal $\gamma \geq \lambda$ with $\text{cof}(\gamma) > \mu$ there is a generic extension of the universe where the following holds:

1. $\kappa$ is a strong limit cardinal with $\text{cof}(\kappa) = \mu$.
2. $\delta = \kappa^+$ and $\lambda = \kappa^{++}$.
3. $2^\kappa \geq \gamma$, hence $\neg \text{SCH}_\kappa$. 
(4) TP($\kappa^+$) and TP($\kappa^{++}$) hold.
(5) There is a very good scale and a bad scale at $\kappa$.

For the proof of this result we shall make use of some ideas developed by Sinapova in [Sin16b] and [Sin08] for the proof of TP($\kappa^+$) and (5), as well as some ideas of [Ung13], [FHS18] and [GP18] for the rest of the claims.

The structure of the paper is as follows: In Section 2 we shall give an overview of Sinapova forcing following [Sin08] and we will prove that some suitable version of Mathias criterion for genericity holds (see e.g. [Git10]).

In Section 3 we will introduce our main forcing construction $R$ and prove some basic facts about the combinatorics of its generic extensions. As the reader will notice, our $R$ is essentially the same forcing as that presented in [FHS18] and [GP18] but here the Prikry/Magidor forcing is replaced for the Sinapova forcing aiming to get the above mentioned scales.

Sections 4 and 5 are devoted, respectively, to the proof of the tree property at $\kappa^{++}$ and $\kappa^+$ in the generic extension given by $R$. In Section 4 we rely on ideas of [Ung13], [FHS18] and [GP18] to show that TP($\kappa^{++}$) and (1)-(3) hold in $V^R$, while in Section 5 we show that the arguments presented in [Sin16b] adapt to our setting.

Any non defined notion is either standard or can be consulted in the references we give.

2. Preliminaries: Sinapova forcing

In this section we shall present a construction due to Dima Sinapova. We will follow Sinapova’s exposition in [Sin08] where the reader may find a complete account of all the relevant results.

Originally, Sinapova forcing (or also Diagonal Supercompact Magidor forcing) was devised to generalize Gitik-Sharon’s theorem on the consistency of $\neg$SCH$_\kappa$ + $\neg$□$_\kappa^+$ (see [GS89]) for uncountable cofinalities. Specifically, inspired also by the subsequent work of Cummings and Foremann [CF], Sinapova used this forcing to get a generic extension with the next properties:

1. There is a strong limit cardinal $\kappa$ of uncountable cofinality,
2. SCH$_\kappa$ fails,
3. There is a very good scale at $\kappa$,
4. There is a bad scale at $\kappa$.

Let $\mu$ be a regular cardinal, and let $\langle \kappa_\xi : \xi < \mu \rangle$ be a continuous sequence of cardinals such that $\kappa_0 > \mu$ is a supercompact cardinal, and for each $\xi < \mu$, $\kappa_{\xi+1}$ is also supercompact. Set $\kappa = \kappa_0$ and assume that there is a weakly compact cardinal $\lambda > \sup_{\xi < \mu} \kappa_\xi$. Let $\gamma \geq \lambda$ be a cardinal with cof($\gamma$) > $\mu$ and set $\varepsilon = \sup_{\xi < \mu} \kappa_\xi$ and $\delta = \varepsilon^+$. All the
supercompact cardinals in the sequence will be assumed to be Laver
indestructible.

For the sake of clarity we shall divide this exposition in three subsections: In the first one we succinctly introduce the notion of scale as well as other related concepts that will become important for latter sections. In the second section we give the definition of Sinapova forcing and analyze some of the combinatorial properties of its generic extension. Finally at the last section we prove a generalization of Röwbottom’s theorem for multiple supercompact measures and from this we get a geometric characterization of genericity for Sinapova forcing similar to that of Prikry forcing.

2.1. Scales. The notion of scale is the cornerstone of Shelah’s PCF theory [She94]. In this section we briefly review some basic definitions from PCF theory that will become necessary for subsequent sections. Despite PCF theory is currently of central interest for the Set theory community we decline to give here an exhaustive exposition of the topic and refer the reader to [She94], [CFM01] or [AM10] for a much more informative exposition of the area as well as for a discussion of its connections with the large cardinals theory.

**Definition 2.1.** Let $\kappa$ be a singular cardinal and $\langle \kappa_\eta : \eta < \text{cof}(\kappa) \rangle$ be an increasing and cofinal sequence of regular cardinals in $\kappa$. Let $\mathcal{I}_{bd}$ be the ideal of bounded subsets of $\text{cof}(\kappa)$. A scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ at $\kappa$ is a sequence of functions $f_\alpha \in \prod_{\eta \in \text{cof}(\kappa)} \kappa_\eta$ such that

1. For each $\alpha < \beta < \kappa^+$, $f_\alpha <_{\mathcal{I}_{bd}} f_\beta$; namely,
   $$\{ \eta \in \text{cof}(\kappa) : f_\alpha(\eta) \geq f_\beta(\eta) \} \in \mathcal{I}_{bd}.$$

2. For each $g \in \prod_{\eta \in \text{cof}(\kappa)} \kappa_\eta$ there is $\alpha < \kappa^+$ such that $g <_{\mathcal{I}_{bd}} f_\alpha$.

Attending to the fact that PCF theory is mainly concerned about the study of powers of singular cardinals it should not seem strange the restriction to them in the aforementioned definition. It is worth to mention that this definition make sense for any singular cardinal since, as noticed by Shelah [She94], all of them admit a scale. Two important concepts linked with the notion of scale are those of good point and of exact upper bound:

**Definition 2.2.** Let $\kappa$ be a singular cardinal and $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a scale at $\kappa$. An ordinal $\alpha < \kappa^+$, $\text{cof}(\kappa) < \text{cof}(\alpha) < \kappa$, is said to be a (very) good point for the scale if there is an (club) unbounded set $A \subseteq \alpha$ and an ordinal $\eta^* < \text{cof}(\kappa)$ such that

$$\forall \beta, \gamma \in A \forall \eta \geq \eta^* (\beta < \gamma \rightarrow f_\beta(\eta) < f_\gamma(\eta)).$$

If a point $\alpha$ is not good is said to be a bad point.

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5Otherwise we would prepare the universe forcing with a $\mu$-iteration $\mathbb{L}_\mu$ of Laver preparation forcing for each $\kappa_{\xi+1}$. 
Definition 2.3. Let $\kappa$ be a singular cardinal and $\langle f_\alpha : \alpha < \beta \rangle$ be an $<\text{I}_{\text{nd}}$-increasing sequence of functions in $\prod_{\eta \in \text{cof}(\kappa)} \kappa_\eta$. A function $g$ in $\prod_{\eta \in \text{cof}(\kappa)} \kappa_\eta$ is called an exact upper bound (eub) for $\langle f_\alpha : \alpha < \beta \rangle$ if

1. For all $\alpha < \beta$, $f_\alpha <_{\text{I}_{\text{nd}}} g$.
2. For all $h \in \prod_{\eta \in \text{cof}(\kappa)} \kappa_\eta$, $h <_{\text{I}_{\text{nd}}} g$, there is $\alpha < \beta$ such that $h <_{\text{I}_{\text{nd}}} f_\alpha$.

It is convenient to mention that the restriction in definition 2.2 to points of cofinality $\geq \text{cof}(\kappa)$ is not restrictive as any $\alpha$ with $\text{cof}(\alpha) < \text{cof}(\kappa)$ is easily seen to be a very good point for the scale as witnessed by $A = \alpha$. On the other hand, it is important to remark that the notion of eub for a sequence $\langle f_\alpha : \alpha < \beta \rangle$ is unique modulo $\text{I}_{\text{nd}}$.

Proposition 2.4. Assume $g, g'$ are eub for a $<_{\text{I}_{\text{nd}}}$-increasing sequence $\langle f_\alpha : \alpha < \beta \rangle$, then $g =_{\text{I}_{\text{nd}}} g'$; namely

$$\{\eta \in \text{cof}(\kappa) : g(\eta) \neq g'(\eta)\} \in \text{I}_{\text{nd}}.$$

Proof. Suppose on the contrary that $g \neq_{\text{I}_{\text{nd}}} g'$. Without loss of generality we may assume that the set $A$ of $\eta < \text{cof}(\kappa)$ such that $g(\eta) > g'(\eta)$ is unbounded. Now define $h$ as $\eta \mapsto g'(\eta) \cdot \chi_A$, where $\chi_A$ is the characteristic function at $A$. Since $h <_{\text{I}_{\text{nd}}} g$ and $g$ is an eup for $\langle f_\alpha : \alpha < \beta \rangle$ there is some $\alpha < \beta$ such that $h <_{\text{I}_{\text{nd}}} f_\alpha$ and thus, since $g'$ is an eub, also $h <_{\text{I}_{\text{nd}}} g'$. Notice however that this is impossible by the definition of $h$, yielding a contradiction with our initial assumption. 

There is a narrow connection between the notion of good point and the existence of exact upper bound for a sequence $\langle f_\alpha : \alpha < \beta \rangle$:

Proposition 2.5. Let $\kappa$ be a singular cardinal and $\beta < \kappa^+$ be an ordinal with $\text{cof}(\kappa) < \text{cof}(\beta) < \kappa$. The point $\beta$ is a good point for a scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ if and only if there is an eup $g$ for $\langle f_\alpha : \alpha < \beta \rangle$ such that $\text{cof}(g(\eta)) = \text{cof}(\beta)$, each $\eta < \text{cof}(\kappa)$.

Proof. Let $g$ be defined as $\eta \mapsto \sup_{\alpha \in A} f_\alpha(\eta)$, where $A \subseteq \beta$ is an unbounded set witnessing that $\beta$ is a good point. It is straightforward to check that $g$ is an eup with the required property so that the first implication follows. For the second implication let $g$ be an eup for $\langle f_\alpha : \alpha < \beta \rangle$ as in the claim and let $B \subseteq \beta$ be a cofinal set in $\beta$. For each $\alpha < \beta$, define $h_\alpha$ as $\eta \mapsto \sup_{\gamma \in B \cap (\alpha + 1)} f_\gamma(\eta)$. Since $\text{otp}(B \cap (\alpha + 1)) < \text{cof}(\beta)$ and for each $\eta < \text{cof}(\kappa)$, $\text{cof}(g(\eta)) = \text{cof}(\beta)$, it is not hard to check that $h_\alpha <_{\text{I}_{\text{nd}}} g$. Therefore, since $g$ is an eup, for each $\alpha < \beta$ there is an ordinal $\gamma < \beta$ such that $h_\alpha <_{\text{I}_{\text{nd}}} f_\gamma$. Let $\phi$ be the map sending each ordinal $\alpha < \beta$ to the least ordinal $\gamma$ witnessing this property. Notice by the definition of $h_\alpha$ that $\phi(\alpha) > \alpha$. Set $A = \{\alpha \in \beta : \phi''\alpha \subseteq \alpha\}$. Since $\text{cof}(\beta) > \text{cof}(\kappa)$, hence $\text{cof}(\beta) \geq \omega_1$, it is not hard to check that $A$ is unbounded. Notice that for each $\alpha < \gamma$ in $A$, $f_\alpha <_{\text{I}_{\text{nd}}} f_\gamma$. Indeed, since $\gamma \in A$ and $\alpha < \gamma$, $\phi(\alpha) < \gamma$ and thus

$$f_\alpha \leq_{\text{I}_{\text{nd}}} h_\alpha <_{\text{I}_{\text{nd}}} f_{\phi(\alpha)} <_{\text{I}_{\text{nd}}} f_\gamma.$$
Now consider a function defined on the diagonal of $A^2$ (i.e. $\{(\alpha, \gamma) \in A^2 : \alpha < \gamma\}$) sending each pair $(\alpha, \gamma)$ to the least $\eta < \text{cof}(\kappa)$ such that for each $\eta' \geq \eta$, $f_\alpha(\eta') < f_\gamma(\eta')$. Once more, since $\text{cof}(\kappa) < \text{cof}(\beta)$, there is an unbounded set $A^* \subseteq A$ where this function takes a constant value $\eta^*$. Finally it is routine to check that $A^*$ and $\eta^*$ witness that $\beta$ is a good point. □

Combining this result with proposition 2.5 it turns out that $\beta$ is a good point for a scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ if and only if every eub $g$ for $\langle f_\alpha : \alpha < \beta \rangle$ takes values with uniform cofinality modulo $\text{Ibd}$. The notion of good and very good point lead to the concept of good and very good scale:

**Definition 2.6.** Let $\kappa$ be a singular cardinal and $\langle f_\alpha : \alpha < \kappa^+ \rangle$ be a scale at $\kappa$. We say that $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is good (very good) if there is a club $C \subseteq \kappa^+$ such that each $\alpha \in C \cap \text{cof}(> \text{cof}(\kappa))$ is a (very) good point. We say that $\text{GS}_\kappa$ ($\text{VGS}_\kappa$) holds if there is a (very) good scale at $\kappa$.

For a singular cardinal $\kappa$ the principles $\square_{\kappa, \lambda}$, $\lambda < \kappa$, imply that any scale at $\kappa$ is very good. This result is optimal as $\square_{\kappa, \lambda}$ is consistent with $\neg\text{VGS}_\kappa$, modulo $\omega$-many supercompact cardinals (see [CFM01, Theorem 21]). There is a connection between the failure of $\square_{\kappa, \lambda}$ above a supercompact cardinal $\kappa$ for cardinals with $\text{cof}(\lambda) < \kappa < \lambda$ and the failure of $\text{GS}_\lambda$. This connection was discovered by Shelah whom in [She94] notice that if $\kappa$ is a supercompact cardinal and $\text{cof}(\lambda) < \kappa < \lambda$ then the principle $\text{GS}_\lambda$ fails and moreover that this failure entails the failure of $\square_{\kappa, \lambda}$. In particular this yields an alternative proof to Solovay’s theorem on the failure of weak square principles at cardinals with small cofinality above a supercompact cardinal. Once more this result is optimal since it is consistent that $\kappa$ is supercompact and $\square_{\lambda, \text{cof}(\lambda)}$ holds, hence $\text{VGS}_\lambda$ holds, for some $\kappa \leq \text{cof}(\lambda) < \lambda$ (see [CFM01, Theorem 17]).

Arguing exactly as in [Sin08, Lemma 2.7] we can prove the next auxiliary lemma:

**Lemma 2.7.** Suppose $\langle G_\beta : \beta < \delta \rangle$ is a bad scale at $\prod_{\xi < \mu} \kappa_\xi+1$. There exists an inaccessible cardinal $\theta < \kappa$ such that there are stationary many bad points $\beta < \delta$ with $\text{cof}(\beta) = \theta^{\mu+1}$.

This result along with proposition 2.11 are used to define the very good and the bad scale in the generic extension given by Sinapova forcing. For a full detailed exposition of this construction see [Sin08, Section 2.5].

### 2.2. Sinapova forcing.

Let $\mathbb{A} = \text{Add}(\kappa, \gamma)$, $G \subseteq \mathbb{A}$ generic and $\langle f_\eta : \eta \in \gamma \rangle$ be an enumeration of the generic functions added by
this filter. The next result can be found in [Sin08] but we add its proof for completeness.

**Proposition 2.8.** In $V[G]$ there is a $\gamma^+$-supercompact embedding $j : V[G] \to M$ such that for each $\eta < \delta$, $j(f_\eta)(\kappa) = \eta$ and $\kappa^{\xi^+} \leq \kappa^+$, for each $\xi < \mu$ limit.

**Proof.** Let $j : V \to M$ be a $\gamma^+$-supercompact embedding with crit($j$) $= \kappa$. Notice that by elementarity $j(\mathbb{A})$ is a $j(\kappa)$-cc and $j(\kappa)$-directed closed forcing in $M$. Since $M$ is $\gamma^+$-closed, hence $j''G$ is a directed set in $M$, and $j(\mathbb{A})$ is $j(\kappa)$-closed in $M$ we can find a master condition $p^*$ for $j''G$. In particular, since $\kappa$ is not in the range of $j$, we may assume without loss of generality that $p^*((\langle \eta \rangle, \kappa)) = \eta$, for each $\eta < \delta$. It thus remains to find a generic filter $H \subseteq j(\mathbb{A})$ such that $p^* \in H$ and $H \in V[G]$.

Since $j(\kappa)$ is inaccessible in $M$ there are at most $j(\kappa)$-many antichains of $j(\mathbb{A})$ in $M$ and thus at most $\gamma^+$-many antichains in $V[G]$. Let $\langle A_\alpha : \alpha < \gamma^+ \rangle$ be an enumeration of all these maximal antichains of $j(\mathbb{A})$ in $V[G]$. Since $M^{\gamma^+} \subseteq M$ and $j(\mathbb{A})$ is $j(\kappa)$-directed closed in $M$ we can define by induction on $\gamma^+$ a sequence of conditions $\langle p_\alpha : \alpha < \gamma^+ \rangle$ such that for each $\alpha < \gamma^+$, $p_\alpha$ is below some condition in $A_\alpha$. Let $H$ be the filter generated by the family $\langle p_\alpha : \alpha < \gamma^+ \rangle$ and notice that our construction guarantees that $H$ is generic for $j(\mathbb{A})$ and besides definable in $V[G]$. Since Cohen forcing is weakly homogeneous, standard arguments yield a generic filter $H^*$ definable in $V[G]$ with $p^* \in H^*$ so that $j$ extends to an embedding $j : V[G] \to M[H^*]$ definable in $V[G]$. Notice that $j$ is $\gamma^+$-supercompact (i.e. $M[H^*]^{\gamma^+} \subseteq M[H^*]$) because $j(\mathbb{A})$ is $j(\kappa)$-closed and $M$ was closed by $\gamma^+$-sequences. By elementarity and our choice of $p^*$ and $H^*$, $j(f_\eta)(\kappa) = \eta$, for each $\eta < \delta$. Finally the last claim follows from GCH in the ground model joint with the $\kappa$-closedness of $\mathbb{A}$. \qed

**Remark 2.9.** The existence of this family of functions $\langle f_\gamma : \eta < \gamma \rangle$ is a technical requirement to prove the next propositions and ultimately for the construction of a very good and the bad scale in the generic extension.

For the ease of notation we will assume along this section that our ground model is $V[G]$ so that on the sequel $V = V[G]$. The proof of the next two propositions can be found in [Sin08]:

**Proposition 2.10.** For each $\xi < \mu$ and each $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}_\kappa(\kappa_\xi))$ there is a supercompact measure $U_\xi$ over $\mathcal{P}_\kappa(\kappa_\xi)$ such that $\mathcal{X} \in \text{Ult}(V, U_\xi)$ and a family of functions $\langle F_\eta : \eta < \delta \rangle$ from $\kappa$ to $\kappa$ such that for each $\eta < \delta$, $j_U(\xi)(\kappa) = \eta$.

\footnote{Here we are using that GCH holds.}
Proposition 2.11. There is a $\prec$-chain of measures $\langle U_\xi : \xi < \mu \rangle$ (i.e. $U_\xi \in \text{Ult}(V, U_\xi^\prime)$, for $\xi < \xi'$) and functions $\langle F^\xi_\eta : \eta < \delta, \xi < \mu \rangle$ from $\kappa$ to $\kappa$ such that each $U_\xi$ is a supercompact measure over $\mathcal{P}_\kappa(\kappa_\xi)$ and for all $\xi < \mu$, $\eta < \delta$, $j_{U_\xi}(F^\xi_\eta)(\kappa) = \eta$.

Notation 2.12.

- If $U_\xi$ is some supercompact measure over $\mathcal{P}_\kappa(\kappa_\xi)$ we write $\mathcal{M}_\xi$ and $j_\xi$ for $\text{Ult}(V, U_\xi)$ and $j_\xi = j_{U_\xi}$, respectively.
- For each $\xi < \mu$ and each $x \in \mathcal{P}_\kappa(\kappa_\xi)$, $\kappa_x$ stands for the ordinal otp($\kappa \cap x$)\(^\#\).
- Let $\xi < \mu$ and $x \in \mathcal{P}_\kappa(\kappa_\xi)$. We denote by $\lambda_x$ the ordinal otp($x$) and not otp($\lambda \cap x$)\(^\#\).
- Let $x, y \in \mathcal{P}_\kappa(\delta)$. We write $x < y$ if $x \subseteq y$ and $\kappa_x < \lambda_y$.

Let $\mathcal{U} = \langle U_\xi : \xi < \mu \rangle$ and $\mathcal{F} = \langle F^\xi_\eta : \xi < \mu, \eta < \delta \rangle$ be respectively the sequences of measures and functions witnessing the former proposition. Since $\mathcal{U}$ is a $\prec$-chain of measures for each $\zeta < \xi < \mu$ there is a function over $\mathcal{P}_\kappa(\kappa_\xi)$, $x \mapsto U^\zeta_{\xi,x}$, representing $U_\xi$ in the ultrapower $\mathcal{M}_\xi$; i.e. $U_\xi = [x \mapsto U^\zeta_{\xi,x}]_{U_\xi}$. Restricting (if necessary) to a $U_\xi$-large set we may assume without loss of generality that each $U^\zeta_{\xi,x}$ is a supercompact measure over $\mathcal{P}_{\kappa_x}(\kappa_{\xi,x})$. To proceed in a coherent fashion along the construction we shall restrict the domain of our sets functions to the following $U_\xi$-large set:

Definition 2.13. For each $\xi < \mu$, let $X_\xi$ be the $U_\xi$-large set of $x \in \mathcal{P}_\kappa(\kappa_\xi)$ such that

(\(\alpha\)) $\kappa_x$ is an $\kappa_{\xi,x}$-supercompact cardinal above $\mu$.

(\(\beta\)) For each $\zeta \leq \xi$, $\kappa_{\xi,x}^{\kappa_{\xi,x}} \leq (\kappa_{\xi,x})^+$ and otp($x \cap \kappa_{\xi,x}^+$) = $(\kappa_{\xi,x})^+$.

(\(\gamma\)) $\kappa_x < \lambda_x$\(^\#\).

Analogously to other Prikry-type forcing, Sinapova forcing is articulated by two components: the first one (stem) is responsible of adding a generic club on $\kappa$ while the second is a sequence of measure one sets (large set part) and plays the role of supplying the stem of new extensions. Within this sort of contexts it is standard to require that all finite subsequence of the potential club (i.e. the stems given by conditions in the generic filter) are $\prec$-increasing: namely, if $x$ and $y$ correspond to two consecutive coordinates in a stem then $x < y$. Broadly speaking this restriction ensures that the stems given by the generic filter are

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\(^7\) Notice that if for any ordinal $\kappa \leq \epsilon \leq \kappa_\xi$ we define $\epsilon_x$ as otp($\epsilon \cap x$) we shall have the value at $x$ of the function representing the ordinal $\epsilon$ within the ultrapower $\mathcal{M}_\xi$.

\(^8\) Notice that the function $x \mapsto \lambda_x$ defined on $\mathcal{P}_\kappa(\kappa_\xi)$ represents $\kappa_\xi$ in the ultrapower by the supercompact measure $U_\xi$.

\(^9\) This is a promise that the selection of the $x$’s is coherent in the sense that it agrees with the fact that $\kappa < \kappa_\xi$. 

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reasonable promises for a potential club in $\kappa$ and further that the different local versions of the forcing do not exhibit interference between them. In the next pages we shall give the definition of Sinapova forcing and we will clarify what we mean by local version of the forcing. Before carrying out a formal discussion of these issues it is convenient to make some preliminary comments on the forcing.

As we have pointed out in the previous paragraph any condition $p$ in Sinapova forcing will be composed by an stem (i.e. a finite sequence $g \in \prod_{\xi \in \mu} P(\kappa_\xi)$) and a family of large sets playing the role of supplying the stem of future extensions. Let $p$ be a condition in Sinapova forcing and assume that the stem of $p$ has only one coordinate, say $\xi$, and set $x \in P(\kappa_\xi)$ be the only element of this stem. If one wants to extend the stem of $p$, say adding a new coordinate $\zeta < \xi$, at the light of the previous comments we have to make sure that the corresponding set $y \in P(\kappa_\zeta)$ satisfies the restriction $y < x$. In other words, the only possible candidates for extending the stem of $p$ from the left-side (i.e. for $\zeta < \xi$) are $y \in P(\kappa_\zeta \cap x)$. Therefore in order to extend the stem of $p$ it becomes necessary to consider not measures over $P(\kappa_\zeta)$ but measures over $P(\kappa_\zeta \cap x)$. Let us now show how these measures can be derived from our former system of measures $(\overline{U}_{\xi,x}^\zeta : \zeta < \xi, x \in X_\xi)$.

Let $\eta < \xi$ and consider $\pi^{\xi,x} : P(\kappa_\eta \cap x) \to P(\kappa_\eta)$ defined by $y \mapsto \overline{y} = \{\text{otp}(x \cap \rho) : \rho \in y\}$. Notice that $\pi^{\xi,x}$ is simply the canonical projection between $P(\kappa_\eta \cap x)$ and $P(\kappa_\eta)$. Now set $\sigma^{\eta,x} = (\pi^{\eta,x})^{-1}$ and $U_{\xi,x}^\eta = \sigma^{\eta,x}(\overline{U}_{\xi,x}^\zeta)$ be the Rudin-Keisler projection of $\overline{U}_{\xi,x}^\zeta$ by $\sigma^{\eta,x}$ (see [Tho78] Lemma 22.3 for definitions). Standard arguments show that $\sigma^{\eta,x}(\overline{U}_{\xi,x}^\zeta)$ yields a supercompact measure on $P(\kappa_\eta \cap x)$ (see [Tho78] Exercise 7.5). It is convenient to mention that the new collection of measures $(U_{\xi,x}^\eta : \eta < \xi \leq \mu, x \in X_\xi)$ forms a coherent system in a similar sense to that of proposition 2.11. More precisely the following is proved in [Sin08] Section 2.2:

$(\mu)$ For each $\rho < \zeta < \xi < \mu$ and for $U_\xi$-many $x$’s, $U_{\xi,x}^\rho < U_{\xi,x}^\zeta$.

$(\mu^*)$ For each $\xi < \mu$,

$$B_\xi = \{x \in X_\xi : (\forall \zeta, \eta) \xi < \eta < \zeta, U_{\xi,x}^\zeta = [y \mapsto \overline{U}_{\eta,y}^\zeta] U_{\xi,x}^\eta \} \in U_\xi.$$

$(\ast)$ For $\zeta < \xi$ and $A \in U_\eta$,

$$\forall U_\xi x (A \cap P(\kappa_\xi \cap \kappa_\zeta)) \in U_{\xi,x}^\eta.$$

$(\diamondsuit)$ For each $\zeta < \eta < \xi$, $z \in B_\xi$ and $A \in U_{\xi,z}^\zeta$,

$$\forall U_{\xi,z} x (A \cap P(\kappa_\xi \cap \kappa_\eta)) \in U_{\eta,x}^\zeta.$$
for extending the stem of \( p \) at a coordinates \( \eta, \zeta < \eta < \xi \), provided \( \zeta \) and \( \xi \) were already coordinates of the stem of \( p \).

At this point we are in conditions to give the definition of Sinapova forcing:

**Definition 2.14** (Sinapova forcing). Under the conditions described so far we define the Sinapova forcing with respect to the triple \( (\kappa, \mu, \mathcal{U}) \) to be the partial order \( S_{(\kappa, \mu, \mathcal{U})} \) of conditions \( \langle g, H \rangle \) such that:

1. \( \text{dom}(g) \in [\mu]^{<\omega} \) and \( \text{dom}(H) = \mu \setminus \text{dom}(g) \).
2. For each \( \xi \in \text{dom}(g) \), \( g(\xi) \in B_\xi \) and \( \kappa_{g(\xi)} > \theta^+ \). Moreover, for each \( \zeta < \xi \) in \( \text{dom}(g) \), \( g(\zeta) < g(\xi) \).
3. For each \( \xi \in \text{dom}(H) \), \( H(\xi) \in U_\xi \) and \( H(\xi) \subseteq B_\xi \). Moreover, if \( \xi < \max \text{dom}(g) \) then setting \( \xi = \min(\text{dom}(g) \setminus \xi + 1) \) and \( x = g(\xi) \), \( H(\xi) \in U_\xi^x \).
4. Say \( \xi = \max(\text{dom}(g)) \). For each \( \zeta \in \text{dom}(H) \) and every element of \( x \in H(\xi) \), \( g(\xi) < x \).

Given a condition \( p = \langle g, H \rangle \) we respectively say that \( g \) is the stem and \( H \) the large set part of \( p \).

The order of \( S \) is defined as follows:

**Definition 2.15.** Let \( p, q \in S \).

(a) \( p \preceq q \) iff

1. \( g^p \supseteq g^q \),
2. If \( \xi \in \text{dom}(g^p) \setminus \text{dom}(g^q) \) then \( g^p(\xi) \in H^q(\xi) \),
3. If \( \xi \notin \text{dom}(g^p) \), \( H^p(\xi) \subseteq H^q(\xi) \),

(b) \( p \preceq^* q \) iff \( p \preceq q \) and both conditions have the same stem.

If \( p \) and \( q \) have the same stem we denote by \( r = p \land q \) the condition with the same stem as \( p \) and whose large set part is given by the function \( H^p \land H^q, \xi \mapsto H^p(\xi) \cap H^q(\xi) \) defined on \( \text{dom}(H^p) \cap \text{dom}(H^q) \).

Let \( S \subseteq S \) be a generic filter for the Sinapova forcing and set \( g^* = \bigcup_{p \in S} g^p \).

**Proposition 2.16** (Some properties of \( S \)).

1. \( S \) is \( \delta \)-cc hence cardinals \( \geq \delta \) are preserved.
2. The function \( g^* \) have domain \( \mu \).
3. For each \( \xi < \mu \), set \( \kappa_\xi = \kappa_{g^*(\xi)} \). Then \( \langle g^*(\xi) : \xi < \mu \rangle \) is a \( \prec \)-increasing and \( \subseteq \)-continuous sequence such that \( \bigcup_{\xi < \mu} g^*(\xi) = \varepsilon \).
4. \( \langle \kappa_\xi : \xi < \mu \rangle \) defines a club subset of \( \kappa \), hence \( \text{cof}^{V[S]}(\kappa) = \text{cof}^{V[S]}(\mu) \),

\(^{10}\text{Formally this definition depends also of the functions representing the different measures, the } B_\zeta\text{'s, etc.}\)

\(^{11}\text{This requirement is technical being necessary for the construction of the bad and the very good scale in the generic extension.}\)
Proof.

(1) Notice that if \( p \) and \( q \) are two conditions with the same stem then they are compatible. Thus the size of any antichain of \( S \) is at most the number of possible stems; namely, \( \delta \).

(2) To prove that \( \text{dom}(g^*) = \mu \) we shall show that for each \( \xi \in \mu \),
\[
D_\xi = \{ \langle g, H \rangle : \xi \in \text{dom}(g) \}\]
is a dense set in \( S \). Let \( p = \langle g, H \rangle \in S \) with \( \xi \notin \text{dom}(g) \) and set \( \zeta = \max(\text{dom}(g) \cap \xi) \). We have to discuss two different cases: either \( \text{dom}(g) \setminus \xi = \emptyset \) or not. Assume we are in the first case. By condition \((\ast)\) for each \( \zeta < \eta < \xi \),
\[
\forall_{U, x} H(\eta) \cap P_{\kappa_\eta}(x \cap \kappa_\eta) \in U^\eta_{\xi, x}.
\]
Let \( Z^\eta_\eta \) be the set of \( x \)'s witnessing the displayed formula and set \( \overline{H}(\eta) = H(\eta) \cap P_{\kappa_\eta}(x \cap \kappa_\eta) \). By the \( \kappa \)-completeness of \( U_\xi \) the set \( H(\xi) \cap \bigcap_{\zeta < \eta < \xi} Z^\eta_\eta \) is \( U_\xi \)-large and thus there is some \( x \in H(\xi) \) such that for each \( \zeta < \eta < \xi \), \( \overline{H}(\eta) \in U^\eta_{\xi, x} \). Now let \( p^* = \langle g^*, H^* \rangle \) with \( g^* = g \cup \{ (\xi, x) \} \), \( H^* \upharpoonright (\text{dom}(g) \setminus (\xi, \xi)) = H \) and \( H^* \upharpoonright (\xi, \xi) = \overline{H} \). Notice that \( p^* \leq p \) and \( p^* \in D_\xi \).

Now assume we are in the opposite situation and set \( \zeta^* = \min(\text{dom}(g) \setminus \xi) \). By condition \((\diamond)\) for each \( \zeta < \eta < \xi \) the following holds:
\[
\forall_{U^\xi_{\eta, g^*(\zeta)}} H(\eta) \cap P_{\kappa_\eta}(\kappa_\eta \cap x) \in U^\eta_{\xi, x}.
\]
Let \( Z^\xi_\eta \) be the \( U^\xi_{\eta, g^*(\zeta^*)} \)-large set of \( x \) witnessing the displayed formula and set \( \overline{H}(\eta) = H(\eta) \cap P_{\kappa_\eta}(\kappa_\eta \cap x) \). Since \( U^\xi_{\eta, g^*(\zeta^*)} \) is a \( \kappa^* \)-complete measure and \( \mu < \kappa^* \) (cf definition 2.13) then the set \( H(\xi) \cap \bigcap_{\zeta < \eta < \xi} Z^\eta_\eta \) is also \( U^\xi_{\eta, g^*(\zeta^*)} \)-large so that there is some \( x \in P_{\kappa_\eta}((\xi \cap g^*(\zeta^*))) \) such that for each \( \zeta < \eta < \xi \), \( \overline{H}(\eta) \in U^\eta_{\xi, x} \). Finally define \( p^* = \langle g^*, H^* \rangle \) as before and notice that \( p^* \leq p \) and \( p^* \in D_\xi \). Altogether this shows that \( D_\xi \) is a dense set and thus that \( g^* \) has domain \( \mu \).

(3) The sequence \( \langle g^*(\xi) : \xi < \mu \rangle \) is clearly \( \leq \)-increasing so it will suffice to show that is \( \subseteq \)-continuous and \( \bigcup_{\xi \in \mu} g^*(\xi) = \varepsilon \).
We shall simply present the proof of the \( \subseteq \)-continuity since \( \bigcup_{\xi \in \mu} g^*(\xi) = \varepsilon \) follows from a similar argument.

Let \( \xi < \mu \) be a limit ordinal, \( p \in S \) with \( \xi \in \text{dom}(g^p) \) and \( \gamma \in g^p(\xi) \). We shall prove that there is some \( \eta < \xi \) such that \( \gamma \in g^*(\eta) \) by showing that
\[
D = \{ \langle g, H \rangle \in S : \exists \eta < \xi, \gamma \in g(\eta) \}.
\]
is a dense set below the condition \( p \). Let \( q = \langle g, H \rangle \) be some condition below \( p \) and set \( \zeta = \max(\text{dom}(g) \cap \xi) \). Since \( \gamma \in g^*(\xi) \) there is some \( \zeta < \eta < \xi \) such that \( \gamma \in \kappa_\eta \cdot g^*(\xi) \). For each
\( \zeta < \rho < \eta \) let \( Z_\rho \) be the \( U^\eta_{\xi, g^*(\xi)} \)-large set of witnesses of

\[
\forall U^\eta_{\xi, g^*(\xi)} x \ H(\rho) \cap P_{\kappa, \rho}(\kappa, \rho) \cap x \in U^\rho_{\eta, x}.
\]

and \( \overline{H}(\rho) = H(\rho) \cap P_{\kappa, \rho}(\kappa, \rho) \cap x \). The measure \( U^\eta_{\xi, g^*(\xi)} \) is \( \kappa^{\eta, \rho} \)-complete, hence \( \eta \)-complete, so that \( Z = \bigcap_{\xi < \rho < \eta} Z_\rho \) is a \( U^\eta_{\xi, g^*(\xi)} \)-large set. On the other hand \( \overline{P}^\eta_{\xi, g^*(\xi)} \) is a supercompact measure on \( P_{\kappa, \rho}(\kappa, \rho) \), hence a fine measure, and \( \gamma \in \kappa^{\eta, \rho} \) so that \( \{ x \in Z : \gamma \in x \} \in U^\eta_{\xi, g^*(\xi)} \). In particular there is a set \( x \) such that \( \gamma \in x \) and such that \( \overline{H}(\rho) \in U^\eta_{\xi, g^*(\xi)} \), each \( \zeta < \rho < \eta \). Defining \( \langle g', H' \rangle \) as \( g' = g \cup \{ \langle \eta, x \rangle \} \), \( H' \upharpoonright (\text{dom}(H) \setminus (\zeta, \mu)) = H \) and \( H' \upharpoonright (\zeta, \mu) = \overline{H} \), we get a condition \( p' \leq q, p' \in D \). This shows that \( D \) is dense below \( p \).

(4) This follows straightforwardly from (3).

\[ \square \]

**Notation 2.17.** Let \( x \in P_{\kappa, \epsilon} \) and \( \kappa \leq \eta \leq \epsilon \). We write \( \eta_{x} \) for the ordinal \( \text{otp}(\eta \cap x) \).

**Corollary 2.18.** Let \( \kappa \leq \eta \leq \epsilon \) be a regular cardinal. The sequence \( \langle \eta_{\gamma^*(\xi) : \xi < \mu} \rangle \) defines a club on \( \eta \). In particular, any cardinal \( \kappa \leq \eta \leq \epsilon \) with \( \text{cof}(\eta) \geq \mu \) changes its cofinality to \( \text{cof}(\text{V}[S])(\mu) \). In particular any cardinal \( \kappa < \eta \leq \epsilon \) collapses.

**Proof.** Since \( \langle g^*(\xi) : \xi < \mu \rangle \) covers \( \epsilon \), \( \bigcup_{\xi < \mu}(g^*(\xi) \cap \eta) = \eta \), hence \( \langle \eta_{\gamma^*(\xi) : \xi < \mu} \rangle \) is unbounded in \( \eta \). On the other hand, it is not hard to check from the of the of the continuity of the sequence \( \subseteq \)-continuity of \( \langle g^*(\xi) : \xi < \mu \rangle \) the continuity of \( \langle \delta_{\gamma^*(\xi) : \xi < \mu} \rangle \). For the moreover part notice that since there are no regular cardinals within the interval \( (\kappa, \epsilon] \) in fact there are no cardinals at all.

One of the key features of Sinapova forcing, shared also with other Prikry-type forcing like Magidor forcing or Radin forcing, is its **reflecting structure**: the forcing below any condition \( p \) (i.e. a local versions of \( S \)) can be resembled from finitely many products of smaller Sinapova forcing. More formally, given any condition \( p \), \( S/p \) may be essentially regarded as a finite product \( \prod_{\langle \xi \rangle \leq \text{dom}(g^p) \rangle} S_\xi \), where \( S_\xi \) is a Sinapova forcing with respect to certain sequence of measures derived from \( p \). This remarkable property of \( S \)- and in fact of any Prikry-type forcing adding a non-countable club- is crucial for carrying out a successful analysis of the combinatorics of \( V[S] \).

Let \( p \in S \) be such that \( g^p = \{ \langle \xi, x \rangle \} \), for some limit \( \xi < \mu \). By definition for each \( \eta < \xi \), \( H(\eta) \in U^\eta_{\xi, x} \). We aim to show that any left-side extensions of \( p \) (i.e. \( q \leq p \) with \( \eta \in \text{dom}(g^q) \cap \xi \)) is a condition in a smaller Sinapova forcing.

By condition \( (\mu) \) the measures \( \langle U^\eta_{\xi, x} \rangle \) are supercompact and \( \mu \)-increasing and thus the same is also true for the collapsed measures
(\overline{\mathcal{U}}_{\xi,x}^n : \eta < \xi). For each \eta < \xi, set \mathcal{V}_\eta = \overline{\mathcal{U}}_{\xi,x}^n and let \overline{H}(\eta) be the natural restriction of \overline{H}(\eta) to \mathcal{P}_{\kappa_x}(\kappa_{\eta} \cap x) (i.e. \overline{H}(\eta) = \{\text{otp}(\zeta \cap x) : \zeta \in y \} : y \in H(\eta)) which is a \mathcal{V}_\eta\text{-large set. Proceeding in the same fashion as before one finds sets } b_\eta \subseteq \overline{H}(\eta) \text{ and a coherent system of measures } \langle \mathcal{V}_{\eta,y}^\zeta : \zeta < \eta < \xi, y \in \mathcal{P}_{\kappa_x}(\kappa_{\eta} \cap x) \rangle \text{ satisfying } (\mu)-(\diamond) \text{ so that letting } \mathfrak{W} = \langle \mathcal{V}_\eta : \eta < \xi \rangle \text{ one can define the corresponding Sinapova forcing, } S_{(\kappa_x,\mu,\mathfrak{W})}.

Given an ordinal \eta < \mu, if \eta < \xi let \overline{H}^*(\eta) be the natural lifting to \mathcal{P}_{\kappa_x}(\kappa_\eta \cap x) of \overline{b}_\eta \text{ while if } \eta > \xi \text{ we set } \overline{H}^*(\eta) = \overline{H}(\eta). \text{ Define } \overline{p} \text{ as the natural } \leq^*\text{-extension of } p \text{ defined using } \overline{H}^\ast. \text{ Letting } k \in \omega \text{ we denote respectively by } \overline{p}^{<\zeta} \text{ and } \overline{p}^{\geq \zeta + k} \text{ the conditions } \langle g \restriction \xi, \overline{H}^\ast \restriction \xi \rangle \text{ and } \langle g \restriction (\mu \setminus \xi + k), \overline{H}^\ast \restriction (\mu \setminus \xi + k) \rangle, \text{ hence } \overline{p} = \overline{p}^{<\zeta} \sim \overline{p}^{\geq \zeta + 1}. \text{ The previous discussion in fact yields to the proof of the next proposition:}

**Proposition 2.19.** Let } p \in S \text{ such that } g = \{(\xi,x)\}, \text{ some } \xi < \mu \text{ limit. Then there is an isomorphism between } S/\overline{p} \text{ and }

\[ S_{(\kappa_x,\mu,\mathfrak{W})}/(\overline{p}^{\xi}) \times S_{(\kappa,\mu,\mathfrak{U}\setminus\xi+1)}/(\overline{p}^{\geq \xi+1}) \]

where

(1) \( S_{(\kappa_x,\mu,\mathfrak{W})} \) is the Sinapova forcing defined using

(a) The measures \( \mathfrak{W} = \langle \mathcal{V}_\eta : \eta < \xi \rangle \),

(b) The sets of coherence \( \langle b_\eta : \eta < \xi \rangle \),

(c) The functions \( y \mapsto \bigtriangleup^\zeta_{\eta,y} \), for \( \zeta < \eta < \xi \),

(2) \( S_{(\kappa,\mu,\mathfrak{U}\setminus\xi+1)} \) is the Sinapova forcing defined using

(a) The measures \( \mathfrak{U}\setminus\xi+1 = \langle \mathcal{U}_\eta : \xi < \eta < \mu \rangle \),

(b) The sets of coherence \( \langle B_\eta : \eta < \xi \rangle \),

(c) The functions \( y \mapsto \bigtriangleup^\zeta_{\xi,y} \), for \( \xi < \zeta < \eta < \mu \).

**Remark 2.20.** In the above stated conditions if \( \xi \) is a successor ordinal a similar factorization holds though the first factor is not a Sinapova forcing. Nonetheless it should be clear that this forcing has cardinality less than \( \kappa_x \).

**Notation 2.21.** Let } p \in S \text{ with } \{\xi + \ell : \ell \leq k\} \subseteq \text{dom } (g^p), \xi < \mu \text{ limit.

- We respectively denote by } S_{\xi,\mathfrak{g}^p(\xi)} \text{ and } \mathcal{S}^{\xi+1,\mathfrak{g}^p(\xi)} \text{ the forcings } \mathcal{S}_{(\kappa_x,\mu,\mathfrak{W})}/(\overline{p}^{<\zeta}) \text{ and } S_{(\kappa,\mu,\mathfrak{U}\setminus\xi+1)}/(\overline{p}^{\geq \xi+1})\text{.}

- We denote by } \mathcal{S}^{\xi+k+1,\mathfrak{g}^p(\xi+k)} \text{ the Sinapova forcing } S_{(\kappa,\mu,\mathfrak{U}\setminus(\xi+k)+1)}/(\overline{p}^{\geq \xi+k+1})\text{.}

- For a generic filter } S \subseteq S, \overline{p} \in S, S \restriction \xi \text{ stands for the image of } S \text{ by the natural projection between } S/\overline{p} \text{ and } S_{\xi,\mathfrak{g}^p(\xi)}\text{.}

The former arguments can be straightforwardly adapted to proof a more general version of the previous factorization:
Proposition 2.22 (Factorization lemma). Let \( p = \langle g, H \rangle \in S \) and \( \xi \) be a limit ordinal with \( \{\xi + \ell : \ell \leq k\} \subseteq \text{dom}(g) \), some \( k \in \omega \). Then there is an isomorphism
\[
\Theta_\xi : S/p \rightarrow S_{\xi,g(\xi)}^{k+1,g(\xi+k)}.
\]

Sinapova forcing satisfies the Prikry property, as shown by the following results whose proofs can be found in Proposition 2.13 and Corollary 2.14 of \cite{Sinapova}:

Proposition 2.23 (Prikry property). Let \( p = \langle g, H \rangle \in S \) and \( \Phi \) be a sentence in the forcing language. Then there is a condition \( \langle g, H \rangle \leq \langle g, H' \rangle \) deciding \( \Phi \).

Proposition 2.24. Let \( \langle g, H \rangle \in S \), \( \xi \in \text{dom}(g) \), \( \xi \) limit, and let \( \Phi \) be a statement in the forcing language. Then there is a condition \( \langle g, H' \rangle \leq \langle g, H \rangle \), such that if \( \langle j, J \rangle \leq \langle g, H \rangle \) decides \( \Phi \), then \( \langle j \upharpoonright \xi, J \upharpoonright \xi \rangle \models \langle g \upharpoonright (\mu \setminus \xi), H' \upharpoonright (\mu \setminus \xi) \rangle \) decides \( \Phi \).

Before analyzing the combinatorics of the generic extension we need to investigate the properties of the previous forcing.

Lemma 2.25. Let \( p = \langle g, H \rangle \) and \( \xi < \mu \) be a limit ordinal with \( \{\xi + \ell : \ell \leq k\} \subseteq \text{dom}(g) \), some \( k \in \omega \). Then,
\[
\begin{align*}
(1) \ S_{\xi,g(\xi)} & \text{ is } \kappa_{\xi,g(\xi)}^+ -\text{cc}. \\
(2) \ S_{\xi[g(\xi)]}^{k+1,g(\xi+k)} & \text{ is } \kappa_{g(\xi+k)}^+ -\text{closed. In particular, since the forcing } S_{\xi[g(\xi)]}^{k+1,g(\xi+k)} \text{ has the Prikry property it does not add new subsets to } \kappa_{g(\xi+k)}^+ \text{ in } V[S \upharpoonright \xi].
\end{align*}
\]

Proof. As pointed out in lemma 2.19, \( S_{\xi,g(\xi)} \) is the Sinapova forcing with respect the sequence \( \mathcal{U} \) below \( \bar{p}^{\xi} \), hence the degree of cc-ness depends on the number of different stems. On this respect notice that there are at most \( \text{sup}_{\eta \in (\kappa_{\eta,g(\eta)})^{<\kappa_{g(\xi)}}} \) many different stems and thus, using (\beta) from definition 2.13 at most \( \kappa_{g(\xi)}^+ \) many of them. In particular, \( S_{\xi,g(\xi)} \) is \( \kappa_{g(\xi)}^+ -\text{cc} \).

For the second claim, let \( \{p_\nu\}_{\nu \in \kappa_{\eta,g(\xi+k)}} \) be a \( \leq^* \)-decreasing family of conditions in \( S_{\xi[k+1,g(\xi+k)]} \), the Sinapova forcing with respect the measure \( \mathcal{U} \setminus \xi + 1 \) below \( \bar{p}^{\xi+k+1} \). Since the argument is similar for any finite stem we shall assume for simplicity that \( g^{p_0} = \{\langle \zeta, x \rangle \} \), some \( \xi + k + 1 \leq \zeta < \mu \). Notice that all the measures defining these conditions are \( \kappa_x \)-complete hence, since \( \kappa_{g(\xi+k)} < \kappa_x \) (c.f. definition 2.14), in particular \( \kappa_{g(\xi+k)}^+ \)-complete measures.

For each \( \eta, \xi + k < \eta < \zeta \) (resp. \( \eta > \zeta \)), define
\[
H'(\eta) = \bigcap_{\nu \in \kappa_{\eta,g(\xi+k)}} H^{p_\nu}(\eta),
\]

where
noticing that \( H'(\eta) \) is a \( U^\eta_{\rho_{\xi}} \)-large set (resp. \( U_\eta \)-large set). Finally defining \( p' = \langle g', H' \rangle \) with \( g' = g^{p_0} \) and \( H' \) as before we get a condition such that \( p' \leq^* p_\nu \), all \( \nu < \kappa_{g(\xi_0 + \eta)} \).

The previous results can be applied to analyze the combinatorics of the generic extension:

**Proposition 2.26.** For each ordinal \( \rho < \kappa \), \( \mathcal{P}(\rho)^{V[S]} = \mathcal{P}(\rho)^{V[S][\xi]} \), where \( \xi \) is the least limit ordinal such that \( \kappa^*_\xi \leq \rho < \kappa^*_\xi + 1 \). Moreover, if \( \rho < \kappa^*_0 \), \( \mathcal{P}(\rho)^{V[S]} = \mathcal{P}(\rho)^V \).

**Proof.** Let \( \rho < \kappa \) and \( \xi \) be the least limit ordinal such that \( \kappa^*_\xi \leq \rho < \kappa^*_\xi + 1 \). This choice of \( \xi \) is possible since \( \langle \kappa^*_\xi : \xi < \mu \rangle \) is a club in \( \kappa \). Let \( \tau \) be a \( \mathcal{S} \)-name for a subset of \( \rho = \langle g, H \rangle \in S \) with \( \{ \xi, \xi + 1 \} \subseteq \text{dom}(g^\rho) \) and \( p \models \mathcal{S} \tau \subseteq \rho \). For each \( \nu < \rho \) denote by \( \Phi_\nu \) the sentence \( "\nu \in \tau" \) and let \( \langle j_\nu, J_\nu \rangle \leq \langle g, H \rangle \) be a condition in \( S \) deciding \( \Phi_\nu \). Using proposition 2.24 for each \( \nu < \rho \) we get a function \( H_\nu \) such that \( \langle g, H_\nu \rangle \leq^* \langle g, H \rangle \) and

\[
\langle j_\nu \restriction \xi, J_\nu \restriction \xi \rangle \models \langle g \restriction (\mu \setminus \xi), H_\nu \restriction (\mu \setminus \xi) \rangle || \Phi_\nu.
\]

In particular, there is some \( p_\nu \in S \restriction \xi \) such that \( p_\nu \models \langle g \restriction (\mu \setminus \xi), H_\nu \restriction (\mu \setminus \xi) \rangle || \Phi_\nu \). For each \( \eta \in \xi \setminus \text{dom}(g) \) set \( H^*(\eta) = H(\eta) \) while for \( \eta > \xi \), \( \eta \notin \text{dom}(g) \), \( H^*(\eta) = \bigcap_{\nu < \rho} H_\nu(\eta) \). This last definition make sense since \( \eta \) is necessarily greater than \( \xi + 1 \) and the forcing \( \mathcal{S}_{\xi+2, g^\xi(\xi+1)} \) is \( (\kappa^*_\xi + 1)^+ \)-closed. This yields a condition \( p^* = \langle g, H^* \rangle \leq^* \langle g, H_\nu \rangle \), for each \( \nu < \rho \).

Let

\[
a = \{ \nu < \rho : \exists p \in S \restriction \xi, \, p \models \langle g \restriction (\mu \setminus \xi), H^* \restriction (\mu \setminus \xi) \rangle || \nu \in \tau \}.
\]

It is clear that \( a \in V[G \restriction \xi] \) and that \( p^* \models \mathcal{S} \tau = a \), by construction. Since \( p^* \leq p \), and the choice of \( p \) was arbitrary, usual density arguments guarantee that \( \tau_S \in V[S \restriction \xi] \) and thus that \( \mathcal{P}(V[S]) \subseteq \mathcal{P}(V[S][\xi])(\rho) \).

The moreover part follows from a similar argument noticing that all the measures defining \( \mathcal{S}/S \) are \( \kappa^*_0 \)-complete. \( \Box \)

**Proposition 2.27** (Preservation of cardinals below \( \kappa \)). In the generic extension \( V[S] \) all the cardinals \( \leq \kappa \) are preserved. Moreover, for each \( \rho < \kappa \), \( \text{cof}(\rho) = \text{cof}^V(\rho) \).

**Proof.** Let \( \rho < \kappa \) be a cardinal. If \( \rho \geq \kappa^*_0 \) by proposition 2.26 it is clear that \( \rho \) remains a cardinal in \( V[S] \) and moreover that \( \text{cof}^V(S)(\rho) = \text{cof}^V(\rho) \). Now assume for the contrary that \( \kappa^*_0 < \rho \) and let \( \xi \) be the least limit ordinal such that \( \kappa^*_\xi \leq \rho < \kappa^*_\xi + 1 \). Again, by proposition 2.26 \( \rho \) is a cardinal in \( V[S] \) if and only if it is a cardinal in \( V[S \restriction \xi] \), hence we shall focus our discussion on whether \( \mathcal{S}_{\xi, g^\xi(\xi)} \) collapses or not \( \rho \). On this regard, we have proved in proposition 2.25 that \( \mathcal{S}_{\xi, g^\xi(\xi)} \) is \( (\kappa^*_\xi)^+ \)-cc so that if \( \kappa^*_\xi < \rho \) then \( \rho \) is preserved. Otherwise, assuming by induction that \( \kappa^*_{\eta+1} \) is preserved for each \( \eta < \xi \), and since \( \kappa^*_\xi = \sup_{\eta \in \xi} \kappa^*_\eta + 1 \), it
turns out that $\rho$ is preserved. A similar argument also proves that $\kappa$ is preserved as well as the claim about the cofinalities.

**Proposition 2.28** (Preservation of cardinals above $\kappa$). Let $\kappa \leq \rho$ be a cardinal in $V$, then the following statements hold in $V[S]$:

1. $\kappa$ is a strong limit cardinal with $\text{cof}(\kappa) = \mu$.
2. If $\delta \leq \rho$, then $\rho$ is a cardinal and $\text{cof}(\rho) = \text{cof}^V(\rho)$.
3. If $\kappa \leq \rho \leq \varepsilon$ is regular, $\text{cof}(\rho) = \mu$.

In particular, in $V[S]$ all cardinals in $(\kappa, \varepsilon]$ are collapsed and $\delta = \kappa^+$.

**Proof.** The result follows combining proposition 2.16, corollary 2.18, proposition 2.26 and proposition 2.27.

The next result summarizes the main properties of $V[S]$:

**Theorem 2.29** (Sinapova). The following statements hold in $V[S]$:

1. $\kappa$ is a strong limit cardinal with $\text{cof}(\kappa) = \mu$ and $2^\kappa \geq \gamma$, hence SCH fails.
2. There is a bad scale at $\kappa$.
3. There is a very good scale at $\kappa$.

**Proof.** (1) has already been proved. For (2) and (3) one has to apply the ideas developed in [Sin08, Section 2.5] to the bad scale $\langle G_\beta : \beta < \delta \rangle$, the functions $\langle F^{\xi}_\eta : \eta < \delta, \xi < \mu \rangle$ (cf propositions 2.7 and 2.11) and the generic sequence $\langle g^*(\xi) : \xi < \mu \rangle$ in order to define a bad and a very good scale in $V[S]$.

2.3. **Mathias criterion for Sinapova forcing:** In this section we shall prove that, similarly to what occurs with other Prikry-type forcings, there is a simple geometric criterion characterizing the generic extensions of Sinapova forcing. Before giving any detail concerning this result let us fix some notation and motivate the need for proving this. Let $\kappa$ be a measurable cardinal, $\mathcal{U}$ be a normal measure on $\kappa$ and let $P_{\mathcal{U}}$ be the Prikry forcing with respect to $\mathcal{U}$ [Git10]. If $G \subseteq P_{\mathcal{U}}^{12}$ is a generic filter

\[ \vec{c} = \bigcup \{ s : \exists A \in \mathcal{U} \langle s, A \rangle \in G \} \]

defines a cofinal $\omega$-sequence in $\kappa$, hence $\text{cof}^{V[G]}(\kappa) = \omega$, called Prikry sequence or Prikry real. Reciprocally from $\vec{c}$ one can recover the generic filter $G$ noticing that

\[ G = \{ \langle s, A \rangle \in P_{\mathcal{U}} : s \triangleleft \vec{c}, \vec{c} \setminus (\text{max}(s) + 1) \subseteq A \}. \]

Altogether this yields $V[\vec{c}] = V[G]$ and thus any extension by $P_{\mathcal{U}}$ is completely determined by a Prikry sequence. Consequently seek for a

\[ ^{12} \text{At previous sections we have used } G \text{ as a generic filter for } \mathcal{A}(\kappa, \gamma) \text{ but since at the present section this forcing is not going to appear we will make this slightly abuse of notation.} \]
criterion that allows to decide whether some sequence is or not a Prikry sequence seems more than reasonable provided one aims to understand Prikry extensions. Analogously to the case of Cohen and Random forcing (see [BJ95]) there is a simple property that characterizes a Prikry sequence: a cofinal $\omega$-sequence $\vec{c} \subseteq \kappa$ is a Prikry sequence for $P_U$ if and only if

$$U = \{ A \subseteq \kappa : \exists n \in \omega \forall m \geq n \ (\vec{c}(m) \subseteq A) \}.$$  

This geometric criterion (Mathias criteria) was firstly noticed by Mathias in [Mat73] and it is very illustrative on what generic extensions of $P_U$ are concerned. For instance, Mathias criteria implies that any infinite subsequence of a Prikry sequence also yields a Prikry sequence and moreover that for any two normal measures $U \subseteq U'$ any Prikry sequence for $P_{U'}$ is also a Prikry sequence for $P_U$. In particular, since any generic extension by Prikry forcing corresponds with some $V[\vec{c}]$, Mathias criterion shows that any generic filter for $P_{U'}$ is a generic filter for $P_U$, hence there is a projection between both posets (see e.g. [Kun14]). This technical result regarded within the context of Sinapova forcing will be crucial along section 3 when we present the main forcing construction of the paper. Similar characterizations for more complex Prikry-type forcing as Magidor or Radin forcing are also known and attributed to Mitchell [CW90]. These kind of geometric criteria are highly useful when working with Prikry-type forcings as them indirectly give an explicit procedure to construct definable generic filters. To be more concrete, one can cook up generic filters for Prikry or Radin forcing using iterated ultrapowers (see [Kan08] and [CW90]) and take profit of this construction for lifting arguments. Examples of applications of this property in the preservation of large cardinals can be found in [CW90] or in [HMP17] where the phenomenon of identity crises is discussed in the region encompassed between the first supercompact cardinal and Vopěnka’s Principle.

The strategy of our proof is summarized as follows: First we shall prove a generalization of the usual Röwbottom’s lemma [Kan08] to the setting of $\mu$-many supercompact measures and afterwards we use this result to prove that $S$ has the Mathias-Prikry property (cf definition 2.33). Finally combining the normality of the measures with the Mathias-Prikry property we shall prove a suitable version of the Mathias criterion for $S$.

2.3.1. Röwbottom’s lemma for supercompact measures. Along this section $\langle \theta_\xi : \xi < \mu \rangle$ will be an increasing sequence of cardinals above $\kappa$.

Notation 2.30. We denote by $[\prod_{\xi \in \mu} P_\kappa(\theta_\xi)]$ the set of all $\prec$-increasing sequences of $\prod_{\xi \in \mu} P_\kappa(\theta_\xi)$. Similarly $[\prod_{\xi \in \mu} P_\kappa(\theta_\xi)]^{[\kappa]}$ will denote the set
Lemma 2.31. Let $\mu < \kappa$ and $\langle U_\xi : \xi < \mu \rangle$ be a collection of supercompact measures over $P_\kappa(\theta_\xi)$, $\xi < \mu$. Then for each function $c : [\prod_{\xi<\mu} P_\kappa(\theta_\xi)]^{[<\omega]} \to \varnothing$, $\varnothing < \kappa$, there is $H^* \in \prod_{\xi<\mu} U_\xi$ homogeneous for $c$; namely, for each $n \in \omega$ and each $s \in [\mu]^n$, $c \restriction [\prod_{\xi \in s} H^*(\xi)]$ takes a constant value.

Proof. We prove by induction over $n \in \omega$ that for each function $\bar{c} : [\prod_{\xi<\mu} P_\kappa(\theta_\xi)]^{[n]} \to \varnothing$ and for each $s \in [\mu]^n$ there is a sequence $H^s = \langle H^s_\xi : \xi \in s \rangle$, $H^s_\xi \in U_\xi$ each $\xi < \mu$, such that $c \restriction [\prod_{\xi \in s} H^s(\xi)]$ is constant.

If $n = 1$ for each $\bar{c} : [\prod_{\xi<\mu} P_\kappa(\theta_\xi)]^{[1]} \to \varnothing$ and each $s \in [\mu]^1$, say $s = \{\xi\}$, we can use the $\kappa$-completedness of $U_\xi$ to get $H^* = \langle H^*_\xi \rangle$ where $\bar{c} \restriction [\prod_{\xi \in s} H^*_\xi]$ is constant. We shall then assume that the result holds for each $1 \leq m \leq n$ and from this we shall infer that it also holds for $n+1$.

Let $\bar{c} : [\prod_{\xi<\mu} P_\kappa(\theta_\xi)]^{[n+1]} \to \varnothing$ be a function and $s \in [\mu]^{n+1}$. Set $\max(s) = \eta_s$. For each $g \in [\prod_{\xi \in s \cap \eta_s} P_\kappa(\theta_\eta_s)]$ let $c_g : P_\kappa(\theta_\eta_s) \to \varnothing$ be the function defined by $x \mapsto \bar{c}(g \cup \{\langle \eta_s, x \rangle\})$, provided $\max(g) < x$, and 0 otherwise. Applying the induction hypothesis, for each such $g$ we get a homogeneous set $H_g \in U_\eta$ and $\varnothing_g \in \varnothing$ be a constant value witnessing it. Let $H^s_{\eta_s} = \triangle\{H_g : g \in [\prod_{\xi \in s \cap \eta_s} P_\kappa(\theta_\xi)]\}$, where this diagonal intersection is defined as

$$\{x \in P_\kappa(\theta_{\eta_s}) : \forall g \in [\prod_{\xi \in s \cap \eta_s} P_\kappa(\theta_\xi)] (\max(g) < x \to x \in H_g)\}.$$  

By normality of $U_{\eta_s}$ it follows that $H^s_{\eta_s} \in U_{\eta_s}$. On the other hand, consider $c^* : [\prod_{\xi<\mu} P_\kappa(\theta_\xi)]^{[n]} \to 2$ be the function sending each $g$ to $\varnothing_g$. By the induction hypothesis for each $s \in [\mu]^{n+1}$ there is $H^{s\cap \eta_s} = \langle H^{s\cap \eta_s}_\xi : \xi \in s \cap \eta_s \rangle$ such that $c^* \restriction [\prod_{\xi \in s \cap \eta_s} H^{s\cap \eta_s}(\xi)]$ is constant with value $\varnothing^*$. We claim that for each $s \in [\mu]^{n+1}$, $\mathcal{H}^s = \mathcal{H}^{s\cap \eta_s} \cup \{\langle \eta_s, H^s_{\eta_s} \rangle\}$ witnesses the claim. Indeed, fix $s \in [\mu]^{n+1}$ and let $f \in [\prod_{\xi \in s} \mathcal{H}^s(\xi)]$. Say $f = g \cup \{\langle \eta_s, x \rangle\}$, where $g \in [\prod_{\xi \in s \cap \eta_s} \mathcal{H}^s(\xi)]$. Since $x \in H^s_{\eta_s}$ and $\max(g) < x$, by definition of diagonal intersection, $x \in H_g$, hence $c_g(x) = \varnothing^*$. On the other hand, $\bar{c}(f) = c_g(x)$, so that $\bar{c}(f) = \varnothing^*$. Altogether this yields the proof of the inductive step.

For each $n \in \omega$ and for each $s \in [\mu]^n$ let $\mathcal{H}^s$ be the sequence of large sets given by the previous argument for the function $c$. Let $H^* \in \prod_{\xi<\mu} U_\xi$ be the sequence defined by

$$\xi \mapsto H^*(\xi) = \bigcap\{H^s_\xi : s \in [\mu]^{<\omega}, \xi \in s\}$$
noticing that this is well-defined since all the measures are \( \kappa \)-complete and we are taking intersections of at most \( \mu \)-many sets. Finally, the definition of the sequences \( H^* \) guarantees that \( H^* \) is a homogeneous set for the function \( c \).

Arguing as in [Git10, Lemma 1.8] one may use lemma 2.31 to get an alternative proof of the Prikry property of \( S \) to that given in [Sin08].

The crucial property of Prikry-type forcings that allows to get a geometric characterization of its generic extensions is the Mathias-Prikry property or (MP-property, for short). Despite, at least as far as we know, this terminology does not appear before we think it is worth to be consider due to its central role in the proof of the Mathias criterion of genericity. As we shall argue in the next lines this property is in essence the best possible generalization of the Prikry property to the context of all open dense subsets and thus it seems reasonable to give it that name. Since the way to express the Mathias-Prikry property will change depending the structure of the forcing we shall first show how it looks like for the Prikry forcing and afterwards we shall extrapolate it to the Sinapova forcing \( S \):

**Lemma 2.32** (see [Git10, Lemma 1.13]). \( \mathcal{P}_U \) has the Mathias-Prikry property; i.e. for every dense open set \( D \subseteq \mathcal{P}_U \) and each \( p = \langle s, A \rangle \in \mathcal{P}_U \) there is an integer \( n_{p,D} \in \omega \) and a \( U \)-large set \( B \subseteq A \) such that for every \( n_{p,D} \leq m \) and every \( t \in [B]^m \), \( \langle s \cup t, B \setminus (\max(t) + 1) \rangle \in D \).

The moral of this property is that any dense open subset \( D \subseteq \mathcal{P}_U \) is the set all the \( \leq \)-minimal extensions of an eventual \( \leq^* \)-extension of a given condition \( p \in \mathcal{P}_U \). More precisely, from some point in advance (for some least length \( n_s \) and some \( B \in U \)) \( D \) is the set of all minimal \( \leq \)-extensions \( q \) of \( p \) with \( |s_q| = |s_p| + n_s \). In symbols,

\[
D = \{ \langle t, B \rangle \in \mathcal{P}_U : \exists \langle s, A \rangle \in \mathcal{P}_U \exists n_s \in \omega \exists B^* \subseteq A \mid |t| \geq |s| + n_s \land t \setminus s \in [B^*]^{\leq \omega} \land B = B^* \setminus \max(t) + 1 \}.
\]

Despite it does not seem that the Mathias-Prikry property implies the Prikry property of \( \mathcal{P}_U \) (nor the converse) it somehow yields a generalization of this latter. Let \( \varphi \) be a sentence in the language of forcing and set \( D_\varphi = \{ q \in \mathcal{P}_U : q \parallel \varphi \} \). It is obvious that for every \( \varphi \) the set \( D_\varphi \) is dense open with respect to \( \leq \). Nonetheless what Prikry property claims is that all these sets are also \( \leq^* \)-dense open. Translating this to the former context, Prikry property says that for each \( p = \langle s, A \rangle \) and each sentence \( \varphi \), there is some \( B \subseteq A \) such that \( 0 \) and \( B \) witness that MP-property holds for \( p \) and \( D_\varphi \). Therefore, regarded in the context of the family \( D = \{ D_{p,\varphi} : p \in \mathcal{P}_U, \varphi \in \text{SENT}(\mathcal{L}_o) \} \), the Prikry property implies the MP-property.

Nevertheless the MP-property somehow yields an optimal generalization of the Prikry property to the setting of all \( \leq \)-dense open sets.
Aiming to argue this let us do the following experiment: let us denote by $D^*$ the family of all $\leq$-dense open sets and say that $\mathcal{P}_U$ has the **Ultimate Prikry property** (UPP) if any set in $D^*$ is $\leq^*$-open dense. It is obvious that UPP implies the MP-property and also the Prikry property. Nonetheless it is not hard to show that UPP does not hold for $\mathcal{P}_U$ as, for instance, the set $D_n = \{ \langle s, A \rangle \in \mathcal{P}_U : |s| \geq n \}$ is $\leq$-open dense but not $\leq^*$-dense. Therefore to have a similar property to Prikry property in the context of all $\leq$-open dense subsets one has to pay the prize of modifying both components of a given condition. Is just in this sense that the MP-property is the optimal generalization of the Prikry property.

**Definition 2.33** (MP-property for $S$). We say that $S$ has the MP-property if for each $\langle g, H \rangle \in S$ and each dense open set $D$ there is $H^*$, $\langle g, H^* \rangle \leq^* \langle g, H \rangle$, and some $s_g \in [\text{dom}(H)]^{<\omega}$ such that for each finite extension $s_g \subseteq t$ with elements of $\text{dom}(H)$ and each $f \in \prod_{\xi \in t} H^*(\xi)$,

$$\langle g \cup f, H^* \setminus \max(f) \rangle \in D,$$

where $H^* \setminus \max(f)$ is the function $F$ with $\text{dom}(F) = \text{dom}(H^*) \setminus \text{dom}(f)$ and for each $\xi \in \text{dom}(F)$, $F(\xi) = \{ x \in H^*(\xi) : \max(f) < x \}$.

**Lemma 2.34.** $S$ has the MP-property.

**Proof.** Let $c \in \prod_{\xi \in \text{dom}(H)} H(\xi) [^{<\omega}] \rightarrow 2$ defined by $f \mapsto 1$ if there is $H_f$ such that $\langle g \cup f, H_f \rangle \in D$ or 0 otherwise. Let $F \in \prod_{\xi \in \text{dom}(H)} H(\xi)$ be a function witnessing the claim of lemma 2.31. Since $D$ is dense open there must exist some $s_g \in [\text{dom}(H)]^{<\omega}$ such that for any $s_g \subseteq t$ the function $c \upharpoonright [\prod_{\xi \in t} F(\xi)]$ has constant value 1. For each $\xi \in \text{dom}(H)$ define

$$H^*(\xi) = F(\xi) \cap \Delta\{H_f(\xi) : f \in \prod_{\xi \in t} F(\xi), s^* \subseteq t \} \in U_\xi$$

where, $\Delta\{H_f(\xi) : f \in \prod_{\xi \in t} F(\xi), s^* \subseteq t \} = \{ P \in \mathcal{P}_n(\theta_\xi) : \forall t \supseteq s_g \forall f \in [\prod_{\xi \in t} F(\xi)] (\max(f) \prec P \rightarrow P \in H_f(\xi)) \}$. We claim that $H^*$ and $s_g$ are as desired. Indeed, let $s^* \subseteq t$, $f \in [\prod_{\xi \in t} H^*(\xi)]$ and notice that by construction $\langle g \cup f, H^* \setminus f \rangle \in D$. Moreover, by definition of diagonal intersection, $\langle g \cup f, H^* \setminus f \rangle \leq \langle g \cup f, H_f \rangle$ hence $\langle g \cup f, H^* \setminus f \rangle \in D$. □

Let $S \subseteq S$ a generic filter and let $g^* = \cup_{\rho \in S} g^\rho$ be the generic sequence added by $S$. It is straightforward to check that

$$S(g^*) = \{ \langle g, H \rangle \in S : g \subseteq g^*, g^* \setminus g \subseteq H \}$$

is a filter containing $S$, hence $S(g^*) = S$, and thus $V[G] = V[g^*]$. Thereby the generic extensions of Sinapova forcing are completely determined by the generic sequence $g^*$ as it happened with Prikry forcing. In the next result we shall give a geometric criterion -analogue to
the result of Mathias for $\mathcal{P}_{\mu^+}$ that characterizes the generic sequences of $\mathcal{S}$. Before proving that we need to set some terminology: for a sequence $g \in \prod_{\xi < \mu} \mathcal{P}_\kappa(\theta_\xi)$ we will say that $g$ is generic for $\mathcal{S}$ if $S(g)$ defines a generic filter for $\mathcal{S}$.

**Proposition 2.35 (Mathias criterion).** Let $\kappa$ be a supercompact cardinal, $\mu < \kappa$ and $(\theta_\xi : \xi < \mu)$ be an increasing sequence of cardinals above $\kappa$. Let $\langle U_\xi : \xi < \mu \rangle$ be a $\langle \xi \langle \mu \rangle$-increasing sequence of supercompact measures over $\mathcal{P}_\kappa(\theta_\xi)$, $\xi < \mu$. Then the following conditions are equivalent for a function $g \in \prod_{\xi < \mu} \mathcal{P}_\kappa(\theta_\xi)$:

1. $g$ is generic for $\mathcal{S}$.
2. For every function $H \in \prod_{\xi < \mu} U_\xi$, there is a finite set of ordinals $s \in [\mu]^{<\omega}$ such that for all $\xi \in \mu \setminus s$, $g(\xi) \in H(\xi)$.

**Proof.** Let $g$ be generic for $\mathcal{S}$, $H \in \prod_{\xi < \mu} U_\xi$ and $D_H = \{\langle h, H' \rangle : \forall \xi \in \text{dom}(H'), H'(\xi) \subseteq H(\xi)\}$. Since $D_H$ is dense and $g$ is generic there is some $\langle h, H' \rangle$ such that $g \restriction h \subseteq H'$ and thus for all $\xi \in \mu \setminus \text{dom}(h)$, $g(\xi) \in H'(\xi) \subseteq H(\xi)$. This proves the implication (1)-(2).

Conversely, let $g$ be a sequence witnessing condition (2) and let us show that $S(g)$ meets any open dense subset of $\mathcal{S}$. Let $D$ be a dense open subset and for each $h \in \prod_{\xi < \mu} U_\xi$ and $D_h = \{\langle h, H' \rangle : \forall \xi \in \text{dom}(H'), H'(\xi) \subseteq H(\xi)\}$. Since $D_h$ is dense and $g$ is generic there is some $\langle h, H' \rangle$ such that $g \restriction h \subseteq H'$ and thus for all $\xi \in \mu \setminus \text{dom}(h)$, $g(\xi) \in H'(\xi) \subseteq H(\xi)$. This proves the implication (1)-(2).

Hence, for each $t \in [\mu \setminus \text{dom}(h)]^{<\omega}$ and every $f \in \prod_{\xi < \mu} H'(\xi)$, $\langle g \cup f, H' \setminus \text{max}(f) \rangle \in D$. Let $H^*$ be the function defined by $\xi \mapsto H^*(\xi)$, where

$$H^*(\xi) = \Delta \{H'_g(\xi) : g \in \prod_{\xi < \mu} \mathcal{P}_\kappa(\theta_\xi)[^{<\omega}]\}.$$ 

Notice that $H^*$ is well-defined since for each $\xi$, $[\prod_{\xi < \mu} \mathcal{P}_\kappa(\theta_\xi)[^{<\omega}]] \leq \kappa$ and $U_\xi$ is a normal measure. By hypothesis, there is $s \in [\mu]^{<\omega}$ such that for every $\xi \in \mu \setminus s$, $g(\xi) \in H^*(\xi)$. Set $g^* = g \restriction s$, $g_* = g \setminus s$ and notice that $\langle g^*, H^* \setminus \text{max}(g^*) \rangle \in \mathcal{S}$. Set $H = H^* \setminus \text{max}(g^*)$. For each $s_{g^*} \subseteq t$ and each $f_t = g_* \restriction t$ we claim that $\langle g^* \cup f_t, H \setminus \text{max}(f_t) \rangle \in S(g) \cap D$.

First notice that for such $s_{g^*} \subseteq t$, $f_t \in \prod_{\xi \in t} H^*_g(\xi)$ and thus $\langle g \cup f_t, H^*_g \setminus \text{max}(f_t) \rangle \in D$. On the other hand, by definition of diagonal intersection, for each ordinal $\xi$, $(\hat{H} \setminus \text{max}(f_t))(\xi) \subseteq H_f(\xi)$, hence $\langle g^* \cup f_t, \hat{H} \setminus \text{max}(f_t) \rangle \leq \langle g^* \cup f_t, H^*_f \setminus \text{max}(f_t) \rangle$. Combining both things we get that $\langle g^* \cup f_t, \hat{H} \setminus \text{max}(f_t) \rangle \in D$. It thus remains to prove that $\langle g^* \cup f_t, \hat{H} \setminus \text{max}(f_t) \rangle \in S(g)$. Indeed, $g^* \cup f_t \subseteq g$ and

\[13\] This follows by a theorem of Solovay which says that $\lambda^{<\kappa} = \lambda$ if $\text{cof}(\lambda) > \kappa$ and $\lambda^+$ otherwise.
\[ g \setminus (g^* \cup f_i) \subseteq \hat{H} \setminus \max(f_i), \text{ hence } \langle g^* \cup f_i, \hat{H} \setminus \max(f_i) \rangle \in S(g). \] This finally yields the genericity of \( S(g) \) as wished. \( \square \)

**Corollary 2.36.** Let \( \kappa \) be a supercompact cardinal, \( \mu < \kappa \) and \( \langle \theta_\xi : \xi < \mu \rangle \) be an increasing sequence of cardinals above \( \kappa \). Let \( \mathfrak{U} \) and \( \mathfrak{V} \) be two \( \triangleleft \)-increasing sequence of measures over \( P_\kappa(\theta_\xi), \xi < \mu \), such that for each \( \xi, \mathfrak{V}(\xi) \subseteq \mathfrak{U}(\xi) \). Then any generic filter for \( S_{(\kappa, \mu, \mathfrak{U})} \) is also a generic filter for \( S_{(\kappa, \mu, \mathfrak{V})} \) and thus there is a projection between \( S_{(\kappa, \mu, \mathfrak{U})} \) and \( S_{(\kappa, \mu, \mathfrak{V})} \).

**Proof.** This easily follows from proposition 2.35 and the fact that the generic extensions of the Sinapova forcing are completely determined by the generic sequences. \( \square \)

**3. The main forcing construction**

Assume GCH holds and let \( \mu \) be a regular cardinal and \( \langle \kappa_\xi : \xi < \mu \rangle \) be a continuous sequence of cardinals such that each \( \kappa_{\xi+1} \) is a supercompact cardinal. Set \( \kappa = \kappa_0, \varepsilon = \sup_{\xi<\mu} \kappa_\xi, \delta = \varepsilon^+ \) and assume there is a weakly compact cardinal \( \lambda > \delta \). Without loss of generality we may assume that for each \( \xi < \mu, \kappa_\xi \) is a Laver indestructible supercompact cardinal. Let also some cardinal \( \gamma \geq \lambda \) with \( \text{cof}(\gamma) > \mu \). Our forcing notion \( \mathbb{R} \) is intended to blow up \( 2^\kappa \) to \( \gamma \), forcing \( \text{TP}(\kappa^+) \) and \( \text{TP}(\kappa^{++}) \) and adding a very good scale joint with a bad scale. For this aim we shall modify the forcing presented in [GP18] exchanging the Magidor forcing for the Sinapova forcing which will produce the aforementioned scales in the resulting generic extension. For the sake of clarity, we shall simply give details of the construction of \( \mathbb{R} \) to produce a 1-gap within the interval \( (\kappa, 2^\kappa) \) (i.e. \( 2^\kappa \geq \lambda^+ \)) whereas the general case will be tackled in section 4.2.

If one looks at Mitchell’s original proof of \( \text{TP}(\aleph_2) \) or Cummings-Foreman proof of \( \text{TP}(\kappa^{++}) \) for some strong limit cardinal \( \kappa \), one will immediately realize that for making the argument work it is crucial that both components of the forcing have the same length. Nevertheless if one pretends to use certain Mitchell-like forcing \( \mathbb{R} \) to force \( 2^\kappa > \lambda \) as well as \( \text{TP}(\kappa^{++}) \) then necessarily appears a disagreement between the length of both components of \( \mathbb{R} \). For instance, if we aim to force \( 2^\kappa \geq \lambda^+ \) and \( \text{TP}(\kappa^{++}) \) we should need a forcing \( \mathbb{R} \) with Cohen component \( \text{Add}(\kappa, \lambda^+) \) and with collapsing component of length at most \( \lambda \); as otherwise \( \lambda \) would be collapsed to \( \kappa^+ \).

Recall that a crucial ingredient for the definition of the main forcing in [CF98] is the coherent family of projections between the forcings \( \langle \text{Add}(\kappa, \alpha) \ast \mathcal{Q}_\alpha : \alpha \in \mathcal{A} \rangle \), described so far. In [FHS18] the authors showed how to modify this construction to embrace the situation where \( 2^\kappa \) is arbitrarily large in the Cummings-Foreman model. In a nutshell, the argument goes as follows: Let \( \mathbb{R} \) be the Mitchell forcing as defined
in [FHS18] Definition 3.9] whose Cohen component is $\text{Add}(\kappa, \lambda^+) * \mathcal{P}(\mathbb{R})$ and its collapsing component collapses the interval $(\kappa^+, \lambda)$ to $\kappa^+$ via $\text{Add}(\kappa, \text{Even}(\alpha)) * \mathcal{Q}_\alpha$-names for conditions in $\text{Add}(\kappa^+, 1)$, $\alpha \in \mathcal{A}$. It can be shown that there is a subforcing $\mathbb{R}^*$ of $\mathbb{R}$ in such a way that both components of $\mathbb{R}^*$ have length $\lambda$, so that classical arguments can be used to prove $\text{TP}(\kappa^{++})$ in the generic extension $V^{\mathbb{R}^*}$, and if $\mathbb{R}$ forces the existence of a $\kappa^{++}$-Aronszajn tree then $\mathbb{R}^*$ also does. Finally, the proof finishes after a detailed analysis of the quotients $\mathbb{R}^* / \mathbb{R}^* \upharpoonright \gamma$ that leads to the non existence of $\kappa^{++}$-Aronszajn trees in $V^{\mathbb{R}^*}$ (see [FHS18] Section 3.2] or section 4 of the present paper).

Throughout this section we shall follow the structure of [FHS18] and [GPT18] in order to introduce our main forcing notion $\mathbb{R}$. Under the hypothesis described on the beginning of the section after forcing with $\text{Add}(\kappa, \lambda^+)$ all the cardinals $\{\kappa\} \cup \{\kappa_{\xi+1} : \xi < \mu\}$ remain supercompact. In particular, there is a $\text{Add}(\kappa, \lambda^+)$-nice name $\dot{U}$ such that

$$\models_{\text{Add}(\kappa, \lambda^+)} "\dot{U} \text{ is a supercompact measure on } \dot{P}_\kappa(\delta)".$$  

On what follows if $x \subseteq \lambda^+$ we shall denote by $\mathbb{A}_x$ the forcing $\text{Add}(\kappa, x)$; namely, the set of all partial functions $p$ with dom $(p) \in [\kappa \times x]^{<\kappa}$ and $p(\alpha, \beta) \in 2$, each $(\alpha, \beta) \in \text{dom}(p)$. Similarly if $H \subseteq \mathbb{A}_x$ is a generic filter and $y \subseteq x$ we denote by $H \upharpoonright y$ the $\mathbb{A}_y$-generic filter induced by $H$ joint with the standard projection between $\mathbb{A}_x$ and $\mathbb{A}_y$.

Working within $V^{\mathbb{A}_{\lambda^+}}$ we first show that there is a big set $A \subseteq \lambda^+$ of intermediate extensions of $V^{\mathbb{A}_{\lambda^+}}$ where the measure $U$ projects:

**Lemma 3.1.** There is an unbounded set of cardinals $A \subseteq \lambda^+$ closed by $\geq \delta^+$-sequences such that for each $\alpha \in A$ and each generic filter $G \subseteq \mathbb{A}_{\lambda^+}$

$$(\ast) \quad \dot{U}_G \cap V[G \upharpoonright \alpha] \in V[G \upharpoonright \alpha]$$

is a supercompact measure on $\mathcal{P}_\kappa(\delta)^{V[G[\alpha]}$.

**Proof.** Let $\alpha < \lambda^+$ and $X_\alpha$ be the set of all $\mathbb{A}_\alpha$-nice names for subsets of $\mathcal{P}_\kappa(\delta)$. Since $\mathbb{A}_\alpha$ is $\kappa^+$-cc, $|\mathcal{A}_\alpha| = \lambda$ and $\lambda$ is inaccessible then $|X_\alpha| = \lambda$, hence we may let $\langle \sigma_\beta : \beta \in \lambda \rangle$ be an enumeration of $X_\alpha$. For each $\beta \in \lambda$ let $A_\beta$ be a maximal antichain of $\mathbb{A}_{\lambda^+}$ deciding the sentence "$\sigma_\beta \in \dot{U}$". Notice that the $\kappa^+$-ccness of $\mathbb{A}_\alpha$ and the regularity of $\lambda^+$ imply that all the conditions in $\bigcup \mathbb{A}_\beta$ have its domain contained in $\kappa \times \gamma$, for some $\alpha < \gamma < \lambda^+$. For each $\alpha \in \lambda^+$ let $F(\alpha)$ be the least ordinal $\gamma$ witnessing this property and define by induction on $\beta \in \delta^+$, $F^*(0, \alpha) = F(\alpha)$, $F^*(\beta + 1, \alpha) = F(F(\beta, \alpha))$ and $F^*(\beta, \alpha) = \sup_{\eta < \beta} F^*(\eta, \alpha)$. Notice that for each $\alpha \in \lambda^+$, $\langle F(\beta, \alpha) : \beta \in \delta^+ \rangle$ is an increasing sequence of ordinals below $\lambda^+$ and thus for each $\beta < \beta'$, $\mathbb{A}_{F(\beta, \alpha)}$ is a complete subposet of $\mathbb{A}_{F(\beta', \alpha)}$. Now let us consider the function $H : \lambda^+ \rightarrow \lambda^+$.

\[\text{Here } \dot{Q} \text{ is a } \text{Add}(\kappa, \lambda^+)-\text{name for the Prikry forcing with respect to some normal measure.}\]
defined by $H(\alpha) = \sup_{\beta < \delta^+} F^*(\beta, \alpha)$ and set $A = \text{ran}(H)$. Clearly $A$ is unbounded and it is not hard to check that it is also closed by $\geq \delta^\tau$-sequences, hence it remains to prove that condition $(\ast)$ is fulfilled. Indeed, let $\tau$ be a $\mathbb{A}_{H(\alpha)}$-nice name for a subset of $\mathcal{P}_\kappa(\delta)$ and notice that since GCH holds, hence $\delta^{< \kappa} = \delta$, and $\text{cof}(H(\alpha)) = \delta^+$ all the conditions in any maximal antichain $X \subseteq \mathbb{A}_{H(\alpha)}$ named in $\tau$ lie in $\mathbb{A}_{F^*(\beta, \alpha)}$, for some $\beta < \delta^+$. Therefore if $A$ is a maximal antichain in $\mathbb{A}_{\lambda^+}$ deciding the sentence “$\tau \in \dot{U}$” it turns out that $A$ was considered in the process for defining $F(\beta + 1, \alpha)$, hence $A \subseteq \mathbb{A}_{H(\alpha)}$. Finally since the map $\pi : p \mapsto p \upharpoonright k \times H(\alpha)$ defines a projection between $\mathbb{A}_{\lambda^+}$ and $\mathbb{A}_{H(\alpha)}$ defining $U_\alpha = \pi(\dot{U})$ is routine to check that for each $\mathbb{A}_{H(\alpha)}$-nice name $\tau$ for a subset of $\mathcal{P}_\delta(\kappa)$,

$$
\tau_G = \tau_{G|\alpha} \in (\hat{U}_\alpha)_{G|\alpha} \iff \tau_G \in \dot{U}_G
$$

which proves that $(\ast)$ holds.

Let $G \subseteq \mathbb{A}_{\lambda^+}$ a generic filter such that $\dot{U}_G = U$ and consider the next family of supercompact measures:

**Definition 3.2.** For each $\alpha \in \mathcal{A}$ we shall denote by $U_\alpha$ the supercompact measure on $\mathcal{P}_\delta(\kappa)^{V[G|\alpha]}$ defined by

$$
U_G \cap V[G \upharpoonright \alpha].
$$

By convention we shall denote by $U_{\lambda^+}$ to the supercompact measure $U$.

For $\alpha \in \mathcal{A} \cup \{\lambda^+\}$ (working in $V[G \upharpoonright \alpha]$) set $U_\alpha = \langle U_{\alpha, \gamma} : \gamma \in \mu \rangle$ the family of Rudin-Keisler projections of $U_\alpha$ onto $\mathcal{P}_\kappa(\kappa_\gamma)$; namely, for each $\gamma < \mu$ if $\sigma_\gamma$ is the standard projection between $\mathcal{P}_\kappa(\delta)$ and $\mathcal{P}_\kappa(\kappa_\gamma)$ then $X \in U_{\alpha, \gamma}$ iff $\sigma_\gamma^{-1}(X) \in U_\alpha$. It is routine to check that since $U_\alpha$ is a supercompact measure on $\mathcal{P}_\kappa(\delta)$ then $U_{\alpha, \gamma}$ is a supercompact measure on $\mathcal{P}_\kappa(\kappa_\gamma)$ and also that $U_{\alpha, \gamma} \preceq U_{\alpha, \gamma'}$, each $\gamma < \gamma' < \mu$. Therefore we can make use of the sequence $U_\alpha$ to define the corresponding Sinapova forcing.

**Notation 3.3.** For each $\alpha \in \mathcal{A}$ set $\mathcal{S}_\alpha$ a $\mathbb{A}_\alpha$-name for which $(\mathcal{S}_\alpha)_{G|\alpha}$ is the Sinapova forcing in $V[G \upharpoonright \alpha]$ with respect to the sequence $U_\alpha$.

Recall from the discussion at the beginning of the section that for defining the main poset $\mathbb{R}$ we first need to consider projections between its Cohen component (i.e. $\mathbb{A}_{\lambda^+} * \mathcal{S}_{\lambda^+}$) and $\lambda$-many subforcings of it. For this aim we will use the ideas of [PHS13, Section 3.1] to show that there is a coherent system of projections between $\mathbb{A}_{\lambda^+} * \mathcal{S}_{\lambda^+}$ and a family of forcings $\langle \mathbb{A}_{\text{Even}(\alpha)} * \mathcal{S}_{\alpha}^\sharp : \alpha \in \mathcal{B} \rangle$ that will take part in the collapsing component of $\mathbb{R}$. So let $\beta_0 \in \mathcal{A}$, $\lambda < \beta_0$ and $\pi : \beta_0 \rightarrow \text{Even}(\lambda)$ be a bijection. \footnote{Here we are identifying $\pi$ with the embedding $\pi^*$ that $\pi$ defines between $V^{\mathbb{A}_{\lambda^+}}$ and $V^{\mathbb{A}_{\text{Even}(\alpha)}}$.} This bijection $\pi$ naturally defines an isomorphism between $\mathbb{R}$ and $V^{\mathbb{A}_{\text{Even}(\alpha)}}$.

\footnote{For an ordinal $\alpha$, Even($\alpha$) stands for the set of all even and limit ordinals $\leq \alpha$.}
Lemma 3.7. Let $A_{\beta_0}$ and $A_{\text{Even}(\lambda)}$ as well as a $\in$-isomorphism between $V^{A_{\beta_0}}$ and $V^{A_{\text{Even}(\lambda)}}$, so that setting $U_0^a = \pi(U_{\beta_0})$ then $(U_0^a)_{(\pi|_{A_{\beta_0}})} = (U_{\beta_0})_{G|_{\beta_0}} = U_{\beta_0}$. For the ease of notation set $H = \pi(G \upharpoonright \beta_0)$. The proof of the next result is analogous to lemma 3.1.

**Lemma 3.4.** There is an unbounded set of cardinals $B \subseteq \lambda$ closed by $\geq \delta^+$-sequences such that for each $\alpha \in B$ and each generic filter $K \subseteq A_{\text{Even}(\lambda)}$

$$(U_0^a)_K \cap V[K \upharpoonright \text{Even}(\alpha)] \in V[K \upharpoonright \text{Even}(\alpha)]$$

is a supercompact measure on $\mathcal{P}_\kappa(\delta)^{V[K \upharpoonright \text{Even}(\alpha)]}$.

**Definition 3.5.** For each $\alpha \in B$ we shall denote by $U_0^\alpha$ the supercompact measure on $\mathcal{P}_\delta(\kappa)^{V[H \upharpoonright \text{Even}(\alpha)]}$ defined by

$$(U_0^\alpha)_K \cap V[K \upharpoonright \text{Even}(\alpha)] \in V[K \upharpoonright \text{Even}(\alpha)].$$

By convention we shall denote by $U_0^\alpha$ to the supercompact measure $U_{\beta_0}$.

As in previous comments for each $\alpha \in B$ (working in $V[H \upharpoonright \text{Even}(\alpha)]$) we can let $U_0^\alpha$ be the sequence of Rudin-Keisler projections of the measure $U_0^\alpha$.

**Notation 3.6.** For each $\alpha \in B \cup \{\lambda\}$ let $\hat{S}_\alpha^\pi$ be a $A_{\text{Even}(\alpha)}$-name for which $(\hat{S}_\alpha^\pi)_{H \upharpoonright \text{Even}(\alpha)}$ is the Sinapova forcing in $V[H \upharpoonright \text{Even}(\alpha)]$ with respect to the sequence $U_0^\alpha$.

The next lemma gives the projection mentioned so far:

**Lemma 3.7.** Let $\hat{A} = (A \cap [\beta_0, \lambda^+)) \cup \{\lambda^+\}$.

1. For every $\gamma, \delta \in \hat{A}$ with $\gamma < \delta$, there is a projection $\sigma^\delta_\gamma : A_\delta * \hat{S}_\delta \to \text{RO}^+(A_\gamma * \hat{S}_\gamma)$.

2. For every $\gamma \in \hat{A}$ and $\alpha \in B$, there is a projection $\sigma^\gamma_\alpha : A_\gamma * \hat{S}_\gamma \to \text{RO}^+(A_{\text{Even}(\alpha)} * \hat{S}_\alpha^\pi)$.

3. For every $\gamma \in \hat{A}$ and $\alpha \in B$, let $\sigma^\gamma_\alpha$ be the extension of $\sigma^\gamma_\alpha$ to the Boolean completion of $A_\gamma * \hat{S}_\gamma$.

Then the projections commute with $\sigma^{\lambda^+}_\alpha$:

$$\sigma^{\lambda^+}_\alpha = \sigma^\gamma_\alpha \circ \sigma^{\lambda^+}_\gamma.$$

**Proof.**

1. It is enough to prove that any generic filter $K \subseteq A_\delta * \hat{S}_\delta$ defines a generic filter $K' \subseteq A_\gamma * \hat{S}_\gamma$. Let $K \subseteq A_\delta * \hat{S}_\delta$ a generic filter. Since any generic filter for Sinapova forcing can be defined via a Sinapova sequence $\hat{C}$, without loss of generality we may assume
that $K$ is of the form $L \ast \vec{C}$, where $L \subseteq A_\delta$ is a generic filter.

On this respect, first notice that $L \upharpoonright \gamma$ gives a generic filter for $A_\gamma$. On the other hand, the sequence $\vec{C}$ is generic for $(\mathcal{S}_\delta)_L$ if and only if $\vec{C}$ satisfies Mathias criterion with respect to $U_\delta$ (see theorem ??). Working in $V[L]$, it is clear that the measures in $U_\delta$ extends the measures in $U_\gamma$ so that $\vec{C}$ satisfies the Mathias criterion as witnessed by the sequence of measures $U_\gamma$ and thus $\vec{C}$ defines a generic filter for $(\mathcal{S}_\gamma)_L$. Altogether this implies that $L \upharpoonright \gamma \ast \vec{C}$ is generic for $A_\gamma \ast \mathcal{S}_\gamma$.

(2) The previous argument show that for any generic filter $L \ast \vec{C} \subseteq A_\gamma \ast \mathcal{S}_\gamma$, $\pi(L \upharpoonright \beta_0) \upharpoonright \text{Even}(\alpha) \ast \vec{C}$ is a generic filter for $A_{\text{Even}(\alpha)} \ast \mathcal{S}_\delta^\pi$ and from this the claim follows.

(3) The proof is the same as in [GHST18 Lemma 3.18 (iii)].

\[ \square \]

The proof of the previous lemma yields to the definition of our main forcing:

**Definition 3.8** (Main forcing). A condition in $\mathbb{R}$ is a triple $(p, \dot{q}, r)$ such that:

1. $(p, \dot{q}) \in A_{\lambda^+} \ast \mathcal{S}_{\lambda^+}$.
2. $r$ is a partial function with $\text{dom}(r) \in [\mathcal{B}]^{<\delta}$.
3. For every $\gamma \in \text{dom}(r)$, $r(\gamma)$ is a $A_{\text{Even}(\gamma)} \ast \mathcal{S}_\gamma^\pi$-name such that

$$\sigma_\gamma^{\lambda^+}(p, \dot{q}) \Vdash_{A_{\text{Even}(\gamma)} \ast \mathcal{S}_\gamma^\pi} r(\gamma) \in \text{Add}(\delta, 1).$$

Let $(p_0, \dot{q}_0, r_0), (p_1, \dot{q}_1, r_1) \in \mathbb{R}$ we shall write $(p_0, \dot{q}_0, r_0) \leq_{A_{\lambda^+} \ast \mathcal{M}} (p_1, \dot{q}_1)$ if and only if $\gamma \in \text{dom}(r_1) \subseteq \text{dom}(r_0)$ and for each $\gamma \in \text{dom}(r_1)$

$$\sigma_\gamma^{\lambda^+}(p_0, \dot{q}_0) \Vdash_{A_{\text{Even}(\gamma)} \ast \mathcal{S}_\gamma^\pi} r_0(\gamma) \leq r_1(\gamma).$$

Set $\mathbb{U} = \{(p, \dot{q}, r) \in \mathbb{R} : p = \emptyset \land \Vdash_{A_{\lambda^+}} \dot{q} = 1\}$ endowed with the inherited order of $\mathbb{R}$ and let $\mathbb{R} = (A_{\lambda^+} \ast \mathcal{S}_{\lambda^+}) \times \mathbb{U}$.

**Proposition 3.9.**

1. $\mathbb{U}$ is $\delta$-closed.
2. The function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ given by $\langle (p, \dot{q}), r \rangle \mapsto (p, \dot{q}, r)$ defines a projection. In particular,

$$V^{A_{\lambda^+} \ast \mathcal{S}_{\lambda^+}} \subseteq V^\mathbb{R} \subseteq V^{\mathbb{R}}.$$

3. $V^{A_{\lambda^+} \ast \mathcal{S}_{\lambda^+}}$ and $V^\mathbb{R}$ have the same $\delta$-sequences.

**Proof.**

1. Let $\eta < \delta$ and $(r_\alpha : \alpha \in \eta)$ be a decreasing sequence of conditions in $\mathbb{U}$. Set $\text{dom}(r^\eta) = \bigcup_{\alpha < \eta} \text{dom}(r_\alpha)$ and for each
γ ∈ dom(r*) let αγ be the first ordinal such that γ ∈ dom(rα) for all α ≥ αγ. Notice that

||_{A_{\text{Even}(γ)} \ast S^*_\eta} (r_\alpha(\gamma) : \alpha \geq \alpha_\gamma) \text{ is decreasing},

so that, since Add(δ,1) is ≤ δ-closed in V^{A_{\text{Even}(γ)} \ast S^*_\eta}, we can choose a A_{\text{Even}(γ)} \ast S^*_\eta-name r^*(γ) for a condition below that sequence. Defining the function r* on dom(r*) as γ ↦ r^*(γ) one gets a lower bound for the sequence \langle r_\alpha : \alpha \in \eta \rangle.

(2) The proof is routine.

(3) By (2) any δ-sequence in V^{A_{\lambda^+} \ast S_{\lambda^+}} is also a δ-sequence in V^R, hence it remains to show the converse inclusion. Since V^{A_{\lambda^+} \ast S_{\lambda^+}} is δ-cc and U is δ-closed the Easton’s lemma (see e.g. [Kun14]) implies that \|_{A_{\lambda^+} \ast S_{\lambda^+}} “U is δ-distributive” so that V^{A_{\lambda^+} \ast S_{\lambda^+}} and V^R have the same δ-sequences. Since V^R ⊆ V^R it thus follows that V^{A_{\lambda^+} \ast S_{\lambda^+}} and V^R have the same δ-sequences.

Let \tilde{R} ⊆ R a generic filter such the its projection onto A_{\lambda^+} generates the filter G. Let R ⊆ R be the generic filter generated by \tilde{R} an the projection R and S ⊆ S_{\lambda^+} be the generic filter over V[G] induced by \tilde{R}. Let \tilde{C} be the Sinapova sequence such that V[G][\tilde{C}] = V[G][S] and set κα = κ_{\tilde{C}(\alpha)} each α ∈ μ.

Recall that a forcing Ω is called η-Knaster for some cardinal η if for every subset of conditions C ⊆ Ω of cardinality η there is a subset C' ⊆ [C]^η of compatible conditions. Some examples of forcings enjoying this property are given for instance by Cohen forcing Add(η,θ) or the Prikry forcing with respect to some normal measure on η.

Is easy to check that η-Knasterness is a productive productive and further that it implies η-ccness. Nonetheless the converse implication is easily refutable since for instance a Suslin line X is ccc but its product not.

We finish the current section with a proposition that collects some of the basic properties of the generic extension V[R]:

**Proposition 3.10.**

(1) R is λ-Knaster. In particular, all the cardinals \geq λ remain cardinals in V[R].

(2) R preserves κ and δ and collapses all the cardinals in (κ, δ) to κ and all the cardinals in (δ, λ) to κ^+. In particular, δ = κ^+ and λ = κ^{++} in V[R].

(3) In V[R] all cardinals \leq κ are preserved.

(4) In V[R], 2^κ = λ^+ = κ^{+3}.

(5) In V[R], κ is a strong limit cardinal with cof(κ) = μ.

(6) In V[R] there is a bad and a very good scale in κ. In particular, \Box^*_κ fails and thus there are no special κ^+-Aronszajn trees.
Proof. (1) First of all if \((p, \check{q}) \in A_{\lambda^+} \ast \hat{S}_{\lambda^+}\) and \(p \Vdash \check{q} = \langle \check{y}, \check{H} \rangle\) we may assume without loss, maybe passing to an extension of \(p\), that \(\check{g}\) is a function in \(V\). Let \(\{(p_\alpha, q_\alpha, r_\alpha)\}_{\alpha \in \lambda}\) be a set of conditions in \(R\) and set \(\check{q}_\alpha = \langle g_\alpha, H_\alpha \rangle\). Since \(A_{\lambda^+}\) has the \(\kappa\)-Knaster property we can find a set \(I \in [\lambda]^\lambda\) such that \(p_\alpha \parallel p_\beta\), all \(\alpha, \beta \in I\). It is not hard to check that

\[
\{(g, f) : \exists p \in A_{\lambda^+} (p \Vdash_{A_{\lambda^+}} \exists H (g, H) \in \hat{S}_{\lambda^+})\}
\]

has cardinality smaller than \(\lambda\). Therefore there is \(I' \subseteq I\) with \(I' \in [\lambda]^\lambda\) and a function \(g\) such that \(g_\alpha = g\) for each \(\alpha \in I'\), hence \(\{(p_\alpha, q_\alpha)\}_{\alpha \in I'}\) are compatible. On the other hand, one can apply the usual \(\Delta\)-system arguments to the set \(\{r_\alpha\}_{\alpha \in I'}\) and find a further refinement \(I^* \subseteq I'\) in such a way that all condition in \(\{r_\alpha\}_{\alpha \in I^*}\) have a fixed value on its common domain. Altogether this entails the compatibility of the family \(\{(p_\alpha, q_\alpha, r_\alpha)\}_{\alpha \in I^*}\).

(2) By proposition \(3.9\) if \(\kappa\) and \(\delta\) are cardinals in the extension given by \((A_{\lambda^+} \ast \hat{S}_{\lambda^+}) \times U\) then \(R\) will force that both are cardinals. On one hand, since \(A_{\lambda^+} \ast \hat{S}_{\lambda^+}\) is \(\delta\)-cc and \(U\) is \(\delta\)-closed the Easton lemma implies that \(\delta\) is preserved (see e.g. \([Kun14]\)). On the other hand, the closure of \(U\) restricts the discussion on whether \(\kappa\) is a cardinal in the extension given by \(A_{\lambda^+} \ast \hat{S}_{\lambda^+}\).

Let us assume aiming for a contradiction there is \(\lambda < \kappa\) and \(f\) a \(A_{\lambda^+}\)-name such that \(\Vdash_{A_{\lambda^+} \ast \hat{S}_{\lambda^+}} f : \lambda \rightarrow \kappa\) is a surjection”. Using the \(\kappa\)-closedness of \(A_{\lambda^+}\), the Prikry property and the \(\kappa\)-closedness of \(\langle \hat{S}_{\lambda^+}, \leq^* \rangle\) we can build a decreasing sequence of conditions \(\{(p_\alpha, \check{q}_\alpha)\}_{\alpha \in \lambda}\) deciding the values of the function. Let \((p, \check{q})\) extending this family of conditions and define

\[
g = \{(\alpha, \beta) \in \lambda \times \kappa : (p, \check{q}) \Vdash_{A_{\lambda^+} \ast \hat{S}_{\lambda^+}} \check{f}(\alpha) = \beta\}.
\]

By construction is straightforward to check that \((p, \check{q}) \Vdash_{A_{\lambda^+} \ast \hat{S}_{\lambda^+}} \check{f} = g^*\”, hence density arguments ensures that any interpretation of the name \(\check{f}\) gives a surjective function in \(V\) between \(\lambda\) and \(\kappa\). Finally this lead us to the desired contradiction.

Let us now address the issue of cardinal preservation in the intervals \((\kappa, \delta) \cup (\delta, \lambda)\). First notice that any cardinal in \((\kappa, \delta)\) is collapsed to \(\kappa\) since there is a projection between \(R\) and \(A_{\lambda^+} \ast \hat{S}_{\lambda^+}\) and \(A_{\lambda^+} \ast \hat{S}_{\lambda^+}\) collapses each cardinal in \((\kappa, \delta)\) to \(\kappa\) (c.f. proposition \(2.28\)). In particular, \(\delta = \kappa^+\). For the second claim notice that for each \(\alpha \in B \cap (\delta, \lambda)\)

\[
\varphi_\alpha \quad \R \quad \rightarrow \quad RO^+(A_{\text{Even}(\alpha)} \ast \hat{S}_\alpha^\delta) \ast \text{Add}(\delta, 1)
\]

\[
(p, \check{q}, r) \quad \mapsto \quad (\sigma_{\alpha, \lambda}^{\kappa^+}(p, \check{q}), \check{r}(\alpha))
\]

defines a projection. Therefore it will be enough to prove that \(RO^+(A_{\text{Even}(\alpha)} \ast \hat{S}_\alpha^\delta) \ast \text{Add}(\delta, 1)\) collapses all cardinal \(\eta\) in the
interval \((\delta, \alpha]\) to \(\delta\). Fix \(\eta \in (\delta, \alpha]\) a \(V\)-cardinal. It is not hard to check using standard arguments that \(A_{\text{Even}(\alpha)} \ast \hat{S}_\eta^\alpha\) preserves \(\alpha\) and forces \(2^\kappa \geq \alpha\). Working in this generic extension, let \(\langle f_\alpha : \alpha \in \xi \rangle\) be a family of different functions in \(2^\kappa\) and \(C \subseteq \text{Add}(\delta, 1)\) a generic filter. For each \(\beta \in \eta\)

\[
D_\beta = \{ p \in \text{Add}(\delta, 1) : \exists \zeta \in \delta \forall \gamma \in \kappa (f_\beta(\gamma) = p(\zeta + \gamma)) \}
\]

is a dense set so that there is some \(p_\beta \in C \cap D_\beta\). For each \(\beta \in \eta\) let \(\zeta_\beta\) be the least ordinal \(\zeta\) among all the conditions \(p \in D_\beta \cap C\) and define the function \(\beta \mapsto \zeta_\beta\). To finish the argument it will suffice to check that this function is injective. Indeed, suppose otherwise that it is not injective and let \(p, p' \in C\) be two conditions such that \(p(\zeta_\beta + \gamma) = f_\beta(\gamma)\) and \(p'(\zeta_\beta + \gamma) = f_{\beta'}(\gamma)\), for all \(\gamma \in \kappa\). Since \(\zeta_\beta = \zeta_\beta'\) by hypothesis and \(p\) and \(p'\) are compatible the previous equalities imply that \(f_\beta = f_{\beta'}\), which gives the desired contradiction.

(3) Notice that proposition \(\mathbf{3.9}\) and the \(\kappa\)-closedness of \(A_{\lambda^+}\) reduces the discussion to what cardinals are preserved by \(S_{\lambda^+}\), which was previously carried out at proposition \(\mathbf{2.27}\).

(4) Follows from the existence of a projection between \(R\) and \(A_{\lambda^+}\) joint with the above claims.

(5) Since \(U\) is \(\delta\)-closed and there is a projection between \(\hat{R}\) and \(R\) it is enough to prove that \(\kappa\) is strong limit in \(V[G \ast \hat{S}]\). Let \(\eta < \kappa\) be a cardinal and use proposition \(\mathbf{2.20}\) to find a limit ordinal \(\xi < \mu\) such that \(P(\eta)^{V[G \ast \hat{S}]} = P(\eta)^{V[G \ast \hat{S}]^\xi}\). Since \(\kappa\) is supercompact in \(V[G]\), in particular strong limit, and \(S_{\xi, \delta^\ast}(\xi)\) has cardinality less than \(\kappa\), then \((2^\eta)^{V[G \ast \hat{S}]^\xi} < \kappa\) and thus \((2^\eta)^{V[G \ast \hat{S}]} < \kappa\).

Regarding the issue of the cofinality, \(\text{cof}^{(A_{\lambda^+} \ast S_{\lambda^+}) \times U}(\kappa) = \text{cof}^{(A_{\lambda^+} \ast S)}(\kappa) = \mu\), as \(U\) is \(\delta\)-closed, and

\[
\text{cof}^{(A_{\lambda^+} \ast S)}(\kappa) \leq \text{cof}^R(\kappa) \leq \text{cof}^{(A_{\lambda^+} \ast S)}(\kappa),
\]

hence \(\text{cof}^R(\kappa) = \mu\).

(6) This is an immediate consequence of the existence of a very good (resp. bad) scale in \(V[G \ast \hat{S}]\) (see theorem \(\mathbf{2.29}\), \((\kappa^+)^{V[G \ast \hat{S}]} = (\kappa^+)^{V[R]} = \delta\) and that \(V[G \ast \hat{S}]\) and \(V[R]\) have the same \(\delta\)-sequences.

\(\square\)

4. Proof of \(\text{TP}(\kappa^{++})\) holds

At the present section we aim to prove that \(\text{TP}(\kappa^{++})\) and \(2^\kappa \geq \gamma\) hold in the generic extension \(V^R\). For doing so we shall divide the proof into two subsections: on the first one we will tackle the issue of forcing \(\text{TP}(\kappa^{++})\) joint with \(2^\kappa \geq \lambda^+\) whilst on the other we will give the ideas
for generalizing the arguments of the former subsection to the general case of $2^\kappa$ arbitrarily large.

### 4.1. Tree property with $2^\kappa \geq \lambda^+$

Along this section we shall show that any counterexample for TP($\lambda$) in $V^R$ will essentially lie in some intermediate extension given by a truncation of $\mathbb{R}$ where both components have length $\lambda$. Afterwards we shall apply the ideas of [Ung13] to these truncation aiming to prove that tree property on $\lambda$ holds in their corresponding generic extensions. Altogether this will yield to the proof of TP($\kappa^{++}$) in $V^R$.

The next definition gives a precise meaning of what we understand by truncations of $\mathbb{R}$:

**Definition 4.1.** Let $\alpha \in \mathcal{A}$, $\beta_0 < \alpha < \lambda^+$ and define $\mathbb{R} \upharpoonright \alpha$ as the set of all triples $(p, q, r)$ such that

1. $(p, q) \in \dot{A}_\alpha \ast \dot{S}_\alpha$,
2. $r$ is a function with $\text{dom}(r) \in [\mathcal{B}]^{<\delta}$ such that for each $\beta \in \text{dom}(r)$,

\[ \models_{A|\text{Even}(\beta) \ast \dot{S}_\beta} \dot{r}(\beta) \in \text{Add}(\delta, 1). \]

We will say that $(p_0, q_0, r_0) \leq (p_1, q_1, r_1)$ if and only if $(p_0, q_0) \leq _{\dot{A}_\alpha \ast \dot{S}_\alpha} (p_1, q_1)$, $\text{dom}(r_1) \subseteq \text{dom}(r_0)$ and for each $\beta \in \text{dom}(r_1)$,

\[ \sigma_\beta^\alpha(p_0, q_0) \models_{A|\text{Even}(\beta) \ast \dot{S}_\beta} \dot{r}_0(\beta) \leq \dot{r}_1(\beta). \]

**Proposition 4.2.** Let $\alpha \in \mathcal{A}$ such that $\beta_0 < \alpha < \lambda^+$. There is a projection between $\mathbb{R}$ and $\text{RO}^+(\mathbb{R} \upharpoonright \alpha)$.

**Proof.** Let us denote by $\hat{\mathbb{R}} \upharpoonright \alpha$ the forcing $\mathbb{R} \upharpoonright \alpha$ but with Cohen part consisting of $\text{RO}^+(\mathbb{P}|\alpha \ast \dot{Q}_\alpha)$-names instead of $\mathbb{P}|\alpha \ast \dot{Q}_\alpha$-names. Notice that both posets $\hat{\mathbb{R}} \upharpoonright \alpha$ and $\mathbb{R} \upharpoonright \alpha$ are isomorphic, hence $\hat{\mathbb{R}} \upharpoonright \alpha$ and $\text{RO}^+(\mathbb{R} \upharpoonright \alpha)$ also, so it will be enough to show that there is such a projection between $\mathbb{R}$ and $\hat{\mathbb{R}} \upharpoonright \alpha$. Letting $\pi : (p, q, r) \mapsto (\sigma_\beta^\alpha(p, q), r)$ it is not hard to show that $\pi$ entails a projection between $\mathbb{R}$ and $\hat{\mathbb{R}} \upharpoonright \alpha$. □

The next proposition shows that a failure of TP($\lambda$) in $V^R$ entails a failure of the same property at some generic extension given by a truncation of $\mathbb{R}$:

**Proposition 4.3.** Assume that $T$ is a $\lambda$-Aronszajn tree in $V^R$. Then there is an ordinal $\beta^* \in \mathcal{A}$, $\beta_0 < \beta^* < \lambda^+$, such that $\mathbb{R} \upharpoonright \beta^*$ forces that $T$ is a $\lambda$-Aronszajn tree in $V^R|\beta^*$.

**Proof.** Let $\hat{T}$ be a $\mathbb{R}$-name for a $\lambda$-Aronszajn tree and assume without loss of generality that $\models_{\mathbb{R}} \hat{T} \subseteq \lambda$ and that $\hat{T}$ is a nice name for a subset of $\lambda$. Let $\{A_\alpha\}_{\alpha \in \lambda}$ be the family of maximal antichains defining $\hat{T}$ and notice that the $\lambda$-Knaster property of $\mathbb{R}$ implies that $A^* = \bigcup_{\alpha \in \lambda} A_\alpha$ has cardinality at most $\lambda$. It turns out that there is some $\beta^* < \lambda^+$
such that for any condition \((p, q, r) \in A^*\), \(\text{dom}(p) \subseteq \kappa \times \beta^*\). As \(A\) is unbounded this ordinal \(\beta^*\) can be picked from it. Finally, since there is a projection between \(\mathbb{R}\) and \(\mathbb{R} \upharpoonright \beta^*\) that does not move the conditions \((p, q, r)\), it turns out that \(\mathbb{R} \upharpoonright \beta^*\) forces that \(\check{T}\) is a \(\lambda\)-Aronszajn tree. □

We have just proved that any counterexample for TP(\(\lambda\)) must have been added by certain truncation \(\mathbb{R} \upharpoonright \beta^*\) hence we can restrict our discussion on this regard to \(\mathbb{R} \upharpoonright \beta^*\). Nonetheless this forcing still exhibits a disagreement between the lengths of the Cohen and the collapsing components and thus to avoid this inconvenience we shall seek for an isomorphic Mitchell’s-like forcing \(\mathbb{R}^*\) where both components have the same length. For this aim let \(\pi^* : \beta^* \rightarrow \lambda\) be a bijection extending \(\pi\) and lift it to an isomorphism between \(\check{A}_{\beta^*}\) and \(A_{\lambda}^{\pi^*}\). In a slightly abuse of notation we shall also denote by \(\pi^*\) the \(\check{\varepsilon}\)-isomorphism between \(V^\check{A}_{\beta^*}\) and \(V^{A_{\lambda}}\) induced by \(\pi^*\). It is clear that:

- \(\pi^*(\check{U}_{\beta^*})_{\pi^*(G|\beta^*)}\) is a supercompact measure on \(\mathcal{P}_\kappa(\delta)^{V[\pi^*(G|\beta^*)]}\),

- \(\pi^*(\check{U}_{\beta^*})_{\pi^*(G|\beta^*)} = (\check{U}_{\beta^*})_{G|\beta^*} = U_{\beta^*}\),

- \(\pi^*(\check{U}_{\check{\beta}^0})_{\pi^*(G|\beta^*)}\) extends \(\pi(\check{U}_{\check{\beta}^0})_{\pi(G|\beta^*)}\).

Set \(\check{S}\) the Sinapova forcing defined in \(V[\pi^*(G | \beta^*)]\) with respect to the measure \(U_{\beta^*}\). There is an obvious candidate for the target forcing \(\mathbb{R}^*\): namely, the forcing \(\mathbb{R} \upharpoonright \beta^*\) where the Cohen component is given by \(\check{A}_{\lambda} \ast \check{S}\) instead of \(\check{A}_{\beta^*} \ast \check{S}\). The next proposition shows there are natural projections between \(\check{A}_{\lambda} \ast \check{S}\) and the forcings \(\check{A}_{\text{Even}(\beta)} \ast \check{S}\) hence one can define \(\mathbb{R}^*\) as indicated so far.

**Proposition 4.4.**

1. **There is an isomorphism** \(\varphi : A_{\beta^*} \ast \check{S}_{\beta^*} \rightarrow A_{\lambda} \ast \check{S}_{\lambda}^{\pi^*}\).  
2. **For each** \(\beta \in \mathcal{B}\) **the function** \(g^\beta_\lambda = g^\beta_{\beta^*} \circ \varphi^{-1}\) **entails a projection between** \(A_{\lambda} \ast \check{S}_{\lambda}^{\pi^*}\) **and** \(\text{RO}^+(A_{\text{Even}(\beta)} \ast \check{S}_{\beta})\).

**Proof.** For (1) it is routine to check that \(\varphi(p, (g, \check{H})) = (\pi^*(p), (g, \pi^*(\check{H}))\) defines an isomorphism, and for (2) is enough to notice that the composition of projections is still a projection. □

**Definition 4.5. A condition in** \(\mathbb{R}^*\) **is a triple** \((p, q, r)\) **such that**

1. \((p, q) \in A_{\lambda} \ast \check{S}_{\lambda}^{\pi^*}\),
2. \(r\) **is a function with** \(\text{dom}(r) \in [\mathcal{B}]^{<\delta}\) **such that for each** \(\beta \in \text{dom}(r)\),

\[
\models_{A_{\text{Even}(\beta)} \ast \check{S}_{\beta}^{\pi^*}} r(\beta) \in \text{Add}(\delta, 1).
\]

*We will say that* \((p_0, q_0, r_0) \leq (p_1, q_1, r_1)\) **if and only if** \((p_0, q_0) \leq_{A_{\lambda} \ast \check{S}_{\lambda}^{\pi^*}} (p_1, q_1)\), **dom** \((r_1) \subseteq \text{dom}(r_0)\) **and for each** \(\beta \in \text{dom}(r_1)\),

\[
g^\beta_\lambda(p_0, q_0) \models_{A_{\text{Even}(\beta)} \ast \check{S}_{\beta}^{\pi^*}} r_0(\beta) \leq r_1(\beta).
\]

---

\(^{17}\) The reason for this is to be coherent with the previous construction.
Proposition 4.6. \( \mathbb{R}^* \) and \( \mathbb{R} \upharpoonright \beta^* \) are isomorphic. In particular, \( \mathbb{R}^* \) forces that \( T \) is a \( \lambda \)-Aronszajn tree.

Proof. It is routine to check that \( (p, q, r) \mapsto (\varphi(p, q), r) \) defines an isomorphism between both forcings. \( \square \)

As we done with \( \mathbb{R} \) we shall need to define truncations of \( \mathbb{R}^* \). For doing so one needs to consider projections of the measure \( U_{\beta^*} \) onto many intermediate generic extensions \( V[\pi^*(G \upharpoonright \beta^*) \upharpoonright \alpha] \), called them \( U_{\alpha}^\pi \) and \( S_{\alpha}^\pi \) the respective Sinapova forcing, and define the corresponding projections between the Cohen components \( A_{\alpha} \ast \hat{S}_{\alpha}^\pi \) and \( A_{\text{Even}(\beta)} \ast \hat{S}_\beta^\pi \).

Given a weakly compact cardinal \( \theta \) the weakly compact filter on \( \theta \), \( \mathcal{F}_\theta \), is the filter defined by all subsets \( X \subseteq \theta \) such that \( \theta \setminus X \) is not \( \Pi^1_1 \)-indescribable in \( \theta \); where a set \( X \subseteq \theta \) is called \( \Pi^1_1 \)-indescribable in \( \theta \) if for any \( R \subseteq V_\theta \) and any \( \Pi^1_1 \) sentence \( \varphi \), provided \( \langle V_\theta, \in, R \rangle \models \varphi \), there is \( \alpha \in X \) such that \( \langle V_\alpha, \in, R \cap V_\alpha \rangle \models \varphi \). Since \( \theta \) is weakly compact, hence \( \Pi^1_1 \)-indescribable, the filter \( \mathcal{F}_\theta \) is proper and normal (see [Kan08, Proposition 6.11]), hence extending the club filter on \( \theta \), and contains the set of Mahlo cardinals below \( \theta \). To prove this last claim let \( S \) be the set of all inaccessible cardinals below \( \theta \), hence a stationary set, and notice that there is a \( \Pi^1_1 \) sentence \( \varphi \) true in the structure \( \langle V_\theta, \in, S, \theta \rangle \) asserting that \( S \) is stationary and that \( \theta \) is an inaccessible cardinal:

\[ \forall C \left( C \subseteq \theta \land C \text{ club} \Rightarrow C \cap S \neq \emptyset \right) \land \theta \text{ is inaccessible} \].

Thereby, since \( \theta \) is weakly compact, the set of inaccessible cardinals \( \eta \in \theta \) for what \( S \cap \eta \) is stationary is in \( \mathcal{F}_\theta \) and thus, noticing this \( \eta \) are Mahlo cardinals, the set of Mahlo cardinals below \( \theta \) lies in the weakly compact filter.

Lemma 4.7. There is a \( \mathcal{B}^* \in \mathcal{F}_\lambda \), \( \mathcal{B}^* \subseteq \mathcal{B} \), such that \( \delta < \min \mathcal{B}^* \) and for every \( \alpha \in \mathcal{B}^* \)

\[ \pi^*(\hat{U}_{\beta^*})_{\pi^*(G \upharpoonright \beta^*)} \cap V[\pi^*(\mathcal{B} \upharpoonright \beta^*) \upharpoonright \alpha] \in V[\pi^*(\mathcal{B} \upharpoonright \beta^*) \upharpoonright \alpha] \]

where \( \pi^*(\mathcal{B} \upharpoonright \beta^*) \upharpoonright \alpha \) is the natural projection onto \( A_{\alpha} \) of \( \pi^*(\mathcal{B} \upharpoonright \beta^*) \).

Proof. The construction of \( \mathcal{B}^* \) is the same as for \( \mathcal{B} \) but starting from \( \mathcal{B} \) instead of \( \lambda \). On the other hand, by construction, \( \mathcal{B}^* \) contains an unbounded set which is \( \geq \kappa^+ \)-closed\(^{18} \), hence a set in the weakly compact filter, and thus \( \mathcal{B}^* \in \mathcal{F}_\lambda \). \( \square \)

Notation 4.8. For each \( \alpha \in \mathcal{B}^* \) we shall denote by \( U_{\alpha}^\pi \) the measure \( \pi^*(\hat{U}_{\beta^*})_{\pi^*(G \upharpoonright \beta^*)} \cap V[\pi^*(\mathcal{B} \upharpoonright \beta^*) \upharpoonright \alpha] \) and by \( S_{\alpha}^\pi \) the Sinapova forcing with respect to \( U_{\alpha}^\pi \) in \( V[\pi^*(\mathcal{B} \upharpoonright \beta^*) \upharpoonright \alpha] \).

Lemma 4.9. Set \( \hat{\mathcal{B}}^* = \mathcal{B}^* \cup \{ \lambda \} \) and let \( \alpha < \gamma \in \hat{\mathcal{B}}^* \). There are projections

\[ \delta_{\alpha}^\gamma : A_\gamma \ast \hat{S}_{\gamma}^\pi \rightarrow \text{RO}^+ (A_{\text{Even}(\alpha)} \ast \hat{S}_{\alpha}^\pi) \]

\(^{18}\)Namely, closed by increasing sequences of ordinals of length \( \geq \kappa^+ \).
We will say that

\( \ell \) was introduced previous to it, we may assume without loss of generality that Let \( R \) be a \( \Pi^1_1 \) formula such that for each \( \alpha \), \( \sigma^\alpha_\ell \). Hence \( \Pi^1_1 \)-counterexample for TP("\( \lambda \)) extension of TP("\( \lambda \)).

\[ \exists \varnothing \subseteq A_\ell^{\Pi^1} \exists \sigma^\alpha_\ell \]

Recall that without loss of generality we may assume that \( \dot{\ell} \), \( \dot{\varnothing} \) and the projections \( \dot{\lambda} \) intended to define the truncations \( R^* \mid \gamma \).

Therefore, we are now in conditions to define the truncations of \( R^* \):

\[ \Phi \]

The moreover clause of the former lemma is important as it points out that there is no disagreement between the projections \( \sigma^\alpha_\ell \) defining \( R^* \) and the projections \( \dot{\varnothing} \_\alpha \) intended to define the truncations \( R^* \mid \gamma \).

**Definition 4.10.** Let \( \gamma \in B^* \). A condition in \( R^* \mid \gamma \) is a triple \((p, \dot{q}, r)\) such that

\[ (1) \ (p, q) \in A^\alpha_\ell \star \dot{S}^\gamma_\ell, \]

\[ (2) r \text{ is a function with } \text{dom}(r) \in [B^* \cap \gamma]^{< \delta} \text{ such that for each } \alpha \in \text{dom}(r), \]

\[ \Vdash_{A^{\Pi^1}_\ell \star S^\delta_\ell} \dot{r}(\alpha) \in \text{Add}(\delta, 1). \]

We will say that \((p_0, \dot{q}_0, r_0) \leq (p_1, \dot{q}_1, r_1)\) if and only if \((p_0, \dot{q}_0) \leq_{A^\gamma_\ell \star S^\ast_\ell}(p_1, \dot{q}_1)\), \( \text{dom}(r_1) \subseteq \text{dom}(r_0) \) and for each \( \alpha \in \text{dom}(r_1), \)

\[ \varnothing^\alpha_\ell(p_0, q_0) \Vdash_{A^{\Pi^1}_\ell \star S^\delta_\ell} \dot{r}_0(\alpha) \leq \dot{r}_1(\alpha). \]

The proof of this proposition is analogous to the proof of proposition ??.

**Proposition 4.11.** For each \( \gamma \in B^* \), there is a projection between \( R^* \) and \( RO^+(R^* \mid \gamma) \). In particular, \( R^* \) can be regarded as the iteration \( R^* \mid \gamma \ast (R^*/R^* \mid \gamma) \).

The ultimate reason to consider these restrictions of \( R^* \) is that any counterexample for TP("\( \lambda \)) in \( V^{R^*} \) must reflect onto some intermediate extension \( V^{R^* \mid \gamma} \). More formally:

**Lemma 4.12.** Assume there is a \( \lambda \)-Aronszajn tree \( T \) in \( V^{R^*} \). There is \( \gamma \in B^* \) such that \( T \cap \gamma \) is a \( \gamma \)-Aronszajn tree in \( V^{R^* \mid \gamma} \).

**Proof.** Let \( \dot{T} \) be a \( R^* \)-name such that \( \Vdash_{R^*} \ "\dot{T} \text{ is a } \lambda \text{-Aronszajn tree}" \). Recall that without loss of generality we may assume that \( \dot{T} \) is a \( R^* \)-name for a subset of \( \lambda \). It is not hard to check that the displayed formula is a \( \Pi^1_1 \) sentence in the extended language \( \{ \in, R^*, \dot{T}, \lambda \} \) and if true then it must holds in \( \langle V_{\lambda}, \in, R^*, \dot{T}, \lambda \rangle \). For convenience of the notation let us denote this formula by \( \Phi \). Since \( \lambda \) is a weakly compact cardinal, hence \( \Pi^1_1 \)-indescribable, there is a set \( X \in F^\lambda \) such that for each \( \gamma \in X \), \( \langle V_\gamma, \in, R^* \cap V_\gamma, \dot{T} \cap \gamma, \gamma \rangle \models \Phi \). By lemma ?? and the discussion previous to it, we may assume without loss of generality that some of the \( \gamma \) witnessing the reflection of \( \Phi \) is a Mahlo cardinal in \( B^* \), hence \( R^* \cap V_\gamma = R^* \mid \gamma \), and thus \( \langle V_\gamma, \in, R^* \mid \gamma, \dot{T} \cap \gamma, \gamma \rangle \models \Phi \). Since \( \Phi \)
is absolute between the universe $V$ and the aforementioned structure altogether this proves that $\forall \gamma \in \mathcal{B}^*$ “$T \cap \gamma$ is a $\gamma$-Aronszajn tree” and thus the lemma follows. $\square$

**Lemma 4.13.** Assume there is a $\lambda$-Aronszajn tree $T$ in $V^{\mathbb{R}^*}$ and let $\gamma \in \mathcal{B}^*$ as in the previous lemma. There is $b_\gamma$ a branch throughout $T \cap \gamma$ that was added by $\mathbb{R}^*/(\mathbb{R}^*/\gamma)$.

**Proof.** Notice that such a branch $b_\gamma$ throughout $T \cap \gamma$ must exists in $V^{\mathbb{R}^*}$ as $T$ is a $\lambda$-tree. Nonetheless $T \cap \gamma$ is $\gamma$-Aronszajn in $V^{\mathbb{R}^*|\gamma}$ so the branch was added by the quotient forcing $\mathbb{R}^*/(\mathbb{R}^*/\gamma)$. $\square$

The previous lemma points out that the status of TP($\lambda$) in $V^{\mathbb{R}^*}$ relies on the combinatorial properties of the quotient forcings $\mathbb{R}^*/(\mathbb{R}^*/\gamma)$, $\gamma \in \mathcal{B}^*$. Specifically if for each $\gamma \in \mathcal{B}^*$ the quotient $\mathbb{R}^*/(\mathbb{R}^*/\gamma)$ does not add subsets to $\gamma$ then there are no $\lambda$-Aronszajn trees in $V^{\mathbb{R}^*}$ and thus TP($\lambda$) holds. Therefore, at this point, the discussion about the tree property in $V^{\mathbb{R}^*}$ may be reduced to a technical investigation of the nature and properties of these quotients.

This situation is very usual in the area and it turns out that all the quotients appearing in the arguments share a common property that link them with the original Mitchell forcing $\mathcal{M}$ (see e.g. [Abr83]): there are forcings $\mathbb{P}$ and $\mathbb{Q}$ with nice combinatorial properties such that $\mathbb{P} \times \mathbb{Q}$ which projects onto the quotient. Here by nice combinatorial properties we mean certain degree of Knasterness/closedness which ultimately imply that $\mathbb{P} \times \mathbb{Q}$ does not add new branches to a tree. Coming back to our current situation, we shall justify in the next series of lemmas that for each $\gamma \in \mathcal{B}^*$ as in lemma 4.12 there are two forcings $\mathbb{P}_\gamma$ and $\mathbb{Q}_\gamma$ such that

- (a) $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma$ projects onto $\mathbb{R}^*/\mathbb{R}^* \upharpoonright \gamma$ in $V^{\mathbb{R}^*|\gamma}$.
- (b) $\mathbb{P}_\gamma \times \mathbb{Q}_\gamma$ does not add new branches to $T \cap \gamma$, and thus the quotient forcing $\mathbb{R}^*/\mathbb{R}^* \upharpoonright \gamma$ either.

For each $\gamma \in \mathcal{B}^* \cup \{\lambda\}$, set $C_\gamma = A_\gamma \ast \check{S}_\gamma^*$ and $\mathbb{P}_\gamma$ for the quotient forcing $A_\lambda \ast \check{S}_\lambda^*/A_\gamma \ast \check{S}_\gamma^*$ and $\mathbb{Q}_\gamma$ for the set $\{r : (1, 1, r) \in \mathbb{R}^* \upharpoonright \gamma\}$ endowed with the order induced from $\mathbb{R}^* \upharpoonright \gamma$. Analogously to what occurs with $\mathbb{R}_\lambda$, $Q_\gamma$ is $\delta$-directed closed and there is a projection between $C_\gamma \times Q_\gamma$ and $\mathbb{R}^* \upharpoonright \gamma$ hence the following result follows from a straightforward adaptation of the proof of proposition 3.11.

**Proposition 4.14.**

1. In $V^{\mathbb{R}^*|\gamma}$, $\kappa$ is a strong limit cardinal with cof($\kappa$) = $\mu$.
2. $\mathbb{R}^* \upharpoonright \gamma$ collapses all the cardinals in the interval $(\kappa, \delta) \cup ((\delta^+)^V, \gamma)$ while it preserves the others. In particular, $\delta = \kappa^+$, $\gamma = \kappa^{++}$.
3. $\mathbb{R}^* \upharpoonright \gamma$ forces $2^\kappa \geq \gamma$

Henceforth we will work within $V^{\mathbb{R}^*|\gamma}$ unless otherwise stated. A condition $\mathbb{R}^*/\mathbb{R}^* \upharpoonright \gamma$ is by definition a triple $(p, \dot{q}, r)$ such that $(\pi^\gamma_\gamma(p, \dot{q}), r \upharpoonright$
Lemma 4.17. Assume that \( P \) and \( Q \) are two forcing notions with a projection \( \pi : P \to Q \) between them. Letting \( p \in P \) and \( q \in Q \), \( q \Vdash_P p \notin P/Q \) if and only if for every generic filter \( G \subseteq P \) with \( p \in G \) then \( q \) does not lie in \( H \), the generic filter generated by \( \pi^*G \).

Proof. The first implication is obvious. For the second assume aiming for a contradiction that \( q \Vdash_Q p \notin P/Q \) and let \( q' \leq q \) be some condition such that \( q' \Vdash_Q p \in P/Q \). Set \( D = \{ p' \leq p : \pi(p') \leq q' \} \) and notice that is dense in \( P \) as \( \pi \) is a projection. Let \( p' \in G \cap D \). On the one
hand, since $\pi(p') \in H$ and $q' \leq q$ then $q \in H$, and on the other $p \in G$ because $p' \leq p$. Altogether we reach a contradiction with our initial assumption proving the remaining implication and thus the lemma. □

**Lemma 4.18.** Let $\gamma \in B^*$, $r = (p, (g, H)) \in C_\lambda$ and $r' = (p', (g', H')) \in C_\gamma$. The condition $r'$ does not force $'r \notin \mathbb{P}_\gamma$ if and only if one of the following assertions is true:

(I) $p' \perp p \upharpoonright \gamma$.

(II) $p' \parallel p \upharpoonright \gamma$ and there is some $\eta \in \text{dom}(g') \cap \text{dom}(g)$ such that $p' \cup p \Vdash_{\lambda_\gamma} g(\eta) \neq g'(\eta)$.

(III) $p' \parallel p \upharpoonright \gamma$ and there is some $\eta \in \text{dom}(g') \setminus \text{dom}(g)$ such that $p' \cup p \Vdash_{\lambda_\gamma} g'(\eta) \notin H(\eta)$.

(IV) $p' \parallel p \upharpoonright \gamma$ and there is some $\eta \in \text{dom}(g) \setminus \text{dom}(g')$ such that $p' \cup p \parallel \text{dom}_{\lambda_\gamma} g(\eta) \notin H'(\eta)$.

**Proof.** Notice that if some of the above displayed assertion fails then $r' \perp g^*_C(r)$ and thus the previous lemma implies that $r' \Vdash_{\lambda_\gamma} r \notin \mathbb{P}_{\bar{\lambda}/\mathbb{P}_\gamma}$, yielding to the proof of the first implication. For the converse, assuming that any of the above assertions hold, we shall prove that $r'$ does not force $r$ in $\mathbb{P}_\gamma$; namely, there is some $\bar{r} \leq_{\mathbb{C}_\gamma} r'$ such that $\bar{r} \Vdash_{\mathbb{P}_{\lambda_\gamma}} r, r \in \mathbb{P}_{\lambda_\gamma}/\mathbb{P}_\gamma$.

Since (I) and (II) are false it makes sense to define the functions $p^* = p' \cup p$ and $g^* = g \cup g'$. Forcing below the condition $p^*$ for each $\eta \in \mu \setminus \text{dom}(g^*)$ let $H^*(\eta)$ be a $\lambda_\eta$-name for the large set $H(\eta) \cap H'(\eta)$ noticing that, since (III) and (IV) fails, $p^* \Vdash_{\lambda_\eta} (g^*, H^*) \in \mathbb{S}_\alpha^*$ and thus $r^* = (p^*, (g^*, H^*)) \in C_\lambda$. Moreover it is clear that $g^*_C(r^*) \leq_{\mathbb{C}_\gamma} r'$ and that $g^*_C(r^*) \Vdash_{\mathbb{C}_\gamma} 'r^* \in \mathbb{P}_\gamma \land \tau^* \leq_{\mathbb{P}_\gamma} r^*$, hence $g^*_C(r^*) \Vdash_{\mathbb{C}_\gamma} r \in \mathbb{P}_\gamma$. □

Now we are interested on the converse direction; namely, we want to seek for sufficient properties warranting that a condition in $C_\lambda$ forces a condition in $C_\gamma$ to be in $\mathbb{P}_\gamma$. We first need to fix some notation:

**Notation 4.19.**

- Let $\gamma \in B^*$ and $\xi < \mu$. We denote by $U^*_{\gamma,\xi}$ the natural projection of $U_{\gamma,\xi}^*$ onto $P_{\alpha}(\kappa_\xi)$ and by $U_{\gamma,\xi}^*$ its corresponding $\lambda_\alpha$-name.
- Let $\gamma \in B^*$ and $F, H \in \prod_{\xi \in \mu} U^*_{\gamma,\xi}$. We denote by $F \wedge H$ to the function $\xi \mapsto F(\xi) \cap H(\xi)$.
- Given $g \in \prod_{\xi \in \mu} P(\kappa_\xi)]^{<\omega}$ we respectively denote by $\max(g)$ and $\min(g)$ the $<\text{-}\text{maximum}$ and $<\text{-}\text{minimum}$ value of $g$.
- Let $(g, H) \in \mathbb{S}$ and $h \in \prod_{\xi \in \text{dom}(H)} H(\xi)]^{<\omega}$ we denote by $\text{add}((g, H), h)$ the condition $p^* = (g^*, H^*) \in \mathbb{S}$ defined by
  
  (a) $g^* = g \cup h$,
  
  (b) $H^*$ is a function with domain $\mu \setminus (\text{dom}(g) \cup \text{dom}(h))$ such that for each $\xi$ in the domain
  
  $H^*(\xi) = \{ P \in H(\xi) : \max(h) < P \}$.

Notice that $\text{add}((g, H), h)$ is well-defined as for each $\xi \in \text{dom}(H)$, $\max(g) < H(\xi)$, hence $\max(g) < \min(h)$.
Lemma 4.20. Let \( \gamma \in B^* \), \( r = (p, (g, \dot{H})) \in C_\lambda \) and \( r' = (p', (g', \dot{H}')) \in C_\gamma \) such that \( p' \leq p \upharpoonright \gamma \). There is a \( \mathbb{A}_\gamma \)-name \( \dot{F} \) for a function with domain \( \mu \setminus (\text{dom}(g) \cup \text{dom}(g')) \) such that:

1. \( p' \Vdash_{\mathbb{A}_\gamma} \forall \xi (\dot{F}(\xi) \in \check{U}_{\gamma, \xi}^{\pi^*_\gamma} \land \dot{F}(\xi) \subseteq \dot{H}'(\xi)) \),
2. The condition \( p' \) forces that for every finite subset \( s \subseteq \text{dom}(\dot{F}) \), each \( h \in \prod_{\xi \in s} \dot{F}(\xi) \cup \omega \) and each \( \xi \in s \)
   - \( (a_h) \) “add\((g', \dot{H}')\), \( h \in \check{S}^\pi_{\gamma} \)”,
   - \( (b_{h, \xi}) \) “\( p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\gamma} h(\xi) \notin \dot{H}(\xi) \)”.

Proof. Working on \( V^{\mathbb{A}_\gamma/\gamma} \) set \( \text{dom}(F) = \mu \setminus (\text{dom}(g) \cup \text{dom}(g')) \) and let \( \Phi : \prod_{\xi \in \text{dom}(F)} H'(\xi) \cup \omega \to 3 \) be defined by

\[
\Phi(h) = \begin{cases} 
0 & \text{if } \exists \xi \in s ((2)(a_h) \text{ is true but } (2)(b_{h, \xi}) \text{ is false}), \\
1 & \text{if } \forall \xi \in s ((2)(a_h) \text{ and } (2)(b_{h, \xi}) \text{ are both true}), \\
2 & \text{if } (2)(a_h) \text{ is false}.
\end{cases}
\]

By Röwbottom’s lemma 2.31 there is a function \( F \in \prod_{\xi \in \text{dom}(F)} U_{\gamma, \xi}^{\pi^*_\gamma} \), \( F \subseteq H' \), such that for each finite subset \( s \subseteq \text{dom}(F) \) the function \( \Psi = \Phi \upharpoonright \prod_{\xi \in s} F(\xi) \cup \omega \) is constant. Thereby if we prove that for each set \( s \), \( \Psi \) is the constant function 1 this will yield to the proof of the lemma. We shall first show that the third possibility (i.e. 2) for \( \Psi \) is impossible. Indeed, assume that for some \( s \) and some \( h \in \prod_{\xi \in s} F(\xi) \cup \omega \), \( \Phi(h) = 2 \).

Nonetheless since \( F \subseteq H' \), add\((g', H'), h\) is well-defined so that we get a contradiction.

Let us now check that the first case (i.e. 0) is also impossible. Aiming for a contradiction, let \( s \) and \( h \in \prod_{\xi \in s} F(\xi) \cup \omega \) be such that \( \Phi(h) = 0 \). Notice that since \( F \) is homogeneous the former claim is already true for all \( h' \). By definition, this means that for each \( h' \) there is an ordinal \( \xi_h \in s \) such that \( p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\gamma} h(\xi_h) \notin \dot{H}(\xi_h) \) and refining if necessary we may assume this is true for some fixed ordinal \( \xi^* \). On the other hand, since \( p' \leq p \upharpoonright \gamma \), \( p' \cup p \) is a condition in \( \mathbb{A}_\lambda \) such that \( p \cup p' \Vdash_{\mathbb{A}_\lambda} \dot{F}(\xi^*) \notin \dot{H}(\xi^*) \) but also \( p \cup p' \Vdash_{\mathbb{A}_\lambda} \dot{F}(\xi^*) \cap \dot{H}(\xi^*) = \emptyset \). This yields to a contradiction with the first case and thus the second option must always hold. \( \square \)

The following lemma shows that under some hypothesis on \( r \) and \( r' \), \( r' \) forces \( r \) to be in \( P_\lambda \).

Lemma 4.21. Let \( \gamma \in B^* \), \( r = (p, (g, \dot{H})) \in C_\lambda \) and \( r' = (p', (g', \dot{H}')) \in C_\gamma \) such that

1. \( p' \leq_{\mathbb{A}_\gamma} p \upharpoonright \gamma \),
2. \( p' \cup p \Vdash_{\mathbb{A}_\lambda} g \subseteq g' \),
3. \( p \cup p' \Vdash_{\mathbb{A}_\lambda} \forall \xi \in \text{dom}(g') \setminus \text{dom}(g) \ (g'(\xi) \in \dot{H}(\xi)) \).
Then there is a pure extension $r^* \leq_{\mathbb{C}_\gamma} r'$ forcing "$r \in \mathbb{P}_\gamma$". Specifically, provided $\hat{F}$ was the function of the previous lemma,

$$(p', (g', \hat{H}' \wedge \hat{F})) \Vdash_{\mathbb{C}_\gamma} "r \in \mathbb{P}_\gamma"$$

Proof. Let us assume on the contrary that the claim was false and let $(q, (j, J)) \leq_{\mathbb{C}_\gamma} (p', (g', \hat{H}' \wedge \hat{F}))$ forcing the opposite. Combining lemma 4.22 with our current hypothesis it immediately follows that condition (III) must fail, hence there is some ordinal $\xi^* \in \text{dom}(j) \setminus \text{dom}(g)$ such that $q \cup p$ forces "$j(\xi^*) \notin \hat{H}(\xi^*)$". Notice that $q \cup p' \Vdash_{\mathbb{A}_\gamma} j(\xi^*) \in \hat{H}'(\xi^*)$, $q \cup p' \leq p \cup p'$, and (3) hold so that $\xi^* \in \text{dom}(j) \setminus \text{dom}(g)$. On the other hand, since $(q, (j, J)) \leq_{\mathbb{C}_\gamma} (p', (g', \hat{H}' \wedge \hat{F}))$, it turns out that $q$ forces "$j \in [\prod \hat{F}(\xi)]^\omega$" so, by construction of $\hat{F}$, $q$ forces "$p \Vdash_{\mathbb{A}_\lambda/\mathbb{A}_\gamma} h(\xi^*) \notin \hat{H}(\xi^*)$". Altogether it follows that $q \cup p$ forces "$j(\xi^*) \notin \hat{H}(\xi^*)$" as well as the contrary, yielding to the desired contradiction.

Lemma 4.22. Let $\gamma \in \mathcal{B}^*$, $(p, (g, \hat{H})) \in \mathbb{C}_\gamma$ and $\bar{r}_0, \bar{r}_1$ be two $\mathbb{C}_\gamma$-names for conditions in $\mathbb{P}_\lambda$. Then there are $(p^*, (g^*, \hat{H}^*)) \in \mathbb{C}_\gamma$, $(p_0, (g_0, \hat{H}_0))$, $(p_1, (g_1, \hat{H}_1))$ and $\bar{p}_0, \bar{p}_1$ such that for $i \in \{0, 1\}$ the following hold:

1. $(p^*, (g^*, \hat{H}^*)) \leq_{\mathbb{C}_\gamma} (p, (g, \hat{H}))$,
2. $(p_i, (g_i, \hat{H}_i)) \Vdash_{\mathbb{C}_\gamma} \bar{r}_i = (p_i, (g_i, \hat{H}_i))$,
3. $(p^*, (g^*, \hat{H}^*))$ and $(\bar{p}_i, (g_i, \hat{H}_i))$ satisfy conditions (1)-(3) of lemma 4.21.

Proof. Let $(p^*, (g^*, \hat{H}^*)) \leq_{\mathbb{C}_\gamma} (p, (g, \hat{H}))$ deciding that $\bar{r}_0$ is some condition in $\mathbb{P}_\lambda$ of the form $(p_0, (g_0, \hat{H}_0))$. By virtue of lemma 4.18, the conditions (I)-(IV) fails, hence in particular (II) and (III) fails, and thus there is some condition $\bar{p}_0 \leq p^* \cup p_0$ forcing "$g^*(\xi) \notin \hat{H}_0(\xi)$", for each ordinal $\xi \in \text{dom}(g^*) \setminus \text{dom}(g_0)$, and moreover $g^*$ and $g_0$ have the same value in their common coordinates. Extending if necessary both $p^*$ and $g^*$ we get that $(p^*, (g^*, \hat{H}^*))$ and $(p_0, (g_0, \hat{H}_0))$ satisfy (a)-(c). Repeating the same argument but this time below the condition $(p^*, (g^*, \hat{H}^*))$ instead of $(p, (g, \hat{H}))$ the result follows.

At this point we are finally in conditions to prove the $\delta$-ccness of the product forcing $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$, for each $\gamma \in \mathcal{B}^*$.

Lemma 4.23. Let $\gamma \in \mathcal{B}^*$. Then the product $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$ is $\delta$-cc in the generic extension $V^{\mathbb{C}_\gamma}$.

Proof. Let $\{\bar{r}_0^\alpha, \bar{r}_1^\alpha\}_{\alpha \in \delta}$ be a collection of $\mathbb{C}_\gamma$-names for a maximal antichain within the product $\mathbb{P}_\gamma \times \mathbb{P}_\gamma$. Applying the previous lemma

---

\textsuperscript{19} Notice that in particular this implies that $p^* \Vdash p_0$ hence $p \cup p \in \mathbb{A}_\lambda$. 
we find families \( \{ (p^0_\alpha, (g^0_\alpha, \check{H}^0_\alpha)), (p^1_\alpha, (g^1_\alpha, \check{H}^1_\alpha)) \}_{\alpha \in \delta} \) and \( \{ (\check{p}^0_\alpha, \check{H}^0_\alpha) \}_{\alpha \in \delta} \) witnessing \((a)-(c)\).

Notice that \( C_\gamma \times C^2_\delta \) is \( \delta \)-Knaster, hence \( \delta \)-cc, as each factor is \( \delta \)-Knaster and this property is productive. Therefore we may assume without loss of generality that all the above displayed conditions are compatible and moreover that \( g^*_\alpha = g^*, g^0_\alpha = g^0 \) and \( g^1_\alpha = g^1 \), for each \( \alpha \in \delta \).

Notice that for each \( \alpha < \alpha' \in \delta \) the conditions \( (p^0_\alpha \cup p^0_{\alpha'}, (g, \check{H}^*_\alpha \land \check{H}^*_{\alpha'})) \) and \( (\check{p}^i_\alpha \cup \check{p}^i_{\alpha'}, (g^i, \check{H}^i_\alpha \land \check{H}^i_{\alpha'})) \), \( i \in \{0,1\} \), satisfies \((a)-(c)\) of lemma \([4.21]\) so that there is an \( r^* \)-extension of \((p^0_\alpha \cup p^0_{\alpha'}, (g, \check{H}^*_\alpha \land \check{H}^*_{\alpha'})) \) forcing that both \((\check{p}^0_\alpha \cup \check{p}^0_{\alpha'}, (g^0, \check{H}^0_\alpha \land \check{H}^0_{\alpha'})) \) and \((\check{p}^1_\alpha \cup \check{p}^1_{\alpha'}, (g^1, \check{H}^1_\alpha \land \check{H}^1_{\alpha'})) \) are in \( P_\lambda \). In particular, \( r^* \) forces the compatibility of \((\check{r}^0_\alpha, \check{r}^1_\alpha)\) and \((\check{r}^0_{\alpha'}, \check{r}^1_{\alpha'})\) and thus \( P_\gamma \times P_\gamma \) is \( \delta \)-cc. \( \square \)

**4.2. Tree property with an arbitrary gap.** In this section we will present the argument for making an arbitrary gap within the interval \( (\kappa, 2^\kappa) \). The ideas for proving this are essentially the same as those presented in the former section so that instead of repeating the proof of each one of the previous results we shall simply present the necessary modifications in the argument.

1. Assume that \( \kappa \) is a supercompact cardinal indestructible by \( \kappa \)-directed closed forcing. After forcing with \( \mathbb{A}_\gamma \), we get a generic extension where \( \kappa \) remains supercompact and \( 2^\kappa \geq \gamma \). Let \( G \subseteq \mathbb{A}_\gamma \), a generic filter and denote by \( G \upharpoonright x \) its natural projection onto \( \mathbb{A}_x \). Working on \( V[G] \) recall that \( \varepsilon = \sup_{\xi < \kappa \xi} \kappa \xi \) and \( \delta = \varepsilon^+ \) and let \( U \) be a supercompact measure on \( P_\kappa(\delta) \). For each \( \xi < \mu \) let \( \pi^\xi \) be the standard projection between \( P_\kappa(\delta) \) and \( P_\kappa(\kappa \xi) \) and \( U^\xi = (\pi^\xi)^*U \) be the corresponding Rudin-Keisler projection of \( U \). Set \( S \) the Sinapova forcing defined using the sequence of measures \( \langle U^\xi : \xi < \mu \rangle \).

Arguing as in lemma \([??]\), we obtain an unbounded family \( \mathbb{A} \subseteq [\gamma]^\lambda \) closed under unions of length \( \geq \kappa^+ \) such that for each \( x \in \mathbb{A} \), \( \lambda + 1 \subseteq x \) and \( U \) reflects below \( x \); namely,

\[
U_x = U_G \cap V[G \upharpoonright x] \subseteq V[G \upharpoonright x].
\]

For each \( x \in \mathbb{A} \) and \( \xi < \mu \) let \( \check{U}_x \) be a \( \mathbb{A}_x \)-name such that \( (\check{U}_x)_{G \upharpoonright x} = U_x \) and \( U^\xi_x \) be the Rudin-Keisler projection of \( U_x \) onto \( P_\kappa(\kappa \xi)^{V[G \upharpoonright x]} \). For each such \( x \in \mathbb{A} \) set \( \check{S}_x \) be a \( \mathbb{A}_x \)-name for the Sinapova forcing defined with respect to the sequence \( \langle U^\xi_x : \xi < \mu \rangle \).

2. Choose \( x_0 \in \mathbb{A} \) joint with an isomorphism \( \pi \) between \( \mathbb{A}_{x_0} \) and \( \mathbb{A}_{\text{Even}(\lambda)} \). Notice that \( \pi(\check{U}_{x_0}) \) is a \( \mathbb{A}_{\text{Even}(\lambda)} \)-name for a supercompact measure on \( P_\kappa(\delta)^{V[\pi(G)_{x_0}]} \) such that \( U_{x_0} = \pi(\check{U}_{x_0})\pi(G_{x_0}) \).

3. Denote \( \check{\mathbb{A}} = \{ x \in \mathbb{A} : x_0 \subseteq x \} \). Now argue as in lemma \([??]\) to get \( \mathcal{B} \subseteq \lambda \) closed under unions of length \( \geq \kappa^+ \) such that
\( \delta < \min B \) and for each \( x \in B \), \( \pi(U_{x_0}) \) reflects below \( \alpha \): namely, setting \( H = \pi(G \upharpoonright x_0) \) for each \( \alpha \in B \),
\[
U^\pi_{\alpha} = \pi(U_{x_0}) \cap V[H \upharpoonright \text{Even}(\alpha)] \in V[H \upharpoonright \text{Even}(\alpha)].
\]
For each \( \alpha \in B \) and \( \xi < \mu \) let \( \dot{U}^\pi_{\alpha} \) be a \( \dot{\lambda}_{\text{Even}(\alpha)} \)-name such that \((\dot{U}^\pi_{\alpha})_H = U^\pi_{\alpha} \) and \( U^\pi_{\xi \alpha} \) be the Rudin-Keisler projection of \( U^\pi_{\alpha} \) onto \( P_\kappa(\kappa^\xi)^{V[H[a]} \). For each \( x \in B \) set \( \dot{S}^\alpha_{\alpha} \) be a \( \dot{\lambda}_{\text{Even}(\alpha)} \)-name for the Sinapova forcing defined with the sequence of measures \( \langle U^\xi_{\alpha} : \xi < \mu \rangle \). Arguing similarly to lemma 4.9 there is a family of projections
\[
\sigma^\gamma_x : A_{\gamma} \ast \dot{S} \to RO^+(A_x \ast \dot{S}_x), \quad \text{each } x \in \dot{A},
\]
\[
\dot{\sigma}^x : RO^+(A_x \ast \dot{S}_x) \to RO^+(A_{\text{Even}(\alpha)} \ast \dot{S}^\alpha_{\alpha}), \quad \text{each } x \in \dot{A} \text{ and } \alpha \in B,
\]
\[
\sigma^\gamma_{\alpha} : A_{\gamma} \ast \dot{S} \to RO^+(A_{\text{Even}(\alpha)} \ast \dot{S}^\alpha_{\alpha}), \quad \text{each } \alpha \in B,
\]
which commutes
\[
\sigma^\gamma_x = \dot{\sigma}^x \circ \sigma^\gamma_{\alpha}, \quad \text{for } x \in \dot{A} \text{ and } \alpha \in B.
\]

(4) Using the former projections define \( \mathbb{R} \) as well as its corresponding truncations \( \mathbb{R} \upharpoonright x \) (see definition 4.11), for each \( x \in \dot{A} \). Notice that both \( \mathbb{R} \) and \( \mathbb{R} \upharpoonright x \) forces that \( \lambda = \kappa^{++} \).

Assuming the existence of a \( \lambda \)-Aronszajn tree \( T \) in \( V^\mathbb{R} \) we can argue similarly to lemma 4.4 than there is a set \( x_0 \subsetneq x^* \in \dot{A} \) such that \( \mathbb{R} \upharpoonright x^* \) forces that \( T \) is a \( \lambda \)-Aronszajn tree. Thus the discussion of the tree property at \( \lambda \) in \( V^\mathbb{R} \) translates to the study of \( TP(\lambda) \) in the different generic extensions given by truncations of \( \mathbb{R} \).

(5) Let \( \pi^* \) be an isomorphism between \( A_{x^*} \) and \( A_{\lambda} \) extending \( \pi \). Once more, notice that \( U_{x^*} = \pi^*(\dot{U}_{x^*})_{x^*}(G[x^*]) \) and that \( \pi^*(\dot{U}_{x^*})_{x^*}(G[x^*]) \) extends the measure \( \pi(\dot{U}_{x^*})_{G[x^*]} \). Let \( \dot{S}^\pi_{\alpha} \) be the Sinapova forcing defined by the corresponding Rudin-Keisler projections of the measure \( \pi^*(\dot{U}_{x^*})_{x^*}(G[x^*]) \). Arguing as in proposition 4.2 we show that \( \pi^* \) extends to an isomorphism between \( A_{x^*} \ast \dot{S}^\pi_{\alpha} \) and \( A_{\lambda} \ast \dot{S}^\pi_{\alpha} \) so we can use it to define \( \mathbb{R}^* \) (see definition 4.5) and furthermore to prove that \( \mathbb{R}^* \) and \( \mathbb{R} \upharpoonright x^* \) are isomorphic. In particular, since \( \mathbb{R} \upharpoonright x^* \) forces \( \text{TP}(\lambda) \), \( \mathbb{R}^* \) adds a \( \lambda \)-Aronszajn tree.

(6) Arguing as in lemma 4.7 there is a set \( B^* \in \mathcal{F}_\lambda \) where for each \( \gamma \in B^* \) the measure \( \pi^*(\dot{U}_{x^*})_{x^*}(G[x^*]) \) projects below \( \gamma \): namely, setting \( H^* = \pi^*(G \upharpoonright x^*) \),
\[
U^\pi_{\gamma} = \pi^*(\dot{U}_{x^*})_H \cap V[H \upharpoonright \gamma] \in V[H \upharpoonright \gamma].
\]
Now with lemma 4.9 one defines the truncations \( \mathbb{R}^* \upharpoonright \gamma \) of \( \mathbb{R}^* \) (see definition 4.10) and proves that any \( \lambda \)-Aronszajn tree in \( V^{\mathbb{R}^*} \) reflects to some \( \gamma \)-Aronszajn tree, \( \gamma \in B^* \), in \( V^{\mathbb{R}^* \upharpoonright \gamma} \).
From this point on all the discussion on the quotient forcings $R^*/R \upharpoonright \gamma$ as well as on $P_\gamma \times Q_\gamma$ is parallel to that carried out so far hence yielding the proof of $\text{TP}(\lambda)$ in $V^R$.

5. TP($\kappa^+$) holds

At the present section we show that TP($\kappa^+$) holds in the generic extension given by $R$. The ideas that we expose here are originally due to Sinapova and the only purpose for exposing them is to convince the reader that the same arguments apply. As the proof we present here runs in parallel to that of [Sin16b] we shall simply give a taste of the main ideas involved in the argument and refer the reader to Sinapova’s article for a complete treatment of the problem. Along the section we shall take profit of the notations and definitions setted in 3.

Broadly speaking, the idea to prove that in $V[R]$ any $\delta$-tree $T$ has a cofinal branch is as follows: Assume that $T$ has no cofinal branches in $V[R]$. In first place we prove that $T$ has a cofinal branch $b$ in $V[\bar{R}]$ and show that this implies, since $b \notin V[R]$, that for each $\alpha < \delta$ there is a splitting at a node $u \in T_\alpha$ (cf proposition 5.13). Provided there is a splitting at a node $u$ we shall show that necessarily there is an $h$-splitting at $u$ for some stem $h$ of a condition in $S$ where $(\dag_h)$ holds (cf definitions 5.16 and 5.12). Finally using arguments in [Sin16b] one can show that provided $(\dag_h)$ holds then the supremum $\alpha_h$ of all levels $\gamma$ of $T$ where an $h$-splitting occurs at some node $u \in T_\gamma$ is less than $\mu$. Thereby the levels of $T$ where some $h$-splitting occur is bounded, yielding a contradiction with the non existence of cofinal branches in $V[R]$.

For implement this argument we first need to define a collection of intermediate forcing between $R$ and $\bar{R}$ whose generic filters “resemble” the generic filter $R$. More formally, for each $A_\lambda^+$-name $\dot{q}$ for a condition in Sinapova forcing we define a forcing $R_{\dot{q}}$ that intuitively corresponds with the subforcing of $R$ obtained when the Sinapova part is $\dot{q}$.

**Notation 5.1.** Let $G \subseteq A_\lambda^+$ a generic filter over $V$. For each $A_\lambda^+$-name $\dot{q}$ for a condition in Sinapova forcing we shall denote by $q$ its interpretation by $G$.

**Definition 5.2.** Let $\dot{q}$ be a $A_\lambda^+$-name for a condition in Sinapova forcing. The forcing $R_{\dot{q}}$ is defined as the set of conditions $(p, \dot{q}, r) \in \mathbb{R}$ endowed with the order $(p_1, \dot{q}_1, r_1) \leq_{R_{\dot{q}}} (p_2, \dot{q}_2, r_2)$ iff

1. $(p_1, \dot{q}_1) \leq_{A_\lambda^+ \times S_\lambda^+} (p_2, \dot{q}_2)$,
2. $\text{dom}(r_2) \subseteq \text{dom}(r_1)$ and for each $\gamma \in \text{dom}(r_2)$,
   $$\sigma_\gamma^{A_\lambda^+} (p_1, \dot{q}) \forces_{A_{\text{Even}}(\gamma) \times S_\gamma} r_1(\gamma) \leq_{\text{Add}(\delta, 1)} r_2(\gamma).$$

As the next proposition shows there is a system of projections between the forcings $\mathbb{R}$, $R$ and $R_{\dot{q}}$. 
Proposition 5.3. Let $\dot{q}$ be a $\mathbb{A}_{\lambda^+}$-name for a condition in Sinapova forcing and set $\dot{q} = \langle (1, \dot{q}), (1, 1, 1) \rangle$, $q^* = (1, \dot{q}, 1)$.

(1) The map $(p, \dot{i}, (1, 1, r)) \mapsto (p, \dot{i}, r)$ defines a projection between $\mathbb{R}$ and $\mathbb{R}_{\dot{q}}$ and further between $\mathbb{R}/\dot{q}$ and $\mathbb{R}_{\dot{q}}/q^*$.

(2) The identity map gives a projection between $\mathbb{R}_{\dot{q}}/q^*$ and $\mathbb{R}/q^*$.

Moreover, $\mathbb{R}_{\dot{q}}/q^*$ is isomorphic to a dense subposet of $\mathbb{R}/q^*$.

Letting $\dot{q}, \dot{i}$ be two $\mathbb{A}_{\lambda^+}$-names for two conditions in $\mathbb{S}$ such that $\Vdash_{\mathbb{A}} \dot{i} \leq \dot{q}$, the identity establishes a projection between $\mathbb{R}_{\dot{q}}$ and $\mathbb{R}_i$.

Proof. We shall give the proof of (2) as well as of the last claim as the argument for (1) is similar.

Let us begin with the proof of the last claim. Since $\Vdash_{\mathbb{A}} \dot{i} \leq \dot{q}$ it is easy to check that the identity map between $\mathbb{R}_{\dot{q}}$ and $\mathbb{R}_i$ is well-defined and order preserving. Now suppose that $(p_1, q_1, r_1) \in \mathbb{R}_i$ and $(p_2, q_2, r_2) \in \mathbb{R}_{\dot{q}}$ are such that $(p_1, q_1, r_1) \leq_{\mathbb{R}_{\dot{q}}} (p_2, q_2, r_2)$. We shall show that there is some condition $(p^*, \dot{q}^*, r^*) \in \mathbb{R}_{\dot{q}}$ such that $(p^*, \dot{q}^*, r^*) \leq_{\mathbb{R}_{\dot{q}}} (p_1, q_1, r_1)$ and $(p^*, \dot{q}^*, r^*) \leq_{\mathbb{R}_{\dot{q}}} (p_2, \dot{q}_2, r_2)$. Let $p^* = p_1$, $\dot{q}^* = \dot{q}_1$ and for each $\gamma \in \text{dom}(r_1)$ let

$$r^*(\gamma) = \{ (\tau, (p, \dot{u})) \in V^{\mathbb{A}_{\lambda^+} \ast S_{\lambda^+}} : (\sigma_{\gamma}^{\lambda^+}((p, \dot{u})) \parallel \sigma_{\gamma}^{\lambda^+}(p_1, \dot{i}) \rightarrow \\
\sigma_{\gamma}^{\lambda^+}((p, \dot{u})) \Vdash_{\mathbb{R}_{\dot{q}}} \tau \in r_1(\gamma) \lor (\sigma_{\gamma}^{\lambda^+}((p, \dot{u})) \perp \sigma_{\gamma}^{\lambda^+}(p_1, \dot{i}) \rightarrow \sigma_{\gamma}^{\lambda^+}((p, \dot{u})) \Vdash_{\mathbb{R}_{\dot{q}}} \tau \in r_2(\gamma)) \}$$

Define $r^* = \{ (\gamma, r^*(\gamma)) : \gamma \in \text{dom}(r_1) \}$. It is easy to check that $(p^*, \dot{q}^*, r^*)$ satisfies the desired property.

On the other hand, since the third coordinate of any condition in $\mathbb{R}_{\dot{q}}/q^*$ is below the empty condition, it is immediate that for each two conditions $s_1$ and $s_2$,

$$s_1 \leq_{\mathbb{R}_{\dot{q}}/q^*} s_2 \iff s_1 \leq_{\mathbb{R}/q^*} s_2.$$

From this equivalence one can proceed as before and check that the identity map establishes a projection between these forcings.

It is not hard to check that $\mathbb{R}_{\dot{q}}$ is isomorphic to $\mathbb{A}_{\lambda^+} \ast (S_{\lambda^+} \times \mathbb{U}_q)$, for some forcing $\mathbb{U}_q \in V[G]$. There is an obvious candidate for $\mathbb{U}_q$ and is that given by the next definition:

Definition 5.4. Work in $V[G]$. For each $q \in \mathbb{S}$ define the forcing $\mathbb{U}_q$ whose conditions are all $r \in \mathbb{U}$ such that $r_1 \leq_{\mathbb{U}_p} r_2$ if and only if $\text{dom}(r_2) \subseteq \text{dom}(r_1)$ and there is $p \in G$ such that for each $\gamma \in \text{dom}(r_2)$,

$$\sigma_{\gamma}^{\lambda^+}((p, \dot{q})) \Vdash_{\mathbb{R}_{\dot{q}}} \tau_1(\alpha) \leq \tau_2(\alpha).$$

Arguing exactly as in proposition 5.3 we can prove the existence of a family of projections between the forcings $\mathbb{U}_q$.
Proposition 5.5. Working in $V[G]$, for each condition $q \in S$ the identity gives a projection between $U$ and $U_q$. Moreover, for each $t \leq q$ the same holds between $U_q$ and $U_t$.

The proof of the next lemma can be found in \cite{Sin16b}:

Lemma 5.6. Let $\hat{q}$ be a $\mathbb{A}_{\lambda^+}$-name for a condition in $\mathbb{S}_{\lambda^+}$, then $R_\hat{q}$ and $\mathbb{A}_{\lambda^+} \ast (\hat{S}_{\lambda^+} \ast U_q)$ are isomorphic. In particular, in $V[G]$ there is a projection between $\mathbb{R}_q$ and $U_q$.

Let $\bar{R} \subseteq \mathbb{R}$ generic whose respective projections onto $\mathbb{R}$, $\mathbb{A}_{\lambda^+}$ and $\mathbb{S}_{\lambda^+}$ yield the generics $R$, $G$ and $S$ as quoted just before proposition 5.10. Let $U \subseteq \mathbb{U}$ be the generic filter induced by $\bar{R}$ and the natural projection between $\mathbb{R}$ and $\mathbb{U}$. Now it is necessary to get generics for the family of forcing $\langle R_p, U_p : p \in S \rangle$. Before doing that we need to prove a technical result:

Lemma 5.7. Let $P$, $Q$, $C$ posets and $\pi : P \to Q$ and $\sigma : Q \to C$ be projections. For any generic filter $H \subseteq C$, the restriction $\pi \restriction P/H$ is a projection between $P/H$ and $Q/H$.

Proof. Let $\psi = \pi \restriction P/H$ and $\varphi = \sigma \circ \pi$. It is not hard to check that $\psi$ is well-defined and order preserving. Now assume $q \leq_{Q/H} \psi(p)$ or some condition $p \in P/H$ and $q \in Q/H$ and let us prove there is $p' \in P/H$ such that $p' \leq_{P/H} p$ and $\psi(p') \leq_{Q/H} q$. Let $D$ be the set of conditions $D = \{ r \in C : \exists p' \in P, p' \leq_{P} p, \pi(p') \leq_{Q} q, \varphi(p') = r \}$.

Since $\pi$ and $\sigma$ are projections one can easily check that $D$ is dense below $\sigma(q)$ and thus, since $\sigma(q) \in H$, there is some $r \in H \cap D$. By definition of $D$ there is some $p' \in P/H$ below $p$ such that $\pi(p') \leq_{Q/H} q$ which leads to the end of the proof.

For any $q \in S$, since $q^* \in R$, $R$ is a generic filter for $\mathbb{R}/q^*$. Since there are projections $\pi_q$ between $\mathbb{R}/\hat{q}$ and $\mathbb{R}_{q^*}/q^*$ and $\pi^q$ between $\mathbb{R}/q^*$ and $\mathbb{R}/q^*$ (cf propositions 3.9 and 5.3) the previous lemma ensures that the restriction of $\pi_q$ to $\mathbb{R}/R$ is a projection between $\mathbb{R}/R$ and $\mathbb{R}/q^*$. For each $q \in S$ let $R_q \subseteq \mathbb{R}_q/q^*$ be the generic filter over $V[R]$ induced by $\bar{R}$ and these projections. Analogously we let $U_q \subseteq \mathbb{U}_q$ the generic filters over $V[G]$ induced by $\bar{R}$ and the corresponding projections.

Remark 5.8.

1. By proposition 5.3 $\bar{R} \subseteq R_q \subseteq R_{q'} \subseteq R$, each $q' \leq q$ in $S$. In particular, $\bar{R} \subseteq \bigcup_{q \in S} R_q \subseteq R$.
2. By proposition 5.3 $U \subseteq U_q \subseteq U_{q'}$, each $q' \leq q$ in $S$. In particular, $U \subseteq \bigcup_{q \in S} U_q$.

Let $T \in V[R]$ be a $\delta$-tree and without loss of generality assume that for each $\alpha < \delta$ the $\alpha$-th level of $T$, $T_\alpha$, is of the form $\{ \alpha \} \times \kappa$. The next notion firstly appeared in \cite{Nee09} and it is the main ingredient to define a branch for $T$: 
Definition 5.9. Working in $V[G][U]$ we say that $(\dagger)$ hold$^{2}$ if there is $\langle p_\alpha, u_\alpha : \alpha \in J \rangle$ such that

1. $J$ is an unbounded subset of $\delta$.
2. For each $\alpha \in J$, $p_\alpha \in S$ with stem $h$.
3. For each $\alpha \in J$, $u_\alpha \in T_\alpha$.
4. For each $\alpha < \beta$ in $J$, $p_\alpha \land p_\beta \Vdash_S \delta \in [\alpha, \beta)$.

(\text{cf \ definition 2}.)

Using ideas of Neeman \cite{Nee09} or of Golshani \cite{Gol17} and that $U$ is $S$-closed, hence all the supercompact cardinals $\kappa_{\xi+1}$ are preserved after forcing with $U$ and $S^V[G] = S^V[G][U]$, one can prove the following:

Theorem 5.10 (Neeman). In $V[G][U]$ the principle $(\dagger)$ holds.

Proposition 5.11. There is a cofinal branch $b$ through $T$ in $V[R]$.

Proof. In first place notice that $J^* = \{ \alpha \in J : p_\alpha \in S \}$ is an unbounded set in $V[R]$. Otherwise, by the $\delta$-cneness of $S$ over $V[G][U]$, there would be an ordinal $\gamma < \delta$ such that $\Vdash_S \delta \in [\alpha, \beta)$.

Set $b = \{ u \in T : \exists \alpha \in J^* (u \leq_T u_\alpha) \} \in V[R]$ and let us show that $b$ is a branch through $T$. Since $J^*$ is unbounded it is clear that for each $\alpha < \delta$, $T_\alpha \cap b \neq \emptyset$.

Let $u, v \in b$ and $\alpha < \beta \in J^*$ be such that $u \leq_T u_\alpha$ and $v \leq_T u_\beta$. Since $p_\alpha \land p_\beta \Vdash_S \delta \in [\alpha, \beta)$, hence $p_\alpha \land p_\beta \Vdash_S [\alpha, \beta) \in \delta$.

The following parametric version of the principle $(\dagger)$ will become necessary in further arguments:

Definition 5.12. Let $h$ be a stem. We will say that $(\dagger)_{h}$ holds if in $V[G][U]$ there are $J \subseteq \delta$ unbounded and $\langle p_\alpha, u_\alpha : \alpha \in J \rangle$ such that

- For each $\alpha \in J$, $p_\alpha$ is a condition in $S$ with stem $h$ and $u_\alpha \in T_\alpha$.
- For each $\alpha \in J$, $p_\alpha \Vdash_S \delta \in [\alpha, \beta)$. $u_\alpha \in b$.

It is worth to point out that the arguments of Theorem 5.10 can be applied to show that for each stem $h$ there is a $h \subseteq h^\ast$ for that one simply has to use the same arguments as in the proof of the theorem but starting with a condition $q \in \mathcal{S}$ with stem $h$.

Let $\tau \in V[G]$ be $R/G$-name for the tree $T$ and assume the empty condition forces this property. Let $\dot{T} \in V[G][U]$ be the $S$-name for the tree $T$ induced by $\tau$; namely,

$q \Vdash_S \delta \in [\alpha, \beta) \iff \exists p \in G \forall r \in U (p, q, r) \Vdash_R \delta \in [\alpha, \beta) \iff u <_\tau v$.

\footnote{Here there is a dependence of the tree $T$ that we omit for the sake of economy in the terminology.}
For each $q \in S$ one analogously defines $T^*_q \in V[G][U_q]$ as the $I$-name for the tree $T$ induced by $\tau$; namely, for each $t \leq q$,
\[ t \Vdash_{S^\mathbb{R}\times U} u <_T v \iff \exists p \in G \exists r \in U \ (p, i, r) \Vdash_{\mathbb{R}/G} u <_\tau v. \]

Let $\dot{b}$ be a $\dot{\mathbb{R}}/\mathbb{G}$-name for the branch of $T$ and assume that
\[ \Vdash_{S^\mathbb{R}\times U} "\dot{b} \mbox{ cofinal branch in } \tau" \]
Since $\dot{\mathbb{R}}/\mathbb{R}$ is a complete subposet of $\dot{\mathbb{R}}/\mathbb{G}$ we may regard $\dot{b}$ and $\tau$ also as $\dot{\mathbb{R}}/\mathbb{R}$-names for the branch and the tree as well as assume that
\[ \Vdash_{\mathbb{R}/\mathbb{R}} "\dot{b} \mbox{ cofinal branch in } \tau". \]

**Proposition 5.13.** Work in $V[R]$. If there is no cofinal branch through $T$ in $V[R]$ then for every $\alpha < \delta$, $s \in \dot{\mathbb{R}}/\mathbb{R}$ and $u \in T_\alpha$ with $s \Vdash_{\dot{\mathbb{R}}/\mathbb{R}} u \in \dot{b}$, there is a splitting at node $u$; namely, there are $\beta > \alpha$, $s_1, s_2 \subseteq_{\dot{\mathbb{R}}/\mathbb{R}} s$ and $v_1, v_2 \in T_\beta$ such that
\[
\begin{align*}
(1) & \ s_\alpha \Vdash_{\dot{\mathbb{R}}/\mathbb{R}} v_k \in \dot{b}, \ k \in \{0, 1\}, \\
(2) & \ v_1 \neq v_2.
\end{align*}
\]

**Proof.** Assume the contrary and let $s^*$ and $\alpha$ witnessing this. Set
\[ d = \{ v \in T : \exists s \in \mathbb{R}/\mathbb{R} (s \leq_{\mathbb{R}/\mathbb{R}} s^*) \land s \Vdash_{\dot{\mathbb{R}}/\mathbb{R}} v \in \dot{b} \} \]
and let us prove that $d$ is a branch through $T$ in $V[R]$. Of course $d \in V[R]$ so that it remains to check that $d$ is a branch. By our assumption, for each $\beta > \alpha$ there is only one node in $d$ at level $\beta$, hence $|d \cap T_\beta| = 1$. Now let us show that any two nodes in $d$ are compatible. Let $u, v \in d$, $s, s' \in \mathbb{R}/\mathbb{R}$ be two conditions witnessing this and let $\gamma$ be an ordinal above $\alpha$ and the levels of $u$ and $v$. Since the trivial condition forces that $\dot{b}$ is a branch there is $\bar{s} \subseteq_{\mathbb{R}/\mathbb{R}} s$ and a node $w \in T_\gamma$ such that $\bar{s} \Vdash_{\mathbb{R}/\mathbb{R}} w \in \dot{b}$. In particular, $w \in T_\gamma \cap d$ and $u <_T w$. Analogously, there is $w' \in T_\gamma \cap d$ such that $v <_T w'$. By our hypothesis $T_\gamma \cap d$ is a singleton, in particular $w = w'$, hence $u, v <_T w$, and thus $u$ and $v$ must be compatible. Altogether this leads to the desired contradiction. \[ \square \]

The previous proposition shows that, provided $V[R]$ does not contain any cofinal branch through $T$, for densely many conditions in $\mathbb{R}/\mathbb{R}$ and for each level $\alpha$ of $T$ there is a splitting at some node $u \in T_\alpha$. In particular if we manage to prove that the amount of levels where a splitting occurs is bounded then we shall conclude that some cofinal branch through $T$ exists in $V[R]$ and thus that TP($\kappa^+$) holds in that generic extension. Along the subsequent results we shall argue that provided there is a splitting at a node $u$ then there is an $h$-splitting at $u$ for some stem $h$ for a condition in $S^{V[G]}$ and latter that the amount

\footnote{This is a consequence of the fact that $\dot{\mathbb{R}}/\mathbb{G}$ projects onto $1 \times \mathbb{U}$ in $V[G]$.}
of levels where some $h$-splitting occurs is bounded. Altogether this will yield the desired proof of $\text{TP}(\kappa^+)$. 

The following proposition will be necessary for future arguments:

**Lemma 5.14.** \( \mathbb{R}/R = \bigcup_{q \in S} R_q \). 

**Proof.** By remark 5.8 it is clear that \( \mathbb{R}/R \supseteq \bigcup_{q \in S} R_q \) so that it remains to prove the converse inclusion. Let \( s \in \mathbb{R}/R, s = ((p, \dot{q}), (1, 1, r)) \). We claim that 
\[
D = \{ s' \in \mathbb{R}/R : \exists q \in S, s' \leq_{\mathbb{R}/R} s \}
\]
is a dense set. Indeed, let \( s' = ((p', \dot{q}'), (1, 1, r')) \) in \( \mathbb{R}/R \). Since \( s, s' \) may be identified with a conditions in \( R \) we may assume without loss of generality that there is some \( s'' \in R, s'' = ((p'', \dot{q}''), (1, 1, r'')) \), such that \( s'' \leq_{\mathbb{R}/R} s, s' \). Clearly \( q'' \in S \) since \( s'' \in R \). In fact by definition of \( R_{q''}, s'' \leq_{\mathbb{R}/R} R_{q''}/R, s, s' \). Using the projection between \( \mathbb{R}/R \) and \( \mathbb{R}_{q''}/R \) we can find a condition \( \bar{s} \in \mathbb{R}/R \) such that \( \bar{s} \leq_{\mathbb{R}/R} s' \) and \( \bar{s} \leq_{\mathbb{R}_{q''}/R} s'' \) which proves that \( D \) is a dense set. Let \( s' \in D \cap R \) and \( q \in S \) witnessing this. In such case \( s' \in R_q \) and thus \( s \in R_q \). Finally as \( s \) was arbitrarily chosen it is the case that \( \mathbb{R}/R \subseteq \bigcup_{q \in S} R_q \).

\[ \square \]

**Notation 5.15.** Work in \( V[G] \) and let \( h \) be a stem for a condition in \( S \). We denote by \( E_h \) the set of nodes \( u \) for which there are \( (q, r) \in S \times U \) such that the stem of \( q \) is \( h \), \( r \in U \) and \( (q, r) \models_{V[G][U]} u \in \hat{b} \).

**Definition 5.16.** Work in \( V[G] \) and let \( h \) be a stem for a condition in \( S \). We say that there is an \( h \)-splitting at a node \( u \in T_\alpha \cap E_h \), some \( \alpha < \delta \), if provided \( (q, r) \) witnesses that \( u \in E_h \) there are \( \beta \geq \alpha, v_1, v_2 \in T_\beta \) and \( r_1, r_2 \leq r \) in \( U_q \) such that 
\[
\cdot \ (q, r) \models_{S \times U} v_k \in \hat{b}, k \in \{0, 1\}, \\
\cdot \ q \models_{S[G][U]} v_1 \mathord{\perp_T} v_2.
\]

**Lemma 5.17.** If there is a splitting at some node \( u \) then there is a \( h \)-splitting at \( u \) and moreover \( (\dagger)_h \) holds.

**Proof.** Let \( u \) be some node where a splitting occurs \( \alpha < \delta \) and \( s \in \mathbb{R}/R \) be such that \( s \models_{V[R]_{\mathbb{R}/R}} u \in \hat{b} \cap T_\alpha \). Let \( \beta > \alpha, v_1, v_2 \in T_\beta \), \( v_1 \neq v_2 \), and \( s_1, s_2 \leq_{\mathbb{R}/R} s \) such that \( s_k \models_{\mathbb{R}/R} v_k \in \hat{b} \cap T_\beta, k \in \{1, 2\} \). Say \( s_1 = ((p_1, q_1), (1, 1, r_1)) \) and \( s_2 = ((p_2, q_2), (1, 1, r_2)) \). Using lemma 5.14 and remark 5.8 we can find some condition \( q \in S \) such that \( s_1, s_2 \in R_q \) and moreover, since \( s_1 \) and \( s_2 \) are compatible and thus \( q_1 \) and \( q_2 \) have the same stem, that \( q \leq S q_1, q_2 \) and \( r_1, r_2 \in U_q \). Since \( (\dagger)_h \) holds for densely many \( h \) (see discussion after definition 5.12) we may take the stem of \( q \) in such a way that this principle holds. Set \( h \) be this stem and \( s'_1 = ((p_1, \dot{q}_1), (1, 1, r_1)), s'_2 = ((p_2, \dot{q}_2), (1, 1, r_2)) \), noticing that
Finally it is not hard to check that \( s'_1, s'_2, v_1 \) and \( v_2 \) witness that there is an \( h \)-splitting at \( u \), for some stem \( h \) where \((\dagger)_h\) holds.

**Notation 5.18.** For a fixed stem \( h \) set

\[
\alpha_h = \sup\{\gamma \in \delta : \exists u \in T_\gamma \cap E_h \text{ "There is an } h \text{-splitting at } u\}\}.
\]

The next proposition provides the crucial argument to get a branch for \( T \) in \( V[R] \). The proof is step-by-step identical to that exposed after Proposition 3.4 of \cite{Sin16b} so that we refer the interested to the original reference for details.

**Proposition 5.19.** Let \( h \) be a stem. If \((\dagger)_h\) holds then \( \alpha_h < \delta \). In particular, \( \Upsilon = \sup_h \alpha_h < \delta \).

Combing the previous proposition with lemma 5.17 one has that there are no splittings at levels \( \alpha \geq \Upsilon \), hence the number of levels where a splitting occurs is bounded, and thus proposition 5.13 guarantees the existence of a cofinal branch through \( T \) in \( V[R] \).

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