REAL ENTROPY RIGIDITY UNDER QUASI-CONFORMAL DEFORMATIONS

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Abstract. We set up a real entropy function $h_R$ on the space $\mathcal{M}_d'$ of Möbius conjugacy classes of real rational maps of degree $d$ by assigning to each class the real entropy of a representative $f \in \mathbb{R}(z)$; namely, the topological entropy of its restriction $f \mid_{\hat{\mathbb{R}}}$ to the real circle. We prove a structure theorem for the real Julia set $\mathcal{J}_R(f) := \mathcal{J}(f) \cap \hat{\mathbb{R}}$ that will be utilized to establish a rigidity result stating that $h_R$ is locally constant on the subspace determined by real maps which are quasi-conformally conjugate to $f$. We also compare the real locus $\mathcal{M}_d(\mathbb{R})$ of the moduli space $\mathcal{M}_d(\mathbb{C})$ of degree $d$ rational maps to the locus $\mathcal{M}_d'$ of classes with a real representative, and we characterize maps of maximal real entropy.

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1. Introduction

This article focuses on the entropy of real rational maps and its relation with the intersection of the Julia set with the real axis. It is well known that a rational map \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of degree \( d \geq 2 \) has topological entropy \( \log(d) \) and admits a unique measure of maximal entropy \( \mu_f \) whose support is the Julia set \( J(f) \) [FLMn83, Mn83]. By contrast, for \( f \in \mathbb{R}(z) \), the topological entropy

\[
(1.1) \quad h_{\mathbb{R}}(f) := h_{\text{top}}(f \mid_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}})
\]

of the induced dynamics on the invariant circle \( \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \) can take any value between 0 and \( \log(\text{deg } f) \). In this paper, we study the real entropy \( h_{\mathbb{R}}(f) \) both in relation with the dynamics of the ambient map \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) and also as \( f \) varies in a family of real rational maps.

Our main goal is to prove a rigidity statement showing that the real entropy is preserved in a family of quasi-conformally conjugate real maps. Let

\[
\mathcal{M}_d(\mathbb{C}) := \text{Rat}_d(\mathbb{C})/\text{PGL}_2(\mathbb{C})
\]

be the moduli space of degree \( d \) rational maps and consider the real subvariety

\[
\mathcal{M}'_d := \text{PGL}_2(\mathbb{C}). (\text{Rat}_d(\mathbb{R})) / \text{PGL}_2(\mathbb{C})
\]

of dimension \( 2d - 2 \) formed by classes with real representatives. The subvariety \( \mathcal{M}'_d \) is contained in the real locus \( \mathcal{M}_d(\mathbb{R}) \) of the moduli space. We have equality \( \mathcal{M}'_d = \mathcal{M}_d(\mathbb{R}) \) if and only if \( d \) is even; see Proposition 2.9.

**Theorem 1.1.** The real entropy (1.1) defines a continuous and surjective function

\[
h_{\mathbb{R}} : \mathcal{M}'_d - S' \to [0, \log(d)]
\]

that for every \( f \in \mathbb{R}(z) \) is constant on connected components of \( \mathcal{M}'_d \cap M(f) - S' \).

Here, \( M(f) \) is the subspace of Möbius conjugacy classes of rational maps which are quasi-conformally conjugate to \( f \) and \( S' \) is the intersection with \( \mathcal{M}'_d \) of the symmetry locus \( S(\mathbb{C}) \) with \( S(\mathbb{C}) \) being the subset of conjugacy classes of degree \( d \) maps that admit non-trivial Möbius symmetries. In Proposition 2.4 we prove that \( S(\mathbb{C}) \) is a closed subvariety of \( \mathcal{M}_d(\mathbb{C}) \) of dimension \( d - 1 \); therefore, the domain \( \mathcal{M}'_d - S' \) of \( h_{\mathbb{R}} \) is a real variety of dimension \( 2d - 2 \); Proposition 2.7. It is irreducible as a real variety but consists of \( d + 1 \) connected components in its analytic topology; Proposition 2.5.

The subvariety \( S' = \mathcal{M}'_d \cap S(\mathbb{C}) \) is excluded so that \( h_{\mathbb{R}} \) is well defined: the dynamics on \( \hat{\mathbb{R}} \) should be independent of the real representative picked from a Möbius conjugacy class so one has to omit real maps admitting twists; that is, real rational maps that are Möbius conjugate only over \( \mathbb{C} \) and it is well known that twists are always associated with non-trivial automorphisms [Sil07, §4.8].
Remark 1.2. If \( f \in \mathbb{R}(z) \) is hyperbolic, then the second statement of Theorem 1.1 can be deduced from the kneading theory of Milnor and Thurston [MT88]. In §4.1 we directly prove a slightly more general result regarding entropy values on hyperbolic components; Theorem 4.2.

Given a degree \( d \) rational map \( f \in \mathbb{R}(z) \), it is only the intersection of the Julia set of \( f \) with \( \hat{\mathbb{R}} \) – the real Julia set \( \mathcal{J}_R(f) := \mathcal{J}(f) \cap \hat{\mathbb{R}} \) – that affects the entropy of \( f \mid_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \). We prove a structure theorem for the system \((f, \mathcal{J}_R(f))\):

**Theorem 1.3.** The non-wandering set of the dynamical system \( f \mid_{\mathcal{J}_R(f)} : \mathcal{J}_R(f) \to \mathcal{J}_R(f) \) is either the whole circle or the union of finitely many intervals and a totally disconnected subsystem. Moreover, the system has a closed subsystem of the same topological entropy in which preperiodic points of \( f \) are dense.\(^1\)

For \( f \in \mathbb{R}(z) \) in Theorem 1.1, if a quasi-conformal deformation \( \{f_t\}_{t \in [0,1]} \) of \( f \) through real maps takes \( \mathcal{J}_R(f) \) onto \( \mathcal{J}_R(f_t) \), then clearly we have \( h_R(f) = h_R(f_t) \). The second statement of Theorem 1.1 would follow immediately if this holds for all such q.c. deformations. As we do not know if this always holds, Theorem 1.3 is used in our proof of Theorem 1.1 instead; it implies that the subset of \( \hat{\mathbb{R}} \) which determines the real entropy remains in \( \hat{\mathbb{R}} \) under perturbations. This idea is reminiscent of the classical \( \lambda \)-lemma from the stability theory of rational maps where one constructs a holomorphic motion of the entire Julia sets from a motion of the dense subset of repelling periodic points [MnSS83].

In another direction, one can also discuss the ramifications of \( h_R(f) \) attaining its maximum \( \log(\deg f) \):

**Theorem 1.4.** Let \( f \in \mathbb{R}(z) \) be of degree \( d \geq 2 \). Then the following are equivalent:

(a) The measure of maximal entropy \( \mu_f \) for \( f \) on \( \hat{\mathbb{C}} \) assigns a positive measure to \( \hat{\mathbb{R}} \);
(b) the Julia set \( \mathcal{J}(f) \) is contained in \( \hat{\mathbb{R}} \);
(c) \( h_R(f) = \log(d) \).

We have included several examples and remarks to exhibit various dynamical behaviors that arise from restricting a real rational map \( f \) to the real circle. Example 2.1 discusses symmetric quadratic maps. Example 2.6 presents cases where \( f \mid_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \) is an unramified covering. Examples 3.12 and 3.13 demonstrate instances of Theorem 1.4 above where the Julia set is completely real. Example 4.9 illustrates Theorem 1.1 for an important family of non-hyperbolic maps, the family of flexible Lattès maps. In all of these examples the real entropy can be calculated explicitly.

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\(^1\)The subsystem is allowed to be empty when the real entropy vanishes. This may occur even if the Julia set intersects the real axis non-trivially along an interval; check Example 2.6 and Remark 3.7 for maps of real entropy zero whose real Julia sets are the whole real circle or an interval of it.
Motivation. An important question regarding the real entropy function on the moduli space of real rational maps is the nature of its level sets, called isentropes. There is an extensive literature on the monotonicity of entropy for various families of polynomials where the central problem is the connectedness of isentropes. The monotonicity was first established by Milnor and Thurston for the quadratic polynomial family (or equivalently, for the logistic family \( x \mapsto \mu x(1-x) : [0,1] \to [0,1] \) with \( 0 \leq \mu \leq 4 \)) in [MT88] and for bimodal cubic polynomials by Milnor and Tresser in [MT00]. The general case of degree \( d \) polynomials with \( d \) real non-degenerate critical points was settled in [BvS15] by van Strien and Bruin. We view this article as a first step towards the study of the level sets of the function \( h_{\mathbb{R}} \) defined on the moduli of real rational maps. The problem is different from the polynomial setting because we are dealing with circle maps \( f \rvert_{\hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}} \) instead of interval maps: In all aforementioned references one essentially deals with a boundary-anchored map of an interval outside which the orbits either escape to infinity or converge to a cycle of period at most two [MT00, Theorem 3.2]. Of course, if the restriction is not surjective, then \( f \rvert_{\hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}} \) can be replaced with an (not necessarily boundary-anchored) interval map of the same entropy; as

\[
(1.2) \quad h_{\mathbb{R}}(f) = h_{\text{top}}\left(f \rvert_{\hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}} \right) = h_{\text{top}}\left(f \rvert_{f(\hat{\mathbb{R}}) : f(\hat{\mathbb{R}}) \to f(\hat{\mathbb{R}})} \right).
\]

For example, when the rational map \( f \) is quadratic; \( f \rvert_{\hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}} \) is either a covering map (and thus of entropy \( \log(2) \) according to [MS80, Theorem 1']) or is not surjective in which case one can work with the interval map \( f \rvert_{f(\hat{\mathbb{R}}) : f(\hat{\mathbb{R}}) \to f(\hat{\mathbb{R}})} \) instead; cf. [Mil93, §10]. It turns out that in general the domain \( \mathcal{M}_d' = \mathcal{S}' \) of the real entropy function is disconnected (Proposition 2.5) and thus the monotonicity has to be studied among maps whose restrictions to \( \hat{\mathbb{R}} \) are of a common topological degree. There are even smaller natural analytic domains in \( \mathcal{M}_d' \) that are dynamically defined according to the degree along with the modality of \( f \rvert_{f(\hat{\mathbb{R}})} \); and the monotonicity of the restriction of \( h_{\mathbb{R}} \) to such regions is still worthy to investigate. As an example, for \( d = 2 \) a natural partition of \( \mathcal{M}_2 = \mathcal{M}_2(\hat{\mathbb{R}}) \cong \mathbb{R}^2 \) to degree \( \pm 2 \), monotonic, unimodal and bimodal regions has been outlined in [Mil93, §10]. The entropy behavior over \( \mathcal{M}_2 \) will be the subject of another article [Fil18] that focuses on the monotonicity problem.

\footnote{It has to be mentioned that in general the smallest subinterval that carries all the “entropy data”, namely the smallest compact interval that includes the real points of the filled Julia set, is not necessarily invariant; see the discussion in [Mil92, §2].}

\footnote{In higher degrees there are rational maps that take the circle \( \hat{\mathbb{R}} \) onto itself via a map which is not covering; e.g., an odd degree polynomial map with at least one real critical point of even multiplicity or a map such as \( x \mapsto \frac{P(x)}{P'(x)} \) where \( P(x) \) is a real polynomial of degree \( d \geq 4 \) with \( P(1).P(-1) < 0 \).

\footnote{Modality can be considered to be finer than degree; for instance a polynomial self-map of the real axis is of topological degree \( 0 \) or \( \pm 1 \) according to the parity of the degree of the polynomial while it can have various numbers of laps. In the literature, the monotonicity problem has been studied mostly for a special class of degree \( d \) real polynomial maps whose lap numbers are \( d \) as well; see [BvS15], [MT00] and [MT88].}
Outline. Putting the real entropy function $h_R$ (as described in Theorem 1.1) on a firm footing is the main goal of §2.1. Another question to be discussed in §2.2 is how $M'_d$ is related to the real locus $M_d(R)$.

The third section is devoted to study the interplay between the real entropy $h_R(f)$ and the real Julia set $J_R(f)$ of a rational map $f \in R(z)$ and heavily relies on several results from the real one-dimensional dynamics. In §3.1 and §3.3 we prove more detailed versions 3.1 and 3.11 of Theorem 1.3 and Theorem 1.4.

In studying the entropy behavior of families the notion of hyperbolicity appears naturally; e.g., the entropy of the quadratic family of polynomials is a devil’s staircase function with plateaus corresponding to “hyperbolic windows”. In §4.1 we prove an analogous statement, Theorem 4.2, indicating that $h_R$ is locally constant on the real locus of a hyperbolic component in $M_d(C)$. The corresponding proposition for a not necessarily hyperbolic real rational map has to be formulated in the context of the theory of Teichmüller and moduli spaces of rational maps developed by McMullen and Sullivan in [MS98]. This is the content of Theorem 4.8 in §4.2 that along with Proposition 2.7 from §2.1 establishes Theorem 1.1.

Notation and Terminology. The non-wandering set of a topological dynamical system $f : X \to X$ is denoted by $NW(f : X \to X)$. Assuming $X$ is compact, it is a standard fact that the subsystem obtained from restricting $f$ to the non-wandering set is of full entropy [ALM93]. When $X$ is an interval, a lap of $f$ is defined to be a maximal monotonic subinterval and the number of laps (i.e. modality + 1) is denoted by $l(f)$ and is called the lap number. It is a standard fact that the entropy of a multimodal (i.e. piecewise monotone or of finite modality) interval map is the exponential growth rate of the number of laps of its iterates [MS80, Theorem 1]:

\begin{equation}
(1.3) \quad h_{top}(f) = \lim_{n \to \infty} \frac{1}{n} \log (l(f^{\circ n})) = \inf_{n} \frac{1}{n} \log (l(f^{\circ n})).
\end{equation}

The result remains valid for a multimodal self-map of a version of the circle $S^1$ as long as “laps” are interpreted as those of the (possibly discontinuous) transformation of $[0, 1)$ obtained from conjugating the transformation of $S^1$ with the bijection $[0, 1) \to S^1 : x \mapsto e^{2\pi i x}$; see [MS80, Theorem 1'] for details. In particular, the $n^{th}$ iterate of a degree $d$ covering $S^1 \to S^1$ lifts to a self-map of $[0, 1)$ with $d^n$ continuous pieces, all of them monotonic. Its entropy is thus $\log(d)$.

A metric space is called a Cantor space if it is compact, perfect (i.e. without an isolated point) and totally disconnected. We refer the reader to the book [Pug02] for a proof and also the terminology of point-set topology that is going to be used in this paper.

Spaces $\hat{C}$ and $\hat{R}$ denote the compactifications $C \cup \{\infty\}$ and $R \cup \{\infty\}$ of $C$ and $R$ respectively and we use $z$ and $x$ for the coordinates on them. When the Riemann sphere is considered as a complex algebraic curve, the notation $\mathbb{P}^1(C)$ is used instead. The degree of a rational map is denoted by $d$ and is always assumed to be higher than one.

For notations related to the moduli space of rational maps, we mainly follow [Mil93],
The moduli space of degree $d$ rational maps is an affine variety $\mathcal{M}_d/\mathbb{Q}$ constructed by Silverman in [Sil98] as the GIT quotient $\text{Rat}_d/\text{PSL}_2$ where $\text{Rat}_d \subset \mathbb{P}^{2d+1}$ is the parameter space for degree $d$ rational maps. For a variety $V/K$, the set of its $K'$-points is written as $V(K')$ for any larger field $K' \supset K$. So here it makes sense to write $\mathcal{M}_d(K)$ for any subfield $K$ of $\mathbb{C}$ and the complex variety $\mathcal{M}_d(\mathbb{C})$ coincides with $\text{Rat}_d(\mathbb{C})/\text{PGL}_2(\mathbb{C})$, i.e. the space of Möbius conjugacy classes of degree $d$ rational maps on the Riemann sphere. The conjugacy class of a rational map $f$ will be shown by $\langle f \rangle$. The subvariety $\mathcal{S}(\mathbb{C})$ of $\mathcal{M}_d(\mathbb{C})$, called the symmetry locus, is the subvariety determined by rational maps $f$ for which the group $\text{Aut}(f)$ of Möbius transformations commuting with $f$ is non-trivial. An orbit of $f$ is called preperiodic if it is eventually periodic. When $f$ is real, its real Julia set, denoted by $\mathcal{J}_\mathbb{R}(f)$, is defined as the intersection of the Julia set $\mathcal{J}(f)$ with the invariant set $\mathbb{R}$.

The real entropy (1.1) can be considered as a well defined continuous function on certain open submanifold of $\mathcal{M}'_d \subset \mathcal{M}_d(\mathbb{C})$ obtained from omitting those $\text{PGL}_2(\mathbb{C})$-conjugacy classes of real maps that include more than one $\text{PGL}_2(\mathbb{R})$-conjugacy class. This is the content of Proposition 2.7 of §2.1. In §2.2 we discuss how the real locus $\mathcal{M}_d(\mathbb{R})$ of $\mathcal{M}_d(\mathbb{C})$ is related to the locus $\mathcal{M}'_d$ determined by rational maps that actually admit a model over $\mathbb{R}$; see Proposition 2.9.

2.1. Symmetries. The real entropy is assigned to rational maps with real coefficients; so it is natural to form the subspace

$$\mathcal{M}'_d := \text{PGL}_2(\mathbb{C}).(\text{Rat}_d(\mathbb{R}))/\text{PGL}_2(\mathbb{C}) = \{ \langle f \rangle \in \mathcal{M}_d(\mathbb{C}) \mid f \in \mathbb{R}(z) \}$$

consisting of conjugacy classes in $\mathcal{M}_d(\mathbb{C})$ that contain a real map. The real entropy $h_{\mathbb{R}}$ is not well defined on the entirety of this real subvariety because the ambiguity in picking a real representative can lead to different entropy values.

Example 2.1. Given $\mu \in \mathbb{R} - \{0\}$, real quadratic rational maps $\frac{1}{\mu}(z \pm \frac{1}{z})$ are conjugate via $z \mapsto iz$ but exhibit quite different dynamical behavior on the real circle. The critical points of $\frac{1}{\mu}(z - \frac{1}{z})$ are not real so it induces a degree two covering $x \mapsto \frac{1}{\mu}(x - \frac{1}{x})$ of $\mathbb{R}$ whose entropy is therefore $\log(2)$. On the other hand, the topological entropy of $x \mapsto \frac{1}{\mu}(x + \frac{1}{x})$.

5For technical reasons, Silverman works with $\text{PSL}_2$ instead of $\text{PGL}_2$. Of course, it does not make a difference over the algebraically closed field $\mathbb{C}$ as $\text{PSL}_2(\mathbb{C}) = \text{PGL}_2(\mathbb{C})$. 


vanishes: for $|\mu| \leq 1$ every orbit is attracted to the fixed point $\infty$ of multiplier $\mu$; when $\mu > 1$ orbits in the invariant interval $(0, \infty)$ tend to the attracting fixed point $\frac{1}{\sqrt{\mu-1}}$ while those in the invariant interval $(-\infty, 0)$ tend to the attracting fixed point $-\frac{1}{\sqrt{\mu-1}}$ and finally; for $\mu < -1$ there is no finite real fixed point and any point of $\hat{\mathbb{R}}$, other than the fixed point $\infty$ and its preimage 0, converges under iteration to the 2-cycle consisting of $\pm \frac{1}{\sqrt{-\mu-1}}$ whose multiplier is $\left(\frac{2+\mu}{\mu}\right)^2 < 1$.

The problem with the pair $\frac{1}{\mu} \left( z \pm \frac{1}{z} \right)$ from the preceding example is that they are conjugate over complex numbers but not over reals. We are interested in the dynamics of $f \restriction \hat{\mathbb{R}}$ and that is invariant only under real conjugacies, i.e. elements of $\text{PGL}_2(\mathbb{R})$. In the literature of arithmetic dynamics, such examples of rational maps over a field $K$ which are conjugate only over a strictly larger field $K'$ are called twists and they happen only if maps admit symmetries in $\text{PGL}_2(K)$; see [Sil07, §§4.7,4.8,4.9] for details. In our context, only twists over $\mathbb{R}$ are relevant.

**Proposition 2.2.** Fix $f \in \mathbb{R}(z)$ with $\text{Aut}(f) = \{1\}$. If $g \in \mathbb{R}(z)$ is $\text{PGL}_2(\mathbb{C})$-conjugate to it, then $f, g$ are in fact $\text{PGL}_2(\mathbb{R})$-conjugate.

**Proof.** Assume the contrary; the complex conjugacy class $\langle f \rangle$ contains at least two distinct real conjugacy classes. This implies the existence of a Möbius transformation $\alpha \in \text{PGL}_2(\mathbb{C}) - \text{PGL}_2(\mathbb{R})$ for which $\alpha \circ f \circ \alpha^{-1} \in \mathbb{R}(z)$. But then taking complex conjugates implies that

$$\bar{\alpha} \circ f \circ \bar{\alpha}^{-1} = \alpha \circ f \circ \alpha^{-1} \Rightarrow (\bar{\alpha}^{-1} \circ \bar{\alpha}) \circ f \circ (\alpha^{-1} \circ \alpha)^{-1} = f.$$  

Hence the non-identity Möbius map $\alpha^{-1} \circ \bar{\alpha}$ lies in $\text{Aut}(f)$; a contradiction. $\square$

Thus, in order to have a well defined real entropy function, we can subtract from the space $\mathcal{M}_d'$ in (2.1) the subspace of complex conjugacy classes of real maps with non-trivial Möbius symmetries; the subspace which will be denoted by

$$S' := \{ \langle f \rangle \in \mathcal{M}_d(\mathbb{C}) \mid f \in \text{Rat}_d(\mathbb{R}), \text{Aut}(f) \neq \{1\} \}$$

herein. This is obviously the intersection of the complex symmetry locus

$$S(\mathbb{C}) = \{ \langle f \rangle \in \mathcal{M}_d(\mathbb{C}) \mid f \in \text{Rat}_d(\mathbb{C}), \text{Aut}(f) \neq \{1\} \}$$

with $\mathcal{M}_d'$. So we arrive at a well defined entropy function:

**Definition 2.3.** For any $d \geq 2$, the real entropy function is defined as

$$h_\mathbb{R} : \mathcal{M}_d' - S' \to [0, \log(d)]$$

$$\langle f \rangle \mapsto h_{\text{top}} \left( f \restriction \hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right)$$

$\left(f \in \mathbb{R}(z)\right)$.

$^6$It is not hard to verify that $\text{PGL}_2(\mathbb{R})$ is the subgroup of fixed elements in the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\text{PGL}_2(\mathbb{C})$. 

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The domain of definition is \(\{ (f) \in \mathcal{M}_d(\mathbb{C}) \mid f \in \mathbb{R}(z), \text{Aut}(f) = \{1\} \}\) and the codomain is \([0, \log(d)]\) since

\[
h_{\mathbb{R}}(f) = h_{\text{top}}(f |_{\hat{\mathbb{R}}}: \hat{\mathbb{R}} \to \hat{\mathbb{R}}) \leq h_{\text{top}}(f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}) = \log(d).
\]

It would be also useful to know the dimension of the symmetry locus:

**Proposition 2.4.** When \(d \geq 2\), the dimension of the symmetry locus (2.3) is \(d - 1\) and, as a subvariety of \(\mathcal{M}_d/\mathbb{Q}\), it has a model over \(\mathbb{Q}\) (so makes sense to talk of its \(\mathbb{R}\)-points). Moreover, \(\dim_{\mathbb{R}} S' = \dim_{\mathbb{R}} S(\mathbb{R}) = d - 1\).

**Proof.** Let \(f \in \mathbb{C}(z)\) be a rational map admitting a non-trivial automorphism \(\alpha \in \text{PGL}_2(\mathbb{C})\). Since \(\text{Aut}(f)\) is finite ([Sil07, Proposition 4.65]), after conjugation with an appropriate M"obius map one may assume that the cyclic subgroup generated by \(\alpha\) has a generator of the form \(\omega_n := z \mapsto e^{\frac{2\pi i}{n}}z\) where \(n \geq 2\). The elements of \(\text{Rat}_d(\mathbb{C})\) commuting with \(\omega_n\) form the following union:

\[
\bigcup_{0 \leq r \leq n} \left\{ \frac{\sum_{0 \leq i \leq d, n \mid i - r - 1} a_i z^i}{\sum_{0 \leq i \leq d, n \mid i - r} b_i z^i} \in \text{Rat}_d(\mathbb{C}) \right\}.
\]

The dimension of each of the sets in this union is at most \(d\): the number of coefficients \(a_i, b_i\) appearing is precisely \(d + 1\) for \(n = 2\) and at most \(2 \left\lceil \frac{d+1}{3} \right\rceil \leq d + 1\) for \(n \geq 3\). Thus the projection from the affine space of coefficients into the quasi-projective variety \(\text{Rat}_d(\mathbb{C})\) yields a \(d\)-dimensional subset of \(\text{Rat}_d(\mathbb{C})\). There is one degree of freedom due to conjugation with scaling maps which preserves the forms appeared above. Hence the symmetry locus \(S(\mathbb{C})\) is of dimension at most \(d - 1\). Indeed, the equality is achieved; otherwise the generic fiber of the projection map from the aforementioned \(d\)-dimensional subspace of \(\text{Rat}_d(\mathbb{C})\) into \(\mathcal{M}_d(\mathbb{C})\) is of dimension at least two. This means that there is a rational map \(f \in \mathbb{C}(z)\) commuting with an \(\omega_n\) such that for a two-dimensional subset \(Z\) of \(\text{PGL}_2(\mathbb{C})\) all maps \(\alpha \circ f \circ \alpha^{-1}\) commute with \(\omega_n\) as well. In particular, there is a morphism

\[
\begin{cases}
Z \to \text{Aut}(f) \\
\alpha \mapsto \alpha^{-1} \circ \omega_n \circ \alpha
\end{cases}
\]

from the two-dimensional variety \(Z\) into the finite set \(\text{Aut}(f)\). So the morphism has to be constant over a two-dimensional subvariety which cannot be the case as the centralizer of \(\omega_n\) in \(\text{PGL}_2(\mathbb{C})\) is the subgroup of scaling maps \(z \mapsto kz\) which is one-dimensional.

For the last part, notice that according to the above union, \(S(\mathbb{C})\) is the image of the algebraic set

\[
\bigcup_{2 \leq n \leq d-1} \bigcup_{0 \leq r \leq n} \{ [a_0 : \cdots : a_d : b_0 : \cdots : b_d] \in \text{Rat}_d(\mathbb{C}) \subset \mathbb{P}^{2d+1}(\mathbb{C}) \mid a_i = 0 \text{ if } n \nmid i - r - 1; \ b_i = 0 \text{ if } n \nmid i - r \}\]
defined over the rationals.

Finally, notice that the dimension count for $S(\mathbb{C})$ implies upper bounds for dimensions of real subvarieties $S' \subseteq S(\mathbb{R})$:

$$\dim_{\mathbb{R}} S' \leq \dim_{\mathbb{R}} S(\mathbb{R}) \leq \dim_{\mathbb{C}} S(\mathbb{C}) = d - 1.$$  

It is not hard to show that the equality is achieved here: The complex dimension of $S(\mathbb{C})$ coincides with the real dimension of its real locus $S(\mathbb{R})$ provided that the complex variety $S(\mathbb{C})$ has a smooth $\mathbb{R}$-point in a highest dimensional irreducible component. For a generic choice of complex or real numbers $a_{2k+1}$ and $b_{2k}$, the automorphism group of the rational map

$$\frac{\sum_{0<2k+1 \leq d} a_{2k+1} z^{2k+1}}{\sum_{0\leq 2k \leq d} b_{2k} z^{2k}}$$

is generated by $z \mapsto -z$ that on coefficients acts via $a_{2k+1} \mapsto -a_{2k+1}$ and $b_{2k} \mapsto b_{2k}$. Hence, assuming that these numbers are furthermore positive, we get a $d$-dimensional submanifold of $\text{Rat}_d(\mathbb{R})$ that bijects onto a $(d-1)$-dimensional real submanifold of the set of smooth points of $S(\mathbb{C})$ which is a subset of $S'$ because every point of it represents the conjugacy class of a real map. Therefore, $\dim_{\mathbb{R}} S(\mathbb{R}) \geq \dim_{\mathbb{R}} S' \geq d - 1$. □

We next investigate the the domain of the function $h_\mathbb{R}$ appeared in Definition 2.3.

**Proposition 2.5.** The domain $M'_d - S' \subseteq M_d(\mathbb{C})$ of the real entropy function (2.4) is an irreducible real variety of dimension $2d - 2$. Nevertheless, in its analytic topology, it decomposes to $d+1$ connected components of the same dimension corresponding to topological degrees of self-maps of the circle $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ that elements of $\text{Rat}_d(\mathbb{R})$ induce:

$$(2.5) \quad M'_d - S' = \bigsqcup_{-d \leq s \leq d, 2d - s} (M'_{d,s} - S') .$$

Here, $M'_{d,s}$ is the subspace of complex conjugacy classes of degree $d$ real rational maps $f$ for which the topological degree of the restriction $f|\hat{\mathbb{R}}: \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ is $s$.

**Proof.** We have

$$\dim_{\mathbb{R}} M'_d \leq \dim_{\mathbb{R}} M_d(\mathbb{R}) \leq \dim_{\mathbb{C}} M_d(\mathbb{C}) = 2d - 2.$$  

The surjective morphism $\text{Rat}_d(\mathbb{R})/\text{PGL}_2(\mathbb{R}) \to M'_d$ is injective on the Zariski open subset of classes without twists (e.g. away from the preimage of the closed subvariety $S'$). The projection also indicates that $M'_d$ is irreducible, being a surjective image of $\text{Rat}_d(\mathbb{R})$ which is irreducible itself as it is the complement of the resultant hypersurface in $\mathbb{P}^{2d+1}(\mathbb{R})$. Therefore, $M'_d$ is irreducible in the Zariski topology and of real dimension $2d - 2$. The same hold for its Zariski open subset $M'_d - S'$.

Since the field $\mathbb{R}$ is not algebraically closed, the irreducibility of $\text{Rat}_d(\mathbb{R})$ or $M'_d - S'$ does not guarantee their connectedness in the analytic topology. As a matter of fact, maps
in \( \text{Rat}_d(\mathbb{R}) \) whose restrictions to the real circle have different topological degrees cannot lie in the same connected component. To see this, notice that the topological degree \( s \in \{-d, \ldots, 0, \ldots, d\} \) of the restriction of \( f \in \text{Rat}_d(\mathbb{R}) \) to \( \hat{\mathbb{R}} \) can be thought of as the value of the integral
\[
\int_{\hat{\mathbb{R}}} f^* \left( \frac{dx}{\pi(1 + x^2)} \right) = \int_{-\infty}^{\infty} \frac{f'(x)dx}{\pi(1 + f(x)^2)}
\]
due to the fact that \( \frac{dx}{\pi(1 + x^2)} \) is a normalized volume form for the circle \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \). If \( f_n \to f \) in the analytic topology of \( \text{Rat}_d(\mathbb{R}) \) (i.e. the usual topology on the space of coefficients), then on the real axis one has
\[
\frac{f'_n}{\pi(1 + f_n^2)} \bigg|_{\mathbb{R}} \Rightarrow \frac{f'}{\pi(1 + f^2)} \bigg|_{\mathbb{R}}
\]
for the integrands of the above integral. Consequently, the integer-valued function
\[
\deg_{\text{top}} : \text{Rat}_d(\mathbb{R}) \to \{-d, \ldots, 0, \ldots, d\}
\]
is continuous and hence its level sets are disjoint unions of connected components of \( \text{Rat}_d(\mathbb{R}) \).

The topological degree of a circle map is preserved under conjugation by a diffeomorphism of the circle and Proposition 2.2 implies that if real maps \( f, g \) give rise to the same class in \( M_d(\mathbb{C}) \) away from \( S'_1 \), then they are conjugate by a Möbius transformation that preserves \( \hat{\mathbb{R}} \). Therefore, the topological degree descends to a continuous function
\[
\deg_{\text{top}} : M'_d - S' \to \{-d, \ldots, 0, \ldots, d\}
\]
and again, each level set \( M'_{d,s} - S' \) is a union of connected components where \( M'_{d,s} \) denotes
\[
(2.6) \quad M'_{d,s} := \left\{ \langle f \rangle \in M_d(\mathbb{C}) \mid f \in \text{Rat}_d(\mathbb{R}), \deg_{\text{top}} \left( f \bigg|_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \right) = s \right\}.
\]

We next show that \( M'_{d,s} \) is non-vacuous if and only if \( d \) and \( s \in \{-d, \ldots, 0, \ldots, d\} \) have the same parity and in that case each \( M'_d - S' \) is connected. This will establish the partition (2.5) of \( M'_d - S' \) to its connected components and will conclude the proof.

Let us proceed with a parametrization of degree \( d \) real rational maps with \( \deg_{\text{top}} = s \). It is more convenient to exchange \( \hat{\mathbb{R}} \) with \( \{z \in \mathbb{C} \mid |z| = 1\} \) via the Cayley transform
\[
(2.7) \quad \begin{cases}
\text{the unit circle } \to \text{the real circle} \\
z \mapsto x = i\frac{1 + z}{1 - z}
\end{cases}
\]
and concentrate on degree \( d \) rational maps \( f \in \mathbb{C}(z) \) that keep the unit circle \( \{z \in \mathbb{C} \mid |z| = 1\} \) invariant instead. The reflection \( z \mapsto \frac{1}{z} \) with respect to the unit circle commutes with such maps.
an $f$, so roots and poles of $f$ on the Riemann sphere occur in (root, pole) pairs like $(q, \frac{1}{q})$ where $q$ is away from the unit circle. Therefore, assuming that $f$ has $k$ roots $a_1, \ldots, a_k$ inside the unit circle and $d-k$ roots $b_1, \ldots, b_{d-k}$ outside of it (both written by multiplicity), its poles would be reciprocals of $\bar{a}_1, \ldots, \bar{a}_k$ outside the unit circle and reciprocals of $b_1, \ldots, \bar{b}_{d-k}$ inside it. The rational maps $f(z)$ is thus a scalar multiple of the Blaschke product

$$
\prod_{i=1}^{k} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right) \prod_{j=1}^{d-k} \left( \frac{z - b_j}{1 - \bar{b}_j z} \right)
$$

that preserves the unit circle as well. We conclude that $f(z)$ has a unique description as (2.8)

$$
f(z) = e^{2\pi i c} \prod_{i=1}^{k} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right) \prod_{j=1}^{d-k} \left( \frac{z - b_j}{1 - \bar{b}_j z} \right) (|a_1|, \ldots, |a_k| < 1; |b_1|, \ldots, |b_{d-k}| > 1; c \in \mathbb{R}/\mathbb{Z}).
$$

Pulling back $\frac{dz}{\pi(1+z^2)}$ with the Cayley transform (2.7) results in the normalized volume form

$$
\frac{1}{2\pi} d\theta = \frac{1}{2\pi i} \frac{dz}{z}
$$

for the unit circle. The topological degree $s$ of the restriction $f\mid_{\{z=1\}}: \{|z|=1\} \to \{|z|=1\}$ of the map in (2.8) then is

$$
\frac{1}{2\pi} \int_{|z|=1} f^*(d\theta) = \frac{1}{2\pi i} \int_{|z|=1} f^* \left( \frac{dz}{z} \right) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} dz \overset{\text{argument principle}}{=} k - (d-k) = 2k - d.
$$

We observe that $k = \frac{d+s}{2}$ and an integer $s$ from $\{-d, \ldots, 0, \ldots, d\}$ can be realized as the topological degree of the restriction precisely when its parity is the same as that of $d$. Now it is easy to verify that there are $d+1$ choices for $s$.

Fixing such an $s$, putting $k$ to be $\frac{d+s}{2}$ in (2.8) and then varying $a_i$’s, $b_j$’s and $c$ respectively inside, outside and on the unit circle, and finally conjugating with the Cayley transform parametrizes a submanifold of $\text{Rat}_d(\mathbb{R})$ diffeomorphic to $\mathbb{R}^{2d} \times S^1$ which is the subspace of real degree $d$ rational maps with $\text{deg}_{\text{top}} = s$. This full-dimensional submanifold of $\text{Rat}_d(\mathbb{R})$ surjects onto the subspace $\mathcal{M}^\prime_{d,s}$ of $\mathcal{M}^\prime_d$ appeared in (2.6); a subspace that is therefore connected and of codimension zero. Finally, we argue that removing $S'$ in (2.5) cannot affect the connectedness of $\mathcal{M}^\prime_{d,s}$: If $d \geq 3$, invoking Proposition 2.4, the codimension of $\mathcal{M}^\prime_{d,s} \cap S'$ in $\mathcal{M}^\prime_{d,s}$ is

$$(2d - 2) - (d - 1) \geq 2;
$$

so $\mathcal{M}^\prime_{d,s} - S'$ will be connected as well. For $d = 2$, we rely on the results of [Mil93]: $S' = S(\mathbb{R})$ is precisely the curve in $\mathcal{M}^\prime_2 = \mathcal{M}_2(\mathbb{R}) = \mathbb{R}^2$ that separates $\mathcal{M}^\prime_{2,\pm 2}$, $\mathcal{M}^\prime_{2,\mp 2}$ and $\mathcal{M}_{2,0}$; check the picture [Mil93, fig. 15].
Example 2.6. For a given \( 0 \neq s \in \{ \pm 1, \ldots, \pm d \} \) with \( s \equiv d \pmod{2} \), the Blaschke product (2.9)
\[
f(z) = e^{2\pi i c} \prod_{i=1}^{d+s} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right) \prod_{j=1}^{d-s} \left( \frac{z - b_j}{1 - \bar{b}_j z} \right) \left( |a_1|, \ldots, |a_{d+s}| < 1; |b_1|, \ldots, |b_{d-s}| > 1; c \in \mathbb{R}/\mathbb{Z} \right)
\]
from the proof of Proposition 2.5 induces a (unramified) degree \( s \) covering of \(|z| = 1\) provided that \( a_i \)'s and \( b_j \)'s are sufficiently close to 0 and \( \infty \) respectively; this is due to the fact that as \( a_i \to 0, b_j \to \infty \), \( f|_{\{ |z| = 1 \}} \) converges uniformly to the covering \( z \mapsto z^s \) of the unit circle. Conjugating with the Cayley transform (2.7) then yields a degree \( d \) rational map that restricts to a degree \( s \) cover \( \mathbb{R} \to \mathbb{R} \); i.e., a degree \( d \) real map of real entropy \( \log(|s|) \). For a generic choice of \( a_i \)'s and \( b_j \)'s, the corresponding map \( f \) will be away from the symmetry locus. We conclude that \( h_{\mathbb{R}} = \log(|s|) \) over some non-empty analytic open subset of the connected component \( \mathcal{M}'_{d,s} \).

For \( s = 1 \) and \( d \) being odd, this type of Blaschke products occurs in constructing rational maps with Herman rings; see [Mil06a, §15]. In such a situation,
\[
f|_{\{ |z| = 1 \}}: \{ |z| = 1 \} \to \{ |z| = 1 \}
\]
would be an orientation-preserving analytic diffeomorphism whose rotation number \( \rho \) can be any desired element in \( \mathbb{R}/\mathbb{Z} \) after a suitable adjustment of \( c \) in (2.9) [Mil06a, §15, Lemma 15.3]. The entropy of this subsystem is then zero and the intersection of \( \mathcal{F}(f) \) with the unit circle is easy to analyze (the same can be said about the real entropy and the real Julia set after switching from the unit circle to the real circle via the Cayley transform). If \( \rho \) is rational, then every orbit on the circle converges to a periodic orbit\(^8\) which is a non-repelling orbit of the rational map \( f \). So the only possible Julia points on the invariant circle can be parabolic points. As an example, consider the rational map
\[
f(z) = z + \frac{1}{z^2 + 2}
\]
that induces an orientation-preserving self-diffeomorphism of \( \mathbb{R} \) where the orbit of any real number tends to the parabolic fixed point \( \infty \) as \( f(x) > x \) for all \( x \in \mathbb{R} \).

For irrational \( \rho \) by Denjoy’s theorem there is a topological conjugacy between \( f|_{\{ |z| = 1 \}}: \{ |z| = 1 \} \to \{ |z| = 1 \} \) and the irrational rotation \( t \in \mathbb{R}/\mathbb{Z} \mapsto t + \rho \in \mathbb{R}/\mathbb{Z} \) in particular every single orbit of this subsystem is dense. So the unit circle is entirely included in either the Julia set or the Fatou set. In the latter situation, the Fatou component of \( f \) having this circle is a fixed rotation domain that has \(|z| = 1\) as a leaf of its natural foliation. In particular, the diffeomorphism \( f|_{\{ |z| = 1 \}}: \{ |z| = 1 \} \to \{ |z| = 1 \} \) is real analytically linearizable. Conversely, if the conjugacy to the rotation map \( t \in \mathbb{R}/\mathbb{Z} \mapsto t + \rho \in \mathbb{R}/\mathbb{Z} \) is real analytic, then it can be

\(^8\)See [BS02, chap. 7] for standard material on rotation theory.
extended to a small annulus around the unit circle so the circle is in a Herman ring.\footnote{There is complete classification based on Diophantine properties due to Yoccoz of rotation numbers for which the existence of a real analytic linearization is guaranteed [Yoc02].} For irrational rotation numbers $\rho$ that are “too well approximated” by rational numbers there are real analytic diffeomorphisms of the unit circle of rotation number $\rho$ that do not admit even $C^\infty$ linearizations. Hence if in (2.9) we fix $a_1, \ldots, a_{d+1}$ near 0 and $b_1, \ldots, b_{d+1}$ near $\infty$ and then adjust $c$ so that we get such a rotation number for the induced self-diffeomorphism of $|z| = 1$, then all points on the unit circle would be Julia despite the fact that the real entropy vanishes.

To conclude this subsection, we establish the claims made about the real entropy functions at the beginning of Theorem 1.1.

**Proposition 2.7.** For any $d \geq 2$, the function $h_{\mathbb{R}} : \mathcal{M}_d' - S' \to [0, \log(d)]$ is surjective and continuous (in the analytic topology).

**Proof.** Fixing an $s \in \{-d, \ldots , 0, \ldots , d\}$ with $s \equiv d \pmod{2}$, by what observed before in the proof of Proposition 2.5 and also in Example 2.6, the component $\deg_{\text{top}} = s$ of $\text{Rat}_d(\mathbb{R})$ can be identified with the space of Blaschke products

$$e^{2\pi i c} \prod_{i=1}^{d+s} \left( \frac{z - a_i}{1 - a_i z} \right) \prod_{j=1}^{d+s} \left( \frac{z - b_j}{1 - b_j z} \right) \left( |a_1|, \ldots , |a_{d+s}| < 1; |b_1|, \ldots , |b_{d+s}| > 1; c \in \mathbb{R}/\mathbb{Z} \right)$$

appeared in (2.9). A convergence of the parameters of the above product inside

$$\{|z| < 1\}^{d+s} \times \{|z| > 1\}^{d+s} \times \mathbb{R}/\mathbb{Z}$$

results in a uniform convergence of the corresponding holomorphic functions over some thin enough annulus around the unit circle $|z| = 1$ and therefore a convergence of the induced maps of the unit circle in the $C^\infty$ topology. Now we merely need to repeat the arguments of the proof of [MT00, Theorem 4.1] but for circle maps instead of interval maps:

- the topological entropy is lower semi-continuous as a function on the space of continuous multimodal circle maps [MS80, Theorem 5]';
- the topological entropy is upper semi-continuous as a function on the space of $C^\infty$ self-maps of any compact differentiable manifold [Yom87].

Consequently, the real entropy is a continuous function on the space $\text{Rat}_d(\mathbb{R})$ of real rational maps of degree $d$. This function, after being factored through the local homeomorphism from the open subset of maps without Möbius symmetry in $\text{Rat}_d(\mathbb{R})$ onto $\mathcal{M}_d' - S'$, descends to the function $h_{\mathbb{R}} : \mathcal{M}_d' - S' \to [0, \log(d)]$ which is thus continuous as well.

Because of the continuity we just established, to obtain surjectivity of $h_{\mathbb{R}}$ it suffices to construct a family of real rational maps of degree $d$ parametrized over a connected space for
which the real entropy gets arbitrarily close to both extremes 0 and \( \log(d) \). We invoke the result [MT00, Theorem 3.2] that allows us to parametrize the class of boundary-anchored polynomial interval maps of full modality via their critical values:

Given numbers \( v_1, \ldots, v_{d-1} \in [-1, 1] \) with \((-1)^i(v_i - v_{i-1}) > 0\) for every \( 0 < i \leq d \) where \( v_0 := 1 \) and \( v_d := (-1)^d \), there is a unique boundary-anchored polynomial map \( f : [-1, 1] \to [-1, 1] \) of degree \( d \) that has distinct critical points

\[
-1 < c_1 < \cdots < c_{d-1} < 1
\]

such that \( f(c_i) = v_i \) for all \( 0 < i < d \) and \( f(-1) = v_0, f(1) = v_d \) on the boundary.

The space of these tuples \((v_1, \ldots, v_{d-1}) \in [-1, 1]^{d-1}\) is obviously connected. As \( v_i \to 0^+ \) for \( 0 < i < d \) even and \( v_i \to 0^- \) for \( 0 < i < d \) odd, the corresponding maps tend to \( x \mapsto (-x)^d \) whose real entropy is zero whereas when \( v_i = (-1)^i \) for each \( 0 < i < d \), the corresponding degree \( d \) polynomial map\(^{10}\) \( f : [-1, 1] \to [-1, 1] \) would have \( d \) surjective monotonic pieces; so the iterate \( f^n \) needs to have \( d^n \) laps and therefore the exponential growth rate of modality of iterates is \( \log(d) \).

\[\square\]

Remark 2.8. It is natural to ask what jumps in the values of the function \( h_R : \mathcal{M}_d' - S' \to [0, \log(d)] \) occur when one crosses a multi-valued point (which necessarily lies on \( S' \)).

It is not hard to check that for \( d = 2 \)

\[
S' = S(\mathbb{R}) = \left\{ \left\langle \frac{1}{\mu} \left( z + \frac{1}{z} \right) \right\rangle = \left\langle \frac{1}{\mu} \left( z - \frac{1}{z} \right) \right\rangle \mid \mu \in \mathbb{R} - \{0\} \right\};
\]

see [Mil93, §5]. Therefore, Example 2.1 indicates that the entropy goes from 0 to \( \log(2) \) or vice versa as we cross the curve \( S(\mathbb{R}) \) in \( \mathcal{M}_2(\mathbb{R}) \) (which turns out to be the affine plane \( \mathbb{R}^2 \); [Mil93, §10]).

A more complicated behavior is anticipated in higher degrees since the modalities of real representatives may differ more drastically.\(^{11}\) As an example, consider the 1-parameter family \( \{z^3 - az\}_{0 \leq a \leq 3} \) of real cubics that all admit the symmetry \( z \mapsto -z \). Conjugating with \( z \mapsto iz \), each \( z^3 - az \) has the alternative real model \(-z^3 - az\) for which the restriction to the real axis is strictly decreasing and hence of zero entropy whereas the original family restricts to 1-parameter family of self-maps of \( \mathbb{R} \) that starts with \( x \mapsto x^3 \) whose entropy is zero and ends with the (monic) Chebyshev polynomial \( x \mapsto x^3 - 3x \) whose entropy is \( \log(3) \).\(^{12}\) We observe that, unlike the case of \( d = 2 \), every value from the range of \( h_R : \mathcal{M}_3' - S' \to [0, \log(3)] \) happens as a jump of values in vicinity of a point of discontinuity.

\(^{10}\)The uniqueness part immediately implies that this particular polynomial \( f \) is closely related to the \( d^{th} \) Chebyshev polynomial; it satisfies \( f(- \cos(\theta)) = \cos(d\theta) \).

\(^{11}\)Another justification is that the complement of the domain of \( h_R \) with respect to \( \mathcal{M}_d' \) is of codimension at least two if \( d \geq 3 \); so one can tend to a point of discontinuity from various directions.

\(^{12}\)The map \( \theta \mapsto 2\cos(2\pi \theta) \) is a finite semi-conjugacy from \( \theta \in \mathbb{R}/\mathbb{Z} \to 3\theta \in \mathbb{R}/\mathbb{Z} \) onto it.
2.2. The Spaces $\mathcal{M}'_d$ and $\mathcal{M}_d(\mathbb{R})$. Next, we are going to elaborate a little bit more on the domain $\mathcal{M}'_d - S' = \mathcal{M}'_d - S(\mathbb{R})$ of $h_\mathbb{R}$ in (2.4); we are going to observe that it differs from $\mathcal{M}_d(\mathbb{R}) - S(\mathbb{R})$ by an irreducible component which is relevant only in odd degrees. The subspace $\mathcal{M}'_d$ was defined in (2.1) by the elements of $\text{Rat}_d(\mathbb{C})$ for which $\mathbb{R}$ is a field of definition. On the other hand, $\mathbb{R}$-points of $\mathcal{M}_d/\mathbb{Q}$ are Möbius conjugacy classes of elements of $\text{Rat}_d(\mathbb{C})$ whose field of moduli is $\mathbb{R}$; see [Sil07, §§4.4, 4.10] for the background material. So $\mathcal{M}'_d \subseteq \mathcal{M}_d(\mathbb{R})$ and in particular $\mathcal{M}'_d - S' \subseteq \mathcal{M}_d(\mathbb{R}) - S(\mathbb{R})$. The latter containment (and consequently the former) can indeed be strict: when $d = 2k + 1$, the automorphism group of the rational map $\phi(z) := i \left( \frac{z-1}{z+1} \right)^{2k+1}$ is trivial and its field of moduli is $\mathbb{Q}$ while it cannot be defined over reals; see [Sil07, Example 4.85, Exercise 4.39]. It is worthy to note that this question of “FoM vs. FoD” is relevant only when $d$ is odd [Sil07, Theorem 4.92].

Let us try to see how the real variety $\mathcal{M}_d(\mathbb{R}) - S(\mathbb{R})$ is related to its (possibly proper) Zariski closed subset $\mathcal{M}'_d - S(\mathbb{R}) = \mathcal{M}'_d - S'$. We claim that classes in the complement $\mathcal{M}_d(\mathbb{R}) - (S(\mathbb{R}) \cup \mathcal{M}'_d) = \mathcal{M}_d(\mathbb{R}) - (S(\mathbb{C}) \cup \mathcal{M}'_d)$ can be represented by antipodal maps; i.e. maps which commute with the anti-holomorphic involution $\gamma(z) := -\frac{1}{\bar{z}}$. Notice that the example $\phi(z) = i \left( \frac{z-1}{z+1} \right)^d$ for $d$ odd that appeared before is indeed antipodal. For more on the dynamics of antipodal preserving maps see [BBM15]. They are not relevant to our treatment of entropy as a generic map of this class does not preserve any circle.

**Proposition 2.9.** The real subvariety $\mathcal{M}'_d/\mathbb{R}$ of $\mathcal{M}_d(\mathbb{C})$ coincides with the real locus $\mathcal{M}_d(\mathbb{R})$ of the moduli space for $d$ even while for $d$ odd, the latter is reducible and has $\mathcal{M}'_d$ and the $(2d - 2)$-dimensional real subvariety of conjugacy classes of antipodal preserving maps as irreducible components. Any other irreducible component of $\mathcal{M}_d(\mathbb{R})$ has to be contained in the symmetry locus $S(\mathbb{R})$ and hence is of dimension at most $d - 1$.

**Proof.** Invoking [Sil07, Proposition 4.86], the obstruction to the field of moduli $\mathbb{R}$ being a field of definition for a map $f \in \text{Rat}_d(\mathbb{C})$ with $\text{Aut}(f) = \{1\}$ is encoded by the Galois cohomology class determined by the cocycle $\text{Gal}(\mathbb{C}/\mathbb{R}) \to \text{PGL}_2(\mathbb{C})$ defined by $\sigma \mapsto \alpha$ with $\sigma$ being the complex conjugation and $\alpha$ a Möbius transformation with $\alpha \circ f \circ \alpha^{-1} = \bar{f}$ that therefore satisfies the cocycle condition $\alpha \circ \bar{\alpha} = 1$. These transformations have to be considered modulo modification via a 1-coboundary, i.e. replacing $\alpha$ with $u \circ \alpha \circ \bar{u}^{-1}$ that amounts to replacing $f$ with $\bar{u} \circ f \circ \bar{u}^{-1}$ from the same conjugacy class. But the group

$$H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{PGL}_2(\mathbb{C})) \cong H^2(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \text{Br}(\mathbb{R})$$

is cyclic of order two and is thus generated by any 1-cocycle non-cohomologous to a coboundary. An example of such is the 1-cocycle associated to the rational map $\phi(z) = i \left( \frac{z-1}{z+1} \right)^{2k+1}$

\[^{13}\text{It is clear (at least in the analytic topology) that the subset of points of } \mathcal{M}_d(\mathbb{C}) \text{ corresponding to maps which admit a model over a prescribed closed subfield of } \mathbb{C} \text{ is closed.}\]

\[^{14}\text{Br}(\mathbb{R}) \text{ denotes the } \textit{Brauer group} \text{ of the field of real numbers.}\]
We conclude that the dimension of the intersection of the whole conjugacy class of a generic map of $S$ is given by

$$\dim \bigcup_{\text{conjugacy class}} \cong \alpha(z) = -\frac{1}{z},$$

that furnishes us with the 1-cocycle determined by $\alpha(z) = -\frac{1}{z}$. It is easy to verify that this is not 1-coboundary. Hence $\mathcal{M}_d(\mathbb{R}) - (S(\mathbb{R}) \cup \mathcal{M}'_d)$ is precisely the subset of classes in $\mathcal{M}_d(\mathbb{R}) - S(\mathbb{R})$ which admit a representative $f \in \mathbb{C}(z)$ with $\tilde{f}(-\frac{1}{z}) = \frac{1}{f(z)}$. Applying the complex conjugation map to both sides, this condition means that $f$ commutes with the anti-holomorphic involution $\gamma(z) = -\frac{1}{z}$; a Zariski closed condition over reals cutting out the antipodal preserving locus of the moduli space. Notice that the constraint automatically guarantees that the field of moduli is $\mathbb{R}$ as the complex conjugate map $\tilde{f}$ is Möbius conjugate to $f$ via $z \mapsto -\frac{1}{z}$ but there might be such maps which cannot be defined over reals; e.g., the example $\phi(z) = i \left(\frac{z-1}{z+1}\right)^d$ for $d$ odd. Consequently, the real variety $\mathcal{M}_d(\mathbb{R}) - S(\mathbb{R})$ is the union of $\mathcal{M}'_d - S(\mathbb{R})$—as discussed before in §2.1—is irreducible and of dimension $2d - 1$ and the antipodal locus.

Here is a simple dimension count for the antipodal preserving locus in $\mathcal{M}_d(\mathbb{C})$. Picking a degree $d$ map $f$ which commutes with $\gamma : z \mapsto -\frac{1}{z}$, after a Möbius conjugation, without any loss of generality we may assume that the roots of $f$ lie in the finite plane. If $q$ is a root of $f$, $-\frac{1}{q}$ has to be a pole. Thus roots and poles can coupled in pairs such as $(q_i, -\frac{1}{q_i})$.

We conclude that $f(z)$ is a scalar multiple of a function in the form of $\prod_{i=1}^d \frac{z-q_i}{1+q_i z}$. Now $f \circ \gamma = \gamma \circ f$ is satisfied for a multiple of such a product if and only if $d$ is odd and the scalar factor is of norm one. This argument indicates that any antipodal preserving map of odd degree $d$, after a suitable conjugation, can be uniquely written as

$$f(z) = \frac{u}{\prod_{i=1}^d (z-q_i) \bar{q}_i z},$$

where $|u| = 1$ and $q_i$’s are complex numbers with $q_i \bar{q}_j \neq -1$. This parametrizes a connected subspace of real dimension $2d + 1$ of $\text{Rat}_d(\mathbb{C})$ that projects onto the antipodal preserving locus in $\mathcal{M}_d(\mathbb{C})$. Consequently, like $\mathcal{M}'_d$, the antipodal locus is irreducible as well, being a surjective image of the irreducible real algebraic subset $S^1 \times (\mathbb{C}^\times)^d$ of $\mathbb{A}^{2d+1}(\mathbb{R})$. Let us find the dimension of the intersection of the whole conjugacy class of a generic map of this kind with the antipodal preserving locus in $\text{Rat}_d(\mathbb{C})$. Given an antipodal map $f$ with

\[ \text{(15) That is, not in the form of $u \circ \bar{u}^{-1}$ for another Möbius transformation; see [Sil07, p. 209] for details.} \]

\[ \text{(16) As a matter of fact, only odd degrees are relevant here since a classical theorem of Borsuk states that an antipodal preserving map $S^n \to S^n$ must be of odd degree. Aside from this topological obstruction, there is an arithmetic obstruction due to a theorem of Silverman which states that "FoM=FoD" in even degrees [Sil07, Theorem 4.92].} \]
\[
\text{Aut}(f) = \{1\}, \text{ if for an } \alpha \in \text{PGL}_2(\mathbb{C}), \alpha \circ f \circ \alpha^{-1} \text{ is antipodal too, then one can write:}
\]
\[
\gamma \circ (\alpha \circ f \circ \alpha^{-1}) \circ \gamma^{-1} = \alpha \circ f \circ \alpha^{-1} = \alpha \circ (\gamma \circ f \circ \gamma^{-1}) \alpha^{-1},
\]
which indicates that the Möbius transformation \((\alpha \circ \gamma)^{-1} \circ (\gamma \circ \alpha)\) is an automorphism of \(f\) and thus \(\alpha\) commutes with \(\gamma\). It is not hard to verify that a Möbius transformation which commutes with \(z \mapsto -\frac{1}{z}\) can be uniquely written either as \(z \mapsto z + ae^{2\pi i r}\) with \(r, s \in \mathbb{R}/\mathbb{Z}, a > 0\) or in one of forms \(z \mapsto v z, z \mapsto \frac{v}{z}\) where \(v\) lies on the unit circle. We conclude that the space of Möbius transformation commuting with the antipodal involution is of real dimension three and thus the antipodal locus in \(\mathcal{M}_d(\mathbb{C})\) is of dimension \((2d + 1) - 3 = 2d - 2\). \qed

3. The Julia Set and the Real Circle

The Julia set \(\mathcal{J}(f)\) of a rational map \(f\) of degree \(d \geq 2\) incorporates the “chaotic” parts of the dynamics of \(f\). So when \(f \in \mathbb{R}(z)\) and the circle \(\mathbb{R}\) is forward-invariant, intuitively the entropy of the restriction \(f \mid \mathbb{R}\) solely depends on how the invariant set \(\mathbb{R}\) meets the Julia set because \(\mathcal{J}(f) \cap \mathbb{R}\) contains the portion of the non-wandering set of \(f \mid \mathbb{R}\) that really matters to entropy.\(^{17}\) The article [Tio15] for instance relates the real entropy of a quadratic polynomial such as \(x \mapsto x^2 + c (-2 \leq c \leq \frac{1}{4})\) to the Hausdorff dimension of the set of rays that land on the intersection of the filled Julia set with the real axis. The interplay between the real Julia set and the real entropy is the main theme of this section. We give a comprehensive description of the real Julia set in Theorem 3.1 that results in a proof of Theorem 1.3 at the end of §3.1. §3.2 is devoted to some remarks and examples regarding the structure theorem 3.1. Finally, Theorem 1.4 will be proven in §3.3.

Certain topics from the dynamics of interval and circle maps will be invoked. See books [dMvS93], [Rue15] or the survey article [vS10] for the background material.

At last, to avoid confusion when going back and forth between real and complex contexts, we remind the reader that in the subsystem obtained by restricting a rational map to an invariant interval or circle, an infinite orbit might tend to a periodic cycle which is merely non-repelling for the rational map;\(^{18}\) we can have a parabolic cycle lying on the interval or circle that as a cycle of the restricted map is weakly attracting or possess a basin which is only 1-sided. Of course, this possibly 1-sided basin of the interval or circle map is contained in a parabolic basin of the ambient rational map.

\(^{17}\)The non-wandering set of \(f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) is the disjoint union of the Julia set with all (finitely many) cycles of rotation domains and (finitely many) attracting periodic points; cf. [Mil06a, Problem 19-a]. But the entropy vanishes restricted to the latter two. Hence, for any closed \(f\)-invariant subset \(A\) of the Riemann sphere, the topological entropy of \(f \mid A\) coincides with that of the subsystem \(f \mid \mathcal{J}(f) \cap A\) as any point outside it is either wandering or in a zero entropy closed subsystem.

\(^{18}\)Keep in mind that a repelling cycle of the complex map is topologically repelling [Mil06a, Lemma 8.10].
3.1. Structure of the Real Julia Set.

**Theorem 3.1.** Let $f \in \mathbb{R}(z)$ be of degree $d \geq 2$ and $\emptyset \neq \mathcal{J}(f) \cap \hat{\mathbb{R}} \subset \hat{\mathbb{R}}$. Then $\mathcal{J}_\mathbb{R}(f) = \mathcal{J}(f) \cap \hat{\mathbb{R}}$ can be uniquely written as the union below of closed $f$-invariant subsets:

$$\mathcal{J}_\mathbb{R}(f) = \left( \bigcup_{I \in \mathcal{I}} I \right) \cup Z;$$

where

1. $\mathcal{I}$ is a countable collection of pairwise disjoint compact non-degenerate intervals on which $f$ acts by taking an interval into another while preserving the boundaries;
2. every $I \in \mathcal{I}$ is eventually periodic under $f$; that is, some iterate of it is contained in an interval $I' \in \mathcal{I}$ satisfying $f^p(I') \subseteq I'$ for some appropriate $p > 0$; furthermore, there are only finitely many such periodic intervals $I' \in \mathcal{I}$;
3. $Z$ is a totally disconnected closed subsystem in which preperiodic points are dense and moreover can intersect an interval $I \in \mathcal{I}$ only at its endpoints;

Furthermore, there is a Cantor subsystem $C$ of $f \mid_Z : Z \to Z$ with the same entropy and a (possibly vacuous) finite set $S$ of isolated periodic points of the system $f \mid_{\mathcal{J}_\mathbb{R}(f)}$ such that $Z$ can be written as the union of $C$ and a closed countable subsystem which consists of backward iterates and the $\alpha$-limit set of $S$:

$$Z = C \bigcup \left( \bigcup_{n \geq 0} f^{-n}(S) \right) \cap \hat{\mathbb{R}}.$$

This emphasis on the density of preperiodic points in this theorem and the succeeding proposition is due to the fact that we intuitively expect $f \mid_{\mathcal{J}_\mathbb{R}(f)} : \mathcal{J}_\mathbb{R}(f) \to \mathcal{J}_\mathbb{R}(f)$ to exhibit “chaotic behavior”, at least in some subsystems, and the density of periodic points is often one of the manifestations of “chaos”. Here, the chaos is intended in the sense of Devaney [Dev89, §1.8]:

A continuous map $f : (X,d) \to (X,d)$ on the metric space $(X,d)$ is called chaotic if:

1. $f$ is transitive;
2. the periodic points of $f$ are dense;
3. $f$ has sensitive dependence on initial conditions.

It turns out that for $X$ infinite, the last condition is always implied by the first two and for interval maps solely the first condition suffices; see [BBC+92] and [Rue15, Proposition 7.2] respectively. Moreover, the density of the countable subset of preperiodic points comes in handy in §3.2 when in Proposition 4.7 we investigate “perturbations” of the real Julia set.
Proof. The real Julia set \( J^R(f) \) is assumed to be a proper non-empty subset of \( \hat{\mathbb{R}} \), and its connected components are thus either non-degenerate compact intervals or points. The continuous transformation \( f \mid_{J^R(f)} : J^R(f) \to J^R(f) \) must take a point component to a point component and an interval component to an interval component because it cannot collapse an interval to a point due to the analyticity of \( f \). Moreover, if \( f(I) \subseteq I' \) for two interval components, then \( f(\partial I) \subseteq \partial I' \) as otherwise, by continuity, there will be an open interval around an endpoint of \( I \) that goes into \( I' \subseteq J(f) \) via \( f \), so it is also a subset of \( J(f) \); this is a contradiction since then the connected component of \( J(f) \cap \hat{\mathbb{R}} \) which has that endpoint would be strictly larger than \( I \).

Therefore, the real Julia set may be written as a disjoint union

\[
J^R(f) = \bigcup_{I \in \mathcal{I}} I \bigcup Z'
\]

of countably many interval components and the totally disconnected set \( Z' \) of point components. The map \( f \) keeps \( Z' \) invariant and maps an element of \( \mathcal{I} \) into another. We claim that every \( I \in \mathcal{I} \) has two distinct iterates that collide. If not, the forward iterates \( I, f(I), f^2(I), \ldots \) are pairwise disjoint. In the terminology of real one-dimensional dynamics, such an interval is called a wandering interval unless \( f \circ n \mid I \) tends to an attracting cycle as \( n \to \infty \). Intervals \( f \circ n \mid I \subseteq J^R(f) \) cannot converge an attracting cycle of \( f \mid_{\hat{\mathbb{R}}} \) since a Julia point cannot lie in an attracting or parabolic basin of \( f \). Now, we invoke a powerful result from real one-dimensional dynamics that immediately rules out existence of wandering intervals:

An analytic map \( S^1 \to S^1 \) does not admit any wandering interval.\(^{19}\)

This establishes that every interval in the union (3.3) is eventually preserved by \( f \): If \( f \circ n \mid I \cap f \circ m \mid I \neq \emptyset \) for \( n > m \), then both of these iterate are contained in the same \( I' \in \mathcal{I} \), a connected component which is then invariant under \( f \circ (n-m) \mid I' \neq \emptyset \).

If \( I \in \mathcal{I} \) is preserved by an iterate \( f \circ p \), then the restriction \( f \circ p \mid I \) is boundary-anchored and possesses a critical point since otherwise \( f \circ p \mid I \) would be monotonic. It is easy to verify that the domain of a continuous monotonic interval map can be decomposed to the disjoint union of a subset consisting of points of period at most two and countably many 1-sided attracting basins of periodic orbits of period at most two. When the map is \( C^4 \) a periodic cycle admitting a 1-sided attracting basin has its multiplier in \([-1, 1]\). But an attracting or parabolic basin of \( f \) does not intersect \( I \subseteq J(f) \) and so the only other possibility is for every point in \( I \) be of period at most two which is absurd. By the chain rule, a critical

\(^{19}\)The non-existence of wandering intervals has been proven by several authors in various generalities and was finally settled in [MdMvS92] in its full strength. The statement above is an immediate corollary of a version of this important result outlined in [dMvS93, p. 260, Theorem A] which is formulated for \( C^2 \) maps without any flat critical point.
point of $f^{op}$ has to be in the backward orbit of a critical point of $f$. So each periodic cycle of intervals among members of $I$ would completely contain a critical orbit of $f$. We deduce that there are at most $2d - 2$ such intervals in $I$.

Next, we decompose $Z'$. Consider the set $Z''$ of points $z \in J_{\mathbb{R}}(f)$ for which there is a non-degenerate interval $K_z := [z, z + \epsilon_z]$ or $K_z := [z - \epsilon_z, z]$ that intersects $J(f)$ only at $z$. Since $Z'$ is formed by the point components of the closed set $J_{\mathbb{R}}(f) = J(f) \cap \hat{\mathbb{R}}$, $Z''$ is clearly dense in $Z'$. We claim that all points of $Z''$ are preperiodic. Without any loss of generality, let $K_z = [z, z + \epsilon]$, the other case is no different. Due to non-existence of wandering Fatou components, the Fatou component of $f$ containing $(z, z + \epsilon_z]$ is eventually periodic and we claim that it cannot end up in a cycle of rotation domains of $f$; in fact it is not hard to show that if a rotation domain of the rational map $f \in \mathbb{R}(z)$ intersects the real line then it must be a Herman ring containing the whole circle $\hat{\mathbb{R}}$ (Lemma 3.2) and this cannot occur here since we have assumed $\hat{\mathbb{R}}$ is not entirely Fatou. Consequently, the points of $(z, z + \epsilon_z]$ under iteration tend to a real attracting or parabolic cycle of the rational map $f$. Without any loss of generality and by replacing $f$ with an iterate if necessary, one can assume that the period is one. As a fixed point of the circle map $f \mid \hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$, this cycle admits a (possibly 1-sided) immediate basin $A_{\mathbb{R}}$. The closure $\overline{A_{\mathbb{R}}}$ of this immediate basin is a closed interval whose endpoints are indeed Julia points. The continuity argument that appeared before shows that $f$ restricts to a boundary-anchored transformation of $\overline{A_{\mathbb{R}}}$; in particular the endpoints of the immediate basin are preperiodic. Sufficiently high iterates of $K_z = [z, z + \epsilon]$ must intersect the open interval $A_{\mathbb{R}} \subset F(f)$ but cannot be contained in that since they have Julia points from the orbit $O(z)$. Therefore, there must be an endpoint of the closure $\overline{A_{\mathbb{R}}}$ contained in an iterate $f^{op}(K_z)$ of $K_z$. But due to the complete invariance of the Fatou and the Julia sets, $f^{op}(z)$ is the unique Julia point of the interval $f^{op}(K_z)$ and so $f^{op}(z)$ is an endpoint of $\overline{A_{\mathbb{R}}}$ and thus preperiodic.

Next, we are going to exploit what has been established about points of $Z''$ being preperiodic. This implies that $Z'$ and therefore its closure, denoted by $Z$, have the dense subset $Z''$ of preperiodic points. Just like $Z'$, the closure $Z$ must be $f$-invariant as well. This is totally disconnected and can intersect each interval components $I \in I$ of $J_{\mathbb{R}}(f)$ only at its endpoints. Therefore, the union from (3.3) can be modified to the union below of closed subsystems:

$$J_{\mathbb{R}}(f) = \left( \bigcup_{I \in I} I \right) \bigcup Z.$$ 

This is the desired decomposition (3.1) predicted by the theorem.

To finish the proof, we merely need to decompose the totally disconnected system $Z$ to a Cantor part and a countable part. Lemma 3.3 provides a canonical way to do so: $Z$ is the union a Cantor set $C$ and the countable closed subspace $D$ which is the closure of the set
of isolated points of $Z$. The proof of Lemma 3.3 establishes that $C$ is in fact the set of condensation points of $Z$ and $f$, due to finiteness of its fibers, takes a condensation point of the subsystem $Z$ to another such point. Hence the Cantor subspace $C$ is indeed a subsystem. Next, as the description of $D$ as the closure of the subset of isolated points of $Z$ suggests, we have to study isolated points of $Z$. Clearly, an isolated point of $Z$ is an isolated point of the real Julia set $J_{\mathbb{R}}(f)$ unless it is an endpoint of an interval component in which case it is already included in $\bigcup_{I \in Z} I$ and excluding it from $Z$ will not affect the compactness of $Z$. Thus we are going to assume that isolated points of $Z$ remain isolated in the bigger space $J_{\mathbb{R}}(f)$. Since the preperiodic points of $Z$ are dense in it, any isolated point $z_0$ of it has to be preperiodic, say $f^{m}(z_0) = f^{m+p}(z_0)$. So $z_0$ ends up in a cycle of period $p$ and it suffices to show that only finitely many such cycles of $f$ can arise in this manner form isolated points of $J_{\mathbb{R}}(f)$. Choose $\delta > 0$ so small that either $(z_0, z_0 + \delta)$ or $(z_0 - \delta, z_0)$ is a subset of the Fatou set. Since $f^{m}(z_0) = f^{o(m+p)}(z_0)$, either the $(m+p)$th or the $(m+2p)$th forward iterate of this interval intersects its $m$th iterate non-trivially based on whether $(f^{o(p)})^{'}(f^{m}(z_0))$ is positive or negative. In particular, there is a Fatou component of period $p$ or $2p$. But there are only finitely many periodic Fatou components and this puts an upper bound on the eventual periods of points like $z_0$; consequently, there are only finitely many of them. This establishes the description (3.2) of $Z$. The entropy of $f \mid_Z$ is the same as that of $f \mid_C$ since the compact subsystem $(\bigcup_{n} f^{o-n}(S)) \cap \hat{\mathbb{R}}$ appeared in (3.2) is countable and thus of zero entropy [BZ99, Proposition 5.1].

To get the countability, notice that otherwise Lemma 3.3 implies that $(\bigcup_{n} f^{o-n}(S)) \cap \hat{\mathbb{R}}$ contains a non-empty closed subspace consisting of condensation points that intersects $(\bigcup_{n} f^{o-n}(S)) \cap \hat{\mathbb{R}}$. So a real point in the backward orbit of a member of $S$ is a condensation point of $(\bigcup_{n} f^{o-n}(S)) \cap \hat{\mathbb{R}}$ and thus that of the bigger space $J_{\mathbb{R}}(f)$; a contradiction because $S$ is a subset of isolated points of $J_{\mathbb{R}}(f)$ and its iterated preimages cannot be condensation points.

Here are the two lemmas we referred to during the proof:

**Lemma 3.2.** A rotation domain of a non-linear rational map with real coefficients which intersects the real axis is a fixed Herman ring that furthermore contains the whole real circle $\hat{\mathbb{R}}$ as one of the leaves of its natural foliation.\(^{21}\)

**Proof.** Suppose a rotation domain $U$ of $f \in \mathbb{R}(z)$ intersects $\hat{\mathbb{R}}$ along an open subset. As $f(\hat{\mathbb{R}}) \subseteq \hat{\mathbb{R}}$, $f(U) \cap U \neq \emptyset$ and thus the period of $U$ is one. Aside from the center of a Siegel disk, the orbit closure of any point from a fixed rotation domain is an analytic Jordan curve which is the leaf passing through that point in the corresponding $f$-invariant foliation. So

\(^{20}\)There are indeed examples where the real Julia set has isolated points; for instance, consider the intersection of the Julia set of $z^2 - 1$ (the basilica) with the real axis.

\(^{21}\)Compare with Example 2.6.
the circle $\hat{\mathbb{R}}$ contains such a leaf. But that leaf is a topological circle as well, therefore they should coincide. Next, it is not hard to deduce that $U$ cannot be a Siegel disk: an open topological disk in $\hat{\mathbb{C}}$ containing the circle $\hat{\mathbb{R}}$ and invariant under the reflection with respect to it has to be the whole Riemann sphere which is absurd. $\square$

Lemma 3.3. Let $X$ be a Hausdorff Baire space with a countable basis for its topology; for instance, a separable complete metric space or a second countable locally compact Hausdorff topological space. Then $X$ can be uniquely written as a union $D \cup P$ of its closed subspaces where $D$ is countable and the closure of isolated points of $X$ and $P$ is perfect. In particular, when $X$ is furthermore compact and totally disconnected, $P$ would be a Cantor space.\footnote{Compare with [Pug02, chap. 2, Exercise 129].}

Proof. Recall that a point of a topological space is called a condensation point if there are uncountably many points in any open neighborhood of it. Obviously, the set of condensation points of $X$ is closed. Denote it by $P$ and let $D$ be the closure of isolated points of $X$. It is easy to show that $D$ is countable: pick a countable basis $U$ for the topology of $X$ such that no two isolated points of $X$ lie in the same element of $U$. If $D$ is uncountable, there will be a $U \in U$ with uncountably many points of $D$ but with precisely one isolated point of $X$. Removing that point yields an open in $X$ disjoint from the subset of isolated points but intersecting its closure $D$ which is impossible.

We need the following elementary fact:

- Each point of a perfect Hausdorff Baire space is a condensation point.

Proof. Otherwise, there exists a point $x$ admitting an open neighborhood containing only countably many points of $X$, say $x_n$’s. Each open subset $X - \{x_n\}$ is dense since $x_n$ is not isolated while $\bigcap_n (X - \{x_n\})$ is not dense because its closure misses $x$; this contradicts the Baire property.

Equipped with this, the uniqueness part is immediate: If one alternatively write $X$ as $D \cup P'$; $P'$, being the complement of a countable subset of the Baire space $X$, would be a Baire space. Hence by the above fact any point of the perfect space $P'$ is a condensation point of $P'$ and thus that of $X$. Conversely, a condensation point $x$ of $X$ has to lie in $P'$ as otherwise $X - P' \subseteq D$ would be a countable open neighborhood of $x$ in $X$.

Next, we establish the remaining properties claimed in the lemma. The union $D \cup P$ is the whole $X$ since if $x \notin D$, then $x$ admits an open neighborhood $U$ with no isolated point of $X$. So in the subspace topology $U$ is a perfect space. Being an open subset of the Baire space $X$, $U$ is a Baire space as well and so the aforementioned fact indicates that $x$ is a condensation point of it and thus a condensation point of the ambient space $X$. Hence $x$ has to lie in $P$. Finally, notice that $P$ is perfect: any open neighborhood in $X$ of a point $x \in P$ has uncountably many points of $X = D \cup P$ with at most countably many of them in $X - P \subseteq D$ so it intersects $P$ in infinitely many points.
To finish the proof, notice that when $X$ is totally disconnected, the same has to be true for the subspace $P$ which then would be a second countable compact Hausdorff (thus metrizable) totally disconnected perfect space and hence homeomorphic to the standard Cantor set [Pug02, chap. 2, Theorem 69].

Going back to the density of preperiodic points established above for the Cantor part of the dynamics $Z$, the same can be said both for those intervals $I$ in (3.3) which are periodic under $f$ and also for the case where $\mathcal{F}_\mathbb{R}(f)$ is the whole circle. This is the content of the proposition below that relies on certain results from real one-dimensional dynamics:

**Proposition 3.4.** Let $f \in \mathbb{R}(z)$ be a rational map and $I$ a non-degenerate compact connected subset of the real Julia set $\mathcal{F}_\mathbb{R}(f)$ which is invariant under (an iterate of) $f$. Then there is a closed subsystem $P \subseteq I$ in which preperiodic points are dense and

$$h_{\text{top}}(f \mid_I: I \to I) = h_{\text{top}}(f \mid_P: P \to P).$$

**Proof.** The proof is based on some of the ideas developed in the proof of Theorem 3.1 and the result below adopted from [Rue15, §7.1]:

For a continuous self-map of an interval or circle the following are equivalent:

- the topological entropy is positive;
- there is an infinite closed subsystem which is chaotic in the sense of Devaney.

Of course, if $f \mid_I$ does not admit any periodic point, the preceding result indicates that its entropy is zero and so $P = \emptyset$ works. Hence, suppose there are periodic orbits. The set of preperiodic points of $f \mid_I: I \to I$ is completely invariant. Its closure $P := \{x \in I \mid x \text{ is preperiodic}\}$ is therefore forward-invariant. If $P = I$ we are done. Otherwise, the proper subset $I - P$ of the circle $\hat{\mathbb{R}}$ is a union of countably many open or half-open subintervals. The image of any of them under $f$ is another non-degenerate subinterval of $I$ whose interior is away from $P$: Otherwise, there would exist an open subinterval $I'$ of $I - P$ with $f(I')$ being an open subinterval that intersects $P$ and thus contains a preperiodic point of $f$. But $I'$ then should have a preperiodic point itself which is impossible. We conclude that $f$ acts on the set of closed arcs of $\mathcal{J}(f) \subseteq \hat{\mathbb{R}}$ that are closures of connected components of $I - P$. Then, the exact same argument as in the proof of Theorem 3.1 based on the non-existence of wandering intervals implies that only finitely many of these closures are periodic (i.e. preserved by an iterate), and under iteration any arbitrary interval component of $I - P$ eventually maps into the closure of a component of $I - P$ which is periodic under $f$. Interval components of $I - P$ which are not periodic are away from the non-wandering set of $f \mid_I$ and hence do not contribute to the entropy. Picking an interval component of $I - P$ whose closure is preserved by an iterate of $f$, and next applying the theorem just quoted to the system obtained by restricting that iterate to the compact arc of the circle which is the closure, we deduce that the entropy of this restriction should be zero; if not, that component of $I - P$
must contain periodic points. Therefore, \( \text{NW}(f |_I : I \to I) \) is contained in the union of \( P \) and the \( f \)-invariant union of finitely many closed arcs that form a subsystem of zero entropy. Consequently, the entropy of \( f |_I \) coincides with that of its subsystem \( f |_P \).\(^{23}\) \hfill \Box

The following corollary is going to be used in §3.2:

**Corollary 3.5.** Given a rational map \( f \in \mathbb{R}(z) \) of degree \( d \geq 2 \), the system \( f |_{\hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}} \) has a closed (possibly vacuous) subsystem of the same topological entropy in which preperiodic points are dense.

**Proof.** This has been established when the whole real circle is Julia in Proposition 3.4 and is an immediate consequence of Theorem 3.1 when the Julia set intersects \( \hat{\mathbb{R}} \) non-trivially: Intervals \( I \in \mathcal{I} \) that are not preserved by any iterate of \( f \) are away from the wandering set of dynamics and aside from them, we have only finitely many intervals in \( \mathcal{I} \), say \( I_1, \ldots, I_n \) and all of them are invariant under some sufficiently divisible iterate \( f^{op} \). Thus the non-wandering set of the system \( f^{op} |_{\mathcal{J}_{\mathbb{R}}(f)} : \mathcal{J}_{\mathbb{R}}(f) \to \mathcal{J}_{\mathbb{R}}(f) \) is contained the union of closed subsystems \( Z \) (admitting a dense subset of preperiodic points according to Theorem 3.1) and \( I_i \)'s \((1 \leq i \leq n)\). Moreover, by Proposition 3.4 the system \( f^{op} |_{I_i} \) has a closed subsystem \( P_i \) of the same entropy in which preperiodic points are dense. So \( A := (\bigcup_{i=1}^{n} P_i) \cup Z \) is a closed subsystem of \( f^{op} |_{\hat{\mathbb{R}}} \) in which preperiodic points are dense and then \( B := \bigcup_{i=1}^{p} f^{-ov}(A) \) would be a closed invariant subsystem of \( f |_{\hat{\mathbb{R}}} \) satisfying the same property. This is the desired subsystem:

\[
\begin{align*}
    h_{\text{top}}(f |_{\hat{\mathbb{R}}}) &= h_{\text{top}}(f |_{\mathcal{J}_{\mathbb{R}}(f) \cap \hat{\mathbb{R}}}) = \max \{ h_{\text{top}}(f |_{\bigcup_{i=1}^{n} I_i}), h_{\text{top}}(f |_{Z}) \} \\
    &= \frac{1}{p} \max \{ h_{\text{top}}(f^{op} |_{\bigcup_{i=1}^{n} I_i}), h_{\text{top}}(f^{op} |_{Z}) \} = \frac{1}{p} \max \left\{ \max_{1 \leq i \leq n} \{ h_{\text{top}}(f^{op} |_{I_i}) \}, h_{\text{top}}(f^{op} |_{Z}) \right\} \\
    &= \frac{1}{p} \max \left\{ \max_{1 \leq i \leq n} \{ h_{\text{top}}(f^{op} |_{I_i}) \}, h_{\text{top}}(f^{op} |_{Z}) \right\} = \frac{1}{p} h_{\text{top}}(f^{op} |_{\bigcup_{i=1}^{n} P_i \cup Z}) = \frac{1}{p} h_{\text{top}}(f^{op} |_{A}) \\
    &\leq \frac{1}{p} h_{\text{top}}(f^{op} |_{\bigcup_{i=1}^{p} f^{-ov}(A)}) = h_{\text{top}}(f |_{\bigcup_{i=1}^{p} f^{-ov}(A)}) = h_{\text{top}}(f |_{B}).
\end{align*}
\]

\hfill \Box

**Proof of Theorem 1.3.** Immediately follows from Theorem 3.1 and Corollary 3.5. \hfill \Box

### 3.2. Examples and Remarks.

**Remark 3.6.** Theorem 3.1 indicates that the real entropy may be written as:

\[
(3.4) \quad h_{\mathbb{R}} = \max \{ h_{\text{Interval}}, h_{\text{Cantor}} \}
\]

\(^{23}\)Here and later in this section we invoke this simple fact that in a compact topological system written as a union of finitely many closed subsystems, the topological entropy coincides with the maximum of the entropies of these subsystems [BS02, Proposition 2.5.5].
where $h_{\text{Interval}}$, $h_{\text{Cantor}}$ are the contributions of interval and Cantor subsystems of $\mathcal{J}_R(f)$ to the real entropy. Unlike $h_R$, the functions

$$h_{\text{Interval}} : \mathcal{M}_d' - S' \to [0, \log(d)] \quad h_{\text{Cantor}} : \mathcal{M}_d' - S' \to [0, \log(d)]$$

are not continuous: The real Julia set of a hyperbolic rational map with real coefficients lacks interval components (cf. Remark 3.7) and so over the points determined by hyperbolic rational maps one has $h_{\text{Interval}} = 0, h_{\text{Cantor}} = h_R$. These points constitute a dense subset of the polynomial locus (the density of hyperbolicity for polynomials [KSvS07]) while there are definitely polynomials whose real Julia sets are union of intervals; e.g. the $d^{th}$ Chebyshev polynomial whose Julia set is the interval $[-2, 2]$ (cf. Example 3.13) where $h_{\text{Interval}} = h_R = \log(d), h_{\text{Cantor}} = 0$.

**Remark 3.7.** The dynamics of the interval subsystems of the real Julia set can be investigated more thoroughly. Suppose $I \subseteq \mathcal{J}_R(f)$ is a compact non-degenerate interval invariant under (an iterate of) $f$. There is a description of “typical” “attractors” of interval maps available from real one-dimensional dynamics that can be applied to $f \mid_I : I \to I$. It has to be mentioned that here “typical” can be interpreted in either the topological sense or the metric (measure-theoretic) sense and “attractors” also arise in both contexts and they might be more complicated than an attracting periodic orbit; see [Mil85] for a treatment of the general notion of an attractor for a continuous self-map of a differentiable manifold.

According to [vS10, Theorems 1.2, 1.3], for points from a “large” subset (residual or of full Lebesgue measure according to the context) $Y$ of $I$, the $\omega$-limit sets $\omega(x) (x \in Y)$ can be described as either of the following:

- $\omega(x)$ is a periodic orbit;
- $\omega(x)$ is a minimal Cantor set of Lebesgue measure zero and coincides with $\omega(c)$ for a critical point $c$;
- $\omega(x)$ is a finite union of intervals on which $f$ acts as a topologically transitive transformation.

The first possibility definitely does not occur in our situation since a periodic orbit to which another orbit accumulates is attracting for the interval map and thus admits a (perhaps parabolic) basin that must be away from the Julia set and hence from the $I$; in particular, for a hyperbolic $f \in \mathbb{R}(z)$ there is no such interval component and the real Julia set is totally disconnected. So a typical $\omega$-limit set fits in one of the last two categories. In the latter, there is a subinterval on which some iterate of $f$ restricts to a transitive (and hence chaotic) interval map. Such a map is of positive entropy [Rue15, Theorem 4.77]; furthermore, it is conjugate to a piecewise linear map [dMvS93, chap. III, Theorem 4.1]. A quadratic polynomial example would be $x \mapsto 4x(1-x)$ from the logistic family or equivalently the Chebyshev polynomial $x \mapsto x^2 - 2$ which on the Julia set $[-2, 2]$ of the $z \mapsto z^2 - 2$ restricts to a unimodal map conjugate to the tent map $x \in [-1, 1] \mapsto 1 - |2x - 1| \in [-1, 1]$; compare with Example 3.13.
As for Cantor attractors, first notice that if an interval subsystem of a real rational map admits either a residual or full Lebesgue measure subset of points whose $\omega$-limit sets are Cantor, all points of the interval are definitely Julia because the $\omega$-limit set of a Fatou point is either finite or a topological circle. If there is a residual subset of points whose $\omega$-limit sets coincide with the Cantor $\omega$-limit set $\omega(c)$ of a critical point, then it can be shown that map $f$ is infinitely renormalizable at $c$ and the attractor is called solenoidal; see [dMvS93, vS10] for details. Such an example from the logistic family is provided by the Feigenbaum map at the end of the first period-doubling cascade whose entropy is zero and its real Julia set is $[0, 1]$. It turns out that there are also examples of the so called wild attractor which are non-solenoidal Cantor metric attractors: There are non-renormalizable unimodal polynomial maps on the unit interval for which $\omega(x)$ is a Cantor set for points from a full Lebesgue measure subset but is the whole interval for points from a residual subset. The entropy is positive due to transitivity. There is no such an example in the quadratic family. See [BKNvS96] for the details.

Example 3.8. For a real quadratic polynomial, the Julia set cannot intersect the real axis along more than one interval. By contrast, the real Julia set of a real cubic polynomial can admit infinitely many interval components: Suppose the Julia set of $f$ is disconnected and there is an invariant interval component $I$ of $\mathcal{J}_R(f)$ for which the dynamics of $f \mid I$ is unimodal. The interval would have infinitely many interval components in its backward orbit due to the fact that a real cubic equation always has an odd number of real roots and also because no component of $f^{-k}(I)$ other than $I$ itself has a critical point (otherwise both critical orbits would be in the Julia set); a fact that shows that for every $l > k > 0$ the subsets $f^{-k}(I) \cap \hat{\mathbb{R}} - I$ and $f^{-ol}(I) \cap \hat{\mathbb{R}} - I$ are disjoint since otherwise there would be an interval component of $\mathcal{J}_R(f)$ other than $I$ invariant under an iterate which then according to the first part of Theorem 3.1 must contain a critical point of $f$. An inductive argument now shows that there are countably many disjoint intervals $\{I_0\}_{k \geq 0}$ with $I_0 = I$ and $f^{-k}(I) \cap \hat{\mathbb{R}} = I_0 \sqcup I_k$ for every $k > 0$. Figure 1 demonstrates such an example.

3.3. Maps of Maximal Real Entropy. In this subsection we study implications of the real entropy achieving its maximum $\log(\deg)$. This culminates in a proof of Theorem 1.4 now rewritten more comprehensively as Theorem 3.11.

First, we are going to try to quantify to what extent the intersection of the Julia set of a degree $d$ map $f \in \mathbb{R}(z)$ with $\hat{\mathbb{R}}$ affects the real entropy:

**Proposition 3.9.** Let $f \in \mathbb{R}(z)$ be of degree $d \geq 2$. Then for any point $x \in \hat{\mathbb{R}}$:

$$h_\mathbb{R}(f) \geq \log(d) + \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{dn} \# \left\{ y \in \hat{\mathbb{R}} \mid f^{on}(y) = x \right\} \right),$$

and the equality is always achieved for some $x$. Moreover, in the absence of post-critical relations, the number of solutions of $f^{on}(y) = x$ can be counted with multiplicities.
Figure 1. The Julia set of $0.19z^3 + z^2 - 1$; the red areas are attracting basins in the filled Julia set.

Proof. First, notice that the right-hand side is just $\limsup_{n \to \infty} \frac{1}{n} \log \left( \# \left\{ y \in \hat{\mathbb{R}} \mid f^n(y) = x \right\} \right)$.

The multimodal circle map $f^n \mid \hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ (after being pulled back to an interval map via a bijection $[0, 1) \to \hat{\mathbb{R}}$) has strictly monotonic pieces due to analyticity. Hence given $x \in \hat{\mathbb{R}}$, no two points of $\left\{ y \in \hat{\mathbb{R}} \mid f^n(y) = x \right\}$ are in the same lap of $f^n$; so

$$h_{\hat{\mathbb{R}}}(f) = h_{\text{top}}(f \mid \hat{\mathbb{R}}) = \lim_{n \to \infty} \frac{1}{n} \log \left( l((f \mid \hat{\mathbb{R}})^n) \right) \geq \limsup_{n \to \infty} \frac{1}{n} \log \left( \# \left\{ y \in \hat{\mathbb{R}} \mid f^n(y) = x \right\} \right),$$

where $l(g)$ is the lap number of $g$. Finally, we claim that the equality can be achieved. Let $c_1, \ldots, c_{l-1} \in \hat{\mathbb{R}}$ be turning points of $f \mid \hat{\mathbb{R}}$. The set of turning points of the iterate $(f \mid \hat{\mathbb{R}})^n$ is

$$\bigcup_{1 \leq i \leq l-1} \bigcup_{0 \leq k \leq n-1} \left\{ y \in \hat{\mathbb{R}} \mid f^{ok}(y) = c_i \right\};$$

which yields the following upper bound for the number of laps of this iterate:

$$l((f \mid \hat{\mathbb{R}})^n) - 1 \leq \sum_{i=1}^{l-1} \sum_{k=0}^{n-1} \# \left\{ y \in \hat{\mathbb{R}} \mid f^{ok}(y) = c_i \right\}. \quad (3.6)$$

Now, aiming for a contradiction, if

$$\forall 1 \leq i \leq l-1 : \limsup_{n \to \infty} \frac{1}{n} \log \left( \# \left\{ y \in \hat{\mathbb{R}} \mid f^n(y) = c_i \right\} \right) < h_{\hat{\mathbb{R}}}(f),$$

then one can find $\alpha < h_{\hat{\mathbb{R}}}(f)$ and $C \gg 0$ with

$$\forall 1 \leq i \leq l-1, \forall n \geq 0 : \# \left\{ y \in \hat{\mathbb{R}} \mid f^n(y) = c_i \right\} \leq C \exp(\alpha n).$$
But then the right-hand side of (3.6) would be at most \( nC(l - 1) \exp(\alpha n) \) and consequently:

\[
    h_{\mathbb{R}}(f) = \lim_{n \to \infty} \frac{1}{n} \log (l ((f |_{\mathbb{R}})^{\circ n})) \leq \lim_{n \to \infty} \frac{1}{n} \log (nC(l - 1) \exp(\alpha n)) = \alpha;
\]
a contradiction. To finish the proof, observe that when there is not any post-critical relation, there is at most one critical point \( c \) in the backward-orbit of \( x \) under \( f |_{\mathbb{R}} \) and counting the number of solutions of \( f^{\circ n}(y) = x \) with multiplicities raises this number through multiplication by a factor which is at most the multiplicity of \( c \); this cannot affect the exponential growth rate. \( \square \)

**Remark 3.10.** Except finitely many cases, the cardinality of a fiber of a \( C^1 \) degree \( s \) self-map \( f : M \to M \) of a compact connected differentiable oriented manifold \( M \) is at least \( \vert s \vert \). Assuming \( s \neq 0 \), in the special case where \( M = \mathbb{R} \) and the degree \( s \) self-map is the restriction of a rational map \( f \in \text{Rat}_d(\mathbb{R}) \), for all but countably many \( x \in \mathbb{R} \) there are at least \( \vert s \vert^n \) points \( y \in \mathbb{R} \) with \( f^{\circ n}(y) = x \). Inequality (3.5) thus implies that \( h_{\mathbb{R}} \geq \log(\vert s \vert) \) over the component \( \mathcal{M}_{d,s}' - \mathcal{S}' \) of the domain of \( h_{\mathbb{R}} \). This is an instant of a well known result asserting that \( h_{\text{top}}(g) \geq \log (\vert \text{deg}_{\text{top}}(g) \vert) \) in the general setting outlined above; see [MP77].

The preceding proposition provides a nice interpretation of real entropy: Given \( f \in \mathbb{R}(z) \) and \( \epsilon > 0 \), for every \( x \in \mathbb{R} \), \( \# \{ y \in \mathbb{R} \mid f^{\circ n}(y) = x \} \) is \( o(\exp ((h_{\mathbb{R}}(f) + \epsilon) n)) \) as \( n \to \infty \) and \( h_{\mathbb{R}}(f) \) is the smallest number for which this holds. In the way that Equation (3.5) is written, the term \( \frac{1}{d} \# \{ y \in \mathbb{R} \mid f^{\circ n}(y) = x \} \) on the right-hand side is reminiscent of one of the constructions of the *measure of maximal entropy* \( \mu_f \) for the rational map \( f : \mathbb{C} \to \mathbb{C} \) as outlined in [FLMn83]: for any non-exceptional point \( x \in \mathbb{C} - \mathcal{E}(f) \), the empirical measures \( \frac{1}{d} \sum_{y \in \mathbb{C} \mid f^{\circ n}(y) = x} \delta_y \) weakly converge to \( \mu_f \) as \( n \to \infty \) where in averaging the Dirac masses the multiplicities have been taken into account. Taking \( x \) in (3.5) to be in \( \mathbb{R} - \mathcal{E}(f) \), we conclude that:

\[
    \limsup_{n \to \infty} \frac{1}{d} \# \{ y \in \mathbb{R} \mid f^{\circ n}(y) = x \} \leq \mu_f(\mathbb{R}).
\]

When \( \mu_f(\mathbb{R}) = 0 \) the argument of the logarithm in (3.5) tends to zero so interesting behavior can be expected from the limit \( \limsup_{n \to \infty} \frac{1}{d} \log \left( \frac{1}{d} \# \{ y \in \mathbb{R} \mid f^{\circ n}(y) = x \} \right) \). This observation promotes us to classify the cases where the measure of maximal entropy assigns a positive value to the real circle. This is the content of the theorem below where we show that these are extreme situations with the Julia set being completely real:

**Theorem 3.11.** Let \( f \in \mathbb{R}(z) \) be of degree \( d \geq 2 \) with \( \mu_f \) its measure of maximal entropy. Then the following are equivalent:

(a) \( \mu_f(\mathbb{R}) > 0 \);

(b) the Julia set \( \mathcal{J}(f) \) is a subset of \( \mathbb{R} \);

(c) $h_{\text{top}}\left( f \mid \mathbb{R} : \hat{\mathbb{R}} \rightarrow \hat{\mathbb{R}} \right) = \log(d)$.

Moreover, in such a situation $\mathcal{J}(f)$ is either the whole circle $\hat{\mathbb{R}}$, a subinterval of it or a Cantor set on it. In all of these cases, a Fatou component is the immediate basin of an attracting or parabolic orbit of period at most two. When $\mathcal{J}(f)$ coincides with $\hat{\mathbb{R}}$, $f$ induces a degree $d$ covering $\hat{\mathbb{R}} \rightarrow \hat{\mathbb{R}}$ and when $\mathcal{J}(f)$ is an interval, $f \mid \mathcal{J}(f)$ is a boundary-anchored $(d - 1)$-modal interval map of entropy $\log(d)$ with surjective monotonic pieces.

**Proof.** It is clear that (b)$\Rightarrow$(a), (c), because if $\mathcal{J}(f) \subseteq \hat{\mathbb{R}}$, using $\text{supp}(\mu_f) = \mathcal{J}(f)$:

$$h_{\text{top}}(f \mid \mathbb{R}) \geq h_{\text{top}}(f \mid \mathcal{J}(f)) \geq h_{\mu_f}(f \mid \mathcal{J}(f)) = h_{\mu_f}(f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}) = h_{\text{top}}(f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}) = \log(d).$$

Next, suppose $\mu_f(\hat{\mathbb{R}}) > 0$. According to Montel’s theorem, for any open neighborhood $U \subseteq \hat{\mathbb{C}}$ of a point of $\mathcal{J}(f)$, $\bigcup_n f^{-n}(U)$ includes a dense subset of $\mathcal{J}(f)$. This union is of full measure with respect to $\mu_f$ since, picking an arbitrary non-exceptional point $x \in U$, this set is of full measure with respect to each of measures $\frac{1}{\delta} \sum_{y \in \mathbb{C} \mid f^{n}(y) = x} \delta_y$ and therefore, with respect to their weak limit $\mu_f$. Now fixing a countable collection of open sets $\{U_m\}_m$ for which $\{U_m \cap \mathcal{J}(f)\}_m$ is an open basis for the topology of $\mathcal{J}(f)$ and then taking the intersection over $U_m$’s, we arrive at the full measure subset $\bigcap_m \bigcup_n f^{-n}(U_m)$ which by the Baire category theorem contains a dense subset of $\mathcal{J}(f)$ and moreover the forward iterates of any of its points form a dense subset of $\mathcal{J}(f)$. As $\mu_f(\hat{\mathbb{R}}) > 0$, this full measure subset intersects $\hat{\mathbb{R}}$: There is a point on $\hat{\mathbb{R}}$ whose orbit is dense in $\mathcal{J}(f)$. But $\hat{\mathbb{R}}$ is closed and forward-invariant so must contain the whole $\mathcal{J}(f)$.

At last, we are going to show (c)$\Rightarrow$(b) that will also yield (c)$\Rightarrow$(a) because we have so far established (a)$\Rightarrow$(b). To this end, we invoke the main result of the article [Hof81] indicating that a multimodal transformation of an interval with positive entropy admits a measure of maximal entropy. In that reference, multimodal maps can be discontinuous at turning points. Thus, identifying $\hat{\mathbb{R}} \cong S^1$ with $[0, 1]$ via the bijection $x \mapsto e^{2\pi i x}$, the same holds for continuous circle maps of positive entropy. So there is a Borel probability measure, say $\nu$, on the circle $\hat{\mathbb{R}}$ with respect to which the metric entropy of $f \mid \mathbb{R} : \hat{\mathbb{R}} \rightarrow \hat{\mathbb{R}}$ coincides with its topological entropy $\log(d)$. Pushing forward via the inclusion $i : \hat{\mathbb{R}} \hookrightarrow \hat{\mathbb{C}}$, one gets a measure $i_*\nu$ on the Riemann sphere with respect to which the measure theoretic entropy of $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is at least $\log(d)$. So $i_*\nu$ is a measure of maximal entropy for this rational map too. The uniqueness of the measure of maximal entropy for rational maps established in [Mn83] then implies that $i_*\nu = \mu_f$. Hence $\mu_f(\hat{\mathbb{R}}) = \nu(\hat{\mathbb{R}}) = 1$.

To obtain the last part of the theorem, recall that the Julia set is always either connected or has uncountably many connected components [Mil06a, Corollary 4.15]. So when $\mathcal{J}(f) \subseteq \hat{\mathbb{R}}$, it is either a non-degenerate subinterval of $\hat{\mathbb{R}}$, the whole circle or a disjoint union of (forcibly at most countably many) compact non-degenerate subintervals of the real circle with a (necessarily uncountable) closed totally disconnected subset of it. We claim that in
the latter case there is not any subinterval and thus \( \mathcal{J}(f) \) is a closed totally disconnected subset of \( \hat{\mathbb{R}} \) and hence (given the fact that it has no isolated point) a Cantor set. Assume the contrary; \( \mathcal{J}(f) \) admits both interval and singleton components. But \( f \) takes a connected component of \( \mathcal{J}(f) \) to another such component and, being a rational map, cannot collapse a non-degenerate interval to a point. So the forward iterates of an interval component never cover a singleton component. This is a contradiction since the former contains an open sub-
set of \( \hat{\mathbb{R}} \) and therefore an open subset of \( \mathcal{J}(f) \) and by Montel’s theorem the union of forward images of any non-empty open subset of the Julia set is the whole Julia set. Consequently, \( \mathcal{J}(f) \subseteq \hat{\mathbb{R}} \) is either \( \hat{\mathbb{R}} \), a subinterval of it or a Cantor set on it; cf. Theorem 3.1.

The claim regarding Fatou components quickly follows from the Sullivan’s classification of Fatou components; when \( \mathcal{J}(f) \subseteq \hat{\mathbb{R}} \) there are at most two Fatou components, so all of them are periodic and only the existence of a rotation domain has to be ruled out; just notice that a rational map with a rotation domain must have infinitely many Fatou components.

To finish the proof, we address the extreme cases where the Julia set is \( \hat{\mathbb{R}} \) or a subinterval or a Cantor subset of it. When \( \mathcal{J}(f) \) is “smooth”, namely is an interval or the whole circle; since the Julia set is backward-invariant, \( f \restriction_{\mathcal{J}(f)} \colon \mathcal{J}(f) \to \mathcal{J}(f) \) would be a degree \( d \) ramified covering whose ramification structure may be readily investigated. First, suppose \( \mathcal{J}(f) = \hat{\mathbb{R}} \). If some critical points of \( f \) lie on \( \hat{\mathbb{R}} \), then \( \hat{\mathbb{R}} - f^{-1} \left( \{ v \in \hat{\mathbb{R}} \mid v \text{ a critical value of } f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \} \right) \) is a disjoint union of \( \# f^{-1} \left( \{ v \in \hat{\mathbb{R}} \mid v \text{ a critical value of } f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \} \right) \) open intervals restricted to which \( f \) yields a covering map onto \( \hat{\mathbb{R}} - \{ v \in \hat{\mathbb{R}} \mid v \text{ a critical value of } f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \} \); a union of \( \# \left\{ v \in \hat{\mathbb{R}} \mid v \text{ a critical value of } f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \right\} \) intervals; so \( f \) takes each of the former intervals bijectively onto one of the latter. But \( \mathcal{J}(f) = \hat{\mathbb{R}} \) is backward-invariant under the degree \( d \) map \( f \) so each of the latter intervals has to be covered by \( d \) of the former ones. This happens only when

\[
\# f^{-1} \left( \{ v \in \hat{\mathbb{R}} \mid v \text{ a critical value of } f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \} \right) = d \left( \# \left\{ v \in \hat{\mathbb{R}} \mid v \text{ a critical value of } f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \right\} \right);
\]

i.e. \( f \) admits \( d \) points over each of its critical values that lies on \( \hat{\mathbb{R}} \), which is absurd and therefore \( f \restriction_{\mathcal{J}(f)} \colon \mathcal{J}(f) \to \mathcal{J}(f) \) is a degree \( d \) (unramified) covering of circles. Finally, when \( \mathcal{J}(f) \) is an interval, \( f \restriction_{\mathcal{J}(f)} \colon \mathcal{J}(f) \to \mathcal{J}(f) \) is a multimodal interval map of topological entropy \( \log(d) \). This entropy cannot be realized if the lap number is less than \( d \); (1.3). Therefore, \( f \restriction_{\mathcal{J}(f)} \) is \((d - 1)\)-modal. On each of its \( d \) laps \( f \restriction_{\mathcal{J}(f)} \) restricts to a surjective map onto \( \mathcal{J}(f) \) because otherwise there would be a non-empty open subinterval of \( \mathcal{J}(f) \) over which \( f \restriction_{\mathcal{J}(f)} \) admits less than \( d \) preimages. The same holds for fibers of \( f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) above these points due to backward-invariance of \( \mathcal{J}(f) \); this cannot happen as the rational map \( f \) is of degree \( d \). The map has to be boundary-anchored too: if \( f \) takes an endpoint of

\[\text{---Footnote---}\]

\[\text{24Another indication of the entropy being } \log(d); \text{ the surjectivity of laps implies that the } n^{\text{th}} \text{ iterate has } d^n \text{ laps and so the exponential growth rate of lap numbers is } \log(d).\]
the interval $\mathcal{J}(f)$ to a point of its interior, by continuity, it maps points outside the interval sufficiently close to that endpoint inside $\mathcal{J}(f)$; a contradiction since the points outside the interval are Fatou points.

**Example 3.12.** Here are three examples of quadratic rational maps with parabolic fixed points adopted from [Mil93, Problem 10-f] that embody all possibilities for the Julia set outlined in the previous theorem. The real map $f(z) = z - \frac{1}{2}$ takes $\mathbb{R}$ to $\mathbb{R}$ via a degree two covering and has a parabolic fixed point at infinity whose multiplicity is $n + 1 = 3$. So there have to be $n = 2$ parabolic basins and thus $\mathcal{J}(f) \subseteq \mathbb{R}$ has to coincide with the real circle and the two components of its complement, namely the upper and the lower half planes, are the Fatou components. Things are different for the real map $f(z) = z - \frac{1}{2} + 1$; although the Julia set is still contained in $\mathbb{R}$ because the critical points are complex imaginary, now the infinity is a parabolic fixed point of multiplicity only $n + 1 = 2$ which permits solely one parabolic basin. So the Fatou set has to be connected and therefore $\mathcal{J}(f)$ is a Cantor set included in $\mathbb{R}$. Finally, consider the map $f(z) = z + \frac{1}{2} + 2$. The critical points $\pm 1$ are real and the subinterval $[0, +\infty]$ of $\mathbb{R}$ is backward-invariant and hence contains $\mathcal{J}(f)$. The point at infinity is of multiplier $+1$ and of multiplicity $n + 1 = 2$ with the attraction vector $\vec{v} = -\frac{1}{2}$. So the unique Fatou component is the parabolic basin of the point at infinity consisting of the points $z$ with $\lim_{k \to \infty} \frac{f^k(z)}{\sqrt{k}} = -\frac{1}{2}$. The limit fails for points $z > 0$ whose iterates are all positive. Thus the Julia set coincides with $[0, +\infty]$.

**Example 3.13.** Here are some examples of the Julia set being a circle or a real interval motivated by [Mil06a, Problems 7-b & 7-d]. A Blaschke product such as

$$f(z) := e^{2\pi i c} \prod_{i=1}^{d} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right)$$

where $d \geq 2$, $|a_1|, \ldots, |a_d| < 1$ and $c \in \mathbb{R}/\mathbb{Z}$ preserves the open unit disk and on its boundary, induces a degree $d$ covering; cf. Example 2.6. So $|z| = 1$ is backward-invariant and hence, by a simple application of Montel’s theorem, must contain $\mathcal{J}(f)$. If one of the $a_i$’s is zero, both $0, \infty$ are fixed and therefore cannot lie in the same Fatou component which implies that there are more than one Fatou components. Therefore, $\mathcal{J}(f)$ is the whole circle. If we furthermore assume that $a_i$’s are real and $e^{2\pi ic} = \pm 1$, then $f(z)$ will commute with $z \mapsto \frac{1}{z}$ and hence through the semi-conjugacy $z \mapsto z + \frac{1}{2}$ induces a real rational map of the same degree whose Julia set is $[-2, 2]$, i.e. the image of $|z| = 1$ under $z \mapsto z + \frac{1}{2}$. This construction yields a positive dimensional family of conformally distinct real degree $d$ rational maps with Julia set $[-2, 2]$. Letting $a_1 = \cdots = a_d = 0$, $e^{2\pi ic} = 1$ results in a prominent member of this family; the $d$th Chebyshev polynomial $T_d(z)$ (normalized to be monic via a linear conjugation) which is defined by $T_d \left( z + \frac{1}{2} \right) = z^d + \frac{1}{z^d}$.
4. Rigidity of Real Entropy

Now equipped with the definition of the function $h_\mathbb{R} : \mathcal{M}'_d - \mathcal{S}' \to [0, \log(d)]$ from §2 and the results established in §3 about the structure of real Julia sets, in this section we finally prove Theorem 1.1. We first treat this rigidity result in the case of hyperbolic maps in §4.1 by utilizing well known techniques of the kneading theory [MT88]. We then proceed to the general formulation in §4.2 based on the theory developed in [MS98].

4.1. Hyperbolic Components. A rational map is called hyperbolic if each of its critical orbits converges to an attracting periodic orbit, see [Mil06a, Theorem 19.1] for equivalent characterizations. Such maps form an open subset of the moduli space $\mathcal{M}_d(\mathbb{C})$ whose connected components are called hyperbolic components. Here we show that every connected component of the intersection with $\mathcal{M}'_d$ of a hyperbolic component in $\mathcal{M}_d(\mathbb{C})$ is included in a single isentrope. Away from the antipodal and symmetry loci, these “real hyperbolic components” can be thought of as the components of the set of fixed points the involution $\langle f \rangle \mapsto \langle \bar{f} \rangle$ induced by conjugating coefficients in $\text{Rat}_d(\mathbb{C})$ when the involution acts on a hyperbolic component in $\mathcal{M}_d(\mathbb{C})$. For future references, we record this involution as acting not only on rational maps but on functions defined on the Riemann sphere:

**Definition 4.1.** For any function $h : \hat{\mathbb{C}} \to \mathbb{C} \cup \{\infty\}$, the function $\tilde{h} : \hat{\mathbb{C}} \to \mathbb{C} \cup \{\infty\}$ is defined as:

\[
(4.1) \quad \tilde{h} : z \mapsto \bar{h}(\bar{z}).
\]

Of course, for a rational map $h \in \mathbb{C}(z)$ this is just $\tilde{h}$ obtained from conjugating the coefficients. It is easy to check that $h \mapsto \tilde{h}$ is an involution which respects the ring structure of the set of $\mathbb{C}$-valued functions on the Riemann sphere; takes homeomorphisms to homeomorphisms and finally commutes with differential operators $\partial_z$ and $\partial_{\bar{z}}$:

\[
(4.2) \quad \frac{\partial \tilde{h}}{\partial z} = \tilde{h} \frac{\partial h}{\partial z}, \quad \frac{\partial \tilde{h}}{\partial \bar{z}} = \tilde{h} \frac{\partial h}{\partial \bar{z}}.
\]

**Theorem 4.2.** Let $d \geq 3$ and $\mathcal{U}$ be a connected component of the intersection of a hyperbolic component of $\mathcal{M}_d(\mathbb{C})$ with the real subvariety $\mathcal{M}'_d$. Then the function $h_\mathbb{R}$ is constant over $\mathcal{U}$ with a value which is the logarithm of an algebraic number.

**Remark 4.3.** There is a thorough classification of hyperbolic components of $\mathcal{M}_2(\mathbb{C})$ [Mil93, Ree90]. In that case Theorem 4.2 still remains valid but requires a little bit of more work as excluding the symmetry locus might cause the real hyperbolic component $\mathcal{U}$ to become disconnected; cf. [Fil18].

**Proof of Theorem 4.2.** The open subset $\mathcal{U} - \mathcal{S}'$ of $\mathcal{M}'_d$ is connected because $\mathcal{U} \cap \mathcal{S}'$ is of codimension at least two; Proposition 2.4. Any real representative $f \in \mathbb{R}(z)$ of a point in it
restricts to a continuous multimodal circle map $f \mid \hat{\mathbb{R}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ that satisfies the hypothesis of Lemma 4.4 below. Therefore, $\log(h_{\mathbb{R}}((f)))$ is always algebraic for a point $\langle f \rangle$ from this open connected set. The continuity of $h_{\mathbb{R}} : \mathcal{M}'_{\delta} - \mathcal{S}' = [0, \log(d)]$ then yields its constancy over $\mathcal{U} - \mathcal{S}'$. □

**Lemma 4.4.** Let $I$ be an interval $[a, b]$ or a circle. Let $f : I \to I$ be a continuous piecewise monotone multimodal map of the interval $I = [a, b]$ whose turning points are attracted by a periodic orbit. Then $\exp(h_{\text{top}}(f))$ is algebraic.

**Proof.** Let us first deal with the interval case. We need to use the classical kneading theory of Milnor and Thurston developed in [MT88]. Suppose $f : [a, b] \to [a, b]$ is multimodal with turning points $c_1 < \cdots < c_{l-1}$ and laps $I_1 = [a, c_1], I_2 = [c_1, c_2], \ldots, I_{l-1} = [c_{l-2}, c_{l-1}], I_l = [c_{l-1}, b]$. The shape of the restriction $f \mid I_n$ of $f$ to the $n^{\text{th}}$ lap $I_n$ will be denoted by $\epsilon_n \in \{\pm 1\}$ which is $+1$ if the restriction is increasing and $-1$ when it is decreasing. To each point $x \in I$ one can assign infinite vectors $(\theta_i(x^+))_{i \geq 0}$ and $(\theta_i(x^-))_{i \geq 0}$ whose components come from the set of symbols $\{\pm I_1, \ldots, \pm I_l\}$. Here is the definition: $\theta_i(x^+) = \epsilon_i I_n$ (respectively $\theta_i(x^-) = \epsilon_i I_n$) means that there is a half-open interval with its left end (resp. right end) at $x$ over which $f^i$ is monotonic of shape $\epsilon \in \{\pm 1\}$ and is mapped by $f^i$ into $I_n$. Associated with them are formal power series $\theta(x^+) := \sum_{i=0}^{\infty} \theta_i(x^+) t^i$ and $\theta(x^-) := \sum_{i=0}^{\infty} \theta_i(x^-) t^i$ from $V[[t]]$ with $V$ being the free abelian group $\mathbb{Z}\{I_1, \ldots, I_l\}$. The next definition is that of the kneading increment $\nu_m := \theta(e_m^+) - \theta(e_m^-)$ for any $1 \leq m \leq l - 1$. One can then form the $(l - 1) \times l$ kneading matrix $N$ of power series in $\mathbb{Z}[[t]]$ with $N_{mn}$ being the coefficient of $I_n$ in $\nu_m$, i.e. $\nu_m = \sum_{n=1}^{l} N_{mn} I_n$. Denoting the orientation of $f \mid I_n$ by $\epsilon_n \in \{\pm 1\}$ and the determinant of the submatrix of $N$ obtained from deleting the $n^{\text{th}}$ column by $D_n(t)$, it can be proven that the formal power series $(-1)^{n+1} D_n(t)/(1 - \epsilon_n t)$ is independent of $1 \leq n \leq l$. This common power series is called the kneading invariant of $f$ and will be denoted by $D(t)$. It is easy to observe that the coefficients of $D(t)$ are bounded integers so $D(t)$ defines an analytic function over the disk $|t| < 1$. Here is the main result ([MT88, Theorem 6.3]):

$$\frac{1}{\exp(h_{\text{top}}(f))}$$

is the smallest root of $D(t)$ in $[0, 1]$.

Thus it suffices to show that under our assumption $D(t)$ is a rational map with integer coefficients whose roots and therefore $\exp(h_{\text{top}}(f))$ are algebraic numbers. In order to do so, one just needs to show that kneading coordinates $(\theta_i(c_m^+))_{i \geq 0}$ and $(\theta_i(c_m^-))_{i \geq 0}$ are eventually periodic. We are going to argue that $(\theta_i(c_m^+))_{i \geq 0}$ is periodic; the case of $(\theta_i(c_m^-))_{i \geq 0}$ is completely similar. Suppose for $i \geq q$, $f^i(c_m)$ is in the immediate basin of the periodic point $x_{i \text{mod } p}$ from the orbit $x_0 \mapsto x_2 \mapsto \cdots \mapsto x_{p-1} \mapsto x_0$. We claim that for $\delta > 0$ small enough, one can take $q' > q$ so large that for $i > q'$ the interval $f^i((c_m, c_m + \delta))$ is in the interior of a lap $I_{n_i \text{mod } p}$ of $f$ which is dependent only on the remainder of $i$ modulo $p$. To see this, fix $0 \leq r < p$. Note that $f^{(jp+r)}(c_m) \to x_r$ as $j \to \infty$ and so if $x_r$ is not a turning point,
$f^{o(jp+r)}(c_m)$ and hence the image under $f^{o(jp+r)}$ of small enough non-degenerate subintervals $[c_m, c_m + \delta]$ of the basin belong to the interior of the lap that has $x_r$ for large enough $j$’s; say for $j > j_r$. The same holds even when $x_r$ is a turning point of $f$; we only have to rule out the possibility of $f^{o(jp+r)}(c_m)$ alternating between the two laps that have $x_r$ in common: if the fixed point

$$x_r = \lim_{j \to \infty} f^{o(jp+r)}\left((c_m, c_m + \delta)\right)$$

of $f^{op}$ is a turning point as well, then for $j$ large enough, $f^{o(jp+r)}\left((c_m, c_m + \delta)\right)$ always lands to the left of $x_r$ if $x_r$ is a local maximum of $f^{op}$ and to the right if it is a local minimum. Repeating this argument for all $r \in \{0, \ldots, p-1\}$, $p. \max\{j_0, \ldots, j_{p-1}\} + 1$ then works as the desired $q’$.

Next, after decreasing $\delta$ if necessary, suppose all iterates $f, \ldots, f^{q'+1}$ restrict to monotonic maps on $[c_m, c_m + \delta]$. We now show that $(\theta_i(c_m^+))_{i \geq 0}$ is periodic of period $2p$ for $i > q'$: $f^i\left((c_m, c_m + \delta)\right)$ is contained in the lap $I_{n_i \mod p}$ and the degree of $f^i \upharpoonright_{[c_m, c_m + \delta]}$ is the product of the degree of the monotonic map $f^{q'+1} \upharpoonright_{[c_m, c_m + \delta]}$ by the degrees of $f$ over laps $I_{n_{(q'+1) \mod p}}, \ldots, I_{n_{(i-1) \mod p}}$. Replacing $i$ with $i + 2p$ does not affect the former while multiplies the latter product by $(\prod_{i=1}^p \epsilon_{n_{i \mod p}})^2 = 1$.

Finally, notice that the kneading theory has also been developed for continuous multimodal circle maps or equivalently for multimodal interval maps with finitely many discontinuities; see [Pre89, appendix]. Therefore, Lemma 4.4 remains valid when $f$ is the transformation of the circle $\hat{\mathbb{R}}$ induced by an element of $\mathbb{R}(z)$.

**Remark 4.5.** Much more can be said about algebraic properties of entropy values of PCF multimodal maps. The paper [Thu14] establishes that a real algebraic integer arises as $\exp\left(h_{\text{top}}(f : I \to I)\right)$ for a critically finite multimodal map $f : I \to I$ if and only if it is a weak Perron number, i.e. at least as large as the absolute values of its Galois conjugates.

**Remark 4.6.** In the preceding theorem it is natural to ask how many distinct entropy values can be taken by $h_{\mathbb{R}}$ over the space of real maps in the same complex hyperbolic component. For instance, when $d = 2$, for real quadratic maps in the escape locus where both critical orbits tend to the same attracting fixed point both extremes $0$, $\log(2)$ occur; consider quadratic polynomials $z \mapsto z^2 + c$ where $c \in \mathbb{R}$ is either to the right or to the left of the Mandelbrot set. In general, following the suggestion on [Mil92, p. 15], if the hyperbolic component is of the homotopy type of a finite CW-complex one can invoke the Smith Theory from algebraic topology which implies that the fixed point set of an involution acting on a finite dimensional topological space with finite Betti numbers has finite mod 2 Čech cohomology groups [Bre72, chap. III, Theorem 7.9]. Therefore, in that case, the intersections of the complex hyperbolic component with $\mathcal{M}_d(\mathbb{R})$ (the fixed point set of the involution $(f) \mapsto (f)$ acting on $\mathcal{M}_d(\mathbb{C})$) and hence with $\mathcal{M}_d' - S'$ both have finitely many connected components; so only finitely many entropy values arise.
4.2. The Teichmüller Space of a Real Rational Map and Isentropes. In the final subsection, we treat the entropy rigidity in the general setting that results in the proof of Theorem 1.1. The main question is if two real degree $d$ rational maps $f, g \in \mathbb{R}(z)$ are quasi-conformally conjugate then what can be deduced regarding topological entropies $h_{\mathbb{R}}(f), h_{\mathbb{R}}(g)$? This is relevant to Theorem 4.2 because in a hyperbolic component of $\mathcal{M}_d(\mathbb{C})$, after excluding subsets defined by post-critical relation, maps from the same component of this complement are quasi-conformally conjugate [MS98, Theorem 2.7].

Let us start with a $f \in \mathbb{R}(z)$ of degree $d$. Recalling the theory developed in [MS98], the quasi-conformal (qc) conjugacy class of $f$, denoted by $\text{qc}(f)$, consists of those rational maps $g$ which are quasi-conformally conjugate to $f$. The space

$$M(f) := \text{qc}(f) / \text{Möbius Equivalence}$$

of Möbius conjugacy classes of maps in $\text{qc}(f)$ is called the moduli space of the rational map $f$ and the natural map $M(f) \hookrightarrow \mathcal{M}_d(\mathbb{C})$ is an injection of complex orbifolds. In analogy with the theory of the moduli spaces of Riemann surfaces, the moduli space $\text{qc}(f)$ is the quotient of a Teichmüller space under a discrete group action:

$$T(f) = \{(g, h) \mid g \text{ a rational map, } h \text{ a qc-homeomorphism with } h \circ f = g \circ h\} / \sim$$

where $(g_1, h_1) \sim (g_2, h_2)$ if in the conjugacy

$$g_2 = (h_2 \circ h_1^{-1}) \circ g_1 \circ (h_2 \circ h_1^{-1})^{-1}$$

the quasi-conformal homeomorphism $h_2 \circ h_1^{-1}$ is isotopic to a Möbius transformation through an isotopy which preserves the conjugacy. In particular, $g_1, g_2$ must be Möbius conjugate. There is an obvious map

$$\begin{cases}
T(f) \rightarrow M(f) = \text{qc}(f) / \text{Möbius Equivalence} \\
[(g, h)] \mapsto \langle g \rangle
\end{cases}$$

sending the class $[(g, h)]$ of a pair to the class $\langle g \rangle$. The fiber above the Möbius class of $f$ can be identified with the group of isotopy classes of qc-homeomorphisms commuting with $f$ where the isotopy has to remain within this space of maps as well. The aforementioned group is called the modular group of the rational map $f$ and will be denoted by $\text{Mod}(f)$. Clearly, $\text{Mod}(f)$ acts (from right) on $T(f)$ by $[(g, h)].[k] = [(g, h \circ k)]$ and the quotient $T(f)/\text{Mod}(f)$ can be identified with $M(f)$ via the projection above. It is known that $\text{Mod}(f)$ is discrete and acts properly discontinuously on $T(f)$. Furthermore, there is a description of the space $T(f)$ as a product of ordinary Teichmüller spaces based on the dynamics of $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$; see [MS98, Theorems 2.2 & 2.3].

There is a naturally defined entropy function $\widetilde{h}_{\mathbb{R}}$ on some appropriate open subset of $T(f)$
sending a class \([(g, h)]\) with \(\text{Aut}(g) = \{1\}\) to \(h_\mathbb{R}(g)\). This is well defined and fits in the commutative diagram below:

\[
\begin{array}{c}
\{(g, h) \in T(f) \mid g \in \mathbb{R}(z), \text{Aut}(g) = \{1\}\} \\
\downarrow_{[g, h] \to (g)}
\end{array}
\]

\[
\mathcal{M}'_d \cap M(f) - S' \xrightarrow{h_{\mathbb{R}}} [0, \log(d)]
\]

Suppose \(g\) is another real rational map quasi-conformally conjugate to \(f\); that is, \((g, h)\) determines a class in \(T(f)\) for an appropriate qc-homeomorphisms \(h\) satisfying \(h \circ f = g \circ h\). In order to compare the topological entropies of \(f \mid \hat{\mathbb{R}}\) and \(g \mid \hat{\mathbb{R}}\), we need to investigate how the quasi-circle \(h(\hat{\mathbb{R}})\) is placed with respect to \(\hat{\mathbb{R}}\). Pulling back the complex conjugate map \(z \mapsto \bar{z}\) via \(h\) yields a reflection \(z \mapsto h^{-1}(\bar{h}(z))\) commuting with \(f\). This differs from the usual reflection \(z \mapsto \bar{z}\) – that also preserves \(f \in \mathbb{R}(z)\) – by a q.c. automorphism of \(f\):

\[
u(z) := h^{-1}(\bar{h}(z));
\]

a map which satisfies \(u\left(\bar{u}(z)\right) = z\) or equivalently, using the notation in (4.1):

\[
u^{-1} = \bar{u}^2
\]

The homeomorphism \(u\) is identity if and only if \(h(\bar{z}) = z\) which in particular indicates that \(h\) preserves \(\hat{\mathbb{R}}\). Now we can form a continuous map taking a \([(g, h)] \in T(f)\) with \(g \in \mathbb{R}(z)\) to the class in \(\text{Mod}(f)\) determined by the corresponding automorphism \(u\) from (4.5). There is a subtlety here: this map being well defined relies on the triviality of \(\text{Aut}(g)\). To see this, pick two equivalent pairs \((g_1, h_1)\) and \((g_2, h_2)\) with \(g_1, g_2 \in \mathbb{R}(z)\).

So there is an isotopy \(\left\{H_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\right\} \subseteq [0,1]\) with \(H_t \circ g_1 = g_2 \circ H_t\), \(H_1\) being a Möbius transformation and \(H_0 = h_2 \circ h_1^{-1}\). Then

\[
(t, z) \mapsto (H_t \circ h_1)^{-1}\left(H_t \circ h_1(z)\right) \quad (t \in [0,1], z \in \hat{\mathbb{C}})
\]

\footnote{The obstruction to a qc-automorphism \(u\) of \(f\) satisfying the functional equation (4.6) being in the form of (4.5) for some appropriate qc-automorphism \(h\) is encoded by the first cohomology of the group \(\mathbb{Z}/2\mathbb{Z}\) with coefficients in the group of qc-automorphisms of \(f\) on which \(\mathbb{Z}/2\mathbb{Z}\) acts by the involution (4.1).}
is an isotopy varying among quasi-conformal automorphisms of $f$

$$f \left( (H_t \circ h_1)^{-1} (H_t \circ h_1(z)) \right) = (f \circ h_1^{-1} \circ H_t^{-1}) \left( H_t \circ h_1(z) \right) = \left( \frac{f \circ h_1^{-1} \circ H_t^{-1}}{h_1^{-1} \circ g_1 \circ H_t^{-1}} \right) \left( H_t \circ h_1(z) \right)$$

$$= (h_1^{-1} \circ H_t^{-1}) \left( \frac{g_2 \circ H_t \circ h_1(z)}{H_t \circ h_1(z)} \right) = (H_t \circ h_1)^{-1} \left( H_t \circ h_1(f(z)) \right);$$

that furthermore starts from $u_2 : z \mapsto h_1^{-1} \left( h_2(z) \right)$ and ends with $(H_t \circ h_1)^{-1} \left( H_t \circ h_1(z) \right)$; a quasi-conformal homeomorphism that we want to coincide with $u_1 : z \mapsto h_1^{-1} \left( h_1(z) \right)$. This holds if the Möbius map $H_1$ lies in $\text{PGL}_2(\mathbb{R})$. But we have a Möbius conjugacy $g_2 = H_1 \circ g_1 \circ H_t^{-1}$ between to real maps $g_1, g_2 \in \mathbb{R}(z)$ and, following the discussion in §§2.1,2.2, failure of $H_1$ to be real amounts to $\text{Aut}(g_1) \cong \text{Aut}(g_2)$ being non-trivial; a possibility that has been ruled out.

So given $f \in \mathbb{R}(z)$ of degree $d \geq 2$, the appropriate domain of definition of the map $[(g, h)] \in T(f) \mapsto [u] \in \text{Mod}(f)$ is that of the function $\tilde{h}_R$ in (4.4):

$$\left\{ [(g, h)] \in T(f) \mid g \in \mathbb{R}(z), \text{Aut}(g) = \{1\} \right\} \rightarrow \text{Mod}(f)$$

$$[g, h] \mapsto \left[ u : z \mapsto h^{-1} \left( \frac{h(z)}{g(z)} \right) \right].$$

This is continuous and attains its values in the discrete group $\text{Mod}(f)$ so must be constant on each connected components of the domain.

**Proposition 4.7.** If $[(g_1, h_1)], [(g_2, h_2)]$ are mapped to the same element of the modular group via (4.8), then $h_R(g_1)$ and $h_R(g_2)$ must coincide.

To verify this, suppose

$$u_1 : z \mapsto \frac{h_1^{-1}(h_1(z))}{g_1}, \quad u_2 : z \mapsto \frac{h_2^{-1}(h_2(z))}{g_2}.$$
differ by a quasi-conformal automorphism \( v \) of \( f \) isotopic to the identity: \( u_2 = u_1 \circ v \). If \( v \mid_{h_1^{-1}(\hat{\mathbb{R}})} \) is identity, then

\[
\forall z \in h_1^{-1}(\hat{\mathbb{R}}) : h_1^{-1}(h_1(z)) = h_2^{-1}(h_2(z));
\]

which by letting \( x \) to be \( h_1(z) \) implies

\[
\forall x \in \hat{\mathbb{R}} : h_2 \circ h_1^{-1}(x) = h_2 \circ h_1^{-1}(x)
\]

meaning that the conjugacy \( h_2 \circ h_1^{-1} \) between \( g_1, g_2 \) preserves \( \hat{\mathbb{R}} \), restricting to a conjugacy between \( g_1 \mid_{\hat{\mathbb{R}}}, g_2 \mid_{\hat{\mathbb{R}}} \) and they are thus of the same entropy. Pulling back via \( h_1 \) and \( h_2 \), this can also be stated as systems \( f = h_1^{-1} \circ g_1 \circ h_1 \mid_{h_1^{-1}(\hat{\mathbb{R}})} \) and \( f = h_2^{-1} \circ g_2 \circ h_2 \mid_{h_2^{-1}(\hat{\mathbb{R}})} \) being of the same entropy. The general case is more complicated; there is an isotopy \( \{v_t\}_{t \in [0,1]} \) from \( v_0 = v \) to \( v_1 = 1 \) through qc-automorphisms of \( f \). Thus \( \{v_t(\hat{h}_1^{-1}(\mathbb{R}))\}_{t \in [0,1]} \) is a 1-parameter family of \( f \)-invariant quasi-circles terminating at \( t = 1 \) with \( h_1^{-1}(\hat{\mathbb{R}}) \). Asking for \( v_t \)'s to restrict to identity on the quasi-circle \( h_1^{-1}(\hat{\mathbb{R}}) \) is too much as there might be some open intervals of Fatou points that can be wiggled within the corresponding Fatou component; think about the natural foliation of a rotation domain. Nevertheless, it would be sufficient if the isotopy fixes the “important” portion of Julia points of \( f \) lying on the quasi-circle \( h_1^{-1}(\hat{\mathbb{R}}) \), namely those forming a subsystem of \( f \mid_{h_1^{-1}(\hat{\mathbb{R}})}: h_1^{-1}(\hat{\mathbb{R}}) \to h_1^{-1}(\hat{\mathbb{R}}) \) of the same entropy. In order to do so, we have to invoke the technical results developed in §3 and the proof of \( h_\mathbb{R}(g_1) = h_\mathbb{R}(g_2) \) intuitively boils done to show that a “real motion” of the Julia set of \( f \) through \( f \)-invariant quasi-circles cannot alter the part of the system that really matters to entropy.

**Proof of Proposition 4.7.** By symmetry, it suffices to argue that the topological entropy of the system \( g_1 = h_1 \circ f \circ h_1^{-1} \mid_{\hat{\mathbb{R}}} \) cannot exceed that of the system \( g_2 = h_2 \circ f \circ h_2^{-1} \mid_{\hat{\mathbb{R}}} \). But these systems are conjugate with \( f \mid_{h_1^{-1}(\hat{\mathbb{R}})} \) and \( f \mid_{h_2^{-1}(\hat{\mathbb{R}})} \), respectively (keep in mind that the quasi-circles \( h_1^{-1}(\hat{\mathbb{R}}), h_2^{-1}(\hat{\mathbb{R}}) \) are \( f \)-invariant as the rational maps \( g_1 = h_1 \circ f \circ h_1^{-1}, g_2 = h_2 \circ f \circ h_2^{-1} \) are with real coefficients.). Therefore, we only need to show that \( h_{\text{top}}(f \mid_{h_1^{-1}(\hat{\mathbb{R}})}) \leq h_{\text{top}}(f \mid_{h_2^{-1}(\hat{\mathbb{R}})}) \).

Applying Corollary 3.5 to \( g_1 \in \mathbb{R}(\hat{\mathbb{R}}) \), the system \( g_1 \mid_{\hat{\mathbb{R}}} : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \) admits a full entropy closed subsystem with a dense subset of preperiodic points of \( g_1 \). Pulling back by \( h_1 \), the system obtained from the restriction of \( f = h_1^{-1} \circ g_1 \circ h_1 \) to the quasi-circles \( h_1^{-1}(\hat{\mathbb{R}}) \) must have a closed subsystem \( P \) of the same entropy containing a dense subset of preperiodic points of \( f \). But for a preperiodic point \( x \in h_1^{-1}(\hat{\mathbb{R}}) \) of \( f \), \( \{v_t(x)\}_{t \in [0,1]} \) is a curve of preperiodic points of \( f \) since \( v_t \)'s commute with \( f \). The set of preperiodic points of \( f \) is countable therefore all of these points coincide with \( v_1(x) = x \). Consequently, \( v_0 = v \) fixes any preperiodic points of \( f \) belonging to \( h_1^{-1}(\hat{\mathbb{R}}) \) meaning that \( v = u_1^{-1} \circ u_2 \) is identity on \( P \). Recalling definitions
of \( u_1, u_2 \) in (4.9), this indicates \( h_1^{-1}(h_1(x)) = h_2^{-1}(h_2(x)) \) for any \( x \in P \). As \( h_1(P) \subseteq \hat{\mathbb{R}} \), this can be rewritten as \( P \subseteq h_2^{-1}(\hat{\mathbb{R}}) \). Therefore, \( P \) is a closed subsystem of \( f \mid_{h_2^{-1}(\hat{\mathbb{R}})} \) as well and then:

\[
 h_{\text{top}} \left( f \mid_{h_1^{-1}(\hat{\mathbb{R}})} \right) = h_{\text{top}} \left( f \mid P \right) \leq h_{\text{top}} \left( f \mid_{h_2^{-1}(\hat{\mathbb{R}})} \right).
\]

This discussion culminates in the following theorem:

**Theorem 4.8.** Given a real rational map \( f \in \mathbb{R}(z) \) of degree \( d \geq 2 \), the real entropy function \( h_\mathbb{R} : \mathcal{M}'_d - S' \rightarrow [0, \log(d)] \) is constant on connected components of \( \mathcal{M}'_d \cap \mathcal{M}(f) - S' \).

**Proof.** Proposition 4.7 says that the lift \( \tilde{h}_\mathbb{R} \) of \( h_\mathbb{R} \) is constant on level sets of the continuous map \( \{(g, h) \in T(f) \mid g \in \mathbb{R}(z), \text{Aut}(g) = \{1\}\} \rightarrow \text{Mod}(f) \) defined in (4.8). Since the target space is discrete, we conclude that \( \tilde{h}_\mathbb{R} : [(g, h)] \mapsto h_\mathbb{R}(g) \) is locally constant on the preceding domain. Now the commutative diagram (4.4) implies that \( h_\mathbb{R} \) is locally constant on the image \( \mathcal{M}'_d \cap \mathcal{M}(f) - S' \) of this space in the moduli space \( \mathcal{M}'_d(\mathbb{C}) \) and this finishes the proof.\( \square \)

**Proof of Theorem 1.1.** An immediate corollary of Proposition 2.7 and Theorem 4.8.\( \square \)

A family of flexible Lattès maps of the same degree is an example of a quasi-conformally trivial family of non-hyperbolic maps (see [Mil06b] for a detailed treatment of Lattès maps.). The family is (conjecturally the only) family that admits invariant line field and forms a single moduli space of complex dimension one. The example below verifies Theorem 4.8 through calculating the real entropy of those flexible Lattès maps that preserve the real circle. It is well known that the Julia set of a Lattès map is the whole Riemann sphere and hence the dynamics of the restriction to the whole real circle must be understood.

**Example 4.9.** Set \( d = m^2 \geq 4 \) and consider a flexible Lattès map \( f \) of degree \( d \) obtained from the multiplication by \( m \) map \( [m] : \left[ z \right] \mapsto \left[ mz \right] \) on the elliptic curve \( E = \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z} \tau} \) with \( \tau \) being in the upper half plane. So \( f \) makes the following diagram commutative where the columns are the two-fold ramified covering \( \pi : E \rightarrow \mathbb{P}^1(\mathbb{C}) \) obtained by taking the quotient of \( E \) modulo the subgroup \( E[2] \) of 2-torsions; the morphism which is induced by the Weierstrass function \( \varphi : \mathbb{C} \rightarrow \hat{\mathbb{C}} \) of the lattice \( \Lambda := \mathbb{Z} + \mathbb{Z} \tau \).

\[
\begin{array}{ccc}
E = \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z} \tau} & \xrightarrow{[m]} & E = \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z} \tau} \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{P}^1(\mathbb{C}) = E[2] & \xrightarrow{f} & \mathbb{P}^1(\mathbb{C}) = E[2]
\end{array}
\]
The map \( f \) preserves \( \hat{\mathbb{R}} \) if the Weierstrass function

\[
\wp(z) = \frac{1}{z^2} + \sum_{0 \neq w \in \mathbb{Z} + Z\tau} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)
\]

commutes with the complex conjugation; for instance, when the lattice \( \mathbb{Z} + Z\tau \) is invariant under the complex conjugation. This happens if and only if \( \text{Re}(\tau) \in \frac{1}{2} \mathbb{Z} \). Up to the action of \( \text{SL}_2(\mathbb{Z}) \), \( \text{Re}(\tau) \) can then be assumed to be 0 or \( \frac{1}{2} \). Let us work with the former and in the case of \( \tau = \frac{1}{2} \) one merely needs to replace \( \tau \) with \( \tau - \frac{1}{2} \) in the subsequent discussion.

Assuming that the period \( \tau \) is purely imaginary, let us investigate the induced map \( f \) and the corresponding dynamics on the real circle. We are going to do so by pulling back to the dynamics on \( \wp^{-1}(\mathbb{H}) \) or on the invariant subset \( \pi^{-1}(\mathbb{H}) \) of the elliptic curve \( E \). Since \( \wp(\bar{z}) = \wp(z) \), \( \wp(z) \) is real if and only if either \( \text{Re}(z) \) or \( \text{Im}(z)i \) belongs to the rectangular lattice \( \Lambda = \mathbb{Z} + Z\tau \). We conclude that \( \wp^{-1}(\mathbb{H}) \) is the countable set of lines parallel to axes in the complex plane whose \( x \) and \( y \) intercepts come from \( \frac{1}{2} \mathbb{Z} \) and \( \text{Im}(\tau)2 \mathbb{Z} \). This collection determines a tessellation of the complex plane with the smaller rectangle with vertices 0, \( \frac{1}{2} \), \( \tau + \frac{1}{2} \) and \( \tau \) (in the counterclockwise order) which is bijectively mapped onto \( \hat{\mathbb{R}} \) via \( \wp \).

The union of lines in \( \mathbb{C} \) that just appeared projects onto the two pairs below of parallel circle on the torus \( E = \mathbb{C}/(\mathbb{Z} + Z\tau) \):

\[
\left( \{ [x] | x \in \mathbb{R} \} , \left\{ \left[ x + \frac{\tau}{2} \right] | x \in \mathbb{R} \right\} \right), \quad \left( \{ [x\tau] | x \in \mathbb{R} \} , \left\{ \left[ x\tau + \frac{1}{2} \right] | x \in \mathbb{R} \right\} \right);
\]

that intersect each other in four points (the 2-torsion points) and constitute \( \pi^{-1}(\hat{\mathbb{R}}) \). So the topological entropy of \( f \mid_{\hat{\mathbb{R}}} \) coincides with that of the restriction of \( [m] \) to this union of circles because \( \pi \) establishes a finite degree semi-conjugacy between these two systems. According to the parity of \( m \), on a circle from the preimage \( \pi^{-1}(\hat{\mathbb{R}}) \) the map \( [m] \) either restricts to the multiplication by \( m \) map or takes it surjectively onto another circle from this union. We conclude that \( h_{\mathbb{R}}(f) \) is the topological entropy of

\[
S^1 = \mathbb{R}/\mathbb{Z} \to S^1 = \mathbb{R}/\mathbb{Z} : [x] \mapsto [mx],
\]

i.e. \( \log(m) \). It should be emphasized that for \( m \geq 2 \) the map \( f \mid_{\hat{\mathbb{R}}} \) is not a covering map since \( f \) has local extrema; e.g., \( \wp \left( \frac{1}{2m} \right) \in \mathbb{R} \) which by analyzing the multiplicities in the diagram (4.10) is a point of multiplicity two for \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) and thus a turning point of \( f \mid_{\hat{\mathbb{R}}} \).

Finally, let us calculate the real entropy of a Lattès map which is not flexible. It may be easily verified that after identifying \( \frac{\mathbb{C}}{Z+Z\tau} \) with the elliptic curve \( y^2 = x^3 - x \), the multiplication map \([1+i] : [z] \mapsto [(1+i)z]\) factors through the quotient modulo 4-torsions and induces the
rational map \( f(z) = -\frac{1}{4} \left( z + \frac{1}{z} - 2 \right) \) which fits into the following diagram:

\[
\begin{align*}
\{ y^2 = x^3 - x \} &= \frac{c}{z + \bar{z}i} \quad \xrightarrow{[1+i]} \quad \{ y^2 = x^3 - x \} = \frac{c}{z + \bar{z}i} \\
\mathbb{P}^1(\mathbb{C}) \xrightarrow{f} \mathbb{P}^1(\mathbb{C})
\end{align*}
\]

The second iterate of \( f \) would be a finite quotient of the endomorphism \([1 + i]^{2} = [2i] \). But in the Legendre form \( y^2 = x^3 - x \), the automorphism defined via the multiplication by \( i \) is just \( (x, y) \mapsto (-x, \pm iy) \) so \( f^{2} \) is in fact induced via the multiplication by two endomorphism and hence, based on the previous discussion, we deduce that:

\[
h_{\mathbb{R}}(f) = \frac{1}{2} h_{\mathbb{R}}(f^{2}) = \frac{1}{2} \log(2) = \log(\sqrt{2}).
\]

Of course, Theorem 4.8 is not interesting unless the intersection \( \mathcal{M}_d' \cap \mathcal{M}(f) \) is of positive dimension or equivalently there are plenty of classes in the Teichmüller space \( T(f) \) which are represented by real maps.

**Corollary 4.10.** For a non-antipodal rational map \( f \in \mathbb{R}(z) \) of degree \( d \geq 2 \) with \( \text{Aut}(f) = \{1\} \), the real entropy function \( h_{\mathbb{R}} : \mathcal{M}_d' - S' \to [0, \log(d)] \) is constant on a submanifold of real dimension \( \dim_{\mathbb{R}} \mathcal{M}(f) \) passing through \( (f) \).

**Proof.** The involution from Definition 4.1 acts on the \( T(f) \) via

\[
(4.11) \quad \iota : [(g, h)] \mapsto [(g, \bar{h})].
\]

Keep in mind that by identities (4.2), if \( h \) is a quasiconformal homeomorphism of dilatation \( \mu \) then \( \bar{h} \) would be another such homeomorphism of dilatation \( \bar{\mu} \). Hence the transformation \( \iota \) from (4.11) acts on both \( T(f) \) and \( \{(g, h) \in T(f) \mid \text{Aut}(g) = \{1\} \} \) in a well defined manner because \( \bar{f} = f \) and the involution in (4.1) respects the composition and preserves the group of Möbius transformations. If \( \text{Aut}(g) = \{1\} \) and \( (g, h) \sim (g, \bar{h}) \) then \( g, \bar{g} = \bar{g} \) should be Möbius conjugate. But, recalling Proposition 2.9, this implies that \( g \) is with real coefficients only if it is away from the antipodal locus. Nonetheless, the conditions of being on the antipodal locus or having non-trivial symmetries are closed. So if one can compute the dimension of \( \text{Fix}(\iota : T(f) \to T(f)) \) around a point \( [(g, h)] = [(\bar{g}, \bar{h})] \) fixed by \( \iota \) where \( g \in \mathbb{R}(z) \) is non-antipodal and \( \text{Aut}(g) = \{1\} \), then something can be inferred about the dimension \( \mathcal{M}_d' \cap \mathcal{M}(f) \).

To do so, suppose \( f \in \mathbb{R}(z) \) is neither antipodal nor admits non-trivial Möbius symmetries. Clearly \( [(f, 1)] \) is a fixed point of the \( C^{\infty} \) involution \( \iota \) of the complex manifold \( T(f) \). The tangent map \( d_{[(f, 1)]} \iota \) of \( \iota \) at this point is a real-linear involution of the complex vector space \( T_{[(f, 1)]}(T(f)) \). Thus the tangent space decomposes to the direct sum of \( +1 \) and \( -1 \) eigenspaces and the former is the tangent space to \( \text{Fix}(\iota : T(f) \to T(f)) \) at \( [(f, 1)] \). We
claim the dimension of these two eigenspaces coincide. Any representative \((g, h)\) of a point in the Teichmüller space \(T(f)\) determines an \(f\)-invariant Beltrami differential \(\mu(z)dz \otimes \frac{\partial}{\partial z}\) where \(\mu\) belongs to the unit ball in \(L^\infty(\hat{\mathbb{C}})\). So there is a \(\mathbb{C}\)-linear surjection from the space

\[
\left\{ \mu \in L^\infty(\hat{\mathbb{C}}) \mid \mu(z)dz \otimes \frac{\partial}{\partial z} \text{ is } f\text{-invariant.} \right\}
\]

onto the tangent space \(T_{[(f,1)]}(T(f))\). Given the way \(\iota\) is defined in (4.11) and remembering that for a qc-homeomorphism \(h\) of dilatation \(\mu\), the map \(\tilde{h}\) is of dilatation \(\tilde{\mu}\), we deduce that the involution \(d_{[(f,1)]}^t\) of \(T_{[(f,1)]}(T(f))\) comes from the involution \(\mu \mapsto \tilde{\mu}\) acting on \(L^\infty(\hat{\mathbb{C}})\). The description of this involution in (4.1) clearly indicates that this map is conjugate-linear and hence so is \(d_{[(f,1)]}^t\). This means scalar multiplication by \(i\) maps the eigenspace of \(+1\) to that of \(-1\) and vice versa. So as real vector spaces they are of dimension

\[
\dim_{\mathbb{C}} T_{[(f,1)]}(T(f)) = \dim_{\mathbb{C}} T(f) = \dim_{\mathbb{C}} M(f).
\]

\[\square\]

The complex dimension of the Teichmüller space \(T(f)\) or equivalently that of the moduli space \(M(f)\) can be determined in terms of certain invariants of the holomorphic system \(f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) [MS98, Theorem 6.9]. Corollary 4.10 thus establishes a lower bound for the real dimension of the isentrope passing through a generic real rational map \(f \in \mathbb{R}(z)\) in terms of the complex dynamics of \(f\).

**References**

[ALM93] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz. *Combinatorial dynamics and entropy in dimension one*, volume 5 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, 1993.

[BBC+92] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney’s definition of chaos. *Amer. Math. Monthly*, 99(4):332–334, 1992.

[BBM15] A. Bonifant, X. Buff, and J. Milnor. Antipode Preserving Cubic Maps: the Fjord Theorem. *ArXiv e-prints*, December 2015.

[BKNvS96] H. Bruin, G. Keller, T. Nowicki, and S. van Strien. Wild Cantor attractors exist. *Ann. of Math. (2)*, 143(1):97–130, 1996.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46.

[BS02] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Cambridge University Press, Cambridge, 2002.

[BvS15] Henk Bruin and Sebastian van Strien. Monotonicity of entropy for real multimodal maps. *J. Amer. Math. Soc.*, 28(1):1–61, 2015.

[BZ99] Jozef Bobok and Ondřej Zindulka. Topological entropy on zero-dimensional spaces. *Fund. Math.*, 162(3):233–249, 1999.
REAL ENTROPY RIGIDITY UNDER QUASI-CONFORMAL DEFORMATIONS

[Dev89] Robert L. Devaney. An introduction to chaotic dynamical systems. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, second edition, 1989.

dMvS93 Welington de Melo and Sebastian van Strien. One-dimensional dynamics, volume 25 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1993.

Fil18 Khashayar Filom. Monotonicity of entropy for real quadratic rational maps. under preparation, 2018.

FLMn83 Alexandre Freire, Artur Lopes, and Ricardo Mañé. An invariant measure for rational maps. Bol. Soc. Brasil. Mat., 14(1):45–62, 1983.

Hof81 Franz Hofbauer. On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. II. Israel J. Math., 38(1-2):107–115, 1981.

KSV07 O. Kozlovski, W. Shen, and S. van Strien. Density of hyperbolicity in dimension one. Ann. of Math. (2), 166(1):145–182, 2007.

MdMvS92 M. Martens, W. de Melo, and S. van Strien. Julia-Fatou-Sullivan theory for real one-dimensional dynamics. Acta Math., 168(3-4):273–318, 1992.

Mm85 John Milnor. On the concept of attractor. Comm. Math. Phys., 99(2):177–195, 1985.

Mm92 John Milnor. Remarks on iterated cubic maps. Experiment. Math., 1(1):5–24, 1992.

Mm93 John Milnor. Geometry and dynamics of quadratic rational maps. Experiment. Math., 2(1):37–83, 1993. With an appendix by the author and Lei Tan.

Mm06a John Milnor. Dynamics in one complex variable, volume 160 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, third edition, 2006.

Mm06b John Milnor. On Lattès maps. In Dynamics on the Riemann sphere, pages 9–43. Eur. Math. Soc., Zürich, 2006.

Mn83 Ricardo Mañé. On the uniqueness of the maximizing measure for rational maps. Bol. Soc. Brasil. Mat., 14(1):27–43, 1983.

MnSS83 R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps. Ann. Sci. École Norm. Sup. (4), 16(2):193–217, 1983.

MP77 Michal Misiurewicz and Feliks Przytycki. Topological entropy and degree of smooth mappings. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 25(6):573–574, 1977.

MS80 Michal Misiurewicz and Wiesław Szlenk. Entropy of piecewise monotone mappings. Studia Math., 67(1):45–63, 1980.

MS98 Curtis T. McMullen and Dennis P. Sullivan. Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. Adv. Math., 135(2):351–395, 1998.

MT88 John Milnor and William Thurston. On iterated maps of the interval. In Dynamical systems (College Park, MD, 1986–87), volume 1342 of Lecture Notes in Math., pages 465–563. Springer, Berlin, 1988.

MT00 John Milnor and Charles Tresser. On entropy and monotonicity for real cubic maps. Comm. Math. Phys., 209(1):123–178, 2000. With an appendix by Adrien Douady and Pierrette Sentenac.

Pre89 Chris Preston. What you need to know to knead. Adv. Math., 78(2):192–252, 1989.

Pug02 Charles Chapman Pugh. Real mathematical analysis. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2002.

Rec90 M. Rees. Components of degree two hyperbolic rational maps. Invent. Math., 100(2):357–382, 1990.
[Rue15] S. Ruette. Chaos on the interval - a survey of relationship between the various kinds of chaos for continuous interval maps. ArXiv e-prints, April 2015.

[Sil98] Joseph H. Silverman. The space of rational maps on $\mathbb{P}^1$. Duke Math. J., 94(1):41–77, 1998.

[Sil07] Joseph H. Silverman. The arithmetic of dynamical systems, volume 241 of Graduate Texts in Mathematics. Springer, New York, 2007.

[Thu14] William P. Thurston. Entropy in dimension one. In Frontiers in complex dynamics, volume 51 of Princeton Math. Ser., pages 339–384. Princeton Univ. Press, Princeton, NJ, 2014.

[Tio15] Giulio Tiozzo. Topological entropy of quadratic polynomials and dimension of sections of the Mandelbrot set. Adv. Math., 273:651–715, 2015.

[vS10] Sebastian van Strien. One-dimensional dynamics in the new millennium. Discrete Contin. Dyn. Syst., 27(2):557–588, 2010.

[Yoc02] Jean-Christophe Yoccoz. Analytic linearization of circle diffeomorphisms. In Dynamical systems and small divisors (Cetraro, 1998), volume 1784 of Lecture Notes in Math., pages 125–173. Springer, Berlin, 2002.

[Yom87] Y. Yomdin. Volume growth and entropy. Israel J. Math., 57(3):285–300, 1987.

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