Supplementary Materials for

Emergent quantum Hall effects below 50 mT in a two-dimensional topological insulator

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This PDF file includes:

Sections S1 to S6
Figs. S1 to S10
References
S1. FURTHER EXPERIMENTAL DATA FOR DEV 1

A. Experimental determination of band gap

The bulk band gap has been determined from an Arrhenius plot of maximum resistance (at the charge neutrality point) as a function of temperature. We measure the longitudinal resistance $R_{xx}$ as a function of normalized gate voltage $V^* = V_g - V_d$, where $V_d$ corresponds to the gate voltage for which $R_{xx}$ is maximum, for different temperatures $T$ (Fig. S1a for $T = 15$ K). The minimum conductance $G_{\text{min}}$ is evaluated as $G_{\text{min}} = 1/R_{xx}^{\text{max}}$, where $R_{xx}^{\text{max}}$ is the maximum resistance in the $R_{xx}$-$V_g$ curve. The band gap can then be evaluated by fitting the function

$$G_{\text{min}} = G_0 \exp \left( -\frac{E_G}{2k_B T} \right), \quad (S1)$$

to the $G_{\text{min}}$-$T$ curve (Fig. S1b). This fitting yields a band gap $\sim 4.5$ meV for Dev 1.

![Graphs showing Rxx vs Vg and Gmin vs 1/T](image)

**FIG. S1.** Determination of band gap of Dev 1. (a) The longitudinal resistance $R_{xx}$ of Dev 1 as a function of normalized gate voltage $V^*$ at temperature $T = 15$ K. (b) The minimum conductance $G_{\text{min}}$ as a function of $1/T$. The red line shows fit according to Eq. (S1).

B. Longitudinal resistance $R_{xx}$ for Dev 1

Figure S2 shows the longitudinal and transverse resistance, $R_{xx}$ and $R_{xy}$ respectively, as a function of perpendicular magnetic field $B_\perp$ for $V_g^* = -1$ V (see Fig. S1 and inset of Fig. 2A in the main text) at 20 mK. $R_{xx}$ shows Shubnikov-de Haas oscillations and $R_{xy}$ shows quantum Hall plateaus. For filling factors $\nu = -1$ and $-2$, we find that $R_{xx} = 0$.

C. Quantum Hall measurements for Dev 1

1. Exceptionally long $\nu = -1$ QH plateau in the $p$-doped regime

Figure S3a shows $R_{xy}$ as a function of $B_\perp$ for $V_g^* = -1$ and $-1.2$ V at 20 mK. The $\nu = -1$ plateau is exceptionally long and persists up to 9 T which cannot be explained by conventional QH physics of a 2D system. The exceptionally long plateau can be explained by the pinning mechanism (see main text and Supplementary Section S6).
FIG. S2. Magnetoresistance for Dev 1. The longitudinal and transverse resistance, \( R_{xx} \) and \( R_{xy} \) respectively, as a function of perpendicular magnetic field \( B_\perp \) for normalized gate voltage \( V_g^* = -1 \) V (see Fig. S1 and inset of Fig. 2A in the main text) at temperature \( T = 20 \) mK.

FIG. S3. Quantum Hall measurements for Dev 1. (a) The transverse resistance \( R_{xy} \) as a function of perpendicular magnetic field \( B_\perp \) for normalized gate voltage \( V_g^* = -1 \) and \(-1.2 \) V at temperature \( T = 20 \) mK. The \( \nu = -1 \) plateau is exceptionally long and persists up to 9 T. (b) \( R_{xy} \) as a function of \( B_\perp \) for \( V_g^* = 2, 2.5 \) and 3 V at \( T = 20 \) mK. The dashed gray lines indicate quantized Hall resistance in units of \( h/\nu e^2 \), where \( \nu \) is the filling factor.

2. Quantum Hall plateaus in the n-doped regime

Figure S3b shows \( R_{xy} \) as a function of \( B_\perp \) for \( V_g^* = 2, 2.5 \) and 3 V at 20 mK. The device is in the n-doped regime and we observe the expected QH plateaus. There is no early onset of QH plateaus when the chemical potential is the n-doped regime.
D. Dependence of the onset fields on temperature for the emergent quantum Hall plateaus

The early onset to a $\nu = -1$ QH plateau in the QSH regime (in the band gap) persists for temperatures up to 2 K with the onset field increasing for higher temperatures (Figs. S4a and S4c). The effect of temperature on the band structure is via the exchange interaction [Eq. (S6)] and hence the magnetization of the Mn atoms. This leads to an increase in the magnetic field required to split the edge states to form the quantum Hall state. In the bulk $p$-regime, the higher QH Hall plateaus ($\nu = -2, -3, -4$ and $-5$) are resolved only for the lowest temperature (20 mK). As shown in Fig. S4b, $\nu = -1, -2$ and a weak $-3$ QH plateau are observed for $T = 280$ mK. At 2 K, no QH plateaus are visible for $V_g^* = -1$ V within the range of magnetic field investigated here.

**FIG. S4.** Temperature dependence of the onset field for quantum Hall plateaus for Dev 1. The transverse resistance $R_{xy}$ as a function of the out-of-plane component of the magnetic field $B_\perp$ at (a) 280 mK for $V_g^*$ ranging from 0.2 to $-0.3$ V. (b) $V_g^*$ ranging from $-0.7$ to $-1.6$ V. (c) $V_g^* = 0$ V. (d) $V_g^* = -1$ V for different temperatures ranging from 0.02 to 2 K.
The carrier density \( n \) has been estimated as \( n = 1/R_He \), where \( R_H \) is the Hall coefficient which is calculated from the slope of the \( R_{xy}-B_\perp \) curve (inset of Fig. S5a,c) and \( e \) is the electron charge. The carrier mobility \( \mu \) is calculated from the Drude model as \( \mu = \sigma_{xx}/ne \), where \( \sigma_{xx} \) is the experimentally measured longitudinal conductivity.

Figure S5a shows that \( n \) is 1.5–3.5 \( \times \) 10\(^{11} \) cm\(^{-2} \) for \( V_g^* \) ranging from 1 to 2 V which corresponds to a gate efficiency of 2 \( \times \) 10\(^{11} \) cm\(^{-2} \) V\(^{-1} \) in the bulk \( n \)-regime. The maximum electron mobility is \( \sim 1.5 \times 10^5 \) cm\(^2 \) V\(^{-1} \) s\(^{-1} \) (Fig. S5b). The estimation of hole density and mobility is more challenging because of the formation of quantum Hall plateaus at very low fields (~20–30 mT). However for \( B_\perp < 20 \) mT, \( R_{xy} \) varies linearly with \( B_\perp \) and hence the slope of the curve in this narrow window allows us to estimate the hole density. The hole density varies from 2–3 \( \times \) 10\(^{10} \) cm\(^{-2} \) for the entire range of \( V_g^* \) measured in the experiment (Fig. S5c), which agrees well with the theoretically estimated value. This weak dependence of \( n \) on gate voltage in the \( p \)-regime agrees with our pinning model as explained in the main text. The maximum hole mobility as calculated from the Drude model is \( \sim 9 \times 10^4 \) cm\(^2 \) V\(^{-1} \) s\(^{-1} \) (Fig. S5d).

We emphasize that the numbers for hole density and mobility mentioned above correspond to mobile carriers. At lowest temperatures of 20 mK, the carriers near the camelback are localized. We can estimate the localized carrier density from the known gate efficiency of 2 \( \times \) 10\(^{11} \) cm\(^{-2} \) V\(^{-1} \) as calculated for the \( n \)-regime (Fig. S5b). In the bulk \( p \)-regime, for \( V_g^* = -2 \) V, the localized carrier density at the camelback is 4 \( \times \) 10\(^{11} \) cm\(^{-2} \). These localized carriers do not contribute to transport at the lowest temperatures (20 mK).

**FIG. S5.** Density and mobility of charge carriers. (a) The carrier density \( n \) as a function of normalized gate voltage \( V_g^* \) at 20 mK in the bulk \( n \)-regime. The dashed red line is a linear fit to the data. The inset shows the transverse resistance \( R_{xy} \) as a function of perpendicular magnetic field \( B_\perp \) for \( V_g^* \) ranging from 1 to 2 V. (b) The electron mobility \( \mu \) as a function of \( n \). (c) The carrier density \( n \) as a function of \( V_g^* \) at 20 mK in the bulk \( p \)-regime. The inset shows \( R_{xy} \) as a function of \( B_\perp \) for \( V_g^* \) ranging from -1 to -2.5 V. (d) The hole mobility \( \mu \) as a function of \( n \). The solid black curves are guides to the eye.
S3. MAGNETORESISTANCE FOR DEV 2

As shown in the band diagram of Fig. 1C in the main text, Dev 2 is an indirect-gap semiconductor with the energy at the camelback being higher than at \( k = 0 \). We study the magneto-resistance of Dev 2 fabricated into a Hall bar of length \( l = 30 \mu m \) and width \( w = 10 \mu m \). The longitudinal resistance \( R_{xx} \) shows a peak as function of gate voltage \( V_g \), with maximum \( R_{xx} \) being much greater than the expected \( \hbar/2e^2 \) (in the QSH regime) due to the large dimensions of the device \((3,20)\). For this device, we do not observe any early onset of the QH plateaus in the full gate voltage regime investigated here. The transverse resistance \( R_{xy} \) shows the QH plateaus at perpendicular magnetic fields \( B_\perp > 1 T \) in contrast to 30 mT for Dev 1.

FIG. S6. Magnetoresistance for Dev 2. (a) The longitudinal resistance \( R_{xx} \) of Dev 2 as a function of gate voltage \( V_g \) at 20 mK. (b) The transverse resistance \( R_{xy} \) of Dev 2 as a function of perpendicular magnetic field \( B_\perp \) for different \( V_g \) at 20 mK (see legend).

S4. \( k \cdot p \) CALCULATIONS FOR A STRIP GEOMETRY

A. Modelling

The band structures in this work are obtained from numerics involving the typical \( k \cdot p \) Hamiltonian for II-VI zincblende semiconductors \((12,25)\). We have used the standard \( 8 \times 8 \) Kane Hamiltonian as described in Ref. \((12)\), involving the eight orbitals
\[
|\Gamma_6, +\frac{1}{2}\rangle, |\Gamma_6, -\frac{1}{2}\rangle, |\Gamma_8, +\frac{3}{2}\rangle, |\Gamma_8, +\frac{1}{2}\rangle, |\Gamma_8, -\frac{1}{2}\rangle, |\Gamma_8, -\frac{3}{2}\rangle, |\Gamma_7, +\frac{1}{2}\rangle, \text{ and } |\Gamma_7, -\frac{1}{2}\rangle. \quad (S2)
\]
We have used the Kane parameters for HgTe and CdTe from Ref. \((12)\) as well. For Hg\(_{1-x}\)Cd\(_x\)Te, the Kane parameters are derived by cubic interpolation \((26)\). For the Kane parameters for Hg\(_{1-x}\)Mn\(_x\)Te, we use the same values as for HgTe, with the exception of the conduction and valence band energies: We assume that the band gap decreases linearly as function of the concentration \( x \), such that the band gap closes at \( x = x_c \approx 0.064 \) \((11)\). In addition to the standard Kane Hamiltonian, we include the appropriate terms for strain and the Zeeman effect \((12)\). We do not consider explicitly the rather small effect of bulk inversion asymmetry (BIA), the strength of which is not known with sufficient precision for (Hg,Mn)Te, to the best of our knowledge. However, we comment briefly on the influence of BIA terms in the last part of Supplementary Material. The exchange coupling describing the interactions between the carriers and the magnetic moments of the Mn is discussed in detail in Supplementary Section S5.

The Kane Hamiltonian is used to calculate the band structures in two geometrical configurations: Firstly, for a bulk geometry, we model the layer stack (see Methods, Material growth and characterization) by discretization in the
growth direction $z$, making the Kane parameters and strain coefficients $z$-dependent. In the remaining two directions ('in-plane'), the system has translational symmetry. Diagonalization of this Hamiltonian thus yields energy eigenvalues $E_n(k_x, k_y)$ and eigenvectors $|\psi_n(k_x, k_y)\rangle$. In view of clarity, we display dispersions $E_n(k) = E_n(k/\sqrt{2}, k/\sqrt{2})$ as a function of the single momentum variable $k$. The choice of this parametrization is necessary in view of the lack of axial symmetry and the locations of the camelback maxima on the lines satisfying $k_x = \pm k_y$. Secondly, for a strip geometry, we discretize in one additional dimension ($y$). Thus, there is only one remaining translational degree of freedom. The energy eigenvalues and eigenvectors are thus $E_n(k_x)$ and $|\psi_n(k_x)\rangle$, respectively.

For all our dispersions, we express energies relative to the zero energy $E = 0 \text{ meV}$, defined as the conduction band edge (four-fold degenerate $\Gamma_8$ orbital at $k = 0$) of intrinsic (unstrained bulk) HgTe. Generically, the edges of the bulk bands and the positions of the bulk gaps may vary depending on the material. Thus, for the sake of comparing samples with various material compositions, we have decided to use a single reference energy rather than to centre all band gaps at $E = 0 \text{ meV}$.

Several properties of the eigenstates can be extracted from the eigenvectors $|\psi_n(k)\rangle$: Given an observable $O$, we calculate the eigenstate expectation values $\langle O \rangle = \langle \psi_n(k) | \hat{O} | \psi_n(k) \rangle$. We represent these expectation values by colouring the dispersion curves. For example, for the strip geometry dispersions in Fig. 2C,D and 4C of the main text, the observable $O = y$ is used to determine whether the states are localized near the edges. Also the orbital character ($O$ being a projection to the orbitals $|\Gamma_6, \pm \frac{1}{2}\rangle$, $|\Gamma_8, \pm \frac{1}{2}\rangle$, etc.) is a useful property that can be extracted from the eigenvectors.

The LL spectra (bulk geometry) are obtained by applying the Peierls substitution, $k_x \rightarrow k_x + eA_x/\hbar$, in the Landau gauge $A_x = -B_1 y$. The calculation follows the standard procedure of replacing the canonical momentum operators by magnetic ladder operators followed by choosing an appropriate ansatz for the wave functions. The ansatz for the LL spinors in axial approximation is given by (12)

$$
|\psi_j(z)\rangle = \begin{cases} 
\left( f_1^{(j)}(z) \ f_2^{(j)}(z) \ |j+1\rangle \ f_3^{(j)}(z) \ |j-1\rangle \ f_4^{(j)}(z) \ f_5^{(j)}(z) \ |j+1\rangle \ f_6^{(j)}(z) \ f_7^{(j)}(z) \ |j+1\rangle \ f_8^{(j)}(z) \ f_9^{(j)}(z) \ |j+1\rangle \right)^T & \text{for } j \geq 1 \\
\left( f_1^{(0)}(0) \ f_2^{(0)}(0) \ |0\rangle \ f_3^{(0)}(0) \ f_4^{(0)}(0) \ f_5^{(0)}(0) \ f_6^{(0)}(0) \ f_7^{(0)}(0) \ f_8^{(0)}(0) \ f_9^{(0)}(0) \right)^T & \text{for } j = 0 \\
\left( 0 \ f_2^{(-1)}(0) \ 0 \ f_3^{(-1)}(0) \ f_4^{(-1)}(0) \ f_5^{(-1)}(0) \ f_6^{(-1)}(0) \ 0 \ f_7^{(-1)}(0) \ f_8^{(-1)}(0) \ f_9^{(-1)}(0) \right)^T & \text{for } j = -1 \\
\left( 0 \ 0 \ 0 \ 0 \ 0 \ f_6^{(-2)}(0) \ 0 \ 0 \ 0 \ 0 \right)^T & \text{for } j = -2,
\end{cases}
$$

where $f_i^{(j)}(z)$ is the $z$-dependent envelope function of the $j$-th LL with orbital component $i = 1, \ldots, 8$ enumerating the eight basis functions [Eq. (S2)]; $|j\rangle$ is the $j$-th eigenstate of a harmonic oscillator. In all bulk LL calculations, we apply the axial approximation which is needed to ensure that the LL index $j$ is a conserved quantity. This approximation reduces the numerical effort substantially, while the approximation error remains sufficiently small [$\approx 1 \text{ meV}$ at the camelback (12)].

**B. Numerical considerations**

The sizes $n \times n$ of the Hamiltonian matrices are given by $n = 8n_z$ in the bulk geometry and by $n = 8n_z n_y$ in the strip geometry, where $n_z$ and $n_y$ are the number of lattice points in the $z$ and $y$ direction, respectively. For the discretization, we typically use a resolution of 0.25 nm, where the discretization error is sufficiently small. In the $z$ direction, the size of the well is 11 nm and we take into account 6 nm of each barrier (which is sufficient for the wave functions to decay to negligible values), so that $n_z \sim 100$. For the strip geometry, we restrict ourselves to a width $w = 500$ nm in order to keep the calculation tractable: There, we have $n_y \sim 2000$, that leads to a matrix size of $n = 1.6 \times 10^6$. We construct these large matrices in a sparse format and diagonalize them using a Lanczos algorithm (27) with the shift-and-invert method, that allows us to target a small number (typically 50–100) of eigenvalues close to a target eigenvalue.

Although the strip width in the numerics ($w = 500$ nm) is significantly smaller than the actual sample size (typically $200 \mu\text{m}$), the numerical results remain suitable for studying the physics of the edge states. The width is chosen to be larger than the localization length of these states, so that hybridization is negligible. A small residual hybridization between edge states remains visible in the numerical results, but can be ignored for the experimental samples in view of their much larger size.
S5. MODEL FOR MAGNETIC RESPONSE OF MANGANESE

The exchange interaction between the magnetic moments of the Mn ions and the carrier spins is typically understood in a mean field picture. The average Mn spin \( \langle m \rangle \) (proportional to the average magnetization) couples isotropically to the carrier spins

\[
H_{\text{ex}} = \sum_{\alpha} C^{(\alpha)} \langle m \rangle \cdot \hat{S}^{(\alpha)},
\]

where \( \alpha \) labels the orbital sector (either \( \Gamma_8 \) or a combination of \( \Gamma_8 \) and \( \Gamma_7 \)) and \( \hat{S}^{(\alpha)} \) is the vector of spin operators in that sector. The coefficients \( C^{(\alpha)} \) are phenomenological coupling constants that differ between the orbitals,

\[
C^{\Gamma_8} = -xN_0 \alpha \quad \text{and} \quad C^{\Gamma_8, \Gamma_7} = -xN_0 \beta,
\]

where \( x \) is the Mn concentration (in Hg\(_{1-x}\)Mn\(_x\)Te), and \( N_0 \alpha = 0.4 \text{ eV} \) and \( N_0 \beta = -0.6 \text{ eV} \) are empirically determined factors \((12)\).

For the response of the average Mn spin to the external magnetic field, we assume the typical empirical law \((9,11,12)\)

\[
\langle m \rangle = -S_0 \frac{B}{B_T} B_{5/2} \left( \frac{5}{2} \frac{g_{\text{Mn}} \mu_B B_T}{k_B (T + T_0)} \right),
\]

where \( B_T \) is the magnitude of the magnetic field and \( B / B_T \) its direction; furthermore, \( B_{5/2} \) is the Brillouin function for spin \( \frac{5}{2} \), \( g_{\text{Mn}} = 2 \) is the \( g \) factor for Mn, \( T + T_0 \) is an effective temperature with an offset \( T_0 = 2.6 \text{ K} \) and \( S_0 = \frac{5}{2} \) is the effective total spin \((12,28)\). We assume that the response is isotropic; the average magnetization is parallel to the magnetic field.

The characteristic magnetic field \( B_{\text{ex}} \) in Eq. (S4), defined by equating \( \frac{5}{2} g_{\text{Mn}} \mu_B B_T / k_B (T + T_0) = B_T / B_{\text{ex}} \), has a lower limit of approximately \( 0.77 \text{ T} \) at zero temperature. In the field regimes discussed in this work, the argument \( B_T / B_{\text{ex}} \) of the Brillouin function is small, which justifies a linear approximation that yields

\[
\langle m \rangle = -S_0 \frac{B}{B_T} B_{5/2} \left( \frac{B_T}{B_{\text{ex}}} \right) \approx -7 \frac{S_0}{15} B \left( \frac{B_T}{B_{\text{ex}}} \right) = -7 \frac{S_0}{6} \frac{g_{\text{Mn}} \mu_B B}{k_B (T + T_0)}.
\]

This argument demonstrates that we may assume that the average Mn spin \( \langle m \rangle \) is proportional to the magnetic field in good approximation. In the context of the rotation experiments, where \( B = (B_x, 0, B_z) \), the exchange interaction is thus proportional to

\[
H^{(\alpha)}_{\text{ex}} \propto B_x \hat{S}^x + B_z \hat{S}^z
\]

within each orbital sector \( \alpha \).

In order to demonstrate that the response of the states to the exchange interaction depends on the orbital content, we explore the following noteworthy scenarios:

- Subbands of heavy hole character, whose orbital character contains \( |\Gamma_8, \pm \frac{3}{2} \rangle \) only, are unaffected by the spin operator \( \hat{S}^x = \frac{1}{2} (S^+ + S^-) \),
  \[
  \langle \Gamma_8, \pm \frac{3}{2} | \hat{S}^x | \Gamma_8, \pm \frac{3}{2} \rangle = \langle \Gamma_8, \pm \frac{3}{2} | \hat{S}^x | \Gamma_8, \mp \frac{3}{2} \rangle = 0.
  \]
  Thus, the effective exchange Hamiltonian in this subspace is proportional to \( B_z \hat{S}^z \) only. In other words, the eigenvalues of \( H_{\text{ex}} \) are proportional to \( \pm \frac{1}{2} B_z \). There is no dependence on the in-plane component \( B_x \). This case applies to the lowest pair of conduction-band subbands near \( k = 0 \) (subband character \( H_1 \)) as well as the other heavy-hole states which reside more deeply in the valence band (subband characters \( H_2, H_3, \ldots \)).

- At \( k = 0 \), the two E1 subbands are mixtures of electronic and light-hole orbitals, i.e., approximately 56% of \( |\Gamma_6, \pm \frac{1}{2} \rangle \) and 44% of \( |\Gamma_8, \pm \frac{1}{2} \rangle \). In the \( \Gamma_6 \) sector, the eigenvalues of \( H_{\text{ex}} \) are isotropic, i.e., proportional to \( B_T = \sqrt{B_x^2 + B_z^2} \). In the projection of the \( \Gamma_8 \) and \( \Gamma_7 \) sector to the subspace of \( |\Gamma_8, \pm \frac{1}{2} \rangle \), we find an angular dependence of the eigenvalues,
  \[
  E_{\text{ex}} \propto \pm \frac{1}{2} \sqrt{4B_x^2 + B_z^2} = \pm \frac{1}{2} B_T \sqrt{1 + 3 \sin^2 \theta}.
  \]
  The combination of the \( \Gamma_6 \) and \( \Gamma_8 \) sectors thus yields a pair of eigenvalues that is explicitly \( \theta \)-dependent, i.e., the exchange splitting is expected to be (weakly) angular dependent.
The states at the camelback are more complicated mixtures of the approximate form $\sqrt{0.7} |E, \pm \rangle + \sqrt{0.3} e^{\pm i \phi} |O, \pm \rangle$, where $\delta$ is a mutual phase and

$$|E, \pm \rangle = \phi_E(z) \left( \sqrt{2/3} |\Gamma_8, \pm \frac{1}{2}\rangle \pm i \sqrt{1/3} |\Gamma_8, \mp \frac{3}{2}\rangle \right),$$

$$|O, \pm \rangle = \phi_O(z) |\Gamma_8, \pm \frac{3}{2}\rangle$$

are the ‘even’ and ‘odd’ components of these states, respectively. Here, $\phi_E(z)$ and $\phi_O(z)$ denote even and odd wave functions in the spatial coordinate $z$, respectively. The even and odd components are orthogonal, so they can be treated independently: The odd sector is purely heavy hole, whose response to the exchange coupling we have discussed before. In the even sector, the eigenvalues of $H_{ex}$ are $E_{ex} \propto \pm \frac{1}{18} \sqrt{40B_z^2 + B_x^2}$. Combination of the two sectors also leads to an angular dependent exchange splitting. The large coefficient for $B_c$ compared to that of $B_z$ suggests that this splitting strongly depends on the in-plane field component, which agrees qualitatively with the camelback splitting shown in Fig. 4F of the main text.

For the latter two cases, we refrain from providing quantitative estimates because of the complicated dependence of the orbital content on other factors as well. The main conclusion is that the different responses to the exchange coupling can indeed be explained from the variation in orbital character.

### S6. FURTHER THEORETICAL DATA FOR DEV 1 AND DEV 2

#### A. Landau level fan charts for Dev 1

The LL spectrum for Dev 1 is shown in Fig. S7a; the colour code indicates the orbital character. At $B_{\perp} = 0 \text{T}$, the band structure is clearly inverted with the H1 subband (blue) being above the E1 subband (brown). While the conduction band looks quite conventional, the appearance of the valence band is dominated by a very dense collection of LLs. As described in the main text, we can attribute these LLs with a large LL index to the camelback (main text, Fig. 1B).

In order to calculate the chemical potential in the presence of a magnetic field $B_{\perp}$, we calculate the density of states $D(E)$ assuming that each LL is broadened with a Gaussian shape,

$$D(E) = \frac{1}{2\pi l_{B_{\perp}}^2} \sum_s \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{(E - E_s)^2}{2\sigma^2} \right),$$

where the summation runs over all LLs $s$, $l_{B_{\perp}} = \sqrt{\hbar/eB_{\perp}}$ is the magnetic length, $E_s$ marks the energy of the respective LL and $\sigma$ determines the width of the LL broadening. Here, we take $\sigma = \sigma_0 \sqrt{B_0/B_{\perp}}$ with $\sigma_0 = 0.85 \text{meV}$ and $B_0 = 1 \text{T}$ (12). Due to the enormous amount of LLs arising from the camelback, the evolution of the chemical potential $\mu(B_{\perp})$ for constant carrier densities is, for $p$-densities, determined by the shape of the camelback as indicated by the dashed lines in Fig. S7a. QH transitions are observed whenever the chemical potential crosses a LL. The LL fan chart, shown in Fig. S7a, is the basis for Fig. 3C of the main text, where, instead of the bulk LL spectrum, the density of states is highlighted.

Analogously to analyzing the evolution of $\mu(B_{\perp})$ for various densities, one can also depict the LL fan chart in a $n(B_{\perp})$ plot (Fig. S7b), where $n$ is the total bulk carrier density and the colour code highlights $d\sigma_{xy}/dn$. Here, it is assumed that only extended states, located in the centre of each broadened LL, can give rise to a change in $\sigma_{xy}$. Since $n \sim V_0$, this form of depicting a LL fan chart is analogous to the experimental results presented in Fig. 3B of the main text. From Fig. S7b, it is apparent that the camelback manifests itself as a large asymmetry between $n$- and $p$-densities.

To study also the high-field features of Dev 1, we show further results in Fig. S7c,d. The characteristic LL crossing of an inverted band structure occurs for this specific sample at $B_{\perp} \approx 3 \text{T}$. Due to pinning of the chemical potential to the camelback, we find that the $\nu = -1$ QH plateau can extend up to very large $B_{\perp}$ (Fig. S7d). This is in good agreement with the experimental results shown in Fig. S3.

#### B. Landau level fan charts and strip band structures for Dev 2

Further evidence in favour of this pinning mechanism can be gained by comparing theory and experiment for other device configurations. For Dev 2, which is an indirect-gap semiconductor device unlike Dev 1, the Dirac point is
FIG. S7. Band structure calculations for Dev 1. (a) Bulk LL energies (no broadening) are shown as function of magnetic field; colour code indicates orbital character. (b) Different representation of the LL fan is presented, where the y-axis shows the carrier density and the colour code highlights $d\sigma_{xy}/dn$. In (a) and (b), dashed lines depict the evolution of the chemical potential as function of $B_{\perp}$ at constant density: $p = 0.05 \times 10^{11}$ cm$^{-2}$ (orange), and $p = 0.2 \times 10^{11}$ cm$^{-2}$ (red). (c) The same LL fan chart for a much larger $B_{\perp}$-range is shown including broadening, where the colour indicates the DOS ($dn/dE$); a white colour implies that the DOS is out-of-scale. (d) As in (b) but for the same magnetic field range as in (c). In (c) and (d), dashed lines depict the evolution of $\mu(B_{\perp})$ at constant density: $p = 0.2 \times 10^{11}$ cm$^{-2}$ (red), and $p = 2 \times 10^{11}$ cm$^{-2}$ (green). In (b) and (d), the shaded area at low magnetic fields marks a region below the computation limit. In all plots, numbers indicate the QH filling factors $\nu$. The label '0' indicates the quantum spin Hall state (edge states not shown).

FIG. S8. Band structure calculations on a strip for Dev 2. The band structures have been calculated on a strip of width $w = 500$ nm of the material of Dev 2, (a) without magnetic field, (b) with $B_{\perp} = 0.1$ T, and (c) with $B_{\perp} = 1$ T. We note the different scaling on the horizontal axis in (c). Buried deeply in the bulk valence band, as a strip calculation without magnetic field demonstrates (see Fig. S8a). For small magnetic fields (e.g., $B_{\perp} = 0.1$ T, see Fig. S8b), the LLs with small filling factors $\nu$ form close to the Dirac point, still deep in the bulk valence band. Even a stronger field of $B_{\perp} = 1$ T (Fig. S8c) is insufficient to lift the $\nu = -1$ QH plateau above the camelback. These theoretical considerations corroborate the absence of the $\nu = -1$ QH plateau at small fields in the QSH regime in experiments (cf. Fig. S6b).

To study also pinning to the camelback in the bulk $p$-regime, we provide Landau fans for Dev 2 in Fig. S9. Because
of the indirect gap, the camelback penetrates into the QSH regime which is enclosed by the two characteristic LLs of an inverted band structure which cross at about 6.5 T. Thus, LLs with small LL indices $j$ are covered completely by the camelback, preventing the early onset of the $\nu = -1$ plateau. The numerics (Fig. S9a–c) demonstrate that the chemical potential is instead pinned to the $\nu = 0$ plateau. An onset to the $\nu = -1$ QH plateau should arise at 3–4 T, where the camelback separates from the QSH regime. We therefore do not show in detail the low-field regime ($B_\perp < 200$ mT) for Dev 2.

C. Influence of bulk inversion asymmetry

Although our theoretical pinning model agrees qualitatively well with experiment for $p$-densities (Fig. 3B of the main text), there is a minor difference between theory and experiment on the quantitative level. In particular, the theory points to a broad $\nu = -3$ and a very narrow $\nu = -2$ QH plateau at low magnetic fields (Fig. 3C of the main text and Fig. S7a), whereas in experiment $\nu = -2$ is the most visible QH plateau. In the numerics, the even-odd difference arises since all valence band LLs with small $j$ come in almost-degenerate (small gap) pairs suggesting that only odd plateaus with $\nu = -1, -3, -5$ should be clearly resolved. Each of these pairs is linked to two different LL spinors, i.e., $\psi_j$ and $\psi_{j+2}$, see Eq. (S3).

This discrepancy may be resolved by considering a perturbation that couples these states and would split the LL pairs, thus leading to an increased width of all even plateaus ($\nu = -2, -4$). Bulk inversion asymmetry, which has been neglected so far for all theoretical figures, is a possible candidate: In linear approximation, the BIA Hamiltonian [see Ref. (29)] couples states with $j$ to $j \pm 2$. Thus, in presence of BIA, a linear combination of LL spinors, Eq. (S3),
**FIG. S10. Influence of bulk inversion asymmetry for Dev 1.** (a) Bulk inversion asymmetry (BIA) lifts the degeneracy already at $B_\perp = 0$, causing a decrease of $\Delta E$ [cf. Fig. S7(a)]. Blue and orange lines correspond to a linear BIA parameter $C = 0$ and $-1.9$ meV nm, respectively [BIA Hamiltonian taken from Ref. (12)]. (b) LL fan is shown including BIA, where numbers indicate QH filling factors $\nu$. The camelback crosses LLs with small LL indices at even smaller magnetic fields. Dashed line extrapolates camelback against $B_\perp = 0$. Note that the $\nu = -2$ QH plateau is well resolved.

is an appropriate ansatz for the Kane Hamiltonian in magnetic fields.

$$|\Psi_E(z)\rangle = \sum_{j=-1}^{\infty} a_{2j} |\psi_{2j}(z)\rangle,$$

$$|\Psi_O(z)\rangle = \sum_{j=-1}^{\infty} a_{2j+1} |\psi_{2j+1}(z)\rangle.$$  

Here, $\Psi$ decomposes into an even and odd subspace labelled by $E$ and $O$, respectively, since only LLs with indices $j$ and $j+2$ hybridize.

Figure S10(a) shows that including BIA in the band structure calculations lifts the degeneracy of the bulk valence bands already at $B_\perp = 0$. This causes a decrease of the energetic difference $\Delta E$ between the valence band edge and the camelback maximum implying even smaller onset fields to QH plateaus in the bulk $p$-regime [cf. Fig. S7(a) and S10(b)]. The most important effect of including BIA is the broadening of all even plateaus, $\nu = -4, -2$. This shows that BIA is a potential candidate to explain the observed small discrepancy between theory and experiment. Here, we have used a realistic estimate for the strength of the BIA for (Hg,Mn)Te. However, since a precise value for the BIA in this material is unknown, an in-depth investigation into BIA is necessary in the future to obtain a better quantitative estimate.
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