A NEKHOROSHEV TYPE THEOREM FOR THE NONLINEAR KLEIN-GORDON EQUATION WITH POTENTIAL

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(Communicated by Susanna Terracini)

Abstract. We study the one-dimensional nonlinear Klein-Gordon (NLKG) equation with a convolution potential, and we prove that solutions with small analytic norm remain small for exponentially long times. The result is uniform with respect to $c \geq 1$, which however has to belong to a set of large measure.

1. Introduction. In this paper we study the real one-dimensional nonlinear Klein-Gordon (NLKG) equation with a convolution potential on the segment,

$$\frac{1}{c^2} u_{tt} - u_{xx} + c^2 u + V * u + f(u) = 0,$$

with $c \in [1, +\infty)$, $x \in I := [0, \pi]$, $f : \mathbb{R} \to \mathbb{R}$ an analytic function with a zero of order 3 at the origin, in the case of Dirichlet boundary conditions.

In this paper we show that the technique developed in [14] allows us to deduce that solutions with small initial data that are analytic in a strip of width $\rho > 0$ remain analytic in a strip of width $\rho/4$ for a timescale which is exponentially long with respect to the size of the initial datum; however, we have to assume that both the parameter $c$ and the coefficients of the potential belong to a set of large measure.

In [19] we proved an almost global existence result uniform with respect to $c \geq 1$ for the NLKG with a convolution potential. Furthermore, we deduced that for any $\delta > 0$ any solution in $H^s$ with initial datum of size $O(c^{-\delta})$ remains of size $O(c^{-\delta})$ up to times of order $O(c^{\delta(r+1/2)})$ for any $r \geq 1$. Here we use normal form techniques in order to establish a result valid for exponentially long times, but we have to use analytic norms instead of Sobolev ones.

The issue of long-time existence for small solutions of the NLKG equation on compact manifolds has been quite studied; see for example [11], [4], [9], [13], [12] and [10]. However, all results in the aforementioned papers rely on a nonresonance condition which is not uniform with respect to $c$. We also point out that the almost global existence for small solutions has been established only for the segment $[0, \pi]$ and in the case of Zoll manifolds, such as the multidimensional spheres $\mathbb{S}^d$, $d \geq 1$.

The proof combines the argument of [14] for the NLS equation on the torus with a diophantine type estimate for the linear frequencies which holds uniformly for $c \geq 1$.

2010 Mathematics Subject Classification. Primary: 35Q40, 37K45; Secondary: 37K55.

Key words and phrases. Nekhoroshev theorem, nonlinear Klein-Gordon equation, Hamiltonian systems, normal forms, long time behaviour.

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We mention that a diophantine estimate for the frequencies uniform with respect to $c$ has been already used in [19] in order to prove the almost global existence.

A further aspect that would deserve future work is the study of Nekhoroshev estimates for the NLKG without potential. This is expected to be a quite subtle problem, since for $c \neq 0$ the frequencies of NLKG are typically non resonant, while the limiting frequencies are resonant.

The paper is organized as follows. In sect. 2 we state the results of the paper, together with some examples and comments. In sect. 3 we introduce the notations and the spaces which we use for our result. In sect. 4 we define a special class of polynomials. In sect. 5 we show that the nonlinearity appearing in the NLKG equation belongs to that class. In sect. 6 we study the resonances of the system. In sect. 7 we introduce the notion of normal form, and in the last section we prove the main theorem.

2. Statement of the main results. In (1) we assume that the potential has the form

$$V(x) = \sum_{k \geq 1} v_k \cos(kx).$$

By using the same approach of [14], we fix a positive $s$, and for any $M > 0$ we consider the probability space

$$\mathcal{V}_s,M = \left\{ (v_k)_{k \geq 1} : v_k' := M^{-1}(1 + |k|)^s v_k \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\},$$

and we endow the product probability measure on the space of $(c, (v_k')_k)$.

It is well known that (1) is Hamiltonian with Hamiltonian

$$H(v, u) = \int_I \frac{c^2|v(x)|^2 + |u_x(x)|^2 + c^2|u(x)|^2 + (V * u)(x)u(x)}{2} \, dx + \int_I F(u) \, dx,$$

where $v := u_t/c^2$ is the momentum conjugated to $u$, and $F(u)$ is such that $\partial_u F = f$.

Consider now the Sturm Liouville problem

$$-\partial_{xx} \phi_k + V * \phi_k = \lambda_k \phi_k,$$

with Dirichlet boundary conditions on $I$: it is well known that all the eigenvalues are distinct, that the solutions $(\phi_k)_{k \geq 1}$ of (5) given by $\phi_k(x) = (2/\pi)^{1/2} \sin(kx)$ form an orthonormal basis of $L^2(I)$. It is useful to expand both $u$ and $v$ with respect to $(\phi_k)_{k \geq 1}$,

$$\begin{cases}
u(t, x) = \sum_{k \geq 1} q_k(t) \phi_k(x), \\
v(t, x) = \sum_{k \geq 1} p_k(t) \phi_k(x),
\end{cases}$$

and to introduce the following change of variables

$$\begin{align*}
\xi_k &= \frac{1}{\sqrt{2}} \left[ q_k \left( \frac{\sqrt{c^2 + \lambda_k}}{c} \right)^{1/2} - ip_k \left( \frac{c}{\sqrt{c^2 + \lambda_k}} \right)^{1/2} \right], \quad k \geq 1, \\
\eta_k &= \frac{1}{\sqrt{2}} \left[ q_k \left( \frac{\sqrt{c^2 + \lambda_k}}{c} \right)^{1/2} + ip_k \left( \frac{c}{\sqrt{c^2 + \lambda_k}} \right)^{1/2} \right], \quad k \geq 1.
\end{align*}$$
where \( \lambda_k = k^2 + v_k \). Indeed, in the coordinates \((\xi, \eta) := ((\xi_k)_{k \geq 1}, (\eta_k)_{k \geq 1})\) the Hamiltonian (4) takes the form
\[
H(\xi, \eta) = H_0(\xi, \eta) + N(\xi, \eta),
\]
where
\[
H_0(\xi, \eta) := \sum_{k \geq 1} \omega_k \xi_k \eta_k,
\]
\[
N(\xi, \eta) := \int_I F \left( \sum_{k \geq 1} \left( \frac{c}{\sqrt{c^2 + \lambda_k}} \right)^{1/2} \xi_k + \eta_k \phi_k(x) \right) \, dx,
\]
where the linear frequencies \((\omega_k)_{k \geq 1}\) of (9) are given by
\[
\omega_k := \omega_k(c) = c\sqrt{c^2 + \lambda_k} = c^2 + \frac{\lambda_k}{1 + \sqrt{1 + \lambda_k/c^2}}
\]
\[
= c^2 + \frac{\lambda_k}{2} - \frac{\lambda_k^2}{2c^2 (1 + \sqrt{1 + \lambda_k/c^2})^2}. \tag{13}
\]
Equation (1) is a semilinear PDE locally well-posed in the energy space \( H^1(I) \times L^2(I) \) (see [17], ch. 2.1). Consider a local solution \((u(t), c^2v(t))\) of (1): a standard computations shows that \((u, c^2v) \in H^1(I) \times L^2(I)\) if and only if \((\xi, \eta) \in l^2(I) \times l^{1/2}(I)\) solves the following system
\[
\begin{cases}
-\partial_t \xi_k &= \omega_k \xi_k + \frac{\partial N}{\partial \xi_k}, \\
\partial_t \eta_k &= \omega_k \eta_k + \frac{\partial N}{\partial \eta_k}.
\end{cases} \tag{14}
\]
Moreover, we will denote by \(\psi\) and \(\bar{\psi}\) the functions given by the following expansions with respect to the eigenfunctions \((\phi_k)_{k \geq 1}\),
\[
\psi(t, x) := \sum_{k \geq 1} \xi_k(t) \phi_k(x), \tag{15}
\]
\[
\bar{\psi}(t, x) := \sum_{k \geq 1} \eta_k(t) \phi_k(x). \tag{16}
\]
It is easy to check that \((u, c^2v) \in H^1 \times L^2\) solve equation (1) if and only if \((\psi, \bar{\psi}) \in H^{1/2} \times H^{1/2}\) solve the following equation,
\[
-\partial_t \psi = c(c^2 - \Delta + \bar{V})^{1/2} \psi
\]
\[
+ \frac{1}{\sqrt{2}} \left( \frac{c}{(c^2 - \Delta + \bar{V})^{1/2}} \right)^{1/2} f \left( \left( \frac{c}{(c^2 - \Delta + \bar{V})^{1/2}} \right)^{1/2} \psi + \bar{\psi} \right), \tag{17}
\]
where \(\bar{V}\) is the operator that maps \(\psi\) to \(V \ast \psi\).

Now, for \(\rho > 0\) we denote by \(A_\rho := A_\rho(I, \mathbb{C} \times \mathbb{C})\) the space of functions that are analytic on the complex neighborhood of \(I\) given by
\[
I_\rho := \{ w = x + iy : x \in I, y \in \mathbb{R}, |y| < \rho \},
\]
and continuous on the closure of this strip. The space \(A_\rho\), endowed with the following norm,
\[
| \langle \varphi_1, \varphi_2 \rangle |_\rho := \sup_{w \in I_\rho} |\varphi_1(w)| + |\varphi_2(w)|, \quad \forall (\varphi_1, \varphi_2) \in A_\rho,
\]
is a Banach space.

Our main result is the following theorem:

**Theorem 2.1.** Consider the equation \((1)\). For any positive \(\beta < 1\) and for any \(\rho > 0\) the following holds: there exist \(\gamma > 0\), \(\tau > 0\), and a set \(\mathcal{R}_\gamma := \mathcal{R}_{\gamma,\tau,M} \subset [1, +\infty) \times V\) satisfying

\[
|\mathcal{R}_\gamma \cap ([n, n + 1] \times V)| = \mathcal{O}(\gamma) \quad \forall n \in \mathbb{N}_0,
\]

such that for any \((c, (v_k)_k) \in ([1, +\infty) \times V) \setminus \mathcal{R}_{\gamma,\tau,M}\) there exist \(K > 0\) and a sufficiently small \(R_0 > 0\) such that, if

\[
(\psi_0, \bar{\psi}_0) \in \mathcal{A}_\rho, \quad |(\psi_0, \bar{\psi}_0)|_\rho = R < R_0,
\]

then the solution of \((1)\) with initial datum \((\psi_0, \bar{\psi}_0)\) exists for times \(|t| \leq e^{-\sigma_\rho|\log R|^{\beta+1}}, \sigma_\rho = \min(1/8, \rho/4)\), and satisfies

\[
|(|\psi(t), \bar{\psi}(t)||_\rho/4 \leq K R, \quad |t| \leq e^{-\sigma_\rho|\log R|^{\beta+1}}.
\]

Furthermore, we have that

\[
\sum_{k \geq 1} e^{\rho k} ||\xi_k(t)| - |\xi_k(0)|| \leq R^{3/2}, \quad |t| \leq e^{-\sigma_\rho|\log R|^{\beta+1}}.
\]

**Remark 1.** In finite dimension \(n\), the standard Nekhoroshev theorem controls the dynamics over timescales of order \(e^{\rho (\alpha R^{-1/(\tau+1)})}\) for some \(\alpha > 0\) and for some \(\tau > n - 1\) (see [18] and [8]; see also [16] and [20] for a more direct proof in the convex and quasi-convex case respectively).

In the infinite-dimensional context there are only few results, mainly due to Bambusi and Pöschel in the one-dimensional case (see [6], [1], [2], [21] and [7]), and by Faou and Grébert in the multidimensional case (see [14]). In particular, in [2] Bambusi proved a Nekhoroshev result for the one-dimensional NLKG: he was able to control the dynamics of analytic solutions in a strip on a timescale of order \(\mathcal{O}(e^{\alpha|\log R|^{\beta+1}})\) for some \(\alpha > 0\) and \(\beta < 1\) (which is the same timescale we cover in Theorem 2.1), but assuming that the parameter that appears in the equation ranges over a compact interval.

**Remark 2.** Actually, our result is slightly different from the one obtained by Faou and Grébert in [14]: indeed, while they proved that there exists a full measure set of potentials for which each solution of the NLS equation corresponding to an initial datum with small analytic norm remains small for exponentially long times, in our result we prove that for “most” of the values of speeds of light and potentials each solution of \((1)\) corresponding to an initial datum with small analytic norm remains small for exponentially long times.

Such a difference is motivated by the non resonance condition we prove below (see Theorem 6.1), which is an adaptation of the uniform diophantine estimates reported in [19].

By exploiting the same argument used to prove Theorem 2.1 one can immediately deduce the following stability result for solutions with small (with respect to \(c\)) initial data.

**Corollary 1.** Consider the equation \((1)\) and fix \(\delta > 0\). Then there exist \(\gamma > 0\), \(\tau > 1\), and a set \(\mathcal{R}_\gamma := \mathcal{R}_{\gamma,\tau,M} \subset [1, +\infty) \times V\) satisfying

\[
|\mathcal{R}_\gamma \cap ([n, n + 1] \times V)| = \mathcal{O}(\gamma) \quad \forall n \in \mathbb{N}_0,
\]
such that for any \((c, (v_k)_k) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\gamma\), for any positive \(\beta < 1\) and for any \(\rho > 0\) the following holds: there exist \(K > 0\) and \(c^* > 0\) such that, if \(c > c^*\) and

\[
(\psi_0, \bar{\psi}_0) \in A_{\rho}, \quad \| (\psi_0, \bar{\psi}_0) \|_\rho = \frac{1}{c},
\]

then the solution of (1) with initial datum \((\psi_0, \bar{\psi}_0)\) the solution of (1) with initial datum \((\psi_0, \bar{\psi}_0)\) exists for times \(|t|\) \leq e^{-\sigma_\rho |\delta \log e|^{\beta + 1}}, \quad \sigma_\rho = \min(1/8, \rho/4), \) and satisfies

\[
\| (\psi(t), \bar{\psi}(t)) \|_{\rho/4} \leq \frac{K}{e^{\delta^2}}, \quad |t| \leq e^{-\sigma_\rho |\delta \log e|^{\beta + 1}}. \tag{20}
\]

3. **Hamiltonian setting.** In the following we set \(Z := \mathbb{N}_0 \times \{\pm 1\}, \mathbb{N}_0 = \{1, 2, 3, \ldots\}\); for any \(j = (k, \delta) \in Z\) we define \([j] := |k|\), and we denote by \(j\) the index \((k, -\delta)\). We also identify a couple \((\xi, \eta)\) with \((z_j)_{j \in Z}\), where

\[
j = (k, \delta) \in Z \rightarrow \begin{cases} z_j = \xi_k & \text{if } \delta = 1, \\
z_j = \eta_k & \text{if } \delta = -1. \end{cases} \tag{21}
\]

By a slight abuse of notation we will often denote by \(z = (z_j)_{j \in Z}\) such an element. The above system (14) may be regarded as an infinite-dimensional Hamiltonian system with coordinates \((\xi_k, \eta_k)_{k \geq 1} \in \mathbb{C}^{\mathbb{N}_0} \times \mathbb{C}^{\mathbb{N}_0}\) and symplectic structure

\[
i \sum_{k \geq 1} d\xi_k \wedge d\eta_k. \tag{22}
\]

For a given \(\rho > 0\), we also consider the Banach space \(L_\rho\) whose elements are those \(z \in \mathbb{C}^Z\) such that

\[
\|z\|_\rho := \sum_{j \in Z} e^{\rho |j|} |z_j| < +\infty.
\]

We say that \(z \in L_\rho\) is real if \(z_j = \bar{z}_j\) for any \(j \in Z\).

**Lemma 3.1.** Let \(\psi, \bar{\psi}\) be complex valued functions analytic on a neighborhood of \(I\), and let \((z_j)_{j \in Z}\) be the sequence defined by (15) and (16). Then for all \(\mu < \rho\) we have

1. if \((\psi, \bar{\psi}) \in A_\rho\), then \(z \in L_\mu\), and \(\|z\|_\mu \leq K_{\rho, \mu}\| (\psi, \bar{\psi})\|_\rho;\)

2. if \(z \in L_\rho\) then \((\psi, \bar{\psi}) \in A_\mu\), and \(\| (\psi, \bar{\psi})\|_\mu \leq K_{\rho, \mu}\|z\|_\mu, \)

where \(K_{\rho, \mu}\) is a positive constant depending only on \(\rho\) and \(\mu\).

**Proof.** Assume that \((\psi, \bar{\psi}) \in A_\rho;\) then by Cauchy formula we have that for all \(j \in Z\)

\[
|z_j| \leq e^{-\rho |j|}\| (\psi, \bar{\psi})\|_\rho.
\]

Hence for \(\mu < \rho\) we have

\[
\|z\|_\mu \leq \| (\psi, \bar{\psi})\|_\rho \sum_{j \in Z} e^{(\mu - \rho) |j|} \leq \| (\psi, \bar{\psi})\|_\rho 2 \sum_{k \geq 1} e^{(\mu - \rho) k} \leq \frac{2 e^{\mu - \rho}}{1 - e^{\mu - \rho}}\| (\psi, \bar{\psi})\|_\rho.
\]

Conversely, assume that \(z \in L_\rho;\) then \(|\xi_k| \leq e^{-\rho k}\| (\psi, \bar{\psi})\|_\mu, \) and thus by (15) we get that for all \(x \in I\) and \(y \in \mathbb{R}\) with \(|y| \leq \mu\)

\[
|\psi(x + iy)| + |\bar{\psi}(x + iy)| \leq \sum_{k \in Z} |z_k| e^{k|y|} \leq \|z\|_\rho 2 \sum_{k \geq 1} e^{-(\rho - \mu) k} \leq \frac{2 e^{\mu - \rho}}{1 - e^{\mu - \rho}}\|z\|_\rho.
\]
hence $\psi$ and $\bar{\psi}$ are bounded on the strip $I_\mu$. \hfill \Box

Consider a function $G \in C^1(L_\rho, \mathbb{C})$ we define its Hamiltonian vector field by $X_G := J \nabla G$, where $J$ is the symplectic operator on $L_\rho$ induced by the symplectic form (22), and $\nabla G(z) := \left( \frac{\partial G}{\partial \bar{z}_j} \right)_{j \in \mathbb{Z}}$, where for $j = (k, \delta) \in \mathbb{Z}$

$$\frac{\partial G}{\partial z_j} = \begin{cases} \frac{\partial G}{\partial \bar{z}_k} & \text{if } \delta = 1, \\ \frac{\partial G}{\partial \bar{\eta}_k} & \text{if } \delta = -1. \end{cases}$$

**Definition 3.2.** For a given $\rho > 0$ we denote by $H_\rho$ the space of real Hamiltonians $G$ satisfying

$$G \in C^1(L_\rho, \mathbb{C}),$$

$$X_G \in C^1(L_\rho, L_\rho).$$

Let $G_1, G_2 \in H_\rho$, then we define the Poisson bracket between $G_1$ and $G_2$ via the formula

$$\{G_1, G_2\} := \nabla G_1^T X_{G_2} = i \sum_{k \geq 1} \frac{\partial G_1}{\partial \bar{\eta}_k} \frac{\partial G_2}{\partial \bar{\xi}_k} - \frac{\partial G_1}{\partial \bar{\xi}_k} \frac{\partial G_2}{\partial \bar{\eta}_k}. \quad (23)$$

We say that the Hamiltonian $H$ is real if $H(z)$ is real for all real $z$.

We associate to a given Hamiltonian $H \in H_\rho$ the corresponding Hamilton equations,

$$\dot{z} = X_H(z) = J \nabla H(z),$$

equivalently

$$\begin{cases} \dot{\xi}_k = -i \frac{\partial H}{\partial \eta_k}, \\ \dot{\eta}_k = i \frac{\partial H}{\partial \xi_k}, \quad k \geq 1. \end{cases}$$

We also denote by $\Phi_H^t(z)$ the time-$t$ flow associated with the previous system. We just remark that if $z = (\xi, \bar{\xi})$ and if $H$ is real, then also the flow $\Phi_H^t(z) = (\xi(t), \bar{\xi}(t))$ is real for all $t$, namely $\xi(t) = \bar{\eta}(t)$ for all $t$. Moreover, the Hamiltonian

$$H(\xi, \eta) = \sum_{k \geq 1} \omega_k \xi_k \eta_k + N(\xi, \eta),$$

with $N$ given by (11), leads to the system (14), that is the NLKG equation (1) written in the coordinates $(\xi, \eta)$.

**Remark 3.** We point out that the Hamiltonian $H_0 = \sum_{k \geq 1} \omega_k \xi_k \eta_k$, which corresponds to the linear part of (9), does not belong to $H_\rho$. However, it generates a flow which maps $L_\rho$ to $L_\rho$, and it is explicitly given by

$$\begin{cases} \xi_k(t) = e^{-i\omega_k t} \xi_k(0), \\ \eta_k(t) = e^{i\omega_k t} \eta_k(0). \end{cases}$$

On the other hand, we will show in sec. 5 that the nonlinearity $N$ belongs to the space $H_\rho$. 

4. The space of polynomials. In this section we define a class of polynomials in $\mathbb{C}^2$. We first introduce some notations about multi-indices: let $l \geq 2$ and $j = (j_1, \ldots, j_l) \in \mathbb{Z}^l$, with $j_l = (k_i, \delta_i)$, we define

- the norm of the multi-index $j$,
  $$\|j\| = \max_{i=1, \ldots, l} |j_i| = \max_{i=1, \ldots, l} |k_i|;$$

- the monomial associated with $j$,
  $$z_j := z_{j_1} \cdots z_{j_l};$$

- the momentum of $j$,
  $$M(j) := k_1 \delta_1 + \cdots + k_l \delta_l;$$

- the divisor associated to $j$,
  $$\Omega(j) := \delta_1 \omega_{k_1} + \cdots + \delta_l \omega_{k_l},$$
  where $\omega_k$ is the $k$-th linear frequency of the system, and is given by (12).

We then define the set of indices with zero momentum by
  $$\mathcal{I}_l = \{ j = (j_1, \ldots, j_l) \in \mathbb{Z}^l : M(j) = 0 \}.$$  
(26)

We also say that an index $j = (j_1, \ldots, j_l) \in \mathbb{Z}^l$ is resonant (we denote it by $j \in \mathcal{N}_l$) if $l$ is even and $j = \bar{i} \cup \bar{i}$ for some choice of $i \in \mathbb{Z}^{l/2}$. In particular, if $j$ is resonant then its associated divisor vanishes, namely $\Omega(j) = 0$, and its associated monomial depends only on the actions.

$$z_j = z_{j_1} \cdots z_{j_l} = \xi_{k_1} \eta_{k_1} \cdots \xi_{k_l} \eta_{k_l},$$
where for all $k \geq 1 I_k(z) = |\xi_k|^2$ denotes the $k$-th action. Finally, we point out that if $z$ is real then $I_k(z) = |\xi_k|^2$; we also remark that for odd $l$ the resonant set $\mathcal{N}_l$ is empty.

**Definition 4.1.** Let $k \geq 2$, we say that a polynomial $P(z) = \sum a_j z_j$ belongs to $\mathcal{P}_k$ if $P$ is real, of degree $k$, which has a zero of order at least 2 in $z = 0$, and if

- $P$ contains only monomials having zero momentum, namely such that $M(j) = 0$ for $a_j \neq 0$, thus $P$ is of the form
  $$P(z) = \sum_{j \in \mathcal{I}_l} a_j z_j,$$
  with $a_j = \bar{a}_j$;

- the coefficients $a_j$ are bounded, namely
  $$\sup_{j \in \mathcal{I}_l} |a_j| < +\infty, \ \forall l = 2, \ldots, k.$$

We endow the space $\mathcal{P}_k$ with the norm
  $$\|P\| = \sum_{l=2}^k \sup_{j \in \mathcal{I}_l} |a_j|.$$  
(28)

**Remark 4.** In the following sections we will crucially use the fact that polynomials in $\mathcal{P}_k$ contain only monomials with zero momentum: indeed, this will allow us to control the largest index by the others.

The zero momentum condition is essential to prove the following result.
Proposition 1. Let \( k \geq 2 \) and \( \rho > 0 \), then \( \mathcal{P}_k \subset \mathcal{H}_\rho \). Moreover, any homogeneous polynomial \( P \) which belongs to \( \mathcal{P}_k \) satisfies

\[
|P(z)| \leq \|P\| \|z\|_\rho^k, \quad \forall z \in \mathcal{L}_\rho \tag{29}
\]

\[
\|XP(z)\|_\rho \leq 2k \|P\| \|z\|_\rho^{k-1}, \quad \forall z \in \mathcal{L}_\rho. \tag{30}
\]

Furthermore, if \( P \in \mathcal{P}_k \) and \( Q \in \mathcal{P}_l \), then \( \{P,Q\} \in \mathcal{P}_k + \mathcal{P}_l - 2 \), and

\[
\|\{P,Q\}\| \leq 2kl \|P\| \|Q\|. \tag{31}
\]

Proof. Let

\[
P(z) = \sum_{j \in \mathcal{I}_k} a_j z_j,
\]

then

\[
|P(z)| \leq \|P\| \sum_{j \in \mathcal{I}_k} |z_{j_1}| \cdots |z_{j_k}| \leq \|P\| \|z\|_\rho^k \leq \|P\| \|z\|_\rho^k,
\]

and hence we get (29).

To prove (30), take \( l \in \mathbb{Z} \), and exploit the zero momentum condition in order to get

\[
\frac{\partial P}{\partial z_l} \leq k \|P\| \sum_{j \in \mathbb{Z}^{k-1}} |z_{j_1} \cdots z_{j_{k-1}}|.
\]

We have

\[
\|XP(z)\|_\rho = \sum_{l \in \mathbb{Z}} e^{\rho |l|} \left| \frac{\partial P}{\partial z_l} \right| \leq k \|P\| \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}^{k-1}} e^{\rho |l|} |z_{j_1} \cdots z_{j_{k-1}}|,
\]

since \( M(j) = -M(l) \),

\[
e^{\rho |l|} \leq \prod_{m=1}^{k-1} e^{\rho |j_m|},
\]

therefore by summing in \( l \) we obtain

\[
\|XP(z)\|_\rho \leq 2k \|P\| \sum_{j \in \mathbb{Z}^{k-1}} e^{\rho |j_1|} |z_{j_1}| \cdots e^{\rho |j_{k-1}|} |z_{j_{k-1}}| \leq 2k \|P\| \|z\|_\rho^{k-1},
\]

which gives (30).

Now let us assume that \( P \) and \( Q \) are homogeneous polynomials of degrees \( k \) and \( l \) respectively, and with coefficients \( a_k, k \in \mathcal{I}_k \), and \( b_l, l \in \mathcal{I}_l \). It is easy to check that \( \{P,Q\} \) is a monomial of degree \( k + l - 2 \) that satisfies the zero momentum condition. Moreover, if we write

\[
\{P,Q\}(z) = \sum_{j \in \mathcal{I}_k + \mathcal{I}_l - 2} c_j z_j,
\]

we have that \( c_j \) is a sum of coefficients \( a_k b_l \) for which there exists \( h \geq 1 \), and \( \delta \in \{\pm 1\} \) such that

\[
(h, \delta) \subseteq k \in \mathcal{I}_k,
\]

\[
(h, -\delta) \subseteq l \in \mathcal{I}_l
\]

(if for example \((h, \delta) = k_1 \) and \((h, -\delta) = l_1 \), then \((k_2, \ldots, k_k, l_2, \ldots, l_l) = j \)). Hence, for a given \( j \) the zero momentum condition on \( k \) and \( l \) determines two
possible values of \((h, \delta)\). This allow us to deduce (31) for monomials; its extension to polynomial follows by the definition of the norm (28).

**Remark 5.** In Proposition 1 we used a \(l^1\)-type norm to estimate Fourier coefficients and vector fields, instead of the usual \(l^2\) norms. This choice does not allow to work in Hilbert spaces, and produces a loss of regularity each time the estimates are transported from the Fourier space to the space of analytic functions \(A_\rho\). However, by using this approach we can control vector fields in a simple way. This argument has been already used in [14].

It may be possible to recover the usual \(l^2\) estimates for Fourier coefficients and vector fields by introducing a more general space of polynomials \(P_{k_1, k_2} \quad (k_1 \geq 2, 1 < k_2 \leq \infty)\), endowed with the following norm

\[
\|P\|_{k_1, k_2} := \sum_{l=2}^{k_1} \left( \sum_{j \in I_l} |a_j|^{k_2} \right)^{1/k_2}, \quad 1 < k_2 < +\infty
\]

\[
\|P\|_{k_1, \infty} := \|P\|_{k_1} = \sum_{l=2}^{k_1} \sup_{j \in I_l} |a_j|.
\]

In this case one should also introduce a more general space of analytic functions \(L_{\rho, k} := \{z \in \mathbb{C}^\mathbb{Z} : \|z\|_{\rho, k} := \sum_{j \in \mathbb{Z}} e^{\rho|j|} |z_j|^{k} < +\infty\}, 1 \leq k < +\infty\).

5. **Nonlinearity.** First we recall that the nonlinearity \(N\) in (11) in the coordinates \((\psi, \bar{\psi})\) takes the form

\[
N(\psi, \bar{\psi}) = \int I F \left( \frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} dx,
\]

where we assume that \(F\) is analytic in a neighbourhood of the origin in \(\mathbb{C} \times \mathbb{C}\). Therefore there exist positive constants \(M\) and \(R_0\) such that the Taylor expansion of the integrand in the nonlinearity

\[
F \left( \frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}}
\]

\[
= \sum_{k_1, k_2 \geq 0} \frac{1}{k_1! k_2!} 2^{-(k_1 + k_2)/2} \left( \frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}} \right)^{(k_1 + k_2)/2} \partial_\psi^{k_1} \partial_{\bar{\psi}}^{k_2} F(0) \psi^{k_1} \bar{\psi}^{k_2}
\]

is uniformly convergent on the ball \(|\psi| + |\bar{\psi}| \leq 2R_0\) of \(\mathbb{C} \times \mathbb{C}\), and is bounded by \(M\).

**Remark 6.** The constant \(M\) in the above estimate can be chosen uniformly with respect to the speed of light \(c\), because the smoothing pseudodifferential operator

\[
\left( \frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}} \right)^{1/2} : \mathcal{A}_\rho \to \mathcal{A}_\rho
\]

can be estimated uniformly with respect to \(c \geq 1\). Indeed it is easy to check that

\[
\left\| \left( \frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}} \right)^{1/2} (\psi, \bar{\psi}) \right\|_\rho
\]
for some constant $K > 0$ that depends only on $\rho$.

Thus, formula (11) defines an analytic function on the ball $||z||_\rho \leq R_0$ of $L_\rho$, and we can write

$$N(z) = \sum_{k \geq 0} N_k(z),$$

where for all $k \geq 0$ $N_k$ is a homogeneous polynomial defined by

$$N_k(\xi, \eta) = \sum_{k_1+k_2=k} \sum_{(a,b) \in N_{k_1}^1 \times N_{k_2}^2} n_{a,b} \xi_1^{a_1} \cdots \xi_{a_k}^{a_k} \eta_{b_1} \cdots \eta_{b_{k_2}},$$

where

$$n_{a,b} = \sum_{k_1,k_2 \geq 0} \frac{1}{k_1!k_2!} 2^{-(k_1+k_2)/2} \left( \frac{c}{(c^2 - \Delta + \tilde{V})^{1/2}} \right)^{(k_1+k_2)/2} \partial_\xi^{k_1} \partial_\eta^{k_2} F(0) \cdot \int I \phi_{a_1}(x) \cdots \phi_{a_k}(x) \phi_{b_1}(-x) \cdots \phi_{b_{k_2}}(-x) dx.$$}

Hence it is easy to check that $N_k$ satisfies the zero momentum condition, thus $N_k \in \mathcal{P}_k$ for all $k$, and $||N_k|| \leq MR_0^{-k}$.

**Remark 7.** One could also extend this argument by considering not only zero momentum monomials, but also monomials with exponentially decreasing or power-law decreasing momentum. This would allow to deal with the NLKG equation with a multiplicative potential and with nonlinearities that depend on $x$,

$$\frac{1}{c^2} u_{tt} - u_{xx} + c^2 u + V(x)u + f(x,u) = 0, \quad x \in I,$$

but this problem would require a more technical discussion.

6. **Non resonance conditions.** In order to prove Theorem 2.1, we need to show some non resonance properties of the frequencies $\omega = (\omega_k)_{k \geq 1}$: it will be crucial that these properties hold uniformly (or at least, up to a set of small probability) in $(1, +\infty) \times V$. The argument is similar to the one reported in [19].

For $r \geq 3$ and $j = (j_1, \ldots, j_r) \in \mathbb{Z}^r$ ($j_i = (a_i, \delta_i)$, $1 \leq i \leq r$), we define $\mu(j)$ as the third largest integer among $j_1, \ldots, j_r$, and we again mention that $j \in \mathbb{Z}^r$ is called resonant if $r$ is even and $j = i \cup \bar{i}$ for some choice of $i \in \mathbb{Z}^{r/2}$.

**Theorem 6.1.** For any sufficiently small $\gamma > 0$ there exists a set $\mathcal{R}_\gamma := \mathcal{R}_{\gamma,\delta} \subset [1, +\infty) \times V$ satisfying

$$|\mathcal{R}_\gamma \cap ([n, n+1] \times V)| = O(\gamma) \quad \forall n \in \mathbb{N}_0,$$
and \( \exists \tau > 0 \) such that \( \forall (c,(v_j)_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\tau \), for any sufficiently large \( N \geq 1 \) and for any \( r \geq 1 \)

\[
|\Omega(j)| \geq \frac{\gamma}{\mu(j)^{\tau r}},
\]

(33)

for any non resonant \( j \in \mathbb{Z}^{r+2} \setminus \mathcal{N}^{r+2} \) with \( \mu(j) \leq N \).

**Remark 8.** The non resonance condition in Theorem 6.1 is slightly weaker from the one proved in sec. 2.4 of [14]; this is the reason for which Faou and Grébert are able to prove a result which is valid for a set of potentials of full measure, while our result is valid only for “most” of the values of speed of light and potentials.

We also mention that a non resonance condition similar to our one has been already used in [3] (see (3.3)) and in [5] (see (r-NR)).

Actually, instead of proving that condition (33) is satisfied, we will prove a more general result. We use the notation “\( a \ll b \)” (resp. “\( a \gg b \)” to mean “there exists a constant \( K > 0 \) independent of \( c \) such that \( a \leq K b \)” (resp. \( a \geq K b \)). We also write \( \mu(j) := \prod_{i=1}^r (1 + |j_i|) \).

**Remark 9.** Observe that \( \mu(j) \leq |j| \leq N(j) \). Furthermore, \( N(j) \leq N \) implies that \( |j| \leq N \), since \( N(j) \geq |j_1| + \ldots + |j_r| \geq |j| \). On the other hand, \( |j| \leq N \) implies that \( N(j) = \prod_{i=1}^r (1 + |j_i|) \leq (1 + N)^r \).

**Proposition 2.** Let \( c \geq 1 \) be fixed. Then for any sufficiently small \( \gamma > 0 \) \( \exists \mathcal{V}'_{s,M,\gamma} \subset \mathcal{V} \) with \( |\mathcal{V} \setminus \mathcal{V}'_{s,M,\gamma}| = \mathcal{O}(\gamma) \), and \( \exists r > 1 \) such that \( \forall (v_j)_j \in \mathcal{V}'_{s,M,\gamma}, \) for any \( N \geq 1 \) and for any \( r \geq 1 \)

\[
|\Omega(j)| + m \geq \frac{\gamma}{N(j)^r}
\]

(34)

\( \forall m \in \mathbb{Z} \), for any \( j \in \mathbb{Z}^r \setminus \mathcal{N}^r \) with \( |j| \leq N \).

**Proof.** Fix \( r > 1 \), \( j \in \mathbb{Z}^r \) non resonant and \( m \in \mathbb{Z} \).

Let \( p_{j,m}((v_j)_j) := \sum_{j=1}^{r} \omega_{\delta_j} \delta_j + m \), and let \( 1 \leq h \leq r \). Then

\[
\left| \frac{\partial p_{j,m}}{\partial v'_h} \right| = \left| \frac{\delta_h a^{-s}_{j} \omega_{\delta_j} \delta_j + m}{2 \sqrt{1 + \lambda_{a_h} / c^2}} \right| \geq \frac{1}{2 \sqrt{1 + \lambda_{a_h} / c^2}} \geq \frac{1}{2 \sqrt{1 + a_h^{\max(s,2)}}} \geq \frac{1}{2 N(j)^r \sqrt{1 + N(j)^{\max(s,2)}}}
\]

since \( a_r \leq \prod_{i=1}^r (1 + a_i) = \prod_{i=1}^r (1 + |j_i|) = N(j) \). Hence, if we define for \( \gamma_0 > 0 \) the set

\[
\text{Res}_{j,m,r}(\gamma_0) := \{(v'_{j})_{j \geq 1} : |p_{j,m}((v_j)_j)_{j \geq 1}| < \gamma_0 \},
\]

we have by Lemma 17.2 of [22]

\[
|\text{Res}_{j,m,r}(\gamma_0)| \leq \gamma_0 N(j)^{s + \max(s,2)/2}.
\]

Now, we estimate the measure of \( \bigcup_{r,m,j} \text{Res}_{j,m,r}(\gamma_0) \); note that the argument in the modulus in (34) can be small only if \( |m| \leq rN \). Therefore

\[
\left| \bigcup_{r \geq 1} \bigcup_{|m| \leq rN} \bigcup_{|j| \leq N} \{(v'_{j})_{j \geq 1} : |p_{j,m}((v_j)_j)_{j \geq 1}| < \gamma_0 \} \right| \leq \sum_{r=1}^{\infty} \gamma_0 N^{r+1}(1 + N)^{r(s + \max(s,2))},
\]

(35)
since \( N(j) \leq (1 + N)^r \), and by choosing \( \gamma_0 = \frac{\gamma}{(1 + N)^{r+1}} \) with \( \tau > s + 2 + \frac{\max(s,2)}{2} \) we get the thesis.

**Proposition 3.** For any sufficiently small \( \gamma > 0 \) there exists a set \( R_\gamma := R_\gamma(s, N) \subset [1, +\infty) \times V \) satisfying

\[
|\{ j \in [n, n+1) \times V | = O(\gamma) \quad \forall n \in \mathbb{N},
\]

and \( \exists \tau > 1 \) such that \( \forall (\nu, \nu_j) \in [(1, +\infty) \times V) \setminus R_\gamma \), for any sufficiently large \( N \geq 1 \) and for any \( r \geq 1 \)

\[
|\Omega(j) + \sigma \omega_1| \geq \frac{\gamma}{N(j)^r}
\]

for any \( j \in Z^r \) with \( |j| \leq N \), \( \sigma = \pm 1 \), \( l \geq N \) such that \( (j, (l, \sigma)) \in Z^{r+1} \setminus N^{r+1} \) is non resonant.

**Proof.** Without loss of generality, we can choose \( \sigma = -1 \).

Now fix \( r \geq 1 \), a non resonant \( j \in Z^r \) with \( |j| \leq N \), and \( l \geq N \). Set \( p_{j,l}(c, (v_j)_{j \geq 1}) := \Omega(j) - \omega_1 \). We can rewrite the function \( p_{j,l} \) in the following way:

\[
p_{j,l}(c, (v_j)_{j \geq 1}) = \alpha c^2 + \sum_{j=1}^r \frac{\delta_j \lambda_{a_j}}{1 + \sqrt{1 + \lambda_{a_j}/c^2}} - \frac{\lambda_l}{1 + \sqrt{1 + \lambda_l/c^2}},
\]

where \( \alpha := (\sum_{j=1}^r \delta_j) - 1 \in \{-r-1, \ldots, -1\} \). Now we have to distinguish some cases:

**Case** \( \alpha = 0 \). in this case we have that \( p_{j,l} \) can be small only if \( l^2 \leq 3(N^2 + N^*)^2 r^2 \). So to obtain the result we just apply Proposition 2 with \( N' := \sqrt{3}(N^2 + N^*) r \), \( r' = r + 1 \).

**Case** \( \alpha 
eq 0 \), \( c \leq \lambda_N^{1/2} r^{-1/2} \). we have that

\[
\sum_{j=1}^r c \sqrt{c^2 + \lambda_{a_j} \delta_j} \leq r \sqrt{c^1 + c^2 \lambda_N} \leq \sqrt{2} r^2 \lambda_N,
\]

so \( |\Omega(j) - \omega_1| \) can be small only for \( l^2 < r N^2 \). Therefore, in order to get the thesis we apply Proposition 2 with \( N' := \sqrt{r} N \), \( r' := r + 1 \).

**Case** \( \alpha > 0 \), \( c > \lambda_N^{1/2} r^{-1/2} \). first notice that if we set \( f(x) := \frac{c^2}{2 \sqrt{1 + \alpha (1 + \sqrt{1 + x})^2}} \), and we put \( x_j := \lambda_{a_j}/c^2 \), in this regime we get

\[
\sum_{j=1}^r \delta_j f(x_j) \leq f(x_N) \leq \frac{1}{2}
\]

Now define \( \tilde{p}_{j,l}(c^2) := \alpha c^2 - \frac{\lambda_l}{1 + \sqrt{1 + \lambda_l/c^2}} \). One can verify that

\[
\tilde{p}_{j,l}(c^2) = 0;
\]

\[
c^2 = c_{\alpha,\alpha}^2 := \frac{\lambda_l}{\alpha (\alpha + 2)},
\]

and that

\[
\frac{\partial \tilde{p}_{j,l}}{\partial (c^2)}(c_{\alpha,\alpha}^2) = \alpha - \frac{\alpha^2 (\alpha + 2)^2}{2 \sqrt{1 + \alpha (\alpha + 2)} (1 + \sqrt{1 + \alpha (\alpha + 2)})^2} > 0.
\]
Besides, there exists a sufficiently small \( \rho > 0 \) such that in the interval
\[
[c^2_{i,\alpha,-}, c^2_{i,\alpha,+}] := \left[ c^2_{i,\alpha} - \frac{\theta}{\alpha(\alpha + 2)}, c^2_{i,\alpha} + \frac{\theta}{\alpha(\alpha + 2)} \right]
\]
we have
\[
\frac{\partial \tilde{p}_{j,I}(c^2)}{\partial(c^2)}(c^2) > \left( \frac{1}{2} + \frac{1}{2(r + 1)} \right) \alpha
\]
(we can also assume that the intervals \( [c^2_{i,\alpha,-}, c^2_{i,\alpha,+}] \)) are pairwise disjoint, by taking \( \rho \) sufficiently small). Thus, by exploiting Lemma 17.2 of [22], we get
\[
\left\{ c^2 \in B \left( c^2_{i,\alpha}, \frac{\theta}{\alpha(\alpha + 2)} \right) : |\tilde{p}_{j,I}(c^2)| \leq \gamma_0 \right\} \leq \gamma_0 \frac{2(r + 1)}{(r + 2)\alpha} \leq \frac{2\gamma_0}{\alpha} \tag{38}
\]
for any \( \gamma_0 > 0 \) s.t. \( \frac{\theta}{\alpha} < \frac{\rho}{\alpha(\alpha + 2)} ; \gamma_0 < \frac{\rho}{2(r + 2)\alpha} \).

Now, since in this regime \( |\frac{\partial p_{j,l}}{\partial c^2}| \leq \frac{1}{2} \), we can deal with \( p_{j,l} \) as in (38), and we get that for each \( n \in \mathbb{N}_0 \)
\[
\left| \bigcup_{l \geq N} (\{ c^2 \in [c^2_{i,\alpha,-}, c^2_{i,\alpha,+}] : |p_{j,l}(c^2)| \leq \gamma_0 \} \cap [n, n + 1]) \right| \leq K_\alpha \gamma_0, \tag{39}
\]
for some \( K_\alpha > 0 \). By taking the union over \( |j| \leq N \) and arguing as in Proposition 2, we can deduce (35).

**Case** \( \alpha < 0 \), \( c > \lambda_N^{1/2} r^{1/2} \), since
\[
\sum_{j=1}^{r} \frac{\delta_j \lambda_{a_j}}{1 + \sqrt{1 + \lambda_{a_j}/c^2}} \leq \frac{r\lambda_N}{2} \leq \frac{c^2}{2},
\]
we have that \( p_{j,l} \) can be small only if \( \lambda_N < r\lambda_N \). So, in order to get the result, we apply Proposition 2 with \( N' := r^{1/2} N, r' := r + 1 \)

**Theorem 6.2.** For any sufficiently small \( \gamma > 0 \) there exists a set \( \mathcal{R}_\gamma := \mathcal{R}_{\gamma,s,M} \subset [1, +\infty) \times \mathcal{V} \) satisfying
\[
|\mathcal{R}_\gamma \cap ([n, n + 1] \times \mathcal{V})| = \mathcal{O}(\gamma) \ \forall n \in \mathbb{N}_0,
\]
and \( \exists \tau > 1 \) such that \( \forall (c_i, (v_j)_j) \in ([1, +\infty) \times \mathcal{V}) \setminus \mathcal{R}_\gamma \), for any sufficiently large \( N \geq 1 \) and for any \( r \geq 1 \)
\[
|\Omega(j) + \sigma_1 \omega_l + \sigma_2 \omega_m| \geq \frac{\gamma}{N(j)} \tag{40}
\]
for any \( j \in \mathcal{Z}^r \) with \( |j| \leq N \), \( \sigma_1, \sigma_2 = \pm 1 \), \( m > l \geq N \) such that \( (j, (l, \sigma_1), (m, \sigma_2)) \in \mathcal{Z}^{r+2} \setminus N^{r+2} \) is non resonant.

**Proof.** If \( \sigma_i = 0 \) for \( i = 1, 2 \), then we can conclude by using Proposition 3.

Now fix \( r \geq 1 \), a non resonant \( j \in \mathcal{Z}^r \) with \( |j| \leq N \), two positive integers \( l \) and \( m \) such that \( m > l \geq N \), and assume that \( \sigma_1 = -1, \sigma_2 = 1 \). Introduce
\[
p_{j,l,m}(c^2) := \Omega(j) - \omega_l(c^2) + \omega_m(c^2).
\]
Now fix \( \delta > 3 \). If \( m \lesssim N^\delta \), then we can conclude by applying Proposition 2 and 3. So from now on we will assume that \( m, l > N^5 \).
We have to distinguish several cases:

Case $c < \lambda_0$. we point out that, since
\[
e c\sqrt{c^2 + \lambda_l} = c\lambda_l^{1/2} \sqrt{1 + \frac{c^2}{\lambda_l}} = c\lambda_l^{1/2} \left(1 + \frac{c^2}{2\lambda_l} + O\left(\frac{1}{\lambda_l^2}\right)\right),
\]
we get (denote $m = l + j$)
\[
\omega_m - \omega_l = j c + \frac{1}{2} \left(\frac{v_m - v_l}{l}\right) + \frac{c^3}{2\lambda_l^{1/2}} - \frac{c^3}{2\lambda_l^{3/2}} + O\left(\frac{1}{m^3}\right) + O\left(\frac{1}{l^3}\right),
\]
that is, the integer multiples of $c$ are accumulation points for the differences between the frequencies as $l, m \to \infty$, provided that $\alpha < \frac{1}{6}$.

Case $c > \lambda_m$. in this case we have (again by denoting $m = l + j$) that $\lambda_m - \lambda_l = j(j + 2l) + (v_m - v_l) = 2jl + j^2 + a_{lm}$, with $|a_{lm}| \leq \frac{C}{l}$, so that
\[
p_{j,l,m} = \Omega(j) \pm 2jl \pm j^2 \pm a_{lm}.
\]
If $l > 2CN^\tau/\gamma$ then the term $a_{lm}$ represents a negligible correction and therefore we can conclude by applying Proposition 2. On the other hand, if $l \leq 2CN^\tau/\gamma$, we can apply the same Proposition with $N' := 2CN^\tau/\gamma$ and $r' := r + 2$.

Case $\lambda_l^{1/6} \leq c \leq \lambda_l^{1/2}$. if we rewrite the quantity to estimate
\[
p_{j,l,m}(c^2) = \alpha c^2 + \sum_{h=1}^r \frac{\lambda_{ah} \delta_h}{1 + \sqrt{1 + \lambda_{ah}/c^2}} + \omega_m - \omega_l,
\]
where $\alpha := \sum_{h=1}^r \delta_h$, we distinguish three cases:

- if $\alpha > 0$, then we notice that
\[
\left|\sum_{h=1}^r \frac{\lambda_{ah} \delta_h}{1 + \sqrt{1 + \lambda_{ah}/c^2}}\right| \leq \frac{r \lambda_N}{1 + \sqrt{1 + \lambda_N/c^2}} \leq \frac{r \lambda_N}{1 + \sqrt{1 + \lambda_N/\lambda_l}} \leq \frac{r \lambda_N}{2},
\]
\[
|\omega_m - \omega_l| = c \frac{\lambda_m - \lambda_l}{\sqrt{c^2 + \lambda_m} + \sqrt{c^2 + \lambda_l}} \geq \frac{c\lambda_l^{1/2}}{\sqrt{N^{25/3} + \lambda_m^{1/2} + N^{25/3} + \lambda_l^{1/2}}} > 0,
\]
thus $|p_{j,l,m}| > |\lambda_l^{1/3} - \frac{5}{2}\lambda_N| > 0$, since $l > N^3$;

- if $\alpha = 0$, then we just notice that
\[
|\omega_m - \omega_l| \geq \gamma (\lambda_m - \lambda_l) \geq \gamma_0 \lambda_l^{1/2},
\]
which is greater than $\gamma_0/N^\tau$ for $\tau > -1$, since $l > N^3$;

- if $\alpha < 0$, then we just recall that $|\omega_m - \omega_l| > \gamma_0 \lambda_l^{1/2}$, and by choosing $\gamma_0$ sufficiently small (actually $\gamma_0 \leq |\alpha|$) we get that also in this case $p_{j,l,m}$ is bounded away from zero.

Now it is easy to deduce Theorem 6.1 by exploiting Theorem 6.2.
7. **Normal form.** Fix $N \geq 1$. For a fixed integer $k \geq 3$, we define
\[ \mathcal{I}_k(N) := \{ j \in \mathcal{I}_k : \mu(j) > N \}. \]

**Definition 7.1.** Let $N$ be an integer. We say that a polynomial $Z \in \mathcal{P}_k$ is in $N$-normal form if it is of the form
\[ Z(z) = \sum_{l=3}^{k} \sum_{j \in \mathcal{I}_k(N)} a_j z_j, \]

namely, if $Z$ contains either monomials that depend only of the actions or monomials whose index $j$ satisfies $\mu(j) > N$, i.e. it involves at least three modes with index greater than $N$.

The reason for introducing such a definition of normal form is given by the observation that the vector field of a monomial of the form $z_{j_1} \ldots z_{j_k}$ containing at least three modes with index larger than $N$ induces a flow whose dynamics can be controlled for exponentially long ($N$-dependent) time scales. This will prevent exchanges of energy between low- and high-index modes for such time scales.

In [3] and [5] such monomials were neglected, since the contribution of their vector fields was to be small in Sobolev norms, and this will keep all the modes (almost) invariant. The point is that, even though the contribution of the vector fields of such monomials is not necessarily small in analytic norm, it can be controlled for exponentially long times. This key property was already used by Faou, Grébert and Paturel in [15] and by Faou and Grébert in [14].

Before we state and prove the aforementioned property we just state an elementary lemma.

**Lemma 7.2.** Let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function, and let $y : \mathbb{R} \to \mathbb{R}_+$ be a differentiable function such that
\[ \frac{d}{dt} y(t) \leq 2f(t) \sqrt{y(t)}, \quad \forall t \in \mathbb{R}. \]

Then
\[ \sqrt{y(t)} \leq \sqrt{y(0)} + \int_0^t f(s) ds, \quad \forall t \in \mathbb{R}. \]

For a given $N$ and $z \in \mathcal{L}_\rho$, we set $\mathcal{R}_\rho^N(z) := \sum_{|j| \geq N} e^{\rho |j|} |z_j|$. Observe that if $z \in \mathcal{L}_{\rho+\mu}$, then
\[ \mathcal{R}_\rho^N(z) \leq e^{-N \mu} \| z \|_{\rho+\mu}. \]  

**Proposition 4.** Let $N \in \mathbb{N}_0$, and $k \geq 3$. Let $Z$ be a homogeneous polynomial of degree $k$ in $N$-normal form. Denote by $z(t)$ the real solution of the flow associated to the Hamiltonian $H_0 + Z$. Then
\[ \mathcal{R}_\rho^N(t) \leq \mathcal{R}_\rho^N(0) + 4k^3 \| Z \| \int_0^t \mathcal{R}_\rho^N(z(s))^2 \| z(s) \|_{\rho}^{k-3} ds, \]  

\[ \| z(t) \|_{\rho} \leq \| z(0) \|_{\rho} + 4k^3 \| Z \| \int_0^t \mathcal{R}_\rho^N(z(s))^2 \| z(s) \|_{\rho}^{k-3} ds. \]

**Proof.** Let $a \geq 1$ be fixed, and let $I_a(t) = \xi_a(t) \eta_a(t)$ be the actions associated to the solution of the Hamiltonian $H_0 + Z$. Recalling that $H_0 = H_0(I)$, and writing
\[ \mathcal{J}_\rho^N(z(t)) \leq \mathcal{R}_\rho^N(z(0)) + 4k^3 \|Z\| \int_0^t \left( \sum_{|j_1|,|j_2| \geq N, j_3, \ldots, j_k \in Z} e^{\rho a_1} |z_{j_1} \ldots e^{\rho a_k-1} z_{jk-1}| \right) ds \]

Ordering the multi-indices in such a way that \( a_1 \) and \( a_2 \) are the largest, and recalling that \( z(t) \) is real, we obtain after a summation in \( a > N \)

\[ \mathcal{R}_\rho^N(z(t)) \leq \mathcal{R}_\rho^N(z(0)) + 4k^3 \|Z\| \int_0^t \mathcal{R}_\rho^N(z(s))^2 \|z(s)\|^{k-3} ds. \]

Remark 10. Estimates (42)-(43) will be fundamental for proving Theorem 2.1. Indeed, take \( z(t) \) the solution of the Hamiltonian in \( N \)-normal form with corresponding initial datum \( z_0 \), and assume that \( \|z_0\|_\rho = R \). Hence, since \( \mathcal{R}_\rho^N(z_0) = \mathcal{O}(Re^{-N/\rho}) \), estimates (42)-(43) ensure that \( \mathcal{R}_\rho^N(z(t)) \) remains of order \( \mathcal{O}(Re^{-N/\rho}) \) and that the norm of \( z(t) \) remains of order \( \mathcal{O}(R) \) up to times \( t \) of order \( \mathcal{O}(e^{N/\rho}) \).

Next we exploit the non resonance condition (33) and the definition of normal forms to estimate the solution of a homological equation.

Proposition 5. Assume that the non resonance condition (33) is fulfilled. Let \( N \) be fixed, and let \( Q \) be a homogeneous polynomial of degree \( k \). Then the homological equation

\[ \{\chi, H_0\} - Z = Q \quad (45) \]

admits a polynomial solution \((\chi, Z)\) homogeneous of degree \( k \) such that \( Z \) is \( N \)-normal form, and such that

\[ \|Z\| \leq \|Q\|, \quad \|\chi\| \leq \frac{N^{rk}}{\gamma} \|Q\|. \quad (46) \]

Proof. Assume that \( Q = \sum_{j \in \mathcal{I}_k} Q_j z_j \), we look for \( Z = \sum_{j \in \mathcal{I}_k} Z_j z_j \) and \( \chi = \sum_{j \in \mathcal{I}_k} \chi_j z_j \) such that the homological equation (45) is satisfied. Then Equation (45) can be rewritten in term of polynomial coefficients

\[ i\Omega(j)\chi_j - Z_j = Q_j, \quad j \in \mathcal{I}_k, \]

where \( \Omega(j) \) is defined as in (25). By setting

\[ \begin{cases} 
  Z_j = Q_j \text{ and } \chi_j = 0, & \text{if } j \in \mathcal{N}_k \text{ or } \mu(j) > N, \\
  Z_j = 0 \text{ and } \chi_j = \frac{Q_j}{i\Omega(j)}, & \text{if } j \notin \mathcal{N}_k \text{ or } \mu(j) \leq N,
\end{cases} \]
and by exploiting (33), we can deduce (46).

8. Proof of the main theorem. We now prove Theorem 2.1 by exploiting the results from the previous sections. The argument is inspired by the one in [14].

8.1. Recursive equation. In this subsection we want to construct a canonical transformation $T$ such that the Hamiltonian $(H_0 + N) \circ T$ is in $N$-normal form, up to a small remainder term. We use Lie transform in order to generate the transformation $T$: hence we look for polynomials $\chi = \sum_{k \geq 3} \chi_k$ and $Z = \sum_{k \geq 3} Z_k$ in $N$-normal form and a smooth Hamiltonian $R$ such that $d^\alpha R(0) = 0$ for all $\alpha \in \mathbb{N}^Z$ with $|\alpha| \geq r$, and such that

$$(H_0 + N) \circ \Phi^1_\chi = H_0 + Z + R.$$  (47)

The exponential estimate will be obtained by optimizing the choice of $r$ and $N$.

We recall that if $\chi$ and $K$ are two Hamiltonian, we have that for all $k \geq 0$

$$\frac{d^k}{dt^k} (K \circ \Phi^t_\chi) = \{\chi, \ldots \{\chi, K \ldots \}(\Phi^t_\chi) = (ad^k_\chi K)(\Phi^t_\chi),$$

where $ad_K \chi := \{\chi, K\}$. On the other hand, if $K$ and $L$ are homogeneous polynomial of degree respectively $k$ and $l$, than $\{K, L\}$ is a homogeneous polynomial of degree $k + l - 2$. Thus, by using Taylor formula

$$(H_0 + N) \circ \Phi^1_\chi - (H_0 + N) = \sum_{k=0}^{r-3} \frac{1}{(k+1)!} ad^k_\chi (\{\chi, H_0 + N\}) + O_r, \quad (48)$$

where by “$+ O_r$” we mean “up to a smooth function $R$ satisfying $\partial^\alpha R(0) = 0$ for all $\alpha \in \mathbb{N}^Z$ with $|\alpha| \geq r$”. Now, recall the following relation

$$\left( \sum_{k=0}^{r-3} B_{k} \xi^k \right) \left( \sum_{k=0}^{r-3} \frac{1}{(k+1)!} \xi^k \right) = 1 + O(|\xi|^{r-2}),$$

where $B_k$ are the Bernoulli numbers defined by the expansion of the generating function $e^{\xi} - 1$. We just recall that there exists $K > 0$ such that $|B_k| \leq k! K^k$ for all $k$.

Hence, defining the two differential operators

$$A_r := \sum_{k=0}^{r-3} \frac{1}{(k+1)!} ad^k_\chi,$$

$$B_r := \sum_{k=0}^{r-3} \frac{B_k}{k!} ad^k_\chi,$$

we obtain

$$B_r A_r = id + C_r,$$

where $C_r$ is a differential operator satisfying $C_r O_3 = O_r$. Applying $B_r$ to both sides of Eq. (48) we get

$$\{\chi, H_0 + N\} = B_r (Z - N) + O_r.$$

Plugging the decomposition in homogeneous polynomials of $\chi$, $Z$ and $N$ in the last equation and comparing the terms with the same degree, we have the following recursive equations

$$\{\chi_m, H_0\} - Z_m = Q_m, \quad m = 3, \ldots, r,$$  (49)
where

\[ Q_m = -N_m - \sum_{k=3}^{m-1} \{ \chi_k, N_{m+2-k} \} \]

\[ + \sum_{k=1}^{m-3} \frac{B_k}{k!} \sum_{l_1 + \ldots + l_{k+1} = m+2k \atop 3 \leq l_i \leq m-k} ad_{\chi_{l_1}} \ldots ad_{\chi_{l_k}} (Z_{l_{k+1}} - N_{l_{k+1}}). \]  

(50)

We point out that in (50) the condition \( l_i \leq m-k \) is a consequence of \( l_i \geq 3 \) and \( l_1 + \ldots + l_{k+1} = m + 2k \). Once these recursive equations are solved, we can define the remainder term \( R = (H_0 + N) \circ \Phi_1 - (H_0 + Z) \). By construction we have that \( R \) is analytic on a neighbourhood of the origin in \( L_\rho \), and that \( R = \mathcal{O}_r \). Hence, by Taylor formula

\[ R = \sum_{m \geq r+1} \sum_{k=1}^{m-3} \frac{1}{k!} \sum_{l_1 + \ldots + l_{k+1} = m+2k \atop 3 \leq l_i \leq m-k} ad_{\chi_{l_1}} \ldots ad_{\chi_{l_k}} H_0 \]

\[ + \sum_{m \geq r+1} \sum_{k=0}^{m-3} \frac{1}{k!} \sum_{l_1 + \ldots + l_{k+1} = m+2k \atop 3 \leq l_i + \ldots + l_{k+1} \leq r} ad_{\chi_{l_1}} \ldots ad_{\chi_{l_k}} N_{l_{k+1}}. \]  

(51)

**Lemma 8.1.** Assume that the non-resonance condition (33) is satisfied. Let \( r \) and \( N \) be fixed. For \( m = 3, \ldots, r \) there exists homogeneous polynomials \( \chi_m \) and \( Z_m \) of degree \( m \) that solve (49), with \( Z_m \) in \( N \)-normal form and such that

\[ \| \chi_m \| + \| Z_m \| \leq (Kn^r)^m, \]  

(52)

where \( K \) is a positive constant that does not depend on \( r \) or \( N \).

**Proof.** We define \( \chi_m \) and \( Z_m \) by induction, via Proposition 5. Note that (52) is trivially satisfied for \( m = 3 \), provided that \( K \) is sufficiently large. Estimate (46), combined with (31) and with the classical estimate on the Bernoulli numbers, we have that for all \( m \geq 3 \)

\[ \frac{\gamma}{N^r} \| \chi_m \| + \| Z_m \| \]

\[ \leq \| N_m \| + 2 \sum_{k=3}^{m-1} k(m + 2 - k) \| N_{m+2-k} \| \| \chi_k \| \]

\[ + 2 \sum_{k=1}^{m-3} (Km)^k \sum_{l_1 + \ldots + l_{k+1} = m+2k \atop 3 \leq l_i \leq m-k} \| \chi_{l_1} \| \ldots \| \chi_{l_k} \| \| Z_{l_{k+1}} - N_{l_{k+1}} \|, \]

for some constant \( K > 0 \). Now we set \( \beta_m := m(\| \chi_m \| + \| Z_m \|) \); using the fact that \( \| N_m \| \leq M/R_0^m \) (see the end of Sec. 5), we get

\[ \beta_m \leq \beta_m^{(1)} + \beta_m^{(2)}, \]

\[ \beta_m^{(1)} := (Kn^r)^m m^3 \sum_{k=3}^{m-1} \beta_k, \]
\[
\beta_m^{(2)} := N^m (Km)^{m-1} \sum_{k=1}^{m-3} \sum_{l_1 + \ldots + l_{k+1} = m+2k, 3 \leq l_i \leq m-k} \beta_{l_1} \ldots \beta_{l_k} (\beta_{l_{k+1}} + \|N_{l_{k+1}}\|),
\]

where \( K \) depends on \( M, R_0, \gamma \). We have to prove by recurrence that \( \beta_m \leq (KmN^\gamma)^m \), for \( m \geq 3 \); this is trivially true for \( m = 3 \), by choosing a sufficiently large \( K \). Now assume that \( \beta_j \leq (KjN^\gamma)^{j^2} \) for \( j = 3, \ldots, m-1 \); we get
\[
\beta_m^{(1)} \leq (KN^\gamma)^m m^4 (KmN^\gamma)^{(m-1)^2} \leq (KmN^\gamma)^{m^2-m+1} \leq \frac{1}{2} (KmN^\gamma)^m,
\]
for \( m \geq 4 \), provided that \( K > 2 \). On the other hand, since \( \|N_m\| \leq M/R_0^m \), we can assume that \( \|N_{l_{k+1}}\| \leq \beta_{l_{k+1}} \), hence
\[
\beta_m^{(2)} \leq N^m (Km)^{m-1} \sum_{k=1}^{m-3} \sum_{l_1 + \ldots + l_{k+1} = m+2k, 3 \leq l_i \leq m-k} (KN^\gamma(m-k))^{l_1+\ldots+l_{k+1}}.
\]
Observe that the maximum of \( l_1^2 + \ldots + l_{k+1}^2 \) when \( l_1 + \ldots + l_{k+1} = m+2k \) and \( 3 \leq l_i \leq m-k \) is obtained for \( l_1 = \ldots = l_k = 3, l_{k+1} = m-k \), and that its value at the maximum is \( (m-k)^2 + 9k \). Furthermore, \( \{(l_1, \ldots, l_k) \in \mathbb{Z}^k : l_1 + \ldots + l_{k+1} = m+2k, 3 \leq l_i \leq m-k\} \) is smaller than \( m^{k+1} \). Therefore
\[
\beta_m^{(2)} \leq \frac{1}{2} (KmN^\gamma)^m
\]
for any \( m \geq 4 \), and for a sufficiently large \( K \).

\[\square\]

8.2. Normal form theorem. Given a positive \( R_0 \), we set \( B_\rho(R_0) := \{ z \in \mathcal{L}_\rho : \|z\|_\rho < R_0 \} \).

**Theorem 8.2.** Let \( N \) be analytic on a ball \( B_\rho(R_0) \) for some \( R_0 > 0 \) and \( \rho > 0 \). Fix \( \beta < 1 \) and \( M > 1 \), and assume that the non resonance condition (33) is fulfilled. Then there exist a sufficiently small \( \epsilon_0 > 0 \) and a positive \( \sigma > 0 \) such that the following holds: for all \( \epsilon < \epsilon_0 \) there exist a polynomial \( \chi \), a polynomial \( Z \) in \( |\log \epsilon|^{1+\beta} \)-normal form, and a Hamiltonian \( R \) analytic on \( B_\rho(M\epsilon) \) such that
\[
(H_0 + N) \circ \Phi^1 = H_0 + Z + R,
\]
and such that for all \( z \in B_\rho(M\epsilon) \)
\[
\|X_Z(z)\|_\rho + \|X_\chi(z)\|_\rho \leq 2\epsilon^3/2, \quad (54)
\]
\[
\|X_R(z)\|_\rho \leq \epsilon \epsilon^{-1/2} |\log \epsilon|^{1+\beta}. \quad (55)
\]

**Proof.** Using Lemma 8.1, we can construct for all \( r \) and \( N \) polynomial Hamiltonians
\[
\chi(z) = \sum_{k=3}^{r} \chi_k(z),
\]
\[
Z(z) = \sum_{k=3}^{r} Z_k(z),
\]
with \( Z \) in \( N \)-normal form such that (53) holds with \( R = \mathcal{O}_r \). Now fix \( \epsilon \), and choose
\[
N := N(\epsilon) = |\log \epsilon|^{1+\beta}, \quad r := r(\epsilon) = |\log \epsilon|^{\beta}. \quad (56)
\]
(56) is motivated by the fact that we can control the error induced by $Z$ by Remark 10, while the error induced by $R$ can be estimated by Lemma 8.1. Indeed, in this way we have
\[
\|\chi_k\| \leq (KkN^\tau)^k \leq \exp(k(\tau k(1 + \beta) \log |\log \epsilon| + k \log (Kk)))
\]
\[
k \leq r \leq \exp(k\tau r(1 + \beta) \log |\log \epsilon| + r \log (Kr)))
\]
\[
(56) \leq \exp(k \log \epsilon |(\tau | \log \epsilon|^{\beta-1}(1 + \beta) \log |\log \epsilon| + |\log \epsilon|^{\beta-1} \log (K | \log \epsilon|^\beta)))
\]
\[
\beta < 1 \leq \epsilon - \frac{k}{8},
\]
provided that $\epsilon_0$ is sufficiently small. Hence, by Proposition 1 we get that for all $z \in B_{\rho}(M\epsilon)$
\[
|\chi_k(z)| \leq \epsilon^{-k/8}(M\epsilon)^k = M^k \epsilon^{7k/8},
\]
and we can deduce that
\[
|\chi(z)| \leq \sum_{k \geq 3} M^k \epsilon^{7k/8} \leq \epsilon^{3/2},
\]
provided that $\epsilon_0$ is sufficiently small. Similarly, we have
\[
\|X_{\chi_k}(z)\|_\rho \leq \frac{2k\epsilon^{-k/8}(M\epsilon)^{k-1}}{2kM^{k-1}\epsilon^{7k/8-1}}, \quad 3 \leq k \leq r,
\]
\[
\|X_{\chi}(z)\|_\rho \leq \sum_{k \geq 3} \frac{2kM^{k-1}\epsilon^{7k/8-1}}{\epsilon^{-1}\epsilon^{21/8}} \leq \epsilon^{3/2},
\]
provided that $\epsilon_0$ is sufficiently small. Similar estimates hold also for $Z$, and therefore we can deduce (54).

In order to estimate the remainder, we recall that by (49) $ad_{\chi_{lk}}H_0 = Z_{lk} + Q_{lk}$, therefore by (50) we obtain
\[
\|ad_{\chi_{lk}}H_0\| \leq (KkN^\tau)^k \leq \epsilon^{-k/8}.
\]
Thus, we can exploit (51) and the fact that $\|N_{lk+1}\| \leq M/R^{l+1}$ in order to get
\[
\|X_{\rho}(z)\|_\rho \leq \sum_{m \geq r+1} \sum_{k=0}^{m-3} m(K\tau)^3m \epsilon^{-\frac{m+2k}{8}} \epsilon^{m-1}
\]
\[
\leq \sum_{m \geq r+1} m^2(K\tau)^3m \epsilon^{m/2}
\]
\[
\leq (K\tau)^3 \epsilon^{r/2}
\]
\[
\leq (K\tau)^3 \epsilon^{r/2},
\]
provided that $\epsilon_0$ is sufficiently small.

Now we can prove Theorem 2.1 by applying Theorem 8.2.

Proof of Thm. 2.1. Let $(\psi_0, \bar{\psi}_0) \in A_\rho$, $\|\psi_0, \bar{\psi}_0\|_\rho = R$, and denote by $z(0)$ the corresponding coefficients, which belong by Lemma 3.1 to $L_{\rho/4}^\sigma$, and satisfy
\[
\|z(0)\|_{\rho/4} \leq \frac{K_\rho}{4} R,
\]
where $K_\rho := \frac{2^3}{1-\epsilon^\rho}$. 

Let $z(t)$ be the local solution in $L_{\rho/2}$ of the Hamiltonian system associated to $H_0 + N$. Let $\chi$, $Z$ and $R$ be given by Theorem 8.2 with $M = K_\rho$, and let $y(t) = \Phi^1_\chi(z(t))$. We recall that $\chi(z) = \mathcal{O}(\|z\|^3)$, that $\Phi^1_\chi$ is close to the identity, $\Phi^1_\chi(z) = z + \mathcal{O}(\|z\|^2)$: hence, for sufficiently small $R$ we get

$$\|y(0)\|_{3\rho/4} \leq \frac{K_\rho}{2}R,$$

and in particular

$$\mathcal{R}_\rho^N(y(0)) \leq \frac{K_\rho}{2} Re^{-\frac{\rho}{2}N} \leq \frac{K_\rho}{2} Re^{-\sigma N},$$

where $\sigma = \sigma_\rho \leq \frac{\rho}{4}$. 

Now let $T_R$ be the maximal time $T$ such that

$$\mathcal{R}_\rho^N(y(t)) \leq K_\rho Re^{-\sigma N}, \ |t| \leq T,$$

$$\|y(t)\|_\rho \leq K_\rho R, \ |t| \leq T.$$

By construction we have

$$y(t) = y(0) + \int_0^t X_{H_0 + Z}(y(s))ds + \int_0^t X_R(y(s))ds,$$

hence by using (43) for the second term and (55) for the third term, we obtain that for $|t| < T_R$

$$\mathcal{R}_\rho^N(y(t)) \leq \frac{1}{2} K_\rho Re^{-\sigma N} + 4|t| \sum_{k=3}^r \|Z_k\| k^3(K_\rho R)^{k-1} e^{-\sigma N} + |t| Re^{-\frac{1}{2} \log \epsilon^r} |t| Re^{-\frac{1}{2} \log \epsilon^r} |t| Re^{-\frac{1}{2} \log \epsilon^r}$$

$$\leq \left(\frac{1}{2} + 4|t| \sum_{k=3}^r \|Z_k\| k^3(K_\rho R)^{k-2} e^{-\sigma N} + |t| Re^{-\frac{1}{2} \log \epsilon^r} \right) K_\rho Re^{-\sigma N},$$

(58)

where in the last inequality we have used that $\sigma = \min(1/8, \rho/4)$ and that $N = |\log \epsilon^r|^{1+\beta}$. Using Lemma 8.1, we get

$$\mathcal{R}_\rho^N(y(t)) \leq \left(\frac{1}{2} + K |t| Re^{-\sigma N} \right) K_\rho Re^{-\sigma N},$$

and thus, for sufficiently small $R$,

$$\mathcal{R}_\rho^N(y(t)) \leq K_\rho Re^{-\sigma N}, \ |t| \leq \min(T_R, e^{-\sigma N}),$$

(59)

and similarly

$$\|y(t)\|_\rho \leq K_\rho R, \ |t| \leq \min(T_R, e^{-\sigma N}).$$

(60)

Now, observe that by (59) and (60) we have $T_R \geq e^{-\sigma N}$; in particular, we have

$$\|z(t)\|_\rho \leq 2K_\rho R, \ |t| \leq e^{-\sigma N} = e^{-\epsilon \log \epsilon^{1+\beta}}.$$

Using Lemma 3.1, we obtain (18) by setting $K = \frac{2^3}{(1-e^{-\sigma N})^2}$. 

Estimate (19) is a consequence of Theorem 8.2 and (43): indeed, it just suffices to remark that $z(t)$ is $R^2$-close to $y(t)$, which in turn is almost invariant, since

$$\sum_{k \geq 1} e^{\rho k} |y_k(t)| - |y_k(0)| \leq 4|t| \sum_{k=3}^r \|Z_k\| k^3(K_\rho R)^{k-1} e^{-\sigma N} + |t| Re^{-\frac{1}{2} \log \epsilon^r} |t| Re^{-\frac{1}{2} \log \epsilon^r}.$$
and arguing as in (58) we can finally deduce (19).

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Received May 2017; revised June 2017.

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