Influence of QED Corrections on the Orientation of Chiral Symmetry Breaking in the NJL model

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Abstract

We study QED corrections to chiral symmetry breaking in the Nambu–Jona-Lasinio (NJL) model with two flavors of quarks. In this model, the isospin symmetry is broken by the differences between the current quark masses and the electromagnetic charges of the up and down quarks. To leading order in the \(1/N\) expansion, we calculate the effective potential of the model with one-loop QED corrections at finite temperature. Evaluating the effective potential, we study the influence of the isospin symmetry breaking on the orientation of chiral symmetry breaking. The current quark mass plays an essential role in maintaining the orientation of the chiral symmetry breaking. If the average of the up and down quark masses is small enough, we find a phase in which the pion field has non-vanishing expectation value and dynamical \(CP\) violation takes place.
§1. Introduction

Quantum chromodynamics (QCD) is the first principle in describing the physics of quarks and gluons. The QCD Lagrangian has the global flavor symmetry $SU_L(N_f) \otimes SU_R(N_f)$ for the $N_f$ flavors of massless quarks. In nature, quarks have different current masses for all flavors, and the global flavor symmetry is an approximate symmetry for light quarks. Owing to asymptotic freedom in the QCD interaction, the coupling constant grows at low energy scales, and the hadronic phase is realized. In the hadron phase, the approximate chiral symmetry is dynamically broken.

The broken chiral symmetry is restored in certain extreme environments, e.g. high temperature and high density. This restoration of chiral symmetry is a non-perturbative phenomenon in QCD. The phase structure of QCD has been studied in some low energy effective theories\(^{(1)}-^{3)}\) and through numerical calculations in lattice QCD. From these results, it is conjectured that the chiral symmetry should be restored near the critical temperature, $T \lesssim 200$ MeV. Recently, an experimental study at RHIC found evidence of symmetry restoration from hadronic matter to a state of deconfined partonic matter.\(^{4)}\)

The effect of an electromagnetic field on chiral symmetry breaking is complicated. Low energy effective theories provide possible approaches for studying non-perturbative QCD phenomena in an external electromagnetic field. The NJL model is the simplest model in which the chiral symmetry is broken dynamically. The dynamical origin of symmetry breaking in the NJL model has been studied in the case that there exists an external electromagnetic field.\(^{5)}\) The combined effect of both an external electromagnetic field and gravity has also been investigated in Refs. 6) and 7). Then it was found that the external electromagnetic fields can induce a very wide variety of phases.

Radiative QED corrections play an important role in the physics of quarks and hadrons, even if we consider a system with no external electromagnetic field. The difference between the electromagnetic charges of up and down quarks breaks the $SU(2)$ isospin symmetry. The difference between the masses of up and down quarks also breaks the isospin symmetry. Because of the isospin breaking, it is conjectured that a variety of phases are realized inside quark matter through QED corrections. If the sum of the up and down quark masses, $m_u + m_d$, is small enough, there is a possibility that a pion field will develop a non-vanishing vacuum expectation value.\(^{8)}\)

We assume that the NJL model is a phenomenological low energy effective theory of QCD and that the fundamental QCD Lagrangian is invariant under the $CP$ transformation. Thus, the quark mass should be real and there should be no $\theta$-term. Because of the anomaly, the chiral $U(1)$ transformation, which flips the sign of the quark mass, produces the $\theta$-term in
QCD. This is only a phase convention for quark fields, and such a $\theta$-term does not induce any $CP$ violating phenomena. The situation may be different after dynamical chiral symmetry breaking. The orientation of the vacuum is fixed, and thus the $CP$ symmetry can also be broken. This is an example of the dynamical $CP$ violation proposed by Dashen.\textsuperscript{9) To observe the phase structure of the theory, it seems more convenient to choose a phase convention for the quark fields in which the Lagrangian has no $\theta$-term. There is no degree of freedom that allows us to set the quark mass to a positive value in this convention. For this reason, both positive and negative quark masses should be considered.

In the present paper, we study the influence of the current quark mass and the quantum corrections of the electromagnetic fields on the chiral symmetry breaking in the NJL model. The order parameters of the chiral symmetry breaking are given by the vacuum expectation values of the $\sigma$ and $\pi^a$ fields. These expectation values are found by determining the minimum of the effective potential. First, we calculate the effective potential of the NJL model to leading order in the $1/N$ expansion. Evaluating it, we derive the phase structure of the NJL model. Next, the gauged NJL model is considered. We calculate the effective potential with one-loop QED corrections and evaluate the phase structure of the gauged NJL model. The charged and neutral pion mass difference is investigated in §4. As is well-known, a higher-order correction in the $1/N$ expansion is essential to elucidate the pion mass difference. We introduce it through phenomenological meson kinetic terms. The phase structure is evaluated in the gauged NJL model with additional meson kinetic terms. In §5 we study the temperature effect in the imaginary time formalism. The behavior of the effective potential is evaluated in both the NJL model and the gauged NJL model near the critical temperature, $T \sim 170$ MeV. In the case of small $|m_u + m_d|$, there is a parameter space in which the pion field acquires a non-vanishing vacuum expectation value. In §6, the dynamical origin of $CP$ violation is studied within the NJL model for small $|m_u + m_d|$. Finally, we give some concluding remarks.

\section{NJL model with finite current quark mass}

We start from the Lagrangian density of the two-flavor NJL model with finite current quark mass. It is defined by

$$
\mathcal{L}_m = \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi + \frac{G}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \tau^a \psi)^2 \right],
$$

where $G$ is the coupling constant of the four-fermion interaction, $N$ represents the number of colors, $M$ is a $2 \times 2$ matrix which contains the current quark masses for the up and down quarks, and $\tau^a \ (a = 1, 2, 3)$ are the Pauli matrices of the isospin vector. We assume
a diagonal form for the mass matrix, \( M \equiv \text{diag}(m_u, m_d) \). In our phase convention it cannot be assumed that the mass matrix is positive. In four dimensions this model is not renormalizable, because the coupling constant \( G \) scales with the mass as \((\text{mass})^{-2}\). The model is defined with a regularization parameter and is regarded as a low energy effective theory of QCD.

In the massless limit, \( M \to 0 \), the Lagrangian density (2.1) is invariant under the isospin symmetry \( SU_V(2) \) transformation, \( \psi(x) \to \exp(i\theta^a \tau^a/2)\psi(x) \), and the chiral transformation, \( \psi(x) \to \exp(i\gamma_5\theta^a \tau^a/2)\psi(x) \). In this case, the chiral charge is defined by \( Q^a_5 = \int d^3x \bar{\psi}\gamma_5(\tau^a/2)\psi \). The commutation relations between the chiral charge and fermion bi-linears are given for the composite scalar operator \( \bar{\psi}\psi \) as

\[
[ iQ^a_5, \bar{\psi}\psi(x) ] = \bar{\psi}i\gamma_5\tau^a\psi(x) \tag{2.2}
\]

and for the composite pseudo-scalar operator \( \bar{\psi}i\gamma_5\tau^a\psi \) as

\[
[ iQ^a_5, \bar{\psi}i\gamma_5\tau^b\psi(x) ] = -\delta^{ab}\bar{\psi}\psi(x). \tag{2.3}
\]

If the composite operators \( \bar{\psi}i\gamma_5\tau^a\psi \) and/or \( \bar{\psi}\psi \) develop non-vanishing expectation values, the chiral symmetry is dynamically broken.

In the real world, up and down quarks have small current masses. The mass term is not invariant under the chiral transformation; it breaks the chiral symmetry explicitly. In the case \( m_u \neq m_d \), the isospin symmetry, \( SU_V(2) \), is also broken.

For simplicity, we rewrite the Lagrangian density (2.1) using the auxiliary fields method. First, we introduce the auxiliary scalar field \( \sigma \simeq -(G/N)\bar{\psi}\psi \) and pseudo-scalar fields \( \pi^a \simeq -(G/N)\bar{\psi}i\gamma_5\tau^a\psi \). Thus the Lagrangian density (2.1) is rewritten as

\[
\mathcal{L}_m = \bar{\psi}(i\gamma^\mu \partial_\mu - M - \sigma - i\gamma_5\tau^a\pi^a)\psi - \frac{N}{2G} \left[ \sigma^2 + (\pi^a)^2 \right]. \tag{2.4}
\]

A neutral pion \( \pi^0 \) and charged pion fields \( \pi^\pm \) are defined by

\[
\begin{pmatrix}
\pi^0 \\
\sqrt{2}\pi^+ \\
\sqrt{2}\pi^- \\
-\pi^0
\end{pmatrix}
\equiv \tau^a\pi^a. \tag{2.5}
\]

To study the phase structure of the model, we evaluate the effective potential. The ground state is found by finding the minimum of the effective potential. Here, we start with the following generating functional of Green functions:

\[
Z_m \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\sigma \mathcal{D}\pi \exp \left( i \int d^4x \mathcal{L}_m \right) = \int \mathcal{D}\sigma \mathcal{D}\pi \exp \left[ iN \left\{ \frac{1}{2G} \int d^4x \left[ \sigma^2 + (\pi^a)^2 \right] - i \ln \det(i\gamma^\mu \partial_\mu - M - \sigma - i\gamma_5\tau^a\pi^a) \right\} \right]. \tag{2.6}
\]
Owing to translational invariance, the expectation value of the auxiliary fields should be constant. Furthermore, the quantum corrections from the internal lines of the auxiliary fields disappear in the large \( N \) limit. To leading order in the \( 1/N \) expansion, the path integral over the auxiliary fields in Eq. (2.6) gives only an overall factor. Thus the effective potential is found to be

\[
V_m(\sigma, \pi^a) \equiv -\frac{1}{2iN} \int d^4x \ln Z_m
\]

\[
= \frac{1}{4G} \left[ \sigma^2 + (\pi^a)^2 \right] + \frac{i}{2} \int d^4x \ln \det(i \gamma^\mu \partial_\mu - M - \sigma - i\gamma_5 \tau^a \pi^a). \tag{2.7}
\]

The current quark masses for up and down quarks are much smaller than the \( \pi \) meson scale. For this reason, we evaluate the effective potential up to second order in the current quark mass and apply the expansion

\[
i \ln \det(i \gamma^\mu \partial_\mu - M - \sigma - i\gamma_5 \tau^a \pi^a)
\]

\[
= itr \ln(i \gamma^\mu \partial_\mu - \sigma - i\gamma_5 \tau^a \pi^a) + \sum_{n=1}^{\infty} I_n
\]

\[
= itr \ln(i \gamma^\mu \partial_\mu - \sigma - i\gamma_5 \tau^a \pi^a) + \text{tr}(I_1 + I_2) + O(m^3), \tag{2.8}
\]

where \( \text{tr} \) represents the trace over the flavor, spinor and space-time coordinates, and \( I_n \) is given by

\[
I_n \equiv \frac{1}{in} \left( \frac{M}{i \gamma^\mu \partial_\mu - \sigma - i\gamma_5 \tau^a \pi^a + i \varepsilon} \right)^n. \tag{2.9}
\]

In four dimensions, Eq. (2.8) is divergent. To obtain a finite result, we introduce the three-momentum cutoff \( \Lambda_f \). After some straightforward calculations, a finite expression for the effective potential is obtained:

\[
V_m(\sigma, \pi^a) = \frac{1}{4G} \left[ \sigma^2 + (\pi^a)^2 \right] - \frac{1}{8\pi^2} f(\sigma^2; \Lambda_f^2)
\]

\[
- \frac{1}{4\pi^2} (m_u + m_d) \sigma g(\sigma^2; \Lambda_f^2)
\]

\[
- \frac{1}{8\pi^2} (m_u^2 + m_d^2) g(\sigma^2; \Lambda_f^2)
\]

\[
- \frac{1}{4\pi^2} \left[ (m_u^2 + m_d^2) \sigma^2 + (m_u - m_d)^2 \pi^+ \pi^- \right]
\]

\[
\times \left( \frac{\Lambda_f}{\sqrt{\Lambda_f^2 + \sigma^2}} - \ln \frac{\Lambda_f + \sqrt{\Lambda_f^2 + \sigma^2}}{\sqrt{\sigma^2}} \right), \tag{2.10}
\]

\(^{*}\) We use a three-momentum cutoff to obtain a finite result, employing the same regularization method even at finite temperature.
where we have defined $\sigma'^2 \equiv \sigma^2 + (\pi^a)^2$, and the functions $f(s^2; t^2)$ and $g(s^2; t^2)$ are given by

$$f(s^2; t^2) \equiv (2t^2 + s^2)\sqrt{t^2(s^2 + t^2)} - s^4 \ln \left(\frac{\sqrt{t^2} + \sqrt{s^2 + t^2}}{\sqrt{s^2}}\right), \quad (2.11)$$

$$g(s^2; t^2) \equiv \sqrt{t^2(s^2 + t^2)} - s^2 \ln \left(\frac{\sqrt{t^2} + \sqrt{s^2 + t^2}}{\sqrt{s^2}}\right). \quad (2.12)$$

The effective potential (2.10) is symmetric with respect to the isospin transformation up to terms linear in the current quark mass. Because the isospin breaking contribution appears only at orders higher than $O(m^2)$, it gives only a small effect. For $m_u = -m_d \neq 0$ the terms of $O(m)$ in Eq. (2.10) vanish and the terms of $O(m^2)$ mainly contribute to the orientation of the chiral symmetry breaking. It induces the pion condensation. To determine the contribution from the isospin breaking, we evaluate the behavior of the effective potential (2.10) as the difference between the masses of the up and down quarks varies. In Table I, we list the values of $\sigma$ and $\pi$ at the local minimum of the effective potential in the $\pi^a = 0$ plane, $V_{\text{up}}(\sigma, \pi^a = 0)$, and the $\sigma = 0$ plane, $V_{\text{up}}(\sigma = 0, \pi^a)$, respectively. We also list the values of the effective potential for both points. The vacuum state corresponds to the point with the smallest value of the effective potential. In our model, this state is located in the $\sigma = 0$ or $\pi^a = 0$ plane.

As seen in Table I, the quark mass mainly contributes to the $\sigma$ direction of the effective potential, while the isospin breaking in the quark mass term induces the $\langle \pi^{1,2} \rangle_{\sigma=0, \pi^3=0} - \langle \pi^3 \rangle_{\sigma=0, \pi^{1,2}=0}$ difference. However, the ratio of the depths of the effective potential,

$$r_V \equiv \left| \frac{V(\sigma = 0, \langle \pi^{1,2} \rangle_{\sigma=0, \pi^3=0}, \pi^3 = 0) - V(\sigma = 0, \pi^{1,2} = 0, \langle \pi^3 \rangle_{\sigma=0, \pi^{1,2}=0})}{V(\sigma = 0, \langle \pi^{1,2} \rangle_{\sigma=0, \pi^3=0}, \pi^3 = 0) + V(\sigma = 0, \pi^{1,2} = 0, \langle \pi^3 \rangle_{\sigma=0, \pi^{1,2}=0})} \right|, \quad (2.13)$$

is approximately 0.008% for $m_u = 2.5$ MeV and $m_d = 6.5$ MeV.

We plot the behavior of the effective potential in the $\pi^a = 0$ plane in Fig. 1. It is seen that increasing the total current quark mass of the up and down quarks, $m_u + m_d$, causes
the effective potential to increase for a negative $\sigma$ and decrease for a positive $\sigma$. Thus the global minimum of the effective potential appears at a positive value of $\sigma$.

There is freedom in the choice of the sign of the fermion mass term. In Fig. 2 we plot the behavior of the effective potential for $m_u = -m_d$. In this case, it depends on $\sigma^2 + (\pi^1)^2 + (\pi^2)^2$ and $(\pi^3)^2$. In the figure, we plot the effective potential in the $\pi^3 = 0$ plane and the $\sigma = \pi^{1,2} = 0$ plane. It is seen that in this case, the effective potential exhibits behavior similar to that the sigma axis, $\pi^a = 0$, and the charged pion axis, $\sigma = \pi^3 = 0$. As seen in Fig. 2, the global minimum of the effective potential lies in the $\sigma = \pi^{1,2} = 0$ plane. This implies that the neutral pion field develops a non-vanishing expectation value, $\langle \pi^3 \rangle \neq 0$. It is found that pion condensation takes place for $|m_u + m_d| \lesssim 0.020$ MeV. Here, we consider only the leading-order in the $1/N$ expansion. Higher order corrections can modify the small difference of the lines in Fig. 2.

If $|m_u| + |m_d|$ is fixed at a realistic value ($\sim 9$ MeV) and both $m_u$ and $m_d$ are positive, the global minimum of the effective potential lies in the $\pi^a = 0$ plane. An enhancement of the symmetry breaking in the $\pi^a = 0$ plane is mainly caused by the isospin symmetric terms, which are proportional to $|m_u + m_d|$ in Eq. (2.10). Only $\sigma$ develops a non-vanishing expectation value. For small values of $|m_u + m_d|$, the situation is different. The higher-order terms, specifically, $O(m^2)$, in Eq. (2.10) break the isospin symmetry and yield non-trivial corrections to the effective potential. These corrections can move the global minimum of the effective potential away from the $\pi^a = 0$ plane. If we take $m_u \simeq -m_d$, the neutral pion develops a non-vanishing vacuum expectation value, and the vacuum state breaks the discrete symmetry under the parity transformation, as pointed out in Ref. 8).

In such calculations, the mass scales of the effective potential are determined from the

| $m_u$ (MeV) | $m_d$ (MeV) | $\langle \sigma \rangle_{\pi^a=0}$ (GeV) | $\langle \pi^a \rangle_{\sigma=0}$ (GeV) | $V_m(\langle \sigma \rangle_{\pi^a=0},\pi^a=0)$ (GeV$^4$) | $V_m(\sigma=0,\langle \pi^a \rangle_{\sigma=0})$ (GeV$^4$) |
|------------|------------|-----------------|-----------------|-----------------|-----------------|
| 4.5 | 4.5 | 0.2829 | 0.2719 | $-9.107 \times 10^{-5}$ | $-6.610 \times 10^{-5}$ |
| 2.5 | 6.5 | 0.2829 | 0.2718 | $-9.105 \times 10^{-5}$ | $\pi^{1,2} : -6.608 \times 10^{-5}$, $\pi^3 : -6.609 \times 10^{-5}$ |
| -4.5 | 4.5 | 0.2718 \(\pi^{1,2} = 0.2718, \pi^3 = 0.2719\) | $-6.604 \times 10^{-5}$ | $\pi^{1,2} : -6.604 \times 10^{-5}$, $\pi^3 : -6.610 \times 10^{-5}$ |
| -2.5 | 6.5 | 0.2769 | 0.2718 | $-7.706 \times 10^{-5}$ | $\pi^{1,2} : -6.603 \times 10^{-5}$, $\pi^3 : -6.609 \times 10^{-5}$ |

Table I. Influence of the isospin breaking on the effective potential. The columns labeled $\langle \sigma \rangle_{\pi^a=0}$ and $\langle \pi^a \rangle_{\sigma=0}$ list the local minima of $V_m(\sigma, \pi^a = 0)$ and $V_m(\sigma = 0, \pi^a)$, respectively.
Fig. 2. Behavior of the effective potential (2.10) for \( m_u + m_d = 0 \) MeV in the \( \pi^3 = 0 \) plane (dashed curve) and the \( \sigma = \pi^{1,2} = 0 \) plane (solid curve).

physical observables. For this purpose, we calculate the pion decay constant \( f_\pi \) and the pion mass \( m_\pi \) and fix the three-momentum cutoff \( \Lambda_f \) and the coupling constant \( G \). Here we set the current quark mass as \( m_u + m_d = 9 \) MeV and \( m \equiv (m_u + m_d)/2 \). In this case, the minimum of the effective potential is at \( \sigma = \langle \sigma \rangle \) and \( \pi^a = 0 \). Taking the partial derivative of the generating functional (2.6) with respect to \( \pi \) twice, we obtain the two-point vertex function for the pion field, \( \Gamma^{(2)}_\pi(p) \). At \( \sigma = \langle \sigma \rangle \) and \( \pi = 0 \), it is given by

\[
\Gamma^{(2)}_\pi(p) = -\frac{N}{G} - N \int \frac{d^4k}{i(2\pi)^4} \text{tr} \left[ -i\gamma_5 \tau^a \gamma^\mu k_\mu - M - \sigma \gamma^\nu (k + p)_\nu - M - \sigma \right] - i\gamma_5 \tau^b \\
- \sqrt{p^2 + (\sigma + m)^2 - p^2} \text{atan} \left( \frac{\Lambda_f \sqrt{p^2}}{4(\sigma + m)^2 - p^2} \right) \\
- \left\{ 2(\sigma + m)^2 - p^2 \right\} \ln \left( \frac{\Lambda_f + \sqrt{\Lambda_f^2 + (\sigma + m)^2}}{\sigma + m} \right),
\]

for \( 0 \leq p^2 < 4(\sigma + m)^2 \). (2.14)

The wave function renormalization, \( Z_\pi \), is defined as \( \Gamma^{(2)}_\pi(p) = Z_\pi^{-1} p^2 + O(p^4) \). Evaluating the coefficient of \( p^2 \) in Eq. (2.14), we obtain

\[
Z_\pi^{-1} = \frac{N}{2\pi^2} \left[ \ln \left( \frac{\Lambda_f + \sqrt{\Lambda_f^2 + (\sigma + m)^2}}{\sigma + m} \right) - \frac{\Lambda_f}{\sqrt{\Lambda_f^2 + (\sigma + m)^2}} \right].
\]

The pion decay constant \( f_\pi \) is given by

\[
f_\pi = Z_\pi^{-1/2} \sigma.
\]

We also use the following model-independent relationship, which is known as, the Gell-Mann–Oakes–Renner relation: (11)

\[
f_\pi^2 m_\pi^2 = \frac{N}{G} m \sigma.
\]

(2.16)
Table II. Influence of the isospin breaking on the charged and neutral pion mass.

| \( m_u \) (MeV) | \( m_d \) (MeV) | Min. of \( V_m(\sigma, \pi^a) \) (MeV) | \( m_\sigma \) (MeV) | \( m_{\pi^\pm} \) (MeV) | \( m_{\pi^0} \) (MeV) | \( m_{\pi^\pm} - m_{\pi^0} \) (MeV) |
|-----------------|-----------------|-----------------|----------------|----------------|----------------|----------------|
| 4.5             | 4.5             | \( \langle \sigma \rangle = 282.9 \) | 590.4          | 134.0          | 134.0          | 0              |
| 2.5             | 6.5             | \( \langle \sigma \rangle = 282.9 \) | 590.4          | 134.0          | 133.9          | 0.1            |
| -4.5            | 4.5             | \( \langle \pi^0 \rangle = 271.9 \) | 8.948          | 8.948          | 538.9          | -530           |
| -2.5            | 6.5             | \( \langle \sigma \rangle = 276.9 \) | 562.4          | 88.75          | 88.29          | 0.46           |

Inserting the experimental values \( f_\pi = 91.9 \text{ MeV} \) and \( m_\pi = 135 \text{ MeV} \), which are neutral pion values, and \( m = 4.5 \text{ MeV} \) into Eqs. (2.16) and (2.17) and solving the gap equation, \( dV_m/d\sigma = 0 \), we fix the cutoff scale and the coupling constant to \( \Lambda_f = 0.697 \text{ GeV} \) and \( G = 24.8 \text{ GeV}^{-2} \). These parameter values were determined without QED corrections. However, the QED scale is much smaller than the typical scale for pions. It is also appropriate to use these parameter values in the gauged NJL model considered below.

Using Eqs. (2.10) and (2.16), we can calculate the mass difference for the charged and neutral pions,

\[
m_{\pi^\pm}^2 - m_{\pi^0}^2 \equiv 2N Z_\pi \left[ \frac{\partial^2 V_m(\sigma, \pi^a)}{\partial (\pi^1)^2} - \frac{\partial^2 V_m(\sigma, \pi^a)}{\partial (\pi^3)^2} \right] \mid_{\langle \sigma \rangle, \langle \pi^a \rangle},
\]

where we evaluate the right-hand side of this equation at the minimum of the effective potential, \( V_m(\sigma, \pi^a) \). We list the pion mass and the mass difference in Table II. Due to the SU(2) isospin symmetry, the charged and neutral pions have the same mass for \( m_u = m_d \).

In the case \( m_u = -2.5 \text{ MeV} \) and \( m_d = 6.5 \text{ MeV} \), the charged and neutral pion masses become smaller than in the positive up and down quark mass case. For \( m_u = -4.5 \text{ MeV} \) and \( m_d = 4.5 \text{ MeV} \), the vacuum expectation value is in the \( \pi^3 \) direction. Non-vanishing expectation value is developed for \( \langle \pi^3 \rangle \). Hence the roles of \( \sigma \) and \( \pi^3 \) are exchanged.

Here we use the cutoff and the coupling constant for \( m_u + m_d = 9 \text{ MeV} \). As seen in Table II, the pion masses are quite different from 135 MeV for \( m_u = -2.5 \) and \(-4.5 \text{ MeV} \). In these cases, Eqs. (2.15) and (2.17) are not satisfied with \( m_\pi = 135 \text{ MeV} \) and \( f_\pi = 91.9 \text{ MeV} \). In order to reproduce realistic values for \( m_\pi \) and \( f_\pi \) in the case with negative \( m_u \), we should modify the condition (2.17) and use suitable values of \( \Lambda_f \) and \( G \).

§3. Gauged NJL model

Since the electric charges of the up and down quarks are different, an electromagnetic interaction breaks the SU(2) isospin symmetry. To elucidate the influence of this isospin breaking on the charged and neutral pion mass.
breaking, we introduce the QED correction. The QED interaction causes the derivative in the Lagrangian density (2.1) to be replaced with a covariant one. Thus, the Lagrangian density is extended to that of the gauged NJL model.

\[ \mathcal{L}_f = \bar{\psi} \left[ i \gamma^\mu (\partial_\mu + ieQA_\mu) - M \right] \psi + \frac{G}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \tau^a \psi)^2 \right], \]  

(3.1)

where \( Q \) represents the charge of quark fields, \( Q = \text{diag}(2/3, -1/3) \). In this Lagrangian density, the isospin symmetry is broken, except along the \( \tau^3 \) direction. Since the QED corrections come from the internal photon lines at lowest order, we should also consider the free Lagrangian for photons,

\[ \mathcal{L}_{ph} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2, \]  

(3.2)

and evaluate the theory with the Lagrangian

\[ \mathcal{L} = \mathcal{L}_f + \mathcal{L}_{ph}. \]  

(3.3)

First, we evaluate the path integral for the quark fields. Employing the auxiliary field method applied in the previous section, we introduce the fields \( \sigma \) and \( \pi \) and calculate the generating functional. Then, replacing the derivative in Eq. (2.6) by a covariant one, we obtain

\[ Z = \int \mathcal{D}\sigma \mathcal{D}\pi \mathcal{D}A \exp \left[ i \int d^4x \mathcal{L}_{ph} + iN \left\{ -\frac{1}{2G} \int d^4x \left[ \sigma^2 + (\pi^a)^2 \right] - i \ln \det \left[ i \gamma^\mu (\partial_\mu + ieQA_\mu) - M - \sigma - i\gamma_5 \tau^a \pi^a \right] \right\} \right]. \]  

(3.4)

The second line in Eq. (3.4) is expanded as

\[ i \ln \det \left[ i \gamma^\mu (\partial_\mu + ieQA_\mu) - M - \sigma - i\gamma_5 \tau^a \pi^a \right] = i \text{tr} \ln (i \gamma^\mu \partial_\mu - \sigma - i\gamma_5 \tau^a \pi^a) + \text{tr} \sum_{n=1}^\infty J_n, \]  

(3.5)

where \( J_n \) is given by

\[ J_n \equiv \frac{1}{in} \left( \frac{M + eQ \gamma^\mu A_\mu}{i \gamma^\mu \partial_\mu - \sigma - i\gamma_5 \tau^a \pi^a + i\varepsilon} \right)^n. \]  

(3.6)

Because the electric charge, \( e \), and the current quark mass, \( M \), are sufficiently small, we evaluate Eq. (3.6) up to \( n = 2 \). At this order, the current quark mass and photon dependent

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\(^*) As discussed in Ref. 12, this model has a large anomalous dimension, and the coupling constant is walking (not running). Hence, it may be possible to renormalize the theory even in four dimensions.
for the photon field, \( \Pi \).

Note that there is no term proportional to the momentum integration for the internal fermion lines, we get

\[
\text{tr} J_2^{ph} = \frac{1}{2} \int d^4 x \int \frac{d^4 p}{(2\pi)^4} \frac{d^D k}{(2\pi)^D} \frac{1}{\gamma_\mu k^\mu - \sigma - i\gamma_5 \tau^a \pi^a} e_0 \mu^\epsilon Q \gamma_\mu A^\mu(p)
\]

\[
\times \frac{1}{\gamma_\sigma (k^\sigma + p^\sigma) - \sigma - i\gamma_5 \tau^a \pi^a} e_0 \mu^\epsilon Q \gamma_\nu A^\nu(-p)
\]

\[
= -\frac{1}{2} \int d^4 x \int \frac{d^4 p}{(2\pi)^4} A_\mu(p) A_\nu(-p) \left[ (p^\mu p^\nu - g^\mu\nu p^2) \Pi(p^2)
\right.
\]

\[
+ g^\mu\nu \pi^+ \pi^- \Pi_{\pi\pi}(p^2) \right].
\]

Note that there is no term proportional to \( A(p)^2 \pi^0 \pi^0 \) in Eq. (3.7). The vacuum self-energies for the photon field, \( \Pi(p^2) \) and \( \Pi_{\pi\pi}(p^2) \), are given by the diagrams in Fig. 3. Performing the momentum integration for the internal fermion lines, we get

\[
\Pi(p^2) \equiv \frac{5\alpha_0}{27\pi} \left[ \frac{1}{\epsilon} - \gamma + \ln \frac{4\pi \mu^2}{\sigma^2} + \frac{8}{3} - h^2 - \frac{h}{2}(3 - h^2) \ln \frac{h + 1}{h - 1} \right], \tag{3.8}
\]

\[
\Pi_{\pi\pi}(p^2) \equiv \frac{2\alpha_0}{\pi} \left( \frac{1}{\epsilon} - \gamma + \ln \frac{4\pi \mu^2}{\sigma^2} + 2 - h \ln \frac{h + 1}{h - 1} \right), \tag{3.9}
\]

where the regularization parameter \( \epsilon \) is given by \( \epsilon = (4 - D)/2 \), and we have defined \( e_0 \mu^\epsilon \equiv e, \ \alpha_0 \equiv e_0^2/(4\pi) \) and \( h \equiv \sqrt{1 - 4\sigma^2/(p^2 + i\varepsilon)} \).

These self-energies are divergent in the four-dimensional limit. To obtain a finite result, we must renormalize the photon self-energy. Here we introduce the following counter-terms into Eqs. (3.7)–(3.9):

\[
\Delta L \equiv -\frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} + Z_{\pi\pi}(\partial_\mu + ieA_\mu)\pi^+(\partial^\mu - ieA^\mu)\pi^-,
\]

where the wave function renormalizations \( Z_3 \) and \( Z_{\pi\pi} \) are defined by \( Z_3 - 1 = -N\Pi(0) \) and \( Z_{\pi\pi} = -(N/2)\Pi_{\pi\pi}(0) \). We shift the pion fields as \( \pi^\pm(x) \rightarrow \langle \pi^\pm \rangle + \tilde{\pi}^\pm(x)/\sqrt{N} \) and normalize the photon field as \( A_\mu \rightarrow \sqrt{N/3} \cdot A_\mu \). Taking the large \( N \) limit, i.e. \( 1/N \ll e_0 \), we obtain

\[
i \int d^4 x (L_{ph} + \Delta L) - iN \text{tr} J_2^{ph} \simeq \frac{iN}{6} \int d^4 x A_\mu \left[ (g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) (1 + N\Pi_R(x)) \right]
\]
\[ + \frac{1}{\xi} \partial_\mu \partial^\nu + Ng^{\mu\nu} \pi^+ \pi^- \Pi_{\pi\pi}^R(x) \] \tag{3.11}

where the renormalized self-energies read

\[ \Pi_{\pi\pi}^R(p^2) \equiv \Pi(p^2) - \Pi(0) = \frac{5\alpha}{2\pi} \left[ \frac{8}{3} - h^2 - \frac{h}{2}(3 - h^2) \ln \frac{h + 1}{h - 1} \right] \tag{3.12} \]

and

\[ \Pi_{\pi\pi}^R(p^2) \equiv \Pi_{\pi\pi}(p^2) - \Pi_{\pi\pi}(0) = \frac{2\alpha}{\pi} \left( 2 - h \ln \frac{h + 1}{h - 1} \right). \tag{3.13} \]

Performing the path integral over \( A_\mu \) and the Wick rotation, \( t \to -ix_4 \), we obtain

\[ \int \mathcal{D}A \exp \left[ i \int d^4x (L_{\text{ph}} + \Delta L) - iN \text{tr} J^2 \right] \]

\[ = \exp \left[ iN \frac{6}{\pi} \text{tr} \left\{ \ln(\delta_{\mu\nu} + N\pi^+ \pi^- \Pi_{\pi\pi}^R D_{\mu\nu}) - \ln D_{\mu\nu} \right\} \right], \tag{3.14} \]

where tr represents the trace operation over the space-time coordinates and \( D_{\mu\nu} \) represents the photon propagator

\[ D_{\mu\nu}(k) = \frac{1}{k^2(1 + N\Pi^R)} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \xi \frac{k_\mu k_\nu}{k^4}. \tag{3.15} \]

Substituting Eqs. (3.12) and (3.13) into Eq. (3.14), we obtain the generating functional in the Landau gauge,

\[ Z \simeq Z_m + \int \mathcal{D}\sigma \mathcal{D}\pi \exp \left[ iN \frac{6}{\pi} \int d^4x \int \frac{d^4p}{(2\pi)^4} \right. \]

\[ \times \left\{ 3 \ln \left[ p^2 + 4\sigma^2 \left( 1 + \frac{4\alpha N}{27\pi} \right) - \frac{4\alpha N}{3\pi} \pi^+ \pi^- \right] \right. \]

\[ + \left. \ln \left[ p^2 + 4\sigma^2 \left( 1 + \frac{4\alpha N}{27\pi} \right) \right] - 4\ln(p^2 + 4\sigma^2) \right\}. \tag{3.16} \]

In Eq. (3.16), we omit terms which do not depend on \( \sigma \) and \( \pi \) and apply the following approximation:

\[ h(p^2) \ln \frac{h(p^2)}{h(p^2) - 1} = 2 \left[ 1 + \frac{1}{3h(p^2)^2} + \frac{1}{5h(p^2)^4} + \cdots \right] \text{ for } p^2 \sim 0. \tag{3.17} \]

After the momentum integration, the effective potential is found to be

\[ V(\sigma, \pi^a) = V_m(\sigma, \pi^a) + V_{\text{gau}}(\sigma, \pi^a), \tag{3.18} \]

where \( V_{\text{gau}}(\sigma, \pi^a) \) is given by

\[ V_{\text{gau}}(\sigma, \pi^a) = \frac{1}{192\pi^2} \left[ 3f \left( 4\sigma^2(1 + 4\alpha N/27\pi) - (4\alpha N/3\pi)\pi^+ \pi^- ; \Lambda_p^2 \right) \right. \]

\[ + \left. f \left( 4\sigma^2(1 + 4\alpha N/27\pi); \Lambda_p^2 \right) - 4f(4\sigma^2; \Lambda_p^2) \right]. \tag{3.19} \]
Here, the function $f(s^2, t^2)$ is defined in Eq. (2.11), and $\Lambda_f$ and $\Lambda_p$ are the three-momentum UV cutoffs for the quark and photon momenta, respectively.

To study the QED corrections for the phase structure, we numerically evaluated the effective potential (3.18). In these calculations, we set $\Lambda_f \equiv \Lambda_p$ and chose the average of the current quark mass to be $m = 4.5$ MeV. The behavior of the effective potential is shown in Fig. 4. It is seen that the QED correction slightly suppresses the symmetry breaking for $V(\sigma, \pi^a = 0)$ and $V(\sigma = 0, \pi^{1,2} = 0, \pi^3)$ but enhances it for $V(\sigma = 0, \pi^{1,2}, \pi^3 = 0)$. This result for $\sigma$ and $\pi^3$ is different from those obtained in previous studies.\(^\text{12}\) This result depends on the procedure used to regularize the theory. However, the global minimum of the effective potential lies on the line $\langle \pi^a \rangle = 0$. Only the scalar composite field $\sigma$ acquires a non-vanishing vacuum expectation value, and the orientation of the chiral symmetry breaking remains in the $\sigma$ direction even with the QED correction at the one loop level. Taking the massless limit, $m \to 0$, the effective potential in the $\sigma$ direction coincides with that in the $\pi^3$ direction.

Because of the QED correction, the second derivative of the effective potential in terms of $\pi^{1,2}$ becomes smaller than that of $\pi^3$. Thus, to leading order in the $1/N$ expansion, we obtain a result for the charged and the neutral pion mass difference whose sign is the opposite of the observed value. We consider this further in the next section.

§4. Charged and neutral pion mass

As is well-known, a pion-photon interaction plays an essential role in determining the mass difference between charged and neutral pions.\(^\text{13}\) Such an interaction is not included...
in the NJL model to leading order in the $1/N$ expansion. Here, we phenomenologically introduce higher-order corrections through the kinetic term for mesons,\(^{14}\)

$$
L_s = \frac{1}{2N} \left[ \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi^0)^2 + (\partial_\mu + ieA_\mu)\pi^+(\partial_\mu - ieA_\mu)\pi^- \right],
$$

(4.1)

where we fix $N = 3$, because this term appears at the next-to-leading order in the $1/N$ expansion. The total Lagrangian is given by

$$
L = L_f + L_{ph} + L_s.
$$

(4.2)

The expectation value of the charged pions induces the photon mass $m_\Lambda^2 \equiv e^2\pi^+\pi^-/3$. This seems to suppress the QED correction for the effective potential $V(\sigma = 0, \pi^{1,2}, \pi^3 = 0)$.

To determine it, we calculate the effective potential for the theory with the Lagrangian (4.2). The leading-order corrections to the vacuum polarization amplitude are represented by diagrams in Fig. 5. Only the charged pions contribute to these corrections. As shown in Refs. 15) and 16), the neutral pion mass should be invariant under the QED correction.

After the normalization $A_\mu \to \sqrt{N/3}A_\mu$, we define the effective action by

$$
\int D A \exp \left[ i\int (L_{ph} + L_s + \Delta L) - iN\text{tr}J^p_{ph} \right] = \exp \left\{ iN\Gamma[\sigma, \pi^a] \right\}.
$$

(4.3)

To leading order in the $1/N$ expansion, it reads

$$
\Gamma[\sigma, \pi^a] \simeq \frac{i}{6} \ln \det \left[ (-g^{\mu\nu}\partial^2 + \partial^\mu \partial^\nu)(1 + N\Pi_R) - \frac{1}{\xi}\partial^\mu \partial^\nu \right.
\left. - Ng^{\mu\nu}\pi^+\pi^-\Pi^R_{\pi\pi} - m_\Lambda^2 \left( g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \right) \right].
$$

(4.4)

In the Landau gauge the effective potential is given by

$$
V_{\text{gau}}(\sigma, \pi^a) \equiv -\frac{1}{2} \int d^4x \Gamma[\sigma, \pi^a]
= \frac{1}{12} \int \frac{d^4p}{i(2\pi)^4} \left\{ 3 \ln \left[ 4\sigma^2 \left( 1 + \frac{4\alpha N}{27\pi} \right) - \frac{4\alpha N}{3\pi} \pi^+\pi^- - p^2 \right] \right.
+ \ln \left[ 4\sigma^2 \left( 1 + \frac{4\alpha N}{27\pi} \right) - p^2 \right] - 4\ln(4\sigma^2 - p^2) + 3\ln(m_\Lambda^2 - p^2) \right\}.
$$

(4.5)
Thus the total effective potential can be written

$$V(\sigma, \pi^a) = V_m(\sigma, \pi^a) + V_{\text{gau+pi}}(\sigma, \pi^a),$$  \hspace{1cm} (4.6)$$

where $V_{\text{gau+pi}}(\sigma, \pi^a)$ is given by

$$V_{\text{gau+pi}}(\sigma, \pi^a) = V_{\text{gau}}(\sigma, \pi^a) + \frac{1}{64\pi^2} f(m_{\Lambda}^2; \Lambda_p^2).$$  \hspace{1cm} (4.7)$$

We numerically analyzed the effective potential for $\Lambda_f \equiv \Lambda_p$ at $m = 4.5$ MeV in this case also. The resulting behavior is plotted along the $\sigma$, $\pi_1^1$, and $\pi_3^3$ axes in Fig. 6. Comparing these results with those in Fig. 4, we find that the effective potential along the $\pi_1^1$, $\pi_2^2$ direction around the minimum. Thus the charged pion acquires the larger mass through the pion-photon interaction. Inserting Eq. (4.6) into Eq. (2.18), the mass difference between the charged and neutral pions is found to be

$$m_{\pi^\pm} - m_{\pi^0} \simeq 7.9 \text{ MeV.}$$  \hspace{1cm} (4.8)$$

This value is larger than the observed one. We note that this result depends on the NJL parameters $\Lambda_f$ and $\Lambda_p$. This implies a non-negligible uncertainty in our result. For this reason, if we consider phenomena in which the isospin breaking of the pion mass may contribute, we must include higher-order corrections in the $1/N$ expansion.

§5. Gauged NJL model at finite temperature

It is conjectured that the order parameters of the symmetry breaking $\langle \sigma \rangle$ and $\langle \pi^a \rangle$ disappear at the critical temperature in the chiral limit, $M \to 0$. The broken chiral symmetry is
restored at higher temperature, and a state of deconfined partonic matter is realized. Here we consider a thermal system below the temperature at which the expectation values $\langle \sigma \rangle$ and $\langle \pi^a \rangle$ disappear. Following the standard procedure of the imaginary time formalism, we introduce the temperature into our model. We then obtain the effective potential for the NJL model to leading order in the $1/N$ expansion at finite temperature as

$$V_{\beta m}(\sigma, \pi^a) = \frac{1}{4G} \left[ \sigma^2 + (\pi^a)^2 \right] - \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln \left\{ \left( \omega_n^F \right)^2 + k^2 + \sigma^2 \right\}$$

$$-2 \frac{(m_u + m_d)\sigma}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left( \omega_n^F \right)^2 + k^2 + \sigma^2}$$

$$- \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left( \omega_n^F \right)^2 + k^2 + \sigma^2} \left\{ (m_u^2 + m_d^2) \left( \omega_n^F \right)^2 + k^2 + \sigma^2 \right\}$$

$$-2 \left\{ (m_u^2 + m_d^2)\sigma^2 + (m_u - m_d)^2 \pi^+ \pi^- \right\},$$

(5.1)

where $\beta$ is the inverse temperature, $\beta = 1/T$, and the discrete variable $\omega_n^F$ is given by $\omega_n^F = (\pi/\beta)(2n+1)$, with anti-periodic boundary conditions for fermions. Using the formulae

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left( \omega_n^F \right)^2 + x^2} = \frac{\beta}{2x} \tanh \frac{\beta}{2} x,$$  \hspace{1cm} (5.2)

$$\sum_{n=-\infty}^{\infty} \ln \left\{ \left( \omega_n^F \right)^2 + x^2 \right\} = 2 \ln \cosh \frac{\beta}{2} x + \text{constant},$$  \hspace{1cm} (5.3)

we carry out the summation in Eq. (5.1) and get

$$V_{\beta m}(\sigma, \pi^a) = \frac{1}{4G} \left[ \sigma^2 + (\pi^a)^2 \right] - \frac{2}{\pi^2\beta} \int_{0}^{A_l} dk \ k^2 \ln \cosh \frac{\beta}{2} \sqrt{k^2 + \sigma^2}$$

$$- \frac{1}{2\pi^2} (m_u + m_d)\sigma \int_{0}^{A_l} dk \ \frac{k^2}{\sqrt{k^2 + \sigma^2}} \tanh \frac{\beta}{2} \sqrt{k^2 + \sigma^2}$$

$$- \frac{1}{4\pi^2} (m_u^2 + m_d^2) \int_{0}^{A_l} dk \ \frac{k^2}{\sqrt{k^2 + \sigma^2}} \tanh \frac{\beta}{2} \sqrt{k^2 + \sigma^2}$$

$$+ \frac{1}{4\pi^2} \left\{ (m_u^2 + m_d^2)\sigma^2 + (m_u - m_d)^2 \pi^+ \pi^- \right\}$$

$$\times \int_{0}^{A_l} dk \ \frac{k^2}{\sqrt{k^2 + \sigma^2}} \left[ \frac{1}{\sqrt{k^2 + \sigma^2}} \tanh \frac{\beta}{2} \sqrt{k^2 + \sigma^2}$$

$$- \frac{\beta}{2} \sech^2 \frac{\beta}{2} \sqrt{k^2 + \sigma^2} \right].$$

(5.4)
For $m_u = 2.5$ MeV and $m_d = 6.5$ MeV, the global minimum of the effective potential lies in the $\pi^a = 0$ plane. In Fig. 7 we plot the behavior of the effective potential $V^\beta_m(\sigma, \pi^a = 0)$ in this plane. In the ground state, only the scalar field $\sigma$ develops a non-vanishing expectation value. As seen in Fig. 7, a small mass difference between up and down quarks induces a larger difference for the effective potential near the critical temperature. Since the effective potential in the $\sigma = 0$ plane, $V^\beta_m(\sigma = 0, \pi^a)$, has only a weak dependence on the current quark masses, $m_u$ and $m_d$, $V^\beta_m(\sigma = 0, \pi^a)$ for $m_u + m_d$ on the order of a few MeV has a shape that is similar to that for $m_u + m_d = 0$. In the limit $m_u + m_d \to 0$, isospin symmetry breaking appears in terms of $O(m^2)$. Thus the shape of the effective potential $V^\beta_m(\sigma = 0, \pi^a)$ is almost the same as that of $V^\beta_m(\sigma, \pi^a = 0)$, which is plotted by the red curve in Fig. 7. The difference between the shapes of the effective potentials in the $\pi^a = 0$ plane and the $\sigma = 0$ plane is extremely small. In fact, we cannot distinguish these shapes in Fig. 7.

It should be noted that the isospin symmetry breaking arises from the difference between the masses $m_u = 2.5$ MeV and $m_d = 6.5$ MeV. This causes the small differences between $\langle \pi^{1,2} \rangle_{\sigma=0,\pi^a=0}$ and $\langle \pi^3 \rangle_{\sigma=0,\pi^{1,2}=0}$. The ratio for the local minimum,

$$ r_\pi \equiv \frac{\langle \pi^{1,2} \rangle_{\sigma=0,\pi^a=0} - \langle \pi^3 \rangle_{\sigma=0,\pi^{1,2}=0}}{\langle \pi^{1,2} \rangle_{\sigma=0,\pi^a=0} + \langle \pi^3 \rangle_{\sigma=0,\pi^{1,2}=0}} , $$

is approximately 0.05% for $m_u = 2.5$ MeV and $m_d = 6.5$ MeV at $T = 170$ MeV. The effective potential at the point $\sigma = 0, \pi^{1,2} = 0, \pi^3 = \langle \pi^3 \rangle_{\sigma=0,\pi^{1,2}=0}$ is slightly smaller than that at $\sigma = 0, \pi^{1,2} = \langle \pi^{1,2} \rangle_{\sigma=0,\pi^a=0}$, $\pi^3 = 0$. The ratio for the depth of the effective potential, $r_V$ given in (2.13), is approximately 0.2% at $T = 170$ MeV. In the case $m_u + m_d \lesssim 0.0038$ MeV, we again observe neutral pion condensation.

In the gauged NJL model, the situation is somewhat complicated. Due to the Debye screening photon mass $m_\beta^2 = (5/54)e^2T^2 + O(T)$, the photon propagator is modified at
finite temperature. In the case of electrostatic shielding, \( p_0 \to 0 \), it is expressed as

\[
D_{\mu \nu} = \frac{1}{p^2} P^T_{\mu \nu} + \frac{1}{p^2 + 2m_\beta^2} P^L_{\mu \nu} + \frac{\xi}{p^2} \frac{p_\mu p_\nu}{p^2},
\]

(5.6)

\[
P^T_{44} \equiv 0, \quad P^T_{4j} \equiv 0, \quad P^T_{ij} \equiv \delta_{ij} - \frac{p_i p_j}{p^2},
\]

(5.7)

\[
P^L_{\mu \nu} \equiv \delta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} - P^T_{\mu \nu}.
\]

(5.8)

Replacing the first two terms of Eq. (4.4) with Eq. (5.6), we obtain the QED corrections to the effective potential at finite temperature in the Landau gauge,

\[
V^\beta_{\text{gaug+pi}}(\sigma, \pi^a) = \frac{1}{12\beta} \int_{-\infty}^{\infty} \frac{d^3 p}{(2\pi)^3} \left[ 3 \ln \left( \frac{(\omega_n^B)^2 + p^2}{\frac{4\alpha N}{3\pi}} \right) + \ln \left( \frac{(\omega_n^B)^2 + p^2}{m_\Lambda^2} \right) \right] -3 \ln \left( \frac{(\omega_n^B)^2 + p^2}{\frac{4\alpha N}{3\pi}} \right) + 2 \ln \left( \frac{(\omega_n^B)^2 + p^2}{m_\Lambda^2} \right) + \ln \left( \frac{(\omega_n^B)^2 + p^2}{m_\Lambda^2} \right),
\]

(5.9)

where the square of the four momentum \( p^2 \) is written \( p^2 = (\omega_n^B)^2 + p^2 \), and the discrete variable \( \omega_n^B \) is given by \( \omega_n^B = (\pi/\beta)2n \), from the periodic boundary conditions for bosons.

From the above, the effective potential of the gauged NJL model is found to be

\[
V^\beta(\sigma, \pi^a) = V^\beta_m(\sigma, \pi^a) + V^\beta_{\text{gaug+pi}}(\sigma, \pi^a).
\]

(5.10)

The photon dependent part \( V^\beta_{\text{gaug+pi}}(\sigma, \pi^a) \) is given by

\[
V^\beta_{\text{gaug+pi}}(\sigma, \pi^a) = \frac{1}{12\pi^2\beta} \int_0^{\Lambda_0} dp \ p^2 \left[ 3 \ln \left( \frac{\beta}{2} \sqrt{p^2 + 4\sigma^2 - \frac{4\alpha N}{3\pi}} \right) + \ln \left( \frac{\beta}{2} \sqrt{p^2 + m_\Lambda^2} \right) \right] -3 \ln \left( \frac{\beta}{2} \sqrt{p^2 + 4\sigma^2} \right) + 2 \ln \left( \frac{\beta}{2} \sqrt{p^2 + m_\Lambda^2} \right) + \ln \left( \frac{\beta}{2} \sqrt{p^2 + m_\Lambda^2} \right),
\]

(5.11)

where we have used the formula

\[
\sum_{n=-\infty}^{\infty} \ln \left( \frac{(\omega_n^B)^2 + x^2}{\frac{4\alpha N}{3\pi}} \right) = 2 \ln \left( \frac{\beta}{2} \right) x + \text{[constant]}
\]

(5.12)

We numerically calculated Eq. (5.10) with Eqs. (5.4) and (5.11) just below the critical temperature. In Fig. 8, we restrict the parameter space to the region in which two of the composite fields \( \sigma, \pi^1, \pi^2 \) and \( \pi^3 \) vanish and plot the effective potential (5.10) for \( T = 170 \) MeV. For \( m_u + m_d = 9 \) MeV, we find the global minimum of the effective potential at \( \pi^1 = \pi^2 = 0 \). Only the composite field \( \sigma \) condenses even at finite temperature. The orientation of the chiral symmetry breaking is again preserved under the QED correction.
Fig. 8. Behavior of the effective potential \[ V(\sigma, \pi, a) \] for \( m_u + m_d = 9 \) MeV at \( T = 170 \) MeV. The black curve represents \( V^\beta(\sigma, a = 0) \), and the red and blue curves represent \( V^\beta(\sigma = 0, \pi^1, \pi^2 = 0) \) and \( V^\beta(\sigma = 0, \pi^1, \pi^2 = 0, \pi^3) \), respectively.

Fig. 9. Behavior of the global and local minima of the effective potential \[ V(\sigma, \pi, a) \] for \( m_u + m_d = 9 \) MeV. The black curve represents the location of the global minimum, \( \langle \sigma \rangle a = 0 \). The red and blue curves represent the local minima \( \langle \pi^1, \pi^2 \rangle a = 0 \) and \( \langle \pi^3 \rangle a = 0 \), respectively.

In Fig. 9 we plot the minimum of the effective potential in the restricted parameter space, \( V^\beta(\sigma, a = 0) \), \( V^\beta(\sigma = 0, \pi^1, \pi^2 = 0) \) and \( V^\beta(\sigma = 0, \pi^1, \pi^2 = 0, \pi^3) \). As is clearly seen in the figure, the expectation value \( \langle \sigma \rangle \) disappears smoothly at higher temperature. Thus, the transition from the broken phase to the symmetric phase is crossover. This is a well-known feature of the theory with a current quark mass. It should be noted that the pion fields have vanishing expectation values in the ground state for \( m_u + m_d = 9 \) MeV. It is conjectured that the chiral symmetry is restored in the early universe. Non-vanishing expectation values of the pion fields \( \langle \pi^1, \pi^2 \rangle a = 0 \) and \( \langle \pi^3 \rangle a = 0 \) may be realized during the transition period of the chiral symmetry breaking, i.e. “disoriented chiral condensation”.

Near the critical temperature we observe that the ratio \( r_V \), which is defined in Eq. (2.13), is much larger than in the \( T = 0 \) case. It is believed that the influence of the isospin breaking may be observed in phenomena near the critical temperature.

Inserting Eq. (5.10) into Eq. (2.18) and using Eq. (2.15), we evaluate the mass for the \( \pi \) meson or a soft mode which corresponds to \( \pi \). In Fig. 10 we plot the behavior of the
pion and sigma masses and the pion mass difference as functions of temperature. As seen in Fig. 10(a), the pion and sigma masses exhibit behavior similar to that found in the previous studies Refs. 3) and 23). A minimum of the sigma mass is found near the temperature $T = 200$ MeV. We call this the critical temperature $T_C$. Above $T_C$, a pion can exist as a soft mode. The QED correction enhances the charged pion mass, even at finite temperature. The pion mass difference has a maximum value below the critical temperature.

§6. Pion condensation and dynamical $CP$ violation

In the present section, we consider the more precise structure of the pion condensation and discuss the dynamical $CP$ violation induced in the four-fermion interaction model.

The QCD Lagrangian is invariant under the $CP$ transformation except for quark mixing terms. Here we consider the NJL model (2.1) as a low energy effective theory of QCD. This model does not contain any quark mixing terms. Thus the Lagrangian density (2.1) is $CP$-invariant. Because a pseudo-scalar operator is $CP$ odd, the expectation value of the pion field transforms under $CP$ as

$$\langle \pi^a \rangle \simeq - \frac{G}{N} \bar{\psi} i \gamma_5 \tau^a \psi \rightarrow - \langle \pi^a \rangle \simeq \frac{G}{N} \bar{\psi} i \gamma_5 \tau^a \psi.$$

(6.1)

Thus, the non-vanishing expectation value of the pion field spontaneously violates the CP invariance. Such a mechanism for violating the $CP$ invariance was suggested long ago by Dashen. A simple four-fermion model to realize such a mechanism was proposed in other contexts in Refs. 25) and 26).

As shown in the previous sections, pion condensation is observed for a vanishing average of the current quark mass, $m_u + m_d = 0$. In Table III we list the values of the effective potential at the minima on the $\sigma$, $\pi^{1,2}$ and $\pi^3$ axes for $m_u + m_d = 0$ at $T = 0$ and $T = 170$
MeV. The global minimum lies on the $\langle \pi^3 \rangle$ axis. The temperature enhances the ratio $r_a$ defined by

$$r_a \equiv \frac{V(\langle \sigma \rangle = 0, \langle \pi^a \rangle) - V(\langle \sigma \rangle, \langle \pi^a \rangle = 0)}{V(\langle \sigma \rangle = 0, \langle \pi^a \rangle) + V(\langle \sigma \rangle, \langle \pi^a \rangle = 0)}.$$  (6.2)

Hence we see that a non-trivial $CP$ violating phase is realized for the NJL model.

We also evaluated the effective potential for small but finite values of $|m_u + m_d| (= 2m)$ and obtained the phase structure of the chiral symmetry breaking in the $m-T$ plane. As seen in Fig. 11 the $CP$ violating phase, $\langle \pi^3 \rangle \neq 0$, appears within a narrow parameter range. In Fig. 11 we cannot observe the restoration of the broken chiral symmetry, except for $m_u + m_d = 0$. The expectation value $\langle \sigma \rangle$ disappears smoothly at higher temperature, e.g. in the case depicted in Fig. 9.

Because our original Lagrangian has no $\theta$-term, the dynamical $CP$ violation takes place in the NJL model with two flavors of light quarks. The $CP$ violating phase is observed for $|m_u + m_d| \lesssim 0.020$ MeV at $T = 0$. Below, we consider two assumptions concerning the current quark mass.

With the assumption $|m_u + m_d| \lesssim 0.020$ MeV and $|m_u| + |m_d| \approx 0.020$ MeV, the up and down quark masses have opposite sign. The sign of the quark mass is reversed by the discrete chiral transformation $u \rightarrow \gamma_5 u$. Thus, the theory with a mass term, $|m_u + m_d| \gtrsim 0.020$ MeV, is obtained through the chiral transformation. Under this transformation the expectation value of the pion field vanishes, and therefore the ground state preserves the $CP$ invariance.

Table III. The depth of the effective potential and the ratio $r_a$ for $m_u + m_d = 0$.

| $T$ (MeV) | $V(\langle \sigma \rangle, 0, 0)$ (GeV$^4$) | $V(0, \langle \pi^{1,2} \rangle, 0)$ (GeV$^4$) | $V(0, 0, \langle \pi^3 \rangle)$ (GeV$^4$) | $r_{1,2}$ | $r_3$ |
|----------|--------------------------|--------------------------|--------------------------|---------|------|
| 0        | $-6.60 \times 10^{-5}$   | $-6.60 \times 10^{-5}$   | $-6.61 \times 10^{-5}$   | 0       | $4.24 \times 10^{-4}$ |
| 170      | $-1.30 \times 10^{-7}$   | $-1.30 \times 10^{-7}$   | $-1.33 \times 10^{-7}$   | 0       | 0.00943 |

Fig. 11. Phase structure in the $m$-$T$ plane.
However, the discrete chiral transformation is anomalous. It produces the CP violating $\theta$-term with the angle $\pi$ in QCD. This is known as a dynamical mechanism violating CP invariance.\textsuperscript{9)} This term is inconsistent with the experimental upper bound for the neutron electric dipole moment. We thus conclude that another contribution to the neutron electric dipole moment, for example contributions from heavier particles, should be introduced.

With the other assumption, $|m_u + m_d| \lesssim 0.020$ MeV and $0 < |m_u| + |m_d| \lesssim 0.020$ MeV, we cannot obtain the Lagrangian with $|m_u + m_d| \gtrsim 0.020$ MeV through any redefinition of the quark fields. In particular, if either the up or down quark is massless, any $\theta$-term can be rotated away. This case has been investigated in Refs. 27) and 8). With the assumption $|m_u| + |m_d| \lesssim 0.020$ MeV, the model is entirely incapable of accounting for the real meson properties. But, nevertheless, it is interesting to suggest the possibility that dynamical CP violation occurs at low temperature without introducing the strong CP problem.

\section*{7. Conclusion}

We have investigated the symmetry properties of the NJL model with two flavors of quarks. The explicit breaking of the global flavor symmetry $SU_L(2) \otimes SU_R(2)$ was introduced from the current quark mass and the QED interaction. We have considered the explicit breaking of this symmetry up to $O(m^2)$ and $O(e^2)$ and calculated the effective potential to leading order in the $1/N$ expansion.

Evaluating the effective potential, we have determined the precise structure of the chiral symmetry breaking. The effective potential has terms proportional to the average of the current quark mass at $O(m)$. We found that a positive quark mass enhances the chiral symmetry breaking for a positive $\sigma$ and suppresses it for a negative $\sigma$. A negative quark mass has the opposite effect. For a realistic quark mass, $m_u + m_d \simeq 9$ MeV, it was found that the ground state is characterized by $\pi^a = 0$. Only the scalar composite state $\sigma$ develops a non-vanishing expectation value. This situation does not change when we introduce the one-loop QED corrections and the finite temperature corrections. Therefore we conclude that the orientation of the chiral symmetry breaking is invariant for a sufficiently large magnitude of the average of the quark mass, $|m_u + m_d| \gtrsim 0.020$ MeV.

The difference between the current quark masses of the up and down quarks begins to affect the effective potential at second order, $O(|m_u - m_d|^2)$. It contributes to the $\pi^3$ direction and causes isospin breaking for the effective potential, but the ratio of $V(\langle \pi^{1,2} \rangle)$ to $V(\langle \pi^3 \rangle)$ is extremely small, 0.008% at $T = 0$ and 0.2% for $m_u = 2.5$ MeV and $m_d = 6.5$ MeV at $T = 170$ MeV.

These effects cannot change the ground state, but there is a possibility that the state
passes through a local minimum of the effective potential at the time of the symmetry breaking in the early universe and the pion fields thus temporarily have a non-vanishing expectation value. Highly excited states are produced near the critical temperature; these excited states may move from the ground state with positive $\langle \sigma \rangle$ to a finite $\langle \pi \rangle$ state. This phenomenon is called “disoriented chiral condensation”.21), 22)

To explain the charged and neutral pion mass difference, we have to consider higher-order contributions in the $1/N$ expansion. Here we phenomenologically introduced a pion kinetic term. Then we obtained the correct sign for $m_{\pi^\pm} - m_{\pi^0}$. The value of $m_{\pi^\pm} - m_{\pi^0}$ increases near the critical temperature. The neutral pion direction is more stable than the charged pion direction, even at finite temperature. It is conjectured that the production number of neutral pions is larger than that of charged pions when the QCD ground state passes from the quark gluon state to the hadron state.

If we assume that the average of the quark mass is sufficiently small, the enhancement of the chiral symmetry breaking in the $\sigma$ direction will not be large. We cannot ignore the isospin breaking terms that come from the mass difference even in the ground state. Neutral pion condensation can be realized in a restricted parameter range. We can consider the case in which only $\sigma$ develops a non-vanishing expectation value through the chiral $SU(2)$ transformation. Then, the only difference is found in the $\theta$-term. Thus the pion condensation studied here affects only phenomena in which the $U_A(1)$ anomaly contributes. Such pion condensation may not be realized in the present universe, but it would be interesting to study the dynamical origin of $CP$ violation and the critical phenomena in the early universe. Especially for $|m_u| + |m_d| \lesssim 0.020$ MeV, we find the possibility of dynamical $CP$ violation.

There is a possibility that the composite operator $\pi^+\pi^-$ also develops a non-vanishing expectation value. In the leading-order analysis of the $1/N$ expansion we cannot evaluate the condensation $\langle \pi^+\pi^- \rangle$. Thus, an extension of the model or the analysis is necessary to elucidate such a phenomenon caused by the pion dynamics.28)

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