THE LAW OF LARGE NUMBERS FOR THE MAXIMUM OF
ALMOST GAUSSIAN LOG-CORRELATED FIELDS COMING
FROM RANDOM MATRICES

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Abstract. We compute the leading asymptotics as \( N \to \infty \) of the
maximum of the field \( Q_N(q) = \log \det |q - A_N| \), \( q \in \mathbb{C} \), for any
unitarily invariant Hermitian random matrix \( A_N \) associated to a
non-critical real-analytic potential. Hence, we verify the leading or-
der in a conjecture of [FS16] formulated for the GUE. The method
relies on a classical upper-bound and a more sophisticated lower-
bound based on a variant of the second-moment method which
exploits the hyperbolic branching structure of the field \( Q_N(q) \),
\( q \in \mathbb{H} \). Specifically, we compare \( Q_N \) to an idealized Gaussian
field by means of exponential moments. In principle, this method
could also be applied to random fields coming from other point pro-
cesses provided that one can compute certain mixed exponential
moments. For unitarily invariant ensembles, we show that these
assumptions follow from the Fyodorov-Strahov formula [FS03] and
asymptotics of orthogonal polynomials derived in [DKMVZ99].

1. Introduction

We consider the following general problem, applicable to the study of the max-
imum of the log-modulus of the determinant of a random matrix with real eigen-
values. Suppose that \( \rho_N \) is the empirical measure of a random collection of \( N \) real
points \( \{\lambda_i\}^N_{i=1} \), i.e.
\[
\rho_N(\lambda) = \sum_{i=1}^N \delta(\lambda - \lambda_i).
\]
We will work under the assumption that \( N^{-1} \rho_N \) is converging to a compactly sup-
ported, deterministic probability measure with compact density, which we denote
by \( \rho \). Without loss of generality, we will assume the support of \( \rho \) is contained in
We will not assume, however, that the limiting measure has a connected support. In terms of these objects, we define $Q_N : \mathbb{H}_\pm \to \mathbb{R}$ as the log-potential of the measure $Nq - \varrho_N$, that is for $q \in \mathbb{H}_\pm$,

$$Q_N(q) = \int_{\mathbb{R}} \log |q - x| \varrho_N(dx) - N \int_{\mathbb{R}} \log |q - x| \varrho(dx).$$

As is known for many classes of random matrices $A_N$, the process $Q_N(q)$ coming from the eigenvalues of $A_N$ satisfies a central limit theorem for fixed $q \in \mathbb{H}_\pm$ (see e.g. [Joh98; BPS95; BS04]), as $N \to \infty$. This gives rise to an explicit, centered Gaussian field $\Lambda : \mathbb{H}_\pm \to \mathbb{R}$. For example, as a consequence of [Joh98], for a wide class of one-cut unitarily invariant random matrices,

$$Q_N(q) \Rightarrow_{N \to \infty} \Lambda(q).$$

The one-cut assumption, meaning that the support of the equilibrium density $\varrho$ is connected, is necessary to get the limiting Gaussian behavior. A similar statement holds for general $\beta$-ensembles as well, but the limiting field is not centered if $\beta \neq 2$.

For multi-cut ensembles, the limiting distribution is no longer Gaussian [She13, Theorem 2] (see also [BG13]).

This field $\Lambda(q)$ is a natural example of a log-correlated Gaussian field, many of whose properties are well understood: in particular, there is work on the geometry of thick points [Dav06; HMP10] in specific cases (which should be expected to generalize naturally) and work on the law of the maximum in great generality [DRZ15]. There is also work on the convergence of exponentials of non-Gaussian log-correlated fields and their convergence to Gaussian multiplicative chaoses [SW16; Web13].

This article is philosophically concerned with determining to what extent predictions about the maximum of such a Gaussian log-correlated field also hold for $Q_N$.

Along this line, [FS16] (see also [FHK12]) have made a prediction for the maximum of the log-determinant of a Gaussian Unitary Ensemble (GUE) matrix, based on a hypothetical analytic continuation of the Selberg integral.

**Conjecture 1.1.** Suppose $A_N$ is the Gaussian Unitary Ensemble. Let

$$M_N = \max_{x \in [-1, 1]} \{\log |\det(x - A_N)| - \mathbb{E}(\log |\det(x - A_N)|)\}.$$ 

Then as $N \to \infty$

$$M_N - \log N + \frac{3}{4} \log \log N \Rightarrow y,$$

where $y$ has an explicit distribution (see [FS16] for more details).

In effect, in line with what is seen for log-correlated Gaussian fields, the maximum grows like $\log N - \frac{3}{4} \log \log N$ plus a fluctuating term. This fluctuating term depends strongly on the details of the model, although some features of it are universal.

There is a parallel story to this which has been much more fully developed for the log-potential of the CUE and its $\beta$-analogue the C$\beta$E. There, the leading order term was first proven to hold by [ABB16] for the CUE, the second order term by [PZ16] and most recently the tightness of the recentered maximum for the C$\beta$E by [CNM16].
Remark 1.2. It is also possible to consider the imaginary part of the characteristic polynomial, defined as

\[ \Im \log p_N(x) = \sum_{i=1}^{N} \Im \log(x - \lambda_i), \]

where for each summand we take the principal branch of the logarithm. The same predictions as for the real part of the logarithm should hold for \( \Im \log p_N(x) \), appropriately adapted. We do not consider this however, as part of our method is limited to the real part of the logarithm (specifically our reliance on the Fyodorov-Strahov formula, \[16\]).

Our results. We show that the first term in the conjectured expansion \[1.1\] holds. Moreover, we show that this holds uniformly over a large class of random matrix ensembles with analytic potentials on \( \mathbb{R} \). We will work under very few assumptions on the potential \( V \) (see \[DKMVZ99\] for background). Specifically, we assume the potential is real-analytic and regular. Under the first assumption, the measure \( \rho \) is supported on finitely many intervals and it has a bounded density. The second assumption implies that the density of \( \rho \) vanishes like a square-root at the edges of these intervals and is strictly positive in the interior of the support.

Theorem 1.3. Let \( V \) be a regular, real analytic potential. Then \( M_N^2/\log N \to 1 \) in probability as \( N \to \infty \). The lower bound holds without the requirement that \( V \) is regular.

We go to great lengths to show that, in some sense, the needed random matrix asymptotics already exist in the literature. The hypotheses on the class of random matrices involved is effectively only limited by the availability of orthogonal polynomial asymptotics. In our setting, we rely on \[DKMVZ99\], which establishes the orthogonal polynomial asymptotics for varying weights on the real line by the Riemann-Hilbert steepest descent method. Further, in some sense, the true assumptions needed are substantially weaker than these full asymptotics. (Moreover, we only assume that \( V \) is regular to get a less technical proof of the upper-bound. We believe that it is not necessary and could be removed using the asymptotics of \[DKMVZ99\] near the singular points.)

General theory. To prove Theorem \[1.3\] we develop an abstract machinery suited for controlling these almost-Gaussian log-correlated fields, which in principle could be applied outside random matrix theory.

Theorem \[1.3\] consists of an upper bound and a lower bound. The upper bound, in a sense, is much easier. In effect, using that \( Q_N \) is harmonic off the real line and arises from an \( N \)-point measure, the problem of controlling \( Q_N \) from above can be reduced to estimating the Laplace transform \( \mathbb{E} e^{2Q_N(q)} \) for \( q \) near the real line. See Section \[4\] for further details.

The more complicated task is to develop a lower bound for \( Q_N \). This is where we nontrivially use the limiting log-correlated structure of the field. We begin by introducing a Gaussian harmonic function on the unit disk. Let \( G \) be the centered Gaussian with covariance

\[ \mathbb{E} [G(z)G(w)] = -\frac{1}{2} \log |1 - z\overline{w}|. \]
This appears as the limiting field for the log-determinant of a Haar–unitary matrix, see \[\text{[PZ10]}\]. The field \(G\) is conformally invariant. Specifically, recall the hyperbolic disk automorphism that for any \(y \in \mathbb{D}\) is given by the map

\[
T_y : \mathbb{D} \to \mathbb{D}, \quad z \mapsto \frac{z - y}{1 - zy},
\]

which is an isometry of the Poincaré disk taking \(y\) to 0. Then for any \(y \in \mathbb{D}\), \(z \mapsto G(T_y(z)) - G(y)\) has the same distribution as \(G\). This leads its structure to naturally be described in terms of hyperbolic geometry. Let \(d_\mathbb{D}\) be the hyperbolic metric on \(\mathbb{D}\). For any point \(z \in \mathbb{D}\), the distance of \(z\) to 0 this can be given by

\[
d_\mathbb{D}(0, z) = \log \left( \frac{1 + |z|}{1 - |z|} \right).
\]

For two arbitrary points \(y, z \in \mathbb{D}\), we can then write \(d_\mathbb{D}(y, z) = d_\mathbb{D}(0, T_y(z))\). A short calculation shows that the covariance structure of \(G\) can alternatively be expressed by

\[
\mathbb{E}[G(z)G(y)] = \frac{1}{2} \log \left( \frac{\cosh(d_\mathbb{D}(0,y)2^{-1}) \cosh(d_\mathbb{D}(z,0)2^{-1})}{\cosh(d_\mathbb{D}(z,y)2^{-1})} \right).
\]

One advantage of this expression is that the function \(x \mapsto \log(cosh(x))\) is uniformly Lipschitz. Hence, for example, the correlation of an increment with any point in the field can be controlled solely in terms of the length of the increment:

\[
\sup_{z,y,x \in \mathbb{D}} \left| \frac{\mathbb{E}[(G(z) - G(y))G(x)]}{d_\mathbb{D}(z,y)} \right| < \infty.
\]

The hyperbolic nature of the field \(G\) leads it to have a natural connection to branching random walk. Let \(\{\zeta_i\}_{0}^\infty\) be points on the positive real axis with \(\zeta_0 = 0\) and \(d_\mathbb{D}(\zeta_i, \zeta_j) = |i - j|\). For \(\theta \in \mathbb{R}\), we wish to estimate the distance \(d_\mathbb{D}(\zeta_i, e^{i\theta}\zeta_j)\). The following lemma, taken from \[\text{[PZ16]}\], exposes the branching structure of the distances and the consequential branching random walk comparison that is possible for the covariances of \(G\).

**Lemma 1.4.** Uniformly in \(h, j \in \mathbb{N}\) and \(\theta \in [-\pi, \pi]\)

\[
d_\mathbb{D}(\zeta_h, e^{i\theta}\zeta_j) = h + j - 2 \min\{ -\log|\sin \frac{\theta}{2}|, h, j\} + O(1).
\]

When \(k = \min\{h, j\} > -\log|\sin \frac{\theta}{2}|\) the error term can be estimated by \(Ce^{-k}|\theta|^{-1}\) for some sufficiently large absolute constant \(C > 0\). For the covariances of \(G\), it follows that

\[
\mathbb{E}G(\zeta_h)G(e^{i\theta}\zeta_j) = \frac{1}{2} \min\{ -\log|\sin \frac{\theta}{2}|, h, j\} - \frac{\log 2}{2} + O(1),
\]

where again the error term can be estimated by \(C\min\{e^{-k}\theta^{-1}, 1\}\).

**Proof.** For a hyperbolic triangle with side lengths \(a, b, c\) with \(\theta\) the angle opposite \(a\), the hyperbolic law of cosines says that

\[
\cosh a = \frac{\cosh(b + c)}{2}(1 - \cos \theta) + \frac{\cosh(b - c)}{2}(1 + \cos \theta).
\]

We apply this with \(a = d_\mathbb{D}(\zeta_h, e^{i\theta}\zeta_j)\), \(b = h\) and \(c = j\). The remainder is a straightforward case-by-case analysis, noting that when \(k = \min\{h, j\} > -\log|\sin \frac{\theta}{2}|\), the first term dominates, and otherwise the second term dominates. Using (4), this estimate can be transferred to the covariances, since for \(x \geq 0\),

\[
\log(cosh(\frac{x}{2})) = \frac{x}{2} - \log 2 + O(e^{-x}).
\]
Connecting $Q_N$ to $G$. The point of introducing the field $G$ is that $Q_N$, at least in the neighborhood of a point in the bulk of the spectrum, is well approximated in law by $G$. In the case that $V$ is one-cut regular, the connection is particularly strong (because of the CLT (1)). If the equilibrium density $\rho$ is scaled so that its support is exactly $[-1, 1]$ and we pull-back the field $Q_N$ in the unit disk using the Joukowsky map

$$J(z) = \frac{z + z^{-1}}{2},$$

then $Q_N \circ J$ will converge as a subharmonic function on $\mathbb{C} \setminus [-1, 1]$ in the local uniform topology to a Gaussian harmonic function $T = \Lambda \circ J$ on $\mathbb{D}$. By an explicit calculation, it can be checked that this field has covariance

$$E[T(z)T(w)] = -\frac{1}{2} \log |1 - zw| - \frac{1}{2} \log |1 - z\overline{w}|.$$

This allows $T$ to be alternately expressed as $z \mapsto (G(z) + G(\overline{z}))/\sqrt{2}$. Moreover, one can see that in any small neighborhood of the boundary of the disk away from the edge points 1 or $-1$ (this points are the only fixed point of the Joukowsky transform), the contribution of $-\frac{1}{2} \log |1 - zw|$ will be uniformly bounded. In effect, the covariance structure is approximately that of $G$ itself.

In the multicut situation, the global Gaussian convergence of $Q_N$ is no longer true, but there is still a type of local Gaussian convergence: it is still possible to compare the fluctuations of $Q_N \circ J$ to $G$ in the neighborhood of a point in the bulk of the spectrum. This leads us to consider a class of Gaussian fields that are locally like $G$.

**Definition 1.5.** Say a centered Gaussian field $W$ is BRW-like in a set $U \subset \mathbb{D}$ if

(a) For any $z \in U$, the function $w \mapsto E[W(z)W(w)]$ is harmonic in $U$.
(b) There is a constant $C > 0$ so that for all $z, w \in U$, with $d_H(z, w) \leq 1$,

$$\text{Var}(W(z) - W(w)) \leq Cd_H(z, w)^2.$$

(c) There is a constant $C > 0$ so that for all $y, z, w \in U$, with $d_H(z, w) \leq 1$,

$$|E[W(y)(W(z) - W(w))]| \leq Cd_H(z, w).$$

(d) There is a function continuous function $K : \mathbb{T}^2 \to \mathbb{R}$ so that for all $\zeta_h e^{i\theta_1} \in U$ and $\zeta_j e^{i\theta_2} \in U$,

$$E[W(\zeta_h e^{i\theta_1})W(\zeta_j e^{i\theta_2})] = \frac{1}{2} \min\{-\log |\sin(\frac{\theta_1 - \theta_2}{2})|, h, j\} + K(\theta_1, \theta_2) + O(1),$$

where the error term goes to 0 uniformly as $\min\{h, j\} + \log |\sin \frac{\theta_1 - \theta_2}{2}| \to \infty$.

**Remark 1.6.** The first condition, that the covariance is harmonic, implies that $W$ is almost surely harmonic in $U$ and it can be easily checked that

$$E\left(\int_\gamma W(z)d\mu_w(z) - W(w)\right)^2 = 0,$$

for smooth, Jordan curves in $U$ with $w$ a point in the interior of the curve and $\mu_w(z)$ the harmonic measure on the curve seen from $w$. 
Lower bound overview. The field $W$, by virtue of being Gaussian and having a log-correlated structure, is amenable to general tools for Gaussian fields: for example very precise information on its maximum on any hyperbolic ball is available using theory in [DRZ15]. In the problems we study here, we are given a sequence of harmonic random fields $W_N$ that are not necessarily Gaussian but are well-approximated by $W$. Our task is to provide simple conditions of comparison that will produce a lower bound for the maximum of $W_N$ in the large $N$ limit. These conditions are given solely in terms of mixed moments of exponentials of $W_N$.

In what follows, we fix $\delta > 0$ a small positive constant, that we will ultimately take to 0. We let $n_0 = \lfloor (1 - \delta)N \rfloor$ and introduce the set

$$\Omega = \left\{ e^{i(\frac{\pi}{2} + h e^{-n_0})} : h \in \mathbb{Z}, |h| < N^{-\delta} e^{n_0} \right\}.$$ 

We will find a lower bound for $W_N$ by considering the values of $W_N$ at the points $\{\omega \zeta_{n_0} : \omega \in \Omega \}$. To do this, however, we will also consider the behavior of $W_N$ at points closer to the origin of $\mathbb{D}$. So, define a domain

$$D_{N,\delta} = \{ ire^{i\theta} | 1 - N^{-\delta} \leq r \leq 1 - N^{-1+\delta}, |\theta| \leq N^{-\delta} \}.$$ 

This is the set in which we compare $W_N$ and $W$.

We will take $b^* = b^*_N$ to be an integral valued sequence so that $b^*_N \approx \delta \log N$ (it will be defined precisely in Section 4). Then, we will define the field statistics, for $\omega \in \Omega$,

$$\mathcal{B}_\omega : F \mapsto 2F(\omega \zeta_{n_0}) - 2F(i \zeta_{b^*}).$$

It will essentially suffice to show that some $\mathcal{B}_\omega(W_N)$ is on the order of $(1 - O(\delta)) \log N$. Define $\mathfrak{M}_{\ell,\delta}(\mathcal{B}_\omega)$ as the set of all cylinder functions

$$\mathfrak{B} : F \mapsto \mathcal{B}_\omega(F) + \sum_{z \in \mathbf{z}} 2F(z) - \sum_{w \in \mathbf{w}} 2F(w),$$

where $\mathbf{z}$ and $\mathbf{w}$ are subsets of $D_{N,\delta}$ with $|\mathbf{z}| = |\mathbf{w}| = \ell$ and the property that: if we denote $Z = \mathbf{z} \cup \{\omega \zeta_{n_0}\}$ and $W = \mathbf{w} \cup \{i \zeta_{b^*}\}$, there is a bijection $\phi : Z \rightarrow W$ so that for all $z \in \mathbf{z}$

$$d_H(z, \phi(z)) \leq \min \left\{ \min_{z' \in Z \setminus \{z\}} d_H(z, z'); \min_{w \in W \setminus \{\phi(z)\}} d_H(w, \phi(z)) \right\}.$$ 

These field statistics are in some sense local perturbations of $\mathcal{B}_\omega(F)$.

We make a quantitative assumption on how well $\mathcal{B}(W_N)$ can be approximated by $\mathcal{B}(W)$ in the sense of exponential moments.

Assumption 1.7. Let $\ell \in \mathbb{N}$. We define the mixed exponential moment assumption $\text{MEM}(\ell)$ to be that the following holds. For all $\delta > 0$ sufficiently small:

1. The function $W_N(z)$ is almost surely harmonic in $D_{N,\delta}$.
2. Uniformly in $\omega \in \Omega$ and $\mathfrak{B} \in \mathfrak{M}_{\ell,\delta}(\mathcal{B}_{\omega_{\zeta_{n_0}}})$,

$$\mathbb{E} \left[ e^{\mathfrak{B}(W_N)} \right] \leq \mathbb{E} \left[ e^{\mathfrak{B}(W)} \right] (1 + O(N^{-\delta})).$$

At first sight, it may not be clear that these assumptions even imply, in any sense, that $W_N$ converges in law to $W$. However, as a corollary of these assumptions, the mixed moments of $W_N$ can be compared to the mixed moments of $W$ with a vanishing error (see Proposition 6.5). Under the assumption $\bigcap_{\ell=1}^\infty \text{MEM}(\ell)$, which we show holds for $Q_N \circ J$, one can deduce Gaussian scaling limits of $W_N$ as a direct consequence of Proposition 6.5.
Theorem 1.8. Under Assumption MEM(2), for any $\delta > 0$,
\[
\limsup_{N \to \infty} \Pr[\max_{z \in D} |W_N(z) - W_N(i\zeta_b)| < (1 - \delta) \log N] = 0.
\]

One can make a comparison between this theorem and four-moment theorems of [TV11] – a small number of moments of the exponential of the field determines the maximum – though the methods of proof could not be more different.

We give the proof of this theorem and an overview of the method in Section 6.

For the field $Q_N \circ J$, a direct application of Corollary 2.5 below and Markov’s inequality shows that with probability going to 1
\[
Q_N(J(i\zeta_b)) > -C\delta \log N
\]
for some sufficiently large $C$. Therefore, upon verifying Assumption 1.7 for $Z_N$, the lower bound in Theorem 1.3 follows.

Characteristic polynomials. Let $A_N$ be a unitarily invariant random matrix (chosen with probability proportional to the weight $e^{-N \operatorname{tr}(V(A_N))}$ on the space of $N \times N$ Hermitian matrices). We consider $\rho_N$ to be the empirical spectral measure of the matrix $A_N$ and take $\rho$ to be the corresponding equilibrium measure. By [Joh98, Theorem 2.1], $\rho$ is the (unique) probability measure which minimize the energy functional
\[
I_V(\mu) = \int \left( \log |t - u|^{-1} + \frac{V(t) + V(u)}{2} \right) \mu(dt)\mu(du).
\]
Moreover, it is absolutely continuous with respect to the Lebesgue measure: $\rho(du) = \rho(u)du$, and if the potential $V$ is real-analytic and regular, by [DKMVZ99, (1.6)–(1.8)], we have
\[
\rho(u) = h(u)1_J(u) \prod_{i=0}^m \sqrt{(a_{i+1} - u)(u - b_i)}
\]
where the function $h$ is real-analytic, strictly positive on
\[
J = (b_0, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_m, a_{m+1}),
\]
and the intervals $(b_0, a_1), \ldots, (b_m, a_{m+1})$ are disjoint.

In the previous sections, we have outlined an approach to controlling the maximum of
\[
Q_N(q) = \log |\det(q - A_N)| - E[\log |\det(q - A_N)|]
\]
which reduces the problem to estimating mixed moments of $|\det(q - A_N)|^{\pm 2}$ for various points $q$ near the bulk $J$. We now elaborate on how to do such estimates when $V$ is a regular, real-analytic potential.

Besides the asymptotics of the monic orthogonal polynomials with respect to measure $e^{-NV(x)}dx$ that are given in [DKMVZ99], we use one other random matrix tool: the Fyodorov-Strahov formula [FS03]. This formula, which we will introduce presently, allows for expectations of ratios of characteristic polynomials to be expressed in terms of determinants involving the orthogonal polynomials and their Cauchy transforms.

For any $\ell, k \geq 1$ and $q = (q_1, \ldots, q_\ell) \in \mathbb{C}^\ell$, we define the matrix
\[
V_k(q) = \begin{pmatrix}
1 & q_1 & \cdots & q_1^{k-1} \\
& \ddots & \ddots & \vdots \\
& & 1 & q_\ell \\
& & & q_\ell^{k-1}
\end{pmatrix}.
\]
In particular, we denote Vandermonde matrix, $V(q) = V_\ell(q)$, and its determinant
\begin{equation}
\Delta(q) := \det(V(q)) = \prod_{1 \leq i < j \leq k} (q_j - q_i).
\end{equation}

Let $\pi_n$ be the monic orthogonal polynomial of degree $n$ with respect to the weight $e^{-NV(x)}$ on $\mathbb{R}$. We define the normalizing constants $(\gamma_n)_{n \geq 0}$ of these polynomials by the formula
\begin{equation}
\int \pi_n(x)\pi_m(x)e^{-NV(x)}dx = \gamma_n^{-2}\delta_{nm}.
\end{equation}

For any $q \in \mathbb{C} \setminus \mathbb{R}$, we introduce the Cauchy transform
\begin{equation}
h_n(q) = \frac{1}{2\pi i} \int \frac{\pi_n(x)}{x-q}e^{-NV(x)}dx.
\end{equation}

The normalizing constants $\gamma_n$ also play a role in determining the 3-term recurrence for $\pi_n$. Specifically, there exists a sequence $\beta_0, \beta_1, \cdots \in \mathbb{R}$ such that
\begin{equation}
\pi_{n+1}(x) + \left(\frac{2n+1}{\gamma_n}\right)^2 \pi_n(x) = (x - \beta_n)\pi_n(x),
\end{equation}
with $\beta_{-1} \equiv 0$. By taking the Cauchy transform of both sides, it can be seen that $h_n$ necessarily satisfies the same recurrence for all $n \geq 0$. One should also keep in mind that, even if we do not write it explicitly, all these sequences $\pi_1, \pi_2, \cdots$ and $h_0, h_1, \ldots$ depend on the dimension $N$ as the weight $e^{-NV(x)}$ is varying.

The Fyodorov-Strahov formula ([FS03, (8)]) states that, for any $\ell, k \geq 0$,
\begin{equation}
E\left[\prod_{i=1}^{\ell} \det(p_i - A_N) \prod_{j=1}^{k} \det(q_j - A_N) \right] = \frac{\prod_{j=1}^{k}(-2\pi i \gamma_{N-j}^2)}{(-1)^{\frac{1}{2}} (\Delta(q)\Delta(p))} \begin{vmatrix}
| h_{N-k}(q_1) & \cdots & h_{N+\ell-1}(q_1) \\
\vdots & \ddots & \vdots \\
| h_{N-k}(q_k) & \cdots & h_{N+\ell-1}(q_k) \\
\pi_{N-k}(p_1) & \cdots & \pi_{N+\ell-1}(p_1) \\
\vdots & \ddots & \vdots \\
| \pi_{N-k}(p_\ell) & \cdots & \pi_{N+\ell-1}(p_\ell)
\end{vmatrix}.
\end{equation}

Care should be taken in the sign conventions in comparing this formula to the original. By applying the three-term recurrence, this determinant can be reduced to an expression involving only $\{\pi_N, \pi_{N-1}, h_N, h_{N-1}\}$, at least in the $k = \ell$ case (see Lemma 5.2 below). Note that, if $k \neq \ell$, this expression would involve a sum of determinants and we will avoid this case by considering only balanced ration (i.e. $\ell = k$). This reduction is useful due to the nature of the asymptotics. While the Riemann-Hilbert steepest descent method can handle joint asymptotics of
\[\{(\pi_n, h_n) : N - \kappa \leq n \leq N\}\]
for fixed $\kappa \in \mathbb{N}$ (see [DKMVZ99, Above Theorem 1.1]), it is typically formulated just for the matrix
\begin{equation}
Y_N(q) = \begin{pmatrix} \pi_N(q) & h_N(q) \\ \bar{\pi}_N(q) & \bar{h}_N(q) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \bar{\pi}_N(p) \\ \bar{h}_N(q) \end{pmatrix} = -2\pi i \gamma_{N-1}^2 \begin{pmatrix} \pi_{N-1}(p) \\ h_{N-1}(q) \end{pmatrix}.
\end{equation}

This is the unique analytic function in $\mathbb{C} \setminus \mathbb{R}$ so that for all $x \in \mathbb{R}$,
\begin{equation}
Y_{N,+}(x) = Y_{N,-}(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}
\end{equation}
and \( Y_N(q) = \begin{pmatrix} q^N & 0 \\ 0 & q^{-N} \end{pmatrix} \left( I + O(q^{-1}) \right) \). The main delicate point in our application of these asymptotics is to ensure the uniformity of the error estimates required by Assumption 1.7.

**Organization.** In Section 2, we give an overview of the asymptotics of orthogonal polynomials in the plane. We closely follow the argument of [DKMVZ99] in the regular case, giving enough details so that the presentation is self-contained. In particular, we deduce from these asymptotics, estimates on exponential moments of the log-potential of the field \( Q_N \). In Section 3, we show how to compare mixed exponential moments of \( Q_N \) to the mixed exponential moments of an idealized Gaussian field \( G \). Hence, verifying the assumption 1.7 and, by applying theorem 1.8, completing the lower-bound in Theorem 1.3. Section 4 contains a key combinatorial estimate needed in Section 3. In Section 5, we show the complete upper bound in Theorem 1.3 using the estimates from Section 2. Finally in Section 6, we prove the general lower bound Theorem 1.8.

**Notation.** For an \( n \times n \) matrix, we will use the norm

\[
\|A\| = \sup \{ |A_{ij}| : i, j \in \{1, 2, \ldots, n\} \}
\]

We use \( \ll, \gg, \asymp \) notation. For two functions \( f \) and \( g \) of some set \( X \to \mathbb{R} \) we say \( f \ll g \) if there is a constant \( c > 0 \) so that \( f(x) \leq cg(x) \) for all \( x \in X \). In some cases, we may allow the constant to have some parameter dependence, in which case we will explicitly state its dependence or write \( f \ll \epsilon \) which is to say that for each \( \epsilon > 0 \) there is a constant so that the inequality holds. The notation \( f \gg g \) means \( g \ll f \), and the notation \( f \asymp g \) means \( f \ll g \) and \( f \gg g \).

We additionally use \( O(\cdot) \) notation. For a function \( f : X \to \mathbb{R}, g = O(f) \) if and only if \( |g| \ll f \). If we wish to restrict or clarify the parameter dependence of \( O(\cdot) \), we will display the parameters beneath the \( O(\cdot) \).

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## 2. OP asymptotics background

The aim of this section is to review the asymptotics of the solution \( Y_N \) of the RHP [15]. We will not need the explicit solution given in [DKMVZ99], but the special form of the solution will be important for us to check the assumption 1.7 in section 3. So we will review in details how to apply the Deift-Zhou steepest descent method to the RHP [15], collecting the estimate which are important to us along the way. Recall that by (8)–(9), the equilibrium measure is supported on the closure of

\[
J = (b_0, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_m, a_{m+1})
\]

For notational convenience, we set \( a_0 = a_{m+1} \) and \( J^* = \mathbb{R} \setminus J \). Viewing these sets on the Riemann sphere, this makes \((a_0, b_0)\) into an interval containing infinity. Let

\[
g(q) = \int_{\mathbb{R}} \log(q - u) \phi(du).
\]
Note that this function appears explicitly in the normalization of the field $Q_N$, (10). Moreover, $g$ is analytic in $C \setminus (-\infty,a_0)$ and it follows from the Euler-Lagrange equation defining the equilibrium measure that there exists a real constant $\ell_V$ so that the function
\[ H(x) := -V(x) - \ell_V + g_+(x) + g_-(x) \]
satisfies the conditions $H(x) = 0$ for all $x \in J$ and, if the potential $V$ is regular, (20)
\[ H(x) < 0, \quad x \in J^*, \]
see [DKMVZ99, (1.10-1.13)]. On the other hand, for any $x \in \mathbb{R}$, we have
\[ g_+(x) - g_-(x) = 2\pi i \int_x^{a_0} g(du). \]
When $V$ is analytic, this function can be analytically continued in both the upper and lower half plane in a neighborhood of $J$: [DKMVZ99, (3.43-3.46)]. To be more specific, we can express $2\pi i \rho(du) = \sqrt{Q(u)} du$ for all $s \in J$ where $Q(s)$ is real-analytic and $\sqrt{Q}$ is given by the principle branch; [DKMVZ99, (3.3-3.5)]. Then, these analytic continuations are given by
\[ G_\pm(z) = \pm \int_z^{a_0} \sqrt{Q(u)} du, \quad z \in \mathbb{H}_\pm. \]
In particular, the potential $V$ is regular if and only if all the real zeros of the function $Q$ are simple and lie at the endpoints of $J$. Then a simple argument shows that for all $x \in J$, there exists a constant $c > 0$ so that
\[ \pm \Re G_\pm(x \pm iy) > cy, \]
when $y > 0$ and sufficiently small. We are now ready to present the asymptotics of the matrix $Y_N$, (17). We begin by introducing
\[ M_N(q) = e^{-N\ell_V/2\sigma_3} Y_N(q) e^{-N(g(q) - \ell_V/2)\sigma_3}, \]
where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This normalization implies that $M_N(q) = 1 + O(q^{-1})$.
Moreover, if we deform the RHP by introducing lens-shaped regions delimit by smooth arcs $\Sigma_\pm$, see [Kui03, Figure 5 p.60], and a new matrix
\[ M_N(q) = S_N(q) U_N(q), \]
where
\[ U_N(q) = \begin{cases} I & \text{if } q \text{ is outside the lens-shaped regions} \\ \begin{pmatrix} 1 \\ e^{\pm N G_\pm(q)} \end{pmatrix} & \text{if } q \text{ lies inside the } \pm \text{ lens-shaped regions} \end{cases}. \]
Note that we open the lenses in such a way that the condition (22) holds inside lens-shaped regions, except in an $\epsilon$-neighborhood of the end-points of $J$. In particular, we obtain the estimate:
\[ \|U_N(q) - I\| \leq e^{-cN|3q|}. \]
for all $q \in \mathbb{C} \setminus \mathbb{R}$, $\epsilon$-away from the end-points of $J$. Using the above definitions, we claim that $S_N$ is the solution of the following RHP:

\[
\begin{align*}
S_N \text{ is analytic in } \mathbb{C} \setminus \{ \mathbb{R} \cup \Sigma \pm \}, \\
S_{N,+}(q) &= S_{N,-}(q)\nu_N(q), \\
S_{N,+}(q) &= S_{N,-}(q)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad q \in J \\
S_{N,+}(q) &= S_{N,-}(q)\begin{pmatrix} 1 & 0 \\ e^{\mp G(q)} & 1 \end{pmatrix}, \quad q \in \Sigma \pm \\
S_N(q) &\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q \to +\infty
\end{align*}
\]  

where

\[
\nu_N(q) = \begin{pmatrix} e^{iN\Omega_j} & e^{-NH(q)} \\ 0 & e^{-iN\Omega_j} \end{pmatrix}, \quad q \in (a_j, b_j), \quad j = 0, \ldots, m
\]

and the parameters $\{\Omega_j\}_{j=0}^m$ are defined by

\[
\Omega_j = 2\pi \int_{b_j}^{\infty} \varrho(du).
\]

In particular, we have $\Omega_0 = 0$, so that the jump matrix $\nu_N$ is exponentially close to the identity on the interval $(a_0, b_0)$ which contains $\infty$. We refer to [DKMVZ99] sections 3.3 and 4.1 for the details of this construction, or to sections 5.1-5.2 in the lecture notes [Kui03] for a comprehensive presentation. In particular, in [Kui03], the constructions of the global and edge parametrices are explained in detail in the one-cut regular case.

We are left with the task of finding a solution of (27). The idea is to disregard the terms which converge to 0 in the jump matrices as $N \to \infty$. By (20) and (22), this leads to consider the solution of the following RHP:

\[
\begin{align*}
M_{N}^\infty \text{ is analytic in } \mathbb{C} \setminus [b_0, a_0], \\
M_{N,+}^\infty(q) &= M_{N,-}^\infty(q)\nu_{N}^\infty(q), \quad q \in J^* \\
M_{N,+}^\infty(q) &= M_{N,-}^\infty(q)\sigma, \quad q \in J \\
M_{N}^\infty(q) &\to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q \to +\infty
\end{align*}
\]

with

\[
\nu_{N}^\infty(q) = \begin{pmatrix} e^{iN\Omega_j} & 0 \\ 0 & e^{-iN\Omega_j} \end{pmatrix}, \quad q \in (a_j, b_j), \quad j = 0, \ldots, m
\]

and

\[
\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The matrix $M_{N}^\infty$ is usually called the global parametric and we expect that $S_N(q) \sim M_{N}^\infty(q)$ as $N \to \infty$ (at least away from the endpoints of $J$ where the terms that we neglected do not tend to 0). In [DKMVZ99], the matrix $M_{N}^\infty$ is constructed in two steps. First, we consider the function

\[
\gamma(q) = \prod_{i=0}^{m} \frac{q - b_i}{q - a_{i+1}}^{1/4}.
\]

The branches are chosen so that $\gamma$ is analytic on $\mathbb{C} \setminus \bar{J}$, $\gamma(q) \sim 1$ as $q \to \infty$ in $\mathbb{H}_+$, and it satisfies a jump condition: $\gamma_+(q) = i\gamma_-(q)$ on $J$, c.f. [DKMVZ99, (4.31)]. Beware that, to remain consistent with existing literature, we use the notation $\gamma$
and \( \gamma_n \), emphasizing that with a subscript, we refer to the normalizing constant \((13)\). In particular, if we define the matrix

\[
H = \begin{pmatrix}
\frac{\gamma + \gamma^{-1}}{2} & \frac{\gamma - \gamma^{-1}}{2} \\
\frac{\gamma - \gamma^{-1}}{2} & \frac{\gamma + \gamma^{-1}}{2}
\end{pmatrix},
\]

then it is the (unique) solution of the RHP:

\[
\begin{align*}
H & \text{ is analytic in } \mathbb{C} \setminus \mathcal{J}, \\
H_+(q) &= H_-(q)\sigma^{-1}, \quad q \in \mathcal{J}^*, \\
H(q) &\to 1, \quad q \to \infty, q \in \mathbb{H}_+ \\
H(q) &\to \sigma, \quad q \to \infty, q \in \mathbb{H}_-.
\end{align*}
\]

Note that, by \([\text{DKMVZ99}, \text{Lemma 4.1}]\), all the zeros of the functions \( \gamma \pm \gamma^{-1} \) are located at points \( z_j \in (a_j, b_j) \) for \( j = 1, \ldots, m \) on the \( \mp \)-side of \( \mathcal{J}^* \) respectively. To complement this object, we need the (unique) solution, denoted

\[
\Theta(q) = \begin{pmatrix}
\vartheta_1(q) \\
\vartheta_2(q) \\
\vartheta_3(q) \\
\vartheta_4(q)
\end{pmatrix},
\]

of the RHP:

\[
\begin{align*}
\Theta & \text{ is analytic in } \mathbb{C} \setminus \mathcal{J}, \\
\Theta_+(q) &= \Theta_-(q)\sigma_1\nu N(q), \quad q \in \mathcal{J}^* \\
\Theta(q) &\to c_\pm + O\left(\frac{1}{q}\right), \quad q \to \infty, q \in \mathbb{H}_\pm
\end{align*}
\]

for some appropriate constant matrices \( c_\pm \) (c.f. \( M^\# \) from \([\text{ DKMVZ99}, \text{ (4.60-4.64)}]\), which is just \( \Theta \) rescaled by a constant diagonal matrix). Using \((31)\) and \((32)\), it is a straightforward computation to check that the global parametric \( M^\infty_N \) is given by

\[
M^\infty_N(q) = \begin{pmatrix}
\frac{\gamma + \gamma^{-1}}{2} & \frac{\gamma - \gamma^{-1}}{2} \\
\frac{\gamma - \gamma^{-1}}{2} & \frac{\gamma + \gamma^{-1}}{2}
\end{pmatrix}, \quad \text{if } q \in \mathbb{H}_+ \\
\frac{\gamma + \gamma^{-1}}{2} & \frac{\gamma - \gamma^{-1}}{2} \\
\frac{\gamma - \gamma^{-1}}{2} & \frac{\gamma + \gamma^{-1}}{2}
\end{pmatrix} \sigma, \quad \text{if } q \in \mathbb{H}_-
\]

The entries \( \vartheta_1, \ldots, \vartheta_4 \) are certain explicit ratio of \( \theta \)-functions whose arguments depend on the parameters \( N \) and \( \{ \Omega_j \}_1^m \). Moreover, by \([\text{DKMVZ99}, \text{Lemma 4.1}]\), the poles of these \( \theta \)-functions are located at the points \( \{ z_j \}_j \) and exactly match with the zeros of the functions \( \gamma \pm \gamma^{-1} \). Thus, besides the jump conditions, the entries of the matrix \( M^\infty_N \) have at worst \( 1/4 \)-root singularity at the end-points of \( \mathcal{J} \); c.f. formulae \((4.73)-(4.73)\) in \([\text{DKMVZ99}]\). In particular, there exists a constant \( C > 0 \) (independent of the dimension \( N \)) so that

\[
\left\| \begin{pmatrix}
\frac{\gamma + \gamma^{-1}}{2} & \frac{\gamma - \gamma^{-1}}{2} \\
\frac{\gamma - \gamma^{-1}}{2} & \frac{\gamma + \gamma^{-1}}{2}
\end{pmatrix}
\right\| \leq C(R(q) + 1)
\]

uniformly over \( \mathbb{C} \setminus \mathcal{J}^* \) with

\[
R(q) := \prod_{i=0}^{m} |(q - a_i)(q - b_i)|^{-1/4}.
\]

An important consequence of this observation is the next powerful identities.

**Lemma 2.1.** All the matrices \( Y_N, M_N \) and \( M^\infty_N \) have determinant identically 1.
Proof. For $Y_N$, this can be seen from the defining Riemann-Hilbert problem, which shows that $\det Y_N$ extends to an entire function and tends to 1 at infinity. The result then holds for $\det M_N$ as well, by formula (23). Similarly, it follows from (28) that the function $q \mapsto \det M_N(q)$ has no jump in $\mathbb{C}$ and the estimate (34) implies that all its plausible singularities at the endpoints of $J$ are removable. Thus, the function $\det M_N(q)$ is also entire and tends to 1 at infinity. □

Let us now give some precise statements about the asymptotics of $M_N$, (23). In fact, since the matrices $U_N$ and $\nu_N$ are not asymptotics to $I$ and $\nu_N^\infty$ uniformly on $\Sigma_\pm$ and $J^*$ respectively (the problem coming from the edge-points), we need to introduce one more auxiliary RHP to get the full solution. Let $\epsilon > 0$ be a small parameter and $\mathcal{E} = \bigcup_{\gamma \in \{a_j, b_j\}} \mathbb{D}(\gamma, \epsilon)$. We also let $\mathcal{C}_\epsilon = \{J^* \cup \Sigma_\pm\} \setminus \mathcal{E}$ and define

$$R_N(q) = \begin{cases} S_N(q)(P_N)^{-1}, & q \in \mathcal{E} \\ S_N(q)(M_N^\infty)^{-1}, & q \in \mathbb{C} \setminus \{\mathbb{R} \cup \Sigma_\pm \cup \mathcal{E}\} \end{cases}.$$  

(36)

In formula (36), $P_N$ is the so-called edge-parametric, it is defined so that it has the same jumps as $S_N(q)$ inside $\mathcal{E}$ and it satisfies

$$M_N^\infty(q)P_N(q)^{-1} = I + O_{N \to \infty}(N^{-1})$$

uniformly for all $q \in \partial \mathcal{E}$. The boundary condition (37) uniquely determine the matrix $P_N$, and the error term is optimal; c.f. formula (42) below. Then, $R_N$ satisfies the RHP:

$$R_N(q) \text{ is analytic in } \mathbb{C} \setminus \{\mathcal{C}_\epsilon \cup \partial \mathcal{E}\},$$

$$R_{N,+}(q) = R_{N,-}(q)E_N(q),$$

$$R_{N,+}(q) = R_{N,-}(q)M_N^\infty(q)P_N(q)^{-1},$$

$$R_N(q) \to I, \text{ as } q \to \infty$$

(38)

where the jump matrix is given by

$$E_N(q) = \begin{cases} M_N^\infty(q) \begin{pmatrix} 1 & 0 \\ e^{i\pi N G_\pm}(q) & 1 \end{pmatrix} M_N^\infty(q)^{-1}, & q \in \Sigma_\pm \\ M_N^\infty(q) \begin{pmatrix} 1 & 0 \\ e^{i\pi N \Omega_j - NH(q)} & 1 \end{pmatrix} M_N^\infty(q)^{-1}, & q \in J^*_\epsilon \end{cases}.$$ 

In particular, using the conditions (20) and (22), an easy computation shows that there exists $\alpha_\epsilon > 0$ so that

$$E_N(q) = I + O_{N \to \infty}(N^{-\alpha_\epsilon}),$$

uniformly for all $q \in \mathcal{C}_\epsilon$. Thus, all the jumps in the RHP (38) converge uniformly to the identity and it follows from the general theory that

$$R_N(q) = I + O_{N \to \infty}(N^{-1}),$$

(39)

uniformly for all $q$ in compact subsets of $\mathbb{C} \setminus J^*$; c.f. [DKMVZ99, section 4.6]. In particular, note that the estimate (39) is valid everywhere except on $J^*$ because we are free to deform slightly the jump contours $\Sigma_\pm$ and $\partial \mathcal{E}$.

At last, we need to say a few words about the edge-parametric. According to [DKMVZ99, section 4.3], it has the form for all $\gamma \in \bigcup_{j=0}^m \{a_j, b_j\}$ and $q \in \mathbb{D}(\gamma, \epsilon)$,

$$P_N(q) = \Psi(\phi_N, \gamma(q)) M_N^\infty(q),$$

(40)
where
\[
\Psi(\zeta) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \zeta^{3/4} A(\zeta) e^{\frac{2\pi}{3} \sigma_3},
\]
and \(A\) is the (unique) solution of the so-called Airy Riemann-Hilbert problem. We refer to the lecture notes [Kui03, section 2.3] for a nice description and an explicit solution of this problem in terms of the Airy function. In particular, by inspection of \(A\), [Kui03, theorem 2.6], it is easy to verify that there exists a constant \(C > 0\) such that for all \(\zeta \in \mathbb{C}\),
\[
\|\Psi(\zeta)\| \leq C.
\]
Moreover, by formula (2.60) therein, we have
\[
\lim_{\zeta \to \infty} \Psi(\zeta) = I + O\left(\zeta^{-3/2}\right)
\]
in the region \(-\pi < \arg \zeta < \pi\). The map \(\phi_{N,\gamma}\) satisfies the relationship
\[
\phi_{N,\gamma}(q) = \left(\frac{3N}{4} \int_{\gamma} \sqrt{Q(s)} ds\right)^{2/3};
\]
c.f. [DKMVZ99, (4.85)] and (21) for the connection to the function \(G\). Notice that in the regular case, \(\gamma\) is a simple zero of the real-analytic function \(Q\) and the map \(\phi_{N,\gamma}\) gives an analytic chart in the disk \(D(\gamma, \epsilon)\). In particular, the matrix \(A\) is chosen so that \(\Psi(\phi_{N,\gamma}(q))\) has the same jumps as \(M_N'(q)\) inside \(D(\gamma, \epsilon)\); c.f. [Kui03, section 5.4]. Moreover, the asymptotic (42) guarantees that uniformly for all \(|q - \gamma| = \epsilon\),
\[
\Psi(\phi_{N,\gamma}(q)) = I + O_{N \to \infty}(N^{-1}).
\]
Hence, by formula (40), the boundary condition (37) holds.

**Proposition 2.2.** There exists a constant \(C > 0\) such that for all \(q \in \mathbb{C} \setminus \mathbb{R}\),
\[
\|M_N(q)\| \leq C(R(q) + 1)
\]
where \(R(q)\) is given by formula (35).

**Proof.** Because of the estimate (33), we only need to prove that there exists \(C > 0\) so that for all \(q \in \mathbb{C} \setminus \mathbb{R}\),
\[
\|M_N(q)\| \leq C\|M_N^\infty(q)\|.
\]
By formulae (24) and (26), we have
\[
M_N(q) = \begin{cases} R_N(q) M_N^\infty(q) U_N(q) & \text{if } q \in \mathbb{C} \setminus \mathbb{R} \cup \Sigma \cup \partial \mathbb{D}, \\ R_N(q) P_N(q) U_N(q) & \text{if } q \in \mathbb{R}. \end{cases}
\]
So, it suffices to observe that, by (39), the error term \(R_N(q)\) is uniformly bounded in both \(q\) and \(N\) and, by (22) and formula (25), \(\|U_N(q)\| \leq 1\) for all \(q \in \mathbb{C} \setminus \mathbb{R}\). \(\square\)

Finally, we can also state the asymptotics of the matrix \(M_N(q)\) in a neighborhood in the upper-half plane (resp. lower-half plane) of a point of the bulk.

**Proposition 2.3.** Let \(0 < \delta < 1/2\) and \(x \in J\). Let \(\mathcal{X}_N^\pm\) be a sequence of compact sets such that
\[
\mathcal{X}_N^\pm \subset \mathbb{D}(x, N^{-\delta}) \quad \text{and} \quad \mathcal{X}_N^\pm \subset \{ \pm (\Im q) > N^{-1+\delta} \}.
\]
Then
\[(45)\]
\[M_N(q) = M_{N,+}\left(x\right) + O_{N^{-\delta}}(N).\]
uniformly in $\mathcal{H}_N^\pm$, where $M_{N,+}^{\infty}$ denotes the boundary values of the matrix $M_{N,+}^{\infty}$ and the error term in formula $(45)$ is uniform for all $x$ in compact subsets of $\mathcal{J}$.

Before proceeding to the proof of proposition 2.3, it is useful to mention the uniform continuity property of the matrix $M_{N,+}^{\infty}$:

**Lemma 2.4.** Let, for all $q \in C\setminus J^*$, 
\[\tilde{M}_{N}^{\infty}(q) = \left(\begin{array}{cc}
\frac{\gamma + \gamma^{-1}}{2} \vartheta_1 & \frac{\gamma - \gamma^{-1}}{-2i} \vartheta_2 \\
\frac{\gamma - \gamma^{-1}}{-2i} \vartheta_3 & \frac{\gamma + \gamma^{-1}}{2} \vartheta_4
\end{array}\right) .\]
For any $x \in C\setminus J^*$, we have
\[\tilde{M}_{N}^{\infty}(q) = \tilde{M}_{N}^{\infty}(x) + O_{x} (N)\]
uniformly for all $q \in D(x, r)$. In particular, the error term is independent of $N$ and uniform for all $x$ in compact subsets of $C\setminus J^*$.

**Proof.** Recall that, by $(30)$ and $(32)$, the matrices $H$ and $\Theta$ are analytic in $C\setminus J^*$. So is the matrix $\tilde{M}_{N}^{\infty}(q)$ and lemma 2.4 follows directly from Cauchy's formula and the estimate $(34)$. \[\square\]

**Proof of Proposition 2.3.** When the parameter $N$ is large, by formula $(44)$, we have 
\[M_N(q) = R_N(q)M_{N,+}^{\infty}(q)U_N(q)\]
for all $q \in \mathcal{H}_N^\pm$. By $(26)$, if $|\Im q| \geq N^{-1+\delta}$, then 
\[U_N(q) = I + O_{N^{-\delta}}(e^{-cN^\delta})\]
uniformly for all $q \in \mathcal{H}_N^\pm$. Notice that, by formula $(33)$, 
\[M_{N,+}^{\infty}(q) = \tilde{M}_{N}^{\infty}(q)\]
for all $q \in \mathcal{H}_+$. Thus, by assumption, Lemma 2.4 implies that 
\[M_{N,+}^{\infty}(q) = \tilde{M}_{N}^{\infty}(x) + O_{x} (N^{-1}),\]
uniformly for all $q \in \mathcal{H}_N^\pm$. Combined this estimate with formula $(46)$ and replacing $\tilde{M}_{N}^{\infty}(x) = M_{N,+}^{\infty}(x)$, this yields formula $(45)$. To get the estimate for $\mathcal{H}_N^-$, we follow the same argument except that we must use that $M_{N}^{\infty}(q) = \tilde{M}_{N}^{\infty}(q)\sigma$ for all $q \in \mathcal{H}_+$ and that $\tilde{M}_{N}^{\infty}(x)\sigma = M_{N,-}^{\infty}(x)$ for all $x \in \mathcal{J}$. \[\square\]

To conclude this section we give an application of Proposition 2.2 to estimate the Laplace transform of the random field $Q_N$, $(10)$.

**Corollary 2.5.** There is a constant $C > 0$ so that for all $q \in C\setminus R$,
\[\mathbb{E} e^{\pm 2Q_N(q)} \leq C\left(1 + |\Im q|\right)R(q)^2 |\Im q| ,\]
where $R(q)$ is given by formula $(35)$.\[\square\]
Proof. Using the Fyodorov-Strahov formula, (10), we have

\[ \mathbb{E}[e^{2Q_N(q)}] = \mathbb{E} \left[ \left| \det(q - A_N) \right|^2 \right] e^{-2N \Re q(q)} = \frac{1}{\overline{q} - q} \left| \frac{\pi_N(q)}{\pi_N(\overline{q})} \frac{\pi_{N+1}(q)}{\pi_{N+1}(\overline{q})} \right| e^{-N(\Re(q) + \Im(q))}. \]

Using the recurrence relation (15), we can express this determinant in terms of the entries of the matrix \( M_N(q) \), (23), we obtain

\[ \mathbb{E}[e^{2Q_N(q)}] = \left| M_N(q)_{11} \right|^2 - \frac{e^{N\ell}}{2\pi i \gamma_N^2(\overline{q} - q)} \left| M_N(p)_{21} M_N(p)_{11} \right|. \]

The asymptotic \( \text{DKMVZ99}, (1.63) \) implies that there exists a universal constant \( C > 0 \) such that

\[ \left| \frac{e^{N\ell}}{2\pi i \gamma_N^2} \right| \leq C, \]

This implies that

\[ \mathbb{E}[e^{2Q_N(q)}] \ll \frac{\left| M_N(q) \right|^2(1 + |\Im q|)}{|\Im q|} \]

and the claim is a direct consequence of Proposition 2.2.

The claim for the negative power is similar, as

\[ \mathbb{E}[e^{-2Q_N(q)}] = \mathbb{E} \left[ \left| \det(q - A_N) \right|^{-2} \right] e^{2N \Re q(q)} = \frac{1}{\overline{q} - q} \left| \frac{h_N(q)}{h_N(\overline{q})} \frac{h_{N+1}(q)}{h_{N+1}(\overline{q})} \right| e^{-N(\Re(q) + \Im(q))}. \]

Proceeding in an analogous way as to the positive power, we are led to the conclusion of the corollary.

\[ \square \]

3. Uniform mesoscopic OP asymptotics

This section is devoted to compute the expectations of balanced ratios of characteristic polynomials, in particular the verification of the assumptions of for the field \( Q_N \). \( \text{DKMVZ99} \). Recall that \( J : \mathbb{D} \to \mathbb{C} \setminus [-1, 1] \) denotes the Joukowski transform given by (8), and we assume that the support of the equilibrium measure \( J \subseteq [-1/2, 1/2] \).

We will compare locally the field \( Z_N = Q_N \circ J \) which is almost surely harmonic in the half-disk \( D_+ \) to the Gaussian field \( G \), (2). Let \( x \in J \) and \( \omega \in \partial D_+ \) be the pre-image of \( x \) under \( J \), then we define for any \( 0 < \delta < 1/2 \),

\[ D_{N, \delta} = \left\{ re^{i\theta} \omega \mid 1 - N^{-\delta} \leq r \leq 1 - N^{-1+\delta}, |\theta| \leq N^{-\delta} \right\}. \]

We slightly generalize the setup from Assumption 1.7 here, as the proof gives slightly more than what is required for Theorem 1.8. For any \( k \in \mathbb{N} \) and \( \epsilon > 0 \), let \( \mathcal{F}_{k, \epsilon, \delta} \) be the collection of cylinder functions from \( \{ \mathbb{D} \to \mathbb{R} \} \to \mathbb{R} \) of the form

\[ \mathbf{F} \mapsto \sum_{z \in z'} 2\mathbf{F}(z) - \sum_{w \in w'} 2\mathbf{F}(w), \]

where \( z' \) and \( w' \) are disjoint subsets of \( D_{N, \delta} \) with \( |z'| = |w'| = k \) and such that

\[ \inf \left\{ d_{\mathbb{R}}(u, v) : u, v \in z' \cup w', u \neq v \right\} \geq \epsilon. \]
We now define a family of perturbed biases. Namely, let $\mathfrak{B}' \in \mathfrak{S}_{\ell,\epsilon,\delta}$ and define $\mathfrak{W}_{\ell,\delta}(\mathfrak{B}')$ as the set of all cylinder functions

$$F \mapsto \mathfrak{B}'(F) + \sum_{z \in \mathfrak{z}} 2F(z) - \sum_{w \in \mathfrak{w}} 2F(w),$$

where $\mathfrak{z}$ and $\mathfrak{w}$ are disjoint subsets of $\mathcal{D}_{N,\delta}$ with $|\mathfrak{z}| = |\mathfrak{w}| = \ell$ and the property that: if we denote $Z = \mathfrak{z} \cup \mathfrak{z}'$ and $W = \mathfrak{w} \cup \mathfrak{w}'$, there is a bijection $\phi: \mathfrak{z} \rightarrow \mathfrak{w}$ so that for all $z \in \mathfrak{z}$:

$$d_H(z, \phi(z)) \leq \min \left\{ \min_{z' \in Z \setminus \{z\}} d_H(z, z'), \min_{w \in W \setminus \{\phi(z)\}} d_H(w, \phi(z)) \right\}.$$

We are now ready to state the main result of this section:

**Proposition 3.1.** For all $k, \ell \geq 0$, all $\epsilon > 0$, and all $\delta > 0$ sufficiently small, we have

$$\mathbb{E} \left[ e^{\mathfrak{B}(Z_N)} \right] = \mathbb{E} \left[ e^{\mathfrak{B}(G)} \right] \left( 1 + O_{N \rightarrow \infty} (N^{-\delta}) \right),$$

uniformly in $\mathfrak{B}' \in \mathfrak{S}_{\ell,\epsilon,\delta}$ and uniformly in $\mathfrak{B} \in \mathfrak{W}_{\ell,\delta}(\mathfrak{B}')$.

As $Z_N$ is harmonic in $\mathcal{D}$, this proposition and Theorem 1.3 (after rotating the field $Z_N$ to move $\omega$ to $i$) implies the lower bound in Theorem 1.3.

The first step in the proof of proposition 3.1 is a reduction of the Fyodorov-Strahov formula, (16), in the case that $k = \ell$. Namely, we show that we can express the expected value of balanced ratio of characteristic polynomials only in terms of the matrix $Y_N$, (17), whose asymptotic we presented in section 2. For notational convenience, if $q = (q_1, \ldots, q_\ell) \in (\mathbb{C} \setminus \mathbb{R})^\ell$ and $f : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ is a continuous function, we let

$$f(q) = \text{diag}(f(q_1), \ldots, f(q_\ell)).$$

Recall also that $V(q)$ and $\Delta(q)$ denote the Vandermonde matrix and determinant of the points $(q_1, \ldots, q_\ell)$; c.f. formulae (11) and (12) respectively. In particular, if $q = (u, v)$, we will also use the notation:

$$V(q) = V(u, v) \quad \text{and} \quad \Delta(q) = \Delta(u, v).$$

**Lemma 3.2.** Let $\ell \geq 1$, $p = (p_1, \ldots, p_\ell)$ and $q = (q_1, \ldots, q_\ell)$ be two tuples of distinct points in $\mathbb{C} \setminus \mathbb{R}$. We have for all $N \geq \ell$,

$$\mathbb{E} \left[ \prod_{i=1}^\ell \frac{\det(p_i - A_N)}{\prod_{j=1}^\ell \det(q_j - A_N)} \right] = \frac{1}{\Delta(q) \Delta(p)} \left| \begin{array}{cc} \hat{h}_N(q) V(q) & \hat{h}_N(q) V(q) \\ \hat{\pi}_N(q) V(p) & \hat{\pi}_N(q) V(p) \end{array} \right|.$$  

**Proof.** For any $n \in \mathbb{N}$, let

$$v_N = (\hat{h}_n(q_1), \ldots, \hat{h}_n(q_\ell), \pi_n(p_1), \ldots, \pi_n(p_\ell))^t.$$

We begin with the Fyodorov-Strahov formula, (16), which we write as

$$\mathbb{E} \left[ \prod_{i=1}^\ell \frac{\det(p_i - A_N)}{\prod_{j=1}^\ell \det(q_j - A_N)} \right] = \mathbb{E} \left[ \prod_{i=1}^\ell \frac{\det(p_i - A_N)}{\prod_{j=1}^\ell \det(q_j - A_N)} \right] v_{N-\ell}, \ldots, v_{N+\ell-1}.$$

Since $\pi_n$ and $h_n$ satisfy the same 3-term recurrence relation, (15), we get

$$v_{n+1} + \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^2 v_{n-1} = (A - \beta_n I) v_n,$$
where $A = \text{diag}(q_1, \ldots, q_\ell, p_1, \ldots, p_\ell)$. Hence, by induction, we can conclude that for each $j \geq 1$

$$v_{N+j} - A_j v_N \in \text{Span}\{v_{N-1}, v_N, A v_N, \ldots, A^{j-1} v_N\}$$

$$v_{N-j} - \left(\frac{\gamma_{N-1}}{\gamma_{N-j}}\right)^2 A^{j-1} v_{N-1} \in \text{Span}\{v_N, v_{N-1}, A v_{N-1}, \ldots, A^{j-2} v_{N-1}\}.$$ 

Therefore, applying column operations to the determinant, we obtain

$$|v_{N-\ell}, \ldots, v_{N+\ell-1}| = \prod_{j=1}^{\ell} \left(\frac{\gamma_{N-1}}{\gamma_{N-j}}\right)^2 |A^{\ell-1} v_{N-1}, \ldots, v_{N-1}, v_N, A v_N, \ldots, A^{\ell-1} v_N|$$

and

$$E \left[ \prod_{i=1}^{\ell} \det(p_i - A_N) \prod_{j=1}^{\ell} \det(q_j - A_N) \right] = \frac{(-2\pi i \gamma_{N-1})^\ell}{(-1)^{\ell} \Delta(q)} M_{\ell-1, v_{N-1}, \ldots, v_{N-1}, v_N, \ldots, A^{\ell-1} v_N}.$$ 

Hence, by definition of $\tilde{h}_N$ and $\tilde{\pi}_N$, c.f. formula (17), scaling the factors $-2\pi i \gamma_{N-1}$ and fully permuting the first $\ell$ columns, we have arrived at the claimed formula. \qed

With these preliminaries, we now turn to the intended application of proving Proposition 3.1. Suppose we have a biasing function

$$\mathcal{B}(F) = \sum_{z \in Z} 2F(z) - \sum_{w \in W} 2F(w),$$

for sets $Z, W \subset D_{N,\delta}$ such that $|Z| = |W| = \ell/2 \in \mathbb{N}$. We define

$$p = J(\overline{Z} \cup Z) \quad \text{and} \quad q = J(W \cup \overline{W})$$

where, as usual, $J$ is the Joukowsky transform. Then, since $Z_N = Q_N \circ J$, we have the identity

$$E \left[ e^{\mathcal{B}(Z_N)} \right] = E \left[ \prod_{i=1}^{\ell} \det(p_i - A_N) e^{-Ng(p_i)} \prod_{j=1}^{\ell} \det(q_j - A_N) e^{-Ng(q_j)} \right].$$

Applying Lemma 3.2 we obtain

$$E \left[ e^{\mathcal{B}(Z_N)} \right] = \frac{1}{\Delta(q) \Delta(p)} \begin{vmatrix} \tilde{h}_N e^{Ng(q)} V(q) & \tilde{h}_N e^{Ng(q)} V(q) \\ \pi_N e^{-Ng}(q) V(p) & \pi_N e^{-Ng}(p) V(p) \end{vmatrix}.$$ 

At this point, it is worth recalling the definition of the normalized matrix $M_N$, (23). Namely, after rescaling the rows and columns of the previous determinant, we have

$$E \left[ e^{\mathcal{B}(Z_N)} \right] = \frac{1}{\Delta(q) \Delta(p)} \begin{vmatrix} (M_N)_{22}(q) V(q) & (M_N)_{12}(q) V(q) \\ (M_N)_{21}(p) V(p) & (M_N)_{11}(p) V(p) \end{vmatrix},$$

where, according to our notation, $(M_N)_{ij}(q)$ are diagonal matrices. Then, we can expand combinatorially the RHS of formula (51) using the following general identity:

**Lemma 3.3.** Let $A, B, C$ and $D$ be $\ell \times \ell$ diagonal matrices. For any $p, q \in \mathbb{C}^\ell$, we have

$$\begin{vmatrix} A V(q) & B V(q) \\
CV(p) & DV(p) \end{vmatrix} = \sum_{S, T \subseteq \{1, \ldots, \ell\}} \Delta(q_S, p_T) \Delta(q_{S^*}, p_{T^*}) \prod_{s \in S} A_{ss} \prod_{s \in S^*} B_{ss} \prod_{t \in T} C_{tt} \prod_{t \in T^*} D_{tt}.$$
where we denote \( T^* = [\ell] \setminus T \), \( S^* = [\ell] \setminus S \), \( p_T = (p_t)_{t \in T} \), and similarly for the tuples \( p_{T^*} \), \( q_S \) and \( q_{S^*} \).

**Proof.** Let \( \tilde{A}, \tilde{B} \in M_{2\ell \times 2\ell} \). For any subset \( X \subseteq [2\ell] \), we denote \( \tilde{A}_X = (\tilde{A}_{ij})_{i \in X, j \in [\ell]} \) and \( \tilde{B}_X = (\tilde{B}_{ij})_{i \in X, j \in [\ell]} \). Using Laplace’s formula, we immediately see that

\[
\det(\tilde{A} \cdot \tilde{B}) = \sum_{X \subseteq [2\ell]} \det(\tilde{A}_X) \det(\tilde{B}_X).
\]

Moreover, for any subsets \( S, T \subseteq [\ell] \), if \( X = S \cup \{ \ell + t : t \in T \} \) and \( \tilde{A} = (AV(q_{S^*}), CV(p_T)) \), since \( A \) and \( C \) are diagonal matrices, we check that

\[
\det(\tilde{A}_X) = \Delta(q_S, p_T) \prod_{s \in S} A_{ss} \prod_{t \in T} C_{tt}.
\]

Similarly, we have

\[
X^* = S^* \cup \{ \ell + t : t \in T^* \} \quad \text{and} \quad \tilde{B} = (BV(q_{S^*}), DV(p_T^*)),
\]

so that

\[
\det(\tilde{B}_{X^*}) = \Delta(q_{S^*}, p_{T^*}) \prod_{s \in S^*} B_{ss} \prod_{t \in T^*} D_{tt}.
\]

To complete the proof, it remains to observe that in formula (52), summing over all subset \( X \subseteq [2\ell] \) with cardinal \(|X| = \ell|\), is equivalent to sum over all pair \((S, T) \subseteq [\ell]^2\) such that \(|S| + |T| = \ell|\). □

The point of this expansion is that all the terms can be controlled uniformly.

**Proposition 3.4.** For all \( k, \ell \geq 0 \), all \( \epsilon > 0 \), and all \( \delta > 0 \) sufficiently small, there is a constant \( C > 0 \) so that uniformly in \( B' \in \mathcal{B}_{k, \ell, \delta} \), uniformly in \( B \in \mathcal{W}_{\ell, \delta}(\mathcal{B}') \) and uniformly in \( S, T \subseteq [2(k + \ell)] \) with \(|S| + |T| = 2(k + \ell)|\),

\[
\left| \frac{\Delta(q_S, p_T) \Delta(q_{S^*}, p_{T^*})}{\Delta(q) \Delta(p)} \right| \leq CE[e^{\text{AB}(G)}].
\]

Once the asymptotics of the matrix \( M_N \) are known, this bound is the main technical task to obtain proposition 3.1. Its proof relies on both conditions (48) and (49) and is based on a rather sophisticated combinatorial matching between the terms in the nominator and denominator of the LHS of (53). For this reason, we postpone it to the section 4.

The relation between the left hand side and right hand side of (53) may not be clear at first sight. Observe that the next lemma shows that there is essentially equality (with constant \( C = 1 \)) in (53) if the sets \( T \) and \( S \) are chosen so that \( q_S = q_+ \) and \( p_T = p_- \) where

\[
p_+ = J(Z), \quad p_- = J(W), \quad q_+ = J(\overline{Z}), \quad q_- = J(\overline{W}).
\]

In particular, these notation are chosen so that \( p_+, q_+ \subset H_+ \) and \( p_-, q_- \subset H_- \).
Lemma 3.5. With the above notation,

\begin{equation}
\mathbb{E}\left[e^{\mathcal{B}(G)}\right] = \frac{\left|\Delta(q_+, p_-)\right|^2}{\left|\Delta(p)\Delta(q)\right|} + O_{N \to \infty} \left(N^{-\delta} \mathbb{E}[e^{\mathcal{B}(G)}]\right).
\end{equation}

Proof. First, if we write \(\Delta(p) = \Delta(p_+; p_-)\) and \(\Delta(q) = \Delta(q_+; q_-)\), then expand the Vandermonde determinants on the RHS of formula (55) into products, we see that

\begin{equation}
\left|\Delta(q_+, p_-)\right|^2 = \frac{\prod_{p_+ \times p_-} |p - q|^2}{\prod_{p_+ \times p_-} |p - p'| \prod_{q_+ \times q_-} |q - q'|}.
\end{equation}

Second, by formula (2),

\begin{align}
\mathbb{E}\left[e^{\mathcal{B}(G)}\right] &= \exp\left(2\mathbb{E}\left[\sum_{z \in Z} G(z)G(z') + \sum_{w \not\in W} G(w)G(w') - 2 \sum_{z \not\in W} G(z)G(w)\right]\right) \\
&= \frac{\prod_{Z \times W} |1 - z\bar{w}|^2}{\prod_{Z \times Z} |1 - z\bar{z}| \prod_{W \times W} |1 - w\bar{w}'|}.
\end{align}

By definition of the Joukowsky transform, (6), we have

\begin{equation}
|p - q| = |J(z) - J(w)| = \frac{|z - w| |1 - z\bar{w}|}{2|z|}.
\end{equation}

In particular, uniformly for all \(z, w \in \mathcal{D}_{N, \delta}\),

\begin{equation}
|p - \bar{q}| = |1 - z\bar{w}| + O_{N \to \infty} (N^{-\delta}).
\end{equation}

Thus, if we replace all the terms in formula (57) using this asymptotic expansion, we obtain

\begin{equation}
\mathbb{E}\left[e^{\mathcal{B}(G)}\right] = \frac{\prod_{p_+ \times p_+} |p - \bar{q}|^2}{\prod_{p_+ \times p_+} |p - \bar{p}| \prod_{q_+ \times q_+} |q - \bar{q}|} + O_{N \to \infty} \left(N^{-\delta} \mathbb{E}[e^{\mathcal{B}(G)}]\right).
\end{equation}

Since \(\bar{q}_+ = q_-\) and \(\bar{p}_+ = p_-\), the claim follows immediately from formula (56). \(\square\)

We are now in position to complete the proof of proposition 3.1. In fact, equipped with the previous lemmas, this just boils down to basic linear algebra.

Proof of Proposition 3.1. We start by applying lemma 3.3 to formula (51), this leads us to

\begin{equation}
\mathbb{E}\left[e^{\mathcal{B}(Z_N)}\right] = \sum_{S, T \subseteq [\ell], |T| + |S| = \ell} \frac{\Delta(q_S, p_T)\Delta(q_S^*, p_{T^*})}{\Delta(q)\Delta(p)} \\
\times \prod_{q \in S} (M_N)_{22}(q) \prod_{q \in S^*} (M_N)_{12}(q) \prod_{p \in T} (M_N)_{21}(p) \prod_{p \in T^*} (M_N)_{11}(p).
\end{equation}

For any \(0 < \delta < 1/2\), the image of the sets \(\mathcal{D}_{N, \delta}\) and \(\overline{\mathcal{D}_{N, \delta}}\) under the Joukowsky transform are compact sets \(\mathcal{X}_{N, \delta}^{\pm}\) which satisfy the assumption of proposition 2.3. Hence, we can replace the entries of \(M_N\) in formula (59) by \(M_{N, \pm}(x)\) up to an error...
of order $N^{-\delta}$. Moreover, by proposition 3.4, this error is uniformly controlled by some constant multiple of $\mathbb{E}[e^{3\delta(B)}]$. Then, using lemma 3.3 backward, we obtain

$$\mathbb{E} \left[ e^{3\delta(Z_N)} \right] = \frac{1}{\Delta(p)\Delta(q)} \begin{vmatrix} M_{22}^+ V_\ell(q_+) & M_{12}^+ V_\ell(q_-) \\ M_{22}^- V_\ell(p_-) & M_{12}^- V_\ell(p_+) \\ M_{21}^+ V_\ell(p_+) & M_{11}^+ V_\ell(q_-) \\ M_{21}^- V_\ell(q_-) & M_{11}^- V_\ell(p_+) \end{vmatrix} + O \left( N^{-\delta} \mathbb{E}[e^{3\delta(B)}] \right).$$

where we let $M_{ij}^\pm = (M_{ij}^\infty(x))_{ij}$, not to overload the notation. Note that $M_{ij}^\pm$ are interpreted as diagonal matrices and, for instance, by formula (11),

$$M_{22}^+ V_\ell(q_+) = \begin{pmatrix} M_{22}^\infty(x) & 0 \\ \vdots & \vdots \\ 0 & M_{22}^\infty(x) \end{pmatrix} \begin{pmatrix} 1 & q_1 & \cdots & q_{\ell-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & q_{\ell/2} & \cdots & q_{\ell-1} \end{pmatrix}.$$

By definition of $M_{ij}^\infty$, formula (83), we have the relations:

$$M_{22}^- = -M_{21}^+, \quad M_{12}^- = -M_{11}^+, \quad M_{21}^- = M_{22}^+, \quad M_{11}^- = M_{12}^+.$$

So, after rearranging (by permuting certain rows) formula (60), we obtain

$$\mathbb{E} \left[ e^{3\delta(Z_N)} \right] = \frac{1}{\Delta(p)\Delta(q)} \begin{vmatrix} M_{22}^+ V_\ell(q_+) & M_{12}^+ V_\ell(q_-) \\ M_{22}^- V_\ell(p_-) & M_{12}^- V_\ell(p_+) \\ M_{21}^+ V_\ell(p_+) & M_{11}^+ V_\ell(q_-) \\ M_{21}^- V_\ell(q_-) & M_{11}^- V_\ell(p_+) \end{vmatrix} + O \left( N^{-\delta} \mathbb{E}[e^{3\delta(B)}] \right).$$

Then, we use the factorization

$$\begin{pmatrix} M_{22}^+ V_\ell(q_+) \\ M_{22}^- V_\ell(p_-) \\ M_{21}^+ V_\ell(p_+) \\ M_{21}^- V_\ell(q_-) \end{pmatrix} = \begin{pmatrix} V_\ell(q_+, p_-) & 0 \\ 0 & V_\ell(p_+, q_-) \end{pmatrix} \begin{pmatrix} M_{22}^+ & M_{12}^+ \\ M_{21}^+ & M_{11}^+ \end{pmatrix}.$$

Note that the last matrix is the Kronecker product $\begin{pmatrix} M_{22}^+ & M_{12}^+ \\ M_{21}^+ & M_{11}^+ \end{pmatrix} \otimes I_\ell$, so that its determinant is $M_{22}^+ M_{21}^+ - M_{21}^+ M_{22}^+$ which is identically equal to 1 by lemma 2.1. Moreover, since $V_\ell(q_+, p_-)$ and $V_\ell(p_+, q_-)$ are exactly Vandermonde matrices, we obtain

$$\begin{vmatrix} M_{22}^+ V_\ell(q_+) & M_{12}^+ V_\ell(q_-) \\ M_{22}^- V_\ell(p_-) & M_{12}^- V_\ell(p_+) \\ M_{21}^+ V_\ell(p_+) & M_{11}^+ V_\ell(q_-) \\ M_{21}^- V_\ell(q_-) & M_{11}^- V_\ell(p_+) \end{vmatrix} = \Delta(q_+, p_-) \Delta(p_+, q_-) = |\Delta(q_+, p_-)|^2$$

since $p_- = \overrightarrow{p_+}$, $q_- = \overrightarrow{q_+}$, and $\ell$ is even. Hence, combining formula (61) and lemma 3.3, we conclude that

$$\mathbb{E} \left[ e^{3\delta(Z_N)} \right] = \mathbb{E} \left[ e^{3\delta(G)} \right] + O_{N \to \infty} \left( N^{-\delta} \mathbb{E}[e^{3\delta(B)}] \right),$$

which is the required asymptotics. □
4. Matching lemma

The purpose of this section is to prove proposition 4.4 Recall that $Z$ and $W$ are two disjoint sets in the domain $\mathcal{D}_{N,\ell}$, such that $|Z| = |W| = k + \ell$ and which satisfy the conditions (58) and (59). For notational convenience, instead of viewing $S, T$ as subsets of the integers $[2(\ell + k)]$, we let $S \subseteq \mathbb{W} \cap W$ and $T \subseteq \mathbb{Z} \cap Z$. We also define

\begin{align}
T_+ &= \{ z \in Z : z \in T \} \\
T_- &= \{ z \in Z : \bar{z} \in T \} \\
T^+_* &= \{ z \in Z : z \in T^* \} \\
T^-_* &= \{ z \in Z : \bar{z} \in T^* \}
\end{align}

and similarly for $S_+, S_-, S^+_*, S^-_*$. Note that all these sets lie in the upper-half disk $D_+$ and that we have the decompositions:

\begin{align}
Z &= T_+ \cup T^+_* = T_- \cup T^-_* \\
W &= S_+ \cup S^+_* = S_- \cup S^-_*
\end{align}

For any function $f : \mathbb{D}^2 \to \mathbb{R}$ and any finite sets $A$ and $B$ of points in $\mathbb{D}$, we denote

$$f(A, B) = \prod_{z \in A, w \in B} f(z, w)$$

and

$$L_f(A, B) = \frac{f(A, B)f(A^*, B^*)}{f(A, A^*)f(B, B^*)}$$

where $A^* = Z \setminus A$ and $B^* = W \setminus B$ respectively. In the following, we let

$$\Gamma(z, w) = |1 - z\bar{w}|,$$

d$E$ be the Euclidean metric on $\mathbb{C}$, and we introduce the pseudo-hyperbolic metric on $\mathbb{D}$ given by

$$d(z, w) = \tanh(d_E(z, w)) = \frac{|z - w|}{|1 - z\bar{w}|}.$$

It is easy to check that $d$ is indeed a metric which is uniformly bounded by 1. As before, we let $p = J(z)$ and $q = J(w)$ for $z \in Z$ and $w \in W$ respectively. Finally, recall that, by formula (58), if the parameter $N$ is large, we have

$$\frac{1}{4} \leq \frac{|z - w|}{|p - q|} \leq 1 \quad \text{and} \quad \frac{1}{4} \leq \frac{|1 - z\bar{w}|}{|p - q|} \leq 1.$$

Note that expanding the Vandermonde determinant on the LHS of (53), we obtain

$$\frac{\Delta(q_S, p_T)\Delta(q_{S'}, p_{T'})}{\Delta(q)\Delta(p)} = \prod_{T \times S} |p - q| \prod_{T^* \times S^*} |p - q|$$

$$\prod_{I \times T} |p - p'| \prod_{S \times S^*} |q - q'|$$

$$= L_{d_E}(T, S),$$

using the notation (64). Moreover, we can rewrite formula (57) as

$$E[e^{\mathcal{B}(G)}] = L_{\Gamma}(Z, W)$$

where by convention: $Z^* = Z$ and $W^* = W$. Thus, we want to demonstrate that there exists a constant $C > 0$ so that uniformly over many choices of bias $\mathcal{B}$ and subsets $S \subseteq \mathbb{W} \cap W$, $T \subseteq \mathbb{Z} \cap Z$, we have

$$L_{d_E}(T, S) \leq C L_{\Gamma}(Z, W).$$
First, using the estimate (66), the definitions of the metric $d$ and (62), it is easy to verify that there exists a constant $C > 0$ which only depends on $|Z|$ such that

$$L_{d_{q}}(T, S) \leq C L_{T}(T, S)L_{d}(T_{+}, S_{+})L_{d}(T_{-}, S_{-}).$$  

The second step is to compare the quantities $L_{T}(T, S)$ and $L_{T}(Z, W)$. Using the relations (62), we have

$$L_{T}(T, S) = \frac{\Gamma(T_{+}, S_{-})\Gamma(T_{-}, S_{+})\Gamma(T_{+}^{*}, S_{+}^{*})\Gamma(T_{-}^{*}, S_{-}^{*})}{\Gamma(T_{+}, T_{-}^{*})\Gamma(T_{-}, T_{+}^{*})\Gamma(S_{+}, S_{+}^{*})\Gamma(S_{-}, S_{-}^{*})}L_{T}(T_{+}, S_{+})L_{T}(T_{-}, S_{-}).$$

On the other hand, using (63), we also see that

$$L_{T}(Z, W) = \frac{\Gamma(T_{+}, S_{-})\Gamma(T_{-}, S_{+})\Gamma(T_{+}^{*}, S_{+}^{*})\Gamma(T_{-}^{*}, S_{-}^{*})}{\Gamma(T_{+}, T_{-}^{*})\Gamma(T_{-}, T_{+}^{*})\Gamma(S_{+}, S_{+}^{*})\Gamma(S_{-}, S_{-}^{*})}L_{T}(T_{+}, T_{-})L_{T}(T_{+}, T_{-})L_{T}(S_{+}, S_{-}).$$

Hence, several terms cancel and we are left with

$$L_{T}(Z, W) = \frac{L_{T}(T_{+}, S_{+})L_{T}(T_{-}, S_{+})}{L_{T}(T_{+}, S_{+})L_{T}(T_{-}, S_{-}) L_{T}(T_{+}, T_{-})L_{T}(S_{+}, S_{-})}.\tag{69}$$

We will ultimately show that this quantity is bounded by 1. To do this, we begin by using the following combinatorial identities to take advantage of cancellation in (69).

**Lemma 4.1.** Let $X$ and $Y$ be any finite sets of (distinct) points in $\mathbb{D}$. For any subsets $A, B \subseteq X, E, F \subseteq Y$, we have

$$\frac{\Gamma(A, E)}{\Gamma(B, E)} = \frac{\Gamma(A \cap B, E \cap F)}{\Gamma(A \cap B, E \cap F)}\tag{70}$$

and

$$\frac{L_{T}(A, E)L_{T}(B, F)}{L_{T}(A, E)L_{T}(B, E)} = \frac{\Gamma(A \cap B, E \cap F)}{\Gamma(A \cap B, E \cap F)}\tag{71}$$

where $A^{*} = X \setminus A$, $E^{*} = Y \setminus E$ and similarly for $B^{*}$ and $F^{*}$.

**Proof.** Both formulae (70) and (71) follow from the simple observations that

$$\frac{\Gamma(A, E)}{\Gamma(B, E)} = \frac{\Gamma(A \cap B, E)}{\Gamma(A \cap B, E)}$$

and

$$\frac{L_{T}(A, E)L_{T}(B, F)}{L_{T}(A, E)L_{T}(B, E)} = \frac{\Gamma(A \cap B, E \cap F)}{\Gamma(A \cap B, E \cap F)}\frac{\Gamma(A \cap B, E)}{\Gamma(A \cap B, E)}\frac{\Gamma(B, E)}{\Gamma(B, F)}.$$

Applying formula (70) with $A = F = T_{+}$ and $B = E = T_{-}$, we obtain

$$L_{T}(T_{+}, T_{-}) = \frac{\Gamma(T_{+} \cap T_{-}, T_{+} \cap T_{-})\Gamma(T_{+}^{*} \cap T_{-}^{*}, T_{+}^{*} \cap T_{-}^{*})}{\Gamma(T_{+} \cap T_{-}, T_{+}^{*} \cap T_{-}^{*})^{2}}.\tag{70}$$

Similarly, applying formula (71) with $A = T_{+}, B = T_{-}$, $E = S_{+}$ and $F = S_{-}$, we get

$$\frac{L_{T}(T_{+}, S_{+})L_{T}(T_{-}, S_{+})}{L_{T}(T_{+}, S_{+})L_{T}(T_{-}, S_{+})} = \frac{\Gamma(T_{+} \cap T_{-}, S_{+} \cap S_{-})\Gamma(T_{+}^{*} \cap T_{-}^{*}, S_{+}^{*} \cap S_{-}^{*})}{\Gamma(T_{+} \cap T_{-}, S_{+}^{*} \cap S_{-}^{*})^{2}}.\tag{71}$$
So, if we now denote \( A = T_+ \cap T_- \), \( A^* = T_+^* \cap T_-^* \), \( B = S_+ \cap S_- \) and \( B^* = S_+^* \cap S_-^* \), by formula (69), this implies that

\[
\frac{L_T(Z,W)}{L_T(T,S)} = \frac{\Gamma(A, B^*)^2 \Gamma(A^*, B)^2 \Gamma(A, A^*)^2 \Gamma(B, B^*)^2}{\Gamma(A, B)^2 \Gamma(A^*, B^*)^2 \Gamma(A, A^*)^2 \Gamma(B, B^*)^2}
\]

\[
= \mathbb{E} \left[ \exp \left( \sum_{z \in A \cap B} G(z) - \sum_{w \in A^* \cap B^*} G(w) \right) \right].
\]

The last equality follows directly from the definition of \( \Gamma \) and the covariance structure of the Gaussian field \( G \), (2). By Jensen’s inequality, this implies that \( L_T(T, S) \leq L_T(Z, W) \) and by formula (68),

\[
\frac{L_{d_{kl}}(T, S)}{L_T(Z, W)} \leq C L_d(T_+, S_+) L_d(T_-, S_-).
\]

Hence, in order to prove (67) to complete the proof of proposition 3.4, it remains to show that there exists a constant \( C > 0 \) which depends only on the parameters \( \epsilon, k, \ell \) such that

\[
\max \left\{ L_d(T_+, S_+); L_d(T_-, S_-) \right\} \leq C.
\]

This is the point of lemma 4.3 below. In particular, note that the hypotheses on \( \mathcal{B} \in \mathcal{B}_{\epsilon, d} \) and \( \mathcal{B} \in \mathcal{B}_{0, d}(\mathbb{D}^\prime) \), (18–19), imply that after correctly re-ordering the sets \( Z \) and \( W \), \( Z \times W \) is a pair-configuration for the pseudohyperbolic metric (65) in the sense of the following definition.

**Definition 4.2.** Let \( d \) be a metric such that \( d(z, w) \leq 1 \) for all \( z, w \in \mathbb{D} \) and let \( \epsilon > 0 \) be a small constant. A **pair-configuration** is a collection \( \mathcal{U} = \{(z_1, w_1), \ldots, (z_{\ell+k}, w_{\ell+k})\} \in (\mathbb{D} \times \mathbb{D})^{\ell+k} \) such that the following conditions hold:

\[
d(z_j, w_j) \leq \left( \min_{w \in W \setminus \{w_j\}} d(w, w_j) \right) \wedge \min_{z \in Z \setminus \{z_j\}} d(z, z_j) \quad \text{if } 1 \leq j \leq \ell,
\]

\[
\begin{cases}
\min_{w \in W \setminus \{w_j\}} d(w, w_j) & \text{if } \ell < i < j \leq \ell + k \\
\min_{z \in Z \setminus \{z_j\}} d(z, z_j) & \text{if } \ell < i, j \leq \ell + k
\end{cases}
\]

where, as usual \( Z = (z_1, \ldots, z_{k+\ell}) \) and \( W = (w_1, \ldots, w_{k+\ell}) \).

**Lemma 4.3.** There exists a constant \( C > 0 \) which only depends on \( \epsilon, \ell, k \) such that for any pair-configuration \( \mathcal{U} = (Z, W) \) and for any subsets \( T \subseteq Z, S \subseteq W \), we have:

\[
L_d(T, S) = \frac{d(T, S) d(T^*, S^*)}{d(T, T^*) d(S, S^*)} \leq C,
\]

where \( T^* = Z \setminus T \) and \( S^* = W \setminus S \).

**Proof.** In this proof, we will use the notation \( d_1 \ll d_2 \) for the existence of a constant \( c > 0 \) which may depend on \( \epsilon > 0, \ell > 0 \), and \( k > 0 \) so that

\[
0 < d_1 < c \cdot d_2,
\]

and \( d_1 \succ d_2 \) if both \( d_1 \ll d_2 \) and \( d_1 \gg d_2 \). Let us reformulate the problem by introducing the complete graph \( G \) with vertex set \( \mathcal{U} \) including a loop attached to
we have and three categories: \( \rho \) and \( \hat{\rho} \). We will proceed by estimating the contribution from the different types of edges \( \rho \), we interpret each vertex. We will denote \( u_j = (z_j, w_j) \) the vertices and \( u_j u_j \) the edges of \( G \). Given the sets \( T \subseteq Z, S \subseteq W \), we define for all \( i \neq j \),

\[
\rho(u_i u_j) = d(z_i, z_j)^{1} (z_i, z_j) \in T \times T \times T \times T \ d(w_i, w_j)^{1} (w_i, w_j) \in S \times S \times S \times S ,
\]

\[
\hat{\rho}(u_i u_j) = d(z_i, w_j)^{1} (z_i, w_j) \in T \times S \times T \times S \ d(z_j, w_i)^{1} (z_j, w_i) \in T \times S \times T \times S ,
\]

as well as \( \rho(u_i u_i) = 1 \) and \( \hat{\rho}(u_i u_i) = d(z_i, w_i)^{1} (z_i, w_i) \in T \times S \times T \times S \). In the following, we interpret \( \rho, \hat{\rho} : E \to \mathbb{R}_+ \) as cost functions defined on the edges of the graph \( G \) such that

\[
L_d(T, S) = \prod_{i,j \in [k+\ell]} \frac{\hat{\rho}(u_i u_j)}{\rho(u_i u_j)}.
\]

Hence, the game is to show that

\[
\prod_{i,j \in [k+\ell]} \hat{\rho}(u_i u_j) \ll \prod_{i,j \in [k+\ell]} \rho(u_i u_j).
\]

We will proceed by estimating the contribution from the different types of edges step by step. Let us first consider the set of vertices

\[
W = \{ u_j \in T \times S^* \cup T^* \times S : 1 \leq j \leq \ell \}.
\]

We claim that if \( u_1 \in W \), then

\[
\prod_{j=1}^{k+\ell} \frac{\hat{\rho}(u_1 u_j)}{\rho(u_1 u_j)} \ll 1.
\] (76)

Without loss of generality, we assume that \( u_1 \in T \times S^* \). Then, for any \( j \in [k+\ell] \), we have

\[
\rho(u_1 u_j) = \begin{cases} 
    d(w_1, w_j) & \text{if } u_j \in T \times S \\
    d(z_1, z_j) & \text{if } u_j \in T^* \times S^* \\
    d(z_1, z_j) d(w_1, w_j) & \text{if } u_j \in T^* \times S \\
    1 & \text{else}
\end{cases}
\]

and

\[
\hat{\rho}(u_1 u_j) = \begin{cases} 
    d(z_1, w_j) & \text{if } u_j \in T \times S \\
    d(z_1, w_1) & \text{if } u_j \in T^* \times S^* \\
    d(z_1, w_j) d(z_j, w_1) & \text{if } u_j \in T^* \times S \\
    1 & \text{else}
\end{cases}.
\]

Using the condition (73) and the triangle inequality:

\[
d(z_1, w_j) \leq d(z_1, w_1) + d(w_1, w_j) \leq 2d(w_1, w_j)
\]

\[
d(z_j, w_1) \leq d(z_j, z_1) + d(z_1, w_1) \leq 2d(z_1, z_1).
\]

This establishes that \( \hat{\rho}(u_1 u_j) \ll \rho(u_1 u_j) \) and we obtain formula (76).

So, if \( G' = (\mathcal{U}', E') \) is the complete graph (including all the loops) with vertex-set \( \mathcal{U}' := \mathcal{U} \setminus W \), formula (76) implies that

\[
\prod_{i,j \in [k+\ell]} \frac{\hat{\rho}(u_i u_j)}{\rho(u_i u_j)} \ll \prod_{u_j \in E'} \rho(u_i u_j).
\] (77)

The next reduction step is more sophisticated. Let us split the vertex-set \( \mathcal{U}' \) in three categories:

\[
\mathcal{A} = \{ u_j \in T \times S : 1 \leq j \leq \ell \} , \quad \mathcal{A}^* = \{ u_j \in T^* \times S^* : 1 \leq j \leq \ell \} ,
\]
and \( \mathfrak{U} = \{ u_j : \ell < j \leq \ell + k \} \) so that the edges of \( G' \) are decomposed as

\[
E' = \left( (\mathfrak{A} \cup \mathfrak{A}^*) \times (\mathfrak{A} \cup \mathfrak{A}^*) \right) \cup \left( (\mathfrak{A} \cup \mathfrak{A}^*) \times \mathfrak{U} \right) \cup \left( \mathfrak{U} \times \mathfrak{U} \right).
\]

We will proceed by induction to estimate the contribution coming from two vertices \( u_1 \in \mathfrak{A} \) and \( u_2 \in \mathfrak{A}^* \). Without loss of generality, \( |\mathfrak{A}| \geq |\mathfrak{A}^*| \) and provided that \( \mathfrak{A}^* \) is not empty, we choose \( u_1 \) and \( u_2 \) so that

\[
d(w_1, w_2) = \min \{ d(w_1, w_j) : u_j \in \mathfrak{A}, u_j \in \mathfrak{A}^* \}.
\]

We will first compare the quantities

\[
\prod_{u_j \in \mathfrak{U} \setminus \mathfrak{W}} \rho(u_1 u_j) \rho(u_2 u_j) = \prod_{u_j \in \mathfrak{W}} d(z_2, z_j) d(w_2, w_j) \prod_{u_j \in \mathfrak{A}^*} d(z_1, z_j) d(w_1, w_j)
\]

and

\[
\prod_{u_j \in \mathfrak{U} \setminus \mathfrak{W}} \tilde{\rho}(u_1 u_j) \tilde{\rho}(u_2 u_j) = \prod_{u_j \in \mathfrak{W}} d(z_1, w_j) d(z_1, w_1) \prod_{u_j \in \mathfrak{A}^*} d(z_2, w_j) d(z_2, w_2).
\]

A straightforward consequence of conditions (73) and (79) is that

\[
d(w_2, z_j) \asymp d(w_1, w_j) \quad \text{and} \quad d(z_2, w_j) \asymp d(z_1, z_j) \quad \forall \ u_j \in \mathfrak{A},
\]

\[
d(w_1, z_j) \asymp d(w_2, w_j) \quad \text{and} \quad d(z_1, w_j) \asymp d(z_2, z_j) \quad \forall \ u_j \in \mathfrak{A}.
\]

Hence, we obtain

\[
\prod_{u_j \in \mathfrak{U} \setminus \mathfrak{W}} \tilde{\rho}(u_1 u_j) \tilde{\rho}(u_2 u_j) \asymp \prod_{u_j \in \mathfrak{U} \setminus \mathfrak{W}} \rho(u_1 u_j) \rho(u_2 u_j).
\]

Now, define the points \( z\# \) and \( w\# \) by

\[
d(z_1, z\#) = \min \{ d(z_1, z_j) : u_j \in \mathfrak{W} \} \quad \text{and} \quad d(w_1, w\#) = \min \{ d(w_1, w_j) : u_j \in \mathfrak{W} \}.
\]

Because of the separation conditions (74), only one of these two distances can be smaller than \( \epsilon/3 \). Without loss of generality, let us assume that it is attained at \( z\# \) and that \( z\# \in T \), in which case we have

\[
\prod_{u_j \in \mathfrak{W}} \rho(u_1 u_j) \rho(u_2 u_j) \asymp d(z_2, z\#).
\]

If \( d(w_1, z\#) \leq 2d(z_2, z\#) \), by (73), we have

\[
d(w_1, z\#) d(z_1, w_1) d(z_2, w_2) \leq 2d(z_2, z\#) d(z_1, z_2) d(w_1, w_2)
\]

On the other hand, if \( d(z_2, z\#) < d(w_1, z\#)/2 \), by the triangle inequality

\[
d(w_1, z\#) \leq d(z\#, z_2) + d(z_2, z_1) + d(z_1, w_1)
\]

and

\[
d(w_1, z\#) \leq 4d(z_2, z_1).
\]

This implies that the estimate (82) still holds with an extra factor of 2. The bottom line is that since

\[
\rho(u_1 u_1) \rho(u_2 u_2) \rho(u_1 u_2) = d(z_1, z_2) d(w_1, w_2)
\]

and

\[
\tilde{\rho}(u_1 u_1) \tilde{\rho}(u_2 u_2) \tilde{\rho}(u_1 u_2) = d(w_1, z\#) d(z_1, w_1) d(z_2, w_2).
\]
By formula \((81)\), this implies that
\[
\hat{\rho}(u_1 u_1) \hat{\rho}(u_2 u_2) \hat{\rho}(u_1 u_3) \hat{\rho}(u_2 u_3) \ll \rho(u_1 u_1) \rho(u_2 u_2) \prod_{u_j \in \mathcal{U}} \rho(u_1 u_j) \rho(u_2 u_j).
\]

Combined with formula \((80)\), this shows that
\[
\prod_{u_j \in \mathcal{U}'} \hat{\rho}(u_1 u_j) \prod_{u_j \in \mathcal{U}'} \hat{\rho}(u_2 u_j) \ll \prod_{u_j \in \mathcal{U}'} \rho(u_1 u_j) \prod_{u_j \in \mathcal{U}'} \rho(u_2 u_j).
\]

Hence, we can disregard the full contribution of the vertices \(u_1, u_2\). By induction, we can repeat this procedure until the set \(\mathcal{A}' = \emptyset\). In this case, by \((78)\), we are left with the edge-set
\[
E' = (\mathcal{A} \times \mathcal{A}) \cup (\mathcal{A} \times \mathcal{V}) \cup (\mathcal{V} \times \mathcal{V}).
\]

By definition,
\[
\prod_{u_i, u_j \in \mathcal{A} \times \mathcal{A}} \rho(u_i u_j) = 1,
\]
and using the separation conditions \((74)\), we have
\[
\prod_{u_i, u_j \in \mathcal{V} \times \mathcal{V}} \rho(u_i u_j) \gg 1.
\]

Thus, by formula \((77)\), we have proved that
\[
\prod_{i,j \in [k+\ell]} \frac{\hat{\rho}(u_i u_j)}{\rho(u_i u_j)} \ll \prod_{u_i, u_j \in \mathcal{A} \times \mathcal{A}} \hat{\rho}(u_i u_j) \prod_{u_i, u_j \in \mathcal{V} \times \mathcal{V}} \hat{\rho}(u_i u_j).
\]

Let \(u_1 \in \mathcal{A}\) and \(u_2 \in \mathcal{V}\). Since \(d(z_2, w_2) \geq \epsilon\) and \(d(z_1, w_1) \leq \rho(z_1, z_2) \wedge d(w_1, w_2)\), we must have
\[
d(z_1, z_2) \geq \epsilon/3 \quad \text{or} \quad d(w_1, w_2) \geq \epsilon/3.
\]

Moreover, also because of the separation condition \((74)\), there is at most one point \(u_2 \in \mathcal{V}\) such that one of this condition is not true. This implies that
\[
\prod_{u_j \in \mathcal{V}} \rho(u_1 u_j) \gg d(z_1, z_2) \wedge d(w_1, w_2) \gg d(z_1, w_1) = \hat{\rho}(u_2 u_1).
\]

So, we have proved that for any vertex \(u_1 \in \mathcal{A}\), we have
\[
\hat{\rho}(u_1 u_1) \prod_{u_j \in \mathcal{V}} \hat{\rho}(u_1 u_j) \ll 1.
\]

Combining the estimates \((84)\) and \((85)\), this completes the proof of the lemma. \(\square\)

5. Upper bound

In this section we prove the upper bound in Theorem 1.3. In fact, to get theorem 5.1 below, it suffices to assume that the equilibrium density \(\hat{\rho}\) has a compact support \(\mathcal{J}\), it is uniformly bounded, and its log-potential
\[
\tilde{g}(x) := -\int_{\mathbb{R}} \log |x - u| g(du)
\]
is Lipschitz continuous on $\mathbb{R}$. Recall also that, up to rescaling the potential $V$, we assume that $J \subseteq [-1/2, 1/2]$.

**Theorem 5.1.** Assume that $V$ is analytic and regular. Then, there exists a constant $C_V > 0$, which depends only on the potential $V$, so that for any sequence $y_N \to \infty$ as $N \to \infty$,

$$
\liminf_{N \to \infty} \Pr \left[ \max_{q \in [-1,1]} Q_N(q) \leq \log N + C_V y_N \right] = 1.
$$

We begin by giving a completely deterministic relaxation of the problem.

**Lemma 5.2.** For any $N \in \mathbb{N}$, the function $Q_N(q)$ is subharmonic in $\mathbb{C}\setminus[-1,1]$ and, almost surely,

$$
\sup_{q \in \mathbb{C}} Q_N(q) = \max_{q \in [-1,1]} Q_N(q) \vee 0.
$$

**Proof.** In fact, the field $Q_N$ is subharmonic on $\mathbb{C}\setminus J$ where $J \subset [-1,1]$ is the support of the equilibrium density $\rho$. Hence the maximum of $Q_N$ is attained either on $J$ or along a sequence of points going to $\infty$. However, we may write

$$
Q_N(q) = \int_{\mathbb{R}} \log |1 - uq^{-1}| \rho_N(du) - N \int_{\mathbb{R}} \log |1 - uq^{-1}| \rho(du).
$$

Hence, $Q_N(q) \to 0$ as $q \to \infty$, and so the lemma follows. \hfill $\square$

Next, on the account that $Q_N$ arises as the modulus of a polynomial, we can reduce the task of bounding $Q_N$ on $[-1,1]$ to bounding $Q_N$ on a deterministic set of cardinality $O(N)$, by losing only an absolute constant. To do so, we may use the following reformulation of Lemma 4.3 in [CNM16].

**Lemma 5.3.** Fix $N \in \mathbb{N}$. Let $P_N$ be a polynomial of degree $N$. Then

$$
\max_{x \in [-1,1]} |P_N(x)| \leq 14 \max_{k \in [2N+1]} |P_N(x_k)|,
$$

where $x_k = \cos(\pi (k-1)/2N)$ for $k \in [2N+1] := \{1, \ldots, 2N+1\}$.

**Proof.** The bound is an immediate consequence of [CNM16, Lemma 4.3] which states that for any polynomial $W_n$ of degree at most $n \in \mathbb{N}$,

$$
\max_{|\omega| = 1} |W_n(\omega)| \leq 14 \max_{k \in [2n]} |W_n(e^{\pi i k/n})|.
$$

If we let

$$
W_{2N}(\omega) = P_N(J(\omega))\omega^N,
$$

then $W_{2N}$ is a polynomial of degree $2N$ such that for any $|\omega| = 1$,

$$
|W_{2N}(\omega)| = |P_N(\cos(\omega))|.
$$

This implies the claim. \hfill $\square$

**Corollary 5.4.** Under the assumption of theorem 5.1 there exists a constant $C_V^* \geq \log 14$, so that almost surely,

$$
\max_{x \in [-1,1]} Q_N(x) \leq \max_{k \in [2N+1]} Q_N(x_k) + C_V^*.
$$
Proof. Let \( x_\star \) be the point where the function \( x \mapsto Q_N(x) \) attains its maximum on \([-1, 1]\). By formula (8) for the equilibrium measure, the function \( \tilde{g} \in C^1(\mathbb{R}) \) and

\[
\tilde{g}'(x) = \int \log |x - u| \varrho'(u) du.
\]

In particular, the function \( \tilde{g}' \) is uniformly bounded on the interval \([-1, 1]\) and, since \( \{x_k = \cos(\pi(k - 1)/2N) : k \in [2N + 1]\} \) is a mesh of \([-1, 1]\) whose size \( \leq N^{-1} \), there exists a constant \( C'_V \) so that, almost surely,

\[
\max_{k \in [2N+1]} |\tilde{g}(x_\star) - \tilde{g}(x_k)| \leq \frac{C'_V}{N}.
\]

If \( P_N \) denotes the characteristic polynomial of the matrix \( A_N \), we may write

\[
Q_N(q) = \log |P_N(q)| - N\tilde{g}(q).
\]

By lemma 5.3 since \( \log(\cdot) \) is continuous increasing on \( \mathbb{R}_+ \), this implies that

\[
\max_{x \in [-1, 1]} Q_N(x) \leq \max_{k \in [2N+1]} \log |P_N(x_k)| + \log 14 - N\tilde{g}(x_\star)
\]

\[
\leq \max_{k \in [2N+1]} \left\{ \log |P_N(x_k)| - N\tilde{g}(x_k) \right\} + C'_V + \log 14.
\]

Letting \( C'_V = C'_V + \log 14 \), this completes the proof. \( \square \)

On the other hand, we claim that to control the maximum value of the field \( Q_N \), we can look slightly off the interval \([-1, 1]\). This will allow us to study more regularized statistics instead.

Proposition 5.5. Under the assumption of theorem 5.1, there exists a constant \( C_V > C'_V \), so that for any \( y \geq 1 \), almost surely,

\[
(87) \quad \max_{x \in [-1, 1]} Q_N(x) \leq \max_{k \in [2N+1]} Q_N(x_k - iy/N) + C_V y.
\]

Proof. First, by monotonicity, for any \( x \in \mathbb{R} \) and \( y > 0 \),

\[
\int_\mathbb{R} \log |x - u| \varrho_N(du) \leq \int_\mathbb{R} \log |x - iy - u| \varrho_N(du).
\]

On the other hand,

\[
\int_\mathbb{R} \log |x - iy - v| \varrho(u) du = \int_\mathbb{R} \log |x - v| \varrho(u) du + \int_\mathbb{R} \int_0^y \frac{t}{(x-u)^2 + t^2} dt \varrho(u) du.
\]

Thus, making a change of variables in the last integral and bounding uniformly the equilibrium density, we obtain

\[
\int_\mathbb{R} \log |x - iy - u| \varrho(u) du \leq \int_\mathbb{R} \log |x - u| \varrho(u) du + y\|\varrho\|_\infty \int_\mathbb{R} \int_0^1 \frac{t}{a^2 + t^2} dt du.
\]

Combining both inequalities for the log potentials of \( \varrho_N \) and \( \varrho \), we conclude that for any \( k \in [2N + 1] \),

\[
Q_N(x_k) \leq Q_N(x_k - iy/N) + y\|\varrho\|_\infty \frac{\pi}{3},
\]

and the bound (87) follows directly from corollary 5.4 with \( C_V = \frac{\pi\|\varrho\|_\infty}{3} + C'_V \). \( \square \)
The inequality (87) and the bound of corollary 2.5 for the exponential moments of the field $Q_N$ allow us to easily complete the proof of theorem 5.1.

**Proof of theorem 5.1.** Given any sequence $y_N \geq 1$, we let for all $k \in [2N+1]$,

$$q_k = x_k - iy_N/N = \cos(\pi(k-1)/2N) - iy_N/N.$$

By a union bound and Markov’s inequality, we have

$$\Pr \left[ \max_{k \in [2N+1]} Q_N(q_k) \geq \log N \right] \leq \frac{1}{N^2} \sum_{k \in [2N+1]} \mathbb{E}[e^{2Q_N(q_k)}].$$

To estimate the RHS of formula (88), we use the bound of corollary 2.5. Namely, we get

(88) $$\Pr \left[ \max_{k \in [2N+1]} Q_N(q_k) \geq \log N \right] \leq C \frac{y_N}{N^2} \sum_{k \in [2N+1]} \{1 + R(q_N)^2\}.$$

On the one hand, by formula (35), we have

(89) $$R(q)^2 = \prod_{i=0}^{m} |(q - a_i)(q - b_i)|^{-1/2} \leq C \sum_{i=0}^{m} \left\{ \frac{1}{\sqrt{|q - a_i|}} + \frac{1}{\sqrt{|q - b_i|}} \right\}.$$

On the other hand, for any $\theta \in [1/3, 2/3]$ and $k = 0, \cdots, 2N$,

$$|q_{k+1} - \cos(\pi\theta)|^2 = |\cos(\pi k/2N) - \cos(\pi\theta)|^2 + y_N^2 N^{-2} \geq |k/2N - \theta|^2 + y_N^2 N^{-2},$$

so that

$$|q_{k+1} - \cos(\pi\theta)|^{-1/2} \leq \begin{cases} \sqrt{N} y_N^{-1/2} & \text{if } |k - 2N\theta| \leq \log N \\ |k/2N - \theta|^{-1/2} & \text{if } |k - 2N\theta| > \log N \end{cases}.$$ 

Note that a simple estimate show that

$$\sum_{k=0, \ldots, 2N \atop |k-2N\theta| > \log N} \frac{1}{\sqrt{|k/2N - \theta|}} \leq 2\sqrt{N} \sum_{j \in [2N]} \sqrt{\frac{1}{j}} \leq 8N,$$

and, since $y_N \geq 1$, we obtain

$$\sum_{k \in [2N+1]} \frac{1}{\sqrt{|q_k - \cos(\pi\theta)|}} \leq 8N + \sqrt{N} \log N.$$

Recall that we assume that the support of the equilibrium measure $J \subseteq [-1/2, 1/2]$, i.e. $a_j, b_j \in [-1/2, 1/2]$ for all $j=0, \ldots, m$. Thus, combining the previous bound and the estimate (89), we have proved that

$$\sum_{k \in [2N+1]} R(q_k)^2 \leq CN.$$

By formula (88), this implies that

$$\Pr \left[ \max_{k \in [2N+1]} Q_N(q_k) \geq \log N \right] \leq \frac{C}{y_N}$$

and, by Proposition 5.5, we conclude that for any $N > 0$,

$$\Pr \left[ \max_{x \in [-1,1]} Q_N(x) \leq \log N + CV y_N \right] \geq 1 - \frac{C}{y_N}.$$
This completes the proof of theorem 5.1. □

6. LOWER BOUND PROOF

This section is concerned with the proof of Theorem 1.8. We will begin by giving an overview of the proof. In what follows we fix \( \delta > 0 \) a small positive constant, that we will ultimately take to 0. We let \( n = \lceil \log N \rceil \), and we define \( n_0 = \lceil (1-\delta) n \rceil \).

Recall that we would like to show, for some \( \omega \in \mathbb{T} \), the unit circle, that \( W_N(\omega \zeta_{n_0}) \) is large. Consider biasing the field \( W_N \) by the Radon-Nikodym derivative \( e^{2W_N(\omega \zeta_{n_0})} \). \( \mathbb{E}[e^{2W_N(\omega \zeta_{n_0})}]^{-1} \). If this were a Gaussian field, this bias would have the effect of changing the means of the field (c.f. Lemma 6.1). In particular, along the ray biased first and second moment of \( W \) however, would be unchanged. In particular, after biasing, it is typical behavior is large. Consider biasing the field \( W \) speak, we must constrain the trajectory of the walk method, which appears frequently in the study of log-correlated fields. Roughly speaking, we need more, as such events are too far from being independent. Hence, we apply a modified second moment bound for the event that \( W_N(\omega \zeta_{n_0}) \approx n_0 \). Moreover, on the event that \( W_N(\omega \zeta_{n_0}) \approx n_0 \), the Radon-Nikodym factor behaves like a constant, which would allow us to produce a lower bound for the event that \( W_N(\omega \zeta_{n_0}) \) is large without the biasing factor.

Hence, this suggests one strategy for showing the field is large: computing a biased first and second moment of \( W_N(\omega \zeta_{n_0}) \). Since we do not have direct access to field moments, we approximate these moments by using linear combinations of exponentials. We refer to these estimates as the field moment calculus (see Section 6.1).

This by itself leads to lower bounds for quantities like \( \Pr[ W_N(\omega \zeta_n) > n] \). To produce a lower bound for the maximum of \( W_N \), we need more, as such events are too far from being independent. Hence, we apply a modified second moment method, which appears frequently in the study of log-correlated fields. Roughly speaking, we must constrain the trajectory of the walk \( W_N(\omega \zeta_j) \approx j \) for many values of \( j \). Moreover, such a constraint turns out to be typical for the direction \( \omega \) along which the maximum is achieved, and hence this condition does not reduce the probability too much.

Let \( \eta = \eta_N \) be a slowly growing sequence to be specified later. Let \( b_k = k \lfloor n_0 / \eta \rfloor \) for all \( k = 0, \ldots, \eta \). Define \( r = r(\delta, N, \eta) \in \mathbb{N} \) to be the smallest integer so that \( z \in \mathbb{D} \) having \( d_{2r}(0, z) > b_r \) have \( |z| > 1 - N^{-2\delta} \). For \( \omega \in \mathbb{T} \), define the subset of fields \( \mathcal{E}(\omega) \) by

\[
\mathcal{E}(\omega) = \{ F : |F(\omega \zeta_{b_k}) - F(i \zeta_{b_k}) - (b_k - b_r)| \leq \eta \sqrt{n}, \forall r < k \leq \eta \}. 
\]

The point \( \zeta_{b_k} \) from the introduction is \( \zeta_{b_k} \).

Recall the set \( \Omega = \{ e^{i(\frac{h}{2} + h e^{\eta_0})} : h \in \mathbb{Z}, |h| < N^{-\delta} e^{n_0} \} \). Roughly, we would like to apply the second moment method to the counting function

\[
\hat{Z} = \sum_{\omega \in \Omega} 1 \{ W_N \in \mathcal{E}(\omega) \}. 
\]

Then, we would estimate \( \Pr[ \hat{Z} \geq 1 ] \geq \frac{(\mathbb{E}[\hat{Z}])^2}{\mathbb{E}[\hat{Z}]} \). One of the subtleties of this strategy is that to get this probability going to 1, one basically needs that for most pairs \((\omega_1, \omega_2)\),

\[
\Pr[ W_N \in \mathcal{E}(\omega_1) \cap \mathcal{E}(\omega_2) ] = \Pr[ W_N \in \mathcal{E}(\omega_1) ] \cdot \Pr[ W_N \in \mathcal{E}(\omega_2) ] (1 + o(1)).
\]

In the case of branching random walk, this is achieved using actual independence of the two events, once a suitably small common ancestral tree is discarded. In our case, no such independence is available. Further, the need to constantly bias
and unbias the measure by exponential factors which are not totally determined by being on the event $\mathcal{E}(\omega)$ causes losses which are not $1 + o(1)$.

We solve this problem by changing the second moment formalism. Define for any $\omega \in \Omega$

$$Y(\omega) = e^{2W_N(\omega_0)} - 2e^{2W_N(\iota_0)} \mathbf{1}\{W_N \in \mathcal{E}(\omega)\}. \tag{91}$$

In terms of this replacement for the indicator, we form the biased counting variable

$$Z = \sum_{\omega \in \Omega} Y(\omega).$$

The advantage of using $Y$ is that a nearly sharp (up to $1 + o(1)$ multiplicative error) estimate for $E[Y(\omega_1)Y(\omega_2)]$ when $\omega_1$ and $\omega_2$ are well-separated is attained by simply dropping the indicator.

We again want to show that this variable is non-negative, as this implies one of the indicated events holds. Applying Cauchy-Schwartz inequality,

$$E[1 \{Z > 0\}] \geq \left(\sum_{\omega} E[Y(\omega)]\right)^2 \sum_{\omega_1, \omega_2} E[Y(\omega_1)Y(\omega_2)], \tag{92}$$

where in both summations, $\omega$ ranges over $\Omega$.

As mentioned, an efficient upper bound for the denominator is simply to drop the indicators and use an estimate for mixed exponential moments. The numerator, however, is less amenable to a direct estimate but the moral here is that with enough uniformity in the estimates, mixed exponential moments are enough to estimate the RHS of (92) from below.

### 6.1. Field moment calculus.

In what follows, we will fix an $\omega \in \Omega$. First, for any finite bias term $\mathfrak{B}$, and any nonnegative measurable function $\phi$, we denote

$$E_{\mathfrak{B}}[\phi(W_N)] = \frac{E[e^{\mathfrak{B}(W_N)}\phi(W_N)]}{E[e^{\mathfrak{B}(W_N)}]}.$$

We will drop the dependence of the notation on $\mathfrak{B}$ when

$$\mathfrak{B}(F) = \mathfrak{B}_\omega(F) := 2F(\omega_0) - 2F(i\iota_0),$$

which plays a special role. This section is concerned with using the exponential moment criteria Assumption 1.7 to produce estimates on biased moments of $W_N$.

Hence, we recall the effect of biasing a jointly Gaussian vector by a linear functional of that vector is to change the mean of the Gaussian vector.

**Lemma 6.1.** Let $z$ and $y$ be finite subsets of $\mathbb{D}$ and let $F \in C(\mathbb{D})$ be a real valued function on $\mathbb{D}$. Let

$$\mathfrak{B} : F \mapsto \sum_{z \in z} 2F(z) - \sum_{y \in y} 2F(y),$$

and let $\mu : \zeta \mapsto E[W(\zeta)\mathfrak{B}(W)]$. Then

$$\frac{E[F(W)e^{\mathfrak{B}(W)}]}{E[e^{\mathfrak{B}(W)}]} = E[F(W + \mu)].$$
Proof. The field $W$ is almost surely continuous (in fact harmonic) in $D$, and so it suffices to assume that $F$ is a function which depends on the value of $W$ at finitely many points of $D$ by a density argument. Let $w$ be a finite set of points containing $z$ and $y$. The vector $(W(w))_{w \in w}$ is jointly Gaussian. Let $\Sigma$ be the covariance matrix of this vector. We can find a vector $v \in \mathbb{R}^k$ representing $\mathcal{B}$ in that

$$E[F(W)e^{\mathcal{B}(W)}] = \int_{\mathbb{R}^k} \frac{F(x)e^{v^t x}}{\sqrt{(2\pi)^k} |\det \Sigma|} dx.$$

We can now set $u = \Sigma v$ and complete the square, to get

$$E[F(W)e^{\mathcal{B}(W)}] = \int_{\mathbb{R}^k} \frac{F(x)e^{-u^t \Sigma^{-1} u}}{\sqrt{(2\pi)^k} |\det \Sigma|} dx.$$

The constant $e^{-u^t \Sigma^{-1} u} = E[e^{\mathcal{B}(W)}]$, as can be verified by setting $F \equiv 1$ and changing variables in the integral. Moreover, the right hand side is exactly the claimed expression in the lemma. \qed

Recall that $\mathcal{B}_{\ell,\delta}(\mathcal{B}_\omega)$ is the set of biases of the form

$$F \mapsto \mathcal{B}_\omega(F) + \sum_{z \in Z} 2F(z) - \sum_{y \in Y} 2F(y)$$

for some $z, y \in D_{N,\delta}$ of cardinalities $\ell$ having the property that $z$ and $y$ are paired: there exists a bijection $\phi: z \to y$ so that (49) holds. We let $\mathcal{B}_{\ell,\delta}(\mathcal{B}_\omega) \subset \mathcal{B}_{\ell,\delta}(\mathcal{B}_\omega)$ be the biases such that in addition to (49), for all $z \in Z$,

$$d_H(z, \phi(z)) \leq 1.$$

This simple change of mean lemma will be used in the following form.

**Corollary 6.2.** Fix $\ell \in \mathbb{N}$. Suppose Assumption $\text{MEM}(\ell)$ (1.7) holds. Let $\mu(\cdot) = E[W(\cdot)\mathcal{B}_\omega(W)]$ be the mean of $W$ under the bias $\mathcal{B}_\omega$. For any $\delta > 0$ there is a constant $C = C(\ell, \delta)$ so that for all $\omega \in \Omega$ and all $\mathcal{B} \in \mathcal{B}_{\ell,\delta}(\mathcal{B}_\omega)$

$$|E_{\mathcal{B}_\omega} [e^{\mathcal{B}(W_N) - \mathcal{B}_\omega(W_N)}] - E[\mathcal{B}(\mu + W) - \mathcal{B}_\omega(\mu + W)]| \leq CN^{-\delta}.$$

**Proof.** By definition of $E_{\mathcal{B}_\omega}$ and an application of Assumption (1.7)

$$E_{\mathcal{B}_\omega} [e^{\mathcal{B}(W_N) - \mathcal{B}_\omega(W_N)}] = \frac{E[e^{\mathcal{B}(W_N)}]}{E[e^{\mathcal{B}_\omega(W_N)}]} = \frac{E[e^{\mathcal{B}(W)}](1 + O(N^{-\delta}))}{E[e^{\mathcal{B}_\omega(W)}](1 + O(N^{-\delta}))} = E[e^{\mathcal{B}(\mu + W) - \mathcal{B}_\omega(\mu + W)}](1 + O(N^{-\delta}))$$

where, at last, we used lemma 6.1 applied to the field $W$. So, to complete the proof, we just need to establish that

$$\frac{E[e^{\mathcal{B}(W)}]}{E[e^{\mathcal{B}_\omega(W)}]} \asymp \ell.$$

As $W$ is Gaussian, this is equivalent to showing that

$$|\text{Var}(\mathcal{B}(W)) - \text{Var}(\mathcal{B}_\omega(W))| \ll \ell.$$
Expanding the variances, we have
\[
\operatorname{Var}(\mathcal{B}(W)) - \operatorname{Var}(\mathcal{B}_\omega(W)) = \sum_{z \in \mathcal{Z}} 4\mathbb{E} \left[ \mathcal{B}_\omega(W)(W(z) - W(\phi(z))) \right] \\
+ \sum_{z, w \in \mathcal{Z}} 4\mathbb{E} \left[ (W(z) - W(\phi(z)))(W(w) - W(\phi(w))) \right].
\]

Using Property (c) of Definition 1.5 and the fact that \(d_\mathcal{B}(z, \phi(z)) \leq 1\), each summand in the above equation is bounded by an absolute constant, from which the desired conclusion on the variances follows.

**Lemma 6.3.** Fix \(\ell \in \mathbb{N}\) and \(\omega \in \Omega\). Suppose Assumption MEM(\(\ell\)) \(\mathbf{[2.7]}\) holds. Let \(\mathcal{B} = \mathcal{B}_\omega\). There is a constant \(C = C(\ell, \delta)\) so that for all \(0 \leq \Delta \leq 1\) the following holds. Let \(\{z_i^\ell\}\) and \(\{y_i^\ell\}\) be points in \(D_{N, \delta}\) with \(d_\mathcal{B}(z_i^\ell, y_i^\ell) \leq \Delta\) for all \(1 \leq i \leq \ell\). Define
\[
\mathcal{B}_i : \mathcal{F} \mapsto 2\mathcal{F}(z_i^\ell) - 2\mathcal{F}(y_i^\ell).
\]
Let \(\mu(\cdot) = \mathbb{E}[W(\cdot)\mathcal{B}(W)]\) be the mean of \(W\) under the bias \(\mathcal{B}\). Then
\[
\left| \mathbb{E}_\mathcal{B} \left[ \prod_{i=1}^\ell \mathcal{B}_i(W_N) \right] - \mathbb{E} \left[ \prod_{i=1}^\ell \mathcal{B}_i(\mu + W) \right] \right| \leq C(\Delta^{\ell+1} \vee N^{-\delta/4}).
\]

**Remark 6.4.** For the field \(Z_N\), in light of Proposition 3.1 this lemma also holds if \(\mathcal{B}_\omega\) is replaced by some \(\mathcal{B} \in \mathcal{G}_{j, \epsilon, \delta}\) for some \(\epsilon\) and \(j\). The proof is completely general and needs no alteration for this case. Further, the errors are uniform in \(\mathcal{G}_{j, \epsilon, \delta}\) for fixed choices of \(\epsilon\) and \(\delta\).

**Proof.** We will write the proof for a general \(\mathcal{B} \in \mathcal{G}_{j, \epsilon, \delta}\) for some \(j \in \mathbb{N}_0\) and some \(\epsilon > 0\). Suppose that
\[
\mathcal{B}(\mathcal{F}) = \sum_{z \in \mathcal{Z}} 2\mathcal{F}(z) - \sum_{y \in \mathcal{Y}} 2\mathcal{F}(y),
\]
where \(\mathcal{Z}\) and \(\mathcal{Y}\) are finite subsets of \(D_{N, \delta}\).

By the symmetry of the problem, it is enough to establish the one-sided bound:
\[
\mathbb{E}_\mathcal{B} \left[ \prod_{i=1}^\ell \mathcal{B}_i(W_N) \right] \leq \mathbb{E} \left[ \prod_{i=1}^\ell \mathcal{B}_i(\mu + W) \right] + C(\Delta^{\ell+1} \vee N^{-\delta/4}).
\]

We will approximate the increments \(\mathcal{B}_i\) by exponentials, in order to apply Assumption 1.7. As we do not suppose anything on the locations of \(\{z_i^\ell\}\) or \(\{y_i^\ell\}\), we use harmonicity in a nontrivial way to reduce the problem to one in which pairs are well-separated, and we give this argument first.

For each \(i = 1, 2, \ldots, \ell\) we will define a contour \(\gamma_i\). Let \(z' = z \cup \{z_j^\ell\}\) and likewise for \(y'\). We define
\[
\gamma_i : \partial \left( \bigcup_{w \in z' \cup y'} B(3\delta) \right),
\]
that is \(\gamma_i\) traces the boundary of the union of these balls with a positive orientation. With this definition \(\gamma_0\) is the set \(z' \cup y'\). As \(\gamma_i\) is a piecewise smooth arc, we have there is a probability measure \(\nu_i\) supported on \(\gamma_i\) so that for any harmonic function \(\varphi\) on \(D_{N, \delta/2}\)
\[
\varphi(z_i) = \int_{\gamma_i} \varphi(w) \nu_i(dw).
\]

Let \(M_i\) be a disk automorphism taking \(z_i\) to \(y_i\). This map is analytic on \(D\) and has the property that \(d_\mathcal{B}(w, M_i(w)) = d_\mathcal{B}(z_i, y_i)\) for all \(w \in D\). In particular, \(W - W \circ M_i\).\]
$M_i$ is harmonic in $D_{N,\delta/2}$ for all $N$ sufficiently large. This allows us to give the representation,

$$W_N(z_i) - W_N(y_i) = \int_{z_i} (W_N(w) - W_N(M_i(w))) \nu_i(dw).$$

Therefore, if we define the biases $\mathfrak{B}_{(i,w)} : F \mapsto 2F(w) - 2F(M_i(w))$, we can write

$$\mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_i(W_N) \right] = \int_{\gamma_i} \cdots \int_{\gamma_i} \mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_{(i,w_i)}(W_N) \right] \nu_i(dw_i). \tag{94}$$

Commuting the contour integration and the $F$-expectation, the representation, $F$

the inequality

$$\mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_i(W_N) \right] \leq \sum_{i=1}^\ell |\mathfrak{B}_{(i,w_i)}(W_N)|^\ell \leq \gamma \sum_{i=1}^\ell \cos(\mathfrak{B}_{(i,w_i)}(W_N)),$$

and Assumption $1.7$ once we verify the provisions of that assumption.

In that direction, we claim that for any subset $S \subseteq [\ell]$, and any points $\{w_i\}_{i \in S}$

in the respective supports of $\{\gamma_i\}$, we are chosen, that for any $w_i \in \gamma_i$ and $w_j \in \gamma_j$ with $i \neq j$,

$$3 \leq d_{\mathbb{E}}(w_i, w_j).$$

As $d_{\mathbb{E}}(w_i, M_i(w_i)) \leq 1$ for all $i$ we also have that

$$2 \leq d_{\mathbb{E}}(M_i(w_i), w_j), 2 \leq d_{\mathbb{E}}(w_i, M_j(w_j)) \text{ and } 1 \leq d_{\mathbb{E}}(M_i(w_i), M_j(w_j)).$$

This implies that $\mathfrak{B} + \sum_{i \in S} \mathfrak{B}_{(i,w_i)} \in \mathfrak{M}_{\ell,\delta/2}(\mathfrak{B})$ as desired.

The bottom line is that if we establish (95) in the case that the pairs $(y_i, z_i)$ are

each at least distance 2 to any other point of $z' \cup y'$, then using (94) we obtain

$$\mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_i(W_N) \right] \leq \int_{\gamma_i} \cdots \int_{\gamma_i} \mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_{(i,w_i)}(W_N) \right] \nu_i(dw_i) \leq \prod_{i=1}^\ell \mathcal{E}_i(\mu + W) \nu_i(dw_i) + C(\Delta^{\ell+1} \vee N^{-\delta/2}).$$

Moreover, by the almost sure harmonicity of $\mu$ and $W$, we conclude:

$$\mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_i(W_N) \right] \leq \mathcal{E}_i(\mu + W) + C(\Delta^{\ell+1} \vee N^{-\delta/2}),$$

as desired.

Therefore, we have reduced the problem to showing (95) in the case that (95) holds. To do this, we will replace the moments we wish to calculate by exponentials, to which our assumption applies. More specifically, we will show that

$$\mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell \mathfrak{B}_i(W_N) \right] - \mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell (e^{\mathfrak{B}_i(W_N)} - 1) \right] \ll_{k,\ell,\delta} \Delta^{\ell+1} \vee N^{-\delta/4}. \tag{96}$$

Having performed this bound, we will be in a position to apply Assumption $1.7$

to the $F$-expectation of exponentials. In particular, a direct application of Corollary $6.2$
shows that

$$\mathcal{F}_{2^\delta} \left[ \prod_{i=1}^\ell (e^{\mathfrak{B}_i(W_N)} - 1) \right] - \mathcal{E}_i(\mu + W) - 1) \ll_{k,\ell,\delta} N^{-\delta/2}. \tag{97}$$
Therefore it will only remain to prove that for the Gaussian field $\mathbf{W}$, we have
\begin{equation}
\mathbb{E}\left[\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mu + \mathbf{W})} - 1)\right] - \mathbb{E}\left[\prod_{i=1}^{\ell}\mathcal{B}_i(\mu + \mathbf{W})\right] \ll_\ell \Delta^{\ell+1}
\tag{98}
\end{equation}
to complete the proof. Indeed, observe that summing \([96], \tag{97} and \tag{98} give \([93].\]

We begin by showing that \([97] holds. It suffices to show that, for any $1 \leq p \leq \ell$,
\begin{equation}
\mathbb{P}_\mathcal{B}\left\{\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mathbf{W}_N)} - 1 - \mathcal{B}_i(\mathbf{W}_N)) \cdot \prod_{i=p+1}^{\ell}\mathcal{B}_i(\mathbf{W}_N)\right\} \ll_{k, \ell, \delta} \Delta^{\ell+p} \vee N^{-\delta/4}.
\tag{99}
\end{equation}

Since $e^x - 1 - x \geq 0$ for all $x \in \mathbb{R}$, the bound \([99] follows from this by adding and subtracting $\mathcal{B}_i(\mathbf{W}_N)$ inside the second product of \([99] and expanding. The terms that result have the form of \([99] up to permutation of the indices of the products.

To establish \([99], we start by applying Cauchy-Schwarz in the following way:
\begin{align*}
&\mathbb{E}\left[\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mathbf{W}_N)} - 1 - \mathcal{B}_i(\mathbf{W}_N)) \cdot \prod_{i=p+1}^{\ell}\mathcal{B}_i(\mathbf{W}_N)\right]^2 \\
\leq&\mathbb{E}\left[\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mathbf{W}_N)} - 1 - \mathcal{B}_i(\mathbf{W}_N))\right] \\
&\cdot\mathbb{E}\left[\prod_{i=p+1}^{\ell}\mathcal{B}_i(\mathbf{W}_N)\right]^2.
\end{align*}

We further bound this expression using the inequalities $e^x - 1 - x \leq 2(\cosh(x) - 1)$ and $x^2 \leq 2(\cosh(x) - 1)$. We simplify notation by writing $\varphi(x) = 2(\cosh(x) - 1)$.

We then apply these bounds by writing
\begin{align*}
&\mathbb{E}\left[\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mathbf{W}_N)} - 1 - \mathcal{B}_i(\mathbf{W}_N)) \cdot \prod_{i=p+1}^{\ell}\mathcal{B}_i(\mathbf{W}_N)\right]^2 \\
\leq&\mathbb{E}\left[\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mathbf{W}_N)} - 1 - \mathcal{B}_i(\mathbf{W}_N))\right] \\
&\cdot\mathbb{E}\left[\prod_{i=1}^{\ell}\varphi(\mathcal{B}_i(\mathbf{W}_N))\right].
\end{align*}

Each of the expectations in this bound can be expanded into expectations of sums of exponentials of biases of the same form as in \([95]. Hence, by Corollary \[6.2,\]
we can replace $\mathbf{W}_N$ by $\mu + \mathbf{W}$ incurring only an additive $N^{-\delta/2}$ error. Moreover, by elementary manipulations, we will establish that for any $1 \leq p \leq \ell$
\begin{equation}
\mathbb{E}\left[\prod_{i=1}^{p}\varphi(\mathcal{B}_i(\mu + \mathbf{W}))\right] \ll_\ell \Delta^{2p}
\tag{100}
\end{equation}

Combining this with the previous displayed equation, we would have that
\begin{align*}
&\mathbb{E}\left[\prod_{i=1}^{\ell}(e^{\mathcal{B}_i(\mathbf{W}_N)} - 1 - \mathcal{B}_i(\mathbf{W}_N)) \cdot \prod_{i=p+1}^{\ell}\mathcal{B}_i(\mathbf{W}_N)\right]^2 \\
\ll_\ell&\Delta^{2p + N^{-\delta/2}}(\Delta^{2\ell} + N^{-\delta/2}) \\
\ll_\ell&\Delta^{2p + 2\ell} \vee N^{-\delta/2}.
\end{align*}

Taking square-roots, the estimates \([99] and \([96] follow provided that we prove \([100]. To do so, we begin by using the arithmetic-geometric mean inequality:
\begin{equation}
\mathbb{E}\left[\prod_{i=1}^{p}\varphi(\mathcal{B}_i(\mu + \mathbf{W}))\right] \leq \frac{1}{p} \sum_{i=1}^{p} \mathbb{E}\left[\varphi(\mathcal{B}_i(\mu + \mathbf{W}))\right]^p.
\end{equation}

By Definition \[13], the variable $\mathcal{B}_i(\mu + \mathbf{W})$ are Gaussian with mean $\mu(z_i) - \mu(y_i) \leq_k d_{\epsilon}(z_i, y_i) \leq \Delta$ and variance $\text{Var}(\mathbf{W}(z_i) - \mathbf{W}(y_i)) \leq d_{\epsilon}(z_i, y_i)^2 \leq \Delta^2$. Hence the variable $\mathcal{B}_i(\mu + \mathbf{W})/\Delta$ satisfies a uniform Gaussian tail bound with constants depending only on $k$. 
The function $\varphi(x)$ vanishes quadratically near 0, so that $\varphi(x)x^{-2}$ can be bounded by $K \cosh(x)$ for some sufficiently large constant $K > 0$. In particular, we have

$$\varphi(x) \leq \Delta^2 \psi(x/\Delta)$$

where $\psi(x) = Kx^2 \cosh(x)$. Hence

$$E[\varphi(\mathcal{B}_i(\mu + W))] \leq \Delta^2 E[\psi(\mathcal{B}_i(\mu + W)/\Delta)] \ll k, \ell \Delta^{2p},$$

which completes the proof of (100).

It just remains to prove the estimate (99). However, the proof here is nearly identical to (96), so we just sketch it. Analogously to (96), we can start by adding and subtracting $\mathcal{B}_i(\mu + W)$ inside the first product and expanding. This reduces the problem to bounding (101)

$$E \left\{ \prod_{i=1}^{p} (e^{\mathcal{B}_i(\mu + W)} - 1 - \mathcal{B}_i(\mu + W)) \right\} \ll k, \ell, \delta \Delta^{\ell + 1}.$$

The exact same bounds used in reducing (99) to (100) can be applied to reduce (101) to (100), which completes the proof. $\Box$

By piecing together various local perturbations, it is therefore possible to remove the requirement that $z_i$ and $y_i$ be close in the previous lemma, at the expense of a larger error term.

**Proposition 6.5.** Fix $\ell \in \mathbb{N}$, $0 < \alpha < 1$ and $\omega \in \Omega$, and let $\mathcal{B} = \mathcal{B}_\omega$. Suppose Assumption MEM($\ell$) (1.7) holds. There is a constant $C = C(\ell, \alpha, \delta)$ so that the following holds. Let $\{z_i\}_{i=1}^{\ell}$ and $\{y_i\}_{i=1}^{\ell}$ be points in $D_{N, \delta}$. Define

$$\mathcal{B}_i : F \mapsto 2F(z_i) - 2F(y_i).$$

Let $\mu(\cdot) = E[W(\cdot)\mathcal{B}(W)]$ be the mean of $W$ under the bias $\mathcal{B}$. Then

$$\left| E_{\mathcal{B}} \left[ \prod_{i=1}^{\ell} \mathcal{B}_i(W_N) \right] - E \left[ \prod_{i=1}^{\ell} \mathcal{B}_i(\mu + W) \right] \right| \leq Ce^{-\alpha \log N}.$$

**Proof.** For each $1 \leq i \leq \ell$, let $\gamma_i$ be the geodesic from $y_i$ to $z_i$ parameterized by hyperbolic arc length. This geodesic necessarily lies in the wedge

$$\left\{ re^{i(\theta + \frac{\pi}{2})} \mid 0 \leq r \leq 1 - N^{-1+\delta}, |\theta| \leq N^{-\delta} \right\},$$

which is geodesically convex (compare with the definition of $D_{N, \delta}$ (17), for which the radius $r \geq 1 - N^{-\delta}$ as well). It may be necessary to deform $\gamma_i$ to stay within $D_{N, \delta}$, and it can be performed by replacing any segment in

$$\left\{ re^{i(\theta + \frac{\pi}{2})} \mid 0 \leq r \leq 1 - N^{-\delta}, |\theta| \leq N^{-\delta} \right\},$$

by a segment that travels along the curve $r = 1 - N^{-\delta}, |\theta| \leq N^{-\delta}$. This inner curved boundary only has length $O(1)$, as its angular length is $O(N^{-\delta})$ and for $z$ on this arc, the hyperbolic metric is the euclidean one scaled by $(1 - |z|)^{-1} = N^{\delta}$. Hence, all curves that arise this way have length $O(\log N)$, as $D_{N, \delta}$ is contained in a hyperbolic ball of radius $\log N$. 


Let $0 = t_1^{(i)} \leq t_2^{(i)} \leq \cdots \leq t_{p_i}^{(i)}$ be evenly spaced points, with spacing $e^{-\log(n)^x}$, save for possibly the last, which may be shorter. For any $1 \leq j \leq p_i$, define $\mathcal{B}_{(i,j)} : F \mapsto 2F(\gamma_j(t_j^{(i)})) - 2F(\gamma_j(t_{j-1}^{(i)}))$. Then we can trivially bound

$$|\mathbb{P}_{\mathcal{B}} \left[ \prod_{i=1}^{\ell} \mathcal{B}_i(W_N) \right] - \mathbb{E} \left[ \prod_{i=1}^{\ell} \mathcal{B}_i(\mu + W) \right]| \leq \sum_{\mu=1}^{p_1} \cdots \sum_{\mu=1}^{p_1} |\mathbb{P}_{\mathcal{B}} \left[ \prod_{i=1}^{\ell} \mathcal{B}_{(i,j)}(W_N) \right] - \mathbb{E} \left[ \prod_{i=1}^{\ell} \mathcal{B}_{(i,j)}(\mu + W) \right]|.$$

To this difference, we now apply Lemma 6.6. Thus the summands can be uniformly controlled by $O(e^{-(\ell+1)(\log N)^x})$. The number of summands, meanwhile, is at most $O(e^{\ell(\log N)^x + \ell \log \log N})$.

6.2. Estimating $\mathbb{E}[\mathbf{1}\{Z > 0\}]$. Using these field moments, we can estimate the conditional probability of $E(\omega)$, (110). For the remainder of this section, we suppose Assumption MEM(2) holds (1.7).

Lemma 6.6. For any $\epsilon > 0$, by making $N$ sufficiently large, we have that uniformly in $\omega \in \Omega$,

$$\mathbb{E}[e^{3\mathcal{B}_\omega(W)}](1 - \epsilon) \leq \mathbb{E}[Y(\omega)] \leq \mathbb{E}[e^{3\mathcal{B}_\omega(W)}](1 + \epsilon).$$

Proof. The upper bound proceeds by a trivial bound and a direct application of Assumption 1.7

$$\mathbb{E}[Y(\omega)] \leq \mathbb{E}[e^{3\mathcal{B}_\omega(W_N)}] = \mathbb{E}[e^{3\mathcal{B}_\omega(W)}](1 + O(N^{-\delta})).$$

For the lower bound, by (111), it suffices that show that, when the parameter $N$ is sufficiently large, $\mathbb{P}[\mathbf{1}\{W_N \in E(\omega)\}] \geq 1 - \epsilon$, where $F = \mathbb{P}_{\mathcal{B}_\omega}$. By a union bound, we simply estimate

$$\mathbb{P}[\mathbf{1}\{W_N \notin E(\omega)\}] \leq \sum_{k=r+1}^{n} \mathbb{P} \left[ \mathbf{1} \left\{ |W_N(\omega_{b_k}) - W_N(\omega_{b_r})| > \eta \cdot n^{1/2} \right\} \right] \leq \sum_{k=r}^{n} \frac{\mathbb{E}[(W_N(\omega_{b_k}) - W_N(\omega_{b_r}) - (b_k - b_r))^2]}{\eta^2 n}.$$

By Proposition 6.5, we can compute the first and second moments of $W_N(\omega_{b_k}) - W_N(\omega_{b_r})$ under $F$. Using this Gaussian comparison, the mean and variance of this increment are $b_k - b_r$ up to a uniformly bounded $O(1)$ error. Therefore, this $F$-expectation is bounded above by $n + O(1)$, which leads to

$$\mathbb{P}[\mathbf{1}\{W_N \notin E(\omega)\}] \ll \sum_{k=r+1}^{n} \frac{n}{\eta^2 n} \leq \eta^{-1}.$$

This concludes the lower bound as $\eta \to \infty$. 

We now turn to estimating $\mathbb{E}[Y(\omega_1)Y(\omega_2)]$ for various values of $(\omega_1, \omega_2)$. There will be two regimes of $|\omega_1 - \omega_2|$ in which we make different estimates. We introduce the midpoint $m = m(\omega_1, \omega_2)$, defined as the integer closest to $-\log(|\omega_1 - \omega_2|)$ that is also at least $n_0$. This is roughly the height at which the $\omega_1$ and $\omega_2$ rays branch. In the first regime, where $m < b_r$, the rays have branched early enough that there is
essentially no correlation between \( Y(\omega_1) \) and \( Y(\omega_2) \). Otherwise, we must appropriately take advantage of the barrier information in \( Y(\omega) \) to assure the correlation is not too high.

The estimate for small \( m \) is no more complicated than the estimates in Lemma 6.6

**Lemma 6.7.** For all \( \epsilon > 0 \) and all \( \omega_1, \omega_2 \in \Omega \) with \( m(\omega_1, \omega_2) \leq (1 - \epsilon) b_r \), if \( N \) is sufficiently large, then

\[
E[Y(\omega_1)Y(\omega_2)] \leq E[Y(\omega_1)]E[Y(\omega_2)](1 + \epsilon).
\]

**Proof.** We estimate the left hand side by the trivial bound

\[
E[Y(\omega_1)Y(\omega_2)] \leq E[e^{B_{\omega_1}(W_N) + B_{\omega_2}(W_N)}].
\]

This bias \( B_{\omega_1} + B_{\omega_2} \) must have all points separated by a distance independent of \( n \) to apply Assumption 1.7.

For a hyperbolic triangle with side lengths \( a, b, c \) with \( \theta \) the angle opposite \( a \), the hyperbolic law of cosines says that

\[
\cosh a = \frac{\cosh(b + c)}{2}(1 - \cos \theta) + \frac{\cosh(b - c)}{2}(1 + \cos \theta).
\]

Hence, we can derive a formula for the cosh of the distance between \( \xi_1 \omega_1 \) and \( \xi_2 \omega_2 \) for any \( \xi_i \in \{\xi_{b_r}, \xi_{m_0}\} \) for \( i = 1, 2 \). Moreover, as \( N^{-\delta} > |\theta| \gg e^{-(1-\epsilon)b_r} \), we have in all cases that

\[
\cosh(d_B(\xi_1 \omega_1, \xi_2 \omega_2)) \gg e^{2b_r} \rightarrow \infty.
\]

Hence we have that

\[
E[Y(\omega_1)Y(\omega_2)] \leq E[e^{B_{\omega_1}(W_N) + B_{\omega_2}(W_N)}](1 + O(N^{-\delta})).
\]

As this is a Gaussian expectation, it suffices to compute covariances of the two biases to show the expectation splits: specifically, the covariance \( E[B_{\omega_1}(W)B_{\omega_2}(W)] \). Using part (c) of Definition 1.5 and the assumption on \( m(\omega_1, \omega_2) \) in the statement of the lemma, we have that

\[
E[W(\xi_1 \omega_1)W(\xi_2 \omega_2)] = -\frac{1}{2} \log |\sin \left(\frac{\arg(\omega_1 \omega_2^{-1})}{2}\right)| + K(\arg \omega_1, \arg \omega_2) + o(1)
\]

for any \( \xi_i \in \{\xi_{b_r}, \xi_{m_0}\} \) for \( i = 1, 2 \). Moreover, the error \( o(1) \) is uniform in \( \omega_i \), and so we get that

\[
E[e^{B_{\omega_1}(W)+B_{\omega_2}(W)}] = E[e^{B_{\omega_1}(W)}]E[e^{B_{\omega_2}(W)}](1 + o(1)).
\]

Hence, applying Lemma 6.6 we can conclude that

\[
E[Y(\omega_1)Y(\omega_2)] \leq E[Y(\omega_1)]E[Y(\omega_2)](1 + o(1))
\]

uniformly in \( (\omega_1, \omega_2) \) satisfying the hypotheses of the lemma. \( \square \)

This lemma covers all but a vanishing fraction of pairs \( (\omega_1, \omega_2) \) we need to consider. However, we must also assure that the remaining terms are not too correlated. This is the content of the following lemma.

**Lemma 6.8.** There is a constant \( C > 0 \) so that for all \( \omega_1, \omega_2 \in \Omega \) with \( m \geq \frac{1}{2} b_r \) and all \( N \) sufficiently large

\[
E[Y(\omega_1)Y(\omega_2)] \leq C E[Y(\omega_1)]E[Y(\omega_2)]e^{m-b_r+n\eta^{-1}+\eta n^{1/2}}.
\]
Proof. Unlike when \( m \) was small, in the setting of the previous lemma, \( \mathfrak{B}_{\omega_1} + \mathfrak{B}_{\omega_2} \), which is the biasing term that appears in the left-hand side, will have much too large a variance. This is because, by analogy with branching random walk, this bias counts the segment before the \( \omega_1 \) and \( \omega_2 \) rays split twice. Hence, we would like to re-bias the exponential weight on the left-hand side. Ideally we would replace \( \omega \) bias counts the segment before the \( \omega_1 \) and \( \omega_2 \) rays split twice. Hence, we would like to re-bias the exponential weight on the left-hand side. Ideally we would replace \( \omega_1 \zeta_{\omega_1} \) and \( \omega_1 \zeta_{\omega_2} \) in the biasing term. However, we do not have exact control on the value of \( \mathfrak{B}_{\omega_1}(\omega_1 \zeta_{\omega_1}) \), and so we instead choose an approximation over which we do. To this end, let \( b_x \in \{b_r, b_{r+1}, \ldots, b_n\} \) be the closest element to \( m \) that is larger than or equal to \( m \), so that \( b_x - m \leq \frac{2}{\eta} \).

Let

\[
\mathfrak{B}_* : F \mapsto \mathfrak{B}_{\omega_1}(F) + \mathfrak{B}_{\omega_2}(F) + 2F(i\zeta_{b_x}) - 2F(\omega_1 \zeta_{b_x})
\]

Recall that

\[
E(\omega) = \{|F(\omega \zeta_{b_x}) - F(i\zeta_{b_x}) - (b_x - b_r)| \leq \eta \cdot n^{1/2}, \; \forall \; r < k \leq \eta\}.
\]

Hence, by (61),

\[
E[Y(\omega_1)Y(\omega_2)] \leq E\left[e^{\mathfrak{B}_*(W_N)+2(b_r-b_r)+2m^{1/2}}\right].
\]

To this exponential moment, we may apply Assumption 1.7, as we have a uniform lower bound on the distance between all points.

Therefore, to complete the evaluation of this exponential moment, we just need to estimate the variance of \( \mathfrak{B}_*(\mathbf{W}) \). By how \( b_x \) was chosen (part (d) of definition 1.5), we have

\[
E[\mathfrak{B}_{\omega_2}(\mathbf{W})(\mathbf{W}(\omega_1 \zeta_{\omega_1}) - \mathbf{W}(\omega_1 \zeta_{b_x}))] = O(1),
\]

as all but the \( O(1) \) terms cancel from the covariance. Hence, we conclude that

\[
\frac{1}{2} \text{Var}(\mathfrak{B}_*(\mathbf{W})) = (n_0 - b_r) + (n_0 - b_x) + O(1),
\]

which leads to the conclusion

\[
E[Y(\omega_1)Y(\omega_2)] \ll e^{2n_0 + b_x - 3b_r + 2m^{1/2}}
\]

\[
\ll e^{2n_0 + m - 3b_r + \eta n^{1/2} + n^{-1}}.
\]

Finally, by Lemma 6.6 for \( i = 1, 2 \)

\[
E[Y(\omega_i)] \approx E\left[e^{\mathfrak{B}_{\omega_i}(\mathbf{W})}\right] = e^{n_0 - b_r} + O(1),
\]

which completes the proof. \( \square \)

We are now able to show the desired lower bound, i.e. the proof of Theorem 1.8.

Proof of Theorem 1.8. Recall that we let

\[
\Omega = \left\{ e^{-\frac{1}{2}+h e^{-\eta}} : h \in \mathbb{Z}, |h| < N^{-\delta} e^{n_0} \right\}, \quad \text{and} \quad Z = \sum_{\omega \in \Omega} Y(\omega).
\]

We bound the probability that \( Z > 0 \) from below using that

\[
E[1 \{ Z > 0 \}] \geq \frac{(\sum_{\omega} E[Y(\omega)])^2}{\sum_{\omega_1, \omega_2} E[Y(\omega_1)Y(\omega_2)]}.
\]
where \( \omega_1, \omega_2 \) run over the set \( \Omega \). We now partition the sum in the denominator according to the value of \( m(\omega_1, \omega_2) \). Specifically, we let \( I_1 \) be the sum

\[
I_1 = \sum_{\omega_1, \omega_2} \mathbb{E} [Y(\omega_1)Y(\omega_2)],
\]

and we let \( I_2 \) be the sum over the remaining pairs \((\omega_1, \omega_2)\).

By Lemma 6.7, for all \( N \) sufficiently large, we have the simple bound

\[
I_1 \leq \sum_{\omega_1, \omega_2} (1 + \delta) \mathbb{E} [Y(\omega_1)] \mathbb{E} [Y(\omega_2)],
\]

\[
\leq (1 + \delta) \left( \sum_{\omega} \mathbb{E} [Y(\omega)] \right)^2.
\]

(104)

To control \( I_2 \), we partition the sum according to the size of \( m(\omega_1, \omega_2) \). For a given \( \omega \) and an integer \( p \), there are at most \( O(e^{n_0 - \ell}) \) many \( \omega' \in \Omega \) so that \( m(\omega, \omega') \geq \ell \).

By the estimate (102), this implies that

\[
I_2 = \sum_{3b_r/4 \leq \ell \leq n_0} \sum_{\omega_1, \omega_2 \in \Omega} \mathbb{E} [Y(\omega_1)Y(\omega_2)]
\]

\[
\ll n_0 |\Omega| e^{3n_0 - 3b_r + n \eta^{-1} + \eta n^{1/2}}
\]

\[
\ll N^{-\delta} e^{4n_0 - 3b_r + n \eta^{-1} + \eta n^{1/2} + \log n}.
\]

For comparison, we need an estimate for the numerator in (103). By Lemma 6.6, we have

\[
\left( \sum_{\omega} \mathbb{E} [Y(\omega)] \right)^2 \gg |\Omega|^2 e^{2n_0 - 2b_r}
\]

\[
\gg N^{-2 \delta} e^{4n_0 - 2b_r}.
\]

By how \( b_r \) was chosen, we have \( b_r = 2 \delta \log N + O(1) \). Recalling that \( n = \log N + O(1) \), we conclude that

\[
I_2 \ll N^{-\delta/2} \left( \sum_{\omega} \mathbb{E} [Y(\omega)] \right)^2.
\]

Thus, applying this and (104) to (103), we get

\[
\mathbb{E} [1 \{ Z > 0 \}] \geq \frac{1}{1 + \delta + o(1)}.
\]

This completes the proof. \( \square \)

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