AN AXIOMATIC CHARACTERIZATION OF THE GABRIEL-ROITER MEASURE

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ABSTRACT. Given an abelian length category \( \mathcal{A} \), the Gabriel-Roiter measure with respect to a length function \( \ell \) is characterized as a universal morphism \( \text{ind} \mathcal{A} \to P \) of partially ordered sets. The map is defined on the isomorphism classes of indecomposable objects of \( \mathcal{A} \) and is a suitable refinement of the length function \( \ell \).

In his proof of the first Brauer-Thrall conjecture \cite{Roiter}, Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories \cite{Gabriel}. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed (see the footnote on p. 91 of \cite{Gabriel}) that the formalism of Gabriel and Roiter works equally well for studying the representations of algebras having unbounded representation type. We refer to recent work \cite{Krause2, Krause3, Krause4} for some beautiful applications.

In this note we present an axiomatic characterization of the Gabriel-Roiter measure which reveals its combinatorial nature. Given a finite dimensional algebra \( \Lambda \), the Gabriel-Roiter measure is characterized as a universal morphism \( \text{ind} \Lambda \to P \) of partially ordered sets. The map is defined on the isomorphism classes of finite dimensional indecomposable \( \Lambda \)-modules and is a suitable refinement of the length function \( \text{ind} \Lambda \to \mathbb{N} \) which sends a module to its composition length.

The first part of this paper is purely combinatorial and might be of independent interest. We study length functions \( \lambda : S \to T \) on a fixed partially ordered set \( S \). Such a length function takes its values in another partially ordered set \( T \), for example \( T = \mathbb{N} \). We denote by \( \text{Ch}(T) \) the set of finite chains in \( T \), together with the lexicographic ordering. The map \( \lambda \) induces a new length function \( \lambda^* : S \to \text{Ch}(T) \), which we call chain length function because each value \( \lambda^*(x) \) measures the lengths \( \lambda(x_i) \) of the elements \( x_i \) occurring in some finite chain \( x_1 < x_2 < \ldots < x_n = x \) of \( x \) in \( S \). We think of \( \lambda^* \) as a specific refinement of \( \lambda \) and provide an axiomatic characterization. It is interesting to observe that this construction can be iterated. Thus we may consider \((\lambda^*)^*\), \(((\lambda^*)^*)^*\), and so on.

The second part of the paper discusses the Gabriel-Roiter measure for a fixed abelian length category \( \mathcal{A} \), for example the category of finite dimensional \( \Lambda \)-modules over some algebra \( \Lambda \). For each length function \( \ell \) on \( \mathcal{A} \), we consider its restriction to the partially ordered set \( \text{ind} \mathcal{A} \) of isomorphism classes of indecomposable objects of \( \mathcal{A} \). Then the Gabriel-Roiter measure with respect to \( \ell \) is by definition the corresponding chain length function \( \ell^* \). In particular, we obtain an axiomatic characterization of \( \ell^* \) and use it to reprove Gabriel’s main property of the Gabriel-Roiter measure. Note that we work with a slight generalization of Gabriel’s original definition. This enables us to characterize the injective objects of \( \mathcal{A} \) as those objects where \( \ell^* \) takes maximal values for some

\[\begin{align*}
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\end{align*}\]
length function $\ell$. This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category $\mathcal{A}$.

1. Chains and length functions

The lexicographic order on finite chains. Let $(S, \leq)$ be a partially ordered set. A subset $X \subseteq S$ is a chain if $x_1 \leq x_2$ or $x_2 \leq x_1$ for each pair $x_1, x_2 \in X$. For a finite chain $X$, we denote by $\min X$ its minimal and by $\max X$ its maximal element, using the convention $\max \emptyset < x < \min \emptyset$ for all $x \in S$.

We write $\text{Ch}(S)$ for the set of all finite chains in $S$ and let $\text{Ch}(S, x) := \{ X \in \text{Ch}(S) \mid \max X = x \}$ for $x \in S$.

On $\text{Ch}(S)$ we consider the lexicographic order which is defined by

$$X \leq Y \iff \min(Y \setminus X) \leq \min(X \setminus Y) \text{ for } X, Y \in \text{Ch}(S).$$

Remark 1.1. (1) $X \subseteq Y$ implies $X \leq Y$ for $X, Y \in \text{Ch}(S)$.

(2) Suppose that $S$ is totally ordered. Then $\text{Ch}(S)$ is totally ordered. We may think of $X \in \text{Ch}(S) \subseteq \{0, 1\}^S$ as a string of 0s and 1s which is indexed by the elements in $S$. The usual lexicographic order on such strings coincides with the lexicographic order on $\text{Ch}(S)$.

Example 1.2. Let $\mathbb{N} = \{1, 2, 3, \cdots \}$ and $\mathbb{Q}$ be the set of rational numbers together with the natural ordering. Then the map

$$\text{Ch}(\mathbb{N}) \rightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}$$

is injective and order preserving, taking values in the interval $[0, 1]$. For instance, the subsets of $\{1, 2, 3\}$ are ordered as follows:

$$\{\} < \{3\} < \{2\} < \{2, 3\} < \{1\} < \{1, 3\} < \{1, 2\} < \{1, 2, 3\}.$$

We need the following properties of the lexicographic order.

Lemma 1.3. Let $X, Y \in \text{Ch}(S)$ and $X^* := X \setminus \{\max X\}$.

(1) $X^* = \max\{X' \in \text{Ch}(S) \mid X' < X \text{ and } \max X' < \max X\}$.

(2) If $X^* < Y$ and $\max X \geq \max Y$, then $X \leq Y$.

Proof. (1) Let $X' < X$ and $\max X' < \max X$. We show that $X' \leq X^*$. This is clear if $X' \subseteq X^*$. Otherwise, we have

$$\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),$$

and therefore $X' \leq X^*$.

(2) The assumption $X^* < Y$ implies by definition

$$\min(Y \setminus X^*) < \min(X^* \setminus Y).$$

We consider two cases. Suppose first that $X^* \subseteq Y$. If $X \subseteq Y$, then $X \leq Y$. Otherwise,

$$\min(Y \setminus X) < \max X = \min(X \set \ Y)$$
and therefore $X < Y$. Now suppose that $X^* \not\subseteq Y$. We use again that $\max X \geq \max Y$, exclude the case $Y \subseteq X$, and obtain

$$\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).$$

Thus $X \leq Y$ and the proof is complete. \QED

**Length functions.** Let $(S, \leq)$ be a partially ordered set. A *length function* on $S$ is by definition a map $\lambda: S \rightarrow T$ into a partially ordered set $T$ satisfying for all $x, y \in S$ the following:

- (L1) $x < y$ implies $\lambda(x) < \lambda(y)$.
- (L2) $\lambda(x) \leq \lambda(y)$ or $\lambda(y) \leq \lambda(x)$.
- (L3) $\lambda_0(x) := \text{card}\{\lambda(x') \mid x' \in S \text{ and } x' \leq x\}$ is finite.

Two length functions $\lambda$ and $\lambda'$ on $S$ are *equivalent* if

$$\lambda(x) \leq \lambda(y) \iff \lambda'(x) \leq \lambda'(y) \text{ for all } x, y \in S.$$

Observe that (L2) and (L3) are automatically satisfied if $T = \mathbb{N}$. A length function $\lambda: S \rightarrow T$ induces for each $x \in S$ a map

$$\text{Ch}(S, x) \rightarrow \text{Ch}(T, \lambda(x)), \quad X \mapsto \lambda(X),$$

and therefore the following *chain length function*

$$S \rightarrow \text{Ch}(T), \quad x \mapsto \lambda^*(x) := \text{max}\{\lambda(X) \mid X \in \text{Ch}(S, x)\}.$$

Note that equivalent length functions induce equivalent chain length functions.

**Example 1.4.** (1) Let $S$ be a poset such that for each $x \in S$ there is a bound $n_x \in \mathbb{N}$ with $\text{card} X \leq n_x$ for all $X \in \text{Ch}(S, x)$. Then the map $S \rightarrow \mathbb{N}$ sending $x$ to $\text{max}\{\text{card} X \mid X \in \text{Ch}(S, x)\}$ is a length function.

(2) Let $S$ be a poset such that $\{x' \in S \mid x' \leq x\}$ is a finite chain for each $x \in S$. Then the map $\lambda: S \rightarrow \mathbb{N}$ sending $x$ to $\text{card}\{x' \in S \mid x' \leq x\}$ is a length function. Moreover, $\lambda^*$ is a length function and equivalent to $\lambda$.

(3) Let $\lambda: S \rightarrow \mathbb{Z}$ be a length function which satisfies in addition the following properties of a *rank function*: $\lambda(x) = \lambda(y)$ for each pair $x, y$ of minimal elements of $S$, and $\lambda(x) = \lambda(y) - 1$ whenever $x$ is an immediate predecessor of $y$ in $S$. Then $\lambda^*$ is a length function and equivalent to $\lambda$.

**Basic properties.** Let $\lambda: S \rightarrow T$ be a length function and $\lambda^*: S \rightarrow \text{Ch}(T)$ the induced chain length function. We collect the basic properties of $\lambda^*$.

**Proposition 1.5.** Let $x, y \in S$.

- (C0) $\lambda^*(x) = \text{max}_{x' < x} \lambda^*(x') \cup \{\lambda(x)\}$.
- (C1) $x \leq y$ implies $\lambda^*(x) \leq \lambda^*(y)$.
- (C2) $\lambda^*(x) = \lambda^*(y)$ implies $\lambda(x) = \lambda(y)$.
- (C3) $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\lambda^*(x) \leq \lambda^*(y)$.

The first property shows that the function $\lambda^*: S \rightarrow \text{Ch}(T)$ can be defined by induction on the length $\lambda_0(x)$ of the elements $x \in S$. The subsequent properties suggest to think of $\lambda^*$ as a refinement of $\lambda$. 
and (iii), we choose for each $x$ by (M3). Thus sending the assertion is true for $S$. For $x, y$ we have
\[
\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y).
\]
To prove (C3), we use (C0) and apply Lemma 1.3 with $X = \lambda^*(x)$ and $Y = \lambda^*(y)$. In fact, $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ implies $X^* < Y$, and $\lambda(x) \geq \lambda(y)$ implies $\max X \geq \max Y$. Thus $X \leq Y$.

**Corollary 1.6.** Let $\lambda: S \to T$ be a length function. Then the induced map $\lambda^*$ is a length function.

**Proof.** (L1) follows from (C1) and (C2). (L2) and (L3) follow from the corresponding conditions on $\lambda$. □

**An axiomatic characterization.** Let $\lambda: S \to T$ be a length function. We present an axiomatic characterization of the induced chain length function $\lambda^*$. Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that $\lambda^*$ refines $\lambda$.

**Theorem 1.7.** Let $\lambda: S \to T$ be a length function. Then there exists a map $\mu: S \to U$ into a partially ordered set $U$ satisfying for all $x, y \in S$ the following:

- (M1) $x \leq y$ implies $\mu(x) \leq \mu(y)$.
- (M2) $\mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y)$.
- (M3) $\mu(x') < \mu(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\mu(x) \leq \mu(y)$.

Moreover, for any map $\mu': S \to U'$ into a partially ordered set $U'$ satisfying the above conditions, we have
\[
\mu'(x) \leq \mu'(y) \iff \mu(x) \leq \mu(y) \text{ for all } x, y \in S.
\]

**Proof.** We have seen in Proposition 1.5 that $\lambda^*$ satisfies (M1) – (M3). So it remains to show that for any map $\mu: S \to U$ into a partially ordered set $U$, the conditions (M1) – (M3) uniquely determine the relation $\mu(x) \leq \mu(y)$ for any pair $x, y \in S$. We proceed by induction on the length $\lambda_0(x)$ of the elements $x \in S$ and show in each step the following for $S_n = \{x \in S \mid \lambda_0(x) \leq n\}$.

(i) $\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\}$ is a finite set for all $x \in S$.
(ii) (M1) – (M3) determine the relation $\mu(x) \leq \mu(y)$ for all $x, y \in S_n$.
(iii) $\mu(x) \leq \mu(y)$ or $\mu(y) \leq \mu(x)$ for all $x, y \in S_n$.

For $n = 1$ the assertion is clear. In fact, $S_1$ is the set of minimal elements in $S$ and $\lambda(x) \geq \lambda(y)$ implies $\mu(x) \leq \mu(y)$ for $x, y \in S_1$, by (M3). Now let $n > 1$ and assume the assertion is true for $S_{n-1}$. To show (i), fix $x \in S$. The map
\[
\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\} \longrightarrow \{\mu(x') \mid x' \in S_{n-1} \text{ and } x' \leq x\} \times \{\lambda(x') \mid x' \leq x\}
\]
by $\mu(x')$ to the pair $(\max y < x' \mu(y), \lambda(x'))$ is well-defined by (i) and (iii); it is injective by (M3). Thus $\{\mu(x') \mid x' \in S_n \text{ and } x' < x\}$ is a finite set. In order to verify (ii) and (iii), we choose for each $x \in S_n$ a Gabriel-Roiter filtration, that is, a sequence
\[
x_1 < x_2 < \ldots < x_{\gamma(x) - 1} < x_\gamma(x) = x
\]
in $S$ such that $x_1$ is minimal and $\max_{x' < x_i} \mu(x') = \mu(x_{i-1})$ for all $1 < i \leq \gamma(x)$. Such a filtration exists because the elements $\mu(x')$ with $x' < x$ form a finite chain, by (i) and (ii). Now fix $x, y \in S$ and let $I = \{i \geq 1 \mid \mu(x_i) = \mu(y_i)\}$. We consider $r = \max I$ and put $r = 0$ if $I = \emptyset$. There are two possible cases. Suppose first that $r = \gamma(x)$ or $r = \gamma(y)$. If $r = \gamma(x)$, then $\mu(x) = \mu(y) = \mu(y_r) \leq \mu(y)$ by (M1). Now suppose $\gamma(x) \neq r \neq \gamma(y)$. Then we have $\lambda(x_{r+1}) \neq \lambda(y_{r+1})$ by (M2) and (M3). If $\lambda(x_{r+1}) > \lambda(y_{r+1})$, then we obtain $\mu(x_{r+1}) < \mu(y_{r+1})$, again using (M2) and (M3). Iterating this argument, we get $\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{\gamma(y)})$. From (M1) we get $\mu(x) < \mu(y_{r+1}) \leq \mu(y)$. Thus $\mu(x) \leq \mu(y)$ or $\mu(x) \geq \mu(y)$ and the proof is complete. □

**Corollary 1.8.** Let $\lambda : S \to T$ be a length function and let $\mu : S \to U$ be a map into a partially ordered set $U$ satisfying (M1) – (M3). Then $\mu$ is a length function. Moreover, we have for all $x, y \in S$

$$\mu(x) = \mu(y) \iff \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) = \lambda(y).$$

**Iterated length functions.** Let $\lambda$ be a length function. Then $\lambda^*$ is again a length function by Corollary 1.6. Thus we may define inductively $\lambda^{(n)} = \lambda$ and $\lambda^{(n-1)} = (\lambda^{(n-1)})^*$ for $n \geq 1$. In many examples, we have that $\lambda^{(1)}$ and $\lambda^{(3)}$ are equivalent. However, this is not a general fact. The author is grateful to Osamu Iyama for suggesting the following example.

**Example 1.9.** The following length functions $\lambda^{(1)}$ and $\lambda^{(3)}$ are not equivalent.

\[
\begin{array}{cccc}
\lambda^{(0)} : & 4 & 5 & 6 \\
\downarrow & \downarrow & \downarrow \\
3 & 2 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
\lambda^{(1)} : & 3 & 6 & 5 \\
\downarrow & \downarrow & \downarrow \\
1 & 2 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
\lambda^{(2)} : & 6 & 4 & 2 \\
\downarrow & \downarrow & \downarrow \\
5 & 3 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
\lambda^{(3)} : & 3 & 5 & 6 \\
\downarrow & \downarrow & \downarrow \\
1 & 2 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
\lambda^{(4)} : & 6 & 4 & 2 \\
\downarrow & \downarrow & \downarrow \\
5 & 3 & 1 \\
\end{array}
\end{array}
\]

2. **Abelian length categories**

In this section we recall the definition and some basic facts about abelian length categories. We fix an abelian category $\mathcal{A}$.

**Subobjects.** We say that two monomorphisms $\phi_1 : X_1 \to X$ and $\phi_2 : X_2 \to X$ in $\mathcal{A}$ are equivalent, if there exists an isomorphism $\alpha : X_1 \to X_2$ such that $\phi_1 = \phi_2 \circ \alpha$. An equivalence class of monomorphisms into $X$ is called a subobject of $X$. Given subobjects $\phi_1 : X_1 \to X$ and $\phi_2 : X_2 \to X$ of $X$, we write $X_1 \subseteq X_2$ if there is a morphism $\alpha : X_1 \to X_2$ such that $\phi_2 = \phi_1 \circ \alpha$. An object $X \neq 0$ is simple if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

**Length categories.** An object $X$ of $\mathcal{A}$ has finite length if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X,$$

that is, each $X_i/X_{i-1}$ is simple. Note that $X$ has finite length if and only if $X$ is both artinian (i.e. it satisfies the descending chain condition on subobjects) and noetherian (i.e. it satisfies the ascending chain condition on subobjects). An abelian category is
called a \textit{length category} if all objects have finite length and if the isomorphism classes of objects form a set.

Recall that an object \( X \neq 0 \) is \textit{indecomposable} if \( X = X_1 \oplus X_2 \) implies \( X_1 = 0 \) or \( X_2 = 0 \). A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

\textbf{Example 2.1.} (1) The finitely generated modules over an artinian ring form a length category.

(2) Let \( k \) be a field and \( Q \) be any quiver. Then the finite dimensional \( k \)-linear representations of \( Q \) form a length category.

3. The Gabriel-Roiter measure

Let \( \mathcal{A} \) be an abelian length category. The definition of the Gabriel-Roiter measure of \( \mathcal{A} \) is due to Gabriel \cite{1} and was inspired by the work of Roiter \cite{5}. We present a definition which is a slight generalization of Gabriel’s original definition. Then we discuss some specific properties.

\textbf{Length functions.} A \textit{length function} on \( \mathcal{A} \) is by definition a map \( \ell \) which sends each object \( X \in \mathcal{A} \) to some real number \( \ell(X) \geq 0 \) such that

\begin{enumerate}
  \item \( \ell(X) = 0 \) if and only if \( X = 0 \), and
  \item \( \ell(X) = \ell(X') + \ell(X'') \) for every exact sequence \( 0 \to X' \to X \to X'' \to 0 \).
\end{enumerate}

Note that such a length function is determined by the set of values \( \ell(S) > 0 \), where \( S \) runs through the isomorphism classes of simple objects of \( \mathcal{A} \). This follows from the Jordan-Hölder Theorem. We write \( \ell_1 \) for the length function satisfying \( \ell_1(S) = 1 \) for every simple object \( S \). Observe that \( \ell_1(X) \) is the usual composition length of an object \( X \in \mathcal{A} \).

\textbf{The Gabriel-Roiter measure.} We consider the set \( \text{ind}\mathcal{A} \) of isomorphism classes of indecomposable objects of \( \mathcal{A} \) which is partially ordered via the subobject relation \( X \subseteq Y \). Now fix a length function \( \ell \) on \( \mathcal{A} \). The map \( \ell \) induces a length function \( \text{ind}\mathcal{A} \to \mathbb{R} \) satisfying (L1) – (L3), and the induced chain length function \( \ell^* : \text{ind}\mathcal{A} \to \text{Ch}(\mathbb{R}) \) is by definition the \textit{Gabriel-Roiter measure} of \( \mathcal{A} \) with respect to \( \ell \). Gabriel’s original definition \cite{1} is based on the length function \( \ell_1 \). Whenever it is convenient, we substitute \( \mu = \ell^* \).

\textbf{An axiomatic characterization.} The following axiomatic characterization of the Gabriel-Roiter measure is the main result of this note.

\textbf{Theorem 3.1.} Let \( \mathcal{A} \) be an abelian length category and \( \ell \) a length function on \( \mathcal{A} \). Then there exists a map \( \mu : \text{ind}\mathcal{A} \to P \) into a partially ordered set \( P \) satisfying for all \( X,Y \in \text{ind}\mathcal{A} \) the following:

\begin{enumerate}
  \item \( X \subseteq Y \) implies \( \mu(X) \leq \mu(Y) \).
  \item \( \mu(X) = \mu(Y) \) implies \( \ell(X) = \ell(Y) \).
  \item \( \mu(X') < \mu(Y) \) for all \( X' \subset X \) and \( \ell(X) \geq \ell(Y) \) imply \( \mu(X) \leq \mu(Y) \).
\end{enumerate}

Moreover, for any map \( \mu' : \text{ind}\mathcal{A} \to P' \) into a partially ordered set \( P' \) satisfying the above conditions, we have

\[ \mu'(X) \leq \mu'(Y) \iff \mu(X) \leq \mu(Y) \quad \text{for all} \quad X,Y \in \text{ind}\mathcal{A}. \]
Proof. Use the axiomatic characterization of the chain length function $\ell^*$ in Theorem 1.7.

Gabriel’s main property. Let $\ell$ be a fixed length function on $A$. The following main property of the Gabriel-Roiter measure $\mu = \ell^*$ is crucial; it is the basis for all applications.

Proposition 3.2 (Gabriel). Let $X, Y_1, \ldots, Y_r \in \text{ind} \ A$. Suppose that $X \subseteq Y = \oplus_{i=1}^r Y_i$. Then $\mu(X) \leq \max \mu(Y_i)$ and $X$ is a direct summand of $Y$ if $\mu(X) = \max \mu(Y_i)$.

Proof. The proof only uses the properties (GR1) – (GR3) of $\mu$. Fix a monomorphism $\phi : X \to Y$. We proceed by induction on $n = \ell_1(X) + \ell_1(Y)$. If $n = 2$, then $\phi$ is an isomorphism and the assertion is clear. Now suppose $n > 2$. We can assume that for each $i$ the $i$th component $\phi_i : X_i \to Y_i$ of $\phi$ is an epimorphism. Otherwise choose for each $i$ a decomposition $Y_i' = \oplus_j Y_{ij}$ of the image of $\phi_i$ into indecomposables. Then we use (GR1) and have $\mu(X) \leq \max \mu(Y_{ij}) \leq \max \mu(Y_i)$ because $\ell_1(X) + \ell_1(Y') < n$ and $Y_{ij} \subseteq Y_i$ for all $j$. Now suppose that each $\phi_i$ is an isomorphism. Thus $\ell(X) \geq \ell(Y_i)$ for all $i$. Let $X' \subset X$ be a proper indecomposable subobject. Then $\mu(X') \leq \max \mu(Y_i)$ because $\ell_1(X') + \ell_1(Y) < n$, and $X'$ is a direct summand if $\mu(X') = \max \mu(Y_i)$. We can exclude the case that $\mu(X') = \max \mu(Y_i)$ because then $X'$ is a proper direct summand of $X$, which is impossible. Now we apply (GR3) and obtain $\mu(X) \leq \max \mu(Y_i)$. Finally, suppose that $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$ for some $k$. We claim that we can choose $k$ such that $\phi_k$ is an epimorphism. Otherwise, replace all $Y_i$ with $\mu(X) = \mu(Y_i)$ by the image $Y_i' = \oplus_j Y_{ij}$ of $\phi_i$ as before. We obtain $\mu(X) \leq \max \mu(Y_{ij}) < \mu(Y_k)$ since $Y_{kj} \subset Y_k$ for all $j$, using (GR1) and (GR2). This is a contradiction. Thus $\phi_k$ is an epimorphism and in fact an isomorphism because $\ell(X) = \ell(Y_k)$ by (GR2). In particular, $X$ is a direct summand of $\oplus_i Y_i$. This completes the proof. □

Gabriel-Roiter filtrations. We keep a length function $\ell$ on $A$ and the corresponding Gabriel-Roiter measure $\mu = \ell^*$. Let $X, Y \in \text{ind} \ A$. We say that $X$ is a Gabriel-Roiter predecessor of $Y$ if $X \subset Y$ and $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. Note that each object $Y \in \text{ind} \ A$ which is not simple admits a Gabriel-Roiter predecessor because $\mu$ is a length function on $\text{ind} \ A$. A Gabriel-Roiter predecessor $X$ of $Y$ is usually not unique, but the value $\mu(X)$ is determined by $\mu(Y)$.

A sequence $X_1 \subset X_2 \subset \ldots \subset X_{n-1} \subset X_n = X$ in $\text{ind} \ A$ is called a Gabriel-Roiter filtration of $X$ if $X_1$ is simple and $X_{i-1}$ is a Gabriel-Roiter predecessor of $X_i$ for all $1 < i \leq n$. Clearly, each $X$ admits such a filtration and the values $\mu(X_i)$ are uniquely determined by $X$. Note that (C0) implies

\begin{equation}
\mu(X) = \{ \ell(X_i) \mid 1 \leq i \leq n \}.
\end{equation}

Injective objects. In order to illustrate Gabriel’s main property, let us show that the Gabriel-Roiter measure detects injective objects. This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category $A$.

Theorem 3.3. An indecomposable object $Q$ of $A$ is injective if and only if there is a length function $\ell$ on $A$ such that $\ell^*(X) \leq \ell^*(Q)$ for all $X \in \text{ind} \ A$. 


We need the following lemma.

**Lemma 3.4.** Let \( \ell \) be a length function on \( \mathcal{A} \) and fix indecomposable objects \( X, Y \in \mathcal{A} \). Suppose that for each pair of simple subobjects \( X' \subseteq X \) and \( Y' \subseteq Y \), we have \( \ell(X') < \ell(Y') \). Then \( \ell^*(X) > \ell^*(Y) \).

**Proof.** We choose Gabriel-Roiter filtrations \( X_1 \subseteq \ldots \subseteq X_n = X \) and \( Y_1 \subseteq \ldots \subseteq Y_m = Y \). Then \( \ell(X_1) < \ell(Y_1) \) and the formula (3.1) implies

\[
\ell^*(X) = \{ \ell(X_i) \mid 1 \leq i \leq n \} > \{ \ell(Y_i) \mid 1 \leq i \leq m \} = \ell^*(Y).
\]

\( \square \)

**Proof of the theorem.** Suppose first that \( Q \) is injective. Then \( Q \) has a unique simple subobject \( S \) and we define a length function \( \ell = \ell_S \) on \( \mathcal{A} \) by specifying its values on each simple object \( T \in \mathcal{A} \) as follows:

\[
\ell(T) := \begin{cases} 
1 & \text{if } T \cong S, \\
2 & \text{if } T \not\cong S.
\end{cases}
\]

Now let \( X \in \text{ind} \mathcal{A} \). We claim that \( \ell^*(X) \leq \ell^*(Q) \). To see this, let \( X' \subseteq X \) be the maximal subobject of \( X \) having composition factors isomorphic to \( S \). Using induction on the composition length \( n = \ell_1(X') \) of \( X' \), one obtains a monomorphism \( X' \to Q^n \), and this extends to a map \( \phi: X \to Q^n \), since \( Q \) is injective. Let \( X/X' = \oplus_i Y_i \) be a decomposition into indecomposables and \( \pi: X \to X/X' \) be the canonical map. Note that \( \ell^*(Y_i) < \ell^*(Q) \) for all \( i \) by our construction and Lemma 3.4. Then \((\pi, \phi): X \to (\oplus_i Y_i) \oplus Q^n \) is a monomorphism and therefore \( \ell^*(X) \leq \ell^*(Q) \) by the main property.

Suppose now that \( \ell^*(X) \leq \ell^*(Q) \) for all \( X \in \text{ind} \mathcal{A} \) and some length function \( \ell \) on \( \mathcal{A} \). To show that \( Q \) is injective, suppose that \( Q \subseteq Y \) is the subobject of some \( Y \in \mathcal{A} \). Let \( Y = \oplus_i Y_i \) be a decomposition into indecomposables. Then the main property implies \( \ell^*(Q) \leq \max \ell^*(Y_i) \leq \ell^*(Q) \) and therefore \( Q \) is a direct summand of \( Y \). Thus \( Q \) is injective and the proof is complete. \( \square \)

Let us mention that there is the following analogous characterization of the simple objects of \( \mathcal{A} \).

**Corollary 3.5.** An indecomposable object \( S \) of \( \mathcal{A} \) is simple if and only if there is a length function \( \ell \) on \( \mathcal{A} \) such that \( \ell^*(S) \leq \ell^*(X) \) for all \( X \in \text{ind} \mathcal{A} \).

**Proof.** Use the property (GR1) of the Gabriel-Roiter measure and apply Lemma 3.4 \( \square \)

**The Kronecker algebra.** Let \( \Lambda = \left[ \begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix} \right] \) be the Kronecker algebra over an algebraically closed field \( k \). We consider the abelian length category which is formed by all finite dimensional \( \Lambda \)-modules. A complete list of indecomposable objects is given by the preprojectives \( P_n \), the regulars \( R_n(\alpha, \beta) \), and the preinjectives \( Q_n \). More precisely,

\[
\text{ind} \Lambda = \{ P_n \mid n \in \mathbb{N} \} \cup \{ R_n(\alpha, \beta) \mid n \in \mathbb{N}, (\alpha, \beta) \in \mathbb{P}^1_k \} \cup \{ Q_n \mid n \in \mathbb{N} \},
\]
and we obtain the following Hasse diagram.

\[ \begin{array}{ccccccc}
7 & & & & \cdot & & \\
6 & & & & \cdot & & \\
5 & & & & \cdot & & \\
4 & & & & \cdot & & \\
3 & & & & \cdot & & \\
2 & & & & \cdot & & \\
1 & & & & \cdot & & \\
\end{array} \]

The set of indecomposables is ordered as follows via the Gabriel-Roiter measure with respect to \( \ell = \ell_1 \).

\[ \ell^* : \quad Q_1 = P_1 < P_2 < P_3 < \ldots < R_1 < R_2 < R_3 < \ldots < Q_4 < Q_3 < Q_2 \]

\[ (\ell^*)^* : \quad Q_1 = P_1 < R_1 < Q_2 < P_2 < R_2 < Q_3 < P_3 < R_3 < Q_4 < \ldots \]

Moreover, \(((\ell^*)^*)^*\) and \(\ell^*\) are equivalent length functions.

**Remark 3.6.** While \(\ell^*\) has been successfully employed for proving the first Brauer-Thrall conjecture, Hubery points out that \(((\ell^*)^*)^*\) might be useful for proving the second. In fact, one needs to find a value \(((\ell^*)^*)^*(X)\) such that the set \(\{X' \in \text{ind } \Lambda \mid (\ell^*)^*(X') = (\ell^*)^*(X)\}\) is infinite. The example of the Kronecker algebra shows that there exists such a value having only finitely many predecessors \(((\ell^*)^*(Y) < (\ell^*)^*(X)\). Note that in all known examples \(((\ell^*)^*)^*\) and \(\ell^*\) are equivalent.

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