The design of complex systems is typically uncertain and ambiguous at early stages. Set-Based Design is a promising approach to complex systems design as it supports alternative exploration and gradual uncertainty reduction. When designing a complex system, functional requirements decomposition is a common and effective approach to progress the design incrementally. However, the current literature on Set-Based Design lacks formal guidance in functional requirements decomposition. To bridge the gap, we propose a formal process to hierarchically decompose the functional requirements for Set-Based Design. A four-step formal process is proposed to systematically define, reason, and narrow the sets, and eventually decompose the functional requirement into the sub-requirements. Such a process can be used by the individual suppliers working in parallel at multiple levels of abstraction and guarantee that the resulting system will eventually satisfy the top-level functional requirements. An example of designing a cruise control system is applied to demonstrate the feasibility of the proposed process.

1 INTRODUCTION

Modern complex systems are usually developed in the OEM-supplier mode. The OEM (or a higher-level supplier) decomposes the top-level functional requirements into lower-level ones and assigns them to the lower-level suppliers that work in parallel. Getting the functional requirements decomposition correct is crucial for the success of product development.

One of the challenges of functional requirements decomposition is that the design of complex systems is typ-
ically uncertain and ambiguous at early stages [1]. Set-Based Design (SBD) [2] is particularly suitable to tackle this challenge as it provides great support in robust design alternative development, uncertainty reduction and resolution [3], and supplier/subsystem autonomy and optimality [4]. In SBD, the overall system design problem is typically decomposed into multiple distinct disciplines, each utilizing their own sets of possibilities [5]. Teams of engineers develop a set of design alternatives in parallel at different levels of abstraction and narrow the prospective set of alternatives based on additional information until a final solution is converged [6].

Our ultimate goal is to automate functional requirements decomposition for SBD because manually decomposing functional requirements is time-consuming and error-prone. This paper, as our first step towards this goal, aims to formalize the functional requirement decomposition process in the context of SBD. Although function decomposition may not be nominally related to SBD, it clearly features set-based reasoning [4].

This paper addresses the following objectives.

- **Objective 1**: Creating methods, leveraging SBD, to formally decompose the functional requirements.
- **Objective 2**: Provide formal guarantees that the resulting component designs are composable, and the system as a whole will satisfy the top-level functional requirements.

Objective 2 is a constraint on Objective 1, and therefore is addressed first in Section 4. We then ensure that this constraint from Objective 2 is considered when reasoning about Objective 1 in Section 5. Achieving these objectives would allow design teams to work at different levels of abstraction in parallel and asynchronously, providing a path towards more seamless integration for OEMs.

For Objective 1: There is little attention in the SBD literature to a general formal process of functional requirements decomposition for complex systems. Shallcross et al. pointed out that “there is limited SBD research contributing to requirements development” [7] and SBD methodologies applied to complex systems are mostly qualitative [3]. Ghosh and Seering [4] posited that SBD had not been formally defined, despite many authors having studied its process inspired by the example of Toyota. Therefore, in response to Objective 1, we propose a four-step formal process by applying set-based reasoning to systematically define, reason, and narrow the sets, and eventually decompose the functional requirement into the sub-requirements. Dullen et al. [8] and Specking et al. [5] observed that “there has been limited (formal) guidance on how to define, reason, and narrow sets while improving the level of abstraction of the design”, which is precisely the proposed process aims to improve.

Furthermore, the proposed process makes two additional contributions to the SBD literature. First, most SBD approaches formulate the functional requirements as the ranges of the elements in the performance spaces. Such a formulation applies to many mechanical components at lower levels. However, for systems at higher levels, the function is usually defined as a transformation between the inputs and outputs. Accordingly, the functional requirements have to be defined as a mapping between the ranges of the inputs and the outputs. Our process focuses on the latter formulation as we focus on the requirements development of complex systems. Therefore, the proposed process solves a different problem than most current SBD approaches and is a complement to the current SBD literature.

Second, according to Eckert et al. [9], there are two ways to address uncertainties: “buffer” as “the portion of parameter values that compensates for uncertainties”, and “excess” as “the value over, and above, any allowances for uncertainties.” The current SBD literature does not make explicit distinctions between buffer and excess when addressing the uncertainties and hence lacks specificity in their robustness claims. In our process, buffer and excess are addressed with a clear distinction in their respective steps, where buffer is practiced to reduce the controllable uncertainties, and excess is practiced to accommodate the possibility of an under-estimated initial characterization of the uncertainty.

For Objective 2: SBD claims autonomy of teams of designers is an advantage of SBD [10], but very little SBD work provides a priori proof of this property. In fact, one cannot rigorously justify such a claim without an explicit underlying formalism. Based on the formalism defined in this paper, we can prove that when the functional requirements are decomposed in a specific way (i.e., composable and refinement), individual design teams can work independently, and the resulting system as a whole will satisfy the top-level functional requirements.

In summary, we propose in this paper a formal process to hierarchically decompose the functional requirements in the context of SBD. Individual design teams can use the proposed process to decompose the functional requirements independently and eventually achieve a system that satisfies the top-level functional requirements. We demonstrate the feasibility of the proposed process with an example of designing a cruise control system based on existing computational tools.
2 BACKGROUND

2.1 Qualitative SBD approaches

There are qualitative procedural models concerning requirements development in the literature. Enhanced function-means modelling (EF-M), a method for function modeling [11] was used in combination with SBD to manage platform-based product family design [12]. Functional decomposition and solutions were generated according to EF-M. At each level of abstraction, there is a mapping from FR to DS, and a mapping from DS to the FR to be assigned to the next level [13]. The two mappings are conceptually aligned with the approach proposed in this paper.

A “wayfaring” model was introduced for set-based requirement generation as a map to discover critical functionalities and create dynamic requirements in [14]. It was found that prototyping critical functionalities could guide the design process from the initial concept idea to arrive at a final product with low tooling and production costs. A novel set-based approach (MBRMA) was developed to filter out weak or costly solutions over time and assess system engineers when adopting trade-off analysis [15]. Although a mathematical formalism containing the requirements, subsystems, activities, and components was defined in this paper, this article did not provide specific guidance on how to decompose the requirements. Another framework based on a set-based engineering approach allows for building re-usable and adaptable engineering methods [16]. The proposed “virtual methods” can be used to create and validate the early phase design requirements, and make sure the introduction of novel technologies to increase the engine subsystem performance can be realized without compromising the requirements on risk and cost. More frameworks exist, such as CONGA [17], DMIV [18], MBSS [19], RR-LeanPD model [20], and a combination of V-model and SBCE [21].

One weakness of the qualitative approaches is the ambiguity about the concrete activities needed to accomplish the requirements development process, which makes it challenging for them to be repeated by the general industry practitioners. Therefore, a formal process is needed to provide precise instructions on decomposing the functional requirements and assigning them to the lower level of abstraction in a transparent and repeatable way.

2.2 Quantitative SBD approaches

There are many quantitative techniques for design space exploration in SBD. We focus on the quantitative approaches that have a discernible feature of hierarchical decomposition to make a meaningful comparison to the approach of this paper.

SBD has been applied to the design of a downhole module to demonstrate whether their method previously developed in a laboratory setting had the same potential when practiced in the actual industry setting [22]. The downhole module was decomposed into a chassis subsystem and a bumper subsystem. The system-level team assigned the design “target” to the subsystem based on a downhole assembly impact model. Mathews et al. applied a set-based approach for a multilevel design problem of negative stiffness metamaterials based on Bayesian Network Classifier [23]. The design process progressed in a top-down fashion from the macro-level to the meso-level to eventually the micro-level, which was a typical hierarchical design approach. Both [22, 23] have distinctive features of “hierarchical decomposition”, but they “walk down” the hierarchy by treating the design space of the higher-level abstraction as the requirements for the lower-level design, which is different from our problem formulation.

Jansson et al. [24] combined Set-Based Design with axiomatic design to manage and evaluate the performance of multiple design alternatives against the established functional requirements. This work considered the mapping from the design space of the higher level of abstraction to the functional requirements for the lower level, but did not show how specifically the functional requirements for the lower level of abstraction are derived. A set-based approach to collaborative design was proposed in [25]. The overall system was decomposed into distributed collaborative subsystems, where each design team built a Bayesian network of his/her local design space and shared their Bayesian network to identify compatibilities and conflicts to improve the efficiency of local design space search. However, the design problem was formulated to optimize an objective function at the top level rather than decomposing the given requirements into the lower level.

A Serious Game was proposed in [26] to illustrate how the customer requirements of an airplane design can be decomposed into the design parameters of the body, tail, wing, and cockpit by applying the principle of SBD. Although the game was only for educational purposes, it showed a clear process of deriving the ranges of the design parameters at the lower level from the ranges of the higher-level design parameters. [27] presented an Interval-based Constraint Satisfaction Method for decentralized, collaborative multifunctional design. A set of interval-based design variables were identified and then reduced systematically to satisfy the design requirements. SBD was also used by [28, 29] to inform system requirements and evaluate design options by iden-
tifying the number of potential feasible designs in the tradespace for each requirement or combination of requirements. The case study was conducted on the design of a UAV to demonstrate how to assess whether the relaxation of the requirements will create better options. The results demonstrated that SBD provides a comprehensive tradespace exploration and valuable insights into requirement development.

In summary, the functional requirements in these approaches are ranges of the variables in the performance space rather than a mapping between ranges of the design space and the performance space as defined in our process. Such a difference has a significant implication in the requirements decomposition process. The decomposition in other approaches is fulfilling a mapping from one set of the ranges (of the performance space) to another set of ranges (of the design space), while the decomposition in our process is accomplishing a mapping from one mapping (between the ranges of the design space and the performance space of the higher level function) to a set of mappings (between the ranges of the design space and the performance space of the sub-functions).

2.3 Design uncertainty

SBD is known for its robustness to design uncertainty [30, 31, 32]. In an introduction about the application of SBD by the U.S. Naval Sea Systems Command, [33] pointed out that SBD was particularly fit for design problems where there were many conflicting requirements and a high level of uncertainty in requirements. This observation was corroborated by [34], “SBD allows designers to develop a set of concepts, so changes in the design requirements are easier to adjust to.” In an approach that incorporated Bayesian network classifiers for mapping design spaces at each level, “design flexibility” was defined as the size of the subspace that produced satisfactory designs, and “performance flexibility” was defined as the size of the feasible performance space relative to the size of the desired performance space [10]. In [35], a Set-Based Design methodology was proposed to obtain scalable optimal solutions that can satisfy changing requirements through remanufacturing. The methodology was demonstrated on a structural aeroengine component remanufactured by direct energy deposition of a stiffener to meet higher loading requirements.

There is another line of work that combines SBD with platform-based design based on Function-Means modeling techniques to preserve design bandwidth, a system’s flexibility that allows its use in different products [36]. A dynamic platform modeling approach based on SBD and a function modeling technique were presented in [37] to represent product production variety streams inherent in a production operation model. Following the SBD processes, inferior alternatives were put aside until new information became available and a new set of alternatives could be reconfigured, which eventually reduced the risk of late and costly modifications that propagated from design to production. Modeling platform concepts in early phases and eliminating undesired regions of the design space was described in [13]. Change was considered in both the requirements space and the design space. By applying set-based concurrent engineering, sets of design solutions were created to cover the bandwidth of each functional requirement. More work on this topic can be found in [38, 39].

Furthermore, [40] conducted a design experiment on how delaying decisions using SBD could cause higher adaptability to requirements changes later in the design process. As a result, the variable and parameter ranges were open enough to accommodate the changes in the requirements. A method called Dynamically Constrained Set-Based-Design efficiently provides a dynamic map for the feasible design space under varying requirements based on parametric constraint sensitivity analysis and convex hull techniques [41]. The methodology ultimately allowed for identifying a robust feasible design space and a flexible family of solutions. A hybrid agent approach was applied for set-based conceptual ship design [42]. It was found that the process was robust to intermediate design errors. After the errors were corrected, the sets were still wide enough that the process could move forward and reach a converged solution without major rework. More work can be found in [43, 44].

According to [45], “excess” is “the quantity of surplus in a system once the necessities of the system are met”. Later, [9] defines “excess” as “the value over, and above, any allowances for uncertainties”, and “buffer” as “the portion of parameter values that compensates for uncertainties.” Robustness achieved by addressing these two concepts has different meanings. Current SBD literature lacks an explicit distinction between buffer and excess when addressing the uncertainties, hence lacking specificity in their robustness claims.

3 PRELIMINARIES

3.1 Notation

We introduce the notation that is used throughout this paper. For a function $f_i$, we make the following definitions. For function $f$, the same concepts can be applied without “$(i)$”.

- $a(i) = (a_1(i), a_2(i), ..., a_j(i), ..., a_n(i))^T$ is a col-
umn vector associated with function $f_i$, where $j = 1, 2, ..., n$.
- $R_{a_j(i)}$ is the range of $a_j(i)$, i.e., $a_j(i) \in R_{a_j(i)}$.
- $\{a(i)\}$ is a set representation of $a(i)$, i.e., $\{a(i)\} = \{a_1(i), a_2(i), ..., a_n(i)\}$.
- $R_{a(i)} = (R_{a_1(i)}, R_{a_2(i)}, ..., R_{a_n(i)})^T$ is a column vector of the ranges of $a(i)$, i.e., $a(i) \in R_{a(i)}$.
- $\{R_{a(i)}\}$ is a set representation of $R_{a(i)}$, i.e., $\{R_{a(i)}\} = \{R_{a_1(i)}, R_{a_2(i)}, ..., R_{a_n(i)}\}$.

Furthermore, we define the following set operations, where $a$ and $b$ are two column vectors.
- $\{a\} \cup \{b\}$ returns the union of the identifiers (instead of the values) of all the elements in $\{a\}$ and $\{b\}$. The identical variable between $\{a\}$ and $\{b\}$ are merged into one element. By “identical”, we mean the elements that have the same physical meaning, the same data source, and hence the same value at all time.
- $\{a\} \cap \{b\}$ returns the identifier of the identical variable(s) between $\{a\}$ and $\{b\}$.
- $\{a\} \subseteq \{b\}$ returns true if the identifiers (instead of the values) of all the variables in $\{a\}$ are included in $\{b\}$.
- $\{R_a\} \cup \{R_b\}$ returns the set of the ranges of $\{a\} \cup \{b\}$. Because the identical elements between $\{a\}$ and $\{b\}$ are subject to the ranges in both $\{R_a\}$ and $\{R_b\}$, the resulting ranges for the identical elements are the intersections ($\cap$) of the respective ranges in $\{R_a\}$ and $\{R_b\}$.
- $a_j|\{R_b\}$ identifies the element in $\{b\}$ that is identical to $a_j$ and returns its range in $\{R_b\}$.
- $\mathcal{TR}(\{R_a\})$ transforms the set $\{R_a\}$ into the column vector $R_a$, i.e., $\mathcal{TR}(\{R_a\}) = R_a$.

### 3.2 The functional requirements

First, functional requirements have different meanings in different contexts. In this paper, we abstract the function of a system in Eq. (1) as an input-output transformation that describes the overall behavior of a system [46, 47].

$$ (x, u, c) \xrightarrow{f} y \quad (1) $$

- $f$ represents the transformation between $(x, u, c)$ and $y$.
- $x$ represents the input variables that the system takes from the environment or other systems.
- $y$ represents the output variables that the system gives in response to the inputs.
- $u$ represents the uncertain design parameters that are out of the designer’s control, such as the weather for an airplane.
- $c$ represents the uncertain design parameters that are under the designer’s control, such as the weight of an airplane.

Second, mathematically, a system must satisfy Eq. (2), where $\left(R_{x}^{*}, R_{u}^{*}, R_{c}^{*}, R_{y}^{*}\right)$ constitute the functional requirements of the system.

$$ \forall x \in R_{x}^{*}, \forall u \in R_{u}^{*}, \exists c \in R_{c}^{*} : (x, u, c) \xrightarrow{f} y \in R_{y}^{*} \quad (2) $$

- $\forall R_{x}^{*}$ implies the system must be able to process all the values from the input set.
- $\forall R_{u}^{*}$ is because $u$ is out of the designer’s control. The system must be able to process all the possible $u$.
- $\exists R_{c}^{*}$ is because $c$ is under the designer’s control. The designer only needs to find one $c$ within $R_{c}^{*}$ to satisfy the functional requirements.
- $R_{y}^{*}$ is the co-domain of $f$. All the possible output values must be bounded within $R_{y}^{*}$.

However, not any random $(R_{x}^{*}, R_{u}^{*}, R_{c}^{*}, R_{y}^{*})$ can be the requirements of $f$. $(R_{x}^{*}, R_{u}^{*}, R_{c}^{*})$ and $R_{y}^{*}$ must satisfy Eq. (3), which implies that the functional requirements of a system are actually a constrained mapping between $(R_{x}^{*}, R_{u}^{*}, R_{c}^{*})$ and $R_{y}^{*}$.

$$ (R_{x}^{*}, R_{u}^{*}, R_{c}^{*}) \rightarrow R_{y}^{*} \text{ s.t. Eq. (2)} \quad (3) $$

Similarly, the requirements of $f_i$ can be represented in Eq. (4), where $f_i$ is the sub-function of $f$ ($i = 1, 2, ..., n$).

$$ (R_{x(i)}^{*}, R_{u(i)}^{*}, R_{c(i)}^{*}) \rightarrow R_{y(i)}^{*} \text{ s.t. Eq. (2)} \quad (4) $$

Therefore, the functional requirements decomposition problem is to decompose the top-level requirements in Eq. (3) into a set of sub-requirements in Eq. (4) at the lower level, repeat the same process independently at each level of abstraction and eventually lead to a system that satisfies the top-level requirements in Eq. (3).

### 3.3 Refinement and composability

**Refinement** defines a relationship between two sets of functional requirements (FR). $FR'$ refining $FR$ implies the reduction of the uncertainties from $FR$ to $FR'$, which can always be reflected in the mapping between the input
and output variables. As shown in Fig. 1(a), $FR'$ refines $FR$ if the relationship in Eq. (5) is satisfied, where the meanings of $x_i|[R^*_x]$ and $y_i|[R^*_y]$ are defined in Section 3.1.

\[
\begin{align*}
\{x\} \subseteq \{x'\} \land \forall x_i \in \{x\} : R^*_x \subseteq x_i \{R^*_x\} \\
\{y\} \subseteq \{y'\} \land \forall y_i \in \{y\} : y_i \{R^*_y\} \subseteq R^*_y
\end{align*}
\] (5)

Intuitively, Eq. (5) means (1) $FR'$ defines all the input/output variables of $FR$, and (2) $FR'$ allows the system to take a larger range of $x$ but will generate a smaller range of $y$ than $FR$.

![Graphical illustration of refinement](image)

(a) Refined by

(b) Ranges of $x$ Ranges of $y$

Fig. 1. Graphical illustration of refinement. The green circles are the ranges defined in $FR$, and the brown circles are the ranges defined in $FR'$.

Note that a system satisfying the functional requirements is a special case of refinement. As implies by Eq. (2), the system can take all the values in $R^*_x$ and provide the output values that are bounded within $R^*_y$, which satisfies the relationship of refinement in Eq. (5).

Composability defines the interface between functional requirements. Therefore, only the input and output variables are concerned. Two sets of functional requirements (denoted as $FR_j$ and $FR_k$ in Fig. 2(a)) are composable if the relationship in Eq. (6) below is satisfied.

\[
\begin{align*}
\{z\} = \{y(j)\} \cap \{x(k)\} \neq \emptyset \\
\forall z_i \in \{z\} : z_i \{R^*_y(j)\} \subseteq z_i \{R^*_x(k)\}
\end{align*}
\] (6)

Intuitively, composability means two things: (1) the output of $FR_j$ and the input of $FR_k$ share at least one common variable (denoted as \{z\}), and (2) $FR_k$ can take in more values of $z$ than $FR_j$ can output (Fig. 2(b)).

Second, based on the definition above, we have the following properties. Property 1 and 2 can be proved by directly applying the definitions of refinement and composability, so we do not provide additional proof. The proof of Property 3 can be found in the Appendix.

**Property 1.** If $FR'$ refines $FR$, and $FR''$ refines $FR'$, then $FR''$ refines $FR$.

**Property 2.** If a set of functional requirements $\{FR_1, FR_2, \ldots, FR_n\}$ are composable, then their refinements $\{FR'_1, FR'_2, \ldots, FR'_n\}$ are also composable.

**Property 3.** Consider a set of composable functional requirements $\{FR_1, FR_2, \ldots, FR_n\}$. If $FR'_i$ refines $FR_i$ (i=1, 2, ..., n), then the composite of $\{FR'_1, FR'_2, \ldots, FR'_n\}$ refines the composite of $\{FR_1, FR_2, \ldots, FR_n\}$.

Because a system satisfying the functional requirements is a special case of refinement, Property 4, 5 and 6 can be derived similar to Property 1, 2 and 3. Therefore, we do not provide additional proof for them.

**Property 4.** If $FR'$ refines $FR$, and a system $S$ satisfies $FR'$, then $S$ satisfies $FR$.

**Property 5.** If a set of functional requirements $\{FR_1, FR_2, \ldots, FR_n\}$ are composable, and a set of systems $\{S_1, S_2, \ldots, S_n\}$ satisfy the functional requirements respectively, then $\{S_1, S_2, \ldots, S_n\}$ are also composable.

**Property 6.** Consider a set of composable functional requirements $\{FR_1, FR_2, \ldots, FR_n\}$ and their composite $FR$. If $S_i$ satisfies $FR_i$ (i=1, 2, ..., n), then the composite of $\{S_1, S_2, \ldots, S_n\}$ satisfies the composite of $\{FR_1, FR_2, \ldots, FR_n\}$.

4 THE HIERARCHICAL INDEPENDENT DECOMPOSITION OF FUNCTIONAL REQUIREMENTS

We address Objective 2 introduced in Section 1. Without loss of generality, we depict a simple system in Fig. 3 to describe the hierarchical functional requirements decomposition process. First, given the top-level functional requirements $FR$, the OEM decomposes $FR$ into $\{FR^1_1, FR^1_2\}$, and assign them to the Level-1 suppliers. The Level-1 suppliers then decompose $FR^1_1$ and $FR^1_2$ independently into $\{FR^2_{1,1}, FR^2_{1,2}\}$ and $\{FR^2_{2,1}, FR^2_{2,2}\}$, and assign them to Level-2 suppliers. The Level-2 suppliers independently implement the functional requirements into individual systems $S^2_{1,1}, S^2_{1,2}, S^2_{2,1}$, and $S^2_{2,2}$ that satisfy the respective functional requirements.

Now, we prove if at each level of abstraction, (1) the sub-requirements are composable, and (2) when composed together the sub-requirements refine the higher-level functional requirements, then $S^2_{1,1}, S^2_{1,2}, S^2_{2,1}$ and $S^2_{2,2}$ are composable and together satisfy $FR$.

**Proof.** First, by definition, $\{FR^1_1, FR^1_2\}$ are composable and refine $FR$; $\{FR^2_{1,1}, FR^2_{1,2}\}$ are composable and refine $FR^1_1$; $\{FR^2_{1,1}, FR^2_{1,2}\}$ are composable and refine $FR^1_2$.
able and refine $FR_2'$; $S_{1,1}^2, S_{1,2}^2, S_{2,1}^2$ and $S_{2,2}^2$ satisfy $FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2$ and $FR_{2,2}^2$ respectively.

Second, $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ are composable according to Property 2. Then, the composite of $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ refines the composite of $\{FR_{1,1}^2, FR_{1,2}^2\}$ according to Property 3. Because $\{FR_{1,1}^2, FR_{1,2}^2\}$ refine $FR$, then the composite of $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ refine $FR$ according to Property 1.

Third, because $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ are composable, and $S_{1,1}^2, S_{1,2}^2, S_{2,1}^2$ and $S_{2,2}^2$ satisfy $FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2$ and $FR_{2,2}^2$ respectively, then $\{S_{1,1}^2, S_{1,2}^2, S_{2,1}^2, S_{2,2}^2\}$ are composable according to Property 5, and the composite of $\{S_{1,1}^2, S_{1,2}^2, S_{2,1}^2, S_{2,2}^2\}$ satisfies the composite of $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ according to Property 6. We have established that the composite of $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ refines $FR$. Then, according to Property 4, the composite of $\{FR_{1,1}^2, FR_{1,2}^2, FR_{2,1}^2, FR_{2,2}^2\}$ satisfies $FR$. In other words, the resulting system satisfies the top-level functional requirements.

Therefore, for the design teams at different levels of abstraction can work in parallel and obtain a system that satisfies the top-level functional requirements, the sub-requirements at each level of abstraction must be composable, and refine the higher-level functional requirements as a whole.

5 SET-BASED APPROACH TO REQUIREMENTS DECOMPOSITION AT EACH LEVEL OF ABSTRACTION

In this section, we apply the set-based approach to functional requirements decomposition (Objective 1) to achieve the sub-requirements required by Objective 2.

5.1 An overall description

Given the higher level functional requirements:

1. Define and analyze the functional architecture (Section 5.2).
2. Explore the sub-functions for initial feasible spaces (Section 5.3).
3. Narrow the initial feasible spaces for the feasible design space and performance space (Section 5.4).
4. Determine the sub-requirements that are composable, and when composed together, can refine the higher-level functional requirements (Section 5.5).

After that, the team for each sub-function will improve the level of detail by independently decomposing the sub-functions into the lower levels iteratively until a final solution can be chosen.

5.2 Define and analyze the functional architecture

A functional architecture is comprised a set of connected sub-functions $(f_1, f_2, ..., f_n)$, and represents a set of potential design solutions of the higher-level requirements. In SBD, there can be multiple alternative functional architectures. However, we focus on a given functional architecture in this paper because different functional architectures can be analyzed in the same way.

To analyze a functional architecture, we extract the following two pieces of information: (1) which variable/parameter is shared with which function/sub-functions, and (2) the constitution of the design space and the performance space. They will be used to narrow the feasible spaces later.

First, we aggregate $x(i), y(i), c(i)$ and $u(i)$ of the sub-functions into $\{x', y', c'\}$ and $\{u'\}$ without considering the functional architecture. As defined in Section 3.1, the $\sqcup$ operation merges the identical vari-
ables/parameters in multiple sub-functions into one variable/parameter in the new set.

\[
\begin{align*}
\{x\}' & = \{x(1)\} \cup \{x(2)\}, \ldots, \cup \{x(n)\} \\
\{y\}' & = \{y(1)\} \cup \{y(2)\}, \ldots, \cup \{y(n)\} \\
\{c\}' & = \{c(1)\} \cup \{c(2)\}, \ldots, \cup \{c(n)\} \\
\{u\}' & = \{u(1)\} \cup \{u(2)\}, \ldots, \cup \{u(n)\}
\end{align*}
\] (7)

**Second**, for the system to implement the higher-level functional requirements, all the elements defined in the higher-level function \(f\) (i.e., \(\{x\}, \{y\}, \{c\} \text{ and } \{u\}\)) must also be defined in the functional architecture, i.e., Eq. (8).

\[
\{x\} \sqsubseteq \{x\}' \land \{c\} \sqsubseteq \{c\}' \land \{u\} \sqsubseteq \{u\}' \land \{y\} \sqsubseteq \{y\}'
\] (8)

**Third**, we examine where the variables and the parameters are defined. The elements of \(\{c\}'\) and \(\{u\}'\) may or may not be defined in \(\{c\}\) and \(\{u\}\). We define the elements of \(\{c\}'\) and \(\{u\}'\) that are not defined in \(\{c\}\) and \(\{u\}\) as \(\overline{c}\) and \(\overline{u}\), and thus have Eq. (9) below.

\[
\begin{align*}
\{c\}' & = \overline{c} \sqcup \{c\} \\
\{u\}' & = \overline{u} \sqcup \{u\}
\end{align*}
\] (9)

Fig. 4. \(\{x\}'\) and \(\{y\}'\) can be divided into six groups to reason about the design space and the performance space. The ranges of each group will be calculated differently in Section 5.4.

For \(\{x\}'\) and \(\{y\}'\), we use Fig. 4 to reason about where the elements can be possibly defined, where \(\{x\}'\) and \(\{y\}'\) are the two outer ovals that include \(\{x\}\) and \(\{y\}\). As shown in the middle of Fig. 4, there are four possible different ways of where \(\{x\}'\) and \(\{y\}'\) can be defined. \(\{x\}'\) and \(\{y\}'\) overlap because the sub-functions are connected, i.e., one’s input is other’s output; \(\{x\}'\) and \(\{y\}'\) overlap if there are feedback loops that feeds some elements of \(\{y\}\) into the sub-functions. \(\{x\}\) does not overlap with \(\{y\}'\) or \(\{y\}\) because \(\{x\}\) are the input variables of the higher-level functional requirements, hence can only come from the external environment rather than being controlled by the system itself. As a result, we identify six different areas (the bottom of Fig. 4) depending on the overlapping between \(\{x\}'\), \(\{y\}'\), \(\{x\}\) and \(\{y\}\).

The variables in Area 1, 2, 3 and 4 are therefore denoted as \(\{y^{(1)}\}\), \(\{y^{(2)}\}\), \(\{y^{(3)}\}\) and \(\{y^{(4)}\}\) in Eq. (10).

\[
\begin{align*}
\{y^{(1)}\} & = \{y\} \cap \neg \{y\}' \cap \neg \{x\}' \\
\{y^{(2)}\} & = \{y\}' \cap \{x\}' \cap \neg \{y\} \\
\{y^{(3)}\} & = \{y\} \cap \{x\}' \\
\{y^{(4)}\} & = \{y\}' \cap \neg \{x\}'
\end{align*}
\] (10)

Area 5 is \(\{x\}\). Area 6, denoted as \(\overline{x}\), can then be represented in Eq. (11).

\[
\overline{x} = \{x\}' \cap \neg \{y\}' \cap \neg \{x\}
\] (11)

Therefore, we can classify all the elements of a functional architecture into the exclusive groups below based on where the elements are defined.

- \(\{x\}\) is defined in \(\{x\}'\) and \(\{x\}\); \(\overline{x}\) is only defined within \(\{x\}'\).
- \(\{c\}\) is defined in \(\{c\}'\) and \(\{c\}\); \(\overline{c}\) is only defined within \(\{c\}'\).
- \(\{u\}\) is defined in \(\{u\}'\) and \(\{u\}\); \(\overline{u}\) is only defined within \(\{u\}'\).
- \(\{y^{(1)}\}\) is only defined in \(\{y\}\); \(\{y^{(2)}\}\) is defined in \(\{x\}'\) and \(\{y\}'\); \(\{y^{(3)}\}\) is defined in \(\{x\}'\), \(\{y\}\) and \(\{y\}'\); \(\{y^{(4)}\}\) is defined in \(\{y\}\) and \(\{y\}'\).

**Finally**, because the design space is the set of the independent variables/parameters of a system and the performance space is the set of the dependent variables \([48, 49]\), the design space of a given functional architecture are comprised of \(\{(x), (c), (u), (\overline{x}), (\overline{c}), (\overline{u})\}\) and the performance space are comprised of \(\{(y^{(1)}), (y^{(2)}), (y^{(3)}), (y^{(4)})\}\).

### 5.3 Explore the initial feasible spaces

The initial feasible spaces of each sub-function is characterized in Eq. (12) after exploring the feasible im-
implementations. The specific initial spaces can only be determined case by case.

\[
\forall x(i) \in R_{x(i)}, \forall u(i) \in R_{u(i)}, \exists c(i) \in R_{c(i)}:
\]

\[
f_i(x(i), u(i), c(i)) \in R_{y(i)} \tag{12}
\]

Note that Eq. (12) is different from Eq. (4). The ranges (without *) in Eq. (12) represent the feasible spaces of the sub-functions while the ranges in Eq. (4) represent the desired spaces (i.e., requirements).

5.4 Narrow the feasible spaces

In this section, we derive the feasible design space and performance space of the system by narrowing from the initial feasible spaces.

First, we aggregate the ranges of the sub-functions \(\{R_{x(i)}\}, \{R_{y(i)}\}, \{R_{a(i)}\}\) and \(\{R_{c(i)}\}\) (\(i = 1, 2, \ldots, n\)) into \(\{R_{x'}\}, \{R_{y'}\}, \{R_{a'}\}\) and \(\{R_{c'}\}\), where \(\{x'\}, \{u'\}, \{c'\}\) and \(\{y'\}\) are defined in Eq. (7). Refer to Section 3.1 for the definition of \(\psi\).

\[
\begin{align*}
\{R_{x'}\} &= \{R_{x(1)}\} \cup \{R_{x(2)}\} \cup \ldots \cup \{R_{x(n)}\} \\
\{R_{y'}\} &= \{R_{y(1)}\} \cup \{R_{y(2)}\} \cup \ldots \cup \{R_{y(n)}\} \\
\{R_{c'}\} &= \{R_{c(1)}\} \cup \{R_{c(2)}\} \cup \ldots \cup \{R_{c(n)}\} \\
\{R_{a'}\} &= \{R_{a(1)}\} \cup \{R_{a(2)}\} \cup \ldots \cup \{R_{a(n)}\}
\end{align*}
\tag{13}
\]

Second, we calculate the ranges of \(\{\bar{x}\}, \{x\}, \{\bar{c}\}, \{c\}, \{\bar{u}\}, \{u\}, \{y^{(1)}\}, \{y^{(2)}\}, \{y^{(3)}\}\) and \(\{y^{(4)}\}\) based on where they are defined. For example, we have established in Section 5.2 that the elements in \(\{x\}\) are defined in both \(\{x\}\) and \(\{x'\}\). Therefore, the range of each element in \(\{x\}\) will be the intersection of the corresponding ranges in \(R_{x'}\) and \(\{R_{x'}\}\) i.e., the first item in Eq. (14). Refer to Section 3.1 for the definition of operation \(x_i|\{R_{x'}\}\). The ranges of \(\{c\}, \{u\}, \{\bar{x}\}, \{\bar{c}\}, \{\bar{u}\}, \{y^{(1)}\}, \{y^{(2)}\}, \{y^{(3)}\}\) and \(\{y^{(4)}\}\) can be calculated in the same way in the rest of Eq. (14).

\[
\begin{align*}
\{R_{x}\} &= \{R_{x(1)}\} \wedge \{R_{x(2)}\} \wedge \ldots \wedge \{R_{x(n)}\} \\
\{R_{a}\} &= \{R_{a(1)}\} \wedge \{R_{a(2)}\} \wedge \ldots \wedge \{R_{a(n)}\} \\
\{R_{c}\} &= \{R_{c(1)}\} \wedge \{R_{c(2)}\} \wedge \ldots \wedge \{R_{c(n)}\} \\
\{R_{y}\} &= \{R_{y(1)}\} \wedge \{R_{y(2)}\} \wedge \ldots \wedge \{R_{y(n)}\}
\end{align*}
\tag{14}
\]

In addition, each range in \(\{R_{x}\}, \{R_{a}\}, \{R_{c}\}, \{R_{y}\}\) cannot be \(\emptyset\) because \(\emptyset\) implies internal conflicts between the initial ranges, unless the corresponding group of elements are not defined. Furthermore, for \(\{R_{x}\}\) and \(\{R_{a}\}\), the conditions are stricter. Recall \(\forall\) is associated with \(\{R_{x}\}\) and \(\{R_{y}\}\) in Eq. (2). The system must operate under all the values defined in \(\{R_{x}\}\) and \(\{R_{a}\}\). For this reason, the ranges of \(\{x\}\) and \(\{u\}\) defined in \(\{R_{x'}\}\) and \(\{R_{a'}\}\) must include \(\{R_{x}\}\) and \(\{R_{a}\}\) respectively. As a result, we can update \(\{R_{x}\}\) and \(\{R_{a}\}\) in Eq. (15) based on Eq. (14).

\[
\{R_{x}\} = \{R_{x}\} \wedge \{R_{a}\} \tag{15}
\]

Now, let \(R_{D}^{1} = \{R_{x}\} \cup \{R_{x}\} \cup \{R_{c}\} \cup \{R_{a}\} \cup \{R_{y}\} \cup \{R_{y}\} \cup \{R_{y}\}\) and \(R_{P}^{1} = \{R_{y(1)}\} \cup \{R_{y(2)}\} \cup \{R_{y(3)}\} \cup \{R_{y(4)}\}\). The feasible design space and performance space can be easily represented in Eq. (16). Refer to Section 3.1 for the definition of operation \(\mathcal{T}\).

\[
FDS^{1} = \mathcal{T}(R_{D}^{1}) \wedge FPS^{1} = \mathcal{T}(R_{P}^{1}) \tag{16}
\]

Third, \(FDS^{1}\) and \(FPS^{1}\) are coupled through the sub-functions, and thus can be further narrowed by exploiting the coupling. Let \(f' = f_{1} \wedge f_{2} \ldots \wedge f_{n}\). \(f'\) can be characterized in Eq. (17).

\[
(x, \bar{x}, c, \bar{c}, u, \bar{u}) \xrightarrow{f'} y' \tag{17}
\]
where \((x, \bar{x}, c, \bar{c}, u, \bar{u}) \in FDS^1\) and \(y' \in FPS^1\).

Based on Eq. (17), we can narrow \(R_c\) and \(R\) to \(R_c^\dagger\) and \(R^\dagger\) so that Eq. (18) is satisfied, which is in fact a Quantified Constraint Satisfaction Problem [50].

For \(\forall x \in R_x, \forall \bar{x} \in R_{\bar{x}}, \forall u \in R_u, \forall \bar{u} \in R_{\bar{u}}, \forall c \in R_c^\dagger,\)
and \(\forall \bar{c} \in R_{\bar{c}}^\dagger:\)
\[
(x, \bar{x}, c, \bar{c}, u, \bar{u}) \mapsto y' \in FPS^1
\]

(18)

Let \(R_D^2 = \{R_x\} \cup \{R_{\bar{x}}\} \cup \{R_u\} \cup \{R_{\bar{u}}\} \cup \{R_c^\dagger\} \cup \{R_{\bar{c}}^\dagger\}\), then the new feasible design space can be represented as \(FDS^2 = TR(R_D^2)\).

The reason that only \(R_c\) and \(R\) are narrowed is because other ranges in Eq. (17) are all associated with \(\forall\), meaning the system must operate under all values in the ranges. Hence ranges associated with \(\forall\) cannot be narrowed. Furthermore, the reason that the ranges of the controllable parameters \(R_c^\dagger\) and \(R_{\bar{c}}^\dagger\) in Eq. (18) are associated with \(\forall\) is that \(R_c^\dagger\) and \(R_{\bar{c}}^\dagger\) will be further narrowed independently by subsystems at the lower levels of abstraction until single values can be determined for the elements in \(\{c\}\) and \(\{\bar{c}\}\). “\(\forall\)” makes sure whatever values chosen for \(c\) and \(\bar{c}\) at the lower levels, Eq. (18) will always be satisfied.

After Eq. (18), because the feasible design space is narrowed from \(FDS^1\) to \(FDS^2\), \(FPS^1\) can be narrowed into a new feasible performance space \(FPS^2\) based on the new \(FDS^2\), such that Eq. (19) holds.

\[
\forall(x, \bar{x}, c, \bar{c}, u, \bar{u}^\top \in FDS^2:
(x, \bar{x}, c, \bar{c}, u) \mapsto y' \in FPS^2
\]

(19)

Reachability analysis [51] can be applied to calculate \(FPS^2\) as an over-approximation so that all the points in \(FDS^2\) will be mapped within the \(FPS^2\). Let the resulting reachable sets for \(y''(b)\) be \(R_y^{(b)}\) and \(R_y^{(P)} = \{R_y^{(i(1))}\} \cup \{R_y^{(i(2))}\} \cup \{R_y^{(i(3))}\} \cup \{R_y^{(i(4))}\}\). The new feasible performance space can be represented as \(FPS^2 = TR(R_y^{(P)})\).

Finally, the feasible design space and performance space of the system are narrowed to \(FDS^2\) and \(FPS^2\).

5.5 Determine the sub-requirements

Uncertainty expansion. Ideally, the ranges in the sub-requirements should be as narrow as possible to reduce the uncertainty to the most extent. The ranges in \(FDS^2\) and \(FPS^2\) are the narrowest ranges that can be calculated at the current level of abstraction. However, it is always possible that the initial characterizations of the uncertainties (i.e., \(R_{u'}\) and \(R_{c'}\)) are under-estimated, which will inevitably lead to an expansion of \(FPS^2\) at a later time.

Naturally, to accommodate the possible unexpected expansion, the desired ranges of \(y'\) in the sub-requirements must also expand from \(FPS^2\). We denote the desired ranges of \(y'\) in the sub-requirements as \(FPS^*\). As shown in Fig. 5, to maximize the capacity of the sub-requirements to accommodate the possible expansion, \(FPS^*\) must be expanded as close to \(FPS^1\) as possible. Note that \(FPS^*\) cannot expand beyond \(FPS^1\) to avoid infeasibility.

![Fig. 5. The boundaries of \(FPS^*\).](image)

However, \(y'\) may also be the input variables of some sub-functions. Expanding the ranges of \(y'\) will inevitably lead to the expansion of the input ranges of some sub-functions, which will in turn makes the system more difficult or expensive to implement. Therefore, there is an inherent tension between accommodating excessive uncertainty and the level of difficulty in implementing the system. This tension dictates that the boundary of \(FPS^*\) in Fig. 5 cannot be too close either to \(FPS^1\) or to \(FPS^2\).

Trade-off study. A trade-off needs to be made to decide the distance of \(FPS^*\) from \(FPS^1\) and \(FPS^2\). This problem highly resembles the barrier function in optimization. Therefore, we construct an optimization problem based on barrier function to conduct the trade-off study.

First, we construct the boundaries of the ranges in \(FPS^*\). For \(y_j(i)\) (i.e., the \(j\)th variable of \(y(i))\), let the corresponding range defined in \(FPS^*\) be \(R_{y_j^*(i)} = [y_j^*(i), y_j^{**}(i)]\). As explained before, the ranges in \(FPS^*\) must be bounded by the ranges in \(FPS^1\) and \(FPS^2\). Therefore, the range of \(y_j(i)\) in \(FPS^*\) is subject to Eq. (20).

\[
\begin{align*}
\{y_j^{**}(i) \leq y_j^*(i) \leq y_j^{**}(i)\} \\
\{y_1^{**}(i) \leq y_1^*(i) \leq y_1^{**}(i)\}
\end{align*}
\]

(20)

- \(y_1^{**}(i)\) and \(y_1^{**}(i)\) are the upper and lower bound of the range of \(y_1(i)\) in \(FPS^1\).
requirements must satisfy the following three conditions:

1. We explain the ranges that meet all three conditions.
2. The sub-requirements.
3. The first condition.

In general, the sub-requirements.

The sub-requirements.

The coefficients \( a_j(i) \) and \( a_k^j(i) \) represent the designers’ preference of the ability to tolerate uncertainty expansion of \( f_x \) and the level of difficulty to implement \( f_x \). Specific values of \( a_j(i) \) and \( a_k^j(i) \) can only be determined case by case.

\[
\begin{align*}
\min \left( \sum_{i=1}^{n} S(i) \sum_{j=1}^{m} h_j(i) + \sum_{i=1}^{n} S(i) \sum_{j=1}^{m} h_k^j(i) \right)
\end{align*}
\]

The objective function is Eq. (22) where \( S(i) \) is the number of variables in \( y(i) \). The optimization problem is to decide \( y_j^{u*}(i) \) and \( y_j^{l*}(i) \) to minimize Eq. (22), subject to the constraints in Eq. (4) and Eq. (20). As a result, \( FPS^* \) can be assembled using the resulting \( y_j^{u*}(i) \) and \( y_j^{l*}(i) \) for \( i = 1, 2, ..., n \).

The sub-requirements. In general, the sub-requirements must satisfy the following three conditions:

1. Must be within the initial feasible spaces of the sub-functions.
2. Must include all the possible values that the elements of the system will encounter.
3. Must be composable and refine the higher-level functional requirements as a whole.

We intend to assign the ranges in \( FDS^2 \) and \( FPS^* \) directly to the sub-functions as the sub-requirements. Now, we explain the ranges satisfy all the three conditions above.

First, \( FDS^1 \) and \( FPS^1 \) are both narrowed from the initial feasible spaces of the sub-functions (see Eq. (14)), and \( FDS^2 \) and \( FPS^* \) are narrowed from \( FDS^1 \) and \( FPS^1 \). Therefore, ranges in \( FDS^2 \) and \( FPS^* \) satisfy the first condition.

Second, \( FDS^2 \) does not narrow the ranges of \( \{x\}, \{\bar{x}\}, \{u\} \) and \( \{\bar{u}\} \), elements that are associated with \( \forall \), thus all the values of the independent variables that the system must accept are included within \( FDS^2 \). Moreover, we have established in Eq. (19) that all the possible values of \( y^* \) are included within \( FPS^* \). Because \( FPS^* \) include \( FDS^2 \), thus all the possible values of \( y \) are also included within \( FPS^* \). Therefore, ranges in \( FDS^2 \) and \( FPS^* \) satisfy the second condition.

Third, the ranges of all the elements are only defined once in \( FDS^2 \) and \( FPS^* \). If they are assigned to the sub-functions, the identical element will only be assigned the same range. Therefore, the sub-requirements will be composable. In addition, the ranges of \( \{x\} \) in \( FDS^2 \) are the same as the ranges in the higher-level functional requirements (see Eq. (15)) and the ranges of \( y \) are narrowed from the ranges in the higher-level functional requirements (see Eq. (14)). Therefore, the sub-requirements as a whole will also refine the higher-level functional requirement, and the third condition is satisfied.

In conclusion, \( R_x^*, R_r^*, R_u^* \) and \( R_y^* \) can be directly obtained from \( FDS^2 \), and \( R_x^*, R_r^*, R_u^* \) can be directly obtained from \( FPS^* \). Together, \( R_x^*, R_r^*, R_u^* \) and \( R_y^* \) are the sub-requirements of \( f_x(i = 1, 2, ..., n) \) decomposed from the \( R_z^*, R_c^*, R_u^* \) and \( R_y^* \) of \( f \).

6 CASE STUDY

We design a cruise control system to demonstrate the feasibility of the four-step formal process proposed in Section 5.

The top-level functional requirements are given as below: for any initial speed \( v_0 \in [22.0, 30.0] \text{ m/s} \) and reference speed between \( v_r \in [35, 36] \text{ m/s} \), the real speed shall be bounded by \( v(t) \in [20, 40] \text{ at all time (we simulated 100 seconds in this paper) and converge} \) to \( v(t) \in [33, 37] \text{ m/s after 20 seconds.} \)

\[
(v_0, v_r) \xrightarrow{f} v(t)
\]

where \( R_z^* \) is \( v_0, v_r \in ([23.0, 30.0], [34, 36])^T \text{ m/s} \) and \( R_z^* \) is \( v(t) \in [20, 40] \text{ m/s for } t \in [0, 100] \text{s} \)

Step 1: Define and analyze the functional architecture. We adopt the functional architecture of a cruise control design from Aström and Murray [52] (Fig. 6) with each sub-function defined as below.
functions. In this example, the only variable we can find the input of the sub-functions and the output of the sub-functions that are defined in the output of the top-level function, $\omega$. To achieve $FDS^1$ and $FPS^1$ by taking intersections of the ranges of the elements defined in multiple places. For example, $m$ is defined in $f_2$ and $f_3$, hence $R_m$ is the intersection of the corresponding ranges defined in $f_2$ and $f_3$. $v_0$ is defined in both top-level function and $f_1$, hence $R_{v_0}$ is the intersection of the corresponding ranges defined in $f_1$ and top-level function.

As a result, the ranges in $FDS^1$ can be computed as below:

\[
R_{v_r} = [23.0, 30.0] \text{m/s}, R_{v_0} = [34, 36] \text{m/s}
\]
\[
R_{\omega_m} = [350, 480] \text{rad/s}, R_m = [990, 1100] \text{kg}
\]

The ranges in $FPS^1$ can be computed as below:

\[
R_{\dot{v}(t)} = [-1.5, 3] \text{m/s}^2, R_{Fr(t)} = [70, 120] \text{N},
\]
\[
R_F(t) = [-250, 3500] \text{N}, R_{Fa(t)} = [0, 1000] \text{N},
\]
\[
R_{T(t)} = [0, 250] \text{Nm}, R_{u(t)} = [-0.5, 2],
\]
\[
R_{\omega(t)} = [0, 450] \text{rad/s}, R_{u(t)} = [20, 40] \text{m/s}
\]

Second, let $f' = f_1 \land f_2 \ldots \land f_6$. $FDS^2$ and $FPS^2$ can be further narrowed into $FDS^2$ and $FPS^2$ using $f'$. To achieve $FDS^2$, the range of the controllable parameter $\omega_m$ must be narrowed into $R^k_{\omega_m}$ based on Eq. (18) such that,

For $\forall v_r \in R_{v_r}, \forall v_0 \in R_{v_0}, \forall m \in R_m, \forall \omega_m \in R^k_{\omega_m}$:

\[
(v_r, v_0, \omega_m, m) \xrightarrow{f'} y' \in FPS^1
\]

where $y' = (v(t), \dot{v}(t), v(t), Fr(t), F(t), Fa(t), T(t), u(t), \omega(t))$. In this example, we apply reachability analysis to help find $R^k_{\omega_m}$. We estimate an over-approximation of $R^k_{\omega_m}$, feed it into a reachability solver (i.e., CORA [53] in this example), and adjust the bounds of the over-approximation until the reachable sets of all the output variables are bounded with $FPS^1$. As a result, we obtain the ranges in $FDS^2$ as below,

\[
R_{v_r} = [23.0, 30.0] \text{m/s}, R_{v_0} = [34, 36] \text{m/s}
\]
\[
R^k_{\omega_m} = [365, 450] \text{rad/s}, R_m = [990, 1100] \text{kg}
\]

After that, $FPS^1$ can be narrowed into $FPS^2$ by calculating the reachable sets according to Eq. (19). We choose CORA as the tool because it can compute the over-approximation of the feasible performance space. The result shows the range of $v(t)$ converges gradually from $R_{v_0}$ to $R_{v_r}$, meaning the reference speed is achieved by the cruise control system (see the Supplemental Material for the plots). Eventually, we obtain the ranges of $FPS^2$.

Step 2: Explore the initial feasible spaces. The initial feasible spaces are provided by the design teams responsible for the sub-functions. The specific ranges can be found in the Supplemental Material.

A detailed explanation of each sub-function can be found in the Supplemental Material. For this step, the input/output variables can be found in Fig. 6; the uncontrollable uncertain parameter is the weight of the car ($m$); the controllable uncertain parameter is the maximum engine speed ($\omega_m$). Other parameters are constant. Note that the constant parameters can also be defined as uncertain parameters and addressed in the same way as $m$ and $\omega_m$. However, we intentionally limit the number of the uncertain parameters to simplify the presentation.

Next, we classify the variables and the parameters based on where they are defined, yielding the following exclusive groups. For example, according to the characterizations in Section 5.2, $\{y^{(3)}\}$ are the output variables that are defined in the output of the top-level function, the input of the sub-functions and the output of the subfunctions. In this example, the only variable we can find is $v(t)$.

- $\{x\} = \{v_r, v_0\}$.
- $\{\bar{c}\} = \{\omega_m\}$.
- $\{\bar{u}\} = \{m\}$.
- $\{y^{(2)}\} = \{\dot{v}(t), Fr(t), F(t), Fa(t), T(t), u(t), \omega(t)\}$.
- $\{y^{(3)}\} = \{v(t)\}$.

As a result, the design space of the cruise control system is $\{v_r, v_0, m, \omega_m\}$ and the performance space is $\{\dot{v}(t), Fr(t), F(t), Fa(t), T(t), u(t), \omega(t), v(t)\}$.

\[
\begin{align*}
\{ f_1 \} & : v(t) = \int_0^t \dot{v} dt + v_0 \\
\{ f_2 \} & : \dot{v}(t) = (F(t) - Fr - Fa(t))/m \\
\{ f_3 \} & : Fr = mgC_r \\
\{ f_4 \} & : F(t) = \alpha * u(t)T(t) \\
\{ f_5 \} & : Fa(t) = \frac{1}{2} \rho C_d A v(t)^2 \\
\{ f_6 \} & : \omega(t) = \alpha * v(t) \\
\{ f_7 \} & : T(t) = T_m(1 - \beta(\omega(t)/\omega_m - 1)^2) \\
\{ f_8 \} & : u(t) = p(v_r - v(t)) + i \int_0^t (v_r - v(t)) dt
\end{align*}
\]
Step 4: Determine the sub-requirements. We compute $R_{st}(i) = [y^s(i), y^{**}(i)]$ for $y(i)$ ($i = 1, 2, \ldots, 8$) by formulating the optimization problem in Eq. (22) subject to the constraints in Eq. (4) and Eq. (20). Note that there is no subscript associated with the output variable $y(i)$, because each $f_i$ in this example only has one variable at its output side.

First, the constraints described in Eq. (20) can be directly obtained from $FDS$, $FPS$. $FDS$ and $FPS$ calculated in the previous step.

Second, Eq. (4) needs to be satisfied for each individual sub-function, which yields the constraints in Eq. (23).

Note that $f_1$ and $f_8$ are not included in the constraints. $f_1$ and $f_8$ are not enough to decide the ranges of the output variables because the operation $f$ makes the constraint range of the output variables correlated with the time $t$. Their ranges can only be determined after the rest of the system is determined. In this example, $f_1$ and $f_8$ belong to the PID controller. These modules are designed after other parts of the car (e.g., engine and tyres) are defined, and hence not part of the constraints in Eq. (23).

$$
\begin{align*}
&y^{**}(6) \leq f_6(y^{**}(1)) \\
&y^{**}(6) \geq f_6(y^{**}(1)) \\
&y^{**}(7) \leq f_7(y^{**}(6), \omega_m) \\
&y^{**}(7) \geq f_7(y^{**}(6), \omega_m) \\
&y^{**}(5) \leq f_5(y^{**}(1)) \\
&y^{**}(5) \geq f_5(y^{**}(1)) \\
&y^s(4) \leq f_4(y^s(7), y^s(8)) \\
&y^s(4) \geq f_4(y^s(7), y^s(8)) \\
&y^s(2) \leq f_2(y^s(4), y^u(3), y^u(5)) \\
&y^s(2) \geq f_2(y^s(4), y^u(3), y^u(5))
\end{align*}
$$

Third, based on Eq. (21) and Eq. (22), we define the objective function below:

$$
\min \left( \sum_{i=1}^{8} h(i) + \sum_{i=1}^{8} h^k(i) \right)
$$

where the coefficients are defined as $a(1) = 0.9, a(2) = 0.6, a(3) = 0.5, a(4) = 0.3, a(5) = 0.4, a(6) = 0.5, a(7) = 0.1, a(8) = 0.9, a^5(1) = 0.5, a^6(1) = 0.1, a^8(1) = 0.8, a^4(2) = 0.6, a^2(3) = 0.4, a^2(4) = 0.8, a^2(5) = 0.7, a^2(6) = 0.6, a^2(7) = 0.9, a^4(8) = 0.9.

Finally, the optimization problem is to decide $y^{**}(i)$ and $y^s(i)$ ($i = 1, 2, \ldots, 8$) by minimizing Eq. (24). As a result, $y^{**}(i)$ and $y^s(i)$ ($i = 1, 2, \ldots, 8$) are calculated. Together with $FDS$ computed in the previous step, the ranges of each sub-function can be summarized in Table 1. Accordingly, the sub-requirements of $f_i$ are the mapping between ($R_{st}(i), R_{c}(i), R_{u}(i)$) and $R_{st}(i)$ in Table 1. Eventually, the four-step process can be repeated to further decompose the sub-requirements independently to the lower levels until a final solution is selected. The resulting system as a whole is guaranteed to satisfy the top-level functional requirements.

| $f$ | $R_{s}(i), R_{c}(i), R_{u}(i)$ | $R_{st}(i)$ |
|-----|-----------------|------------|
| $f_1$ | $v_0 \in [34.36, m/s]$ | $v(t) \in [21.8, 38.4, m/s]$ |
| $f_2$ | $F(r) \in [88.2, 107.8, N]$ | $v(t) \in [-1.1, 3, m/s^2]$ |
| $f_3$ | $m \in [990, 1100, kg]$ | $F(r) \in [88.2, 107.8, N]$ |
| $f_4$ | $T(t) \in [150, 200.1, Nm]$ | $F(t) \in [-159.1, 3024.0, N]$ |
| $f_5$ | $v(t) \in [21.8, 38.4, m/s]$ | $F(t) \in [-159.1, 3024.0, N]$ |
| $f_6$ | $v(t) \in [21.8, 38.4, m/s]$ | $\omega(t) \in [109.8, 406.1, rad/s]$ |
| $f_7$ | $\omega(t) \in [109.8, 406.1, rad/s]$ | $T(t) \in [150, 200.1, Nm]$ |
| $f_8$ | $v_0 \in [23.0, 30.0, m/s]$ | $u(t) \in [-0.1, 1.511]$ |

7 CONCLUSION

7.1 Summary

SBD has excellent potential in assisting functional requirements decomposition for complex systems, especially under the OEM-supplier development mode. This paper formalizes the functional requirements decomposition process for SBD. We first proved that as long as the
sub-requirements are composable and refine the higher-level functional requirements, the design teams at multiple levels can decompose their functional requirements independently, and the final system will still satisfy the top-level functional requirements. After that, we proposed a set-based formal process for the design teams at each level of abstraction to decompose the functional requirements independently so that the sub-requirements are composable and refine the higher-level functional requirements. Finally, a case study on designing a cruise control system was conducted to demonstrate the feasibility of the proposed set-based process.

7.2 Discussion

Toyota’s set-based approach. In one of the seminal works of SBD, Ward et al. [54] explained Toyota’s qualitative SBD process in five steps. Later, Specking et al. [5] observed that many SBD literature shared the similar characteristics with Toyota’s process. We compare our process with Toyota’s approach below.

1. The team defines a set of solutions at the system level.
2. It defines sets of possible solutions for various subsystems.
3. It explores these possible subsystems in parallel to characterize a possible set of solutions.
4. It gradually narrows the set of solutions, converging slowly toward a single solution. In particular, the team determines the appropriate specifications to impose on the subsystems.
5. Once the team established the single solution for any part of the design, it does not change it unless absolutely necessary.

The first step above is the same as Step 1 in our process, where the “solution” in our case is the functional architecture. The second step and third step above correspond to Step 2 of our process, which are essentially defining the feasible implementation of the subsystems individually. The fourth step above corresponds to Step 3 and 4 in our process. However, our formal process provides more details without ambiguity. With the underlying formalism defined in the process, we are able to explain how to narrow the sets, determine the specifications on the subsystems, and gradually converge to a single solution using hierarchy. The fifth step above is related to the uncertainty expansion addressed in our process. Specifically, Toyota applies “rigorous effort to avoid changes that expand, rather than contract, the space of possible designs” [54]. We believe expansion sometimes is unavoidable due to epistemic uncertainties. Step 4 of our process for “uncertainty expansion” is motivated for this reason and designed to address this possible impacts of the unexpected uncertainty expansions.

Scalability. One possible question about the proposed process relates to scalability. The hierarchical nature of our process makes it scalable for complex systems, as the complexity can always be reduced through hierarchical abstraction, i.e., an iterative “divide-and-conquer” approach at multiple levels of abstraction that still provides mathematical guarantees that the composed system meets its requirements. However, at one specific level of abstraction, the scalability of our process is subject to the effectiveness of the computational techniques used in the process. Specifically, Step 3 requires computational tools to compute \( FDS^2 \) and \( FPS^2 \). Too many sub-functions or uncertain parameters may make the computation challenging. This suggests that creating appropriate abstractions and architectures is vital to the approach, as it is in any complex system design.

Future work. Our approach is only applicable to the systems that can be abstracted as input-output transformations (e.g., dynamic systems), while general SBD does not assume any formalism of the system. This is a limitation of the paper. The applicability of other formalisms within the framework should be investigated in the future.

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APPENDIX A

Proof of Property 3. Let $FR$ be the composite of $\{FR_1, FR_2, ..., FR_n\}$. Replace one $FR_i$ with $FR'_i$ ($i = 1, 2, ..., n$). According to Property 2, $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$ are composable, meaning $FR_i$ can be replaced with $FR'_i$ without causing changes to other functional requirements.

Therefore, if the input of $FR_i$ is the input of $FR$, then the composite of $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$ will have a larger input space. If the output of $FR_i$ is the output of $FR$, then the composite of $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$ will have a smaller input space. Otherwise, the composite of $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$ will have the same input space and output space as $FR$. As a result, we can conclude the composite of $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$ will always refine $FR$.

Next, replace another element in $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$ with its refinement. We will obtain a refinement of $\{FR_1, FR_2, ..., FR'_i, ..., FR_n\}$. Repeat the process until all the functional requirements are replaced with the respective refinement. Eventually, we will obtain $\{FR'_1, FR'_2, ..., FR'_n\}$ that refines $FR$. □
SUPPLEMENTAL MATERIAL
A FURTHER DETAILS OF THE SUBFUNCTIONS FROM THE CASE STUDY

\( f_1 \) is a mathematical definition of the speed, and hence will be allocated to no subsystem. \( \dot{v}(t) \) has to be bounded to avoid violent acceleration/deceleration by the cruise control; \( v(t) \) is bounded to constrain the fastest speed that the cruise control can be engaged due to safety concerns.

- \( f_1: v(t) = \int_0^t \dot{v} dt + v_0 \)
- \( x(1) = (v_0, \dot{v})^\top \) and,
  \[ R_{x(1)} = ([0, 40][m/s], [-1.5, 3][m/s^2])^\top. \]
- \( y(1) = v(t) \) and \( R_{y(1)} = [0, 50][m/s]. \)

\( f_2 \) combines all the forces and yields the acceleration of the car as a whole, where \( F(t) \) is the driving force, \( Fa(t) \) is the aerodynamic drag, and \( Fr \) is the tyre friction. The weight of the car \( m \) is defined as an uncontrollable parameter.

- \( f_2: \ddot{v}(t) = (F(t) - Fr - Fa(t))/m. \)
- \( x(2) = (F(t), Fr, Fa(t))^\top \) and,
  \[ R_{x(2)} = ([−250, 3500][N], [60, 130][N], [0, 1000][N])^\top. \]
- \( y(2) = \ddot{v}(t) \) and \( R_{y(2)} = [−2, 4][m/s^2]; \)
- \( u(2) = m \) and \( R_{u(2)} = [990, 1100][kg]. \)

\( f_3 \) represent the rolling friction of the tyre. No input is defined for this function. \( C_r \) is the rolling friction coefficient.

- \( f_3: Fr = mgC_r. \)
- \( y(3) = Fr \) and \( R_{y(3)} = [70, 120][N]. \)
- \( u(3) = m \) and \( R_{u(3)} = [990, 1100][kg]. \)

\( f_4 \) represents the ability of the drivetrain to convert the torque to the driving force, where \( \alpha \) is the gear ratio, \( u(t) \) is the control input from the cruise controller (not the uncontrollable uncertain parameters defined in this paper), \( T(t) \) is the torque.

- \( f_4: F(t) = \alpha \cdot u(t)T(t). \)
- \( x(4) = (u(t), T(t))^\top \) and,
  \[ R_{x(4)} = ([−0.5, 2], [0, 250][N \cdot m])^\top. \]
- \( y(4) = F(t) \) and \( R_{y(4)} = [−1250, 5000][N]. \)

\( f_5 \) represents the aerodynamic drag, where \( \rho \) is the air density, \( C_d \) is the shape-dependent aerodynamic drag coefficient and \( A \) is the frontal area of the car.

- \( f_5: Fa(t) = \frac{1}{2}\rho C_d A v(t)^2; \)
- \( x(5) = v(t) \) and \( R_{x(5)} = [0, 60][m/s]. \)
- \( y(5) = Fa(t) \) and \( R_{y(5)} = [0, 2000][N]. \)

\( f_6 \) represents the ability of the drivetrain to convert from the car speed \( v(t) \) to the engine speed \( \omega(t) \).

- \( f_6: \omega(t) = \alpha \cdot v(t); \)
- \( x(6) = v(t) \) and \( R_{x(6)} = [0, 55][m/s]. \)
- \( y(6) = \omega(t) \) and \( R_{y(6)} = [0, 560][rad/s]. \)

\( f_7 \) represents the correlation between the torque, the engine speed \( \omega(t) \), the maximum torque \( T_m \) and the maximum engine speed \( \omega_m \); \( \omega_m \) is the controllable uncertain design parameter that starts with a rough estimation but will eventually be determined by the designers.

- \( f_7: T(t) = T_m(1 - \beta(\omega(t)/\omega_m - 1)^2). \)
- \( x(7) = \omega(t) \) and \( R_{x(7)} = [0, 450][rad/s]. \)
- \( y(7) = T(t) \) and \( R_{y(7)} = [0, 250][NM]. \)
- \( c(7) = \omega_m \) and \( R_{c(7)} = [350, 480][rad/s]. \)

\( f_8 \) is the cruise controller, in this case based on PID control. \( p \) and \( i \) are the proportional and integral parameters for the PID controller.

- \( f_8: u(t) = p(v_r - v(t)) + i \int_0^t (v_r - v(t))dt. \)
- \( x(8) = (v_r, v(t))^\top \) and,
  \[ R_{x(8)} = ([0, 60][m/s], [0, 60][m/s])^\top. \]
- \( y(8) = u(t) \) and \( R_{y(8)} = [−0.5, 4]. \)

The coefficients mentioned above are \( \alpha = 10, g = 9.8, C_r = 0.01, C_d = 0.32, \rho = 1.3, A = 2.4, \beta = 0.4, T_m = 200, p = 0.1, i = 0.5. \)

B ADDITIONAL RESULTS OF THE CASE STUDY

We mentioned in the case study that after applying CORA for the reachability analysis, the result shows the range of \( v(t) \) converges gradually from \( R_{v_0} \) to \( R_v \) during the simulated time, meaning the reference speed is achieved by the cruise control system. Here are the plots of the converging trend from CORA.

Fig. 1 is a continuous plot of the reachable set of \((v(t), u(t))^\top \) for \( t \in [0, 100] \) seconds. The blue line at the left bottom corner is the initial condition of \((v(0), u(0))^\top \). It converges to the right and becomes stable at the darker red area as the time approaches to \( t = 100 \).

Fig. 2 shows the convergence of \((v(t), u(t))^\top \) through time. In the beginning, it is bounded by the large blue area during \([1, 2]\)s; it then gradually migrates to the right and is bounded in the smaller green area for \([10, 11]\)s; during \([20, 100]\)s, it is bounded in the smallest magenta area. This discrete change in different sections of time coincides the narrowing and converging trend in Fig. 1.
Fig. 1. Continuous evolution of the reachable sets of \((v(t), u(t))^\top\): given a set of known dynamics (Table 1), all the possible values of \((v(t), u(t))^\top\) during \(t \in [0, 100]\) are properly bounded within the red shape.

Fig. 2. Discrete sampling of the continuous evolution of the reachable sets (Figure 1) at different time intervals. Instead of only showing the trend of the convergence, it compares the reachable sets of \((v(t), u(t))^\top\) at different time intervals. Compared with the blue area during \([1, 2]\) seconds and the green area during \([10, 11]\) seconds, the magenta area of the last 80 seconds is much smaller.