JANTZEN FILTRATION OF WEYL MODULES FOR $GL(m|n)$

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Abstract. Let $G = GL(m|n)$ be a general linear supergroup over an algebraically closed field $k$ of odd characteristic $p$. In this paper we construct Jantzen filtration of Weyl modules $V(\lambda)$ of $G$ when $\lambda$ is a typical weight in the sense of Kac’s definition, and consequently obtain a sum formula for their characters. By Steinberg’s tensor product theorem, it is enough for us to study typical weights with aim to formulate irreducible characters. As an application, it turns out that an irreducible $G$-module $L(\lambda)$ can be realized as a Kac module if and only if $\lambda$ is $p$-typical.

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Introduction

0.1. As is well-known, it is a very important but very difficult problem to formulate irreducible characters for reductive algebraic groups in prime characteristic (see [1], [10] §II.C], [28], etc.). Establishing Jantzen filtration and its sum formula of characters for Weyl modules is an approach to understanding the question (see [10] §II.8]. As a counterpart, it’s a significant task to establish Jantzen filtration in the study of representations of algebraic supergroups. In this paper, we initiate

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to study Jantzen filtration for algebraic supergroups, beginning with the case of the general linear supergroup $GL(m|n)$.

0.2. Let $k$ be an given algebraically closed field of characteristic $p > 2$. Let $G$ be an algebraic supergroup of Chevalley type over $k$ in the sense of [8], $G_{ev}$ is the purely-even subgroup of $G$, $T$ be a maximal torus of $G$ with $X(T)$ being the character group, and $B$ the Borel subgroup corresponding to negative roots. Denote by $W$ the Weyl group of $G$, by $w_0$ the longest elements in $W$ and $l(w_0)$ is the length of $w_0$. Let $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{b}^- = \text{Lie}(B)$.

0.3. When considering a connected reductive algebraic group $G_{ev}$ (here we abuse some notations like $G_{ev}$ with subscript $ev$ for the time being to avoid any confusions), one has “standard modules” ($H^0_{ev}(\lambda) := H^0(G_{ev}/B_{ev}, \mathcal{L}_{G_{ev}/B_{ev}}(k_{\chi}))$) and “costandard modules” ($V(\lambda) := H^0_{ev}(-w_0\lambda)^*$) with $\lambda \in X^+(T)$ (the set of dominant weights). The Weyl module $V_{ev}(\lambda)$ is isomorphic to $H^0_{ev}(w_0, \lambda)$ under the Serre duality (cf. [10 §II.4.2]). The character formulas for $V_{ev}(\lambda)$ and $H^0_{ev}(\lambda)$ are the same. The theory of Jantzen filtrations and the sum formulas for Weyl modules plays a very important role in representations of $G_{ev}$ (cf. [10 §II.8]).

0.4. The standard modules $H^0(\lambda)$ for algebraic supergroup $G$ have been studied in different versions (see [21 31 18 24 30 32]). In this paper, we will focus on $G = GL(m|n)$ unless other stated. Define the Weyl supermodule $V(\lambda) := H^l(w_0)\lambda (w_0, \lambda)$, the Serre duality for superschemes (see Theorem 2.2) makes $V(\lambda)$ isomorphic to $H^0(G/B, k_{-w_0\lambda} \otimes \wedge^*(\mathfrak{g}/\mathfrak{b}^-)^*)_t$ (see (7.1)). Although not as satisfactory with $V(\lambda)$ as in the case of reductive algebraic groups, it turns out that we have the following axioms (see Lemma 7.1).

(W1) The socle of $H^0(G/B, k_{-w_0\lambda} \otimes \wedge^*(\mathfrak{g}/\mathfrak{b}^-)^*)_t$ is isomorphic to $L(-w_0\lambda + 2\rho_1)$ where $\rho_1$ is half the sum of positive odd roots.
(W2) Correspondingly, the head of $V(\lambda)$ is isomorphic to $L(-w_0\lambda + 2\rho_1)^*$.
(W3) The following character formula holds

$$\text{ch}(V(\lambda)) = \text{ch}(H^0(\lambda)).$$

So, the Weyl modules are still important as in the ordinary case although it becomes more complicated.

Indeed, different from the case of reductive algebraic groups, the head of $V(\lambda)$ is no more isomorphic to the socle of $H^0(\lambda)$ in general. There is no longer enough information from $H^0(\lambda)$ providing for the construction of Jantzen filtration because only considering the Weyl group $W$ it is not enough for us to take the whole picture, due to the emergence of odd parts.

So, in order to establish the Jantzen filtration of $V(\lambda)$, we need to consider the influence caused by the odd reflections. Especially, we need to introduce a so-called totally-odd induced module $H^0(G/w_1(B), \mathcal{L}_{G/w_1(B)}(k_{\lambda-2\rho_1}))$ with respect to the Borel subgroup $w_1(B)$ defined by total negative odd roots where $\rho_1$ stands for half the sum of positive odd roots, and $w_1$ is the “longest” element of the part generated odd reflections of the super Weyl group $\hat{W}$ introduced in [20] for
construction of Jantzen filtration in modular representations of basic classical Lie superalgebras. We will denote this totally-odd induced module by $H^0_{\text{total}}(\lambda)$. In our construction, $H^0_{\text{total}}(\lambda)$ will take the place of $H^0(\lambda)$ in the situation of reductive algebraic groups, which turns out to satisfy the following axiom (see Lemma 6.4).

(W4) $\text{ch}(H^0_{\text{total}}(\lambda)) = \text{ch}(H^0_{\text{ev}}(\lambda - 2\rho_1))\Xi_{mn}$ with $\Xi_{mn} = \prod_{\beta \in \Phi_+^w}(1 + e^\beta)$.

0.5. Now we make a rough explanation on the above. In contrast with reductive algebraic groups, the critical difference in the super case is that there exist non-conjugate Borel subgroups even under the circumstance that they contain the same maximal torus. Correspondingly, the role of Weyl groups of purely-even subgroups is not enough powerful in describing Weyl modules. We need to add the role of odd reflections integrated into the so-called super Weyl groups. Then, in some sense, for example, under the action of super Weyl groups (see §2.1.2.2), all Borel subgroups containing the same maximal torus are conjugate. So it enables us to have the following pictures involving $V(\lambda)$. Take $T$ to be the standard maximal torus of $GL(m|n)$ and denote by $\Phi$ the root system associated with $T$. One can make an order of the positive odd roots with respect to the stand Borel subgroup $B^+$ as $\{\beta_1, \beta_2, \ldots, \beta_{mn}\}$ (note that all odd roots for $GL(m|n)$ are isotropic). Denote by $\tilde{r}_\beta$ the corresponding odd reflections (see §1.2.2). Then a sequence of Borel subgroups $\{B^{(i)} | i = 1, \ldots, mn\}$ arise. Naturally, with respect to $w_1(B) = B^{(mn)}$ we already have talked about the totally-odd induced module $H^0_{\text{total}}(\lambda) = H^0(G/w_1(B), L_{G/w_1(B)}(k_{\lambda - 2\rho_1}))$. The socle of $H^0_{\text{total}}(\lambda)$ turns out isomorphic to $L(-w_0\lambda + 2\rho_1)^*$ which coincides with the head of $V(\lambda)$, up to isomorphisms (see §7.2).

Next, it is revealed that there exists connection $V(\lambda)$ with $H^0_{\text{total}}(\lambda)$. We construct a series of homomorphisms arising from an ordered odd reflections along with the ordinary longest element $w_0$ in $W$ (see Lemma 1.2). Composing the odd parts, we first establish the nonzero homomorphism

$$\tilde{T}_{A,w_1} : H^0_A(\lambda) \to H^0(G_A/w_1(B)A, L_{G/w_1(B)}(A_{\lambda - 2\rho_1}))$$

for any commutative $\mathbb{Z}$-algebra $A$ (see §6.5). For the other part arising from $w_0$, one can mimic the arguments for reductive algebraic groups. In summary, the construction is based on the reduced expression of the longest element $\tilde{w}_\ell = w_0w_1$ of the super Weyl group $\tilde{W}$. So our arguments on the chain of homomorphisms mentioned above are divided into two parts. One part is from real reflections, which is actually the same as in [10] §II.8.16, giving rise to a homomorphism

$$\tilde{T}_{A,w_0} : V_A(\lambda) \to H^0_A(\lambda).$$

The other part are completely the new phenomenon, from odd reflections, which reflects the cohomological information arising from the transition of Borels by odd reflections (see §7.3). The final composite

$$\tilde{T}_{A,\tilde{w}_\ell} = \tilde{T}_{A,w_1} \circ \tilde{T}_{A,w_0} :$$

$$H^{(w_0)}(G_A/B_A, L_{G_A/B_A}(A_{w_0,\lambda})) \to H^0(G_A/w_1(B)A, L_{G_A/w_1(B)}(A_{\lambda - 2\rho_1}))$$
turns out to be nonzero when $\lambda$ is typical\footnote{In this present paper, the notion “typical weight” is identical to Kac’s definition over complex numbers in \cite{12} (see Definition 5.1), different from the one in \cite{29} and \cite{32}, the latter of which is identical another notion “$p$-typical weight” here (see Remark 8.5).}. In particular, when taking $A$ to be a principal ideal domain such that the fractional field $K$ has characteristic 0, then $\tilde{T}_{k,\bar{\omega}_t}$ is an isomorphism for typical $\lambda$. By localization of $A$ at a special prime ideal, one has that $\tilde{T}_{k,\bar{\omega}_t}$ maps the head of $V(\lambda)$ to the socle of $H^0_{\text{total}}(\lambda)$ (see Proposition 7.10).

0.6. Based on the above results, we successfully establish the Jantzen filtration of the Weyl module $V(\lambda)$ and the sum formula for characters when $\lambda$ is typical (see Theorem 8.4). As to atypical weights, Steinberg’s tensor theorem enables us to reduce the question of formulating irreducible characters to the case of typical weights (see Theorem 5.4). As an application, we investigate the question of realizing irreducible modules via Kac modules, the latter of which are induced modules from irreducible $G_{ev}$-module $L_{ev}(\lambda)$, regarded $B^+G_{ev}$-modules (see §9.1). It is deduced that an irreducible module $L(\lambda)$ is isomorphic to a Kac module if and only if $\lambda$ is a $p$-typical, i.e. $(\lambda + \rho, \beta) \not\equiv 0 \mod p$ for all $\beta \in \Phi^+_1$ (see Theorem 9.3). This is a modular version of typical irreducible modules over complex numbers (see Remarks 5.3 and 9.4).

0.7. The paper is organized as follows. In Section 1, we list the notations and some preliminary results. In Section 2, we present the induction and restriction functors on the quotients of supergroup schemes and associated sheaves in the spirit of \cite[§2]{3}, in particular, some material is devoted to the super version of Serre duality. In Sections 3 and 4, on the basis of computation on induced modules of $GL(1|1)$, we study homomorphisms between two induced modules arising from two Borel subgroups which are adjacent by odd reflections. Here the arguments on $GL(1|1)$ in \cite{14} \cite{32} are much used. In Section 5, we introduce typical weights and Steinberg’s tensor product theorem, showing that it is enough for us to understand the case of typical weights (see Theorem 5.4). In Section 6, we show a nonzero homomorphism from the induced module $H^0(G/B, \lambda)$ to the totally-odd induced module $H^0(G/w_1(B), \lambda - 2\rho_1)$. In Section 7, with the help of Serre duality we analyze the Weyl module $V(\lambda)$. In Section 8, we establish the Jantzen filtration of $V(\lambda)$ and a sum formula (Theorem 8.4). In the concluding section, we introduce Kac modules, and obtain a result on Kac module realization of $p$-typical irreducible modules (Theorem 9.3).

1. Preliminaries

1.1. Basic notations and conventions. Throughout the paper (particularly from Section 2.4 on), the notions of vector spaces (resp. modules, subgroups) means vector superspaces (resp. super-modules, super-subgroups). For simplicity, we often omit the adjunct word “super.” Preliminarily, by a commutative superalgebra $R = R_0 \oplus R_1$ it means that $ab = (-1)^{|a||b|}ba$ for any $\mathbb{Z}_2$-homogeneous elements $a \in R_{|a|}, b \in R_{|b|}$ where $|a|, |b| \in \mathbb{Z}_2$. We will denote by $\text{salg}_{k}$ the category of
commutative $k$-superalgebras. Furthermore, we keep the following notations and assumptions unless other stated.

1. Recall that the general linear supergroup $GL(m|n)$ can be defined as an (affine) algebraic supergroup scheme over $\mathbb{Z}$ (for example, see [8, Chapter 5] or [24, §3]). In this sense, we write $G_{\mathbb{Z}} = GL(m|n)_{\mathbb{Z}}$. Set $G_{\mathcal{A}} = G(\mathcal{A})_A$ for any (commutative) $\mathbb{Z}$-algebra $A$ and $G = G_k$ in the same spirit as in [10, §I.1.1]. This is to say, $G_{\mathcal{A}}$ is a representable functor from the category of commutative $\mathcal{A}$-superalgebras to the category of groups. Furthermore, its closed subgroups corresponding to the above subgroups appearing above are well defined as split algebraic $\mathbb{Z}$-supergroups (see [8, Chapter 4]). Similarly, we have $T_{\mathcal{A}}, B_{\mathcal{A}},$ and $(G_{ev})_{\mathcal{A}}$. In particular, $G = (G_{\mathbb{Z}})_k$. Denote by $\text{Dist}(G)$ the distribution algebra of $G$ (see [4] or [10]).

2. $T$: the standard maximal torus of $G = GL(m|n)$ which is actually the standard maximal torus of $GL(m) \times GL(n)$ consisting of diagonal matrices.

3. $B = B^-$: Borel subgroup with $B(R)$ consisting of lower triangular matrices of $GL(m|n)(R)$ for $R \in \text{salg}_k$; $B^+$: Borel subgroup opposite to $B$.

4. $\Phi, \Phi^\pm, \Phi_0, \Phi_1$: the root system of $G$, the positive/negative root system of $G$ associated with the standard Borel subgroup, the even root sets of $\Phi$, the odd root sets of $\Phi$ respectively.

5. $\Pi, \Pi_0, \Pi_1$: the fundamental simple root system, the even simple roots system, the odd simple roots system.

6. $\rho_0, \rho_1$: half the sum of positive even roots, half the sum of positive odd roots, respectively; $\rho := \rho_0 - \rho_1$.

7. For $\alpha \in \Phi$ and a weight $\lambda \in X(T)$, we often write $(\lambda, \alpha)$ for $\lambda(\alpha^\vee)$ for the coroot $\alpha^\vee \in T$ (see for example [8, §3]).

8. $r_\alpha, W$: even reflection with $r_\alpha(\mu) = \mu - (\mu, \alpha^\vee)\alpha, \alpha \in \Pi_0$, the Weyl group generated by $\{r_\alpha \mid \alpha \in \Phi_0\}$.

9. $X^+(T) := \{\lambda \in X(T) \mid (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Phi_0^+\}$ the set of dominant weight in $X(T)$.

10. For simplicity of notations, set $H^i(G/B', \lambda) := H^i(G/B', \mathcal{L}_{G/B'}(k\lambda))$ (see Convention 2.5). Associated with a commutative $\mathbb{Z}$-algebra $A$, the sheaf cohomology $H^i(G_A/B_A', \mathcal{L}_{G_A/B_A'}(A\lambda))$ will be written as $H^i(G_A/B_A', \lambda)$ or $H^i_A(G/B', \lambda)$.

Some other conventions for induced modules over purely-even groups are also introduced in [2.6].

11. For a finite-dimensional $T$-module $M$, $M$ is decomposed into a direct sum of weight spaces $M = \sum_{\lambda \in X(T)} M_\lambda$.

12. For a superscheme $X$ over $k$, we denote by $k[X]$ the global section of the structural sheaf of $X$. 
1.2. Keep $G = \text{GL}(m|n)$ in the remainder of this section. For $\lambda \in X(T)$, we have $\lambda = \sum_{1 \leq i \leq m+n} \lambda_i \sigma_i$, where $\sigma_i(t) = t_i, t = \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_n \end{pmatrix} \in T(R)$. Then
$$\Phi = \{\sigma_i - \sigma_j \mid 1 \leq i \neq j \leq m + n\}$$ is the root system for $G$.

Call $B^+$ the standard Borel subgroup. Associated with $B^+$ the (standard) positive root system $\Phi^+ = \{\sigma_i - \sigma_j \mid 1 \leq i < j \leq m + n\}$ and the (standard) simple root system $\Pi = \{\sigma_i - \sigma_{i+1} \mid 1 \leq i \leq m + n - 1\}$.

In order to distinguish even and odd roots, we change the notations by setting $\delta_i = \sigma_i, 1 \leq i \leq m$ and $\epsilon_j = \sigma_{j+m}, 1 \leq j \leq n$. Then $\Pi_1 = \{\delta_{m} - \epsilon_{1}\}$ and $\Pi_0 = \{\delta_{i} - \delta_{i+1}, \epsilon_{j} - \epsilon_{j+1} \mid 1 \leq i \leq m-1; 1 \leq j \leq n-1\}$. There exists a bilinear form on $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ with
$$(\delta_i, \delta_j) = \begin{cases} 1 & \text{if } 1 \leq i = j \leq m, \\ 0 & \text{if } i \neq j; \end{cases}$$
$$(\epsilon_i, \epsilon_j) = \begin{cases} -1 & \text{if } 1 \leq i = j \leq n, \\ 0 & \text{if } i \neq j. \end{cases}$$

1.2.1. **Odd reflections.** In the standard positive root system mentioned above, there are $mn$ positive odd roots: $\delta_s - \epsilon_t$ with $s$ ranging through $\{1,\ldots,m\}$ and $t$ ranging through $\{1,\ldots,n\}$. We make them in an order by defining $\delta_s - \epsilon_t < \delta_k - \epsilon_l$ if and only if either $s > k$, or $s = k$ but $t < l$. This order is identical to the one defined in [4 §4]. By this order, the set of positive odd roots can be parameterized by the positive integers from 1 to $mn$ as below, $\beta_1 := \delta_m - \epsilon_1, \beta_2 := \delta_m - \epsilon_2, \ldots, \beta_n := \delta_m - \epsilon_n, \beta_{n+1} := \delta_{m-1} - \epsilon_1, \beta_{n+2} := \delta_{m-1} - \epsilon_2, \ldots, \beta_{2n} := \delta_{m-1} - \epsilon_n, \beta_{2n+1} := \delta_{m-2} - \epsilon_1, \beta_{2n+2} := \delta_{m-2} - \epsilon_2, \ldots, \beta_{3n} := \delta_{m-2} - \epsilon_n, \ldots, \beta_{(m-1)n+1} := \delta_1 - \epsilon_{1}, \ldots, \beta_{mn} := \delta_1 - \epsilon_n$. This is to say, $\Phi_1 = \{\beta_1, \beta_2, \ldots, \beta_{mn}\}$ (there will be a concise presentation of $\beta_i$ in (1.1)). In contrast with the ordinary algebraic groups, there are different Borel subgroups which are not mutually conjugate. Recall that a Borel subgroup is dependent on its positive root system, equivalently, its simple root system. Note that for any simple root system $\Pi'$ of $G$, there exists at least one odd root in $\Pi'$.

Fix an odd root $\beta \in \Pi'$ which is certainly isotropic in the case of $G = \text{GL}(m|n)$. Then, there is a new simple root system $\Pi''$ as below:
$$\Pi'' = \{\alpha \in \Pi' \setminus \{\beta\} \mid (\alpha, \beta) = 0\} \cup \{\alpha + \beta \mid \alpha \in \Pi', (\alpha, \beta) \neq 0\} \cup \{-\beta\}.$$ This operation changing $\Pi'$ into $\Pi''$ is just the odd reflection associated with $\beta$, denoted by $\hat{r}_\beta$ (see [11] or [11 §1.3]).

If a simple root system arises from some other one via an odd reflection, then the corresponding Borel subgroups are not conjugate.

1.2.2. **Adjacent simple root systems via an ordered chain of odd reflections.** For $\text{GL}(1|1)$, the root system consists of two odd roots $\pm(\delta_1 - \epsilon_1)$. It is simple and clear. Let us consider the situation when $\text{GL}(m|n)$ is not of type $(1|1)$. Recall that $\Pi$ contains $\beta_1 = \delta_m - \epsilon_1$. Set $\Phi^+_1 := \{-\beta_1\} \cup \Phi^+ \setminus \{\beta_1\}$. Then $\Phi^+_1$ is a new positive root system (see for example [6, Lemma 1.30]), whose fundamental root system is
Lemma 1.1. There are $mn$ simple root systems $\Pi_{\beta_i}$, $i = 1, \ldots, mn$ satisfying that for each $i$, $\Pi_{\beta_i+1}$ is arising from $\Pi_{\beta_i}$ by odd reflection $\hat{r}_{\beta_i+1}$.

Proof. For the type (1|1), there is nothing to say. Suppose $G = \text{GL}(m|n)$ is not of type (1|1). If $n = 1$ (consequently, $m \geq 2$), then we can list $\beta_i = \delta_{i'} - \epsilon_1$ for $i = 1, \ldots, m$ and $i' := m - i + 1$. And $\Pi_{\beta_1} = \{\delta_1 - \delta_2, \ldots, \delta_{i'-2} - \delta_i, -\beta_i, \beta_{i+1}, \delta_{i'} - \delta_{i'+1}, \ldots, \delta_{m-1} - \delta_m\}$ for $i = 1, \ldots, m - 1$, and $\Pi_{\beta_m} = \{-\beta_m, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_m\}$.

In this case, the statement obviously holds.

In the following, suppose $n \geq 2$. We only need to show that $\Pi_{\beta_i}$ contains $\beta_{i+1}$ and $-\beta_i$ (when $i = mn$, $\beta_{i+1}$ is redundant). Note that for $i \in \{1, \ldots, mn\}$, we can write $i = kn$ with $1 \leq k \leq m$, or $i = kn + l$ with $0 \leq k < m$ while $1 \leq l < n$.  Set $k' := m - (k - 1)$ for $1 \leq k < m$. Then we can rewrite $\beta_i$ as below

$$\beta_i = \begin{cases} 
\delta_{k'} - \epsilon_n, & \text{if } i = kn; \\
\delta_{k'-1} - \epsilon_l, & \text{if } i = kn + l.
\end{cases} \quad (1.1)$$

We will inductively present $\Pi_{\beta_i}$. Hence the lemma follows from such a precise presentation. When $i = 1$ and $i = 2$, it is clear by the arguments in the paragraph before the lemma. For the general case, we proceed by dividing into cases of variant forms of $i$.

When $i = kn$ with $1 \leq k \leq m$, then

$$\Pi_{\beta_i} = \{\delta_1 - \delta_2, \ldots, \delta_{k'-2} - \delta_{k'-1}, \delta_{k'-1} - \epsilon_1 (= \beta_{i+1}), \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n; \epsilon_n - \delta_{k'} (= -\beta_i), \delta_{k'} - \delta_{k'+1}, \ldots, \delta_{m-1} - \delta_m\}$$

if $k < m,$
and

\[ \Pi_{\beta_{mn}} = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \\
\epsilon_n - \delta_1(= -\beta_{mn}), \\
\delta_1 - \delta_2, \ldots, \delta_{m-1} - \delta_m \}. \]

Now we proceed to work with \( i = kn + l \) with \( 0 \leq k < m \) and \( 1 \leq l < n \). We need some additional convention. When \( k = 0 \) we additionally appoint \( k' - 2 = (k + 1)' - 1 = m - 1 \), \( k' - 1 = (k + 1)' = m \) (note that \( k' \) for \( k = 0 \) does not make sense. Naturally, any term indicated by \( k' \) is redundant). Then

\[ \Pi_{\beta_i} = \{ \delta_1 - \delta_2, \ldots, \delta_{k' - 2} - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{l-1} - \epsilon_l, \\
\epsilon_l - \delta_{k'-1}(= -\beta_i), \\
\delta_{k'-1} - \epsilon_{l+1}(= \beta_{i+1}), \\
\epsilon_{l+1} - \epsilon_{l+2}, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n - \delta_{k'}, \delta_{k'} - \delta_{k'+1}, \ldots, \delta_{m-1} - \delta_m \} \text{ if } l + 1 < n, \]

and

\[ \Pi_{\beta_i} = \{ \delta_1 - \delta_2, \ldots, \delta_{k' - 2} - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{l-1} - \epsilon_l, \\
\epsilon_l - \delta_{k'-1}(= -\beta_i), \\
\delta_{k'-1} - \epsilon_n(= \beta_{i+1}), \\
\epsilon_n - \delta_{k'}, \delta_{k'} - \delta_{k'+1}, \ldots, \delta_{m-1} - \delta_m \} \text{ if } l + 1 = n. \]

By the above arguments, it has been verified that \( \Pi_{\beta_i} \) contains \( -\beta_i \) and \( \beta_{i+1} \). The proof is completed. \( \square \)

The positive root system corresponding to \( \Pi_{\beta_i} \) will be denoted by \( \Phi^+_{\beta_i} \) which contains the subset \( (\Phi^+_{\beta_i})_1 \) of odd roots. The corresponding Borel subgroup is denoted by \( B^{(i)^+} \). The opposite one will be denoted by \( B^{(i)^-} \) as \( = \).

### 1.2.3. Super Weyl groups and their longest elements.

The super Weyl group \( \hat{W} \) is introduced in [20], which is by definition the subgroup of transformation groups of all Borels containing \( T \) generated by all odd reflections defined previously along with real (ordinary) reflections (called even reflections later). Then \( \hat{W} \) contains the ordinary Weyl group \( W \) as a subgroup.

Set \( w_1 = \hat{r}_{mn} \hat{r}_{mn-1} \cdots \hat{r}_1 \) where \( \hat{r}_i = \hat{r}_{\beta_i} \). Denote by \( w_0 \) the longest element of \( W \). By Theorem [20, Theorem 3.10], the following result is readily deduced.

**Lemma 1.2.** The longest element \( \hat{w}_\ell \) of \( \hat{W} \) can be written as \( w_0 w_1 \), which changes the standard Borel group \( B^+ \) associated with \( \Phi^+ \) into \( B^- \) associated with \( -\Phi^+ \).

### 2. Induced modules and Serre Duality

In this section, we first introduce the general notations and conventions on algebraic supergroups and induced modules, which is referred to [2 §2]) and [3 §2]). Then we introduce Serre duality for higher cohomology arising from induced functor. Suppose \( G \) is a given algebraic supergroup, and \( H \) its closed subsupergroup.
2.1. Let $\mu$ be the multiplication in $G$. Denote by $\mu^\#$ the comorphism from $\mathcal{O}_G \to \mu_*(\mathcal{O}_G \otimes \mathcal{O}_G)$. We denote by $G\text{-mod}$ and $H\text{-mod}$ the supermodule categories of $G$ and $H$ respectively. As an analogue of algebraic group case, there are induction and restriction functors:

$$\text{ind}^G_H : H\text{-mod} \to G\text{-mod}, \quad \text{res}^G_H : G\text{-mod} \to H\text{-mod}$$

(see [10, §1.3], [3, §2], [2, §6]). Then $\text{res}$ is exact. And $\text{ind}^G_H$ is left exact, and right adjoint to $\text{res}^G_H$. Denote by $R^i\text{ind}^G_H$ by the $i$-th right derived functor. Suppose $M$ (resp. $N$) is a $(left)$ $H$-supermodule (resp. $G$-supermodule). This is equivalent to say, $M$ is a (right) comodule over $k[H]$ with structure map $\eta : M \to M \otimes k[H]$. Then there is a natural isomorphism (a generalized tensor identity):

$$R^i\text{ind}^G_H(\text{res}^G_H M \otimes N) \cong M \otimes R^i\text{ind}^G_H N \quad (2.1)$$

2.2. Suppose there is a quotient $X$ of $G$ by $H$ with the defining morphism: $\pi : G \to X$ which satisfies (Q1)-(Q6) in [3, §2]. This is to say, $G/H$ is locally decomposable and $G_{ev}/H_{ev}$ is projective; etc. Under these assumptions, we further have the underlying purely-even scheme $X_{ev}$ which is the scheme over $k$ equal to $X$ itself as a topological space, with structure sheaf $\mathcal{O}_X/\mathcal{J}_X$. Here $\mathcal{J}_X$ is the quasicoherent sheaf of superideas on $X$ which is defined via the presheaf sending any open subset $U$ to $\mathcal{O}_X(U)/\mathcal{O}_X(U)_1$. Let $\rho : X \times G \to X$ be the right action of $G$ on $X$ induced by multiplication in $G$. As an ordinary case, one can define a $G$-equivariant $\mathcal{O}_X$-supermodule (see [3]). By definition, a quasicoherent $\mathcal{O}_X G$-supermodule means a quasicoherent $\mathcal{O}_X$-supermodule $\mathcal{M}$ equipped with an even $\mathcal{O}_X$-supermodule map $\iota : \mathcal{M} \to \rho_*(\mathcal{M} \otimes \mathcal{O}_G)$ satisfying the compatibility axioms with respect to the $G$-action (see [3, §2]). Denote by $\mathcal{O}_X G\text{-mod}_{\text{qcoh}}$ the category of $G$-equivariant quasicoherent $\mathcal{O}_X$-supermodules. For example, $\rho_*\mathcal{O}_G$ is a quasicoherent $\mathcal{O}_X G$-module with structure map $\pi_*\mu^\# : \pi_*\mathcal{O}_G \to \rho_*(\pi_*\mathcal{O}_G \otimes \mathcal{O}_G)$. Also, $k[H]_{\text{trivial}} \otimes \pi_*\mathcal{O}_G$ is a quasicoherent $\mathcal{O}_X G$-module with structure map $\text{id} k[H] \otimes \pi_*\mu^\#$.

Furthermore, as the ordinary case, there is a functor $\mathcal{L}$ from $H\text{-mod}$ to the category $\mathcal{O}_X G\text{-mod}_{\text{qcoh}}$ of quasicoherent $\mathcal{O}_X G$-supermodules as blow. Suppose $M \in H\text{-mod}$, then $M$ is a (right) cosupermodule over $k[H]$ with structure map $\eta : M \to M \otimes k[H]$. Define the quasicoherent $\mathcal{O}_X G$-supermodule $\mathcal{L}(M)$ to be the kernel of the map $\eta \otimes \text{id}_{\pi_*\mathcal{O}_G} - \text{id}_M \otimes \delta$ where $\delta = \pi_*\mu^\#$ is a natural $\mathcal{O}_X G$-supermodule map from $\pi_*\mathcal{O}_G$ to $k[H]_{\text{trivial}} \otimes \pi_*\mathcal{O}_G$, and $\mu$ is the restriction of $\mu$ to $H \times G$. Actually, $\mathcal{L}(M)$ is just the associated sheaf of $X$ with respect to $M \in H\text{-mod}$ (see [10, §I.5.8] and [3, §2]).

2.3. By [3, Corollary 2.4], we have

$$R^i\text{ind}^G_H(-) \cong H^i(X, -) \circ \mathcal{L}. \quad (2.2)$$

Write $i : X_{ev} \to X$ for the canonical closed immersion which is $G_{ev}$-equivariant. There is a natural restriction functor $\text{res}^G_{X_{ev}} : \mathcal{O}_X G\text{-mod}_{\text{qcoh}} \to \mathcal{O}_{X_{ev}} G_{ev}\text{-mod}_{\text{qcoh}}$. And the functor $\mathcal{L}_{ev}$ can be defined as a restriction of $\mathcal{L}$ to $H_{ev}\text{-mod}$. Then $\mathcal{L}_{ev}$ is a functor from $H_{ev}\text{-mod}$ to the category $\mathcal{O}_{X_{ev}} G_{ev}\text{-mod}_{\text{qcoh}}$. 
Lemma 2.1. ([8 §2(6) and Lemma 2.5]) For $M \in H\text{-mod}$, and $M \in \mathcal{O}_X G\text{-mod}$, the following statements hold.

1. $i^*(\text{res}^G_{\text{ev}} L(M)) \cong \mathcal{L}_\text{ev}(\text{res}^H_{\text{ev}} M)$.
2. $H^i(X, \text{res}^G_{\text{ev}} M) \cong \text{res}^G_{\text{ev}} H^i(X, M)$.

2.4. In the following we suppose $G$ is an algebraic supergroup of Chevalley type (cf. [8 §5]). Consider $X = G/B$ for any Borel subgroup $B$ of $G$. Then $G_{ev}/B_{ev}$ is projective and $G/B$ is locally decomposable and Brundan’s assumptions (Q1-Q6) satisfy (cf. [16, 17]), we have a quotient $\pi : G \to X$. In this case, the purely-even scheme $X_{\text{ev}}$ is just a quotient of $G_{\text{ev}}$ by $B_{\text{ev}}$. We remind again, for simplicity, we will omit the adjunctword “super” for super-modules, super-subgroups, etc.

By [19] or [27 §2], we have the following super version of Serre duality theorem.

Theorem 2.2. Keep the notations as above. There is a natural isomorphism

$$(R^i \text{ind}_B^G M)^* \cong R^{l(w_0) - i} \text{ind}_B^G (M^* \otimes \text{Ber}(X)).$$

As to $\text{Ber}(X)$, it can be described below.

Proposition 2.3. Keep the notations as above. Then

$$\text{Ber}(X) \cong \omega_X \cong \mathcal{L}_\text{ev}(\bigwedge^* (g/b^-)_1 \otimes k_{-\rho_0}).$$

Proof. By [19] Lemma 1 and Corollary 2], we have

$$\text{Ber}(X) \cong \omega_X \cong \bigwedge^* (E^*) \otimes \mathcal{O}_{X_{\text{ev}}} \Omega^{l(w_0)}(X_{\text{ev}}),$$

where $E^*$ is the dual space of $\mathcal{F}_X/\mathcal{F}_X^2$, and $\Omega^{l(w_0)}(X_{\text{ev}})$ is the differential form of degree $l(w_0) = \dim X_{\text{ev}}$. By [3 Lemma 2.6], $E^* \cong \mathcal{L}_\text{ev}((\text{Lie}G/\text{Lie}B)_1)$. Hence $\bigwedge^* (E^*) \cong \mathcal{L}_\text{ev}(\bigwedge^* (g/b^-)_1)$. By a known result in the case of algebraic groups, we have $\Omega^{l(w_0)}(X_{\text{ev}}) \cong \mathcal{L}_\text{ev}(k_{-\rho_0})$ (cf. [10] §II.4.2(6)).

2.5. Super analogue of Mackey imprimitivity theorem. Keep the notations and assumptions in §2.2. Additionally, suppose $L$ is an affine supergroup scheme which is a closed subsupergroup of $G$, there is a morphism of supergroup schemes $L \to X = G/H$ giving rise to epimorphism of structural sheaves. Then one has a super analogue of Mackey imprimitivity theorem [7 Theorem 4.1] for algebraic groups.

Theorem 2.4. ([32 Theorem 10.1]) Keep the above notations and assumptions. Then for any $H$-supermodule $M$, there is an isomorphism of $L$-supernodes

$$\text{Res}^G_{L}(R^i \text{ind}_B^G M) \cong R^i \text{ind}_B^L (\text{Res}^H_{L \cap H}(M)).$$

2.6. Induced modules and their socles. Keep the notations and assumptions as above. For a $B$-module $M$ we by convention write $H^i(M)$ and $H^i_{\text{ev}}(M)$ for $H^i(G/B, \mathcal{L}(M))$ and for $H^i(G_{\text{ev}}/B_{\text{ev}}, \mathcal{L}_{\text{ev}}(M))$ respectively.

Let $k_\lambda$ be the one-dimensional $B$-module of weight $\lambda$ for $\lambda \in X(T)$. As a usual way we write $H^i(\lambda) = H^i(k_\lambda)$ and $H^i_{\text{ev}}(\lambda) = H^i_{\text{ev}}(k_\lambda)$. According to [22], we have
$H^i(\lambda) \cong R^i\text{ind}_B^G(k\lambda)$. Especially, $H^0(\lambda) \cong \text{ind}_B^G(k\lambda)$. In the subsequent, we will identify $\text{ind}_B^G(k\lambda)$ with $H^0(\lambda)$. Set $L(\lambda) := \text{Soc}_G(\text{ind}_B^G(k\lambda))$.

Furthermore, we will use the following convention.

**Convention 2.5.** Often simply write $H^i(G/B, \lambda)$ for $H^i(G/B, \mathcal{L}(k\lambda))$, even write $H^i(G/B', \lambda)$ for $H^i(G/B', \mathcal{L}_{G/B'}(k\lambda))$ where $B'$ is a given Borel subgroup, and $\mathcal{L}_{G/B'}(k\lambda)$ is an associated sheaf on $G/B'$.

The following results are important for the subsequent arguments.

**Theorem 2.6.** (1) Up to an isomorphism of $G_{ev}$-modules, $H^i_{ev}(\lambda)$ can be regarded a submodule of $H^i(\lambda)$ for any $i$ and $\lambda \in X(T)$. This statement holds for any Borel subgroup containing $T$.

(2) (cf. [24, Theorem 5.1], etc.) Let $B'^+$ be a Borel subgroup corresponding to a positive root system $\Phi'^+$, $B' = B'^-$ its opposite Borel. Set $\mathfrak{b}'^- = \text{Lie}(B')$. Then for any $B'$-module $M$, there is a $T$-module isomorphism

$$R^i\text{ind}_B^{G_{ev}}(M) \cong R^i\text{ind}_B^{G_{ev}}(M) \otimes \bigwedge^* (\mathfrak{g}/\mathfrak{b}'^-)_1^*.$$  

(2.3)

**Proof.** We make an account of the first part of (1). By the same arguments for any Borel subgroup containing $T$, the second part (1) can be verified. Note that $\text{ind}_{B_{ev}}^{G_{ev}}$ is left exact. It is sufficient to show there is an injective homomorphism of $G_{ev}$-modules from $H^0_{ev}(\lambda)$ into $H^0(\lambda)$. Recall

$$H^0(\lambda) = \{ f \in k[G] \mid f(gb) = \lambda(b)^{-1}f(g) \ \forall g \in G(R), b \in B(R) \$$

for any $R \in \text{salg}_k$.

And $G_{ev}$ is a closed subgroup of $G$, and there exist set-theoretic factorization $G(R) = G_{ev}(R)G_1(R)$ for any $R \in \text{salg}_k$ where $G_1(R)$ is a normal subgroup of $G(R)$ generated by one-parameter subgroups arising from odd root vectors in somewhat “odd”-way (see [8, Theorem 5.15]). So there is a natural way to define a homomorphism of $G_{ev}$-modules from $H^0_{ev}(\lambda)$ to $H^0(\lambda)$. Note that as a topological space, open subsets of $G$ are by definition just the ones of $G_{ev}$. So it follows that this homomorphism is injective. \hfill \Box

As in the case of reductive algebraic groups, one has that $H^0(\lambda)$ is finite-dimensional (cf. [30, Corollary 5.2] or [24, Proposition 4.17]). As mentioned above, $L(\lambda) = \text{soc}(H^0(\lambda))$. Regarded as a $B^+$-module, $\text{soc}_{B^+}(L(\lambda))$ is precisely the $\lambda$-weight space $H^0(\lambda)\lambda$ coinciding with $k\lambda$ (cf. [24, Proposition 4.11]). Consequently, $L(\lambda)$ is simple $G$-module. Furthermore, irreducible modules $\{ L(\lambda) \mid \lambda \in X^+(T) \}$ form a representative set of isomorphism classes of irreducible modules of $G$ (see [4, Theorem 4.5] or [24, Theorems 4.12 and 5.5, Example 5.9]). One can also describe irreducible $G$-modules by considering a kind of $T$-compatible $\text{Dist}(G)$-module category (see [4, 5, 23], etc.). Furthermore, all these modules $\{ L(\lambda) \mid \lambda \in X^+(T) \}$ are of type $\mathcal{M}$ (i.e. $\text{End}_G(L(\lambda))$ is 1-dimensional).

We sum up with the following theorem.

**Theorem 2.7.** Suppose that $\lambda \in X(T)$. The following statements hold.

(1) $H^0(\lambda)$ is finite-dimensional.
(2) $H^0(\lambda)$ is non-zero if and only if $\lambda \in X^+(T)$.

(3) The socle of $B^+$-module $H^0(\lambda)$ is precisely its $\lambda$-weight space $H^0(\lambda)_\lambda$ coinciding with $k_\lambda$. And $L(\lambda)$ is simple $G$-module.

(4) The modules $\{L(\lambda) \mid \lambda \in X^+(T)\}$ form a complete set of isomorphism classes of irreducible $G$-modules.

(5) All modules $\{L(\lambda) \mid \lambda \in X^+(T)\}$ are of type $M$.

3. Induced modules: Case GL$_A(1|1)$

In this section, we assume $G = GL(1|1)$. In this case, there are two Borel subgroups containing the standard torus. Both of them are not mutually conjugate under $G$-conjugation. In this section, we introduce the construction of induced modules associated with different Borel subgroups. The material here mainly comes from [14 §7.3] and [32], which will be important to the subsequent sections.

3.1. We are given a commutative $k$-superalgebra $R$. Recall that $G(R)$ consists of invertible matrix $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$ with $a, b \in R_0$ and $m, n \in R_1$. Then $G(R)$ is generated by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ n & b \end{pmatrix}$. In this case, $G$ has only two Borel subgroups $B^+_\beta$ and $B^-_{-\beta}$, corresponding to $\Phi^+_\beta = \{ \beta : = \delta_1 - \epsilon_1 \}$ and $\Phi^-_{-\beta} = \{-\beta = \epsilon_1 - \delta_1\}$, respectively. Denote $B_{-\beta} = B^-_{\beta}$, $B_\beta = B^+_{-\beta}$. Let $k_\lambda$ be the one-dimensional $B_{\pm\beta}$-module via the torus action by weight $\lambda$. We will simply write a weight $\lambda = i\delta_1 + j\epsilon_1$ as $(i|j)$ in the following.

Recall $H^i(\lambda) \cong R^i\text{ind}^G_{B_{\pm\beta}}(k_\lambda)$. For GL$(1|1)$, in order to avoid confusion, we denote $H^i_{\pm\beta}(\lambda) = R^i\text{ind}^G_{B_{\pm\beta}}(k_\lambda)$. Denote by $L_{B_{\pm\beta}}(\lambda)$ the socle of $H^0_{\pm\beta}(\lambda)$.

3.2. We define a function $c_{ij}^k$ on $2 \times 2$-matrices via $c_{ij}^k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{ij}^k$. Then we have

$$c_{ij}^k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = (a_{11}b_{1j} + a_{12}b_{2j})^k.$$ 

Recall that the coordinate ring $k[G]$ is the localization of the polynomial ring $k[c_{ij} \mid i, j = 1, 2]$ at $\text{det} = c_{11}c_{22}$. It is a Hopf superalgebra with comultiplication $\Delta$ via $\Delta(c_{st}) = \sum_{k=1,2} c_{sk} \otimes c_{kt}$, $s, t = 1, 2$.

Set $k[G]_\lambda$ to be the subspace of $k[G]$ of weight $\lambda$, which is spanned by all monomials of weight $\lambda$. For example, take $F = c_{11}^ac_{12}^bc_{21}^dc_{22}^e$, $a, b, c, d \in \mathbb{Z}$ and $0 \leq b, c \leq 1$. The weight of $F$ is $\lambda = (a + b|c + d)$.

3.3. For $\lambda = (i|j)$, we have $(\lambda, \beta) = i + j$, denoted by $|\lambda|$. For $k[G] = k[c_{ij}]_1 \leq i, j \leq 2|c_{11}c_{22}$, set

$$A_\lambda = c_{11}^ic_{22}^j, B_\lambda = c_{11}^{i-1}c_{12}^jc_{22}^j, C_\lambda = c_{11}^ic_{21}^j, D_\lambda = c_{11}^{i-1}c_{12}c_{21}^{j-1}.$$
By computation,
\[ \Delta(A_\lambda) = \Delta(c_{11}^i \cdot \Delta(c_{22}^j) \\
= (c_{11}^i \otimes c_{11}^i + c_{11}^i \otimes c_{11}^i \cdot c_{22}^j)(c_{22}^j \otimes c_{22}^j + j c_{22}^j \otimes c_{11}^i \cdot c_{11}^i \cdot c_{22}^j) \\
= c_{11}^i c_{22}^j \otimes c_{11}^i c_{22}^j + j c_{11}^i c_{22}^j \otimes c_{11}^i c_{22}^j + c_{11}^i c_{12} c_{22}^j \otimes c_{11}^i c_{21} c_{22}^j \\
+ i j c_{11}^i c_{12} c_{21} c_{22}^j \otimes c_{11}^i c_{21} c_{22}^j \).
\]

Thus \( \Delta(A_\lambda) = A_\lambda \otimes A_\lambda + i B_\lambda \otimes C_{\lambda-\beta} + j C_\lambda \otimes B_{\lambda+\beta} + ij D_\lambda \otimes D_\lambda \). Similarly, we have
\[
\begin{align*}
\Delta(B_\lambda) &= B_\lambda \otimes Y_{\lambda-\beta} + X_\lambda \otimes B_\lambda; \\
\Delta(C_\lambda) &= C_\lambda \otimes X_{\lambda+\beta} + Y_\lambda \otimes C_\lambda; \\
\Delta(D_\lambda) &= D_\lambda \otimes A_\lambda - C_\lambda \otimes B_{\lambda+\beta} + B_\lambda \otimes C_{\lambda-\beta} + (A_\lambda + (j-i) D_\lambda) \otimes D_\lambda
\end{align*}
\]
where \( X_\lambda = A_\lambda + j D_\lambda, Y_\lambda = A_\lambda - i D_\lambda \).

3.4. There is a natural action of \( G(R) \) for \( R \in \mathfrak{salg}_k \) on \( k[G] \otimes_k R \) in the same spirit as in the ordinary algebraic group case. By the computations in \[3.2\] and \[3.3\]
\[
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} C_\lambda = X_{\lambda+\beta} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} C_\lambda + C_\lambda \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Y_\lambda = a^{i+1} b^{j-1} C_\lambda.
\]
Similarly, \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} B_\lambda = a^{-i} b^{j+1} B_\lambda; \) \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} X_\lambda = a^i b^j X_\lambda; \) and \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Y_\lambda = a^i b^j Y_\lambda \). Correspondingly, the weights of \( C_\lambda, B_\lambda, X_\lambda \) and \( Y_\lambda \) are \( \lambda + \beta, \lambda - \beta, \lambda \) and \( \lambda \), respectively.

Further computations show
\[
\begin{align*}
\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} B_\lambda &= a^{-i} b^{j+1} B_\lambda + ma^{-i} b^{-1} X_\lambda, \\
\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} C_\lambda &= a^{i+1} b^{j-1} C_\lambda, \\
\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} X_\lambda &= a^i b^j X_\lambda, \\
\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} Y_\lambda &= a^i b^j Y_\lambda + (i+j) ma^i b^{j-1} C_\lambda;
\end{align*}
\]
and
\[
\begin{align*}
\begin{pmatrix} a & 0 \\ n & b \end{pmatrix} B_\lambda &= a^{-i} b^{j+1} B_\lambda, \\
\begin{pmatrix} a & 0 \\ n & b \end{pmatrix} C_\lambda &= a^{i+1} b^{j-1} C_\lambda + na^i b^{j-1} Y_\lambda, \\
\begin{pmatrix} a & 0 \\ n & b \end{pmatrix} X_\lambda &= a^i b^j X_\lambda + (i+j) na^{-i} b^{j-1} B_\lambda, \\
\begin{pmatrix} a & 0 \\ n & b \end{pmatrix} Y_\lambda &= a^i b^j Y_\lambda.
\end{align*}
\]

3.5. Summarizing up, we have the following lemma.

**Lemma 3.1.** The following statements hold.

1. The induced modules \( H^0_{\beta}(\lambda) = kC_\lambda + kY_\lambda; \) and \( H^0_{-\beta}(\lambda + \beta) = kB_{\lambda+\beta} + kX_{\lambda+\beta} \).

2. Furthermore,
   (2.1) when \( p = 0 \) or \( p \nmid |\lambda| \), \( H^0_{\beta}(\lambda + \beta) \) and \( H^0_{-\beta}(\lambda) \) are mutually isomorphic. Both of them are irreducible.
(2.2) When $p \neq 0$ and $p \mid |\lambda|$, then $X_\lambda = Y_\lambda$. Consequently, $L_{B_{-\beta}}(\lambda) \cong L_{B_{\beta}}(\lambda) \cong kX_\lambda$, and

$$H^0_{\beta}(\lambda)/L_{\beta}(\lambda) \cong (kC_\lambda + kX_\lambda)/kX_\lambda \cong L_{B_{\beta}}(\lambda + \beta)$$

along with

$$H^0_{\beta}(\lambda + \beta)/L_{\beta}(\lambda + \beta) \cong (kB_{\lambda+\beta} + kX_{\lambda+\beta})/kX_{\lambda+\beta} \cong L_{B_{\beta}}(\lambda).$$

(3) When $p \neq 0$ with $p \mid |\lambda|$, there is a homomorphism

$$\Upsilon_{\lambda,\beta} : H^0_{-\beta}(\lambda + \beta) \rightarrow H^0_{-\beta}(\lambda)$$

satisfying $\text{im}(\Upsilon_{\lambda,\beta}) \cong kX_\lambda \cong L_{B_{\beta}}(\lambda)$ and $\ker(\Upsilon_{\lambda,\beta}) \cong kB_{\lambda+\beta} \cong L_{B_{\beta}}(\lambda + \beta).$

Proof. (1) and (2) follows from \[7.3\].

(3) As $H^0_{\beta}(\lambda + \beta)$ (resp. $H^0_{-\beta}(\lambda)$) admits a $k$-basis consisting of $B_{\lambda+\beta}$ and $X_{\lambda+\beta}$ (resp. $B_\lambda$ and $X_\lambda$), we can define a map $\Upsilon_{\beta}$ from $H^0_{\beta}(\lambda + \beta)$ to $H^0_{-\beta}(\lambda)$ via $\Upsilon_{\beta}(B_{\lambda+\beta}) = X_\lambda$ and $\Upsilon_{\beta}(X_{\lambda+\beta}) = 0$. From \[3.4\] it follows that $\Upsilon_{\beta}$ is really a $G$-module homomorphism satisfying $\text{im}(\Upsilon_{\beta}) \cong kX_\lambda \cong L_{B_{\beta}}(\lambda)$ and $\ker(\Upsilon_{\beta}) \cong kB_{\lambda+\beta} \cong L_{B_{\beta}}(\lambda + \beta).$

3.6. From now on, we always take $A$ to be a principal ideal domain (PID for short). Denote by $K$ the fractional field of $A$.

As in \[1.1\], let $G_A = \text{GL}_A(1|1)$. In the same sense, we can talk about $(B_{\pm \beta})_A$. More generally, for a closed subgroup $H$ of $G$ we can talk about $H_A$ as long as $H$ can be defined over $\mathbb{Z}$. Then one can define the induced modules over $A$

$$H^0_{\pm \beta, A}(A) := \text{ind}^{G_A}_{(B_{\pm \beta})_A}(A).$$

Here and further the notation $A_\lambda$ indicates the rank-one $(B_{\pm \beta})_A$-module of weight $\lambda$ over $A$. By the same arguments, an analogue of Lemma \[3.4\] yields

$$H^0_{\pm \beta, A}(-) = H^0_{\pm \beta, \mathbb{Z}}(-) \otimes_{\mathbb{Z}} A$$

with $H^0_{\pm \beta, \mathbb{Z}}(-) = H^0_{\pm \beta, \mathbb{Z}}(-) \otimes_{\mathbb{Z}} k$.

Let $\lambda = (i|j)$. According to \[3.4\], we can define the following homomorphisms

$$T_{A, \lambda, \beta} : H^0_{-\beta, A}(\lambda + \beta) \rightarrow H^0_{\beta, A}(\lambda)$$

with

$$T_{A, \lambda, \beta}(v) = \begin{cases} Y_\lambda, & v = B_{\lambda+\beta} \\ (i + j)C_\lambda, & v = X_{\lambda+\beta}. \end{cases} \quad (3.5)$$

And

$$T'_{A, \lambda, \beta} : H^0_{\beta, A}(\lambda) \rightarrow H^0_{-\beta, A}(\lambda + \beta)$$

with

$$T'_{A, \lambda, \beta}(v) = \begin{cases} X_{\lambda+\beta}, & v = C_\lambda \\ (i + j)B_{\lambda+\beta}, & v = Y_\lambda. \end{cases} \quad (3.6)$$
4. Induced modules: Case \( \text{GL}_A(m|n) \)

From now on we always suppose \( G = \text{GL}(m|n) \). In this section, we keep the notations and assumptions in §3.6 for some Levi subgroups of \( G \) isomorphic to \( \text{GL}(1|1) \). In particular, \( A \) is a given PID, and \( \mathbb{K} \) is the fractional field of \( A \). Suppose \( \beta \) is a given odd root which is naturally isotropic, i.e. \( (\beta, \beta) = 0 \).

4.1. Suppose \( K^+ \) and \( K^+_{\beta} \) are a given pair of adjacent Borel subgroups, the latter of which are produced by an odd reflection \( \bar{r}_\beta \) from the former as in §1.2.1.

We uniformly write \( K^+_{\beta} \) for \( K^+ \). Then the purely-even groups of \( K^+_{\pm} \) are just \( B_{ev}^+ \). There is a minimal parabolic subgroup \( P^+(\beta) \) of \( G \) containing \( K^+_\pm \). The opposite Borel subgroups are denoted by \( K^{\mp}_\beta \) respectively, this is to say, \( K^{-}_\beta \) is opposite to \( K^+_{\pm} \), and \( K_\beta \) is opposite to \( K^+_\beta \). The opposite parabolic subgroup is denoted by \( P(\beta, \mp) \).

Set \( \mathfrak{k}_{\mp} = \text{Lie}(K_{\pm \beta}) \), and write \( \mathfrak{k}^+ = \mathfrak{h} + \sum_{\gamma \in S} \mathfrak{g}_{\gamma} \) with \( S = S_0 \cup S_1 \) being a closed subset of \( \Phi \) corresponding to \( K^+_{\pm} \). Then \( \mathfrak{k}^+ = \mathfrak{h} + \sum_{\gamma \in S_1} \mathfrak{g}_{\gamma} \) with \( S_1 \) being a closed subset of \( \Phi \) corresponding to \( K^+_{\pm} \). Thus, by Lemma 3.1 the induced module \( \text{ind}_{K_{A, \mp}}^{P_A(\beta)} A_\beta \equiv \text{ind}_{\text{GL}_{A}(1|1)}^{K_{A, \mp \beta}} A_\beta \).

This, by Lemma 3.1, the induced module \( \text{ind}_{K_{A, \mp \beta}}^{P_A(\beta)} A_\beta \) (resp. \( \text{ind}_{K_{A, \beta}}^{P_A(\beta)} A_\beta \)) has an \( A \)-basis consisting of \( B_\lambda \) and \( X_\lambda \) (resp. \( C_\lambda \) and \( Y_\lambda \)). Owing to (2.2), we write \( H^0_{P_A(\beta)}(\lambda) \) for \( \text{ind}_{K_{A, \pm \beta}}^{P_A(\beta)} k_\lambda \) respectively when working over \( \mathfrak{k} \).

On the other hand, it turns out that the unipotent radical of \( P_A(\beta) \) acts on \( \text{ind}_{K_{A, \pm \beta}}^{P_A(\beta)} A_\beta \) trivially (cf. [32, Proposition 11.5]). Similar to §3.6, we can define \( P_A(\beta) \)-module homomorphisms \( T_{A, \beta} : \text{ind}_{K_{A, \mp}}^{P_A(\beta)} A_{\lambda + \beta} \rightarrow \text{ind}_{K_{A, \beta}}^{P_A(\beta)} A_\beta \) with

\[
T_{A, \beta}(v) = \begin{cases} Y_\lambda, & \text{when } v = B_{\lambda + \beta} \\ (a + b)C_\lambda, & \text{when } v = X_{\lambda + \beta} \end{cases}
\]
and
\[ T'_{A,\beta} : \text{ind}_{K_{A,\beta}}^{P_{A}(\beta)} A_{\lambda} \to \text{ind}_{K_{A,\beta}}^{P_{A}(\beta)} A_{\lambda+\beta} \]
with
\[ T'_{A,\alpha}(v) = \begin{cases} X_{\lambda+\beta}, & \text{when } v = C_{\lambda} \\ (a+b)B_{\lambda+\beta}, & \text{when } v = Y_{\lambda}. \end{cases} \tag{4.3} \]

Then we have the following lemma.

**Lemma 4.1.** Keep the notations as above. Both \( T_{A,\lambda,\beta} \circ T'_{A,\lambda,\beta} \) and \( T'_{A,\lambda,\beta} \circ T_{A,\lambda,\beta} \) are multiplication by \( a+b \). Furthermore, when \((\lambda, \beta) = a+b \neq 0\) in \( \mathbb{K} \), \( T_{A,\lambda,\beta} \) (resp. \( T'_{A,\lambda,\beta} \)) is injective and \( T_{\mathbb{K},\lambda,\beta} = T_{A,\lambda,\beta} \otimes \mathbb{K} \) (resp. \( T'_{\mathbb{K},\lambda,\beta} = T'_{A,\lambda,\beta} \otimes \mathbb{K} \)) is an isomorphism.

4.2. As in the case of ordinary algebraic groups, the following statements still hold (see for example [32 Proposition 11.3]):

\[ H^i(G/A/K_{A,\pm,\beta}, \lambda) \cong R^i\text{ind}_{P_{A}(\beta)}^{G_{A}}(\text{ind}_{K_{A,\beta}}^{P_{A}(\beta)} A_{\lambda}). \tag{4.4} \]

So we can apply the functor \( \text{ind}_{P_{A}(\alpha)}^{G_{A}}(-) \) to the maps \( T_{A,\lambda,\beta} \) and \( T'_{A,\lambda,\beta} \) (see [33.6]). Then we have two homomorphisms

\[ \tilde{T}_{A,\lambda,\beta} : H^0(G/A/K_{A,\pm,\beta}, \lambda + \beta) \to H^0(G/A/K_{A,\beta}, \lambda) \]
and
\[ \tilde{T}'_{A,\lambda,\beta} : H^0(G/A/K_{A,\beta}, \lambda) \to H^0(G/A/K_{A,\beta}, \lambda + \beta). \]

satisfying that

both \( \tilde{T}_{A,\lambda,\beta} \circ \tilde{T}'_{A,\lambda,\beta} \) and \( \tilde{T}'_{A,\lambda,\beta} \circ \tilde{T}_{A,\lambda,\beta} \) are multiplication by \( a+b \). \tag{4.5} \]

Consequently, by Lemma 4.1 both \( \tilde{T}_{A,\lambda,\beta} \) and \( \tilde{T}'_{A,\lambda,\beta} \) are injective if \( a+b \neq 0 \) in \( \mathbb{K} \). Furthermore, \( \tilde{T}_{\mathbb{K},\beta} = \tilde{T}_{A,\lambda,\beta} \otimes \mathbb{K} \) (resp. \( \tilde{T}'_{\mathbb{K},\beta} = \tilde{T}'_{A,\lambda,\beta} \otimes \mathbb{K} \)) is an isomorphism whenever \((\lambda, \beta) = a+b \neq 0\).

4.3. Similarly we can define as above

\[ \tilde{\Upsilon}_{\mathbb{K},\beta} : H^0(G/K_{-\beta}, \lambda + \beta) \to H^0(G/K_{-\beta}, \lambda) \]

via applying the functor \( \text{ind}_{P_{A}(\alpha)}^{G_{A}}(-) \) to the homomorphism \( \Upsilon_{\lambda,\beta} \) defined in (3.4).

4.4. Denote by \( L_{K_{\pm,\beta}}(\lambda) \) the socle of \( H^0(G/K_{\pm,\beta}, \lambda) \). We have the following lemma (compatible with [20] Lemma 5.2 in the case of Lie superalgebras).

**Lemma 4.2.** The following statements hold.

1. When \((\lambda, \beta) \neq 0 \text{ mod } p\), both \( \tilde{T}_{\mathbb{K},\beta} \) and \( \tilde{T}'_{\mathbb{K},\beta} \) are isomorphisms. And \( L_{K_{-\beta}}(\lambda + \beta) \cong L_{K_{\beta}}(\lambda) \).
2. When \((\lambda, \beta) \equiv 0 \text{ mod } p\),
   1. (2.a.1) \( \ker(\tilde{T}_{\mathbb{K},\beta}) = \text{im}(\tilde{T}'_{\mathbb{K},\beta}) \cong \text{coker}(\tilde{T}_{\mathbb{K},\beta}) \);
   2. (2.a.2) \( \ker(\tilde{T}'_{\mathbb{K},\beta}) = \text{im}(\tilde{T}_{\mathbb{K},\beta}) \cong \text{coker}(\tilde{T}'_{\mathbb{K},\beta}) \).
3. Furthermore, \( L_{K_{\pm,\beta}}(\lambda) \cong L_{K_{\beta}}(\lambda) \).
4. \( \text{im}(\tilde{\Upsilon}_{\mathbb{K},\beta}) \) is \( \text{ker}(\tilde{\Upsilon}_{\mathbb{K},(i-1)\beta}) \) for all positive integers \( i \).
Proof. (1) Suppose \((\lambda, \beta) \not\equiv 0 \mod p\). The statement follows from (4.5).

(2) Suppose \((\lambda, \beta) \equiv 0 \mod p\). For (2.a), keep in mind that both \(K_{\pm \beta}\) have purely-even group \(B_{ev}\). Hence by Theorem 2.6(2), we have isomorphisms as \(T\)-modules

\[
H^0(G/K_{-\beta}, \lambda + \beta) \cong H^0(G_{ev}/B_{ev}, \lambda + \beta) \otimes \bigwedge^\bullet \left( \sum_{\gamma \in S_1} \mathfrak{g}_\gamma \right)^* \quad \text{and}
\]

\[
H^0(G/K_\beta, \lambda) \cong H^0(G_{ev}/B_{ev}, \lambda) \otimes \bigwedge^\bullet \left( \sum_{\gamma \in (S_-)_1} \mathfrak{g}_\gamma \right)^*.
\]

By the representation theory of algebraic groups (see [10], Proposition II.2.2], \(H^0(G_{ev}/B_{ev}, \lambda)\) (resp. \(H^0(G_{ev}/B_{ev}, \lambda + \beta)\)) has one-dimensional weight space of the \(B_{ev}^+\)-highest weight \(\lambda\) (resp. \(\lambda + \beta\)). In the sense of (1.4) along with (1.2) and (1.3), the height weight spaces are clearly spanned by \(Y_\lambda^+\) and \(X_{\lambda + \beta}\), respectively. On the other hand, by the arguments in (1.1) we know \(S_0 = (S_-)_0 = \Phi_0^-\). Consequently, the even homogenous spaces of the second parts in the tensor products of the \(T\)-module decomposition formula (1.7) are all of negative roots, i.e. in \(\Phi_0^-\). Hence the \(B_{ev}^+\)-highest weight space of \(H^0(G_{ev}/B_{ev}, \diamond)\) remains the ones of \(H^0(G/K_\beta, \diamond)\) for \(\diamond \in \{\lambda, \lambda + \beta\}\). Hence, (1.2) and (1.3) are still true for \(\widetilde{\mathcal{T}}_{\lambda, \beta}\) and \(\widetilde{T}_{\lambda, \beta}\), respectively.

Under the assumption \((\lambda, \beta) \equiv 0 \mod p\), by (1.2) and (1.3) we have \(\ker(T_{k_\lambda, \beta}) = kX_{\lambda + \beta}\) and \(\im(T_{k_\lambda, \beta}) = kY_\lambda\), and then \(\coker(T_{k_\lambda, \beta}) \cong kX_{\lambda + \beta} = \ker(T_{k_\lambda, \beta})\). Similarly, \(\ker(T'_{k_\lambda, \beta}) = kY_\lambda\) and \(\im(T'_{k_\lambda, \beta}) = kX_{\lambda + \beta}\), and then \(\coker(T'_{k_\lambda, \beta}) \cong kY_\lambda = \ker(T_{k_\lambda, \beta})\). Clearly, \(\im(T_{k_\lambda, \beta}) = \ker(T'_{k_\lambda, \beta})\), and \(\im(T'_{k_\lambda, \beta}) = \ker(T_{k_\lambda, \beta})\).

On the other hand, by definition \(\widetilde{T}_{k_\lambda, \beta} = \ind_{\mathcal{P}(\beta)}^G(T_{k_\lambda, \beta})\), and \(\widetilde{T}_{k_\lambda, \beta} = \ind_{\mathcal{P}(\beta)}^G(T'_{k_\lambda, \beta})\). So \(\im(\widetilde{T}_{k_\lambda, \beta}) = \ind_{\mathcal{P}(\beta)}^G(\im(T_{k_\lambda, \beta}))\), \(\ind_{\mathcal{P}(\beta)}^G(\ker(T_{k_\lambda, \beta})) \subset \ker(\widetilde{T}_{k_\lambda, \beta})\). On the other hand, the functor \(\ind_{\mathcal{P}(\beta)}^G(-)\) is left exact. Hence \(\ind_{\mathcal{P}(\beta)}^G(-)\) preserves the exact sequence

\[
0 \to \ker(T_{k_\lambda, \beta}) \to \ind_{\mathcal{P}(\beta)}^G(k_{\lambda + \beta}) \to \im(T_{k_\lambda, \beta}) \to 0.
\]

The similar result also holds for \(\widetilde{T}_{k_\lambda, \beta}'\). Combining with the above arguments, the two statements in (2.a) are proved.

We now check (2.b). Keep (1.1) in mind. Turn back to the results in the case of \(GL(1|1)\). Recall when \((\lambda, \beta) \equiv 0 \mod p\), \(H^0_\beta(\lambda)\) (resp. \(H^0_\beta(\lambda + \beta)\)) has one-dimensional socle \(L_\beta(\lambda)\) (resp. \(L_{-\beta}(\lambda + \beta)\)) which is isomorphic to the head of \(H^0_{-\beta}(\lambda + \beta)\) (resp. \(H^0_\beta(\lambda)\)) as \(GL(1|1)\)-module in the sense of Lemma 3.1. By the trivial action of the unipotent subgroup of \(P(\beta)\), \(L_\beta(\lambda)\) can be extended to a \(P(\beta)\)-module. Simply write \(\ind_{\mathcal{P}(\beta)}^G(\lambda)\) for \(\ind_{\mathcal{P}(\beta)}^G(L_\beta(\lambda))\). By (3.3), there is an short exact sequence of \(P(\beta)\)-modules

\[
0 \to L_\beta(\lambda) \to H^0_\beta(\lambda) \to L_\beta(\lambda + \beta) \to 0.
\]

Applying the exact functor \(\ind_{\mathcal{P}(\beta)}^G(-)\), we have

\[
0 \to \ind_{\mathcal{P}(\beta)}^G(\lambda) \to H^0(G/K_\beta, \lambda) \to \ind_{\mathcal{P}(\beta)}^G(\lambda + \beta) \to R^1\ind_{\mathcal{P}(\beta)}^G(\lambda) \to \cdots \tag{4.7}
\]

Thus, it follows that the socle of \(\ind_{\mathcal{P}(\beta)}^G(\lambda)\) is isomorphic to \(L_{K_\beta}(\lambda)\).
Similarly, we have
\[ 0 \to \text{ind}_{P(\beta)}^G(\lambda) \to H^0(G/K_{-\beta}, \lambda) \to \text{ind}_{P(\beta)}^G(\lambda - \beta) \to R^1\text{ind}_{P(\beta)}^G(\lambda) \to \cdots \] (4.8)
This implies the socle of \( \text{ind}_{P(\beta)}^G(\lambda) \) is isomorphic to \( L_{K_{-\beta}}(\lambda) \). It is proved that
\[ L_{K_{-\beta}}(\lambda) \cong L_{K_{\beta}}(\lambda). \]

As to (2.c), \( \langle \lambda, \beta \rangle \equiv 0 \mod p \) implies \( \langle \lambda + i\beta, \beta \rangle \equiv 0 \mod p, \forall i \in \mathbb{N} \). Similar to Lemma \([3.1](3)\), there exists \( \Upsilon_{\lambda,i\beta}: H^0_{-\beta}(\lambda + i\beta) \to H^0_{-\beta}(\lambda + (i - 1)\beta), \forall i \in \mathbb{N} \) such that \( \text{im}(\Upsilon_{\lambda,i\beta}) \cong kX_{\lambda+i(1-\beta)} \) and \( \ker(\Upsilon_{\lambda,i\beta}) \cong kX_{\lambda+i\beta} \). Then \( \text{im}(\Upsilon_{\lambda,i\beta}) \cong kX_{\lambda+i\beta} \cong \ker(\Upsilon_{\lambda,i\beta}) \). In particular, \( \text{im}(\Upsilon_{\lambda,i\beta}) \cong kX_{\lambda+i\beta} \cong \ker(\Upsilon_{\lambda,i\beta}) \).

Define
\[ \tilde{\Upsilon}_{\lambda,i\beta}: H^0(G/K_{-\beta}, \lambda + i\beta) \to H^0(G/K_{-\beta}, \lambda + (i - 1)\beta) \]
via \( \tilde{\Upsilon}_{\lambda,i\beta} = \text{ind}_{P(\beta)}^G(\Upsilon_{\lambda,i\beta}) \). Then we have
\[ \text{ind}_{P(\beta)}^G(\text{im}(\Upsilon_{\lambda,i\beta})) \cong \text{ind}_{P(\beta)}^G(\ker(\Upsilon_{\lambda,i\beta})). \] (4.9)

Keep it in mind that
\[ \text{ind}_{P(\beta)}^G(\ker(\Upsilon_{\lambda,i\beta})) \subseteq \ker(\tilde{\Upsilon}_{\lambda,i\beta}) \text{ and } \text{im}(\tilde{\Upsilon}_{\lambda,i\beta}) = \text{ind}_{P(\beta)}^G(\text{im}(\Upsilon_{\lambda,i\beta})) \]
for all \( i \). Hence, the left exactness of \( \tilde{\Upsilon}_{\lambda,i\beta}(-) \) ensures that it preserves the following short exact sequence coming from Lemma \([3.1](3)\)
\[ 0 \to \ker(\Upsilon_{\lambda,i\beta}) \to \text{ind}_{K_{-\beta}}^P(\kappa_{\lambda,i\beta}) \to \text{im}(\Upsilon_{\lambda,i\beta}) \to 0. \]
Hence, we have \( \text{ind}_{P(\beta)}^G(\ker(\Upsilon_{\lambda,i\beta})) = \ker(\tilde{\Upsilon}_{\lambda,i\beta}) \). In summary, (4.9) gives rise to the desired equality \( \text{im}(\tilde{\Upsilon}_{\lambda,i\beta}) = \ker(\tilde{\Upsilon}_{\lambda,i\beta}) \).

The proof is completed. \( \square \)

As a consequence, we have

**Corollary 4.3.** Let \( \lambda \in X^+(T) \) satisfying \( \langle \lambda, \gamma \rangle \neq 0 \) for any odd root \( \gamma \). If \( \beta \) is an odd root with \( \langle \lambda, \beta \rangle \equiv 0 \mod p \), the following equation holds in the Grothendieck group of \( G \)-module category
\[ \text{[coker}\tilde{\Upsilon}_{\lambda,i\beta}] = [H^0(G/K_{\beta}, \lambda)] + \sum_{k=1}^{\infty} (-1)^k [H^0(G/K_{-\beta}, \lambda + k\beta)]. \] (4.10)

**Proof.** Note that all \( \lambda + k\beta \) still satisfy \( \langle \lambda + k\beta, \beta \rangle \neq 0 \) but \( \langle \lambda + k\beta, \beta \rangle \equiv 0 \mod p \), and consequently they all lie in \( X^+(T) \) (see the forthcoming Lemma \([5.2](2)\) where more general notions related will be introduced). According to Lemma \([4.2](2)\), there is a long exact sequence
\[ \cdots \to \tilde{\Upsilon}_{\lambda,2\beta} \to H^0(G/K_{-\beta}, \lambda + 3\beta) \to \tilde{\Upsilon}_{\lambda,2\beta} \to H^0(G/K_{-\beta}, \lambda + 2\beta) \to \tilde{\Upsilon}_{\lambda,2\beta} \to H^0(G/K_{-\beta}, \lambda + \beta) \to \tilde{\Upsilon}_{\lambda,\beta} \to \text{im}\tilde{\Upsilon}_{\lambda,\beta} \cong \text{coker}\tilde{\Upsilon}_{\lambda,\beta} \to 0. \] (4.11)
The formula (4.10) follows. \( \square \)
4.5. **Characters.** For any finite-dimensional $T$-module $M$, $M$ can be decomposed into a sum of weight spaces as $M = \sum_{\mu \in X(T)} M_{\mu}$. As usual, we adopt the formal character of $M$, that is, $\text{ch}(M) = \sum_{\mu \in X(T)}\text{dim}(M_{\mu})e^{\mu}$ in $\mathbb{Z}[X(T)]$ where $\{e^{\mu}\}$ with $\mu$ running through $X(T)$ are the standard basis of the group ring $\mathbb{Z}[X(T)]$ over $\mathbb{Z}$, satisfying $e^{\mu_1}e^{\mu_2} = e^{\mu_1+\mu_2}$. For example, one has for $\lambda \in X^+(T)$,

$$
\text{ch}(H^0(\lambda)) = \frac{A(\lambda+\rho)}{A(\rho)}\Xi
$$

where $A(\mu) := \sum_{w \in W}(-1)^{l(w)}e^{w(\mu)}$ and $\Xi := \prod_{\beta \in \Phi^+_1}(1+e^{-\beta})$ (see for example, [24 Corollary 5.8]). For finite-dimensional $G$-modules $M_i$ ($i = 1, 2$) and $N$ one has

$$
\text{ch}(M_1 \otimes M_2) = \text{ch}(M_1)\text{ch}(M_2)
$$

and has further

$$
\text{ch}(N) = \text{ch}(M_1) + \text{ch}(M_2)
$$

if there exists a short exact sequence of $G$-modules: $M_1 \hookrightarrow N \twoheadrightarrow M_2$.

5. **Typical weights and Steinberg’s tensor product theorem**

5.1. **Typical weights.**

**Definition 5.1.** A dominant weight $\lambda \in X^+(T) := \{\mu = \sum_{i=1}^m a_i\delta_i + \sum_{j=1}^n b_j\epsilon_j \in X(T) \mid a_1 \geq a_2 \geq \cdots \geq a_m; b_1 \geq b_2 \geq \cdots \geq b_n\}$ is called typical if $(\lambda + \rho, \beta_i) \neq 0$ for all $i = 1, \ldots, mn$. Otherwise, a dominant weight $\mu \in X^+(T)$ which is not typical is called atypical.

By a straightforward computation, it is readily verified that

$$
(\lambda + \rho, \beta_i) = (\lambda_{i-1}, \beta_i).
$$

Hence $\lambda \in X^+(T)$ is typical if and only if $(\lambda_{i-1}, \beta_i) \neq 0$ for all $i = 1, \ldots, n$. The following observation is due to [22 or 23].

**Lemma 5.2.** Suppose $\lambda \in X^+(T)$ is a typical weight. Then all $\lambda_i$ lie in $X^+(T)$.

**Proof.** When $\lambda \in X^+(T)$ is typical, $\lambda_i$ is $B^{(i)+}$-dominant. Note that the purely-even subgroup of $B^{(i)+}$ is $B^{(i)+}_{ev}$. Consequently, $\lambda_i$ lies in $X^+(T)$. \qed

By the above lemma, Theorem [2.6(2)] along with the classical result [10 Proposition II.2.6] on induced modules of reductive algebraic groups ensure that all $H^0(G/B^{(i-1)+}, \lambda_i)$ are nonzero.

**Remark 5.3.** When considering representations of $\text{GL}(m|n)$ over $\mathbb{C}$, one has that a typical weight $\lambda \in X^+(T)$ gives rise to a Kac module realization of typical irreducible modules $L(\lambda)$ where a Kac module is an induced module from an irreducible $G_{ev}$-module (see [6, 12, 21] etc.). In this paper, we will present a modular version of this result in the concluding section. What is the most important is that typical weights will give rise to Jantzen filtration and consequently a sum formula of characters, which is the main purpose of the present paper.
5.2. Steinberg’s tensor product theorem and the reduction from atypicals to typicals. There is a natural question: how to deal with atypical weights in characteristic $p > 0$ in contrast with the privilege of typical weights mentioned in Remark 5.3. Steinberg’s tensor product theorem will tell us that it is enough to understand typical irreducible characters along with even irreducible characters (for purely-even subgroups).

Set

$$X_p^+(T) := \{ \sum_{i=1}^m a_i \delta_i + \sum_{j=1}^n b_j \epsilon_j \in X^+(T) \mid 0 \leq (a_i - a_{i+1}), (b_j - b_{j+1}) \leq p - 1, \forall i = 1, \ldots, m - 1; j = 1, \ldots, n - 1 \}.$$ 

For any atypical weight $\mu \in X^+(T)$, we can write it in $p$-adic expression

$$\mu = \mu_0 + p \mu_1 + \cdots + p^r \mu_r$$

such that all $\mu_i \in X_p^+(T)$. Denote by $\varpi_i$ the $i$th fundamental weight of $\text{GL}(m)$ ($i = 1, \ldots, m - 1$) which means $\varpi_i \in X^+(T)$ satisfying $(\varpi_i, \delta_k - \delta_{k+1}) = \delta_{ik}$ for $k = 1, \ldots, m - 1$. There exists great enough positive number $l (> r)$ such that $\nu := p^l \sum_{i=1}^{m-1} \varpi_i$ satisfies that $\lambda := \mu + \nu$ is a typical weight. Obviously, $\varpi := \sum_{i=1}^{m-1} \varpi_i \in X_p^+(T)$.

Owing to Kujawa’s work [13], we have the following Steinberg’s tensor product theorem for $\text{GL}(m|n)$.

**Theorem 5.4.** Keep the notation as above. There is an isomorphism of $G$-modules

$$L(\lambda) \cong L(\mu) \otimes L_{\text{ev}}(\varpi)[l]$$

where $L_{\text{ev}}(\varpi)[l]$ stands for the $l$-th Frobenius twist of $L_{\text{ev}}(\varpi)$ (see [13] for more details), and $L_{\text{ev}}(\varpi)$ stands for the irreducible module of $G_{\text{ev}}$ with highest weight $\varpi$.

By §4.5, this theorem ensures that the question of formulating irreducible characters of atypical weights can be reduced to the study of irreducible characters of typical weights.

6. Totally-odd induced modules and related homomorphisms

Recall that we already have a series of Borel subgroups $B^{(i)}$ ($1 \leq i \leq mn$) in §1.2.2. We now introduce the totally-odd induced modules which have been mentioned in §1.2.4.

6.1. Keep in mind the notations and assumptions in §1.2.4, §1.2.3 and §2.6. We have that the purely-even parts of $B^{(i)}$, $i = 0, 1, \ldots, mn$ are the same one $B_{\text{ev}}$. For $1 \leq i \leq mn$, let $P(\beta_i)$ is the minimal parabolic subgroups containing $B^{(i-1)}$ and $B^{(i)}$. By induction on $i$, $i = 1, \ldots, mn$ starting with $\lambda_0 := \lambda$, set $\lambda_i = \lambda_{i-1} - \beta_i$.

By the conventional notations in §2.6, the totally-odd induced modules are naturally introduced as below

$$H^0_{\text{total}}(\lambda) := H^0(G/B^{(mn)}; \lambda_{mn}).$$
From now on, we always suppose \( \lambda \) is a given typical weight in \( X^+(T) \).

6.2. On the other hand, for \( \mu \in X^+(T) \), inductively set for \( i = 1, \ldots, mn \) with \( \mu^{(0)} := \mu \),

\[
\mu^{(i)} = \begin{cases} 
\mu^{(i-1)}, & \text{if } (\lambda_i, \beta_i) \equiv 0 \mod p; \\
\mu^{(i-1)} - \beta_i, & \text{if } (\lambda_i, \beta_i) \not\equiv 0 \mod p.
\end{cases}
\]

(6.1)

Recall that irreducible \( G \)-modules coincides with finite-dimensional irreducible integrable \( \text{Dist}(G) \)-modules (see [4]). By [4, Theorems 4.3 and 4.5], the terminal one \( \mu^{(mn)} \in X^+(T) \) with \( L_{B^{(mn)}}(\mu^{(mn)}) \cong L_{B^{(0)}}(\mu^{(0)}) \). This \( L_{B^{(0)}}(\mu) \) is exactly \( L(\mu) \) which is by definition the simple socle of \( H^0(\mu) \).

More generally, by Lemma 4.2

\[
H^0(G/B^{(i-1)}, \mu) \cong H^0(G/B^{(i)}, \mu - \beta_i) \text{ and } L_{B^{(i-1)}}(\mu) \cong L_{B^{(i)}}(\mu - \beta_i)
\]

whenever \( (\mu, \beta_i) \not\equiv 0 \mod p \). So we have

Lemma 6.1. (cf. [4, Lemma 4.2]) Suppose \( \mu \in X^+(T) \) is given. For \( 1 \leq i \leq mn \),

\[
L_{B^{(i-1)}}(\mu) \cong \begin{cases} 
L_{B^{(i)}}(\mu), & (\mu, \beta_i) \equiv 0 \mod p, \\
L_{B^{(i)}}(\mu - \beta_i), & (\mu, \beta_i) \not\equiv 0 \mod p.
\end{cases}
\]

(6.2)

Remark 6.2. (1) By the above lemma, there exists weight \( \tilde{\mu} \in X^+(T) \) such that

\[
L(\mu) \cong L_{B^{(mn)}}(\tilde{\mu}).
\]

(6.3)

Furthermore, this \( \tilde{\mu} \) is exactly \( \mu^{(mn)} \) (see [4, Theorem 4.3]). By [4, Theorem 4.5],

\[
L(\mu)^* \cong L(-w_0 \tilde{\mu}).
\]

(6.4)

(2) A weight \( \lambda \in X^+(T) \) will be called \( p \)-typical if \( (\lambda_{i-1}, \beta_i) \not\equiv 0 \mod p \) for all \( i \in \{1, \ldots, mn\} \). Note that by [5,7], \( (\lambda_{i-1}, \beta_i) \not\equiv 0 \mod p \) if and only if \( (\lambda + \rho, \beta_i) \not\equiv 0 \mod p \). So the notion of \( p \)-typical weights here is identical to the one of “typical weights” introduced in [29] and [32, §12]. By Lemma 4.2 \( H^0(\lambda) \) and \( H^0(G/B^{(mn)}, \lambda_{mn}) \) are isomorphic if and only if \( \lambda \) is \( p \)-typical. In this case \( H^0(\lambda) \) is irreducible if additionally \( \lambda \) lies in the fundamental alcove (see [32, Proposition 12.10]).

As a consequence, we have

Corollary 6.3. For \( \lambda \in X^+(T) \), the following statements hold.

(1) There exists \( \mu \in X^+(T) \) such that \( L_{B^{(mn)}}(\lambda_{mn}) \cong L(\mu) \), equivalent to say, \( \lambda_{mn} = \tilde{\mu} \).

(2) Set \( \gamma = -w_0 \lambda + 2\rho_1 \). Then \( L(\gamma)^* \cong L_{B^{(mn)}}(\lambda_{mn}) \).

Proof. (1) It follows from (6.3).

(2) Note that \( L(\gamma) = L(-w_0(\lambda - 2\rho_1)) \cong L(-w_0 \lambda_{mn}) \), which is isomorphic to \( L(\mu)^* \) by (6.4). The desired isomorphism follows from (1). \( \square \)
6.3. Keep the notations and assumptions in §3.6. As in §3.5, we take \( C_\lambda, Y_\lambda \) as a \( k \)-basis of \( H^0_{\beta_1}(\lambda_i) \), and take \( B_{\lambda_{i-1}}, X_{\lambda_{i-1}} \) as a \( k \)-basis of \( H^0_{-\beta_1}(\lambda_{i-1}) \).

Similar to (4.2), (4.3) and (4.4) we have \( P_A(\beta_i) \)-module homomorphisms

\[
T_{A,\lambda_i, \beta_i} : \text{ind}_{B_{\lambda_{i-1}}}^{P_A(\beta_i)} A_{\lambda_{i-1}} \rightarrow \text{ind}_{B_{\lambda_{i-1}}}^{P_A(\beta_i)} A_{\lambda_i};
\]

\[
T'_{A,\lambda_i, \beta_i} : \text{ind}_{B_{\lambda_{i-1}}}^{P_A(\beta_i)} A_{\lambda_{i-1}} \rightarrow \text{ind}_{B_{\lambda_{i-1}}}^{P_A(\beta_i)} A_{\lambda_{i-1}}.
\]

And \( G_A \)-module homomorphisms

\[
\tilde{T}_{A,\lambda_i, \beta_i} : H^0(G_A/B^{(i)}_A, \lambda_{i-1}) \rightarrow H^0(G_A/B^{(i)}_A, \lambda_i);
\]

\[
\tilde{T}'_{A,\lambda_i, \beta_i} : H^0(G_A/B^{(i)}_A, \lambda_{i-1}) \rightarrow H^0(G_A/B^{(i-1)}_A, \lambda_{i-1}).
\]

6.4. Keep the notations in §4.5. Now we adopt the Euler characteristic for each finite dimensional \( B \)-module \( M \),

\[
\chi(M) := \sum_{i \geq 0} (-1)^i \text{ch}H^i(M).
\]

Then we have the following facts.

**Proposition 6.4.** Suppose \( \lambda \in X^+(T) \). The following statements hold.

1. \( \text{ch}(H^0_{\text{total}}(\lambda)) = \text{ch}(H^0(\lambda - 2\rho_1))\Xi_{mn}, \) where \( \Xi_{mn} := \prod_{\beta \in \Phi^+}(1 + e^{\beta}) \).

2. Set \( \chi(\lambda) = \chi(k_\lambda), \) and \( \chi_0(\lambda) := \chi_0(k_\lambda). \) Then

\[
\chi(\lambda) = \chi_0(\lambda)\Xi.
\]

\[
\chi(w.\lambda) = \det(w)\chi(\lambda) \text{ for any } w \in W \text{ with determinant } \det(w).
\]

**Proof.** (1) It follows from Theorem 2.6(2).

(2) By Theorem 2.6(2) again, for any \( B \)-module \( M \) there is a \( T \)-module isomorphism

\[
R^i \text{ind}_{B}^G(M) \cong R^i \text{ind}_{B_{ev}}^G(M) \otimes \bigwedge^i (g/b^-)^*_T.
\]

Note \( \text{ch}(\bigwedge^i (g/b^-)^*_T) = \Xi. \) The first part follows. As to the second one, note that \( w(\rho_1) = \rho_1 \). It follows from the first part along with the result on Euler characters of \( G_{ev} \)-modules (see [10, §II.5.9(1)]).

6.5. Homomorphisms arising from odd reflections. According to 6.3, we have a sequence of homomorphisms

\[
H^0(G_A/B^{(0)}_A, \lambda_0) \xrightarrow{T_{A,\beta_1}} H^0(G_A/B^{(1)}_A, \lambda_1) \xrightarrow{T_{A,\beta_2}} \cdots
\]

\[
\xrightarrow{T_{A,\beta_{mn}}} H^0_{\text{total}}(\lambda) := H^0(G_A/B^{(mn)}_A, \lambda_{mn}).
\]

Denote the composite of these sequence of homomorphisms by \( \tilde{T}_{A,w_1} \), i.e.

\[
\tilde{T}_{A,w_1} = T_{A,\beta_{mn}} \circ T_{A,\beta_{mn-1}} \circ \cdots \circ T_{A,\beta_1}.
\]

Similarly, consider \( \tilde{T}'_{A,w_1} = \tilde{T}'_{A,\beta_1} \circ \tilde{T}'_{A,\beta_2} \circ \cdots \circ \tilde{T}'_{A,\beta_{mn}} \). Then we have

\[
\tilde{T}_{A,w_1} : H^0(G_A/B^{(0)}_A, \lambda_0) \rightarrow H^0(G_A/B^{(mn)}_A, \lambda_{mn}).
\]

(6.5)
and
\[ \widetilde{T}_{A,w_{i}} : H^{0}(G_{A}/B_{A}^{(mn)}), \lambda_{mn}) \rightarrow H^{0}(G_{A}/B_{A}^{(0)}), \lambda_{0}). \]

Keep it in mind that \( \mathbb{K} \) is the fractional field of \( A \). Let \( H^{0}_{\text{total}, \mathbb{K}}(\lambda) := H^{0}_{\text{total}, A}(\lambda) \otimes_{A} \mathbb{K} \). Lemma 4.1 yields the following result.

**Lemma 6.5.** Suppose \( \text{char}(\mathbb{K}) = 0 \), and \( \lambda \) is typical. Then the natural extension of \( T_{A,w_{i}} \)
\[ \widetilde{T}_{\mathbb{K},w_{i}} : H^{0}_{\mathbb{K}}(\lambda) \rightarrow H^{0}_{\text{total}, \mathbb{K}}(\lambda) \] is a \( \mathbb{K} \)-isomorphism.

6.6. We turn to the case over \( k \).

6.6.1. We first prove the following basic result. Recall that for an algebraic supergroup scheme and its module, one can talk about the fixed point space (cf. [10, §I.2.10] with change of any commutative \( k \)-algebra \( A \) into any commutative \( k \)-superalgebra \( R \)). And one can talk about unipotent supergroup scheme over \( k \).

By a result of Zubkov, an affine group scheme \( U \) is unipotent if and only if its purely-even subgroup scheme \( U_{ev} \) is unipotent (see [15, Theorem 41]). Any irreducible module of a unipotent supergroup scheme \( U \) is one-dimensional and trivial (see [15], [31], etc.). Therefore, for a unipotent supergroup \( U \), its fixed-point space of a non-zero \( U \)-module is nonzero.

**Lemma 6.6.** Let \( U^{(mn)}^{+} \) be the unipotent radical of \( B^{(mn)}^{+} \). Then the \( U^{(mn)}^{+} \)-fixed point subspace \( H^{0}(G/B^{(i-1)}, \lambda_{i-1}) U^{(mn)}^{+} \) for any \( i = 1, \ldots, mn \) is one-dimensional, which is exactly the weight space \( H^{0}(G/B^{(i-1)}, \lambda_{i-1}) \lambda_{mn} \).

**Proof.** Keep it in mind that the purely-even subgroup of \( B^{(mn)}^{+} \) is \( B_{ev}^{+} \) which has unipotent radical \( U_{ev}^{+} \), and for any nonzero \( U^{(mn)}^{+} \)-module \( M \), the fixed-point space \( M U^{(mn)}^{+} \) must be nonzero. For simplicity of notations, we set \( \mathcal{K} = H^{0}(G/B^{(i-1)}, \lambda_{i-1}) \) which can be described as
\[ \mathcal{K} = \{ \phi \in k[G] | \phi(xb) = \lambda_{i-1}^{-1}(b)^{-1} \phi(x) \text{ for any } x \in G(R), b \in B^{(i-1)}(R) \} \]
and for any \( R \in \text{salg}_{k} \).

The action of \( G(R) \) is given by left translation. For any \( f \in \mathcal{K} U^{(mn)}^{+} \), it satisfies \( f(ub) = b^{-1} f(1) \) for all \( u \in U^{(mn)}^{+}(R) \) with \( f(1) \in k_{\lambda_{i-1}} \). Note that \( U^{(mn)}^{+} \) and \( B^{(i-1)} \) have purely-even subgroups identical to \( U_{ev}^{+} \) and \( B_{ev} \), respectively. Hence, \( f |_{U^{(mn)}_{ev} B_{ev}} \) is determined by \( f(1) \). Recall that as a topological space, open subsets of \( \tilde{G} \) are just the ones of \( G_{ev} \) while \( U_{ev}^{+} B_{ev} \) is open dense (see [26, §8.3] or [10, §II.1.9]). Then we can conclude that \( f(1) \) determines \( f \). Correspondingly, \( \mathcal{K} U^{(mn)}^{+} \) is one-dimensional. On the other hand, as mentioned previously \( \mathcal{K} U^{(mn)}^{+} \neq 0 \). So we have \( \dim \mathcal{K} U^{(mn)}^{+} = 1 \).

According to Theorem 2.6, \( H^{0}(G/B^{(i-1)}, \lambda_{i-1}) \) has a \( G_{ev} \)-submodule isomorphic to \( H_{ev}^{0}(\lambda_{i-1}) \) which admits one-dimensional weight space of the \( B_{ev}^{+} \)-highest weight.
\( \lambda_{i-1} = \lambda - (\beta_1 + \ldots + \beta_{i-1}) \). What’s more, by Theorem 2.6(2) there is a \( T \)-module isomorphism

\[
H^0(G/B^{(i-1)}, \lambda_{i-1}) \cong H^0_{ev}(\lambda_{i-1}) \otimes \bigwedge^* (g/(b^{(i-1)^-}))^*_1
\]

where \( b^{(i-1)^-} = \text{Lie}(B^{(i-1)}) \). It is readily known that the set of \( T \)-weights of \( (g/b^{(i-1)^-})^*_1 \) is exactly

\[
\{-\beta_1, \ldots, -\beta_{mn-(i-1)}, \beta_{mn-(i-1)+1}, \ldots, \beta_{mn}\}.
\]

Hence, \( (g/b^{(i-1)^-})^*_1 \) has one-dimensional weight space of the \( B^{(mn)^+} \)-highest weight \( \beta_i + \beta_{i+1} + \ldots + \beta_{(mn)} \). Therefore, \( H^0(G/B^{(i-1)}, \lambda_{i-1}) \) has one-dimensional weight space of the \( B^{(mn)^+} \)-highest weight \( \lambda_{mn} = \lambda - \sum_{i=1}^{mn} \beta_i \).

Combining the above, we have

\[
H^0(G/B^{(i-1)}, \lambda_{i-1})^{U^{(mn)^+}} = H^0(G/B^{(i-1)}, \lambda_{i-1})_{\lambda_{mn}}.
\]

The proof is completed. \( \square \)

6.6.2. Now we are in a position to introduce the following important result.

**Proposition 6.7.** Let \( \lambda \in X^+(T) \) be a typical weight. The composite of the above homomorphisms

\[
\widetilde{T}_{k,\lambda':} : H^0(\lambda) \rightarrow H^0_{\text{total}}(\lambda).
\]

is nonzero.

**Proof.** Applying Lemma 4.2 (its statement(1) and its statement (2.1.2)) to the case

\[
\widetilde{T}_{k,\beta_i} : H^0(G/B^{(i-1)}, \lambda_{i-1}) \rightarrow H^0(G/B^{(i)}, \lambda_i),
\]

we have that \( \text{im}(\widetilde{T}_{k,\beta_i}) \) is not zero. Hence for the unipotent group \( U^{(mn)^+} \), the fixed point subspace \( \text{im}(\widetilde{T}_{k,\beta_i})^{U^{(mn)^+}} \) is surely nonzero. By Lemma 6.6, \( \text{im}(\widetilde{T}_{k,\beta_i})^{U^{(mn)^+}} = H^0(G/B^{(i)}, \lambda_i)_{\lambda_{mn}} \). Hence all one-dimensional \( U^{(mn)^+} \)-fixed point subspaces concerned are preserved by the sequence of nonzero morphisms \( \widetilde{T}_{k,\beta_i} \). Consequently, the composite of them are nonzero. The proof is completed. \( \square \)

**Remark 6.8.** Lemma 6.6 and Proposition 6.7 are true for the case of base fields of characteristic zero.

7. **Weyl modules and related homomorphisms**

Keep the notations and assumptions as before. Especially, \( w_0 \) is the longest element of \( W \). For \( \lambda \in X^+(T) \), inductively set \( \lambda_i \) with the initial one \( \lambda_0 = \lambda \), and the terminal one \( \lambda_{mn} = \lambda - 2p_1 \). For \( w \in W \), denote \( w.\lambda = w(\lambda) - \rho \) for \( \rho = \rho_0 - \rho_1 = \frac{1}{2}(\sum_{\alpha \in \Phi^+_0} \alpha - \sum_{\beta \in \Phi^+_1} \beta) \).
7.1. **Weyl modules.** Recall the length of \( w_0 \) is \( l(w_0) = |\Phi^+_0| \). For \( \lambda \in X(T) \), define Weyl module

\[
V(\lambda) := R^{l(w_0)} \text{ind}^G_B(k_{w_0,\lambda}) \cong H^{l(w_0)}(G/B, \mathcal{L}(k_{w_0,\lambda})).
\]

As usual, we write \( V(\lambda) = H^{l(w_0)}(w_0,\lambda) \). Furthermore, it can be defined over any commutative \( \mathbb{Z} \)-algebra \( A \), written as

\[
V_A(\lambda) = R^{l(w_0)} \text{ind}^G_A(A_{w_0,\lambda}).
\]

Then

\[
V_A(\lambda) = H^{l(w_0)}_A(w_0,\lambda).
\]

Recall we have (cf. Theorem 2.2 and Proposition 2.3)

\[
(R^i \text{ind}^G_B M)^* \cong R^{(l(w_0)) - i} \text{ind}^G_B (M^* \otimes \text{Ber}(X))
\]

with \( X = G/B \), and

\[
\text{Ber}(X) \cong \mathcal{L}_{ev}(\bigwedge^* (g/b^-)_1 \otimes k_{-2\rho_b}).
\]

Hence, we have

\[
V(\lambda)^* \cong \text{ind}^G_B (k_{w_0,\lambda} \otimes \text{Ber}(X))) \cong \text{ind}^G_H (k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1).
\]

Thus

\[
V(\lambda) \cong \text{ind}^G_B (k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1)^*.
\] (7.1)

In the following arguments we denote \( M = k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1 \) for the time being.

**Lemma 7.1.** Assume \( \lambda \in X^+(T) \). The following statements hold.

1. \( H^0(G/B, k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1) \) is nonzero.
2. The module \( H^0(G/B, k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1) \) has \( B^+ \)-highest weight \( -w_0\lambda + 2\rho_b \).
3. Denote \( \gamma = -w_0\lambda + 2\rho_b \). Then \( H^0(G/B, k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1) \) has simple socle isomorphic to \( L(\gamma) \). Correspondingly, \( V(\lambda) \) has simple head isomorphic to \( L(\gamma)^* \cong L(-w_0\gamma) \).
4. \( \text{ch}(V(\lambda)) = \text{ch}(H^0(\lambda)) \).

**Proof.** (1) Set \( \mathcal{H} := H^0(G/B, k_{-w_0,\lambda} \otimes \bigwedge^* (g/b^-)_1) \). By the above arguments, \( \mathcal{H}^* \cong H^{l(w_0)}(G/B, w_0,\lambda) \). Note that as a \( T \)-module there is an isomorphism as below (see Theorem 2.6(2))

\[
H^{l(w_0)}(w_0,\lambda) \cong H^{l(w_0)}_{ev}(w_0,\lambda) \otimes \bigwedge^* (g/b^-)_1^*.
\] (7.2)

By the classical Serre duality, \( H^{l(w_0)}(G_{ev}/B_{ev}, w_0,\lambda) \cong H^0(G_{ev}/B_{ev}, -w_0\lambda)^* \neq 0 \). Hence Theorem 2.6(1) ensures that \( \mathcal{H} \) is nonzero.

(2) We will prove that \( \mathcal{H}^{U^+} = \mathcal{H}_{-w_0,\lambda + 2\rho_b} \) by the same argument as in the proof of Lemma 6.6 with suitable change (in the present situation, the associated sheaf over \( G/B \) has rank greater than one). In order to show this, by definition we have for any commutative \( k \)-superalgebra \( R \)

\[
\mathcal{H} = \{ \phi \in \text{Mor}(G, M) \mid \phi(xb) = b^{-1}\phi(x) \text{ for all } x \in G(R), b \in B(R) \}
\]
where \( \text{Mor}(G, M) \) stands for the morphism set, \( M \) is regarded as an additive supergroup scheme, i.e. \( M(R) \) is identified with \( M \otimes_k R \). The action of \( G(R) \) is given by left translation. For any \( f \in \mathcal{H}^U \), it satisfies \( f(ub) = b^{-1}f(1) \) for all \( u \in U(R) \) where by definition \( f(1) \in M(R) = R_{-w_0\lambda} \otimes \wedge^*(g/b^-)_1 \). Hence, \( f \in U(R)B(R) \) is determined by \( f(1) \). Recall again that as a topological space, open subsets of \( G \) are by definition just ones of \( G_{ev} \) while \( U_{ev}^+B_{ev} \) is open dense (see [26, §II.1.9]). Then we can conclude that \( f(1) \) determines \( f \). Correspondingly, \( \mathcal{H}^U \) is of dimension one. On the other hand, \( \mathcal{H}^U \neq 0 \). So we have \( \dim \mathcal{H}^U = 1 \). Taking the generalized tensor identity into an account (see (2.1) and [10, §II.5 and §II.8]), by the above arguments we have the following \( T \)-module decomposition

\[
\mathcal{H} \cong H^0_{ev}(-w_0\lambda) \otimes \wedge^*(g/b^-)_1. \tag{7.3}
\]

By comparing the weights, we know that \(-w_0\lambda + 2\rho_1 \) is a \( B^+ \)-maximal weight, and its weight space is one dimensional here, as below

\[
\mathcal{H}_{-w_0\lambda + 2\rho_1} \cong H^0_{ev}(-w_0\lambda)_{-w_0\lambda} \otimes \wedge^m(g/b^-)_1.
\]

As in the case of reductive algebraic groups, if \( \mu \) is a \( B^+ \)-maximal one among the weights of \( \mathcal{H} \), then \( \mathcal{H}_\mu \subset \mathcal{H}^U \). Hence we can conclude \( \mathcal{H}^U \cong H^0_{ev}(-w_0\lambda)_{-w_0\lambda} \otimes \wedge^m(g/b^-)_1 \). Correspondingly, \( \mathcal{H}^U = \mathcal{H}_{-w_0\lambda + 2\rho_1} \), and any weight of \( \mathcal{H} \) is smaller than \(-w_0\lambda + 2\rho_1 \) in the sense of the standard positive root system.

(3) For the last statement, if \( L_1 \) and \( L_2 \) are two distinct simple submodules of \( \mathcal{H} \), then \( L_1^U \oplus L_2^U \subset \mathcal{H}^U \). There would be a contradiction with rank 1 of \( \mathcal{H}^U(R) \) over \( R \). Hence \( H^0(G/B, k_{-w_0\lambda} \otimes \wedge^*(g/b^-)_1) \) has simple socle \( L(\gamma) \). As \( V(\lambda) \cong H^0(G/B, k_{-w_0\lambda} \otimes \wedge^*(g/b^-)_1)^* \), we have that \( V(\lambda) \) has simple head \( L(\gamma)^* \) which is isomorphic to \( L(-w_0\gamma) \) by [4, Theorem 4.5].

(4) Note that \( \text{ch}(V(\lambda)) = \text{ch}(\mathcal{H}) \). This statement is a consequence of (2.3) and (7.3).

The proof is completed. \( \square \)

7.2. Recall \( L(\gamma)^* \) is isomorphic to the head of \( V(\lambda) \) (see Lemma 7.1). We have the following observation.

**Lemma 7.2.** Keep the notations and assumptions as before. The head of \( V(\lambda) \) is isomorphic to the socle of \( H^0_{\text{total}}(\lambda) = H^0(G/B^{(mn)}, \lambda_{mn}) \).

**Remark 7.3.** Unlike the case of reductive algebraic groups, in general, the head of Weyl module \( V(\lambda) \) is not isomorphic to \( L(\lambda) \). When \( \lambda \) is \( p \)-typical (see Remark 6.2), by Lemma 4.2(1), \( L(\lambda) \cong L_{B^{(mn)}}(\lambda_{mn}) \). On the other hand, in this case \( \lambda = \lambda \). It follows that \( L(\lambda) \cong L_{B^{(mn)}}(\lambda_{mn}) \cong L(\gamma)^* \). So in this case, the head of \( V(\lambda) \) is isomorphic to \( L(\lambda) \).

7.3. **Homomorphisms arising from Weyl groups.** Now we will present a homomorphism from \( V(\lambda) \) to \( H^0(\lambda) \), by exploiting the arguments for reductive algebraic groups (see [10, §II.5 and §II.8]) to our case.
7.3.1. Let’s first recall some necessary structural information for $H^i(\lambda)$ over $\mathbb{F}$ which is any given ground field. We temporarily suppose $G$ is over $\mathbb{F}$ for the following lemma.

Let $\alpha$ be a given simple even root with even simple reflection $r_\alpha \in W$. Consider the minimal parabolic subgroup $P(\alpha)$ containing $B = B^-$ and the purely-even root subgroup $G_\alpha$ (see [10 §II.1.3]). We look at the structure $R^i ind_B^{P(\alpha)} F_\lambda$. Note that $\alpha$ is an even root. By Theorem 2.4, there is an isomorphism of $\mathbb{R}$-superalgebra $R^i ind_B^{P(\alpha)} F_\lambda \cong R^i ind_{B_{ev}}^{P(\alpha)} F_\mu$ (with trivial $R_\alpha$-action on the left hand side for the unipotent radical $R_\alpha$ of $P(\alpha)$). So we can mimic [10 Proposition II.5.2] as below, with a little modification.

**Lemma 7.4.** The following statements hold.

1. The unipotent radical of $P(\alpha)$ acts trivially on each $R^i ind_B^{P(\alpha)} F_\lambda, \forall i \geq 0$.
2. If $(\lambda, \alpha) = -1$, then $R^i ind_B^{P(\alpha)} F_\lambda = 0$.
3. (a) If $(\lambda, \alpha) = s \geq 0$, then $R^i ind_B^{P(\alpha)} F_\lambda = 0, \forall i \neq 0$ and $ind_B^{P(\alpha)} F_\lambda$ has a basis $\{v_i \mid i = 0, 1, \ldots, s\}$ such that for all $i, 0 \leq i \leq s$ and any commutative $\mathbb{F}$-superalgebra $R = R_0 \oplus R_1$:
   1. $tv_i = (\lambda - i\alpha)(t)v_i, \forall t \in T(R_0)$;
   2. $x_\alpha(a)v_i = \sum_{j=0}^{s} (\binom{s}{j}) a^{s-j}v_j, \forall a \in R_0$;
   3. $x_{-\alpha}(a)v_i = \sum_{j=1}^{s} (\binom{s}{j-1}) a^{s-j+1}v_j, \forall a \in R_0$.
   Here and later $x_{\pm\alpha}(a)$ are Chevalley generators of Chevalley supergroups in the same sense of reductive algebraic groups (see [8 §5.2], [10 §II.1.19]).
3. (b) If $(\lambda, \alpha) \leq -2$, then $R^i ind_B^{P(\alpha)} F_\lambda = 0, \forall i \neq 1$ and $R^1 ind_B^{P(\alpha)} F_\lambda$ has a basis $\{v'_i \mid i = 0, 1, \ldots, s\}$ such that for all $i, 0 \leq i \leq s$ and each super commutative $\mathbb{F}$-superalgebra $R = R_0 \oplus R_1$:
   1. $tv'_i = (r_\alpha \lambda - i\alpha)(t)v'_i, \forall t \in T(R_0)$;
   2. $x_\alpha(a)v'_i = \sum_{j=0}^{s} (\binom{s-j}{s-j}) a^{s-j}v'_j, \forall a \in R_0$;
   3. $x_{-\alpha}(a)v'_i = \sum_{j=1}^{s} (\binom{s-j}{s-j}) a^{s-j+1}v'_j, \forall a \in R_0$.
4. For any $P(\alpha)$-module $M$ and any $i \geq 0$, $R^i ind_B^{P(\alpha)} M \cong R^i ind_{P(\alpha)} M$.
5. (a) If $(\lambda, \alpha) \geq 0$, then $H^i(\lambda) \cong H^i(\lambda)$ for all $i$. Furthermore, if $\text{ch}(\mathbb{F}) = 0$, or $\text{ch}(\mathbb{F}) = p > 0$ with $(\lambda, \alpha) = rp^{m-1}, r, m \in \mathbb{N}, 0 < r < p$, then for all $i$
   $H^{i+1}(r_\alpha \lambda) \cong H^i(\lambda)$.
5. (b) If $(\lambda, \alpha) \leq -2$, then for all $i$
   $H^i(\lambda) \cong H^{i-1}(R^1 ind_{B_{ev}}^{P(\alpha)} \lambda)$.
5. (c) If $(\lambda, \alpha) = -1$, then $H^\bullet(\lambda) = 0$. 
7.3.2. Turn to the PID $A$ and its fractional field $\mathbb{K}$. In the remaining part of this section, we assume that $\text{char}(\mathbb{K}) = 0$. Then we have $G_A$, $B_A$, $T_A$ and $P(\alpha)_A$. For a $B_A$-module $M$, we set

$$H^i_{\alpha,A}(M) = R^i \text{ind}^P_{B_A}(M)$$

and

$$H^i_A(\lambda) := R^i \text{ind}^G_{B_A}(A_\lambda)$$

for natural $B_A$-module $A_\lambda$.

**Lemma 7.5.** The following statements hold.

1. If $(\lambda, \alpha) = -1$, then $H^\bullet_{\alpha,A}(\lambda) = 0$.
2. If $(\lambda, \alpha) \geq 0$, then $H^i_{\alpha,A}(\lambda) = 0$ for all $i \neq 0$, and $H^i_{\alpha,A}(r_\alpha \cdot \lambda) = 0$ for all $j \neq 1$. More precisely, for $s := (\lambda, \alpha) \geq 0$.
   
   a. $H^0_{\alpha,A}(r_\alpha \cdot \lambda) \cong \sum_{i=0}^s A v'_i$ is $A$-torsion free of rank $(r + 1)$, including $A$-free basis elements \{\(v'_i \mid i = 0, 1, \ldots, r\}\}.
   
   b. $H^0_{\alpha,A}(\lambda) \cong \sum_{i=0}^s A v_i$ is $A$-torsion free of rank $(r + 1)$, including $A$-free basis elements \{\(v_i \mid i = 0, 1, \ldots, r\}\}.
   
   c. There exists $P(\alpha)_A$-module homomorphism
   $$T_\alpha(r_\alpha \cdot \lambda) : H^1_{\alpha,A}(r_\alpha \cdot \lambda) \to H^0_{\alpha,A}(\lambda)$$
   
   with $v'_i \mapsto \binom{r}{i} v_i$. And
   $$T_\alpha(\lambda) : H^0_{\alpha,A}(\lambda) \to H^1_{\alpha,A}(r_\alpha \cdot \lambda)$$
   
   with $v_i \mapsto (r - i)!i!v'_i$.
3. Furthermore, if $(\lambda, \alpha) \geq 0$, then for any $i$,

$$H^i_A(\lambda) \cong R^i \text{ind}^G_{P(\alpha)_A}(H^0_{\alpha,A}(\lambda)) \cong H^i_A(H^0_{\alpha,A}(\lambda)).$$

**Proof.** All statements follows from the previous lemma. In particular, (3) follows from its fourth statement. \hfill \Box

7.3.3. More generally, for simple even root $\alpha$, and $\mu \in X(T)$ with $(\mu, \alpha) \geq 0$, we have the following lemma.

**Lemma 7.6.** $H^i_A(r_\alpha \cdot \mu) \cong R^{i-1} \text{ind}^G_{P(\alpha)_A}(H^1_{\alpha,A}(r_\alpha \cdot \mu)) \cong H^{i-1}_A(H^1_{\alpha,A}(r_\alpha \cdot \mu)).$

**Proof.** For supergroup schemes, the following spectral sequence still holds as happening in the case of algebraic group schemes (see [10, §4.5])

$$R^i \text{ind}^G_{P(\alpha)_A}(R^j \text{ind}^P_{B_A}(A_{r_\alpha \cdot \mu})) \Longrightarrow R^{i+j} \text{ind}^G_{B_A}(A_{r_\alpha \cdot \mu}).$$

On the other hand, by Lemma 7.3.2 (2) we have $H^j_{\alpha,A}(r_\alpha \cdot \mu) = 0$ for all $j \neq 1$ when $(\mu, \alpha) \geq 0$. Hence, $H^i_A(r_\alpha \cdot \mu) \cong R^{i-1} \text{ind}^G_{P(\alpha)_A}(H^1_{\alpha,A}(r_\alpha \cdot \mu)) \cong H^{i-1}_A(H^1_{\alpha,A}(r_\alpha \cdot \mu)).$ \hfill \Box
Furthermore, under the assumption $(\mu, \alpha) \geq 0$ there is some $j \in \mathbb{N}$ such that $H^j_A(\mu)$ is not torsion module. Then this $j$ is unique and $H^{j+1}(r_\alpha \cdot \mu)$ is not a torsion module. By Lemmas 7.5 and 7.6 we have the following homomorphisms

$$\tilde{T}(r_\alpha \cdot \mu) : H^{j+1}_A (r_\alpha \cdot \mu) \to H^j_A(\mu)$$

and

$$\tilde{T}(\mu) : H^j_A(\mu) \to H^{j+1}_A (r_\alpha \cdot \mu)$$

such that the composite of them are the multiplication by $(\mu, \alpha)!$.

7.3.4. Choose a reduced expression $w_0 = r_{\alpha_N} r_{\alpha_{N-1}} \cdots r_{\alpha_1}$ for all $\alpha_i \in \Pi_0$ with $l(w_0) = N := |\Phi_0^+|$. For $\lambda \in X^+(T)$ and any $i$ we have the following homomorphism

$$\tilde{T}_{\alpha_i}(r_{\alpha_i} r_{\alpha_{i-1}} \cdots r_{\alpha_1} \cdot \lambda) : H^i_A (r_{\alpha_i} r_{\alpha_{i-1}} \cdots r_{\alpha_1} \cdot \lambda) \to H^{i-1}_A (r_{\alpha_{i-1}} \cdots r_{\alpha_1} \cdot \lambda).$$

Set $H^i_{\lambda}(\mu) := H^j_A(\mu) \otimes_A \mathbb{K}$. By composing all these homomorphisms we have

$$\tilde{T}_{A,w_0} : H^{l(w_0)}(w_0 \cdot \lambda) \to H^0(\lambda).$$

Set $H^i_{\lambda}(\lambda) = H^j_A(\lambda) \otimes_A \mathbb{K}$. Using the arguments in §7.3.3 we further have the following observation.

**Lemma 7.7.** Suppose $\mathbb{K}$ is of characteristic 0. Then the natural extension of $T_{A,w_0}$

$$\tilde{T}_{\mathbb{K},w_0} : H^{l(w_0)}(w_0 \cdot \lambda) \to H^0_{\mathbb{K}}(\lambda).$$

is a $G_{\mathbb{K}}$-module isomorphism.

7.3.5. Now we turn back to the base field $\mathbb{k}$. The following result holds.

**Lemma 7.8.** Suppose $\lambda \in X^+(T)$. Then the composed homomorphism

$$\tilde{T}_{k,w_0} : V(\lambda) \to H^0(\lambda)$$

is nonzero.

**Proof.** From Lemma 7.3 and the result on the composed homomorphism from Weyl module $V_{ev}(\lambda) := H^{l(w_0)}_{ev}(G/B, \mathcal{L}(k_{w_0} \cdot \lambda))$ to $H^0_{ev}(\lambda)$ is nonzero (see [10 §II.6.16]). On the other side, by Theorem 2.6(1), $V_{ev}(\lambda)$ and $H^0_{ev}(\lambda)$ are $G_{ev}$-submodules of $V(\lambda)$ and $H^0(\lambda)$, respectively. So the lemma can be deduced from Theorem 2.6(2). □

**Remark 7.9.** In general, the image of $\tilde{T}_{k,w_0}$ contains $L(\lambda)$ as a proper submodule. It will be seen that both of them coincide if and only if $\lambda$ is $p$-typical (see Theorem 9.3).
Proposition 7.10. Let \( \lambda \in X^+(T) \) be a typical weight. The following statements hold.

1. The composite \( \widehat{T}_{k,\lambda} \) is a nonzero homomorphism which maps the head of \( V(\lambda) \) onto the socle of \( H^0(\lambda) = H^0(G/B^{(\lambda)} \lambda_{mn}) \), both of which are isomorphic to \( L(-w_0\lambda + 2\rho_1)^* \).

2. The image \( \widehat{T}_{k,\lambda} \) is exactly the simple socle of \( H^0(\lambda) \).

Proof. (1) By Lemma 6.3, \( T_{k,\lambda} \) is nonzero. Hence \( \text{im}((T_{k,\lambda})^+ U^{(\lambda)}) \) is nonzero, which by Lemma 6.4 coincides with \( H^0(\lambda)^+ \). Furthermore, \( H^0(\lambda)^+ \) is identical to the one-dimensional weight space \( H^0(\lambda)_\lambda \). Combining with Proposition 6.7 and its proof, we have that \( \widehat{T}_{k,\lambda} \) must be a nonzero homomorphism.

(2) Thanks to Corollary 6.3, the irreducible head of \( V(\lambda) \) is isomorphic to \( L(\gamma)^* \), which is exactly isomorphic to \( L_{B^{(\lambda)}}(\lambda_{mn}) = \text{soc}(H^0(\lambda)) \). By Lemma 7.2, the \( B^{(\lambda)} \)-highest weight space is exactly \( H^0(\lambda)_\lambda \). Combining with Proposition 6.7 and its proof, we have that \( \widehat{T}_{k,\lambda} \) is a nonzero homomorphism.

The proof is completed.

8. Jantzen filtration of Weyl modules and sum formulas

8.1. General construction of Jantzen filtration. In this subsection, we recall some general construction of Jantzen filtration for the reader's convenience. More details can be referred to [10, §II.8].
$V_{A'}(\lambda) \cong V_Z(\lambda) \otimes_Z A'$. In particular, as $V(\lambda) \cong V_{F_p}(\lambda) \otimes_{F_p} k$, we have
\[ \text{ch} V(\lambda) = \text{ch} V_{F_p}(\lambda) = \text{ch} V_Z(\lambda)_{\text{fr}}. \] (8.1)

8.1.2. Set $\nu_p(N)$ to be the length of the $A_p$-module $N_p = N \otimes_A A_p$ for any $A$-module $N$. For $M \in G_A\text{-mod}$, set
\[ \nu_p^c(M) = \sum_{\mu} \nu_p(M_\mu)e(\mu). \] (8.2)

Let $M$ and $M'$ be torsion free $A$-modules in $G_A\text{-mod}$. Consider a homomorphism $\psi : M \to M'$ in $G_A\text{-mod}$ which satisfies $\psi \otimes k : M \otimes_A k \cong M' \otimes_A k$. Then we have $\coker(\psi) = M'/\psi(M) \in G_A\text{-mod}$ is a finitely generated $A$-torsion module. Set
\[ \nu_p(\psi) = \nu_p(\coker(\psi)) \quad \text{and} \quad \nu_p^c(\varphi) = \nu_p^c(\coker(\psi)). \] (8.3)

Note that $\overline{\psi} : \overline{M} \to \overline{M}'$ is the homomorphism induced by $\varphi$ where $\overline{M} = M/pM$ and $\overline{M}' = M'/pM'$. Set
\[ M^i := \{ m \in M | \psi(m) \in p^iM'; \forall i \in \mathbb{N} \}. \]

Set $\overline{M}^i$ to be the image of $M^i$ in $\overline{M}$. Then all $M^i$ are $G_A$-submodules of $M$ and all $\overline{M}^i$ are $G_{A/p}$-submodule of $\overline{M}$. Furthermore, we have (cf. [10, II.8.18])
\[ \overline{M}/\overline{M}^1 \cong \im \overline{\psi} \] (8.4)

\[ \overline{M}^i = \ker \overline{\psi}. \] (8.5)

And
\[ \sum_{i>0} \text{ch}(\overline{M}^i) = \nu_p^c(\psi). \] (8.6)

8.1.3. Suppose there are two homomorphism $\varphi$ and $\varphi' : M' \to M''$ in $G_A\text{-mod}$ satisfying the above assumptions. Consider $\psi = \varphi' \circ \varphi : M \to M''$.

By the same arguments as in [10, §II.8.11] we have
\[ \nu_p^c(\varphi' \circ \varphi) = \nu_p^c(\varphi') + \nu_p^c(\varphi). \] (8.7)

In order to make a distinction between different filtrations arising from different homomorphisms, we adopt an additional subscript like $\{M^i_\psi\}$ for the above filtration associated with $\varphi$. Then by (8.6) and (8.7) we have the following observation.

**Lemma 8.1.** \[ \sum_{i>0} \text{ch}(\overline{M}^i_\psi) = \sum_{i>0} \text{ch}(\overline{M}^i_\varphi) + \sum_{i>0} \text{ch}(\overline{M}^i_{\varphi'}). \]
8.2. Arguments for the part arising from even reflections. From now on, we take $A = \mathbb{Z}$, and take $p = p\mathbb{Z}$. Then $F_p = A/p$, and $k = F_p \otimes_{\mathbb{Z}} k$.

Set $\varphi := \tilde{T}_{A,w_0}$ in (7.3). We already have the following homomorphism

$$\varphi : H_{A_w}^0(w_0, \lambda) \rightarrow H_A^0(\lambda).$$

By Lemmas 7.7 and 7.8 the construction in § Proposition 8.2. Let $\lambda \in X^+(T)$. There is a filtration of $G$-modules

$$V(\lambda) = V_{\varphi}(\lambda)^0 \supset V_{\varphi}(\lambda)^1 \supset \cdots$$

such that the following sum formula holds

$$\sum_{i>0} \text{ch} V_{\varphi}(\lambda)^i = \sum_{\alpha \in \Phi^+_0 \ 0 < mp < (\lambda + \rho_0, \alpha)} \nu_p(mp) \chi(r_{\alpha, mp}, \lambda)$$

where $\nu_p(mp)$ means the $p$-adic valuation of $mp$.

8.3. Arguments for the part arising from odd reflections. Set $\varphi' := \tilde{T}_{A,w_1}$ in (6.3). We already have the following homomorphism

$$\varphi' : H_A^0(\lambda) \rightarrow H_{A_w}^0(\lambda).$$

Lemma 6.5 and Proposition 6.7 ensure the construction in § Proposition 8.3. Keep the notations and assumptions as above. Then the following result.

$$\sum_{i>0} \text{ch} V_{\varphi'}(\lambda)^i = \sum \nu_p(coker(\tilde{T}_{k, \beta_i})). \tag{8.8}$$

Note that by Lemma 4.2(1), $coker(\tilde{T}_{k, \beta_i}) = 0$ when $(\lambda, \beta_i) \equiv 0 \ mod \ p$. By Lemma 4.2(2), Corollary 4.3 along with 4.5 Equation (8.8) gives rise to the following formula.

$$\sum_{i>0} \text{ch} V_{\varphi'}(\lambda)^i = \sum_{\beta_i \in \Phi^+_0 : p(\lambda_{i-1}, \beta_i)} (\text{ch}(H^0(G/K_{\beta_i}, \lambda_{i-1})) + \sum_{k=1}^{\infty} (-1)^k \text{ch}(H^0(G/K_{-\beta_i}, \lambda_{i-1} + k\beta_i))). \tag{8.9}$$

Set $\mathcal{W}(\lambda) := \text{ch}(H^0_{ev}(\lambda))$ which is computed via the Weyl character formula. Denote $\Xi_i = \Pi_{\beta_i \in \Phi^+_0}(1 + e^{-\beta})$. Note that $K^+_{\beta_i}$ corresponds to the positive root systems $\Phi^+_{\beta_{i-1}}$ and $\Phi^+_{\beta_i}$ respectively. Here $\Phi^+_{\beta_{i-1}}$ when $i = 1$ just stands for the standard positive root system $\Phi^+$. Keep it in mind that $(\lambda_{i-1}, \beta_i) \equiv 0 \ mod \ p$ is equivalent to $(\lambda + \rho, \beta_i) \equiv 0 \ mod \ p$. Theorem 2.6(2) and (8.9) give rise to the following result.

Proposition 8.3. Keep the notations and assumptions as above. Then the following formula holds

$$\sum_{i>0} \text{ch} V_{\varphi'}(\lambda)^i = \sum_{\beta_i \in \Phi^+_0 : p(\lambda + \rho, \beta_i)} (\Xi_i \mathcal{W}(\lambda_{i-1}) + \Xi_{i-1} \sum_{k>0} (-1)^k \mathcal{W}(\lambda_{i-1} + k\beta_i)).$$
Here we appoint that $\Xi_0$ is just $\Xi$ in Proposition 6.4.

8.4. **Total arguments.** We are in a position to introduce our main result. Set $\psi := \tilde{T}_{A,\tilde{a}_0}$ in (7.8). Then we already have

$$\psi : V_A(\lambda) \to H^0_{\text{total},A}(\lambda)$$

with $\psi = \phi' \circ \phi$. By Lemma 8.1 along with Propositions 8.2 and 8.3 we have

**Theorem 8.4.** Keep the notations as above. Let $\lambda \in X^+(T)$ be a typical weight. There is a filtration of $G$-modules

$$V(\lambda) = V_\psi(\lambda)^0 \supset V_\psi(\lambda)^1 \supset \cdots$$

such that

1. $V(\lambda)/V_\psi(\lambda)^1 \cong L(-w_0\lambda + 2\rho)$.
2. The following sum formula holds

$$\sum_{i>0} \chi V_\psi(\lambda)^i = \sum_{\alpha \in \Phi^+_0} \sum_{0 < mp < (\lambda + \rho_0, \alpha^\vee)} \nu_p(mp) \chi(r_{a,mp}, \lambda) +$$

$$+ \sum_{\beta_i \in \Phi^+_1 : p(\lambda + \rho, \beta_i)} (\Xi_i \mathcal{W}(\lambda_{i-1}) + \Xi_{i-1} \sum_{k>0} (-1)^k \mathcal{W}(\lambda_{i-1} + k\beta_i)).$$

**Proof.** The first statement follows from Lemma 7.2, Corollary 7.10 and (8.4) along with Proposition 7.10(2). The second one follows from the above two propositions.

\[ \square \]

**Remark 8.5.** When $\lambda$ is $p$-typical, i.e. $(\lambda + \rho, \beta_i) \not\equiv 0 \mod p$ for $1 \leq i \leq mn$, we have

$$\sum_{i>0} \chi V_\psi(\lambda)^i = \sum_{\alpha \in \Phi^+_0} \sum_{0 < mp < (\lambda + \rho_0, \alpha)} \nu_p(mp) \chi(r_{a,mp}, \lambda),$$

and $V(\lambda)/V_\psi(\lambda)^1 \cong L(\lambda)$ (cf. Remark 7.3). In this case, the statement is the same as in the case of reductive algebraic groups (see [10, §II.8.19]).

9. **Kac modules and realizations of $p$-typical irreducible modules**

9.1. **Kac modules.** Let $L_{ev}(\lambda)$ denote the socle of $H^0_{ev}(\lambda)$. Considering the closed subgroup scheme $B^+G_{ev}$ of $G$, we define a $\text{Dist}(G)$-module $\text{Dist}(G) \otimes_{\text{Dist}(B^+G_{ev})} L_{ev}(\lambda)$ with trivial $\text{Dist}(U^+)_1$-action on $L_{ev}(\lambda)$, which is called a Kac module. We denote it by $\mathcal{K}(\lambda)$. Here $\text{Dist}(\otimes)$ stands for the distribution algebra of a (super)group scheme $\otimes$ (see [10], or [4] for details). Recall that the category of finite-dimensional rational $G$-modules is equivalent to the category of finite-dimensional integrable $\text{Dist}(G)$-modules (see [4, Corollary 3.5]). Here by an integrable $\text{Dist}(G)$-module $M$ it means there is a $T$-module structure on $M$ compatible with $\text{Dist}(G)$-module structure (see $\text{Dist}(G)$-$T$-modules for algebraic groups in [10, Page 171]). So both will be identified in the following.

**Lemma 9.1.** There is a nontrivial homomorphism from $\mathcal{K}(\lambda)$ onto $L(-w_0\lambda)^*$ for any $\lambda \in X^+(T)$. This homomorphism becomes an isomorphism if and only if $L(-w_0\lambda)^*$ has weight space of $w_0\lambda - 2\rho$. 

Proof. It is readily seen that $G$ has a closed subgroup scheme $B^{(mn)}G_{ev}$. Now we exploit the structure theory of algebraic supergroups of Chevalley type (see [8 §5]). As mentioned in the proof of Theorem [2.6.1] where we have already a normal subgroup functor $L_1$ of $G$, it can be shown that both $B^{(mn)}G_{ev}$ and $B^{(mn)}$ have a normal subgroup functor $L_1$ such that

$$B^{(mn)}G_{ev}/L_1 \cong G_{ev} \text{ and } B^{(mn)}/L_1 \cong B_{ev}. $$

Such a normal subgroup functor $L_1$ can be described as $L_1(R) = B^{(mn)}(R) \cap G_1(R)$ for any $R \in \text{salg}_k$, where $G_1(R)$ is the same as in the proof of Theorem [2.6.1]. By the same arguments as in the proof of [32 Lemma 10.4], a super analogue of the result [10 Proposition 6.11] yields the following isomorphism of $B^{(mn)}G_{ev}$-modules

$$H^0(B^{(mn)}G_{ev}/B^{(mn)}, \lambda) \cong H^0(G_{ev}/B_{ev}, \lambda).$$

Recall that $H^0(G/B^{(mn)}, \lambda)$ is finite-dimensional (see Theorem [2.7]), and it has a $B^{(mn)}G_{ev}$-submodule $H^0(B^{(mn)}G_{ev}/B^{(mn)}, \lambda) \cong H^0(G_{ev}/B_{ev}, \lambda)$ which can be regarded a $\text{Dist}(B^{(mn)}G_{ev})$-module with $\text{Dist}(B^{(mn)1})$-trivial action. Note that $\text{Dist}(U^{(mn)})_1 = \text{Dist}(U^+)_1$. By a classical result of reductive algebraic groups, $H^0(G_{ev}/B_{ev}, \lambda)$ has simple socle $L_{ev}(\lambda)$. So the socle $L_{ev}(\lambda)$ of $H^0(B^{(mn)}G_{ev}/B^{(mn)}, \lambda)$ is actually an irreducible $\text{Dist}(B^{(mn)}G_{ev})$-module with trivial $\text{Dist}(U^+_1)$-action. This one is actually an irreducible $\text{Dist}(B^+G_{ev})$-irreducible module with trivial $\text{Dist}(U^+_1)$-action. On the other side, this $L_{ev}(\lambda)$ has one-dimensional $B^+_1$-highest weight space of weight $\lambda$. By the same arguments as in the proof of Lemma [6.6] it is known that there is one-dimensional subspace

$$H^0(G/B^{(mn)}, \lambda) = L_{B^{(mn)}}(\lambda) = H^0(G/B^{(mn)}, \lambda)B^{(mn)+}. $$

Hence the $G$-submodule generated by this $L_{ev}(\lambda)$ in $H^0(G/B^{(mn)}, \lambda)$ is exactly $L_{B^{(mn)}}(\lambda)$. Note that $\text{Dist}(G) = \text{Dist}(U^-)_1 \text{Dist}(B^+G_{ev})$. This means, $L_{B^{(mn)}}(\lambda) = \text{Dist}(U^-)_1 L_{ev}(\lambda)$. By the universality of tensor products, there is a nontrivial homomorphism from $\mathcal{X}(\lambda)$ onto $L_{B^{(mn)}}(\lambda)$ sending $1 \otimes L_{ev}(\lambda)$ to the socle of $H^0(B^{(mn)}G_{ev}/B^{(mn)}, \lambda)$. Keep it in mind that $L_{B^{(mn)}}(\lambda) \cong L(-w_0\lambda)^*$ (see Lemma [7.1]). The first part is proved.

For the second part, consider $\mathfrak{g} = \text{Lie}(G)$ which has a direct-sum decomposition of root spaces $\mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with $\mathfrak{h} = \text{Lie}(T)$. By the routine arguments for highest weight categories, $\mathcal{X}(\lambda)$ has a unique maximal submodule $N$ which is the direct sum of all proper submodules. Note that as a vector space, $\mathcal{X}(\lambda) = \text{Dist}(U^-)_1 \otimes L_{ev}(\lambda)$. The latter is exactly equal to $\wedge^* (\sum_{\beta \in \Phi^+} \mathfrak{g}_{-\beta}) \otimes L_{ev}(\lambda)$ (see Lemma [4 Lemma 3.1]). If $N$ is nonzero, by some trivial but a little tedious arguments $N$ must contain one-dimensional $B^+$-lowest weight space $\mathcal{X}(\lambda)_{w_0\lambda-2\rho_1}$. In this case, the irreducible quotient of $\mathcal{X}(\lambda)$ does not contain nonzero weight space of weight $\lambda_{mn}$. Hence $\mathcal{X}(\lambda)$ is irreducible if and only if $L(\lambda)$ does not contain nonzero weight space of weight $w_0(\lambda) - 2\rho_1$. The second statement follows. \hfill $\square$

Remark 9.2. Note that $B^{(mn)} = w_1(B)$ for $G = \text{GL}(m|n)$. It is not hard to see for $G = \text{GL}(m|n)$ or $G = \text{OSp}(m|2n)$, $w_1(B)G_{ev} = B^+G_{ev}$, and $\mathcal{X} \cong \text{ind}_{B^+,G_{ev}}^G L_{ev}(\lambda)$
by using the structural properties of $\text{Dist}(G)$ (see [4], [25]) and of algebraic supergroups of Chevalley type (see [8], §5.3). So one can expect an alternative definition of a Kac module $\mathcal{X}(\lambda)$ for basic classical supergroups via induced modules.

9.2. An application.

**Theorem 9.3.** The following statements are equivalent for $G = \text{GL}(m|n)$ over $k$.

1. $\lambda \in X^+(T)$ is $p$-typical.
2. $L(\lambda) \cong L(-w_0\lambda + 2\rho_1)^*$.
3. The irreducible module $L(\lambda)$ is isomorphic to $\mathcal{X}(\lambda - 2\rho_1)$

**Proof.** Note that (1) is equivalent to say $L(\lambda) = L_B(mn)(\lambda_{mn})$. The latter is equivalent to say $L(\lambda)$ has one-dimensional $B^+$-lowest weight space of weight $\lambda_{mn}$. So the equivalence of the first two follows from Lemma 4.2 and Remark 7.3.

By the above arguments, the equivalence of the second and third ones is deduced from Lemma 9.1. □

**Remark 9.4.** This theorem can be regarded a modular version of the result on Kac module realization of typical irreducible modules over complex numbers (see Remark 5.3).

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