CONFORMALLY EXACT METRIC AND DILATON IN STRING THEORY ON CURVED SPACETIME *

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ABSTRACT

Using a Hamiltonian approach to gauged WZW models, we present a general method for computing the conformally exact metric and dilaton, to all orders in the $1/k$ expansion, for any bosonic, heterotic, or type-II superstring model based on a coset $G/H$. We prove the following relations: (i) For type-II superstrings the conformally exact metric and dilaton are identical to those of the non-supersymmetric *semi-classical* bosonic model except for an overall renormalization of the metric obtained by $k \rightarrow k - g$. (ii) The exact expressions for the heterotic superstring are derived from their exact bosonic string counterparts by shifting the central extension $k \rightarrow 2k - h$ (but an overall factor $(k - g)$ remains unshifted). (iii) The combination $e^\Phi \sqrt{-G}$ is independent of $k$ and therefore can be computed in lowest order perturbation theory as required by the correct formulation of a conformally invariant path integral measure. The general formalism is applied to the coset models $SO(d-1,2)_-k/SO(d-1,1)_-k$ that are relevant for string theory on curved spacetime. Explicit expressions for the conformally exact metric and dilaton for the cases $d = 2, 3, 4$ are given. In the semiclassical limit $(k \rightarrow \infty)$ our results agree with those obtained with the Lagrangian method up to 1-loop in perturbation theory.

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1. Introduction

During the past year there has been extensive investigations of curved space-time string backgrounds generated by non-compact cosets \( G/H \). All models with space-time dimension \( d \leq 4 \) require the non-compact current algebra coset \( SO(d - 1, 2)/SO(d - 1, 1)_{-k} \) as part of, or as the full, conformal field theory \([1]\). The action is written in the form of a gauged WZW model \([2]\). For models involving more than four space-time coordinates there are other possibilities which have been classified \([3]\), but so far not investigated. The semi-classical analysis \([4]\) for \( k \to \infty \) has shown that these are useful models for learning more about string and particle propagation in gravitational singular spaces such as black holes and more interesting singularities in various dimensions. By now essentially all models up to dimension four have been subjected to the semi-classical analysis \([5]\) \([6]\) \([7]\) \([8]\) \([9]\) \([10]\) \([11]\) \([12]\). A cosmological interpretation has also been found \([13]\) \([14]\) \([15]\). A group theoretical method for the global analysis of these semi-classical geometries, including an explicit solution of the geodesics, has been formulated and explicitly applied to some cases \([14]\).

As in \([8]\) heterotic and type-II superstring actions can be constructed in exactly 4 dimensions in the form of \( N = 1 \) superconformal gauged WZW model. We believe that a heterotic string model of this type, perhaps with some variations, taken with a cosmological interpretation, provides the kind of setting suitable for a discussion of the physics of the early Universe in the context of string theory.

The principal method of semi-classical investigation followed ref. \([4]\) that used a Lagrangian method. Quantum corrections which were necessary to obtain the dilaton and satisfy the perturbative equations for conformal invariance \([16]\) were limited to one loop. In practical terms one cannot carry out the quantum computation of the sigma-model like theory to all orders with this method. However, the main interest in these models stem from the fact that they are conformally exact current algebra theories, which are in principle exactly solvable quantum theories. In order to take advantage of this fact it is desirable to go back to the Hamiltonian method and use the algebraic properties of the current algebra. The model can then be investigated via the coset methods for non-compact current algebras \([17]\) \([18]\).

In this paper we will show how to use the Hamiltonian approach to compute the gravitational metric and dilaton backgrounds to all orders in the quantum theory (all orders in the central extension \( k \)). These will then provide a more accurate representation of the
conformally exact vacuum configuration of the string at the “classical” level (i.e. no string loops). We have managed to obtain these quantities for bosonic, type-II supersymmetric, and heterotic string theories in $d \leq 4$.

The main idea is the following: the conformally exact Hamiltonian is the sum of left and right Virasoro generators $L_0^L + L_0^R$ that may be written purely in terms of Casimir operators of $G$ and $H$ at the tachyon level. The exact dependence on the central extension $k$ is included in this form. If we investigate the exact quantum eigenstates in configuration space, then the Casimir operators become Laplacians constructed as differential operators in group parameter space ($\dim G$). If the state $\psi$ is a singlet under the gauge group $H$ (acting simultaneously on left and right movers), then gauge invariance requires that it is a function of singlet combinations of group parameters. There are exactly $\dim(G/H)$ such invariants which we choose as our string coordinates $X^a$. We have recently shown [14] that these invariants provide a global description of the geometry. In this way we can write the conformally exact Hamiltonian $L_0^L + L_0^R$ as a differential operator in the global curved space-time manifold involving only the string coordinates $X^a$. By comparing to the expected general form $(L_0^L + L_0^R)\psi = \frac{1}{e^{\Phi} \sqrt{-G}} \partial_a (e^{\Phi} \sqrt{-G} G^{ab} \partial_b \psi)$ for the singlet $\psi$, we read off the exact global metric and dilaton.

We have applied this program to the general bosonic, heterotic and type-II superstrings and derived relationships among the exact quantities of these theories as announced in the abstract of this paper. For the specific cosets of interest $SO(d-1,2)/SO(d-1,1)$ explicit expressions are given below. The large $k$ limit of our results agree with the previous semi-classical computations. In the special case of two dimensions we also agree with another previous derivation of the exact metric and dilaton for the $SO(2,1)/SO(1,1)$ bosonic string [19].

2. **Algebraic formalism for computing the exact metric and dilaton**

Let us consider a bosonic string theory for closed strings in $d$ curved space-time dimensions, based on a sigma model conformal field theory with string coordinates $X^a, \ a = 0, 1 \cdots d -1$. The space-time metric and dilaton fields are $G_{ab}(X)$ and $\Phi(X)$

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1 The corresponding results are also given for a particle theory whose WZW-like action was defined in [14]. The particle theory can be thought of as a string shrunk to a point which has no interactions with string excitations. For this case the semi-classical result is actually exact.
respectively. We begin with the effective action for the tachyon field $T(X)$ in $d$ space-time dimensions. The most general form of this effective action is

$$S[T] = \int d^d X \sqrt{-G} e^\Phi (G^{ab} \partial_a T \partial_b T - V(T))$$

$$(2.1)$$

where $V(T)$ is the tachyon potential whose precise form is not necessary for the analysis that follows. From the point of view of conformal field theory the tachyon is completely defined through the action of the zero modes $L_0^L$ and $L_0^R$ of the stress tensors for the left and the right movers. Therefore (2.1) must be equivalent to the following action

$$S[T] = \int d^d X (T(L_0^L + L_0^R)T - V(T)) .$$

(2.2)

Comparison of (2.1) with (2.2) determines the form of $L_0^L + L_0^R$ as a differential operator in configuration space

$$(L_0^L + L_0^R)T = -\frac{1}{e^\Phi \sqrt{-G}} \partial_a (G^{ab} e^\Phi \sqrt{-G} \partial_b T) .$$

(2.3)

Now let us consider the sigma model like action which results from an exact conformal theory based on the gauged WZW action. Using the equivalent current algebra coset model $G/H$ we can write $L_0^L$ in terms of the quadratic Casimir operators $\Delta_L^G$ and $\Delta_L^H$ for the group and the subgroup, as follows

$$L_0^L T = (\frac{\Delta_L^G}{k-g} - \frac{\Delta_L^H}{k-h}) T$$

$$(2.4)$$

where $J^G_L$ and $J^H_L$ are antihermitian group and subgroup generators obeying the appropriate Lie algebras, and $g$, $h$ are the Coxeter numbers for the group and the subgroup. For the cases of interest in this paper $g = d-1$, $h = d-2$ for $d \geq 3$, and $g = 2$, $h = 0$ for $d = 2$.\footnote{For the particle theory of footnote 1 the Hamiltonian contains no Coxeter numbers since the higher string excitations are absent. Then $L_0^L = (\Delta_L^G - \Delta_L^H)/k$.} An expression similar to (2.4) can also be written for $L_0^R$. As shown below, we construct the generators $J^G_L, J^G_R, J^H_L, J^H_R$ as first order differential operators acting on
group parameter space. Then the Casimir operators $\Delta_G^L, \Delta_H^L, \Delta_G^R, \Delta_H^R$, contain single and double derivatives with respect to all $\text{dim}G$ parameters in $G$. Gauging the subgroup $H$ means that we have to impose the following gauge invariance conditions on the tachyon $T$.

\[ (J_H^L + J_H^R) T = 0 . \] (2.5)

The number of conditions is $\text{dim}H$ and therefore $T$ can depend only on $d = \text{dim}(G/H)$ parameters, $X^a$ (string coordinates), which are $H$-invariants. The fact that there are exactly $\text{dim}(G/H)$ such independent invariants is not immediately obvious but it should become apparent to the reader by considering a few specific examples. As discussed in [14] these are in fact the coordinates that globally describe the sigma model geometry. Consequently, using the chain rule, we reduce the derivatives in (2.4) to only derivatives with respect to the $d$ string coordinates $X^a$. Moreover, using the fact that $\Delta_G^L = \Delta_G^R$ for any group, together with the fact that the gauge invariance condition (2.3) leads to $(\Delta_H^L - \Delta_H^R) T = 0$ (see (2.14) below), we ensure the physical condition for closed bosonic strings $(L_0^L - L_0^R) T = 0$. Then using (2.3) and (2.4) one can deduce uniquely the expression for the inverse metric $G^{ab}$ by comparing the coefficients of the double derivatives $\partial_a \partial_b T$. Comparison of the single derivative terms $\partial_a T$ will give a system of $d$ coupled linear partial differential equations, whose solution determines the dilaton field $\Phi$.

The general $k$ dependence of the exact expressions takes a particular form that can be seen as follows. In the large $k$ limit (2.4) becomes proportional to $\frac{1}{k} (\Delta_G^L - \Delta_H^L)$ from which we can read off the semi-classical metric and dilaton according to the above procedure. Therefore, we may rewrite (2.4) in the following form

\[ L_0^L T = \frac{1}{k-g}(\Delta_G^L + g \Delta_H^L) T \] (2.6)

3 Since we have defined our Casimir operators as the square of antihermitian generators we differ by a minus sign from usual conventions. For example for $SU(2)$ we would get the eigenvalues $\Delta_G = -j(j+1)$ instead of $+j(j+1)$.

4 If $H$ contains an abelian $U(1)$ or $\mathbb{R}$ factor there is the alternative of imposing the axial gauging condition $(J_H^L - J_H^R) T = 0$ for the currents associated with the abelian factor. For an application see [20].

5 This follows from $J_G^R = -g^{-1} J_G^L g + \frac{1}{D} Tr(g^{-1} J_G^L g)$, where $D$ is the dimension of the matrix $g$. The second term is present because $g$ and $J_G^L$ do not commute as quantum operators and $J_G^R$ has to be traceless. But despite its presence the relation $\Delta_G^L = \Delta_G^R$ is derived from it.
where $\Delta_{G/H}^L = \Delta_{G}^L - \Delta_{H}^L$. Then it is evident that, except for the overall factor $(k - g)$, all dependence on $k$ has the form $\frac{a - h}{k}$. This applies to the bosonic string. For the heterotic and type-II superstrings the $k$ dependence can be derived by the same technique as will be seen below. It is evident that for the particle theory of footnotes 1,2 there are no such corrections to the semi-classical result.

Let us specialize to the coset models $SO(d - 1, 2)/SO(d - 1, 1)$ with $d = 2, 3, 4$ since these are the ones of interest for a theory in four dimensions. We want to find the currents appropriate for right or left transformations of the group elements of $SO(d - 1, 2)$ in a $SO(d - 1, 1)$ basis. It is convenient to parametrize the group element of $SO(d - 1, 2)$ as the product $g = h t$, where $h \in SO(d - 1, 1)$ and $t \in SO(d - 1, 2)/SO(d - 1, 1)$. The $h, t$ are given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & h^\mu_\nu \end{pmatrix}, \quad t = \begin{pmatrix} b \quad (b + 1)x^\nu \\ -(b + 1)x_\mu \quad (\eta_\mu^\nu - (b + 1)x_\mu x^\nu) \end{pmatrix}; \quad (2.7)$$

Furthermore $h$ can be written in the form $h^\mu_\nu = [(1 + a)(1 - a)^{-1}]^\mu_\nu$, with $a_{\mu\nu} = -a_{\nu\mu}$ when both indices are lowered. To insure that $t$ is a $SO(d - 1, 2)$ group element we take $b = \frac{1 - x^2}{1 + x^2}$. By considering the infinitesimal left transformations $\delta_L g = \epsilon_L g$ we can read off the form of the generators

$$J^L_\mu^\nu = \frac{1}{2}(1 + a)_{\mu\alpha}(1 + a)_{\nu\beta} \frac{\partial}{\partial a_{\alpha\beta}}$$

$$J^L_\mu^\nu = -\frac{1}{2}(1 + x^2)(\frac{1 + a}{1 - a})_{\mu}^\nu \frac{\partial}{\partial x^\nu} + \frac{1}{2}(1 + a)_{\mu\alpha}(1 + a)_{\beta\gamma}x^\gamma \frac{\partial}{\partial a_{\alpha\beta}}. \quad (2.8)$$

It can be shown that the above generators obey the commutation rules of the $SO(d - 1, 2)$ algebra. Namely

$$[J^L_\mu^\nu, J^L_{\alpha\beta}] = J^L_{\mu\alpha}(\eta_{\nu\beta} - \eta_{\nu\beta}) + J^L_{\nu\beta}(\eta_{\mu\alpha} - \eta_{\mu\alpha})$$

$$[J^L_\mu^\nu, J^L_\nu] = \eta_{\mu\alpha}J^L_\alpha - \eta_{\nu\alpha}J^L_\mu$$

$$[J^L_\mu, J^L_\nu] = J^L_{\mu\nu}. \quad (2.9)$$

If we consider instead, the infinitesimal right transformations $\delta_R g = g \epsilon_R$ we find the following expressions

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6 We follow the notation of [14]. As explained there, to compare with [14] where another vector $X^\mu$ was used, one should set $X^\mu = 2x^\mu/(x^2 - 1)$. These Lorentz vectors should not be confused with the Lorentz invariant string coordinates $X^a$ even though they have the same dimension $d = \text{dim}(G/H)$. 

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\[ J_{\mu\nu}^R = -\frac{1}{2} (1-a)_{\mu\alpha} (1-a)_{\nu\beta} \frac{\partial}{\partial a_{\alpha\beta}} - x[\mu \frac{\partial}{\partial x^\nu}] \]
\[ J_{\mu}^R = \frac{1}{2} (x^2 - 1) \frac{\partial}{\partial x^\mu} - x^{\mu} x^{\nu} \frac{\partial}{\partial x^\nu} - \frac{1}{2} (1-a)_{\mu\alpha} (1-a)_{\nu\beta} x^\gamma \frac{\partial}{\partial a_{\alpha\beta}}. \] \quad (2.10)

These currents obey the same commutation rules as in (2.9) and moreover commute with the left currents \([J^L, J^R] = 0\). Now we construct the quadratic Casimir associated with the left and right currents. We find

\[ \Delta^L_G = \frac{1}{2} (J^L)_{\mu\nu} (J^L)^{\mu\nu} + (J^L)_{\mu} (J^L)^{\mu} \]
\[ = \frac{1}{4} (1 + x^2)^2 \frac{\partial^2}{\partial x^{\mu} x^\mu} - \frac{d - 2}{2} (1 + x^2) x^\mu \frac{\partial}{\partial x^\mu} \]
\[ + \frac{1}{4} (1 + a)_{\mu\gamma} x^\gamma (1 + a)_{\alpha\delta} x^\delta (1 - a^2)_{\nu\beta} \frac{\partial^2}{\partial a_{\mu\nu} \partial a_{\alpha\beta}} \]
\[ + \frac{1}{2} (1 + a)_{\mu\alpha} x^\alpha (1 - a^2)_{\nu\beta} \frac{\partial}{\partial a_{\mu\nu}} \]
\[ + \frac{1}{2} (1 + x^2)^2 (1 + a)_{\mu\alpha} x^\alpha (1 + a)_{\nu\beta} \frac{\partial^2}{\partial a_{\mu\nu} \partial x^\beta} \]
\[ + \frac{1}{8} (1 - a^2)_{\mu\alpha} (1 - a^2)_{\nu\beta} \frac{\partial^2}{\partial a_{\mu\nu} \partial a_{\alpha\beta}} + \frac{1}{4} (a - a^3)_{\mu\nu} \frac{\partial}{\partial a_{\mu\nu}}. \] \quad (2.11)

The expression for \(\Delta^R_G\) is identical for reasons explained in footnote 5. The quadratic Casimir operators corresponding to the subgroup \(H = SO(d-1,1)\) are

\[ \Delta^L_H = \frac{1}{2} (J^L)_{\mu\nu} (J^L)^{\mu\nu} \]
\[ = \frac{1}{8} (1 - a^2)_{\mu\alpha} (1 - a^2)_{\nu\beta} \frac{\partial^2}{\partial a_{\mu\nu} \partial a_{\alpha\beta}} + \frac{1}{4} (a - a^3)_{\mu\nu} \frac{\partial}{\partial a_{\mu\nu}}. \] \quad (2.12)

and

\[ \Delta^R_H = \frac{1}{2} (J^R)_{\mu\nu} (J^R)^{\mu\nu} \]
\[ = \frac{1}{8} (1 - a^2)_{\mu\alpha} (1 - a^2)_{\nu\beta} \frac{\partial^2}{\partial a_{\mu\nu} \partial a_{\alpha\beta}} + \frac{1}{4} (a - a^3)_{\mu\nu} \frac{\partial}{\partial a_{\mu\nu}} \]
\[ + (x^2 \eta_{\mu\nu} - x^{\mu} x^{\nu}) \frac{\partial^2}{\partial x^\mu \partial x^\nu} - (d - 1) x^{\mu} \frac{\partial}{\partial x^\mu} + (1 + a)_{\mu\alpha} x^\alpha (1 + a)_{\nu\beta} \frac{\partial^2}{\partial a_{\mu\nu} \partial x^\beta}. \] \quad (2.13)

The two expressions differ by the last line in (2.13) which equals to
\[ \Delta^R_H - \Delta^L_H = \frac{1}{2} ((J^R)_{\mu\nu} - (J^L)_{\mu\nu}) ((J^R)_{\mu\nu} + (J^L)_{\mu\nu}) . \] (2.14)

Using the expressions (2.8) (2.10) the gauge invariance conditions (2.5) take the form

\[ (a_{[\mu\lambda} \frac{\partial}{\partial a_{\lambda\nu]} } - x_{[\mu} \frac{\partial}{\partial x_{\nu]}} ) T = 0 , \] (2.15)

where the brackets indicate antisymmetrization of the \( \mu, \nu \) indices. This form is recognized as the global Lorentz generator and it requires that \( T \) be constructed only from Lorentz invariants that can be formed from \( x_\mu \) and \( a_{\mu\nu} \). Next we specialize to the cases \( d = 3 \) and \( d = 4 \). The \( d = 2 \) case corresponding to the 2d black hole is discussed in the Appendix.

3. 3d Bosonic String

The 3d model based on the coset model \( SO(2, 2)/SO(2, 1) \) was discussed semi-classically from the gauged WZW model Lagrangian point of view in [3][4][5]. This model may be viewed as the 3d submanifold of a four dimensional model which is constructed by adjoining a factor of \( U(1) \) or \( \mathbb{R} \) to the coset. The global structure of the 3d manifold was analyzed in [14] by finding the global coordinates and examining the particle trajectories. In particular, it was found that the space consists of two topologically distinct sectors. There is a curvature singularity with the topology of “pinched double trousers” in one sector and that of a “double saddle” in the other. In this case the antisymmetric matrix \( a_{\mu\nu} \) has three parameters, therefore it is possible to reparametrize it in terms of a three dimensional vector \( y^\mu \), as \( a_{\mu\nu} = \epsilon_{\mu\nu\lambda} y^\lambda \). Then the gauge condition (2.15) takes the form

\[ (y_{[\mu} \frac{\partial}{\partial y_{\nu]}} + x_{[\mu} \frac{\partial}{\partial x_{\nu]}} ) T = 0 . \] (3.1)

The constraint (3.1) requires that \( T \) depend only on the three Lorentz invariants \( x^2, y^2, x \cdot y \), or their combinations. In fact, in order to make correspondence with previous results we choose the same invariants as in [14] with \( T(v, u, b) \) where

\[ b = \frac{1 - x^2}{1 + x^2}, \quad v = \frac{2}{1 + y^2}, \quad u = -2 \frac{(x \cdot y)^2}{x^2(1 + y^2)} . \] (3.2)

Using the chain rule we transform the derivatives with respect to the vectors \( x^\mu \) and \( y^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda} a_{\nu\lambda} \) in (2.11) (2.12) to derivatives with respect to the \( H \)-invariants \( v, u, b \), e.g.

\[ \frac{\partial}{\partial x_{\mu}} T = 2v(x \cdot y)/x^4((x \cdot y)x^\mu - x^2y^\mu) \frac{\partial}{\partial u} T - (b + 1)^2 x^\mu \frac{\partial}{\partial b} T, \] etc. Finally, with the dot
products in the Laplacians, \( L_0 \) is written only in terms of \((v, u, b)\) when acting on \( T \). Then comparison of the double derivatives terms in (2.3) and (2.4) gives the non-zero elements of the inverse of the metric (we omit an overall factor of \( \frac{1}{2(k-2)} \) which will be restored later in (3.5))

\[
G^{bb} = 4(b^2 - 1) \\
G^{vv} = -4 \frac{b-1}{b+1} v(v - u - 2) + \frac{4}{k-1} v(v - 2) \\
G^{uu} = \frac{4}{b-1} u(v - u - 2) + \frac{4}{k-1} u(u + 2) \\
G^{vu} = \frac{4}{k-1} vu ,
\]

and comparison of the single derivative terms yields a system of linear partial differential equations which determine the dilaton

\[
\frac{\partial}{\partial b} \ln (\sqrt{-G} e^\Phi) = \frac{b}{b^2 - 1} \\
\frac{\partial}{\partial v} (G^{vv} \sqrt{-G} e^\Phi) + \frac{\partial}{\partial u} (G^{uv} \sqrt{-G} e^\Phi) = 2 \sqrt{-G} e^\Phi \left[ \frac{b-1}{b+1} (u + 2 - 3v) + \frac{4v - 2}{k-1} \right] \\
\frac{\partial}{\partial u} (G^{uu} \sqrt{-G} e^\Phi) + \frac{\partial}{\partial v} (G^{vu} \sqrt{-G} e^\Phi) = 2 \sqrt{-G} e^\Phi \left[ \frac{b+1}{b-1} (v - 2 - 3u) + \frac{4u + 2}{k-1} \right] .
\]

Note that without a dilaton these equations have no solutions. Therefore, even without the hindsight of general string arguments, a dilaton must be introduced in our approach in order to have a solution to these equations. If we invert the inverse metric we get for the line element the following expression (we also restore the overall factor \( 2(k-2) \))

\[
ds^2 = 2(k-2)(G_{bb} db^2 + G_{vv} dv^2 + G_{uu} du^2 + 2G_{vu} dv du) .
\]

with

\[
G_{bb} = \frac{1}{4(b^2 - 1)} \\
G_{vv} = -\frac{\beta(v, u, b)}{4v(v - u - 2)} \left( \frac{b+1}{b-1} + \frac{1}{k-1} \frac{u+2}{v-u-2} \right) \\
G_{uu} = \frac{\beta(v, u, b)}{4u(v - u - 2)} \left( \frac{b-1}{b+1} - \frac{1}{k-1} \frac{v-2}{v-u-2} \right) \\
G_{vu} = \frac{1}{4(k-1)} \frac{\beta(v, u, b)}{(v - u - 2)^2} ,
\]

and where the function \( \beta(v, u, b) \) is defined as follows
\[ \beta^{-1}(v,u,b) = 1 + \frac{1}{k-1} \frac{1}{v-u-2} \left( \frac{b-1}{b+1}(u+2) - \frac{b+1}{b-1}(v-2) - \frac{2}{k-1} \right). \quad (3.7) \]

It remains to solve the system of differential equations (3.4). Although the solution to those equations is straightforward, it is illuminating to guess the solution by recalling some of the results of [7][8]. There it was found that conformal invariance required a path integral measure for gauged WZW models that includes, in addition to the Haar measure for the group \( G \), an extra purely group theoretical factor which must be \( H \)-invariant.

Moreover, it was observed that this factor was equal to \( 1/(e^{\Phi} \sqrt{-G}) \). This result was true in the semi-classical limit \( k \to \infty \). Because of the group theoretic nature of the derivation it was conjectured that the combination \( e^{\Phi} \sqrt{-G} \) is \( k \)-independent and equal to the 1-loop result, although individually the metric and the dilaton can receive \( 1/k \) corrections. If the conjecture is not correct it implies that the path integral measure for the gauged WZW model is not defined once and for all at the outset, before one begins a perturbative analysis. In the notation of [14] the one loop semi-classical result is

\[ e^{\Phi} \sqrt{-G} \bigg|_{k \to \infty} = \left( \frac{b^2 - 1}{vu} \right)^{1/2}. \quad (3.8) \]

One can now check that this expression indeed satisfies the system of differential equations (3.4) for all values of \( k \). Therefore, the conjecture is correct as we expected on the basis of a well defined path integral quantization of the model. So we have proven the theorem

\[ e^{\Phi} \sqrt{-G} = e^{\Phi} \sqrt{-G} \bigg|_{k \to \infty} \quad \text{for all} \quad k. \quad (3.9) \]

This is also true in all string and superstring models we consider in the present paper. We are convinced that this is a general feature of gauged WZW models. After calculating \( \sqrt{-G} \) from (3.3) or (3.6) the result for the dilaton is the following

\[ \Phi = \ln \left( \frac{(b^2 - 1)(v-u-2)}{\sqrt{\beta(v,u,b)}} \right) + \Phi_0, \quad (3.10) \]

\footnote{The appropriate factor can be obtained in a patch of the manifold by choosing a unitary gauge. It must be designed to cancel the Haar measure and the Faddeev-Popov determinant so that the dilaton is associated only with the determinant that emerges from the integration of the gauge fields. This dilaton then solves the one-loop conformal invariance conditions. The extra factor is extended from the patch in the unitary gauge to the \( H \)-invariant global manifold by the methods of [14].}
where $\Phi_0$ is the constant of integration. In the limit $k \to \infty$, $\beta \to 1$ both the metric and the dilaton tend to their semi-classical expressions of $[14]$.

One might ask the question: how does the finite value of $k$ modify the manifold? In figures 1.ab, 2.ab, and 3.ab the allowed regions in the $v - u$ plane are indicated at fixed values of $b$. As in $[14]$, the three signs inside the parentheses in the various regions are the signs of the coefficients of $dv^2$, $du^2$ and $db^2$ in the semi-classical metric. A minus (plus) sign corresponds to a time (space) coordinate, thus indicating the signature of the region. The regions with one time coordinate correspond to the $SO(2,2)/SO(2,1)$ coset. The remaining regions correspond to the analytic continuations $SO(3,1)/SO(3)$ $(+ + +)$, $SO(4)/SO(3,3)$ $(− − −)$, and $SO(3,1)/SO(2,1)$ (one plus). Thus, by specializing to each one of these regions our exact metric and dilaton describe those cosets as well. The $45^\circ$ line $u = v - 2$ is a curvature singularity in the semi-classical limit, the other two being at $b = \pm 1$.

The way the $b$-fixed planes are sliced up by the lines at $u = 0$, $v = 0$, $v - u - 2 = 0$ into regions of various signatures is a purely group theoretical result about the coset manifolds that are listed above. That is, the coset manifold $SO(2,2)/SO(2,1)$ lives in the ranges of $(v, u, b)$ parameters indicated in the figures, independently than any metric (similarly for the other manifolds). As it turns out, the full region coincides with the properties of the semi-classical metric. However, quantum corrections may require additional constraints on the acceptable regions in order to maintain the signature. This is indeed what happens, and how the $k$-dependence of the exact metric shows up. The second line, with varying slope (which depends on $b$) is a singularity of the function $\beta(v, u, b)$. For the case of the coset $SO(2,2)/SO(2,1)$ and $k > 2$ one must demand that $\beta(v, u, b) > 0$ so that the exact metric has the correct signature, as seen from the determinant of the metric (one time coordinate requires $detG < 0$). This leads to further restrictions for the allowed regions. The result is shown as the shaded areas in the figures: they have switched signature due to the quantum corrections. Therefore, although they were previously allowed, they are now off limits since a classical geodesic in the $SO(2,2)/SO(2,1)$ white regions cannot enter the shaded areas. Therefore quantum amplitudes are expected to decay off and tunnel in these regions. This implies that quantum effects have created a screening of the classical

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8 The signature depends crucially on the sign of $(k - 2)$ as seen from (3.5). Demanding $c = 26$ for the bosonic string gives $k \simeq 2.48$ or $k \simeq 0.91$, and $c = 15$ for the superstring gives $k = 20/7$. It is believed that a consistent quantum theory requires $k > 2$ [13]. Nevertheless one could perform a similar analysis even when $k < 1$.  

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singularity, although not everywhere. In particular, using the figures of [14], we indicate the 3-dimensional view of the screening that occurs. The completely screened singularities are: (i) the central blob of the double trouser singularity in region II, (ii) the pinched region and the $b \sim \pm 1$ neighborhoods on the inside of the trousers, and (iii) the entire sheets forming the saddle singularities in regions I’ and III’. The unscreened regions are: (i) Part of the inside of the trousers in regions I and III when $|b| > k/(k - 2)$, (ii) the farther out parts of the outside of the trousers in region II, and (iii) the region II’ between the saddles.

4. 4d Bosonic String

The 4d model based on the coset $SO(3,2)/SO(3,1)$ was analyzed in [8] where expressions for the perturbative metric and dilaton were given in some patches of the manifold. We will see that the method we have been following in the present paper leads to the discovery of the global coordinates as well. The four invariants one can construct and which satisfy the gauge condition (2.15) are

$$x^2, \quad z_1 = \frac{1}{4} Tr(a^2), \quad z_2 = \frac{1}{4} Tr(a^* a), \quad z_3 = x a^2 x/x^2,$$  \hspace{1cm} (4.1)

where $a^*_\mu^{\nu} = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} a^{\alpha \beta}$ is the dual of $a^{\mu \nu}$. However, the metric written in these coordinates is non-diagonal and very complicated. Instead we use a different set of four invariants $b, u, v, w$ for which the semi-classical metric is diagonal,

$$b = \frac{1 - x^2}{1 + x^2}, \quad u = \frac{1 + z_1^2 + 2(z_1 - z_3)}{1 - 2z_1 - z_2^2}, \quad v = \frac{1 + z_1 + \sqrt{z_1^2 + z_2^2}}{1 - z_1 - \sqrt{z_1^2 + z_2^2}}, \quad w = \frac{1 + z_1 - \sqrt{z_1^2 + z_2^2}}{1 - z_1 + \sqrt{z_1^2 + z_2^2}}.$$  \hspace{1cm} (4.2)

To find the ranges in which the above global coordinates take their values we consider a Lorentz frame that can cover all possibilities without loss of generality. First we notice that by Lorentz transformations the antisymmetric matrix $a_{\mu \nu}$ can always be transformed, as in [8], to a block diagonal matrix, with the non-zero elements

$$a_{01} = \tanh t \text{ or } \coth t, \quad a_{23} = \tan \phi.$$  \hspace{1cm} (4.3)
Then using (4.2) one can deduce the form of the global variables: $v = \pm \cosh 2t$, $w = \cos 2\phi$, and $u = \frac{1}{x^2} (w(x_0^2 - x_1^2) - v(x_2^2 + x_3^2))$. Therefore the string variables can take values in the following regions

region – I : \hspace{1cm} b^2 > 1 \ , \hspace{1cm} w < u < v \hspace{1cm} or \hspace{1cm} v < u < w (4.4)

region – II : \hspace{1cm} b^2 < 1 \ , \hspace{1cm} -\infty < u < w < v \hspace{1cm} or \hspace{1cm} v < w < u < +\infty .

In both regions $v^2 > 1$ and $-1 < w < 1$. Then by considering states of the type $T = T(b, u, v, w)$ and following a procedure similar to the 3d case we find the line element

$$\begin{align*}
ds^2 &= 2(k - 3) \left( G_{bb} b^2 + G_{uu} du^2 + G_{vv} dv^2 + G_{ww} dw^2 \\
&\quad + 2G_{uv} du dv + 2G_{uw} du dw + 2G_{vw} dv dw \right),
\end{align*}$$

where

$$\begin{align*}
G_{bb} &= \frac{1}{4(b^2 - 1)} \\
G_{uu} &= \frac{\beta(b, u, v, w)}{4(u - w)(v - u)} \left( \frac{b - 1}{b + 1} - \frac{1}{k - 2} \frac{(v - w)^2}{(v - u)(u - w)} \right) \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \right) \\
G_{vv} &= -\frac{(v - w) \beta(b, u, v, w)}{4(v^2 - 1)(v - u)} \left( \frac{b + 1}{b - 1} - \frac{1}{k - 2} \frac{1}{(v - u)(u - w)} \right) [1 - u^2] \\
&\quad + \left( \frac{b + 1}{b - 1} \right)^2 (v - u)(v - w) + \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{(1 + w^2)(u + v) - 2v(1 + uw)}{v - w} \\
G_{ww} &= \frac{(v - w) \beta(b, u, v, w)}{4(1 - w^2)(u - w)} \left( \frac{b + 1}{b - 1} - \frac{1}{k - 2} \frac{1}{(v - u)(u - w)} \right) [1 - u^2] \\
&\quad + \left( \frac{b + 1}{b - 1} \right)^2 (u - w)(v - w) - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{(1 + w^2)(u + v) - 2w(1 + uw)}{v - w} \\
G_{uv} &= \frac{\beta(b, u, v, w)}{4(k - 2)(v - u)^2} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{v - w}{u - w} \right) \\
G_{uw} &= \frac{\beta(b, u, v, w)}{4(k - 2)(u - w)^2} \left( 1 - \frac{1}{k - 2} \frac{b + 1}{b - 1} \frac{v - w}{v - u} \right) \\
G_{vw} &= \frac{1}{(k - 2)^2 b - 1} \frac{\beta(b, u, v, w)}{4(v - u)(u - w)},
\end{align*}$$

where the function $\beta(b, u, v, w)$ is defined by

$$\begin{align*}
\beta^{-1} &= 1 + \frac{1}{k - 2} \frac{(v - w)^2}{(v - u)(w - u)} \left( \frac{b + 1}{b - 1} + \frac{b - 1}{b + 1} \frac{1}{(v - w)^2} \right) \\
&\quad + \frac{1}{k - 2} \frac{vw + u(v + w) - 3}{(v - w)^2} \left( \frac{b + 1}{b - 1} \right)^2 + \frac{2}{(k - 2)^2 b - 1} \frac{b + 1}{b - 1} \frac{vw - 1}{(v - w)(u - w)}. (4.7)
\end{align*}$$
The dilaton field is

\[ \Phi = \ln \left( \frac{(b^2 - 1)(b - 1)(v - u)(w - u)}{\sqrt{\beta(b, u, v, w)}} \right) + \Phi_0. \]  

(4.8)

It would be instructive to write the expression for the metric in the semi-classical limit and verify that the range for the string parameters (4.4) is such that there is only one time coordinate in every region. From (4.8) we obtain

\[ \frac{ds^2}{2(k - 2)} \bigg|_{k \to \infty} = \frac{db^2}{4(b^2 - 1)} + \frac{b - 1}{b + 1} \frac{du^2}{4(v - u)(u - w)} + \frac{b + 1}{b - 1} \frac{dv^2}{4(1 - w^2)(u - w)} - \frac{dw^2}{4(v^2 - 1)(v - u)}. \]  

(4.9)

This metric was derived in [8] in region–I. It can be easily seen that in region–I of (4.4) “time” = v whereas in region–II “time” = b. The presence of a finite k modifies the manifold in a similar way to the 3d case, as considered in the previous section.

Again we can check that \( e^{\Phi} \sqrt{-G} \) is independent of k, which proves the theorem for \( d = 4 \).

### 5. Type–II and Heterotic superstrings

In this section we consider superconformal extensions of the bosonic models we have been considering in the previous sections. The general type–II and heterotic superstring model in curved space-time was defined in [8] in the form of supersymmetric \( N = 1 \) gauged WZW model. This corresponds to a Kazama-Suzuki model with a non-compact group \( SO(d - 1, 2) \) [1] and therefore can be analyzed with current algebra techniques [17][18].

The 4d case was worked out explicitly, to leading order in perturbation theory, using the Lagrangian method. As we shall see, the exact metric and dilaton for these superstring models are closely related to the corresponding expressions for the bosonic strings. It was pointed out in footnote 5 that for any WZW model \( \Delta^L_G = \Delta^R_G \equiv \Delta_G \). Restricting to H–invariant tachyon states, as we were instructed to do by (2.5), gives another condition \( \Delta^L_H T = \Delta^R_H T \equiv \Delta_H T \), or equivalently \( \Delta^L_{G/H} T = \Delta^R_{G/H} T \equiv \Delta_{G/H} T \). For the bosonic string we saw that these remarks led to the \( k \) dependence exhibited in (2.6). Now we turn to the superstrings.
For the type–II superstring the coset model with $N = 1$ superconformal symmetry ($N = 2$ if $G/H$ is kählerian) is described by \[ (5.1) \]

\[
\frac{SO(d-1,2)_{-k} \otimes SO(d-1,1)_1}{SO(d-1,1)_{-k+g-h}},
\]

for both the left and the right movers. For the tachyon the fermionic factor $SO(d-1,1)_1$ makes no contribution. However, the shifting of the level in the denominator in (5.1), i.e. $(-k + g - h) = -k + 1$ instead of $-k$ for the bosonic case, has a profound effect and we obtain the quantum Hamiltonian with the exact dependence on $k$

\[
(L^L_0 + L^R_0) T_{II} = \left( \frac{\Delta^L_G}{k-g} - \frac{\Delta^L_H}{k-g} + \frac{\Delta^R_G}{k-g} - \frac{\Delta^R_H}{k-g} \right) T_{II} \]

\[
= \frac{2}{k-g} \Delta_{G/H} T_{II} .
\]

Except for an overall renormalization of $k \rightarrow k - g$, this is exactly the expression in the semi-classical limit. Therefore, almost trivially we have proven a theorem: For any type-II superstring based on a Kazama-Suzuki coset, as in (5.1), the exact metric and dilaton are given by the 1–loop perturbative result except for the overall normalization of the metric!! In the special case of $d = 2$ available field theoretic perturbative computations verify this result up to 5 loops \[21\].

For a heterotic superstring only the left sector is supersymmetric whereas the right sector is not. Therefore, for the tachyon we can write

\[
(L^L_0 + L^R_0) T_{het} = \left( \frac{\Delta^L_G}{k-g} - \frac{\Delta^L_H}{k-g} + \frac{\Delta^R_G}{k-g} - \frac{\Delta^R_H}{k-h} \right) T_{het} + L^R_0(int)T_{het}
\]

\[
= \frac{2}{k-g} \left( \Delta_{G/H} + \frac{g-h}{2(k-h)} \Delta_H \right) T_{het} + L^R_0(int)T_{het} ,
\]

where $L^R_0(int)$ is the internal part which does not contribute to the spacetime metric and dilaton. Comparing (2.6) to (5.3) we see that we can obtain the exact metric and dilaton for the heterotic superstring if we replace $k$ in the bosonic expressions by $2k - h$ (except for the overall factor in the line element $ds^2$ which is the same in both cases). In the $k \rightarrow \infty$ limit these fields tend to their bosonic counterparts, as in \[8\], in agreement with the general arguments of \[10\].
6. Concluding remarks

We found a general method for obtaining the conformally exact metric and dilaton fields for any theory which can be formulated as a coset model. Since the value of $k$ that yields $c = 26$ (or $c = 15$) is actually small, our results represent substantial deviations from the semi-classical computations. We have applied our method to the coset models $SO(d - 1, 2)/SO(d - 1, 1)$. Our expressions hold true also for all the models that are obtained from this coset by appropriate analytic continuations, e.g. $SO(d, 1)/SO(d)$, $SO(d + 1)/SO(d)$, etc., by simply specializing to the appropriate region of our global space. For the cases $d = 2, 3, 4$ we gave explicit results. We have also derived results that apply to the general gauged WZW model with or without supersymmetry. For any type–II superstring the perturbative 1–loop results were shown to be also exact except for an overall factor in the metric. We have also shown that for a heterotic superstring the conformally exact metric and dilaton fields can be obtained by a simple shifting of $k$ in the corresponding bosonic expressions. Finally, we have shown that the combination $e^{\Phi} \sqrt{-G}$ is indeed $k$–independent as conjectured in earlier work - an crucial result for a meaningful path integral formulation.

Our exact results may be verified in perturbation theory, as was the case in $d = 2$ (the 2d heterotic case has not yet been verified). However, this should be regarded as a challenge for perturbation theory which is beset with uncertainties over renormalization schemes. It is generally believed, but only tentatively proved that coset models and gauged WZW models are equivalent. This is certainly the case at the classical level, as can be seen from the equations of motion in the axial gauge $[3][7][8][14]$. Furthermore, our Hamiltonian approach should leave no doubt that semi-classically these two theories are equivalent. The Hamiltonian versus the path integral can be regarded as two possible approaches to quantization which may differ in higher orders of $\hbar$, unless one insures that they are equivalent by choosing the correct measure of integration, as we have suggested. If only to strengthen the relationship between these two formulations, it may be of interest to study the perturbative approach to verify our results for the more challenging cases in $d = 3, 4$. This should also be helpful in pinning down the appropriate renormalization scheme which may be useful for other computations in these models.

We have pointed out the screening effects of the quantum corrections in the neighborhoods of the singularities. It is not yet clear how to use these conformally exact results in physical applications. One needs to know how this “classical” string vacuum configuration competes with higher genus string loop effects in specific physical situations. It
may be that, together with the small-large duality properties pointed out in [14], one may arrive at believable physical conclusions in some regimes even from the zero genus exact computation presented in this paper.

APPENDIX

The reader who is familiar with the exact results for the metric and dilaton of the 2d Minkowski or Euclidean black hole, obtained in [19], may wonder how these can be deduced in our formalism. This appendix serves exactly this scope. In the 2d case, for the coset model $SO(2,1)/SO(1,1)$, the Casimir operators (2.11) (2.12) take the form

$$
\Delta^L_G = \frac{1}{4} (1 + x^2)^2 \frac{\partial^2}{\partial x^\mu \partial x^\mu} - \frac{1}{4} (1 + x^2) \frac{\partial^2}{\partial t^2} + \frac{1}{2} (1 + x^2) \frac{\partial}{\partial t} \epsilon_{\mu \nu} x^\mu \frac{\partial}{\partial x^\nu},
$$

$$
\Delta^L_H = -\frac{1}{4} \frac{\partial^2}{\partial t^2},
$$

where we have used $a_{\mu \nu} = a \epsilon_{\mu \nu}$, and $(1 - a^2) \frac{\partial}{\partial a} = \frac{\partial}{\partial t}$ for $a = \cosh t$ or $\sinh t$. The gauge condition (2.15) is

$$
\epsilon_{\mu \nu} x^\mu \frac{\partial}{\partial x^\nu} T = 0.
$$

For gauge invariant tachyon states of the form $T = T(b, t)$, where $b$ is defined by $b = \frac{x^2 - 1}{x^2 + 1}$, the Casimirs reduce to the simpler equations

$$
\Delta^L_G T = (-b^2 - 1) \partial_b^2 - 2b \partial_b + \frac{1}{2(b - 1)} \partial_t^2) T
$$

$$
\Delta^L_H T = -\frac{1}{4} \partial_t^2 T.
$$

Proceeding as before we obtain the line element

$$
ds^2 = 2(k - 2) \left( \frac{db^2}{4(b^2 - 1)} - \beta(b) \frac{b - 1}{b + 1} dt^2 \right), \quad \beta^{-1}(b) = 1 - \frac{2}{k} \frac{b - 1}{b + 1}.
$$

For the dilaton the corresponding expression is

$$
\Phi = \ln \left( \frac{b + 1}{\sqrt{\beta(b)}} \right) + \Phi_0.
$$

\footnote{We have here a sign difference with (3.2) which we used in higher dimensions. This allows us to agree with fig.2 in a previous paper [14] for 2d.}
The scalar curvature reads

\[ R = \frac{2k}{k-2} \frac{(k-2)b + k - 4}{((k-2)b + k + 2)^2}. \] (6.6)

The scalar curvature is singular at \( b = -(k+2)/(k-2) \) which is exactly the point of singularity for the function \( \beta(b) \). To make contact with the results of [19], one needs to reparametrize \( b \) in the various patches. For instance in the black hole region outside the horizons at \( b = 1 \) one can set \( b = \cosh 2r \), in the naked singularity region \( b = -\cosh 2r' \), and in the inside the horizons regions \( b = \cos 2r'' \). The corresponding expressions for the Euclidean black hole follow from (6.4) and (6.5) if we analytically continue \( t \to i\theta \), where \( \theta \) is compact, \( 0 < \theta < 2\pi \). The cigar \((b = \cosh 2r)\) and trumpet \((b = -\cosh 2r')\) correspond to the \( SO(2,1)/SO(2) \) coset, whereas the cymbal region \((b = \cos 2r'')\) to the \( SO(3)/SO(2) \) coset.
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