Boundary states, matrix factorisations and correlation functions for the E-models

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Abstract

The open string spectra of the B-type D-branes of the \( N = 2 \) E-models are calculated. Using these results we match the boundary states to the matrix factorisations of the corresponding Landau-Ginzburg models. The identification allows us to calculate specific terms in the effective brane superpotential of \( E_6 \) using conformal field theory methods, thereby enabling us to test results recently obtained in this context.

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1 Introduction

$N = 2$ minimal models with an $ADE$ classification play a central role in the description of certain Calabi-Yau compactifications. In particular, they form the building blocks of Gepner models \[1\]. For this reason, there has been great interest in the branes of these models and their spectrum.

In the language of abstract conformal field theory, branes are given by boundary states. They are linear combinations of Ishibashi states and must satisfy the Cardy condition. On the other hand, these models can also be described as Landau Ginzburg models. In this case branes correspond to matrix factorisations of the superpotential \[2, 3, 4, 5, 6, 7, 8\]. An interesting problem is thus to compare these descriptions by matching boundary states to matrix factorisations. For the $A$ and $D$ models, this has been done in \[3, 5\] and \[9\], respectively.

In this paper, we perform the match for the $N = 2 E$ models. For these models the complete set of matrix factorisations has been known to mathematicians for some time \[10, 11\]. On the CFT side, the boundary states have been constructed in \[12, 13, 14\]. We calculate their spectrum and match the two descriptions.

We then use the identification to discuss obstructions to brane deformations. The critical loci of the effective superpotential $W_{\text{eff}}$ describe the directions in which a given matrix factorisation can be deformed, and nonvanishing potential terms describe obstructions to deformations \[15, 16\]. On the other hand, $W_{\text{eff}}$ is also the generating functional of open string topological disk correlators \[17\]. Using our identification, we show that certain specific correlators do not vanish, so that the brane deformation in these directions is obstructed. This calculation can then be used to test results obtained using other approaches \[18\].

This paper is organised as follows: In section 2 we recall the $ADE$ classification for affine $su(2)$ models and the construction of their boundary states. For later use we list some basic properties of $N = 2$ minimal models, the exceptional Lie groups $E_n$, and matrix factorisations. In section 3 for each model and each choice of GSO-projection, we first assemble all information on matrix factorisations and boundary states. We then calculate their spectrum and match the boundary states to their corresponding matrix factorisations. In section 4 we use this identification to calculate topological correlators to get certain specific terms of the effective superpotential. We then draw our conclusions in section 5.

2 Basics

2.1 Matrix factorisations

The topological part of a $N = 2$ minimal model can also be described in terms of a Landau Ginzburg model. The superpotential $W$ is a weighted homogeneous polynomial in $x_i$. For $E_n$, the superpotentials and the charges $q_i$ of the variables are given in table 1. Note that for each model there are two different superpotentials which correspond to the two choices of GSO-projections \[3\]: the two variable potentials give type $0B$ projection, the three variable potentials type $0A$. 

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B-type branes in the Landau Ginzburg description are given by square matrices $E, J$ with polynomial entries, and a charge matrix $R$. $E, J$ satisfy

$$E J = J E = W \mathbf{1},$$

or equivalently,

$$Q^2 = W \mathbf{1} \quad \text{where } Q = \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix}. \quad (2.2)$$

In our conventions $W$ has $U(1)$ charge 2 and $Q$ has charge 1:

$$e^{i\lambda R} Q (e^{i\lambda q_i x_i}) e^{-i\lambda R} = e^{i\lambda} Q(x_i). \quad (2.3)$$

To determine $R$ uniquely, one must in addition fix $\text{tr} R$ (see [11] for details).

Define the operator $D$ by

$$D(\phi) := Q_2 \phi - (-1)^{\text{deg}(\phi)} \phi Q_1, \quad (2.4)$$

where $\text{deg}(\phi)$ is the natural $\mathbb{Z}_2$-grading of $\phi$: even for bosons, odd for fermions. The topological spectrum between $Q_1, Q_2$ is given by morphisms $\phi(x_i)$ in the cohomology of $D$. The charge $q$ of $\phi$ is given by

$$e^{i\lambda R_2} \phi(e^{i\lambda q_i x_i}) e^{-i\lambda R_1} = e^{i\lambda q} \phi(x_i). \quad (2.5)$$

The antibrane $\bar{Q}$ of $Q$ is obtained by interchanging $E$ and $J$. Note that the even spectrum between two branes is equivalent to the odd spectrum between brane and antibrane and vice versa.

### 2.2 The affine $su(2)$ case

In this subsection we start the CFT description of $N = 2$ minimal models. In view of their construction as cosets (see 2.3) we will first consider $su(2)$ models. The ADE classification gives all possible modular invariant partition functions obtained from combinations of $su(2)_k$ characters. Each such partition function corresponds to a simply laced Lie algebra $A_n$, $D_n$, or $E_n$. 

---

| $h$ | $E$ | $GSO$ | $q_i$ |
|-----|-----|-------|-------|
| $E_6$ | 12 | 1,4,5,7,8,11 | \begin{align*} W &= x^3 + y^4 \\ W &= x^3 + y^4 + z^2 \end{align*} | \begin{align*} [x] &= \frac{2}{3}, [y] = \frac{1}{2}, [z] = 1 \end{align*} |
| $E_7$ | 18 | 1,5,7,9,11,13,17 | \begin{align*} W &= x^3 + x y^3 \\ W &= x^3 + x y^3 + z^2 \end{align*} | \begin{align*} [x] &= \frac{2}{3}, [y] = \frac{4}{9}, [z] = 1 \end{align*} |
| $E_8$ | 30 | 1,7,11,13,17,19,23,29 | \begin{align*} W &= x^3 + y^5 \\ W &= x^3 + y^5 + z^2 \end{align*} | \begin{align*} [x] &= \frac{2}{7}, [y] = \frac{2}{5}, [z] = 1 \end{align*} |

Table 1: Exceptional groups and their superpotential
Here we are interested only in the exceptional groups $E_n$. Their Dynkin diagrams and other properties can be found in tables 1 and 2. The corresponding partition functions are given by:

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2 \quad (k = 10)$$
$$Z_{E_7} = |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 + \chi_8(\chi_2 + \chi_{14} + (\chi_2 + \chi_{14})\chi_8 \quad (k = 16)$$
$$Z_{E_8} = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2 \quad (k = 28)$$

where the $\chi_\lambda$ are $su(2)_k$ characters, and $k$ is related to the Coxeter number $h$ of $E_n$ by $h = k + 2$. The boundary states of these models have been constructed some time ago [12]:

To each node $L$ of the Dynkin diagram there corresponds a boundary state given by

$$|L\rangle = \sum_{l+1 \in \mathcal{E}} \frac{\psi_L^{(l)}}{\sqrt{S_0^l}} |[l]| \rangle. \quad (2.6)$$

Here $l + 1$ runs over the Coxeter exponents of $E_n$. The $\psi_L^{(l)}$ for each model are listed in appendix A. The modular transformation matrix is

$$S_L^l = \sqrt{\frac{2}{h}} \sin \left(\pi \frac{(L + 1)(l + 1)}{h}\right). \quad (2.7)$$

The overlap of two boundary states is then given by

$$\langle[L_1] | q^{(L_0 + L_0)/2 - c/24} | [L_2]\rangle = \sum_{l=0}^{k} \chi_l(\tilde{q}) n_{L_1}^{L_2}. \quad (2.8)$$

The matrices $(n_i)^{L_1}_{L_2}$ are the so-called fused adjacency matrices [12]. They can be obtained recursively by applying $su(2)_k$ fusion rules

$$n_{i+1} = n_1 n_i - n_{i-1}, \quad i \leq k - 1, \quad (2.9)$$

where $n_0$ is the identity matrix and $n_1$ is the adjacency matrix of the Dynkin diagram. By construction the $n_i$ form an integer valued representation of the fusion algebra, and explicit calculation shows that it is non-negative as well. The $|L\rangle$ thus satisfy the Cardy condition.

### 2.3 The $N = 2$ minimal model

We consider now $N = 2$ minimal models. Their bosonic subalgebra can be described as the coset

$$su(2)_k \oplus u(1)_4 \quad u(1)_{2k+4}. \quad (2.10)$$

The representations of the coset are labelled by triples $(l, m, s)$, where $l = 0, \ldots, k$ is twice the spin of $su(2)$, $m \in \mathbb{Z}_{2k+4}$, and $s \in \mathbb{Z}_4$. The representations must obey $l + m + s = 0$ mod 2 and are subject to the identification

$$(l, m, s) \sim (k - l, m + k + 2, s + 2). \quad (2.11)$$
The conformal weights and $U(1)$ charges of the highest weight states are up to integers given by

$$h(l, m, s) = \frac{l(l + 2) - m^2}{4(k + 2)} + \frac{s^2}{8}, \quad (2.12)$$

$$q(l, m, s) = \frac{s}{2} - \frac{m}{k + 2}. \quad (2.13)$$

In the NS sector ($s$ even), the chiral primaries appear in the representations $(l, l, 0)$. In the R sector ($s$ odd), the R ground states appear in $(l, l + 1, 1)$.

The characters $\chi_{[l,m,s]}(q)$ transform under the modular $S$-transformation as

$$\chi_{[L,M,S]}(q) = \sum_{[l,m,s]} S_{LMS}^{lms} \chi_{[l,m,s]}(\tilde{q}), \quad (2.14)$$

where the sum is over distinct equivalence classes. The $S$-matrix is given by

$$S_{LMS}^{lms} = \frac{1}{\sqrt{2h}} S_L^l e^{\frac{i\pi}{8}mM} e^{-\frac{i\pi}{8}sS}, \quad (2.15)$$

where $S_L^l$ is the $S$-matrix of $su(2)$ ($2.7$). Let

$$Z = \sum_{l,t} A_{l,l} \chi_{l,l} \bar{\chi}_{l,l} \quad (2.16)$$

be an ADE-modular invariant of $su(2)$. Then we can construct two different $N = 2$ modular invariants by $[19]$

$$Z = \sum_{l,t} A_{l,l} \chi_{[l,m,s]} \bar{\chi}_{[l,m,s]} \quad (2.17)$$

Physically, the choice $s = \bar{s}$ corresponds to the type 0B GSO-projection, and $s = -\bar{s}$ to type 0A. See $[20]$ for the complete list of all possible modular invariants of $N = 2$ superconformal minimal models.
We want to construct boundary states \(|B\rangle\) that satisfy B-type gluing conditions
\[
(L_n - \bar{L}_{-n})|B\rangle = 0, \\
(J_n + \bar{J}_{-n})|B\rangle = 0, \\
(G^\pm_r + i\eta \bar{G}^\pm_r)|B\rangle = 0,
\]
where \(\eta = \pm 1\) determines the spin structure. The boundary states of the E-models are then given by \([13]\)
\[
|L, M, S\rangle = K \sum_{[l, m, s]} \frac{\psi^{(l)}_L}{\sqrt{S^{lms}_{000}}} e^{\frac{i\pi}{2} M m} e^{-\frac{i\pi}{2} s S} [l, m, s]\rangle,
\]
where \(h\) is the Coxeter number of the group and \(\psi^{(l)}_L\) are the coefficients of the corresponding \(su(2)\) model. The overall normalisation \(K\) depends on the model and the type of GSO-projection.

The Ishibashi states \(|l, m, s\rangle\) live in sectors with \(m = -\bar{m}\) and \(s = -\bar{s}\), and the sum in (2.19) is over distinct equivalence classes. \(|L, M, S\rangle\) satisfies (2.18) with \(\eta = 1\) \((\eta = -1)\) for \(S\) even \((S\) odd\). In section 3 we will discuss the exact ranges of \(l, m, s\) and \(L, M, S\) for each case individually.

The chiral primaries \((l, l, 0)\) in the overlap between two boundary states \(|B_1\rangle\) and \(|B_2\rangle\) should then correspond one-to-one to the morphisms in the cohomology between the two corresponding matrix factorisations \(Q_1, Q_2\) — in particular, their \(U(1)\) charges given by (2.13) and (2.14) respectively, must be equal. By calculating and comparing the spectra, we can thus match matrix factorisations to boundary states.

3 The exceptional models: \(E_6, E_7, E_8\)

3.1 Branes of \(E_6\)

3.1.1 Type 0B: \(W = x^3 + y^4\)

This case corresponds to \(m = \bar{m}, s = \bar{s}\) in (2.17). There are 12 Ishibashi states \(|l, m, s\rangle\), \(l + 1 \in \mathcal{E}(E_6)\), \(l + m + s\) even, and \(m = 0\) or \(6\) depending on the value of \(l\):
\[
|[0, 0, 0]\rangle, \quad |[4, 0, 0]\rangle, \quad |[6, 0, 0]\rangle, \quad |[10, 0, 0]\rangle, \quad |[3, 6, 1]\rangle, \quad |[7, 6, 1]\rangle, \\
|[0, 0, 2]\rangle, \quad |[4, 0, 2]\rangle, \quad |[6, 0, 2]\rangle, \quad |[10, 0, 2]\rangle, \quad |[3, 6, -1]\rangle, \quad |[7, 6, -1]\rangle.
\]

The boundary states are given by
\[
|L, M, S\rangle = \frac{1}{\sqrt{2}} \sum_{[l, m, s]} \frac{\psi^{(l)}_L}{\sqrt{S^{lms}_{000}}} e^{\frac{i\pi}{2} M m} e^{-\frac{i\pi}{2} s S} [l, m, s]\rangle,
\]
where \(L = 1, \ldots, 6\) and \(S, M \in \mathbb{Z}_4\) with \(L + M + S\) even, and the sum runs over the Ishibashi states (3.1).

The map \(\tau : S \mapsto S + 2\) maps branes to antibranes, as it changes the sign of the coupling to RR states. Note that in this case there is the symmetry
\[
|2, S\rangle = \tau(|4, S\rangle), \quad |1, S\rangle = \tau(|5, S\rangle), \\
|3, S\rangle = \tau(|3, S\rangle), \quad |6, S\rangle = \tau(|6, S\rangle).
\]

6
Moreover, we have $\langle L, M, S \rangle = \langle L, M + 2, S + 2 \rangle$. $M$ is thus fixed by demanding that $L + M + S$ be even, and by (3.3) we can restrict $S$ to 0,1. This means that we are left with 12 different boundary states, 6 for each choice of spin structure. Their spectrum is

\[
\langle \langle L_1, M_1, S_1 || q^{(L_0 + L_0)/2 - e/24} || L_2, M_2, S_2 \rangle \rangle = \frac{1}{2} \sum_{[l,m,s]} \chi_{[l,m,s]}(\tilde{q}) \delta^{(2)}(S_1 - S_2 - s)
\times \left( n_{lL_2}^{L_1} \left( 1 + e^{\pm \frac{e}{24} (S_2 - S_1 + S + M_2 - M_1 + m)} \right) + n_{10 - l L_2}^{L_1} \left( 1 - e^{\pm \frac{e}{24} (S_2 - S_1 + S + M_2 - M_1 + m)} \right) \right),
\]

(3.4)

where $n_{lL_2}^{L_1}$ are the fused adjacency matrices for $E_6$.

There are six matrix factorisation for this model, listed in appendix B.1. Their spectrum has been calculated in [18]. It agrees with the chiral primary fields of (3.4) if we make the identifications:

\[
Q_L \equiv \langle L, M, 0 \rangle
\]

(3.5)

with $M \in \{0, 1\}$ such that $L + M$ even for the spin structure $S = 0$, and

\[
Q_L \equiv \langle L, M, 1 \rangle
\]

(3.6)

with $M \in \{1, 2\}$ such that $L + M$ odd for $S = 1$.

3.1.2 Type 0A: $W = x^3 + y^4 + z^2$

There are 12 Ishibashi states

\[
\langle [l, 0, s] \rangle \quad l + 1 \in \mathcal{E}(E_6),
\]

(3.7)

with $s \in \mathbb{Z}_4$ such that $l + s$ even. The boundary states are given by

\[
\langle [L, S] \rangle = \langle [L, 0, S] \rangle = \frac{1}{\sqrt{2}} \sum_{[l,m,s]} \frac{\psi_{L}^{(l)}}{\sqrt{S_{lms}^{000}}} e^{-\frac{i e}{2} s S} \langle [l, m, s] \rangle
\]

(3.8)

the sum running over the Ishibashi states (3.7). We have $L = 1, \ldots, 6$ and $S \in \mathbb{Z}_4$, but again the symmetry under $\tau$ allows us to restrict $S \in \{0, 1\}$, so that we have 6 boundary states per spin structure.

Their overlap is

\[
\langle \langle L_1, S_1 || q^{(L_0 + L_0)/2 - e/24} || L_2, S_2 \rangle \rangle = \sum_{[l,m,s]} \chi_{[l,m,s]}(\tilde{q}) \left( n_{lL_2}^{L_1} \delta^{(4)}(S_1 - S_2 - s) + n_{10 - l L_2}^{L_1} \delta^{(4)}(S_1 - S_2 + 2 - s) \right).
\]

(3.9)

The matrix factorisations of $W = x^3 + y^4 + z^2$ are listed in appendix B.1 and their spectrum has been calculated in [11] (beware of the difference in labelling!) It agrees with (3.9) if we identify

\[
Q_L \equiv \langle [L, S] \rangle.
\]

(3.10)
3.2 Switching between GSO-projections

3.1.1 and 3.1.2 illustrate nicely how one can change between one GSO-Projection and the other: One constructs the new branes out of the old branes by orbifolding by $\tau$. For instance, if we start out with the type 0A theory, we take the orbits of all branes that are not invariant,

$$||3, M, S\rangle\rangle = \frac{1}{\sqrt{2}}(||2, S\rangle\rangle + ||4, S\rangle\rangle)$$

$$||6, M, S\rangle\rangle = \frac{1}{\sqrt{2}}(||1, S\rangle\rangle + ||5, S\rangle\rangle) .$$

We have thus projected out the Ramond part of these branes.

On the other hand, a fixed point $||B\rangle\rangle$ of $\tau$ corresponds to a fractional brane which must be resolved by adding linear combinations of the new Ramond Ishibashi states, i.e.

$$||B_1\rangle\rangle = \frac{1}{\sqrt{2}}||B\rangle\rangle + \text{linear combination of new states}$$

$$||B_2\rangle\rangle = \frac{1}{\sqrt{2}}||B\rangle\rangle - \text{linear combination of new states}$$

It can be checked that by this procedure we really obtain the boundary states (3.2) of the type 0B theory.

3.3 Branes of $E_7$

3.3.1 Type 0B: $W = x^3 + xy^3$

$E_7$ is insofar different from $E_6$ as the two GSO-projections have a different number of boundary states. For type 0B projection, there are 28 Ishibashi states,

$$||[l, 0, s]\rangle\rangle \quad l + 1 \in \mathcal{E}(E_7), \quad s \in \{0, 2\} ,$$

and

$$||[l, 9, s]\rangle\rangle \quad l + 1 \in \mathcal{E}(E_7), \quad s \in \{-1, 1\} .$$

The boundary states are

$$||L, M, S\rangle\rangle = \frac{1}{2} \sum_{l+1 \in \mathcal{E}, \ m=0,9 \ \text{even}} \frac{\psi^{(j)}_L}{\sqrt{S^{lm}}_{00}} e^{i\pi m M} e^{-i\pi S S} ||[l, m, s]\rangle\rangle$$

where $L = 1, \ldots, 7, S = 0, 1, 2, 3$ with $L+M+S$ even. This time the $\psi^{(j)}_L$ are the coefficients for the affine $E_7$ model given in appendix A.2. Again, $S$ odd and $S$ even give two different spin structures with 14 boundary states each.

The overlap is

$$\langle\langle L_1, M_1, S_1||q^{(L_0+\bar{L}_0)/2-e/24}||L_2, M_2, S_2\rangle\rangle =$$

$$\frac{1}{2} \sum_{[l,m,s]} \chi^{[l,m,s]}(q) n_{L_1} L_2 S_2 (S_1 - S_2 - s) \left(1 + e^{i\pi (M_2+2M_1-S_1-M_1+s)}\right) ,$$

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where the $n_{L_1 L_2}$ are now the fused adjacency matrices for $E_7$.

The matrix factorisations are given in appendix [B.2]. Their spectrum agrees with (3.14) if we make the identification

$$Q_L \equiv ||L, M, 0||, \bar{Q}_L \equiv ||L, M, 2||,$$

with $M \in \{0, 1\}$ such that $L + M$ even, and

$$Q_L \equiv ||L, M, 1||, \bar{Q}_L \equiv ||L, M, 3||,$$

with $M \in \{1, 2\}$ such that $L + M$ odd.

### 3.3.2 Type 0A: $W = x^3 + xy^3 + z^2$

In this case we only have 14 Ishibashi states,

$$||[l, 0, s]|| \quad l + 1 \in \mathcal{E}(E_7), \quad s \in \{0, 2\}.$$

(3.17)

For the type 0B case, the map $\tau : S \mapsto S + 2$ had no fixed points. It is thus straightforward to construct the boundary states for the 0A projection by

$$||L, S|| = \frac{1}{\sqrt{2}}(||L, M, S|| + ||L, M, S + 2||).$$

(3.18)

This gives the required 14 states. We could also have obtained these boundary states by using (2.19) with $K = \frac{1}{\sqrt{2}}$.

The overlap is

$$\langle \langle L_1, S_1 | q^{(L_0 + L_0)/2 - e/24} | L_2, S_2 \rangle \rangle = \sum_{[l, m, s]} n_{L_1 L_2} \left( \delta^{(4)}(S_1 - S_2 - s) + \delta^{(4)}(S_1 - S_2 + 2 - s) \right) \chi_{[l, m, s]}(\tilde{q}).$$

(3.19)

The identification with the matrix factorisations of appendix [B.2] is

$$\hat{Q}_L \equiv ||L, S||.$$

(3.20)

### 3.4 Branes of $E_8$

#### 3.4.1 Type 0B: $W = x^3 + y^5$

The $E_8$ model is completely analogous to the $E_7$ model. For the 0B projection there are 32 Ishibashi states

$$||[l, 0, s]|| \quad l + 1 \in \mathcal{E}(E_8), \quad s \in \{0, 2\},$$

$$||[l, 9, s]|| \quad l + 1 \in \mathcal{E}(E_8), \quad s \in \{-1, 1\},$$

and 32 boundary states $||L, M, S||), L = 1 \ldots 8, S = 0, 1, 2, 3, M = 0, 1, L + M + S$ even, given by (2.19) with $K = \frac{1}{2}$. Their spectrum is identical to (3.14) with $n_{L_1 L_2}$ replaced by the fused adjacency matrices of $E_8$. The identification with the matrix factorisations of appendix [B.3] is

$$Q_L \equiv ||L, M, 0||, \bar{Q}_L \equiv ||L, M, 2||,$$

(3.21)

and

$$Q_L \equiv ||L, M, 1||, \bar{Q}_L \equiv ||L, M, 3||,$$

(3.22)

with $M$ as in (3.15) and (3.16).
3.4.2 Type 0A: \( W = x^3 + y^5 + z^2 \)

Again, we only have 16 Ishibashi states. The 16 boundary states are constructed just as in (3.18), their spectrum is as in (3.19) and they are identified with the matrix factorisations of appendix B.3 by

\[
\hat{Q}_L \equiv \langle |L, S \rangle \rangle .
\] (3.23)

4 Correlators and the effective superpotential

4.1 Introduction and motivation

In this chapter we make use of the previous match between boundary states and matrix factorisations to calculate specific correlators of the \( E_6 \) model. These explicit calculations of correlators are to be viewed as checks for results that were obtained by other methods. On the one hand, one can try to determine open-closed topological disk amplitudes for minimal models by solving the consistency conditions that these correlators have to satisfy [17], in particular, the \( A_\infty \)-relations and the homotopy version of bulk-boundary crossing symmetry. These two conditions give rise to an underdetermined set of equations for the correlators. In [17] it was proposed that a generalised Cardy condition should be imposed. For the A-series of minimal models this method yields unique correlation functions that agree with previous results. For the E models, however, it appears to be unapplicable [18], as in these cases the Cardy condition seems incompatible with the other sewing conditions. The same incompatibility has been observed for the torus [21].

A second approach, the Massey product algorithm, was illustrated in [18]. Here brane deformations are considered in the context of topological Landau-Ginzburg theories ([8],[22]), i.e as deformations of the superpotential and its factorisations:

\[
Q_{\text{def}} = Q + \sum_{\vec{m} \in \mathbb{N}^i} \alpha_{\vec{m}} u_{\vec{m}} + \sum_{\vec{n} \in \mathbb{N}^i} \tilde{\alpha}_{\vec{n}}(u) s_{\vec{n}} ,
\] (4.1)

\[
W_{\text{def}} = W + \sum_i s^i \phi_i ,
\] (4.2)

where the \( \phi_i \) are a basis of the space of bulk chiral primary fields, and \( \alpha_{\vec{m}}, \tilde{\alpha}_{\vec{n}} \) are boundary fields.

\( N = 2 \)-worldsheet supersymmetry requires

\[
Q_{\text{def}}^2 = W_{\text{def}} ,
\] (4.3)

that is

\[
\{ Q, \alpha_{\vec{m}} \} = - \sum_{\vec{n} + \vec{m}' = \vec{m}} \{ \alpha_{\vec{m}'}, \alpha_{\vec{m}''} \} ,
\] (4.4)

\[
\left\{ Q + \sum_{\vec{m} \in Q'} \alpha_{\vec{m}} u_{\vec{m}}, \tilde{\alpha}_n(u) \right\} = - \sum_{\vec{n} + \vec{n}' = \vec{n}} \{ \tilde{\alpha}_{\vec{n}'}(u), \tilde{\alpha}_{\vec{n}''}(u) \} + \sum_{i \vec{e}_i} \delta_{\vec{n}, \vec{e}_i} \phi_i .
\] (4.5)

If for some \( \vec{m} \) and \( \vec{n} \) the r.h.s of (4.1) and (4.5) are non-trivial elements in the cohomology of \( Q \) and \( Q' \) respectively, then the deformations are obstructed. This means that (4.3)
is satisfied only if \( u \) and \( s \) satisfy some analytic constraints \( p_i(u, s) = 0 \). For the cases considered in \cite{18} these expressions were found to be integrable, \( p_i(u, s) \sim \partial W_{\text{eff}} \). Since the conditions \( p_i(u, s) = 0 \) are related to the \( N = 2 \)-supersymmetry on the worldsheet, they are interpreted as F-term equations \cite{8} for the deformation parameters, viewed as \( N = 1 \) chiral fields in the low-energy theory. Therefore \( W_{\text{eff}} \) is interpreted as the spacetime effective superpotential and, up to reparametrisations, as the generating function of (symmetrised) open-closed topological correlators \cite{17}.

Thirdly, \cite{18} also proposed a "mixed" approach. In this approach the \( A_\infty \)-relations are first solved for the bulk insertions set to zero. One then gets rid of the underdetermination of the problem by requiring agreement with the results of the Massey product algorithm. The correlators so obtained are subsequently used as input for solving the problem with bulk deformations.

For the factorisation \( W = x^3 + y^4 - z^2 = E_1 J_1 \) of the \( E_6 \)-model, this mixed prescription leads to a uniquely determined solution. After reparametrisation, it corresponds to the effective superpotential calculated using the Massey product algorithm, but only if certain deformation parameters are set to zero. In particular, the superpotentials in this case are \cite{18}:

\[
\begin{align*}
W_{\text{eff}}^{\text{mixed}}(u; s) &= W_{\text{eff}}^{\text{Massey}}(u_1, u_4; s_2 = 0, s_5 = 0, s_6, s_8 = 0, s_9, s_{12}) \\
W_{\text{eff}}^{\text{Massey}}(u; s) &= \frac{5}{832} u_1^{13} + \frac{1}{8} u_4 u_1^9 + \frac{3}{4} u_4^2 u_1^5 + u_4^3 u_1 + \frac{1}{352} s_2 u_1^{11} \\
&\quad + \frac{1}{192} s_2^2 u_1^9 - \frac{3}{64} s_5 u_1^8 + \frac{3}{56} s_6 u_1^7 + \frac{3}{448} s_8^2 u_1^7 \\
&\quad + \frac{1}{16} s_9 u_1^4 - \frac{1}{10} (s_8 + \frac{1}{4} s_6 s_2) u_1^3 - \frac{1}{2} s_5 u_4 u_1^4 \\
&\quad + \frac{1}{8} s_9 u_1^4 - \frac{1}{4} s_2 u_4 u_1^3 + \frac{1}{2} s_6 u_4 u_1^3 - \frac{1}{12} (s_8 s_2 - s_5^2) u_1^3 \\
&\quad + \frac{1}{4} s_5 s_2 u_4 u_1^2 - \frac{1}{4} s_6 s_5 u_1^2 + \frac{1}{4} s_7^2 u_1^2 - \frac{1}{2} s_2 s_6 u_4 u_1 \\
&\quad - s_8 u_1 + (s_{12} + \frac{1}{4} s_6) u_1 - \frac{1}{2} s_5 u_1^3 + s_9 u_1^2 + \text{const}
\end{align*}
\]

In this section we want to focus on the discrepancy between the two results. In particular, we shall concentrate on the term \( \frac{1}{2} s_5 u_1^2 \) which we have underlined in \cite{18}. The presence of this term in \( W_{\text{eff}}^{\text{Massey}} \) implies that the corresponding 4-point disk correlator does not vanish. On the other hand, \( W_{\text{eff}}^{\text{mixed}} \) would imply that it vanishes. Note that even though \( W_{\text{eff}}^{\text{Massey}} \) corresponds to the generating function only up to reparametrisation of the deformation parameters, R-charge considerations show that the term we are considering cannot be eliminated by such a reparametrisation.

Our task is thus to check if the topological disk correlator \( \langle \phi_7 \psi_4 \int [G, \psi_4] \rangle_1 \) vanishes, where \( \phi_i \) (\( \psi_i \)) is the bulk (boundary) field of R-charge \( i \), and the label 1 indicates that we impose boundary conditions corresponding to \( Q_1 \).
4.2 Decomposition of $E_6$

For later use we make the following observation: The fact that $c_{k=10} = c_{k=1} + c_{k=2}$ suggests that we can decompose $E_6$ into the simpler models $A_1$ and $A_2$. In terms of the LG potential, this corresponds to the observation that $W = x^3 + y^4$ is the sum of two $A$-model potentials. By decomposing $k = 10$ characters we can identify

\[ ||0,0\rangle_1 \otimes ||1,0\rangle_2 \sim ||1,0\rangle_{E_6} , \]
\[ ||0,0\rangle_1 \otimes ||1,2\rangle_2 \sim ||5,0\rangle_{E_6} , \]
\[ ||0,0\rangle_1 \otimes ||0,0\rangle_2 \sim ||6,0\rangle_{E_6} . \]  

(4.7)

Here $||L,S\rangle_{1,2}$ are boundary states of the A-model, see e.g. [9] for details of the notation. The other $E_6$ boundary states cannot be written as tensor products of $A_1$ and $A_2$ boundary states. This is confirmed by looking at the matrix factorisation of $Q_1$, $Q_5$, and $Q_6$ are tensor products, all the other $Q$ contain terms of the form $xy$ and cannot be decomposed.

4.3 Topological correlators

To obtain a topological conformal field theory, one can twist a $N = 2$ superconformal model. On the sphere, this leads to a $U(1)$ background charge of $-\frac{x_3}{3}$. This means that all topological correlators vanish unless their total charge is equal to $\frac{x_3}{3}$. If we want to calculate such correlators in the original $N = 2$ theory, we must introduce by hand additional fields of charge $\frac{x_3}{3}$. We will do this by inserting one unit of spectral flow $\rho(\xi)$ on the boundary. To get a topological correlator, we then multiply the result by $\xi^{x_3/3}$ and let $\xi \rightarrow \infty$ [24].

4.4 Calculating $\langle \phi_7 \psi_4 \int [G, \psi_4] \rangle_1$

The matrix factorisation $Q_1$ of the three-variable case factorises as \(^{\dagger}\)

\[ Q_1 = \begin{pmatrix} 0 & x \\ x^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & y^2 - iz \\ y^2 + iz & 0 \end{pmatrix} , \]

(4.8)

where $\otimes$ is the graded tensor product [8]. For its fermionic spectrum we get

\[ \psi := \begin{pmatrix} 0 & 1 \\ -x & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \psi_2 = y \psi . \]

By comparing charges, we find that

\[ \phi_7 \longleftrightarrow xy \]
\[ \psi_4 \longleftrightarrow \psi . \]  

(4.9)

Therefore the superpotential term we are interested in corresponds to the correlator

\[ D = \langle xy \psi \int dt \left( G_{-1/2}^{-} \psi \right)(t) \rangle . \]  

(4.10)

\(^{\dagger}\)Note that here we use $W = x^3 + y^4 + z^2$, in agreement with our earlier conventions.
After factorising we obtain
\[
\int dt \langle x \left( \begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) \left( G_{-1/2} \left( \begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) \right) (t) \rangle_{A_1} \langle y \left( \begin{array}{c} 1 \\ 1(t) \end{array} \right) \rangle_{A_2} + \int dt \langle x \left( \begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) \rangle_{A_1} \langle y \left( G_{-1/2} 1 \right) (t) \rangle_{A_2} . \tag{4.11} \]

In the second term, \( \langle \cdots \rangle_{A_2} \) vanishes because its total charge is \( \frac{1}{2} - 1 = -\frac{1}{2} \) instead of the required \( \frac{1}{2} \). On the other hand, the \( A_2 \) correlator of the first term is independent of \( t \). As it contains no integrated operator insertions, we can evaluate it using [5]:
\[
\langle y \rangle_{A_2} = \frac{1}{2(2\pi i)^2} \oint dy dz \frac{y \cdot \text{STr}(\partial_y Q \partial_z Q)}{\partial_y W_{A_2} \partial_z W_{A_2}} = \frac{i}{4}. \tag{4.12} \]

To evaluate the \( A_1 \) correlator, we write it as a coset model CFT correlator. By comparing \( U(1) \) charges, we can identify the fields
\[
x \leftrightarrow \phi_{110}(z) \phi_{110}(\bar{z}) ,
\left( \begin{array}{cc} 0 & 1 \\ -x & 0 \end{array} \right) \leftrightarrow \psi_{110}(s) .
\]
Moreover we insert one unit of spectral flow \( \psi_{1-10}(\xi) \). We thus have to calculate the correlator
\[
\int dt \langle \phi_{110}(z) \phi_{110}(\bar{z}) \psi_{112}(t) \psi_{110}(s) \psi_{1-10}(\xi) \rangle , \tag{4.13} \]
where we have used \( G_{-1/2} \psi_{110} = \psi_{112} \). Our task is simplified further since the \( A_1 \) model is really just the free boson,
\[
\frac{su(2)_1 \oplus u(1)_2}{u(1)_3} = u(1)_6 , \tag{4.14} \]
and we can identify (see e.g. [25])
\[
\phi_{110} \leftrightarrow e^{\frac{1}{\sqrt{3}} x} ,
\psi_{112} \leftrightarrow e^{\frac{\sqrt{3}}{2} x} ,
\psi_{1-10} \leftrightarrow e^{\frac{\sqrt{3}}{3} x} .
\]
Our original boundary state is a B-type brane and corresponds thus to Neumann boundary conditions for the free boson. We can use an explicit expression for (4.13) [23],
\[
2\pi i C |z - \bar{z}|^{1/3} |z - s|^{2/3} |z - \xi|^{-2/3} |\xi - s|^{-1/3} \int dt |\xi - t|^{2/3} |s - t|^{-2/3} |z - t|^{-4/3} , \tag{4.15} \]
where \( C \) is a regularised functional determinant. To obtain the topological correlator, we have to multiply by \( |\xi|^{1/3} \) and let \( \xi \to \infty \). Exchanging limit and integral, the result is
\[
\langle \cdots \rangle_{A_1} = 2\pi i C |z - \bar{z}|^{1/3} |z - s|^{2/3} \int \frac{dt}{|z - t|^{4/3} |s - t|^{2/3}} \neq 0 . \tag{4.16} \]
The result of these calculations is thus that \( D \) does not vanish. In a similar way, one can show that the correlator corresponding to \( s_8 u_4 u_1 \) does not vanish either.
Our results therefore agree with those obtained with the Massey product algorithm and not with those calculated with the mixed approach.
5 Conclusion

Our results for the exceptional models conclude the program started in [3, 5, 9]: For all ADE models, the match between matrix factorisations and boundary states is now known. We have also confirmed that the different GSO-projections correspond to superpotentials with and without additional $z^2$ terms.

The identification of matrix factorisations with boundary states allows one to calculate topological correlators using conformal field theory methods. In this paper we have demonstrated this for one of the correlators of the $E_6$ model. While in general this approach is likely to be complicated, there are cases (for example the correlator studied in this paper) where this is actually an efficient method. In any case, it allows one to check terms of the effective superpotential that characterise the obstructions of matrix factorisations under deformations. A good general method to determine the effective superpotential in minimal models is, however, still missing.

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A Boundary states for $su(2)$

A.1 Coefficients for $E_6$

| $l$ | 0   | 3   | 4   | 6   | 7   | 10  |
|-----|-----|-----|-----|-----|-----|-----|
| $\psi^{(l)}_1$ | $(\frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, 0, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6})$ |
| $\psi^{(l)}_2$ | $(\frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6})$ |
| $\psi^{(l)}_3$ | $(\frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, 0, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6})$ |
| $\psi^{(l)}_4$ | $(\frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, -\frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6})$ |
| $\psi^{(l)}_5$ | $(\frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6})$ |
| $\psi^{(l)}_6$ | $(\frac{1}{2} \sqrt{3-\sqrt{3}} \frac{3+\sqrt{3}}{6}, 0, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6}, \frac{1}{2} \sqrt{3+\sqrt{3}} \frac{3+\sqrt{3}}{6})$ |
A.2 Coefficients for $E_7$

\[
\begin{align*}
\psi^{(l)}_1 &= (a, c, b, \frac{1}{\sqrt{3}}, b, c, a) \\
\psi^{(l)}_2 &= (e, f, d, 0, -d, -f, -e) \\
\psi^{(l)}_3 &= (c, b, -a, -\frac{1}{\sqrt{3}}, -a, b, c) \\
\psi^{(l)}_4 &= (f, -d, -e, 0, e, d, -f) \\
\psi^{(l)}_5 &= (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \\
\psi^{(l)}_6 &= (d, -e, f, 0, -f, e, -d) \\
\psi^{(l)}_7 &= (b, -a, -c, \frac{1}{\sqrt{3}}, -c, -a, b)
\end{align*}
\]

where

\[a = \left(18 + 12\sqrt{3} \cos \frac{\pi}{18}\right)^{-\frac{1}{2}}, \quad b = \left(18 + 12\sqrt{3} \cos \frac{11\pi}{18}\right)^{-\frac{1}{2}},\]
\[c = \left(18 + 12\sqrt{3} \cos \frac{13\pi}{18}\right)^{-\frac{1}{2}}, \quad d = \left(12(1 + \cos \frac{\pi}{9})\right)^{-\frac{1}{2}},\]
\[e = \left(12(1 + \cos \frac{5\pi}{9})\right)^{-\frac{1}{2}}, \quad f = \left(12(1 + \cos \frac{7\pi}{9})\right)^{-\frac{1}{2}}.\]

A.3 Coefficients for $E_8$

\[
\begin{align*}
\psi^{(l)}_1 &= (a, f, c, d, d, c, f, a) \\
\psi^{(l)}_2 &= (b, e, h, g, -g, -h, -e, -b) \\
\psi^{(l)}_3 &= (c, d, -a, -f, -f, -a, d, c) \\
\psi^{(l)}_4 &= (d, a, -f, -c, c, f, -a, -d) \\
\psi^{(l)}_5 &= (e, -h, -g, b, b, -g, -h, e) \\
\psi^{(l)}_6 &= (f, -c, d, -a, a, -d, c, -f) \\
\psi^{(l)}_7 &= (g, -b, e, -h, -h, e, -b, g) \\
\psi^{(l)}_8 &= (h, -g, -b, e, -e, b, g, -h)
\end{align*}
\]

where

\[
\begin{align*}
a &= \left[\frac{15(3+\sqrt{5})+\sqrt{15(130+58\sqrt{5})}}{2}\right]^{-1/2}, \quad b = \left[15 + \sqrt{75 - 30\sqrt{5}}\right]^{-1/2},
\end{align*}
\]
\[
\begin{align*}
c &= \left[\frac{15(3+\sqrt{5})-\sqrt{15(130+58\sqrt{5})}}{2}\right]^{-1/2}, \quad e = \left[15 - \sqrt{75 + 30\sqrt{5}}\right]^{-1/2},
\end{align*}
\]
\[
\begin{align*}
d &= \left[\frac{15(3-\sqrt{5})-\sqrt{15(130-58\sqrt{5})}}{2}\right]^{-1/2}, \quad g = \left[15 + \sqrt{75 + 30\sqrt{5}}\right]^{-1/2},
\end{align*}
\]
\[
\begin{align*}
f &= \left[\frac{15(3-\sqrt{5})+\sqrt{15(130-58\sqrt{5})}}{2}\right]^{-1/2}, \quad h = \left[15 - \sqrt{75 - 30\sqrt{5}}\right]^{-1/2}.
\end{align*}
\]
B Matrix factorisations

B.1 Matrix factorisations for $E_6$

The matrix factorisations for $W = x^3 + y^4$ are

$$E_1 = J_5 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix} \quad E_2 = J_4 = \begin{pmatrix} x^2 & -xy & y^2 \\ y^3 & x^2 & -xy \\ -xy^2 & y^3 & x^2 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} x & y^2 & 0 & 0 \\ y^2 & -x^2 & 0 & 0 \\ 0 & -xy & x^2 & y^2 \\ y & 0 & y^2 & -x \end{pmatrix} \quad J_3 = \begin{pmatrix} x^2 & y^2 & 0 & 0 \\ 0 & y^2 & x & y^2 \\ xy & 0 & y^2 & -x^2 \end{pmatrix}$$

$$E_4 = J_2 = \begin{pmatrix} x & y & 0 \\ x^3 & -x \\ y^3 & -x \end{pmatrix}$$

The matrix factorisations for $W = x^3 + y^4 + z^2$ are

$$E_1 = J_5 = \begin{pmatrix} -y^2 + iz & x \\ x^2 & y^2 + iz \end{pmatrix}$$

$$E_2 = J_4 = \begin{pmatrix} -y^2 + iz & 0 & xy \\ -xy & y^2 + iz & x^2 & 0 \\ 0 & x & iz & y \\ x^2 & -xy & y^3 & iz \end{pmatrix}$$

$$E_3 = \begin{pmatrix} -iz & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & -iz & 0 & 0 & 0 & x \\ 0 & 0 & -iz & -x & 0 & y \\ 0 & xy & -x^2 & -iz & y^3 & 0 \\ x & 0 & 0 & y & -iz & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & -iz \end{pmatrix}$$

$$J_3 = \begin{pmatrix} iz & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & iz & 0 & 0 & 0 & x \\ 0 & 0 & iz & -x & 0 & y \\ 0 & xy & -x^2 & iz & y^3 & 0 \\ x & 0 & 0 & y & iz & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & iz \end{pmatrix}$$
B.2 Matrix factorisations $E_7$

For $W = x^3 + xy^3$, the matrix factorisations are given by \[10\]

$$
E_1 = \begin{pmatrix} x \end{pmatrix}, \quad J_1 = \begin{pmatrix} x^2 + y^3 \end{pmatrix}
$$

$$
E_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix}, \quad J_2 = \begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}
$$

$$
E_3 = \begin{pmatrix} x & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} -xy & x^2 & 0 \\ 0 & -y & x \end{pmatrix}
$$

$$
E_4 = \begin{pmatrix} x & y & -y & 0 \\ y^2 & -x & 0 & -y \\ 0 & 0 & x^2 & xy \\ 0 & 0 & xy^2 & -x^2 \end{pmatrix}, \quad J_4 = \begin{pmatrix} x^2 & xy & y & 0 \\ xy^2 & -x^2 & 0 & y \\ 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \end{pmatrix}
$$

$$
E_5 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix}, \quad J_5 = \begin{pmatrix} x^2 & -x^2 & -x^2 y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}
$$

$$
E_6 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}, \quad J_6 = \begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}
$$

$$
E_7 = \begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \quad J_7 = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}
$$

The other factorisations $\bar{Q}_i$ correspond to their antibranes and are given by $\hat{E}_i = J_i$, $\bar{J}_i = E_i$.

For $W = x^3 + xy^3 + z^2$, the factorisations are constructed out of the above by

$$
\hat{E}_i = \bar{J}_i = \begin{pmatrix} z \mathbf{1} & J_i \\ E_i & -z \mathbf{1} \end{pmatrix},
$$

so that $\bar{Q}_i$ is equal to its own antibrane.
### B.3 Matrix factorisations $E_8$

For $W = x^3 + y^5$ the matrix factorisations are given by [10]

\[
E_1 = \begin{pmatrix}
    x^2 & y \\
    y^4 & -x
\end{pmatrix} \quad J_1 = \begin{pmatrix}
    x & y \\
    y^4 & -x^2
\end{pmatrix}
\]

\[
E_2 = \begin{pmatrix}
    x^2 & y^3 x^2 \\
    y^4 & xy \\
    -xy & y^2 x^2
\end{pmatrix} \quad J_2 = \begin{pmatrix}
    y & -x \\
    0 & y \\
    0 & -y^3
\end{pmatrix}
\]

\[
E_3 = \begin{pmatrix}
    -x^2 & xy & -y^3 \\
    0 & -y^2 & -x \\
    y^2 & 0 & y - x
\end{pmatrix} \quad J_3 = \begin{pmatrix}
    y & -x \\
    0 & -y^3 \\
    -y^2 & 0 & -x^2
\end{pmatrix}
\]

\[
E_4 = \begin{pmatrix}
    y^4 & xy^3 x^2 & 0 & 0 & xy \\
    -x^2 & y^3 xy & -x & 0 & 0 \\
    -xy^2 & -x^2 & y^3 & 0 & -xy \\
    0 & 0 & 0 & -y^2 & -x \\
    0 & 0 & 0 & y^2 & -x
\end{pmatrix} \quad J_4 = \begin{pmatrix}
    y & -x & 0 & 0 & 0 \\
    0 & y^2 & -x \\
    -xy^2 & y^3 & 0 & x \\
    0 & 0 & 0 & -y^2 \\
    0 & 0 & 0 & -xy^2 & -x^2
\end{pmatrix}
\]

\[
E_5 = \begin{pmatrix}
    x^2 & y^2 & 0 & xy \\
    y^3 & -x & -y^2 & 0 \\
    0 & 0 & x & y^2 \\
    0 & 0 & y^3 & -x^2
\end{pmatrix} \quad J_5 = \begin{pmatrix}
    x & y^2 & 0 \\
    y^3 & -x^2 & -xy^2 & 0 \\
    0 & 0 & x^2 & y^2 \\
    0 & 0 & y^3 & -x
\end{pmatrix}
\]

\[
E_6 = \begin{pmatrix}
    x & y^2 \\
    y^3 & -x^2
\end{pmatrix} \quad J_6 = \begin{pmatrix}
    y & x^2 \\
    y^3 & -x
\end{pmatrix}
\]

\[
E_7 = \begin{pmatrix}
    x^2 & y^2 \\
    y^3 & -x^2
\end{pmatrix} \quad J_7 = \begin{pmatrix}
    y & -x \\
    0 & y^2 \\
    x & 0
\end{pmatrix}
\]

\[
E_8 = \begin{pmatrix}
    y^4 & xy^2 x^2 \\
    -x^2 & y^2 x^2 \\
    -xy^2 & y^2 x^2
\end{pmatrix} \quad J_8 = \begin{pmatrix}
    y & -x \\
    0 & y^2 \\
    x & 0
\end{pmatrix}
\]

and their respective antibranes.

The factorisations for $W = x^3 + y^5 + z^2$ are constructed in the same way as for $E_7$. 

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