1. Introduction

Let $L$ be a finite extension of $\mathbb{Q}_p$ and let $G = G(L)$ be the group of $L$-valued points of a split connected reductive algebraic group $G$ over $L$. This paper is about the Jordan-Hölder series of locally analytic $G$-representations which are induced from locally algebraic representations of a parabolic subgroup $P \subset G$. The theory of locally analytic representations was introduced by Schneider and Teitelbaum [ST1]. They come up in the study of vector bundles on $p$-adic period domains [ST3, O] and in the $p$-adic Langlands program. In [S2, Tc] the authors pose the question of determining the irreducible parabolically induced locally-analytic representations. More generally, one like to know the Jordan-Hölder series of such an object. Here we give a partial answer to these questions. As an application we consider the equivariant filtration constructed in [O] on the space of sections $H^0(\mathcal{X}, \mathcal{L})$ where $\mathcal{L}$ is a homogeneous line bundle on projective space $\mathbb{P}^d$ and $\mathcal{X} \subset \mathbb{P}^d$ is the Drinfeld half space. We prove that it is “essentially” a Jordan-Hölder series.
In our paper, we consider more generally locally analytic representations which are in the image of certain bi-functors

$$\mathcal{F}_P^G(\cdot, \cdot) : \mathcal{O}_\text{alg}^p \times \text{Rep}_{\infty,a}^\infty(L_P) \to \text{Rep}_{K,\text{alg}}^\ella(G)$$
depending on a standard parabolic subgroup $P \subset G$. Let us explain the notation used. We fix once and for all a finite extension $K$ of $L$ which will be our field of coefficients. Let $\text{Rep}_{K,\infty,a}^\infty(L_P)$ be the category of smooth admissible representations of a Levi subgroup $L_P \subset P$ on $K$-vector spaces. Let $\text{Rep}_{K,\text{alg}}^\ella(G)$ be the category of locally analytic representations of $G$ on $K$-vector spaces.

By $\mathcal{O}$ we denote the $BGG$-category of $U(g \otimes K)$-modules where $g = \text{Lie}(G)$. Then we let $\mathcal{O}_{\text{alg}}$ be the subcategory consisting of those modules $M$ which are of type $p = \text{Lie}(P)$ (i.e. $p_K$ acts locally finite, cf. [III, ch. 9]), and whose weights are integral. Given such a module $M$, there is a finite-dimensional $K$-subspace $W \subset M$ which is stable under $U(p_K)$ and which generates $M$ as $U(g_K)$-module. Let $d$ be the kernel of the canonical surjection $U(g_K) \otimes U(p_K) W \to M$, and let $W'$ be the dual representation of $p$. Our assumption on the weights implies that $W'$ lifts uniquely to an algebraic representation of $P$. Then we consider the locally analytic induced representation

$$\text{Ind}_P^G(W') = \{ f \in C^\text{an}_L(G,W') \mid \forall g \in G, p \in P : f(gp) = p^{-1} \cdot f(g) \}.$$ 

The action of $G$ on $\text{Ind}_P^G(W')$ is given by left translation: $(g \cdot f)(x) = f(g^{-1}x)$. There is a $C^\text{an}(G,K)$-valued paring

$$\langle \cdot, \cdot \rangle_{C^\text{an}(G,K)} : (U(g_K) \otimes U(p_K) W) \otimes_K \text{Ind}_P^G(W') \to C^\text{an}(G,K)$$

$$(\eta \otimes w) \otimes f \mapsto [g \mapsto ((\eta \cdot \tau f)(g))(w)]$$

where, for $\tau \in g$ we have, by definition,

$$(\tau \cdot f)(g) = \frac{d}{dt} f(g \exp(t\tau))|_{t=0}.$$ 

The common kernel of all $\langle \zeta, \cdot \rangle$, with $\zeta \in \mathfrak{d}$, is a subrepresentation

$$\mathcal{F}_P^G(M) = \text{Ind}_P^G(W')^\mathfrak{d} = \{ f \in \text{Ind}_P^G(W') \mid \text{for all } \zeta \in \mathfrak{d} : \langle \zeta, f \rangle_{C^\text{an}(G,K)} = 0 \}.$$ 

More generally, we consider smooth admissible $L_P$-representations $V$, and define the representation

$$\mathcal{F}_P^G(M,V) = \text{Ind}_P^G(W' \otimes_K V)^\mathfrak{d} = \{ f \in \text{Ind}_P^G(W' \otimes_K V) \mid \text{for all } \zeta \in \mathfrak{d} : \langle \zeta, f \rangle_{C^\text{an}(G,V)} = 0 \}$$

similarly as before (here, the pairing $\langle \cdot, \cdot \rangle_{C^\text{an}(G,V)}$ takes values in $C^\text{an}(G,V)$).
Let $M$ be an object of $\mathcal{O}^p$. We call the parabolic subgroup $P$, or its Lie algebra $\mathfrak{p}$, maximal for $M$, if $M$ does not lie in a subcategory $\mathcal{O}^q$ with a parabolic subalgebra $\mathfrak{q}$ properly containing $\mathfrak{p}$. It follows from [H2], sec. 9.4, that for every object $M$ of $\mathcal{O}^p$ there is unique parabolic subalgebra $\mathfrak{q} \supset \mathfrak{p}$ which is maximal for $M$. The same definition applies for objects in the subcategory $\mathcal{O}_{\text{alg}}^p$.

The main result of this paper is the following:

**Theorem 1.** $\mathcal{F}^G_P$ is functorial in both arguments: contravariant in $M$ and covariant in $V$.

2. $\mathcal{F}^G_P$ is exact in both arguments.

3. If $Q \supset P$ is a parabolic subgroup and $M$ is an object of $\mathcal{O}_{\text{alg}}^q$ (and hence, in particular, an object of $\mathcal{O}_{\text{alg}}^p$), then

$$
\mathcal{F}^G_P(M,V) = \mathcal{F}^G_Q(M,\text{ind}_{L^p(L_Q \cap U^P)}^Q(V))
$$

for all smooth admissible representations $V$ of $L^p$. Here, $\text{ind}_{L^p(L_Q \cap U^P)}^Q(V) = \text{ind}_P^Q(V)$ denotes the smooth induction of the representation $V$.

4. $\mathcal{F}^G_P(M,V)$ is topologically irreducible if and only if the following conditions are both satisfied:

- $M$ is simple,
- if $Q \supset P$ is maximal for $M$, then the smooth representation $\text{ind}_{L^p(L_Q \cap U^P)}^Q(V)$ is irreducible as $L_Q$-representation.

Here we assume that the residue field characteristic $p$ of $L$ is odd, if the root system $\Phi = \Phi(\mathfrak{g},t)$ has irreducible components of type $B$, $C$ or $F_4$. If irreducible components of type $G_2$ occur, we furthermore assume that $p > 3$.

Remark. We expect that the assumptions on the residue field characteristic $p$ of $L$ can eventually be removed, by a refinement of our methods.

Our main result covers a particular case of the irreducibility result shown in [OS]. In loc.cit. we considered arbitrary reductive groups and arbitrary locally analytic finite-dimensional $L^p$-representations $W$, but where, on the other hand, no differential equations appear. Using as an input Jordan-Hölder series for $M$ and $V$, and possibly for occurring smooth induced representations of the form $\text{ind}_{L^p(L_Q \cap U^P)}^Q(V)$, it is a rather formal exercise to determine a Jordan-Hölder series for $\mathcal{F}^G_P(M,V)$, cf. Section 6. Indeed, we give a recipe how to compute a Jordan-Hölder series for locally analytic representations of the form $\text{Ind}_P^G(W')$. Finally, we consider for a homogenous line bundle $\mathcal{L}$ on projective space $\mathbb{P}^d$ the filtration $H^0(X,\mathcal{L}) = \mathcal{L}(X)^0 \supset \mathcal{L}(X)^1 \supset \cdots \supset \mathcal{L}(X)^{d-1} \supset \mathcal{L}(X)^d = H^0(\mathbb{P}^d,\mathcal{L})$ consisting of closed $G$-subspaces with respect to Drinfeld’s half space $X \subset \mathbb{P}^d$. The duals of the graded pieces of
this filtration are locally analytic representations which lie in the image of the functors \( F_P^G \) (for \( G = \text{GL}_{d+1} \) and \( P \) a proper maximal standard parabolic subgroup) or which are extensions of such a locally analytic representation by a smooth generalized Steinberg representation. By using our main result, we prove that the locally analytic representations above are irreducible. This proves that \( \mathcal{L}(X)^\bullet \) is “essentially” a Jordan-Hölder series.

**Notation:** We denote by \( p \) a prime number and consider fields \( L \subset K \) which are both finite extensions of \( \mathbb{Q}_p \). Let \( O_L \) and \( O_K \) be the rings of integers of \( L \), resp. \( K \), and let the norm on \( K \) be such that \( |p|_K = p^{-1} \). For a locally convex \( K \)-vector space \( V \), we denote by \( V' \) its strong dual, i.e., the \( K \)-vector space of continuous linear forms equipped with the strong topology of bounded convergence.

We denote by \( G_0, P_0 \) etc. reductive groups over \( O_L \) and by \( G_0, P_0 \) etc. their \( p \)-adic groups of \( O_L \)-valued points. We use bold letters \( G, P \) etc. to denote their generic fibres over \( L \), whereas we use normal letters \( G, P \) etc. for their \( p \)-adic groups of \( L \)-valued points. Finally, Gothic letters \( g, p \) etc. will denote their Lie algebras.

We make the general convention that we denote by \( U(g), U(p) \) etc. the corresponding enveloping algebras, after base change to \( K \), i.e., what would be usually denoted by \( U(g) \otimes_L K, U(p) \otimes_L K \) etc. Similarly, we use the abbreviations \( D(G) = D(G, K), D(P) = D(P, K) \) etc. for the locally \( L \)-analytic distributions with values in \( K \).

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2. Preliminaries on \( \mathfrak{g} \)-modules and locally analytic representations

2.1. **Locally analytic representations.** We start with recalling some basic facts on locally analytic representations introduced by Schneider and Teitelbaum [ST1], [ST2].

For a locally \( L \)-analytic group \( H \), let \( C^{an}(H, K) \) be the locally convex vector space of locally \( L \)-analytic \( K \)-valued functions. More generally, if \( V \) is a Hausdorff locally convex \( K \)-vector space, let \( C^{an}(H, V) \) be the \( K \)-vector space consisting of locally analytic functions with values in \( V \). It has the structure of a Hausdorff locally convex vector space, as well. The dual \( D(H) = C^{an}(H, K)' \) is a topological \( K \)-algebra which has the structure of a Fréchet-Stein algebra when \( H \) is compact [ST2].

Let \( V \) be a Hausdorff locally convex \( K \)-vector space. A homomorphism \( \rho : H \rightarrow \text{GL}_K(V) \) (or simply \( V \)) is called a **locally analytic representation** of \( H \) if the topological \( K \)-vector space \( V \) is barrelled, the action of \( H \) on \( V \) is continuous, and the orbit maps \( \rho_v : H \rightarrow V, h \mapsto \rho(h)(v) \), are elements in \( C^{an}(H, V) \) for all \( v \in V \). If \( V \) is of compact type, i.e., it is a compact inductive limit
of Banach spaces, the strong dual \( V' \) is a nuclear Fréchet space and a separately continuous left \( D(H, K) \)-module. The duality functor gives an equivalence of categories

\[
\begin{align*}
\text{locally analytic } H\text{-representations on } K\text{-vector spaces} & \quad \longrightarrow \quad \text{separately continuous } D(H, K)\text{-modules on nuclear Fréchet spaces}
\end{align*}
\]

of compact type with continuous linear \( H \)-maps

In particular, \( V \) is topologically irreducible if \( V' \) is a simple \( D(H, K) \)-module.

For any closed subgroup \( H' \) of \( H \) and any locally analytic representation \( V \) of \( H' \), we denote by \( \text{Ind}_{H'}^{H}(V) \) the induced locally analytic representation. It is given by

\[
\begin{align*}
\text{Ind}_{H'}^{H}(V) & := \left\{ f \in C^\text{an}(H, V) \mid f(h \cdot h') = (h'^{-1} \cdot f(h)) \forall h', \forall h \in H \right\}.
\end{align*}
\]

The group \( H \) acts on this vector space by \((h \cdot f)(x) = f(h^{-1}x)\). For a finite-dimensional representation \( V \), the following map is an isomorphism of \( D(H, K) \)-modules:

\[
(2.1.2) \quad \text{Ind}_{H'}^{H}(V) \otimes_{D(H', K)} V' \to (\text{Ind}_{H'}^{H}(V))', \quad \delta \otimes \varphi \mapsto \delta \cdot \varphi,
\]

with \((\delta \cdot \varphi)(f) = \delta(h \mapsto \varphi(f(h)))\) (cf. \cite{K2}, identity (53) in the proof of Prop. 5.1, Prop. 5.3, and Remark 5.4). Finally we recall that for a reductive group \( G \), every rational \( G \)-representation and every smooth admissible \( G(L) \)-representation may be considered as locally analytic representations, cf. \cite{ST4, §2].

\subsection*{2.2. The category \( O \) and its parabolic variants \( O^p \).}

Let \( G \) be a connected split reductive group over \( L \). We fix a Borel subgroup \( B \) of \( G \) and denote its unipotent radical by \( U \). Further we fix a maximal split torus \( T \subset B \). The category \( O \) in the sense of Bernstein, Gelfand, Gelfand \cite{BGG}, \cite{HI} is the full subcategory of all \( g \)-representations whose objects \( M \) satisfy the following properties:

1. The action of \( u \) on \( M \) is locally finite, i.e. for all \( m \in M \), the subspace \( U(u) \cdot m \subset M \) is finite-dimensional.
2. The action of \( t \) on \( M \) is semi-simple and locally finite.
3. \( M \) is finitely generated as \( U(g) \)-module.

\footnote{Traditionally, the category \( O \) is defined for semi-simple Lie algebras. Here we extend their definition to general reductive Lie algebras.}
It is known that $O$ is a $K$-linear, abelian, noetherian, artinian category which is closed under submodules and quotients, cf. [11]. In particular, any $U(g)$-module in $O$ has a Jordan-Hölder series in $O$.

In our paper we are rather interested in a subcategory of $O$. The reason will become clear later on. Note that by property (2), we may write any object $M$ in $O$ as a direct sum

\[(2.2.3)\quad M = \bigoplus_{\lambda \in t^*} M_\lambda\]

where $M_\lambda = \{ m \in M \mid t \cdot m = \lambda(t)m \ \forall t \in t \}$ is the $\lambda$-eigenspace attached to $\lambda \in t^*$. Let $X^*(T)$ be the group of characters of the torus $T$ which we consider via the derivative as a subgroup of $t^*$.

**Definition 2.3.** We denote by $O_{\text{alg}}$ the full subcategory of $O$ whose objects are $U(g)$-modules such that all $\lambda$ appearing in (2.2.3) are integral, i.e., are contained in $X^*(T) \subset t^*$.

Thus $M \in O$ is an object of $O_{\text{alg}}$ if the $t$-module structure on $M_\lambda$ lifts to an algebraic action of $T$. Again, $O_{\text{alg}}$ is an abelian noetherian, artinian category which is closed under submodules and quotients. The Jordan-Hölder series of a given $U(g)$-module lying in $O_{\text{alg}}$ is the same as the one considered in the category $O$.

**Example 2.4.** For $\lambda \in t^*$, let $K_{\lambda} = K$ be the 1-dimensional $t$-module where the action is given by $\lambda$. Let

$M(\lambda) = U(g) \otimes_{U(b)} K_{\lambda} \in O$

be the corresponding Verma module. Denote by $L(\lambda) \in O$ be its simple quotient. Then $M(\lambda)$ resp. $L(\lambda)$ is an object of $O_{\text{alg}}$ if and only if $\lambda \in X^*(T)$.

Let $P$ be a standard parabolic subgroup of $G$ with Levi decomposition $P = L_P \cdot U_P$ where $T \subset L_P$. Let $O^p$ be the category consisting of $U(g)$-modules of type $p = \text{Lie}(P)$, cf. [11, ch. 9]. Recall that its objects are $U(g)$-modules $M$ satisfying the following properties:

1. The action of $u_P$ on $M$ is locally finite.
2. The action of $t_P$ on $M$ is semi-simple and locally finite.
3. $M$ is finitely generated as $U(g)$-module.

Clearly the category $O^p$ is a full subcategory of $O$. Further it is $K$-linear, abelian and which is closed under submodules and quotients, cf. [11]. Hence the Jordan-Hölder series of every $U(g)$-module in $O^p \subset O$ lies in $O^p$, as well. Finally, if $Q$ is a standard parabolic subgroup with $Q \supset P$, then $O^q \subset O^p$. In particular, for $p = g$ the category $O^g$ consists of all finite-dimensional semi-simple $g$-modules.

Similarly, as before we define a subcategory $O^p_{\text{alg}}$ of $O^p$ as follows. Let $\text{Irr}(l_P)^{\text{fd}}$ be the set of finite-dimensional irreducible $l_P$-modules. Again any object in $O^p$ has by property (2) a
decomposition into $l_P$-modules

\[ M = \bigoplus_{a \in \text{Irr}(l_P)^{\text{id}}} M_a \]

where $M_a \subset M$ is the $a$-isotypic part of the representation $a$. We denote by $O_{\text{alg}}^p$ the full subcategory of $O^p$ given by objects of $O^p$ such that all $l_P$-representations appearing in (2.4.4) are integral i.e., are induced by finite-dimensional algebraic $L_P$-representations. Again, the category $O_{\text{alg}}^p$ is contained in $O_{\text{alg}}$ and contains all finite-dimensional $g$-modules which are induced by $G$-modules. Every object in $O_{\text{alg}}^p$ has a Jordan-Hölder series which coincides with the Jordan-Hölder series in $O_{\text{alg}}$. It is easily verified that $O_{\text{alg}}^p = O_{\text{alg}}^p \cap O^p$.

**Example 2.5.** Let $\Delta$ be the set of simple roots of $G$ with respect to $T \subset B$. Let $\lambda \in X(T)^*$ and set $I = \{ \alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \}$. Let $P = P_I$ be the standard parabolic subgroup of $G$ attached to $I$. Then $\lambda$ is dominant with respect to the Levi subgroup $L_P$. Denote by $V_I(\lambda)$ the corresponding irreducible finite-dimensional algebraic $L_P$-representation, cf. [Ja]. We consider it as a $P$-module by letting act $U_P$ trivially on it. The generalised parabolic Verma module (in the sense of Lepowsky [Le]) attached to the weight $\lambda$ is given by

\[ M_I(\lambda) = U(g) \otimes_{U(P_I)} V_I(\lambda). \]

Then $M_I(\lambda)$ is an object of $O_{\text{alg}}^p$. Further, there is a surjective map

\[ M(\lambda) \twoheadrightarrow M_I(\lambda), \]

where the kernel is given by the image of $\bigoplus_{\alpha \in I} M(s_\alpha \cdot \lambda) \to M(\lambda)$. It follows that $L(\lambda)$ is an object of $O_{\text{alg}}^p$, as well.

3. From $O_{\text{alg}}$ to locally analytic representations

**3.1. The representation associated to an object of $O_{\text{alg}}$.** We fix as in the previous chapter a standard parabolic subgroup $P$ with Levi decomposition $P = L_P \cdot U_P$ where $T \subset L_P$. Let $M$ be an object of $O_{\text{alg}}^p$. By the defining axioms (1)-(3) for $O_{\text{alg}}^p$, we may choose a finite-dimensional representation $W \subset M$ of $p$ which generates $M$ as $U(g)$-module. Thus we have a short exact sequence of $U(g)$-modules

\[ 0 \to \mathfrak{d} \to U(g) \otimes_{U(p)} W \to M \to 0, \]

where, by definition, $\mathfrak{d}$ is the kernel of the canonical map $U(g) \otimes_{U(p)} W \to M$. We denote the representation of $p$ on $W$ by $\rho$. It is induced by an algebraic $P$-representation by the following lemma.

**Lemma 3.2.** The representation $\rho$ lifts uniquely to an algebraic $P$-representation on $W$, which we denote again by $\rho$. 
Proof. As $M$ is an object of $O^p$, the $U(p)$-module $W$, considered as a $U(p)$-module, decomposes as a direct sum of isotypic modules $W_a$. Each module $W_a$ (whose weights are integral) lifts uniquely to an algebraic representation of $L_p$. We denote this action of $L_p$ on $W = \bigoplus_a W_a$ by $\rho_L$. On the other hand, since the unipotent radical $U_p$ of $P$ is simply connected, the action of the Lie algebra $u_p$ integrates uniquely to give an algebraic action of $U_p$ on $W$. More precisely, a given element $u = \exp(x) \in U_p(K)$ acts as $\rho(u) := \sum_{n \geq 0} \frac{\rho(x)^n}{n!}$, where $\rho(x)^n = 0$ for $n \gg 0$.

These two representations are compatible in the sense that $\rho_L(h) \circ \rho(u) \circ \rho_L(h^{-1}) = \rho(\text{Ad}(h)(u))$, for $h \in L_p(K)$, $u \in U_p(K)$. This shows that $\rho$ lifts uniquely to an algebraic representation of $P$ on $W$. $\square$

The induced locally analytic algebraic action of $P$ on the dual space $W' = \text{Hom}_K(W, K)$ will be denoted by $\rho'$. We consider the locally analytic induced representation $\text{Ind}^G_P(W')$. By (2.1.2) the canonical map of $D(G)$-modules

$$D(G) \otimes_{D(P)} W \longrightarrow (\text{Ind}^G_P(W'))', \quad \delta \otimes w \mapsto [f \mapsto \delta(f)(w)],$$

is an isomorphism of topological vector spaces. We thus have a pairing

$$(3.2.6) \langle \cdot, \cdot \rangle : (D(G) \otimes_{D(P)} W) \otimes_K \text{Ind}^G_P(W') \longrightarrow K,$$

which identifies the left hand side with the topological dual of the right hand side and vice versa. We remark that the Iwasawa decomposition $G = G_0 \cdot B$ shows that as $D(G_0)$-modules, the restriction of $D(G) \otimes_{D(P)} W$ to $D(G_0)$ is isomorphic to $D(G_0) \otimes_{D(P)} W$. The pairing (3.2.6) is obtained from the following $C^{an}(G, K)$-valued pairing by composition with the evaluation map $C^{an}(G, K) \to K$, $f \mapsto f(1)$.

$$(3.2.7) \langle \cdot, \cdot \rangle_{C^{an}(G, K)} : \quad (D(G) \otimes_{D(P)} W) \otimes_K \text{Ind}^G_P(W') \longrightarrow C^{an}(G, K)$$

$$(\delta \otimes w) \otimes f \quad \mapsto \quad \left[ g \mapsto ( ((\delta \cdot r f)(g))(w) \right]$$

where, by defintion, we have $(\delta \cdot r f)(g) = \delta(x \mapsto f(gx))$. Furthermore, there is a canonical injective map of $U(g)$-modules

$$U(g) \otimes_{U(p)} W \to D(G) \otimes_{D(P)} W.$$

We denote by $\text{Ind}^G_P(W')^\delta$ the subspace of $\text{Ind}^G_P(W')$ annihilated by $\delta$ via the pairing $\langle \cdot, \cdot \rangle_{C^{an}(G, K)}$:

$$\text{Ind}^G_P(W')^\delta = \{ f \in \text{Ind}^G_P(W') \mid \text{for all } \delta \in \delta : \langle \delta, f \rangle_{C^{an}(G, K)} = 0_{C^{an}(G, K)} \}.$$
Then Ind^G_P(W')^0 is clearly a closed G-invariant subspace, since the action of U(g) is smooth on Ind^G_P(W').

Lemma 3.3. i) The representation Ind^G_P(W')^0 is an admissible locally analytic G-representation (in the sense of [ST2, sec. 6]).

ii) The annihilator of Ind^G_P(W')^0 in D(G) ⊗ D(P) W, i.e., the set

\{ψ ∈ D(G) ⊗ D(P) W | for all f ∈ Ind^G_P(W')^0 : ⟨ψ, f⟩ = 0 \}

is equal to D(G) diarr. We therefore have

\((Ind^G_P(W')^0)' \simeq (D(G) ⊗ D(P) W) / D(G) diarr.\)

Proof. (i) The representation Ind^G_P(W') is (strongly) admissible. Since closed subspaces of admissible representations are admissible again [ST2, Prop. 6.4 (iii)], the assertion follows.

(ii) By [ST2, Theorem 6.3], the admissible subrepresentations of Ind^G_P(W') are in bijection with the coadmissible quotients of D(G) ⊗ D(P) W. By [ST2, Lemma 3.6] these are given by the closed submodules of D(G) ⊗ D(P) W. The bijection is given by associating to a closed submodule J ⊂ D(G) ⊗ D(P) W the representation

\{f ∈ Ind^G_P(W') | for all ψ ∈ J : ⟨ψ, f⟩ = 0 \}

By definition, for f ∈ Ind^G_P(W') to lie in Ind^G_P(W')^0 is equivalent to satisfying

\[ \sum_{i=1}^{m} \left( (η_i \cdot_r f)(g) \right)(w_i) = 0 \]

for all g ∈ G and all \( \sum_i η_i \otimes w_i \in diarr \). By the definition of \( \cdot_r \) and the definition of the convolution product in D(G) this is equivalent to

\[ \sum_{i=1}^{m} \left( (δ_g \cdot η_i)(f) \right)(w_i) = 0 \]

for all g ∈ G and all \( \sum_i η_i \otimes w_i \in diarr \). Hence f is in Ind^G_P(W')^0 if and only if \( ⟨δ_g \cdot η_i, f⟩ = 0 \), for all g ∈ G and all η ∈ diarr, where \( ⟨·,·⟩ \) is the K-valued pairing in (3.2.6). Because the delta distributions δ are dense in D(G), this is equivalent to saying that \( ⟨δ, f⟩ = 0 \), for all δ ∈ D(G) diarr. As diarr is again an object of \( O^p_{alg} \), it is finitely generated as \( U(g) \)-module. Hence D(G) diarr is finitely
generated as $D(G)$-module and therefore closed by [ST2 Cor. 3.4, Lemma 3.6]. The assertion follows. □

3.4. Another description. We proceed by giving another description of the dual space of $\text{Ind}^G_F(W')^0$. The action of $p$ integrates uniquely to a locally analytic representation of $P$ on $M$. Indeed, we may write $M$ as a union of finite-dimensional algebraic $p$-modules. Each individual term has by Lemma 3.2 the structure of an algebraic locally analytic $P$-representation. By considering the locally convex limit topology, it inherits the structure of a locally analytic $P$-representation. Thus we have on $M$ a $D(P)$-module structure which extends the $p$-module structure, and is hence compatible with the action of $g$ on $M$, cf. [ST1 Prop. 3.2].

Consider the subring $U(g, P)$ generated by $U(g)$ and $D(P)$ inside $D(G)$. It follows from the proposition below that this ring is equal to $U(g)D(P)$, i.e., every element of $U(g, P)$ can be written as a finite sum of elements of the form $\mathfrak{z} \cdot \delta$ with $\mathfrak{z} \in U(g)$ and $\delta \in D(P)$.

Proposition 3.5. Let $H \subset G$ be a closed analytic subgroup, and let $\delta \in D(H)$. Then $\delta \cdot U(g) \subset U(g) \cdot D(H)$. In particular, the smallest subring of $D(G, K)$ containing $U(g)$ and $D(H, K)$ consists of finite sums $\sum_j \mathfrak{z}_j \cdot \delta_j$ with $\mathfrak{z}_j \in U(g)$ and $\delta_j \in D(H, K)$.

Proof. Of course, it is enough to show that for any $\mathfrak{r} \in g$ we have $\delta \cdot \mathfrak{r} \in g \cdot D(H, K)$. Let $f \in C^{\text{an}}(G, K)$. For $g \in G$ we denote by $(g.f)(x) = f(g^{-1}x)$. Recall that the convolution product of two distributions $\lambda_1, \lambda_2 \in D(G)$ is given by

$$(\lambda_1 \cdot \lambda_2)(f) = \lambda_1(g \mapsto \lambda_2(g^{-1}f)) = \lambda_1(g \mapsto \lambda_2(h \mapsto f(gh)) = \lambda_2(h \mapsto \lambda_1(g \mapsto f(gh))).$$

The last equality might be called “Fubini’s Theorem”, and this also used below in one instance. Furthermore, the image of $\mathfrak{r}$ in $D(G, K)$ is given by the formula

$$\mathfrak{r}(f) = \lim_{t \to 0} \frac{1}{t}(f(\exp(t\mathfrak{r})) - f(1)).$$

For $h \in H$ and $\mathfrak{r}$ as above write $\text{Ad}(h)(\mathfrak{r}) = \sum_{i=1}^d c_i(h)\mathfrak{r}_i$, where $(\mathfrak{r}_i)_i$ is some basis for $g$, and the $c_i$ are locally analytic functions on $H$. Define distributions $\delta_i \in D(H, K)$ by $\delta_i(f) = \delta(h \mapsto c_i(h)f(h))$. Then we compute:

$$(\delta \cdot \mathfrak{r})(f) = \delta(h \mapsto \mathfrak{r}(h^{-1}.f))$$

$$= \delta(h \mapsto \lim_{t \to 0} \frac{1}{t}((h^{-1}.f)(\exp(t\mathfrak{r})) - (h^{-1}.f)(1)))$$

$$= \lim_{t \to 0} \frac{1}{t}(f(h \exp(t\mathfrak{r})) - f(h)))$$
\[
\delta(h \mapsto \lim_{t \to 0} \frac{1}{t}(f(h \exp(t \tau) h^{-1}) - f(h)))
\]
\[
= \delta(h \mapsto \lim_{t \to 0} \frac{1}{t}(f(\exp(t \text{Ad}(h)(\tau))h) - f(h)))
\]
\[
= \delta(h \mapsto \text{Ad}(h)(\tau)(g \mapsto f(gh)))
\]
\[
= \delta(h \mapsto (\sum_{i=1}^{d} c_i(h)\tau_i)(g \mapsto f(gh)))
\]
\[
= \sum_{i=1}^{d} \delta(h \mapsto c_i(h)\tau_i)(g \mapsto f(gh)))
\]
\[
= \sum_{i=1}^{d} \delta(h \mapsto \tau_i(g \mapsto c_i(h)f(gh)))
\]
\[
= \sum_{i=1}^{d} \tau_i(g \mapsto \delta(h \mapsto c_i(h)f(gh)))
\]
\[
= \sum_{i=1}^{d} \delta(h \mapsto \tau_i(g \mapsto c_i(h)(g^{-1}.f)(h)))
\]
\[
= \sum_{i=1}^{d} \delta_i(g \mapsto \tau_i(g \mapsto c_i(h)(g^{-1}.f)))
\]
\[
= \sum_{i=1}^{d} (\tau_i \cdot \delta_i)(f)
\]

And this shows that \( \delta \cdot \tau = \sum_{i=1}^{d} \tau_i \cdot \delta_i \) is in \( g \cdot D(H, K) \).

Thus \( M \) becomes a \( U(g, P) \)-module. Moreover, any morphism \( M' \to M \) in \( \mathcal{O}_{\text{alg}}^P \) is automatically a homomorphism of \( U(g, P) \)-modules.

**Proposition 3.6.** The canonical map

\[
\iota : M = (U(g) \otimes_{U(P)} W) / \mathfrak{d} \longrightarrow (D(G) \otimes_{D(P)} W) / D(G)\mathfrak{d}
\]

extends to an isomorphism of \( D(G) \)-modules

\[
D(G) \otimes_{U(g, P)} M \simeq (D(G) \otimes_{D(P)} W) / D(G)\mathfrak{d}
\]
and therefore we get

\[ D(G) \otimes_{U(g,P)} M \simeq (\text{Ind}^{G}_P(W'))' \, . \]

The analogous isomorphism holds for the compact subgroup \( G_0 \):

\[ D(G_0) \otimes_{U(g,P_0)} M \simeq (\text{Ind}^{G}_P(W'))' \, , \]

with \( U(g,P_0) = U(g)D(P_0) \).

Proof. First we have to show that \( \iota \) is \( U(g,P) \)-linear. But this follows from the \( U(g,P) \)-linearity of the natural maps \( q : U(g) \otimes_{U(p)} W \to M \) and \( U(g) \otimes_{U(p)} W \to D(G) \otimes D(P) W \) together with the fact that \( \mathfrak{d} \) is a \( U(g,P) \)-submodule of \( U(g) \otimes_{U(p)} W \). This shows that \( \iota \) extends to \( D(G) \)-module homomorphism

\[ \Phi : D(G) \otimes_{U(g,P)} M \to (D(G) \otimes_{D(P)} W) / D(G) \mathfrak{d} \]

by setting \( \Phi(\delta \otimes v) = \delta \iota(v) \). In the other direction we define

\[ \Psi : (D(G) \otimes_{D(P)} W) / D(G) \mathfrak{d} \to D(G) \otimes_{U(g,P)} M \]

by \( \Psi((\delta \otimes w) + D(G) \mathfrak{d}) = \delta \otimes (w + \mathfrak{d}) \). It is immediate that \( \Psi \) is well-defined and easily checked that \( \Phi \) and \( \Psi \) are inverse to each other. \( \Box \)

3.7. Yet another description. In this paragraph we present another approach to the representation \( \text{Ind}^{G}_P(W')\mathfrak{d} \) which appeared already in [O] in the case of \( G = \text{GL}_n \). Here we correct certain group actions which were formulated in loc.cit..

Let \( G_0 \) be a split reductive group model of \( G \) over \( O_K \). Let \( T_0 \subset G_0 \) be a maximal torus and fix a Borel subgroup \( B_0 \subset G_0 \) containing \( T_0 \). We fix a standard parabolic subgroup \( P_0 \) of \( G_0 \). We denote by \( U_{P,0} \) its unipotent radical, by \( U_{-P,0} \) its opposite unipotent radical and by \( L_{P,0} \) its Levi component containing \( T_0 \).

Let \( \pi \in O_L \) be a uniformizer. For any positive integer \( n \in \mathbb{N} \), we consider the reduction map

(3.7.1) \[ p_n : G_0(O_L) \to G_0(O_L/(\pi^n)) \]

Set

\[ P^n = p_n^{-1}(P(O_L/(\pi^n))), \quad U^n = p_n^{-1}(U_P(O_L/(\pi^n))), \quad L^n = p_n^{-1}(L_P(O_L/(\pi^n))) \]

and

\[ U^{-n} = \ker(U_P(O_L) \to U_P(O_L/(\pi^n))). \]
These are compact open subgroups of $G_0 = G(O_L)$. The Levi decomposition on $P(O_L/(\pi^n))$ induces a decomposition

\[(3.7.2)\quad P^n = L^n_P \cdot U^n_P.\]

Further we have equalities

\[
P^n = U^{n,0}_P \cdot P_0 = P_0 \cdot U^{-n}_P,
\]
\[
U^n_P = U^{n,0}_P \cdot (U_P)_0 = (U_P)_0 \cdot U^{-n}_P,
\]
\[
L^n_P = U^{n,0}_P \cdot (L_P)_0 = (L_P)_0 \cdot U^{-n}_P.
\]

We may interpret $U^{n,0}_P \subset (U_P)_0$ as the $L$-valued points of an open $L$-affinoid polydisc, since all non-diagonal entries $x$ in $U^{n,0}_P$ have norm $|x| \leq |\pi^n|$. Then the ring of $K$-valued rigid-analytic functions $\mathcal{O}(U^{n,0}_P)$ is a $K$-Banach algebra equipped with the following norm. Let

\[
\Phi_{U^P} = \{\beta_1, \ldots, \beta_r\}
\]

be the set of roots appearing in $u_P$. We consider the ring $\mathcal{O}(U^P)$ of $K$-valued algebraic functions on $U^P$ as the polynomial $K$-algebra in the indeterminates $X_{\beta_1}, \ldots, X_{\beta_r}$. For $n \in \mathbb{N}$, set $\epsilon_n = |\pi|^n$. Then

\[(3.7.3)\quad \left| \sum_{(i_1, \ldots, i_r) \in \mathbb{N}_0^n} a_{i_1, \ldots, i_r} X_{\beta_1}^{i_1} \cdots X_{\beta_r}^{i_r} \right|_n := \sup_{(i_1, \ldots, i_r) \in \mathbb{N}_0^n} |a_{i_1, \ldots, i_r}|_K \epsilon_n^{i_1 + \cdots + i_r}.
\]

defines a norm on the $K$-algebra $\mathcal{O}(U^P)$ so that $\mathcal{O}(U^{n,0}_P)$ becomes the completion of it. This $K$-Banach space is contained in the larger ring of bounded functions on $U^{n,0}_P$

\[
\mathcal{O}_b(U^{n,0}_P) := \left\{ \sum_{(i_1, \ldots, i_r)} a_{i_1, \ldots, i_r} X_{\beta_1}^{i_1} \cdots X_{\beta_r}^{i_r} \mid a_{i_1, \ldots, i_r} \in K, \sup_{(i_1, \ldots, i_r)} |a_{i_1, \ldots, i_r}|_K \epsilon_n^{i_1 + \cdots + i_r} < \infty \right\}
\]

which is a $K$-Banach space with the same norm, as well.

Let $W$ be a finite-dimensional locally analytic $P$-representation. The open subgroups $P^n \subset G, n \in \mathbb{N}$, form a cofinal system of $P_0$-stable neighbourhoods of the identity. This gives rise to an identification

\[
\text{Ind}_{P_0}^{G_0}(W') \cong \lim_{\longrightarrow \, n \in \mathbb{N}} \text{Ind}_{P^n}^{G_0}(V_n),
\]

where

\[
V_n = \{\text{rigid-analytic maps } f : P^n \to W' \mid f(gp) = p^{-1} \cdot f(g)\}
\]

and where $\text{Ind}_{P^n}^{G_0}$ denotes the ordinary induction of group representations. On $V_n$ the subgroup $P^n$ acts via left translation. There is a natural identification

\[
V_n \cong \mathcal{O}(U^{n,0}_P) \otimes W'.
\]

The action of $P^n = U^{n,0}_P \cdot (L_P)_0 \cdot U^{n,0}_P$ on the latter object translates as follows. The subgroup $(L_P)_0$ acts via conjugation on $U^{n,0}_P$ on the first factor and on $W'$ by the given one. The subgroup
$U_P^{-n}$ acts by translations on the first factor and trivially on $W'$. We omit the description of the action of $(U_P)_0$ since we won’t use it explicitly.

On the other hand, we consider $W$ as a Lie algebra representation of $\mathfrak{p}$. Since the universal enveloping algebra of $\mathfrak{g}$ splits (by the PWB-theorem) into a tensor product $U(\mathfrak{g}) = U(\mathfrak{u}_P) \otimes_K U(\mathfrak{p})$, we get $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W' = U(\mathfrak{u}_P) \otimes_K W'$. Similarly as above, there is an action of $P_0$ on this space. On elements $x \otimes w \in U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W'$, a given $p \in P_0$ acts by

$$p \cdot (x \otimes w) = \text{Ad}(p)(x) \otimes pw.$$  

Here $\text{Ad}$ denotes the adjoint action. Under the identification $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W' = U(\mathfrak{u}^-) \otimes W'$ the action of the Levi factor $L_P$ is again via the adjoint action on $U_P^{-n}$ and the natural one on $W'$. We stress that there is no natural action of $U_P^{-n}$ on the above space since it not complete in a suitable sense, cf. Step 2 in the proof of Theorem 3.1.

Now we consider the $L_P$-equivariant pairing

$$O(U_P) \times U(u_P) \to K$$

$$(f, z) \mapsto z \cdot f(1).$$

This is a non-degenerate pairing and induces therefore a $K$-linear $L_P$-equivariant injection

$$O(U_P) \hookrightarrow \text{Hom}_K(U(u_P), K).$$

given by

$$X^{i_1}_{\beta_1} \cdots X^{i_r}_{\beta_r} \mapsto (i_1)! \cdots (i_r)! \cdot (L^{i_1}_{\beta_1} \cdots L^{i_r}_{\beta_r})^*.$$  

Here $\{(L^{i_1}_{\beta_1} \cdots L^{i_r}_{\beta_r})^* \mid (i_1, \ldots, i_r) \in \mathbb{N}_0^r\}$ is the dual basis of $\{L^{i_1}_{\beta_1} \cdots L^{i_r}_{\beta_r} \mid (i_1, \ldots, i_r) \in \mathbb{N}_0^r\}$. The pairing (3.7.3) extends by continuity to a non-degenerate $(L_P)_0$-equivariant pairing

$$O(U_P^{-n}) \times U(u_P) \to K.$$  

which induces a $P_0$-equivariant pairing

$$(f \otimes n, z \otimes \phi) \mapsto \phi(n) \cdot z \cdot f(1).$$  

For $\epsilon \in [K^*]$, we consider the norm $\| \cdot \|_\epsilon$ on $U(u_P)$ given by

$$\| \sum_{(i_1, \ldots, i_r) \in \mathbb{N}_0^r} a_{i_1, \ldots, i_r} L^{i_1}_{\beta_1} \cdots L^{i_r}_{\beta_r} \|_\epsilon := \sup_{(i_1, \ldots, i_r) \in \mathbb{N}_0^r} |(i_1)! \cdots (i_r)! \cdot a_{i_1, \ldots, i_r}| \epsilon^{i_1 + \cdots + i_r}.$$  

The completion of $U(u_P)$ with respect to this norm yields the $K$-algebra

$$U(u_P)_\epsilon := \left\{ \sum_{(i_1, \ldots, i_r) \in \mathbb{N}_0^r} a_{i_1, \ldots, i_r} L^{i_1}_{\beta_1} \cdots L^{i_r}_{\beta_r} \mid a_{i_1, \ldots, i_r} \in K, \right.$$  

$$\left. |(i_1)! \cdots (i_r)! \cdot a_{i_1, \ldots, i_r}| \epsilon^{i_1 + \cdots + i_r} \to 0, i_1 + \cdots + i_r \to \infty \right\}.$$
We abbreviate $U(u_p^-)_n := U(u_p^-)_{p^n}$. By the example given in [S2] ch. I, §3 we see that the pairing [3.7.4] extends to

\[ \mathcal{O}_b(U_p^{-,n}) \times U(u_p^-)_n \to K \]

such that $\mathcal{O}_b(U_p^{-,n})$ becomes the topological dual of $U(u_p^-)_n$. It follows that $\mathcal{O}_b(U_p^{-,n}) \otimes W'$ is the topological dual of $U(u_p^-)_n \otimes W$. In particular, we get an action of $U_p^{-,n}$ and hence of $P^n$ on $U(u_p^-)_n \otimes W$.

The following statement has been proved for $G = \text{GL}_n$ in [O].

**Proposition 3.8.** There is an isomorphism of (Hausdorff) locally convex $K$-vector spaces

\[ \lim_{n \to \infty} \mathcal{O}(U_p^{-,n}) \otimes W' \cong \left( \lim_{n \to \infty} U(u_p^-)_n \otimes W \right)' \]

compatible with the action of $\lim_{n \to \infty} P^n = P_0$.

**Proof.** The proof is the same as in loc.cit. and proceeds as follows. There are identifications

\[ \lim_{n \to \infty} \mathcal{O}_b(U_p^{-,n}) = \lim_{n \to \infty} \mathcal{O}(U_p^{-,n}) \]

resp.

\[ \lim_{n \to \infty} \mathcal{O}_b(U_p^{-,n}) \otimes W' = \lim_{n \to \infty} \mathcal{O}(U_p^{-,n}) \otimes W' \]

of locally convex $K$-vector spaces. As we saw above, the space $\mathcal{O}_b(U_p^{-,n}) \otimes W'$ is the topological dual of $U(u_p^-)_n \otimes W$. The claim follows now from [Mo1] Theorem 3.4 respectively [S2] Prop. 16.10 on the duality of projective limits of $K$-Fréchet spaces and injective limits of $K$-Banach spaces.

Now we consider more generally a surjective map

\[ \phi : U(\mathfrak{g}) \otimes_{U(p)} W \to M \]

of $U(\mathfrak{g})$-modules as in (3.1.5). Let $\mathfrak{d} = \ker \phi$ be its kernel. We may consider $\mathfrak{d}$ by PWB as a submodule of $U(u_p^-) \otimes_K W$. We denote by

\[ \mathfrak{d}_n \subset U(u_p^-)_n \otimes_K W \]

its topological closure in $U(u_p^-)_n \otimes W$. Finally, we put

\[ (\mathcal{O}(U_p^{-,n}) \otimes W')^\mathfrak{d} = \{ f \in \mathcal{O}(U_p^{-,n}) \otimes W' | \langle \mathfrak{z}, f \rangle = 0 \forall \mathfrak{z} \in \mathfrak{d} \}. \]

Then $U(u_p^-)_n \otimes W'/\mathfrak{d}_n$ and $(\mathcal{O}(U_p^{-,n}) \otimes W')^\mathfrak{d}$ are $K$-Banach spaces equipped with a natural $P^n$-action, as well. The following statement generalises Proposition 3.8.

**Proposition 3.9.** There is an isomorphism of (Hausdorff) locally convex $K$-vector spaces

\[ \lim_{n \to \infty} (\mathcal{O}(U_p^{-,n}) \otimes W')^\mathfrak{d} \cong \left( \lim_{n \to \infty} U(u_p^-)_n \otimes W/\mathfrak{d}_n \right)' \]

compatible with the action of $\lim_{n \to \infty} P^n = P_0$. 
Proof. We define similarly \((O(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d}, (O_b(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d}\) and \((O_b(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d}\). By the density of \(\mathfrak{d}\) in \(\mathfrak{d}\), we have
\[(O(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d} = (O(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d}\]
resp.
\[(O_b(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d} = (O_b(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d}\].
Now, the \(K\)-Banach space \((O_b(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d}\) is the topological dual of \(U(u^\mathfrak{p}_{n}) \otimes W/\mathfrak{d}\). Then we proceed with the argumentation as in the proof of Proposition 3.8. \(\square\)

Corollary 3.10. There is an isomorphism of locally convex \(K\)-vector spaces compatible with the action of \(G_0\).

\[\text{Ind}^G_{G_0}(W')^\mathfrak{d} = \lim_{\leftarrow} \text{Ind}^G_0(U(U^\mathfrak{p}_{n}^{-n}) \otimes W')^\mathfrak{d} \cong \left( \lim_{\leftarrow} \text{Ind}^G_0(U(u^\mathfrak{p}_{n}) \otimes W/\mathfrak{d}) \right)'\]

4. The functor \(\mathcal{F}_P^G\) and its properties

4.1. The functor \(\mathcal{F}_P^G\). Denote by \(\text{Rep}^\mathfrak{d}_K(G)\) the category of locally analytic representations of \(G\) on \(K\)-vector spaces. We saw in the previous section that for any object \(M \in \mathcal{O}_\mathfrak{p}^{\text{alg}}\), the \(D(G_0)\)-module \((\text{Ind}^G_0(W')^\mathfrak{d})'\) is finitely generated and hence admissible. In particular, \(\text{Ind}^G_0(W')^\mathfrak{d} = (D(G) \otimes_{U(\mathfrak{g}, P)} M)'\) is reflexive, and we can thus define a functor

\[\mathcal{F}_P^G : \mathcal{O}_\mathfrak{p}^{\text{alg}} \rightarrow \text{Rep}^\mathfrak{d}_K(G)\]

by

\[\mathcal{F}_P^G(M) = (D(G) \otimes_{U(\mathfrak{g}, P)} M)'\].

Proposition 4.2. The functor \(\mathcal{F}_P^G\) is exact.

Proof. Let \(0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0\) be an exact sequence in the category \(\mathcal{O}_\mathfrak{p}^{\text{alg}}\). Then there are finite-dimensional algebraic generating \(\mathfrak{p}\)-representations \(W_i \subset M_i, i = 1, 2, 3,\) such that we have an induced exact sequence \(0 \rightarrow W_1 \rightarrow W_2 \rightarrow W_3 \rightarrow 0\). Indeed, let \(W_2\) be a \(\mathfrak{p}\)-submodule which generates \(M_2\). Then the image \(W_3\) of \(W_2\) in \(M_3\) generates \(M_3\). Consider \(X := \ker(W_2 \rightarrow W_3)\). If it does not generate \(M_1\), then we let \(Y\) be any generating system and put \(W_1 = X + Y\) resp. replace \(W_2\) by \(W_2 + Y\). The resulting sequence satisfies the claim.

We get a commutative diagram with exact rows:
Note that exactness of the middle row follows from the exactness of the functor $U(g) \otimes U(p) \cdot$. The exactness of the last row is a consequence of the snake lemma. If we apply our functor $F = F_p^G$ to the above diagram we get a new diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \Leftarrow F(M_1) & \Leftarrow F(M_2) & \Leftarrow F(M_3) & \Leftarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Leftarrow \text{Ind}_p^G(W_1^\prime) & \Leftarrow \text{Ind}_p^G(W_2^\prime) & \Leftarrow \text{Ind}_p^G(W_3^\prime) & \Leftarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Leftarrow F(\mathfrak{d}_1) & \Leftarrow F(\mathfrak{d}_2) & \Leftarrow F(\mathfrak{d}_3) & \Leftarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & \\
\end{array}
\]

The middle row is obviously exact. The first row coincides by definition with the sequence

\[0 \leftarrow \text{Ind}_p^G(W_1^\prime)^{\mathfrak{d}_1} \leftarrow \text{Ind}_p^G(W_2^\prime)^{\mathfrak{d}_2} \leftarrow \text{Ind}_p^G(W_3^\prime)^{\mathfrak{d}_3} \leftarrow 0.\]

A simple computation shows that it is right exact. Hence our functor $F_p^G$ is in particular left exact. So all we have to show by using the snake lemma is to prove that the columns of the diagram are exact. For simplicity of notation, we consider the middle column. We abbreviate $\mathfrak{d} = \mathfrak{d}_2$ and $W = W_2$. The submodule $\mathfrak{d}$ is again an object of the category $O_{\text{alg}}^P$. Thus we have a short exact sequence

\[0 \rightarrow \mathfrak{d} \rightarrow U(g) \otimes U(p) \bar{W} \rightarrow \bar{\mathfrak{d}} \rightarrow 0\]

for some finite-dimensional algebraic $\mathbf{P}$-representation $\bar{W}$. Hence $F(\mathfrak{d}) = \text{Ind}_p^G(\bar{W})^{\bar{\mathfrak{d}}}$. In order to show the surjectivity of $\text{Ind}_B^G(W') \rightarrow F(\mathfrak{d})$, it is enough to show the injectivity of its dual map. But the dual is by Proposition 3.8 given by the map

\[\lim_{\leftarrow n} \text{Ind}_{p,n}^G(U(u^-)_n \otimes \bar{W}/\bar{\mathfrak{d}}_n) \rightarrow \lim_{\leftarrow n} \text{Ind}_{p,n}^G(U(u^-)_n \otimes W).\]

Now the functor $\lim_{\leftarrow n}$ is exact on projective systems consisting of Fréchet spaces where the transition maps have dense image (topological Mittag-Leffler property), cf. [EGA 13.2.4]. Hence it suffices to prove that the maps $U(u^-)_n \otimes \bar{W}/\bar{\mathfrak{d}}_n \rightarrow U(u^-)_n \otimes W$ are injective. But these maps
are just the “completions” of the map $U(u^-) \otimes \bar{W} / \delta \rightarrow U(u^-) \otimes W$ with respect to the norm (3.7.6). By using [BGR] Cor. 6, 1.19 we see that the completion functor is exact. Further we have $U(u^-) \otimes \bar{W} / \delta \cong \delta \subset U(u^-) \otimes W$ by construction. The claim follows.

4.3. Extending the functor $\mathcal{F}_G^P$. Let as before $P \subset G$ be a standard parabolic subgroup. Let $V$ be a $K$-vector space, equipped with a smooth admissible representation of the Levi subgroup $L_P \subset P$. We consider $V$ via inflation as a smooth representation of $P$. We equip $V$ with the finest locally convex $K$-vector space topology, and consider $V$ as a locally analytic $P$-representation, i.e. we write $V = \lim_{\rightarrow \mathcal{H}} V^H$ as the strict inductive limit of the finite-dimensional Banach spaces $V^H$ consisting of $H$-fix vectors [ST1 §2]. Here $H$ ranges over all compact open subgroups of $L_P$.

Let $M$ be an object of $\mathcal{O}_{\text{alg}}^P$ and write it as before as a quotient of a generalized Verma module as in (3.1.5):

$$0 \rightarrow \delta \rightarrow U(g) \otimes_{U(p)} W \rightarrow M \rightarrow 0.$$ 

We consider the (injective) tensor product $W' \otimes_{K,\ell} V$. Since $W$ is a finite-dimensional Banach space, it coincides with the projective tensor product $W' \otimes_{K,\pi} V$. Therefore we simply write $W' \otimes K V$ for it. We also remark that it is automatically complete and that the identity $W' \otimes K V = \lim_{\rightarrow \mathcal{H}} W' \otimes V^H$ holds as locally convex vector spaces. Then we set

$$(4.3.1) \quad \mathcal{F}_G^P(M, V) = \text{Ind}_G^P(W' \otimes_K V)^0 = \{ f \in \text{Ind}_G^P(W' \otimes_K V) \mid \text{for all } \delta \in \delta : \langle \delta, f \rangle = 0_V \}.$$ 

As for the case when $V$ is the trivial representation, we will see below that it is independent of the chosen $L_P$-representation $W$. First we state the following proposition which follows from results due to Emerton.

**Proposition 4.4.** The representation $\mathcal{F}_G^P(M, V)$ is admissible for all smooth admissible $L_P$-representations $V$ and for all $M \in \mathcal{O}_{\text{alg}}^P$.

**Proof.** Since $W'$ and $V$ are admissible [ST2] Thm. 6.6 (i)], it follows by [Em1] Prop. 6.1.5 that the $P$-representation $W' \otimes V$ is admissible, as well. Further the parabolic induction process preserves admissibility, cf. [Em2] Prop 2.1.2], at least, if the action of $P$ on $W'$ factors through $L_P$. In general, we can find a filtration of $W'$ by $P$-representations such that the action of $P$ on the associated graded space acts through $L_P$. Using the fact that an extension of two admissible representations is again admissible, cf. [ST5 p. 304], [ST2] Remark 3.2], we conclude that $\text{Ind}_G^P(W' \otimes_K V)$ is again admissible. Thus the claim follows, since $\mathcal{F}_G^P(M, V)$ is a closed subspace of this admissible representation. \qed
In order to describe the dual space $F_G^G(M,V)'$, we proceed as follows. The completed tensor product

$$M \hat{\otimes}_K V'$$

has then the structure of an $D(P \times P)$-module. Note that $D(P \times P) = D(P) \hat{\otimes}_{\mathbb{N},K} D(P)$. And as $V'$ is a (trivial) $p$-module, the tensor product $M \hat{\otimes}_K V'$ is naturally a $U(g \times g)$-module. These two module structures are compatible (cf. the argument in section 3.3) and $M \hat{\otimes}_K V'$ is thus a module for the subring

$$U^{(2)}(g,P) := U(g \times g) D(P \times P)$$

of $D(G \times G)$. The diagonal morphism

$$\theta : G \rightarrow G \times G, \quad \theta(g) = (g,g),$$

induces a $K$-algebra homomorphism $\theta_* : D(G) \rightarrow D(G \times G)$ by $\theta_*(\delta)(f) = \delta(f \circ \theta)$, and $\theta_*$ maps $U(g,P)$ into $U^{(2)}(g,P)$. We then give $M \hat{\otimes}_K V'$ the structure of an $U(g,P)$-module via $\theta_*$. Then we get

$$F_G^G(M,V) = (D(G) \hat{\otimes}_{U(g,P),\theta_*}(M \hat{\otimes}_K V'))'.$$

Indeed, the same proof as in Section 3 applies. First one verifies that Prop. 3.6 generalizes to obtain an isomorphism

$$D(G) \hat{\otimes}_{U(g,P),\theta_*}(M \hat{\otimes}_K V') \cong (D(G) \hat{\otimes}_{D(P)}(W \hat{\otimes}_K V')) / D(G)\mathfrak{d}.$$

Now write $V = \varinjlim_H V^H$ where the subgroups $H$ range over all normal compact open subgroups of $P_0$. Then we get

$$(\text{Ind}_{P_0}^G(W' \otimes V)^{\mathfrak{d}}')' = (\text{Ind}_{P_0}^{G_0}(W' \otimes_K \varinjlim_H V^H)^{\mathfrak{d}}')'$$

$$= (\text{Ind}_{P_0}^{G_0}(\varinjlim_H W' \otimes_K V^H)^{\mathfrak{d}}')'$$

$$= (\varinjlim_H \text{Ind}_{P_0}^{G_0}(W' \otimes_K V^H)^{\mathfrak{d}}')'$$

$$= \varinjlim_H (D(G_0) \hat{\otimes}_{D(P_0)}(W \otimes_K (V^H)^{\mathfrak{d}}) / D(G_0)\mathfrak{d})$$

$$= (D(G_0) \hat{\otimes}_{D(P_0)}(W \otimes_K V')) / D(G_0)\mathfrak{d}$$

$$= (D(G) \hat{\otimes}_{D(P)}(W \otimes_K V')) / D(G)\mathfrak{d}.$$

The fourth equality follows from [S1, Prop. 16.10] and our discussion in sec. 3. For the fifth equality, we refer to the proof of [ST2, Theorem 6.6 (i)] and to [Em1, Prop. 1.1.29, 1.1.31].

We thus get a bi-functor
\[ \mathcal{F}_P^G(\cdot, \cdot) : \mathcal{O}_\text{alg}^p \times \text{Rep}_K^{\infty, \alpha}(L_P) \longrightarrow \text{Rep}_K^{\delta, \alpha}(G), \]

which is contravariant in the first argument and covariant in the second. Here, \( \text{Rep}_K^{\infty, \alpha}(L_P) \) denotes the category of smooth admissible \( L_P \)-representations on \( K \)-vector spaces.

**Remark 4.5.** One can also extend the functor \( \mathcal{F}_P^G \) to some category of \( U(\mathfrak{g}, P) \)-modules which contains all objects of \( \mathcal{O}_\text{alg}^p \), as well as the dual spaces of all smooth admissible \( L_P \)-representations, and tensor products of those. In this paper we do not need to consider such a larger category and have therefore decided not to introduce it.

**Proposition 4.6.**

a) The bi-functor \( \mathcal{F}_P^G \) is exact in both arguments.

b) If \( Q \supset P \) is a parabolic subgroup, \( q = \text{Lie}(Q) \), and \( M \) an object of \( \mathcal{O}_\text{alg}^p \), then

\[ \mathcal{F}_P^G(M, V) = \mathcal{F}_Q^G(M, \iota_{L_P(L_Q \cap U_P)}^Q(V)), \]

where \( \iota_{L_P(L_Q \cap U_P)}^Q(V) = \iota_P^Q(V) \) denotes the corresponding induced representation in the category of smooth representations.

**Proof.** a) On the level of \( K \)-vector spaces we have \( \mathcal{F}_P^G(M, V) = \mathcal{F}_P^G(M) \otimes_K V \), and \( \mathcal{F}_P^G \) is thus exact in the second argument. That it is exact in the first argument is Prop. \ref{exactness_of_FG}.

b) We note that this formula is analogous to the projection formula \( \text{Ind}_H^G(W) = \text{Ind}_H^K(K) \otimes W \) in the representation theory of finite groups, when \( W \) is a \( G \)-representations. The analogous proof can be checked by hand, using the description \ref{projection_formula}. A more ring-theoretic argument goes as follows. Without loss of generality we may assume that all smooth representations involved are of finite length. Then we have that \( \mathcal{F}_P^G(M, V)' = D(G) \otimes_{U(\mathfrak{g}, P) \otimes} (M \hat{\otimes} V') \). Define

\[ \tilde{\theta} : G \rightarrow G \times G, \text{ by } \tilde{\theta}(g) = (g, g^{-1}). \]

Then \( \tilde{\theta} \) induces a \( K \)-linear map \( \tilde{\theta} : D(G) \rightarrow D(G \times G) \) (which is not a ring homomorphism if \( G \) is not abelian). Both \( \theta \) and \( \tilde{\theta} \) map \( U(\mathfrak{g}, Q) \) into \( U(\mathfrak{g}, Q) \). Give \( M \hat{\otimes}_K (U(\mathfrak{g}, Q) \otimes_{U(\mathfrak{g}, P)} V') \) the structure of a \( U(\mathfrak{g}, Q) \)-module via \( \theta \).

On the other hand, there is a unique \( U(\mathfrak{g}, Q) \)-module structure on \( U(\mathfrak{g}, Q) \otimes_{U(\mathfrak{g}, P)} (M \hat{\otimes} V') \) with the property that a tensor product of distributions \( \delta_1 \otimes \delta_2 \) (with \( \delta_1, \delta_2 \in U(\mathfrak{g}, Q) \)) acts on a tensor \( \delta \otimes (v \otimes \phi) \) by \( (\delta_1 \delta) \otimes ((\delta_2 v) \otimes \phi) \). Then define maps

\[ \Phi : U(\mathfrak{g}, Q) \otimes_{U(\mathfrak{g}, P)} (M \hat{\otimes} V') \longrightarrow M \hat{\otimes}_K (U(\mathfrak{g}, Q) \otimes_{U(\mathfrak{g}, P)} V'), \]
\[ \Psi : M \hat{\otimes}_K (U(g, Q) \otimes_{U(g, P)} V') \longrightarrow U(g, Q) \otimes_{U(g, P)} \theta_* (M \hat{\otimes}_K V') \]

by
\[ \Phi (\delta \otimes (v \otimes \phi)) = \theta_* (\delta) \cdot (v \otimes (1 \otimes \phi)) , \quad \text{and} \quad \Psi (v \otimes (\delta \otimes \phi)) = \check{\theta}_* (\delta) \cdot (1 \otimes (v \otimes \phi)) . \]

Using the fact that the delta-distributions \( \delta_g \) are dense in \( D_G \) (cf. [ST1, Lemma 3.1]), it is an easy matter to verify that these maps are well-defined and \( U(g, Q) \)-module homomorphisms which are inverse to each other. (Remark: \( \delta \mapsto \check{\theta}_* (\delta) \) does not turn \( U(g, Q) \otimes U(g, P) \theta_* (M \hat{\otimes}_K V') \) into a \( U(g, Q) \)-module; but this is not needed here.) We thus have a canonical isomorphism of \( U(g, Q) \)-modules
\[ U(g, Q) \otimes_{U(g, P)} \theta_* (M \hat{\otimes}_K V') \cong M \hat{\otimes}_K (U(g, Q) \otimes U(g, P) V') , \]
where on the right hand side the ring \( U(g, Q) \) acts via \( \theta_* \). The claim follows by the following lemma.

**Lemma 4.7.** With the above notation we have
\[ U(g, Q) \otimes_{U(g, P)} V' = i^Q_P(V)' . \]

**Proof.** We start with the observation that \( i^Q_P(V) \) is just the subspace of \( \text{Ind}^Q_P(V) \) which is annihilated by \( q = \text{Lie}(Q) \). The dual space of \( \text{Ind}^Q_P(V) \) coincides by (2.1.2) with \( D(Q) \otimes D(P) V' \). Let \( I(q) \subset D(Q) \) be the two-sided ideal generated by \( q \). Then the dual space of \( i^Q_P(V) \) is
\[ (D(Q) \otimes D(P) V')/(I(q) \otimes D(P) V') \]
which is naturally isomorphic to
\[ D(Q) \otimes_{U(q, P)} V' . \]
Here \( V' \) has the structure of a \( U(q) \)-module via the trivial action. But the latter object is isomorphic to
\[ U(g, Q) \otimes_{U(g, P)} V' . \]

**Remark 4.8.** We remark that also the description in Cor. [3.10] has a counterpart in the relative version. Indeed, let \( V \) be as before a smooth admissible \( L_P \)-representation. By letting act \( U_P^{-\alpha} \) trivially on \( V \), it has even the structure of a \( P^n \)-representation. Then we get isomorphisms
\[ \text{Ind}_{P_0}^G (W' \otimes V)^\emptyset = \lim_{\to} \text{Ind}_{P_n}^G (O(U_P^{-\alpha}) \otimes W' \otimes V)^\emptyset \]
\[ \cong \left( \lim_{\to} \text{Ind}_{P_n}^G (U(P_n) \otimes (W' \otimes V)/\mathfrak{g}_n) \right)' . \]
Indeed, for the proof we write again \( W' \otimes V = \lim_{\longrightarrow} H' W' \otimes V^H \). Here \( H \) runs through the set of all compact open subgroups of \((L_P)_0\). As any such subgroup \( H \) normalises \((U_P)_0\) and \( U_P^{-n} \), and since the commutativity relation (3.7.2) is satisfied, we deduce that \( U_P^{-n} \cdot H \cdot (U_P)_0 \) is an open subgroup of \( P \). Then

\[
\text{Ind}^{G_0}_{P_0}(O(U_P^{-n}) \otimes W' \otimes V)^0 = \lim_{\longrightarrow} H \text{Ind}^{G_0}_{U_P^{-n} \cdot H \cdot (U_P)_0}(O(U_P^{-n}) \otimes W' \otimes V^H)^0.
\]

By the finite-dimensionality of \( V^H \) we get by Prop. 3.9

\[
\text{Ind}^{G_0}_{P_0}(W' \otimes V)^0 = \lim_{\longrightarrow} \lim_{\longrightarrow} H \text{Ind}^{G_0}_{U_P^{-n} \cdot H \cdot (U_P)_0}(O(U_P^{-n}) \otimes W' \otimes V^H)^0.
\]

4.9. \textit{Locally analytic BGG-resolutions.} Recall that for a character \( \lambda \in X^*(\mathbf{T}) \), we denote by \( M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K_{\lambda} \) the corresponding Verma module and by \( L(\lambda) \) its simple quotient. Let \( \Delta \) be the set of simple roots and \( \Phi \) the set of roots of \( G \) with respect to our data \( \mathbf{T} \subset \mathbf{B} \). Let

\[
X_+ = \{ \lambda \in X^*(\mathbf{T}) \mid (\lambda, \alpha^\vee) \geq 0 \ \forall \alpha \in \Delta \}
\]

be the set of dominant weights in \( X^*(\mathbf{T}) \). If \( \lambda \in X_+ \), then \( L(\lambda) \) is finite-dimensional and comes from an irreducible algebraic \( G \)-representation. In this situation, we also write \( V(\lambda) \) for \( L(\lambda) \).

Denote by \( \cdot \) the dot action of \( W \) on \( t^* = X^*(\mathbf{T})_\mathbb{R} \) given by

\[
w \cdot \chi = w(\chi + \rho) - \rho,
\]

where \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi} \alpha \). If \( \lambda \in X^*(\mathbf{T}) \), then \( w \cdot \lambda \) is contained in \( X^*(\mathbf{T}) \), as well. For a dominant weight \( \lambda \in X_+ \), the BGG-resolution of the finite-dimensional \( G \)-module \( V(\lambda) \) has the following shape

\[
(4.9.1) \quad 0 \rightarrow M(w_0 \cdot \lambda) \rightarrow \bigoplus_{\ell(w) = \ell(w_0) - 1} M(w \cdot \lambda) \rightarrow \cdots \rightarrow \bigoplus_{\ell(w) = 1} M(w \cdot \lambda) \rightarrow M(\lambda) \rightarrow V(\lambda) \rightarrow 0.
\]

We refer to [Ku] for the choice of differentials in this complex. By applying our exact functor \( \mathcal{F}_B^G \), we get a resolution

\[
0 \leftarrow \mathcal{F}_B^G(M(w_0 \cdot \lambda)) \leftarrow \bigoplus_{w \in W, \ell(w) = \ell(w_0) - 1} \mathcal{F}_B^G(M(w \cdot \lambda)) \leftarrow \cdots
\]

\[
\leftarrow \bigoplus_{w \in W, \ell(w) = 1} \mathcal{F}_B^G(M(w \cdot \lambda)) \leftarrow \mathcal{F}_B^G(M(\lambda)) \leftarrow \mathcal{F}_B^G(V(\lambda)) \leftarrow 0.
\]
which coincides with
\[
0 \leftarrow \text{Ind}_B^G((w_0 \cdot \lambda)^{-1}) \leftarrow \bigoplus_{w \in W} \text{Ind}_B^G((w \cdot \lambda)^{-1}) \leftarrow \ldots
\]
\[
\leftarrow \bigoplus_{w \in W} \text{Ind}_B^G((w \cdot \lambda)^{-1}) \leftarrow \text{Ind}_B^G(\lambda^{-1}) \leftarrow V(\lambda) \otimes i_B^G \leftarrow 0.
\]

There is also a parabolic version of the BGG-resolution due to Lepowsky \cite{Le}. Let \( P = P_I \subset G, I \subset \Delta \) be a standard parabolic subgroup. Let
\[
X_I^+ = \{ \lambda \in X^*(T) \mid (\lambda, \alpha) \geq 0 \ \forall \alpha \in I \}
\]
be the set of \( I \)-dominant weights. In particular \( X^+ \subset X_I^+ \). Let \( ^I W = W_I \backslash W \) which we identify with the representatives of shortest length in \( W \). Let \( ^I w_0 \) be the element of maximal length in \( ^I W \). If \( \lambda \) is in \( X^+ \) and \( w \in ^I W \) then \( w \cdot \lambda \in X_I^+ \), cf. \cite{Le} p. 502. The \( I \)-parabolic BGG-resolution of \( V(\lambda) \) is given by a sequence
\[
0 \rightarrow M_I(^I w_0 \cdot \lambda) \rightarrow \bigoplus_{w \in ^I W} M_I(w \cdot \lambda) \rightarrow \ldots \rightarrow \bigoplus_{w \in ^I W} M_I(w \cdot \lambda) \rightarrow M_I(\lambda) \rightarrow V(\lambda) \rightarrow 0.
\]
Again, we refer to \cite{Ku} for the definition of the differentials in this complex.

By applying our exact functors \( F_B^G \), we get a resolution
\[
0 \leftarrow F_B^G(M_I(^I w_0 \cdot \lambda)) \leftarrow \bigoplus_{w \in ^I W} F_B^G(M_I(w \cdot \lambda)) \leftarrow \ldots
\]
\[
\leftarrow \bigoplus_{w \in ^I W} F_B^G(M_I(w \cdot \lambda)) \leftarrow F_B^G(M_I(\lambda)) \leftarrow F_B^G(V(\lambda)) \leftarrow 0
\]
which coincides by Proposition \ref{16} with
\[
0 \leftarrow \text{Ind}_P^G(V_I(^I w_0 \cdot \lambda)^{'}) \leftarrow \bigoplus_{w \in ^I W} \text{Ind}_P^G(V_I(w \cdot \lambda)^{'}) \leftarrow \ldots
\]
\[
\leftarrow \bigoplus_{w \in ^I W} \text{Ind}_P^G(V_I(w \cdot \lambda)^{'}) \leftarrow \text{Ind}_P^G(V_I(\lambda)^{'}) \leftarrow V(\lambda) \otimes i_P^G \leftarrow 0.
\]

**Example 4.10.** Let \( G = \text{SL}_2 \) and identify characters of \( T \) with tuples of integers in the usual way. Let \( \lambda = (0, 0) \in X^*(T) \). Thus \( L(\lambda) = K \) is the trivial representation. Let \( \lambda' = (-1, 1) \in X^*(T) \). The BGG resolution of the trivial representation is the short exact sequence
\[
0 \rightarrow M(\lambda') \rightarrow M(\lambda) \rightarrow K \rightarrow 0.
\]
By applying the functor \( F_B^G \) to this sequence, we obtain an exact sequence
\[
0 \rightarrow i_B^G(K) \rightarrow \text{Ind}_B^G(K) \rightarrow \text{Ind}_B^G(\lambda'^{-1}) \rightarrow 0
\]
The last map coincides with the derivative map, cf. \cite{ST1} §6, \cite{Mo2}
Remark. In a recent preprint [Jo], Owen Jones constructs a similar exact sequence for locally analytic principal series representations of subgroups of \( G \) possessing an Iwahori decomposition. His exact sequence is based on the BGG-resolution as well.

5. Irreducibility results

In this section we suppose that the residue characteristic is at least five$^2$

Theorem 5.1. Let \( M \in \mathcal{O}_{\text{alg}} \) be simple and assume that \( p \) is maximal among all standard parabolic subalgebras \( q \) with the property that \( M \) is contained in \( \mathcal{O}_q^{\text{alg}} \). Then

i) \( D(G_0) \otimes U(g,P_0) M \) is simple as \( D(G_0) \)-module.

ii) \( F_G^G(M) = \text{Ind}_P^G(W')^0 \) is topologically irreducible as \( G_0 \)-representation.

iii) \( F_G^G(M) = \text{Ind}_P^G(W')^0 \) is topologically irreducible as \( G \)-representation.

Proof. Statement ii) follows from i) by the equivalence \( \text{(2.1.1)} \) and iii) is a consequence of ii).

We are going to prove the first assertion now.

Let \( r \) always denote a real number in \((0,1) \cap p^\mathbb{Q}\). We choose an open normal uniform pro-
\( p \) subgroup \( H \subset G_0 \) (below we will choose \( H \) specifically, but here it can be any open normal uniform pro-
\( p \) subgroup.), cf. [DDMS, 8.34]. Define the norm \( \| \cdot \|_r \) on \( D(H) \) via the canonical \( p \)-valuation on \( H \) as explained in [ST2, §4] resp. [OS, 2.2.3, 2.2.4]. We use the decomposition

\[
D(G_0) = \bigoplus_{g \in G_0/H} \delta_g D(H)
\]

to define the maximum norm, denoted by \( \| \cdot \|_r \) too, on \( D(G_0) \), i.e.,

\[
\| \sum_{g \in G_0/H} \delta_g \lambda_g \|_r = \max \{ \| \lambda_g \|_r \}.
\]

Put \( M = D(G_0) \otimes U(g,P_0) M \). It suffices to show that

\[
M_r := D_r(G_0) \otimes D(G_0) M = D_r(G_0) \otimes U(g,P_0) M
\]
is simple as \( D_r(G_0) \)-module for \( r \) sufficiently close to 1, cf. [ST2 Lemma 3.9].

Let \( U_r(\mathfrak{g}) \) be the topological closure of \( U(\mathfrak{g}) \) in \( D_r(G_0) \), and let

\[
m_r = U_r(\mathfrak{g}) M
\]

$^2$In a later version we will make this restriction more precise, or possibly eliminate it entirely. In case that the root system \( \Phi(\mathfrak{g},1) \) has only components of type \( A, D \) or \( E \), the present proof holds for any residue characteristic.
be the $U_r(\mathfrak{g})$-submodule of $\mathbf{M}_r$ generated by $M$. It follows from [K1] that $D_r(P_0)$ is generated as a module over $U_r(p_0) = D_r(P_0) \cap U_r(\mathfrak{g})$ by Dirac distributions $\delta_{g_1}, \ldots, \delta_{g_m}$, with $g_i \in P_0$. Because $P_0$ acts on $M$, the $U_r(\mathfrak{g})$-module $\mathfrak{m}_r$ is actually a module over the subring

$$U_r(\mathfrak{g}, P_0) = U_r(\mathfrak{g})D_r(P_0) = \sum_{i=1}^m U_r(\mathfrak{g}) \cdot \delta_{g_i}$$

generated by $U_r(\mathfrak{g})$ and $D_r(P_0)$ inside $D_r(G_0)$. Moreover, $\mathfrak{m}_r$ is a finitely generated $U_r(\mathfrak{g})$-module (in fact, generated by a single vector of highest weight), hence carries a canonical $U_r(\mathfrak{g})$-module topology. The module $M$ is clearly dense in $\mathfrak{m}_r$ with respect to this topology. It follows from [E] or [OS 3.4.5] that $\mathfrak{m}_r$ is a simple $U_r(\mathfrak{g})$-module, and in particular a simple $U_r(\mathfrak{g}, P_0)$-module.

Identifying elements of $G$ with Dirac distributions, we put

$$P_{0,r} = G_0 \cap U_r(\mathfrak{g}, P_0) \subset D_r(G),$$

which is an open compact subgroup of $G_0$ containing $P_0$. Then $D_r(G_0)$ is a finitely generated and free $U_r(\mathfrak{g}, P_0)$-module and

$$D_r(G_0) = \bigoplus_{g \in G_0/P_{0,r}} \delta_g U_r(\mathfrak{g}, P_0).$$

We have $\mathfrak{m}_r = U_r(\mathfrak{g}, P_0) \otimes_{U(\mathfrak{g}, P_0)} M \subset D_r(G_0) \otimes_{U(\mathfrak{g}, P_0)} M = \mathbf{M}_r$, and

$$\mathbf{M}_r = D_r(G_0) \otimes_{U(\mathfrak{g}, P)} M = D_r(G_0) \otimes_{U_r(\mathfrak{g}, P_0)} \mathfrak{m}_r = \bigoplus_{G_0/P_{0,r}} \delta_g \mathfrak{m}_r.$$

With $\mathfrak{m}_r$ all $U_r(\mathfrak{g})$-submodules $\delta_g \mathfrak{m}_r$ are simple as $U_r(\mathfrak{g})$-modules, as well. Hence in order to show that $\mathbf{M}_r$ is simple it suffices to prove that there is no isomorphism of $U_r(\mathfrak{g})$-modules $\phi : \delta_g \mathfrak{m}_r \cong \delta_{g_2} \mathfrak{m}_r$ for $g_1 P_{0,r} \neq g_2 P_{0,r}$. Defining $\psi(v) = \delta_{g_2}^{-1} \phi(\delta_{g_2} v)$ gives an $U_r(\mathfrak{g})$-module isomorphism $\psi : \delta_{g_2}^{-1} \mathfrak{m}_r \cong \mathfrak{m}_r$, so that we may assume $g_2 = 1$. Let $I \subset G_0$ be the Iwahori subgroup whose image in $G(O_L/(\pi_L))$ is $B(O_L/(\pi_L))$. Using the Bruhat decomposition

$$G_0 = \coprod_{w \in W/W_p} I w P_0$$

we may write $g = g_1 = h^{-1} w p_1$ with $h \in I$, $w \in W$ and $p_1 \in P_0$. By the Iwahori decomposition

$$I = (I \cap U_p^{-}) \cdot (I \cap P_0)$$

we can write $h = u p_2$ with $p_2 \in I \cap P_0$ and $u \in I \cap U_p$. We recall that $U_p^{-}$ is the unipotent radical of the parabolic subgroup opposite to $P$. The same reasoning as before then shows that an isomorphism $\delta_g \mathfrak{m}_r \cong \mathfrak{m}_r$ induces an isomorphism of $U_r(\mathfrak{g})$-modules

$$\phi : \delta_w \mathfrak{m}_r \cong \delta_u \mathfrak{m}_r.$$
Step 1. We show first that this can only happen if \( w \in W_P \). Let \( \lambda \in X(T)^* \) be the highest weight of \( M \), i.e. \( M = L(\lambda) \), and \( I = \{ \alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \} \), cf. Example 2.3. Beware of the double notation.\(^3\)

Then by Cor. 8.5 and the maximality condition w.r.t. \( \mathfrak{p} \), the parabolic subalgebra \( \mathfrak{p} \) is \( \mathfrak{p}_I \), where the root system of the Levi subalgebra of \( \mathfrak{p}_I \) has \( I \) as a basis of simple roots. Suppose \( w \) is not contained in \( W_I = W_P \). Then there is a positive root \( \beta \notin \Phi^+_I \) such that \( w^{-1}\beta < 0 \), hence \( w^{-1}(-\beta) > 0 \), cf. \cite{Car}. Consider a non-zero element element \( y \in \mathfrak{g}_{-\beta} \), and let \( v^+ \in M \) be a weight vector of weight \( \lambda \) (uniquely determined up to a non-zero scalar). Then we have the following identity in \( M_r \),

\[
y \delta_w \cdot v^+ = \delta_w \text{Ad}(w^{-1})(y) \cdot v^+ = 0
\]
as \( \text{Ad}(w^{-1})(y) \in \mathfrak{g}_{-w^{-1}\beta} \) annihilates \( v^+ \). We have \( \phi(\delta_w \cdot v^+) = \delta_u \cdot v \) for some nonzero \( v \in m_r \).

And therefore

\[
0 = \phi(y \delta_w \cdot v^+) = y \phi(\delta_w \cdot v^+) = y \delta_u \cdot v = \delta_u \cdot \text{Ad}(u^{-1})(y) \cdot v.
\]

And hence \( \text{Ad}(u^{-1})(y) \cdot v = 0 \) since \( \delta_u \) is a unit in \( D(G_0) \). On the other hand, \( y' := \text{Ad}(u^{-1})(y) \) is an element of \( u_P^{-} \) and as such acts injectively on \( M \), cf. Corollary 8.5. We show that \( y' \) actually acts injectively on \( m_r \) too. Write \( v \) as a convergent sum \( v = \sum_{\mu \in \Lambda(v)} u_{\lambda - \mu} \) where \( \Lambda(v) \) is a (non-empty) subset of \( \mathbb{Z}_{\geq 0} \alpha_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0} \alpha_\ell \) (\( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \)), and \( v_{\lambda - \mu} \in M_{\lambda - \mu} \setminus \{0\} \) is a vector of weight \( \lambda - \mu \). Write \( y' = \sum_{\gamma \in B} y_{\gamma} \), where \( B \) is a non-empty subset of \( \Phi^+ \setminus \Phi^+_I \) and \( y_{\gamma} \) is a non-zero element of \( \mathfrak{g}_{-\gamma} \subset \mathfrak{u}_P^{-} \). Then we have

\[
0 = y' \cdot v = \sum_{\nu \in \mathbb{Z}_{\geq 0} \alpha_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0} \alpha_\ell} \left( \sum_{\mu \in \Lambda(v), \gamma \in B, \nu = \mu + \gamma} y_{\gamma} \cdot v_{\lambda - \mu} \right),
\]

where

\[
\sum_{\mu \in \Lambda(v), \gamma \in B, \nu = \mu + \gamma} y_{\gamma} \cdot v_{\lambda - \mu}
\]

lies in \( M_{\lambda - \mu} \) and is thus zero. Define on \( \mathbb{Q} \alpha_1 \oplus \cdots \oplus \mathbb{Q} \alpha_\ell \) the lexicographic ordering as in the proof of Prop. 8.3. Choose \( \gamma^+ \in B \) and \( \mu^+ \in \Lambda(v) \), both minimal with respect to this ordering. With \( \nu^+ = \gamma^+ + \mu^+ \) we then have

\[
0 = \sum_{\mu \in \Lambda(v), \gamma \in B, \nu^+ = \mu + \gamma} y_{\gamma} \cdot v_{\lambda - \mu} = y_{\gamma^+} \cdot v_{\lambda - \mu^+}.
\]

This contradicts Cor. 8.5 and we can thus conclude that \( w \) must be an element of \( W_P \).

\(^3\)We won’t use the Iwahori subgroup anymore
From now on we may even assume that \( w = 1 \) since \( W_P \subset P_{0,r} \).

**Step 2.** Now let \( \phi : m_r \rightarrow \delta_u m_r \) be an isomorphism of \( U_\tau(\mathfrak{g}) \)-modules. Let \( v^+ \in M_\lambda \setminus \{0\} \) be a vector of highest weight as above. Put \( \phi(v^+) = \delta_u \cdot v \) with some non-zero \( v \in m_r \). Then we have for any \( x \in \mathfrak{t} \), the following identity in \( \delta_u m_r \):

\[
\lambda(x) \delta_u \cdot v = \phi(x \cdot v^+) = x \cdot \phi(v^+) = \delta_u \text{Ad}(u^{-1})(x) \cdot v.
\]

We have thus for all \( x \in \mathfrak{t} \), the following identity in \( m_r \):

\[
(5.1.1) \quad \lambda(x)v = \text{Ad}(u^{-1})(x) \cdot v.
\]

Note that this equation only involves the action of elements of \( u^- \oplus \mathfrak{t} \), because \( \text{Ad}(u^{-1})(x) \) is in \( u^- \oplus \mathfrak{t} \). Next we embed \( m_r \) into the "formal completion" of \( M \), i.e.,

\[
\hat{M} = \prod_\mu M_\mu
\]

by mapping the weight spaces \( M_\mu \subset m_r \) to the corresponding space \( M_\mu \subset \hat{M} \). Then \( \hat{M} \) is in an obvious way a module for \( U(u^- \oplus \mathfrak{t}) \). This module structure extends to a representation of \( U^- \) as follows. Since \( U^- \) is a nilpotent group, the exponential \( \exp_{U^-} : u^- \rightarrow U^- \) is actually defined on the whole Lie algebra, so that we have a map \( \exp_{U^-} : u^- \rightarrow U^- \). Let \( \log_{U^-} : U^- \rightarrow u^- \) be the inverse map. Then we can define for \( u \in U^- \) and \( v = \sum_\mu v_\mu \in \hat{M} : \)

\[
\delta_u \cdot v = \sum_\mu \sum_{n \geq 0} \frac{1}{n!} \log_{U^-}(u)^n \cdot v_\mu.
\]

Note that this sum is well-defined in \( \hat{M} \), because \( \log_{U^-}(u) \) in \( u^- \), and there are hence only finitely many terms of given weight in this sum. (We continue to write the action of an element \( u \in U^- \) on \( \hat{M} \) by \( \delta_u \).) Moreover, it gives an action of \( U^- \) on \( \hat{M} \) because

\[
\exp_{U^-}(\log_{U^-}(u_1)) \cdot \exp_{U^-}(\log_{U^-}(u_2)) = \exp_{U^-}(\mathcal{H}(\log_{U^-}(u_1), \log_{U^-}(u_2)))
\]

\[
= \exp_{U^-}(\log_{U^-}(u_1 u_2)),
\]

where \( \mathcal{H}(X,Y) = \log(\exp(X) \exp(Y)) \) is the Baker-Campbell-Hausdorff series (which converges on \( u^- \), as \( u^- \) is nilpotent). It is then immediate that this action is compatible with the action of \( U(u^- \oplus \mathfrak{t}) \). The identity (5.1.1) implies then the following identity in \( \hat{M} \)

\[
\delta_{u^{-1}} \cdot (x \cdot (\delta_u \cdot v)) = \text{Ad}(u^{-1})(x) \cdot v = \lambda(x)v,
\]
for all $x \in \mathfrak{t}$, and thus, multiplying both sides of the previous equation with $\delta_u$,

\[ x \cdot (\delta_u \cdot v) = \lambda(x) \delta_u \cdot v. \]

Hence $\delta_u \cdot v$ is a weight vector (in $\hat{M}$) of weight $\lambda$ and must therefore be equal to a non-zero scalar multiple of $v^+$. After scaling $v$ appropriately we have $\delta_u \cdot v = v^+$ or $v = \delta_{u^{-1}} \cdot v^+$ with

\[
\delta_{u^{-1}} \cdot v^+ = \sum_{n \geq 0} \frac{1}{n!} (- \log_{U^-}(u))^n \cdot v^+ =: \Sigma. 
\]

Our final goal is to show that the series $\Sigma$, which is an element of $\hat{M}$, does in fact not lie in the image of $\mathfrak{m}_r$ in $\hat{M}$, thus achieving a contradiction.

We begin by remarking that the uniform pro-$p$ subgroup $H \triangleleft G_0$ used above to introduce the norms $\| \cdot \|_r$ can be chosen in the following way: $H = H^-H^+$ (i.e., every element of $h \in H$ is the product $h = h^-h^+$ with uniquely determined elements $h^- \in H^-$ and $h^+ \in H^+$) with open uniform pro-$p$ subgroups $H^- \triangleleft U^-$ and $H^+ \triangleleft P_0$. This follows from a consideration of the $\mathbb{Z}_p$-Lie algebra (in the sense of [DDMS 4.5]) of some open normal uniform pro-$p$ subgroup of $G_0$, as in [OS 3.3]. Moreover, $H^-$ can be chosen such that its $\mathbb{Z}_p$-Lie algebra $\text{Lie}_{\mathbb{Z}_p}(H^-)$ is a direct sum of $O_L$-lattices in the root groups $\mathfrak{g}_{-\beta}$ with $\beta \in \Phi^+ \setminus \Phi^+_I$. Write

\[
\text{Lie}_{\mathbb{Z}_p}(H^-) = \bigoplus_{\beta \in \Phi^+ \setminus \Phi^+_I} O_L y_{\beta},
\]

where $y_{\beta}$ is a generator of $\mathfrak{g}_{-\beta}$. The closure of $U(u_r^>)$ in $D_r(G_0)$ is then the same as the closure $U_r(u_r^>)$ of $U(u_r^>)$ in $D_r(H^-)$. This closure has an explicit description in terms of the logarithms $\log(b_n + 1)$, where $b_\beta = h_{1,\beta} - 1$, and $h_{1,\beta} = \exp_{H^-}(y_{\beta})$, cf. [Kl]. Let now $r \in \left(\frac{1}{p}, 1\right) \cap \mathbb{Q}$ be an element of the set $S$, as defined before [Sch 7.4]. Then there is some $m \in \mathbb{Z}_{\geq 0}$ such that $s = r^p^m$ has the property that $s > \frac{1}{p}$, but $s^n < p^{-\frac{1}{p^{n-1}}}$, where $\kappa = 1$ if $p > 2$ and $\kappa = 2$ if $p = 2$. Denote by $H^{-,m}$ the $m$th group in the lower $p$-series of $H^-$. Then norm $\| \cdot \|_r$ on $D_r(H^-)$ restricts to a norm on $D(H^{-,m})$ which defines the same topology as the norm $\| \cdot \|_s^{(m)}$ defined on $D(H^{-,m})$ by means of its canonical $p$-valuation, cf. [Sch 7.4]. In this case, however, the algebra $U(u_r^>)$ is dense in $D_s(H^{-,m})$ for the norm $\| \cdot \|_s^{(m)}$. As both $\| \cdot \|_r$ and $\| \cdot \|_s^{(m)}$ define the same topology on $U(u_r^>)$, we get that $U_r(u_r^>) = D_s(H^{-,m})$. Moreover, the $\mathbb{Z}_p$-Lie algebra of $H^{-,m}$ is

\[
\text{Lie}_{\mathbb{Z}_p}(H^{-,m}) = \bigoplus_{\beta \in \Phi^+ \setminus \Phi^+_I} O_L p^m y_{\beta},
\]

and elements in $U_r(u_r^>)$ have a description of power series in $y_{\beta}^{(m)} = p^m y_{\beta}$.

\[^4\text{In this preliminary version we always have } \kappa = 1.\]
\[ U_p(u_1) = \left\{ \sum_{n=0}^{\infty} d_n(y^{(m)})^n \mid \lim_{|n| \to \infty} |d_n| s^n|n| = 0 \right\}, \]

where \((y^{(m)})^n\) is the product of the \((y^{(m)})^\beta\) over all \(\beta \in \Phi^+ \setminus \Phi_j^+\), taken in some fixed order. The \(s\)-norm of any generator \(y^{(m)}_\beta\) is

\[ ||y^{(m)}_\beta||_s = s^\kappa. \tag{5.1.3} \]

The group which we denoted by \(P_{0,r}\) is with this notation \(H^{-m}P_0\). Write

\[ \log U_p(u) = \sum_{\beta \in B(u)} z_\beta, \]

with a non-empty set \(B(u) \subset \Phi^+ \setminus \Phi_j^+\) and non-zero elements \(z_\beta \in g_{-\beta}\). Put

\[ B^+(u) = \{ \beta \in B(u) \mid z_\beta \notin O_L y^{(m)}_\beta \}. \]

This is a non-empty subset of \(B(u)\). Put \(B'(u) = B(u) \setminus B^+(u)\),

\[ z^+ = \sum_{\beta \in B^+(u)} z_\beta \quad \text{and} \quad z' = \sum_{\beta \in B'(u)} z_\beta = \log U_p(u) - z^+. \]

Then \(z' \in \text{Lie}_{Z_p}(H^{-m})\), and thus \(\exp(z') \in H^{-m}\). The element \(u_1 = u \exp(z')^{-1} = u \exp(-z')\) does not lie in \(H^{-m}\), and \(\delta_u m_r = \delta_{u_1} m_r\). We may hence replace \(u\) by \(u_1 = u \exp(-z')\). Then we compute \(\log U_-(u_1) = \log U_-(u \exp(-z'))\) by means of the Baker-Campbell-Hausdorff formula. Because of the commutators appearing in this formula we have

\[ \log U_-(u_1) \in z^+(u) + \sum \left( \text{iterated commutators of } g_{-B^+(u)} \text{ with } g_{-B'(u)} \right), \tag{5.1.4} \]

where \(g_{-B^+(u)} = \bigoplus_{\beta \in B^+(u)} g_{-\beta}\) and analogous for \(g_{-B'(u)}\). Put \(ht'(u) = \min\{ht(\beta) \mid \beta \in B'(u)\}\). The height \(ht(\beta)\) of the root \(\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha\) is defined to be the sum \(\sum_{\alpha \in \Delta} n_\alpha\). It follows from the preceding formula (5.1.4) that \(ht'(u_1) > ht'(u)\). The process of passing from \(u\) to \(u_1\) can be repeated finitely many times, but will finally produce an element \(u_N \in (I \cap U^-) \setminus H^{-m}\) which has the property that \(B'(u_N) = \emptyset\) (and hence \(u_{N+1} = u_N\)). We will denote this element again by \(u\).

Next we chose an extremal element \(\beta^+\) among the \(\beta \in B(u) = B^+(u)\), i.e. one of the minimal generators of the cone \(\sum_{\beta \in B(u)} \mathbb{R}_{\geq 0} \beta\) inside \(\bigoplus_{\alpha \in \Delta} \mathbb{R} \alpha\). Then we have
\[ \mathbb{R}_{>0}\beta^+ \cap \sum_{\beta \in B(u), \beta \neq \beta^+} \mathbb{R}_{\geq 0}\beta = \emptyset. \]

This means in particular that no positive multiple of \( \beta^+ \) can be written as a linear combination \( \sum_{\beta \in B(u), \beta \neq \beta^+} c_\beta \beta \) with non-negative integers \( c_\beta \in \mathbb{Z}_{\geq 0} \). It follows that if \( n \) is a positive integer and

\[
n\beta^+ = \gamma_1 + \ldots + \gamma_m \quad \text{with not necessarily distinct} \quad \gamma_i \in B(u) \]

\[
\Rightarrow \quad [m = n \quad \text{and} \quad \gamma_1 = \ldots = \gamma_n = \beta^+] .
\]

After these intermediate considerations we return to the series

\[
\Sigma = \sum_{n \geq 0} \frac{1}{n!} (-\log U_p(u))^n \cdot v^+ = \sum_{n \geq 0} \frac{(-1)^n}{n!} (z_{\beta_1} + z_{\beta_2} + \ldots + z_{\beta_s})^n \cdot v^+
\]

where \( B(u) = \{ \beta^+, \beta_2, \ldots, \beta_s \} \). It follows from (5.1.5) and Cor. 8.5 that if we write \( \Sigma \) as a (formal) sum of weight vectors, there is for any \( n \in \mathbb{Z}_{\geq 0} \) a non-zero weight vector in this series which is of weight \( \lambda - n\beta^+ \), and this element is \( \frac{(-1)^n}{n} z_{\beta^+} \cdot v^+ \).

The last part of the proof is to show that the formal sum of weight vectors

\[
\sum_{n \geq 0} \frac{(-1)^n}{n!} z_{\beta^+}^n \cdot v^+
\]

cannot be the corresponding sum of weight components of an element of \( m_r \), when considered as an element of \( \hat{M} \) and written as a sum of weight vectors.

We start by fixing a Chevalley basis \( (x_\gamma, y_\gamma, h_\alpha \mid \gamma \in \Phi^+, \alpha \in \Delta) \) of \( [g, g] \). By this we mean that the structural constants are in \( \mathbb{Z} \). We have \( x_\gamma \in g_\gamma, \ y_\gamma \in g_{-\gamma}, \) and \( h_\alpha = [x_\alpha, y_\alpha] \in t \), for \( \alpha \in \Delta \). Write \( \Phi^+ \setminus \Phi_I = \{ \beta_1, \ldots, \beta_r \} \) (the \( \beta_i \) here are not necessarily the same as above).

Choose finitely many weight vectors \( v_j \in M \) of weight \( \lambda - \mu_j, \ j = 1, \ldots, r \), generating \( M \) as \( U(\mathfrak{u}_P) \)-module. Put \( y_i = y_{\beta_i}, \ i = 1, \ldots, r \). To show that (5.1.6) does not converge to an element in \( m_r \), we write

\[
\frac{y_{\beta^+}^n}{n!} \cdot v^+ = \sum_{1 \leq j \leq r} \sum_{\nu \in \mathcal{I}_{n,j}} c_{\nu,j}^{(n)} y_1^{\nu_1} \cdot \ldots \cdot y_r^{\nu_r} \cdot v_j,
\]

where \( \mathcal{I}_{n,j} \subset \mathbb{Z}_{\geq 0}^r \) consists of those \( \nu = (\nu_1, \ldots, \nu_r) \) such that \( \mu_j + \nu_1 \beta_1 + \ldots + \nu_r \beta_r = n\beta^+ \). It follows from the Lemma 8.10 that there is at least one index \( (\nu, j) \) with the property that
\[ |c_{\nu,j}^{(n)}|_K \geq \left| \frac{1}{n!} \right|_K. \]

We will use this inequality to estimate the \(| \cdot |_r\) - norm of any lift of \(\frac{1}{n!} z_{\beta^+}^n, v^+\) to

\[ \bigoplus_{1 \leq j \leq t} U_r(u_P) \otimes_K K v_j. \]

Here the \(\| \cdot \|_r\) - norm on this free \(U_r(u_P)\)-module is the supremum norm of the \(| \cdot |_r\) - norms on each summand. Let \(g_Z\) be a \(\mathbb{Z}\)-form of \(g\), i.e. \(g_Z \otimes \mathbb{Z} L = g\). We may choose \(g_Z\) in such a way that its derived algebra has the Chevalley basis which we fixed before. We choose the uniform pro-p subgroup \(H\) such that its \(Z_p\)-Lie algebra is equal to \(p^{m_0} g_{O_L}\), where \(g_{O_L} = g_Z \otimes \mathbb{Z} O_L\). For \(m_0\) big enough the \(Z_p\)-Lie algebra \(p^{m_0} g_{O_L}\) will be powerful in the sense of [DDMS, sec. 9.4]. The group \(H = \exp(p^{m_0} g_{O_L})\) is then a uniform pro-p group [DDMS, 9.10]. Furthermore, the group \(H^{-m}\) has the \(Z_p\)-Lie algebra \(p^{m_0+m}(u_P)_{O_L}\). The elements \(y_{m,i} = p^{m_0+m} y_i, i = 1, \ldots, r\), which are a basis of \(\text{Lie}_{Z_p}(H^{-m})\), have all \(\| \cdot \|_s\) - norm equal to \(s\), cf. [S.13]. And since \(B'(u) = \emptyset\), we have \(z_{\beta^+} = p^{-a} y_{m,\beta^+}\) for some \(a > 0\). We then get:

\[ \frac{z_{\beta^+}^n}{n!} v^+ = p^{-a} \frac{y_{m,\beta^+}^n}{n!} \cdot v^+ = \sum_{1 \leq j \leq t} \sum_{\nu \in I_{n,j}} c_{\nu,j}^{(n)} p^{-a + (m_0+m)(n-\nu_1-\ldots-\nu_r)} y_{m,1}^{\nu_1} \ldots y_{m,r}^{\nu_r}. \]

We are now going to estimate the \(\| \cdot \|_s\) - norm of the term

\[ c_{\nu,j}^{(n)} p^{-a + (m_0+m)(n-\nu_1-\ldots-\nu_r)} y_{m,1}^{\nu_1} \ldots y_{m,r}^{\nu_r}. \]

Using the estimate for \(c_{\nu,j}^{(n)}\) we find that the \(\| \cdot \|_s\) - norm of this term is greater or equal to

\[ \left| \frac{1}{n!} \right|_K p^{na + (m_0+m)(\nu_1+\ldots+\nu_r)} y_{m,1}^{\nu_1} \ldots y_{m,r}^{\nu_r}. \]

\[ = \left| \frac{1}{n!} \right|_K p^{na - n + (m_0+m-1)(\nu_1+\ldots+\nu_r)} (ps)^{\nu_1+\ldots+\nu_r}. \]

Note that \(\mu_j + \nu_1 \beta_1 + \ldots + \nu_r \beta_r = n \beta^+\) implies that \(n - (\nu_1 + \ldots + \nu_r)\) is bounded from above by some constant \(C_1\), cf. [S.11]. Hence we get

\[ p^{na - n + (m_0+m-1)(\nu_1+\ldots+\nu_r)} \geq p^{n(a-1)} \cdot p^{-(m_0+m-1)C_1} \]

is bounded away from 0 (recall that \(a \in \mathbb{Z}_{>0}\)). And because \(s > \frac{1}{p}\) we have \(ps > 1\), and the term \((ps)^{\nu_1+\ldots+\nu_r}\) is unbounded as \(n \to \infty\). Altogether we get that any lift of \(\frac{1}{n!} z_{\beta^+}^n, v^+\) to \(\bigoplus_{1 \leq j \leq t} U_r(u^-) \otimes_K K v_i\) has an \(\| \cdot \|_s\) - norm which exceeds \(C_2(ps)^p\), where \(C_2 > 0\) is some constant.

The sum \(\sum_{a \geq 0} \frac{1}{n!} z_{\beta^+}^n, v^+\), which is an element of \(\hat{M}\) is therefore not contained in the image of
m_r. And as we have seen before, this in turn proves that
\[ \sum_{n \geq 0} \left( \frac{(-\log(u))^n}{n!} \right) v^+ = \delta_{u^{-1}} v^+ \in \hat{M} \] is not in the image of m_r.

**Theorem 5.2.** Let M ∈ O_{alg} be simple and assume that p is maximal among all standard parabolic subalgebras q with the property that M is contained in O_{alg}. Let V be a smooth and irreducible L_P-representation. Then \( \mathcal{F}_P^G(M, V) = \text{Ind}_P^G(W' \otimes_K V)^\circ \) is topologically irreducible as G-representation.

**Outline of proof.** Put \( X = \mathcal{F}_P^G(M, V) \). The proof proceeds as follows. To any G-subrepresentation U ⊂ X there is a smooth representation \( U_s \) and a natural G-morphism \( \text{Ind}_P^G(W')^\circ \otimes_K U_s \to U \). \( U_s \) is a subrepresentation of \( X_s \) which is shown to be isomorphic to the smoothly induced representation \( \text{ind}_P^G(V) \). We prove that if U is closed and non-zero, then \( U_s \) is non-zero as well, and the composite map

\[ \text{Ind}_P^G(W) \otimes_K U_s \hookrightarrow \text{Ind}_P^G(W) \otimes_K \text{ind}_P^G(V) \twoheadrightarrow X \]

is surjective. As this map factors through U we necessarily have \( U = X \).

**Step 1: the smooth representation \( U_s \).** For a G-subrepresentation U ⊂ X, we put

\[ U_s = \lim_{\to H} \text{Hom}_H(\text{Ind}_P^G(W')^\circ |_H, U|_H), \]

where the limit is taken over all compact open subgroups H of G, and the homomorphisms are continuous homomorphisms of topological vector spaces. Of course, it suffices to take the limit over a set of compact open subgroups which is a neighborhood basis of 1 ∈ G. There is an action of G defined on \( U_s \) as follows: for \( \phi \in U_s \), \( g \in G \) put

\[ (g \phi)(f) = \pi(g)(\phi(\pi(g^{-1})(f))). \]

Note that \( U_s \) is a smooth representation. Taking for U the whole space X gives us a smooth G-representation \( X_s \). Note that there is always a continuous map of G-representations

\[ \Phi_U : \text{Ind}_P^G(W')^\circ \otimes_K U_s \to U \ , \ f \otimes \phi \mapsto \phi(f). \]

In order to analyze \( U_s \) we will need to understand the restriction of \( \text{Ind}_P^G(W')^\circ \) to compact-open subgroups H.

**Step 2: restricting \( \text{Ind}_P^G(W')^\circ \) to compact open subgroups.** Let \( G_0 \subset G \) be a maximal compact open subgroup such that \( G = G_0 P \), and put \( P_0 = G_0 \cap P \). By [Ber, sec. II.2] ('Bruhat’s Theorem') there exists a fundamental system of neighborhoods of 1 consisting of compact open
subgroups $H$ of $G_0$, with each $H$ being normal in $G_0$, and such that every $H$ has an Iwahori decomposition

$$H = (H \cap U^-) \cdot (H \cap P)$$

where $U^-$ is the unipotent radical of the parabolic subgroup $P^-$ opposite to $P$. As $H$ is normal is $G_0$, we have for every $\gamma \in G_0$

$$H = (H \cap \gamma U^- \gamma^{-1}) \cdot (H \cap \gamma P \gamma^{-1}).$$

We have

$$\text{Ind}_{P}^{G}(W')^O|_{H} = \text{Ind}_{P_0}^{G_0}(W')^O|_{H} = \bigoplus_{\gamma \in H \setminus G_0/P_0} \text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^{H \cap \gamma P_0 \gamma^{-1}}((W')^O).$$

The $G_0$-representation $\text{Ind}_{P_0}^{G_0}(W')^O$ is topologically irreducible by Theorem 5.1. Therefore, all $H$-representations on the right hand side are irreducible too (here we use that $H$ is normal and has an Iwahori decomposition). Between two such representations $\text{Ind}_{H \cap \gamma_1 P_0 \gamma_1^{-1}}^{H \cap \gamma_1 P_0 \gamma_1^{-1}}(W)$ and $\text{Ind}_{H \cap \gamma_2 P_0 \gamma_2^{-1}}^{H \cap \gamma_2 P_0 \gamma_2^{-1}}(W)$ with $H \gamma_1 P_0 \neq H \gamma_2 P_0$, there is no non-zero continuous $H$-equivariant morphism. This fact is implicitly contained in the proof of Thm. 5.1 (the subgroups $H$ used here can be taken to be the subgroups $P_0, r$ from the proof of Thm. 5.1).

Similarly, for the induced representation $\text{Ind}_{P}^{G}(W' \otimes_K V)^O$ we have

$$\text{Ind}_{P}^{G}(W' \otimes_K V)^O|_{H} = \text{Ind}_{P_0}^{G_0}(W' \otimes_K V)^O|_{H} = \bigoplus_{\gamma \in H \setminus G_0/P_0} \text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^{H \cap \gamma P_0 \gamma^{-1}}((W')^O \otimes_K V^O).$$

Denote by $V^{H \cap P_0} \subset V$ the subspace on which $H \cap P_0$ acts trivially. Then

$$\text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^{H \cap \gamma P_0 \gamma^{-1}}((W')^O \otimes_K (V^{H \cap P_0})^O)$$

is an $H$-subrepresentation of $\text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^{H \cap \gamma P_0 \gamma^{-1}}((W')^O \otimes_K V^O)$. And

$$\text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^{H \cap \gamma P_0 \gamma^{-1}}((W')^O \otimes_K (V^{H \cap P_0})^O)$$

is canonically isomorphic to

$$\left(\text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^{H \cap \gamma P_0 \gamma^{-1}}((W')^O)\right)^O \otimes_K V^{H \cap P_0}.$$

Let

$$\phi : \text{Ind}_{P_0}^{G_0}(W')^O|_{H} \to \text{Ind}_{P_0}^{G_0}(W' \otimes_K V)^O|_{H}$$

\footnote{In a forthcoming version we will make this argument self-contained.}
be a continuous homomorphism of $H$-representations. For each $\gamma$, let $f_\gamma$ be a non-zero vector in

$$\text{Ind}_{H\cap\gamma P_0\gamma^{-1}}^H((W')\gamma)^0.$$

Because of the irreducibility of the representations $\text{Ind}_{H\cap\gamma P_0\gamma^{-1}}^H((W')\gamma)^0$, the map $\phi$ is uniquely determined by the images $\phi(f_\gamma) \in \text{Ind}_{P_0}^G(W' \otimes_K V)^0$. By the very definition of the representation $\text{Ind}_{P_0}^G(W' \otimes_K V)^0$, the function $\phi(f_\gamma) : G_0 \to W' \otimes_K V$ is rigid-analytic on the components of some covering of $G_0$ by 'polydiscs' $\Delta_i$, and takes values in a $BH$-subspace of $W' \otimes_K V$. But any such $BH$-subspace is finite-dimensional, hence contained in a subspace of the form $W' \otimes_K V_{H' \cap P_0}$ for some (small enough) open subgroup $H' \subset H$. Shrinking $H'$ we may further suppose that each $\Delta_i$ is $H'$-stable (by multiplication from the left), and that each $\phi(f_\gamma)$ takes values in $W' \otimes_K V_{H' \cap P_0}$. For any $h \in H$, we have

$$\phi(h.f_\gamma)(g) = [h.\phi(f_\gamma)](g) = \phi(f_\gamma)(h^{-1}g) \in W' \otimes_K V_{H' \cap P_0}.$$

Because the subspace of $\text{Ind}_{H\cap\gamma P_0\gamma^{-1}}^H((W')\gamma)^0$ generated by the functions $h.f_\gamma$ is dense in $\text{Ind}_{H\cap\gamma P_0\gamma^{-1}}^H((W')\gamma)^0$, we have $\phi(f)(g) \in W' \otimes_K V_{H' \cap P_0}$ for all $f \in \text{Ind}_{H\cap\gamma P_0\gamma^{-1}}^H((W')\gamma)^0$. It follows that $\phi$ induces a continuous map of $H'$-representations

$$\phi : \text{Ind}_{P_0}^G(W')^0|_{H'} \to \text{Ind}_{P_0}^G(W' \otimes_K V_{H' \cap P_0})^0|_{H'} = \left[\text{Ind}_{P_0}^G(W')^0\right]|_{H' \otimes_K V_{H' \cap P_0}}.$$

The irreducible $H'$-representation $\text{Ind}_{H' \cap \gamma P_0 \gamma^{-1}}^H((W')\gamma)^0$ on the left hand side is mapped to the corresponding summand $\text{Ind}_{H' \cap \gamma P_0 \gamma^{-1}}^H((W')\gamma)^0|_{H'} \otimes_K V_{H' \cap P_0}$, and is thus of the form $f \mapsto f \otimes v$ for some $v \in V_{H' \cap P_0}$. This shows that any element in $X_s$ is "locally" (i.e. on the subrepresentations $\text{Ind}_{H' \cap \gamma P_0 \gamma^{-1}}^H((W')\gamma)^0$ given by vectors in $V_{H' \cap P_0}$ (for sufficiently small subgroups $H'$). It is now an easy matter to verify that the $G$-action on $X_s$ identifies $X_s$ with the smoothly induced representation $\text{ind}_F^G(V)$. Thus we have shown the

**Lemma:** The canonical map $\text{ind}_F^G(V) \to X_s$, $\varphi \mapsto [f \mapsto \Phi(f \otimes \varphi)]$, is an isomorphism.

**Step 3:** $U_s$ is non-zero for non-zero $U$. Let $U \subset X$ be a non-zero closed $G$-invariant subspace. Let $f \in U$ be a non-zero element. Then, as we have seen before, $f$ takes values in a vector space $W' \otimes_K V_{H' \cap P_0}$ for a sufficiently small compact open subgroup $H$. Therefore, $f$ is contained in

$$\bigoplus_{\gamma \in H' \cap G_0 / P_0} \text{Ind}_{H' \cap \gamma P_0 \gamma^{-1}}^H((W')\gamma \otimes_K (V_{H' \cap P_0})^\gamma)^0 = \bigoplus_{\gamma \in H' \cap G_0 / P_0} \text{Ind}_{H' \cap \gamma P_0 \gamma^{-1}}^H((W')^0 \otimes_K V_{H' \cap P_0})^\gamma.$$

It follows from this that $U$ contains a non-zero $H$-invariant subspace of
As we have seen above, the set
\[ \bigoplus_{\gamma \in H \setminus G_0/P_0} \text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^H ((W')^\gamma) \otimes_K V^H. \]
Because the representations \( \text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^H ((W')^\gamma) \) are irreducible, and because there is no non-zero continuous intertwiner between any two such (with \( H_{\gamma_1}P_0 \neq H_{\gamma_2}P_0 \)), any such subspace is isomorphic to
\[ \bigoplus_{\gamma \in H \setminus G_0/P_0} \text{Ind}_{H \cap \gamma P_0 \gamma^{-1}}^H ((W')^\gamma) \otimes_K V_\gamma, \]
with a subspace \( V_\gamma \) of \( V^{H \cap P_0} \). Thus, any vector in some \( V_\gamma \) gives rise to a non-zero \( H \)-equivariant homomorphism from \( \text{Ind}^{G_0}_{P_0}(W')^\gamma \) to \( U \). Therefore, \( U_s \neq \{0\} \).

**Step 4**: \( \text{Ind}^G_{P_0}(W')^\otimes_K U_s \) surjects onto \( X \). Let \( \phi_0 \in U_s \) be a non-zero element which we identify with an element of the smoothly induced representation \( \text{ind}^G_P(V) \). Without loss of generality we may assume \( \phi_0(1) \neq 0 \). Consider the \( P \)-morphism \( \Pi : U_s \rightarrow V \) given by \( \phi \mapsto \phi(1) \). As \( \phi_0(1) \neq 0 \) we deduce that the image of \( \Pi \) is a non-zero subrepresentation of \( V \), and is hence equal to \( V \) (because \( V \) was supposed to be irreducible). Therefore, for any \( v \in V \) there is some \( \phi \in U_s \) with \( \phi(1) = v \). As \( U_s \) is a \( G \)-representation we find that for any \( g \in G \) and any \( v \in V \) there is some \( \phi \in U_s \) with \( \phi(g) = v \). Then, on a neighborhood \( N \) of \( g \), the map \( \phi \) is constant with value \( v \) on this neighborhood. Let \( S \subset G_0 \) be a compact locally analytic subset such that \( G = S \cdot P \) and hence \( G_0 = S \cdot P_0 \). (That is, every \( g \in G \) is the product \( s \cdot p \) with uniquely determined \( s \in S \) and \( p \in P \).) Then there is a compact open subset \( S' \subset S \) and a compact open neighborhood of the identity \( P' \subset P_0 \) such that \( S' \cdot P' \subset N \). Let \( f : G_0 \rightarrow W' \) be a function whose support is contained in \( S' \cdot P_0 \). Then the function \( S \rightarrow W' \otimes_K V \), \( s \mapsto f(s) \otimes \phi(s) \), is equal to the function \( S \rightarrow W' \otimes_K V \), \( s \mapsto f(s) \otimes v \).

Let \( f \in \text{Ind}^G_{P_0}(W' \otimes_K V)^0 \) be any element. \( f \) is uniquely determined by its restriction to \( S \). As we have seen above, the set \( f(G_0) \) is contained in a finite-dimensional vector space \( W' \otimes_K V_0 \) with \( V_0 \subset V \). Let \( v_1, \ldots, v_r \) be a basis of \( V_0 \). Write \( f(s) = f_1(s) \otimes v_1 + \ldots + f_r(s) \otimes v_r \) with locally analytic \( W' \)-valued functions \( f_i \) on \( S \). Extend each function \( f_i \) to \( G_0 \) by \( f_i(s \cdot p) = \rho'(p^{-1})(f_i(s)) \), where \( \rho' \) is the representation of \( P \) on \( W' \). Then \( f_i \in \text{Ind}^G_{P_0}(W') \) for all \( i \). In fact, examining the action of the differential operators in \( \mathfrak{d} \) shows that \( f_i \in \text{Ind}^G_{P_0}(W' \otimes_K V)^0 \) for all \( i \). For any \( s \in S \) choose some \( \phi_{i,s} \in U_s \) such that \( \phi_{i,s}(x) = v_i \) for all \( x \) in a compact open neighborhood \( N_{i,s} \subset S \) of \( s \). As \( S \) is compact, finitely many of the \( N_{i,s} \) will cover \( S \). We can then choose a (finite) disjoint refinement \( \{N_{i,j}\}_j \) of the finite covering. Then we restrict \( f_i \) to each of these \( N_{i,j} \cdot P \), and extend it by 0 to \( (S \setminus N_{i,j}) \cdot P \). Denote the function thus obtained by \( f_{i,j} \). Again, all \( f_{i,j} \) lie in \( \text{Ind}^G_{P_0}(W') \), and \( x \mapsto f_{i,j}(x) \otimes \phi_{i,s(j)}(x) = f_{i,j}(x) \otimes v_i \), for a suitably chosen \( s(i,j) \in S \). Obviously, we have: \( f = \sum_{i,j} f_{i,j} \otimes \phi_{i,s(j)} \). Indeed, by construction, both \( f \mid_S \) and the restriction of the sum to \( S \) coincide. And both are functions in \( \text{Ind}^G_{P_0}(W' \otimes_K V)^0 \); therefore, they are equal.
This shows that \( f \) can be written as a sum of “tensor products” of functions in \( \text{Ind}_P^G(W')^p \) and functions in \( U_s \) hence, the map \( \text{Ind}_P^G(W')^p \otimes_K U_s \to \text{Ind}_P^G(W' \otimes_K V)^p = X \) is surjective. \( \square \)

**Example 5.3.** a) Let \( M \) be a finite-dimensional irreducible algebraic \( G \)-module, so that in the above theorem we may choose \( P = G \). Then we deduce that \( F_G^G(M, V) = M \otimes V \) is topologically irreducible for any smooth irreducible representation \( V \) of \( G \). This result was already proved by D. Prasad [Pr], and our proof is modeled after his approach.

b) Let \( M = U(g) \otimes U(p) \) be an irreducible generalized Verma module for some irreducible \( L_P \)-representation \( W \). Then by Corollary 8.6 the parabolic subalgebra \( p \) is maximal in the above sense. It follows that \( F_G^G(M) = \text{Ind}_P^G(W') \) is topologically irreducible. This result was proved in [OS] for an arbitrary irreducible finite-dimensional locally analytic \( L_P \)-representation \( W \).

### 6. Composition series

**6.1.** Before we formulate our main result in this section, we recall the definition of generalized Steinberg representations, cf. [BW], [Ca]. Let \( P \) be a standard parabolic subgroup of \( G \) and denote by \( i_P^G = \text{ind}_P^G(1) = C^\infty(G/P, K) \) the smooth induced representation of locally constant functions on \( G/P \). The generalized Steinberg representation to \( P \) is the quotient \( v_P^G = i_P^G/ \sum_{P \subsetneq Q \subset G} i_Q^G \).

This is an irreducible \( G \)-representation and all irreducible subquotients of \( i_P^G \) are of the form \( v_Q^G \) with \( P \subset Q \). Each such representation occurs with multiplicity one, cf. loc.cit.

**6.2. How to obtain a composition series for \( F_P^G(M, V) \).** We are going to describe a method for finding a composition series of the representation \( F_P^G(M, V) \). Let \( M \) be an object of \( \mathcal{O}_p \text{alg} \) and \( V \) an object of \( \text{Rep}_K^\infty(L_P) \) of finite length. Let \( 0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_r = V \) be a composition series of smooth \( L_P \)-representations for \( V \). By the exactness of \( F_P^G \) in the second argument, cf. Prop. 4.6 we get a chain of inclusions

\[
0 = F_P^G(M, V_0) \subsetneq F_P^G(M, V_1) \subsetneq \ldots \subsetneq F_P^G(M, V_r) = F_P^G(M, V).
\]

Further the quotient \( F_P^G(M, V_{i+1})/F_P^G(M, V_i) \) is isomorphic to \( F_P^G(M, V_{i+1}/V_i) \). We may hence assume that \( V \) is irreducible.

Let \( M \) be an object of the category \( \mathcal{O}_p \text{alg} \). Then it has a Jordan-Hölder series

\[
M = M^0 \supsetneq M^1 \supsetneq M^2 \supsetneq \ldots \supsetneq M^r \supsetneq M^{r+1} = (0)
\]

in the same category. Thus we may apply our functor \( F_P^G \) to the pair \( (M, V) \). We get a sequence of surjections
\[ \mathcal{F}_P^G(M, V) = \mathcal{F}_P^G(M^0, V) \xrightarrow{p_0} \mathcal{F}_P^G(M^1, V) \xrightarrow{p_1} \mathcal{F}_P^G(M^2, V) \xrightarrow{p_2} \ldots \xrightarrow{p_{r-1}} \mathcal{F}_P^G(M^r, V) \xrightarrow{p_r} (0). \]

For any integer \( i \) with \( 0 \leq i \leq r \), we put

\[ q_i := p_i \circ p_{i-1} \circ \cdots \circ p_1 \circ p_0. \]

and

\[ \mathcal{F}^i := \ker(q_i) \]

which is a closed subrepresentation of \( \mathcal{F}_P^G(M, V) \). We obtain a filtration

\[ \mathcal{F}^{-1} = (0) \subset \mathcal{F}^0 \subset \ldots \subset \mathcal{F}^{r-1} \subset \mathcal{F}^r = \mathcal{F}_P^G(M, V) \] (6.2.1)

by closed subspaces with

\[ \mathcal{F}^i / \mathcal{F}^{i-1} \simeq \mathcal{F}_P^G(M^i / M^{i+1}, V). \]

Let \( Q_i = L_i \cdot U_i \supset P \) be the standard parabolic subgroup which is maximal with the property that \( M^i / M^{i+1} \) is an object of \( \mathcal{O}_{\text{alg}} \). Then

\[ \mathcal{F}_P^G(M^i / M^{i+1}, V) = \mathcal{F}_{Q_i}^G(M^i / M^{i+1}, i_{L_i}^{L_i} L_i \subseteq P (V)). \]

Then we can choose a Jordan-Hölder series for \( i_{L_i}^{L_i} L_i \subseteq P (V) \) and we obtain a Jordan-Hölder series for

\[ \mathcal{F}_{Q_i}^G(M^i / M^{i+1}, i_{L_i}^{L_i} L_i \subseteq P (V)). \]

By refining the filtration (6.2.1) we get thus a Jordan-Hölder series of \( \mathcal{F}_P^G(M) \). In case \( V \) is actually the trivial representation the irreducible subquotients of the smooth representation \( i_{L_i}^{L_i} L_i \subseteq P (1) \) are the generalized Steinberg representations \( v_{Q_i}^Q \) recalled above, and the irreducible subquotients of \( \mathcal{F}_P^G(M, 1) = \mathcal{F}_P^G(M) \) are of the form \( \mathcal{F}_{Q_i}^G(M^i / M^{i+1}, v_{Q_i}^Q) \).

**Remark 6.3.** Independently of our work, B. Schraen determined the Jordan-Hölder series for certain subquotients of parabolically induced locally analytic representations for the group \( \text{GL}_3(\mathbb{Q}_p) \), including the locally analytic Steinberg representation.
7. Applications to equivariant vector bundles on Drinfeld’s half space

Let $\mathcal{X}$ be Drinfeld’s half space of dimension $d \geq 1$ over $K$. This is a rigid-analytic variety over $K$ given by the complement of all $K$-rational hyperplanes in projective space $\mathbb{P}^d_K$, i.e.,

$$\mathcal{X} = \mathbb{P}^d_K \setminus \bigcup_{H \subseteq K^{d+1}} \mathbb{P}(H).$$

There is natural action of $G = \text{GL}_{d+1}(K)$ on $\mathcal{X}$ induced by the algebraic action $m : G \times \mathbb{P}^d_K \to \mathbb{P}^d_K$ of $G = \text{GL}_{d+1}/K$ defined by

$$g \cdot [q_0 : \cdots : q_d] := m(g, [q_0 : \cdots : q_d]) := [q_0 : \cdots : q_d]g^{-1}.$$

It is known that $\mathcal{X}$ is a Stein space. Moreover, for every homogeneous vector bundle $\mathcal{F}$ on $\mathbb{P}^d_K$, the space of sections $\mathcal{F}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{F})$ has the structure of a $K$-Fréchet space equipped with a continuous $G$-action. Its topological dual $\mathcal{F}(\mathcal{X})'$ is a locally analytic $G$-representation, cf. [ST1].

In [O] one of us studied the structure of $\mathcal{F}(\mathcal{X})$ for homogeneous vector bundles $\mathcal{F}$ on $\mathbb{P}^d_K$. The theorem below generalizes the result of Schneider and Teitelbaum [ST3] for the canonical bundle $\mathcal{F} = \Omega^d$ resp. Pohlkamp [Po] for the structure sheaf $\mathcal{F} = \mathcal{O}$.

**Theorem 7.1.** Let $\mathcal{F}$ be a homogeneous vector bundle on $\mathbb{P}^d_K$. There is a $G$-equivariant filtration by closed $K$-subspaces of $\mathcal{F}(\mathcal{X})$

$$\mathcal{F}(\mathcal{X}) = \mathcal{F}(\mathcal{X})^0 \supset \mathcal{F}(\mathcal{X})^1 \supset \cdots \supset \mathcal{F}(\mathcal{X})^{d-1} \supset \mathcal{F}(\mathcal{X})^d = H^0(\mathbb{P}^d_K, \mathcal{F}),$$

such that for $j = 0, \ldots, d - 1$, there are extensions of locally analytic $G$-representations

$$0 \to v_{P_{(j+1,d-j)}}^G(H^{d-j}(\mathbb{P}^d_K, \mathcal{F})) \to (\mathcal{F}(\mathcal{X})^j / \mathcal{F}(\mathcal{X})^{j+1})' \to \text{Ind}^{\text{an,G}}_{P_{(j+1,d-j)}}(U_j')^\mathcal{O} \to 0. \tag{7.1.3}$$

Here, for a decomposition $(n_1, \ldots, n_s)$ of $d + 1$, the symbol $P_{(n_1, \ldots, n_s)}$ denotes the corresponding lower standard parabolic subgroup of $G$. The module $v_{P_{(j+1,\ldots,1)}}^G(H^{d-j}(\mathbb{P}^d_K, \mathcal{F}))$ is a generalized Steinberg representation with coefficients in the finite-dimensional algebraic $G$-module $H^{d-j}(\mathbb{P}^d_K, \mathcal{F})'$. The $P_{(j+1,d-j)}$-representation $U_j'$ is a tensor product $N_j' \otimes v_{L_{(j+1,d-j)}}^{L_{(j+1,d-j)}}$ of an algebraic representation $N_j'$ and the Steinberg representation $v_{L_{(j+1,d-j)}}^{L_{(j+1,d-j)}}$ of its Levi factor $L_{(j+1,d-j)}$. The first one is characterized by the property that it generates the kernel $\mathcal{H}_{\mathbb{P}^d_K}^{d-j}(\mathbb{P}^d_K, \mathcal{F})$ of the natural homomorphism

$$\mathcal{H}_{\mathbb{P}^d_K}^{d-j}(\mathbb{P}^d_K, \mathcal{F}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{F})$$

as a module with respect to $U(\mathfrak{g})$. Here $\mathbb{P}^d_K = V(X_{j+1}, \ldots, X_d)$ is the linear subvariety of $\mathbb{P}^d_K$ defined by the vanishing of the coordinates $X_{j+1}, \ldots, X_d$. The module $\mathfrak{g}_j$ is just the kernel of the induced surjection $U(\mathfrak{g}) \otimes_U(\mathbb{P}^d_{(j+1,d-j)}) N_j' \to \mathcal{H}_{\mathbb{P}^d_K}^{d-j}(\mathbb{P}^d_K, \mathcal{F})$. 
In the case where \( F \) arises from an irreducible representation of the Levi subgroup \( L_{(1,d)} \), we could make our result more precise, i.e., concerning the structure of \( N_j \). Rather than recalling this result in full generality, we restrict our attention from now on to homogeneous line bundles on \( \mathbb{P}_K^d \), where we will get even a more precise formula, cf. Proposition 7.5.

Let \( s \in \mathbb{Z} \) and denote by \( \lambda' = (s, \ldots, s) \in \mathbb{Z}^d \) the constant integral weight of \( \text{GL}_d \). Let \( r = \lambda_0 \in \mathbb{Z} \) and set

\[
\lambda := (r, s, \ldots, s) \in \mathbb{Z}^{d+1}.
\]

We denote by \( L_{\lambda} \) the homogeneous line bundle on \( \mathbb{P}_K^d \) such that its fibre in the base point is the irreducible algebraic \( L_{(1,d)} \)-representation corresponding to \( \lambda \). Then we obtain

\[
L_{\lambda} = \mathcal{O}(r - s)
\]

where the \( G \)-linearization is given by the tensor product of the natural one on \( \mathcal{O}(r - s) \) with the character \( \det^s \).

Put \( w_j := s_j \cdots s_1 \), where \( s_i \in W \) is the (standard) simple reflection in the Weyl group \( W \cong S_{d+1} \) of \( G \). Recall that \( \cdot \) denotes the dot action of \( W \) on \( X^*(T)_\mathbb{Q} \). If \( \chi = (\chi_0, \ldots, \chi_d) \in \mathbb{Z}^{d+1} \) we get

\[
w_i \cdot \chi = (\chi_1 - 1, \chi_2 - 1, \ldots, \chi_i - 1, \chi_0 + i, \chi_{i+1}, \ldots, \chi_d).
\]

Hence for \( \chi = \lambda = (r, s, \ldots, s) \), we compute

\[
w_0 \cdot \lambda = \lambda
\]

\[
w_1 \cdot \lambda = (s - 1, r + 1, s, \ldots, s)
\]

\[
\vdots
\]

\[
w_i \cdot \lambda = (s - 1, s - 1, \ldots, s - 1, r + i, s, \ldots, s)
\]

\[
\vdots
\]

\[
w_d \cdot \lambda = (s - 1, s - 1, \ldots, s - 1, r + d).
\]

In particular, there is at most one integer \( 0 \leq i \leq d \), such that \( w_i \cdot \lambda \) is dominant with respect to the Borel subgroup \( B^+ \) of upper triangular matrices. In fact, this integer is characterized by the non-vanishing of \( H^i(\mathbb{P}_K^d, L_{\lambda}) \), cf. [Bo, Theorem IV], resp. [Ha, Theorem II.5.1]. Further one has \( i_0 = 0 \) for \( r \geq s \) and \( i_0 = d \) for \( s \geq r + d + 1 \). We denote this integer by \( i_0 \) if it exists. Otherwise, there is a unique integer \( i_0 < d \) with \( w_{i_0} \cdot \lambda = w_{i_0 + 1} \cdot \lambda \). This is the case for \( 0 < i_0 = s - r - 1 < d + 1 \). We get

\[
(7.1.5) \quad w_i \cdot \lambda \succ w_{i+1} \cdot \lambda
\]
for all $i \geq i_0$ (resp. $i > i_0$ if $w_i \cdot \lambda = w_{i_0+1} \cdot \lambda$), and

\[(7.1.6) \quad w_i \cdot \lambda < w_{i+1} \cdot \lambda \]

for all $i < i_0$, with respect to the dominance order $\succ$ on $X^*(T)_\mathbb{Q}$. We put

$$
\mu_{i,\lambda} := \begin{cases} 
  w_{i-1} \cdot \lambda & : i \leq i_0 \\
  w_i \cdot \lambda & : i > i_0
\end{cases}
$$

This is a $L_{(i,d-i+1)}$-dominant weight (with respect to the Borel subgroup $L_{(i,d-i+1)} \cap B^+$).

Let $V_{i,\lambda}$ be the finite-dimensional irreducible $L_{(i,d-i+1)}$-module with highest weight $\mu_{i,\lambda}$. Let $P^+_{(i,d-i+1)}$ be the upper triangular parabolic subgroup to the decomposition $(i,d+1-i)$ and denote by $U^+_{(i,d-i+1)}$ its unipotent radical. By considering the trivial action of $U^+_{(i,d-i+1)}$ on $V_{i,\lambda}$, we may view it as a $P^+_{(i,d-i+1)}$-module.

In the following we identify the linear subvarieties $\mathbb{P}^{d-i}_K$, $0 \leq i \leq d$, with the closed subschemes $V(X_0,\ldots,X_{i-1}) \subset \mathbb{P}^d_K$ defined by the vanishing of the first $i$ coordinate functions. Note that the stabilizer of this subvariety in $G$ is just $P^+_{(i,d+1-i)}$. In loc.cit. we saw that we can realize $V_{i,\lambda}$ as a submodule of $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{L}_\lambda)$. In fact, there is an action of $P^+_{(i,d-i+1)} \times U(g)$ on $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{L}_\lambda)$. The one of $g$ is induced by the homogeneous line bundle. The second one is induced by functoriality since it is the stabilizer of $\mathbb{P}^{d-i}_K$.

**Proposition 7.2.** For $1 \leq i \leq d$, the $P^+_{(i,d-i+1)} \times U(g)$-module $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{L}_\lambda)$ coincides with the module $\bigoplus_{k_0,\ldots,k_{i-1} \geq 0} K \cdot X_0^{k_0}X_1^{k_1} \cdots X_d^{k_d} : V_{i,\lambda}$.

**Proof.** This was proved in loc.cit. Proposition 1.4.2. $\Box$

From this latter statement we deduce immediately the following corollary.

**Corollary 7.3.** For $1 \leq i \leq d$, the $U(g)$-module $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{L}_\lambda)$ lies in the category $\mathcal{O}^g_{\text{alg}}_{(i,d-i+1)}$.

**Remark 7.4.** It is well-known that the above objects lie in $\mathcal{O}$, although, mostly the case of the full flag variety is considered, cf. e.g. [Ku], [AL]. Indeed this can be seen by using the Grothendieck-Cousin complex which was used in loc.cit. to compute $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{L}_\lambda)$. Here all contributions of this complex are objects in $\mathcal{O}$. Hence the cohomology of this complex which computes $\tilde{H}^i_{\mathbb{P}^{d-i}_K}(\mathbb{P}^d_K,\mathcal{L}_\lambda)$ lies in $\mathcal{O}$, as well.

Recall that we denote for a character $\mu \in X^*(T)$ by $L(\mu)$ the irreducible highest weight $U(g)$-module of weight $\mu$.

---

\[6\text{We note that for formulating Theorem 7.1 we have used in loc.cit. the identification of $\mathbb{P}^{d-i}_K$ with $V(X_{d-i+1},\ldots,X_d)$. Therefore the standard parabolic subgroups used are lower (block) triangular. Afterwards, we used the conjugacy of $V(X_0,\ldots,X_{i-1})$ and $V(X_{d-i+1},\ldots,X_d)$ within $\mathbb{P}^d_K$ via the action of $G$ on $\mathbb{P}^d_K$.}\]
Proposition 7.5. For $1 \leq i \leq d$, the $\mathbf{P}^{d-1}_{(d-i+1)}(\mathbb{P}^d_K, \lambda)$ is isomorphic to $L(\mu_{i,\lambda})$.

Proof. In order to show the statement it suffices to check the following conditions.

1) $\tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ is a highest weight module of weight $\mu_{i,\lambda}$.

2) $\tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ is irreducible.

Because of identity (7.1.4) it suffices to check the conditions in the case where $s = 0$. Then $\mathcal{L}_\lambda = \mathcal{O}(r)$ is the $r$-th Serre twist with its natural $G$-linearization. All cohomology groups are $K$-subspaces of $\bigoplus_{k_0,...,k_d \in \mathbb{Z}} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}$. Here the action of $g$ is given as follows. For a root $\alpha = \alpha_{i,j} = \epsilon_i - \epsilon_j \in \Phi$, let

$$L_\alpha := L(i,j) \in \mathfrak{g}_\alpha$$

be the standard generator of the weight space $\mathfrak{g}_\alpha$ in $\mathfrak{g}$. Then the action of $\mathfrak{g}$ is determined by

$$L(i,j) \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d} = k_j \cdot X_i \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}$$

and

$$t \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d} = (\sum_i k_i t_i) \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}$$

for $t = (t_0, \ldots, t_d) \in t$.

We prove statements 1) and 2) case by case.

a) Case $r \geq 0$. Then $\mu_{i,\lambda} = w_i \cdot \lambda$ and

$$V_{i,\lambda} = \bigoplus_{k_i + \cdots + k_d \geq 0, k_i + \cdots + k_d = r+i} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d}.$$ 

Further $v_{i,\lambda} = X_0^{-1} X_1^{-1} \cdots X_{i-1}^{-1} X_i^{r+i}$ is a maximal vector in $\tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ of weight $\mu_{i,\lambda}$.

It is easy to see by formula (7.5.0) that $v_{i,\lambda}$ generates $\tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ as $U(\mathfrak{g})$-module, so condition 1) is satisfied. For condition 2), let $N \subset \tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ be a submodule, $N \neq (0)$. Then $N$ lies in the category $\mathcal{O}$ as a submodule, hence $t$ acts semi-simply. Let $X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d} \in N$, where $k_0, \ldots, k_{i-1} < 0$, $k_i \ldots, k_d \geq 0$ and $\sum_{j=0}^d k_j = r$. Then by repeatedly multiplication with $L_{(k,l)}$ with $0 \leq k \leq i-1$ and $l \geq i$, we can achieve that - up to scalar - $k_0 = \cdots = k_{i-1} = -1$. By further multiplying with $L_{(k,l)}$ where $k = i$ and $l \geq i + 1, \ldots, d$, we see that $v_{i,\lambda} \in N$. Thus $N = \tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$.

b) Case $r \leq -d-1$. Then $\mu_{i,\lambda} = w_{i-1} \cdot \lambda$ and

$$V_{i,\lambda} = \bigoplus_{k_0, \ldots, k_{i-1} < 0, k_0 + \cdots + k_{i-1} = r} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_{i-1}^{k_{i-1}}.$$ 

Further $X_0^{-1} X_1^{-1} \cdots X_{i-2}^{-1} X_{i-1}^{r+i-1}$ is a maximal vector in $\tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ of weight $\mu_{i,\lambda}$.

It is easy to see by formula (7.5.0) that $v_{i,\lambda}$ generates $\tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ as $U(\mathfrak{g})$-module, so condition 1) is satisfied. For condition 2), let $N \subset \tilde{H}^i_{\mathbb{P}^d_K}(\mathbb{P}^d_K, \lambda)$ be a submodule, $N \neq (0)$. Let
$X_0^{k_0} X_1^{k_1} \cdots X_d^{k_d} \in N$, where $k_0, \ldots, k_{i-1} < 0$, $k_i, \ldots, k_d \geq 0$ and $\sum_{j=0}^{d} k_j = r$. Then by repeatedly multiplication with $L_{(k,l)}$ with $0 \leq k \leq i-1$ and $l \geq i$, we can achieve that - up to scalar - $k_i = \cdots = k_d = 0$. By further multiplying with $L_{(k,l)}$ where $k < i - 1$ and $l = i - 1$, we see that $v_{i, \lambda} \in N$. Thus $N = \hat{H}^i_{\mathbb{P}^d_{K}}(\mathbb{P}^d_{K}, \mathcal{L}_{\lambda})$.

c) Case $0 > r > -d - 1$. Then for $i \leq i_0 = s - r - 1 \mu_{i, \lambda} = w_{i-1} \cdot \lambda$ and

$$V_{i, \lambda} = \bigoplus_{k_0, \ldots, k_{i-1} < 0} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_{i-1}^{k_{i-1}}.$$ 

Further $X_0^{-1} X_1^{-1} \cdots X_{i-2}^{-1} X_{i-1}^{r+i-1}$ is a maximal vector in $\hat{H}^i_{\mathbb{P}^d_{K}}(\mathbb{P}^d_{K}, \mathcal{L}_{\lambda})$ of weight $\mu_{i, \lambda}$.

For $i > i_0 = s - r - 1$, we have $\mu_{i, \lambda} = w_{i} \cdot \lambda$ and

$$V_{i, \lambda} = \bigoplus_{k_0, \ldots, k_{i-1} < 0} K \cdot X_0^{k_0} X_1^{k_1} \cdots X_{i-1}^{k_{i-1}}.$$ 

Further $X_0^{-1} X_1^{-1} \cdots X_{i-2}^{-1} X_{i-1}^{1+r+i-1}$ is a highest weight vector in $\hat{H}^i_{\mathbb{P}^d_{K}}(\mathbb{P}^d_{K}, \mathcal{L}_{\lambda})$ of weight $\mu_{i, \lambda}$.

Here the reasoning is a mixture of the previous cases.

Now we translate the above result for the computation of $\hat{H}^i_{\mathbb{P}^d_{K}}(\mathbb{P}^d_{K}, \mathcal{F})$ where $\mathbb{P}^d_{K}$ is identified with the closed subscheme $V_i = V(X_{d-i+1}, \ldots, X_d)$ of $\mathbb{P}^d_{K}$. Consider the block matrix

$$z_i := \begin{pmatrix} \ 0 & I_i \\ I_{d+1-i} & 0 \end{pmatrix} \in G,$$

where $I_j \in \text{GL}_j(K)$ denotes the $j \times j$-identity matrix. Then $V(X_0, \ldots, X_{i-1})$ is transformed into $V(X_{d-i+1}, \ldots, X_d)$ under the action of $z_i$ on $\mathbb{P}^d_{K}$. We have

$$z_i \cdot P_{(d-i+1,i)} \cdot z_i^{-1} = P_{(1,d+1-i)}$$

and on the Levi subgroups the conjugacy map is given by

$$L_{(d-i+1,i)} \ni \begin{pmatrix} \ A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} \ B & 0 \\ 0 & A \end{pmatrix} \in L_{(1,d-i+1)}.$$ 

Hence the $P_{(d-i+1,i)} \ltimes U(\mathfrak{g})$-module $\hat{H}^i_{V_i}(\mathbb{P}^d_{K}, \mathcal{F})$, is given by $\hat{H}^i_{\mathbb{P}^d_{K}}(\mathbb{P}^d_{K}, \mathcal{F})$ twisted with the action of $z_i$. In particular, we can choose $V_{i, \lambda}$ - equipped with its action of $P_{(d-i+1,i)}$ via the isomorphism (7.5.0) - to be the representation $N_{d-i}$ of Theorem 7.1. Its highest weight is $z_i^{-1} \cdot \mu_{i, \lambda}$.

**Corollary 7.6.** If $\mathcal{F} = \mathcal{L}_{\lambda}$ is a line bundle then the outer contributions in (7.1.3) are topologically irreducible. Consequently, (7.1.3) is a "essentially" Jordan-Hölder series of $H^0(X, \mathcal{L}_{\lambda})$.

---

7By refining the filtration in the naive way we get a Jordan-Hölder series.
Proof. By Theorem 5.2 all contributions of the right hand side in the extensions (7.1.3) are topologically irreducible since the standard parabolic subgroups are (proper) maximal and the modules $L(\mu, \lambda)$ do not lie in $\mathcal{O}^\theta$ (since they are not finite-dimensional). On the other hand, for the line bundle $L_{\lambda}$, we compute

$$H^*(\mathbb{P}^d_K, L_{\lambda}) = H^{i_0}(\mathbb{P}^d_K, L_{\lambda}) = \begin{cases} K & \text{for } r - s \geq 0 \text{ and } i_0 = 0 \\ K & \text{for } r - s \leq -d - 1 \text{ and } i_0 = d \\ 0 & \text{otherwise} \end{cases}$$

so $v_{\mathbb{P}^d(i+1,1,\ldots,1)}(H^{d-j}(\mathbb{P}^d_K, \mathcal{F})')$ vanishes or coincides with $v_{\mathbb{P}^d(i+1,1,\ldots,1)}(K)$ - the generalized Steinberg representation - which is irreducible. \qed

8. Appendix: Some Properties of Highest Weight Modules in $\mathcal{O}$

This section is about relations in a simple $U(\mathfrak{g})$-module $M$ of the form

$$y^\gamma \cdot v^+ = \sum_{\nu \in I_n} c_{\nu} y_{\beta_1}^{\nu_1} \cdots y_{\beta_r}^{\nu_r} \cdot v^+,$$

where the elements $y_{\gamma}, y_{\beta_i} \in \mathfrak{g}_{-\beta_i}$ are part of a Chevalley basis, $v^+$ is a highest weight vector, and the coefficients $c_{\nu}$ are in $K$. We assume that standard parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_I$ is maximal for $M$, and that $\gamma \in \Phi^+ \setminus \Phi^+_I$. Our aim is to show that there is at least one $\nu$ with $|c_{\nu}| \geq 1$. This, of course, implies that $y_{\gamma}$ acts injectively on $M$ (which is the case when all $c_{\nu}$ vanish). This is, as was kindly pointed out to us by V. Mazorchuk, already shown in [DMP, 3.6]. Our proof is completely different from the proof given in [DMP], but the basic strategy extends to the more general problem considered here (where the $c_{\nu}$ are arbitrary).\footnote{We still have some conditions on the residue characteristic of $K$. However, we hope to remove these restrictions in a later version.}

We begin with a standard commutator relation valid in any associative unital algebra $A$. For elements $x, z \in A$ and $i \in \mathbb{Z}_{\geq 0}$ define inductively $[x^{(i)}, z]$ by $[x^{(0)}, z] = z$ and $[x^{(i+1)}, z] = [x, [x^{(i)}, z]] = x [x^{(i)}, z] - [x^{(i)}, z] x$.

Lemma 8.1. Let $x, z_1, \ldots, z_n \in A$. For all $k \in \mathbb{Z}_{\geq 0}$ one has

$$x^k \cdot z_1 z_2 \cdots z_n = \sum_{i_1 + \cdots + i_{n+1} = k} \left( \begin{array}{c} k \\ i_1 \ldots i_{n+1} \end{array} \right) [x^{(i_1)}, z_1] \cdots [x^{(i_n)}, z_n] x^{i_{n+1}}$$

where, as usual,
\[
\binom{k}{i_1 \ldots i_{n+1}} = \frac{k!}{(i_1)! \ldots (i_{n+1})!}
\]

Similarly,
\[
[x^{(k)}, z_1 z_2 \ldots z_n] = \sum_{i_1 + \ldots + i_n = k} \binom{k}{i_1 \ldots i_n} [x^{(i_1)}, z_1] \cdot \ldots \cdot [x^{(i_n)}, z_n]
\]

Proof. This is straightforwardly proved by induction. \qed

Lemma 8.2. Let \( x \in \mathfrak{g} \), let \( M \) be a \( U(\mathfrak{g}) \)-module and \( v \in M \).

a) If \( x \) acts locally finite on \( v \) (i.e., the \( K \)-vector space generated by \( (x^i.v)_{i \geq 0} \) is finite-dimensional), then \( x \) acts locally finite on \( U(\mathfrak{g}).v \).

b) If \( x.v = 0 \) and \( [x, [x, y]] = 0 \) for some \( y \in \mathfrak{g} \), then
\[
x^n y^n . v = n! \cdot [x, y]^n . v
\]

Proof. a) It suffices to show that \( x \) acts locally finite on any element of the form \( z_1 \ldots z_n.v \) (for arbitrary \( n \) and arbitrary elements \( z_1, \ldots, z_n \in \mathfrak{g} \)). Any element of \( U(\mathfrak{g}) \) of the form \( [x^{(i_1)}, z_1] \cdot \ldots \cdot [x^{(i_n)}, z_n] \) is contained in \( \mathfrak{g} \cdot \ldots \cdot \mathfrak{g} \), and \( \mathfrak{g} \cdot \ldots \cdot \mathfrak{g} \) is a finite-dimensional \( K \)-subspace of \( U(\mathfrak{g}) \). As the elements \( x^i.v, i \geq 0 \), are all contained in a finite-dimensional vector space, it follows from the formula in Lemma 8.1 all elements \( x^i \cdot z_1 \ldots z_n.v \) are contained in a finite-dimensional \( K \)-vector space.

b) In the formula of Lemma 8.1 we let \( z_i = y, 1 \leq i \leq n \), and get that the term
\[
[x^{(i_1)}, y] \cdot \ldots \cdot [x^{(i_n)}, y] x^{i_{n+1}}.v
\]
vanishes as soon as \( i_{n+1} > 0 \) or some \( i_j > 1 \) for \( 1 \leq j \leq n \). Therefore, the only non-zero term corresponds to \( (i_1, \ldots, i_{n+1}) = (1, \ldots, 1, 0) \). \qed

Lemma 8.3. Write \( \gamma \in \Phi^+ \) as \( \gamma = \alpha + \beta \) with \( \alpha \in \Delta \) and \( \beta \in \Phi^+ \). Suppose that \( i\beta - j\alpha \) is in \( \Phi^+ \) for some \( i, j \in \mathbb{Z}_{>0} \).

a) Then \( (i\beta - j\alpha) - \gamma = (i-1)\beta - (j+1)\alpha \) is either a positive root or not in \( \Phi \cup \{0\} \). Therefore: either \( [\mathfrak{g}_{i\beta - j\alpha}, \mathfrak{g}_{-\gamma}] \) is equal to a root space \( \mathfrak{g}_\chi \) with \( \chi = i'\beta - j'\alpha \in \Phi^+ \) for some \( i', j' \in \mathbb{Z}_{>0} \) or \( [\mathfrak{g}_{i\beta - j\alpha}, \mathfrak{g}_{-\gamma}] = 0 \).
b) Moreover, \((i\beta - j\alpha) - \alpha = i\beta - (j + 1)\alpha\) is either a positive root or not in \(\Phi \cup \{0\}\). Therefore: either \([g_{i\beta - j\alpha}, g_{-\alpha}]\) is equal to a root space \(g_\chi\) with \(\chi = i'\beta - j'\alpha \in \Phi^+\) for some \(i', j' \in \mathbb{Z}_{>0}\) or \([g_{i\beta - j\alpha}, g_{-\alpha}] = 0\).

c) Let \(M\) be a \(U(g)\)-module and \(v \in M\) be annihilated by \(u\). Let \(x \in g_\beta\) and \(y \in g_{-\gamma}\). Then, for any sequence of non-negative integers \(i_1, \ldots, i_n\) we have

\[
[x^{(i_1)}, y] \cdot \ldots \cdot [x^{(i_n)}, y]. v = 0
\]

if there is at least one \(i_j > 1\).

Proof. Assertions a) and b) follow from the fact that \(\beta\) is not a multiple of \(\alpha\) (the root system \(\Phi\) is reduced), and \(\beta\) must then contain a simple root \(\alpha' \neq \alpha\), i.e. \(\beta - \alpha' \in \sum_{\tau \in \Delta} \mathbb{Z}_{\geq 0} \tau\).

c) We may clearly assume that \(i_1 > 1\) and that all \(i_j \in \{0, 1\}\) for \(j > 1\). We will show by induction over \(k\) that \([x^{(i_1)}, y] \cdot \ldots \cdot [x^{(i_k)}, y]\) is contained in

\[
\sum_{\substack{i > 0, j > 0 \\
i \beta - j\alpha \in \Phi^+}} U(g) g_{i\beta - j\alpha} [x^{(i_k)}, y] \cdot \ldots \cdot [x^{(i_n)}, y]
\]

The assertion then follows if we take \(k = n + 1\). To begin with, notice that \([x^{(i_1)}, y]\) is in \(g_{i_1\beta - \gamma} = g_{(i_1 - 1)\beta - \alpha}\), and as \(i_1 - 1 > 0\) this space is either zero or a root space \(g_\chi\) with \(\chi \in \Phi^+\). The assertion is hence true for \(k = 2\). Now consider \(z \in g_{i\beta - j\alpha}\) with \(i\beta - j\alpha \in \Phi^+\) and the product \(z[x^{(i_k)}, y]\). Because we assume \(i_k \in \{0, 1\}\) we have \([x^{(i_k)}, y] = y \in g_{-\gamma}\) or \([x^{(i_k)}, y] = [x, y] \in g_{-\alpha}\).

\begin{itemize}
  \item[i)] If \(i_k = 0\) then \([x^{(i_k)}, y] = y \in g_{-\gamma}\) and \(z[x^{(i_k)}, y] = zy = [z, y] + yz\). If \(i\beta - j\alpha - \gamma = (i - 1)\beta - (j + 1)\alpha\) is a positive root, we are done, because then \([z, y]\) is in \(g_{i'\alpha - j'\beta}\) with \(i'\alpha - j'\beta \in \Phi^+\). Otherwise, by a), \(i\beta - j\alpha - \gamma\) is not in \(\Phi \cup \{0\}\), and then \([z, y] = 0\).
  \item[ii)] If \(i_k = 1\) then \([x^{(i_k)}, y] = [x, y] \in g_{-\alpha}\) and \(z[x^{(i_k)}, y] = z[x, y] = [z, [x, y]] + [x, y] z\). If \(i\beta - j\alpha - \alpha = i\beta - (j + 1)\alpha\) is a positive root, we are done, because then \([z, [x, y]]\) is in \(g_{i'\alpha - j'\beta}\) with \(i'\alpha - j'\beta \in \Phi^+\). Otherwise, by b), \(i\beta - j\alpha - \alpha\) is not in \(\Phi \cup \{0\}\), and then \([z, [x, y]] = 0\). \(\Box\)
\end{itemize}

**Proposition 8.4.** Let \(p = p_I\) for some \(I \subset \Delta\). Suppose \(M \in \mathcal{O}^p\) is a highest weight module with weight \(\lambda\) and \(I\) is maximal with respect to \(\lambda\), i.e.
Then no non-zero element of $u_p^-$ acts locally finite on $M$.

Proof. Let $v^+$ be a weight vector with weight $\lambda$. Let $y \in u_p^-$ be a non-zero element which acts locally finite on $M$. Write $y = \sum_{\gamma \in \Phi^+ \setminus \Phi_I} y_\gamma$ with elements $y_\gamma \in g_{-\gamma}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and define on $\mathbb{Z} \alpha_1 \oplus \ldots \oplus \mathbb{Z} \alpha_\ell$ the lexicographic ordering: $\sum_{i=1}^\ell n_i \alpha_i \geq \sum_{i=1}^\ell n'_i \alpha_i$ if and only if there is $k \geq 1$ such that $n_i \geq n'_i$ for $1 \leq i \leq k$ and $n_{k+1} > n'_{k+1}$. Put $B = \{\gamma \in \Phi^+ \setminus \Phi_I \mid y_\gamma \neq 0\}$ (this set is non-empty) and choose $\gamma^+ \in B$ to be maximal among the elements in $B$ (for the lexicographic ordering just defined). Write

$$y^n.v^+ = \sum_{1 \leq i_1, \ldots, i_n \leq s} y_{i_1}^n \cdot \ldots \cdot y_{i_n}^n.v^+.$$ 

where $\Phi^+ \setminus \Phi_I = \{\gamma_1, \ldots, \gamma_s\}$. Using the total ordering just introduced it is easily seen that among the elements $y_{i_1}^n \cdot \ldots \cdot y_{i_n}^n.v^+$ only $y_{\gamma^+}^n.v^+$ can have weight $\lambda - n\gamma^+$. But as all $y^i.v^+$, $i \geq 0$, are contained in a finite-dimensional subspace, we must have $y_{\gamma^+}^n.v^+ = 0$ for some $n \in \mathbb{Z}_{\geq 0}$. It follows from Lemma 8.2 that $y_{\gamma^+}$ acts then locally finite on $M$.

We can therefore assume that $y = y_\gamma \in g_{-\gamma} \setminus \{0\}$ is a root element with $\gamma \in \Phi^+ \setminus \Phi_I$. Write $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ (with non-negative integers $c_\alpha$). We show by induction on the height of $\gamma$, $ht(\gamma) = \sum_{\alpha \in \Delta} c_\alpha$, that $y_\gamma$ can not act locally finite. By Lemma 8.2 this is equivalent to the statement that $y_\gamma^n.v^+ \neq 0$ for all positive integers $n$. If $ht(\gamma) = 1$, then $\gamma$ is an element of $\Delta \setminus I$. Rescaling $y_\gamma$ we can choose $x_\gamma \in g_\gamma$ such that $[x_\gamma, y_\gamma] = h_\gamma$ and $[h_\gamma, x_\gamma] = 2x_\gamma$ and $[h_\gamma, y_\gamma] = -2y_\gamma$. A well-known formula (which is easy to prove by induction) gives

$$x_\gamma^n y_\gamma^n.v^+ = n! \prod_{i=0}^{n-1} (\lambda(h_\gamma) - i).v^+ = n! \prod_{i=0}^{n-1} (\langle \lambda, \gamma^+ \rangle - i).v^+. \quad (8.4.1)$$

As $I$ is supposed to be maximal with respect to $\lambda$, it follows that $\langle \lambda, \gamma^+ \rangle \notin \mathbb{Z}_{\geq 0}$ and the term on the right of (8.4.1) does not vanish (for any $n$).

Now suppose $ht(\gamma) > 1$. Then we can write $\gamma = \alpha + \beta$ with $\alpha \in \Delta$ and $\beta \in \Phi^+$. Clearly, not both $\alpha$ and $\beta$ can be contained in $\Phi_I$. We distinguish two cases.

i) Suppose $\beta - \alpha$ is not in $\Phi$. As $\beta \neq \alpha$ (the root system $\Phi$ is reduced) we have $[g_\alpha, g_{-\beta}] = [g_{-\alpha}, g_\beta] = \{0\}$. Then, if $\alpha \notin I$, we let $x_\beta$ be a non-zero element of $g_\beta$ and have by Lemma 8.2
$$x^n \beta y^n \gamma^+ = n! [x \beta, y \gamma]^n \cdot v^+.$$ 

But as $[x \beta, y \gamma]$ is a non-zero element of $\mathfrak{g}_{-\alpha}$ we can conclude by induction. If, on the other hand, $\alpha \in I$, then $\beta \notin \Phi_I$, and if we let $x_\alpha$ be a non-zero element of $\mathfrak{g}_\alpha$ and have by Lemma 8.2

$$x^n \alpha y^n \gamma^+ = n! [x_\alpha, y \gamma]^n \cdot v^+.$$ 

And as $[x_\alpha, y \gamma]$ is a non-zero element of $\mathfrak{g}_{-\beta}$ we can again conclude by induction.

i) Suppose $\beta - \alpha$ is in $\Phi$. Then it must be in $\Phi^+$, and we have $\gamma - k\alpha \in \Phi^+$ for $0 \leq k \leq k_0$ (with $k_0 \leq 3$), and $\gamma - k\alpha \notin \Phi \cup \{0\}$ for $k > k_0$. This implies $[x^{(i)}_\alpha, y \gamma] = 0$ for $i > k_0$. By Lemma 8.1 we have

$$x^n k_0 \gamma \gamma^+ = \sum_{i_1 + \ldots + i_{n+1} = n k_0} \left( \begin{array}{c} n k_0 \\ i_1 \ldots i_n i_{n+1} \end{array} \right) [x^{(i_1)}_\alpha, y \gamma] \cdot \ldots \cdot [x^{(i_n)}_\alpha, y \gamma] \cdot x^{i_{n+1}} \cdot v^+$$

and as $x$ annihilates $v^+$ this reduces to

$$\sum_{i_1 + \ldots + i_n = n k_0} \left( \begin{array}{c} n k_0 \\ i_1 \ldots i_n \end{array} \right) [x^{(i_1)}_\alpha, y \gamma] \cdot \ldots \cdot [x^{(i_n)}_\alpha, y \gamma] \cdot v^+$$

By what we have just observed, the corresponding term vanishes if there is one $i_j > k_0$. Therefore, only the term with all $i_j = k_0$ contributes, and this sum is hence equal to

$$\left( \begin{array}{c} n k_0 \\ k_0 \ldots k_0 0 \end{array} \right) [x^{(k_0)}_\alpha, y \gamma]^n \cdot v^+ = \frac{(n k_0)!}{(k_0)!^n} [x^{(k_0)}_\alpha, y \gamma]^n \cdot v^+$$

If $\gamma - k_0 \alpha$ is not in $\Phi_I$ we are done, because $[x^{(k_0)}_\alpha, y \gamma]$ is a non-zero element of $\mathfrak{g}_{-(\gamma - k_0 \alpha)}$. Otherwise we necessarily have $\alpha \notin I$. In this case, if we choose some $x_\beta \in \mathfrak{g}_\beta \setminus \{0\}$, we have by Lemma 8.2 and Lemma 8.3

$$x^n \beta y^n \gamma^+ = n! [x_\beta, y \gamma]^n \cdot v^+,$$

and $[x_\beta, y \gamma]$ is a non-zero element of $\mathfrak{g}_{-\alpha}$. And we can thus conclude again. \hfill \Box

**Corollary 8.5.** Let $p = p_I$ for some $I \subset \Delta$. Suppose $M \in \mathcal{O}^p$ is a simple module with weight $\lambda$ and $I$ is maximal with respect to $\lambda$, i.e.

$$I = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \}.$$
Then the action of any non-zero element of \( u_p^- \) on \( M \) is injective.

**Proof.** By Lemma 8.2 a), the set \( N \) of elements \( v \in M \) on which \( \mathfrak{r} \in u_p^- \setminus \{0\} \) acts locally finite is a \( U(\mathfrak{g}) \)-submodule. Clearly \( N \) contains \( \ker(M \xrightarrow{f} M) \). Because \( M \) is assumed to be simple, and as \( \mathfrak{r} \) is not acting locally finite on \( M \) by Proposition 8.4, this submodule must be the zero. \( \square \)

**Corollary 8.6.** Let \( M \in \mathcal{O} \) be a highest weight module. Then the set of elements in \( \mathfrak{g} \) which act locally finite on \( M \) is a standard parabolic Lie subalgebra of \( \mathfrak{g} \). If \( M \) has highest weight \( \lambda \), then this standard parabolic subalgebra is \( p_I \) where

\[
I = \{ \alpha \in \Delta \mid (\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0} \}.
\]

**Proof.** So suppose \( M \) has weight \( \lambda \). Let \( v^+ \) be a weight vector with weight \( \lambda \) and define \( I \) as above. Then \( \lambda \) is in \( \Lambda^+_I \) and \( M \) is in \( \mathcal{O}^p \) with \( p = p_I \), cf. [H1, Theorem in sec. 9.4]. Suppose \( z \in \mathfrak{g} \setminus p \) acts locally finite on \( M \). Then there is \( n \in \mathbb{Z}_{>0} \) and \( c_1, \ldots, c_n \in K \) such that

\[
(8.6.2) \quad z^n.v^+ + c_1z^{n-1}.v^+ + \ldots + c_{n-1}z.v^+ + c_nv^+ = 0.
\]

Write \( z = x + \sum_{\gamma \in \Phi_+ \setminus \Phi_I} y_\gamma \) with elements \( y_\gamma \in \mathfrak{g}_{-\gamma} \). As \( z \notin p \) there is at least one \( \gamma \in \Phi^+_I \setminus \Phi_I \) with \( y_\gamma \neq 0 \). Let \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \) and define on \( \mathbb{Z}_{\alpha_1} \times \ldots \times \mathbb{Z}_{\alpha_\ell} \) the lexicographic ordering:

\[
\sum_{i=1}^\ell n_i \alpha_i \geq \sum_{i=1}^\ell n'_i \alpha_i \text{ if and only if there is } k \geq 1 \text{ such that } n_i \geq n'_i \text{ for } 1 \leq i \leq k \text{ and } n_{k+1} > n'_{k+1}.
\]

Put \( B = \{ \gamma \in \Phi^+_I \setminus \Phi_I \mid y_\gamma \neq 0 \} \) (this set is non-empty) and choose \( \beta \in B \) to be maximal among the elements in \( B \) for the lexicographic ordering just defined. By expanding \( z^n \) as a sum of products of \( x \)'s and \( y_\gamma \)'s it is then easily seen that only \( y_\gamma^n.v^+ \) can have weight \( \lambda - n\beta \). From equation (8.6.2) we deduce that \( y_\gamma^n.v^+ = 0 \). This however contradicts Proposition énotlocnilp . \( \square \)

**8.7. On certain relations in highest weight modules.** Let \( M \) be a highest weight module in the category \( \mathcal{O}^p \), and assume that \( p \) is maximal among all standard parabolic subalgebras \( q \) with the property that \( M \) is an object of \( \mathcal{O}^q \). Denote by \( v^+ \) a vector of highest weight \( \lambda \). In this section we consider relations

\[
y^1_\gamma.v^+ = \sum_{\nu \in \mathcal{I}_n} c_{\nu} y^1_\nu \cdot \ldots \cdot y^r_\nu.v^+
\]
where $\mathcal{I}_n$ consists of all tuples $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}_{\geq 0}^r$ satisfying $\nu_1 \beta_1 + \ldots + \nu_r \beta_r = n\gamma$, and $c_\nu$ are coefficients in $K$. Our aim is to show that there is at least one $\nu \in \mathcal{I}_n$ having both of the following properties:

- i. $\nu_1 + \ldots + \nu_r \geq n$,
- ii. $|c_\nu|_K \geq 1$.

We start by showing the existence of some $\nu \in \mathcal{I}_n$ with the second property.

**Lemma 8.8.** Let $M$ and $p$ be as above (8.7). Suppose the residue characteristic of $K$ does not divide any of the non-zero numbers among $\langle \beta, \alpha \rangle$, $\alpha, \beta \in \Phi, \alpha \neq \beta$. Write $\Phi^+ = \{\beta_1, \ldots, \beta_r\}$, and let $\gamma \in \Phi^+ \setminus \Phi^+_I$. Let $(x_\beta, y_\beta, h_\alpha \mid \beta \in \Phi^+, \alpha \in \Delta)$ be a Chevalley basis of $[g, g]$, as in the proof of Thm. 5.1. Denote by $\mathcal{I}_n$ the set of all $\nu \in \mathbb{Z}_{\geq 0}^r$ such that $\nu_1 \beta_1 + \ldots + \nu_r \beta_r = n\gamma$. Then, for any $n \in \mathbb{Z}_{\geq 0}$ and any expression

$$y_\gamma^n.v^+ = \sum_{\nu \in \mathcal{I}_n} c_\nu y_\nu^\alpha \cdot \ldots \cdot y_\nu^\beta.v^+, \tag{8.8.1}$$

(where $y_i = y_{\beta_i}$, $i = 1, \ldots, r$) there is at least one index $\nu \in \mathcal{I}_n$ such that $|c_\nu|_K \geq 1$.

**Proof.** We start by recalling that for any $i \geq 0$ and $\beta \in \Phi^+$ the endomorphism of $g$:

$$\frac{1}{i!} \{x_\beta^{(i)}, \cdot \} = \frac{1}{i!} \text{ad}(x_\beta)^i$$

preserves the $\mathbb{Z}$-form $g_\mathbb{Z}$ of $g$, cf. [Ch], [Kos]. We put $g_{O_L} = g_\mathbb{Z} \otimes \mathbb{Z} O_L$. Moreover, whenever we have two positive roots $\beta, \beta'$ such that $\beta + \beta'$ is a root, then $[y_{\beta'}^\alpha, y_\beta] = \pm (m + 1)y_{\beta' + \beta}^\alpha$, where $m$ is the greatest integer such that $\beta - m\beta'$ is a root, cf. [Ch].

The proof proceeds by induction over $ht(\gamma)$. Suppose $ht(\gamma) = 1$ and let $\beta_i = \gamma$. Then the set $\mathcal{I}_n$ consists of a single element $\nu$ which is the $r$-tuple that has the entry $n$ in the $i$th place and zeros elsewhere. The right hand side of (8.8.1) is thus $c_\nu y_\gamma^n.v^+$. By Cor. 8.3 the element $y_{\gamma}$ acts injectively on $M$, and we therefore get $c_\nu = 1$.

Now we assume that $ht(\gamma) > 1$. Write $\gamma = \alpha + \beta$ with a simple root $\alpha \in \Delta$ and a positive root $\beta$. Clearly, not both $\alpha$ and $\beta$ can be contained in $\Phi_I$. We distinguish two cases.

i) Suppose $\beta - \alpha$ is not in $\Phi$. As $\beta \neq \alpha$ (the root system $\Phi$ is reduced) we have $[g_{\alpha}, g_{-\beta}] = [g_{-\alpha}, g_\beta] = \{0\}$. Then, if $\alpha \notin I$, we consider $x_\beta$, the element of the Chevalley basis which generates $(g_\mathbb{Z})_{\beta}$. We have by Lemma 8.2
\[ x_{β}^{n} y_{γ}^{n} . v^{+} = n! [x_{β}, y_{γ}]^{n} . v^{+} . \]

We have \( y_{γ} = ± [y_{α}, y_{β}] \), and therefore

\[ ± [x_{β}, y_{γ}] = [x_{β}, [y_{α}, y_{β}]] = − [y_{α}, [y_{β}, x_{β}]] − [y_{β}, [x_{β}, y_{α}]] = − [y_{α}, − h_{β}] = − [h_{β}, y_{α}] = − ⟨α, β⟩ y_{α} . \]

Hence

\[ x_{β}^{n} y_{γ}^{n} . v^{+} = n! (− ⟨α, β⟩)^{n} y_{α}^{n} . v^{+} . \]

As \( [g_{β}, g_{γ}] = g_{−α} \) the element \( [x_{β}, y_{γ}] \) is non-zero. Thus \( ⟨α, β⟩ \neq 0 \). And because the residue characteristic of \( K \) does not divide the non-zero integer \( ⟨α, β⟩ \), this integer is invertible in \( O_{K} \).

On the other hand, we have

\[ x_{β}^{n} z . v^{+} = \text{ad}(x_{β})^{n}(z) . v^{+} = \sum_{ν \in I_{n}} c_{ν} \text{ad}(x_{β})^{n}(y_{1}^{ν_{1}} \cdot \ldots \cdot y_{r}^{ν_{r}}) . v^{+} . \]

But as \( \text{ad}(x_{β})^{n}(y_{1}^{ν_{1}} \cdot \ldots \cdot y_{r}^{ν_{r}}) \) is in \( n! U(g_{O_{K}}) \) we find that \( x_{β}^{n} z . v^{+} \) is of the form

\[ n! \sum_{ν \in I_{n}'} c'_{ν} y_{1}^{τ_{1}} \cdot \ldots \cdot y_{r}^{τ_{r}} . v^{+} , \]

where \( I_{n}' \) consists of all \( τ \in \mathbb{Z}_{≥0} \) such that \( τ_{1} β_{1} + \ldots + τ_{r} β_{r} = nα = n(γ − β) \), and the numbers \( c'_{ν} \) are linear combinations with integral coefficients. We therefore get

\[ y_{α}^{n} . v^{+} = \frac{1}{(− ⟨α, β⟩)^{n}} \sum_{ν \in I_{n}'} c'_{ν} y_{1}^{τ_{1}} \cdot \ldots \cdot y_{r}^{τ_{r}} . v^{+} . \]

The induction hypothesis shows that at least one of the coefficients \( c'_{ν} \) must be of absolute value at least 1, and this implies that at least one of the coefficients \( c_{ν} \) is of absolute value at least 1.

Now we consider the case in which \( α ∈ I \). Then \( β \notin Φ_{I} \). If we let \( x_{α} \) be the member of the Chevalley basis which generates \( g_{α} \), we get by Lemma [8.2]

\[ x_{α}^{n} y_{γ}^{n} . v^{+} = n! [x_{α}, y_{γ}]^{n} . v^{+} . \]

And as \( [x_{α}, y_{γ}] = cy_{β} \) with \( |c|_{K} = 1 \) (cf. the reasoning above), we can again conclude by induction.
ii) Suppose $\beta - \alpha$ is in $\Phi$. Then it must be in $\Phi^+$, and we have $\gamma - k\alpha \in \Phi^+$ for $0 \leq k \leq k_0$ (with $k_0 \leq 3$), and $\gamma - k\alpha \notin \Phi \cup \{0\}$ for $k > k_0$. This implies $[x^{(i)}_{\alpha}, y_{\gamma}] = 0$ for $i > k_0$. By Lemma \ref{lem:8.1} we have

$$x^{nk_0}_{\alpha} y_{\gamma}.v^+ = \sum_{i_1 + \ldots + i_n+1 = nk_0} \binom{nk_0}{i_1 \ldots i_n} [x^{(i_1)}_{\alpha}, y_{\gamma}] \cdots [x^{(i_n)}_{\alpha}, y_{\gamma}] x^{i_{n+1}}.v^+$$

and as $x$ annihilates $v^+$ this reduces to

$$\sum_{i_1 + \ldots + i_n = nk_0} \binom{nk_0}{i_1 \ldots i_n} [x^{(i_1)}_{\alpha}, y_{\gamma}] \cdots [x^{(i_n)}_{\alpha}, y_{\gamma}] .v^+$$

By what we have just observed, the corresponding term vanishes if there is one $i_j > k_0$. Therefore, only the term with all $i_j = k_0$ contributes, and this sum is hence equal to

$$\binom{nk_0}{k_0 \ldots k_0} [x^{(k_0)}_{\alpha}, y_{\gamma}]^n.v^+ = \frac{(nk_0)!}{(k_0)!} [x^{(k_0)}_{\alpha}, y_{\gamma}]^n .v^+$$

Note that $[x^{(k_0)}_{\alpha}, y_{\gamma}] = k_0! \cdot c \cdot y_{\gamma-k_0\alpha}$ with an integer $c$ which is a unit in $O_K$. We conclude that

$$x^{nk_0}_{\alpha} y_{\gamma}.v^+ = (nk_0)! \cdot c^n \cdot y_{\gamma-k_0\alpha}.$$

If $\gamma - k_0\alpha$ is not in $\Phi_I$ we are done. Otherwise we necessarily have $\alpha \notin I$. In this case we have by Lemma \ref{lem:8.2} and Lemma \ref{lem:8.3}

$$x^n_{\beta} y^n_{\gamma}.v^+ = n! [x_{\beta}, y_{\gamma}]^n .v^+, \quad [x_{\beta}, y_{\gamma}] = cy_{\alpha} \text{ with an integer } c \text{ which is a unit in } O_K. \quad \Box$$

**Lemma 8.9.** Let $\Phi$ be a root system which has the property that none of its irreducible components is of type $G_2$. Let $\Phi^+ = \{\beta_1, \ldots, \beta_r\}$ be its positive roots (with respect to some fixed chosen basis of simple roots), and let $\gamma \in \Phi^+$. Consider a relation

\begin{equation}
(8.9.1) \quad n\gamma = \nu_1 \beta_1 + \ldots + \nu_r \beta_r
\end{equation}

with non-negative integers $n, \nu_1, \ldots, \nu_r \in \mathbb{Z}_{\geq 0}$. Then $n \leq \nu_1 + \ldots + \nu_r$.\footnote{This must be checked by an explicit calculation}
Proof. Of course, we may assume that \( \Phi \) is irreducible and that \( n \) is positive (there is nothing to show when \( n = 0 \)). It is known that for irreducible reduced roots systems other than \( G_2 \) the square of the ratio between the lengths of any two roots \( \beta, \gamma \) is among \( \{ \frac{1}{2}, 1, 2 \} \). Root systems of type \( A, D, E \) (so-called simply laced root systems) have the property that its roots are all of equal length, whereas for root systems of type \( B, C, \) and \( F_4 \) the ratio can also be \( \frac{1}{2} \) or 2. The relations

\[
\langle \beta, \gamma \rangle = \frac{2\langle \beta, \gamma \rangle}{\langle \gamma, \gamma \rangle} = 2 \frac{\| \beta \|}{\| \gamma \|} \cos(\theta) \quad \text{and} \quad \langle \beta, \gamma \rangle \langle \gamma, \beta \rangle = 4 \cos(\theta)^2
\]

show that if \( \frac{\| \beta \|^2}{\| \gamma \|^2} \in \{ \frac{1}{2}, 1, 2 \} \) then \( \langle \beta, \gamma \rangle \leq 2 \), i.e., \( \langle \beta, \gamma \rangle \leq \langle \gamma, \gamma \rangle \), cf. [H2, 9.4]. Taking the scalar product of both sides of (8.9.1) with \( \gamma \) and dividing by \( \langle \gamma, \gamma \rangle \) we get

\[
n = \nu_1 \frac{\langle \beta_1, \gamma \rangle}{\langle \gamma, \gamma \rangle} + \ldots + \nu_r \frac{\langle \beta_r, \gamma \rangle}{\langle \gamma, \gamma \rangle} \leq \nu_1 + \ldots + \nu_r.
\]

And this is what we asserted. \( \square \)

Now we generalize the preceding lemma so as to assure the existence of some \( \nu \in I_n \) satisfying both conditions \( i. \) and \( ii. \) of 8.7.

**Proposition 8.10.** Let \( M \) and \( p \) be as above (8.7). Suppose the residue characteristic of \( K \) does not divide any of the non-zero numbers among \( \langle \beta, \alpha^\vee \rangle, \alpha, \beta \in \Phi, \alpha \neq \beta \). Write \( \Phi^+ = \{ \beta_1, \ldots, \beta_r \} \), and let \( \gamma \in \Phi^+ \setminus \Phi^+_f \). Let \( (x_\beta, y_\beta, h_\alpha | \beta, \alpha \in \Phi^+, \alpha \in \Delta) \) be a Chevalley basis of \([g, g] \), as in the proof of Thm. 5.4. Denote by \( I_n \) the set of all \( \nu \in \mathbb{Z}_{\geq 0}^r \) such that

\[
n = \nu_1 \beta_1 + \ldots + \nu_r \beta_r = n \gamma.
\]

Then, for any \( n \in \mathbb{Z}_{\geq 0} \) and any expression

\[
y_n \gamma, v^+ = \sum_{\nu \in I_n} c_{\nu} y_{\nu_1} \beta_1^{\nu_1} \cdots y_{\nu_r} \beta_r^{\nu_r} v^+,
\]

(8.10.1)

(where \( y_i = y_{\beta_i}, i = 1, \ldots, r \) there is at least one index \( \nu \in I_n \) such that \( \nu_1 + \ldots + \nu_r \geq n \) and \( |c_{\nu}|_K \geq 1 \).

Proof. The basic idea of the proof is as follows. There is nothing to show if \( ht(\gamma) = 1 \). Now suppose that \( ht(\gamma) > 1 \). Then there is \( \gamma' \in \Phi^+ \) and \( k_0 \in \mathbb{Z}_{>0} \) such that \( \gamma - k_0 \gamma' \in \Phi^+ \setminus \Phi^+_f \) and

\[
x_{\gamma'}^{nk_0} \cdot y_{\gamma}^n v^+ = n! [x_{\gamma'}^{(k_0)}, y_{\gamma}]^n v^+ = n! \cdot c \cdot (y_{\gamma - k_0 \gamma'})^n v^+,
\]

with \( c \in O_K^* \). Considering the right hand side of (8.10.1), we aim to show that for any \( \nu \in I_n \) with \( \nu_1 + \ldots + \nu_r < n \) the term
(8.10.2) \[ x_{\gamma}^{nk_0} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_r^{\nu_r} \cdot v^+ \]

in

(8.10.3) \[ x_{\gamma'}^{nk_0} \cdot \left( \sum_{\nu \in I_n} c_{\nu} y_1^{\nu_1} \cdot \ldots \cdot y_r^{\nu_r} \cdot v^+ \right) = \sum_{\nu \in I_n} c_{\nu} \cdot x_{\gamma'}^{nk_0} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_r^{\nu_r} \cdot v^+ \]

vanishes. This means that the sum on the right hand side of the equation above (8.10.3) is equal to

(8.10.4) \[ \sum_{\nu \in J_n} c_{\nu} \cdot x_{\gamma'}^{nk_0} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_r^{\nu_r} \cdot v^+ , \]

where \( J_n \subset I_n \) consists only of those \( \nu \in I_n \) for which \( \nu_1 + \ldots + \nu_r \geq n \). We can then rewrite (8.10.4) as

\[ \sum_{\tau \in I'_n} c'_{\tau} \cdot y_1^{\tau_1} \cdot \ldots \cdot y_r^{\tau_r} \cdot v^+ , \]

where \( I'_n \) consists of all \( \tau \in \mathbb{Z}_{\geq 0}^{r} \) such that \( \tau_1 \beta_1 + \ldots + \tau_r \beta_r = n(\gamma - k_0 \gamma') \), and the numbers \( c'_{\tau} \) are linear combinations with integral coefficients of the numbers \( c_{\nu} \), with \( \nu \in J_n \). Applying Lemma 8.8, we find that there is \( \tau \in I'_n \) such that \( |c'_{\tau}|_K \geq 1 \). Hence there is at least one \( \nu \in J_n \) such that \( |c_{\nu}|_K \geq 1 \). Thus proving our assertion.

We need to show that the term (8.10.2) actually vanishes if \( \nu_1 + \ldots + \nu_r < n \).

Suppose \( \Phi \) is \( G_2 \). Let \( \Delta = \{ \alpha, \beta \} \) with \( \alpha \) being the short and \( \beta \) being the long root. We let

\[ \beta_1 = \alpha, \quad \beta_2 = \beta, \quad \beta_3 = \alpha + \beta, \quad \beta_4 = 2\alpha + \beta, \quad \beta_5 = 3\alpha + \beta, \quad \beta_6 = 3\alpha + 2\beta. \]

Consider the equation \( n\gamma = \nu_1 \beta_1 + \ldots + \nu_6 \beta_6 \), i.e.,

(8.10.5) \[ n\gamma = \nu_1 \alpha + \nu_2 \beta + \nu_3 (\alpha + \beta) + \nu_4 (2\alpha + \beta) + \nu_5 (3\alpha + \beta) + \nu_6 (3\alpha + 2\beta) \]

We consider all possible cases for \( \gamma \) and \( I \).
Case when $\gamma = \alpha$ or $\gamma = \beta$. There is nothing to show in this case, as we can write a multiple of a simple root in one and only one way as a sum of positive roots.

Case when $\gamma = \beta_5$ or $\gamma = \beta_6$. Comparing the coefficients of $\alpha$, the equation (8.10.5) implies in this case

$$3n = \nu_1 + \nu_3 + 2\nu_4 + 3\nu_5 + 3\nu_6.$$ 

On the other hand, assuming $\nu_1 + \ldots + \nu_6 < n$ we get that $3\nu_1 + \ldots + 3\nu_6 < \nu_1 + \nu_3 + 2\nu_4 + 3\nu_5 + 3\nu_6$ which implies $2\nu_1 + 3\nu_2 + 2\nu_3 + \nu_4 < 0$ which is impossible. It remains to discuss the cases when $\gamma = \alpha + \beta$ or $\gamma = 2\alpha + \beta$.

Case when $\gamma = \alpha + \beta$. In this case equation (8.10.5) implies

$$n = \nu_1 + \nu_3 + 2\nu_4 + 3\nu_5 + 3\nu_6 \quad \text{and} \quad n = \nu_2 + \nu_3 + \nu_4 + \nu_5 + 2\nu_6.$$ 

Subtracting the equation on the right from the equation on the left and adding $\nu_2$ on both sides gives $\nu_2 = \nu_1 + \nu_4 + 2\nu_5 + \nu_6$.

a) Suppose $I = \{\alpha\}$. Then we let $\gamma' = \alpha$. Consider

$$x^n y_1^{\nu_1} \cdot \ldots \cdot y_6^{\nu_6} v^+ = \text{ad}(x_\alpha)^n (y_1^{\nu_1} \cdot \ldots \cdot y_6^{\nu_6}) v^+ = \sum_{i_1 + \ldots + i_6 = n} \binom{n}{i_1 \ldots i_6} \text{ad}(x_\alpha)^{i_1} (y_1^{\nu_1}) \cdot \ldots \cdot \text{ad}(x_\alpha)^{i_6} (y_6^{\nu_6}) v^+$$

Moreover,

(8.10.6) \[ \text{ad}(x_\alpha)^{i_j} (y_j^{\nu_j}) = \sum_{k_1 + \ldots + k_{\nu_j} = i_j} \binom{i_j}{k_1 \ldots k_{\nu_j}} \text{ad}(x_\alpha)^{k_1} (y_j^{\nu_j}) \cdot \ldots \cdot \text{ad}(x_\alpha)^{k_{\nu_j}} (y_j^{\nu_j}) . \]

Therefore, if $\beta_j - \alpha$ is not in $\Phi \cup \{0\}$, the term (8.10.6) vanishes if $i_j > 0$. Hence $i_2 = i_6 = 0$.

Moreover, if $i_j > \nu_j$, then there must be at least one $k_i \geq 2$ in the formula (8.10.6). However, if $\beta_j - 2\alpha$ is not in $\Phi \cup \{0\}$, the term (8.10.6) vanishes then. So we get $i_3 \leq \nu_3$. Similarly we see that $i_1 \leq 2\nu_1$. Now suppose $n > \nu_1 + \ldots + \nu_6$ and consider the inequality

$$n = i_1 + \ldots + i_6 > \nu_1 + \ldots + \nu_6 = 2\nu_1 + \nu_3 + 2\nu_4 + 3\nu_5 + 2\nu_6 .$$ 

Here we have used that $\nu_2 = \nu_1 + \nu_4 + 2\nu_5 + \nu_6$. As $i_2 = i_6 = 0$, $i_3 \leq \nu_3$ and $i_1 \leq 2\nu_1$ we get
\[ i_4 + i_5 > (2 \nu_1 - i_1) + (3 \nu_3 - i_3) + 2 \nu_4 + 3 \nu_5 + 2 \nu_6 \geq 2 \nu_4 + 3 \nu_5. \]

This shows that either \( i_4 > 2 \nu_4 \) or \( i_5 > 3 \nu_5 \). But in each of these cases the corresponding term \( \text{ad}(x_\alpha)^4_i(y_4^{\nu_4}) \) or \( \text{ad}(x_\alpha)^5_i(y_5^{\nu_5}) \) vanishes. This proves our assertion in this case.

b) Suppose \( I = \{ \beta \} \). Then we let \( \gamma' = \beta \). Consider

\[
x^n_\beta \cdot y_1^{\nu_1} \cdot \ldots \cdot y_6^{\nu_6} v^+ = \text{ad}(x_\beta)^n(y_1^{\nu_1} \cdot \ldots \cdot y_6^{\nu_6}) v^+
\]

Moreover,

\[
(8.10.7) \quad \text{ad}(x_\beta)^{ij}(y_j^{\nu_j}) = \sum_{k_1 + \ldots + k_{\nu_j} = i_j} \left( \begin{array}{c} i_j \\ k_1 \ldots k_{\nu_j} \end{array} \right) \text{ad}(x_\beta)^{k_1}(y_j) \cdot \ldots \cdot \text{ad}(x_\beta)^{k_{\nu_j}}(y_j).
\]

Therefore, if \( \beta_j - \beta \) is not in \( \Phi \cup \{0\} \), the term \( (8.10.7) \) vanishes if \( i_j > 0 \). Hence \( i_1 = i_4 = i_5 = 0 \). Moreover, if \( i_j > 0 \), then there must be at least one \( k_l \geq 2 \) in the formula \( (8.10.7) \). However, if \( \beta_j - 2 \beta \) is not in \( \Phi \cup \{0\} \), the term \( (8.10.7) \) vanishes then. So we get \( i_3 \leq \nu_3 \) and \( i_6 \leq \nu_6 \).

Similarly we see that \( i_2 \leq 2 \nu_2 \). Assuming \( n > \nu_1 + \ldots + \nu_6 \) we get

\[ n = i_1 + \ldots + i_6 > \nu_1 + \ldots + \nu_6, \]

and hence

\[ i_2 > \nu_1 + \nu_2 + (\nu_3 - i_3) + \nu_4 + \nu_5 + (\nu_6 - i_6) \geq \nu_2. \]

The left hand side of \( (8.10.7) \), for \( j = 2 \), is of weight \( (i_2 - \nu_2)\beta \), and \( \text{ad}(x_\beta)^{i_2}(y_2^{\nu_2}) v^+ \) is therefore 0 or of weight \( \lambda + (i_2 - \nu_2)\beta \), which shows that it must vanish. Assuming, as we may, that we had ordered the positive roots such that \( \beta \) comes last (i.e., \( \beta = \beta_6 \) instead of \( \beta = \beta_2 \)), we see that \( x^n_\beta \cdot y_1^{\nu_1} \cdot \ldots \cdot y_6^{\nu_6} v^+ \) vanishes.

**Case when \( \gamma = 2 \alpha + \beta \).** In this case equation \( (8.10.5) \) implies

\[
(8.10.8) \quad 2n = \nu_1 + 3 \nu_3 + 2 \nu_4 + 3 \nu_5 + 3 \nu_6
\]

\[
(8.10.9) \quad n = \nu_2 + 3 \nu_3 + \nu_4 + \nu_5 + 2 \nu_6
\]
We assume that \( n > \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6 \). By (8.10.9), this implies
\[
\nu_2 + \nu_3 + \nu_4 + \nu_5 + 2\nu_6 > \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6,
\]
and therefore \( \nu_6 > \nu_1 \). In particular,
\[
(8.10.10) \quad \nu_6 > 0.
\]

Subtracting (8.10.9) from (8.10.8) we get
\[
n = \nu_1 - \nu_2 + \nu_4 + 2\nu_5 + \nu_6,
\]
and using again our assumption that \( n > \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6 \) we find
\[
\nu_1 - \nu_2 + \nu_4 + 2\nu_5 + \nu_6 > \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6,
\]
or, equivalently, \( \nu_5 > 2\nu_2 + \nu_3 \). In particular
\[
(8.10.11) \quad \nu_5 > 0.
\]

a) Suppose \( I = \{\alpha\} \). Then we let \( \gamma' = \alpha \). We arrange the positive roots in this order: \( \beta_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \), and write elements of \( U(u^-_b) \) as sums of monomials of the form \( y_6^{\nu_6} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_5^{\nu_5} \). Consider
\[
(8.10.12) \quad x_\alpha^{2n} \cdot y_6^{\nu_6} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_5^{\nu_5} \cdot v^+
\]
\[
= \text{ad}(x_\alpha)^{2n}(y_6^{\nu_6} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_5^{\nu_5}) \cdot v^+
\]
\[
= \sum_{i_1 + \ldots + i_6 = 2n} \binom{2n}{i_1 \ldots i_6} \text{ad}(x_\alpha)^{i_6}(y_6^{\nu_6}) \cdot \text{ad}(x_\alpha)^{i_1}(y_1^{\nu_1}) \cdot \ldots \cdot \text{ad}(x_\alpha)^{i_5}(y_5^{\nu_5}) \cdot v^+
\]

Moreover,
\[
(8.10.13) \quad \text{ad}(x_\alpha)^{ij}(y_j^{\nu_j}) = \sum_{k_1 + \ldots + k_{\nu_j} = i_j} \binom{i_j}{k_1 \ldots k_{\nu_j}} \text{ad}(x_\alpha)^{k_1}(y_j) \cdot \ldots \cdot \text{ad}(x_\alpha)^{k_{\nu_j}}(y_j) .
\]
Since \( \text{ad}(x_\alpha)(y_6) = 0 \), it follows from (8.10.13) that we only need to consider tuples \((i_1, \ldots, i_6)\) for which \(i_6 = 0\). For those tuples we have \(2n = i_1 + i_2 + i_3 + i_4 + i_5\). On the other hand, (8.10.8) implies

\[
(8.10.14) \quad \nu_1 + \nu_3 + 2\nu_4 + 3\nu_5 = 2n - 3\nu_6 < 2n,
\]

because \(\nu_6 > 0\) (cf. (8.10.10)). Now consider the term

\[
\text{ad}(x_\alpha)^{i_1}(y_1^{\nu_1}) \cdot \ldots \cdot \text{ad}(x_\alpha)^{i_5}(y_5^{\nu_5}).v^+
\]

in the third line of (8.10.12). It has weight

\[
(i_1 + i_2 + i_3 + i_4 + i_5)\alpha - \nu_1\beta_1 - \nu_2\beta_2 - \nu_3\beta_3 - \nu_4\beta_4 - \nu_5\beta_5 + \lambda
\]

\[
= (2n - \nu_1 + \nu_3 + 2\nu_4 + 3\nu_5)\alpha + (2n - \nu_2 + \nu_3 + \nu_4 + \nu_5)\beta + \lambda
\]

By (8.10.14), this weight is not of the form \(\lambda - \text{(sum of positive roots)}\), and must therefore vanish. Hence

\[
x_{\alpha}^{2n} \cdot y_6^{\nu_6} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_5^{\nu_5}.v^+ = 0
\]

for all \(\nu\) with \(n > \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6\).

b) Suppose \(I = \{\beta\}\). Then we let \(\gamma' = \alpha + \beta\). In this case we order the positive roots as follows: \(\beta_5, \beta_1, \beta_2, \beta_3, \beta_4, \beta_6\), and write elements of \(U(u^-)\) as sums of monomials of the form

\[
y_5^{\nu_5} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_4^{\nu_4} \cdot y_6^{\nu_6}.
\]

Consider

\[
(8.10.15) \quad x_{\alpha+\beta}^{n} \cdot y_5^{\nu_5} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_4^{\nu_4} \cdot y_6^{\nu_6}.v^+
\]

\[
= \text{ad}(x_{\alpha+\beta})^n(y_5^{\nu_5} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_4^{\nu_4} \cdot y_6^{\nu_6}).v^+
\]

\[
= \sum_{i_1 + \ldots + i_6 = n} (i_1 \ldots i_6) \text{ad}(x_{\alpha+\beta})^{i_1}(y_5^{\nu_5}) \times
\]

\[
\times \text{ad}(x_{\alpha+\beta})^{i_2}(y_1^{\nu_1}) \cdot \ldots \cdot \text{ad}(x_{\alpha+\beta})^{i_4}(y_4^{\nu_4}) \cdot \text{ad}(x_{\alpha+\beta})^{i_6}(y_6^{\nu_6}).v^+
\]

Moreover,
\[(8.10.16)\] \(\text{ad}(x_{\alpha+\beta})^{i_j}(y_{\nu_j}) = \sum_{k_1 + \ldots + k_{\nu_j} = i_j} \left( {i_j \atop k_1 \ldots k_{\nu_j}} \right) \text{ad}(x_{\alpha+\beta})^{k_1}(y_j) \cdot \ldots \cdot \text{ad}(x_{\alpha+\beta})^{k_{\nu_j}}(y_j).\)

Since \(\text{ad}(x_{\alpha+\beta})(y_5) = 0\), it follows from (8.10.16) that we only need to consider tuples \((i_1, \ldots, i_6)\) for which \(i_5 = 0\). For those tuples we have \(n = i_1 + i_2 + i_3 + i_4 + i_6\). On the other hand, (8.10.9) implies

\[(8.10.17)\] \(\nu_2 + \nu_3 + \nu_4 + 2\nu_6 = n - \nu_5 < n,\)

because \(\nu_5 > 0\) (cf. (8.10.11)). Now consider the term

\(\text{ad}(x_\alpha)^{i_1}(y_1^{\nu_1}) \cdot \ldots \cdot \text{ad}(x_\alpha)^{i_4}(y_4^{\nu_4}) \cdot \text{ad}(x_\alpha)^{i_6}(y_6^{\nu_6}).v^+\)

in the last line of (8.10.15). It has weight

\[(i_1 + i_2 + i_3 + i_4 + i_6)(\alpha + \beta) - \nu_1\beta_1 - \nu_2\beta_2 - \nu_3\beta_3 - \nu_4\beta_4 - \nu_6\beta_6 + \lambda\]

\[= (n - \nu_1 + \nu_3 + 2\nu_4 + 3\nu_5)\alpha + (n - \nu_2 + \nu_3 + \nu_4 + 2\nu_6)\beta + \lambda\]

By (8.10.17), this weight is not of the form \(\lambda - \text{sum of positive roots}\), and must therefore vanish. Hence

\[x_n^{\alpha+\beta} \cdot y_5^{\nu_5} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_4^{\nu_4} \cdot y_6^{\nu_6}.v^+ = 0\]

for all \(\nu\) with \(n > \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6\). This completes the proof. \(\square\)

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