An approach for the calculation of one-loop effective actions, vacuum energies, and spectral counting functions

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ABSTRACT: In this paper, we provide an approach for the calculation of one-loop effective actions, vacuum energies, and spectral counting functions and discuss the application of this approach in some physical problems. Concretely, we construct the equations for these three quantities; this allows us to achieve them by directly solving equations. In order to construct the equations, we introduce shifted local one-loop effective actions, shifted local vacuum energies, and local spectral counting functions. We solve the equations of one-loop effective actions, vacuum energies, and spectral counting functions for free massive scalar fields in \( \mathbb{R}^n \), scalar fields in three-dimensional hyperbolic space \( H_3 \) (the Euclidean Anti-de Sitter space \( AdS_3 \)), in \( H_3/Z \) (the geometry of the Euclidean BTZ black hole), and in \( S^1 \), and the Higgs model in a \((1+1)\)-dimensional finite interval. Moreover, in the above cases, we also calculate the spectra from the counting functions. Besides exact solutions, we give a general discussion on approximate solutions and construct the general series expansion for one-loop effective actions, vacuum energies, and spectral counting functions. In doing this, we encounter divergences. In order to remove the divergences, renormalization procedures are used. In this approach, these three physical quantities are regarded as spectral functions in the spectral problem.

KEYWORDS: Thermal Field Theory, Black Holes.
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1. Introduction

The main aim of this paper is to provide an approach for calculating one-loop effective actions, vacuum energies, and spectral counting functions by constructing their equations. The effective action plays an important role in quantum field theory [1], which contains all the information of quantized fields. The vacuum energy comes from the quantum fluctuation arising from the uncertainty principle, which can be observed in, e.g., the Casimir effect, and has consequences for the behavior of the universe on cosmological scales [2, 3, 4]. The spectral counting function describes the number of the eigenstates whose eigenvalues are smaller than a given number, which is the core issue in the problem formulated by Kac as "Can one hear the shape of a drum?" [5].

The regularized one-loop effective action \( W_s \), the regularized vacuum energy \( E_0 (\epsilon) \), and the spectral counting function \( N (\lambda) \) are global functions, i.e., they are not functions of space coordinates. In practice, the global functions are very difficult to calculate. An effective method for calculating the global functions is to first calculate the corresponding local functions and then to achieve the global ones from the local functions. The reason why it is relatively easy to obtain the local function is that the local function has its own equation and can be obtained by solving the equation. A typical example is the heat kernel: the local heat kernel \( K (t; x, y) \) can be obtained by solving the heat equation, and the global heat kernel \( K (t) \) can be obtained by taking trace of \( K (t; x, y) \).

Concretely, we first introduce the shifted local versions of these three global quantities: corresponding to \( W_s, E_0 (\epsilon), \) and \( N (\lambda) \), we introduce the shifted local one-loop effective action \( W (s; q; x, y) \), the shifted local vacuum energy \( E_0 (\epsilon; q; x, y) \), and the shifted local spectral counting function \( N (\lambda; q; x, y) \), respectively. Then we construct the equations for these local ones; the global ones, \( W_s, E_0 (\epsilon), \) and \( N (\lambda) \), can be obtained by taking trace of the corresponding shifted local ones with \( q = 0 \). As a bridge, we construct the equation for the local Hurwitz zeta function \( \zeta (s; q; x, y) \) at first; \( \zeta (s; x, y) = \zeta (s; 0; x, y) \) is the known local zeta function [1, 2, 3, 4].

Some exact solutions of the shifted local one-loop effective action \( W (s; q; x, y) \), the shifted local vacuum energy \( E_0 (\epsilon; q; x, y) \), and the local spectral counting function \( N (\lambda; x, y) \) are solved from their equations in this paper. Such as a free massive scalar field in \( \mathbb{R}^n \), scalar fields in three-dimensional hyperbolic space \( H_3 \) (the Euclidean Anti-de Sitter space...
AdS$_3$) and in $H_3/Z$ (the geometry of the Euclidean BTZ black hole), a scalar field in $S^1$, and the Higgs model in a $(1 + 1)$-dimensional finite interval with the Dirichlet boundary condition. Based on the solved local results, we, then, by taking trace, achieve the global ones, $W$, $E_0$, and $N(\lambda)$. In order to obtain finite results, renormalization procedures are used for removing the divergences. Moreover, starting from a counting function, we calculate the eigenvalue spectrum of the operator $D$ using the approach given in Ref. [9].

Besides exact solutions, we give a general discussion on series solutions, which is a starting point for seeking approximate solutions. In order to achieve an approximate solution, the first thing is to construct a proper series expansion. In this paper, we construct the series expansions for shifted local one-loop effective action $W(s;q;x,y)$, shifted local vacuum energy $E_0(\epsilon;q;x,y)$, and local spectral counting function $N(\lambda;x,y)$. Concretely, we first construct the series expansion for the solution of general second-order differential operators of Laplace type with local boundary conditions; such a case is often related to the interaction case, such as gauge interactions, in physical problems. Then, we construct the general form of the series expansions for $W(\epsilon;q;x,y)$, $E_0(\epsilon;q;x,y)$, and $N(\lambda;x,y)$. In finding the series solutions, divergences are encountered and removed by renormalization procedures.

From a mathematical viewpoint, for an operator $D$ on a manifold $M$, the character of $D$ and the geometry of $M$ are embodied in the spectrum $\{\lambda_n\}$ determined by the eigenequation

$$D\phi_n = \lambda_n \phi_n.$$  \hfill (1.1)

In principle, one can extract the information of $D$ and $M$ from the spectrum $\{\lambda_n\}$. In modern researches, the study of spectrum is often not through studying the eigenequation, but turns to the study of the corresponding heat-type equation,

$$\partial_t \phi + D\phi = 0,$$  \hfill (1.2)

the wave-type equation,

$$\partial^2_t \phi + D\phi = 0,$$  \hfill (1.3)

the Schrödinger-type equation,

$$i\partial_t \phi - D\phi = 0$$  \hfill (1.4)

and, in principle, other equations in more general forms, by introducing auxiliary variables (in the above mentioned three cases, the auxiliary variable is the time $t$). For a given spectrum $\{\lambda_n\}$, different equations define different spectral functions, e.g., the spectral function for heat-type equations is the fundamental solution

$$K(t;x,y) = \sum_n e^{-\lambda_n t} \phi_n(x) \phi_n^*(y)$$  \hfill (1.5)

(the heat kernel), for wave-type equations is the fundamental solution

$$\omega(t;x,y) = \sum_n e^{-i\sqrt{\lambda_n} t} \phi_n(x) \phi_n^*(y),$$  \hfill (1.6)
and for Schrödinger-type equations is the fundamental solution

\[ h(t; x, y) = \sum_n e^{-i\lambda_n t} \phi_n(x) \phi_n^*(y). \] (1.7)

For the spectrum \( \{\lambda_n\} \), we can in principle define other spectral functions which also embody the information of both \( D \) and \( M \). Effective actions, vacuum energies, and spectral counting functions are all defined by the spectrum \( \{\lambda_n\} \) and embody the information of the operator \( D \) and the manifold \( M \), so they can serve as spectral functions. That is to say, the local functions \( N(\lambda; q; x, y) \), \( W(s; q; x, y) \), \( E_0(\epsilon; q; x, y) \), and \( \zeta(s; q; x, y) \) are all spectral functions for a spectral problem. These spectral functions are physical meaningful and allow us to investigate the geometry of a manifold through physical measures.

Many researches have been devoted to the study of the one-loop effective action [10, 11, 12, 13, 14]. The zeta function has many applications in spectrum problems [15]. Some effective methods for calculating vacuum energies with the help of heat kernels have been developed [16, 17, 18, 19, 20, 21, 22, 23, 24]. There are many studies on the heat kernel [25, 26, 27, 28] and on its applications [29, 30, 31, 32]. For spectral counting functions, in mathematics, the study sets off researches into spectral theory, with the idea of recovering geometry of a manifold from the knowledge of the eigenvalues of a differential operator [33]. In physics, for example, one may seek to reconstruct the shape of the universe from the eigenproblem [34]. There are also experimental studies on spectral counting functions [35].

In section 2, we construct the equations. In sections 3, 4, and 5, we first solve one-loop effective actions, vacuum energies, and spectral counting functions for free massive scalar fields. Then we discuss the proper series expansion for these three quantities. In sections 6, 7, and 8 we solve the local and global one-loop effective actions, vacuum energies, and counting functions for scalar fields in \( H_3 \), \( H_3/Z \), and \( S^1 \), and the Higgs model in a (1+1)-dimensional finite interval, respectively. A discussion of spectra in such cases is also given in these sections based on the result of the counting functions. The conclusions are summarized in section 9.

2. Hurwitz zeta functions, one-loop effective actions, vacuum energies, spectral counting functions, and heat kernels

In this section, we construct the equations for one-loop effective actions, vacuum energies, and spectral counting functions. For this purpose, we introduce a shifted local regularized one-loop effective action \( W(s; q; x, y) \), a shifted local regularized vacuum energy \( E_0(\epsilon; q; x, y) \), and a shifted local spectral counting function \( N(\lambda; q; x, y) \), and generalize the local heat kernel \( K(t; x, y) \) to a shifted local heat kernel \( K(t; q; x, y) \). The corresponding unshifted global ones, \( W_s \), \( E_0(\epsilon) \), \( N(\lambda) \), and \( K(t) \), can be obtained from these local ones by taking trace and setting \( q = 0 \).

To introduce these local functions, we start with the corresponding operators, the zeta operator, the shifted counting operator, and the shifted heat kernel operator; the shifted regularized one-loop effective action operator and the shifted regularized vacuum energy...
operator can be directly achieved from the zeta operator. The local functions are defined as the matrix elements of these operators. The relations among these functions can be immediately obtained from the definitions of the operators.

As a bridge, we first construct an equation for the local Hurwitz zeta function, \( \zeta (s; q; x, y) \) (The local zeta function also plays an important role in many problems \([1, 2, 3]\)). Then, based on the relations among \( W (s; q; x, y) \), \( E_0 (\epsilon; q; x, y) \), \( N (\lambda; q; x, y) \), and \( \zeta (s; q; x, y) \), we construct equations for the other three quantities.

Moreover, from the mathematical point of view, these local functions can be regarded as spectral functions of a spectral problem.

### 2.1 Definitions

For an operator \( D \) with spectrum \( \{\lambda_n\} \), the global heat kernel \( K (t) = \sum_n e^{-\lambda_n t} \), the zeta function \( \zeta (s) = \sum_n \lambda_n^{-s} \), and the spectral counting function \( N (\lambda) = \sum_n \theta (\lambda - \lambda_n) \), where \( \theta (x) \) denotes the step function, etc., can be viewed as various spectral functions of the spectral problem of \( D \). To construct the equations, we need the corresponding local functions. We start with the corresponding operators. For an operator \( D \), the zeta operator is defined as \( \zeta = (D + q)^{-s} \). Then the global Hurwitz zeta function is \( \zeta (s; q) = tr \zeta = \sum_n (\lambda_n + q)^{-s} \).

The one-loop effective action operator reads \( W = \ln \sqrt{D} \), and the one-loop effective action is \( W = tr W = \frac{1}{2} \ln \det D = \sum_n \ln \sqrt{\lambda_n} \). From the zeta operator, we can define a regularized one-loop effective action operator: \( W_s = -\frac{i}{2} \mu^2 \Gamma (s) \zeta \), where \( \mu \) is a constant. The global regularized one-loop effective action is \( W_s = tr W_s \big|_{q=0} = -\frac{i}{2} \mu^2 \Gamma (s) \sum_n \lambda_n^{-s} \). From the zeta operator, we can also define a shifted regularized vacuum energy operator: \( E_0 = 2 \mu^2 (D + q)^{1/2-\epsilon} = 2 \zeta |_{s=-1/2+\epsilon} \). Its trace with \( q = 0 \) gives the regularized vacuum energy: \( E_0 (\epsilon) = tr E_0 |_{q=0} = \frac{1}{2} \mu^2 \zeta (-1/2 + \epsilon) |_{q=0} \); a renormalized vacuum energy \( E_0 \) can be directly obtained from \( E_0 (\epsilon) \).

The shifted counting operator is defined as \( N = \theta (\lambda - (D + q)) \). The spectral counting function reads \( N (\lambda) = tr N |_{q=0} = \sum_n \theta (\lambda - \lambda_n) \). We also introduce a shifted heat kernel operator, \( K = e^{-(D+q)t} \). The trace of \( K \) gives the shifted global heat kernel: \( K (t; q) = tr K = \sum_n e^{-(\lambda_n+q)t} \).

The local functions are defined as the matrix elements of the operators. The local Hurwitz zeta function is the matrix element of the zeta operator,

\[
\zeta (s; q; x, y) = \langle x \mid \zeta \mid y \rangle = \sum_n (\lambda_n + q)^{-s} \phi_n (x) \phi_n^* (y) .
\]  

(2.1)

\( \zeta (s; x, y) = \zeta (s; 0; x, y) \) is just the known local zeta function \([1, 2, 3]\). The shifted local regularized one-loop effective action is the matrix element of the regularized one-loop effective action operator,

\[
W (s; q; x, y) = \langle x \mid W_s \mid y \rangle = -\frac{1}{2} \mu^2 \Gamma (s) \zeta (s; q; x, y) .
\]  

(2.2)

The shifted local vacuum energy is the matrix element of the shifted regularized vacuum energy operator,

\[
E_0 (\epsilon; q; x, y) = \frac{1}{2} \langle x \mid E_0 \mid y \rangle = \frac{\mu^{2\epsilon}}{2} \zeta \left( -\frac{1}{2} + \epsilon; q; x, y \right) .
\]  

(2.3)
Note that \( E_0 (x) = E_0 (0; 0; x, x) \), the unshifted diagonal case of \( E_0 (\epsilon; q; x, y) \), is just the vacuum energy density \( \mathbb{E} \).

The matrix element of the shifted counting operator defines the shifted local spectral counting function:
\[
N (\lambda; q; x, y) = \langle x | N | y \rangle = \sum_n \theta (\lambda - (\lambda_n + q)) \phi_n (x) \phi_n^* (y); \tag{2.4}
\]
the matrix element of the shifted heat kernel operator defines the shifted local heat kernel:
\[
K (t; q; x, y) = \langle x | K | y \rangle = \sum_n e^{-(\lambda_n + q)t} \phi_n (x) \phi_n^* (y). \tag{2.5}
\]

For the case of \( q = 0 \), \( N (\lambda; x, y) = N (\lambda; 0; x, y) \) defines the local spectral counting function, and \( K (t; x, y) = K (t; 0; x, y) \) is the local heat kernel. We have \( K (t; q; x, y) = e^{-qt} K (t; x, y) \) and \( N (\lambda; q; x, y) = N (\lambda - q; x, y) \). The trace of an operator, taking the spectral counting function as an example, can be taken as
\[
N (\lambda) = tr \ N_{\mid q=0} = \int d^n x \sqrt{g} N (\lambda; x, x) = \sum_n \theta (\lambda - \lambda_n) = \sum_{\lambda_n < \lambda} 1. \tag{2.6}
\]

2.2 Relations

Mathematically speaking, all the functions mentioned above are essentially various spectral functions for a given operator \( D \). The relations among them can be formally deduced from their definitions. From eqs. (2.1), (2.4), and (2.5), we can achieve the relations among \( \zeta (s; q; x, y) \), \( N (\lambda; q; x, y) \), and \( K (t; q; x, y) \).

By the representation
\[
D^{-s} = s \int_0^\infty d\lambda \frac{1}{\lambda^{s+1}} \theta (\lambda - D), \tag{2.7}
\]
we achieve
\[
\zeta (s; q; x, y) = s \int_0^\infty d\lambda \frac{1}{\lambda^{s+1}} N (\lambda; q; x, y), \tag{2.8}
\]
and by the representation
\[
D^{-s} = \frac{1}{\Gamma (s)} \int_0^\infty dt t^{s-1} e^{-tD}, \tag{2.9}
\]
we achieve
\[
\zeta (s; q; x, y) = \frac{1}{\Gamma (s)} \int_0^\infty dt t^{s-1} K (t; q; x, y). \tag{2.10}
\]
The corresponding inverse transformations can be obtained directly, e.g.,

\[
N (\lambda; q; x, y) = \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} ds \frac{\lambda^s}{s} \zeta (s; q; x, y). \tag{2.11}
\]

We then obtain the relation between \( N (\lambda; q; x, y) \) and \( K (t; q; x, y) \):
\[
K (t; q; x, y) = t \int_0^\infty d\lambda N (\lambda; q; x, y) e^{-\lambda t}, \tag{2.12}
\]
\[
N (\lambda; q; x, y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \frac{e^{\lambda t}}{t} K (t; q; x, y). \tag{2.13}
\]
The relations among $N(\lambda)$, $K(t)$, and $\zeta(s)$ can be immediately obtained from the corresponding local relations:

$$
\zeta(s) = s \int_0^\infty d\lambda \frac{N(\lambda)}{\lambda^{s+1}} = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}K(t)}{(s-1)!}.
$$

(2.14)

An in-depth discussion of the global heat kernel and the spectral counting function has been provided in Refs. [36, 37].

Moreover, the relations among $\zeta(s; q; x, y)$, $W(s; q; x, y)$, $E_0(\epsilon; q; x, y)$, and $N(\lambda; q; x, y)$ can be obtained straightforwardly. For example, from eqs. (2.2) and (2.8), we have

$$
W(s; q; x, y) = -\frac{1}{2} \tilde{\mu}^2 s \Gamma(s+1) \int_0^\infty d\lambda \frac{N(\lambda; q; x, y)}{\lambda^{s+1}}.
$$

(2.15)

2.3 Equations

The equation for Hurwitz zeta functions. The equation for $\zeta(s; q; x, y)$ can be constructed as

$$(s - 1) \int_0^q dq \zeta(s; q; x, y) + (D_x + q) \zeta(s; q; x, y) = 0,$$

(2.16)

with the condition

$$
\lim_{t \to 0} \left[ \left( \int_{c-i\infty}^{c+i\infty} ds \right) \zeta(s; 0; x, y) \right] = i2\pi \delta(x - y),
$$

where the order of the integrals is arbitrary. This is a partial integro-differential equation [38]. Such an equation can be translated into a partial differential equation by taking derivative with respect to $q$:

$$
D_x \frac{\partial}{\partial q} \zeta(s; q; x, y) + q \frac{\partial}{\partial q} \zeta(s; q; x, y) + s \zeta(s; q; x, y) = 0.
$$

(2.17)

The reason why we use the local Hurwitz zeta function $\zeta(s; q; x, y)$ which can be viewed as a shifted local zeta function rather than the local zeta function $\zeta(s; x, y)$ is that though for the aim of constructing an equation for the zeta function, only a local zeta function $\zeta(s; x, y)$ is sufficient, the equation for $\zeta(s; x, y)$ is a recurrence differential equation,

$$
D_x \zeta(s; x, y) - \zeta(s - 1; x, y) = 0,
$$

(2.18)

which is difficult to deal with. If we adopt the local Hurwitz zeta function, then by the relation $\frac{\partial}{\partial q} \zeta(s; q; x, y) = -s \zeta(s + 1; q; x, y)$, we achieve eq. (2.16) instead of the recurrence differential equation.

The equation for regularized one-loop effective actions. The equation for $W(s; q; x, y)$ can be obtained from eqs. (2.2) and (2.16):

$$(s - 1) \int_0^q dq W(s; q; x, y) + (D_x + q) W(s; q; x, y) = 0;
$$

(2.19)

with the condition

$$
\lim_{t \to 0} \left[ \left( \int_{c-i\infty}^{c+i\infty} ds W(s; 0; x, y) \right) / (\tilde{\mu}^2 t^s) \right] = -i\pi \delta(x - y).
$$

Taking derivative with respect to $q$ gives the corresponding partial differential equation:

$$
D_x \frac{\partial}{\partial q} W(s; q; x, y) + q \frac{\partial}{\partial q} W(s; q; x, y) + s W(s; q; x, y) = 0.
$$

(2.20)
By the way, if we start with an unshifted local regularized one-loop effective action \( W(s; x, y) = W(s; 0; x, y) \), we will obtain a recurrence differential equation:

\[
D_x W(s; x, y) - (s - 1) \tilde{\mu}^2 W(s - 1; x, y) = 0. \tag{2.21}
\]

The equation for regularized vacuum energies. The equation for the shifted local vacuum energy can be obtained directly from eqs. (2.3) and (2.16):

\[
\frac{3}{2} \int dq E_0(\epsilon; q; x, y) - \epsilon \int dq E_0(\epsilon; q; x, y) - (D_x + q) E_0(\epsilon; q; x, y) = 0, \tag{2.22}
\]

with the condition \( \lim_{t \to 0} \left[ \int_{c-i\infty}^{c+i\infty} d\epsilon \Gamma(-1/2 + \epsilon) t^{1/2 - \epsilon} \tilde{\mu}^{-2 \epsilon} E_0(\epsilon; 0; x, y) \right] = i \pi \delta(x - y) \).

Taking derivative with respect to \( q \) gives the corresponding partial differential equation:

\[
D_x \frac{\partial}{\partial q} E_0(\epsilon; q; x, y) + q \frac{\partial}{\partial q} E_0(\epsilon; q; x, y) - \frac{1}{2} E_0(\epsilon; q; x, y) + \epsilon E_0(\epsilon; q; x, y) = 0. \tag{2.23}
\]

The regularized vacuum energy can be achieved by taking trace with \( q = 0 \).

The equation for spectral counting functions. The equation for \( N(\lambda; x, y) \) can be obtained from the equation of \( \zeta(s; q; x, y) \) with \( q = 0 \) by using the relation (2.8):

\[
\int \lambda d\lambda N(\lambda; x, y) + (D_x - \lambda) N(\lambda; x, y) = 0 \tag{2.24}
\]

with the condition \( \lim_{t \to 0} \left[ \int_0^\infty d\lambda N(\lambda; x, y) te^{-\lambda t} \right] = \delta(x - y) \). Eq. (2.24) is a partial integro-differential equation. Taking derivative with respect to \( \lambda \) to both sides of eq. (2.24) will give the corresponding partial differential equation:

\[
D_x \frac{\partial}{\partial \lambda} N(\lambda; x, y) - \lambda \frac{\partial}{\partial \lambda} N(\lambda; x, y) = 0. \tag{2.25}
\]

Defining a local state density \( \rho(\lambda; x, y) \equiv \frac{\partial}{\partial \lambda} N(\lambda; x, y) \), we achieve

\[
D_x \rho(\lambda; x, y) = \lambda \rho(\lambda; x, y). \tag{2.26}
\]

The global state density \( \rho(\lambda) \) can be obtained by taking trace of \( \rho(\lambda; x, y) \).

3. Solutions of local and global one-loop effective actions: Renormalization

3.1 The free-field solution

In this section, we first solve the shifted local regularized one-loop effective action for a free massive scalar field in \( \mathbb{R}^n \) from eq. (2.19). In this case, \( D_0 = -\nabla^2 + m^2 \), where \( m \) is the mass. The solution of eq. (2.19) reads

\[
W_0(s; q; x, y) = -\tilde{\mu}^{2s} \left( \frac{2\sqrt{m^2 + q}}{|x - y|} \right)^{n/2 - s} K_{n/2 + s} \left( \sqrt{m^2 + q} |x - y| \right), \tag{3.1}
\]
where \( K_n(z) \) is the modified Bessel function.

As a function of \( x \) and \( y \), \( W_0(s; q; x, y) \) has a singular point at \( |x - y| = 0 \), or, \( W_0(s; q; x, y) \) is analytic except for \( |x - y| = 0 \). However, when seeking a global one-loop effective action, what we concern is just the point \( |x - y| = 0 \) since the global one-loop effective action is obtained through taking trace of the local one. This means that when achieving a global one-loop effective action from the corresponding local one-loop effective action, we need a renormalization procedure to remove the divergence.

In order to extract the divergence led by \( |x - y| = 0 \), by use of the expansion of \( K_\nu(z) \) around \( z = 0 \),

\[
K_\nu(z) = \frac{1}{2} \Gamma(\nu) \left( \frac{2}{z} \right) \sum_{p=0}^{\infty} \frac{(z/2)^{2p}}{(1 - \nu)p!} + \frac{1}{2} \Gamma(-\nu) \left( \frac{z}{2} \right) \sum_{p=0}^{\infty} \frac{(z/2)^{2p}}{(\nu + 1)p!},
\]

(3.2)

where \((a)_p = a(a+1)(a+2)\cdots(a+p-1)\), we expand the local one-loop effective action \[\text{(3.1)}\] around \( |x - y| = 0 \):

\[
W_0(s; q; x, y) = \frac{-\tilde{\mu}^{2s}}{2(4\pi)^{n/2}} \left[ \Gamma \left( s - \frac{n}{2} \right) \sum_{p=0}^{\infty} \frac{(\sqrt{m^2 + q})^{2p-n-2s}}{(1 + n/2-s)_p} \frac{\Gamma \nu \left( \frac{|x - y|}{2} \right)^{2p}}{p!} \right] + \frac{\Gamma \left( \frac{n}{2} - s \right)}{2(4\pi)^{n/2}} \sum_{p=0}^{\infty} \frac{(\sqrt{m^2 + q})^{2p-n+2s}}{(1 - n/2+s)_p} \frac{\Gamma \nu \left( \frac{|x - y|}{2} \right)^{2p-n+2s}}{p!}.
\]

(3.3)

The negative power terms in the expansion of \( W_0(s; q; x, y) \) will diverge when \( |x - y| = 0 \). That is to say, by such a procedure, we have extracted the divergent part of \( W_0(s; q; x, y) \). In order to achieve a finite result, we drop the negative power term of \( |x - y| \). Then taking trace gives

\[
W_0(s; q) = TrW_0(s; q; x, y) = -Vol \frac{-\tilde{\mu}^{2s}}{2(4\pi)^{n/2}} (m^2 + q)^{n/2-s} \Gamma \left( s - \frac{n}{2} \right).
\]

(3.4)

In even-dimensional space-times, \( s = 0 \) is a singular point, which is a simple pole of \( W_0(s; q) \). In order to remove the divergence coming from \( s = 0 \), for 2\( \nu \)-dimensional cases, we Laurent expand \( W_0(s; 0) \) around \( s = 0 \).

\[
W_0(s; 0)
= -Vol \frac{(−1)^\nu}{2(4\pi)^{\nu} \mu^{2\nu}} \left\{ \frac{1}{s} + \psi(\nu + 1) - \ln \frac{m^2}{\mu^2} \right\}
\sum_{p=2}^{\infty} \hat{\mathcal{D}}_{s, p} \left\{ \sum_{\beta=0}^{\nu} \sum_{\alpha=0}^{\nu} \left( -1 \right)^{\frac{\alpha + \beta - 1}{\alpha} \left( \frac{\beta}{\alpha} - 2 \right)} B_{\beta-\alpha} \frac{\Gamma(\nu + 1)}{\alpha! (\beta - \alpha)! (p - \beta)!} \left[ \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{1}{\Gamma(\xi)} \right]_{\xi=\nu+1} \left( \ln \frac{\mu^2}{m^2} \right)^{p-\beta} \right\} \right\},
\]

(3.5)

where \( B_\nu \) is the Bernoulli number. The finite physical observable does not contain the parameter \( s \), which is embodied in \( W_0(s; 0) \) with \( s = 0 \), i.e., \( W_0(0; 0) \). Take \( s = 0 \). Then,
a regularized unshifted $2\nu$-dimensional one-loop effective action without the regularization parameter $s$ is the remaining part of the expansion of $W_0(s;0)$ after dropping the divergent negative power term of $s$, \(^1\)

\[
W_0 = -Vol \frac{(-1)^\nu}{2(4\pi)^{\nu+1/2}} m^{2\nu} \left[ \psi(\nu + 1) - \ln \frac{m^2}{\bar{\mu}^2} \right], \quad (3.6)
\]

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. The 4-dimensional case ($\nu = 2$) agrees with the result in Refs. [1, 10].

In odd-dimensional space-times, $s = 0$ is not a singular point, so the one-loop effective action given by eq. (3.4) does not need to drop divergent terms. The expansion of $W_0(s;0)$ reads

\[
W_0(s;0) = -Vol \frac{1}{2(4\pi)^{\nu+1/2}} m^{2\nu+1} \Gamma(- (\nu + 1/2)) \left( \ln \frac{\bar{\mu}^2}{m^2} \right)^{p-\beta} s^p. \quad (3.7)
\]

For $(2\nu + 1)$-dimensional cases, from eq. (3.4), taking $s = 0$, we arrive at

\[
W_0 = -Vol \frac{1}{2(4\pi)^{\nu+1/2}} m^{2\nu+1} \Gamma(- (\nu + 1/2)). \quad (3.8)
\]

### 3.2 The series solution: the Laplace-type operator with local boundary conditions

Exact solutions are rare, so in more general cases, such as interaction cases, we turn to find perturbation solutions for $W(s;q;x,y)$. When seeking a perturbation solution, the first thing that we need to do is to find a proper series expansion for $W(s;q;x,y)$. This is, in principle, a difficult task. Fortunately, a thorough study on the expansion of heat kernels has already been made [39, 40]. We can construct a proper series for one-loop effective actions by starting from the series of heat kernels, based on the transformation relation between one-loop effective actions and heat kernels.

In the section, we first consider a special case: the case of a second-order differential operator of Laplace type with a local boundary condition. In next section, we discuss the general form of the series expansion of a one-loop effective action.

In the case of a second-order differential operator of Laplace type with a local boundary condition, the $n$-dimensional heat kernel corresponding to the operator $D$ can be expanded as [41]

\[
K(t;q;x,y) = K_0(t;q;x,y) \sum_{k=0,\frac{1}{2},1,\ldots} b_k(x,y) t^k, \quad (3.9)
\]

where $K_0(t;q;x,y) = (4\pi t)^{-n/2} e^{-(x-y)^2/(4t) - (m^2+q)t}$ is the heat kernel for a $n$-dimensional free massive scalar field and $b_k(x,y)$ is the heat kernel coefficient.

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\(^1\)In the published version (JHEP06(2010)070), we have used a misleading usage: we call, e.g., $W(s;0)$ or $W_s$, etc., as regularized one-loop effective action, call $W$, the regularized one-loop effective action without the regularized parameter $s$ (the remaining part of $W_s$ of taking $s = 0$ and dropping the divergent terms of $s$) as renormalized one-loop effective action, but call the common renormalized quantities as finite physical quantities.
The shifted local regularized one-loop effective action can be achieved by performing the transformation (2.10) to the heat kernel $K (t; q; x, y)$ given by eq. (3.9):

$$
W (s; q; x, y) = - \frac{\tilde{\mu}^{2s}}{(4\pi)^{n/2}} \sum_{k=0, \frac{1}{2}, 1, \ldots} b_k (x, y) \left( \frac{2\sqrt{m^2 + q}}{|x - y|} \right)^{n/2 - k - s} \times K_n/2 - k - s \left( \sqrt{m^2 + q} |x - y| \right).
$$

(3.10)

$W (s; q; x, y)$ is analytic except for $|x - y| = 0$.

In order to achieve the global regularized one-loop effective action, we need to take trace of eq. (3.10) with $q = 0$. The divergence coming from the singular point $|x - y| = 0$ of the local one-loop effective action can be removed by the same procedure used in the case of free fields. Using eq. (3.2), we achieve the local regularized one-loop effective action

$$
W (s; q; x, y) = - \frac{\tilde{\mu}^{2s}}{2 (4\pi)^{n/2}} \sum_{k=0, \frac{1}{2}, 1, \ldots} b_k (x, y) \left[ \Gamma \left( k - \frac{n}{2} + s \right) \sum_{p=0}^{\infty} \frac{(m^2 + q)^{p-k+n/2-s} (|x - y| / 2)^{2p}}{p! (1 - k + n/2 - s)_p} \right] + \Gamma \left( -k + \frac{n}{2} - s \right) \sum_{p=0}^{\infty} \frac{(m^2 + q)^p (|x - y| / 2)^{2(p-n/2+k+s)}}{p! (1 + k - n/2 + s)_p}.
$$

(3.11)

In this result, the divergent part of the one-loop effective action has been extracted. Dropping the negative power term of $|x - y|$ and taking trace gives the global result:

$$
W (s; q) = Tr W (s; q; x, y) = - \frac{\tilde{\mu}^{2s}}{2 (4\pi)^{n/2}} \sum_{k=0, \frac{1}{2}, 1, \ldots} B_k \frac{\Gamma (s - n/2 + k)}{(m^2 + q)^{k-n/2+s}},
$$

(3.12)

where $B_k = tr b_k (x, y) = \int d^n x \sqrt{g} b_k (x, x)$.

To obtain a regularized unshifted one-loop effective action, we Laurent expand $W (s; 0)$ with respect to $s$ around $s = 0$.

$$
W (s; 0) = - \frac{Vol}{2 (4\pi)^{n/2}} \left\{ \sum_{k=0,1/2,1,\ldots} B_k (\frac{1}{n/2 - k})! m^{n-2k} \left\{ \frac{1}{s} + \psi \left( \frac{n}{2} - k + 1 \right) - \ln \frac{m^2}{\mu^2} \right\} \right\}
$$

$$
+ \sum_{p=2}^{\infty} \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\alpha+\beta-1} \frac{(-1)^{\alpha+\beta-1} (2^{\beta-a} - 2 B_{\beta-a} \Gamma (n/2 - k + 1))}{\alpha! (\beta-a)! (p-\beta)!} \left[ \frac{\partial^{p-a}}{\partial \xi^{p-a}} \frac{1}{\Gamma (\xi)} \right]_{\xi=n/2-k+1} \left( \ln \frac{\mu^2}{m^2} \right)^{p-\beta} s^{p-1}
$$

$$
+ \sum_{k=0,1/2,1,\ldots} B_k m^{n-2k} \sum_{p=0}^{\infty} \left\{ \sum_{\beta=0}^{\beta-1} \frac{\Gamma (\beta) (-n/2 + k)}{\beta! (p-\beta)!} \left( \ln \frac{\mu^2}{m^2} \right)^{p-\beta} s^{p} \right\}.
$$

(3.13)
Take $s = 0$ and drop the divergent term:

$$W = \frac{Vol}{2 (4\pi)^{n/2}} \left\{ \sum_{k=0, \frac{1}{2}, 1, \ldots}^{N} b_k (x, y) t^k + \sum_{k=N+\frac{1}{2}}^{\infty} t^k \left[ b'_k (x, y) \ln t + b''_k (x, y) \right] \right\},$$

(3.14)

For manifolds without boundaries, the half-integer power terms vanish, i.e., $B_{m/2} = 0$. The result (3.14) with $B_{m/2} = 0$ and $n = 4$ agrees with the result given by [1, 10].

### 3.3 The series solution: general cases

In this section, we give a discussion on the general form of series expansion for local one-loop effective actions, $W (s; q; x, y)$. In order to achieve a proper series for one-loop effective actions, we start from the series of heat kernels, based on the transformation relation between one-loop effective actions and heat kernels.

General form of the heat kernel expansion contains logarithmic terms, which can be written as [10, 11]

$$K (t; q; x, y) = K_0 (t; q; x, y) \left\{ \sum_{k=0, \frac{1}{2}, 1, \ldots}^{N} b_k (x, y) t^k + \sum_{k=N+\frac{1}{2}}^{\infty} t^k \left[ b'_k (x, y) \ln t + b''_k (x, y) \right] \right\},$$

(3.15)

where $b_k (x, y)$, $b'_k (x, y)$ and $b''_k (x, y)$ are heat kernel coefficients.

Starting from the expansion of heat kernels, we can achieve a series of the local regularized one-loop effective action by performing the transformation (2.10):

$$W (s; q; x, y) = -\frac{\mu^{2s}}{(4\pi)^{n/2}} \left\{ \sum_{k=0, \frac{1}{2}, 1, \ldots}^{N} b_k (x, y) + \sum_{k=N+\frac{1}{2}}^{\infty} b'_k (x, y) \ln \frac{|x-y|}{2 \sqrt{m^2 + q}} + \sum_{k=N+\frac{1}{2}}^{\infty} b''_k (x, y) \right\}$$

$$\times \left( \frac{|x-y|}{2 \sqrt{m^2 + q}} \right)^{k-n/2+s} K_{k-n/2+s} \left( \sqrt{m^2 + q} |x-y| \right)$$

$$- \sum_{k=N+\frac{1}{2}}^{\infty} b'_k (x, y) \left( \frac{|x-y|}{2 \sqrt{m^2 + q}} \right)^{k-n/2+s} K_{(k-n/2+s)}^{(1)} \left( \sqrt{m^2 + q} |x-y| \right),$$

(3.16)

where $K_{\nu}^{(1)} (z) = \frac{\partial}{\partial \nu} K_{\nu} (z)$. Based on this series expansion, one can in principle achieve a perturbation solution of $W (s; q; x, y)$. 

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- 12 -
In order to achieve a series expansion for the global regularized one-loop effective action, we take trace of eq. (3.16). The divergence coming from the singular point \(|x - y| = 0\) of the local one-loop effective action can be removed by the same procedure used in the case of free fields. The series of the shifted global regularized one-loop effective action then reads

\[
W(s; q) = tr W(s; q; x, y) = -\frac{\hat{\mu}^{2s}}{2(4\pi)^{n/2}} \left[ \sum_{k=0, \frac{1}{2}, 1, \cdots}^N B_k + \sum_{k=N+\frac{1}{2}}^\infty B_k'' + \sum_{k=N+\frac{1}{2}}^\infty B_k' \psi \left(s - \frac{n}{2} + k\right) \right] \frac{\Gamma(s - n/2 + k)}{(m^2 + q)^{k-n/2+s}},
\]

(3.17)

where the relation

\[
K^{(1)}_\nu (z) = \frac{\pi \csc (\nu \pi)}{2} \sum_{p=0}^\infty \left\{ \psi (p - \nu + 1) - \frac{\pi}{\tan (\nu \pi)} - \ln \left(\frac{z}{2}\right) \right\} \frac{1}{\Gamma (p - \nu + 1) p!} \left(\frac{z}{2}\right)^{2p-\nu}
\]

\[+ \left[ \psi (p + \nu + 1) + \frac{\pi}{\tan (\nu \pi)} - \ln \left(\frac{z}{2}\right) \right] \frac{1}{\Gamma (p + \nu + 1) p!} \left(\frac{z}{2}\right)^{2p+\nu} \}
\]

(3.18)

is used.

To obtain a regularized series expansion without the regularization parameter \(s\), we Laurent expand \(W(s; 0)\),

\[
W(s; 0) = -\frac{Vol}{2(4\pi)^{n/2}} \left\{ \left( \sum_{k=0, \frac{1}{2}, 1, \cdots}^N B_k + \sum_{k=N+\frac{1}{2}}^\infty B_k'' \right) \frac{(-1)^{n/2-k}}{(n/2 - k)!} m^{n-2k} \right\} + \frac{m^2}{\hat{\mu}^2} + \sum_{p=2}^\infty \left\{ \sum_{\beta=0}^p \sum_{\alpha=0}^\beta \frac{(-1)^{\beta+1}}{\alpha! (\beta - \alpha)!} \frac{\Gamma(n/2-k+1)}{(n/2-k)!} \left(\ln \frac{\hat{\mu}^2}{m^2}\right)^{p-\beta} \right\} s^{p-1}
\]

\[+ \left( \sum_{k=0, \frac{1}{2}, 1, \cdots}^N B_k + \sum_{k=N+\frac{1}{2}}^\infty B_k'' \right) m^{n-2k} \left\{ \sum_{\beta=0}^p \frac{\Gamma(n/2-k+1)}{\beta! (p-\beta)!} \left(\ln \frac{\hat{\mu}^2}{m^2}\right)^{p-\beta} \right\} s^p \]

\[-13-\]
\[ + \sum_{k=N+1/2}^{n/2-k \neq 0,1,2,\ldots} B'_km^{n-2k}(1)^{n/2-k} \left(1\right)^{n/2-k} \]

\[ \times \left\{ \sum_{k=0,1/2,\ldots}^{N} B_k + \sum_{k=N+1/2}^{n/2-k \neq 0,1,2,\ldots} B''_k \right\} \Gamma \left( k - \frac{n}{2} \right) m^{n-2k} \]

\[ + \left( \sum_{k=0,1/2,\ldots}^{N} B_k + \sum_{k=N+1/2}^{n/2-k \neq 0,1,2,\ldots} B''_k \right) \frac{(1)^{-n/2+k}}{(n/2-k)!} m^{n-2k} \left[ \psi \left( 1 - k + \frac{n}{2} \right) - \ln \frac{m^2}{\mu^2} \right] \]

\[ + \sum_{k=N+1/2}^{n/2-k \neq 0,1,2,\ldots} B'_k \frac{(1)^{-n/2+k}}{(n/2-k)!} m^{n-2k} \left[ \frac{1}{2} \psi^2 \left( 1 - k + \frac{n}{2} \right) - \frac{1}{2} \psi' \left( 1 - k + \frac{n}{2} \right) + \frac{\pi^2}{6} - \frac{1}{2} \left( \ln \frac{m^2}{\mu^2} \right)^2 \right] \]

(3.20)

where \( \psi' (z) = \frac{d}{dz} \psi (z) \).

4. Solutions of local and global vacuum energies: Renormalization

4.1 The free-field solution

For a free massive scalar field in \( \mathbb{R}^n \), \( D_0 = -\nabla^2 + m^2 \). The shifted local vacuum energy can be solved from eq. (2.23):

\[ E_0 (\epsilon; q; x, y) = \frac{\tilde{\mu}^2}{(4\pi)^{n/2} \Gamma (-1/2 + \epsilon)} \left( \frac{2\sqrt{m^2 + q}}{|x - y|} \right)^{(n+1)/2-\epsilon} K_{-(n+1)/2+\epsilon} \left( \sqrt{m^2 + q} |x - y| \right) \]

(4.1)
The shifted local vacuum energy $E_0 (\epsilon; q; x, y)$ has a singular point at $|x - y| = 0$, corresponding to the divergence in the global vacuum energy which is the trace of the local one. In order to extract the divergence, we expand $E_0 (\epsilon; q; x, y)$ around the singularity $|x - y| = 0$,

$$E_0 (\epsilon; q; x, y) = \frac{\hat{\mu}^{2\epsilon}}{2 (4\pi)^{n/2}} \frac{\pi}{\Gamma (-1/2 + \epsilon)} \sin ((- (n + 1)/2 + \epsilon) \pi) \times \sum_{p=0}^{\infty} \frac{1}{p!} \left[ \frac{(m^2 + q)^{(n+1)/2+p-\epsilon}}{\Gamma (1 + (n+1)/2 - \epsilon + p)} - \frac{(m^2 + q)^p (|x - y|/2)^{2p-n-1+2\epsilon}}{\Gamma (1 - (n+1)/2 + \epsilon + p)} \right].$$

Taking trace and dropping the divergent negative power term gives

$$E_0 (\epsilon; q) = Vol \frac{\hat{\mu}^{2\epsilon}}{2 (4\pi)^{n/2}} \frac{\Gamma (- (n + 1)/2 + \epsilon)}{\Gamma (-1/2 + \epsilon)} (m^2 + q)^{(n+1)/2-\epsilon}.$$  \hspace{1cm} (4.2)

In odd-dimensional space-times, $\epsilon = 0$ is a singular point of $E_0 (\epsilon; q)$. To remove the divergence, we Laurent expand $E_0 (\epsilon; 0)$ with respect to $\epsilon$ around $\epsilon = 0$,

$$E_0 (\epsilon) = -Vol \frac{m^{2\nu}}{2 (4\pi)^{n/2}} \frac{(-1)^{\nu}}{\nu!} \left\{ \frac{1}{\epsilon} + \left[ \ln \frac{4\hat{\mu}^2}{m^2} + \psi (\nu + 1) + \gamma_E - 2 \right] \right\} + \epsilon \left[ 12 \left( \ln \frac{4\hat{\mu}^2}{m^2} + \gamma_E - 2 \right) \psi (\nu + 1) + 6 \left( \ln \frac{4\hat{\mu}^2}{m^2} + \gamma_E - 2 \right)^2 + 6\psi^2 (\nu + 1) - 6\psi (1) (\nu + 1) - \pi^2 - 24 \right]$$

$$+ \epsilon^2 \left\{ \frac{1}{12} \left[ 6 \left( \ln \frac{4\hat{\mu}^2}{m^2} + \gamma_E - 2 \right) \psi^2 (\nu + 1) + \psi (\nu + 1) \left[ 6 \ln \frac{4\hat{\mu}^2}{m^2} \left( \ln \frac{4\hat{\mu}^2}{m^2} + 2\gamma_E - 4 \right) - 6 \psi (1) (\nu + 1) \right. \right. 

$$

$$+ \ln \frac{4\hat{\mu}^2}{m^2} \left[ 2 \ln \frac{4\hat{\mu}^2}{m^2} \left( \ln \frac{4\hat{\mu}^2}{m^2} + 3\gamma_E - 6 \right) + 6\gamma_E (\gamma_E - 4) - \pi^2 \right] 

$$

$$+ 2\psi^3 (\nu + 1) + 2\psi^2 (\nu + 1) + 28\zeta (3) + 2\pi^2 + \gamma_E \left[ 2 (\gamma_E - 6) \gamma_E - \pi^2 \right] \right\} + \cdots \right\}. \hspace{1cm} (4.4)$$

A regularized unshifted $(2\nu - 1)$-dimensional vacuum energy without the regularization parameter $s$ can be obtained by taking $\epsilon = 0$ and dropping the divergent negative power term of $\epsilon$:

$$E_0 = Vol \frac{(-1)^{\nu}}{2 (4\pi)^{n/2} \nu!} m^{2\nu} \left[ 2 - \gamma_E - \psi (\nu + 1) + \ln \frac{m^2}{4\hat{\mu}^2} \right], \hspace{1cm} (4.5)$$

where $\gamma_E$ is the Euler constant.

In even-dimensional space-times, $\epsilon = 0$ is not a singular point, so the $2\nu$-dimensional vacuum energy can be achieved directly by setting $\epsilon = 0$ in eq. (4.3) without dropping
divergent terms. The expansion of $E_0(\epsilon)$ is

$$E_0(\epsilon) = -Vol \frac{m^{2\nu+1}}{2(4\pi)^{\nu+1/2}} \frac{\Gamma(-1/2) \Gamma(- (\nu + 1/2) + \epsilon)}{\Gamma(-1/2 + \epsilon)} \left( \frac{\tilde{\mu}^2}{m^2} \right)^{\epsilon}$$

$$= -Vol \frac{m^{2\nu+1}}{2(4\pi)^{\nu+1/2}} \Gamma(- (\nu + 1/2)) \left\{ 1 + \epsilon \left[ \ln \frac{4\tilde{\mu}^2}{m^2} + H_{-(\nu+3/2)} - 2 \right] \right. 

+ \frac{\epsilon^2}{4} \left[ 2\psi(- (\nu + 1/2)) \left( 2\ln \frac{4\tilde{\mu}^2}{m^2} + H_{-(\nu+3/2)} + \gamma_E - 4 \right) 

+ 2\psi^{(1)}(- (\nu + 1/2)) + 2\gamma_E (\gamma_E - 4) - \pi^2 \right] + \cdots \right\}.$$  (4.6)

Taking $\epsilon = 0$ gives

$$E_0 = -Vol \frac{1}{2(2\sqrt{\pi})^{2\nu+1}} \Gamma(- (\nu + 1/2)) m^{2\nu+1}. \quad (4.7)$$

### 4.2 The series solution: the Laplace-type operator with local boundary conditions

To find a perturbation solution for the vacuum energy, we need to first construct a proper series for $E_0(\epsilon; q; x, y)$. In this section, we first consider the case of a second-order differential operator of Laplace type $D$ with a local boundary condition.

The series expansion of a $n$-dimensional vacuum energy can be obtained by performing the transformation (2.10) to eq. (3.9):

$$E_0(\epsilon; q; x, y) = \frac{\tilde{\mu}^{2\epsilon}}{(4\pi)^{n/2}} \left( \frac{\sqrt{m^2 + q}}{2} \right)^{1/2} \left( \frac{|x - y|}{2 \sqrt{m^2 + q}} \right)^{k-(n+1)/2+\epsilon} K_{-(n+1)/2+k+\epsilon} \left( \sqrt{m^2 + q} |x - y| \right). \quad (4.8)$$

Taking trace and dropping the divergent negative power term gives

$$E_0(\epsilon; q) = \frac{\tilde{\mu}^{2\epsilon}}{2(4\pi)^{n/2}} \Gamma(-1/2 + \epsilon) \sum_{k=0, \frac{1}{2}, 1, \ldots} B_k \Gamma \left( -\frac{n+1}{2} + k + \epsilon \right) \left( m^2 + q \right)^{(n+1)/2 - k - \epsilon}. \quad (4.9)$$
To achieve an unshifted regularized vacuum energy, we Laurent expand $E_0 (\epsilon; 0)$,

$$
E_0 (\epsilon) = - \frac{Vol}{2 (4\pi)^{(n+1)/2}} \left\{ \frac{1}{\epsilon} \sum_{k=0, \frac{1}{2}, 1, \ldots}^{(n+1)/2 - k} B_k m^{(n+1)-2k} \frac{(-1)^{(n+1)/2-k}}{((n+1)/2 - k)!} \right. \\
+ \left. \sum_{k=0, \frac{1}{2}, 1, \ldots}^{(n+1)/2 - k} B_k m^{(n+1)-2k} \frac{(-1)^{(n+1)/2-k}}{((n+1)/2 - k)!} \left[ \ln \frac{4\bar{\mu}^2}{m^2} + \psi \left( \frac{n+3}{2} - k \right) + \gamma_E - 2 \right] \right. \\
+ \left. \frac{1}{12} \sum_{k=0, \frac{1}{2}, 1, \ldots}^{(n+1)/2 - k} B_k m^{(n+1)-2k} \frac{(-1)^{(n+1)/2-k}}{((n+1)/2 - k)!} \left[ 12 \left( \ln \frac{4\bar{\mu}^2}{m^2} + \gamma_E - 2 \right) \psi \left( \frac{n+3}{2} - k \right) \right. \\
+ 6 \left( \ln \frac{4\bar{\mu}^2}{m^2} + \gamma_E - 2 \right)^2 + 6\psi^2 \left( \frac{n+3}{2} - k \right) - 6\psi(1) \left( \frac{n+3}{2} - k \right) - \pi^2 - 24 \left. \right] \\
+ \left. \sum_{k=0, \frac{1}{2}, 1, \ldots}^{(n+1)/2 - k} B_k m^{(n+1)-2k} \Gamma \left( - \frac{n+1}{2} + k \right) \left[ \ln \frac{4\bar{\mu}^2}{m^2} + H_{-(n+3)/2+k} - 2 \right] \right\} + \cdots \\
(4.10)
$$

Taking $\epsilon = 0$ and dropping the divergent negative power term gives

$$
E_0 = \frac{1}{2 (4\pi)^{(n+1)/2}} \left\{ \sum_{k=0, \frac{1}{2}, 1, \ldots}^{n+1/2 - k} B_k \frac{(-1)^{(n+1)/2-k} m^{n+1-2k}}{((n+1)/2 - k)!} \left[ 2 - \gamma_E - \psi \left( \frac{n+3}{2} - k \right) + \ln \frac{m^2}{4\bar{\mu}^2} \right] \\
- \sum_{k=0, \frac{1}{2}, 1, \ldots}^{n+1/2 - k} B_k \Gamma \left( k - \frac{n+1}{2} \right) m^{n+1-2k} \right\}. \\
(4.11)
$$

### 4.3 The series solution: general cases

To construct the general expansion for vacuum energies, we start from the general form of
the expansion of heat kernels, eq. (3.15). By eqs. (2.3) and (2.10), we arrive at

\[ E_0 (\epsilon; q; x, y) = \frac{\tilde{\mu}^{2\epsilon}}{(4\pi)^{n/2} \Gamma (-1/2 + \epsilon)} \left\{ \sum_{k=0, 1/2, 1, \ldots}^{N} b_k (x, y) + \sum_{k=N+\xi, 1/2}^{\infty} b'_k (x, y) \ln \frac{|x - y|}{2\sqrt{m^2 + q}} + \sum_{k=N+\xi, 1/2}^{\infty} b''_k (x, y) \right\} \times \left( \frac{|x - y|}{2\sqrt{m^2 + q}} \right)^{k - (n+1)/2 + \epsilon} K_{k-(n+1)/2+\epsilon} \left( \sqrt{m^2 + q} |x - y| \right) \]

\[ - \sum_{k=N+\xi, 1/2}^{\infty} b'_k (x, y) \left( \frac{|x - y|}{2\sqrt{m^2 + q}} \right)^{k - (n+1)/2 + \epsilon} K_{-k+(n+1)/2-\epsilon} \left( \sqrt{m^2 + q} |x - y| \right) \right\} . \quad (4.12) \]

Taking trace and dropping the divergent term gives the shifted global vacuum energy:

\[ E_0 (\epsilon; q) = Tr E_0 (\epsilon; q; x, y) = \frac{\tilde{\mu}^{2\epsilon}}{2 (4\pi)^{n/2} \Gamma (-1/2 + \epsilon)} \left[ \sum_{k=0, 1/2, 1, \ldots}^{N} B_k + \sum_{k=N+\xi, 1/2}^{\infty} B'_k + \sum_{k=N+\xi, 1/2}^{\infty} B''_k \psi \left( k - \frac{n+1}{2} + \epsilon \right) \right] \frac{\Gamma (k - (n+1)/2 + \epsilon)}{(m^2 + q)^{k-(n+1)/2+\epsilon}} . \quad (4.13) \]

In order to extract the divergence corresponding to \( \epsilon = 0 \), we Laurent expand \( E_0 (\epsilon; 0) \),

\[ E_0 (\epsilon) = -Vol \frac{1}{2 (4\pi)^{(n+1)/2}} \left\{ \frac{-1}{\epsilon^2} \sum_{k=N+1/2}^{(n+1)/2-k=0, 1, 2, \ldots} B'_k m^{(n+1)-2k} \left( -1 \right)^{-(n+1)/2+k} \frac{(-1)^{(n+1)/2-k}}{(n+1)/2-k)!} \right\} \]

\[ + \frac{1}{\epsilon} \left\{ \sum_{k=0, 1/2, 1, \ldots}^{N} B_k + \sum_{k=N+1/2}^{(n+1)/2-k=0, 1, 2, \ldots} B''_k \right\} m^{(n+1)-2k} \left( -1 \right)^{-(n+1)/2+k} \frac{(-1)^{(n+1)/2-k}}{(n+1)/2-k)!} \]

\[ - \sum_{k=N+1/2}^{(n+1)/2-k=0, 1, 2, \ldots} B'_k m^{(n+1)-2k} \left( -1 \right)^{-(n+1)/2+k} \frac{(-1)^{(n+1)/2-k}}{(n+1)/2-k)!} \left( \ln \frac{4\tilde{\mu}^2}{m^2} + \gamma_E - 2 \right) \right\} \]

\[ + \left\{ \sum_{k=0, 1/2, 1, \ldots}^{N} B_k + \sum_{k=N+1/2}^{(n+1)/2-k=0, 1, 2, \ldots} B''_k \right\} m^{(n+1)-2k} \left( -1 \right)^{-(n+1)/2-k} \frac{(-1)^{(n+1)/2-k}}{(n+1)/2-k)!} \left( \ln \frac{4\tilde{\mu}^2}{m^2} + \psi \left( \frac{n+3}{2} - k \right) + \gamma_E - 2 \right) \]
\[
+ \left[ \sum_{k=0,1/2,1,\ldots}^{N} B_k + \sum_{k=N+1/2}^{\infty} B_k' \right] m^{(n+1)-2k} \Gamma \left( -\frac{n+1}{2} + k \right)
- \frac{1}{2} \sum_{k=N+1/2}^{\infty} B_k' m^{(n+1)-2k} \frac{(-1)^{-(n+1)/2+k}}{((n+1)/2-k)!}
\times \left[ 4 \ln \frac{2 \mu}{m} \left( \ln \frac{2 \mu}{m} + \gamma_E - 2 \right) - \psi^2 \left( \frac{n+3}{2} - k \right) + \psi^{(1)} \left( \frac{n+3}{2} - k \right) - \frac{5 \pi^2}{6} + (\gamma_E - 4) \gamma_E \right]
+ \sum_{k=N+1/2}^{\infty} B_k' m^{(n+1)-2k} \Gamma \left( -\frac{n+1}{2} + k \right) \psi \left( -\frac{n+1}{2} + k \right) \right] 
+ \epsilon \left\{ \left[ \sum_{k=0,1/2,1,\ldots}^{N} B_k + \sum_{k=N+1/2}^{\infty} B_k'' \right] m^{(n+1)-2k} \frac{(-1)^{-(n+1)/2-k}}{((n+1)/2-k)!} \right. 
\times \left\{ \left( \ln \frac{4 \mu^2}{m^2} + \gamma_E - 2 \right) \psi \left( \frac{n+3}{2} - k \right) + \frac{1}{2} \left( \ln \frac{4 \mu^2}{m^2} + \gamma_E - 2 \right)^2 
+ \frac{1}{2} \psi^2 \left( \frac{n+3}{2} - k \right) - \frac{1}{2} \psi^{(1)} \left( \frac{n+3}{2} - k \right) - \frac{\pi^2}{12} - 2 \right\} 
+ \left[ \sum_{k=0,1/2,1,\ldots}^{N} B_k + \sum_{k=N+1/2}^{\infty} B_k'' \right] m^{(n+1)-2k} \Gamma \left( -\frac{n+1}{2} + k \right) 
\times \left[ \ln \frac{4 \mu^2}{m^2} + H_{-(n+3)/2+k} - 2 \right] - \frac{1}{12} \sum_{k=N+1/2}^{\infty} B_k' m^{(n+1)-2k} \frac{(-1)^{-(n+1)/2+k}}{((n+1)/2-k)!} 
\times \left\{ -6 \left( \ln \frac{4 \mu^2}{m^2} + \gamma_E - 2 \right) \psi^2 \left( \frac{n+3}{2} - k \right) + 6 \left( \ln \frac{4 \mu^2}{m^2} + \gamma_E - 2 \right) \psi^{(1)} \left( \frac{n+3}{2} - k \right) 
+ \ln \frac{4 \mu^2}{m^2} \left[ 2 \ln \frac{4 \mu^2}{m^2} + 3 \gamma_E - 6 \right] + 6 \gamma_E (\gamma_E - 4) - 5 \pi^2 \right] 
+ 28 \zeta(3) - 4 \psi^3 \left( \frac{n+3}{2} - k \right) - 4 \left[ \pi^2 - 3 \psi^{(1)} \left( \frac{n+3}{2} - k \right) \right] \psi \left( \frac{n+3}{2} - k \right) - 4 \psi^{(2)} \left( \frac{n+3}{2} - k \right) \right\} 
+ 10 \pi^2 + \gamma_E \left[ 2 (\gamma_E - 6) \gamma_E - 5 \pi^2 \right] 
+ \sum_{k=N+1/2}^{\infty} B_k' m^{(n+1)-2k} \Gamma \left( -\frac{n+1}{2} + k \right) 
\times \left[ \psi \left( -\frac{n+1}{2} + k \right) \left( \ln \frac{4 \mu^2}{m^2} + \gamma_E - 2 + \psi \left( -\frac{n+1}{2} + k \right) \right) + \psi^{(1)} \left( -\frac{n+1}{2} + k \right) \right] + \cdots \right\}.
\]
Taking $\epsilon = 0$ and dropping the divergent term gives the series expansion of the unshifted regularized vacuum energy without the regularization parameter $\epsilon$,

$$E_0 = -\frac{1}{2 (4\pi)^{(n+1)/2}} \left\{ \sum_{k=0, \frac{1}{2}, 1, \ldots}^{N} B_k + \sum_{k=N+\frac{1}{2}}^{\infty} B_k' \psi \left( k - \frac{n+1}{2} \right) \right\}$$

$$\times \frac{\Gamma(k - (n + 1)/2)}{m^{2k-n-1}} + \sum_{k=0, \frac{1}{2}, 1, \ldots}^{N} B_k + \sum_{k=N+\frac{1}{2}}^{\infty} B_k' \left( -1 \right)^{(n+1)/2+k} \frac{m^{n+1-2k}}{((n+1)/2 - k)!}$$

$$\times \left[ H_{(n+1)/2-k} - 2 - \ln \frac{m^2}{4\mu^2} \right] + \sum_{k=N+\frac{1}{2}}^{\infty} B_k' \left( -1 \right)^{(n+1)/2+k} \frac{m^{n+1-2k}}{((n+1)/2 - k)!}$$

$$\times \left\{ \frac{1}{2} \psi^2 \left( \frac{n + 3}{2} - k \right) - \frac{1}{2} \left( \ln \frac{m^2}{4\mu^2} - \gamma_E + 2 \right)^2 - \frac{1}{2} \psi^{(1)} \left( \frac{n + 3}{2} - k \right) + \frac{5\pi^2}{12} + 2 \right\}.$$  \hspace{1cm} (4.15)

5. Solutions of local and global spectral counting functions: Renormalization

5.1 The free-field solution

We now solve the spectral counting function from eq. (2.24) for a free massive scalar field in $\mathbb{R}^n$; in this case, $D_0 = -\nabla^2 + m^2$.

The solution of eq. (2.24) for $D_0$ reads

$$N_0 (\lambda; x, y) = \left( \frac{\sqrt{\lambda - m^2}}{2\pi |x - y|} \right)^{n/2} J_{n/2} \left( |x - y| \sqrt{\lambda - m^2} \right), \hspace{1cm} (5.1)$$

where $J_k (z)$ is the Bessel function of the first kind.

The spectral counting function can be obtained by taking trace of $N_0 (\lambda; x, y)$:

$$N_0 (\lambda) = \text{Vol} \frac{(\lambda - m^2)^{n/2}}{(4\pi)^{n/2} \Gamma (1 + n/2)}. \hspace{1cm} (5.2)$$

The case of $m = 0$ and $n = 2$ recovers Weyl’s famous result [3].

5.2 The series solution: the Laplace-type operator with local boundary conditions

In order to seek an approximation solution for the counting function, we need to construct a proper series for $N (\lambda; x, y)$. When the problem is the Laplace-type operator with local boundary conditions, we can start from the expansion of the heat kernel, eq. (3.9).
At the first sight, it seems that one can achieve the expansion of \( N(\lambda; x, y) \) by performing the integral transformation \((2.13)\) to eq. \((3.9)\) with \( q = 0 \) directly. However, the series \((3.9)\) is not uniformly convergent, so the integral transformation cannot be applied term by term, i.e., the order of integral and summation cannot be exchanged. As a result, when performing the integral transformation term by term, some of the terms will diverge. Concretely, when applying the transformation \((2.13)\) to each term of eq. \((3.9)\), one encounters the integral

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda t}}{t} (4\pi t)^{-n/2} e^{-(x-y)^2/(4t)-m^2 t} k^d dt; \tag{5.3}
\]

when \( k \geq n/2 + 1 \), the integral diverges. To make sense of these divergent integrals, we need a renormalization procedure for removing the divergence.

When \( k < n/2 + 1 \), the integral which equals the Bessel function \( J_{n/2-k}(z) \) is convergent. Analytically continuing the integral \((5.3)\) to \( J_\nu(z) \), where \( \nu \) can take on any complex value, we achieve a finite result,

\[
N(\lambda; x, y) = \sum_{k=0}^{\infty} \frac{b_k(x, y)}{2^k(2\pi)^{n/2}} \left( \frac{\sqrt{\lambda-m^2}}{|x-y|} \right)^{n/2-k} J_{n/2-k} \left( \sqrt{\lambda-m^2} |x-y| \right)
\]

\[
+ \sum_{k=0}^{\infty} \frac{b_k(x, y)}{2^k(2\pi)^{n/2}} \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{2p}p!} |x-y|^{2p} \delta^{(k-(n/2+1)-p)} (\lambda-m^2),
\]

where \( \delta^{(m)}(z) = \frac{\partial^m}{\partial z^m} \delta(z) \). In this expansion, the terms with \( k < n/2 + 1 \) are convergent and need not to be renormalized, and the terms with \( k \geq n/2 + 1 \) are the renormalized terms.

The spectral counting function \( N(\lambda) \) is the trace of \( N(\lambda; x, y) \):

\[
N(\lambda) = \sum_{k=0}^{\infty} \frac{B_k}{(4\pi)^{n/2}} \frac{(\lambda-m^2)^{n/2-k}}{\Gamma(n/2-k+1)} + \sum_{k=0}^{\infty} \frac{B_k}{(4\pi)^{n/2}} \delta^{(k-(n/2+1))} (\lambda-m^2)
\]

\[
= \left( \sum_{k=0,\frac{1}{2},\frac{3}{2},...}^{\infty} + \sum_{k=0,\frac{1}{2},\frac{3}{2},...}^{\infty} + \sum_{k=0,\frac{1}{2},\frac{3}{2},...}^{\infty} \right) \frac{B_k}{(4\pi)^{n/2}} \left( \frac{\lambda-m^2}{\Gamma(n/2-k+1)} \right)
\]

\[
+ \sum_{k=0,\frac{1}{2},\frac{3}{2},...}^{\infty} \frac{B_k}{(4\pi)^{n/2}} \delta^{(k-(n/2+1))} (\lambda-m^2). \tag{5.5}
\]

The case of \( m = 0 \) recovers the result of Ref. \([8]\), in which the renormalization procedure is based on the analytical continuation of the gamma function.

### 5.3 The series solution: general cases

To construct the general expansion for local counting functions, we perform the transfor-
motion (2.13) to the expansion of heat kernels, eq. (3.15) and, then, we achieve

\[ N (\lambda; x, y) \]

\[ = \frac{1}{(4\pi)^{n/2}} \left\{ \sum_{k=0}^{N} b_k (x, y) + \sum_{k=N+\frac{1}{2}}^{\infty} b''_k (x, y) + \sum_{k=N+\frac{1}{2}}^{\infty} b'_k (x, y) \left[ \psi \left( \frac{n}{2} + 1 - k \right) - \ln (\lambda - m^2) \right] \right\} \]

\[ \times \left( \frac{|x - y|}{2} \right)^{k-n/2} \left( \lambda - m^2 \right)^{n/4-k/2} J_{n/2-k} \left( \sqrt{\lambda - m^2} |x - y| \right) + \sum_{k=N+\frac{1}{2}}^{\infty} b_k (x, y) \]

\[ + \sum_{k=N+\frac{1}{2}+2}^{\infty} b''_k (x, y) \left[ \frac{(-1)^p}{2^{2p}p!} |x - y|^{2p} \delta^{(k-(n/2+1)-p)} (\lambda - m^2) \right] \]

\[ \times \Gamma \left( \frac{n+1}{2} + k \right) \left( \lambda - m^2 \right)^{n/2-k} \left( 1 \tilde{F}_2 \right)_{a_1} \left( \begin{array}{c} n/2 + 1 - k; n/2 + 1 - k \\ n/2 + 1 - k; n/2 + 1 - k; -\frac{(\lambda - m^2)(x - y)^2}{4} \end{array} \right) \right\}, \]

(5.6)

where \( (1 \tilde{F}_2)_{a_1} (a_1; b_1, b_2; z) = \frac{\partial}{\partial a_1} i \tilde{F}_2 (a_1; b_1, b_2; z), 1 \tilde{F}_2 (a_1; b_1, b_2; z) = \frac{i F_2(a_1; b_1, b_2; z)}{\Gamma(b_1)\Gamma(b_2)}, \) and \( 1 \tilde{F}_2 (a_1; b_1, b_2; z) \) is the generalized hypergeometric function.

The expansion of the global counting function is the trace of \( N (\lambda; x, y) \)

\[ N (\lambda) = \frac{1}{(4\pi)^{n/2}} \left\{ \sum_{k=0}^{N} B_k + \sum_{k=N+\frac{1}{2}}^{\infty} B''_k \right\} \left( \lambda - m^2 \right)^{n/2-k} \frac{1}{\Gamma (n/2 + 1 - k)} \]

\[ + \sum_{k=N+\frac{1}{2}+2}^{\infty} B_k \left( \lambda - m^2 \right)^{n/2-k} \left[ \psi \left( \frac{n}{2} + 1 - k \right) - \ln (\lambda - m^2) \right] \right\}. \]

(5.7)

6. Scalar fields in \( H_3 \) (Euclidean \( AdS_3 \)) and \( H_3/Z \) (geometry of Euclidean BTZ black hole): one-loop effective actions, vacuum energies, counting functions, and spectra

In this section, we present the local and global regularized one-loop effective actions, vacuum energies, counting functions, and spectra of scalar fields in \( H_3 \) and \( H_3/Z \). \( H_3 \), the three-dimensional hyperbolic space, or, the Euclidean Anti-de Sitter space \( AdS_3 \), is a subspace of the four-dimensional space with metric \( ds^2 = dX_1^2 - dT_1^2 + dX_2^2 + dT_2^2 \) satisfying the constraint \( X_1^2 - T_1^2 + X_2^2 + T_2^2 = -l^2 \). \( H_3/Z \) is the geometry of the Euclidean BTZ black hole [43], which is a quotient space of \( H_3 \). A clear description of \( H_3 \) and \( H_3/Z \) can be found in Ref. [44]. Moreover, a series depth studies on spectral functions of hyperbolic spaces are given in Refs. [45, 46, 47, 48].
6.1 The one-loop effective action in $H_3$

For a scalar field in $H_3$, $D_x = -\nabla^2 + m^2$ with $\nabla^2 = \partial_x^2 + 2\coth r \partial_r$, where $r(x,y) = \arccosh [1 + u(x,y)]$ is the geodesic distance between $x = (\xi, \eta)$ and $y = (\xi', \eta')$ and $u(x,y) = \left[ (\xi - \xi')^2 + |\eta - \eta'|^2 \right] / (2\xi')$ [44]. The solution of eq. (2.20) gives the shifted local regularized one-loop effective action:

$$W(s; q;x, y) = -\frac{\tilde{\mu}^{2s}}{2 s^{3/2} \pi^{3/2} r (x, y)} \frac{(m^2 + 1 + q)^{3/2}}{(m^2 + 1 + qr (x, y))} K_{3/2-s} \left( \sqrt{m^2 + 1 + qr (x, y)} \right). \tag{6.1}$$

The unshifted global regularized one-loop effective action can be achieved by taking trace of $W(s; 0; x, y)$. Dropping the divergent negative power term gives the global regularized one-loop effective action,

$$W_s = TrW(s; 0; x, y) = -Vol(H_3) \frac{\tilde{\mu}^{2s}}{16 s^{3/2} \pi^{3/2}} \Gamma \left( s - \frac{3}{2} \right) \left( m^2 + 1 \right)^{3/2-s}. \tag{6.2}$$

Here $s = 0$ is not a singular point and the regularized one-loop effective action without the regularization parameter $s$ is just $W_s|_{s=0}$:

$$W = -Vol(H_3) \frac{1}{12 \pi} \left( m^2 + 1 \right)^{3/2}. \tag{6.3}$$

This agrees with the result given by Ref. [44].

6.2 The vacuum energy in $H_3$

With $D_x = -\nabla^2 + m^2$ and $\nabla^2 = \partial_x^2 + 2\coth r \partial_r$ [44], the solution of eq. (2.23) gives the shifted local regularized vacuum energy in $H_3$,

$$E_0(\epsilon; q; x, y) = \frac{\tilde{\mu}^{2\epsilon}}{2^{1+\epsilon} \pi^{3/2} \Gamma (-1/2 + \epsilon)} \frac{1}{r (x, y)^{1-\epsilon}} \frac{(m^2 + 1 + q)^{1-\epsilon/2}}{\sinh r (x, y)} K_{2-\epsilon} \left( \sqrt{m^2 + 1 + qr (x, y)} \right). \tag{6.4}$$

The global regularized vacuum energy is the trace of $E_0(\epsilon; 0; x, y)$:

$$E_0(\epsilon; 0) = TrE_0(\epsilon; 0; x, y) = Vol(H_3) \frac{\tilde{\mu}^{2\epsilon}}{16 \pi^{3/2} \Gamma (-1/2 + \epsilon)} \left( m^2 + 1 \right)^{2-\epsilon}. \tag{6.5}$$

Laurent expanding $E_0(\epsilon; 0)$,

$$E_0(\epsilon) = -Vol \left( \frac{(m^2 + 1)^2}{64 \pi^2} \right) \times \left\{ \frac{1}{\epsilon} + \left( \ln \frac{4\tilde{\mu}^2}{m^2 + 1} - \frac{1}{2} \right) + \epsilon \left[ \frac{1}{2} \left( \ln \frac{4\tilde{\mu}^2}{m^2 + 1} \right)^2 - \frac{1}{2} \ln \frac{4\tilde{\mu}^2}{m^2 + 1} - \frac{5}{4} - \frac{\pi^2}{6} \right] \right. \right.

+ \epsilon^2 \left[ \frac{1}{6} \ln \left( \frac{4\tilde{\mu}^2}{m^2 + 1} \right)^3 - \frac{1}{4} \left( \ln \frac{4\tilde{\mu}^2}{m^2 + 1} \right)^2 - \left( \frac{5}{4} + \frac{\pi^2}{6} \right) \ln \frac{4\tilde{\mu}^2}{m^2 + 1} - \frac{13}{8} + \frac{\pi^2}{12} + 2 \zeta(3) \right] + \cdots \right\}, \tag{6.6}$$
taking $\epsilon = 0$, and dropping the divergent negative power term, we achieve a regularized vacuum energy without the regularization parameter $\epsilon$,

$$E_0 = Vol (H_3) \left( \frac{m^2 + 1}{64\pi^2} \right)^2 \left( \frac{1}{2} + \ln \frac{m^2 + 1}{4\mu^2} \right).$$ (6.7)

### 6.3 The counting function and the spectrum in $H_3$

The solution of eq. (2.24) gives the local counting function in $H_3$,

$$N (\lambda; x, y) = \frac{\sin \left( \sqrt{\lambda - (m^2 + 1)} r(x, y) \right) - r(x, y) \sqrt{\lambda - (m^2 + 1)} \cos \left( \sqrt{\lambda - (m^2 + 1)} r(x, y) \right)}{2\pi^2 r(x, y)^2 \sinh r(x, y)}. \quad (6.8)$$

Taking trace gives the global counting function,

$$N (\lambda) = Vol (H_3) \left[ \lambda - (m^2 + 1) \right]^{3/2} \left/ 6\pi^2 \right. \quad (6.9)$$

From a counting function, one can immediately achieve the eigenvalue spectrum of the operator $D$. By $N (\lambda_n) = n$, we can obtain the spectrum,

$$\lambda_n = m^2 + 1 + \left[ \frac{6\pi^2 n}{Vol (H_3)} \right]^{2/3} \quad (6.10)$$

### 6.4 The one-loop effective action in $H_3/Z$

$H_3/Z$ is a quotient of $H_3$. The solution in the quotient space $H_3/Z$ can be represented as a linear combination of the solutions in the space $H_3$ once the equation is linear [44, 49]. Concretely, for one-loop effective actions, the solution of eq. (2.19) on $V/\Gamma$ can be expressed as a linear combination of the solutions of eq. (2.20) on $V$:

$$W^{V/\Gamma} (s; q; x, y) = \sum_{\alpha \in \Gamma} W^V (s; q; x, \alpha y) \quad (6.11)$$

Then for $H_3$ and its quotient $H_3/Z$, we have

$$W^{H_3/Z} (s; q; x, y) = \sum_{n=-\infty}^{\infty} W^{H_3} (s; q; x, \gamma^n y) \quad (6.12)$$

where $y = (\xi, \eta)$ and $\gamma y = \gamma (\xi, \eta) \to \left( |\lambda|^{-1} \xi, \lambda^{-1} \eta \right)$ with $\lambda = e^{i2\pi \tau}$ and $\tau = (\theta + i\beta) / 2\pi$; here $\beta$ is the temperature and $\theta$ is the angular potential of a thermal Anti-de Sitter space [44].

In the present case, the shifted local regularized one-loop effective action of a scalar field in space $H_3/Z$ is

$$W^{H_3/Z} (s; q; x, y) = -\tilde{\mu}^{2s} \frac{2^{1/2+2s} \pi^{3/2}}{2^{1/2+2s} \pi^{3/2}} \sum_{n=-\infty}^{\infty} \left( \frac{m^2 + 1}{q} \right)^{3/2-s} K_{3/2-s} \left( \sqrt{m^2 + 1 + qr(x, \gamma^n y)} \right) \frac{r(x, \gamma^n y)^{1/2-s} \sinh r(x, \gamma^n y)}{r(x, y)^{1/2-s} \sinh r(x, y)}. \quad (6.13)$$
The corresponding global regularized one-loop effective action can be achieved by taking trace of $W^{H_3/Z}(s; 0; x, y)$:

$$W_s = Tr W(s; 0; x, y) = \int_{H_3/Z} d^3 x \sqrt{g} W(s; 0; x, x)$$

$$= -Vol(H_3/Z) \frac{\tilde{\mu}^{2s}}{16\pi^{3/2}} \Gamma\left(s - \frac{3}{2}\right) \left(\sqrt{m^2 + 1}\right)^{3-s/2}$$

$$- \frac{\tilde{\mu}^{2s} \beta^{1/2+s}}{2^{1/2+s} \sqrt{\pi}} \left(\sqrt{m^2 + 1}\right)^{1/2-s} \sum_{n=1}^{\infty} \frac{K_{1/2-s}}{n^{1/2-s} \left[cosh (n\beta) - cos (n\theta)\right]}, \quad (6.14)$$

where we have used $\int_{H_3/Z} d^3 x \sqrt{g} = \int_0^{\infty} \frac{d r}{r^3} \int_0^{2\pi} d \phi \int_0^1 d \rho \frac{\sinh r}{2 \left[cosh (n\beta) - cos (n\theta)\right]}$. This agrees with the result given by Ref. [44].

Here $s = 0$ is not a singular point, so the one-loop effective action can be directly obtained by substituting $s = 0$ into eq. (6.14):

$$W = -Vol(H_3/Z) \frac{1}{12 \pi} (m^2 + 1)^{3/2} - \frac{\sqrt{\beta}}{2 \pi} (m^2 + 1)^{1/4} \sum_{n=1}^{\infty} \frac{K_{1/2}}{\sqrt{n} \left[cosh (n\beta) - cos (n\theta)\right]}, \quad (6.15)$$

6.5 The vacuum energy in $H_3/Z$

Based on the result of the vacuum energy in $H_3$, using the treatment that we have used in obtaining the one-loop effective action in $H_3/Z$, we can achieve the local vacuum energy in $H_3/Z$,

$$E_0(\epsilon; q; x, y) = \frac{\tilde{\mu}^{2\epsilon}}{2^{\epsilon} \pi^{3/2} \Gamma(-1/2 + \epsilon)} (m^2 + 1 + q)^{1-\epsilon/2} \sum_{n=-\infty}^{\infty} \frac{K_{2-\epsilon}}{r(x, y^{n})^{1-\epsilon} \sinh r(x, y^{n})}.$$

(6.16)

Taking trace of $E_0(\epsilon; 0; x, y)$ gives the global vacuum energy,

$$E_0(\epsilon) = Tr E_0(\epsilon; 0; x, y) = \int_{H_3/Z} d^3 x \sqrt{g} E_0(\epsilon; 0; x, x)$$

$$= Vol(H_3/Z) \frac{\tilde{\mu}^{2\epsilon}}{16\pi^{3/2} \Gamma(-1/2 + \epsilon)} (m^2 + 1)^{2-\epsilon}$$

$$+ \sum_{n=1}^{\infty} \frac{\tilde{\mu}^{2\epsilon} \beta^{\epsilon}}{2^{\epsilon} \pi^{1/2} \Gamma(-1/2 + \epsilon)} \left(\sqrt{m^2 + 1}\right)^{1-\epsilon} \frac{K_{1-\epsilon}}{n^{1-\epsilon} \left[cosh (n\beta) - cos (n\theta)\right]}, \quad (6.17)$$

- 25 -
Laurent expanding $E_0(\epsilon)$,

$$E_0(\epsilon) = \frac{-1}{\epsilon} Vol \frac{(m^2 + 1)^2}{64\pi^2} - \epsilon \left\{ \frac{Vol (m^2 + 1)^2}{64\pi^2} \left( \ln \frac{4\mu^2}{m^2 + 1} - \frac{1}{2} \right) + \sum_{n=1}^{\infty} \frac{\sqrt{m^2 + 1}}{4n\pi |\sin (n\pi \tau)|^2} K_1 \left( 2n\pi \tau_2 \sqrt{m^2 + 1} \right) \right\}$$

$$- \epsilon^2 \left\{ \frac{Vol (m^2 + 1)^2}{64\pi^2} \left[ \frac{1}{2} \left( \ln \frac{4\mu^2}{m^2 + 1} \right)^2 - \frac{1}{2} \ln \frac{4\mu^2}{m^2 + 1} - \frac{5}{4} - \frac{\pi^2}{6} \right] \right\}$$

$$+ \sum_{n=1}^{\infty} \frac{\sqrt{m^2 + 1}}{4n\pi |\sin (n\pi \tau)|^2} \left[ \left( \ln \frac{4n\pi \tau_2 \mu^2}{\sqrt{m^2 + 1}} + \gamma_E - 2 \right) K_1 \left( 2n\pi \tau_2 \sqrt{m^2 + 1} \right) - K_1^{(1)} \left( 2n\pi \tau_2 \sqrt{m^2 + 1} \right) \right]$$

$$+ \epsilon^2 \left\{ \frac{Vol (m^2 + 1)^2}{64\pi^2} \left[ \frac{1}{6} \left( \ln \frac{4\mu^2}{m^2 + 1} \right)^3 - \frac{1}{4} \left( \ln \frac{4\mu^2}{m^2 + 1} \right)^2 - \frac{5}{4} + \frac{\pi^2}{6} \right] \ln \frac{4\mu^2}{m^2 + 1} - \frac{17}{8} + \frac{\pi^2}{12} - 2\zeta(3) \right\}$$

$$+ \frac{1}{4} \sum_{n=1}^{\infty} \frac{\sqrt{m^2 + 1}}{4n\pi |\sin (n\pi \tau)|^2} \left\{ 2K_2^{(1)} \left( 2n\pi \tau_2 \sqrt{m^2 + 1} \right) - 2 \left( \ln \frac{4n\pi \tau_2 \mu^2}{\sqrt{m^2 + 1}} + \gamma_E - 2 \right) K_0 \left( 2n\pi \tau_2 \sqrt{m^2 + 1} \right) \right\}$$

$$+ \left\{ 2 \ln \frac{4n\pi \tau_2 \mu^2}{\sqrt{m^2 + 1}} \left( \ln \frac{4n\pi \tau_2 \mu^2}{\sqrt{m^2 + 1}} + 2\gamma_E - 2 \right) + 2\gamma_E (\gamma_E - 2) - \pi^2 \right\} K_1 \left( 2n\pi \tau_2 \sqrt{m^2 + 1} \right) \right\} + \cdots ,$$

taking $\epsilon = 0$, and dropping the divergent negative power term gives a regularized vacuum energy without the regularization parameter $\epsilon$,

$$E_0 = Vol (H_{3/Z}) \frac{(m^2 + 1)^2}{64\pi^2} \left( \frac{1}{2} + \ln \frac{m^2 + 1}{4\mu^2} \right) - \sum_{n=1}^{\infty} \frac{\sqrt{m^2 + 1}}{2\pi n |\cosh (n\beta) - \cos (n\theta)|} K_1 \left( n\beta \sqrt{m^2 + 1} \right).$$

(6.19)

### 6.6 The counting function and the spectrum in $H_{3/Z}$

From the local counting function in $H_3$, eq. (6.8), we can obtain the local counting function in $H_{3/Z}$,

$$N (\lambda; x, y) = \sum_{n=-\infty}^{\infty} \sin \left( \sqrt{\lambda \cdot (m^2 + 1)r (x, \gamma^n y)} - \sqrt{\lambda - (m^2 + 1)r (x, \gamma^n y)} \cos \left( \sqrt{\lambda - (m^2 + 1)r (x, \gamma^n y)} \right) \right) \frac{\sqrt{\lambda - (m^2 + 1)r (x, \gamma^n y)}}{2\pi^2 r^2 (x, \gamma^n y) \sinh r (x, \gamma^n y)}.$$

(6.20)

Taking trace gives the global counting function,

$$N (\lambda) = Vol (H_{3/Z}) \frac{\left( \lambda - (m^2 + 1) \right)^{3/2}}{6\pi^2} + \sum_{n=1}^{\infty} \frac{\sin \left( \sqrt{\lambda - (m^2 + 1)n\beta} \right)}{\pi n |\cosh (n\beta) - \cos (n\theta)|}.$$

(6.21)
The sum in eq. (6.21) is difficult to be solved, so we turn to consider an approximate result. When $\beta$ is very large, only the first term, corresponding to $n = 1$ is important. Then we approximately achieve

$$N(\lambda) \approx \frac{\text{Vol}(H_3/Z)}{6\pi^2} \left[ \lambda - (m^2 + 1) \right]^{3/2} + \frac{1}{\pi} \sin \left( \sqrt{\lambda - (m^2 + 1)} \beta \right) \cosh \beta - \cos \theta. \quad (6.22)$$

The spectrum is determined by $N(\lambda_n) = n$; approximately solving this equation gives

$$\lambda_n \approx m^2 + 1 + \left[ \frac{6\pi^2 n}{\text{Vol}(H_3/Z)} \right]^{2/3} \left( 1 - \frac{2\sin \alpha}{3n\pi \cosh \beta - 3n\pi \cos \theta + \alpha \cos \alpha} \right), \quad (6.23)$$

where $\alpha = \left[ \frac{6\pi^2 n}{\text{Vol}(H_3/Z)} \right]^{1/3} \beta$.

7. Massless scalar fields in $S^1$: one-loop effective actions, vacuum energies, counting functions, and spectra

In this section, we consider the local and global one-loop effective actions, vacuum energies, counting functions, and spectra of a massless scalar field in $S^1 = \mathbb{R}^1/Z$, the quotient of $\mathbb{R}^1$.

7.1 The one-loop effective action

As discussed above, the shifted local regularized one-loop effective action, the solution of eq. (2.20), in the quotient space $\mathbb{R}^1/Z$ is the linear combination of the solutions in space $\mathbb{R}^1$. The solution in space $\mathbb{R}^1$ can be solved from eq. (2.20): $W_{\mathbb{R}^1}(s; q; x, y) = -\frac{1}{27\sqrt{2\pi}} \bar{\mu}^{2s} (\sqrt{q/|x - y|})^{1/2-s} K_{1/2-s}(\sqrt{q}|x - y|)$. Then the shifted local regularized one-loop effective action reads:

$$W(s; q; x, y) = \sum_{Z} W_{\mathbb{R}^1}(s; q; x, y) = q^{1/2-s} \frac{\bar{\mu}^{2s}}{2^{s+1/2} \sqrt{\pi}} \sum_{n=\infty} K_{1/2-s}(\sqrt{q}|y + nL - x|) \left( \sqrt{q} \right)^{1/2-s}, \quad (7.1)$$

where $L$ is the perimeter of $S^1$.

The shifted global regularized one-loop effective action can be directly achieved by taking trace. After dropping the divergent power term, we have

$$W(s; q) = Tr W(s; q; x, y) = -L \frac{q^{1/2-s} \bar{\mu}^{2s}}{4\sqrt{\pi}} \left[ \Gamma \left( s - \frac{1}{2} \right) + 2^{5/2-s} \sum_{n=1}^{\infty} K_{1/2-s} \left( n\sqrt{q}L \right) \right]. \quad (7.2)$$

The unshifted regularized one-loop effective action is

$$W_s = \lim_{q \to 0} W(s; q) = -\left( \frac{\bar{\mu}L}{2} \right) \frac{2s}{\sqrt{\pi} \Gamma \left( \frac{1}{2} - s \right)} \zeta(1 - 2s). \quad (7.3)$$
The regularized one-loop effective action can be obtained by Laurent expanding $W_s$ around $s = 0$,

$$
W_s = \frac{1}{2s} - \frac{1}{2} \gamma_E + \ln L \bar{\mu} \\
+ \frac{1}{2 \sqrt{\pi}} \sum_{p=2}^{\infty} \left[ \sum_{\beta=0}^{p} \frac{(-1)^{p-\beta} \Gamma(p-\beta) \Gamma(1/2)}{\beta! (p-\beta)!} \left( \ln \frac{L^2 \bar{\mu}^2}{4} \right)^{\beta} \right] s^{p-1} \\
- \frac{1}{\sqrt{\pi}} \sum_{p=1}^{\infty} \left[ \sum_{\beta=0}^{p} \sum_{\rho=0}^{p} \frac{(-1)^{p-\beta-\rho} \Gamma(p) \Gamma(1/2) \gamma_{p-\beta-\rho}}{\beta! \rho! (p-\beta-\rho)!} \left( \ln \frac{L^2 \bar{\mu}^2}{4} \right)^{\beta} \right] s^p.
$$

(7.4)

Taking $s = 0$ and dropping the divergent negative power term gives the regularized result without the regularization parameter $\epsilon$,

$$
W = -\frac{1}{2} \gamma_E + \ln (\bar{\mu} L).
$$

(7.5)

Moreover, if we approximately replace the sum $\sum_{n=1}^{\infty}$ by the integral $\int_{1}^{\infty} dn$, the shifted local and global regularized one-loop effective actions can be calculated analytically, respectively,

$$
W (s; q; x, y) \\
\simeq -\frac{1}{2L} \frac{\tilde{\mu}^{2s}}{q^s} \Gamma (s) + \frac{1}{4 \sqrt{\pi} L} \frac{\tilde{\mu}^{2s}}{q^s} \left\{ 2 \Gamma \left( s - \frac{1}{2} \right) \left( \sqrt{q} \frac{y-x}{2} \right)^{2s+1} \mathbf{F}_2 \left( \frac{1}{2}; \frac{3}{2}, \frac{3}{2} - s; q \frac{(y-x)^2}{4} \right) + \frac{1}{s} \Gamma \left( \frac{1}{2} - s \right) \left\{ \left( \sqrt{q} \frac{y-x}{2} \right)^{2s} \times \mathbf{F}_2 \left( s; s + \frac{1}{2}, s + 1; q \frac{L-(y-x)^2}{4} \right) \right\} \right\}
$$

(7.6)

and

$$
W (s; q) \simeq -\frac{1}{2L} \frac{\tilde{\mu}^{2s} q^{1/2-s}}{4 \sqrt{\pi}} \Gamma \left( s - \frac{1}{2} \right) - \frac{\tilde{\mu}^{2s}}{q^s} \left\{ \sqrt{qL} \Gamma \left( s - \frac{1}{2} \right) \mathbf{F}_2 \left( \frac{1}{2}; \frac{3}{2}, \frac{3}{2} - s; \left( \frac{\sqrt{qL}}{2} \right)^2 \right) \right\} + \left( \frac{\sqrt{qL}}{2} \right)^{2s} \frac{1}{2s} \Gamma \left( \frac{1}{2} - s \right) \mathbf{F}_2 \left( s; s + \frac{1}{2}, s + 1; \left( \frac{\sqrt{qL}}{2} \right)^2 \right),
$$

(7.7)

where we assume that $y \geq x$,

7.2 The vacuum energy

The shifted local regularized vacuum energy of a massless scalar field in $\mathbb{R}^1$ can be solved from eq. (2.23), $E_0^1 (\epsilon; q; x, y) = \frac{\tilde{\mu}^{2s}}{1-(-1/2+\epsilon)} \frac{q^{1-\epsilon}}{2 \sqrt{\pi}} K_{1-\epsilon} \left( \sqrt{q} |y-x| \right) / \left( \sqrt{q} |y-x| \right)^{1-\epsilon}$. Then, the shifted local regularized vacuum energy in $S^1$ is

$$
E_0 (\epsilon; q; x, y) = \frac{\tilde{\mu}^{2s} q^{1-\epsilon}}{2 \epsilon \sqrt{\pi} \Gamma (-1/2 + \epsilon)} \sum_{n=-\infty}^{\infty} \frac{K_{1-\epsilon} \left( \sqrt{q} |y + nL - x| \right)}{\left( \sqrt{q} |y + nL - x| \right)^{1-\epsilon}},
$$

(7.8)
since $S^1$ is the quotient space of $\mathbb{R}^1$.

The shifted global regularized vacuum energy can be obtained by taking trace:

$$E_0 (\epsilon; q) = \text{Tr} E_0 (\epsilon; q; x, y)$$

$$= L \frac{\tilde{\mu}^{2\epsilon}}{\Gamma (-1/2 + \epsilon) 2^\epsilon \sqrt{\pi}} \left[ \frac{\Gamma (-1 + \epsilon)}{2^{2\epsilon - \epsilon}} + 2 \sum_{n=1}^{\infty} \frac{K_{1 - \epsilon} (\sqrt{q} n L)}{(\sqrt{q} n L)^{1 - \epsilon}} \right]. \quad (7.9)$$

Laurent expanding $E_0 (\epsilon; q)$ with respect to $\epsilon$ gives

$$E_0 (\epsilon; q) = \frac{1}{\epsilon} q L 8 \pi \left( 1 + \ln \frac{q}{4 \tilde{\mu}^2} \right) - \frac{q L}{8 \pi} \sum_{n=1}^{\infty} \frac{\sqrt{\pi} K_1 (\sqrt{q} n L)}{n \pi}$$

$$+ \epsilon \left\{ \frac{q L}{48 \pi} \left[ 3 \left( \ln \frac{4 \tilde{\mu}^2}{q} \right)^2 - 6 \ln \frac{4 \tilde{\mu}^2}{q} - 6 - \pi^2 \right] \right.$$  

$$+ \sum_{n=1}^{\infty} \frac{1}{\pi L n^2} \left[ K_0 (\sqrt{q} n L) - \sqrt{q} n L K_1 (\sqrt{q} n L) \left( \ln \frac{2 n L \tilde{\mu}^2}{\sqrt{q}} + \gamma - 2 \right) \right] \right\} + \cdots. \quad (7.10)$$

Taking $\epsilon = 0$ and dropping the divergent negative power term gives the regularized shifted global vacuum energy without the regularization parameter $\epsilon$,

$$E_0 (0; q) = - \frac{q L}{8 \pi} \left( 1 + \ln \frac{q}{4 \tilde{\mu}^2} \right) - \frac{q L}{8 \pi} \sum_{n=1}^{\infty} \frac{K_1 (\sqrt{q} n L)}{\sqrt{q} n L}. \quad (7.11)$$

The unshifted vacuum energy is then given by taking the limit $q \to 0$,

$$E_0 = - \sum_{n=1}^{\infty} \frac{1}{\pi L n^2} = - \frac{\pi}{6 L}. \quad (7.12)$$

By the way, by approximately replacing the sum $\sum_{n=1}^{\infty}$ with the integral $\int_1^{\infty} dn$, we can achieve an analytical expression of the shifted local regularized vacuum energy,

$$E (\epsilon; q; x, y)$$

$$\approx \frac{\tilde{\mu}^{2\epsilon} q^{1/2 - \epsilon}}{2 L} + \frac{\tilde{\mu}^{2\epsilon}}{\Gamma (-1/2 + \epsilon) 2^\epsilon \sqrt{\pi} L} \left\{ \frac{\Gamma (1 - \epsilon)}{(1 - 2\epsilon) 2^{1 - \epsilon}} \right.$$  

$$\times \left\{ 1 F_2 \left( \epsilon - 1/2; \epsilon, \epsilon + 1/2; q (y - x)^2 / 4 \right) \right.$$  

$$\left. + \frac{1}{\sqrt{q} (L - (y - x))^2} F_2 \left( \epsilon - 1/2; \epsilon, \epsilon + 1/2; q L^2 / (y - x)^2 / 4 \right) \right\} \right.$$  

$$- \frac{\gamma - 1}{2^{1 - \epsilon}} \left\{ \frac{\sqrt{q} (y - x)}{2} F_2 \left( \frac{1}{2} 1/2; 3/2, 2 - \epsilon; q (y - x)^2 / 4 \right) \right.$$  

$$+ \frac{\sqrt{q} (L - (y - x))}{2} F_2 \left( \frac{1}{2} 3/2, 2 - \epsilon; q L^2 / (y - x)^2 / 4 \right) \right\} \right\}. \quad (7.13)$$
The corresponding regularized vacuum energy is

\[
E_0 (0; q) \simeq - \frac{q L}{8 \pi} \left[ 1 + \ln \frac{q}{4 \mu^2} \right] - \sqrt{q} \frac{G_{3,0}^{1,3}}{4} \left( \frac{q L^2}{4} \right)^{1/2} \left[ 0, 0, \frac{1}{2}, -\frac{1}{2} \right],
\]

(7.14)

where \( G_{p,q}^{m,n} \left( a_1, \cdots a_n, a_{n+1}, \cdots, a_p \right) \) is Meijer’s G-function. Then, \( q \to 0 \) gives

\[
E_0 \simeq - \frac{1}{\pi L}.
\]

(7.15)

### 7.3 The counting function and the spectrum

The local counting function of a massless scalar field in \( \mathbb{R}^1 \) can be solved from eq. (2.25),

\[
N (\lambda; x, y) = \frac{1}{\pi} \sin \left( \sqrt{\lambda} |y - x| \right) / |y - x|.
\]

Then, the local counting function in \( S^1 \), a quotient space of \( \mathbb{R}^1 \), reads

\[
N (\lambda; x, y) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sin \left( \sqrt{\lambda} |y + nL - x| \right) / |y + nL - x|.
\]

(7.16)

The global counting function \( N (\lambda) \) is the trace of \( N (\lambda; x, y) \),

\[
N (\lambda) = tr N (\lambda; x, y) = 2k + 1, \quad \left( \frac{2k \pi}{L} \right)^2 < \lambda < \left[ \frac{2 (k + 1) \pi}{L} \right]^2, \quad k = 0, 1, 2, \cdots.
\]

(7.17)

From the counting function, we can achieve the eigenvalue spectrum of the operator \( D_x \):

\[
\lambda_0 = 0,
\]

\[
\lambda_{2k+1} = \lambda_{2k+2} = \left[ \frac{2 (k + 1) \pi}{L} \right]^2, \quad k = 0, 1, 2, \cdots.
\]

(7.18)

### 8. The Higgs model in a (1+1)-dimensional finite interval with the Dirichlet boundary condition: one-loop effective actions, vacuum energies, counting functions, and spectra

In the (1 + 1)-dimensional Higgs model \([50]\), we concern ourselves with the fluctuation \( \delta H (x^\mu) \) which is a linearized shift of a scalar field from the homogeneous stable solution. Here, the shift of a scalar field is defined as \( H (x^\mu) = \phi (x^\mu) - 1 \) and the action of the scalar field \( \phi (x^\mu) \) is

\[
S = \frac{m^2}{2} \int dx^2 \left[ \frac{1}{2} \left( \partial \phi \right)^2 - \frac{1}{2} \left( \phi^2 (x_0, x) - 1 \right)^2 \right].
\]

In this model, we consider the second-order fluctuation operator \( D_x = - \frac{d^2}{dx^2} + 4 \) in a finite interval \( I = [0, l] \), \( l = mL/\sqrt{2} \) with the Dirichlet boundary condition.
8.1 The one-loop effective action

The shifted local regularized one-loop effective action reads

\[ W(s; q; x, y) = -\mu^{2s} \Gamma(s) \frac{1}{4l} \sum_{n=-\infty}^{\infty} \frac{e^{i\pi n(x-y)/l} - e^{i\pi n(x+y)/l}}{(n^2 \pi^2/l^2 + 4 + q)^s}. \]  

(8.1)

The sum in eq. (8.1) can be exactly converted into an integral:

\[ W(s; q; x, y) = -\mu^{2s} \frac{2}{2^{3/2+s} \sqrt{\pi}} \int_{0}^{\infty} dt \left( \frac{\pi \sqrt{4 + q}}{l} \right)^{1/2-s} J_{-1/2+s} \left( \frac{1}{\pi} \sqrt{4 + qt} \right) \times \frac{(\cos \frac{x-y}{t} - \cos \frac{x+y}{t})}{(\cosh t - \cos \frac{x-y}{t})(\cosh t - \cos \frac{x+y}{t})}. \]  

(8.2)

The global regularized one-loop effective action is the trace of \( W(s; 0; x, y) \):

\[ W_s = TrW(s; 0; x, y) = \int_{0}^{l} dxW(s; 0; x, x) = -\frac{1}{2} \mu^{2s} \Gamma(s) \left( \frac{l}{\pi} \right)^{2s} Z \left( s; \frac{2l}{\pi} \right), \]  

(8.3)

where \( Z(s; a) \) is the Epstein-Hurwitz zeta function [13].

The regularized one-loop effective action without the regularization parameter \( s \) can be achieved by Laurent expanding \( W_s \) around \( s = 0 \),

\[ W_s = \frac{1}{4s} - \frac{1}{4} \gamma_E + l + \frac{1}{2} \ln \frac{2}{\mu (1 - e^{-4l})} \]

\[ + \frac{1}{4} \sum_{p=2}^{\infty} \sum_{\beta=0}^{\alpha} \left\{ \frac{1}{\alpha!} \Gamma(\alpha) \left( \frac{1}{2} \right)^{-\alpha} \right\} \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!(p - \alpha)!} \left( \ln \frac{\mu^{2}}{4} \right)^{p-\alpha} K_{\alpha}^{(4n) 1/2} \left( 4n \right), \]  

(8.4)

Then taking \( s = 0 \) and dropping the divergent negative power term gives

\[ W = -\frac{1}{4} \gamma_E + l + \frac{1}{2} \ln \left( \frac{1 - e^{-4l}}{2} \right) \tilde{\mu}. \]  

(8.5)

Moreover, besides the exact expression (8.2), we can also find an approximate solution but somewhat simple expression for \( W(s; q; x, y) \) and \( W_s \). Replacing approximately the sum \( \sum_{n=-\infty}^{\infty} \) by the integral \( \int_{-\infty}^{\infty} dn \) gives

\[ W(s; q; x, y) \simeq -\mu^{2s} \frac{2^{s}}{2^{s+1} \sqrt{2\pi}} \left[ \left( \frac{\sqrt{4 + q}}{|x - y|} \right)^{1/2-s} K_{1/2-s} \left( \sqrt{4 + q \, |x - y|} \right) - \left( \frac{\sqrt{4 + q}}{|x + y|} \right)^{1/2-s} K_{1/2-s} \left( \sqrt{4 + q \, |x + y|} \right) \right] \]  

(8.6)
and
\[ W_s \simeq \frac{\hat{\mu}^{2s}}{2} \frac{\Gamma(s)\Gamma^2(s)}{(1 - 2s)\pi^{2s}} \cdot 2F_1 \left( -\frac{1}{2} + s; s; \frac{1}{2} + s; -\frac{4l^2}{\pi^2} \right). \] (8.7)

Laurent expanding \( W_s \) around \( s = 0 \), taking \( s = 0 \) and dropping the divergent part gives
\[ W \simeq 1 - \frac{\gamma E}{2} + \frac{2l}{\pi} \tan^{-1} \left( \frac{2l}{\pi} \right) - \ln \frac{\sqrt{\pi^2/l^2 + 4}}{\hat{\mu}}. \] (8.8)

### 8.2 The vacuum energy

By a similar treatment, through solving eq. (2.23) with \( D_x = -\frac{d^2}{dx^2} + 4 \), we can obtain the shifted local regularized vacuum energy:
\[ E_0 (\epsilon; q; x, y) = \frac{\hat{\mu}^{2\epsilon}}{4l} \sum_{n=-\infty}^{\infty} \left[ e^{i\pi n(x-y)/l} - e^{i\pi n(x+y)/l} \right] \left[ \frac{n^2\pi^2}{l^2} + 4 + q \right]^{1/2-\epsilon}. \] (8.9)

In addition, the sum in eq. (8.9) can be exactly converted into an integral:
\[ E_0 (\epsilon; q; x, y) = \frac{\hat{\mu}^{2\epsilon}}{\Gamma(-1/2 + \epsilon)2^{1+\epsilon}\sqrt{\pi}} \times \int_0^\infty dt \left( \frac{\pi \sqrt{4 + q}}{t} \right)^{1-\epsilon} J_{-1+\epsilon} \left( \frac{l}{\pi} \sqrt{4 + qt} \right) \frac{\left( \cos \frac{x-y}{t} - \cos \frac{x+y}{t} \right) \sinh t}{\left( \cosh t - \cos \frac{x-y}{t} \right) \left( \cosh t - \cos \frac{x+y}{t} \right)}. \] (8.10)

The global regularized vacuum energy is the trace of \( E_0 (\epsilon; 0; x, y) \):
\[ E_0 (\epsilon) = \frac{\hat{\mu}^{2\epsilon}}{2} \left( \frac{\pi}{l} \right)^{1-2\epsilon} \cdot Z \left( \epsilon - \frac{1}{2}, \frac{2l}{\pi} \right). \] (8.11)

To remove the divergence, we Laurent expand \( E_0 (\epsilon) \),
\[ E_0 (\epsilon) = \frac{l}{2\pi\epsilon} - \frac{l}{2\pi} \left( \frac{\pi}{l} + \ln \frac{1}{\mu^2} + 1 \right) - \sum_{n=1}^{\infty} \frac{1}{n\pi} K_1 (4nl) + \epsilon \left\{ \frac{1}{2} \ln \frac{4}{\mu^2} + \frac{l}{2\pi} \left[ \frac{1}{2} \left( \ln \frac{1}{\mu^2} \right)^2 + \ln \frac{1}{\mu^2} - \frac{\pi^2}{6} - 1 \right] + \sum_{n=1}^{\infty} \left[ \frac{1}{4n^2\pi l} K_0 (4nl) + \frac{1}{n\pi} K_1 (4nl) \left( \ln \frac{1}{2nl\mu^2} - \gamma_E + 2 \right) \right] \right\} + \cdots. \] (8.12)

Taking \( \epsilon = 0 \) and dropping the divergent negative power term, we arrive at the regularized vacuum energy without the regularization parameter \( \epsilon \):
\[ E_0 = -\frac{l}{2\pi} \left( 1 + \frac{\pi}{l} + \ln \frac{1}{\mu^2} \right) - \sum_{n=1}^{\infty} \frac{1}{n\pi} K_1 (4nl). \] (8.13)

Besides, we can also approximately work out the summation in eq. (8.9) by replacing the sum \( \sum_{n=-\infty}^{\infty} \) with an integral \( \int_0^\infty \cdot dn \). This gives a somewhat simple expression for \( E_0 (q; x, y) \)
\[ E_0 (\epsilon; q; x, y) \simeq \frac{\hat{\mu}^{2\epsilon} (4 + q)^{1/2-\epsilon/2}}{2\sqrt{\pi} \Gamma (\epsilon - 1/2)} \left[ \frac{K_{1-\epsilon} (\sqrt{4 + q} \cdot |x - y|)}{|x - y|^{1-\epsilon}} - \frac{K_{1-\epsilon} (\sqrt{4 + q} \cdot |x + y|)}{|x + y|^{1-\epsilon}} \right], \] (8.14)
which is an asymptotic expression for eq. (8.10) for large $l$. The global regularized vacuum energy then reads

$$E_0 (\epsilon) \simeq \tilde{\mu}^2 \left[ \frac{l}{4\epsilon \sqrt{\pi}} \Gamma (-1+\epsilon) - \frac{1}{4^{1/2+\epsilon}} \right].$$

(8.15)

An approximate expression of the regularized vacuum energy without the regularization parameter $\epsilon$ reads

$$E_0 \simeq -\frac{l}{2\pi} \left[ \left( 1 + \frac{\pi}{l} + \ln \frac{1}{\tilde{\mu}^2} \right) - \frac{1}{4\pi} G^{3,0}_{1,3} \left( 4l^2 \left| \left( \frac{1}{2}, 0, -\frac{1}{2} \right) \right. \right. \right].$$

(8.16)

### 8.3 The counting function and the spectrum

The solution of eq. (2.24) gives the local counting function,

$$N (\lambda; x, y) = \frac{1}{l} \left[ e^{i\pi n(x-y)/l} - e^{i\pi n(x+y)/l} \right] \sum_{n=1}^{\infty} \theta \left( \lambda - \left( \frac{n^2 \pi^2}{l^2} + 4 \right) \right).$$

(8.17)

Taking trace gives the global counting function,

$$N (\lambda) = Tr N (\lambda; x, y) = \int_0^l dx \frac{1}{l} \left( 1 - e^{i2\pi nx/l} \right) \sum_{n=1}^{\infty} \theta \left( \lambda - \left( \frac{n^2 \pi^2}{l^2} + 4 \right) \right)

= \sum_{n=1}^{\infty} \theta \left( \lambda - \left( \frac{n^2 \pi^2}{l^2} + 4 \right) \right).$$

(8.18)

From the counting function, one can achieve the eigenvalue spectrum:

$$\lambda_n = \frac{n^2 \pi^2}{l^2} + 4, \quad n = 1, 2, \cdots.$$  

(8.19)

### 9. Conclusions

In this paper, we suggest an approach for calculating one-loop effective actions, vacuum energies, and spectral counting functions: constructing the equations for them so that they can be obtained by solving equations.

We solve some exact solutions for one-loop effective actions, vacuum energies, and spectral counting functions, including a free massive scalar field in $\mathbb{R}^n$, scalar fields in three-dimensional hyperbolic space $H_3$ and $H_3/Z$, a scalar field in $S^1$, and the Higgs model in a $(1+1)$-dimensional finite interval.

We construct the series expansion for local one-loop effective actions, vacuum energies, and spectral counting functions. The result can be used to find approximate solutions. In order to remove the divergence, renormalization procedures are used.

In our treatment, the physical quantities such as one-loop effective actions, vacuum energies, and spectral counting functions play the roles as spectral functions in spectral problems. This suggests us that the physical quantities like the one-loop effective actions and vacuum energies also can be used as tools in spectral problems.
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