\[ Z_c(4430) \text{ and } Z_c(4200) \text{ as triangle singularities} \]

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\[ Z_c(4430) \text{ discovered by the Belle and confirmed by the LHCb in } B^0 \to \psi(2S)K^-\pi^+ \text{ is generally considered to be a charged charmonium-like state that includes minimally two quarks and two antiquarks. } Z_c(4200) \text{ found in } B^0 \to J/\psi K^-\pi^+ \text{ by the Belle is also a good candidate of a charged charmonium-like state. We demonstrate that kinematical singularities in triangle loop diagrams induce a resonance-like behavior that can consistently explain the properties (mass, width, and Argand plot) of } Z_c(4430) \text{ and } Z_c(4200) \text{ from the experimental analyses. The triangle diagrams include only experimentally well-established hadrons. Applying this idea to } \Lambda_b^0 \to J/\psi p\pi^- \text{, we also identify triangle singularities that behave like } Z_c(4200), \text{ but no triangle diagram is available for } Z_c(4430). \text{ This is consistent with the LHCb’s finding that their description of the } \Lambda_b^0 \text{ contribution while } Z_c(4430) \text{ seems to hardly contribute. Even though the proposed mechanisms have uncertainty in the absolute strengths which are currently difficult to estimate, they are certainly a compelling alternative to tetraquark-based interpretations of } Z_c(4430) \text{ and } Z_c(4200). \]

Charged quarkonium-like states, so-called \( Z_c \) and \( Z_b [1] \), occupy a special position in the contemporary hadron spectroscopy. This is because, if they do exist, they clearly consist of at least four valence (anti)quarks, being different from the conventional \( q\bar{q} \) structure. The QCD phenomenology would become significantly richer by establishing their existence. Among \( \sim 10 \) of such states that have been claimed to exist as of 2018, we focus on \( Z_c(4430) \) and \( Z_c(4200) \).

\( Z_c(4430) \) was discovered by the Belle Collaboration as a bump in the \( \psi(2S)\pi^+ \) invariant mass distribution of \( B^0 \to \psi(2S)K^-\pi^+ \) [3]; charge conjugate modes are implicitly included throughout. Many theoretical interpretations of \( Z_c(4430) \) have been proposed: diquark-antidiquark, hadronic molecule, hadro-charmonium, hybrid, and kinematical cusp, as summarized in reviews [4–7]. The experimental determination of the spin-parity \( (J^P = 1^+ \) ruled out many of the scenarios [8–10]; in particular, the threshold cup has been eliminated. After the LHCb Collaboration found a resonance behavior in the \( Z_c(4430) \) Argand plot [9], a consensus is that \( Z_c(4430) \) is a genuine tetraquark state [10]. \( Z_c(4200) \) is also a good tetraquark candidate. It was observed by the Belle in \( B^0 \to J/\psi K^-\pi^+ \) [11]. The LHCb also found \( Z_c(4200) \)-like contributions in \( B^0 \to J/\psi K^-\pi^+ \) [12] and \( \Lambda_b^0 \to J/\psi p\pi^- \) [13].

Meanwhile, triangle singularities (TS) [14–16] have been considered to interpret several resonance(-like) states such as the hidden charm pentaquark \( P_c(4450)^+ \) [17–19] and \( a_{1}(1420) \) [20, 21]. The TS is a kinematical effect that arises in a triangle diagram like Fig. 1 when a special kinematical condition is reached: three intermediate particles are allowed to be on-shell at the same time. A mathematical detail how the singularity shows up is well illustrated in Ref. [19]. Although it was claimed in Ref. [22, 23] that a kinematical effect from a triangle diagram can induce a spectrum bump of \( Z_c(4430) \), this effect has nothing to do with the above TS and relies on the existence of an experimentally unobserved hadron.

In this work, we give a new insight into \( Z_c(4430) \) and \( Z_c(4200) \) by showing that these exotic candidates can be consistently interpreted as TS if the TS have absolute strengths detectable in the experiments. First we point out that triangle diagrams in Fig. 2 formed by experimentally well-established hadrons, meet the kinematical condition to cause the TS (in the zero width limit of unstable particles). Then we demonstrate that the diagram of Fig. 2a [Fig. 2b, c] creates a \( Z_c(4430) [Z_c(4200)] \)-like bump in the \( \psi J^P \) (\( \psi J^P = J/\psi, \psi(2S) \) invariant mass distribution of \( B^0 \to \psi(2S)K^-\pi^+ \) [\( B^0 \to J/\psi K^-\pi^+ \) and \( \Lambda_b^0 \to J/\psi p\pi^- \)]. The Breit-Wigner masses and widths extracted from the spectra turn out to be in very good agreement with those of \( Z_c(4430) \) and \( Z_c(4200) \). The \( Z_c(4430) \) Argand plot from the LHCb [9] is also well reproduced by the triangle diagram. Finally, we give a natural explanation for the absence of \( Z_c(4430) \) in \( \Lambda_b^0 \to J/\psi p\pi^- \) and \( e^+e^- \) annihilations in terms of the TS.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{triangle_diagram.png}
\caption{Triangle diagram. Particle labels and their momenta (in parentheses) are defined.}
\end{figure}
where the particles 1 and 2 for which $\Gamma_j$ is important to consider the vector charmonium width in Fig. 2(a) where $\psi(4260)$ and $K^*(892)$ have comparable widths. We take the mass and width values from Ref. [2]. An exception is applied to unstable intermediate particles and their momenta in Fig. 1 to generally express the triangle diagrams of Fig. 2. Let us use labeling of $\psi(4260)$ and $K^*(892)$ for $1$ and $2$ for which $\Gamma_j$ is the width. It is particularly important to consider the vector charmonium width in Fig. 2(a) where $\psi(4260)$ and $K^*(892)$ have comparable widths. We take the mass and width values from Ref. [2].

Regarding the $23 \to ab$ interaction $\nu_{ab;23}$ in Eq. (1), the particles $2$ and $a$ are vector charmoniums while $3$ and $b$ are pions, we use an $s$-wave interaction:

$$\nu_{ab;23}(p_a, p_b; p_2, p_3) = f_{ij}^{01}(p_{ab}) f_{23}^{01}(p_{23}) e_a \cdot e_2,$$  

where $e_a$ and $e_2$ are polarization vectors for the particles $a$ and $2$, respectively. The form factors $f_{ij}^{01}(p_{ab})$ and $f_{23}^{01}(p_{23})$ will be defined in Eq. (4); the momentum of the particle $i$ in the $ij$-CM frame is denoted by $p_{ij}$ and $p_{ij} = |p_{ij}|$. An $s$-wave pair of $\psi_f \pi$ coming out from this interaction has $J^P = 1^+$, which is consistent with the experimentally determined spin-parity of $Z_c(4430)$ and $Z_c(4200)$, and also with the insignificant $d$-wave contribution in the $Z_c(4430)$-region [3].

The $R \to ij$ decay vertex $\Gamma_{ij,R}$ in Eq. (1) is explicitly given as

$$\Gamma_{ij,R}(p_i, p_j; p_R) = \sum_{LS} f_{ij}^{LS}(p_{ij})(s_i s_j |S| S^2) \times (LMSS^* S_R S_R^* Y_{LM}(\hat{p}_{ij})),$$  

where $Y_{LM}$ is spherical harmonics. Clebsch-Gordan coefficients are written as $\langle ab | ef \rangle$, and the spin and its $z$-component of a particle $x$ are denoted by $s_x$ and $s^z_x$, respectively. The form factor $f_{ij}^{LS}(p_{ij})$ is parameterized as

$$f_{ij}^{LS}(p) = g_{ij}^{LS} \frac{p^L}{E_i(p) E_j(p)} \left( \frac{\Lambda^2}{\Lambda^2 + p^2} \right)^{1+(L/2)},$$  

where we use the cutoff $\Lambda = 1$ GeV throughout; main conclusions in this work are essentially determined by the kinematical singularities and are robust in a reasonable cutoff range: $\Lambda = 0.7 - 1.3$ GeV. For the $1 \to 3c$ and $23 \to ab$ interactions, there is only one available set of $\{L, S\}$ for which we set $g_{ij}^{LS} = 1$. Regarding the $H \to 12$ decay, meanwhile, several sets of $\{L, S\}$ are available. The $H \to 12$ decay vertices are currently unknown but details would not change the main conclusions. Thus we assume simple structures and detectable strengths. For the $B^0$ decays, we set $g_{ij}^{LS} = 1$ only for $S = |s_1 - s_2|$ and the lowest allowed $L$: $g_{ij}^{LS} = 0$ for the other $\{L, S\}$. Because of using the above $\nu_{ab;23}$, the $B^0$ decays are necessarily parity-violating. For the $\Lambda_b^0$ decays, on the other hand, both parity-conserving and -violating interactions are possible. We choose the parity-conserving one and set $g_{ij}^{LS} = 1$ only for $S = |s_1 - s_2|$ and the lowest allowed $L$: $g_{ij}^{LS} = 0$ otherwise.

We present the $\psi_f \pi$ invariant mass distributions for $\bar{B}^0 \to \psi(2S)K^- \pi^+$ and $\bar{B}^0 \to J/\psi K^- \pi^+$. The red solid curves in Figs. 3(a) and (b) are solely from the triangle diagrams of Figs. 2(a) and (b), respectively. For comparison, we also plot the phase-space distributions by the black dotted curves. A clear resonance-like peak appears at $m_{\psi(2S)\pi} \sim 4.45$ GeV in panel (a) ($m_{J/\psi \pi} \sim 4.2$ GeV in panel (b)) due to the TS. We also calculated the $m_{J/\psi \pi}$ spectrum for $\bar{B}^0 \to J/\psi K^- \pi^+$ from the triangle diagram of Fig. 3(a), and obtained a result very similar to Fig. 3(a) after the normalization explained in the caption.

We interpret the peaks associated with the TS in terms of $Z_c$-excitation mechanisms. We fit the Dalitz plot distributions from the triangle diagrams of Figs. 2(a) and (b) using the mechanism of $\bar{B}^0 \to Z_c K^-$ followed by $Z_c \to \psi_f \pi^+$; the $Z_c$ propagation is expressed by the Breit-Wigner form used in Ref. [3]. In the fit, we include the kinematical region where the magnitude of the Dalitz plot distribution is larger than 20% of the peak.
FIG. 3. Distributions of the $\psi_f \pi$ ($\psi_f = J/\psi, \psi(2S)$) invariant mass for $B^0 \to \psi(2S)K^-\pi^+$ (a), $B^0 \to J/\psi K^-\pi^+$ (b), and $\Lambda_b \to J/\psi \pi^-\pi^+$ (c). The red solid curves in panels (a) and (b) are obtained from triangle diagrams Fig. 2(a) and (b), respectively. The blue dash-dotted curves are from Breit-Wigner amplitudes fitted to the red solid curves. In panel (c), the red solid, green dashed, and magenta dash-two-dotted curves are obtained from Fig. 2(c) with $N^0 = N(1440) \frac{1}{2}^+$, $N(1520) \frac{3}{2}^-$, and $N(1680) \frac{5}{2}^+$, respectively. The dotted curves are the phase-space distributions. Each curve, except for the blue dash-dotted, is normalized to give unity when integrated with respect to $m_{\psi_f \pi}$.

height. The obtained fits of reasonable quality are shown by the blue dash-dotted curves in Figs. 3(a) and (b). Because the spectrum shape from the triangle diagrams is somewhat different from the Breit-Wigner, their peak positions are slightly different. The Breit-Wigner parameters resulting from the fits are given in Table 1 along with those from experimental data. Their agreement is remarkable.

Next we confront the triangle amplitude with the $Z_c(4430)$ Argand plot from the LHCb [9]. Because $Z_c$ and $K^−$ are relatively in p-wave, the angle-independent part of the amplitude (A) to be compared with the Argand plot is

$$A(m_{ab}) = c_{bg} + c_{norm} \int d\Omega_{p_c} Y_{s_{Z_c}}^* (\hat{p}_c) M_{abc,H}, \quad (5)$$

where $s_{Z_c}$ is the z-component of the $Z_c$ spin and $m_{ab}$ the $ab$ invariant mass. The invariant amplitude $M_{abc,H}$ is related to $T_{abc,H}$ of Eq. (11) through Eq. (B3) of Ref. [24]. Complex constants $c_{norm}$ and $c_{bg}$ are adjusted to fit the empirical Argand plot; $c_{bg}$ represents a background.

In the LHCb analysis, a complex value representing the $Z_c(4430)$ amplitude is fitted to dataset in a $m_{\psi(2S) \pi}$ bin with a bin size $\Delta$. To take account of the bin size, we simply average our amplitude without pursuing a theoretical rigor:

$$\bar{A}(m_{ab}^2(i)) \equiv \frac{1}{\sqrt{\Delta}} \int_{m_{ab}^2(i) - \Delta/2}^{m_{ab}^2(i) + \Delta/2} A(m_{ab}^2) dm_{ab}^2, \quad (6)$$

where $m_{ab}^2(i)$ is the central value of an i-th bin. As shown in Fig. [4] the empirical $Z_c(4430)$ Argand plot is fitted well with $A(m_{ab}^2(i))$ from the triangle diagram of Fig. 2(a); $c_{norm} = -0.16 - 0.79i, c_{bg} = 0.16 + 0.02i$ in Eq. (5). This demonstrates that the counterclockwise behavior found in Ref. [3] does not necessarily indicate the existence of a resonance state. A similar statement has also been made in Ref. [17]. We also confirmed a counterclockwise behavior of the Argand plot from the triangle diagram of Fig. 2(b), as the Belle [11] found the $Z_c(4200)$ amplitude to behave so.

A puzzle about $Z_c(4430)$ is its large branching to $\psi(2S)\pi$ compared with $J/\psi\pi$: $R_{Z_c(4430)}^{exp} \equiv B[Z_c^+(4430) \to \psi(2S)\pi^+] / B[Z_c^+(4430) \to J/\psi\pi^+] \sim 11 \pm 8 \pm 11$. This can be qualitatively understood if $Z_c(4430)$ is due to the TS, and the coupling strength ratio ($c_\psi^R$) of $\psi(260)\pi^+\to \psi(2S)\pi^+$ to $\psi(260)\pi^+\to J/\psi\pi^+$ interactions of Eq. (2) is fixed by $R_{\psi(260)}^{exp} \equiv B[\psi(260) \to \psi(2S)\pi^+\pi^-] / B[\psi(260) \to J/\psi\pi^+\pi^-] \sim 0.1 - 0.5$. Because of the large difference in the phase-space available to the final states, $R_{\psi(260)}^{model} \sim 0.2 \times c_\psi^R$ is obtained by using Eq. (2). In addition, the larger phase-space allows resonance(-like) $f_0(980)$ [26] and $Z_c(3900)$ [27] to contribute to $B[\psi(260) \to J/\psi\pi^+\pi^-]$.  

| $Z_c(4430)$ | $Z_c(4200)$ |
|-------------|-------------|
| $Z_c(4430)$ | $Z_c(4200)$ |
| (a) Belle [8] | LHCb [9] | (b) Belle [11] |
| $M_{BW}$ | 4469 | 4485 $\pm 22_{-11}^{+28}$ | 4475 $\pm 7_{-25}^{+15}$ | 4307 | 4196 $\pm 41_{-29}^{+17}$ |
| $\Gamma_{BW}$ | 190 $\pm 46_{-35}^{+26}$ | 172 $\pm 13_{-34}^{+37}$ | 307 | 370 $\pm 70_{-70}^{+132}$ |

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of 18 GeV are from fitting data in six bins equally-separating the range from 12 GeV to 21 GeV. Six data points from Ref. [9] are from fitting data in six bins equally-separating the range from 12 GeV to 21 GeV. A curved segment and a data point of the same color belong to the same bin. A solid circle is an average of the curved segment of the same color. See Eq. (6) for averaging.

by $\sim 40\%$, and thus $R_{\psi(4260)}^{\text{model}} \sim 0.1 \times |c_{\psi\pi}|^2$. Therefore, the model reproduces $R_{\psi(4260)}^{\exp} \sim 0.3$ with $|c_{\psi\pi}| \sim 1.7$, and the puzzling $R_{Z_c(4430)}^{\exp} \sim 11$ is also reproduced with the same $|c_{\psi\pi}|$.

Now we discuss the $J/\psi\pi$ invariant mass distributions for $\Lambda_b^0 \rightarrow J/\psi p\pi^-$. The model reproduces $R_{\psi(4260)}^{\text{exp}}$ and the puzzling $R_{Z_c(4430)}^{\exp}$ is not. In future, we look for more TS that would be responsible for the other $Z_c$ and other seemingly exotic hadrons.

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References:

[1] We follow Ref. [2] on the particle notations.

[2] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).

[3] S.K. Choi et al. (Belle Collaboration), Phys. Rev. Lett. 100, 142001 (2008).

[4] A. Hosaka, T. Iijima, K. Miyabayashi, Y. Sakai, and S. Yasui, PTEP 2016, 062C01 (2016).

[5] H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Phys. Rept. 639, 1 (2016).

[6] R.F. Lebed, R.E. Mitchell, and E.S. Swanson, Prog. Part.

FIG. 4. $Z_c(4430)$ Argand plot. Six curved segments are from triangle diagram Fig. 2(a). Six data points from Ref. [9] are from fitting data in six bins equally-separating the range from 12 GeV to 21 GeV. A curved segment and a data point of the same color belong to the same bin. A solid circle is an average of the curved segment of the same color. See Eq. (6) for averaging.

Another important finding in the LHCb analysis [13] is that $Z_c(4430)$ seems to hardly contribute to $\Lambda_b^0 \rightarrow J/\psi p\pi^-$. If $Z_c(4430)$ found in $B^0 \rightarrow \psi(2S)K^-\pi^+$ is associated with the TS, a natural explanation follows: within experimentally observed hadrons, no combination of a charmonium and a nucleon resonance is available to form a triangle diagram like Fig. 2(c) that causes TS at the $Z_c(4430)$ position. This idea can be further generalized. At present, a puzzling situation about $Z_c$ is that those observed in $e^+e^-$ annihilations and in $B$ decays are mutually exclusive. If the $Z_c$ states are due to TS, the answer is simple: a TS in a $B$ decay does not exist or is highly suppressed in $e^+e^-$ annihilations, and vice versa. Therefore, a key to establishing a genuine tetraquark state is to identify it in different processes including different initial states. However, there are still cases where, as we have seen in Figs. 2(b) and (c), different TS could induce similar resonance-like behaviors.

In summary, we demonstrated that $Z_c(4430)$ and $Z_c(4200)$, which are often regarded as genuine tetraquark states, can be consistently interpreted as singularities from the triangle diagrams we identified. The Breit-Wigner parameters extracted from the TS-induced spectrum bumps of $B^0 \rightarrow \psi(2S)K^-\pi^+$ are in very good agreement with those of $Z_c(4430)$ and $Z_c(4200)$ from the Belle and LHCb analyses. The $Z_c(4430)$ Argand plot from the LHCb is also well reproduced. We also explained in terms of TS why $Z_c(4200)$-like contribution was observed in $\Lambda_b^0 \rightarrow J/\psi p\pi^-$ but $Z_c(4430)$ was not. In future, we look for more TS that would be responsible for the other $Z_c$ and other seemingly exotic hadrons.

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[7] R.M. Albuquerque, J.M. Dias, K.P. Khemchandani, A. Martinez Torres, F.S. Navarra, M. Nielsen, and C.M. Zanetti. [arXiv:1812.08207] [hep-ph].

[8] K. Chilikin et al. (Belle Collaboration), Phys. Rev. D 88, 074026 (2013).

[9] R. Aaij et al. (LHCb Collaboration), Phys. Rev. Lett. 112, 222002 (2014).

[10] https://physics.aps.org/synopsis-for/10.1103/PhysRevLett.112.222002

[11] K. Chilikin et al. (Belle Collaboration), Phys. Rev. D 90, 112009 (2014).

[12] R. Aaij et al. (LHCb collaboration), [arXiv:1901.05745]

[13] R. Aaij et al. (LHCb Collaboration), Phys. Rev. Lett. 117, 082003 (2016).

[14] L.D. Landau, Nucl. Phys. 13, 181 (1959).

[15] S. Coleman and R.E. Norton, Nuovo Cim. 38, 438 (1965).

[16] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, The Analytic S-Matrix, Cambridge University Press, Cambridge, 1966

[17] F.-K. Guo, U.-G. Meißner, W. Wang, and Z. Yang, Phys. Rev. D 92, 071502 (2015).

[18] X.-H. Liu, Q. Wang, and Q. Zhao, Phys. Lett. B757, 231 (2016).

[19] M. Bayar, F. Aceti, F.-K. Guo, and E. Oset, Phys. Rev. D 94, 074039 (2016).

[20] M. Mikhasenko, B. Ketzer, and A. Sarantsev, Phys. Rev. D 91, 094015 (2015).

[21] F.-K. Guo, U.-G. Meißner, W. Wang, and Z. Yang, Phys. Rev. D 92, 071502 (2015).

[22] P. Pakhlov, Phys. Lett. B702, 139 (2011).

[23] P. Pakhlov and T. Uglov, Phys. Lett. B748, 183 (2015).

[24] H. Kamano, S.X. Nakamura, T.-S.H. Lee, and T. Sato, Phys. Rev. D 84, 114019 (2011).

[25] J. Zhang and L. Yuan, Eur. Phys. J. C 77, 727 (2017).

[26] J.P. Lees et al. (BaBar Collaboration), Phys. Rev. D 86, 051102 (2012).

[27] M. Ablikim et al. (BESIII Collaboration), Phys. Rev. Lett. 110, 252001 (2013).