On weak uniqueness for some degenerate SDEs by global $L^p$ estimates

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Abstract

We prove uniqueness in law for possibly degenerate SDEs having a linear part in the drift term. Diffusion coefficients corresponding to non-degenerate directions of the noise are assumed to be continuous. When the diffusion part is constant we recover the classical degenerate Ornstein-Uhlenbeck process which only has to satisfy the Hörmander hypoellipticity condition. In the proof we also use global $L^p$-estimates for hypoelliptic Ornstein-Uhlenbeck operators recently proved in Bramanti-Cupini-Lanconelli-Priola (Math. Z. 266 (2010)) and adapt the localization procedure introduced by Stroock and Varadhan. Appendix contains a quite general localization principle for martingale problems.

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1 Introduction and notation

In this paper we prove existence and weak uniqueness (or uniqueness in law) for possibly degenerate SDEs like

$$dZ_t = AZ_t dt + b(Z_t) dt + B(Z_t) dW_t, \quad t \geq 0, \ Z_0 = z_0 \in \mathbb{R}^d,$$

(1)

where $A$ is a $d \times d$ real matrix, $W = (W_t)$ is a standard $r$-dimensional Wiener process, $r \geq 1$, $B(z) = \begin{pmatrix} B_0(z) \\ 0 \end{pmatrix}$, with $B_0(z) \in \mathbb{R}^{d_0} \otimes \mathbb{R}^r$ (i.e., $B_0(z)$ is a real $d_0 \times r$-matrix, for any $z \in \mathbb{R}^d$), $1 \leq d_0 \leq d$, and $B(z) \in \mathbb{R}^d \otimes \mathbb{R}^r$, $z \in \mathbb{R}^d$. Moreover, we suppose that

$$b(z) = \begin{pmatrix} b_0(z) \\ 0 \end{pmatrix},$$

where $b_0 : \mathbb{R}^d \to \mathbb{R}^{d_0}$ ($\mathbb{R}^{d_0} \simeq \mathbb{R}^{d_0} \otimes \mathbb{R}$) is a Borel and locally bounded function.
Writing \( z \in \mathbb{R}^d \) in the form \( z = \begin{pmatrix} x \\ y \end{pmatrix} \simeq (x, y) \in \mathbb{R}^d \), with \( x \in \mathbb{R}^{d_0} \) and \( y \in \mathbb{R}^{d_1} \) (if \( d_1 = d - d_0 = 0 \) then \( z = x \)) and, similarly, \( Z_t = (X_t, Y_t) \), we may rewrite (1) as

\[
\begin{align*}
\frac{dX_t}{dY_t} &= A \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} b_0(X_t, Y_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} B_0(X_t, Y_t) \\ 0 \end{pmatrix} dW_t,
\end{align*}
\]

(2)

\( t \geq 0, (X_0, Y_0) = z_0 = (x_0, y_0) \in \mathbb{R}^d \). We assume that \( B_0 \) is continuous from \( \mathbb{R}^d \) into \( \mathbb{R}^{d_0} \otimes \mathbb{R}^r \) and also that the \( d_0 \times d_0 \) symmetric matrix \( Q_0(z) = B_0(z)B_0(z)^* \) (here \( B_0(z)^* \) denotes the adjoint matrix of \( B_0(z) \)) is uniformly positive definite (see Hypothesis \( \textbf{H} \) for more details).

Moreover, for any \( z_0 \in \mathbb{R}^d \), the Ornstein-Uhlenbeck process \( dZ_t = AZ_t dt + B(z_0)dW_t \) must satisfy a hypoellipticity type condition (see (ii) in Hypothesis \( \textbf{H} \)). Finally, we suppose that there exists a smooth Lyapunov function \( \phi : \mathbb{R}^d \to \mathbb{R}_+ \) which controls the growth of coefficients (cf. Chapter 10 in \( \textbf{[30]} \)). In the standard case of \( \phi(z) = 1 + |z|^2 = 1 + |x|^2 + |y|^2 \) (\( | \cdot | \) denotes the euclidean norm) this means that there exists \( C > 0 \) such that

\[
\text{Tr}(Q_0(x, y)) + 2 \langle A(x, y), (x, y) \rangle + 2|b_0(x, y, x)|_{\mathbb{R}^{d_0}} \leq C(1 + |x|^2), \ z = (x, y) \in \mathbb{R}^d
\]

(3)

(here \( \text{Tr} \) denotes the trace and \( \langle \cdot, \cdot \rangle \) the inner product).

Solutions to equation (2) appear as a natural generalization of OU processes. On the other hand degenerate Kolmogorov operators \( L \) associated to (2) (see \( \textbf{[8]} \)) arise in Kinetic Theory (see \( \textbf{[10]} \) and the references therein) and in Mathematical Finance (see the survey paper \( \textbf{[22]} \)). In addition diffusion processes like \( (Z_t) \) appear in stochastic motion of particles according to the Newton law (see, for instance, \( \textbf{[14]} \)).

If \( d = d_0 \), i.e., we are in the case of a non-degenerate diffusion, weak uniqueness (or uniqueness in law) has been proved in \( \textbf{[29]} \) even in the case of time dependent coefficients (see \( \textbf{[19]} \) for a different proof of uniqueness when the coefficients are independent of time). This has been done by introducing the important localization principle. It states that uniqueness is a local result in that it suffices to show that each starting point has a neighbourhood on which the coefficients of our SDE equal other coefficients for which uniqueness holds (cf. Theorem 6.6.1 in \( \textbf{[30]} \)). This principle combined with global \( L^p \)-estimates for heat equations has been used in \( \textbf{[29]} \) to prove the uniqueness result.

The results in \( \textbf{[29]} \) have been generalized in several papers about non-degenerate diffusions (see \( \textbf{[2, 20]} \) and the references therein) by allowing some discontinuous coefficients \( B_0(z) \) (see \( \textbf{[27]} \) for a counterexample to uniqueness with \( d \geq 3 \) and \( B_0(z) \) measurable).

Weak uniqueness results are also available for some degenerate SDEs with non locally Lipschitz coefficients (see \( \textbf{[1, 3, 6, 13, 23, 25]} \)). Such results do not cover equations like (2) under our assumptions. In particular related degenerate SDEs with \( d_0 < d \) are considered in \( \textbf{[6, 27]} \). In \( \textbf{[25]} \) the non-degenerate diffusion part has bounded Hölder continuous coefficients but it is not assumed that the drift term has a linear part like \( AZ_t dt \). In \( \textbf{[6]} \) degenerate SDEs with time-dependent coefficients which growth at most linearly are considered; these equations have a linear part in the drift which has to satisfy a lower-diagonal block form.

To explain better our assumptions let us consider the following three-dimensional example

\[
\begin{cases}
\begin{align*}
\frac{dx_t}{dt} &= (-x_t^3 + \frac{y_t}{|y_t|}) dt + a(x_t, y_t, z_t) dW_t \\
\frac{dy_t}{dt} &= (x_t + y_t) dt \\
\frac{dz_t}{dt} &= (y_t + z_t) dt,
\end{align*}
\end{cases}
\]

(4)
where \((x_t, y_t, z_t) \in \mathbb{R}^3, (x_0, y_0, z_0) = \xi\). Here \(W = (W_t)\) is a one-dimensional Wiener process. Thus \(d_0 = 1, b_0(x, y, z) = -x^3 + \frac{y}{|y|}\), \(A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\) and we can assume that \(a\) is continuous and bounded and that \(a^2\) is uniformly positive on \(\mathbb{R}^3\). The associated degenerate Kolmogorov operator is

\[
\mathcal{L} = \frac{1}{2} a^2(x, y, z) \partial_{xx}^2 - x^3 \partial_x + \frac{y}{|y|} \partial_x + (x + y)\partial_y + (y + z) \partial_z.
\]

We will prove well-posedness for \((1)\) or, equivalently, well-posedness of the martingale problem for \(\mathcal{L}\) starting from any initial distribution on \(\mathbb{R}^3\). Note that this implies the Markov property for the diffusion process.

To establish our main result on well-posedness (see Theorem 15) we first prove in Section A.3 of appendix a variant of the localization principle of Stroock and Varadhan (see, in particular, Theorems 23 and 27 and Lemma 24 which provide extensions of some related results in Chapter 4 of \([12]\)). We cannot apply directly the localization principle (see, in particular, Theorems 23 and 27 and Lemma 24 which provide extensions of some related results in Chapter 4 of \([12]\)). We cannot apply directly the localization principle (see, in particular, Theorems 23 and 27 and Lemma 24 which provide extensions of some related results in Chapter 4 of \([12]\)).

The regularity results in \([5]\) are proved using that \(L\) with respect to the Lebesgue measure recently proved in \([5]\) (see, in particular, Theorem 8).

The plan of the paper is as follows. In Section 2 we start with basic definitions and preliminary results about well-posedness of \((2)\) (see in particular Theorem 6). In Section 3 we prove a uniqueness result for \((2)\) assuming an additional hypothesis on the coefficients \((\text{see } (22))\). In that section we also prove some necessary analytic results for OU hypoelliptic operators \(\mathcal{L}_0\). The complete uniqueness result is proved in Section 4 where we remove the additional hypothesis using the localization procedure. Finally Appendix contains a quite general localization principle for martingale problems.

We collect our assumptions on SDE \((2)\). Recall that \((e_i)_{i=1,\ldots,d}\) denotes the canonical basis on \(\mathbb{R}^d\). Moreover, \(\langle \cdot, \cdot \rangle\) indicates the inner product in any \(\mathbb{R}^n, n \geq 1\), and \(|\cdot|\) denotes the euclidean norm in \(\mathbb{R}^n\).

**Hypothesis 1**

(i) The symmetric \(d_0 \times d_0\) matrix \(Q_0(z) = B_0(z)B_0(z)^*\) is positive definite and there exists \(\eta > 0\) such that

\[
\langle Q_0(z)h, h \rangle \geq \eta |h|^2, \quad h \in \mathbb{R}^{d_0}, \quad z \in \mathbb{R}^d.
\]  

(ii) There exists a non-negative integer \(k\), such that the vectors

\[
\{e_1, \ldots, e_{d_0}, Ae_1, \ldots, Ae_{d_0}, \ldots, A^k e_1, \ldots, A^k e_{d_0}\} \text{ generate } \mathbb{R}^d;
\]

we denote by \(k\) the *smallest* non-negative integer such that \((5)\) holds (one has \(0 \leq k \leq d - 1\)).

(iii) \(b_0 : \mathbb{R}^d \to \mathbb{R}^{d_0}\) is Borel and locally bounded; \(B_0 : \mathbb{R}^d \to \mathbb{R}^{d_0} \otimes \mathbb{R}^r\) is continuous.

(iv) There exists a smooth Lyapunov function \(\phi\) for \((2)\), i.e., there exists a \(C^2\)-function \(\phi : \mathbb{R}^d \to (0, +\infty)\) such that \(\phi \to +\infty\) as \(|z| \to +\infty\) and

\[
\mathcal{L}\phi(z) \leq C\phi(z), \quad z \in \mathbb{R}^d,
\]
for some $C > 0$; $\mathcal{L}$ is the possibly degenerate Kolmogorov operator related to $\mathbb{P}$,

$$
\mathcal{L}f(z) = \frac{1}{2} \text{Tr}(Q_0(z)D^2f(z)) + \langle Az, Df(z) \rangle 
+ \langle b_0(z), D_xf(z) \rangle, \quad f \in C^2_b(\mathbb{R}^d), \ z \in \mathbb{R}^d,
$$

where $Df(z) = (D_xf(z), D_yf(z)) \in \mathbb{R}^d$ indicates the gradient of $f$ in $z$ and

$$
D^2f(z) = \begin{pmatrix}
D^2_{xx}f(z) & D^2_{xy}f(z) \\
D^2_{yx}f(z) & D^2_{yy}f(z)
\end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^d
$$

denotes the Hessian matrix of $f$ in $z$.

Note that $d_1 = 0$ if and only if $k = 0$. In this case $d = d_0$ and we have a non-degenerate SDEs with $B(z) = B_0(z)$ for which weak uniqueness is already known (see [30] in the example [1] we have $k = 2$.

By the Hörmander condition on commutators, (6) is equivalent to the hypoellipticity of the operator $\mathcal{L}_0 - \partial_t$ in $(d + 1)$ variables $(t, z_1, \ldots, z_d)$; here $\mathcal{L}_0$ is the OU operator

$$
\mathcal{L}_0 u(z) = \frac{1}{2} \sum_{i,j=1}^{d_0} q_{ij} \partial^2_{x_i x_j} u(z) + \sum_{i,j=1}^{d} a_{ij} z_j \partial_{z_i} u(z), \ z \in \mathbb{R}^d,
$$

where $Q_0 = (q_{ij})_{i,j=1,\ldots,d_0}$ is symmetric and positive definite on $\mathbb{R}^{d_0}$ and the $a_{ij}$ are the components of the $d \times d$-matrix $A$; further $\partial_{x_i}$ and $\partial^2_{x_i x_j}$ denote partial derivatives.

It is also well-known (see Section 1.3 in [31]) that (6) is equivalent to the fact that the symmetric $d \times d$ matrix

$$
Q_t = \int_0^t e^{sA} Q e^{sA^*} ds \text{ is positive definite for all } t > 0, \quad \text{with } Q = \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix};
$$

here $e^{sA}$ denotes the exponential matrix of $A$.

We will use the letter $c$ or $C$ with subscripts for finite positive constants whose precise value is unimportant.

For a matrix $B \in \mathbb{R}^r \otimes \mathbb{R}^d$, $r \geq 1$, $d \geq 1$, $\|B\|$ denotes its Hilbert-Schmidt norm.

The space $B_b(\mathbb{R}^d)$ denotes the Banach space of all real bounded and Borel functions $f : \mathbb{R}^d \to \mathbb{R}$ endowed with the supremum norm $\| \cdot \|_\infty$; its subspace of all continuous functions is indicated by $C_b(\mathbb{R}^d)$. Moreover $C^2_K = C^2_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the space of functions of class $C^2$ with compact support and similarly $C^\infty_K(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the space of functions of class $C^\infty$ with compact support. In addition we consider the space $C^2_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ consisting of all functions of class $C^2$ having first and second partial derivatives which are bounded on $\mathbb{R}^d$.

We also consider standard $L^p$-spaces $L^p(\mathbb{R}^d)$ with respect to the Lebesgue measure and indicate by $\| \cdot \|_p$ (or $\| \cdot \|_{L^p}$) the usual $L^p$-norm, $p \geq 1$. For measurable matrix-valued functions $u : \mathbb{R}^d \to \mathbb{R}^r \otimes \mathbb{R}^d$ we also consider $\|u\|_p = (\int_{\mathbb{R}^d} \|u(z)\|^p dz)^{1/p}$.

Finally by $\mathcal{P}(\mathbb{R}^d)$ we denote the set of all Borel probability measures on $\mathbb{R}^d$. A probability space will be indicated with $(\Omega, \mathcal{F}, P)$ and $E$ (or $E^P$) will denote expectation with respect to $P$. 

2 Basic definitions and preliminary results

Our definitions will mainly follow Chapter 4 in [15] (see also [12, 30]). Let us consider the SDE

\[ Z_t = Z_0 + \int_0^t b(Z_s)ds + \int_0^t B(Z_s)dW_s, \quad t \geq 0. \]  
(11)

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( B : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r \) are Borel and locally bounded functions and \( W = (W_t) \) denotes a \( r \)-dimensional Wiener process.

The corresponding Kolmogorov operator (generator) is

\[ \tilde{\mathcal{L}} f(z) = \frac{1}{2} \text{Tr} (B(z)B^*(z)D^2 f(z)) + \langle b(z), Df(z) \rangle, \quad f \in C_0^2(\mathbb{R}^d), \ z \in \mathbb{R}^d. \]

Let \( \mu \in \mathcal{P}(\mathbb{R}^d) \). Let us recall two related notion of solutions.

A weak solution \( Z = (Z_t) = (Z_t)_{t \geq 0} \) with initial condition \( \mu \) is a continuous \( d \)-dimensional process (i.e., it has continuous paths with values in \( \mathbb{R}^d \)) defined on a probability space \( (\Omega, \mathcal{F}, P) \) endowed with a reference filtration \( (\mathcal{F}_t) \) such that

(i) there exists a \( r \)-dimensional \( \mathcal{F}_t \)-Wiener process \( W = (W_t) \);

(ii) \( Z \) is \( \mathcal{F}_t \)-adapted and the law of \( Z_0 \) is \( \mu \);

(iii) \( Z \) solves \(\mathcal{L} \mu \)-a.s.

A solution of the martingale problem for \( (\tilde{\mathcal{L}}, \mu) \) is a continuous \( d \)-dimensional process \( Z = (Z_t) \) defined on some probability space \( (\Omega, \mathcal{F}, P) \) such that, for any \( f \in C_0^2(\mathbb{R}^d), \)

\[ M_t(f) = f(Z_t) - \int_0^t \tilde{\mathcal{L}} f(Z_s)ds, \quad t \geq 0, \]  
(12)

(with respect to the natural filtration \( (\mathcal{F}_t^Z) \), where \( \mathcal{F}_t^Z = \sigma(Z_s : 0 \leq s \leq t) \), i.e., \( \mathcal{F}_t^Z \) is the \( \sigma \)-algebra generated by the random variables \( Z_s, 0 \leq s \leq t \), and moreover, the law of \( Z_0 \) is \( \mu \).

Note that \( \tilde{\mathcal{L}} : \mathcal{D}(\tilde{\mathcal{L}}) \subset C_b^2(\mathbb{R}^d) \) satisfies Hypothesis [18] in Appendix. This fact is quite standard; we sketch the proof in the next remark.

Remark 2 There exists a countable set \( H_0 \subset C_0^2(\mathbb{R}^d) \) such that for any \( f \in C_0^2(\mathbb{R}^d), \) we can find a sequence \( (f_k) \in H_0 \) satisfying

\[ \lim_{k \to \infty} (\|f - f_k\|_\infty + \|\tilde{\mathcal{L}} f_k - \tilde{\mathcal{L}} f\|_\infty) = 0. \]  
(13)

To prove the assertion consider the separable Banach space \( V = C_0(\mathbb{R}^d) \subset C_b(\mathbb{R}^d) \) consisting of all continuous functions vanishing at infinity (it is endowed with \( \| \cdot \|_\infty \)).

Then introduce \( \Lambda_n = \{(f, Df, D^2 f)\}_{f \in C_0^2(B_n)} \), where \( C_0^2(B_n) = \{f \in C_0^2(\mathbb{R}^d) \text{ with support}(f) \subset B_n\} \); \( B_n = B(0, n) \) is the open ball of center 0 and radius \( n \geq 1 \).

Identifying \( \mathbb{R}^d \otimes \mathbb{R}^d \) with \( \mathbb{R}^{2d} \) we see that each \( \Lambda_n \) is contained in the product metric space \( V^{1+d+d^2} \) which is also separable. It follows that \( \Lambda_n \) is separable and so there exists a countable set \( \Gamma_n \subset C_0^2(B_n) \) such that \( \{(f, Df, D^2 f)\}_{f \in \Gamma_n} \) is dense in \( \Lambda_n \). For any \( f \in C_0^2(B_n) \) we can find a sequence \( (f_k^n)_{k \geq 1} \subset \Gamma_n \) such that

\[ \|f - f_k^n\|_\infty + \|Df - Df_k^n\|_\infty + \|D^2 f - D^2 f_k^n\|_\infty \to 0, \quad \text{as} \ k \to \infty. \]  
(14)

Define \( H_0 = \bigcup_{n \geq 1} \Gamma_n \). If \( g \in C_0^2(\mathbb{R}^d) \) then \( g \in C_0^2(B_{n_0}) \), for some \( n_0 \geq 1 \), and we can consider \( (f_k^{n_0}) \subset C_0^2(B_{n_0}) \) such that \( \text{(14)} \) holds with \( f \) and \( f_k^{n_0} \) replaced by \( g \) and \( f_k^{n_0} \). Then we obtain easily \(\text{(14)}\) with \( f \) and \( f_k \) replaced by \( g \) and \( f_k^{n_0} \) (note that \( \|\tilde{\mathcal{L}} f_k^{n_0} - \tilde{\mathcal{L}} g\|_\infty = \sup_{|z| \leq n_0} |\tilde{\mathcal{L}} f_k^{n_0}(z) - \tilde{\mathcal{L}} g(z)| \)).
If $Z$ is a weak solution on $(\Omega, \mathcal{F}, P)$ an application of Itô’s formula shows that $Z$ is also a martingale solution for $(\tilde{\mathcal{L}}, \mu)$.

Conversely, if there exists a martingale solution $Z$ for $(\tilde{\mathcal{L}}, \mu)$ on $(\Omega, \mathcal{F}, P)$ then there exists a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ on which there exists an $r$-dimensional $\mathcal{F}_t$-Wiener process and a weak solution $Y = (Y_t)$ for (11) such that the law of $Y$ coincides with the one of $Z$ (for more details see Section IV.2 in [15] or Section 5.3 in [12]). Thus we have (cf. Proposition IV.2.1 in [15])

**Theorem 3** The existence of a weak solution to (11) with initial condition $\mu$ is equivalent to the existence of a martingale solution for $(\tilde{\mathcal{L}}, \mu)$.

The following result is essentially due to Skorokhod (for a proof one can argue as in the proofs of Theorems IV.2.3 and IV.2.4 in [15]; see also Theorem 5.3.10 in [12]).

**Theorem 4** If the coefficients $b$ and $B$ are continuous functions on $\mathbb{R}^d$ and we assume the existence of a Lyapunov function $\phi$ as in (7) (i.e., $\tilde{\mathcal{L}}\phi \leq C \phi$ on $\mathbb{R}^d$, $\phi : \mathbb{R}^d \to (0, +\infty)$ is a $C^2$-function and $\phi \to +\infty$ as $|z| \to +\infty$) then there exists at least one weak solution to (11) for any initial condition $\mu \in \mathcal{P}(\mathbb{R}^d)$.

If the drift $b$ is not continuous (as it happens in (1) where $b(z) = Az + \left(\frac{b_0(z)}{0}\right)$, $z \in \mathbb{R}^d$) to get existence of solution in general one needs additional non-degeneracy of the noise.

We say that *weak uniqueness or uniqueness in law holds for* (11) *with initial condition* $\mu \in \mathcal{P}(\mathbb{R}^d)$ *if given two weak solutions* $Z$ *and* $Z'$ *even defined on different stochastic bases* such that the law of $Z_0$ and $Z'_0$ is $\mu$ they have the same finite dimensional distributions. Similarly we say that *uniqueness in law holds for the martingale problem for* $(\tilde{\mathcal{L}}, \mu)$ (cf. Section A.1).

It is clear that uniqueness in law for $(\tilde{\mathcal{L}}, \mu)$ implies uniqueness in law for (11); also the converse holds (see Corollary 3.3.5 in [12]). Indeed we have

**Theorem 5** Uniqueness in law for (11) holds with initial condition $\mu$ if and only if uniqueness in law for the martingale problem for $(\tilde{\mathcal{L}}, \mu)$ holds.

Finally, we say that the *martingale problem for* $\tilde{\mathcal{L}}$ is well-posed if, for any $\mu \in \mathcal{P}(\mathbb{R}^d)$, there exists a martingale solution for $(\tilde{\mathcal{L}}, \mu)$ and, moreover, uniqueness in law holds for the martingale problem for $(\tilde{\mathcal{L}}, \mu)$. Similarly, we can define well-posedness for (11).

Let us come back to our SDE (1) associated to $\mathcal{L}$ given in (8).

The next result shows that the study of existence and uniqueness of solutions for (1) may be reduced to the case in which $b_0 = 0$ and $Q_0$ is also a bounded function from $\mathbb{R}^d$ into $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$.

For any $k \geq 1$ define $\psi_k \in C^\infty_K(\mathbb{R}^d)$ such that $0 \leq \psi_k \leq 1$, $\psi_k(z) = 1$ for $|z| \leq k$ and $\psi_k(z) = 0$ for $|z| \geq 2k$. Define the $d_0 \times d_0$-matrix

$$Q^k_0(z) = \psi_k(z)Q_0(z) + (1 - \psi_k(z))Q_0(0), \quad z \in \mathbb{R}^d.$$ 

It is clear that each $Q^k_0(z)$ is a bounded function on $\mathbb{R}^d$. Moreover, we have

$$\langle Q^k_0(z)h, h \rangle \geq \eta|h|^2, \quad h \in \mathbb{R}^{d_0}, \text{ and } \quad Q^k_0(z) = Q_0(z), \quad |z| \leq k.$$  

**Theorem 6** Under Hypothesis [2] the martingale problem for $\mathcal{L}$ given in (8) is well-posed if for any $k \geq 1$ the martingale problem for $\mathcal{L}^k$,

$$\mathcal{L}^k f(z) = \frac{1}{2} \text{Tr}(Q^k_0(z)D_z^2 f(z)) + \langle Az, D f(z) \rangle, \quad f \in C^2_K, \quad z \in \mathbb{R}^d,$$ 

is well-posed.
Proof. We suppose that the martingale problem for $\mathcal{L}^{(k)}$ is well-posed, for any $k \geq 1$, and prove that the martingale problem for $\mathcal{L}$ is well-posed as well.

The proof is divided into two steps. In the first step we will use a well-known argument based on the Girsanov theorem; in the second one we will apply Corollary 29.

I Step. We prove that, for any $k \geq 1$, the martingale problem for $A_k$

$$
A_k f(z) = \frac{1}{2} \text{Tr}(Q_k^0(z) D_x^2 f(z)) + \langle b_k(z), D_x f(z) \rangle + \langle Az, D f(z) \rangle,
$$

$f \in C^2_k$, $z \in \mathbb{R}^d$, is well posed. Here $b_k = b_0 \cdot 1_{B(0,k)}$ ($1_{B(0,k)}$ is the indicator function of the open ball $B(0,k)$ of center 0 and radius $k$).

Let us fix $k \geq 1$. By Theorem 20 it is enough to show that, for any $z \in \mathbb{R}^d$, the martingale problem for $(A_k, \delta_z)$ is well-posed. Let us fix $z_0 \in \mathbb{R}^d$ and consider the SDE

$$
dZ_t = AZ_t dt + \left( \sqrt{Q_k^0(z_t)} 0 \right) dt + \left( \sqrt{Q_k^0(z_t)} 0 \right) dW_t, \quad Z_0 = z_0,
$$

(16)

where $\sqrt{Q_k^0(z)}$ denotes the unique symmetric $d_0 \times d_0$ square root of $Q_k^0(z)$; note that $\sqrt{Q_k^0(z)}$ is a continuous functions of $z$. Moreover $W = (W_t)$ is a standard Wiener process with values in $\mathbb{R}^d$. By Theorems 3 and 5 it is enough to prove the well-posedness of the SDE (16).

Since the martingale problem for $\mathcal{L}^{(k)}$ is well-posed, we know the well-posedness of the SDE

$$
dZ_t = AZ_t dt + \left( \sqrt{Q_k^0(z_t)} 0 \right) dW_t, \quad Z_0 = z_0.
$$

(17)

An application of the Girsanov theorem (see Theorem IV.4.2 in [15]) allows to deduce that there exists a unique weak solution to

$$
dZ_t = \left( AZ_t + \left( \sqrt{Q_k^0(z_t)} 0 \right) \gamma(Z_t) \right) dt
$$

$$
+ \left( \sqrt{Q_k^0(z_t)} 0 \right) dW_t, \quad Z_0 = z_0,
$$

(18)

if $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ is any Borel and bounded function. By defining

$$
\gamma(z) = \left((Q_k^0(z))^{-1/2} b_k(z)\right), \quad z \in \mathbb{R}^d,
$$

we obtain that $\gamma$ is bounded by [15] and moreover equation (18) becomes equation (16). This proves the assertion.

II Step. We prove well-posedness of the martingale problem for $\mathcal{L}$.

Consider the previous operators $A_k$, $k \geq 1$. By the previous step the martingale problem for each $A_k$ is well-posed. In order to apply Corollary 29 we note that $U_k = B(0,k)$ form an increasing sequence of open sets in $\mathbb{R}^d$. Moreover by (15), for any $f \in C^2_k(\mathbb{R}^d)$,

$$
\mathcal{L} f(z) = A_k f(z), \quad z \in U_k, \quad k \geq 1.
$$
Let us fix $z_0 \in \mathbb{R}^d$ and denote by $Z_k^k = (Z_k^k)$ a solution to the martingale problem for $(\mathcal{A}_k, \delta_{z_0})$ defined on some probability space $(\Omega^{(k)}, \mathcal{F}^{(k)}, P^{(k)})$ (this solution is unique in law). Define the stopping times

$$\tau_k = \tau_{k}^{z_0} = \inf\{t \geq 0 : Z_t^k \notin U_k\}, \; k \geq 1 \tag{19}$$

(where $\inf \emptyset = \infty$). To prove the assertion, according to (20) we need to show that, for any $t > 0$,

$$\lim_{k \to \infty} P^{(k)}(\tau_k \leq t) = 0. \tag{19}$$

Let $k$ large enough such that $z_0 \in U_k$ and consider the Lyapunov function $\phi$ (see (7)). It is easy to see that there exists $\phi_k \in C^2_F(\mathbb{R}^d)$ such that $\phi(z) = \phi_k(z)$, $z \in U_k$. By the optional sampling theorem we know that

$$\phi_k(Z_{t\wedge \tau_k}^k) - \int_0^{t\wedge \tau_k} \mathcal{A}_k \phi_k(Z_s^k)ds$$

is a martingale. Denoting by $E^{(k)}$ expectation with respect to $P^{(k)}$, we find, for $t \geq 0$,

$$E^{(k)}[\phi(Z_{t\wedge \tau_k}^k)] = \phi(z_0) + E^{(k)}[\int_0^{t\wedge \tau_k} \mathcal{L} \phi(Z_s^k)ds] \leq \phi(z_0) + C \int_0^t E^{(k)}[\phi(Z_{s\wedge \tau_k}^k)]ds. \tag{21}$$

By the Gronwall lemma we get $E^{(k)}[\phi(Z_{t\wedge \tau_k}^k)]1_{\{\tau_k \leq t\}} \leq \phi(z_0)e^{Ct}$, so that

$$\min_{y \in \mathbb{R}^d} \{\phi(y)\} \cdot P^{(k)}(\tau_k \leq t) \leq \phi(z_0)e^{Ct}, \tag{21}$$

$t \geq 0$. Since $\phi \to \infty$ as $|z| \to \infty$ we obtain (21) and this finishes the proof. \hfill \blacksquare

According to Theorem 6 in the sequel we concentrate on proving that the martingale problem for $\mathcal{L}_1$,

$$\mathcal{L}_1 f(z) = \frac{1}{2} \text{Tr}(Q_0(z)D^2 f(z)) + \langle Az, Df(z) \rangle, \quad f \in C^2_F, \; z \in \mathbb{R}^d, \tag{20}$$

is well-posed assuming (6) and in addition that

$$\eta |h|^2 \leq \langle Q_0(z)h, h \rangle \leq \frac{1}{\eta} |h|^2, \quad h \in \mathbb{R}^{d_0}, \quad \text{for some } \eta > 0. \tag{21}$$

Indeed if we prove well-posedness for such martingale problem then we also have well-posedness of the martingale problem for each $\mathcal{L}^{(k)}$ (note that each $\mathcal{L}^{(k)}$ verifies (6) and also (21) with some $\eta = \eta(k) > 0$) and by Theorem 6 we obtain well-posedness of the martingale problem for $\mathcal{L}$.

### 3 The martingale problem for $\mathcal{L}_1$ under an additional hypothesis

**Theorem 7** Let us consider $\mathcal{L}_1$ assuming (i) and (ii) in Hypothesis 7 and also (21) for some $\eta > 0$. There exists a positive constant $\gamma = \gamma(A, d_0, \eta, \hat{d})$ such that if

$$\sup_{z \in \mathbb{R}^d} \|Q_0(z) - \hat{Q}_0\| < \gamma, \tag{22}$$

for some positive define symmetric matrix $\hat{Q}_0 \in \mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$ such that $\eta |h|^2 \leq \langle \hat{Q}_0 h, h \rangle \leq \frac{1}{\eta} |h|^2, \; h \in \mathbb{R}^{d_0}$, then the martingale problem for $\mathcal{L}_1$ is well-posed.

To prove the result we need some analytic regularity results for $\mathcal{L}_1$ when $Q_0(z)$ is constant.
3.1 Analytic regularity results for hypoelliptic OU operators

Let us consider the OU operator

\[ \mathcal{L}_0 f(z) = \frac{1}{2} \text{Tr}(Q D^2 f(z)) + \langle Az, Df(z) \rangle = \frac{1}{2} \text{Tr}(Q_0 D^2 f(z)) + \langle Az, Df(z) \rangle, \quad f \in C^2_0, \]

(23)

for some \( z \in \mathbb{R}^d \), where \( Q = \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix} \), and \( Q_0 \) is a symmetric positive definite \( d_0 \times d_0 \) matrix such that

\[ \eta |h|^2 \leq \langle Q_0 h, h \rangle \leq \frac{1}{\eta} |h|^2, \quad h \in \mathbb{R}^{d_0}, \]

(24)

for some \( \eta > 0 \). The associated OU process starting at \( z \in \mathbb{R}^d \) solves the SDE

\[ Z^\lambda_t = z + \int_0^t AZ^\lambda_s ds + \int_0^t \sqrt{Q} dW_s, \quad t \geq 0. \]

(25)

The corresponding Markov semigroup is given by

\[ P_t f(z) = E[f(Z^\lambda_t)] = \int_{\mathbb{R}^d} f(e^{tA}z + y)N(0, Q_t)dy, \]

(26)

where \( f \in B_b(\mathbb{R}^d) \), \( z \in \mathbb{R}^d \) and \( N(0, Q_t) \) is the Gaussian measure with mean 0 and covariance operator \( Q_t \)

\[ Q_t = \int_0^t e^{sA}Q e^{sA^*} ds, \quad t \geq 0. \]

(27)

We assume that \( Q_t \) is positive definite, for any \( t > 0 \) (cf. (10)).

We will investigate regularity properties of the resolvent \( R(\lambda, \mathcal{L}_0) \) which is defined by

\[ R(\lambda, \mathcal{L}_0) f(z) = \int_0^{+\infty} e^{-\lambda t} E[f(Z^\lambda_t)]dt = \int_0^{+\infty} e^{-\lambda t} P_t f(z)dt, \quad f \in C^2_0(\mathbb{R}^d), \]

(28)

\( \lambda > 0, z \in \mathbb{R}^d \). Our starting point is the following regularity result proved in [5] (a previous result for non-degenerate OU operators was established in [25]).

**Theorem 8** Let \( p \in (1, \infty) \). Let us consider the hypoelliptic OU operator \( \mathcal{L}_0 \) (i.e., we are assuming (24) and (6) or (10)). There exists \( C = C(\eta, A, d_0, d, p) \) such that, for any \( v \in C^\infty(\mathbb{R}^d) \), we have

\[ \|D^2 v\|_p \leq C(\|\mathcal{L}_0 v\|_p + \|v\|_p). \]

(29)

The previous result allows to prove

**Theorem 9** Let us consider the hypoelliptic OU operator \( \mathcal{L}_0 \). Let \( p \in (1, \infty) \). There exists \( \lambda_0 = \lambda_0(A, p, d) > 0 \) and \( C = C(\eta, A, d_0, d, p) \) such that, for any \( f \in C^2_0(\mathbb{R}^d) \), \( \lambda > \lambda_0 \), we have

\[ \|D^2 R(\lambda, \mathcal{L}_0) f\|_p \leq C\|f\|_p. \]

(30)

Before proving the theorem we establish two lemmas of independent interest.

**Lemma 10** Let us consider the OU resolvent given in (28) with \( Q \) as in (24) and \( A \) which satisfies (6). Let \( f \in C^2(\mathbb{R}^d) \). There exists \( \tilde{p} = \tilde{p}(\eta, d, d_0, A) \geq 1 \) such that if \( p > \tilde{p} \) then

\[ \sup_{z \in \mathbb{R}^d} |R(\lambda, \mathcal{L}_0) f(z)| \leq \sup_{z \in \mathbb{R}^d} \int_0^{+\infty} e^{-\lambda t} |P_t f(z)|dt \leq C\|f\|_p, \quad \lambda > 0, \]

(31)

with \( C = C(p, \eta, d, d_0, A) > 0 \) independent of \( f \).
Proof. (i) By changing variable and using Hölder inequality we find, for \( p \geq 1, t > 0, z \in \mathbb{R}^d \),
\[
|P_tf(z)| = \left| c_d \int_{\mathbb{R}^d} f(e^{tA}z + \sqrt{Q}y)e^{-\frac{|y|^2}{2}} dy \right|
\leq c_p \left( \int_{\mathbb{R}^d} |f(e^{tA}z + \sqrt{Q}y)|^p dy \right)^{1/p} = \frac{c_p}{(\det(Q_t))^{1/2p}} \left( \int_{\mathbb{R}^d} |f(e^{tA}z + w)|^p dw \right)^{1/p}
= \frac{c_p}{(\det(Q_t))^{1/2p}} \|f\|_p.
\]
with \( c_p \) independent of \( z \). Setting \( u_\lambda = R(\lambda, L_0)f \) we find
\[
\|u_\lambda\|_\infty \leq \sup_{z \in \mathbb{R}^d} \int_0^{+\infty} e^{-\lambda t}|P_tf(z)|dt \leq c_p \|f\|_p \int_0^{+\infty} e^{-\lambda t} \frac{1}{(\det(Q_t))^{1/2p}} dt.
\]
Now we need to estimate \( \det(Q_t) \), for \( t > 0 \), with a constant possibly depending on \( \eta \) (see (24)). We have
\[
\langle Q_t h, h \rangle = \int_0^t \langle Q e^{sA} h, e^{sA} h \rangle ds \geq \int_0^t \langle I_\eta e^{sA} h, e^{sA} h \rangle ds = \langle Q_\eta h, h \rangle, \ h \in \mathbb{R}^d,
\]
where \( I_\eta = \begin{pmatrix} \eta I_0 & 0 \\ 0 & 0 \end{pmatrix} \), with \( I_0 \) the \( d_0 \times d_0 \)-identity matrix, and
\[
Q_\eta^t = \int_0^t e^{sA} I_\eta e^{sA}^* ds.
\]
Condition (ii) in Hypothesis (I) is equivalent to the controllability Kalman condition
\[
\text{rank}[B, AB, \ldots, A^k B] = d,
\]
with \( B = I_\eta \). This is also equivalent to the fact that \( Q_\eta^t \) is positive definite for any \( t > 0 \) (see, for instance, Chapter I.1 in [21]).

Now we use a result in [24] (see also Lemma 3.1 in [21]). According to formulae (1.4) and (2.6) in [24] (in [24] \( Q_\eta^t \) is denoted by \( W_t \)) we have
\[
\|(Q_\eta^t)^{-1}\| \sim \frac{C_1}{t^{2k+1}} \quad \text{as} \ t \to 0^+.
\]
It follows that \( \langle Q_\eta^t h, h \rangle \geq ct^{2k+1}, \ t \in (0, 1), |h| = 1 \). Using (32) we easily obtain
\[
\det(Q_t) \geq C t^{2k+1}, \ t \in (0, 1),
\]
where \( C = C(\eta, A, d_0, d) \). On the other hand, \( \det(Q_t) \geq \det(Q_1) \geq C, \ t \geq 1 \). It follows that
\[
\|u_\lambda\|_{\infty} \leq \sup_{z \in \mathbb{R}^d} \int_0^{+\infty} e^{-\lambda t}|P_tf(z)|dt \leq c_p \|f\|_p \int_0^{+\infty} \frac{C' e^{-\lambda t}}{(t^{2k+1} + \lambda 1)^{1/2p}} dt,
\]
\( C' = C'(p, \eta, A, d_0, d) \). By choosing \( p \) large enough we get easily assertion (31). ■

**Lemma 11** Assume the same assumptions of Lemma (I) and let \( f \in C^2_K(\mathbb{R}^d) \). Then, for any \( p \geq 1 \) there exists \( \lambda_0 = \lambda_0(p, d, A) > 0 \), and \( C = C(p, d, A) > 0 \) such that
\[
\|R(\lambda, L_0)f\|_p \leq \frac{C}{\lambda} \|f\|_p,
\]

(34)
\[
\|DR(\lambda, L_0)f\|_p \leq \frac{C}{\lambda}\|Df\|_p, \quad \|D^2R(\lambda, L_0)f\|_p \leq \frac{C}{\lambda}\|D^2f\|_p, \quad \lambda > \lambda_0. \tag{35}
\]

Moreover, for any \(\lambda > \lambda_0\) the function \(u_\lambda = R(\lambda, L_0)f \in C^2_c(\mathbb{R}^d)\) is the unique bounded classical solution to
\[
\lambda u - L_0u = f \tag{36}
\]
on \(\mathbb{R}^d\). Finally, we have, for \(\lambda > \lambda_0\), with \(C = (p,d,A),\)
\[
\lambda\|u_\lambda\|_p + \|L_0u_\lambda\|_p \leq C\|f\|_p. \tag{37}
\]

**Proof.** Set \(g_t(z) = f(e^{tA}z), t \geq 0, z \in \mathbb{R}^d\). By changing variable we find
\[
P_tf(z) = \int_{\mathbb{R}^d} g_t(z + e^{-tA}y)N(0,Q_t)dy = \int_{\mathbb{R}^d} g_t(z + w)N(0,e^{-tA}Q_te^{-tA^*})dw.
\]

By the Young inequality we get, for \(\lambda > \lambda_0\),
\[
\|P_tf\|_p \leq \|g_t\|_p = e^{-\lambda Tr(A)}\|f\|_p.
\]

Hence, by using the Jensen inequality, we have for \(\lambda > -Tr(A)\)
\[
\|u_\lambda\|_p = \int_{\mathbb{R}^d} \frac{1}{\lambda} \int_0^{+\infty} \lambda e^{-\lambda t}P_tf(z)dt \|^p dz
\]
\[
\leq \frac{1}{\lambda^p} \int_{\mathbb{R}^d} dz \int_0^{+\infty} \lambda e^{-\lambda t}|P_tf(z)|^p dt \leq \lambda^{1-p} \int_0^{+\infty} e^{-\lambda t}e^{-tTr(A)}dt \|f\|_p^p
\]
\[
\leq \frac{\lambda^{1-p}}{\lambda + Tr(A)} \|f\|_p
\]
and so (34) follows easily.

Concerning (35) note that, for any \(h \in \mathbb{R}^d,\)
\[
\langle Du_\lambda(z), h \rangle = \int_0^{+\infty} e^{-\lambda t}P_t((Df(\cdot), e^{tA}h))(z)dt. \tag{38}
\]

Indeed we have the following straightforward formulae
\[
\langle DP_tf(z), h \rangle = P_t(\langle Df(\cdot), e^{tA}h \rangle)(z),
\]
\[
\langle D^2P_tf(z)[h,k], k \rangle = P_t(\langle D^2f(\cdot)[e^{tA}h], e^{tA}k \rangle)(z), \quad h,k \in \mathbb{R}^d, \quad t \geq 0,
\]
z \in \(\mathbb{R}^d\). Starting from (35) the first estimate in (35) can be proved arguing as in the proof of (34). In a similar way we get also the second estimate in (35).

Let us prove the final assertion. It is easy to see that there exists \(\lambda_0 = \lambda_0(A,d) > 0\) such that for \(\lambda > \lambda_0\) we have that \(u_\lambda \in C^2_c(\mathbb{R}^d)\). Moreover, for any \(z \in \mathbb{R}^d\), differentiating under the integral sign we get
\[
\mathcal{L}_0u_\lambda(z) = \int_0^{+\infty} e^{-\lambda t}\mathcal{L}_0(P_tf)(z)dt
\]
\[
= \int_0^{+\infty} e^{-\lambda t} \frac{d}{dt}(P_tf)(z)dt = -f(z) + \lambda u_\lambda(z),
\]
so that \( u_\lambda \) is a classical solution to \( \lambda u_\lambda - \mathcal{L}_0 u_\lambda = f \) (\( u_\lambda \) is the unique bounded classical solution by the maximum principle). Finally, writing
\[
\mathcal{L}_0 u_\lambda = -f + \lambda u_\lambda
\]
and using (34) we obtain (37).

**Proof of Theorem 9.** The proof is divided into two steps.

**Step 1.** We show that (29) holds even if \( v \in C^2_K(\mathbb{R}^d) \).

To this purpose take any \( v \in C^2_K(\mathbb{R}^d) \) and consider standard mollifiers \( (\rho_n) \subset C^\infty(\mathbb{R}^d) \) (i.e., \( 0 \leq \rho_n \leq 1 \), \( \rho_n(z) = 0 \) if \( |z| > \frac{2}{n} \), \( \int \rho_n = 1 \), \( \rho_n(z) = \rho_n(-z) \)). Define \( v_n = v * \rho_n \in C^\infty(\mathbb{R}^d) \). According to (29) we have
\[
\|D^2v_n\|_p \leq C(\|\mathcal{L}_0 v_n\|_p + \|v_n\|_p).
\]

It is not difficult to show that \( L_0 v_n \to L_0 v \) in \( L^p(\mathbb{R}^d) \) as \( n \to \infty \), \( p \geq 1 \). We only show that \( (Az, Dv_n(z)) \to (Az, Dv(z)) \) in \( L^p(\mathbb{R}^d) \) as \( n \to \infty \) (similarly, one can check that \( \frac{1}{2} \text{Tr} (Q_0 D^2v_n) \to \frac{1}{2} \text{Tr} (Q_0 D^2v) \) in \( L^p(\mathbb{R}^d) \)). We have
\[
(Az, Dv_n(z)) = g_n(z) + h_n(z),
\]
\[
g_n(z) = \int_{\mathbb{R}^d} \langle Az - Aw, Dv(w) \rangle \rho_n(z - w) dw,
\]
\[
h_n(z) = \int_{\mathbb{R}^d} \langle Aw, Dv(w) \rangle \rho_n(z - w) dw.
\]

By standard properties of mollifiers, \( h_n \to \langle Az, Dv(z) \rangle \) in \( L^p(\mathbb{R}^d) \) as \( n \to \infty \). Concerning \( g_n \), we find
\[
\int_{\mathbb{R}^d} |g_n(z)|^p dz \leq \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} |(Aw, Dv(z - w))|^p \rho_n(w) dw \leq \frac{2p}{n^p} \|A\|_p \|Dv\|_p,
\]

which tends to 0 as \( n \to \infty \). Since \( L_0 v_n \to L_0 v \) in \( L^p(\mathbb{R}^d) \), we can pass to the limit in (39) as \( n \to \infty \) and get, for \( p > 1 \),
\[
\|D^2v\|_p \leq C(\|\mathcal{L}_0 v\|_p + \|v\|_p).
\]

**Step 2.** We consider \( \lambda_0 \) from Lemma [7] and prove that \( u = u_\lambda = R(\lambda, L_0)f \) verifies (30) for \( \lambda > \lambda_0 \).

From Lemma [11] we already know several regularity properties of \( u \). We will use these properties in the sequel.

Let \( \phi \in C^\infty(\mathbb{R}^d) \) be such that \( 0 \leq \phi \leq 1 \) and \( \phi(z) = 1 \), \( |z| \leq 1 \). Define \( w_n(z) = u(z) \cdot \psi_n(z) \), \( z \in \mathbb{R}^d \), where \( \psi_n(z) = \phi(\frac{z}{n}) \), for \( n \geq 1 \). It is clear that each \( w_n \in C^2_K \).

Applying the first step we have
\[
\|D^2w_n\|_p \leq \hat{C}(\|\mathcal{L}_0 w_n\|_p + \|w_n\|_p),
\]

which becomes (for \( h, k \in \mathbb{R}^{d_0}, h \otimes k \in \mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0} \), with \( h \otimes k \psi = h(k, w) \), \( w \in \mathbb{R}^{d_0} \))
\[
\int_{\mathbb{R}^d} \left| \frac{1}{n^2} u(z) D^2 \phi \frac{z}{n} + \frac{1}{n} D_z u(z) \otimes D_x \phi \frac{z}{n} + \frac{1}{n} D_z \phi \frac{z}{n} \otimes D_x u(z) + D^2_z u(z) \phi \frac{z}{n} \right| \|dz\|
\leq C' \left( \|L_0 u\|_p + \sup_{z \in \mathbb{R}^d} |(Az, D \phi(z))| \cdot \|u\|_p + \frac{1}{n^2} \|D^2 \phi\|_\infty \|u\|_p + \frac{1}{n} \|D_x \phi\|_\infty \|D_x u\|_p + \|u\|_p \right),
\]

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with \( C' = C'(\eta, A, d_0, d, p) > 0 \). Now by the Fatou lemma (using also \(35\) in Lemma \(11\)) as \( n \to \infty \) we find
\[
\|D_x^2u\|_p \leq C_1(\|L_0u\|_p + \|u\|_p) \leq C_1(\|L_0u - \lambda u\|_p + \|u\|_p)
\]
with \( C_1 \) independent of \( \lambda \). Using \(35\) we get (recall that \( u = u_\lambda \))
\[
\|D_x^2u_\lambda\|_p \leq C_1(\|f\|_p + C\|f\|_p + \frac{C_{\lambda_0}}{\lambda_0}\|f\|_p),
\]
for \( \lambda > \lambda_0 \) and this gives the assertion. \( \blacksquare \)

### 3.2 An estimate for the resolvent of a martingale solution

Next we generalize estimate \((31)\) to the case in which we have a martingale solution for the operator \( L_1 \) given in \((20)\) assuming \((21)\).

**Theorem 12** Let us consider \( L_1 \) assuming (i) and (ii) in Hypothesis \( 1 \) and also \((21)\) for some \( \eta > 0 \). Consider \( \hat{p} \) from Lemma \(10\). There exists a positive constant \( \gamma = \gamma(A, d_0, \eta, d) \) such that if \( Q_0(z) \) in \((20)\) verifies
\[
\sup_{z \in \mathbb{R}^d} \|Q_0(z) - \hat{Q}_0\| < \gamma,
\]
for some positive definite matrix \( \hat{Q}_0 \in \mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0} \) such that \( \eta |h|^2 \leq \langle \hat{Q}_0 h, h \rangle \leq \frac{1}{\eta} |h|^2 \), \( h \in \mathbb{R}^{d_0} \), then any solution \( Y = (Y_t) = (Y_t^z) \) to the martingale problem for \( (L_1, \delta_z) \) verifies, for any \( f \in C^2_K(\mathbb{R}^d), p > \hat{p}, \lambda > \lambda_0 > 0 \), with \( \lambda_0 = \lambda_0(A, p, d) \) given in Theorem \(2\)
\[
\sup_{z \in \mathbb{R}^d} \left| \int_0^{+\infty} e^{-\lambda t} E[f(Y_t^z)]dt \right| \leq C\|f\|_p,
\]
for some constant \( C = C(p, \eta, d, d_0, A) > 0 \).

**Proof.** The proof is inspired by the one of Theorem IV.3.3 in \[15\] (see also Chapter 7 in \[30\]) and uses Theorem \[3\], Lemmas \[10\] and \[11\].

Given a martingale solution \( Y \) there exists a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) on which there exists a \( d_0 \)-dimensional \( \mathcal{F}_t \)-Wiener process \( W = (W_t) \) and a solution \( Z = (Z_t) = (Z_t^z) \) to
\[
Z_t = z + \int_0^t AZ_s ds + \int_0^t \sqrt{Q(Z_s)} dW_s, \quad t \geq 0, \quad Q(z) = \left( \begin{array}{c} Q_0(z) \\ 0 \\ 0 \end{array} \right),
\]
(42)
such that the law of \( Y \) coincides with the one of \( Z \) (for more details see Section IV.2 in \[15\] or Section 5.3 in \[12\]). It is not difficult to prove that we have
\[
Z_t = e^t A z + \int_0^t e^{(t-s)A} \sqrt{Q(Z_s)} dW_s, \quad t \geq 0
\]
(43)
(see Proposition 6.3 in \[9\] for a more general result). In the sequel to simplify notation we write \( Z_t \) instead of \( Z_t^z \). Thus it is enough to show that, for a fixed \( \lambda > \lambda_0 \) we have
\[
\left| \int_0^{+\infty} e^{-\lambda t} E[f(Z_t)]dt \right| \leq C\|f\|_p, \quad f \in C^2_K.
\]
Let us define new adapted processes \( X^m = (X^m_t), m \geq 1, \)
\[
X^m_t = Z_{\frac{m}{\gamma} \wedge m},
\]

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for $t \in \left[ \frac{k}{2m}, \frac{k+1}{2m} \right]$ and $k = 0, 1, \ldots$; moreover consider

$$Z^m_t = e^{tA}z + \int_0^t e^{(t-s)A} \sqrt{Q(X^m_s)} \, dW_s, \quad t \geq 0.$$ 

Since, for any $T > 0$, $\lim_{m \to \infty} E[\sup_{t \in [0,T]} |Z^m_t - Z_t|^2] = 0$, it is easy to check that

$$\int_0^{+\infty} e^{-\lambda t} E[f(Z^m_t)] \, dt \to \int_0^{+\infty} e^{-\lambda t} E[f(Z_t)] \, dt$$

as $m \to \infty$, for any $f \in C^2_K(\mathbb{R}^d)$, $\lambda > 0$. Therefore the assertion follows if we prove that

$$\left| \int_0^{+\infty} e^{-\lambda t} E[f(Z_t^m)] \, dt \right| \leq C \|f\|_p, \quad f \in C^2_K, \quad \lambda > \lambda_0, \quad (45)$$

with $C = C(p, \eta, d, d_0, A)$ independent of $m$. This will be achieved into three steps.

**Step 1.** We show that, for any $m \geq 1$, (45) holds with $C$ possibly depending on $m$.

We fix $f \in C^2_K$, $m \geq 1$, $\lambda > 0$ and consider

$$V_m(\lambda, z) := E \left[ \int_0^{+\infty} e^{-\lambda t} f(Z_t^m) \, dt \right]$$

(66)

$$= \sum_{k=0}^{m2^m-1} E \left[ \int_{\frac{k}{2m}}^{\frac{k+1}{2m}} e^{-\lambda t} f(Z_t^m) \, dt \right] + E \left[ \int_{\frac{m}{2m}}^{+\infty} e^{-\lambda t} f(Z_t^m) \, dt \right].$$

Let us fix $k \in \{0, \ldots, m2^m - 1\}$ and define

$$J_k = E \left[ \int_{\frac{k}{2m}}^{\frac{k+1}{2m}} e^{-\lambda t} f(Z_t^m) \, dt \right] = E \left[ \int_{\frac{k}{2m}}^{\frac{k+1}{2m}} e^{-\lambda t} E \left[ f(Z_t^m) \mid \mathcal{F}_{\frac{k}{2m}} \right] \, dt \right]$$

(we are using conditional expectation with respect to $\mathcal{F}_{\frac{k}{2m}}$). If we set

$$U = e^{k/2m}A_z + \int_0^{k/2m} e^{(k/2m-s)A} \sqrt{Q(X^m_s)} \, dW_s$$

then, by a well-known property of conditional expectation (using also that

$$\int_{k/2m}^{t} e^{(t-r)A} \sqrt{Q(y)} \, dW_r$$

is independent of $\mathcal{F}_{\frac{k}{2m}}$ for any $y \in \mathbb{R}^d$) we have, for $t \in \left[ \frac{k}{2m}, \frac{k+1}{2m} \right]$, 

$$E \left[ f(Z_t^m) \mid \mathcal{F}_{\frac{k}{2m}} \right] = E \left[ f \left( e^{(t-k/2m)A} U + \int_{k/2m}^{t} e^{(t-s)A} \sqrt{Q(Z_{\frac{k}{2m}}^m)} \, dW_s \right) \mid \mathcal{F}_{\frac{k}{2m}} \right] = F(t - \frac{k}{2m}, U, Z_{\frac{k}{2m}})$$

where

$$F(s, y_1, y_2) = E \left[ f \left( e^{sA}y_1 + \int_0^{s} e^{(s-r)A} \sqrt{Q(y_2)} \, dW_r \right) \right]$$

(note that

$$F(t - \frac{k}{2m}, y_1, y_2) = E \left[ f \left( e^{(t-k/2m)A}y_1 + \int_{k/2m}^{t} e^{(t-r)A} \sqrt{Q(y_2)} \, dW_r \right) \right].$$
It follows that
\[ J_k = E \left[ \int_k^{k + 1} e^{-\lambda t} E \left[ f \left( Z_t^m \right) \right] dt \right] = \int_k^{k + 1} e^{-\lambda t} E \left[ f \left( t - \frac{k}{m}, U, Z_t^m \right) \right] dt = \int_0^1 e^{-\lambda \left(s + \frac{k}{m}\right)} E \left[ f(s, U, Z_s^m) \right] ds. \]

Therefore, for any \( k = 0, \ldots, m2^m - 1, \)
\[ |J_k| \leq \int_0^1 e^{-\lambda \left(s + \frac{k}{m}\right)} E[|f(s, U, Z_s^m)|] ds \leq \int_0^{+\infty} e^{-\lambda s} E[|f(s, U, Z_s^m)|] ds. \]

Now it is crucial to observe that by Lemma 10 we have, for any \( y_1, y_2 \in \mathbb{R}^d, p > \hat{p}, \lambda > \lambda_0, \)
\[ \int_0^{+\infty} e^{-\lambda s}|F(s, y_1, y_2)| ds \leq C \|f\|_p, \tag{47} \]
where \( C = C(\eta, d, d_0, A, p) > 0 \) is independent of \( y_1 \) and \( y_2. \) Indeed \( F(t, y_1, y_2) \) coincides with the OU semigroup in (26) with \( y_1 = z \) and \( Q \) replaced by \( Q(y_2) = \begin{pmatrix} Q_0(y_2) & 0 \\ 0 & 0 \end{pmatrix} ; \)
note that \( Q_0(y_2) \) verifies (24) by (21).

It follows that \( |J_k| \leq C \|f\|_p, \) for any \( k = 0, \ldots, m2^m - 1. \) Similarly, using that
\[ I = E \left[ \int_m^{+\infty} e^{-\lambda f} E \left( Z_t^m \right) dt \right] = E \left[ \int_m^{+\infty} e^{-\lambda f} E \left( f \left( Z_t^m \right) \right) dt \right], \]
we find the estimate \( |I| \leq C \|f\|_p. \) Returning to (16) we get
\[ \left| E \left[ \int_0^{+\infty} e^{-\lambda f} E \left( Z_t^m \right) dt \right] \right| \leq \sum_{k=0}^{m2^m - 1} |J_k| + \left| E \left[ \int_m^{+\infty} e^{-\lambda f} E \left( Z_t^m \right) dt \right] \right| \leq m2^m C \|f\|_p \]
which shows (15) with a constant possibly depending on \( m. \)

**Step 2.** We establish the following identity, for any \( f \in C^2_b(\mathbb{R}^d), \lambda > 0, \)
\[ \lambda \int_0^{+\infty} e^{-\lambda t} E \left[f \left( Z_t^m \right) \right] dt = f(z) + E \int_0^{+\infty} e^{-\lambda \mathcal{L}_{m} f} \left(t, Z_t^m \right) dt, \tag{48} \]
with a suitable operator \( \mathcal{L}_m. \)

Consider first \( f \in C^2_b(\mathbb{R}^d) \) and fix \( m \geq 1. \) Writing Itô’s formula for \( f(Z_t^m) \) and taking expectation we find
\[ E f(Z_t^m) = f(z) + E \int_t^\infty \langle A Z_s^m, D f(Z_s^m) \rangle ds + \frac{1}{2} \int_0^t E[\text{Tr}(Q(X_s^m)D^2 f(Z_s^m))] ds, \]
\( t \geq 0. \) Defining the operator
\[ \mathcal{L}_m f(s, z) = \frac{1}{2} \text{Tr}(Q(X_s^m)D^2 f(z)) + \langle Az, D f(z) \rangle, \quad f \in C^2_b(\mathbb{R}^d), \quad z \in \mathbb{R}^d, \]
with random coefficients, we see that \( E[f(Z_t^m)] = f(z) + E \int_0^t \mathcal{L}_m f(s, Z_t^m) ds. \) Using the Fubini theorem we find
\[ \int_0^{+\infty} e^{-\lambda t} E \left[ \int_0^t \mathcal{L}_m f(s, Z_s^m) ds \right] dt = E \left[ \int_0^{+\infty} \mathcal{L}_m f(s, Z_s^m) ds \int_s^{+\infty} e^{-\lambda t} dt \right] = \frac{1}{\lambda} E \left[ \int_0^{+\infty} e^{-\lambda t} \mathcal{L}_m f(t, Z_t^m) dt \right]. \tag{49} \]
It follows \([45]\) for \(f \in C^2_{\mathcal{K}}(\mathbb{R}^d)\). Now a simple approximation argument shows that \([45]\) holds even for \(f \in C^2_{\mathcal{B}}(\mathbb{R}^d)\). To this purpose note also that \(E[\sup_{t \in [0,T]} |Z_t^m|^2] < +\infty\), for any \(T > 0, m \geq 1\).

**Step 3.** We prove assertion \([45]\) with \(C\) independent of \(m\).

Using hypothesis \([40]\) let \(\hat{\mathcal{L}}_0\) be the hypoelliptic OU operator associated to \(A\) and \(\hat{\mathcal{Q}}\) where

\[
\hat{\mathcal{Q}} = \begin{pmatrix}
Q_0 & 0 \\
0 & 0
\end{pmatrix}.
\]

We write

\[
\mathcal{L}_m f(s, z) = \mathcal{L}_0 f(z) + \mathcal{R}_m f(s, z),
\]

\[
\mathcal{R}_m f(s, z) = \frac{1}{2} \text{Tr}([Q_0(X_s^m) - \hat{Q}_0] D_z^2 f(z)), \quad f \in C^2_{\mathcal{B}}(\mathbb{R}^d), \quad z \in \mathbb{R}^d, \quad s \geq 0.
\]

Recall that

\[
V_m(\lambda, z) f = \int_0^\infty e^{-\lambda t} E[f(Z_t^m)] dt, \quad f \in C^2_{\mathcal{B}}(\mathbb{R}^d);
\]

we can rewrite \([45]\) as

\[
\lambda V_m(\lambda, z) f = f(z) + E \int_0^\infty e^{-\lambda t} \mathcal{L}_0 f(Z_t^m) dt - \lambda E \int_0^\infty e^{-\lambda t} f(Z_t^m) dt
\]

\[
+ \lambda E \int_0^\infty e^{-\lambda t} D_z \mathcal{R}_m f(t, Z_t^m) dt.
\]

By taking

\[
f = R(\lambda, \hat{\mathcal{L}}_0) g = R(\lambda) g,
\]

for \(g \in C^2_{\mathcal{K}}(\mathbb{R}^d)\) \((R(\lambda, \hat{\mathcal{L}}_0) g)\) is defined as in \([28]\) with \(\mathcal{L}_0\) replaced by \(\hat{\mathcal{L}}_0\) and using that \((\lambda - \hat{\mathcal{L}}_0) R(\lambda, \hat{\mathcal{L}}_0) g = g\) (see \([36]\)), we obtain from the above identity

\[
\lambda V_m(\lambda, z) [R(\lambda) g] = R(\lambda) g(z) - V_m(\lambda, z) g
\]

\[
+ \lambda V_m(\lambda, z) [R(\lambda) g] + E \int_0^\infty e^{-\lambda t} \mathcal{R}_m [R(\lambda) g] (t, Z_t^m) dt.
\]

We find, for any \(g \in C^2_{\mathcal{K}}(\mathbb{R}^d), m \geq 1, \lambda > 0, z \in \mathbb{R}^d\),

\[
V_m(\lambda, z) g = R(\lambda) g(z) + E \int_0^\infty e^{-\lambda t} \mathcal{R}_m [R(\lambda) g] (t, Z_t^m) dt.
\]

Now by the first step we know that for \(p > \hat{p}, \lambda > \lambda_0, z \in \mathbb{R}^d, m \geq 1,\)

\[
\|V_m(\lambda, z)\|_{L(L^p; \mathbb{R})} = \sup_{g \in C^2_{\mathcal{K}}; \|g\|_{L^p(\mathbb{R}^d)} \leq 1} |V_m(\lambda, z) g| < +\infty.
\]

Using Lemma \([10]\) and condition \([40]\), we find that

\[
|V_m(\lambda, z) g| \leq |R(\lambda) g(z)|
\]

\[
+ \frac{1}{2} E \int_0^\infty e^{-\lambda t} \text{Tr}([Q_0(X_s^m) - \hat{Q}_0] D_z^2 R(\lambda) g(Z_t^m)) dt
\]

\[
\leq C\|g\|_p + \frac{1}{2} E \int_0^\infty e^{-\lambda t} \|D_z^2 R(\lambda) g(Z_t^m)\| dt \leq C\|g\|_p + \frac{1}{2} V_m(\lambda, z) \|D_z^2 R(\lambda) g\|
\]
(we are considering \( V_m(\lambda, z) \) applied to the function \( z \mapsto \|D_2^2 R(\lambda)g(z)\| \) with \( C = C(d, d_0, \eta, A, p) \). By taking the supremum over \( \Lambda_1 = \{ g \in C^2_R, \|g\|_{L^p(\mathbb{R}^d)} \leq 1 \} \), we find

\[
\|V_m(\lambda, z)\|_{L^p(\mathbb{R}^d)} \leq C + \frac{\gamma}{2} \|V_m(\lambda, z)\|_{L^p(\mathbb{R}^d)} \cdot \sup_{g \in \Lambda_1} \|D_2^2 R(\lambda)g\|_{L^p(\mathbb{R}^d)}.
\]

Now we use Theorem 9 to deduce that, for any \( \lambda > \lambda_0 \), we have

\[
\sup_{g \in \Lambda_1} \|D_2^2 R(\lambda)g\|_{L^p} \leq C'
\]

with \( C' = C'(d, d_0, \eta, A, p) \). By choosing \( \gamma \) small enough (\( \gamma < \frac{1}{C} \)) we get that

\[
\|V_m(\lambda, z)\|_{L^p(\mathbb{R}^d)} \leq 2C, \quad \lambda > \lambda_0,
\]

with \( C \) which is also independent of \( m \geq 1 \). This proves (55) and finishes the proof. \( \blacksquare \)

### 3.3 Proof of Theorem 7

Since existence of martingale solutions follows from Theorem 4 let us concentrate on uniqueness of martingale solutions.

We will use Theorems 9 and 12. The constant \( \gamma \) appearing in (22) will be the same constant as in Theorem 12.

According to Corollary 22 to prove that the martingale problem for \( L_1 \) is well-posed it is enough to fix any \( z \in \mathbb{R}^d \) and prove that if \( X_1 = (X_1(t)) \) and \( X_2 = (X_2(t)) \) are two solutions for the martingale problem for \( (L_1, \delta_z) \) (defined, respectively, on \( (\Omega_1, \mathcal{F}_1, P_1) \) and \( (\Omega_2, \mathcal{F}_2, P_2) \)) then they have the same one dimensional marginal distributions.

To this purpose we first consider \( \hat{p} \) from Theorem 12 and fix any \( p > \hat{p} \). Then we take \( \lambda_0 = \lambda_0(A, p, d) > 0 \) from Theorems 9 and 12 and define

\[
G_i(\lambda, z)f = \int_0^\infty e^{-\lambda t} E_i[f(X_i(t))] dt, \quad i = 1, 2, \quad f \in C^2_K(\mathbb{R}^d), \quad \lambda > \lambda_0.
\]

If we prove that for \( \lambda > \lambda_0 \) we have

\[
G_1(\lambda, z)f = G_2(\lambda, z)f,
\]

for \( f \in C^2_K(\mathbb{R}^d) \), then by a well-known property of the Laplace transform we get that \( E[f(X_1(t))] = E[f(X_2(t))] \), \( t \geq 0, \ f \in C^2_K(\mathbb{R}^d) \) and this shows that \( X_1 \) and \( X_2 \) have the same one dimensional marginal distributions.

To check (55) we will also use some arguments from the proof of Theorem 12.

Let us fix \( i = 1, 2 \). By the martingale property we deduce that

\[
E_i[f(X_i(t))] = f(z) + E_i\int_0^t L_1 f(X_i(s)) ds, \quad f \in C^2_K, \quad t \geq 0.
\]

Arguing as in the proof of (48) we obtain

\[
\lambda \int_0^\infty e^{-\lambda t} E_i[f(X_i(t))] dt = f(z) + E_i\int_0^\infty e^{-\lambda t} L_1 f(X_i(t)) dt
\]

or, equivalently,

\[
\lambda G_i(\lambda, z)f = f(z) + G_i(\lambda, z)L_1 f.
\]
Note that (55) holds even for $f \in C_b^2(\mathbb{R}^d)$ (see the comment after (49)). Using hypothesis (22) let $\hat{Q}_0$ be the OU operator associated to $A$ and $\hat{Q}$ where

$$\hat{Q} = \left( \hat{Q}_0 \ 0 \right).$$

We write, similarly to (50),

$$L_1 f(z) = \hat{L}_0 f(z) + \mathcal{R} f(z),$$

$$\mathcal{R} f(z) = \frac{1}{2} \text{Tr}([Q_0(z) - \hat{Q}_0]D_2^2 f(z)), \ f \in C_b^2(\mathbb{R}^d), \ z \in \mathbb{R}^d.$$ We can rewrite (55) as

$$G_i(\lambda, z)(\lambda f - \hat{L}_0 f) = f(z) + G_i(\lambda, z)\mathcal{R} f, \ f \in C_b^2(\mathbb{R}^d).$$

By taking $f = R(\lambda, \hat{L}_0)g = R(\lambda)g$, $g \in C_b^2(\mathbb{R}^d)$ ($R(\lambda, \hat{L}_0)g$ is defined as in (28) with $L_0$ replaced by $\hat{L}_0$) we obtain from the above identity

$$G_i(\lambda, z)g = R(\lambda)g(z) + G_i(\lambda, z)\mathcal{R}(R(\lambda)g), \ (56)$$

$g \in C_b^2(\mathbb{R}^d)$, $\lambda > \lambda_0$, $i = 1, 2$. Define $T(\lambda, z) : C_b^2(\mathbb{R}^d) \to \mathbb{R}$,

$$T(\lambda, z)g = G_1(\lambda, z)g - G_2(\lambda, z)g.$$ We have by (56)

$$T(\lambda, z)g = T(\lambda, z)(\mathcal{R}[R(\lambda)g]). \ (57)$$

By using Theorem 12 we know that $T(\lambda, z)$, for any $\lambda > \lambda_0$, can be extended to a bounded linear operator from $L^p(\mathbb{R}^d)$ into $\mathbb{R}$. By (57) we find, using also (22),

$$\|T(\lambda, z)\|_{L^p(\mathbb{R}^d)} = \sup_{g \in \Lambda_1} \|T(\lambda, z)g\| \leq \frac{\gamma}{2} \|T(\lambda, z)\|_{L^p(\mathbb{R}^d)} \cdot \sup_{g \in \Lambda_1} \|D_2^2[R(\lambda)g]\|_{L^p}. \hspace{1cm} (57)$$

where $\Lambda_1 = \{g \in C_b^2(\mathbb{R}^d), \|g\|_{L^p(\mathbb{R}^d)} \leq 1\}$. Now by Theorem 9 we know that, for any $\lambda > \lambda_0$,

$$\sup_{g \in \Lambda_1} \|D_2^2[R(\lambda)g]\|_{L^p} \leq C',$$

with $C' = C'(d, d_0, \eta, A, p)$. By choosing $\gamma$ small enough ($\gamma = \frac{1}{2C'}$) we get that

$$\|T(\lambda, z)\|_{L^p(\mathbb{R}^d)} = 0, \ \lambda > \lambda_0. \ (58)$$

Note that it is important that $C'$ is independent of $\lambda$ (at least for $\lambda$ large enough); otherwise we should choose for any $\lambda$ a suitable constant $\gamma = \gamma(\lambda)$ and we could not conclude the argument.

Formula (58) shows that (54) holds and this finishes the proof.
4 The main result

Let us consider the operator $L_1$.

$$L_1 f(z) = \frac{1}{2} \text{Tr}(Q_0(z)D^2_z f(z)) + \langle Az, Df(z) \rangle, \quad f \in C^2_K, \ z \in \mathbb{R}^d.$$ 

Combining Theorems \ref{thm:wellposed} and \ref{thm:regularity} we obtain

Theorem 13 Assume (i) and (ii) in Hypothesis \ref{hyp:main} and also (21) for some $\eta > 0$. Then the martingale problem for $L_1$ is well-posed. 

**Proof.** Since existence of martingale solutions follows from Theorem \ref{thm:existence}, let us concentrate on uniqueness of martingale solutions.

In order to apply Theorem \ref{thm:regularity} we set $A = L_1$ and $D(A) = C^2_K(\mathbb{R}^d)$. By using the continuity of $Q_0(z)$ it is easy to construct a set of points $(z_j) \subset \mathbb{R}^d$, $j \geq 1$, and numbers $\delta_j > 0$ such that the open balls $B(z_j, \delta_j)$ of center $z_j$ and radius $\delta_j$ form a covering for $\mathbb{R}^d$ and moreover in each $B(z_j, 2\delta_j)$ we have $\|Q_0(z) - Q_0(z_j)\| < \gamma$ for any $z \in B(z_j, 2\delta_j)$ ($\gamma > 0$ is defined in Theorem \ref{thm:wellposed}).

The balls $\{B(z_j, \delta_j)\}_{j \geq 1}$ give the covering $\{U_j\}_{j \geq 1}$ used in Theorem \ref{thm:regularity}. Let us define operators $A_j$ such that

$$A_j f(z) = Af(z), \quad z \in U_j = B(z_j, \delta_j), \quad f \in C^2_K(\mathbb{R}^d),$$

and such that the martingale problem for each $A_j$ is well-posed.

We fix $j \geq 1$ and consider $\rho_j \in C^2_K(\mathbb{R}^d)$ with $0 \leq \rho_j \leq 1$, $\rho_j = 1$ in $B(z_j, \delta_j)$ and $\rho_j = 0$ outside $B(z_j, 2\delta_j)$. Now define

$$Q_0^j(z) := \rho_j(z)Q_0(z) + (1 - \rho_j(z))Q_0(z_j).$$

We see that, for any $h \in \mathbb{R}^d$, we have

$$\langle Q_0^j(z)h, h \rangle = \rho_j(z)\langle Q_0(z)h, h \rangle + (1 - \rho_j(z))\langle Q_0(z_j)h, h \rangle \geq \frac{\eta}{2}|h|^2,$$

for any $z \in \mathbb{R}^d$, and also $\langle Q_0^j(z)h, h \rangle \leq \frac{1}{\eta}|h|^2$. Moreover $Q_0^j(z) = Q_0(z)$, $z \in U_j$, and

$$\|Q_0^j(z) - Q_0(z_j)\| < \gamma,$$

for any $z \in \mathbb{R}^d$. Let us consider

$$A_j f(z) = \frac{1}{2} \text{Tr}(Q_0^j(z)D^2_z f(z)) + \langle Az, Df(z) \rangle.$$ 

Such operators verifies (59) and moreover they satisfy (i) and (ii) in Hypothesis \ref{hyp:main} and also (21). By Theorem \ref{thm:wellposed} the martingale problem for each $A_j$ is well-posed. Applying Theorem \ref{thm:regularity} we finish the proof. ■

**Remark 14** In the proof of the previous result we can not apply directly the results in Section 6.6 of \cite{Sato} instead of Theorem \ref{thm:regularity}. Indeed the mentioned results in \cite{Sato} would require to truncate both coefficients $Az$ and $Q_0(z)$ on balls in order to deal with diffusions with bounded coefficients. The problem is that if we truncate in the previous way and then consider the truncated mapping $z \mapsto Az$ it becomes difficult to prove the analytic regularity results of Sections 3.1 which are needed to prove well-posedness.

Combining Theorems \ref{thm:existence} and \ref{thm:wellposed} we obtain the main result.

**Theorem 15** Assume Hypothesis \ref{hyp:main}. Then the martingale problem for $L$ given in (8) is well-posed.
A Appendix: the localization principle for martingale problems

The localization principle introduced by Stroock and Varadhan (see [29] and [30]) says, roughly speaking, that to prove uniqueness in law it suffices to show that each starting point has a neighbourhood on which the diffusion coefficients equal other coefficients for which uniqueness holds (see also [11, 18]). Martingale problems and localization principle have been extensively investigated in Chapter 4 of [12] in the setting of a complete and separable metric space $E$. This generality allows applications of the martingale problem to branching processes (see Chapter 9 in [12]) and to SPDEs (see, for instance, [7] and the references therein).

In this appendix we present some extensions and modifications of theorems given in Sections 4.5 and 4.6 of [12]. Our main results are in Section A.3 (see in particular Theorem 23 and Lemma 24). As a consequence we get the localization principle (see Theorem 27) which is an extension of Theorem 4.6.2 in [12] and of Theorem 6.6.1 in [30].

Unlike Sections 4.5 and 4.6 of [12] which mainly deal with càdlàg martingale solutions here we always work with martingale solutions with continuous paths. It is not straightforward to extend results in [12] about the localization principle from càdlàg to continuous martingale solutions; see in particular Lemma 4.5.16 in [12]. On the other hand, proving well-posedness can be more difficult in the class of càdlàg solutions than in the class of continuous solutions. Another difference with respect to [12], is that we always assume that the linear operator $A$ appearing in the martingale problem is countably pointwise determined (see Hypothesis 18). This assumption is usually satisfied in applications and allows to improve some results from [12] (see, in particular, Section A.2).

A.1 Basic definitions

In this appendix $E$ will denote a complete and separable metric space endowed with its $\sigma$-algebra of Borel sets $\mathcal{B}(E)$. The space of all real bounded and Borel functions on $E$ is indicated with $B_b(E)$. Its closed subspace $C_b(E)$ is the space of all real bounded and continuous functions on $E$. We will also consider the space $C_b(E)[0,\infty)$ of all continuous functions from $[0,\infty)$ into $E$. This is a complete and separable metric space endowed with the metric of uniform convergence on compact sets of $[0,\infty)$. In addition $\mathcal{P}(E)$ denotes the metric space of all Borel probability measures on $E$ endowed with the Prokhorov metric which induces the weak convergence of measures. It is a complete and separable metric space (see Chapter 3 in [12]). Its Borel $\sigma$-algebra is denoted by $\mathcal{B}(\mathcal{P}(E))$.

Let us fix a linear operator $A$ with domain $\operatorname{Dom}(A) \subset C_b(E)$ taking values in $B_b(E)$, i.e.,

$$A : \operatorname{Dom}(A) \subset C_b(E) \rightarrow B_b(E) \text{ is linear.}$$  \hfill (60)

Let $\mu \in \mathcal{P}(E)$. An $E$-valued stochastic process $X = (X_t) = (X_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ with continuous trajectories is a solution of the martingale problem for $(A, \mu)$ if, for any $f \in \operatorname{Dom}(A)$,

$$M_t(f) = f(X_t) - \int_0^t Af(X_s)ds, \quad t \geq 0, \quad \text{is a martingale}$$ \hfill (61)

(with respect to the natural filtration $\mathcal{F}_t^X$, where $\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$ is the $\sigma$-algebra generated by the random variables $X_s$, $0 \leq s \leq t$), and moreover, the law of $X_0$ is $\mu$. 
Comparing with [12] we only consider solutions $X$ to the $C_E[0, \infty)$-martingale problem for $(A, \mu)$ (see also Remark [10]).

It is also convenient to call a Borel probability $P$ on $C_E[0, \infty)$ (i.e., $P \in \mathcal{P}(C_E[0, \infty])$) a (probability) solution of the martingale problem for $(A, \mu)$ if the canonical process $X = (X_t)$ defined on $(C_E[0, \infty), \mathcal{B}(C_E[0, \infty]), P)$ by

$$X_t(\omega) = \omega(t), \quad \omega \in C_E[0, \infty), \quad t \geq 0,$$

is a solution of the martingale problem for $(A, \mu)$.

The martingale property (61) only concerns the finite dimensional distribution of $X$. In fact it is equivalent to the following property: for arbitrary $0 \leq t_1 < \ldots < t_n < t_{n+1}$, $f \in D(A)$ and arbitrary $h_1, \ldots, h_n \in C_b(E)$, we have

$$E\left[(M_{t_{n+1}}(f) - M_{t_n}(f)) \cdot \prod_{k=1}^n h_k(X_{t_k})\right] = 0.$$  

(63)

Hence $X$ is a martingale solution for $(A, \mu)$ if and only if its law on $(C_E[0, \infty), \mathcal{B}(C_E[0, \infty]))$ is a martingale solution for $(A, \mu)$.

**Remark 16** We give additional comments motivated by [12].

i) We have required that a solution has sample paths in $C_E[0, \infty)$. On the other hand as in [12] one can also consider martingale solutions $X$ which have càdlàg trajectories, that is, they have sample paths in $D_E[0, \infty)$ ($D_E[0, \infty)$ denotes the complete and separable metric space of all càdlàg functions from $[0, \infty)$ into $E$ endowed with the Skorokhod metric).

The book [12] treats even more general martingale solutions $X$ without càdlàg trajectories. Moreover in [12] the reference filtration $(\mathcal{G}_t)$ can be larger than $(\mathcal{F}^X_t)$; this allows to obtain the Markov property with respect to $(\mathcal{G}_t)$ when the martingale problem is well-posed.

ii) Recall that, for any $x \in E$, $\delta_x \in \mathcal{P}(E)$ is defined by

$$\delta_x(A) = 1_A(x), \quad x \in E, \quad A \in \mathcal{B}(E).$$

(64)

(where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$). According to Theorem 4.3.5 in [12] if there exists a solution $X_x$ of the martingale problem for $(A, \delta_x)$ for any $x \in E$ then $A$ is dissipative, i.e., $\lambda \|f\|_\infty \leq \|\lambda f - Af\|_\infty$, $\lambda > 0$, $f \in D(A)$. Further relations between the martingale problem and semigroup theory of linear operators are investigated in [12].

**Definition 17** Let $\mu \in \mathcal{P}(E)$. We say that uniqueness holds for the martingale problem for $(A, \mu)$ if all the solutions $X$ have the same finite dimensional distributions (i.e., all the solutions $X$ have the same law on $C_E[0, \infty)$, i.e., all (probability) martingale solutions $P$ coincide on $\mathcal{B}(C_E[0, \infty)))$.

The martingale problem for $(A, \mu)$ is well-posed if there exists a martingale solution for $(A, \mu)$ and, moreover, uniqueness holds for the martingale problem for $(A, \mu)$.

Finally, the martingale problem for $A$ is well-posed if the martingale problem for $(A, \mu)$ is well-posed for any $\mu \in \mathcal{P}(E)$.

Next we consider boundedly and pointwise convergence for multisequences of functions similarly to [12], page 111, and [8].

**Hypothesis 18** A linear operator $A : D(A) \subset C_b(E) \rightarrow B_b(E)$ is countably pointwise determined (c.p.d.) if there exists a countable subset $H_0 \subset D(A)$ such that for any $f \in$
There exists an $m$-sequence of functions $(f_{n_1,\ldots,n_m}) \subset H_0$, $(n_1,\ldots,n_m) \in \mathbb{N}^m$, $m \geq 1$, such that $(f_{n_1,\ldots,n_m})$ and $(Af_{n_1,\ldots,n_m})$ converge boundedly and pointwise respectively to $f$ and $Af$. This means that there exists $M > 0$ such that $\|f_{n_1,\ldots,n_m}\|_\infty + \|Af_{n_1,\ldots,n_m}\|_\infty \leq M$, for any $(n_1,\ldots,n_m) \in \mathbb{N}^m$, and moreover
\[
\lim_{n_1 \to \infty} \ldots \lim_{n_m \to \infty} (\lim_{n_m \to \infty} f_{n_1,\ldots,n_m}(x)) = f(x), \quad x \in E.
\]
\[
\lim_{n_1 \to \infty} \ldots \lim_{n_m \to \infty} (\lim_{n_m \to \infty} Af_{n_1,\ldots,n_m}(x)) = Af(x), \quad x \in E. \quad \blacksquare
\]

In particular $A$ is c.p.d. if there exists a separable subspace $M$ of $C_0(E)$ such that $\{(f,Af)\}_{f \in D(A)} \subset M \times M$.

It is easy to verify that if Hypothesis \[18\] holds for $A$ then it is enough to check the martingale property \[61\] only for $f \in H_0$ in order to have a martingale solution.

### A.2 Preliminary results

Results and arguments of this section are quite similar to those given in Chapter 6 of \[30\] (see also \[16\], \[17\]) even if here we are in the general setting of martingale solutions with values in a Polish space. We include self-contained proofs for the sake of completeness.

Assuming Hypothesis \[18\] to prove well-posedness we only have to check that the martingale problems is well-posed for any initial distribution $\delta_x$, $x \in E$ (see \[61\]).

The first result deals with uniqueness of the martingale problem for $(A,\delta_x)$ for any $x \in E$ (cf. Theorem 6.2.3 in \[30\] and Theorem 4.27 in \[17\]). It is a variant of Theorem 4.4.6 in \[12\] which considers the case when, starting from any initial distribution $\mu \in \mathcal{P}(E)$, any two martingale solutions have the same marginals.

**Theorem 19** Suppose that the operator $A$ satisfies Hypothesis \[18\]. Suppose that, for any $x \in E$, any two (probability) martingale solutions $P^x_1$ and $P^x_2$ for $(A,\delta_x)$ have the same one dimensional marginal distributions, i.e.,
\[
P^x_1(X_t \in B) = P^x_2(X_t \in B), \quad t \geq 0, \quad B \in \mathcal{B}(E),
\]
where $(X_t)$ denotes the canonical process in \[62\]. Then, for any $x \in E$, there exists at most one martingale solution for $(A,\delta_x)$.

**Proof.** Let $P^x_1 = P_1$ and $P^x_2 = P_2$ and set $\Omega = C_E[0,\infty)$ endowed with the Borel $\sigma$-algebra $\mathcal{F} = \mathcal{B}(C_E[0,\infty))$. Take any sequence $(t_k) \subset [0,\infty)$, $0 \leq t_1 < \ldots < t_n < \ldots$. It is enough to show that, for any $n \geq 1$, $P_1$ and $P_2$ coincide on the $\sigma$-algebra $\sigma(X_{t_1},\ldots,X_{t_n})$ generated by $X_{t_1},\ldots,X_{t_n}$. To show this we use induction on $n$. For $n = 1$ the assertion follows from \[63\]. We assume that the assertion holds for $n - 1$ with $n \geq 2$ and prove it for $n$. Set
\[
\mathcal{G} = \sigma(X_{t_1},\ldots,X_{t_{n-1}}).
\]
We know that $P_1$ and $P_2$ coincide on $\mathcal{G}$. Since $\Omega = C_E[0,\infty)$ is a complete and separable metric space, by applying Theorem 3.18, page 307 in \[17\] there exists a regular conditional probability $Q^\omega_1$ for $P_1$ given $\mathcal{G}$; this satisfies:

a) for any $\omega \in \Omega$, $Q^\omega_1$ is a probability on $(\Omega,\mathcal{F})$;

b) for any $A \in \mathcal{F}$, the map: $\omega \mapsto Q^\omega_1(A)$ is $\mathcal{G}$-measurable;

c) for any $A \in \mathcal{F}$, $Q^\omega_1(A) = P_1(A|\mathcal{G})(\omega) := E^{P_1}[1_A|\mathcal{G}](\omega)$, $P_1$-a.s. $\omega \in \Omega$.

By $E^{P_1}[1_A|\mathcal{G}]$ we have indicated the conditional expectation of $1_A$ with respect to $\mathcal{G}$ in $(\Omega,\mathcal{F},P_1)$. Moreover, since $\mathcal{G}$ is countable determined (i.e., there exists a countable set
\[ M \subset G \text{ such that whenever two probabilities agree on } M \text{ they also agree on } G \] we also have that there exists \( N' \in G \) with \( P_1(N') = 0 \) and

\[ Q^\omega_1(A) = 1_A(\omega), \quad A \in G, \ \omega \notin N'. \tag{66} \]

Now the proof continues in two steps.

**I Step.** We show that there exists a \( P_1 \)-null set \( N_1 \in G \) such that, for any \( \omega \notin N_1 \), the probability measure \( R^\omega_1 = Q^\omega_1 \circ \theta_{t_n}^{-1} \), i.e.,

\[ R^\omega_1(B) = Q^\omega_1((\theta_{t_n})^{-1}(B)), \quad B \in \mathcal{F}, \]
solves the martingale problem for \( (A, \delta_{\omega(t_n)}) \).

Here \( \theta_{t_n} : \Omega \to \Omega \) is a shift operator, i.e., \( \theta_{t_n}(\omega)(s) = \omega(s + t_n), s \geq 0 \). It is clear by \( 66 \) that there exists a \( P_1 \)-null set \( N' \in G \) such that for any \( \omega \notin N' \),

\[ R^\omega_1(\omega' \in \Omega : \omega'(0) = \omega(t_n-1)) = Q^\omega_1(\omega' \in \Omega : \omega'(t_n-1 + 0) = \omega(t_n-1)) = 1. \]

To prove the martingale property \( [53] \) we first introduce the family \( \mathcal{S} \) of all finite intersections of open balls \( B(x_i, 1/k) \subset E \), where \( k \geq 1 \) and \( x_i \in E_0 \) with \( E_0 \) a fixed countable and dense subset of \( E \), and then consider the countable set \( \Gamma \) of bounded random variables \( \eta : \Omega \to \mathbb{R} \) of the form

\[ \eta = (M_{s_{m+1}} - M_{s_m}) \cdot \prod_{k=1}^{m} h_k(X_{s_k}) \]

where \( f \in H_0 \) (see Hypothesis \( [18] \). \( 0 \leq s_1 < \ldots < s_m < s_{m+1}, m \geq 1 \), are arbitrary rational numbers, \( h_k \) are indicator functions of sets in \( \mathcal{S} \) and \( (X_t) \) is the canonical process.

By using a monotone class argument it is not difficult to see that \( R^\eta_1 \) solves the martingale problem for \( (A, \delta_{\omega(t_n)}) \) if and only if \( \int_{\Omega} \eta(\omega') R^\omega_1(\omega' \, d\omega') = 0 \) for any \( \eta \in \Gamma \).

Therefore the claim follows if we prove that for a fixed \( \eta \in \Gamma \) there exists a \( P_1 \)-null set \( N \in G \) (possibly depending on \( \eta \)) such that for any \( \omega \notin N \),

\[ \int_{\Omega} \eta(\omega') R^\omega_1(\omega' \, d\omega) = 0. \]

To show that the \( G \)-measurable random variable \( \omega \mapsto \int_{\Omega} \eta(\omega') R^\omega_1(\omega' \, d\omega') \) is 0, \( P_1 \)-a.s., it is enough to prove that, for any \( G \in \mathcal{G} = \sigma(X_{t_1}, \ldots, X_{t_{n-1}}) \),

\[ \int_{\Omega} \left[ 1_G(\omega) \int_{\Omega} \eta(\omega') R^\omega_1(\omega' \, d\omega') \right] P_1(\omega) = 0. \]

We have

\[ \int_{\Omega} \left[ 1_G(\omega) \int_{\Omega} \eta(\omega') R^\omega_1(\omega' \, d\omega') \right] P_1(\omega) = \int_{\Omega} \left[ 1_G(\omega) \int_{\Omega} \left( (M_{s_{m+1+t_n}} - M_{s_m+t_n}) \cdot \prod_{k=1}^{m} h_k(X_{s_k+t_n}) \right) \right] \eta(\omega') Q^\omega_1(\omega' \, d\omega') P_1(\omega) = \]

\[ = E^{P_1} \left[ 1_G E^{P_1} \eta \circ \theta_{t_n} / \mathcal{G} \right] = E^{P_1} \left[ E^{P_1} (\eta \circ \theta_{t_n}) / \mathcal{G} \right] = E^{P_1} \left[ (M_{s_{m+1+t_n}} - M_{s_m+t_n}) \cdot \prod_{k=1}^{m} h_k(X_{s_k+t_n}) \cdot 1_G \right] = 0 \]

(in the last passage we have used that \( P_1 \) is a martingale solution).
II Step. We show that $P_1$ and $P_2$ coincide on $\sigma(X_1, \ldots, X_n)$.

Repeating the previous step for the measure $P_2$ we define $Q^\omega_2$ (the regular conditional probability for $P_2$ given $\mathcal{G}$) and $R^\omega_2 = Q^\omega_2 \circ \theta_{t_{n-1}}^{-1}$. We find that there exists a $P_2$-null set $N_2 \in \mathcal{G}$ such that for any $\omega \notin N_2$, the probability measure $R^\omega_2$ solves the martingale problem for $(A, \delta_{\omega(t_{n-1})})$.

Since $P_1$ and $P_2$ coincide on $\mathcal{G}$, the set $N' = N_1 \cup N_2$ verifies $P_k(N') = 0$, $k = 1, 2$. By hypothesis, for any $\omega \notin N'$ we know that $R^\omega_1$ and $R^\omega_2$ have the same one-dimensional marginals. Therefore, for any $A \in \mathcal{B}(\mathbb{E}^{n-1})$, $B \in \mathcal{B}(E)$, we find

$$P_1(\omega \in \Omega : (\omega(t_1), \ldots, \omega(t_{n-1})) \in A, \omega(t_n) \in B) = E(P_1[1_{\{1 \leq \omega(t_1), \ldots, \omega(t_{n-1})\}} \in A] \cdot \omega(t_n) \in B]) = E(P_1[1_{\{1 \leq \omega(t_1), \ldots, \omega(t_{n-1})\}} \in A, \omega(t_n) \in B]) = P_2(\omega \in \Omega : (\omega(t_1), \ldots, \omega(t_{n-1})) \in A, \omega(t_n) \in B).$$

This finishes the proof. □

Recall that a family of measures $(P^x) = (P^x)_{x \in E} \subset \mathcal{P}(C_E[0, \infty))$ depends measurable on $x$ (cf. Lemma 1.40 in [19]) if for any $B \in \mathcal{B}(C_E[0, \infty))$, the mapping:

$$x \mapsto P^x(B)$$

is measurable from $E$ into $[0, 1]$. (67)

Suppose that, for any $x \in E$, there exists a martingale solution $P^x$ on $\mathcal{B}(C_E[0, \infty))$ for $(A, \delta_x)$. If $(P^x)$ depends measurable on $x$ then it is easy to check that, for any initial distribution $\mu \in \mathcal{P}(E)$, there exists a martingale solution $P^\mu$ for $(A, \mu)$ which is given by

$$P^\mu(B) = \int_E P^x(B)\mu(dx), \quad B \in \mathcal{B}(C_E[0, \infty)).$$

(68)

Usually, $(P^x)$ depends measurable on $x$ if one provides a constructive proof for existence of martingale solutions. On the other hand, the next theorem shows that uniqueness implies this measurability property. This result is a kind of extension of Theorem 4.4.6 in [12] (in fact in [12] it is required that the martingale problem is well-posed for any initial $\mu \in \mathcal{P}(E)$).

**Theorem 20** Suppose that $A$ satisfies Hypothesis [13]. Suppose that, for any $x \in E$, there exists a unique (probability) martingale solution $P^x$ for $(A, \delta_x)$.

Then $(P^x)$ depends measurable on $x$ and for any initial distribution $\mu \in \mathcal{P}(E)$ there exists a unique (probability) martingale solution $P^\mu$ given by (68). In particular the martingale problem for $A$ is well-posed.

**Proof.** We combine ideas from the proofs of Theorem 21.10 in [16] and that of Theorem 4.4.6 in [12]. In the sequel $\Omega = C_E[0, \infty)$ and we denote with $\mathcal{F}$ its Borel $\sigma$-algebra. Recall that $\mathcal{P}(E)$ and $\mathcal{P}(\Omega)$ are complete and separable metric spaces with the Prokhorov metric.

**I Step.** We consider the countable family $\Gamma$ of random variables $\eta$ defined in (67) by means of the canonical process $(X_t)$. Recall that by a monotone class argument, $P \in \mathcal{P}(\Omega)$ is a martingale solution for $(A, \delta_x)$ if and only if $P(X_0 \in A) = P(X_0^{-1}(A)) = \delta_x(A)$, $A \in \mathcal{B}(E)$, and

$$\int_{\Omega} \eta(\omega)P(d\omega) = 0, \quad \eta \in \Gamma. \quad (69)$$
II Step. We prove that the set \((P^x)_{x \in E}\) of all martingale solutions (each \(P^x\) is the unique martingale solution for \((A, \delta_x)\)) belongs to \(\mathcal{B}(\mathcal{P}(\Omega))\).

To this purpose we consider the following measurable mapping

\[ G : \mathcal{P}(\Omega) \to \mathcal{P}(E), \quad G(P) = P \circ X_0^{-1}, \quad P \in \mathcal{P}(\Omega), \]

where \(P \circ X_0^{-1}(A) = P(X_0 \in A), A \in \mathcal{B}(E)\). By (69) we deduce that

\[ (P^x)_{x \in E} = \Lambda_1 \cap \Lambda_2, \quad \text{where} \quad \Lambda_1 = \bigcap_{\eta \in \Gamma} \{ P \in \mathcal{P}(\Omega) : \int_\Omega \eta(\omega)P(d\omega) = 0 \}, \quad \Lambda_2 = G^{-1}(\{ \delta_x \}_{x \in E}). \]

Note that for any \(\eta \in \mathcal{B}_b(\Omega)\), the mapping: \(P \mapsto \int_\Omega \eta(\omega)P(d\omega)\) is Borel on \(\mathcal{P}(\Omega)\) (this is easy to verify if in addition \(\eta \in C_b(\Omega)\); the general case follows by a monotone class argument). It follows that \(\Lambda_1 \in \mathcal{B}(\mathcal{P}(\Omega))\).

On the other hand, \(D = \{ \delta_x \}_{x \in E} \in \mathcal{B}(\mathcal{P}(E))\) (this follows from Lemma 1.39 in [16]) and so \(\Lambda_2 \in \mathcal{B}(\mathcal{P}(\Omega))\). The claim is proved.

III Step. Considering the restriction \(G_0\) of \(G\) to \((P^x)_{x \in E}\) we find that the measurable mapping \(G_0 : (P^x)_{x \in E} \to \{ \delta_x \}_{x \in E}\) is one to one and onto. By a result of Kuratowski (see Theorem A.1.3 in [16]) the inverse function \(G_0^{-1} : \{ \delta_x \}_{x \in E} \to (P^x)_{x \in E}\) is also measurable. Finally to show that \(x \mapsto P^x(A) = \int_\Omega 1_A(\omega)P^x(d\omega)\) is Borel on \(E\), for any \(A \in \mathcal{B}(E)\), we observe that the mapping \(x \mapsto \delta_x\) from \(E\) into \(\{ \delta_x \}_{x \in E}\) is a measurable isomorphism.

IV Step. We fix \(\mu \in \mathcal{P}(E)\) and show that there exists a unique martingale solution \(P^\mu\) given by (68).

We have only to prove uniqueness since it is clear that \(P^\mu\) in (68) is a martingale solution for \((A, \mu)\). Let \(P\) be a martingale solution for \((A, \mu)\). We prove that it coincides with \(P^\mu\). Similarly to the first step in the proof of Theorem [19] we consider the regular conditional probability \(Q^\omega\) for \(P\) given \(\sigma(X_0)\) (the \(\sigma\)-algebra generated by \(X_0\)). We see that there exists a \(\bar{P}\)-null set \(N \in \sigma(X_0)\) such that for any \(\omega \notin N\), the probability measure \(Q^\omega\) solves the martingale problem for \((A, \delta_{\omega(0)}) = (A, \delta_{X_0(\omega)})\).

By the uniqueness assumption we deduce that \(Q^\omega = P^{X_0(\omega)}, \omega \notin N\). Setting \(\bar{E} = E^P\) and using also the measurability property, we finish with

\[ P(A) = \bar{E}[E[1_A \mid \sigma(X_0)]] = \bar{E}[Q^\omega(A)] = \bar{E}[P^{X_0(\omega)}(A)] = \int_{\bar{E}} P^\mu(A)\mu(dx), \quad A \in \mathcal{B}(E). \]

Remark 21 Under the assumptions of Theorem [20] one can introduce the semigroup \((P_t)\), \(P_t : \mathcal{B}_b(E) \to \mathcal{B}_b(E), P_tf(x) = \int_{C_\mathbb{R}(0, \infty)} f(\omega(t))P^x(d\omega), f \in \mathcal{B}_b(E), t \geq 0, x \in E\). Combining Theorem [20] and Theorem 4.4.2 in [12] one proves the strong Markov property for a martingale solution \(X\) for \((A, \mu)\). This means that, for any a.s. finite \(\mathcal{F}_t^X\) - stopping time \(\tau\) one has: \(E[f(X_{t+\tau}) \mid \mathcal{F}_t] = P_t f(X_\tau), t \geq 0, f \in \mathcal{B}_b(E)\).

By the previous theorems we get the following useful result.

Corollary 22 Suppose that the operator \(A\) satisfies Hypothesis [18] and assume the following two conditions:

(i) for any \(x \in E\), there exists a (probability) martingale solution \(P^x\) for \((A, \delta_x)\);

(ii) for any \(x \in E\), any two (probability) martingale solutions \(P^x_t\) and \(P^x_t\) for \((A, \delta_x)\) have the same one dimensional marginal distributions (see [55]).

Then the martingale problem for \(A\) is well-posed. In addition, \((P^x)\) depends measurably on \(x\) and so formula (68) holds for any \(\mu \in \mathcal{P}(E)\).
A.3 The localization principle

Let us first introduce the stopped martingale problem following Section 4.6 in [12].

Let $A$ be a linear operator, $A : D(A) \subset C_b(E) \to B_b(E)$. Consider $\mu \in \mathcal{P}(E)$ and an open set $U \subset E$.

An $E$-valued stochastic process $Z = (Z_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ with continuous trajectories is a solution of the stopped martingale problem for $(A, \mu, U)$ if, the law of $Z_0$ is $\mu$ and the following conditions hold:

(i) $Z_t = Z_{t \wedge \tau} = Z_{t \wedge \tau} \in U$, $P$-a.s, where

$$
\tau = \tau^Z_U = \inf \{ t \geq 0 : Z_t \notin U \} \quad (70)
$$

($\tau = +\infty$ if the set is empty; it turns out that this exit time $\tau$ is an $\mathcal{F}_t^Z$-stopping time);

(ii) for any $f \in D(A)$,

$$
M_{t \wedge \tau}(f) = f(Z_t) - \int_0^{t \wedge \tau} Af(Z_s)ds, \quad t \geq 0,
$$

is a martingale with respect to the natural filtration $(\mathcal{F}_t^Z)$.

The next key result shows that if the (global) martingale problem for $A$ is well-posed then also the stopped martingale problem for $(A, \mu, U)$ is well-posed for any choice of $(U, \mu)$.

A related statement is given in Theorem 4.6.1 of [12] which is based on Lemma 4.5.16. However such theorem requires uniqueness for the (global) martingale problem in the class of all càdlàg martingale solutions; actually, it is not clear how to modify the proof of Lemma 4.5.16 in order to have the same statement of the lemma but in the case of continuous martingale solutions.

**Theorem 23** Assume that $A$ verifies Hypothesis [18] and that the martingale problem for $A$ is well-posed.

Then also the stopped martingale problem for $(A, \mu, U)$ is well-posed for any $\mu \in \mathcal{P}(E)$ and for any open set $U$ of $E$.

The proof is based on the following technical lemma which provides a kind of extension property for solutions to the stopped martingale problem (a related result is Lemma 4.5.16 in [12] which is proved in the class of càdlàg martingale solutions).

We denote by $\tau_U : C_E[0, \infty) \to [0, \infty]$ the exit time from $U$.

**Lemma 24** Let $A$ be a linear operator as in [60]. Suppose that for any $x \in E$ there exists a (probability) martingale solution $P^x$ for $A$ and that $(P^x)$ depends measurably on $x$ (see [61]). Let $\mu \in \mathcal{P}(E)$ and $U$ be an open set of $E$. Let $Z = (Z_t)$ be a martingale solution for the stopped martingale problem for $(A, \mu, U)$.

Then, for any $T > 0$, there exists a (probability) martingale solution $P_T$ for $(A, \mu)$ such that if $X$ is the canonical process on $(C_E[0, \infty), \mathcal{B}(C_E[0, \infty]), P_T)$ (see [62]) then $(X_{t \wedge \tau_U \wedge T})_{t \geq 0}$ and $(Z_{t \wedge \tau^Z_U \wedge T})_{t \geq 0} = (Z_{t \wedge T})_{t \geq 0}$ have the same law.

**Proof.** I Step. Construction of $P_T$.

Our construction is inspired by page 271 of [11]. Let $Z$ be defined on some probability space $(\Omega, \mathcal{F}, P)$ and introduce

$$
\tau = \tau^Z_U \wedge T. \quad (72)
$$
We consider the measurable space $\Omega_* = \Omega \times C_E[0, \infty)$ endowed with the product $\sigma$-algebra $\mathcal{F}_* = \mathcal{F} \otimes \mathcal{B}(C_E[0, \infty])$. On this product space, using the measurability of $x \mapsto P^x$, we consider a probability measure $P_*$ defined by the formula
\[
\int f(\omega, \omega') P_*(d\omega, d\omega') := \int f(\omega) \int_{C_E[0, \infty)} f(\omega, \omega') P_{Z(\omega)}(d\omega'),
\]
for any real bounded and measurable function $f$ on $\Omega \times C_E[0, \infty)$ (according to pages 19-20 in [16], $P_{Z(\omega)}(d\omega')$ is a kernel from $\Omega$ into $C_E[0, \infty]$). Note that if $f(\omega, \omega') = f(\omega)$ then $E^{P_*}[f] = E^P[f]$ (here $E^P$ and $E^{P_*}$ denote expectations on $(\Omega, \mathcal{F}, P)$ and $(\Omega_*, \mathcal{F}_*, P_*)$ respectively). Then define
\[
J = \{(\omega, \omega') \in \Omega_* : Z(\omega)(\omega) = \omega'(0)\}.
\]
Since $\omega \mapsto Z(\omega)(\omega)$ is $\mathcal{F}$-measurable, it is clear that $J \in \mathcal{F}_*$. Moreover we have $P_*(J) = 1$ since $P^x(\omega' : \omega'(0) = x) = 1$, $x \in E$. We restrict the events of $\mathcal{F}_*$ to $J$ and consider the probability space $(J, \mathcal{F}_*, P_*)$.

Using that $\tau < \infty$, we define a measurable mapping $\phi : J \to C_E[0, \infty)$ as follows
\[
\phi_t(\omega, \omega') = \begin{cases} Z(\omega), & t \leq \tau(\omega) \\ \omega'(t - \tau(\omega)), & t > \tau(\omega) \end{cases}, \quad \omega \in \Omega, \ \omega' \in C_E[0, \infty), \ t \geq 0
\]
(or $\phi_t(\omega, \omega') = Z(\omega)1_{\{t \leq \tau(\omega)\}} + \omega'(t - \tau(\omega))1_{\{t > \tau(\omega)\}}, \ t \geq 0$). Equivalently, $\phi = (\phi_t)$ is an $E$-valued continuous stochastic process. Note that $\tau(t)(\omega) = \tau(t)(\omega, \omega')$, for any $(\omega, \omega') \in \Omega_*$. The required measure $P_T$ will be the image probability distribution of $P_*$ under $\phi$, i.e.,
\[
P_T(B) = P_*(\phi^{-1}(B)), \quad B \in \mathcal{B}(C_E[0, \infty)).
\]
By the previous construction the fact that $(X_{t \wedge \tau_U \wedge T})_{t \geq 0}$ and $(Z_{t \wedge T})_{t \geq 0}$ have the same law can be easily proved. Indeed, for any $B \in \mathcal{B}(C_E[0, \infty))$,
\[
P_T(X_{t \wedge \tau_U \wedge T} \in B) = P_T(\omega' \in C_E[0, \infty) : \omega' (\cdot \wedge \tau_U \wedge T) \in B)
= P_*((\phi_t^\wedge \tau_U \wedge T) \in B) = E^{P_*}\{1_{B}(Z_{t \wedge \tau_U \wedge T})| \} = P(Z_{t \wedge \tau_U \wedge T} \in B).
\]

II Step. The measure $P_T$ is a martingale solution for $(A, \mu)$.

First we have $P_T(X_0 \in C) = P(Z_0 \in C) = \mu(C)$, for any $C \in \mathcal{B}(E)$.

Now check the martingale property. For fixed $0 \leq t_1 < \ldots < t_{n+1}$, $f \in D(A)$ and $h_1, \ldots, h_n \in C_b(E)$, we have to show that (using the canonical process $X$ defined in (22))
\[
E^{P_T}[\left(M_{t_{n+1}}(f) - M_{t_n}(f)\right) \cdot \prod_{k=1}^n h_k(X_{t_k})] = 0,
\]
where $M_t(f)(\omega') := \omega'(t) - \int_0^t A(f(\omega'(s))ds, \ t \geq 0, \ \omega' \in C_E[0, \infty)$.

Note that $\left(M_{t_{n+1}}(f) - M_{t_n}(f)\right) \cdot \prod_{k=1}^n h_k(X_{t_k}) = R_1 + R_2$, where $R_i : C_E[0, \infty) \to \mathbb{R}$, $i = 1, 2$,
\[
R_1 = \left(M_{t_{n+1}}(\wedge \tau_U \wedge T)(f) - M_{t_n}(\wedge \tau_U \wedge T)(f)\right) \cdot \prod_{k=1}^n h_k(X_{t_k}),
R_2 = \left(M_{t_{n+1}}(\vee \tau_U \wedge T)(f) - M_{t_n}(\vee \tau_U \wedge T)(f)\right) \cdot \prod_{k=1}^n h_k(X_{t_k}).
\]

As for $R_1$ we note that if $t_n \geq \tau_U \wedge T$, then $R_1 = 0$; so with $\tau = \tau_U \wedge T$ as in (22) we find
\[
E^{P_T}[R_1] = E^{P_*}[R_1(\phi) 1_{\{t_n < \tau\}}] = E^{P_*}\left[\left(f(Z_{t_{n+1}}) - f(Z_{t_n}) - \int_{t_n \wedge \tau}^{t_{n+1} \wedge \tau} A(f(Z_t))dt\right) \cdot \prod_{k=1}^n h_k(Z_{t_k} \wedge \tau) \cdot 1_{\{t_n < \tau\}}\right].
\]
Since \(\prod_{k=1}^{\tau_n} h_k(Z_{tk} \wedge \tau) \cdot 1_{\{t_k < \tau\}}\) is bounded and \(F_{t_n}^Z\)-measurable, using the martingale property (71) we find that \(E^{P^\tau}[R_1] = 0\).

Let us consider \(R_2\) and note that \(R_2 = 0\) if \(\tau_n \wedge T \geq t_{n+1}\). Set \(C_E = C_E[0, \infty)\) and define

\[\Lambda(\omega, \omega') = f(\omega'(t_{n+1} + \tau(\omega) - \tau(\omega))) - f(\omega'(t_n + \tau(\omega) - \tau(\omega))) - \int_{t_n \vee \tau(\omega)}^{t_{n+1} \vee \tau(\omega)} Af(\omega'(r - \tau(\omega)))dr, \quad \omega \in \Omega, \omega' \in C_E.\]

Since \((P^x)\) are martingale solutions, we have

\[\int_{C_E} \Lambda(\omega, \omega') F(\omega, \omega') P^x(d\omega') = 0, \quad \omega \in \Omega, x \in E, \quad (74)\]

for any \(F : \Omega \times C_E \to \mathbb{R}\), bounded and \(F_\ast\)-measurable and such that \(F(\cdot, \cdot)\) is \(F_{t_n \vee \tau(\omega) - \tau(\omega)}\) measurable, for any \(\omega \in \Omega\). Hence

\[E^{P^\tau}[R_2] = E^{P^\ast}[R_2(\phi) 1_{\{t_{n+1} > \tau\}}] = E^{P^\ast}\left[\Lambda \cdot \prod_{k=1}^{n} h_k(\phi_{tk}) \cdot 1_{\{t_{n+1} > \tau\}}\right] = \int_{\Omega} 1_{\{t_{n+1} > \tau(\omega)\}} \prod_{t_k \leq \tau(\omega)} h_k(Z_{tk}(\omega)) P(d\omega) \int_{C_E} \Lambda(\omega, \omega') F(\omega, \omega') P^Z_{\tau(\omega)}(d\omega')\]

with \(F(\omega, \omega') = \prod_{t_k > \tau(\omega)} h_k(\omega'(t_k - \tau(\omega)))\) and so by (74) we get \(E^{P^\tau}[R_2] = 0\). We have found that (73) holds and this completes the proof. ■

**Proof of Theorem 23.** *Existence.* Consider a martingale solution \(X\) for \((A, \mu)\) and set \(Z_t = X_{t \wedge \tau_1^Z}, t \geq 0\). Note that \(\tau_1^Z = \tau_1^X\). By the optional sampling theorem we deduce that \(Z = Z_t\) is a solution of the stopped martingale problem for \((A, \mu, U)\).

*Uniqueness.* Since \(A\) satisfies Hypothesis 18 we know by Theorem 20 that the martingale solutions \(P^x\) depend measurably on \(x\).

Let \(Z^1\) and \(Z^2\) be two solutions for the stopped martingale problem for \((A, \mu, U)\). To show that they have the same law it is enough to prove that, for any \(T > 0\), the processes \((Z^1_{t \wedge T})\) and \((Z^2_{t \wedge T})\) have the same law.

Fix \(T > 0\). By Lemma 24 there exist martingale solutions \(P^1\) and \(P^2\) for \((A, \mu)\) such that if \(X\) is the canonical process on \((C_E[0, \infty), B(C_E[0, \infty)), P^k)\), then \((X_{t \wedge \tau_{1 \wedge \tau_1}^X})_{t \geq 0}\) and \((Z^k_{t \wedge T})_{t \geq 0}, k = 1, 2\), have the same law. Since by hypotheses \(P^1 = P^2\) we obtain easily the assertion. ■

From Theorem 23 we get

**Corollary 25** Let \(A_1\) and \(A_2\) be linear operators with common domain \(D(A_1) = D(A_2) = D \subset C_b(E)\) with values in \(B_b(E)\). Suppose that Hypothesis 18 is satisfied. Let \(U\) be an open subset of \(E\) such that

\[A_1 f(x) = A_2 f(x), \quad x \in U, \quad f \in D. \quad (75)\]

If the martingale problem for \(A_1\) is well-posed then the stopped martingale problem for \((A_2, \mu, U)\) is well-posed for any \(\mu \in \mathcal{P}(E)\).

**Proof.** *Existence.* If \(X\) is a solution of the martingale problem for \((A_1, \mu)\) defined on \((\Omega, \mathcal{F}, P)\) then \(Z = (X_{t \wedge \tau})\) is a solution for the stopped martingale problem for \((A_1, \mu, U)\), with \(\tau = \tau_1^X\). Since, for any \(f \in D, t \geq 0\),

\[f(X_{t \wedge \tau}) - \int_0^{t \wedge \tau} A_1 f(X_s) ds = f(X_{t \wedge \tau}) - \int_0^{t \wedge \tau} A_2 f(X_s) ds\]

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we see that $Z$ is also a solution for the stopped martingale problem for $(A_2, \mu, U)$ (note that $X_0(\omega) \notin U$ implies $\tau(\omega) = 0$ and $X_0(\omega) \in U$ implies $\tau(\omega) > 0$, $\omega \in \Omega$).

**Uniqueness.** Assume now that $Z$ and $W$ are both solutions for the stopped martingale problem for $(A_2, \mu, U)$. It follows that they are also solutions for the stopped martingale problem for $(A_1, \mu, U)$. By Theorem 23 we deduce that $Z$ and $W$ have the same law. ■

The following result is a kind of converse of Theorem 23 and gives conditions under which uniqueness for stopped martingale problems implies uniqueness for the global martingale problem. It is a modification of Theorem 4.6.2 in [12].

**Theorem 26** Assume that $A$ verifies Hypothesis 18 and that for any $x \in E$ there exists a martingale solution for $(A, \delta_x)$.

Suppose that there exists a sequence of open sets $U_k \subset E$ with $\cup_{k \geq 1}U_k = E$ such that for any $\mu \in \mathcal{P}(E)$, for any $k \geq 1$, we have uniqueness for the stopped martingale problem for $(A, \mu, U_k)$.

Then the martingale problem for $A$ is well-posed.

**Proof.** By Corollary 22 we have to prove that for a fixed $x \in E$ any two martingale solutions $P^1$ and $P^2$ for $(A, \delta_x)$ have the same one dimensional marginal distribution. Thus using the canonical process $(X_t)$ given in (62) and a uniqueness result for the Laplace transform, it is enough to show that, for any $\lambda > 0$, $f \in C_b(E)$,

$$E^1 \left[ \int_0^{\infty} e^{-\lambda t} f(X_t)dt \right] = E^2 \left[ \int_0^{\infty} e^{-\lambda t} f(X_t)dt \right],$$

(76)

with $E^j = E^{\mu^j}$, $j = 1, 2$. We first introduce $\mathcal{S} = \{U_k^{(j)}\}_{k \geq 1, j \geq 1}$, where $U_k^{(j)} = U_k$, $k$, $j \geq 1$. Then we enumerate $\mathcal{S}$ using positive integers and find $\mathcal{S} = (V_i)_{i \geq 1}$ (so each $U_k$ appears infinitely many times in $(V_i)_{i \geq 1}$).

To prove (76) we show that for any $\lambda > 0$ there exist $\mu_i \in \mathcal{P}(E)$, $i \geq 1$, such that, for any (probability) martingale solution $P$ for $(A, \delta_x)$, we have that

$$g(\lambda, f) := E^P \left[ \int_0^{\infty} e^{-\lambda t} f(X_t)dt \right]$$

can be computed, for any $f \in C_b(E)$, using the (unique) laws of solutions of the stopped martingale problems for $(A, \mu_i, V_i)$, $i \geq 1$.

The previous claim can be proved adapting the proof of Theorem 4.6.2 in [12]; we give a sketch of proof for the sake of completeness.

Define, for any $\omega \in C_E[0, \infty) = C_E$, $\tau_0(\omega) = 0$ and, for $i \geq 1$,

$$\tau_i(\omega) = \inf\{t \geq \tau_{i-1}(\omega) : \omega(t) \notin V_i\}$$

(where $\inf \emptyset = \infty$). By Proposition 2.1.5 in [12] each $\tau_i$ is an $\mathcal{F}^X_t$-stopping time. Moreover, for any $\omega \in C_E$, $\tau_i(\omega) \to +\infty$, as $i \to \infty$.

Indeed let $\tau = \sup_i \tau_i$ and suppose that for some $\omega \in C_E$ we have $\tau(\omega) < +\infty$. Then there exists $U(\omega)$ such that $\omega(\tau(\omega)) \notin U(\omega)$. It follows that for $s \in [0, \tau(\omega)]$ close enough to $\tau$ we have $\omega(s) \in U(\omega)$. Then we can find an integer $i = i(\omega)$ large enough such that $\omega(\tau_i(\omega)) \notin U(\omega)$ and also $V_i(\omega) = U(\omega)$; this is a contradiction since by construction $\omega(\tau_i(\omega)) \notin V_i(\omega)$.

Let $P$ be any martingale solution for $(A, \delta_x)$ on $(C_E, \mathcal{B}(C_E))$ and fix $\lambda > 0$. We find, setting $E = E^P$,

$$g(\lambda, f) = \sum_{i \geq 1} E \left[ 1_{\{\tau_{i-1} < \infty\}} \int_{\tau_{i-1}}^{\tau_i} e^{-\lambda t} f(X_t)dt \right],$$

(77)

$$\sum_{i \geq 1} E \left[ e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}} \int_0^{\eta_i} e^{-\lambda t} f(X_{t \wedge \eta_i + \tau_{i-1}})dt \right].$$
where on \( \{ \tau_{i-1} < \infty \} \), we define \( \eta_i := \tau_i - \tau_{i-1} \) so that \( \eta_i = \inf \{ t \geq 0 : X_t + \tau_{i-1} \notin V_i \} \). For any \( i \geq 1 \) such that \( P(\tau_{i-1} < \infty) > 0 \) define \( \mu_i \in \mathcal{P}(E) \),
\[
\mu_i(B) = \frac{E[e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}} 1_B(X_{\tau_{i-1}})]}{E[e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}}]}, \quad B \in \mathcal{B}(E),
\]
and the stochastic process \( Y^i = (Y^i_t) \), \( Y^i_t := X_{t \land \eta_i} + \tau_{i-1} \), \( t \geq 0 \), defined on \( (C_E, \mathcal{B}(C_E), P_i) \) where \( P_i(C) = \frac{E[e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}} 1_C]}{E[e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}}]} \), \( C \in \mathcal{B}(C_E) \). It follows that \( \mu_i = \delta_x \). We need to show that \( Y^i \) is a solution of the stopped martingale problem for \((A, \mu_i, V_i)\). Note that
\[
\tau^Y_{V_i} = \eta_i, \quad i \geq 1. \tag{78}
\]
It is also clear that the law of \( Y^0 \) is \( \mu_0 \) and also that \( Y_i = Y_{t \land \eta_i}, \ t \geq 0 \). It remains to check the martingale property \( \{Y_t \} \). To this purpose it is enough to prove that \( \bar{X} = (X_t + \tau_{i-1})_{t \geq 0} \) defined on \( (C_E, \mathcal{B}(C_E), P_i) \) is a (global) martingale solution for \((A, \mu_i)\).

We fix \( t_2 > t_1 \geq 0 \) and consider \( G \in \mathcal{F}_{\tau_{i-1} + t_1} \) that \( \mathcal{F}_{\tau_{i-1} + t_1} \)-measurable. Note that the last quantity is zero by the optional sampling theorem (see also Remark 2.2.14 in [12]). Now we pass to the limit as \( T \to \infty \) and get \( E^{P_i} \left[ \int_{0}^{t_2} e^{-\lambda t} f(X_t) dt \right] = 0 \).

To justify such limit procedure one can use the estimate
\[
e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}} \int_0^T e^{-\lambda t} |Af(X_t)| dt \leq Z_0, \quad T > 0,
\]
where \( Z_0 := \|Af\|_{\infty} (t_2 + \tau_{i-1})e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}} \) is bounded.

Let us denote by \( Q_i \) the law of \( Y^i \) on \((C_E, \mathcal{B}(E))\). We have (using [13])
\[
g(\lambda, f) = \sum_{i \geq 1} \alpha_i E^{P_i} \left[ \int_0^t e^{-\lambda t} f(Y^i_t) dt \right] = \sum_{i \geq 1} \alpha_i E^{Q_i} \left[ \int_0^{\tau^Y_{V_i}} e^{-\lambda t} f(X_t) dt \right]. \tag{79}
\]
Note that, for any \( B \in \mathcal{B}(E) \),
\[
\mu_{i+1}(B) = \frac{1}{\alpha_i} E^{P_i} \left[ e^{-\lambda \tau_{i-1}} 1_{\{\tau_{i-1} < \infty\}} e^{-\lambda \eta_i} 1_{\{\eta_i < \infty\}} 1_B(X_{\tau_i}) \right] \tag{80}
\]
and, for \( i \geq 1 \),
\[
\alpha_i = \alpha_{i-1} E^{P_i} \left[ e^{-\lambda \eta_i} 1_{\{\eta_i < \infty\}} \right] = \alpha_{i-1} E^{Q_i} \left[ e^{-\lambda \tau^Y_{V_i}} 1_{\{\tau^Y_{V_i} < \infty\}} \right] = \prod_{k=1}^{i} E^{Q_k} \left[ e^{-\lambda \tau^Y_{V_k}} 1_{\{\tau^Y_{V_k} < \infty\}} \right].
\]
Now \( \mu_1 = \delta_x \) determines \( Q_1 \) by uniqueness of the stopped martingale problem and then \( Q_1 \) determine \( \mu_2 \) by (50). Proceeding in this way, \( Q_1, \ldots, Q_1 \) determine \( \mu_{i+1} \) and again by uniqueness this characterize \( Q_{i+1}, i \geq 1 \). By (74), for any \( \lambda > 0, \) for any \( f \in C_b(E), \) \( g(\lambda, f) \) is completely determined independently of the martingale solution \( P \) for \( (A, \delta_x) \) we have chosen. This completes the proof. \( \blacksquare \)

Combining Theorems 23 and 26 and using Corollary 25 we get the following localization principle. It extends Theorem 6.6.1 in [30] and shows that to perform the localization procedure it is enough to have existence of (global) martingale solutions of any \( x \in E \).

**Theorem 27** Assume that \( A \) verifies Hypothesis 18 and that for any \( x \in E \) there exists a martingale problem for \( (A, \delta_x) \). Suppose that there exists a family \( \{U_j\}_{j \in J} \) of open sets \( U_j \subset E \) with \( \sqcup_{j \in J} U_j = E \) and linear operators \( A_j \) with the same domain of \( A, i.e., \ A_j : D(A) \subset C_b(E) \rightarrow B_b(E), \ j \in J \) such that

i) for any \( j \in J \), the martingale problem for \( A_j \) is well-posed.

ii) for any \( j \in J, f \in D(A) \), we have \( A_j f(x) = A f(x), \ x \in U_j \).

Then the martingale problem for \( A \) is well-posed. In addition, \( (P^\mu) \) depends measurably on \( x \) and so formula (58) holds for any \( \mu \in \mathcal{P}(E) \).

**Proof.** Since \( E \) is a separable metric space we can consider a countable sub-covering of \( \{U_j\}_{j \in J} \) that we denote by \( \{U_k(k \geq 1 \text{ i.e. }, (U_k)_{k \geq 1} \subset \{U_j\}_{j \in J} \text{ and } \sqcup_{k \geq 1} U_k = E \} \).

By Corollary 25 we deduce that the stopped martingale problem for \( (A, \mu, U_k) \) is well-posed for any \( \mu \in \mathcal{P}(E) \) and for any open set \( U_k \). Applying Theorem 20 we obtain the first assertion. The measurability assertion follows from Corollary 22. \( \blacksquare \)

We state another result on well-posedness in which one considers an increasing sequence of open sets (cf. Theorem 6.6.3 in [12]). It extends Corollary 10.1.2 in [30].

**Theorem 28** Let \( \mu \in \mathcal{P}(E) \) and let \( (U_k)_{k \geq 1} \) be an increasing sequence of open sets in \( E \), i.e., \( U_k \subset U_{k+1}, k \geq 1 \). Suppose that, for any \( k \geq 1 \), there exists a unique (in law) solution for the stopped martingale problem for \( (A, \mu, U_k) \).

Let \( Z^k \) be a solution for the stopped martingale problem for \( (A, \mu, U_k) \) defined on a probability space \( (\Omega^k, \mathcal{F}^k, P^k) \) and consider

\[
\tau_k = \tau_{\delta^k}^R = \inf \{t \geq 0 : Z_t^k \not\in U_k \}.
\]

There exists a unique solution for the martingale problem for \( (A, \mu) \) if, for any \( t > 0, \)

\[
\lim_{k \rightarrow \infty} P^k(\tau_k \leq t) = 0. \tag{81}
\]

**Proof.** One can adapt without difficulties the proof of Theorem 6.6.3 in [12] which deals with càdlàg martingale solutions. To this purpose, using (51), one first proves that there exists a continuous process \( Z_\infty \) with values in \( E \) such that the law of \( Z^k \) converges in the Prokhorov distance to the law of \( Z_\infty \). One checks that \( Z_\infty \) is a solution of the martingale problem for \( (A, \mu) \). Also the uniqueness part can be proved as in [12]. \( \blacksquare \)

Applying Theorems 28 and 23 we obtain

**Corollary 29** Assume that \( A \) verifies Hypothesis 18. Suppose that there exists an increasing sequence of open sets \( (U_k)_{k \geq 1} \) in \( E \) and linear operators \( A_k \) with the same domain of \( A \). Moreover, assume:

i) for any \( k \geq 1 \), the martingale problem for \( A_k \) is well-posed;

ii) for any \( k \geq 1 \), \( f \in D(A) \), we have \( A_k f(x) = A f(x), \ x \in U_k \).
For \( x \in E \), let \( X^k = X^{k,x} \) be a martingale solution for \((A_k, \delta_x)\) defined on a probability space \((\Omega^k, \mathcal{F}^k, P^k)\); define
\[
\tau_k = \tau_{x}^k = \inf\{t \geq 0 : X^k_t \notin U_k\}.
\]
Then the martingale problem for \( A \) is well-posed if, for any \( x \in E \), for any \( t > 0 \),
\[
\lim_{k \to \infty} P^k(\tau_k \leq t) = 0.
\] (82)

**Proof.** By Theorem 20 it is enough to prove that for any \( x \in E \), the martingale problem for \((A, \delta_x)\) is well-posed. Let us fix \( x \in E \). By Corollary 25 the stopped martingale problems for \((A, \delta_x, U_k)\) are well-posed, \( k \geq 1 \).

If \( X^k \) is a solution of the martingale problem for \((A_k, \delta_x)\) defined on \((\Omega^k, \mathcal{F}^k, P^k)\) then \( Z^k := (X^k_t \wedge \tau_k^x)_{t \geq 0} \) is a solution for the stopped martingale problem for \((A_k, \delta_x, U_k)\), with \( \tau_k = \tau_{Z^k}^U \). If follows that (82) is just (81). By Theorem 28 there exists a unique martingale solution for \((A, \delta_x)\) and this finishes the proof.

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