ON CONVERGENCE IN THE SUBPOWER HIGSON CORONA OF METRIC SPACES

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Abstract. The subpower Higson corona of a proper metric space is defined in [5]. We prove that, unlike to the Higson corona, the closure of a σ-compact subset of the subpower Higson corona of a proper unbounded metric space does not necessarily coincide with its Stone-Čech corona.

1. Introduction

The notions of Higson compactification and Higson corona play an important role in coarse geometry and, in particular, in the asymptotic dimension theory (see, e.g., [6, 2, 1]).

The sublinear Higson corona $\nu_{L}(X)$ was introduced and investigated in [3] in connection with the asymptotic Assouad-Nagata dimension theory. The subpower Higson corona of a proper metric space is defined in [5]. This is a counterpart of the sublinear Higson corona. In some sense, the subpower Higson corona is an intermediate corona between the sublinear Higson corona and the Higson corona.

J. Keesling proved that the closure of a $\sigma$-compact subset in the Higson corona of a proper metric space is homeomorphic to its Stone-Čech compactification [4, Theorem 1]. In particular, the Higson corona does not contain non-trivial convergent sequences. This is not the case for the sublinear corona as it is proved in [3] that the sublinear corona $\nu_{L}(X \times \mathbb{R}_{+})$ contains a topological copy of $\nu_{L}(X) \times [0, 1]$.

In the first version of this note the authors erroneously claimed that the subpower corona behaves similarly to the remainder of the Stone-Čech compactification, i.e., the sublinear Higson corona of a noncompact proper metric space does not contain nontrivial convergent sequences. The authors express their gratitude to Roman Pol and Yutaka Iwamoto for indicating the gap in the proof.

2. Preliminaries

In the sequel, $(X, d)$ is assumed to be an unbounded, proper metric space with the basepoint $x_0$. Recall that a metric space is said to be proper if every its bounded and closed subset is compact.

Let $|x| = d(x, x_0)$ for $x \in X$.

The following notion is more general than that introduced in [5].

Definition 2.1. A function $p: \mathbb{R}_{+} \to \mathbb{R}_{+}$ is called asymptotically subpower if for every $\alpha > 0$ there exists $t_0 > 0$ such that $p(t) < t^\alpha$ for all $t > t_0$.

Definition 2.2. A bounded and continuous function $f: X \to \mathbb{R}$ is called Higson subpower if for every asymptotically subpower function $p$ we have $\lim_{|x| \to \infty} \text{diam}(f(B_{p(|x|)}(x))) = 0$, more precisely if for any asymptotically subpower function $p$ and $\epsilon > 0$ there exists $r_0 > 0$ that $\text{diam}(f(B_{p(|x|)}(x))) < \epsilon$ for all $x \in X$ such that $|x| > r_0$.

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It is easy to verify that Higson subpower functions form a closed subalgebra in the algebra of all continuous and bounded functions on a proper metric space $X$. This algebra contains the constant functions and separates points and closed sets. The compactification of $X$ that corresponds to this subalgebra is called the Higson subpower compactification and is denoted by $h_{P}X$. The remainder of this compactification $h_{P}X \setminus X$ is denoted by $\nu_{P}X$ and is called the subpower Higson corona of $X$.

A subset $D$ of a space $X$ is called $C$-embedded if any continuous bounded function on $D$ can be continuously extended over $X$.

3. Results

**Lemma 3.1.** For any subsets $A, B$ of a space $X$ such that $\nu_{P}X \cap \overline{A \cap B} = \emptyset$ there exist constants $\alpha, r_{0} > 0$ such that $\max\{d(x, A), d(x, B)\} \geq |x|^{\alpha}$ for every $x \in X$ such that $|x| \geq r_{0}$.

**Proof.** Let $A, B \subseteq X$ and let $\nu_{P}X \cap \overline{A \cap B} = \emptyset$. Suppose, that the condition does not hold, i.e. for any $\alpha, r_{0} > 0$ there exists $x \in X$, such that $|x| \geq r_{0}$, but $\max\{d(x, A), d(x, B)\} < |x|^{\alpha}$. Therefore there exist a sequence $(x_{n})$ of elements of $X$ with the property, that for all $n \in \mathbb{N}$ it is $|x_{n}| \geq 2n^{2}$ and $\max\{d(x_{n}, A), d(x_{n}, B)\} < |x_{n}|^{\frac{1}{10}}$. Thus, there exist sequences $(a_{n})$ and $(b_{n})$ of elements of $A$ and $B$ respectively, such that $d(x_{n}, a_{n}) < |x_{n}|^{\frac{1}{10}}$ and $d(x_{n}, b_{n}) < |x_{n}|^{\frac{1}{10}}$. Let us define continuous function $F : h_{P}X \to \mathbb{R}$ such that $F_{\overline{A \cap \nu_{P}X}} = 1$ and $F_{\overline{B \cap \nu_{P}X}} = 0$. Existence of such a function is guaranteed by normality of the space $h_{P}X$ and fact that sets $\overline{A \cap \nu_{P}X}$ and $\overline{B \cap \nu_{P}X}$ are closed and disjoint. Since $F$ is continuous, it is an extension of some Higson subpower function $f : X \to \mathbb{R}$. Moreover there exist open neighborhoods $U$ and $V$ of $\overline{A \cap \nu_{P}X}$ and $\overline{B \cap \nu_{P}X}$ respectively, which contains every but finite elements of sequences $(a_{n})$ and $(b_{n})$ respectively, such that $f_{U} > 3/4$ and $f_{V} < 1/4$. Indeed, every infinite sequence in a compact Hausdorff space has at least one cluster point and for the sequences $(a_{n})$ (respectively $(b_{n})$) all the cluster points can belong only to $\overline{A \cap \nu_{P}X}$ and (respectively $\overline{B \cap \nu_{P}X}$). No point of $X$ can be a cluster point of these sequences, because unboundedness of $(|x_{n}|)$ easily implies unboundedness of $(|a_{n}|)$ and $(|b_{n}|)$.

Now let us consider the function $p : \mathbb{R}_{+} \to \mathbb{R}_{+}$ defined as follows: $p(t) = 1$ for $t < |x_{1}|$ and $p(t) = |x_{n}|^{1/n}$ for $t \in [|x_{n}|, |x_{n+1}|]$, $n = 1, 2, \ldots$. The function $p$ is asymptotically subpower, as for any $\alpha > 0$ there exists $n \in \mathbb{N}$, such that $1/n < \alpha$, and therefore taking $r_{0} = |x_{n}| \geq 1$ for $t > r_{0}$ with the property that $t \in [|x_{m}|, |x_{m+1}|)$ the condition $p(t) = |x_{m}|^{1/m} \leq t^{1/m} \leq t^{1/n} < t^{\alpha}$ holds.

Moreover, for any $n \in \mathbb{N}$ we have $a_{n}, b_{n} \in B_{p(|x_{n}|)}(x_{n})$, as $|x_{n}|^{\frac{1}{10}} < |x_{n}|^{\frac{1}{10}} = p(|x_{n}|)$ and so $\text{diam}(B_{p(|x_{n}|)}(x_{n})) > 1/2$ for every but finite elements of $(x_{n})$.

Therefore $\lim_{|x_{n}| \to \infty} \text{diam}(B_{p(|x_{n}|)}(x_{n})) \neq 0$. Contradiction.

The following statement is a counterpart of the corresponding property of the sublinear corona [3].

**Lemma 3.2.** For every subsets $E_{1}, E_{2}$ of a space $X$ the fact that there exist $r_{0}, \alpha > 0$ such that $\max\{\rho(x, E_{1}), \rho(x, E_{2})\} \geq |x|^\alpha$ for all $x \in X$ with $|x| \geq r_{0}$ implies $\nu_{P}X \cap \overline{E_{1} \cap E_{2}} = \emptyset$.

**Proof.** Let $E_{1}, E_{2} \subseteq X$ and let $r_{0}$ and $\alpha$ satisfy the assumptions. Let $F_{i} = E_{i} \setminus B_{r_{0}+r_{0}^{\alpha}}(x_{0})$ for $i \in \{1, 2\}$. Let also $f : X \to \mathbb{R}$ be defined by the formula $f(x) = \rho(x, F_{1}) + \rho(x, F_{2})$, $x \in X$. Remark that $f(x) \geq |x|^\alpha$ for all $x$ with $|x| \geq r_{0}$, and also $f(x) \geq r_{0}^{\alpha} \geq |x|^\alpha$ for all $x$ with $|x| \leq r_{0}$. Thus, for every $x \in X$ we have $f(x) \geq |x|^\alpha$ and $f(x) > 0$. For $i \in \{1, 2\}$, define a function $g_{i} : X \to \mathbb{R}$, $g_{i}(x) = \rho(x, F_{i})/f(x)$. Let $p$ be an arbitrary asymptotically subpower function and let $\epsilon > 0$. Let $\tilde{r}_{0} > (3/\epsilon)^{2/\alpha}$ be large enough so that for every $x \in X$ with $|x| > \tilde{r}_{0}$
we have $p(|x|) < |x|^a/2$ and let $y \in B_{p(|x|)}(x)$. Then for $i \in \{1, 2\}$ the function $g_i$ is continuous, bounded and

$$|g_i(y) - g_i(x)| = \left| \frac{\rho(y, F_i)}{f(y)} - \frac{\rho(y, F_i)}{f(x)} + \frac{\rho(y, F_i)}{f(x)} - \frac{\rho(x, F_i)}{f(x)} \right| \leq \rho(y, F_i) \left| \frac{1}{f(y)} - \frac{1}{f(x)} \right| + \frac{\rho(y, F_i) - \rho(x, F_i)}{f(x)} \leq \frac{\rho(y, F_i)}{f(x)f(y)}|f(x) - f(y)| + \frac{\rho(y, x)}{f(x)} \leq \frac{2\rho(y, x)}{f(x)} + \frac{\rho(y, x)}{f(x)} \leq 3\frac{\rho(x, y)}{|x|^a}.$$

Then for every $x \in X$ with $|x| > r_0$ we have $|g_i(y) - g_i(x)| < \epsilon$, and thus $g_i$ is a subpower Higson function for $i \in \{1, 2\}$. Let $\bar{g}_i : h_pX \to \mathbb{R}$ be the unique extension of $g_i$ onto the subpower Higson compactification, $i \in \{1, 2\}$. Since $g_1(x) + g_2(x) = 1$ for every $x \in X$, we see that $\bar{g}_1(x) + \bar{g}_2(x) = 1$ for every $x \in h_pX$. Moreover, $\bar{F}_i \subseteq \bar{g}_i^{-1}(\{0\})$ and $\nu_pX \cap \bar{F}_i = \nu_pX \cap \bar{E}_i$ for $i \in \{1, 2\}$. Therefore

$$\nu_pX \cap \bar{E}_1 \cap \bar{E}_2 = \nu_pX \cap \bar{F}_1 \cap \bar{F}_2 = \nu_pX \cap \bar{g}_1^{-1}(\{0\}) \cap \bar{g}_2^{-1}(\{0\}) = \emptyset.$$

The proof of the following result is suggested by Roman Pol.

**Theorem 3.3.** There exists a proper unbounded metric space whose subpower corona contains a $\sigma$-compact subset which is not $C^\ast$-embedded.

**Proof.** Let $\mathbb{N}^* = \mathbb{N} \setminus \{1\}$ and let

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \in [0, \sqrt{y_1}]\}.$$

Let $y_0 = (0, 0)$. We endow $Y$ with the max-metric $d$. Note that for $y = (y_1, y_2) \in Y$ such that $|y| \geq 1$ we have $|y| = y_1$. For $i \in \mathbb{N}^*$ let

$$Y_i = \{(y_1, y_1^{1/i}) \mid y_1 \geq 2\}$$

and

$$K_i = \bar{Y}_i \cap \nu_pY = \nu_pY_i.$$

Let also $K = \bigcup_{i \in \mathbb{N}} K_i$, $K' = \bigcup_{i \in 2\mathbb{N}+1} K_i$, and $K'' = \bigcup_{i \in 2\mathbb{N}} K_i$.

Remark that $K = K' \cup K''$ is a $\sigma$-compact subspace of $\nu_pY$ and consider a function $f : K \to [0, 1]$ such that $f|_{K'} \equiv 1$ and $f|_{K''} \equiv 0$. Clearly, $f$ is bounded. Let us verify that $f$ is well defined and continuous. To this end, let us show that for every $i \in \mathbb{N}^*$ there exists a set $W \subseteq Y$ such that $Y_i \subseteq W$, $\bigcup_{j \in \mathbb{N}^* \setminus \{i\}} Y_j \subseteq Y \setminus W$ and there exists $\alpha, r_0 > 0$ such that for every $y = (y_1, y_2) \in Y$ with $|y| \geq r_0$ we have

$$\max\{d(y, Y_i), d(y, Y \setminus W)\} \geq |y|^\alpha.$$

Let $i \in \mathbb{N}^*$ and

$$W = \{(y_1, y_2) \in Y \mid y_1 \geq 2, \ y_2 \in \left(y_1^{1/(i+1)}, y_1^{1/(i-1)}\right)\}.$$

Clearly, $Y_i \subseteq W$ and $\bigcup_{j \in \mathbb{N}^* \setminus \{i\}} Y_j \subseteq Y \setminus W$. Without loss of generality, one may consider the points $y \in Y$ with $|y| > 2$ and $y_1^{1/(i+1)} \leq y_2 \leq y_1^{1/(i-1)}$. Let $\alpha = \frac{1}{2(i+1)}$ and $r_0 = 8^{2i+1}$. Assume that $y_1^{1/(i+1)} \leq y_2 \leq y_1^{1/i}$. Then

$$\max\{d(y, Y_i), d(y, Y \setminus W)\} = \max\{d(y, Y_i), \min\{d(y, Y_{i+1}), d(y, Y_{i-1})\}\}.$$
Consider the following two cases.

(1) If \( \min \{d(y, Y_{j+1}), d(y, Y_{j-1})\} = d(y, Y_{j+1}) \), then
\[
\max \{d(y, Y_i), d(y, Y \setminus W)\} \geq \frac{1}{4} \left( y_1^{\frac{1}{i}} - y_1^{\frac{1}{i+1}} \right) \geq \frac{1}{4} \left( y_1^{\frac{1}{i+1}} - \frac{1}{i} - 1 \right) \geq \frac{1}{8} y_1^{\frac{1}{i+1}} \geq y_1^{\frac{1}{2(i+1)}} = |y|^\alpha.
\]

(2) If \( \min \{d(y, Y_{j+1}), d(y, Y_{j-1})\} = d(y, Y_{j-1}) \), then
\[
\max \{d(y, Y_i), d(y, Y \setminus W)\} \geq \frac{1}{4} \left( y_1^{\frac{1}{i}} - y_1^{\frac{1}{i-1}} \right) \geq \frac{1}{4} \left( y_1^{\frac{1}{i+1}} - \frac{1}{i} - 1 \right) \geq \frac{1}{8} y_1^{\frac{1}{i+1}} \geq y_1^{\frac{1}{2(i-1)}} \geq |y|^\alpha.
\]

Similar calculations can be made also for the case \( y_1^{\frac{1}{i}} \leq y_2 \leq y_1^{\frac{1}{(i-1)}} \). Using Lemma 3.2 we obtain
\[
\nu_p Y \cap \overline{Y_i} \cap \overline{Y \setminus W} = \emptyset.
\]
Moreover, for every \( j \in \mathbb{N}^* \), \( j \neq i \), we have
\[
\overline{Y_j} \cap \nu_p Y \subset \overline{Y \setminus W} \cap \nu_p Y =: F,
\]
i.e., \( K_j \subset F \). Since \( j \neq i \) is arbitrary, we also have \( \bigcup_{j \neq i} K_j \subset F \). At the same time \( K_i \cap F = \emptyset \), thus \( f \) is well-defined. Therefore, \( K_i \subset \nu_p Y \setminus F \), and, since \( F \) is closed in \( \nu_p Y \), we obtain that \( K_i \) is open in \( K \). Since \( i \) is arbitrary, we obtain that the sets \( K' \) and \( K'' \) are simultaneously open and closed in \( K \), which implies the continuity of \( f \).

Now, let \( Y_0 = \{ (y_1, 0) \in Y \mid y_1 \geq 2 \} \) and let \( K_0 = \nu_p Y_0 \). Show that \( K_0 \subset \overline{K} \). Assume the contrary. Then there exists \( k \in K_0 \setminus \overline{K} \). By the normality of \( h_p Y \), there exist disjoint open neighborhoods \( U_k \) and \( V_k \) of the point \( k \) and the set \( \overline{K} \) in the space \( h_p Y \) respectively. There are neighborhoods \( U_k' \) and \( V_k' \) of \( k \) and \( \overline{K} \) respectively such that \( U_k' \subset U_k \) and \( V_k' \subset \overline{K} \). Let \( U = \overline{U_k'} \cap Y \) and \( V = \overline{V_k'} \cap Y \). Remark that
\[
\overline{U} \cap \overline{V} \cap \nu_p Y \subset \overline{U_k'} \cap \overline{V_k'} \cap \nu_p Y \subset U_k \cap \overline{V_k'} = \emptyset,
\]
therefore by Lemma 3.1 there exist constants \( \tilde{\alpha}, \tilde{r}_0 > 0 \) such that, for every \( y \in Y \) with \( |y| > \tilde{r}_0 \) we have
\[
\max \{d(y, U), d(y, V)\} \geq |y|^\tilde{\alpha}.
\]

Now, remark that, for every \( \alpha > 0 \), there exist \( i_\alpha \in \mathbb{N}^* \) and \( r_\alpha > 0 \) such that, for every \( y \in Y_0 \) with \( |y| > r_\alpha \) we have
\[
\max \{d(y, Y_{i_\alpha}), d(y, Y_0)\} < |y|^\alpha.
\]

Indeed, let \( \alpha > 0 \). Take \( i_\alpha \in \mathbb{N}^* \) such that \( i_\alpha > \frac{2}{\alpha} \) and \( r_\alpha = 2^i \alpha \). At the same time, for \( y = (y_1, 0) \in Y_0 \) such that \( |y| = y_1 > r_\alpha \) we have
\[
\max \{d(y, Y_{i_\alpha}), d(y, Y_0)\} = d(y, Y_{i_\alpha}) \leq 2y_1^{\frac{1}{i_\alpha}} < y_1^{\frac{2}{\alpha}} < |y|^\alpha.
\]

Let \( A = Y_0 \cap U \). Since \( k \in \overline{A} \), the set \( A \) is unbounded in the metric space \( Y \). Consider the set \( A_0 = \{ (a_n, 0) \in A \mid a_n > n, n \in \mathbb{N}^* \} \), and, for \( i \in \mathbb{N}^* \) the sets \( A_i = \{ (a_n, a_n^{\frac{1}{i}}) \mid (a_n, 0) \in A \} \). Clearly, \( A_i \subset Y_i \) for every \( i \in \mathbb{N}^* \cup \{0\} \). Also, \( A_0 \subset U \). Moreover, since \( K_i = \nu_p Y_i \) for every \( i \in \mathbb{N}^* \), and the set \( V_{\overline{K}} \) is an open neighborhood of \( \overline{K} \) (in particular, for every \( K_i z i \in \mathbb{N}^* \)), for every \( i \in \mathbb{N}^* \) there exist \( r_i' > 0 \) such that for every \( y \in Y_i \), \( |y| > r_i' \), we have \( y \in V_{\overline{K}} \cap Y \subset V \). Indeed, \( Y_i \setminus V_{\overline{K}} = \overline{Y_i} \setminus V_{\overline{K}} \) is compact and therefore bounded subset of \( Y \) for every \( i \in \mathbb{N}^* \).
Let $\alpha, r_0 > 0$. Choose $i_\alpha \in \mathbb{N}^*$ and $r_{i_\alpha} > 0$ as is described above. Choose a point $y' = (a, 0) \in A_0 \subset Y$ so that $|y'| = a > \max \{r_0, r_{i_\alpha}, 2r_{i_\alpha}'\}$, where $r_{i_\alpha}'$ is picked for $i_\alpha$ as described above. At the same time

$$\max \{d(y', U), d(y', V)\} \leq \max \{d(y', Y_0), d(y', Y_{i_\alpha})\} < |y'|^\alpha$$

and we obtain a contradiction.

One can similarly prove that $K_0 \subset \overline{K'}$ and $K_0 \subset \overline{K''}$. By the definition, $f$ cannot be continuously extended over $\nu_\mathbb{P}Y$.

\[\Box\]

**Corollary 3.4.** The closure of a $\sigma$-compact subset of the subpower corona of a proper unbounded metric space does not necessarily coincide with its Stone-Čech corona.

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