Condensate mass of QCD vacuum, a comparison between mass production via Wilson line approach and Schwinger effect

Sara Tahery, \(^a\) Xurong Chen \(^b\)

\(^a\) Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China
\(^b\) University of Chinese Academy of Sciences, Beijing 100049, China

\(^b\) Guangdong Provincial Key Laboratory of Nuclear Science, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China

Abstract

By using duality approach, we discuss condensate mass of QCD vacuum via dilaton wall background in presence of gluon condensation parameter \(c\). First from Wilson line calculation we find \(m_0^2\) whose behavior mimics that of QCD. The \(m_0^2\) value in our first step, is in agreement with QCD data. In the next step we consider produced mass \(m\) via Schwinger effect mechanism in presence of gluon condensation. Deriving an analytic relation between these two is our final interest such in presence of gluon condensation the ratio of \(\frac{m}{m_0}\) as a function of distance will be found. We will show that generally produced mass via Schwinger effect in presence of gluon condensation parameter, is not considerable in comparison with what is obtained via vacuum condensate directly.
1 Introduction

Studying strong interactions in QCD, shows importance of vacuum condensate in both theory and phenomenology. The QCD sum rules is known as a basic tool in this issue, but a strong alternative is needed for studying strongly coupled gauge theories in a non-perturbative formulation.

Many studies represent a holographic description of AdS/CFT in which, a strongly coupled field theory on the boundary of the AdS space is mapped to the weakly coupled gravity theory in the bulk of AdS [1,2]. AdS/QCD, is an approach in which one starts from a five-dimensional effective field theory somehow motivated by string theory and tries to fit it to QCD as much as possible. Although the first conjecture is based on conformal field theories, by considering some modifications in gravity duals, mass gap, confinement, and supersymmetry breaking could be included [3–5].

A holographic model which represents the gluon condensation parameter in a gravity background with Euclidean signature is known with the following action,

\[ S = -\frac{1}{2k^2} \int d^5x \sqrt{g} \left( R + \frac{12}{L^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right), \]  

where \( k \) is the 5-dimensional gravitational coupling, \( L \) is the radius of the asymptotic \( AdS_5 \) spacetime, \( R \) denotes the Ricci scalar and \( \phi \) is a massless scalar coupled with the gluon operator. To solve the Einstein equation and the dilaton equation of motion a suitable ansatz is the following dilaton-wall solution in Euclidean spacetime,

\[ ds^2 = \frac{L^2}{z^2} \left( \sqrt{1 - c^2 z^8} (dt^2 + dx^2) + dz^2 \right), \]

\[ \phi(z) = \sqrt{\frac{3}{2}} \log \frac{1 + cz^4}{1 - cz^4} + \phi_0, \]

where \( \phi_0 \) is a constant and \( x = x_1, x_2, x_3 \) are orthogonal spatial boundary coordinates. \( z \) denotes the 5th dimension, radial coordinate and \( z = 0 \) sets the boundary. \( c \) is gluon condensation with value \( 0 < c \leq 0.9\text{GeV}^4 \) [6,7]. The dilaton wall solution is related to the zero temperature case so this is the appropriate candidate to study condensate of vacuum.

By using the above solution we can study gluon condensate physics. An interesting phenomenon which could be studied in presence of gluon condensation parameter, is Schwinger effect. By definition pair production in presence of an external electric field is known as Schwinger effect in non-perturbative QED [8]. Due to this phenomenon when the external field is strong enough the virtual electron-positron pair become real particles. In other words vacuum is destroyed in presence of such a field. Although this context had been considered in QED first, it is not restricted to it any more and it has been extended to QCD [9].

According to this mechanism, the vacuum decay can be considered in presence of a deformation parameter [10], also according to the solution (2) potential analysis of Schwinger effect in presence of gluon condensation has been done in [11]. Therefore we skip this part and interested reader can...
refer to the mentioned references. But produced mass in such a Schwinger effect mechanism is one of our interest in this study.

This paper is organized as follows. In section 2 we will follow Wilson line calculation method based on [12] to find $m_0^2$ as produced mass from condensation.

In section 3 considering standard form of brane embedding in Schwinger effect holographic set up, we will calculate $m$ as produced mass where the effect of gluon condensation parameter will be taken into account. Having results from both above approaches we will compare them. In section 4 conclusions are given.

2 Mass from gluon condensate

Before starting our calculation we review method of [12], so if the reader is familiar with this reference can skip this part and continue from (11).

Condensation is associated with a dimension-5 operator constructed from the quark and gluon field as $q\sigma^{\mu\nu}G_{\mu\nu}\bar{q}$ where $\sigma^{\mu\nu}$ is an antisymmetric combination of matrices. So,

$$<g\bar{q}\sigma^{\mu\nu}G_{\mu\nu}q> = m_0^2 <\bar{q}q>,$$

where $g$ is a gauge coupling constant, $<\bar{q}q>$ is a quark condensate, and $m_0^2$ appears as a constant of proportionality in the conventional parametrization. Mixed condensate and the parameter $m_0^2$ both appear in a non-perturbative gauge invariant correlator.

$$\Psi(x_1,x_2) = <\bar{q}U_P(x_1,x_2)q>,$$

where $U_P(x_1,x_2)$ is a path-ordered Wilson line defined as,

$$U_P(x_1,x_2) = \text{P} \exp[ig \int_0^1 ds dx^\mu \frac{d A_\mu(x(s))}{ds}],$$

and $s$ is a parameter of the path running from 0 at $x = x_1$ to 1 at $x = x_2$. The path is taken to be a straight line. If one sets $U_P(x_1,x_2) = <\bar{q}q> Q(r)$, then $m_0^2$ is given by the coefficient of $r_2$, with $r = |x_1 - x_2|$, in the expansion of the function $Q$ as $r \rightarrow 0$,

$$Q(r) = 1 - \frac{1}{16} m_0^2 r^2 + \mathcal{O}(r^4),$$

which holds in Euclidean space and in Minkowski space it is modified by $r^2 \rightarrow -r^2$.

One can set an ansatz for computing the function $Q$ within gauge/string duality. The quark operators, the Wilson line on a four-manifold (which is the boundary of a 5-dimensional
manifold) and the function $Q$ are given in terms of the area (in string units) of a surface in the 5-dimensional manifold by

$$Q(r) = e^{-S}. \quad (9)$$

After warming up with above review, it is understandable if we find $Q$ as a function of $r$ then we can approximate $m_0^2$ parameter.

In continue we will find shape of the string describing the quark source. Let’s consider the Nambu-Goto action,

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\sigma^2 \sqrt{\det G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu}, \quad (10)$$

where,

$$X^\mu(\sigma) = (t, X_1, X_2, X_3, z). \quad (11)$$

Then from the background metric (2) and in static gauge $\sigma_1 = t$, $\sigma_2 = X_1 = x$ one may obtain

$$S = \frac{1}{2\pi\alpha'} \int_0^T dt \int_{-r/2}^{r/2} dx \frac{\sqrt{1 + cz^4}}{z^2} \left( \sqrt{1 - c^2z^8} + \left( \frac{dz}{dx} \right)^2 \right)^{\frac{1}{2}} \cdot (12)$$

where we set $L = 1$. Then the following relation should be satisfied,

$$\frac{\partial \mathcal{L}}{\partial(\partial_z z)} \partial_z z - \mathcal{L} = \text{Const.} \quad (13)$$

\[\text{In this calculation we considered only the first digit after the decimal point so } (1 - cz^4)^{0.016} \approx 1 \text{ and } (1 + cz^4)^{0.516} \approx \sqrt{1 + cz^4} \text{ are taken into account. In addition one can set } \phi_0 = 0 \text{ in [3] according to } [6, 7].\]
And from the boundary condition\(^4\)

\[ \text{at } z = z_*, \quad \Rightarrow \quad \frac{dz}{dx} = 0, \quad (14) \]

the constant value in that right hand side of \((13)\) could be found. Thus the solution of \((13)\) is,

\[
\int_{-\frac{z}{2}}^{z} \frac{dz}{\sqrt{\left(\frac{z}{z_*}\right)^4 \left(1+cz_*^4\right) \left(1-c^2z_*^8\right) - \sqrt{1-c^2z_*^8}}}.
\]

\[
(15)
\]

With change of variable \(u = \frac{z}{z_*}\), shape of the string appears as,

\[
r = z_* \int_{0}^{1} \frac{u^2 du}{\sqrt{\frac{(1+cz_*^4)(1-c^2z_*^8)}{(1+c^4z_*^4)\sqrt{1-c^2z_*^8}} - u^4\sqrt{1-c^2z_*^8} u^8}}.
\]

\[
(16)
\]

After describing the string shape, now we need to calculate the renormalized area of the surface in figure 1. So we choose the gauge \(\sigma_1 = X_1 = x\) and \(\sigma_2 = z\), then from the action \((10)\) we lead to\(^5\)

\[
S = \frac{1}{2\pi\alpha'} \int_{-\frac{z}{2}}^{\frac{z}{2}} dx \int_{0}^{z_*} dz \sqrt{\frac{1 + cz_*^4}{z_*^2}}.
\]

\[
(17)
\]

From \((17)\) the regularized answer of the integral is,

\[
S_{\text{reg}} = \frac{r}{2\pi\alpha'z_*} \left\{ \frac{1}{\epsilon} - 2F_1\left[ -\frac{1}{2}, -\frac{1}{4}, \frac{3}{4}, -cz_*^4 \right] \right\}.
\]

\[
(18)
\]

We subtract \(\left\{ \frac{r}{2\pi\alpha'z_*} \left( \frac{1}{\epsilon} - a \right) \right\}\) to deal with the power divergence where \(a\) is a constant must be specified from renormalization conditions, so the answer of the integral \((17)\) given by the hypergeometric function is,

\[
S_{\text{reg}} = -\frac{r}{2\pi\alpha'z_*} 2F_1\left[ -\frac{1}{2}, -\frac{1}{4}, \frac{3}{4}, -cz_*^4 \right] + a.
\]

\[
(19)
\]

Up to now, we have found both string shape \((16)\) and the action \((19)\) describing the renormalized area. Relating them to each other, we will have behavior of function \(Q\) versus \(r\). It is not clear how to do this but by considering two important limiting cases, long distances and short distances, we can analyze the behavior of the function.

We begin with \((16)\). By considering \(c_1 = cz_*^4\), one can check \(c_1 \rightarrow 0\) leads to long \(r\) and \(c_1 \rightarrow 1\) leads to short \(r\).

Expanding right hand side of \((16)\) around these two values of \(c_1\) will show behavior of \(r\) at limiting cases.

So at short distances the asymptotic behavior of \((16)\) is given by,

\[
r \approx \sqrt[4]{cz_*^4},
\]

\[
(20)
\]

\(^4\)From the string configuration it is easy to find that \(z_*\), turning point of the string shows maximum of \(z\).

\(^5\)Same approximations with what have been done in calculation of \((12)\).
and at long distances the asymptotic behavior of (10) is given by,

$$ r \approx \frac{\sqrt{\pi}}{6} z^* \left(2 + cz^*_4\right) \frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}. $$

(21)

In the same manner we expand (19) around $c_1 \to 0$ and $c_1 \to 1$ to find behavior of the action at long distances and short distances respectively.

Then at short distances, the action (19) behaves as,

$$ S = \frac{r}{2\pi\alpha'} \left(-1 + \frac{\sqrt{\pi}}{8} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} cz^*_4 + a\right), $$

(22)

and at long distances, the action (19) behaves as,

$$ S = \frac{r}{2\pi\alpha'} \left(-1 + \frac{cz^*_4}{6} + a\right). $$

(23)

First we should fix the value of $a$. According to the standard normalization of $Q$ we may impose the condition $Q(0) = 1$ [13], which gives $a = 0$.

Combining (22) with (20) we find the desired behavior of the function $Q$ at short distances as,

$$ Q = 1 - \frac{1}{2\pi\alpha'} \left(\frac{3}{10}\right) \sqrt{c} R^2 + O(r^4), $$

(24)

where $R = r - C_2$ and $C_2$ is a constant value. Finally comparing (8) with (24) we may approximate mass at short distances as,

$$ m^2_0 \approx \frac{12}{5} \frac{\sqrt{c}}{\pi\alpha'}. $$

(25)

Considering $\frac{1}{\alpha'} = 0.94$ and $c = 0.9 GeV^4$, we can estimate $m^2_0$ as,

$$ m^2_0 \approx 0.63 GeV^2. $$

(26)

According to the original phenomenological estimate based on the QCD sum rules [14] it is given by $m^2_0 = 0.8 \pm 0.2 GeV^2$, therefore (26) shows an acceptable result.

To deal with long distances we combine (23) with (21) and consider leading order of $r$, shows that at long distances the function $Q$ decays exponentially, as it is expected in QCD.

$$ Q = e^{-S}, $$

$$ = e^{-M^2 r^2}, $$

(27)

where,

$$ M \approx \frac{1}{\sqrt{12\pi\alpha'}} \sqrt{c}. $$

(28)
Let us estimate the above $M$, with $\frac{1}{\alpha'} = 0.94$ and $c = 0.9 GeV^4$, it is given by,

$$M \approx 0.15 GeV,$$

whose value is close to pion mass. Same with result of [12], it is understood that at long distances the correlator is dominated by the lightest meson contribution.

Having found an estimate for function $Q$ at short distances and behavior of that at long distances, we close this section and in the next step we will study produced mass by Schwinger effect in presence of gluon condensation parameter.

3 Mass from Schwinger effect

Recall that Schwinger effect is about vacuum decay in presence of an external electric field. In fact before vacuum decay, there is a potential barrier as $V_{CP+SE}$ which is the sum of the Coulomb potential (CP) and static energy (SE). Once the external field is turned on, the total potential is $V_{tot} = V_{CP+SE} - E_0 x$ that includes electrostatic potential between test particles. In this relation $E$ is external electric field and $x$ is the distance between particles. It is clear that increasing field will suppress the potential barrier after a critical value, so vacuum decays and virtual test particles become real [15].

![Figure 2: The holographic set up to consider Schwinger effect test particles.](image)

As we mentioned before, the Schwinger effect potential analysis in presence of gluon condensation parameter has been studied in [11]. In this section we follow mass production in the same mechanism. To avoid the divergent mass in a holographic way, a probe D3-brane attached to the
test particles is located at an intermediate position \( z_0 \). Then the mass formula is given by,
\[
m = \frac{1}{2\pi \alpha'} \int_{z_0}^{\infty} dz \sqrt{\det G_{\mu\nu} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu},
\] (30)
thus with the induced metric on the string world-sheet and in static gauge one may obtain,
\[
m = \frac{1}{2\pi \alpha'} \int_{z_0}^{\infty} dz \sqrt{1 + \frac{cz^4}{z^2}} \left( \sqrt{1 - c^2 z^8} + \left( \frac{dz}{dx} \right)^2 \right)^{\frac{1}{2}}.
\] (31)
Combining this with (15) one may find,
\[
m = \frac{1}{2\pi \alpha'} \int_{z_0}^{\infty} dz \sqrt{1 + \frac{cz^4}{z^2}} \sqrt{1 - \frac{c^2}{1 - c^2 z^8}}.
\] (32)
With changing variable \( y = \frac{z}{z_0} \) and \( b = \frac{z_0}{z} \), (32) is,
\[
m = \frac{1}{2\pi \alpha'} \frac{1}{b^2 z_0} \frac{1}{\sqrt{(1 + c_1)^{1/2}} \sqrt{(1 - c_1^2)^{1/2}}} \int_{1}^{\infty} dy \ y^{-4}(1 + c_1 b^4 y^4)^{3/2}(1 - c_1 b^4 y^4)^{1/2},
\] (33)
recall that we defined \( c_1 = cz^4 \). Integral (33) is solvable analytically, however since \( b < 1 \) we consider the leading term of \( b \) in the answer and ignore others. Then the answer is,
\[
m = \frac{1}{6\pi \alpha' z_0 b^2} \frac{1}{\sqrt{(1 + c_1)^{1/2}} \sqrt{(1 - c_1^2)^{1/2}}}.
\] (34)
Again we expand (34) near \( c_1 \to 0 \) and \( c_1 \to 1 \) to consider mass at long distances and short distances respectively.
At long distances (34) behaves as,
\[
m = \frac{1}{6\pi \alpha' z_0 b^2} \left( 1 - \frac{c_1}{2} \right) + \mathcal{O}(c_1^2),
\] (35)
and at short distances,
\[
m = \frac{1}{6\sqrt{2\pi \alpha' z_0 b^2}} \sqrt{c_1^2}.
\] (36)
Combining (36) with (20) and (35) with (21) we may study mass at short distances and long distances respectively.
At long distances produced mass in Schwinger effect behaves as,
\[
m = \frac{1}{6\pi \alpha' z_0 b^2} \left( 1 - \frac{\sqrt{c r^4}}{2} \right),
\] (37)
which shows with increasing \( r \) mass goes to zero.
And at short distances mass is given by,
\[
m = \frac{1}{6\sqrt{2\pi \alpha' z_0 b^2}} \sqrt{\frac{c r^4}{3}},
\] (38)
according to which, any discussion on produced mass in Schwinger effect significantly depends on
brane position \(z_0\) in the bulk, (in fact since \(b = \frac{m_0}{z_0}\), having one of \(b\) or \(z_0\) is enough). So, although we have phenomenological value for parameter \(c\) which can relate distance of produced pair and mass, but \(z_0 b^2\) always needs to be considered as some coefficient.

After deriving mass formula in Schwinger effect in presence of gluon condensation, it is interesting to compare the result with that of previous section. Recall that what we have found in last section is very different in its nature with Schwinger effect produced mass. \(m_0^2\) which appears in non-local and mixed condensates is a constant of proportionality in the conventional parametrization \(6\). What we have up to now, are two different values by two different approaches. \(m\) from Schwinger effect is exactly mass of quark antiquark which have been produced by vacuum decay. Although the responsible for vacuum decay and pair production is the external field \(E\), but in the background there is parameter of gluon condensation \(c\). On one hand if one accepts that in any mass production from vacuum some kind of condensation appears, then existence of parameter \(c\) in holographic Schwinger effect makes sense. On the other hand we have \(m_0^2\) from Wilson line calculation directly, means the condensation is happening based on \(c\). So these two approaches are different in -what is responsible for mass production- also in string and/ or brane configuration which leads to different computations, but they have parameter \(c\) which plays important role in both.

All above motivations make us compare \(m\) and \(m_0\). Considering both (36) and (25) we can find the following ratio approximately,

\[
\frac{m_0}{m} \approx 20z_0 b^2 r^2 \frac{\sqrt{c^3}}{r^4} .
\]  

(39)

As an example lets consider position of probe brane in the bulk as \(z_0 = \frac{z^*}{2}\), knowing the mentioned position we can find the fraction as,

\[
\frac{m_0}{m} \approx \frac{9}{4} \frac{\sqrt{(cr^4)^3}}{r^2} .
\]  

(40)

If we consider the intermediate position of the embedded brane in the bulk, then it is interesting to study the behavior of \(\frac{m_0}{m}\) schematically.

Figure 3 shows \(\frac{m_0}{m}\) as a function of distance and gluon condensation parameter. The distance has been considered \(0 < r < 1 \text{fm}\) or \(5(\text{GeV})^{-1}\), also \(c\) axis mentions value \(0 < c \leq 0.9\text{GeV}^4\). Considering parameter \(c\) around our desired value \(0.9\text{GeV}^4\) with distance \(r \approx 0.14 \text{fm} = 0.7\text{GeV}^{-1}\) lead to \(\frac{m_0}{m} = 1\). Increasing \(r\) after this crucial value, increases the ratio \(\frac{m_0}{m}\) significantly.

Notice that \(r\) in string configuration corresponds to diameter of meson. As a case, lets consider the ratio \(\frac{m_0}{m} = 1\) and \(r \approx 2.6\text{GeV}^{-1} = 0.52\text{fm}\) corresponds to \(J/\psi\) diameter. It leads to very small value of the parameter \(c \approx 0.005\text{GeV}^4\). This value of \(c\) gives us the near zero value of

\(\text{Notice that even considering } m_0 \text{ as exact value of quark mass could not be correct , as we mentioned in last section the value of this parameter has been estimated by original phenomenological estimate based on the QCD sum rules }[14]. \text{ it is given by } m_0^2 = 0.8 \pm 02\text{GeV}^2.\)
\[ m_0^2 \approx 0.003 \text{GeV}^2 \text{ from (25). In comparison with (26) it is far from acceptable value which fit QCD data. But } J/\psi \text{ with } c \approx 0.9 \text{GeV}^4 \text{ results in } \frac{m_0}{m} \approx 18. \text{ Therefore as we expected parameter } m_0 \text{ suppresses Schwinger effect mass obviously.} \]

4 Conclusions

Gluon condensation is an important parameter since it represents many phenomenological aspects of QCD. In this work we considered dilaton wall background related to zero temperature to study vacuum condensation. First we found \( m_0^2 \) by Wilson line calculation. Our results satisfy QCD behavior at both short distances and long distances. In the second step we studied another mass production mechanism, Schwinger effect in presence of gluon condensation in which an external electric field is responsible for vacuum decay. These two mechanisms naturally are different in approaches and physics both. However since gluon condensation parameter plays role in both of them, it was our interest to compare their results. We ended up by finding ratio of \( \frac{m_0}{m} \) as a function of \( r \) and brane position. We have found that generally produced mass via Schwinger effect in presence of gluon condensation parameter, is not considerable in comparison with what has been obtained via vacuum condensate directly.

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