Spatial risk measures and rate of spatial diversification

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Abstract

An accurate assessment of the risk of extreme environmental events is of great importance for populations, authorities and the banking/insurance industry. Koch (2017) introduced a notion of spatial risk measure and a corresponding set of axioms which are well suited to analyze the risk due to events having a spatial extent, precisely such as environmental phenomena. The axiom of asymptotic spatial homogeneity is of particular interest since it allows one to quantify the rate of spatial diversification when the region under consideration becomes large. In this paper, we first investigate the general concepts of spatial risk measures and corresponding axioms further. We also explain the usefulness of this theory for the actuarial practice. Second, in the case of a general cost field, we especially give sufficient conditions such that spatial risk measures associated with expectation, variance, Value-at-Risk as well as expected shortfall and induced by this cost field satisfy the axioms of asymptotic spatial homogeneity of order 0, −2, −1 and −1, respectively. Last but not least, in the case where the cost field is a function of a max-stable random field, we mainly provide conditions on both the function and the max-stable field ensuring the latter properties. Max-stable random fields are relevant when assessing the risk of extreme events since they appear as a natural extension of multivariate extreme-value theory to the level of random fields. Overall, this paper improves our understanding of spatial risk measures as well as of their properties with respect to the space variable and generalizes many results obtained in Koch (2017).

Key words: Central limit theorem; Insurance; Max-stable random fields; Rate of spatial diversification; Risk management; Risk theory; Spatial dependence; Spatial risk measures and corresponding axiomatic approach.

1 Introduction

Hurricane Irma, which affected many Caribbean islands and parts of Florida in September 2017 caused at least 134 deaths and catastrophic damage exceeding 64.8 billion USD in value. Such an example shows the prime importance for civil authorities and for the insurance industry of the accurate assessment of the risk of natural disasters, particularly

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†Throughout the paper, insurance also refers to reinsurance.
as, in a climate change context, certain types of extreme events become more and more frequent (see, e.g., Bevere and Mueller, 2014).

Motivated by the spatial feature of natural disasters, Koch (2017) introduced a new notion of spatial risk measure, which makes explicit the contribution of the space and enables one to account for at least part of the spatial dependence in the risk measurement. He also introduced a set of axioms describing how the risk is expected to evolve with respect to the space variable, at least under some conditions. These notions constitute relevant tools for risk assessment. For instance, the knowledge of the order of asymptotic spatial homogeneity allows the quantification of the rate of spatial diversification. Hence, they might be of interest for the banking/insurance industry. It should be highlighted that the literature about risk measures in a spatial context is very limited. To the best of our knowledge, the paper by Koch (2017) constitutes the first attempt to establish a theory about risk measures in a spatial context where the risks spread over a continuous geographical region.

In the following, the spatial risk measure associated with a classical risk measure \( \Pi \) and induced by a cost random field \( C \) (e.g., modelling the cost due to damage caused by a natural disaster) consists in the function of space arising from the application of \( \Pi \) to the normalized integral of \( C \) on various geographical areas. The contribution of this paper is threefold. First, we investigate further the notions of spatial risk measure and corresponding axioms introduced in Koch (2017). Especially, we show that, for a given region, the distribution of the normalized spatially aggregated loss is entirely determined by the finite-dimensional distributions of the cost field. Moreover, we propose alternative definitions of the concepts developed in Koch (2017) and show how the latter can be used by insurance companies to tackle concrete issues. Second, in the case of a general cost field, we especially give sufficient conditions such that spatial risk measures associated with expectation, variance, Value-at-Risk (VaR) as well as expected shortfall (ES) and induced by this cost field satisfy the axiom of asymptotic spatial homogeneity of order \( 0, -2, -1 \) and \( -1 \), respectively. Last but not least, we focus on the case where the cost field is a function of a max-stable random field. We mostly give sufficient conditions on both the function and the max-stable field such that spatial risk measures associated with expectation, variance, VaR as well as ES and induced by the resulting cost field satisfy the axiom of asymptotic spatial homogeneity of order \( 0, -2, -1 \) and \( -1 \), respectively. Max-stable random fields naturally appear when we are interested in extreme events having a spatial extent since they constitute an extension of multivariate extreme-value theory to the level of random fields (in the case of stochastic processes, see, e.g., de Haan, 1984; de Haan and Ferreira, 2006). They are particularly well suited to model the temporal maxima of a given variable (for instance a meteorological variable) at all points in space since they arise as the pointwise maxima taken over an infinite number of appropriately rescaled independent and identically distributed random fields. On the whole, this paper improves our understanding of spatial risk measures as well as of their properties with respect to the space variable. Moreover, it generalizes many results in Koch (2017).

The remainder of the paper is organized as follows. In Section 2, we recall and investigate further the notion of spatial risk measure and the corresponding set of axioms introduced in Koch (2017). Then, we introduce some concepts about mixing and central limit theorems for random fields. Finally, we provide some insights about max-stable random fields. Then, Section 3 presents our results relating to the properties of some spatial risk measures. Finally, Section 4 contains a short summary as well as some perspectives.
Throughout the paper, \((\Omega, \mathcal{F}, P)\) is an adequate probability space and \(d\) and \(\rightarrow\) designate equality and convergence in distribution, respectively. In the case of random fields, distribution has to be understood as the set of all finite-dimensional distributions. Finally, we denote by \(\nu\) the Lebesgue measure.

2 Spatial risk measures and other concepts

2.1 Spatial risk measures and corresponding axioms

First, we describe the setting required for a proper definition of spatial risk measures. Let \(\mathcal{A}\) be the set of all compact subsets of \(\mathbb{R}^2\) with a positive Lebesgue measure and \(\mathcal{A}_c\) be the set of all convex elements of \(\mathcal{A}\). Denote by \(\mathcal{C}\) the set of all real-valued and measurable random fields on \(\mathbb{R}^2\) having almost surely (a.s.) locally integrable sample paths. Let \(\mathcal{P}\) be the family of all possible distributions of random fields belonging to \(\mathcal{C}\). Each random field represents the economic or insured cost caused by the events belonging to specified classes and occurring during a given time period, say \([0, T_L]\). In the following, \(T_L\) is considered as fixed and does not appear anymore for the sake of notational parsimony. Each class of events (e.g., a European windstorm or a hurricane) will be referred to as a hazard in the following. Let \(\mathcal{L}\) be the set of all real-valued random variables defined on \((\Omega, \mathcal{F}, P)\). A risk measure typically will be some function \(\Pi : \mathcal{L} \rightarrow \mathbb{R}\). This kind of risk measure will be referred to as a classical risk measure in the following. A classical risk measure \(\Pi\) is termed law-invariant if, for all \(\tilde{X} \in \mathcal{L}\), \(\Pi(\tilde{X})\) only depends on the distribution of \(\tilde{X}\).

We first remind the reader of the definition of the normalized spatially aggregated loss, which enables one to disentangle the contribution of the space and the contribution of the hazards and underpins our definition of spatial risk measure.

**Definition 1** (Normalized spatially aggregated loss as a function of the distribution of the cost field). For \(A \in \mathcal{A}\) and \(P \in \mathcal{P}\), the normalized spatially aggregated loss is defined by

\[
L_N(A, P) = \frac{1}{\nu(A)} \int_A C_P(x) \, \nu(dx),
\]

(1)

where the random field \(\{C_P(x)\}_{x \in \mathbb{R}^2}\) belongs to \(\mathcal{C}\) and has distribution \(P\).

The quantity

\[
L(A, P) = \int_A C_P(x) \, \nu(dx)
\]

(2)

corresponds to the total economic or insured loss over region \(A\) due to specified hazards. For technical reasons and to favour a more intuitive understanding, we choose to consider \(L_N(A, P)\), which is the loss per surface unit and can be interpreted, in a discrete setting and in an insurance context, as the mean loss per insurance policy. Among other advantages, this normalization enables a fair comparison of the risks associated with regions having different sizes.

Since the field \(C_P\) is measurable, \(L(A, P)\) and \(L_N(A, P)\) are well-defined random variables. Moreover, they are a.s. finite as \(A\) is compact and \(C_P\) has a.s. locally integrable

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2Unless otherwise stated, by a.s., we mean \(P\)-a.s.
3See Section 2.2.
sample paths. The following proposition gives a sufficient condition for a random field to have a.s. locally integrable sample paths.

**Proposition 1.** Let \( d \geq 1 \) and \( \{Q(x)\}_{x \in \mathbb{R}^d} \) be a measurable random field. If the function

\[
E : \mathbb{R}^d \rightarrow \mathbb{R}, \quad x \mapsto E[|Q(x)|]
\]

is locally integrable, then \( Q \) has a.s. locally integrable sample paths.

**Proof.** Let \( A \) be a compact subset of \( \mathbb{R}^d \). First, since \( Q \) is measurable, \( \int_A |Q(x)| \nu(dx) \) is a well-defined random variable. By Fubini’s theorem, we have

\[
E \left[ \int_A |Q(x)| \nu(dx) \right] = \int_A E[|Q(x)|] \nu(dx) < \infty,
\]

which necessarily implies that

\[
\int_A |Q(x)| \nu(dx) < \infty \text{ a.s.}
\]

Since this is true for all \( A \) being a compact subset of \( \mathbb{R}^d \), we obtain the result. \( \square \)

We now recall the notion of spatial risk measure introduced by Koch (2017), which makes explicit the contribution of the space in the risk measurement.

**Definition 2** (Spatial risk measure as a function of the distribution of the cost field). A spatial risk measure is a function \( \mathcal{R}_\Pi \) that assigns a real number to any region \( A \in \mathcal{A} \) and distribution \( P \in \mathcal{P} \):

\[
\mathcal{R}_\Pi : \mathcal{A} \times \mathcal{P} \rightarrow \mathbb{R}, \quad (A, P) \mapsto \Pi(L_N(A, P)),
\]

where \( \Pi \) is a classical and law-invariant risk measure and \( L_N(A, P) \) is defined in (1).

Note that the assumption that \( \Pi \) is law-invariant is necessary for spatial risk measures to be defined in this way; see below for more details. For a given \( \Pi \) and \( P \in \mathcal{P} \), the quantity \( \mathcal{R}_\Pi(\cdot, P) \) is referred to as the spatial risk measure associated with \( \Pi \) and induced by the distribution \( P \). A nice feature is that, for many useful classical risk measures \( \Pi \) such as, e.g., variance, VaR and ES, this notion of spatial risk measure allows one to take (at least) part of the spatial dependence structure of the field \( C_P \) into account. Nevertheless, this is not true in the trivial case of expectation.

Now, we remind the reader of the set of axioms for spatial risk measures developed in Koch (2017). It concerns the spatial risk measures properties with respect to the space and not to the cost distribution, the latter being considered as given by the problem at hand. For any \( A \in \mathcal{A} \), let \( b_A \) denote its barycenter.

**Definition 3** (Set of axioms for spatial risk measures induced by a distribution). Let \( \Pi \) be a classical and law-invariant risk measure. For a fixed \( P \in \mathcal{P} \), we define the following axioms for the spatial risk measure associated with \( \Pi \) and induced by \( P \), \( \mathcal{R}_\Pi(\cdot, P) \):

1. **Spatial invariance under translation:** for all \( v \in \mathbb{R}^2 \) and \( A \in \mathcal{A} \), \( \mathcal{R}_\Pi(A + v, P) = \mathcal{R}_\Pi(A, P) \), where \( A + v \) denotes the region \( A \) translated by the vector \( v \).
2. **Spatial sub-additivity:**
   
   for all $A_1, A_2 \in \mathcal{A}$, $\mathcal{R}_\Pi(A_1 \cup A_2, P) \leq \min\{\mathcal{R}_\Pi(A_1, P), \mathcal{R}_\Pi(A_2, P)\}$.

3. **Asymptotic spatial homogeneity of order $-\alpha, \alpha \geq 0$:**
   
   for all $A \in \mathcal{A}_c$,
   
   $$\mathcal{R}_\Pi(\lambda A, P) = K_1(A, P) + \frac{K_2(A, P)}{\lambda^\alpha} + o\left(\frac{1}{\lambda^\alpha}\right),$$
   
   where $\lambda A$ is the area obtained by applying to $A$ a homothety with center $b_A$ and ratio $\lambda > 0$, and $K_1(\cdot, P) : \mathcal{A}_c \rightarrow \mathbb{R}$, $K_2(\cdot, P) : \mathcal{A}_c \rightarrow \mathbb{R}\setminus\{0\}$ are functions depending on $P$.

It is also reasonable to introduce the axiom of **spatial anti-monotonicity:** for all $A_1, A_2 \in \mathcal{A}$, $A_1 \subset A_2 \Rightarrow \mathcal{R}_\Pi(A_2, P) \leq \mathcal{R}_\Pi(A_1, P)$. The latter is equivalent to the axiom of spatial sub-additivity. These axioms appear natural and make sense at least under some conditions on the cost field $C_P$ and for some classical risk measures $\Pi$. The axiom of spatial sub-additivity indicates spatial diversification. If it is satisfied with strict inequality, an insurance company would be well advised to underwrite policies in both regions $A_1$ and $A_2$ instead of only one of them. The axiom of asymptotic spatial homogeneity of order $-\alpha$ quantifies the rate of spatial diversification when the region becomes large. Consequently, determining the value of $\alpha$ is of interest for the insurance industry. We refer to Section 2.2 for more details.

There are obviously some links between our notion of spatial risk measures and classical risk measures as for instance summarized in Föllmer and Schied (2004). Nevertheless, the inclusion of the space and the cost field in our definition sets our approach rather aside. Of course, the fact that the axioms of Definition 3 are satisfied depends on both the classical risk measure $\Pi$ and the cost field $C_P$. It might be of interest to determine for which classical risk measures the axioms are satisfied for the broadest class of cost fields. These classical risk measures could be considered as “adapted” to the spatial context.

**Remark 1.** Although the concept of spatial risk measure and related axioms naturally apply in an insurance context (see Section 2.2 for further details), they can also be used in the banking industry and financial markets. A potential application might be the assessment of the risk associated with event-linked securities such as CAT bonds. Furthermore, they can be used for a wider class of risks than those linked with damage due to environmental events. These concepts are actually insightful as soon as the risks spread over a geographical region. One might think, e.g., about the loss in value of real estate due to adverse economic conditions.

We close this section by deeply commenting on the previous concepts and giving slightly modified and more natural versions of previous definitions. First, we need the following useful result.

**Theorem 1.** Let $d \geq 1$ and $\{H(x)\}_{x \in \mathbb{R}^d}$ be a measurable random field having a.s. locally integrable sample paths. Moreover, let $A$ be a compact subset of $\mathbb{R}^d$ with positive Lebesgue measure. Then the distribution of

$$L_N(A, H) = \int_A H(x) \nu(dx)$$

only depends on $A$ and the finite-dimensional distributions of $H$. 

Proof. The proof is partly inspired from the proof of Theorem 11.4.1 in Samorodnitsky and Taqqu (1994). We assume that the random field $H$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fixed $\omega \in \Omega$, we denote by $H_\omega$ the corresponding realization of $H$ on $\mathbb{R}^d$ and by $H_\omega(x)$ the realization of $H$ at location $x$. By definition, we have, for almost every $\omega \in \Omega$,

$$L_N(A, H_\omega) = \frac{1}{\nu(A)} \int_A H_\omega(x) \, \nu(dx). \tag{3}$$

Now, let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be a probability space different from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $U$ be a random vector defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and following the uniform distribution on $A$, with density $f_U(x) = 1_{\{x \in A\}}/\nu(A), x \in \mathbb{R}^d$. From (3), it directly follows that, for almost every $\omega \in \Omega$,

$$L_N(A, H_\omega) = \int_{\mathbb{R}^d} H_\omega(x) f_U(x) \, \nu(dx). \tag{4}$$

Let us denote by $E_1$ the expectation with respect to the probability measure $\mathbb{P}_1$. We have

$$E_1[H_\omega(U)] = \int_{\mathbb{R}^d} H_\omega(x) f_U(x) \, \nu(dx),$$

giving, using (4), that

$$L_N(A, H_\omega) = E_1[H_\omega(U)].$$

Now, let $U_1, \ldots, U_n$ be independent replications of $U$ (which are independent of the random field $H$). The strong law of large numbers gives that, for almost every $\omega \in \Omega$,

$$L_N(A, H_\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n H_\omega(U_i) \mathbb{P}_1\text{-a.s.} \tag{5}$$

Therefore, using Fubini’s theorem, we deduce that, for $\mathbb{P}_1$-almost every $\omega_1 \in \Omega_1$,

$$L_N(A, H_\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n H_\omega(U_i(\omega_1)) \mathbb{P}\text{-a.s.} \tag{6}$$

Now, we choose $\omega_0 \in \Omega_1$ such that the (non-random) sequence $(U_1(\omega_0), U_2(\omega_0), \ldots)$ satisfies (6). We obtain

$$L_N(A, H_\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n H_\omega(U_i(\omega_0)) \mathbb{P}\text{-a.s.} \tag{7}$$

Equation (7) says that the distribution of $L_N(A, H)$ is determined by the finite-dimensional distributions at the points belonging to the set $\{U_i(\omega_0) : i \in \mathbb{N}\}$. This yields the result. \hfill \square

It is more natural, especially in terms of interpretation, to introduce the normalized spatially aggregated loss as a function of the cost field instead of its distribution, as shown immediately below.

**Definition 4** (Normalized spatially aggregated loss as a function of the cost field). The normalized spatially aggregated loss function is defined by

$$L_N : \mathcal{A} \times \mathcal{C} \to \mathbb{R}$$

$$(A, C) \mapsto \frac{1}{\nu(A)} \int_A C(x) \, \nu(dx). \tag{8}$$
Let $C_P \in \mathcal{C}$ be a random field with distribution $P$. Although a particular realization of $L_N(A, C_P)$ obviously depends on $C_P$ (through its corresponding realization), we know from Theorem 1 that its distribution is entirely characterized by $A$ and $P$. This explains our notation $L_N(A, P)$ instead of $L_N(A, C_P)$ in Definition 1. More precisely, let $C_P^{(1)}, C_P^{(2)} \in \mathcal{C}$ be random fields having the same distribution $P$. Then, $C_P^{(1)}$ and $C_P^{(2)}$ have the same finite-dimensional distributions, which implies that $L_N(A, C_P^{(1)}) \overset{d}{=} L_N(A, C_P^{(2)})$.

Similarly, it can appear more natural to define spatial risk measures as functions of the cost field instead of its distribution. Moreover, this allows spatial risk measures to be defined even in the case where the classical risk measure $\Pi$ is not law-invariant.

**Definition 5** (Spatial risk measure as a function of the cost field). A spatial risk measure is a function $R_{\Pi}$ that assigns a real number to any region $A \in \mathcal{A}$ and random field $C \in \mathcal{C}$:

$$R_{\Pi} : \mathcal{A} \times \mathcal{C} \to \mathbb{R}, \quad (A, C) \mapsto \Pi(L_N(A, C)),$$

where $\Pi$ is a classical risk measure.

For a given classical and law-invariant risk measure $\Pi$ and a given region $A \in \mathcal{A}$, the value of the spatial risk measure of Definition 5 is completely determined by the distribution of $L_N(A, C)$ by law-invariance of $\Pi$. Consequently, using Theorem 1, it is completely determined by $A$ and the distribution of $C$. This explains why Koch (2017) has introduced the notion of spatial risk measure as a function of the distribution of $C$ (see the reminder in Definition 2). Actually, if $\Pi$ is law-invariant, the spatial risk measures described in Definitions 2 and 5 refer to the same notion. For a given $\Pi$ and $C \in \mathcal{C}$, the quantity $R_{\Pi}(\cdot, C)$ is referred to as the spatial risk measure associated with $\Pi$ and induced by the cost field $C$.

Of course, we can also express the axioms recalled in Definition 3 for the spatial risk measures induced by a cost field $C \in \mathcal{C}$ introduced in Definition 5. On top of being more natural, it enables one to leave out the assumption of law-invariance for the classical risk measure $\Pi$.

**Definition 6** (Set of axioms for spatial risk measures induced by a cost field). Let $\Pi$ be a classical risk measure. For a fixed $C \in \mathcal{C}$, we define the following axioms for the spatial risk measure associated with $\Pi$ and induced by $C$, $R_{\Pi}(\cdot, C)$:

1. **Spatial invariance under translation:**
   for all $v \in \mathbb{R}^2$ and $A \in \mathcal{A}$, $R_{\Pi}(A + v, C) = R_{\Pi}(A, C)$, where $A + v$ denotes the region $A$ translated by the vector $v$.

2. **Spatial sub-additivity:**
   for all $A_1, A_2 \in \mathcal{A}$, $R_{\Pi}(A_1 \cup A_2, C) \leq \min\{R_{\Pi}(A_1, C), R_{\Pi}(A_2, C)\}$.

3. **Asymptotic spatial homogeneity of order $-\alpha$, $\alpha \geq 0$:**
   for all $A \in \mathcal{A}$,
   $$R_{\Pi}(\lambda A, C) \xrightarrow[\lambda \to \infty]{} K_1(A, C) + \frac{K_2(A, C)}{\lambda^\alpha} + o\left(\frac{1}{\lambda^\alpha}\right),$$
   where $\lambda A$ is the area obtained by applying to $A$ a homothety with center $b_A$ and ratio $\lambda > 0$, and $K_1(\cdot, C) : \mathcal{A}_c \to \mathbb{R}$, $K_2(\cdot, C) : \mathcal{A}_c \to \mathbb{R}\setminus\{0\}$ are functions depending on $C$.
For the reasons mentioned above, our opinion is that Definitions 4-6 rather than Definitions 1-3 should be used. All interpretations remain the same. This is what is done in the following.

2.2 Concrete applications to insurance

This section is dedicated to the potential connection between the concepts described above and real insurance practice. We especially show how they can be used for concrete purposes. In an insurance context, the quantity

\[ L(A, C) = \int_A C(x) \, \nu(dx) \]  

appearing in Definition 4 (or equivalently (2)) can be seen as a continuous and more complex version of the classical actuarial individual risk model. The latter can be formulated as

\[ S = \sum_{i=1}^{N} X_i, \]  

where \( S \) is the total loss, \( N \) denotes the number of insurance policies and, for \( i = 1, \ldots, N \), \( X_i \) is the claim associated with the \( i \)-th policy. The \( X_i \) are generally assumed to be independent but not necessarily identically distributed. In \( L(A, C) \), each location \( x \) corresponds to a specific insurance policy and thus each \( C(x) \) is equivalent to a \( X_i \) in (10). By the way, by choosing \( \nu \) to be a counting measure instead of the Lebesgue measure, the integral in (9) can be reduced to a sum, e.g., \( \sum_{x \in A'} C(x) \), where \( A' \) is a finite set of locations in \( \mathbb{R}^2 \) (e.g., part of a lattice in \( \mathbb{Z}^2 \)). It is worth mentioning that the ideas of this paper can easily be applied to such a framework.

Even if dependence between the \( X_i, i = 1, \ldots, N, \) in (10) was allowed, considering \( L(A, C) \) (see (4)) would appear more promising. Indeed, the geographical information of each risk (i.e., insurance policy) is explicitly taken into account and, consequently, the dependence between all risks can be modelled in a more realistic way than in (10). The dependence between the risks directly inherits from their respective associated geographical positions and, thus, ignoring their localizations as in (10) makes the modelling of their dependence more arbitrary and much less reliable. In our approach, this dependence is fully characterized by the spatial dependence structure of the cost field \( C \). Potential central limit theorems (see below) would have stronger implications because the dependence is more realistic. For these reasons, Models (8) and (9) allow a more accurate assessment of spatial diversification. The same remarks hold if we compare our loss models with the classical actuarial collective risk model.

Our risk models (8) and (9) and more generally our theory about spatial risk measures may be particularly relevant for an insurance company willing to adapt its policies portfolio. E.g., the axioms of spatial sub-additivity and asymptotic spatial homogeneity can help it to assess the potential relevance of extending its activity to a new geographical region. Such an analysis requires the company to have an accurate view of the dependence between its risks (especially between the possible new risks and those already present in the portfolio), as allowed by Models (8) and (9) through the cost field \( C \). Model (10) would not enable the insurer to precisely account for the dependence between the new risks and those already in the portfolio and hence to properly quantify the impact of a geographical expansion, i.e., of an increase of the number of contracts \( N \).
At present, we show that, consistently with our intuition, considering the risk associated with the normalized spatially aggregated loss is also insightful when the insurer is interested in the risk associated with its non-normalized counterpart. Let $\Pi$ be a positive homogeneous and translation invariant classical risk measure and $p_r$ denote either the claims reserves, revenues or any relevant related quantity per surface unit (possibly the mean premium per surface unit) of an insurance company Ins.

We first consider the axiom of spatial sub-additivity, which is assumed to be satisfied.

Ins covers region $A_1$ for a given hazard and potentially aims at covering also a region $A_2$ disjoint of $A_1$. We assume that Ins properly hedges its risk on $A_1$, i.e.,

$$\nu(A_1)p_r \geq \Pi(L(A_1, C)), \quad \text{i.e.,} \quad p_r \geq \Pi(L_N(A_1, C)), \tag{11}$$

by positive homogeneity. Using again the same property,

$$\Pi(L(A_1 \cup A_2, C)) = \nu(A_1 \cup A_2)\Pi(L_N(A_1 \cup A_2, C)).$$

Combined with

$$\Pi(L_N(A_1 \cup A_2, C)) \leq \Pi(L_N(A_1, C)),$$

this yields

$$\Pi(L(A_1 \cup A_2, C)) \leq \frac{\nu(A_1 \cup A_2)}{\nu(A_1)}\Pi(L(A_1, C)).$$

Hence, by translation invariance,

$$\Pi(L(A_1 \cup A_2, C) - \nu(A_1 \cup A_2)p_r) = \Pi(L(A_1 \cup A_2, C)) - \nu(A_1 \cup A_2)p_r$$

$$\leq \frac{\nu(A_1 \cup A_2)}{\nu(A_1)}\Pi(L(A_1, C)) - \nu(A_1 \cup A_2)p_r. \tag{12}$$

It follows from (11) that

$$p_r[\nu(A_1 \cup A_2) - \nu(A_1)] \geq \frac{\Pi(L(A_1, C))}{\nu(A_1)}[\nu(A_1 \cup A_2) - \nu(A_1)],$$

which gives

$$\frac{\nu(A_1 \cup A_2)}{\nu(A_1)}\Pi(L(A_1, C)) - \nu(A_1 \cup A_2)p_r \leq \Pi(L(A_1, C)) - \nu(A_1)p_r. \tag{13}$$

The combination of (12) and (13) yields that

$$\Pi(L(A_1 \cup A_2, C) - \nu(A_1 \cup A_2)p_r) \leq \Pi(L(A_1, C) - \nu(A_1)p_r).$$

The last inequality is strict if that in the axiom of spatial sub-additivity or in (11) is so. Thus, if Ins suitably hedges its risk on $A_1$, the risk is even better hedged on $A_1 \cup A_2$. Exactly the same reasoning holds for $A_2$.

Remark 2. We could have proposed another axiom of spatial sub-additivity: for all disjoint $A_1, A_2 \in \mathcal{A}$, $\Pi(L(A_1 \cup A_2, C)) \leq \Pi(L(A_1, C)) + \Pi(L(A_2, C))$. Nevertheless, this property is trivially satisfied as soon as the classical risk measure $\Pi$ is sub-additive and therefore its validity does not depend on the properties of the cost field $C$. Basing the axiom of spatial sub-additivity on the normalized spatially aggregated loss as we do is more appealing since it allows a diversification effect coming from $C$ (and not only from $\Pi$).

\footnotetext{We do not enter into accounting details in this study.}
We now consider the axiom of asymptotic spatial homogeneity of order $-\alpha$. Assume that it is satisfied with $\alpha > 0$ (e.g., we will see that in the cases of VaR and ES, $\alpha$ typically equals 1). It follows from Definition [3] Point 3, that

$$
\Pi(L(\lambda A, C) - \nu(\lambda A)p_r) = \lambda^2 \nu(A)K_1(A, C) + \nu(A)K_2(A, C)\lambda^{2-\alpha} + o\left(\lambda^{2-\alpha}\right) = \lambda^2 \nu(A)(K_1(A, C) - p_r) + \nu(A)K_2(A, C)\lambda^{2-\alpha} + o\left(\lambda^{2-\alpha}\right).
$$

Since $\alpha > 0$, the dominant term as $\lambda \to \infty$ is $\lambda^2 \nu(A)(K_1(A, C) - p_r)$. Consequently, for $\lambda$ large enough, the total risk of the company, $\Pi(L(\lambda A, C) - \nu(\lambda A)p_r)$, is a decreasing function of $\lambda$ as soon as the revenue per surface unit (or claims reserves, ...) satisfies $p_r > K_1(A, C)$. For instance, for VaR and ES, $K_1(A, C) = E[C(0)] = E[C(x)]$ for all $x \in \mathbb{R}^2$, under conditions given in Section [3]. Therefore, the latter inequality entails that the revenue (e.g., the premium) exceeds the expected cost at each location, which appears natural.

Finally, we discuss a possible way for a company to develop an adequate model for the cost field $C$ in regions where it is still inactive. The general model for the cost field introduced in [Koch (2017)], Section 2.3, is written

$$
\{C(x)\}_{x \in \mathbb{R}^2} = \{E(x) D(Z(x))\}_{x \in \mathbb{R}^2}, \quad (14)
$$

where $\{E(x)\}_{x \in \mathbb{R}^2}$ is the exposure field, $D$ a damage function and $\{Z(x)\}_{x \in \mathbb{R}^2}$ the random field of the environmental variable generating risk. The cost is assumed to be only due to a unique class of events, i.e., to a unique natural hazard. The latter (e.g., a heat wave or a hurricane) is described by the random field of an environmental variable (e.g., the temperature or the wind speed, respectively), $Z$. We assume that $Z$ is representative of the risk during the whole period $[0, T_L]$. The application of the damage function (also referred to as vulnerability curve in the literature) $D$ to the natural hazard random field gives the destruction percentage at each location. Finally, multiplying the destruction percentage by the exposure gives the cost at each location. For more details, we refer the reader to [Koch (2017)], Section 2.3. In order to obtain an adequate model $C$ in regions where it has no policies yet, the company can for instance consider crude estimates of the exposure field in the new region, develop a detailed statistical model for the environmental field $Z$ responsible of the risk insured (e.g., wind speed in the case of hurricanes) using appropriate data and apply the same damage functions as in the region it already covers. The company can then simulate from this cost model, hence obtaining an empirical distribution of the loss appearing in (8) and (9). This makes it possible to check whether the axiom of spatial sub-additivity is satisfied or not. Furthermore, if the spatial domain is large, considering asymptotic spatial homogeneity and potential central limit theorems (see below) is useful.

Remark 3. Strictly speaking, the terms of the insurance policies should be accounted for in Model (14). By the way, the latter model can be interpreted differently as done here. For instance, we can imagine that $Z$ represents the random field of the real cost and $D$ accounts for the terms of the policies.

---

5Potentially different from those developed in the natural catastrophes industry: e.g., a max-stable model.
Remark 4. In this paper, time is fixed and we look at what happens when the spatial domain varies. Potentially interesting research would consist in developing a ruin theory with respect to space (when the spatial domain grows) instead of time, using Models \( \mathfrak{S} \) or \( \mathfrak{Q} \).

2.3 Mixing and central limit theorems for random fields

We first remind the reader of the definition of the \( \alpha \)- and \( \beta \)-mixing coefficients which will be used in Section 3. Let \( \{X(x)\}_{x \in \mathbb{R}^d} \) be a real-valued random field. For \( S \subset \mathbb{R}^d \) a closed subset, we denote by \( \mathcal{F}_S^X \) the \( \sigma \)-field generated by the random variables \( \{X(x) : x \in S\} \). Let \( S_1, S_2 \subset \mathbb{R}^d \) be disjoint closed subsets. The \( \alpha \)-mixing coefficient (introduced by Rosenblatt, 1956) between the \( \sigma \)-fields \( \mathcal{F}_S^X \) and \( \mathcal{F}_S^X \) is defined by

\[
\alpha^X(S_1, S_2) = \sup \left\{ \left| \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \right| : A \in \mathcal{F}_{S_1}^X, B \in \mathcal{F}_{S_2}^X \right\}.
\]  

(15)

The \( \beta \)-mixing coefficient or absolute regularity coefficient (attributed to Kolmogorov in Volkonskii and Rozanov, 1959) between the \( \sigma \)-fields \( \mathcal{F}_S^X \) and \( \mathcal{F}_S^X \) is given by

\[
\beta^X(S_1, S_2) = \frac{1}{2} \sup \left\{ \sum_{i=1}^I \sum_{j=1}^J \left| \mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j) \right| : A \in \mathcal{F}_{S_1}^X, B \in \mathcal{F}_{S_2}^X \right\},
\]

where the supremum is taken over all partitions \( \{A_1, \ldots, A_I\} \) and \( \{B_1, \ldots, B_J\} \) of \( \Omega \) with the \( A_i \)'s in \( \mathcal{F}_{S_1}^X \) and the \( B_j \)'s in \( \mathcal{F}_{S_2}^X \). These coefficients satisfy the useful inequality

\[
\alpha^X(S_1, S_2) \leq \frac{1}{2} \beta^X(S_1, S_2), \quad \text{for all } S_1, S_2 \subset \mathbb{R}^d.
\]  

(16)

Now, we recall the concepts of Van Hove sequence and central limit theorem (CLT) in the case of random fields. This will be useful, since, for instance, asymptotic spatial homogeneity of order \(-1\) of spatial risk measures associated with VaR (at a level \( \alpha \in (0, 1) \setminus \{1/2\} \)) and induced by a cost field \( C \) is satisfied as soon as \( C \) fulfills the CLT (with a positive variance) and has a constant expectation (see below). For \( V \subset \mathbb{R}^d \) and \( r > 0 \), we denote \( V^r = \{x \in \mathbb{R}^d : \text{dist}(x, V) \leq r\} \), where dist stands for the Euclidean distance. Additionally, we denote \( \partial V \) the boundary of \( V \). A Van Hove sequence in \( \mathbb{R}^d \) is a sequence \( (V_n)_{n \in \mathbb{N}} \) of bounded measurable subsets of \( \mathbb{R}^d \) satisfying \( V_n \uparrow \mathbb{R}^d \), \( \lim_{n \to \infty} \nu(V_n) = \infty \), and \( \lim_{n \to \infty} \nu((\partial V_n)^{++})/\nu(V_n) = 0 \) for all \( r > 0 \). The assumption “bounded” does not always appear in the definition of a Van Hove sequence. Let \( \text{Cov} \) denote the covariance. In the following, we say that a random field \( \{X(x)\}_{x \in \mathbb{R}^d} \) such that, for all \( x \in \mathbb{R}^d \), \( \mathbb{E}[[X(x)]^2] < \infty \), satisfies the CLT if

\[
\int_{\mathbb{R}^d} |\text{Cov}(X(0), X(x))| \nu(dx) < \infty,
\]

\[
\sigma_X = \left( \int_{\mathbb{R}^d} \text{Cov}(X(0), X(x)) \nu(dx) \right)^{\frac{1}{2}} > 0,
\]

and, for any Van Hove sequence \( (V_n)_{n \in \mathbb{N}} \) in \( \mathbb{R}^d \),

\[
\frac{1}{\nu(V_n)^{1/2}} \int_{V_n} (X(x) - \mathbb{E}[X(x)]) \nu(dx) \overset{d}{\to} \mathcal{N}(0, \sigma_X^2), \quad \text{as } n \to \infty,
\]

where \( \mathcal{N}(\mu, \sigma^2) \) denotes the normal distribution with expectation \( \mu \in \mathbb{R} \) and variance \( \sigma^2 > 0 \). In the case of a random field satisfying the CLT, we have the following result.
**Theorem 2.** Let \( \{C(x)\}_{x \in \mathbb{R}^2} \in \mathcal{C} \). Assume moreover that \( C \) has a constant expectation (i.e., for all \( x \in \mathbb{R}^2 \), \( \mathbb{E}[C(x)] = \mathbb{E}[C(0)] \)) and satisfies the CLT. Then, we have, for all \( A \in \mathcal{A}_c \), that

\[
\lambda (L_N(\lambda A, C) - \mathbb{E}[C(0)]) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\sigma_C^2}{\nu(A)} \right), \text{ for } \lambda \to \infty.
\]

**Proof.** The result is essentially based on part of the proof of Theorem 4 in [Koch (2017)]. We refer the reader to this proof for details and only provide the main ideas here. First, we show (see Koch, 2017, third paragraph of the proof of Theorem 4) that, for any \( A \in \mathcal{A}_c \) and any positive non-decreasing sequence \( (\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R} \) such that \( \lim_{n \to \infty} \lambda_n = \infty \), the sequence \( (\lambda_n A)_{n \in \mathbb{N}} \) is a Van Hove sequence. Therefore, since \( C \) satisfies the CLT, we obtain

\[
\lambda_n (L_N(\lambda_n A, C) - \mathbb{E}[C(0)]) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\sigma_C^2}{\nu(A)} \right), \text{ for } n \to \infty.
\]

Second, we deduce (see Koch, 2017, proof of Theorem 4, after (44)) that, for all \( A \in \mathcal{A}_c \),

\[
\lambda (L_N(\lambda A, C) - \mathbb{E}[C(0)]) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\sigma_C^2}{\nu(A)} \right), \text{ for } \lambda \to \infty.
\]

This concludes the proof. \( \square \)

This theorem will be useful in the following since it will allow us to prove asymptotic spatial homogeneity of order respectively \(-2\), \(-1\) and \(-1\) for spatial risk measures associated with variance, VaR as well as ES and induced by a cost field satisfying the CLT and additional conditions. Moreover, if \( \lambda \) is large enough, it gives an approximation of the distribution of the normalized spatially aggregated loss:

\[
L_N(\lambda A, C) \approx \mathcal{N} \left( \mathbb{E}[C(0)], \frac{\sigma_C^2}{\lambda^2 \nu(A)} \right),
\]

where \( \approx \) means “approximately follows”. Such an approximation can be fruitful in practice, e.g., for an insurance company.

### 2.4 Max-stable random fields

This concise introduction to max-stable fields is partly based on [Koch et al. (2018)], Section 2.2. Below, \( \bigvee \) denotes the supremum when the latter is taken over a countable set. In any dimension \( d \geq 1 \), max-stable random fields are defined as follows.

**Definition 7** (Max-stable random field). A real-valued random field \( \{Z(x)\}_{x \in \mathbb{R}^d} \) is said to be max-stable if there exist sequences of functions \((a_T(x), x \in \mathbb{R}^d)_{T \geq 1} > 0 \) and \((b_T(x), x \in \mathbb{R}^d)_{T \geq 1} \in \mathbb{R} \) such that, for all \( T \geq 1 \),

\[
\left\{ \frac{\bigvee_{t=1}^T \{Z_t(x)\} - b_T(x)}{a_T(x)} \right\}_{x \in \mathbb{R}^d} \overset{d}{=} \{Z(x)\}_{x \in \mathbb{R}^d},
\]

where the \( \{Z_t(x)\}_{x \in \mathbb{R}^d}, t = 1, \ldots, T, \) are independent replications of \( Z \).
A max-stable random field is termed simple if it has standard Fréchet margins, i.e., for all \( x \in \mathbb{R}^d \), \( P(Z(x) < z) = \exp(-1/z) \), \( z > 0 \).

Now, let \( \{ \tilde{T}_i(x) \}_{x \in \mathbb{R}^d}, i = 1, \ldots, n \), be independent replications of a random field \( \{ \tilde{T}(x) \}_{x \in \mathbb{R}^d} \). Let \( (c_n(x), x \in \mathbb{R}^d)_{n \geq 1} > 0 \) and \( (d_n(x), x \in \mathbb{R}^d)_{n \geq 1} \in \mathbb{R} \) be sequences of functions. If there exists a non-degenerate random field \( \{ G(x) \}_{x \in \mathbb{R}^d} \) such that

\[
\left\{ \frac{\sum_{i=1}^{n} \tilde{T}_i(x) - d_n(x)}{c_n(x)} \right\}_{x \in \mathbb{R}^d} \xrightarrow{d} \{ G(x) \}_{x \in \mathbb{R}^d}, \text{ for } n \to \infty,
\]

then \( G \) is necessarily max-stable; see, e.g., de Haan (1984). This explains the relevance and significance of max-stable random fields in the modelling of spatial extremes.

Any simple max-stable random field \( Z \) can be written (see, e.g., de Haan, 1984) as

\[
\{ Z(x) \}_{x \in \mathbb{R}^d} = \left\{ \sqrt[n]{\sum_{i=1}^{n} U_i Y_i(x)} \right\}_{x \in \mathbb{R}^d}, \tag{17}
\]

where the \( (U_i)_{i \geq 1} \) are the points of a Poisson point process on \((0, \infty)\) with intensity \( u^{-2} \nu(du) \) and the \( Y_i, i \geq 1 \), are independent replications of a random field \( \{ Y(x) \}_{x \in \mathbb{R}^d} \) such that, for all \( x \in \mathbb{R}^d \), \( E[Y(x)] = 1 \). The field \( Y \) is not unique and is called a spectral random field of \( Z \). Conversely, any random field of the form (17) is a simple max-stable random field. Hence, (17) enables the building up of models for max-stable fields. We now present one of the most famous among such models, the Brown–Resnick random field, which is defined in Kabluchko et al. (2009) as a generalization of the stochastic process introduced in Brown and Resnick (1977). We recall that a random field \( \{ W(x) \}_{x \in \mathbb{R}^d} \) is said to have stationary increments if the distribution of the random field \( \{ W(x + x_0) - W(x_0) \}_{x \in \mathbb{R}^d} \) does not depend on \( x_0 \in \mathbb{R}^d \). Provided the increments of \( W \) have a finite second moment, the variogram of \( W \), \( \gamma_W \), is defined by

\[
\gamma_W(x) = \text{Var}(W(x) - W(0)), \quad x \in \mathbb{R}^d,
\]

where \( \text{Var} \) denotes the variance. The Brown–Resnick random field is specified as follows.

**Definition 8** (Brown–Resnick random field). Let \( \{ W(x) \}_{x \in \mathbb{R}^d} \) be a centred Gaussian random field with stationary increments and with variogram \( \gamma_W \). Let us consider the random field \( Y \) defined by

\[
\{ Y(x) \}_{x \in \mathbb{R}^d} = \left\{ \exp \left( W(x) - \frac{\text{Var}(W(x))}{2} \right) \right\}_{x \in \mathbb{R}^d}.
\]

Then the simple max-stable random field defined by (17) with \( Y \) is referred to as the Brown–Resnick random field associated with the variogram \( \gamma_W \). In the following, we will call this field the Brown–Resnick random field built with \( W \).

The Brown–Resnick field is stationary (see Kabluchko et al., 2009, Theorem 2) and its distribution only depends on the variogram (see Kabluchko et al., 2009, Proposition 11).

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6Throughout the paper, stationarity refers to strict stationarity.
7In the following, when \( W \) is sample-continuous, what we refer to as the Brown–Resnick random field built with \( W \) is obtained by taking replications of \( W \) (see (17)) which are also sample-continuous.
Now, let \((U_i, C_i)_{i \geq 1}\) be the points of a Poisson point process on \((0, \infty) \times \mathbb{R}^d\) with intensity function \(u^{-2} \nu(du) \times \nu(dc)\). Independently, let \(f_i, i \geq 1\), be independent replicates of some non-negative random function \(f\) on \(\mathbb{R}^d\) satisfying \(E \left[ \int_{\mathbb{R}^d} f(x) \nu(dx) \right] = 1\). Then, it is known that the Mixed Moving Maxima (M3) random field

\[
\{Z(x)\}_{x \in \mathbb{R}^d} = \left\{ \bigvee_{i=1}^{\infty} \{U_i f_i(x - C_i)\} \right\}_{x \in \mathbb{R}^d}
\]  

(18)

is a stationary and simple max-stable field. The so-called Smith random field introduced by Smith (1990) is a specific case of M3 random field and is defined immediately below.

**Definition 9 (Smith random field).** Let \(Z\) be written as in (18) with \(f\) being the density of a \(d\)-variate Gaussian random vector with mean \(0\) and covariance matrix \(\Sigma\). Then, the field \(Z\) is referred to as the Smith random field with covariance matrix \(\Sigma\).

Finally, we briefly present the extremal coefficient (see, e.g., Schlather and Tawn, 2003) which is a well-known measure of spatial dependence for max-stable random fields. Let \(\{Z(x)\}_{x \in \mathbb{R}^d}\) be a simple max-stable random field. In the case of two locations, the extremal coefficient function \(\theta\) is defined by

\[
P(Z(x_1) \leq u, Z(x_2) \leq u) = \exp \left( \frac{-\theta(x_1, x_2)}{u} \right), \quad x_1, x_2 \in \mathbb{R}^d,
\]

where \(u > 0\).

3 Properties of some induced spatial risk measures

In this section, we provide sufficient conditions on the cost field such that some induced spatial risk measures satisfy the axioms presented in Definition 6. First, we consider the case of a general cost field before investigating the relevant case of a cost field being a function of a max-stable random field. In the following, for \(\alpha \in (0, 1)\), \(q_\alpha\) and \(\phi\) denote the quantile at the level \(\alpha\) and the density of the standard Gaussian distribution, respectively.

We recall that for a random variable \(\tilde{X}\) with distribution function \(F\), its Value-at-Risk at the level \(\alpha \in (0, 1)\) is written \(\text{VaR}_\alpha(\tilde{X}) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}\). Moreover, provided \(E[|\tilde{X}|] < \infty\), its expected shortfall at the level \(\alpha \in (0, 1)\) is defined as

\[
\text{ES}_\alpha(\tilde{X}) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} \text{VaR}_\alpha(\tilde{X}) \nu(du).
\]

Typical values for \(\alpha\) are 0.95 and 0.99. It should be noted that in the actuarial literature, ES is sometimes referred to as Tail Value-at-Risk (see, e.g., Denuit et al., 2005, Definition 2.4.1). In the following, we mainly consider the spatial risk measures

\[
R_1(A, C) = E[L_N(A, C)], \quad A \in \mathcal{A}, C \in \mathcal{C},
\]

\[
R_2(A, C) = \text{Var}(L_N(A, C)), \quad A \in \mathcal{A}, C \in \mathcal{C},
\]

\[
R_{3,\alpha}(A, C) = \text{VaR}_\alpha(L_N(A, C)), \quad A \in \mathcal{A}, C \in \mathcal{C},
\]

\[
R_{4,\alpha}(A, C) = \text{ES}_\alpha(L_N(A, C)), \quad A \in \mathcal{A}, C \in \mathcal{C}.
\]

As a classical risk measure, the expectation is not very satisfying since it does not provide any information about variability. Moreover, as will be seen, the associated spatial risk
measures do not take into account the spatial dependence of the cost field. An advantage of variance, VaR and ES lies in the fact that their associated spatial risk measures all take into account (at least) part of this spatial dependence. Historically, the variance has been the dominating risk measure in finance, mainly due to the huge influence of the portfolio theory of Markowitz which uses variance as a measure of risk. However, it does not provide any information about the severity of losses which occur with a probability lower than the most widely used risk measure in the finance/insurance industry. However, it does not which are approximately symmetric around the expectation. Currently, VaR is probably the second moment. Moreover, since it allocates the same weight to positive and negative deviations from the expectation, variance is a good risk measure only for distributions that have a finite second moment. Historically, the variance has been the dominating risk measure in finance, mainly due to the huge influence of the portfolio theory of Markowitz which uses variance as a measure of risk. However, using variance is only possible when the normalized spatially aggregated loss has a finite second moment. Moreover, since it allocates the same weight to positive and negative deviations from the expectation, variance is a good risk measure only for distributions which are approximately symmetric around the expectation. Currently, VaR is probably the most widely used risk measure in the finance/insurance industry. However, it does not provide any information about the severity of losses which occur with a probability lower than 1 − α, which is obviously a serious shortcoming. Moreover, VaR is in general not sub-additive and hence not coherent in the sense of Artzner et al. (1999). ES overcomes these two drawbacks of VaR. Pertaining to the first one, it can be seen from the fact that, if a random variable X has a continuous distribution function, then

\[ \text{ES}_\alpha (X) = E \left[ X \left| X > \text{VaR}_\alpha (X) \right. \right]. \]

Hence, the Basel Committee on Banking Supervision proposed the use of ES instead of VaR for the internal models-based approach (Basel Committee on Banking Supervision, 2012, Section 3.2.1). However, contrary to VaR, ES is not elicitable (Gneiting, 2011), implying that backtesting for ES is more difficult than for VaR.

3.1 General cost field

Next result provides sufficient conditions on the cost field C such that the induced spatial risk measure \( R_1(\cdot, C) \) satisfies the axioms presented in Definition 6.

**Theorem 3.** Let \( \{C(x)\}_{x \in \mathbb{R}^2} \) be a measurable random field having a constant expectation and such that, for all \( x \in \mathbb{R}^2 \), \( E[|C(x)|] = E[|C(0)|] < \infty \). Then, we have, for all \( A \in \mathcal{A} \), that \( R_1(A, C) = E[C(0)] \). Hence, the spatial risk measure induced by \( C \) \( R_1(\cdot, C) \) satisfies the axioms of spatial invariance under translation and spatial sub-additivity. If, moreover, \( E[C(0)] \neq 0 \), then \( R_1(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order 0 with \( K_1(A, C) = 0 \) and \( K_2(A, C) = E[C(0)], A \in \mathcal{A}_c \).

**Proof.** By assumption, the function \( x \mapsto E[|C(x)|] \) is constant and hence obviously locally integrable. Consequently, as \( C \) is measurable, Proposition 4 gives that \( C \) has a.s. locally integrable sample paths. Using Fubini’s theorem and the fact that \( C \) has a constant expectation, we have, for all \( A \in \mathcal{A} \), that

\[ R_1(A, C) = \frac{1}{\nu(A)} \int_A E[C(0)] \nu(dx) = E[C(0)]. \]

Thus, for all \( v \in \mathbb{R}^2 \) and \( A \in \mathcal{A} \), \( R_1(A + v, C) = R_1(A, C) \). Moreover, for all \( A_1, A_2 \in \mathcal{A} \),

\[ R_1(A_1 \cup A_2, C) = R_1(A_1, C) = R_1(A_2, C) = \min\{R_1(A_1, C), R_1(A_2, C)\}. \]

Finally, for all \( A \in \mathcal{A}_c \) and \( \lambda > 0 \), we have that \( R_1(\lambda A, C) = E[C(0)] \). As \( |E[C(0)]| \leq E[|C(0)|] < \infty \), we have \( |E[C(0)]| \in (0, \infty) \), which concludes the proof. \( \square \)
Next result is a generalization of Theorem 2 in Koch (2017) and will be useful in the following.

**Theorem 4.** Let \( \{ C(x) \}_{x \in \mathbb{R}^2} \in \mathcal{C} \) and such that, for all \( x \in \mathbb{R}^2 \), \( \mathbb{E} [(C(x))^2] < \infty \). Then, for all \( A \in \mathcal{A} \) and \( \lambda > 0 \), we have

\[
R_2(\lambda A, C) = \frac{1}{\lambda^2 [\nu(A)]^2} \int_A \int_A \text{Cov}(C(x), C(y)) \, \nu(dx) \, \nu(dy).
\]

**Proof.** For all \( A \in \mathcal{A} \), we consider \( L(A, C) = \nu(A) L_N(A, C) \). Thus, using Fubini’s theorem, we obtain

\[
\mathbb{E} [(L(A, C))^2] = \mathbb{E} \left[ \left( \int_A C(x) \, \nu(dx) \right)^2 \right] = \mathbb{E} \left[ \int_A C(x) \, \nu(dx) \int_A C(y) \, \nu(dy) \right] = \int_A \int_A \mathbb{E} [C(x)C(y)] \, \nu(dx) \, \nu(dy). \tag{19}
\]

Moreover, it is clear that

\[
(\mathbb{E} [L(A, C)])^2 = \int_A \int_A \mathbb{E} [C(x)] \mathbb{E} [C(y)] \, \nu(dx) \, \nu(dy). \tag{20}
\]

The combination of (19) and (20) gives that

\[
\mathbb{E} [(L(A, C))^2] - (\mathbb{E} [L(A, C)])^2 = \int_A \int_A \text{Cov}(C(x), C(y)) \, \nu(dx) \, \nu(dy),
\]

which implies that

\[
R_2(A, C) = \frac{1}{[\nu(A)]^2} \int_A \int_A \text{Cov}(C(x), C(y)) \, \nu(dx) \, \nu(dy).
\]

The result is obtained by replacing \( A \) with \( \lambda A \). \( \square \)

We recall that for a random field \( \{ C(x) \}_{x \in \mathbb{R}^2} \) such that, for all \( x \in \mathbb{R}^2 \), \( \mathbb{E} [(C(x))^2] < \infty \), we note

\[
\sigma_C = \left( \int_{\mathbb{R}^2} \text{Cov}(C(0), C(x)) \, \nu(dx) \right)^{\frac{1}{2}}.
\]

Next theorem provides the main results of this subsection. It especially gives sufficient conditions on the cost field \( C \) such that the induced spatial risk measures \( R_2(\cdot, C) \), \( R_{4,0}(\cdot, C) \) and \( R_{4,0}(\cdot, C) \) satisfy the axioms of asymptotic spatial homogeneity of order \(-2\), \(-1\) and \(-1\), respectively.

**Theorem 5.** Let \( \{ C(x) \}_{x \in \mathbb{R}^2} \in \mathcal{C} \).

1. Assume that \( C \) is stationary. Then, provided it exists, any spatial risk measure associated with a law-invariant classical risk measure \( \Pi \) and induced by \( C \) satisfies the axiom of spatial invariance under translation.
2. Assume that $C$ is such that, for all $x \in \mathbb{R}^2$,
\[
\mathbb{E} \left[ (C(x))^2 \right] < \infty, \tag{21}
\]
for all $x, y \in \mathbb{R}^2$,
\[
\text{Cov}(C(x), C(y)) = \text{Cov}(C(0), C(x - y)), \tag{22}
\]
and
\[
\int_{\mathbb{R}^2} |\text{Cov}(C(0), C(x))| \, \nu(dx) < \infty. \tag{23}
\]
Then, we have, for all $A \in \mathcal{A}_c$, that
\[
\mathcal{R}_2(\lambda A, C) = \frac{\sigma_C^2}{\lambda^2 \nu(A)} + o \left( \frac{1}{\lambda^2} \right). \tag{24}
\]
Hence, if $\sigma_C > 0$, $\mathcal{R}_2(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-2$ with $K_1(A, C) = 0$ and $K_2(A, C) = \sigma_C^2 / \nu(A)$, $A \in \mathcal{A}_c$.

3. Assume that $C$ has a constant expectation and satisfies the CLT. Then, we have, for all $A \in \mathcal{A}_c$, that
\[
\mathcal{R}_{3,\alpha}(\lambda A, C) = \frac{\sigma_C q_{\alpha}}{\lambda \nu(A)} + o \left( \frac{1}{\lambda} \right). \tag{25}
\]
Hence, if $\alpha \in (0, 1) \setminus \{1/2\}$, $\mathcal{R}_{3,\alpha}(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-1$ with $K_1(A, C) = \mathbb{E}[C(0)]$ and $K_2(A, C) = \sigma_C q_{\alpha} / \{\nu(A)\}^{\frac{1}{2}}(1 - \alpha)$, $A \in \mathcal{A}_c$.

4. Assume that $C$ has a constant expectation, satisfies the CLT and is such that the random variables $\lambda(L_N(\lambda A, C) - \mathbb{E}[C(0)])$, $\lambda > 0$, are uniformly integrable. Then, we have, for all $A \in \mathcal{A}_c$, that
\[
\mathcal{R}_{4,\alpha}(\lambda A, C) = \mathbb{E}[C(0)] + \frac{\sigma_C \phi(q_{\alpha})}{\lambda [\nu(A)]^{\frac{1}{2}}(1 - \alpha)} + o \left( \frac{1}{\lambda} \right). \tag{26}
\]
Hence, $\mathcal{R}_{4,\alpha}(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-1$ with $K_1(A, C) = \mathbb{E}[C(0)]$ and $K_2(A, C) = \sigma_C \phi(q_{\alpha}) / \{\nu(A)\}^{\frac{1}{2}}(1 - \alpha)$, $A \in \mathcal{A}_c$.

Proof. 1. Let $A \in \mathcal{A}$, $v \in \mathbb{R}^2$ and $\Pi$ be a classical risk measure. Using the fact that $\nu(A + v) = \nu(A)$ and a change of variable, we obtain
\[
\mathcal{R}_{\Pi}(A + v, C) = \Pi \left( \frac{1}{\nu(A + v)} \int_{A + v} C(x) \, \nu(dx) \right) = \Pi \left( \frac{1}{\nu(A)} \int_A C(y + v) \, \nu(dy) \right). \tag{27}
\]
Due to the stationarity of $C$, we have, for all $v \in \mathbb{R}^2$, that $\{C(x)\}_{x \in \mathbb{R}^2} \overset{d}{=} \{C(x + v)\}_{x \in \mathbb{R}^2}$, yielding, since $\Pi$ is law-invariant,
\[
\Pi \left( \frac{1}{\nu(A)} \int_A C(y + v) \, \nu(dy) \right) = \Pi \left( \frac{1}{\nu(A)} \int_A C(x) \, \nu(dx) \right) = \mathcal{R}_{\Pi}(A, C). \tag{28}
\]
The combination of (27) and (28) provides the result.
2. We know from Theorem 1 that, for all $A \in \mathcal{A}$ and $\lambda > 0$, $R_2(\lambda A, C)$ is well-defined. The result follows from an adapted version of the proof of Theorem 3, Point 3, in [Koch (2017)]. We refer the reader to this proof for the technical parts. We only highlight some of the main steps as well as the main differences here.

The first part consists in showing that

$$\lim_{\lambda \to \infty} \lambda^2 \nu(A)R_2(\lambda A, C) = \sigma_C^2. \quad (29)$$

Using Theorem 1 and (22), it follows that

$$R_2(\lambda A, C) = \frac{1}{\lambda^2 \nu(A)^2} \int_{\lambda A \setminus \lambda A} \int_{\lambda A} \text{Cov}(C(0), C(x - y)) \, \nu(dx) \, \nu(dy).$$

Let $A_\lambda = \lambda A, \lambda > 0$. Then, we consider the quantity

$$T_\lambda = \frac{1}{\lambda^2 \nu(A)} \int_{A_\lambda \setminus A_\lambda} \int_{A_\lambda} k(x - y) \, \nu(dx) \, \nu(dy), \quad \lambda > 0,$$

where

$$k(x) = \text{Cov}(C(0), C(x)), \quad x \in \mathbb{R}^2.$$

The next step consists in showing that

$$\lim_{\lambda \to \infty} T_\lambda = \sigma_C^2. \quad (30)$$

For this purpose, we proceed similarly as in [Koch (2017)], proof of Theorem 3, Point 3. The only difference consists in the fact that here $k$ is not necessarily non-negative. Hence, in order to bound $|T_{1,\lambda}|$ and $|T_{3,\lambda}|$ (these quantities are defined in Koch (2017)) from above, $k$ must be replaced with its absolute value in the corresponding integrals. This is where Condition (23) plays a role. Finally, since, for all $\lambda > 0$,

$$T_\lambda = \lambda^2 \nu(A)R_2(\lambda A, C),$$

(29) follows from (30).

In a second part, we easily derive (24) from (29). Now, as a compact subset of $\mathbb{R}^2$, $A$ is bounded, giving that $\nu(A) \in (0, \infty)$. Since, moreover, $\sigma_C^2 \in (0, \infty)$, $\sigma_C^2/\nu(A) \in (0, \infty)$. Hence, the second part of the result follows from (24).

3. Theorem 2 gives that, for all $A \in \mathcal{A}$,

$$\lambda(L_N(\lambda A, C) - E[C(0)]) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_C^2}{\nu(A)}\right), \quad \text{for } \lambda \to \infty.$$

Hence, the fact that the quantile function of a normal random variable is continuous on $(0, 1)$, Proposition 0.1 in [Resnick (1987)] and easy computations (see Koch, 2017, proof of Theorem 5) yield (24). Since $C$ satisfies the CLT, we have $E[(C(0)]^2] < \infty$ and thus $E[C(0)] < \infty$. Additionally, as $\alpha \neq 0.5$, we have $q_\alpha \neq 0$. Moreover, as $\sigma_C > 0$ (because $C$ satisfies the CLT) and $\nu(A) > 0$, we obtain that $\sigma_Cq_\alpha/\nu(A)^{\frac{1}{2}} \neq 0$. Finally, since $\alpha \notin \{0, 1\}$, $\|q_\alpha\| < \infty$. Furthermore, $\sigma_C < \infty$ and $\nu(A) < \infty$, giving that $|\sigma_Cq_\alpha/\nu(A)^{\frac{1}{2}}| < \infty$. The result follows by definition.

4. Since $C$ satisfies the CLT, we have, for all $x \in \mathbb{R}^2$, $E[(C(x)]^2] < \infty$. Thus, we know from Theorem 2 that, for all $A \in \mathcal{A}$ and $\lambda > 0$, $L_N(\lambda A, C)$ has a finite second moment.
We deduce that $E[L_N(\lambda A, C)]$ is finite, and, therefore, that $\mathcal{R}_{4, \alpha}(\lambda A, C)$ is well-defined. Theorem 2 gives that, for all $A \in \mathcal{A}_c$,

$$\lambda \left( L_N(\lambda A, C) - E[C(0)] \right) \overset{d}{\to} \mathcal{N} \left( 0, \frac{\sigma_C^2}{\nu(A)} \right), \text{ for } \lambda \to \infty.$$  

Now, ES is known to be continuous with respect to convergence in distribution in the case of uniformly integrable random variables. For details, we refer for instance to Wang et al. (2018), Theorem 3.2 and Example 2.2, Point (ii); the authors’ results concern bounded random variables but the mentioned result can be extended to the case of integrable random variables. Hence, it follows from the fact that the random variables $\lambda \left( L_N(\lambda A, C) - E[C(0)] \right)$, $\lambda > 0$, are uniformly integrable, that

$$\lim_{\lambda \to \infty} \frac{1}{1 - \alpha} \int_0^1 \text{Var}_u(\lambda[L_N(\lambda A, C) - E[C(0)]]) \nu(du) = \frac{\sigma_C \phi(q_\alpha)}{[\nu(A)]^2(1 - \alpha)}.$$  

(31)

Moreover, we have

$$\frac{1}{1 - \alpha} \int_0^1 \text{Var}_u(\lambda[L_N(\lambda A, C) - E[C(0)]]) \nu(du) = \frac{1}{1 - \alpha} \int_0^1 \lambda(\text{Var}_u(L_N(\lambda A, C)) - E[C(0)]) \nu(du) = \lambda(\mathcal{R}_{4, \alpha}(\lambda A, C) - E[C(0)]).$$

Thus, (31) gives, for all $A \in \mathcal{A}_c$,

$$\lambda \left( \mathcal{R}_{4, \alpha}(\lambda A, C) - E[C(0)] \right) \overset{\lambda \to \infty}{\to} \frac{\sigma_C \phi(q_\alpha)}{[\nu(A)]^2(1 - \alpha)} + o(1),$$

which yields (26). Now, we have $E[C(0)] < \infty$. Moreover, using the fact that, for all $\alpha \in (0, 1)$, $\phi(q_\alpha) \in (0, \infty)$, and arguments stated at the end of the proof of Point 3, we obtain $|\sigma_C \phi(q_\alpha)/{[\nu(A)]^2(1 - \alpha)}| \in (0, \infty)$. Consequently, the result follows by definition.

**Remark 5.** In order to establish Points 3 and 4, we take advantage of the fact that both VaR and ES are continuous with respect to convergence in distribution under appropriate assumptions. Hence, similar results might hold for other classical risk measures satisfying continuity with respect to convergence in distribution.

Theorem 5 entails the following important result.

**Corollary 1.** Let $\{C(x)\}_{x \in \mathbb{R}^2} \in \mathcal{C}$. Moreover, assume that $C$ satisfies (22) and the CLT. Then, we have, for all $A \in \mathcal{A}_c$, that

$$\mathcal{R}_2(\lambda A, C) = \frac{\sigma_C^2}{\lambda^2 \nu(A)} + o \left( \frac{1}{\lambda^2} \right).$$

Hence, $\mathcal{R}_2(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-2$ with $K_1(A, C) = 0$ and $K_2(A, C) = \sigma_C^2/\nu(A)$, $A \in \mathcal{A}_c$.

**Proof.** Since $C$ satisfies the CLT, it satisfies (21), (23) and $\sigma_C > 0$. Thus, the result follows from Theorem 5, Point 2. 

\[ \square \]
Next result provides a convenient condition ensuring the uniform integrability required in Theorem \ref{thm:uniform_integrability} Point 4.

**Proposition 2.** Let \( \{C(x)\}_{x \in \mathbb{R}^2} \in \mathcal{C} \). Assume moreover that \( C \) has a constant expectation and satisfies the CLT. If \( C \) satisfies \eqref{eq:clt}, then the random variables \( \lambda (L_N(\lambda A, C) - \mathbb{E}[C(0)]) \), \( \lambda > 0 \), are uniformly integrable.

**Proof.** Let, for \( \lambda > 0 \), \( M_\lambda = \lambda (L_N(\lambda A, C) - \mathbb{E}[C(0)]) \). Theorem \ref{thm:uniform_integrability} gives that, for all \( A \in \mathcal{A}_c \), \( M_\lambda \xrightarrow{d} M \), for \( \lambda \to \infty \), where \( M \sim \mathcal{N}(0, \sigma^2 \lambda^2 / \nu(A)) \). Therefore, by the continuous mapping theorem, we obtain

\[
M_\lambda^2 \xrightarrow{d} M^2, \quad \text{for} \quad \lambda \to \infty.
\]

Now, it is clear that, for all \( \lambda > 0 \), \( \text{Var}(M_\lambda) = \lambda^2 \mathcal{R}_2(\lambda A, C) \). Hence, it follows from \eqref{eq:variance} that \( \text{Var}(M_\lambda) \xrightarrow{\lambda \to \infty} \mathbb{E}[M^2] \). Additionally, \( M^2 \) is non-negative and integrable. Furthermore, the \( M_\lambda^2 \) are non-negative and, for all \( \lambda > 0 \), \( \mathbb{E}[M_\lambda^2] = \lambda^2 \mathcal{R}_2(\lambda A, C) \), which is finite according to Theorem \ref{thm:uniform_integrability}. Therefore, the \( M_\lambda^2 \) are integrable. Consequently, using \eqref{eq:uniform_integrability} and Theorem 3.6 in \cite{Billingsley1999}, we know that the random variables \( M_\lambda^2, \lambda > 0 \), are uniformly integrable. This directly yields that the random variables \( M_\lambda \), \( \lambda > 0 \), are uniformly integrable. \( \square \)

### 3.2 Cost field being a function of a max-stable random field

We now consider a cost field model written as in \eqref{eq:cost_field}, i.e.,

\[
\{C(x)\}_{x \in \mathbb{R}^2} = \{E(x) \cdot D(Z(x))\}_{x \in \mathbb{R}^2},
\]

where \( Z \) is max-stable and the exposure is uniformly equal to unity. The relevance of using max-stable random fields has been previously highlighted.

We first give sufficient conditions on the damage function \( D \) and the field \( Z \) such that the spatial risk measure \( \mathcal{R}_1(\cdot, D(Z)) \) induced by the cost field \( D(Z) \) satisfies the axioms presented in Definition \ref{def:spatial_risk_measure}.

**Corollary 2.** Let \( \{Z(x)\}_{x \in \mathbb{R}^2} \) be a simple max-stable random field and \( D \) a measurable function such that \( \{C(x)\}_{x \in \mathbb{R}^2} = \{D(Z(x))\}_{x \in \mathbb{R}^2} \in \mathcal{C} \) and \( \mathbb{E}[|C(0)|] < \infty \). Then, for all \( A \in \mathcal{A} \), \( \mathcal{R}_1(A, C) = \mathbb{E}[C(0)] \). Hence, \( \mathcal{R}_1(\cdot, C) \) satisfies the axioms of spatial invariance under translation and spatial sub-additivity. If, moreover, \( \mathbb{E}[C(0)] \neq 0 \), then \( \mathcal{R}_1(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order 0 with \( K_1(A, C) = 0 \) and \( K_2(A, C) = \mathbb{E}[C(0)], A \in \mathcal{A}_c \).

**Proof.** The result directly follows from Theorem \ref{thm:uniform_integrability} \( \square \)

The result below gives sufficient conditions on \( D \) and \( Z \) such that the spatial risk measure \( \mathcal{R}_2(\cdot, D(Z)) \) induced by the cost field \( D(Z) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\).

**Theorem 6.** Let \( \{Z(x)\}_{x \in \mathbb{R}^2} \) be a simple and sample-continuous max-stable random field and \( D \) a measurable function such that \( \{C(x)\}_{x \in \mathbb{R}^2} = \{D(Z(x))\}_{x \in \mathbb{R}^2} \in \mathcal{C} \) and such that there exist \( p, q > 0 \) satisfying \( 2/p + 1/q = 1 \) such that

\[
\mathbb{E}[|C(0)|^p] < \infty
\]

\( \text{(34)} \)
Since $p, q > 0$, therefore, using (34) and (35), we obtain

$$
\int_{\mathbb{R}^2} [2 - \theta(0, x)]^{\frac{1}{p}} \nu(dx) < \infty,
$$

where $\theta$ is the extremal coefficient function of $Z$. Then, we have

$$
\int_{\mathbb{R}^2} |\text{Cov}(C(0), C(x))| \nu(dx) < \infty.
$$

Additionally, assume that, for all $x, y \in \mathbb{R}^2$,

$$
\text{Cov}(C(x), C(y)) = \text{Cov}(C(0), C(x - y)),
$$

and $\sigma_C > 0$. Then $R_2(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-2$ with $K_1(A, C) = 0$ and $K_2(A, C) = \sigma_C^2/\nu(A)$, $A \in \mathcal{A}_e$.

**Proof.** Since $Z$ has identical margins, (34) yields that, for all $x \in \mathbb{R}^2$, $E[|C(x)|^p] < \infty$. Thus, using the fact that $2/p + 1/q = 1$, Davydov’s inequality (Davydov, 1968, Equation (2.2)) gives that

$$
|\text{Cov}(C(0), C(x))| \leq 12 \left(\alpha_C(\{0\}, \{x\})\right)^{\frac{1}{p}} (E[|C(0)|^p])^{\frac{1}{p}} (E[|C(x)|^p])^{\frac{1}{p}}.
$$

(36)

For all $x \in \mathbb{R}^2$, since $D$ is measurable, $C(x) = D(Z(x))$ is $\mathcal{F}_{\{x\}}^Z$-measurable. Hence, $\mathcal{F}_{\{x\}}^C \subset \mathcal{F}_{\{x\}}^Z$, which gives by (15) that, for all $x \in \mathbb{R}^d$,

$$
\alpha_C(\{0\}, \{x\}) \leq \alpha_Z(\{0\}, \{x\}).
$$

(37)

Now, using (16) and Corollary 2.2 in Dombry and Eyi-Minko (2012), we obtain that, for all $x \in \mathbb{R}^d$,

$$
\alpha_Z(\{0\}, \{x\}) \leq 2[2 - \theta(0, x)].
$$

(38)

Thus, the combination of (37) and (38) gives that

$$
\alpha_C(\{0\}, \{x\}) \leq 2[2 - \theta(0, x)].
$$

Consequently, (36) gives that

$$
|\text{Cov}(C(0), C(x))| \leq 12 2^{\frac{1}{p}} (E[|C(0)|^p]) E[|C(x)|^p]^{\frac{1}{p}} [2 - \theta(0, x)]^{\frac{1}{p}}.
$$

Therefore, using (34) and (35), we obtain

$$
\int_{\mathbb{R}^2} |\text{Cov}(C(0), C(x))| \nu(dx) < \infty.
$$

Since $p, q > 0$ and $2/p + 1/q = 1$, we have $p > 2$. Consequently, for all $x \in \mathbb{R}^2$, $E[|C(x)|^2] < \infty$. Thus, Theorem 5, Point 2, gives the result. \qed

Until the end, the following results provide sufficient conditions on $D$ and $Z$ such that the induced spatial risk measures $R_2(\cdot, D(Z))$, $R_{3,\alpha}(\cdot, D(Z))$ and $R_{4,\alpha}(\cdot, D(Z))$ satisfy the axioms of asymptotic spatial homogeneity of order $-2$, $-1$ and $-1$, respectively. In order to establish them, we take advantage of the results in Koch et al. (2018) about the existence of a CLT for functions of stationary max-stable random fields. Let $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}((0, \infty))$ be the Borel $\sigma$-fields on $\mathbb{R}$ and $(0, \infty)$, respectively, and let $'$ designate transposition. For $h = (h_1, \ldots, h_d)' \in \mathbb{Z}^d$, we adopt the notation $[h, h + 1] = [h_1, h_1 + 1] \times \cdots \times [h_d, h_d + 1]$. Next theorem considers a general simple, stationary and sample-continuous max-stable random field.
Theorem 7. Let \( \{Z(x)\}_{x \in \mathbb{R}^2} \) be a simple, stationary and sample-continuous max-stable random field and \( D \) be a measurable function from \( ((0, \infty), \mathcal{B}((0, \infty))) \) to \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) satisfying
\[
\mathbb{E} \left[ |D(Z(0))|^{2+\delta} \right] < \infty,
\]
for some \( \delta > 0 \). Furthermore, assume that, for all \( h \in \mathbb{Z}^d \),
\[
\mathbb{E} \left[ \min \left\{ \sup_{x \in [0,1]^d} \{Y(x)\}, \sup_{x \in [h,h+1]} \{Y(x)\} \right\} \right] \leq K\|h\|^{-b},
\]
for some \( K > 0 \), \( b > d \max\{2, (2 + \delta)/\delta\} \) and \( \{Y(x)\}_{x \in \mathbb{R}^d} \) is a spectral random field of \( Z, (\text{see (17)}) \). Let \( \{C(x)\}_{x \in \mathbb{R}^2} = \{D(Z(x))\}_{x \in \mathbb{R}^2} \). Then, if \( \sigma_C > 0 \):
1. \( \mathcal{R}_2(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\) with \( K_1(A, C) = 0 \) and \( K_2(A, C) = \sigma_C^2/\nu(A) \), \( A \in \mathcal{A}_c \).
2. For all \( \alpha \in (0,1) \backslash \{1/2\} \), \( \mathcal{R}_{4,\alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_C q_\alpha/\{\nu(A)\}^{1/2} \), \( A \in \mathcal{A}_c \).
3. For all \( \alpha \in (0,1) \), \( \mathcal{R}_{4,\alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_C \phi(q_\alpha)/\{\nu(A)\}^{1/2}(1 - \alpha) \), \( A \in \mathcal{A}_c \).

Proof. Since \( Z \) is sample-continuous, it is measurable. Thus, the function \( D \) being measurable from \( ((0, \infty), \mathcal{B}((0, \infty))) \) to \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \), we obtain that \( C \) is measurable. Moreover, it follows from the stationarity of \( C \) (due to the stationarity of \( Z \) and Condition (39)) that, for all \( x \in \mathbb{R}^2 \), \( \mathbb{E} \|C(x)\| = \mathbb{E} \|C(0)\| < \infty \). Therefore, the function \( x \mapsto \mathbb{E} \|C(x)\| \) is constant and hence obviously locally integrable. Consequently, Proposition 1 gives that \( C \) has a.s. locally integrable sample paths. Therefore, \( C \in \mathcal{C} \).

Furthermore, the assumptions enable us to apply Theorem 2 in Koch et al. (2018). The latter yields that the random field \( C \) satisfies the CLT. Finally, since \( C \) is stationary, it satisfies \( (22) \) and has a constant expectation. Hence, Corollary 1 gives the first result. The second result follows from Theorem 5. Point 3. The combination of Proposition 2 and Point 4 in Theorem 5 yields the third result.

Theorem 7 directly entails the following result.

Corollary 3. Let \( Z, D \) and \( C \) be as in Theorem 7. Moreover, assume that \( D \) is non-decreasing and non-constant. Then:
1. \( \mathcal{R}_2(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\) with \( K_1(A, C) = 0 \) and \( K_2(A, C) = \sigma_C^2/\nu(A) \), \( A \in \mathcal{A}_c \).
2. For all \( \alpha \in (0,1) \backslash \{1/2\} \), \( \mathcal{R}_{4,\alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_C q_\alpha/\{\nu(A)\}^{1/2} \), \( A \in \mathcal{A}_c \).
3. For all \( \alpha \in (0,1) \), \( \mathcal{R}_{4,\alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_C \phi(q_\alpha)/\{\nu(A)\}^{1/2}(1 - \alpha) \), \( A \in \mathcal{A}_c \).

\(^8\)Without the condition \( \sigma_C > 0 \).
The combination of Proposition 2 and Point 4 in Theorem 5 gives the third result. Corollary 1 yields the first result. The second result follows from Theorem 5, Point 3.

Theorem 8. Let \( \{Z(x)\}_{x \in \mathbb{R}^2} \) be the Brown–Resnick random field associated with the variogram \( \gamma_W(x) = \eta \|x\|^\alpha \), where \( \eta > 0 \) and \( \alpha \in (0, 2] \), and \( D \) be as in Theorem 7. Let \( \{C(x)\}_{x \in \mathbb{R}^2} = \{D(Z(x))\}_{x \in \mathbb{R}^2} \). Then, if \( \sigma_C > 0 \):

1. \( \mathcal{R}_2(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\) with \( K_1(A, C) = 0 \) and \( K_2(A, C) = \sigma_C^2/\nu(A), A \in \mathcal{A}_c \).

2. For all \( \alpha \in (0, 1) \setminus \{1/2\} \), \( \mathcal{R}_{3, \alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_Cq_\alpha/\{\nu(A)\}^{\frac{1}{\alpha}}, A \in \mathcal{A}_c \).

3. For all \( \alpha \in (0, 1) \), \( \mathcal{R}_{4, \alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_C\phi(q_\alpha)/\{\nu(A)\}^{\frac{1}{\alpha}}(1 - \alpha) \), \( A \in \mathcal{A}_c \).

Proof. As previously mentioned, the Brown–Resnick random field is stationary. Thus, \( C \) is stationary and hence satisfies (22) and has a constant expectation. Moreover, we can see from the proof of Theorem 3 in Koch et al. (2018) that \( Z \) is sample-continuous. Consequently, the same arguments as in the proof of Theorem 7 yield that \( C \in \mathcal{C} \). Furthermore, Theorem 3 in Koch et al. (2018) gives that \( C \) satisfies the CLT. Therefore, Corollary 4 yields the first result. The second result follows from Theorem 5, Point 3. The combination of Proposition 2 and Point 4 in Theorem 5 gives the third result.

Next corollary easily follows from Theorem 8.

Corollary 4. Let \( Z, D \) and \( C \) be as in Theorem 5. Moreover, assume that \( D \) is non-decreasing and non-constant. Then:

1. \( \mathcal{R}_2(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\) with \( K_1(A, C) = 0 \) and \( K_2(A, C) = \sigma_C^2/\nu(A), A \in \mathcal{A}_c \).

2. For all \( \alpha \in (0, 1) \setminus \{1/2\} \), \( \mathcal{R}_{3, \alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_Cq_\alpha/\{\nu(A)\}^{\frac{1}{\alpha}}, A \in \mathcal{A}_c \).

3. For all \( \alpha \in (0, 1) \), \( \mathcal{R}_{4, \alpha}(\cdot, C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A, C) = \mathbb{E}[C(0)] \) and \( K_2(A, C) = \sigma_C\phi(q_\alpha)/\{\nu(A)\}^{\frac{1}{\alpha}}(1 - \alpha) \), \( A \in \mathcal{A}_c \).

Proof. As explained in the proof of Theorem 8, \( Z \) is stationary and sample-continuous. Furthermore, it is simple max-stable. Thus, Proposition 1 in Koch et al. (2018) gives that \( \sigma_C > 0 \). Hence, Theorem 8 yields the result.
Let \( \| \cdot \| \) denote the Euclidean distance in \( \mathbb{R}^2 \). We introduce \( B_1 = \{ x \in \mathbb{R}^2 : \| x \| = 1 \} \), the unit ball of \( \mathbb{R}^2 \). For two functions \( g_1 \) and \( g_2 \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \), the notation \( g_1(h) = o(g_2(h)) \) means that \( \lim_{h \to \infty} \sup_{u \in B_1} \left\{ \left| g_1(hu) / g_2(hu) \right| \right\} = 0 \). Moreover, \( \lim_{\| h \| \to \infty} g_1(h) = \infty \) must be understood as \( \lim_{h \to \infty} \inf_{u \in B_1} \{ g_1(hu) \} = \infty \).

**Theorem 9.** Let \( \{ Z(x) \}_{x \in \mathbb{R}^2} \) be the Brown–Resnick random field built with a random field \( \{ W(x) \}_{x \in \mathbb{R}^2} \) which is sample-continuous and whose variogram satisfies
\[
\sup_{x \in [0,1]^2} \{ \gamma_W(h) - \gamma_W(x + h) \} \quad = \quad o(\gamma_W(h)),
\]
and
\[
\lim_{\| h \| \to \infty} \frac{\gamma_W(h)}{\ln(\| h \|)} = \infty.
\]
Moreover, let \( D \) be as in Theorem \( \square \). Let \( \{ C(x) \}_{x \in \mathbb{R}^2} = \{ D(Z(x)) \}_{x \in \mathbb{R}^2} \). Then, if \( \sigma_C > 0 \):

1. \( R_2(\cdot,C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\) with \( K_1(A,C) = 0 \) and \( K_2(A,C) = \sigma_C^2 / \nu(A) \), \( A \in \mathcal{A}_c \).

2. For all \( \alpha \in (0,1) \backslash \{1/2\} \), \( R_{4,\alpha}(\cdot,C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A,C) = E[C(0)] \) and \( K_2(A,C) = \sigma_C q_\alpha / [\nu(A)]^{1/\alpha} \), \( A \in \mathcal{A}_c \).

3. For all \( \alpha \in (0,1) \), \( R_{4,\alpha}(\cdot,C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A,C) = E[C(0)] \) and \( K_2(A,C) = \sigma_C \phi(q_\alpha) / [\nu(A)]^{1/\alpha} (1 - \alpha) \), \( A \in \mathcal{A}_c \).

**Proof.** The same arguments as in the proof of Theorem \( \square \) show that \( C \) satisfies \( \square \text{22} \) and has a constant expectation. As \( W \) is sample-continuous, Proposition 13 in Kabluchko et al. (2009) gives that \( Z \) is sample-continuous. Thus, the same arguments as in the proof of Theorem \( \square \) show that \( C \in \mathcal{C} \). Moreover, Remark 3 in Koch et al. (2018) gives that \( C \) satisfies the CLT. Hence, Corollary \( \square \) gives the first result. The second result follows from Theorem \( \square \) Point 3. The combination of Proposition \( \square \) and Point 4 in Theorem \( \square \) yields the third result. 

The following result is a direct consequence of Theorem \( \square \).

**Corollary 5.** Let \( Z, D \) and \( C \) be as in Theorem \( \square \). Moreover, assume that \( D \) is non-decreasing and non-constant. Then:

1. \( R_2(\cdot,C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-2\) with \( K_1(A,C) = 0 \) and \( K_2(A,C) = \sigma_C^2 / \nu(A) \), \( A \in \mathcal{A}_c \).

2. For all \( \alpha \in (0,1) \backslash \{1/2\} \), \( R_{4,\alpha}(\cdot,C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A,C) = E[C(0)] \) and \( K_2(A,C) = \sigma_C q_\alpha / [\nu(A)]^{1/\alpha} \), \( A \in \mathcal{A}_c \).

3. For all \( \alpha \in (0,1) \), \( R_{4,\alpha}(\cdot,C) \) satisfies the axiom of asymptotic spatial homogeneity of order \(-1\) with \( K_1(A,C) = E[C(0)] \) and \( K_2(A,C) = \sigma_C \phi(q_\alpha) / [\nu(A)]^{1/\alpha} (1 - \alpha) \), \( A \in \mathcal{A}_c \).
Proof. The random field $Z$ is simple, stationary, sample-continuous (see the proof of Theorem 9) and max-stable. Thus, Proposition 1 in Koch et al. (2018) gives that $\sigma_C > 0$. Consequently, Theorem 9 yields the result.

Finally, the last results of this paper concern the Smith random field.

**Theorem 10.** Let $\{Z(x)\}_{x \in \mathbb{R}^2}$ be the Smith random field with covariance matrix $\Sigma$ and $D$ be as in Theorem 7. Let $\{C(x)\}_{x \in \mathbb{R}^2} = \{D(Z(x))\}_{x \in \mathbb{R}^2}$. Then, if $\sigma_C > 0$:

1. $R_2(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-2$ with $K_1(A, C) = 0$ and $K_2(A, C) = \sigma_C^2 / \nu(A)$, $A \in \mathcal{A}_c$.

2. For all $\alpha \in (0, 1) \setminus \{1/2\}$, $R_{3,\alpha}(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-1$ with $K_1(A, C) = E[C(0)]$ and $K_2(A, C) = \sigma_C q_{\alpha} / [\nu(A)]^{1/2}$, $A \in \mathcal{A}_c$.

3. For all $\alpha \in (0, 1)$, $R_{4,\alpha}(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-1$ with $K_1(A, C) = E[C(0)]$ and $K_2(A, C) = \sigma_C \phi(q_{\alpha}) / [\nu(A)]^{1/2} (1 - \alpha)$, $A \in \mathcal{A}_c$.

**Proof.** The Smith random field is stationary as an instance of M3 random field. Thus, $C$ is stationary and consequently satisfies [22] and has a constant expectation. Moreover, as the Smith random field is sample-continuous, the same arguments as in the proof of Theorem 7 yield that $C \in C$. Additionally, Theorem 4 in Koch et al. (2018) gives that $C$ satisfies the CLT. Therefore, Corollary 6 yields the first result. The second result follows from Theorem 5, Point 3. The combination of Proposition 2 and Point 4 in Theorem 5 gives the third result.

Corollary 6 easily implies the following corollary.

**Corollary 6.** Let $Z$, $D$ and $C$ be as in Theorem 10. Moreover, assume that $D$ is non-decreasing and non-constant. Then:

1. $R_2(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-2$ with $K_1(A, C) = 0$ and $K_2(A, C) = \sigma_C^2 / \nu(A)$, $A \in \mathcal{A}_c$.

2. For all $\alpha \in (0, 1) \setminus \{1/2\}$, $R_{3,\alpha}(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-1$ with $K_1(A, C) = E[C(0)]$ and $K_2(A, C) = \sigma_C q_{\alpha} / [\nu(A)]^{1/2}$, $A \in \mathcal{A}_c$.

3. For all $\alpha \in (0, 1)$, $R_{4,\alpha}(\cdot, C)$ satisfies the axiom of asymptotic spatial homogeneity of order $-1$ with $K_1(A, C) = E[C(0)]$ and $K_2(A, C) = \sigma_C \phi(q_{\alpha}) / [\nu(A)]^{1/2} (1 - \alpha)$, $A \in \mathcal{A}_c$.

**Proof.** As stated in the proof of Theorem 10, the Smith random field is stationary and sample-continuous. Moreover, it is simple max-stable. Hence, Proposition 1 in Koch et al. (2018) gives that $\sigma_C > 0$. Thus, Theorem 10 yields the result.

We conclude this section by commenting on the damage function $D(z) = 1_{\{z > u\}}, z > 0$, for $u > 0$, which is considered in Koch (2017). This function is measurable from $((0, \infty), \mathcal{B}((0, \infty)))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Moreover, it is bounded and hence obviously satisfies $\sigma_C$ for every random field $Z$. Additionally, this function is non-decreasing and non-constant. Consequently, the results of Theorem 3, Point 3 and Theorem 5, Point 2 in
Koch (2017) concerning the Brown–Resnick random field associated with the variogram
\( \gamma_W(x) = \eta \|x\|^\alpha \), where \( \eta > 0 \) and \( \alpha \in (0, 2] \), and the Smith random field, are particular
cases of Corollary 4 and Corollary 6 respectively.

**Remark 6.** Although the Smith field is a limiting case of the Brown–Resnick field, corresponding results are here presented separately because it is usual in the spatial extremes literature to distinguish both models, the Smith field being defined using the M3 representation (18) and the Brown–Resnick field being written using the random fields-based representation (17). Furthermore, our results involve findings in Koch et al. (2018), where the results for the Smith and Brown–Resnick fields are stated separately.

### 4 Conclusion

In this paper, we first investigate the notions of spatial risk measure and corresponding axioms introduced in Koch (2017) further as well as describe their utility for actuarial practice. Second, in the case of a general cost field, we especially give sufficient conditions such that spatial risk measures associated with expectation, variance, VaR as well as ES and induced by this cost field satisfy the axiom of asymptotic spatial homogeneity of order 0, −2, −1 and −1, respectively. Finally, in the case where the cost field is a function of a max-stable random field, we give sufficient conditions on both the function and the max-stable field such that spatial risk measures associated with expectation, variance, VaR as well as ES and induced by the resulting cost field satisfy the axiom of asymptotic spatial homogeneity of order 0, −2, −1 and −1, respectively. Hence, these conditions allow one to know the rate of spatial diversification when the region under study becomes large. This can be of interest for the banking/insurance industry. Overall, this paper improves our understanding of the concept of spatial risk measure as well as of their properties with respect to the space variable. Among others, it generalizes several results found in Koch (2017). Ongoing work consists in the study of concrete examples of spatial risk measures involving max-stable fields and relevant damage functions. Future work will include the study of spatial risk measures associated with other classical risk measures (e.g., more general distortion risk measures than VaR or ES and expectile risk measures) and/or induced by cost fields involving other kinds of random fields than max-stable fields. For instance, it would be of interest to investigate whether spatial risk measures associated with VaR and ES still satisfy the axiom of asymptotic spatial homogeneity of order −1 in the case where the cost field does not satisfy the CLT.

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9 Note that Ahmed et al. (2016) study the properties of spatial risk measures associated with expectation and variance and induced by excesses of Gaussian random fields.
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