Reconstruction from $k$-decks for graphs with maximum degree 2

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Abstract

The $k$-deck of a graph is its multiset of induced subgraphs on $k$ vertices. We prove that $n$-vertex graphs with maximum degree 2 have the same $k$-decks if each cycle has at least $k+1$ vertices, each path component has at least $k-1$ vertices, and the number of edges is the same. Using this for lower bounds, we obtain for each graph with maximum degree at most 2 the least $k$ such that it is determined by its $k$-deck. For the $n$-vertex cycle this value is $\lfloor n/2 \rfloor$, and for the $n$-vertex path it is $\lfloor n/2 \rfloor + 1$. Also, the least $k$ such that the $k$-deck of an $n$-vertex graph always determines whether it is connected is at least $\lfloor n/2 \rfloor + 1$.

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1 Introduction

The famous Reconstruction Conjecture of Kelly [5, 6] and Ulam [15] has been open for more than 50 years. A card of a graph $G$ is a subgraph of $G$ obtained by deleting one vertex. Cards are unlabeled, so only the isomorphism class of a card is given. The deck of $G$ is the multiset of all cards of $G$. A graph is reconstructible if it is uniquely determined by its deck.

Conjecture 1.1 (The Reconstruction Conjecture; Kelly [5, 6], Ulam [15]). Every graph having more than two vertices is reconstructible.

We require more than two vertices since both graphs on two vertices have the same deck. Graphs in many families are known to be reconstructible; these include disconnected graphs, trees, regular graphs, and perfect graphs. Surveys on graph reconstruction include [2, 3, 7, 8].

Various parameters have been introduced to measure the difficulty of reconstructing a graph. Harary and Plantholt [4] defined the reconstruction number of a graph to be the
minimum number of cards from its deck that suffice to determine it, meaning that no other graph has the same multiset of cards in its deck. All trees with at least five vertices have reconstruction number 3 (Myrvold [11]), and almost all graphs have reconstruction number 3 (Bollobás [1]). Since $K_{r,r}$ and $K_{r+1,r-1}$ have $r + 1$ common cards, the reconstruction number of an $n$-vertex graph can be at least as large as $\frac{n}{2} + 2$ (Myrvold [10]).

Kelly looked in another direction, considering cards obtained by deleting more vertices. He conjectured a more detailed version of the Graph Reconstruction Conjecture.

**Conjecture 1.2** (Kelly [6]). For $\ell \in \mathbb{N}$, there is an integer $f(\ell)$ such that any graph with at least $f(\ell)$ vertices is reconstructible from its deck of cards obtained by deleting $\ell$ vertices.

The Graph Reconstruction Conjecture is the claim $f(1) = 3$ in this conjecture.

A $k$-card of a graph is an induced subgraph having $k$ vertices. The $k$-deck of $G$, denoted $D_k(G)$, is the multiset of all $k$-cards. Since each induced subgraph with $k - 1$ vertices arises exactly $n - k + 1$ times by deleting one vertex from a member of $D_k(G)$, we have the following.

**Observation 1.3.** For any graph $G$, the $k$-deck $D_k(G)$ determines the $(k-1)$-deck $D_{k-1}(G)$.

Thus decks of larger cards provide at least as much information as decks of smaller cards. Graphs are “easier” to reconstruct if they can be reconstructed from smaller cards.

**Definition 1.4.** A graph $G$ is $k$-deck reconstructible if no other graph has the same $k$-deck. Let $\rho(G)$ denote the least $k$ such that $G$ is $k$-deck reconstructible.

In light of Observation 1.3, it is useful to know what information about a graph can be reconstructed from the $k$-deck for small fixed $k$. Such information is also available when considering larger $k$. For example, only the numbers of edges and vertices are reconstructible from the 2-deck. At the other end, Manvel [9] proved that if $|V(G)| = n \geq 6$, then one can determine from the $(n-2)$-deck whether or not $G$ is connected, acyclic, regular, or bipartite. This has recently been improved in [13], where the authors showed that connectedness can always be determined from the $(n-3)$-deck.

For a graph $G$, the maximum degree $\Delta(G)$ is reconstructible from the $(\Delta(G) + 2)$-deck, since some $(\Delta(G) + 2)$-card has a vertex of degree $\Delta(G)$, but no $(\Delta(G) + 2)$-card has a vertex of degree $\Delta(G) + 1$. This was strengthened by Manvel:

**Theorem 1.5** (Manvel [9]). The degree list of a graph $G$ with maximum degree $\Delta(G)$ is reconstructible from $D_{\Delta(G)+2}$.

Manvel [9] also showed that the result is sharp in a strong sense; the maximum degree is not always determined by $D_{\Delta(G)+1}(G)$. Let $G_k$ be the forest $\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} K_{1,k-2i}$ (that is, $\binom{k}{2i}$ stars with $k-2i$ edges for $0 \leq i \leq \lfloor k/2 \rfloor$). Also, let $H_k = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} K_{1,k-2i-1}$. Note that $\Delta(G_k) = k$ and $\Delta(H_k) = k - 1$. Nevertheless, the two graphs have the same $k$-deck, and hence $\Delta(H)$ cannot always be determined from $D_{\Delta(H)+1}(H)$. 
With Theorem 1.5, we already recognize from the \( k \)-deck whether a graph has maximum degree 2 (when \( k \geq 4 \)). However, we will show that much larger cards are needed to guarantee determining whether a graph with maximum degree 2 is connected. In Problem 11898 of the American Mathematical Monthly, Richard Stanley posed a question that begins to suggest the difficulty of reconstructing 2-regular graphs from their \( k \)-decks.

**Problem 1.6** (Stanley [14]). Let \( n \) and \( k \) be integers, with \( n \geq k \geq 2 \). Let \( G \) be a graph with \( n \) vertices whose components are cycles of length greater than \( k \). Let \( f_k(G) \) be the number of \( k \)-element independent sets of vertices of \( G \). Show that \( f_k(G) \) depends only on \( k \) and \( n \).

Let \( s(G, H) \) denote the number of induced subgraphs of \( G \) isomorphic to \( H \). Graphs \( G \) and \( G' \) have the same \( k \)-deck if and only if \( s(G, H) = s(G', H) \) for all \( H \) with \( k \) vertices. In the language of reconstruction, Stanley’s problem asserts \( s(G, K_k) = s(G', K_k) \) for \( n \)-vertex 2-regular graphs \( G \) and \( G' \) whose components have length greater than \( k \), where \( K_k \) is the complete graph with \( k \) vertices and \( \overline{H} \) denotes the complement of \( H \). Stanley’s proposed solution of Problem 1.6 used generating functions. Our proof and generalization are bijective and relate to reconstruction.

Problem 1.6 considers only subgraphs having no edges. We will prove the same conclusion for all subgraphs with \( k \) vertices. That is, \( n \)-vertex 2-regular graphs whose components have more than \( k \) vertices all have the same \( k \)-deck. Our technique of proof further generalizes to graphs with maximum degree 2.

**Theorem 1.7.** Let \( G \) and \( G' \) be graphs with maximum degree 2 having the same number of vertices and the same number of edges. If every component in each graph is a cycle with more than \( k \) vertices or a path with at least \( k - 1 \) vertices, then \( D_k(G) = D_k(G') \).

The essence of the theorem, and in fact the way we prove it, is what it says for graphs with one or two components. Let \( G + H \) denote the disjoint union of graphs \( G \) and \( H \), and let \( C_n \) and \( P_n \) denote the \( n \)-vertex cycle and path. The theorem includes

1. \( D_k(C_{q+r}) = D_k(C_q + C_r) \) if \( q, r \geq k + 1 \),
2. \( D_k(P_{q+r}) = D_k(C_q + P_r) \) if \( q \geq k + 1 \) and \( r \geq k - 1 \), and
3. \( D_k(P_{q-1} + P_r) = D_k(P_q + P_{r-1}) \) if \( q, r \geq k \).

These statements yield the following result.

**Corollary 1.8.** For \( n \geq 3 \), the least \( k \) such that connectedness of an \( n \)-vertex graph \( G \) can always be determined from its \( k \)-deck is at least \( \lfloor n/2 \rfloor + 1 \) (even when given \( \Delta(G) = 2 \)). Furthermore, \( \rho(P_n) = \lfloor n/2 \rfloor + 1 \) and \( \rho(C_n) = \lfloor n/2 \rfloor \) when \( n \geq 6 \).

*Proof.* By (2), \( D_k(P_n) = D_k(C_{\lfloor n/2 \rfloor + 1} + P_{\lfloor n/2 \rfloor - 1}) \) when \( k \leq \lfloor n/2 \rfloor \). This proves the claim about connectedness and also \( \rho(P_n) \geq \lfloor n/2 \rfloor + 1 \).
Consider $D_k(P_n)$ with $k = \lfloor n/2 \rfloor + 1$. If $n \geq 6$, then $k \geq 4$, and by Theorem 1.5 we can reconstruct the degree list. The components of any reconstruction $G$ are cycles and one path. Since the $k$-deck has no cycle, $G$ can only have one cycle, and its length must exceed $k$. Now the path component has fewer than $\lfloor n/2 \rfloor - 1$ vertices. In $D_k(P_n)$, there are $n - k + 1$ copies of $P_k$. However, when $l < k - 1$, in $D_k(P_l + C_{n-k-l})$ there are $n - l$ copies of $P_k$, which is larger than in $P_n$. Hence the deck differs from $D_k(P_n)$ unless $G = P_n$.

By (1), $D_k(C_n) = D_k(C_{\lfloor n/2 \rfloor} + C_{\lfloor n/2 \rfloor})$ when $k < \lfloor n/2 \rfloor$. Suppose $k = \lfloor n/2 \rfloor$. If $n \geq 8$, then $\lfloor n/2 \rfloor \geq 4$ and by Theorem 1.5 we can reconstruct the degree list. Any 2-regular graph other than $C_n$ has a cycle of length at most $\lfloor n/2 \rfloor$, and this can be seen in the $\lfloor n/2 \rfloor$-deck.

For $n \in \{6, 7\}$, reconstruction of $C_n$ from the 3-deck requires a different argument. We know the number of edges of any reconstruction $G$ from the 2-deck, and we know the number of incidences (corresponding to edges in the line graph) from the 3-deck. This yields $\sum_{v \in V(G)} \binom{d(v)}{2} = n = \sum_{v \in V(G)} \frac{d(v)}{2}$. Now it is a standard exercise by convexity that $G$ is 2-regular. Again a cycle will appear in the 3-deck if $G \neq C_n$.

When $n = 5$, the graphs $P_5$ and $C_4 + P_1$ have the same 3-deck, so the condition $n \geq 6$ in Corollary 1.8 cannot be weakened. There are also three pairs of 7-vertex graphs that have the same 4-deck, but all six graphs are connected. Possibly the threshold $k \geq \lfloor n/2 \rfloor + 1$ for guaranteed recognizability of connectedness is sharp when $n \geq 6$.

**Question 1.9.** For $n \in \mathbb{N}$, what is the least $k$ such that for every $n$-vertex graph $G$, it can be determined from $D_k(G)$ whether $G$ is connected? In particular, does $\lfloor n/2 \rfloor + 1$ suffice when $n \geq 6$? Does $n - 4$ suffice?

Nýdl [12] proved that for any $n_0 \in \mathbb{N}$ and $0 < q < 1$, there exist nonisomorphic graphs of some order $n$ larger than $n_0$ that have the same $[qn]$-deck. However, connectedness is much less information to request than the isomorphism class, and it remains possible that $\lfloor n/2 \rfloor + 1$ is a threshold for $k$ such that $D_k(G)$ always determines whether $G$ is connected.

Sections 2 and 3 are devoted to the proof of Theorem 1.7, which via facts (1,2,3) yield lower bounds on $\rho(G)$ whenever $\Delta(G) = 2$. In Section 4 we prove that these lower bounds are optimal, giving procedures to reconstruct $G$ from its $\rho(G)$-deck in all cases. Here we give only a simplified statement of the result. The parameter $\epsilon'$ in this statement depends on which paths are components of $G$, as detailed in Theorem 1.8.

**Theorem 1.10.** If $\Delta(G) = 2$, then $\rho(G) = \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}$, where $m$ is the number of vertices in a largest component $H$ of $G$, $m'$ is the number of vertices in a largest component of $G - V(H)$ (possibly $m' = 0$), $\epsilon$ is 1 if $G$ has $P_m$ as a component and otherwise 0, and $\epsilon' \in \{0, 1, 2\}$.

In particular, if $G$ is 2-regular, then the full statement yields $\epsilon' = 0$ and $\rho(G) = \max\{\lfloor m/2 \rfloor, m'\}$. 

4
2 Common $k$-Decks for Linear Forests

A useful technical lemma implies that when two graph have the same $k$-deck, taking the disjoint union of either with a third graph again yields two graphs with the same $k$-deck. This will allow us to change one or two components of a graph while keeping the rest of the graph unchanged. Note that $G[X]$ denotes the subgraph of $G$ induced by a vertex subset $X$.

Lemma 2.1. If $G$, $G'$, and $H$ are graphs, then $\mathcal{D}_k(G) = \mathcal{D}_k(G')$ if and only if $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$.

Proof. Given a graph $F$, let $S_k(F)$ denote the set of labeled induced subgraphs with at most $k$ vertices. If $\mathcal{D}_k(G) = \mathcal{D}_k(G')$, then there is a bijection $g$ from $S_k(G)$ to $S_k(G')$ that pairs isomorphic subgraphs. It suffices to find such a bijection $h$ from $S_k(G + H)$ to $S_k(G' + H)$. Given a set $U \subseteq V(G) \cup V(H)$, let $X = U \cap V(G)$ and $Y = U \cap V(H)$. Note that $|X|, |Y| \leq k$. Hence we may define $h(U) = g(G[X]) + H[Y]$. In fact, $h$ is a bijection, and $G[X] + H[Y] \cong g(G[X]) + H[Y]$, so $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$.

Conversely, suppose that $\mathcal{D}_k(G + H) = \mathcal{D}_k(G' + H)$. By Observation 1.3, we also have $\mathcal{D}_j(G + H) = \mathcal{D}_j(G'+ H)$ for $j \leq k$. Let $X$ be a graph with $k$ vertices and $r$ components. We claim $s(G, X) = s(G', X)$, by induction on $k + r$. If $r = 1$, then $s(G, X) = s(G + H, X) - s(H, X) = s(G' + H, X) - s(H, X) = s(G', X)$. Let $[r] = \{1, \ldots, r\}$. For $r > 1$, let $X_1, \ldots, X_r$ be the components of $X$. For $T \subseteq [r]$, let $X_T$ denote the disjoint union of $\{X_i: i \in T\}$, and let $\overline{T} = [r] - T$. Using the induction hypothesis, we compute

$$s(G, X) = s(G + H, X) - \sum_{\emptyset \neq T \subseteq [r]} s(H, X_T)s(G, X_{\overline{T}}) = s(G' + H, X) - \sum_{\emptyset \neq T \subseteq [r]} s(H, X_T)s(G', X_{\overline{T}}) = s(G', X).$$

Thus $\mathcal{D}_k(G) = \mathcal{D}_k(G')$. \qed

We will use this lemma in both directions. In one direction, it tells us that any lower bound on $\rho(G)$ is also a lower bound on $\rho(G + H)$. In the other, it tells us that when two graphs with the same $k$-deck have a common component, deleting the shared component leaves two smaller graphs with the same $k$-deck.

When we consider only graphs where every cycle has length larger than $k$, every $k$-card is a linear forest, meaning a disjoint union of paths. It will be simpler to prove the equal-deck result first for linear forests. To discuss linear forests precisely, we introduce helpful notation.

Definition 2.2. Let $L$ denote a list $\ell_1, \ldots, \ell_p$ of distinct positive integers, let $m$ denote $m_1, \ldots, m_p$, and let $L^m$ denote the linear forest having $m_i$ components isomorphic to $P_{\ell_i}$, for $1 \leq i \leq p$. Let $L_i^m$ denote the linear forest obtained from $L^m$ by deleting a component
isomorphic to \( P_{\ell_i} \), and let \( L^m_{i,j} \) denote the result of deleting components isomorphic to \( P_{\ell_i} \) and \( P_{\ell_j} \) (we allow \( i = j \) when \( m_i \geq 2 \)). Again \( s(G, H) \) is the number of induced subgraphs of \( G \) isomorphic to \( H \), and let \( s'(G, H) \) be the number of induced subgraphs of \( G \) isomorphic to \( H \) in which a specified vertex of \( G \) is used as an isolated vertex in \( H \).

We consider \( s'(G, H) \) only when \( H \) has an isolated vertex. The vertex specified in \( G \) does not appear in the notation \( s'(G, H) \), because we will prove next that under appropriate conditions the value is the same for a range of vertices. For the remainder of this section, let the vertices of \( P_n \) be \( w_1, \ldots, w_n \) in order.

**Lemma 2.3.** Let \( L^m \) be a linear forest with \( k \) vertices. For each specified vertex \( w_h \) such that \( k \leq h \leq n - k + 1 \), the quantity \( s'(P_n, L^m) \) has the same value.

**Proof.** We use induction on \( k \). When \( k = 1 \), there is exactly one copy of \( P_1 \) containing any specified vertex. For \( k > 1 \), the value is 0 unless \( L^m \) has an isolated vertex.

We compare \( s'(P_n, L^m) \) with \( s'(C_n, L^m) \), where \( C_n \) is obtained by adding the edge \( w_n w_1 \). By symmetry, \( s'(C_n, L^m) \) is independent of the specified vertex. Note that \( s'(C_n, L^m) \) does not count copies of \( L^m \) in \( P_n \) in which some path starts with \( w_1 \) and another ends with \( w_n \). On the other hand, it does count unwanted subgraphs that use the edge \( w_n w_1 \).

Note that \( w_h \) is far enough from the ends of \( P_n \) that there is room for \( P_{\ell_i} \) containing \( w_1 \) and \( P_{\ell_j} \) containing \( w_n \) without touching \( w_h \). Also, in \( C_n \) the edge \( w_n w_1 \) may occupy any of \( \ell_i - 1 \) positions within a copy of \( P_{\ell_i} \). Summing over all the possible orders of the paths or path using \( w_1 \) and \( w_n \), we thus obtain the following relation.

\[
s'(P_n, L^m) = s'(C_n, L^m) + \sum_{i,j} s'(P_{n-(\ell_i+\ell_j+2)}, L^m_{i,j}) - \sum_i (\ell_i - 1) s'(P_{n-(\ell_i+2)}, L^m_i)
\]

Two extra vertices are deleted in each term to separate components of \( L^m \). In the middle sum, \( i = j \) is allowed when \( m_i \geq 2 \), and the set \( \{i, j\} \) yields two terms when \( i \neq j \); this sum is empty when \( L^m \) consists of only one path. The final sum is actually a double-sum; we will show that the summand in the inner sum with \( \ell_i - 1 \) terms is constant.

By symmetry, \( s'(C_n, L^m) \) is independent of \( h \). To obtain the same conclusion for the other terms, we check the conditions in the statement of the induction hypothesis.

For terms in the double sum, deleting \( P_{\ell_i} \) and the neighboring vertex from the beginning of \( P_n \) leaves the vertex \( w_h \) with a new index \( h' \) in \( P_{n-\ell_i-\ell_j-2} \). With \( P_{\ell_i} \) containing \( w_1 \), we obtain \( h' = h - \ell_i - 1 \). We have \( h - \ell_i - 1 \geq k - (\ell_i + \ell_j) \) since \( h \geq k \) and \( \ell_j \geq 1 \). Similarly,

\[
h - \ell_i - 1 \leq n - (\ell_i + \ell_j + 2) - (k - (\ell_i + \ell_j)) + 1,
\]

since \( h \leq n - k + 1 \) and \( \ell_i \geq 1 \).

The last sum is actually also a double sum, but the induction hypothesis guarantees that the terms in the inner sum are equal. When considering the terms involving \( \ell_i \), we lose at
most \((\ell_i - 1) + 1\) vertices at the beginning of the path, yielding \(h' \geq h - \ell_i \geq k - \ell_i\). Similarly, we lose at most \(\ell_i\) vertices from the end of the path and the index must decrease at least by 2, so \(h' \leq h - 2 \leq (n - \ell_i - 2) - (k - \ell_i) + 1\).

By the induction hypothesis, all contributions are independent of the choice of the specified vertex when it is in the given range.

Note that we never need the value of \(s'(P_n, L^m)\). Lemma \ref{lem:2.3} enables us to prove the special case of Theorem \ref{thm:1.7} for linear forests.

**Theorem 2.4.** Let \(L^m\) be a linear forest with \(k\) vertices. For an \(n\)-vertex graph \(G\) that is a disjoint union of paths, each with at least \(k - 1\) vertices, the number of induced copies of \(L^m\) depends only on \(L^m\), \(n\), and \(|E(G)|\).

**Proof.** Given \(n\), fixing \(|E(G)|\) is equivalent to fixing the number of components. By keeping all but two components fixed and applying Lemma \ref{lem:2.1} it therefore suffices to show \(s(P_{q-1} + P_r, L^m) = s(P_q + P_{r-1}, L^m)\) for \(q, r \geq k\).

Consider \(P_{q+r+2}\) with \(V(P_{q+r+2}) = \{w_1, \ldots, w_{q+r+2}\}\). Deleting \(\{w_q, w_{q+1}, w_{q+2}\}\) yields \(P_{q-1} + P_r\), while deleting \(\{w_{q+1}, w_{q+2}, w_{q+3}\}\) yields \(P_q + P_{r-1}\). Thus \(s(P_{q-1} + P_r, L^m) = s'(P_{q+r+2}, L^m + P_1)\) when specifying \(w_{q+1}\), while \(s(P_q + P_{r-1}, L^m) = s'(P_{q+r+2}, L^m + P_1)\) when specifying \(w_{q+1}\). By Lemma \ref{lem:2.3} we need only capture \(q + 1\) and \(q + 2\) in the given range.

We have \(|V(L^m + P_1)| = k + 1\) and apply Lemma \ref{lem:2.3} with \(n = q + r + 2\). Since \(q, r \geq k\),

\[|V(L^m + P_1)| = k + 1 \leq q + 1 < q + 2 = n - r \leq n - k = n - |V(L^m + P_1)| + 1,\]

as desired. \qed

**Corollary 2.5.** If \(G\) and \(G'\) are linear forests with the same number of vertices and same number of edges whose components have at least \(k - 1\) vertices, then \(\mathcal{D}_k(G) = \mathcal{D}_k(G')\).

### 3 Common \(k\)-Decks for Maximum Degree 2

We can extend the results to allow cycles because deleting any vertex of a cycle leaves the same path. Again the problem will reduce to working with just two components.

**Lemma 3.1.** Let \(L^m\) be a linear forest with \(k\) vertices. If \(q \geq k + 1\) and \(r \geq k - 1\), then \(s(P_{q+r}, L^m) = s(C_q + P_r, L^m)\)

**Proof.** Let \(u_1, \ldots, u_{q+r}\) be the vertices of \(V(P_{q+r})\) in order. Consider an induced copy of \(L^m\). Either \(u_q\) is not used, or it appears in a path of some length \(\ell_i\). In the latter case let \(t\) be the number of vertices starting with \(u_q\) that lie in the copy of \(P_{\ell_i}\); the hypotheses on \(q\) and \(r\) allow \(t\) to run from 1 to \(\ell_i\). These possibilities yield

\[s(P_{q+r}, L^m) = s(P_{q-1} + P_r, L^m) + \sum_{i=1}^{p} \sum_{t=1}^{\ell_i} s(P_{q-(\ell_i-t)-2} + P_{r-t}, L_i^m).\]
Now consider a vertex \( x \) on \( C_q \) in \( C_q + P_r \). By symmetry, the choice of \( x \) does not matter. As above, in a copy of \( L^m \) the vertex \( x \) may be omitted or appear in a copy of \( P_{\ell_i} \) for some \( i \). By symmetry, the position of \( x \) in its copy of \( P_{\ell_i} \) does not matter, since deleting \( V(P_{\ell_i}) \) and two additional unused vertices always leaves \( P_{q-\ell_i-2} \). Thus

\[
s(C_q + P_r, L^m) = s(P_{q-1} + P_r, L^m) + \sum_{i=1}^{p} \ell_i s(P_{q-\ell_i-2} + P_r, L^m_i)
\]

It suffices to prove that the right sides of these two equations are equal. The first term is identical. It remains to show

\[
s(P_{q-\ell_i-2} + P_r, L^m_i) = s(P_{q-(\ell_i-t)}-2 + P_{r-t}, L^m_i)
\]

for \( 1 \leq i \leq p \) and \( 1 \leq t \leq \ell_i \). Adding vertices \( w_{h-1}, w_h, w_{h+1} \) to connect the two given paths shows that each such value is \( s'(P_n, L^m_i + P_1) \) for the specified vertex \( w_h \) along the host path with vertices \( w_1, \ldots, w_n \), where \( n = q + r + 1 - \ell_i \). Theorem 2.4 states that the value does not depend on \( h \) as long as \( k' = h \leq n - k' + 1 \), where \( k' \) is the number of vertices in the desired linear forest.

Here \( k' = k - \ell_i + 1 \) and \( n = q + r + 1 - \ell_i \), so we seek \( k - \ell_i + 1 \leq h \leq q + r - k + 1 \). The lowest value taken by \( h \) is \( q - \ell_i \), and the highest is \( q \) (when \( t = \ell_i \)). Since \( q \geq k + 1 \) and \( r \geq k - 1 \), the desired inequalities hold (and we cannot weaken the hypotheses).

Lemma 3.1 and Lemma 2.3 yield the desired result for graphs that are not 2-regular.

**Corollary 3.2.** Let \( G \) and \( G' \) be non-regular graphs with maximum degree 2 that have the same number of vertices and same number of edges. If all cycles in \( G \) and \( G' \) have more than \( k \) vertices and all path components have at least \( k - 1 \) vertices, then \( \mathcal{D}_k(G) = \mathcal{D}_k(G') \).

**Proof.** Since \( G \) and \( G' \) are not regular, each has at least one path component. Using Lemma 3.1 to absorb cycles into paths, each has the same \( k \)-deck as some linear forest with the same numbers of vertices and edges as it and with at least \( k - 1 \) vertices in each component. By Corollary 3.2, the resulting linear forests \( H \) and \( H' \) have the same \( k \)-deck.  

It remains only to consider 2-regular graphs, which was our original motivation. The results from the earlier cases simplify the proof here.

**Theorem 3.3.** Let \( L^m \) be a linear forest with \( k \) vertices. For \( n \)-vertex graphs whose components are cycles with at least \( k + 1 \) vertices, the number of induced copies of \( L^m \) depends only on \( L^m \) and \( n \).

**Proof.** In particular, for each such graph, we show that the number of induced copies of \( L^m \) is the same as in \( C_n \). It suffices to show \( s(C_{q+r}, L^m) = s(C_q + C_r, L^m) \) when \( q, r \geq k + 1 \); we can then iteratively reduce the number of components without changing the \( k \)-deck.
Choose \(x \in V(C_{q+r})\) and \(y \in V(C_r)\). We expand the two needed quantities by considering the usage of \(x\) and \(y\) in induced copies of \(L^m\). In each case, the specified vertex may be omitted, or it may occur in a copy of some path \(P_{\ell_i}\). In the latter case, it may occur with any position in \(P_{\ell_i}\), but the resulting number of subgraphs is the same for each position, since deleting any \(\ell_i\)-vertex path from a cycle leaves a path of the same length. We thus have the following two expansions.

\[
s(C_{q+r}, L^m) = s(P_{q+r-1}, L^m) + \sum_{i=1}^{p} \ell_i s(P_{q+r-\ell_i-2}, L_i^m)
\]

\[
s(C_q + C_r, L^m) = s(C_q + P_{r-1}, L^m) + \sum_{i=1}^{p} \ell_i s(C_q + P_{r-\ell_i-2}, L_i^m)
\]

It suffices to use Lemma 3.1 to show that corresponding terms on the right are equal. Equality of the first terms follows from \(q \geq k+1\) and \(r-1 \geq k-1\), which hold by assumption. For the other case it suffices to have \(q \geq k-\ell_i+1\) and \(r-\ell_i-2 \geq k-\ell_i-1\). The first inequality holds since \(q \geq k + 1\). The second simplifies to \(r \geq k + 1\), which holds by assumption.

**Corollary 3.4.** Any two \(n\)-vertex graphs whose components are cycles with at least \(k + 1\) vertices have identical \(k\)-decks.

With Corollaries 3.2 and 3.4, we have now proved Theorem 1.7, our main result.

### 4 \(\rho(G)\) for Graphs with Maximum Degree 2

We first reduce the problem of \(k\)-deck reconstruction to the problem of finding all components with more than \(k\) vertices. This generalizes classical reconstruction of disconnected graphs, and it applies to all graphs.

**Lemma 4.1.** If all the components with more than \(k\) vertices in a graph \(G\) can be determined from \(\mathcal{D}_k(G)\), then \(G\) is \(k\)-deck reconstructible.

**Proof.** It suffices to show that all the components with exactly \(k\) vertices can be determined, since we have already observed that \(\mathcal{D}_k(G)\) determines \(\mathcal{D}_{k-1}(G)\). We then iterate to find all smaller components.

Let \(H_1, \ldots, H_r\) be the components of \(G\) with more than \(k\) vertices. Let \(F\) be a component with exactly \(k\) vertices. The number of components of \(G\) isomorphic to \(F\) is obtained by subtracting \(\sum_{i=1}^{r} s(H_i, F)\) from the number of cards in \(\mathcal{D}_k(G)\) isomorphic to \(F\).  

**Lemma 4.2.** If \(\Delta(G) = 2\), then the number of components of \(G\) that are paths with at least \(k - 1\) vertices is \(s(G, P_{k-1}) - s(G, P_k) - ks(G, C_k)\).
Proof. Each path component with at least \( k - 1 \) vertices contributes exactly 1 to \( s(G, P_{k-1}) - s(G, P_k) \). Each \( m \)-cycle with \( m > k \) contributes \( m \) to both \( s(G, P_{k-1}) \) and \( s(G, P_k) \). Each \( k \)-cycle contributes \( k \) to both \( s(G, P_{k-1}) \) and \( k s(G, C_k) \). No smaller component contributes. Hence each component is counted correctly. \( \square \)

**Lemma 4.3.** If \( \Delta(G) = 2 \), then the number of components of \( G \) that are paths with at least \( k - 1 \) vertices is determined by \( D_k(G) \).

Proof. Each subgraph of \( G \) having \( k \) vertices appearances exactly once as a card in \( D_k(G) \). Hence counting the cards that are paths and cycles yields \( s(G, P_k) \) and \( s(G, C_k) \). Each induced subgraph of \( G \) that is a copy of \( P_{k-1} \) occurs as an induced subgraph of a \( k \)-card exactly \( n - k + 1 \) times, where \( n = |V(G)| \). Thus \( s(G, P_{k-1}) = s(J, P_{k-1})/(n - k + 1) \), where \( J \) is the disjoint union of all the \( k \)-cards of \( G \). Hence we can determine all the terms in the computation in Lemma 4.2. \( \square \)

**Lemma 4.4.** Let \( G \) be a graph with maximum degree 2. If \( G \) has no path components with at least \( k \) vertices, and \( 0 < s(G, P_k) \leq 2k + 1 \), then \( G \) has exactly one component with more than \( k \) vertices, and it is a cycle with \( s(G, P_k) \) vertices.

Proof. By hypothesis, no components are paths with more than \( k \) vertices, so such components are cycles, each contributing at least \( k + 1 \) cards that are \( P_k \). With \( s(G, P_k) \leq 2k + 1 \), there is at most one such component. With \( s(G, P_k) > 0 \), there is at least one. \( \square \)

**Lemma 4.5.** Let \( G \) be a graph with maximum degree 2. If \( G \) has exactly one path component with at least \( k - 1 \) vertices, and \( 0 \leq s(G, P_k) \leq k \), then \( G \) has no cycle with more than \( k \) vertices, and its one path component with at least \( k - 1 \) vertices has \( s(G, P_k) + k - 1 \) vertices.

Proof. Since \( s(G, P_k) \leq k \), no component is a cycle with more than \( k \) vertices. Since \( s(C_k, P_k) = 0 \), all copies of \( P_k \) come from paths, of which by hypothesis there is only one. Now \( s(P_m, P_k) = m - k + 1 \) for \( m \geq k - 1 \) completes the proof. \( \square \)

In order to use the lemmas above to prove the upper bounds, we need to determine from \( D_k(G) \) that \( G \) has maximum degree 2. When \( k \geq 4 \), this follows from Manvel’s result, but we will need it also sometimes when \( k = 3 \). The cases in the next lemma will suffice.

**Lemma 4.6.** If \( \Delta(G) = 2 \), then every reconstruction from \( D_3(G) \) has maximum degree 2 in the following cases: \( G \) has no isolated vertices, \( G = P_4 + aP_1 \) with \( a \geq 0 \), or \( G = aP_3 + bC_3 + cP_2 + dP_1 \) with \( \min\{b, d\} \leq 3 \) and \( a \leq 1 \). When \( G \) has an isolated vertex, there are alternative reconstructions with maximum degree 3 in the following cases: \( G \) has a component with at least five vertices, or a 4-cycle, or three components forming \( P_4 + C_3 + P_1 \), or eight components forming \( 4C_3 + 4P_1 \). Let \( \mathcal{F} \) denote the family of such graphs \( G \).
Proof. We first exhibit the alternative reconstructions for \( G \in F \). Let \( Y_r \) be any tree with \( r \) vertices and three leaves. Note that \( \Delta(Y_r) = 3 \).

For \( m \geq 4 \), the graph \( C_m + P_1 \) has the same 3-deck as \( Y_{m+1} \). The 3-deck has no triangles, \( m \) copies of \( P_3 \), and \( m(m-4) \) copies of \( P_2 + P_1 \), with the other cards being \( 3P_1 \).

For \( m \geq 5 \), the graph \( P_m + P_1 \) has the same 3-deck as \( Y_{m-1} + P_2 \). The 3-deck has no triangles, \( m-2 \) copies of \( P_3 \), and \( (m-2)^2 + 1 \) copies of \( P_2 + P_1 \); the other cards are \( 3P_1 \).

In addition, \( D_3(P_4 + C_3 + P_1) = D_3(K_{1,3}^+ + 2P_2) \), where \( K_{1,3}^+ \) is the “paw”, obtained from \( K_{1,3} \) by adding one edge (the 3-deck has one triangle, two copies of \( P_3 \), and 29 copies of \( P_2 + P_1 \)). Also, \( D_3(4C_3 + 4P_1) = D_3(K_4 + 6P_2) \) (the 3-deck has four triangles, no copies of \( P_3 \), and 156 copies of \( P_2 + P_1 \)).

For the remaining cases, let \( H \) be a reconstruction from \( D_3(G) \). We know \(|V(H)|\) from \( D_1(G) \) (call it \( n \)) and \(|E(H)|\) from \( D_2(G) \). Also \( D_3(G) \) tells us the number of incidences between edges, which equals \( \sum_{v \in V(H)} \binom{d(v)}{2} \). If \( G \) has no isolated vertices, then \( G \) has \( n-t/2 \) edges and \( n-t \) incidences, where \( t \) is the number of vertices of degree 1. Among all lists \( d_1, \ldots, d_n \) of nonnegative integers summing to \( 2n-t \), by convexity \( \sum \binom{d_i}{2} \) is minimized (and equals \( n-t/2 \)) precisely when all entries are 1 or 2. Hence in this case we know the maximum degree (and degree list) of \( H \).

When \( G = P_4 + aP_1 \), every reconstruction \( H \) from \( D_3(G) \) has three edges. Thus \( H \) consists of \( P_4 \), \( K_{1,3} \), \( C_3 \), \( P_3 + P_1 \), or \( 3P_2 \) plus isolated vertices. Among these, only \( G \) has exactly two copies of \( P_3 \) in its 3-deck.

It remains to consider \( G = aP_3 + bC_3 + cP_2 + dP_1 \) with \( a \leq 1 \). If \( a = 1 \), then \( D_3(G) \) has exactly one copy of \( P_3 \). Being connected, it comes from one component of \( H \), and the only connected graph with exactly one copy of \( P_3 \) in its 3-deck is \( P_3 \). Hence \( H \) has \( P_3 \) as one component. By Lemma 2.1, we therefore need only consider \( G = bC_3 + cP_2 + dP_1 \). Let \( H \) be an alternative reconstruction from the 3-deck of a minimal such graph \( G \). By Lemma 2.1, each graph is a component in at most one of \( G \) and \( H \).

Since \( P_3 \) is not a 3-card, \( H \) is a disjoint union of complete graphs. When \( b > 0 \), we have that \( C_3 \) is not a component of \( H \). Hence \( b \) counts the triangles in the components of \( H \) with more than three vertices. In \( G \), we have three edges per triangle. In \( H \) the components generating triangles have fewer than three edges per triangle. Hence \( H \) has isolated edges, and \( G \) does not. A copy of \( K_3 \) in \( H \) with \( m > 3 \) uses \( \binom{m}{2} \) edges to generate \( \binom{m}{3} \) triangles, which in \( G \) use \( 3\binom{m}{3} \) edges. Hence \( H \) has \( 3\binom{m}{3} - \binom{m}{2} \) isolated edges for each such component. Associated with each such component in \( H \), we thus have \( m + 6\binom{m}{3} - 2\binom{m}{2} \) vertices in \( H \) and \( 3\binom{m}{3} \) edges in \( G \). This requires at least \( 3\binom{m}{3} - m(m-2) \) isolated vertices in \( G \). If \( \Delta(H) \neq 2 \), then \( H \) has a component with \( m \geq 4 \), which requires that \( G \) has at least four isolated vertices and at least four components that are triangles.

Finally, if \( G = cP_2 + dP_1 \), then we know \( G \) is reconstructible from \( D_3(G) \).

These exceptions in Lemma 4.6 will yield exceptions to the general formula we now define.
**Definition 4.7.** Given a graph $G$ with $n$ vertices and maximum degree at most 2, let $m$ and $m'$ be the numbers of vertices in two largest components of $G$, with $m \geq m'$ (possibly $m' = 0$). Let $\epsilon = 1$ if $G$ has $P_m$ as a component; otherwise $\epsilon = 0$. Let $\epsilon' = 2$ if $m' < m - 1$ and $G$ has $P_{m'}$ as a component. Let $\epsilon' = 1$ if $m' = m - 1$ and $G$ has $P_{m'}$ as a component, if $m' < m$ and $G$ has $P_{m'-1}$ but not $P_{m'}$ as a component, or if $m' = m$ and at least two components of $G$ equal $P_m$. Otherwise, let $\epsilon' = 0$. Now define

$$k_G = \max\{\lfloor m/2 \rfloor + \epsilon, m' + \epsilon'\}.$$  

(*)

Now we can determine $\rho(G)$.

**Theorem 4.8.** Let $G$ be a graph with $n$ vertices and maximum degree at most 2, using notation $m, m', \epsilon, \epsilon', k_G$ as in Definition 4.7. Always $\rho(G) = k_G$, except that $\rho(G) = 4$ when $k_G = 3$ and $G \in \mathcal{F}$.

Proof. Lower bounds. We first use facts (1),(2),(3) listed after Theorem 1.7. When we provide another graph having the same $k$-deck, we obtain $\rho(G) > k$.

Consider first a largest component, and let $k = \lfloor m/2 \rfloor + \epsilon - 1$.

(1) yields $D_k(C_m) = D_k(C_{\lfloor m/2 \rfloor} + C_{\lfloor m/2 \rfloor})$ when $k < \lfloor m/2 \rfloor$, and

(2) yields $D_k(P_m) = D_k(C_{\lfloor m/2 \rfloor} + 1 + P_{\lfloor m/2 \rfloor - 1})$ when $k \leq \lfloor m/2 \rfloor$.

Combined with Lemma 2.1, we obtain $\rho(G) \geq \lfloor m/2 \rfloor + \epsilon$.

Now consider two large components, and let $k = m' + \epsilon' - 1$. Suppose first that $G$ has $P_q$ as a component, where $q \in \{m', m'-1\}$.

(2) yields $D_k(C_m + P_q) = D_k(P_{m+q})$ when $k < m$ and $k \leq q + 1$,

(3) yields $D_k(P_m + P_q) = D_k(P_{m-1} + P_{q+1})$ when $k \leq \min\{m, q+1\}$.

Depending on whether the unique largest component of $G$ is a path, these observations yield $\rho(G) \geq m' + \epsilon'$ in these cases: $\epsilon' = 2$ (using $q = m'$), and $\epsilon' = 1$ (when $m' < m$ using $q = m' - 1$ or $m' = m - 1$ using $q = m'$, and when $m' = m$ using $q = m' = m$).

When $\epsilon' = 0$, every component with $m'$ vertices is a cycle, except when $G$ contains $C_m + P_m$. This we can also write as $P_m + C_m$, since then $m' = m$. Let $k = m' - 1$. Now

(1) yields $D_k(C_m + C_m) = D_k(C_{m+m})$ when $k < m' \leq m$,

(2) yields $D_k(P_m + C_m) = D_k(P_{m+m})$ when $k < m'$ (since also $k \leq m + 1$).

Combined with Lemma 2.1, we obtain $\rho(G) \geq m' + \epsilon'$ in each case.

Thus $\rho(G) \geq k_G$. When $k_G = 3$ and $G \in \mathcal{F}$, the alternative reconstructions in Lemma 4.6 show that $\rho(G) \geq 4$.

Upper bounds. If $|E(G)| \leq 1$, then $k_G = 2$, and indeed $G$ is determined by its 2-deck. If $|E(G)| \geq 2$ and $\Delta(G) = 1$, then $k_G = 3$, and by Manvel’s result $D_3(G)$ determines the
degree list, which in turn determines $G$. In all other cases, $\Delta(G) = 2$ and $k_G \geq 3$. If $k_G = 3$ and $G$ has an isolated vertex with $m \geq 4$ (except $P_4 + aP_1$) or with $G$ containing $4C_3 + 4K_1$, then set $k = 4$. Otherwise, set set $k = k_G$.

When $k_G = 3$ and $G \notin \mathcal{F}$, every reconstruction from $\mathcal{D}_3(G)$ has maximum degree 2, by Lemma 4.6. In all other cases, $k \geq 4$ and Manvel’s result implies that every reconstruction has maximum degree 2. This fact is all we need for the main argument.

By Lemma 4.1 it suffices to show that $\mathcal{D}_k(G)$ determines the components of $G$ with more than $k$ vertices. Since $k \geq k_G$, we have $k \geq \lceil m/2 \rceil + \epsilon$ and $k \geq m' + \epsilon'$. The key claim that allows us to apply the lemmas is this:

Claim: If $k \geq m' + \epsilon'$ and $m' < m - 1$ (or $m' = m - 1$ and $G$ does not have $P_m'$ as a component), then at most one path component has at least $k - 1$ vertices.

We check cases. If $\epsilon' = 2$, then $G$ has $P_m'$ as a component and at most one component with more vertices, which suffices since $m' < k - 1$. If $\epsilon' = 1$ and $G$ has $P_{m'-1}$ but not $P_{m'}$ as a component, then at most one component that is a path has at least $m'$ vertices, which suffices since $m' \leq k - 1$. If $m' = m - 1$ and $G$ does not have $P_{m'}$ or $P_{m'-1}$ as a component, then $\epsilon' = 0$ and $G$ has at most one path component with at least $k - 1$ vertices. The claim applies to all cases with $m' < m$ except when $m' = m - 1$ and $G$ has $P_{m'}$ as a component.

For all these cases, $G$ has at most one path component having at least $k - 1$ vertices. Since $\epsilon' = 2$ only when $m' < m - 1$, whenever $m' < m$ we also have $k \leq m$. Hence there is one such path component if $P_m$ is a component, in which case $s(G, P_k) = m - k + 1$, and there are none if $C_m$ is a component and $P_m$ is not, in which case $s(G, P_k) = m$.

Now consider a reconstruction $H$ from $\mathcal{D}_k(G)$. By Lemma 4.3, the number of components of $H$ that are paths with at least $k - 1$ vertices is the same as in $G$. Furthermore, $s(H, P_k) = s(G, P_k)$; this just counts the $k$-cards isomorphic to $P_k$.

When $G$ has no components that are paths with at least $k - 1$ vertices, cards that are paths arise only from cycles with more than $k$ vertices. In particular, since $k \geq m'$, no such cards arise from $m'$-cycles, and $m = s(G, P_k) = s(H, P_k)$. Since $k \geq \lceil m/2 \rceil + \epsilon$ and here $\epsilon = 0$, we have $m \leq 2k + 1$. Now Lemma 4.4 implies that $H$ has exactly one component with more than $k$ vertices, and it is $C_m$.

When $G$ has exactly one component that is a path with at least $k - 1$ vertices, and it is $P_m$, the same holds for $H$. Again $k \geq m'$ implies that no copies of $P_k$ arise from $m'$-cycles, so $m - k + 1 = s(G, P_k) = s(H, P_k)$. $k \geq \lceil m/2 \rceil + \epsilon \leq m' + \epsilon'$ and $\epsilon = 1$, we have $m \leq 2k - 1$, and hence $m - k + 1 \leq k$. Now Lemma 4.5 implies that $H$ has exactly one component with more than $k$ vertices, and it is $P_m$.

In each case above the components of $H$ having more than $k$ vertices are the same as in $G$, which suffices. In the remaining cases we show that both have no such components. These cases are when $m' = m$ or when $m' = m - 1$ with $P_m'$ being a component of $G$.

If $G$ has at least two components isomorphic to $P_m$, then $\epsilon' = 1$ and $k = m + 1$. Since
no component of $G$ has at least $k$ vertices, no card is connected; hence $H$ has no component with at least $k$ vertices. Otherwise, $k = m$. Since $G$ has no component with more than $k$ vertices, at most one $k$-card is $P_k$. Thus $s(H, P_k) \leq 1$. Since $\Delta(H) = 2$, we again conclude that $H$ has no component with more than $k$ vertices.

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