Abstract

In this paper, we consider the solutions of the relaxed Q-tensor flow in $\mathbb{R}^3$ with small parameter $\epsilon$. Firstly, we show that the limiting map is the so-called harmonic map flow; Secondly, we also present a new proof for the global existence of weak solution for the harmonic map flow in three dimensions as in [23] and [18], where Ginzburg-Landau approximation approach was used.

1 Introduction

Liquid crystals are a state of matters that have properties between those of a conventional liquid and those of a solid crystal. One of the most common liquid crystal phases is the nematic. The nematic liquid crystals are composed of rod-like molecules with the long axes of neighboring molecules approximately aligned to one another. There are three different kinds of theories to model the nematic liquid crystals: Doi-Onsager theory, Landau-de Gennes theory and Ericksen-Leslie theory. The first is the molecular kinetic theory, and the later two are the continuum theory. In the spirit of Hilbert sixth problem, it is very important to explore the relationship between these theories.

Ball-Majumdar [1] define a Landau-de Gennes type energy functional in terms of the mean-field Maier-Saupe energy. Majumdar-Zarnescu [14] consider the Oseen-Frank limit of the static Q-tensor model. Their results show that the predictions of the Oseen-Frank theory and the Landau-de Gennes theory agree away from the singularities of the limiting Oseen-Frank global minimizer.

In [11, 5], Kuzzu-Doi and E-Zhang formally derive the Ericksen-Leslie equation from the Doi-Onsager equations by taking small Deborah number limit. In [20, 21], Wang-Wang-Zhang present a rigorous derivation from Doi-Onsager theory and Landau-de Gennes theory. In [9], a systematical approach was proposed to derive the continuum theory from the molecular kinetic theory in both static and dynamic case.
Different with the above results on the static or local case, the goal of this work is to investigate the global convergence problem of the solutions $Q_\varepsilon$ of the Q-tensor flow in $\mathbb{R}^3$. We will show that $Q_\varepsilon$ convergences weakly to the weak solution of the harmonic map flow.

1.1 The relaxed Q-tensor flow

In Landau-de Gennes theory, the state of the nematic liquid crystals is described by the macroscopic Q-tensor order parameter, which is a symmetric, traceless $3 \times 3$ matrix. Physically, it can be interpreted as the second-order moment of the orientational distribution function $f$, that is

$$Q = \int_{S^2} (mm - \frac{1}{3}Id) f dm.$$

When $Q = 0$, the nematic liquid crystal is said to be **isotropic**. When $Q$ has two equal non-zero eigenvalues, it is said to be **uniaxial** and $Q$ can be written as

$$Q = s(nn - \frac{1}{3}Id), \quad n \in S^2.$$

When $Q$ has three distinct eigenvalues, it is said to be **biaxial** and $Q$ can be written as

$$Q = s(nn - \frac{1}{3}Id) + \lambda(n'n' - \frac{1}{3}Id), \quad n, n' \in S^2, \quad n \cdot n' = 0.$$

The general Landau-de Gennes energy functional takes the form

$$F_{LG}(Q, \nabla Q) = \int_{\mathbb{R}^3} \left\{ \begin{array}{c}
-\frac{a}{2}tr Q^2 - \frac{b}{3}tr Q^3 + \frac{c}{4}(tr Q^2)^2 \\
+ \frac{1}{2}(L_1|\nabla Q|^2 + L_2Q_{ij,j}Q_{ik,k} + L_3Q_{ij,k}Q_{ik,j} + L_4Q_{ij}Q_{kl,i}Q_{kl,j})
\end{array} \right\} dx,$$

where $a, b, c$ are material-dependent and temperature-dependent nonnegative constant and $L_i (i = 1, 2, 3, 4)$ are material dependent elastic constants. We refer to [8, 13] for more details.

There are several dynamic Q-tensor models to describe the flow of the nematic liquid crystal, which are either derived from the molecular kinetic theory for the rigid rods by various closure approximation such as [6, 7, 9], or directly derived by variational method such as Beris-Edwards model [3] and Qian-Sheng’s model [17].

In [21], the authors consider the following Beris-Edwards model

$$\left\{ \begin{array}{l}
\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \nabla \cdot (\sigma^s + \sigma^a + \sigma^d), \\
\nabla \cdot v = 0,
\end{array} \right\} \quad (Q)$$

$$\frac{\partial Q}{\partial t} + v \cdot \nabla Q + Q \cdot \Omega - \Omega \cdot Q = \frac{1}{\Gamma} H + S_Q(D).$$

Here $\Gamma$ is a collective rotational diffusion constant, and

$$D = \frac{1}{2}(\nabla v + (\nabla v)^T), \quad \Omega = \frac{1}{2} (\nabla v - (\nabla v)^T).$$
Moreover, \( \sigma^s, \sigma^a \) and \( \sigma^d \) are symmetric viscous stress, antisymmetric viscous stress and distortion stress respectively defined by

\[
\sigma^s = \eta D - S_Q(H), \quad \sigma^a = Q \cdot H - H \cdot Q, \quad \sigma^d = -\frac{\partial F_{LG}}{\partial Q_{klj}} Q_{kl,i},
\]

where \( \eta > 0 \) is the viscous coefficient, \( H \) is the molecular field given by

\[
H(Q) = -\frac{\delta F_{LG}}{\delta Q},
\]

and \( S_Q(M) \) is defined by

\[
S_Q(M) = \xi \left( M \cdot (Q + \frac{1}{d} I) + (Q + \frac{1}{d} I) \cdot M - 2(Q + \frac{1}{d} I)Q : M \right)
\]

for symmetric and traceless matrix \( M \), where \( \xi \) is a constant depending on the molecular details of a given liquid crystals. The well-posedness results of the Q-tensor model are studied in [15, 16].

Wang-Zhang-Zhang [21] justify the limit from Beris-Edwards system with a small parameter \( \epsilon \) to the Ericksen-Leslie system before the first singularity time of the limit system. The limit behavior of the solution after the singularity remains unknown. In this paper, we are interested in the global convergence from the Q-tensor flow to the harmonic map flow in \( \mathbb{R}^3 \).

Let us begin with the simplest form \( L_2 = L_3 = L_4 = 0 \), i.e.,

\[
\frac{\partial Q}{\partial t} = -\frac{1}{\epsilon \Gamma} J(Q) + \frac{L_1}{\Gamma} \Delta Q,
\]

where

\[
J(Q) := \frac{\delta f_B(Q)}{\delta Q} = -aQ - bQ^2 + c|Q|^2Q + \frac{1}{3} b|Q|^2 I.
\]

Let \( \bar{b} \in S^2 \) is a constant vector, \( n_0 : \mathbb{R}^3 \to S^2 \) such that \( n_0 - \bar{b} \in H^{s+1}(\mathbb{R}^3)(s > 0) \). Moreover, \( Q_0(x) = s_+(n_0(x) \otimes n_0(x) - \frac{Id}{3}) \). We consider the following relaxed Q-tensor equations with a small parameter \( \epsilon \):

\[
(Q_\epsilon) \quad \begin{cases} 
\frac{\partial Q_\epsilon}{\partial t} = \frac{(a - c|Q_\epsilon|^2)Q_\epsilon + bQ_\epsilon^2 - b\frac{|Q_\epsilon|^2}{3}I}{\epsilon \Gamma} + \frac{L_1}{\Gamma} \Delta Q_\epsilon, \\
Q_\epsilon(\cdot, 0) = Q_0(x),
\end{cases}
\]

which has a unique strong solution \( Q_\epsilon(t, x) \) satisfying \( Q_\epsilon(t, x) - Q_0(x) \in L^\infty_{loc}([0, \infty), H^{s+1}(\mathbb{R}^3)) \). We will study the global convergence of \( Q_\epsilon \) as \( \epsilon \) tends to zero.

### 1.2 Main result

The initial data \( Q_0 \) of the Q-tensor flow equations (1.4) lies in a special space, which contains the minimizers of the bulk energy \( f_B(Q) \). To begin this, we introduce some notations and known results.
Let $M_{3 \times 3}^{\text{sym}}$ denote the set of real $3 \times 3$ symmetric matrices and $Q_0 \subset M_{3 \times 3}^{\text{sym}}$ denote the space of $Q$-tensors defined by

$$Q_0 := \{ Q \in M_{3 \times 3}^{\text{sym}}, Q_{ii} = 0 \},$$

where we have used the Einstein summation convention. The matrix norm is defined as

$$|Q| := \sqrt{ \text{tr} Q^2 } = \sqrt{ Q_{ij} Q_{ij} }.$$

We also write

$$|\nabla Q|^2 = \partial_\alpha Q_{ij} \partial_\alpha Q_{ij}.$$

The bulk energy density can be written as

$$f_B(Q) = -\frac{a}{2} |Q|^2 - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} |Q|^4. \quad (1.5)$$

One can verify that $f_B$ is bounded from below (for example, see [14, Proposition 8]), thus $f_B$ has the corresponding non-negative bulk energy density $\tilde{f}_B$ defined by

$$\tilde{f}_B(Q) = f_B(Q) - \min_{Q \in Q_0} f_B(Q). \quad (1.6)$$

In [14, Proposition 8], it was proved that $\tilde{f}_B$ attains its minimum on the uniaxial $Q$-tensors with constant order parameter $s_+$ as shown below

$$\tilde{f}_B(Q) = 0 \iff Q \in \mathcal{N} \quad \text{where} \quad \mathcal{N} = \left\{ Q \in Q_0, Q = s_+ \left( n \otimes n - \frac{1}{3} \text{Id} \right), n \in S^2 \right\}, \quad (1.7)$$

with

$$s_+ = \frac{b + \sqrt{b^2 + 24ac}}{4c}. \quad (1.8)$$

For a matrix $Q \in \mathcal{N}$, we use $T_Q \mathcal{N}$ to denote the tangent space to $\mathcal{N}$ at $Q$ in $Q_0$, and $(T_Q \mathcal{N})_{Q_0}^\perp$ to denote the orthogonal complement of $T_Q \mathcal{N}$ to $Q_0$.

Let \{A, B\} = AB + BA for $A, B \in M_{3 \times 3}^{\text{sym}}$. It was described in [22] Lemma 2] that

$$T_Q \mathcal{N} = \left\{ \dot{Q} \in M_{3 \times 3}^{\text{sym}} : \frac{1}{3} s_+ \dot{Q} = \{ \dot{Q}, Q \} \right\},$$

$$= \left\{ \dot{n} \otimes \dot{n} + \dot{n} \otimes \dot{n} : \dot{n}, \dot{n} \in T_n S^2 \right\}, \quad (1.9)$$

and

$$(T_Q \mathcal{N})_{Q_0}^\perp = \left\{ Q^\perp \in Q_0 : Q^\perp Q = QQ^\perp \right\}. \quad (1.10)$$

We will specifically describe the orthogonal basis for the tangent and normal space in the following Lemma 2.2. It is obvious from (1.10) that $J(Q) \in (T_Q \mathcal{N})_{Q_0}^\perp$ for $Q \in \mathcal{N}$, and we will show in Lemma 2.4 that for the approximating $Q_\epsilon$ near $\mathcal{N}$, we still have

$$J(Q_\epsilon) \in (T_{\pi_\mathcal{N}(Q_\epsilon)} \mathcal{N})_{Q_0}^\perp,$$
where $\pi_{\mathcal{N}}$ denotes the projection operator on $\mathcal{N}$.

Let $z = (x, t)$ denote points in $\mathbb{R}^3 \times \mathbb{R}_+$. For $z_0 = (x_0, t_0)$, $R > 0$, let

\[
P_R(z_0) = \{ z = (x, t) \mid |x - x_0| < R, |t - t_0| < R^2 \},
\]

\[
S_R(z_0) = \{ z = (x, t) \mid t = t_0 - R^2 \},
\]

\[
T_R(z_0) = \{ z = (x, t) \mid t_0 - 4R^2 < t < t_0 - R^2 \}.
\]

Denote the scaled fundamental solution to the heat equation

\[
G_{z_0}(z) = \tilde{G}_{(x_0, t_0)} \left( \frac{\Gamma}{L_1}, \frac{\Gamma}{L_1} t \right),
\]

where

\[
\tilde{G}_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{3/2}} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}}, t < t_0.
\]

Then we have

\[
\nabla G_{z_0} = \frac{\Gamma}{2L_1} \cdot \frac{x - x_0}{t - t_0} G
\]

Also we write $P_R(0) = P_R$, $T_r(0) = T_r$, and $G_0(z) = G(z)$.

Similar to the harmonic map flow, the limiting Q-tensor flow takes as follows

\[
\partial_t Q - \frac{L_1}{\Gamma} \Delta Q + \lambda(x, t)\gamma_{\mathcal{N}}(Q) = 0, \quad Q(x, t)|_{t=0} = Q_0(x),
\]

(1.11)

where $\lambda(x, t)$ is a function of $L_{\text{loc}}^2$, and $\gamma_{\mathcal{N}}(Q)$ is unit normal vector to $(T_{Q_N})_{\frac{1}{2}}$ at $Q$.

**Definition 1.1** A Q-tensor $Q(x, t): \mathbb{R}^3 \times \mathbb{R}_+ \to \mathcal{N}$ is called a weak solution to (1.11) if $Q(x, t)|_{t=0} = Q_0(x)$ a.e., $\partial_t Q, \nabla Q \in L^2_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}_+)$, and there holds

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}_+} (\partial_t Q : \phi + \frac{L_1}{\Gamma} \nabla Q : \nabla \phi + \lambda \gamma_{\mathcal{N}}(Q) : \phi) dx dt = 0,
\]

for all $\phi \in C^0_0(\mathbb{R}^3 \times \mathbb{R}_+, \mathbb{R}^{3 \times 3})$.

Our main theorem is stated as follows.

**Theorem 1.2** Let $Q_{\epsilon}$ satisfy the equations of the relaxed Q-tensor flow equations (1.4) with the data $Q_0(x) = s_+ \left( n_0 \otimes n_0 - \frac{1}{3} \text{Id} \right) \in \mathcal{N}$ as in (1.7). Then

1. there exists a subsequence of $\epsilon$ (also denoted $\epsilon$) such that

\[
Q_{\epsilon} \rightharpoondown Q = s_+ \left( n \otimes n - \frac{1}{3} \text{Id} \right) \in \mathcal{N},
\]

and $Q$ solves weakly the equations (1.11).
2. The director field \( n \) weakly solves
\[
\partial_t n - \Delta n = |\nabla n|^2 n, \quad n(x,t)|_{t=0} = n_0(x).
\] (1.12)

3. \( n \) is regular on a dense open set \( \Omega_0 \subset \mathbb{R}^3 \times \mathbb{R}_+ \), whose complement \( \Sigma \) has locally finite 3-dimensional Hausdorff-measure (with respect to the parabolic metric).

**Remark 1.3** Compared with [20, 21], the above theorem makes it reasonable that the global weak convergence from the Q-tensor flow to the simplified Oseen-Frank map flow (i.e., harmonic map flow). Different from Ginzburg-Landau approximation used in [23] and [18], we consider the Q-tensor approximation, and the difficulty is that the properties of the limit manifold \( N \) is unclear as stated in [22]. In the next section, we give a careful study for the geometry of \( N \) (see Lemma 2.2).

## 2 Technical lemmas and interior regularity estimates

In this section, we will introduce the properties of Q-tensor matrix, the tangent space, the normal space and the equivalence of the bulk energy. Using these estimates and exploring monotonicity inequalities as in [23], we can obtain the interior regularity criteria of Q-tensor equations.

First of all, for the matrix of \( 3 \times 3 \), we have the following properties.

**Lemma 2.1** Let \( A, B \) be matrices of \( 3 \times 3 \).

(i) If \( A \) is symmetric, then
\[
A : B = A : \tilde{B}, \quad \tilde{B} = \frac{B + B^T}{2}
\]
where \( \tilde{B} \) is the symmetrization for \( B \).

(ii) If \( A \) is antisymmetric, then
\[
A : B = A : \bar{B}, \quad \bar{B} = \frac{B - B^T}{2}
\]
where \( \bar{B} \) is the antisymmetrization for \( B \).

(iii) If \( A \) is symmetric and \( B \) is antisymmetric, then
\[
A : B = 0.
\]

For \( Q \in N \), it is easy to verify that the orthogonal basis of \( T_Q N \) and \( (T_Q N)^\perp_{Q_0} \) is as follows.

**Lemma 2.2** Let \( Q = s_+ (n_3 \otimes n_3 - \frac{1}{3} I d) \in N \), and \( n_1, n_2 \) be unit perpendicular vectors in \( V_{n_3} = \{ n^\perp \in \mathbb{R}^3 : n^\perp \cdot n_3 = 0 \} \). Then it holds that

1. 
\[
T_Q N = \text{Span} \left\{ \frac{1}{\sqrt{2}} (n_3 \otimes n_2 + n_2 \otimes n_3), \frac{1}{\sqrt{2}} (n_3 \otimes n_1 + n_1 \otimes n_3) \right\}.
\] (2.1)
\[ (T_{Q}N)^{\perp}_{Q_0} = \text{Span} \left\{ \frac{1}{\sqrt{2}}(n_2 \otimes n_1 + n_1 \otimes n_2), \frac{1}{\sqrt{2}}(n_1 \otimes n_1 - n_2 \otimes n_2), \sqrt{6} \left( \frac{1}{2}n_1 \otimes n_1 + \frac{1}{2}n_2 \otimes n_2 - \frac{Id}{3} \right) \right\} \] (2.2)

3. Moreover, \( Q_0 = T_{Q}N \oplus (T_{Q}N)^{\perp}_{Q_0} \).

Proof: The tangent space at \( s_+ (n_3 \otimes n_3 - \frac{1}{3}Id) \) of \( N \) is a direct result from \[21\] (2.3). The others can be deduced by direct computations. \( \square \)

**Lemma 2.3** For \( Q \in Q_0 \), there exists \( \epsilon_0 > 0 \) such that if \( \text{dist}(Q, N) < \epsilon_0 \), then

\[ J(Q) \in (T_{\pi_N(Q)}(N))^{\perp}_{Q_0}. \] (2.3)

Proof: Denote the eigenvectors of \( Q(x, t) \) by \( n_1(x, t), n_2(x, t), n_3(x, t) \) corresponding to its eigenvalues \( \lambda_1(x, t), \lambda_2(x, t), \lambda_3(x, t) = -\lambda_1(x, t) - \lambda_2(x, t) \). Then we have

\[ Q(x, t) = \lambda_1 n_1 \otimes n_1 + \lambda_2 n_2 \otimes n_2 + \lambda_3 n_3 \otimes n_3, \] (2.4)

especially,

\[ I = n_1 \otimes n_1 + n_2 \otimes n_2 + n_3 \otimes n_3. \] (2.5)

Choose \( \epsilon_0 \) small enough such that \( \text{dist}(Q, N) < \epsilon_0 \), then

\[ \text{dist}(Q, N)^2 = (\lambda_1 + \frac{s_+}{3})^2 + (\lambda_2 + \frac{s_+}{3})^2 + (\lambda_1 + \lambda_2 + 2\frac{s_+}{3})^2, \] (2.6)

furthermore,

\[ \text{dist}(Q, N)^2 = |Q - \pi_N(Q)|^2, \quad \pi_N(Q) = s_+(n_3 \otimes n_3 - \frac{Id}{3}) \]

where \( \pi_N(Q) \) is unique and depends continuously on \( Q \). See \[22\] Lemma 8 for more details.

It is easy to see the projection of \( Q \) on \( T_{s_+(n_3 \otimes n_3 - \frac{1}{3}Id)}N \) is 0. Using (2.5), we have

\[
Q = \lambda_1 n_1 \otimes n_1 + \lambda_2 n_2 \otimes n_2 - (\lambda_1 + \lambda_2) n_3 \otimes n_3 \\
= (2\lambda_1 + \lambda_2) \left( n_1 \otimes n_1 - \frac{Id}{3} \right) + (\lambda_1 + 2\lambda_2) \left( n_2 \otimes n_2 - \frac{Id}{3} \right),
\]

\[
Q^2 = \lambda_1^2 n_1 \otimes n_1 + \lambda_2^2 n_2 \otimes n_2 + (\lambda_1 + \lambda_2)^2 n_3 \otimes n_3 \\
= -\lambda_2(2\lambda_1 + \lambda_2) \left( n_1 \otimes n_1 - \frac{Id}{3} \right) - \lambda_1(\lambda_1 + 2\lambda_2) \left( n_2 \otimes n_2 - \frac{Id}{3} \right) \\
+ \frac{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{3} Id,
\]
Lemma 2.4 which is stated as the following Lemma.

where we have used the orthogonal basis $e$ where

\[
Q^2 - \frac{|Q|^2}{3} Id = -\lambda_2 (2\lambda_1 + \lambda_2) \left( n_1 \otimes n_1 - \frac{Id}{3} \right) - \lambda_1 (\lambda_1 + 2\lambda_2) \left( n_2 \otimes n_2 - \frac{Id}{3} \right).
\]

Hence, we get

\[
J(Q) = (a - c|Q|^2)Q + b \left( Q^2 - \frac{|Q|^2}{3} Id \right)
\]

\[
= (a - c|Q|^2 - b\lambda_2)(2\lambda_1 + \lambda_2)(n_1 \otimes n_1 - \frac{Id}{3}) + (a - c|Q|^2 - b\lambda_1)(\lambda_1 + 2\lambda_2)(n_2 \otimes n_2 - \frac{Id}{3})
\]

\[
= \frac{1}{\sqrt{2}} (a - c|Q|^2 - b\lambda_3)(\lambda_1 - \lambda_2) e_2
\]

\[
+ \frac{1}{\sqrt{6}} ((a - c|Q|^2 - b\lambda_2)(2\lambda_1 + \lambda_2) + (a - c|Q|^2 - b\lambda_1)(\lambda_1 + 2\lambda_2)) e_3,
\]

where we have used the orthogonal basis $e_2$ and $e_3$ in the normal space $(T_{\pi_N(Q)}(N))_{\varnothing_0}$ (see 2.2), and

\[
e_2 = \frac{1}{\sqrt{2}} (n_1 \otimes n_1 - n_2 \otimes n_2), \quad e_3 = \sqrt{6} \left( \frac{1}{2} n_1 \otimes n_1 + \frac{1}{2} n_2 \otimes n_2 - \frac{Id}{3} \right).
\]

Thus, 2.3 is an immediate result. □

In fact, the nonnegative bulk energy $\tilde{f}_B(Q)$ is equivalent to the distance from $Q$ to $N$, which is stated as the following Lemma.

Lemma 2.4 There exists $\epsilon_0 > 0$ such that if $\text{dist}(Q,N) < \epsilon_0$, then

\[
\frac{1}{C} \text{dist}(Q,N)^2 \leq \tilde{f}_B(Q) \leq C \left( \text{dist}(Q,N) \right)^2,
\]

(2.7)

where $C$ is independent of $Q$, but depends on $a, b, c$.

Proof: Assume that the eigenvalues of $Q$ are $x, y, -x - y$. If $\text{dist}(Q,N) < \epsilon_0$ and $\epsilon_0$ is small enough, similar to (2.3), we have

\[
\text{dist}(Q,N)^2 = (x + \frac{s_+}{3})^2 + (y + \frac{s_+}{3})^2 + (x + y + 2\frac{s_+}{3})^2 \leq \epsilon_1.
\]

(2.8)

Let

\[
(x + \frac{s_+}{3})^2 + (y + \frac{s_+}{3})^2 + (x + y + \frac{2s_+}{3})^2 \triangleq G(x, y).
\]

(2.9)

On the other hand, for the nonnegative bulk energy, we have

\[
\tilde{f}_B(Q) = -\frac{a}{2}|Q|^2 - \frac{b}{3} tr Q^3 + \frac{c}{4} |Q|^4 - \min_{Q \in \varnothing_0} f_B(Q)
\]

\[
= -a(x^2 + y^2 + xy) + b(x^2 y + xy^2) + c(x^2 + y^2 + xy)^2 - \min_{Q \in \varnothing_0} f_B(Q)
\]

\[
\triangleq H(x, y),
\]

(2.10)
where \( H : \mathbb{R}^2 \to \mathbb{R} \) is the 2-dimensional function as \( G(x,y) \). Note that \( H(x,y) = 0 \) only at three pairs \((x,y)\) namely \((-\frac{s_+}{3}, -\frac{s_+}{3})\), \((\frac{s_+}{3}, -\frac{s_+}{3})\) and \((-\frac{s_+}{3}, 2\frac{s_+}{3})\), c.f. \[\text{[14]}\] Lemma 5. This fact together with \( (2.9) \) and \( (2.10) \) gives

\[
\begin{align*}
H\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) &= \frac{\partial H}{\partial x}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) = \frac{\partial H}{\partial y}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) = 0, \\
G\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) &= \frac{\partial G}{\partial x}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) = \frac{\partial G}{\partial y}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) = 0. \quad (2.11)
\end{align*}
\]

Careful computations show that

\[
\begin{align*}
\frac{\partial H}{\partial x} &= (2x + y)[-a + by + 2c(x^2 + y^2 + xy)], \\
\frac{\partial^2 H}{\partial x \partial y} &= [-a + by + 2c(x^2 + y^2 + xy)] + (2x + y)[b + 2c(2y + x)], \\
\frac{\partial^2 H}{\partial x^2} &= 2[-a + by + 2c(x^2 + y^2 + xy)] + 2c(2x + y)^2.
\end{align*}
\]

Noting \( 2cs_+^2 - bs_+ - 3a = 0 \), we have

\[
\begin{align*}
\frac{\partial^2 H}{\partial x^2}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) &= \frac{\partial^2 H}{\partial y^2}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) = 2cs_+^2 = bs_+ + 3a, \\
\frac{\partial^2 H}{\partial x \partial y}\left(-\frac{s_+}{3}, -\frac{s_+}{3}\right) &= -bs_+ + 2cs_+^2 = 3a, \\
\frac{\partial^2 G}{\partial x^2} &= 4 = \frac{\partial^2 G}{\partial y^2}, \quad \frac{\partial^2 G}{\partial x \partial y} = 2.
\end{align*}
\]

Then, for \((x,y) \neq (-\frac{s_+}{3}, -\frac{s_+}{3})\), by the above computations and \( (2.11) \), we have

\[
\frac{H(x,y)}{G(x,y)} = \frac{H_1(x,y) + R_H(x,y)}{G_1(x,y)}, \quad (2.12)
\]

where

\[
\begin{align*}
H_1(x,y) &= (bs_+ + 3a) \left((x + \frac{s_+}{3})^2 + (y + \frac{s_+}{3})^2\right) + 6a \left(x + \frac{s_+}{3}\right) \left(y + \frac{s_+}{3}\right), \\
G_1(x,y) &= 4 \left(x + \frac{s_+}{3}\right)^2 + 4 \left(x + \frac{s_+}{3}\right) \left(y + \frac{s_+}{3}\right) + 4 \left(y + \frac{s_+}{3}\right)^2, \quad (2.13)
\end{align*}
\]

and \( R_H \) is the remainder in the Taylor expansions of \( H(x,y) \) at \((-\frac{s_+}{3}, -\frac{s_+}{3})\). Thus for \( \epsilon_1 \) sufficiently small in \( (2.8) \), we get

\[
|R_H(x,y)| \leq \frac{bs_+}{2} \left((x + \frac{s_+}{3})^2 + (y + \frac{s_+}{3})^2\right). \quad (2.15)
\]

Summing up the inequalities \( (2.12) \) \( (2.15) \), we conclude that

\[
\frac{bs_+}{12} \leq \frac{H(x,y)}{G(x,y)} \leq (3a + \frac{3bs_+}{4}).
\]

The proof is completed. \( \square \)
Now we consider the evolution of the energy, and we will follow the same line as in [23]. First of all, we define the energy density by

\[
e_{\epsilon}(Q, \nabla Q) = \frac{1}{\epsilon} \tilde{f}_B(Q) + \frac{L_1}{2\Gamma} |\nabla Q|^2. \tag{2.16}
\]

For the equations of (1.4) with initial data \(Q_0 \in \mathcal{N}\), there exists a global weak solution \(Q_\epsilon(x,t)\) (denoted by \(Q\) for simplicity), see [16]. The solution \(Q\) is regular indeed by usual energy estimates, and we have the following basic estimates.

**Lemma 2.5 (Energy Inequality)** Suppose that \(Q(x,t)\) solves (1.4) with initial data \(Q_0 \in \mathcal{N}\), then it holds that

\[
\sup_{t \geq 0} \int_{\mathbb{R}^3} |\partial_t Q|^2 \, dx + \int_{\mathbb{R}^3} e_{\epsilon}(Q(\cdot,t), \nabla Q(\cdot,t)) \, dx \leq \int_{\mathbb{R}^3} |\nabla Q_0|^2 \, dx. \tag{2.17}
\]

**Proof:** Multiplying (1.4) by \(Q_t\) and integration by parts yield that

\[
\int_{\mathbb{R}^3} |\partial_t Q|^2 = \frac{1}{\epsilon} \int_{\mathbb{R}^3} -\frac{\delta \tilde{f}_B}{\delta Q} : \partial_t Q \, dx + \frac{L_1}{\Gamma} \int_{\mathbb{R}^3} \Delta Q : \partial_t Q \, dx.
\]

Noting that \(\frac{\delta \tilde{f}_B}{\delta Q} : \partial_t Q = \partial_t \tilde{f}_B(Q)\) and \(\tilde{f}_B(Q_0) = 0\), the lemma follows. \(\Box\)

The following parabolic maximal principle lemma is similar to Proposition 3 in [14], where the elliptic case was considered. We omitted the proof.

**Lemma 2.6 (Maximal Principle)** Suppose that \(Q(x,t)\) solves (1.4) with initial data \(Q_0 \in \mathcal{N}\), then it holds that

\[
|Q| \leq \sqrt{\frac{2}{3}} s_+.
\]

For the case of \(L_2 = L_3 = L_4 = 0\), we also have the monotonicity properties of the level energy as in [23].

**Lemma 2.7** Suppose that \(Q(x,t)\) solves (1.4) with initial data \(Q_0 \in \mathcal{N}\). For any point \(z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+\), the functions

\[
\Phi(R, Q, \epsilon) = \frac{1}{\Gamma} R^2 \int_{S_R(z_0)} \left[ \frac{L_1}{2} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{\epsilon} \right] G_{z_0} \, dx, \tag{2.18}
\]

\[
\Psi(R, Q, \epsilon) = \frac{1}{\Gamma} \int_{T_R(z_0)} \left[ \frac{L_1}{2} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{\epsilon} \right] G_{z_0} \, dx \, dt \tag{2.19}
\]

are non-decreasing for \(0 < R < \sqrt{t_0}/2\).

**Proof:** We first note that \(Q(x+x_0, t+t_0)\) satisfies the equations of (1.4) with initial data \(Q(\cdot, -t_0) = Q_0(x)\). Thus, we may assume that \(z_0 = (0,0)\). By scale invariance \(Q_R(x,t) = Q(Rx, R^2t)\) satisfying (1.4) with constant \(\epsilon_R = \frac{\epsilon}{R^2}\), we have

\[
\Phi(R, Q, \epsilon) = \frac{1}{\Gamma} R^2 \int_{S_R} \left[ \frac{L_1}{2} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{\epsilon} \right] G \, dx
\]

\[
= \frac{1}{\Gamma} \int_{S_1} \left[ \frac{L_1}{2} |\nabla Q_R|^2 + \frac{\tilde{f}_B(Q_R)}{\epsilon/R^2} \right] G \, dx \triangleq \Phi(1, Q_R, \epsilon/R^2).
\]
It suffices to consider the case of $R = 1$. Direct computations and the equations (1.4) show that
\[
\frac{d}{dR} \Phi(R, Q, \epsilon)|_{R=1} = \frac{d}{dR} \Phi(1, Q, \epsilon/R^2)|_{R=1}
\]
\[
= \frac{1}{\Gamma} \int_{S_1} \left( L_1(-\Delta Q) : (x \cdot \nabla Q + 2t \partial_t Q) \right) G(x, -1)dx \\
- \frac{1}{\Gamma} \int_{S_1} L_1(x \cdot \nabla Q + 2t \partial_t Q) \nabla_k Gdx + \frac{2}{\epsilon \Gamma} \int_{S_1} \tilde{f}_B G(x, -1)dx \\
= \int_{S_1} -\partial_t Q : (x \cdot \nabla Q + 2t \partial_t Q)Gdx + \frac{1}{\Gamma} \int_{S_1} \frac{\Gamma G}{2t}(x \cdot \nabla Q) : (x \cdot \nabla Q + 2t \partial_t Q)dx \\
+ \frac{2}{\epsilon \Gamma} \int_{S_1} \tilde{f}_B G(x, -1)dx \\
= \int_{S_1} \frac{1}{2|t|} (2t \partial_t Q + x \cdot \nabla Q)^2 Gdx + \frac{2}{\epsilon \Gamma} \int_{S_1} \tilde{f}_B Gdx \geq 0,
\]
which implies the first inequality (2.18) for $R < \sqrt{t_0}$.

For $0 < R < R_1 < \sqrt{t_0}/2$, we consider the term $\Psi(R, Q, \epsilon)$ with $r'/r = R_1/R$, then
\[
\Psi(R, Q, \epsilon) = \frac{1}{\Gamma} \int_{T_R} \left[ \frac{L_1}{2} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{\epsilon} \right] Gdxdt \\
= \frac{1}{\Gamma} \int_{-R^2}^{R^2} \int_{\mathbb{R}^3} \left[ \frac{L_1}{2} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{\epsilon} \right] Gdxdt \\
= 2 \int_{R}^{2R} r^{-1} \Phi(r, Q, \epsilon)dr \\
= 2 \int_{R}^{2R} \Phi(r, Q, \epsilon) \frac{r^{-1} \Phi(r', Q, \epsilon)}{\Phi(r', Q, \epsilon)} dr' \leq \Psi(R_1, Q, \epsilon),
\]
where we used the monotonicity inequality (2.18).}

**Remark 2.8** The above lemma indicates that the monotonic radius of $\Phi$ and $\Psi$ depends on $t_0$, which is reasonable since we have no definition for $t < 0$. Similarly, if we consider the $Q$-tensor flow in $\mathbb{R}^3 \times (-4R_0^2, R_0^2)$, then $\Phi(R, Q, \epsilon)$ is nondecreasing for $0 < R < 2R_0$ and $\Psi(R, Q, \epsilon)$ is nondecreasing for $0 < R < R_0$.

We have the following Bochner-type inequality.

**Lemma 2.9** Suppose that $Q(x, t)$ solves (1.4) with initial data $Q_0 \in N$. There exist $\epsilon_0 > 0$ and a constant $C > 0$, independent of $\epsilon$, such that
\[
(\partial_t - \Delta)e_\epsilon(Q, \nabla Q)(x, t) \leq Ce_\epsilon(Q, \nabla Q)^2(x, t),
\] (2.20)

provided that there exists a ball $B_{\rho(x)}(x)$ with $\rho(x) > 0$ such that $\sup_{y \in B_{\rho(x)}(x)} \text{dist}(Q(y, t), N) < \epsilon_0$. 11
Proof: Direct calculation shows that
\[
\left( \partial_t - \frac{L_1}{\Gamma} \Delta \right) \left( \frac{L_1}{2\Gamma} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{c^2} \right) = \partial_t \left( \frac{L_1}{2\Gamma} |\nabla Q|^2 + \frac{\tilde{f}_B(Q)}{c^2} \right) - \frac{L_1^2}{2\Gamma^2} \Delta (|\nabla Q|^2) - \frac{L_1}{\Gamma^2} \Delta (\tilde{f}_B(Q))
\]
\[
= \frac{L_1}{\Gamma} \nabla : \partial_t \nabla Q + \frac{1}{c^2} \delta \tilde{f}_B : \partial_t Q - \frac{L_1^2}{2\Gamma^2} (2 \nabla \Delta \nabla Q + 2 |\nabla^2 Q|^2)
\]
\[
- \frac{L_1}{c^2} \delta \tilde{f}_B : \Delta Q - \frac{L_1}{\Gamma^2} \nabla \left( \frac{\delta \tilde{f}_B}{\delta Q} \right) : \nabla Q
\]
\[
= - \frac{1}{c^2\Gamma^2} \left| \delta \tilde{f}_B \right|^2 - \frac{2L_1}{c^2} \nabla \left( \frac{\delta \tilde{f}_B}{\delta Q} \right) : \nabla Q - \frac{L_1^2}{\Gamma^2} \left| \nabla^2 Q \right|^2.
\]

(2.21)

It suffices to estimate the second term of the above equality.

Denote the eigenvectors of \( Q(x, t) \) by \( n_1(x, t), n_2(x, t), n_3(x, t) \) corresponding to \( \lambda_1(x, t), \lambda_2(x, t), \lambda_3(x, t) = -\lambda_1(x, t) - \lambda_2(x, t) \). Then
\[
\pi_N(Q) = s_+(n_3(x, t) \otimes n_3(x, t) - \frac{1}{3} Id),
\]
which is a minimizer of the bulk energy \( \tilde{f}_B \). By the Taylor expansion of \( \frac{\partial^2 \tilde{f}_B}{\partial Q_{ij} \partial Q_{mn}} \) near \( \pi_N(Q) \), we get
\[
\frac{\partial^2 \tilde{f}_B}{\partial Q_{ij} \partial Q_{mn}}(Q(x, t)) = \frac{\partial^2 \tilde{f}_B}{\partial Q_{ij} \partial Q_{mn}}(\pi_N(Q)) + \frac{\partial^3 \tilde{f}_B}{\partial Q_{ij} \partial Q_{mn} \partial Q_{pq}}(\pi_N(Q))(Q_{pq}(x, t) - \pi_N(Q))_{pq} + O(|Q_{pq}(x, t) - \pi_N(Q)|^2),
\]
where we have used the formula (1.6) of \( \tilde{f}_B(Q) \).

Using the convex property of \( f_B(Q) \) at \( \pi_N(Q) \) and the maximum of \( Q \) in Lemma 2.6, we have
\[
-\nabla \left( \frac{\delta \tilde{f}_B}{\delta Q} \right) : \nabla Q = -\nabla \left( \frac{\partial^2 \tilde{f}_B}{\partial Q_{ij} \partial Q_{mn}}(Q(x, t))Q_{mn,k}(x, t)Q_{ij,k}(x, t) \right)
\leq C|Q(x, t) - \pi_N(Q)|^2 + C|\nabla Q|^4.
\]

Moreover, due to Lemma 2.4, we get
\[
-\nabla \left( \frac{\delta \tilde{f}_B}{\delta Q} \right) : \nabla Q \leq C\tilde{f}_B(Q(x, t)) + C|\nabla Q|^4.
\]

(2.22)

Combining (2.21) and (2.22), we obtain
\[
\left( \partial_t - \frac{\Gamma}{L_1} \Delta \right) e_\epsilon(Q) + \frac{1}{c^2 \Gamma^2} \left| \delta \tilde{f}_B \right|^2 \leq C e_\epsilon(Q)^2.
\]

(2.23)

The proof of the lemma is completed. \( \Box \)
We consider the local uniform regularity property of the solution $Q_\epsilon$, which follows from the monotonicity Lemma 2.7 and Schoen’s trick, c.f. [23, Theorem 5.1] or [19, Theorem 2.2].

**Lemma 2.10** Suppose that $Q_\epsilon(x, t)$ solves (1.4) in $\mathbb{R}^3 \times [-4R_0^2, R_0^2]$, and there exist positive constants $\epsilon'_0$ and $\epsilon_1$, such that when $\epsilon < \epsilon'_0$, for some $0 < R < R_0$ the following inequality holds

$$
\Psi(R) = \Psi(R, Q_\epsilon, \epsilon) = \int_{TR} \left( \frac{L_1}{2T} |\nabla Q_\epsilon|^2 + \frac{f_B(Q_\epsilon)}{\epsilon} \right) G dx dt < \epsilon_1,
$$

(2.24)

then

$$
\sup_{P_R} \left( \frac{L_1}{2} |\nabla Q_\epsilon|^2 + \frac{f_B(Q_\epsilon)}{\epsilon} \right) \leq C(\delta R)^{-2}
$$

(2.25)

where the constant $\delta > 0$ depends only on $\epsilon_c(Q_0, \nabla Q_0)$ and $\min\{R, 1\}$.

Proof: We follow the same line as in [23, Theorem 5.1]. Let $r_1 = \delta R$, $\delta \in (0, 1/2)$ to be determined later. For $r, \sigma \in (0, r_1)$, $r + \sigma < r_1$, and any $z_0 = (x_0, t_0) \in P_r$, we have

$$
I(\sigma, z_0) = \sigma^{-3} \int_{P_\sigma(z_0)} e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) dx dt \leq C \int_{P_\sigma(z_0)} e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) G_{(x_0, t_0 + 2\sigma^2)} dx dt
$$

$$
\leq C \int_{T_\sigma(t_0 + 2\sigma^2)} e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) G_{(x_0, t_0 + 2\sigma^2)} dx dt.
$$

Moreover, apply Remark 2.8, choose $\delta$ small enough, and take $t_0 + 2\sigma^2 - 4R_1^2 = -R^2$ and $t_0 + 2\sigma^2 - 4R_2^2 = -4R^2$, then we deduce that

$$
I(\sigma, z_0) \leq C \min \left( \int_{t_0 + 2\sigma^2 - 3R_1^2}^{t_0 + 2\sigma^2 - R_1^2}, \int_{t_0 + 2\sigma^2 - 4R_2^2}^{t_0 + 2\sigma^2 - 4R_2^2} \right) \left( e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) G_{(x_0, t_0 + 2\sigma^2)} \right) dx dt
$$

$$
\leq C \int_{T_R} e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) G_{(x_0, t_0 + 2\sigma^2)} dx dt.
$$

Direct calculation shows that for given $\epsilon_2 > 0$, if $\delta > 0$ is small enough, then we have

$$
G_{(x_0, t_0 + 2\sigma^2)}(x, t) \leq C \exp \left( C\delta^2 \frac{|x|^2}{4|t|} \right) G(x, t)
$$

$$
\leq \begin{cases} 
CG(x, t), & \text{if } |x| \leq \frac{R}{\delta} \\
CR^{-3}\exp(-C\delta^{-2}), & \text{if } |x| \geq \frac{R}{\delta} 
\end{cases}
$$

$$
\leq CG(x, t) + CR^{-2}\exp(-\ln R - C\delta^2)
$$

$$
\leq CG(x, t) + \epsilon_2 R^{-2},
$$

which holds on $T_R$, and here $C$ is independent of $\delta$ and $R$. Select $\delta \sim (\ln |R| + \ln \epsilon_2)^{-1/2}$ for a small $R$ and independent of $R$ if $R \geq 1$. Thus, it follows that

$$
\sigma^{-3} \int_{P_\sigma(z_0)} e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) dx dt \leq C\Psi(R) + C\epsilon_2 e_\epsilon(Q_0, \nabla Q_0) \leq C(\epsilon_1 + \epsilon_2 e_\epsilon(Q_0, \nabla Q_0)).
$$

(2.26)
For simplicity, we let 

\[ e_\epsilon(Q_\epsilon, \nabla Q_\epsilon) = e_\epsilon(Q_\epsilon) \]  

Since \( Q_\epsilon \) is regular, there exists \( \sigma_\epsilon \in (0, r_1) \) such that

\[ (r_1 - \sigma_\epsilon)^2 \sup_{P_{r_\epsilon}} e_\epsilon(Q_\epsilon) = \max_{0 \leq \sigma \leq r_1} (r_1 - \sigma)^2 \sup_{P_\sigma} e_\epsilon(Q_\epsilon). \] (2.27)

Also, there exists a point \((x_\epsilon, t_\epsilon) \in \bar{P}_{r_\epsilon}\) such that

\[ \sup_{\bar{P}_{r_\epsilon}} e_\epsilon(Q_\epsilon) = e_\epsilon(Q_\epsilon)(x_\epsilon, t_\epsilon) = e_\epsilon. \] (2.28)

Set \( \rho_\epsilon = \frac{1}{2}(r_1 - \sigma_\epsilon) \). Then it follows from (2.27) and (2.28) that

\[ \sup_{P_{\rho_\epsilon}(x_\epsilon, t_\epsilon)} e_\epsilon(Q_\epsilon) \leq \sup_{P_{\rho_\epsilon} + \rho_\epsilon} e_\epsilon(Q_\epsilon) \leq 4 e_\epsilon. \]

Denote

\[ \tilde{Q}_\epsilon(x, t) = Q_\epsilon\left(\frac{x}{\sqrt{e_\epsilon}} + x_\epsilon, \frac{t}{e_\epsilon} + t_\epsilon\right), \]

which solves the equation (1.4) in \( P_{r_\epsilon} \) with \( \tilde{\epsilon} = \epsilon e_\epsilon \) and \( r_\epsilon = \sqrt{e_\epsilon} \rho_\epsilon \). Moreover, \( \tilde{Q}_\epsilon \) satisfies

\[ e_\tilde{\epsilon}(\tilde{Q}_\epsilon)(0, 0) = 1, \quad \sup_{P_{r_\epsilon}} e_\tilde{\epsilon}(\tilde{Q}_\epsilon) \leq 4. \]

If \( r_\epsilon \geq 1 \), \( \text{dist}(\tilde{Q}_\epsilon, N) \) convergence uniformly to 0 on \( P_1 \), and there exists \( \epsilon'_0 \), such that \( \text{dist}(\tilde{Q}_\epsilon, N) < \epsilon_0 \) for \( \epsilon < \epsilon'_0 \). Thus, Lemma 2.9 implies that

\[ (\partial_t - \Delta) e_\tilde{\epsilon}(\tilde{Q}_\epsilon) \leq c_1 e_\tilde{\epsilon}(\tilde{Q}_\epsilon), \quad \text{on} \quad P_1. \]

Moser’s Harnack inequality shows that

\[ 1 = e_\tilde{\epsilon}(\tilde{Q}_\epsilon)(0, 0) \leq C \int_{P_1} e_\tilde{\epsilon}(\tilde{Q}_\epsilon) dx dt, \]

while, (2.26) tells us

\[ \int_{P_1} e_\tilde{\epsilon}(\tilde{Q}_\epsilon) dx dt = (\sqrt{e_\epsilon})^3 \int_{P_{\sqrt{e_\epsilon}(x, t)}} e(Q_\epsilon) dx dt \leq c(e_1 + \epsilon_2 e_0(Q_0)), \]

which leads to a contradiction if \( \epsilon_1 \) and \( \epsilon_2 \) are suitably small.

Hence, we may assume that \( r_\epsilon \leq 1 \). Then

\[ 1 = e_\tilde{\epsilon}(\tilde{Q}_\epsilon)(0, 0) \leq C r_\epsilon^{-5} \int_{P_{r_\epsilon}} e_\tilde{\epsilon}(\tilde{Q}_\epsilon) dx dt = C r_\epsilon^{-2} \rho_\epsilon^{-3} \int_{P_{\rho_\epsilon}} e_\epsilon(Q_\epsilon) dx dt, \]

and using (2.26), we get

\[ \rho_\epsilon^2 e_\epsilon = r_\epsilon^2 \leq C, \]

then

\[ \max_{0 < \sigma < r_\epsilon} (r_\epsilon - \sigma)^2 \sup_{P_\sigma} e(Q_\epsilon) \leq 4 \rho_\epsilon^2 e_\epsilon \leq C, \]

which implies the required result by choosing \( \sigma = \frac{1}{2} r_\epsilon = \delta R \). □
3 Proof of the main theorem and the equation of $n$

By Lemma 2.5, we know that for given smooth data $Q_0 : \mathbb{R}^3 \to \mathcal{N}$ with $\nabla Q_0 \in L^2(\mathbb{R}^3)$, there exist a subsequence of $Q_\epsilon$ (also denoted by $Q_\epsilon$) and a function $Q(x,t)$, such that as $\epsilon \to 0$, we have

$$\nabla Q_\epsilon \to \nabla Q \quad \text{weakly* in } L^\infty([0,\infty); L^2(\mathbb{R}^3)),$$

$$\partial_t Q_\epsilon \to \partial_t Q \quad \text{weakly in } L^2(\mathbb{R}^3 \times \mathbb{R}_+),$$

$$Q_\epsilon \to Q \quad \text{weakly in } H^{1,2}(\mathbb{R}^3 \times \mathbb{R}_+),$$

$$\tilde{f}_B(Q_\epsilon) \to 0, \quad \text{in } L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}_+), \quad (3.29)$$

which yield that

$$\partial_t Q \in L^2(\mathbb{R}^3 \times \mathbb{R}_+), \nabla Q \in L^\infty([0,\infty), L^2(\mathbb{R}^3)),$$

and hence also $Q_\epsilon \to Q$ a.e. on $\mathbb{R}^3 \times \mathbb{R}_+$. Also there is a lifting map $n \in \dot{H}^1(\mathbb{R}^3)$ such that

$$Q = s_+ \left( n \otimes n - \frac{1}{3} \text{Id} \right) \in \mathcal{N}. \quad (3.31)$$

3.1 Proof of Theorem 1.2: the limit Q-tensor equations

We follow the standard arguments as in [23] or [4]. Define the singular set by

$$\Sigma = \cap_{R>0} \{ z_0 \in \mathbb{R}^3 \times \mathbb{R}_+ \mid \lim \inf_{\epsilon \to 0} \int_{T_R(z_0)} e^\epsilon(Q_\epsilon) dx dt \geq \epsilon_1 \}. \quad (3.30)$$

Then as in [23, Theorem 6.1](see also [4]), one can show that $\Sigma$ is closed and has locally finite 3-dimensional Hausdorff-measure with respect to the parabolic metric.

For $z_0 \notin \Sigma$, there exists a $R_0 > 0$, and a subsequence of $\epsilon$, which is still denoted by $\epsilon$, such that

$$R^{-3} \int_{T_R(z_0)} e^\epsilon(Q_\epsilon) G_{z_0} dx dt \leq \epsilon_1. \quad (3.32)$$

It follows from Lemma 2.10 that

$$|\nabla Q_\epsilon|, \frac{\tilde{f}_B(Q_\epsilon)}{\epsilon} \leq C$$

hold uniformly in a uniform neighborhood $\Omega$ of $z_0$. Let $Q$ be the weak limit of $Q_\epsilon$. Then there exists a subsequence which we denote as $Q_\epsilon$ again, such that

$$Q_\epsilon \to Q, \quad \text{in } C^0_{loc}(\Omega),$$

$$\nabla Q_\epsilon \to \nabla Q, \quad \text{weakly* in } L^\infty_{loc}(\Omega).$$

Note that $\tilde{f}_B(Q_\epsilon) \leq C \epsilon$ is a polynomial of $Q_\epsilon$, then the convergence shows $\tilde{f}_B(Q) = 0$, i.e., $Q \in \mathcal{N}$. Also $\text{dist}^2(Q_\epsilon, \mathcal{N}) \leq |Q_\epsilon - Q|^2 < \epsilon_0$ for $\epsilon$ small enough. Then [2.23] shows that

$$\left( \partial_t - \frac{L_1}{\Gamma} \Delta \right) e^\epsilon(Q_\epsilon) + \frac{1}{\epsilon^2 \Gamma^2} \left| \frac{\delta \tilde{f}_B}{\delta Q} \right|^2(Q_\epsilon) \leq C \quad \text{on } \Omega, \quad (3.32)$$
which implies that
\[
\int_{\Omega} \frac{1}{c^2T^2} \left| \frac{\delta f_B}{\delta Q} \right|^2 (Q_{\epsilon})(x) \phi dxdt \leq C(\phi),
\]
for all \( \phi \in C_0^\infty(\Omega) \). Moreover, \( \frac{1}{c^2T^2} \frac{\delta f_B}{\delta Q}(Q_{\epsilon}) = \frac{1}{c^2T^2} \mathcal{J}(Q_{\epsilon}) \) is uniformly bounded in \( L^2_{loc}(\Omega) \), also \( (\partial_t - \frac{L_1}{\Gamma}\Delta)Q_{\epsilon} \) is uniformly bounded in \( L^2_{loc}(\Omega) \), and similar arguments hold for \( \partial_t Q_{\epsilon} \) and \( \nabla^2 Q_{\epsilon} \). Then we may assume that
\[
(\partial_t - \frac{L_1}{\Gamma}\Delta)Q_{\epsilon} \rightarrow (\partial_t - \frac{L_1}{\Gamma}\Delta)Q, \quad \text{weakly in } L^2_{loc}(\Omega),
\]
\[
\frac{1}{c^2T^2} |\mathcal{J}(Q_{\epsilon})| = \frac{1}{c^2T^2} \left| \frac{\delta f_B}{\delta Q}(Q_{\epsilon}) \right| \rightarrow \bar{\lambda}, \quad \text{weakly in } L^2_{loc}(\Omega).
\]
(3.33)

The convergence (3.32) shows that \( Q(x, t) \in C(\Omega, \mathcal{N}) \), then the lifting \( \mathbf{n} \) in (3.31) satisfies \( \mathbf{n} \in C(\Omega, S^2) \). Lemma 2.3 tells us that
\[
\mathcal{J}(Q_{\epsilon}) \perp T_{s+} (\mathbf{n}_3^3 \otimes \mathbf{n}_3^3 - \frac{I_d}{3}) \mathcal{N}, \quad s+(\mathbf{n}_3^3 \otimes \mathbf{n}_3^3 - \frac{I_d}{3}) = \pi\mathcal{N}(Q_{\epsilon}),
\]
where \( \mathbf{n}_3^3 \) is the main eigenvector of \( Q_{\epsilon} \).

Now we want to prove that the limit \( Q \) satisfies the equations of the Q-tensor flow (1.11). By the definition of the matrix norm, the vector \( \mathbf{n}_3^3 \) can be estimated as follows:
\[
\|\nabla \mathbf{n}_3^3\|_{L^\infty} \leq c\|\nabla \pi(Q_{\epsilon})\|_{L^\infty} \leq c\|\nabla Q_{\epsilon}\|_{L^\infty},
\]
As in [2] Theorem 2], we can assume \( \mathbf{n}_3^3 \rightarrow \mathbf{n} \) uniformly on \( \Omega' \subset \Omega \).

Let \( \mathbf{n}_1^i \) and \( \mathbf{n}_2^i \) be unit perpendicular vectors in \( T_{\mathbf{n}_3^3} S^2 \), which also continuously depend on \( Q_{\epsilon} \). Then the following three vectors are the basis of \( (T_{\pi\mathcal{N}(Q_{\epsilon})})^\perp \):
\[
e_1^i \quad = \frac{1}{\sqrt{2}} (\mathbf{n}_2^i \otimes \mathbf{n}_1^i + \mathbf{n}_1^i \otimes \mathbf{n}_2^i),
\]
\[
e_2^i \quad = \frac{1}{\sqrt{2}} (\mathbf{n}_1^i \otimes \mathbf{n}_1^i - \mathbf{n}_2^i \otimes \mathbf{n}_2^i),
\]
\[
e_3^i \quad = \sqrt{6} \left( \frac{1}{2} \mathbf{n}_1^i \otimes \mathbf{n}_1^i + \frac{1}{2} \mathbf{n}_2^i \otimes \mathbf{n}_2^i - \frac{I_d}{3} \right) = -\frac{\sqrt{6}}{2} (\mathbf{n}_3^3 \otimes \mathbf{n}_3^3 - \frac{I_d}{3}).
\]

Then
\[
\mathcal{J}(Q_{\epsilon}) = f_1^i(x, t) e_1^i(x, t) + f_2^i(x, t) e_2^i(x, t) + f_3^i(x, t) e_3^i(x, t).
\]
Note that (3.33) also shows that
\[
\frac{f_1^i(x, t)}{cT} \rightarrow \bar{\lambda}_1^i(x, t), \quad \text{weakly in } L^2_{loc}(\Omega),
\]
for \( i = 1, 2, 3 \). Due to \( Q_{\epsilon} \rightarrow Q \) in \( C_0^\infty(\Omega) \) uniformly, we can assume \( \mathbf{n}_2^i \rightarrow \mathbf{n}_2 \) and \( \mathbf{n}_1^i \rightarrow \mathbf{n}_1 \), where \( \mathbf{n}_2 \) and \( \mathbf{n}_1 \) are perpendicular on \( T_{\mathbf{n}_3^3} S^2 \). Thus,
\[
e_1^i(x, t) \rightarrow e_i(x, t), \quad \text{in } C_0^\infty(\Omega), \quad i = 1, 2, 3,
\]
and
\[
\int_{\Omega'} \mathcal{J}(Q_{\epsilon}) \cdot \phi dx dt = \int_{\Omega'} \frac{f_1(x,t)}{e_1} (e_1 - e_1) \cdot \phi dx dt + \int_{\Omega'} \frac{f_2(x,t)}{e_2} (e_2 - e_2) \cdot \phi dx dt + \int_{\Omega'} \frac{f_3(x,t)}{e_3} (e_3 - e_3) \cdot \phi dx dt \rightarrow 0
\]
for any vector field \( \phi \in L^2_{\text{loc}}(\Omega) \) such that \( \phi(z) \in T_QN \) a.e. on \( \Omega \), and any \( \Omega' \subset \Omega \), which and (3.33) yield that
\[
(\partial_t - \frac{L_1}{1} \Delta)Q \perp T_QN, \quad \text{a.e. in } \Omega.
\]
Then there exists a unit normal vector field \( \gamma_N(Q) \in (T_QN)^{\perp}_{Q_0} \) along \( Q \) and a scalar function \( \lambda \in L^2_{\text{loc}}(\Omega) \) such that
\[
\partial_t Q - \frac{L_1}{1} \Delta Q + \lambda \gamma_N(Q) = 0 \quad (3.34)
\]
a.e. on \( \Omega \) and in the sense of distribution.

Standard arguments show that \( Q \) also weakly solves (3.34) on \( \mathbb{R}^3 \times \mathbb{R}_+ \) (for example, see [23]), and we omitted the details.

3.2 Proof of Theorem 1.2: the harmonic map flow

Note that \( n \in \dot{H}^1(\mathbb{R}^3 \times \mathbb{R}, S^2) \) is a lifting such that \( Q = s_+(n \otimes n - \frac{Ld}{\delta}) \) and \( |Q| \) is a constant. For \( \phi \in C^\infty_0(\mathbb{R}^3 \times \mathbb{R}_+, \mathbb{R}^3) \),
\[
0 = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} ((\phi \cdot n) \partial_t Q : Q + \nabla Q : \nabla((\phi \cdot n)Q)) dx dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} (\phi \cdot n) \lambda \gamma_N(Q) : Q dx dt \right)
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} |\nabla Q|^2 (\phi \cdot n) dx dt + \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} (\phi \cdot n) \lambda \gamma_N(Q) : Q dx dt. \quad (3.35)
\]
We use \( \phi^n \) to denote \( (n \cdot \phi)n \). Then
\[
n \otimes (\phi - \phi^n) + (\phi - \phi^n) \otimes n \in T_QN. \quad (3.36)
\]
On the other hand, by (3.34), (3.36) and Lemma 2.1 we derive

\[
0 = \int_{\mathbb{R}_+^3} (\partial_t Q - \Delta Q + \lambda \gamma_N(Q)) : (n \otimes \phi + \phi \otimes n)
\]

\[
= -\int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} Q : \partial_t (n \otimes \phi + \phi \otimes n) \ dx dt + \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \nabla Q : \nabla (n \otimes \phi + \phi \otimes n) \ dx dt
\]

\[
 + \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \lambda \gamma_N(Q) : (n \otimes \phi + \phi \otimes n) \ dx dt
\]

\[
= -2 \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} Q : (\partial_t n \otimes \phi + n \otimes \partial_t \phi) \ dx dt + 2 \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \nabla Q : (\nabla n \otimes \phi + n \otimes \nabla \phi) \ dx dt
\]

\[
 + \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \lambda \gamma_N(Q) : (n \otimes \phi^n + \phi^n \otimes n) \ dx dt
\]

\[
= -2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \frac{2}{3} n \cdot \partial_t \phi - \frac{1}{3} \partial_t n \cdot \phi \ dx dt + 2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} |\nabla n|^2 n \cdot \phi + \nabla n \cdot \nabla \phi \ dx dt
\]

\[
 + 2 \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \phi \cdot n \lambda \gamma_N(Q) : n \otimes n dx dt.
\]

\[
-2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \frac{2}{3} n \cdot \partial_t \phi - \frac{1}{3} \partial_t n \cdot \phi \ dx dt + 2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} |\nabla n|^2 n \cdot \phi + \nabla n \cdot \nabla \phi \ dx dt
\]

\[
 + 2 \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \phi \cdot n \frac{\lambda}{s_+} \gamma_N(Q) : Q dx dt.
\]

\[
= 2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \partial_t n \cdot \phi dx dt - 2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} |\nabla n|^2 n \cdot \phi dx dt + 2s_+ \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+^3} \nabla n \cdot \nabla \phi dx dt,
\]

where in the last step we used (3.35).

Thus the proof of the Theorem 1.2 is completed. \( \square \)

Acknowledgments. Part of this work is carried out when the first author is visiting Princeton university. Meng Wang is partially supported by NSFC 10931001. Wendong Wang is supported NSFC 11301048 and "the Fundamental Research Funds for the Central Universities". Zhifei Zhang is partially supported by NSF of China under Grants 11371037 and 11425103.

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