STRONG MORITA EQUIVALENCE FOR COMPLETELY POSITIVE LINEAR MAPS ON $C^*$-ALGEBRAS

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Abstract. We will introduce the notion of strong Morita equivalence for completely positive linear maps and study its basic properties. Also, we will discuss the relation between strong Morita equivalence for bounded $C^*$-bimodule linear maps and strong Morita equivalence for completely positive linear maps. Furthermore, we will show that if two unital $C^*$-algebras are strongly Morita equivalent, then there is a $1-1$ correspondence between the two sets of all strong Morita equivalence classes of completely positive linear maps on the two unital $C^*$-algebras and we will show that the corresponding two classes of the completely positive linear maps are also strongly Morita equivalent.

1. Introduction

In the previous paper [7], we introduced the notions of strong Morita equivalence for bounded $C^*$-bimodule linear maps and the Picard group of a bounded $C^*$-bimodule linear map.

In this paper, we will introduce the notion of strong Morita equivalence for completely positive linear maps applying its minimal Stinespring representation and [4, Definition 2.1] introduced by Echterhoff and Raeburn.

To do this, following [4, Definition 2.1], in Section 3 we will introduce the notion of strong Morita equivalence for non-degenerate representations of $C^*$-algebras and we will discuss its basic properties.

In Section 3 we will define strong Morita equivalence for completely positive linear maps and discuss its basic properties.

In Section 5 we will consider inclusions of $C^*$-algebras $A \subset C$ with $\overline{AC} = C$ and conditional expectations. Conditional expectations are regarded as bounded $C^*$-bimodule linear maps. Also, they are regarded as completely positive linear maps. We will discuss the relation between strong Morita equivalence for bounded $C^*$-bimodule linear maps and strong Morita equivalence for completely positive linear maps.

In Section 6 we will consider two unital $C^*$-algebras $A$ and $B$, which are strongly Morita equivalent. We will construct a $1-1$ correspondence between the set of all strong Morita equivalence classes of completely positive linear maps from $A$ to the $C^*$-algebra of all bounded linear operators on a Hilbert space and the set of all strong Morita equivalence classes of completely positive linear maps from $B$ to the $C^*$-algebra of all bounded linear operators on a Hilbert space.

2. Preliminaries

Let $A$ be a $C^*$-algebra. We denote by $\text{id}_A$ the identity map on $A$. When $A$ is unital, we denote by $1_A$ the unit element in $A$. We simply them by $\text{id}$ and $1$, respectively if no confusion arises.

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For each $n \in \mathbb{N}$, let $M_n(A)$ be the $n \times n$-matrix algebra over $A$. We identify $M_n(A)$ with $A \otimes M_n(C)$. Let $1_n$ be the unit element in $M_n(C)$. For each $a \in M_n(A)$, we denote by $a_{ij}$, the $i \times j$-entry of the matrix $a$.

Let $M(A)$ be the multiplier $C^\ast$-algebra of $A$ and for any automorphism $\alpha$ of $A$, let $\alpha$ be the automorphism of $M(A)$ extending $\alpha$ to $M(A)$, which is defined in Jensen and Thomsen [5, Corollary 1.1.15].

Let $A$ and $B$ be $C^\ast$-algebras and $X$ an $A \times B$-equivalence bimodule. We denote its left $A$-action and right $B$-action on $X$ by $a \cdot x$ and $x \cdot b$ for any $a \in A$, $b \in B$, $x \in X$, respectively. Let $\tilde{X}$ be the dual $B \times A$-equivalence bimodule of $X$ and let $\tilde{x}$ denote the element in $\tilde{X}$ associated to an element $x \in X$.

For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $B(\mathcal{K}, \mathcal{H})$ be the space of all bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$ and if $\mathcal{H} = \mathcal{K}$, we denote $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. For a Hilbert space $\mathcal{H}$, we denote by $\langle \cdot, \cdot \rangle_\mathcal{H}$ the inner product of $\mathcal{H}$.

Let $\mathbf{K}$ be the $C^\ast$-algebra of all compact operators on a countably infinite dimensional Hilbert space $\mathcal{H}_0$. Let $\{\epsilon_i\}_{i=1}^\infty$ be an orthogonal basis of $\mathcal{H}_0$ and $\{e_{ij}\}_{i,j=1}^\infty$ the system of matrix units of $\mathbf{K}$ with respect to $\{\epsilon_i\}_{i=1}^\infty$.

3. Definition and properties of Strong Morita equivalence for non-degenerate representations of $C^\ast$-algebras

Following [4] Definition 2.1, we give the definition of a representation of equivalence bimodule.

**Definition 3.1.** Let $A$ and $B$ be $C^\ast$-algebras. A representation of an $A \times B$-equivalence bimodule $X$ on the pair of Hilbert spaces $(\mathcal{H}, \mathcal{K})$ is a triple $(\pi_A, \pi_X, \pi_B)$ consisting of non-degenerate representations $\pi_A : A \to B(\mathcal{H}), \pi_B : B \to B(\mathcal{K})$ and a linear map $\pi_X : X \to B(\mathcal{K}, \mathcal{H})$ satisfying the following: for any $a \in A$, $b \in B$, $x, y \in X$,

1. $\pi_X(x)\pi_X(y)^* = \pi_A(A(x, y))$,
2. $\pi_X(x)^*\pi_X(y) = \pi_B((x, y)b)$,
3. $\pi_X(a \cdot x \cdot b) = \pi_A(a)\pi_X(x)\pi_B(b)$.

Let $(\pi_A, \mathcal{H})$ and $(\pi_B, \mathcal{K})$ be non-degenerate representations of $C^\ast$-algebras $A$ and $B$, respectively.

**Definition 3.2.** The non-degenerate representation $(\pi_A, \mathcal{H})$ of $A$ is strongly Morita equivalent to the non-degenerate representation $(\pi_B, \mathcal{K})$ of $B$ if there are an $A \times B$-equivalence bimodule $X$ and a linear map $\pi_X : X \to B(\mathcal{K}, \mathcal{H})$ such that $(\pi_A, \pi_X, \pi_B)$ is a representation of $X$ on the pair of Hilbert spaces $(\mathcal{H}, \mathcal{K})$.

**Lemma 3.1.** With the above notation, strong Morita equivalence for non-degenerate representations of $C^\ast$-algebras is equivalence relation.

**Proof.** Let $(\pi_A, \mathcal{H})$ be a non-degenerate representation of a $C^\ast$-algebra $A$. We regard $A$ as the trivial $A \times A$-equivalence bimodule in the usual way. We denote it by $X_0$. Let $\pi_{X_0}$ be the linear map from $X_0$ to $B(\mathcal{H})$ defined by $\pi_{X_0}(x) = \pi_A(x)$ for any $x \in X_0$. Then we can see that $\pi_{X_0}$ satisfies Conditions (1)-(3) in Definition 3.1. Hence $(\pi_A, \mathcal{H})$ is strongly Morita equivalent to itself.

Let $(\pi_B, \mathcal{K})$ be a non-degenerate representation of a $C^\ast$-algebra $B$ and we suppose that $(\pi_A, \mathcal{H})$ is strongly Morita equivalent to $(\pi_B, \mathcal{K})$. Then there are an $A \times B$-equivalence bimodule $X$ and a linear map $\pi_X$ from $X$ to $B(\mathcal{K}, \mathcal{H})$ satisfying Conditions (1)-(3) in Definition 3.1. Let $\pi_{X_0}$ be the linear map from $X$ to $B(\mathcal{K}, \mathcal{H})$ defined by $\pi_{X_0}(x) = \pi_X(x)^*$ for any $x \in X$. Then we can see that $\pi_{X_0}$ satisfies Conditions (1)-(3) in Definition 3.1. Thus, $(\pi_B, \mathcal{K})$ is strongly Morita equivalent to $(\pi_A, \mathcal{H})$.

Let $(\pi_C, \mathcal{L})$ be a non-degenerate representation of a $C^\ast$-algebra $C$. We suppose that $(\pi_A, \mathcal{H})$ is strongly Morita equivalent to $(\pi_B, \mathcal{K})$ with respect to an
A – B-equivalence bimodule \( X \) and a linear map \( \pi_X : X \to \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \((\pi_A, \pi_X, \pi_B)\) is a representation of \( X \) on the pair of Hilbert spaces \((\mathcal{H}, \mathcal{K})\). Also, we suppose that \((\pi_B, \mathcal{K})\) is strongly Morita equivalent to \((\pi_C, \mathcal{L})\) with respect to a \( B – C\)-equivalence bimodule \( Y \) and a linear map \( \pi_Y : Y \to \mathcal{B}(\mathcal{L}, \mathcal{K}) \) such that \((\pi_B, \pi_Y, \pi_C)\) is a representation of \( Y \) on the pair of Hilbert spaces \((\mathcal{K}, \mathcal{L})\). Let \( \pi_{X \otimes_B Y} \) be the linear map from \( X \otimes_B Y \) to \( \mathcal{B}(\mathcal{L}, \mathcal{H}) \) defined by
\[
\pi_{X \otimes_B Y}(x \otimes y) = \pi_X(x)\pi_Y(y)
\]
for any \( x \in X, \ y \in Y \). Then by routine computations, we can see that \( \pi_{X \otimes_B Y} \) satisfies Conditions (1)-(3) in Definition 3.1. Thus \((\pi_A, \pi_{X \otimes_B Y}, \pi_C)\) is a representation of \( X \otimes Y \) on a pair of Hilbert spaces \((\mathcal{H}, \mathcal{L})\) and \((\pi_A, \mathcal{H})\) is strongly Morita equivalent to \((\pi_C, \mathcal{L})\). Therefore, we obtain the conclusion.

Lemma 3.2. Let \((\pi_1, \mathcal{H}_1)\) and \((\pi_2, \mathcal{H}_2)\) be non-degenerate representations of a \( C^* \)-algebra \( A \). If \((\pi_1, \mathcal{H}_1)\) and \((\pi_2, \mathcal{H}_2)\) are unitarily equivalent, they are strongly Morita equivalent.

Proof. Since \((\pi_1, \mathcal{H}_1)\) and \((\pi_2, \mathcal{H}_2)\) are unitarily equivalent, there is an isometry \( u \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that \( \pi_2 = \text{Ad}(u) \circ \pi_1 \). Let \( X_0 \) be the trivial \( A – A \)-equivalence bimodule defined in the proof of Lemma 3.1. Let \( \pi_{X_0} \) be the linear map from \( X_0 \) to \( \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) defined by \( \pi_{X_0}(x) = \pi_1(x)u^* \). Then by easy computations, we can see that \((\pi_1, \pi_{X_0}, \pi_2)\) is a representation of \( X_0 \) on the pair of Hilbert spaces \((\mathcal{H}_1, \mathcal{H}_2)\). Hence \((\pi_1, \mathcal{H}_1)\) and \((\pi_2, \mathcal{H}_2)\) are strongly Morita equivalent.

Let \((\pi_A, \mathcal{H})\) and \((\pi_B, \mathcal{K})\) be non-degenerate representations of \( C^* \)-algebras \( A \) and \( B \), respectively. We suppose that \((\pi_A, \mathcal{H})\) and \((\pi_B, \mathcal{K})\) are strongly Morita equivalent, that is, there are an \( A – B \)-equivalence bimodule \( X \) and a linear map \( \pi_X \) from \( X \) to \( \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that \((\pi_A, \pi_X, \pi_B)\) is representation of \( X \) on the pair of Hilbert spaces \((\mathcal{H}, \mathcal{K})\).

Let \( L_X \) be the linking \( C^* \)-algebra for \( X \), that is,
\[
L_X = \{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} \mid a \in A, \ b \in B, \ x, y \in X \}.
\]

Let \( \rho \) be the representation of \( L_X \) on \( \mathcal{H} \otimes \mathcal{K} \) defined by
\[
\rho \left( \begin{bmatrix} a & x \\ y & b \end{bmatrix} \right) = \begin{bmatrix} \pi_A(a) & \pi_X(x) \\ \pi_X(y)^* & \pi_B(b) \end{bmatrix}
\]
for any \( a \in A, \ b \in B, \ x, y \in X \). Since \((\pi_A, \mathcal{H})\) and \((\pi_B, \mathcal{K})\) are non-degenerate, so is \((\rho, \mathcal{H} \otimes \mathcal{K})\) by \[^2\] §2, Remark (3). Let \( p = \begin{bmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{bmatrix}, q = \begin{bmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{bmatrix} \). Then \( p \) and \( q \) are projections in \( M(L_X) \) with
\[
L_X p L_X = L_X, \quad L_X q L_X = L_X, \quad p L_X p \cong A, \quad q L_X q \cong B
\]
as \( C^* \)-algebras, respectively. We identify \( p L_X p \) and \( q L_X q \) with \( A \) and \( B \), respectively.

Lemma 3.3. With the above notation, \((\rho, \mathcal{H} \otimes \mathcal{K})\) is strongly Morita equivalent to \((\pi_A, \mathcal{H})\) and \((\pi_B, \mathcal{K})\).

Proof. We have only to show that \((\rho, \mathcal{H} \otimes \mathcal{K})\) is strongly Morita equivalent to \((\pi_A, \mathcal{H})\) by Lemma 3.1. Let \( Y = p L_X \). Since we identify \( p L_X p \) with \( A \), \( Y \) can be regarded as an \( A – L_X \)-equivalence bimodule in the usual way. Let \( \pi_Y \) be the linear map from \( Y \) to \( \mathcal{B}(\mathcal{H} \otimes \mathcal{K}, \mathcal{H}) \) defined by
\[
\pi_Y \left( \begin{bmatrix} a & x \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \pi_A(a) & \pi_X(x) \\ 0 & 0 \end{bmatrix}
\]
for any \( a \in A, x \in X \), where we identify \( pL_X \) with the space
\[
\begin{bmatrix} a & x \\ 0 & 0 \end{bmatrix} | a \in A, \quad x \in X .
\]
Then by routine computations, we can see that \((\pi_A, \pi_Y, \rho)\) is a representation of \( Y \) on the pair of Hilbert spaces \((\mathcal{H}, \mathcal{H} \oplus \mathcal{K})\). Therefore, \((\rho, \mathcal{H} \oplus \mathcal{K})\) is strongly Morita equivalent to \((\pi_A, \mathcal{H})\).

Since \((\rho, \mathcal{H} \oplus \mathcal{K})\) is a non-degenerate representation of \( L_X \), by Pedersen [10] Theorem 3.7.7, there is a unique normal homomorphism \( \rho'' \) of \( L''_X \) onto \( \rho(L_X)' \), which extends \( \rho \), where \( L''_X \) is the enveloping von Neumann algebra of \( L_X \). Also, by [10] 3.12.1 we can regard \( \pi \) equivalent to \(( \pi, \mathcal{H} \oplus \mathcal{K})\). Furthermore, we see that \( \rho \) is the restriction of \( \rho'' \) to \( \rho(L_X)' \). Hence \( \rho(p) \) and \( \rho(q) \) are the projections in \( \mathcal{B}(\mathcal{H} \oplus \mathcal{K}) \) with their ranges are \( \mathcal{H} \oplus 0 \) and \( 0 \oplus \mathcal{K} \), respectively. That is, \( P_X = \rho(p) \) and \( P_K = \rho(q) \), where \( P_X \) and \( P_K \) are projections from \( \mathcal{H} \oplus \mathcal{K} \) onto \( \mathcal{H} \oplus 0 \) and \( 0 \oplus \mathcal{K} \), respectively. Furthermore, we assume that \( A \) and \( B \) are \( \sigma \)-unitable \( C^* \)-algebras. Then in the same way as in the proof of Brown, Green and Rieffel [2] Theorem 3.4], there is a partial isometry \( w \in M(L_X) \) such that \( w^*w = p, \) \( ww^* = q \).

Let \( \theta \) be the map from \( pL_Xp \) to \( qL_Xq \) defined by
\[
\theta \begin{bmatrix} a \\ 0 \end{bmatrix} = w \begin{bmatrix} a \\ 0 \end{bmatrix} w^*
\]
for any \( a \in A \). Then \( \theta \) is an isomorphism of \( pL_Xp \) onto \( qL_Xq \). Identifying \( A \) and \( B \) with \( pL_Xp \) and \( qL_Xq \), respectively, we can regard \( \theta \) as an isomorphism of \( A \) onto \( B \).

Let \( W = \tilde{W}(w) \). Then \( W \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}) \) and
\[
W^*W = \rho(w^*w) = \rho(p) = P_X, \quad WW^* = \rho(ww^*) = \rho(q) = P_K.
\]

**Lemma 3.4.** With the above notation, there is an isometry \( \tilde{W} \) from \( \mathcal{H} \) onto \( \mathcal{K} \) such that
\[
\pi_B(\theta(a)) = \tilde{W} \pi_A(a) \tilde{W}^*
\]
for any \( a \in A \).

**Proof.** Let \( \tilde{W} = W|_\mathcal{H} = WP_X \). Then for any \( \xi \in \mathcal{H} \),
\[
\tilde{W} \xi = WP_X \xi = W\xi = WW^*W\xi = P_KW\xi \in \mathcal{K}.
\]
Hence \( \tilde{W} \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \). For any \( \eta \in \mathcal{K} \),
\[
W^*\eta = WW^*\eta = P_KW^*\eta \in \mathcal{H}.
\]
Then
\[
\tilde{W}W^*\eta = WW^*\eta = P_K\eta = \eta.
\]
Thus \( \tilde{W} \) is surjective. Furthermore, for any \( \xi_1, \xi_2 \in \mathcal{H} \),
\[
(\tilde{W} \xi_1, \tilde{W} \xi_2)_{\mathcal{K}} = (W\xi_1, W\xi_2)_{\mathcal{H} \oplus \mathcal{K}} = (\xi_1, W^*W\xi_2)_{\mathcal{H} \oplus \mathcal{K}} = (\xi_1, P_K\xi_2)_{\mathcal{H} \oplus \mathcal{K}} = (\xi_1, \xi_2)_{\mathcal{H}}.
\]
Hence \( \tilde{W} \) is an isometry from \( \mathcal{H} \) onto \( \mathcal{K} \). Finally, for any \( a \in A \),
\[
\pi_B(\theta(a)) = \pi_B(w \begin{bmatrix} a \\ 0 \end{bmatrix} w^*) = \pi_B(ww^*w \begin{bmatrix} a \\ 0 \end{bmatrix} w^*ww^*)
\]
\[
= \pi_B(qw \begin{bmatrix} a \\ 0 \end{bmatrix} w^*q) = \rho(qw) \begin{bmatrix} a \\ 0 \end{bmatrix} w^*q
\]
\[
= \rho(qw) \begin{bmatrix} \pi_A(a) \\ 0 \end{bmatrix} \rho(w^*q) = P_K W \begin{bmatrix} \pi_A(a) \\ 0 \end{bmatrix} W^* P_K
\]
\[
= WP_X \pi_A(a) P_K W^* = \tilde{W} \pi_A(a) \tilde{W}^*.
\]
Therefore, we obtain the conclusion.

Next, we will give an easy example of non-degenerate representations of $C^*$-algebras which are strongly Morita equivalent.

Let $(\pi, \mathcal{H})$ be a non-degenerate representation of a $C^*$-algebra $A$. Let $A^s = A \otimes K$ and let $\pi^s = \pi \otimes \text{id}_K$ and $\mathcal{H}^s = \mathcal{H} \otimes \mathcal{H}_0$. Then $(\pi^s, \mathcal{H}^s)$ is a non-degenerate representation of $A^s$ on $\mathcal{H}^s$.

**Example 3.5.** With the above notation, $(\pi, \mathcal{H})$ and $(\pi^s, \mathcal{H}^s)$ are strongly Morita equivalent.

**Proof.** Let $X = A^s(1 \otimes e_{11})$. Then $X$ is an $A^s - A$-equivalence bimodule in the usual way, where we identify $A$ with $(1_{M(A)} \otimes e_{11})(A \otimes K)(1_{M(A)} \otimes e_{11})$. Let $i_H$ be the linear map from $\mathcal{H}$ to $\mathcal{H}^s$ defined by $i_H \xi = \xi \otimes e_1$ for any $\xi \in \mathcal{H}$. Then $i_H$ is an isometry and

$$i_H^*(\xi \otimes e_i) = \begin{cases} \xi & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

for any $\xi \in \mathcal{H}$ by routine computations. Thus $i_H i_H^* = 1 \otimes e_{11}$ and $i_H^* i_H = \text{id}_\mathcal{H}$ on $\mathcal{H}$. Let $\pi_X$ be the linear map from $X$ to $B(\mathcal{H}, \mathcal{H}^s)$ defined by

$$\pi_X(a(1_{M(A)} \otimes e_{11})) = \pi^s(a(1_{M(A)} \otimes e_{11})) i_H = \pi^s(a(1_{M(A)} \otimes e_{11}) i_H$$

for any $a \in A^s$. Then for any $a, b \in A^s$,

$$\pi^s(A \cdot (a(1_{M(A)} \otimes e_{11}), b(1_{M(A)} \otimes e_{11}))) = \pi^s(a(1_{M(A)} \otimes e_{11} b)^*).$$

On the other hand,

$$\pi_X(a(1_{M(A)} \otimes e_{11}))(\pi_X(b(1_{M(A)} \otimes e_{11})))^* = \pi^s(a(1 \otimes e_{11})) i_H i_H^*(\pi^s(1 \otimes e_{11}))^*$$

since $i_H i_H^* = 1 \otimes e_{11}$. Thus

$$\pi^s(A \cdot (a(1_{M(A)} \otimes e_{11}), b(1_{M(A)} \otimes e_{11}))) = \pi_X(a(1_{M(A)} \otimes e_{11})\pi_X(b(1_{M(A)} \otimes e_{11})))^*$$

Also, for any $a, b \in A^s$,

$$\pi((a(1 \otimes e_{11}), b(1 \otimes e_{11}))) = \pi^s((1 \otimes e_{11}) a b(1 \otimes e_{11})) = (1 \otimes e_{11}) \pi^s(a^s b)(1 \otimes e_{11}).$$

Since we identify $A$ with $(1 \otimes e_{11}) A^s (1 \otimes e_{11})$, we regard $\mathcal{H}$ as the closed subspace $\mathcal{H} \otimes e_1$ of $\mathcal{H}^s$. Hence we can regard $(1 \otimes e_{11}) \pi^s(a^s b)(1 \otimes e_{11})$ as an element $i_H^*(1 \otimes e_{11}) \pi^s(a^s b)(1 \otimes e_{11}) \pi^s(1 \otimes e_{11}) i_H$ in $B(\mathcal{H})$. On the other hand,

$$\pi_X(a(1 \otimes e_{11}))^* \pi_X(b(1 \otimes e_{11})) = (\pi^s(a)(1 \otimes e_{11}) i_H)^* (\pi^s(b)(1 \otimes e_{11}) i_H)$$

Thus

$$\pi((a(1 \otimes e_{11}), b(1 \otimes e_{11}))) = \pi_X(a(1 \otimes e_{11}))^* \pi_X(b(1 \otimes e_{11}))$$

for any $a, b \in A^s$. Furthermore, for any $a \in A^s$, $b \in A$, $x \in A^*$,

$$\pi_X(a \cdot x(1 \otimes e_{11}) \cdot b) = \pi_X(a x(b \otimes e_{11})) = \pi^s(a x(b \otimes e_{11})) i_H$$

$$= \pi^s(a) \pi^s(x)(1 \otimes e_{11})(i_H i_H^* i_H) i_H.$$

For any $\xi \in \mathcal{H}$,

$$(\pi(b) \otimes e_{11}) i_H \xi = (\pi(b) \otimes e_{11})(\xi \otimes e_1) = (\pi(b)(\xi \otimes e_1) = i_H \pi(b) \xi.$$.

Hence $\pi(b) \otimes e_{11}) i_H = i_H \pi(b)$. Thus

$$\pi_X(a \cdot x(1 \otimes e_{11}) \cdot b) = \pi^s(a) \pi^s(x)(1 \otimes e_{11}) i_H \pi(b) = \pi^s(a) \pi_X(x(1 \otimes e_{11})) i_H.$$
4. Definition and properties of strong Morita equivalence for completely positive linear maps

In this section, we define strong Morita equivalence for completely positive linear maps from a $C^*$-algebra to a $C^*$-algebra of all bounded linear operators on a Hilbert space.

Let $\phi$ be a completely positive linear map from a $C^*$-algebra $A$ to $B(H)$, where $H$ is a Hilbert space $\mathcal{H}$. Then by a Stinespring dilation theorem, there is a Hilbert space $\mathcal{H}_\phi$, a representation $\pi_\phi$ of $A$ on $\mathcal{H}_\phi$ and $V_\phi \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\phi)$ with $\|V_\phi\| = \|\phi\|$ such that

$$\phi(a) = V_\phi^* \pi_\phi(a) V_\phi$$

for any $a \in A$ and such that $\mathcal{H}_\phi = \pi_\phi(A) V_\phi \mathcal{H}$. For more information, see Blackadar [10 II. 6.9.7], Paulsen [9 Theorem 4.1] and Stinespring [11]. We call the above $(\pi_\phi, V_\phi, \mathcal{H}_\phi)$ a minimal Stinespring representation for $\phi$. Furthermore, we can see that $(\pi_\phi, V_\phi, \mathcal{H}_\phi)$ is unique in the sense of [9 Proposition 4.2], that is,

**Proposition 4.1.** ([9 Proposition 4.2]) With the above notation, if $(\rho, W, K)$ is another minimal Stinespring representation for $\phi$, then there is an isometry $U$ from $\mathcal{H}_\phi$ onto $K$ satisfying that $UV_\phi = W$ and that $U \pi_\phi(a) U^* = \rho(a)$ for any $a \in A$.

**Remark 4.2.** We note that a minimal Stinespring representation for $\phi$ is non-degenerate. Indeed, let $(\pi_\phi, V_\phi, \mathcal{H}_\phi)$ be a minimal Stinespring representation for $\phi$. Then since $V_\phi \mathcal{H} \subset \mathcal{H}_\phi$,

$$\mathcal{H}_\phi = \pi_\phi(A) V_\phi \mathcal{H} \subset \pi_\phi(A) \mathcal{H}_\phi \subset \mathcal{H}_\phi.$$

Thus $\mathcal{H}_\phi = \pi_\phi(A) \mathcal{H}_\phi$, that is, $(\pi_\phi, \mathcal{H}_\phi)$ is non-degenerate.

Let $\phi$ and $\psi$ be completely positive linear maps from $C^*$-algebras $A$ and $B$ to $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces.

**Definition 4.1.** We say that $\phi$ is strongly Morita equivalent to $\psi$ if a minimal Stinespring representation for $\phi$ is strongly Morita equivalent to that for $\psi$.

**Proposition 4.3.** Strong Morita equivalence for completely positive linear maps on $C^*$-algebras is equivalence relation.

**Proof.** This is immediate by Lemmas 3.1 and Proposition 4.1. $\square$

Let $\phi$ and $\psi$ be completely positive linear maps from $C^*$-algebras $A$ and $B$ to $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively, which are strongly Morita equivalent. Let $(\pi_\phi, V_\phi, \mathcal{H}_\phi)$ and $(\pi_\psi, V_\psi, \mathcal{H}_\psi)$ be minimal Stinespring representations for them. Then since $(\pi_\phi, \mathcal{H}_\phi)$ and $(\pi_\psi, \mathcal{H}_\psi)$ are strongly Morita equivalent, there are an $A - B$-equivalence bimodule $X$ and a linear map $\pi_X$ from $X$ to $\mathcal{B}(\mathcal{H}_\phi, \mathcal{H}_\psi)$ such that $(\pi_\phi, \pi_X, \pi_\psi)$ is a representation of $X$ on the pair of Hilbert spaces $(\mathcal{H}_\phi, \mathcal{H}_\psi)$. Let $L_X$ be the linking $C^*$-algebra for $X$ and let $\rho$ be the representation of $L_X$ on $\mathcal{H}_\phi \oplus \mathcal{H}_\psi$ induced by the representation $(\pi_\phi, \pi_X, \pi_\psi)$ of $X$, which is defined in Section 3. Let $\tau$ be the completely positive linear map from $L_X$ to $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ defined by

$$\tau(\begin{bmatrix} a & x \\ y & b \end{bmatrix}) = \begin{bmatrix} V_\phi^* & 0 \\ 0 & V_\psi^* \end{bmatrix} \rho(\begin{bmatrix} a & x \\ y & b \end{bmatrix}) \begin{bmatrix} V_\phi & 0 \\ 0 & V_\psi \end{bmatrix}$$

for any $\begin{bmatrix} a & x \\ y & b \end{bmatrix} \in L_X$.

**Lemma 4.4.** With the above notation, $(\rho, V_\phi \oplus V_\psi, \mathcal{H}_\phi \oplus \mathcal{H}_\psi)$ is a minimal Stinespring representation for $\tau$. 

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Proof. It suffices to show that $\rho(L_X)(V_\phi \oplus V_\psi)(H \oplus K)$ is dense in $H_\phi \oplus H_\psi$. Indeed, 

$$\pi_\phi(A)V_\phi H \oplus \pi_\psi(B)V_\psi K \subset \rho(L_X)(V_\phi \oplus V_\psi)(H \oplus K).$$

Since $\pi_\phi(A)V_\phi H = H_\phi$ and $\pi_\psi(B)V_\psi K = H_\psi$,

$$\rho(L_X)(V_\phi \oplus V_\psi)(H \oplus K) = H_\phi \oplus H_\psi.$$ 

Therefore, we obtain the conclusion. \hfill \Box

**Proposition 4.5.** Let $\phi$, $\psi$ and $\tau$ be as above. Then $\tau$ is strongly Morita equivalent to $\phi$ and $\psi$ and

$$\phi(a) = P_H \tau \left( \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right), \quad \psi(b) = P_K \tau \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

for any $a \in A$ and $b \in B$, respectively, where we identify $H$ and $K$ with $H \oplus 0$ and $0 \oplus K$, respectively and $P_H$ and $P_K$ are projections from $H \oplus K$ onto $H$ and $K$, respectively.

Proof. This is immediate by Lemmas 3.3 and 3.4 and the definition of $\tau$. \hfill \Box

Furthermore, we assume that $A$ and $B$ are $\sigma$-unital stable $C^*$-algebras. Then by the discussions before Lemma 3.3 and Lemma 3.4, there are an isomorphism $\theta$ of $A$ onto $B$ and an isometry $\tilde{W}$ from $H_\phi$ onto $H_\psi$ such that 

$$\pi_\psi(\theta(a)) = \tilde{W}\pi_\phi(a)\tilde{W}^*$$

for any $a \in A$. By easy computations, we obtain that 

$$\psi(\theta(a)) = V_\psi^*\pi_\phi(\theta(a))V_\psi = V_\psi^*\tilde{W}\pi_\phi(a)\tilde{W}^*V_\psi$$

for any $a \in A$.

**Proposition 4.6.** Let $A$ and $B$ be $\sigma$-unital stable $C^*$-algebras. Let $\phi$ and $\psi$ be completely positive linear maps from $C^*$-algebra $A$ and $B$ to $B(H)$ and $B(K)$, respectively. Let $(\pi_\phi, V_\phi, H_\phi)$ and $(\pi_\psi, V_\psi, H_\psi)$ be minimal Stinespring representations for $\phi$ and $\psi$, respectively. Then there are an isomorphism $\theta$ of $A$ onto $B$ and an isometry $\tilde{W}$ from $H_\phi$ onto $H_\psi$ satisfying that $(\pi_\phi, \tilde{W}^*V_\psi, H_\psi)$ is a minimal Stinespring representation for $\psi \circ \theta$.

Proof. Let $\tilde{W}$ and $\theta$ be as before Lemma 3.3 and in the proof of Lemma 3.4. Then it suffices to show that $(\pi_\phi, \tilde{W}^*V_\psi, H_\psi)$ is a minimal Stinespring representation for $\psi \circ \theta$. Since $(\pi_\psi, V_\psi, H_\psi)$ is a minimal Stinespring representation for $\psi$,

$$\tilde{W}\pi_\phi(A)\tilde{W}^*V_\psi K = \pi_\psi(\theta(A))V_\psi K = \pi_\psi(B)V_\psi K = H_\psi.$$

Since $\tilde{W}$ is an isometry of $H_\phi$ onto $H_\psi$,

$$H_\phi = \tilde{W}^*H_\phi = \tilde{W}^*(\tilde{W}\pi_\phi(A)\tilde{W}^*V_\psi K) = \pi_\phi(A)\tilde{W}^*V_\psi K.$$

Thus, $(\pi_\phi, \tilde{W}^*V_\psi, H_\psi)$ is a minimal Stinespring representation for $\psi \circ \theta$. \hfill \Box

Let $\phi$ be a completely positive linear map from $A$ to $B(H)$. We will show that $\phi$ and $\phi \otimes \text{id}_K$ are strongly Morita equivalent.

**Example 4.7.** With the above notation, $\phi$ and $\phi \otimes \text{id}_K$ are strongly Morita equivalent.

Proof. Let $(\pi_\phi, V_\phi, H_\phi)$ be a minimal Stinespring representation for $\phi$. Then $(\pi_\phi^*, V_\phi \otimes 1_{H_\phi}, H_\phi^*)$ is a minimal Stinespring representation for $\phi \otimes \text{id}_K$, where $1_{H_\phi}$ is the identity operator on $H_\phi$. Indeed, for any $a \in A$, $k \in K$,

$$(V_\phi \otimes 1_{H_\phi})^*\pi_\phi^*(a \otimes k)(V_\phi \otimes 1_{H_\phi}) = V_\phi^*\pi_\phi(a)V_\phi \otimes k = (\phi \otimes \text{id}_K)(a \otimes k).$$
Also, 
\[ \pi^*_\phi(A^*)(V_\phi \otimes 1_{H_\phi})(H^*_\phi) = \pi_\phi(A)V_\phi H_\phi \otimes K H_0. \]
Since \( \pi_\phi(A)V_\phi H_\phi = H_\phi \), we can see that 
\[ \pi^*_\phi(A^*)(V_\phi \otimes 1_{H_\phi})(H^*_\phi) = H^*_\phi. \]
Since by Example 3.5, \( (\pi_\phi, H_\phi) \) and \( (\pi^*_\phi, H^*_\phi) \) are strongly Morita equivalent, we obtain the conclusion.

\[ \square \]

5. Relation between strong Morita equivalence for bimodule linear maps and strong Morita equivalence for completely positive linear maps

Let \( A \subset C \) and \( B \subset D \) be inclusions of \( C^* \)-algebras with \( AC = C \) and \( BD = D \). Let \( E^A \) and \( E^B \) be conditional expectations from \( C \) and \( D \) onto \( A \) and \( B \), respectively. We assume that \( E^A \) and \( E^B \) are strongly Morita equivalent with respect to a \( C \)-\( D \)-equivalence bimodule \( Y \) and its closed subspace \( X \) as bimodule linear maps (See [7, Definition 3.1]). We note that \( A \subset C \) and \( B \subset D \) are strongly Morita equivalent with respect to \( Y \) and its closed subspace \( X \) (See [5, Definition 2.1]) and that \( E^A \) and \( E^B \) are completely positive linear maps from \( C \) and \( D \) onto \( A \) and \( B \), respectively. It is natural that we consider whether \( E^A \) and \( E^B \) are strongly Morita equivalent as completely positive linear maps.

In this section, first, we will give the following result: We consider \( E^A \) and \( E^B \) which are strongly Morita equivalent as bimodule linear maps. For any non-degenerate representation \( (\pi_B, \mathcal{K}_B) \) of \( B \), there is a non-degenerate representation \( (\pi_A, \mathcal{H}_A) \) of \( A \) such that \( \pi_A \circ E^A \) and \( \pi_B \circ E^B \) are strongly Morita equivalent as completely positive linear maps. Also, we will consider its inverse direction.

We will use the same notation as above. We note that \( \pi_B \circ E^B \) is a completely positive linear map from \( D \) to \( \mathcal{B}(\mathcal{K}_B) \). Hence by [1, II. 6.9.7] or [5, Theorem 4.1], there is a minimal Stinespring representation \( (\pi_D, V_D, \mathcal{K}_D) \) for \( \pi_B \circ E^B \) such that
\[ (\pi_B \circ E^B)(d) = V_D^{\ast} \pi_D(d)V_D \]
for all \( d \in D \), where \( V_D \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_D) \). Modifying the proof of [5, Theorem 4.1], we define \( \mathcal{K}_D \) and \( V_D \). Let \( D \otimes \mathcal{K}_B \) be the algebraic tensor product of \( D \) and \( \mathcal{K}_B \).

We define a symmetric bilinear function \( \langle \cdot, \cdot \rangle \) on \( D \otimes \mathcal{K}_B \) by setting
\[ \langle d \otimes \xi, d_1 \otimes \eta \rangle = \langle \xi, (\pi_B \circ E^B)(d^\ast d_1)\eta \rangle_{\mathcal{K}_B} \]
for any \( d, d_1 \in D \), \( \xi, \eta \in \mathcal{K}_B \) and extending linearly. Let \( \mathcal{N}_D \) be the subspace of \( D \otimes \mathcal{K}_B \) defined by
\[ \mathcal{N}_D = \{ u \in D \otimes \mathcal{K}_B \mid \langle u, u \rangle = 0 \}. \]
The induced bilinear form on the quotient space \( (D \otimes \mathcal{K}_B)/\mathcal{N}_D \) defined by
\[ \langle u + \mathcal{N}_D, v + \mathcal{N}_D \rangle = \langle u, v \rangle, \]
is an inner product, where \( u, v \in D \otimes \mathcal{K}_D \). We denote by \( \mathcal{K}_D \) and \( \langle \cdot, \cdot \rangle_{\mathcal{K}_D} \) the Hilbert space, the completion of the inner product space \( (D \otimes \mathcal{K}_B)/\mathcal{N}_D \) and its inner product, respectively.

We define \( V_D : \mathcal{K}_B \rightarrow \mathcal{K}_D \) by setting
\[ V_D(\xi) = \lim_{\lambda} (d_\lambda \otimes \xi + \mathcal{N}_D) \]
for any \( \xi \in \mathcal{K}_B \), where \( \{d_\lambda\} \) is an approximate unit of \( D \) with \( 0 \leq d_\lambda \) and \( \|d_\lambda\| \leq 1 \) and the limit is taken under the weak topology in \( \mathcal{K}_D \). Also, we note that \( \lim_\lambda (d_\lambda \otimes \xi + \mathcal{N}_D) \) is independent of the choice of an approximate unit of \( D \). Furthermore,
\[ V_D^\ast (d \otimes \xi + \mathcal{N}_D) = (\pi_B \circ E^B)(d)\xi \]
for any \( d \in D, \xi \in K_B \). Indeed, for any \( d \in D, \xi, \eta \in K_B \),
\[
(V_B^*(d \otimes \xi + N_D), \eta)_{K_B} = \langle d \otimes \xi + N_D, \lim_{\lambda}(d_\lambda \otimes \eta + N_D) \rangle_{K_D}
= \lim_{\lambda}(\langle \pi_B \circ E^B(d_\lambda \xi), \eta \rangle_{K_B})
= \langle (\pi_B \circ E^B)(d_\xi), \eta \rangle_{K_B}.
\]

Next, following the proof of [3, Lemma 2.2], we define non-degenerate representations \((\pi_A, H_A)\) and \((\pi_C, H_C)\) of \( A \) and \( C \), which are strongly Morita equivalent to \((\pi_B, K_B)\) and \((\pi_D, K_D)\), respectively.

We regard \( K_B \) as a Hilbert \( B \)-\( C \)-bimodule using the representation \((\pi_B, K_B)\) and let \( H_A = X \otimes_B K_B \), a Hilbert space, where its inner product \( \langle \cdot, \cdot \rangle_{H_A} \) is defined by
\[
\langle x \otimes \xi, y \otimes \eta \rangle_{H_A} = \langle \xi, \pi_B((x, y)B)\eta \rangle_{K_B}
\]
for any \( x, y \in X, \xi, \eta \in K_B \). We define \( \pi_A \) by setting
\[
\pi_A(a)(x \otimes \xi) = (a \cdot x) \otimes \xi
\]
for all \( a \in A, x \in X, \xi \in K_B \). Furthermore, we define a linear map \( \pi_X \) from \( X \) to \( B(K_B, H_A) \) by setting
\[
\pi_X(x) = x \otimes \xi
\]
for any \( x \in X, \xi \in K_B \). By [3, Lemma 2.2], \((\pi_A, \pi_X, \pi_B)\) is a representation of \( X \) on the pair of Hilbert spaces \((H_A, K_B)\). Thus \((\pi_A, H_A)\) and \((\pi_B, K_B)\) are strongly Morita equivalent. Similarly we define a representation of \( Y \) on the pair of Hilbert spaces \((H_C, K_D)\) as follows:
\[
H_C = Y \otimes_D K_D, \quad \pi_Y(y)\xi = y \otimes \xi, \quad \pi_C(c)(y \otimes \xi) = (c \cdot y) \otimes \xi
\]
for any \( c \in C, y \in Y, \xi \in K_D \).

We consider \( \pi_A \circ E^A \), which is a completely positive linear map from \( C \) to \( B(H_A) \). By [3, Theorem 4.1] or [1, II. 6.9.7], there is a minimal Stinespring representation \((\pi_C', V_C', H_C')\). We will show that \( H_C \cong H_C' \) as Hilbert spaces. In order to do this, we introduce the following Hilbert space \( \mathcal{E} \). We regard \( Y \) as a Hilbert \( C \)-\( B \)-bimodule in the following way: We define the left \( C \)-action, the right \( B \)-action and the left \( C \)-valued inner product in the usual way. We define the right \( B \)-valued inner product by setting
\[
\langle x, y \rangle_B = E^B((x, y)_D)
\]
for any \( x, y \in Y \). We denote it by the symbol \( Y_B \). We define the Hilbert space \( \mathcal{E} \) by
\[
\mathcal{E} = Y_B \otimes_B K_B
\]
in the same way as the definition of Hilbert space \( H_A = X \otimes_B K_B \).

First, we show that \( H_C \) is isomorphic to \( \mathcal{E} \) as Hilbert spaces. Before doing it, we prepare the following lemma.

**Lemma 5.1.** With the above notation, let \( \{d_\lambda\}_{\lambda \in \Lambda} \) be an approximate unit of \( D \) with \( 0 \leq d_\lambda \) and \( \|d_\lambda\| \leq 1 \) for any \( \lambda \in \Lambda \). For any \( y \in Y \), \( \|y - y \cdot d_\lambda\| \to 0 \) (\( \lambda \to \infty \)).

**Proof.** Let \( y \) be any element in \( Y \). For any \( \epsilon > 0 \), there are \( y_1, y_2, \ldots, y_n \in Y, d_1, d_2, \ldots, d_n \in D \) such that
\[
\|y - \sum_{i=1}^n y_i \cdot d_i\| < \epsilon,
\]
by [3 Proposition 1.7]. Then
\[ ||y - y \cdot d\lambda|| \]
\[ \leq ||y - \sum_{i=1}^{n} y_i \cdot d_i|| + ||\sum_{i=1}^{n} y_i \cdot d_i - \sum_{i=1}^{n} y_i \cdot d_id\lambda|| + ||\sum_{i=1}^{n} y_i \cdot d_id\lambda - y \cdot d\lambda|| \]
\[ < 2\epsilon + ||\sum_{i=1}^{n} y_i \cdot (d_i - d_id\lambda)||. \]

Since \( ||d_i - d_id\lambda|| \to 0 \) \((\lambda \to \infty)\) for \( i = 1, 2, \ldots, n \), there is a \( \lambda_0 \in \Lambda \) such that
\[ \||\sum_{i=1}^{n} y_i \cdot (d_i - d_id\lambda)|| < \epsilon \]
for any \( \lambda \geq \lambda_0 \). Thus \( ||y - y \cdot d\lambda|| \to 0 \) \((\lambda \to \infty)\).

\[ \square \]

**Lemma 5.2.** With the above notation, \( \mathcal{H}_C \cong E \) as Hilbert spaces.

**Proof.** Let \( \Phi \) be the linear map from \( Y \otimes K_D \) to \( E \) defined by
\[ \Phi(y \otimes (d \otimes \xi)) = (y \cdot d) \otimes \xi \]
for any \( y \in Y, \ d \in D, \ \xi \in K_B \), where \( Y \otimes K_D \) is the algebraic tensor product of \( Y \) and \( K_D \). Let \( y, y_1 \in Y, \ d, d_1 \in D, \ \xi, \xi_1 \in K_B \). Then
\[ \langle y \otimes (d \otimes \xi), y_1 \otimes (d_1 \otimes \xi_1) \rangle_{\mathcal{H}_C} = \langle d \otimes \xi, \pi_D((y \cdot y_1)D)(d_1 \otimes \xi_1) \rangle_{K_D} \]
\[ = \langle d \otimes \xi, (y \cdot y_1)Dd_1 \otimes \xi_1 \rangle_{K_D} \]
\[ = \langle \xi, (\pi_B \circ E_B)(d^*\langle y, y_1 \rangle Dd_1)\xi_1 \rangle_{K_B}. \]

On the other hand,
\[ \langle \Phi(y \otimes (d \otimes \xi)), \Phi(y_1 \otimes (d_1 \otimes \xi_1)) \rangle_E = \langle (y \cdot d) \otimes \xi, (y_1 \cdot d_1) \otimes \xi_1 \rangle_E \]
\[ = \langle \xi, \pi_B((y \cdot d, y_1 \cdot d_1)B)\xi_1 \rangle_E \]
\[ = \langle \xi, (\pi_B \circ E_B)(d^*\langle y, y_1 \rangle Dd_1)\xi_1 \rangle_{K_B}. \]

Thus \( \Phi \) preserves the inner products on the algebraic tensor products. We can extend \( \Phi \) to \( \mathcal{H}_C \). We denote it by the same symbol \( \Phi \). Then \( \Phi \) is an isometry from \( \mathcal{H}_C \) to \( E \). Next, we show that \( \Phi \) is surjective. Let \( y \) and \( \xi \) be elements in \( Y_B \) and \( K_D \), respectively. Let \( \{d_\lambda\}_{\lambda \in \Lambda} \) be an approximate units of \( D \) with \( d_\lambda \geq 0 \) and \( ||d_\lambda|| \leq 1 \). Then by Lemma 5.1 \( y = \lim_{\lambda} y \cdot d_\lambda \). Also, \( y \otimes (d_\lambda \otimes \xi) \) is an element in \( Y \otimes D K_D \) and
\[ \Phi(y \otimes (d_\lambda \otimes \xi)) = y \cdot d_\lambda \otimes \xi \to y \otimes \xi \ \ (\lambda \to \infty). \]

Since \( \Phi \) is isometric, \( \{y \otimes (d_\lambda \otimes \xi)\}_{\lambda \in \Lambda} \) is a Cauchy net in \( Y \otimes D K_D \). Hence there exists an element \( z \in \mathcal{H}_C \) such that \( y \otimes (d_\lambda \otimes \xi) \to z \) \((\lambda \to \infty)\). Thus \( \Phi(z) = y \otimes \xi \), that is, \( \Phi \) is surjective. Therefore \( \Phi \) is an isometry from \( \mathcal{H}_C \) onto \( E \). \( \square \)

By the proof of Lemma 5.2
\[ \Phi^*(y \otimes \xi) = \lim_{\lambda} y \otimes (d_\lambda \otimes \xi) \]
for any \( y \in Y, \ \xi \in K_B \), where \( \{d_\lambda\}_{\lambda \in \Lambda} \) is an approximate units of \( D \) with \( 0 \leq d_\lambda \) and \( ||d_\lambda|| \leq 1 \) for any \( \lambda \in \Lambda \).

Next, we will show that \( \mathcal{H}_C' \cong E \) as Hilbert spaces.

**Lemma 5.3.** With the above notation \( \mathcal{H}_C' \cong E \) as Hilbert spaces.
With the above notation, the non-degenerate representations extend $\Psi$ to $H$ defined by
$$\Psi(c \otimes (x \otimes \xi)) = (c \cdot x) \otimes \xi$$
for any $c \in C$, $x \in X$, $\xi \in K_B$, where we note that $X$ is a closed subspace of $Y$. Let $c, c_1 \in C$, $x, x_1 \in X$, $\xi, \xi_1 \in K_B$. Then
$$(c \otimes (x \otimes \xi), c_1 \otimes (x_1 \otimes \xi_1))_{H_C} = (x \otimes \xi, (\pi_A \circ E^A)(c^*c_1)(x_1 \otimes \xi_1))_{H_A}
= (x \otimes \xi, [E^A(c^*c_1) \cdot x_1] \otimes \xi_1)_{H_A}
= (\xi, \pi_B((x, E^A(c^*c_1) \cdot x_1))_{K_B}.$$

Since $E^A$ and $E^B$ are strongly Morita equivalent with respect to $Y$ and its closed subspace $X$ as bimodule linear maps, we have the equation
$$(z, E^A(a \cdot z_1)) = E^B((z, a \cdot z_1))$$
for any $z, z_1 \in X$, $a \in C$. Hence we obtain that
$$(c \otimes (x \otimes \xi), c_1 \otimes (x_1 \otimes \xi_1))_{H_C} = (\xi, (\pi_B \circ E^B)((x, c^*c_1) \cdot x_1))_{K_B}.$$

On the other hand,
$$(\Psi(c \otimes (x \otimes \xi)), \Psi(c_1 \otimes (x_1 \otimes \xi_1)))_C = ((c \cdot x) \otimes \xi, (c_1 \cdot x_1) \otimes \xi_1)
= (\xi, (\pi_B \circ E^B)((c \cdot x, c_1 \cdot x_1))_{K_B}.$$n
Thus $\Psi$ preserves the inner products on the algebraic tensor products. We can extend $\Psi$ to $H_C$. We denote it by the same symbol $\Psi$. Then $\Psi$ is an isometry from $H_C$ to $E$. We show that $\Psi$ is surjective. Let $y$ and $\xi$ be any elements in $Y_B$ and $K_B$, respectively. Brown, Mingo and Shen [3] Proposition 1.7, $Y = Y \cdot D$. Hence $K$ definition 2.1
$$Y = Y \cdot D = Y \cdot (Y, X)_{D} = \pi_1(Y) = C \cdot X.$$n
For any $m \in N$, there are elements $c_1, c_2, \ldots, c_m \in C$ and $x_1, x_2, \ldots, x_m \in X$ such that
$$||y - \sum_{i=1}^{m} c_i \cdot x_i|| < \frac{1}{m}.$$n
Let $z_m = \sum_{i=1}^{m} c_i \otimes (x_i \otimes \xi)$. Then $z_m \in H_C$ for any $m \in N$ and $\Psi(z_m) = \sum_{i=1}^{m} (c_i \cdot x_i) \otimes \xi \rightarrow y \otimes \xi (m \rightarrow \infty)$. Since $\Psi$ is isometric, $\{z_m\}$ is a Cauchy sequence in $H_C$. Thus there exists an element $z \in H_C$ such that $z_m \rightarrow z (m \rightarrow \infty)$. Hence $\Psi(z) = y$, that is, $\Psi$ is surjective. Therefore, $\Psi$ is an isometry from $H_C$ onto $E$. $
$
**Lemma 5.4.** With the above notation, the non-degenerate representations $(\pi_C, H_C)$ and $(\pi'_C, H'_C)$ are unitarily equivalent.

**Proof.** Let $c \in C$, $x \in X$ and $\xi \in K_B$. Then
$$(\Phi^*\Psi)(c \otimes (x \otimes \xi)) = \Phi^*((c \cdot x) \otimes \xi) = \lim_{\lambda}(c \cdot x) \otimes (d_\lambda \otimes \xi),$$n
where $\{d_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit of $D$ with $d_\lambda \geq 0$ and $||d_\lambda|| \leq 1$ for any $\lambda \in \Lambda$. Thus for any $c_1 \in C$,
$$(\Phi^*\Psi_{\pi'_C}(c_1))(c \otimes (x \otimes \xi)) = \lim_{\lambda}(c_1 \cdot x) \otimes (d_\lambda \otimes \xi).$$n
On the other hand,
$$(\pi_C(c_1)\Phi^*\Psi)(c \otimes (x \otimes \xi)) = \lim_{\lambda} \pi_C(c_1)((c \cdot x) \otimes (d_\lambda \otimes \xi)) = \lim_{\lambda}(c_1 \cdot x) \otimes (d_\lambda \otimes \xi).$$
Hence \( \pi_C(c_1) \Phi^* \Psi = \Phi^* \Psi \pi_C'(c_1) \) for any \( c_1 \in C \). Therefore, we obtain the conclusion. \( \Box \)

Let \( U = \Phi^* \Psi \). Then \( U \) is an isometry from \( \mathcal{H}'_C \) onto \( \mathcal{H}_C \) and \( \pi_C(c) = U \pi_C'(c) U^* \) for any \( c \in C \).

**Lemma 5.5.** With the above notation, \( (\pi_C, UV'_C, \mathcal{H}_C) \) is a minimal Stinespring representation for \( \pi_A \circ E^A \).

**Proof.** For any \( c \in C \),
\[
(\pi_C(c)') UV'_C = V'_C \pi_C(c) UV'_C = V'_C \pi_C(c) V'_C = (\pi_A \circ E^A)(c).
\]
Also,
\[
\frac{\pi_C(c) UV'_C \mathcal{H}_A}{U \pi_C(c) V'_C \mathcal{H}_A} = \frac{\mathcal{H}_C}{\mathcal{H}_C}. \]
Hence \( (\pi_C, UV'_C, \mathcal{H}_C) \) is a minimal Stinespring representation for \( \pi_A \circ E^A \). \( \Box \)

**Lemma 5.6.** With the above notation, \( \pi_A \circ E^A \) and \( \pi_B \circ E^B \) are strongly Morita equivalent as completely positive linear maps.

**Proof.** By the definition of \( (\pi_C, \mathcal{H}_C) \), \( (\pi_C, \mathcal{H}_C) \) and \( (\pi_D, \mathcal{K}_D) \) are strongly Morita equivalent. Also, since \( (\pi_C, UV'_C, \mathcal{H}_C) \) is a minimal Stinespring representation for \( \pi_A \circ E^A \) by Lemma 5.5, \( \pi_A \circ E^A \) and \( \pi_B \circ E^B \) are strongly Morita equivalent as completely positive linear maps. \( \Box \)

Combining the above discussions, we obtain the following theorem.

**Theorem 5.7.** Let \( A \subset C \) and \( B \subset D \) be inclusions of \( C^* \)-algebras with \( \overline{AC} = C \) and \( \overline{BD} = D \). Let \( E^A \) and \( E^B \) conditional expectations from \( C \) and \( D \) onto \( A \) and \( B \), respectively. We assume that \( E^A \) and \( E^B \) are strongly Morita equivalent with respect to a \( C - D \)-equivalence bimodule \( Y \) and its closed subspace \( X \). Then for any non-degenerate representation \( (\pi_B, \mathcal{K}_B) \) of \( B \), there exists a non-degenerate representation \( (\pi_A, \mathcal{H}_A) \) of \( A \) such that \( \pi_A \circ E^A \) and \( \pi_B \circ E^B \) are strongly Morita equivalent as completely positive linear maps.

**Proof.** This is immediate by Lemma 5.6. \( \Box \)

Next, we will consider the inverse direction. Let \( A \subset C \) and \( B \subset D \) be as above. We suppose that \( A \subset C \) and \( B \subset D \) are strongly Morita equivalent with respect to a \( C - D \)-equivalence bimodule \( Y \) and its closed subspace \( X \). Let \( E^A \) and \( E^B \) be conditional expectations from \( C \) and \( D \) onto \( A \) and \( B \), respectively. Let \( (\pi_B, \mathcal{K}_B) \) be a non-degenerate representation \( B \) and \( (\pi_A, \mathcal{H}_A) \) be the non-degenerate representation of \( A \) induced by \( X \) and \( (\pi_B, \mathcal{K}_B) \), which is defined in [4, Lemma 2.2]. Let \( (\pi_D, \mathcal{K}_D) \) be a minimal Stinespring representation for \( \pi_B \circ E^B \) and \( (\pi_C, \mathcal{H}_C) \) the non-degenerate representation of \( C \) induced by \( Y \) and \( (\pi_D, \mathcal{K}_D) \). First, we show the following lemma.

**Lemma 5.8.** Let \( \{d_\lambda\}_{\lambda \in \Lambda} \) be an approximate unit of \( D \) with \( d_\lambda \geq 0 \) and \( \|d_\lambda\| \leq 1 \) for any \( \lambda \in \Lambda \). Then \( \{y \circ (d_\lambda \otimes \xi)\}_{\lambda \in \Lambda} \) is a Cauchy net in \( \mathcal{H}_C \) with respect to the weak topology of \( \mathcal{H}_C \) for any \( y \in Y \), \( \xi \in \mathcal{K}_B \).

**Proof.** Since the linear span of the set
\[ \{y \circ (d \otimes \xi) \in \mathcal{H}_C \mid y \in Y, d \in D, \xi \in \mathcal{K}_B\} \]
is dense in $H_C$, it suffices to show that for any $y_1 \in Y$, $\xi_1 \in K_B$, $d_1 \in D$, the net 
\{$(y \otimes (d_\lambda \otimes \xi), y_1 \otimes (d_1 \otimes \xi_1))_{H_C}$\}$\lambda$ is a Cauchy net. For any $\lambda, \mu \in \Lambda$,
\[
\langle y \otimes ((d_\lambda - d_\mu) \otimes \xi), y_1 \otimes (d_1 \otimes \xi_1) \rangle_{H_C} = \langle (d_\lambda - d_\mu) \otimes \xi, \pi_D((y, y_1) D)(d_1 \otimes \xi_1) \rangle_{K_D} \\
= \langle (d_\lambda - d_\mu) \otimes \xi, (y, y_1) D d_1 \otimes \xi_1 \rangle_{K_D} \\
= \langle \xi, (\pi_B \circ E^B)((d_\lambda - d_\mu) (y, y_1) D d_1) \xi_1) \rangle_{K_B} \to 0 \quad (\lambda, \mu \to \infty).
\]
Thus, we obtain the conclusion. 

Let $V_C$ be the linear map from $H_A$ to $H_C$ defined by

\[ V_C(x \otimes \xi) = \lim_{\lambda} x \otimes (d_\lambda \otimes \xi) \]

for any $x \in X$, $\xi \in K_B$, where \{$d_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit of $D$ with $d_\lambda \geq 0$ and $\|d_\lambda\| \leq 1$ for any $\lambda \in \Lambda$ and the limit is taken under the weak topology of $H_C$. By Lemma 5.9, the above limit is convergent with respect to the weak topology of $H_C$ and by routine computations, $V_C$ is well-defined and independent of the choice of an approximate unit of $D$.

**Lemma 5.9.** With the above notation, $V_C$ is an isometry from $H_A$ to $H_C$.

**Proof.** For any $x, x_1 \in X$, $\xi, \xi_1 \in K_B$,
\[
\langle V_C(x \otimes \xi), V_C(x_1 \otimes \xi_1) \rangle_{H_C} = \lim_{\lambda, \mu} \langle x \otimes (d_\lambda \otimes \xi), x_1 \otimes (d_\mu \otimes \xi_1) \rangle_{H_C} \\
= \lim_{\lambda, \mu} \langle d_\lambda \otimes \xi, \pi_D((x, x_1) B)(d_\mu \otimes \xi_1) \rangle_{K_D} \\
= \lim_{\lambda, \mu} \langle d_\lambda \otimes \xi, (x, x_1) B d_\mu \otimes \xi_1 \rangle_{K_D} \\
= \lim_{\lambda, \mu} \langle \xi, (\pi_B \circ E^B)(d_\lambda (x, x_1) B d_\mu) \xi_1 \rangle_{K_B} \\
= \langle \xi, \pi_B((x, x_1) B) \xi_1 \rangle_{K_B} \\
= \langle x \otimes \xi, x_1 \otimes \xi_1 \rangle_{H_A}.
\]
Thus, we obtain the conclusion. 

**Proposition 5.10.** Let $V_C$ be the isometry from $H_A$ to $H_C$ defined by

\[ V_C(x \otimes \xi) = \lim_{\lambda} x \otimes (d_\lambda \otimes \xi) \]

for any $x \in X$, $\xi \in K_B$, where \{$d_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit of $D$ with $d_\lambda \geq 0$ and $\|d_\lambda\| \leq 1$ for any $\lambda \in \Lambda$ and the limit is taken under the weak topology of $H_C$. We suppose that $(\pi_B, K_B)$ is faithful and that
\[
(\pi_A \circ E^A)(c) = V_C^* \pi_C(e) V_C
\]
for any $c \in C$. Then $E^A$ and $E^B$ are strongly Morita equivalent as bimodule linear maps.

**Proof.** For any $c \in C$, $x, x_1 \in X$ and $\xi, \xi_1 \in K_B$,
\[
\langle (\pi_A \circ E^A)(c) (x \otimes \xi), x_1 \otimes \xi_1 \rangle_{H_A} = \langle (E^A(c) \cdot x) \otimes \xi, x_1 \otimes \xi_1 \rangle_{H_A} \\
= \langle \xi, \pi_B((E^A(c) \cdot x) \otimes \xi_1) \rangle_{K_B}.
\]
Also,
\[
(V_C^* \pi_C(c)V_C)(x \otimes \xi), x_1 \otimes \xi_1)_{H_A} = \langle \pi_C(c)V_C(x \otimes \xi), V_C(x_1 \otimes \xi_1) \rangle_{H_C}
\]
\[
= \lim_{\lambda, \mu}(c \otimes (d_\lambda \otimes \xi)), x_1 \otimes (d_\mu \otimes \xi_1)_{H_C}
\]
\[
= \langle \pi_B((c \otimes x), x_1)D, d_\mu \otimes \xi_1 \rangle_{K_B}
\]
\[
= \langle \xi, \langle \pi_B \circ \rho_B((c \otimes x), x_1)D \rangle \rangle_{K_B}
\]

Since \((\pi_A \circ \rho_A)(c) = V_C^* \pi_C(c)V_C\) for any \(c \in C\) and \(\pi_B\) is faithul,
\[
E_B((c \otimes x), x_1), D) = \langle (\rho_A(c) \otimes x), x_1 \rangle_B.
\]
for any \(c \in C, x, x_1 \in X\). By [1] Lemma 2.5, \(E^A\) and \(E^B\) are strongly Morita equivalent as bimodule linear maps.

\[\square\]

6. A CORRESPONDENCE OF STRONG MORITA EQUIVALENCE CLASSES OF COMPLETELY POSITIVE LINEAR MAPS

Let \(A\) and \(B\) be \(C^*\)-algebras, which are strongly Morita equivalent with respect to an \(A - B\)-equivalence bimodule \(X\). In this section, we will construct a 1-1 correspondence between the set of all strong Morita equivalence classes of completely positive linear maps on \(A\) and the set of all strong Morita equivalence classes of completely positive linear maps on \(B\) and we will show that the corresponding positive linear maps are strongly Morita equivalent.

Let \(\psi\) be a completely positive linear map from \(B\) to \(B(\mathcal{H})\), where \(\mathcal{H}\) is a Hilbert space. Let \((\pi_\psi, V_\psi, K_\psi)\) be a minimal Stinespring representation for \(\psi\). Let \((\pi_A, H_A)\) be the non-degenerate representation of \(A\) induced by \(X\) and \((\pi_\psi, V_\psi, K_\psi)\). Let \(\{u_i\}_{i=1}^n\) be a finite subset of \(X\). Let \(\phi\) be the linear map from \(A\) to \(B(\mathcal{K}) \otimes M_n(\mathcal{C})\) defined by
\[
[\phi(a)]_{i,j=1}^n = [\psi((u_i, a \cdot u_j)B)]_{i,j=1}^n
\]
for any \(a \in A\). Since \(\psi(b) = V_\psi^* \pi_\psi(b) V_\psi\) for any \(b \in B\),
\[
[\phi(a)]_{i,j=1}^n = [V_\psi^* \pi_\psi((u_i, a \cdot u_j)B) V_\psi]_{i,j=1}^n
\]
\[
= (V_\psi^* \otimes I_n)[\pi_\psi((u_i, a \cdot u_j)B)]_{i,j=1}^n (V_\psi \otimes I_n).
\]

First, we will show that \(\phi\) is a completely positive linear map from \(A\) to \(B(\mathcal{K}) \otimes M_n(\mathcal{C})\).

Let \(X^n\) be the \(n\)-times direct sum of \(X\) and we regard \(X^n\) as an \(A - M_n(B)\)-equivalence bimodule as follows: For any \(a \in A, [b_{ij}]_{i,j=1}^n \in M_n(B), [x_1, \ldots, x_n], [z_1, \ldots, z_n] \in X^n\), we define the left \(A\)-action, right \(M_n(B)\)-action and the left \(A\)-valued inner product, the right \(M_n(B)\)-valued inner product on \(X^n\) by setting
\[
a \cdot [x_1, \ldots, x_n] = [a \cdot x_1, \ldots, a \cdot x_n],
\]
\[
[x_1, \ldots, x_n] \cdot [b_{ij}]_{i,j=1}^n = \sum_{i=1}^n x_i \cdot b_{ij} \cdot \sum_{i=1}^n x_i \cdot b_{ij},
\]
\[
A([x_1, \ldots, x_n], [z_1, \ldots, z_n]) = \sum_{i=1}^n A(x_i, z_i),
\]
\[
\langle [x_1, \ldots, x_n], [z_1, \ldots, z_n] \rangle_{M_n(B)} = \langle [x_1, z_j]_{i,j=1}^n \rangle.
\]
Let $Y = X^n$ and $D = M_n(B)$. We regard $Y$ as an $A - D$-equivalence bimodule in the above way. For each $m \in \mathbb{N}$, let $M_m(Y)$ be the $\mathbb{C}$-linear space of all matrices over $Y$. We regard $M_m(Y)$ as an $M_m(A) - M_m(D)$-equivalence bimodule as follows: For any $[a_{kl}]_{k,l=1}^m \in M_m(A)$, $[d_{kl}]_{k,l=1}^m \in M_m(D)$, $[y_{kl}]_{k,l=1}^m \in M_m(Y)$, we define the left $M_m(A)$-action, the right $M_m(D)$-action and the left $M_m(A)$-valued inner product, the right $M_m(D)$-valued inner product on $M_m(Y)$ by setting

$$
[a_{kl}]_{k,l=1}^m \cdot [y_{kl}]_{k,l=1}^m = \sum_{t=1}^m a_{kt} \cdot [b_{tl}]_{k,l=1}^m,
[y_{kl}]_{k,l=1}^m \cdot [d_{kl}]_{k,l=1}^m = \sum_{t=1}^m y_{kt} \cdot [d_{tl}]_{k,l=1}^m,
M_m(A)\langle [y_{kl}]_{k,l=1}^m, [z_{kl}]_{k,l=1}^m \rangle = \sum_{t=1}^m A\langle y_{kt}, z_{tt} \rangle_{k,l=1}^m,
M_m(D)\langle [y_{kl}]_{k,l=1}^m, [z_{kl}]_{k,l=1}^m \rangle = \sum_{t=1}^m y_{lk} \cdot \langle z_{tt} \rangle_{k,l=1}^m.
$$

Lemma 6.1. With the above notation, $\phi$ is a completely positive linear map from $A$ to $\mathcal{B}(K) \otimes M_n(\mathbb{C})$.

**Proof.** Since $\phi$ is clearly linear, we have only to show that for any $[a_{kl}]_{k,l=1}^m \in M_m(A)$ with $[a_{kl}]_{k,l=1}^m \geq 0$,

$$
[\phi(a_{kl})]_{k,l=1}^m \geq 0.
$$

Let $[a_{kl}]_{k,l=1}^m$ be any positive element in $M_m(A)$. Then by the definition of $\phi$,

$$
\phi(a_{kl})_{ij} = \psi((u_i, a_{kl} \cdot u_j)_B).
$$

Thus

$$
(\phi \otimes \text{id}_{M_m(\mathbb{C})})([a_{kl}]_{k,l=1}^m) = \begin{bmatrix}
\psi((u_i, a_{11} \cdot u_j)_B))_{i,j=1}^n & \cdots & \psi((u_i, a_{1m} \cdot u_j)_B))_{i,j=1}^n \\
\vdots & \ddots & \vdots \\
\psi((u_i, a_{m1} \cdot u_j)_B))_{i,j=1}^n & \cdots & \psi((u_i, a_{mm} \cdot u_j)_B))_{i,j=1}^n \\
\end{bmatrix}_{k,l=1}^m = \begin{bmatrix}
\psi((u_i, a_{kl} \cdot u_j)_B))_{i,j=1}^n \\
\end{bmatrix}_{k,l=1}^m.
$$

Since $\psi$ is a completely positive linear map from $B$ to $\mathcal{B}(K)$, we have only to show that the element

$$
\begin{bmatrix}
([u_i, a_{kl} \cdot u_j])_{k,l=1}^n \\
\end{bmatrix}_{k,l=1}^m
$$

is positive in $B \otimes M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. Let $y = [u_1, \ldots, u_n] \in Y$. Let $[y_{kl}]_{k,l=1}^m$ be an element in $M_m(Y)$ defined by

$$
y_{kl} = \begin{cases} 
y & \text{if } k = l \\
0 & \text{if } k \neq l
\end{cases}
$$

Then

$$
\langle [y_{kl}]_{k,l=1}^m, [a_{kl}]_{k,l=1}^m \cdot [y_{kl}]_{k,l=1}^m \rangle_{M_m(D)} \geq 0
$$
since \( |a_{kl}|_{k,l=1}^m \geq 0 \). On the other hand, by the definition of \( |y_{kl}|_{k,l=1}^m \) and \( y \),

\[
\left\langle |y_{kl}|_{k,l=1}^m, [a_{kl}]_{k,l=1}^m \cdot |y_{kl}|_{k,l=1}^m \right\rangle_{M_n(D)} = \left( \left\langle |y_{kl}|_{k,l=1}^m, \left( \sum_{k,l=1}^m a_{kl} \cdot y_{kl} \right) \right\rangle_{k,l=1}^m \right)_{M_n(D)}
\]

\[
= \left\langle |y_{kl}|_{k,l=1}^m, [y_{kl}]_{k,l=1}^m \right\rangle_{M_n(D)}
\]

\[
= \left( \sum_{k,l=1}^m \langle y_{kl}, a_{kl} \cdot y \rangle D \right)_{k,l=1}^m
\]

\[
= \left( \sum_{k,l=1}^m \langle y_{kl}, a_{kl} \cdot y \rangle D \right)_{k,l=1}^m
\]

\[
= \left( \sum_{k,l=1}^m \langle u_{n1}, \ldots, u_{nm}, [a_{kl} \cdot u_{1}, \ldots, a_{kl} \cdot u_{n}] \right\rangle_{M_n(B)} \right)_{k,l=1}^m
\]

Therefore, we obtain the conclusion. \( \square \)

Let \((\pi_\phi, V_\phi, \mathcal{H}_\phi)\) be a minimal Stinespring representation for \( \phi \). Let \( A \odot (\mathcal{K} \otimes \mathbb{C}^n) \) be the algebraic tensor product of \( A \) and \( \mathcal{K} \otimes \mathbb{C}^n \). Let \( \{b_p\}_{p \in P} \) be an approximate unit of \( B \) with \( b_p \geq 0 \) and \( \|b_p\| \leq 1 \) for any \( p \in P \). We define a map \( U \) from 
\( A \oplus (\mathcal{K} \otimes \mathbb{C}^n) \) to \( \mathcal{H}_A \) by setting

\[
U(a \otimes \xi \otimes \lambda) = \lim_{p \rightarrow \infty} \sum_{i=1}^n \lambda_i (a \cdot u_i) \otimes b_p \otimes \xi
\]

for any \( a \in A \), \( \xi \in \mathcal{K} \), \( \lambda \in \mathbb{C}^n \) and \( \lambda = \left[ \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \right] \), and extending linearly, where we identify \( B(\mathcal{K}) \otimes M_n(\mathbb{C}) \) with \( B(\mathcal{K} \otimes \mathbb{C}^n) \) and the limit is taken under the weak topology of \( \mathcal{H}_A \).

**Lemma 6.2.** With the above notation, \( U \) is an isometry from \( A \oplus (\mathcal{K} \otimes \mathbb{C}^n) \) to \( \mathcal{H}_A \). Hence we can extend \( U \) to an isometry from \( \mathcal{H}_\phi \) to \( \mathcal{H}_A \).

**Proof.** Let \( a, b \in A \), \( \xi, \eta \in \mathcal{K} \), \( \lambda = \left[ \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \right] \), \( \mu = \left[ \begin{array}{c} \mu_1 \\ \vdots \\ \mu_n \end{array} \right] \) \( \in \mathbb{C}^n \). Then

\[
\left\langle (U(a \otimes \xi \otimes \lambda), U(b \otimes \xi \otimes \mu) \right\rangle_{\mathcal{H}_A}
\]

\[
= \left( \lim_{p \rightarrow \infty} \sum_{i=1}^n \lambda_i (a \cdot u_i) \otimes b_p \otimes \xi \right), \lim_{q \rightarrow \infty} \sum_{j=1}^n \mu_j (b \cdot u_j) \otimes b_q \otimes \eta \right\rangle_{\mathcal{H}_A}
\]

\[
= \lim_{p, q \rightarrow \infty} \sum_{i, j=1}^n \langle b_p \otimes \xi, \pi_\phi((\lambda_i(a \cdot u_i), \mu_j(b \cdot u_j))_B) (b_q \otimes \eta) \rangle_{\mathcal{K}_\phi}
\]

\[
= \sum_{i, j=1}^n \langle V_\psi \xi, \pi_\psi((\lambda_i(a \cdot u_i), \mu_j(b \cdot u_j))_B) V_\psi \eta \rangle_{\mathcal{K}_\psi}
\]

\[
= \sum_{i, j=1}^n \langle (\xi, V_\psi \pi_\psi((\lambda_i(a \cdot u_i), \mu_j(b \cdot u_j))_B) V_\psi \eta \rangle_{\mathcal{K}}
\]

\[
= \sum_{i, j=1}^n \langle (\xi, \psi((\lambda_i(a \cdot u_i), \mu_j(b \cdot u_j))_B) \eta) \rangle_{\mathcal{K}}.
\]
On the other hand,

\[
\langle a \otimes \xi \otimes \lambda, b \otimes \eta \otimes \mu \rangle_{\mathcal{H}_\phi} = \langle \begin{bmatrix} \lambda_1 \xi \\ \vdots \\ \lambda_n \xi \end{bmatrix}, \begin{bmatrix} \phi(a^*b)_{i,j} & \cdots & \phi(a^*b)_{i,n} \\ \vdots & \ddots & \vdots \\ \phi(a^*b)_{n,j} & \cdots & \phi(a^*b)_{n,n} \end{bmatrix} \rangle_{\mathcal{K} \otimes \mathbb{C}^n} = \sum_{i,j=1}^n \langle \lambda_i \xi, \phi(a^*b)_{i,j} \mu_j \eta \rangle_{\mathcal{K}} = \sum_{i,j=1}^n \langle \lambda_i \xi, \phi(a^*b)_{i,j} \mu_j \eta \rangle_{\mathcal{K}}.
\]

Hence \(U\) preserves the inner products on the algebraic tensor products. We can extend \(U\) to an isometry from \(\mathcal{H}_\phi\) to \(\mathcal{H}_A\). □

We denote by the same symbol the above isometry from \(\mathcal{H}_\phi\) to \(\mathcal{H}_A\). From now on, we assume that \(A\) and \(B\) are strongly Morita equivalent unital \(\mathcal{C}^*\)-algebras. Since \(X\) is an \(A-B\)-equivalence bimodule, by Kajiwara and Watatani \[6, Corollary 1.19\], there is a left \(A\)-basis in \(X\), which is a finite subset of \(X\). We will show that \(\phi\) is strongly Morita equivalent to \(\psi\) if \(\{u_i\}_{i=1}^n\) is a left \(A\)-basis in \(X\).

**Lemma 6.3.** With the above notation, we assume that \(\{u_i\}_{i=1}^n\) is a left \(A\)-basis in \(X\). Then \(U\) is surjective.

**Proof.** Since \(\{u_i\}_{i=1}^n\) is a left \(A\)-basis in \(X\), for any \(x \in X\), \(x = \sum_{i=1}^n a \cdot u_i \cdot u_i\). Thus the set \(\{\sum_{i=1}^n a \cdot u_i | a \in A\}\) is equal to \(X\). Also, by \[3\ Proposition 1.7\], \(X \cdot B = X\). Hence we can see that the set

\[
\{\sum_{i=1}^n (a \cdot u_i) \otimes 1_B \otimes \xi | a \in A, \xi \in \mathcal{K}\}
\]

is dense in \(\mathcal{H}_A\). Therefore, \(U\) is surjective. □

**Lemma 6.4.** With the above notation, we assume that \(\{u_i\}_{i=1}^n\) is a left \(A\)-basis in \(X\). Then

\[
\pi_\phi(c) = U^* \pi_A(c) U
\]

for any \(c \in A\).
Proof. Let $a, b, c \in A$, $\xi, \eta \in \mathcal{K}$, $\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \in \mathbb{C}^n$. Then

$$\langle U^* \pi_A(c)U(a \otimes \xi \otimes \lambda), b \otimes \eta \otimes \mu \rangle_{\mathcal{H}_\phi}$$

$$= \langle \pi_A(c) \sum_{i=1}^n \lambda_i(a \cdot u_i) \otimes 1_B \otimes \xi, \sum_{j=1}^n \mu_j(b \cdot u_j) \otimes 1_B \otimes \eta \rangle_{\mathcal{H}_A}$$

$$= \sum_{i,j=1}^n \langle \lambda_i(a \cdot u_i) \otimes 1_B \otimes \xi, \mu_j(b \cdot u_j) \otimes 1_B \otimes \eta \rangle_{\mathcal{H}_A}$$

$$= \sum_{i,j=1}^n \langle 1_B \otimes \xi, \pi_\phi(\langle \lambda_i(a \cdot u_i), \mu_j(b \cdot u_j) \rangle_B) \rangle_{\mathcal{K}_\phi}$$

$$= \sum_{i,j=1}^n \langle V_\psi \xi, \pi_\phi(\langle \lambda_i(a \cdot u_i), \mu_j(b \cdot u_j) \rangle_B) V_\psi \eta \rangle_{\mathcal{K}_\phi}$$

$$= \sum_{i,j=1}^n \langle \xi, \psi(\langle \lambda_i(a \cdot u_i), \mu_j(b \cdot u_j) \rangle_B) \eta \rangle_{\mathcal{K}}.$$ 

On the other hand,

$$\langle \pi_\phi(c)(a \otimes \xi \otimes \lambda), b \otimes \eta \otimes \mu \rangle_{\mathcal{H}_\phi}$$

$$= \langle \xi \otimes \lambda, \phi(a_*c^*b)(\eta \otimes \mu) \rangle_{\mathcal{K} \otimes \mathbb{C}^n}$$

$$= \langle \begin{bmatrix} \lambda_1 \xi \\ \vdots \\ \lambda_n \xi \end{bmatrix}, [\psi(\langle u_1, a^*b \cdot u_j \rangle_B)]_{i,j=1}^n \begin{bmatrix} \mu_1 \eta \\ \vdots \\ \mu_n \eta \end{bmatrix} \rangle_{\mathcal{K} \otimes \mathbb{C}^n}$$

$$= \langle \begin{bmatrix} \lambda_1 \xi \\ \vdots \\ \lambda_n \xi \end{bmatrix}, \begin{bmatrix} \sum_{j=1}^n \psi(\langle u_1, a^*b \cdot u_j \rangle_B) \mu_j \eta \\ \vdots \\ \sum_{j=1}^n \psi(\langle u_n, a^*b \cdot u_j \rangle_B) \mu_j \eta \end{bmatrix} \rangle_{\mathcal{K} \otimes \mathbb{C}^n}$$

$$= \sum_{i,j=1}^n \langle \xi, \psi(\langle \lambda_i u_i, a^*b \cdot u_j \rangle_B \mu_j \eta \rangle_{\mathcal{K}}.$$

Thus we obtain the conclusion. □

We give the main theorem in the paper.

**Theorem 6.5.** Let $A$ and $B$ be unital $C^*$-algebras, which are strongly Morita equivalent with respect to an $A - B$-equivalence bimodule $X$. Let $\{u_i\}_{i=1}^n$ be a left $A$-basis in $X$. Let $\psi$ be a completely positive linear map from $B$ to $B(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space. Let $\phi$ be a map from $A$ to $B(\mathcal{K} \otimes \mathbb{C}^n)$ defined by

$$[\phi(a)_{ij}]_{i,j=1}^n = [\psi((u_i, a \cdot u_j)_B)]_{i,j=1}^n \in B(\mathcal{K} \otimes \mathbb{C}^n),$$

for any $a \in A$. Then $\phi$ is a completely positive linear map from $A$ to $B(\mathcal{K} \otimes \mathbb{C}^n)$, which is strongly Morita equivalent to $\psi$.

**Proof.** By Lemma 5.1 we can see that $\phi$ is a completely positive linear map from $A$ to $B(\mathcal{K} \otimes \mathbb{C}^n)$. Also, by Lemmas 6.2, 6.3 and 6.4 and the definition of strong Morita equivalence for completely positive maps, we can see that $\phi$ is strongly Morita equivalent to $\psi$. □

**Corollary 6.6.** Let $A$ and $B$ be unital $C^*$-algebras, which are strongly Morita equivalent. Then there is a $1 - 1$ correspondence between the set of all strong...
Morita equivalence classes of completely positive linear maps on $A$ and the set of all strong Morita equivalence classes of completely positive linear maps on $B$.

Proof. This is immediate by Theorem 6.5. □

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