Combined efficiency margin in the implementation of the DEA method

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Abstract. DEA (Data Envelopment Analysis) method and its modification are considered in the paper. An inefficient frontier is suggested to be introduced. The intersection of the efficient frontier and inefficient frontier is researched. The intersection of both frontiers helps to construct a hyper plane to separate a sample in to two classes.

1. Introduction

In recent years an estimation problem of efficient operation of business arises in spheres of product manufacture and production distribution. Frequently appears the problem of comparison and ranking organization departments and firms or organizations on the whole by some characteristics that can’t be directly measured [1-3]. And a general idea of demonstration the degree of analyzing latent quality is formed as a result of particular measured characteristics this quality depends on. Beyond all question efficiency is the main concept. “Efficiency is a most general property of any purposeful activity that from the cognitive point of view uncovers itself by way of the target category and is objectively expressed by the degree of a goal achievement taking into account time and resources expenses” [1]. That’s why the efficiency estimation of business and organizations operation is very important for making right management decisions.

The given work concerns a DEA-method [4]. It is based on constructing an efficiency frontier. This frontier has a form of a convex hull. The efficiency frontier is used as a standard (or a peer) to obtain numerical values of efficiency measure of each object in the research system [4-6].

The paper introduces a notion «a loss frontier» by analogy with the efficiency frontier. The loss frontier is used as a peer to obtain the numerical values of the loss measure for each object. The intersection of two frontiers helps to construct a hyper plane to separate a sample into two classes.

2. DEA-method

The given method estimates the production function that in reality is unknown. A DEA-method is based on constructing the efficiency frontier that is an analogy of the production function in the case when the output is not a scalar but a vector, i.e. when there exist some kinds of output. This frontier has a form of a convex hall or a convex con in the space of input and output variables describing each decision-making units (DMU) in the system. As follows from method’s name, the efficiency frontier
envelopes or covers over a scatter of points in the multidimensional space. The efficiency frontier is used as a standard (or a peer) to obtain numerical value of efficiency measure of each object in the researching set. The efficiency degree for the i-th DMU is determined with degree of proximity to the efficiency frontier in the multidimensional space of inputs and outputs. The way of constructing an efficiency frontier is to solve the linear programming problem N times.

Let’s describe the main idea of the DEA-method on the example. A firm uses two inputs \((x_1 \text{ and } x_2)\) to produce a single output \((y)\) (refer to Figure 1) [7].

If we assume constant returns to scale, we can represent the technology by a unit production possibility curve on a two-dimensional plot. The axes reflect costs for a unit, i.e. the volume of resources \(x_1\) and \(x_2\) per a unit of the output. Consequently, we have a unit isoquant (refer to figure 1).

Let the given firm uses quantities of inputs, corresponding to the point P, to produce a unit of the output. Then the technical inefficiency of that firm could be represented by the segment QP [8]. A point Q is a point P projection on the efficiency frontier. At the same time the projection is made towards the coordinate origin. The segment QP is a number by which all inputs \((x_1 \text{ and } x_2)\) could be proportionally reduced without a reduction in the output \((y)\). This approach of the efficiency measurement is called input-oriented [9].

![Figure 1. Production technology with two inputs and a single output.](image)

The technical efficiency (TE) of the firm is measured as follows:

\[
TE_p = \frac{OQ}{OP}.
\]

In Figure 1 A, B, C and D are efficiency points. They form the efficiency frontier. A point P is not technically efficient because it doesn’t lie on the efficient isoquant.

If an object A can produce the definite volume of output from the definite volume of resources, an object B can also produce the same volume of output from the same volume of resources. That’s why it is right to project the points corresponding to inefficient objects on the efficiency frontier.

Figure 1 shows that the value of technical efficiency can’t exceed 1. Projecting the inefficient object on the efficiency frontier, a hypothetic target object is formed. It is to be efficient. In mathematical sense this hypothetic target object corresponds to a linear combination of the real efficient objects (in this case a real object is a point in multidimensional space). The number of objects included by this combination depends on a number of factors like the input and output variables describing the objects, and values of these variables. The input and output variables of the target object are targets for the inefficient objects.

Let’s describe an idea of DEA-method using an example. Assume there exist data for \(K\) inputs and \(M\) outputs for each of \(N\) objects or DMU’s. They could be firms, banks, universities etc. For the \(i\)-th
DMU they are represented by vectors $x_i$ and $y_i$, respectively. Then matrix $X$ of $K \times N$ dimension and matrix $Y$ of $M \times N$ dimension are matrices of input and output parameters for all the $i$-th DMU. The model is formulated as a linear programming problem:

$$\begin{align*}
\min_{\theta, \lambda}(\theta), \\
-y_i + Y \lambda &\geq 0, \\
\theta x_i - X \lambda &\geq 0, \\
\lambda &\geq 0,
\end{align*}$$

(1)

where $\theta$ is a scalar and $\lambda$ is a $N \times 1$ a vector of constants. The obtained value $\theta$ will be the efficiency score for the $i$-th DMU. It will satisfy $\theta \leq 1$. The same linear programming problem must be solved $N$ times, once for each DMU in the sample.

The presented model (1) is constructed under assumption of constant returns to scale and after its $N$ times solving the efficiency frontier is formed as a convex cone. The conical hull of the efficiency frontier is determined by the absence in the model of a constraint on the element sum of the vector $\lambda$ like $\sum_{i=1}^{N} \lambda_i = 1$.

Let’s explain the meaning of the vector $\lambda$. From convex analysis it’s known that each point belonging to the convex cone covered over the set of points can be represented as a nonnegative linear combination of these points $(X\lambda, Y\lambda)$. A part of vector $\lambda$ has nonzero values. These elements correspond to object that is a peer for examining objects. The linear combination of peer objects forms hypothetic object that is on the efficiency frontier and is a projection of a real inefficient object. In the case represented in Figure 1 the peers for the object $P$ are objects $B$ and $C$ that’s why $\lambda_B \neq 0$, $\lambda_C \neq 0$, $\lambda_A \neq 0$, $\lambda_D \neq 0$.

If the value of $\theta$ is equal to 1 that object lie on the efficiency frontier. If the value of $\theta$ is less than 1, then there can be given recommendations to such objects to move to the efficiency frontier. It means the input quantities can be proportionally reduced without changing the output quantities. That’s why this model is called an input-oriented model. The recommended input values are calculated by a formula:

$$x_{i, \text{recommend}} = \theta x_i,$$

where $\theta$ is a efficiency score for the $i$-th DMU, $x_i$ is input values for the $i$-th DMU.

Let’s present a similar model to (1) modified for variable returns to scale:

$$\begin{align*}
\min_{\theta, \lambda}(\theta), \\
-y_i + Y \lambda &\geq 0, \\
\theta x_i - X \lambda &\geq 0, \\
\sum_{i=1}^{N} \lambda_i = 1, \\
\lambda &\geq 0.
\end{align*}$$

(2)

Solving this problem the formed efficiency frontier has a shape of a convex hull because as a result of condition $\sum_{i=1}^{N} \lambda_i = 1$ the hypothetic objects $(X\lambda, Y\lambda)$ are equal to linear combination of efficient points. In Figure 2 both models are shown for the same system of DMU.
3. Loss frontier and set separation into two classes

Let’s take a model of a DEA-method for variable returns to scale (VRS) situation. The efficiency frontier envelopes data points corresponding to efficient objects. Now let’s build the same frontier, but vice versa. It envelopes the data points inside out. It can be called an inefficiency/loss frontier. The loss frontier shows bad or loser points of other data points.

Reasoning by analogy with [7] for each DMU we take a measure of the ratio of all inputs over all outputs, such as $\frac{v'x_i}{u'y_j}$, where $u$ is an $M \times 1$ vector of outputs weights, $v$ is a $K \times 1$ vector of input weights. To select optimal weights we specify a mathematical programming problem:

$$\max_{u,v} \left( \frac{v'x_i}{u'y_j} \right),$$

$$\frac{v'x_j}{u'y_j} \leq 1, \quad j = 1,2,...,N,$$

$$u,v \geq 0.$$  

This involves finding values for $u$ and $v$, such that the efficiency measure of the $i$-th DMU is maximized, subject to the constraint that all efficiency measures must be less than or equal to one. A problem with this particular ratio formulation is that it has an infinite number of solutions. To avoid this one can impose the constraint $u'y_j = 1$, which provides:

$$\max_{\mu,v} (v'x_i),$$

$$\mu'y_j = 1,$$

$$v'x_j - \mu'y_j \leq 0, \quad j = 1,2,...,N,$$

$$\mu,v \geq 0.$$  

Let’s find a dual of the problem:
Let’s add a constraint for the element sum of the vector $\lambda$. Then a model of loss frontier is

$$\min_{\theta, \lambda}(\theta),$$

$$\theta y_i - Y \lambda \geq 0,$$

$$-x_i - X \lambda \geq 0,$$

$$\lambda \geq 0,$$

$$\sum_{i=1}^{N} \lambda_i = 1,$$

$$\lambda \geq 0,$$

where $\theta$ is a measure of loser objects. If $\theta = 1$, DMU forms the loss frontier and it is a loser object. The index $\theta$ can vary from 0 to 1.

In Figure 3 an efficiency frontier and a loss frontier are shown.

Figure 3. Efficiency and loss frontier.

Figure 3 shows two frontiers intersecting in two points. A line can join both points of intersections and divide DMU set into two classes. The first class contains efficient and normal DMU and the second class contains loser DMU. These two classes are shown in Figure 4.
Figure 4. The line divide DMU system; two classes of objects are got.

If we extend this model for a multivariate case (some inputs and some outputs), a hyper plane will be a separating surface.

4. Conclusion
In the given work a DEA-method is concerned; a loss frontier is offered and constructed. Also a separation of the system into two classes is offered.

The suggested approach lets the decision maker classify DMU easier because the existence only of two classes (bad and good) is assumed. It is convenient when the set of DMU is large. If a decision maker needs an exact numerical value of efficiency, he gets from the direct decision of the problem with a DEA-method.

References
[1] Morgunov E P 2003 Multivariate classification based on DEA-method (Krasnoyarsk: NII SUVPT)
[2] Seiford L M and Thrall R M 1990 Recent Developments in DEA: The Mathematical Programming Approach to Frontier Analysis Journal of Econometrics 46 7-38
[3] Hoque R and Rayhan I 2012 Data Envelopment Analysis of Banking Sector in Bangladesh Russian Journal of Agricultural and Socio-Economic Sciences 5(5) 17-22
[4] Banker R D, Charnes A B and Cooper W W 1984 Some Models for Estimating Technical and Scale Efficiencies in Data Envelopment Analysis Management Science 30(9) 1078-92
[5] Kovalev I et al 2015 IOP Conf. Ser.: Mater. Sci. Eng. 70 012007
[6] Kovalev I et al 2015 The Efficiency Analysis of Automated Lines of Companies Based on DEA Method Lecture Notes in Economics and Mathematical Systems 675 107-15
[7] Coelli T et al 1998 An introduction to efficiency and productivity analysis (Boston: Kluwer Academic Publishers)
[8] Belton V and Vickers S P 1993 Demystifying DEA - a visual interactive approach based on multiple criteria analysis Journal of the Operational Research Society 44(9) 883-96
[9] Fu Ya, Liu L, Hu R and Jialin C 2007 Modified Interval DEA Method for Economic Evaluation of Distribution Network, Proceedings of Second International Conference on Innovative Computing, Informatio and Control (ICICIC 2007)