A more efficient way of finding Hamiltonian cycle

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August 25, 2022

1 Introduction

Algorithm tests if a Hamiltonian cycle exists in directed and undirected graphs, if it exists - algorithm can often show found Hamiltonian cycle. If you want to test an undirected graph, such a graph should be converted to the form of directed graph. Algorithm’s goal is to solve NP-complete problems of Directed Hamiltonian cycle and Undirected Hamiltonian cycle in polynomial time. Previously known algorithm solving Hamiltonian cycle problem - brute-force search can’t handle relatively small graphs. Algorithm presented here is referred to simply as "algorithm" in this paper.

Why algorithm is more efficient than brute-force search?

In order to find Hamiltonian cycle, algorithm should find edges that creates a Hamiltonian cycle. Higher number of edges creates more possibilities to check to solve the problem. Both brute-force search and algorithm use recursive depth-first search. The reason why brute-force search often fails to solve this problem in reasonable amount of time is too large number of possibilities to check in order to solve the problem.

Algorithm rests on analysis of original graph and opposite graph to it. Algorithm prefers "to think over" which paths should be checked than check many wrong paths. Algorithm is more efficient than brute-force search because it can:

- remove unnecessary edges from graph
- test when Hamiltonian cycle can’t exist
- choose most optimal path

2 Definitions

1. **Graph** - set of vertices \( v_1, v_2, \ldots, v_N \)
   and edges \( (a_1, a_2), (a_3, a_4), \ldots, (a_{M-1}, a_M) \) where \( \{a_1, a_2, \ldots, a_M\} \in \{v_1, v_2, \ldots, v_N\} \)

2. **Path** - set of edges \( (v_1, v_2), (v_2, v_3), \ldots, (v_{K-1}, v_K) \) where \( v_1 \neq v_2 \neq \ldots \neq v_K \) and graph contains edges \( (v_1, v_2), (v_2, v_3), \ldots, (v_{K-1}, v_K) \)

3. **Cycle** - Path from graph which contains an edge \( (v_K, v_1) \) where \( v_K \) is last vertex in cycle and \( v_1 \) is first vertex in cycle.

4. **Hamiltonian cycle** - cycle of length equal to the number of vertices in graph

5. **Hamiltonian graph** - graph which contains Hamiltonian cycle

6. **Vertex degree** - Vertex \( v_1 \) has degree equal to \( W \) if graph contains following \( W \) edges \( (v_1, b_1), (v_1, b_2), \ldots, (v_1, b_W) \)
7. **Opposite graph** - Graph $G$ is set of vertices $v_1, v_2, \ldots, v_N$ and edges $(a_1, a_2), (a_3, a_4), \ldots, (a_{M-1}, a_M)$.

Graph opposite to $G$ is set of vertices $v_1, v_2, \ldots, v_N$ and edges $(a_2, a_1), (a_4, a_3), \ldots, (a_M, a_{M-1})$

Example:

![Figure 1: Original graph](image)

![Figure 2: Opposite graph](image)

3 **No loops and multiple edges**

Algorithm removes multiple edges and loops from graph because they are irrelevant in process of finding Hamiltonian cycle.

4 **Unique neighbours**

Problem of finding Hamiltonian cycle in graph with $N$ vertices can be described as a problem of finding following edges:

\[
\begin{align*}
    a_1 &\rightarrow x_1 \\
    a_2 &\rightarrow x_2 \\
    &\vdots \\
    a_N &\rightarrow x_N
\end{align*}
\]

where $x_1, x_2, \ldots, x_N \in \{a_1, a_2, \ldots, a_N\}$; $x_1 \neq x_2 \neq \ldots \neq x_N$ and edges $a_1 \rightarrow x_1, a_2 \rightarrow x_2, \ldots, a_N \rightarrow x_N$ don’t create a cycle that is not Hamiltonian.

For graph with $N$ vertices algorithm should find $N$ unique neighbours.

Let’s consider example graph presented with adjacency list:

1 : 3, 4
2 : 3, 4
3 : 2, 4
4 : 1, 3, 5
5 : 2, 3
6 : 1
Let’s test which neighbours have vertices 1 and 2. They are: 3, 4, 3 and 4. 
Let’s test how many and which unique neighbours are on the list. 
There are 2 vertices: 3 and 4.

When for \( M \) tested vertices the number of unique neighbours equals \( M \) then algorithm 
can remove these unique neighbours from vertices, that were not tested. \( M \) is lower than \( N \).

Graph after removal of unnecessary edges presented with adjacency list:

1 : 3, 4  
2 : 3, 4  
3 : 2, 4  
4 : 1, 3, 5  
5 : 2, 3  
6 : 1

”Unique neighbours” test is used for original graph and opposite graph.

Let’s consider another example graph presented with adjacency list:

1 : 2, 4  
2 : 1, 2  
3 : 2  
4 : 1, 4, 5  
5 : 2, 3  
6 : 1, 4

Let’s test which neighbours vertex 3 has. It’s only 2. 
Let’s test how many and which unique neighbours are on the list. There is only one vertex: 2. 
Graph after removal of unnecessary edges presented with adjacency list:

1 : 2, 4  
2 : 1, 2  
3 : 2  
4 : 1, 4, 5  
5 : 2, 3  
6 : 1, 4

4.1 \hspace{1pt} Method of accomplishing ”unique neighbours” test

For graph \( G \), for every vertex \( V \) in \( G \) algorithm takes its adjacency list and checks how many and which 
of others adjacency lists are subsets of \( V \)’s adjacency list. Let’s consider example graph presented with 
adjacency list:

1 : 4, 8  
2 : 1, 3, 5, 7  
3 : 4, 5, 6  
4 : 1, 3, 7  
5 : 1, 4, 8  
6 : 1, 3, 4, 5  
7 : 3, 5  
8 : 5, 7

To analyze vertex 2 with ”unique neighbours” test algorithm takes its adjacency list - 1, 3, 5, 7 and checks 
how many and which of others adjacency lists are subsets of 2’s adjacency list, they are:

4 : 1, 3, 7  
7 : 3, 5  
8 : 5, 7

Algorithm can remove edges marked with red color:

1 : 4, 8  
2 : 1, 3, 5, 7  
5 : 1, 4, 8  
6 : 1, 3, 4, 5
4.2 Add edges to remove edges

Let's consider example graph presented with adjacency list:

1 : 5, 6, 7  5 : 2, 7
2 : 1, 4  6 : 3, 2, 5
3 : 2, 5, 7  7 : 1, 4, 5
4 : 2, 5, 6

When algorithm tests graph with method 4.1 it will not find any edges that it could remove. However with addition of edge $1 \rightarrow 2$ algorithm can remove edges marked with red color:

1 : 2, 5, 6, 7  5 : 2, 7
2 : 1, 4  6 : 3, 2, 5
3 : 2, 5, 7  7 : 1, 4, 5
4 : 2, 5, 6

Temporary addition of new edges may allow algorithm to remove some edges.

4.2.1 Which edges should be added?

Let's consider graph with $N$ vertices: $x_1, x_2, \ldots, x_N$. To test which edges should be added while testing vertex $x_1$ with method 4.1 algorithm compares adjacency list of $x_1$ with adjacency lists of $x_2, x_3, \ldots, x_N$:

If lists $x_1$ and $x_2$ have at least one common vertex then added edges are these that are on the list $x_2$ but that are not on list $x_1$.

If lists $x_1$ and $x_3$ have ... 

If lists $x_1$ and $x_N$ have ...

Let's consider example graph presented with adjacency list:

1 : 2, 5, 6, 7  5 : 2, 4, 7, 8
2 : 3, 5, 7  6 : 3, 4, 5, 7, 8
3 : 2, 5, 7  7 : 2, 5, 8
4 : 1, 2, 7, 8  8 : 1, 3, 5

Added edges(edges marked with red color are not tested):

|   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 |     | 1 → 3 |     | 1 → 8 | 1 → 4 | 1 → 8 | 1 → 8 | 1 → 3 |
| 2 | 2 → 8 |     |     |     |     |     |     |     |
| 3 | 3 → 8 | 3 → 1 |     |     |     |     |     |     |
| 4 | 4 → 5 | 4 → 3 | 4 → 5 |     |     |     |     |     |
| 5 |     |     |     |     |     |     |     |     |
| 6 | 6 → 1 | 6 → 2 | 6 → 1 | 6 → 2 | 6 → 2 | 6 → 1 |     |     |
| 7 |     |     |     |     |     |     | 7 → 1 |     |
| 8 |     |     |     |     |     |     | 7 → 3 |     |

4
4.2.2 Combination of added edges

Let's consider example graph presented with adjacency list:

Let's consider example graph presented with adjacency list:

| Nodes | Adjacency List |
|-------|----------------|
| 1     | 4, 5, 7, 8, 9  |
| 2     | 6, 9, 13, 14  |
| 3     | 5, 8, 9, 15   |
| 4     | 1, 13, 14     |
| 5     | 1, 3, 12, 15  |
| 6     | 2, 7, 9, 10, 13, 14 |
| 7     | 1, 6, 12, 13, 14 |
| 8     | 1, 3, 9, 12, 13 |
| 9     | 1, 2, 3, 6, 8  |
| 10    | 6, 13, 14     |
| 11    | 1, 3, 6, 9, 12, 13, 14 |
| 12    | 2, 4, 6, 7, 8, 10, 15 |
| 13    | 2, 4, 6, 7, 10, 11 |
| 14    | 2, 4, 6, 7, 10, 11 |
| 15    | 3, 5, 6, 13   |

Algorithm can temporary add a combination of added edges. Algorithm can temporary add following set of edges: 11 → 1, 11 → 3, 11 → 6, 11 → 9, 11 → 13 which allows algorithm to remove edges marked with red color:

How to select combination of added edges:

Algorithm tries to select added edges in "greedy" way. It tries to reduce difference between:
1) A's degree with addition of number of added edges
2) amount of vertices which adjacency list is subset of A's adjacency list with added edges
to 0 - to remove edges and below 0 - to determine that graph is not Hamiltonian.

All possible subsets that can be added to 11 in example graph are:

4, 5, 7, 8, 9 for 1
6, 9, 13 for 2
5, 8, 9, 15 for 3
1, 13 for 4
1, 3, 15 for 5
2, 7, 9, 10, 13, 16 for 6
1, 6, 13 for 7
1, 3, 9, 13 for 8
1, 2, 3, 6, 8 for 9
6, 13 for 10
5, 7, 8, 11 for 12
2, 4, 6, 7, 8, 10, 15 for 13
2, 4, 6, 7, 10, 11 for 14
3, 5, 6, 13 for 15

Out of these subsets algorithm selects following combination: 1, 3, 6, 9, 13.

4.3 Not enough unique neighbours

Graph is not Hamiltonian when number of unique neighbours checked for any M vertices is lower than M.

Example presented with adjacency list:

1 : 4, 5
2 : 4, 5
3 : 4, 5
4 : 1, 2
5 : 1, 3
Another example presented with adjacency list:
1 : 1, 4, 5
2 :
3 : 1, 4
4 : 2, 3
5 : 1, 2

4.4 Looped neighbours
Graph is not Hamiltonian when:
1) for M tested vertices number of unique neighbours equals M
2) subsets of: a) M tested vertices and b) theirs unique neighbours, are equal.
because in such a case, each possible assignment of edges in M vertices, forms a cycle that is not a Hamiltonian cycle.

Example graph presented with adjacency list:
1 : 5, 6
2 : 3, 4
3 : 6
4 : 2, 3
5 : 1, 6
6 : 1, 5

4.4.1 Almost looped neighbours
When:
1) for M tested vertices number of unique neighbours equals M + 1 and
2) M tested vertices contains only one unique neighbour that was not in M tested vertices
then this one unique neighbour that was not in M tested vertices, must be in Hamiltonian cycle, if Hamiltonian cycle exists in graph. Therefore algorithm can delete edges other than this one unique neighbor.

Example graph presented with adjacency list:
1 : 4, 5, 6, 7
2 : 3, 4
3 : 1
4 : 1, 2
5 : 4, 6, 7
6 : 5, 7
7 : 5, 6

4.5 ”Looped neighbours” effect
To analyze vertex x with ”unique neighbours” test algorithm takes its adjacency list and checks how many and which of others adjacency lists are subsets of x’s adjacency list. Algorithm also checks how many and which of others adjacency lists can be subsets of x’s adjacency list - rest of it’s neighbours are not on adjacency list of x. Then algorithm only takes under consideration those neighbours which are not on adjacency list of x - if they create effect of ”looped neighbours”, than one of those vertices will not be in Hamiltonian cycle, meaning this vertex will definitly be a subset of x’s adjacency list. Because of that algorithm can remove edges or determine that graph is not Hamiltonian.

Example 1 - graph presented with adjacency list:
1 : 2, 3, 7
2 : 1, 3, 8
To analyze vertex 3 with "unique neighbours" test algorithm takes its adjacency list and checks how many and which of others adjacency lists are subsets of 3’s adjacency list. They are: 5 and 6. Then algorithm checks how many and which of others adjacency lists are can be subsets of 3’s adjacency list - they are 4 and 7. Than algorithm checks if 4 and 7 create an effect of "looped neighbours" - it is true. In summary, there are following subsets of 3’s adjacency list:
1. 3’s adjacency list.
2. 5’s adjacency list.
3. 6’s adjacency list.
4. 4’s or 7’s adjacency list.
therefore algorithm can remove edges: 1 → 2, 2 → 8, 8 → 6.

Example 2 - graph presented with adjacency list:

There are following subsets of 6’s adjacency list:
1. 6’s adjacency list.
2. 9’s adjacency list.
3. 13’s adjacency list.
4. 2’s or 5’s or 7’s adjacency list.
5. 3’s or 8’s or 10’s or 11’s adjacency list.
therefore algorithm can remove edges: 1 → 12, 1 → 15, 12 → 1, 15 → 1.

4.6 Recursion effect - ”unique neighbours among unique neighbours”

In addition to the above-described method of finding which adjacency lists can be subsets of x’s adjacency list, algorithm also recursively uses the ”unique neighbors” principle.
Let’s consider following graph presented with adjacency list:

1: 2, 3, 5, 7
2: 5, 7
3: 4, 5
4: 3, 5
5: 2, 3, 8
6: 5, 7, 8
Thanks to ”unique neighbors” principle, algorithm knows that if Hamiltonian cycle exists in graph, only one of following statements is true:

- Hamiltonian cycle contains edge $5 \rightarrow 8$
- Hamiltonian cycle contains edge $6 \rightarrow 8$
- Hamiltonian cycle doesn’t contain both $5 \rightarrow 8$ and $6 \rightarrow 8$ edges

That’s why, there are following subsets of 1’s adjacency list:
1. 1’s adjacency list.
2. 2’s adjacency list.
3. 3’s or 4’s adjacency list, because of ”looped neighbours” effect.
4. 5’s or 6’s adjacency list, because of ”unique neighbours among unique neighbours” effect.

Therefore algorithm can remove edges: $7 \rightarrow 4, 7 \rightarrow 5, 8 \rightarrow 4, 8 \rightarrow 5$.

## 5 Single edge in only one direction

When in original graph or in opposite graph the only one neighbour of vertex $X$ is vertex $Y$, it means that for Hamiltonian cycle to exist edge $X \rightarrow Y$ must be in this cycle, when graph also contains edge $Y \rightarrow X$ then algorithm can remove edge $Y \rightarrow X$.

Example graph presented with adjacency list:

1: 2, 3, 4
2: 3
3: 2, 4
4: 1

Algorithm can remove edge 3 → 2.

### 5.1 Path that must be in Hamiltonian cycle

When in original graph or in opposite graph:

- the only one neighbour of vertex $A$ is vertex $B$, when graph also contains edge $B \rightarrow A$ then algorithm can remove edge $B \rightarrow A$
- when also the only one neighbour of vertex $B$ is vertex $C$, when graph also contains edge $C \rightarrow A$ then algorithm can remove edge $C \rightarrow A$
  
  ...
- when also the only one neighbour of vertex $Y$ is vertex $Z$, when graph also contains edge $Z \rightarrow A$ then algorithm can remove edge $Z \rightarrow A$

In summary, if path $A \rightarrow B \rightarrow C \cdots \rightarrow Z$ must be in Hamiltonian cycle than algorithm can remove edges:

- $B \rightarrow A$, if it exists in graph
- $C \rightarrow A$, if it exists in graph
  
  ...
- $Z \rightarrow A$, if it exists in graph
Example 1 - graph presented with adjacency list:
1 : 5
2 : 6, 7
3 : 2
4 : 1
5 : 3, 4, 6
6 : 1, 4, 7
7 : 3
If graph contains Hamiltonian cycle, than path 7 → 3 → 2 must be in it, therefore algorithm can remove edge 2 → 7.

Example 2 - graph presented with adjacency list:
1 : 2
2 : 4, 6
3 : 4, 5, 7
3 : 1
4 : 3
5 : 4, 6, 7
6 : 4, 5, 7
7 : 4, 5, 6
If graph contains Hamiltonian cycle, than path 4 → 3 → 1 → 2 must be in it, therefore algorithm can remove edges 3 → 4 and 2 → 4.

6  Cycle that is not Hamiltonian

Graph is not Hamiltonian when:

6.1 Vertices with 1 neighbour

In original graph or in opposite graph among vertices that have 1 neighbour exists cycle, that is not Hamiltonian. Example graph:

![Figure 3: Cycle created by vertices: 2, 3 and 4](image)
6.2 Vertices with more than 1 neighbour

Let’s consider graph with \( N \) vertices. Graph contains vertex \( X \) that has \( M \) neighbours, \( 1 < M < N/2 \), \( X \)’s list of in-neighbours is a superset of \( X \)’s list of out-neighbours. Graph also contains at least \( M - 1 \) vertices which their list of in-neighbours are supersets of out-neighbours and list of in-neighbours which is a subset of list of neighbours of vertex \( X \).

Example 1. presented with adjacency list:
Original and opposite graph:
X: A B
Y: A B

There are only 2 paths that can be created, they are:

- \( A \rightarrow X \rightarrow B \rightarrow Y \rightarrow A \)
- \( B \rightarrow X \rightarrow A \rightarrow Y \rightarrow B \)

Every possible path creates cycle that is not Hamiltonian.

Example 2. presented with adjacency list:
Original and opposite graph:
X: A B C
Y: A B C
Z: A B C

There are 12 paths that can be created, they are:

- \( A \rightarrow X \rightarrow B \rightarrow Y \rightarrow C \rightarrow Z \rightarrow A \)
- \( A \rightarrow X \rightarrow B \rightarrow Z \rightarrow C \rightarrow Y \rightarrow A \)
- \( A \rightarrow X \rightarrow C \rightarrow Y \rightarrow B \rightarrow Z \rightarrow A \)
- \( A \rightarrow X \rightarrow C \rightarrow Z \rightarrow B \rightarrow Y \rightarrow A \)
- \( B \rightarrow X \rightarrow A \rightarrow Y \rightarrow C \rightarrow Z \rightarrow B \)
- \( B \rightarrow X \rightarrow A \rightarrow Z \rightarrow C \rightarrow Y \rightarrow B \)
- \( B \rightarrow X \rightarrow C \rightarrow Y \rightarrow A \rightarrow Z \rightarrow B \)
- \( B \rightarrow X \rightarrow C \rightarrow Z \rightarrow A \rightarrow Y \rightarrow B \)
- \( C \rightarrow X \rightarrow A \rightarrow Y \rightarrow B \rightarrow Z \rightarrow C \)
- \( C \rightarrow X \rightarrow A \rightarrow Z \rightarrow B \rightarrow Y \rightarrow C \)
- \( C \rightarrow X \rightarrow B \rightarrow Y \rightarrow A \rightarrow Z \rightarrow C \)
- \( C \rightarrow X \rightarrow B \rightarrow Z \rightarrow A \rightarrow Y \rightarrow C \)

Every possible path creates cycle that is not Hamiltonian.

Example 3. presented with adjacency list:
Original and opposite graph:
X: A B C
Y: A B
Z: A C

There are 2 paths that can be created, they are:

- $B \rightarrow X \rightarrow C \rightarrow Z \rightarrow A \rightarrow Y \rightarrow B$
- $C \rightarrow X \rightarrow B \rightarrow Y \rightarrow A \rightarrow Z \rightarrow C$

Every possible path creates cycle that is not Hamiltonian.

7 Closed alleys

If in the graph any of the vertices has a neighbour, the selection of which creates a path that is not a Hamiltonian cycle, and which creates a path that ends back at the same vertex, the edge with that neighbour can be deleted.

Example presented with adjacency list:

1 : 7
2 : 4, 5, 6
3 : 2
4 : 1
5 : 3, 4, 6
6 : 3, 4
7 : 3, 4, 5, 6

If edge $7 \rightarrow 4$ would be in Hamiltonian cycle, it would create a path: $7 \rightarrow 4 \rightarrow 1 \rightarrow 7$, such a path would create a cycle that is not Hamiltonian, therefore algorithm can remove edge $7 \rightarrow 4$.

8 Graph is disconnected

Let’s consider graph with $N$ vertices: $a_1, a_2, \ldots, a_N$. If a path doesn’t exist between vertices: $a_1$ to $a_2$ or $a_1$ to $a_3$ ... or $a_1$ to $a_N$, then graph is not Hamiltonian.

Figure 4: A path doesn’t exist between vertices 1 and 5
9 Never only 1 Hamiltonian cycle

In undirected graph for each edge \( a \rightarrow b \) there is an edge \( b \rightarrow a \), which means that if there is a Hamiltonian cycle in an undirected graph: \((v_1, v_2), (v_2, v_3), \ldots, (v_{K-1}, v_K)\) the following Hamiltonian cycle must also exist in this graph: \((v_K, v_{K-1}), \ldots, (v_3, v_2)(v_2, v_1)\). This means that if in an undirected graph, which contains Hamiltonian cycle, there is a vertex that has exactly 2 degree, then both edges coming from this vertex must be in Hamiltonian cycle.

9.1 Too low degree

Graph is not Hamiltonian when in an undirected graph there is any vertex that has degree lower than 2.

Example - graph presented with adjacency list:

\begin{align*}
0 : & 2, 3 \\
1 : & 2 \\
2 : & 0, 1, 4 \\
3 : & 0, 4 \\
4 : & 2, 3 \\
\end{align*}

9.2 Too many similar edges

If among above-described necessary edges for the existence of a Hamiltonian cycles, there are more than two edges with the same out-neigbour, it means that graph is not Hamiltonian.

Example - graph presented with adjacency list:

\begin{align*}
0 : & 5, 6 \\
1 : & 3, 4, 6 \\
2 : & 4, 5 \\
3 : & 1, 5 \\
4 : & 1, 2, 6 \\
5 : & 0, 2, 3 \\
6 : & 0, 1, 4 \\
\end{align*}

It is impossible for Hamiltonian cycles to contain all of the following edges simultaneously: \(0 \rightarrow 5, 2 \rightarrow 5\) and \(3 \rightarrow 5\).

9.3 Remove edges

If graph has a Hamiltonian cycle and graph has a vertex \( x \), which has exactly 2 degree, \( x : a, b \), it means that the graphs have at least two Hamiltonian cycles containing edges: \(x \rightarrow a, x \rightarrow b, a \rightarrow x, b \rightarrow x\). Based on this observation, algorithm can remove edges, assuming that the above-indicated edges must be in the Hamiltonian cycle.

Example - graph presented with adjacency list:

\begin{align*}
0 : & 4, 5, 6 \\
1 : & 3, 4, 5 \\
2 : & 3, 4, 6 \\
3 : & 1, 2, 4 \\
4 : & 0, 1, 2, 3 \\
5 : & 0, 1 \\
6 : & 0, 2 \\
\end{align*}

If above graph has Hamiltonian cycles, then these cycles contain edges \(5 \rightarrow 0, 5 \rightarrow 1, 6 \rightarrow 0\) and \(6 \rightarrow 1\). Since these cycles contain edges: \(5 \rightarrow 0\) and \(6 \rightarrow 0\), then edge \(4 \rightarrow 0\) can be deleted from the graph, and due to fact that the graph is undirected, then edge \(0 \rightarrow 4\) can also be deleted.
9.4 Edges constantly removed

If undirected graph has Hamiltonian cycle, and this graph has a vertex $x$ with a degree greater than 2 - $x : a, b, \ldots, z$ where only $a$ has a degree 2, it means that possibly existing Hamiltonian cycles in this graph, will definitely contains an edge $x \to a$ and one of the following edges:

$x \to b$

\ldots

$x \to z$

Example - graph presented with adjacency list:



| Vertices | Neighbours |
|----------|------------|
| 0        | 2, 3       |
| 1        | 4, 5       |
| 2        | 0, 6       |
| 3        | 0, 4, 6    |
| 4        | 1, 3, 5    |
| 5        | 1, 4, 6    |
| 6        | 2, 3, 5    |

If above graph has Hamiltonian cycle, it definitely contains an edge $0 \to 3$ and one of the following edges: $3 \to 4$ or $4 \to 6$.

Based on this observation, algorithm can remove edges, with methods described in 10.

10 Edges constantly removed

10.1 Remove edge

Algorithm performs following test for every edge in graph: if after removal of a single edge in graph analysis of graph proves that graph is not Hamiltonian, then for a Hamiltonian cycle to exist in graph, such an edge must be in it.

Therefore if vertex $X$’s adjacency list consists of $A, B, C, \ldots Z$ and edge $X \to A$ is a "neccessary edge" then algorithm can remove edges: $X \to B$, $X \to C$, \ldots $X \to Z$.

Example presented with adjacency list:



| Vertices | Neighbours |
|----------|------------|
| 1        | 2, 4, 5, 6 |
| 2        | 3, 4, 6, 8 |
| 3        | 4, 6, 7, 8 |
| 4        | 1, 6, 8    |
| 5        | 1, 2, 4, 6 |
| 6        | 1, 4, 8    |
| 7        | 1, 2       |
| 8        | 4, 5, 6, 7 |

After removal of edge $2 \to 3$ analysis of graph proves that graph is not Hamiltonian, therefore algorithm can remove edges: $2 \to 4$, $2 \to 6$, $2 \to 8$.

10.2 Select edge

Graph with $N$ vertices: $a_1, a_2, \ldots, a_N$. For Hamiltonian cycle to exist each of these vertices must contain an edge that is in this cycle. Vertex $V$ has $M$ neighbours: $b_1, b_2, \ldots, b_M; M \geq 2; b_1, b_2, \ldots, b_M \in \{a_1, a_2, \ldots, a_N\}$. After removal of certain edges, algorithm may be able to remove more unnessary edges. Following tests are made for every vertex that has more than 1 neigbour, tests are based on an assumption that chosen edge must be in Hamiltonian cycle:

- Chosen edge: $V \to b_1$. Test which edges will be removed after removal of edges: $V \to b_2, V \to b_3, \ldots, V \to b_M$, all removed edges are saved on list $l_1$
Figure 5: $I \rightarrow R$ is an edge constantly removed

- Chosen edge: $V \rightarrow b_2$. Test which edges will be removed after removal of edges: $V \rightarrow b_1, V \rightarrow b_3, \ldots, V \rightarrow b_M$, all removed edges are saved on list $l_2$

- ...  

- Chosen edge: $V \rightarrow b_M$. Test which edges will be removed after removal of edges: $V \rightarrow b_1, V \rightarrow b_2, \ldots, V \rightarrow b_{M-1}$, all removed edges are saved on list $l_M$

If test of following assumption: edge $V \rightarrow b_x$, where $x \in \{1, 2, \ldots, M\}$, must be in Hamiltonian cycle, proves that Hamiltonian cycle can’t exist if edge $V \rightarrow a_x$ is in it, then algorithm can remove edge $V \rightarrow b_x$ and doesn’t save removed edges on list $l_x$. Example: edge $V \rightarrow B$ shown in Figure 5.

When for all of the edges: $V \rightarrow b_1, V \rightarrow b_2, \ldots, V \rightarrow b_M$ test will prove that Hamiltonian cycle can’t exist if chosen edge is in it, then graph does not contain Hamiltonian cycle.

Edges that are on all of the lists: $l_1, l_2, l_3, \ldots, l_M$ can be removed from graph because for Hamiltonian cycle to exist at least one of the tested edges must be chosen and regardless which edge will it be, the edges that are on all of the lists will be constantly removed. Example: edge $I \rightarrow R$ shown in Figure 5.

10.3 Compare removed edges

For each edge in the graph, the following statement is true: if there is a Hamiltonian cycle in the graph, then the edge will either be in it or not. So the algorithm checks for each edge $A \rightarrow B$: if there are any edges that will be deleted, regardless of whether the edge $A \rightarrow B$ will be in the Hamiltonian cycle or not, then these edges can be deleted from the graph.
11 Connected edges

If in undirected graph vertex $x$ has 2 degree - $X : A, B$ and graph has a Hamiltonian cycle, it means that:

1. Edges $X \rightarrow A$ and $X \rightarrow B$ are parts of Hamiltonian cycles
2. Edge $X \rightarrow A$ and edge $X \rightarrow B$ are not part of the same Hamiltonian cycle - they are in separate Hamiltonian cycles

In any undirected graph having a Hamiltonian cycle, there are at least 2 Hamiltonian cycles. In order to conclude that a graph has a Hamiltonian cycle, it is enough to find only 1 Hamiltonian cycle, it is not necessary to find a second Hamiltonian cycle, nor all of the Hamiltonian cycles in graph.

Edges can be "connected" by establishing which edges must exist simultaneously in the same Hamiltonian cycle. The attempt to connect the edges only occurs between the edges coming out of the vertices of degree 2. Let’s consider following graph:

```
A : X_1, X_2
B : X_3, X_4
...  
C : X_{N-1}, X_N
Y : X_1, X_2, X_3,...
Z : X_1, X_2, X_3,...
... 
```

Algorithm will try to connect edges: $A \rightarrow X_1$, $A \rightarrow X_2$, $B \rightarrow X_3$, $B \rightarrow X_4$, ..., $C \rightarrow X_{N-1}$ and $C \rightarrow X_N$

Consider the edges $A \rightarrow X_1$, $A \rightarrow X_2$, $B \rightarrow X_3$ and $B \rightarrow X_4$:

1. If, as a result of analysis, algorithm finds that $A \rightarrow X_1$ and $B \rightarrow X_3$ edges cannot exist simultaneously in the same Hamiltonian cycle, it means that the $A \rightarrow X_1$ and $B \rightarrow X_4$ edges can be connected.
2. If ... $A \rightarrow X_1$ and $B \rightarrow X_4$ ... it means ... $A \rightarrow X_1$ and $B \rightarrow X_3$ edges can be connected.
3. If ... $A \rightarrow X_2$ and $B \rightarrow X_3$ ... it means ... $A \rightarrow X_2$ and $B \rightarrow X_4$ edges can be connected.
4. If ... $A \rightarrow X_2$ and $B \rightarrow X_4$ ... it means ... $A \rightarrow X_2$ and $B \rightarrow X_3$ edges can be connected.

Algorithm tries to connect as many edges as possible, knowing that if graph has a Hamiltonian cycle, it would connect edges that are in Hamiltonian cycle.

Example of connected edges:
```
0 : 3, 4, 5
1 : 2, 4
2 : 1, 6
3 : 0, 4, 5
4 : 0, 1, 3, 5, 6
5 : 0, 3, 4, 7
6 : 2, 4, 7
7 : 5, 6
```

12 Skipping search

For some graphs, algorithm is able to determine that graph is Hamiltonian based on:
12.1 Bondy-Chvátal theorem

"The Bondy–Chvátal theorem operates on the closure cl(G) of a graph G with n vertices, obtained by repeatedly adding a new edge uv connecting a nonadjacent pair of vertices u and v with \( \deg(v) + \deg(u) \geq n \) until no more pairs with this property can be found. ”

Bondy–Chvátal theorem

"A graph is Hamiltonian if and only if its closure is Hamiltonian.” [1].

12.2 Dirac’s theorem

"A simple graph with n vertices (\( n \geq 3 \)) is Hamiltonian if every vertex has degree \( \frac{n}{2} \) or greater.” [1].

12.3 Adding edges

Let’s consider graph G with M edges. If G is Hamiltonian, it will always remain Hamiltonian, regardless of how many edges could be added to G.

12.4 Directed graphs

Bondy-Chvátal theorem and Dirac’s theorem apply to undirected graphs. These theorems can be applied to directed graphs, but only if those graphs are changed into their undirected versions, by removing all edges \( A \to B \), if graph doesn’t contain edge \( B \to A \).

12.5 Summary

Algorithm uses Dirac’s theorem on closure of:

1. undirected graphs - if graph is ”fully undirected”
2. ”undirected version” of directed graph - if tested graph is not ”fully undirected”

References

[1] Wikipedia

13 Most optimal path

13.1 Visiting a path

Both brute-force search and algorithm use recursive depth-first search to test paths - test possibilities, however the second one use it differently. Algorithm’s goal is to find edges described in the beginning of 4

Let’s consider graph G with N vertices and path P from graph G which consists of following edges: \( a_1 \to a_2, a_2 \to a_3, \ldots, a_{M-1} \to a_M \).

Brute-force search will visit P in following way:
1. Visit vertex \( a_1 \)
2. Visit vertex \( a_2 \)

\vdots
M. Visit vertex \( a_M \)
M+1. Check if \( N == M \) and if G contains edge \( a_M \to a_1 \)
Algorithm will visit $P$ in following way:
1. Visit edge $a_{x_1} \rightarrow a_{x_2}$
2. Remove unnecessary edges and test if Hamiltonian cycle can exist in graph
3. Test if Hamiltonian cycle was found
   If Hamiltonian cycle was not found:
4. Visit edge $a_{x_3} \rightarrow a_{x_4}$
5. Remove unnecessary edges . . .
   ...

$x_1, x_2, \ldots, x_N \in \{1, 2, \ldots, M\}$

Algorithm doesn’t visit a path by visiting one vertex after another.

Algorithm doesn’t necessarily need to visit every edge in path to know if it is not a Hamiltonian cycle or if it is a Hamiltonian cycle.

Figure 6: Example of usage of algorithm
13.2 Correct order
Let’s consider graph $G$. $G$ contains vertices: 2 and 3, vertex 2 has 5 neighbours and vertex 3 has 3 neighbours. Graph $G$ has only one Hamiltonian cycle. Algorithm will test first vertex 3 because it will give algorithm probability equal to $1/3$ of choosing edge that is in Hamiltonian cycle, which is better than probability equal to $1/5$.

![Diagram of graph G with vertices 2 and 3 and their neighbors]

When algorithm decide which vertex should it test first, it will decide to choose vertex with smaller degree.

Algorithm creates correct order, it is a list of vertices in graph ordered by their degrees in ascending order.

13.3 Start vertex
Algorithm starts the search in graph with:

1. first vertex in correct order from graph with degree greater than 1, if such a vertex doesn’t exist it will start with first vertex in graph
2. first vertices in correct order

13.4 Next edge
When algorithm tests vertex $A$, it have to decide which of $A$’s edges should it test first. Let’s consider following situation: vertex $A$ has 3 neighbours: $N$, $M$ and $P$, their degrees in opposite graph are: $N - 5$, $M - 4$ and $P - 8$. Algorithm will test first edge $A \rightarrow M$ because it will give algorithm probability equal to $1/4$ of choosing edge that is in Hamiltonian cycle. To check edges in such an order, adjacency list of currently visited vertex is sorted by correct order from opposite graph.

13.5 Next vertex
Next visited vertex is selected:

1. as in 13.3.1
2. second vertex in currently tested edge - as in brute-force search
14 Features

1. Attempt of removal of unnecessary edges and testing if Hamiltonian cycle can exist in graph occurs:
   (a) before search for Hamiltonian cycle begins.
   (b) with every recursive call of function that searches for Hamiltonian cycle - with visiting every edge in tested path.
2. When algorithm makes the decision to test edge $A \rightarrow X$ it removes all of the other edges from $A$.
3. Edges removed with choosing the wrong edge are restored.
4. Edges removed with choosing the wrong path are restored.

15 Algorithm

Algorithm 1 Algorithm

```
START
Remove multiple edges and loops
Dirac’s theorem test
for stage = 1 to 8 do
    Initialize variables
    Analyze graph
    if Hamiltonian cycle can’t exist then
        END, Answer = ”NO”
    end if
    FindHamiltonianCycle(13.3)
    if Answer was found then
        END
    end if
end for
```

15.1 Stages

$N$ - number of vertices in tested graph. Except for stage 6, algorithm uses $13.5.1$.

Stage 0:

1. Check the closure of graph using Dirac’s theorem to see if the graph is a Hamiltonian - 12.

Stage 1:

1. Graph analysis using basic rules (4.1 4.3 5.1 6.1 6.2 7 8 9.1 9.2 9.3).
2. Check $2 \times N$ paths by DFS in graph, using basic rules.

Stage 2:

1. For ”fully undirected” graphs only - analysis of closure of graph with basic rules.

Stage 3:

1. Graph analysis using basic rules and:
2. Check 4 * N paths by DFS in graph, using basic rules.

Stage 4:

1. Graph analysis using basic rules and:
   (a) 4.2.1
   (b) 4.4
   (c) 4.4.1
   (d) 9.4 - used only for "not very irregular" and "very undirected" graphs.

2. Check up to $N^2$ paths by DFS in graph, using basic rules.

Stage 5:

1. Graph analysis using basic rules and:
   (a) 4.2.1
   (b) 4.4
   (c) 4.4.1
   (d) 4.5

2. Check 2 * N paths by DFS in closure of graph, using basic rules.

Stage 6:

1. Check 2 * N paths by DFS in graph, using basic rules, using 13.3.2.

Stage 7:

1. Check N paths by DFS in the graph, starting from N/2 different vertices - 13.3.2, using basic rules.

Stage 8:

1. Graph analysis using basic rules and:
   (a) 4.2.1
   (b) 4.2.2
   (c) 4.4
   (d) 4.4.1
   (e) 4.5
   (f) 4.6

2. Check 2 * N paths by DFS in closure of graph, using basic rules and:
   (a) 4.2.1
   (b) 4.5
The algorithm deliberately does not use all its capabilities at once, because for the vast majority of tested graphs, it would only unnecessarily extend the time of determining whether the graph has a Hamiltonian cycle.

If the algorithm is unable to determine whether the graph has a Hamiltonian cycle using the above-mentioned stages, it returns the information that it has not found an answer.

The above-described limits of the paths to be checked exist so that the algorithm has polynomial complexity.

Algorithm 2 FindHamiltonianCycle(Vertex A)

\[
\text{level} \leftarrow \text{level} + 1 \\
\text{if Hamiltonian cycle was found then} \\
\quad \text{Answer} = "\text{Hamiltonian cycle was found}" \\
\text{end if} \\
\text{Sort neighbours of A /}^{13.4} \\
\text{for } i = 0; i < A\text{'s degree}; i \leftarrow i + 1 \text{ do} \\
\quad B \leftarrow A\text{'s i-th neighbour} \\
\quad C \leftarrow \text{current graph} \\
\quad \text{Remove edge } A \rightarrow B \\
\quad D \leftarrow \text{edges: } A \rightarrow \text{all neighbours of } A \text{ in current graph} \\
\quad \text{Remove edges } D /^{8.2} \\
\quad \text{Add edge } A \rightarrow B \\
\quad \text{AnalyzeGraph(D) /}^{8.1b} \\
\quad \text{if Hamiltonian cycle can exist then} \\
\quad \quad \text{FindHamiltonianCycle(13.5)} \\
\quad \quad \text{Restore graph to } C /^{8.4} \\
\quad \text{else} \\
\quad \quad \text{Add edges } D /^{8.3} \\
\quad \text{end if} \\
\text{end for} \\
\text{level} \leftarrow \text{level} - 1 \\
\text{if level == 0 then} \\
\quad \text{Answer} = "\text{Hamiltonian cycle doesn’t exist in graph}" \\
\text{end if}
\]

15.2 Which vertices should be analyzed?

Before search for Hamiltonian cycle begins every vertex is analyzed. In every recursive call of function that searches for Hamiltonian cycle two types of vertices should be analyzed:

1. vertices whose adjacency list where changed in current call of function
2. vertices whose adjacency list are supersets of any of adjacency list of

16 Examination of algorithm

Algorithm was tested on 4 types of graphs:

- "directed regular"
- "directed irregular"
Graph "undirected regular" is a graph which has many edges $X \rightarrow Y$ and also $Y \rightarrow X$.
Graph "irregular" is a graph with median of all vertices degrees being much different than degree of vertex with highest number of neighbours.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{directed_regular_graph.png}
\caption{Example of "directed regular" graph}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{directed_irregular_graph.png}
\caption{Example of "directed irregular" graph}
\end{figure}
Figure 10: Example of "undirected regular" graph

Figure 11: Example of "undirected irregular" graph
16.1 Algorithms correctness

Algorithms correctness was checked by passing the algorithm graphs with Hamiltonian cycle and testing if algorithm would confirm existence of Hamiltonian cycle. Graphs with 50 vertices were tested.

Tested graphs can be downloaded from:

http://figshare.com/articles/Correctness_Test/1057640

10 000 "directed regular" graphs are located in directory "CT_50_T_T".
10 000 "directed irregular" graphs are located in directory "CT_50_T_F".
10 000 "undirected regular" graphs are located in directory "CT_50_F_T".
10 000 "undirected irregular" graphs are located in directory "CT_50_F_F".

For every tested graph, algorithm confirmed existence of Hamiltonian cycle.

16.2 Algorithms efficiency

Following results were given on AMD FX-8350 4.00 Ghz CPU, only on a single CPU core, on Windows 7. Algorithm was tested with every one of 4 types of graphs.

I tested the algorithm performance on a total of 10 000 000 graphs:

1. 4 000 000 graphs with 25 vertices, in a graph with 25 vertices there can be a maximum of 600 edges.
2. 3 000 000 graphs with 50 vertices, in a graph with 50 vertices there can be a maximum of 2 450 edges.
3. 2 000 000 graphs with 75 vertices, in a graph with 75 vertices there can be a maximum of 5 550 edges.
4. 1 000 000 graphs with 100 vertices, in a graph with 100 vertices there can be a maximum of 9 900 edges.

The use of only a part of the algorithm’s capabilities - described as stage 0 and 1, made it possible to determine whether the graph has a Hamiltonian cycle for 9 997 927 graphs, which is over 99.97% of all examined graphs.

The time to examine graphs in stages 0 and 1 took up to:

1. for graphs with 25 vertices - 0.05 seconds
2. for graphs with 50 vertices - 0.20 seconds
3. for graphs with 75 vertices - 0.95 seconds
4. for graphs with 100 vertices - 2.07 seconds

The number of graphs that required the use of full capabilities of the algorithm was 2 073. Among these graphs, the algorithm was able to determine whether the graph has a Hamiltonian cycle for 2 051 graphs in time:

1. for graphs with 25 vertices - in an average time of 0.063 seconds and maximum time of 0.33 seconds
2. for graphs with 50 vertices - in an average time of 0.33 seconds and a maximum time of 5.60 seconds
3. for graphs with 75 vertices - in an average time of 1.17 seconds and maximum time of 19.52 seconds
4. for graphs with 100 vertices - in an average time of 2.52 seconds and maximum time of 33.4 seconds
Only for 22 graphs, the algorithm was unable to find a solution in described above stages - meaning, in a relatively short time.

"Directed regular" graphs are located in directory "E_A_T_T".
"Directed irregular" graphs are located in directory "E_A_T_F".
"Undirected regular" graphs are located in directory "E_A_F_T".
"Undirected irregular" graphs are located in directory "E_A_F_F"
where A is number of vertices.

All tested graphs can be downloaded from:
http://figshare.com/authors/Pawe_Kaftan/568545
2 073 graphs that needed more than basic capabilities of algorithm, can be downloaded from:
https://figshare.com/articles/dataset/Graphs_not_solved_in_stages_0_1_v6_zip/20562648
22 graphs that algorithm was unable to find a solution, can be downloaded from:
https://figshare.com/articles/dataset/Graphs_not_solved_in_all_stages_v6_zip/20562600.

16.3 How the examined graphs look like

Figure 12: Example of graph with 25 vertices and 313 edges
Figure 13: Example of graph with 50 vertices and 1246 edges

Figure 14: Example of graph with 75 vertices and 2824 edges
17 Algorithm implementation

Algorithm implementation in C#:
https://figshare.com/articles/software/Algorithm_v6_zip/20563158

18 Contact

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