A paradox of diffusion market model related with existence of winning combinations of options*

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We consider strategies of investments into options and diffusion market model. It is shown that there exists a correct proportion between "put" and "call" in the portfolio such that the average gain is almost always positive for a generic Black and Scholes model. This gain is zero if and only if the market price of risk is zero. It is discussed a paradox related to the corresponding loss of option's seller.

Key words: Diffusion market model, investment in options, Black and Scholes, winning strategy

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1 Introduction

There are many works devoted investment strategies dealing with risky assets (stocks). We consider strategies of investments into options for a generic stochastic diffusion model of a financial market. It is assumed that there is a risky stock and a risk-free asset (bond), and that it is available European options "put" and "call" on that stock at the initial time. We consider only strategies of selecting options portfolio at the initial time. The selection of this portfolio is the only action of the investor; after that, he/she waits until the expiration time to accept gain or loss.

We show that there exist a correct proportion between "put" and "call" options in the portfolio such that the average gain is almost always positive. This gain is zero if and only

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if the market price of risk is zero, i.e. when the appreciation rate of the stock is equal to
the interest rate of the risk free asset (i.e. \( a \neq r \) in (2.1)–(2.2) below). It is discussed a
paradox related to the corresponding loss of option’s seller.

\section{Definitions}

Consider the generic model of financial market consisting of a risk-free asset (bond, or bank
account) with price \( B_t \), and a risky asset (stock) with price \( S_t \). We are given a standard
probability space with a probability measure \( P \) and a standard Brownian motion \( w_t \). The
bond and stock prices evolve as

\[ B_t = e^{rt} B_0, \quad (2.1) \]
\[ dS_t = a S_t dt + \sigma S_t dw_t. \quad (2.2) \]

Here \( r \geq 0 \) is the risk-free interest rate, \( \sigma > 0 \) is the volatility, and \( a \in \mathbb{R} \) is the
appreciation rate. We assume that \( t \in [0, T] \), where \( T > 0 \) is a given terminal time. The
equation (2.2) is Itō’s equation, and can be rewritten as

\[ S_t = S_0 \exp \left( at - \frac{\sigma^2 t}{2} + \sigma w_t \right). \]

We assume that it is available the options ”put” and ”call” on that stock for that price
defined by the Black-Scholes formula (see e.g. Strong (1994)).

Further, we assume that \( \sigma, r, B_0 > 0 \) and \( S_0 > 0 \) are given, but the constant \( a \) is
unknown.

Let \( p_{BS}(S_0, K, r, T, \sigma) \) denote Black-Scholes price for
"put" option, and \( c_{BS}(S_0, K, r, T, \sigma) \) denote Black-Scholes price for "call" option. Here
\( S_0 \) is the initial stock price, \( K \) is the strike price, \( r \) is the risk-free interest rate, \( \sigma \) is the
volatility, \( T \) is the expiration time.

We remind the Black-Scholes formula. Let

\[ \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \]
\[ d \triangleq \frac{\ln(S_0/K) + T (r + \sigma^2/2)}{\sigma \sqrt{T}}, \quad d^- = d - \sigma \sqrt{T}. \quad (2.3) \]
Then
\[ c_{BS}(S_0, K, r, T, \sigma) = S_0 \Phi(d) - Ke^{-rT} \Phi(d^-), \quad (2.4) \]
\[ p_{BS}(S_0, K, r, T, \sigma) = c_{BS}(S_0, K, r, T, \sigma) - S_0 + Ke^{-rT} \quad (2.5) \]
(see e.g. Strong (1994)-Duffie (1988)).

We denote by \( X_0 \) the initial wealth of an investor (i.e. at the initial \( t = 0 \)), and we denote by \( X_T \) the wealth of the investor at the terminal time \( t = T \).

Consider a vector \((K_p, \mu_p, K_c, \mu_c)\), such that \( K_p > 0, \mu_p \geq 0, K_c > 0, \mu_c \geq 0 \). We shall use this vector to describe the following strategy: buy a portfolio of options which consists of \( \mu_p \) ”put” options with the strike price \( K_p \) and \( \mu_c \) options with the strike price \( K_c \), with the same expiration time \( T \); thus,

\[ X_0 = \mu_p p_{BS}(S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(S_0, K_p, r, T, \sigma), \quad (2.6) \]
(We assume that the options are available for the Black-Scholes price). We have assumed that the investor does not take any other actions until the expiration time. In that case, the terminal wealth at time \( t = T \) will be

\[ X_T = \mu_p (K_p - S_T)^+ + \mu_c (S_T - K_c)^+. \quad (2.7) \]

**Definition 2.1** The vector \((K_p, \mu_p, K_c, \mu_c)\) is said to be a strategy.

For a case of risk-free ”hold-only-bond” strategy, \( X_T = e^{rT} X_0 \). It is natural to compare the results of any investment with the risk-free investment.

**Definition 2.2** The difference \( E X_T - e^{rT} X_0 \) is said to be the average gain.

Note that the appreciation rate \( a \) in this definition is fixed but unknown. The average gain for a strategy depends on \( a - r \). For example, for a ”call” option holder, when \( \mu_p = 0 \), the average gain is positive if \( a > r \).

3 The result

Let \( d_p \) and \( d_c \) be defined by

\[ \frac{2}{\sqrt{T}} d \quad (2.3) \]
where \( d \) is defined after substituting \( K = K_p \) or \( K = K_c \) correspondingly.
Theorem 3.1 Let \( \mu_p > 0, \mu_c > 0 \) and

\[
\frac{\mu_c}{\mu_p} = \frac{1 - \Phi(d_p)}{\Phi(d_c)},
\]

(3.1)

Then the average gain for the strategy \((K_p, \mu_p, K_c, \mu_c)\) is positive for any \( a \neq r \), i.e.

\[
EX_T > e^{rT}X_0 \quad \forall a \neq r.
\]

(3.2)

Moreover,

\[
EX_T = e^{rT}X_0 \quad \text{if} \quad a = r.
\]

(3.3)

For any \((K_p, K_c)\), the proportion (3.3) is the only one which ensures (3.1): for any other proportion \( \mu_c/\mu_p \) there exists \( \in \mathbb{R} \) such that the average gain is negative.

Corollary 3.1 Let the variable \( a \) be random, independent of \( w(\cdot) \) and such that \( P(a \neq r) > 0 \). Then \( EX_T > e^{rT}X_0 \) for the strategy from Theorem 3.1.

Set

\[
Q(x,t) \triangleq \mu_p p_{BS}(x,K_p,r,T-t,\sigma) + \mu_c c_{BS}(x,K_c,r,T-t,\sigma),
\]

\[
\Delta(x,t) = \frac{dQ(x,t)}{dx}.
\]

(3.4)

Corollary 3.2 Let \((K_p, \mu_p, K_c, \mu_c)\) be such as in Theorem 3.1, then

\[
\Delta(S_0,0) = 0.
\]

(3.5)

In terms of Strong (1994), the equality (3.5) means that the position is risk-neutral at time \( t = 0 \).

Example. Consider example with same parameters as in Strong (1994)[p. 109]. Let \( K_p = K_c = $25, S_0 = $30, T = 0.25 \) (i.e. the expiration time is 3 months=25 years); \( r = 0.05 \) (i.e 5% annual), \( \sigma = 0.45 \) (i.e. 45% annual). Then \( d = d_p = d_c = 0.978, \Phi(d) = 0.836, \mu_c/\mu_p = 164/836 \). Let \( S_0 \) be arbitrary, \( K_p = K_c = S_0 \), then \( d = d_p = d_c = 0.168, \Phi(d) = 0.567, \mu_c/\mu_p = 433/567 \).
4 A consequence for the seller and a paradox

Consider the result for a seller (writer) who has sold the options portfolio described in
Theorem 3.1. The seller has received the premium $X_0$ and must pay $X_T$ at the time $t = T$.

Let $Y_t$ be the wealth, which was obtained by the seller from $Y_0 = X_0$ by some self-
financing strategy. Let $\hat{Y}_T = Y_T - X_T$ be the terminal wealth after paying obligations to
options holder at the expiration time $T$.

Let us consider possible actions of the sellers after receiving the premium. The following
strategy is most commonly presented in textbooks devoted to mathematical aspects of
option pricing:

**Strategy I**: To replicate the claim $X_T$ using the replicating strategy.

As is known, the Black-Scholes price is defined as a minimum initial wealth such that
the option’s random claim can be replicated. For the Strategy I, the number of shares is
$\Delta(S_t, t)$ at any time $t \in [0, T]$, and $\hat{Y}_T = 0$ a.s., i.e. there is not neither risk nor any gain.
Thus, it is doubtful that the seller uses this strategy in practice.

Furthermore, it was mentioned in Strong (1994)[c. 53] that in practice the option writer
rather just keeps premium as a really compensation for bearing added risk of for foregoing
future price appreciation or depreciation. Thus, the second strategy is the following:

**Strategy II**: To invest the premium $X_0$ into bonds, take no further actions and wait
the outcome of the price movement similarly as the option’s holder.

The seller who sells only put (or only call ) options and uses the Strategy II puts
his/her stake on the random events $K_p \leq S_T$ (or $S_T \leq K_c$ correspondingly). A gain of the
holder implies a loss for the writer. It looks as a fair game in a case of selling either put
or call separately, because we know that the Black-Scholes price is fair and chances for
gain should be equal for buyer and seller (otherwise, either ask or bid will prevail). But
Theorem 3.1 implies that that the seller will receive non-positive in average gain which
is negative in average for any $a \neq r$ if he/she sells the combination described in Theorem
3.1. In other words, we have a paradox:

*A combination of two “fair” deals of selling puts and calls gives an ”unfair” deal.*

The second paradox can be formulated as following:
We know that the Black-Scholes price is fair for buyers as well as for sellers (otherwise, either ask or bid will prevail). However, we found an options combination such that buying is preferable, because the buyer has non-negative and almost always positive average gain, but the seller has either zero or negative average gain.

In practice, unlike as for our generic model, brokers use sophisticated measures such as insurance, long positions in stocks or other options, etc. to reduce the risk, but that does not affect the core of the paradoxes.

A possible explanation is that writer also use the premium $X_0$ to can receive a gain from $a \neq r$ by using some other strategies such as Merton’s strategies which are possibly more effective then options portfolio. In other words, a rational option’s seller does not use neither risk-free replicating of claims nor ”keep-only-bonds” strategy; he/she rather uses strategies which are able to explore $a \neq r$.

5 Proofs

Proof of Theorem 3.4. Let $P_a$ be the conditional probability measure given $a$. Let $E_a$ be the corresponding expectation. We denote $E_*$ the expectation which corresponds the risk neutral measure, when $a = r$. Set

$$h(a) \triangleq E_a X_T.$$  

By the definitions of $X_T$, it follows that

$$h(a) = \mu_p E_* (K_p - e^{(a-r)T} S_T)^+ + \mu_c E_* (e^{(a-r)T} S_T - K_c)^+. \quad (5.1)$$

As is known (see e.g. Strong (1994) and Duffie (1988)), the Black-Scholes price can be presented as

$$p_{BS}(S_0, K, r, T, \sigma) = e^{-rT} E_* (K - S_T)^+, \quad c_{BS}(S_0, K, r, T, \sigma) = e^{-rT} E_* (S_T - K)^+. \quad (5.2)$$

Then

$$e^{-rT} h(a) = \mu_p p_{BS}(e^{(a-r)T} S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(e^{(a-r)T} S_0, K_c, r, T, \sigma).$$
By the put and call parity formula, it follows that
\[
e^{-rT} h(a) = \mu_p \left[ c_{BS}(e^{(a-r)T} S_0, K_p, r, T, \sigma) - e^{(a-r)T} S_0 + K_p \right] + \mu_c c_{BS}(e^{(a-r)T} S_0, K_c, r, T, \sigma).
\]

The following proposition is well known (see e.g. Strong (1994)[p. 100]).

**Proposition 5.1** For any \( T > 0, K > 0 \), the following holds:
\[
\frac{\partial}{\partial x} c_{BS}(x, K, r, T, \sigma) = \Phi(d),
\]
where \( d \) is defined by (2.4).

Let \( d_p(a) \) and \( d_c(a) \) be defined as \( d_p \) and \( d_c \) correspondingly with substituting \( e^{(a-r)T} S_0 \) as \( S_0 \). Set
\[
y = y(a) \triangleq e^{(a-r)T}, \quad R(y) \triangleq h(T^{-1} \ln y).
\]

We have that
\[
R(y(a)) \equiv h(a),
\]
then
\[
\frac{dR(y)}{dy} = e^{rT} S_0 \left[ (\mu_p(\Phi(d_p(a)) - 1) + \mu_c \Phi(d_c(a)) \right].
\]

By (3.1) it follows that
\[
\frac{dR}{dy}(y) \big|_{y=1} = 0.
\]
(Note that \( y = 1 \) if and only if \( a = 1 \).) It is know that
\[
\frac{\partial^2}{\partial x^2} p_{BS}(x, K_p, 0, T, \sigma) > 0, \quad \frac{\partial^2}{\partial x^2} c_{BS}(x, K_p, 0, T, \sigma) > 0
\]
and the derivatives exist. Then \( R''(y) > 0 \ (\forall y) \), i.e. \( R(\cdot) \) is strongly convex. It follows that \( a = r \) is the only one which minimize \( h \). By (2.6), (5.1) it follows that
\[
R(1) = e^{rT} X_0.
\]
The uniqueness of the proportion (3.1) follows from the uniqueness of \( \mu_c/\mu_p \) which ensures (5.4). This completes the proof of Theorem 3.1.

Corollary 3.1 follows from (3.2)–(3.3). Corollary 3.2 follows from (3.3).
References

[1] Strong R. Speculative markets. 2nd ed. Harper Collins College Publishers, 1994.

[2] Duffie D. Security markets. Academic Press. Inc, 1988.