Another connection of harmonic maps to gravity is presented. Using 1-soliton and anti-soliton solutions of the sine-Gordon equation, we construct a pair of harmonic maps that we express in terms of a particular dilaton field in Jackiw-Teitelboim gravity. This field satisfies a linearized sine-Gordon equation. We use it also to construct an explicit transformation that relates the corresponding solitonic metric to a two dimensional black hole metric.

1 Introduction

The theory of harmonic maps provides a pleasant, unifying setting in which various field equations can be viewed and discussed. The equations of motion of a Bosonic string, for example, coincide with the requirement that the map of its world sheet to 26-dimensional space should be harmonic. The solution of the $O(3)_\sigma$-model is provided by a harmonic map from the unit 2-sphere $S^2$ to itself. Certain Einstein equations are given by harmonic maps. A broad overview of the role of harmonic maps in Yang-Mills theory, general relativity, and quantum field theory is presented in the inspiring papers of C. Misner and N. Sánchez, for example.

We consider a pair of harmonic maps from the plane $\mathbb{R}^2$ to $S^2$ that we relate to sine-Gordon solitons and to two dimensional Jackiw-Teitelboim dilaton gravity. An intriguing observation of J. Gegenberg and G. Kunstatter connects such solitons with two dimensional black holes. We construct an explicit transformation $\Psi$ that takes the solitonic metric to a black hole metric, and we express the harmonic maps in terms of the dilaton field.

2 Harmonic maps from $\mathbb{R}^2$ to $S^2$

A smooth map $\Phi : (M, g) \to (N, h)$ of Riemannian (or pseudo Riemannian) manifolds is called harmonic if it satisfies the following (local) conditions; one can also formulate harmonicity by a global condition. Let $(U, \phi = (x_1, \ldots, x_m))$ and $(V, \psi = (y_1, \ldots, y_n))$ be coordinate systems on $M, N$ with $U \subset \Phi^{-1}(V)$, let $\Phi^j = y_j \circ \Phi \circ \phi^{-1}$ ($1 \leq j \leq n$) denote the $j$th component of $\Phi$ relative to these systems, let $\Gamma^k_{ij}$ denote the Christoffel symbols...
of \((N, h)\), and let \(\partial_i = \frac{\partial}{\partial x_i}(1 \leq i \leq m)\). If
\[
B = \frac{1}{\sqrt{|\det(g \circ \phi^{-1})|}} \sum_{i,j=1}^{m} \partial_i \sqrt{|\det(g \circ \phi^{-1})|} (g^{ij} \circ \phi^{-1}) \partial_j
\]
is the Laplacian of \((M, g)\) on \(\phi(U)\), then we require that
\[
(\tilde{B}_s \Phi)(p) \equiv \sum_{i,j=1}^{m} (g^{ij} \circ \phi^{-1}) \left( \frac{\partial_i \Phi_k \partial_j \Phi_r}{\phi(p)} \right) \Gamma_{kr}^s(\Phi(p)) + (B \Phi_s)(p) = 0
\]
for \(p \in U, 1 \leq s \leq n\). We construct harmonic maps \(\Phi = \Phi^\pm : R^2 \to S^2\) as follows. \(\Phi = (\cos \beta \sin \alpha, \sin \beta \sin \alpha, \cos \alpha)\) where \(\alpha, \beta : R^2 \to R\) are given by \(\alpha(x, t) = u(x, t)/2, \beta(x, t) = m(vx + t)/a\) for parameters \(m, v > 0, a = a(v) \equiv \sqrt{1 + v^2}\) where for \(\rho(x, t) \equiv m(x - vt)/a, u(x, t) = u^\pm(x, t) = 4 \tan^{-1} e^{\pm \rho(x, t)}\) are one-soliton solutions of the Euclidean sine-Gordon equation (SGE)
\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = m^2 \sin u.
\]
for \(\rho(x, t) = 4 \tan^{-1} e^{\pm \rho(x, t)}\), \(v = \sqrt{1 + v^2}\).

The harmonicity condition on \(\Phi\) in fact reduces to condition \(\Phi\) - which contrasts the point of view in 3 where \(\Phi\) is obtained by variation of the dilaton field \(\tau\) in the Jackiw-Teitelboim (J-T) action
\[
I(\tau, u) = \frac{1}{2G} \int dx \int dt \left[ \Delta u - m^2 \sin u \right].
\]
\(m\) is revealed as a mass parameter and \(v\) as a soliton velocity parameter. As pointed out in 3, the linearised SGE \(\Delta \tau = (m^2 \cos u) \tau\) is satisfied by the field
\[
\tau(x, t) = a(v)\text{sech} \rho(x, t).
\]

3 Statement of the main result

The solitonic metric
\[
ds^2 = \cos^2 \alpha(x, t) dx^2 - \sin^2 \alpha(x, t) dt^2
\]
has scalar curvature \(R = \frac{2\Delta u}{\sin u}\), which is therefore constant by 3: \(R = 2m^2\); or the Gaussian curvature \(K = -\frac{R}{2} = -m^2\). Recall that \(a = a(v) = \sqrt{1 + v^2}\).

**Theorem**

Let \(\Psi = (\psi_1, \psi_2) : R^2 \to R^2\) be the transformation defined as follows:
\[
\psi_1(T, r) = vT + \frac{1}{m} \coth^{-1} \left[ \sqrt{\frac{a^2}{m^2} + m^2 r^2} \right],
\]
\[
\psi_2(T, r) = \frac{\psi_1(T, r)}{v} - \frac{a v}{m^2} \log \left[ a + \sqrt{a^2 - m^2 r^2} \right]
\]
on the domain

\[ C^\dagger = \left\{ (T, r) \in \mathbb{R}^2 \mid 0 < r < \frac{a}{m}, \sqrt{a^2 - m^2 r^2} > 1 \right\}, \]  

and let \( \Theta = (\theta_1, \theta_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by

\[
\begin{align*}
\theta_1(x, t) &= -\frac{1}{mv} \coth^{-1}[a \tanh \rho(x, t)] + \frac{x}{v}, \\
\theta_2(x, t) &= \frac{a}{m} \sech \rho(x, t) = \frac{\tau(x,t)}{m}
\end{align*}
\]

(9)
on the domain

\[ D^\dagger = \left\{ (x, t) \in \mathbb{R}^2 \mid \left| \tanh \rho(x, t) \right| > 1 \right\}. \]  

(10)

Then \( \Psi : C^\dagger \to D^\dagger \) and \( \Theta : D^\dagger \to C^\dagger \) are bijections and inverses of each other: \( \Theta \circ \Psi = 1 \) on \( C^\dagger \), \( \Psi \circ \Theta = 1 \) on \( D^\dagger \). Also \( \Psi \) transforms the solitonic metric \( \Box \) to the black hole metric

\[ ds^2 = (M - m^2 r^2) dT^2 - (M - m^2 r^2)^{-1} dr^2 \]  

(11)
with mass \( M = v^2 \). Thus, conversely, \( \Theta \) takes \( 11 \) back to \( 6 \). Also the harmonic maps \( \Phi^\pm \) constructed in the preceding section are expressed in terms of the dilaton field \( \Theta \) as follows:

\[ \Phi^\pm = \frac{1}{a}(\tau \cos \beta, \tau \sin \beta, \pm a \tanh \rho). \]  

(12)

The interesting connection of sine-Gordon solitons to black hole solitons in J-T gravity is the remarkable observation of the paper 3, although the transformation \( \Psi \) that we have presented here does not explicitly appear there. We have considered another connection of harmonic maps to gravity.

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