Approximate counting of regular hypergraphs

Andrzej Dudek∗  Alan Frieze†  Andrzej Ruciński‡  Matas Šileikis§

November 11, 2018

Abstract

In this paper we asymptotically count $d$-regular $k$-uniform hypergraphs on $n$ vertices, provided $k$ is fixed and $d = d(n) = o(n^{1/2})$. In doing so, we extend to hypergraphs a switching technique of McKay and Wormald.

1 Introduction

We consider $k$-uniform hypergraphs (or $k$-graphs, for short) on the vertex set $V = [n] := \{1, \ldots, n\}$. A $k$-graph $H = (V, E)$ is $d$-regular, if the degree of every vertex $v \in V$, $\deg_H(v) := \deg(v) := |\{e \in E : v \in e\}|$ equals $d$.

Let $\mathcal{H}^{(k)}(n, d)$ be the class of all $d$-regular $k$-graphs on $[n]$. Note that each $H \in \mathcal{H}^{(k)}(n, d)$ has $m := nd/k$ edges (throughout, we implicitly assume that $k|nd$). We treat $d$ as a function of $n$, possibly constant.

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*Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA. Research supported in part by Simons Foundation Grant #244712.
†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA. Research supported in part by NSF Grant CCF2013110.
‡Department of Discrete Mathematics, Adam Mickiewicz University, Poznań, Poland. Research supported by the Polish NSC grant N201 604940 and the NSF grant DMS-1102086. Part of research performed at Emory University, Atlanta.
§Department of Mathematics, Uppsala University, Sweden. Research supported by the Polish NSC grant N201 604940. Part of research performed at Adam Mickiewicz University, Poznań.
A result of McKay [7] contains an asymptotic formula for the number of $n$-vertex $d$-regular graphs, when $d \leq \varepsilon n$ for any constant $\varepsilon < 2/9$. In this paper we present an asymptotic enumeration of all $d$-regular $k$-graphs on a given set of $n$ vertices, where $k \geq 3$ and $d = d(n)$ is either a constant or does not grow with $n$ too quickly. Let $\kappa = \kappa(k) = 1$ for $k \geq 4$ and $\kappa(3) = 1/2$.

**Theorem 1.** For every $k \geq 3$, $1 \leq d = o(n^\kappa)$, and

$$|\mathcal{H}^{(k)}(n, d)| = \frac{(nd)!}{(nd/k)!(k!)^{nd/k}(d!)^n} \exp\left\{ -\frac{1}{2} (k-1)(d-1) + O\left((d/n)^{1/2} + d^2/n\right) \right\}. $$

The error term in the exponent tends to zero (thus giving the asymptotics of $|\mathcal{H}^{(k)}(n, d)|$) if and only if $d = o(n^{1/2})$. Cf. an analogous formula for $k = 2$ by McKay [7], which gives the asymptotics if and only if $d = o(n^{1/3})$. Recently, Blinovsky and Greenhill [2] obtained more general results counting sparse uniform hypergraphs with given degrees.

Theorem 1 extends a result from [5] where Cooper, Frieze, Molloy and Reed proved that formula for $d$ fixed using the by now standard configuration model (see [1, 3, 10] for the graph case). Already for graphs, in [7], and later in [8] and [9], this technique was combined with the idea of switchings, a sequence of operations on a graph which eliminate loops and multiple edges, while keeping the degrees unchanged and leading to an almost uniform distribution of the simple graphs obtained as the ultimate outcome (but see Remark 3 in Section 3).

To prove Theorem 1 we apply these ideas together with a modification from [4], where instead of configurations, permutations were used to generate graphs with a given degree sequence. To describe this modification, consider a generalization of a $k$-graph in which edges are multisets of vertices rather than just sets. By a $k$-multigraph we mean a pair $H = (V, E)$ where $V$ is a set and $E$ is a multiset of $k$-element multisubsets of $V$. Thus we allow both multiple edges and loops, a loop being an edge which contains more than one copy of a vertex. We call an edge proper if it is not a loop. We say that a $k$-multigraph is simple if it is a $k$-graph, that is, if it contains neither multiple edges nor loops. Henceforth, for brevity
of notation, we denote an edge of a $k$-multigraph by $v_1 \ldots v_k$ rather than $\{v_1, \ldots, v_k\}$.

Given a sequence $x \in [n]^{ks},$ $s \in \mathbb{N},$ let $H(x)$ stand for the $k$-multigraph with edge multiset $E = \{x_{ki+1}, \ldots, x_{ki+k} : i = 0, \ldots, s - 1\}$ and let $\lambda(x)$ be the number of loops in $H(x)$.

Let $P = P(n, d) \subset [n]^{nd}$ be the family of all permutations of the sequence

\[
\begin{pmatrix}
\underbrace{1,\ldots,1}_{d}, \underbrace{2,\ldots,2}_{d}, \ldots, \underbrace{n,\ldots,n}_{d}
\end{pmatrix}.
\]

Note that $|P| = (nd)!/(d!)^n$. Let $Y = (Y_1, \ldots, Y_{nd})$ be chosen uniformly at random from $P$.

In the next section we sketch a proof of Theorem 1 together with some auxiliary results.

2 Proof of Theorem 1

2.1 Setup

Let $\mathcal{E}$ be the family of those permutations $y \in P$ for which the $k$-multigraph $H(y)$ has no multiple edges and contains at most

\[L := \sqrt{nd}\]

loops, but no loops with less than $k - 1$ distinct vertices. Let

\[\mathcal{E}_l = \{y \in \mathcal{E} : \lambda(y) = l\}, \quad l = 0, \ldots, L.\]

Note that

\[\mathcal{E}_0 = \left\{ y \in P : H(y) \in \mathcal{H}^{(k)}(n,d) \right\}\]

is precisely the family of those permutations from $P$ which represent simple $k$-graphs. In turn, for each $H \in \mathcal{H}^{(k)}(n,d)$ there are $(nd/k)!/(k!)^{nd/k}$ permutations $y \in \mathcal{E}_0$ with $H(y) = H$. 
Therefore, in order to prove Theorem 1, it suffices to show that
\[
|\mathcal{P}|/|\mathcal{E}|_0 = \exp \left\{ \frac{1}{2} (k - 1)(d - 1) + O\left( \sqrt{d/n} + d^2/n \right) \right\}.
\] (1)

Our plan is as follows. First, in Proposition 2, we prove that
\[
|\mathcal{P}| \sim \left( 1 + O \left( \sqrt{d/n} + d^2/n^{k-2} \right) \right) |\mathcal{E}|.
\] (2)

Note that for \( d = o(n^\kappa) \), the error term in (2) tends to zero and is at most the error term in (1). Thus, it is enough to show (1) with \(|\mathcal{E}|_0|\) in place of \(|\mathcal{P}|\), which we do by writing
\[
|\mathcal{E}|_0/|\mathcal{E}_i| = \sum_{l=0}^L \frac{\prod_{i=1}^l |\mathcal{E}_i|}{|\mathcal{E}_{i-1}|},
\] (3)

and estimating the ratio \(|\mathcal{E}_i|/|\mathcal{E}_{i-1}|\) uniformly for every \( 1 \leq l \leq L \).

In what follows it will be convenient to work directly with permutation \( \mathbf{Y} \) rather than with the \( k \)-multigraph \( H(\mathbf{Y}) \) generated by it. Recycling the notation, we still call consecutive \( k \)-tuples \( (Y_{ki+1}, \ldots, Y_{ki+k}) \) of \( \mathbf{Y} \) edges, proper edges, or loops, whatever appropriate. E.g., we say that \( \mathbf{Y} \) contains multiple edges, if \( H(\mathbf{Y}) \) contains multiple edges, that is, some two edges of \( \mathbf{Y} \) are identical as multisets. We use the standard notation \((x)_a = x(x-1)\cdots(x-a+1)\).

The following proposition implies (2), because \( \mathbb{P}(\mathbf{Y} \in \mathcal{E}) = |\mathcal{E}|/|\mathcal{P}| \).

**Proposition 2.** If \( k \geq 3 \), then \( \mathbb{P}(\mathbf{Y} \in \mathcal{E}) = 1 - O(\sqrt{d/n} + d^2/n^{k-2}) \).

A simple proof of Proposition 2 (details can be found in Appendix A) is based on the first moment method. In particular, the expected numbers of pairs of multiple edges, loops with less than \( k - 1 \) distinct vertices, and all loops are, respectively, \( O(d^2/n^{k-2}) \), \( O(d/n) \), and \( \mathbb{E}\lambda(\mathbf{Y}) \sim \frac{k-1}{2}(d-1) \). The last formula implies that \( \mathbb{P}(\lambda(\mathbf{Y}) > L) \leq \frac{\mathbb{E}\lambda(\mathbf{Y})}{L} = O(\sqrt{d/n}) \).
2.2 Switchings

Now we define an operation, called switching, which generalizes to $k$-graphs a graph switching introduced in [8] (see also [9]). Permutations $y \in E_i$, $z \in E_{i-1}$ are said to be switchable, if $z$ can be obtained from $y$ by the following operation. From the edges of $y$, choose a loop $f$ and two proper edges $e_1, e_2$ that are disjoint from $f$ and share at most $k - 2$ vertices (see Figure 1(a)). Letting $s = |e_1 \cap e_2|$, write

$$f = vv x_1 \ldots x_{k-2}, \quad e_1 = w_1 \ldots w_s y_1 \ldots y_{k-s}, \quad e_2 = w_1 \ldots w_s z_1 \ldots z_{k-s}.$$ 

Select vertices $y_s \in \{y_1, \ldots, y_{k-s}\}$ and $z_s \in \{z_1, \ldots, z_{k-s}\}$, and replace $f, e_1,$ and $e_2$ by three proper edges

$$e'_1 = e_1 \cup \{v\} - \{y_s\}, \quad e'_2 = e_2 \cup \{v\} - \{z_s\}, \quad e'_3 = f \cup \{y_s, z_s\} - \{v, v\}$$

as in Figure 1(b). Since we are dealing with permutations, for definiteness let us assume that the procedure is performed by swapping with $y_s$ the copy of $v$ which appears in $y$. 

Figure 1: Switching (a) before and (b) after.
further to the left and with $z_*$ the one further to the right.

We can reconstruct permutations in $E_{l+1}$ which are switchable with $y$ as follows. Pick a vertex $v \in [n]$, two edges $e_1', e_2'$ containing $v$, and one more edge $e_3'$ (consult with Figure 1 again). Choose a pair $\{y_*, z_*\}$ of vertices from $e_3'$; replace $e_i'$, $i = 1, 2, 3$, by a loop and two edges defined as

$$f = e_3' \cup \{v, v\} \setminus \{y_*, z_*\}, \quad e_1 = e_1' \cup \{y_*\} \setminus \{v\}, \quad e_2 = e_2' \cup \{z_*\} \setminus \{v\}.$$ 

Given $y \in E_l$, let $F(y)$ and $B(y)$ stand, respectively, for the number of ways to perform the forward and backward switching, or, in other words, the number of permutations $x \in E_{l-1}$ and $z \in E_{l+1}$ which are switchable with $y$. Recall that $L = \sqrt{n\ell}$ and set $F_l = d^2n^2l$, $l = 1, \ldots, L$, and $B = \frac{k-1}{2}n^2d^2(d-1)$.

**Proposition 3.** There is a sequence $\delta = \delta(n) = O((L + d^2)/dn)$ such that for all $y \in E_l$, $0 < l \leq L$

$$(1 - \delta)F_l \leq F(y) \leq F_l \quad \text{and} \quad (1 - \delta)B \leq B(y) \leq B.$$

**Proof.** Clearly $F(y) \leq ln^2k^2 = n^2d^2l$. We say that two edges $e', e''$ of a $k$-graph are *distant* from each other if their distance in the intersection graph of $H(y)$ is at least three. Note that given $f, e_1, \text{and } e_2$, some choice of $y_*$ and $z_*$ might not yield a permutation $z \in E_{l-1}$, because one or more of $e_i'$'s might already be present in $y$. However, all $k^2$ choices of $(y_*, z_*)$ are allowed, if $e_1 \cap e_2 = \emptyset$ and both $e_1$ and $e_2$ are distant from $f$. Therefore,

$$F(y) \geq k^2(m - l - 2k^2d^2)^2l = k^2m^2l(1 - O((L + d^2)/m)).$$

Clearly $B(y) \leq n(d)2m^2 = B$. To bound $B(y)$ from below, we estimate the number of choices of $(v, e_1', e_2', e_3')$, for which at least one pair $\{y_*, z_*\}$ does not yield a permutation in $E_{l+1}$. This can only happen when one of $e_1', e_2', e_3'$ is a loop, which occurs for at most $2kldm + ln(d)2$ choices, or when $e_3'$ is not distant from both $e_1'$ and $e_2'$, which occurs for at
most \(n(d) \cdot 2k^2d^2\) choices. We have \(B = \Theta(n^2d^3)\), therefore

\[
B(y) \geq B - \binom{k}{2} (2kldm + ln(d) + 2k^2nd^4) = B \left(1 - O\left(\frac{L + d^2}{nd}\right)\right).
\]

**Proof of Theorem**

Counting the switchable pairs \(y \in E_l, z \in E_{l-1}\) in two ways, from Proposition 3 we conclude that

\[
\frac{(1 - \delta)B}{F_l} \leq \frac{|E_l|}{|E_{l-1}|} \leq \frac{B}{(1 - \delta)F_l}.
\]

(4)

Since \(B/F_l = (k - 1)(d - 1)/2l\), from (3) and (4) we get

\[
\sum_{l=0}^{L} x^l \frac{x}{l!} \leq \frac{|E|}{|E_0|} \leq \sum_{l=0}^{L} y^l \frac{y}{l!}
\]

where \(x = \frac{1}{2}(1 - \delta)(k - 1)(d - 1)\) and \(y = \frac{1}{2}(k - 1)(d - 1)/(1 - \delta)\). Therefore by Taylor’s theorem \(|E|/|E_0|\) is at most \(e^y\) and at least

\[
e^x(1 - x^L/L!) \geq e^x(1 - (ex/L)^L) = \exp\left\{x - o\left(\sqrt{d/n}\right)\right\},
\]

the inequality following from a standard fact \(L! \geq (L/e)^L\). Since \(x, y = (k - 1)(d - 1)/2 + O(\sqrt{d/n} + d^2/n)\), we get

\[
\frac{|E|}{|E_0|} = \exp\left\{\frac{1}{2}(k - 1)(d - 1) + O(\sqrt{d/n} + d^2/n)\right\}
\]

which together with (2) implies (1), hereby completing the proof.

**3 Concluding remarks**

**Remark 1.** We believe that for \(k = 3\) the constraint \(d = o(n^{1/2})\) in Theorem 1 can be relaxed to \(d = o(n)\) by allowing \(O(d^2/n)\) multiple edges in \(y \in E\) and applying an appropriate
switching technique to eliminate them along with the loops.

Remark 2. In a forthcoming paper we apply the switching technique presented here to embed asymptotically almost surely (a.a.s.) an ordinary Erdős-Rényi random $k$-graph $\mathbb{H}^{(k)}(n, m')$, $k \geq 3$, into a random $d$-regular $k$-graph $\mathbb{H}^{(k)}(n, d)$ for $d = \Omega(\log n)$, $d = o(\sqrt{n})$ and $m' = cnd/k$, for some constant $c > 0$. Consequently, a.a.s. $\mathbb{H}^{(k)}(n, d)$ inherits from $\mathbb{H}^{(k)}(n, m')$ all increasing properties held by the latter model.

Remark 3. An algorithm of McKay and Wormald can be easily adapted to $k$-graphs, yielding an expected polynomial time uniform generation of $d$-regular $k$-graphs in $\mathcal{H}^{(k)}(n, d)$. The algorithm keeps selecting a random permutation $y \in \mathcal{P}$ until $y \in \mathcal{E}$. Then, iteratively, a random switching is applied $\lambda(y)$ times to eliminate all loops and finally yield a random element of $\mathcal{E}_0$. This leads to an almost uniform distribution over $\mathcal{H}^{(k)}(n, d)$. To make it exactly uniform, McKay and Wormald applied an ingenious trick of restarting the whole algorithm after every iteration of switching, say from $y \in \mathcal{E}_l$ to $z \in \mathcal{E}_{l-1}$, with probability $1 - (F(y)(1 - \delta_1)B)/(B(z)F_l) \leq 2\delta_1$. However, the assumption on $d$ has to be strengthened, so that the reciprocal of the probability of not restarting the algorithm before its successful termination, or $(1 - \phi_k(n))^{-1}(1 - 2\delta_1(n))^{-L} = e^{O(\delta_1(n)L)}$, is at most a polynomial function of $n$. With our choice of $L$ this imposes the bound $d = O(n^{1/3}(\log n)^{2/3})$. We may push it up to $d = O(\sqrt{n} \log n)$ by redefining $L = kd + \omega(n)$ for any (sufficiently slow) sequence $\omega(n) \to \infty$. This change requires that in the last part of the proof of Proposition 2 instead of the first moment, Chebyshev’s inequality is used (see Appendix A).

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A Appendix

Proof of Proposition 2. We will show that each of the following four statements holds with probability $1 - O(\sqrt{d/n} + d^2/n^{k-2})$:

(i) $Y$ has no multiple edges,

(ii) $Y$ has no edge with a vertex of multiplicity at least 3,

(iii) $Y$ has no edge with two vertices of multiplicity at least 2,

(iv) $\lambda(Y) \leq L$.

(i) The probability that two particular edges of $Y$ are identical as multisets equals

$$\sum_{k_1, \ldots, k_n = k} \binom{k}{k_1, \ldots, k_n} \binom{d_n - 2k}{d_{k_1, \ldots, d_{k_n}}} \leq k!^2 \sum\frac{d^{2k}}{(dn)^{2k}} = O\left(\frac{n^k d^{2k}}{(dn)^{2k}}\right) = O(n^{-k}),$$

therefore, by our assumption on $d$, the expected number of pairs of multiple edges does not exceed

$$O\left(\binom{m}{2} n^{-k}\right) = O(d^2 n^{2-k}).$$

(ii) The expected number of edges of $Y$ having a vertex of multiplicity at least 3 is at most

$$m \times \binom{k}{3} \times n \times \binom{d_{n-3,d_{n-3},d_{n-3},d}}{d_{n,n,n,d}} = m \binom{k}{3} n^2 \frac{(d)^3}{(dn)^3} = O(d/n).$$

(iii) Similarly, the expected number of edges of $Y$ having at least two vertices of multiplicity at least 2 is at most

$$m \times k^4 \times n^2 \times \binom{d_{n-4,d_{n-4},d_{n-4},d_{n-4},d}}{d_{n,n,n,n,d}} = mk^4 n^2 \frac{(d)^2}{(dn)^4} = O(d/n).$$

(iv) In view of (ii) and (iii), it is enough to show that the number of loops of the form $x_1 x_2 x_3 \ldots x_{k-1}$ does not exceed $L$. For $i = 1, \ldots, m$, let $\mathbb{I}_i$ be the indicator of the event
that the \(i\)'th edge of \(Y\) is such a loop. Hence, \(\lambda(Y) = \sum_{i=1}^{m} I_i\). For every \(i\) we have

\[
E I_i = \left(\begin{array}{c} k \\ 2 \end{array}\right) \frac{(n)_{k-1}(d)_{2}d^{k-2}}{(nd)_k} \sim \left(\begin{array}{c} k \\ 2 \end{array}\right) \frac{d - 1}{d} n^{-1}.
\]

Therefore

\[
E \lambda(Y) \sim \frac{k - 1}{2}(d - 1), \quad (5)
\]

and by Markov’s inequality,

\[
P(\lambda(Y) > L) \leq \frac{E \lambda(Y)}{L} = O(d^{1/2}n^{-1/2})
\]

\[
\square
\]

**Proof that** \(P(\lambda(Y) > kd + \omega(n)) = o(1)\). Let \(L := kd + \omega(n)\). We will show that \(\text{Var} \lambda(Y) = O(d)\), from which the desired fact follows by (5) and Chebyshev’s inequality:

\[
P(\lambda(Y) > L) \leq \frac{\text{Var} \lambda(Y)}{(L - E \lambda(Y))^2} = O\left(\frac{d}{(d + \omega(n))^2}\right) = O((d + \omega(n))^{-1}) = o(1).
\]

Recall that \(I_i\) is the indicator that the \(i\)'th edge of \(Y\) is a loop with only one repetition, \(\lambda(Y) = \sum_{i=1}^{m} I_i\), and for every \(i\) we have \(E I_i \sim \left(\begin{array}{c} k \\ 2 \end{array}\right) \frac{d - 1}{d} n^{-1}\). If \(i \neq j\), then

\[
E I_i I_j \leq \left(\begin{array}{c} k \\ 2 \end{array}\right)^2 \frac{(n)_{k-1}(d)_{2}d^{2k-4}}{(nd)_{2k}}
\]

therefore

\[
\text{Cov}(I_i, I_j) = E I_i I_j - E I_i E I_j
\]

\[
\leq \left(\begin{array}{c} k \\ 2 \end{array}\right)^2 \frac{(n)_{k-1}(d)_{2}d^{2k-4}}{(nd)_{2k}(nd)_k} (nd)_k - (nd - k)_k = O(n^{-3}d^{-1}).
\]
Finally we get

$$\text{Var } \lambda(Y) = \sum_{1\leq i \leq m} \text{Var } I_i + \sum_{1 \leq i \neq j \leq m} \text{Cov}(I_i, I_j) = O(mn^{-1} + m^2n^{-3}d^{-1}) = O(d).$$