On the linearity of HNN-extensions with abelian base groups

V. Metaftsis, E. Raptis and D. Varsos

Abstract
We show that an HNN-extension with finitely generated abelian base group is \( \mathbb{Z} \)-linear if and only if it is residually finite.

1 Introduction
A group \( G \) is linear if it admits a faithful representation into some matrix group \( \text{GL}_n(k) \) for some commutative ring \( k \). The linearity of groups seems to be a difficult property to recognize and in most cases there is no certain method to do so. The first that systematically studied linearity of groups was Mal’cev [8]. Since then many authors have shown the linearity of certain families of groups but the list of linear groups remains short. We must mention here that in 1988, Lubotzky (see [7]) gave necessary and sufficient conditions for a group to be linear over \( \mathbb{C} \). Unfortunately, proving that certain groups satisfy the Lubotzky’s criterion, appears to be not an easy job.

In case we choose \( k \) to be the ring of integers \( \mathbb{Z} \), the range of examples of linear groups shortens even further. The purpose of the present note is to investigate the linearity of HNN-extensions of the form \( G = \langle t, K \mid tAt^{-1} = B \rangle \) with \( K \) a finitely generated abelian group. Our main result shows that those groups are \( \mathbb{Z} \)-linear if and only if are residually finite.

Moreover, an interesting side result shows that when a certain isolated subgroup of \( K \) is trivial and \( A \cap B \neq 1 \) then there is a finite index subgroup \( K' \) of \( K \) such that the isomorphism between \( A \) and \( B \) is induced by an isomorphism of \( K' \) of finite order. This allows us to embed a certain finite index subgroup of the HNN-extension into a larger group which in turn has a finite index subgroup which is a right-angled Artin group. That is enough to prove the linearity of the original HNN-extension. At the end of the paper we also give various consequences of our main result concerning HNN-extensions.

We conjecture that fundamental groups of trees with finitely generated abelian vertex groups are always linear and even that fundamental groups of graphs with finitely generated abelian vertex groups are linear if and only if are residually finite, but we have not been able to prove such result with the techniques at hand.
2 On the isomorphisms of subgroups of finitely generated free abelian groups

Let $G$ be a group and $H$ a subgroup of $G$. The subgroup $H$ is isolated in $G$, if whenever $g^n H \in H$ for $g \in G$ and $n > 0$, then $g \in H$. By $i_G(H)$ we denote the isolated closure of $H$ in $G$, that is, the intersection of all isolated subgroups of $G$ that contain $H$. For more on isolated subgroups and the isolated closure of a group, the author should consult [3].

For the sequel, $G$ is always the HNN-extension $G = \langle t, K \mid tAt^{-1} = B, \phi \rangle$ where $K$ is a finitely generated free abelian group and $A, B$ isomorphic subgroups of $K$ with $\phi : A \rightarrow B$ the isomorphism induced by $\phi$. Let $D$ be the subgroup of $G$ with

$$D = \{x \in K \mid \text{for each } \nu \in \mathbb{Z} \text{ there exists } \lambda = \lambda(\nu) \in \mathbb{N} \text{ such that } t^{-\nu}x^{\lambda}t^\nu \in K\}.$$ 

Then $D$ is an isolated subgroup of $K$ (see Proposition 2 in [1]) and therefore a direct factor of $K$. Moreover $D \leq i_K(A \cap B)$ and $i_K(A \cap B)/D$ is a free abelian group. If fact $D$ plays a central rôle in the proofs of the main results of [1, 2] and apparently in the present work as well.

We can also describe $D$ as follows: let $M_0 = A \cap B$, $M_1 = \varphi^{-1}(M_0) \cap M_0 \cap \varphi(M_0)$ and inductively $M_{i+1} = \varphi^{-1}(M_i) \cap M_i \cap \varphi(M_i)$. Then $M_{i+1} \leq M_i$ and since $K$ is finitely generated, there is $k \in \mathbb{N}$ such that the rank $\text{rank}(M_{k+1}) = \text{rank}(M_k)$. Consequently, $D = i_K(M_k)$ (see again [1, 2]). Notice also that if $H = \bigcap_{i=0}^\infty M_i$, then $H$ contains every subgroup $L$ of $K$ with $\varphi(L) = L$.

Finally, we can give an alternative description of $D$ using standard Bass-Serre theory (see [13]). Let $T$ be the standard tree on which $G$ acts. Then $D$ is the subgroup of $K$ such that for every finite subtree $T'$ of $T$, there is a positive integer $n \in \mathbb{N}$ such that $D^n$ stabilizes $T'$ pointwise. Notice that the subgroup $H$ defined above is the subgroup of $K$ that stabilizes the entire tree $T$ pointwise.

In this section we show that if $D = 1$ and $A \cap B \neq 1$, then there is an algorithm that allows us to consider a certain finite index subgroup of $K$, such that the isomorphism induced to it by $\varphi$, has finite order.

Step I. Take a non-trivial $c_1 \in A \cap B$ and the powers

$$\ldots, \varphi^{-2}(c_1), \varphi^{-1}(c_1), c_1, \varphi(c_1), \varphi^2(c_1), \ldots.$$ 

Since $D = 1$ there are $\lambda_1, \mu_1 \in \mathbb{N}$ with the property

$$\mathcal{S}_1 = \{\varphi^{-\lambda_1+1}(c_1), \ldots, \varphi^{-1}(c_1), c_1, \varphi(c_1), \ldots, \varphi^{\mu_1-1}(c_1)\} \subseteq A \cap B,$$

but $\varphi^{-\lambda_1}(c_1) \in A \setminus (A \cap B)$ and $\varphi^{\mu_1}(c_1) \in B \setminus (A \cap B)$.

Here we can suppose that $\langle \varphi^{-\lambda_1}(c_1) \rangle \cap (A \cap B) = 1$ and $\langle \varphi^{\mu_1}(c_1) \rangle \cap (A \cap B) = 1$. For if $\langle \varphi^{-\lambda_1}(c_1) \rangle^\kappa \in A \cap B$ for some $\kappa \in \mathbb{N}$ and $\langle \varphi^{\mu_1}(c_1) \rangle^\nu \in A \cap B$ for some $\nu \in \mathbb{N}$, then we can replace $c_1$ by $c'_1 = (c_1)^{\kappa \nu}$ and obtain (possibly some others) $\lambda_1, \mu_1 \in \mathbb{N}$ with the property that they are the greatest positive integers such that

$$\{\varphi^{-\lambda_1+1}(c'_1), \ldots, \varphi^{-1}(c'_1), c'_1, \varphi(c'_1), \ldots, \varphi^{\mu_1-1}(c'_1)\} \subseteq A \cap B,$$

but $\varphi^{-\lambda_1}(c'_1) \in A \setminus (A \cap B)$ and $\varphi^{\mu_1}(c'_1) \in B \setminus (A \cap B)$.
\langle \varphi^{-\lambda_1}(c'_1) \rangle \cap (A \cap B) = 1 \text{ and } \langle \varphi^{\mu_1}(c'_1) \rangle \cap (A \cap B) = 1.

The elements of the set

\[ \tilde{S}_1 = \{ \varphi^{-\lambda_1}(c_1), \ldots, \varphi^{-1}(c_1), c_1, \varphi(c_1), \ldots, \varphi^{\mu_1}(c_1) \} = \{ \varphi^{-\lambda_1}(c_1) \} \cup S_1 \cup \{ \varphi^{\mu_1}(c_1) \} \]

are \((Z-)\) linearly independent in \(K\). Indeed, let \(\xi_1 \varphi^{-\lambda_1}(c_1) + \cdots + \xi_n \varphi^{-1}(c_1) + \xi_0 c_1 + \xi_1 \varphi(c_1) + \cdots + \xi_{\mu_1} \varphi^{\mu_1}(c_1) = 0\) with \(\xi_j \in \mathbb{Z}\). If \(j_0\) with \(-\lambda_1 \leq j_0 \leq \mu_1\) is the smallest subscript such that \(\xi_{j_0} \neq 0\) we have that \(\xi_{j_0} \varphi^{\mu_1}(c_1) = \xi_{j_0+1} \varphi^{\mu_1+1}(c_1) + \cdots + \xi_n \varphi^{-1}(c_1) + \xi_0 c_1 + \xi_1 \varphi(c_1) + \cdots + \xi_{\mu_1-1} \varphi^{\mu_1}(c_1)\) and from this (by applying \(\varphi^{-\lambda_1-j_0}\)) we have that \(\xi_{j_0} \varphi^{-\lambda_1}(c_1) = \xi_{j_0+1} \varphi^{\mu_1+1}(c_1) + \cdots + \xi_n \varphi^{-1}(c_1) + \xi_0 c_1 + \xi_1 \varphi(c_1) + \cdots + \xi_{\mu_1-1} \varphi^{\mu_1}(c_1)\) \(\varphi^{-\lambda_1-j_0} \in B\), absurd by the previous observation.

So the set \(\tilde{S}_1\), as linearly independent, generates a free abelian group \(S_1 = \langle \tilde{S}_1 \rangle\) of rank \(\lambda_1 + 1 + \mu_1\) and the (cyclic) permutation \(j \rightarrow j+1\) for \(-\lambda_1 \leq j \leq \mu_1\) taken mod \((\lambda_1 + 1 + \mu_1)\) on the exponents of \(\varphi(c)\) defines an automorphism, say \(\varphi_1\), of \(S_1\), of order \(\lambda_1 + 1 + \mu_1\).

Notice that, by the definition of \(\varphi_1\), we have that \(\varphi_1 |_{A \cap S_1} = \varphi\). Moreover, we have that \(\varphi_1(A \cap S_1) = \varphi(A \cap S_1) = B \cap S_1\). Indeed, if \(a \in A \cap \tilde{S}_1\) then \(a\) is a linear combination of \(\varphi^{-\lambda_1}(c_1), \ldots, \varphi^{-1}(c_1), c_1, \varphi(c_1), \ldots, \varphi^{\mu_1}(c_1)\).

Moreover, \(\varphi(a)\) is a linear combination of \(\varphi^{-\lambda_1+1}(c_1), \ldots, \varphi^{-1}(c_1), c_1, \varphi(c_1), \ldots, \varphi^{\mu_1}(c_1)\).

Step II. Let \(c_2 \in A \cap B\) such that \(\langle c_2 \rangle \cap S_1 = 1\). We repeat the above process in order to construct the set \(\tilde{S}_2 = \{ \varphi^{-\lambda_2+1}(c_2), \ldots, \varphi^{-1}(c_2), c_2, \varphi(c_2), \ldots, \varphi^{\mu_2-1}(c_2) \} \subseteq A \cap B\) and the set \(\tilde{S}_2\). Again, \(\tilde{S}_2\) has similar properties, namely \(\tilde{S}_2\) is \((Z-)\) linearly independent in \(K\) and there are \(\lambda_2, \mu_2 \in \mathbb{N}\) being the greatest positive integers such that

\[ \{ \varphi^{-\lambda_2+1}(c_2), \ldots, \varphi^{-1}(c_2), c_2, \varphi(c_2), \ldots, \varphi^{\mu_2-1}(c_2) \} \subseteq A \cap B, \]

\(\langle \varphi^{-\lambda_2}(c_2) \rangle \cap (A \cap B) = 1\) and \(\langle \varphi^{\mu_2}(c_2) \rangle \cap (A \cap B) = 1\).

Also, the set \(\tilde{S}_2\), as linearly independent in \(K\), generates the free abelian group \(S_2 = \langle \tilde{S}_2 \rangle\) of rank \(\lambda_2 + 1 + \mu_2\) and the (cyclic) permutation \(j \rightarrow j + 1\) for \(-\lambda_2 \leq j \leq \mu_2\) taken mod \((\lambda_2 + 1 + \mu_2)\) on the exponents of \(\varphi(c)\) defines the automorphism \(\varphi_2\), of \(S_2\) of order \(\lambda_2 + 1 + \mu_2\).

Notice that \(\tilde{S}_1 \cap \tilde{S}_2 = \emptyset\). For if \(\varphi_2(c_2) = \varphi_1(c_1)\), then \(c_2 = \varphi_2^j(c_1)\) with \(-\lambda_1 \leq j - i \leq \mu_1\) (the only powers of \(\varphi\) that can be applied to \(c_1\) are integers between \(-\lambda_1\) and \(\mu_1\)), and this contradicts the choice of \(c_2\).

Step III. Repeat the above procedure, for \(c_3 \in A \cap B\) such that \(\langle c_3 \rangle \cap (S_1 + S_2) = 1\), in order to construct the sets \(\tilde{S}_3\) and \(\tilde{S}_3\) with

\[ \tilde{S}_3 = \{ \varphi^{-\lambda_3+1}(c_3), \ldots, \varphi^{-1}(c_3), c_3, \varphi(c_3), \ldots, \varphi^{\mu_3-1}(c_3) \} \subseteq A \cap B. \]
and
\[ \mathcal{S}_3 = \{ \varphi^{-\lambda_3}(c_3), \ldots, \varphi^{-1}(c_3), c_3, \varphi(c_3), \ldots, \varphi^{\mu_3}(c_3) \} = \{ \varphi^{-\lambda_1}(c_3) \} \cup \mathcal{S}_3 \cup \{ \varphi^{\mu_1}(c_3) \}. \]

Since the free rank of \( A \cap B \) is finite, there will be an \( n \in \mathbb{N} \) such that the sets \( \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n \), constructed in finitely many steps, by the above procedure, almost generate the subgroup \( A \cap B \) in the sense that \( \langle \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n \rangle \) is of finite index in \( A \cap B \).

For each \( \mathcal{S}_i \) and \( \tilde{\mathcal{S}}_i \), let \( S_i = \langle \tilde{\mathcal{S}}_i \rangle \). If we take the equivalence relation up to commensurability on the subgroups \( S_1, S_2, \ldots, S_n \), we can suppose that for \( i \neq j, S_i \cap S_j \) is of infinite index in both \( S_i \) and \( S_j \). This means that the intersection \( S_i \cap S_j \) belongs to \( A \cap B \) for all \( i \neq j \). By their definitions, the automorphisms \( \varphi_i \) act on the elements of \( A \cap B \) as the isomorphism \( \varphi \). So, for all \( i = 1, \ldots, n \) all \( \varphi_i \) coincide on the elements of the intersections \( S_i \cap S_j \). Therefore all \( \varphi_i \) can be extended to a common automorphism \( \vartheta \) of the subgroup \( S = \langle S_1, S_2, \ldots, S_n \rangle \), which is of finite order, since it acts as a permutation on the generators of \( S_i \).

So we can now show the following.

**Proposition 2.1.** Let \( G = \langle t, K \mid t A t^{-1} = B, \varphi \rangle \) where \( \varphi \) is the isomorphism induced by \( t \). Suppose that \( D \) is defined as above and that \( D = 1 \). Then there exists a finite index subgroup \( K \) of \( K \) in which \( \varphi \) induces an automorphism \( \bar{\varphi} \), i.e., \( \bar{\varphi} |_{K \cap A} = \varphi \). Moreover, \( \bar{\varphi} \) has finite order.

**Proof.** Since \( D = 1 \), the subgroups \( A \) and \( B \) must be of infinite index in \( K \).

Suppose that \( A \cap B = 1 \). Then for the isolator closures \( i_K(A) \) and \( i_K(B) \) of \( A \) and \( B \) we have \( i_K(A) \cap i_K(B) = 1 \) and therefore, there is a subgroup \( N \subseteq K \) such that \( K = N \oplus i_K(A) \oplus i_K(B) \). The group \( \bar{K} = N \oplus A \oplus B \) has the required property. Indeed, we can define the automorphism \( \bar{\varphi} = 1_N \oplus \varphi \oplus \varphi^{-1} \) which has order two and \( \bar{\varphi}|_A = \varphi \).

Suppose now that \( A \cap B \neq 1 \). Let \( S = \langle S_1, S_2, \ldots, S_n \rangle \) be the subgroup constructed in the above procedure. Then \( S \) is a subgroup of \( \langle A, B \rangle \). If \( S \) is of finite index in \( \langle A, B \rangle \), then we take a (direct) complement of \( K \) in the sense \( K = N \oplus i_K(\langle A, B \rangle) \). Then the group \( \bar{K} = N \oplus S \) has the required property since we can define the automorphism \( \bar{\varphi} = 1_N \oplus \vartheta \) which has finite order, where \( \vartheta \) is the automorphism defined above.

If \( S \) is of infinite index in \( \langle A, B \rangle \), then there are (free) generators \( \{ \bar{v}_1, \ldots, \bar{v}_m \} \) of \( A \) such that \( A = \langle \bar{v}_1, \ldots, \bar{v}_m \rangle \oplus i_A(\langle S_1, S_2, \ldots, S_n \rangle) \). If we take \( b_1 = i_A \varphi \bar{v}_1, \ldots, b_m = i_A \varphi \bar{v}_m \), we obtain the group \( C = \langle \bar{v}_1, \ldots, \bar{v}_m \rangle \oplus \langle S_1, S_2, \ldots, S_n \rangle \oplus \langle b_1 = \varphi(\bar{v}_1), \ldots, b_m = \varphi(\bar{v}_m) \rangle \), which by its construction is of finite index in \( \langle A, B \rangle \). Evidently, the automorphism \( \vartheta \) can be extended to an automorphism of \( C \), and then we proceed as above taking \( C \) in place of \( S \).

\( \square \)

### 3 Linearity of HNN-extensions

**Theorem 3.1.** Let \( K \) be a finitely generated free abelian group, \( \varphi \) an automorphism of \( K \) of finite order and \( A \) a subgroup of \( K \). Then, the multiple
HNN-extension

\[ G = \langle t_1, \ldots, t_n, K \mid t_i a t_i^{-1} = \varphi(a), a \in A, i = 1, \ldots, n \rangle \]

is \( \mathbb{Z} \)-linear.

**Proof.** We take the HNN-extension \( \tilde{G} \) generated by the elements \( \xi_1, \ldots, \xi_n, \zeta, K \), satisfying the following relations \( \xi_i k \xi_i^{-1} = \varphi(k), k \in K, i = 1, \ldots, n, [\zeta, a] = 1 \) for all \( a \in A \). Notice that each \( \xi_i \) acts on \( K \) as an automorphism of finite order, the same order for all \( i = 1, \ldots, n \).

The map \( f : G \to \tilde{G} \) with \( f(k) = k \) for every element of \( K \) and \( f(t_i) = \xi_i \zeta \) defines a monomorphism. Indeed, the relations in \( G \) are preserved by \( f \), so it is a homomorphism, and if \( 1 \neq g \in G \), then by a \( t \)-length argument we can see that \( f(g) \neq 1 \), namely \( f \) is 1-1.

Let \( \nu \) be the order of the automorphism \( \varphi \). Since \( \langle \xi_i, i = 1, \ldots, n \rangle \) generate a free group, we can consider the epimorphism \( \psi : \langle \xi_1, \ldots, \xi_n \rangle \to \mathbb{Z}_\nu \) with \( \psi(\xi_i) = 1 \) for all \( i = 1, \ldots, n \). This epimorphism, extends to an epimorphism of \( \tilde{G} \), which for simplicity we also denote \( \psi, \psi : G \to \mathbb{Z}_\nu \) by sending all elements of \( K \) and \( \zeta \) to zero. Let \( H \) be the kernel of \( \psi \). Obviously, \( H \) is a subgroup of finite index in \( \tilde{G} \). In order to find a presentation of \( H \) we choose a Schreier transversal \( U \) for \( H \) to be the set \( U = \{1, \xi_1, \xi_1^2, \ldots, \xi_1^{\nu-1}\} \). Then \( H \) is generated by \( \{\xi_i, \xi_1^k, \xi_1^{k-2}, \xi_1^2 \xi_1 \xi_1^{-3}, \ldots, \xi_1^{\nu-2}, \xi_1^\nu, \xi_1^{-\nu} \xi_1 \} \) for all \( i = 1, \ldots, n \) along with the generators of \( K, \zeta \) and all \( \xi_1^j \) conjugates of the generators of \( K \) and \( \zeta \xi_1^{-j} \) for all \( j = 1, \ldots, \nu - 1 \).

Now, from the Schreier rewriting process, the relations of \( H \) are the relations \( u r u^{-1} \) where \( u \in U \) and \( r \in R \), where \( R \) is the set of relations of \( \tilde{G} \). One can easily see that the relations that do not involve any \( \xi \) are commutators and therefore all its conjugates are also commutators in the generating set of \( H \). On the other hand, any relation of the form

\[ \xi_1^k \xi_i a \xi_i^{-1} \xi_1^k = \xi_1^k \varphi(a) \xi_1^{-k} \]

transformed into the generating set of \( H \), becomes

\[ \xi_1^k \xi_i a \xi_i^{-1} \xi_1^{k-1} . \xi_1^{-(k-1)} a \xi_1^{k-1} . \xi_1^{-(k-1)} \xi_i^{-1} \xi_1^{-k} = \xi_1^k \varphi(a) \xi_1^{-k} \]

or equivalently

\[ \xi_1^k \xi_i a \xi_1^{k-1} . \xi_1^{-(k-1)} a \xi_1^{k-1} . \xi_1^{-(k-1)} \xi_i^{-1} \xi_1^{-k} = \xi_1^k (\xi_1^{-1} a \xi_1) \xi_1^{-k} = \xi_1^{-(k-1)} a \xi_1^{k-1}, \]

which is again a commutator in the generators of \( H \). Therefore, all relations that involve \( \xi \), rewritten in the generators of \( H \) become commutators.

But the above implies that \( H \) has a presentation where all relations are commutators of the generators and therefore is a right-angled Artin group. Consequently \( H \) is \( \mathbb{Z} \)-linear. (For more on right-angled Artin groups and its linearity the reader can see [4, 6].) But it is known that the linearity is closed under taking finite extensions or subgroups. Therefore the group \( G \) is \( \mathbb{Z} \)-linear. \( \Box \)
The above technique can actually show that if $K$ is a right-angled Artin group with standard generating set $S$, and $S_1, S_2$ are subsets of $S$ then any HNN-extension $\langle t, S \mid tS_i t^{-1} = S_j \rangle$ is $\mathbb{Z}$-linear. This is only a special case of a more general result by Hsu and Leary [5].

**Proposition 3.2.** Let $K$ be a finitely generated free abelian group and $A, B$ isomorphic subgroups of $K$ with $\varphi : A \to B$ an isomorphism. Suppose that $D = 1$, then the HNN-extension

$$G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A, \rangle$$

is $\mathbb{Z}$-linear.

**Proof.** From the proposition 2.1 there exists a finite index subgroup $\bar{K}$ in $K$ and an automorphism $\check{\varphi}$ of $\bar{K}$ of finite order such that $\check{\varphi} |_{K \cap A} = \varphi$. Let $G_1 = \langle t, \bar{K} \rangle^G$ be the normal closure of $\langle t, K \rangle$ in $K$. Evidently $G_1$ is of finite index in $G$. Let $k_1, k_2, \ldots, k_n$ be representatives of $\bar{K}$ in $K$. We take $t_i = k_i t k_i^{-1}$, $i = 1, 2, \ldots, n$, then using the Schreier rewriting process we obtain for $G_1$ the presentation $G_1 = \langle t_1, \ldots, t_n, \bar{K} \mid t_i a t_i^{-1} = \check{\varphi}(a), a \in K \cap A, i = 1, \ldots, n \rangle$. The group $G_1$ is $\mathbb{Z}$-linear by the previous theorem. So $G$ is, as finite extension of $G_1$. \hfill \Box

**Proposition 3.3.** Let $K$ be a finitely generated abelian group and $A, B$ isomorphic subgroups of $K$ with $\varphi : A \to B$ an isomorphism. Suppose that $D$ is finite, then the HNN-extension

$$G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A \rangle$$

is $\mathbb{Z}$-linear.

**Proof.** The group $G$ is residually finite, since $D$ is finite (see [1]). Therefore there exists a finite index normal subgroup $N$ such that $N \cap D = 1$. Let $K_1 = K \cap N$, $A_1 = A \cap N, B_1 = B \cap N$ and $\varphi_1 : A_1 \to B_1$ the induced isomorphism (the group $N$ is normal in $G$). It is clear that the corresponding subgroup $D_1 = \{ x \in K_1 \mid \text{for each } \nu \in \mathbb{Z} \text{ there exists } \lambda = \lambda(\nu) \in \mathbb{N} \text{ such that } t^{-\nu} x^\lambda t^\nu \in K_1 \}$ is trivial. So the hypotheses of Proposition 2.1 are satisfied, consequently there is a finite index subgroup $\bar{K}_2$ of $K_1$ and an automorphism $\check{\varphi}_1$ of $\bar{K}_2$ of finite order which extends $\varphi_1$. The normal closure $\langle t, \bar{K}_2 \rangle^G$ is of finite index in $G$ and, as in the previous proposition, we get that $\langle t, \bar{K}_2 \rangle^G$ (and therefore the group $G$) is $\mathbb{Z}$-linear. \hfill \Box

Let $K$ be a finitely generated abelian group and $A, B$ isomorphic subgroups of $K$ with $\varphi : A \to B$ an isomorphism and

$$G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A \rangle$$

be the corresponding HNN-extension. Let also $H$ be the largest subgroup of $K$ such that $\varphi(H) = H$. Then, since $K$ is abelian, it is easy to see that $H$ is the larger normal subgroup of $G$ contained in $K$, in other words $H = K_G$, the core of $K$ in $G$. 

6
Theorem 3.4. Let $K$ be a f.g. abelian group and $A, B$ proper, isomorphic subgroups of $K$ with $\varphi : A \rightarrow B$ an isomorphism and

$$G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A \rangle$$

be the corresponding HNN-extension. The group $G$ is $\mathbb{Z}$-linear if and only if the group $\bar{G} = G/H$ is $\mathbb{Z}$-linear.

Proof. Assume $G$ to be linear. Then $G$ is residually finite, therefore, by the main Theorem in [1], we have that $H$ is of finite index in $D$. The quotient $G/H$ has the HNN-presentation $G/H = \langle t, K \mid tA/Ht^{-1} = B/H, \varphi_H \rangle$, where $\varphi_H$ is the induced isomorphism (since $\varphi(H) = H$). The corresponding subgroup $D_H$ is equal to $\bar{D} = D/H$ which is finite. Therefore by previous proposition $G/H$ is linear.

For the converse, since the group $\bar{G} = G/H$ is linear, there is a homomorphism $\vartheta : G \rightarrow R$ to a linear group $R$ with $\text{Ker}\vartheta \leq H$. On the other hand the linearity of $\bar{G} = G/H$ implies the residually finiteness of it, but the corresponding $\bar{H} = (K/H)\bar{G}$ is trivial. Therefore, by the main Theorem in [1] the group $\bar{D} = D/H$ must be finite, which, by the Theorem 2 in [2], implies that there exists a finitely generated abelian group $X$ such that $K \leq X$ and an automorphism $\varphi$ of $X$ with $\varphi|_A = \varphi$. Now the obvious homomorphism $\varrho : G \rightarrow X \rtimes \langle \varphi \rangle$ is an embedding on $K$, so $\text{Ker}\varrho \cap K = 1$. The linearity of $G$ follows from the linearity of the groups $R$, $X \rtimes \langle \varphi \rangle$ and the fact that $\text{Ker}\vartheta \cap \text{Ker}\varrho = 1$.

Corollary 3.5. Let $K$ be a f.g. abelian group and $A, B$ proper isomorphic subgroups of $K$ with $\varphi : A \rightarrow B$ an isomorphism and

$$G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A \rangle$$

be the corresponding HNN-extension. The group $G$ is $\mathbb{Z}$-linear if and only if it is residually finite.

Proof. Suppose that $G$ is residually finite. By the main Theorem in [1] we have that $H$ is of finite index in $D$, where $H$ and $D$ are the subgroups of $K$ defined above. Therefore the group $G/H$ is linear by proposition 3.3. The result now follows from the previous theorem.

For the converse, it is well known that a finitely generated linear group is residually finite.

Remark 1. In the proof of Theorem 3.4, in order to prove that the linearity of $G$ implies the linearity of $G/H$ it is worth remarking that we used only that $G$ is residually finite and proposition 3.3, which in turn depends on proposition 2.1. We can give a direct proof using heavily that $G$ is linear as follows.

The quotient $G/H$ has the HNN-presentation $G/H = \langle t, K \mid tA/Ht^{-1} = B/H, \varphi_H \rangle$, where $\varphi_H$ is the induced isomorphism (since $\varphi(H) = H$). The corresponding subgroup $D_H$ is equal to $\bar{D} = D/H$ which is finite, so $G/H$ is residually finite. Therefore there exists a normal subgroup $\bar{N}$ of finite index in $G$ such that $H \leq \bar{N}$ and $\bar{N} \cap \bar{D} = \bar{H}$. By the structure theorem of Bass-Serre
theory (see [13]), the structure of the subgroups of an HNN-extension $N$ is the fundamental group of a finite graph of groups with vertex groups of the form $N \cap gKg^{-1}$, $g \in G$ and edge groups of the form $N \cap gAg^{-1}$, $g \in G$. Since $H = K_G$ and $N$ is normal in $G$, $H$ is contained in every edge (and vertex) group. Now the group $D$ is isolated in $K$, which implies that $D \cap N$ is isolated in $K \cap N$. So $H = gHg^{-1}$ is isolated (therefore a direct factor) in every vertex group. This means, by the normal form of elements of $N$, that $N = H \ltimes M$ for some subgroup $M$ of the linear group $G$. Consequently $N/H \simeq M$ is linear. But $N/H$ is of finite index in $G/H$, so $G/H$ is linear.

**Remark 2.** The value of the propositions 2.1 and 3.3 consists in the fact that exhibiting an internal property of finitely generated abelian groups we obtain a tangible criterion for the linearity of HNN-extensions with base group a finitely generated abelian group.

**Corollary 3.6.** Let $K$ be a finitely generated abelian group, $A,B$ proper subgroups of $K$ and $\varphi : A \longrightarrow B$ an isomorphism.

1. The subgroup $D$ is finite.

2. There exists a finite index subgroup $K_1$ of $K$ and an automorphism $\varphi_1$ of $K_1$ of finite order which extends $\varphi$ in the sense that $\varphi_1|_{A \cap K_1} = \varphi$.

3. There exists an abelian group $X$ which contains the group $K$ as a subgroup of finite index and an automorphism $\vartheta$ of $X$ of finite order such that $\vartheta|_A = \varphi$.

For the above statements we have the following:

1) implies 2).

2) is equivalent to 3).

1) implies 3).

**Proof.** 1) implies 2). This is the proposition 3.3.

2) is equivalent to 3). Assuming 2), it is easy to construct the group $X$ adding roots to the group $K_1$.

Assuming 3), we take $K_1 = X^n$, where $n$ is the index of $K$ in $X$.

1) implies 3) is a consequence of the above two. \qed

**Remark 3.** In account of Theorem 2 in [2], we see that there is a (weak) equivalence where there is no needed for the automorphism $\vartheta$ to have finite order. On the other hand, the statements 2) or 3) do not imply 1). For example the free abelian group $K = \langle a,b \mid [a,b] \rangle$ with $A = \langle a \rangle = B$ and $\varphi$ the identity satisfies (trivially) both 2) and 3), but not 1).

In Corollary 3.5 it is assumed that the associated subgroups $A$ and $B$ are proper subgroups in the base group $K$. In the case where one of them is all the base group (e.g. $K = A$), then the HNN-extension $G = \langle t, K \mid tKt^{-1} = B \rangle$ is a residually finite group, as a solvable constructible group, it is $\mathbb{Q}$-linear. (This result and a good account of basic properties of solvable constructible groups can be found in [14]). Then $G$ is not a $\mathbb{Z}$-linear group. This is concluded form
the fact that the subgroup $B$ is not closed in the profinite topology of $G$ (see e.g. in [10]), on the other hand if $G$ was $\mathbb{Z}$-linear, then by Theorem 5 p. 61 in [11], all subgroups of $K$ must be closed in the profinite topology of $G$.

In the case where the base group of an HNN-extension is not a f.g. abelian group only miscellaneous cases are known for the linearity of these groups.

**Proposition 3.7.** Let $K$ be any finitely generated linear group and $\varphi$ an isomorphism between finite subgroups $A$ and $B$ of $K$. The HNN-extension $G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A \rangle$ is linear.

*Proof.* At first, $K$ is residually finite and since $A$ and $B$ are finite groups, the group $G$ is residually finite. Therefore there exists a normal subgroup $N$ of finite index in $G$ such that $N \cap A = N \cap B = 1$. From the structure theorem of Bass-Serre theory, the subgroup $N$ is a free product of a free group and a finite family of subgroups of kind $N \cap gKg^{-1}$, $g \in G$. Then the result of Nisnevich ([9], see also [15]) implies that if the group $K$ is linear of degree $d \geq 2$, then $N$ is linear of degree at most $d + 1$. So $G$ is linear.

**Proposition 3.8.** (cf. Theorem 1.3. in [11]) The HNN-extension $G = \langle t, K \mid tat^{-1} = \varphi(a), a \in A \rangle$ with base group $K$ a polycyclic-by-finite group and proper associated subgroups $A$ and $B = \varphi(A)$ of finite index in $K$ is $\mathbb{Z}$-linear if and only if it is subgroup separable.

In the case where the associated subgroups are not of finite index we have a simple (very) special result.

**Proposition 3.9.** Let $K$ be the split extension of a polycyclic-by-finite group $A$ by a polycyclic-by-finite group $C$ ($K = A \rtimes C$). The HNN-extension $G = \langle t, K \mid tAt^{-1} = A, \varphi \rangle$ is linear.

*Proof.* Evidently $G = A \rtimes \langle t, C \rangle$. Therefore $G/A \simeq \langle t \rangle \ast C$, so it is linear. On the other hand the map $\vartheta : G \to A \rtimes \langle \varphi \rangle$ which sends every element $a \in A$ to itself, every element of $C$ to the trivial element and $t$ to $\varphi$ is a well defined homomorphism with $ker\vartheta \cap A = 1$. Therefore, $G$ is linear.

**References**

[1] Andreadakis, S., Raptis, E., Varsos, D., *A characterization of residually finite HNN-extensions of finitely generated abelian groups*, Archiv Math. 50 (1988), 495-501.

[2] Andreadakis, S., Raptis, E., Varsos, D., *Extending isomorphisms to automorphisms*, Archiv Math. 53 (1989), 121-125.

[3] Baumslag, G., Lecture notes on nilpotent groups. Regional Conference Series in Mathematics, No. 2 American Mathematical Society, Providence, R.I. 1971

[4] Brown, K.S., Buildings. Springer-Verlag, New York, 1989.
[5] Hsu T. and Leary, I.J., Artin HNN-extensions virtually embed in Artin groups. Bull. London Math. Soc., 40 (2008), 715–719.

[6] Hsu T. and Wise, D.T., On linear and residual properties of graph products. Michigan Math. J. 46 (1999), no. 2, 251–259.

[7] Lubotzky, A., A group theoretic characterization of linear groups. J. Algebra 113 (1988), no. 1, 207–214.

[8] Mal’cev, A.I., On the isomorphic representations of infinite groups of matrices, Mat. Sb. 9 (1940), 405–422.

[9] Nisnevč, V.I., Über gruppen die durch matrizen über einem kommutativen feld isomorph darstellbar sind, Mat. Sb. 8 (1940), 395–540.

[10] Raptis, E., Varsos, D., On the subgroup separability of the fundamental group of a finite graph of groups, Demonstratio Mathematica Vol. XXIX (1996), 43-52.

[11] Raptis, E., Talelli, O., Varsos, D., On finiteness conditions of certain graphs of groups, Int. J. Algebra and Computation 5 (1995), 719-724.

[12] Segal, D., Polycyclic groups, Cambridge University Press, 1983.

[13] Serre, J-P., Trees. Springer-Verlag 1980.

[14] Strebel, R., Finitely presented soluble groups, Group Theory, essays for Philip Hall (ed. K. W. Gruenberg and J. E. Roseblade), Academic Press 1984.

[15] Wehrfritz, B.A.F., Generalized free products of linear groups, Proc. London Math. Soc. (3) 27 (1973), 425-439.

Department of Mathematics, University of the Aegean, Karlovasi 832 00, Samos, Greece. Email: vmet@aegean.gr.

Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece.

Email of E. Raptis: eraptis@math.uoa.gr.

Email of D. Varsos: dvarsos@math.uoa.gr.