The shadow of a collapsing dark star

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Abstract

The shadow of a black hole is usually calculated, either analytically or numerically, on the assumption that the black hole is eternal, i.e., that it has existed for all time. Here we ask the question of how this shadow comes about in the course of time when a black hole is formed by gravitational collapse. To that end we consider a star that is spherically symmetric, dark and non-transparent and we assume that it begins, at some instant of time, to collapse in free fall like a ball of dust. We analytically calculate the dependence on time of the angular radius of the shadow, first for a static observer who is watching the collapse from a certain distance and then for an observer who is falling towards the centre following the collapsing star.

1 Introduction

When a black hole is viewed against a backdrop of light sources, the observer sees a black disc in the sky which is known as the shadow of the black hole. Points inside this black disc correspond to past-oriented light rays that go from the observer towards the horizon of the black hole, while points outside this black disc correspond to past-oriented light rays that are more or less deflected by the black hole and then meet one of the light sources. For a Schwarzschild black hole, which is non-rotating the shadow is circular and its boundary corresponds to light rays that asymptotically spiral towards circular photon orbits that fill the so-called photon sphere at 1.5 Schwarzschild radii around the black hole. For a Kerr black hole, which is rotating, the shadow is flattened on one side and its boundary corresponds to light rays that spiral towards spherical photon orbits that fill a 3-dimensional photon region around the black hole. For the supermassive black hole that is assumed to sit at the centre of our Galaxy, the predicted angular diameter of the shadow is about 53 microarcseconds which is within reach of VLBI observations. There is an ongoing effort to actually observe this shadow, and also the one of the second-best black-hole candidate at the centre of M87, see [http://www.eventhorizontelescope.org](http://www.eventhorizontelescope.org).
When calculating the shadow one usually considers an *eternal* black hole, i.e., a black hole that is static or stationary and exists for all time. For a Schwarzschild black hole, there is a simple analytical formula for the angular radius of the shadow which goes back to Synge [1]. (Synge did not use the word “shadow” which was introduced much later. He calculated what he called the *escape cones* of light. The opening angle of the escape cone is the complement of the angular radius of the shadow.) For a Kerr black hole, the shape of the shadow was calculated for an observer at infinity by Bardeen [2]. More generally, an analytical formula for the boundary curve of the shadow was given, for an observer anywhere in the domain of outer communication of a Plebański-Demiański black hole, by Grenzebach et al. [3, 4]. For the Kerr case, this formula was further evaluated by Tsupko [5]. These analytical results are complemented by ambitious numerical studies, performing ray tracing in black hole spacetimes with various optical effects taken into account. We mention in particular a paper by Falcke et al. [6] where the perspectives of actually observing black-hole shadows were numerically investigated taking the presence of emission regions and scattering into account, and a more recent article by James et al. [7] which focusses on the numerical work that was done for the movie *Interstellar* but also reviews earlier work.

As we have already emphasised, in all these analytical and numerical works an eternal black hole is considered. Actually, we believe that black holes are not eternal: They have come into existence some finite time ago by gravitational collapse (and are then possibly growing by accretion or mergers with other black holes). This brings us to the question of how an observer who is watching the collapse would see the shadow coming about in the course of time. This is the question we want to investigate in this paper.

The visual appearance of a star undergoing gravitational collapse has been studied in several papers, beginning with the pioneering work of Ames and Thorne [8]. In this work, and in follow-up papers e.g. by Jaffe [9], Lake and Roeder [10] and Frolov et al. [11], the emphasis is on the frequency shift of light coming from the surface of the collapsing star. More recent papers by Kong et al. [12] [13] and by Ortiz et al. [14] [15] investigated the frequency shift of light passing through a collapsing transparent star, thereby contrasting the collapse to a black hole with the collapse to a naked singularity. In contrast to all these earlier articles, here we consider a *dark* and *non-transparent* collapsing star which is seen as a black disc when viewed against a backdrop of light sources and we ask how this black disc changes in the course of time.

For the collapsing star we use a particularly simple model: We assume that the star is spherically symmetric and that it begins to collapse, at some instant of time, in free fall like a ball of dust until it ends up in a point singularity at the centre. The metric inside such a collapsing ball of dust was found in a classical paper by Oppenheimer and Snyder [16]. However, for our purpose, as we assume the collapsing star to be non-transparent, we do not need this interior metric. All we need to know is that a point on the surface follows a timelike geodesic in the ambient Schwarzschild spacetime. We will demonstrate that in this situation the time dependence of the shadow can be given analytically. We do this first for a static observer
who is watching the collapse from a certain distance, and then also for an observer who is falling towards the centre and ending up in the point singularity after it has formed. The latter situation is (hopefully) not of relevance for practical astronomical observations but we believe that the calculation is quite instructive from a conceptual point of view.

The paper is organised as follows. In Section 2 we review some basic facts on the Schwarzschild solution in Painlevé-Gullstrand coordinates. These coordinates are particularly well suited for our purpose because they are regular at the horizon, so they allow to consider worldlines of observers or light signals that cross the horizon without the need of patching different coordinate charts together. In Section 3 we rederive in Painlevé-Gullstrand coordinates the equations for the shadow of an eternal Schwarzschild black hole. We do this both for a static and for an infalling observer. The results of this section will then be used in the following two sections for calculating the shadow of a collapsing star. We do this first for a static observer in Section 4 and then for an infalling observer in Section 5. We summarise our results in Section 6.

2 Schwarzschild metric in Painlevé-Gullstrand coordinates

Throughout this paper, we work with the Schwarzschild metric in Painlevé-Gullstrand coordinates [17, 18],

\[ g_{\mu \nu} dx^\mu dx^\nu = -\left(1 - \frac{2m}{r}\right)c^2dT^2 + 2\sqrt{\frac{2m}{r}}c\,dT\,dr + dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2). \] (1)

Here

\[ m = \frac{GM}{c^2} \] (2)

is the mass parameter with the dimension of a length; \(M\) is the mass of the central object in SI units, \(G\) is Newton’s gravitational constant and \(c\) is the vacuum speed of light.

The Painlevé-Gullstrand coordinates \((T, r, \vartheta, \varphi)\) are related to the standard text-book Schwarzschild coordinates \((t, r, \vartheta, \varphi)\) by

\[ c\,dT = c\,dt + \sqrt{\frac{2m}{r}}\frac{dr}{\left(1 - \frac{2m}{r}\right)}. \] (3)

As a historical side remark, we mention that both Painlevé [17] and Gullstrand [18] believed that they had found a new solution to Einstein’s vacuum field equation before Lemaître [19] demonstrated that it is just the Schwarzschild solution in other coordinates. Whereas in the
standard Schwarzschild coordinates the metric has a coordinate singularity at the horizon at $r = 2m$, in the Painlevé-Gullstrand coordinates the metric is regular on the entire domain $0 < r < \infty$.

On the domain $2m < r < \infty$ we will use the tetrad

$$
e_0 = \frac{1}{c\sqrt{1 - \frac{2m}{r}}} \frac{\partial}{\partial T}, \quad e_1 = \sqrt{1 - \frac{2m}{r}} \frac{\partial}{\partial r} + \frac{\sqrt{2m}}{r} \frac{\partial}{\partial T},$$
$$
e_2 = \frac{1}{r} \frac{\partial}{\partial \vartheta}, \quad e_3 = \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi}.
$$

From (1) we read that this tetrad is orthonormal, $g_{\mu\nu} e^\mu e^\nu = \eta_{\rho\sigma}$ with $(\eta_{\rho\sigma}) = \text{diag}(-1, 1, 1, 1)$, for $2m < r < \infty$. Up to a factor of $c$, the vector field $e_0$ is the four-velocity field of observers that stay at fixed spatial coordinates $(r, \vartheta, \varphi)$. We refer to them as to the static observers.

From (1) we find that, for a static observer at radius coordinate $r (> 2m)$, proper time $\tau$ is related to the Painlevé-Gullstrand time coordinate $T$ by

$$
\frac{dT}{d\tau} = \frac{1}{\sqrt{1 - \frac{2m}{r}}}.\tag{5}
$$

In the Painlevé-Gullstrand coordinates, the geodesics in the Schwarzschild spacetime are the solutions to the Euler-Lagrange equations of the Lagrangian

$$
\mathcal{L}(x, \dot{x}) = \frac{1}{2} \left( - \left(1 - \frac{2m}{r}\right) c^2 \dot{T}^2 + 2 \sqrt{\frac{2m}{r}} c \dot{T} \dot{r} + r^2 + r^2 \left( \sin^2 \vartheta \dot{\varphi}^2 + \dot{\vartheta} \right) \right).
$$

Here the overdot means derivative with respect to an affine parameter.

The $T$ and $\varphi$ components of the Euler-Lagrange equations give us the familiar constants of motion $E$ and $L$ in Painlevé-Gullstrand coordinates,

$$
E = - \frac{\partial \mathcal{L}}{\partial \dot{T}} = \left(1 - \frac{2m}{r}\right) c^2 \dot{T} - \sqrt{\frac{2m}{r}} c \dot{r} \tag{7}
$$

and

$$
L = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2 \sin^2 \vartheta \dot{\varphi}.\tag{8}
$$

For the purpose of this paper we will need the radial timelike geodesics and the lightlike geodesics in the equatorial plane.
2.1 Radial timelike geodesics

We consider massive objects in radial free fall, i.e., radial geodesics ($\dot{\varphi} = 0$ and $\dot{\vartheta} = 0$) which are timelike. Then we may choose the affine parameter equal to proper time $\tau$,

$$- c^2 = - \left(1 - \frac{2m}{r}\right) c^2 \left(\frac{dT}{d\tau}\right)^2 + 2 \sqrt{\frac{2m}{r}} c \frac{dT}{d\tau} \frac{dr}{d\tau} + \left(\frac{dr}{d\tau}\right)^2. \quad (9)$$

In this notation (7) can be rewritten as

$$\varepsilon := \frac{E}{c^2} = \left(1 - \frac{2m}{r}\right) \frac{dT}{d\tau} - \sqrt{\frac{2m}{r}} \frac{1}{c} \frac{dr}{d\tau} \quad (10)$$

whereas (8) requires $L = 0$.

In the following we restrict to the case that the parametrisation by proper time is future oriented with respect to the Painlevé-Gullstrand time coordinate, $dT/d\tau > 0$, and we consider only ingoing motion, $dr/d\tau < 0$, that starts in the domain $r > 2m$. Then $\varepsilon > 0$ and (10) and (9) imply

$$\frac{dr}{d\tau} = - c \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}}, \quad \frac{dT}{d\tau} = \frac{\varepsilon - \sqrt{\frac{2m}{r} \varepsilon^2 - 1 + \frac{2m}{r}}}{1 - \frac{2m}{r}} \quad (11)$$

We distinguish three cases, see Fig. 1:

(a) $0 < \varepsilon < 1$: Then $dr/d\tau = 0$ at a radius coordinate $r_i$ given by $\varepsilon^2 = 1 - 2m/r_i$, i.e., this case describes free fall from rest at $r_i$. Clearly, the possible values of $r_i$ are $2m < r_i < \infty$.

(b) $\varepsilon = 1$: This is the limit of case (a) for $r_i \to \infty$, i.e., free fall from rest at infinity. It is usual to refer to such freely falling observers as to the Painlevé-Gullstrand observers. In this case the two equations (11) reduce to

$$\frac{dr}{d\tau} = - c \sqrt{\frac{2m}{r}}, \quad \frac{dT}{d\tau} = 1. \quad (12)$$

The second equation shows that the coordinate $T$ gives proper time along the worldlines of the Painlevé-Gullstrand observers.

(c) $1 < \varepsilon < \infty$: These are freely falling observers that come in from infinity with a non-zero inwards-directed initial velocity and then fall towards the centre.

Choosing a value of $\varepsilon > 0$ defines a family of infalling observers. We associate with this family the tetrad.
Figure 1: Radially infalling observers with $\varepsilon < 1$ (dotted), $\varepsilon = 1$ (solid) and $\varepsilon > 1$ (dashed). For the four worldlines with $\varepsilon \geq 1$ we have chosen the initial conditions such that they arrive simultaneously at $r = 0$. For the three worldlines with $\varepsilon < 1$ we have assumed that they start from rest at the same Painlevé-Gullstrand time; then these worldlines do not arrive simultaneously at $r = 0$. As an alternative, one could have considered worldlines that start from rest at the same Schwarzschild time; they would arrive simultaneously at $r = 0$.

$$
\tilde{e}_0 = \frac{\varepsilon - \sqrt{\frac{2m}{r}} \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}}}{\left(1 - \frac{2m}{r}\right)c} \partial_T - \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}} \frac{\partial}{\partial r},
$$

$$
\tilde{e}_1 = \varepsilon \frac{\partial}{\partial r} + \frac{\varepsilon \sqrt{\frac{2m}{r}} - \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}}}{\left(1 - \frac{2m}{r}\right)c} \frac{\partial}{\partial T},
$$

$$
\tilde{e}_2 = \frac{1}{r} \frac{\partial}{\partial \vartheta}, \quad \tilde{e}_3 = \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi}.
$$

(13)

For $\varepsilon \geq 1$ this tetrad is well-defined and orthonormal on the entire domain $0 < r < \infty$. For $0 < \varepsilon < 1$ the tetrad is restricted to the domain $0 < r < r_i = \frac{2m}{(1 - \varepsilon^2)}$.

The relative velocity $v$ of a radially infalling observer with respect to the static observer at the same event can be calculated from the special relativistic formula
\[ g_{\mu\nu} e^\mu_0 e^\nu_0 = \frac{-1}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

This results in

\[ \frac{v}{c} = \frac{1}{\varepsilon} \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}}. \]  

(15)

Clearly, this formula makes sense only on the domain on which both families of observers are defined. For \( \varepsilon \geq 1 \) this is true on the domain \( 2m < r < \infty \) whereas for \( 0 < \varepsilon < 1 \) it is true on the domain \( 2m < r < r_i = 2m/(1 - \varepsilon^2) \).

## 2.2 Lightlike geodesics in the equatorial plane

We will now rederive some results on lightlike geodesics in the Schwarzschild spacetime, using Painlevé-Gullstrand coordinates. Because of spherical symmetry, it suffices to consider geodesics in the equatorial plane, \( \vartheta = \pi/2 \). For lightlike geodesics the Lagrangian is equal to zero,

\[ 0 = -\left(1 - \frac{2m}{r}\right)c^2 \dot{T}^2 + 2 \sqrt{\frac{2m}{r}} c \dot{T} \dot{r} + r^2 \dot{\varphi}^2. \]  

(16)

Dividing by \( \dot{\varphi}^2 \) and using (7) and (8) yields

\[ \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \pm \frac{\sqrt{E^2 r^4 c^2 L^2 - r^2 + 2mr}}{r}. \]  

(17)

Reinserting this result into (16) gives us the equation for the Painlevé-Gullstrand travel time of light,

\[ c \frac{dT}{dr} = c \frac{\dot{T}}{\dot{r}} = \sqrt{\frac{2mr}{r - 2m}} \pm \frac{r}{(r - 2m) \sqrt{1 - (r - 2m) \frac{c^2 L^2}{E^2 r^5}}}. \]  

(18)

For all \( r > 2m \), in the last expression the second term is bigger than the first. Therefore, in this domain the upper sign has to be chosen if \( dT/dr > 0 \) and the lower sign has to be chosen if \( dT/dr < 0 \).

By differentiating (17) with respect to \( \varphi \) we find

\[ \frac{d^2r}{d\varphi^2} = \frac{4E^2 r^3}{c^2 L^2} - 2r + 2m. \]  

(19)
If along a lightlike geodesic the radius coordinate goes through an extremum at value $r_m$, (17) and (19) imply

$$0 = \frac{E^2 r_m^3}{c^2 L^2} - r_m + 2m,$$

(20)

$$\frac{d^2 r}{d\varphi^2} \bigg|_{r=r_m} = r_m - 3m.$$  

(21)

This demonstrates that only local minima may occur in the domain $r > 3m$ and only local maxima may occur in the domain $r < 3m$. The sphere at $r = 3m$ is filled with circular photon orbits that are unstable with respect to radial perturbations. These well-known facts will be crucial for the following analysis.

For a lightlike geodesic with an extremum of the radius coordinate at $r_m$, (20) may be used for expressing $E^2/L^2$ in (17) and (18) in terms of $r_m$. This results in

$$\frac{dr}{d\varphi} = \pm \sqrt{\frac{(r_m - 2m)r^4}{r_m^3} - r^2 + 2mr}$$

(22)

and

$$c \frac{dT}{dr} = \frac{\sqrt{2mr}}{(r - 2m)} \pm \frac{\sqrt{(r_m - 2m)r^5}}{(r - 2m)\sqrt{(r_m - 2m)r^3 - (r - 2m)r_m^3}}$$

(23)

### 3 The shadow of an eternal Schwarzschild black hole

In this section we rederive the formulas for the angular radius of the shadow of an eternal Schwarzschild black hole, both for a static and for an infalling observer, using Painlevé-Gullstrand coordinates. The results of this section will then be used for calculating the shadow of a collapsing star in the following sections.

We consider a lightlike geodesic $(T(s), r(s), \varphi(s))$ in the equatorial plane, where $s$ is an affine parameter. As before, we denote the derivative with respect to $s$ by an overdot. We may then expand the tangent vector of the lightlike geodesic with respect to the static tetrad $\mathbf{(1)}$ and also, as an alternative, with respect to the infalling tetrad $\mathbf{(13)}$ for some chosen $\varepsilon > 0$. Of course, the resulting equations are restricted to the domain where the respective tetrad is well-defined and orthonormal. As the tangent vector is lightlike, these expansions may be
written in terms of two angles $\alpha$ and $\tilde{\alpha}$,

$$
\dot{T} \frac{\partial}{\partial T} + \dot{r} \frac{\partial}{\partial r} + \dot{\varphi} \frac{\partial}{\partial \varphi} = \chi \left( e_0 + \cos \alpha \, e_1 - \sin \alpha \, e_3 \right) \\
= \tilde{\chi} \left( \tilde{e}_0 + \cos \tilde{\alpha} \, \tilde{e}_1 - \sin \tilde{\alpha} \, \tilde{e}_3 \right).
$$

(24)

If the parametrisation of the lightlike geodesic is future-oriented with respect to the $T$ coordinate, the scalar factors $\chi$ and $\tilde{\chi}$ are positive; otherwise they are negative. $\alpha$ is the angle between the lightlike geodesic and the radial direction in the rest system of the static observer, whereas $\tilde{\alpha}$ is the analogously defined angle in the rest system of the infalling observer. $\alpha$ and $\tilde{\alpha}$ may take all values between 0 and $\pi$. Of course, $\alpha$ is well-defined on the domain where the static observer exists (i.e., for $2m < r < \infty$) whereas $\tilde{\alpha}$ is well-defined on the domain where the infalling observer exists (i.e., for $0 < r < r_i = 2m/(1 - \varepsilon^2)$ if $0 < \varepsilon < 1$, and for $0 < r < \infty$ if $1 \leq \varepsilon < \infty$).

Comparing coefficients of $\partial/\partial r$ and $\partial/\partial \varphi$ in (24) yields

$$
\dot{r} = \chi \sqrt{1 - \frac{2m}{r}} \cos \alpha = \tilde{\chi} \left( \varepsilon \cos \tilde{\alpha} - \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}} \right),
$$

(25)

$$
\dot{\varphi} = -\chi \frac{\sin \alpha}{r} = -\tilde{\chi} \frac{\sin \tilde{\alpha}}{r}.
$$

(26)

From (25) and (26) we find

$$
\frac{\dot{\varphi}}{\dot{r}} = -\frac{\sin \alpha}{\sqrt{1 - \frac{2m}{r}} \cos \alpha} = \frac{-\sin \tilde{\alpha}}{r \left( \varepsilon \cos \tilde{\alpha} - \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}} \right)}.
$$

(27)

Now we apply these results to the case of a lightlike geodesic that goes through an extremum of the radius coordinate at some value $r_m$. If we evaluate (27) at a radius value $r > 3m$ this extremum is necessarily a local minimum, whereas it is necessarily a local maximum if we evaluate (27) at a radius value $r < 3m$. In either case (22) implies that the angles $\alpha$ and $\tilde{\alpha}$ at $r$ satisfy

$$
\frac{1}{r^2(r_m - 2m)} - \frac{2m}{r^3} = \frac{\sin^2 \alpha}{r^2 \left( 1 - \frac{2m}{r} \right)^2 \cos^2 \alpha} = \frac{\sin^2 \tilde{\alpha}}{r^2 \left( \varepsilon \cos \tilde{\alpha} - \sqrt{\varepsilon^2 - 1 + \frac{2m}{r}} \right)^2}.
$$

(28)

From the second equality sign in (28) we find
\[
\sin \alpha = \frac{\sqrt{1 - \frac{2m}{r} \frac{1}{\varepsilon} \sin \tilde{\alpha}}}{1 - \frac{1}{\varepsilon} \sqrt{\varepsilon^2 - 1 + \frac{2m}{r} \cos \tilde{\alpha}}}. \tag{29}
\]

By (15), this just demonstrates that \(\alpha\) and \(\tilde{\alpha}\) are related by the standard aberration formula.

From the first equality sign in (28) we find

\[
\sin \alpha = \sqrt{\frac{r_m^3(r - 2m)}{r^3(r_m - 2m)}} \tag{30}
\]

and equating the first to the third expression in (28) yields

\[
\sin \tilde{\alpha} = \frac{\sqrt{1 - \frac{2m}{r} \sqrt{\frac{(r - 2m)r_m^3}{(r_m - 2m)r^3}}}}{\varepsilon \pm \sqrt{\varepsilon^2 - 1 + \frac{2m}{r} \sqrt{1 - \frac{(r - 2m)r_m^3}{(r_m - 2m)r^3}}}}. \tag{31}
\]

In (31) the upper sign is valid if \(dr/d\varphi > 0\) and the lower sign is valid if \(dr/d\varphi < 0\) at \(r\).

From (30) and (31) we can now easily determine the angular radius of the shadow. The latter is defined in the following way: Consider an observer at radius coordinate \(r = r_O > 3m\).

Then a lightlike geodesic issuing from the observer position into the past may either go to infinity, possibly after passing through a minimum of the radius coordinate at some \(r_m > 3m\), or it may go to the horizon. Similarly, for an observer position \(2m < r_O < 3m\) there are lightlike geodesics that go to the horizon, possibly after passing through a maximum of the radius coordinate at some \(r_m < 3m\), and lightlike geodesics that go to infinity. In either case the borderline between the two classes consists of lightlike geodesics that asymptotically spiral towards a circular lightlike geodesic at \(r = 3m\). If we assume that there are light sources distributed in the spacetime anywhere but not between the observer and the black hole, then we have to associate darkness with the initial directions of lightlike geodesics that go to the horizon and brightness with those that go to infinity. This results in a circular black disc in the sky which is called the shadow of the black hole. The boundary of the shadow corresponds to lightlike geodesics that spiral towards \(r = 3m\). Therefore, we get the angular radius of the shadow for an observer at \(r = r_O\) if we send \(r_m \to 3m\) in (30) and (31). This results in

\[
\sin \alpha_{sh} = \frac{\sqrt{27m}}{r_O} \sqrt{1 - \frac{2m}{r_O} \sqrt{1 - \frac{2m}{r_O}}} \tag{32}
\]

and

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Figure 2: Angular radius $\alpha_{sh}$ of the shadow of a Schwarzschild black hole for a static observer. What we have plotted here is Synge’s formula (32). For an observer at $r_O = 3m$, we have $\alpha = \pi/2$, i.e., half of the sky is dark. For $r_O \to 2m$, we have $\alpha \to \pi$, i.e., in this limit the entire sky becomes dark.

respectively. (32) gives us the angular radius $\alpha_{sh}$ as it is seen by a static observer at $r_O$, see Figure 2. This formula is known since Synge [1]. It is meaningful only for observer positions $2m < r_O < \infty$ because the static observers exist on this domain only. By contrast, (33) gives us the angular radius $\tilde{\alpha}_{sh}$ of the shadow as it is seen by an infalling observer at momentary radius coordinate $r_O$, see Figure 3. A similar formula was derived by Bakala et al. [20], even for the more general case of a Schwarzschild-deSitter (Kottler) black hole. (33) is meaningful for $0 < r_O < r_i = 2m/(1 - \varepsilon^2)$ if $0 < \varepsilon < 1$ and for $0 < r_O < \infty$ if $1 \leq \varepsilon < \infty$. We have to choose the upper sign for $3m < r_O < \infty$ and the lower sign for $0 < r_O < 3m$. Nothing particular happens if the infalling observer crosses $r_O = 3m$ or $r_O = 2m$.

$$\sin \tilde{\alpha}_{sh} = \frac{\sqrt{27m^2}}{r_O} \left( \varepsilon \mp \sqrt{\varepsilon^2 - 1 + \frac{2m}{r_O} \sqrt{1 - \frac{27m^2}{r_O^2} \left( \frac{1 - 2m}{r_O} \right)}} \right),$$

(33)

$$\sin \tilde{\alpha}_{sh} \bigg|_{r_O=3m} = \frac{1}{\sqrt{3\varepsilon}}, \quad \sin \tilde{\alpha}_{sh} \bigg|_{r_O=2m} = \frac{\sqrt{27\varepsilon}}{1 + \frac{27}{4}\varepsilon^2}.$$  (34)
4 The shadow of a collapsing star for a static observer

In this section we consider a spherically symmetric star that undergoes gravitational collapse and a static observer who is watching the collapse. The star is assumed to be dark and non-transparent. In analogy to the black-hole case, we assume that there are light sources distributed anywhere in the spacetime but not in the region between the observer and the star. By the latter we mean the region covered by past-oriented light rays from the observer that reach the surface of the star (before the black hole has formed) or go to the horizon (after the black hole has formed). Under these assumptions the star will cast a circular shadow on the observer’s sky. It is our goal to determine the angular radius of this shadow as a function of
time.

For the collapsing star we use the simplest model: We assume that the star has constant radius \( r_S = r_i \) up to Painlevé-Gullstrand time \( T_S = 0 \) and then collapses in free fall like a ball of dust, i.e., such that each point on the surface of the star follows a radial timelike geodesic. Here and in the following we use the index \( S \) for the Painlevé-Gullstrand coordinates of the surface of the star, i.e., the star has radius \( r_S \) at time \( T_S \). For times \( T_S > 0 \) the worldline of an observer on the surface of the star is then given by one of the dotted lines in Figure 1. From (11) with \( \varepsilon^2 = 1 - \frac{2}{r_i} \frac{m}{r} \) we find that for \( T_S > 0 \)

\[
cT_S = \int_{r_S}^{r_i} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{2m(r_i - r)}} - \int_{r_S}^{r_i} \frac{\sqrt{2mr} \, dr}{(r - 2m)}.
\]

(35)

Equating \( r_S \) to zero gives the collapse time, \( T_S^{coll} \), i.e., the Painlevé-Gullstrand time when the star has collapsed to a point singularity at the centre, see Figure 4.

\[
cT_S^{coll} = \int_{0}^{r_i} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{2m(r_i - r)}} - \int_{0}^{r_i} \frac{\sqrt{2mr} \, dr}{(r - 2m)}.
\]

(36)

Note that necessarily \( 2m < r_i \). If \( 2m < r_i \leq 3m \), the star casts the same shadow as an eternal black hole, for any observer position outside the star. The reason is that then the past-oriented light rays from the observer position separate into the same two classes as in the case of an eternal black hole: there is the class of light rays that go to infinity and the class of light rays that do not, with the borderline corresponding to light rays that asymptotically spiral towards the light sphere at \( r = 3m \). So the formulas of the preceding section apply to this case as well. Of course, here it is crucial that the star is assumed to be dark and non-transparent.

We will, thus, assume from now on that the star collapses from an initial radius \( r_i > 3m \). For calculating the shadow we have to consider lightlike geodesics that graze the surface of the collapsing star. If such a lightlike geodesic passes through a minimum radius value \( r_m \), we may determine \( r_m \) by equating the first and the last expression in (28) with \( r = r_S, \tilde{\alpha} = \pi/2 \) and \( \varepsilon^2 = 1 - 2m/r_i \). This results in

\[
\frac{r_m^3}{r_m - 2m} = \frac{r_i r_S^2}{r_i - 2m}.
\]

(37)

Recall that a minimum value is possible only for \( r_m > 3m \), i.e., in (37) \( r_S \) must satisfy the inequality \( r_S > r_S^{(2)} \) where

\[
r_S^{(2)} = 3m \sqrt{3 - \frac{6m}{r_i}}.
\]

(38)
Figure 4: Spacetime diagram of a collapsing star and a static observer. The star begins to collapse at Painlevé-Gullstrand time $T = 0$ with radius $r_i$. The observer is static at radius $r_O$.

As $r_i > 3m$, (38) implies that

$$3m < r_s^{(2)} < r_i.$$  \hfill (39)

If $r_i$ varies over its allowed values from $3m$ to infinity, $r_s^{(2)}$ monotonically increases from $3m$ to $\sqrt{27}m$.

By (35), the star passes through the critical radius value $r_s^{(2)}$ at

$$cT_s^{(2)} = \int_{r_s^{(2)}}^{r_i} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{2m(r_i - r)}} - \int_{r_s^{(2)}}^{r_i} \frac{\sqrt{2mr} \, dr}{(r - 2m)}.$$  \hfill (40)

We divide the collapse of the star into three phases, see again Figure 4. In the first phase from $T_s = -\infty$ to $T_s = 0$ the star has a constant radius $r_i$. In the second phase from $T_s = 0$ to $T_s = T_s^{(2)}$ the star collapses to the critical radius value $r_s^{(2)}$. In the third phase from $T_s = T_s^{(2)}$ to $T_s = T_s^{\text{coll}}$ the star completes the collapse.

In Figure 4 we have indicated the worldline of an observer who is static at radius coordinate $r_O$. We will now discuss the shadow of the collapsing star as seen by this observer. As necessarily $2m < r_O$, we have to distinguish the following three cases, in accordance with (39): (a) $r_i < r_O$, (b) $r_s^{(2)} < r_O < r_i$ and (c) $2m < r_O < r_s^{(2)}$.

In case (a), we distinguish three phases of the development of the shadow, corresponding to the three phases of the collapse. In the first phase the observer sees a static star of radius $r_i$. As
the star is assumed to be dark, the observer sees a shadow whose angular radius is determined by light rays grazing the surface of the star, i.e., by light rays going through a minimum of the radius coordinate at \( r_m = r_i \). From (30) we read that the angular radius \( \alpha_{sh} \) of this shadow is given by

\[
\sin \alpha_{sh} = \sqrt{\frac{r_i^3 (r_O - 2m)}{r_O^3 (r_i - 2m)}}. \tag{41}
\]

This first phase ends when the observer sees the beginning of the collapse, i.e., at an observer time \( T_O^{(1)} \) when a light signal that has gone through its minimum radius value \( r_m = r_i \) at time \( T_S = 0 \) reaches the observer at \( r_O \). From (23) with the plus sign we find that

\[
c T_O^{(1)} = \int_{r_i}^{r_O} \frac{\sqrt{2mr} \, dr}{(r - 2m)} + \int_{r_i}^{r_O} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{(r_i - 2m)r^3 - (r - 2m)r_S^3}}. \tag{42}
\]

During the second phase the observer sees a collapsing star. The boundary of the shadow is determined by light rays that graze the surface of the collapsing star. The minimum radius value \( r_m \) of such light rays is given by (37). Inserting this value into (30), with \( r = r_O \), gives us the angular radius of the shadow in the second phase as a function of the parameter \( r_S \),

\[
\sin \alpha_{sh} = \sqrt{\frac{r_i^3 (r_O - 2m)}{r_O^3 (r_i - 2m)}}. \tag{43}
\]

The time \( T_O \) at which the shadow with this angular radius is seen is found by integrating (23),

\[
c(T_O - T_S) = \int_{r_S}^{r_O} \frac{\sqrt{2mr} \, dr}{(r - 2m)} + \int_{r_S}^{r_O} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{(r_i - 2m)r^3 - (r - 2m)r_S^3}} \tag{44}
\]

where again we have chosen the plus sign in (23) because \( T_O > T_S \). With (35) this results in

\[
c T_O = \int_{r_S}^{r_i} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{2m(r_i - r)}} + \int_{r_i}^{r_O} \frac{\sqrt{2mr} \, dr}{(r - 2m)} + \int_{r_S}^{r_O} \frac{\sqrt{r_i - 2m} \sqrt{r^3} \, dr}{(r - 2m) \sqrt{(r_i - 2m)r^3 - (r - 2m)r_S^3}}. \tag{45}
\]
Figure 5: Angular radius $\alpha_{\text{sh}}$ of the shadow of a collapsing dark star for a static observer. The observer is at $r_O = 10m$, the surface of the star is assumed to collapse from $r_i = 5m$. As a function of the Painlevé-Gullstrand time coordinate $T_O$ of the observer, $\alpha_{\text{sh}}$ is constant until $T_O^{(1)}$, which is shown as a dashed line, then it decreases according to the parametric description given by (45) and (43) until time $T_O^{(2)}$, and from $T_O^{(2)}$ on it is given by Synge’s formula (32) which is shown as a dotted line.

If $r_i$ and $r_O$ are given, with $3m < r_i < r_O$, (45) and (43) give us the relation between $T_O$ and $\alpha_{\text{sh}}$ in parametric form, $T_O = f_1(r_S)$ and $\alpha_{\text{sh}} = f_2(r_S)$, i.e., they give us the angular radius of the shadow in analytic form. This relation is valid in the second phase which lasts from $T_O^{(1)}$ up to a time $T_O^{(2)}$. In this time interval, $r_S$ runs down from $r_S^{(1)} = r_i$ to the value $r_S^{(2)}$ given in (38). From (45) we find that

$$cT_O^{(2)} = \int_{r_S^{(2)}}^{r_i} \frac{\sqrt{r_i - 2m} \sqrt{r_S^3} \, dr}{(r - 2m) \sqrt{2m(r_i - r)}} + \int_{r_i}^{r_O} \frac{\sqrt{2mr} \, dr}{(r - 2m)} + \int_{r_S^{(2)}}^{r_O} \frac{\sqrt{r_S^3} \, dr}{(r - 2m) \sqrt{r_S^3 - (r - 2m)27m^2}}.$$  

(46)

In the third phase, i.e., for times $T_O > T_O^{(2)}$, the angular radius of the shadow is given by
Synge’s formula (32). Past-oriented light rays grazing the surface of the star cannot escape to infinity anymore, i.e., they do not give the boundary of the shadow; the latter is determined by light rays that spiral asymptotically to $r = 3m$.

$$cT_O^{(2)} - cT_O^{(1)}$$

Figure 6: Time $T_O^{(2)} - T_O^{(1)}$ over which the static observer sees the star collapse, plotted against the observer position $r_O$. The star is collapsing from an initial radius $r_i = 5m$ (dotted), $r_i = 10m$ (solid) or $r_i = 15m$ (dashed), respectively.

We summarise our analysis in the following way. In the first phase, which lasts from $T_O = -\infty$ to $T_O = T_O^{(1)}$ given by (42), the observer sees a shadow of constant angular radius given by (41). In the second phase, which lasts from $T_O = T_O^{(1)}$ until $T_O = T_O^{(2)}$ given by (46), the observer sees a shrinking shadow whose angular radius as a function of observer time $T_O$ is given in parametric form by (45) and (43). The parameter $r_S$ runs down from $r_S^{(1)} = r_i$ to $r_S^{(2)} = 3m\sqrt{3} - 6m/r_i$. The third phase lasts from $T_O = T_O^{(2)}$ to $T_O = \infty$. In this period the observer sees a shadow of constant angular radius given by Synge’s formula (32). The angular radius of the shadow is plotted against $T_O$, over all three periods, for $r_i = 5m$ and $r_O = 10m$ in Figure 5.

In Fig. 6 we plot the time $T_O^{(2)} - T_O^{(1)}$ over which the observer sees the star collapse against the observer position $r_O$. We see that this time is largely independent of $r_O$, unless the observer is very close to the star. For a star collapsing from an initial radius of 5 Schwarzschild radii, $r_i = 10m$, we see that $T_O^{(2)} - T_O^{(1)} \approx 34 m/c$ for a sufficiently distant observer. For a stellar black hole, a typical value would be $m \approx 15 km$, resulting in $T_O^{(2)} - T_O^{(1)} \approx 0.001 sec$, so such a collapse would happen quite quickly. Even for a supermassive black hole of $m \approx 10^6 km$, the observer would see the collapse happen in less than 2 minutes. For the case of a collapsing cluster of galaxies the formation of the shadow would take longer, but in this case it is more reasonable.
to model the collapsing object as transparent. Note that on the worldline of a distant static observer Painlevé-Gullstrand time $T_O$ is practically the same as proper time $\tau_O$ because, by (5),

$$\tau_O = \sqrt{1 - \frac{2m}{r_O}} T_O + \text{constant.}$$

(47)

Figure 7: Angular radius $\alpha_{sh}$ of the shadow of a collapsing dark star for a static observer. The observer is at $r_O = 4.5m$, the surface of the star is collapsing from $r_i = 5m$. The observation begins at Painlevé-Gullstrand time $T_O^{(0)}$ when the surface of the star passes through the radius value $r_O$. At this moment the angular radius of the shadow takes a value $\alpha_{sh}^{(0)}$ which is smaller than $\pi/2$, because of aberration. As a function of the Painlevé-Gullstrand time coordinate $T_O$, the angular radius $\alpha_{sh}$ then decreases until it becomes a constant at time $T_O^{(2)}$. This constant value, which is again shown as a dotted line, is given by Synge’s formula (32).

A very similar analysis applies to case (b). The only difference is that then in the beginning the observer is inside the star. The observation can begin only at the time when the surface of the star passes through the radius value $r_O$ which, by assumption, is bigger than $r_S^{(2)}$. From that time on, the angular radius of the shadow is given by the same equations as before for the second and the third phase. A plot of the angular radius of the shadow against $T_O$ is shown in Figure 7 for $r_O = 4.5m$ and $r_i = 5m$.

In case (c) the observer is initially inside the star, as in case (b). The difference is in the
fact that now the radius of the star is smaller than \( r_S^{(2)} \) at the moment when the observation begins. Therefore, the shadow is never determined by light rays that graze the surface of the star; it is always determined by light rays that spiral towards \( r = 3m \), i.e., the angular radius of the shadow is constant from the beginning of the observation and given by Synge’s formula.

5 The shadow of a collapsing star for an infalling observer

We consider the same collapsing dark star as in the preceding section, but now we want to calculate the shadow as it is seen by an infalling observer. The relation between the coordinates \( r_O \) and \( T_O \) of the infalling observer can be found by integrating (11),

\[
ct_O = \int_{r_O}^{r^*_O} \frac{\varepsilon \sqrt{r^3} - \sqrt{2mr^2r - r + 2m}}{(r - 2m)\sqrt{\varepsilon^2r - r + 2m}} \, dr. \tag{48}
\]

Here \( r^*_O \) is an integration constant that gives the position of the observer at \( T = 0 \) which is the time when the star begins to collapse. We assume that \( r_i > 3m \), hence \( 3m < r_S^{(2)} < r_i \), and that \( r^*_O \) has been chosen big enough such that the observer is outside the star for all times, see Figure 8. For the time being we leave the constant of motion \( \varepsilon \) unspecified.

We will determine the angular radius of the shadow as a function of the observer position \( r_O \). As before, we distinguish three phases. In the first phase the observer sees a star of constant radius \( r_i \). The angular radius of the shadow can be read from (31) with the lower sign where we have to insert \( r = r_O \) and \( r_m = r_i \),

\[
\sin \tilde{\alpha}_{sh} = \frac{(r_O - 2m)\sqrt{r_i^3}}{\varepsilon r_O^{2\sqrt{r_i} - 2m} - \sqrt{\varepsilon^2r_O - r_O + 2m}\sqrt{(r_i - 2m)r_O^3 - (r_O - 2m)r_i^3}}. \tag{49}
\]

If \( r_i \) is given, this gives us explicitly \( \tilde{\alpha}_{sh} \) as a function of \( r_O \) for the first phase.

In the second phase we may again use (45). In combination with (48) this implies

\[
\int_{r_S}^{r_i} \frac{\sqrt{r_i - 2m}\sqrt{r^5} \, dr}{(r - 2m)^2\sqrt{2m(r_i - r)}} + \int_{r_S}^{r_O} \frac{\sqrt{r_i - 2m}\sqrt{r^5} \, dr}{(r - 2m)^2\sqrt{(r_i - 2m)r^3 - (r - 2m)r_i r_S^3}}.
\]

\[
= \int_{r_S}^{r_i} \frac{\sqrt{r_i - 2m}\sqrt{r^5} \, dr}{(r - 2m)^2\sqrt{2m(r_i - r)}} + \int_{r_S}^{r_O} \frac{\sqrt{r_i - 2m}\sqrt{r^5} \, dr}{(r - 2m)^2\sqrt{(r_i - 2m)r^3 - (r - 2m)r_i r_S^3}}.
\]
Figure 8: Spacetime diagram of a collapsing star and an infalling observer. The star begins to collapse at Painlevé-Gullstrand time $T = 0$ with radius $r_i$. The freely falling observer passes at this time through the radius value $r^*_O$.

The angular radius of the shadow is again given by (31) with the lower sign where now we have to insert $r = r_O$ and $r_m$ from (37),

$$\sin \tilde{\alpha}_{sh} = \frac{(r_O - 2m)r_S \sqrt{r_i}}{\varepsilon r_O^2 \sqrt{r_i - 2m - \sqrt{\varepsilon^2 r_O - r_O + 2m \sqrt{(r_i - 2m)r_O^2 - (r_O - 2m)r_S^2}}}}. \quad (51)$$

This equation can be solved for $r_S$,

$$r_S = \frac{\sqrt{r_i - 2m} \sqrt{r_O^3 \sin \tilde{\alpha}_{sh}}}{\sqrt{r_i} \left( \varepsilon \sqrt{r_O^2 - \sqrt{\varepsilon^2 r_O - r_O + 2m \cos \tilde{\alpha}_{sh}}} \right)}. \quad (52)$$

Inserting (52) into (50) gives us the desired relation between $r_O$ and $\tilde{\alpha}_{sh}$ in implicit but fully analytical form, provided that $r_i$ and $r^*_O$ are prescribed. The second phase begins when the observer passes through a radius value $r_O = r_O^{(1)}$ such that (50) holds with $r_S = r_i$. It ends at a radius value $r_O = r_O^{(2)}$ such that (50) holds with $r_S = 3m$.

Finally, in the third phase the boundary of the shadow is determined by light rays that spiral asymptotically to $r = 3m$, i.e., $\tilde{\alpha}_{sh}$ is given by (33).
Figure 9: Angular radius $\tilde{\alpha}_{\text{sh}}$ of the shadow of a collapsing dark star for an infalling observer. The observer is a Painlevé-Gullstrand observer ($\varepsilon = 1$) passing at $T_O = 0$ through the radius value $r_O^* = 10 \, m$. At this time the surface of the star starts collapsing from $r_i = 5 \, m$. The angular radius of the shadow is plotted against the radius coordinate of the observer. We distinguish three phases: In the first phase (dashed), the observer sees a star of constant radius $r_i$ and the angular radius of the shadow is given by (49). In the second phase (solid), the observer sees a collapsing star and the angular radius of the shadow is implicitly given by (50) with $r_S$ inserted from (52). In the third phase (dotted), the boundary of the shadow is no longer given by light rays grazing the surface of the star but rather by light rays spiralling towards the photon sphere at $r = 3 \, m$, so $\tilde{\alpha}_{\text{sh}}$ is given by (33), cf. Fig. 3.

6 Conclusions

In this paper we have demonstrated that, for a spherically symmetric dark and non-transparent star that collapses in free fall like a ball of dust, the development of the shadow can be calculated analytically, both for a static and for an infalling observer. In particular we have shown that for a static observer the black-hole shadow according to Synge’s formula forms in a finite time which, for a stellar black hole, is in the order of fractions of a second. This result could not have been easily anticipated before doing the calculation: Intuitively, one might have expected that the black-hole shadow forms asymptotically. The situation is similar for an infalling observer (provided that the observer is sufficiently far behind not to catch up with the star): Also in this case the surface of the star determines the shadow only over a finite time; during the last stage of the infall, the observer sees the same shadow as when infalling into an eternal black hole.

Admittedly, getting analytical results was possible only because we used a somewhat over-
simplified model for a collapsing star. More realistically, instead of a spherically symmetric ball of dust one should consider a rotating star with pressure which would probably make the calculations so complicated that only a numerical treatment would be possible. However, we believe that the simple model considered here gives a good idea of all the relevant qualitative features of how the black-hole shadow comes about in the course of time.

In this paper we have concentrated on the formation of the shadow during gravitational collapse. However, we mention that some of our results may also be useful for investigating the temporal change of the shadow of an already existing black hole. If a black hole is surrounded by matter its mass will grow by accretion, so its shadow will become bigger in the course of time. We have not investigated this problem in detail, but we believe that the Painlevé-Gullstrand approach pursued in this paper may be appropriate also for calculating the growth of the shadow of an accreting black hole.

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