\(L^p\) \((p \geq 1)\) solutions of multidimensional BSDEs with monotone generators in general time intervals

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Abstract

In this paper, we are interested in solving general time interval multidimensional backward stochastic differential equations in \(L^p\) \((p \geq 1)\). We first study the existence and uniqueness for \(L^p\) \((p > 1)\) solutions by the method of convolution and weak convergence when the generator is monotonic in \(y\) and Lipschitz continuous in \(z\) both non-uniformly with respect to \(t\). Then we obtain the existence and uniqueness for \(L^1\) solutions with an additional assumption that the generator has a sublinear growth in \(z\) non-uniformly with respect to \(t\).

Keywords: Backward stochastic differential equation, General time interval, Existence and uniqueness, Monotone generator, General growth

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1. Introduction

In this paper, we consider the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

\[
y_t = \xi + \int_t^T g(s, y_s, z_s) \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T],
\]

where \(T\) satisfies \(0 \leq T \leq +\infty\) called the terminal time; \(\xi\) is a \(k\)-dimensional random vector called the terminal condition; the random function \(g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k\) is progressively measurable for each \((y, z)\), called the generator of BSDE (1); and \(B\) is a \(d\)-dimensional Brownian motion. The solution \((y_t, z_t)_{t \in [0, T]}\) is a pair of adapted processes. The triple \((\xi, T, g)\) is called the parameters of BSDE (1). We denote also by BSDE \((\xi, T, g)\) the BSDE with the parameters \((\xi, T, g)\).

The nonlinear case of multidimensional BSDEs has been introduced by Pardoux and Peng [20]. They proved an existence and uniqueness result under the assumptions that the generator \(g\) is uniformly Lipschitz continuous in both \(y\) and \(z\). Their terminal time \(T\) is a finite constant and the terminal condition \(\xi\) and the process \(\{g(t, 0, 0)\}_{t \in [0, T]}\) are square-integrable. Since then, the research of BSDEs in both theory and application has been widely made by more and more people. Many applications of BSDEs have been found in mathematical finance, stochastic control, partial differential equations and so on (See El Karoui et al. [7] for details). In both theory and application of BSDEs, it is essential to relax the Lipschitz conditions on the generator \(g\), improve the terminal time into the general case and study the solutions under non-square integrable parameters.

Many works including Bahlali [1], Briand et al. [4], Fan et al. [11], Hamadène [14], Jia [15], Kobylanski [16], Lepeltier and San Martin [17], Mao [18], Wang and Huang [23], see also the references therein, have weakened the Lipschitz condition on the generator \(g\). These works dealt only with the BSDEs with square-integrable parameters. But the terminal condition \(\xi\) and the process \(\{g(t, 0, 0)\}_{t \in [0, T]}\) are not necessarily square-integrable in some practical applications. Then \(L^p\) \((p \geq 1)\) solutions of BSDEs when \(\xi\) and \(\{g(t, 0, 0)\}_{t \in [0, T]}\) are \(p\)-integrable attracted a lot of attention of many researchers. Briand and Carmona [2], Briand et al. [3], Chen [5], El Karoui et al. [7], Fan and Jiang [10], for instance,

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proved some existence and uniqueness results for $L^p$ ($p > 1$) solutions of BSDEs respectively in different conditions. In particular, Briand et al. [3] proved the existence and uniqueness for $L^p$ ($p > 1$) solutions of multidimensional BSDEs when the generator $g$ is monotonic in $y$ and is Lipschitz continuous in $z$, and with an additional assumption that $g$ has a kind of sublinear growth in $z$ they obtained the existence and uniqueness for $L^1$ solutions. To our knowledge, only few papers solved BSDEs with only integrable parameters besides Briand et al. [3], such as Peng [21] and Fan and Liu [13]. Particularly, Fan and Liu [13] obtained the existence and uniqueness for $L^1$ solutions of one-dimensional BSDEs when the generator $g$ is Lipschitz continuous in $y$ and $\alpha$-Hölder ($0 < \alpha < 1$) continuous in $z$, which may be a basic result for $L^1$ solutions of BSDEs. However, all these works talked above dealt only with the BSDEs with a finite time interval. BSDEs with general time intervals have not been researched widely.

In the sequel, we introduce some papers which studied the existence and uniqueness of solutions of BSDEs with general time intervals. Chen and Wang [6] first improved the terminal time into the general case and proved the existence and uniqueness for $L^2$ solutions of one-dimensional BSDEs by the fixed point theorem under the assumptions that the generator $g$ is Lipschitz continuous in $(y, z)$ non-uniformly with respect to $t$, which actually extended the result of Pardoux and Peng [20] into the general time interval case. Fan and Jiang [8] and Fan et al. [12] respectively relaxed the Lipschitz condition of Chen and Wang [6], and obtained the existence and uniqueness result for $L^2$ solutions of BSDEs with general time intervals. Recently, Fan and Jiang [9] investigated the existence and uniqueness for $L^p$ ($p > 1$) solutions of multidimensional BSDEs with general time intervals under some weaker assumptions. However, all these works need a linear-growth condition of the generator $g$ with respect to $y$ to guarantee the existence of $L^p$ ($p > 1$) solutions. On the other hand, to our knowledge, there are no papers which have studied the $L^1$ solutions of BSDEs with general time intervals.

In this paper, under a monotonicity condition and a general growth condition for the generator $g$ with respect to $y$ we establish a general existence and uniqueness result for $L^p$ ($p \geq 1$) solutions of multidimensional BSDEs with general time intervals (see Theorem 9 in Section 3 and Theorem 17 in Section 4). In particular, the first part of this paper is devoted to proving the existence and uniqueness for $L^p$ ($p > 1$) solutions when the generator $g$ is monotonic and has a general growth in $y$ and is Lipschitz continuous in $z$, which are both non-uniform with respect to $t$ (see (H3) – (H5) in Section 3). After that, we study the existence and uniqueness for $L^1$ solutions under the same conditions together with an additional sublinear growth assumption in $z$ (see (H6) in Section 4). Note that the $u(t)$, $v(t)$ and $\gamma(t)$ appearing in assumptions (H4) – (H6) may be unbounded and their integrability is the only requirement (see Remarks 7 and 19 for details). Our results actually extend and improve the results of Briand et al. [3] into the general time interval case when the assumptions on the generator $g$ is not necessarily uniform with respect to $t$. Besides, our results also include the corresponding results of Pardoux and Peng [20], Pardoux [19] and Chen and Wang [6] as its particular cases.

The rest of this paper is organized as follows. Section 2 introduces some notations and lemmas used in the whole paper, and also establishes some important apriori estimates for solutions of BSDE (1). Section 3 puts forward and proves the existence and uniqueness result for the $L^p$ ($p > 1$) solutions. Section 4 shows the existence and uniqueness for the $L^1$ solutions. Appendix A gives some detailed proofs of lemmas.

2. Preliminaries and apriori estimates

Although many researchers use the same notations in studying BSDEs, we will still introduce the following notations in order to make the paper easy to read.

First of all, let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$ and let $(\mathcal{F}_t)_{t \geq 0}$ be the natural $\sigma$-algebra filtration generated by $(B_t)_{t \geq 0}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and complete. In this paper, the Euclidean norm of a vector $y \in \mathbb{R}^k$ will be defined by $|y|$, and for a $k \times d$ matrix $z$, we define $|z| = \sqrt{Tr(z^*z)}$, where $z^*$ is the transpose of $z$.

Let $(x, y)$ represent the inner product of $x, y \in \mathbb{R}^k$.

For each real number $p > 0$, let $L^p(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$ be the set of $\mathbb{R}^k$-valued and $\mathcal{F}_T$-measurable random variables $\xi$ such that $\|\xi\|_{L^p} := \mathbb{E}[|\xi|^p] < +\infty$ and let $S^p(0, T; \mathbb{R}^k)$ (or $S^p$ for notation convenience) denote the set of $\mathbb{R}^k$-valued, adapted and continuous processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{S^p} := \left(\mathbb{E}\left[\sup_{t \in [0, T]} |Y_t|^p\right]\right)^{1/p} < +\infty.$$
If $p \geq 1$, $\| \cdot \|_{S^p}$ is a norm on $S^p$ and if $p \in (0,1)$, $(Y, Y') \mapsto \|Y - Y'\|_{S^p}$ defines a distance on $S^p$. Under this metric, $S^p$ is complete. Moreover, let $M^p(0,T; \mathbb{R}^{k \times d})$ (or $M^p$ for notation convenience) denote the set of (equivalent classes of) $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^{k \times d}$-valued processes $(Z_t)_{t \in [0,T]}$ such that

$$
\|Z\|_{M^p} := \left\{ \mathbb{E} \left[ \left( \int_0^T |Z_t|^2 \, dt \right)^{\frac{p}{2}} \right] \right\}^{1/p} < +\infty.
$$

For any $p \geq 1$, $M^p$ is a Banach space endowed with this norm and for any $p \in (0,1)$, $M^p$ is a complete metric space with the resulting distance. We also denote by $\| \cdot \|_{S^p \times M^p}$ the norm in the space $S^p \times M^p$ for any $p > 1$ with the following definition

$$
\|(y, z)\|_{S^p \times M^p} := \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} |y_t|^p + \left( \int_0^T |z_t|^2 \, dt \right)^{\frac{p}{2}} \right] \right\}^{\frac{1}{p}}.
$$

Let us recall that a continuous process $(Y_t)_{t \in [0,T]}$ belongs to the class (D) if the family $\{Y_\tau : \tau \in \Sigma_T\}$ is uniformly integrable, where $\Sigma_T$ stands for the set of all stopping times $\tau$ such that $\tau \leq T$. For a process $(Y_t)_{t \in [0,T]}$ belonging to the class (D), we define

$$
\|Y\|_1 = \sup \{ \mathbb{E} \|Y_\tau\| : \tau \in \Sigma_T \}.
$$

The space of $(\mathcal{F}_t)$-progressively measurable continuous processes which belong to the class (D) is complete under this norm.

As mentioned above, we will deal only with the multidimensional BSDE which is an equation of type (1), where the terminal condition $\xi$ is $\mathcal{F}_T$-measurable, the terminal time $T$ satisfies $0 \leq T \leq +\infty$ and the generator $g$ is $(\mathcal{F}_t)$-progressively measurable for each $(y, z)$.

**Definition 1.** Let $T$ satisfy $0 \leq T \leq +\infty$. A pair of processes $(y_t, z_t)_{t \in [0,T]}$ taking values in $\mathbb{R}^k \times \mathbb{R}^{k \times d}$ is called a solution of BSDE (1), if $(y_t, z_t)_{t \in [0,T]}$ is $(\mathcal{F}_t)$-adapted and satisfies that $d\mathbb{P} - a.s., t \mapsto y_t$ is continuous, $t \mapsto z_t$ belongs to $L^2(0,T)$, $t \mapsto g(t, y_t, z_t)$ belongs to $L^1(0,T)$ and $d\mathbb{P} - a.s.,$ BSDE (1) holds true for each $t \in [0,T]$.

Let us introduce the following Lemma 2 which comes from Lemma 1.1 and Theorem 1.2 of Chen and Wang [6]. Note that Lemma 2 holds also in the multidimensional case since the proofs of Lemma 1.1 and Theorem 1.2 of Chen and Wang are done via a standard contraction argument combined with apriori estimates without using comparison theorem.

**Lemma 2.** Assume that $0 \leq T \leq +\infty$, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^k)$ and the following assumptions hold:

1. $(C1)$ $\mathbb{E} \left[ \left( \int_0^T \|g(t,0,0)\|^2 \, dt \right)^{\frac{1}{2}} \right] < +\infty$;

2. $(C2)$ There exist two deterministic functions $u(t), v(t) : [0,T] \mapsto \mathbb{R}^+$ with $\int_0^T (u(t) + v^2(t)) \, dt < +\infty$ such that $d\mathbb{P} \times dt$ - a.e., for each $(y_t, z_t) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$, $i = 1, 2,$

$$
|g(t, y_t, z_t) - g(t, y_{t+}, z_t)| \leq u(t)|y_t - y_{t+}| + v(t)|z_t - z_{t+}|.
$$

Then BSDE (1) has a unique solution in the space $S^2 \times M^2$.

Throughout this paper we will use the Corollary 2.3 in Briand et al. [3] several times. So we list it as a lemma. Note that this conclusion holds still true for $T = +\infty$.

**Lemma 3.** If $(y_t, z_t)_{t \in [0,T]}$ is a solution of BSDE (1), $p \geq 1$, $c(p) = \frac{p((p-1) \wedge 1)}{2}$ and $0 \leq t \leq u \leq T \leq +\infty$, then

$$
\int_t^u |y_s|^p + c(p) \int_t^u |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \leq |y_t|^p + p \int_t^u |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, g(s, y_s, z_s) \rangle \, ds
$$

$$
- p \int_t^u |y_s|^{p-2} 1_{|y_s| \neq 0} \langle y_s, z_s \, dB_s \rangle.
$$
Next we will establish some apriori estimates which play an important role in proving our main results. In stating them, it is useful to introduce the following assumption on the generator \( g \), where \( p > 0 \) and \( 0 \leq T \leq +\infty \).

(A) There exist two nonnegative functions \( \mu(t), \lambda(t) : [0, T] \to \mathbb{R}^+ \) with \( \int_0^T (\mu(t) + \lambda^2(t)) \, dt < +\infty \) such that \( dP \times dt \) a.e., for each \((y, z) \in \mathbb{R}^k \times \mathbb{R}^k \times d \),

\[
\langle y, g(t, y, z) \rangle \leq \mu(t)|y|^2 + \lambda(t)|y||z| + f_t|y|,
\]

where \( (f_t)_{t \in [0, T]} \) is a nonnegative and \((\mathcal{F}_t)\)-progressively measurable processes with \( \mathbb{E}[(\int_0^T f_t \, dt)^p] < +\infty \).

The following Lemmas 4 and 5 give some estimates for \( L^p \) solutions of BSDE (1) with \( 0 \leq T \leq +\infty \) in the spirit of the work in Briand et al. [3], whose proofs are given in Appendix A.

**Lemma 4.** Assume that \( 0 \leq T \leq +\infty \), \( g \) satisfies assumption (A), \((y_t, z_t)_{t \in [0, T]} \) is a solution of BSDE (1) such that \((y_t)_{t \in [0, T]} \in \mathcal{S}^p \) with \( p > 0 \), \( \beta(t) : [0, T] \to \mathbb{R}^+ \) with \( \int_0^T \beta(t) \, dt < +\infty \) and \( \beta(t) \geq 2(\mu(t) + \lambda^2(t)) \). Then there exists a constant \( C_p^1 > 0 \) depending only on \( p \) such that for each \( 0 \leq r \leq t \leq T \),

\[
\mathbb{E}\left[\left(\int_t^T e^{\int_s^t \beta(u) \, du} |y_s|^2 \, ds\right)^{\frac{p}{2}} \bigg| \mathcal{F}_r\right] \leq C_p^1 \mathbb{E}\left[\left(\sup_{s \in [t, T]} \left(e^{\int_s^t \beta(u) \, du} |y_s|^p\right)\right)^p \bigg| \mathcal{F}_r\right].
\]

**Lemma 5.** Let the assumptions of Lemma 4 hold and assume further that \( p > 1 \) and \( \beta(t) \geq 2\mu(t) + \lambda^2(t)/[1 \wedge (p - 1)] \). Then there exists a constant \( C_p^2 > 0 \) depending only on \( p \) such that for each \( 0 \leq r \leq t \leq T \),

\[
\mathbb{E}\left[\left(\sup_{s \in [t, T]} \left(e^{\int_s^t \beta(u) \, du} |y_s|^p\right)\right)^p \bigg| \mathcal{F}_r\right] \leq C_p^2 \mathbb{E}\left[\left(e^{\int_t^T \beta(u) \, du} \left|\int_t^T e^{\int_s^t \beta(u) \, du} f_s \, ds\right|^p\right)^p \bigg| \mathcal{F}_r\right].
\]

Combing Lemma 4 and Lemma 5 we can obtain the following Proposition 6.

**Proposition 6.** Let the assumptions in Lemma 5 hold and \( p > 1 \), then there exists a constant \( C_p > 0 \) depending only on \( p \) such that for each \( 0 \leq r \leq t \leq T \),

\[
\mathbb{E}\left[\left(\int_t^T e^{\int_s^t \beta(u) \, du} \beta(s)|y_s|^p \, ds\right)^p \bigg| \mathcal{F}_r\right] + \mathbb{E}\left[\left(\sup_{s \in [t, T]} \left(e^{\int_s^t \beta(u) \, du} |y_s|^p\right)\right)^p \bigg| \mathcal{F}_r\right] + \mathbb{E}\left[\left(\left|\int_t^T e^{\int_s^t \beta(u) \, du} f_s \, ds\right|^p\right)^p \bigg| \mathcal{F}_r\right] \leq C_p \left(\mathbb{E}\left[\left(e^{\int_t^T \beta(u) \, du} \left|\int_t^T e^{\int_s^t \beta(u) \, du} f_s \, ds\right|^p\right)^p \bigg| \mathcal{F}_r\right] + \mathbb{E}\left[\left(\int_t^T e^{\int_s^t \beta(u) \, du} f_s \, ds\right)^p \bigg| \mathcal{F}_r\right]\right).
\]

**3.** \( L^p \) (\( p > 1 \)) solution

This section will give an existence and uniqueness result for \( L^p \) (\( p > 1 \)) solutions of BSDE (1) with \( 0 \leq T \leq +\infty \) under the assumptions that the generator \( g \) is monotonic and has a general growth in \( y \), and is Lipschitz continuous in \( z \), which are both non-uniform with respect to \( t \).

First, we introduce the following assumptions with respect to the generator \( g \) of BSDE (1) where \( p > 1 \) and \( 0 \leq T \leq +\infty \). In stating them we always suppose that \( u(t), v(t) : [0, T] \to \mathbb{R}^+ \) are two deterministic functions such that \( \int_0^T (u(t) + v^2(t)) \, dt < +\infty \).

(H1) \( \mathbb{E}\left[\left(\int_0^T |g(t, 0, 0)| \, dt\right)^p\right] < +\infty \);

(H2) \( dP \times dt \) a.e., for each \( z \in \mathbb{R}^k \), \( g(t, y, z) \) is continuous;

(H3) \( g \) has a general growth in \( y \), i.e., for each \( r' \in \mathbb{R}^+ \), we have

\[
\psi_r(t) := \sup_{|y| \leq r'} |g(t, y, 0) - g(t, 0, 0)| \in L^1([0, T] \times \Omega);
\]
(H4) \( g \) is monotonically increasing in \( y \) non-uniformly with respect to \( t \), i.e., \( dP \times dt - \text{a.e.} \), for each \( y_1, y_2 \in \mathbb{R}^k \), \( z \in \mathbb{R}^{k \times d} \), we have

\[
\langle y_1 - y_2, g(t, y_1, z) - g(t, y_2, z) \rangle \leq u(t)|y_1 - y_2|^2;
\]

(H5) \( g \) is Lipschitz continuous in \( z \) non-uniformly with respect to \( t \), i.e., \( dP \times dt - \text{a.e.} \), for each \( y \in \mathbb{R}^k \), \( z_1, z_2 \in \mathbb{R}^{k \times d} \), we have

\[
|g(t, y, z_1) - g(t, y, z_2)| \leq v(t)|z_1 - z_2|.
\]

Moreover, we need the following assumption.

(H3') There exists a continuous increasing function \( \varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) such that \( dP \times dt - \text{a.e.} \), for each \( y \in \mathbb{R}^k \), \( z \in \mathbb{R}^{k \times d} \), we have

\[
|g(t, y, z)| \leq |g(t, 0, z)| + u(t)\varphi(|y|).
\]

**Remark 7.** It is clear that assumption (H3) is weaker than assumption (H3'). In addition, it should be noted that in the corresponding assumptions of Briand et al. [3] and Pardoux [19], the \( u(t) \) and \( v(t) \) in (H3'), (H4) and (H5) are assumed to be bounded by a constant \( c > 0 \) since they work with continuous functions in a compact time interval. In our framework, they may be unbounded.

**Remark 8.** It is not hard to verify that \((y_t, z_t)_{t \in [0,T]}\) is a solution of BSDE \((\xi, T, g)\) iff

\[
\langle \zeta_t, \gamma_t \rangle := \left( e^{\int_0^t u(s) \, ds} y_t, e^{\int_0^t u(s) \, ds} z_t \right)
\]

is a solution of BSDE \((e^{\int_0^T u(s) \, ds} \xi, T, \gamma)\), where

\[
\gamma(t, y, z) := e^{\int_0^t u(s) \, ds} g(t, e^{-\int_0^t u(s) \, ds} y, e^{-\int_0^t u(s) \, ds} z) - u(t)y.
\]

We can check that \( \gamma \) satisfies the previous assumptions as \( g \), but with (H4) replaced by

(H4') \( \langle y_1 - y_2, g(t, y_1, z) - g(t, y_2, z) \rangle \leq 0 \).

Therefore, without loss of generality, we can assume that \( g \) satisfies (H4') provided that \( g \) satisfies (H4).

The main result of this section is as follows.

**Theorem 9.** Assume that \( 0 \leq T \leq +\infty \), \( p > 1 \) and \( g \) satisfies assumptions (H1) – (H5). Then for each \( \xi \in L^p(\Omega, \mathcal{F}_T, P ; \mathbb{R}^k) \), BSDE (1) has a unique solution \((y_t, z_t)_{t \in [0,T]}\) in \( \mathcal{S}^p \times \mathcal{M}^p \).

Next we will prove Theorem 9. By Remark 8 we shall always assume that \( g \) satisfies (H1) – (H3), (H4') and (H5). Let us prove the uniqueness part first and then the existence part.

**Proof of the uniqueness part of Theorem 9.** Let \((y_1^{t, z_1^{t}}, z_1^{t})_{t \in [0,T]}\) and \((y_2^{t, z_2^{t}}, z_2^{t})_{t \in [0,T]}\) be two solutions of BSDE (1) such that both \((y_1^{t, z_1^{t}}, z_1^{t})_{t \in [0,T]}\) and \((y_2^{t, z_2^{t}}, z_2^{t})_{t \in [0,T]}\) belong to \( \mathcal{S}^p \times \mathcal{M}^p \). We set \( \tilde{y} := y^1 - y^2 \) and \( \tilde{z} := z^1 - z^2 \), then \((\tilde{y}_t, \tilde{z}_t)_{t \in [0,T]}\) is a solution of the following BSDE in \( \mathcal{S}^p \times \mathcal{M}^p \),

\[
\tilde{y}_t = \int_t^T \tilde{g}(s, \tilde{y}_s, \tilde{z}_s) \, ds - \int_t^T \tilde{z}_s \, dB_s, \quad t \in [0, T],
\]

(2)

where \( \tilde{g}(t, y, z) := g(t, y + y_2^2, z + z_2^2) - g(t, y_1^2, z_1^2) \) for each \((y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d} \). It follows from assumptions (H4') and (H5) on \( g \) that

\[
\langle y, \tilde{g}(t, y, z) \rangle = \langle y, \tilde{g}(t, y, z + z_1^2) - \tilde{g}(t, y^2, z + z_1^2) \rangle + \langle y, \tilde{g}(t, y^2, z + z_1^2) - \tilde{g}(t, y_2^2, z_1^2) \rangle \leq v(t)|y||z|,
\]

which means that assumption (A) is satisfied for the generator \( \tilde{g}(t, y, z) \) of BSDE (2) with \( \mu(t) \equiv 0 \), \( \lambda(t) = v(t) \) and \( f_t \equiv 0 \). Thus, by Proposition 6 with \( r = t = 0 \) we know that

\[
\mathbf{E} \left[ \sup_{s \in [0,T]} |\tilde{y}_s|^p + \left( \int_0^T |\tilde{z}_s|^2 \, ds \right)^{\frac{1}{2}} \right] \leq 0.
\]

Therefore, \((\tilde{y}_t, \tilde{z}_t)_{t \in [0,T]} = (0, 0)\). The uniqueness part is then complete. \(\square\)
Next we begin to prove the existence part of Theorem 9. The proof method is enlightened by Pardoux [19], Briand et al. [3], and Briand and Carmona [2]. More precisely, the techniques applied in the first step, the convolution and the weak convergence, are lent from Pardoux [19], and the truncation techniques are taken partly from Briand et al. [3] and Briand and Carmona [2]. It should be mentioned that since we have changed the temporal term from the finite case to the general case, the space of \((y_t)_{t \in [0,T]}\) from the square-integrable space to \(S^p(0,T; \mathbb{R}^k)\), and the \(p\)-integrable condition of \(\{g(t,0,0)\}_{t \in [0,T]}\) from \(E[\int_0^T |g(t,0,0)|^p \, dt] < +\infty\) to \(E[\int_0^T |g(t,0,0)| \, dt]^p] < +\infty\), some new troubles come up naturally when we combine those techniques mentioned above together. For example, in the case of \(T = +\infty\), the integration of a constant over \([0,T]\) is not finite anymore: \(\|X\|_{L^p} \leq C\|X\|_{S^p}\) may not hold any longer; and the condition \(\int_0^T v^2(s) \, ds < +\infty\) can not imply \(\int_0^T v(s) \, ds < +\infty\). Additionally, from the point of technique view, in order to prove the existence part of Theorem 9, we need an existence result under assumption \((H_3')\), but it has not been proved in the general time interval case. All these troubles will be solved using different procedures.

**Proof of the existence part of Theorem 9.** The proof will be done by four steps as follows:

- With the help of Lemma 2 and Proposition 6, by applying an approximation method via convolution smoothing as well as an argument on weak convergence borrowed from Pardoux [19], we prove the existence of a solution in \(S^2 \times M^2\) for the following BSDE:
  \[
y_t = \xi + \int_t^T g(s,y_s,V_s) \, ds - \int_t^T z_s \, dB_s, \quad t \in [0,T],
\]
  under assumptions \((H2), (H3'), (H4')\) and \((H5)\), provided that \(V \in M^p\) and there exists a nonnegative constant \(K\) such that
  \[
  |\xi| \leq K, \quad dP - a.s. \quad \text{and} \quad |g(t,0,V_t)| \leq K e^{-t}, \quad dP \times dt - a.e.. \quad (4)
  \]

- By using a particular truncation technique, we prove that the assumption \((H3)\) in the above step can be weakened to \((H3)\).

- With the help of Proposition 6, by a similar truncation argument to that in Briand et al. [3], we prove that for each \(\xi \in L^p(\Omega, F_T, P; \mathbb{R}^k)\) and \(V \in M^p\), BSDE \((3)\) has a solution in \(S^p \times M^p\) under assumptions \((H1) - (H3), (H4')\) and \((H5)\).

- We construct a strict contraction by subdividing the time interval \([0,T]\) to show the existence of a solution to BSDE \((1)\) in the space \(S^p \times M^p\) for each \(\xi \in L^p(\Omega, F_T, P; \mathbb{R}^k)\) under assumptions \((H1) - (H3), (H4')\) and \((H5)\), which is the desired result.

On the whole, the first three steps deal with the case where the generator \(g\) is independent of \(z\), and the last step considers the general case.

**First step:** Now we assume that \(\xi \in L^p(\Omega, F_T, P; \mathbb{R}^k), V \in M^p(0,T; \mathbb{R}^{k \times d})\) and that \((H2), (H3'), (H4'), (H5)\) and \((4)\) hold true. For notational convenience, in this step we set, for each \(y \in \mathbb{R}^k\),
\[
f(t,y) := g(t,y,V_t).
\]
Clearly, we have that
\[
E[|\xi|^2] < +\infty \quad \text{and} \quad E \left[ \left( \int_0^T |f(t,0)| \, dt \right)^2 \right] < +\infty.
\]

Let \(\rho_n(x) := n^k \rho(nx)\), where \(\rho : \mathbb{R}^k \mapsto \mathbb{R}^+\) is a nonnegative \(C^\infty\) function with the unit ball for compact support and which satisfies \(\int_{\mathbb{R}^k} \rho(x) \, dx = 1\). We define for each \((\omega,t,y) \in \Omega \times [0,T] \times \mathbb{R}^k\),
\[
f_n(t,y) := (\rho_n(\cdot) * f(t,\cdot))(y) = \int_{\mathbb{R}^k} \rho_n(x) f(t,y-x) \, dx. \quad (5)
\]
Then \(f_n\) is a \((F_t)\)-progressively measurable process for each \(y \in \mathbb{R}^k\) and
\[
f_n(t,y) = \int_{\mathbb{R}^k} \rho(x) f(t,y-x) \, dx = \int_{|x| \leq 1} \rho(x) f(t,y-x) \, dx. \quad (6)
\]
Concerning \(f_n(t,y)\), we have the following Lemma 10, whose proof is given in Appendix A.
Lemma 10. Take driver \( g \) under \((H3')-(H4')\) and define \( \phi(u') := \varphi(u' + 1) \) for each \( u' \in \mathbb{R}^+ \). Then for each \( n \in \mathbb{N} \), \( f_n(t,y) \) satisfies \((H2), (H3')\) with \( \varphi \) replaced by \( \phi \) and \((H4')\). Furthermore, we have
\[
|f_n(t,0)| \leq Ke^{-t} + u(t)\phi(0), \tag{7}
\]
and for each \( y_1, y_2 \in \mathbb{R}^k \) with \(|y_1| \leq m \) and \(|y_2| \leq m \), there exists a constant \( C_{m} \) depending on \( m \) and \( n \)
such that
\[
|f_n(t, y_1) - f_n(t, y_2)| \leq C_{m}(e^{-t} + u(t)|y_1 - y_2|), \tag{8}
\]
which means that \( f_n(t,y) \) is locally Lipschitz continuous in \( y \) non-uniformly with respect to \( t \).

We now define for each \( q \in \mathbb{N} \),
\[
f_n,q(t,y) := f_n(t,\pi_q(y)),
\]
where and hereafter for each \( u' \in \mathbb{R}^+ \) and \( x \in \mathbb{R}^k \),
\[
\pi_u(x) := \frac{u'x}{u' \vee |x|}.
\]
By \((7)\) we know that \( f_n,q(t,0) = f_n(t, 0) \) satisfies \((C1)\) in Lemma 2. Furthermore, it follows from \((8)\) that there exists a constant \( K_{n,q} \) depending on \( n \) and \( q \) such that for each \( y_1, y_2 \in \mathbb{R}^k \), in view of \(|\pi_q(y)| \leq q\) for each \( y \in \mathbb{R}^k \) and \(|\pi_q(y_1) - \pi_q(y_2)| \leq |y_1 - y_2|\),
\[
|f_n,q(t, y_1) - f_n,q(t, y_2)| \leq K_{n,q}(e^{-t} + u(t)|y_1 - y_2|),
\]
which implies that \( f_n,q \) satisfies \((C2)\) in Lemma 2. It then follows from Lemma 2 that BSDE \((\xi, T, f_n,q)\) has a unique solution \((y_{1n,q}, z_{1n,q})_{t \in [0,T]}\) in the space \( S^2 \times M^2 \) for each fixed \( n, q \).

In the sequel, it follows from \((H4')\) on \( f_n \) and \((7)\) that \( d\mathbb{P} \times dt \) a.e., for each \( n, q \in \mathbb{N} \) and \( y \in \mathbb{R}^k \),
\[
\langle y, f_n,q(t,y) \rangle = \frac{q \vee |y|}{q}(\pi_q(y), f_n(t,\pi_q(y)) - f_n(t,0)) + \langle y, f_n(t,0) \rangle \leq |y||f_n(t,0)| \leq (Ke^{-t} + u(t)\phi(0))|y|.
\]
Then assumption \((A)\) is satisfied for the generator \( f_n,q(t,y) \) with \( p = 2, \mu(t) = \lambda(t) \equiv 0 \) and \( f_t = Ke^{-t} + u(t)\phi(0) \). Thus, since \((y_{1n,q}, z_{1n,q})_{t \in [0,T]}\) is the unique solution of BSDE \((\xi, T, f_n,q)\) in \( S^2 \times M^2 \), it follows from \((4)\) and Proposition 6 with taking \( r = t \) that there exists a universal positive constant \( C_1 \) such that for each \( n, q \in \mathbb{N} \) and \( t \in [0,T] \),
\[
|y_{1n,q}|^2 + \mathbb{E} \left[ \int_t^T |z_{1n,q}|^2 \, ds \mid \mathcal{F}_t \right] \leq C_1 \mathbb{E} \left[ \left( \int_t^T |\phi(u(s))| \, ds \right)^2 \right] \mathbf{1}_t
\]
\[
\leq C_1 \left[ 3K^2 + 2\phi^2(0) \left( \int_0^T u(s) \, ds \right)^2 \right] := a^2.
\]
Consequently, for any \( q > a \), \( (y_{1n,q}, z_{1n,q})_{t \in [0,T]} \) does not depend on \( q \). We then denote it by \((y_{1n}, z_{1n})_{t \in [0,T]}\) and it is a solution of the following BSDE:
\[
y_{1n} = \xi + \int_t^T f_n(s, y_{1n}) \, ds - \int_t^T z_{1n} \, dB_s, \quad t \in [0,T]. \tag{10}
\]
Furthermore, by \((9)\) we have, for each \( n \in \mathbb{N} \),
\[
d\mathbb{P} \times dt \text{ a.e., } \ |y_{1n}|^2 \leq a^2 \quad \text{and} \quad \mathbb{E} \left[ \int_0^T |z_{1n}|^2 \, dt \right] \leq a^2. \tag{11}
\]
Assumption \((H3')\) on \( f_n \) and \((7)\) yield that for each \( n \in \mathbb{N} \),
\[
|f_n(t, y_{1n})| \leq |f_n(t, 0)| + u(t)\phi(a) \leq Ke^{-t} + u(t)(\phi(0) + \phi(a)).
\]
Thus, we know that
\[
\sup_n \mathbb{E} \left[ \sup_{t \in [0,T]} |y_{1n}|^2 + \left( \int_0^T |f_n(t, y_{1n})| \, dt \right)^2 + \int_0^T |z_{1n}|^2 \, dt \right] < +\infty. \tag{12}
\]
Set $U^n_t := f_n(t, y^n_t)$ for each $t \in [0, T]$. By (12) we can conclude that there exists a subsequence of the sequence $\{(y^n_t, U^n_t, z^n_t)_{t \in [0, T]}\}_{n=1}^{\infty}$, still denoted by $\{(y^n_t, U^n_t, z^n_t)_{t \in [0, T]}\}_{n=1}^{\infty}$, that converges weakly in $S^2(0, T; R^k) \times L^2(0, T; R^k) \times M^2(0, T; R^{k \times d})$ to a limit $(y, U, z)_{t \in [0, T]}$, where $L^2(0, T; R^k)$ denotes the set of $(F_t)$-progressively measurable $R^k$-valued processes $(U_t)_{t \in [0, T]}$ such that

$$
\|U\|_{L^2} := \left\{ E \left[ \left( \int_0^T |U_t| \, dt \right)^2 \right] \right\}^{1/2} < +\infty.
$$

In view of (11), we have

$$
|y_t| \leq a, \quad dP \times dt - a.e. \quad \text{and} \quad E \left[ \sup_{t \in [0, T]} |y_t|^2 \right] < +\infty. \quad (13)
$$

In the sequel, take any bounded linear functional $\Phi(\cdot)$ defined on $L^2(\Omega, F_T, P; R^k)$. Then there exists a positive constant $b$ such that for each $(y^n_t, U^n_t, z^n_t)_{t \in [0, T]} \in S^2(0, T; R^k) \times L^2(0, T; R^k) \times M^2(0, T; R^{k \times d})$ and $t \in [0, T]$, the following three inequalities hold true:

- $|\Phi(y^n_t)| \leq b \|y^n_t\|_{L^2} \leq b \|y_t\|_{L^2}$,
- $|\Phi \left( \int_t^T U^n_s \, ds \right)| \leq b \left\| \int_t^T U^n_s \, ds \right\|_{L^2} \leq b \|\mathcal{T}\|_{L^2}$,
- $|\Phi \left( \int_t^T z^n_s \, dB_s \right)| \leq b \left\| \int_t^T z^n_s \, dB_s \right\|_{L^2} \leq b \|\mathcal{M}\|_{L^2}$.

This means that for each $t \in [0, T]$,

$$
\Phi(y_t), \quad \Phi \left( \int_t^T \cdot \, ds \right), \quad \Phi \left( \int_t^T \cdot \, dB_s \right)
$$

are bounded linear functionals defined respectively on $S^2(0, T; R^k) \times L^2(0, T; R^k)$ and $M^2(0, T; R^{k \times d})$. Consequently, in view of the fact that $\{(y^n_t, U^n_t, z^n_t)_{t \in [0, T]}\}_{n=1}^{\infty}$ converges weakly in $S^2 \times L^2 \times M^2$ to the process $(y, U, z)_{t \in [0, T]}$, we have that for each $t \in [0, T]$,

$$
\lim_{n \to \infty} \Phi(y^n_t) = \Phi(y_t), \quad \lim_{n \to \infty} \Phi \left( \int_t^T U^n_s \, ds \right) = \Phi \left( \int_t^T U_s \, ds \right), \quad \lim_{n \to \infty} \Phi \left( \int_t^T z^n_s \, dB_s \right) = \Phi \left( \int_t^T z_s \, dB_s \right).
$$

That is, for each $t \in [0, T]$, in the space $L^2(\Omega, F_T, P; R^k)$, $y^n_t$, $\int_t^T U^n_s \, ds$ and $\int_t^T z^n_s \, dB_s$ converge weakly to $y_t$, $\int_t^T U_s \, ds$ and $\int_t^T z_s \, dB_s$ respectively. Thus, taking weak limit in $L^2(\Omega, F_T, P; R^k)$ for BSDE (10) yields that for each $t \in [0, T]$,

$$
y_t = \xi + \int_t^T U_s \, ds - \int_t^T z_s \, dB_s, \quad \text{dP \ - \ a.s.}.
$$

Then, noticing that $(y_t)_{t \in [0, T]} \in S^2(0, T; R^k)$ and the process $(\xi + \int_t^T U_s \, ds + \int_t^T z_s \, dB_s)_{t \in [0, T]}$ is also continuous, we have, $\text{dP \ - \ a.s.,}$

$$
y_t = \xi + \int_t^T U_s \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T].
$$

Finally, by the following Lemma 11 we complete the proof of the first step.

**Lemma 11.** $\text{dP \times dt \ - \ a.e., } U_t = f(t, y_t) = g(t, y_t, V_t)$.

The proof of Lemma 11 will be given in Appendix A.

**Second step:** In this step we will prove that provided that $\xi \in L^p(\Omega, F_T, P; R^k)$, $V \in M^p(0, T; R^{k \times d})$ and that (H2), (H3), (H4'), (H5) and (4) hold true, BSDE (3) has a solution in $S^2 \times M^2$.

Assume now that $\xi \in L^p(\Omega, F_T, P; R^k)$, $V \in M^p(0, T; R^{k \times d})$ and that (H2), (H3), (H4'), (H5) and (4) hold true. For some positive real $r' > 0$, which will be chosen later, let $\theta_{r'}$ be a smooth function such that $0 \leq \theta_{r'}(y) \leq 1$, $\theta_{r'}(y) = 1$ for $|y| \leq r'$ and $\theta_{r'}(y) = 0$ as soon as $|y| \geq r' + 1$. Now we define for each $(\omega, t, y) \in \Omega \times [0, T] \times R^k$

$$
h_n(t, y, V_t) := \theta_{r'}(y) \left( g(t, y, \pi_{n_{r'}^{-1}}(V_t)) - g(t, 0, \pi_{n_{r'}^{-1}}(V_t)) \right) \frac{ne^{-t}}{\psi_{r'+1}(t) \vee (ne^{-t})} + g(t, 0, V_t).
$$

Then we have the following Lemma 12, whose proof will be given in Appendix A.
Lemma 12. $h_n$ satisfies assumptions (H2), (H3'), (H4) and $|h_n(t,0,V_i)| \leq Ke^{-t}$.

By Lemma 12 and Remark 8, it follows from the first step that BSDE $(\xi,T,h_n)$ has a solution $(y^n_t,z^n_t)_{t\in[0,T]}$ in the space $S^2 \times M^2$. Thanks to assumption (H4') on $g$ and (4), we get that for each $y \in \mathbb{R}^k$,

$$\langle y,h_n(t,y,V_i) \rangle \leq \langle y,g(t,0,V_i) \rangle \leq |y||g(t,0,V_i)| \leq Ke^{-t}.$$ 

which means that assumption (A) holds true for the generator $h_n(t,y,V_i)$ with $\mu(t) = \lambda(t) \equiv 0$ and $f_t = Ke^{-t}$. Then it follows from Proposition 6 with $p = 2$ that there exists a universal positive constant $C_2$ such that for each $n \in \mathbb{N}$ and $0 \leq r \leq t \leq T$,

$$E \left[ |y^n_t|^2 \mid F_r \right] + E \left[ \int_r^T |z^n_s|^2 \, ds \mid F_r \right] \leq C_2 K^2 := r'^2.$$ 

Then we know that for each $n \in \mathbb{N}$,

$$dP \times dt - a.e., \quad |y^n_t| \leq r' \quad \text{and} \quad E \left[ \int_0^T |z^n_s|^2 \, ds \right] \leq r'^2. \quad (14)$$ 

Hence, $(y^n_t,z^n_t)_{t \in [0,T]}$ is a solution of BSDE $(\xi,T,h'_n)$ where $h'_n(t,y,V_i) := (g(t,y,\pi_{n\to e}(V_i)) - g(t,0,\pi_{n\to e}(V_i))) \frac{ne^{-t}}{\psi_{r+1}(t) \vee (ne^{-t})} + g(t,0,V_i)$. It is clear that $h'_n$ satisfies (H4') since $g$ satisfies it.

In the sequel, for each $i,n \in \mathbb{N}$, we set $\hat{y}^n_{t,i} := y^n_{t+i} - y^n_t$ and $\hat{z}^n_{t,i} := z^n_{t+i} - z^n_t$. Ito’s formula and assumption (H4') on $h'_{n+1}$ yield that for each $t \in [0,T]$,

$$|\hat{y}^n_{t,i}| + \int_t^T |\hat{z}^n_{s,i}|^2 \, ds \leq 2 \int_t^T |\hat{y}^n_{s,i}| |h'_{n+1}(s,y^n_s,V_s) - h'_n(s,y^n_s,V_s)| \, ds - 2 \int_t^T \langle \hat{y}^n_{s,i}, \hat{z}^n_{s,i} \rangle dB_s,$$

from which it follows that

$$E \left[ \int_0^T |\hat{z}^n_{s,i}|^2 \, ds \right] \leq 2E \left[ \int_0^T |\hat{y}^n_{s,i}| |h'_{n+1}(s,y^n_s,V_s) - h'_n(s,y^n_s,V_s)| \, ds \right], \quad (15)$$

and

$$E \left[ \sup_{t \in [0,T]} |\hat{y}^n_{t,i}|^2 \right] \leq 2E \left[ \int_0^T |\hat{y}^n_{s,i}| |h'_{n+1}(s,y^n_s,V_s) - h'_n(s,y^n_s,V_s)| \, ds \right] + 2E \left[ \sup_{t \in [0,T]} \left| \int_t^T \langle \hat{y}^n_{s,i}, \hat{z}^n_{s,i} \rangle dB_s \right| \right]. \quad (16)$$

Moreover, the Burkholder-Davis-Gundy (BDG for short in the remaining) inequality yields the existence of a constant $k$ such that

$$2E \left[ \sup_{t \in [0,T]} \left| \int_t^T \langle \hat{y}^n_{s,i}, \hat{z}^n_{s,i} \rangle dB_s \right| \right] \leq 2kE \left[ \int_0^T |\hat{y}^n_{s,i}|^2 |\hat{z}^n_{s,i}|^2 \, ds \right]^\frac{1}{2} \leq 2kE \left[ \sup_{s \in [0,T]} |\hat{y}^n_{s,i}|^2 \right] + 2kE \left[ \int_0^T |\hat{z}^n_{s,i}|^2 \, ds \right].$$

Putting the previous inequality into (16) we get

$$E \left[ \sup_{s \in [0,T]} |\hat{y}^n_{s,i}|^2 \right] \leq 4E \left[ \int_0^T |\hat{y}^n_{s,i}| |h'_{n+1}(s,y^n_s,V_s) - h'_n(s,y^n_s,V_s)| \, ds \right] + 4k^2E \left[ \int_0^T |\hat{z}^n_{s,i}|^2 \, ds \right]. \quad (17)$$

Combining with (17) and (15) and noticing by (14) that $dP \times dt - a.e., \quad |\hat{y}^n_{t,i}| \leq 2r'$, we get that there exists a constant $K > 0$ such that

$$E \left[ \sup_{s \in [0,T]} |\hat{y}^n_{s,i}|^2 + \int_0^T |\hat{z}^n_{s,i}|^2 \, ds \right] \leq r' K E \left[ \int_0^T |h'_{n+1}(s,y^n_s,V_s) - h'_n(s,y^n_s,V_s)| \, ds \right]. \quad (18)$$

Furthermore, we have the following Lemma 13, whose proof will be provided in Appendix A.
Lemma 13. For each $n, i \in \mathbb{N}$, the following inequality holds true:

$$|h_n^{i+1}(s, y_n^i, V_s) - h_n^{i}(s, y_n^i, V_s)| \leq 2\nu(s)|V_s|^1|V_s|^{\nu-1} + 2\nu(s)|V_s|^1|\psi_{i+1}(s)|^{\nu-1} + \psi_{i+1}(s)|\psi_{i+1}(s)|^{\nu-1}.$$

In view of $V \in \mathcal{M}^p(0, T; \mathbb{R}^{k \times d})$, $\psi_{i+1}(t) \in L^1([0, T] \times \Omega)$, $\int_0^T v^2(s) \, ds < +\infty$ and Hölder’s inequality, Lebesgue’s dominated convergence theorem yields that the right term of (18) converges to 0 as $n \to +\infty$. So $\{(y_n^i, z_n^i)_{t \in [0, T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in the space $\mathcal{S}^2 \times \mathcal{M}^2$. Finally, passing to the limit in both sides of BSDE $(\xi, T, h_0)$ under ucp (uniform convergence in probability) yields a desired solution of BSDE (3) in $\mathcal{S}^2 \times \mathcal{M}^2$. The second step is then completed.

**Third step:** In this step we will eliminate the condition (4) used in the second step. Under assumptions (H1) – (H3), (H4') and (H5), we first define for each $n \in \mathbb{N}$,

$$\xi^n := \pi_n(\xi), \quad g^n(t, y, V_t) := g(t, y, V_t) - g(t, 0, V_t) + \pi_{ne^{-t}}(g(t, 0, V_t)).$$

Then we can deduce that $|\xi^n| \leq n, |g^n(t, 0, V_t)| \leq ne^{-t}$ and assumptions (H2), (H3), (H4') hold true for $g^n$. It follows from the second step that the following BSDE has a solution $(y^n_t, z^n_t)_{t \in [0, T]}$ in $\mathcal{S}^2 \times \mathcal{M}^2$ such that $(y^n_t)_{t \in [0, T]}$ is a bounded process:

$$y^n_t = \xi^n + \int_t^T g^n(s, y^n_s, V_s) \, ds - \int_t^T z^n_s \, dB_s, \quad t \in [0, T]. \quad (19)$$

Obviously, $(y^n_t)_{t \in [0, T]} \in \mathcal{S}^p$. It follows from assumption (H4') on $g$ that for each $y \in \mathbb{R}^k$,

$$\langle y, g^n(t, y, V_t) \rangle \leq |y||\pi_{ne^{-t}}(g(t, 0, V_t))| \leq ne^{-t}|y|,$$

which means that the generator $g^n$ satisfies assumption (A) with $\mu(t) = \lambda(t) \equiv 0, f_t = ne^{-t}$. It then follows from Lemma 4 that $(z^n_t)_{t \in [0, T]}$ belongs to $\mathcal{M}^p$. Next we will prove the sequence $\{(y^n_t, z^n_t)_{t \in [0, T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in the space $\mathcal{S}^p \times \mathcal{M}^p$.

For each $n, m \in \mathbb{N}$, let $\tilde{\xi}^{n,m} := \xi^n - \xi^m, \tilde{\gamma}^{n,m} := y^n - y^m$ and $\tilde{z}^{n,m} := z^n - z^m$. Then

$$\tilde{y}_t^{n,m} = \tilde{\xi}_t^{n,m} + \int_t^T \tilde{\gamma}_t^{n,m}(s, \tilde{y}_s^{n,m}, V_s) \, ds - \int_t^T \tilde{z}_s^{n,m} \, dB_s, \quad t \in [0, T],$$

where the generator $\tilde{\gamma}_t^{n,m}(t, y, V_t) := g^n(t, y + y_t^{n,m}, V_t) - g^m(t, y_t^{n,m}, V_t)$ for each $y \in \mathbb{R}^k$. In view of assumption (H4') on $g^n$, we know that for each $y \in \mathbb{R}^k$ and $t \in [0, T]$,

$$\langle y, \tilde{\gamma}_t^{n,m}(t, y, V_t) \rangle \leq \langle y, g^n(t, y_t^{n,m}, V_t) - g^m(t, y_t^{n,m}, V_t) \rangle \leq |y||\pi_{ne^{-t}}(g(t, 0, V_t)) - \pi_{ne^{-t}}(g(t, 0, V_t))|.$$

Thus, assumption (A) is satisfied for the generator $\tilde{\gamma}_t^{n,m}$ with $u(t) = v(t) \equiv 0$ and $f_t = |\pi_{ne^{-t}}(g(t, 0, V_t)) - \pi_{ne^{-t}}(g(t, 0, V_t))|$. Therefore, it follows from Proposition 6 with $r = t = 0$ that there exists a positive constant $C_p$ depending only on $p$ such that for each $n, m \in \mathbb{N}$,

$$E \left[ \sup_{s \in [0, T]} |\tilde{y}_s^{n,m}|^p + \left( \int_0^T |\tilde{z}_s^{n,m}|^2 \, ds \right)^{\frac{p}{2}} \right] \leq C_p^2 E \left[ |\tilde{\xi}_t^{n,m}|^p \right] + C_p^2 E \left[ \left( \int_0^T |\pi_{ne^{-t}}(g(s, 0, V_s)) - \pi_{ne^{-t}}(g(s, 0, V_s))| \, ds \right)^p \right]. \quad (20)$$

Note that $E[|\xi^n - \xi|^p] \to 0$ as $n \to +\infty$ by Lebesgue’s dominated convergence theorem. By assumption (H5), we have that $|g(t, 0, V_t)| \leq |g(t, 0, 0)| + |v(t)|V_t$, then Hölder’s inequality yields that

$$E \left[ \left( \int_0^T |g(t, 0, V_t)| \, dt \right)^p \right] \leq 2^{p-1} E \left[ \left( \int_0^T |g(t, 0, 0)| \, dt \right)^p \right]^{\frac{p}{2}} + 2^{p-1} \left( \int_0^T v^2(t) \, dt \right)^{\frac{p}{2}} E \left[ \left( \int_0^T |V_t|^2 \, dt \right)^{\frac{p}{2}} \right] < +\infty.$$

Thus, by using Lebesgue’s dominated convergence theorem once again, we get that as $n \to +\infty$,

$$E \left[ \left( \int_0^T |\pi_{ne^{-t}}(g(t, 0, V_t)) - g(t, 0, V_t)| \, dt \right)^p \right] \to 0.$$
Consequently, the right terms of (20) converges to 0 as \( n, m \to +\infty \). Hence, \( \{(y^n, z^n)_{t\in[0,T]}\}_{n=1}^{\infty} \) is a Cauchy sequence in the space \( S^p \times M^p \). Finally, by passing to the limit in both sides of (19) under ucp we can obtain a desired solution of BSDE (3) in \( S^p \times M^p \).

**Fourth step:** In this step, we will finally complete the proof of the existence part of Theorem 9. Assume now that \( \xi \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}, \mathbb{R}^k) \) and that (H1) – (H3), (H4') and (H5) hold true. It follows from the third step that for each \((V_t)_{t\in[0,T]} \in M^p\), BSDE (3) has a solution \((y_t, z_t)_{t\in[0,T]} \) in \( S^p \times M^p \). Take any \((U_t)_{t\in[0,T]} \in S^p\) and consider the following operator \( \Psi : (U, V) \in S^p \times M^p \mapsto (y, z) \) defined by

\[
y_t = \xi + \int_t^T g(s, y_s, V_s) \, ds - \int_t^T z_s \, dB_s, \quad t \in [0, T].
\]

Thus, we have constructed a mapping \( \Psi \) from \( S^p \times M^p \) to \( S^p \times M^p \). Take another \((U', V') \) in the space \( S^p \times M^p \) and set \( (y', z') := \Psi(U', V') \). Let us set \((\overline{U}, \overline{V}) := (U - U', V - V') \) and \((\overline{y}, \overline{z}) := (y - y', z - z') \) for notational convenience. Then \((\overline{y}, \overline{z})_{t\in[0,T]} \) is a solution of the following BSDE:

\[
\overline{y}_t = \int_t^T \overline{g}(s, \overline{y}_s, \overline{z}_s) \, ds + \int_t^T \overline{z}_s \, dB_s, \quad t \in [0, T],
\]

where the generator \( \overline{g}(t, y) := g(t, y + y', V_t) - g(t, y', V_t') \) for each \( y \in \mathbb{R}^k \). It follows from assumptions (H4') and (H5) on \( g \) that

\[
\langle y, \overline{g}(t, y) \rangle \leq \langle y, g(t, y' + y', V_t) - g(t, y', V_t') \rangle \leq v(t)\|\overline{v}_t\|\|y\|,
\]

which means that \( \overline{g} \) satisfies assumption (A) with \( \mu(t) = \lambda(t) = 0 \) and \( f_t = v(t)\|\overline{v}_t\| \). Thus, it follows from Proposition 6 and Hölder’s inequality that there exists a positive constant \( C_p \) depending only on \( p \) such that

\[
\mathbb{E}\left[ \sup_{t\in[0,T]} \|\overline{y}_t\|^p + \left( \int_0^T \|\overline{z}_t\|^2 \, dt \right)^{p/2} \right] \leq C_p^4 \mathbb{E}\left[ \left( \int_0^T v(t)\|\overline{v}_t\| \, dt \right)^p \right] \leq C_p^4 \left( \int_0^T v^2(t) \, dt \right)^{p/2} \mathbb{E}\left[ \left( \int_0^T \|\overline{v}_t\|^2 \, dt \right)^{p/2} \right].
\]

Hence, if \( \delta := C_p^4 \left( \int_0^T v^2(t) \, dt \right)^{p/2} < 1 \), we have the strict contraction in the space \( S^p \times M^p \) as follows:

\[
\|(\overline{y}, \overline{z})\|_{S^p \times M^p} < \delta \|(U, V)\|_{S^p \times M^p}.
\]

Then by the fixed point theorem BSDE (1) has a unique solution in \( S^p \times M^p \).

In the general case about \( \delta \), in view of the fact that \( \int_0^T v^2(t) \, dt < +\infty \) and Proposition 6, we can follow exactly the proof procedure of the existence part in Fan and Jiang [9]. That is, we can subdivide the interval \([0, T]\) into a finite number of small intervals like \([T_i, T_{i+1}] \) \( (i = 0, 1, \cdots, N - 1) \) such that \( 0 = T_0 < T_1 < \cdots < T_{N-1} < T_N = +\infty \) and for each \( i = 0, 1, \cdots, N - 1 \),

\[
C_p^4 \left( \int_{T_i}^{T_{i+1}} v^2(t) \, dt \right)^{p/2} < 1.
\]

And in every small interval there exists a strict contraction in the space \( S^p \times M^p \). Then we complete the proof of the existence part of Theorem 9. \( \square \)

**Remark 14.** Motivated by Remark 9.3 in Touzi [22] we can also consider Picard’s iteration procedure to show the fourth step. Set \((y^0, z^0) := (0, 0) \) and define \((y^n, z^n)_{t\in[0,T]} \) recursively, in view of the third step, for each \( n \geq 0 \),

\[
y_t^{n+1} = \xi + \int_t^T g(s, y^{n+1}_s, z^n_s) \, ds - \int_t^T z^{n+1}_s \, dB_s, \quad t \in [0, T].
\]

Set \( \delta y^n := y^{n+1} - y^n \), \( \delta z^n := z^{n+1} - z^n \), then we have

\[
\delta y_t = \int_t^T \delta g^n(s, \delta y^n_s) \, ds - \int_t^T \delta z^n_s \, dB_s, \quad t \in [0, T],
\]
where $\delta g^n(s,y) := g(s, y + y^n, z^n_1) - g(s, y^n, z^n_1)$ for each $y \in \mathbb{R}^k$. Since $\delta g^n$ satisfies assumption (A) with $\mu(t) = u(t), \lambda(t) \equiv 0$ and $f_i = v(t)|\delta z_i^{n-1}$, it follows from Proposition 6 and an induction argument that

$$\| (\delta g^n, \delta z^n_1) \|_{SP \times MP}^p \leq \left[ C_p \left( \int_0^T v^2(s) \, ds \right) \right]^{n-1} \| (\delta g^1, \delta z^1_1) \|_{SP \times MP}^p.$$  

Then by subdividing the time interval $[0, T]$ into a finite number of small intervals like $[T_i, T_{i+1}]$ ($i = 0, 1, \ldots, N - 1$) such that $0 = T_0 < T_1 < \cdots < T_{N-1} < T_N = +\infty$, and for each $i = 0, 1, \ldots, N - 1$, $C_p \left( \int_{T_{i+1}}^{T_{i+2}} v^2(s) \, ds \right)^{p/2} < 1$, we can deduce that $\{(y^n_i, z^n_i)_{t\in[T_i, T_{i+1}]} \}_{n=1}^\infty$ is a Cauchy sequence in $S^p(T_{i-1}, T_i; \mathbb{R}^k) \times MP(T_{i-1}, T_i; \mathbb{R}^{k \times d})$, and then the limit process $(y_t, z_t)_{t\in[0, T]}$ is a solution of BSDE (1) in $S^p \times MP$.

**Example 15.** Let $0 \leq T < +\infty, k = 1$ and

$$g(t,y,z) = |\ln t| \{ -e^y + |y| \} + \frac{|z|}{\sqrt{t}} + |B_t|.$$  

It is not difficult to check that $g$ satisfies assumptions (H1) – (H5) with $u(t) = |\ln t|$ and $v(t) = 1/\sqrt{T}$. Then by Theorem 9 we know that for each $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^k)$, BSDE $(\xi, T, g)$ has a unique solution in $S^p \times MP$. But because $u(t)$ and $v(t)$ are unbounded, this conclusion can not be obtained by Theorem 4.2 in Briand et al. [3].

**Example 16.** Let $0 \leq T \leq +\infty, k = 2$ and

$$g(t, y, z) = t^2 e^{-t} \left\{ \frac{-y_1^2 + y_2}{-y_2 - y_1} + \frac{1}{\sqrt{1 + t^2}} \left[ \frac{z_1}{z_2} \right] + \frac{t^2}{t^4 + 1} \right\},$$  

where $y_i$ and $z_i$ ($i = 1, 2$) stand for the $i$th component of the vector $y$ and the $i$th row of the matrix $z$, respectively. It is not hard to verify that $g$ satisfies assumptions (H1) – (H5) with $u(t) = t^2 e^{-t}$, $v(t) = 1/\sqrt{1 + t^2}$. Thus, by Theorem 9 we know that for each $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^k)$, BSDE $(\xi, T, g)$ has a unique solution in $S^p \times MP$. It should be noted that this conclusion can not be obtained by Theorem 4.2 in Briand et al. [3] when $T = +\infty$.

4. $L^1$ solution

In this section we will give an existence and uniqueness result for $L^1$ solutions of BSDE (1). Here, we suppose that the generator $g$ is monotonic and has a general growth in $y$, and is Lipschitz continuous and has a kind of sublinear growth in $z$, which are both non-uniform with respect to $t$.

We first introduce the following assumptions on the generator $g$, where $0 \leq T \leq +\infty$.

(H1') $E \left[ \int_0^T |g(t,0,0)| \, dt \right] < +\infty$;

(H6) There exist an $\alpha \in (0, 1)$ and a deterministic function $\gamma(t) : [0, T] \to \mathbb{R}^+$ with $\int_0^T (\gamma(t) + \gamma^{1/(1-\alpha)}(t) + \gamma^{2/(2-\alpha)}(t)) \, dt < +\infty$ such that $d\mathbb{P} \times dt$-a.e., for each $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k \times d},$

$$|g(t, y, z) - g(t, y, 0)| \leq \gamma(t) (|y| + |y| + |z|)\alpha,$$

where $(y_t)_{t\in[0, T]}$ is a nonnegative and $(\mathcal{F}_t)$-progressively measurable process with $E[\int_0^T g_t \, dt] < +\infty$.

The following Theorem 17 is the main result of this section.

**Theorem 17.** Let $0 \leq T < +\infty$ and assumptions (H1'), (H2) – (H6) on the generator $g$ hold. Then for each $\xi \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^k)$, BSDE (1) has a solution $(y_t, z_t)_{t\in[0, T]}$ in $S^p \times MP$ for $\beta \in (0, 1)$ with $(y_t)_{t\in[0, T]}$ belonging to the class (D), which is unique in $S^p \times MP$ for $\beta \in (\alpha, 1)$.

The proof of Theorem 17 is completed with the help of Theorem 9 in the spirit of Theorems 6.2 and 6.3 in Briand et al. [3]. In view of Remark 8, we shall always assume that $g$ satisfies (H1'), (H2), (H3), (H4'), (H5) and (H6). As usual, let us first show the uniqueness part.
Proof of the uniqueness part of Theorem 17. Assume that both \((y_t, z_t)_{t \in [0, T]}\) and \((y'_t, z'_t)_{t \in [0, T]}\) are solutions of BSDE (1) such that both \((y_t)_{t \in [0, T]}\) and \((y'_t)_{t \in [0, T]}\) belong to the class (D), and both \((y_t, z_t)_{t \in [0, T]}\) and \((y'_t, z'_t)_{t \in [0, T]}\) belong to \(S^\beta \times M^\beta\) for some \(\beta \in (\alpha, 1)\). Let us fix \(n \in \mathbb{N}\) and denote \(\tau_n\) the stopping time
\[
\tau_n = \inf \left\{ t \in [0, T] : \int_0^t |z_s|^2 + |z'_s|^2 \, ds \geq n \right\} \wedge T.
\]
Lemma 3 leads to the following inequality with setting \(\hat{g} := y - y'\) and \(\hat{z} := z - z'\),
\[
|\hat{y}_{t \land \tau_n}| \leq |\hat{y}_{\tau_n}| + \int_{t \land \tau_n}^T |\hat{y}_s|^{-1} \begin{cases} 1_{\hat{y}_s \neq 0}(\hat{y}_s, g(s, y_s, z_s) - g(s, y'_s, z'_s)) \, ds \\
- \int_{t \land \tau_n}^T |\hat{y}_s|^{-1} 1_{\hat{y}_s \neq 0} \langle \hat{y}_s, z_s \rangle \, dB_s \end{cases}, \tag{22}
\]
Putting the previous inequality into (22) and then taking conditional expectation with respect to \(\mathcal{F}_t\) in both sides, we have that for each \(n \in \mathbb{N}\),
\[
E \left[ \frac{\int_0^T \gamma(s) (g_s + |y'_s| + |z'_s| + |z_s|)^\alpha \, ds}{\hat{y}_{t \land \tau_n}} \right] \leq E \left[ |\hat{y}_{\tau_n}| + 2 \int_0^T \gamma(s) (g_s + |y'_s| + |z'_s| + |z_s|)^\alpha \, ds \right] \mathcal{F}_t.
\]
Now sending \(n \to \infty\) and noticing that \(\tau_n \to T\), \((\hat{y}_t)_{t \in [0, T]}\) belongs to the class (D) and \(d\mathbb{P} = \text{a.s.}, \hat{y}_T = 0\), we know that for each \(t \in [0, T]\),
\[
|\hat{y}_t| \leq 2 E \left[ \int_0^T \gamma(s) (g_s + |y'_s| + |z'_s| + |z_s|)^\alpha \, ds \right] \mathcal{F}_t.
\]
Moreover, noticing that \(\beta/\alpha > 1\), Doob’s inequality yields that there exists a positive constant \(C_\beta^\alpha\) depending on \(\alpha\) and \(\beta\) such that
\[
E \left[ \sup_{t \in [0, T]} |\hat{y}_t|^{\beta/\alpha} \right] \leq C_\beta^\alpha E \left[ \left( \int_0^T \gamma(s) (g_s + |y'_s| + |z'_s| + |z_s|)^\alpha \, ds \right)^{\beta/\alpha} \right]. \tag{23}
\]
We set the random variable \(G := \int_0^T \gamma(s) (g_s + |y'_s| + |z'_s| + |z_s|)^\alpha \, ds\). Then \(G \in L^{\beta/\alpha}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})\). Indeed, Hölder’s inequality yields that
\[
\int_0^T \gamma(s)|g_s|^\alpha \, ds \leq \left( \int_0^T \gamma^{1-\alpha}(s) \, ds \right)^{1-\alpha} \left( \int_0^T g_s \, ds \right)^\alpha,
\]
and \(z'_s\) has the similar estimate. Besides,
\[
\int_0^T \gamma(s)|y'_s|^\alpha \, ds \leq \int_0^T \gamma(s) \, ds \cdot \sup_{t \in [0, T]} |y_t|^\alpha.
\]
Consequently, noticing that \(E[\int_0^T g_s \, ds] < +\infty\), \((y_t)_{t \in [0, T]}\) belongs to the space \(S^\beta\), \((z_t)_{t \in [0, T]}\) belongs to the space \(M^\beta\), we have that \(G \in L^{\beta/\alpha}\). It then follows from (23) that \((\hat{y}_t)_{t \in [0, T]}\) belongs to the space \(S^{\beta/\alpha}\). Furthermore, note that \((\hat{y}_t, \hat{z}_t)_{t \in [0, T]}\) is a solution of the following BSDE:
\[
\hat{y}_t = \int_t^T \hat{g}(s, \hat{y}_s, \hat{z}_s) \, ds - \int_t^T \hat{z}_s \, dB_s, \quad t \in [0, T],
\]
where the generator \( \hat{g}(t, \hat{y}, \hat{z}) := g(t, y + y'_t, z + z'_t) - g(t, y'_t, z'_t) \) for each \((y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}\). It follows from assumptions (H4') and (H5) on the generator \( g \) that
\[
(y, \hat{g}(t, \hat{y}, \hat{z})) \leq (y, g(t, y'_t, z + z'_t) - g(t, y'_t, z'_t)) \leq v(t)|y||z|,
\]
which means that assumption (A) is satisfied for the generator \( \hat{g} \) with \( \mu(t) \equiv 0, \lambda(t) = v(t) \) and \( f_t \equiv 0 \). It then follows from Proposition 6 with \( p = \beta/\alpha \) that \( (\hat{y}_t, \hat{z}_t)_{t \in [0, T]} = (0, 0) \in \mathcal{S}^{3/\alpha} \times \mathcal{M}^{3/\alpha} \). By now the uniqueness part is proved completely.

**Proof of the existence part of Theorem 17.** The proof will be split into two steps. The first step deals with the case that the generator \( g \) is independent of the variable \( z \) under assumptions (H1'), (H2), (H3) and (H4'), and the second step considers the general case.

**First step:** We assume that \( g \) is independent of \( z \). For each \( n \in \mathbb{N} \), we set
\[
\xi^n := \pi_n(\xi), \quad g^n(t, y) := g(t, y) - g(t, 0) + \pi_{n^{-1}}(g(t, 0)).
\]
Note that \( |\xi^n| \leq n \) and \( g^n(t, y) \) satisfies assumptions (H1') - (H3) and (H4'). It follows from Theorem 9 that the following BSDE (24) has a unique solution \((\hat{y}^n_t, \hat{s}^n_t)_{t \in [0, T]} \in \mathcal{S}^2 \times \mathcal{M}^2\).
\[
g^n_t = \xi^n + \int_t^T g^n(s, y^n_s) \, ds - \int_t^T \hat{z}^n_s \, dB_s, \quad t \in [0, T]. \tag{24}
\]
For each \( i, n \in \mathbb{N} \), we set \( \hat{y}^{n,i}_t := y^{n,i} - y^n, \hat{z}^{n,i}_t := z^{n,i} - z^n \) and \( \hat{\xi}^{n,i} := \xi^{n,i} - \xi^n \). It then follows from Lemma 3 with taking \( p = 1 \) that for each \( t \in [0, T], \)
\[
|\hat{y}^{n,i}_t| \leq |\hat{\xi}^{n,i}| + \int_t^T |\hat{y}^{n,i}_s|^{-1} 1_{|\hat{y}^{n,i}_s| \neq 0} \langle \hat{y}^{n,i}_s, g^{n,i}(s, y^{n,i}_s) - g^n(s, y^n_s) \rangle \, ds
\]
\[
- \int_t^T |\hat{y}^{n,i}_s|^{-1} 1_{|\hat{y}^{n,i}_s| \neq 0} \langle \hat{y}^{n,i}_s, \hat{z}^{n,i}_s \rangle \, dB_s. \tag{25}
\]
As before, the inner product including \( g \) can be enlarged via assumption (H4') on \( g^{n,i} \) as follows:
\[
|\hat{y}^{n,i}_t|^{-1} 1_{|\hat{y}^{n,i}_s| \neq 0} \langle \hat{y}^{n,i}_s, g^{n,i}(s, y^{n,i}_s) - g^n(s, y^n_s) \rangle \leq |g^{n,i}(s, y^{n,i}_s)| = |g^{n,i}(s, y^{n,i}_s)|.
\]
Putting the previous inequality into (25) and taking conditional expectation with respect to \( \mathcal{F}_t \) in both sides, we can get that for each \( t \in [0, T], \)
\[
\mathbb{E} \left[ |\hat{y}^{n,i}_t| \right] \leq \mathbb{E} \left[ |\hat{\xi}^{n,i}| + \int_t^T |g^{n,i}(s, y^{n,i}_s)| - g^n(s, y^n_s) \right] \, ds \right|_{\mathcal{F}_t}
\]
\[
\leq \mathbb{E} \left[ |\xi| 1_{|\xi| > n} + \int_0^T |g(s, 0)| 1_{|g(s, 0)| > n^{-\epsilon}} \, ds \right] \right|_{\mathcal{F}_t}.
\]
Thus, we have
\[
\left\| \hat{y}^{n,i} \right\|_1 \leq \mathbb{E} \left[ |\xi| 1_{|\xi| > n} + \int_0^T |g(s, 0)| 1_{|g(s, 0)| > n^{-\epsilon}} \, ds \right].
\]
And according to Lemma 6.1 in Briand et al. [3] we know that for any \( \beta \in (0, 1), \)
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |\hat{y}^{n,i}_t|^\beta \right] \leq \frac{1}{1 - \beta} \left( \mathbb{E} \left[ |\xi| 1_{|\xi| > n} + \int_0^T |g(s, 0)| 1_{|g(s, 0)| > n^{-\epsilon}} \, ds \right] \right)^\beta.
\]
Therefore, note that \( \mathbb{E}[|\xi| + \int_0^T |g(s, 0)| \, ds] < +\infty \), the process sequence \( \{(y^n_t)_{t \in [0, T]}\}_{n=1}^\infty \) is a Cauchy sequence both under \( \| \cdot \|_1 \) and in the space \( \mathcal{S}^2 \). Let \((y_t)_{t \in [0, T]}\) denote the limit of this process sequence, then \((y_t)_{t \in [0, T]}\) belongs to the class (D) and \( \mathcal{S}^2 \) for any \( \beta \in (0, 1) \).

Furthermore, note that \((\hat{y}^{n,i}_t, \hat{s}^{n,i}_t)_{t \in [0, T]}\) is a solution of the following BSDE:
\[
\hat{y}^{n,i}_t = \xi^{n,i} + \int_t^T \hat{g}(s, \hat{y}^{n,i}_s) \, ds - \int_t^T \hat{z}^{n,i}_s \, dB_s, \quad t \in [0, T],
\]
where the generator \( \hat{g}(t, y) := g^{n+1}(t, y + y^n) - g^n(t, y^n) \) for each \( y \in \mathbb{R}^k \). It follows from assumption (H4') on \( g^{n+1} \) that

\[
\langle y, \hat{g}(t, y) \rangle \leq \langle y, g^{n+1}(t, y^n) - g^n(t, y^n) \rangle \leq |y||g(t, 0)|1_{|g(t, 0)| > ne^{-t}},
\]

which means that \( \hat{g} \) satisfies assumption (A) with \( \mu(t) = \lambda(t) \equiv 0 \) and \( f_t = |g(t, 0)|1_{|g(t, 0)| > ne^{-t}} \). It then follows from Lemma 4 with \( p = \beta \) that for any \( \beta \in (0, 1) \), there exists a constant \( C_{\beta} \) depending on \( \beta \) such that

\[
E \left[ \left( \int_0^T |\hat{y}_t^{n,i}|^2 \, dt \right)^{\frac{\gamma}{2}} \right] \leq C_{\beta} E \left[ \sup_{t \in [0,T]} |\hat{y}_t^{n,i}|^{\beta} + \left( \int_0^T |g(t, 0)|1_{|g(t, 0)| > ne^{-t}} \, dt \right)^{\frac{\gamma}{2}} \right].
\]

In view of the previous inequality, we know that for any \( \beta \in (0, 1) \), the process sequence \( \{(\hat{z}^n_t)_{t \in [0,T]}\}_{n=1}^{\infty} \) is a Cauchy sequence in the space \( M^3 \). Let \( (z_t)_{t \in [0,T]} \) denote the limit which belongs to the space \( M^3 \) for any \( \beta \in (0, 1) \), then \( \int_0^T z^n_s \, dB_s \) converges to \( \int_0^T z_s \, dB_s \) under ucp.

Finally, since \( y \mapsto g(t, y) \) is continuous, we can obtain that \( (y_t, z_t)_{t \in [0,T]} \) is a desired solution of BSDE (24) under ucp.

**Second step:** The general case, i.e., \( g \) may depend on \( z \).

The next proof procedure will use Picard’s iterative procedure. Let us set \((y^0, z^0) := (0, 0)\). It is not hard to verify that for each \((z_t)_{t \in [0,T]} \in M^3 \) for any \( \beta \in (0, 1) \), the generator \( g(t, y, z_t) \) satisfies (H1'), (H2), (H3) and (H4'). Thus, we can define the process sequence \( \{(y^n_t, z^n_t)_{t \in [0,T]}\}_{n=1}^{\infty} \) recursively, in view of the first step, for each \( n \geq 0 \),

\[
y_t^{n+1} = \xi + \int_t^T g(s, y_s^{n+1}, z^n_s) \, ds - \int_t^T z_s^{n+1} \, dB_s, \quad t \in [0, T],
\]

where for each \( n \geq 0 \), \((y^n_t, z^n_t)_{t \in [0,T]} \) belongs to the class (D) and \((y^n_t, z^n_t)_{t \in [0,T]} \) belongs to \( \mathcal{S}^3 \times M^3 \) for any \( \beta \in (0, 1) \).

For each \( n \in \mathbb{N} \), arguing as in the proof of the uniqueness part of Theorem 17, we can deduce, in view of assumptions (H4') and (H6), that for each \( t \in [0, T] \),

\[
|y_t^{n+1} - y_t^n| \leq E \left[ \int_0^T |g(s, y_s^n, z^n_s) - g(s, y_s, z_s^{n-1})| \, ds \right] \leq 2E \left[ \int_0^T \gamma(s) \left( g_s + |y_s^n| + |z_s^n| + |z_s^{n-1}| \right)^\alpha \, ds \right].
\]

We set the random variable

\[
I_n := \int_0^T \gamma(s) \left( g_s + |y_s^n| + |z_s^n| + |z_s^{n-1}| \right)^\alpha \, ds.
\]

Similar to the proof procedure of the uniqueness part of Theorem 17, we can prove that \( I_n \) belongs to \( L^q(\Omega, \mathcal{F}_T; \mathbb{P}; \mathbb{R}) \) as soon as \( aq < 1 \) with \( q > 1 \). Furthermore, in view of Doob’s inequality, for some \( q > 1 \) such that \( aq < 1 \), by (27) we can deduce that there exists a positive constant \( c_q \) depending only on \( q \) such that

\[
E \left[ \sup_{t \in [0,T]} |y_t^{n+1} - y_t^n|^q \right] \leq c_q E \left[ I_n^q \right] < +\infty,
\]

which implies that \((y_t^{n+1} - y_t^n)_{t \in [0,T]} \) belongs to the space \( \mathcal{S}^q \) for some \( q > 1 \). In the sequel, for each \( n \in \mathbb{N} \), we set \( \bar{y}^n := y_t^{n+1} - y^n \) and \( \bar{z}^n := z_t^{n+1} - z^n \), then \((\bar{y}^n_t, \bar{z}^n_t)_{t \in [0,T]} \) is a solution of the following BSDE:

\[
\bar{y}_t^n = \int_t^T g^n(s, \bar{y}_s^n) \, ds - \int_t^T \bar{z}_s^n \, dB_s, \quad t \in [0, T],
\]

where the generator \( g^n(t, y) := g(t, y + y^n_t, z^n_t) - g(t, y^n_t, z^n_t) \) for each \( y \in \mathbb{R}^k \). It follows from assumptions (H4') and (H6) that

\[
\langle y, g^n(t, y) \rangle \leq |y||g(t, y^n_t, z^n_t) - g(t, y^n_t, z^n_t)| \leq 2 |y| \gamma(t) \left( g_s + |y_s^n| + |z_s^n| + |z_s^{n-1}| \right)^\alpha,
\]
which means that the generator $g^n$ satisfies assumption (A) with $\mu(t) = \lambda(t) \equiv 0$, $f_t = 2\gamma(t)(g_t + |y^n_t| + |z^n_t| + |z^{n-1}_{t-}|)^p$ and $p = q$ since $I_n$ belongs to $L^q$. Thus, Lemma 4 yields that $(\tilde{z}^n_t)_{t \in [0,T]}$ belongs to the space $M^q$. Besides, in view of assumptions (H4') and (H5) we have $(y,g^n(t,y)) \leq v(t)|y^n|z^n_{t-1}$. Thus, Proposition 6 with $p = q$, $\mu(t) = \lambda(t) \equiv 0$ and $f_t = v(t)|z^n_{t-1}|$ yields that there exists a constant $C_q > 0$ depending only on $q$ such that for each $n \geq 2$,

$$
\| (y^n, z^n) \|_{S^q \times M^q} \leq C_q \mathbb{E} \left[ \left( \int_0^T y^n(t) \| z^n_{t-1} \| dt \right)^q \right].
$$

Then by Hölder’s inequality we get

$$
\| (y^n, z^n) \|_{S^q \times M^q} \leq C_q \left( \int_0^T v^n(t) dt \right)^{\frac{q}{2}} \mathbb{E} \left[ \left( \int_0^T |z^n_{t-1}|^2 dt \right)^{\frac{q}{2}} \right],
$$

from which it follows that for each $n \geq 2$,

$$
\| (y^n, z^n) \|_{S^q \times M^q} \leq \left[ C_q \left( \int_0^T v^n(t) dt \right)^{\frac{q}{2}} \right]^{n-1} \| (y^n_1, z^n_1) \|_{S^q \times M^q}.
$$

We first assume that

$$
C_q \left( \int_0^T v^n(t) dt \right)^{\frac{q}{2}} < 1.
$$

Since $\| (y^n_1, z^n_1) \|_{S^q \times M^q} < +\infty$, it follows immediately that $\{(y^n_t - y^1_t, z^n_t - z^1_t)_{t \in [0,T]}\}_{n=1}^{\infty}$ converges to some $(Y_t, Z_t)_{t \in [0,T]}$ in the space $S^q \times M^q$. Then we can deduce that $\{(y^n_t, z^n_t)_{t \in [0,T]}\}_{n=1}^{\infty}$ converges to $(y_\infty := Y_t + y^1_t, z_\infty := Z_t + z^1_t)_{t \in [0,T]}$ in the space $S^q \times M^q$ for any $\beta \in (0,1)$ since $(y^n_t, z^n_t)_{t \in [0,T]}$ belongs to it. Moreover, since $(y^1_t)_{t \in [0,T]}$ belongs to the class (D) and the convergence in $S^q$ with $q > 1$ is stronger than the convergence under the norm $\| \cdot \|_1$, we know that $\{(y^n_t)_{t \in [0,T]}\}_{n=1}^{\infty}$ converges to $(y_\infty)_{t \in [0,T]}$ under the norm $\| \cdot \|_1$. Then by taking the limit in both sides of BSDE (26) under ucp we can see that $(y_\infty, z_\infty)_{t \in [0,T]}$ is the desired solution of BSDE (1).

For the general case, in view of $\int_0^T v^n(t) dt < +\infty$, we can as before subdivide the time interval $[0, T]$ into a finite number of small intervals like $[T_i, T_{i+1}]$ such that

$$
C_q \left( \int_{T_i}^{T_{i+1}} v^n(t) dt \right)^{\frac{q}{2}} < 1.
$$

This completes the proof of the existence part of Theorem 17.

**Remark 18.** According to the proof procedure of the uniqueness and existence part of Theorem 17, we know that if assumption (H6) is satisfied as follows:

$$
g(t, y, z) = g(t, y, 0) \leq \gamma(t) |z|^{\alpha},
$$

then $\gamma(t)$ in (H6) need only to satisfy $\int_0^T \gamma^{2/(2-\alpha)}(t) dt < +\infty$.

**Remark 19.** In the case that $0 \leq T < +\infty$, the $u(t)$, $v(t)$ and $\gamma(t)$ appearing in (H4), (H5) and (H6) are all bounded by a constant $c > 0$ in the corresponding assumptions in Briand et al. [3], and they do not deal with the case $T = +\infty$. But in our framework, $u(t)$, $v(t)$ and $\gamma(t)$ may be unbounded, and their integrability is the only requirement.

**Example 20.** Let $0 \leq T < +\infty$, $k = 1$ and

$$
g(t, y, z) = \frac{1}{\sqrt{T}} \left( e^{-y}1_{y \leq 0} + (1 - g^2)1_{y > 0} \right) + \frac{t + \frac{1}{\sqrt{T}} (|z|^2 + \sqrt{|z|})}{1 + t^2}.
$$

It is not hard to check that the generator $g$ satisfies assumptions (H1'), (H2) – (H6) with $\alpha = 1/2$. Then Theorem 9 leads to that for each $x \in L^2(\Omega, F_T, P, \mathbb{R})$, BSDE $(\xi, T, g)$ has a unique solution $(y_t, z_t)_{t \in [0, T]}$ in $S^3 \times M^3$ for $\beta \in (1/2, 1)$ with $(y_t)_{t \in [0, T]}$ belonging to the class (D). Remark 19 applies.
Example 21. Let $0 \leq T \leq +\infty$, $k = 2$ and
\[
g(t, y, z) = \frac{1}{1 + t^2} \left[ e^{-y_1} + 3y_2 \right] e^{-t} \sin \left( \frac{|z_1|}{|z_2|} \right) + \frac{e^{-t} \sin t}{t}.
\]
where $y_i$ and $z_i$ $(i = 1, 2)$ stand for the $i$th component of the vector $y$ and the $i$th row of the matrix $z$ respectively. It is not hard to verify that $g$ satisfies assumptions $(H1^*)$, $(H2) - (H6)$ with $u(t) = 1/(1 + t^2)$, $v(t) = e^{-t}$ and for any $\alpha \in (0, 1)$. Thus, by Theorem 9 we know that for each $\xi \in L^1(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, BSDE $(\xi, T, g)$ has a unique solution $(y_t, z_t)_{t \in [0, T]}$ in $\mathcal{S}^2 \times \mathcal{M}^2$ for $\beta \in (\alpha, 1)$ with $(y_t)_{t \in [0, T]}$ belonging to the class $(D)$. Remark 19 applies.

Appendix A. Complement proofs of some lemmas

This appendix gives the detailed proofs of some lemmas.

Proof of Lemma 4. Let us fix the nonnegative function $\beta(t) : [0, T] \rightarrow \mathbb{R}^+$ with $\int_0^T \beta(t) \, dt < +\infty$ and $\beta(t) \geq 2(\mu(t) + \lambda^2(t))$ for each $t \in [0, T]$. Similar to the change of variables in Remark 8, we define\[
\bar{y}_t = e^{\int_0^t \beta(s) \, ds} y_t, \quad \bar{z}_t = e^{\int_0^t \beta(s) \, ds} z_t.
\]
Then $(\bar{y}_t, \bar{z}_t)_{t \in [0, T]}$ solves BSDE $(e^{\int_0^t \beta(s) \, ds} \xi, T, \bar{g})$ where\[
\bar{g}(t, y, z) := e^{\int_0^t \beta(s) \, ds} g(t, e^{-t} \int_0^t \beta(s) \, ds y, e^{-t} \int_0^t \beta(s) \, ds z) - \frac{1}{2} \beta(t) y,
\]
which satisfies assumption $(A)$ with $\bar{\eta}(t) = \mu(t) - \frac{1}{2} \beta(t)$, $\bar{\lambda}(t) = \lambda(t)$ and $\bar{f}_t = e^{t} \int_0^t \beta(s) \, ds f_t$. The integrability conditions are equivalent with or without the superscript $\text{sup}$ since $\int_0^T \beta(t) \, dt < +\infty$. Thus, with this change of variables we reduce to the case $\beta(t) \equiv 0$ and $\mu(t) + \lambda^2(t) \leq 0$. With omitting the superscript $\text{sup}$ for notational convenience, we need to prove that there exists a constant $C_p^0 > 0$ such that for each $0 \leq r \leq T$,\[
\mathbb{E} \left[ \left( \int_r^T |z_s|^2 \, ds \right)^\frac{p}{2} \bigg| \mathcal{F}_r \right] \leq C_p^1 \mathbb{E} \left[ \sup_{s \in [r, T]} |y_s|^p + \left( \int_r^T f_s \, ds \right)^p \bigg| \mathcal{F}_r \right].
\]
(A.1)

For each $n \in \mathbb{N}$, let us introduce the following stopping time:\[
\tau_n = \inf \left\{ t \in [0, T] : \int_0^t |z_s|^2 \, ds \geq n \right\} \wedge T.
\]
Applying Itô’s formula to $|y_s|^2$ yields that\[
|y_{t \wedge \tau_n}|^2 + \int_{t \wedge \tau_n}^{t \wedge \tau} |z_s|^2 \, ds = |y_{\tau_n}|^2 + 2 \int_{t \wedge \tau_n}^{\tau_n} \langle y_s, g(s, y_s, z_s) \rangle \, ds - 2 \int_{t \wedge \tau_n}^{\tau_n} \langle y_s, z_s \, dB_s \rangle.
\]
(A.2)
The inner product term including $g$ can be enlarged via assumption $(A)$ (stated between Lemma 3 and Lemma 4), $2ab \leq a^2 + b^2/2$, $2ab \leq a^2 + b^2$ and $\mu(t) + \lambda^2(t) \leq 0$ as follows:\[
2 \int_{t \wedge \tau_n}^{\tau_n} \langle y_s, g(s, y_s, z_s) \rangle \, ds \leq \sup_{s \in [t \wedge \tau_n, \tau_n]} |y_s|^2 + 2 \mu(s)|y_s|^2 + 2 \lambda^2(s)|y_s|^2 + \frac{1}{2} |z_s|^2 + 2f_s |y_s| \, ds \\
\leq \sup_{s \in [t \wedge \tau_n, \tau_n]} |y_s|^2 + \frac{1}{2} \int_{t \wedge \tau_n}^{\tau_n} |z_s|^2 \, ds + \left( \int_{t \wedge \tau_n}^{\tau_n} f_s \, ds \right)^2.
\]
Putting the previous inequality into (A.2) and noticing that $\tau_n \leq T$, we can deduce that there exists a constant $c_p > 0$ depending only on $p$ such that for each $n \in \mathbb{N}$,\[
\left( \int_{t \wedge \tau_n}^{\tau_n} |z_s|^2 \, ds \right)^\frac{p}{2} \leq c_p \left[ \sup_{s \in [t \wedge \tau_n, T]} |y_s|^p + \left( \int_{t \wedge \tau_n}^{\tau_n} f_s \, ds \right)^p + \left( \int_{t \wedge \tau_n}^{\tau_n} \langle y_s, z_s \, dB_s \rangle \right)^p \right].
\]
(A.3)
Moreover, the BDG inequality yields that there exists a constant $d_p > 0$ depending only on $p$ such that for each $n \in \mathbb{N}$ and $0 \leq r \leq t \leq T,$

\[
c_p \mathbb{E} \left[ \int_{t \wedge \tau_n}^{r} \langle y_s, z_s \rangle dB_s \right]^\frac{p}{2} \leq d_p \mathbb{E} \left[ \left( \int_{t \wedge \tau_n}^{r} |y_s|^2 |z_s|^2 \, ds \right)^\frac{p}{2} \right] F_r
\]

\[
\leq \frac{d^2}{2} \mathbb{E} \left[ \sup_{s \in [t \wedge \tau_n, T]} |y_s|^p \right] F_r + \frac{1}{2} \mathbb{E} \left[ \left( \int_{t \wedge \tau_n}^{r} |z_s|^2 \, ds \right)^\frac{p}{2} \right] F_r.
\]

Thus, by taking conditional expectation with respect to $F_r$ in both sides of the inequality (A.3) and then making use of Fatou’s lemma, we can deduce that there exists a constant $C_p^1 > 0$ depending only on $p$ which satisfies estimate (A.1). This completes the proof of Lemma 4.

**Proof of Lemma 5.** Let the assumptions of Lemma 4 hold and $p > 1$. Fix the nonnegative function $\beta(t) : [0, T] \mapsto \mathbb{R}^+$ with $\int_0^T \beta(t) \, dt < +\infty$ and $\beta(t) \geq \rho [\mu(t) + \lambda^2(t)/[1 \wedge (p - 1)]]$. As in the proof of Lemma 4, we also make the change of variables $\mathcal{F}_t = e^{\frac{1}{2} \int_0^t \beta(s) \, ds} y_t$, $\mathcal{F}_t = e^{\frac{1}{2} \int_0^t \beta(s) \, ds} z_t$. This reduces to the case $\beta(t) \equiv 0$ and $\mu(t) + \lambda^2(t)/[1 \wedge (p - 1)] \leq 0$. With omitting the superscript $\gamma$, we have to prove that there exists a constant $C_p^2 > 0$ depending only on $p$ such that for each $0 \leq r \leq t \leq T,$

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |y_s|^p \right] F_r \leq C_p^2 \mathbb{E} \left[ |\xi|^p + \left( \int_t^T f_s \, ds \right)^p \right] F_r. 
\]

(4.4)

It follows from Lemma 3 that for each $t \in [0, T]$

\[
|y_t|^p + c(p) \int_t^T |y_s|^{p-2} 1_{[y_s \neq 0]} |z_s|^2 \, ds \leq |\xi|^p + p \int_t^T |y_s|^{p-2} 1_{[y_s \neq 0]} (y_s, g(s, z_s)) \, ds
\]

\[
- p \int_t^T |y_s|^{p-2} 1_{[y_s \neq 0]} (y_s, z_s \, dB_s). 
\]

(5.5)

In view of assumption (A), we can get that

\[
c(p) \int_0^T |y_s|^{p-2} 1_{[y_s \neq 0]} |z_s|^2 \, ds \leq |\xi|^p + p \int_0^T (\mu(s)|y_s|^p + \lambda(s)|y_s|^{p-1}|z_s| + f_s|y_s|^{p-1}) \, ds
\]

\[
- p \int_0^T |y_s|^{p-2} 1_{[y_s \neq 0]} (y_s, z_s \, dB_s),
\]

from which we have $dP$ - a.s.,

\[
\int_0^T |y_s|^{p-2} 1_{[y_s \neq 0]} |z_s|^2 \, ds < +\infty.
\]

Now enlarge the inner product term including $g$ in (A.5) with $ab \leq \alpha a^2/4 + b^2/\alpha$ ($\alpha = (p - 1) \wedge 1$) and $\mu(t) + \lambda^2(t)/[1 \wedge (p - 1)] \leq 0$ as follows:

\[
p \int_t^T |y_s|^{p-2} 1_{[y_s \neq 0]} (y_s, g(s, z_s)) \, ds
\]

\[
\leq \int_t^T \left[ p \mu(s)|y_s|^p + \frac{\lambda^2(s)}{1 \wedge (p - 1)}|y_s|^p + c(p) \frac{p}{2} |y_s|^{p-2} 1_{[y_s \neq 0]} |z_s|^2 + p |y_s|^{p-1} f_s \right] \, ds
\]

\[
\leq \int_t^T \left[ \frac{c(p)}{2} |y_s|^{p-2} 1_{[y_s \neq 0]} |z_s|^2 + p |y_s|^{p-1} f_s \right] \, ds.
\]

Putting the previous inequality into (A.5) we can get that for each $t \in [0, T]$

\[
|y_t|^p + \frac{c(p)}{2} \int_t^T |y_s|^{p-2} 1_{[y_s \neq 0]} |z_s|^2 \, ds \leq X_t - p \int_t^T |y_s|^{p-2} 1_{[y_s \neq 0]} (y_s, z_s \, dB_s),
\]

(6.6)

where $X_t := |\xi|^p + p \int_t^T |y_s|^{p-1} f_s \, ds$. Note that $\{M_t := \int_0^t |y_s|^{p-2} 1_{[y_s \neq 0]} (y_s, z_s \, dB_s) \}_{t \in [0, T]}$ is a uniformly integrable martingale. In fact, it follows from the BDG inequality and Young’s inequality (for any
nonnegative constant $a$ and $b$, $ab \leq a^p/p + b^q/q$ holds true with $q = p/(p - 1)$ that

$$E \left[ (M, M)^{1/2} \right] \leq E \left[ \left( \int_0^T |y_s|^{2p-2} |z_s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq E \left[ \sup_{s \in [0,T]} |y_s|^{p-1} \cdot \left( \int_0^T |z_s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq \frac{p - 1}{p} E \left[ \sup_{s \in [0,T]} |y_s|^p \right] + \frac{1}{p} E \left[ \left( \int_0^T |z_s|^2 \, ds \right)^{\frac{p}{2}} \right].$$

Since $(y_t)_{t \in [0,T]}$ belongs to $S^p$ and then $(z_s)_{t \in [0,T]}$ belongs to $M^p$ by Lemma 4, the right term in the previous inequality is finite. Therefore, for each $0 \leq r \leq t \leq T$ the inequality (A.6) yields both

$$\frac{c(p)}{2} E \left[ \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \right] \leq E[X_t | F_r],$$

and

$$E \left[ \sup_{s \in [t,T]} |y_s|^p \big| F_r \right] \leq E[X_t | F_r] + pE \left[ \sup_{s \in [t,T]} \left| \int_t^s |y_u|^{p-2} 1_{|y_u| \neq 0} (y_u, z_u \, dB_u) \right| \big| F_r \right],$$

$$\leq E[X_t | F_r] + k_p E \left[ \left( \int_t^T |y_s|^{2p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \right)^{\frac{p}{2}} \right],$$

$$\leq E[X_t | F_r] + \frac{1}{2} E \left[ \sup_{s \in [t,T]} |y_s|^p \big| F_r \right] + \frac{k_p^2}{2} E \left[ \int_t^T |y_s|^{p-2} 1_{|y_s| \neq 0} |z_s|^2 \, ds \right] \leq 2 \left[ 1 + \frac{k_p^2}{c(p)} \right] E[X_t | F_r].$$

where the constant $k_p > 0$ depending only on $p$ follows by the BDG inequality. Combining the previous two inequalities we can get that for each $t \in [0, T]$,

$$E \left[ \sup_{s \in [t,T]} |y_s|^p \big| F_r \right] \leq 2 \left[ 1 + \frac{k_p^2}{c(p)} \right] E[X_t | F_r]. \quad (A.7)$$

Let $l_p := 2p(1 + k_p^2/c(p))$. It follows from Young’s inequality that

$$l_p E \left[ \int_t^T |y_s|^{p-1} f_s \, ds \big| F_r \right] \leq E \left[ \left( \frac{p}{2} \right)^{\frac{p}{2}} \sup_{s \in [t,T]} |y_s|^{p-1} \cdot l_p \left( \frac{p}{2p-2} \right)^{\frac{p}{2}} \int_t^T f_s \, ds \big| F_r \right] \leq \frac{1}{2} E \left[ \sup_{s \in [t,T]} |y_s|^p \big| F_r \right] + K_p E \left[ \left( \int_t^T f_s \, ds \right)^p \big| F_r \right],$$

where $K_p := (l_p)^p (p/(2p - 2))^{1-r}/p$. Now combining (A.7) with the definition of $X_t$ and the previous inequality, we deduce that there must exist a constant $C_2^p > 0$ depending only on $p$ such that (A.4) holds true. The proof of Lemma 5 is then completed. \hfill \Box

**Proof of Lemma 10.** It is clear that $f_n(t, y)$ satisfies assumption (H2). By assumption (H3') on $g$, (5) and (6) we can obtain that

$$|f_n(t, y)| \leq \int_{\mathbb{R}^k} \rho_n(x) |f(t, y - x)| \, dx \leq \int_{\mathbb{R}^k} \rho_n(x) (|f(t, 0)| + u(t) \varphi(|y - x|)) \, dx \leq |f(t, 0)| + u(t) \int_{\|x\| \leq 1} \rho(x) \varphi(|y - \frac{x}{n}|) \, dx \leq |f(t, 0)| + u(t) \varphi(|y| + 1).$$

Thus, $f_n(t, y)$ satisfies (H3') with $\varphi$ replaced by $\phi$, and then (7) follows from (4).

Furthermore, for each $y_1, y_2 \in \mathbb{R}^k$, we have, in view of (H4') on $g$,

$$\langle y_1 - y_2, f_n(t, y_1) - f_n(t, y_2) \rangle = \int_{\mathbb{R}^k} \rho_n(x) (y_1 - y_2, f(t, y_1 - x) - f(t, y_2 - x)) \, dx \leq 0,$$

which means that (H4') holds true for $f_n$.
Finally, fix \((\omega, t) \in \Omega \times [0, T]\), for the gradient of \(f_n(t, y)\) with respect to \(y\), we have

\[
|\nabla f_n(t, y)| \leq \int_{\mathbb{R}^k} |\nabla \rho_n(y - x)||f(t, x)| \, dx, \quad \forall y \in \mathbb{R}^k.
\]

It then follows from assumption (H3') on \(g\) and \((4)\) that for each \(y \in \mathbb{R}^k\),

\[
|\nabla f_n(t, y)| \leq \int_{\mathbb{R}^k} |\nabla \rho_n(y - x)||f(t, 0) + u(t)\varphi(|x|)| \, dx
\]

\[
= \int_{\{x:|x-\bar{x}| \leq 1/n\}} |\nabla \rho_n(y - x)||f(t, 0) + u(t)\varphi(|x|)| \, dx
\]

\[
\leq (Ke^{-t} + u(t)\varphi(|\bar{x}|) + 1) \int_{\mathbb{R}^k} |\nabla \rho_n(x)| \, dx.
\]

Then \((8)\) follows immediately. That is, \(f_n(t, y)\) is locally Lipschitz continuous in \(y\) non-uniformly with respect to \(t\). Lemma 10 is then proved. \(\square\)

**Proof of Lemma 11.** For any real number \(\varepsilon > 0\), we set

\[
X^\varepsilon_t := y_t - \frac{\varepsilon(U_t - f(t, y_t))}{|U_t - f(t, y_t)|} 1_{|U_t - f(t, y_t)| \neq 0}.
\]

In view of \((13)\), it is clear that \(dP \times dt - a.e., |X^\varepsilon_t| \leq |y_t| + \varepsilon \leq a + \varepsilon\). It then follows from \((6)\) that \(dP \times dt - a.e., f_n(t, X^\varepsilon_t) \to f(t, X^\varepsilon_t)\) as \(n \to \infty\) and from \((4)\) and assumption (H3') on \(f_n\) and \(f\) that

\[
|f_n(t, X^\varepsilon_t) - f(t, X^\varepsilon_t)| \leq 2Ke^{-t} + u(t)(\phi(0) + \varphi(a + \varepsilon) + \phi(a + \varepsilon)).
\]

Then Lebesgue’s dominated convergence theorem leads to

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |f_n(t, X^\varepsilon_t) - f(t, X^\varepsilon_t)| \, dt \right] = 0. \quad \text{(A.8)}
\]

Furthermore, we can deduce that

\[
\lim_{n \to \infty} \sup_{t < T} \mathbb{E} \left[ \int_0^T |y_t^n - X^\varepsilon_t, f_n(t, y_t^n) - f(t, X^\varepsilon_t)| \, dt \right] \leq 0. \quad \text{(A.9)}
\]

Indeed, for each \(n \in \mathbb{N}\), by (H4') on \(f_n\) we know that \(dP \times dt - a.e.,\)

\[
|y_t^n - X^\varepsilon_t, f_n(t, y_t^n) - f(t, X^\varepsilon_t)| \leq (y_t^n - X^\varepsilon_t, f_n(t, X^\varepsilon_t) - f(t, X^\varepsilon_t)).
\]

Then by the previous inequality and \((A.8)\) and noticing that \(dP \times dt - a.e., |y_t^n - X^\varepsilon_t| \leq 2a + \varepsilon\), we have

\[
\lim_{n \to \infty} \sup_{t < T} \mathbb{E} \left[ \int_0^T |y_t^n - X^\varepsilon_t, f_n(t, y_t^n) - f(t, X^\varepsilon_t)| \, dt \right] \leq (2a + \varepsilon) \lim_{n \to \infty} \sup_{t < T} \mathbb{E} \left[ \int_0^T |f_n(t, X^\varepsilon_t) - f(t, X^\varepsilon_t)| \, dt \right] = 0.
\]

In the sequel, applying Itô’s formula to \(|y_t^n|^2\) we can get that

\[
2E \left[ \int_0^T \langle y_t^n, f_n(t, y_t^n) \rangle \, dt \right] = |y_0^n|^2 - E \left[ |\xi|^2 \right] + E \left[ \int_0^T |z_t^n|^2 \, dt \right].
\]

Then since the mapping \(z \mapsto E[\int_0^T |z_t|^2 \, dt] \) is weakly lower semi-continuous and \(y_0^n\) converges to \(y_0\) in \(\mathbb{R}^k\), we have

\[
\lim_{n \to \infty} \inf 2E \left[ \int_0^T \langle y_t^n, f_n(t, y_t^n) \rangle \, dt \right] \geq |y_0|^2 - E \left[ |\xi|^2 \right] + E \left[ \int_0^T |z_t|^2 \, dt \right] = 2E \left[ \int_0^T \langle y_t, U_t \rangle \, dt \right]. \quad \text{(A.10)}
\]
The equal sign in the previous equation follows from applying Itô’s formula to $|y_t|^2$. Combining the weak convergences with (A.10) and (A.9), we can deduce that
\[
E \left[ \int_0^T (y_t - X_t^\varepsilon, U_t - f(t, X_t^\varepsilon)) \, dt \right] \leq \liminf_{n \to \infty} E \left[ \int_0^T (y_t^n - X_t^n, f_n(t, y_t^n) - f(t, X_t^\varepsilon)) \, dt \right] \leq 0.
\]

Thus, noticing the definition of $X_t^\varepsilon$, we have, for each $\varepsilon > 0$,
\[
E \left[ \int_0^T \left( \frac{U_t - f(t, y_t)}{|U_t - f(t, y_t)|} \right) \mathbb{1}_{|U_t - f(t, y_t)| \neq 0}, U_t - f(t, X_t^\varepsilon) \right) \, dt \leq 0.
\]

Sending $\varepsilon$ to 0 yields that $dP \times dt - a.e., X_t^\varepsilon \to y_t$. Then noticing that (H2) holds true for $f$, we have $dP \times dt - a.e., f(t, X_t^\varepsilon) \to f(t, y_t)$. Moreover, since $E[\int_0^T |U_t|^2 \, dt] < +\infty$ and $|f(t, X_t^\varepsilon)| \leq Ke^{-t} + u(t)\varphi(a + \varepsilon)$, Lebesgue’s dominated convergence theorem leads to that
\[
E \left[ \int_0^T |U_t - f(t, y_t)| \, dt \right] \leq 0,
\]
from which we have $dP \times dt - a.e., U_t = f(t, y_t)$.

\[\square\]

**Proof of Lemma 12.** It follows from the definition of $h_n$ that $h_n$ satisfies (H2) and $h_n(t, 0, V_t) = g(t, 0, V_t)$. Therefore, $|h_n(t, 0, V_t)| \leq Ke^{-t}$. Next we check that $h_n$ satisfies (H4). For each $y_1, y_2 \in \mathbb{R}^k$, if $|y_1| > r' + 1$ and $|y_2| > r' + 1$, (H4) is trivially satisfied and thus we reduce to the case where $|y_2| \leq r' + 1$. For notation convenience, we set, for each $n \in \mathbb{N}$ and $t \in [0, T]$,
\[
\pi^n(t) := \pi_{ne^{-t}}(V_t), \quad \psi^n(t) := \frac{ne^{-t}}{\psi_{r+1}(t) \vee (ne^{-t})}.
\]

By adding and subtracting $\theta_r(y_1)g(t, y_2, \pi^n(t))$ we can deduce that
\[
\langle y_1 - y_2, h_n(t, y_1, V_t) - h_n(t, y_2, V_t) \rangle = \psi^n(t)\theta_r(y_1)(y_1 - y_2, g(t, y_1, \pi^n(t)) - g(t, y_2, \pi^n(t))) \\
+ \psi^n(t)(\theta_r(y_1) - \theta_r(y_2))(y_1 - y_2, g(t, y_2, \pi^n(t)) - g(t, 0, 0, \pi^n(t))).
\]

The first term on the right side is non-positive since $g$ satisfies (H4'). For the second term, since $\theta_r$ is $C(r')$-Lipschitz and $|y_2| \leq r' + 1$, we can get that
\[
(\theta_r(y_1) - \theta_r(y_2))(y_1 - y_2, g(t, y_2, \pi^n(t)) - g(t, 0, 0, \pi^n(t))) \leq C(r')|y_1 - y_2|^2|g(t, y_2, \pi^n(t)) - g(t, 0, 0, \pi^n(t))|.
\]

And assumption (H5) on $g$ and the definition of $\psi_r(t)$ in (H3) yield that
\[
\begin{align*}
|g(t, y_2, \pi^n(t)) - g(t, 0, 0, \pi^n(t))| &\leq v(t)|\pi^n(t)| + v(t)|g(t, y_2, 0) - g(t, 0, 0, \pi^n(t))| \\
&\leq 2v(t)\pi^n(t) + |g(t, y_2, 0) - g(t, 0, 0, \pi^n(t))| \leq 2v(t)ne^{-t} + \psi_{r+1}(t).
\end{align*}
\]

Then, in view of $|\psi^n(t)| \leq 1$ and $|\psi^n(t)|\psi_{r+1}(t) = |\pi_{ne^{-t}}(\psi_{r+1}(t))| \leq ne^{-t}$, we have
\[
\langle y_1 - y_2, h_n(t, y_1, V_t) - h_n(t, y_2, V_t) \rangle \leq nc(2v(t)ne^{-t} + e^{-t})|y_1 - y_2|^2.
\]

Thus, note that $2\int_0^T v(t)e^{-t} \, dt \leq \int_0^T v^2(t) \, dt + \int_0^T e^{-2t} \, dt < +\infty$, we know that $h_n$ satisfies (H4).

Finally, we check that $h_n$ satisfies (H3'). It follows from the definition of $h_n$ that
\[
|h_n(t, y, V_t)| \leq \psi^n(t)\theta_r(y)|g(t, y, \pi^n(t)) - g(t, 0, 0, \pi^n(t))| + |h_n(t, 0, V_t)|.
\]

And similar to (A.12), we know that for each $y \in \mathbb{R}^k$ with $|y| \leq r' + 1$,\[
|g(t, y, \pi^n(t)) - g(t, 0, 0, \pi^n(t))| \leq 2nv(t)e^{-t} + \psi_{r+1}(t).
\]

Hence, we deduce that for each $y \in \mathbb{R}^k$,
\[
|h_n(t, y, V_t)| \leq |h_n(t, 0, V_t)| + n(2v(t)e^{-t} + e^{-t}),
\]
which means that (H3') is satisfied for $h_n$. \[\square\]
Proof of Lemma 13. We will use the same notations in (A.11). Let

\begin{align*}
A(s) &= \left(g(s, y^n_s, \pi^{n+i}(s)) - g(s, y^n_s, \pi^n(s))\right)\psi^{n+i}(s), \\
B(s) &= \left(g(s, y^n_s, \pi^n(s)) - g(s, y^n_s, 0)\right)\left(\psi^{n+i}(s) - \psi^n(s)\right), \\
C(s) &= \left(g(s, y^n_s, 0) - g(s, 0, 0)\right)\left(\psi^{n+i}(s) - \psi^n(s)\right), \\
D(s) &= \left(g(s, 0, 0) - g(s, 0, \pi^n(s))\right)\left(\psi^{n+i}(s) - \psi^n(s)\right), \\
E(s) &= \left(g(s, 0, \pi^n(s)) - g(s, 0, \pi^{n+i}(s))\right)\psi^{n+i}(s).
\end{align*}

Then we have

\[|h'_{n+1}(s, y^n_s, V_s) - h'_n(s, y^n_s, V_s)| = |A(s) + B(s) + C(s) + D(s) + E(s)|.\]

It follows from assumption (H5) on \(g\), the fact \(|\psi^{n+i}(s)| \leq 1\) and the definitions of \(\pi^n\) and \(\pi^{n+i}\) that

\[|A(s)| \leq v(s)|\pi^{n+i}(s) - \pi^n(s)| = v(s)|V_s| \left(\frac{(n + i)e^{-s}}{|V_s| \vee \{(n + i)e^{-s}\}} - \frac{ne^{-s}}{|V_s| \vee \{ne^{-s}\}}\right) \leq v(s)|V_s|1_{|V_s| > ne^{-s}}.\]

Similar to the previous proof procedure, we also have that \(|E(s)| \leq v(s)|V_s|1_{|V_s| > ne^{-s}}\).

Next, in view of assumption (H5) on \(g\) and the fact \(|\psi^{n+i}(s) - \psi^n(s)|\leq 1_{\psi_{r+1}(s)>ne^{-s}}\), we can get

\[|B(s)| \leq v(s)|\pi^n(s)|1_{\psi_{r+1}(s)>ne^{-s}} \leq v(s)|V_s|1_{\psi_{r+1}(s)>ne^{-s}}.\]

Similarly, we also have \(|D(s)| \leq v(s)|V_s|1_{\psi_{r+1}(s)>ne^{-s}}\).

Finally, since \(\mathbb{dP} \times d\mathbb{f} - a.e., [g^n_s] \leq r^, \) it follows from assumption (H3) that

\[|C(s)| \leq |g(s, y^n_s, 0) - g(s, 0, 0)| |\psi^{n+i}(s) - \psi^n(s)| \leq \psi_{r+1}(s)1_{\psi_{r+1}(s)>ne^{-s}}.\]

Then we can obtain the result as follows:

\[|h'_{n+1}(s, y^n_s, V_s) - h'_n(s, y^n_s, V_s)| \leq 2v(s)|V_s|1_{|V_s| > ne^{-s}} + 2v(s)|V_s|1_{\psi_{r+1}(s)>ne^{-s}} + \psi_{r+1}(s)1_{\psi_{r+1}(s)>ne^{-s}},\]

which completes this proof. \(\square\)

References

[1] K. Balbali, Backward stochastic differential equations with locally Lipschitz coefficient, C. R. Acad. Sci. Paris, Ser. I, 333(2001) 481–486.
[2] P. Briand and R. Carmona, BSDEs with polynomial growth generators, J. Appl. Math. Stoch. Anal., 13(2000) 207–238.
[3] P. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, \(L^p\) solutions of backward stochastic differential equations, Stoch. Proc. Appl., 108(2003) 109–129.
[4] P. Briand, J.-P. Lepeltier and J. San Martin, One-dimensional backward stochastic differential equations whose coefficient is monotonic in \(y\) and non-Lipschitz in \(z\), Bernoulli, 13(2007) 80–91.
[5] S. Chen, \(L^p\) solutions of one-dimensional backward stochastic differential equations with continuous coefficients, Stoch. Anal. Appl., 28(2010) 820–841.
[6] Z. Chen and B. Wang, Infinite time interval BSDEs and the convergence of \(g\)-martingales, J. Austral. Math. Soc. (Series A), 69(2000) 187–211.
[7] N. El Karoui, S. Peng and M. C. Quenez, Backward stochastic differential equations in finance, Math. Finance., 7(1997) 1–71.
[8] S. Fan and L. Jiang, Finite and infinite time interval BSDEs with non-Lipschitz coefficients, Stat. Probabil. Lett., 80(2010) 962–968.
[9] S. Fan and L. Jiang, \(L^p\) solutions of finite and infinite time interval BSDEs with non-lipschitz coefficients, Stochastics An Interna. J. of Prob. and Stoch. Proc., 84(2012) 487–506.
[10] S. Fan, and L. Jiang \(L^p\) (\(p > 1\)) solutions for one-dimensional BSDEs with linear-growth generators, J. Appl. Math. Comput., 38(2012) 295–304.
[11] S. Fan, L. Jiang and M. Davison, Uniqueness of solutions for multidimensional BSDEs with uniformly continuous generators, *C. R. Acad. Sci. Paris, Ser. I*, 348(2010) 683–686.

[12] S. Fan, L. Jiang and D. Tian, One-dimensional BSDEs with finite and infinite time horizons, *Stoch. Proc. Appl.*, 121(2011) 427–440.

[13] S. Fan and D. Liu, A class of BSDEs with integrable parameters, *Stat. Probabil. Lett.*, 80(2010) 2024–2031.

[14] S. Hamadène, Multidimensional backward stochastic differential equations with uniformly continuous coefficients, *Bernoulli*, 9(2003) 517–534.

[15] G. Jia, A class of backward stochastic differential equations with discontinuous coefficients, *Stat. Probabil. Lett.*, 78(2008) 231–237.

[16] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth, *Ann. Probab.*, 28(2000) 558–602.

[17] J.-P. Lepeltier and J. San Martin, Backward stochastic differential equations with continuous coefficient, *Stat. Probabil. Lett.*, 32(1997) 425–430.

[18] X. Mao, Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients, *Stoch. Proc. Appl.*, 58(1995) 281–292.

[19] E. Pardoux, BSDEs, weak convergence and homogenization of semilinear PDEs. In F. Clarke and R. Stern (Eds.), *Nonlinear Analysis, Differential Equations and Control*, pp. 503–549. (New York: Kluwer Academic, 1999).

[20] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation, *Syst. Control Lett.*, 14(1990) 55–61.

[21] S. Peng, Backward SDE and related $g$-expectation. In N. El Karoui and L. Mazliak (Eds.), *Backward stochastic differential equations (Paris,1995–1996)*, pp. 141–159. (Harlow: Longman, volume 364 of *Pitman Research Notes Mathematical Series*, 1997).

[22] N. Touzi, *Optimal stochastic control, stochastic target problems, and backward SDE*. Fields Institute Monographs, vol. 29. Springer, New York, 2013.

[23] Y. Wang and Z. Huang, Backward stochastic differential equations with non-Lipschitz coefficients, *Stat. Probabil. Lett.*, 79(2009) 1438–1443.