Wick’s theorem for \(q\)-deformed boson operators

Toufik Mansour\(^1\), Matthias Schork\(^2\) and Simone Severini\(^3\)

\(^1\) Department of Mathematics, University of Haifa, Haifa 31905, Israel
\(^2\) Alexanderstr. 76, 60489 Frankfurt, Germany
\(^3\) Institute for Quantum Computing and Department of Combinatorics and Optimization, University of Waterloo, N2 L 3G1 Waterloo, Canada

E-mail: toufik@math.haifa.ac.il, mschork@member.ams.org and simoseve@gmail.com

Received 18 April 2007, in final form 31 May 2007
Published 3 July 2007
Online at stacks.iop.org/JPhysA/40/8393

Abstract
In this paper, combinatorial aspects of normal ordering of arbitrary words in the creation and annihilation operators of the \(q\)-deformed boson are discussed. In particular, it is shown how, by introducing appropriate \(q\)-weights for the associated ‘Feynman diagrams’, the normally ordered form of a general expression in the creation and annihilation operators can be written as a sum over all \(q\)-weighted Feynman diagrams, representing Wick’s theorem in the present context.

PACS number: 02.10.Ox
Mathematics Subject Classification: 05A10, 05A30, 05C99

1. Introduction

Since the seminal work of Katriel [1], the combinatorial aspects of normal ordering of arbitrary words in the creation and annihilation operators \(b^\dagger\) and \(b\) of a single-mode boson have been studied intensively and many generalizations have also been considered; see, e.g., [2–18] and the references given therein. For an important discussion and references to the earlier literature on normal ordering of noncommuting operators, see [19]. For the creation and annihilation operators \(f^\dagger\) and \(f\) of a single-mode fermion, respectively, the analogous combinatorial problem does not exist due to the nilpotency of the operators, i.e. \((f^\dagger)^2 = 0 = f^2\). However, if one considers instead of a single-mode fermion a multi-mode fermion (i.e. several sets of operators \(f_i, f_i^\dagger\)), then interesting combinatorial connections to rook numbers exist (note that the general normal ordering problem of a single-mode boson has also straightforward connections to rook numbers [14]). This was noted by Navon [20] even before Katriel demonstrated that normal ordering powers of the bosonic number operator, i.e. \((b^\dagger b)^n\), involve the Stirling numbers of the second kind [1]. Very recently, combinatorial aspects of multi-mode boson operators (where the different modes interact due to a nontrivial commutation relation) have also been investigated [18]. Starting with the paper of Katriel and Kibler [21],
the combinatorial aspects of normal ordering arbitrary words in the creation and annihilation operators \( c \) and \( c^\dagger \) of a single-mode \( q \)-boson having the commutation relations

\[
[c, c^\dagger]_q = cc^\dagger - qc^\dagger c = 1, \quad [c, c] = 0, \quad [c^\dagger, c^\dagger] = 0 \tag{1}
\]

have also been considered [8, 14, 21–23]. Recall that normal ordering is a functional representation of operator functions in which all the creation operators stand to the left of the annihilation operators. Let an arbitrary operator function \( F(c, c^\dagger) \) be given; a function \( F(c, c^\dagger) \) can be seen as a word on the alphabet \( \{c, c^\dagger\} \). We denote by \( \mathcal{N}_q[F(c, c^\dagger)] \) the normal ordering of the function \( F(c, c^\dagger) \). Clearly, the operation of normal ordering \( \mathcal{N}_q \) is a linear map, that is, \( \mathcal{N}_q(F(c, c^\dagger) + G(c, c^\dagger)) = \mathcal{N}_q(F(c, c^\dagger)) + \mathcal{N}_q(G(c, c^\dagger)) \) for any two operator functions \( F(c, c^\dagger) \) and \( G(c, c^\dagger) \) respectively. Using the commutation relations (1), it is clear that their normally ordered form \( \mathcal{N}_q[F(c, c^\dagger)] = F(c, c^\dagger) \) can be written as

\[
\mathcal{N}_q[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{k,l} C_{k,l}(q)(c^\dagger)^k c^l \tag{2}
\]

for some coefficients \( C_{k,l}(q) \), and the main task consists of determining the coefficients as explicitly as possible.

For example, Katriel and Kibler showed [21] that normal ordering the powers of \( c^\dagger c \) involves the \( q \)-deformed Stirling numbers of the second kind \( S_q(n, k) \) in the version of Milne [24], i.e.

\[
\mathcal{N}_q[(c^\dagger c)^n] = \sum_{k=0}^n S_q(n, k)(c^\dagger)^k c^k. \tag{3}
\]

Note that when introducing Fock space representations one has \( c^\dagger c = [N]_q \), where we have denoted by \( N \) the associated number operator and by \( [x]_q = \frac{1 - q^x}{1 - q} = 1 + q + \cdots + q^{x-1} \) the \( q \)-deformed version of \( x \) (where \( x \) is a number or an operator). To describe the normally ordered form of an arbitrary expression is, therefore, an interesting problem. Varvak has shown [14] that the general coefficients can be interpreted as \( q \)-rook numbers. We will describe in the present paper a different approach associated with \('q\)-weighted Feynman diagrams'. Note that in [25–27], very similar results have been shown in slightly different situations.

The structure of the paper is as follows: In section 2 we introduce 'Feynman diagrams' associated with our problem (following closely the terminology of [27]) and introduce \( q \)-weights for these Feynman diagrams. In section 3 we state and prove Wick's theorem adapted to the present situation, i.e. describe the coefficients of (2) in terms of \( q \)-weighted Feynman diagrams. In section 4, some examples and consequences are discussed.

2. Feynman diagrams and associated \( q \)-weights

Recall that in the undeformed case (i.e. \( q = 1 \)) one has

\[
\mathcal{N}[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{\pi \in \mathcal{C}(F(c, c^\dagger))} : \pi :, \tag{4}
\]

where we have denoted by \( \mathcal{C}(F(c, c^\dagger)) \) the multiset of contractions of the word \( F(c, c^\dagger) \), and the double dot operation changes the order of the operators such that all creation operators precede the annihilation operators [17]. In comparison to [17, 18] we now slightly switch the terminology to match that of [27] (and also [25, 26, 28]) which we will follow closely. Let \( S = \{s_1, \ldots, s_n\} \) be a finite linearly ordered set consisting of two types of elements, i.e. there exists a 'type-map' \( \tau : S \to \{\mathcal{A}', \mathcal{C}'\} \) which associates with each letter \( s_i \) its type, i.e. \( \tau(s_i) \in \{\mathcal{A}', \mathcal{C}'\} \). We call elements \( s_j \) with \( \tau(s_j) = \mathcal{A}' \) 'annihilators' and elements \( s_j \) with
\(\tau(s_j) = \mathcal{G}' \) \('creators'\). We also denote by \(S^r\) (respectively \(S^s\)) the set of \(j\) with \(\tau(s_j) = \mathcal{G}'\) (respectively \(i\) with \(\tau(s_i) = \mathcal{A}\)). A Feynman diagram \(\gamma\) on \(S\) is a partition of \(S\) into one- and two-element sets, where the two-element sets have the special property that the two elements are of a different type (i.e. contain exactly one creator and one annihilator) and where the element of type \(\mathcal{G}'\) is the one with a larger index. We also regard \(\gamma\) as a set of ordered pairs \([(i_1, j_1), \ldots, (i_p, j_p)]\) with \(i_k < j_k\) and \(i_k \neq i_l, j_k \neq j_l\) and \(s_{i_0} \in S^r, s_{j_0} \in S^s\). We also assume with this notation that \(i_1 < i_2 < \cdots < i_p\).

**Remark 2.1.** Before continuing let us draw the connection to the terminology used in \([17, 18]\). In our concrete model, the set \(S\) is given by the word \(F(c, c^\dagger)\), the two types are given by \(\mathcal{G}' = c^\dagger\) and \(\mathcal{A} = c\), respectively, and a Feynman diagram \(\gamma\) corresponds precisely to a contraction. In fact, the two-element sets \((i_k, j_k)\) correspond to the edges of the contraction connecting a creator \(c^\dagger\) with a preceding annihilator \(c\). Thus, the Feynman diagram \(\gamma = [(i_1, j_1), \ldots, (i_p, j_p)]\) corresponds to a contraction of degree \(p\).

Let us introduce some further terminology following \([27]\). Given \(S\), we call the elements of \(S\) vertices. A Feynman diagram with representation \(\gamma = [(i_1, j_1), \ldots, (i_p, j_p)]\) is said to have degree \(p\) and the two-element sets \((i_k, j_k)\) are called edges. We denote the set of all Feynman diagrams on \(S\) by \(\mathcal{F}(S)\) and the set of Feynman diagrams of degree \(p\) by \(\mathcal{F}_p(S)\). If \(|S| = n\), then \(\mathcal{F}_p(S) = 0\) for \(p > \frac{n}{2}\). Given a Feynman diagram \(\gamma \in \mathcal{F}_p(S)\) with \(2p \leq n\), there will be \(n - 2p\) unpaired indices in \(\gamma\) to which we refer as singletons (in the terminology of \([17, 18]\), these are the vertices of degree 0). The set of singletons of \(\gamma\) will be denoted by \(S(\gamma)\). Let us now introduce the double dot operation for a Feynman diagram \(\gamma\) on \(S\). Intuitively, it means that we omit all vertices contained in the two-element sets of \(\gamma\) and order the remaining singletons in such a fashion that all creators precede the annihilators. More formally, let \(\gamma \in \mathcal{F}_p(S)\) and assume that \(\gamma\) has \(r\) singletons of type \(\mathcal{G}'\) (respectively \(s\) singletons of type \(\mathcal{A}\) with \(s = n - 2p - r\)). Then we define \(\gamma^r : = \mathcal{G}'^r \mathcal{A}^s\). Using this terminology, we can write the undeformed case (4) as

\[
N[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{\gamma \in \mathcal{F}(F(c, c^\dagger))} \gamma^r : \gamma :,
\]

for any word \(F(c, c^\dagger)\).

We now introduce a \(q\)-weight for Feynman diagrams such that we can write the normally ordered form of words \(F(c, c^\dagger)\) in the \(q\)-boson operators in a form analogous to (5). For this we have to introduce some more terminology following \([27]\). Let \(\gamma = [(i_1, j_1), \ldots, (i_p, j_p)]\) be a Feynman diagram. We say that a pair \((i_k, j_k)\) is a left crossing for \((i_m, j_m)\) if \(i_k < i_m < j_k < j_m\), and we define \(c(i, j)\) to be the number of such left crossings for \((i, j)\). We define \(c(\gamma) := \sum_{(i, j) \in \gamma} c(i, j)\) as the crossing number of \(\gamma\) (Biane calls it a restricted crossing number \([28]\)); it counts the intersections in the corresponding graph (the linear representation of \(\gamma\)). We also need to count degenerate crossings. These are the triples \(i < k < j\), where \(k\) is not paired (i.e. a singleton), and \((i, j) \in \gamma\). Letting \(d(i, j)\) be the number of such unpaired \(k\) for the edge \((i, j)\), then \(d(\gamma) := \sum_{(i, j) \in \gamma} d(i, j)\) counts the number of such triples in \(\gamma\). The total crossing number of \(\gamma\) is defined by

\[
tc(\gamma) := c(\gamma) + d(\gamma).
\]

It accounts for the ‘interaction’ between edges (the crossings) and the ‘interaction’ between singletons and edges (covering of singletons by edges). We now need in addition a measure which accounts for the ‘interaction’ between singletons. For a singleton \(s_k\) of \(\gamma\), we define its length (to the right) \(l_k(s_k)\) as follows: If the singleton \(s_k\) is of type \(\mathcal{G}'\), then \(l_k(s_k) = 0\); if the singleton \(s_k\) is of type \(\mathcal{A}\), then \(l_k(s_k)\) is given by the number of singletons of type \(\mathcal{G}'\) to the
right of $s_k$. The length of $\gamma$ is defined to be the sum of the lengths of all its singletons, i.e. $l(\gamma) := \sum_{c \in \gamma} l_s(c)$. After these lengthy preparations, we can now define the $q$-weight of a Feynman diagram $\gamma$ to be

$$W_q(\gamma) := q^{l(\gamma) + l(\gamma)}.$$  

**Example 2.2.** Let $F(c, c^\dagger) = c c^\dagger c c^\dagger c c^\dagger c^\dagger$ and consider the Feynman diagram $\gamma = \{(1, 3), (2, 6), (4, 9), (5, 7), (8, 12)\}$ of degree 5. In its linear representation in figure 1 the vertices of type $\mathcal{A} = c$ are depicted by an empty circle, while the vertices of type $\mathcal{G} = c^\dagger$ are depicted by a black circle. The crossing number of $\gamma$ is given by $c(\gamma) = 4$ and there are two degenerate crossings, i.e. $d(\gamma) = 2$, yielding the total crossing number $tc(\gamma) = 6$. There is only one singleton of type $\mathcal{A} = c$ having length $l_s(s_{10}) = 1$, yielding the length $l(\gamma) = 1$. Thus, the $q$-weight of $\gamma$ is given by $W_q(\gamma) = q^{6+1} = q^7$.

3. Wick’s theorem for the $q$-deformed boson

We can now state the generalization of (5) to the $q$-deformed case.

**Theorem 3.1.** Let $F(c, c^\dagger)$ be an operator function of the annihilation and creation operators of the $q$-boson (1). Then the normally ordered form $N_q[F(c, c^\dagger)]$ can be described with $q$-weighted Feynman diagrams and the double dot operation as follows:

$$N_q[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{\gamma \in \mathcal{F}(F(c, c^\dagger))} W_q(\gamma) : \gamma : .$$

**Proof.** We show this by induction in the length of the words $F(c, c^\dagger)$. Thus, assume that the relation holds for all words of length less than or equal to $n$. A word $G(c, c^\dagger)$ of length $n + 1$ can be written either as (I) $c^\dagger F(c, c^\dagger)$ or as (II) $c F(c, c^\dagger)$. Let us start with case (I). From the definitions, it follows that

$$N_q[c^\dagger F(c, c^\dagger)] = c^\dagger N_q[F(c, c^\dagger)] = \sum_{\gamma \in \mathcal{F}(c^\dagger F(c, c^\dagger))} W_q(\gamma) c^\dagger : \gamma : = \sum_{\gamma \in \mathcal{F}(F(c^\dagger, c))} W_q(\gamma) : \gamma :, $$

where we have used in the last equation the fact that there is a bijection between $\mathcal{F}(F(c, c^\dagger))$ and $\mathcal{F}(c^\dagger F(c, c^\dagger))$ such that we can identify $\gamma \in \mathcal{F}(F(c, c^\dagger))$ with the corresponding $\gamma \in \mathcal{F}(c^\dagger F(c, c^\dagger))$ having the same weight. This shows case (I). Let us turn to case (II). By the induction hypothesis, we assume that $N_q[F(c, c^\dagger)] = \sum_{\gamma \in \mathcal{F}(F(c, c^\dagger))} W_q(\gamma) : \gamma :$. Thus, from the definition of the $q$-normal ordering we can state

$$N_q[c F(c, c^\dagger)] = N_q[c N_q[F(c, c^\dagger)]] = N_q[c \sum_{\gamma \in \mathcal{F}(F(c, c^\dagger))} W_q(\gamma) : \gamma :] = c \sum_{\gamma \in \mathcal{F}(F(c, c^\dagger))} W_q(\gamma) N_q[c : \gamma : ] .$$

which is equivalent to (write $: \gamma : = (c^\dagger)^{a_\gamma} c b_\gamma)$

$$N_q[c F(c, c^\dagger)] = \sum_{\gamma \in \mathcal{F}(F(c, c^\dagger))} W_q(\gamma) N_q[c (c^\dagger)^{a_\gamma} c b_\gamma] .$$

Figure 1. The linear representation of the Feynman diagram $\gamma$. 
Using \( N_q[c(c^\dagger)^e c^b] = q^{a\nu}(c^\dagger)^e c^b + [a^\nu]_q (c^\dagger)^{a\nu-1} c^b \), we obtain

\[
N_q[cF(c, c^1)] = \sum_{\gamma \in \mathcal{F}(cF(c, c^1))} \mathcal{W}_q(\gamma) (q^{a\nu}(c^\dagger)^e c^b + [a^\nu]_q (c^\dagger)^{a\nu-1} c^b).
\] (8)

This is the explicit expression for the left-hand side of (7) in the present case. We will now study the right-hand side of (7) in the present case and show that it yields the same result as (8), thus proving the assertion for case (II). As the first step, we write \( \mathcal{F}(cF(c, c^1)) = \mathcal{F}^+(cF(c, c^1)) \cup \mathcal{F}^-(cF(c, c^1)) \) where \( \mathcal{F}^+(cF(c, c^1)) \) denotes the subset of Feynman diagrams where an edge starts at the most left vertex \( c \) and where \( \mathcal{F}^-(cF(c, c^1)) \) denotes the set of the remaining Feynman diagrams where the most left vertex \( c \) is a singleton. Thus,

\[
\sum_{\gamma \in \mathcal{F}(cF(c, c^1))} \mathcal{W}_q(\gamma) : \gamma : = \sum_{\delta \in \mathcal{F}^-(cF(c, c^1))} \mathcal{W}_q(\delta) : \delta : + \sum_{\beta \in \mathcal{F}^+(cF(c, c^1))} \mathcal{W}_q(\beta) : \beta : .
\] (9)

Note that there is a bijection between \( \mathcal{F}^-(cF(c, c^1)) \) and \( \mathcal{F}(F(c, c^1)) \) associating with \( \delta \in \mathcal{F}^-(cF(c, c^1)) \) the Feynman diagram \( \delta' \in \mathcal{F}(F(c, c^1)) \) by deleting the most left vertex \( c \) in the word \( cF(c, c^1) \). However, the \( q \)-weights of the two Feynman diagrams \( \delta \) and \( \delta' \) are not equal. The total crossing numbers are equal, i.e. \( tc(\delta) = tc(\delta') \), whereas the lengths are related by \( l(\delta) = l(\delta') + a_\beta \). Using 6, this yields for the \( q \)-weights \( \mathcal{W}_q(\delta) = q^{a\nu} \mathcal{W}_q(\delta') \), implying \( \mathcal{W}_q(\delta) : \delta : = q^{a\nu} \mathcal{W}_q(\delta') : \delta' : \). Thus, the first sum in (9) can be written as

\[
\sum_{\delta \in \mathcal{F}^-(cF(c, c^1))} \mathcal{W}_q(\delta) : \delta : = \sum_{\gamma \in \mathcal{F}(F(c, c^1))} \mathcal{W}_q(\gamma) q^{a\nu}(c^\dagger)^{a\nu}c^b + 1,
\] (10)

where we have switched to the more convenient notation \( \delta' \leadsto \gamma \). Let us turn to the second sum in (9). In analogy to an argument made above, we introduce the map

\( \mathcal{B} : \mathcal{F}^+(cF(c, c^1)) \rightarrow \mathcal{F}(F(c, c^1)) \),

which (i) deletes the edge beginning at the most left vertex \( c \) in the word \( cF(c, c^1) \) and (ii) deletes the most left vertex \( c \) clearly, this map is well defined. In contrast to above this is not a bijection, since there can be many \( \beta \in \mathcal{F}^+(cF(c, c^1)) \) which are mapped onto the same \( \beta' \in \mathcal{F}(F(c, c^1)) \). Let us denote the preimage of \( \beta' \) under this map by \( \mathcal{F}_p^+(cF(c, c^1)) \), i.e.

\( \mathcal{F}_p^+(cF(c, c^1)) = \{ \beta \in \mathcal{F}^+(cF(c, c^1)) | \mathcal{B}(\beta) = \beta' \in \mathcal{F}(F(c, c^1)) \} \).

Since these preimages are disjoint, we can write

\[
\sum_{\beta \in \mathcal{F}(cF(c, c^1))} \mathcal{W}_q(\beta) : \beta : = \sum_{\beta' \in \mathcal{F}(F(c, c^1))} \sum_{\beta \in \mathcal{F}_p^+(cF(c, c^1))} \mathcal{W}_q(\beta) : \beta : = \sum_{\beta \in \mathcal{F}(F(c, c^1))} \left\{ \sum_{\beta' \in \mathcal{F}(cF(c, c^1))} \mathcal{W}_q(\beta) \right\} (c^1)^{-1} : \beta' :,
\]

where we have used the fact that for all \( \beta \in \mathcal{F}_p^+(cF(c, c^1)) \) one has \( \beta : = (c^1)^{-1} : \beta' : \). Note that this formal notation is not meaningless in indicating that the degree of the creation operator has to be decreased by 1, since, by definition, there is at least one singleton of type \( C \) in every \( \beta' \), namely the one which becomes ‘free’ after deleting the edge in step (i) from above. Let us write as above : \( \beta' : = (c^1)^{e\nu} c^{b\nu} \). Assuming for the moment

\[
\sum_{\beta \in \mathcal{F}_p^+(cF(c, c^1))} \mathcal{W}_q(\beta) = [a_{\beta^\dagger}]_q \mathcal{W}_q(\beta'),
\] (11)
we have, therefore, shown that
\[ \sum_{\beta \in \mathcal{F}^+(cF(c,c^1))} \mathcal{W}_q(\beta) : \beta := \sum_{\beta' \in \mathcal{F}^+(cF(c,c^1))} \mathcal{W}_q(\beta') [a_{\beta'}]_q (c^1)^{a_{\beta'} - 1} c^{b_{\beta'}}. \] (12)

Switching to the more convenient notation $\beta' \sim \gamma$ and inserting (10) and (12) into the right-hand side of (9) yields
\[ \sum_{\gamma \in \mathcal{F}^+(cF(c,c^1))} \mathcal{W}_q(\gamma) : \gamma := \sum_{\gamma \in \mathcal{F}^+(cF(c,c^1))} \mathcal{W}_q(\gamma) [q^{a_{\gamma}'} (c^1)^{a_{\gamma}'} c^{b_{\gamma}'} + [a_{\gamma'}]_q (c^1)^{a_{\gamma'} - 1} c^{b_{\gamma'}}]. \] (13)

Comparing (8) and (13) shows that $N_q[cF(c,c^1)] = \sum_{\gamma \in \mathcal{F}^+(cF(c,c^1))} \mathcal{W}_q(\gamma) : \gamma$, provided (11) holds true. We are now going to show (11). First note that $|\mathcal{F}^+_{\beta_i}(cF(c,c^1))| = d_{\beta_i}$. The Feynman diagrams $\beta_i \in \mathcal{F}^+_{\beta_i}(cF(c,c^1))$ with $1 \leq i \leq a_{\beta_i}$ are easy to describe: $\beta_i$ arises from $\beta_i'$ by adjoining the most left vertex $c$ to the word $F(c,c^1)$ and connecting it via an edge with the $i$th singleton of type $C$ in the word $cF(c,c^1)$. We are now going to show that
\[ \mathcal{W}_q(\beta_i) = q^{l(\beta_i)} W_q(\beta_i). \] (14)

This implies
\[ \sum_{\beta \in \mathcal{F}^+(cF(c,c^1))} \mathcal{W}_q(\beta) = \sum_{i=1}^{a_{\beta_i}} \mathcal{W}_q(\beta_i) = \sum_{i=1}^{a_{\beta_i}} q^{l(\beta_i)} W_q(\beta_i) = [a_{\beta_i}]_q \mathcal{W}_q(\beta_i), \]
i.e. the sought-for equality (11). To show (14), we depict the word $cF(c,c^1)$ in the following fashion:
\[ cF(c,c^1) \equiv v_0 R_1 v_1 R_2 v_2 \cdots R_{a_{\beta_i}} v_{a_{\beta_i}} R_{a_{\beta_i} + 1}, \]
where $v_0$ denotes the most left vertex $c$, the vertices $v_i$ with $1 \leq i \leq a_{\beta_i}$ denote the singletons of type $C$ and (possibly empty) $R_i$ the ‘blocks’ in between. Let us denote by $\alpha_i$ the number of singletons of type $\beta_i$ in $R_i$. For the Feynman diagram $\beta_i$, we denote by $v_{i,1}$ the number of edges starting in $R_i$ and ending behind the vertex $v_i$. Let us now consider the Feynman diagrams $\beta_i$ and start with the case $i = 1$. Due to the additional edge between $v_0$ and $v_1$, one has the following relations:
\[ c(\beta_1) = c(\beta') + v_{1,1}, \quad \ell(\beta_1) = \ell(\beta') + 1, \quad l(\beta_1) = l(\beta') - \alpha_1, \]
showing that $tc(\beta_1) + l(\beta_1) = tc(\beta') + l(\beta')$ and, therefore, $W_q(\beta_1) = W_q(\beta')$. Let us consider the case $i = 2$. Due to the additional edge between $v_0$ and $v_2$ there will be $v_{1,2} + v_{2,2}$ additional crossings in $\beta_2$ compared to $\beta'$, i.e. $c(\beta_2) = c(\beta') + v_{1,2} + v_{2,2}$. Turning to the degenerate crossings, there are—in comparison to $\beta'$—three effects which have to be considered.

1. Since the vertex $v_2$ is no more a singleton there will be $v_{1,2} + v_{2,2}$ degenerate crossings less.
2. Since there is a new edge between $v_0$ and $v_2$ there will be $\alpha_1 + \alpha_2$ degenerate crossings more, coming from the covered blocks $R_1$ and $R_2$.
3. Since $v_1$ lies in between $v_0$ and $v_2$, the triple $v_0 < v_1 < v_2$ also accounts for an additional degenerate crossing.

This shows that $d(\beta_2) = d(\beta') + (\alpha_1 + \alpha_2) - (v_{1,2} + v_{2,2}) + 1$. The length of $\beta_2$ results by decreasing the length of $\beta'$ by $\alpha_1 + \alpha_2$ since the vertex $v_2$ is no more a singleton. Collecting the above results shows that $tc(\beta_2) + l(\beta_2) = tc(\beta') + l(\beta') + 1$, implying $W_q(\beta_2) = q W_q(\beta')$. Let us now turn to the case of general $i$ (with $1 \leq i \leq a_{\beta_i}$) where one has the analogous ‘trade off’ between the different parts which go into the $q$-weight (leaving, in effect, only the
vertices $v_1, \ldots, v_{i-1}$ as contributors to the difference between $\beta_i$ and $\beta'$. The same argument as above shows that

$$c(\beta_i) = c(\beta') + \sum_{k=1}^{i} v_{k,i}, \quad d(\beta_i) = d(\beta') + \sum_{k=1}^{i} a_{k} - \sum_{k=1}^{i} v_{k,i} + (i - 1),$$

$$l(\beta_i) = l(\beta') + \sum_{k=1}^{i} a_{k}.$$  

It follows that $\tau c(\beta_i) + l(\beta_i) = \tau c(\beta') + l(\beta') + (i - 1)$ and, therefore, that $W_q(\beta_i) = q^{i-1}W_q(\beta')$. But this is exactly (14) which was to be shown. Thus, the proof for case (II) is complete.

Clearly, letting $q = 1$ reduces (7) to the undeformed case (5). Note that very similar results have been derived in [25–27] with a slightly different point of view. In [12] and [29], different graphical means for normal ordering bosonic operators are discussed.

4. Examples and consequences

Let us consider a simple example for theorem 3.1 before we draw some connections to $q$-rook numbers and Stirling numbers.

**Example 4.1.** Let $F(c, c^\dagger) = c^2 c^1 c^2 \dagger = c c^1 c^1 \dagger$. Since there are only two creators in the word, the Feynman diagrams can have degree at most 2. The trivial Feynman diagram $\gamma$ of degree 0 yields $:v: = (c^1)^2 c^\dagger$ and has $q$-weight $W_q(\gamma) = q(\gamma) = q^{2+2+0+1+1+0}$. Thus, the Feynman diagram of degree 0 yields the contribution $q^6(c^1)^2 c^\dagger$. There are six Feynman diagrams of degree 1, namely

$$\{ (1, 3), (1, 6), (2, 3), (2, 6), (4, 6), (5, 6) \},$$

and their $q$-weights are given by (same order) $\{ q^4, q^3, q^4, q^3, q^2 \}$. Thus, the Feynman diagrams of degree 1 yield the contribution $(q^2 + 2q^3 + 2q^4 + q^5)c^1 c^\dagger$. There are six Feynman diagrams of degree 2, namely

$$\{ (1, 3)(2, 6), (1, 3)(4, 6), (1, 3)(5, 6), (1, 6)(2, 3), (2, 3)(4, 6), (2, 3)(5, 6) \}$$

with $q$-weights $\{ q^3, q^2, q, q^2, q, 1 \}$, yielding the contribution $(1 + 2q + 2q^2 + q^3)c^2$. Thus, $N_q[c^2 c^1 c^\dagger] = q^6(c^1)^2 c^4 + (q^2 + 2q^3 + 2q^4 + q^5)c^1 c^3 + (1 + 2q + 2q^2 + q^3)c^2$, (15) which may also be calculated by hand from (1).

We now want to draw a connection between theorem 3.1 and the results of Varvak [14]. Given a word $w = F(c, c^\dagger)$ containing $m$ creation operators $c^1$ and $n$ annihilation operators $c$ (with $n \leq m$), she associates with $w$ a certain Ferrers board $B_w$ outlined by $w$. Denoting by $R_k(B_w, q)$ the $k$th $q$-rook number of the board $B_w$, she shows that ([14], theorem 6.1)

$$w = \sum_{k=0}^{n} R_k(B_w, q) (c^\dagger)^{m-k} c^{n-k}.$$ 

Using theorem 3.1, we first note that the set of Feynman diagrams is the disjoint union of Feynman diagrams of degree $k$, i.e. $\mathcal{F}(w) = \bigcup_{k=0}^{n} \mathcal{F}_k(w)$, and that $\gamma \in \mathcal{F}_k(w)$ implies
\[ \gamma = (c^\dagger)^{m-k}c^{n-k}, \] yielding
\[ w = \sum_{k=0}^{n} \left\{ \sum_{\gamma \in \mathcal{F}(w)} \mathcal{W}_{q}(\gamma) \right\} (c^\dagger)^{m-k}c^{n-k}. \]

Comparing the two expressions yields the following corollary.

**Corollary 4.2.** Given a word \( w = F(c, c^\dagger) \), the \( k \)th q-rook number of the associated Ferrers board \( B_w \) equals the sum of the q-weights of all Feynman diagrams of degree \( k \) on \( w \), i.e.
\[ R_k(B_w, q) = \sum_{\gamma \in \mathcal{F}_{n-k}(c^\dagger c)^n} \mathcal{W}_{q}(\gamma). \] (16)

Let us return to the undeformed case \( (q=1) \) and consider the particular word \( F(c, c^\dagger) = (c^\dagger c)^n \). The same argument as above shows that
\[ N[(c^\dagger c)^n] = \sum_{k=0}^{n} \sum_{\gamma \in \mathcal{F}_{n-k}(c^\dagger c)^n} (c^\dagger)^k c^k. \]

Comparing this with the undeformed case of (3) shows that the conventional Stirling numbers of the second kind can also be interpreted as the number of Feynman diagrams of degree \( n-k \) on the particular word \( c^\dagger c c^\dagger c \cdots c^\dagger c \) of length \( 2n \), i.e.
\[ S(n, k) = |\mathcal{F}_{n-k}(c^\dagger c)^n|. \] (17)

Turning to the \( q \)-deformed situation, the same argument shows that
\[ S_q(n, k) = \sum_{\gamma \in \mathcal{F}_{n-k}(c^\dagger c)^n} \mathcal{W}_{q}(\gamma) = \sum_{\gamma \in \mathcal{F}_{n-k}(c^\dagger c)^n} q^{\ell(\gamma)+\ell(\gamma)}. \] (18)

**Remark 4.3.** In [18] the study of combinatorial aspects of the normal ordering of multi-mode boson operators was begun, and interesting combinatorial questions were addressed. In view of the above \( q \)-Wick’s theorem, it is natural to consider the analogous problem for the \( q \)-deformed variant of the multi-mode boson operator. However, this does not seem to be easy as the following example will show. For concreteness, we consider the deformed two-mode boson having the commutation relations
\[ [a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad [a, b^\dagger] = 1 \]
and all other commutators vanish (note, in particular, that the two modes interact nontrivially!). Here, the deformation parameters \( q_a, q_b, q_{ab} \) are in the moment arbitrary. Let us consider the simple example \( F(a, a^\dagger, b, b^\dagger) = abb^\dagger \). First commuting \( b \) and \( b^\dagger \) using the above commutation relations yields \( abb^\dagger = q_{ab}b^\dagger ab + a \); then commuting \( a \) and \( b^\dagger \) yields \( abb^\dagger = q_{ab}q_{ab}b^\dagger ab + q_{ab}b^\dagger ab + a \). On the other hand, first commuting \( a \) and \( b \) and then commuting \( b^\dagger \) to the left yields \( abb^\dagger = q_{ab}q_{ab}b^\dagger ba + q_{ab}ba + b \). Clearly, for the two results to coincide we have to assume that \( q_{ab} = 1 \). A similar computation for \( baa^\dagger \) also shows that \( q_a = 1 \). Thus, it seems that a \( q \)-deformed version of the multi-mode boson considered in [18] does not exist.

**Acknowledgment**

The authors would like to thank an anonymous referee for carefully reading the paper and for helpful comments.
References

[1] Katriel J 1974 Combinatorial aspects of boson algebra Lett. Nuovo Cimento 10 565–7
[2] Witschel W 1975 Ordered operator expansions by comparison J. Phys. A: Math. Gen. 8 143–54
[3] Mikhailov V V 1983 Ordering of some boson operator functions J. Phys. A: Math. Gen. 16 3817–27
[4] Katriel J 2000 Bell numbers and coherent states Phys. Lett. A 237 159–61
[5] Katriel J 2002 Coherent states and combinatorics J. Opt. B: Semiclass. Opt. 4 S200–3
[6] Blasiak P, Penson K A and Solomon A I 2003 The general boson normal ordering problem Phys. Lett. A 309 198–205
[7] Blasiak P, Penson K A and Solomon A I 2004 Boson normal ordering problem and generalized bell numbers Ann. Comb. 7 127–39
[8] Schork M 2003 On the combinatorics of normal ordering bosonic operators and deformations of it J. Phys. A: Math. Gen. 36 4651–65
[9] Blasiak P, Penson K A and Solomon A I 2004 Combinatorial coherent states via normal ordering of bosons Lett. Math. Phys. 67 13–23
[10] Fuji K and Suzuki T 2004 A new symmetric expression of Weyl ordering Mod. Phys. Lett. A 19 827–40
[11] Blasiak P, Horzela A, Penson K A, Duchamp G H E and Solomon A I 2005 Boson normal ordering via substitutions and Sheffer type polynomials Phys. Lett. A 338 108–16
[12] Blasiak P, Penson K A, Solomon A I, Horzela A and Duchamp G H E 2005 Some useful combinatorial formulas for bosonic operators J. Math. Phys. 46 052110
[13] Mendez M A, Blasiak P and Penson K A 2005 Combinatorial approach to generalized Bell and Stirling numbers and boson normal ordering problem J. Math. Phys. 46 083511
[14] Varvak A 2005 Rook numbers and the normal ordering problem J. Combin. Theory Ser. A 112 292–307
[15] Witschel W 2005 Ordering of boson operator functions by the Hausdorff similarity transform Phys. Lett. A 334 140–3
[16] Mansour T and Severini S 2006 Noncrossing normal ordering for functions of bosons Preprint quant-ph/0607074
[17] Mansour T, Schork M and Severini S 2007 A generalization of the boson normal ordering Phys. Lett. A 364 (3–4) 214–20 (Preprint quant-ph/0608081)
[18] Mansour T and Schork M 2007 On the normal ordering of multi-mode boson operators Preprint quant-ph/0701185
[19] Wilcox R M 1967 Exponential operators and parameter differentiation in quantum physics J. Math. Phys. 8 962–82
[20] Navon A M 1973 Combinatorics and fermion algebra Nuovo Cimento 16 324–30
[21] Katriel J and Kibler M 1992 Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers J. Phys. A: Math. Gen. 25 2683–91
[22] Blasiak P, Horzela A, Penson K A and Solomon A I 2004 Deformed bosons: combinatorics of normal ordering Czech. J. Phys. 54 1179–84
[23] Schork M 2006 Normal ordering q-bosons and combinatorics Phys. Lett. A 355 293–7
[24] Milne S 1978 A q-analog of restricted growth functions, Dobinski’s equality, and Charlier polynomials Trans. Am. Math. Soc. 245 89–118
[25] Bozejko M, Kümmerer B and Speicher R 1997 q-Gaussian processes: non-commutative and classical aspects Commun. Math. Phys. 185 129–54
[26] Anshelevich M 2001 Partition-dependent Stochastic Measures and q-deformed cumulants Doc. Math. 6 343–84
[27] Effros E G and Popa M 2003 Feynman diagrams and Wick products associated with q-Fock space Proc. Natl Acad. Sci. 100 8629–33
[28] Biane P 1997 Some properties of crossings and partitions Discrete Math. 175 41–53
[29] Gough J 2003 Quantum statistical field theory and combinatorics Preprint quant-ph/0311161