Relative Invariants, Contact Geometry and Open String Invariants

An-Min Li
Department of Mathematics, Sichuan University
Chengdu, PRC

Li Sheng
Department of Mathematics, Sichuan University
Chengdu, PRC

Abstract

In this paper we propose a theory of contact invariants and open string invariants, which are generalizations of the relative invariants. We introduce two moduli spaces $\overline{M}_A(M^+, C, g, m + \nu, y, p, (k, \epsilon))$ and $\overline{M}_A(M, L; g, m + \nu, y, p, \mu)$, prove the compactness of the moduli spaces and the existence of the invariants.

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1 Introduction

Open string invariant theory have been studied by many mathematicians and physicists (see [1, 2, 11, 16, 17]). This theory closely relates to relative Gromov-Witten theory and contact geometry. In this paper we propose a theory of contact invariants and open string invariants, which are generalizations of the relative invariants. We outline the idea as follows.

1). Let \((M, \omega)\) be a compact symplectic manifold, \(L \subset M\) be a compact Lagrangian submanifold. Let \((x_1, \ldots, x_n)\) be a local coordinate system in \(O \subset L\), there is a canonical coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) in \(T^*L|_O\). Suppose that given a Riemannian metric \(g\) on \(L\), and consider the unit sphere bundle \(\widetilde{M}\). Let \(\Lambda\) be the Liouville form on \(T^*L\), denote \(\lambda = -\partial |_{\widetilde{M}}\). Then \((\widetilde{M}, \lambda)\) is a contact manifold with contact form \(\lambda\). The Reeb vector field is given by \(X = \sum_{i} \frac{\partial}{\partial x_i}\). By Lagrangian Neighborhood Theorem we can write \(M - L\) as

\[ M^+ = M_0 \cup \{[0, \infty) \times \widetilde{M}\}, \]
where $M_0$ is a compact symplectic manifold with boundary. We choose an almost complex structure $J$ on $M^+$ such that $J$ is tamed by $\omega$ and over the cylinder end $J$ is given by

$$J|_\xi = \tilde{J}, \quad JX = -\frac{\partial}{\partial a}, \quad J(\frac{\partial}{\partial a}) = X,$$

where $\xi = ker\lambda$, $a$ is the canonical coordinate in $\mathbb{R}$, and $\tilde{J}$ is a $d\lambda$-tame almost complex structure in $\xi$. Assume that the periodic orbit sets of $X$ are either non-degenerate or of Bott-type, and $\tilde{J}$ can be chosen such that $L_XJ = 0$ along every periodic orbit.

Let $\tilde{\Sigma}$ be a Riemann surface with a puncture point $p$. We use the cylinder coordinates $(s,t)$ near $p$, i.e., we consider a neighborhood of $p$ as $(s_0, \infty) \times S^1$. Let $u : \tilde{\Sigma} \to M^+$ be a $J$-holomorphic map with finite energy. Suppose that

$$[\lim_{s\to \infty} u(s, S^1)] = \sum \mu^i[c_i],$$

where $[c_i], i = 1, ..., a$, is a bases in $H_1(L, \mathbb{Z})$ and $\mu^i \in \mathbb{Z}$. Then $u(s, t)$ converges to a periodic orbit $x \subset \tilde{M}$ of the Reeb vector field $X$ as $s \to \infty$. We can view $\lim_{s \to \infty} u(s, S^1)$ as a loop in $L$, representing $\sum \mu^i[c_i]$. In this way, we can control the behaviour at infinity of $J$-holomorphic maps with finite energy.

2). To compactify the moduli space we need study the $J$-holomorphic maps into $\{(-\infty, \infty) \times \tilde{M}\}$. There are two global vector fields on $\mathbb{R} \times \tilde{M}$: $\frac{\partial}{\partial a}$ and $X$. Similar to the situation of relative invariants, there is a $\mathbb{R}$ action, which induces a $\mathbb{R}$-action on the moduli space of $J$-holomorphic maps. We need mod this action. Since there is a vector field $X$ with $|X| = 1$ on $\tilde{M}$, the Reeb vector field, there is a one parameter group $\varphi_\theta$ action on $\tilde{M}$ generated by $-X$. In particular, there is a $S^1$-action on every periodic orbit, corresponding to the freedom of the choosing origin of $S^1$. Along every periodic orbit we have $L_X\lambda = 0$, on the other hand, we assume that on every periodic orbit $L_XJ = 0$, then we can mod this action.

On the other hand, we choose the Li-Ruan’s compactification in [18], that is, we firstly let the Riemann surfaces degenerate in Delingne-Mumford space and then let $M^+$ degenerate compatibly as in the situation of relative invariants. At any node, the Riemann surface degenerates independently boundaries of codimension 2 or more in the moduli space.

3). Another core technical issue in this paper is to define invariants using virtual techniques. As we know, there had been several different approaches, such as Fukaya-Ono [10], Li-Tian [21], Liu-Tian [22], Ruan [27], Siebert [29] and etc. In [18], Li and Ruan provide a completely new approach to this issue: they show that the invariants can be defined via the integration on the top stratum virtually. In order to achieve this goal, they provide refined estimates of differentiations for gluing parameters: $\partial/\partial r$. In [18] the estimates for $\frac{\partial}{\partial r}$ is of order $r^{-2}$ when $r \to \infty$, that is enough to define invariants. In this paper these estimates achieve to be of exponential decay order $\exp(-cr)$. Then we can use the estimates to define the invariants and prove the smoothness of the moduli space. The same method can be applied to GW-invariant for compact symplectic manifold.

In this paper we introduce two moduli spaces $\overline{MA}(M^+, C, g, m+\nu, y, p, (k, \nu))$ and $\overline{MA}(M, L; g, m+\nu, y, p, \vec{m})$, prove the compactness of the moduli spaces and the existence of the contact invariants.
\[ \Psi^{(C)}_{(A,g,m+\nu,k,\ell)}(\alpha_1, \ldots, \alpha_m; \beta_{m+1}, \ldots, \beta_{m+\nu}) \] and the open string invariants \( \Psi^{(L)}_{(A,g,m+\nu,\gamma)}(\alpha_1, \ldots, \alpha_m) \). In our next paper [20] we will prove the smoothness of the two moduli spaces. Our open string invariants \( \Psi^{(L)}_{(A,g,m+\nu,\gamma)} \) can be generalized to \( L \) which is a disjoint union of compact Lagrangian sub-manifolds \( L_1, \ldots, L_d \).

We consider a neighborhood of Lagrangian sub-manifold \( L \) as \( \mathbb{R} \times \widetilde{M} \). By the same method above we can define a local open string invariant. We will discuss this problem and calculate some examples in our next paper.

2 Symplectic manifolds with cylindrical ends

2.1 Contact manifolds

Let \((Q, \lambda)\) be a \((2n-1)\)-dimensional compact manifold equipped with a contact form \( \lambda \). We recall that a contact form \( \lambda \) is a 1-form on \( Q \) such that \( \lambda \wedge (d\lambda)^{n-1} \) is a volume form. Associated to \((Q, \lambda)\) we have the contact structure \( \xi = \ker(\lambda) \), which is a \((2n-2)\)-dimensional subbundle of \( TQ \), and \((\xi, d\lambda|_{\xi})\) defines a symplectic vector bundle. Furthermore, there is a unique nonvanishing vector field \( X = X_{\lambda} \), called the Reeb vector field, defined by the condition

\[
ix \lambda = 1, \quad ix d\lambda = 0.
\]

We have a canonical splitting of \( TQ \),

\[
TQ = \mathbb{R}X \oplus \xi,
\]

where \( \mathbb{R}X \) is the line bundle generated by \( X \).

2.2 Neighbourhoods of Lagrangian submanifolds

Let \((M, \omega)\) be a compact symplectic manifold, \( L \subset M \) be a compact Lagrangian sub-manifold. The following Theorem is well-known.

**Theorem 2.1.** Let \((M, \omega)\) be a symplectic manifold of dimension \( 2n \), and \( L \) be a compact Lagrangian submanifold. Then there exists a neighbourhood \( U \subset T^*L \) of the zero section, a neighbourhood \( V \subset M \) of \( L \), and a diffeomorphism \( \phi: U \to V \) such that

\[
\phi^*\omega = -d\Lambda, \quad \phi|_L = \text{id},
\]

where \( \Lambda \) is the canonical Liouville form.

Let \((x_1, \ldots, x_n)\) be a local coordinate system on \( O \subset L \), there is a canonical coordinates

\[
(x_1, \ldots, x_n, y_1, \ldots, y_n)
\]

on \( T^*O = T^*L|_O \). In terms of this coordinates the Liouville form can be written as

\[
\Lambda = \sum y_idx_i.
\]

Let \( \pi: T^*L \to L \) be the canonical projection. There is a global defined vector field \( W \) in \( T^*L \) such that in the local coordinates of \( \pi^{-1}(O) \), \( W \) can be written as

\[
W|_{\pi^{-1}(O)} = -\sum_{i=1}^{n} y_i \frac{\partial}{\partial y_i}.
\]
Suppose that given a Riemannian metric on \( L \), in terms of the coordinates \( x_1, \ldots, x_n \),
\[
g_L = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j.
\]
It naturally induced a metric on \( T^*L \). Let
\[
V = \frac{W}{\|W\|}.
\]
\( V \) is a global defined vector field on \( T^*L - L \).

Denote by \( S^{n-1}(1) \) (resp. \( B_1(0) \)) the Euclidean unit sphere (resp. the Euclidean unit ball). Consider the coordinates transformation between the sphere coordinates and the Cartesian coordinate
\[
\Psi : (0,1] \times S^{n-1}(1) \rightarrow B_1(0)
\]
\[(r, \theta_1, \cdots, \theta_{n-1}) \rightarrow (y_1, \cdots, y_n) . \tag{3} \]
Consider the unit sphere bundle \( \overline{M} \) and the unit ball bundle \( D_1(T^*L) \) in \( T^*L \), in terms of the coordinates \((x_1, \cdots, x_n, y_1, \cdots, y_n)\)
\[
\overline{M}|_{\pi^{-1}(O)} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \in \pi^{-1}(O) | \sum_{i,j=1}^{n} g^{ij}(x)y_i y_j = 1 \}, \tag{4} \]
\[
D_1(T^*L)|_{\pi^{-1}(O)} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \in \pi^{-1}(O) | \sum_{i,j=1}^{n} g^{ij}(x)y_i y_j \leq 1 \}. \tag{5} \]
Denote \( \lambda = -\Lambda|_{\overline{M}}. \) We have
\[
\Lambda = -\|y\|\lambda.
\]
\( \lambda \) is a contact form, i.e., \((\overline{M}, \lambda)\) is a contact manifold. Put \( \xi = \ker(\lambda) \). Then \( X|_{\overline{M}} = -\sum g^{ij} y_i \frac{\partial}{\partial x_j} \)
is the Reeb vector field, and \( V|_{\overline{M}} = \sum_{i=1}^{n} y_i \frac{\partial}{\partial x_i} \).

The map \( \Psi \) induced a map \( \tilde{\Psi} : (0,1] \times \overline{M} \rightarrow D_1(T^*L) \). Through \( \tilde{\Psi} \) we consider \( D_1(T^*L)|_{\pi^{-1}(O)} - L \) as \((0,1] \times \overline{M} \). By Theorem 2.1 we consider \( M - L \) as
\[
M^+ = M_0^+ \bigcup \{(0,1] \times \overline{M} \}
\]
with the symplectic form
\[
\omega_\phi = -d\Lambda = \|y\|d\lambda + d\|y\| \wedge \lambda, \tag{6} \]
where \( M^+ := M - L \) and \( M_0^+ \) is a compact symplectic manifold with boundary.

We choose the neck stretching technique.

Let \( \phi : [0, \infty) \rightarrow (0, \ell] \) be a smooth function satisfying, for any \( k > 0 \),
\[
(1) \ \phi' < 0, \ \phi(0) = \ell, \ \phi(a) \rightarrow 0 \ \text{as} \ a \rightarrow \infty,
\]
\[
(2) \ \lim_{a \rightarrow 0^+} \frac{\partial^k \phi}{\partial a^k} = 0. \]

Through \( \phi \) we consider \( M^+ \) to be \( M^+ = M_0^+ \bigcup \{(0, \infty) \times \overline{M} \} \) with symplectic form \( \omega_\phi|_{M_0^+} = \omega \), and over the cylinder \([0, \infty) \times \overline{M} \)
\[
\omega_\phi = -d\Lambda = \phi d\lambda + \phi' da \wedge \lambda. \tag{7} \]
Moreover, if we choose the origin of $\mathbb{R}$ tending to $\infty$, we obtain $\mathbb{R} \times \widetilde{M}$ in the limit.

Choose $\ell_0 < \ell$ and denote

$$\Phi^+ = \{ \phi : [0, \infty) \to (0, \ell_0) | \phi' < 0 \}.$$ 

Let $\ell_1 < \ell_2$ be two real numbers satisfying $0 < \ell_1 < \ell_2 \leq \ell_0$. Let $\Phi_{\ell_1, \ell_2}$ be the set of all smooth functions $\phi : \mathbb{R} \to (\ell_1, \ell_2)$ satisfying

$$\phi' < 0, \ \phi(a) \to \ell_1 \text{ as } a \to \infty, \ \phi(a) \to \ell_2 \text{ as } a \to -\infty.$$ 

To simplify notations we use $\Phi$ to denote both $\Phi^+$ and $\Phi_{\ell_1, \ell_2}$, in case this does not cause confusion.

We fixed $\phi \in \Phi$ and consider the symplectic manifold $(M_0^+ \cup \{(0, \infty) \times \widetilde{M}\}, \omega_{\phi})$. For any different $\phi_1 \in \Phi$, we have $\omega_{\phi}|M_0^+ = \omega_{\phi_1}|M_0^+$ and $\phi \circ \phi_1^{-1}$ is a symplectic diffeomorphic over cylinder part $\{(0,1) \times \widetilde{M}\}$.

### 2.3 Cylindrical almost complex structures

Let

$$M^+ = M_0^+ \cup \{[0, \infty) \times \widetilde{M}\}$$

be a symplectic manifold with cylindrical end, where $\widetilde{M}$ be a compact contact manifold with contact form $\lambda$. Denote by $\omega_{\phi}$ the symplectic form of $M^+$ such that $\omega_{\phi}|M_0^+ = \omega$, and over the cylinder $[0, \infty) \times \widetilde{M}$

$$\omega_{\phi} = -d\Lambda = \phi d\lambda + \phi' da \wedge \lambda.$$  

(9)

We also consider $\mathbb{R} \times \widetilde{M}$. Denote by $N$ one of $M^+$ and $\mathbb{R} \times \widetilde{M}$.

Put $\xi = \ker(\lambda)$, and denote by $X$ the Reeb vector field defined by

$$\lambda(X) = 1, \ d\lambda(X) = 0.$$  

We choose a $d\lambda$-tame almost complex structure $\widetilde{J}$ for the symplectic vector bundle $(\xi, d\lambda) \to \widetilde{M}$ such that

$$g_{\widetilde{J}(x)}(h, k) = \frac{1}{2} \left( d\lambda(x)(h, \widetilde{J}(x)k) + d\lambda(x)(k, \widetilde{J}(x)h) \right),$$

(10)

for all $x \in \widetilde{M}$, $h, k \in \xi_x$, defines a smooth fibrewise metric for $\xi$. We assume that we can choose $\widetilde{J}$ such that on every periodic orbit $L_X \widetilde{J} = 0$. Denote by $\Pi : T\widetilde{M} \to \xi$ the projection along $X$. We define a Riemannian metric $\langle , \rangle$ on $\widetilde{M}$ by

$$\langle h, k \rangle = \lambda(h)\lambda(k) + g_{\widetilde{J}}(\Pi h, \Pi k)$$

(11)

for all $h, k \in T\widetilde{M}$.

Given a $\widetilde{J}$ as above there is an associated almost complex structure $J$ on $\mathbb{R} \times \widetilde{M}$ defined by

$$J|_{\xi} = \widetilde{J}, \quad JX = \frac{\partial}{\partial a}, \quad J\left(\frac{\partial}{\partial a}\right) = -X.$$  

(12)
where \( a \) is the canonical coordinate in \( \mathbb{R} \). It is easy to check that \( J \) defined by (12) is \( \omega \phi \)-tame over the cylinder end. We can choose an almost complex structure \( J \) on \( M^+ \) such that \( J \) is tamed by \( \omega \) and over the cylinder end \( J \) is given by (12).

There is a canonical coordinate system for \( \mathbb{R} \times \tilde{M} \) and for cylinder end of \( M^+ \), but still there are some freedom of choosing the coordinates:

When we write \( M^+ \) as (8) we have chosen a coordinate \( a \) over the cylinder part. We can choose different coordinate \( \hat{a} \) over the cylinder part such that \( a = \hat{a} + C \) for some constant \( C > 0 \). Similarly, for \( \mathbb{R} \times \tilde{M} \) we can choose \( \hat{a} \) such that \( a = \hat{a} \pm C \) for some constant \( C > 0 \).

For any \( \phi \in \Phi \)

\[
\langle v, w \rangle_{\omega \phi} = \frac{1}{2} (\omega \phi(v, Jw) + \omega \phi(w, Jv)) \quad \forall \ v, w \in TN
\]

defines a Riemannian metric on \( N \). Note that \( \langle \ , \ \rangle_{\omega \phi} \) is not complete. We choose another metric \( \langle \ , \ \rangle \) on \( M_0^+ \) and over the tubes

\[
\langle (a, v), (b, w) \rangle = ab + \lambda(v)\lambda(w) + g_{\tilde{J}}(\Pi v, \Pi w),
\]

where we denote by \( \Pi : T\tilde{M} \to \xi \) the projection along \( X \). It is easy to see that \( \langle \ , \ \rangle \) is a complete metric on \( N \).

2.4 \( J \)-holomorphic maps with finite energy

Let \( (\Sigma, i) \) be a compact Riemann surface and \( P \subset \Sigma \) be a finite collection of puncture points. Denote \( \tilde{\Sigma} = \Sigma \setminus P \). Let \( u : \tilde{\Sigma} \to N \) be a \( J \)-holomorphic map, i.e., \( u \) satisfies

\[
\text{du} \circ i = J \circ \text{du}.
\]

Following [14] we impose an energy condition on \( u \). For any \( J \)-holomorphic map \( u : \tilde{\Sigma} \to N \) and any \( \phi \in \Phi \) the energy \( E_\phi(u) \) is defined by

\[
E_\phi(u) = \int_{\Sigma} u^* \omega_\phi.
\]

Let \( z = e^{s+2\pi \sqrt{-1} t} \). One computes over the cylindrical part

\[
u^* \omega_\phi = (\phi d\lambda ((\pi \bar{u})_s, (\pi \bar{u})_t)) - \phi'(a^2_s + a^2_t) ds \wedge dt,
\]

which is a nonnegative integrand. A \( J \)-holomorphic map \( u : \tilde{\Sigma} \to N \) is called a finite energy \( J \)-holomorphic map if over the cylinder end

\[
\sup_{\phi \in \Phi} \left\{ \int_{\Sigma} u^* \omega_\phi \right\} + \int_{\Sigma} u^* d\lambda < \infty.
\]
For a $J$-holomorphic map $u : \Sigma \to \mathbb{R} \times \tilde{M}$ we write $u = (a, \tilde{u})$ and define
\[ \tilde{E}(u) = \int_{\Sigma} \tilde{u}^* \lambda. \] (22)
Denote
\[ \tilde{E}(s) = \int_{s}^{\infty} \int_{S^1} |\tilde{u}_t|^2 dsdt, \]
Then
\[ \frac{d\tilde{E}(s)}{ds} = -\int_{S^1} |\tilde{u}_t|^2 dt. \] (23)
Here and later we use $| \cdot |$ denotes the norm with respect to the metric defined by (17).

The following two lemmas are well-known (see [13]):

**Lemma 2.2.** (1) Let $u = (a, \tilde{u}) : \mathbb{C} \to \mathbb{R} \times \tilde{M}$ be a $J$-holomorphic map with finite energy. If $\int_{\mathbb{C}} \tilde{u}^*(\pi^* \lambda) = 0$, then $u$ is a constant.

(2) Let $u = (a, \tilde{u}) : \mathbb{R} \times S^1 \to \mathbb{R} \times \tilde{M}$ be a $J$-holomorphic map with finite energy. If $\int_{\mathbb{R} \times S^1} \tilde{u}^*(\pi^* \lambda) = 0$, then $(a, \tilde{u}) = (kTs + c, kt + d)$, where $k \in \mathbb{Z}^+$, $c$ and $d$ are constants.

**Lemma 2.3.** Let $u = (a, \tilde{u}) : \mathbb{C} - D_1 \to \mathbb{R} \times \tilde{M}$ be a nonconstant $J$-holomorphic map with finite energy. Put $z = e^{s + 2\pi t^{\sqrt{-1}}}$. Then for any sequence $s_i \to \infty$, there is a subsequence, still denoted by $s_i$, such that
\[ \lim_{i \to \infty} \tilde{u}(s_i, t) = x(kTt) \]
in $C^\infty(S^1)$ for some $kT$-periodic orbit $x(kTt)$.

### 2.5 Periodic orbits of Bott-type

Let $\mathcal{F} \subset \tilde{M}$ be the locus of minimal periodic orbits with $\mathcal{F} = \bigcup_{i=1}^{l} \mathcal{F}_i$, where each $\mathcal{F}_i$ is a connected component of $\mathcal{F}$ with minimal periodic $T_i$. We assume that

**Assumption 2.4.** (1) every $\mathcal{F}_i$ is either non-degenerate or of Bott-type;

(2) let $\mathcal{F}_i$ be of Bott-type, then there exists a free $S^1$-action on $\mathcal{F}_i$ such that $Z_i = \mathcal{F}_i/S^1$ is a closed, smooth manifold. Set $n_i = \dim(\mathcal{F}_i)$;

(3) for every periodic orbit $x$, there is a smooth submanifold $\mathcal{R}_x \subset \tilde{M}$ of dimension $> 2$ such that $d\lambda |_{\mathcal{R}_x} = 0$ and $x \subset \mathcal{R}_x$, and the almost complex structure $\tilde{J}$ can be chosen such that $L_X \tilde{J} = 0$ along $x$.

Let $u = (a, \tilde{u}) : \mathbb{C} - D_1 \to \mathbb{R} \times \tilde{M}$ be a $J$-holomorphic map with finite energy. Put $z = e^{s + 2\pi it}$. Assume that there exists a sequence $s_i \to \infty$ such that $\tilde{u}(s_i, t) \to x(kTt)$ in $C^\infty(S^1, \tilde{M})$ as $i \to \infty$ for some $k \in \mathbb{Z}$, where $T$ is the minimal periodic. Following Hofer (see [14]) we introduce a convenient local coordinates near the periodic orbit $x$. Since $S^1 = \mathbb{R}/\mathbb{Z}$, we work in the covering space $\mathbb{R} \to S^1$. 

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Lemma 2.5. Let \( \tilde{M}, \lambda \) be a \((2n-1)\)-dimensional compact manifold, and let \( x(kt) \) be a \( k \)-periodic orbit with the minimal periodic \( T \). Then there is an open neighborhood \( U \subset S^1 \times \mathbb{R}^{2n-2} \) of \( S^1 \times \{0\} \) with coordinates \((\vartheta, w_1, \ldots, w_{2n-2})\) and an open neighborhood \( \mathcal{O} \subset \tilde{M} \) of \( \{x(t)| t \in \mathbb{R}\} \) and a diffeomorphism \( \psi : U \rightarrow \mathcal{O} \) mapping \( S^1 \times \{0\} \) onto \( \{x(t)| t \in \mathbb{R}\} \) such that

\[
\psi^* \lambda = g \lambda_0,
\]

where \( \lambda_0 = d\vartheta + \sum w_i dw_{n+i-1} \) and \( g : U \rightarrow \mathbb{R} \) is a smooth function satisfying

\[
g(\vartheta, 0) = T, \quad dg(\vartheta, 0) = 0
\]

for all \( \vartheta \in S^1 \).

Remark 2.6. We call the coordinate system \((\vartheta, w)\) in Lemma 2.5 a pseudo-Darboux coordinate system, and call the following transformation of two local pseudo-Darboux coordinate systems

\[
(\vartheta, w) \rightarrow (\tilde{\vartheta}, \tilde{w}), \quad \tilde{\vartheta} = \vartheta + \vartheta_0
\]
a canonical coordinate transformation, where \( \vartheta_0 \) is a constant.

The following theorem is well-known (see [5, 18, 32]).

Theorem 2.7. Suppose that \( \mathcal{F} \) satisfies Assumption 2.4. Let \( u : \mathcal{O} = D_1 \rightarrow \mathbb{R} \times \tilde{M} \) be a \( J \)-holomorphic map with finite energy. Put \( z = e^{s+2\pi \sqrt{-1}t} \). Then

\[
\lim_{s \to \infty} \tilde{u}(s,t) = x(kT)
\]
in \( C^\infty(S^1) \) for some \( kT \)-periodic orbit \( x \), and there are constants \( \ell_0, \vartheta_0 \) such that for any \( 0 < \epsilon < \min\{\frac{1}{2}, \frac{\epsilon^2}{2}\} \) and for all \( \mathbf{n} = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2 \)

\[
|\partial^n[a(s,t) - kT \ell - \ell_0]| \leq C_n e^{-c|s|}
\]

(27)

\[
|\partial^n[\vartheta(s,t) - kT \vartheta - \vartheta_0]| \leq C_n e^{-c|s|}
\]

(28)

\[
|\partial^n[w(s,t)]| \leq C_n e^{-c|s|},
\]

(29)

where \( C_n \) are constants. Here \( (\vartheta, w) \) is a pseudo-Darboux coordinate near the periodic orbit \( x \).

Let \( x_0 \) be a minimal periodic orbit. We choose a local pseudo-Darboux coordinate on an open set \( \mathcal{O} \subset \tilde{M} \) near \( x_0 \). Let \( \mathcal{O} \) be an open set such that \( \overline{\mathcal{O}} \subset \mathcal{O} \) is compact and \( x_0 \subset \mathcal{O} \). We fix a positive constant \( C \). Denote by \( \mathcal{S} \) the class of \( J \)-holomorphic maps with finite energy \( u : [s'_0, \infty) \times S^1 \rightarrow \mathbb{R} \times \tilde{M} \) satisfying

(i) \( \sup_{\varphi \in \mathcal{S}} E_{\varphi}(u) \leq \frac{1}{2} h \), where \( h \) is the constant in Theorem 5.2 and Lemma 5.3

(ii) \( \lim_{s \to \infty} \tilde{u}(s,t) = x(kT) \) in \( C^\infty(S^1) \) with \( x \subset \mathcal{O} \),

(iii) \( \tilde{u}([s'_0, \infty) \times S^1) \) lie in the pseudo-Darboux coordinate system on \( \mathcal{S} \),

(iv) there is a ball \( D_1(s_0, t_0) \subset [s'_0, \infty) \times S^1 \) such that \( \tilde{u}(D_1(s_0, t_0)) \subset \mathcal{O} \) and \( |a(D_1(s_0, t_0))| \leq C \).

In our next paper [19] we will prove the following theorem.
Theorem 2.8. Let \( u \in \mathcal{S} \). Then the constants \( C_n \) in Theorem 2.4 depend only on \( C_1, s_0, C, n, h, c \) and \( O \). Moreover, we have

\[ |\ell_0| \leq C_1, \tag{30} \]

where \( C_1 \) depends only on \( h, C_n, c, s_0, C \) and \( O \).

Following [14] we introduce functions

\[ a^\circ(s, t) = a(s, t) - ks, \quad \vartheta^\circ(s, t) = \vartheta(s, t) - kt. \tag{31} \]

Denote

\[ \mathcal{L} = (a^\circ, \vartheta^\circ). \tag{32} \]

Set \( b_i = 0, b_{n-1+i}(w) = w_i, \forall i = 1, \ldots, n-1 \), and

\[ e_i = \frac{\partial}{\partial w_i} - b_i \frac{\partial}{\partial \vartheta}, \quad i = 1, \ldots, 2n-2. \]

Then \( \xi = \text{span}\{e_1, \ldots, e_{2n-2}\} \). Denote \( \tilde{J}e_i = \sum \tilde{J}_{ij}e_j \). In the basis \( \vartheta, e_1, \ldots, e_{2n-2} \), the Reeb vector field can be re-written as

\[ X = \frac{1}{g} \vartheta + \frac{1}{g^2} \left( \sum_{i \leq n-1} e_{n-1+i}(g)e_i - \sum_{i \geq n} e_{n+i}(g)e_i \right). \tag{33} \]

Let \( \tilde{X} = \frac{1}{g^2}(e_ng, \ldots, e_{2n-2}g, -e_1g, \ldots, -e_ng) \). By [18], we have

\[ \mathcal{L}_s + J\mathcal{L}_t = h, \quad w_s + \tilde{J}w_t + a_t \tilde{X} - a_s \tilde{J} \tilde{X} = 0. \]

where

\[ h = (\Sigma b_i(w_i)^t + (\vartheta_t + \Sigma b_i(w_i)(w_i)_t)(\Sigma f_i(w_i)) - \Sigma b_i(w_i)(w_i)_s - (\vartheta_t + \Sigma b_i(w_i)(w_i)_s)(\Sigma f_i(w_i)) \]

and \( f_i = \int_0^1 \partial w_i g(\vartheta, \tau w)d\tau \). Let \( V = \mathcal{L}_t \) and \( g = h_t \). Denote

\[ \dot{E}(\mathcal{L}) = \int_\Sigma \|\mathcal{L}_s\|^2 + \|\mathcal{L}_t\|^2 ds dt. \tag{34} \]

In our next paper [19] we will prove the following theorem.

Theorem 2.9. Suppose that \( \tilde{M} \) satisfies the Assumption 2.4. Let \( u : [-R, R] \times S^1 \to \mathbb{R} \times \tilde{M} \) be a \( J \)-holomorphic maps with finite energy. Assume that

(i) \( \sup_{\phi \in \Phi} E_{\phi}(u, -R \leq s \leq R) + \dot{E}(\mathcal{L}) \leq \frac{1}{2} h \),

(ii) \( \tilde{u}([-R, R] \times S^1) \) lie in a pseudo-Darboux coordinate system \((\vartheta, w)\) on \( \mathcal{S} \),

(iii) \( \sum_{n_1, n_2 \leq 2} \|\nabla^n u(-R, \cdot)\|_{L^2(S^1)} \leq C_2, \quad \sum_{n_1, n_2 \leq 2} \|\nabla^n u(R, \cdot)\|_{L^2(S^1)} \leq C_2 \), where \( n = (n_1, n_2) \),

Then there exist constants \( C_1 > 0 \) and \( 0 < c < \frac{1}{2} \) depending only on \( \tilde{J} \) and \( C_2 \) such that

\[ |\nabla w|(s, t) \leq C_1 e^{-c(R-|s|)}, \quad |\nabla \mathcal{L}| \leq C_1 e^{-c(R-|s|)}, \quad \forall |s| \leq R - 1, \tag{35} \]
In this paper, we also need the following implicit function theorem (see [23]).

**Lemma 2.10.** Let $X$ and $Y$ be Banach spaces, $U \subset X$ be an open set, and $\ell$ be a positive integer. If $F : U \to Y$ is of class $C^\ell$. Let $x_0 \in U$ be such that $D := dF(x_0) : X \to Y$ is surjective and has a bounded linear right inverse $Q : Y \to X$. Choose positive constants $h_1$ and $C$ such that $\|Q\| \leq C$, $B_{h_1}(x_0, X) \subset U$ and

$$\|dF(x) - D\| \leq \frac{1}{2C}, \quad \forall x \in B_{h_1}(x_0)$$

where $B_{h_1}(x_0) = \{x \in X | \|x - x_0\| \leq h_1\}$. Suppose that $x_1 \in X$ satisfies

$$\|F(x_1)\| \leq \frac{h_1}{4C}, \quad \|x_1 - x_0\| \leq \frac{h_1}{8}. \quad (37)$$

Then there exists a unique $x \in X$ such that

$$F(x) = 0, \quad x - x_1 \in \text{Im} Q, \quad \|x - x_0\| \leq h_1, \quad \|x - x_1\| \leq 2C\|F(x_1)\|. \quad (38)$$

Moreover, write $x := x_0 + \xi + Q \circ f(\xi)$, $\xi \in \ker D$, then $f$ is of class $C^\ell$.

### 3 Weighted sobolev norms

Consider $\mathbb{R} \times \widetilde{M}$ and $M^+ = M_0^+ \cup \left\{ [0, \infty) \times \widetilde{M} \right\}$. Let $N$ be one of $\mathbb{R} \times \widetilde{M}$, $M^+$. Suppose that $\Sigma = \bigcup \Sigma_u$ is Riemann surface with nodal points $\{q_1, \ldots, q_3\}$, puncture points $\{p_1, \ldots, p_\nu\}$ and $u : \Sigma \to \bigcup N_i$ is a continuous map such that the restriction of $u$ to each smooth component is smooth, where $\bigcup N_i$ denotes the union of some copy of $N$. We choose cylinder coordinates $(s, t)$ on $\Sigma$ near each nodal point and each puncture point. We choose a local pseudo-Darboux coordinate system near each periodic orbit on $N$. Let $\Sigma = \Sigma - \{q_1, q_3, p_1, \ldots, p_\nu\}$.

Over each tube the linearized operator $D_u$ takes the following form

$$D_u = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S = \bar{\partial}_J + S. \quad (39)$$

By exponential decay we have

$$\left| \frac{\partial^{k+l}}{\partial s^k \partial t^l} S \right| \leq C_{k,l} e^{-cs}$$

for some constant $C_{k,l} > 0$ for $s$ big enough. Therefore, the operator $H_s = J_0 \frac{d}{dt} + S$ converges to $H_\infty = J_0 \frac{d}{dt}$. Obviously, the operator $D_u$ is not Fredholm operator because over each puncture and node the operator $H_\infty = J_0 \frac{d}{dt}$ has zero eigenvalue. The $\ker H_\infty$ consists of constant vectors. To recover a Fredholm theory we use weighted function spaces. We choose a weight $\alpha$ for each end. Fix a positive function $W$ on $\Sigma$ which has order equal to $e^{\alpha|s|}$ on each end, where $\alpha$ is a small constant such that $0 < \alpha < \epsilon$ and over each end $H_\infty - \alpha = J_0 \frac{d}{dt} - \alpha$ is invertible. We will write the weight function simply as $e^{\alpha|s|}$. Denote by $C(\Sigma; u^* T(\bigcup N_i))$ all tangent vector fields $h$ on $\bigcup N_i$ along $u$ satisfying

(a) $h \in C^0(\Sigma, u^* T(\bigcup N_i))$,

(b) the restriction of $h$ to each smooth component is smooth.

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Denote by $D$ the symplectic manifold with cylindrical end. We define the weighted Sobolev norm on $W$ as follows:

$$
\|h\|_{1,p,\alpha} = \sum_v \left( \int_{\Sigma_v} (|h|^p + |\nabla h|^p) \, d\mu \right)^{1/p} + \sum_v \left( \int_{\Sigma_v} e^{2\alpha s} |(h|^2 + |\nabla h|^2) \, d\mu \right)^{1/2}
$$

$$
\|\eta\|_{p,\alpha} = \sum_v \left( \int_{\Sigma_v} |\eta|^p \, d\mu \right)^{1/p} + \sum_v \left( \int_{\Sigma_v} e^{2\alpha s} |\eta|^2 \, d\mu \right)^{1/2}
$$

for $p \geq 2$, where all norms and covariant derivatives are taken with respect to the metric $\langle \cdot \rangle$ on $u^*T(\bigcup N_i)$ defined in (17), and the metric on $\Sigma$. Denote

$$
C(\Sigma; u^*T(\bigcup N_i)) = \{ h \in C(\Sigma; u^*T(\bigcup N_i)); \|h\|_{1,p,\alpha} < \infty \},
$$

$$
C(u^*T(\bigcup N_i) \otimes \wedge^{0,1}) = \{ \eta \in \Omega^{0,1}(u^*T(\bigcup N_i)); \|\eta\|_{p,\alpha} < \infty \}.
$$

Denote by $W^{1,p,\alpha}(\Sigma; u^*T(\bigcup N_i))$ and $L^{p,\alpha}(u^*T(\bigcup N_i) \otimes \wedge^{0,1})$ the completions of $C(\Sigma; u^*T(\bigcup N_i))$ and $C(u^*T(\bigcup N_i) \otimes \wedge^{0,1})$ with respect to the norms (40) and (41) respectively. Then the operator $D_u : W^{1,p,\alpha} \rightarrow L^{p,\alpha}$ is a Fredholm operator.

For each puncture point $p_j, j = 1, \ldots, \nu$, let $h_{j0} \in (T_{p_j}(F_{\epsilon_j}) \oplus (\text{span}\{e\}))$. For each bounded nodal $q_i$, denote $\mathbb{H}_{q_i} = T_{q_i}N_i$, let $h_{(i+\nu)0} \in \mathbb{H}_{q_i}$; for each unbounded nodal $q_i$, denote $\mathbb{H}_{q_i} = (T_{q_i}(F_{\epsilon_j}) \oplus (\text{span}\{e\}))$ and let $h_{(i+\nu)0} \in \mathbb{H}_{q_i}$, where $u : \Sigma \rightarrow N$ converges to $kt$ periodic orbit $x(kt) \subset F_{\epsilon_j}$ as $z \rightarrow q_i$. Put $\mathbb{H} = \left( \bigoplus_{j=1}^\nu (T_{p_j}(F_{\epsilon_j}) \oplus (\text{span}\{e\})) \right) \bigoplus \left( \bigoplus_{i=1}^{\nu} \mathbb{H}_{q_i} \right)$, $h_0 = (h_{10}, \ldots, h_{i0}, h_{(i+\nu)0}, \ldots, h_{(2+\nu)0})$.

$h_0$ may be considered as a vector field in the coordinate neighborhood. We fix a cutoff function $\rho$:

$$
\rho(s) = \begin{cases} 
1, & \text{if } |s| \geq d, \\
0, & \text{if } |s| \leq \frac{d}{2}
\end{cases}
$$

where $d$ is a large positive number. Put

$$
\hat{h}_0 = \rho h_0.
$$

Then for $d$ big enough, $\hat{h}_0$ is a section in $C^\infty(\Sigma; u^*TN)$ supported in the tube $\{(s,t)||s| \geq \frac{d}{2}, t \in S^1\}$. Denote

$$
W^{1,p,\alpha} = \{ h + \hat{h}_0 | h \in W^{1,p,\alpha}, h_0 \in \mathbb{H} \}.
$$

We define the weighted Sobolev norm on $W^{1,p,\alpha}$ by

$$
\|(h, \hat{h}_0)||_{1,\alpha} = \|h\|_{\Sigma, 1,\alpha} + |\hat{h}_0|.
$$

Obviously, the operator $D_u : W^{1,p,\alpha} \rightarrow L^{p,\alpha}$ is also a Fredholm operator.

4 Moduli spaces of $J$-holomorphic maps

4.1 Boundary conditions

Consider the symplectic manifold with cylindrical end

$$
M^+ = M_0^+ \bigcup \left\{ (0, \infty) \times \tilde{M} \right\}.
$$
Let \((\Sigma, j); y, p)\) be a connected semistable curve with \(m\) marked points \(y = (y_1, \ldots, y_m)\) and \(\nu\) puncture points \(p = (p_1, \ldots, p_\nu)\), and \(u : \Sigma \to M^+\) be a \(J\)-holomorphic map. Let \(\Sigma = \bigcup_{v=1}^d (\Sigma_v, j_v)\) where \((\Sigma_v, j_v)\) is a smooth Riemann surface and \(\pi_v : \Sigma_v \to \Sigma\) is a continuous map. To describe the boundary conditions we consider two different cases separately:

**Case A.** Moduli space of \(J\)-holomorphic maps in contact geometry.

**Definition 4.1.** Let \(p = (p_1, \ldots, p_\nu)\) be the puncture points. We assign two weights \((k, \epsilon)\) to \(p:\)

1. \(k : p \to \mathbb{Z}_{>0}\) assigning a \(k_i\) to each puncture point \(p_i\), denote \(k = (k_1, \ldots, k_\nu)\).
2. \(\epsilon : p \to \mathbb{Z}_{>0}\) assigning a number \(\epsilon_i\), \(1 \leq \epsilon_i \leq \ell\) to each puncture point \(p_i\).

We call a \(J\)-holomorphic map \(u\) satisfies \((k, \epsilon)\) boundary condition if \(u(z)\) converges to a \(k_i \cdot T_{\epsilon_i}\)-periodic orbit \(x(k_i \cdot T_{\epsilon_i} t) \subset F_{\epsilon_i}\) as \(z\) tends to \(p_i\).

**Case B.** Moduli space of \(J\)-holomorphic maps in \((M, L)\).

As we show in section \[\text{[22]}\] that \(M - L\) can be considered as \(M^+ = M_0^+ \cup \left\{0, \infty\right\} \times \tilde{M}\). Let \([c_i], i = 1, \ldots, \ell\) is a bases in \(H_1(L; \mathbb{Z})\).

**Definition 4.2.** Let \(p = (p_1, \ldots, p_\nu)\) be the order puncture points. We assign a weight \(\overrightarrow{\mu}\) to \(p:\)

\(\overrightarrow{\mu} : p \to \mathbb{Z}_{>0}^\nu\) assigning a \(\mu_i = \sum_{l=1}^\nu \mu_{il}[c_l]\) to each puncture point \(p_i\), where \(\mu_{il} \in \mathbb{Z}\). Choose the cylinder coordinates \((s_i, t_i)\) near \(p_i\). We call a \(J\)-holomorphic map \(u\) satisfies \((\overrightarrow{\mu})\) boundary condition if \(u\) satisfies

\[\pi(\lim_{s_i \to \infty} u(s_i, S^1)) = \mu_i, \quad \forall 1 \leq i \leq \nu,\]  

where \(\pi : T^*L \to L\) is the canonical projection.

### 4.2 Homology

We fix \(A \in H^2(M^+, \mathbb{R}; \mathbb{Z})\) satisfying \(\partial A = \sum [x(k_i T_{\epsilon_i} t)]\), where \(x(k_i T_{\epsilon_i} t) \subset F_{\epsilon_i}\) is a \(k_i T_{\epsilon_i}\) periodic orbit. Consider a \(J\)-holomorphic map \(u\) satisfying

\[\lbrack u_*(\Sigma) \rbrack = A.\]  

(45)

We show that the homology class \(A\) give a bound of Energy. To simplify notation we let \((u, (\Sigma, j), y, p)\) be a \(J\)-holomorphic map converging to a \(kT\)-periodic orbit \(x(kT t)\) as \(z\) tends to \(p\), where \(x(kT t)\) lie in \(F_{\epsilon_i}\) and \(T\) is the minimal periodic. For the case \(A\), by the assumption (3) of \(F\), we construct a connected surface \(W \subset \mathbb{R}_x\) with boundary \(x(kT t)\) (including the \(kT\)-periodic). Then \([u_*(\Sigma) \cup W] \in H^2(M^+, \mathbb{Z})\).

Denote \(A = [u_*(\Sigma) \cup W]\). By \(d\lambda|_{\mathbb{R}_x} = 0\) and \(W \subset \tilde{M}\) we have

\[\omega(A) = \int_{u_*(\Sigma)} \omega + \int_W \omega = E_\phi(u) + \int_W d\lambda = E_\phi(u).\]  

(46)

Let \(W' \subset \mathbb{R}_x\) be another surface with boundary \(x\), denote \(A' = [u_*(\Sigma) \cup W'] \in H^2(M^+, \mathbb{Z})\). We have \(\omega(A) = \omega(A') = E_\phi(u)\), that is, \(E_\phi(u)\) is independent of the choice of \(W\) in \(\mathbb{R}_x\).

For case \(B\) let \(A \in H^2(M, L; \mathbb{Z})\) be a fixed homology class satisfying \(\partial A = \sum \mu_i\). We have the same results.
4.3 Holomorphic blocks in $M^+$

Let $((\Sigma, j); y, p)$ be a connected semistable curve with $m$ marked points $y = (y_1, ..., y_m)$ and $\nu$ puncture points $p = (p_1, ..., p_\nu)$. Let $u : \Sigma \to M^+$ be a $J$-holomorphic map. Suppose that $u(z)$ converges to a $k_i \cdot T_{\epsilon_i}$-periodic orbit $x(k_i T_{\epsilon_i} t) \subset F_{\epsilon_i}$ as $z$ tends to $p_i$.

**Definition 4.3.** A $J$-holomorphic map $(u; ((\Sigma, j), y, p))$ is said to be stable if for each $v$ one of the following conditions holds:

1. $u \circ \pi_{\Sigma_v} : \Sigma_v \to M^+$ is not a constant map.
2. Let $\text{val}_v$ be the number of special points on $\Sigma_v$ which are nodal points, marked points or puncture points. Then $\text{val}_v + 2g \geq 3$.

**Definition 4.4.** Two stable $J$-holomorphic maps $\Gamma = (u, ((\Sigma, j), y, p))$ and $\tilde{\Gamma} = (\tilde{u}, ((\tilde{\Sigma}, \tilde{j}), \tilde{y}, \tilde{p}))$ is called equivalent if there exists a diffeomorphism $\varphi : \Sigma \to \tilde{\Sigma}$ such that it can be lifted to biholomorphic isomorphisms $\varphi_{vw} : (\Sigma_v, j_v) \to (\tilde{\Sigma}_w, \tilde{j}_w)$ for each component $\Sigma_v$ of $\Sigma$, and

1. $\varphi(y_i) = \tilde{y}_i$, $\varphi(p_j) = \tilde{p}_j$ for any $1 \leq i \leq m, 1 \leq j \leq \nu$,
2. $\tilde{u} \circ \varphi = u$.
3. near every periodic orbit $x$, $\tilde{u}$ and $\tilde{u} \circ \varphi$ may differ by a canonical coordinate transformation.

**Definition 4.5.** Put 

$$\text{Aut}(u; ((\Sigma, j), y, p)) = \{ \phi : \Sigma \to \tilde{\Sigma} | \phi \text{ is an automorphism satisfying (1), (2) and (3) in Definition 4.4} \}.$$ 

We call it the automorphism group of $(u; ((\Sigma, j), y, p))$.

The following Lemma is obvious.

**Lemma 4.6.** A $J$-holomorphic map $(u; ((\Sigma, j), y, p))$ is stable if and only if $\text{Aut}(u; ((\Sigma, j), y, p))$ is a finite group.

Denote by $\mathcal{M}_A(M^+, C, g, m + \nu, y, p, (k, \epsilon))$ the moduli space of equivalence classes of all $J$-holomorphic curves in $M^+$ representing the homology class $A$ and satisfying $(k, \epsilon)$ boundary condition.

Fix $A \in H^2(M, L; \mathbb{Z})$, denote by $\mathcal{M}_A(M, L; g, m + \nu, y, p, \overline{\mu})$ the moduli space of equivalence classes of all $J$-holomorphic curves in $M^+$ representing the homology class $A$ and satisfying $(\overline{\mu})$ boundary condition.

**Lemma 4.7.** There is a constant $C > 0$ depending on $A$ and $k$ such that for any $b = (u; (\Sigma, j), y, p) \in \mathcal{M}_A(M^+, C, g, m + \nu, y, p, (k, \epsilon))$ we have, over the cylinder end,

$$E_\phi(u) + \int_\Sigma u^* d\lambda \leq C.$$  \hspace{1cm} (47)
**Proof.** By the Stokes formula we have \( \int_\Sigma u^*d\lambda \leq \sum_{i=1}^{\nu} k_i \cdot T_{\epsilon_i} \) over the cylinder end. Then the lemma follows from (46). □

Similarly, we also have

**Lemma 4.8.** There is a constant \( C > 0 \) depending on \( A \) and \( \overrightarrow{t} \) such that for any \( b = (u; (\Sigma, j), y, p) \in \mathcal{M}_A(M, L; g, m + \nu, y, p, \overrightarrow{t}) \) we have, over cylinder end,

\[
E_b(u) + \int_\Sigma u^*d\lambda \leq C. \tag{48}
\]

We call \( \mathcal{M}_A(M^+, C, g, m + \nu, y, p, (k, \epsilon)) \) a holomorphic block in \( M^+ \).

Let \( b = (u; (\Sigma, j), y, p) \in \mathcal{M}_A(M^+, C, g, m + \nu, y, p, (k, \epsilon)) \), \( D_u : \mathcal{W}_{1,p,\alpha}^\perp \rightarrow L_{p,\alpha}^\perp \) be a Fredholm operator with \( ind = dim(\ker D_u) - dim(\text{coker} D_u) \). Put

\[
Ind^C = ind + 6(g - 6) + 2(m + \nu).
\]

The virtual dimension of \( \mathcal{M}_A(M^+, C, g, m + \nu, y, p, (k, \epsilon)) \) is \( Ind^C \).

Similarly, let \( b = (u; (\Sigma, j), y, p) \in \mathcal{M}_A(M, L; g, m + \nu, y, p, \overrightarrow{t}) \). Suppose that \( \lim_{s_j \rightarrow \infty} u(s_j, S^1) \subset \mathcal{F}_{ij} \) and

\[
[\pi(\lim_{s_j \rightarrow \infty} u(s_j, S^1))] = \mu_j, \quad \forall 1 \leq j \leq \nu. \tag{49}
\]

Then \( D_u : \mathcal{W}_{1,p,\alpha}^\perp \rightarrow L_{p,\alpha}^\perp \) be a Fredholm operator with \( ind = dim(\ker D_u) - dim(\text{coker} D_u) \). Put

\[
Ind^L = ind + 6(g - 6) + 2(m + \nu).
\]

The virtual dimension of \( \mathcal{M}_A(M, L; g, m + \nu, y, p, \overrightarrow{t}) \) is \( Ind^L \).

It is possible that there are finite many combinations such that (49) holds. In this case \( \mathcal{M}_A(M, L; g, m + \nu, y, p, \overrightarrow{t}) \) is the union of some \( \mathcal{M}_A(M^+, C, g, m + \nu, y, p, \overrightarrow{t}) \).

**Remark 4.9.** As we mod the \( S^1 \) action on every periodic orbit of Reeb vector field, the situation is very similar to the symplectic cutting. For example, let \( b = (u; (\Sigma, j, p) \in \mathcal{M}_A(M^+, C, g, m + 1, (k, \epsilon_1)) \). Roughly speaking:

We collapse the \( S^1 \)-action on the orbit \( x(kT_{\epsilon_1}t) \) at infinity to get a “manifold” \( \tilde{M}^+ \), locally, such that \( Z_{\epsilon_1} \) is a “submanifold” of \( \tilde{M}^+ \), choose a local pseudo-Darboux coordinate \( (a, \vartheta, w) \) around \( x(kT_{\epsilon_1}t) \). Our estimates [27], [28], [29] show that the puncture point \( p \) can be “removed”, we get a \( J \)-holomorphic map \( \tilde{u} \) from \( \Sigma \) into \( \tilde{M}^+ \). The condition that \( u \) converges to a \( k \)-multiple periodic orbit at \( p \) is naturally interpreted as \( \tilde{u} \) being tangent to \( Z_{\epsilon_1} \) at \( p \) with order \( k \).

We will study the local geometry in other paper.

### 4.4 Holomorphic blocks in \( \mathbb{R} \times \tilde{M} \)

Note that the space \( \mathcal{M}_A(M^+, C; g, m + \nu, (k, \epsilon)) \) is not large enough to compactify the moduli space of all \( J \)-holomorphic maps into \( M^+ \), we need consider \( \mathcal{M}_A(\mathbb{R} \times M, C; (k^-, \epsilon^-), (k^+, \epsilon^+)) \), which will be studied in this section.

Let \( ((\Sigma, j); y, p^+, p^-) \) be a connected semistable curve with \( m \) marked points \( y = (y_1, \ldots, y_m) \) and \( \nu^\pm \) puncture points \( p^+ = (p^+_1, \ldots, p^+_{\nu^+}), p^- = (p^-_1, \ldots, p^-_{\nu^-}) \), and \( u : \Sigma \rightarrow \mathbb{R} \times M \) be a \( J \)-holomorphic map. Suppose that \( u(z) \) converges to a \( k^\pm_1 \cdot T_{\epsilon^\pm_1} \) periodic orbit \( x_{k^\pm_1} \subset \mathcal{F}_{\epsilon^\pm_1} \) as \( z \) tends to \( p^\pm_1 \).
Similar to the situation of relative invariants, there is a \( \mathbb{R} \) action, which induces a \( \mathbb{R} \)-action on the moduli space of \( J \)-holomorphic maps. We need mod this action. Since there is a vector field \( X \) with \( |X| = 1 \) on \( \hat{M} \), the Reeb vector field, there is a one parameter group \( \varphi_\theta \) action on \( \hat{M} \) generated by \( -X \). In particular, there is a \( S^1 \)-action on every periodic orbit, corresponding to the freedom of the choosing origin of \( S^1 \). Along every periodic orbit we have \( L_X \lambda = 0 \). Recall that by (3) of Assumption \ref{assumption2.24} \( L_X J = 0 \) along every periodic orbit. We can mod this action.

**Definition 4.10.** Two \( J \)-holomorphic maps \( \Gamma = (u, (\Sigma, j), y, p^+, p^-) \) and \( \tilde{\Gamma} = (\tilde{u}, (\Sigma, j), \tilde{y}, \tilde{p}^+, \tilde{p}^-) \) are called equivalent if there exists a diffeomorphism \( \varphi : \Sigma \to \Sigma \) such that it can be lifted to bi-holomorphic isomorphisms \( \varphi_v : (\Sigma_v, j_v) \to (\Sigma_v, j_v) \) for each component \( \Sigma_v \) of \( \Sigma \), and

1. \( \varphi(y_i) = \tilde{y}_i, \varphi(p^+_j) = \tilde{p}^+_j, \varphi(p^-_l) = \tilde{p}^-_l \) for any \( 1 \leq i \leq m, 1 \leq j \leq \nu^+, 1 \leq l \leq \nu^-; u \) and \( \tilde{u} \circ \varphi \) converges to the same periodic orbit \( x_{k^\pm} \) at \( z \) tends to \( p^\pm_k \);

2. \( \tilde{a} \circ \varphi = a + C, \tilde{u} \circ \varphi = \tilde{u} \) for some constant \( C \);

3. near every periodic orbit \( x, \tilde{u} \) and \( \tilde{u} \circ \varphi \) may differ by a canonical coordinate transformation \( \tilde{u} \).

**Definition 4.11.** A \( J \)-holomorphic map \((u; ((\Sigma, j), y, p))\) is said to be stable if for each \( v \) one of the following conditions holds:

1. \( \tilde{E}(u \circ \pi_{\Sigma_v}) \neq 0 \),

2. Let \( \text{val}_v \) be the number of special points on \( \Sigma_v \) which are nodal points, marked points or puncture points. Then \( \text{val}_v + 2g_v \geq 3 \).

**Definition 4.12.** Put

\[
\text{Aut}(u; ((\Sigma, j), y, p^+, p^-)) = \{ \phi : \Sigma \to \Sigma | \phi \text{ is an automorphism satisfying (1), (2) and (3) in Definition 4.10} \}.
\]

We call it the automorphism group of \((u; ((\Sigma, j), y, p^+, p^-))\).

For any \( A \in H^2(\mathbb{R} \times \hat{M}, \mathbb{R}; \mathbb{Z}) \) we define \( d\lambda(A) \) as following: let \( v : \mathbb{R} \times S^1 \to \mathbb{R} \times \hat{M} \) be a \( C^\infty \) map such that \([v(\mathbb{R} \times S^1)] = A\), we define \( d\lambda(A) := \int_{\mathbb{R} \times S^1} v^*(d\lambda) \).

We fix \( A \in H^2(\mathbb{R} \times \hat{M}, \mathbb{R}; \mathbb{Z}) \) and \( k^\pm = (k^\pm_1, \ldots, k^\pm_{\nu^\pm}), e^\pm = (e^\pm_1, \ldots, e^\pm_{\nu^\pm}) \) satisfying

\[
d\lambda(A) = \sum_{i=1}^{\nu^+} k^+_i \cdot T_{x^+_i} - \sum_{i=1}^{\nu^-} k^-_i \cdot T_{x^-_i}. \tag{50}
\]

We define \( \mathcal{M}_A(\mathbb{R} \times \hat{M}, g, m + \nu^+ + \nu^-, (k^-, e^-), (k^+, e^+)) \) to be the space of equivalence classes of all stable \( J \)-holomorphic maps in \( \mathbb{R} \times M \) representing \( A \) and converging to a \( k^\pm \cdot T_{x^\pm} \)-periodic orbit \( x^\pm_k \subset \mathcal{F}_{x^\pm} \) as \( z \) tends to \( p^\pm_k \). For any \((u, \Sigma, y, p^+, p^-) \in \mathcal{M}_A(\mathbb{R} \times \hat{M}, g, m + \nu^+ + \nu^-, (k^-, e^-), (k^+, e^+))\), by Stoke’s formula we have

\[
d\lambda(A) = \int_{\Sigma} u^*d\lambda = \sum_{i=1}^{\nu^+} \lambda(k^+_ix^+_i) - \sum_{i=1}^{\nu^-} \lambda(k^-_ix^-_i) = \sum_{i=1}^{\nu^+} k^+_i \cdot T_{x^+_i} - \sum_{i=1}^{\nu^-} k^-_i \cdot T_{x^-_i}.
\]

We call \( \mathcal{M}_A(\mathbb{R} \times \hat{M}, C; g, m + \nu^+ + \nu^-, (k^-, e^-), (k^+, e^+)) \) a holomorphic rubber block in \( \mathbb{R} \times \hat{M} \).
Remark 4.13. Let \( \Sigma_v \subset \Sigma \) be a smooth connected component of \( \Sigma \). Suppose that \( u(\Sigma_v) \) lie in a compact set \( K \subset \mathbb{R} \times M \). Then every nodal point of \( \Sigma_v \) is a removable singular point. As \( \omega_\phi = d(\phi \lambda) \) we conclude that \( E_\phi(u) |_{\Sigma_v} = 0 \), then \( u(\Sigma_v) \) is one point, then \( \bar{E}(u) |_{\Sigma_v} = 0 \).

4.5 Local coordinate system of holomorphic blocks

Let \( b = (u, \Sigma, y, p^+, p^-) \in \mathcal{M}_A(\mathbb{R} \times \tilde{M}, g, m + \nu^+ + \nu^-, (k^-, \epsilon^-), (k^+, \epsilon^+)) \) and \( h \in W^{1,p,\alpha}(\Sigma, u^* T(\mathbb{R} \times \tilde{M})) \) such that \( D_u(h) = 0 \). Under the transformation

\[
\hat{\nu} = \nu + C
\]

and the canonical transformation of local pseudo-Daubeaux coordinate system

\[
\hat{\nu} = \nu + \vartheta_0
\]

\( h \) is invariant (see subsection 7.1.2 Case 2 below), so we can view \( h \) as a element in \( T_b(\mathcal{M}_A(\mathbb{R} \times \tilde{M}, g, m + \nu^+ + \nu^-, (k^-, \epsilon^-), (k^+, \epsilon^+))) \). We assume that \( D_u \) is surjective. Let \( e_1, ..., e_a \) be a base of \( \ker D_u \). Then for any \( h \in \ker D_u \) we have \( h = \sum_i x_i e_i \). Let \( w_1, ..., w_d \) be the local coordinates of \( \mathcal{M}_{g,m+\nu^++\nu^-} \) around \( \Sigma \), where \( d = 6g - 6 + 2(m + \nu^+ + \nu^-) \). Then the \( (w_1, ..., w_d, x_1, ..., x_a) \) is a local coordinate system of \( \mathcal{M}_A(\mathbb{R} \times \tilde{M}, g, m + \nu^+ + \nu^-, (k^-, \epsilon^-), (k^+, \epsilon^+)) \).

Similarly, let \( b = (u, \Sigma, y, p^+, p^-) \in \mathcal{M}_A(M^+, g, m + \nu, (k, \epsilon)) \), \( h \in W^{1,p,\alpha}(\Sigma, u^* T(M^+)) \) with \( D_u(h) = 0 \), we view \( h \) as a element in \( T_b(\mathcal{M}_A(M^+, g, m + \nu, (k, \epsilon))) \). Assume that \( D_u \) is surjective. We can choose a local coordinate system in \( \mathcal{M}_{g,m+\nu} \) around \( \Sigma \) together with a coordinate system in \( \ker D_u \) as a local coordinate system of \( \mathcal{M}_A(M^+, g, m + \nu, (k, \epsilon)) \).

5 Compactness theorems

We will use Li-Ruan’s Compactification to our setting. Roughly speaking, the idea of Li-Ruan’s Compactification is that firstly let the Riemann surfaces degenerate in Delingne-Mumford space and then let \( M^+ \) degenerate compatibly. We will give a detail description of the Li-Ruan’s idea and give the definition of the stable compactification using language of graphs. This is one of the key parts of this paper.

5.1 Bubble phenomenon

The proofs of the following two theorems are standard (see [6]).

**Theorem 5.1.** There is a constant \( h > 0 \) such that for every \( J \)-holomorphic map \( u = (a, \bar{u}) : \mathbb{C} \rightarrow \mathbb{R} \times \tilde{M} \) with finite energy,

1. if \( E_\phi(u) \neq 0 \), we have \( E_\phi(u) \geq h \);
2. if \( \bar{E}(u) \neq 0 \), we have \( \bar{E}(u) \geq h \).

**Theorem 5.2.** There is a constant \( h > 0 \) such that for every \( J \)-holomorphic map \( u \in \mathcal{M}_A(\mathbb{R} \times \tilde{M}, g, m + \nu^\pm, (k^-, \epsilon^-), (k^+, \epsilon^+)) \) with finite energy, if \( \bar{E}(u) \neq 0 \), we have

\[
\bar{E}(u) = \sum_{i=1}^{\nu^+} k_i^+ - \sum_{i=1}^{\nu^-} k_i^- \geq h.
\]
Let \( \Gamma^{(i)} = (u^{(i)}, \Sigma^{(i)}; y^{(i)}, p^{(i)}) \in M_A(M^+, C, g, m + \nu, A, (k, \epsilon)) \) be a sequence. Then there is a constant \( C > 0 \) such that
\[
\overline{E}(u^{(i)}) + E_{\partial}(u^{(i)}) < C \quad \forall i.
\]

Suppose that \((\Sigma^{(i)}; y^{(i)}, p^{(i)})\) is stable and converges to \((\Sigma; y, p)\) in \( \overline{M}_{g, m + \nu} \). On a neighborhood of \((\Sigma; y, p)\) in \( \overline{M}_{g, m + \nu} \) we construct a smooth family of metrics on each \((\Sigma^{(i)}; y^{(i)}, p^{(i)})\) such that near each marked point the metric is the Euclidean metric on disc in \( \mathbb{C} \), and near each nodal point the metric is the standard cylinder metric.

### 5.1.1 Bound of the number of singular points

Following McDuff and Salamon [23] we introduce the notion of singular points for a sequence \( u^{(i)} \) and the notion of mass of singular points. We show that there is a constant \( h > 0 \) such that the mass of every singular point is larger than \( h \). Let \( q \) be a singular point and \( q^{(i)} \in \Sigma^{(i)} \), \( q^{(i)} \rightarrow q \). In case \( u^{(i)}(q^{(i)}) \in M_0^+ \) the argument is standard (see [23]). We only consider \( \overline{E} \) over the cylinder end. Since the metrics near nodal points are the standard cylinder metrics, we have, in the cylinder coordinates \( D_1(q^{(i)}) \subset \Sigma^{(i)} - \{ \text{nodal points} \} \), where \( D_1(q^{(i)}) = \{ (s^{(i)}, t^{(i)}) \mid (s^{(i)} - s^{(i)}(q^{(i)}))^2 + (t^{(i)} - t^{(i)}(q^{(i)}))^2 \leq 1 \} \). We identify \( q^{(i)} \) with 0 and consider \( J \)-holomorphic maps \( u^{(i)} : D_1(0) \rightarrow N \).

The proof of the following lemma is similar to Theorem 4.6.1 in [23]. We give the proof here for the reader’s convenience.

**Lemma 5.3.** Let \( u^{(i)} : D_1(0) \rightarrow \mathbb{R} \times \overline{M} \) be a sequence of \( J \)-holomorphic maps with finite energy such that
\[
\sup_i \overline{E}(u^{(i)}) < \infty, \quad |du^{(i)}(0)| \longrightarrow \infty, \quad as \ i \rightarrow \infty.
\]

Then there is a constant \( h > 0 \) independent of \( u^{(i)} \) such that, for every \( \epsilon > 0 \)
\[
\liminf_{i \rightarrow \infty} \overline{E}(u^{(i)}; D_\epsilon(0)) \geq h. \tag{53}
\]

**Proof:** Consider the function
\[
F^{(i)}(z) = |du^{(i)}(z)|d^2(z, \partial D_1(0))
\]
where \( d(z, \partial D_1(0)) \) denotes the distance from \( z \) to \( \partial D_1(0) \) with respect to the standard Euclidean metrics. Obviously, \( F^{(i)} \) attains its maximum at some interior point \( q^*_i \in D_1(0) \) and \( \lim_{i \rightarrow \infty} F^{(i)}(q^*_i) = \infty \). Set \( \delta_i = \frac{1}{2}d(q^*_i, \partial D_1(0)) \). Then for any \( q \in D_{\delta_i}(q^*_i) \),
\[
|du^{(i)}|(q) \leq 4|du^{(i)}|(q^*_i) := 4A_i.
\]

Consider the re-scaling sequence
\[
u^{(i)}(z) = u^{(i)} \left(q^*_i + \frac{z}{A_i}\right)
\]
As \( \lim_{i \rightarrow \infty} F_i(q^*_i) = \infty \), we have \( \lim_{i \rightarrow \infty} \delta_i A_i = \infty \). Then in \( D_{\delta_i A_i}(0) \)
\[
\sup |dv^{(i)}| \leq 4, \quad |dv^{(i)}|(0) = 1, \quad \overline{E}(v^{(i)}, D_{\delta_i A_i}(0)) = \overline{E}(u^{(i)}, D_{\delta_i}(q^*_i)).
\]
By choosing a subsequence we conclude that $v_i$ locally uniformly converges to a nonconstant $J$-holomorphic map with finite energy $v : \mathbb{C} \to \mathbb{R} \times \tilde{M}$. Then the lemma follows from Lemma 5.1.

Recall that $N$ denotes one of $M^+$ and $\mathbb{R} \times \tilde{M}$. By Lemma 5.3 we conclude that the rigid singular points are isolated and the limit

$$m_\epsilon(q) = \lim_{i \to \infty} \tilde{E}(u^{(i)}; D_{q^{(i)}}(\epsilon, h^{(i)}))$$

exists for every sufficiently small $\epsilon > 0$. The mass of the singular point $q$ is defined to be

$$m(q) = \lim_{\epsilon \to 0} m_\epsilon(q).$$

Denote by $P \subset \Sigma$ the set of singular points for $u^{(i)}$, the double points and the puncture points. By Lemma 5.3 and (10), $P$ is a finite set. By definition, $|du^{(i)}|_{h^{(i)}}$ is uniformly bounded on every compact subset of $\Sigma - P$. By a possible translation along $\mathbb{R}$ and passing to a subsequence we may assume that $u^{(i)}$ converges uniformly with all derivatives on every compact subset of $\Sigma - P$ to a $J$-holomorphic map $u : \Sigma - P \to N$. Obviously, $u$ is a finite energy $J$-holomorphic map.

We need to study the behaviour of the sequence $u^{(i)}$ near each singular point for $u^{(i)}$. Let $q \in \Sigma$ be a rigid singular point for $u^{(i)}$. We have two cases.

(a) $q \in \Sigma - \{nodal \ points\}$. We consider $J$-holomorphic maps $u^{(i)} : D_1(0) \to N$.

(a-1) there are $\epsilon > 0$ and a compact set $K \subset N$ such that $u^{(i)}(D_\epsilon(q)) \subset K$.

(a-2) $q$ is a nonremovable singularity.

(b) $q \in \{nodal \ points\}$. In this case a neighborhood of $q$ is two discs $D_1(0)$ joint at 0, where $D_1(0) = \{|z|^2 \leq 1\}$.

For (a-1) we construct bubbles as usual for a compact symplectic manifold (see [23,25,28]). We call this type of bubbles (resp. bubble tree) the normal bubbles (resp. normal bubble tree).

5.1.2 Construction of the bubble tree for (a-2)

We use cylindrical coordinates $z = e^{-s-2\pi\sqrt{-1}t}$ and write

$$u^{(i)}(s,t) = (a^{(i)}(s,t), \tilde{u}^{(i)}(s,t))$$

$$u(s,t) = (a(s,t), \tilde{u}(s,t)).$$

Note that the graduate $|du^{(i)}|$ depends not only on the metric $<,>$ on $N$ but also depends on the metric on $\Sigma^{(i)}$. The energy don’t depend on the metric on $\Sigma^{(i)}$. To construct bubbling in present case it is more convenient to take the Euclidean metric $|dz|^2$ on the disk $D_1(0)$ and pullback to the coordinate $(s,t)$ through $z = e^{-s-2\pi\sqrt{-1}t}$. Obviously, if $q$ is a singular point with respect to the cylinder metric, then it is also a singular point with respect to the disk metric.
By Theorem \textbf{2.7} we have
\[ \lim_{s \to \infty} \tilde{u}(s,t) = x(kTt) \]
in \( C^\infty(S^1) \), where \( x(\ , \ ) \) is a \( kT \)-periodic orbit on \( M \). Choosing \( \epsilon \) small enough we have
\[ |m_\epsilon(q) - m(q)| \leq \frac{1}{10} \min\{h,T\} \]
For every \( i \) there exists \( \delta_i > 0 \) such that
\[ \tilde{E}(u^{(i)}; D\delta_i(0)) = m(q) - \frac{1}{2} \min\{h,T\}, \quad (54) \]
where \( T \) is the minimal positive periodic. By definition of the mass \( m(q) \), the sequence \( \delta_i \) converges to 0. Since \( u^{(i)} \) converges uniformly with all derivatives to \( u \) on any compact set of \( D_i(0) - \{0\} \), \( \delta_i \) must converge to 0. Put
\[ \hat{s}(i) = s + \log \delta_i, \quad \hat{t}(i) = t \quad \text{for} \quad |s^{(i)}_i| > 2R_0. \quad (55) \]
\[ \hat{a}(i) = a - kT(\log \delta_i). \quad (56) \]
Define the \( J \)-holomorphic curve \( v^{(i)}(\hat{s},t) \) by
\[ v^{(i)}(\hat{s},t) = (\hat{a}^{(i)}(\hat{s},t), \tilde{v}^{(i)}(\hat{s},t)) = \left( a^{(i)}(-\log \delta_i + \hat{s},t) - kT(\log \delta_i), \tilde{u}^{(i)}(-\log \delta_i + \hat{s},t) \right). \quad (57) \]

**Lemma 5.4.** Suppose that 0 is a nonremovable singular point of \( u \). Define the \( J \)-holomorphic map \( v^{(i)} \) as above. Then there exists a subsequence (still denoted by \( v^{(i)} \)) such that

1. The set of singular points \( \{Q_1, \cdots, Q_d\} \) for \( v^{(i)} \) is finite and tame, and is contained in the disc \( D_1(0) = \{z \mid |z| \leq 1\} \);

2. The subsequence \( v^{(i)} \) converges with all derivatives uniformly on every compact subset of \( \C \setminus \{Q_1, \cdots, Q_d\} \) to a nonconstant \( J \)-holomorphic map \( v : \C \setminus \{Q_1, \cdots, Q_d\} \to \R \times \tilde{M} \);

3. \( \tilde{E}(v) > \frac{1}{2} \min\{h,T\} \);

4. \( \tilde{E}(v) + \sum_1^d m(Q_i) = m(0). \)

5. \( \lim_{s \to \infty} \tilde{u}(s,t) = \lim_{\hat{s} \to -\infty} \tilde{v}(\hat{s},t). \)

**Proof:** The proofs of (1), (2) and (4) are standard (see \textbf{23}), we omit them here. The proof of (5) will be given in our next paper (see \textbf{19}). We only prove (3).

(3) Note that
\[ \left| \int_{\tilde{u}^{(i)}(-\log \delta_i,S^1)} \lambda - \int_{\tilde{u}^{(i)}(-\log \epsilon,S^1)} \lambda \right| = \tilde{E}(u; -\log \epsilon \leq s \leq -\log \delta_i) = \frac{1}{2} \min\{h,T\}. \quad (58) \]

Since \( \lim_{s \to \infty} \tilde{u}(s,t) = x(kTt) \) we have
\[ \left| \int_{\tilde{u}^{(i)}(-\log \epsilon,S^1)} \lambda - kT \right| \leq \frac{1}{8} \min\{h,T\}. \]
when $\epsilon$ small enough and $i$ big enough. Then by the locally uniform convergence of $v^{(i)}$

$$\tilde{E}(v) \geq \left| kT - \int_{\tilde{v}(0, S^{1})} \lambda \right| \geq \frac{1}{3} \min \{ h, T \}. \quad (59)$$

We can repeat this again to construct bubble tree.

We introduce a terminology. Let $\Sigma$, $u$, $w$, $\lambda$, $c$, $\epsilon$, $\nu$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{\lambda}$, $\tilde{\epsilon}$, $\tilde{\nu}$ be the local complex coordinates of $\Sigma$. Without loss of generality we assume that there is a subsequence, still denoted by $i$, such that $|\lambda^{(i)}| = 1$. Let $\Sigma^{(i)}$ be the $i$-th bubble tree. We can repeat this again to construct bubble tree.

We say $u_1$ and $u_2$ converge to a same periodic orbit, if $k_1 = k_2$, and in the pseudo-Daubaux coordinates $(a, \vartheta, w)$, (see Lemma 2.3) $w(x_1) = w(x_2)$ (in this case $T_1 = T_2$ holds naturally).

### 5.1.3 Construction of bubble tree for (b)

Let $\Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ are smooth Riemann surfaces of genus $g_1$ and $g_2$ joining at $q$. Let $z_1$, $z_2$ be the local complex coordinates of $\Sigma_1$ and $\Sigma_2$ with $z_1(q) = z_2(q) = 0$. Recall that a neighborhood of $\Sigma_1 \cap \Sigma_2 \in \mathcal{M}_{g,m+\nu}$ is given by

$$z_1z_2 = w = e^{-R-2\pi \sqrt{-1} \tau}, \quad w \in \mathbb{C},$$

where $R = 2lr$, $l \in \mathbb{Z}^+$. We will use $(r, \tau)$ as the local coordinates in the neighborhood of $\Sigma \in \mathcal{M}_{g,m+\nu}$. Let

$$z_1 = e^{-s_1-2\pi \sqrt{-1} t_1}, \quad z_2 = e^{s_2+2\pi \sqrt{-1} t_2}.$$

$(s_1, t_1)$ are called the holomorphic cylindrical coordinates near $p$. In terms of the holomorphic cylindrical coordinates we write

$$\Sigma_1 - \{p\} = \Sigma_{10} \bigcup \{ [0, \infty) \times S^{1} \},$$

$$\Sigma_2 - \{p\} = \Sigma_{20} \bigcup \{ (\infty, 0] \times S^{1} \}.$$

Then $\Sigma^{(r)} \to \Sigma_1 \cap \Sigma_2$ as $r \to \infty$ in the $\mathcal{M}_{g,m+\nu}$.

We consider the case that $u^{(i)} : \Sigma^{(i)} \to N$ is a sequence of $J$-holomorphic maps, where $\Sigma^{(i)} := \Sigma^{(r^{(i)})}$, $q^{(i)} \in \Sigma^{(i)}$, $q^{(i)} \to q$.

Without loss of generality we assume that there is a subsequence, still denoted by $i$, such that $|s^{(i)}_1(q^{(i)})| \leq l r^{(i)}$. In this case we may identify a neighborhood of $q^{(i)}$ in $\Sigma^{(i)}$ with $D_1(0) \setminus D_{r^{(i)}}(0) \subset \Sigma_1$, where $\lim \epsilon^{(i)} = 0$. The sequence $u^{(i)}$ is considered to be a sequence of $J$-holomorphic maps from...
Let \( D_1(0) \setminus D_{\epsilon(0)} \) into \( N \), and \( q^{(i)} \in D_1(0) \setminus D_{\epsilon(0)} \), \( q^{(i)} \to 0 \). In terms of the cylinder coordinates we can identify the coordinates \((s^{(i)}, t^{(i)})\) in \( \Sigma^{(i)} \) with the coordinates \((s_1, t_1)\) of \( \Sigma_1 \).

By Theorem 2.7 we have

\[
\lim_{s_1 \to \infty} \tilde{u}(s_1, t_1) = x(kT t_1)
\]

in \( C^\infty(S^1) \), where \( x(\cdot, \cdot) \) is a \( kT \)-periodic orbit on \( M \). Choosing \( \epsilon \) small enough we have

\[
|m_\epsilon(q) - m(q)| \leq \frac{1}{10} \min\{h, T\}
\]

For every \( i \) there exists \( \delta_i > 0 \) such that

\[
\tilde{E}(u^{(i)}; D_{\delta_i}(0)) = m(q) - \frac{1}{2} \min\{h, T\},
\]

where \( T \) is the minimal positive periodic. We choose a smooth conformal transformation \( \psi^{(i)} : \Sigma^{(i)} \to \Sigma^{(i)} \) such that

\[
\psi^{(i)}|_{D(R_0)} = I,
\]

\[
z^{(i)} = \delta_i z^{(i)} = e^{2\pi\sqrt{-1} t^{(i)}} e^{-s^{(i)} - 2\pi\sqrt{-1} t^{(i)}}, \text{ for } |s^{(i)}| > 2R_0
\]

where \((\hat{s}^{(i)}, \hat{t}^{(i)}) = \hat{v}_i(s^{(i)}, t^{(i)})\), i.e.,

\[
\hat{s}^{(i)} = s^{(i)} + log\delta_i = s_1 + log\delta_i, \quad \hat{t}^{(i)} = t^{(i)} = t_1 \text{ for } |s_1^{(i)}| > 2R_0.
\]

We call coordinates rescaling of Riemann surface a \( D \)-rescaling. In our present case we need not only a \( D \)-rescaling, but also a translation along \( \mathbb{R} \), called a \( T \)-rescaling. We call such composition a \( DT \)-rescaling. Put

\[
\hat{a}^{(i)} = a - kT(-\log\delta_i).
\]

Define the \( J \)-holomorphic curve \( \nu^{(i)}(\hat{s}, t) \) by

\[
\nu^{(i)}(\hat{s}, t) = (\hat{a}^{(i)}(\hat{s}, t), \hat{v}^{(i)}(\hat{s}, t)) = \left( a^{(i)}(-\log\delta_i + \hat{s}, t) - kT(-\log\delta_i), \tilde{u}^{(i)}(-\log\delta_i + \hat{s}, t) \right).
\]

By the same argument as in subsection 5.1.2 we construct a bubble \( S^2 \) with \( E(v)|_{S^2} > \frac{1}{3} \min\{h, T\} \) or \( \tilde{E}(v)|_{S^2} > \frac{1}{3} \min\{h, T\} \), inserted between \( \Sigma_1 \) and \( \Sigma_2 \). The same results as Lemma 5.4 still hold.

We can repeat this again to construct bubble tree.

### 5.1.4 For the case of genus 0

Let \( \Gamma^{(i)} = (u^{(i)}, \Sigma^{(i)}, y^{(i)}, p^{(i)}) \in \mathcal{M}_A(M^+, C, g, m + \nu, y, p, (k, c)) \) be a sequence. Let \( \Sigma^{(i)} = \bigcup_{\nu=1}^{N} \Sigma^{(i)}_\nu \). Assume that there is one component of \( \Sigma^{(i)} \) that has genus 0 and is unstable. Let \( \Sigma^{(i)}_1 \) is such a component. We identify \( \Sigma^{(i)}_1 \) with a sphere \( S^2 \), and consider \( u^{(i)} : S^2 \to N \). We discuss several cases:

1. \( u^{(i)}|_{S^2} \) has no singular point. Then \( ||\nabla u^{(i)}||_{S^2} \) are uniformly bounded above. As \((u^{(i)}, S^2, y^{(i)}, p^{(i)}) \) is stable, \( E(u^{(i)})|_{S^2} \geq h \) or \( \tilde{E}(u^{(i)})|_{S^2} \geq h \). Then \( u^{(i)}|_{S^2} \) locally uniformly converges to \( u|_{S^2} \) with \( E(u)|_{S^2} \geq h \) or \( \tilde{E}(u)|_{S^2} \geq h \), so \( u; S^2 \) is stable.
2). \( S^2 \) joints with \( \Sigma^{(i)}_2 \) at a nodal point \( q \). Then the number of the special points, including the marked points and the puncture points, is \( \leq 1 \).

2a) There is one singular point \( p \) and one special point \( \mathcal{X} \).

(2a-1) \( p \neq q \) and \( p \neq \mathcal{X} \),
(2a-2) \( p = \mathcal{X} \) or \( p = q \)

2b) There are two singular points \( p_1, p_2 \).

(2b-1) \( p_1 \neq q \) and \( p_2 \neq q \),
(2b-2) \( p_1 = q \) or \( p_2 = q \)

2c) There is only one singular point \( p \) and no special point.

3). \( S^2 \) joint with \( \Sigma^{(i)}_2 \) and \( \Sigma^{(i)}_3 \) at \( q_1 \) and \( q_2 \) respectively. There is only one singular point \( p \).

(3a) \( p \neq q_1 \) and \( p \neq q_2 \),
(3b) \( p = q_1 \) or \( p = q_2 \).

For the cases (2a-1), (2b-1), and (3a) we construct bubble tree at singular points as in subsection §5.1.2 to get a stable map \((u, S^2)\). For the cases (2a-2), (2b-2), (2c) and (3b) we forget the map \( u_i \big|_{S^2} \) and contract \( S^2 \) as a point, then we construct a bubble as in subsection §5.1.3.

By (3) of Lemma 5.4 we get a stable map \((v, S^2)\). We can repeat the procedure to construct bubble tree.

5.2 \( \mathcal{T} \)-rescaling

Let \( \Gamma^{(i)} = (u^{(i)}, \Sigma^{(i)}, y^{(i)}, p^{(i)}) \in \mathcal{M}_A(M^+, C_g, m + \nu, y, p, (k, e)) \) be a sequence. If there is some \( \Sigma_v \) of genus 0, we treat it as in subsection §5.1.4. In the following we assume that \((\Sigma^{(i)}, y^{(i)}, p^{(i)})\) is stable and converges to \((\Sigma; y, p)\) in \( \mathcal{M}_{g,m+\nu} \).

Note that we have fixed the degeneration of Riemann surfaces now, we are concern with the \( \mathcal{T} \)-rescaling. We explain our procedure of \( \mathcal{T} \)-rescaling. We discuss maps into \( M^+ \), for maps into \( \mathbb{R} \times \bar{M} \) the situation are the same.

Let \( \Sigma \) be a Riemann surface with \( \nu \) puncture points \( p = (p_1, \ldots, p_\nu) \). We choose holomorphic coordinates \((s_i, t_i)\) near \( p_i \). Let \( u : \bar{\Sigma} \to M^+ \) be a \( J \)-holomorphic map. Denote \( u = (a, \tilde{u}) \). Suppose that

\[
\lim_{|s_j| \to \infty} \tilde{u}(s_j, t_j) = x(k_j T_{t_j} t_j)
\]

in \( C^\infty(S^1) \) for some \( k_j T_{t_j} \)-periodic orbit \( x(k_j T_{t_j} t_j) \). For each \( j \) we choose a local pseudo-Darboux coordinates \((a_j, \vartheta_j, w_j)\) near \( p_j \) such that \( x(k_j T_{t_j} t_j) = (k_j T_{t_j} t_j + \vartheta_j, 0) \) for some constant \( \vartheta_j \). Note that for any \( 0 < l, j \leq \nu \),

\[
a_l = a_j + C_{ij},
\]

where \( C_{ij} \) are constants.

For simplicity, we consider the case as in Figure 1, the other cases is similar. Let \( \Sigma = \bigcup_{j=1}^{6} \Sigma_j \), where each \( \Sigma_j \) is a smooth connected component. Let \( T_j \subset \Sigma_j - P \) be compact sets, where \( P \subset \Sigma \).
denotes the set of singular points for \( u^{(i)} \), the double points and the puncture points. Denote \( I = \{1, \cdots , 6\} \). We write
\[
u^{(i)}(z) = (a^{(i)}(z), \bar{u}^{(i)}(z)).
\]

For each \( j \in I, \) \( T_j^{(i)} \subset \Sigma^{(i)} \) converges to \( T_j \). We fix points \( z_j \in T_j \). Suppose that \( z_j^{(i)} \in T_j^{(i)} \) is a sequence of points which converges to \( z_j \). Without loss of generality, we assume that
\[
(1) \quad \left| a^{(i)}(z_1^{(i)}) \right| \leq \min_{j \in I \setminus \{1\}} \left| a^{(i)}(z_j^{(i)}) \right|, \forall i,
\]
\[
(2) \quad \sup_i \left| a^{(i)}(z_1^{(i)}) \right| < \infty,
\]

Restricting to \( T_1 \), \( u^{(i)} \) uniformly converges to a map \( u_1 \). For any compact \( K \subset \Sigma_1 - P, u^{(i)} \) also converges to \( u_1 \). Then \( u \) is naturally defined on \( \Sigma_1 - P \).

For \( \Sigma_2 \) we consider two different cases:

**Case 1.** \( \sup_i \left| a^{(i)}(z_2^{(i)}) \right| < \infty \). Restricting to \( T_2 \), \( u^{(i)} \) uniformly converges to a map \( u_2 \). Then \( u_2 \) is naturally defined on \( \Sigma_2 - P \). Then \( (u_1; \Sigma_1, y_1, p_1) \) and \( (u_2; \Sigma_2, y_2, p_2) \) are belong to the same holomorphic block in \( M^+ \).

**Case 2.** \( \lim_{k \to \infty} a^{(i)}(z_2^{(i)}) = \infty \). In this case we take a coordinate transformation:
\[
(a^*)^{(i)} = a - a^{(i)}(z_2^{(i)}).
\]

Define
\[
(u^*)^{(i)}(z) = ((a^*)^{(i)}(z), (\bar{u}^*)^{(i)}(z)), \quad (\bar{u}^*)^{(i)}(z) = \bar{u}^{(i)}(z).
\]

Note that \( |d(u^{(i)})| \) is invariant under translation along \( \mathbb{R} \). As above, restricting to \( T_2^{(i)}, (u^*)^{(i)} \) locally uniformly converges to a map \( u^* = (a^*, \bar{u}^*): T_2 \to \mathbb{R} \times \tilde{M} \) with \( a^*(z_2) = 0 \), which extends to \( u^*: \Sigma_2 - P \to \mathbb{R} \times \tilde{M} \). If \( \lim_{\epsilon \to 0} m_\epsilon(q_2) \geq h \), then we can construct bubbles as in subsection 5.1.3

Otherwise, \( u \) and \( u^* \) converges to the same periodic orbit as \( z \to q_2 \). Note that, near \( q_2 \), in terms of the local cylinder coordinates the degeneration is given by
\[
(s^*)^{(i)} = s - 2l^{(i)}, \quad (t^*)^{(i)} = t - \tau^{(i)}.
\]
In local pseudo-Dauboux coordinate system \((a, \vartheta, w)\) and the cylinder coordinates of Riemann surface near \(q_2\) we write

\[
 u(s, t) = (a(s, t), \tilde{u}(s, t)), \quad u^*(s^*, t^*) = (a^*(s^*, t^*), \tilde{u}^*(s^*, t^*)). \tag{71}
\]

Then

\[
 \lim_{|s| \to \infty} \tilde{u}(s, t) = x(kTt), \quad \lim_{|s^*| \to \infty} \tilde{u}^*(s^*, t^*) = x(kTt^* + \vartheta_0^*) \tag{72}
\]
in \(C^\infty(S^1)\) for some \(kT\)-periodic orbit \(x(kTt)\).

We call \(u^* : \Sigma_2 \to \mathbb{R} \times \tilde{M}\) a rubber component.

By the same argument we can construct maps on \(\Sigma_j - P, j \geq 3\).

**Remark 5.5.** We use notations above. Suppose that \(\sup_i |a_1^{(i)}(z_1^{(i)})| \to \infty\) and there is only one singular point \(p \in \Sigma_1\) with

\[
 p^{(i)} \to p, \quad \sup_i \left| u^{(i)}(p^{(i)}) \right| < \infty.
\]

We re-scale as above and construct bubble tree as usual. Then all main part of \(u(\Sigma)\) are rubber component and there is a bubble tree with \(u(p) \in M^+\).

**Remark 5.6.** For Figure 1, it is possible that some \(u^{(i)}(T_j^{(i)}) \subset M^+\) for some \(j \geq 2\). We assume that

\[
 u^{(i)}(T_1^{(i)}) \subset M^+, \quad u^{(i)}(T_6^{(i)}) \subset M^+,
\]

and

\[
 \sup_{i \to \infty} |a^{(i)}(z_1^{(i)})| < \infty, \quad \sup_{i \to \infty} |a^{(i)}(z_6^{(i)})| < \infty. \tag{73}
\]

In this case there is a relation between \(a^{(i)}(z_j^{(i)}), 1 \leq j \leq 6\). To see this and to simplify notations, we omit the index \((i)\) and let

\[
 l_1 = a(z_3) - a(z_1), \quad l_2 = a(z_4) - a(z_3),
\]

\[
 l_3 = a(z_5) - a(z_4), \quad l_4 = a(z_6) - a(z_5).
\]

Then

\[
 l_1 + l_2 + l_3 + l_4 = a(z_6) - a(z_1). \tag{74}
\]

This means that \(l_1 + l_2 + l_3 + l_4\) is a \(T\) re-scaling invariant. By (73), there is only three independent parameter.

If we start from \(\Sigma_6\) to re-scale by the above procedure, we get the same result. Similarly, we start from any component \(\Sigma_i, i \geq 3\), we get also the same result. This suggests a alternative way to do \(T\)-re-scaling for \(\Sigma\): we first re-scale for both \(\Sigma_1\) and \(\Sigma_6\) by

\[
 \hat{a}^{(i)}(z) = a^{(i)}(z) - a^{(i)}(z_3) = a^{(i)}(z) - l_4^{(i)} - a^{(i)}(z_1),
\]

\[
 \hat{a}^{(i)}(z) = a^{(i)}(z) - a^{(i)}(z_5) = a^{(i)}(z) + l_4^{(i)} - a^{(i)}(z_6),
\]

then we re-scale for both \(\Sigma_3\) and \(\Sigma_5\) by

\[
 a^{\ast(i)}(z) = \hat{a}^{(i)}(z) - \hat{a}^{(i)}(z_4) = \hat{a}^{(i)}(z) - l_2^{(i)},
\]

\[
 a^{\ast(i)}(z) = \hat{a}^{(i)}(z) - \hat{a}^{(i)}(z_4) = \hat{a}^{(i)}(z) + l_3^{(i)}.
\]

In view of (75) we find that the above re-scalings are equivalent.
In the following we consider the singular points. Let \( \tilde{q} \) be a singular point of \( u^{(i)} \). We discuss two cases:

**Case a.** \( \tilde{q} \) is not a node of \( \Sigma \). In this case we construct bubble tree as usual.

**Case b.** \( \tilde{q} \) is a node of \( \Sigma \). Suppose that \( \tilde{q} = q_1 \). We construct bubble as in subsection 5.2 to get \( S^2 \), inserted between \( \Sigma_1 \) and \( \Sigma_3 \), with \( \tilde{E}(v) \mid_{s^2 > \frac{1}{3}} \min \{ h, T \} \). Denote \( q_1' = \Sigma_1 \cap S^2, q_1 = \Sigma_3 \cap S^2 \).

We fix a point \( \tilde{z} \) in the compact set of \( S^2 - \{ q_1, q_1' \} \). Let \( \tilde{z}^{(i)} \in \Sigma^{(i)} \) such that \( \tilde{z}^{(i)} \rightarrow \tilde{z} \). Then \( \Sigma' = \Sigma \cup S^2 \) is a semi-stable curve and there exists a fixed degeneration \( \Sigma^{(i)} \rightarrow \Sigma' \).

Consider the coordinate transformations

\[
\tilde{a}^{(i)} = a - kT(- \log \delta),
\]

and

\[
(a_\circ)^{(i)} = \tilde{a}^{(i)} - \tilde{a}^{(i)}(z_3^{(i)}).
\]

Note that for any \( z \in \Sigma_3 \)

\[
\tilde{a}^{(i)}(z) - \tilde{a}^{(i)}(z_3^{(i)}) = a^{(i)}(z) - a^{(i)}(z_3^{(i)}).
\]

Then we have

\[
(a_\circ)^{(i)}(z) = (\tilde{a})^{(i)}(z).
\]

To simplify notations, we omit the index \( (i) \) and let

\[
l_1 = a(\tilde{z}) - a(z_1), \quad l_1^* = a(z_3) - a(\tilde{z}), \quad l_2 = a(z_4) - a(z_3),
\]

\[
l_3 = a(z_5) - a(z_4), \quad l_4 = a(z_6) - a(z_5).
\]

Then

\[
l_1^* + l_1 + l_2 + l_3 + l_4 = a(z_6) - a(z_1).
\]

We conclude that the \( T \)-rescaling based on \( \Sigma^{(i)} \rightarrow \Sigma \) and the \( T \)-rescaling based on \( \Sigma^{(i)} \rightarrow \Sigma' \) are equivalent.

### 5.3 Equivalent DT - rescaling

We consider the example in subsection 5.2 (see Figure 1). For simplicity we only consider the degeneration of \( \Sigma^{(r)} \) at \( q_2 \) and consider the **Case 2** here. Let \( q^{(i)} \in \Sigma^{(i)} \), \( q^{(i)} \rightarrow q_2 \). Suppose that there exists a constant \( N > 0 \) such that

\[
\lim_{i \rightarrow \infty} \sup \tilde{E}(u^{(i)}; N - 1 \leq s \leq 2l^{(i)} - N + 1) \leq \frac{1}{2} \min \{ h, T \}.
\]

We have two DT - rescaling:

**A.** The degeneration of Riemann surfaces in the Delingue-Mumford space, together with the \( T \)-re-scaling in Section 5.2 we have a DT - rescaling given by (68), (69) and (70).

**B.** There is another re-scaling as following. Assume that \( \Sigma^{(i)} \) degenerate at \( q_2 \) with the formula (70). We choose a new coordinate system \( \tilde{a}^{(i)} \) by

\[
\tilde{a}^{(i)} = a - 2kTl^{(i)}.
\]
Define
\[ \tilde{u}^{(i)}(z) = (\tilde{a}^{(i)}(z), \tilde{u}^{(i)}(z)) = (a^{(i)}(z) - 2kT l^{(i)}, \tilde{u}^{(i)}(z)). \] (78)

Then the two coordinate systems \( \tilde{a}^{(i)} \) and \((a^*)^{(i)} \) satisfies
\[ \tilde{a}^{(i)} = (a^*)^{(i)} + a^{(i)}(z_2^{(i)}) - 2kT l^{(i)}. \] (79)

Choose two sequences of points \( \delta_1^{(i)}, \delta_2^{(i)} \) such that \((s, t)(\delta_1^{(i)}) = (N, 0) \) and \((s^*)(t^*)(\delta_2^{(i)}) = (-N, 0) \). Obviously, \( \delta_j^{(i)} \to \delta_j \in \Sigma_j - P, j = 1, 2 \). It follows from the convergence of \( u^{(i)} \) and \((u^*)^{(i)} \) on compact sets that
\[ |a^{(i)}(\delta_1^{(i)}) - kT s(\delta_1^{(i)})| \leq C_2, \quad |a^{(i)}(\delta_2^{(i)}) - a^{(i)}(z_2^{(i)})| = |(a^*)^{(i)}(\delta_2^{(i)}) - (a^*)^{(i)}(z_2^{(i)})| \leq C_3. \] (80)

By the same argument of (5) in Lemma 5.4 we have
\[ |a^{(i)}(\delta_2^{(i)}) - kT s(\delta_2^{(i)})| \leq C_3'. \]

By (70) we have
\[ s(\delta_2^{(i)}) - 2l^{(i)} = s^*(\delta_2^{(i)}) = -N. \]

Then
\[ |a^{(i)}(\delta_2^{(i)}) - 2kT l^{(i)}| \leq |a^{(i)}(\delta_2^{(i)}) - kT s(\delta_2^{(i)})| + |kT s(\delta_2^{(i)}) - 2kT l^{(i)}| \leq C_3 + kTN \leq C_4. \] (81)

It follows from (79), (80) and (81) that for any point \( z \in \Sigma_2 - P \) we have
\[ |\tilde{a}^{(i)}(z) - (a^*)^{(i)}(z)| \leq C \] (82)

for some positive constant \( C \).

We call two \( \mathcal{DT} \)-rescalings are equivalent if there exists a constant \( C > 0 \) independent of \( i \) such that (82) holds. We have proved

**Lemma 5.7.** Let \( \Sigma = \Sigma_1 \wedge \Sigma_2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are smooth Riemann surfaces of genus \( g_1 \) and \( g_2 \) joining at \( q \). Assume that \( \Sigma^{(i)} \) is a sequence of smooth Riemann surface which converges to \( \Sigma \) in the Delingne-Mumford space as \( i \to \infty \). Suppose that
\[ \lim_{i \to \infty} \sup E(u^{(i)}; D_\epsilon(q)) \leq \frac{1}{2} \min\{h, T\}, \] (83)

and restricting on \( \Sigma_2 \setminus D_\epsilon(q) \) there is no singular points of \( u^{(i)} \). Then the two \( \mathcal{DT} \)-rescalings \( A \) and \( B \) are equivalent.

The above discussion can be immediately generalized to the case of several nodal points.

**Lemma 5.8.** Let \( \Sigma = \Sigma_1 \wedge \Sigma_2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are smooth Riemann surfaces of genus \( g_1 \) and \( g_2 \) joining at \( q_1, \ldots, q_\nu \). Assume that \( \Sigma^{(i)} = \Sigma_1 \#_{(i)} \Sigma_2 \) be a sequence of smooth Riemann surface which converges to \( \Sigma \) in the Delingne-Mumford moduli space as \( i \to \infty \). Let \( u^{(i)} : \Sigma^{(i)} \to M^+ \) be a sequence \( J \)-holomorphic maps. Suppose that for any \( j \in \{1, \ldots, \nu\}, \)
\[ \lim_{i \to \infty} \sup E(u^{(i)}; D_\epsilon(q_j)) \leq \frac{1}{2} \min\{h, T\}, \] (84)

and restricting on \( \Sigma_2 - \bigcup_{j=1}^\nu D_\epsilon(q_j) \) there is no singular points of \( u^{(i)} \). Then the \( \mathcal{DT} \)-rescaling of type \( A \) and any \( \mathcal{DT} \) re-scalings of type \( B \) are equivalent.
5.4 Procedure of re-scaling

To sum up, for any sequence \( \Gamma^{(i)} = (u^{(i)}, \Sigma^{(i)}; y^{(i)}, p^{(i)}) \in \mathcal{M}_A(M^+, C, g, m+\nu, (k, \epsilon)) \) our procedure is following:

1. If there is some \( \Sigma_\nu \) of genus 0, we treat it as in subsection 5.1.4. In the following we assume that \( (\Sigma^{(i)}; y^{(i)}, p^{(i)}) \) is stable and converges to \( (\Sigma; y, p) \) in \( \overline{\mathcal{M}}_{g, m+\nu} \).

2. By Lemma 5.3 the number of singular points of \( \Sigma \) is finite. Denote by \( P \subset \Sigma \) the set of singular points for \( u^{(i)} \), the nodal points and the puncture points.

3. We first find a component \( \Sigma_k \) of \( \Sigma \), for example \( \Sigma_1 \) in Figure 1, such that

\[
\left| a_1^{(i)}(z_1^{(i)}) \right| \leq \min_{j \in I \setminus \{1\}} \left| a_j^{(i)}(z_j^{(i)}) \right|, \quad \forall i.
\]

Without loss of generality we suppose that \( \sup_i \left| a_1^{(i)}(z_1^{(i)}) \right| < \infty \), that is, \( u^{(i)}(z_1^{(i)}) \subset M^+ \). Find a set \( J \subset I \) such that \( j \in J \) if and only if \( u(S_j) \subset M^+ \), for example \( \Sigma_1 \) and \( \Sigma_6 \) in Figure 1, i.e., \( J = \{1, 6\} \). Let \( \Sigma^{1\#} = \Sigma - \bigcup_{j \in J} \Sigma_j \). \( \Sigma^{1\#} \) may have several connected components. For every connected component of \( \Sigma^{1\#} \) we do \( \tau \)-rescaling independently as in 5.2.

We repeat the procedure. Finally we will stop after finite steps.

4. Then we construct bubble tree for every singular point independently to get \( \Sigma' \), where \( \Sigma' \) is obtained by joining chains of \( P^1 \)s at some double points of \( \Sigma \) to separate the two components, and then attaching some trees of \( P^1 \)'s. Then we have

(a) for every nodal point \( q \in \mathcal{N} \) there is a neighborhood \( D_t(q) \) so that

\[
\lim_{i \to \infty} \sup E(u^{(k)}; D_t(q)) \leq \frac{\hbar}{2},
\]

where \( \mathcal{N} \) denotes the set of all nodal points of \( \Sigma' \);

(b) restricting on \( \Sigma' \setminus \bigcup_{q \in \mathcal{N}} D_t(q) \) there is no singular points of \( u^{(i)} \). \( \square \)

For every sequence \( \Gamma^{(i)} = (u^{(i)}, \Sigma^{(i)}; y^{(i)}, p^{(i)}) \in \mathcal{M}_A(M^+, C, g, m+\nu, (k, \epsilon)) \), using our procedure we get \( \Gamma = (u, \Sigma', y, p) \), where

(A-1) \( \Sigma' \) is obtained by joining chains of \( P^1 \)s at some double points of \( \Sigma \) to separate the two components, and then attaching some trees of \( P^1 \)'s. \( \Sigma' \) is a connected curve with normal crossings. We call components of \( \Sigma \) principal components and others bubble components.

(A-2) \( u : \Sigma' \to (M^+)\) is a continuous map, where \( (M^+)\) is obtained by attaching some \( \mathbb{R} \times \widetilde{M} \) to \( M^+ \). Let \( \Sigma_1 \) be a connected component of \( \Sigma' \) and \( u|_{\Sigma_1} : \Sigma_1 \to \mathbb{R} \times \widetilde{M} \). We call \( (u; \Sigma_1) \), modulo the translations \( \mathbb{R} \times \widetilde{M} \) and the \( S^1 \)-action on periodic orbits, a rubber component.

(A-3) If we attach a tree of \( P^1 \) at a marked point \( y_i \) or a puncture point \( p_i \), then \( y_i \) or \( p_i \) will be replaced by a point different from the intersection points on a component of the tree. Otherwise, the marked points or puncture points do not change.
(A-4) Let $val_v$ be the number of points on $\Sigma_v$ which are nodal points or marked points or puncture points. In case $u(\Sigma_v) \subset M^+$, if $u|_{\Sigma_v}$ is constant then $val_v + 2g_v \geq 3$; in case $u : \Sigma_v \to \mathbb{R} \times \bar{M}$, if $\tilde{u}|_{\Sigma_v}$ is constant then $val_v + 2g_v \geq 3$.

(A-5) $u$ converges exponentially to $(k_1T_1, \cdots, k_vT_v)$ periodic orbits $(x_{k_1}, \ldots, x_{k_v})$ as the variable tends to the puncture $(p_1, \ldots, p_v)$; more precisely, $u$ satisfies $|27| - 29$.

(A-6) The restriction of $u$ to each component of $\Sigma'$ is $J$-holomorphic. Let $q$ be a nodal point of $\Sigma'$.
Suppose $q$ is the intersection point of $\Sigma_v$ and $\Sigma_w$. If $q$ is a removable singular point of $u$, then $u$ is continuous at $q$; If $q$ is a nonremovable singular point of $u$, then $u|_{\Sigma_v}$ and $u|_{\Sigma_w}$ converge exponentially to the same periodic orbit on $\bar{M}$ as the variables tend to the nodal point $q$.

5.5 Weighted dual graph with an oriented decomposition

It is well-known that the moduli space of stable maps in a compact symplectic manifold has a stratification indexed by the combinatorial type of its decorated dual graph. In this section we generalize this construction to our setting and in the next section we state Li-Ruan’s compatification by using weighted dual graphs.

Let $G$ be a graph. Denote $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set of vertices and $E(G)$ is a finite set of edges. Suppose that $V = \{v_1, \ldots, v_N\}$. Given a partition of $\{1, 2, \ldots, N\}$
$$v : \{1, \cdots, N\} = (\cup_{i=1}^c I_i) \cup \left(\cup_{\alpha=1}^d J_\alpha\right),$$
it induces a decomposition of $V$, still denoted by $v$,
$$v : V = A \cup B,$$
where
$$A = \bigcup_{i=1}^c W_{I_i}, \quad B = \bigcup_{\alpha=1}^d W_{J_\alpha}, \quad W_{I_i} = \{v_k | k \in I_i\}, \quad W_{J_\alpha} = \{v_k | k \in J_\alpha\}.$$
Obviously, $A \cap B = \emptyset$. Every subset $W_{I_i}$ (resp. $W_{J_\alpha}$) determines an induced subgraph $G_{I_i}$ (resp. $G_{J_\alpha}$) of $G$.

Assumption 1. For any $W_{I_i}, W_{I_j} \subset A, i \neq j$, there is no edge in $E(G)$ connecting $G_{I_i}$ and $G_{I_j}$.

Denote by $R(G)$ all the edges which connect two subgraphs above. Obviously,
$$R(G) = E(G) - (\cup_{i=1}^c E(G_{I_i})) \cup \left(\cup_{\alpha=1}^d E(G_{J_\alpha})\right).$$
Let $\ell \in R(G)$ be an edge connecting $v_i$ and $v_j$. We give an orientation to $\ell$, denoted by
$$\ell : v_i \overset{\ell}{\rightarrow} v_j.$$ (85)
Sometimes we denote simply by $v_i \ell v_j$.

We call $\ell \in R(G)$ an oriented edge, the edges in $E(G) - R(G)$ are called normal edges, or simply edges. If for all $\ell \in R(G)$ we have an orientation, we say that $v$ is an oriented decomposition.

Assumption 2. There is an orientation of $v$ such that
(1) For any edge $\ell \in R(G)$ connecting $v_i \in G_{I_i}$ and $v_j \in G_{J_{\alpha_i}}$, we have

$$\ell : v_i \xrightarrow{\ell} v_j,$$

and we denote $G_{I_i} \to G_{J_{\alpha_i}}$.

(2) For any connected subgraph $G_{J_{\alpha}}$ and $G_{J_{\beta}}$, all edges between $G_{J_{\alpha}}$ and $G_{J_{\beta}}$ have the same orientation, that is, if there is an edge $\ell \in R(G)$ connecting $v_i \in G_{J_{\alpha}}$ and $v_j \in G_{J_{\beta}}$ satisfying $v_i \xrightarrow{\ell} v_j$, then any edge $\ell' \in R(G)$ connecting $v_i' \in G_{J_{\alpha}}$ and $v_j' \in G_{J_{\beta}}$ satisfies $v_i' \xrightarrow{\ell'} v_j'$. We denote $G_{J_{\alpha}} \to G_{J_{\beta}}$.

(3) For any subgraph sequence $G_{J_{\alpha_1}}, \ldots, G_{J_{\alpha_i}}$ satisfying

$$G_{J_{\alpha_1}} \to G_{J_{\alpha_2}} \to \cdots \to G_{J_{\alpha_i}},$$

if there is an edge $\ell \in R(G)$ connecting $v_i \in G_{J_{\alpha_i}}$ and $v_j \in G_{J_{\alpha_j}}$ with $j > i$ then we have

$$v_i \xrightarrow{\ell} v_j, \quad G_{J_{\alpha_i}} \to G_{J_{\alpha_j}}.$$

(4) For any vertices $v_{j_0}, v_{j_1} \in G$, if there exist a walk of the form

$$v_{j_0}, e_{j_1}, v_{j_1}, e_{j_2}, \ldots, e_{j_i}, v_{j_i},$$

where each edge $e_{j_l}$, $1 \leq l \leq i$, is the normal edge, then $v_{j_0}, v_{j_i}$ belong to the same subgraph; otherwise $v_{j_0}, v_{j_i}$ belong to different subgraphs.

Let $v_k \xrightarrow{\ell} v_j$, $v_k \in G_{I_i}$ (or $G_{J_{\alpha}}$), $v_j \in G_{J_{\beta}}$. When we consider the subgraphs $G_{I_i}, G_{J_{\beta}}$ we attach a half edge $\ell^+$ to $v_k$ and attach $\ell^-$ to $v_j$.

Let $(V(G), E(G))$ be a graph and $\mathfrak{d}$ be an oriented decomposition satisfying Assumption 1 and Assumption 2. We call the graph $G$ a graph with a oriented decomposition $\mathfrak{d}$, denoted by $(V(G), E(G), \mathfrak{d})$.

**Definition 5.9.** Let $g$, $m$, and $\nu$ be nonnegative integers. A $(g, m + \nu, \mathfrak{d})$-weighted dual graph $G$ consists of $(V(G), E(G), \mathfrak{d})$ together with three weights, where

(1) $(V(G), E(G))$ is a graph, and $\mathfrak{d}$ is an oriented decomposition of $V(G),$

(2) $g : V(G) \to \mathbb{Z}_{\geq 0}$ assigning a nonnegative integer $g_v$ to each vertex $v$ such that

$$g = \sum_{v \in V(G)} g_v + b_1(G)$$

where $b_1(G)$ is the first Betti number of the graph $G$;

(3) assign $m$ ordered tails $m = (t_1, \ldots, t_m)$ to $V(G)$ : attach $m_v$ tails to $v$ for each $v \in V(G),$

(4) assign $\nu$ ordered half edges $l = (e_1, \ldots, e_\nu)$ to $V(G)$ : attach $l_v$ half edges to $v$ for each $v \in V(G).$
We denote the \((g, m + \nu, \varnothing)\)-weighted dual graph \(G\) by \((V(G), E(G), g, m, l, \varnothing)\).

We introduce a terminology: a vertex \(v\) is called an interior vertex if there is no half edge attached it, it is called boundary vertex if there are some half edges attached it.

**Definition 5.10.** Let \(g, m\) and \(\nu\) be nonnegative integers. A \(H-(g, m + \nu, \varnothing)\) weighted dual graph \(G\) consists of \((V(G), E(G), g, m, l, \varnothing)\) together with three weights:

1. \(\mathfrak{h} : V(G) \to H_2(M^+, \mathbb{R}, \mathbb{Z}) \cup H_2(M^+, \mathbb{R}, \mathbb{Z})\) assigning a \(A_v \in H_2(M^+, \mathbb{R}, \mathbb{Z})\) to each boundary vertex, assigning a \(A_v \in H_2(M^+, \mathbb{R}, \mathbb{Z})\) to each interior vertex \(v \in \mathfrak{A}\), assigning a 0 to each interior vertex \(v \in \mathfrak{B}\). For any \(G_{I_i}\) and \(G_{J_{\alpha}}\), denote

\[
A_{I_i} = \sum_{v \in V(G_{I_i})} A_v \in H_2(M^+, \mathbb{R}, \mathbb{Z}), \quad A_{J_{\alpha}} = \sum_{v \in V(G_{J_{\alpha}})} A_v \in H_2(M^+, \mathbb{R}, \mathbb{Z}),
\]

and

\[
A = \sum_{i=1}^{c} A_{I_i} + \sum_{\alpha=1}^{d} A_{J_{\alpha}}.
\]

2. assign \(\nu\) ordered weights \(k = (k_1, \ldots, k_{\nu})\) and weights \(\epsilon = (\epsilon_1, \ldots, \epsilon_{\nu})\) to the half edges \(l = (e_1, \ldots, e_{\nu})\) such that \(l\) becomes weighted half edges \((k, \epsilon) = (k_1, \epsilon_1, \ldots, (k_{\nu}, \epsilon_{\nu})\).

3. \(\mathfrak{t} : R(G) \to \mathbb{Z}^+\), for each \(\ell \in R(G)\) with \(v_k \xrightarrow{\ell} v_j\), \(v_k \in G_{I_i}\) (or \(G_{J_{\alpha}}\)), \(v_j \in G_{J_{\beta}}\), assigning \(\mathfrak{t}(\ell) = k_{\ell} > 0\) such that

\[
\sum_{e_j \in G_{I_i}} k_j + \sum_{\ell^+ \in G_{I_i}} k_{\ell} > 0, \text{ for any } G_{I_i}
\]

and

\[
d\lambda(A_{J_{\alpha}}) = \sum_{e_j \in G_{J_{\alpha}}} k_j \cdot T_{e_j} + \sum_{\ell^+ \in G_{J_{\alpha}}} k_{\ell^+} \cdot T_{\ell^+} - \sum_{\ell^- \in G_{J_{\alpha}}} k_{\ell^-} \cdot T_{\ell^-}, \text{ for any } G_{J_{\alpha}}.
\]

and for each boundary vertex \(v \in G_{J_{\alpha}}\)

\[
d\lambda(A_v) = \sum_{e_i^+ \in \ell^+_v} k_{i^+} \cdot T_{e_i^+} - \sum_{e_i^- \in \ell^-_v} k_{i^-} \cdot T_{e_i^-}, \quad (86)
\]

where \(\ell^+_v\) is the subset of \(\ell^+_v\), the half edges attached to \(v\).

We denote the \(H-(g, m + \nu, \varnothing)\) weighted dual graph \(G\) by \((V(G), E(G), g, m, l(k, \epsilon), \mathfrak{t}, \mathfrak{h}, \varnothing)\).

By a leg of \(G\) we mean either a tail or a half-edge.

**Definition 5.11.** Let \(G\) be a \((V(G), E(G), g, m, l(k, \epsilon), \mathfrak{t}, \mathfrak{h}, \varnothing)\) graph. A vertex \(v\) is called stable if one of the following holds:

1. \(2g_v + \text{val}(v) \geq 3\), where \(\text{val}(v)\) denotes the sum of the number of legs attached to \(v\);
2. \(A_v \neq 0\), when \(v \in \mathfrak{A}\);
3. \(d\lambda(A_v) \neq 0\) when \(v \in \mathfrak{B}\).

\(G\) is called stable if all vertices are stable.
Let $G$ be a stable $(V(G), E(G), g, m, ℓ^{(k, ϵ)}, ℓ, h, δ)$ graph. Then all subgraphs $G_1, G_{J_0}$ are stable.

Two $H$-$(g, m + ν, δ)$ weighted dual graphs $G_1$ and $G_2$ are called isomorphic if there exists a bijection $T$ between their vertices and edges keeping oriented decomposition and all weights.

Let $S_{g,m,ℓ^{(k, ϵ)},ℓ,h,δ}$ be the set of isomorphic classes of $H$-$(g, m + ν, δ)$ weighted dual graphs. Given $g, m, ν, A ∈ H_2(M^+, ℜ, ℤ)$ and two weights $k = (k_1, ..., k_ν), ϵ = (ϵ_1, ⋯, ϵ_ν)$, denote by $S_{g,m+ν,A,ℓ^{(k, ϵ)}}$ the union of all possible $S_{g,m,ℓ^{(k, ϵ)},ℓ,h,δ}$.

5.6 Li-Ruan’s Compactification

In this section we state Li-Ruan’s compactification on the moduli space of maps in terms of language of graphs.

Let $G$ be a stable $(V(G), E(G), g, m, ℓ^{(k, ϵ)}, ℓ, h, δ)$ graph with $N$ vertices $(v_1, ..., v_N), m$ tails and $ν$ half edges, and $(Σ, y, p)$ be a semi-stable curve with $m$ marked points and $ν$ puncture points. Let $A ∈ H_2(M^+, ℜ, ℤ)$. A stable $J$-holomorphic map of type $G$ is a quadruple

$$(u; Σ, y, p)$$

where $u : Σ → (M^+)′$ is a continuous map, $(M^+)′$ is obtained by attaching some $ℜ × Ā$ to $M^+$, satisfying the following conditions:

[A-1] $Σ = ∪_{v=1}^{N} Σ_v$, where each $v ∈ V(G)$ represents a smooth component $Σ_v$ of $Σ$.

[A-2] for the $i$-th tail attached to the vertex $v$ there exists the $i$-th marked point $y_i ∈ Σ_v$, $m_v$ is equal to the number of the marked points on $Σ_v$;

[A-3] for the $j$-th half edge attached to the vertex $v$ there exists $j$-th puncture point $p_j ∈ Σ_v$, $l_v$ is equal to the number of puncture points on $Σ_v$;

[A-4] if there is an edge connected the vertices $v$ and $w$, then there exists a node between $Σ_v$ and $Σ_w$, the number of edges between $v$ and $w$ is equal to the number of node points between $Σ_v$ and $Σ_w$;

[A-6] the restriction of $u$ to each component $Σ_v$ is $J$-holomorphic.

[A-7] $u$ converges exponentially to $(k_1 · T_{y_1}, ⋯, k_ν · T_{y_ν})$ periodic orbits $(x_{k_1}, ..., x_{k_ν})$ satisfying $x_{k_i} ∈ F_{y_i}$ as the variable tends to the puncture $(p_1, ..., p_ν)$; more precisely, $u$ satisfies $[27] - [29]$;

[A-8] let $q$ be a nodal point of $Σ$. Suppose $q$ is the intersection point of $Σ_v$ and $Σ_w$ associated to the edge $ℓ ∈ R(G)$. Then $k_ℓ > 0$, $u|Σ_v$ and $u|Σ_w$ converge exponentially to the same $k_ℓ$ periodic orbit $x_{k_ℓ}$ on $Ā$ as the variables tend to the nodal point $q$.

[A-9] For any $v ∈ V(G)$, $[u(Σ_v)] = A_v ∈ H_2(M^+, ℜ, ℤ)$ when $v$ is a boundary vertex; $[u(Σ_v)] = A_v ∈ H_2(M^+, ℜ, ℤ)$ when $v ∈ A$ is an interior vertex, $[u(Σ_v)] = 0$ when $v ∈ B$ is an interior vertex; $A = ∑_{i=1}^{v} A_v$.
Definition 5.12. Two stable $J$-holomorphic maps $\Gamma = (u, (\Sigma, j), y, p)$ and $\tilde{\Gamma} = (\tilde{u}, (\tilde{\Sigma}, \tilde{j}), \tilde{y}, \tilde{p})$ of type $G$ are called equivalent if there exists a diffeomorphism $\varphi : \Sigma \to \tilde{\Sigma}$ such that it can be lifted to bi-holomorphic isomorphisms $\varphi_v : (\Sigma_v, j_v) \to (\tilde{\Sigma}_w, \tilde{j}_w)$ for each component $\Sigma_v$ of $\Sigma$, and

1. $\varphi(y_i) = \tilde{y}_i$, $\varphi(p_j) = \tilde{p}_j$ for any $1 \leq i \leq l$, $1 \leq j \leq \nu$,
2. $u(\Sigma_v)$ and $\tilde{u} \circ \varphi(\Sigma_v)$ lie in the same holomorphic block for any $\Sigma_v$,
3. In case $v \in \mathfrak{A}$ we have $\tilde{u} \circ \varphi = u$ on $\Sigma_v$; In case $v \in \mathfrak{B}$, we have $\tilde{u} \circ \varphi = a + C$ on $\Sigma_v$.

Moreover, near every periodic orbit $x$, $\tilde{u}$ and $\tilde{u} \circ \varphi$ may differ by a canonical coordinate transformation.

Denote by $M_G$ the space of the equivalence class of stable $J$-holomorphic maps of type $G$.

Remark 5.13. Every subgraph $G_{I_i}$ determines a holomorphic block of type $G_{I_i}$ in $M^+$, every subgraph $G_{J_\alpha}$ determines a holomorphic rubber block of type $G_{J_\alpha}$ in $\mathbb{R} \times \tilde{M}$. Roughly speaking, $M_G$ is the gluing of several holomorphic blocks.

We define the automorphism group of $u$:

$$\text{Aut}(u) = \{ \varphi \mid \varphi : \Sigma \to \tilde{\Sigma} \text{ is a holomorphic isomorphism such that (1), (2) and (3) hold in Definition 5.12} \}$$

It is easy to see

Lemma 5.14. For any stable holomorphic map $u$ of type $G$ the automorphism group $\text{Aut}(u)$ is finite.

Given $g$, $m$, $\nu$, $A \in H_2(M^+, \mathbb{R}, \mathbb{Z})$, two weights $k = (k_1, ..., k_\nu)$ and $e = (e_1, \cdots, e_\nu)$, we define

$$\overline{M}_A(M^+, C; g, m + \nu, (k, e)) = \bigcup_{G \in S_{g, m + \nu, A, (k, e)}} M_G.$$ 

This gives a stratification of $\overline{M}_A(M^+, C; g, m + \nu, (k, e))$. Denote by $\mathcal{D}_{g, m + \nu, k, e}$ the number of all possible $S_{g, m + \nu, (k, e), f, h, \beta}$. The following Lemma is obvious.

Lemma 5.15. $\mathcal{D}_{g, m + \nu, k, e}$ is finite.

We immediately obtains

Theorem 5.16. $\overline{M}_A(M^+, C; g, m + \nu, (k, e))$ is compact.

Remark 5.17. Let $u : S^2 \to \mathbb{R} \times \tilde{M}$ be a $J$-holomorphic map with $\tilde{E}(u) = 0$. Then $u$ must be $u = (kT^s + d, x(kT + \vartheta_0))$. If the number of the special points (including nodal points, puncture points and marked points) $\leq 2$, $u$ is called an unstable second class ghost bubble. In our compactification there is no unstable second class ghost bubble.
6 Gluing theory-Pregluing

The gluing is the inverse of the degeneration. In section §3 we see that for a sequence \( \Gamma^{(i)} = (u^{(i)}, \Sigma^{(i)}; y^{(i)}, p^{(i)}) \in \mathcal{M}_A(M^+, C; g, m + \nu, (k, e)) \), each component degenerates independently, and the bubble trees are also constructed independently, furthermore each node of a connected component degenerates also independently. So we also glue independently for each nodal points. Recall that there are some freedoms of choosing the coordinates \( a, \vartheta \) for \( M^+ \) and \( \mathbb{R} \times \tilde{M} \) (see subsection §2.3).

For any \( r \) we glue \( M^+ \) and \( \mathbb{R} \times \tilde{M} \) with parameter \( r \) to get again \( M^+ \). We cut off the part of \( M^+ \) and \( \mathbb{R} \times \tilde{M} \) with cylindrical coordinate \( |a| > \frac{3r}{2} \) and glue the remainders along the collars of length \( lr \) of the cylinders with the gluing formulas:

\[
a_1 = a_2 + 2lr. \tag{87}
\]

6.1 Gluing one relative node for \( M^+ \cup (\mathbb{R} \times \tilde{M}) \)

Let \( b = (u_1, u_2; \Sigma_1 \land \Sigma_2, j_1, j_2) \in \mathcal{M}_A(M^+, C; g_1, m_1 + 1, (k, e_1)) \times \mathcal{M}_A(\mathbb{R} \times \tilde{M}, g_2, m_2 + 1, (k, e_1), (k, e_2)) \), where \((\Sigma_1, j_1)\) and \((\Sigma_2, j_2)\) are smooth Riemann surfaces of genus \( g_1 \) and \( g_2 \) joining at \( q \) and \( u_1 : \Sigma_1 \to M^+, u_2 : \Sigma_2 \to \mathbb{R} \times \tilde{M} \) are \( J \)-holomorphic maps such that \( u_i(z) \) converge to the same \( kT \)-periodic orbit \( x \) as \( z \to q \). To describe the maps \( u_i \) we choose a local pseudo-Darboux coordinate system \((a_i, \vartheta_i, w), i = 1, 2\), near \( x \). Suppose that

\[
a_i(s_i, t_i) - kT s_i - \ell_i \to 0 \quad \vartheta_i(s_i, t_i) - k t_i - \vartheta_0 \to 0.
\]

If we choose a different origin in the periodic orbit \( x \), we have a different coordinate system \( \vartheta_i^x \). Suppose that

\[
\vartheta_i^x = \vartheta_i + \tau_i.
\]

Obviously,

\[
\vartheta_i^x(s_i, t_i) - k t_i - (\vartheta_0 + \tau_i) \to 0.
\]

Without loss of generality we assume that \( \vartheta_0 = 0 \). Set \( \tau = -\vartheta_20 \). We consider \( \tau \) to be a parameter satisfying \( 0 \leq \tau < 1 \). Then

\[
\vartheta_1 = \vartheta_2 + \tau. \tag{88}
\]

Given gluing parameters \((r) = (r, \tau)\) we construct a surface \( \Sigma_{(r)} = \Sigma_{1\#(r)} \Sigma_2 \) with gluing formulas:

\[
s_1 = s_2 + \frac{2r}{Tk} \tag{89}
\]

\[
t_1 = t_2 + \frac{\tau + n}{k} \tag{90}
\]

for some \( n \in \mathbb{Z}_k \).

To get a pregluing map \( u_{(r)} \) from \( \Sigma_{(r)} \) we set

\[
u_{(r)} = \begin{cases} 
  u_1 & \text{on } \Sigma_{10} \cup \{(s_1, t_1)|0 \leq s_1 \leq \frac{jr}{2Tk}, t_1 \in S^1\} \\
  (kTs_1, x(kTt_1)) & \text{on } \{(s_1, t_1)|\frac{3jr}{4Tk} \leq s_1 \leq \frac{5jr}{4Tk}, t_1 \in S^1\} \\
  u_2 & \text{on } \Sigma_{20} \cup \{(s_2, t_2)|0 \geq s_2 \geq -\frac{jr}{2Tk}, t_2 \in S^1\},
\end{cases}
\]

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where \( x(kTt_1) = (kt_1, 0) \) as \( \vartheta_{10} = 0 \).

To define the map \( u_{(r)} \) in the remaining part we fix a smooth cutoff function \( \beta : \mathbb{R} \to [0, 1] \) such that

\[
\beta(s) = \begin{cases} 
1 & \text{if } s \geq 1 \\
0 & \text{if } s \leq 0
\end{cases}
\]

and \( \sqrt{1 - \beta} \) is a smooth function, \( |\beta'(s)| \leq 2 \). We assume that \( r \) is large enough such that \( u_i \) maps the tube \( \{(s_i, t_i) | |s_i| \geq \frac{r + \epsilon}{2}, t_i \in S^1 \} \) into a domain with pseudo-Darboux coordinates \( (a_i, \vartheta_i, w_i) \). We write \( u_{(r)} = (a_{(r)}, \tilde{u}_{(r)}) \) and define

\[
a_{(r)} = kT s_1 + \left( 3 - \frac{4Tks}{T} \right) (a_1(s_1, t_1) - kT s_1) + \beta(\frac{4Tks}{T} - 5)(a_2(s_2, t_2) - kT s_2),
\]

\[
\tilde{u}_{(r)} = x(kT t_1) + \left( 3 - \frac{4Tks}{T} \right) (\tilde{u}_1(s_1, t_1) - x(kT t_1)) + \beta(\frac{4Tks}{T} - 5)(\tilde{u}_2(s_2, t_2) - x(kT t_2)),
\]

where \( x(kT t_2) = (kt_2, 0) \) as \( (90) \) and \( (88) \). It is easy to check that \( u_{(r)} \) is a smooth function.

### 6.2 Gluing two relative nodal points for \( M^+ \cup (\mathbb{R} \times \tilde{M}) \)

Recall Lemma 5.8 we glue two relative nodal points independently.

Let \( b = (u_1, u_2; \Sigma_1 \setminus \Sigma_2, (j_1, j_2)) \in \mathcal{M}_A(M^+, C; g_1, m_1 + 2, (k, \epsilon)) \times \mathcal{M}_A(\mathbb{R} \times \tilde{M}; g_2, m_2 + 2, (k', \epsilon'), (k', \epsilon')) \), where \( k = k^-(k_1, k_2), \epsilon = \epsilon^- = (\epsilon_1, \epsilon_2) \), and \( k' \) satisfying \( \sum_{j=1}^{2} k_j < k' \).

Here \( (\Sigma_1, j_1) \) and \( (\Sigma_2, j_2) \) are smooth Riemann surfaces of genus \( g_1 \) and \( g_2 \) joining at \( q_1, q_2 \), and \( u_1 : \Sigma_1 \to M^+, u_2 : \Sigma_2 \to \mathbb{R} \times \tilde{M} \) are \( J \)-holomorphic maps such that \( u_j(z) \) converge to the same \( k_j T_{\epsilon_j} \)-periodic orbit \( x_j(k_j T_{\epsilon_j}) \) as \( z \to q_j, j = 1, 2 \) (see Figure 2). Choose cylinder coordinates \( (s_{1j}, t_{1j}) \) and \( (s_{2j}, t_{2j}) \) on \( \Sigma_1 \) and \( \Sigma_2 \) near node \( q_j \). We choose local pseudo-\( J \)-holomorphic coordinate systems \( (a_1, \vartheta_{1j}, w_j) \) on the cylinder end of \( M^+ \), \( (a_2, \vartheta_{2j}, w_j) \) on \( \mathbb{R} \times \tilde{M} \) near \( x_j \), where \( w_j \) is a local coordinates near \( x_j \). Suppose that

\[
a_1(s_{1j}, t_{1j}) - kT s_{1j} - \ell_{1j} \to 0, \quad \vartheta_{1j}(s_{1j}, t_{1j}) - kT_{\epsilon_{1j}} - \vartheta_{1j0} \to 0, \quad j = 1, 2,
\]

\[
a_2(s_{2j}, t_{2j}) - kT s_{2j} - \ell_{2j} \to 0, \quad \vartheta_{2j}(s_{2j}, t_{2j}) - kT_{\epsilon_{2j}} - \vartheta_{2j0} \to 0, \quad j = 1, 2.
\]

Without loss of generality we assume that \( \ell_{1j} = \ell_{2j} = 0 \), for \( j = 1, 2 \). Denote \( T_j = T_{\epsilon_j}, j = 1, 2 \).
For any parameter \( \varrho > 0 \), we can glue \( M^+ \) and \( \mathbb{R} \times \tilde{M} \) to get \( M^+_\varrho \) with gluing formula:

\[
a_1 = a_2 + 2l\varrho. \tag{93}
\]

For each \( x_j \) we take \( T \) translation:

\[
a_{1j} = a_1 - c_{1j}, \quad a_{2j} = a_2 - c_{2j} \tag{94}
\]

for some constants \( c_{1j} \geq 0 \). Then \((a_{1j}, \vartheta_{1j}, w_j)\) and \((a_{2j}, \vartheta_{2j}, w_j)\) are local coordinate systems near \( x_j \) over cylindrical end of \( M^+ \) and over \( \mathbb{R} \times \tilde{M} \).

Choose \( R_0 \) such that

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} \tilde{E}(u_i; |s_{ij}|) \geq \frac{R_0}{2} \leq \min\{h, T\}, \quad 2C_1c_1^{-1}e^{-\frac{\varrho_0}{4}} \leq \frac{h_1}{8}, \tag{95}
\]

where \( h, C_1, c \) and \( h_1 \) are the constants in Theorem 2.9 and Theorem 2.10.

For any \( r_j \geq R_0 \) we glue \( M^+ \) and \( \mathbb{R} \times \tilde{M} \) in coordinates \( a_{1j} \) and \( a_{2j} \) with the gluing formula:

\[
a_{1j} = a_{2j} + 2lr_j. \tag{96}
\]

Now we return to the coordinates \( a_i, i = 1, 2 \). By relation (94) the gluing formula can be re-written as

\[
a_1 = a_2 + c_{1j} - c_{2j} + 2lr_j. \tag{97}
\]

Choose

\[
c_{1j} = -c_{2j} = l(\rho - r_j). \tag{98}
\]

By (98), in order to get \( M^+_\varrho \), we need

\[
R_0 \leq r_j \leq \varrho, \quad j = 1, 2. \tag{99}
\]

Now we glue \( J \)-holomorphic maps. We express \((u_1, u_2)\) in terms of the coordinates \((a_{1j}, \vartheta_{1j})\) and \((a_{2j}, \vartheta_{2j})\):

\[
a_{1j}(s_{1j}, t_{1j}) - k_j T_j s_{1j} + c_{1j} \to 0, \quad \vartheta_{1j}(s_{1j}, t_{1j}) - k_j t_{1j} \to \vartheta_{1j0} \to 0, \quad j = 1, 2,
\]

\[
a_{2j}(s_{2j}, t_{2j}) - k_j T_j s_{2j} + c_{2j} \to 0, \quad \vartheta_{2j}(s_{2j}, t_{2j}) - k_j t_{2j} \to \vartheta_{2j0} \to 0, \quad j = 1, 2.
\]

As in subsection 6.1 we assume \( \vartheta_{1j0} = 0 \) and consider \( \tau_j = -\vartheta_{2j0} \) as parameters, and construct a surface \( \Sigma_{(r_j)} = \Sigma_1 \#_{(r_j)} \Sigma_2 \) with gluing formulas:

\[
s_{1j} = s_{2j} + \frac{2lr_j}{k_j}, \quad j = 1, 2, \tag{100}
\]

\[
t_{1j} = t_{2j} + \frac{\tau_j + n_j}{k_j}, \quad j = 1, 2, \tag{101}
\]

for some \( n_j \in Z_{k_j}, j = 1, 2 \). By (98), in terms of \( \varrho \) the gluing formulas (100) and (101) can be written as

\[
s_{1j} = s_{2j} + \frac{2\varrho - c_{1j} + c_{2j}}{k_j T_j}, \quad j = 1, 2, \tag{102}
\]

\[
t_{1j} = t_{2j} + \frac{\tau_j + n_j}{k_j}, \quad j = 1, 2, \tag{103}
\]

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for some $n_j \in Z_{k_j}, j = 1, 2$. We construct pre-gluing map $u_{(r)}$ as in subsection 6.1. Then we have

$$u_{(r1)} = u_{(r2)} = u_1, \quad \text{in } \bigcap_{j=1}^2 \{|s_{1j}| \leq \frac{lr_j}{4T_j k_j} \},$$

$$u_{(r1)} = u_{(r2)} = u_2, \quad \text{in } \bigcap_{j=1}^2 \{|s_{2j}| \leq \frac{lr_j}{4T_j k_j} \}.$$  

The pre-gluing above can be generalized immediately to the case of gluing several nodal points.

6.3 Norms on $\mathcal{C}^\infty(\Sigma_{(r)}; u_{(r)}^* TM_{(r)}^+)$

We only consider the case of gluing one node, the other cases are the same. For any $\eta \in C^\infty(\Sigma_{(r)}; u_{(r)}^* TM_{(r)}^+ \otimes \wedge^{0,1})$, let $\eta_i$ be its restriction to the part $\Sigma_{i0} \cup \{(s_i, t_i) \mid |s_i| \leq \frac{3}{2} T k_s \}$, extended by zero to yield a section over $\Sigma_i$. Define

$$\|\eta\|_{p,\alpha, r} = \|\eta_1\|_{\Sigma_{1, p, \alpha}} + \|\eta_2\|_{\Sigma_{2, p, \alpha}}. \quad (104)$$

Denote the resulting completed spaces by $L^{p, \alpha}_{r, \Sigma}$. We define a norm $\| \cdot \|_{1, p, \alpha, r}$ on $\mathcal{C}^\infty(\Sigma_{(r)}; u_{(r)}^* TM_{(r)}^+)$. For any section $h \in C^\infty(\Sigma_{(r)}; u_{(r)}^* TM_{(r)}^+)$, denote

$$h_0 = \int_{\Sigma_1} h \left( \frac{lr}{T k \cdot t} \right) dt, \quad (105)$$

$$h_1 = (h - h_0) \beta \left( \frac{3}{2} - \frac{T k s}{l r} \right), \quad (106)$$

$$h_2 = (h - h_0) \left[ 1 - \beta \left( \frac{3}{2} - \frac{T k s}{l r} \right) \right]. \quad (107)$$

We define

$$\|h\|_{1, p, \alpha, r} = \|h_1\|_{\Sigma_{1,1, p, \alpha}} + \|h_2\|_{\Sigma_{2,1, p, \alpha}} + |h_0|. \quad (108)$$

Denote the resulting completed spaces by $W^{1, p, \alpha}_{r, \Sigma}$. We introduce some notations. Choose the cylinder coordinates near puncture points and nodal points. Denote

$$\mathcal{D}_i(R_0) = \Sigma_{i0} \cup \{(s_i, t_i) \mid |s_i| \leq R_0 \}, \quad i = 1, 2,$$

$$\mathcal{D}(R_0) = \mathcal{D}_1(R_0) \cup \mathcal{D}_2(R_0).$$

Choose $R_0$ such that

$$\sum_{i=1}^2 \mathcal{E}(u_i; |s_i| \geq \frac{R_0}{2}) \leq \frac{\min\{h, T\}}{8}, \quad 4c_1 \epsilon_1^{-1} e^{-\frac{\epsilon_0}{4}} \leq \frac{h_1}{8}, \quad (109)$$

We let $\frac{h}{T} >> 4R_0$.

Lemma 6.1. There exists $\epsilon > 0$ such that for any $J$-holomorphic map $v : \Sigma_{(r)} \to M_{(r)}^+$ with $v = \exp u_{(r)}(h)$, if

$$|h|_{\mathcal{D}_i(R_0)} \mid_{C^1} \leq \epsilon \text{ for } i = 1, 2,$$

then for any $0 < \alpha < \epsilon$

$$\|h\|_{1, p, \alpha, r} \leq \frac{h_1}{4},$$

where $\epsilon$ is the constant in Theorem 2.3.
Proof. Note that
\[
\bar{E}(v; |s_i| \geq R_0/2) = \int_{R_0/2 \leq s_1 \leq \frac{R_0}{2}} v^* d\lambda = \int_{v(R_0/2, S^1)} \lambda - \int_{v(R_0/2, S^1)} \lambda.
\]
Denote \(s_0 = R_0/2\). A direct calculation gives us
\[
\left| \int_{v(s_0, S^1)} \lambda - \int_{u_1(s_0, S^1)} \lambda \right| = \left| \int_{S^1} \lambda(v_t)(s_0, t) dt - \int_{S^1} \lambda((u_1)_t)(s_0, t) dt \right|
\leq \left| \int_{S^1} \lambda(P_{u_1, v}(u_1)_t)(s_0, t) dt - \int_{S^1} \lambda((u_1)_t)(s_0, t) dt \right| + \left| \int_{S^1} \lambda(d\exp_{u_1} h_t)(s_0, t) dt \right|
\leq C|h|D_1(R_0)|C^1.
\]
Then
\[
\bar{E}(v; |s_i| \geq R_0/2) \leq \bar{E}(u_i; |s_i| \geq R_0/2) + C \sum |h|D_1(R_0)|C^1.
\]
It follows that
\[
\bar{E}(v; \Sigma(v) - \mathcal{D}(R_0/2)) \leq \frac{\min\{h, T\}}{4},
\]
when \(0 < \epsilon < \frac{\min\{h, T\}}{8C}\). Then by Theorem 2.11 we have
\[
|\nabla v |_{\Sigma(v) - \mathcal{D}(R_0)} | \leq C_1 e^{-\epsilon R_0/2}
\]
for all \(R_0 \leq s_1 \leq \frac{R_0}{2} - R_0\). Together with (110) we have
\[
\|h |_{\mathcal{D}(R_0)} \|_{1,p,\alpha} \leq \frac{h_1}{8}
\]
when \(\epsilon\) small enough. Then lemma follows.\(\square\)

The lemma can be generalized to the case of gluing several Riemann surfaces and several nodal points.

7 Gluing theory–Regularization

7.1 Local regularization

We fist discuss the top strata, then consider lower stratas.

7.1.1 Top strata.

Let \(b = (u, (\Sigma, j); y, p) \in \overline{\mathcal{M}}_A(M^+, C; g, m + \nu, (k, \epsilon))\). Here \(\Sigma\) is a smooth Riemann surface of genus \(g\), \(j\) is a complex structure (including marked points ), which is standard near each puncture point. We introduce the holomorphic cylindrical coordinates on \(\Sigma\) near each puncture points \(p_i\).

We discuss two different cases:

(1.1) \(\Sigma\) is stable.
Denote by $O_j$ a neighborhood of complex structures on $(\Sigma, j)$. Note that we change the complex structure in the compact set $\mathfrak{D}(R_0)$ of $\Sigma$ away from the puncture points. A neighborhood $U_b$ of $b$ can be described as

$$O_j \times \{ \exp_u(h + \hat{h}_0); h \in C(\Sigma; u^*TM^+), h_0 \in \mathbb{H}, \|h\|_{1,p,\alpha} + |h_0| < \epsilon \}/\text{stab}_b,$$

where $\mathbb{H} = \bigoplus_{j=1}^r (T_{p_j}(\mathcal{F}_j) \oplus \text{span}\{ \frac{\partial}{\partial \theta_j} \}).$

(1.2) $\Sigma$ is unstable.

In this case the automorphism group $\text{Aut}_\Sigma$ is infinite. One must construct a slice of the action $\text{Aut}_\Sigma$ and construct a neighborhood $U_b$ of $b$. This is done in [7]. We omit it here.

There is a neighborhood $U$ of $b$ such that $\mathcal{E}$ is trivialized over $U$, more precisely, $P(u, v)$ gives an isomorphism $\mathcal{E}|_u \to \mathcal{E}|_v$ for any $v \in U$. Choose $K_b \subset \mathcal{E}|_u$ to be a finite dimensional subspace such that

$$K_b + \text{image}D_u = \mathcal{E}|_u,$$

where $D_u : W^{1,p,\alpha}(u^*TM^+) \to \mathcal{E}|_u$. Without loss of generality we may assume that every element of $K_b$ is smooth along $u$ and supports in the common compact subset $\mathfrak{D}(R_0)$. Define a thickened Fredholm system $(K_b \times U, K_b \times \mathcal{E}|_U, S_b)$, where

$$S(\kappa, v) = \tilde{\partial}_f v + P_{u,\nu\kappa} \in \mathcal{E}|_v, \forall (\kappa, v) \in K_b \times U.$$

By (113), there exists a smaller neighborhood of $b$, still denoted by $U$, such that $DS_{(\kappa, v)} : K_b \times W^{1,p,\alpha} \to L^{p,\alpha}$ is surjective for any $(\kappa, v) \in K_b \times U$. Then for each $b' = (\kappa, v) \in K_b \times U$ there exists a right inverse

$$Q_{b'} : L^{p,\alpha} \to K_b \times W^{1,p,\alpha}$$

such that $\|Q_{b'}\| \leq C_1$. Obviously, $DS_{b'} : K_b \times W^{1,p,\alpha} \to L^{p,\alpha}$ is also surjective, $Q_{b'} : L^{p,\alpha} \to K_b \times W^{1,p,\alpha}$ is also a right inverse of $DS_{b'}$.

A pair $(\kappa, v) \in K_b \times U$ is called a perturbed $J$-holomorphic map if

$$\tilde{\partial}_f v + P_{u,\nu\kappa} = 0.$$

7.1.2 Lower strata.

We shall consider two cases for simplicity, the discussion for general cases are the same. Let $D$ be a strata whose domain has two components $(\Sigma_1, j_1)$ and $(\Sigma_2, j_2)$ joining at $p$.

**Case 1.** Let $b = (u_1, u_2; \Sigma_1 \wedge \Sigma_2, j_1, j_2)$, where $(\Sigma_1, j_1)$ and $(\Sigma_2, j_2)$ are smooth Riemann surfaces of genus $g_1$ and $g_2$ joining at $p$ and $u_i : \Sigma_i \to M^+$ are $J$-holomorphic maps with $u_1(p) = u_2(p)$. Suppose that both $(u_i, \Sigma_i, j_i)$ are stable. A neighborhood $U_b^D$ of $b$ in the strata $D$ can be described as

$$O_{j_1} \times O_{j_2} \times \left\{ (\exp_{u_1}(h_1 + \hat{h}_{10}), \exp_{u_2}(h_2 + \hat{h}_{20})) \mid h_i \in W^{1,p,\alpha}(\Sigma; u_i^*TM^+), h_{10} = h_{20} \in T_{u_1(p)}M^+, \|h_i\|_{1,p,\alpha} + |h_0| < \epsilon \}/\text{stab}_u \right\}.$$

The fiber of $\mathcal{E}_D(M^+, g, m+\nu) \to B_D(M^+, g, m+\nu)$ at $b$ is $L^{p,\alpha}(u_1^*TM^+ \otimes \wedge^{0,1}) \times L^{p,\alpha}(u_2^*TM^+ \otimes ^{0,1})$.
Case 2. Let $b = (u_1, u_2; \Sigma_1 \cup \Sigma_2, j_1, j_2)$, where $(\Sigma_1, j_1)$ and $(\Sigma_2, j_2)$ are smooth Riemann surfaces of genus $g_1$ and $g_2$ joining at $q$, $u_1 : \Sigma_1 \to M^+$ and $u_2 : \Sigma_2 \to \mathbb{R} \times \tilde{M}$ are $J$-holomorphic maps such that $u_1$ and $u_2$ converge to a same $kT$-periodic orbit $x(kTt) \subset \tilde{F}_j$ for some $j$ when $z_1$ and $z_2$ converge to $q$. Suppose that both $(u_1, \Sigma_1, j_1)$ and $(u_2, \Sigma_2, j_2)$ are stable.

We first describe the neighborhood of $u_2 : \Sigma_2 \to \mathbb{R} \times \tilde{M}$ in the space of $W^{1,p,\alpha}$-maps. The $R$-action on $\mathbb{R} \times \tilde{M}$ and $S^1$-action on every periodic orbit naturally induce the actions on $W^{1,p,\alpha}(\Sigma_2, \mathbb{R} \times \tilde{M})$, which we need mod. Write $u(s, t) = (a(s, t), \tilde{u})$. For any $C$, the $R$-action is defined as

$$
\mathbb{R}_C \circ u(s, t) = (C + a(s, t), \tilde{u}).
$$

We construct a local $R$-action slice as follows. We fix a point $p \in \Sigma_2$ and choose coordinates $a$ such that the $u_2(p) = (0, \star)$. Let $h$ be a vector field along $u_2$. We write

$$
h = b_1 \frac{\partial}{\partial a} + \tilde{h},
$$

where $\tilde{h} \in W^{1,p,\alpha}(\tilde{u}_2^*(\tilde{M}))$ is a vector field along $\tilde{u}_2$. Then $h = b_1 \frac{\partial}{\partial a} + \tilde{h}$ can be considered in a natural way as a $\mathbb{R}$-equivalent vector field along $\mathbb{R} \circ u$. We can construct an equivalent tubular neighborhood around the orbit $\mathbb{R} \circ u$. The point-wise exponential map give us a Banach manifold structure on the space of $\mathbb{R}$ equivalence class of $W^{1,p,\alpha}(\Sigma_2, \mathbb{R} \times \tilde{M})$. To simplify notations we will use $W^{1,p,\alpha}(\Sigma_2, \mathbb{R} \times \tilde{M})$ to denote the space of $\mathbb{R}$ equivalence class if no danger of confusion.

On the other hand, we need mod $S^1$-action on every periodic orbit. Let $x$ be a periodic orbit. We choose a local pseudo-Darboux coordinate system $(a, \vartheta, w)$, near $x$. If we choose another origin in $S^1$ we get another local pseudo-Darboux coordinate system $(a', \vartheta', w)$, which differ by a canonical coordinates transformation $[\mathbb{R}]$. The vector $h = b_1 \frac{\partial}{\partial a} + \tilde{h}$ is independent of the local pseudo-Darboux coordinates, so independent of the choice of the origin.

For simplicity we assume that $\Sigma_i, i = 1, 2$ has no puncture points. A neighborhood $\mathcal{U}_b^D$ of $b$ in $D$ can be described as

$$
O_{j_1} \times O_{j_2} \times \left\{ (\exp_{u_1}(h_1 + \tilde{h}_{10}), \exp_{u_2}(h_2 + \tilde{h}_{20}))\big| h_1 \in W^{1,p,\alpha}(\Sigma; u_1^* TM^+), \right.

\left. h_2 \in W^{1,p,\alpha}(\Sigma; u_2^* T(\mathbb{R} \times \tilde{M})), h_{10} = h_{20} \in \mathbb{R}, \quad \sum_{i=1}^2 \|h_i\|_{1,p,\alpha} + |h_{10}| < \epsilon \right\} / \text{stab}_{u_1}.
$$

where $\mathbb{H} = T_q(\mathcal{F}_j) \oplus (\text{span}\{ \frac{\partial}{\partial \vartheta} \})$. The fiber of $\mathcal{E}_D \to \mathcal{B}_D$ at $b$ is $L^{p,\alpha}(u_1^* TM^+ \otimes \Lambda^{0,1}) \times L^{p,\alpha}(u_2^*(\mathbb{R} \times \tilde{M}) \otimes \Lambda^{0,1})$.

We use the gluing argument to describe the neighborhoods $\mathcal{U}_b$ of $b$ in $\mathcal{B}_A(M^+, C; g, m + \nu, (k, \epsilon))$. Denote $Gl_{\mathcal{U}_b^D} = \{(r, \tau)| 0 \leq \tau < 1, T_0 \leq r \leq \infty \}$. Then

$$
\mathcal{U}_b = \bigcup_{T_0 \leq r < \infty} B(u_{(r)}), \epsilon
$$

where $B(u_{(r)}, \epsilon) = \{ u \in \mathcal{B} | u = \exp_{u_{(r)}}(h + \tilde{h}_0), \quad \|h\|_{1,p,\alpha} + |h_0| < \epsilon \}$. Choose $K_b = (K_1, K_2) \subset \mathcal{E}_b = (\mathcal{E}_{u_1}, \mathcal{E}_{u_2})$ to be a finite dimensional subspace such that

$$
K_1 + \text{image}D_{u_1} = \mathcal{E}_{u_1}, \quad K_2 + \text{image}D_{u_2} = \mathcal{E}_{u_2}.
$$
where \( D_{u_i} : W^{1,p,\alpha} \to \mathcal{E}_{u_i} \). We may assume that every element of \( K_i \) is smooth along \( u_i \) and supports in the compact subset \( \Omega(R_0) \) of \( \Sigma_i \). For any \((r, \tau) \in \text{Gl}_w T_0 \) there is a neighborhood \( U \) of \((\Sigma(r), u(r)) \) in \( B_A(M^+, C; g, m + \nu, (k, \epsilon)) \) such that \( \mathcal{E} \) is trivialized over \( U \). We choose \( T_0 > R_0 \) so large that for \( r > T_0 \)

\[ K_b \subset \mathcal{E}|_U. \]

Note that \( u(r) \|_{s_i} \leq R_0 = u_i \|_{s_i} \leq R_0 \) and \( \text{supp}(K_b) \subset \{|s_i| \leq R_0\} \). Then \( P(u(r), v)|_{s_i} \leq R_0 \) is independent of \( r \). Hence \( \kappa_u \) is well defined for any \( u \in B(u(r), \epsilon) \). We can naturally identify \( K_b \) and \( K_{b(r)} \).

We define a thickened Fredholm system and regularization equation as in (1). Then there exists a neighborhood \( U \) of \( b \), such that \( DS_{b(r)} : K_b \times W^{1,p,\alpha} \to L^{p,\alpha} \) is surjective for any \( b^r = (\kappa, v) \in K_b \times U \). Then \( DS_{b(r)} : K_b \times W^{1,p,\alpha} \to L^{p,\alpha} \) is also surjective.

### 7.2 Global regularization

Denote

\[ U_b^\epsilon = \{ v \in B | v = \exp_u(h + \hat{h}_0), \|h\|_{1,p,\alpha} + |h_0| < \epsilon \}. \]

As \( \overline{\mathcal{M}_A}(M^+, C; g_1, m_1 + \nu, (k, \epsilon)) \) is compact, there exist finite points \( b_i, \ 1 \leq i \leq n \) such that the collection \( \{U_{b_i}^\epsilon\} \) is an open cover of \( \overline{\mathcal{M}_A}(M^+, C; g_1, m_1 + \nu, (k, \epsilon)) \). For any \( I \subset \{1, 2, \ldots, n\} \), setting

\[ K_I = \prod_{i \in I} K_{b_i}, \quad S_I((\kappa_i)_{i \in I}, v) = \overline{\theta}_I(v) + \sum_{i \in I} \beta_{b_i} P_{u,v}(\kappa_i) \in \mathcal{E}_v. \]

One can construct a global regularization \((C_I, F_I, S_I)\) of the original Fredholm system \((B, \mathcal{E}, \overline{\theta}_I)\). From the global regularization we can obtain that (see \[78\]):

**Lemma 7.1.** There exists a finite dimensional virtual orbifold system for \((B, \mathcal{E}, \overline{\theta}_I)\) which is a collection of triples

\[ \{(U_I, E_I, \sigma_I)|I \subset \{1, 2, \ldots, n\}\} \]

indexed by a partially ordered set \((I = 2^{\{1,2,\ldots,n\}}, \subset)\), where

1. \( \{U_I|I \subset \{1, 2, \ldots, n\}\} \) is a finite dimensional proper étale virtual groupoid, where \( U_I = S_I^{-1}(0) \),
2. \( \{E_I\} \) is a finite rank virtual vector bundle over \( \{U_I\} \),
3. \( \{\sigma_I\} \) is a section of the virtual vector bundle \( \{E_I\} \) whose zeros \( \{\sigma_I^{-1}(0)\} \) form a cover of \( \overline{\mathcal{M}_A}(M^+, C; g_1, m_1 + \nu, (k, \epsilon)) \).

Let \( U_{I,\epsilon} \) be the set of points \((\kappa_i)_{i \in I}, v) \in U_I \) such that \( \sum_{i \in I} \|\kappa_i\|_{p,\alpha} \leq \epsilon \), where \( I \subset \{1, 2, \ldots, n\} \). We define \( U_{I,\epsilon}(\mathbb{R} \times \widetilde{M}), U_{I,\epsilon}(M^+ \cup (\mathbb{R} \times \widetilde{M})) \) in the same way.

By the same argument of Theorem 5.19 we conclude that:

**Theorem 7.2.** \( \{U_{I,\epsilon}(M^+)\} \) is a compact virtual manifold.
8 Gluing theory– analysis estimates

8.1 Gluing theory for 1-nodal case

We consider the case of gluing one node in Subsection 8.1. The general cases are similar.

Recall that we have a global regularization \{((C,F,S))\}. Consider the regular Fredholm system \((C,F,S_1)\). Let \(b = ((u_1,u_2), \Sigma_1 \wedge \Sigma_2; j_1, j_2) \in C\), where \((\Sigma_1, j_1)\) and \((\Sigma_2, j_2)\) are smooth Riemann surfaces of genus \(g_1\) and \(g_2\) joining at \(q\) and \(u_1 : \Sigma_1 \to M^+, u_2 : \Sigma_2 \to \mathbb{R} \times \tilde{M}\) are \(J\)-holomorphic maps such that \(u_i(z)\) converge to the same \(kT\)-periodic orbit \(x(kTt) \subset F_j\) for some \(j\) as \(z \to p\).

For any \((\kappa,h,h_0) \in \text{Ker} D_b\), where \(h \in W^{1,p,\alpha}(\Sigma; u^*TN)\), we define

\[
\| (\kappa,h) \|_{1,p,\alpha} = \| \kappa \|_{p,\alpha} + \| h \|_{1,p,\alpha}, \quad \| (\kappa,h,h_0) \| = \| (\kappa,h) \|_{1,p,\alpha} + |h_0|.
\]

For any \((\kappa,h,(r)) \in \text{Ker} D_{b(r)}\), we define

\[
\| (\kappa,h,(r)) \| = \| \kappa \|_{p,\alpha} + \| h_{(r)} \|_{1,p,\alpha}.
\]

By using the exponential decay of \(u_i\) one can easily prove that \(u_{(r)}\) are a family of approximate \(J\)-holomorphic map, precisely the following lemma holds.

**Lemma 8.1.** For any \(r > R_0\), we have

\[
\| \tilde{\partial}_J(u_{(r)}) \|_{p,\alpha} \leq Ce^{-(\epsilon-\alpha)r}. \tag{114}
\]

The constants \(C\) in the above estimates are independent of \(r\).

8.2 Estimates of right inverse

**Lemma 8.2.** Let \(D_b : K_b \times W^{1,p,\alpha} \to L^{p,\alpha}\) be a Fredholm operator defined in section 8.2. Suppose that \(D_b|_{K_b \times W^{1,p,\alpha}} : K_b \times W^{1,p,\alpha} \to L^{p,\alpha}\) is surjective. Denote by \(Q_b : L^{p,\alpha} \to K_b \times W^{1,p,\alpha}\) a right inverse of \(D_b\). Then \(D_{b(r)}\) is surjective for \(r\) large enough. Moreover, there are a right inverses \(Q_{b(r)}\) such that

\[
\begin{align*}
D_{b(r)} Q_{b(r)} &= \text{Id} \tag{115} \\
\|Q_{b(r)}\| &\leq C \tag{116}
\end{align*}
\]

for some constant \(C > 0\) independent of \(r\).

**Proof:** We first construct an approximate right inverse \(Q_{b(r)}'\) such that the following estimates holds

\[
\|Q_{b(r)}'\| \leq C_1 \tag{117}
\]

\[
\|D_{b(r)} \circ Q_{b(r)}' - \text{Id}\| \leq \frac{1}{2}. \tag{118}
\]

Then the operator \(D_{b(r)} \circ Q_{b(r)}'\) is invertible and a right inverse \(Q_{b(r)}\) of \(D_{b(r)}\) is given by

\[
Q_{b(r)} = Q_{b(r)}' (D_{b(r)} \circ Q_{b(r)}')^{-1} \tag{119}
\]
Since \( u \) by the exponential decay of \( \eta \), we have
\[
\| (\eta_1, \eta_2) \| = \eta_1 \eta, \quad \eta_2 = \eta_2 \eta.
\]
Let \( Q_b(\eta_1, \eta_2) = (\kappa_b, h) \). We may write \( h \) as \( (h_1, h_2) \), and define
\[
h(r) = h_1 + h_2 b_2.
\]
(120)

Note that on \( \left\{ \frac{lr}{2T} \leq s_1 \leq \frac{3lr}{2T} \right\} \), \( \kappa = 0 \) and on \( \{|s_2| \leq \frac{lr}{2T} \} \) we have \( u(r) = u_i, \kappa(r) = \kappa_b \), so along \( u(r) \) we have \( \kappa(r) = \kappa_b \). Then we define
\[
Q_i(\eta) = (\kappa(r), h(r)) = (\kappa_b, h(r)).
\]
(121)

Since \( |\beta_1| \leq 1 \) and \( |\frac{\partial \beta_i}{\partial s_1}| \leq CT \), (117) follows from \( \|Q_0\| \leq C \). We prove (118). Since \( \kappa + D_u h = \eta \), we have
\[
DS_{b(r)} \circ Q_b(\eta) = \eta, \quad |s_1| \leq \frac{lr}{2T}.
\]
(122)

It suffices to estimate the left hand side in the left annulus \( \frac{lr}{2T} \leq |s_2| \leq \frac{3lr}{2T} \). Note that in this annulus
\[
\beta_1 + \beta_2 = 1, \quad \kappa_b = 0, \quad D_u h_i = \eta_i,
\]
\[
\beta_1 D_u h_1 + \beta_2 D_u h_2 = (\beta_1^2 + \beta_2^2) \eta.
\]

Since near \( u_1(p) = u_2(p) \) (or near the periodic orbit \( x(kT) \)), \( D_u h_i = \delta_{b_i} + S u_i \), we have
\[
DS_{b(\eta)} \circ Q_{b(\eta)} - (\beta_1^2 + \beta_2^2) \eta = \kappa_{b(\eta)} + D_{b(\eta)} h(\eta) - (\beta_1^2 + \beta_2^2) \eta
\]
\[
= (\delta \beta_1) h_1 + \beta_1 (S_{u_1} - S_{u_2}) h_1 + (\delta \beta_2) h_2 + \beta_2 (S_{u_2} - S_{u_2}) h_2.
\]
(123)

By the exponential decay of \( S \) and \( \beta_1^2 + \beta_2^2 = 1 \) we get
\[
\left\| DS_{b(\eta)} \circ Q_{b(\eta)} - \eta \right\|_{p, a, r} \leq \left\| DS_{b(\eta)} \circ Q_{b(\eta)} - (\beta_1^2 + \beta_2^2) \eta \right\|_{p, a, r}
\]
\[
\leq \frac{C_1}{r} (\|h_1\|_{p, a} + \|h_2\|_{p, a}) \leq \frac{C_2}{r} \| \eta \|_{p, a, r}
\]
(124)

In the last inequality we used that \( \|Q_0\| \leq C \) and \( (h_1, h_2) = \pi_2 \circ Q_b(\eta_1, \eta_2) \). Then (118) follows by choosing \( r \) big enough. The estimate (118) implies that
\[
\frac{1}{2} \leq \| DS_{b(\eta)} \circ Q_{b(\eta)} \| \leq \frac{3}{2}.
\]
(125)

Then (116) follows. \( \square \)

### 8.3 Isomorphism between \( \text{Ker} DS_b \) and \( \text{Ker} DS_{b(r)} \)

Put
\[
E_1 := \text{Ker} DS_{u_1}, \quad E_2 := \text{Ker} DS_{u_2}, \quad \mathbb{H} = T_q(F) \oplus (\text{span}\{ \frac{\partial}{\partial a} \}),
\]

Denote
\[
\text{Ker} DS_b := E_1 \bigoplus E_2.
\]

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For a fixed gluing parameter \((r) = (r, \tau)\) we define a map \(I_r : Ker DS_b \rightarrow Ker DS_{b(r)}\). For any \((\kappa, h, h_0) \in Ker DS_b\), where \(h \in W^{1,p,\alpha}(\Sigma; u^*TN)\), we write \(h = (h_1, h_2)\), and define
\[
h_{(r)} = \hat{h}_0 + h_1 \beta_1 + h_2 \beta_2, \tag{126}
\]
\[
I_r(\kappa, h; h_0) = (\kappa, h_{(r)}) - Q_{b(r)} \circ DS_{b(r)}(\kappa, h_{(r)}). \tag{127}
\]

**Lemma 8.3.** \(I_r\) is an isomorphisms for \(r\) big enough.

**Proof:** The proof is basically a similar gluing argument as in [9]. The proof is divided into 2 steps.

**Step 1.** We define a map \(I'_r : Ker DS_{b(r)} \rightarrow Ker DS_b\) and show that \(I'_r\) is injective for \(r\) big enough. For any \((\kappa, h) \in Ker DS_{b(r)}\) we denote by \(h_i\) the restriction of \(h\) to the part \(|s_i| \leq \frac{lr}{kT} + 1\), we get a pair \((h_1, h_2)\). Let
\[
h_0 = \int_{S^1} h \left( \frac{lr}{kT}, t \right) dt. \tag{128}
\]
We denote
\[
\beta[h] = \left( (h_1 - \hat{h}_0) \beta \left( \frac{alr}{kT} + 1 - \alpha s_1 \right) + \hat{h}_0, (h_2 - \hat{h}_0) \beta \left( \frac{alr}{kT} + 1 + \alpha s_2 \right) + \hat{h}_0 \right)
\]
and define \(I'_r : Ker DS_{b(r)} \rightarrow Ker DS_b\) by
\[
I'_r(\kappa, h) = (\kappa, \beta[h]) - Q_b \circ DS_b(\kappa, \beta[h]), \tag{129}
\]
where \(Q_b\) denotes the right inverse of \(DS_b|_{K_b \times W^{1,p,\alpha}} : K_b \times W^{1,p,\alpha} \rightarrow L^{p,\alpha}\). Since \(DS_b \circ Q_b = DS_b|_{K_b \times W^{1,p,\alpha}} \circ Q_b = I\), we have \(I'_r(Ker DS_{b(r)}) \subset Ker DS_b\).

Since \(\kappa\) and \(D_u(\beta(h - \hat{h}_0))\) have compact support and \(S_u \in L^{p,\alpha}\), we have \(DS_b(\kappa, \beta[h]) \in L^{p,\alpha}\). Then \(Q_b \circ DS_b(\kappa, \beta[h]) \in K_b \times W^{1,p,\alpha}\).

Let \((\kappa, h) \in Ker DS_{b(r)}\) such that \(I'_r(\kappa, h) = 0\). Since \(\beta(h - \hat{h}_0) \in W^{1,p,\alpha}\) and \(Q_b \circ DS_b(\kappa, \beta[h]) \in K_b \times W^{1,p,\alpha}\), then \(I'_r(\kappa, h) = 0\) implies that \(h_0 = 0\). From \([129]\) we have
\[
\|I'_r(\kappa, h) - (\kappa, \beta[h])\|_{1,p,\alpha} \leq C_1 \|\kappa + D_u(\beta h)\|_{p,\alpha}
\]
\[
= C_1 \|\kappa + \beta \left( D_u h + D_u(\kappa) h + \kappa - D_u(\kappa) h - \kappa \right) + (\hat{\partial}\beta) h\|_{p,\alpha}
\]
Since \((\kappa, h) \in Ker DS_{b(r)}\), we have \(\kappa + D_u(\kappa) h = 0\). We choose \(\frac{lr}{2kT} > R_0\). As \(\kappa|_{s_1| \geq R_0} = 0\) and \(\beta|_{s_1| \leq \frac{lr}{kT}} = 1\) we have \(\kappa = \beta \kappa\). Therefore
\[
\|I'_r(\kappa, h) - (\kappa, \beta[h])\|_{1,p,\alpha} \leq C_1 \|(\hat{\partial}\beta)h + \beta(S_u - S_{u(r)}) h\|_{p,\alpha}.
\]
Note that
\[
S_u = S_{u(r)} \quad if \quad s_1 \leq \frac{lr}{2kT}, or \quad s_2 \geq -\frac{lr}{2kT}.
\]
By exponential decay of \(S\) we have
\[
\|(S_u - S_{u(r)}) \beta h\|_{p,\alpha} \leq C e^{-\frac{lr}{2kT}} \|\beta h\|_{1,p,\alpha}
\]

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for some constant \( C > 0 \). Since \( \partial \beta(\frac{alr}{kT} + 1 - \alpha s_1)h_1 \) supports in \( \frac{lr}{kT} \leq s_1 \leq \frac{lr}{kT} + \frac{1}{\alpha} \), and over this part

\[
|\partial \beta(\frac{alr}{kT} + 1 - \alpha s_1)| \leq 2|\alpha|
\]

\[
\beta(\frac{alr}{kT} + 1 + \alpha s_2) = 1, \quad e^{2|s_1|} \leq e^{4|s_2|},
\]

we obtain

\[
\|\partial \beta(\frac{alr}{kT} + 1 - \alpha s_1)\|_{p, \alpha} \leq 2|\alpha|e^{4\|\beta h\|_{p, \alpha}}.
\]

Similar inequality for \( \partial \beta(\frac{alr}{kT} + 1 + \alpha s_2)h_2 \) also holds. So we have

\[
\|\partial \beta h\|_{p, \alpha} \leq 4|\alpha|e^{4\|\beta h\|_{1, p, \alpha}}.
\]

Hence

\[
\|I'_r(\kappa, h) - (\kappa, \beta h)\|_{1, p, \alpha} \leq C_3(|\alpha| + e^{-\frac{dr}{4\alpha}})\|\beta h\|_{1, p, \alpha} \leq 1/2\|\beta h\|_{1, p, \alpha}
\]

for some constant \( C_3 > 0 \), here we chose \( 0 < \alpha < \frac{1}{4\alpha} \) and \( r \) big enough such that \( \frac{lr}{kT} > \frac{1}{\alpha} \) and \( C_3e^{-\frac{dr}{4\alpha}} < 1/4 \).

Then \( I'_r(\kappa, h) = 0 \) and (130) gives us

\[
\|\kappa\|_{p, \alpha} = 0, \quad \|\beta h\|_{1, p, \alpha} = 0.
\]

It follows that \( \kappa = 0, \ h = 0 \). So \( I'_r \) is injective.

**Step 2.** Since \( \|Q_{b(i)}\| \) is uniformly bounded, from (127) and (116), we have

\[
\|I_r((\kappa, h), h_0) - (\kappa, h)\|_{1, p, \alpha, r} \leq C\|DS_{b(i)}((\kappa, h), h)\|.
\]

By a similar calculation as in the proof of Lemma 5.2 we obtain

\[
\|I_r((\kappa, h), h_0) - (\kappa, h)\|_{1, p, \alpha, r} \leq \frac{C}{r}(\|h\|_{p, \alpha} + |h_0|).
\]

In particular, it holds for \( p = 2 \). It remains to show that \( \|h(\kappa, h, h_0)\|_{2, \alpha, r} \) is close to \( \|h\|_{2, \alpha} \). Denote \( \pi \) the projection into the second component, that is, \( \pi((\kappa, h), h_0) = h \). Then \( \pi(\ker DS_b) \) is a finite dimensional space. Let \( f_i, \ i = 1, ..., d \) be an orthonormal basis. Then \( F = \sum f_i^2 e^{2\|s_i|} \) is an integrable function on \( \Sigma \). For any \( \epsilon > 0 \), we may choose \( R_0 \) so big that

\[
\int_{|s_i| \geq R_0} F \leq \epsilon.
\]

Then the restriction of \( h \) to \( |s_i| \geq R_0 \) satisfies

\[
\|h\|_{|s_i| \geq R_0} \leq \epsilon\|h\|_{2, \alpha},
\]

therefore

\[
\|h(\kappa, h, h_0)\|_{2, \alpha, r} \geq \|h\|_{|s_i| \leq R_0} \geq |h_0| \geq (1 - \epsilon)\|h\|_{2, \alpha} + |h_0|,
\]

for \( r > R_0 \). Suppose that \( I_r((\kappa, h), h_0) = 0 \). Then (131) and (132) give us \( h = 0 \) and \( h_0 = 0 \), and so \( \kappa = 0 \). Hence \( I'_r \) is injective.

The **step 1** and **step 2** together show that both \( I_r \) and \( I'_r \) are isomorphisms for \( r \) big enough. □
8.4 Gluing maps

There is a neighborhood $U$ of $b_{(r)}$ in $B_{A}(M^{+}, C; g, m + \nu, (k, e))$ such that $E$ is trivialized over $U$, more precisely, $P_{u_{(r)}},v$ gives an isomorphism $E|_{u_{(r)}} \rightarrow E|_{v}$ for any $v \in U$.

Consider a map
\[
F_{(r)} : K_{b_{(r)}} \times \mathcal{W}_{r}^{1,p,\alpha}(\Sigma_{(r)}; u_{(r)}TM^{+}) \rightarrow L_{r}^{p,\alpha}(u_{(r)}TM^{+} \otimes \wedge^{0,1})
\]
\[
F_{(r)}(\kappa, h) = P_{\exp_{\kappa, h}}(\partial_{J} \exp_{\kappa, h} + \kappa).
\]

Let $(\kappa_{0}, h_{0}) : [0, \delta] \rightarrow K_{b_{(r)}} \times \mathcal{W}_{r}^{1,p,\alpha}(\Sigma_{(r)}; u_{(r)}TM^{+})$ be a curve satisfying
\[
\kappa_{0} = \kappa, \quad h_{0} = h, \quad \left. \frac{d}{d\tau} \kappa_{\tau} \right|_{\tau=0} = \eta, \quad \left. \frac{d}{d\tau} h_{\tau} \right|_{\tau=0} = g
\]

By the same method of [23] we can prove

Lemma 8.4. 1. $dF_{(r)}(0) = DS_{b_{(r)}}$.

2. There is a constant $h_{2} > 0$ such that for any $(\kappa, h) \in K_{b_{(r)}} \times \mathcal{W}_{r}^{1,p,\alpha}(\Sigma_{(r)}; u_{(r)}TM^{+})$ with $\| (\kappa, h) \| < h_{2}$ and $\| F_{\kappa_{0}, h_{0}}(\kappa, h) \|_{\rho, p, r} \leq 2h_{2}$, the following inequality holds
\[
\| dF_{(r)}(\kappa, h) - dF_{(r)}(0) \| \leq \frac{1}{4C_{2}}.
\]  \hspace{1cm} (133)

Let $h_{3} = \min\{h_{1}, h_{2}\}$. Note that $M^{+} = M_{0}^{+} \times ([0, \infty) \times \tilde{M})$. By the definition of the metrics $(.,.)$ (see (16) and (17)) the curvature $|Rm|$ is uniform bounded above. Then there exists a constant $C_{2} > 0$ such that $|\exp_{p}| \leq C_{2}$ for any $p \in M^{+}$. For any $\| (\kappa, h) \| \leq \frac{h_{3}}{8C_{2}}$,
\[
\| F_{(r)}(\kappa, h) \|_{\rho, p, r} = \| \partial_{J} \exp_{\kappa, h} + \kappa \|_{\rho, p, r}
\]
\[
\leq \| \partial_{J} u_{(r)} \|_{\rho, p, r} + \| \exp_{\kappa, h} \partial_{J} h \|_{\rho, p, r} + \| \exp_{\kappa, h} \partial_{J} h \|_{\rho, p, r} + \| \kappa \|_{\rho, p, r}
\]
\[
\leq \| \partial_{J} u_{(r)} \|_{\rho, p, r} + C_{2}\| (\kappa, h) \| \leq C_{1} e^{-(\varepsilon - \alpha)r} + \frac{h_{3}}{8C_{2}} \leq \frac{h_{3}}{4C_{2}}
\]  \hspace{1cm} (134)

when $r$ big enough. It follows from (2) of Lemma 8.4 and (134) that
\[
\| dF_{(r)}(\kappa, h) - DS_{b_{(r)}} \| \leq \frac{1}{4C_{2}}.
\]  \hspace{1cm} (135)

Then $F_{(r)}$ satisfies the conditions in Lemma 2.10. It follows that for any $(\kappa, h) \in \ker DS_{b_{(r)}}$ with $\| (\kappa, h) \| \leq \frac{h_{3}}{8C_{2}}$ there exists a unique $(\kappa_{v}, \zeta)$ such that $F_{(r)}(\kappa_{v}, \zeta) = 0$. Since $K_{b_{(r)}} \times \mathcal{W}_{r}^{1,p,\alpha}(\Sigma_{(r)}; u_{(r)}TM^{+}) = \operatorname{im} Q_{b_{(r)}} + \ker DS_{b_{(r)}}$ and $Q_{b_{(r)}}$ is injective, then there exists a unique smooth map
\[
f_{(r)} : \operatorname{Ker} DS_{b_{(r)}} \rightarrow L_{r}^{p,\alpha}(u_{(r)}TM^{+} \otimes \wedge^{0,1})
\]
such that $(\kappa_{v}, \zeta) = ((\kappa, h) + Q_{b_{(r)}} \circ f_{(r)}(\kappa, h))$.

On the other hand, suppose that $(\kappa_{v}, \zeta)$ satisfies
\[
F_{(r)}(\kappa_{v}, \zeta) = 0, \quad \| (\kappa_{v}, \zeta) \| \leq \frac{h_{3}}{8(1 + C_{2}\| DS_{b_{(r)}} \|)C_{2}}.
\]  \hspace{1cm} (136)
We write \((\kappa_v, \zeta) = (\kappa, h) + Q_{b(r)} \circ \eta\) where \((\kappa, h) \in \ker DS_{b(r)}\) and \(\eta \in L^p_{\alpha}(u^*_r TM^+ \otimes \wedge^{0,1})\). Since \((\kappa, h) = (I - Q_{b(r)} \circ DS_{b(r)})(\kappa_v, \zeta)\), we have \(\| (\kappa, h) \| \leq \frac{h^3}{8e_2} \); then \(\eta = f_r(\kappa, h)\), i.e.,

\[(\kappa_v, \zeta) = (\kappa, h) + Q_{b(r)} \circ f_r(\kappa, h).\] (137)

Hence the zero set of \(F_r\) is locally the form \(((\kappa, h), Q_{b(r)} \circ f_r(\kappa, h))\), i.e

\[F_r((\kappa, h) + Q_{b(r)} \circ f_r(\kappa, h)) = 0\] (138)

where \((\kappa, h) \in \ker DS_{b(r)}\).

For fixed \((r)\) denote

\[U_{b(r)} = \left\{ (\kappa, h) \in K_{b(r)} \times W^{1,p,\alpha,r}(\Sigma_{(r)}; u^*_r TM^+) \mid \partial f_r \exp_{\nu_{b(r)}} h + \kappa = 0 \right\},\]

\[(\kappa_r, \zeta_r) := I_r(\kappa, \zeta) + Q_{b(r)} \circ f_r \circ I_r(\kappa, \zeta)).\]

Define \(\pi_1, \pi_2\) by

\[\pi_1(\kappa, h) = \kappa, \text{ and } \pi_2(\kappa, h) = h, \text{ for any } (\kappa, h).\]

Since \(I_r : \ker DS_b \to \ker DS_{b(r)}\) is an isomorphism, we have proved the following

**Lemma 8.5.** There is a neighborhood \(O\) of 0 in \(\ker DS_b\) and a neighborhood \(O_j\) of \((j_1, j_2)\) and \(R_0 > 0\) such that

\[\text{glu}_{b(r)} : O_j \times O \times \mathbb{Z}_k \to U_{b(r)}\]

defined by

\[\text{glu}_{b(r)}((j_1, j_2, \kappa, \zeta, n) = (\pi_1(\kappa_r, \zeta_r), \exp_{b(r)}(\pi_2(\kappa_r, \zeta_r))))\]

where for \(r > R_0\) is a family of orientation preserving local diffeomorphisms.

We may choose \(((j_1, j_2), r, \tau, \kappa, \zeta)\) as a local coordinate system around \(b\) in \(U_{I, \epsilon}\). We write \(f_r \circ I_r\) as a function \(f((j_1, j_2, r, \tau, \kappa, \zeta))\). As \(I_r\) is a smooth map we can see that \(f((j_1, j_2, r, \tau, \kappa, \zeta)\) is smooth (see p.131 in [30]).

### 8.5 Surjectivity and injectivity

**Proposition 8.6.** The gluing map in Lemma [8.3] is surjective and injective in the sense of the Gromov-Uhlenbeck convergence.

**Proof.** The injectivity follows immediately from the implicit function theorem. We prove the surjectivity. Let \(\Gamma = ((\kappa_{v}, v), \Sigma_{(r)}) \in U_{b(r)}\) and \(\zeta \in W^{1,p,\alpha}_r\) such that \(\exp_{u_{b(r)}}(\zeta) = v\). Suppose that \(\|\kappa_v\|_{p,\alpha} + \|\zeta\|_{C^1(\Sigma_{(r)})} \leq \epsilon\). By the same argument of Lemma [6.1] we have \((\kappa, \zeta)\) satisfies (136). Then there exists a unique \((\kappa, h) \in \ker DS_b\) such that

\[(\kappa_v, \zeta) = I_r(\kappa, h) + Q_{b(r)} \circ f_r \circ I_r(\kappa, h)).\]

The surjectivity follows.
9 Estimates of differentiations for gluing parameters

In the section, let $\alpha << \epsilon$ be a constant. We estimate the differentiations for gluing parameters. This is another key point of this paper.

9.1 Linear analysis on weight sobolov spaces

Let $\Sigma = \mathbb{R} \times S^1$, $E = \Sigma \times \mathbb{R}^{2n}$. Let $\pi : E \to \Sigma$ be the trivial vector bundle. Consider the Cauchy-Riemannian operator $D$ on $E$ defined by

$$D = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S = \bar{\partial}_0 + S,$$

where $S(s,t) = (S_{ij}(s,t))_{2n \times 2n}$ is a matrix such that for any $k > 0$,

$$\sum_{i+j=k} \left| \frac{\partial^k S}{\partial^i s \partial^j t} \right| \leq C_k e^{-cs}.$$(139)

for some constant $C_k > 0$. Therefore, the operator $H_s = J_0 \frac{d}{dt} + S$ converges to $H_\infty = J_0 \frac{d}{dt}$ as $s \to \infty$.

We choose a small weight $\alpha$ for each end such that $H_\infty - \alpha = J_0 \frac{d}{dt} - \alpha$ is invertible. Let $W$ be the function in section 43. We can define the space $W^{1,p,\alpha}(\Sigma; E)$ and $L^{p,\alpha}(E \otimes \wedge^{0,1})$. Then the operator $D : W^{1,p,\alpha}(\Sigma; E) \to L^{p,\alpha}(E \otimes \wedge^{0,1})$ is a Fredholm operator so long as $\alpha$ does not lie in the spectrum of the operator $H_{i\infty}$ for all $i = 1, \ldots, \nu$.

By the observation of Donaldson we know that the multiplication by $W$ gives isometries from $W^{1,2,\alpha}$ to $W^{1,2}$ and $L^{2,\alpha}$ to $L^2$, and

$$\|f\|_{2,\alpha} = \|Wf\|_{L^2}, \quad C^{-1}\|Wf\|_{W^{1,2}} \leq \|f\|_{1,2,\alpha} \leq C\|Wf\|_{W^{1,2}}.$$(140)

Consider the operator

$$D : W^{1,2,\alpha} \to L^{2,\alpha}.$$

It is equivalent to the map (see [3], P59)

$$D_W := D - \frac{\partial}{\partial s} \log W = WDW^{-1} : W^{1,2} \to L^2.$$

Let $h \in W^{1,2,\alpha}$ be the solution of the equation $Dh = \eta$, where $\eta \in L^{2,\alpha}$. Obviously,

$$D_W(Wh) = WD(W^{-1}Wh) = W\eta.$$

Let $\rho = W\eta$, $f = Wh$. Then $f$ satisfies the equation $D_W f = \rho$.

When $s \geq R_0$ we consider the operator $D - \alpha : W^{1,2} \to L^2$. We write $D := D - \alpha = \frac{\partial}{\partial s} + L + S$, where $L := J_0 \frac{d}{dt} - \alpha$ is an invertible elliptic operator on $S^1$ when $0 < \alpha < 1$.

The space $L^2(S^1)$ can be decomposed into two infinite spaces $H^+$ and $H^-(\text{see P33 in [3]})$, where

$$H^+ = \left\{ \sum a_n e^{i\alpha n} | n \geq 0 \right\}, \quad H^- = \left\{ \sum a_n e^{i\alpha n} | n < 0 \right\}.$$

Since $L$ is an invertible operator, there exists a complete eigen-function space decomposition such that $L \phi_\lambda = \lambda \phi_\lambda$ where $|\lambda| > \delta$ for some $\delta > 0$. Obviously, $\lambda \in \mathbb{Z} - \alpha$. By choosing $0 < \alpha < 1/2$ we have $\alpha = \min |\lambda|$.
Using the same method of Donaldson in [9] we can prove the following lemmas (for details please see our next paper [20].)

For $A > 1$ we consider the finite tube $(-A, A) \times S^1$ and denote 

$$B_A^- = (-A, -A + 1) \times S^1, \quad B_A^+ = (A - 1, A) \times S^1.$$ 

**Lemma 9.1.** Let $h$ be a solution of $Dh = 0$ over the finite tube $(-A,A) \times S^1$. Then for any $0 < \alpha < \min\{\epsilon, \frac{1}{2}\}$, there exists a constant $C_1 > 0$ such that for any $0 < |s_o| + 1 < A$

$$\|h\|_{[-|s_o|+1,|s_o|-1]}S^1 \|_{1,p,\alpha} \leq C_1 e^{-\alpha(A-|s_o|)} \left( \|h\|_{B_A^+} \|_{2,\alpha} + \|h\|_{B_A^-} \|_{2,\alpha} \right). \quad (141)$$

Using this lemma we can prove the following corollary.

**Corollary 9.2.** Suppose that $D$ is surjective. Denote by $Q$ a bounded right inverse of $D$. Let $h = Q\eta$ be a solution of $Dh = \eta$ over the finite tube $(0, \frac{2\pi}{kT}) \times S^1$. Then for any $0 < \alpha < \min\{\epsilon, \frac{1}{2}\}$, there exists a constant $C_2 > 0$ such that for any $0 < A < \frac{2\pi}{kT}$

$$\|h\|_{(\frac{2\pi}{kT}, \frac{2\pi}{kT}) \times S^1} \|_{2,\alpha} \leq C_2 \left( e^{-\alpha A} \|\eta\|_{2,\alpha} + \|\eta\|_{\frac{2\pi}{kT}, \alpha} \leq \frac{2\pi}{kT} + A} \|_{2,\alpha} \right). \quad (142)$$

**Proof.** Denote by $\tilde{\eta}$ the restriction of $\eta$ to the part $\frac{2\pi}{kT} - A \leq s \leq \frac{2\pi}{kT} + A$. Let $\hat{h} = Q\tilde{\eta}$. Then $D(h - \hat{h}) = 0$ in $\frac{2\pi}{kT} - A \leq s \leq \frac{2\pi}{kT} + A$. By (141) and

$$\|\tilde{\eta}\|_{1,p,\alpha} = \|\eta\|_{\frac{2\pi}{kT}, \alpha} \leq \frac{2\pi}{kT} + A} \|_{2,\alpha} \quad \|\hat{h}\|_{1,p,\alpha} = \|Q\tilde{\eta}\|_{1,p,\alpha} \leq C\|\tilde{\eta}\|_{2,\alpha}.$$

we have

$$\|h\|_{(\frac{2\pi}{kT}, \frac{2\pi}{kT}) \times S^1} \|_{2,\alpha} \leq \|h - \hat{h}\|_{(\frac{2\pi}{kT}, \frac{2\pi}{kT}) \times S^1} \|_{2,\alpha} + \|\tilde{\eta}\|_{(\frac{2\pi}{kT}, \frac{2\pi}{kT}) \times S^1} \|_{2,\alpha} \leq C_1 \left( e^{-\alpha A} \|h - \hat{h}\|_{\frac{2\pi}{kT}, \alpha} \leq \frac{2\pi}{kT} + A} \|_{2,\alpha} \right).$$

Then the corollary follows from $\|h - \hat{h}\|_{1,2,\alpha} = \|Q(\eta - \tilde{\eta})\|_{1,2,\alpha} \leq C\|\tilde{\eta}\|_{2,\alpha}$. □

**9.2. Estimates of $\frac{\partial Q_{b(\cdot)}(\eta_r)}{\partial r}$**

**Lemma 9.3.** Let $DS_b : K_b \times W^{1,p,\alpha} \to L^{p,\alpha}$ be a Fredholm operator. Suppose that $DS_b|_{K_b \times W^{1,p,\alpha}} : K_b \times W^{1,p,\alpha} \to L^{p,\alpha}$ is surjective. Denote by $Q_b : L^{p,\alpha} \to K_b \times W^{1,p,\alpha}$ a right inverse of $DS_b$. Let $Q_{b(r)}$ be an right inverse of $DS_{b(r)}$ defined in (119). Then there exists a constant $C_3 > 0$, independent of $r$, such that $\left\| \frac{\partial}{\partial r} Q_{b(r)} \right\| \leq C_3$ and for any $0 < s_0 \leq \frac{2\pi}{kT}$

$$\left\| \frac{\partial}{\partial r} \left( Q_{b(r)}(\eta_r) \right) \right\| \leq C_3 \left[ \left\| \frac{\partial}{\partial r} \eta_r \right\|_{p,\alpha,r} + \frac{1}{r} \left\| \eta_r \right\|_{\frac{2\pi}{kT}, \leq |s| \leq \frac{2\pi}{kT}} \right] \left\| \eta_r \right\|_{p,\alpha,r} + e^{-\alpha \frac{2\pi}{kT}} \left\| \eta_r \right\|_{2,\alpha,r}, \quad (143)$$

for any $\eta_r \in L^{2,\alpha}_r$.

**Proof.** Given $\eta_r \in L^{2,\alpha}_r$, denote $\tilde{\eta}_r = (DS_{b(r)} \circ Q'_{b(r)})^{-1}\eta_r$, we have a pair $(\tilde{\eta}_1, \tilde{\eta}_2)$, where $\tilde{\eta}_1 = \beta_1 \eta_r$, $\tilde{\eta}_2 = \beta_2 \eta_r$. Set

$$Q_b(\tilde{\eta}_1, \tilde{\eta}_2) = (\tilde{a}_b, \tilde{b}_1, \tilde{b}_2), \quad \tilde{h}(r) = \tilde{h}_1 \beta_1 + \tilde{h}_2 \beta_2.$$
Then $Q_{b(r)}'\eta_r = Q_{b(r)}'\tilde{\eta}_r = (\tilde{\kappa}_b, \tilde{h}(r))$. So

$$\frac{\partial}{\partial r} \left( Q_{b(r)}' \eta_r \right) \bigg|_{s_i \leq \frac{\nu r}{2kr \alpha}} = \left( \frac{\partial \tilde{\kappa}_b}{\partial r} \sum \frac{\partial \tilde{h}_i}{\partial r} \beta_1 + \sum \tilde{h}_i \frac{\partial \beta_i}{\partial r} \right) \bigg|_{s_i \leq \frac{\nu r}{2kr \alpha}} = Q_b \left( \frac{\partial}{\partial r} \tilde{\eta}_1, \frac{\partial}{\partial r} \tilde{\eta}_2 \right) \bigg|_{s_i \leq \frac{\nu r}{2kr \alpha}},$$

where we used $\beta \bigg|_{s_i \leq \frac{\nu r}{2kr \alpha}} = 1$. As $\frac{\partial}{\partial r} \tilde{\eta}_1 = \beta_1 \frac{\partial}{\partial r} \tilde{\eta}_r + \frac{\partial \beta_1}{\partial r} \tilde{\eta}_r$, we have

$$\left\| \left( \frac{\partial \kappa}{\partial r}, \frac{\partial \tilde{h}_1}{\partial r}, \frac{\partial \tilde{h}_2}{\partial r} \right) \right\| = \left\| Q_b \left( \frac{\partial}{\partial r} \tilde{\eta}_1, \frac{\partial}{\partial r} \tilde{\eta}_2 \right) \right\| \leq C_1 \left\| \frac{\partial \tilde{\eta}_r}{\partial r} \right\|_{p, \alpha, r} + \frac{C_1}{r} \left\| \tilde{\eta}_r \bigg|_{s_i \leq \frac{\nu r}{2kr \alpha}} \right\|_{1, 2, \alpha, r}. \tag{144}$$

By the same calculation as (123) we have

$$\eta_r - \tilde{\eta}_r = DS_{b(r)}' \cdot Q_{b(r)}' \tilde{\eta}_r - (\beta_1^2 + \beta_2^2)\tilde{\eta}_r = \kappa_{b(r)} + D_{u(r)} \tilde{h}(r) - (\beta_1^2 + \beta_2^2)\tilde{\eta}_r \tag{145}$$

By the exponential decay $S_{u(r)}, S_{u_i}$ and

$$\tilde{\beta}|_{s_i \leq \frac{\nu r}{2kr \alpha}} = 0, \ |\tilde{\beta}| \leq \frac{C}{r}, S_{u(r)}|_{s_i \leq \frac{\nu r}{2kr \alpha}} = S_{u_i}|_{s_i \leq \frac{\nu r}{2kr \alpha}} \tag{146}$$

we conclude from (145) that

$$\left\| \tilde{\eta}_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} \leq \left\| \eta_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} + C_2 \left\| \tilde{h}_i \right\|_{\nu r \leq |s_i| \leq 3\nu r} \tag{147}$$

Applying Corollary 9.2 to (147) we have

$$\left( 1 - \frac{C_3}{r} \right) \left\| \tilde{\eta}_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} \leq \left\| \eta_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} + e^{-\alpha \frac{r}{kr \alpha}} \left\| \eta_r \right\|_{2, \alpha, r} \tag{148}$$

where we used $\| \tilde{\eta}_r \|_{p, \alpha, r} \leq 2 \| \eta_r \|_{p, \alpha, r}$. Taking the derivative of (145), by the exponential decay of $S_{u(r)}, S_{u_i}$, and $\left\| \frac{\partial}{\partial r} \tilde{\beta} \right\| \leq \frac{C}{r}$, using (146), (142) and (145), we have

$$\left\| \frac{\partial}{\partial r} \eta_r \right\|_{p, \alpha, r} \leq \left\| \frac{\partial}{\partial r} \tilde{\eta}_r \right\|_{p, \alpha, r} + C_4 \left\| \frac{\partial}{\partial r} \tilde{h}_i \right\|_{\nu r \leq |s_i| \leq 3\nu r} \left\| 1, p, \alpha \right\| + \frac{C_4}{r} \left\| \frac{\partial \tilde{h}_i}{\partial r} \right\|_{\nu r \leq |s_i| \leq 3\nu r} \left\| 1, p, \alpha \right\| 
\leq \left\| \frac{\partial}{\partial r} \eta_r \right\|_{p, \alpha, r} + \frac{C_5}{r} \left\| \tilde{\eta}_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} \left\| 1, p, \alpha \right\| + e^{-\alpha \frac{r}{kr \alpha}} \left\| \tilde{\eta}_r \right\|_{2, \alpha} \tag{149}$$

Following from (149) and (148) we obtain

$$C_1 \left\| \frac{\partial}{\partial r} \tilde{\eta}_r \right\|_{p, \alpha, r} + \frac{C_1}{r} \left\| \tilde{\eta}_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} \leq \left\| \eta_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} + e^{-\alpha \frac{r}{kr \alpha}} \left\| \tilde{\eta}_r \right\|_{2, \alpha} \tag{150}$$

(144) and (150) gives us

$$\left\| \frac{\partial}{\partial r} \left( Q_{b(r)}' \eta_r \right) \right\|_{s_i \leq s_0 \leq \frac{\nu r}{2kr \alpha}} \leq \left\| \frac{\partial}{\partial r} \eta_r \right\|_{p, \alpha, r} + \frac{C_1}{r} \left\| \tilde{\eta}_r \right\|_{\nu r \leq |s_i| \leq 7\nu r} \left\| 1, p, \alpha \right\| + e^{-\alpha \frac{r}{kr \alpha}} \left\| \eta_r \right\|_{2, \alpha} \tag{151}$$
Using (152), we have
\[\| \sum \tilde{h}_i \frac{\partial^2 h_i}{\partial r^2} \|_{1,p,a} \leq C_8 \left( e^{-\alpha \frac{lr}{8kt}} \| \eta \|_{1,2,a} + \| \eta \|_{\frac{lr}{8kt} \leq |s| \leq \frac{3lr}{8kt}} \| 2,a \right) \]
\[\leq C_9 \frac{r}{e^{-\alpha \frac{lr}{8kt}}} \| \eta \|_{1,2,a} + \| \eta \|_{\frac{lr}{8kt} \leq |s| \leq \frac{3lr}{8kt}} \| 2,a \right) \]
(152)

Note that \( \frac{\partial}{\partial r} (Q_{b(r)} \eta_r) = \left( \frac{\partial h_i}{\partial r}, \sum \frac{\partial h_i}{\partial r} \beta_i + \sum \tilde{h}_i \frac{\partial h_i}{\partial r} \right) \). Then (143) follows from (144), (150) and (152). \( \square \)

### 9.3 Estimates of \( \frac{\partial h_r}{\partial r} \)

**Lemma 9.4.** Let \( I_r : \text{Ker}DS_b \to \text{Ker}DS_{b(r)} \) be an isomorphisms defined in (127). Then there exists a constant \( C > 0 \), independent of \( r \), such that
\[\left\| \frac{\partial}{\partial r} (I_r(\kappa, h, h_0)) \right\|_{1,2,a,r} \leq C \frac{r}{e^{-\alpha \frac{lr}{8kt}}} \| \eta \|_{1,2,a} \leq C \frac{r}{e^{-\alpha \frac{lr}{8kt}}} \| \eta \|_{\frac{lr}{8kt} \leq |s| \leq \frac{3lr}{8kt}} \| 2,a \right) \]
(153)
for any \((\kappa, h, h_0) \in \text{Ker}DS_b \).

**Proof.** By definition \( \kappa \) is independent of \( r \). Then \( \frac{\partial}{\partial r} h_r = \sum \frac{\partial}{\partial r} \beta_i \eta_i \) and
\[\frac{\partial}{\partial r} I_r(\kappa, h, h_0) = (0, \frac{\partial h_r}{\partial r}) + \frac{\partial}{\partial r} \left( Q_{b(r)} \right) DS_{b(r)}(\kappa, h(r)) + Q_{b(r)} \frac{\partial}{\partial r} (DS_{b(r)}(\kappa, h(r))) \]

A direct calculation gives us
\[DS_{b(r)}(\kappa, h(r)) = (\tilde{\partial} \beta_1) h_1 + \beta_1 (S_{u(r)} - S_u) h_1 + (\tilde{\partial} \beta_2) h_2 + \beta_2 (S_{u(r)} - S_u) h_2 \]
(154)

By (146) we have \( \text{supp} DS_{b(r)}(\kappa, h(r)) \subset \{ \frac{lr}{8kt} \leq |s| \leq \frac{3lr}{8kt} \} \). Since \( |S_{u(r)}| + |S_u| \leq \frac{lr}{8kt} \leq |s| \leq \frac{3lr}{8kt} \leq Ce^{-\alpha \frac{lr}{8kt}} \), the lemma follows from \( \| Q_{b(r)} \| \leq C \), \( \| \frac{\partial}{\partial r} Q_{b(r)} \| \leq C \). \( \square \)

### 9.4 Estimates of \( \frac{\partial}{\partial r} \left[ I_r(\kappa, \zeta) + Q_{b(r)} f_r(I_r(\kappa, \zeta)) \right] \)

First we prove the following lemma.

**Lemma 9.5.** Let \((\kappa, \zeta) \in \text{ker}DS_b \) with \( \| (\kappa, \zeta) \| \leq h_3 \) such that
\[(\kappa, h(r)) = I_r(\kappa, \zeta) + Q_{b(r)} \circ f_r \circ I_r(\kappa, \zeta) \]
(155)
\[\tilde{\partial} J \text{exp}_{u(r)}(h_r) + \kappa = 0. \]
(156)

Then there exists a positive constants \( C \) such that
\[\| f_r \circ I_r(\kappa, \zeta) \|_{\frac{lr}{8kt} \leq |s| \leq \frac{3lr}{8kt}} \| p,a,r \leq C e^{-(\alpha \frac{lr}{8kt})} \| (\kappa, \zeta) \|, \quad \forall \frac{lr}{8kt} \geq R_0 \]
(157)

**Proof.** From (155) and \( \kappa \) \( |s| \geq R_0 = 0 \) we have
\[f_r(I_r(\kappa, \zeta)) = \left( \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} \right) h_r + S_{u(r)} h_r, \quad \tilde{\partial} J \text{exp}_{u(r)} h_r = 0, \]
(158)
for any $|s| \geq R_0$. By the assumption $u_i$ converges to $kT$-periodic orbit $x(kTt)$ as $z \to p$. In the given local pseudo-Dauboux coordinate $(a, \theta, w)$, we denote

$$\exp_{u(r)}(h(r)) = \left( \hat{a}(r)(s, t), \hat{\theta}(r)(s, t), \hat{w}(r)(s, t) \right).$$

By Theorem 2.7, (91) and (92) we have for any $|s| \geq R_0$

$$|\nabla(a(r) - kTs_i)| \leq Ce^{-\frac{c}{4kT}}, \quad |\nabla(\vartheta(r) - kt_i)| \leq Ce^{-\frac{c}{4kT}},$$

$$|\nabla(w(r))| \leq Ce^{-\frac{c}{4kT}}. \tag{159}$$

Let

$$a^0(r) = a(r) - kTs_i, \quad \vartheta^0(r) = \vartheta(r) - kt_i, \quad \hat{a}^0(r) = \hat{a}(r) - kTs_i, \quad \hat{\vartheta}^0(r) = \hat{\vartheta}(r) - kt_i.$$

By (159) and (160) we have

$$\int_{|s| \geq s_0} \|\nabla a^0(r)\|^2 + \|\nabla \vartheta^0(r)\|^2 + \|\nabla w(r)\|^2 dsdt \leq Ce^{-c s_o}, \quad \forall s_o \geq R_0. \tag{161}$$

Since $\|(\kappa, h(r))\| \leq Ch_3$ and $\kappa|_{|s| \geq R_0} = 0$, we have

$$\|h(r)|_{|s| \geq R_0}\|_{1, 2, \alpha} \leq Ch_3. \tag{162}$$

By the elliptic regularity we can get

$$|h(r)| \leq Ce^{-\alpha \min\{|s_1|, |s_2|\}}, \quad \forall |s| \geq R_0.$$ \hspace{1cm}

Then $\hat{w}(r)|_{|s| \geq R_0}$ is also in the pseudo-Dauboux coordinates when $\frac{lr}{4kT} > R_0$ and $h_3$ small. By (161)

$$\int_{|s| \geq s_0} \|\nabla \hat{a}^0(r)\|^2 + \|\nabla \hat{\vartheta}^0(r)\|^2 + \|\nabla \hat{w}(r)\|^2 dsdt \leq Ce^{-\alpha s_o}, \quad \forall s_o \geq R_0. \tag{163}$$

Then by Theorem 2.9 we have for any $\frac{lr}{4kT} \leq |s| \leq \frac{7lr}{4kT}$

$$|\nabla(\hat{a}(r) - kTs_i)| \leq Ce^{-\frac{c r}{4kT}}, \quad |\nabla(\hat{\vartheta}(r) - kt_i)| \leq Ce^{-\frac{c r}{4kT}},$$

$$|\nabla(\hat{w}(r))| \leq Ce^{-\frac{c r}{4kT}}. \tag{164}$$

Since $\nabla \exp_{u(r)}(h(r)) = P_{u(r), \exp_{u(r)}(h(r))}(\nabla u(r)) + d\exp_{u(r)}(\nabla h(r))$, (159), (160), (164) and (165) gives us

$$\left| \frac{d}{dt} \exp_{u(r)} \nabla h(r) \right| \frac{lr}{4kT} \leq |s| \leq \frac{7lr}{4kT} \leq Ce^{-\frac{c r}{4kT}}. \tag{166}$$

Note that $d\exp^{-1}_{u(r)}$ is uniform bounded as $\|h(r)\|_{1, p, \alpha, r}$ small. Then

$$\|\nabla h(r)\| \frac{lr}{4kT} \leq |s| \leq \frac{7lr}{4kT} \|p, \alpha \leq Ce^{-\alpha \frac{lr}{4kT}}. \tag{167}$$

The lemma follows from (158), (167) and the exponential decay of $S_{u(r)}$. □

**Lemma 9.6.** There exists a constant $C > 0$ such that,

$$\left| \frac{\partial}{\partial r} \left( I_r(\kappa, \zeta) + Q_{b(r)} \circ f_r \circ I_r(\kappa, \zeta) \right) \right|_{1, 2, \alpha, r} \leq Ce^{-\alpha \frac{lr}{4kT}} \|(\kappa, \zeta)\|_{1, 2, \alpha}. \tag{168}$$

Restricting to the compact set $\{|s| \leq R_0\}$, we have

$$\left| \frac{\partial}{\partial r} \left( I_r(\kappa, \zeta) + Q_{b(r)} \circ f_r \circ I_r(\kappa, \zeta) \right) \right| \leq Ce^{-\alpha \frac{lr}{4kT}} \|\zeta\|_{L^\infty}. \tag{169}$$
To simplify the expressions of formulas we denote

\[ \kappa \]

where \( \kappa_r \in K_{b(r)}, h_r \in C^\infty(\Sigma; u^*_r TM^+) \). We have

\[ \bar{\partial}_j \exp u_r h_r + \kappa_r = 0. \]  

To simplify the expressions of formulas we denote

\[ \frac{\partial}{\partial r} \exp u_r (h_r) = G, \quad \frac{\partial}{\partial r} \kappa_r = \Psi. \]

Taking differentiation of (171) with respect to \( r \) we get

\[ D_{\exp u_r (h_r)} G + \Psi = 0. \]  

We may write (172) as

\[ P_{\exp u_r (h_r)} \left( D S_{\exp u_r (h_r)} (\Psi, G) - D S_{b(r)} \left( \frac{\partial}{\partial r} (\kappa_r, h_r) \right) \right) \]

\[ + D S_{b(r)} \left( \frac{\partial}{\partial r} (\kappa_r, h_r) \right) = 0. \]

We estimate the difference

\[ P_{\exp u_r (h_r)} \left( D S_{\exp u_r (h_r)} (\Psi, G) - D S_{b(r)} \left( \frac{\partial}{\partial r} (\kappa_r, h_r) \right) \right). \]

We choose \( \|\eta\|_{1, p, \alpha, r} \) very small. From the Implicit function Theorem we have

\[ \| f_r (I_r (\kappa, \zeta)) \|_{p, \alpha, r} \leq C \|(\kappa, \zeta)\|_{1, p, \alpha} + \| \bar{\partial}_j u_r (\kappa) \| \]

for some constant \( C > 0 \). For any small \( \zeta \in \ker DS_{b}, exp u_r \zeta \) converges to a periodic orbit as \( |s| \to \infty \). It follows that \( S_{exp u_r \zeta} \) converges to zero exponentially. Since \( (\kappa, h_r) \) satisfies (174) with \( \kappa|_{|s|\geq \frac{r}{2kT}} = 0 \), by elliptic estimate we conclude that in the part \( \frac{r}{2kT} \leq |s| \leq \frac{3r}{2kT} \)

\[ |S_{\exp u_r (h_r)}| \leq C_0 e^{-\frac{|r|}{2kT}}, \quad |S_{u_r (h_r)}| \leq C_0 e^{-\frac{|r|}{2kT}} \]  

for \( r \) big enough. Moreover, we may choose \( r \) very large and \( |(\kappa, \zeta)| \) very small such that

\[ \| Q_{b(r)} \| P_{\exp u_r (h_r)} \left( D S_{\exp u_r (h_r)} (\Psi, G) - D S_{b(r)} \left( \frac{\partial}{\partial r} (\kappa_r, h_r) \right) \right) \|_{1, p, \alpha, r} \]

\[ \leq \epsilon \left\| \frac{\partial}{\partial r} (\kappa_r, h_r) \right\|_{1, p, \alpha, r} + C e^{-(\alpha)\frac{|r|}{2kT}}, \]

where \( \epsilon > 0 \) is a constant such that \( 4\epsilon C_4 \leq 1 \). Note that

\[ \frac{\partial}{\partial r} (\kappa_r, h_r) = \frac{\partial}{\partial r} I_r (\kappa, \zeta) + \frac{\partial}{\partial r} \left( Q_{b(r)} \circ f_r (I_r (\kappa, \zeta)) \right) + \left( Q_{b(r)} \frac{\partial}{\partial r} \circ f_r (I_r (\kappa, \zeta)) \right). \]
Then
\[ DS_{b(r)} \left( \frac{\partial}{\partial r} (\kappa_r, h_{(r)}) \right) = DS_{b(r)} \left( \frac{\partial}{\partial r} I_r(\kappa, \zeta) \right) + DS_{b(r)} \left( \frac{\partial}{\partial r} (Q_{b(r)}) f_r(I_r(\kappa, \zeta)) \right) + \frac{\partial}{\partial r} (f_r(I_r(\kappa, \zeta))). \]

It follows together with (174), (176) that
\[
\left\| \frac{\partial}{\partial r} (f_r(I_r(\kappa, \zeta))) \right\|_{p, \alpha, r} \leq \epsilon \left\| \frac{\partial}{\partial r} (\kappa_r, h_{(r)}) \right\|_{1, p, \alpha, r} + \left\| DS_{b(r)} \left( \frac{\partial}{\partial r} I_r(\kappa, \zeta) \right) \right\|_{p, \alpha, r} + \left\| DS_{b(r)} \circ \frac{\partial}{\partial r} (Q_{b(r)}) \circ f_r(I_r(\kappa, \zeta)) \right\|_{p, \alpha, r}. \tag{178} \]

For any \((\kappa_1, h_1)\), we have
\[
\frac{\partial}{\partial r} \left( DS_{b(r)} \right) (\kappa_1, h_1) = \frac{\partial}{\partial r} \left( D_{u(r)} h_1 + \kappa_1 \right) = \frac{\partial S_{u(r)}}{\partial r} h_1 = \left( 0, \frac{\partial S_{u(r)}}{\partial r} \right) \cdot (\kappa_1, h_1). \]

Then
\[
\frac{\partial}{\partial r} \left( DS_{b(r)} \right) = \left( 0, \frac{\partial S_{u(r)}}{\partial r} \right) \tag{179} \]

As \(I_r(\kappa, \zeta) \in \text{ker} DS_{b(r)}\) we have
\[ DS_{b(r)} \circ I_r(\kappa, \zeta) = 0. \]

Then
\[ DS_{b(r)} \left( \frac{\partial}{\partial r} I_r(\kappa, \zeta) \right) + \left( 0, \frac{\partial S_{u(r)}}{\partial r} \right) I_r(\kappa, \zeta) = 0. \]

Since \(\frac{\partial S_{u(r)}}{\partial r}\) supports in the part \(\frac{l_r}{2kT} \leq |s| \leq \frac{3l_r}{2kT}\), by the exponential decay of \(S_{u_i}\) we get
\[
\left\| DS_{b(r)} \left( \frac{\partial}{\partial r} I_r(\kappa, \zeta) \right) \right\|_{p, \alpha, r} \leq C_1 e^{-(c-\alpha) \frac{l_r}{2kT}}. \tag{180} \]

Taking the differentiation of the equality
\[ DS_{b(r)} \circ Q_{b(r)} = I \]
we get
\[ DS_{b(r)} \left( \frac{\partial}{\partial r} (Q_{b(r)}) \right) = - \left( 0, \frac{\partial S_{u(r)}}{\partial r} \right) \cdot Q_{b(r)}. \]

Together with (174) and (175) we get
\[
\left\| Q_{b(r)} \right\| \left\| DS_{b(r)} \frac{\partial}{\partial r} (Q_{b(r)}) f_r(I_r(\kappa, \zeta)) \right\|_{1, p, \alpha, r} \leq C_2 e^{-(c-\alpha) \frac{l_r}{2kT}} \tag{181} \]
and
\[
\left\| \frac{\partial}{\partial r} (f_r(I_r(\kappa, \zeta))) \right\|_{p, \alpha, r} \leq C_2 e^{-(c-\alpha) \frac{l_r}{2kT}} + \epsilon \left\| \frac{\partial}{\partial r} (\kappa_r, h_{(r)}) \right\|_{1, p, \alpha, r}. \tag{182} \]

By (170) we have
\[
\left\| \frac{\partial}{\partial r} (\kappa_r, h_{(r)}) \right\|_{1, p, \alpha, r} \leq \left( \left\| \frac{\partial}{\partial r} I_r(\kappa, \zeta) \right\|_{1, p, \alpha, r} + \left\| \frac{\partial}{\partial r} (Q_{b(r)} \circ f_r(I_r(\kappa, \zeta))) \right\|_{1, p, \alpha, r} \right) \tag{183} \]

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Together with (143) we have
\[
\Big\| \frac{\partial}{\partial r} (\kappa_r, h_{(r)}) \Big\|_{1,p,\alpha,r} \\
\leq C_3 \left( e^{-\alpha \frac{lr}{4kT}} + \left\| \frac{\partial}{\partial r} f_r(\kappa, \zeta) \right\| + \left\| \frac{\partial}{\partial r} (f_r(I_r(\kappa, \zeta))) \right\|_{1,p,\alpha,r} + \| f_r \circ I_r(\kappa, \zeta) \|_{\frac{1}{4kT} \leq |s_i| \leq \frac{7lr}{4kT}} \right) \\
\leq C_4 e^{-\alpha \frac{lr}{4kT}} + C_3 \alpha \left\| \frac{\partial}{\partial r} (\kappa_r, h_{(r)}) \right\|_{1,p,\alpha,r}.
\]
where we used (153), (157) and (182) in last inequality. Then the Lemma follows from $4C_3 \epsilon < 1$ and (184).

\[\square\]

10 Contact invariants and Open string invariants

\textbf{Proposition 10.1.} $U_I$ has the property

1. Each strata of $U_I$ is a smooth manifold.

2. If $U_{I,D'} \subset U_{I,D}$ is a lower stratum, it is a submanifold of codimension at least 2.

\textbf{Proof.} The proof of (1) for the top strata is standard, we omit it here. The proof for the lower strata will be given in our next paper [20]. It is well-known that an interior node corresponds to codimension 2 stratum. It suffices to consider those nodal corresponding to periodic orbits. Note that:

1. we mod the freedoms of choosing the origin in $\mathbb{R}$ and the origin of the periodic orbits;

2. we choose the Li-Ruan’s compactification in [18], that is, we firstly let the Riemann surfaces degenerate in Delingne-Mumford space and then let $M^+$ degenerate compatibly. At any node, the Riemann surface degenerates independently with two parameters, which compatible with those freedoms of choosing the origins (see section $\S$ for degeneration and section $\S$ for gluing);

Then both blowups at interior and at infinity lead boundaries of codimension 2 or more. \[\square\]

From section $\S$ we have a finite dimensional virtual orbifold system $\{U_I, E_I, \sigma_I\}$ and a finite dimensional virtual orbifold $\{U_I\}$ indexed by a partially ordered set $(I = 2^{\{1;2;\cdots;N\}}, \subset)$. By the same argument of Theorem 5.10 we conclude that $\{U_{I,\epsilon}(M^+)\}$ is a compact virtual manifold. As we have mod the $S^1$-action on periodic orbit, and $S^1$-action on the puncture points on Riemann surfaces, by the same method of [23] we can show that $\{U_{I,\epsilon}(M^+)\}$ is oriented (see [20]).

Let $\Lambda = \{\Lambda_I\}$ be a partition of unity and $\{\Theta_I\}$ be a virtual Euler form of $\{E_I\}$ such that $\Lambda_I \Theta_I$ is compactly supported in $U_{I,\epsilon}$.

Recall that we have two natural maps
\[
e_{\epsilon_I} : U_{I,\epsilon} \longrightarrow M^+ \\
(u; \Sigma, y, p) \longrightarrow u(y_i)
\]
for $i \leq m$ defined by evaluating at marked points and
\[ e_j : U_{i,e} \rightarrow Z_{i,j} \]
\[ (u; \Sigma, y, p) \rightarrow u(p_j) \]
for $j > m$ defined by projecting to its periodic orbit. The contact invariant can be defined as
\[ \Psi^{(C)}_{(A,g,m+\nu,k,e)}(\alpha_1, \ldots, \alpha_m; \beta_{m+1}, \ldots, \beta_{m+\nu}) = \sum_I \int_{U_{I,e}} \prod_i e_i^* \alpha_i \wedge \prod_j e_j^* \beta_j \wedge \Lambda_I \Theta_I. \tag{185} \]
for $\alpha_i \in H^*(M^+, \mathbb{R})$ and $\beta_j \in H^*(Z_{i,j}, \mathbb{R})$ represented by differential form. Clearly, $\Psi^C = 0$ if $\sum \deg(\alpha_i) + \sum \deg(\beta_i) \neq \text{Ind}^C$.

Similarly, the open string invariant can be defined as
\[ \Psi^{(L)}_{(A,g,m+\nu,\mu)}(\alpha_1, \ldots, \alpha_m) = \sum_I \int_{U_{I,e}} \prod_i e_i^* \alpha_i \wedge \Lambda_I \Theta_I. \tag{186} \]
for $\alpha_i \in H^*(M^+; \mathbb{R})$. Clearly, $\Psi^{(L)} = 0$ if $\sum \deg(\alpha_i) \neq \text{Ind}^L$.

It is proved that these integrals are independent of the choices of $\Theta_I$ and the choices of the regularization (see \textbf{3}). We must prove the convergence of the integrals (185) and (186) near each lower strata.

Let $\alpha \in H^*(M^+, \mathbb{R})$ and $\beta \in H^*(Z, \mathbb{R})$ represented by differential form. We may write
\[ \prod_i e_i^* \alpha_i \wedge \prod_j e_j^* \beta_j \wedge \Lambda_I \Theta_I = y dr \wedge d\tau \wedge dj \wedge d\zeta, \]
where $d\zeta$ and $dj$ denote the volume forms of $\ker DS_b$ and the space of complex structures respectively and $y = y(r, \tau, j, \zeta)$ is a function. Then (169) implies that $|y| \leq C_1 e^{-C_2 r}$ for some constants $C_1 > 0$, $C_2 > 0$. Then the convergence of the integral (185) follows. Similarly, we can prove the convergence of the integral (186).

Obviously, both $\Psi^C$ and $\Psi^L$ are generalizations of the relative GW invariants.

One can easily show that

**Theorem 10.2.** (i). $\Psi^{(C)}_{(A,g,m+\nu,k,e)}(\alpha_1, \ldots, \alpha_m; \beta_{m+1}, \ldots, \beta_{m+\nu})$ is well-defined, multi-linear and skew symmetric.

(ii). $\Psi^{(C)}_{(A,g,m+\nu,k,e)}(\alpha_1, \ldots, \alpha_m; \beta_{m+1}, \ldots, \beta_{m+\nu})$ is independent of the choice of forms $\alpha_i, \beta_j$ representing the cohomology classes $[\beta_j], [\alpha_i]$, and the choice of virtual neighborhoods.

(iii). $\Psi^{(C)}_{(A,g,m+\nu,k,e)}(\alpha_1, \ldots, \alpha_m; \beta_{m+1}, \ldots, \beta_{m+\nu})$ is independent of the choice of $\tilde{J}$ and $J$ over $M_0^+$.

**Theorem 10.3.** (i). $\Psi^{(L)}_{(A,g,m+\nu,\mu)}(\alpha_1, \ldots, \alpha_m)$ is well-defined, multi-linear and skew symmetric.

(ii). $\Psi^{(L)}_{(A,g,m+\nu,\mu)}(\alpha_1, \ldots, \alpha_m)$ is independent of the choice of forms $\alpha_i$ representing the cohomology classes $[\alpha_i]$, and the choice of virtual neighborhoods.

(iii). $\Psi^{(L)}_{(A,g,m+\nu,\mu)}(\alpha_1, \ldots, \alpha_m)$ is independent of the choice of $\tilde{J}$ and $J$ over $M_0^+$. 

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