A note on negative $\lambda$-binomial distribution

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\section*{Abstract}
In this paper, we introduce one discrete random variable, namely the negative $\lambda$-binomial random variable. We deduce the expectation of the negative $\lambda$-binomial random variable. We also get the variance and explicit expression for the moments of the negative $\lambda$-binomial random variable.

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\section*{1 Introduction}
In a sequence of independent Bernoulli trials, let the random variable $X$ denote the trial at which the $r$th success occurs, where $r$ is a fixed nonnegative integer. Then

$$P(X = x) = \binom{x - 1}{r - 1} p^r (1 - p)^{x - r}, \quad x = r, r + 1, r + 2, \ldots,$$

and we say that $X$ has a negative binomial distribution with parameters $(r, p)$ (see [1–3, 12, 13]).

The negative binomial distribution is sometimes defined in terms of the random variable $Y$, the number of failures before the $r$th success. This formulation is statistically equivalent to one given above in terms of $X$ denoting the trial at which the $r$th success occurs, since $Y = X - r$. The alternative form of the negative binomial distribution is

$$p(k) = P(Y = k) = \binom{r + k - 1}{k} p^k (1 - p)^r, \quad k = 0, 1, 2, \ldots,$$

where $p$ is the probability of success in the trial (see [1, 3, 12, 13]).

It is known that the degenerate exponential function is defined by

$$e^\lambda_x(t) = (1 + \lambda t)^\lambda = \sum_{n=0}^{\infty} (x)_{n, \lambda} \frac{t^n}{n!}, \quad \lambda \in \mathbb{R},$$

(1)

where

$$(x)_{0, \lambda} = 1, \quad (x)_{n, \lambda} = x(x - \lambda) \cdots (x - (n - 1)\lambda) \quad (n \geq 1) \text{ (see [5–7, 10, 11]).}$$

(2)
Recently, \(\lambda\)-analogue of binomial coefficients was considered by Kim to be
\[
\binom{x}{0}_\lambda = 1, \quad \binom{x}{n}_\lambda = \frac{x(x-\lambda) \cdots (x-(n-1)\lambda)}{n!} \quad (n \geq 1) \quad \text{(see [6, 8, 9])}. \tag{3}
\]

In this paper, we consider the negative \(\lambda\)-binomial distribution and obtain expressions for its moments.

### 2 Negative \(\lambda\)-binomial distribution

**Definition 2.1** \(Y_\lambda\) is the negative \(\lambda\)-binomial random variable if the probability mass function of \(Y_\lambda\) with parameters \((r, p)\) is given by
\[
p_\lambda(k) = P_\lambda(Y_\lambda = k) = \binom{r + (k - 1)\lambda}{k}_\lambda e_\lambda^r(p - 1)(1 - p)^k, \tag{4}
\]
where \(\lambda \in (0, 1)\) and \(p\) is the probability of success in the trials.

Note that
\[
\binom{r + (k - 1)\lambda}{k}_\lambda = (-1)^k \binom{-r}{k}_\lambda, \quad k \geq 0 \quad \text{(see [4])} \tag{5}
\]
and
\[
\sum_{n=0}^{\infty} p_\lambda(k) \sum_{n=0}^{\infty} \binom{r + (k - 1)\lambda}{k}_\lambda \left(1 - p\right)^k e_\lambda^r(p - 1) = e_\lambda^r(p - 1)e_\lambda^{-r}(p - 1) = 1. \tag{6}
\]

From (4), we note that
\[
\lim_{\lambda \to 1} p_\lambda(k) \tag{7}
\]
is the probability mass function of negative binomial random variable with parameters \((r, p)\), and
\[
\lim_{\lambda \to 0} p_\lambda(k) \tag{8}
\]
is the probability mass function of Poisson random variable with parameters \(r(1 - p)\).

Let \(X\) be a discrete random variable, and let \(f(x)\) be a real-valued function. Then we have
\[
E(f(X)) = \sum_x f(x)p(x), \tag{9}
\]
where \(p(x)\) is the probability mass function.

From (9), we note that
\[
E(Y_\lambda) = \sum_{k=0}^{\infty} kp_\lambda(k) = \sum_{k=0}^{\infty} k \left(\frac{r + (k - 1)\lambda}{k}\right)_\lambda (1 - p)^k e_\lambda^r(p - 1) \tag{10}
\]

\[
= \frac{r}{e_\lambda^r(p - 1)} \sum_{k=1}^{\infty} \frac{(r + (k - 1)\lambda) \cdots (r + \lambda)}{(k - 1)!} (1 - p)^k e_\lambda^{r+k}(p - 1)
\]
\[
\begin{align*}
&= \frac{r}{e^\lambda(p - 1)} \sum_{k=0}^\infty \frac{(r + k\lambda) \cdots (r + \lambda)}{k!} (1 - p)^{k+1} e^{\lambda+k\lambda}(p - 1) \\
&= \frac{r(1 - p)}{e^\lambda(p - 1)} \sum_{k=0}^\infty \binom{r + \lambda + (k - 1)\lambda}{k} (1 - p)^k e^{\lambda+k\lambda}(p - 1) \\
&= \frac{r(1 - p)}{e^\lambda(p - 1)} e^{(r+\lambda\lambda)}(p - 1)e^{\lambda+k\lambda}(p - 1) \\
&= \frac{r(1 - p)}{e^\lambda(p - 1)}.
\end{align*}
\]

Therefore, by (10), we obtain the following theorem.

**Theorem 2.1** Let \( Y_\lambda \) be a negative \( \lambda \)-binomial random variable with parameters \((r, p)\).

Then we have

\[
E(Y_\lambda) = \frac{r(1 - p)}{e^\lambda(p - 1)}.
\]

**Note 2.1**

\[
\lim_{\lambda \to 1} E(Y_\lambda) = \frac{r(1 - p)}{p} = E(Y),
\]

where \( Y \) is the negative binomial random variable with parameters \((r, p)\).

**Note 2.2**

\[
\lim_{\lambda \to 0} E(Y_\lambda) = r(1 - p) = E(Y),
\]

where \( Y \) is the Poisson random variable with parameter \( r(1 - p) \).

Now, we observe that

\[
E(Y_\lambda^2) = \sum_{k=0}^\infty k^2 p_\lambda(k) = \sum_{k=0}^\infty k(k + 1 - 1)p_\lambda(k) \tag{11}
\]

\[
= \sum_{k=0}^\infty k(k - 1)p_\lambda(k) + \sum_{k=0}^\infty kp_\lambda(k)
\]

\[
= \sum_{k=0}^\infty k(k - 1) \left( \frac{r + (k - 1)\lambda}{k} \right) (1 - p)^k e^{\lambda+k\lambda}(p - 1) + E(Y_\lambda)
\]

\[
= \frac{r(r + \lambda)}{e^\lambda(p - 1)} \sum_{k=2}^\infty \frac{(r + (k - 1)\lambda) \cdots (r + 2\lambda)}{(k - 2)!} (1 - p)^{k+2} e^{\lambda+2\lambda}(p - 1) + E(Y_\lambda)
\]

\[
= \frac{r(r + \lambda)(1 - p)^2}{e^\lambda(p - 1)} \sum_{k=0}^\infty \frac{(r + 2\lambda + (k - 1)\lambda)}{k} (1 - p)^k e^{\lambda+2\lambda}(p - 1) + E(Y_\lambda)
\]
\begin{align*}
&= \frac{r(r + \lambda)(1-p)^2}{e_\lambda^2(p-1)} e_\lambda^{-r(r+2\lambda)}(p-1)e_\lambda^{r+2\lambda}(p-1) + E(Y_\lambda) \\
&= \frac{r(r + \lambda)(1-p)^2}{e_\lambda^2(p-1)} + \frac{r(1-p)}{e_\lambda^2(p-1)}
\end{align*}

The variance of random variable \( X \) is defined by

\[
\text{Var}(X) = E(X^2) - [E(X)]^2 \quad \text{(see [1, 3]).} \quad (12)
\]

From Theorem 2.1, (11), and (12), we note that

\[
\text{Var}(Y_\lambda) = \lambda r(1-p)^2 + \frac{r(1-p)}{e_\lambda^2(p-1)}.
\]

Therefore, we obtain the following theorem.

**Theorem 2.2** Let \( Y_\lambda \) be a negative \( \lambda \)-binomial random variable with parameters \((r, p)\). Then we have

\[
\text{Var}(Y_\lambda) = \lambda r(1-p)^2 + \frac{r(1-p)}{e_\lambda^2(p-1)}.
\]

**Note 2.3**

\[
\lim_{\lambda \to 1} \text{Var}(Y_\lambda) = \frac{r(1-p)}{p^2} = \text{Var}(Y),
\]

where \( Y \) is the negative binomial random variable with parameters \((r, p)\).

**Note 2.4**

\[
\lim_{\lambda \to 0} \text{Var}(Y_\lambda) = r(1-p) = \text{Var}(Y),
\]

where \( Y \) is the Poisson random variable with parameter \( r(1-p) \).

Note that

\[
k^n = \sum_{l=0}^{n} S_2(n, l)(k)_l, \quad (13)
\]

where \( S_2(n, l) \) is the Stirling number of the second kind, and

\[
(k)_0 = 1, \quad (k)_l = k(k-1) \cdots (k-l+1) \quad (l \geq 1) \text{ (see [14, 15]).}
\]
From (13), we note that

\[
E(Y^n_\lambda) = \sum_{k=0}^{\infty} k^n p_\lambda(k) = \sum_{l=0}^{n} S_2(n, l) \sum_{k=l}^{\infty} \frac{(r+(k-l)\lambda)}{k^l} (1-p)^k e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \sum_{k=l}^{\infty} \frac{(r+(k-l)\lambda)}{k^l} \frac{(1-p)^k e^{r\lambda}(p-1)}{e^{r\lambda}(p-1)}
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)}{e^{r\lambda}(p-1)}
\]

\[
\times \sum_{k=0}^{\infty} \frac{(r+(k-l)\lambda)(r+l\lambda)}{k!} (1-p)^k e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)}{e^{r\lambda}(p-1)}
\]

\[
\times \sum_{k=0}^{\infty} \frac{(r+(k-l)\lambda)(r+l\lambda)}{k!} (1-p)^k e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e^{r\lambda}(p-1)}
\]

\[
\times \sum_{k=0}^{\infty} \frac{(r+l\lambda+(k-l)\lambda)}{k!} (1-p)^k e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e^{r\lambda}(p-1)}
\]

\[
\times \sum_{k=0}^{\infty} \frac{(r+l\lambda+(k-l)\lambda)}{k!} (1-p)^k e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e^{r\lambda}(p-1)}
\]

\[
\times e^{-r\lambda}(p-1) e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{r(r+\lambda)\cdots(r+(l-1)\lambda)(1-p)^l}{e^{r\lambda}(p-1)}
\]

\[
\times e^{-r\lambda}(p-1) e^{r\lambda}(p-1)
\]

\[
= \sum_{l=0}^{n} S_2(n, l) \frac{(r+(l-1)\lambda)\lambda (1-p)^l}{e^{r\lambda}(p-1)}
\]

Therefore, we obtain the following theorem.

**Theorem 2.3** Let \( Y_\lambda \) be a negative \( \lambda \)-binomial random variable with parameters \((r, p)\). Then we have

\[
E(Y^n_\lambda) = \sum_{l=0}^{n} S_2(n, l) \frac{(r+(l-1)\lambda)\lambda (1-p)^l}{e^{r\lambda}(p-1)}
\]
Note 2.5

\[
\lim_{\lambda \to 1} E\left( Y_n^\lambda \right) = \sum_{l=0}^{n} S_2(n, l) \frac{(r + (l - 1))l(1-p)^l}{p^l} = E(\lambda^n),
\]

where \( Y \) is the negative binomial random variable with parameters \((r, p)\) (see [4, 12]).

Note 2.6

\[
\lim_{\lambda \to 0} E\left( Y_n^\lambda \right) = \sum_{l=0}^{n} S_2(n, l) (r(1-p))^l = E(\lambda^n),
\]

where \( Y \) is the Poisson random variable with parameter \( r(1-p) \) (see [16]).

Note that

\[
E(\lambda^\lambda) = \sum_{k=0}^{\infty} k^n p_\lambda(k) = \sum_{k=0}^{\infty} k^n \frac{\left( r + (k - 1)\lambda \right)}{k!} (1-p)^k e_\lambda'(p-1)
\]
\[
= \sum_{k=1}^{\infty} k^{n-1} \frac{(r + (k - 1)\lambda)\cdots(r + \lambda)}{(k-1)!} (1-p)^k e_\lambda'(p-1)
\]
\[
= \sum_{k=0}^{\infty} (k+1)^n \frac{(r + k\lambda)\cdots(r + \lambda)}{k!} (1-p)^{k+1} e_\lambda'(p-1)
\]
\[
= r(1-p) \sum_{k=0}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} k^i \frac{(r + k\lambda)\cdots(r + \lambda)}{k!} (1-p)^k e_\lambda'(p-1)
\]
\[
= \frac{r(1-p)}{e_\lambda'(p-1)} \sum_{i=0}^{n-1} \binom{n-1}{i} \sum_{k=0}^{\infty} k^i \frac{(r + \lambda + (k - 1)\lambda)}{k!} (1-p)^k e_\lambda'(p-1)
\]
\[
= \frac{r(1-p)}{e_\lambda'(p-1)} \sum_{i=0}^{n-1} \binom{n-1}{i} E(Z_i),
\]

where \( Z_i \) is the negative \( \lambda \)-binomial random variable with parameters \((r + \lambda, p)\).

Therefore, we obtain the following theorem.

**Theorem 2.4** Let \( Y_\lambda, Z_\lambda \) be two negative \( \lambda \)-binomial random variables with parameters \((r, p), (r + \lambda, p)\) respectively. Then we have

\[
E(\lambda^\lambda) = \frac{r(1-p)}{e_\lambda'(p-1)} \sum_{i=0}^{n-1} \binom{n-1}{i} E(Z_i).
\]

3 Conclusion

In this paper, we introduced one discrete random variable, namely the negative \( \lambda \)-binomial random variable. The details and results are as follows. We defined the negative \( \lambda \)-binomial random variable with parameter \((r, p)\) in (4) and deduced its expectation in The-
orem 2.1. We also obtained its variance in Theorem 2.2 and derived explicit expression for the moment of the negative $\lambda$-binomial random variable in Theorem 2.3.

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