ILL-POSEDNESS OF NAVIER-STOKES EQUATIONS AND CRITICAL BESOV-MORREY SPACES

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Abstract. The blow up phenomenon in the first step of the classical Picard’s scheme was proved in this paper. For certain initial spaces, Bourgain-Pavlović and Yoneda proved the ill-posedness of the Navier-Stokes equations by showing the norm inflation in certain solution spaces. But Chemin and Gallagher said the space $\dot{B}^{-1}_{\infty,\infty}$ seems to be optimal for some solution spaces best chosen. In this paper, we consider more general initial spaces than Bourgain-Pavlović and Yoneda did and establish ill-posedness result independent of the choice of solution space. Our result is a complement of the previous ill-posedness results on Navier-Stokes equations.

1. Introduction and main result

We consider the incompressible Navier-Stokes equations

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u - \nabla p &= 0, & \text{in } [0, T) \times \mathbb{R}^n; \\
\nabla \cdot u &= 0, & \text{in } [0, T) \times \mathbb{R}^n; \\
u|_{t=0} &= u_0, & \text{in } \mathbb{R}^n;
\end{aligned}
\end{equation}

where $u(t, x)$ and $p(t, x)$ denote the velocity vector field and the pressure of fluid at the point $(t, x) \in [0, T) \times \mathbb{R}^n$ respectively. While $u_0$ is a given initial velocity vector field, Cannone [2] established the well-posedness for Besov spaces, Koch-Tataru [10] obtained the well-posedness for $\text{BMO}^{-1}$. These spaces are special critical Besov-Morrey spaces. Later on, Li-Xiao-Yang [13] showed the well-posedness for more general critical Besov-Morrey spaces. In this paper, we consider the rest critical Besov-Morrey spaces, and we will show blow up phenomenon in the first step of the classical Picard’s scheme for these initial spaces.

2000 Mathematics Subject Classification. Primary 35Q30; 76D03; 42B35; 46E30.
Key words and phrases. Navier-Stokes equations, Meyer wavelets, Besov-Morrey spaces, ill-posedness, blow up.

This work was supported by NSF of China under grant numbers 11571261, 11371295, 11471041.
The solutions of the above Cauchy problem can be obtained via the integral equation:
\[(1.2) \quad u(t, x) = e^{t\Delta}u_0(x) - B(u, u)(t, x),\]
where
\[(1.3) \quad \begin{cases} 
  B(u, u)(t, x) & \equiv \int_0^t e^{(t-s)\Delta}B\nabla(u \otimes u)\,ds, \\
  \mathcal{P}\nabla(u \otimes u) & \equiv \sum_\ell \mathcal{P}(u_\ell u) - \sum_\ell \sum_r (-\Delta)^{-1/2}(\partial_{x_\ell} \cdot \partial_{x_r})\nabla(u_\ell u_r).
\end{cases}
\]
The equation (1.2) can be solved by a fixed-point method whenever the convergence is suitably defined in certain function spaces. For any \(l, l', l'' \in \{1, \cdots, n\}\), denote
\[(1.4) \quad \begin{cases} 
  B_l(u, v)(t, x) & \equiv \int_0^t e^{(t-s)\Delta} \frac{\partial}{\partial x_l}(u(s, x)v(s, x))\,ds; \\
  B_{l,l',l''}(u, u)(t, x) & \equiv \int_0^t e^{(t-s)\Delta}(-\Delta)^{-1/2} \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_{l'}} \frac{\partial}{\partial x_{l''}}(u(s, x)v(s, x))\,ds.
\end{cases}
\]
For \(u_0\) belongs to some initial space \(X^n = (X(\mathbb{R}^n))^n\), denote
\[(1.5) \quad \begin{cases} 
  u^{(0)}(t, x) = e^{t\Delta}u_0; \\
  u^{(j+1)}(t, x) = u^{(0)}(t, x) - B(u^{(j)}, u^{(j)})(t, x), \forall j = 0, 1, 2, \cdots;
\end{cases}
\]
where \(e^{t\Delta}u_0\) belongs to some solution space \(Y^n = (Y([0, T] \times \mathbb{R}^n))^n\). To prove
\[B(u, v)\] are bounded from \(Y^n \times Y^n\) to \(Y^n\),
the traditional method is to prove that the following conclusion is true:
\[(1.6) \quad B_l(u, v), B_{l,l',l''}(u, v) \text{ are bounded from } Y \times Y \text{ to } Y, \quad \forall l, l', l'' \in \{1, \cdots, n\}.
\]
The above iteration process convergence for \(\|u_0\|_{X^n}\) small enough. Such solutions of (1.2) are called mild solutions of (1.1). The notion of such a mild solution was pioneered by Kato-Fujita [9] in 1960s. During the latest decades, many important results about mild solutions to (1.1) have been established. See, for example, Cannone [2, 3], Germin-Pavlovic-Stacli [6], Giga-Miyakawa [7], Lemarié [11, 12], Kato [8], Koch-Tataru [10], Wu [23, 24, 25, 26] and the references therein.

Morrey spaces were introduced in 1938 by Morrey [19]. Many authors extended them to two kinds of oscillation spaces: Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. They consider the well-posedness results for initial value in generalized Morrey spaces, see [13, 14, 16, 28]. In this paper, we consider the ill-posedness in critical Besov-Morrey spaces. Choose
\[(1.7) \quad \phi(x) \in C_0^\infty(B(0, 2n)) \text{ such that } \phi(x) \text{ takes value } 1 \text{ on the ball } B(0, 2\sqrt{n}).\]
Let \( Q(x_0, r) \) be a cube parallel to the coordinate axis, centered at \( x_0 \) and with side length \( r \), \( \phi_Q(x) = \phi(\frac{x - x_0}{r}) \), and \( \mathbb{C} = \{ \phi_Q(x_0, r), x_0 \in \mathbb{R}^n, r > 0 \} \).

For \( 1 \leq p, q \leq \infty \) and \( \gamma_1, \gamma_2 \in \mathbb{R} \), let \( m_0 = m_{p,q}^{\gamma_1,\gamma_2} \) be a positive constant large enough. For arbitrary function \( f(x) \), let \( S_{p,q,f}^{\gamma_1,\gamma_2} \) be the set of polynomial functions \( P_{Q,f}(x) \) of order less than \( m_0 \).

Now we introduce the definition of Besov-Morrey spaces which includes Besov spaces, Morrey spaces, BMO\(^{-1} \) and \( Q \) spaces etc (see [16][31]).

**Definition 1.1.** Given \( 1 \leq p, q \leq \infty \) and \( \gamma_1, \gamma_2 \in \mathbb{R}, 0 \leq \gamma_2 \leq \frac{n}{p} \). Besov-Morrey spaces \( B_{p,q}^{\gamma_1,\gamma_2}(\mathbb{R}^n) \) are defined as follows:

\[
\sup_{Q \in \mathbb{C}} |Q|^\frac{\gamma_1 - \frac{n}{p}}{p} \inf_{P_{Q,f} \in S_{p,q,f}^{\gamma_1,\gamma_2}} \|\phi_Q(f - P_{Q,f})\|_{B_{p,q}^{\gamma_1,\gamma_2}} < \infty.
\]

Critical spaces occupied a significant place for Navier-Stokes equations (1.1). For the above oscillation spaces, according to Lemma 2.6 if \( \gamma_1 - \gamma_2 = -1 \), then they are critical spaces. Further, one pays attention to the largest critical spaces. According to Lemma 2.7, \( B_{p,q}^{1,0}(1 \leq p, q \leq \infty) \) are the relative large spaces among the above critical oscillation spaces.

For initial values in some critical initial spaces \( X^n \), one has proved norm inflation in certain solution spaces \( Y^n \) which are include in \( C([0, T], X^n) \). In fact, Bourgain-Pavlović [11] considered integral Bloch space \( X = \dot{B}_{1,\infty}^{-1} \). For \( q > 2 \), Yoneda [30] considered Besov spaces \( X = \dot{B}_{\infty,q}^{-1} \) and Triebel-Lizorkin spaces \( X = \dot{F}_{\infty,q}^{-1} \) which are special Besov-Morrey spaces \( X = \dot{B}_{q,q}^{-1,0} \). Chemin and Gallagher [4] said the space \( \dot{B}_{\infty,q}^{-1,0} \) seems to be optimal for certain solution space well chosen. For all the critical Besov-Morrey spaces, according to Lemmas 2.7 and 2.8 only \( B_{\infty,q}^{-1,0}(q > 2) \) or \( B_{2,q}^{-1,1}(q' > 2) \) or \( B_{p,p'}^{-1,0}(p > 2, p' \geq 1) \) can not contained in BMO\(^{-1} \). For these three classes of spaces, we will prove the equation (1.6) is not true. Hence the classical Picard’s process can not be applied to these function spaces. Let

\[
e = (1, \cdots, 1).
\]

(1.8)
\[
g(t, x) = \exp\{-\frac{x_2^2}{4t}\}, \forall t > 0, x \in \mathbb{R}^n.
\]

Precisely, our main results can be formulated as follows.

**Theorem 1.2.** For any \( \delta > 0, \) there exists \( u_0 = (u_1, u_2, 0, \cdots, 0) \) satisfying

(i) \( \text{div} \, u_0 = 0 \).

(ii) \( \forall p > 2, p' \geq 1, q > 2, q' > 2, \)

\[
\max\{\|u_0\|_{B_{p,q}^{-1,0}}, \|u_0\|_{B_{2,q'}^{-1,0}}, \|u_0\|_{B_{p',p}^{-1,0}}\} \leq \delta.
\]

(iii) \( u_0 \) belongs to \( (S(\mathbb{R}^n))^n \) outside some ball: \( (1 - \phi(x - \frac{1}{2}e))u_0 \in (S(\mathbb{R}^n))^n \).
\[ B_1(e^{i\lambda}u_1, e^{i\lambda}u_1) \not\in S'(\mathbb{R}^n). \]

In fact, for any \( 0 < t < 1 \), we have
\[ \langle B_1(e^{i\lambda}u_1, e^{i\lambda}u_1), g(t, x) \rangle = -\infty. \]

Our ill-posedness result does not depend on the choice of solution spaces.

Remark 1.3. (i) Bourgain-Pavlović [1] and Yoneda [30] used some special periodic function in some function space \( X \) to show the norm inflation in certain solution space belong to \( C([0, T], X) \). Cui [5] obtained some norm inflation phenomena for logarithmic some Besov spaces. Cui’s skills are based on refining the arguments of Wang [21] and Yoneda [30]. All these ill-posedness results depend on the choice of solution spaces.

(ii) Besov-Morrey spaces can be found in [13, 16, 31]. In this paper, we consider only the critical Besov-Morrey spaces. Cui’s Besov type spaces are logarithmically refined Besov space which are not critical spaces. Further, our skills can be applied also to Cui’s spaces.

(iii) We don’t impose any restriction on solution spaces in Theorem 1.2. Our result shows that we can not apply the classical Picard’s process to get any solution for any \( t > 0 \). Further, our initial value satisfies \((1 - \phi(x - \frac{3}{2}e))u_0 \in (S(\mathbb{R}^n))^n\), which should have global weak solution.

(iv) In [29], Yang-Zhu considered the multiplier operators and hence we know how the product of two functions produce the blow-up phenomena.

Remark 1.4. By [13], \( B_{p,q}^{\gamma,-1}(1 \leq p < \infty, 1 \leq q \leq \infty, 0 < \gamma \leq \frac{4}{p}) \) have well-posedness property. By [10], BMO\(^{-1}\) has well-posedness. Hence, by Lemma 2.8, \( B_{\infty,q}^{\gamma,-1}(q \leq 2) \) or \( B_{2,q}^{\gamma,-1}(q' \leq 2) \) or \( B_{p,p'}^{\gamma,-1,0}(p < 2, p' \geq 1) \) all have well-posedness property. That is to say, for all the critical Besov-Morrey spaces \( B_{p,q}^{\gamma,-1}(1 \leq p < \infty, 1 \leq q \leq \infty, 0 \leq \gamma \leq \frac{4}{p}) \), according to Lemma 2.8 in the next section, only the three classes of spaces in our Theorem 1.2 cause ill-posedness.

By the equation (4.2), the value of left hand of the equation (1.11) is some multiple of the integration of correlation functions \( h_s, t \) defined in (3.7). We will use wavelets, vaguelets and the special property of Gauss function to prove the theorem 1.2. The rest of this paper is organized as follows: In section 2, we will present some preliminaries about Meyer wavelets, vaguelets and Calderón-Zygmund operators, and we will further construct some special functions in Besov-Morrey spaces. In section 3, we will compute some integration related to Gauss functions and present some properties of two flows related to the equation (1.11). Finally, we will prove Theorem 1.2 in Section 4.
2. Wavelets, Special Functions and Operators

2.1. Meyer wavelets. First of all, we indicate that we will use tensorial product real valued orthogonal Meyer wavelets. We refer the reader to \cite{17, 22, 27} for further information. Let $\Psi^0$ be an even function in $C_0^\infty((-\frac{4\pi}{3}, \frac{4\pi}{3}))$ with

\[
\begin{cases}
0 \leq \Psi^0(\xi) \leq 1; \\
\Psi^0(\xi) = 1 \text{ for } |\xi| \leq \frac{2\pi}{3}.
\end{cases}
\]

Write

\[
\Omega(\xi) = \sqrt{(\Psi^0(\frac{\xi}{2}))^2 - (\Psi^0(\xi))^2}.
\]

Then $\Omega(\xi)$ is an even function in $C_0^\infty((-\frac{8\pi}{3}, \frac{8\pi}{3}))$. Clearly,

\[
\begin{cases}
\Omega(\xi) = 0 \text{ for } |\xi| \leq \frac{2\pi}{3}; \\
\Omega^2(\xi) + \Omega^2(2\xi) = 1 = \Omega^2(\xi) + \Omega^2(2\pi - \xi) \text{ for } \xi \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right].
\end{cases}
\]

Let $\Psi^0(\xi) = \Omega(\xi)e^{-\frac{\xi^2}{2}}$. For any $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0, 1\}^n$, define $\Phi^\epsilon(x)$ by $\hat{\Phi^\epsilon}(\xi) = \prod_{i=1}^{n} \Psi_{\epsilon_i}(\xi_i)$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $\Phi^\epsilon_{j,k}(x) = 2^j \Phi^\epsilon(2^j x - k)$. For any $\epsilon \in \{0, 1\}^n$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and distribution $f(x)$, denote $f^\epsilon_{j,k} = \langle f, \Phi^\epsilon_{j,k} \rangle$. Furthermore, we put

\[\Lambda_n = \{(\epsilon, j, k), \epsilon \in \{0, 1\}^n \backslash \{0\}, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.
\]

Then, the following result is well-known.

**Lemma 2.1.** The Meyer wavelets $\{\Phi^\epsilon_{j,k}(x)\}_{(\epsilon, j, k) \in \Lambda_n}$ form an orthogonal basis in $L^2(\mathbb{R}^n)$. Consequently, for any $f \in L^2(\mathbb{R}^n)$, the following wavelet decomposition holds in the $L^2$ convergence sense:

\[f(x) = \sum_{(\epsilon, j, k) \in \Lambda_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).\]

2.2. Properties of Besov-Morrey spaces. We recall first wavelet characterization of oscillation spaces (see \cite{13, 16, 31}). Denote $\mathcal{D} = \{Q_{j,k} = 2^{-j}k + 2^{-j}[0, 1]^n, \forall j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. We have

**Lemma 2.2.** Given $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 \leq \gamma_2 \leq \frac{n}{p}$.

(i) $f(x) = \sum_{\epsilon, j, k} a_{\epsilon, j, k} \Phi^\epsilon_{j, k}(x) \in B^{\gamma_1, \gamma_2}_{\infty, q}(\mathbb{R}^n) \iff$

\[\left[ \sum_j 2^{2j\gamma_2} \left( \sup_{\epsilon, k,j} |a_{\epsilon, j, k}| \right)^q \right]^\frac{1}{q} < \infty,
\]

(2.1)
Lemma 2.7. Inclusion relation: Besov-Morrey spaces \( u(1.1) \) and denote \( (2.3) \)

\[
\sup_{Q \in \mathbb{D}} |Q|^\frac{2}{p} \sum_{j \geq -\log_2 |Q|} 2^{j(\gamma_1 + \frac{2}{p} - \frac{d}{2})} \left( \sum_{(\epsilon,k):Q,\mathbb{Q} \subset Q} |a_{\epsilon,k}|^p \right)^{\frac{1}{p}} < +\infty.
\]

For \( j \geq 1 \), take

\[
h_j(x) = \sum_{i=1,\ldots,n,2^j \leq |x| \leq 2^{j+1}} \frac{2^j}{j^2} \Phi^j(2^j x - l)
\]

and denote

\[
h(x) = \sum_{j \in \mathbb{N}} h_j(x).
\]

By wavelet characterization Lemma 2.2, we have

**Corollary 2.3.**

\( h(x) \in \dot{B}^{-1,q}_{\infty} \bigcap \dot{B}^{-1,0}_{p,q'} \bigcap \dot{B}^{-1,0}_{p,p'} (\forall p, q, q' > 2, p' \geq 1). \)

As a generalization of Morrey spaces, Besov-Morrey spaces cover many important function spaces, for example, Besov spaces, Morrey spaces, BMO spaces and so on. For an overview, we refer to Li-Xiao-Yang \[13\], Li-Yang-Zheng \[15\], Lin-Yang \[16\], Yang \[27\] and Yuan-Sickel-Yang \[31\].

**Lemma 2.4.** (i) If \( 1 \leq p < \infty, 1 \leq q \leq \infty, \gamma_1 \in \mathbb{R}, \gamma_2 = \frac{d}{p}, \) then the above Besov-Morrey spaces \( \dot{B}^{\gamma_1,\gamma_2}_{p,q} \) become the relative Besov spaces \( \dot{B}^{\gamma_1,q}_{p,q} \).

(ii) If \( p = \infty \) and \( 1 \leq q \leq \infty, \) then \( \dot{B}^{\gamma_1,\gamma_2}_{\infty,q} (\mathbb{R}^n) = \dot{B}^{\gamma_1,q}_{\infty,q} (\mathbb{R}^n) \).

(iii) If \( 0 < \alpha < \min\left(\frac{d}{2}, 1\right), \dot{B}^{\alpha,\alpha}_{2,2} = Q_n. \)

Critical spaces occupied a significant place for Navier-Stokes equations (1.1). If \( u(t, x) \) is a solution of (1.1) with initial value \( u_0(x) \), we replace \( u(t, x), p(t, x) \) and \( u_0(x) \) by \( u_0(t, x) = \lambda u(\lambda^2 t, \lambda x), p_0(t, x) = \lambda^2 p(\lambda^2 t, \lambda x) \) and \( u_0(0, x) = \lambda u_0(\lambda x), \) then \( u_0(t, x) \) is a solution of (1.1) with initial value \( u_0^\lambda(x). \)

**Definition 2.5.** If \( ||u_0(x)||_{L^\infty} \sim ||u_0^\lambda(x)||_{L^\infty} \) for any \( \lambda > 0, \) then \( X \) is called to be a critical space.

For the function spaces defined above, if \( \gamma_1 - \gamma_2 = -1, \) then they are critical spaces.

**Lemma 2.6.** For \( 1 \leq p, q \leq \infty \) and \( 0 \leq \gamma_2 \leq \frac{d}{p}, \dot{B}^{\gamma_2,1}_{p,q} \) are critical spaces.

For \( 1 \leq p, q \leq \infty, \) the critical Besov-Morrey spaces have the following inclusion relation:

**Lemma 2.7.** Given \( 1 \leq p, q \leq \infty, 0 \leq \gamma'_2 \leq \gamma_2 \leq \frac{d}{p}. \)

\[ \dot{B}^{\gamma_1,q}_{p,q'} \subset \dot{B}^{\gamma_2,1}_{p,q} \subset \dot{B}^{\gamma'_2,1}_{p,q} \subset \dot{B}^{-1,0}_{\infty,q} (\mathbb{R}^n). \]
The above Lemma shows that $\hat{B}_{p,q}^{-1,0}(1 \leq p, q \leq \infty)$ are the relative large spaces among the critical Besov-Morrey spaces. For critical Besov-Morrey spaces $\hat{B}_{p,q}^{-1,0}(1 \leq p, q \leq \infty)$, by the wavelet characterization Lemma 2.2, we have

**Lemma 2.8.**

\[
\bigcup_{q \geq 2} \hat{B}_{q}^{-1,q} \bigcup_{q' \leq 2} \hat{B}_{2,q'}^{-1,0} \bigcup_{p < 2, p' \geq 1} \hat{B}_{p,p'}^{-1,0} \subset \text{BMO}^{-1} \subset \bigcap_{q > 2} \hat{B}_{q}^{-1,q} \bigcap_{q' > 2} \hat{B}_{2,q'}^{-1,0} \bigcap_{p > 2, p' \geq 1} \hat{B}_{p,p'}^{-1,0}.
\]

2.3. **Vaguelets and Calderón-Zygmund operators.** Now we introduce some preliminaries on Calderón-Zygmund operators, see [17] and [20]. For $x \neq y$, let $K(x, y)$ be a smooth function such that

\[
|\partial_\alpha^\alpha \partial_\beta^\beta K(x, y)| \leq \frac{C}{|x - y|^{\frac{d}{2} + |\alpha| + |\beta|}}, \forall |\alpha| + |\beta| \leq N_0.
\]

In this paper, we assume that $N_0$ is a large enough constant.

**Definition 2.9.** A linear operator $T$ is said to be a Calderón-Zygmund operator in $\text{CZO}(N_0)$ if

1. $T$ is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))^\prime$;
2. There exists a kernel $K(\cdot, \cdot)$ satisfying (2.4) and for $x \notin \text{supp} f$,

\[
Tf(x) = \int K(x, y)f(y)dy;
\]
3. $Tx^\alpha = T^*x^\alpha = 0, \forall \alpha \in \mathbb{N}^n$ and $|\alpha| \leq N_0 - 1$.

According to Schwartz kernel theorem, the kernel $K(x, y)$ of a linear continuous operator $T$ is only a distribution in $S^\prime(\mathbb{R}^{2n})$. Meyer-Yang [18] proved the continuity of Calderón-Zygmund operators on Besov spaces and Triebel-Lizorkin spaces. Lin-Yang [16] considered the relative continuity on Besov-Morrey spaces. In fact, we have

**Lemma 2.10.** Given $1 \leq p, q \leq \infty$ and $T \in \text{CZO}(2)$, then $T$ is continuous from $\hat{B}_{p,q}^{-1,0}$ to $\hat{B}_{p,q}^{-1,0}$.

In this paper, we will use Calderón-Zygmund operators generated by vaguelets. Denote $\psi^i(x) = x_2x_3e^{-x^2}$ and $\psi^2(x) = -x_1x_3e^{-x^2}$. We have

\[
\int x^\alpha \psi^i(x)dx = 0, \forall i = 1, 2, 0 \leq |\alpha| \leq 1.
\]

Denote $\psi^i_{jk}(x) = 2^{nij} \psi^i(2^jx - k)$. Then $\{\psi^i_{jk}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^n}$ are vaguelets generated by the derivatives of Gauss functions. The definition of vaguelets can be found in Meyer [17]. For $i = 1, 2$, define

\[
\left\{
\begin{array}{ll}
T_i : \Phi^i_{jk}(x) = \psi^i_{jk}(x), & \text{if } \epsilon = e; \\
T_i : \Phi^\epsilon_{jk}(x) = 0, & \text{if } \epsilon \neq e.
\end{array}
\right.
\]
It is easy to see

**Lemma 2.11.** For \( n \geq 3 \), we have

(i) \( T_i \in CZO(2), \forall i = 1, 2; \)

(ii) the divergence of \((T_1 h, T_2 h, 0, \cdots, 0)\)' is zero.

By Corollary 2.3 and Lemma 2.10, we have

**Corollary 2.12.**

\[ T_i h(x) \in B^{-1,q}_\infty \cap B^{-1,0}_{2,q'} \cap B^{-1,0}_{p,p'} \ \forall \ i = 1, 2, p, q, q' > 2, p' \geq 1. \]

For \( i = 1, 2 \), \( T_i h(x) \) are distributions concentrated on the cube \([1, 2]^n\) and \( T_i f(x) \) are good functions outside of the cube \([\frac{1}{2}, \frac{5}{2}]^n\). Particulary,

**Lemma 2.13.**

(2.6) \quad (1 - \phi(x - \frac{3}{2}e))T_i h(x) \in S(\mathbb{R}^n). \]

**Proof.** Denote \( x^2 = \sum_{i=1}^n x_i^2 \) and for \( \gamma \in \mathbb{N}^n \), denote \( x^\gamma = \prod_{i=1}^n x_i^{\gamma_i} \). For \( j \geq 1 \) and \( i = 1, \cdots, n, 2^j \leq l_i \leq 2^{j+1} \), we have

(2.7) \quad (2^{-j}l - \frac{3}{2}e)^2 \leq \frac{n}{4}.

If \( (x - \frac{3}{2}e)^2 \geq 4n \), then

(2.8) \quad (x - 2^{-j}l)^2 \geq \frac{1}{2}(x - \frac{3}{2}e)^2 - (2^{-j}l - \frac{3}{2}e)^2 \geq 2n - \frac{n}{4} \geq \frac{7n}{4}.

(2.9) \quad (x - \frac{3}{2}e)^2 \leq 2(x - 2^{-j}l)^2 + 2(2^{-j}l - \frac{3}{2}e)^2 \leq 3(x - 2^{-j}l)^2.

For \( \beta, \gamma \in \mathbb{N}^n \), we have

(2.10) \quad |\partial_x^\beta ((2^j x - l)^\gamma e^{-2^j(x - l)^2})| \leq C2^{j|\beta|}(2^j x - l)^{|\beta|+|\gamma|}e^{-2^j(x - l)^2}.

Combine the equations from (2.7) to (2.10), for any \( N \geq 0, \alpha, \beta, \gamma \in \mathbb{N}^n \), we have

(2.11) \quad |(x - \frac{3}{2}e)^2 \partial_x^\beta (1 - \phi(x - \frac{3}{2}e))\partial_x^\beta ((2^j x - l)^\gamma e^{-2^j(x - l)^2})| \\
\leq C2^{j|\beta|+2N}(2^j x - l)^{2N+|\beta|+|\gamma|}e^{-2^j(x - l)^2} \\
\leq C2^{-j(2N+n+2)}(2^j x - l)^{2N+2|\beta|+|\gamma|+n+2}e^{-2^j(x - l)^2} \\
\leq C2^{-j(2N+n+2)}.

Hence, for any \( N \geq 0, \alpha, \beta \in \mathbb{N}^n \), we have

(2.12) \quad |(x - \frac{3}{2}e)^2 \partial_x^\beta (1 - \phi(x - \frac{3}{2}e))\partial_x^\beta |T_i h(x)|| \\
\leq C \sum_{j \in \mathbb{N}} 2^{-j(2N+n+2)}2^{n/2} \leq C.

The last equation implies the equation (2.6).
3. Gauss function and two flows related

In this section, we first consider four integrations related to Gauss function, then we consider some properties of two flows which will be used to consider the blow up phenomenon in the next section.

3.1. Four integrations relative to Gauss function. In this subsection, we compute four integrations related to Gauss function and their derivatives which will be used to compute the expression of two flows in the next subsection. For any $0 < s < t$, $l = 1, \cdots, n$, $j \in \mathbb{N}$, $x_l \in \mathbb{R}$ and $k_l \in \mathbb{Z}$, denote

$$I(x_l, t, s, 0) = (t-s)^{-\frac{1}{2}} \int \exp \{ -\frac{(x_l - y_l)^2}{4(t-s)} \} \exp \{ -\frac{y_l^2}{4t} \} dy_l;$$

$$I(x_l, t, s, 1) = (t-s)^{-\frac{1}{2}} \int (x_l - y_l) \exp \{ -\frac{(x_l - y_l)^2}{4(t-s)} \} \exp \{ -\frac{y_l^2}{4t} \} dy_l;$$

$$I(x_l, t, j, k_l, 0) = t^\frac{1}{2} \int \exp \{ -\frac{(x_l - y_l)^2}{4t} \} \exp \{ -2jy_l - k_l^2 \} dy_l;$$

$$I(x_l, t, j, k_l, 1) = t^\frac{1}{2} \int \exp \{ -\frac{(x_l - y_l)^2}{4t} \} (2jy_l - k_l) \exp \{ -2jy_l - k_l^2 \} dy_l.$$

Then we have

**Lemma 3.1.** For $l = 1, \cdots, n$, $0 \leq s \leq t$, $j \in \mathbb{N}$, $x_l \in \mathbb{R}$, $k_l \in \mathbb{Z}$, we have

(i) $I(x_l, t, s, 0) = c(2t-s)^{-\frac{1}{2}} \exp \{ -\frac{y_l^2}{4(2t-s)} \}$;

(ii) $I(x_l, t, s, 1) = cx_l(2t-s)^{-\frac{1}{2}} \exp \{ -\frac{y_l^2}{4(2t-s)} \}$;

(iii) $I(x_l, t, j, k_l, 0) = c(1 + 4j^2t)^{-\frac{1}{2}} \exp \{ -\frac{(2s/x_l - k_l)^2}{1 + 4j^2t} \}$;

(iv) $I(x_l, t, j, k_l, 1) = c(2jx_l - k_l)(1 + 4j^2t)^{-\frac{1}{2}} \exp \{ -\frac{(2s/x_l - k_l)^2}{1 + 4j^2t} \}$.

**Proof.** (i) We regroup the function inside the integration and get

$$I(x_l, t, s, 0) = (t-s)^{-\frac{1}{2}} \int \exp \{ -\frac{1}{4} \left( \frac{1}{t-s} + \frac{1}{t} \right) y_l^2 \} \left( x_l - \frac{y_l}{2(t-s)} \right)^2 dy_l 
\times \exp \left\{ -\frac{y_l^2}{4(t-s)} + \frac{t}{4(t-s)(t-s)} \right\}.$$

Then, by changing variable, we have

$$I(x_l, t, s, 0) = (t-s)^{-\frac{1}{2}} \int \exp \{ -\frac{1}{4} \left( \frac{1}{t-s} + \frac{1}{t} \right) y_l^2 \} dy_l \exp \{ -\frac{y_l^2}{4(2t-s)} \}.$$

Since \( \int \exp \{ -\frac{1}{4} \left( \frac{1}{t-s} + \frac{1}{t} \right) y_l^2 \} dy_l = c(t-s)^{\frac{1}{2}}(2t-s)^{-\frac{1}{2}} \), we get

$$I(x_l, t, s, 0) = (2t-s)^{-\frac{1}{2}} \exp \{ -\frac{x_l^2}{4(2t-s)} \}.$$

(ii) We make variable substitution $z_l = y_l - \frac{tx_l}{2(t-s)}$. We have then $x_l - y_l = x_l - \frac{tx_l}{2(t-s)} - z_l$ and get
3.2. Expression of two flow functions. In this subsection, we compute the expression of two flow functions. Let 

\[ I(x_1, t, s, 1) = (t - s)^{\frac{1}{2}} \int \exp \left\{ -\frac{1}{4} \left( \frac{1}{t - s} + \frac{1}{(2t - s)^{\frac{1}{2}}} \right) z_i \right\} dz_i \exp \left\{ -\frac{x_i^2}{4(2t - s)} \right\}. \]

Note that \( \int \exp \left\{ -\frac{1}{4} \left( \frac{1}{t - s} + \frac{1}{(2t - s)^{\frac{1}{2}}} \right) z_i \right\} dz_i = c(t - s)^{\frac{1}{2}}(2t - s)^{-\frac{1}{2}} \) and \( \int z_i \exp \left\{ -\frac{1}{4} \left( \frac{1}{t - s} + \frac{1}{(2t - s)^{\frac{1}{2}}} \right) z_i^2 \right\} dz_i = 0 \), we obtain

\[ I(x_1, t, s, 1) = cx_1(2t - s)^{-\frac{1}{2}} \exp \left\{ -\frac{x_i^2}{4(2t - s)} \right\}. \]

(iii) Regrouping the function inside the integration, we get

\[ I(x_1, t, j, k_i, 0) = r^{-\frac{1}{2}} \int \exp \left\{ -(\frac{1}{4t} + 4j) y_i \right\} dy_i \exp \left\{ -\frac{[2x_i - k_i]^2}{1 + 4j + 1} \right\}. \]

Applying \( \int \exp \left\{ -(\frac{1}{4t} + 4j) y_i^2 \right\} dy_i = c (1 + 4j + 1)^{-\frac{1}{2}} \), we get

\[ I(x_1, t, j, k_i, 0) = c (1 + 4j + 1)^{-\frac{1}{2}} \exp \left\{ -\frac{[2x_i - k_i]^2}{1 + 4j + 1} \right\}. \]

(iv) Based on (iii) and the fact \( \int y_i \exp \left\{ -(\frac{1}{4t} + 4j) y_i^2 \right\} dy_i = 0 \), we have

\[ I(x_1, t, j, k_i, 1) = c (2jx_i - k_i)(1 + 4j + 1)^{-\frac{1}{2}} \exp \left\{ -\frac{[2x_i - k_i]^2}{1 + 4j + 1} \right\}. \]

\[ \square \]

3.2. Expression of two flow functions. In this subsection, we compute the expression of two flow functions. Let \( A_i^t = \frac{\partial}{\partial s} e^{t\Delta} \). We first consider the flow \( A_i^{t-s} g(t, x) \). The kernel of \( A_i^t \) is \( C(x_i - y_i) \)

\[ \frac{C(x_i - y_i)}{t^{s+1}} \exp \left\{ \frac{(x - y)^2}{4t} \right\}. \]

By Lemma 3.1, we have

**Lemma 3.2.**

\[ A_i^{t-s} g(t, x) = \frac{cx_1}{(2t - s)^{\frac{n+2}{2}}} \exp \left\{ -\frac{x^2}{4(2t - s)} \right\}. \]

**Proof.**

\[ A_i^{t-s} g(t, x) = \frac{1}{\sqrt{t-s}} \int \frac{y_1 - y_i}{t-s} \exp \left\{ -\frac{(y_1 - y_i)^2}{4(t-s)} \right\} \exp \left\{ -\frac{y_i^2}{4t} \right\} dy_1 \]

\[ \times \prod_{l=2}^{n} \frac{1}{\sqrt{t-s}} \int \frac{y_l - y_i}{t-s} \exp \left\{ -\frac{(y_l - y_i)^2}{4(t-s)} \right\} \exp \left\{ -\frac{y_i^2}{4t} \right\} dy_l \]

\[ = cx_1 \frac{3^n}{\sqrt{(2t-s)^{\frac{n+2}{2}}}} \prod_{l=2}^{n} \exp \left\{ -\frac{y_i^2}{4(2t-s)} \right\} \]

\[ = \frac{cx_1}{(2t-s)^{\frac{n+2}{2}}} \exp \left\{ -\frac{x^2}{4(2t-s)} \right\}. \]
Next we consider another flow. Let
\begin{equation}
v_1(x) = \sum_{j \in \mathbb{N}} v_{1,j}(x), \quad \text{where}
\end{equation}
\begin{equation}
v_{1,j}(x) = \sum_{i=1, \ldots, n, 2^j i \leq 2^{j+1}} 2^j j^{-\frac{1}{2}} \psi^1(2^j x - i).
\end{equation}

The kernel of $e^{tA}$ is $\frac{e}{\pi} \exp\{-\frac{(x-y)^2}{4r}\}$. By Lemma 5.1, we have

**Lemma 3.3.** For any $j \in \mathbb{N}$, we have
\begin{equation}
\sum_{i=1, \ldots, n, 2^j i \leq 2^{j+1}} e^{tA} v_{1,j}(x) = 2^j j^{-\frac{1}{2}} \frac{(2/x_2 - k_2)(2/x_3 - k_3)}{(1+4^j r_l)^{j+2}} \exp\{-\frac{(2/x - k)^2}{1+4^j r_l}\}.\end{equation}

Hence,
\begin{equation}
\sum_{j \in \mathbb{N}} \sum_{i=1, \ldots, n, 2^j i \leq 2^{j+1}} e^{tA} v_1(x) = \frac{e^{tA} v_1(x)}{\frac{(2/x_2 - k_2)(2/x_3 - k_3)}{(1+4^j r_l)^{j+2}} \exp\{-\frac{(2/x - k)^2}{1+4^j r_l}\}}.
\end{equation}

**Proof.** The conclusions of Lemma 5.2 are obtained by the following equality:
\begin{equation}
e^{tA} \{(2j x_2 - k_2)(2j x_3 - k_3) \exp\{-2j x - k\})
= \frac{e^{tA} v_1(x)}{(1+4^j r_l)^{j+2}} \exp\{-\frac{(2/x - k)^2}{1+4^j r_l}\}.
\end{equation}

\hfill \Box

### 3.3. Some properties of flows.

In this subsection, we consider the following four properties of the above two flow functions: symmetry, monotonicity, positivity and zero point. To present well the symmetry properties of flow functions $A_t^{-s} g(t, x)$, we introduce the following notations:
\begin{equation}
x_{i}^{-0} = (x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n).
\end{equation}

For $A_t^{-s} g(t, x)$, we have

**Lemma 3.4.** (i) **Positivity.** $A_t^{-s} g(t, x) > 0, \forall x > 0$;
(ii) **Anti-symmetry.** $A_t^{-s} g(t, x) = -A_t^{-s} g(t, x_{-0})$;
(iii) **Symmetry.** $A_t^{-s} g(t, x) = A_t^{-s} g(t, x_{-0})$, $\forall i = 2, \cdots, n$.

To present well the properties of flow functions $e^{tA} u_{1,j}(x)$, we introduce the following notations:
\begin{equation}
x_{i}^{-3} = (x_1, \cdots, x_{i-1}, 3 - x_i, x_{i+1}, \cdots, x_n),
\end{equation}
\begin{equation}
x_{i}^{0} = (x_1, \cdots, x_{i-1}, \frac{3}{2}, x_{i+1}, \cdots, x_n).
\end{equation}

For $e^{tA} v_{1,j}(x)$, we have
**Lemma 3.5.** (i) Anti-symmetry. \( e^{\Delta}v_{1,j}(x) = -e^{\Delta}v_{1,j}(x_{i}^{-3}), \forall i = 2, 3. \)  
(ii) Symmetry. \( e^{\Delta}v_{1,j}(x) = e^{\Delta}v_{1,j}(x_{i}^{-3}), \forall i = 1, 4, \cdots, n. \)  
(iii) Zero point. \( e^{\Delta}v_{1,j}(x_{0}^{2}) = e^{\Delta}v_{1,j}(x_{0}^{2}) = 0. \)  
(iv) Positivity. \( e^{\Delta}v_{1,j}(x) > 0, \text{ if } (x_{2} - \frac{3}{2})(x_{3} - \frac{1}{2}) > 0. \)  
(v) Negativity. \( e^{\Delta}v_{1,j}(x) < 0, \text{ if } (x_{2} - \frac{3}{2})(x_{3} - \frac{1}{2}) < 0. \)

**Proof.** Applying the expression of flow functions in Lemma 3.2, we can get the above results. The details are omitted. \( \square \)

For \( v_{1}(x) \) defined in (3.2) and \( e^{\Delta}v_{1}(x) \) defined in (3.4), denote \( V(t, x) = (e^{\Delta}v_{1}(x))^{2} \), and \( x' = (x_{2}, \cdots, x_{n}) \). Then

**Lemma 3.6.** For any \( x_{1} > 0, x'_{1} \in \mathbb{R}^{n-1}, \)

\[(3.5) \quad |e^{\Delta}v_{1,j}(s, x)| \geq |e^{\Delta}v_{1,j}(s, x_{1} + 3, x'_{1})|.\]

Hence,

\[(3.6) \quad V(s, x) - V(s, x_{1} + 3, x'_{1}) \geq 0.\]

**Proof.** For \( j \in \mathbb{N}, x_{1} \geq 0 \) and \( 2^{j} \leq k_{1} \leq 2^{j+1} \), we have

\[3 \cdot 2^{2j} + 2(2^{j}x_{1} - k_{1}) \geq 3 \cdot 2^{2j} - 2k_{1} \geq 3 \cdot 2^{2j} - 2^{j+2} \geq 0.\]

That is to say,

\[(2^{j}(x_{1} + 3) - k_{1})^{2} \geq (2^{j}x_{1} - k_{1})^{2}.\]

By applying Lemma 3.5 we get

\[|e^{\Delta}v_{1,j}(x)| \geq |e^{\Delta}V_{1,j}(x_{1} + 3, x'_{1})|.\]

\( \square \)

### 3.4. Correlation function and non-integrability.

Let \( h_{s,t} \) be the correlation function between function \( V(s, x) \) and function \( A_{1}^{-s}g(t, x) \), i.e.,

\[(3.7) \quad h_{s,t} = \langle V(s, x), A_{1}^{-s}g(t, x) \rangle.\]

Then we have

**Theorem 3.7.** \( \forall 0 < s < t < 1, h_{s,t} \geq 0. \)

**Proof.** Denote \( x'_{1} = (x_{2}, \cdots, x_{n}) \) and \( x_{1}^{-0} = (-x_{1}, x_{2}, \cdots, x_{n}). \) Then

\[h_{s,t} = \int_{0}^{\infty} \int_{\mathbb{R}^{n-1}} V(s, x) \frac{x_{1}}{(2t-s)^{2}s} \exp \left\{ -\frac{x^{2}}{4(2s-t)} \right\} dx_{1} dx'_{1} - \int_{0}^{\infty} \int_{\mathbb{R}^{n-1}} V(s, x_{1}^{-0}) \frac{x_{1}}{(2t-s)^{2}s} \exp \left\{ -\frac{x^{2}}{4(2s-t)} \right\} dx_{1} dx'_{1} = \int_{0}^{\infty} \int_{\mathbb{R}^{n-1}} [V(s, x) - V(s, x_{1} + 3, x'_{1})] \frac{x_{1}}{(2t-s)^{2}s} \exp \left\{ -\frac{x^{2}}{4(2s-t)} \right\} dx_{1} dx'_{1} = \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{n-1}} [V(s, x) - V(s, x_{1} + 3, x'_{1})] \frac{x_{1}}{(2t-s)^{2}s} \exp \left\{ -\frac{x^{2}}{4(2s-t)} \right\} dx_{1} dx'_{1} + \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^{n-1}} [V(s, x) - V(s, x_{1} + 3, x'_{1})] \frac{x_{1}}{(2t-s)^{2}s} \exp \left\{ -\frac{x^{2}}{4(2s-t)} \right\} dx_{1} dx'_{1} \]
By the equation (3.6) of \( V(s, x) \), we have

\[
h_{s,t} \geq \int_{\mathbb{R}^n} [V(s, x) - V(s, x_1 + 3, x'_1)] \frac{x_i}{(2^{s-t})^{n+1}} \exp \left\{-\frac{x^2}{4(2^{s-t})}\right\} dx_1 dx'_1
\]

\[
\geq \int_{[2,3]^n} [V(s, x) - V(s, x_1 + 3, x'_1)] \frac{x_i}{(2^{s-t})^{n+1}} \exp \left\{-\frac{x^2}{4(2^{s-t})}\right\} dx_1 dx'_1
\]

\[
\geq 0.
\]

\[\Box\]

**Theorem 3.8.** For any \( j \in \mathbb{N} \), \( 0 < s < t < 1 \) and \( 2^{-2j} \leq s < 2^{2^{-2j}} \),

\[
h_{s,t} \geq \frac{c2^{2j}}{j}.
\]

**Proof.** By (3.8),

\[
h_{s,t} \geq \int_{[2,3]^n} [V(s, x) - V(s, x_1 + 3, x'_1)] \frac{x_i}{(2^{s-t})^{n+1}} \exp \left\{-\frac{x^2}{4(2^{s-t})}\right\} dx_1 dx'_1.
\]

By (3.5), for any \( j \in \mathbb{N} \) and \( x \in [2,3]^n \), we have

\[
V(s, x) - V(s, x_1 + 3, x'_1) \geq (e^{s \Delta} v_{1,j}(x))^2 - (e^{s \Delta} v_{1,j}(x_1 + 3, x'_1))^2.
\]

Further, if \( 2^{-2j} \leq s < 2^{2^{-2j}} \) and \( x \in [2,3]^n \), then

\[
(e^{s \Delta} v_{1,j}(x))^2 - (e^{s \Delta} v_{1,j}(x_1 + 3, x'_1))^2 \geq \frac{1}{2}(e^{s \Delta} v_{1,j}(x))^2 \geq \frac{c2^{2j}}{j}.
\]

Hence, for \( 2^{-2j} \leq s < 2^{2^{-2j}} \) and \( x \in [2,3]^n \),

\[
[V(s, x) - V(s, x_1 + 3, x'_1)] \geq \frac{c2^{2j}}{j}.
\]

Consequently,

\[
h_{s,t} \geq \frac{c2^{2j}}{j}.
\]

\[\Box\]

**Theorem 3.9.** \( \int_0^t h_{s,t} ds = \infty \), \( \forall 0 < t < 1 \).

**Proof.** For any \( 0 < t < 1 \), there exists \( j_t \in \mathbb{N} \) such that \( 2^{-2j_t} \leq t < 2^{2^{-2j_t}} \). Then,

\[
\int_0^t h_{s,t} ds \geq \sum_{j \geq j_t} \frac{c2^{2j}}{j} 2^{-2j} = \sum_{j \geq j_t} \frac{c}{j} = +\infty.
\]

\[\Box\]
4. Proof of Theorem 1.2

Proof. We define $u_0$ as follows:

$$u_0 = (u_1, u_2, u_3, \ldots, u_n)^T = (\delta T_1 h, \delta T_2 h, 0, \cdots, 0)^T.$$  

According to Lemmas 2.11, 2.13 and Corollary 2.12, we have

(i) $\text{div} u_0 = 0$;

(ii) for any $p, q, q' > 2, p' \geq 1$,

$$\max\{\|u_0\|_{B^{-1, q'}_1}, \|u_0\|_{B^{-1, q}_2}, \|u_0\|_{B^{-1, 0}_p}, \|u_0\|_{F^{-1, 0}_p}\} \leq c\delta;$$

(iii) $(1 - \phi(x - \frac{3}{2} e))u_0 \in (S(\mathbb{R}^n))^n$.

By definition of $B_1$, we get

$$\langle B_1(e^{t\Delta} u_1, e^{t\Delta} u_1), g(t, x) \rangle = -\int_0^t \langle (e^{s\Delta} u_1)^2, \frac{\partial}{\partial x_1} e^{(t-s)\Delta} g(t, x) \rangle ds.$$ 

Hence,

$$\langle B_1(e^{t\Delta} u_1, e^{t\Delta} u_1), g(t, x) \rangle = -\delta^2 \int_0^t h_{x_1} ds.$$ 

Invoking Theorem 3.9 leads to

$$\langle B_1(e^{t\Delta} u_1, e^{t\Delta} u_1), g(t, x) \rangle = -\infty.$$ 

This completes the proof of Theorem 1.2. □

Acknowledgement: The authors would like thank professor Yves Meyer for his interesting of this work and many useful advices.

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