Observable measure of quantum coherence in finite dimensional systems

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Quantum coherence is the key resource for quantum technology, with applications in quantum optics, information processing, metrology and cryptography. Yet, there is no universally efficient method for quantifying coherence either in theoretical or in experimental practice. I introduce a framework for measuring quantum coherence in finite dimensional systems. I define a theoretical measure which satisfies the reliability criteria established in the context of quantum resource theories. Then, I present an experimental scheme implementable with current technology which evaluates the quantum coherence of an unknown state of a \( d \)-dimensional system by performing two programmable measurements on an ancillary qubit, in place of the \( O(d^2) \) direct measurements required by full state reconstruction. The result yields a benchmark for monitoring quantum effects in complex systems, e.g. certifying non-classicality in quantum communication and simulation protocols and probing the quantum behaviour of biological complexes.

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Introduction – While harnessing quantum coherence is matter of routine in delivering quantum technology \([1–5]\), and the quantum optics rationale rests on creation and manipulation of coherence \([6]\), there is no universally efficient route to measure the amount of quantum coherence carried by the state of a system in dimension \( d > 2 \). It is customary to employ a quantifier tailored to the scenario of interest. If the density matrix of the state is available, then a function of its (off-diagonal) coefficients is employed. Model-dependent strategies, expressed in terms of entropic functions, correlators and state space metrics, have been recently proposed \([7–9]\).

Quantum information theory provides the framework to address the problem. Physical laws are interpreted as restrictions on the accessible quantum states and operations, while the properties of physical systems are the resources that one must consume to perform a task under such laws \([10]\). An algorithmic characterization of quantum coherence as a resource and a set of \textit{bona fide} criteria for coherence monotones have been identified \([7, 11, 12]\). Also, coherence has been shown to be related to the \textit{asymmetry} of a quantum state \([13, 14]\). On the experimental side, the scalability of the detection scheme is a major criterion in developing witnesses and coherence measures. We are interested in exploring the quantum features of highly complex macrosystems, e.g. multipartite quantum registers and networks, of which one expects that only partial information and control is achievable, because of technical imperfections and environmental degrees of freedom.

Here I introduce a measure of quantum coherence for states of finite dimensional systems. The quantity satisfies the properties of reliable coherence quantifiers and it is easy to compute, not involving any optimization. Also, I define a lower bound which is experimentally observable. The detection of quantum coherence does not require to reconstruct the full density matrix of the state, but it relies upon the estimation of quadratic functionals of the density matrix co-efficients. I propose a scheme which is readily implementable with current quantum technology, e.g. in all-optical setup \([6]\). Regardless of the dimensionality \( d \) of the system, the protocol requires to realize two programmable measurements \([15–20]\), which are basic operations in quantum information, on an ancillary qubit which has been let interact by a unitary dynamics with the system under scrutiny. An alternative scheme requiring \( O(d) \) measurements overcomes the implementation of multipartite controlled gates.

Main Results – Quantum coherence manifests in the quantum measurement apparatus \textit{par excellence}, i.e. the interferometer, where we observe wave-like probability distributions of detecting quantum particles. The coherence of the particle state embodies the quantum contribution to the uncertainty about which detector will be hit by the particle injected into the interferometer. In fact, the uncertainty on the outcome of a measurement is twofold \([21, 22]\). First, an inherently classical uncertainty is brought about by the ignorance about the particle state \( \rho \) and it is measured by the state mixedness. Second, a quantum kind of uncertainty is due to the fact that \( \rho \) is changed by the measurement of an observable \( K \), which I assume to be bounded and nondegenerate. The state is left invariant by the measurement if and only if it is an eigenstate or a mixture of eigenstates of the observable, i.e. \( [\rho, K] = 0 \) \([23]\). Absence of coherence means that only classical uncertainty plays a role and no interference is observed. Quantum coherence entails quantum uncertainty and an interference pattern.

Since quantum coherence effects appear in any measurement scheme, the paradigm of coherence as uncertainty can be universally applied. I define the \( K \)-coherence of a \( d \)-dimensional state \( \rho \) as the quantum coherence it carries when measuring \( K \). A class of functions which quantifies coherence, satisfying the intuitive requirements of non-negativity, faithfulness, i.e. vanishing if and only if state and observable commute, and convexity, is given by the Wigner-Yanase-Dyson skew informations \([24]\).
The skew information is therefore a full-fledged theoretical measure of coherence. However, quantum technologies demand to design practical schemes to experimentally detect quantum effects. In laboratory, functional of the state density matrix are estimated by implementing quantum programmable measurements on an ancillary qubit [15–20]. The method has been applied to measure entanglement and general quantum correlations without state reconstruction [31, 32]. Here I employ it to evaluate the quantum coherence of a state whose density matrix is unknown.

The square root terms prevent from recasting the skew information as a function of observables. Nevertheless, it is possible to set a non-trivial lower bound. One has $1/2\text{Tr}[|\rho|, K^2] \geq \text{Tr}[|\sqrt[p]{\rho}|, K^2], \forall \rho, K$, and therefore

$$I(\rho, K) \geq I^L(\rho, K) \geq 0,$$

$$I^L(\rho, K) = -1/4 \text{Tr}[|\rho|, K^2].$$

Given the spectral decomposition $\rho = \sum_i |\psi_i\rangle \langle \psi_i|$, the two quantities read $I(\rho, K) = 1/2 \sum_i (\sqrt{|K_i|^2} - \sqrt{|T_i|^2} K_i^2, I^L(\rho, K) = 1/4 \sum_i (\lambda_i - \lambda_i) K_i^2, K_i = |\langle \psi_i| |\psi_i\rangle|$. The inequality is satisfied if $(\sqrt{|T_i|} - \sqrt{|T_i|} \geq 1/2(\lambda_i - \lambda_i)^2, \forall i, j$. Simplifying, one obtains $\sqrt{|T_i|} + \sqrt{|T_j|} \leq \sqrt{2}$, which is always true. Also, $I^L(\rho, K) = 0 \Rightarrow I(\rho, K) = 0$. Note that for pure states $\forall \rho, K = I(\rho, K) = 2I^L(\rho, K)$, while for two-dimensional systems (qubits) the inequality $2I^L(\rho, K) \geq I(\rho, K)$ holds.

The lower bound is experimentally measurable. By defining the unitary transformation $U_K(t) = e^{iKt}$ and using the Taylor expansion about $t = 0$, one has $\text{Tr}[\rho U_K(t)\mu U_K(t)^\dagger] = \text{Tr}[\rho] - (\text{Tr}[\rho^2 K^2] - \text{Tr}[\rho K^2 K])t^2 + O(t^4)$, and then $I^L(\rho, K) = 1/2 \text{Tr}[\rho^2] - (\text{Tr}[\rho^2 K^2] - \text{Tr}[\rho K^2 K])t^2 + O(t^4)$. The two terms admit an expression in terms of observables. The purity equals the mean value of the SWAP operator $V_{AB} = \sum_{ij} |\alpha_i\beta_j\rangle \langle \alpha_j\beta_i|$ applied to two space copies $\rho_{1,2} \equiv \rho$: $\text{Tr}[\rho^2] = \text{Tr}[V_{12}(\rho_1 \otimes \rho_2) \text{Tr}[V_{12}(\rho_1 \otimes \rho_2)]^{1/2}]$. On the same hand, the overlap is given by $\text{Tr}[\rho U_K(t)U_{K^\dagger}(t)] = \text{Tr}[V_{12}(\rho_1 \otimes U_{2,K} \rho_2 U_{2,K}^\dagger)]$. The mean value of the SWAP is estimated by implementing the interferometers in Fig. 2, where an ancillary qubit prepared in the arbitrary states $\rho_{1,2}$ acts as the control state. Adding a controlled-SWAP gate, the polarisation of ancilla at the output gives the mean value of the SWAP: $\langle \sigma_z \rangle_{\rho_{1,2}} = \text{Tr}[\sigma_z \rho_{1,2}] = \delta_{\rho_1} \delta_{\rho_2} - (\text{Tr}[\rho^2] - (\text{Tr}[\rho K^2 K])t^2 + O(t^4)$, and then $I^L(\rho, K) = 1/2 \text{Tr}[\rho^2] - \text{Tr}[\rho U_K(0)\rho U_K(0)^\dagger]$. Hence,

**Result 2.** The experimental evaluation of (a lower bound of the) quantum coherence of an unknown state in a d-dimensional system requires two programmable measurements on an ancillary qubit.
polarisation is then the unitary gate normalized \( \rho = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \langle i| \), followed by an interacting controlled-V gates applied to the ancilla and the state copies:

\[ C_{V_{12}} = \begin{pmatrix} 0 & e^{i\theta} \sqrt{d} \\ 0 & 0 \end{pmatrix} \], \quad \rho_{12} = \frac{1}{d} (|i\rangle \langle i| + \sum_{i=1}^{d} |i\rangle \langle i| \otimes |i\rangle \langle i|), \quad \text{where} \ [\tau_{i}] \text{are the normalized } d \text{-dimensional Gell-Mann matrices} \ (|\hat{\sigma}|) \text{; } \tau_{i} = \sqrt{\frac{d-1}{2}} \hat{\sigma}_{i}.

The SWAP can be recast in terms of projectors \( P_{12}^{\pm} = \frac{1}{d} (|i\rangle \langle i| \pm V_{12}) = \frac{d+1}{2d} |i\rangle \langle i| \pm \frac{1}{d} \sum_{i=1}^{d} |i\rangle \langle i| \otimes |i\rangle \langle i| \text{ on the (anti)-symmetric subspaces, which are employable observables in optical setups. Note also that any } d \text{-gate is decomposable in a sequence of one-qubit and two-qubit controlled-NOT transformations} [33]. A second Hadamard gate is finally applied to the ancilla. The mean value of the ancilla polarisation, which corresponds to the visibility of the interferometer, is given by

\[ \langle \sigma_{1} \rangle_{\text{dep}} = \text{Tr} [\alpha_{m}^{2} \sigma_{1} |U_{K} \otimes |P_{12}^{+} \otimes |P_{12}^{+} \rangle \langle U_{K} \otimes |P_{12}^{+} \otimes |P_{12}^{+}| \sigma_{1} |U_{K} \rangle] = \text{Tr} [\alpha_{m}^{2} \sigma_{1} |U_{K} \rangle \langle U_{K}|] \text{.} \]

The strategy still enjoys a polynomial advantage against state tomography.

\[ \text{Discussion} – \text{I introduced a model-independent quantitative characterization of quantum coherence for states of finite dimensional systems. The formalism is powerful enough to explore the interplay between quantum resources. In [30], I discuss the relation between quantum coherence, correlations and asymmetry. Also, some insights on the observation of coherence dynamics in open systems are provided. The significance of the proposed experimental schemes rests on their scalability, outperforming protocols based on state reconstruction, and generality. The result promises to propel a quantum phenomenology of macro-systems. It may help test the quantum speed up in high dimensional quantum registers, e.g. in quantum computing machines [35], and other deliverables of bottom-up quantum technology. Also, the quantum coherence measure could benchmark the efficiency of energy transport mechanisms in biological and artificial...} \]

The experimental detection of (a lower bound of the) quantum coherence of an unknown state in a d-dimensional system requires O(d) projective measurements on an ancillary qudit and the system itself.

**Result 2/bis.** The experimental detection of (a lower bound of the) quantum coherence of an unknown state in a d-dimensional system requires O(d) projective measurements on an ancillary qudit and the system itself.
intelligent systems [5, 36], becoming a standard tool for novel routes of investigation based on the top-down perspective. This consists in grasping the key features of complex macro-systems built up by Nature and then implement them in bio-inspired devices, which would be ipso facto robust against high temperature-high energy environments.

Proof of Result 1.

a) The skew information is a faithful measure of coherence. Indeed, it is convex and non-negative [24]. Also, it vanishes if the only if the state is incoherent. The latter is defined as a state whose density matrix is diagonal in a given basis. By definition, \( I(\rho, K) = 0 \iff \rho, K = 0 \), i.e. state and observable diagonalize in the same eigenbasis q.e.d.

b) The skew information is monotonically non-increasing under incoherent operations, which are expressed by a set of Kraus operators \( \{K_i\} \) such that \( \sum_i K_i^\dagger K_i = \mathbb{I}_A \) and \( K_i^\dagger K_j \subseteq \mathcal{L}(\mathbb{C}^n) \), where \( I_K \) is the set of incoherent states w.r.t. \( \{|K_i\rangle\} \).

The skew information \( I(\rho, K) \) does not increase on average by a von Neumann measurement of \( K \): \( \sum_i p_i I(K_i\rho_i K_i^\dagger, K) \leq I(\rho, K) \) [37]. The result and the convexity of the skew information proves the monotonicity for completely positive trace preserving (CPTP) incoherent maps, and that for any incoherent state \( \rho_{I_K} \) one has \( I(K_i\rho_{I_K} K_i^\dagger, K) = 0 \). I provide an alternative constructive argument for the class of K-invariant operations, which is a subset of the CPTP ones [7]. The skew information of a bipartite state \( \rho_{AB} \) satisfies \( I(\rho_{AB}, K_A \otimes I_B) \geq I(\mathcal{T}_{\rho_{AB}}, K_A, 4K) \).

A K-invariant channel on system A takes the form \( \mathcal{E}_A^{K}(\rho_A) = \mathcal{T}_{\rho_{AB}}(V_{AB}^{\dagger}(\rho_A \otimes \tau_B)V_{AB}) \), where \( V_{AB}^{\dagger} \) is a K-invariant unitary, i.e. \( V_{AB}^{\dagger}(K_A \otimes I_B + I_A \otimes K_B)V_{AB}^{\dagger} = K_A \otimes I_B + I_A \otimes K_B \), and \( \tau_B \in I_K \). One then obtains \( I(\rho_{AB}, K_A) = I(\mathcal{T}_{\rho_{AB}}, K_A \otimes I_B + I_A \otimes K_B) = I(\mathcal{T}_{\rho_{AB}}, V_{AB}^{\dagger}(K_A \otimes I_B + I_A \otimes K_B)V_{AB}^{\dagger}) \geq I(\rho_{AB}, K_A) \) q.e.d. One may further demand monotonicity under classical encoding: \( \sum_i p_i I(K_i^{\dagger} \rho A \otimes P_A K_A, n) \otimes \{|n\rangle\otimes \rho_{I_B} \otimes I_B, K \} \leq I(\rho_{AB}, K_A) \) (criterion C2c of [7]). The property is satisfied, since \( \sum_i p_i I(K_i^{\dagger} \rho A \otimes P_A K_A, n) \otimes \{|n\rangle\otimes \rho_{I_B} \otimes I_B, K \} \leq \sum_i p_i I(K_i^{\dagger} \rho_A P_A K_A \otimes \{|n\rangle\otimes \rho_{I_B} \otimes I_B, K \} \leq I(\rho_{AB}, K_A) \).

Proof of Result 2/2bis.

Here I prove that the schemes in Fig. 3 evaluate the overlap of two arbitrary density matrices \( \rho_{AB} \), generalizing the method of Ref. [34]. Result 2/2bis is then obtained as a case study with \( \rho_{AB} = \rho \) (TOP scheme), \( \rho_A = \rho, \rho_B = \rho U_K \rho U_K^\dagger \) (CENTRE) and \( \rho_A = \rho U_K \rho U_K^\dagger, \rho_B = \rho \) (BOTTOM). The steps of the protocol are:

a) Preparation of the input states: a d-dimensional ancilla (in a pure state, for simplicity) and two d-dimensional states whose density matrices are respectively \( \beta = \frac{1}{d}(I_d + x_B \cdot \hat{\tau}) \), \( |x_B| = 1 \), \( \rho_A = \frac{1}{d}(I_d + x_A \cdot \hat{\tau}) \), \( \rho_B = \frac{1}{d}(I_d + x_B \cdot \hat{\tau}) \). The goal is to determine \( \text{Tr}[\rho_A \rho_B] = \frac{1}{d}(1 + (d-1)x_A \cdot x_B) \).

b) Application of the gate \( \sqrt{V_{AB}} = \frac{1}{\sqrt{2}}(I_d - iV_{AB}) \) to the state \( \rho_A \) and the ancilla \( \beta \). The resulting marginal state of the ancilla at this intermediate stage is given by \( \beta_{out} = \frac{1}{d}(I_d + y_B \cdot \hat{\tau}) \), where \( y_B = \frac{1}{2}(x_A + x_B + (d-1)x_B \cdot y_B) \), where \( \wedge \) represents the exterior product.

c) Implementation of the second \( \sqrt{V_{AB}} \) gate to the ancilla and the state \( \rho_B \). The output state of the ancilla reads \( \beta_{out} = \frac{1}{d}(I_d + y_B \cdot \hat{\tau}) \), with \( y_B = \frac{1}{2}(x_B + y_B + (d-1)y_B \cdot x_B) \).

d) Performing a complete set of \( d \) projective measurements over a basis \( \{|ii\rangle\} = \rho_i, i = 1, 2, \ldots, d \) on the output state of the ancilla. A clever choice is such that the pure state \( \beta = |\beta\rangle\langle\beta| \) is an element of the basis: \( \text{Tr}[eta |ii\rangle\langle ii|] = \delta_{ii} \). The outcome of each measurement is \( S_i^{\dagger} \rho_B = \text{Tr}[^{\dagger} \beta_{out} \cdot \hat{\tau}] = (d-1)^{-1} x_B \cdot x_B + \frac{d-1}{4} \langle x_A \cdot x_B \rangle + \frac{d-1}{4} \langle x_A \cdot x_B \rangle \cdot x_B \cdot x_B \).

e) Repetition of the protocol by interchanging \( \rho_A, \rho_B \), obtaining the term \( S_A^{\dagger} \rho_A \cdot S_B \). One then has \( S_A^{\dagger} \rho_A + S_B^{\dagger} \rho_B = \frac{d}{4}(3(x_A \cdot x_B + x_B \cdot x_A) + 2\delta_{ii} \beta) + \frac{d-1}{4} \langle x_A \cdot x_B \rangle + \frac{d-1}{4} \langle x_A \cdot x_B \rangle \cdot (x_B \cdot x_B) \).

After some algebra (see appendix of [34] for the case \( A = B \)), one obtains \( \langle x_A \cdot x_B \rangle + \langle x_B \cdot x_A \rangle + \langle x_B \cdot x_B \rangle \langle x_A \cdot x_B \rangle + \langle x_A \cdot x_B \rangle \cdot (x_B \cdot x_B) \).

f) Additional \( d \) projective measurements on \( \rho_A, \rho_B \) have outcomes \( S_A^{\dagger} \rho_A \cdot S_B \) for the purity. \( \langle x_A \cdot x_B \rangle + \langle x_B \cdot x_A \rangle + \langle x_B \cdot x_B \rangle \langle x_A \cdot x_B \rangle + \langle x_A \cdot x_B \rangle \cdot (x_B \cdot x_B) \).

The method requires 5d measurements to obtain \( S_B \cdot S_{BA}, S_A, S_B \), for the overlap and \( S_{AA} \) for the purity. Allowing for interacting gates between \( \rho_{AB} \), the task requires 4d measurements. In such a case, the protocol has to be run setting \( \beta = \rho_B = \rho, \rho_B = U_K \rho U_K^\dagger \), then switching to \( \beta = \rho_B = U_K \rho U_K^\dagger, \rho_A = \rho \) and finally making \( d \) projective measurements on \( \rho_{AB} \).

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A. Quantum coherence in multipartite systems

The uncertainty on a quantum measurement is affected by the correlations shared by the system of interest with other parties. While absence of entanglement does not entail classicality, as it is not necessary to ensure coherence, the concepts of quantum coherence and quantum discord [1, 2], are entwined. The latter is defined as the least amount of disturbance experienced by a compound system due to a local measurement on one of the subsystems, say A. If and only if discord-like correlations are shared among parts of a compound system, then quantum coherence is guaranteed in any local basis [3]. Indeed, a class of bona fide measures of quantum discord of state \( \rho_{AB} \) is defined by \( \min_{K_A} I(\rho_{AB}, K_A \otimes I_B) \), where \( K_A \) indexes the spectrum of \( K_A^T \) [4]. The minimization is made over same spectrum observables, and each choice of the spectrum pinpoints a specific measure. Furthermore, the skew information framework highlights a to date unexplored (to my knowledge) kind of statistical dependence. Here I define the residual \( K_A \)-coherence as the difference between the quantum coherence of global and marginal states:

\[
C(\rho_{A_1A_2...A_n}, I_{A_1A_2...A_{n-1}} \otimes K_A) = I(\rho_{A_1A_2...A_n}, I_{A_1A_2...A_{n-1}} \otimes K_A \otimes I_{A_{n-1}A_{n-2}...A_1}) - I(\rho_A, K_A),
\]

where \( \rho_A \) is the marginal state of subsystem \( A_i \). The quantity is nonnegative, vanishing for states whose density matrix is block-diagonal in the eigenbasis of \( I_{A_1A_2...A_{n-1}} \otimes K_A \otimes I_{A_{n-1}A_{n-2}...A_1} \). The operational power and a full algorithmic characterization of the residual \( K_A \)-coherence is worthy of investigation.

B. Quantum coherence dynamics in open systems

FIG. 4: (Colors online) Characterizing quantum memory effects by coherence revivals. In an open quantum system, the system \( S \) under scrutiny (blue ellipse) interacts with a second system or a bunch of systems, called environment \( E \) (red ellipse). The interaction entails a two-way exchange of information: from system to environment (red arrow and sphere) and vice versa (green arrow and sphere). A scalable model-independent description quantum coherence in open quantum systems consists in studying the behaviour of the skew information (Result 2) and the signature of coherence revivals given by Result 1A. An experimentalist may track the behaviour of the lower bound \( I^L \) and a coherence revival measure induced by the lower bound \( R^L \).

Here I benchmark the dynamics of quantum coherence in open systems by introducing a signature of coherence revivals, which is able to capture the coherence induced by the environment back-flow of information in the system. No information on
In order to develop universal methods to measure, sustain and shield coherence, one needs to investigate the dynamical evolution of quantum open systems [5]. In such a paradigm, the system interacts with an environment, which both destroys and injects coherence in the system by a two-way flow of information (Fig. 4). The interaction is studied along the evolution of a set of parameters \( \vec{\theta} = \{ \theta^i \} \) (e.g. time, temperature, etc.). The back-flow of information from the environment has been linked to the non-Markovian character of the evolution in time [6–9]. Yet, measures of non-Markovianity or non-reversibility are usually cast in form of state space distance functions or entropies. Thus, they evaluate revivals from decay of the distinguishability between density matrices, which is arguably given by both quantum and classical effects. Also, they demand full information on the density matrix to be worked out. I identify a class of memory effects characterized by revivals of a genuinely quantum resource, i.e. coherence, under the evolution of a single parameter \( \theta^i \). This allows to recover information on potentially useful environment back-action and filters out detrimental noise.

One evaluates coherence revivals through the dynamics of the skew information:

**Result 1A.** The quantum coherence revivals of a state \( \rho \) for the interval \( (\theta^i_0, \theta^i) \) are quantified by

\[
R(\theta^i) = \int_{\theta^i_0}^{\theta^i} \left| \frac{\partial I(\rho(\theta^i), K)}{\partial \theta^i} \right| d\theta^i - (I(\rho(\theta^i_0), K) - I(\rho(\theta^i), K)).
\]  

(A.2)

The quantity is positive whenever the decay of coherence properties of the open systems experiences a revival, being inspired by the signature of entanglement revivals introduced in Ref. [8]. An experimentally convenient measure is given by \( \mathcal{R}_G(\theta^i) = \int_{\theta^i_0}^{\theta^i} \left| \frac{\partial I(\rho(\theta^i), K)}{\partial \theta^i} \right| d\theta^i - (I(\rho(\theta^i_0), K) - I(\rho(\theta^i), K)) \), but the correspondence between revivals of the skew information and the lower bound has to be checked case by case.

C. A common framework for quantum resources: coherence and asymmetry

**Introduction**

The quantum asymmetry of a state is the quantum coherence emerging in the measurement of the charge under a superselection rule (SSR) [10, 11]. I introduce a new measure of asymmetry based on the skew information, which evaluates the ability of a quantum state to act as a reference frame under SSR. A piece of information is said unspeakable if it cannot be encoded into words or bits [12]. For example, the transformation law between the coordinates of two distant laboratories is not determined without a shared reference frame. This is an ancillary system which singles out a Cartesian triad, e.g. a gyroscope. Unspeakable information could be an impediment to reach efficient communication processing. In particular, multipartite quantum protocols are usually designed taking for granted that all the players align with an agreed reference frame. In fact, a routine operation as a phase shift has no meaning without uniquely determining the phase direction [13]. The problem has been intensively investigated in recent years [10, 14–24]. The absence of a reference frame has been proven equivalent to constrain quantum dynamics by SSR [10, 14, 25–28], while the ability of a system to act as reference frame is the quantum resource known as asymmetry or frameness [10]. A series of results highlighted the limitations to entanglement creation and ensemble quantum information processing under a SSR [14]. Reference alignment strategies have been developed [10], and experimental studies have discussed and implemented toolkits for alignment-free quantum information [23, 24].

**A measure of quantum asymmetry**

I recall the technical definition of SSR [14, 25]. A G-SSR for a quantity \( Q \) (charge) is defined as a law of invariance of the state of a system with respect to a transformation group \( G \). Given a system with Hilbert space \( \mathcal{H} \) and a unitary representation \( U : G \to \mathcal{B}(\mathcal{H}) \) mapping the group to the set of bounded observables on the Hilbert space, any operation \( \mathcal{E}^Q \) is said \( G \)-covariant if it satisfies \( \mathcal{E}^Q(U(g)\rho U(g)^\dagger) = U(g)\mathcal{E}^Q(\rho) U(g)^\dagger, \forall g \in G \). There is no way to distinguish by means of a \( G \)-covariant operation, without violating the G-SSR, the state \( \rho \) from \( U(g)\rho U(g)^\dagger, \forall g \). Thus, for finite groups, the physical states are described by the density matrices obtained by averaging over the group transformations through the \( G \)-twirling operation \( \mathcal{G}[\rho] = \frac{1}{\text{dim} G} \sum_{g \in G} U(g)\rho U(g)^\dagger \) (an equivalent definition holds for Lie groups) [14]. Any density matrix \( \rho \) with off-diagonal entries (coherence) in the basis of the eigenstates of \( Q \) is projected by the average over the group transformations into the diagonal state \( \mathcal{G}[\rho] \). Consequently, it is not distinguishable, by allowed physical operations, from \( \mathcal{G}[\rho] \), and it cannot be exploited for quantum information tasks, unless one could overcome the limitations imposed by the SSR by accessing a reference frame, i.e. an ancillary system \( R \) which...
quantum coherence and asymmetry in the thermodynamics of quantum systems certainly deserves to be explored.

An entropic measure of asymmetry is the relative entropy of \( G \)-frameness or \( G \)-asymmetry \( S(G[\rho]) - S(\rho) \), being \( S \) the von Neumann entropy [18, 20, 21].

I introduce a new measure of quantum asymmetry:

**Result 2A.** Given a \( G \)-SSR with related charge \( Q \), the skew information

\[
I(\rho, Q) = -\frac{1}{2} \text{Tr} \left[ \sqrt{\rho} Q^2 \right] - \text{Tr} [\sqrt{\rho} Q \sqrt{\rho} Q]
\]

(A.3)

satisfies the criteria identifying an asymmetry measure of the state \( \rho \) [10, 29].

**Proof.**

a) The skew information is a faithful measure of asymmetry. It is convex, non-negative and \( I(\rho, Q) = 0 \) \( \iff \rho = G[\rho] \).

Under a SSR, a physical state \( \rho \) is either eigenstate \( |q\rangle \) of \( Q \) or mixture of its eigenstates \( \sum_q c_q |q\rangle \langle q| \). In the first case, it is trivial to see that the skew information is zero, while for the mixture, by exploiting the convexity of \( I \), one obtains \( I(\sum_q c_q |q\rangle \langle q|, Q) \leq \sum_q c_q I(|q\rangle \langle q|, Q) = 0 \). Also, by construction one has \( I(\rho, Q) = 0 \iff [\rho, Q] = 0 \) and \( [\rho, Q] = 0 \iff [\rho, U(g)] = 0, \forall g \in G \), which is true if and only if the state is symmetric, i.e. \( \rho = G[\rho] \).

b) It is monotonically non-increasing under \( G \)-covariant operations: \( I(\mathcal{E}^G(\rho), Q) \leq I(\rho, Q), \forall \mathcal{E}^G \). \( G \)-covariant operations correspond to a subset of incoherent operations with respect to the basis \( |q\rangle \).

The proof b) of Res. 1 works here as well. In the very same way, one builds \( G \)-covariant operations \( \mathcal{E}^G_A \) and shows that \( I(\rho_A, Q_A) \geq I(\mathcal{E}^G_A(\rho_A), Q_A) \) (see Theorem II.1 of [30]).

Accessing unspeakable information means displaying coherence: quantum asymmetry represents the amount of coherence in the eigenbasis of the charge [11]. Under a SSR on the system \( S \), one has \( I(\rho_S, Q_S) = 0 \). It is known that the SSR is broken by introducing a reference frame \( R \), which has access to asymmetric states, and then coupling it by a global \( G \)-invariant operation with \( S \) [10]. I give a general quantitative prescription for symmetry breaking transformations in terms of quantum uncertainties, by exploiting the properties of the skew information. For any observables \( Q_S, Q_R \) and states \( \rho_S, \tau_R \), one has \( I(\rho_S \otimes \tau_R, Q_S \otimes |\psi_S\rangle \langle \psi_S|) = I(\rho_S, Q_S) + I(\tau_R, Q_R) \) and \( I(\rho_S, Q_S \otimes |\psi_S\rangle \langle \psi_S|) = 0 \iff I(\text{Tr}[\rho_S \otimes Q_S], Q_S) = I(\text{Tr}[\rho_S \otimes |\psi_S\rangle \langle \psi_S|], Q_S) = 0 \). Given the \( G \)-invariant transformation \( \mathcal{U}_{SR}(\rho_S \otimes \tau_R) \mathcal{U}_{SR}^\dagger = \tilde{\rho}_{SR}, \) one obtains \( I(\tilde{\rho}_{SR}, Q_S \otimes |\psi_S\rangle \langle \psi_S|) = I(\rho_S \otimes \tau_R, Q_S \otimes |\psi_S\rangle \langle \psi_S|) = I(\tau_R, Q_R) \). Thus, if \( I(\tau_R, Q_R) > 0 \), one can obtain asymmetric states of the system \( S : I(\text{Tr}[\tilde{\rho}_{SR}], Q_S) > 0 \).

I finally remark that if the observable is the Hamiltonian of the system, then the skew information turns out to be the Hessian matrix of the relative entropy between the state and the equilibrium state, which equals the free energy [31]. The role played by quantum coherence and asymmetry in the thermodynamics of quantum systems certainly deserves to be explored.

**Unspeakable quantum information on a state eigenbasis**

![FIG. 5](Colors online) Transformation of coordinates between \( A \) and \( B \). The observer in \( A \) (blue) describes a two-dimensional system (qubit) by the density matrix \( \rho_A \). A second observer (purple) has a different description expressed by the density matrix \( \rho_B \). The transformation law \( U \) between the two systems reference frames is not known. Yet, one can evaluate the quantum uncertainty about the eigenbasis of the system. This corresponds to the unspeakable information on the eigenbasis \( |\psi_i\rangle \), which is given by the relative skew information \( R I(\rho_A, \rho_B) \).
The knowledge of an observer Alice about a quantum state is formally represented by a density matrix, whose spectral decomposition reads \( \rho^A = \sum \lambda_i |\psi^A_i \rangle \langle \psi^A_i | \). The \( \{ \lambda_i \} \) is the spectrum of the state. This piece of information is \emph{speakeable}: Alice may transmit it to a second observer Bob by sending bits, regardless of the physical conveyer. Conversely, the information about the eigenbasis \( \{ |\psi^A_i \rangle \} \) is \emph{unspeakable}: it cannot be recovered by a set of measurement outcomes or encoded in bits, and in general it cannot be teleported [32]. Bob’s description of the state is \( \rho^B = \sum \lambda_i |\phi^B_i \rangle \langle \phi^B_i | \). Without knowing the transformation law \( U^A \rightarrow^B \langle \phi^A_i | = \langle \phi^B \rangle \), i.e. without sharing a reference frame, the information on the eigenbasis is not accessible to Bob. I here introduce the concept of \emph{relative} unspeakable information captured by the following measure:

**Result 3A.** The unspeakable information hidden to Bob on the state eigenbasis \( \{ |\psi^A_i \rangle \} \) is quantified by the relative skew information

\[
RI(\rho^A, \rho^B) = -\frac{1}{2} Tr [\sqrt{\rho^A \rho^B \rho^A \rho^B}] = Tr[\rho^A \rho^B - \sqrt{\rho^A \rho^B} \sqrt{\rho^A \rho^B}].
\]

The expanded expression in terms of spectral decompositions of the two states is

\[
RI(\rho^A, \rho^B) = \sum_{i,j} \lambda_i \lambda_j |\psi^A_i \rangle \langle \psi^A_j | - \sum_{i,k,j} \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l} \langle \psi^A_i | \langle \psi^A_j | \langle \psi^A_k | \langle \psi^A_l |
\]

\[
\times |\langle \psi^B_i | \langle \psi^B_j | \langle \psi^B_k | \langle \psi^B_l |\rangle\rangle| 
\]

\[
= \sum_{i \neq j \neq k \neq l} \sqrt{\lambda_i \lambda_j \lambda_k \lambda_l} \langle \psi^A_i | \langle \psi^A_j | \langle \psi^A_k | \langle \psi^A_l |
\]

\[
\times |\langle \psi^B_i | \langle \psi^B_j | \langle \psi^B_k | \langle \psi^B_l |\rangle\rangle|.
\]

The quantity is nonnegative, convex and vanishing if and only if the density matrices commute.

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