Lagrangian symmetries and supersymmetries depending on derivatives. Global analysis

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Abstract: Generalized symmetries and supersymmetries depending on derivatives of dynamic variables are treated in a most general setting. Studying cohomology of the variational bicomplex, we state the first variational formula and conservation laws for Lagrangian systems on fiber bundles and graded manifolds under generalized symmetries and supersymmetries of any order. Cohomology of nilpotent generalized supersymmetries are obtained.

1 Introduction

Symmetries of differential equations under transformations of dynamic variables depending on their derivatives have been intensively investigated (see [3, 27, 29, 33] for a survey). Following [3, 33], we agree to call them the generalized symmetries in contrast with the classical (point) ones. In mechanics, conservation laws corresponding to generalized symmetries are well known [33]. In field theory, BRST transformations provide the most interesting example of generalized supersymmetries [16, 17].

Generalized symmetries of Lagrangian systems on a local coordinate domain have been described in detail [33]. We aim to provide the global analysis of Lagrangian systems on fiber bundles and graded manifolds under generalized symmetries and supersymmetries of any order.

Let us note that an $m$-order differential equation on a finite-dimensional smooth fiber bundle $\pi : Y \rightarrow X$ is conventionally defined as a closed subbundle of the $m$-order jet bundle $J^mY \rightarrow X$ of sections of $Y \rightarrow X$ [11, 29]. Euler–Lagrange equations need not satisfy this condition, unless an Euler–Lagrange operator is of constant rank. Therefore, we regard infinitesimal symmetry transformations as derivations of the graded differential algebra (henceforth GDA) $\mathcal{O}_\infty$ of exterior forms on jet manifolds, but not as manifold maps. This approach is straightforwardly extended to Lagrangian systems on graded manifolds.

We use the first variational formula of the calculus of variations in order to obtain Lagrangian conservation laws [18, 31, 38]. Recall that an $r$-order Lagrangian on a fiber bundle...
$Y \to X$ is defined as a horizontal density $L : J^r Y \to \wedge^n T^* X$, $n = \dim X$, on the $r$-order jet manifold $J^r Y$. If $X = \mathbb{R}$, we are in the case of non-relativistic time-dependent mechanics. A classical symmetry is represented by a projectable vector field $u$ on $Y \to X$ seen as an infinitesimal generator of a local one-parameter group of bundle automorphisms of $Y \to X$. Let $L_{J^r u} L$ be the Lie derivative of $L$ along the jet prolongation $J^r u$ of $u$ onto $J^r Y$. The first variational formula provides its canonical decomposition

$$L_{J^r u} L = u_V \delta L + d_H(h_0(J^{2r-1} u|\Xi_L)), \quad (1.1)$$

where $\delta L$ is the Euler–Lagrange operator, $\Xi_L$ is a Lepagean equivalent (e.g., a Poincaré–Cartan form) of $L$, $u_V$ is the vertical part of $u$, $d_H$ is the total differential, and $h_0$ is the horizontal projection (see all the definitions below). Let $u$ be a divergence symmetry of $L$, i.e., the Lie derivative $L_{J^r u} L$ is a total differential $d_H \sigma$. Then the first variational formula (1.1) on the kernel $\text{Ker} \delta L$ of the Euler–Lagrange operator $\delta L$ leads to the weak conservation law

$$0 \approx d_H(h_0(J^{2r-1} u|\Xi_L) - \sigma). \quad (1.2)$$

If $u$ is a variational symmetry of $L$, i.e., $L_{J^r u} L = 0$, the conservation law (1.2) comes to the conservation law of the Noether current $J_u = h_0(J^{2r-1} u|\Xi_L)$.

In the case of a classical symmetry, the first variational formula (1.1) and the existence of a globally defined symmetry current $J_u$ issue from the existence of a global Lepagean equivalent $\Xi_L$ of $L$ [23]. In order to extend the first variational formula to generalized symmetries (formula (3.7)) and generalized supersymmetries (formula (6.4)), we derive it from the decomposition

$$dL = \delta L - d_H(\Xi), \quad (1.3)$$

provided by the global exactness of the subcomplex of one-contact forms of the variational bicomplex on fiber bundles and graded manifolds (Propositions 2.3 and 4.3). As a consequence, the existence of a global finite order Lepagean equivalent of a graded Lagrangian is stated.

A vector field $u$ in the first variational formula (1.1) represents a derivation of the $\mathbb{R}$-ring $C^\infty(Y)$ of smooth real functions on $Y$. Accordingly, a $k$-order generalized vector field $\vartheta$ can be defined as a derivation of the $\mathbb{R}$-ring $C^\infty(Y)$ with values into the ring $C^\infty(J^kY)$ of smooth real functions on the jet manifold $J^kY$. This definition recovers the geometric notion of a generalized vector field as a section of the pull-back bundle $TY \times J^kY \to J^kY$ in [16]. The prolongation $J^r \vartheta$ of $\vartheta$ onto any finite order jet manifold $J^r Y$ is that one calls a generalized symmetry. We give the intrinsic definition of a generalized symmetry as a derivation $\nu$ of the $\mathbb{R}$ ring $O^n_0$ such that the Lie derivative $L_{\nu}$ preserves the ideal of contact forms of the above mentioned GDA $O^n_\infty$ (Propositions 3.1 – 3.3). The key point is that the Lie derivative $L_{J^r \vartheta}$ along a generalized symmetry $J^r \vartheta$ sends exterior forms on the jet manifold $J^r Y$ onto those on the jet manifold $J^{r+k} Y$. By virtue of the well-known Bäcklund theorem, $L_{J^r \vartheta}$ preserves $O^*_r$ iff either $\vartheta$ is a vector field on $Y$ or $Y \to X$ is a one-dimensional bundle and $\vartheta$ is a
generalized vector field at most of first order [27]. Thus, considering generalized symmetries, one deals with Lagrangian systems of unspecified finite order.

Infinite order jet formalism provides a convenient tool for studying these systems [2, 20, 29, 31, 38, 41]. With the inverse system of finite order jet manifolds

\[ X \leftarrow \pi Y \leftarrow \pi_1 J^1 Y \leftarrow \cdots \leftarrow J^{r-1} Y \pi_r J^r Y \leftarrow \cdots, \tag{1.4} \]

we have the direct system

\[ O^*(X) \xrightarrow{\pi^*} O^*(Y) \xrightarrow{\pi_1^*} \cdots \xrightarrow{\pi_r^*} O^*_r \longrightarrow \cdots \tag{1.5} \]

of GDAs of exterior forms on these manifolds with respect to the pull-back monomorphisms \( \pi_r^* \). Its direct limit is the above mentioned GDA \( \mathcal{O}_\infty^* \) consisting of all the exterior forms on finite order jet manifolds modulo the pull-back identification. The exterior differential on \( \mathcal{O}_\infty^* \) is decomposed into the sum \( d = d_H + d_V \) of the total and the vertical differentials. These differentials and the variational operator \( \delta \) split \( \mathcal{O}_\infty^* \) into the variational bicomplex (2.4), which provides the algebraic description of Lagrangian systems on a fiber bundle \( Y \to X \).

Restricted to a coordinate domain of \( Y \), this bicomplex, except the terms \( R \), is exact. One refers to this fact as the algebraic Poincaré lemma (e.g., [33]). Recently, we have stated cohomology of the variational bicomplex for an arbitrary \( Y \) [19, 20, 39]. The key point is that this cohomology provides a topological obstruction to local divergence symmetries to be the global ones. For instance, if a generalized symmetry \( \vartheta \) is a divergence symmetry of a Lagrangian \( L \), the equality \( \delta(L J_r \vartheta L) = 0 \) holds, but the converse is not true because of the de Rham cohomology group \( H^n(Y) \).

Remark 1.1. Let us point out the following technical detail repeatedly met in the sequel. The de Rham cohomology of \( \mathcal{O}_\infty^* \) is easily proved to equal the de Rham cohomology \( H^*(Y) \) of \( Y \) [2]. However, one has to enlarge the GDA \( \mathcal{O}_\infty^* \) in order to find its \( d_H \)- and \( \delta \)-cohomology. Let \( \mathfrak{O}_r^* \) be the sheaf of germs of exterior forms on the \( r \)-order jet manifold \( J^r Y \), and let \( \mathfrak{O}_r^* \) be its canonical presheaf. We throughout follow the sheaf terminology of [26]. There is the direct system of presheaves

\[
\mathfrak{O}_X^* \xrightarrow{\pi^*} \mathfrak{O}_0^* \xrightarrow{\pi_1^*} \mathfrak{O}_1^* \cdots \xrightarrow{\pi_r^*} \mathfrak{O}_r^* \longrightarrow \cdots
\]

Its direct limit \( \mathfrak{O}_\infty^* \) is a presheaf of GDAs on the projective limit of the inverse system (1.4) of jet manifolds. This projective limit, called the infinite order jet space \( J^\infty Y \), is endowed with the weakest topology such that surjections \( \pi_r^\infty : J^\infty Y \to J^r Y \) are continuous. This topology makes \( J^\infty Y \) into a paracompact Fréchet manifold [41]. Let \( \mathfrak{O}_r^* \) be a sheaf constructed from \( \mathfrak{O}_\infty^* \). The module \( \mathcal{Q}_\infty^* = \Gamma(\mathfrak{O}_\infty^*) \) of sections of \( \mathfrak{O}_\infty^* \) is a GDA such that, given an element \( \phi \in \mathfrak{O}_\infty^* \) and a point \( z \in J^\infty Y \), there exist an open neighbourhood \( U \) of \( z \) and an exterior form \( \phi^{(k)} \) on some finite order jet manifold \( J^k Y \) so that \( \phi|_U = \phi^{(k)} \circ \pi_k^\infty|_U \). In particular, there is the monomorphism \( \mathcal{O}_\infty^* \rightarrow \mathcal{Q}_\infty^* \). The key point is that the paracompact space \( J^\infty Y \) admits a partition of unity by elements of the ring \( \mathcal{Q}_\infty^0 \) [41] and \( Y \) is a strong deformation.
retract of $J^\infty Y$ [2, 20]. These facts have enabled one to obtain $d_H$- and $\delta$-cohomology of $Q^*_\infty$ [1, 2, 41]. Recently, we have shown that its subalgebra $O^*_\infty$ possesses the same $d_H$- and $\delta$-cohomology as $Q^*_\infty$ [19, 39].

The following two peculiarities of generalized supersymmetries should be additionally noted. Firstly, generalized supersymmetries are expressed into jets of odd variables, and we should define an algebra where they act. Secondly, generalized supersymmetries can be nilpotent.

Stimulated by BRST theory, we do not focus on particular geometric models of ghost fields in gauge theories (e.g., [7, 40]), but consider Lagrangian systems of odd variables in a general setting. For this purpose, one calls into play fiber bundles over graded manifolds and supermanifolds [12, 13, 32]. However, the antifield BRST theory on $X = \mathbb{R}^n$ [4, 5, 8, 9] involves jets of odd fields only with respect to space-time coordinates. Therefore, we describe odd variables on a smooth manifold $X$ as generating elements of the structure ring of a graded manifold, whose body is $X$. By the well-known Batchelor theorem [6], such a graded manifold is isomorphic to the one whose structure sheaf $\mathfrak{A}_Q$ is formed by germs of sections of the exterior product

$$\wedge Q^* = \mathbb{R} \oplus Q^* \oplus \frac{2}{X} \wedge Q^* \oplus \cdots,$$

(1.6)

where $Q^*$ is the dual of some real vector bundle $Q \to X$. In physical models, a vector bundle $Q$ is usually given from the beginning. Therefore, we restrict our consideration to graded manifolds $(X, \mathfrak{A}_Q)$ where the Batchelor isomorphism holds fixed, i.e., automorphisms of $(X, \mathfrak{A}_Q)$ are restricted to those induced by bundle automorphisms of $Q$. This restriction enables us to handle the structures which are not preserved by general automorphisms of a graded manifold. We agree to call $(X, \mathfrak{A}_Q)$ a simple graded manifold constructed from $Q$. Accordingly, $r$-order jets of odd variables are defined as generating elements of the structure ring of the simple graded manifold $(X, \mathfrak{A}_{J^rQ})$ constructed from the jet bundle $J^rQ$ of $Q$ [31, 36]. Let $C^*_{J^rQ}$ be the bigraded differential algebra (henceforth BGDA) of graded exterior forms on the graded manifold $(X, \mathfrak{A}_{J^rQ})$. Since $\pi^r_{r-1} : J^rQ \to J^{r-1}Q$ is a linear bundle morphism over $X$, it yields the morphism of graded manifolds $(X, \mathfrak{A}_{J^rQ}) \to (X, \mathfrak{A}_{J^{r-1}Q})$ and the monomorphism of the BGDA$s$ $C^*_{J^{r-1}Q} \to C^*_{J^rQ}$ [6, 31]. Hence, there is the direct system of BGDA$s$

$$C^*_Q \xrightarrow{\pi^1_0} C^*_{J^1Q} \longrightarrow \cdots C^*_{J^rQ} \xrightarrow{\pi^{r+1}_r} \cdots,$$

(1.7)

whose direct limit $C^*_\infty$ consists of graded exterior forms on graded manifolds $(X, \mathfrak{A}_{J^rQ})$, $0 \leq r$, modulo the pull-back identification.

This definition of odd jets differs from that of jets of a graded fiber bundle in [25], but reproduces the heuristic notion of jets of ghosts in the above mentioned antifield BRST theory on $\mathbb{R}^n$. Moreover, it enables one to describe odd and even variables (e.g., classical fields, ghosts, ghosts-for-ghosts and antifields in BRST theory) on the same footing. Namely, let a smooth fiber bundle $Y \to X$ be affine. Then its de Rham cohomology equals that of $X$. Let $\mathcal{P}^*_\infty$ be the $C^\infty(X)$-subalgebra of the GDA $O^*_\infty$ which consists of exterior forms
whose coefficients are polynomial in the fiber coordinates on $J^\infty Y \to X$. Let us consider the product $\mathcal{S}_\infty^*$ of graded algebras $\mathcal{O}_\infty^*$ and $\mathcal{P}_\infty^*$ over their common subalgebra $\mathcal{O}^*(X)$. It is a BGDA. For the sake of brevity, we continue to call its elements the graded forms.

Similarly to $\mathcal{O}_\infty^*$, the BGDA $\mathcal{S}_\infty^*$ is split into the graded variational bicomplex, which provides the algebraic description of Lagrangian systems of even and odd variables indexed by elements of the fiber bundles $Y$ and $Q$ over a smooth manifold $X$. Following a procedure similar to that in Remark 1.1, we obtain cohomology of some complexes of the BGDA $\mathcal{S}_\infty^*$. These are the short variational complex of horizontal (local in the terminology of [5, 8]) graded exterior forms (4.4), the complex of one-contact graded exterior forms (4.5) and the de Rham complex (4.6). Cohomology of the first and third complexes is proved to equal the de Rham cohomology $H^*(X)$ of $X$, while the second one is globally exact (Theorem 4.1). This exactness provides the decomposition (1.3) of a graded Lagrangian and leads to the first variational formula and conservation law under generalized supersymmetries (Propositions 4.3 and 6.2). Cohomology $H^{<n}(X)$ of the short variational complex is the main ingredient in a computation of the iterated cohomology of nilpotent generalized supersymmetries.

By analogy with a generalized symmetry, a generalized supersymmetry $\upsilon$ is defined as a graded derivation of the $\mathbb{R}$-ring $\mathcal{S}_0^\infty$ such that the Lie derivative $\mathbf{L}_\upsilon$ preserves the contact ideal of the BGDA $\mathcal{S}_\infty^*$ (Proposition 6.1). The BRST transformation $\upsilon$ (6.6) in gauge theory on a principal bundle exemplifies such a generalized supersymmetry. Its peculiarity is that the Lie derivative $\mathbf{L}_\upsilon$ of horizontal graded forms is nilpotent. This fact motivates us to study nilpotent generalized supersymmetries in a general setting.

Note that nilpotent generalized supersymmetries are necessarily odd, i.e., there are no nilpotent generalized symmetries. The key point is that the Lie derivative $\mathbf{L}_\upsilon$ along a generalized supersymmetry and the total differential $d$ mutually commute. If $\mathbf{L}_\upsilon$ is nilpotent, let us suppose that the $d_H$-complex $\mathcal{S}_\infty^{0,*}$ of horizontal graded forms is split into a complex of complexes $\{\mathcal{S}_{k,m}\}$ with respect to $\mathbf{L}_\upsilon$ and $d_H$. In order to make it into a bicomplex, let us introduce the nilpotent operator

$$\mathbf{s}_\upsilon \phi = (-1)^{|\phi|} \mathbf{L}_\upsilon \phi, \quad \phi \in \mathcal{S}_0^{0,*},$$

such that $d_H \circ \mathbf{s}_\upsilon + \mathbf{s}_\upsilon \circ d_H = 0$. In the case of the BRST transformation $\upsilon$ (6.6), $\mathbf{s}_\upsilon$ (1.8) is the BRST operator. The bicomplex $S^{*,*}$ is graded by the form degree $0 \leq m \leq n$ and an integer $k \in \mathbb{Z}$. Let us consider horizontal graded forms $\phi \in \mathcal{S}_0^{0,*}$ such that a nilpotent generalized supersymmetry $\upsilon$ is their divergence symmetry, i.e. $\mathbf{L}_\upsilon \phi = d_H \phi$. We come to the relative and iterated cohomology of $\mathbf{s}_\upsilon$ with respect to the total differential $d_H$. In the antifield BRST theory, relative cohomology is known as the local BRST cohomology [5, 8] (see [15] for the BRST cohomology modulo the exterior differential $d$). Relative and iterated cohomology groups coincide with each other on horizontal densities, and they naturally characterize graded Lagrangians $L$, for which $\upsilon$ is a divergence symmetry, modulo the Lie derivatives $\mathbf{L}_\upsilon \xi$, $\xi \in \mathcal{S}_0^{0,*}$, and the $d_H$-exact graded forms.

We obtain the iterated cohomology $H^{*,m<n}(\mathbf{s}_\upsilon|d_H)$ (Theorem 7.2) and state the relation between the iterated cohomology $H^{*,n}(\mathbf{s}_\upsilon|d_H)$ and the total $(\mathbf{s}_\upsilon + d_H)$-cohomology of the
bicomplex $S^{*,*}$ (Theorem 7.3). This relation plays a prominent role, e.g., in the antifield BRST theory [5, 8]. Note that relative cohomology of form degree $m < n$ fails to be related to the total cohomology. For instance, in Section 9.6 of [5], iterated BRST cohomology in fact is considered.

2 Lagrangian systems of unspecified finite order on fiber bundles

This Section addresses the basic formulae for finite order Lagrangian systems on a smooth fiber bundle $Y \to X$ in the framework of infinite order jet formalism. The similar formulae for graded Lagrangian systems will be stated in Section 4. Our main goal is the decomposition (1.3).

Remark 2.1. Smooth manifolds throughout are real, finite-dimensional, Hausdorff, second-countable (hence, paracompact) and connected.

Any bundle coordinate atlas $\{(U_Y; x^\lambda, y^i)\}$ of $\pi : Y \to X$ yields the coordinate atlas

$$\{((\pi^\infty)^{-1}(U_Y); x^\lambda, y^i)\}, \quad y^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x^\lambda} d\mu y^i_{\Lambda}, \quad 0 \leq |\Lambda|,$$

(2.1)
of the Fréchet manifold $J^\infty Y$, where $\Lambda = (\lambda_k...\lambda_1)$ is a symmetric multi-index of length $k$, $\lambda + \Lambda = (\lambda_k...\lambda_1)$, and

$$d_{\lambda} = \partial_{\lambda} + \sum_{|\Lambda|\geq 0} y^i_{\lambda+\Lambda} \partial^\lambda_i, \quad d_{\Lambda} = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}, \quad \Lambda = (\lambda_r...\lambda_1),$$

(2.2)
are the total derivatives. Hereafter, we fix an atlas of $Y$ and, consequently, that of $J^\infty Y$ containing a finite number of charts $(U_Y; x^\lambda, y^i)$ (their branches $U_Y$ however need not be domains) [24].

Restricted to a coordinate chart (2.1), elements of the GDA $O^*_\infty$ can be written in a coordinate form; horizontal forms $\{dx^\lambda\}$ and contact one-forms $\{\theta^i = dy^i - y^i_{\lambda+\Lambda} dx^\lambda\}$ make up a local basis for the $O^0_\infty$-algebra $O^*_\infty$. There is the canonical decomposition

$$O^*_\infty = \oplus O^{k,m}_\infty$$
of this algebra into $O^0_\infty$-modules $O^{k,m}_\infty$ of $k$-contact and $m$-horizontal forms together with the corresponding projections $h_k : O^*_\infty \to O^{k,*}_\infty$ and $h^m : O^*_\infty \to O^{*m}_\infty$. Accordingly, the exterior differential on $O^*_\infty$ is split into the sum $d = d_H + d_V$ of the total and vertical differentials

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi),$$

$$d_V \circ h^m = h^m \circ d \circ h^m, \quad d_V(\phi) = \theta^i \wedge \partial^\lambda_i \phi, \quad \phi \in O^*_\infty.$$

One also introduces the $\mathbb{R}$-module projector

$$\varrho = \sum_{k>0} \frac{1}{k} h_k \circ h^k, \quad \overline{\varrho}(\phi) = \sum_{|\Lambda|\geq 0} (-1)^{|\Lambda|} \theta^i \wedge [d_{\Lambda}(\partial^\lambda_i \phi)], \quad \phi \in O^{0,n}_\infty,$$

(2.3)
of $\mathcal{O}_\infty^*$ such that $\varrho \circ d_H = 0$ and the nilpotent variational operator $\delta = \varrho \circ d$ on $\mathcal{O}_\infty^{k,n}$. Put $E_k = \varrho(\mathcal{O}_\infty^{k,n})$. As a consequence, the GDA $\mathcal{O}_\infty^*$ is split into the above mentioned variational bicomplex

$$
0 \rightarrow \mathcal{O}_\infty^{1,0} \xrightarrow{d_H} \mathcal{O}_\infty^{1,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{O}_\infty^{1,m} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{O}_\infty^{1,n} \xrightarrow{\varrho} E_1 \rightarrow 0
$$

$$(2.4)$$

The second row from the bottom and the last column of this bicomplex assemble into the variational complex

$$
0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}_\infty^0 \xrightarrow{d_H} \mathcal{O}_\infty^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{O}_\infty^{0,m} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{O}_\infty^{0,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \rightarrow \cdots.
$$

$$(2.5)$$

One can think of its elements

$$L = \mathcal{L}\omega \in \mathcal{O}_\infty^{0,n}, \quad \delta L = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} d_A(\partial_A^L)\theta^i \wedge \omega \in E_1, \quad \omega = dx^1 \wedge \cdots \wedge dx^n,$$

as being a finite order Lagrangian and its Euler–Lagrange operator.

**Theorem 2.1.** Cohomology of the variational complex (2.5) is isomorphic to the de Rham cohomology of a fiber bundle $Y$ [19, 39].

**Outline of proof.** We have the complex of sheaves of $\mathcal{Q}_\infty^0$-modules

$$
0 \rightarrow \mathbb{R} \rightarrow \mathcal{Q}_\infty^0 \xrightarrow{d_H} \mathcal{Q}_\infty^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}_\infty^{0,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \rightarrow \cdots, \quad \mathcal{E}_k = \varrho(\mathcal{Q}_\infty^{k,n}),
$$

$$(2.6)$$

on $J_\infty Y$ and the complex of their structure modules

$$
0 \rightarrow \mathbb{R} \rightarrow \mathcal{Q}_\infty^0 \xrightarrow{d_H} \mathcal{Q}_\infty^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}_\infty^{0,n} \xrightarrow{\delta} \mathcal{E}_1 \xrightarrow{\delta} \mathcal{E}_2 \rightarrow \cdots.
$$

$$(2.7)$$

Since the paracompact space $J_\infty Y$ admits a partition of unity by elements of the ring $\mathcal{Q}_\infty^0$, the sheaves $\mathcal{Q}_\infty^{0,k}$ in the complex (2.6) are fine. The sheaves $\mathcal{E}_k$ are also proved to be fine [19, 39]. Consequently, all sheaves, except $\mathbb{R}$, in the complex (2.5) are acyclic. Then, by virtue of the above mentioned algebraic Poincaré lemma, the complex (2.6) is a resolution of the constant sheaf $\mathbb{R}$ on $J_\infty Y$. In accordance with the abstract de Rham theorem [26], cohomology of the complex (2.7) equals the cohomology of $J_\infty Y$ with coefficients in $\mathbb{R}$. The latter, in turn, is isomorphic to the de Rham cohomology of $Y$, which is a strong deformation retract of
Finally, the $d_H$- and $\delta$-cohomology of $Q^*_\infty$ is proved to equal that of its subalgebra $O^*_\infty$ [19, 39]. □

**Corollary 2.2.** Every $d_H$-closed form $\phi \in O^{0,m-n}_\infty$ is the sum

$$\phi = h_0 \varphi + d_H \xi, \quad \xi \in O^{0,m-1}_\infty,$$

(2.8)

where $\varphi$ is a closed $m$-form on $Y$. Every $\delta$-closed form (a variationally trivial Lagrangian) $L \in O^{0,n}_\infty$ is the sum

$$L = h_0 \varphi + d_H \xi, \quad \xi \in O^{0,n-1}_\infty,$$

(2.9)

where $\varphi$ is a closed $n$-form on $Y$.

**Remark 2.2.** The formulae (2.8) – (2.9) have been stated in [1] by computing cohomology of the fixed order variational sequence, but the proof of the local exactness of this sequence requires rather sophisticated *ad hoc* techniques. The proof of Theorem 4.1 below on cohomology of the graded variational bicomplex follows the above proof of Theorem 2.1.

**Proposition 2.3.** For any Lagrangian $L \in O^{0,n}_\infty$, there is the decomposition (1.3), where $\Xi \in O^{1,n-1}_\infty$.

**Proof.** Let us consider the third row

$$0 \to O^{1,0}_\infty \xrightarrow{d_H} O^{1,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} O^{1,n}_\infty \xrightarrow{\rho} E_1 \to 0$$

(2.10)

from the bottom of the variational bicomplex (2.4). Similarly to the proof of Theorem 2.1, one can show that the complex (2.10) is exact. Its exactness at the term $O^{1,n}_\infty$ relative to the projector $\rho$ provides the $\mathbb{R}$-module decomposition

$$O^{1,n}_\infty = E_1 \oplus d_H(O^{1,n-1}_\infty).$$

It leads to the splitting (1.3) of $dL \in O^{1,n}_\infty$. The form $\Xi$ in this splitting is not uniquely defined. It reads

$$\Xi = \sum_{s=0}^r F^{\lambda\nu_k\ldots\nu_1}_i \rho^{\nu_k\ldots\nu_1}_i \wedge \omega_\lambda, \quad \omega_\lambda = \partial_\lambda \omega,$$

(2.11)

$$F^{\nu_k\ldots\nu_1}_i = \partial^{\nu_k\ldots\nu_1}_i L - d_\lambda F^{\lambda\nu_k\ldots\nu_1}_i + h^{\nu_k\ldots\nu_1}_i,$$

where local functions $h \in O^0_\infty$ obey the relations $h^{\nu}_i = 0$, $h^{(\nu_k\ldots\nu_{k-1})\nu_1}_i = 0$. It follows that $\Xi_L = \Xi + L$ is a Lepagean equivalent, e.g., a Poincaré–Cartan form of a finite order Lagrangian $L$ [23]. □
3 Generalized Lagrangian symmetries

A derivation $v \in \mathfrak{d}O^0_{\infty}$ of the $\mathbb{R}$-ring $O^0_{\infty}$ is said to be a generalized symmetry if the Lie derivative $L_v$, being a derivation of the GDA $O^*_{\infty}$, preserves its ideal of contact forms. Forthcoming Propositions 3.1 – 3.3 confirm the contentedness of this definition.

Proposition 3.1. The derivation module $\mathfrak{d}O^0_{\infty}$ is isomorphic to the $O^0_{\infty}$-dual $O^1_{\infty}$ of the module of one-forms $O^1_{\infty}$.

Proof. At first, let us show that $O^*_{\infty}$ is generated by elements $df$, $f \in O^0_{\infty}$. It suffices to justify that any element of $O^1_{\infty}$ is a finite $O^0_{\infty}$-linear combination of elements $df$, $f \in O^0_{\infty}$. Indeed, every $\phi \in O^1_{\infty}$ is an exterior form on some finite order jet manifold $J^rY$. By virtue of the de Rham theorem extended to vector bundles over non-compact manifolds [34, 37], the $C^\infty(J^rY)$-module $O^r_{\infty}$ of one-forms on $J^rY$ is a projective module of finite rank, i.e., $\phi$ is represented by a finite $C^\infty(J^rY)$-linear combination of elements $df$, $f \in C^\infty(J^rY) \subset O^0_{\infty}$. Any element $\Phi \in (O^1_{\infty})^*$ yields a derivation $v_\Phi(f) = \Phi(df)$ of the $\mathbb{R}$-ring $O^0_{\infty}$. Since the module $O^1_{\infty}$ is generated by elements $df$, $f \in O^0_{\infty}$, different elements of $(O^1_{\infty})^*$ provide different derivations of $O^0_{\infty}$, i.e., there is a monomorphism $(O^1_{\infty})^* \to \mathfrak{d}O^0_{\infty}$. By the same formula, any derivation $v \in \mathfrak{d}O^0_{\infty}$ sends $df \mapsto v(f)$ and, since $O^0_{\infty}$ is generated by elements $df$, it defines a morphism $\Phi_v : O^1_{\infty} \to O^0_{\infty}$. Moreover, different derivations $v$ provide different morphisms $\Phi_v$. Thus, we have a monomorphism and, consequently, an isomorphism $\mathfrak{d}O^0_{\infty} \to (O^1_{\infty})^*$. □

Remark 3.1. As follows from Proposition 3.1, the de Rham complex of the GDA $O^*_{\infty}$ is both the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{d}O^0_{\infty}$ with coefficients in $O^0_{\infty}$ and the universal differential calculus over the $\mathbb{R}$-ring $O^0_{\infty}$.

Proposition 3.2. Relative to an atlas (2.1), a derivation $v \in \mathfrak{d}O^0_{\infty}$ is given by the expression

$$v = v^\lambda \partial_\lambda + v^i \partial_i + \sum_{|\lambda| > 0} v^i_\lambda \partial_i^\lambda,$$

(3.1)

where $v^\lambda$, $v^i$, $v^i_\lambda$ are local smooth functions of finite jet order obeying the transformation law

$$v^\lambda_j = \frac{\partial x^\lambda}{\partial x^\mu} v^\mu, \quad v^i_j = \frac{\partial y^i}{\partial y^j} v^j + \frac{\partial y^i}{\partial x^\mu} v^\mu, \quad v^i_\lambda = \sum_{|\Sigma| \leq |\lambda|} \frac{\partial y^i_\Sigma}{\partial x^\mu} v^\mu_\Sigma + \frac{\partial y^i_\lambda}{\partial x^\mu} v^\mu.$$

(3.2)

Proof. Restricted to a coordinate chart (2.1), $O^1_{\infty}$ is a free $O^0_{\infty}$-module generated by the exterior forms $dx^\lambda$, $\theta^\lambda_A$. Then $\mathfrak{d}O^0_{\infty} = (O^1_{\infty})^*$ restricted to this chart consists of elements (3.1), where $\partial_\lambda$, $\partial^\lambda_i$ are the duals of $dx^\lambda$, $\theta^\lambda_A$. The transformation rule (3.2) results from the transition functions (2.1). Since the atlas (2.1) is finite, a derivation $v$ (3.1) preserves $O^*_{\infty}$. □
The interior product \( v\hat{\cdot}\phi \) and the Lie derivative \( L_v \phi, \phi \in \mathcal{O}_\infty^* \), obey the standard formulae.

**Proposition 3.3.** A derivation \( v \) (3.1) is a generalized symmetry iff

\[
v^i_\Lambda = d_\Lambda (v^i - y^i_\mu v^\mu) + y^i_{\mu + \Lambda} v^\mu, \quad 0 < |\Lambda|. \tag{3.3}
\]

*Proof.* The expression (3.3) results from a direct computation similarly to the first part of the above mentioned Bäcklund theorem. Then one can justify that local functions (3.3) fulfill the transformation law (3.2). \(\square\)

Any generalized symmetry admits the horizontal splitting

\[
v = v_H + v_V = v^\lambda d_\lambda + \left( \vartheta^i \partial_i + \sum_{|\Lambda| > 0} d_\Lambda \vartheta^i \partial^\Lambda \right), \quad \vartheta^i = v^i - y^i_\mu v^\mu, \tag{3.4}
\]

relative to the canonical connection \( \nabla = dx^\lambda \otimes d_\lambda \) on the \( C^\infty(X) \)-ring \( \mathcal{O}_\infty^0 \) [31]. For instance, let \( \tau \) be a vector field on \( X \). Then the derivation \( \tau \hat{\cdot}(d_H f), f \in \mathcal{O}_\infty^0 \), of \( \mathcal{O}_\infty^0 \) is a horizontal generalized symmetry \( v = \tau^\mu d_\mu \). A direct computation shows that any vertical generalized symmetry \( v = v_V \) obeys the relations

\[
v \hat{\cdot} d_H \phi = -d_H (v \hat{\cdot} \phi), \tag{3.5}
\]

\[
L_v (d_H \phi) = d_H (L_v \phi), \quad \phi \in \mathcal{O}_\infty^* \tag{3.6}
\]

**Proposition 3.4.** Given a Lagrangian \( L \in \mathcal{O}_\infty^{0,n} \), its Lie derivative \( L_v L \) along a generalized symmetry \( v \) (3.4) fulfils the first variational formula

\[
L_v L = v \hat{\cdot} \delta L + d_H (h_0 (v \hat{\cdot} \Xi_L)) + \mathcal{L} (v_H \hat{\cdot} \omega), \tag{3.7}
\]

where \( \Xi_L \) is a Lepagean equivalent, e.g., a Poincaré–Cartan form of \( L \).

*Proof.* The formula (3.7) comes from the splitting (1.3) and the relation (3.5) as follows:

\[
L_v L = v \hat{\cdot} dL + d_\nu (v \hat{\cdot} L) = v \hat{\cdot} dL + d_H (v_H \hat{\cdot} L) + \mathcal{L} (v_H \hat{\cdot} \omega) =
\]

\[
v \hat{\cdot} \delta L - v \hat{\cdot} d_\nu \Xi + d_H (v_H \hat{\cdot} L) + \mathcal{L} (v_H \hat{\cdot} \omega) =
\]

\[
v \hat{\cdot} \delta L + d_\nu (v \hat{\cdot} \Xi + v_H \hat{\cdot} L) + \mathcal{L} (v_H \hat{\cdot} \omega),
\]

where we put \( \Xi_L = \Xi + L \). \(\square\)

In comparison with the first variational formula (1.1) for classical symmetries, the right-hand side of the first variational formula (3.7) contains an additional contact term which vanishes if a generalized symmetry \( v \) is projected onto \( X \), i.e., its components \( v^\lambda \) depend only on coordinates on \( X \).
Let $v$ be a divergence symmetry of $L$, i.e., $L_u L = d_H \sigma$, $\sigma \in \mathcal{O}_{\infty}^{0,n-1}$. By virtue of the expression (3.8), this condition implies that a generalized symmetry $v$ is projected onto $X$. Then the first variational formula (3.7) takes the form

$$d_H \sigma = v_V \delta L + d_H (h_0(v) \Xi_L).$$

(3.9)

Restricted to $\text{Ker} \delta L$, it leads to the weak conservation law

$$0 \approx d_H (h_0(v) \Xi_L) - \sigma.$$

(3.10)

A glance at the expression (3.8) shows that a generalized symmetry $v$ (3.4), projected onto $X$, is a divergence symmetry of a Lagrangian $L$ iff its vertical part $v_V$ is so. Moreover, $v$ and $v_V$ lead to the same conservation law (3.10). Thus, we can restrict our consideration to vertical divergence symmetries $v$. In this case, the conservation law (3.10) takes the form

$$0 \approx d_H (v) \Xi_L - \sigma,$$

where $\mathcal{J}_v = v \Xi_L$ is the Noether current along $v$.

It should be noted that a generalized symmetry is almost never a variational symmetry of a Lagrangian. Let us obtain the characteristic equation for divergence symmetries of a Lagrangian $L$. Let $v$ be a vertical generalized symmetry. Then the Lie derivative $L_u L$ (3.8) is a horizontal density. Let us require that it is a $\delta$-closed form, i.e., $\delta (L_u L) = 0$. In accordance with the equality (2.9), this condition is fulfilled iff

$$L_u L = h_0 \varphi + d_H \sigma,$$

(3.11)

where $\varphi$ is a closed $n$-form on $Y$, i.e., $v$ at least locally is a divergence symmetry of $L$. It is readily observed that the topological obstruction $h_0 \varphi$ (3.11) for $v$ to be a global divergence symmetry is at most of first order. If $Y \to X$ is an affine bundle, its de Rham cohomology equals that of $X$ and, consequently, the topological obstruction $h_0 \varphi = \varphi$ (3.11) reduces to a non-exact $n$-form on $X$. Recall that, by virtue of the master identity

$$L_{f^2 u} \delta L = \delta (L_{f^2 u} L),$$

any classical divergence symmetry of a Lagrangian is also a symmetry of its Euler–Lagrange operator. However, this equality is not true for generalized symmetries [33]. It comes to the relation

$$\delta (L_u L) = L_u \delta L + \sum_{|\lambda| > 0} (-1)^{|\lambda|} d_\lambda (\partial^\lambda_k v^i \delta_i L dy^k) \wedge \omega.$$

4 Graded Lagrangian systems

Let $(X, \mathfrak{A}_Q)$ be the simple graded manifold constructed from a vector bundle $Q \to X$ of fiber dimension $m$. Its structure ring $\mathcal{A}_Q$ of sections of $\mathfrak{A}_Q$ consists of sections of the exterior
bundle (1.6) called graded functions. Given bundle coordinates \((x^\lambda, q^a)\) on \(Q\) with transition functions \(q^a = \rho_b^aq^b\), let \(\{c^a\}\) be the corresponding fiber bases for \(Q^* \to X\), together with transition functions \(c^a = \rho^b_c b\). Then \((x^\lambda, c^a)\) is called the local basis for the graded manifold \((X, \mathcal{A}_Q)\) [6, 31]. With respect to this basis, graded functions read

\[
f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1...a_k} c^{a_1} \cdots c^{a_k},
\]

where \(f_{a_1...a_k}\) are local smooth real functions on \(X\).

Given a graded manifold \((X, \mathcal{A}_Q)\), by the sheaf \(\mathfrak{A}_Q\) of graded derivations of \(\mathcal{A}_Q\) is meant a subsheaf of endomorphisms of the structure sheaf \(\mathcal{A}_Q\) such that any section \(u\) of \(\mathfrak{A}_Q\) over an open subset \(U \subset X\) is a graded derivation of the graded ring \(\mathcal{A}_Q(U)\) of graded functions on \(U\), i.e.,

\[
u(ff') = u(f)f' + (-1)^{[u][f]} fu(f'), \quad f, f' \in \mathcal{A}_Q(U),
\]

for homogeneous elements \(u \in \mathfrak{A}_Q(U)\) and \(f, f' \in \mathcal{A}_Q(U)\), where \([\cdot]\) denotes the Grassmann parity. One can show that sections of \(\mathfrak{A}_Q\) over \(U\) exhaust all graded derivations of the graded ring \(\mathcal{A}_Q(U)\) [6]. Let \(\mathfrak{d}_Q\) be the Lie superalgebra of graded derivations of the \(\mathbb{R}\)-ring \(\mathcal{A}_Q\). Its elements are called graded vector fields on \((X, \mathcal{A}_Q)\). Due to the canonical splitting \(VQ = Q \times Q\), the vertical tangent bundle \(VQ \to Q\) of \(Q \to X\) can be provided with the fiber bases \(\{\partial_a\}\) which is the dual of \(\{c^a\}\). Then a graded vector field takes the local form \(u = u^\lambda \partial_\lambda + u^a \partial_a\), where \(u^\lambda, u^a\) are local graded functions. It acts on \(\mathcal{A}_Q\) by the rule

\[
u(f_a...b c^a \cdots c^b) = u^\lambda \partial_\lambda (f_a...b)c^a \cdots c^b + u^af_a...b\partial_a](c^a \cdots c^b).
\]

(4.1)

This rule implies the corresponding transformation law

\[
u^\lambda = u^\lambda, \quad u^a = \rho^a_j u^j + u^\lambda \partial_\lambda (\rho^a_j)c^j.
\]

Then one can show [31, 35] that graded vector fields on a simple graded manifold can be represented by sections of the vector bundle \(V_Q \to X\) which is locally isomorphic to the vector bundle

\[
V_Q|_U \approx \Lambda^Q|_U \otimes (Q \oplus TX)|_U,
\]

and is equipped with the bundle coordinates \((x^\lambda_{a_1...a_k}, v^b_{b_1...b_k})\), \(k = 0, \ldots, m\), together with the transition functions

\[
x^\mu_{ii...i_k} = \rho_a^{-1} a_k \cdots \rho_a^{-1} a_k x^\lambda_{a_1...a_k},
\]

\[
\nu^j_{j_1...j_k} = \rho_{b_1}^{-1} j_k \cdots \rho_{b_1}^{-1} j_k \left[ \rho_j^j v^b_{b_1...b_k} + \frac{k!}{(k-1)!} x^\lambda_{b_1...b_{k-1}} \partial_\lambda \rho_b^j \right].
\]

Using this fact, one can introduce graded exterior forms on the graded manifold \((X, \mathcal{A}_Q)\) as sections of the exterior bundle \(\wedge V_Q^\ast\), where \(V_Q^* \to X\) is the pointwise \(\wedge Q^*\)-dual of \(V_Q\).
Relative to the dual bases \( \{dx^\lambda\} \) for \( T^*X \) and \( \{dc^b\} \) for \( Q^* \), sections of \( \mathcal{V}_Q \to X \) (graded one-forms) read
\[
\phi = \phi_\lambda dx^\lambda + \phi_a dc^a, \quad \phi_a' = \rho^{-1b}_a \phi_b, \quad \phi'_\lambda = \phi_\lambda + \rho^{-1b}_a \partial_\lambda (\rho^a_j) \phi u c^j.
\]
The duality morphism is given by the interior product
\[
u| \phi = u^\lambda \phi_\lambda + (-1)^{|\phi_a|} u^a \phi_a.
\]
Graded exterior forms constitute the BGDA \( C^*_Q \) with respect to the graded exterior product \( \wedge \) and the even exterior differential \( d \). Recall the standard formulae
\[
\phi \wedge \sigma = (-1)^{|\phi||\sigma|} \phi \wedge \sigma \wedge \phi, \quad d(\phi \wedge \sigma) = (d\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (d\sigma),
\]
\[
u| (\phi \wedge \sigma) = (\nu| \phi) \wedge \sigma + (-1)^{|\phi|+|\nu|} \phi \wedge (\nu| \sigma),
\]
\[
\mathbf{L}_u \phi = \nu| d\phi + d(\nu| \phi), \quad \mathbf{L}_u (\phi \wedge \sigma) = \mathbf{L}_u (\phi) \wedge \sigma + (-1)^{|\nu|} \phi \wedge \mathbf{L}_u (\sigma).
\]

Since the jet bundle \( J^rQ \to X \) of a vector bundle \( Q \to X \) is a vector bundle, let us consider the simple graded manifold \((X, \mathfrak{A}_{J^rQ})\) constructed from \( J^rQ \to X \). Its local basis is \( \{x^\lambda, c^a_\lambda\} \), \( 0 \leq |\lambda| \leq r \), together with the transition functions
\[
c^a_{\lambda+\Lambda} = d_\lambda (\rho^a_j c^b_\Lambda), \quad d_\lambda = \partial_\lambda + \sum_{|\Lambda|<r} c^a_{\lambda+\Lambda} \partial_a^\Lambda, \quad (4.2)
\]
where \( \partial_a^\Lambda \) are the duals of \( c^a_\lambda \). Let \( C^*_rQ \) be the above mentioned BGDA of graded exterior forms on the graded manifold \((X, \mathfrak{A}_{J^rQ})\). It is locally a free \( C^\infty(X) \)-algebra finitely generated by the elements \((1, c^a_\lambda, dx^\lambda, \theta^a_\lambda = dc^a_\lambda - c^a_{\lambda+\Lambda} dx^\lambda), 0 \leq |\lambda| \leq r \). The direct limit \( C^*_\infty \) of the direct system \((1.7)\) inherits the BGDA operations which commute with the monomorphisms \( \pi^r_{r+1} \). It is locally a free \( C^\infty(X) \)-algebra countably generated by the elements \((1, c^a_\lambda, dx^\lambda, \theta^a_\lambda), 0 \leq |\lambda| \).

It should be emphasized that, in contrast with the GDA \( O^*_\infty \), the BGDA \( C^*_\infty \) consists of sections of sheaves on \( X \). In order to regard these algebras on the same footing, let us consider the open surjection \( \pi^\infty : J^\infty \to X \) and the direct image \( \pi^\infty_* \mathfrak{T}^*_\infty \) on \( X \) of the sheaf \( \mathfrak{T}^*_\infty \) of exterior forms on \( J^\infty Y \). Its stalk at a point \( x \in X \) consists of the equivalence classes of sections of the sheaf \( \mathfrak{T}^*_\infty \) which coincide on the inverse images \((\pi^\infty)^{-1}(U_x)\) of open neighbourhoods \( U_x \) of \( X \). Since \((\pi^\infty)^{-1}(U_x)\) is the infinite order jet space of sections of the fiber bundle \( \pi^{-1}(U_x) \to X \), every point \( x \in X \) has a base of open neighbourhoods \( \{U_x\} \) such that the sheaves \( \mathfrak{T}^*_\infty \) of \( \mathcal{Q}_0^* \)-modules and the sheaves \( \mathfrak{E}_k \) in Theorem \((2.1)\) are acyclic on the inverse images \((\pi^\infty)^{-1}(U_x)\) of these neighbourhoods. Then, in accordance with the Leray theorem \([21]\), cohomology of \( J^\infty Y \) with coefficients in the sheaves \( \mathfrak{T}^*_\infty \) and \( \mathfrak{E}_k \) is isomorphic to that of \( X \) with coefficients in their direct images \( \pi^\infty_* \mathfrak{T}^*_\infty \) and \( \pi^\infty_* \mathfrak{E}_k \), i.e., the sheaves \( \pi^\infty_* \mathfrak{T}^*_\infty \) and \( \pi^\infty_* \mathfrak{E}_k \) on \( X \) are acyclic. Hereafter, let \( Y \to X \) be an affine bundle. Then \( X \) is a strong deformation retract of \( J^\infty Y \). In this case, the inverse images \((\pi^\infty)^{-1}(U_x)\) of contractible neighbourhoods \( U_x \) are contractible and \( \pi^\infty_* \mathbb{R} = \mathbb{R} \). Then, by virtue of the
algebraic Poincaré lemma, the variational bicomplex $\mathcal{F}_\infty$ of sheaves on $(\pi_\infty)^{-1}(U_x)$, except the terms $\mathbb{R}$, is exact, and the variational bicomplex $\pi_\infty^*\mathcal{F}_\infty$ of sheaves on $X$ is so. There is the $\mathbb{R}$-algebra isomorphism of the GDA of sections of the sheaf $\pi_\infty^*\mathcal{F}_\infty$ on $X$ to the GDA $\mathcal{Q}_\infty^*$. Thus, the GDA $\mathcal{Q}_\infty^*$ and its subalgebra $\mathcal{O}_\infty^*$ can be regarded as algebras of sections of sheaves on $X$, and they keep their $d_-$, $d_H$- and $\delta$-cohomology expressed into the de Rham cohomology $H^*(X) = H^*(Y)$ of $X$ [19].

Let us restrict our consideration to the above mentioned polynomial subalgebra $\mathcal{P}_\infty^*$ of $\mathcal{O}_\infty^*$. Let us consider the graded product $\mathcal{S}_\infty^* = \mathcal{C}_\infty^* \wedge \mathcal{P}_\infty^*$ of graded algebras $\mathcal{C}_\infty^*$ and $\mathcal{P}_\infty^*$ over their common graded subalgebra $\mathcal{O}_\infty^*(X)$. It consists of the elements

$$
\psi \otimes \phi, \quad (\psi \wedge \sigma) \otimes \phi = \psi \otimes (\sigma \wedge \phi), \quad \psi \in \mathcal{C}_\infty^*, \quad \phi \in \mathcal{P}_\infty^*, \quad \sigma \in \mathcal{O}_\infty^*,
$$

$$
\phi \otimes \psi, \quad (\phi \wedge \sigma) \otimes \psi = \phi \otimes (\sigma \wedge \psi), \quad \psi \in \mathcal{C}_\infty^*, \quad \phi \in \mathcal{P}_\infty^*, \quad \sigma \in \mathcal{O}_\infty^*,
$$

of the tensor products $\mathcal{C}_\infty^* \otimes \mathcal{P}_\infty^*$ and $\mathcal{P}_\infty^* \otimes \mathcal{C}_\infty^*$ of the $\mathcal{O}_\infty^*$-modules $\mathcal{C}_\infty^*$ and $\mathcal{P}_\infty^*$ which are subject to the commutation relation

$$
\psi \otimes \phi = (-1)^{|\psi||\phi|} \phi \otimes \psi
$$

and the multiplication

$$
(\psi \otimes \phi) \wedge (\psi' \otimes \phi') = (-1)^{|\psi||\phi'|}(\psi \wedge \psi') \otimes (\phi \wedge \phi'),
$$

written for homogeneous elements of graded algebras $\mathcal{C}_\infty^*$ and $\mathcal{P}_\infty^*$. Introducing the notation

$$
\psi \otimes 1 = 1 \otimes \psi = \psi, \quad 1 \otimes \phi = \phi \otimes 1 = \phi, \quad \psi \otimes \phi = (\psi \otimes 1) \wedge (1 \otimes \phi) = \psi \wedge \phi,
$$

one can think of $\mathcal{S}_\infty^*$ as being a bigraded algebra generated by elements of $\mathcal{C}_\infty^*$ and $\mathcal{P}_\infty^*$ and provided with the total form degree $|\psi \wedge \phi| = |\psi| + |\phi|$ and the total Grassmann parity $[\psi \wedge \phi] = [\psi]$. For instance, elements of the ring $S_\infty^0$ are polynomials of $c^a_A$ and $y^i_A$ with coefficients in $\mathcal{C}_\infty^*(X)$. The sum of exterior differentials on $\mathcal{C}_\infty^*$ and $\mathcal{P}_\infty^*$ makes $\mathcal{S}_\infty^*$ into a BGDA with the standard rules

$$
\varphi \wedge \varphi' = (-1)^{|\varphi||\varphi'|+|\varphi||\varphi'|} \varphi' \wedge \varphi, \quad d(\varphi \wedge \varphi') = d\varphi \wedge \varphi' + (-1)^{|\varphi|} \varphi \wedge d\varphi'.
$$

It is locally generated by the elements $(1, c^a_A, y^i_A, dx^\lambda, \theta^a_A, \theta^i_A), |A| \geq 0$.

Remark 4.1. If $Y \to X$ is a vector bundle, one can get the BGDA $\mathcal{S}_\infty^* = \mathcal{C}_\infty^* \wedge \mathcal{P}_\infty^*$ in a different way. Let us consider the Whitney sum $S = Q \oplus Y$ of vector bundles $Q \to X$ and $Y \to X$ regarded as a bundle of graded vector spaces $Q_x \oplus Y_x, x \in X$. Let us define the quotient $\overline{S}^k$ of the tensor product

$$
\overline{S}^k = \mathbb{R} \oplus S_x^* \otimes \bigoplus_{k}^{\infty} \otimes \bigoplus_{X}^{k} S_x^*
$$

by the elements

$$
q \otimes q' + q' \otimes q, \quad y \otimes y' - y' \otimes y, \quad q \otimes y - y \otimes q
$$
for all \( q, q' \in Q_x, y, y' \in Y_x \), and \( x \in X \). The \( C^\infty(X) \)-modules \( \mathcal{A}_S^k \) of sections of the vector bundles \( \mathcal{S}^k \rightarrow X \) make up a direct system with respect to the natural monomorphisms \( \mathcal{A}_S^k \rightarrow \mathcal{A}_S^{k+1} \). Its direct limit \( \mathcal{A}_S^\infty \) is endowed with a structure of a graded commutative \( C^\infty(X) \)-ring generated by odd and even elements. Generalizing the above technique for a graded manifold \((X, \mathcal{A}_Q)\) to \((X, \mathcal{A}_S^\infty)\), one obtains the BGDA isomorphic to \( \mathcal{S}_\infty^\ast \) [31, 36].

**Remark 4.2.** In physical applications, one can think of \( \mathcal{S}_\infty^\ast \) as being a graded algebra of even and odd variables on a smooth manifold \( X \). In particular, this is the case of the above mentioned antifield BRST theory on \( X = \mathbb{R}^n \) [4, 5, 8, 9]. Recall that, in gauge theory on a principal bundle \( P \rightarrow X \) with a structure Lie group \( G \), principal connections on \( P \rightarrow X \) are represented by sections of the quotient \( C = J^1P/G \rightarrow X \) [18, 31, 38]. The connection bundle \( C \rightarrow X \) is affine. It is coordinated by \((x^\lambda, a^\lambda_\lambda)\) such that, given a section \( A \subset C \rightarrow X \), its components \( A_\lambda^\lambda = a^\lambda_\lambda \circ A \) are coefficients of the familiar local connection form (i.e., gauge potentials). Let \( J^\infty C \) be the infinite order jet space of \( C \rightarrow X \) coordinated by \((x^\lambda, a^\lambda_\lambda), 0 \leq |\lambda|\), and let \( \mathcal{P}_\infty^\ast(C) \) be the polynomial subalgebra of the GDA \( \mathcal{O}_\infty^\ast(C) \) of exterior forms of finite jet order on \( J^\infty C \) whose coefficients are polynomials of \( a^\lambda_\lambda \). Infinitesimal generators of one-parameter groups of vertical automorphisms (gauge transformations) of a principal bundle \( P \) are \( G \)-invariant vertical vector fields on \( P \rightarrow X \). They are associated to sections of the vector bundle \( \mathcal{V}_G P = VP/G \rightarrow X \) of right Lie algebras of the group \( G \). Let us consider the simple graded manifold \((X, \mathcal{A}_{V_G P})\) constructed from this vector bundle. Its local basis is \((x^\lambda, C^\nu)\). Let \( \mathcal{C}_{V Q} \) be the BGDA of graded exterior forms on the graded manifold \((X, \mathcal{A}_{J^rV_G P})\), and \( \mathcal{C}_\infty^\ast(V_G P) \) the direct limit of the direct system (1.7) of these algebras. Then the graded product

\[
\mathcal{S}_\infty^\ast(V_G, C) = \mathcal{C}_\infty^\ast(V_G P) \wedge \mathcal{P}_\infty^\ast(C)
\]

describe gauge potentials, odd ghosts and their jets in the BRST theory. A generic basis for BRST theory contains the following three sectors: (i) even classical fields of vanishing ghost number (e.g., the above mentioned gauge potentials), (ii) odd ghosts, ghosts-for-ghosts and antifields, (iii) even ghosts-for-ghosts and antifields [5, 9, 22]. From the physical viewpoint, it seems more natural to describe odd and even elements of the non-classical sectors (ii) – (iii) of this basis in the framework of the unified construction in Remark 4.1 [31, 36].

Hereafter, let the collective symbols \( s^\lambda_\lambda \) and \( \theta^\lambda_\lambda \) stand both for even and odd generating elements \( c^\lambda_\lambda, y^\lambda_\lambda, \theta^\lambda_\lambda, \theta^\lambda_\lambda \) of the \( C^\infty(X) \)-algebra \( \mathcal{S}_\infty^\ast \). Similarly to \( \mathcal{O}_\infty^\ast \), the BGDA \( \mathcal{S}_\infty^\ast \) is decomposed into \( \mathcal{S}_\infty^0 \)-modules \( \mathcal{S}_\infty^{k,r} \) of \( k \)-contact and \( r \)-horizontal graded forms, together with the corresponding projections \( h_k \) and \( h^r \). Accordingly, the graded exterior differential \( d \) on \( \mathcal{S}_\infty^\ast \) is split into the sum \( d = d_H + d_V \) of the total and vertical differentials

\[
d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad d_V(\phi) = \theta^\lambda_\lambda \wedge \partial^\lambda_\lambda(\phi), \quad \phi \in \mathcal{S}_\infty^\ast
\]

The projection endomorphism \( \varrho \) of \( \mathcal{S}_\infty^\ast \) is given by the expression

\[
\varrho = \sum_{k>0} \frac{1}{k} h_k \circ h^n, \quad \varrho(\phi) = \sum_{|\lambda| \geq 0} (-1)^{|\lambda|} \theta^\lambda_\lambda \wedge [d_\lambda(\partial^\lambda_\lambda) \phi], \quad \phi \in \mathcal{S}_\infty^{0,n}
\]
similar to (2.3). The graded variational operator \( \delta = \rho \circ d \) is introduced. Then the BGDA \( S^*_{\infty} \) is split into the graded variational bicomplex, analogous to the bicomplex (2.4).

The key point is that, in contrast with the variational bicomplex (2.4), the algebraic Poincaré lemma has been stated only for the short variational complex

\[
0 \rightarrow \mathbb{R} \rightarrow S^0_{\infty} \xrightarrow{d_H} S^1_{\infty} \rightarrow \cdots \xrightarrow{\delta} S^{n}_{\infty} \rightarrow 0,
\]

of the BGDA \( S^*_{\infty} \), i.e., this complex on \( X = \mathbb{R}^n \) is exact at all terms, except \( \mathbb{R} \) [5, 14]. We also consider the complex

\[
0 \rightarrow S^1_{\infty} \xrightarrow{d_H} S^1_{\infty} \rightarrow \cdots \xrightarrow{\delta} S^k_{\infty} \rightarrow \cdots
\]

of graded one-forms, analogous to the complex (2.10), and the de Rham complex

\[
0 \rightarrow \mathbb{R} \rightarrow S^0_{\infty} \xrightarrow{d} S^1_{\infty} \rightarrow \cdots
\]

One can think of elements

\[
L = L \omega \in S^0_{\infty}, \quad \delta(L) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda (\partial^A L) \in E_1
\]

of the complexes (4.4) – (4.5) as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

**Theorem 4.1.** The cohomology of the complexes (4.4) and (4.6) equals the de Rham cohomology \( H^*(X) \) of \( X \). The complex (4.5) is exact.

**Proof.** Next Section is devoted to the proof. \( \square \)

**Corollary 4.2.** Every \( d_H \)-closed graded form \( \phi \in S^m_{\infty} \) falls into the sum

\[
\phi = \varphi + d_H \xi, \quad \xi \in S^{m-1}_{\infty},
\]

where \( \varphi \) is a closed \( m \)-form on \( X \). Every \( \delta \)-closed graded form (a variationally trivial graded Lagrangian) \( L \in S^0_{\infty} \) is the sum

\[
\phi = \varphi + d_H \xi, \quad \xi \in S^{n-1}_{\infty},
\]

where \( \varphi \) is a non-exact \( n \)-form on \( X \).

The global exactness of the complex (4.5) at the term \( S^1_{\infty} \) results in the following.

**Proposition 4.3.** Given a graded Lagrangian \( L \), there is the decomposition

\[
dL = \delta L - d_H(\Xi), \quad \Xi \in S^{n-1}_{\infty},
\]

\[
\Xi = \sum_{s=0}^{n-1} \theta^A_{\nu_s \ldots \nu_1} \wedge F^A_{\lambda \nu_s \ldots \nu_1} \omega_\lambda, \quad F^A_{\lambda \nu_s \ldots \nu_1} = \partial^A_{\lambda \nu_s \ldots \nu_1} L - d_\lambda F^A_{\nu_s \ldots \nu_1} + h^A_{\nu_s \ldots \nu_1},
\]

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where local graded functions \( h \) obey the relations 
\[ h_a^{\nu_1 \nu_2 \cdots \nu_k} = 0, \quad h_a^{(\nu_1 \nu_2 \cdots \nu_k)} = 0. \]

**Proof.** The proof of the decomposition (4.9) repeats that in Proposition 2.3. The coordinate expression (4.10) results from a direct computation. \( \square \)

Proposition 4.3 states the existence of a global finite order Lepage equivalent \( \Xi_L = \Xi + L \) of any graded Lagrangian \( L \). Locally, one can always choose \( \Xi \) (4.10) where all functions \( h \) vanish.

### 5 Proof of Theorem 4.1

The proof of Theorem 4.1 follows a scheme of the proof of Theorem 2.1, but all sheaves are set on \( X \).

We start from the exactness of the complexes (4.4) – (4.6). The exactness of the short variational complex (4.4) (the algebraic Poincaré lemma) and the de Rham complex (4.6) (the Poincaré lemma) on \( X = \mathbb{R}^n \) at all terms, except \( \mathbb{R} \), has been stated [5, 8, 14]. Let us extend the algebraic Poincaré lemma to the complex (4.5).

**Lemma 5.1.** The complex (4.5) on \( X = \mathbb{R}^n \) is exact.

**Proof.** The fact that a \( d_H \)-closed graded form \( \phi \in S_{\infty}^{1,m<n} \) is \( d_H \)-exact results from the algebraic Poincaré lemma for horizontal graded forms \( \phi \in S_{\infty}^{0,m<n} \). Indeed, let us formally associate to a graded \( m \)-form \( \phi = \sum \phi_A^A \wedge ds_A^A \) the horizontal graded \( (m-1) \)-form \( \overline{\phi} = \sum \phi_A^A \overline{s}_A^A \) depending on additional variables \( \overline{s}_A^A \) of the same Grassmann parity as \( s_A^A \), and let us introduce the modified total differential

\[ d_H \overline{\phi} = d_H \overline{\phi} + dx^\Lambda \wedge (\overline{s}_A^A \overline{\partial}_A \overline{\phi} + \overline{s}_A^A \overline{\partial}_\mu \overline{\phi} + \cdots), \quad \overline{\partial}_A = \partial / \partial s_A^A. \]

It is easily justified that \( d_H \overline{\phi} = d_H \overline{\psi} \). If \( d_H \phi = 0 \), then \( d_H \overline{\phi} = 0 \) and, consequently, \( \overline{\phi} = d_H \overline{\psi} \) where \( \overline{\psi} = \sum \psi_A^A \overline{s}_A^A \) is linear in \( \overline{s}_A^A \). Then \( \phi = d_H \psi \) where \( \psi = \sum \psi_A^A \wedge ds_A^A \). It remains to show that, if

\[ \varrho(\phi) = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^A \wedge [d_A(\partial^A_\Lambda) \phi] = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} \theta^A \wedge [d_A \phi^A_\Lambda] = 0, \quad \phi \in S_{\infty}^{1,n}, \]

then \( \phi \) is \( d_H \)-exact. A direct computation gives

\[ \phi = d_H \psi, \quad \psi = - \sum_{|\Lambda| \geq 0} \sum_{\Sigma + \Xi = \Lambda} (-1)^{|\Sigma|} \theta^A_\Sigma \wedge d_A \phi^{A+\mu}_\Lambda \omega_\mu. \]

\( \square \)

Let us associate to each open subset \( U \subset X \) the \( \mathbb{R} \)-module \( S_{U}^* \) of elements of the \( C^\infty(X) \)-algebra \( S_{\infty}^* \) whose coefficients are restricted to \( U \). These modules make up a presheaf on \( X \). Let \( \mathcal{S}_{\infty}^* \) be the sheaf constructed from this presheaf and \( \Gamma(\mathcal{S}_{\infty}^*) \) its structure module of sections. One can show that \( \mathcal{S}_{\infty}^* \) inherits the bicomplex operations, and \( \Gamma(\mathcal{S}_{\infty}^*) \) does so.
For short, we say that $\Gamma(\mathcal{S}^*_\infty)$ consists of polynomials in $s^*_\Lambda$, $ds^*_\Lambda$ of locally bounded jet order $|\Lambda|$. There is the monomorphism $\mathcal{S}^*_\infty \to \Gamma(\mathcal{S}^*_\infty)$. Let us consider the complexes of sheaves of $C^\infty(X)$-modules

$$
0 \to \mathbb{R} \to \mathcal{G}^0_\infty \xrightarrow{d_H} \mathcal{G}^0_{1,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{G}^{0,n}_\infty \xrightarrow{\delta} 0, 
$$

(5.1)

$$
0 \to \mathcal{G}^{1,0}_\infty \xrightarrow{d_H} \mathcal{G}^{1,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{G}^{1,n}_\infty \xrightarrow{\varphi} \mathcal{E}_1 \to 0, \quad \mathcal{E}_1 = \varrho(\mathcal{G}^{1,n}_\infty), 
$$

(5.2)

$$
0 \to \mathbb{R} \to \mathcal{G}^0_\infty \xrightarrow{d} \mathcal{G}^1_\infty \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{G}^k_\infty \xrightarrow{\cdots} 
$$

(5.3)

on $X$ and the complexes of their structure modules

$$
0 \to \mathbb{R} \to \Gamma(\mathcal{G}^0_\infty) \xrightarrow{d_H} \Gamma(\mathcal{G}^{0,1}_\infty) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Gamma(\mathcal{G}^{0,n}_\infty) \xrightarrow{\delta} 0, 
$$

(5.4)

$$
0 \to \Gamma(\mathcal{G}^{1,0}_\infty) \xrightarrow{d_H} \Gamma(\mathcal{G}^{1,1}_\infty) \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Gamma(\mathcal{G}^{1,n}_\infty) \xrightarrow{\varphi} \Gamma(\mathcal{E}_1) \to 0, 
$$

(5.5)

$$
0 \to \mathbb{R} \to \Gamma(\mathcal{G}^0_\infty) \xrightarrow{d} \Gamma(\mathcal{G}^1_\infty) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\mathcal{G}^k_\infty) \xrightarrow{\cdots} .
$$

(5.6)

The terms $\mathcal{S}^*_\infty$ of these complexes are sheaves of $C^\infty(X)$-modules. Therefore, they are fine and, consequently, acyclic. Turn to the sheaf $\mathcal{E}_1$.

**Lemma 5.2.** The sheaf $\mathcal{E}_1$ on $X$ is fine.

*Proof.* We use the fact that the sheaf $\mathcal{E}_1$ is a projection $\varrho(\mathcal{G}^{1,n}_\infty)$ of the sheaf $\mathcal{G}^{1,n}_\infty$ of $C^\infty(X)$-modules. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite open covering of $X$ and $\{f_i \in C^\infty(X)\}$ the associated partition of unity. For any open subset and any section $\phi$ of the sheaf $\mathcal{G}^{1,n}_\infty$ over $U$, let us put $h_i(\phi) = f_i \phi$. The endomorphisms $h_i$ of $\mathcal{G}^{1,n}_\infty$ yield the $\mathbb{R}$-module endomorphisms

$$
\overline{h}_i = \varrho \circ h_i : \mathcal{E}_1 \xrightarrow{\text{in}} \mathcal{G}^{1,n}_\infty \xrightarrow{h_i} \mathcal{G}^{1,n}_\infty \xrightarrow{\varphi} \mathcal{E}_1
$$

of the sheaf $\mathcal{E}_1$. They possess the properties for $\mathcal{E}_1$ to be a fine sheaf. Indeed, for each $i \in I$, $\text{supp} f_i \subset U_i$ provides a closed set such that $\overline{h}_i$ is zero outside it, while the sum $\sum_{i \in I} \overline{h}_i$ is the identity morphism. □

Consequently, the sheaf $\mathcal{E}_1$ is acyclic. The above mentioned exactness of the complexes (4.4) – (4.6) on $X = \mathbb{R}^n$ implies the exactness of the complexes of sheaves (5.1) – (5.3) at all terms, except $\mathbb{R}$. It follows that the complexes (5.1) and (5.3) are resolutions of the constant sheaf $\mathbb{R}$, while the complex (5.2) is exact. By virtue of the abstract de Rham theorem, the cohomology of the complexes (5.4) and (5.6) equals the de Rham cohomology $H^*(X)$ of $X$, whereas the complex (5.5) is globally exact. It remains to prove the following.

**Theorem 5.3.** Cohomology of the complexes (4.4) – (4.6) equals that of the complexes (5.4) – (5.6).

The rest of this Section is the proof of Theorem 5.3. Let the common symbols $\Gamma^*_\infty$ and $D$ stand for all the modules and the coboundary operators, respectively, in the complexes (5.4) – (5.6). With this notation, one can say that any $D$-closed element $\phi \in \Gamma^*_\infty$ takes the form

$$
\phi = \varphi + D\xi,
$$

(5.7)
where $\varphi$ is a non-exact closed exterior form on $X$. For the proof, it suffices to show that, if an element $\phi \in S^*_\infty$ is $D$-exact in the module $\Gamma^*_\infty$, then it is so in $S^*_\infty$. By virtue of the above mentioned Poincaré lemmas, if $X$ is contractible and a $D$-exact element $\phi$ is of finite jet order $[\phi]$ (i.e., $\phi \in S^*_\infty$), there exists an element $\varphi \in S^*_\infty$ such that $\phi = D\varphi$. Moreover, a glance at the corresponding homotopy operators shows that the jet order $[\varphi]$ is bounded by an integer $N([\phi])$, depending only on $[\phi]$. We agree to call this fact the finite exactness of an operator $D$. Given an arbitrary manifold $X$, the finite exactness takes place on any domain $U \subset X$. Let us state the following.

**Lemma 5.4.** Given a family $\{U_\alpha\}$ of disjoint open subsets of $X$, let us suppose that the finite exactness of an operator $D$ takes place on every subset $U_\alpha$. Then it holds on the union $\bigcup_\alpha U_\alpha$.

**Proof.** Let $\phi \in S^*_\infty$ be a $D$-exact graded form on $X$. The finite exactness on $\bigcup_\alpha U_\alpha$ holds since $\phi = D\varphi_\alpha$ on every $U_\alpha$ and all $[\varphi_\alpha] < N([\phi])$. $\square$

**Lemma 5.5.** Suppose that the finite exactness of an operator $D$ takes place on open subsets $U, V$ of $X$ and their non-empty overlap $U \cap V$. Then it is also true on $U \cup V$.

**Proof.** Let $\phi = D\varphi \in S^*_\infty$ be a $D$-exact graded form on $X$. By assumption, it can be brought into the form $D\varphi_U$ on $U$ and $D\varphi_V$ on $V$, where $\varphi_U$ and $\varphi_V$ are graded forms of bounded jet order. Due to the decomposition (5.7), one can choose the forms $\phi_U, \phi_V$ such that $\varphi - \varphi_U$ on $U$ and $\varphi - \varphi_V$ on $V$ are $D$-exact. Let us consider the difference $\varphi_U - \varphi_V$ on $U \cap V$. It is a $D$-exact graded form of bounded jet order which, by assumption, can be written as $\varphi_U - \varphi_V = D\sigma$ where $\sigma$ is also of bounded jet order. Lemma 5.6 below shows that $\sigma = \sigma_U + \sigma_V$ where $\sigma_U$ and $\sigma_V$ are graded forms of bounded jet order on $U$ and $V$, respectively. Then, putting

$$
\varphi'_U = \varphi_U - D\sigma_U, \quad \varphi'_V = \varphi_V + D\sigma_V,
$$

we have the graded form $\phi$, equal to $D\varphi'_U$ on $U$ and $D\varphi'_V$ on $V$, respectively. Since the difference $\varphi'_U - \varphi'_V$ on $U \cap V$ vanishes, we obtain $\phi = D\varphi'$ on $U \cup V$ where

$$
\varphi' \stackrel{\text{def}}{=} \begin{cases} 
\varphi'|_U = \varphi'_U \\
\varphi'|_V = \varphi'_V
\end{cases}
$$

is of bounded jet order. $\square$

**Lemma 5.6.** Let $U$ and $V$ be open subsets of $X$ and $\sigma$ a graded form of bounded jet order on $U \cap V$. Then $\sigma$ splits into the sum $\sigma_U + \sigma_V$ of graded exterior forms $\sigma_U$ on $U$ and $\sigma_V$ on $V$ of bounded jet order.

**Proof.** By taking a smooth partition of unity on $U \cup V$ subordinate to its cover $\{U, V\}$ and passing to the function with support in $V$, we get a smooth real function $f$ on $U \cup V$ which
is 0 on a neighborhood $U_{U-V}$ of $U-V$ and 1 on a neighborhood $U_{V-U}$ of $V-U$ in $U \cup V$. The graded form $f \sigma$ vanishes on $U_{U-V} \cap (U \cap V)$ and, therefore, can be extended by 0 to $U$. Let us denote it $\sigma_U$. Accordingly, the graded form $(1-f)\sigma$ has an extension $\sigma_V$ by 0 to $V$. Then $\sigma = \sigma_U + \sigma_V$ is a desired decomposition because $\sigma_U$ and $\sigma_V$ are of finite jet order which does not exceed that of $\sigma$. □

By virtue of Lemmas 5.4 and 5.5, the finite exactness of an operator $D$ on a manifold $X$ takes place because one can choose the corresponding cover of $X$ (see Lemma 9.5 in [10], Chapter V).

6 Generalized Lagrangian supersymmetries

A graded derivation $v \in \mathfrak{d}S_\infty^0$ of the $\mathbb{R}$-ring $S_\infty^0$ is said to be a generalized supersymmetry if the Lie derivative $L_v \phi$ preserves the ideal of contact graded forms of the BGDA $S_\infty^*$.

**Proposition 6.1.** With respect to the local basis $(x^\lambda, s^A_\lambda, dx^\lambda, \theta^A_\lambda)$ for the BGDA $S_\infty^*$, any generalized supersymmetry takes the form

$$v = v_H + v_V = v^\lambda d_\lambda + (v^A \partial_A + \sum_{|\lambda|>0} d_\lambda v^A \sigma^A_\lambda),$$

(6.1)

where $v^\lambda$, $v^\sigma$ are local graded functions.

**Proof.** The key point is that any element of the $C^\infty(X)$-algebra $S_\infty^*$ is a section of a finite-dimensional vector bundle over $X$ and any graded form is a finite composition of $df$, $f \in S_\infty^0$. Therefore, the proof follows those of Propositions 3.1 – 3.3. □

The interior product $v] \phi$ and the Lie derivative $L_v \phi$, $\phi \in S_\infty^*$ obey the same formulae

$$v] \phi = v^\lambda \phi_\lambda + (-1)^{|\phi|} v^A \sigma^A_\lambda,$$

($\phi \in S_\infty^1$),

$$v] (\phi \wedge \sigma) = (v] \phi) \wedge \sigma + (-1)^{|\phi|+|\sigma|} v] \phi \wedge (v] \sigma),$$

($\phi, \sigma \in S_\infty^*$)

$$L_v \phi = v] d \phi + d (v] \phi),$$

$$L_v (\phi \wedge \sigma) = L_v (\phi) \wedge \sigma + (-1)^{|\phi|+|\sigma|} \phi \wedge L_v (\sigma).$$

as those on a graded manifold. In particular, it is easily justified that any vertical generalized supersymmetry $v$ (6.1) satisfies the relations

$$v] d_H \phi = -d_H (v] \phi),$$

(6.2)

$$L_v (d_H \phi) = d_H (L_v \phi),$$

(6.3)

**Proposition 6.2.** Given a graded Lagrangian $L \in S_\infty^n$, its Lie derivative $L_v L$ along a generalized supersymmetry $v$ (6.1) fulfills the first variational formula

$$L_v L = v_V] \delta L + d_H (h_0 (v] \Xi_L)) + d_V (v_H] \omega) L,$$

(6.4)

where $\Xi_L = \Xi + L$ is a Lepagean equivalent of $L$ given by the coordinate expression (4.10).
Proof. The proof follows that of Proposition 3.4 and results from the decomposition (4.9) and the relation (6.2). □

In particular, let \( \nu \) be a divergence symmetry of a graded Lagrangian \( L \), i.e., \( L_\nu L = d_H \sigma \), \( \sigma \in S_0^{0,n-1} \). Then the first variational formula (6.4) restricted to Ker \( \delta L \) leads to the weak conservation law

\[
0 \approx d_H (h_0 (\nu \mid \Xi_L) - \sigma).
\]

(6.5)

Similarly to the case of generalized symmetries, one can justify that a generalized supersymmetry is a divergence symmetry of a Lagrangian \( L \) iff its vertical part \( \nu_V \) (6.1) is so. In this case the conservation law (6.5) takes the form

\[
0 \approx d_H (\nu_V \mid \Xi_L - \sigma'),
\]

where \( J_\nu = \nu_V \mid \Xi_L \) is the graded Noether current along a generalized supersymmetry \( \nu_V \).

It should be emphasized that a Lepagean equivalent \( \Xi_L \) in the conservation law (6.5) is not uniquely defined. One can always choose it of the local form (4.10) where graded functions \( h \) vanish. There is a global Lepagean equivalent of this form if either \( \dim X = 1 \) or \( L \) is of first order.

The BRST transformation in gauge theory on a principal bundle \( P \to X \) with a structure group \( G \) in Remark 4.2 exemplifies a vertical generalized supersymmetry. With respect to a local basis \( (x^\lambda, a^r_\lambda, C^r) \) for the BGDA \( S^*_\infty (V_G, C) \) (4.3), it is given by the expression

\[
v = v^r_\lambda \frac{\partial}{\partial a^r_\lambda} + v^r C^r + \sum_{|\Lambda| > 0} \left( \Lambda \nu^r_\lambda \frac{\partial}{\partial a^r_{\Lambda, \lambda}} + \Lambda \nu^r \frac{\partial}{C^r} \right),
\]

(6.6)

\[
v^r_\lambda = C^r_\lambda + c^r_{pq} a^p_\lambda C^q, \quad v^r = \frac{1}{2} c^r_{pq} C^p C^q,
\]

where \( c^r_{pq} \) are structure constants of the right Lie algebra of \( G \). A remarkable peculiarity of this generalized supersymmetry is that the Lie derivative \( L_\nu \) along \( \nu \) (6.6) is nilpotent on the module \( S_0^{0,*} \) of horizontal graded forms.

One says that a vertical generalized supersymmetry \( \nu \) (6.1) is nilpotent if

\[
L_\nu (L_\nu \phi) = \sum_{|\Sigma| \geq 0, |\Lambda| \geq 0} (v^r_\Sigma \partial^\Sigma_E (v^A_\Lambda) \partial^A_{\Lambda} + (-1)^{|B||\nu|} v^B_\Sigma v^A_\Lambda \partial^\Sigma_E \partial^A_{\Lambda}) \phi = 0
\]

(6.7)

for any horizontal graded form \( \phi \in S_0^{0,*} \). A glance at the second term in the expression (6.7) shows that a nilpotent generalized supersymmetry is necessarily odd. The following is an important criterion of a nilpotent generalized supersymmetry.

**Lemma 6.3.** A generalized supersymmetry \( \nu \) is nilpotent iff the equality

\[
L_\nu (v^A) = \sum_{|\Sigma| \geq 0} v^B_\Sigma \partial^\Sigma_E (v^A) = 0
\]

holds for all \( v^A \).
Proof. The proof results from a direct computation. □

Remark 6.1. A useful example of a nilpotent generalized supersymmetry is the supersymmetry

\[ \nu = \nu^A(x)\partial_A + \sum_{|A|>0} \partial_A \nu^A \partial_A^A, \]  

(6.8)

where all \( \nu^A \) are smooth real functions on \( X \), but all \( s^A \) are odd.

7 Cohomology of nilpotent generalized supersymmetries

Let \( \nu \) be a nilpotent generalized supersymmetry. Since the Lie derivative \( \mathbf{L}_\nu \) obeys the relation (6.3), let us assume that the \( \mathbb{R} \)-module \( S^{0,*}_\infty \) of graded horizontal forms is split into a bicomplex \( \{ S^{k,m} \} \) with respect to the nilpotent operator \( s_\nu \) (1.8) and the total differential \( d_H \). This bicomplex

\[ d_H : S^{k,m} \rightarrow S^{k,m+1}, \quad s_\nu : S^{k,m} \rightarrow S^{k+1,m} \]

is graded by the form degree \( 0 \leq m \leq n \) and an integer \( k \in \mathbb{Z} \), though it may happen that \( S^{k,*} = 0 \) starting from some number. For the sake of brevity, let us call \( k \) the charge number.

For instance, the BRST bicomplex \( S^{0,*}_\infty(\mathbb{C}, V_G P) \) is graded by the charge number \( k \) which is the polynomial degree of its elements in odd variables \( C^r_A \). In this case, \( s_\nu \) (1.8) is the BRST operator. Since the ghosts \( C^r_A \) are characterized by the ghost number 1, \( k \) is the ghost number. The bicomplex defined by the supersymmetry (6.8) in Remark 6.1 has the similar gradation, but the nilpotent operator \( s_\nu \) decreases the odd polynomial degree.

Let us consider horizontal graded forms \( \phi \in S^{0,*}_\infty \) such that a nilpotent generalized supersymmetry \( \nu \) is their divergence symmetry, i.e., \( s_\nu \phi = d_H \sigma \). As was mentioned above, we come to the relative and iterated cohomology of the nilpotent operator \( s_\nu \) (1.8) with respect to the total differential \( d_H \).

Recall that a horizontal graded form \( \phi \in S^{*,*}_\infty \) is said to be a relative (i.e., \( (s_\nu/d_H) \)-) closed form if \( s_\nu \phi \) is a \( d_H \)-exact form. This form is called exact if it is a sum of an \( s_\nu \)-exact form and a \( d_H \)-exact form. Accordingly, we have the relative cohomology \( H^{*,*}(s_\nu/d_H) \). If a \( (s_\nu/d_H) \)-closed form \( \phi \) is also \( d_H \)-closed, it is called an iterated \( (s_\nu|d_H) \)-closed form. This form \( \phi \) is said to be exact if \( \phi = s_\nu \xi + d_H \sigma \), where \( \xi \) is a \( d_H \)-closed form. Thus, we come to the iterated cohomology \( H^{*,*}(s_\nu|d_H) \) of the \( (s_\nu,d_H) \)-bicomplex \( S^{*,*} \). It is the term \( E^{*,*}_2 \) of the spectral sequence of this bicomplex [30]. There is an obvious isomorphism \( H^{*,n}(s_\nu/d_H) = H^{*,n}(s_\nu|d_H) \) of relative and iterated cohomology groups on horizontal graded densities. Forthcoming Theorems 7.2 and 7.3 extend our results on iterated cohomology in [19] to an arbitrary nilpotent generalized supersymmetry.

Note that, with respect to the total differential \( d_H \), the bicomplex \( S^{*,*}_\infty \) is the complex

\[ 0 \longrightarrow \mathbb{R} \longrightarrow S^0_\infty \xrightarrow{d_H} S^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} S^{0,n}_\infty \xrightarrow{d_H} 0, \]

(7.1)
which differs from the short variational complex (4.4) in the last morphism.

**Proposition 7.1.** Cohomology groups $H^{m<n}(d_H)$ of the complex (7.1) equal the de Rham cohomology groups $H^{m<n}(X)$ of $X$, while the cohomology group $H^n(d_H)$ fulfills the relation

$$H^n(d_H)/H^n(X) = E_1.$$  

(7.2)

**Proof.** Cohomology of the complex (7.1) is determined similarly to that of the short variational complex (4.4). The only difference is that the horizontal complex (7.1) on $X = \mathbb{R}^n$ is not exact at the last term $S_0^0\to\cdots\to S_0^n\to 0$. Accordingly, the corresponding complex of sheaves

$$0 \to \mathbb{R} \to S_0^0 \to S_0^1 \to \cdots \to S_0^n \to 0$$

on $X$ fails to be a resolution of the constant sheaf $\mathbb{R}$ at the last term. Therefore, one should use a minor modification of the abstract de Rham theorem [20, 41] in order to obtain cohomology of the corresponding complex of structure modules

$$0 \to \mathbb{R} \to \Gamma(S_0^0) \to \Gamma(S_0^1) \to \cdots \to \Gamma(S_0^n) \to 0$$

at all the terms, except the last one. Then Theorem 5.3 shows that this cohomology coincides with that of the complex (7.1). The relation (7.2) results from the formula (4.8). □

**Remark 7.1.** Let us mention that, in the antifield BRST theory [5, 8] extended to an arbitrary $X$, the antibracket is defined on elements of the quotient $H^n(d_H)/H^n(X)$. They correspond to local functionals up to surface integrals. At the same time, the antibracket on local functionals implies rather intricate geometric interpretation of antifields [28, 42].

**Theorem 7.2.** There is an epimorphism

$$\zeta : H^{m<n}(X) \to H^{*,m<n}(s_v|d_H)$$  

(7.3)

of the de Rham cohomology $H^m(X)$ of $X$ of form degree less than $n$ onto the iterated cohomology $H^{*,m<n}(s_v|d_H)$.

**Proof.** Since a nilpotent generalized supersymmetry $v$ is vertical, all exterior forms $\phi$ on $X$ are $s_v$-closed. It follows that they are $(s_v|d_H)$-closed. Since any $d_H$-exact horizontal graded form is also $(s_v|d_H)$-exact, we have a morphism $\zeta (7.3)$. By virtue of Corollary 4.2 (and, equivalently, Proposition 7.1), any $d_H$-closed horizontal graded $(m<n)$-form $\phi$ is split into the sum $\phi = \varphi + d_H\xi$ (4.7) of a closed $m$-form $\varphi$ on $X$ and a $d_H$-exact graded form. It follows that the morphism $\zeta (7.3)$ is an epimorphism. □

In particular, if $X = \mathbb{R}^n$, the iterated cohomology $H^{*,0<m<n}(s_v|d_H)$ is trivial in contrast with the relative ones.
The kernel of the morphism $\zeta$ (7.3) consists of elements whose representatives are $s_\nu$-exact closed exterior forms on $X$. For instance, a glance at the the BRST transformation $\nu$ (6.6) shows that, in BRST theory, exterior forms on $X$ are never $s_\nu$-exact. However, this is not the case of the supersymmetry (6.8). Note that, in the both examples, exterior forms on $X$ are only of zero charge number. In this case, we have the trivial iterated cohomology $H^{\neq 0, m<n}(s_\nu|d_H)$ and an epimorphism (in particular, an isomorphism) of $H^{m<n}(X)$ to $H^{0, m<n}(s_\nu|d_H)$.

Turn now to the iterated cohomology $H^{*, m}(s_\nu|d_H)$. It requires a particular analysis because, by virtue of Proposition 7.1, the cohomology $H^n(d_H)$ of the complex (7.1) fails to equal the de Rham cohomology $H^n(X)$ of $X$.

The bicomplex $S^{*, *}$ is a complex with respect to the total coboundary operator $\tilde{s}_\nu = s_\nu + d_H$. We aim to state the relation between the iterated cohomology $H^{*, m}(s_\nu|d_H)$ and the total $\tilde{s}_\nu$-cohomology $H^*(\tilde{s}_\nu)$ of the bicomplex $S^{*, *}$. Similarly to the morphism (7.3), there exists the morphism

$$\gamma : H^{<n}(X) \to H^*(\tilde{s}_\nu)$$

(7.4)

of the de Rham cohomology $H^{<n}(X)$ of $X$ of form degree $< n$ to the total cohomology $H^*(\tilde{s}_\nu)$. Its kernel consists of elements whose representatives are $\tilde{s}_\nu$-exact closed exterior forms on $X$. Put $\overline{H}^* = H^*(\tilde{s}_\nu)/\text{Im} \gamma$.

**Theorem 7.3.** There is the isomorphism

$$H^{*, n}(s_\nu|d_H)/\overline{H}^* = \text{Ker} \gamma.$$  

(7.5)

**Proof.** The proof falls into the following three steps.

(i) At first, we state the morphism

$$\eta : H^{*, n}(s_\nu|d_H) \to \text{Ker} \gamma$$

(7.6)

of the iterated cohomology $H^{*, n}(s_\nu|d_H)$ to Ker $\gamma$. Let a horizontal graded $n$-form $\phi_n$ be $(s_\nu|d_H)$-closed. Then, by definition, $s_\nu \phi_n$ is $d_H$-exact, i.e.,

$$s_\nu \phi_n + d_H \phi_{n-1} = 0.$$  

(7.7)

Acting on this equality by $s_\nu$, we observe that $s_\nu \phi_{n-1}$ is a $d_H$-closed graded form, i.e.,

$$s_\nu \phi_{n-1} + d_H \phi_{n-2} = \varphi_{n-1},$$  

(7.8)

where $\varphi_{n-1}$ is a closed $(n-1)$-form on $X$ in accordance with Corollary 4.2. Since $s_\nu \varphi_{n-1} = 0$, an action of $s_\nu$ on the equation (7.8) shows that $s_\nu \phi_{n-2}$ is a $d_H$-closed graded form, i.e.,

$$s_\nu \phi_{n-2} + d_H \phi_{n-3} = \varphi_{n-2},$$

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where \( \varphi_{n-2} \) is a closed \((n - 2)\)-form on \( X \). Iterating the arguments, one comes to the system of equations

\[
\begin{align*}
\mathbf{s}_v \phi_{n-k} + d_H \phi_{n-k-1} &= \varphi_{n-k}, & 0 \leq k < n, \\
\mathbf{s}_v \phi_0 &= \varphi_0 = \text{const},
\end{align*}
\]  
which assemble into the descent equation

\[
\begin{align*}
\tilde{\mathbf{s}}_v \phi_n &= \tilde{\varphi}_{n-1}, \\
\tilde{\phi}_n &= \phi_n + \phi_{n-1} + \cdots + \phi_0, \\
\tilde{\varphi}_{n-1} &= \varphi_{n-1} + \cdots + \varphi_0.
\end{align*}
\]

Thus, any \((\mathbf{s}_v|d_H)\)-closed horizontal graded form defines a descent equation \((7.10)\) whose right-hand side \( \tilde{\varphi}_{n-1} \) is a closed exterior form on \( X \) such that its de Rham class belongs to the kernel \( \text{Ker} \gamma \) of the morphism \((7.4)\). For the sake of brevity, let us denote this descent equation by \( \langle \tilde{\varphi}_{n-1} \rangle \). Accordingly, we say that a horizontal graded form \( \tilde{\phi}_n \) \((7.11)\) is a solution of the descent equation \( \langle \tilde{\varphi}_{n-1} \rangle \) \((7.11)\). A descent equation defined by a \((\mathbf{s}_v|d_H)\)-closed horizontal graded form \( \phi_n \) is not unique. Let \( \tilde{\phi}' \) be another solution of another descent equation \( \langle \tilde{\varphi}'_{n-1} \rangle \) such that \( \phi_n = \phi'_n \). Let us denote \( \Delta \phi_k = \phi_k - \phi'_k \) and \( \Delta \varphi_k = \varphi_k - \varphi'_k \). Then the equation \((7.7)\) leads to the equation \( d_H(\Delta \phi_{n-1}) = 0 \). It follows that

\[
\Delta \phi_{n-1} = d_H \xi_{n-2} + \alpha_{n-1},
\]

where \( \alpha_{n-1} \) is a closed \((n - 1)\)-form on \( X \). Accordingly, the equation \((7.9)\) leads to the equation

\[
\mathbf{s}_v(\Delta \phi_{n-1}) + d_H(\Delta \phi_{n-2}) = \Delta \varphi_{n-1}.
\]

Substituting the equality \((7.12)\) into this equation, we obtain the equality

\[
d_H(-\mathbf{s}_v \xi_{n-2} + \Delta \phi_{n-2}) = \Delta \varphi_{n-1}.
\]

It follows that

\[
\Delta \phi_{n-2} = \mathbf{s}_v \xi_{n-2} + d_H \xi_{n-3} + \alpha_{n-2}, \quad \Delta \varphi_{n-1} = d \alpha_{n-2}
\]

where \( \alpha_{n-2} \) is an exterior form on \( X \). Iterating the arguments, one comes to the relations

\[
\Delta \phi_{n-k} = \mathbf{s}_v \xi_{n-k} + d_H \xi_{n-k-1} + \alpha_{n-k}, \quad \Delta \varphi_{n-k} = d \alpha_{n-k-1}, \quad 1 < k < n, \quad (7.13)
\]

where \( \alpha_{n-k-1} \) are exterior forms on \( X \) and, finally, to the equalities \( \Delta \phi_0 = 0, \Delta \varphi_0 = 0 \). Then it is easily justified that

\[
\begin{align*}
\tilde{\phi}_n - \tilde{\phi}'_n &= \tilde{\mathbf{s}} \tilde{\sigma} + \tilde{\alpha}, \\
\tilde{\varphi}_{n-1} - \tilde{\varphi}'_{n-1} &= d \tilde{\alpha},
\end{align*}
\]

\[
\begin{align*}
\bar{\phi}_n &= \bar{\phi}'_n = \bar{\mathbf{s}} \bar{\sigma} + \bar{\alpha}, \\
\bar{\varphi}_{n-1} &= \bar{\varphi}'_{n-1} = d \bar{\alpha}, \\
\bar{\sigma} &= \Delta \phi_{n-1} + \cdots + \Delta \phi_1, \\
\bar{\alpha} &= \alpha_{n-1} + \cdots + \alpha_1.
\end{align*}
\]

It follows that right-hand sides of any two descent equations defined by a \((\mathbf{s}_v|d_H)\)-closed horizontal graded form \( \phi_n \) differ from each other in an exact form on \( X \). Moreover, let
\(\phi_n\) and \(\phi'_n\) be representatives of the same iterated cohomology class in \(H^{*\cdot n}(s_\nu|d_H)\), i.e., 
\(\phi_n = \phi'_n + s_\nu \xi_n + d_H \sigma_{n-1}\). Let \(\phi_n\) provide a solution \(\bar{\phi}_n\) of a descent equation \(\langle \bar{\phi}_{n-1} \rangle\). Then \(\phi'_n\) defines a solution \(\bar{\phi}' = \bar{\phi} + \bar{s}_\nu(\xi_n + \sigma_{n-1})\) of the same descent equation. Thus, the assignment \(\phi_n \mapsto \langle \bar{\phi}_{n-1} \rangle\) yields the desired morphism \(\eta\) (7.6).

(ii) Let \(\bar{\varphi}_{n-1}\) be a closed exterior form on \(X\) whose de Rham cohomology class belongs to \(\text{Ker} \gamma\). Then \(\bar{\varphi}_{n-1}\) yields some descent equation \(\langle \bar{\varphi}_{n-1} \rangle\) (7.10). Let \(\bar{\varphi}'_{n-1}\) differ from \(\bar{\varphi}_{n-1}\) in an exact form, i.e., let the relation (7.15) hold. Then any solution \(\bar{\phi}_n\) of the equation \(\langle \bar{\varphi}_{n-1} \rangle\) yields a solution \(\bar{\phi}'_n = \bar{\phi}_n - \bar{\alpha}\) (7.14) of the equation \(\langle \bar{\varphi}'_{n-1} \rangle\) such that \(\phi'_n = \phi_n\). It follows that the morphism \(\eta\) (7.6) is an epimorphism.

(iii) The kernel of the morphism \(\eta\) (7.6) is represented by \((s_\nu|d_H)\)-closed horizontal graded forms \(\phi_n\) which define the homogeneous descent equation

\[\bar{s}_\nu \bar{\phi}_n = 0.\]  

(7.16)

Its solutions \(\bar{\phi}_n\) are \(\bar{s}_\nu\)-closed horizontal graded forms. Let us assign to a solution \(\bar{\phi}_n\) of the descent equation (7.16) its higher term \(\phi_n\). Running back the arguments at the end of item (i), one can show that, if solutions \(\bar{\phi}_n\) and \(\bar{\phi}'_n\) of the descent equation (7.16) belong to the same total cohomology class, then its higher terms \(\phi_n\) and \(\phi'_n\) belong to the same iterated cohomology class. It follows that the assignment \(\bar{\phi}_n \mapsto \phi_n\) provides an epimorphism of the total cohomology \(H^*(\bar{s}_\nu)\) onto \(\text{Ker} \eta\). The kernel of this epimorphism is represented by solutions \(\bar{\phi}_n\) of the descent equation (7.16) whose higher term vanishes. Following item (i), one can easily show that these solutions take the form \(\bar{\phi}_n = \bar{s}_\nu \sigma + \tilde{\alpha}\), where \(\tilde{\alpha}\) is a closed exterior form on \(X\) of form degree \(< n\). Cohomology classes of these solutions exhaust the image of the morphism \(\gamma\) (7.4), i.e., \(\text{Im} \gamma = \text{Ker} \eta\). □

In particular, if the morphism \(\gamma\) (7.4) is a monomorphism (i.e., no non-exact closed exterior form on \(X\) is \(\bar{s}_\nu\)-exact), the isomorphism (7.5) gives the isomorphism

\[H^{*\cdot n}(s_\nu|d_H) = H^*(\bar{s}_\nu)/H^\leq n(X).\]

For instance, this is the case of the BRST transformation (6.6) [19].

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