Corrigendum: The Plebanski sectors of the EPRL vertex

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Abstract

We correct what amounts to a sign error in the proof of part (i) of theorem 3. The Plebanski sectors isolated by the linear simplicity constraints do not change—they are still the three sectors (deg), (II+) and (II−). What changes is the characterization of the continuum Plebanski two-form corresponding to the first two terms in the asymptotics of the EPRL vertex amplitude for Regge-like boundary data. These two terms do not correspond to the Plebanski sectors (II+) and (II−), but rather to the two possible signs of the product of the sign of the sector—+1 for (II+) and −1 for (II−)—and the sign of the orientation \( \epsilon_{IJKL} B^{IJ} \wedge B^{KL} \) determined by \( B^{IJ} \). This is consistent with what one would expect, as this is exactly the sign which classically relates the BF action, in the Plebanski sectors (II+) and (II−), to the Einstein–Hilbert action, whose discretization is the Regge action appearing in the asymptotics.

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The error and the corrected final result

The error lies in part (i) of theorem 3 of the paper. In order to state this error, let us define a numbered 4-simplex to be a geometrical 4-simplex with vertices numbered arbitrarily, and each tetrahedron numbered by the vertex it does not contain. An ‘ordered 4-simplex’ as defined in definition 3 is then a numbered 4-simplex that additionally satisfies a condition relating the numbering to orientation. In order for the argument for part (i) of theorem 3 to be valid, the numbered 4-simplex guaranteed by the reconstruction theorem must be ‘ordered’, because it is then used to calculate the Plebanski sector of the geometrical bivectors, whose well-definition requires this. But, in general, the reconstructed 4-simplex will not be ordered.

This is the error in the paper. As we will see, it can be easily corrected, and upon correction, the interpretation of the terms in the asymptotics of the vertex amplitude will no longer involve
only Plebanski sectors, but also the orientation \( \varepsilon_{IJKL} B_{I}^{J} \wedge B_{K}^{L} \) determined by the continuum two-form \( B_{\mu \nu} \), reconstructed from the discrete data at the critical points. Specifically, let

\[
\omega(B_{\mu \nu}) := \text{sgn}(\hat{\varepsilon}^{a b \gamma \delta} \varepsilon_{IJKL} B_{a \mu}^{I} B_{b \nu}^{J} B_{\gamma \delta}^{KL}),
\]

where \( \hat{\varepsilon}^{a b \gamma \delta} \) is the fixed orientation on \( M \cong \mathbb{R}^4 \), and let \( \nu(B_{\mu \nu}) = +1, -1 \) if \( B_{\mu \nu} \) is in the Plebanski sector (II+) or (II−), respectively, and let \( \nu(B_{\mu \nu}) = 0 \) otherwise. Then the first and second terms in the asymptotics of equation (3.10) correspond to critical points, where \( \omega \nu = +1 \) or \( -1 \), respectively.

Note that this modified result is exactly what one would expect. The first and second terms in equation (3.10) are respectively \( e_{\text{Regge}} \) and \( e^{\text{Regge}} \), where \( S_{\text{Regge}} \) is the Regge action. The Regge action is a discretization of the Einstein–Hilbert action \( S_{\text{EH}} \), and the relation of the BF action \( S_{\text{BF}} \) to the Einstein–Hilbert action, in the Plebanski sectors (II+) and (II−), is precisely \( S_{\text{BF}} = \omega \nu S_{\text{EH}} \).

### Details of the correction

In the following, \( \{B_{a b}\} \) shall always denote a ‘discrete Plebanski field’ in the sense of [1]—that is, a set of \( \mathfrak{so}(4) \) algebra elements \( B_{a b}^{I} = -B_{b a}^{I} \) satisfying closure (\( \sum_{a b c d} B_{abcd}^{I} = 0 \)) and orientation \( (B_{a b}^{I} = -B_{b a}^{I}) \). The algebra indices \( I J \) will usually be suppressed. The algebra elements \( B_{a b} \) are also referred to as bivectors due to the antisymmetry of the algebra indices. Let \( B_{\mu \nu}(\{B_{a b}\}, \sigma) \) denote the unique \( \mathfrak{so}(4) \)-valued two-form, constant with respect to \( \partial_{\sigma} \), such that its integral over each triangle \( \Delta_{a b c}(\sigma) \) of the numbered 4-simplex \( \sigma \) is equal to the algebra element \( B_{a b} \). The existence and uniqueness of the two-form \( B_{\mu \nu} \) satisfying these conditions is ensured by lemma 1 of [1]. The proof of lemma 1 does not depend on \( \sigma \) being ordered; see also the related work in [2]. When defining the Plebanski sector and the orientation of a set of algebra elements \( \{B_{a b}\} \), however, we will see that it is necessary to restrict \( \sigma \) to be ordered, but for the mere reconstruction of \( B_{\mu \nu} \) itself, we can and do omit this restriction.

We begin by noting that the proof of theorem 1 in [1] actually succeeds in proving the following much stronger statement.

**Theorem 1, Stronger statement.** For any numbered 4-simplex \( \sigma \), \( B_{\mu \nu}(B_{a b}^{\text{geom}}(\sigma), \sigma) \) is in the Plebanski sector (II+) and has orientation \( \omega = +1 \).

Let us next prove two lemmas, which will make the corrected proof of part (i) of theorem 3 a single line. For these two lemmas, let \( P \) denote any orientation-reversing diffeomorphism such that \( P \circ P \) is the identity.

**Lemma 3.** Given any discrete Plebanski field \( \{B_{a b}\} \) and any numbered 4-simplex \( \sigma \),

\[
B_{\mu \nu}(\{B_{a b}\}, P \sigma) = -P^{a} B_{\mu \nu}(\{B_{a b}\}, \sigma).
\]

**Proof.** As mentioned in [1], the only background structures used in the construction of the continuum two-form \( B_{\mu \nu}(\{B_{a b}\}, \sigma) \) are the flat connection \( \partial_{\sigma} \) and the fixed orientation \( \hat{\varepsilon}^{a b \gamma \delta} \). We begin by making the fixed orientation an explicit argument in the construction \( B_{\mu \nu}(B_{a b}, \sigma, \hat{\varepsilon}) \), so that, given \( \{B_{a b}^{I}\}, (\sigma, \hat{\varepsilon}) \mapsto B_{\mu \nu}^{I} \) is covariant under the symmetry group of \( \partial_{\sigma} \), that is, under all of \( \text{GL}(4) \). In particular, for \( P \in \text{GL}(4) \), it follows that

\[
B_{\mu \nu}(\{B_{a b}\}, P, P \hat{\varepsilon}) = P^{a} B_{\mu \nu}(\{B_{a b}\}, \sigma, \hat{\varepsilon}).
\]
Furthermore, by definition of the reconstructed two-forms (and introducing the orientation \( \hat{e} \) as an explicit argument also of each oriented triangle \( \Delta_{ab}(\sigma, \hat{e}) \)), one has

\[
\int_{\Delta_{ab}(\langle P\sigma, \hat{e} \rangle)} B(\{B_{ab}'\}, P\sigma, \hat{e}) = B_{ab} := \int_{\Delta_{ab}(\langle P\sigma, \hat{e} \rangle)} B(\{B_{ab}'\}, P\sigma, P\hat{e})
\]

\[
= -\int_{\Delta_{ab}(\langle P\sigma, \hat{e} \rangle)} B(\{B_{ab}'\}, P\sigma, P\hat{e})
\]

\[
= -\int_{\Delta_{ab}(\langle P\sigma, \hat{e} \rangle)} P^* B(\{B_{ab}'\}, \sigma, \hat{e})
\]

where the second to last equality holds because the sole effect of replacing \( P\hat{e} \) with \( \hat{e} \) in the argument for triangle \( \Delta_{ab} \) is to reverse the orientation of the triangle and hence negate the value of the integral, and the last equality holds because of (2). Because the continuum two-forms are constant with respect to \( \check{a} \) and \( \check{b} \) [1], it follows that the integrands of the first and last expressions are equal, which, combined with \( B_{\mu\nu}(\{B_{ab}\}, \sigma) := B_{\mu\nu}(\{B_{ab}\}, \sigma, \hat{e}) \), implies the claimed result (1). □

In order to understand the significance of the above lemma, we first note that, for \( B_{\mu\nu} \) in the Plebanski sector (II+) or (II−), the action of \( P \) on \( B_{\mu\nu} \) flips the orientation of \( B_{\mu\nu} \) while leaving its Plebanski sector unchanged and negation of \( B_{\mu\nu} \) flips its Plebanski sector while leaving its orientation unchanged. These facts, together with the above lemma imply

\[
\omega(B_{\mu\nu}(\{B_{ab}\}), P\sigma) = -\omega(B_{\mu\nu}(\{B_{ab}\}), \sigma) \quad \text{and} \quad v(B_{\mu\nu}(\{B_{ab}\}), P\sigma) = -v(B_{\mu\nu}(\{B_{ab}\}), \sigma).
\]

Because of the above equations, if we wish to use \( B_{\mu\nu}(\{B_{ab}\}, \sigma) \) to define a Plebanski sector and orientation for a given set of algebra elements \( \{B_{ab}\} \), a restriction must be placed on the numbered 4-simplex \( \sigma \) such that it not possible to use both a 4-simplex \( \sigma' \) and its parity reversal \( P\sigma' \); otherwise, the Plebanski sector and the orientation of \( \{B_{ab}\} \) will be ill-defined. The restriction used is precisely that \( \sigma \) be ordered in the sense of [1]. Once this restriction is made, \( v(B_{\mu\nu}(\{B_{ab}\}), \sigma) \) and \( \omega(B_{\mu\nu}(\{B_{ab}\}), \sigma) \) are independent of the remaining freedom in \( \sigma \). This was proven for \( v(B_{\mu\nu}(\{B_{ab}\}), \sigma) \) in lemma 2 of [1]. For \( \omega(B_{\mu\nu}(\{B_{ab}\}), \sigma) \), the proof follows from the same argument, together with the fact that, for any orientation-preserving diffeomorphism \( \varphi \), \( \omega(\varphi^* B_{\mu\nu}) = \omega(B_{\mu\nu}) \). Thus, one may define \( \omega(B_{\mu\nu}(\{B_{ab}\}), \sigma) \) and \( \omega(B_{\mu\nu}(\{B_{ab}\})) := \omega(B_{\mu\nu}(\{B_{ab}\}), \sigma) \) where any ordered \( \sigma \) may be used. (The significance of the ordering condition on \( \sigma \) in this context is essentially that, by imposing a certain compatibility between the numbering and the orientation, the ordering condition endows the numbering of vertices with orientation information which turns out to be essential in extracting the Plebanski sector and dynamical orientation from the algebra elements \( \{B_{ab}\} \).

**Lemma 5.** For any numbered 4-simplex \( \sigma \), \( \omega(\{B_{ab}^{\text{geom}}(\sigma)\}) v(\{B_{ab}^{\text{geom}}(\sigma)\}) = 1. \)

**Proof.** Case 1. \( \sigma \) is ordered: then \( v(\{B_{ab}^{\text{geom}}(\sigma)\}) := v(B_{\mu\nu}(\{B_{ab}^{\text{geom}}(\sigma)\}, \sigma)) = +1 \) and \( \omega(\{B_{ab}^{\text{geom}}(\sigma)\}) := \omega(B_{\mu\nu}(\{B_{ab}^{\text{geom}}(\sigma)\}, \sigma)) = +1 \) where, in each of these equations, the first equality follows by definition and the final equality is implied by the above stronger version of theorem 1.

Case 2. \( \sigma \) is not ordered: then \( P\sigma \) is an ordered 4-simplex, so that

\[
v(\{B_{ab}^{\text{geom}}(\sigma)\}) := v(B_{\mu\nu}(\{B_{ab}^{\text{geom}}(\sigma)\}, P\sigma)) = -v(B_{\mu\nu}(\{B_{ab}^{\text{geom}}(\sigma)\}, \sigma)) = -1
\]

and

\[
\omega(\{B_{ab}^{\text{geom}}(\sigma)\}) := \omega(B_{\mu\nu}(\{B_{ab}^{\text{geom}}(\sigma)\}, P\sigma)) = -\omega(B_{\mu\nu}(\{B_{ab}^{\text{geom}}(\sigma)\}, \sigma)) = -1
\]
where, in each of the above equations, the first equality follows by definition, the second equality follows from equation (3) and the final equality is implied by the above stronger version of theorem 1.

In both cases, one has $\omega(B_{ab}^{\text{geom}}(\sigma)) \nu(B_{ab}^{\text{geom}}(\sigma)) = 1$, as claimed. $\square$

The corrected statement and proof of part (i) of theorem 3 are then as follows.

**Theorem 3, Part (i), corrected.** Suppose \(\{A_{ab}, n_{ab}\}\) is a set of non-degenerate reduced boundary data satisfying closure and \(\{X^+\}\) are such that the orientation is satisfied. If \(\{X^-\} \neq \{X^+\}\), then \(B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X^\pm)\) is either in the Plebanski sector (II+) or (II−). Furthermore, the sign $\mu$ in the reconstruction theorem equals $\omega \nu$.

**Proof.** Let $\sigma$ denote the numbered 4-simplex guaranteed by the reconstruction theorem, unique up to translation, rotation and inversion. Using the relation $B_{ab}^{\text{phys}} = \mu B_{ab}^{\text{geom}}(\sigma)$ between the physical and geometrical bivectors in the reconstruction theorem, and using lemma 5, one has

$$\omega(B_{ab}^{\text{phys}}) \nu(B_{ab}^{\text{phys}}) = \omega(B_{ab}^{\text{geom}}(\sigma))(\mu \cdot \nu(B_{ab}^{\text{geom}}(\sigma))) = \mu.$$ $\square$

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The Plebanski sectors of the EPRL vertex

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Abstract
Modern spin-foam models of four-dimensional gravity are based on a discrete version of the Spin(4) Plebanski formulation. Beyond what is already in the literature, we clarify the meaning of different Plebanski sectors in this classical discrete model. We show that the linearized simplicity constraint used in the EPRL and FK models are not sufficient to impose a restriction to a single Plebanski sector, but rather, three Plebanski sectors are mixed. We propose this as the reason for certain extra ‘undesired’ terms in the asymptotics of the EPRL vertex analyzed by Barrett et al. This explanation for the extra terms is new and different from that sometimes offered in the spin-foam literature thus far.

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1. Introduction

Spin foams [1–3] provide a path integral framework for defining the dynamics of loop quantum gravity (LQG) [1,3–5], a background-independent, canonical quantization of general relativity. Ever since the Barrett–Crane model [6], spin foams have been based on the Spin(4) Plebanski formulation of gravity [7, 8], in which the co-tetrad $e$ is replaced by an $\mathfrak{so}(4)$-valued two-form $B = \star e \wedge e$, where $\star$ denotes Hodge duality in the internal tetrad indices; at the same time, one introduces a constraint, called the simplicity constraint, to ensure that $B$ is of this form for some $e$. However, the original constraint function used in [6, 7, 9] (equation (2.12)) does not impose $B = \star e \wedge e$ for some $e$. Rather, it imposes that $B$ belongs to one of five possible sectors:

(I$\pm$) $B = \pm e \wedge e$ for some $e$

(II$\pm$) $B = \pm \star e \wedge e$ for some $e$

(deg) $B$ is degenerate ($\text{tr}(\star B \wedge B) = 0$),

where the terminology for the sectors has been taken from [9]. We call these Plebanski sectors.

The only spin-foam model thus far to match the kinematics of LQG is the EPRL model [3, 10]. This, together with the FK model [11], forms the so-called new models. The common feature of both of these models is a certain reformulation of the simplicity constraint that is linear in the $B$ variables. The existing literature does not directly address the question of which
of the Plebanski sectors are still included by the linear constraint. In this paper, we answer this question, and show in a precise sense that sectors (II±) and (deg) are included. We furthermore show that this mixing of Plebanski sectors is precisely the reason for the ‘undesired’ terms in the semiclassical limit of the EPRL vertex computed in [12]. This explanation is new and different from the explanation in terms of a sum over orientations of simplices mentioned, for example, in [13], and implied by remarks given already in [12]. We discuss this issue in the conclusion section. As section 2 is classical in nature and is an analysis of the linear simplicity constraint which the FK model also uses, all discussion and conclusions in section 2 are equally relevant for FK as for EPRL.

The paper is structured as follows. We begin by reviewing the classical discrete framework underlying the EPRL and FK models. We then clarify the meaning of Plebanski sectors in this discrete context, and prove that the linear simplicity constraint imposes a restriction to precisely Plebanski sectors (II±) and (deg). Finally, we review the asymptotics of the EPRL vertex, and identify the Plebanski sector of the critical point of each term. We then close with a discussion.

2. Classical analysis of the linear simplicity constraints

2.1. Discrete classical framework

2.1.1. Generalities. First, let us introduce some conventions and definitions. Elements of \( su(2) \) will be bold. We use the normalized basis \( \tau^i \) of \( su(2) \), where \( \sigma^i \) are the standard Pauli matrices, so that \( [\tau^i, \tau^j] = \epsilon^{ij} k \tau^k \). Given an element \( \lambda \in su(2) \), \( \lambda^i \in \mathbb{R}^3 \) denotes its components with respect to \( \tau^i \). Let \( I \) denote the \( 2 \times 2 \) identity matrix. In the following, we will freely use the natural isomorphism between \( so(4) \) and \( spin(4) \):

\[
JIJ \leftrightarrow (J^i_+, J^i_-) \equiv (J^i + \tau^i, J^i - \tau^i),
\]

effectively identifying these two algebras. Here \( J^i_\pm \) and \( I, J_0 \) are 0, 1, 2, 3. Explicitly, the isomorphism is

\[
J^i_\pm = \frac{1}{2} \epsilon^{ijk} J^j_\pm \pm \frac{1}{2} J^0_i,
\]

with inverse

\[
J^{ij} = \epsilon^{ijk} (J^k_+ + J^k_-) \quad J^{0i} = J^i_+ - J^i_-.
\]

2.1.2. Continuum theory. The EPRL model, as all spin-foam models of gravity, takes as its starting point the formulation of gravity as a constrained BF theory, following the ideas of Plebanski [8]. The basic continuum variables are an \( so(4) \)-valued two-form \( B^\mu_{\nu} \) and an \( so(4) \) connection \( \omega^\mu_{ij} \). We call \( B^\mu_{\nu} \), the Plebanski two-form. The action is

\[
S = -\frac{1}{2\kappa} \int \text{tr} \left[ \left( B + \frac{1}{\gamma} B \right) \wedge F \right],
\]

where \( F := d\omega + \omega \wedge \omega \) is the curvature of \( \omega \), and \( \kappa := 8\pi G \). The variable conjugate to \( \omega \) is thus

\[
J^i := \frac{1}{\kappa} \left( B^i + \frac{1}{\gamma} B \right).
\]

In terms of the anti-self-dual and self-dual parts of \( J \) and \( B \),

\[
(J^\pm)^i = \left( \frac{\gamma \pm 1}{\kappa \gamma} \right) (B^\pm)^i.
\]
Action (2.3) is a BF action, and leads to a topological field theory with no local degrees of freedom. To turn it into gravity, one imposes the simplicity constraints, reviewed here in equations (2.12) and (2.13).

2.1.3. Discrete variables. To construct the spin-foam model, one introduces a discretization of spacetime using a triangulation into 4-simplices. But for the purpose of analyzing the vertex amplitude, it suffices to focus on a single 4-simplex. We therefore do so. In the past years [14, 15], the EPRL model has also been generalized to arbitrary cell-complexes; however, for simplicity, and because we heavily use the work [12], we restrict ourselves to the case in which the cell-complex is a simplicial triangulation.

Consider an oriented 4-simplex $S$. Number the tetrahedra $a = 0, \ldots, 4$, and label the triangles by the unordered pair $(ab)$ of tetrahedra that contain it. One thinks of each tetrahedron, as well as the 4-simplex itself, as having its own ‘frame’ [16]. The connection $\omega$ is discretized by specifying a parallel transport map from each tetrahedron to the 4-simplex frame—thus, in our case, there are five parallel transport maps $G_a = (X_a^-, X_a^+)$, $a = 1, \ldots, 5$. Let $\Delta_{ab}$ denote the triangle $(ab)$, endowed with the orientation induced on it as part of the boundary of tetrahedron $a$, whose orientation in turn is induced from it being part of the boundary of $S$.

The two-forms $B$ and $J$ are then discretized as the elements

$$B_{ab} = \int_{\Delta_{ab}} B, \quad J_{ab} = \int_{\Delta_{ab}} J,$$

where one thinks of these elements as being ‘in the frame at $a$’. For each $ab$, these algebra elements are related, in terms of their self-dual and anti-self-dual parts, by

$$(J^\pm_{ab})^i = \left(\frac{\gamma \pm 1}{\kappa \gamma}\right)(B^\pm_{ab})^i.$$  

(2.5)

Because the bivectors $J_{ab}$ ‘in the frame at $a$’ are key in the canonical theory in section 3.1, we call them the canonical bivectors.

However, in the rest of section 2, we will focus on reconstructing the four-dimensional continuum Plebanski two-form $B^\gamma_{\mu\nu}$ from this discrete data, and proving related results. For this purpose, it is necessary to parallel transport all of the bivectors $B_{ab}$ to a common frame. If one parallel transports them to the 4-simplex frame, one obtains

$$B_{ab}(S) := G_a \triangleright B_{ab},$$

where $\triangleright$ denotes the adjoint action of Spin(4) on spin(4) $\cong so(4)$. If one parallel transports them to some tetrahedron frame $c$, one has

$$B_{ab}(t_c) := G_{ca} \triangleright B_{ab},$$

where $G_{ca} := G_c^{-1} G_a$ is the parallel transport map from tetrahedron $a$ to tetrahedron $c$. Throughout the rest of this section, in order to use a presentation closer to that in [12], we will work in the 4-vertex frame; however, all constructions and analyses in the rest of this section can be equally done in any of the tetrahedron frames—when a tetrahedron frame is used, only the parallel transports $G_{ab} := G_a^{-1} G_b$ between tetrahedra are needed.

Let us now turn to some expected properties of the bivectors $B_{ab}$. Note that in the 4-simplex frame, because $\Delta_{ab}$, $\Delta_{ba}$ differ by only a change of orientation, one should have

$$B_{ab}(S) \equiv -B_{ba}(S).$$  

(2.6)

This is the ‘orientation constraint’. As was shown in [12], this is imposed by the EPRL vertex amplitude itself: If it is not satisfied, the vertex amplitude is exponentially suppressed. Furthermore, classically, in each tetrahedron frame $a$, the four algebra elements $B_{ab}$ must
satisfy the closure relation \( \sum_{a \neq b} B_{ab} = 0 \). This is similarly imposed by the vertex amplitude being exponentially suppressed if it is not satisfied. Finally, if the closure relation is satisfied in each tetrahedron, then the algebra elements for each tetrahedron uniquely determine a tetrahedron geometry, and hence a shape for each triangle \( \Delta_{ab} \) in the tetrahedron. Then, one would expect that the shape of the triangle \( \Delta_{ab} \) as determined in tetrahedra \( a \) and \( b \) would be the same—the gluing constraint \([17, 18]\). As we will see, when this is not satisfied, either the vertex amplitude is exponentially suppressed or one is in the degenerate Plebanski sector.

A constraint that holds from the start in both the classical and quantum frameworks is \( |B^\pm_{ab}|^2 = |B^\pm_{ba}|^2 \). It is consequently convenient to introduce separate variables for the norms and directions of the self-dual and anti-self-dual parts of these bivectors. In the tetrahedron frames, for each pair \((ab)\), we then have a pair of norms \( B^\pm_{ab} \equiv B^\pm_{ba} \equiv |B^\pm_{ab}| \) and two pair of directions (equivalently, unit \( su(2) \) elements) \( u^\pm_{ab} := B^\pm_{ab} / |B^\pm_{ab}(S)|, u^\pm_{ba} := B^\pm_{ba} / |B^\pm_{ba}(S)| \).

One has

\[
B_{ab} = (B^-_{ab}u^+_{ab}, B^+_{ab}u^-_{ab}).
\]

From (2.5), also

\[
J_{ab} = \left( \frac{\gamma - 1}{\kappa} \right) B^-_{ab}u^+_{ab}, \left( \frac{\gamma + 1}{\kappa} \right) B^+_{ab}u^-_{ab} \right). \tag{2.7}
\]

The bivectors in the simplex frame are then given by

\[
B_{ab}(S) = (B^-_{ab}u^+_{ab}(S), B^+_{ab}u^-_{ab}(S)), \tag{2.8}
\]

where \( u^\pm_{ab}(S) := X^\pm_{ab} \rhd u^\pm_{ab} \). The orientation constraint then takes the form \( u^+_{ab}(S) = -u^-_{ab}(S) \).

### 2.2. Discrete Plebanski sectors defined

We now define what we call a 'discrete Plebanski field'. As will be seen in definition 8, it plays the role of a precursor to the bivector geometry definition used in \([12]\).

**Definition 1** (Discrete Plebanski field). A discrete Plebanski field is a set of bivectors \( \{B^I_{ab}\}, a, b = 0, 1, 2, 3, 4 \), such that

(i) \( B^I_{ab} = -B^I_{ba} \) (orientation)

(ii) \( \sum_{a \neq b} B^I_{ab} = 0 \) (closure).

In the following, we will see that every discrete Plebanski field determines in a certain sense a continuum \( so(4) \) -valued two-form field in the simplex. This will then be used to define the notion of a discrete Plebanski field being in a certain 'Plebanski sector'.

Let \( M \) denote \( \mathbb{R}^4 \) as an oriented manifold, equiped with the canonical flat connection \( \partial_a \) on \( \mathbb{R}^4 \). This is the arena where we will define the simplex and reconstruct the continuum \( so(4) \)-valued two-form from given discrete data \( \{B^I_{ab}\} \). The symmetry group of \( (M, \partial_a) \) is the proper inhomogeneous \( GL(4) \) group, \( IGL(4)^+ \). \( \partial_a \) defines notions of straight line segments and planes in \( M \) in the usual way, and the notion of convex hull is defined in the usual way using straight line segments.

Furthermore, let \( V \) denote the tangent space of \( M \) at any point. Because \( \partial_a \) is flat, its parallel transport maps provide (1) a natural isomorphism between \( V \) and every other tangent space, allowing us to identify every tangent space with \( V \), and (2) a natural isomorphism between \( V \) and the space of constant vector fields on \( M \), obtained by parallel transporting any given element of \( V \) throughout \( M \). We will let the same symbol denote a given vector in \( V \) and the corresponding vector at any other point, as well as the corresponding constant vector field on \( M \). In like manner, \( \partial_a \) defines a natural isomorphism between tensors of a given type over
V and tensors of the same type at any other point, as well as with the space of constant tensors on \( M \) of the same type. Again, we will let the same symbol denote a given tensor over V and the corresponding tensor at any other point, as well as the associated constant tensor.

**Definition 2 (Geometrical 4-simplex).** A geometrical 4-simplex in \( M \) is the convex hull of five points, called vertices, in \( M \), not all of which lie in the same 3-plane.

**Definition 3 (Ordered 4-simplex).** We define an ordered 4-simplex \( \sigma \) to be a geometrical 4-simplex in \( M \) with an assignment of labels 0, 1, 2, 3, 4 to its five vertices such that the ordered set of four vectors \((01, 02, 03, 04)\) in \( V \) has positive orientation. Each tetrahedron is then labeled by the number of the one vertex it does not contain.

A standard way to specify orientation of a manifold is to specify a nowhere vanishing volume form modulo rescaling by a positive function. However, through the one-to-one relation between volume forms and inverse volume forms, \( \epsilon_{\alpha_1 \cdots \alpha_n} \epsilon_{\alpha_1 \cdots \alpha_n} = n! \), it is just as easy to specify orientation by a nowhere vanishing inverse volume form modulo rescaling by a positive function. Because it is more convenient, throughout this section we will specify orientation in the latter way.

**Definition 4 (Oriented triangle \( \Delta_{ij} \)).** Given an ordered 4-simplex \( \sigma \), let \( \Delta_{ij} = \Delta_{ij}(\sigma) \) denote the triangle between tetrahedra \( a \) and \( b \). Let \( (N_a)_a, (N_b)_b \) (covectors on \( M \)) be any outward-pointing normals to tetrahedra \( a \) and \( b \), respectively, and let \( \epsilon^{\mu \nu \rho \sigma} \) be any oriented inverse volume form. Then, the inverse two-form

\[
\epsilon^{[\mu \nu}_{(ab)]} : = \epsilon^{\gamma \delta \mu \nu} (Na)^\gamma (Nb)^\delta \quad (2.9)
\]

is non-zero and tangent to \( \Delta_{ij} \), and well defined up to rescaling by a positive function. We let \( \epsilon^{[\mu \nu}_{(ab)]} \) define the orientation of \( \Delta_{ij} \).

The above-defined orientation of \( \Delta_{ij} \) is simply the orientation induced on \( \Delta_{ij} \) as part of the boundary of tetrahedron \( a \), considered as part of the boundary of \( \sigma \).

**Lemma 1.** Given a discrete Plebanski field \( \{ B^{IJ}_{ab} \} \) and any choice of ordered 4-simplex \( \sigma \) in \( M \), there exists a unique constant Lie algebra-valued two-form \( B^{IJ}_{\mu \nu} \) such that

\[
B^{IJ}_{ab} = \int_{\Delta_{ij}} B^{IJ}_{\mu \nu}.
\]

**Proof.** For each \( \Delta_{ij} \) in \( \sigma \), define the bivector \( I^{\mu \nu}_{ij} \) by

\[
I^{\mu \nu}_{ij} : = \int_{\Delta_{ij}} \lambda,
\]

for all constant \( \lambda_{ij} \). This set of bivectors satisfies the same closure and orientation conditions as \( B^{IJ}_{ij} \). In addition, \( \{ I^{\mu \nu}_{ij} \} \) satisfies a non-degeneracy condition: for \( i < j \), \( i, j = 0, 1, 2, 3 \), \( I^{\mu \nu}_{ij} \) form a basis of \( V^* \otimes \text{skew} V \). Let \( I^{\mu \nu}_{ij} \) denote the corresponding dual basis of \( V^* \otimes \text{skew} V^* \). Then for each \( I, J = 0, 1, 2, 3 \), define

\[
B^{IJ}_{\mu \nu} : = \sum_{i \neq j} B^{IJ}_{ij} I^{\mu \nu}_{ij}.
\]

We then have

\[
\int_{\Delta_{ij}} B^{IJ} = I^{\mu \nu}_{ij} B^{IJ}_{\mu \nu} = B^{IJ}_{ij}
\]

This is due to the fact that the pullback of \( (N_b)_b \) to the plane of tetrahedron \( a \) is an outward-pointing normal to \( \Delta_{ij} \).
for all \( i < j, j, i = 0, 1, 2, 3 \). Using that both \( B_{ab} ^{ij}, I_{ab} ^{\mu\nu} \) satisfy closure and orientation, it follows

\[
\int _{\Delta_a} B_{ab} ^{ij} I_{ab} ^{\mu\nu} B_{ab} ^{ij} = B_{ab} ^{ij}
\]

(2.11)
for all \( a, b \), proving existence. To prove uniqueness, suppose (2.11) is satisfied. Then, from the fact that it holds in particular for \( a < b, a, b = 0, 1, 2, 3 \), it follows immediately that \( B_{ab} ^{ij} \) is as in (2.10).

**Definition 5.** Given a discrete Plebanski field \( \{ B_{ab} ^{ij} \} \) and any choice of ordered 4-simplex \( \sigma \) in \( M \), we call the unique BIJ field \( \{ B_{ab} ^{ij} \} \) adapted to \( \sigma \).

If \( B_{ab} ^{ij} \) satisfies the Plebanski constraint,

\[
\epsilon _{ijkl}( B_{ab} ^{ij} B_{pq} ^{kl} - \frac{1}{4} \eta _{ijpq} \eta ^{\alpha \beta \gamma \delta} B_{ab} ^{ij} B_{pq} ^{kl}) = 0,
\]

(2.12)
where \( \eta _{ijpq} \) and \( \eta ^{\alpha \beta \gamma \delta} \) denote the Levi-Civita tensors of density weight \(-1\) and \(1\), respectively; then, for example from [9], it must be of one of the five forms

\[
\begin{align*}
(I\pm) & B_{ab} ^{ij} = \pm \epsilon _{\alpha \beta} \wedge \epsilon _{\gamma \delta} \\
(II\pm) & B_{ab} ^{ij} = \pm \frac{1}{4} \epsilon _{\alpha \beta \gamma \delta} \epsilon _{\epsilon \eta \kappa \lambda} \\
(deg) & \epsilon _{ijkl} \eta _{\mu
u\rho\sigma} B_{ab} ^{ij} B_{pq} ^{kl} = 0 \quad \text{(degenerate case)}. \end{align*}
\]

Each of these forms defines a particular sector, which we refer to as (I\(\pm\)), (II\(\pm\)) or (deg). These five sectors are disjoint [9].

**Definition 6.** If, for a given choice of ordered 4-simplex \( \sigma \), a given discrete Plebanski field \( \{ B_{ab} ^{ij} \} \) has two-form in the Plebanski sector (I\(\pm\)), (II\(\pm\)) or (deg), we say that \( \{ B_{ab} ^{ij} \} \), relative to \( \sigma \), is also in the Plebanski sector (I\(\pm\)), (II\(\pm\)) or (deg), respectively.

In fact, if \( \{ B_{ab} ^{ij} \} \) is in a given Plebanski sector, this property is independent of the choice of \( \sigma \), as we will now prove, so that the qualification ‘relative to \( \sigma \)’ is not necessary.

**Lemma 2.** If a given discrete Plebanski field \( \{ B_{ab} ^{ij} \} \) is in a given Plebanski sector—(I\(\pm\)), (II\(\pm\)) or (deg)—relative to a given ordered 4-simplex \( \sigma \), then \( \{ B_{ab} ^{ij} \} \) is in the same Plebanski sector relative to any ordered 4-simplex.

**Proof.** Let a discrete Plebanski field \( \{ B_{ab} ^{ij} \} \) be given. Let two ordered simplices \( \sigma, \sigma' \) be given, and let \( \phi, B_{ab} ^{ij}, \sigma B_{ab} ^{ij} \) denote the corresponding two-forms.

There exists a unique element \( G \) of the inhomogeneous \( GL(4) \) group, \( IGL(4) \), mapping the five vertices of \( \sigma \) into the five vertices of \( \sigma' \), in order, so that \( G \) maps \( \sigma \) into \( \sigma' \). Furthermore, because the vertices of \( \sigma \) and \( \sigma' \) are each numbered with positive orientation in the sense of definition 3, \( G \) is in the proper inhomogeneous \( GL(4) \) group, \( IGL(4)^+ \). To construct, from \( \{ B_{ab} ^{ij} \} \) and \( \sigma \), the two-form \( \sigma B_{ab} ^{ij} \), on the manifold \( M \), one only uses the orientation of \( M \) and the flat connection \( \partial _a \); these are invariant under \( IGL(4)^+ \). Therefore, for a given discrete Plebanski field \( \{ B_{ab} ^{ij} \} \), the map from ordered 4-simplices \( \sigma \) to two-forms \( \sigma B_{ab} ^{ij} \) is \( IGL(4) \) covariant. Thus one concludes

\[
G \cdot \sigma B_{ab} ^{ij} = \sigma B_{ab} ^{ij}.
\]

But the action of \( G \) in this equation is the action of a particular diffeomorphism, and all diffeomorphisms preserve each of the continuum Plebanski sectors (as is immediate from the diffeomorphism covariance of the equation defining each one). Thus, if either of \( \phi, \sigma, B_{ab} ^{ij} \) is in one of the Plebanski sectors, then both of them must be in the same Plebanski sector.

Thus, the notion of the discrete Plebanski field \( \{ B_{ab} ^{ij} \} \) being in a given Plebanski sector is independent of the ordered 4-simplex used to define the continuum two-form.
2.3. Plebanski sectors of the linear simplicity constraints

Here, we review the linear simplicity constraint used in the ‘new’ spin-foam models, and show that it restricts the bivectors \( B_{ab} \) to be in the Plebanski sector \((II \pm)\) or \((\text{deg})\). The linear simplicity constraint imposes that there exists an assignment of an \( N^i_a \) to each \( a \), such that

\[
C^i_{ab} := N^i_a (B_{ab})^I = 0 \quad \forall \ b \neq a.
\]  

(2.13)

Using this, we now define ‘weak bivector geometry’ and ‘bivector geometry’. The latter is the same as the definition in [12], except that the \( B_{ab}^I \) algebra elements here are not to be identified with the \( B_{ab}^I \) algebra elements in [12], but rather with their Hodge duals. This is because we have chosen instead to be consistent with the convention for \( B_{ab}^I \) used in [9, 10, 16, 19].

**Definition 7** (Weak bivector geometry). A discrete Plebanski field \( \{B_{ab}^I\} \) is additionally called a weak bivector geometry if

(i) for each \( a \) there exists \( N^i_a \in \mathbb{R}^4 \), such that \( N^i_a (B_{ab})^I = 0, \forall b \neq a \) (linear simplicity);

(ii) for all distinct \( a, b, c, d \), \( \text{tr}(B_{ab}[B_{cd}, B_{de}]) \neq 0 \) (tetrahedron non-degeneracy).

**Definition 8** (Bivector geometry). A weak bivector geometry \( \{B_{ab}^I\} \) is additionally called a bivector geometry if it satisfies (full) non-degeneracy. For \( i < j, i, j = 0, 1, 2, 3 \), \( B_{ij}^I \) form a basis of \( \mathbb{R}^4 \) @ skew \( \mathbb{R}^4 \).

The above definitions are intended to be applied to bivectors in the 4-simplex frame. The canonical variables, by contrast, are defined in the tetrahedron frames, where we impose a gauge-fixed version of equation (2.13) in which \( N^i_a \) is fixed to be \( N^i_a := (1, 0, 0, 0) \), following [10]. In each tetrahedron frame, the simplicity constraint then becomes

\[
C^i_{ab} := \frac{1}{2} \epsilon^{ijk} B^k_{ab} = 0.
\]  

(2.14)

In terms of \( B_{ab}^\pm, u_{ab}^\pm \), this is equivalent to

\[
B_{ab}^+ = B_{ab}^- \quad \text{and} \quad u_{ab}^+ = -u_{ab}^-.
\]  

(2.15)

The first of these equations is equivalent to what is called the ‘diagonal simplicity constraint’; as this is \( SO(4) \) invariant, it is clear that it also follows from the non-gauge-fixed condition (2.13).

Thus, although, in the original works, the constraint (2.13) was presented as a reformulation of cross-simplicity only, in fact it also contains in it the diagonal simplicity constraint. This fact is also reflected in the quantum theory in section 3.1. This is what allows us to omit diagonal simplicity as a separate condition in definitions 7 and 8.

Equation (2.15) implies that the solution space of (2.14) can be parameterized by what we call reduced boundary data \( \{A_{ab}, n_{ab}\} \):

\[
B_{ab}^\pm = \frac{1}{2} A_{ab} \quad \text{and} \quad u_{ab}^\pm = -u_{ab}^- = n_{ab}.
\]  

(2.16)

\(A_{ab}\) and \( n_{ab}\) have direct geometrical significance: \( A_{ab} \) is the area of triangle \( \Delta_{ab} \), and \( n_{ab}^i\) is the outward normal to \( \Delta_{ab} \) in the frame \( a \). Lastly, let \( B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_{ab}^\pm) \) denote the corresponding bivectors \( B_{ab}(\sigma) \) in the 4-simplex frame (2.8):

\[
B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_{ab}^\pm) := B_{ab}(S) = \frac{1}{2} A_{ab} (\pm - X_{ab}^- n_{ab}, X_{ab}^+ n_{ab}).
\]  

(2.17)

Note that all weak bivector geometries are of the form \( \{B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_{ab}^\pm)\} \) for some data set \( \{A_{ab}, n_{ab}, X_{ab}^\pm\} \).

Next, we prove that a weak bivector geometry—i.e. a discrete Plebanski field in which the linear simplicity constraint and tetrahedron non-degeneracy are imposed—is in the Plebanski sector \((II^-), (II^+)\) or (deg), and derive the conditions for each of these. We do this by first
proving a simpler theorem, quoting a theorem from [12] and then proving the main result. Remember here again that the $B_{ab}$ algebra elements used here are not the same as the $B_{ab}$ algebra elements used in [12], but are rather related by the Hodge dual.

We will use a canonical tetrad on $M \equiv \mathbb{R}^4$, defined in the canonical chart as $\ell_a^I := \delta_a^I$, with associated metric $\hat{g}_{ab} = \delta_a^I \delta_b^J$, and oriented volume form $\hat{e} := \hat{e}^I \wedge \hat{e}^J \wedge \hat{e}^K \wedge \hat{e}^L$ given, respectively, by $\hat{g}_{ab}$ and the fully skew array with $\hat{e}_{0123} = 1$. Manifold indices will be raised and lowered with $\hat{g}_{ab}$.

**Definition 9 (Geometrical bivectors).** Given an ordered 4-simplex in $M \equiv \mathbb{R}^4$, the associated geometrical bivectors $(B_{ab}^{\text{geom}})^{IJ}$ are

$$(B_{ab}^{\text{geom}})^{IJ} := A(\Delta_{ab}) \frac{(N_a \wedge N_b)^{IJ}}{|N_a \wedge N_b|},$$

where $A(\Delta_{ab})$ is the area of $\Delta_{ab}$ relative to $\hat{g}_{ab}$. $N_a^I$ is the outward-pointing unit normal to the $a$th tetrahedron using $\hat{g}_{ab}$, $N_a^I = \hat{e}_a^I N_a^\alpha$, $(N_a \wedge N_b)^{IJ} := 2N_a^J N_b^I$, and $|X^{IJ}|^2 := \frac{1}{2}X^{IJ}X_{IJ}$.

As pointed out in [12], for any ordered 4-simplex $\sigma$ in $M \equiv \mathbb{R}^4$, $(B_{ab}^{\text{geom}}(\sigma))$ form a bivector geometry. Furthermore,

**Theorem 1.** For any ordered 4-simplex $\sigma$, $(B_{ab}^{\text{geom}}(\sigma))$ is in the Plebanski sector ($H^+$).

**Proof.** Let an ordered 4-simplex $\sigma$ be given. Let $N_a^I$ denote the unit outward-pointing normal to the $a$th tetrahedron using $\hat{g}_{ab}$. Let $\hat{e}_a^{\alpha I}$ denote the oriented metric area form on $\Delta_{ab}$ and $\hat{e}_a^{\alpha I}$ its inverse. From (2.9),

$$(\hat{e}_a^{\alpha I})^{ab} = \lambda(N_a) \gamma(N_b) \delta^{\alpha \beta},$$

for some positive function $\lambda$. Let $\hat{q}_{ab}$ denote the pullback of the metric $\hat{g}_{ab}$ to $\Delta_{ab}$. Then

$$2 = \hat{q}_{ab} \hat{q}_{\alpha \beta} \hat{e}_a^{\alpha I} \hat{e}_b^{\beta I} = \hat{g}_{ab} \hat{q}_{\alpha \beta} \hat{e}_a^{\alpha I} \hat{e}_b^{\beta I},$$

and lowered with $\hat{g}_{ab}$.

where, in the first line, we have used that $\hat{e}_a^{\alpha I}$ is tangent to $\Delta_{ab}$. Thus

$$\lambda = |N_a \wedge N_b|^{-1}.$$

So (suppressing the two-form indices of $\hat{e}_a^{\alpha I}$),

$$\int_{\Delta_{ab}} \hat{e}_a^{\alpha I} \wedge \hat{e}_b^{\beta I} = \int_{\Delta_{ab}} \hat{e}_a^{\alpha I} \hat{e}_b^{\beta I} \delta_a^\alpha \delta_b^\beta$$

$$= |N_a \wedge N_b|^{-1} \int_{\Delta_{ab}} \hat{e}_a^{\alpha I} (N_a) \gamma(N_b) \delta^{\alpha \beta} \delta_a^\alpha \delta_b^\beta$$

$$= |N_a \wedge N_b|^{-1} \int_{\Delta_{ab}} \hat{e}_a^{\alpha I} (N_a) \gamma(N_b) \delta^{\alpha \beta}$$

$$= \frac{1}{2} \epsilon^{IJ} K \lambda A(\Delta_{ab}) (N_a \wedge N_b)^{KL}$$

whence

$$B_{ab}^{\text{geom}}(\sigma)^{IJ} = \int_{\Delta_{ab}} \frac{1}{2} \epsilon^{IJ} K \lambda \epsilon^{KL} \wedge \hat{e}_a^{\alpha I}.$$
with \( \frac{1}{2} e^{IJ} K L e^K \wedge e^L \) constant. This proves that \( \{ B_{ab}^{\text{geom}}(\sigma) \} \) is in the Plebanski sector (II+).

Let us review what can be called a partial version of theorem 3 in [12] (the ‘reconstruction theorem’). For the following, we say that a set of reduced boundary data \( \{ A_{ab}, n_{ab} \} \) is non-degenerate if for each \( a \), every set of three vectors \( n_{ab} \) with \( b \neq a \) is linearly independent, \( \{ A_{ab}, n_{ab} \} \) satisfies closure if \( \sum_{j=0}^5 A_{ab} n_{ab} = 0 \) and a set \( \{ X_a^\pm \} \subset SU(2) \) satisfies the orientation constraint if \( X_a^\pm n_{ab} = -X_b^\pm n_{ab} \). Furthermore,

**Definition 10.** Two sets of SU(2) group elements \( \{ U_a^1 \}, \{ U_a^2 \} \) are called equivalent, \( \{ U_a^2 \} \sim \{ U_a^1 \} \), if \( \exists Y \in SU(2) \) and a set of five signs \( \varepsilon_j \) such that

\[
U_a^2 = \varepsilon_j Y U_a^1.
\]

(2.19)

**Theorem 2** (Partial version of the reconstruction theorem). Let a set of non-degenerate reduced boundary data \( \{ A_{ab}, n_{ab} \} \) satisfying closure be given, as well as a set \( \{ X_a^\pm \} \subset SU(2) \), \( a = 1, \ldots, 5 \), solving the orientation constraint, such that for each \( a \), \( X_a^- \not\sim X_a^+ \). Then, there exists an ordered 4-simplex \( \sigma \) in \( \mathbb{R}^4 \), unique up to inversion and translation, such that

\[
B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) = \mu B_{ab}^{\text{geom}}(\sigma)
\]

(2.20)

for some \( \mu = \pm 1 \), with \( \mu \) independent of the ambiguity in \( \sigma \).

**Proof.** From the discussion in section V.C.1 in [12], because \( \{ A_{ab}, n_{ab} \} \) is a set of non-degenerate boundary data and \( X_a^- \not\sim X_a^+ \), \( \{ B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) \} \) satisfies non-degeneracy and hence is a bivector geometry in the sense of [12], and the result follows from theorem 3 in [12]. The fact that \( \mu \) is independent of which ordered 4-simplex in \( \mathbb{R}^4 \) is used follows from the invariance of \( B_{ab}^{\text{geom}}(\sigma) \) under inversion and translation of \( \sigma \).

**Theorem 3.** Suppose \( \{ A_{ab}, n_{ab} \} \) is a set of non-degenerate reduced boundary data satisfying closure and \( \{ X_a^\pm \} \) are such that orientation is satisfied.

(i) If \( \{ X_a^- \} \not\sim \{ X_a^+ \} \), then \( \{ B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) \} \) is either in the Plebanski sector (II+) or (II−), according to whether the sign \( \mu \) in theorem 2 is 1 or −1, respectively.

(ii) If \( \{ X_a^- \} \sim \{ X_a^+ \} \), then \( \{ B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) \} \) is in the degenerate Plebanski sector.

**Proof.**

**Proof of (i).**

This is immediate from theorems 1 and 2.

**Proof of (ii).**

As \( \{ X_a^- \} \sim \{ X_a^+ \} \), there exists sign \( \varepsilon_a \) and an \( SU(2) \) element \( Y \) such that \( X_a^+ = \varepsilon_a Y X_a^- \), so that (2.17) becomes

\[
B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_a^\pm) = \frac{1}{2} A_{ab} (Y^{-1} X_a^- n_{ab}, (Y X_a^-) n_{ab}),
\]

where the sign \( \varepsilon_a \) could be dropped from the action in the self-dual component because it is the adjoint action. Let \( H := (I, Y), N^{IJ} = (1, 0, 0, 0, 0) \) and \( N := H \cdot N \). Then

\[
(B_{ab}^{\text{phys}})_{IJ} = (B_{ab}^{\text{phys}})_{IJ} (H \cdot N) = (B_{ab}^{\text{phys}})_{KL} H^K M^I (H^{-1})^JM^J N^J = (H^{-1} \cdot B_{ab}^{\text{phys}})_{MN} (H^{-1})^MN = 0,
\]

where (2.2) has been used in the last line. Let \( \sigma \) be any ordered 4-simplex in \( M \), and \( B_{IJ}^{\text{th}} \) be the associated unique constant two-form determined by \( B_{ab}^{\text{phys}} \). Then, we have

\[
0 = \left( \int_{\Delta_{ab}} (B)^{IJ} \right) N_j = \int_{\Delta_{ab}} ((B)^{IJ} N_j)
\]

9
for all \( a \neq b \). Because \((\mathcal{B})^{\mu \nu} \mathcal{N}_j = 0\), it follows that \((\mathcal{B})^{\mu \nu} \mathcal{N}_j = 0\). Completing \( \mathcal{N}_j \) to an oriented orthonormal basis \((u_j, v_j, w_j, \mathbf{1}_j)\) of the internal space \( \mathbb{R}^4 \), one then sees
\[
\eta^{\mu \nu \rho \sigma} \epsilon_{IJKL} B^{I \mu}_{\rho \sigma} B^{K \nu}_{\rho \sigma} = \eta^{\mu \nu \rho \sigma} \epsilon_{IJKL} (\mathcal{B})^{I \mu}_{\rho \sigma} = \eta^{\mu \nu \rho \sigma} (4!N\mathcal{N}_j^I u_j v_k w_l) (\mathcal{B})^{I \mu}_{\rho \sigma} (\mathcal{B})^{K \nu}_{\rho \sigma} = 0
\]
so that \( \{B_{\text{phys}}^{ab}\} \) is in the degenerate Plebanski sector. \( \square \)

3. Interpretation of the asymptotics of EPRL

3.1. Review of EPRL

Below, we give a brief review of the quantization leading to the EPRL vertex. We do this both in order to clearly establish the meaning of the variables involved in its definition, as well as to briefly remind the reader of the role of linear simplicity.

3.1.1. Notation for SU(2) and Spin(4) structures. Given \( g \in SU(2) \) and \( x \in su(2) \), let \( \rho_j(g) \) and \( \rho_x(x) \) denote their representation on the spin \( j \) carrying space \( V_j \). When it is clear from the context, the \( j \) subscript will be dropped. Let \( \hat{L}^j := i \rho(\tau^i) \) denote the spinors in the spin \( j \) representation, so that \([\hat{L}^i, \hat{L}^j]\) = \( i \epsilon^{ij} \hat{L}^k \).

Let \( \epsilon : V_j \times V_j \rightarrow \mathbb{C} \) denote the standard skew-symmetric bilinear epsilon inner product on \( V_j \). Let \( \epsilon(\rho(g) \psi, \rho(g) \phi) = \epsilon(\psi, \phi) \) (and defined using the alternating spinor \( \epsilon_{ab} \) when \( V_j \) is realized as the symmetrized tensor product of \( 2k \) fundamental representations), and \( \langle \cdot, \cdot \rangle \) denote the Hermitian inner product on \( V_j \), the spin for these inner products being inferred from the arguments. These inner products are related by the antilinear structure map \( J : V_j \rightarrow V_j^* \):
\[
\langle \psi, \phi \rangle = \epsilon(J \psi, \phi).
\]
\( J \) satisfies \( \rho_j(g) = \rho_j(g) J \), which implies \( \hat{L}^j J = -J \hat{L}^j \).

Let \( V^+_{j^-, j^+} \) denote the carrying space for the spin \((j^-, j^+)\) representation of \( Spin(4) \equiv SU(2) \times SU(2) \), and \( \rho_j \) the representation of \( (X^-, X^+) \in Spin(4) \) thereon. Again, when it is clear from the context, the \( j^- \) and \( j^+ \) subscript will be dropped. Let \( \hat{J}_- := i \rho(\tau^i) \otimes I_j \) and \( \hat{J}_+ := iI_j \otimes \rho(\tau^i) \) denote the anti-self-dual and self-dual generators, respectively, and let \( \hat{L} := \hat{J}_- \otimes \hat{J}_+ \) denote the spatial rotations on \( V^+_{j^-, j^+} \). Define the bilinear form \( \epsilon : V_{j^-, j^+} \times V_{j^-, j^+} \rightarrow \mathbb{C} \) by
\[
\epsilon(\psi^+ \otimes \psi^-, \phi^+ \otimes \phi^-) := \epsilon(\psi^+, \phi^+) \epsilon(\psi^-, \phi^-).
\]
Finally, let \( \hat{J}^{j^-, j^+}_i \) denote the intertwining map from \( V_k \) to \( V_{j^-, j^+} \), unique up to scaling, with scaling fixed such that the intertwining map is isometric in the Hilbert space inner products.

3.1.2. Canonical data and phase space. In the general boundary formulation of quantum mechanics [1], to the boundary of any four-dimensional region one associates a phase space, which is then quantized to obtain the boundary Hilbert space of the theory formulated in that region [1]. In the present case, the region is the 4-simplex \( S \). The boundary data are trivially constructed from the data introduced in section 2.1.3—one has the algebra elements \( B_{ab} \) and the related \( J_{ab} \) in the frame of each tetrahedron \( a \), and for each pair of tetrahedra \( a, b \) one constructs a parallel transport map \( G_{ab} \) from the frame \( b \) to the frame \( a \). These are related to the variables \( G_{a} \) introduced in section 2.1.3 by \( G_{ab} = (G_{a})^{-1}G_{b} \).

These boundary data are assembled into a classical phase space which may be identified with the cotangent bundle over any choice of five independent parallel transport maps.
\[ G_{ab} = (X_{ab}^+, X_{ab}^-), \quad L = \left( \begin{array}{c} \hat{L}^1 \\hat{L}^2 \end{array} \right) = T^* (\text{Spin}(4))^5 = T^* (SU(2) \times SU(2))^5. \]

Without loss of generality, we choose these to be \( G_{ab} = (X_{ab}^+, X_{ab}^-) \) with \( a < b \). For \( a < b \), \( J_{ab} = \left( J_{ab}^+, J_{ab}^- \right) \) and \( J_{ba} = \left( J_{ba}^+, J_{ba}^- \right) \) respectively generate right and left translations on \( G_{ab} \). For any \( a \neq b \), furthermore have the generators of internal spatial rotations, \( L_{ab} := \left( L_{ab}^1 \right)^\dagger + \left( L_{ab}^2 \right)^\dagger \). Note that when linear simplicity is satisfied (2.14), in terms of the reduced boundary data we have

\[ L_{ab}^i = \frac{1}{\kappa'} A_{ab} n_{ab}^i. \]  

(3.1)

3.1.3. Kinematical quantization and the vertex. The boundary Hilbert space of states \( \mathcal{H}_{\text{Spin}}^{(4)} \) is the \( L^2 \) space over the five \( G_{ab} = (X_{ab}^+, X_{ab}^-) \) \( \in \text{Spin}(4) \) with \( a < b \). The momenta operators \((\hat{J}_{ab}^1, \hat{J}_{ab}^2)\) and \((\hat{J}_{ba}^2, \hat{J}_{ba}^1)\) act respectively by \( i \) times the right and left invariant vector fields, associated with \( \tau' \in \mathfrak{su}(2) \), on the elements \( X_{ab}^\pm \). A (generalized) projected spin-network state (see [20, 21]) in \( \mathcal{H}_{\text{Spin}}^{(4)} \) is labeled by a choice of four spins \( j^+_{ab}, k_{ab}, k_{ba} \) and two states \( \psi_{ab} \in V_{k_{ab}}, \psi_{ba} \in V_{k_{ba}} \) per triangle:

\[ \Psi_{(j_{ab}, k_{ab}, \psi_{ab})}(X_{ab}^+, X_{ab}^-) := \prod_{a < b} \epsilon(i^{+}_{k_{ab}, j_{ab}}\psi_{ab}, \rho(X_{ab}^+, X_{ab}^-; j^+_{ab}, j^-_{ab})\psi_{ba}). \]  

(3.2)

This is an eigenstate of \((\hat{J}_{ab}^1)^2, (\hat{J}_{ab}^2)^2\) and \(\hat{L}_{ab}^2\) with eigenvalues \(j_{ab}^2(j_{ab}^2 + 1), k_{ab}(k_{ab} + 1)\) and \(k_{ba}(k_{ba} + 1)\), respectively, where \(\hat{L}_{ab}^2 = \sum \left( \hat{L}_{ab}^i \right)^2\). The generalized projected spin network states form an (overcomplete) basis of \( \mathcal{H}_{\text{Spin}}^{(4)} \).

To impose the linear simplicity constraint in quantum theory, one takes the sum of the squares of the constraints (2.14) for each \( ab \) to form a master constraint [22–24]:

\[ \hat{M}_{ab} := \sum_{i=1}^3 (C_{ab}^i)^2. \]  

(3.3)

The ordering is determined by the stringent condition that solutions exist. The generalized projected spin networks (3.2) are eigenstates of the resulting operator \( M_{ab} \) with eigenvalue \( M_{ab} \) given by

\[ M_{ab} = \lambda \left[ \left( 1 - \frac{1}{\gamma} \right) k_{ab}^2 + \frac{2}{\gamma} \left( 1 - \frac{1}{\gamma} \right) \left( j_{ab}^+ - 1 \right)^2 + \frac{2}{\gamma} \left( \frac{1}{\gamma} + 1 \right) \left( j_{ab}^- \right)^2 \right], \]

where \( \lambda \) is an unimportant positive constant. From the constraints \( \hat{M}_{ab}\psi = 0 \), one derives

\[ k_{ab} = \frac{2j_{ab}^+}{|1 - \gamma|} = \frac{2j_{ab}^-}{|1 + \gamma|} = k_{ba}, \]  

(3.4)

for all \( a \neq b \). The generalized projected spin networks with labels satisfying quantum simplicity (3.4) are thus parameterized by a choice of one spin \( k_{ab} \) and two states \( \psi_{ab}, \psi_{ba} \in V_{k_{ab}} \) per triangle—exactly the parameters specifying a (generalized) \( SU(2) \) spin-network state of LQG:

\[ \Psi_{(k_{ab}, \psi_{ab})}(X_{ab}) := \prod_{a < b} \epsilon (\psi_{ab}, \rho(X_{ab})\psi_{ba}) \in \mathcal{H}_{\text{Spin}}^{(4)} \equiv L^2(SU(2))^5. \]  

(3.5)

The condition that \( j_{ab}^+ = \frac{1}{2} \lfloor 1 \pm \gamma \rfloor \) be half-integer imposes a restriction on the spins \( k_{ab} \); let \( K_{\gamma} \) be the set of allowable values of \( k_{ab} \), and let \( \mathcal{H}_{\text{Spin}}^{(4)} \) be the span of the generalized \( SU(2) \) spin networks (3.5) with \( \{k_{ab}\} \subset K_{\gamma} \). One has an isomorphism \( \iota : \mathcal{H}_{\text{Spin}}^{(4)} \rightarrow \mathcal{H}_{\text{Spin}}^{(4)} \) between \( \mathcal{H}_{\text{Spin}}^{(4)} \) of the original linear simplicity constraints \( C_{ab} \) (2.14): for all \( \Psi, \Psi' \in \mathcal{H}_{\text{Spin}}^{(4)} \), \( \langle \Psi, \hat{C}_{ab}'(\psi, \check{C}_{ab}')\rangle = 0 \) [27].

2 The solution space to the master constraints also satisfies the Gupta-Bleuler criterion [25, 26] for the quantization \( \check{C}_{ab} \) of the original linear simplicity constraints \( C_{ab} \) (2.14): for all \( \Psi, \Psi' \in \mathcal{H}_{\text{Spin}}^{(4)} \), \( \langle \Psi, \check{C}_{ab}(\psi, \check{C}_{ab}')\rangle = 0 \) [27].
and the solution space to the master constraints in \( \mathcal{H}^{\text{Spin}(4)}_{\partial S} \). Here, and throughout the rest of the paper, we set \( s^\pm := \frac{1}{2} |1 \pm \gamma|k \).

The EPRL vertex for a given LQG boundary state \( \Psi^{\text{LQG}}_{\partial S} \in \mathcal{H}^{\text{LQG}}_{\partial S} \) is then

\[
A_{\gamma}(\Psi^{\text{LQG}}_{\partial S}) = \int_{\text{Spin}(4)^s} |m_{ab}| \prod_a dX^+_a X^+_a (\Psi^{\text{LQG}}_{\partial S})(X^+_a, X^+_a),
\]

where \( X^+_a := (X^+_a)^{-1} X^+_a \).

### 3.2. Boundary coherent states and integral expressions

**Definition 11** (Coherent state). Given a unit 3-vector \( n \) and a spin \( j \), let \( |\Gamma(n)\rangle_j \in V_j \) denote the unit norm state determined by the equation \( n \cdot \hat{L}|\Gamma(n)\rangle_j = j|\Gamma(n)\rangle_j \), with phase ambiguity fixed arbitrarily for each \( n \) and \( j \). For each \( \theta \), define \( |n, \theta\rangle_j := e^{\theta |\Gamma(n)\rangle_j} \). These are the coherent states. The \( \theta \) argument represents a phase ambiguity that will usually be suppressed.

**Definition 12** (Quantum boundary data). We call an assignment of one spin \( k_{ab} \in K \) and two unit 3-vectors \( n^a_{ab}, n^b_{ab} \) per triangle \( ab \) in \( S \) a set of quantum boundary data.

**Definition 13** (Boundary state corresponding to a set of quantum boundary data). Given a set of quantum boundary data and a choice of phase \( \theta \), one defines a corresponding state in the \( SU(2) \) boundary Hilbert space of the simplex:

\[
\Psi_{(k_{ab}, n_{ab}), \theta} := \Psi_{(k_{ab}, \psi_{ab})} \quad \text{with} \quad |\psi_{ab}\rangle := |n_{ab}, \theta_{ab}\rangle_{k_{ab}},
\]

where the \( \theta_{ab} \) are any phases summing to \( \theta \) modulo \( 2\pi \). The phase \( \theta \) will usually be omitted from the notation.

In order to derive the asymptotics of the vertex, [12] first cast the vertex in an appropriate integral form, separately for the cases \( \gamma < 1 \) and \( \gamma > 1 \):

\[
A_{\gamma}(\Psi_{(k_{ab}, n_{ab})}) = \int \left( \prod_a dX^+_a dX^+_a \right) \exp(S_{\gamma < 1}) \quad \text{for} \quad \gamma < 1
\]

and

\[
A_{\gamma}(\Psi_{(k_{ab}, n_{ab})}) = \int \left( \prod_a dX^+_a dX^+_a \right) \left( \prod_a (-1)^{s^+_a} (2s^+_a + 1) dm_{ab} \right) \exp(S_{\gamma > 1}) \quad \text{for} \quad \gamma > 1,
\]

where \( dm_{ab} \) is the measure on the metric 2-sphere normalized to unit volume and where \( S_{\gamma < 1} \) and \( S_{\gamma > 1} \) are ‘actions’ [12]. These actions are generally complex. As in [12], we are interested only in critical points whose contributions are not exponentially suppressed, and for this reason define ‘critical point’ to mean points where the action is stationary and its real part is maximal and non-negative. The critical point equations for \( \gamma < 1 \) are

\[
X^+_a n_{ab} = -X^+_b n_{ba} = 0
\]

for all \( a, b \). The critical point equations for \( \gamma < 1 \) are again (3.8) and (3.9) plus equations determining \( n_{ab} \) in terms of \( n_{ab} \). Thus, the non-trivial critical point equations are always (3.8) and (3.9), allowing both cases to be treated in a unified way.

---

3 If we compose \( \iota \) with projection onto Spin(4)-gauge invariant states, one additionally has an isomorphism between the \( SU(2) \)-gauge invariant part of \( \mathcal{H}^{\text{Spin}(4)}_{\partial S} \) and the Spin(4)-gauge-invariant part of the solution space to the master constraints (3.3) [14, 28].
3.3. Interpretation of the asymptotics and critical points

Before interpreting the critical points in terms of Plebanski sectors, we make clear the meaning of the data \( \{ k_{ab}, \bm{n}_{ab}, X^+_a \} \) in terms of classical discrete geometry. The data \( \{ k_{ab}, \bm{n}_{ab} \} \) label the coherent boundary state \( \Psi_{\{k_{ab}, \bm{n}_{ab}\}} \in \mathcal{L}^{1 \text{LOG}} \), which, in the definition of the vertex, is mapped by \( i \) into a Spin(4) boundary state in \( \mathcal{H}_{\text{Spin}(4)} \). By construction, \( i \Psi_{\{k_{ab}, \bm{n}_{ab}\}} \) satisfies linear simplicity (\( \tilde{M}_{ab} \Psi = 0 \)). Combined with equation (3.1), this leads to the conclusion that \( i \Psi_{\{k_{ab}, \bm{n}_{ab}\}} \) is a quantum state approximating a Spin(4) classical boundary state satisfying linear simplicity with reduced boundary data \( \Lambda_{ab} = A(k_{ab}) := \kappa \gamma k_{ab} \) and \( \bm{n}_{ab} \). Lastly, as [12] do, we identify the group variables \( X^+_a \) in the definition of the vertex (3.6) with the discrete connection introduced in section 2.1. This identification is consistent with the relation between the covariant and canonical transport variables presented in section 3.1.

We say that \( \{ k_{ab}, \bm{n}_{ab} \} \) is non-degenerate or satisfies closure if \( \{ A(k_{ab}), \bm{n}_{ab} \} \) is non-degenerate or satisfied closure, respectively.

**Definition 14** (Regge-like boundary data). Let a non-degenerate quantum boundary data set \( \{ k_{ab}, \bm{n}_{ab} \} \) satisfying closure be given. Then, for each tetrahedron \( a \), there exists a geometrical tetrahedron in \( \mathbb{R}^3 \), unique up to translations, such that each of the four quantities \( \{ A(k_{ab}) \}_b \neq a \) is equal to the area of one of the triangular faces, and each of the four vectors \( \{ \bm{n}_{ab} \}_b \neq a \) is equal to the outward-pointing normal of the corresponding triangular face. If these five geometrical tetrahedra can be glued together consistently to form a 4-simplex, we say that the boundary data \( \{ k_{ab}, \bm{n}_{ab} \} \) are Regge-like.

If the data \( \{ k_{ab}, \bm{n}_{ab} \} \) are Regge-like, in particular this means that, for each pair of tetrahedra \( a, b \), the triangle \( ab \) in \( a \) is congruent to the triangle \( ba \) in \( b \). It follows that, for each pair of tetrahedra, there exists a unique \( SU(2) \) element \( g_{ab} \) such that (1) the adjoint action of \( g_{ab} \) on \( \mathbb{R}^3 \) maps the triangle \( ab \) into the triangle \( ba \), and (2) \( g_{ab} \bm{n}_{ab} = -\bm{n}_{ab} \), where \( g_{ab} \) acts via the adjoint action. It follows that \( g_{ab} = \tilde{g}_{ba}^{-1} \).

**Definition 15** (Regge state). If a quantum boundary data set \( \{ k_{ab}, \bm{n}_{ab} \} \) is Regge-like, then the phase ambiguity in the state (3.7) can be uniquely resolved [12] by requiring that the phase of the coherent states be chosen such that
\[
g_{ab} \bm{n}_{ba} k_{ab} = J(\Lambda_{ab}) k_{ag},
\]
where \( J \) is the antilinear map defined in section 3.1.1. The resulting state \( \Psi_{\{k_{ab}, \bm{n}_{ab}\}} \) is called the Regge state corresponding to \( \{ k_{ab}, \bm{n}_{ab} \} \), and we denote it by \( \Psi_{\{k_{ab}, \bm{n}_{ab}\}}^{\text{Regge}} \).

We are now ready to quote the EPRL asymptotics from [12]. The statement of the asymptotics uses the fact that the boundary geometry of a 4-simplex is sufficient to determine the geometry of the 4-simplex itself [12, 29] and hence, in particular, the dihedral angles \( \Theta_{ab} \) between adjacent tetrahedra—if \( N_a \) and \( N_b \) denote the outward-pointing normals to the \( a \)th and \( b \)th tetrahedra, respectively, \( \Theta_{ab} \) is defined to be the unique angle in \([0, \pi]\) such that \( N_a \cdot N_b = \cos \Theta_{ab} \). For the following, we also need the notion of a *vector geometry*. A set of boundary data \( \{ k_{ab}, \bm{n}_{ab} \} \) is called a vector geometry if it satisfies closure and there exists \( \{ \hat{h}_a \} \subset SO(3) \) such that \( \hat{h}_a \cdot \bm{n}_{ab} = -(\hat{h}_a \cdot \bm{n}_{ba})^\dagger \) for all \( a \neq b \). The notion of asymptotic here is the same as that in [12].

\[4\] By looking instead at the operator \( \hat{L}^2 \), one alternatively concludes \( \Lambda_{ab} = \tilde{A}(k_{ab}) := x y \sqrt{k_{ab} + t} \). These two possibilities for relating \( \Lambda_{ab} \) and \( k_{ab} \) are equivalent in the semiclassical limit, which is what concerns us here.
Theorem 4 (EPRL asymptotics). Let non-degenerate quantum boundary data $B = \{k_{ab}, n_{ab}\}$ satisfying closure be given.

1. If $B$ is Regge-like, then in the limit $\lambda \to \infty$,

$$A_v(\Psi_{\text{Regge}}^{(\lambda k_{ab}, n_{ab})}) \sim \left(\frac{2\pi}{\lambda}\right)^{12} N^\gamma_{-+} \exp \left(\frac{i}{\lambda} \sum_{a < b} A(\lambda k_{ab}) \Theta_{ab}\right) + N^\gamma_{++} \exp \left(\frac{i}{\gamma} \sum_{a < b} A(\lambda k_{ab}) \Theta_{ab}\right) + N^\gamma_{--} \exp \left(-\frac{i}{\gamma} \sum_{a < b} A(\lambda k_{ab}) \Theta_{ab}\right),$$

(3.10)

where $N^\gamma_{-+}, N^\gamma_{++}, N^\gamma_{--}$ are the Hessian factors given in [12].

2. If $B$ is not Regge-like, but forms a vector geometry, then in the limit $\lambda \to \infty$,

$$A_v(\Psi_{\text{Regge}}^{(\lambda k_{ab}, n_{ab})}) \sim \left(\frac{2\pi}{\lambda}\right)^{12} N,$$

(3.11)

where $N$ is as defined in [12].

3. If $B$ is not a vector geometry, then $A_v(\Psi_{(\lambda k_{ab}, n_{ab}), \theta})$ decays exponentially with large $\theta$ for any $\theta$.

Classification of the critical points according to the Plebanski sector.

We now come to the interpretation, in terms of Plebanski sectors, of the critical points giving rise to the different terms in theorem 4.

- At the critical points giving rise to the first two terms of (3.10), from [12], the data $\{A(k_{ab}), n_{ab}, X^\pm\}$ satisfy $\{X^+\} \neq \{X^-\}$, and at the first term, $\mu = +1$, while at the second term $\mu = -1$. Therefore, by theorem 3, the first two terms of (3.10) correspond to bivectors in the Plebanski sectors (II+) and (II−), respectively.

- At the critical points giving rise to the rest of the non-exponentially suppressed terms in theorem 4, from [12], the data $\{A(k_{ab}), n_{ab}, X^\pm\}$ satisfy $\{X^+\} \sim \{X^-\}$. Therefore, by theorem 3, the rest of the non-exponentially suppressed terms in the asymptotics—namely, terms 3 and 4 of (3.10), and (3.11)—correspond to the degenerate Plebanski sector.

The above statements constitute the principal conclusions of this work.

4. Conclusions

In the foregoing work, we have clarified what it means for the discrete classical data involved in the semiclassical interpretation of spin foams to be in different Plebanski sectors. We then proved that the simplicity constraint used in both EPRL and FK—the linear simplicity constraint—restricts to the Plebanski sectors (II+), (II−) and the degenerate sector, mixing these three sectors. Finally, after reviewing the asymptotics of the EPRL vertex, we have identified the Plebanski sector of the data associated with each term in the asymptotics. This allowed us to see that the presence of terms other than the desired $e^{i\mathcal{S}_{\text{Regge}}}$ term is directly due to the mixing of these three Plebanski sectors by linear simplicity. Although these conclusions have been drawn for the Euclidean signature, we expect similar arguments to hold in the Lorentzian case.

In the literature until now, when an interpretation of the different terms is given, it is a different one. In the paper [13], the viewpoint is mentioned that the presence of terms in

14
the asymptotics with actions differing only by a sign are to be interpreted as a sum over orientations of the 4-simplex. This is based on an interpretation, first given in [12] itself, that the $\mu$ parameter in the reconstruction theorem (theorem 2) is to be interpreted as measuring the orientation of the 4-simplex. Although an interesting proposal, we believe this is not the natural interpretation: for, what is relevant in distinguishing these critical points is the value of the discrete Plebanski field $B^{\mu}_{\nu\rho}(S)$ in the 4-simplex frame (or equivalently, in any one of the tetrahedron frames). That is, based on the presentation in [12] and here, the distinction between critical points with actions of equal and opposite value lies in the dynamical variables themselves, and not in the orientation of the 4-simplex as a manifold. Rather, we have argued that the interpretation of such critical points is that of elements of two distinct Plebanski sectors that are being mixed in the EPRL model.

For the purpose of semiclassical calculations with the spin-foam model, it is important that all terms in the asymptotics other than $e^{iS_{\text{Regge}}}$ be eliminated. The only proposal so far in the literature for this is to eliminate the extra terms by selecting the boundary state to be peaked on the group variables as well as the conjugate canonical bivectors [30–32]. It is clear why this works: as mentioned, the different critical points in the terms of the asymptotics differ in the values of the discrete Plebanski field in a chosen frame. For the purpose of talking about boundary states it is most convenient to use a tetrahedron frame so that the discrete field depends only on the canonical data $G_{ab}$ and $J^{IJ}_{ab}$. No matter which tetrahedron frame is used, the discrete field will depend on both the group elements $G_{ab}$ and the conjugate canonical bivectors $J^{IJ}_{ab}$ so that by choosing a boundary state peaked on both $J^{IJ}_{ab}$ and $G_{ab}$, one is able to select a single discrete Plebanski field, and hence a single Plebanski sector, and thus in particular to select the single term $e^{iS_{\text{Regge}}}$ in the asymptotics, if desired. Although this works for a single simplex, because the strategy is based on specifying a boundary state, it is not immediately clear if this solution will work for simplicial complexes with interior tetrahedra.

The conclusions of this work suggest another possible solution. If one could modify the vertex in such a way as to restrict to only the Plebanski sector (II+)—something which is necessary anyway in order to unambiguously describe general relativity in the usual sense—then the asymptotics of the vertex should be simply $e^{iS_{\text{Regge}}}$, as desired. Such an avenue might be interesting to pursue.

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