Background

Let $\Sigma$ be a closed, smooth surface in $\mathbb{R}^3$. For any two sets $X, Y \subset \mathbb{R}^3$, let $d(X, Y)$ denote the Euclidean distance between $X$ and $Y$. The local feature size $f(x)$ at a point $x \in \Sigma$ is defined to be the distance $d(x, M)$ where $M$ is the medial axis of $\Sigma$. Let $n_p$ denote the unit normal (inward) to $\Sigma$ at point $p$. Amenta and Bern in their paper [1] claimed the following:

Claim 1 Let $q$ and $q'$ be any two points in $\Sigma$ so that $d(q, q') \leq \varepsilon \min\{f(q), f(q')\}$ for $\varepsilon \leq \frac{1}{3}$. Then, $\angle n_q, n_{q'} \leq \varepsilon$.

Unfortunately, the proof of this claim as given in Amenta and Bern [1] is wrong; it also appears in the book by Dey [2]. In this short note, we provide a correct proof with an improved bound of $\frac{\varepsilon}{1-\varepsilon}$.

Theorem 2 Let $q$ and $q'$ be two points in $\Sigma$ with $d(q, q') \leq \varepsilon f(q)$ where $\varepsilon \leq \frac{1}{3}$. Then, $\angle n_q, n_{q'} \leq \frac{\varepsilon}{1-\varepsilon}$.

Definitions and Preliminaries

For any point $p \in \mathbb{R}^3$, let $\tilde{p}$ denote the closest point of $p$ in $\Sigma$. When $p$ is a point in $\Sigma$, the normal to $\Sigma$ at $p$ is well defined. We extend this definition to any point $p \in \mathbb{R}^3$. Define the normal $n_p$ at $p \in \mathbb{R}^3 \setminus M$ as the normal to $\Sigma$ at $\tilde{p}$. Similarly, we extend the definition of local feature size $f$ to $\mathbb{R}^3$. For any point $p \in \mathbb{R}^3$, let $f(p)$ be the distance of $p$ to the medial axis of $\Sigma$. Notice that $f$ is 1-Lipschitz. If two points $x$ and $y$ lie on a surface $F \subset \mathbb{R}^3$, let $d_F(x, y)$ denote the geodesic distance between $x$ and $y$. The following facts are well known in differential geometry.

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Proposition 3 Let $F$ be a smooth surface in $\mathbb{R}^3$. Let $q$ and $q'$ be two points in $F$. Then,
\[
\lim_{d \to 0} \frac{d_F(q, q')}{d(q, q')} = 1.
\]

Proposition 4 Consider the geodesic path between $q, q'$ on a smooth surface $F$ in $\mathbb{R}^3$. Let $\kappa_m$ be the maximum curvature on this geodesic path. Then $\angle n_q, n_{q'} \leq \kappa_m d_F(q, q')$.

The Proof

We are to measure $\angle n_q, n_{q'}$ for two points $q$ and $q'$ in $\Sigma$. One approach would be to use the propositions above to bound the length of a path from $p$ to $q$ on $\Sigma$ and then use that length to bound the change in normal direction, but we can get a better bound by considering the direct path from $p$ to $q$.

Let $\Sigma_\omega$ denote an offset of $\Sigma$, that is, each point in $\Sigma_\omega$ has distance $\omega$ from $\Sigma$.

Formally, consider the distance function $h : \mathbb{R}^3 \to \mathbb{R}$, $h(x) \mapsto d(x, \Sigma)$.

Then, $\Sigma_\omega = h^{-1}(\omega)$.

Claim 5 For $\omega \geq 0$ let $p$ be a point in $\Sigma_\omega$ where $\omega < f(\tilde{p})$. There is an open set $U \subset \mathbb{R}^3$ so that $\sigma_p = \Sigma_\omega \cap U$ is a smooth 2-manifold which can be oriented so that $n_x$ is the normal to $\sigma_p$ at any $x \in \sigma_p$.

PROOF. Since $\omega < f(\tilde{p})$, $p$ is not a point on the medial axis. Therefore, the distance function $h$ is smooth at $p$. One can apply the implicit function theorem to claim that there exists an open set $U \subset \mathbb{R}^3$ where
\[
\sigma_p = h^{-1}(\omega) \cap U
\]
is a smooth 2-manifold. The unit gradient $\left(\frac{\nabla h}{\|\nabla h\|}\right)_x = \frac{x - \tilde{x}}{\|x - \tilde{x}\|}$ which is precisely $n_x$ up to orientation is normal to $\sigma_p$ at $x \in \sigma_p$.

PROOF. [Proof of Theorem] Consider parameterizing the segment $qq'$ by the length of $qq'$. Take two arbitrarily close points $p = p(t)$ and $p' = p(t + \Delta t)$ in $qq'$ for arbitrarily small $\Delta t > 0$. Let $\theta(t) = \angle n_q, n_{p(t)}$ and $\Delta \alpha = \angle n_p, n_{p'}$. Then, $|\theta(t + \Delta t) - \theta(t)| \leq \Delta \alpha$ giving
\[
|\theta'(t)| \leq \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t}.
\]
If we show that
\[ \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} \leq \frac{1}{(1 - \varepsilon)f(q)} \]
we are done since then
\[ \angle n_q, n_{q'} \leq \int_{q'q} |\theta'(t)| \, dt \]
\[ \leq \int_{q'q} \frac{1}{(1 - \varepsilon)f(q)} \, dt \]
\[ = \frac{d(q, q')}{(1 - \varepsilon)f(q)} \]
\[ \leq \frac{\varepsilon}{(1 - \varepsilon)}. \]

We have \( d(q, \tilde{p}) \leq d(q, p) + d(p, \tilde{p}) \) and \( d(q, p) \leq \varepsilon f(q) \). Since also \( \omega = d(p, \tilde{p}) \leq d(p, q) \leq \varepsilon f(q) \) (by a standard argument using the fact that the function \( f \) is 1-Lipschitz), we have \( \omega < f(\tilde{p}) \) for \( \varepsilon < 1/3 \), and there is a smooth neighborhood \( \sigma_p \subset \Sigma_\omega \) of \( p \) satisfying Claim 5.

Let \( r \) be the closest point to \( p' \) in \( \Sigma_\omega \), and let \( \Delta t \) be small enough so that \( r \) and the geodesic between \( p \) and \( r \) in \( \sigma_p \) lies in \( \sigma_p \). Notice that, by Claim 5,
\[ \Delta \alpha = \angle n_p, n_{p'} = \angle n_p, n_r. \]

**Claim 6** \( \lim_{\Delta t \to 0} \frac{d(p, r)}{\Delta t} \leq 1. \)

**Proof.** Consider the triangle \( prp' \). If the tangent plane to \( \sigma_p \) at \( r \) separates \( p \) and \( p' \), the angle \( \angle prp' \) is obtuse. It follows that \( d(p, r) \leq d(p, p') = \Delta t \). In the other case when the tangent plane to \( \sigma_p \) at \( r \) does not separate \( p \) and \( p' \), the angle \( \angle prp' \) is non-obtuse. Let \( x \) be the foot of the perpendicular dropped from \( p \) on the line of \( p'r \). We have \( d(p, r) \cos \alpha \leq d(p, p') \) where \( \alpha \) is the acute angle \( \angle rpx \). Combining the two cases we have \( d(p, r) / \Delta t \leq \frac{1}{\cos \alpha} \). Since \( \alpha \) goes to 0 as \( \Delta t \) goes to 0, we have \( \lim_{\Delta t \to 0} \frac{d(p, r)}{\Delta t} \leq 1. \)

Now consider the geodesic between \( p \) and \( r \) in \( \sigma_p \), and let \( m \) be the point on the geodesic at which the maximum curvature \( \kappa_m \) is realized. Recall that \( d_{\sigma_p}(p, r) \) denotes the geodesic distance between \( p \) and \( r \) on \( \sigma_p \). Let \( r_m \) be the radius of curvature corresponding to \( \kappa_m \), i.e., \( \kappa_m = 1/r_m \). Clearly, \( f(m) \leq r_m \). So, Proposition 4 tells us that
\[ \Delta \alpha \leq \frac{d_{\sigma_p}(p, r)}{f(m)}. \]

Therefore,
\[ \lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} \leq \lim_{\Delta t \to 0} \frac{d_{\sigma_p}(p, r)}{\Delta t f(m)} \]
In the limit when $\Delta t$ goes to zero, $d_{\sigma_p}(p, r)$ approaches $d(p, r)$ which in turn approaches $\Delta t$ (Proposition 3 and Claim 6). Meanwhile, $d(q, m) \leq d(q, p) + d(p, r)$ approaches $d(q, p) \leq \varepsilon f(q)$ as $\Delta t$ goes to zero (again by Claim 6). So, in the limit, $f(m) > (1 - \varepsilon) f(q)$ (again using the fact that $f$ is 1-Lipshitz). Therefore,

$$\lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} \leq \frac{1}{(1 - \varepsilon) f(q)}$$

which is what we need to prove.

Remark: The bound on normal variation can be slightly improved to $-\ln(1 - \varepsilon)$ by observing the following. We used that $d(q, p) \leq \varepsilon f(q)$ to arrive at the bound $f(m) > (1 - \varepsilon) f(q)$. In fact, one can observe that $d(q, p) \leq \varepsilon t f(q)$ giving $f(m) > (1 - \varepsilon t) f(q)$. This gives $|\theta'(t)| \leq \frac{1}{(1 - \varepsilon t)f(q)}$. We have

$$\angle n_q, n_{q'} \leq \int_{qq'} \frac{1}{(1 - \varepsilon t)f(q)} \, dt = \frac{d(q, q') \ln(1 - \varepsilon)}{\varepsilon f(q)} = -\ln(1 - \varepsilon).$$

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References

[1] N. Amenta and M. Bern. Surface reconstruction by Voronoi filtering. *Discr. Comput. Geom.* **22** (1999), 481–504.

[2] T. K. Dey. Curve and surface reconstruction : Algorithms with mathematical analysis. Cambridge University Press, New York, 2006.