Mass formula for 2 dimensional flavorful mesons

Osamu Abe(1) (*) and Nobuaki Watanabe(1)

(1) Laboratory of Physics

Asahikawa Campus, Hokkaido University of Education

9 Hokumoncho, Asahikawa 070-8621, Japan

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Summary. — We analytically and numerically investigate the 't Hooft equations, the lowest order mesonic Light-Front Tamm-Dancoff equations for SU(NC) and U(NC)gauge theories, generalized to flavor non singlet mesons. We find the wave function can be well approximated by new basis functions and obtain an analytic and an empirical formulae for the mass of the lightest bound state. Its value is consistent with the precedent results.

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The light front Tamm-Dancoff (LFTD) method [1, 2, 3] has been introduced

(*) e-mail: abeosamu@asa.hokkyodai.ac.jp
as an alternative tool to lattice gauge theory to investigate relativistic bound states nonperturbatively. In the LF coordinate, the physical vacuum is equivalent to the bare vacuum, since all constituents must have non-negative longitudinal momenta defined by $k^+ = (k^0 + k^3)/\sqrt{2}$. Because of this simple structure of the true vacuum, we can avoid the serious problems which appeared in the Tamm-Dancoff (TD) approximation [4] in the equal time frame. Therefore, the TD approximation is commonly used in the context of the LF quantization.

The techniques have been developed [5, 6, 7, 8, 9, 10] for solving LFTD equations in several models such as the massive Schwinger model [11], which is the extension of the simplest (1+1)-dimensional QED$_2$ [12]. Bergknoff [13] first applied LFTD approximation to the massive Schwinger model. In the most of above references, as they concentrated mainly on taking account of the higher Fock state contributions systematically in the context of LFTD approximation, they analyzed LFTD equations assuming that all the masses of quarks are degenerated in order to avoid complexities of numerical treatment. Mo and Perry [6] and Harada and coworkers [7] introduced basis functions to treat the massive Schwinger model in the context of LFTD approximation. One of the present author [10] generalized their basis functions. But, all the basis functions are applicable only in the case where all quark masses are degenerated.

In the real world, as six quark have their inherent masses, there are many mesons consist of quark and anti-quark with different masses. In this short note, we will attempt to generalize basis functions so as to treat the masses of mesons consist of different flavors with different masses. We will neglect the contributions
from higher Fock states, then we are led to the generalized ’t Hooft-Bergknoff-Eller
equation [13, 14]

\[
M^2 - \frac{m^2_f - 1}{x} - \frac{m^2_f' - 1}{1 - x} \Phi(x) = -\wp\int_0^1 \frac{\Phi(y)}{(y - x)^2}dy + \alpha \int_0^1 \Phi(y)dy.
\]

Here, parameter \(\alpha\) specifies the model under consideration, i.e., \(\alpha = 0\) for \(SU(N_C)\)
and \(\alpha = 1\) for \(U(N_C)\), \(\wp\) stands for the Hadamard’s finite part. \(M\) is the dimensionless meson mass, \(m_f\) and \(m_{f'}\) are the dimensionless quark mass of flavor \(f\) and \(f'\). They are related to the coupling constant \(g\) and bare masses \(\bar{M}\) and \(\bar{m}_f\) as follows:

\[
M^2 = \frac{2\pi N_C \bar{M}^2}{(N_C^2 + \alpha - 1)g^2}, \quad m^2_f = \frac{2\pi N_C \bar{m}_f^2}{(N_C^2 + \alpha - 1)g^2}.
\]

One of present authors (O.A.) [10] pointed out that the wave function can be expanded in terms of \((x(1-x))^{\beta_n+j}\) in case \(m_{f'} = m_f\). Here, \(\beta_n\) is the \((n+1)\)-th smallest positive solution of Eq. (6) which will be given bellow. Main interest in the present paper is to extend the basis functions so that we can treat mesons with \(m_f \neq m_{f'}\). One may expect it is enough to extend above basis function to \(x^{\beta_n+j}(1-x)^{\beta_n'+j}\). As we will see shortly, it is not the case.

At first, according to ’t Hooft [14], we put

\[
\Phi(x) = x^\beta (1-x)^{\beta'}.
\]

The most singular part of the left hand side of Eq. (1) at the end point \(x = \epsilon\) is given by \(-m^2_f - 1)\epsilon^{\beta-1}. One of the right hand side of Eq. (1) is given by

\[
-\beta \pi \cot(\pi \beta) \epsilon^{\beta-1}.
\]

Here, we have used

\[
\wp\int_0^1 \frac{y^a(1-y)^b}{(y - x)^2} = B(a - 1, b + 1)F(2, 1 - a - b; 2 - a; x)
\]
\[-\pi \cot(\pi a) \left\{ ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} \right\} \equiv f_{ab}(x), \]

where $B$ is the beta function and $F$ denotes the Gauss’s hypergeometric function.

Thus, we are led to

\[
m_f^2 - 1 + \beta \pi \cot(\pi \beta) = 0. \quad (6)
\]

Analogously, we also have at another end point $x = 1 - \epsilon'$,

\[
m_{f'}^2 - 1 - \beta' \pi \cot(\pi \beta') = 0. \quad (7)
\]

Here, we have used $f_{ab}(x) = f_{ba}(1-x)$. If we assume $\beta \simeq O(m_f)$ for small $m_f$, we obtain

\[
\beta = \frac{\sqrt{3}}{\pi} m_f + \mathcal{O}(m_f^2),
\]

\[
\beta' = \frac{\sqrt{3}}{\pi} m_{f'} + \mathcal{O}(m_{f'}^2). \quad (8)
\]

We multiply both sides of Eq. (1) by $\Phi(x)$ and integrate them, we have

\[
M^2B(1 + 2 \beta, 1 + 2 \beta') = (m_f^2 - 1)B(2\beta, 1 + 2 \beta') + (m_{f'}^2 - 1)B(1 + 2 \beta, 2 \beta)
\]

\[-I(\beta, \beta', \beta, \beta') + \alpha B(1 + 2 \beta, 1 + 2 \beta'). \quad (9)
\]

Here,

\[
I(a, b, c, d) \equiv \int_{x=0}^{1} \int_{y=0}^{1} \frac{y^a(1-y)^b x^c(1-x)^d}{(y-x)^2} dy dx
\]

\[
= -\pi a \cot(\pi a) B(a + c, b + d)
\]

\[
+ \pi (a + b) \cot(\pi a) B(1 + a + c, b + d)
\]

\[
+ B(-1 + a, 1 + b) B(1 + c, 1 + d) \times
\]

\[
3F_2(2; 1 - a - b, 1 + c; 2 - a, 2 + c + d; 1). \quad (10)
\]
In the above equation, \( _3F_2 \) denotes the generalized hypergeometric function. Thus, for small \( m_f \) and small \( m_f' \), we are led to

\[
M^2 = \alpha + \frac{\pi}{\sqrt{3}} (m_f + m_f') + \mathcal{O}(m_f^2, m_f m_f', m_f'^2).
\]

As a next step, we consider a higher order correction to Eq. (3). The most general form of the wave function is given by

\[
\Phi(x) = \lim_{N \to \infty} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n} C^{j_1,j_2}_{n_1 n_2} x^{\beta_{n_1} + j_1} (1 - x)^{\beta_{n_2} + j_2},
\]

where \( \beta_n \) and \( \beta'_n \) are \((n + 1)\)-th smallest positive solution of Eq. (6) and Eq. (7), respectively. The reason why we cannot introduce a term other than one in Eq. (12) will be presented later.

If we substitute Eq. (12) into Eq. (1), we have, at the end point \( x = \epsilon \),

\[
0 = - \sum_{n_1=0}^{\infty} (m_f^2 - 1 + \pi \beta_{n_1} \cot \beta_{n_1}) \sum_{n_2,j_2} C^{0,j_2}_{n_1 n_2} \epsilon^{\beta_{n_1} - 1} \\
+ \sum_{n_1,j=0}^{\infty} \left[ M^2 \sum_{k=0}^{J} \sum_{n_2,j_2=0}^{\infty} C^{J-k,j_2}_{n_1 n_2} \frac{(-\beta'_{n_2} - j_2)_k}{k!} \\
- (m_f^2 - 1) \sum_{k=0}^{J+1} \sum_{n_2,j_2=0}^{\infty} C^{J+1-k,j_2}_{n_1 n_2} \frac{(-\beta'_{n_2} - j_2)_k}{k!} \\
- (m_f'^2 - 1) \sum_{k=0}^{J} \sum_{n_2,j_2=0}^{\infty} C^{J-k,j_2}_{n_1 n_2} \frac{(-\beta'_{n_2} + j_2 + 1)_k}{k!} \\
- \sum_{k=0}^{J+1} \sum_{n_2,j_2=0}^{\infty} \pi (\beta_{n_1} + J + 1 - k) \cot \beta_{n_1} C^{J+1-k,j_2}_{n_1 n_2} \frac{(-\beta'_{n_2} - j_2)_k}{k!} \\
+ \sum_{k=0}^{J} \sum_{n_2,j_2} \pi (\beta'_{n_2} + j_2) \cot \beta_{n_1} C^{J-k,j_2}_{n_1 n_2} \frac{(-\beta'_{n_2} - j_2 + 1)_k}{k!} \right] \epsilon^{\beta_{n_1} + J} \\
+ \sum_{k=0}^{\infty} \sum_{n_1,j_1,n_2,j_2} C^{j_1,j_2}_{n_1 n_2} \left[ B(\beta_{n_1} + j_1 - 1, 1 + \beta'_{n_2} + j_2) \times \\
\frac{(2)_k(1 - \beta_{n_1} - j_1 - \beta'_{n_2} - j_2)_k}{(2 - \beta_{n_1} - j_1)_k k!} \\
- \alpha \delta_{k0} B(1 + \beta_{n_1} + j_1, 1 + \beta'_{n_2} + j_2) \right] \epsilon^k.
\]
Here, \((a)_n \equiv \frac{\Gamma(a + n)}{\Gamma(a)}\) is the Pochhammer symbol. The first line in Eq. (14) vanishes automatically, because of the definition of \(\beta_n\).

Now, it becomes clear that Eq. (12) is the most general form. Suppose that we introduce the term like \(cx^\gamma(1 - x)^{\gamma'}\) with \(\gamma \neq \beta_n + j\), then it requires

\[
0 = c(m_f^2 - 1 + \pi \gamma \cot \pi \gamma)\epsilon^{\gamma'-1}
\]

to hold. Thus, the coefficient \(c\) should vanish. Analogously, if \(\gamma' \neq \beta'_n + j\) then \(c = 0\).

We also have similar equation to Eq. (14) at another end point \(x = 1 - \epsilon'\). If we truncate Eq. (12) to given finite \(N\). The total number of free parameters is \((N + 1)^2(N + 2)^2/4\). On the other hand, if we require Eq. (14) and similar equation to hold upto of order \(O(\epsilon^{\beta N-1})\) or \(O(\epsilon'^{\beta'_N-1})\), we have \(N(N + 3)\) independent equations.

Thus, we cannot solve the equations in general. We have to reduce the degree of freedom. We put

\[
C_{n_1 n_2}^{j_1 j_2} = \delta_{n_1 0} \delta_{n_2 0} \delta_{j_1 j_2} d_{j_1} + \delta_{j_1 0} \delta_{j_2 0} e_{n_1 n_2},
\]

that is we assume

\[
\Phi(x) = \sum_{j=0}^{N} d_j x^{\beta_0 + j}(1 - x)^{\beta'_0 + j} + \sum_{n_1, n_2 = 0}^{N} e_{n_1 n_2} x^{\beta_{n_1}} (1 - x)^{\beta_{n_2}},
\]

with \(d_{00} = 1\) and \(e_{00} = 0\).

For a given \(m_f\) and \(m'_f\), we put \(M_f^2 = M_i^2\). We can then solve Eq. (14) for \(d_j\) and \(e_{n_1 n_2}\) in terms of \(M_i\). We thus obtain the \(M_i\) dependent truncated wave function, say, \(\Phi(x; M_i)\). We can calculate a new mass eigenvalue \(M_{i+1}\) using this
wave function as
\[
M_{i+1}^2 = \frac{\langle \Phi(M_i)|H|\Phi(M_i)\rangle}{\langle \Phi(M_i)|\Phi(M_i)\rangle}.
\]

We can use Eq.(11) as $M_0^2$. For $N \leq 3$, mass eigenvalue $M^2$ converges in 5 iterations.

For $N = 3$ and $0 < m_f, m_{f'} \leq 0.1$, we obtain $M^2$’s by the use of Mathematica. The results are summarized in Tables I and II. We can fit them by polynomials:

\[
M^2(\alpha = 0, m) = 1.8139(m_f + m_{f'}) + 0.892(m_f + m_{f'})^2 + 0.008m_fm_{f'} - 0.041(m_f + m_{f'})^3 - 0.18m_fm_{f'}(m_f + m_{f'}) + \cdots
\]

\[
M^2(\alpha = 1, m) = 1 + 1.8092(m_f + m_{f'}) + 0.497(m_f + m_{f'})^2 + 1.564m_fm_{f'} + 0.95(m_f + m_{f'})^3 - 4.40m_fm_{f'}(m_f + m_{f'}) + \cdots.
\]

We cannot proceed this procedure beyond $N = 3$, because we cannot calculate $\,_{3}F_{2}(2, a; b; c, d; 1)$ in desired precision. The results with $m_{f'} = m_f$ are consistent with the previous results in [10].

Finally, we will discuss a so-called “2% discrepancy” problem. In $U(N_C)$, single flavor model, we expect the dimensionless meson mass squared can be expanded as follows,

\[
M^2 = 1 + b_1 m_f + b_2 m_f^2 + \cdots.
\]

By the use of the bosonization method, Banks and his coworker [15] found $b_1 = 2 \exp(\gamma_E) = 3.56214 \cdots$. Bergknoff [13], however, found $b_1 = 2\pi/\sqrt{3} = 3.62759 \cdots$. Two results differ each other by 2%. Our result, Eq. (20), is consistent with Bergknoff’s result rather than Banks et al.’s. We may expect that $b_1 = 2\pi/\sqrt{3}$
is the minimum value in the context of ’t Hooft-Bergknoff-Eller equation and higher Fock sector should be included to solve the problem.

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Table I. – Numerical results for bound state mass $M^2$ in SU($N_C$) model as a function of quark masses $m_f$ and $m_f'$.

| $m_f$ | 0.01   | 0.02   | 0.04   | 0.06   | 0.08   | 0.10   |
|-------|--------|--------|--------|--------|--------|--------|
| 0.01  | 0.03663| 0.05522| 0.09293| 0.13136| 0.17051| 0.21036|
| 0.02  | 0.05522| 0.07398| 0.11205| 0.15084| 0.19034| 0.23056|
| 0.04  | 0.09293| 0.11205| 0.15083| 0.19033| 0.23055| 0.27149|
| 0.06  | 0.13136| 0.15084| 0.19033| 0.23056| 0.27147| 0.31312|
| 0.08  | 0.17051| 0.19034| 0.23055| 0.27147| 0.31317| 0.35546|
| 0.10  | 0.21036| 0.23056| 0.27149| 0.31312| 0.35546| 0.39866|

Table II. – Numerical results for bound state mass $M^2$ in U($N_C$) model as a function of quark masses $m_f$ and $m_f'$.

| $m_f$ | 0.01   | 0.02   | 0.04   | 0.06   | 0.08   | 0.10   |
|-------|--------|--------|--------|--------|--------|--------|
| 0.01  | 1.03660| 1.05506| 1.09233| 1.13012| 1.16847| 1.20738|
| 0.02  | 1.05506| 1.07387| 1.11159| 1.14988| 1.18867| 1.22799|
| 0.04  | 1.09233| 1.11159| 1.15040| 1.18954| 1.22922| 1.26940|
| 0.06  | 1.13012| 1.14988| 1.18954| 1.22965| 1.27018| 1.31122|
| 0.08  | 1.16847| 1.18867| 1.22922| 1.27018| 1.31154| 1.35348|
| 0.10  | 1.20738| 1.22799| 1.26940| 1.31122| 1.35348| 1.39622|