A Group Theoretical Identification of Integrable Cases of the Liénard Type Equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$: Part I: Equations having Non-maximal Number of Lie point Symmetries

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(Dated: July 31, 2009)

Abstract

We carry out a detailed Lie point symmetry group classification of the Liénard type equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$, where $f(x)$ and $g(x)$ are arbitrary smooth functions of $x$. We divide our analysis into two parts. In the present first part we isolate equations that admit lesser parameter Lie point symmetries, namely, one, two and three parameter symmetries, and in the second part we identify equations that admit maximal (eight) parameter Lie-point symmetries. In the former case the invariant equations form a family of integrable equations and in the latter case they form a class of linearizable equations (under point transformations). Further, we prove the integrability of all of the equations obtained in the present paper through equivalence transformations either by providing the general solution or by constructing time independent Hamiltonians. Several of these equations are being identified for the first time from the group theoretical analysis.
I. INTRODUCTION

In this set of two papers we perform a Lie symmetry analysis for the Liénard type equation

\[ A(x, \dot{x}, \ddot{x}) \equiv \ddot{x} + f(x)\dot{x} + g(x) = 0, \]  

where over dot denotes differentiation with respect to time and \( f(x) \) and \( g(x) \) are arbitrary smooth functions of \( x \). Notable equations from class \([1]\) include a large number of physically important nonlinear oscillators such as the anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing and Duffing-van der Pol oscillators, modified Emden-type equation (MEE), and its hierarchy, and generalized Duffing-van der Pol oscillator equation hierarchy. These equations arise naturally in several physical applications. The outstanding representative of the class of equations \([1]\) is the modified Emden equation (also called Painlevé-Ince equation), \( \ddot{x} + \alpha x\dot{x} + \beta x^3 = 0 \), where \( \alpha \) and \( \beta \) are arbitrary parameters, which has received considerable attention from both mathematicians and physicists for more than a century (see for example Ref. \([1]\) and references therein).

During the past three decades, immense interest has been shown towards the search for symmetry generators of nonlinear ordinary differential equations (ODEs) and classification of low dimensional Lie algebras and linearization. Even though Lie himself had shown that the second order ODE of the form, \( \ddot{x} + f(t, x, \dot{x}) = 0 \), can admit a maximum of eight symmetry generators, the recent impetus came only when Wulfman and Wybourne \([2]\) showed that the maximal Lie group of point transformations for the simple harmonic oscillator is eight and the associated group is \( SL(3, R) \). Subsequently, Cervero and Villarroel \([3]\) showed that the damped linear harmonic oscillator also admits eight symmetry generators. Thereafter, several studies were made to isolate the equations which admit rich Lie point symmetries by exploring their symmetry algebras and their applications in physics and mathematics \([4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]\). For a recent survey of results one may refer Ref. \([15]\) and references therein.

Specific equations of the form \([1]\) have also been investigated from other points of view. For example, Noether symmetries for certain physically important systems were also studied in Refs. \([3, 16, 17, 18]\). Also contact symmetries for the harmonic oscillator were also explicitly constructed by Cervero and Villarroel \([3]\). The nonlocal symmetries for the MEE have also been studied in several papers \([19, 20]\). Recently much interest has also been shown towards
exploring generalized symmetries, namely, $\lambda$-symmetries (also called $c^\infty$-symmetries) for certain nonlinear ODEs (see for example Ref. 21).

The main goal of the present set of papers is to present a detailed Lie point symmetry analysis of (1) and isolate integrable and linearizable cases explicitly. At this point we mention here that the study of group classification is interesting not only from a purely mathematical point of view, but is also important for applications. Physical models are often constrained with apriori requirements to symmetry properties specified by physical laws, for example, from the Galilean or special relativity principles. In this work we not only isolate the invariant equations and symmetries but also present the integrals of motion and/or general solutions wherever possible. The motivation for the present study comes from our recent work in which we have considered a rather general second-order nonlinear ODE of the form $\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda x = 0$, where $k_i$'s, $i = 1, 2, 3, 4$, $\lambda$ and $q$ are arbitrary parameters, and identified several interesting integrable cases and for certain equations we have also derived explicit solutions of both oscillatory and non-oscillatory types\textsuperscript{22}. Interestingly, we demonstrated that a system which is very close to the MEE, namely $\ddot{x} + \alpha x\dot{x} + \frac{\alpha^2}{9} x^3 + \lambda x = 0$, possesses certain basic properties which is very uncommon to nonlinear oscillators\textsuperscript{23}. It is a conserved Hamiltonian system (of nonstandard type) admitting amplitude independent harmonic oscillations. While the above mentioned equations are specific examples of (1), the question arises as to whether there exist other integrable/linearizable second order ODEs belonging to this class. In this and the accompanying paper (referred to as II), we present a systematic analysis towards this goal. In this way we classify both integrable and linearizable equations which belong to the Liénard type system (1).

Even though the Lie’s algorithm is in principle a straightforward one, the group classification for the present problem is reduced to integration of a complicated overdetermined system of partial differential equations for the infinitesimal symmetry functions for arbitrary forms of $f(x)$ and $g(x)$ in (1). While solving these determining equations we have to choose all the symmetry functions (see Eqs. 12 and 13 below) not equal to zero in order to obtain the maximal Lie point symmetries. On the other hand considering the special case of one or more of the symmetry functions to be equal to zero also, we obtain a spectrum of integrable equations. In this way we are able to classify (i) systems with non-maximal symmetries (less than eight) (ii) systems with maximal symmetries (eight). Note that for
second order ODEs the dimension of the Lie vector space cannot be four, five, six and seven.
In the present paper we focus our attention only on the equations associated with lesser parameter Lie-point symmetry groups, that is one, two and three parameter Lie-point symmetry groups. However, within this classification, we identify a wider class of important integrable equations which are invariant under 2-parameter point symmetry group. Many of them are being identified for the first time from the group theoretical analysis. In the second paper we identify all the equations admitting maximal number of symmetries which also turn out to be linearizable. The forms of \( f \) and \( g \) which lead to only two symmetry generators are as follows:

(i) \[ f = k_2 + f_1 x^q, \quad g = \frac{k_2^2 (q + 1)}{(q + 2)^2} x + \frac{k_2 f_1}{(q + 2)} x^{q+1} + g_1 x^{2q+1}, \] (2a)

(ii) \[ f = -f_1 + \lambda_2 \log(x), \quad g = g_1 x - \left( \frac{\lambda_2 f_1}{2} + \frac{\lambda_2^2}{4} \right) x \log x + \frac{\lambda_2^2}{4} x (\log x)^2, \] (2b)

(iii) \[ f = \frac{f_1}{x^2}, \quad g = -\frac{A^2}{4} x + \frac{Af_1}{2x} + \frac{g_1}{x^3}, \] (2c)

(iv) \[ f = \frac{\lambda_1}{\lambda_2} + f_1 e^{-\lambda_2 x}, \quad g = -\frac{\lambda_1}{\lambda_2^2} f_1 e^{-\lambda_2 x} + g_1 e^{-2\lambda_2 x} - \frac{\lambda_1^2}{\lambda_2^3}. \] (2d)

(Here \( f_1, g_1, k_2, \lambda_1, \lambda_2, q, A \) are all constants). All the above equations are pointed out to be integrable through equivalence transformations. We also identify that the only system which admits a three parameter symmetry group is the Pinney-Ermakov equation where \( f = 0 \) and \( g = \omega^2 x - \frac{\tilde{g}}{x^3} \) (\( \omega, \tilde{g} \): constants).

It is well known that the infinitesimal generators of a given Lie group form a Lie algebra. The Lie algebras constituted by Lie vector fields are widely used in the integration of differential equations\(^5\), group classification of ODEs and PDES\(^{24}\), in geometric control theory and in the theory of systems with superposition principles\(^{25}\) and in different schemes for numerical solution of differential equations\(^{26}\). A vast amount of works is available in the literature on the classification of realizations of finite dimensional Lie algebras on the real and complex planes. For example, the realizations of all possible complex Lie algebras of dimensions no greater than four were listed by Lie himself\(^{27}\). Recently Gonzalez-Lopez et al have provided the Lie’s classification of realizations of complex Lie algebras\(^{28}\) and extended it to the real case. A complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in the vector fields on a space of an arbitrary (finite) number of variables was constructed in Ref. \(^29\). For more details on the classification of Lie vector fields one may refer the recent work of Ref. \(^{30}\) and references therein. On the other hand, in
our paper we focus our attention on constructing Lie vector fields for the class of equations resulting out of Eq. (1) alone and discuss their integrability.

The plan of the paper is as follows. In the following section, we present the Lie’s algorithm for Eq. (1) and discuss the solvability of the determining equations. A careful analysis of our investigations show that one should consider two separate cases, namely (i) the symmetry function \( b = 0 \) and (ii) \( b \neq 0 \), while solving the determining equations. Since the former case admits three symmetry functions, \( a(t, x), c(t, x), d(t, x) \), while classifying the integrable equations, we consider the possibilities (i) \( d = 0, a, c \neq 0 \), and (ii) \( c = 0, a, d \neq 0 \) separately and bring out the equations that are invariant under both the possibilities in Sec. III. We also discuss sub-cases in both the cases (i) and (ii). Further, we prove that the system (1) does not admit a three parameter Lie point symmetry group when both \( f(x), g(x) \neq 0 \). In Sec. IV we investigate the equivalence transformations for Eq. (1) and show that they lead to integrable forms. In Sec. V we consider the special case in which either \( f(x) \) or \( g(x) \) is equal to zero and identify the associated integrable equations in this class. The notable example in this class includes Pinney-Ermakov equation. In Appendix A, we present some details on the Hamiltonian structure of an integrable equation that arises in the case (i). In Appendix B, we point out briefly some notable equations that are included in the most general equation (corresponding to (2a)). In Appendix C, we discuss the method of solving the integrable equation identified as integrable in this paper corresponding to (2b). The Liouville integrability of the two other integrable equations identified in the category \( c = 0, a, d \neq 0 \) are presented in Appendices D and E. Finally, we present our conclusions in Sec. VI.

II. DETERMINING EQUATIONS FOR THE INFINITESIMAL SYMMETRIES

We consider the one dimensional nonlinear Liénard type system of the form (1). Let the evolution equation be invariant under the one parameter Lie group of infinitesimal transformations

\[
\begin{align*}
\tilde{t} &= t + \epsilon \xi(t, x) + O(\epsilon^2), \\
\tilde{x} &= x + \epsilon \eta(t, x) + O(\epsilon^2), \\
\epsilon &\ll 1,
\end{align*}
\]  

(3)
where $\xi$ and $\eta$ represent the infinitesimal symmetries associated with the variables $t$ and $x$ respectively. The associated infinitesimal generator can be written as

$$X = \xi(t,x) \frac{\partial}{\partial t} + \eta(t,x) \frac{\partial}{\partial x}. \quad (4)$$

Eq. (1) is invariant under the action of (4) iff

$$X^{(2)}(A)|_{A=0} = 0, \quad (5)$$

where

$$X^{(2)} = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \eta^{(1)} \frac{\partial}{\partial x} + \eta^{(2)} \frac{\partial}{\partial x} \quad (6)$$

is the second prolongation in which

$$\eta^{(1)} = \dot{\eta} - \dot{x}\xi, \quad \eta^{(2)} = \ddot{\eta} - \dot{x}\ddot{\xi} - 2\dot{x}\dot{\xi}, \quad (7)$$

and dot denotes total differentiation. By analysing Eq. (5) we get the following determining equations:

$$\xi_{xx} = 0, \quad (8)$$

$$\eta_{xx} - 2\xi_{tx} + 2f\xi_x = 0, \quad (9)$$

$$2\eta_{tx} - \xi_{tt} + f\xi_t + 3g\xi_x + \eta f_x = 0, \quad (10)$$

$$\eta_{tt} - (\eta_x - 2\xi_t)g + f\eta + \eta g_x = 0, \quad (11)$$

where subscripts denote partial derivatives.

Solving Eqs. (8) and (9) we obtain

$$\xi = a(t) + b(t)x \quad (12)$$

and

$$\eta = bx^2 - 2b\mathfrak{H}(x) + c(t)x + d(t), \quad (13)$$

where

$$\mathfrak{H}_x = F(x) = \int_0^x f(x')dx' \quad \text{and} \quad \mathfrak{H}_{xx} = f(x), \quad (14)$$

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and \(a(t), b(t), c(t)\) and \(d(t)\) are arbitrary functions of \(t\). With these forms of \(\xi\) and \(\eta\), Eqs. (10) and (11) can be rewritten as

\[
(bx^2 - 2b\ddot{\xi} + cx + d)f_x + (a + bx)f + 3bg - 4b\ddot{F} + 3\dddot{x} + 2c - \dddot{a} = 0,
\]

and

\[
(bx^2 - 2b\ddot{\xi} + cx + d)g_x - (c - 2\dddot{a} - 2bF)g - 2\dddot{\xi} \\
+ (\dddot{b}x^2 + \dddot{c}x + \dddot{d})f + \dddot{b}x^2 + \dddot{c}x + \dddot{d} = 0,
\]

respectively. Solving Eqs. (15) and (16) for the given forms of \(f(x)\) and \(g(x)\) we can get the infinitesimal symmetries.

The foremost and simplest solution for (15) and (16) for any form of \(f\) and \(g\) is \(a = \text{constant}, b, c, d = 0\). In other words, one immediately gets the time translation generator \(X = \frac{\partial}{\partial t}\) irrespective of the form of \(f\) and \(g\). Our motivation here is to find explicit forms of \(f\) and \(g\) which admit more number of symmetries. For this purpose we solve Eqs. (15) and (16) in the following way. Rewriting Eq. (15), we get

\[
g = \frac{1}{3b}[-(bx^2 - 2b\ddot{\xi} + cx + d)f_x - (a + bx)f + 4b\dddot{F} - 3\dddot{x} - 2c + \dddot{a}], \quad b \neq 0.
\]

Thus the existence of Lie point symmetries of the general form (12) and (13) with \(b \neq 0\) introduces an interrelation between the functions \(f\) and \(g\). However, this relation has to be compatible with the second determining equation (16). Thus using (17) into (16), one can obtain an equation for \(f\), (which also involves all the four symmetry functions) which fixes its form as well as the associated symmetries systematically. This is carried out in the following paper II, where we show explicitly that maximal (eight) number of Lie point symmetries exists only for the case \(f_{xx} = 0\) and for \(f_{xx} \neq 0\) necessarily requires the symmetry function \(b = 0\). Consequently, one has to consider the case \(b = 0\) in Eqs. (15) - (16) separately because of the condition \(b \neq 0\) in Eq. (17). Thus it is of interest to consider two separate cases associated with Eqs. (15)-(16):

Case (i) \(b = 0\) : Since we assume one of the symmetry functions to be zero the determining equations lead us to lesser Lie point symmetries alone (one, two and three symmetries).

Case (ii) \(b \neq 0\) : In this case we solve the full determining equations which in turn lead us to the maximal (eight) Lie-point symmetry group (as well as other lesser point symmetries while \(b \neq 0\)) when \(f_{xx} = 0\).
In this paper, we analyze in detail only the Case (i) and present the results of the other case in the subsequent paper II.

### A. Alternate Way

One may also note here that one can proceed in an alternate way to analyze the determining equations (15) and (16) for compatibility to determine the forms of $f(x)$ and $g(x)$ and the associated symmetries. For example, Eq. (15) can be rewritten as

$$\ddot{a} = (\dot{b}x^2 - 2b\Im + cx + d)f_x + (\dot{a} + \dot{bx})f + 3bg - 4bF + 3bx + 2\dot{c}. \quad (18)$$

Differentiating the above equation with respect to $x$ and rearranging one can express $\dddot{b}$ in terms of $\dot{a}, \dot{b}, c$ and $d$. Differentiating again the resultant equation with respect to $x$ and simplifying the latter we find

$$\dot{a}f_{xx} = -[(\dot{b}x^2 - 2b\Im + cx + d)_{xx}f_x - 2(\dot{b}x^2 - 2b\Im + cx + d)_x f_{xx}] - (\dot{b}x^2 - 2b\Im + cx + \dot{d}) f_{x} f_{xx} - 3bg_{xx} + 2bf_{x}. \quad (19)$$

Similarly from Eq. (16) one can obtain expressions for $\dddot{c}$ and $\dddot{d}$ and finally arrive at an expression of the form

$$0 = -[(\dot{b}x^2 - 2b\Im + cx + d)g_x - (c - 2\dot{a} - 2bF)g - 2b\Im] + (\dot{b}x^2 - 2b\Im + \dot{c}x + \dot{d}) f + \dddot{b}x^2. \quad (20)$$

To find a compatible solution of Eqs. (19) and (20) one may substitute for $\dot{a}$ from (19) into (20) and analyze the resultant equation to find the allowed forms of $f$ and $g$ and the associated symmetries. However, in practice we find the method leads to very lengthy and laborious calculations. On the other hand in the procedure we adopt we find that for the case $f_{xx} \neq 0$, one necessarily requires $b = 0$, see Sec. V in the following paper II. Consequently the analysis given in Sec. III follows naturally.

On the other hand, for the case $f_{xx} = 0$, Eq. (19) leads to the condition

$$0 = 2bf_{x} - 3bg_{xx}. \quad (21)$$

which is consistent with (20). From (21) one can immediately write $f = f_1 + f_2 x$ and $g = \frac{1}{3}f_1 f_2 x^2 + \frac{1}{6}f_2 f_2 x^3 + g_1 x + g_2$. To fix the corresponding symmetries one has to resubstitute the forms of $f$ and $g$ in the original determining equations and solve them consistently. This is carried out in Paper II.
III. LIE SYMMETRIES OF LIÉNARD TYPE SYSTEMS - LESSER PARAMETER SYMMETRIES: CASE $b = 0$

We now explore the nature of the evolution equations which possess lesser parameter Lie point symmetries. Considering Eqs. (15) and (16), we now assume the function $b = 0$ to obtain the following determining equations,

$$\begin{align*}
(cx + d)f_x + \dot{a}f + 2\dot{c} - \ddot{a} &= 0 \quad (22) \\
(cx + d)g_x - (c - 2\dot{a})g + (\dot{c}x + \ddot{d})f + \ddot{c}x + \ddot{d} &= 0, \quad (23)
\end{align*}$$

respectively. We note here that the determining equation (22) for the function $f$ does not involve the function $g$. As a consequence an explicit form for $f$ can be determined by direct integration. Now substituting this form of $f$ into equation (23) we can derive the corresponding form of $g$.

It is a well known fact\(^\text{5,6}\) that a second order ODE admits only 1, 2, 3 or 8 parameter Lie point symmetries (which we will see explicitly for Eq. (1) also in the present as well as in the follow up paper II). In Sec.2 we noted that the most general equation which is invariant under the one parameter Lie point symmetry group is the general equation (1) itself with arbitrary form of $f(x)$ and $g(x)$ since there is no explicit appearance of $t$ in the equation and the associated symmetry generator is $\frac{\partial}{\partial t}$. But for $b = 0$, we explicitly show in the following that only two parameter symmetries exist when both $f \neq 0$ and $g \neq 0$ and obtain their specific forms, while three parameter symmetries can also exist only when $f = 0, g \neq 0$. Finally, in the case $f \neq 0$, $g = 0$, the second order ODE (1) can be rewritten as a first order equation which in turn can be integrated by quadratures straightforwardly. So we do not investigate this last category in this work.

A. 2-parameter Lie point symmetries

Rewriting Eq. (22), we have

$$f_x + \frac{\dot{a}}{c(x + \frac{d}{c})}f = \frac{\ddot{a} - 2\dot{c}}{c(x + \frac{d}{c})}, \quad (24)$$
Since $f$ should be a function of $x$ alone (vide Eq. (1)), we choose

$$\frac{\dot{a}}{c} = \lambda_1, \quad \frac{\ddot{a} - 2\dot{c}}{c} = \lambda_2, \quad \frac{d}{c} = \lambda_3,$$  \hspace{1cm} (25)

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are constants. Then Eq. (24) becomes

$$f_x + \frac{\lambda_1}{(x + \lambda_3)} f = \frac{\lambda_2}{(x + \lambda_3)}.$$  \hspace{1cm} (26)

Integrating Eq. (26) one obtains

$$f = \frac{\lambda_2}{\lambda_1} + f_1(x + \lambda_3)^{-\lambda_1},$$  \hspace{1cm} (27)

where $f_1$ is also an arbitrary constant. Note that since $f_1, \lambda_1, \lambda_2$ and $\lambda_3$ now occur in the form for $f(x)$ which determine the ODE (1), they are not symmetry parameters but rather they are the system parameters.

One can proceed further by constructing the associated form of $g$ by solving Eq. (23) and classify the invariant equations. To start with, let us first classify for convenience the equations which are invariant under 2-parameter Lie-point symmetry group. To deduce these equations, we consider the two possibilities $(i) d = 0; \ a, c \neq 0, \ (ii) c = 0; \ a, d \neq 0$ and also the sub-cases in both of them. We further note that due to the fact the system always admits translational symmetry, from Eqs. (11) and (12), it is clear that $a$ cannot be zero. So we need not consider the case $a = 0, c, d \neq 0$. On the other hand if both $c$ and $d$ are simultaneously zero, then $g = 0$ as may be inferred from Eq. (23). Finally, at the end of this section we consider the cases where none of the functions $a, b, c$ are zero and show that even here only 2-parameter symmetries exist.

1. **Case 1** $d = 0; \ a, c \neq 0 \ (\lambda_1 \neq 0, \lambda_2 \neq 0)$

The choice $d(t) = 0$ with $a(t), c(t) \neq 0$ leads us to several interesting new integrable equations, as we see below. Solving Eq. (25), with $d(t) = 0$ and so $\lambda_3 = 0$, one can obtain explicit forms for the functions $a$ and $c$ as

$$a = a_1 + \frac{\lambda_1}{\lambda_2} (\lambda_1 - 2) c_1 e^{(\frac{\lambda_2}{\lambda_1 - 2})t}, \quad c = c_1 e^{(\frac{\lambda_2}{\lambda_1 - 2})t},$$  \hspace{1cm} (28)

where $a_1$ and $c_1$ are two arbitrary (symmetry) parameters, which lead to a two parameter Lie-point symmetry group. Substituting Eqs. (27) and (28) into (23) with $d = 0$, we get

$$g_x + \frac{(2\lambda_1 - 1)}{x} g = \frac{-2\lambda_2^2 (\lambda_1 - 1)}{\lambda_1 (\lambda_1 - 2)^2} \frac{\lambda_2 f_1}{(\lambda_1 - 2)} x^{-\lambda_1}.$$  \hspace{1cm} (29)
Integrating Eq. (29), we obtain

\[ g = \frac{\lambda_2^2}{\lambda_1^2}(1 - \lambda_1) + \frac{\lambda_2 f_1}{(2 - \lambda_1) \lambda_1} x^{1-\lambda_1} + g_1 x^{1-2\lambda_1}, \tag{30} \]

where \( g_1 \) is another integration constant. The above forms of \( f \) and \( g \) (vide Eqs. (27) and (30), respectively) fix Eq. (1) to the form

\[ \ddot{x} + \left( \frac{\lambda_2}{\lambda_1} + f_1 x^{-\lambda_1} \right) \dot{x} + \frac{\lambda_2^2}{\lambda_1^2(\lambda_1 - 2)^2} x + \frac{\lambda_2 f_1}{(2 - \lambda_1) \lambda_1} x^{1-\lambda_1} + g_1 x^{1-2\lambda_1} = 0. \tag{31} \]

For the sake of neatness, we rewrite \( \lambda_1 = -q, q \neq 0, \) and \( \lambda_2 = -k_2 q, \) where \( k_2 \) is an arbitrary parameter, in the above equation so that we obtain

\[ \ddot{x} + \left( k_2 + f_1 x^q \right) \dot{x} + \frac{k_2^2(q + 1)}{(q + 2)^2} x + \frac{k_2 f_1}{(q + 2)} x^{q+1} + g_1 x^{2q+1} = 0, \tag{32} \]

where \( k_2, f_1 \) and \( q \) are nothing but system parameters. Eq. (32) is the most general equation that is invariant under the two parameter Lie point symmetry group with infinitesimal symmetries

\[ \xi = a_1 - \frac{1}{k_2} (q + 2)c_1 e^{(q + 2) t}, \quad \eta = c_1 e^{(q + 2) t} x. \tag{33} \]

The corresponding infinitesimal generators read

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = e^{(q + 2) t} \left[ -\frac{(q + 2)}{k_2} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right]. \tag{34} \]

The commutation relation between the vector fields \( X_1 \) and \( X_2 \) is given by

\[ [X_1, X_2] = \frac{k_2 q X_2}{(q + 2)}. \tag{35} \]

2. Integrability of Eq. (32) for arbitrary values of \( q \)

Eq. (32) is the most general integrable equation which is invariant under the two parameter symmetry group (33). Now we discuss the integrability of (32) briefly here. By introducing the transformation

\[ w = x e^{\frac{k_2}{(q + 2)} t}, \quad z = -\frac{(q + 2)}{q k_2} e^{-\frac{q k_2}{(q + 2)} t}, \tag{36} \]
where \( w \) and \( z \) are new dependent and independent variables, respectively, one can transform (32) to the form

\[
w'' + \alpha w^q w' + \beta w^{2q+1} = 0,
\]

where \( \alpha = \frac{(q+2)^2 f_1}{2k_2q^2} \) and \( \beta = \frac{(q+2)q_1}{4k_2q^2} \). Eq. (37) has been analyzed from different perspectives. For example, Lemmer and Leach\(^{20}\) have studied the hidden symmetries of Eq. (37). Feix et al.\(^{31}\) have shown that through a direct transformation to a third order equation the above Eq. (37) can be integrated to obtain the general solution for the specific choice of the parameter \( \beta \), namely \( \beta = \frac{\alpha}{(q+2)^2} \). For this choice of \( \beta \), the general solution of (37) can be written as

\[
x(t) = \left( \frac{(2 + 3q + I^2)(t + I_1)^l}{(l + I_1)^{q+1} + (2 + 3q + q^2)I_2} \right)^{\frac{1}{l}}, \quad I_1, I_2: \text{arbitrary constants.} \quad (38)
\]

For the same parametric choice recently we have shown that this equation can be linearized to a free particle equation through a generalized linearizing transformation so that the solution of the nonlinear equation can be constructed from the solution of the linearized equation.\(^{41}\) However, our very recent studies show that Eq. (37) admits time independent Hamiltonian description for all values of \( \alpha \) and \( \beta \). By introducing appropriate canonical transformation to the Hamilton’s canonical equation of motion one can integrate the resultant equations straightforwardly and obtain the general solution (for more details one may see Refs. \(^{22},^{34}\)). For convenience, the Hamiltonian structure of the above equation is indicated in the Appendix A. We also point out briefly other notable equations included in (32) in Appendix B.

3. **Integrable equations with** \( d = 0, \ a, c \neq 0 \) \( (\lambda_1 = 0 \ (q = 0), \lambda_2 \neq 0) \)

Earlier while deriving the form (27) for \( f(x) \) we assumed that \( \lambda_1 \neq 0 \). Now let us consider the case \( \lambda_1 = 0 \). From Eq. (25) we find that in this case

\[
a = a_1, \quad c = c_1 e^{-\lambda_2 t},
\]

where \( a_1 \) and \( c_1 \) are two arbitrary symmetry parameters which again lead us to a two parameter symmetry group. Solving Eq. (24), with the above forms of \( a \) and \( c \), we obtain

\[
f(x) = -f_1 + \lambda_2 \log(x), \quad (40)
\]
where \( f_1 \) is an integration constant. Substituting Eq. (40) into Eq. (23) with \( d = 0 \), we get

\[
g_x - \frac{g}{x} + \frac{\lambda_2}{2} (f_1 - \lambda_2 \log x) + \frac{\lambda_2^2}{4} = 0. \tag{41}
\]

Integrating Eq. (41), we obtain the following specific form for \( g \),

\[
g = g_1 x - \left( \frac{\lambda_2 f_1}{2} + \frac{\lambda_2^2}{4} \right) x \log x + \frac{\lambda_2^2}{2} x (\log x)^2, \tag{42}
\]

where \( g_1 \) is an integration constant. Using Eqs. (40) and (42) in Eq. (1), we have the following nonlinear ODE,

\[
\ddot{x} + (-f_1 + \lambda_2 \log(x)) \dot{x} + g_1 x - \left( \frac{\lambda_2 f_1}{2} + \frac{\lambda_2^2}{4} \right) x \log x + \frac{\lambda_2^2}{2} x (\log x)^2 = 0, \tag{43}
\]

which is invariant under the following infinitesimal symmetries

\[
\xi = a_1, \quad \eta = c_1 e^{-\frac{\lambda_1 t}{2}} x. \tag{44}
\]

The associated symmetry generators take the form

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = e^{\frac{\lambda_1 t}{2}} x \frac{\partial}{\partial x}. \tag{45}
\]

The integrability of Eq. (43) can be proved straightforwardly which we indicate in Appendix C.

4. **Integrable equations with** \( d = 0, a, c \neq 0 (\lambda_1 \neq 0, \lambda_2 = 0) \)

While deriving (27) we assumed that \( \lambda_2 \neq 0 \). Now we analyse the case \( \lambda_2 = 0 \) with \( \lambda_1 \neq 0 \). In this case we find that the compatible solution exists for either \( \lambda_1 \neq 2 \) or \( \lambda_1 = 2 \). In the first case by repeating the previous analysis we find that \( f = f_1 x^{-\lambda_1} \) and \( g = g_1 x^{(1-2\lambda_1)} \), where \( f_1 \) and \( g_1 \) are two arbitrary parameters so that Eq. (1) becomes

\[
\ddot{x} + f_1 x^{-\lambda_1} \dot{x} + g_1 x^{(1-2\lambda_1)} = 0. \tag{46}
\]

The associated infinitesimal generators turn out to be

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \dot{t} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}. \tag{47}
\]

Eq. (46) exactly coincides with (37) by redefining \( \lambda_1 = -q \), and so the integrability of (46) can be extracted from (37).
In the second case, namely, $\lambda_1 = 2$, we obtain that the following form of equation for (1),
\[ \ddot{x} + \frac{f_1}{x^2} \dot{x} - \frac{A^2}{4} x + \frac{A f_1}{2 x} + \frac{g_1}{x^3} = 0, \] (48)
where $A$ is an arbitrary parameter, which is invariant under the two parameter infinitesimal symmetry generators,
\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = e^{-At} \left( \frac{-2}{A} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right). \] (49)
We discuss the integrability of Eq. (49) in Appendix D.

In this and previous sub-sections we discussed the cases (i) $\lambda_1 = 0$, $\lambda_2 \neq 0$ and (ii) $\lambda_2 = 0$, $\lambda_1 \neq 0$. Finally, for the third case, namely (iii) $\lambda_1 = 0$, $\lambda_2 = 0$, one gets $a = constant = a_1$, $c = constant = c_1$. The invariant equation turns out to be the linear damped harmonic oscillator equation $\ddot{x} + f_1 \dot{x} + g_1 x = 0$, where $f_1$ and $g_1$ are arbitrary parameters. The associated infinitesimal vector fields are $X_1 = \frac{\partial}{\partial t}$, $X_2 = x \frac{\partial}{\partial x}$. It is known that damped harmonic oscillator equation admits eight parameter symmetry group. Since one of the symmetry functions is zero we obtained only a two parameter symmetry group. The full symmetry group of the damped harmonic oscillator will be discussed in paper II.

5. Case 2 $c = 0$, $a, d \neq 0$: Integrable equation

In the previous sub-section we considered the case $d = 0, a, c \neq 0$. Now we focus our attention on the case $c = 0, a, d \neq 0$ and fix the forms of $f$ and $g$ which are invariant under the corresponding symmetry transformations. Restricting to $c = 0$ and $a, d \neq 0$ in (22), we have
\[ f_x + \frac{\dot{a}}{d} f = \ddot{a}. \] (50)
As before, since $f$ has to be a function of $x$ alone, we choose
\[ \frac{\dot{a}}{d} = constant = \lambda_1, \quad \frac{\ddot{a}}{d} = \lambda_2 = constant, \] (51)
so that Eq. (50) becomes
\[ f_x + \lambda_2 f = \lambda_1. \] (52)
Solving (51) we obtain
\[ a = a_1 + \frac{\lambda_2^2}{\lambda_1} d_1 e^{\frac{\lambda_2}{\lambda_1} t}, \quad d = d_1 e^{\frac{\lambda_2}{\lambda_1} t}, \] (53)
where $a_1$ and $d_1$ are two integration constants which are also the two symmetry parameters. Integration of Eq. (52) leads us to

$$f = \frac{\lambda_1}{\lambda_2} + f_1 e^{-\lambda_2 x}, \quad (54)$$

where $f_1$ is an arbitrary constant. Substituting (54) into (23), with $c = 0$, we get

$$g_x + 2\lambda_2 g + \frac{\lambda_1}{\lambda_2} f_1 e^{-\lambda_2 x} + \frac{2\lambda_1^2}{\lambda_2^2} = 0. \quad (55)$$

Integrating (55) we obtain

$$g = -\frac{\lambda_1}{\lambda_2} f_1 e^{-\lambda_2 x} + g_1 e^{-2\lambda_2 x} - \frac{\lambda_1^2}{\lambda_2^3}, \quad (56)$$

where $g_1$ is an integration constant. Eqs. (54) and (56) fix the equation (1) to the specific form

$$\ddot{x} + \left[\frac{\lambda_1}{\lambda_2} + f_1 e^{-\lambda_2 x}\right] \dot{x} - \frac{\lambda_1}{\lambda_2} f_1 e^{-\lambda_2 x} + g_1 e^{-2\lambda_2 x} - \frac{\lambda_1^2}{\lambda_2^3} = 0. \quad (57)$$

We note that in the above $\lambda_1$, $\lambda_2$, $g_1$, and $f_1$ are system parameters. Eq. (57) is invariant under the following two parameter Lie point symmetries

$$\xi = a(t) = a_1 + \frac{\lambda_2^2}{\lambda_1} d_1 e^{\frac{\lambda_2}{\lambda_1} t}, \quad \eta = d_1 e^{\frac{\lambda_2}{\lambda_1} t}, \quad (58)$$

where $d_1$ and $a_1$ are the symmetry parameters. The associated infinitesimal generators are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = e^{\frac{\lambda_2}{\lambda_1} t} \left[\frac{\lambda_2^2}{\lambda_1} \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right]. \quad (59)$$

The integrability of the Eq. (57) is discussed in Appendix E.

B. Non-existence of 3-parameter symmetry group in the general case $a, c, d \neq 0$

Finally, we consider the general case in which none of the functions $a$, $c$ and $d$ are zero. In this case, the function $f$ takes the form given in Eq. (27). The functions $a, c$ and $d$ can be fixed by solving the Eq. (25). Doing so we find

$$a = a_1 + \frac{\lambda_1(\lambda_1 - 2)}{\lambda_2} c_1 e^{\frac{\lambda_2}{\lambda_1} t}, \quad c = c_1 e^{\frac{\lambda_2}{\lambda_1} t}, \quad d = \lambda_3 c_1 e^{\frac{\lambda_2}{\lambda_1} t}, \quad (60)$$

where only $a_1$ and $c_1$ are the symmetry parameters. So in this case also only 2-parameter symmetries exist and no 3-parameter symmetry group is possible.
Substituting the forms \( f, a, c \) and \( d \) into Eq. (23) and simplifying the resultant equation one obtains

\[
g_x + \frac{(2\lambda_1 - 1)}{(x + \lambda_3)} g = \frac{2\lambda_2^2(1 - \lambda_1)}{\lambda_1(\lambda_1 - 2)^2} + \frac{\lambda_2 f_1}{(2 - \lambda_1)(x + \lambda_3)^{\lambda_1}},
\]

where \( g_1 \) is an integration constant. One can directly integrate (61) to obtain

\[
g = \left( \frac{\lambda_2}{\lambda_1} \right)^2 \frac{(1 - \lambda_1)}{(2 - \lambda_1)^2} (x + \lambda_3) + \left( \frac{\lambda_2}{\lambda_1} \right) \frac{f_1(x + \lambda_3)^{(1-\lambda_1)}}{(2 - \lambda_1)} + g_1(x + \lambda_3)^{(1-2\lambda_1)}. \tag{62}
\]

Inserting the forms (27) and (62) in (1) we get

\[
\ddot{x} + \left( \frac{\lambda_2}{\lambda_1} \right) \frac{f_1}{(x + \lambda_3)^{\lambda_1}} \dot{x} + \left( \frac{\lambda_2}{\lambda_1} \right)^2 \frac{(1 - \lambda)}{(2 - \lambda_1)^2} (x + \lambda_3) + \left( \frac{\lambda_2}{\lambda_1} \right) \frac{f_1(x + \lambda_3)^{(1-\lambda_1)}}{(2 - \lambda_1)} + g_1(x + \lambda_3)^{(1-2\lambda_1)} = 0. \tag{63}
\]

It is interesting to note that the system possesses only two parameter Lie point symmetries. The infinitesimal symmetries and generators are

\[
\xi = a_1 + \frac{\lambda_1(\lambda_1 - 2)}{\lambda_2} c_1 e^{\frac{\lambda_2}{\lambda_1-2} t}, \quad \eta = c_1 e^{\frac{\lambda_2}{\lambda_1-2} t} (x + \lambda_3), \tag{64}
\]

and

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = e^{\frac{\lambda_2}{\lambda_1-2} t} \left( \frac{\lambda_1(\lambda_1 - 2)}{\lambda_2} \frac{\partial}{\partial t} + (x + \lambda_3) \frac{\partial}{\partial x} \right) \tag{65}
\]

respectively.

Redefining \( X = x + \lambda_3 \) in (63), the resultant equation coincides exactly with the integrable Eq. (31). The symmetry generators also coincide with the ones given in Eq. (33). So effectively no new nonlinear ODE is identified even when all the three symmetry functions are simultaneously nonzero.

Thus we conclude that the system (1) does not admit a three parameter Lie-point symmetry group when \( f(x), g(x) \neq 0 \) while the symmetry function \( b(x) = 0 \) in (8)-(11). Further, the only equations which admit two parameter symmetry group alone are the four nonlinear ODEs given by Eqs. (32), (43), (48) and (57).

**IV. EQUIVALENCE TRANSFORMATIONS**

We have shown in the above section that the identified evolution equations, namely Eqs. (32), (43), (48) and (57), admitting two parameter Lie point symmetries can be transformed
into integrable equations (32), (C1), (D1) and (E1) respectively, through appropriate trans-
formations. In this section we give a group theoretical interpretation for these results through
equivalence transformations (ETs). We invoke the equivalence transformations and give an
explanation for the results since the group classification problem is closely related to the
concept of equivalence of equations of the above forms with respect to transformations, see
for example Ref. 32.

Considering our original differential equation (1), let us consider a set of smooth, locally
one-to-one transformations 
\[ T : (t, x, f, g) \longrightarrow (T, X, f_1, g_1) \]
of the space \( \mathbb{R}^4 \) that act by the
formulas
\[ T = F(t, x), \quad X = G(t, x), \quad f_1 = H(t, x, f), \quad g_1 = L(t, x, g) \] (66)

A transformation is called an Equivalence Transformation (ET) of the equality \( \ddot{x} = -f(x)\dot{x} - g(x) \) if it transforms the equation

\[ \ddot{x} = -f(x)\dot{x} - g(x) \] (67)
to an equation of the same form

\[ \ddot{X} = -f_1(X)\dot{X} - g_1(X). \] (68)

In this case, Eqs. (67) and (68) and the functions \{f(x), g(x)\} and \{f_1(X), g_1(X)\} are equivalent\(^{32}\).

It is a proven fact that equivalent equations admit similar groups (for local transformations) and ET is a similarity transformation. That is, if (67) admits the group \( E \) then (68) also admits a group similar to it for local transformations.

Substituting the transformation (66) into Eq. (68) we get

\[
\begin{align*}
(G_t^2 F_t f_1 + G_x^3 g_1) + \dot{x} [(G_t^2 F_x + 2 G_t F_t G_x) f_1 + 3 G_t^2 G_x g_1] + \dot{x}^2 [3 G_x^2 G_t g_1] \\
+ (F_t G_x^2 + 2 G_t F_x G_x) f_1 + \dot{x}^3 [F_x G_x^2 f_1 + G_x^3 g_1] = -(G_t + \dot{x} G_x) [(F_{tt} \\
+ 2 \dot{x} F_{tx} + \dot{x}^2 F_{xx} - F_x (\dot{x} f + g)] + (F_t + \dot{x} F_x) [(G_{tt} + 2 \dot{x} G_{tx} \\
+ \dot{x}^2 G_{xx} - G_x (\dot{x} f + g)],
\end{align*}
\] (69)

where the subscripts denote partial derivative with respect to that variable. Equating the
coefficients of different powers of \( \dot{x}^n \), \( n = 0, 1, 2, 3 \), we get

\[
F_x G_x^2 f_1 + G_x^3 g_1 = F_x G_{xx} - G_x F_{xx}, \quad (70)
\]

\[
(F_t G_x^2 + 2G_t F_x G_x) f_1 + 3G_x^2 G_t g_1 = F_t G_{xx} + 2F_x G_{tx} - G_t F_{xx} - 2G_x F_{tx}, \quad (71)
\]

\[
(G_t^2 F_x + 2G_t F_x G_x) f_1 + 3G_t^2 G_x g_1 = -2G_t F_{tx} - G_x F_{tt} + f F_x G_t + 2F_t G_{tx}
\]

\[
- F_t G_x - F_x G_t, \quad (72)
\]

\[
G_t^2 F_t f_1 + G_t^3 g_1 = -G_t F_{tt} + g F_x G_t + F_t G_{tt} - g G_x F_t. \quad (73)
\]

Solving Eqs. (70) and (71) consistently we find \( G_x = 0 \) and \( F_{xx} = 0 \). As a result one gets

\[
G = \alpha(t), \quad F = \beta(t)x + \gamma(t), \quad (74)
\]

where \( \alpha, \beta \) and \( \gamma \) are arbitrary functions of \( t \). Substituting Eq. (74) in (72) and (73) and simplifying the resultant equations we get

\[
\dot{\alpha}^2 \beta f_1 = f \beta \dot{\alpha} + \beta \ddot{\alpha} - 2 \dot{\alpha} \dot{\beta}, \quad (75)
\]

\[
\dot{\alpha}^2 (\dot{\beta}x + \dot{\gamma}) f_1 + \dot{\alpha}^3 g_1 = -\dot{\alpha} (\dot{\beta}x + \dot{\gamma}) + g \beta \dot{\alpha} + \ddot{\alpha} (\dot{\beta}x + \dot{\gamma}). \quad (76)
\]

From Eq. (75) we can obtain an expression which connects the transformed function \( f_1 \) with the original function \( f \) of the form

\[
f_1 = \frac{f}{\dot{\alpha}} + \frac{\ddot{\alpha}}{\dot{\alpha}^2} - \frac{2 \dot{\beta}}{\beta \dot{\alpha}}. \quad (77)
\]

Substituting (77) in (76) and simplifying the resultant equation we arrive at

\[
g_1 = \frac{\beta g}{\dot{\alpha}^2} - \frac{(\dot{\beta}x + \dot{\gamma})}{\dot{\alpha}^2} f + \frac{2 \dot{\beta} (\dot{\beta}x + \dot{\gamma})}{\beta \dot{\alpha}^2} - \frac{(\dot{\beta}x + \dot{\gamma})}{\dot{\alpha}^2}. \quad (78)
\]

Thus we obtain the general ET

\[
T = \alpha(t), \quad X = \beta(t)x + \gamma(t), \quad f_1 = \frac{f}{\dot{\alpha}} + \frac{\ddot{\alpha}}{\dot{\alpha}^2} - \frac{2 \dot{\beta}}{\beta \dot{\alpha}},
\]

\[
g_1 = g_1 = \frac{\beta g}{\dot{\alpha}^2} - \frac{(\dot{\beta}x + \dot{\gamma})}{\dot{\alpha}^2} f + \frac{2 \dot{\beta} (\dot{\beta}x + \dot{\gamma})}{\beta \dot{\alpha}^2} - \frac{(\dot{\beta}x + \dot{\gamma})}{\dot{\alpha}^2}. \quad (79)
\]

Since we have already identified only four equations (vide Eqs. (32), (43), (48) and (57)) that are invariant under two parameter Lie point symmetries within the class of equations (1) we consider only these four equations and present our result. Now solving Eqs. (79)
with the given form of $f$ and $g$ one obtains the following result

Case 1 (Eq. (32))

\[ \alpha = -\frac{(q + 2)}{q k_2} e^{-\frac{q k_2}{(q + 2)^2} t}, \quad \beta = e^{-\frac{q k_2}{(q + 2)^2} t}, \quad \gamma = 0 \]

so that $f_1 = \alpha X^q, \quad g_1 = \beta X^{2q + 1}$

Case 2 (Eq. (43))

\[ \alpha = t, \quad \beta = e^{-\frac{\lambda_2}{2} \int \log[x(t)] dt}, \quad \gamma = 0 \]

so that $f_1 = \text{constant}, \quad g_1 = X$

Case 3 (Eq. (48))

\[ \alpha = \frac{1}{A} e^{At}, \quad \beta = e^{At}, \quad \gamma = 0 \]

so that $f_1 = \frac{1}{X^2}, \quad g_1 = \frac{1}{X^3}$

Case 4 (Eq. (57))

\[ \alpha = -\frac{\lambda_2}{\lambda_1} e^{-\frac{\lambda_2}{2}}, \quad \beta = 1, \quad \gamma = -\frac{\lambda_1}{\lambda_2^2} t \]

so that $f_1 = e^{\lambda_2 U}, \quad g_1 = e^{-2\lambda_2 U}$

(80)

It directly follows that with the above form of $f_1$ and $g_1$, Eq. (68) takes the form of (32), (C1), (D1) and (E1) respectively, which were shown to be integrable.

V. LIE SYMMETRIES OF EQ. (11) WITH $f(x) = 0$ OR $g(x) = 0$

Next we consider the special case of Eq. (11) with $f(x) = 0$, that is,

\[ \ddot{x} + g(x) = 0. \]  

(81)

In the following we focus our attention only on the case $b = 0$ so that we have $\xi = a(t), \eta = c(t)x + d(t)$. Eqs. (11) and (11) with $b(x) = 0$ and $f(x) = 0$ give rise to the following conditions, respectively,

\[ \ddot{a} - 2\dot{c} = 0, \]  

(82)

and

\[ g_x + \left( \frac{2\dot{a} - c}{cx + d} \right) g + \frac{\ddot{c} x + \ddot{d}}{cx + d} = 0. \]  

(83)

Since $g(x)$ should be a function of $x$ alone, we choose

\[ \frac{2\dot{a}}{c} - 1 = \lambda_1, \quad \frac{d}{c} = \lambda_2, \quad \frac{\ddot{c}}{c} = -\lambda_3, \]  

(84)

where $\lambda_1, \lambda_2$ and $\lambda_3$ are constant parameters. Note that the above implies $\frac{\ddot{d}}{c} = \lambda_2 \frac{\ddot{c}}{c} = \lambda_2 \lambda_3$. 

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Solving (84) we find that the solution exists either for the parametric choice \( \lambda_1 \neq 3 \) or \( \lambda_1 = 3 \). The respective infinitesimal symmetries are

\[
\xi = a(t) = a_1 + a_2 t, \quad \eta = c(t)x + d(t) = \frac{2a_2x}{(1 + \lambda_1)} + \frac{2\lambda_2a_2}{(1 + \lambda_1)}, \quad \lambda_1 \neq 3 \tag{85}
\]

\[
\xi = a(t) = a_1 - \frac{(1 + \lambda_1)}{2\sqrt{\lambda_3}}(c_2 \cos \sqrt{\lambda_3}t - c_1 \sin \sqrt{\lambda_3}t), \nonumber
\]

\[
\eta = c(t)x + d(t) = a_1 - (c_1 \cos \sqrt{\lambda_3}t + c_2 \sin \sqrt{\lambda_3}t)(x + \lambda_2), \quad \lambda_1 = 3 \tag{86}
\]

The respective invariant equations turn out to be

\[
\ddot{x} + \frac{g_1}{(x + \lambda_2)^{\lambda_1}} = 0, \quad \lambda_1 \neq 3, \tag{87}
\]

and

\[
\ddot{x} + \frac{\lambda_3}{4}(x + \lambda_2) + \frac{g_1}{(x + \lambda_2)^3} = 0, \quad \lambda_1 = 3. \tag{88}
\]

Thus Eq. (87) admits a two parameter symmetry group with the generators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = t \frac{\partial}{\partial t} + \frac{2(x + \lambda_2)}{(1 + \lambda_1)} \frac{\partial}{\partial x}. \tag{89}
\]

On the other hand Eq. (88) admits a three parameter symmetry group with the symmetry generators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \sin \sqrt{\lambda_3}t \left( \frac{(1 + \lambda_1)}{2\sqrt{\lambda_3}} \frac{\partial}{\partial t} + (x + \lambda_2) \frac{\partial}{\partial x} \right), \nonumber
\]

\[
X_3 = \cos \sqrt{\lambda_3}t \left( \frac{(1 + \lambda_1)}{2\sqrt{\lambda_3}} \frac{\partial}{\partial t} + (x + \lambda_2) \frac{\partial}{\partial x} \right). \tag{90}
\]

Redefining \( x + \lambda_2 = X \) in Eqs. (87) and (88) we get

\[
\ddot{X} + \frac{g_1}{X^{\lambda_1}} = 0, \quad \lambda_1 \neq 3. \tag{91}
\]

\[
\ddot{X} + \omega^2 X - \frac{\tilde{g}}{X^3} = 0, \quad \omega^2 = \frac{\lambda_3}{4}, \quad \tilde{g} = -g_1, \quad \lambda_1 = 3. \tag{92}
\]

Equation in (91) corresponds to a conservative Hamiltonian system \((H = \frac{p^2}{2} + \frac{g_1}{(1 - \lambda_1)} X^{1 - \lambda_1})\) and so the Liouville integrability is assured. On the other hand Eq. (92) is nothing but the Pinney-Ermakov equation, whose origin, properties and the method of finding its general solution have been discussed widely in the contemporary nonlinear dynamics literature (see for example Ref. 33 and references therein). For the sake of completeness we give the general solution of this equation as

\[
X = \frac{1}{A\omega} \sqrt{\left(\omega^2 A^4 - \tilde{g}\right) \sin^2(\omega t + \phi) + \tilde{g}}, \tag{93}
\]
It has also been shown that Eq. (92) can be transformed to harmonic oscillator equation through suitable nonlocal transformation and from the solution of the latter one can construct the solution for the nonlinear equation. For more details one may refer\textsuperscript{41}.

Finally, for \( g(x) = 0 \), Eq. (11) can be written as

\[
\ddot{x} + f(x)\dot{x} = 0.
\]

Eq. (94) can be transformed to a first order equation by a trivial change of variable which in turn can also be integrated trivially. So we do not discuss the symmetries of this equation here.

VI. CONCLUSIONS

In the present paper we have investigated the Liénard type equation (1) in the framework of modern group analysis of differential equations. Even though the integrability properties of some of the specific equations coming under the Liénard type have been discussed in the literature, we have identified all those equations which admit only two and three parameter symmetry groups.

To identify the integrable equations belonging to the class (1) we have deduced all the equations that are invariant under one, two and three parameter Lie point symmetries. Obviously the general Eq. (11) does not contain the variable \( t \) explicitly and so it always admits a time translational generator. However, we have demonstrated that several equations admit two parameter Lie point symmetry groups. In particular these equations correspond to four specific forms of the functions \( f(x) \) and \( g(x) \) in (1), see Eq. (2), namely Eqs. (32), (43), (48) and (57). These equations have been deduced here through a group theoretical point of view alone. We have also discussed the integrability properties of these equations briefly and shown the existence of equivalence transformations. After analyzing the Lie point symmetries we have also shown that Liénard type equation does not admit a three parameter symmetry group when both \( f(x), g(x) \neq 0 \) in Eq. (1). However, in the sub-case, \( f(x) = 0 \), one can find that the well known Pinney-Ermakov equation is the only equation which is invariant under a three parameter Lie point symmetry group.

In this paper we have restricted our attention only on the non-maximal Lie point symmetry groups. The question which naturally arises is what happens if one considers the more
general case, \( b \neq 0 \), vide Eq. (17). Such an analysis allows us to isolate a class of equations admitting eight parameter symmetries. We will present the results in the follow-up paper II.

Acknowledgments

One of us (SNP) is grateful to the Centre for Nonlinear Dynamics, Bharathidasan University, Tiruchirappalli, for warm hospitality. The work of SNP forms part of a Department of Science and Technology, Government of India sponsored research project. The work of MS forms part of a research project sponsored by National Board for Higher Mathematics, Government of India. The work of ML forms part of a Department of Science and Technology (DST), Ramanna Fellowship and is also supported by a DST-IRHPA research project.

In the following, we briefly discuss the integrability properties of the equations derived in Sec. III. To begin with let us consider the Liouville integrability of Eqs. (33).

APPENDIX A: TIME INDEPENDENT HAMILTONIAN FOR (37)

Recently, we have studied the integrability of (32) or equivalently (37) and found that it admits time independent integrals for all values of the parameters \( \alpha \) and \( \beta^{22,34} \). From the time independent integrals we have identified the following time independent Hamiltonian for (37), namely,

\[
H = \begin{cases} 
\frac{(r-1)}{r-2} \rho \frac{(r-2)}{r-1} \dot{w} + \frac{(r-1)}{r} \dot{x} w^{q+1}, & \alpha^2 > 4\beta(q+1) \\
\frac{1}{2} p w^{q+1} + \log\left(\frac{1}{p}\right), & \alpha^2 = 4\beta(q+1) \\
\frac{1}{2} \log\left[\frac{w^{2(q+1)}}{(q+1)^2} \sec^2\left(\frac{\omega}{(q+1)} w^{q+1} p\right)\right] - \frac{1}{2} p w^{q+1}, & \alpha^2 < 4\beta(q+1),
\end{cases}
\]  

(A1)

where the corresponding canonically conjugate momentum is defined by

\[
p = \begin{cases} 
\left(\dot{w} + \frac{(r-1)}{r} \dot{x} w^{q+1}\right)^{(1-r)}, & \alpha^2 \geq 4\beta(q+1) \\
\frac{(q+1)}{\omega^{q+1}} \tan^{-1}\left[\frac{\alpha w^{q+1} + 2(q+1) \dot{w}}{2w^{q+1}}\right], & \alpha^2 < 4\beta(q+1),
\end{cases}
\]  

(A2)

where \( r = \frac{\alpha}{2\beta(q+1)} (\alpha \pm \sqrt{\alpha^2 - 4\beta(q+1)}) \), \( \omega = \frac{1}{2} \sqrt{4\beta(q+1) - \alpha^2} \) and \( \dot{\alpha} = \frac{\alpha}{q+1} \). For more details about the derivation of the above Hamiltonian one may refer to Ref. 34. The time independent Hamiltonian ensures the Liouville integrability of (32) or (37).
APPENDIX B: NOTABLE INTEGRABLE EQUATIONS IN (32)

Besides the general case, \( q = \text{arbitrary} \), Eq. (32) encompasses several known integrable equations of contemporary interest. The interesting equations can be identified by appropriately choosing the parameter \( q \) as we demonstrate briefly in the following.

For example, choosing \( q = 1 \) in (32) one gets the generalized MEE,

\[
\ddot{x} + (k_2 + f_1 x)\dot{x} + \frac{2k_2^2}{9} x + \frac{k_2f_1}{3} x^2 + g_1 x^3 = 0, \tag{B1}
\]

Eq. (B1) can be transformed into the MEE, \( w'' + f_1 ww' + g_1 w^3 = 0 \), by introducing a transformation \( w = xe^{\frac{k_2}{2} t} \) and \( z = -\frac{3}{k_2} e^{-\frac{k_2}{3} t} \). The Hamiltonian structure for this equation can be extracted from (A1) by restricting \( q = 1 \) in the latter relations. The restriction \( f_1 = 0 \) in (B1) provides us the force-free Duffing oscillator whose invariance and integrability properties have been discussed in Refs. 14,35. With the choice \( f_1 = 3, g_1 = 1, k_2 = 0 \), the resultant equation becomes a linearizable one whose invariance and integrability properties have been discussed in detail in Refs. 36,37,38,39.

The case \( q = 2 \) in (32) gives us

\[
\ddot{x} + (k_2 + f_1 x^2)\dot{x} + \frac{3k_2^2}{16} x + \frac{k_2f_1}{4} x^3 + g_1 x^5 = 0. \tag{B2}
\]

The explicit form of the Hamiltonian can be fixed from (A1) by restricting \( q = 2 \) in the latter relations. We note here that Eq. (B2) also includes several known integrable equations. The notable examples are force-free Duffing-van der Pol oscillator equation \((g_1 = 0)\) and the second equation in the MEE hierarchy \((f_1 = 0, g_1 = \frac{1}{10})\).

Finally, we note that one may also recover specific equations like the force-free Helmholtz oscillator and the associated Lie symmetries can be obtained by appropriately choosing the value of the parameter \( q \). Choosing \( q = \frac{1}{2} \) and \( f_1 = 0 \) in (32) one gets the force-free Helmholtz oscillator. The symmetries of (33) with \( q = \frac{1}{2} \) coincide exactly with the one reported in Ref. 40.

APPENDIX C: METHOD OF INTEGRATING EQ. (43)

The solution of Eq. (43) can be constructed from the solution of the damped harmonic oscillator using a general procedure given by us sometime ago in Ref. 41. For example, let
us consider a linear ODE of the form
\[ \ddot{U} + \alpha \dot{U} + g_1 U = 0, \quad (C1) \]
where \( \alpha \) and \( g_1 \) are arbitrary parameters. By introducing a nonlocal transformation of the form
\[ U = x e^{\frac{\lambda_2}{2} \int^x \log(t') dt'} \]
in the linear ODE (C1) the latter can be brought to the form
\[ \ddot{x} + (\alpha + \frac{\lambda_2}{2} + \lambda_2 \log(x)) \dot{x} + \frac{\alpha \lambda_2}{2} x \log x + \frac{\lambda_2^2}{4} x (\log x)^2 = 0. \quad (C2) \]
Now redefining the constants \( (\alpha + \frac{\lambda_2}{2}) = -f_1 \) in (C2) one exactly ends up with (43).

Following the procedure given in Ref. 41 one can obtain the general solution for (C2) from the linear equation.

**APPENDIX D: METHOD OF INTEGRATING EQ. (48)**

Eq. (48) can be transformed to the equation of the form
\[ \ddot{U} + \frac{f_1}{U^2} \dot{U} + \frac{g_1}{U^3} = 0, \quad (D1) \]
through the transformation \( U = x e^{\frac{A}{2} t}, \ Z = \frac{1}{2} e^{At} \). Eq. (D1) admits Hamiltonian structure for all values of \( f_1 \) and \( g_1 \). The underlying Hamiltonian reads
\[ H = \begin{cases} \frac{(r-1)}{(r-2)} \frac{p^{(r-2)}}{p} + \frac{(r-1)}{r} f_1, & f_1^2 > -4g_1 \\ \log(\frac{1}{p}) - \frac{f_1}{2} \frac{p}{U}, & f_1^2 = -4g_1 \\ \frac{f_1}{2} \frac{p}{U} + \frac{1}{2} \log \left[ \frac{1}{2} \sec^2 \left( \frac{\omega^2}{2} \right) \right], & f_1^2 < -4g_1, \end{cases} \quad (D2) \]
where the canonical conjugate momentum is defined by
\[ p = \begin{cases} \frac{1}{(r-1)} \left( \dot{U} - \frac{(r-1)f_1}{rU} \right)^{(1-r)}, & f_1^2 \geq 4g_1 \\ -\frac{U}{\omega} \tan^{-1} \left[ \frac{f_1-2U}{2\omega} \right], & f_1^2 < 4g_1 \end{cases} \quad (D3) \]
where \( r = \frac{f_1^2}{2g_1} (f_1 \pm \sqrt{f_1^2 + 4g_1}) \) and \( \omega = \frac{1}{2} \sqrt{-4g_1 - f_1^2} \).

The time independent Hamiltonian given above ensures the Liouville integrability of Eq. (D1).

**APPENDIX E: METHOD OF INTEGRATING EQ. (57)**

By introducing a transformation \( U = x - \frac{\lambda_1}{\lambda_2} t \) and \( z = \frac{\lambda_2}{\lambda_1} e^{-\frac{\lambda_1}{\lambda_2} t} \) in (57) the latter can be transformed into the form
\[ U'' + f_1 e^{\lambda_2 z} U' + g_1 e^{-2\lambda_2 z} U = 0. \quad (E1) \]
Eq. (E1) can be rewritten in the form
\[ U'' + f_1 f(U) U' + \tilde{g}_1 f(U) \int f(U) dU = 0, \]  
(E2)
where \( f(U) = e^{-\lambda_2 U} \) and \( \tilde{g}_1 = -\lambda_2 g_1 \). Eq. (E2) admits time independent Hamiltonian for all values of \( f_1 \) and \( g_1 \). The respective Hamiltonians are
\[
H = \begin{cases} 
\frac{(r-1)}{(r-2)} p^{(r-2)} + \frac{(r-1) f_1}{\lambda_2} pe^{-\lambda_2 U}, & f_1^2 > 4 \tilde{g}_1 \\
\log[p] + \frac{f_1 p}{2 \lambda_2} e^{-\lambda_2 U}, & f_1 = 4 \tilde{g}_1 \\
\frac{1}{2} \log \left[ \frac{e^{-2 \lambda_2 U}}{\lambda_2^2} \sec^2 \left( \frac{\omega p e^{-\lambda_2 U}}{-\lambda_2^2} \right) \right] + \frac{f_1}{2 \lambda_2} pe^{-\lambda_2 U}, & f_1 < 4 \tilde{g}_1,
\end{cases}
\]  
(E3)
where the canonically conjugate momentum is defined by
\[
p = \begin{cases} 
[U' + \frac{(1-r)}{r} \frac{f_1 e^{-\lambda_2 U}}{\lambda_2}]^{1-r}, & f_1^2 \geq 4 \tilde{g}_1 \\
-\frac{\lambda_2 e^{-\lambda_2 U}}{\omega} \tan^{-1} \left( \frac{2 \lambda_2 U' - f_1 e^{-\lambda_2 U}}{2 \omega e^{-\lambda_2 U}} \right), & f_1^2 < 4 \tilde{g}_1
\end{cases}
\]  
(E4)
where \( r = \frac{f_1}{2 g_1} (f_1 \pm \sqrt{f_1^2 - 4 \tilde{g}_1}) \), \( \omega = \frac{1}{2} \sqrt{4 \tilde{g}_1 - f_1^2} \).

One may note that the above Hamiltonian resembles the Hamiltonian structure of (A1). The reason for this is that both the Hamiltonians (E3) and (A1) can be generated from the time independent Hamiltonian of the damped harmonic oscillator by suitable nonlocal transformation. For more details about this nonlocal transformation one may refer to Ref. 34.

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