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On the Solvability of Some Boundary Value Problems for the Nonlocal Poisson Equation with Boundary Operators of Fractional Order

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Abstract: In this paper, in the class of smooth functions, integration and differentiation operators connected with fractional conformable derivatives are introduced. The mutual reversibility of these operators is proved, and the properties of these operators in the class of smooth functions are studied. Using transformations generalizing involutive transformations, a nonlocal analogue of the Laplace operator is introduced. For the corresponding nonlocal analogue of the Poisson equation, the solvability of some boundary value problems with fractional conformable derivatives is studied. For the problems under consideration, theorems on the existence and uniqueness of solutions are proved. Necessary and sufficient conditions for solvability of the studied problems are obtained, and integral representations of solutions are given.

Keywords: involution; nonlocal Laplace operator; Poisson equation; fractional conformable derivatives; Dirichlet problem; Neumann problem

1. Introduction

Differential equations in which, along with the desired function \( u(t) \), there is a value \( u(S(t)) \), where \( S^2 t = t \), are called equations with Carleman shifts [1] or equations with involution. The theory of equations with involutively transformed arguments and their applications are described in detail in monographs [2–5]. To date, differential equations with various types of involution; the well-posedness of boundary and initial-boundary value problems; and the qualitative properties of solutions and spectral questions have been studied quite well [6–13]. The series of papers by Cabada and Tojo (see [14,15] for an expanded list of citations) pioneered in creating a comprehensive theory of Green’s functions for the one-dimensional differential equations with involution. In [16–20], the solvability of inverse problems for partial differential equations with involution is studied. In this paper, the solvability of some boundary value problems with fractional-order boundary operators for a nonlocal analogue of the Poisson equation is studied. For this purpose, using some transformations from \( \mathbb{R}^n \) generalizing involutive transformations, we introduce the corresponding nonlocal Laplace operator.

Let us consider the formulation of the problems studied in this work.

Let \( \Omega = \{ x \in \mathbb{R}^n : |x| < 1 \} \) be a unit ball, \( n \geq 2 \), \( \partial \Omega = \{ x \in \mathbb{R}^n : |x| = 1 \} \) be a unit sphere and \( S \) be a real orthogonal matrix \( S \cdot S^T = E \). Suppose also that there exists a natural number \( J \) such that \( S^J = E \). Note that if \( x \in \Omega \), or \( x \in \partial \Omega \), then for any \( k \), the inclusions \( S^k x \in \Omega \) or \( S^k x \in \partial \Omega \) hold true.

Let us give one simple example of such a matrix \( S \).
Example 1. In the case of \( n = 2 \), the matrix \( S \) can be chosen in the form:

\[
S = \begin{pmatrix}
\cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\
\sin \frac{\pi}{4} & \cos \frac{\pi}{4}
\end{pmatrix}
\]

It is clear that \( S \cdot S^T = E \). In this case, \( l = 8 \).

Let \( a_1, a_2, \ldots, a_l \) be some real numbers. Introduce the nonlocal operator

\[
L_Iu(x) \equiv -\sum_{k=1}^{l} a_k \Delta u \left( S^{k-1}x \right).
\]

In the case \( a_1 = 1, a_k = 0, k = 2, 3, \ldots, l \) the operator \( L_I \) coincides with the classical Laplace operator \( \Delta \).

Further, to define the boundary conditions, we have to define the integral and fractional derivatives.

Let the function \( f(x) \in C^1[a, b] \). Define the operator as

\[
T^\beta f(x) = x^{1-\beta}f'(x), \beta > 0.
\]

In the case \( \beta \in (0, 1) \), this operator corresponds to an integral operator of the form

\[
I^\beta f(x) = \int_0^x f(t) \frac{dt}{t^{1-\beta}}.
\]

Let \( 0 < a, \beta \) and \( a^\alpha = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n} \), \( n = 1, 2, \ldots \). In [21,22], the following integro-differential operators were considered:

\[
j^{\alpha, \beta} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left( \frac{x^\beta - t^\beta}{\beta} \right)^{n-1} f(t) \frac{dt}{t^{1-\beta}}.
\]

\[
D^{\alpha, \beta} f(x) = \frac{n}{\Gamma(n-\alpha)} \int_0^x \left( \frac{x^\beta - t^\beta}{\beta} \right)^{n-\alpha-1} f(t) \frac{dt}{t^{1-\beta}}, n-1 < \alpha \leq n
\]

Suppose that \( J^{0, \beta} f(x) = f(x) \).

In the case \( \beta = 1 \), the operator \( J^{\alpha, \beta} \) coincides with the operator of the order of integration in the Riemann–Liouville sense, whereas \( D^{\alpha, \beta} \) and \( cD^{\alpha, \beta} \) coincide with the operators of the order of differentiation \( \alpha \) in the Riemann–Liouville and Caputo senses, respectively [23].

Let \( x \in \Omega, r = |x|, \theta = \frac{x}{|x|}, \frac{d}{dr} = \sum_{j=1}^{n} \frac{x_j}{r} \frac{\partial}{\partial x_j} \), and the function \( u = u(x) \) be defined in the domain \( \Omega \).

For \( \alpha \in (0, 1] \), we introduce the operators

\[
j^{\alpha, \beta}_r u(x) = \frac{1}{\Gamma(\alpha)} \int_0^r \left( \frac{r^\beta - t^\beta}{\beta} \right)^{n-1} u(t\theta) \frac{dt}{t^{1-\beta}}, \quad \frac{d}{dr}
\]

\[
D^{\alpha, \beta}_r u(x) = \frac{1}{\Gamma(n-\alpha)} \left( r^{1-\beta} \frac{d}{dr} \right) \int_0^r \left( \frac{r^\beta - t^\beta}{\beta} \right)^{n-\alpha} u(t\theta) \frac{dt}{t^{1-\beta}}.
\]
\[ D^{\alpha,\beta}_x u(x) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \int_0^x \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{-\alpha} \left( \frac{\tau^{1-\beta} \, d\tau}{\tau^{1-\beta}} \right) u(\tau) \, d\tau. \]

Further, we will also consider the operators
\[ B^{\alpha,\beta}[u](x) = r^{\alpha\beta} D^{\alpha,\beta}_x, \quad B^{-\alpha\beta}[u](x) = r^{\alpha\beta} D^{-\alpha,\beta}_x. \]

**Example 2.** Let \( 0 < \alpha < 1, \beta > 0, \) \( H_k(x) \) be a homogeneous harmonic polynomial of order \( k. \) Consider the actions of operators \( B^{-\alpha\beta} \) and \( B^{\alpha,\beta} \) on \( H_k(x). \) We get
\[ B^{-\alpha\beta} H_k(x) = \frac{H_k(x)}{\beta^\alpha \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{\beta-1} d\tau = \frac{H_k(x) \Gamma(\alpha) \Gamma\left(\frac{k}{\beta} + 1\right)}{\beta^\alpha \Gamma\left(\frac{k}{\beta} + 1\right)} = \frac{\Gamma\left(\frac{k}{\beta} + 1\right)}{\beta^\alpha \Gamma\left(\frac{k}{\beta} + 1\right)} H_k(x), \]
\[ B^{\alpha\beta}[H_k](x) = r^{\alpha\beta} D^{\alpha,\beta}_x H_k(x) = r^{\alpha\beta} \frac{H_k(\theta)}{\Gamma(1-\alpha)} \left( \frac{r^{1-\beta} \, d\tau}{\beta} \right) \int_0^\tau \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{-\alpha} \tau^k \, d\tau \]
\[ = r^{\alpha\beta} \beta^{\alpha-1} \frac{H_k(\theta)}{\Gamma(1-\alpha)} \int_0^{1-\xi} (1-\xi)^{-\alpha} \xi^\frac{k}{\beta} d\xi \xi^k \left( \frac{r^{1-\beta} \, d\tau}{\beta} \right) \xi^{k-\alpha\beta+k} \]
\[ = r^{\alpha\beta} \beta^{\alpha-1} \frac{H_k(\theta)}{\Gamma(1-\alpha)} \frac{\Gamma\left(\frac{k}{\beta} + 1\right)}{\Gamma\left(\frac{k}{\beta} + 2 - \alpha\right)} (k - \alpha\beta + \beta)r^{k-\alpha\beta} = \beta^\alpha \frac{\Gamma\left(\frac{k}{\beta} + 1\right)}{\Gamma\left(\frac{k}{\beta} + 1 - \alpha\right)} H_k(x). \]

Hence,
\[ B^{\alpha,\beta} \left[ B^{-\alpha\beta}[H_k]\right](x) = B^{-\alpha\beta} \left[ B^{\alpha,\beta}[H_k]\right](x) = H_k(x). \]

Consider the following problems in the domain \( \Omega. \)

**Problem 1.** Let \( 0 < \alpha \leq 1, \beta > 0. \) Find a function \( u(x) \in C^2(\Omega) \cap C(\bar{\Omega}), \) for which \( D^{\alpha,\beta}_x u(x) \in C(\bar{\Omega}), \) satisfying the conditions
\[ L_i u(x) = f(x), \; x \in \Omega, \quad (1) \]
\[ D^{\alpha,\beta}_x u(x)|_{\partial\Omega} = g(x). \quad (2) \]

**Problem 2.** Let \( 0 < \alpha \leq 1, \beta > 0. \) Find a function \( u(x) \in C^2(\Omega) \cap C(\bar{\Omega}), \) for which \( D^{\alpha,\beta}_x u(x) \in C(\bar{\Omega}), \) satisfying Equation (1) and the boundary condition
\[ D^{\alpha,\beta}_x u(x)|_{\partial\Omega} = g(x). \quad (3) \]

The issue of the solvability of boundary value problems with boundary operators of fractional order for classical equations of elliptic type has been studied in a number of papers (see, for example, \([24–33]\)). In \([25],\) the boundary value problem with boundary
operators of the Marchaud, Grünwald–Letnikov and Liouville–Weyl type was studied in Sobolev spaces, where it was established that in order to make the considered problems solvable, starting from the value of the boundary operator $\frac{1}{2}$, the requirement of satisfaction of an additional orthogonality condition for the boundary functions must be introduced. Later, in [29, 30], similar results were obtained in the Hölder classes for boundary value problems with Riemann–Liouville and Caputo boundary operators.

Applications of boundary value problems to elliptic equations with fractional-order boundary operators are considered in [34, 35].

In this paper, similar studies are carried out for a nonlocal analogue of the Poisson equation. The properties of the nonlocal operator $L_1$ and the main boundary value problems (with the Dirichlet, Neumann and Robin conditions) for the corresponding nonlocal Poisson equation were studied in [36]. The paper [37] considers Bitsadze–Samarskii type boundary value problems for a nonlocal analogue of the Helmholtz equation.

The present work is a continuation of these investigations for the case of boundary operators of fractional order.

Studying Problems 1 and 2, we obtain an “interpolation” of the classical Dirichlet and Neumann problems for the case of non-integer orders of boundary operators. Moreover, for $\beta = 1$, we get $D^{\alpha_1} = DRL^{\alpha_1}$ (the Riemann–Liouville operator) and $C D^{\alpha_1} = C D^{\alpha_1}$ (the Caputo operator). Therefore, by studying these problems for parameter values $\alpha \in (0, 1)$ and $\beta > 0$, we obtain new classes of well-posed boundary value problems for Equation (1) and even for the classical Poisson equation, i.e., for the case $\alpha = 1, \beta = 0$. It follows from [36] that for the case $\alpha = 0$, the corresponding problem (the Dirichlet problem) is unconditionally solvable, whereas for the case $\alpha = 1$ (the Neumann problem), the problem is solvable, only if the connection conditions between the right-hand side of Equation (1) and the boundary function (3) are satisfied. In this paper, we establish that for all values of $\alpha \in (0, 1), \beta > 0$, Problem 1 is unconditionally solvable; i.e., the solution of this problem has the properties of the solution of the Dirichlet problem, whereas Problem 2, as in the case of the Neumann problem, is conditionally solvable.

Problems 1 and 2 are studied using the research technique developed in [36] and the properties of operators $B^{\alpha, \beta}$ and $B^{-(\alpha, \beta)}$ in the class of smooth functions and functions from the Hölder class.

As $J_{\alpha, \beta} u(x) = u(x)$ for $\alpha, \beta = 1$ for points $x \in \partial \Omega$, the following equality holds

$$
D^{1, \beta}_r u(x)|_{\partial \Omega} = D^{1, \beta}_r u(x)|_{\partial \Omega} = \frac{\partial u(x)}{\partial r}|_{\partial \Omega} = \frac{\partial u(x)}{\partial v}|_{\partial \Omega},
$$

where $v$ is the normal vector to the sphere $x \in \partial \Omega$. Hence, in the case, $\alpha = 1$ in Problems 1 and 2; the boundary conditions coincide with the Neumann condition. This case was studied in [29], and therefore, further we study Problems 1 and 2 for all values of the parameters $\alpha \in (0, 1)$ and $\beta > 0$.

The results of this work are presented in the following order. Section 2 contains auxiliary statements obtained by one of the authors and related to the properties of transformations $S$. Section 3 deals with the properties of operators $B^{-(\alpha, \beta)}$, $B^{\alpha, \beta}$ and $B^{\alpha, \beta}$. Lemma 2 proves that the operators $B^{-(\alpha, \beta)}$ and $B^{\alpha, \beta}$ are mutually inverse. Lemma 3 of the section establishes a connection between operators $B^{\alpha, \beta}$ and $\beta^{\alpha, \beta}$. The actions of these operators in the class of continuous functions and in the Hölder class are also studied. In Section 4, Problems 1 and 2 are studied for the case of the classical Poisson equation. The existence and uniqueness theorems for solutions of the problems under consideration are proved. It is established that for all values $\alpha \in (0, 1), \beta > 0$, the solution of Problem 1 has the properties of the solution of the Dirichlet problem, and the solution of Problem 2 has the properties of the solution of the Neumann problem. In Section 5, Problems 1 and 2 are studied in the general case. Conditions for solvability of the studied problems are found, and representations of solutions are obtained.
2. Some Auxiliary Statements Related to Transformations $S$

In this section, we present some auxiliary statements regarding a special form of matrices. Consider the matrix $A$, consisting of the coefficients $a_1, a_2, \ldots, a_i$ of the operator $L_i$:

$$A = \begin{pmatrix}
    a_1 & a_2 & \cdots & a_i \\
    a_1 & a_1 & \cdots & a_{i-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_2 & a_3 & \cdots & a_1
\end{pmatrix}$$

In [36] the following assertions are proved.

Lemma 1. Let $\lambda_1 = e^{\frac{2\pi}{l}}$ be the primitive $l$-th root of unity. Then

$$\det A = \prod_{k=1}^{l} (a_1\lambda_0^k + \cdots + a_i\lambda_{l-1}^k),$$

where $\lambda_k = e^{\frac{2\pi k}{l}}, k = 1, \ldots, l$.

Clearly, $\lambda_k^n = (\lambda_1^n)^k = \lambda_n^k$. Denote

$$\mu_k = \left(a_1\lambda_0^k + \cdots + a_i\lambda_{l-1}^k\right) = \sum_{q=1}^{l} a_q\lambda_{q-1}^k = \sum_{q=1}^{l} a_q\lambda_{q-1}^k$$

(4)

Lemma 2. Let the numbers $\mu_k$ in (4) be nonzero $\mu_k \neq 0, k = 1, 2, \ldots, l$. Then, there exists an inverse to the matrix $A$, which is given by the formula

$$A^{-1} = \frac{1}{l} M_+ \text{diag}^{-1}(\mu_1, \ldots, \mu_l) M_-^T,$$

where

$$M_+ = \left(\lambda_{j-1}^{i} \right)_{i,j=1,2,\ldots,d}, M_- = \left(\lambda_{i-1}^{-j} \right)_{i,j=1,2,\ldots,d}.$$

Lemma 3. The operator $I_S u(x) = u(Sx)$ and the Laplace operator $\Delta$ commute: $\Delta I_S u(x) = I_S \Delta u(x)$. Operators $\Lambda = \sum_{i=1}^{n} x_i u_i(x)$ and $I_S$ also commute: $L \Lambda I_S u(x) = I_S \Lambda u(x)$, and the equality $\nabla I_S = I_S S^T \nabla$ holds true.

Corollary 1. If a function $u(x)$ is harmonic in $\Omega$, then the function $u(Sx) = I_S u(x)$ is also harmonic in $\Omega$.

Corollary 2. If a function $u(x)$ is harmonic in $\Omega$, then it satisfies the homogeneous equation (Equation (1)) in $\Omega$.

Lemma 4. Let the function $u(x) \in C^2(\Omega)$ satisfy the homogeneous equation (Equation (1)) and $\mu_k \neq 0, k = 1, 2, \ldots, l$. Then, the function $u(x)$ is harmonic in $\Omega$.

3. Properties of Integro-Differentiation Operators

Lemma 5. Let $0 < \alpha, \beta$ and $u(x) \in C(\Omega)$. Then, $f_{r}^{\alpha, \beta} u(x) \in C(\Omega)$ and $f_{r}^{\alpha, \beta} u(0) = 0$.

Proof. By the definition of the operator $f_{r}^{\alpha, \beta}$, we get

$$\left| f_{r}^{\alpha, \beta} u(x) \right| \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{r} \left( \frac{r^\beta - t^\beta}{t^\beta} \right)^{\alpha-1} |u(t\theta)| \frac{dt}{t^\beta} \leq \|u\|_{C(\Omega)} f_{r}^{\alpha, \beta}[1] = \frac{\beta^{-\alpha} \|u\|_{C(\Omega)}}{\Gamma(\alpha + 1)} r^\beta$$

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Then we get
\[
\max_{\Omega} \left| f_{r}^{\alpha, \beta} u(x) \right| \leq \frac{\beta^{\alpha-n}}{\Gamma(\alpha + 1)} \|u\|_{C(\Omega)}.
\]

Hence, \( f_{r}^{\alpha, \beta} u(x) \in C(\Omega) \). Moreover, \( \lim_{x \to 0} f_{r}^{\alpha, \beta} u(x) = 0 \). The Lemma is proved. \( \square \)

**Corollary 3.** Let \( 0 < \alpha, \beta \) and \( u(x) \in C(\Omega) \). Then \( B^{-(\alpha, \beta)} u(x) \in C(\Omega) \).

**Lemma 6.** Let \( 0 < \alpha < 1, \beta > 0 \), \( u(x) \) be a smooth function in the domain \( \Omega \). Then the equalities
\[
B^{-(\alpha, \beta)} \left[ B^{\alpha, \beta} [u] \right](x) = B^{\alpha, \beta} \left[ B^{-(\alpha, \beta)} [u] \right](x) = u(x), \ x \in \Omega
\]
are valid.

**Proof.** By the definition of operators \( B^{-(\alpha, \beta)} \) and \( B^{(\alpha, \beta)} \), we get
\[
B^{-(\alpha, \beta)} \left[ B^{\alpha, \beta} [u] \right](x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{r} \left( \frac{r^{\beta} - t^{\beta}}{\beta} \right)^{\alpha-1} l_{1}^{\alpha-\beta} B^{\alpha, \beta} [u](t \theta) \frac{dt}{t^{1-\beta}}
\]
\[
= \frac{1}{\beta^{\alpha-1} \Gamma(\alpha + 1)} \int_{0}^{r} \left( \frac{r^{\beta} - t^{\beta}}{\beta} \right)^{\alpha-1} \frac{d}{dt} l_{1}^{\alpha-\beta} B^{\alpha, \beta} [u](t \theta) dt
\]
\[
= \frac{1}{\beta^{\alpha-1} \Gamma(\alpha + 1)} \int_{0}^{r} \left( \frac{r^{\beta} - t^{\beta}}{\beta} \right)^{\alpha-1} l_{1}^{\alpha-\beta} B^{\alpha, \beta} [u](t \theta) \frac{dt}{t^{1-\beta}}
\]
\[
= r^{1-\beta} \frac{d}{dr} \left[ \frac{1}{\Gamma(\alpha)} \int_{0}^{r} \left( \frac{r^{\beta} - t^{\beta}}{\beta} \right)^{\alpha-1} l_{1}^{\alpha-\beta} B^{\alpha, \beta} [u](t \theta) \frac{dt}{t^{1-\beta}} \right] = r^{1-\beta} \frac{d}{dr} l_{1}^{\alpha, \beta} [u](x)
\]
\[
= r^{1-\beta} \frac{d}{dr} l_{1}^{\alpha, \beta} [u](x) = r^{1-\beta} \frac{d}{dr} \int_{0}^{r} u(t \theta) \frac{dt}{t^{1-\beta}} = u(x).
\]

On the other hand,
\[
B^{\alpha, \beta} \left[ B^{-(\alpha, \beta)} [u] \right](x) = r^{\alpha+1-\beta} \frac{d}{dr} l_{1}^{\alpha-\beta} B^{\alpha, \beta} [r^{-\alpha \beta} u](x) = r^{\alpha+1-\beta} \frac{d}{dr} l_{1}^{\alpha, \beta} [r^{-\alpha \beta} u](x)
\]
\[
= r^{\alpha+1-\beta} r^{\beta-1-\alpha \beta} u(x) = u(x).
\]

The Lemma is proved. \( \square \)

**Lemma 7.** Let \( 0 < \alpha < 1, \beta > 0 \) and \( u(x) \) be a smooth function in the domain \( \Omega \). Then the equalities
\[
B_{r}^{\alpha, \beta} [u](x) = B^{\alpha, \beta} [u](x) - \frac{\beta^{\alpha}}{\Gamma(1-\alpha)} u(0), \ x \in \Omega.
\]
\[ B_{a,b}^\alpha[u](0) = 0 \]  

are valid.

**Proof.** Let \( 0 < \alpha \leq 1 \). By the definition of the operator \( B_{a,b}^\alpha \), we obtain

\[
B_{a,b}^\alpha[u](x) = \frac{r^{\alpha+1-b}}{\Gamma(2-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha+1-b}}{\Gamma(2-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha+1-b}}{\Gamma(2-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha+1-b}}{\Gamma(2-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

Therefore, equality (6) is valid. Further, after change of variables \( \tau = r_\xi^\beta \), the function \( B_{a,b}^\alpha[u](x) \) can be represented as

\[
B_{a,b}^\alpha[u](x) = \frac{r^{\alpha-\beta}}{\Gamma(1-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha-\beta}}{\Gamma(1-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha-\beta}}{\Gamma(1-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha-\beta}}{\Gamma(1-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha-\beta}}{\Gamma(1-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

\[
= \frac{r^{\alpha-\beta}}{\Gamma(1-a)} \int_0^r \left( \frac{r^\beta - \tau^\beta}{\beta} \right)^{1-a} \frac{d\tau}{d\tau} u(\tau\theta) \frac{d\tau}{\tau^{1-b}}
\]

If \( u(x) \) is a smooth function, then

\[
\lim_{x \to 0} \left( \frac{d}{d\tau} \int_0^1 (1-\xi)^{-\alpha} u\left( \xi^\beta x \right) d\xi \right) = 0.
\]

Besides,
Let Lemma 8 hold true.

For an arbitrary smooth function $f \in \Omega$, Lemma 6 holds true.

Hence

$$B_+^{a,\beta}[u](0) = \lim_{x \to 0} B_+^{a,\beta}[u](x) = \lim_{x \to 0} B_+^{a,\beta}[u](x) - \frac{\beta}{\Gamma(1-\alpha)} u(0) = 0,$$

i.e., equality (7) is also true. The Lemma is proved. □

The assertions of Lemmas 6 and 7 imply the following corollary.

**Corollary 4.** Let $0 < \alpha \leq 1, \beta > 0$, $u(x)$ be a smooth function in the domain $\Omega$. Then the equalities

$$B_+^{a,\beta}\left[B^{-\alpha,\beta}[u]\right](x) = u(x) - u(0) \tag{8}$$

hold true.

**Lemma 8.** Let $\Delta u(x) = f(x), x \in \Omega$. Then the equality

$$\Delta B_+^{a,\beta}[u](x) = |x|^{-2}B_+^{a,\beta}[|x|^2f](x), x \in \Omega. \tag{9}$$

holds true.

**Proof.** For an arbitrary smooth function $v(x)$, the equality

$$\Delta \left[ r \frac{d}{dr} v(x) \right] = \left( r \frac{d}{dr} + 2 \right) \Delta v(x)$$

holds true.

As $\Delta u(x) = f(x)$ and

$$B_+^{a,\beta}[u](x) = \left( r \frac{d}{dr} + (1-\alpha)\beta \right) \frac{\beta}{\Gamma(1-\alpha)} \frac{1}{0} (1-\xi)^{-a} u \left( \xi^\beta x \right) d\xi,$$

then

$$\Delta B_+^{a,\beta}[u](x) = \left( r \frac{d}{dr} + (1-\alpha)\beta + 2 \right) \frac{\beta}{\Gamma(1-\alpha)} \frac{1}{0} (1-\xi)^{-a} \xi^\beta f \left( \xi^\beta x \right) d\xi.$$

After change of variables $\xi = \frac{r^\beta}{\alpha}$ in the last integral, we obtain

$$\left( r \frac{d}{dr} + (1-\alpha)\beta + 2 \right) \frac{\beta}{\Gamma(1-\alpha)} \frac{1}{0} (1-\xi)^{-a} \xi^\beta f \left( \xi^\beta x \right) d\xi$$

$$= ((1-\alpha)\beta + 2) \frac{r^{(a-1)\beta-2}}{\Gamma(1-\alpha)} \int_0^r \left( \frac{r - \tau^{\beta}}{\beta} \right) \frac{-a}{\tau^2 f(\tau)} d\tau + \frac{r^{(a-1)\beta-2}}{\Gamma(1-\alpha)} \int_0^{r^{\beta}} \left( \frac{r^{\beta} - \tau^{\beta}}{\beta} \right) \frac{-a}{\tau^2 f(\tau)} d\tau.$$
We transform the last integral as follows.

\[
\left( \frac{r}{dr} \right) \frac{r^{(\alpha-1)\beta-2}}{\Gamma(1-\alpha)} \int_0^r \left( \frac{r^\beta - r^\beta}{\beta} \right)^{-\alpha} t^2 f(\tau\theta) \frac{d\tau}{\tau^{1-\beta}}
\]

\[
= ((\alpha - 1)\beta - 2) \frac{r^{(\alpha-1)\beta-2}}{\Gamma(1-\alpha)} \int_0^r \left( \frac{r^\beta - r^\beta}{\beta} \right)^{-\alpha} t^2 f(\tau\theta) \frac{d\tau}{\tau^{1-\beta}}
\]

\[
+ \frac{\rho^{\alpha-2}}{\Gamma(1-\alpha)} \left( \frac{r^{1-\alpha} d}{dr} \right) \int_0^r \left( \frac{r^\beta - r^\beta}{\beta} \right)^{-\alpha} t^2 f(\tau\theta) \frac{d\tau}{\tau^{1-\beta}}
\]

Then,

\[
\Delta B^{\alpha,\beta}[u](x) = ((\alpha - 1)\beta + 2) \frac{r^{(\alpha-1)\beta-2}}{\Gamma(1-\alpha)} \int_0^r \left( \frac{r^\beta - r^\beta}{\beta} \right)^{-\alpha} t^2 f(\tau\theta) \frac{d\tau}{\tau^{1-\beta}}
\]

\[
+ ((\alpha - 1)\beta - 2) \frac{r^{(\alpha-1)\beta-2}}{\Gamma(1-\alpha)} \int_0^r \left( \frac{r^\beta - r^\beta}{\beta} \right)^{-\alpha} t^2 f(\tau\theta) \frac{d\tau}{\tau^{1-\beta}}
\]

\[
+ \frac{\rho^{\alpha-2}}{\Gamma(1-\alpha)} \left( \frac{r^{1-\alpha} d}{dr} \right) \int_0^r \left( \frac{r^\beta - r^\beta}{\beta} \right)^{-\alpha} t^2 f(\tau\theta) \frac{d\tau}{\tau^{1-\beta}} = \rho^{\alpha-2} D^{\alpha,\beta}[r^2 f](x) = r^{-2} B^{\alpha,\beta}[r^2 f](x)
\]

Thus, equality (9) is valid. The Lemma is proved. □

Corollary 5. If a function \( u(x) \) is harmonic in the domain \( \Omega \), then the function \( B^{\alpha,\beta}[u](x) \) will also be harmonic in the domain \( \Omega \).

Remark 1. The function \( |x|^{-2} B^{\alpha,\beta}[|x|^{2}f](x) \) can also be represented as

\[
|x|^{-2} B^{\alpha,\beta}[|x|^{2}f](x) = \left( r \frac{d}{dr} + 2 + (1 - \alpha)\beta \right) f_{a,\beta}(x)
\]

where \( f_{a,\beta}(x) = |x|^{(\alpha-1)\beta-2} f^{1-a,\beta}[|x|^{2}f](x) \). If \( \alpha = 1 \), then \( f_{1,\beta}(x) = f(x) \).

Lemma 9. Let \( \Delta u(x) = F(x), x \in \Omega \). Then the equality

\[
\Delta B^{-(a,\beta)}[v](x) = |x|^{-2} B^{-(a,\beta)} \left[ |x|^{2}F \right](x), x \in \Omega
\]

holds true.

Proof. Let \( 0 < \alpha < 1 \). After the changing of variables \( \tau = r_{0}^{\alpha}/\xi \), the function \( B^{\alpha,\beta}[u](x) \) can be represented as

\[
\Delta B^{-(a,\beta)}[v](x) = \frac{1}{\Gamma(\alpha)} \int_0^r \left( \frac{r^\beta - t^\beta}{\beta} \right)^{a-1} t^{-a\beta} u(t\theta) \frac{dt}{t^{1-\beta}}
\]
The Lemma is proved. \[\square\]

**Lemma 10.** Let $0 < \alpha < 1, \beta > 0, 0 < \lambda < 1$ and $f(x) \in C^{\lambda+p}(\Omega), p = 0, 1, \ldots$. Then $B^{-(\alpha, \beta)} f(x) \in C^{\lambda+p}(\Omega)$.

**Proof.** Let $x, y$ be arbitrary points in the domain $\Omega$. Denote $h(x) = B^{-(\alpha, \beta)} f(x)$. Then

$$|h(x) - h(y)| \leq \frac{1}{\beta^a}\int_0^1 (1 - \tau)^{a-1} \tau^{-\alpha} |u(\tau^{1/\beta} x) - u(\tau^{1/\beta} y)| d\tau$$

$$\leq \frac{C|x - y|^{\lambda}}{\beta^a}\int_0^1 (1 - \tau)^{a-1} \tau^{1/\beta - a} d\tau \leq C|x - y|^{\lambda}.$$

Here and further on, the symbol $C$ will denote a constant whose value does not interest us.

Further, if $i = (i_1, \ldots, i_n)$ is a multi-index $\partial^i_x = \frac{\partial^{||i||}}{\partial x_1^{i_1} \cdots x_n^{i_n}}$, then for all $i$ with length $|i| \leq p$ and for any points $x, y \in \Omega$, we get

$$|\partial^i_x h(x) - \partial^i_x h(y)| \leq \frac{1}{\beta^a}\int_0^1 (1 - \tau)^{a-1} \tau^{1/\beta - a} |\partial^i_x u(z_x) - \partial^i_x u(z_y)| ds \leq C|x - y|^{\lambda},$$

where $\partial^i_x = \frac{\partial^{||i||}}{\partial x_1^{i_1} \cdots x_n^{i_n}}, z_x = \tau^{1/\beta} x, z_y = \tau^{1/\beta} y$.

This means that functions $h(x), \partial^i_x h(x), |i| \leq p$ belong to the class $C^{\lambda}(\Omega)$, and hence $B^{-(\alpha, \beta)} f(x) \in C^{\lambda+p}(\Omega)$. The Lemma is proved. \[\square\]

The following assertion is proved similarly.

**Lemma 11.** Let $0 < \alpha < 1, \beta > 0, 0 < \lambda < 1$ and $f(x) \in C^{\lambda+p}(\Omega), p = 1, 2, \ldots$. Then $B^{\alpha, \beta} f(x) \in C^{\lambda+p-1}(\Omega)$.
**Corollary 6.** Let $0 < \alpha < 1, \beta > 0, 0 < \lambda < 1$ and $f(x) \in C^{\lambda+p}(\Omega), p = 1, 2, \ldots$. Then $|x|^{-2B^{\alpha,\beta}[|x|^2f]}(x) \in C^{\lambda+p-1}(\Omega)$.

### 4. Boundary Value Problems for the Classical Poisson Equation

In this section, we consider Problems 1 and 2 for the case of the classical Poisson equation; i.e., we will consider the following problems.

**Problem 3.** Let $0 < \alpha \leq 1, \beta > 0$. Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, for which $B^{\alpha,\beta}u(x) \in C(\bar{\Omega})$, satisfying the conditions

$$\nabla u(x) = f(x), x \in \Omega, \tag{12}$$

$$D^\alpha_B u(x)|_{\partial \Omega} = g(x). \tag{13}$$

**Problem 4.** Let $0 < \alpha \leq 1, \beta > 0$. Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, for which $B^{\alpha,\beta}u(x) \in C(\bar{\Omega})$, satisfying Equation (12) and the condition

$$D^\alpha_B u(x)|_{\partial \Omega} = g(x). \tag{14}$$

Note that Problems 3 and 4 for the case $\beta = 1$ were studied in [29].

First, let us study the uniqueness of solutions to Problems 3 and 4.

**Theorem 1.** Let the solution to Problems 3 and 4 exist. Then

(1) the solution to Problem 3 is unique;

(2) the solution to Problem 4 is unique up to a constant term.

**Proof.** Let $u(x)$ be a solution of the homogeneous problem, Problem 3. Let us show that $u(x) \equiv 0, x \in \bar{\Omega}$. Apply the operator $B^{\alpha,\beta}$ to $u(x)$ and denote $v(x) = B^{\alpha,\beta}u(x)$. If we apply the Laplace operator to this equality, then we get $\Delta v(x) = |x|^{-2B^{\alpha,\beta}[|x|^2\Delta u]}(x) = 0$, i.e., $v(x)$ is a harmonic function. Under homogeneous boundary conditions, we also obtain $v(x)|_{\partial \Omega} = B^{\alpha,\beta}u(x)|_{\partial \Omega} = 0$. Hence, the function $v(x)$ is a solution to the following Dirichlet problem: $\Delta v(x) = 0, x \in \Omega, v(x)|_{\partial \Omega} = 0$. Due to the uniqueness of the solution of the Dirichlet problem, $v(x) \equiv 0, x \in \bar{\Omega}$. Then $B^{\alpha,\beta}u(x) \equiv 0, x \in \bar{\Omega}$. Let us apply the operator $B^{-\alpha,\beta}$ to this equality and get $u(x) \equiv 0, x \in \bar{\Omega}$. Therefore, the solution to Problem 3 is unique. In the case of problem 4, by analogous reasoning, we get the equality $B^{\alpha,\beta}u(x) \equiv 0, x \in \bar{\Omega}$. Hence, taking into account (8), we have

$$B^{-(\alpha,\beta)}B^{\alpha,\beta}u(x) \equiv 0 \Rightarrow u(x) - u(0) \equiv 0 \Rightarrow u(x) \equiv \text{Const.}$$

The theorem is proved. $\square$

Let us consider the existence of a solution to Problem 3. Suppose that a solution to Problem 3 exists. Denote it as $v(x) = B^{\alpha,\beta}u(x)$. Then, using equality (9) for the function $v(x)$, we obtain the problem

$$-\Delta v(x) = F(x), x \in \Omega, v(x)|_{\partial \Omega} = g(x). \tag{15}$$

where $F(x) = |x|^{-2B^{\alpha,\beta}[|x|^2f]}(x)$.

For smooth $F(x)$ and $g(x)$, a solution to problem (15) exists, is unique and can be represented as (see, for example, [38], p. 35.)

$$v(x) = \int_{\Omega} G(x,y)F(y)dy - \int_{\partial \Omega} \frac{\partial G(x,y)}{\partial v}g(y)dS_y, \tag{16}$$

where $G(x,y)$ is Green’s function of problem (15).

Let us apply the operator $B^{-\alpha,\beta}$ to equality $v(x) = B^{\alpha,\beta}u(x)$. Then, by virtue of the first equality in (5), the function $u(x)$ is uniquely determined by the formula
\( u(x) = B^{-(\alpha, \beta)} v(x) \). Let us show that if a function \( v(x) \) is a solution to problem (15), then the function \( u(x) = B^{-(\alpha, \beta)} v(x) \) satisfies the conditions of problem 3. Indeed, applying the Laplace operator to this function, by virtue of equality (11), we obtain

\[
-\Delta u(x) = -\Delta B^{-(\alpha, \beta)} v(x) = |x|^{-2} B^{-(\alpha, \beta)} \left[ |x|^2 F \right](x)
\]

\[
= |x|^{-2} B^{-(\alpha, \beta)} \left[ |x|^2 \right] F(x) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f \right](x) = |x|^{-2} f(x) = f(x).
\]

Therefore, Equation (12) will be fulfilled. Let us check the fulfillment of the boundary condition (13):

\[
D^{\alpha, \beta}_x u(x)|_{\partial \Omega} = B^{\alpha, \beta} \left[ B^{-(\alpha, \beta)} v(x) \right]|_{\partial \Omega} = v(x)|_{\partial \Omega} = g(x).
\]

From this we obtain that the function \( u(x) = B^{-(\alpha, \beta)} v(x) \) formally satisfies the conditions of Problem 3. Consider the smoothness of this function.

If \( f(x) \in C^{1,1}(\Omega), 0 < \lambda < 1 \), then the function \( F(x) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f \right](x) \) belongs to the class \( C^1(\Omega) \). Then, if \( g(x) \in C^{1,1+2}(\partial \Omega) \), then the solution to problem (15) belongs to the class \( C^{1,2}(\Omega) \) (see, for example, [39]). Therefore, according to Lemma 10, the function \( u(x) = B^{-(\alpha, \beta)} v(x) \) also belongs to the class \( C^{1,2}(\Omega) \).

Thus, we have proved the following assertion.

**Theorem 2.** Let \( 0 < \alpha < 1, \beta > 0, f(x) \in C^{1,1}(\Omega), g(x) \in C^{1,2}(\partial \Omega), 0 < \lambda < 1 \). Then, a solution to Problem 3 exists, is unique, belongs to the class \( C^{1,2}(\Omega) \) and can be represented as

\[
u(x) = B^{-(\alpha, \beta)} v(x),
\]

where \( v(x) \) is a solution of problem (15) with the function \( F(x) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f \right](x) \).

Regarding Problem 4, the following assertion holds true.

**Theorem 3.** Let \( 0 < \alpha < 1, \beta > 0, f(x) \in C^{1,1}(\Omega), g(x) \in C^{1,2}(\partial \Omega), 0 < \lambda < 1 \). Then, for the solvability of Problem 4, it is necessary and sufficient that the condition

\[
\int_{\Omega} f_{\alpha, \beta}(y)dy + (1 - \alpha) \beta \int_{\Omega} f_{\alpha, \beta}(y)dy + \int_{\partial \Omega} g(y)dS_y = 0,
\]

be satisfied.

If a solution to the problem exists, then it is unique up to a constant term, belongs to the class \( C^{1,2}(\Omega) \), and can be represented as

\[
u(x) = C + B^{-(\alpha, \beta)} v(x)
\]

where \( v(x) \) is a solution of problem (15) with the function \( F(x) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f \right](x) \), satisfying the additional condition \( v(0) = 0 \).

**Proof.** If a solution to Problem 4 exists, then by denoting \( v(x) = B^{\alpha, \beta}_x u(x) \), and taking into account equalities (6) and (9), we obtain

\[
\Delta B^{\alpha, \beta}_x [u](x) = \Delta B^{\alpha, \beta}_x [u](x) - \Delta \left( \frac{\beta^\alpha}{\Gamma(1 - \alpha)} u(0) \right) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f \right](x) \equiv F(x).
\]
Further, as \( v(x)\mid_{\partial \Omega} = B^\alpha_\beta u(x) \mid_{\partial \Omega} = D^\alpha_\beta v(x) \mid_{\partial \Omega} = g(x) \), the function \( v(x) \) also satisfies the boundary condition of problem (15). In addition, due to equality (6), it follows that \( v(0) = 0 \). If \( f(x) \in C^{1,1}(\Omega) \) and \( g(x) \in C^{1,2}(\partial \Omega) \), then a solution to problem (15) exists, is unique and can be represented as (16). Now we find the condition under which the equality \( v(0) = 0 \) holds. It is known that for \( n \geq 3 \), the Green’s function of problem (15) is written as

\[
G(x, y) = \frac{1}{\omega_n (n - 2)} \left| x - y \right|^{2-n} - \left| x \right| - \frac{y}{|y|} \left| y \right|^{2-n},
\]

where \( \frac{\partial G(x, y)}{\partial n} \) is defined by the equality

\[
\frac{\partial G(x, y)}{\partial n} = -\frac{1}{\omega_n} \frac{1}{|x - y|^n}.
\]

Here \( \omega_n \) is the area of a unit sphere.

Further, from equality (16), it follows that

\[
v(0) = \frac{1}{\omega_n (n - 2)} \int_{\Omega} \left[ |y|^{2-n} - 1 \right] |y|^{-2} B^\alpha_\beta |y|^2 f(y) dy + \frac{1}{\omega_n} \int_{\partial \Omega} g(y) dS_y.
\]

If we denote

\[
g_{\alpha, \beta}(y) = \frac{1}{(n-2)} \left| y \right|^{2-n} - 1 \right] |y|^{-2} B^\alpha_\beta |y|^2 f(y),
\]

then to satisfy the condition \( v(0) = 0 \) it is necessary to satisfy the equality

\[
\int_{\Omega} g_{\alpha, \beta}(y) dy + \int_{\partial \Omega} g(y) dS_y = 0. \tag{19}
\]

It was proved in [29] that if a function is represented in the form \( g(x) = (r \frac{d}{dr} + 2) g_1(x) \), then the equality

\[
\frac{1}{(n-2)} \int_{\Omega} \left[ |x|^{2-n} - 1 \right] g(x) dx = \frac{1}{(n-2)} \int_{\Omega} \left[ |x|^{2-n} - 1 \right] \left( r \frac{d}{dr} + 2 \right) g_1(x) dx = \int_{\Omega} g_1(x) dx
\]

is valid.

Then, taking into account equality (10), we obtain

\[
\int_{\Omega} g_{\alpha, \beta}(y) dy = \int_{\Omega} f_{\alpha, \beta}(y) dy + (1 - \alpha) \beta \int_{\Omega} f_{\alpha, \beta}(y) dy
\]

Therefore, condition (19) can be rewritten in the form (17). Thus, if there is a solution to Problem 4, then condition (17) must be satisfied.

Let us show that this condition is also sufficient for the existence of a solution to Problem 4. Indeed, if \( f(x) \in C^{1,1}(\Omega) \), then the function \( F(x) = |x|^{-2} B^\alpha_\beta |x|^2 f(x) \) belongs to the class \( C^1(\Omega) \). If \( g(x) \in C^{1,2}(\partial \Omega) \), then the solution to problem (15) exists, and under condition (17), it satisfies the equality \( v(0) = 0 \). Let us consider the function \( u(x) = C + B^{-(\alpha, \beta)} [v](x) \). Then,

\[
-\Delta u(x) = -\Delta [C] - \Delta B^{-(\alpha, \beta)} [v](x) \mid x = |x|^{-2} B^{-(\alpha, \beta)} |x|^2 (-\Delta) [v](x) = \mid x \mid^{-2} \left[ B^{\alpha, \beta} |x|^2 f \right](x) = f(x).
\]
Moreover,
\[ D_{\alpha}^{\alpha,\beta} u(x) \bigg|_{\partial \Omega} = B_{\alpha}^{\alpha,\beta} u(x) \bigg|_{\partial \Omega} = B_{\alpha}^{\alpha,\beta} \left[ C + B^{-(\alpha,\beta)} v(x) \right] \bigg|_{\partial \Omega} \]
\[ = B_{\alpha}^{\alpha,\beta} [C] + B_{\alpha}^{\alpha,\beta} \left[ B^{-(\alpha,\beta)} v(x) \right] \bigg|_{\partial \Omega} = B_{\alpha}^{\alpha,\beta} \left[ B^{-(\alpha,\beta)} v(x) \right] \bigg|_{\partial \Omega}. \]

Further, due to (6), we have the equality
\[ B_{\alpha}^{\alpha,\beta} \left[ B^{-(\alpha,\beta)} v(x) \right] = B_{\alpha}^{\alpha,\beta} \left[ B^{-(\alpha,\beta)} v(x) \right] - \frac{\beta^\alpha}{\Gamma(1-\alpha)} B^{-(\alpha,\beta)} [v](0). \]

If the condition \( v(0) = 0 \) is satisfied, then the equality \( B^{-(\alpha,\beta)} [v](0) = 0 \) is fulfilled. Hence,
\[ D_{\alpha}^{\alpha,\beta} u(x) \bigg|_{\partial \Omega} = B_{\alpha}^{\alpha,\beta} \left[ B^{-(\alpha,\beta)} v(x) \right] \bigg|_{\partial \Omega} = v(x) \big|_{\partial \Omega} = g(x). \]

Thus, the function \( u(x) = C + B^{-(\alpha,\beta)} [v](x) \) satisfies the conditions of Problem 4. The smoothness of this function is verified as in the case of solving Problem 3. The theorem is proved. \( \square \)

5. The Main Problem

First, let us study the uniqueness of the solutions of Problems 1 and 2.

**Theorem 4.** Let solutions to Problems 1 and 2 exist and for all the inequalities hold. Then,

1. the solution to Problem 1 is unique;
2. the solution to Problem 2 is unique up to a constant term.

**Proof.** Let \( u(x) \) be a solution to the homogeneous problem, Problem 1. Since the function \( u(x) \) satisfies the homogeneous equation (Equation (1)), by virtue of Lemma 4, it is a harmonic function in the domain \( \Omega \). This means that the function \( u(x) \) is a solution to the homogeneous problem, Problem 3. Since the solution to Problem 3 is unique, so is the solution to Problem 1.

If \( u(x) \) is a solution to Problem 2, then by analogous reasoning, we obtain that it is also a solution to Problem 4. Then \( u(x) \equiv C \). The theorem is proved. \( \square \)

**Theorem 5.** Let \( 0 < \alpha < 1, \beta > 0 \). Then, numbers \( \{a_k : k = 1, \ldots, l\} \) are such that \( \mu_k = a_1 \lambda_0^k + \cdots + a_l \lambda_1^k \neq 0 \) for \( k = 1, \ldots, l \), where \( \{\lambda_k\} \) are roots of the \( l \)-th degree of one and \( f \in C^{\varepsilon+1}(\Omega), g \in C^{\varepsilon+2}(\partial \Omega), 0 < \varepsilon < 1. \) Then a solution to Problem 1 exists, is unique and can be represented as
\[ u(x) = B^{-(\alpha,\beta)} v(x), \]
where \( v(x) \) is the solution to the problem
\[ L v(x) = F(x), x \in \Omega, v(x) \big|_{\partial \Omega} = g(x). \]

**Proof.** Let \( u(x) \) be a solution to problem 1 and \( v(x) = B^{\alpha,\beta} u(x) \). Using equality (9) for the function \( v(x) \), we obtain
\[ -\Delta v(x) = |x|^{-2} B^{\alpha,\beta} \left[ |x|^2 (-\Delta) u \right] (x), x \in \Omega. \]

Hence, in points \( S^k x, k = 1, 2, \ldots, l - 1 \) we have
\[ -\Delta v(S^k x) = F(S^k x), k = 1, 2, \ldots, l - 1. \]
Then, using Equation (1) we make sure that the function \( v(x) \) satisfies the equation

\[
L_t[v](x) = -\sum_{k=0}^{l-1} a_k \Delta v \left( S^k x \right) = |x|^{-2} B^{a,\beta} \left[ |x|^2 \left( -\sum_{k=0}^{l-1} a_k \Delta u \left( S^k x \right) \right) \right]
\]

\[
= |x|^{-2} B^{a,\beta} \left[ |x|^2 f(x) \right] = -|x|^{-2} B^{a,\beta} \left[ |x|^2 f(x) \right] = F(x).
\]

If we use the condition (2), then \( v(x) \) satisfies the boundary condition \( v(x)|_{\partial \Omega} = B^{a,\beta} u(x)|_{\partial \Omega} = g(x) \). Thus, if \( u(x) \) is a solution to Problem 1, then the function \( v(x) = B^{a,\beta} u(x) \) will satisfy the conditions of the Dirichlet problem (21).

In [36] it is proved that if \( \mu_k \neq 0, k = 1, \ldots, l \), \( F(x) \) and \( g(x) \) are sufficiently smooth functions, then a solution to problem (20) exists, is unique, and can be represented as

\[
v(x) = \int_{\Omega} G_S(x,y) F(y) \, dy + \int_{\partial \Omega} P_S(x,y) \sum_{k=1}^{l} a_k g(S^{k-1} y) \, ds_y,
\]

where

\[
G_S(x,y) = \sum_{q=1}^{l} b_q G(S^{q-1} y, x), \quad P_S(x,y) = \sum_{q=1}^{l} b_q P(S^{q-1} y, x).
\]

Here

\[
b_q = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_{k}^{q-1} \mu_k} , \quad P(x,y) = \frac{1}{\omega_d} \frac{1 - |x|^2}{|x-y|^n}.
\]

Applying operator \( B^{-\mu,\beta} \) to the equality \( v(x) = B^{a,\beta} u(x) \) operator \( B^{-\mu,\beta} \), we get the function \( u(x) = B^{-\mu,\beta} v(x) \). Let us show that this function satisfies the conditions of Problem 1. Indeed, due to equality (11), we have \( \Delta u(x) = |x|^{-2} B^{-\mu,\beta} \left[ |x|^2 \Delta v \right](S^k x), k = 1, 2, \ldots, l - 1 \). Hence,

\[
L_t[u](x) = -\sum_{k=0}^{l-1} a_k \Delta u \left( S^k x \right) = |x|^{-2} B^{-\mu,\beta} \left[ |x|^2 \left( -\sum_{k=0}^{l-1} a_k \Delta v \left( S^k x \right) \right) \right]
\]

\[
= |x|^{-2} B^{-\mu,\beta} \left[ |x|^2 f(x) \right] = -|x|^{-2} B^{-\mu,\beta} \left[ |x|^2 f(x) \right] = f(x).
\]

Thus, the function \( u(x) = B^{-\mu,\beta} v(x) \) satisfies Equation (1). Further, from the equalities

\[
D^{\mu,\beta}_t u(x)|_{\partial \Omega} = B^{\mu,\beta} \left[ B^{-\mu,\beta} v(x) \right]|_{\partial \Omega} = v(x)|_{\partial \Omega} = g(x)
\]

we find that the boundary condition (2) is also satisfied. The smoothness of the function \( u(x) = B^{-\mu,\beta} v(x) \) is checked in the same way as in the case of problem 3. The theorem is proved.

**Theorem 6.** Let \( 0 < \alpha < 1, \beta > 0 \), the numbers \( \{a_k : k = 1, \ldots, l\} \) are such that \( \mu_k = a_1 \lambda_1^{\alpha_k} + \cdots + a_l \lambda_l^{\alpha_k} \neq 0, k = 1, \ldots, l \), and \( f \in C^{l+1}(\Omega) \), \( g \in C^{l+2}(\partial \Omega) \), \( 0 < \varepsilon < 1 \). Then, for the solvability of problem 2 it is necessary and sufficient that the condition

\[
\int_{\Omega} f_{a,\beta}(y) \, dy + (1 - \alpha) \beta \int_{\Omega} f_{a,\beta}(y) \, dy + \left( \sum_{k=1}^{l} a_k \right) \int_{\partial \Omega} g(y) \, dS_y = 0,
\]

where \( f_{a,\beta}(x) \) is determined from (18), be fulfilled. If a solution to the problem exists, then it is unique up to a constant term, belongs to the class \( C^{l+2}(\Omega) \) and can be represented as
\[ u(x) = C + B^{-(\alpha, \beta)} v(x), \]

where \( v(x) \) is a solution of problem (21) with the function \( F(x) = |x|^{-2} B^{\alpha, \beta}(|x|^2 f)(x) \) satisfying the additional condition \( v(0) = 0 \).

**Proof.** Let \( u(x) \) be a solution to the problem 2 and \( v(x) = B^{\alpha, \beta} u(x) \). Then, using equalities (6) and (9), we obtain

\[
\Delta v(x) = \Delta B^{\alpha, \beta} f(x) = \Delta B^{\alpha, \beta} u(x) = \Delta \left( \frac{\beta^\alpha}{\Gamma(1-\alpha)} u(0) \right) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 \Delta u \right](x), x \in \Omega.
\]

Hence, we get that the function \( v(x) \) satisfies the equation:

\[
L_l[v](x) = \sum_{k=1}^{l} a_k \Delta v \left( S^k x \right) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 L_l u(x) \right] = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f(x) \right] = F(x).
\]

Moreover, \( v(x)|_{\partial \Omega} = B^{\alpha, \beta} u(x)|_{\partial \Omega} = D^{\alpha, \beta} u(x)|_{\partial \Omega} = g(x) \); i.e., the function \( v(x) \) also satisfies the boundary condition of Problem (21). By virtue of equality (7), the condition \( v(0) = 0 \) is also satisfied. If \( f(x) \in C^{1+1} \Omega \), then there is an inclusion \( F(x) = |x|^{-2} B^{\alpha, \beta} \left[ |x|^2 f(x) \right] \in C^{1+1} \Omega \), and if \( g(x) \in C^{2+2} \partial \Omega \), then the solution to problem (2) exists, is unique and can be represented as (22).

Further, for the functions \( G_S(x, y) \) and \( P_S(x, y) \) from (23) for \( n > 2 \), we have

\[
G_S(0, y) = \sum_{q=1}^{l} b_q G(0, y) = G(0, y) \sum_{q=1}^{l} b_q = C_1 \left[ |y|^{2-n} - 1 \right],
\]

\[
P_S(0, y) = \sum_{q=1}^{l} b_q P(0, y) = P(0, y) \sum_{q=1}^{l} b_q = C_2,
\]

where \( C_1 = \frac{1}{(n-2) \lambda_0} \mu_1(b) \), \( C_2 = \frac{1}{(n-2) \lambda_0} \mu_1(b) = \sum_{q=1}^{l} b_q \).

In [40], the equality is proved:

\[
\int_{\partial \Omega} g(Sy) \ ds_y = \int_{\partial \Omega} g(y) \ ds_y.
\]

Then, from (22) we get

\[
v(0) = \int_{\Omega} G_S(0, y) F(y) \ dy + \int_{\partial \Omega} P_S(0, y) \sum_{k=1}^{l} a_k g(S^{k-1} y) \ ds_y
\]

\[
= C_1 \int_{\Omega} \left[ |y|^{2-n} - 1 \right] F(y) \ dy + C_2 \sum_{k=1}^{l} a_k \int_{\partial \Omega} g(S^{k-1} y) \ ds_y
\]

\[
= C_1 \int_{\Omega} \left[ |y|^{2-n} - 1 \right] F(y) \ dy + C_2 \mu(a) \int_{\partial \Omega} g(y) \ ds_y,
\]

where \( \mu(a) = \sum_{k=1}^{l} a_k \).

If \( \mu_k = a_1 \lambda^b_0 + \cdots + a_l \lambda^b_{l-1} \neq 0, k = 1, \ldots, l \), then \( \mu(a) \neq 0 \) and \( \mu(b) \neq 0 \). Hence

\[
v(0) = \frac{\mu_l(b)}{\omega_n} \left( \frac{1}{(n-2)} \int_{\Omega} \left[ |y|^{2-n} - 1 \right] F(y) \ dy + \mu(a) \int_{\partial \Omega} g(y) \ ds_y \right)
\]
Hence we obtain that for the condition $v(0) = 0$ to be satisfied it is necessary and sufficient that the equality

$$
\frac{1}{(n - 2)} \int_{\Omega} (|y|^{2-n} - 1) F(y) \, dy + \mu(a) \int_{\partial\Omega} g(y) \, ds_y = 0 \tag{25}
$$

is satisfied. Further, using the representation of the function $F(y)$, as in Problem 4, we can rewrite condition (25) as

$$
\int_{\Omega} f_{s,\beta}(y) dy + (1 - \alpha) \beta \int_{\Omega} f_{s,\beta}(y) dy + \mu(a) \int_{\partial\Omega} g(y) dS_y = 0.
$$

Thus, if there is a solution to Problem 1, then condition (24) must be satisfied. Let us show that this condition is also sufficient for the existence of a solution to Problem 1. Indeed, if $f(x) \in C^{k,1}(\Omega)$, then the function $F(x) = |x|^{-2} B^{a,\beta} |x|^2 f(x)$ belongs to the class $C^1(\Omega)$. Then, for the functions $F(x) \in C^1(\Omega)$ and $g(x) \in C^{k+2}(\partial\Omega)$, the solution of problem (21) exists. If condition (24) is satisfied, the condition $v(0) = 0$ is also satisfied for this solution. Further, let us consider the function $u(x) = C + B^{-(a,\beta)} |v|(x)$. For it, as in the case of Problem 1, we have the equalities

$$
L_1[u](x) = -\sum_{k=0}^{l-1} a_k \Delta u(S_k^x) = |x|^{-2} B^{-(a,\beta)} \left( |x|^2 - \sum_{k=0}^{l-1} a_k \Delta v(S_k^x) \right) = |x|^{-2} B^{-(a,\beta)} \left( |x|^2 F(x) \right) = |x|^{-2} B^{-(a,\beta)} \left( |x|^2 |x|^{-2} B^{a,\beta} |x|^2 f \right)(x) = |x|^{-2} B^{-(a,\beta)} \left( B^{a,\beta} |x|^2 f \right)(x) = f(x),
$$

and

$$
D_{sr}^{a,\beta} u(x) \big|_{\partial\Omega} = B^{a,\beta} u(x) \big|_{\partial\Omega} = B^{a,\beta} \left( C + B^{-(a,\beta)} v(x) \right) \big|_{\partial\Omega} = B^{a,\beta} \left( B^{-(a,\beta)} v(x) \right) \big|_{\partial\Omega} = B^{a,\beta} \left( B^{-(a,\beta)} v(x) \right) - \frac{\beta^s}{\Gamma(1 - \alpha)} B^{-(a,\beta)} [v](0) = v(x) \big|_{\partial\Omega} = g(x).
$$

Thus, the function $u(x) = C + B^{-(a,\beta)} |v|(x)$ satisfies the conditions of Problem 2. The smoothness of this function is verified as in the case of solving Problem 3. The theorem is proved. \(\blacksquare\)

6. Conclusions

Summarizing the results of the study, it should be noted that using the invertibility of integro-differential operators $B^{a,\beta}$ and $B^{-(a,\beta)}$ studied in Lemma 6 and Corollary 4, as well as smoothness of functions $B^{-(a,\beta)} f(x)$ and $B^{-(a,\beta)} f(x)$ in the Hölder classes, in Theorems 1 and 2 the uniqueness and existence of the solution to Problem 3 for the classical Poisson equation is proved, and in Theorem 3 the necessary and sufficient condition for solvability of Problem 4 is found. Further, using the properties of the matrix $A$, homogeneous Problems 1 and 2 were reduced to Problems 3 and 4, respectively, and the uniqueness theorem was proved. Moreover, the properties of the matrix $A$ enabled us to reduce Problems 1 and 2 to equivalent Dirichlet problems for Equation (1) and determine solvability conditions for the main problems. If we consider further possible applications of the proposed method, we should note that a similar method can be used to study boundary value problems with fractional-order boundary operators for nonlocal analogues of higher order elliptic equations. These problems are the subject of further work, and they will be considered in the other papers.

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