Relaxed super self-duality and $N = 4$ SYM at two loops

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Abstract

A closed-form expression is obtained for a holomorphic sector of the two-loop low-energy effective action for the $N = 4$ super Yang-Mills theory on its Coulomb branch where the gauge group $SU(N)$ is spontaneously broken to $SU(N-1) \times U(1)$ and the dynamics is described by a single Abelian $\mathcal{N} = 2$ vector multiplet. In the framework of the background-field method, this holomorphic sector is singled out by computing the effective action for a background $\mathcal{N} = 2$ vector multiplet satisfying a relaxed super self-duality condition. At the two-loop level, the $N = 4$ SYM effective action is shown to possess no $F^4$ term (with $F$ the $U(1)$ field strength), in accordance with the Dine-Seiberg non-renormalization conjecture hep-th/9705057 and its generalized form given in hep-th/0310025. An unexpected outcome of our calculation is that no (manifestly supersymmetric invariant generating) $F^6$ quantum correction occurs at two loops. This is in conflict with previous results.
1 Introduction

A year ago, we developed a manifestly covariant multi-loop scheme\(^\dagger\) for computing finite quantum corrections to low-energy effective actions, within the background-field method, in supersymmetric Yang-Mills theories [1]. Since then, we have given two important applications [6, 7] of the approach suggested. In ref. [6], the two-loop Euler-Heisenberg effective action for \(\mathcal{N} = 2\) supersymmetric QED was derived. The subject of [7] was a detailed two-loop scrutiny of the Dine-Seiberg non-renormalization conjecture [8] that the quantum corrections with four derivatives (or supersymmetric \(F^4\) terms) are one-loop exact on the Coulomb branch of \(\mathcal{N} = 2, 4\) superconformal field theories in four space-time dimensions. To test the Dine-Seiberg proposal, in [7] we compared the two-loop \(F^4\) quantum corrections in two different superconformal theories with the gauge group \(SU(N)\): (i) \(\mathcal{N} = 4\) SYM; (ii) \(\mathcal{N} = 2\) SYM with \(2N\) hypermultiplets in the fundamental. According to the Dine-Seiberg conjecture, these theories should yield identical two-loop \(F^4\) contributions from all the supergraphs involving quantum hypermultiplets, since the pure \(\mathcal{N} = 2\) SYM and ghost sectors are identical provided the same gauge conditions are chosen. We explicitly evaluated the relevant two-loop supergraphs and observed that the \(F^4\) corrections generated have different large \(N\) behaviour in the two theories under consideration, in obvious conflict with [8].

Inspired by the AdS/CFT correspondence [9, 10, 11], it was further conjectured in [7] that the Dine-Seiberg proposal should hold for those \(\mathcal{N} = 2\) superconformal theories which possess supergravity duals, including the \(\mathcal{N} = 4\) SYM theory (no supergravity dual exists for \(\mathcal{N} = 2\) \(SU(N)\) SYM with \(2N\) hypermultiplets in the fundamental). In order to test this conjecture at two loops in the case of \(\mathcal{N} = 4\) SYM, the hypermultiplet results of [7] should be complemented with the two-loop \(F^4\) contribution from the pure \(\mathcal{N} = 2\) SYM and ghost sectors. To compute this contribution is one of the aims of the present work.

Among the primary motivations for the present work was the desire to gather more evidence for the conjectured correspondence [9, 12, 13, 14] between the D3-brane action in \(AdS_5 \times S^5\) and the low-energy action for \(\mathcal{N} = 4\) \(SU(N)\) SYM on its Coulomb branch, with the gauge group \(SU(N)\) spontaneously broken to \(SU(N-1) \times U(1)\). Let us explain the final form [14] of this remarkable conjecture in some more detail.

On the SYM side, the conjecture of [14] deals with the \(\mathcal{N} = 4\) SYM effective action for

\(^\dagger\)The approach of [1] is a natural generalization of methods developed in the past in [2, 3, 4, 5], and is equally applicable for computing effective actions in non-supersymmetric gauge theories.
an Abelian $\mathcal{N} = 4$ vector multiplet corresponding to a particular direction in the moduli space of vacua specified by the conditions (here we consider the bosonic sector only)

$$
\mathcal{X}_i = X_i H_0, \quad \mathcal{F}_{ab} = F_{ab} H_0, \quad H_0 = \frac{1}{\sqrt{N(N-1)}} \text{diag}(N - 1, -1, \ldots, -1), \quad (1.1)
$$

where $i = 1, 2, \ldots, 6$. Such a background induces spontaneous breaking of $SU(N)$ to $SU(N - 1) \times U(1)$, and therefore it admits a stringy interpretation in terms of a D3-brane separated from a stack of $(N - 1)$ branes. In an approximation of slowly varying fields, when the scalar fields $X_i$ and the $U(1)$ gauge field strength $F_{ab}$ are considered to be effectively constant, the $\mathcal{N} = 4$ SYM effective action should have the following general form

$$
\Gamma = \frac{1}{g_s^2} \int d^4x \sum_{l=0}^{\infty} f_l(g_s^2, N) \frac{F^{2l+2}}{|X|^4}, \quad |X|^2 = X_i X_i, \quad (1.2)
$$

In the planar (large $N$, fixed $\lambda = g_s^2 N$) approximation, the functions $f_l$ should depend on $\lambda$ only, $f_l(g_s^2, N) \to f_l(\lambda)$. As conjectured in [14] on the basis of the AdS/CFT correspondence, these functions should possess the following large $\lambda$ limit:

$$
f_l(\lambda) \xrightarrow{\lambda \gg 1} a_l \lambda^l, \quad (1.3)
$$

with some coefficients $a_l$.

On the supergravity side, the conjecture of [14] deals with the action for a D3-brane probe in the $AdS_5 \times S^5$ space (oriented parallel to the boundary of $AdS_5$), which in the case $X_i = \text{const}$ becomes

$$
S = -T_3 \int d^4x \frac{|X|^4}{Q} \left\{ -\det \left( \eta_{ab} + \frac{Q^{1/2}}{|X|^2} F_{ab} \right) - 1 \right\}
= \frac{1}{g_s^2} \int d^4x \sum_{l=0}^{\infty} c_l (g_s N)^l \frac{F^{2l+2}}{|X|^4}, \quad (1.4)
$$

where $T_3 = (2\pi g_s)^{-1}$, $Q = g_s N/\pi$. The coefficients $c_l$ can be easily computed by expanding the Born-Infeld action in powers of $F$.

The AdS/CFT correspondence is known to require $g_s^2 = 2\pi g_s$. According to the conjecture of [14], the coefficients $a_l$ in (1.3) should be directly related to the coefficients $c_l$ in (1.4). The simplest possibility to satisfy this condition [14] is that the functions of the coupling, $f_l(g_s^2, N)$, in front of some of the terms in (1.2) receive contributions only from the particular orders in perturbation theory, and are not renormalized by all higher-loop corrections, see [14] for a more detailed discussion.
Let us now re-iterate the conjecture of [14] in terms of superfields. It is well known that $\mathcal{N} = 4$ SYM can be viewed as $\mathcal{N} = 2$ SYM coupled to a hypermultiplet in the adjoint representation of the gauge group. In $\mathcal{N} = 1$ superfield notation, the $\mathcal{N} = 2$ Yang-Mills supermultiplet is described by two chiral gauge-covariant superfields $(\Phi, \mathcal{W}_\alpha)$ and their conjugates, with $\mathcal{W}_\alpha$ the field strength of the $\mathcal{N} = 1$ vector multiplet. The hypermultiplet is described by two chiral scalars $(\mathcal{Q}, \bar{\mathcal{Q}})$ and their conjugates. On the Coulomb branch, one is interested in the dynamics of light degrees of freedom when $\mathcal{Q} = \bar{\mathcal{Q}} = 0$ and the $\mathcal{N} = 2$ vector multiplet takes values in the Cartan subalgebra of $SU(N)$. Of special interest is the low-energy dynamics of a single $U(1)$ vector multiplet corresponding to a particular direction in the moduli space of vacua specified by the conditions

$$\Phi = \phi H_0, \quad \mathcal{W}_\alpha = W_\alpha H_0, \quad (1.5)$$

with the generator $H_0$ given in (1.1). Let $\Gamma[W, \phi]$ denote the $\mathcal{N} = 4$ SYM low-energy effective action for slowly varying fields. Its generic structure is [15]

$$g_{YM}^2 \Gamma[W, \phi] = \frac{1}{2} \int d^6 z W^2 + \int d^8 z \frac{W^2 W^2}{\phi^2 \bar{\phi}^2} \Omega(\Psi^2, \bar{\Psi}^2), \quad (1.6)$$

where

$$\bar{\Psi}^2 = \frac{1}{4} D^2 \left( \frac{W^2}{\phi^2 \bar{\phi}^2} \right), \quad \Psi^2 = \frac{1}{4} \bar{D}^2 \left( \frac{\bar{W}^2}{\phi^2 \bar{\phi}^2} \right), \quad (1.7)$$

and $\Omega$ is a real analytic function. The conjecture of [14] now implies that, in the planar approximation (large $N$, fixed $\lambda = g_{YM}^2 N$) accompanied by the large $\lambda$ limit, the $\mathcal{N} = 4$ SYM effective action $\Gamma[W, \phi]$ should reduce to the superconformal extension [17] of the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [18]

$$g_{YM}^2 S_{BI} = \frac{1}{2} \int d^6 z W^2 + \frac{1}{2} \kappa \int d^8 z \frac{W^2 W^2 (\phi \bar{\phi})^{-2}}{1 + \frac{1}{2} A + \sqrt{1 + A + \frac{1}{4} B^2}}, \quad (1.8)$$

where

$$A = \kappa (\Psi^2 + \bar{\Psi}^2), \quad B = \kappa (\Psi^2 - \bar{\Psi}^2), \quad (1.9)$$

with $\kappa \propto \lambda = g_{YM}^2 N$.

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2The effective action of $\mathcal{N} = 4$ SYM is invariant under quantum-corrected superconformal transformations [16] which differ from the ordinary linear superconformal transformations that leave the classical action invariant. The low-energy action (1.6) corresponds to an approximation when all terms with derivatives of $\phi$ and $\bar{\phi}$ are ignored. For such an approximation, the quantum-corrected superconformal transformations can be shown to reduce to the classical ones, and the latter leave invariant the action (1.6). The superfields $\Psi^2$ and $\bar{\Psi}^2$ are scalars with respect to the $\mathcal{N} = 1$ superconformal group [15].
The conjecture of [14] is highly nontrivial. It implies that the quantum corrections to a term of the form $\Psi^{2n}\bar{\Psi}^{2m}$, in the Taylor decomposition of $\Omega(\Psi^2,\bar{\Psi}^2)$, can occur only at loop orders not higher than $L = 1 + n + m$, and the dominant $L$-loop contribution comes with a coefficient proportional to $(Ng_{YM}^2)^L$; the latter coefficient should match the one corresponding to the same structure in the Born-Infeld action.

The first consequence of the conjecture is that the $F^4$ quantum correction

$$\int d^8z \frac{\bar{W}^2W^2}{\phi^2\bar{\phi}^2}$$

(1.10)

is one-loop exact, and its coefficient fixes the Born-Infeld coupling constant $\lambda^2$ in (1.8). Until recently, the Dine-Seiberg non-renormalization theorem [8] was considered to be the firm justification for this. As we now understand, there is a subtle flaw in the proof of the theorem given in [8]. This proof is claimed to be applicable to all $\mathcal{N} = 2$ superconformal theories, and implies the absence of $F^4$ quantum corrections beyond one loop in such theories. Unfortunately, this does not hold at two loops for some $\mathcal{N} = 2$ superconformal theories, as was demonstrated in [7] by explicit calculations. Nevertheless, it is most likely, on symmetry grounds, that the $F^4$ correction is indeed one-loop exact in $\mathcal{N} = 4$ SYM. As will be shown below, there is no two-loop $F^4$ correction in $\mathcal{N} = 4 SU(N)$ SYM (in contrast with $\mathcal{N} = 2 SU(N)$ SYM with $2N$ hypermultiplets in the fundamental).

The second consequence of the above conjecture is that the $F^6$ structure

$$\int d^8z \frac{\bar{W}^2W^2}{\phi^2\bar{\phi}^2} \left(\Psi^2 + \bar{\Psi}^2\right)$$

(1.11)

does not receive quantum corrections beyond two loops,\(^3\) and the corresponding two-loop coefficient, proportional in the large $N$ limit to $(Ng_{YM}^2)^2$, should coincide with that coming from the Born-Infeld action (1.8). The problem of computing the two-loop $F^6$ quantum correction was recently addressed in [14] on the basis of the background field formulation in $\mathcal{N} = 2$ harmonic superspace [20]. The authors of [14] reported complete agreement between the $F^6$ terms which occur in the two-loop $\mathcal{N} = 4$ SYM effective action and in the supersymmetric Born-Infeld action (1.8).

One of the aims of the present work is to provide an independent calculation of the two-loop $F^6$ quantum correction in $\mathcal{N} = 4$ SYM using the background field formulation in $\mathcal{N} = 1$ superspace [21] and the multi-loop techniques developed in [1, 6, 7]. Actually, we compute a holomorphic sector of the two-loop effective action\(^4\), of which the $F^6$ term is

\(^3\)No $F^6$ quantum correction occurs in the one-loop effective action for $\mathcal{N} = 4$ SYM [19, 15].

\(^4\)Similarly to [14], in this paper we are interested in the 1PI effective action for $\mathcal{N} = 4$ SYM on its Coulomb branch.
simply a special sub-sector. The $F^6$ correction is associated with the linear homogeneous term of the holomorphic function $\Omega(\Psi^2, 0)$. At the one-loop level, this function is known to be trivial \[15\]
\[\Omega_{\text{one-loop}}(\Psi^2, 0) = \Omega_{\text{one-loop}}(0, 0).\]
(1.12)

What happens at two loops?

The main result of the present paper is a closed-form expression for $\Omega_{\text{two-loop}}(\Psi^2, 0)$ in the case of $\mathcal{N} = 4\ SU(N)$ SYM. This function turns out to be remarkably simple:

\[\Omega_{\text{two-loop}}(\Psi^2, 0) = -\frac{4g_\text{YM}^4N(N-1)}{(4\pi)^4}\left(\frac{\Psi^2}{e^2} - 2\int_0^\infty ds s^2\left\{\frac{\Psi}{2e}\coth\frac{s\Psi}{2e} - \frac{1}{s}\right\}e^{-s}\right),\]
(1.13)

As can be seen, no $F^6$ term is generated since $\Omega_{\text{two-loop}}(\Psi^2, 0) = O(\Psi^4)$. More precisely, the first term in the expression for $\Omega_{\text{two-loop}}(\Psi^2, 0)$, that is

\[-\frac{(2(N-1)g_\text{YM}^2)^2}{(4\pi)^4}\Psi^2,\]

matches exactly, in the large $N$ limit, the $F^6$ structure in the Born-Infeld action. However, the similar structure in the second term in $\Omega_{\text{two-loop}}(\Psi^2, 0)$ differs only in sign from the first, and hence the total two-loop $F^6$ correction is zero.

To compute quantum corrections to the holomorphic function $\Omega(\Psi^2, 0)$, it is sufficient (and extremely advantageous, as far as computational efforts are concerned) to work with covariant supergraphs in a vector multiplet background satisfying a relaxed super self-duality condition \[22\]. The latter can be defined by

\[W_\alpha \neq 0,\quad D_\alpha W_\beta = 0,\quad \bar{D}_{(\bar{\alpha})} \bar{W}_\beta \neq 0\]
(1.14)

in the case of $\mathcal{N} = 1$ supersymmetry, or

\[D^i_\alpha W \neq 0,\quad D^i_\alpha D^j_\beta W = 0,\quad D^{i(\bar{\alpha})} D^{j(\bar{\beta})} W \neq 0\]
(1.15)

in the case of $\mathcal{N} = 2$ supersymmetry. Here $W$ is the chiral superfield strength describing an $\mathcal{N} = 2$ Abelian vector multiplet. Ordinary (Euclidean) super self-duality \[23\] corresponds to setting $W = 0$ while keeping $\bar{W}$ non-vanishing. From the point of view of $\mathcal{N} = 1$ supersymmetry, the $\mathcal{N} = 2$ vector multiplet strength $W$ consists of two $\mathcal{N} = 1$ superfields: (i) a chiral scalar $\phi$; and (ii) the $\mathcal{N} = 1$ vector multiplet strength $W_\alpha$. The conditions on $W_\alpha$ which follow from (1.15) coincide with (1.14).
In Minkowski space-time, the conditions (1.14) and (1.15) are purely formal, as they are obviously inconsistent with the structure of a single real vector multiplet. Nevertheless, their use is completely legitimate if we are only interested in computing some special sector of the effective action with holomorphic structure.

This paper is organized as follows. Section 2 contains the necessary setup regarding the \( \mathcal{N} = 4 \) super Yang-Mills theory and its background field quantization (for supersymmetric 't Hooft gauge) in \( \mathcal{N} = 1 \) superspace. The structure of the one-loop effective action is briefly discussed in section 3. In section 4 we work out useful functional representations for the two-loop supergraphs. In section 5 we specify the superfield background and collect some important group-theoretical results, following [7]. Section 6 is devoted to the evaluation of the two-loop supergraphs that come from the hypermultiplet sector of the theory. The two-loop supergraphs from the pure \( \mathcal{N} = 2 \) super Yang-Mills and ghost sectors are evaluated in section 7. A discussion of the results obtained is given in section 8. In appendix A the main properties of the parallel displacement propagator are given. In appendix B we discuss simplifications in the structure of the \( U(1) \) superfield heat kernel which occur under the relaxed super self-duality condition. In appendix C we prove two identities given in section 4. Finally, appendix D contains details of the group-theoretical manipulations leading to the expressions (5.17) and (5.20).

## 2 \( \mathcal{N} = 4 \) SYM setup

From the point of view of \( \mathcal{N} = 2 \) supersymmetry, the classical action of the \( \mathcal{N} = 4 \) super Yang-Mills theory, \( S = S_{\text{SYM}} + S_{\text{hyper}} \), consists of two parts: (i) the pure \( \mathcal{N} = 2 \) SYM action (with \( g^2 = 2g_{\text{YM}}^2 \))

\[
S_{\text{SYM}} = \frac{1}{g^2} \text{tr} \left( \int d^8 z \, \Phi^\dagger \Phi + \int d^6 z \, W^\alpha W_\alpha \right);
\] (2.1)

(ii) the hypermultiplet action

\[
S_{\text{hyper}} = \frac{1}{g^2} \text{tr} \left( \int d^8 z \, (\bar{Q}^\dagger Q + \bar{\bar{Q}}^\dagger \bar{Q}) - i \int d^6 z \, \bar{Q} \lbrack \Phi, Q \rbrack - i \int d^6 \bar{z} \, \bar{Q}^\dagger \lbrack \Phi^\dagger, Q^\dagger \rbrack \right). \] (2.2)

Here \( \Phi = \Phi^\mu(z) T_\mu, Q = Q^\mu(z) T_\mu \) and \( \bar{Q} = \bar{Q}^\mu(z) T_\mu \) are covariantly chiral superfields in the adjoint representation of the gauge group, with the latter chosen to be \( SU(N) \) throughout this paper. It is assumed in (2.1) and (2.2) that the trace is taken in the fundamental representation of \( SU(N) \), \( \text{tr} = \text{tr}_F \), with the corresponding generators normalized such
that \( \text{tr} (T_\mu T_\nu) = \delta_{\mu\nu} \). The covariantly chiral superfield strength \( W_\alpha \) is associated with the gauge covariant derivatives

\[
D_A = (D_a, D_\alpha, \bar{D}^\dot{\alpha}) = D_A + i \Gamma_A , \quad \Gamma_A = \Gamma^\mu_A(z) T_\mu , \quad (T_\mu)^\dagger = T_\mu ,
\]

(2.3)

where \( D_A \) are the flat covariant derivatives\(^5\), and \( \Gamma_A \) the superfield connection taking its values in the Lie algebra of the gauge group. In any representation of the gauge group, the gauge covariant derivatives satisfy the following algebra:

\[
\{ D_\alpha, D_\beta \} = \{ \bar{D}^\dot{\alpha}, \bar{D}^\dot{\beta} \} = 0 \quad , \quad \{ D_\alpha, \bar{D}^\dot{\beta} \} = \{-2i D_\alpha \bar{\beta},
\]

\[
[D_\alpha, D_\beta] = 2i \varepsilon_{\alpha\beta} W_\beta \quad , \quad [\bar{D}^\dot{\alpha}, D_\beta] = 2i \varepsilon_{\dot{\alpha}\beta} \bar{W}^{\dot{\beta}} ,
\]

\[
[D_\alpha, \bar{D}^\dot{\beta}] = i F_{\alpha\dot{\alpha}\dot{\beta}} = -\varepsilon_{\alpha\dot{\beta}} \bar{D}_\dot{\alpha} \bar{W}^{\dot{\beta}} - \varepsilon_{\dot{\alpha}\beta} D_\alpha W_\beta .
\]

(2.4)

The spinor field strengths \( W_\alpha \) and \( \bar{W}^{\dot{\alpha}} \) obey the Bianchi identities

\[
\bar{D}_{\dot{\alpha}} W_\alpha = 0 \quad , \quad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} .
\]

(2.5)

To quantize the theory, we will use the \( \mathcal{N} = 1 \) background field formulation\(^6\) and split the dynamical variables into background and quantum ones,

\[
\Phi \rightarrow \Phi + \varphi \quad , \quad Q \rightarrow Q + q \quad , \quad \bar{Q} \rightarrow \bar{Q} + \bar{q} ,
\]

\[
D_\alpha \rightarrow e^{-v} D_\alpha e^v \quad , \quad \bar{D}_{\dot{\alpha}} \rightarrow \bar{D}_{\dot{\alpha}} ,
\]

(2.6)

with lower-case letters used for the quantum superfields. In this paper, we are not interested in the dependence of the effective action on the hypermultiplet superfields, and therefore we set \( Q = Q = 0 \) in what follows. After the background-quantum splitting, the action (2.1) turns into

\[
S_{\text{SYM}} = \text{tr} \int d^8 z \left( (\Phi^\dagger + \varphi^\dagger) e^v (\Phi + \varphi) e^{-v} + \text{tr} \int d^6 z \ W^\alpha W_\alpha \right) \equiv S_{\text{scal}} + S_{\text{vect}} ,
\]

(2.7)

where

\[
W_\alpha = -\frac{1}{8} \bar{D}^2 (e^{-v} D_\alpha e^v \cdot 1) = W_\alpha
\]

(2.8)

\[
-\frac{1}{8} \bar{D}^2 (D_\alpha v - \frac{1}{2}[v, D_\alpha v] + \frac{1}{6}[v, [v, D_\alpha v]] - \frac{1}{24}[v, [v, [v, D_\alpha v]]]) + O(v^5) .
\]

\(^5\)Our \( \mathcal{N} = 1 \) notation and conventions correspond to \([24]\).

\(^6\)To simplify the notation, we set \( g^2 = 1 \) at the intermediate stages of the calculation. The explicit dependence on the coupling constant will be restored in the final expression for the effective action.
The hypermultiplet action (2.2) takes the form

\begin{equation}
S_{\text{hyper}} = \text{tr} \int d^8z \left( q^\dagger e^v q e^{-v} + \bar{q}^\dagger e^\varphi \bar{q} e^{-v} \right) - \text{tr} \left( i \int d^6\bar{z} \bar{q} [\Phi + \varphi, q] + i \int d^6\bar{z} \bar{q}^\dagger [\Phi^\dagger + \varphi^\dagger, q^\dagger] \right). \tag{2.9}
\end{equation}

It is advantageous to use \( \mathcal{N} = 1 \) supersymmetric ’t Hooft gauge (a special case of the supersymmetric \( R_\xi \)-gauge introduced in [25] and further developed in [26]) which is specified by the nonlocal gauge conditions

\begin{align*}
-4\chi &= D^2v + [\Phi, (\square_+)^{-1}D^2\varphi^\dagger] = D^2v + [\Phi, D^2(\square_-)^{-1}\varphi^\dagger], \\
-4\chi^\dagger &= D^2v - [\Phi^\dagger, (\square_-)^{-1}D^2\varphi] = D^2v - [\Phi^\dagger, D^2(\square_+)^{-1}\varphi]. \tag{2.10}
\end{align*}

Here the covariantly chiral d’Alembertian, \( \square_+ \), is defined by

\begin{align*}
\square_+ &= D^aD_a - W^aD_a - \frac{1}{2} (D^aW_a), \quad \square_+ \Psi = \frac{1}{16} D^2\bar{D}^2\Psi, \quad \bar{D}_a \Psi = 0, \tag{2.11}
\end{align*}

for a covariantly chiral superfield \( \Psi \). Similarly, the covariantly antichiral d’Alembertian, \( \square_- \), is defined by

\begin{align*}
\square_- &= D^aD_a + \bar{W}_a\bar{D}^a + \frac{1}{2} (\bar{D}_a\bar{W}^a), \quad \square_- \bar{\Psi} = \frac{1}{16} D^2\bar{D}^2\bar{\Psi}, \quad \bar{D}_a \bar{\Psi} = 0, \tag{2.12}
\end{align*}

for a covariantly antichiral superfield \( \bar{\Psi} \). The gauge conditions chosen lead to the following Faddeev-Popov ghost action

\begin{align*}
S_{\text{gh}} &= \text{tr} \int d^8z \left( \bar{c} - \bar{c^\dagger} \right) \left\{ L_{v/2} (c + c^\dagger) + L_{v/2} \coth(L_{v/2})(c - c^\dagger) \right\} \\
- \text{tr} \int d^8z \left\{ [\bar{c}, \Phi] (\square_-)^{-1}[c^\dagger, \Phi^\dagger + \varphi^\dagger] + [\bar{c}^\dagger, \Phi^\dagger] (\square_+)^{-1}[c, \Phi + \varphi] \right\}, \tag{2.13}
\end{align*}

with \( L_X Y = [X, Y] \). Here the ghost (Grassmann) superfields \( c \) and \( \bar{c} \) are background covariantly chiral.

A convenient gauge-fixing functional is

\begin{equation}
S_{\text{gf}} = -\text{tr} \int d^8z \chi^\dagger \chi. \tag{2.14}
\end{equation}

Its introduction is accompanied by the presence of the Nielsen-Kallosh ghost action

\begin{equation}
S_{\text{NK}} = \text{tr} \int d^8z b^\dagger b, \tag{2.15}
\end{equation}

where the third (Grassmann) ghost superfield \( b \) is background covariantly chiral. The Nielsen-Kallosh ghosts lead to a one-loop contribution only.

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The quantum quadratic part of $S_{\text{SYM}} + S_{\text{gf}}$ is
\[
S^{(2)}_{\text{SYM}} + S_{\text{gf}} = \text{tr} \int d^8z \left( \varphi^\dagger \varphi - [\Phi^\dagger, [\Phi, \varphi^\dagger]] \right) (\Box^+)^{-1} \varphi
- \frac{1}{2} \text{tr} \int d^8z \left( \Box_v \varphi - [\Phi^\dagger, [\Phi, \varphi]] \right) + \ldots
\]
(2.16)
where the dots stand for the terms with derivatives of the background (anti)chiral superfields $\Phi^\dagger$ and $\Phi$. The vector d’Alembertian, $\Box_v$, is defined by
\[
\Box_v = D^\alpha D_\alpha - W^\alpha D_\alpha + \bar{W}_\dot{\alpha} \bar{D}^{\dot{\alpha}}
= -\frac{1}{8} D^\alpha D^2 D_\alpha + \frac{1}{16} \{D^2, D^2\} - W^\alpha D_\alpha - \frac{1}{2} (D^\alpha W_\alpha)
= -\frac{1}{8} \bar{D}_\dot{\alpha} D^2 \bar{D}^{\dot{\alpha}} + \frac{1}{16} \{D^2, D^2\} + \bar{W}_\dot{\alpha} \bar{D}^{\dot{\alpha}} + \frac{1}{2} (\bar{D}_\dot{\alpha} \bar{W}^{\dot{\alpha}})
\]
(2.17)
The quantum quadratic part of $S_{\text{hyper}}$ is
\[
S^{(2)}_{\text{hyper}} = \text{tr} \int d^8z \left( q^\dagger \varphi \Box q + \varphi^\dagger \Box q + q^\dagger \varphi \Box q^\dagger \right),
\]
(2.18)
where the mass operator $\mathcal{M}$ is defined by
\[
\mathcal{M} \Sigma = -i [\Phi, \Sigma],
\]
(2.19)
for a superfield $\Sigma$ in the adjoint.

In what follows, the background superfields will be chosen to satisfy the conditions:
\[
[\Phi, \Phi^\dagger] = 0, \quad D_\alpha \Phi = 0, \quad D^\alpha W_\alpha = 0.
\]
(2.20)
Some additional conditions will be imposed on the background superfields later on. Such an on-shell background configuration is convenient for computing those corrections to the effective action which do not contain derivatives of $\Phi$ and $\Phi^\dagger$. The above conditions imply that the background superfields belong to the Cartan subalgebra of the gauge group.

Now, the action (2.16) becomes
\[
S^{(2)}_{\text{SYM}} + S_{\text{gf}} = \text{tr} \int d^8z \left( \varphi^\dagger (\Box^+)^{-1} (\Box^+ - |\mathcal{M}|^2) \varphi - \frac{1}{2} \nu(\Box_v - |\mathcal{M}|^2) \varphi \right).
\]
(2.21)
Similarly, the quadratic part of the Faddeev-Popov ghost action takes the form
\[
S^{(2)}_{\text{gh}} = \text{tr} \int d^8z \left( c^\dagger (\Box^+)^{-1} (\Box^+ - |\mathcal{M}|^2) c - \bar{c}^\dagger (\Box^+)^{-1} (\Box^+ - |\mathcal{M}|^2) \bar{c} \right).
\]
(2.22)

The cubic and quartic parts of $S_{\text{scal}}$ are:
\[
S^{(3)}_{\text{scal}} = \text{tr} \int d^8z \left( \varphi^\dagger [v, \varphi] + \frac{1}{6} \Phi^\dagger [v, [v, \varphi]] + \frac{1}{2} \Phi^\dagger [v, [v, \Phi]] + \text{h.c.} \right),
\]
(2.23)
\[
S^{(4)}_{\text{scal}} = \frac{1}{2} \text{tr} \int d^8z \left( \varphi^\dagger [v, [v, \varphi]] + \frac{1}{3} \Phi^\dagger [v, [v, \varphi]] + \text{h.c.} \right)
+ \frac{1}{12} [v, [v, \Phi^\dagger]] [v, [v, \Phi]]
\]
(2.24)
The cubic and quartic parts of $S_{\text{vect}}$ are:

\begin{align*}
S^{(3)}_{\text{vect}} & = \frac{1}{2} \text{tr} \int d^8z \, [v, D^a v] \left( \frac{1}{8} \bar{D}^2 D_\alpha v + \frac{1}{3} [W_\alpha, v] \right) \equiv S^{(3)}_{\text{vect,I}} + S^{(3)}_{\text{vect,II}}, \\
S^{(4)}_{\text{vect}} & = -\frac{1}{8} \text{tr} \int d^8z \, [v, D^a v] \left( \frac{1}{8} \bar{D}^2 [v, D_\alpha v] - \frac{1}{6} [v, \bar{D}^2 D_\alpha v] + \frac{1}{3} [v, [v, W_\alpha]] \right).
\end{align*}

It is an instructive exercise to show, using the algebra of covariant derivatives (2.4), that the functionals $S^{(3)}_{\text{vect}}$ and $S^{(4)}_{\text{vect}}$ are real modulo total derivatives. However, it turns out advantageous for loop calculations [27] to keep these interaction terms in the complex form given.

The cubic and quartic parts of $S_{\text{hyper}}$ are:

\begin{align*}
S^{(3)}_{\text{hyper}} & = \text{tr} \left\{ \int d^8z \left( q^\dagger [v, q] + \bar{q}^\dagger [v, \bar{q}] \right) - i \int d^6z \, \bar{q} [\varphi, q] - i \int d^6z \, \bar{q}^\dagger [\varphi^\dagger, q^\dagger] \right\}, \\
S^{(4)}_{\text{hyper}} & = \frac{1}{2} \text{tr} \int d^8z \left( q^\dagger [v, [v, q]] + \bar{q}^\dagger [v, [v, \bar{q}]] \right).
\end{align*}

The cubic and quartic parts of $S_{\text{gh}}$ are:

\begin{align*}
S^{(3)}_{\text{gh}} & = \frac{1}{2} \text{tr} \int d^8z \left( \bar{c} - \bar{c}^\dagger \right) [v, (c + c^\dagger)] \\
& \quad - \text{tr} \int d^8z \left\{ \left[ \bar{c}, \Phi \right] (\Box_+)^{-1}[c^\dagger, \varphi^\dagger] + \left[ \bar{c}^\dagger, \Phi^\dagger \right] (\Box_-)^{-1}[c, \varphi] \right\}, \\
S^{(4)}_{\text{gh}} & = \frac{1}{12} \text{tr} \int d^8z \left( \bar{c} - \bar{c}^\dagger \right) [v, [v, (c - c^\dagger)]]).
\end{align*}

For the background chosen, the Feynman propagators associated with the quadratic actions (2.18), (2.21) and (2.22) can be expressed via a single Green’s function. Such a Green’s function, $G(z, z')$, originates in the following auxiliary model

\begin{equation}
S = \frac{1}{2} \int d^8z \, \Sigma^T (\Box_v - |M|^2) \Sigma,
\end{equation}

which describes the dynamics of an unconstrained real superfield $\Sigma = (\Sigma^\mu)$ transforming in the adjoint representation of the gauge group. The relevant Feynman propagator reads

\begin{equation}
G(z, z') = i \langle 0 | T \left( \Sigma(z) \Sigma^T(z') \right) |0 \rangle \equiv i \langle \Sigma(z) \Sigma^T(z') \rangle
\end{equation}

and satisfies the equation

\begin{equation}
(\Box_v - |M|^2) G(z, z') = -i \delta^8(z - z').
\end{equation}

In listing the Feynman propagators associated with the actions (2.18), (2.21) and (2.22), all the dynamical adjoint superfields will be treated as column-vectors (e.g., $\varphi = (\varphi^\mu)$
and \( \bar{\phi} = (\bar{\phi}^\mu) \), with \( \varphi^\dagger \) being a row-vector), and not as Lie-algebra-valued objects, say \( v = v^\mu T_\mu \), as they have been understood so far. The Feynman propagators for the action (2.21) are:

\[
\begin{align*}
    \langle v(z) v^T(z') \rangle &= -G(z, z') , \\
    \langle \varphi(z) \varphi^\dagger(z') \rangle &= \frac{1}{16} \bar{D}^2 D'^2 G(z, z') , \quad \langle \varphi(z) \varphi^T(z') \rangle = \langle \bar{\varphi}(z) \varphi^\dagger(z') \rangle = 0 .
\end{align*}
\]  

(2.34)

The Feynman propagators for the action (2.18) are:

\[
\begin{align*}
    \langle q(z) q^\dagger(z') \rangle &= \langle \bar{q}(z) \bar{q}^\dagger(z') \rangle = \frac{1}{16} \bar{D}^2 D'^2 G(z, z') , \\
    \langle q(z) \bar{q}^T(z') \rangle &= \mathcal{M}^\dagger \left( -\frac{1}{4} \bar{D}^2 \right) G(z, z') = \mathcal{M}^\dagger \left( -\frac{1}{4} D'^2 \right) G(z, z') , \\
    \langle \bar{q}(z) q^\dagger(z') \rangle &= \mathcal{M} \left( -\frac{1}{4} D^2 \right) G(z, z') = \mathcal{M} \left( -\frac{1}{4} D'^2 \right) G(z, z') .
\end{align*}
\]  

(2.35)

The Feynman propagators for the action (2.22) are:

\[
\begin{align*}
    \langle \tilde{c}(z) c^\dagger(z') \rangle &= -i \langle c(z) \tilde{c}^\dagger(z') \rangle = \frac{1}{16} \bar{D}^2 D'^2 G(z, z') .
\end{align*}
\]  

(2.36)

3 One-loop effective action

The one-loop contribution to the effective action is determined by the quantum quadratic actions (2.15), (2.18), (2.21) and (2.22). It can be shown that the contributions coming from the quantum chiral superfields (i.e. \( b, c, \tilde{c}, q, \tilde{q} \) and \( \varphi \)) and their conjugates, cancel each other\(^7\) (see the second reference in [26]). As a result, the one-loop effective action is solely generated by the contribution coming from the quantum gauge superfield in (2.21):

\[
\Gamma_{\text{one-loop}} = -\frac{i}{2} \text{Tr} \ln G = -\frac{i}{2} \int_0^\infty \frac{ds}{s} \text{Tr} \{ K(s) e^{-i(|\mathcal{M}|^2 - i\varepsilon)s} \} .
\]  

(3.1)

Here \( K(s) \exp(-i |\mathcal{M}|^2 s) \) is the heat kernel associated with the Green’s function \( G \) satisfying the equation (2.33) and the Feynman boundary conditions,

\[
G(z, z') = \int_0^\infty ds K(z, z'|s) e^{-i(|\mathcal{M}|^2 - i\varepsilon)s} , \quad \varepsilon \to +0 .
\]  

(3.2)

\(^7\)The relation (3.1) was first derived in [28] using the background field formulation in \( \mathcal{N} = 2 \) harmonic superspace.
The functional trace in (3.1) is defined as follows
\[
\text{Tr} \left\{ K(s) e^{-i(|M|^2 - i\varepsilon)s} \right\} = \text{tr}_{\text{Ad}} \int d^8 z K(z, z|s) e^{-i(|M|^2 - i\varepsilon)s},
\]
where \( \text{tr}_{\text{Ad}} \) denotes the operation of trace in the adjoint representation of the gauge group. The background superfields are chosen to satisfy the conditions (2.20).

For the case of a covariantly constant background vector multiplet,
\[
\mathcal{D}^\alpha \mathcal{W}_\alpha = 0, \quad \mathcal{D}_\alpha \mathcal{W}_\beta = 0,
\]
the exact expression for the heat kernel is known\(^8\) [1, 6]:
\[
K(z, z'|s) = -\frac{i}{(4\pi s)^2} \left( \frac{s\mathcal{F}}{\sinh(s\mathcal{F})} \right) U(s) \zeta^2(\zeta) \bar{\zeta}^2(s) e^{\frac{i}{2} s \rho F \coth(s\mathcal{F})} I(z, z'),
\]
where the determinant is computed with respect to the Lorentz indices,
\[
U(s) = \exp \left\{ -is(\mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}) \right\},
\]
and \( I(z, z') \) is the so-called parallel displacement propagator, see Appendix A for its definition and basic properties. The supersymmetric two-point function \( \zeta^A(z, z') = -\zeta^A(z', z) = (\rho^A, \zeta^A, \bar{\zeta}_{\dot{A}}) \) is defined as follows:
\[
\rho^A = (x - x')^a - i(\theta - \theta')\sigma^a\bar{\theta}' + i\theta'\sigma^a(\bar{\theta} - \bar{\theta}'), \quad \zeta^A = (\theta - \theta')^a, \quad \bar{\zeta}_{\dot{A}} = (\bar{\theta} - \bar{\theta}')_{\dot{a}}.
\]
With the notation\(^9\)
\[
\mathcal{N}_\alpha^\beta = \mathcal{D}_\alpha \mathcal{W}^\beta, \quad \bar{\mathcal{N}}_{\dot{\alpha}}^{\dot{B}} = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{B}}, \quad \text{tr} \mathcal{N} = \text{tr} \bar{\mathcal{N}} = 0,
\]
for special proper-time dependent variables of the form \( \Psi(s) \equiv U(s) \Psi U(-s) \) one gets
\[
\mathcal{W}^\alpha(s) = (\mathcal{W} e^{-i\mathcal{N}})^\alpha, \quad \bar{\mathcal{W}}^{\dot{\alpha}}(s) = (\bar{\mathcal{W}} e^{-i\bar{\mathcal{N}}})^{\dot{\alpha}},
\]
\[
\zeta^\alpha(s) = \zeta^\alpha + \left( \mathcal{W} e^{-i\mathcal{N}} - 1 \right)^\alpha, \quad \bar{\zeta}^{\dot{\alpha}}(s) = \bar{\zeta}^{\dot{\alpha}} - \left( \bar{\mathcal{W}} e^{-i\bar{\mathcal{N}}} - 1 \right)^{\dot{\alpha}},
\]
\[
\rho_{\alpha\dot{\alpha}}(s) = \rho_{\alpha\dot{\alpha}} - 2 \int_0^s dt \left( \mathcal{W}_{\alpha}(t) \bar{\zeta}_{\dot{\alpha}}(t) + \zeta^\alpha(t) \bar{\mathcal{W}}^{\dot{\alpha}}(t) \right).
\]

---

\(^8\)This heat kernel was first found in Fock-Schwinger gauge in [29].

\(^9\)The symbol \( \text{tr} \) denotes the trace with respect to spinor indices.
Here $\rho^a(s)$, $\zeta^\alpha(s)$ and $\bar{\zeta}^\dot{\alpha}(s)$ are the building blocks which appear in the second line of (3.5). The explicit expression for $U(s) I(z, z')$ is given in [1].

The parallel displacement propagator is the only building block for the supersymmetric heat kernel which involves the naked gauge connection. In covariant supergraphs, however, the parallel displacement propagators that come from all possible internal lines ‘annihilate’ each other through the mechanism sketched in [1]. At one loop, the parallel displacement propagator disappears because of the identity (A.4). At two and higher loops, it is the identity (A.5) and its generalizations which are responsible for annihilation of the parallel displacement propagators.

Since the background fields take their values in the Cartan subalgebra, then

$$W_\alpha W_\beta W_\gamma = 0, \quad \bar{W}_\dot{\alpha} \bar{W}_\dot{\beta} \bar{W}_\dot{\gamma} = 0,$$

and therefore the heat kernel at coincident points is

$$K(z, z|s) = -\frac{i}{(4\pi s)^2} \sqrt{\det \left( \frac{s F}{\sinh(s F)} \right)} \left( \frac{\sin^2(s N/2)}{(N'/2)^2} \right) \left( \frac{\sin^2(s \bar{N}/2)}{(\bar{N}'/2)^2} \right) \times \frac{1}{4} W^2 \bar{W}^2.$$  (3.11)

Introducing the (anti) self-dual components of $F$,

$$F_\pm = \frac{1}{2} (F \mp i \tilde{F}) , \quad \tilde{F}_\pm = \pm i F_\pm ,$$  (3.12)

with $\tilde{F}$ the Hodge-dual of $F$, the above result can be rewritten as follows

$$K(z, z|s) = -\frac{i}{(4\pi s)^2} \sqrt{\det \left( \frac{s F}{\sinh(s F)} \right) \frac{\sinh(s F_+)}{F_+} \frac{\sinh(s F_-)}{F_-}} W^2 \bar{W}^2 .$$  (3.13)

Imposing the condition of relaxed super self-duality discussed in the Introduction,

$$F_- = 0, \quad F_+ \neq 0 ,$$  (3.14)

the expression for $K(z, z|s)$ simplifies drastically,

$$K(z, z|s) = -\frac{i s^2}{(4\pi)^2} W^2 \bar{W}^2 .$$  (3.15)

The result (1.12) follows from this. If the $N = 2$ vector multiplet is aligned along a generic direction in the moduli space of vacua (with the Cartan-Weyl basis for $SU(N)$ specified in eqs. (5.1) and (5.2)),

$$\Phi = \phi^I H_I , \quad W_\alpha = W_\alpha^I H_I ,$$  (3.16)
then one can show that the one-loop effective action coincides with the result given in [28] (see also [30]). For a special field configuration, eq. (5.6), that corresponds to the spontaneous breakdown of \(SU(N)\) to \(SU(N-1) \times U(1)\), one obtains

\[
\Gamma_{\text{one-loop}} = \frac{N-1}{(4\pi)^2} \int d^8z \frac{W^2W^2}{\phi^2\phi^2},
\]

(3.17)

and therefore

\[
\Omega_{\text{one-loop}}(\Psi^2, 0) = \frac{g^2YM(N-1)}{(4\pi)^2}.
\]

(3.18)

4 Functional representation for two-loop supergraphs

We now turn to obtaining a useful functional representation for the two-loop supergraphs. Most of the consideration of this section is valid for an arbitrary gauge group.

Even a quick glance at the structure of the cubic and quartic interactions is sufficient to recognize that the supergraphs generated by the pure \(\mathcal{N} = 1\) SYM vertices (2.25) and (2.26) are the most laborious ones. Their evaluation turns out to simplify significantly if the background superfields are subject to an additional restriction of the form

\[
\mathcal{D}_\alpha \mathcal{W}_\beta = 0, \quad \mathcal{D}_{(a)\dot{\alpha}} \mathcal{W}_{\dot{\beta}} \neq 0.
\]

(4.1)

As discussed in the Introduction, these conditions are rather formal, since they are incompatible with a real vector supermultiplet in Minkowski space. However, their use is completely legitimate if we are only interested in computing a special holomorphic sector of the effective action.

To explain why the first condition in (4.1) is useful, let us point out that without this condition the heat kernel satisfies the identity

\[
\mathcal{D}'_\alpha K(z, z'|s) = -(e^{i\mathcal{N}})^{\beta}_{\alpha} \mathcal{D}_\beta K(z, z'|s) \, , \quad \mathcal{N}_\alpha^{\beta} = \mathcal{D}_\alpha \mathcal{W}^\beta.
\]

(4.2)

Therefore

\[
\mathcal{D}_a \mathcal{W}_\beta = 0 \quad \Longrightarrow \quad \mathcal{D}'_\alpha K(z, z'|s) = -\mathcal{D}_\alpha K(z, z'|s),
\]

(4.3)

and similarly for the corresponding Green’s function,

\[
\mathcal{D}_a \mathcal{W}_\beta = 0 \quad \Longrightarrow \quad \mathcal{D}'_\alpha G(z, z') = -\mathcal{D}_\alpha G(z, z').
\]

(4.4)

The latter identity proves to be invaluable when evaluating the supergraphs generated by the pure \(\mathcal{N} = 1\) SYM vertices (2.25) and (2.26).
4.1 Hypermultiplet ‘fish’ supergraphs

The two-loop supergraphs generated by the hypermultiplet vertices (2.27) and (2.28) have been analyzed in detail in [7], and therefore we simply reproduce the results.

The cubic interaction (2.27) generates the following fish-type contributions to the effective action

\[
\Gamma^{(3)}_I + \Gamma^{(3)}_{II} \equiv \Gamma_1 + \Gamma_II, \quad (4.5)
\]

where

\[
\Gamma_I = \frac{1}{2^9} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_A \left( T_\mu [\bar{D}^2, D^2]G(z, z') T_\nu [D^2, \bar{D}^2]G(z', z) \right),
\]

\[
\Gamma_{II} = -\frac{1}{2^4} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_A \left( T_\mu \Phi^\dagger \bar{D}^2G(z, z') T_\nu \Phi D^2G(z', z) \right). \quad (4.6)
\]

The quantum corrections \( \Gamma_1 \) and \( \Gamma_{II} \) were denoted in [7] as \( \Gamma_{I+II} \) and \( \Gamma_{III} \), respectively.

4.2 Hypermultiplet ‘eight’ supergraphs

The quartic interaction (2.28) generates an ‘eight’ supergraph,

\[
\langle S^{(4)}_{\text{hyper}} \rangle_{1PI} \equiv \Gamma_{III}, \quad (4.7)
\]

which has the following structure [7]

\[
\Gamma_{III} = \frac{1}{2^4} \int d^8z \lim_{z' \to z} G^{\mu\nu}(z, z') \text{tr}_A \left( T_\mu D^2D^2G(z, z') T_\nu \right). \quad (4.8)
\]

This quantum correction was denoted in [7] as \( \Gamma_{IV} \).
4.3 Vector ‘fish’ supergraphs

The cubic interaction (2.25) generates the following fish-type contributions to the effective action

$$\frac{i}{2} \left\langle \mathcal{S}^{(3)}_{\text{vec,I}} \mathcal{S}^{(3)}_{\text{vec,I}} \right\rangle_{\text{1PI}} + i \left\langle \mathcal{S}^{(3)}_{\text{vec,I}} \mathcal{S}^{(3)}_{\text{vec,II}} \right\rangle_{\text{1PI}} \equiv \Delta \Gamma_1 + \Delta \Gamma_2 ,$$

where

$$\Delta \Gamma_1 = \frac{1}{2 \pi^2} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} T_\mu \left( 2 \mathcal{D}^{2} G(z, z') T_\nu \mathcal{D}^{2} G(z', z) + [\mathcal{D}^{2}, \mathcal{D}^{2}] G(z, z') T_\nu D^\alpha D^2 D^2 G(z', z) - D^\alpha G(z, z') T_\nu \mathcal{D}^{2} D^2 G(z', z) \right) ,$$

$$\Delta \Gamma_2 = \frac{1}{3 \cdot 25} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} T_\mu \left( \mathcal{D}^{2} G(z, z') T_\nu \mathcal{W}^{\alpha} D^\alpha D^2 G(z', z) - \frac{1}{2} D^\alpha G(z, z') T_\nu \mathcal{W}^{\alpha} D^2 G(z', z) \right) .$$

One can readily show that

$$\left\langle \mathcal{S}^{(3)}_{\text{vec,II}} \mathcal{S}^{(3)}_{\text{vec,II}} \right\rangle_{\text{1PI}} = 0$$

for the background configuration, specified in the next section, eq. (5.6).

The expression for $\Delta \Gamma_2$ can be brought to a simpler form,

$$\Delta \Gamma_2 = \frac{1}{26} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^\alpha G(z, z') T_\nu \mathcal{W}^{\alpha} D^\alpha D^2 G(z', z) - T_\mu D^\alpha G(z, z') \mathcal{W}^{\alpha} T_\nu D^2 G(z', z) \right) ,$$

if one notices the following identities

$$0 = \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^\alpha G(z, z') T_\nu \mathcal{W}^{\alpha} D^2 G(z', z) \right) ,$$

$$0 = \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^\alpha G(z, z') \mathcal{W}^{\alpha} T_\nu D^2 G(z', z) \right) + 2 T_\mu D^2 G(z, z') \mathcal{W}^{\alpha} T_\nu D^\alpha D^2 G(z', z) - T_\mu D^2 G(z, z') T_\nu \mathcal{W}^{\alpha} D^\alpha D^2 G(z', z) ,$$

which hold for the background chosen. The proofs of these identities are given in Appendix C.

The contribution (4.10) requires more work to simplify. Let us first note the algebraic symmetry property

$$\left( G(z, z') \right)^T = G(z', z) ,$$

(4.16)
and the differential identities [6]

\[ \overline{\mathcal{D}}^2 G(z, z') = \mathcal{D}^2 G(z, z') , \quad \mathcal{D}^2 G(z, z') = \mathcal{D}^2 G(z, z') . \] (4.17)

Taken together, they imply

\[ \int d^8 z \int d^8 z' G^\mu \nu (z, z') \text{tr}_{\text{Ad}} \left( T_\mu \left[ \overline{\mathcal{D}}^2, \mathcal{D}^2 \right] G(z, z') T_\nu \left( \mathcal{D}^2, \overline{\mathcal{D}}^2 \right) G(z', z) \right) = 0 . \] (4.18)

Therefore, one can replace

\[ \mathcal{D}^2 \overline{\mathcal{D}}^2 \rightarrow \frac{1}{2} \{ \mathcal{D}^2, \overline{\mathcal{D}}^2 \} \]

in the third term on the right of (4.10). The number of spinor covariant derivatives in the first and fourth terms on the right of (4.10) can be reduced by making use of the identity

\[ \frac{1}{16} \mathcal{D}^2 \mathcal{D}^2 \overline{\mathcal{D}}^2 = \Box + \overline{\mathcal{D}}^2 = \mathcal{D}^2 \Box_\nu = \mathcal{D}^2 D_\nu \]

along with the equation

\[ \left( \Box_\nu - \Phi \Phi^\dagger \right) G(z, z') = -4 \delta^8(z - z') \] (4.20)

which the Green’s function under consideration obeys. Those contributions, in which a Green’s function turns into the delta-function \( \delta^8(z - z') \), are no longer ‘fish’ supergraphs; rather, they become ‘eight’ diagrams. Finally, it turns out that the second term on the right of (4.10) is identically zero,

\[ \int d^8 z \int d^8 z' G^\mu \nu(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \left[ \overline{\mathcal{D}}^2, \mathcal{D}^2 \right] G(z, z') T_\nu \mathcal{D}^\alpha \mathcal{D}^2 \mathcal{D}^\alpha G(z', z) \right) = 0 , \] (4.21)

for the background chosen.

The above considerations lead to the following expression for \( \Delta \Gamma_1 \):

\[ \Delta \Gamma_1 = -\frac{1}{26} \int d^8 z \int d^8 z' G^\mu \nu(z, z') \text{tr}_{\text{Ad}} \left( \frac{1}{25} T_\mu \{ \mathcal{D}^2, \overline{\mathcal{D}}^2 \} G(z, z') T_\nu \{ \mathcal{D}^2, \mathcal{D}^2 \} G(z', z) \right) + T_\mu \mathcal{D}^\alpha G(z, z') T_\nu \Phi\Phi^\dagger \mathcal{D}^\alpha \overline{\mathcal{D}}^2 G(z', z) - 2 T_\mu \overline{\mathcal{D}}^2 G(z, z') T_\nu \Phi\Phi^\dagger \mathcal{D}^2 G(z', z) \right) + \frac{1}{26} \int d^8 z \lim_{z' \to z} \overline{\mathcal{D}}^2 \left\{ \mathcal{D}^\alpha G^\mu \nu(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \mathcal{D} \mathcal{D} G(z, z') T_\nu \right) - G^\mu \nu(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \mathcal{D}^2 G(z, z') T_\nu \right) \right\} . \] (4.22)

This may be further simplified by noting the identity

\[ \int d^8 z \int d^8 z' G^\mu \nu(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \mathcal{D}^\alpha G(z, z') T_\nu \Phi\Phi^\dagger \mathcal{D}^\alpha \overline{\mathcal{D}}^2 G(z', z) \right) = \frac{1}{2} \int d^8 z \int d^8 z' G^\mu \nu(z, z') \text{tr}_{\text{Ad}} \left( T_\mu G(z, z') T_\nu \Phi\Phi^\dagger \overline{\mathcal{D}}^2 \mathcal{D}^2 G(z', z) \right) . \] (4.23)
The final expression for $\Delta \Gamma_1$ is:

$$\Delta \Gamma_1 = -\frac{1}{26} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( \frac{1}{25} T_\mu \{ \bar{D}^2, D^2 \} G(z, z') T_\nu \{ \bar{D}^2, D^2 \} G(z', z) \right)$$

$$+ \frac{1}{2} T_\mu G(z, z') T_\nu \Phi \Phi^\dagger D^2 \bar{D}^2 G(z', z) - 2 T_\mu D^2 G(z, z') T_\nu \Phi \Phi^\dagger D^2 G(z', z)$$

$$+ \frac{1}{26} \int d^8 z \lim_{z' \rightarrow z} \bar{D}^2 \left\{ D^\alpha G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D_\alpha G(z, z') T_\nu \right) \right. - G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') T_\nu \right) \right\}. \quad (4.24)$$

4.4 Vector ‘eight’ supergraphs

The quartic interaction (2.26) generates a number of ‘eight’ supergraphs,

$$\left\langle S^{(4)}_{\text{vect}} \right\rangle_{1PI} \equiv \Delta \Gamma_3. \quad (4.25)$$

Their total contribution to the effective action is

$$\Delta \Gamma_3 = \frac{1}{24} \int d^8 z \lim_{z' \rightarrow z} \left\{ D^\alpha G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \bar{D}_\alpha D_\alpha D^\alpha G(z, z') T_\nu \right) \right. - \frac{1}{3} G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') T_\nu \right) \right\}$$

$$- \frac{1}{26} \int d^8 z \lim_{z' \rightarrow z} \bar{D}^2 \left\{ D^\alpha G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D_\alpha G(z, z') T_\nu \right) \right. - G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') T_\nu \right) \right\}. \quad (4.26)$$

The contributions in the third and fourth lines of (4.24) and (4.26) cancel each other.

4.5 Scalar ‘fish’ supergraphs

The cubic interaction (2.23) generates the following fish-type contributions to the effective action

$$\frac{1}{2} \left\langle S^{(3)}_{\text{scal}} S^{(3)}_{\text{scal}} \right\rangle_{1PI} \equiv \Delta \Gamma_4. \quad (4.27)$$

A direct evaluation gives

$$\Delta \Gamma_4 = \frac{1}{26} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( \frac{1}{8} T_\mu D^2 D^2 G(z, z') T_\nu \bar{D}^2 \bar{D}^2 G(z', z) \right)$$

$$+ T_\mu \Phi \bar{D}^2 D^2 G(z, z') T_\nu \Phi^\dagger G(z', z) - 2 T_\mu D^2 D^2 G(z, z') T_\nu \Phi \Phi^\dagger D^2 G(z', z) \Phi \right). \quad (4.28)$$
With the use of (4.18), we can transform $\Delta \Gamma_4$ to the form

$$
\Delta \Gamma_4 = \frac{1}{2^5} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( \frac{1}{2^5} T_\mu \{D^2, D^2\} G(z, z') T_\nu \{D^2, D^2\} G(z', z) \right)
$$

\begin{align*}
&+ \frac{1}{2^5} T_\mu [D^2, D^2] G(z, z') T_\nu [D^2, D^2] G(z', z) \\
&+ T_\mu \Phi D^2 D^2 G(z, z') T_\nu \Phi^\dagger G(z', z) - 2 T_\mu D^2 D^2 G(z, z') T_\nu \Phi^\dagger G(z', z) \Phi \right),
\end{align*}

(4.29)

As can be seen, the contributions in the first lines of (4.24) and (4.29) cancel each other.

### 4.6 Scalar ‘eight’ supergraphs

The two-loop contribution to the effective action from the quartic interaction (2.24) is extremely simple:

$$
\Delta \Gamma_5 \equiv \left\langle S_{\text{scal}}^{(4)} \right\rangle_{1\text{PI}} = \frac{1}{2^5} \int d^8 z \lim_{z' \to z} G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 D^2 G(z, z') T_\nu \right).
$$

(4.30)

### 4.7 Vector-scalar cross ‘fish’ supergraphs

A fish-type contribution to the effective action,

$$
i \left\langle S_{\text{vect}}^{(3)} S_{\text{scal}}^{(3)} \right\rangle_{1\text{PI}} \equiv \Delta \Gamma_6 ,
$$

(4.31)

is generated by both the cubic vector and scalar interactions (2.25) and (2.23). Its direct evaluation leads to

$$
\Delta \Gamma_6 = \frac{1}{2^5} \int d^8 z \int d^8 z' D^\alpha G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') X_\nu D'_\alpha G(z', z) \right)
$$

\begin{align*}
&- \frac{1}{2^5} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') X_\nu D'^2 G(z', z) \right),
\end{align*}

(4.32)

where

$$
X_\nu = \Phi T_\nu \Phi^\dagger - \Phi^\dagger T_\nu \Phi .
$$

(4.33)

It can be shown that the first term in $\Delta \Gamma_6$ vanishes, and thus

$$
\Delta \Gamma_6 = - \frac{1}{2^5} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') X_\nu D'^2 G(z', z) \right).$$

(4.34)
4.8 Ghost ‘fish’ supergraphs

The next step is to collect the two-loop fish-type supergraphs generated by the cubic ghost action (2.29),

\[
\frac{i}{2} \left\langle S_{gh}^{(3)} S_{gh}^{(3)} \right\rangle_{\text{1PI}} \equiv \Delta \Gamma_7 .
\] (4.35)

It is easy to recognize that the second line in (2.29) does not contribute to \( \Delta \Gamma_7 \). With this observation at our disposal, the remaining calculation is rather simple. The results is

\[
\Delta \Gamma_7 = -\frac{1}{24 \pi} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z')
\times \text{tr}_{\text{Ad}} \left( T_\mu [\bar{D}^2, D^2] G(z, z') T_\nu [\bar{D}^2, D^2] G(z', z) \right) .
\] (4.36)

It can be seen that \( \Delta \Gamma_7 \) cancels the contribution in the second line of (4.29).

4.9 Ghost ‘eight’ supergraphs

The two-loop contribution to the effective action from the quartic interaction (2.30) is extremely simple:

\[
\Delta \Gamma_8 \equiv \left\langle S_{gh}^{(4)} \right\rangle_{\text{1PI}} = -\frac{1}{3} \cdot \frac{2}{5} \int d^8 z \lim_{z' \to z} G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \bar{D}^2 D^2 G(z, z') T_\nu \right) .
\] (4.37)

4.10 The pure \( \mathcal{N} = 2 \) super Yang-Mills sector

In this subsection, we would like to present the complete two-loop contribution to the effective action from the pure \( \mathcal{N} = 2 \) super Yang-Mills sector, including the ghosts. Several additional simplifications occur in the case of the background (5.6) we are actually interested in. First of all, it can be seen that the quantum correction (4.34) vanishes for the background (5.6),

\[
\Delta \Gamma_6 = 0 .
\] (4.38)

Second, the Green’s function commutes with \( \Phi, \Phi^\dagger \) and \( \mathcal{W}_\alpha \) (but not with \( \bar{\mathcal{W}}_\dot{\alpha} \)) for the background chosen,

\[
[\Phi, G(z, z')] = [\Phi^\dagger, G(z, z')] = 0 , \quad [\mathcal{W}_\alpha, G(z, z')] = 0 .
\] (4.39)

One can now prove the following identity

\[
\int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \Phi \bar{D}^2 D^2 G(z, z') T_\nu G(z', z) \Phi^\dagger \right)
= \frac{1}{2} \int d^8 z \int d^8 z' G^{\mu \nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu \Phi^\dagger \Phi \bar{D}^2 D^2 G(z, z') T_\nu G(z', z) \right) .
\] (4.40)
This identity implies that the first term in the second line of (4.24) cancels the first term in the third line of (4.29).

The complete two-loop contribution to the effective action from the pure $\mathcal{N} = 2$ super Yang-Mills sector, including the ghosts, is

$$\Gamma_{\text{SYM}} = \Gamma_{\text{IV}} + \Gamma_{\text{V}} + \Gamma_{\text{VI}} + \Gamma_{\text{VII}} + \Gamma_{\text{VIII}} ,$$

(4.41)

where

$$\Gamma_{\text{IV}} = \frac{1}{26} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') T_\nu W^\alpha D'_\alpha D'^2 G(z', z) \right) ,$$

$$\Gamma_{\text{V}} = -\frac{1}{26} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') W'_\alpha T_\nu D'^2 G(z', z) \right) ,$$

$$\Gamma_{\text{VI}} = \frac{1}{25} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') T_\nu \Phi^i \Phi^j D'^2 G(z', z) \right) ,$$

$$\Gamma_{\text{VII}} = -\frac{1}{25} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 G(z, z') T_\nu \Phi^j G(z', z) \Phi^i \right) ,$$

$$\Gamma_{\text{VIII}} = \frac{1}{24} \int d^8z \lim_{z' \to z} D^\alpha G^{\mu\nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D_\alpha D'^\alpha G(z, z') T_\nu \right) .$$

(4.42)

As follows from the relations (4.26), (4.30) and (4.37), all ‘eight’ supergraphs of the form

$$\int d^8z \lim_{z' \to z} G^{\mu\nu}(z, z') \text{tr}_{\text{Ad}} \left( T_\mu D^2 D'^2 G(z, z') T_\nu \right)$$

(4.43)

cancel each other.

5 Specification of the background and related group-theoretical results

At this stage, it is necessary to describe the $SU(N)$ conventions adopted in the paper. Lower-case Latin letters from the middle of the alphabet, $i, j, \ldots$, will be used to denote matrix elements in the fundamental, with the convention $i = 0, 1, \ldots, N - 1 \equiv 0, \bar{I}$. We choose a Cartan-Weyl basis to consist of the elements:

$$H_I = \{ H_0, H_I \} , \quad I = 1, \ldots, N - 2 , \quad E_{ij} , \quad i \neq j .$$

(5.1)

The basis elements in the fundamental representation are defined similarly to [31],

$$(E_{ij})_{kl} = \delta_{ik} \delta_{jl} ,$$

$$(H_I)_{kl} = \frac{1}{\sqrt{(N - I)(N - I - 1)}} \left\{ (N - I) \delta_{kl} \delta_{II} - \sum_{i=I}^{N-1} \delta_{kl} \delta_{ii} \right\} .$$

(5.2)
and are characterized by the properties
\[ \text{tr}(H_I H_J) = \delta_{IJ}, \quad \text{tr}(E_{ij} E_{kl}) = \delta_{il} \delta_{jk}, \quad \text{tr}(H_I E_{kl}) = 0. \] (5.3)

A generic element of the Lie algebra \( su(N) \) is
\[ v = v^I H_I + v^{ij} E_{ij} \equiv v^\mu T_\mu, \quad i \neq j, \] (5.4)

For \( SU(N) \), the operation of trace in the adjoint representation, \( \text{tr}_{\text{Ad}} \), is related to that in the fundamental, \( \text{tr} \), as follows
\[ \text{tr}_{\text{Ad}} v^2 = 2N \text{tr} v^2, \quad v \in su(N). \] (5.5)

The \( \mathcal{N} = 2 \) background vector multiplet is chosen to be
\[ \Phi = \phi H_0, \quad W_\alpha = W_\alpha H_0, \] (5.6)

Its characteristic feature is that it leaves the subgroup \( U(1) \times SU(N-1) \subset SU(N) \) unbroken, where \( U(1) \) is associated with \( H_0 \) and \( SU(N-1) \) is generated by \( \{ H_I, E_{ij} \} \). In what follows, it is assumed that the gauge freedom associated with the broken generators has been used to bring the superfield connection \( \Gamma_A = \Gamma^\mu_A(z)T_\mu \) in (2.3) to the form \( \Gamma_A = \Gamma^0_A(z)H_0 \).

The mass matrix is
\[ |\mathcal{M}|^2 = \bar{\phi} \phi (H_0)^2, \] (5.7)

and therefore a superfield’s mass is determined by its \( U(1) \) charge with respect to \( H_0 \). With the notation
\[ e = \sqrt{N/(N-1)}, \] (5.8)

the \( U(1) \) charges of all quantum superfields are given in the table.

| superfield | \( v^0 \) | \( v^0 \) | \( v^I \) | \( v^{i\bar{j}} \) |
|-----------|--------|--------|------|--------|
| \( U(1) \) charge | \( e \) | \( -e \) | \( 0 \) | \( 0 \) |

Table 1: \( U(1) \) charges of superfields

Among the quantum gauge superfields, eq. (5.4), there are \( 2(N-1) \) massive superfields \( (v^{0i} \text{ and their conjugates } v^{0\bar{i}}) \) coupled to the background, while the remaining \( (N-1)^2 \) superfields \( (v^I \text{ and } v^{i\bar{j}}) \) do not interact with the background and, therefore, are free massless. This follows from the identity
\[ [H_0, E_{ij}] = \sqrt{\frac{N}{N-1}} (\delta_{0i} E_{0j} - \delta_{0j} E_{0i}). \] (5.9)
Since the basis (5.1) is not orthonormal, \( \text{tr}_F(T_\mu T_\nu) = g_{\mu\nu} \neq \delta_{\mu\nu} \), it is now necessary to keep track of the Cartan-Killing metric when working with adjoint vectors. For any elements \( u = u^\mu T_\mu \) and \( v = v^\mu T_\mu \) of the Lie algebra, we have \( u \cdot v = \text{tr}_F(u v) = u^\mu v_\mu \), where \( v_\mu = g_{\mu\nu} u^\nu \) (\( v_I = v^I, v_{ij} = v^{ij} \)).

For the background chosen, the Green’s function \( \mathcal{G} = (G^\mu_\nu, \quad \mathcal{G}^\mu_\nu(z, z') = G^{\mu\lambda}(z, z') g_{\lambda\nu}, \quad \mathcal{G}^{\mu\nu}(z, z') = -i \langle v^\mu(z) v^\nu(z') \rangle \),

is diagonal. Relative to the basis \( T_\mu = (H_I, E_\alpha, E_\beta, E_\gamma) \), this Green’s function has the form

\[
\mathcal{G} = \text{diag}(G^{(0)}_{1_{N-1}}, G^{(e)}_{1_{N-1}}, G^{(-e)}_{1_{N-1}}, G^{(0)}_{1_{(N-1)(N-2)}}).\tag{5.11}
\]

Here \( G^{(e)}(z, z') \) denotes a \( U(1) \) Green’s function of charge \( e \) under the equation

\[
\left( D^a D_a - e W^a D_a + e \tilde{W}_a \bar{D}^a - m^2 \right) G^{(e)}(z, z') = -\delta^8(z - z'), \quad m^2 = e^2 \bar{\phi} \phi, \tag{5.12}
\]

with \( D_A = D_A + ie \Gamma^a_A(z) \) the \( U(1) \) gauge covariant derivatives. Apart from the specific choice of a gauge group, its representation and a mass term, this equation coincides with (2.33). The heat kernel associated with \( G^{(e)}(z, z') \) will be denoted \( K^{(e)}(z, z'|s) \).

It follows from the analysis in section 4 that all the two-loop ‘fish’ supergraphs have the following general form

\[
\Gamma_{\odot} = \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_A \left( T_\mu \mathcal{G}(z, z') T_\nu \mathcal{G}(z', z) \right), \tag{5.13}
\]

where

\[
\mathcal{G}(z, z') = A \mathcal{G}(z, z') A' \equiv \hat{\mathcal{G}}, \quad \mathcal{G}(z', z) = B \mathcal{G}(z, z') B' \equiv \hat{\mathcal{G}}', \tag{5.14}
\]

with \( A, A' \) and \( B, B' \) some diagonal operators, with respect to their \( SU(N) \) indices, of the form

\[
A = \text{diag}(A^{(0)}_{1_{N-1}}, A^{(e)}_{1_{N-1}}, A^{(-e)}_{1_{N-1}}, A^{(0)}_{1_{(N-1)(N-2)}}), \tag{5.15}
\]

and so on. We thus have

\[
\hat{\mathcal{G}} = \text{diag}(\hat{G}^{(0)}_{1_{N-1}}, \hat{G}^{(e)}_{1_{N-1}}, \hat{G}^{(-e)}_{1_{N-1}}, \hat{G}^{(0)}_{1_{(N-1)(N-2)}}), \tag{5.16}
\]

and similarly for \( \hat{\mathcal{G}}' \). In the ‘fish’ supergraphs listed in the previous section, the operators \( A \) and \( B \) are either the covariant derivatives or some of the background superfields \( \Phi, \mathcal{W}_a \) and their conjugates, and products of these.
Using the results of [7], one obtains
\[
\frac{\Gamma_\Theta}{N(N-1)} = 2(N-2) \int d^8 z \int d^8 z' \ G^{(0)}(z, z') \hat{G}^{(0)}(\hat{G}^{(0)}) \\
+ \int d^8 z \int d^8 z' \ G^{(0)}(z, z') \left\{ \hat{G}^{(e)}(\hat{G}^{(e)}) + \hat{G}^{(-e)}(\hat{G}^{(-e)}) \right\} \\
+ \int d^8 z \int d^8 z' \ (G^{(e)}(z, z')) \left\{ \hat{G}^{(e)}(\hat{G}^{(0)}) + \hat{G}^{(0)}(\hat{G}^{(e)}) \right\} \\
+ \hat{G}^{(-e)}(z, z') \left\{ \hat{G}^{(e)}(\hat{G}^{(0)}) + \hat{G}^{(0)}(\hat{G}^{(-e)}) \right\} ,
\]
(5.17)
see Appendix D for the derivation. For the ‘fish’ supergraphs under consideration, the expression in the first line is background-independent, and therefore
\[
\frac{\Gamma_\Theta}{N(N-1)} = \int d^8 z \int d^8 z' \ G^{(0)}(z, z') \left\{ \hat{G}^{(e)}(\hat{G}^{(e)}) + \hat{G}^{(-e)}(\hat{G}^{(-e)}) \right\} \\
+ \int d^8 z \int d^8 z' \ (G^{(e)}(z, z')) \left\{ \hat{G}^{(e)}(\hat{G}^{(0)}) + \hat{G}^{(0)}(\hat{G}^{(e)}) \right\} \\
+ \hat{G}^{(-e)}(z, z') \left\{ \hat{G}^{(e)}(\hat{G}^{(0)}) + \hat{G}^{(0)}(\hat{G}^{(-e)}) \right\} = \Gamma^{(0)} + \Gamma^{(e)} .
\]
(5.18)

The generic contribution to the effective action from an ‘eight’ supergraph is of the form
\[
\Gamma_\infty = \int d^8 z \lim_{z' \to z} \hat{G}^{\mu \nu}(z, z') \ tr_{Ad} \left( T_\mu T_\nu \hat{G}(z, z') \right) , \quad \hat{G}^{\mu \nu} = \hat{G}^{\mu \lambda} g^{\lambda \nu} ,
\]
(5.19)
where \( \hat{G} \) and \( \hat{G} \) are of the type discussed above. Using the group theoretic results obtained in [7] and letting \( G' = G(z, z) \), we get
\[
\Gamma_\infty = 2N(N-1)(N-2) \int d^8 z \hat{G}^{(0)}(\hat{G}^{(0)}) + N(N-1) \int d^8 z \left\{ \hat{G}^{(0)}(\hat{G}^{(e)}(\hat{G}^{(e)}) + \hat{G}^{(-e)}) \right\} \\
+ \hat{G}^{(e)}(\hat{G}^{(0)} + \hat{G}^{(e)}) + \hat{G}^{(-e)}(\hat{G}^{(0)} + \hat{G}^{(-e)}) ,
\]
(5.20)
see Appendix D for the derivation. For the ‘eight’ supergraphs under consideration, either \( \hat{G}^{(0)} = 0 \) or \( \hat{G}^{(0)} = 0 \), and therefore
\[
\Gamma_\infty = N(N-1) \int d^8 z \left\{ \hat{G}^{(0)}(\hat{G}^{(e)}(\hat{G}^{(e)}) + \hat{G}^{(-e)}) \right\} \\
+ \hat{G}^{(e)}(\hat{G}^{(0)} + \hat{G}^{(e)}) + \hat{G}^{(-e)}(\hat{G}^{(0)} + \hat{G}^{(-e)}) .
\]
(5.21)
As follows from (5.18) and (5.21), the two-loop effective action contains a common factor \( N(N-1) \).

6 The hypermultiplet sector

In this section, we focus on evaluating the two-loop contributions to the effective action from the hypermultiplet sector.
6.1 Evaluation of $\Gamma_1$

The hardest quantum correction to compute is

$$\Gamma_1 = \frac{1}{2\pi} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_{Ad}(T_\mu [\slashed{D}^2, \slashed{D}^2]G(z, z') T_\nu [\slashed{D}^2, \slashed{D}^2]G(z', z)) \quad (6.1)$$

Following the notation introduced at the end of the previous section, we now have

$$\hat{G} = [\slashed{D}^2, \slashed{D}^2]G(z, z') \quad \text{and} \quad \hat{G}' = [\slashed{D}^2, \slashed{D}^2]G(z', z) \quad (6.2)$$

and the relevant $U(1)$ components of $\hat{G}$ and $\hat{G}'$ are

$$\hat{G}^{(e)} = [\slashed{D}^2, \slashed{D}^2]G^{(e)}(z, z') \quad \text{and} \quad \hat{G}'^{(e)} = [\slashed{D}^2, \slashed{D}^2]G^{(e)}(z', z) = -[\slashed{D}^2, \slashed{D}^2]G^{(e)}(z, z) = -[\slashed{D}^2, \slashed{D}^2]G^{(-e)}(z, z'). \quad (6.3)$$

In computing $\Gamma_1^{(0)}$ and $\Gamma_1^{(e)}$, a key technical observation is the identity [6]

$$\frac{1}{16} [\slashed{D}^2, \slashed{D}^2]K^{(e)}(z, z'|s) \approx \frac{i}{(4\pi s)^2} \left| \det \left( \sinh(s eF) \right) \right| \frac{2eF}{\rho} \frac{\rho eF \coth(s eF) - 1}{\rho eF \coth(s eF) + 1} ^{\alpha \dot{\alpha}}$$

$$\times \zeta^{(e)}(s) \bar{\zeta}^{(e)}(s) e^{\frac{i}{\rho} eF \coth(s eF)} \rho I(z, z'), \quad (6.4)$$

where we have omitted all terms of at least third order in the Grassmann variables $\zeta_\alpha$, $\bar{\zeta}_{\dot{\alpha}}$ and $W_\alpha$, $\bar{W}_{\dot{\alpha}}$ as they do not contribute to $\Gamma_1$. In particular, it makes the evaluation of one of the two Grassmann integrals trivial, say, the integral over $\theta'$. The term $\Gamma_1^{(0)}$ involves a Grassmann integral of the form

$$\int d^4\theta' \delta^2(\zeta) \delta^2(\bar{\zeta}) \zeta^{(e)}(s) \bar{\zeta}^{(e)}(s) \zeta^{(-e)}(t) \bar{\zeta}^{(-e)}(t) = \left( \zeta^{(e)}(s) \bar{\zeta}^{(e)}(s) \zeta^{(-e)}(t) \bar{\zeta}^{(-e)}(t) \right) \bigg|_{\zeta=\bar{\zeta}=0},$$

where the Grassmann delta-function $\delta^2(\zeta) \delta^2(\bar{\zeta})$ comes from the heat kernel corresponding to $G^{(0)}(z, z')$. The expression obtained is proportional to $W^2 \bar{W}^2$. The $\Gamma_1^{(e)}$ involves a Grassmann integral of the form

$$\int d^4\theta' \delta^2(\zeta^{(e)}(s)) \delta^2(\bar{\zeta}^{(e)}(s)) \zeta^{(e)}(s) \bar{\zeta}^{(e)}(s) \zeta^{(-e)}(t) \bar{\zeta}^{(-e)}(t) = \left( \zeta^{(e)}(s) \bar{\zeta}^{(e)}(s) \zeta^{(-e)}(t) \bar{\zeta}^{(-e)}(t) \right) \bigg|,$$

where the symbol $\big|$ indicates that one should set

$$\zeta^{(e)}(s) = \bar{\zeta}^{(e)}(s) = 0,$$

with $\zeta^{(e)}(s)$ and $\bar{\zeta}^{(e)}(s)$ defined in eq. (B.7).
The next step in evaluating $\Gamma^{(0)}_1$ and $\Gamma^{(e)}_1$ is to do one of the space-time integrals, say, the integral over $x'$. Replacing the bosonic integration variables by the rule $\{x, x'\} \rightarrow \{x, \rho\}$, we end up with a Gaussian integral of the form
\[
\frac{1}{(4\pi)^2} \int d^4 \rho \rho^2 e^{i \rho^2 A/4} = -\frac{8}{A^3},
\] (6.5)
where
\[
A = \frac{1}{u} + \Upsilon(s, t), \quad \Upsilon(s, t) = \frac{eB}{2} \frac{\sin(eB(s + t)/2)}{\sin(s eB/2) \sin(t eB/2)}.
\] (6.6)
Here we have used the explicit structure of the heat kernel, described in Appendix B. The proper-time parameter $u$ in (6.6) corresponds to a free kernel, $K^{(0)}(z, z'|u)$, while the proper-time parameters $s$ and $t$ correspond to charged kernels, $K^{(e)}(z, z'|s)$ and $K^{(-e)}(z, z'|t)$.

Let us briefly describe the evaluation\(^\text{10}\) of $\Gamma^{(0)}_1$. The two terms in
\[
\Gamma^{(0)}_1 = \frac{1}{2^9} \int d^8 z \int d^8 z' G^{(0)}(z, z') \left\{ \hat{G}^{(e)}(z) \hat{G}^{(e)}(z') + \hat{G}^{(-e)}(z) \hat{G}^{(-e)}(z') \right\}
\] (6.7)
turn out to produce the same contribution, therefore
\[
\Gamma^{(0)}_1 = \frac{1}{2^8} \int d^8 z \int d^8 z' G^{(0)}(z, z') \hat{G}^{(e)}(z) \hat{G}^{(e)}(z').
\] (6.8)

Direct calculations lead to
\[
\Gamma^{(0)}_1 = -\frac{4e^4}{(4\pi)^4} \int d^8 z W^2 W^2 \int_0^\infty ds dt du \frac{st}{(stu)^2} \frac{st}{u^{-1} + \Upsilon(s, t)} e^{-ie^2(\phi - i\epsilon)(s+t)} \times \Lambda(s e\hat{B}/2) \Lambda(t e\hat{B}/2),
\] (6.9)
with $\Upsilon(s, t)$ and $\Lambda(s e\hat{B}/2)$ defined in (6.6) and (B.5), respectively. The $u$-integral here is elementary. Making use of the explicit expressions for $\Upsilon(s, t)$ and $\Lambda(s e\hat{B}/2)$, one obtains
\[
\Gamma^{(0)}_1 = -\frac{2e^4}{(4\pi)^4} \int d^8 z W^2 W^2 \int_0^\infty ds \int_0^\infty dt \frac{st (e\hat{B}/2)^2}{\sin^2(eB(s + t)/2)} e^{-ie^2(\phi - i\epsilon)(s+t)}.\] (6.10)

In this paper, we are only interested in the real part of the effective action, and therefore the Wick rotation $s \rightarrow -is$ and $t \rightarrow -it$ can be implemented naively. After re-scaling the proper-time variables, one obtains
\[
\Gamma^{(0)}_1 = \frac{2}{(4\pi)^4} \int d^8 z W^2 W^2 \int_0^\infty ds \int_0^\infty dt \frac{st (\Psi/2e)^2}{\sinh^2(\Psi(s + t)/2e)} e^{-(s+t)}.\] (6.11)

\(^{10}\)The quantum correction $\Gamma^{(0)}_1$ actually coincides with a special sector (denoted by $\Gamma_{I+II}$ in [6]) of the two-loop effective action for $N = 2$ SQED computed in [6].
The double proper-time integral in (6.11) can be reduced to a single integral, by introducing new integration variables, $\alpha$ and $\tau$, defined as follows (see, e.g., [32])

$$s + t = \tau, \quad s - t = \tau \alpha, \quad \tau \in [0, \infty), \quad \alpha \in [-1, 1],$$

(6.12)

such that

$$\int_0^\infty ds \int_0^\infty dt L(s, t) = \frac{1}{2} \int_0^\infty d\tau \int_{-1}^1 d\alpha \ \tau L(s(\alpha, \tau), t(\alpha, \tau)).$$

(6.13)

This leads to

$$\Gamma_{1}^{(0)} = \frac{1}{3(4\pi)^4} \int d^8z \frac{W^2 W'^2}{(\phi \phi)^2}$$

$$+ \frac{1}{3(4\pi)^4} \int d^8z \frac{W^2 W'^2}{(\phi \phi)^2} \int_0^\infty ds \int_0^\infty s^3 \left\{ \frac{(\Psi/2e)^2}{\sinh^2(s \Psi/2e)} - \frac{1}{s^2} \right\} e^{-s}.$$

(6.14)

We now turn to

$$\Gamma_{1}^{(e)} = \frac{1}{2^4} \int d^8z \int d^8z' \left( \mathbf{G}^{(e)}(z, z') \left\{ \hat{\mathbf{G}}^{(-e)} \hat{\mathbf{G}}^{(0)} + \hat{\mathbf{G}}^{(0)} \hat{\mathbf{G}}^{(-e)} \right\} ight.$$

$$+ \mathbf{G}^{(-e)}(z, z') \left\{ \mathbf{G}^{(e)} \mathbf{G}^{(0)} + \mathbf{G}^{(0)} \mathbf{G}^{(-e)} \right\} \right),$$

(6.15)

with $\hat{\mathbf{G}}^{(e)}$ and $\hat{\mathbf{G}}^{(-e)}$ defined in (6.3). It can be shown that the four terms in $\Gamma_{1}^{(e)}$ produce identical contributions, hence

$$\Gamma_{1}^{(e)} = \frac{1}{2^7} \int d^8z \int d^8z' \mathbf{G}^{(e)}(z, z') \hat{\mathbf{G}}^{(-e)} \hat{\mathbf{G}}^{(0)}.$$

(6.16)

Unlike $\Gamma_{1}^{(0)}$, no quantum corrections of this type occur in the case of $\mathcal{N} = 2$ SQED [6]. Therefore, the quantum correction $\Gamma_{1}^{(e)}$ is non-abelian in origin.

The evaluation of $\Gamma_{1}^{(e)}$ follows the steps outlined above. The result is

$$\Gamma_{1}^{(e)} = \frac{4e^4}{(4\pi)^4} \int d^8z W^2 W'^2 \int_0^\infty ds \int_0^\infty dt \frac{s(s + t)(e \bar{B}/2)^2}{\sinh^2(t e \bar{B}/2)} e^{-e^2 \phi(t)}$$

$$+ \frac{4}{(4\pi)^4} \int d^8z \frac{W^2 W'^2}{(\phi \phi)^2} \int_0^\infty dt \frac{(e \bar{B}/2)^2}{\sinh^2(t e \bar{B}/2)} \left\{ \frac{2}{e^2 \phi} + t \right\} e^{-e^2 \phi(t)}.$$

(6.17)

This quantum correction involves a divergent proper-time integral. Let us separate its finite and divergent pieces,

$$\Gamma_{1}^{(e)} = \frac{4}{(4\pi)^4} \int d^8z W^2 W'^2 \int_0^\infty ds \left( \frac{(\Psi/2e)^2}{\sinh^2(s \Psi/2e)} - \frac{1}{s^2} \right) e^{-s}$$

$$+ \frac{4}{(4\pi)^4} \int d^8z \frac{W^2 W'^2}{(\phi \phi)^2} \int_0^\infty dt \left\{ \frac{2}{e^2 \phi} + \frac{1}{t} \right\} e^{-e \phi(t)}.$$

(6.18)
The expression in the first line is finite, while that in the second line contains a divergent proper-time integral. We are going to show that its divergence is cancelled against the divergent part of $\Delta\Gamma_{III}$.

### 6.2 Cancellation of divergences

On the basis of direct calculations, one finds

\[
\frac{\Gamma_{III}}{N(N-1)} = -\frac{2\epsilon^4}{(4\pi)^4} \int d^8z W^2 \bar{W}^2 \int_0^\infty ds s^2 e^{-\epsilon^2 \phi s} \int dt \left\{ \frac{(e\bar{B}/2)^2}{\sinh^2(t e\bar{B}/2)} e^{-\epsilon^2 \phi t} + \frac{1}{t^2} \right\}
\]

\[
= -\frac{4}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds \left\{ \frac{(\Psi/2\epsilon)^2}{\sinh^2(s \Psi/2\epsilon)} - \frac{1}{s^2} \right\} e^{-s}
\]

\[
-\frac{4}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty dt \frac{2}{e^2 \phi \bar{\phi} t^2} \left\{ e^{-\epsilon^2 \phi t} + 1 \right\} . \tag{6.19}
\]

Here the expression in the second line is finite, while the one in the third line contains a divergent proper-time integral. Let us combine this divergent contribution with that in the second line of (6.18):

\[
\int_0^\infty dt \left\{ \frac{2}{e^2 \phi \bar{\phi} t^2} + \frac{1}{t^2} \right\} e^{-\epsilon^2 \phi t} - \int_0^\infty dt \frac{2}{e^2 \phi \bar{\phi} t^2} \left\{ e^{-\epsilon^2 \phi t} + 1 \right\}
\]

\[
= \int_0^\infty dt \frac{d}{dt} \left\{ \frac{1 - e^{-\epsilon^2 \phi t}}{e^2 \phi \bar{\phi} t} \right\} = -1 . \tag{6.20}
\]

This finite contribution will prove in Section 8 to be vital for the absence of $F^4$ quantum corrections at two loops.

### 6.3 Evaluation of $\Gamma_{II}$

The remaining hypermultiplet quantum correction is

\[
\Gamma_{II} = -\frac{1}{24} \int d^8z \int d^8z' G^w(z, z') tr_{Ad} \left( T_\mu \Phi^\dagger D^2 \mathcal{G}(z, z') T_\nu \Phi D^2 \mathcal{G}(z', z) \right) . \tag{6.21}
\]

Following the notation introduced at the end of section 5, we now have

\[
\hat{\mathcal{G}} = \Phi^\dagger D^2 \mathcal{G}(z, z') , \quad \hat{\mathcal{G}}' = \Phi D^2 \mathcal{G}(z', z) , \tag{6.22}
\]

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and the relevant $U(1)$ components of $\hat{G}$ and $\tilde{G}'$ are

$$\dot{G}^{(e)} = e\bar{\phi}D^2G^{(e)}(z, z') \quad \dot{G}'^{(e)} = e\phi D^2G^{(e)}(z', z) = e\phi D^2G^{(-e)}(z, z'),$$

$$\dot{G}^{(0)} = \dot{G}'^{(0)} = 0. \quad \text{(6.23)}$$

We thus have

$$\Gamma^{(e)}_{\Pi} = 0, \quad \Gamma^{(0)}_{\Pi} = -\frac{1}{2\Delta} \int d^8z \int d^8z' G^{\mu\nu}(z, z') \dot{G}^{(e)} \dot{G}'^{(e)}. \quad \text{(6.24)}$$

As compared with the evaluation of $\Gamma^{(0)}_{\Pi}$, the procedure for computing $\Gamma^{(0)}_{\Pi}$ differs in three points. First of all, the quantum correction $\Gamma^{(0)}_{\Pi}$ involves a Grassmann integral of the form

$$\int d^4\theta' \delta^2(\zeta') \delta^2(\zeta(s)) \delta^2(\bar{\zeta}'(t)) = (\delta^2(\zeta(s)) \delta^2(\bar{\zeta}'(t))) \Big|_{\zeta = \bar{\zeta} = 0}.$$ 

Second, instead of the Gaussian integral (6.5), $\Gamma^{(0)}_{\Pi}$ involves the following integral

$$\frac{1}{4\pi^2} \int d^4\rho \rho^2 e^{i\rho^2 A/4} = \frac{i}{A^2}, \quad \text{(6.25)}$$

with $A$ defined in (6.6). Finally, in contrast to the $u$-integral in (6.9), $\Gamma^{(0)}_{\Pi}$ involves the following proper-time integral

$$\int_0^\infty \frac{du}{u^2 [u^{-1} + \Upsilon(s, t)]^2} = \frac{1}{\Upsilon(s, t)}. \quad \text{(6.26)}$$

Direct calculations lead to

$$\Gamma^{(0)}_{\Pi} = \frac{2e^4}{(4\pi)^4} \int d^8z W^2 \bar{W}^2 \int_0^\infty ds \int_0^\infty d\tau \frac{s^2 \phi^2 (eB/2) \sinh(t eB/2)}{\sinh(s eB/2) \sinh(eB(s + t)/2)} e^{-s^2 \phi(s + t)}$$

$$= \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi^2)} \int_0^\infty ds \int_0^\infty d\tau \frac{s^2 (\Psi/2e) \sinh(t \Psi/2e)}{\sinh(s \Psi/2e) \sinh(s + t/2e)} e^{-(s + t)}. \quad \text{(6.27)}$$

Using the identity

$$\frac{\sinh t}{\sinh s \sinh(s + t)} = \coth s - \coth(s + t), \quad \text{(6.28)}$$

we can rewrite $\Gamma^{(0)}_{\Pi}$ as follows

$$\Gamma^{(0)}_{\Pi} = \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi^2)} \int_0^\infty d\tau e^{-t} \int_0^\infty ds s^2 (\Psi/2e) \coth(s \Psi/2e) e^{-s}$$

$$- \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi^2)} \int_0^\infty ds \int_0^\infty d\tau s^2 (\Psi/2e) \coth(s + t/2e) e^{-(s + t)}. \quad \text{(6.29)}$$
In the expression in the first line, the proper-time integrals are factorized, and one of them is elementary. For the expression in the second line, the corresponding double proper-time integral can be reduced to a single integral, by implementing the change of variables (6.12). We end up with

\[
\Gamma_{II}^{(0)} = \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds \left( s^2 - \frac{1}{3} s^3 \right) \left\{ \left( \frac{\Psi}{2e} \right) \coth(s \Psi/2e) - \frac{1}{s} \right\} e^{-s}
\]

\[
+ \frac{2}{3(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \cdot \tag{6.30}
\]

### 6.4 Hypermultiplet effective action

Now, as the quantum corrections \( \Gamma_I, \Gamma_{II} \) and \( \Gamma_{III} \) have been computed, we can put these parts together to obtain the total contribution, \( \Gamma_{\text{hyper}} = \Gamma_I + \Gamma_{II} + \Gamma_{III} \), to the effective action from all the two-loop supergraphs generated by \( S_{\text{hyper}}^{(3)} \) and \( S_{\text{hyper}}^{(4)} \). There still remains, however, one technical point to take care of. Looking at (6.30), we see that \( \Gamma_{II} \) involves the function

\[
\left\{ \left( \frac{\Psi}{2e} \right) \coth(s \Psi/2e) - \frac{1}{s} \right\} , \tag{6.31}
\]

while both \( \Gamma_I \) and \( \Gamma_{III} \) involve a slightly different but related function

\[
\left\{ \frac{(\Psi/2e)^2}{\sinh^2(s \Psi/2e)} - \frac{1}{s^2} \right\} = -\frac{d}{ds} \left\{ \left( \frac{\Psi}{2e} \right) \coth(s \Psi/2e) - \frac{1}{s} \right\} . \tag{6.32}
\]

Using this identity, we can integrate by parts all the proper-time integrals in \( \Gamma_I + \Gamma_{III} \) in order to convert the function (6.32) into (6.31). We thus end up with

\[
\frac{\Gamma_{\text{hyper}}}{N(N-1)} = -\frac{3}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \cdot \tag{6.33}
\]

\[
-\frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds \left( 4s - 3s^2 + s^3 \right) \left\{ \left( \frac{\Psi}{2e} \right) \coth(s \Psi/2e) - \frac{1}{s} \right\} e^{-s} .
\]

In the second term, the proper-time integral involves the cubic polynomial \( P(s) = 4s - 3s^2 + s^3 \) such that \( P(0) = 0 \). The latter property is simply a consequence of the fact that there are no divergences.
7 The super Yang-Mills sector

Evaluation of the supergraphs $\Gamma_{IV}, \Gamma_{V}, \Gamma_{VI}$ and $\Gamma_{VII}$ is very similar to that of $\Gamma_{II}$ described in the previous section. We therefore give only the final results:

$$\Gamma_{IV}^{(0)} = -\frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \int dt \left( s \Psi/2e \right) \coth(s + t) e^{-(s+t)}$$

$$+ \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( s \Psi/2e \right) \coth(s \Psi/2e) e^{-s},$$

$$\Gamma_{IV}^{(e)} = -\frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \int dt \left( s + t \right) \left( \Psi/2e \right) \coth(s + t) e^{-(s+t)}$$

$$+ \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( 1 + s \right) \left( \Psi/2e \right) \coth(s \Psi/2e) e^{-s}. \quad (7.1)$$

$$\Gamma_{V}^{(0)} = \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \int dt \left( s \Psi/2e \right) \coth(s + t) e^{-(s+t)}$$

$$- \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( s \Psi/2e \right) \coth(s \Psi/2e) e^{-s},$$

$$\Gamma_{V}^{(e)} = \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( 1 + s \right) \left( \Psi/2e \right) \coth(s \Psi/2e) e^{-s}. \quad (7.2)$$

$$\Gamma_{VI}^{(0)} = \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \int dt \left( s^2 \Psi/2e \right) \coth(s + t) e^{-(s+t)}$$

$$- \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( s^2 \Psi/2e \right) \coth(s \Psi/2e) e^{-s},$$

$$\Gamma_{VI}^{(e)} = \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \int dt \left( s + t \right) \left( \Psi/2e \right) \coth(s + t) e^{-(s+t)}$$

$$- \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( 1 + s + \frac{1}{2} s^2 \right) \left( \Psi/2e \right) \coth(s \Psi/2e) e^{-s}. \quad (7.3)$$

$$\Gamma_{VII}^{(0)} = -\frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \int dt \left( s^2 \Psi/2e \right) \coth(s + t) e^{-(s+t)}$$

$$+ \frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int ds \left( \Psi/2e \right) \coth(s \Psi/2e) e^{-s},$$
\[ \Gamma^{(e)}_{\text{VIII}} = \frac{4}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds \left(1 + s + \frac{1}{2} s^2\right) (\Psi/2e) \coth(s\Psi/2e) e^{-s}. \] (7.4)

As concerns \( \Gamma_{\text{VIII}} \), this is the only ‘eight’ supergraph in the pure \( \mathcal{N} = 2 \) super Yang-Mills sector. Its direct evaluation leads to the following divergent contribution:

\[ \frac{\Gamma_{\text{VIII}}}{N(N-1)} = -\frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds \frac{\cosh(s\Psi/e)}{\sinh^2(s\Psi/2e)} e^{-s} \]
\[ = -\frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds s \left\{ \frac{(\Psi/2e)^2}{\sinh^2(s\Psi/2e)} - \frac{1}{s^2} \right\} e^{-s} \]
\[ -\frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds \frac{1}{s} e^{-s} - \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} (\Psi/e)^2. \] (7.5)

The divergence is now isolated, and is given by the second term. The first term can be integrated by parts using the identity (6.32). This gives

\[ \frac{\Gamma_{\text{VIII}}}{N(N-1)} = -\frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds (\Psi/2e) \coth(s\Psi/2e) e^{-s} \]
\[ +\frac{2}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds s \left\{ (\Psi/2e) \coth(s\Psi/2e) - \frac{1}{s} \right\} e^{-s} \]
\[ -\frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} (\Psi/e)^2. \] (7.6)

Here the divergence has been absorbed into the first term.

The quantum corrections (7.1) – (7.4) and (7.6) produce a divergent \( F^4 \) contribution contained in

\[ \frac{1}{(4\pi)^4} \int d^8z \frac{W^2 \bar{W}^2}{(\phi \bar{\phi})^2} \int_0^\infty ds (\Psi/2e) \coth(s\Psi/2e) e^{-s}. \] (7.7)

But the total contribution of this functional form is multiplied by

\[ 1 - 1 - 2 - 2 + 2 + 4 - 2 = 0, \]

and therefore no divergence is present.

It only remains to convert the double proper-time integrals in (7.1) – (7.4) into single ones by implementing the change of variables (6.12). Then, the complete two-loop contribution (4.41) to the effective action from the pure \( \mathcal{N} = 2 \) super Yang-Mills sector,
including the ghosts, is

\[
\frac{\Gamma_{\text{SYM}}}{N(N-1)} = \frac{3}{(4\pi)^4} \int d^8 z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} - \frac{1}{(4\pi)^4} \int d^8 z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} (\Psi/e)^2
\]

\[
+ \frac{1}{(4\pi)^4} \int d^8 z \frac{W^2 \bar{W}^2}{(\phi \phi)^2} \int_0^\infty ds \left(4s - s^2 + s^3\right) \left\{ (\Psi/2e) \coth(s\Psi/2e) - \frac{1}{s} \right\} e^{-s}.
\]

(7.8)

8 Conclusion

Under the relaxed super self-duality condition, the two-loop effective action for the $\mathcal{N} = 4$ super Yang-Mills theory is the sum of the two contributions given in eqs. (6.33) and (7.8).

These results lead to the expression (1.13) for $\Omega_{\text{two-loop}}(\Psi^2, 0)$, upon restoring the explicit dependence on the coupling constant $g^2 = 2g^2_{\text{YM}}$.

It should be noted that we have not restricted ourselves to the planar limit, in which $N \to \infty$. The expression (1.13) for $\Omega_{\text{two-loop}}(\Psi^2, 0)$ includes all subleading contributions, and such a result is of interest in itself.

As a consequence of (1.13), there is no two-loop $F^4$ quantum correction for the $\mathcal{N} = 4$ $SU(N)$ theory, in accordance with the conjectures of [8, 7]. This is a nontrivial result. Our analysis above shows that non-vanishing $F^4$ contributions are generated by several two-loop supergraphs. But their total contribution turns out to be zero. In this respect, it is worth pointing out that a non-vanishing two-loop $F^4$ quantum correction does appear in the case of $\mathcal{N} = 2$ $SU(N)$ SYM with $2N$ hypermultiplets in the fundamental [7]. The latter theory is a finite $\mathcal{N} = 2$ supersymmetric field theory, but it possesses no supergravity dual in the framework of the AdS/CFT correspondence.

It follows from the analysis of two-loop supergraphs that, at intermediate stages, there appear contributions to $\Omega_{\text{two-loop}}(\Psi^2, 0)$ of the form

\[
\int_0^\infty ds \sum_{n=0}^3 c_n s^n \left\{ (\Psi/2e) \coth(s\Psi/2e) - \frac{1}{s} \right\} e^{-s},
\]

with $c_n$ numerical coefficients. In the final expression for $\Omega_{\text{two-loop}}(\Psi^2, 0)$, eq. (1.13), it is only the coefficient $c_2$ which does not vanish. It is easy to understand why $c_0 = 0$ — this is equivalent to the cancellation of divergences which are present in some (and only in) $F^4$ quantum corrections. We do not understand, however, why $c_1$ and $c_3$ vanish. In the case of $\mathcal{N} = 2$ super QED, one has $c_1 = 0$ and $c_2 = -3c_3 \neq 0$ at two loops [22].
An unexpected outcome of our analysis is that no $F^6$ quantum correction occurs at two loops, and this is in contradiction with the $\mathcal{N} = 2$ harmonic superspace calculation of [14]. If our calculation is correct\(^\text{11}\) (all the technical steps of the calculation have been checked several times), then there should be a reason why no $F^6$ quantum corrections appear both at one and two loops.\(^\text{12}\) Although we presently have no solid explanations, it is worth speculating about possible reasons.

The $F^6$ term turns out to be rather special from the point of view of extended supersymmetry. In $\mathcal{N} = 3$ harmonic superspace (see [34] for a review), one can construct super-extensions of component structures $F^{4k}$, where $k = 1, 2, \ldots$, in terms of the $\mathcal{N} = 3$ superfield strengths, but no manifestly supersymmetric extension of the $F^6$ term exists [35]. As shown by Ivanov and Zupnik [35], a non-vanishing $F^6$ term is then generated at the component level (provided the $\mathcal{N} = 3$ supersymmetric $F^4$ term is present) only upon elimination of some bispinor auxiliary fields belonging to the off-shell $\mathcal{N} = 3$ vector multiplet. The $\mathcal{N} = 3$ SYM theory is known to be classically equivalent to the $\mathcal{N} = 4$ SYM theory [34], but only three supersymmetries are manifestly realized in the $\mathcal{N} = 3$ harmonic superspace formulation [34]. In the case of $\mathcal{N} = 4$ SYM, when one tries to keep control of an additional supersymmetry, it is clear that the constraints imposed by $\mathcal{N} = 3$ supersymmetry cannot be relaxed, but new constraints may appear.

Independent of the $\mathcal{N} = 3$ harmonic superspace story, it would be interesting to understand whether there exists an $\mathcal{N} = 4$ supersymmetric completion of the $F^6$ term, both in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superspace settings.\(^\text{13}\) Such an analysis seems to be easier in $\mathcal{N} = 2$ superspace, where the $F^6$ term is [15]

\[
\int d^4 x \, d^8 \theta \, \frac{1}{W^2} \ln W \, D^4 \ln W , \tag{8.2}
\]

compare with (1.11). To obtain an $\mathcal{N} = 4$ vector multiplet, we should add to $W$ and $\bar{W}$ a single hypermultiplet described by some constrained $\mathcal{N} = 2$ superfields $Q_i^j$ and $\bar{Q}_i$, where $i = 1, 2$ (or, in the $\mathcal{N} = 2$ harmonic superspace approach [34], by unconstrained analytic superfields $Q^+$ and $\tilde{Q}^+$). Now, the idea is to look for a completion of this functional

\(^{11}\)It is worth pointing out that our work is the first calculation of such complexity in supersymmetric gauge theories.

\(^{12}\)One hint that a non-vanishing $F^6$ term may be expected is the agreement [33] between the $v^6/|X|^{14}$ term in the interaction potential between D0-branes in the supergravity description and the corresponding two-loop term in the effective action of the Matrix model.

\(^{13}\)The low-energy effective action for the $\mathcal{N} = 4$ SYM theory should possess some kind of (deformed) $\mathcal{N} = 4$ on-shell supersymmetry provided one retains all ($\mathcal{N} = 0$, or $\mathcal{N} = 1$, or $\mathcal{N} = 2$) components of the $\mathcal{N} = 4$ vector multiplet.
by hypermultiplet dependent contributions such that the resultant functional is invariant under two additional supersymmetries (a similar problem has recently been solved for the $F^4$ term in [36]). If this proves to be impossible to realize, then indeed the absence of $F^6$ quantum corrections should be natural.

The problem of $\mathcal{N} = 4$ on-shell completion of the $\mathcal{N} = 2$ superconformal $F^6$, $F^8$ and higher order terms [15] has recently been analyzed by Banin and Pletnev [38]. Regarding the $F^6$ term, their conclusion is that its $\mathcal{N} = 4$ on-shell completion is feasible provided (i) a non-vanishing $F^4$ term is present; (ii) $\mathcal{N} = 4$ supersymmetry is deformed (as compared with the supersymmetry transformations of the classical action) in a special manner. So, the fate of the $F^6$ term depends on the explicit structure of the quantum deformation of two hidden supersymmetries. If the quantum theory supports the deformation advocated in [38], then there must be a non-vanishing $F^6$ term in the effective action; otherwise it must vanish. But the issue of quantum supersymmetry deformation in the $\mathcal{N} = 4$ super Yang-Mills theory is still open.

In our opinion, the absence of two-loop $F^6$ contributions to the $\mathcal{N} = 4$ SYM effective action does not mean that the conjectured correspondence [9, 12, 13, 14] between the D3-brane action in $AdS_5 \times S^5$ and the low-energy action for $\mathcal{N} = 4 \text{SU}(N)$ SYM (on its Coulomb branch) is in doubt. Rather, it is an indication that this correspondence is more subtle than previously thought. The vanishing of the two-loop $F^6$ contribution involves the cancellation of two terms, in eq. (1.13), one of which generates the Born-Infeld $F^6$ term. These two contributions have very different origins at the level of supergraphs, and the one which generates the Born-Infeld $F^6$ term (the first term on the right of (1.13)) is not accompanied by higher powers of $F$. This would be consistent with an idea that the superconformal Born-Infeld action (1.8) should correspond only to a sub-sector of the $\mathcal{N} = 4$ SYM effective action in the large $N$, fixed $Ng_{YM}^2 \gg 1$ approximation.

In the large $N$ limit, the expression (1.13) takes the form

$$\Omega_{\text{two-loop}}(\Psi^2, 0) = -\frac{(2g_{YM}^2N)^2}{(4\pi)^4} \left(\Psi^2 - 2\int_0^\infty ds s^2 \left\{\Psi \coth \frac{s\Psi}{2} - \frac{1}{s}\right\} e^{-s}\right) + O(N).$$

It generates quantum corrections of the form $F^{8+2k}$, where $k = 0, 1, \ldots$, proportional to $\lambda^2$, with $\lambda = g_{YM}^2N$. This result is in fact in accord with the conjecture of [14]. As applied to the $F^8$ terms, the conjecture (i) allows sub-leading one- and two-loop contributions.

---

14One of us (SMK) was informed by I. Buchbinder and A. Tseytlin that, together with A. Petrov and O. Solomina, they had computed two-loop $F^8$ corrections for the $\mathcal{N} = 4 \text{SU}(N)$ SYM theory. Their $F^8$ results have the same qualitative structure as ours, but there seem to be quantitative differences.
proportional to $\lambda$ and $\lambda^2$; (ii) requires the dominant contribution, proportional to $\lambda^3$, to appear at three loops.

The $F^6$ puzzle raised in our paper seems to be similar in nature to that uncovered by Stieberger and Taylor [39] who computed the two-loop perturbative $F^6$ interactions in $SO(32)$ heterotic superstring theory and compared them, guided by heterotic – type I string duality [40, 41], with those predicted by the expansion of (any form of) non-Abelian generalization of the Born-Infeld action (see [13] for a review). Quite unexpectedly, even when restricted to the Cartan subalgebra of $SO(32)$, their two-loop action does not agree with the weak field expansion of the Abelian Born-Infeld action. As argued in [39], what makes $F^6$ truly different from $F^4$ is that the corresponding amplitudes are not “BPS-saturated.” This indicates that the $F^6$ terms are sensitive to the full spectrum of superstring theory, and therefore the comparison of dual descriptions may be rather nontrivial.

We believe that this paper sheds some light on the structure of perturbative expansions in the $\mathcal{N} = 4$ super Yang-Mills theory on its Coulomb branch, the issue recently raised by Witten [37]. What is more certain is that our paper raises more questions than answers.

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A Parallel displacement propagator

In this appendix we describe, following [1], the salient properties of the $\mathcal{N} = 1$ parallel displacement propagator $I(z, z')$. This object is uniquely specified by the following requirements:

(i) the gauge transformation law

$$ I(z, z') \rightarrow e^{i\tau(z)} I(z, z') e^{-i\tau(z')} \quad \text{(A.1)} $$

with respect to an arbitrary gauge ($\tau$-frame) transformation of the covariant derivatives

$$ D_A \rightarrow e^{i\tau(z)} D_A e^{-i\tau(z)} , \quad \tau^\dagger = \tau , \quad \text{(A.2)} $$

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with the gauge parameter \( \tau(z) \) being arbitrary modulo the reality condition imposed;

(ii) the equation

\[
\zeta^A D_A I(z, z') = \zeta^A \left( D_A + i \Gamma_A(z) \right) I(z, z') = 0 ;
\]

(A.3)

(iii) the boundary condition

\[
I(z, z) = 1 .
\]

(A.4)

These imply the important relation

\[
I(z, z') I(z', z) = 1 ,
\]

(A.5)

as well as

\[
\zeta^A D' A I(z, z') = \zeta^A \left( D_A' I(z, z') - i I(z, z') \Gamma_A(z') \right) = 0 .
\]

(A.6)

Under Hermitian conjugation, the parallel displacement propagator transforms as

\[
\left( I(z, z') \right)^\dagger = I(z', z) .
\]

(A.7)

For a covariantly constant vector multiplet,

\[
D_a W_\beta = 0 ,
\]

(A.8)

the covariant differentiation of \( D_A I(z, z') \) gives [1]

\[
D_{\beta\bar{\beta}} I(z, z') = I(z, z') \left( -\frac{i}{4} \rho^{\alpha\alpha} F_{a\alpha, \beta\bar{\beta}}(z') - i \zeta_\bar{\beta} \bar{W}_\beta(z') + i \bar{\zeta}_\beta W_\beta(z') \\
+ \frac{2i}{3} \bar{\zeta}_\beta \zeta^a D_a W_\beta(z') + \frac{2i}{3} \zeta_\beta \bar{\zeta}^a D_a W_\beta(z') \right)
\]

\[
= \left( -i \frac{1}{4} \rho^{\alpha\alpha} F_{a\alpha, \beta\bar{\beta}}(z) - i \zeta_\bar{\beta} \bar{W}_\beta(z) + i \bar{\zeta}_\beta W_\beta(z) \\
- \frac{1}{3} \zeta_\beta \bar{\zeta}^\alpha D_a W_\beta(z) - \frac{1}{3} \zeta_\beta \bar{\zeta} \bar{W}_\beta(z) \right) I(z, z') ;
\]

(A.9)

\[
D_\beta I(z, z') = I(z, z') \left( \frac{1}{12} \bar{\zeta}^\beta \rho^{\alpha\alpha} F_{a\alpha, \beta\bar{\beta}}(z') - i \rho_{\beta\bar{\beta}} \left\{ \frac{1}{2} \bar{W}_\beta(z') - \frac{1}{3} \bar{\zeta}^\alpha D_a \bar{W}_\beta(z') \right\} \\
+ \frac{1}{3} \zeta_\beta \bar{\zeta}_\bar{\beta} \bar{W}_\beta(z') + \frac{1}{3} \zeta^2 \left\{ W_\beta(z') + \frac{1}{2} \zeta^\alpha D_a W_\beta(z') - \frac{1}{4} \zeta_\beta \bar{\zeta} \bar{W}_\beta(z') \right\} \right)
\]

\[
= \left( \frac{1}{12} \bar{\zeta}^\beta \rho^{\alpha\alpha} F_{a\alpha, \beta\bar{\beta}}(z) - \frac{1}{2} \rho_{\beta\bar{\beta}} \left\{ \bar{W}_\beta(z) + \frac{1}{3} \bar{\zeta}^\alpha D_a \bar{W}_\beta(z) \right\} \right. \\
+ \frac{1}{3} \zeta^2 \left\{ W_\beta(z) - \frac{1}{2} \zeta^\alpha D_a W_\beta(z) + \frac{1}{4} \zeta_\beta \bar{\zeta} \bar{W}_\beta(z) \right\} \right) I(z, z') ;
\]

(A.10)

\[
D_{\bar{\beta}} I(z, z') = I(z, z') \left( -\frac{i}{12} \zeta_\beta \rho^{\alpha\alpha} F_{a\alpha, \beta\bar{\beta}}(z') - i \rho_{\beta\bar{\beta}} \left\{ \bar{W}_\beta(z') - \frac{1}{3} \zeta^\alpha D_a \bar{W}_\beta(z') \right\} \\
- \frac{1}{3} \bar{\zeta}_\beta \zeta^\beta W_\beta(z') - \frac{1}{3} \zeta^2 \left\{ W_\beta(z') - \frac{1}{2} \zeta^\alpha D_a W_\beta(z') + \frac{1}{4} \zeta_\beta \bar{\zeta} \bar{W}_\beta(z') \right\} \right)
\]

\[
= \left( -\frac{1}{12} \zeta_\beta \rho^{\alpha\alpha} F_{a\alpha, \beta\bar{\beta}}(z) - \frac{1}{2} \rho_{\beta\bar{\beta}} \left\{ W_\beta(z) - \frac{1}{3} \zeta^\alpha D_a W_\beta(z) \right\} - \frac{1}{3} \bar{\zeta}_\beta \zeta^\beta W_\beta(z) \\
- \frac{1}{3} \zeta^2 \left\{ W_\beta(z) + \frac{1}{2} \zeta^\alpha D_a W_\beta(z) - \frac{1}{4} \zeta_\beta \bar{\zeta} \bar{W}_\beta(z) \right\} \right) I(z, z') .
\]

(A.11)
\section*{B \hspace{1em} U(1) heat kernel in a self-dual background}

In this appendix, we describe simplifications in the structure of the $U(1)$ heat kernel

\begin{equation}
K^{(e)}(z, z'|s) = -\frac{i}{(4\pi s)^{2}} \sqrt{\text{det} \left( \frac{s eF}{\sinh(s eF)} \right)} \delta^{2}(\zeta^{(e)}(s)) \delta^{2}(\bar{\zeta}^{(e)}(s)) \\
\times U(s) \ e^{\frac{ie}{2} \epsilon \gamma eF \coth(s eF) \rho} \ I(z, z').
\end{equation}

with

\begin{align}
\zeta^{(e)\alpha}(s) &= U(s) \zeta^{\alpha} U(-s), \quad \bar{\zeta}^{(e)\dot{\alpha}}(s) = U(s) \bar{\zeta}^{\dot{\alpha}} U(-s), \\
U(s) &= \exp \left\{ -is e(W^{\alpha}D_{\alpha} + \bar{W}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}}) \right\},
\end{align}

which occur when the background vector multiplet satisfies the relaxed self-duality condition

\begin{equation}
W_{\alpha} \neq 0, \quad D_{\alpha}W_{\beta} = 0, \quad \bar{D}_{(\dot{\alpha}} W_{\dot{\beta})} \neq 0.
\end{equation}

With the notation

\begin{equation}
\bar{N}_{\dot{\alpha} \dot{\beta}} = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\beta}}, \quad \bar{B}^{2} = \frac{1}{2} \text{tr} \bar{N}^{2} = \frac{1}{4} \bar{D}^{2} \bar{W}^{2},
\end{equation}

one obtains

\begin{equation}
\sqrt{\text{det} \left( \frac{s eF}{\sinh(s eF)} \right)} = \left( \frac{s e\bar{B}/2}{\sin(s e\bar{B}/2)} \right)^{2} \equiv \Lambda(s e\bar{B}/2).
\end{equation}

Since the field strength is self-dual, the matrix $eF \coth(s eF)$, which appears in the exponential in (B.1), becomes a multiple of the unit matrix,

\begin{equation}
eF \coth(s eF) = \frac{e\bar{B}}{2} \cot(s e\bar{B}/2)1_{4}.
\end{equation}

One also obtains

\begin{align}
\zeta^{(e)\alpha}(s) &= \zeta^{\alpha} - is eW^{\alpha}, \quad \bar{\zeta}^{(e)\dot{\alpha}}(s) = \bar{\zeta}^{\dot{\alpha}} - \left( \bar{W} \frac{e^{-iseN}}{N} - 1 \right)^{\dot{\alpha}}.
\end{align}

\section*{C \hspace{1em} Proof of some identities I}

In this appendix, we illustrate the types of manipulations which can be performed on the functional expressions for supergraphs by proving the identities (4.14) and (4.15). These
manipulations have also been used extensively in section 4 to bring the expressions for individual supergraphs into a relatively standard form.

Let us denote by $I$ the right hand side of (4.14),

$$I = \int d^8z \int d^8z' G^{\mu\nu}(z, z') \text{tr}_\text{Ad} \left(T_\mu D^\alpha G(z, z') T_\nu W_\alpha' D^2D^2 G(z', z)\right).$$

One can use the identity (4.4) to replace $D^\alpha G(z, z')$ by $-D^{\alpha} G(z, z')$. Integrating by parts with respect to the derivative $D^\alpha$, one arrives at

$$I = \int d^8z \int d^8z' (D^\alpha G^{\mu\nu}(z, z')) \text{tr}_\text{Ad} \left(T_\mu G(z, z') T_\nu W_\alpha' D^2D^2 G(z', z)\right)$$

$$= \int d^8z \int d^8z' (D^\mu G^{\nu\gamma}(z, z')) (T_\mu)^{\rho \sigma} G^{\sigma\gamma}(z, z') (T_\gamma)^{\delta \rho} (W_\alpha' D^2D^2 G(z', z))_{\rho \nu}. \quad (C.1)$$

where the matrices $(T_\mu)^\lambda_{\nu}$ in the adjoint representation are defined by

$$[T_\mu, T_\nu] = T_\lambda (T_\mu)^\lambda_{\nu}, \quad (C.2)$$

with $T_m$ the generators of $SU(N)$ in any representation. This can be re-arranged as

$$I = -\int d^8z \int d^8z' G^{\sigma\gamma}(T_\sigma)^{\rho \mu} D^{\alpha} G^{\mu\nu}(z, z') (T_\gamma)^{\nu \delta} (W_\rho' D^2D^2 G(z', z))_{\rho \nu}$$

$$= \int d^8z \int d^8z' G^{\sigma\gamma}(z, z') \text{tr}_\text{Ad} \left(T_\sigma D^\alpha G(z, z') T_\gamma W_\rho' D^2D^2 G(z', z)\right). \quad (C.3)$$

Using (4.4) again, the right hand side of (C.3) is equivalent to $-I$, proving that $I$ vanishes.

To prove (4.15), one begins with the identity

$$0 = \int d^8z \int d^8z' D^\alpha \left(G^{\mu\nu}(z, z') \text{tr}_\text{Ad} \left(T_\mu D^\beta G(z, z') W_\beta' T_\nu D_\alpha' D^2G(z', z)\right)\right)$$

$$= \int d^8z \int d^8z' \left\{ -(D^\mu G^{\nu\gamma}(z, z')) \text{tr}_\text{Ad} \left(T_\mu D^\beta G(z, z') W_\beta' T_\nu D_\alpha' D^2G(z', z)\right) \right.$$

$$+ \frac{1}{2} G^{\mu\nu}(z, z') \text{tr}_\text{Ad} \left(T_\mu D^2 G(z, z') W_\beta' T_\nu D_\alpha' D^2G(z', z)\right)$$

$$\left. + G^{\mu\nu}(z, z') \text{tr}_\text{Ad} \left(T_\mu D^\beta G(z, z') W_\beta' T_\nu D^2D^2G(z', z)\right)\right\}. \quad (C.4)$$

The first term on the right hand side of (C.4) is

$$-\int d^8z \int d^8z' (D^\alpha G^{\mu\nu}(z, z')) (T_\mu)^{\omega \mu} D^\beta G(z, z') (W_\beta')_{\sigma \tau} (T_\nu)^{\rho \sigma} D_\alpha' D^2G(z', z)_{\rho \omega}.$$ 

Using the commutation relations and the symmetry properties of the generators in the adjoint representation, this can be re-expressed in the form

$$-\int d^8z \int d^8z' (D^\beta G^{\mu\nu}(z, z')) \text{tr}_\text{Ad} \left(T_\mu D^\alpha G(z, z') [T_\nu, W_\beta'] D_\alpha' D^2G(z', z)\right).$$

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Integrating by parts with respect to the derivative \( \mathcal{D}^{\alpha} \) yields the expression

\[
\int d^8 z \int d^8 z' G^{\mu\nu}(z, z') \text{tr}_A \left( -\frac{1}{2} T_{\mu} \mathcal{D}^2 G(z, z') [T_{\nu}, \mathcal{W}^{\alpha}] \mathcal{D}_{\alpha} \mathcal{D}^2 G(z', z) \right) 
+ T_{\mu} \mathcal{D}^{\alpha} G(z, z') [T_{\nu}, \mathcal{W}^{\alpha}] \mathcal{D}_{\alpha} \mathcal{D}^2 G(z', z).
\]  

(C.5)

Substituting this expression for the first term in (C.4) back in, and using the identity (4.14), yields the required result (4.15).

Other identities given in section 4 can be proved by similar means.

**D Proof of some identities II**

In this appendix, details of the group-theoretical manipulations leading to the expressions (5.17) and (5.20) are presented. They make use of the following identities for the generators of \( SU(N) \) in the adjoint representation:

\[
\sum_{I} \sum_{I=\perp} (H_I)^{0}_{\dot{0}} (H_I)^{0}_{0} = 2(N - 1),
\]  

(D.1)

\[
\sum_{I \neq I=\perp} (E_{Ij})^{0}_{0} (E_{Ij})^{0}_{0} = (N - 1)(N - 2),
\]  

(D.2)

\[
\sum_{I \neq I=\perp} (E_{0j})^{0}_{0} (E_{0j})^{0}_{0} = 2(N - 1),
\]  

(D.3)

\[
\sum_{I \neq I=\perp} (E_{0j})^{kl}_{0} (E_{0j})^{0}_{0} = (N - 1)(N - 2),
\]  

(D.4)

\[
\sum_{I} \sum_{I=\perp} (H_I)^{0}_{0} (H_I)^{0}_{0} = 2(N - 1)(N - 2),
\]  

(D.5)

\[
\sum_{I \neq I=\perp} \sum_{I=\perp} (E_{Ij})^{kl}_{0} (E_{Ij})^{0}_{0} = 2(N - 1)(N - 2)(N - 3),
\]  

(D.6)

\[
\sum_{I=\perp} (E_{0j} E_{0j})^{0}_{0} = N(N - 1),
\]  

(D.7)

\[
\sum_{I=\perp} (E_{0j} E_{0j})^{0}_{0} = 0,
\]  

(D.8)

\[
\sum_{I} \sum_{I=\perp} (H_I H_I)^{0}_{0} = 2(N - 1),
\]  

(D.9)

\[
\sum_{I} \sum_{I=\perp} (H_I H_I)^{0}_{0} = 2(N - 1)(N - 2),
\]  

(D.10)

\[
\sum_{I \neq I=\perp} \sum_{I=\perp} (E_{Ij} E_{Ij})^{kl}_{0} = \sum_{I \neq I=\perp} \sum_{I=\perp} (E_{Ij} E_{Ij})^{0}_{0} = (N - 1)(N - 2).
\]  

(D.11)
The above identities are easily proved by using the definition

\[ [T_\mu, T_\nu] = T_\rho (T_\mu)_{\rho \nu} \]

of the adjoint representation and evaluating the commutators in the fundamental representation, which is detailed in section 5.

Using (5.10) and (5.11), the expression (5.13) for the generic ‘fish’ supergraph can be decomposed as

\[
\Gamma_{\Phi} = \int d^8 z \int d^8 z' \left( G^{(e)}(z, z') \sum_I \text{tr}_{Ad} (E_{0I} \hat{G}(z, z') E_{0']} \hat{G}(z', z) ) \\
+ G^{(-e)}(z, z') \sum_I \text{tr}_{Ad} (E_{0I} \hat{G}(z, z') E_{0'} \hat{G}(z', z) ) \\
+ G^{(0)}(z, z') \sum_I \text{tr}_{Ad} (H_I \hat{G}(z, z') H_I \hat{G}(z', z) ) \\
+ G^{(0)}(z, z') \sum_{I \neq \tilde{I}} \text{tr}_{Ad} (E_{0I} \hat{G}(z, z') E_{0'} \hat{G}(z', z) ) \right)
\equiv \Gamma_{\Phi}^{(1)} + \Gamma_{\Phi}^{(2)} + \Gamma_{\Phi}^{(3)} + \Gamma_{\Phi}^{(4)}.
\]  

(D.12)

The calculation of \( \Gamma_{\Phi}^{(1)} \) will be outlined. The remaining terms in \( \Gamma_{\Phi} \) follow in a similar manner. Due to the presence of \( G^{(e)} \) in \( \Gamma_{\Phi}^{(1)} \), the considerations of \( U(1) \) charge conservation at each vertex dictate that either \( \hat{G} \) has charge \(-e\) and \( \hat{G} \) has charge 0, or \( \hat{G} \) has charge 0 and \( \hat{G} \) has charge \( e \). As a result,

\[
\Gamma_{\Phi}^{(1)} = \int d^8 z \int d^8 z' G^{(e)}(z, z') \left( \sum_I \sum_{k \neq \tilde{k}} (E_{0I}^I)^0 (E_{0I}^0) \hat{G}^{(-e)}(z, z') (E_{0I}^0) \hat{G}^{(0)}(z', z) \\
+ \sum_{k \neq \tilde{k}} (E_{0I}^I)^0 (E_{0I}^0) \hat{G}^{(-e)}(z, z') (E_{0I}^0) \hat{G}^{(0)}(z', z) \\
+ \sum_{I \neq \tilde{I}} \sum_{j \neq \tilde{j}} (E_{0I}^I)^0 \hat{G}^{(0)}(z, z') (E_{0I}^0) \hat{G}^{(0)}(z', z) \\
+ \sum_{I \neq \tilde{I}} \sum_{j \neq \tilde{j}} (E_{0I}^I)^0 \hat{G}^{(0)}(z, z') (E_{0I}^0) \hat{G}^{(0)}(z', z) \right).
\]  

(D.13)

Making use of the identities (D.3) and (D.4), this yields

\[
\Gamma_{\Phi}^{(1)} = N(N-1) \int d^8 z \int d^8 z' G^{(e)}(z, z') \\
\times \left( \hat{G}^{(-e)}(z, z') \hat{G}^{(0)}(z', z) + \hat{G}^{(0)}(z, z') \hat{G}^{(e)}(z', z) \right).
\]  

(D.14)

In a similar manner, \( \Gamma_{\Phi}^{(2)}, \Gamma_{\Phi}^{(3)} \) and \( \Gamma_{\Phi}^{(4)} \) can be computed using the above identities:

\[
\Gamma_{\Phi}^{(2)} = N(N-1) \int d^8 z \int d^8 z' G^{(-e)}(z, z')
\]
\[ \times \left( \tilde{\mathcal{G}}(e)(z, z') \; \tilde{\mathcal{G}}(0)(z', z) + \tilde{\mathcal{G}}(0)(z, z') \; \tilde{\mathcal{G}}(-e)(z', z) \right), \]

\[ \Gamma^{(3)}_\varnothing = 2(N - 1) \int d^8z \int d^8z' \mathcal{G}(0)(z, z') \left( \tilde{\mathcal{G}}(e)(z, z') \; \tilde{\mathcal{G}}(e)(z', z) + \tilde{\mathcal{G}}(-e)(z, z') \; \tilde{\mathcal{G}}(-e)(z', z) \right) \]

\[ + \tilde{\mathcal{G}}(0)(z, z') \; \mathcal{G}(0)(z', z) + (N - 2) \tilde{\mathcal{G}}(0)(z, z') \; \tilde{\mathcal{G}}(0)(z', z) \]

\[ \Gamma^{(4)}_\varnothing = (N - 1)(N - 2) \int d^8z \int d^8z' \mathcal{G}(0)(z, z') \left( \tilde{\mathcal{G}}(e)(z, z') \; \tilde{\mathcal{G}}(e)(z', z) + \tilde{\mathcal{G}}(-e)(z, z') \; \tilde{\mathcal{G}}(-e)(z', z) + 2(N - 1) \tilde{\mathcal{G}}(0)(z, z') \; \tilde{\mathcal{G}}(0)(z', z) \right) \] (D.15)

Together with \( \Gamma^{(1)}_\varnothing \), these yield the final result (5.17) for \( \Gamma_\varnothing \).

We now turn to the derivation of the result (5.20) for the generic ‘eight’ supergraph. Using (5.16), the expression (5.19) for these supergraphs can be decomposed as

\[ \Gamma_\infty = \int d^8z \lim_{z' \to z} \left( \tilde{\mathcal{G}}(e)(z, z') \sum_i \text{tr}_\text{Ad}(E_{0i} E_{0j} \tilde{\mathcal{G}}(z, z')) \right. \]

\[ + \tilde{\mathcal{G}}(-e)(z, z') \sum_i \text{tr}_\text{Ad}(E_{0j} E_{0i} \tilde{\mathcal{G}}(z, z')) \]

\[ + \tilde{\mathcal{G}}(0)(z, z') \sum_l \text{tr}_\text{Ad}(H_l H_l \tilde{\mathcal{G}}(z, z')) \]

\[ + \tilde{\mathcal{G}}(0)(z, z') \sum_{i \neq j} \text{tr}_\text{Ad}(E_{ij} E_{ij} \tilde{\mathcal{G}}(z, z')) \]

\[ \equiv \Gamma^{(1)}_\infty + \Gamma^{(2)}_\infty + \Gamma^{(3)}_\infty + \Gamma^{(4)}_\infty. \] (D.16)

Again, we detail only the calculation of \( \Gamma^{(1)}_\infty \). The remaining terms in \( \Gamma_\infty \) follow by similar manipulations. Expanding out the trace,

\[ \Gamma^{(1)}_\infty = \int d^8z \lim_{z' \to z} \tilde{\mathcal{G}}(e)(z, z') \left( \sum_{i \neq j} E_{0i} E_{0j} \tilde{\mathcal{G}}(e)(z, z') \right. \]

\[ \left. + \sum_l E_{0i} E_{0j} H_l H_l \tilde{\mathcal{G}}(0)(z, z') \right) \] (D.17)

This can be rearranged in the form

\[ \Gamma^{(1)}_\infty = \int d^8z \lim_{z' \to z} \tilde{\mathcal{G}}(e)(z, z') \left( \sum_i \text{tr}_\text{Ad}(E_{0i} E_{0j}) \tilde{\mathcal{G}}(0)(z, z') \right. \]

\[ + \sum_{i \neq j} (E_{0i} E_{0j}) \tilde{\mathcal{G}}(e)(z, z') \]

\[ + \sum_{i \neq j} (E_{0i} E_{0j}) \tilde{\mathcal{G}}(-e)(z, z') \] (D.18)

Using \( \text{tr}_\text{Ad}(E_{ij} E_{kl}) = 2N \delta_{ij} \delta_{jk} \), the relation \( \lim_{z' \to z} \tilde{\mathcal{G}}(e)(z, z') = \lim_{z' \to z} \tilde{\mathcal{G}}(-e)(z, z') \), and the identities (D.7) and (D.8), one obtains

\[ \Gamma^{(1)}_\infty = N(N - 1) \int d^8z \lim_{z' \to z} \tilde{\mathcal{G}}(e)(z, z') \left( \tilde{\mathcal{G}}(e)(z, z') + \tilde{\mathcal{G}}(0)(z, z') \right). \] (D.19)
In a similar manner, $\Gamma^{(2)}_\infty$, $\Gamma^{(3)}_\infty$ and $\Gamma^{(4)}_\infty$ can be computed:

\[
\Gamma^{(2)}_\infty = N(N-1) \int d^8z \lim_{z' \to z} \hat{\mathcal{G}}^{(-e)}(z, z') \left( \hat{\mathcal{G}}^{(-e)}(z, z') + \hat{\mathcal{G}}^{(0)}(z, z') \right), \quad (D.20)
\]

\[
\Gamma^{(3)}_\infty = 2(N-1) \int d^8z \lim_{z' \to z} \hat{\mathcal{G}}^{(0)}(z, z') \\
\times \left( \hat{\mathcal{G}}^{(e)}(z, z') + \hat{\mathcal{G}}^{(-e)}(z, z') + (N-2) \hat{\mathcal{G}}^{(0)}(z, z') \right), \quad (D.21)
\]

\[
\Gamma^{(4)}_\infty = (N-1)(N-2) \int d^8z \lim_{z' \to z} \hat{\mathcal{G}}^{(0)}(z, z') \\
\times \left( \hat{\mathcal{G}}^{(e)}(z, z') + \hat{\mathcal{G}}^{(-e)}(z, z') + 2(N-1) \hat{\mathcal{G}}^{(0)}(z, z') \right). \quad (D.22)
\]

Combining these results with that for $\Gamma^{(1)}_\infty$ yields the required expression (5.20) for $\Gamma_\infty$.

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