Response to: Comment on "Orbital precession of the S2 star in scalar-tensor-vector gravity"

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Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT
The explicit derivation for the orbital precession of the S2 star in the Galactic Center in the Scalar–Tensor–Vector Gravity is discussed and compared with previous research. The two different predictions are validated by numerically integrating the geodesic equations for a test particle.

Key words: gravitation – stars, black hole – stars, kinematics and dynamics – Galaxy, centre – dark matter

Motivated by the recent advancement in the astrometric measurements of the proper motion of individual stars belonging to the S-stars cluster around the supermassive black hole (SMBH) Sagittarius A* (SgrA*) in the Galactic Center, in our original work, Della Monica et al. (2022), we computed stellar orbits around a SMBH in the Scalar-Tensor-Vector Gravity (STVG), an extension to the theory of General Relativity (GR) first proposed in Moffat (2006). Making use of the pre-pericenter data for the S2 star, recorded over more than two decades up to 2016 (Gillissen et al. 2017), and of the more recent detection of its post-Newtonian orbital precession (GRAVITY Collaboration et al. 2020), for the first time we placed constraints on the deviations of STVG from GR on the very small scales (100–1000 AU) of the Galactic Center (Della Monica et al. 2022). More recently, Turimov (2022) discussed the calculations for the periastron precession of a test particle around a SMBH in STVG, that apparently contradict the relation for the orbital precession given in Eq. (16) of Della Monica et al. (2022). In this Response, we show how an erroneous assumption made in Turimov (2022) leads to the seemingly contradictory results and how the two works can be reconciled.

The STVG is a generally covariant alternative theory of gravity based on a modification of the Hilbert-Einstein action of GR. The theory introduces, in addition to the metric tensor field $g_{\mu\nu}$, a Proca-type massive vector field $\phi^\mu$ and elevates Newton’s gravitational constant $G$ and the mass $m$ of the vector field to dynamical scalar fields that allow for an effective description of the variation of these “constants” with space and time. The STVG field equations, obtained by minimization of the action, can be solved in vacuum assuming spherical symmetry. Further assumptions include that the proportionality constant of the gravitational coupling $G$ is constant, $\partial_\nu G = 0$, and has an enhanced (w.r.t. GR) value $G = G_N (1 + \alpha)$ depending on a free dimensionless parameter $\alpha$, and that the mass of the vector field $\phi, \bar{\mu}$, can be neglected when solving field equations for compact objects as black holes (BHs), because its effects manifest on kpc scales from the source. These assumptions lead to the expression of the space-time metric (Moffat & Toth 2015)

$$ds^2 = \frac{\Delta}{r^2} dt^2 - \frac{\Delta}{\Delta} dr^2 - r^2 d\Omega^2,$$ (1)

where, assuming $c = 1$ (the speed of light in vacuum),

$$\Delta = r^2 - 2GM + aG_N GM^2,$$ (2)

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$ (3)

which describes exactly the space-time around a point-like source with mass $M$. One of the key features of STVG is that the geodesic equations possess a non-null right-hand side

$$\left(\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda}\right) = \frac{\mu}{m} \frac{\partial^\mu}{\partial \lambda} \frac{dx^\lambda}{d\lambda},$$ (4)

due to the interaction of massive particles with the vector field $\phi^\mu$. As a matter of fact, $q$ acts as a coupling constant between the point-like particles of mass $m$ and the vector field $\phi^\mu$, resulting in an extra Lorentz-type force, usually called fifth force, which depends on the velocity of the particle. The constant $q$ is called the fifth-force charge of the particle and its sign is postulated to be positive (implying a repulsive fifth force) in order to describe physically stable stars, galaxies, galaxy clusters, and agreement with solar system observational data (Brownstein & Moffat 2006, 2007; Moffat & Toth 2009; Moffat & Rahvar 2013, 2014; Moffat 2015; Moffat & Toth 2015; De Martino & De Laurentis 2017; de Martino 2020) and its value is assumed to be proportional to the mass of the particle itself, $q = \kappa m$, so that the weak equivalence principle can be recovered. The proportionality constant $\kappa$ is defined by

$$\kappa = \sqrt{\alpha G_N},$$ (5)

so that when $\alpha$ vanishes, both the metric and the geodesic equations in STVG reduce to their GR counterpart.

Under the same spherical symmetry assumptions, the dynamical
equations for the vector field $\phi^\mu$ reads

\[
\nabla_\mu B^{\nu\mu} = 0, \\
\nabla_\sigma B^{\mu\nu} + \nabla_\mu B_{\nu\sigma} + \nabla_\nu B_{\sigma\mu} = 0,
\]

where $\nabla_\mu$ is the covariant derivative operator related to the metric tensor and $B^{\mu\nu} = \nabla_\mu \phi_\nu - \nabla_\nu \phi_\mu$, can be solved, resulting in (e.g., Lopez Armengol & Romero (2017))

\[
\phi_\mu = \left(-\frac{\sqrt{a}G_NM}{r},0,0,0\right). 
\]

The vector field $\phi_\mu$ behaves like a purely electrical radial field generated by a point source located at the BH position, whose charge is $Q = \sqrt{a}G_NM$ and whose action on massive test particles is a repulsive force counteracting the enhanced gravitational constant $G$. Due to the presence of this vector field in the geodesic equations, the description of the motion of test particles in terms of conserved quantities has to be modified accordingly. While the Lagrangian function per unit mass of the test particle can be defined in the usual way from

\[
2L = g_{\mu\nu}x^\mu x^\nu = \begin{cases} 
0 & \text{(mass-less particle)} \\
n & \text{(massive test particle)} 
\end{cases}
\]

one has to define generalized momenta, $\pi_\mu$, to incorporate the presence of the vector field. Due to the particular form of $\phi_\mu$ in Eq. (8), only the time component of the 4-momentum is affected by the vector field (Misner et al. 2017, see pages 898-900):

\[
\pi_t = \frac{\partial L}{\partial \dot{t}} - \frac{q}{m} \phi_t, \\
\pi_r = \frac{\partial L}{\partial \dot{r}}, \\
\pi_\theta = \frac{\partial L}{\partial \dot{\theta}}, \\
\pi_\phi = \frac{\partial L}{\partial \dot{\phi}}.
\]

Due to the symmetry of the Lagrangian with respect to the coordinates $r$ and $\phi$, the two quantities

\[
E = \pi_t = \frac{\Delta}{r^2} t + \frac{MG_N}{r} \alpha, \\
L = -\pi_\phi = r^2 \sin^2 \theta \dot{\phi}
\]

are constants of motion and can be regarded as the specific energy and specific angular momentum of the particle at infinity (the term specific refers to the fact that these quantities have to be regarded as energy and angular momentum per unit mass, i.e. for $m = 1$). Moreover, from

\[
\pi_\theta = -r^2 \dot{\theta}, \\
\frac{\partial L}{\partial \theta} = -r^2 \dot{\theta} \sin \theta \cos \theta,
\]

one can see that upon setting initial conditions so that the test particle initially lies on the equatorial plane $\theta = \pi/2$ with $\dot{\theta} = 0$, the entire motion will take place on this plane and thus $\theta(r) = \pi/2$ identically.

We can now use equations Eq. (14) and Eq. (15) to express $\phi$ and $i$

\[
\Delta \phi_{\text{STVG}} = \Delta \phi_{\text{GR}} \left(1 + \frac{5}{6} \alpha\right).
\]

In order to double-check our results, we have computed a numerical integration of the geodesic equations (4) for a time-like geodesic
(whose orbital data correspond to those of the S2 star) for the case of a repulsive vector field $\phi$. For the sake of completeness, we have repeated the numerical computations also for the case of an attractive $\phi$, as erroneously considered in Turimov (2022). The orbital precession is quantified from the numerically integrated orbits as the difference from $2\pi$ of the angle spanned during one orbital period (i.e. between two radial turning points). For this reason, in Figure 1 we report the radial velocity profile of the S2 star as a function of the angular coordinate $\phi$. The plot illustrates both the $\alpha = 0$ (dashed orange line) case and the $\alpha = 1$ (solid red line) one, during one orbital period. Points in which $t = 0$ correspond to radial turning points, i.e. those points along the orbit where either a maximum or a minimum radial distance from the central source are reached (the apocentre and pericentre, respectively). As shown in the inset figure, which zooms into the apocentre passage, the GR orbit has gained a periastron shift of 12.2' after one orbital period, which is compatible with values reported in literature (Gillessen et al. 2017; GRAVITY Collaboration et al. 2020). The STVG orbit (with the correctly assumed repulsive vector field $\phi$), on the other hand, is characterised by a greater value of the orbital precession of 22.3' per orbital period is observed. This is greater than the Schwarzschild precession in GR by a factor 5/6, which is in perfect agreement with our analytical expression (19) for $\alpha = 1$. In Figure 2, on the other hand, we report the ratio to the GR precession of the numerically computed orbital precession of S2 for both the correct repulsive case for $\phi$ (black circles) and the erroneous attractive field (violet squares), as compared to the values predicted by the analytical formula Eq. (19), originally presented in Della Monica et al. (2022), and the one reported in Turimov (2022) for values of $\delta = 1$ and $\delta = 2$ (solid blue lines). In the range $\alpha \in [0, 0.5]$, the numerically computed values for the precession in the repulsive case agree with the prediction from the analytical expression. Moreover, the numerically computed precession in the attractive case agrees with the $\delta = 2$ profile from Turimov (2022).

The two results can be reconciled upon considering the correct signs in the energy definition in Turimov (2022) which, on turn, would imply a different definition for the extra parameter $\delta$ thereby introduced. In this new definition, when particularized for the S2 star, one would find $\delta \ll 1$ (see Appendix A) thus reducing the formula for the orbital precession found in Turimov (2022) to our originally proposed expression.

**ACKNOWLEDGEMENTS**

RDM acknowledges support from Consejería de Educación de la Junta de Castilla y León and from the Fondo Social Europeo. IDM acknowledges support from Ayuda JCI2018-036198-I funded by MCIN/AEI/10.13039/501100011033 and: FSE “El FSE invierte en tu futuro” o “Financiado por la Unión Europea “NextGenerationEU”/PRTR. IDM is also supported by the project PID2021-122938NB-I00 funded by the Spanish “Ministerio de Ciencia e Innovación” and FEDER “A way of making Europe”, and by the project SA096P20 Junta de Castilla y León.

**DATA AVAILABILITY STATEMENT**

No new data were generated or analysed in support of this research.

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**APPENDIX A: DETAILED CALCULATIONS OF THE ORBITAL PRECESSION**

In order to compute a first-order expression for the orbital precession for the S2 star in STVG (considering a repulsive vector field $\phi$), we apply the usual textbook calculations starting from the dynamical equation regulating the polar motion, $r(\phi)$, in the equatorial plane for a massive test particle, already given in Eq. (18) of the main text,
where for simplicity of notation we adopted \( G = c = 1 \),

\[
\left( \frac{dr}{d\phi} \right)^2 = \frac{r^4}{L^2} \left( E - \frac{Ma}{r} \right)^2 - \Delta(r) \left( 1 + \frac{r^2}{L^2} \right).
\]  
(A1)

Let’s make the variable change \( \xi = L^2/Mr \), so that the equation becomes

\[
\left( \frac{L^2}{M\xi^2} \right)^2 \frac{d\xi}{d\phi} = \frac{L^6}{M^4\xi^4} \left( E - \frac{M^2\alpha}{L^2} \xi \right) - \Delta(r(\xi)) \left( 1 + \frac{L^2}{M^2\xi^2} \right).
\]  
(A2)

If we now define the parameter \( \sigma = (M/L)^2 \), the latter equation can be rewritten as

\[
\frac{d\xi}{d\phi} = \frac{1}{\sigma} \left( E - \alpha \sigma \xi \right)^2 + (1 - 2\sigma(1 + \alpha)\xi + \sigma^2(1 + \alpha)\xi^2) \left( \xi^2 + \frac{1}{\sigma} \right)
\]  
(A3)

where we have used the definition of \( \Delta \) given in Eq. (2). The latter equation can be differentiated with respect to \( \xi \) resulting in

\[
\xi'' + \xi(1 + \sigma\alpha) = 1 + 3\sigma(1 + \alpha)\xi - 2\alpha(1 + \alpha)\xi^2 - 2\sigma(1 + \alpha)\xi^2.
\]  
(A4)

Where primes stand for derivatives with respect to the angle \( \phi \). The non-homogeneous term \( \delta = 1 - E \) appearing in this equation can be neglected for the purpose of this computation. In fact, if we restore all the dimensional constants and particularize it for the orbit of S2 (e.g. at pericenter), we obtain

\[
\delta = 1 - \frac{\Delta(r)}{r^2} \approx \frac{GNMa}{c^2r} \approx 2.10 \times 10^{-5} \quad \text{(for } \alpha = 0 \text{)}
\]  
(A5)

\[
\approx 2.13 \times 10^{-5} \quad \text{(for } \alpha = 2 \text{)}
\]  
(A6)

Neglecting the contribution of \( \delta \), Eq. (A4) becomes

\[
\xi'' + \xi(1 + \sigma\alpha) = 1 + 3\sigma(1 + \alpha)\xi^2 - 2\alpha(1 + \alpha)\sigma^2.
\]  
(A7)

By applying a perturbative expansion of \( \xi \) using \( \sigma \) as a perturbation parameter (this is possible because, generally, \( \sigma \ll 1 \), e.g. for the S2 star \( \sigma = 1.88 \times 10^{-4} \)), we can express the solution as

\[
\xi = \xi_0 + \sigma\xi_1 + O(\sigma^2).
\]  
(A8)

If we substitute this solution in Eq. (A7) stopping at first order, we get

\[
\xi''_0 + 
\xi_0 = 1,
\]  
(A9)

\[
\xi''_1 + \xi_0 = -3(1 + \alpha)\xi_1.
\]  
(A10)

for the zero-th and first order, respectively. The zero-th order solution is the well known keplerian orbit, given by

\[
\xi_0(\phi) = 1 + e \cos \phi
\]  
(A11)

where \( e \) is the orbital eccentricity. Plugging this solution into the first order equation and neglecting all the subdominant contributions (i.e. terms of higher order in the eccentricity), we obtain:

\[
\xi''_1 + \xi_1 = 3 + 2\alpha + (5\alpha + 6)e \cos \phi.
\]  
(A12)

Due to the presence of the cosine at the right-hand side we look for solutions of the form \( \xi_1(\phi) = A + B\phi \sin \phi \). Upon inserting this solution in Eq. (A12) one easily finds:

\[
A = 3 + 2\alpha; \quad B = \frac{(5\alpha + 6)e}{2}.
\]  
(A13)

The full solution thus reads

\[
\xi(\phi) = 1 + e \cos \phi + (3 + 2\alpha)\sigma + \frac{(5\alpha + 6)e}{2} \sigma \phi \sin \phi
\]  
(A14)

Using the perturbative expansion of the cosine function, i.e.

\[
\cos(\phi + \beta \phi) \approx \cos(\phi) - \beta \phi \sin \phi + O(\beta^2)
\]  
(A15)

with \( \beta \ll 1 \), one can rewrite the solution in the form

\[
\xi(\phi) \approx 1 + (3 + 2\alpha)\sigma + \cos \phi \left( 1 - 3\sigma \left( 1 + \frac{5}{6} \right) \right).
\]  
(A16)

During one orbital period, thus, the orbit preceeds by an angle

\[
\Delta\phi_{\text{STVG}} = 6\pi\sigma \left( 1 + \frac{5}{6} \right)
\]  
(A17)

and, remembering that the orbital precession predicted in GR is given by \( \Delta\phi_{\text{GR}} = 6\pi\sigma = 6\pi G_N M/a(1 - e^2)c^2 \) (where \( a \) is the semi-major axis of the orbit), we finally obtain the departure of the orbital precession in STVG from the one predicted by GR, given by

\[
\Delta\phi_{\text{STVG}} = \Delta\phi_{\text{GR}} \left( 1 + \frac{5}{6} \right).
\]  
(A18)