$N_T = 8, D = 2$ Hodge–type cohomological gauge theory
with global $SU(4)$ symmetry

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Abstract

We show that the partially topological twisted $N = 16, D = 2$ super Yang–Mills theory gives rise to a $N_T = 8$ Hodge–type cohomological gauge theory with global $SU(4)$ symmetry.

1. Introduction

Some very enlightening, but preliminary attempts have been made to incorporate into the gauge–fixing procedure of general gauge theories besides the basic ingredient of BRST cohomology $\Omega$ also a co–BRST cohomology $\star \Omega$ which, together with the BRST Laplacian $W$, form the same kind of superalgebra as the de Rham cohomology operators in differential geometry (for a review, see, e.g., Ref. [1]). This would allow, according to the Hodge–type decomposition $\psi = \omega + \Omega \chi + \star \Omega \phi$ of a general quantum state, by imposing both the BRST condition $\Omega \psi = 0$ and the co–BRST condition $\star \Omega \psi = 0$ upon $\psi$, to select the uniquely determined harmonic state $\omega$ thereby projecting onto the subspace of physical states.

It has been a long–standing problem to present a non–abelian field theoretical model obeying such a Hodge–type cohomological structure. Recently, the authors have shown [2] that the dimensional reduced Blau–Thompson model [3] — the novel $N_T = 2$ topological twist of the $N = 4, D = 3$ super Yang–Mills theory (SYM) — gives a prototype example of a $N_T = 4, D = 2$ Hodge–type cohomological gauge theory. The conjecture, that topological gauge theories could be possible candidates for Hodge–type cohomological theories was already asserted by van Holten [4]. In fact, $D = 2$ topological gauge theories [5] are of particular interest because of their relation to $N = 2$ superconformal theories [6] and Calabi–Yau moduli spaces [7].

Here we present another example of a Hodge–type cohomological gauge theory. It is obtained by a $N_T = 8$ topological twist of the Euclidean $N = 16, D = 2$ SYM, and its action localizes onto the moduli space of complexified flat connections. The $N_T = 8$ scalar supercharges $Q^a$ and $\star Q^a$ of that theory form a topological superalgebra which is completely analogous to the de Rham cohomology. Both supercharges are interrelated by a discrete Hodge–type $\star$ operation and generate the topological shift and co–shift symmetries. In accordance with the group theoretical description of some classes of topologically twisted low–dimensional supersymmetric world–volume theories [8], it is shown that this $N_T = 8$ cohomological theory has actually the global symmetry group $SU(4)$. Such effective low–energy world–volume theories appear quite naturally in the study of curved D–branes and D–brane instantons wrapping around supersymmetric cocycles for special Lagrangian submanifolds of Calabi–Yau $n$–folds (see, e.g., [6, 8, 3]).

The paper is organized as follows: In Sec. 2 we briefly describe the BRST complex of general gauge theories based on harmonic gauges. In Sec. 3 we obtain the Euclidean $N = 16, D = 2$
SYM theory with R–symmetry group $SO(8)$ from the $N = 4$, $D = 4$ SYM via dimensional reduction to $D = 2$. In Sec. 4 we perform the partial $N_T = 8$ topological twist of this SYM theory thereby getting the looked for $N_T = 8$ Hodge–type cohomological theory with global symmetry group $SU(4)$. A more detailed version will be presented elsewhere [10].

2. BRST complex and Hodge decomposition

In order to select uniquely the physical states from the ghost–extended quantum state space some attempts [1] have been made to incorporate into the gauge–fixing procedure of general gauge theories besides the BRST cohomology $\Omega$ also a co–BRST cohomology $\star \Omega$ which, together with the BRST Laplacian $W$, obeys the following BRST–complex:

\[
\begin{align*}
\Omega^2 &= 0, & \star \Omega^2 &= 0, & [\Omega, W] &= 0, & [\star \Omega, W] &= 0, & W &= \{\Omega, \star \Omega\} \neq 0,
\end{align*}
\]

where $\Omega$ and $\star \Omega$ have ghost number +1 and −1, respectively. Obviously, $\star \Omega$ can not be identified with the anti–BRST operator $\bar{\Omega}$ which anticommutes with $\Omega$.

Representations of this algebra for the first time have been considered by Nishijima [11]. However, since $\Omega$ and $\star \Omega$ are nilpotent hermitian operators they cannot be realized in a Hilbert space. Instead, the BRST complex has to be represented in a Krin space $K$ [12]. $K$ is obtained from a Hilbert space $H$ with non–degenerate positive inner product $(\chi, \psi)$ if $H$ will be endowed also with a self–adjoint metric operator $J \neq 1$, $J^2 = 1$, allowing for the introduction of another non–degenerate, but indefinite scalar product $\langle \chi | \psi \rangle := (\chi, J \psi)$. With respect to the inner product $\Omega$ and $\star \Omega = \pm J \Omega J$ are adjoint to each other, $(\chi, \star \Omega \psi) = (\Omega \chi, \psi)$, however they are self–adjoint with respect to the indefinite scalar product of $K$. Notice, that different inner products $(\chi, \psi)$ lead to different co–BRST operators!

From these definitions one obtains a remarkable correspondence between the BRST cohomology and the de Rham cohomology:

| BRST operator | $\Omega$, \quad \text{differential} | $d$, |
| co–BRST operator | $\star \Omega = \pm J \Omega J$, \quad \text{co–differential} | $\delta = \pm \star d \star$, |
| duality operation | $J$, \quad \text{Hodge star} | $\star$, |
| BRST Laplacian | $W = \{\Omega, \star \Omega\}$, \quad \text{Laplacian} | $\Delta = \{d, \delta\}$.

Because of this correspondence one denotes a state $\psi$ to be BRST (co–)closed iff $\Omega \psi = 0$ ($\star \Omega \psi = 0$), BRST (co–)exact iff $\psi = \Omega \chi$ ($\psi = \star \Omega \phi$) and BRST harmonic iff $W \psi = 0$. Completely analogous to the Hodge decomposition theorem in differential geometry there exists a corresponding decomposition of any state $\psi$ into a harmonic, an exact and a co–exact state, $\psi = \omega + \Omega \chi + \star \Omega \phi$. The physical properties of $\psi$ lie entirely within the BRST harmonic part $\omega$ which is given by the zero modes of the operator $W$; thereby $W \omega = 0$ implies $\Omega \omega = 0 = \star \Omega \omega$, and vice versa. The cohomologies of the (co–)BRST operator are given by equivalence classes:

\[
\begin{align*}
H(\Omega) &= \frac{\text{Ker} \Omega}{\text{Im} \Omega}, & \psi \sim \psi' &= \psi + \Omega \chi \quad (\text{equivalence class}), \\
H(\star \Omega) &= \frac{\text{Ker} \star \Omega}{\text{Im} \star \Omega}, & \psi \sim \psi' &= \psi + \star \Omega \phi \quad (\text{equivalence class}).
\end{align*}
\]

By imposing only the BRST gauge condition, $\Omega \psi = 0$, within the equivalence class of BRST–closed states $\psi = \omega + \Omega \chi$ besides the harmonic state $\omega$ there occur also spurious BRST–exact states, $\Omega \chi$, which have zero physical norm. On the other hand, by imposing also the co–BRST gauge condition, $\star \Omega \psi = 0$, one gets for each BRST cohomology class the uniquely determined harmonic state, $\psi = \omega$. 
3. Dimensional reduction of the $N=4$, $D=4$ super Yang–Mills theory

Our final aim is to show that by a partial topological twist of $N=16$, $D=2$ SYM one gets a $N_T=8$ Hodge–type cohomological theory with global symmetry group $SU(4)$. However, since the relationship between the twisted and untwisted fields is rather complex, let us first introduce the $N=16$, $D=2$ SYM. This theory can be obtained by dimensional reduction to $D=2$ from either $N=1$, $D=10$ SYM or $N=4$, $D=4$ SYM. Because the latter theory is well known, we choose the last possibility.

The field content of $N=4$, $D=4$ SYM consists of an anti–hermitean gauge field $A_\mu$, two Majorana spinors $\lambda_{A\alpha}$ and $\bar{\lambda}_A^\alpha$ ($\alpha=1,2,3,4$) which transform as the fundamental and its complex conjugate representation of $SU(4)$, respectively, and a set of complex scalar fields $G_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} G^{\gamma\delta}$, which transform as the second–rank complex selfdual representation of $SU(4)$. All the fields take their values in the Lie algebra $Lie(G)$ of some compact gauge group $G$.

In Euclidean space this theory has the following invariant action $[13]$

$$S^{(N=4)} = \int_E d^4x \text{tr} \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \bar{\lambda}_A^\alpha (\sigma_\mu)^{AB} D^\mu \lambda_{B\alpha} + \frac{1}{64} [G_{\alpha\beta}, G_{\gamma\delta}][G^{\alpha\beta}, G^{\gamma\delta}] \right. $$

$$\left. - \frac{1}{2} i \lambda_{A\alpha}[G^{\alpha\beta}, \lambda^A_{\beta}] - \frac{1}{2} i \bar{\lambda}_A^\alpha [G_{\alpha\beta}, \bar{\lambda}^\beta_A] + \frac{1}{8} D_\mu G_{\alpha\beta} D^\mu G^{\alpha\beta} \right\}, \quad (1)$$

where the numerically invariant tensors $(\sigma_\mu)^{AB}$ and $(\sigma_\mu)_{AB}$ are the Clebsch–Gordan coefficients relating the representation $(1/2,1/2)$ of $SL(2,C)$ to the the vector representation of $SO(4)$,

$$(\sigma_\mu)^{AB} = (-i \sigma_1, -i \sigma_2, -i \sigma_3, I_2), \quad (\sigma_\mu)_{AB} \equiv (\sigma_\mu)^{CD} \epsilon_{C\bar{A}}\epsilon_{D\bar{B}} = (\sigma^*_\mu)^{\bar{A}\bar{B}}, \quad (\sigma_\mu)^{\bar{A}\bar{B}} = (i \sigma_1, i \sigma_2, i \sigma_3, I_2), \quad (\sigma_\mu)_{\bar{A}\bar{B}} \equiv \epsilon^{AC} \epsilon^{\bar{B}\bar{D}} (\sigma_\mu)_{C\bar{D}} = (\sigma^*_\mu)^{\bar{A}\bar{B}}, \quad (2)$$

$(\sigma_\mu)_{AB}$ and $(\sigma_\mu)^{AB}$ being the corresponding complex conjugate coefficients, respectively. Here, $\sigma_a$ ($a=1,2,3)$ are the Pauli matrices. The selfdual and anti–selfdual generators of the $SO(4)$ rotations, $(\sigma_{\mu\nu})_{AB}$ and $(\sigma_{\mu\nu})_{\bar{A}\bar{B}}$, obey the relations

$$(\sigma_{\mu\nu})^{AC}_{\bar{B}} (\sigma_{\nu})^B_C = (\sigma_{\mu\nu})^{AB} - \delta_{\mu\nu} \sigma^{AB},$$

$$(\sigma_{\mu\nu})_{\bar{A}\bar{B}} (\sigma_{\nu})_{\bar{C}\bar{D}} = \delta_{\mu\nu} (\sigma_{\mu\nu})^{AB} - \delta_{\mu\nu} (\sigma_{\mu\nu})_{\bar{A}\bar{B}} - \epsilon_{\mu\nu\rho\sigma} (\sigma^{\rho\sigma})_{\bar{A}\bar{B}}, \quad (3)$$

The spinor index $A$ (and analogously $\bar{A}$) is raised and lowered as follows: $\epsilon^{AC} \varphi^{\bar{B}}_C = \varphi^{AB}$ and $\varphi^C_A \epsilon_{CB} = \varphi_{AB}$, where $\epsilon_{AB}$ (and analogous $\epsilon_{\bar{A}\bar{B}}$) is the invariant tensor of the group $SU(2)$, $\epsilon_{12} = \epsilon_{12}^* = \epsilon_{12} = 1$.

The action $[\mathbf{13}]$ is manifestly invariant under hermitean conjugation:

$$ (A_\mu, \lambda_{A\alpha}, \bar{\lambda}_A^\alpha, G^{\alpha\beta}) \to (A_\mu, \bar{\lambda}_A^\alpha, \lambda^A_{\alpha}, G_{\alpha\beta}).$$

Furthermore, making use of $[\mathbf{13}]$ and $[\mathbf{14}]$, one verifies that $[\mathbf{13}]$ is invariant also under the following on–shell supersymmetry transformations,

$$ Q^A_{\alpha} A_\mu = -i (\sigma_\mu)_{\bar{A}\bar{B}} \bar{\lambda}^{\bar{B}\alpha},$$

$$ Q^A_{\bar{A}} \bar{\lambda}^{\bar{B}\beta} = (\sigma^\mu)_{\bar{A}\bar{B}} D_\mu G^{\alpha\beta},$$

$$ Q^A_{\alpha} G_{\beta\gamma} = 2 i (\delta^\alpha_{\beta} \lambda_{A\gamma} - \delta^\alpha_{\gamma} \lambda_{A\beta}),$$

$$ Q^A_{\alpha} \lambda_{B\beta} = -\frac{1}{2} \delta^\alpha_{\beta} (\sigma^\mu_{AB}) F_{\mu\nu} - \frac{1}{2} \epsilon_{AB}[G^{\alpha\gamma}, G_{\gamma\beta}].$$
and

\[ Q_{\dot{A}\alpha} A_\mu = i(\sigma_\mu)^{\dot{A}B}_\alpha, \]

\[ Q_{\dot{A}\alpha} \lambda_{B\beta} = (\sigma^{\mu\nu})^{\dot{A}B}_\beta D_\mu G_{\alpha\beta}, \]

\[ Q_{\dot{A}\alpha} G^{\beta\gamma} = 2i(\delta_\alpha^{\beta\gamma} \lambda^{\dot{A}} - \delta_\alpha^{\gamma\dot{B}} \lambda^{\dot{A}}), \]

\[ Q_{\dot{A}\alpha} \lambda^{\dot{B}}_B = -\frac{1}{2} \delta_\alpha^{\beta\gamma} (\sigma^{\mu\nu})^{\dot{A}B}_\delta F_{\mu\nu} + \frac{1}{2} \epsilon_{\dot{A}B}[G_{\alpha\gamma}, G^{\gamma\beta}]. \]

Let us recall that it is not possible to complete this superalgebra off–shell with a finite number of auxiliary fields \[ \text{[14]} \].

In order to perform in \[ \text{[14]} \] the dimensional reduction to \( D = 2 \) we re–name the third and fourth component of \( A_\mu \) according to

\[ A_3 = \frac{1}{2} (\phi + \bar{\phi}), \quad A_4 = \frac{1}{2} i(\phi - \bar{\phi}), \quad (5) \]

reserving the notation \( A_\mu \) (\( \mu = 1, 2 \)) for the gauge field in \( D = 2 \). Moreover, we decompose the components of \( (\sigma_\mu)^B_A \), \((\sigma_{\mu\nu})^B_A\) and \((\sigma_{\mu\nu})_A^B\) in the following manner,

\[ (\sigma_\mu)^B_A \rightarrow i(\sigma_\mu)^B_A, \quad (\sigma_{\mu\nu})^B_A \rightarrow i(\sigma_{\mu\nu})^B_A, \quad (\sigma_{\mu\nu})_A^B \rightarrow \delta_A^B, \]

\[ (\sigma_{\mu\nu})^{\dot{A}}_B \rightarrow -i(\sigma_{\mu\nu})^{\dot{A}}_B, \quad (\sigma_{\mu\nu})^{\dot{A}}_A \rightarrow i(\sigma_{\mu\nu})^{\dot{A}}_A, \quad (\sigma_{\mu\nu})_{\dot{A}}^B \rightarrow \delta_{\dot{A}}^B, \]

\[ (\sigma_{\mu\nu})_{\dot{A}}^B \rightarrow -i(\sigma_{\mu\nu})_{\dot{A}}^B, \quad (\sigma_{\mu\nu})_{\dot{A}}^B \rightarrow i(\sigma_{\mu\nu})_{\dot{A}}^B, \quad (\sigma_{\mu\nu})_{\dot{A}}^B \rightarrow -i(\sigma_{\mu\nu})_{\dot{A}}^B, \quad (\sigma_{\mu\nu})_{\dot{A}}^B \rightarrow i(\sigma_{\mu\nu})_{\dot{A}}^B, \]

such that both the relations \[ \text{[13]} \] and \[ \text{[14]} \] become the algebra of the Pauli matrices,

\[ (\sigma_\mu)^C_A (\sigma_\nu)_{CB} = \delta_{\mu
u} \epsilon_{AB} + i \epsilon_{\mu\nu} (\sigma_3)_{AB}, \quad (\sigma_\mu, \sigma_3)_{AB} = (\sigma_1, \sigma_2, \sigma_3), \quad (\sigma_{\mu\nu})^C_A (\sigma_{\mu\nu})_{CB} = \epsilon_{AB}, \]

Then, from \[ \text{[14]} \] we obtain the Euclidean action of the \( N = 16, D = 2 \) SYM

\[ S^{(N=16)} = \int_E d^2 x \text{tr} \left\{ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} D_\mu \bar{\phi} D^\mu \phi - \frac{1}{8} [\bar{\phi}, \phi]^2 \right. \]

\[ - \frac{1}{2} \bar{\lambda}^{\alpha}_A (\sigma^\nu)^{AB}_\alpha [\phi + \bar{\phi}, \lambda_{BA}] + \frac{1}{2} \bar{\lambda}^{\alpha}_A [\phi - \bar{\phi}, \lambda_{A\alpha}] \]

\[ + \bar{\lambda}^{\alpha}_A (\sigma^\nu)^{AB}_\alpha D_\mu \lambda_{BA} + \frac{1}{2} i \lambda_{A\alpha} (G^\alpha^\beta, \lambda^\beta_A) - \frac{1}{2} i \bar{\lambda}^{\alpha}_A [G_{\alpha\beta}, \bar{\lambda}^\beta_A] \]

\[ + \frac{1}{8} D_\mu G_{\alpha\beta} D^\mu G^{\alpha\beta} + \frac{1}{16} [\bar{\phi}, G_{\alpha\beta}] [\phi, G^{\alpha\beta}] + \frac{1}{32} [G_{\alpha\beta}, G_{\gamma\delta}] [G^{\alpha\beta}, G^{\gamma\delta}] \right\}. \quad (7) \]

Since the decompositions \[ \text{[13]} \] explicitly include various factors of \( i \), the action \[ \text{[7]} \] is no longer manifestly invariant under hermitean conjugation. Rather, it is invariant under the following \( Z_2 \) symmetry,

\[ Z_2 : \quad (A_\mu, \phi, \bar{\phi}, \lambda_{A\alpha}, \bar{\lambda}^{A\alpha}, G^{\alpha\beta}) \rightarrow (A_\mu, \bar{\phi}, \phi, -\bar{\lambda}^{A\alpha}, -\lambda_{A\alpha}, G^{\alpha\beta}). \quad (8) \]
Denoting the $N = 16$ spinorial supercharges in $D = 2$ by $Q_A^\alpha$ and $\tilde{Q}_{A\alpha}$, which are interchanged by the $Z_2$ symmetry \( \mathcal{S} \), the transformation rules of the re–named fields are:

\[
\begin{align*}
Q_A^\alpha A_\mu &= (\sigma_\mu)_{AB} \tilde{\lambda}^B, \\
Q_A^\alpha \phi &= -(\sigma_3)_{AB} \tilde{\lambda}^B - \bar{\lambda}_A^\alpha, \\
Q_A^\alpha \bar{\phi} &= -(\sigma_3)_{AB} \tilde{\lambda}^B + \bar{\lambda}_A^\alpha, \\
Q_A^\alpha \bar{\lambda}_B^\beta &= \frac{1}{2} i \epsilon_{\mu} \delta_{AB} D_{\mu} G_{\alpha\beta} - \frac{1}{2} (\sigma_3)_{AB} [\phi + \bar{\phi}, G_{\alpha\beta}] - \frac{1}{2} i \epsilon_{AB} [\phi - \bar{\phi}, G_{\alpha\beta}], \\
Q_A^\alpha G_{\beta\gamma} &= 2 i (\delta_{\alpha} \lambda_{A\gamma} - \bar{\delta}_{\alpha} \lambda_{A\beta}), \\
Q_A^\alpha \lambda_{B\beta} &= \frac{1}{2} i \epsilon_{\mu} \epsilon^{\mu\nu} (\sigma_\nu)_{AB} D_{\mu} (\phi + \bar{\phi}) + \frac{1}{2} \delta_{\alpha} (\sigma_\mu)_{AB} D_{\mu} (\phi - \bar{\phi}) \\
&+ \frac{1}{2} \delta_{\alpha} (\sigma_3)_{AB} [\phi, \bar{\phi}] - \frac{1}{2} i \epsilon_{\mu} \epsilon^{\mu\nu} (\sigma_3)_{AB} F_{\mu\nu} - \frac{1}{2} i \epsilon_{AB} [G_{\alpha\gamma}, G_{\beta\delta}].
\end{align*}
\]

(9)

4. $N_T = 8$ topological twist of the $N = 16$, $D = 2$ super Yang–Mills theory

Let us now perform the $N_T = 8$ topological twist of the $N = 16$, $D = 2$ SYM (for the group theoretical description of that topological twist we refer to \( (4) \)). For that purpose we introduce the following set of twisted fields: A $SU(4)$–quartet of Grassmann–odd vector fields $\psi_1^\alpha$, two $SU(4)$–quartets of Grassmann–odd scalar fields, $\bar{\eta}_\alpha$ and $\bar{\zeta}_\alpha$ which transform as the fundamental and its complex conjugate representation of $SU(4)$, respectively, and a $SU(4)$–sextet of Grassmann–even complex scalar fields $M_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} M^{\gamma\delta}$, which transform as the second–rank complex selfdual representation of $SU(4)$.

Explicitly, the relationships between the original and the twisted fields are chosen as follows:

\[
\begin{align*}
\lambda_{A\alpha} &= \frac{1}{2} \left( i (\sigma_\mu)_{AB} (\psi_1^\alpha - \epsilon_{\mu\nu} \psi_2^\alpha) + (\sigma_3)_{AB} (\bar{\eta}_4 + \bar{\eta}_2) + i \epsilon_{AB} (\bar{\zeta}_4 - \bar{\zeta}_2) \right), \\
\bar{\lambda}^{A\alpha} &= \frac{1}{2} \left( i (\sigma_\mu)_{AB} (\psi_2^\alpha + \epsilon_{\mu\nu} \psi_1^\alpha) + (\sigma_3)_{AB} (\bar{\eta}_1 - \bar{\eta}_3) + i \epsilon_{AB} (\bar{\zeta}_3 + \bar{\zeta}_1) \right),
\end{align*}
\]

(10)

between $\lambda_{A\alpha}$, $\bar{\lambda}^{A\alpha}$ and the twisted vector and scalar fields $\psi_1^\alpha$, $\bar{\eta}_\alpha$, $\bar{\zeta}_\alpha$, as well as

\[
\begin{align*}
\phi &= M_1 - i M_2, \\
\bar{\phi} &= M_1 + i M_2, \\
G_{\alpha\beta} &= \left( (\sigma_\mu)_{AB} V_\mu - (\sigma_3)_{AB} M_3 - i \epsilon_{AB} M_4 \right), \\
G^{\alpha\beta} &= \left( (\sigma_\mu)_{AB} V_\mu + (\sigma_3)_{AB} M_3 + i \epsilon_{AB} M_4 \right),
\end{align*}
\]

(11)

where

\[
\begin{align*}
M_1 &= \frac{1}{2} (M^{12} + M^{34}), & M_3 &= \frac{1}{2} i (M^{12} - M^{34}), & M_5 &= M^{24}, \\
M_2 &= \frac{1}{2} (M^{14} + M^{23}), & M_4 &= \frac{1}{2} i (M^{14} - M^{23}), & M_6 &= M^{31},
\end{align*}
\]

between $\phi$, $\bar{\phi}$, $G_{\alpha\beta}$ and the twisted vector and scalar fields $V_\mu$ and $M_{\alpha\beta}$, respectively.

Thereby, the assignment between the index $\alpha$ of the internal group and the spinor index $A$ is the following: In \( (11) \) the spinor indices $B = 1, 2$ at the top and at the bottom of both columns correspond to the values $\alpha = 1, 2$ and $\alpha = 3, 4$ of both spinors $\lambda_{A\alpha}$ and $\bar{\lambda}^{A\alpha}$, respectively. Similarly, in \( (12) \) the spinor indices $A = 1, 2$ (resp. $B = 1, 2$) at the upper and at the lower raw
(resp. at the left and at the right column) of the both matrices correspond to the values $\alpha = 1, 2$ and $\alpha = 3, 4$ (resp. $\beta = 1, 2$ and $\beta = 3, 4$) of the scalar fields $G_{\alpha \beta}$, respectively. By using the explicit form of the Clebsch–Gordon coefficients one establishes that $G_{\alpha \beta}$ and $G^{\alpha \beta}$ in (12) are actually dual to each other, $G_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} G^{\gamma \delta}$.

The relationship between the spinorial supercharges $Q^{A \alpha}$ and $\tilde{Q}_{A \alpha}$, being interrelated by the $Z_2$ symmetry $\tilde{S}$, and the twisted scalar and vector supercharges $Q^\alpha$, $\ast Q^\alpha$ and $\tilde{Q}_{\mu \alpha}$, being interchanged by a discrete Hodge–type $\ast$ operation (see Eq. (15) below), is quite similar to the ones of the spinor fields, Eq. (10), namely

$$Q^{A \alpha} = \frac{1}{2} \left( i (\sigma^\mu)_{AB} (\tilde{Q}_{\mu 1} - \epsilon_{\mu \nu} \tilde{Q}_{\nu 2}) - (\sigma_3)_{AB} (Q^4 - i \ast Q^2) - e^{AB} (\ast Q^4 - i Q^2) \right),$$

$$\tilde{Q}_{A \alpha} = \frac{1}{2} \left( i (\sigma^\mu)_{AB} (\epsilon_{\mu \nu} Q^\nu - \tilde{Q}_{\mu 2}) + (\sigma_3)_{AB} (i \ast Q^2 - Q^4) + e_{AB} (i Q^4 - \ast Q^2) \right).$$

(13)

After performing in (9) the topological twist (10) – (12) and introducing the Grassmann–even auxiliary fields $B$, $\tilde{B}$, $Y$ and $E_{\mu \alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} E^{\gamma \delta}_{\mu}$ one gets the following $N_T = 8$ Hodge–type cohomological gauge theory with global symmetry group $SU(4)$:

$$S^{(N_T=8)} = \int_E d^2 x \text{tr} \left\{ \frac{1}{2} i e^{\mu \nu} B F_{\mu \nu} (A + i V) - \frac{1}{4} i e^{\mu \nu} \tilde{B} F_{\mu \nu} (A - i V) - \frac{1}{2} \tilde{B} B \\
- e^{\mu \nu} \tilde{\zeta}_\alpha D_\mu (A + i V) \psi^\alpha_{(\nu)} - \tilde{\eta}_\alpha D_\mu (A - i V) \psi^\alpha_{(\nu)} - \frac{1}{8} e_{\mu \alpha \beta} E^{\alpha \beta} \\
+ \frac{1}{2} i e^{\mu \nu} M_{\alpha \beta} \{ \psi^\alpha_{(\nu)}, \psi^\beta_{(\nu)} \} + i M^{\alpha \beta} \{ \tilde{\eta}_\alpha, \tilde{\zeta}_\beta \} - Y D_\mu (A) V^\mu_\alpha - \frac{1}{2} Y^2 \\
+ \frac{1}{8} D^\mu (A + i V) M_{\alpha \beta} D_\mu (A - i V) M^{\alpha \beta} + \frac{1}{64} [M_{\alpha \beta}, M_{\gamma \delta}] [M^{\alpha \beta}, M^{\gamma \delta}] \right\}.$$  

(14)

In this $SU(4)$ symmetric form the action (14) is manifestly invariant under the following Hodge–type $\ast$ symmetry, defined by the replacements

$$
\varphi \equiv \begin{bmatrix} \bar{\partial} \mu A^\mu & V_\mu \\ \bar{\eta}_\alpha & \bar{\zeta}_\alpha & M^{\alpha \beta} \\ B & \bar{B} & Y & E^{\alpha \beta} \end{bmatrix} \quad \Rightarrow \quad \ast \varphi = \begin{bmatrix} \epsilon_{\mu \nu} \bar{D}^{\nu} & \epsilon_{\mu \nu} A^{\nu} & -\epsilon_{\mu \nu} V^{\nu} \\ -i \epsilon_{\mu \nu} & -i \bar{\zeta}_\alpha & i \bar{\eta}_\alpha & -M^{\alpha \beta} \\ -B & -\bar{B} & -Y & \epsilon_{\mu \nu} E^{\nu \alpha \beta} \end{bmatrix}.
$$

(15)

with the property $\ast (\ast \varphi) = -P \varphi$. Here, $P$ is the operator of Grassmann–parity whose eigenvalues are defined by

$$P \varphi = \begin{cases} +\varphi & \text{if } \varphi \text{ is Grassmann-odd,} \\ -\varphi & \text{if } \varphi \text{ is Grassmann-even.} \end{cases}$$

Hence, after twisting the $Z_2$ symmetry $\tilde{S}$ changes into the Hodge–type $\ast$ symmetry $\ast S$.

The transformations rules for the topological shift symmetry, generated by $Q^\alpha$, are

$$Q^\alpha A^\mu = \psi^\alpha_{(\mu)};$$

$$Q^\alpha V^\mu = -i \psi^\alpha_{(\mu)};$$

$$Q^\alpha M_{\beta \gamma} = 2 i (\delta^\alpha_{\beta} \bar{\zeta}_\gamma - \delta^\alpha_{\gamma} \bar{\zeta}_\beta);$$

$$Q^\alpha \psi^\beta_{(\mu)} = E^{\alpha \beta} - i \epsilon_{\mu \nu} \bar{D}^{\nu} (A - i V) M^{\alpha \beta};$$

$$Q^\alpha \bar{\zeta}_\beta = i \delta^\alpha_{\beta} B;$$

$$Q^\alpha B = 0;$$

$$Q^\alpha \bar{\eta}_\beta = i \delta^\alpha_{\beta} Y + \frac{1}{2} [M^{\alpha \gamma}, M_{\gamma \beta}];$$

$$Q^\alpha Y = [M^{\alpha \beta}, \bar{\zeta}_\beta];$$

$$Q^\alpha \bar{B} = -2 [M^{\alpha \beta}, \bar{\eta}_\beta];$$

$$Q^\alpha E^{\mu \nu}_{\beta \gamma} = \delta^\alpha_{[\beta} (\epsilon^{\mu \nu} D_{(\nu} \bar{\zeta}_{\gamma)} - D^\mu (A - i V) \bar{\eta}_{\gamma]} - i \epsilon^{\mu \nu} [M_{\gamma \delta}, \psi^\delta_{(\nu)}].$$

(16)
From combining \(Q^\alpha\) with the above displayed Hodge–type \(\star\) symmetry one gets the corresponding transformations rules for the topological co–shift symmetry: \(*Q^\alpha = P \star Q^\alpha\).

By a straightforward calculation one verifies that both the supercharges \(Q^\alpha\) and \(*Q^\alpha\) provide an off–shell realization of the following topological superalgebra,

\[
\{Q^\alpha, Q^\beta\} = 0, \quad \{Q^\alpha, *Q^\beta\} = -2\delta_G(M^{\alpha\beta}), \quad \{*Q^\alpha, *Q^\beta\} = 0,
\]

where the field–dependent gauge transformations are defined by \(\delta_G(M^{\alpha\beta})A_\alpha = -D_\alpha M^{\alpha\beta}\) and \(\delta_G(M^{\alpha\beta})X = [M^{\alpha\beta}, X]\) for all the other fields.

Obviously, the structure of this superalgebra is directly analogous to the de Rham cohomology in differential geometry: The exterior and the co–exterior derivatives \(d\) and \(\delta = \pm \star d\star\), being interrelated by the duality \(\star\) operation, correspond to the nilpotent topological shift and co–shift operators \(Q^\alpha\) and \(*Q^\alpha = P \star Q^\alpha\star\), respectively. Moreover, the Laplacian \(\Delta = \{d, \delta\}\) corresponds to the field–dependent gauge generator \(\delta_G(M^{\alpha\beta})\), so that we have indeed a perfect example of a Hodge–type cohomological gauge theory.

Furthermore, by an explicit calculation one can verify that the action (14) is also invariant under the following on–shell vector supersymmetries,

\[
\begin{align*}
\bar{Q}_{\mu\alpha} A_\nu &= \delta_{\mu\nu} \bar{\eta}_\alpha - \epsilon_{\mu\nu} \bar{\zeta}_\alpha, \\
\bar{Q}_{\mu\alpha} V_\nu &= -i \delta_{\mu\nu} \bar{\eta}_\alpha - i \epsilon_{\mu\nu} \bar{\zeta}_\alpha, \\
\bar{Q}_{\mu\alpha} M^{\gamma\nu} &= 2i \epsilon_{\mu\nu} (\delta_{\alpha}^\beta \psi^{\nu\gamma} - \delta_{\alpha}^\gamma \psi^{\nu\beta}), \\
\bar{Q}_{\mu\alpha} \bar{\zeta}_\beta &= \epsilon_{\mu\nu} E_{\alpha\beta} + i D_\mu (A - i V) M_{\alpha\beta}, \\
\bar{Q}_{\mu\alpha} \bar{\eta}_\beta &= E_{\mu\alpha\beta} + i \epsilon_{\mu\nu} D^\nu (A + i V) M_{\alpha\beta}, \\
\bar{Q}_{\mu\alpha} \psi^\beta_\nu &= -2i \delta^\alpha_\beta F_{\mu\nu} (A) - 2i \delta^\alpha_\beta D_\mu (A) V_\nu - i \delta^\alpha_\beta \delta_\mu_\nu Y - i \delta^\alpha_\beta \epsilon_{\mu\nu} B + \frac{1}{2} \delta_{\mu\nu} [M_{\alpha\gamma}, M^{\gamma\beta}], \\
\bar{Q}_{\mu\alpha} B &= 2i \epsilon_{\mu\nu} D^\nu (A + i V) \bar{\eta}_\alpha, \\
\bar{Q}_{\mu\alpha} Y &= 2i D_\mu (A - i V) \bar{\eta}_\alpha - \epsilon_{\mu\nu} [M_{\alpha\beta}, \psi^{\nu\beta}], \\
\bar{Q}_{\mu\alpha} B &= 2i \epsilon_{\mu\nu} D^\nu (A - i V) \bar{\eta}_\alpha + 4i D_\mu (A) \bar{\zeta}_\alpha + 2 [M_{\alpha\beta}, \psi^{\mu\beta}], \\
\bar{Q}_{\mu\alpha} E^{\gamma\nu}_\nu &= -\delta_\alpha^\beta \delta_\mu_{[\nu} [M_{\mu\alpha\nu}, \psi^{\gamma\nu}] - \delta_\alpha^\beta \delta_\mu_\nu D^\nu (A - i V) \psi^{\gamma\nu} + i \delta_\alpha^\beta [M^{\gamma\beta}] \epsilon_{\mu\nu} \bar{\eta}_\beta - \delta_{\mu\nu} \bar{\zeta}_\beta. \tag{18}
\end{align*}
\]

In addition, this action is also invariant under the co–vector supersymmetries

\[*Q_{\mu\alpha} = P \star \bar{Q}_{\mu\alpha} \star = i \bar{Q}_{\mu\alpha},\]

which on–shell, i.e., by using only the equations of motion of the auxiliarly fields, become \(i\) times the vector supersymmetries! Hence, it holds

\[
(Q^\alpha, *Q^\alpha, \bar{Q}_{\mu\alpha}) S^{(N_T=8)} = 0,
\]

and the total number of (real) supercharges is actually \(N = 16\).

Finally, let us mention that there is also a \(N_T = 4\) topological twist of \(N = 16\), \(D = 2\) SYM with global symmetry group \(SO(4) \otimes SU(2)\). This topological theory can be regarded as the \(N_T = 4\) super–BF theory coupled to a spinorial hypermultiplet. Another way of obtaining the action of this theory is to dimensionally reduce either the higher dimensional analogue of the Donaldson–Witten theory in \(D = 8\) \([13, 16]\) to \(D = 2\) or to dimensionally reduce the \(N_T = 1\) half–twisted theory \([17]\) in \(D = 4\) to \(D = 2\). However, that topological twist does not lead to another Hodge–type cohomological theory, since the underlying cohomology is only equivariantly nilpotent and not strictly nilpotent as in Eqs. (17).
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