Shannon entropy for intuitionistic fuzzy information

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Abstract

The paper presents an extension of Shannon fuzzy entropy for intuitionistic fuzzy one. This extension uses a new formula for distance between two intuitionistic fuzzy pairs.

Keywords: Intuitionistic fuzzy information, escort fuzzy information, intuitionistic fuzzy entropy, Shannon entropy.

1 Introduction

The intuitionistic fuzzy representation of information was proposed by Atanassov [1], [2], [3] and it is defined by the pair \((\mu, \nu)\) where \(\mu \in [0, 1]\) is the degree of truth while \(\nu \in [0, 1]\) is the degree of falsity. Also, Atanassov considered the following condition for the intuitionistic fuzzy pair \((\mu, \nu)\):

\[\mu + \nu \leq 1\]  (1)

The condition (1) permits to consider the third parameter, the degree of incompleteness \(\pi \in [0, 1]\) defined by:

\[\pi = 1 - \mu - \nu\]  (2)

In addition to parameter \(\pi\), we define the net truth \(\tau \in [-1, 1]\), by:

\[\tau = \mu - \nu\]  (3)
On this way, we have two systems representation of intuitionistic fuzzy information: the primary space \((\mu, \nu)\) or the explicit space and the secondary space \((\tau, \pi)\) or the implicit space. Also, there exists a supplementary condition for the secondary space, namely:

\[ |\tau| + \pi \leq 1 \] (4)

Taking into account the condition (4), using the secondary space \((\tau, \pi)\) we define the third parameter \([8]\), degree of ambiguity \(\alpha \in [0, 1]\), by:

\[ \alpha = 1 - |\tau| - \pi \] (5)

The formulae (2) and (3) represent the transform from the primary space \((\mu, \nu)\) to the secondary space \((\tau, \pi)\) while the next two formulae represent the inverse transform:

\[ \mu = \frac{1 - \pi + \tau}{2} \] (6)

\[ \nu = \frac{1 - \pi - \tau}{2} \] (7)

For the intuitionistic fuzzy information \(X = (\mu, \nu)\), it was defined the complement \(\bar{X}\) by:

\[ \bar{X} = (\nu, \mu) \] (8)

After presentation of the main parameters that will be used in this approach, the next will have the following structure: section two presents a new distance for intuitionistic fuzzy information; section three presents formulae for evaluating of some feature of intuitionistic fuzzy information like certainty, score, uncertainty; section four presents the escort fuzzy information; section five presents the Shannon entropy formula for intuitionistic fuzzy information; section six presents the conclusion while the last is the references section.

### 2 A distance for intuitionistic fuzzy information

In this section we define a new distance for intuitionistic fuzzy pairs. For two intuitionistic fuzzy pairs \(P = (\mu_p, \nu_p)\) and \(Q = (\mu_q, \nu_q)\), we consider the \(L1\) distance \(d(P, Q) \in [0, 2]\) define by:
The $L_1$ distance \cite{10, 11} is a metric and considering the auxiliary point $C = (1,1)$ (see Figure 1), there exists the triangle inequality, namely:

$$d(P,C) + d(C,Q) \geq d(P,Q)$$  \hspace{1cm} (10)

Because

$$d(P,C) + d(C,Q) \neq 0$$  \hspace{1cm} (11)

we can transform (10) into (12):

$$1 \geq \frac{d(P,Q)}{d(P,C) + d(C,Q)}$$  \hspace{1cm} (12)

The right term represents the new distance or the new dissimilarity, namely:

$$D(P,Q) = \frac{d(P,Q)}{d(P,C) + d(C,Q)}$$  \hspace{1cm} (13)

From (13) and (9) it results the distance formula:

$$D(P,Q) = \frac{|\mu_p - \mu_q| + |\nu_p - \nu_q|}{2 + \pi_p + \pi_q}$$  \hspace{1cm} (14)

Figure 1: The geometrical framework for the proposed intuitionistic fuzzy distance.
Using the classical negation it results the following similarity formula:

\[
S(P, Q) = 1 - \frac{|\mu_p - \mu_q| + |\nu_p - \nu_q|}{2 + \pi_p + \pi_q}
\]  \hspace{1cm} (15)

We notice that distance (14) and its similarity (15) take values in the interval \([0, 1]\). Also, this distance is not a metric because it does not verify the triangle inequality.

3 The certainty, the score and the uncertainty for intuitionistic fuzzy information

Starting from the proposed distance defined by (14), we will construct some measures for the following three features of intuitionistic fuzzy information: the certainty, the score and the uncertainty.

3.1 The intuitionistic fuzzy certainty

For any intuitionistic fuzzy pair \(X = (\mu, \nu)\) we consider its complement \(\bar{X} = (\nu, \mu)\) and we define the certainty as dissimilarity between \(X\) and \(\bar{X}\), namely:

\[
g(X) = D(X, \bar{X})
\]  \hspace{1cm} (16)

with its equivalent form:

\[
g(X) = \frac{|\mu - \nu|}{2 - \mu - \nu}
\]  \hspace{1cm} (17)

In the space \((\mu, \nu)\) we identify the following properties for intuitionistic fuzzy certainty:

1. \(g(1, 0) = g(0, 1) = 1\)
2. \(g(x, x) = 0\)
3. \(g(\mu, \nu) = g(\nu, \mu)\)
4. \(g(\mu_1, \nu_1) \leq g(\mu_2, \nu_2)\) if \(|\mu_1 - \nu_1| \leq |\mu_2 - \nu_2|\) and \(\mu_1 + \nu_1 \leq \mu_2 + \nu_2\)

The property (4) shows that the certainty increases with \(|\tau|\) and decreases with \(\pi\).

From property (4) it results that \(g(\mu, \nu) \in [0, 1]\) because \(g(\mu, \nu) \geq g(0, 0)\) and \(g(\mu, \nu) \leq g(1, 0)\).
3.2 The intuitionistic fuzzy score

From (17) came the idea to define the intuitionistic fuzzy score by:

\[ r(X) = \frac{\mu - \nu}{2 - \mu - \nu} \]  

(18)

with the equivalent form in the space \((\tau, \pi)\), [6]:

\[ r(X) = \frac{\tau}{1 + \pi} \]  

(19)

In the space \((\mu, \nu)\) the properties for the intuitionistic fuzzy score derive from the certainty properties, namely:

1. \( r(1, 0) = 1; r(0, 1) = -1 \)
2. \( r(x, x) = 0 \)
3. \( r(\mu, \nu) = -r(\nu, \mu) \)
4. \( r(\mu_1, \nu_1) \leq r(\mu_2, \nu_2) \) if \( \mu_1 - \nu_1 \leq \mu_2 - \nu_2 \) and \( \mu_1 + \nu_1 \leq \mu_2 + \nu_2 \)

The property (4) shows that the intuitionistic fuzzy score increases with \( \tau \) and decreases with \( \pi \). From property (4) it results that \( r(\mu, \nu) \in [-1, 1] \) because \( r(\mu, \nu) \geq r(0, 1) \) and \( r(\mu, \nu) \leq r(1, 0) \).

3.3 The intuitionistic fuzzy uncertainty

Finally, we define the intuitionistic fuzzy uncertainty using the negation of the certainty:

\[ e(X) = 1 - \frac{|\mu - \nu|}{2 - \mu - \nu} \]  

(20)

with the equivalent form in the space \((\tau, \pi)\) [7]:

\[ e(X) = 1 - \frac{|\tau|}{1 + \pi} \]  

(21)

In the space \((\mu, \nu)\) the intuitionistic fuzzy uncertainty verifies the following conditions [6]:

1. \( e(1, 0) = e(0, 1) = 0 \)
2. \( e(x, x) = 1 \)
3. \( e(\mu, \nu) = e(\nu, \mu) \)
4. \( e(\mu_1, \nu_1) \leq e(\mu_2, \nu_2) \) if \( |\mu_1 - \nu_1| \geq |\mu_2 - \nu_2| \) and \( \mu_1 + \nu_1 \geq \mu_2 + \nu_2 \)

The property (4) shows that the uncertainty decreases with \( |\tau| \) and increases with \( \pi \).

From property (4) it results that \( e(\mu, \nu) \in [0, 1] \) because \( e(\mu, \nu) \geq e(1, 0) \) and \( e(\mu, \nu) \leq e(0, 0) \)

## 4 The escort fuzzy information

We will associate to any intuitionistic fuzzy information \( X = (\mu, \nu) \) a fuzzy one \( \hat{X} = (\hat{\mu}, \hat{\nu}) \) that we call *escort fuzzy information*. The escort fuzzy pair \((\hat{\mu}, \hat{\nu})\) will be determined in order to preserve the score of the intuitionistic fuzzy pair \((\mu, \nu)\). It will be obtained by solving the following system:

\[
\hat{\mu} + \hat{\nu} = 1 \quad (22)
\]

\[
\hat{\mu} - \hat{\nu} = r(\mu, \nu) \quad (23)
\]

It results the following values for the escort fuzzy pair \((\hat{\mu}, \hat{\nu})\):

\[
\hat{\mu} = \frac{\mu + \pi}{1 + \pi} \quad (24)
\]

\[
\hat{\nu} = \frac{\nu + \pi}{1 + \pi} \quad (25)
\]

There exist the following two inequalities:

\[
\mu + \pi \geq \hat{\mu} \geq \mu \quad (26)
\]

\[
\nu + \pi \geq \hat{\nu} \geq \nu \quad (27)
\]

The escort fuzzy pair \((\hat{\mu}, \hat{\nu})\) can be used to extend existing results from fuzzy theory \cite{12, 13} to intuitionistic fuzzy one. An example could be the cardinal’s calculation of an intuitionistic fuzzy set using its escort fuzzy set \cite{6}. In this paper we will use the escort pair for extending the Shannon fuzzy entropy to Shannon intuitionistic fuzzy entropy.
5 The Shannon entropy for intuitionistic fuzzy information

For any intuitionistic fuzzy pair $X = (\mu, \nu)$, using the escort fuzzy pair $\hat{X} = (\hat{\mu}, \hat{\nu})$ we will define the Shannon entropy \cite{9, 5} by the formula:

$$E_S(X) = e_S(\hat{X})$$

(28)

where $E_S$ represents the Shannon entropy for intuitionistic fuzzy information while $e_S$ represents the Shannon entropy for fuzzy one \cite{5}. For fuzzy information, De Luca and Termini \cite{5} extended the Shannon formula for calculating the fuzzy entropy by:

$$e_S(\mu) = -\mu \ln(\mu) - (1 - \mu) \ln(1 - \mu)$$

(29)

From (29) it results:

$$e_S(\hat{X}) = -\hat{\mu} \ln(\hat{\mu}) - \hat{\nu} \ln(\hat{\nu})$$

(30)

From (24), (25), (28) and (30) it results the Shannon variant for intuitionistic fuzzy entropy:

$$E_S(X) = -\frac{\mu + \pi}{1 + \pi} \ln \left( \frac{\mu + \pi}{1 + \pi} \right) - \frac{\nu + \pi}{1 + \pi} \ln \left( \frac{\nu + \pi}{1 + \pi} \right)$$

(31)

There are the next four equivalent formulae:

Using (6) and (7) it results:

$$E_S(X) = -\frac{1 + \tau}{2} \ln \left( \frac{1 + \tau}{2} \right) - \frac{1 - \tau}{2} \ln \left( \frac{1 - \tau}{2} \right)$$

(32)

Using (19) it results:

$$E_S(X) = -\frac{1 + r}{2} \ln \left( \frac{1 + r}{2} \right) - \frac{1 - r}{2} \ln \left( \frac{1 - r}{2} \right)$$

(33)

Because in (33) there exists symmetry between $r$ and $-r$, it results:

$$E_S(X) = -\frac{1 + |r|}{2} \ln \left( \frac{1 + |r|}{2} \right) - \frac{1 - |r|}{2} \ln \left( \frac{1 - |r|}{2} \right)$$

(34)

From (17), (18) and (34) it results:
\[ E_S(X) = -\frac{1+g}{2} \ln \left( \frac{1+g}{2} \right) - \frac{1-g}{2} \ln \left( \frac{1-g}{2} \right) \] (35)

We notice that:

\[ \frac{\partial g}{\partial |\tau|} = \frac{1}{1+\pi} \] (36)

\[ \frac{\partial g}{\partial \pi} = -\frac{|\tau|}{(1+\pi)^2} \] (37)

\[ \frac{\partial E_S}{\partial g} = \frac{1}{2} \ln \left( \frac{1-g}{1+g} \right) \] (38)

\[ \frac{\partial E_S}{\partial |\tau|} = \frac{1}{2} \frac{1}{1+\pi} \ln \left( \frac{1-g}{1+g} \right) \] (39)

\[ \frac{\partial E_S}{\partial \pi} = -\frac{1}{2} \frac{|\tau|}{(1+\pi)^2} \ln \left( \frac{1-g}{1+g} \right) \] (40)

Because

\[ \frac{1-g}{1+g} \leq 1 \] (41)

it results:

\[ \ln \left( \frac{1-g}{1+g} \right) \leq 0 \] (42)

From (39), (40) and (42) it results:

\[ \frac{\partial E_S}{\partial |\tau|} \leq 0 \] (43)

\[ \frac{\partial E_S}{\partial \pi} \geq 0 \] (44)

As conclusion, it results that the Shannon entropy for intuitionistic fuzzy information defined by (31) verifies the condition (4) from subsection 3.3, namely it decreases with |\tau| and increases with \( \pi \).

Also the function \( E_S \) defined by (31) verifies the conditions (1) and (3). In order to verify the condition (2) it necessary to multiply by the well-known normalization factor:

\[ \lambda = \frac{1}{\ln(2)} \] (45)
Finally it results the normalized variant for Shannon entropy, namely:

$$E_{SN}(X) = -\frac{1}{\ln(2)} \left[ \frac{\mu + \pi}{1 + \pi} \ln \left( \frac{\mu + \pi}{1 + \pi} \right) + \frac{\nu + \pi}{1 + \pi} \ln \left( \frac{\nu + \pi}{1 + \pi} \right) \right]$$

(46)

We can decompose the normalized Shannon entropy $E_{SN}$ in a sum with two terms, fuzziness $E_A$ and incompleteness $E_U$, namely:

$$E_{SN}(X) = E_A(X) + E_U(X)$$

(47)

where

$$E_A(X) = -\frac{(\mu + \pi) \ln(\mu + \pi) + (\nu + \pi) \ln(\nu + \pi)}{(1 + \pi) \ln(2)}$$

(48)

$$E_U(X) = \frac{\ln(1 + \pi)}{\ln(2)}$$

(49)

From Jensen inequality [4] it results that:

$$E_A(X) \leq -\frac{1}{\ln(2)} \ln \left( \frac{1 + \pi}{2} \right)$$

(50)

From [50] it results that $E_A(X)$ is maximum for $X = (0.5, 0.5)$ while from (49) it results that $E_U(X)$ is maximum for $X = (0, 0)$. Thus, it is highlighted that the uncertainty of intuitionistic fuzzy information has two sources: fuzziness (or ambiguity) that is the similarity of the pairs $(\mu, \nu)$ with $(0.5, 0.5)$ and incompleteness (or ignorance) that is the similarity of the pairs $(\mu, \nu)$ with $(0, 0)$.

6 Conclusion

In this paper, we presented a new formula for calculating the distance and similarity of intuitionistic fuzzy information. Then, we constructed measures for information features like score, certainty and uncertainty. Also, a new concept was introduced, namely escort fuzzy information. Then, using the escort fuzzy information, Shannon’s formula for intuitionistic fuzzy information was obtained. It should be underlined that Shannon’s entropy for intuitionistic fuzzy information verifies the four defining conditions of intuitionistic fuzzy uncertainty. The measures of its two components were also identified: fuzziness (ambiguity) and incompleteness (ignorance).
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