Numerical solution of an equilibrium problem for a membrane with a delaminated thin rigid inclusion

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Abstract. The paper deals with the numerical solution of an equilibrium problem for an elastic membrane with a thin rigid inclusion. The thin inclusion is supposed to delaminate, therefore a crack between the inclusion and the membrane is considered. The boundary conditions for nonpenetration of the crack faces are fulfilled. We provide the relaxation of the problem and propose an iterative method for the numerical solution of the approximated problem. The method is based on a domain decomposition and the Uzawa algorithm for finding a saddle point of the Lagrangian. Examples of the numerical solution of the initial problem are presented.

1. Introduction
Rigid inclusions (also called stiffeners or anticracks) are used in solid mechanics to describe fibres embedded in a matrix material. Under compression, rigid inclusions may delaminate from the matrix material, thereby introducing cracks. There are different approaches to model cracks in solids. The classical formulation of the problems with cracks implies linear boundary conditions on the crack faces [1]. It is well known that such models have shortcoming because there can be situations when the crack faces penetrate each other. It is natural to impose boundary conditions that exclude mutual penetration of the crack faces. The book [2] and papers [3, 4] contain results for crack models with the nonpenetration conditions for a wide class of constitutive laws.

In the present paper, we propose an iterative method of the numerical solution an equilibrium problem for an elastic membrane with a delaminated thin rigid inclusion. We assume that the nonpenetration conditions are imposed on the crack faces. To construct an effective numerical algorithm, we provide the relaxation of the initial problem. Then for the approximated problem, we apply the domain decomposition method [5] and the Uzawa algorithm for finding a saddle point of the Lagrangian [6]. To this end, the initial domain is divided into two subdomains. Two linear problems are solved in each subdomain at every iteration. One of problems describes the equilibrium of a membrane having a rigid inclusion on the external boundary. The presence of the rigid inclusion imposes constraints on a solution. In order to find the solution of such problem, we introduce three auxiliary Dirichlet problems without constraints and solve its. For “gluing” the solutions of both linear problems and providing the nonpenetration conditions, we use the Lagrange multipliers. Numerical experiments illustrate the performance of our algorithm.
2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial \Omega$. Assume that $\Omega$ is divided into two subdomains $\Omega_1$ and $\Omega_2$ with Lipschitz boundaries $\partial \Omega_1$ and $\partial \Omega_2$, respectively, such that $\text{meas}(\partial \Omega \cap \partial \Omega_i) > 0$, $i = 1, 2$; and denote by $\Sigma$ the interface of $\Omega_i$. Let $\gamma_c$ be a curve lying on the interface $\Sigma$ such that $\gamma_c \cap \partial \Omega = \emptyset$, and let $\Omega_c = \Omega \setminus \gamma_c$. Denote by $\Sigma$ the rest part of $\gamma_g$, by $\nu_i$ the outward unit normals to $\Omega_i$, $i = 1, 2$. In particular, we have $\nu_2 = -\nu_1 = \nu$ on $\Sigma$, see Figure 1 for an illustration of the geometry.

In what follows, an elastic membrane occupies the domain $\Omega_c$, and $\gamma_c$ corresponds to a crack in the membrane. In the domain $\Omega_c$, we intend to consider a mixed boundary value problem for the displacement field $u$. Namely, denote by $u_i$ the restriction of $u$ to $\Omega_i$, $i = 1, 2$. We suppose that the traces $u_1$ and $u_2$ on $\gamma_c$ are, in general, different and satisfying nonlinear boundary conditions that prevent mutual penetration of the crack faces. Furthermore, due to the presence of a thin rigid inclusion the function $u_2$ is a constant function on $\gamma_c$. The equilibrium problem for the elastic membrane having the crack and the thin rigid inclusion reads as follows. For a given external force $f \in L^2(\Omega)$ acting on the membrane, we have to find a displacement $u$ and a real number $A$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega_c,$$
$$u = 0 \quad \text{on} \quad \partial \Omega,$$
$$[u] \geq 0 \quad \text{on} \quad \gamma_c,$$
$$\frac{\partial u_1}{\partial \nu} \geq 0, \quad \frac{\partial u_1}{\partial \nu} [u] = 0 \quad \text{on} \quad \gamma_c,$$
$$u_2 = A \quad \text{on} \quad \gamma_c,$$
$$\int_{\gamma_c} [\frac{\partial u}{\partial \nu}] ds = 0,$$

where $[u] = u_2 - u_1$ is a jump of the function $u$ on $\gamma_c$.

To give a weak formulation of the problem (1)–(6), we introduce the functional space

$$V = \{ v \in H^1(\Omega_c) \mid v = 0 \text{ on } \partial \Omega \}$$

and the set of admissible displacements

$$K_c = \{ v \in V \mid [v] \geq 0 \text{ on } \gamma_c, \quad v_2 \text{ is a constant on } \gamma_c \},$$

where $v_2 = v|_{\Omega_2}$. The potential energy of the system is represented by the functional

$$\Pi(v) = \frac{1}{2} \int_{\Omega_c} |\nabla v|^2 dx - \int_{\Omega_c} fv dx.$$

Then the weak formulation of the problem (1)–(6) is the following minimization problem:

Find $u \in K_c$ such that $\Pi(u) = \inf_{v \in K_c} \Pi(v)$.

Note that the functional $\Pi$ is coercive and weakly lower semicontinuous on the space $V$. Moreover, the set $K_c$ is weakly closed. Hence, the minimization problem (7) has a unique solution $u \in K_c$ (see, e.g., [6, 2]), which satisfies the variational inequality

$$\int_{\Omega_c} \nabla u (\nabla v - \nabla u) dx \geq \int_{\Omega_c} f (v - u) dx \quad \forall v \in K_c.$$
3. Domain decomposition

Define the functional spaces \( V_i = \{ v_i \in H^1(\Omega_i) \mid v_i = 0 \text{ on } \partial \Omega_i \} \) that we endow with the norms \( \| v_i \|_{V_i}^2 = \int_{\Omega_i} |\nabla v_i|^2 \, dx, \quad i = 1, 2 \), which, due to the Poincaré inequality are equivalent to the usual \( H^1 \)-norms. Moreover, we introduce the one-dimensional subspace \( V_2^g \) of \( V_2 \) given by

\[
V_2^g = \{ v_2 \in V_2 \mid v_2 \text{ is a constant on } \gamma_c \}
\]

and the set \( K_{gc} \subset V_1 \times V_2^g \) given by

\[
K_{gc} = \{ (v_1, v_2) \in V_1 \times V_2^g \mid v_2 - v_1 \geq 0 \text{ on } \gamma_c, \ v_2 - v_1 = 0 \text{ on } \gamma_g \}.
\]

Let us consider now the constrained minimization problem:

Find \( (u_1, u_2) \in K_{gc} \) such that \( \Pi_1(u_1) + \Pi_2(u_2) = \inf_{(v_1, v_2) \in K_{gc}} (\Pi_1(v_1) + \Pi_2(v_2)), \) (9)

with the energy functionals \( \Pi_i \) are defined on \( V_i \) by the relations

\[
\Pi_i(v_i) = \frac{1}{2} \int_{\Omega_i} |\nabla v_i|^2 \, dx - \int_{\Omega_i} f v_i \, dx, \quad i = 1, 2.
\]

Theorem 1. Let \( u \) be a solution of the minimization problem (7). Then (9) has a unique solution \( (u_1, u_2) \in K_{gc} \), which satisfies the conditions

\[
u_i = u|_{\Omega_i}.
\]

We omit details, the essential ideas behind the proof of Theorem 1 are the same as in [7].

4. Regularized problem

For the efficient numerical solution of (7), we propose a domain decomposition method applied to a properly regularized problem.

For any positive number \( p \), let us introduce the sets

\[
U_1^p = \{ v_1 \in V_1 \mid \| v_1 \|_{V_1} \leq p \}, \quad U_2^p = \{ v_2 \in V_2^g \mid \| v_2 \|_{V_2^g} \leq p \},
\]

\[
\Lambda_c^p = \{ \lambda_c \in L^2(\gamma_c) \mid 0 \leq \lambda_c \leq p \text{ on } \gamma_c \}, \quad \Lambda_g^p = \{ \lambda_g \in L^2(\gamma_g) \mid -p \leq \lambda_g \leq p \text{ on } \gamma_g \},
\]

and the Lagrange function \( L \) defined on \( U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p \) by

\[
L(v_1, v_2, \lambda_c, \lambda_g) = \Pi_1(v_1) + \Pi_2(v_2) + \int_{\gamma_c} \lambda_c (v_1 - v_2) \, ds + \int_{\gamma_g} \lambda_g (v_1 - v_2) \, ds.
\]

Consider the family of saddle point problems depending on the parameter \( p \):

Find \( (u_1^p, u_2^p, \mu_c^p, \mu_g^p) \in U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p \) such that

\[
L(u_1^p, u_2^p, \lambda_c, \lambda_g) \leq L(u_1^p, u_2^p, \mu_c^p, \mu_g^p) \leq L(v_1, v_2, \mu_c^p, \mu_g^p) \quad \forall (v_1, v_2, \lambda_c, \lambda_g) \in U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p.
\]

The existence theory of saddle points implies that, for every \( p > 0 \), the problem (10) has a solution \( (u_1^p, u_2^p, \mu_c^p, \mu_g^p) \in U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p \).
Theorem 2. There exists a constant \( c > 0 \) such that, for all \( p > c \), the saddle point \((u_1^p, u_2^p, \mu_c^p, \mu_g^p)\) satisfies the system of variational equalities and inequalities

\[
\int_{\Omega_1} \nabla u_1^p \nabla v_1 \, dx + \int_{\gamma_c} \mu_c^p v_1 \, ds + \int_{\gamma_g} \mu_g^p v_1 \, ds = \int_{\Omega_1} f v_1 \, dx \quad \forall v_1 \in V_1, \tag{11}
\]

\[
\int_{\Omega_2} \nabla u_2^p \nabla v_2 \, dx - \int_{\gamma_c} \mu_c^p v_2 \, ds - \int_{\gamma_g} \mu_g^p v_2 \, ds = \int_{\Omega_2} f v_2 \, dx \quad \forall v_2 \in V_2^g, \tag{12}
\]

\[
\int_{\gamma_c} \lambda_c (u_1^p - u_2^p) \, ds \leq \int_{\gamma_c} \mu_c^p (u_1^p - u_2^p) \, ds \quad \forall \lambda_c \in \Lambda_c^p, \tag{13}
\]

\[
\int_{\gamma_g} \lambda_g (u_1^p - u_2^p) \, ds \leq \int_{\gamma_g} \mu_g^p (u_1^p - u_2^p) \, ds \quad \forall \lambda_g \in \Lambda_g^p. \tag{14}
\]

Remark. Theorem 2 means that the saddle point of the Lagrangian \( L \) over the set \( U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p \) coincides with the saddle point of \( L \) over the set \( V_1 \times V_2 \times \Lambda_c^p \times \Lambda_g^p \) for all \( p > c \).

Applying the same arguments as in [7], we can prove the following theorem.

Theorem 3. The sequence \((u_1^p, u_2^p)\) is such that

\[
(u_1^p, u_2^p) \to (u_1, u_2) \quad \text{strongly in} \quad V_1 \times V_2 \quad \text{as} \quad p \to \infty.
\]

5. Linear equilibrium problem for a membrane with a thin rigid inclusion

The function \( u_2 \) satisfying the variational equality (13) is a solution of a linear equilibrium problem for an elastic membrane with a rigid inclusion located on the external boundary. Despite the fact that this problem is linear, it is a constraint problem. Now we specify the method of solving of (13).

Let \( \omega \) be a bounded domain with Lipschitz boundary \( \partial \omega \), which is divided on two disjoint parts \( \gamma_N \) and \( \gamma_D \). Let \( \gamma \) be a part of \( \gamma_N \) such that \( \gamma \cap \gamma_D = \emptyset \). Denote by \( n \) the outward normal vector to \( \partial \omega \).

In the domain \( \omega \), let us consider the boundary value problem with nonlocal boundary conditions. For given functions \( f \in L_2(\omega) \) and \( g \in H^{-1/2}(\gamma_N) \), we have to find a function \( u \) and a real number \( A \) such that

\[-\Delta u = f \quad \text{in} \quad \omega, \tag{15}\]

\[u = 0 \quad \text{on} \quad \gamma_D, \tag{16}\]

\[
\frac{\partial u}{\partial n} = g \quad \text{on} \quad \gamma_N \setminus \gamma, \tag{17}\]

\[u = A \quad \text{on} \quad \gamma, \quad \int_{\gamma} \frac{\partial u}{\partial n} \, ds = \int_{\gamma} g \, ds. \tag{18}\]

The problem (15)–(18) corresponds to the equilibrium state of an elastic membrane occupying the domain \( \omega \) and containing a thin rigid inclusion \( \gamma \) on the external boundary \( \partial \omega \). The volume force \( f \) and the boundary traction \( g \) act on the membrane. In particular, the traction \( g \) acts on the rigid inclusion \( \gamma \).

Let us give a weak formulation of (15)–(18). For this purpose, we introduce the set of admissible displacements \( K = \{ v \in H_\gamma^1(\omega) \mid v \text{ is a constant on } \gamma \} \), where \( H_\gamma^1(\omega) = \{ v \in \)}
A function \( u \in K \) is a weak solution of (15)–(18), if the variational equality

\[
\int_{\omega} \nabla u \nabla v \, dx = \int_{\omega} fv \, dx + \int_{\gamma_N} gv \, ds \quad \forall v \in K
\]  

holds.

The space \( H^1_{\gamma_D}(\omega) \) is a Hilbert space with the inner product

\[
(u, v) = \int_{\omega} \nabla u \nabla v \, dx
\]

in virtue of the Poincaré inequality. Let us consider the following problem:

Find \( u_f \in H^1_{\gamma_D}(\omega) \) such that

\[
\int_{\omega} \nabla u_f \nabla v \, dx = \int_{\omega} fv \, dx + \int_{\gamma_N} gv \, ds \quad \forall v \in H^1_{\gamma_D}(\omega).
\]  

(20)

There exist a unique solution \( u_f \) of the problem (20); therefore we can rewrite the variational equality (19) in the form

\[
(u - u_f, v) = 0 \quad \forall v \in K,
\]

which means that the function \( u \) is an orthogonal projection of \( u_f \) onto the set \( K \).

Since \( K \) is a subspace of \( H^1_{\gamma_D}(\omega) \), the decomposition

\[
H^1_{\gamma_D}(\omega) = K \oplus K^\perp
\]

holds, where \( K^\perp \) is an orthogonal complement for \( K \) in \( H^1_{\gamma_D}(\omega) \).

Let \( u_f \in H^1_{\gamma_D}(\omega) \) be an arbitrary function. Describe the algorithm that allow us to find the decomposition \( u_f = u + w \), where \( u \in K, w \in K^\perp \).

It is possible to show that any function \( w \in K^\perp \) satisfies the following nonlocal boundary value problem:

\[
-\Delta w = 0 \text{ in } \omega, \tag{21}
\]

\[
w = 0 \text{ on } \gamma_D, \tag{22}
\]

\[
\frac{\partial w}{\partial n} = 0 \text{ on } \gamma_N \setminus \gamma, \tag{23}
\]

\[
\int_{\gamma} \frac{\partial w}{\partial n} \, ds = 0. \tag{24}
\]

Obviously, the converse is also true, i.e., any function satisfying (21)–(24) belongs to \( K^\perp \).

Choose a real number \( a \in R \) and consider the variational problem:

Find \( y^a \in H^1_{\gamma_D}(\omega) \) such that \( y^a = u_f|_{\gamma} - a \) on \( \gamma \) and

\[
\int_{\omega} \nabla y^a \nabla v \, dx = 0 \quad \forall v \in H^1_{\gamma_D \cup \gamma}(\omega). \tag{25}
\]

For any \( a \), the problem (25) has a unique solution. Furthermore, \( y^a \) is a weak solution to the boundary value problem

\[
-\Delta y^a = 0 \text{ in } \omega, \tag{26}
\]

\[
y^a = 0 \text{ on } \gamma_D, \tag{27}
\]

\[
y^a = u_f|_{\gamma} - a \text{ on } \gamma. \tag{28}
\]
\[ \frac{\partial y^a}{\partial n} = 0 \text{ on } \gamma_N \setminus \gamma. \]

Denote by
\[ h(a) = \int_{\gamma} \frac{\partial y^a}{\partial n} ds, \]
and show that there exists \( A \in \mathbb{R} \) such that \( h(A) = 0 \). It means that the function \( y^A \) satisfies the nonlocal condition (24). Since the solution \( y^a \) of (25) is an affine function of \( a \), we have the decomposition
\[ y^a = ay + y^0, \quad (26) \]
where \( y^0 \) is the solution of (25) corresponds to \( a = 0 \) and \( y \in H^1_{\gamma_D}(\omega) \) satisfies the variational equality
\[ (y, v) = 0 \quad \forall v \in H^1_{\gamma_D \cup \gamma}(\omega), \quad (27) \]
while \( y = -1 \) on \( \gamma \). The differential formulation of the problem (27) reads as follows. We have to find a function \( y \) such that
\[ -\Delta y = 0 \quad \text{in } \omega, \]
\[ y = 0 \quad \text{on } \gamma_D, \]
\[ y = -1 \quad \text{on } \gamma, \]
\[ \frac{\partial y}{\partial n} = 0 \quad \text{on } \gamma_N \setminus \gamma. \]

In turn, the function \( h \) is an affine function of \( a \), and the representation
\[ h(a) = ka + b \]
takes place with
\[ k = \int_{\gamma} \frac{\partial y}{\partial n} ds, \quad b = \int_{\gamma} \frac{\partial y^0}{\partial n} ds. \]
Due to the Green formula, we have
\[ k = -(y_1, y_1) < 0, \quad b = -(y_0, y_1), \]
therefore the function \( h \) vanishes when \( a = A := -b/k \).

Finally, we obtain that any function \( u_f \in H^1_{\gamma_D}(\omega) \) can be decomposed into a sum of two functions \( u \) and \( w \), where
\[ w = y^A \in K^\perp, \quad u = u_f - y^A \in K \]
and \( y^A \) is the solution of (25) with \( a = A \).

**Remark.** Due to (26), we have \( y^A = Ay + y^0 \). It means that we do not need to solve the problem (25) for \( a = A \) again. It suffices to use the functions \( y \) and \( y^0 \).
6. Domain decomposition algorithm

With the result of the previous section, we present a domain decomposition method to solve (7) numerically. Let us introduce the space

$$V_{2, \gamma} = \{ v_2 \in V_2 \mid v_2 = 0 \text{ on } \gamma \} = \{ v_2 \in H^1(\Omega_2) \mid v_2 = 0 \text{ on } \gamma \cup (\partial \Omega \cap \partial \Omega_2) \}.$$

**Algorithm.**

1. Initialization. \( \mu_{c,0} \in \Lambda_0^p \) and \( \mu_{g,0} \in \Lambda_0^p \) are given.
2. Iteration \( k \geq 0 \). Compute successively \( u_{1,k}, u_{2,k} \) as follows:
   - find \( u_{1,k} \in V_1 \) such that
     $$\int_{\Omega_1} \nabla u_{1,k} \nabla v_1 \, dx + \int_{\gamma_c} \mu_{c,k} v_1 \, ds + \int_{\gamma_g} \mu_{g,k} v_1 \, ds = \int_{\Omega_1} f v_1 \, dx \quad \forall v_1 \in V_1.$$
   - find \( w_k \in V_2 \) such that
     $$\int_{\Omega_2} \nabla w_k \nabla v_2 \, dx = \int_{\Omega_2} f v_2 \, dx + \int_{\gamma_c} \mu_{c,k} v_2 \, ds + \int_{\gamma_g} \mu_{g,k} v_2 \, ds, \quad \forall v_2 \in V_2.$$
   - find \( y_{k}^{0} \in V_2 \) such that \( y_{k}^{0} = w_k \) on \( \gamma \) and
     $$\int_{\Omega_2} \nabla y_{k}^{0} \nabla v_2 \, dx = 0 \quad \forall v_2 \in V_{2, \gamma}.$$
   - find \( y_{k} \in V_2 \) such that \( y_{k} = -1 \) on \( \gamma \) and
     $$\int_{\Omega_2} \nabla y_{k} \nabla v_2 \, dx = 0 \quad \forall v_2 \in V_{2, \gamma}.$$
   - compute the auxiliary real numbers \( A_k, b_k, c_k \) as follows:
     $$b_k = - \int_{\Omega_2} \nabla y_{k} \nabla y_{k} \, dx, \quad c_k = - \int_{\Omega_2} \nabla y_{k} \nabla y_{k}^{0} \, dx, \quad A_k = - c_k / b_k.$$
   - set \( y_{A,k} = A_k y_{k} + y_{k}^{0} \) and \( u_{2,k} = w_k - y_{A,k} \).
   - update the Lagrange multipliers
     $$\mu_{c,k+1} = P_{\Lambda_0^p}(\mu_{c,k} + \theta(u_{1,k} - u_{2,k})), \quad \mu_{g,k+1} = P_{\Lambda_0^p}(\mu_{g,k} + \theta(u_{1,k} - u_{2,k})).$$
3. Iteration continue until the relative error

$$\varepsilon = \max\left( \frac{\| u_{1,k} - u_{1,k-1} \|^2_{V_1}}{\| u_{1,k} \|^2_{V_1}}, \frac{\| u_{2,k} - u_{2,k-1} \|^2_{V_2}}{\| u_{2,k} \|^2_{V_2}} \right)$$

becomes sufficiently small.

The convergence of Algorithm follows from the general convergence theorem for the Uzawa algorithm (see [6]). Moreover, note that the proof of the convergence is similar to that in the articles [7].

**Theorem 4.** There exists \( \theta^* \) such that, for all \( \theta \in (0, \theta^*) \), the sequence \( (u_{1,k}, u_{2,k}) \) generated in Algorithm 1 (and, respectively, in Algorithm 2) is such that

$$\lim_{k \to \infty} (u_{1,k}, u_{2,k}) \to (u_1, u_2) \quad \text{strongly in} \quad V_1 \times V_2$$

as \( k \to \infty. \)
7. Numerical experiments

Using piecewise linear finite elements (P1-Lagrange elements), Algorithm was implemented in FreeFEM++. The test problems with different external forces are used to illustrate the behavior of our algorithm. Dimensionless units are used, and deformed configurations are plotted with amplification factors.

In all numerical experiments, we set $\mu_{c,0} = 0$ and $\mu_{g,0} = 0$ for the initialization of the algorithm, $\varepsilon = 10^{-12}$ for the stopping criterion; the regularized parameter $p$ equals $10^5$ and the relaxation parameter $\theta$ equals 2.

We consider a special domain configuration. Let the domain $\Omega$ consist of two subdomains $\Omega_1 = (-1, 1) \times (-1, 0)$ and $\Omega_2 = (-1, 1) \times (0, 1)$ with the interface $\Sigma = (-1, 1) \times \{0\}$, while the crack is $\gamma_c = (-0.5, 0.5) \times \{0\}$.

The subdomains $\Omega_1$ and $\Omega_2$ are partitioned into 5700 triangles with 2971 nodes and into 5674 triangles with 2958 nodes, respectively. In particular, 80 nodes are on $\gamma_c$ and $\gamma_g$ for both domains. The maximal sizes of the triangles (in a neighborhood of $\Sigma$) are 0.077 and 0.078 in $\Omega_1$ and $\Omega_2$, respectively, while the minimal sizes are equal to 0.012 for both domains.

Next, we consider four types of the external force $f$.

Example 1. Let $f$ be equal to 1 in $\Omega$. The resulting deformations are shown in Figure 1 (left).

Example 2. Assume that $f$ is given by

$$f(x_1, x_2) = \begin{cases} -1, & (x_1, x_2) \in \{(-1, 0) \times (0, 1)\} \cup \{(0, 1) \times (-1, 0)\}, \\ 1, & (x_1, x_2) \in \{(-1, 0) \times (-1, 0)\} \cup \{(0, 1) \times (0, 1)\}. \end{cases}$$

In this case, the crack faces contact with each other near the crack tip $(-1, 0)$. The resulting deformations and plots of displacements and jump of $u$ along the interface $\Sigma$ are shown in Figure 1 (right).

Example 3. Assume that $f$ is given by

$$f(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in (-1, 1) \times (0, 1), \\ -1, & (x_1, x_2) \in (-1, 1) \times (-1, 0). \end{cases}$$

Such external force provides an opening mode of the crack $\gamma_c$. The resulting deformations are shown in Figure 2 (left).

Example 4. If $f$ is given by

$$f(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in (-1, 0) \times (-1, 1), \\ -1, & (x_1, x_2) \in (0, 1) \times (-1, 1). \end{cases}$$
then the crack faces contact near the crack tip \((-1, 0)\). The resulting deformations are shown in Figure 2 (right).

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