EXAMPLES OF ANOSOV LIE ALGEBRAS

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Abstract. We construct new families of examples of (real) Anosov Lie algebras starting with algebraic units. We also give examples of indecomposable Anosov Lie algebras (not a direct sum of proper Lie ideals) of dimension 13 and 16, and we conclude that for every $n \geq 6$ with $n \neq 7$ there exists an indecomposable Anosov Lie algebra of dimension $n$.

1. Introduction

A diffeomorphism $f$ of a compact differentiable manifold $M$ is called Anosov if it has a global hyperbolic behavior, i.e. the tangent bundle $TM$ admits a continuous invariant splitting $TM = E^+ \oplus E^-$ such that $df$ expands $E^+$ and contracts $E^-$ exponentially. This kind of diffeomorphism plays an important and beautiful role in dynamics since they give examples of dynamical systems with very nice properties, and it is then a natural problem to understand which are the manifolds supporting them (see [16]).

Up to now, the only known examples are hyperbolic automorphisms on infranilmanifolds (manifolds finitely covered by nilmanifold) which are called Anosov automorphisms. Moreover, it is conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of a infranilmanifold (see [15]). The conjecture is known to be true in many particular cases, for example, J. Franks [6] and A. Manning [12] proved it for Anosov diffeomorphisms on infranilmanifolds themselves.

We will say that an $n$-dimensional rational Lie algebra is Anosov if it admits a hyperbolic automorphism $\tau$ (i.e. none of the eigenvalues of $\tau$ are of modulus 1) such that $[\tau]_\beta \in GL_n(\mathbb{Z})$ for some basis $\beta$ of $\mathfrak{n}$, where $[\tau]_\beta$ denotes the matrix of $\tau$ with respect to $\beta$. We say that a real Lie algebra is Anosov if it admits a rational form which is Anosov. It is easy to observe that a real Lie algebra $\mathfrak{n}$ is Anosov if and only if it admits a hyperbolic automorphism $\tau$ such that $[\tau]_\beta \in GL_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis $\beta$ of $\mathfrak{n}$ (i.e. with integer structure constants).

It is well known that any Anosov Lie algebra is necessarily nilpotent, and it is easy to see that the classification of nilmanifolds which admit an Anosov automorphism is essentially equivalent to that of Anosov Lie algebras (see [9, 2, 7, 4]).

Therefore, if one is interested in finding those Lie groups which are simply connected covers of Anosov infranilmanifold, then the objects to find are real nilpotent Lie algebras $\mathfrak{n}$ supporting an Anosov automorphism.

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Concerning the known examples, beside the case of free nilpotent Lie algebras (see [2]), there were only sporadic examples of Anosov Lie algebras before [9], where it is proved that \( \tilde{n} = n + \ldots + n \) is a real Anosov Lie algebra for any graded Lie algebra \( n \) admitting a rational form. Also, in [3] other kind of examples are given in the context of certain two-step nilpotent Lie algebras attached to graphs. In this way, there are in the literature examples of nonabelian Anosov real Lie algebra for each dimension \( n \geq 6 \) but \( 7 \) and \( 13 \). Moreover, in [3] the existence of indecomposable \( n \)-dimensional 2-step Anosov Lie algebra is proved for \( n \geq 6 \), except for \( n = 7, 9, 12, 13, 16 \). We recall that a Lie algebra is said to be \emph{indecomposable} if it can not be expressed as a direct sum of proper Lie ideals. It is known that there is no 7-dimensional Anosov Lie Algebra [10] and for \( n = 9, 12 \), there exists an indecomposable Anosov Lie algebra of dimension \( n \) (see [9]). In fact, [9] gives a family of indecomposable \((3r + 3)\)-dimensional Anosov Lie algebras, \( r \geq 1 \).

In this paper we will give explicit families of examples of Anosov (real) Lie algebras to illustrate a general procedure to construct Anosov Lie algebras and, as an application we will give an indecomposable 13-dimensional Anosov Lie algebra. In fact, for each pair of algebraic integers \( \lambda, \mu \) of degree \( p \) and \( q \) respectively which satisfy

1. they are units,
2. if we denote by \( \{ \lambda = \lambda_1, \ldots, \lambda_p \} \) and \( \{ \mu = \mu_1, \ldots, \mu_q \} \) the conjugates to \( \lambda \) and \( \mu \) respectively, then \( |\lambda_i| \neq 1 \neq |\mu_j| \), and
3. \( |\lambda_i \lambda_j| \neq 1 \),

we will exhibit a type \((p, q)\) Anosov Lie algebra. This first construction is quite easy to extend and we are able to show examples of 3-step (and in fact of \( k \)-step) Anosov Lie algebras, and also in the special case of \( p = 2 \) we give another example of type \((3q, q + 2)\) for any \( q \).

Finally, we also give an example of an indecomposable 16-dimensional Anosov Lie algebra, which allows us to conclude that for \( n \geq 6, n \neq 7 \) there exists an indecomposable \( n \)-dimensional Anosov Lie algebra.

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2. Examples

Given a nilpotent Lie algebra \( n \), we call the \emph{type} of \( n \) to the \( r \)-tuple \((n_1, \ldots, n_r)\), where \( n_i = \dim C^{i-1}(n)/C^{i}(n) \) and \( C^i(n) \) is the central descending series. It is proven in [10] that if \( n \) is a real Anosov Lie algebra of type \((n_1, \ldots, n_r)\), then there exist a hyperbolic \( A \in \text{Aut}(n) \) such that

1. \( A n_i = n_i \) for all \( i = 1, \ldots, r \),
2. \( A \) is semisimple (in particular \( A \) is diagonalizable over \( \mathbb{C} \)),
3. For each \( i \), there exists a basis \( \beta_i \) of \( n_i \) such that \( [A]_{\beta_i} \in SL_{n_i}(\mathbb{Z}) \), where \( n_i = \dim n_i \) and \( A_i = [A]_{\beta_i} \).

It is important to mention that the existence of an Anosov automorphism is a really strong condition on an infranilmanifold and also in a Lie algebra, and therefore, our approach is to start with a hyperbolic automorphism.
In this context, to show an example of an Anosov Lie algebra, we are going to construct a complex Lie algebra (to be able to work with eigenvalues) in such a way that it admits a hyperbolic automorphism $A$ such that $[A]_{\beta} \in \text{GL}_n(\mathbb{Z})$ for some $\mathbb{Z}$-basis $\beta$ of $\mathfrak{n}$.

We begin by noting that if $\lambda$ and $\mu$ are algebraic units of degree $p$ and $q$ respectively, and we denote by $\{\lambda = \lambda_1, \ldots, \lambda_p\}$ and $\{\mu = \mu_1, \ldots, \mu_q\}$ the sets of conjugates of $\lambda$ and $\mu$ over $\mathbb{Q}$ respectively, it is not hard to see that $\{\lambda_i \mu_j\}$ are also algebraic units and moreover, the matrix $\begin{bmatrix} \lambda_1 \mu_1 & \cdots & \cdots & \cdots \\ \cdots & \lambda_1 \mu_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \lambda_p \mu_q \end{bmatrix}$ is conjugated to a matrix in $\text{GL}_{pq}(\mathbb{Z})$ with determinant $\pm 1$.

Bearing this in mind, for each pair of non negative integers $p \neq q$, we take the Lie algebra $\mathfrak{n}$ with basis $\beta = \{X_1, \ldots, X_{pq}, Y_1, \ldots, Y_p, Z_1, \ldots, Z_q\}$ and Lie bracket among given by:

\begin{equation}
[X_{ip+j}, Y_j] = Z_{i+1} \quad 0 \leq i < q, \ 1 \leq j \leq p.
\end{equation}

It is clear that $\mathfrak{n}$ is a two-step nilpotent Lie algebra, a basis of $\mathfrak{n}_1$ is $\{X_i, Y_j : 1 \leq i \leq pq, \ 1 \leq j \leq p\}$ and $\{Z_k : 1 \leq k \leq q\}$ is a basis for $\mathfrak{n}_2$. Now, let $A$ be an automorphism such that $[A]_{\beta} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ where

\begin{align*}
A_1 &= \begin{bmatrix}
\lambda_1 \mu_1 \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\lambda_p \mu_q \\
\lambda_1^{-1} \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\lambda_p^{-1}
\end{bmatrix} \quad \text{and} \quad A_2 &= \begin{bmatrix} 
\mu_1 \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\mu_q
\end{bmatrix}.
\end{align*}

We note that $\beta$ is a basis of eigenvectors of $A$. Also, if we take $\lambda$ and $\mu$ as above and such that $|\lambda_1| \neq 1, |\mu_j| \neq 1$ and $|\lambda_i \mu_j| \neq 1$ for all $i,j$ then $A$ is a hyperbolic automorphism.

In what follows, we are going to show that $\mathfrak{n}$ is an Anosov Lie algebra by constructing a $\mathbb{Z}$-basis of $\mathfrak{n}$ preserved by $A$. In order to make the calculation more clear we will make a small change in the notation. Let $X_{(i,j)}$ be the eigenvector of $A$ corresponding to the eigenvalue $\lambda_i \mu_j$. Note that this is only a reordering of the $\{X_i\}$. In fact, $X_{(i,j)} = X_{(j-1)p+i}$, and therefore we may say that $\beta = \{X_{(i,j)}, Y_k, Z_l : 1 \leq i, k \leq p, \ 1 \leq j, l \leq q\}$ and (1) is now given by

\begin{equation}
[X_{(i,j)}, Y_j] = Z_j.
\end{equation}
Let \( \beta' = \{ \chi_{(k,l)}, \gamma_r, \zeta_s : 0 \leq i, k < p, 0 \leq j, l < q \} \) be the new basis of \( n \) given by

\[
\chi_{(k,l)} = \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i^k \mu_j^l X_{(i,j)} \quad 0 \leq k < p, 0 \leq l < q,
\]

\[
\gamma_r = \sum_{k=1}^{p} \lambda_{r-k} Y_k \quad 0 \leq r < p,
\]

\[
\zeta_s = \sum_{l=1}^{q} \mu_l^s Z_l \quad 0 \leq s < q.
\]

To see that this is actually a basis of \( n \), it is enough to check that the sets \( \{ \chi_{(k,l)} \}, \{ \gamma_r \} \) and \( \{ \zeta_s \} \) are linearly independent over \( \mathbb{Q} \). Since all the calculations are similar, we are only going to show how to proceed with \( \{ \chi_{(k,l)} \} \). Suppose \( a'_{kl} \) such that

\[
0 = \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} a_{kl} \chi_{(k,l)}
\]

\[
= \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} a_{kl} \left( \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i^k \mu_j^l X_{(i,j)} \right)
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{q} \left( \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} a_{kl} \lambda_i^k \mu_j^l \right) X_{(i,j)}.
\]

Hence, for \( 1 \leq i \leq p, 1 \leq j \leq q \) we have that

\[0 = \sum_{k=0}^{p-1} \left( \sum_{l=0}^{q-1} a_{kl} \lambda_i^k \mu_j^l \right) \lambda_i^k.
\]

This can be seen, for each \( 1 \leq j \leq q \) fixed, as a polynomial in \( \lambda_i \). This polynomial has degree \( p - 1 \) and it vanish on each one of the \( \lambda_i \), so by our choice of \( \lambda_i \) it has \( p \) different roots and therefore is identically zero. Hence for \( 1 \leq j \leq q \) we have that its coefficients are zero. That is, for each \( 0 \leq k < p \)

\[0 = \sum_{l=0}^{q-1} a_{kl} \mu_j^l
\]

which is again a polynomial in \( \mu_j \) of degree \( q - 1 \) with \( q \) different roots, and therefore we can conclude that \( a_{kl} = 0 \) for all \( k, l \) as we wanted to show.

If \( x^p + a_{p-1} x^{p-1} + \cdots + a_0 \) and \( x^q + b_{q-1} x^{q-1} + \cdots + b_0 \) are the minimal polynomial of \( \lambda^{-1} \) and \( \mu \) respectively, it is not hard to check that

\[
A\gamma_r = \begin{cases} 
\gamma_{r+1} & r < p - 1, \\
-\sum_{j=0}^{p-1} a_j \gamma_j & r = p - 1,
\end{cases}
\]

\[
A\zeta_s = \begin{cases} 
\zeta_{s+1} & s < q - 1, \\
-\sum_{l=0}^{q-1} b_l \zeta_l & s = q - 1.
\end{cases}
\]

Note that \( a_i \) and \( b_j \) are all integer numbers.
Concerning $X_{(k,l)}$, by the definition we have that for each $i, j$,

$$A(\lambda_i^k \mu_j^l X_{(i,j)}) = \lambda_i^{k+1} \mu_j^{l+1} X_{(i,j)},$$

and therefore, for $k < p - 1$ and $l < q - 1$, $A X_{(k,l)} = X_{(k+1,l+1)}$. In the same line of the calculation done above, we have that

$$AX_{(k,l)} = \begin{cases} 
- \sum_{k=1}^{p-1} c_k X_{(k,l+1)} & k = p - 1, l < q - 1, \\
- \sum_{l=1}^{q-1} b_l X_{(k+1,l)} & k < p - 1, l = q - 1
\end{cases}$$

$$AX_{(p-1,q-1)} = - \sum_{k=1}^{p-1} \sum_{l=1}^{q-1} c_k b_l X_{(k,l)}$$

where $c_j \in \mathbb{Z}$ are the coefficients of the minimal polynomial of $\lambda$.

On the other hand, to see that the Lie bracket of any two elements of $\beta'$ is a linear combination of elements of $\beta'$ with integer coefficients, it is enough to check it for $[X_{(k,l)}, Y_r]$. Using (2) we have that

$$[X_{(k,l)}, Y_r] = \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i^{k-r} \mu_j^l [X_{(i,j)}, Y_i]$$

$$= \left( \sum_{i=1}^{p} \lambda_i^{k-r} \right) \left( \sum_{j=1}^{q} \mu_j^l \right) Z_j$$

$$= M(k, l) Z_l.$$ 

Here $M(k, l) = \text{tr} A_{\lambda}^{k-l}$ where $A_{\lambda} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_p \end{bmatrix}$. Due to our choice of $\lambda$, $A_{\lambda}$ is conjugated to a matrix in $\text{GL}_p(\mathbb{Z})$ and therefore so is $A_{\lambda}^m$ for any $m \in \mathbb{N}$. Hence $M(k, l)$ is an integer number for any $k, l$ as we wanted to show.

**Remark 2.1.** Note that the Lie algebra $\mathfrak{n}$ we have constructed does not depend on the algebraic numbers $\lambda$ and $\mu$, it only depends on $p$ and $q$, and moreover it is easy to see (by looking at the dimension of the center for example) that the Lie algebra associated to $(p, q)$ is not isomorphic to the one corresponding to $(q, p)$ unless $p = q$.

We have obtained in this way two non isomorphic Anosov Lie algebra of dimension $n$ for all $n = p.q + p + q$ for any non negative integers $p, q$. It is easy to check that for $p = 2 = q$, we obtain the two step nilpotent Lie algebra $\mathfrak{g}$, of type $(6, 2)$ given in [10].

Concerning the existence of algebraic numbers as we need, we refer to [11].

Finally we would like to point out, for further use, that $\mathfrak{n}$ can be viewed as $V_0 \oplus V_1 \oplus Z$ where $V_0$ is the subspace generated by the $\{X_{(i,j)}\}$, $V_1$ is the one spanned by the $\{Y_k\}$ and $Z$ is the center. In this setup, $V_0$ acts on $V_1 \oplus Z$, as it is stated in (2).

**Example 2.2.** As a new example, we can carry out the calculations for $p = 2$, $q = 3$ to obtain the 11-dimensional Lie algebra with basis

$$\beta = \{X_1, \ldots, X_6, Y_1, Y_2, Y_3, Z_1, Z_2\}$$
and Lie bracket among them given by

\[
\begin{align*}
[X_1, Y_1] &= Z_1 & [X_3, Y_2] &= Z_1 & [X_5, Y_3] &= Z_1 \\
[X_2, Y_1] &= Z_2 & [X_4, Y_2] &= Z_2 & [X_6, Y_3] &= Z_2.
\end{align*}
\] (4)

The hyperbolic automorphism \( A \) is given by \([A]_\beta = \begin{bmatrix} A_0 & A_1 \\ A_2 & \end{bmatrix}\) where

\[
A_0 = \begin{bmatrix}
\lambda^{-1}_{\mu_1} & \lambda^{-1}_{\mu_2} & \lambda^{-1}_{\mu_3} \\
\lambda^{-1}_{\mu_2} & \lambda^{-1}_{\mu_3} & \lambda^{-1}_{\mu_4} \\
\lambda^{-1}_{\mu_3} & \lambda^{-1}_{\mu_4} & \lambda^{-1}_{\mu_5}
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
\mu_1^{-1} & \mu_2^{-1} & \mu_3^{-1} \\
\mu_2^{-1} & \mu_3^{-1} & \mu_4^{-1} \\
\mu_3^{-1} & \mu_4^{-1} & \mu_5^{-1}
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
\lambda & & \\
& \lambda & \\
& & \lambda
\end{bmatrix}.
\]

In this case we have obtained a Lie algebra of type \((9, 2)\), and note that for \(p = 3, q = 2\) we obtain a Lie algebra of type \((8, 3)\). We would like to point out that here and in general, we can add non zero constant to the Lie brackets in (4) but it is easy to see that this leads to isomorphic Lie algebras.

Once we have stated the general picture, let us consider an analogous procedure by starting from two algebraic units \(\lambda, \mu\) and \(p, q\) of degree \(\lambda, \mu\) in the same procedure as above, we can construct a two step nilpotent Anosov Lie algebra of type \((p + q, pq)\), where the eigenvalues of the corresponding \(A_1\) are the conjugated numbers to \(\lambda, \mu\) and the ones corresponding to \(A_2\) are all the products among them, \(\{\lambda, \mu\}_k\). It is not hard to see that this algebra is isomorphic to the one associated to a bipartite graph \((p, q)\) which is proved to be Anosov in [3]. As in this case, a lot of changes can be made to this procedure to obtain a variety of new examples. Among them we are now going to mention a few more, and since the proofs are essentially the same, they will be omitted.

**Example 1.** One can start be taking three algebraic units as above \(\lambda, \mu\) of degree \(p, q\) and \(r\) respectively, such that the conjugate numbers to each of them satisfies \(|\lambda i j | \neq 1, |\lambda i j | \neq 1, \) and \(|\mu i j | \neq 1, \). It is not hard to see that we can proceed analogously considering the pair \(\lambda \mu, \nu, \) \((pq, r)\) in spite of which is the degree of \(\lambda \mu\). In fact, in the proof of the linear independence of the new basis, and also in (3), we only use the fact that we are adding over all the conjugated numbers to \(\lambda\) and \(\mu\). Following the lines of the above procedure, we then obtain an Anosov Lie algebra, \(n_{pq,r}\) of type \((pq + r, pq)\). Moreover, once we have stated this, it is clear that it is also true for \(\lambda \nu, \mu, (pr, q)\) and in this case, our procedure leads to a Lie algebra of type \((pq + q, pr)\).

Now, it is clear that in each of these algebras, \(n_{pq,r}\) and \(n_{pr,q}\), the eigenvalues of the associated automorphism corresponding to \(X_{(k,l)}\) are the same, that is \(\lambda_{i} \mu_{j} \nu_{s}\) for some \(i, j, s\). Therefore, the corresponding subspaces \(V_{0}\) can be identifying (see remark 2.1). In this case, it is easy to see that a new algebra can be constructed from this two by identifying the \(V_{0}\). Explicitly, if \(n_{pq,r} = V_{0} \oplus V_{1} \oplus Z_{1}\) and \(n_{pr,q} = V_{0} \oplus V_{2} \oplus Z_{2}\), let \(n\) be the Lie algebra with vector space \(n = (V_{0} \oplus V_{1} \oplus V_{2}) \oplus (Z_{1} \oplus Z_{2})\), where the action is as before: \([V_{0}, V_{i}] \subset Z_{i}, i = 1, 2\). This is a two step nilpotent Lie algebra of type \((pq + q + r, pr + pq)\). In this framework, there is a natural way to define an automorphism in \(n\), using the ones in \(n_{pq,r}\) and \(n_{pr,q}\) : \([A]_\beta = \begin{bmatrix} A_1 & A_2 \\ A_2 & \end{bmatrix}\)
where

\[
A_1 = \begin{pmatrix}
\lambda_1 \mu_1 \nu_1 \\
\vdots & \ddots & \ddots \\
\lambda_p \mu_q \nu_r \\
\mu_1^{-1} & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\rho_1 & \ddots & \ddots \\
\nu_1^{-1} & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\nu_r^{-1} & \ddots & \ddots \\
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\lambda_1 \nu_1 \\
\vdots & \ddots & \ddots \\
\lambda_p \nu_r \\
\lambda_1 \mu_1 & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\nu_1 & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\nu_r & \ddots & \ddots \\
\end{pmatrix}.
\]

It is easy to check that due to our choice of \( \lambda, \mu \) and \( \nu \), it is hyperbolic. On the other hand, note that in both cases the lattice we have constructed in \( V_0 = \mathcal{X}(k,l) \), are the same (is just a matter of notation) and therefore as with the automorphism, the natural extension of the lattice we have constructed in \( \mathcal{A} \) is just a matter of notation and therefore as with the automorphism, the natural extension of the lattice we have constructed in \( \mathcal{A} \), it is easy to check that \( \mathfrak{a} \) is an Anosov Lie Algebra.

In this way we obtain two step Anosov Lie algebras of dimension \( n = pq + pr + q + r \) for any \( p, q, r \). Distinguishing them by the type, it is easy to see that in general, if \( p \neq q \neq r \) the Lie algebra one obtains by interchanging the roll of \( p, q \) and \( r \) are not isomorphic. The smallest one we can construct corresponds to \( p = q = r = 2 \) is 18 dimensional and its type is \( (10,8) \).

It is not hard to see that this procedure extends in a natural way to consider \( k \) algebraic numbers as above \( \lambda, \mu \) and \( \nu \) of degree \( p \), \( q \), and \( r \) respectively.

As before we take algebraic numbers as above \( \lambda, \mu \) and \( \nu \) of degree \( p \), \( q \), and \( r \) respectively.

In this case we have in mind \( A = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \), where \( A_1 \) and \( A_2 \) are as in the previous example, and \( A_3 = \begin{pmatrix} \lambda_1 & & \\
& \ddots & \\
& & \lambda_p \end{pmatrix} \). As before, we are going to make a small change in the notation in order to be consistent with the eigenvalues. Let \( \mathfrak{n} \) be the Lie algebra with basis

\[
\beta = \{ X_{(i,j,k)}, Y_j, Z_k, V_{(i,k)}, W_{(i,j)}, U_i : 1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r \}
\]

and the Lie bracket among them given by

\[
[X_{(i,j,k)}, Y_m] = \delta_{(j,m)} V_{(i,k)} \quad \quad [X_{(i,j,k)}, Z_n] = \delta_{(k,n)} W_{(i,j)}
\]

\[ [Z_n, V_{(i,k)}] = \delta_{(n,k)} U_i \quad \quad [Y_m, W_{(i,j)}] = \delta_{(m,j)} U_i. \]

It is easy to see that \( \mathfrak{n} \) is a three-step nilpotent Lie algebra, that is, it satisfies Jacobi identities, and the type of \( \mathfrak{n} \) is \((pq + q + r, pr + pq, p)\). Let \( A \) denote the linear transformation of \( \mathfrak{n} \) such that \( [A]_\beta = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \) hence, \( A \) is a hyperbolic automorphism of \( \mathfrak{n} \) and \( \beta \) is a basis of eigenvectors of \( A \). To construct a \( \mathbb{Z} \)-basis,
we proceed similarly as before:

\[ X_{(m,l,s)} = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \lambda_{i}^{m} \mu_{j}^{l} \nu_{k}^{s} X_{(i,j,k)} \quad 0 \leq m < p, \quad 0 \leq l < q, \quad 0 \leq k < r, \]

\[ Y_{l} = \sum_{k=1}^{q} \mu_{-l}^{k} Y_{k} \quad 0 \leq l < q, \]

\[ Z_{s} = \sum_{i=1}^{r} \nu_{s}^{i} Z_{l} \quad 0 \leq s < r, \]

\[ V_{(m,s)} = \sum_{i=1}^{p} \sum_{k=1}^{r} \lambda_{i}^{m} \nu_{k}^{s} V_{(i,k)} \quad 0 \leq m < p, \quad 0 \leq s < r, \]

\[ W_{(m,l)} = \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_{i}^{m} \mu_{j}^{l} W_{(i,j)} \quad 0 \leq m < p, \quad 0 \leq l < q, \]

\[ U_{i} = \sum_{n=1}^{p} \lambda_{i}^{n} U_{n} \quad 0 \leq i < p. \]

Straightforward calculation, using the same techniques as before, shows that this is a \( \mathbb{Z} \)-basis preserved by \( A \), and therefore \( n \) is an Anosov Lie algebra. The smallest example we obtain in this way is a tree step nilpotent Lie algebra of dimension 20 of type \((10, 8, 2)\).

It is not hard to see that if \( n = n_{1} \oplus n_{r} \) is a real Anosov Lie algebra of type \((n_{1}, n_{2}, \ldots, n_{r})\) then \( n/n_{r} \) is also an Anosov Lie algebra (see [14]). Note that in this case, this fact is what we have showed in the previous construction.

Also, it is not hard to prove by induction, that this procedure extends in a natural way to the case of consider \( k \)-algebraic units, to obtain a \( k \)-step Anosov Lie algebra. Moreover, by the above observation, we have all the quotients one has in between.

**Example 3.** As a last application of our procedure we are going to consider the special case of \( p = 2 \). In this case, in addition to \( n_{(2,q)} \) and \( n_{(q,2)} \), we can define others Lie algebras, for example by adding Lie brackets among the \( \{X_{(i,j)}\} \). That is, we take

\[ \beta = \{X_{i}, Y_{k}, Z_{l} : 1 \leq i \leq 2q, 1 \leq k \leq q, 1 \leq l \leq q + 2\} \]

as a basis of \( n \) with the Lie bracket given by

\[ [X_{iq+j}, Y_{j}] = Z_{q+i+1} \quad i = 0, 1 : j = 1 \ldots, q \]

\[ [X_{j}, X_{j+q}] = Z_{j} \quad j = 1 \ldots, q \]
This is a two-step nilpotent Lie algebra of type $(3q, q + 2)$. Let $A$ be the automorphism of $\mathfrak{n}$ such that $[A]_\beta = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, where

$$A_1 = \begin{bmatrix}
\lambda \mu_1 \\
& \ddots \\
& & \ddots \\
& & & \mu_1^{-1} \\
\lambda^{-1} \mu_q & \ddots & & \\
& \mu_2^{-1} & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \mu_q^{-1} \\
& & & & \lambda^{-1} 
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
\mu_1^2 \\
& \ddots \\
& & \ddots \\
& & & \mu_2^2 \\
& & & & \mu_q^2 \\
\lambda^{-1} & \ddots & & \\
& \lambda & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \lambda^{-1}
\end{bmatrix}.$$  

Concerning the lattice, we will take $\mathcal{Y}_k$ as before, and let

$$\mathcal{X}_i = \begin{cases}
\sum_{k=1}^{q} \mu_k^i (X_k + X_{q+k}) & 0 \leq i < q \\
\sum_{k=1}^{q} \mu_k^{i-q} (\lambda X_k + \lambda^{-1} X_{q+k}) & q \leq i < 2q 
\end{cases}$$

(6)

$$Z_l = \begin{cases}
(\lambda^{-1} - \lambda) \sum_{k=1}^{q} \mu_k^l Z_k & 0 \leq l < q \\
Z_q + Z_{q+1} & l = q \\
\lambda Z_q + \lambda^{-1} Z_{q+1} & l = q + 1.
\end{cases}$$

One can see that this is also a basis of $\mathfrak{n}$ preserved by $A$, and moreover one can check that

$$[\mathcal{X}_i, \mathcal{X}_j] = 0 \quad i, j < q \text{ or } i, j \geq q$$

(7)

$$[\mathcal{X}_i, \mathcal{X}_j] = Z_{i+j-q}, \quad 0 \leq i < q \text{ and } q \leq j < 2q.$$  

Also, as in (3), one can see that

$$[\mathcal{X}_{i+j}, \mathcal{Y}_k] = N(j, k) Z_{q+i}, \quad i = 0, 1 : 0 \leq j < q,$$

where $N(j, k) = \text{tr}(A_{i-k}^j)$, such that $A_{i} = \begin{bmatrix} \mu_1 & & \\
& \ddots & \\
& & \mu_2 \\
& & & \ddots \\
& & & & \ddots \\
& & & & & \mu_q 
\end{bmatrix}$, is an integer number

and therefore, we can conclude that $\mathfrak{n}$ is an Anosov Lie algebra. Note that the dimension of $\mathfrak{n}$ is $n = 4q + 2$ and the type is $(3q, q + 2)$. The small one we can obtain corresponds to $q = 2$, is of type $(6, 4)$, dimension 10.

**Remark 2.3.** Note that the subalgebra generated by the set $\{X_i, Z_l : 1 \leq i \leq 2q, 1 \leq l \leq q\}$ is an Anosov Lie subalgebra of $\mathfrak{n}$

3. 13-DIMENSIONAL EXAMPLE

This last example is rather different. We will take $p = 2, q = 3$ and we will split the basis in the center. So let $\lambda$ and $\mu$ two algebraic numbers of degree 2 and 3.
respectively such that $|\lambda_i\mu_j| \neq 1$ for all its conjugated numbers of $\lambda$ and $\mu$. Then we take $\mathfrak{n}$ the complex vector space with basis

$$\beta = \{X_1, \ldots, X_6, Y_1, Y_2, Y_3, Z_1, Z_2, W_1, W_2\}$$

and define the Lie bracket among them by

$$[X_1, Y_1] = Z_1 \quad [X_4, Y_1] = W_1$$

(8) $$[X_2, Y_2] = Z_2 \quad [X_5, Y_2] = W_2$$

$$[X_3, Y_3] = -(Z_1 + Z_2) \quad [X_6, Y_3] = -(W_1 + W_2).$$

This is a two-step nilpotent Lie algebra of type $(9, 4)$. Let $A$ be the automorphism of $\mathfrak{n}$ such that $[A]_\beta = [A_1 \quad A_2]$, where

$$A_1 = \begin{bmatrix} \lambda & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \lambda^{-1} & \mu_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mu_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mu_3^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu_1^{-1} \end{bmatrix} \quad A_2 = \begin{bmatrix} \lambda & \lambda^{-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \lambda & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$ 

Concerning the lattice, we will take $\{\mathfrak{X}_i\}, 0 \leq i \leq 5$ and $\{\mathfrak{Y}_k\}, 0 \leq k \leq 2$ as in the previous example, and let

$$Z_l = \sum_{i=1}^{2} (\mu_i^l - \mu_3^l) (Z_i + W_i) \quad l = -1, 1,$$

$$W_l = \sum_{i=1}^{2} (\mu_i^l - \mu_3^l) (\lambda Z_i + \lambda^{-1} W_i) \quad l = -1, 1.$$

To see that this is a basis of $\mathfrak{n}$, as we have pointed out before, it is enough to check that each one of $\{\mathfrak{X}_i\}, \{\mathfrak{Y}_j\}$ and $\{\mathfrak{Z}_k \mathfrak{W}_l\}$ are linearly independent sets. We also note that the calculations for the first two sets have been already done, and then we are only going to show how to proceed in the center. Suppose then that

$$0 = a_1 Z_{-1} + a_1 Z_1 + b_{-1} W_{-1} + b_{1} W_{1}$$

$$= \sum_{i=1}^{2} \left[ (\mu_i^1 - \mu_3^1) (a_{-1} + \lambda b_{-1}) + (\mu_i^1 - \mu_3^3) (a_{1} + \lambda b_{1}) \right] Z_i$$

$$+ \sum_{i=1}^{2} \left[ (\mu_i^1 - \mu_3^3) (a_{-1} + \lambda^{-1} b_{-1}) + (\mu_i^1 - \mu_3^3) (a_{1} + \lambda^{-1} b_{1}) \right] W_i$$

Hence, for $i = 1, 2$ we have that

$$0 = (\mu_i^1 - \mu_3^1) (a_{-1} + \lambda b_{-1}) + (\mu_i^1 - \mu_3^3) (a_{1} + \lambda b_{1}),$$

(9) $$0 = (\mu_i^1 - \mu_3^3) (a_{-1} + \lambda^{-1} b_{-1}) + (\mu_i^1 - \mu_3^3) (a_{1} + \lambda^{-1} b_{1}).$$

If we denote by

$$P_{\lambda}(x) = x^{-1} (a_{-1} + \lambda b_{-1}) + x (a_{1} + \lambda b_{1}),$$

then by the first equation we have that $P_{\lambda}(\mu_i) = P_{\lambda}(\mu_j) = C$ for $i, j = 1, 2, 3$. Hence,

(10) $$a_{-1} + \lambda b_{-1} + a^2 (a_{1} + \lambda b_{1}) = Cx,$$
is a degree two polynomial annulated by each one of the $\mu_i$. Since these are three
different algebraic numbers, we have that (10) is identically zero. In particular
(11) $0 = (a_{-1} + \lambda b_{-1})$ and $0 = (a_1 + \lambda b_1)$.

We can do the same for the second equation in 9 and we will obtain
(12) $0 = (a_{-1} + \lambda^{-1} b_{-1})$ and $0 = (a_1 + \lambda^{-1} b_1)$.

Finally, from (11) and (12) we can conclude that
\[ a_{-1} = b_{-1} = a_1 = b_1 = 0 \]
as was to be shown.

One can also see for $i = 1$ or 2, let say for simplicity $i = 1$, that
\[
\mu_1^2 - \mu_3^2 = (\mu_1 - \mu_3)(\mu_1 + \mu_3) = (\mu_1 - \mu_3) (n_1 - (\mu_1\mu_3)^{-1}) = n_1 (\mu_1 - \mu_3) + (\mu_1^{-1} - \mu_3^{-1})
\]
where $n_1 = \text{tr} A_\mu^j$ is an integer number for all $j \in \mathbb{Z}$ and we have also used that
$\mu_2 = (\mu_1\mu_3)^{-1}$. Therefore,
\[
\sum_{i=1}^{2} (\mu_i^2 - \mu_3^2) (Z_i + W_i) = n_1 Z_1 + Z_{-1}.
\]

In the same way, we also have that
\[
\sum_{i=1}^{2} (\mu_i^2 - \mu_3^2) (\lambda Z_i + \lambda^{-1} W_i) = n_1 W_1 + W_{-1}
\]
It is also easy to see that this is also valid for $\mu_i^{2 - 2} - \mu_3^{2 - 2}$, that is we have similar
formulas to these ones for
\[
\sum_{i=1}^{2} (\mu_i^{2 - 2} - \mu_3^{2 - 2}) (\lambda^j Z_i + \lambda^{-j} W_i),
\]
for $j = 0, 1$.

Also, as in the previous examples, it is not hard to see that this is a basis of $\mathfrak{n}$
preserved by $A$. With all this, one can check that
\[
[X_0, Y_0] = 0 \quad [X_3, Y_0] = 0
\]
\[
[X_1, Y_0] = Z_1 \quad [X_4, Y_0] = W_1.
\]
\[
[X_2, Y_0] = n_1 Z_1 + Z_{-1} \quad [X_5, Y_0] = n_1 W_1 + W_{-1}
\]
\[
[X_0, Y_1] = Z_{-1} \quad [X_0, Y_2] = n_{-1} Z_1 + Z_1.
\]
Using this and the fact that $A$ is an automorphism, it is easy to prove, that this is a $\mathbb{Z}$-basis of $n$. For example,

$$[\mathcal{X}_2, \mathcal{Y}_3] = [A\mathcal{X}_1, A\mathcal{Y}_2]$$

$$= A[A\mathcal{X}_0, A\mathcal{Y}_1]$$

$$= A(AZ_{-1})$$

$$= A(W_{-1}) = -aW_{-1} - Z_{-1}$$

where $x^2 + ax + 1$ is the minimal polynomial of $\lambda$. Therefore, we can conclude that this is a Anosov Lie algebra, as desired. In the following we will prove by using similar arguments as in Lemma 6.6 of [3], that it is also indecomposable.

**Proposition 3.1.** $n$, defined by the relations given by (8), is indecomposable.

**Proof.** Suppose on the contrary that $n = n_1 \oplus n_2$ is the sum of two nontrivial ideals of $n$. By definition, we have that $n = V \oplus W$ where $V$ is the subspace spanned by the set $S = \{X_1, \ldots, X_6, Y_1, Y_2, Y_3\}$, and $W$ is the subspace of $n$ spanned by $\{Z_1, Z_2, W_1, W_3\}$. Let $p : n \to V$ be the projection onto $V$ with respect to this decomposition $n = V \oplus W$ and let $V_1 = p(n_1)$ and $V_2 = p(n_2)$. For $i = 1, 2$ let

$$S_i = \{v \in S : a_v v + \sum_{v' \in S, v' \neq v} a_{v'} v' \in V_i \text{ for some } a_v \neq 0\}.$$ 

Then $S = S_1 \cup S_2$, and since $n_1$ and $n_2$ are nontrivial ideals, it is easy to see that $S_1$ and $S_2$ are nonempty sets. Moreover, we have that $S_2 \setminus S_1$ is empty. In fact, if $S_2 \setminus S_1$ is nonempty, we can either have that for all $v \in S_2 \setminus S_1$ and $v' \in S_1$, $[v, v']$ is zero, or there exists $v \in S_2 \setminus S_1$ and $v' \in S_1$ such that $[v, v']$ is nonzero.

In the first situation, as $S = S_1 \cup S_2$, we may assume that $Y_1$ is contained in $S_1$. Then there exists nonzero $a$ such that $aY_1 + x \in V_1$ where $x$ is contained in the span of $S \setminus \{Y_1\}$. Since $n_1$ is an ideal $[aY_1 + x, X_1]$ and $[aY_1 + x, X_4]$ are contained in $n_1$. This means $Z_1, W_1 \in n_1$. We notice that not all $Y_1's$ are contained in $S_1$ because if all $Y_i's$ are contained in $S_1$ then (by our assumption) all $X_i's$ are contained in $S_1$ and then $S_2 \setminus S_1$ is empty. Now either $Y_2 \in S_1$ or $Y_2 \in S_2 \setminus S_1$. If $Y_2 \in S_1$ (similar argument works for the other case) then $Y_3$ must be contained in $S_2 \setminus S_1$. In that case $Z_2$ and $W_2$ are contained in $n_1$ (by considering Lie brackets with $X_2$ and $X_3$), and similarly $Z_1 + Z_2$ and $W_1 + W_2$ are contained in $n_2$. This is a contradiction.

On the other hand, if there exists $v \in S_2 \setminus S_1$ and $v' \in S_1$ such that $[v, v']$ is nonzero, it is easy to see that if $v \in \{Y_1, Y_2, Y_3\}$ then $v' \in \{X_1, \ldots, X_6\}$ and if $v \in \{X_1, \ldots, X_6\}$ then $v' \in \{Y_1, Y_2, Y_3\}$. So, since it is entirely equivalent, we can assume then that $v \in \{Y_1, Y_2, Y_3\}$ and $v = Y_1$. Moreover, either $v' = X_1$ or $v' = X_4$, and therefore, we may assume that $v' = X_1$. From our definition of $S_1$, there exist nonzero scalars $s$ and $t$ such that $sX_1 + x$ is contained in $V_1$ and $tY_1 + y$ is contained in $V_2$, where $x$ is in the subspace of $V$ spanned by $Y_1, Y_2, Y_3, X_2, \ldots, X_6$ and $y$ is in the subspace of $V$ spanned by $Y_2, Y_3, X_1, X_2, \ldots, X_6$. Hence $[sX_1 + x, Y_1] \in n_1$ and $[tY_1 + y, X_1], [tY_1 + y, X_4] \in n_2$ since $n_1$ and $n_2$ are Lie ideals of $n$. This implies that $sZ_1 + s'W_1 \in n_1$ where $s'$ is a scalar, and $Z_1, W_1 \in n_2$. This is a contradiction since $s$ is nonzero and then we can conclude that $S_2 \setminus S_1$ is empty.
Therefore, we have that \( S = S_1 \) and moreover we can see that \( Z_1, Z_2, W_1, W_2 \) are contained in \( n_1 \). Hence \([n_1, n_1] = [n, n]\) and from this, one has that \( n_2 \) is in the center of \( n \).

On the other hand, it is easy to see that the center is equal to \([n, n] = W\) and hence \( n_2 \) is contained in \( n_1 \), contradicting our assumption that \( n_2 \) is nontrivial. Hence \( n \) can not be seen as a sum of two proper ideals as we wanted to show. \( \Box \)

4. 16-DIMENSIONAL EXAMPLE

Let \((S, E)\) denote the complete bipartite graph on a set \( S \) of 5 elements partitioned into subsets \( S_1 \) and \( S_2 \) of 2 and 3 elements respectively. Following for example [3] we can define from this (and any graph) a 2-step nilpotent Lie algebra. Let \( \mathcal{N} = V \oplus W \) denote the 2-step nilpotent Lie algebra associated with \((S, E)\). We recall that in this case we obtain an Anosov Lie algebra of type \((5, 6)\) (see [3]). Using this algebra we are going to construct a 16-dimensional Anosov Lie algebra as follows.

Let \( n \) be that Lie algebra with linear space \( n = V \oplus V \oplus W \) and Lie bracket defined by \([x_1, x_2, w], (y_1, y_2, w') = [x_1, y_1] + [x_2, y_2]\) where \( x_i \)'s and \( y_i \)'s are vectors in \( V \), \( w, w' \in W \) and \([x_1, y_1] \) denotes the Lie bracket in \( \mathcal{N} \). To see that it is an Anosov Lie algebra, let us consider \( \Phi \) the additive subgroup of \( n \) generated by the elements of the type \((v, 0, 0), (v', 0), (0, 0, [\gamma, \delta])\) where \( v, v', \gamma, \delta \in S \). It is easy to see that \( \Phi \) is a \( \mathbb{Z} \)-subalgebra of \( n \) (i.e. \( \Phi \) is the set of all \( \mathbb{Z} \) linear combinations of the basis of \( n \) with integer structure coefficients) and moreover, \( n \) admits a hyperbolic automorphism \( \tau \) such that \( \tau(\Phi) = \Phi \). In fact, if \( \Phi' \) is a subgroup of \( \mathcal{N} \) generated by \( S \cup \{[v, v'] : v, v' \in S\} \), \( \mathcal{N} \) admits a hyperbolic automorphism \( \tau' \) such that \( \tau'(\Phi') = \Phi' \). (see [3], Theorem 1.1). We take \( \tau \) to be the natural extension of \( \tau' \) to \( n \). Hence \( n \) is an Anosov Lie algebra.

Proposition 4.1. \( n \), defined as above, is indecomposable.

Proof. Let \( S_1 = \{\alpha, \beta\} \) and \( S_2 = \{\gamma, \delta, \eta\} \). Suppose that \( n_1 \) and \( n_2 \) are two proper ideals of \( n \) such that \( n = n_1 \oplus n_2 \). As \([n, n] = [n_1, n_1] \oplus [n_2, n_2] \) and \([n, n] \) is 6-dimensional, we may assume that \( \dim [n_1, n_1] \leq 3 \). Let \( X = (v, v', w) \in n_1 \) where \( v, v' \in V \) and \( w \in W \). Let \( v = \sum_{\xi \in S} a_\xi \xi \). As \( n_1 \) is an ideal, \([X, (\xi, 0, 0)] \in n_1 \) for all \( \xi \in S \). Hence \( a_\gamma \gamma \xi + a_\delta \delta \xi + a_\eta \eta \xi + a_\alpha \alpha \xi + a_\beta \beta \xi + a_\xi \xi \xi \) are contained in \([n_1, n_1]\) for all \( \xi \in S_1 \) and \( n_2 \). As \( \dim [n_1, n_1] \leq 3 \), we see that either \( a_\xi = 0 \) for all \( \xi \in S_1 \) or \( a_\xi = 0 \) for all \( \xi \in S_2 \). Let \( V_1 \) (respectively \( V_2 \)) denote the subspace of \( V \) spanned by \( S_1 \) (respectively \( S_2 \)). Then by the above observation, \( v \in V_1 \) or \( v \in V_2 \).

Suppose \( v \in V_1 \). We will prove that \( n_1 \) is contained in \( V_1 \oplus V \oplus W \). Suppose that \( v \neq 0 \). Then the vectors \( a_\alpha \alpha \xi + a_\beta \beta \xi \in [n_1, n_1] \) for all \( \xi \in S_2 \) are linearly independent. Hence \( \dim [n_1, n_1] = 3 \). Suppose \((v_1, v'_1, w_1) \in n_1 \) be such that \( v_1 \in V_2 \). Let \( v_1 = a_\gamma \gamma + a_\delta \delta + a_\eta \eta \). Now as \( \dim [n_1, n_1] = 3 \) and \( a_\gamma \gamma \xi + a_\delta \delta \xi + a_\eta \eta \xi \) and \( a_\alpha \alpha \xi + a_\beta \beta \xi \) are contained in \([n_1, n_1]\), \( a_\gamma = a_\delta = a_\eta = 0 \). Hence \( v_1 = 0 \). Thus we have proved that if \( v \in V_1 \), then \( n_1 \) is contained in \( V_1 \oplus V \oplus W \). Similarly we prove that if \( v \in V_2 \), then \( n_1 \) is contained in \( V_2 \oplus V \oplus W \). Suppose \( v \in V_2 \) and \( v \neq 0 \). Let \((v_1, v'_1, w_1) \in n_1 \) be such that \( v_1 \in V_1 \) and write \( v_1 = a_\alpha \alpha + a_\beta \beta \). We note that \( a_\alpha \alpha \xi + a_\beta \beta \xi \in [n_1, n_1] \) for all \( \xi \in S_2 \). If the vectors \( a_\alpha \alpha \xi + a_\beta \beta \xi \) are linearly independent for all \( \xi \in S_2 \), then \( \dim [n_1, n_1] = 3 \). This is a contradiction.
as \(a \gamma \wedge \alpha + a_3 \delta \wedge \alpha + a_\eta \eta \wedge \alpha\) and \(a \gamma \wedge \beta + a_3 \delta \wedge \beta + a_\eta \eta \wedge \beta\) are contained in \([n_1, n_1]\). Hence \(a'_\alpha = a'_\beta = 0\), and so \(v_1 = 0\). Thus if \(v \in V_2\), then \(n_1\) is contained in \(V_2 \oplus V \oplus W\). Similarly we can prove that \(n_1\) is contained in \(V \oplus V_1 \oplus W\) or \(V \oplus V_2 \oplus W\). Hence \(n_1\) is contained in \(V_i \oplus V_j \oplus W\) for some \(i, j \in \{1, 2\}\). But then \([n_1, n_1] = 0\) which is a contradiction. This proves the proposition.

\[\square\]

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