AN EQUIVARIANT NON-COMMUTATIVE RESIDUE

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Abstract. Let $\Gamma$ be a finite group acting on a compact manifold $M$ and $\mathcal{A}(M)$ denote the algebra of classical complete symbols on $M$. We determine all traces on the cross product algebra $\mathcal{A}(M) \rtimes \Gamma$. These traces appear as residues of certain meromorphic 'zeta' functions and can be considered as equivariant generalization of the non-commutative residue trace. The local formula for these traces depends on more than one component of the complete asymptotic expansion. For instance, the local formula for these traces depends also on derivatives in the normal directions to fixed point manifolds of higher order components. As an application, we obtain a formula for the asymptotic occurrence of an irreducible representation of $\Gamma$ in the eigenspaces of an invariant positive elliptic operator. We also obtain a new construction for Dixmier trace of an invariant operator.

1. Introduction

The spectrum of a geometric differential operator, for instance that of the Laplace operator, is of significant interest in analysis and geometry. It is well known that the Laplace operator on a bounded domain has a discrete spectrum. In a classical paper [23] Hermann Weyl derived estimates on asymptotic growth of eigenvalues. In 1980s, Guillemin [11] has obtained Weyl’s estimates by using the non-commutative residue.

In [24], Wodzicki introduced a trace on the algebra of pseudodifferential operators on a smooth, compact manifold, called the residue trace. A similar result was obtained in [11]. Many interesting invariants can be expressed in terms of traces on algebras. The space of all traces on an algebra forms its 0-th Hochschild cohomology. Hochschild and cyclic homology are important tools in non-commutative geometry. Results on the cyclic homology for algebras of complete symbols over compact manifolds were obtained in [3, 24]. In particular, these homology calculations recover the non-commutative residue of Guillemin [11] and Wodzicki [24]. The Atiyah-Singer index theorem can also be formulated in terms of the non-commutative residue. Residues form the basic ingredient in the non-commutative index theorem of Connes and Moscovici [7]. An homological approach to the equivariant version of the non-commutative residue is discussed in [8].

We construct an equivariant, delocalized generalization of the noncommutative residue for manifolds endowed with a group action. This result amounts, in particular, to the computation of the traces on the cross-product

Supported by FWF grant Y237-N13 of the Austrian Science Fund.
algebra of pseudodifferential operators by a finite group. As an application, we then obtain results on the asymptotic properties of eigenspaces of operators that are invariant with respect to the given group action. This application is to count the occurrence of an irreducible representation asymptotically in the eigenspaces.

Let $V$ be a smooth, compact, $n$ dimensional Riemannian manifold, and let $\Delta := d^*d$ be the (positive) Laplace operator on $V$. Weyl’s theorem asserts that the spectral counting function

$$N_\Delta(\lambda) := \#\{\lambda_i : \lambda_i \in \text{sp}(\Delta), \lambda_i \leq \lambda\}$$

grows asymptotically as $C\lambda^{n/2}$, where the constant $C$ depends only on the dimension and volume of $M$.

In view of the results of Guillemin and Wodzicki, in order to generalize Weyl’s result to the equivariant case, we can proceed as follows. Let $\Gamma$ be a finite group, acting on a closed manifold $M$ by diffeomorphisms. For instance, $\Gamma$ could be a finite group of isometries. Let $D$ be a positive differential operator on $M$ that is preserved under the $\Gamma$ action. A good choice is the Laplace operator $D = \Delta$ associated to an invariant metric. More generally, $D$ could be an elliptic pseudodifferential operator of positive order. Let $\{\lambda_i\}$ be the eigenvalues of $D$ and let $V_i = \ker(D - \lambda_i I)$ be the corresponding eigenspaces. Then each eigenspace $V_i$ acquires a representation of the group $\Gamma$. One natural problem is to asymptotically count the occurrence of any particular representation in these eigenspaces. More precisely, if we start with a representation $\pi$ of $\Gamma$ and we count the multiplicity of $\pi$ in each eigenspace $V_i$, we obtain the refined counting function

$$N_{\pi,D}(\lambda) := \sum_{\lambda_i < \lambda} \text{"multiplicity of } \pi \text{ in } V_i."$$

Then a natural question is to estimate the asymptotic growth of $N_{\pi,D}(\lambda)$. One might also want to compare asymptotically the relative occurrence of various representations of $\Gamma$ in these eigenspaces. The answer that we propose to these questions is based on the construction of “equivariant, delocalized” noncommutative residues.

Recall that the classical pseudodifferential operators on a closed manifold $M$ form an algebra $\Psi^\infty(M)$. The space of smoothing operators $\Psi^{-\infty}(M)$ is an ideal of this algebra. The algebra of complete symbols is then the quotient $\Psi^\infty(M)/\Psi^{-\infty}(M)$, and we will denote it by $A(M)$. If the cosphere bundle $S^*M$ is connected, then the noncommutative residue is the unique (up to a constant) trace on the algebra $A(M)$. Moreover, if $P$ is an order one, positive operator, then, for any $A \in \Psi^\infty(M)$, we can define $\zeta_A(z) := Tr(P^{-z}A)$ for $\Re(z)$ large enough. This function turns out to be meromorphic and the residue at zero, $Tr_R(A) = \text{res}_{z=0} \zeta_A(z)$, then defines a trace on $\Psi^\infty(M)$, called the non-commutative residue $[11, 24]$. This trace descends to a trace on $A(M)$. 
To obtain an equivariant version, one should consider traces on the cross-product algebra $\mathcal{A}(M) \rtimes \Gamma$ obtained from an action of $\Gamma$ on $M$. As a set, the cross product algebra is

$$\mathcal{A}(M) \rtimes \Gamma := \{ \sum_{g \in \Gamma} A_g g : A_g \in \mathcal{A}(M) \}.$$ 

Its product structure comes from the action of $\Gamma$, namely,

$$A_g g \cdot B_h h = A_{gh}.$$

For a fixed, $\Gamma$ invariant operator $D$ of positive order and any element $g$ in $\Gamma$, we consider the function $z \mapsto \text{Tr}(D^{-z} A_g)$. This function is a priori defined only on a half plane $Re(z) > N$, with $N$ large (see below).

We prove that this new “zeta” function $z \mapsto \text{Tr}(D^{-z} A_g)$ has a meromorphic extension to the complex plane with only simple poles. These poles can be only at $z = d/m, (d-1)/m, (d-2)/m, \ldots$, where $d = \text{ord}(A) + \dim(M^g)$ and $m = \text{ord}(D)$. Furthermore, since all the poles are simple for any conjugacy class $\langle \gamma \rangle \in \langle \Gamma \rangle$, the sum of residues

$$\text{Tr}_R^{\langle \gamma \rangle} (A) = \sum_{g \in \langle \gamma \rangle} \text{res}_{z=0} \text{Tr}(D^{-z} A_g),$$

defines a trace on $\mathcal{A}(M) \rtimes \Gamma$ that is independent of choice of the invariant, positive operator $D$. This is the first version of our equivariant noncommutative residue.

Given an irreducible representation $\pi$ of the group $\Gamma$, the asymptotic of the counting function $N_{\pi,D}$ for an order $m$ operator $D$ can now be obtained by applying the standard Tauberian theorem to the functions $z \mapsto \text{Tr}(D^{-z} g)$. One can then prove that if the group action on $M$ is faithful, then $N_{\pi,D}(\lambda)$ grows asymptotically as $C \dim(\pi) \lambda^{n/m}$, where $n = \dim(M)$ and $m = \text{order}(D)$.

Wodzicki shows that the noncommutative residue of an operator $P$ can be locally be obtained by integrating the $-n$ component $\sigma_{-n}(P)$ of the asymptotic expansion in local coordinates, over the cosphere bundle $\text{Tr}_R(P) = \int_{S^* M} \sigma_{-n}(P)$ with respect to volume measure coming from the symplectic structure on $T^* M$. This is remarkable because the local expansion is not diffeomorphism invariant. A similar formula for the equivariant case will involve not just the $-n$th term in the local asymptotic expansion, but also normal derivations over fixed point submanifolds of higher order (i.e. $>-n$) terms from the expansion (see Proposition 3.1.)

Also we can now work back on Connes Trace formula that tells us that the noncommutative residue is an extension of the Dixmier trace to all pseudodifferential operators. We thus obtain another equivalent construction for of the Dixmier trace of an invariant operator.

We begin with the analysis of the zeta functions $z \mapsto \text{Tr}(D^{-z} A_g)$. The proof that these functions are meromorphic requires a careful use of the stationary phase principle. We prove, in fact, a slightly stronger version of
the stationary phase principle needed to obtain our local formula for the noncommutative residues. Finally we use the Tauberian theorem to obtain the asymptotics of representations in eigenspaces.

Acknowledgment. I would like to thank my supervisor Victor Nistor for all his suggestions and comments and above all his patience with my work.

2. Explicit description of Traces

For a finite group $\Gamma$ acting on a closed manifold $M$, the description of the Hochschild homology of the crossed product algebra $A(M) \rtimes \Gamma$ determines the dimension of the space of traces on the cross-product algebra $A(M) \rtimes \Gamma$ (see [8]). These traces correspond to conjugacy classes $\langle \gamma \rangle$ of the group $\Gamma$, for when the elements $\gamma \in \langle \gamma \rangle$ have a nonempty fixed point set on the cosphere bundle or $S^*M^\gamma \neq \emptyset$. We wish to describe these traces more explicitly.

By a constant order holomorphic family $A(z) \in \Psi^m(M)$ of pseudodifferential operators on a compact, smooth manifold $M$, we mean a family that can be obtained from a holomorphic family of complete symbols of fixed order and a holomorphic family of regularizing operators. (The regularizing operators are given the Fréchet topology of $C^\infty(M \times M)$.) By a holomorphic family $A(z) \in \Psi^z(M)$, we mean that $A(z) = B(z)D^z + C(z)$, where $B$ is a holomorphic family of order zero operators, $D$ is a positive first order pseudodifferential operator, and $C(z)$ is a holomorphic family of regularizing operators. See [1, 15, 16, 18] and the references therein for more details.

Let us fix a positive, elliptic, order one operator $D \in \Psi^1(M)$ that is invariant under $\Gamma$ action. This is to say $Dg = gD$ as operators on $C^\infty(M)$ for all $g \in \Gamma$. The complex powers $D^z$ can be defined easily by spectral theorem, but it is much harder to prove that each $D^z$ is a pseudo-differential operator of order $z$. This was proved by Seeley [22]. Another proof was given by Guillemin in [11].

In particular, Seeley result implies that for $Re(z) < -n$, $D^z$ is a trace class, and hence the map

$$z \to tr(D^z)$$

is holomorphic on the half plane $Re(z) < -n$ of complex numbers.

For $A$ in $\Psi^\infty(M)$ and any group element $g$, we define

$$\zeta_{g,A}(z) := Tr(D^{-z}Ag).$$

This function is a priori defined only when $D^{-z}A$ is a trace class, that is, when $Re(z) > n + \text{order}(A)$. We wish to establish the meromorphic continuation of the function $\zeta_{g,A}(z)$. This would involve the use of the stationary phase principle. We recall the following two theorems from [12].

**Theorem 2.1.** For an open domain $U \subset \mathbb{R}^n$, let $u(x) \in \mathcal{C}^\infty_c(U)$. Let $f \in \mathcal{C}^\infty(U)$ be such that $f'(x) \neq 0$ in $\text{Supp}(u)$. Then for every $k \in \mathbb{Z}$ we have

$$\left| \int e^{irf(x)}u(x)dx \right| < r^{-k}$$
for $r$ large enough. That is, the integral is a rapidly decreasing function of $r$.

Let $U \subset \mathbb{R}^n$ be an open domain. Let $f \in \mathcal{C}^\infty(U)$ and $x_0$ be an isolated non-degenerate critical point of $f$ such that $f'(x_0) = 0$, $\text{Im} f(x_0) \geq 0$ and $f''(x_0) \neq 0$. Let
\[
g(x) = f(x) - f(x_0) - (f''(x_0)(x - x_0), x - x_0)
\]
and for $j \geq 0$ define the operators,
\[
(2) \quad L_j(u) := (\det(f''(x_0)/2\pi i))^{\frac{1}{2}} \sum_{p-q=j, 2p \geq 3q} i^{-j} 2^{-p} (f''(x_0)^{-1}D,D)^p(g^q u).
\]

The operators $L_j$ are order $2j$ constant coefficient operators, for a given function $f(x)$.

**Theorem 2.2** (Stationary Phase Principle). For an open domain $U \subset \mathbb{R}^n$, let $u(x) \in \mathcal{C}_c^\infty(U)$, $f \in \mathcal{C}^\infty(U)$, and $r > 0$. If the point $x_0 \in U$ is such that $f'(x_0) = 0$, $\text{Im} f(x_0) \geq 0$, and $\det f''(x_0) \neq 0$ is the unique stationary point of $f$ in $\text{Supp}(u)$, then there exist constants $M_j$ such that, for every $k \in \mathbb{Z}$,
\[
\left| \int e^{irf(x)}u(x)dx - \sum_{j<k} e^{irf(x_0)} M_j r^{-\frac{n}{2} - j} \right| < r^{-k}
\]

for $r$ large enough. Furthermore, each $M_j$ is of the form $M_j = L_j(u)(x_0)$, where $L_j$ is a differential operator defined by (2).

Thus a function $\rho(r) = \int e^{irf(x)}u(x)dx$, where $u$ and $f$ satisfy the above conditions, is asymptotically of order $-\frac{n}{2}$ in the highest term as $r \to \infty$.

The following corollary states the same result for a holomorphic family of such integrals.

**Corollary 2.3.** If $u^z(x)$ is a holomorphic family of functions in $\mathcal{C}_c^\infty(U)$, and $x_0$ is the only critical point of $f$ in $\text{supp}(u^z)$. Then for $z$ in a compact set, there exist holomorphic functions $M_j(z)$ and a constant $C > 0$, such that
\[
\left| \int e^{irf(x)}u^z(x)dx - \sum_{j<k} e^{irf(x_0)} M_j(z) r^{-\frac{n}{2} - j} \right| < r^{-k}
\]

for $r$ large enough.

**Proof.** By the Theorem 2.2, we can define $M_j(z) = L_j(u^z)(x_0)$. Now we only have to check that $M_j(z)$ is holomorphic. Since $u^z$ is a holomorphic family of functions $z \mapsto u^z(x_0)$ is a holomorphic function. By the definition of holomorphic family, $\partial_j u^z$ is a holomorphic family of functions, and therefore each $L_j(u^z)$ is, since $L_j$ are constant coefficient operators. Thus $M_j(z)$ are holomorphic functions as desired. \(\square\)

We are now ready to extend $\zeta_{\mathcal{O}, A}$ to a meromorphic function. For any fixed $g \in \Gamma$, we shall denote by $M^g$ the set of fixed points of $g$ acting on $\mathcal{M}$. 

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**TRACES 5**
Proposition 2.4. Let \( k_g = \dim(M^g) \) and \( D \) be a positive definite elliptic operator invariant under \( \Gamma \). Then the function \( z \mapsto \text{Tr}(D^{-z}Ag) \) is holomorphic on the half plane \( \Re(z) > d \), \( d = k_g + \text{order}(A) \), and has a meromorphic extension with possible simple poles at \( z = d, d - 1, d - 2, \ldots \).

Proof. For the purpose of simplifying the notation, we say that two functions over an open domain in the complex plane are equivalent \( f \simeq g \) if the difference \( f - g \) extends to an entire holomorphic function.

Given any neighborhood of the diagonal \( U \), the operator \( D^{-z}A \) can be written as a sum of \( D^{-z}A = P(z) + S(z) \), where the operator \( P(z) \) has its kernel supported in \( U \) and \( S(z) \) is a family of smoothing operators. Since \( \text{Tr}(S(z)g) \) is a holomorphic function, the zeta function \( \text{Tr}(D^{-z}Ag) \simeq \text{Tr}(P(z)g) \). Thus we could assume without any loss of generality that the family \( D^{-z}A \) is supported sufficiently close to the diagonal.

For \( \Re(z) > n + \text{order}(A) \), the operators \( D^{-z}A \) are trace class, and hence the map \( z \mapsto \text{Tr}(D^{-z}Ag) \) is holomorphic on the half plane \( \Re(z) > n + \text{order}(A) \). We wish to show that it has a meromorphic continuation.

Let us fix a \( \Gamma \) invariant metric on \( M \). For every point \( x \in M \), choose a normal coordinate chart \( U_x \). We can assume that the closures \( \overline{U_x} \) and \( g\overline{U_x} \) are either equal or disjoint. Thus, when \( x \in M^g \), we assume that \( U_x \) is \( g \) invariant. We choose a finite subcover \( U_i = U_{x_i} \) for finitely many \( x_i \in M \). We choose \( \psi_i \prec U_i \) a partition of unity subordinate to this cover.

For any trace class pseudo-differential operator \( P \),

\[
\text{Tr}(Pg) = \sum_{j,k} \text{Tr}(\psi_k P \psi_j g)
= \sum_{j,k} \text{Tr}(\psi_j^\frac{1}{2} g \psi_k P \psi_j^\frac{1}{2}).
\]

If the kernel of \( P \) is supported in the open set \( \Omega \supset \Delta \) such that \( \overline{\Omega} \cap g\overline{U_i} \times \overline{U_i} = \emptyset \) whenever \( x_i \) is not a fixed point, then the sum above reduces to sum about the fixed point manifold namely,

\[
\text{Tr}(Pg) = \sum_{U_i \cap g \neq \emptyset} \text{Tr}(\psi_j^\frac{1}{2} g P \psi_j^\frac{1}{2}).
\]

The trace of a smoothing family is an entire function, and hence we could subtract a suitable family of smoothing operators from \( D^{-z}A \), and we assume that its supported in a small enough neighborhood of the diagonal as above. We can thus restrict ourselves to coordinate neighborhoods of the fixed points.

Similarly, we only need to consider operators of the form \( \psi_j^\frac{1}{2} D^{-z}A \psi_j^\frac{1}{2} \) when we work on local coordinates with \( \psi_j \) as partition of unity subordinate to a cover as above. Then \( D^{-z}A \) restricted to the functions supported in such a neighborhood \( U \) could be given in terms of quantization of a symbol.
\[ p^z(x, \xi) \in S^{-z+a}(U) \]

(3) \[ \hat{q}(p^z)u(x) = \int e^{i(x-y,\xi)}p^z(x, \xi)u(y)dyd\xi = \psi^zD^zA\psi^z u(x). \]

Thus by \( \Gamma \) invariance of the metric we get
\[
\psi^zD^{-z}A\psi^z g(u(x)) = \int e^{i(x-y,\xi)}p^z(x, \xi)g \cdot u(y)dyd\xi
= \int e^{i(x-y,\xi)}p^z(x, \xi)u(y)dyd\xi.
\]

By abuse of notation, we shall continue to denote the operator of the form \( \psi^zD^{-z}A\psi^z \) as \( D^{-z}A \).

The distributional kernel of \( D^{-z}A \) restricted to \( U \times U \) is therefore given by
\[
K(x, y) = \int e^{i(x-y,\xi)}p^z(x, \xi)d\xi.
\]

Thus when the operator \( D^{-z}A \) is trace class (for sufficiently large values of \( z \)), the kernel \( K(x, y) \) is integrable and the trace is given by
\[
(4) \quad \zeta_{g, A, U}(z) = \int K(x, y)dx = \int e^{i(x-y,\xi)}p^z(x, \xi)d\xi dx.
\]

The integral on the right hand side is absolutely convergent for \( \text{Re}(z) > n + a \), and we would like to establish the continuation of this integral to a meromorphic function on the complex plane.

To apply Stationary phase to the Integral (4), we notice first that in \( \Gamma \) invariant normal coordinates, \( g \) becomes an orthogonal linear map with \( g^* = g^{-1} \). Then the derivative of the phase function \( f(x, \eta) = \langle x - gx, \eta \rangle \) would be
\[
(5) \quad Df(x, \eta)(\alpha_1, \beta_1) = \langle x - gx, \beta_1 \rangle + \langle \alpha_1 - ga_1, \eta \rangle
\]

that implies that the stationary points \( (x, \eta) \) of the phase function correspond to the points \( (x, \eta) \) such that \( x = gx \) and \( \eta = gy \). Furthermore,
\[
(6) \quad (g^*)'(x, \eta) = D^2f(x, \eta)(\alpha_2, \beta_2) = \langle \alpha_2 - g\alpha_2, \beta_1 \rangle + \langle \alpha_1 - g\alpha_1, \beta_2 \rangle.
\]

We choose a special normal coordinate chart. Since we are only interested in the neighborhoods of \( x \in M^g \), the tangent space here decomposes as \( T_xM = T_xM^g \oplus (1-g)T_xM \). We choose an orthonormal basis for \( T_xM^g \) and \( (1-g)T_xM \) to determine a normal coordinates at \( x \). Set \( U^9 = \text{exp}(T_xM^g) \) and \( (1-g)U = \text{exp}((1-g)T_xM) \). That is,
\[
U \ni x = (x_1, x_2) \in U^9 \times (1-g)U.
\]

After trivializing the the cotangent bundle \( T^*U = U \times \mathbb{R}^n \) one can also rewrite
\[
\mathbb{R}^n = \mathbb{R}^n^g \times (1-g)\mathbb{R}^n,
\]

and henceforth \( \xi = (\xi_1, \xi_2) \). Set \( \pi_x(x) = x_1 \) and \( \pi_\xi(\xi) = \xi_1 \). And for each \( (x', \xi') \in T^*U^9 \), set \( F(x', \xi') := \pi_{x'}^{-1}(x') \times \pi_{\xi'}^{-1}(\xi') \subset T^*U \).
Now the symbol $p^z(x, \xi) \in S^{-z+\alpha}(\mathcal{U})$ has an asymptotic expansion of the form
\[ p^z(x, \xi) \simeq p^z_{-z+a}(x, \xi) + p^z_{-z+a-1}(x, \xi) + \ldots + p^z_{-z+a-j}(x, \xi) \ldots, \]
where each symbol $p^z_l(x, \xi)$ is homogeneous of degree $l$ for $|\xi| \geq 1$. By the holomorphicity of the family $\mathcal{D}^{-z}A$, each $p^z_l(x, \xi)$ is also holomorphic in $C^\infty(T^*\mathcal{U}/\{0\})$. Further this expansion determines the operator $\mathcal{D}^{-z}A$ up to a smoothing operator. If the $\text{Re}(z)$ is in the interval $[J-1, J)$, then the operator
\[ Q(z) := \hat{q} \left( p^z(x, \xi) - \sum_{j \leq J+n} p^z_{-z+a-j}(x, \xi) \right), \]
is trace class and hence the function $\text{Tr}(Q(z))$ is entire. Thus
\[ \zeta_{g,A,\mathcal{U}}(z) \simeq \sum_{\text{Re}(l) \geq -n} \text{Tr}(\mathcal{D}^{-z}\hat{q}(p^z_l)g). \]

Now it suffices to check that $z \mapsto \text{Tr}(\mathcal{D}^{-z}\hat{q}(p^z_l)g)$ has a meromorphic continuation for the operators defined by each term in the right-hand side, that is, for $l = -z + a, -z + a - 1, -z + a - 2 \ldots$ so on till $\text{Re}(l) \geq -n$. (Because as we have just shown only finitely many of these terms contribute to the trace.) For this one must show that the integral in (4) for $\hat{q}(p^z_l)$ has a meromorphic continuation. Since this integral on any compact subset in $\xi$ would yield an entire function, we could restrict the domain where $p^z_l$ is homogeneous, which is $|\xi| > 1$.

Let $r = |\xi|$. Let $dS^n(r)$ be the volume form on a sphere of radius $r$. Then
\[ \zeta_{g,\hat{q}(p^z_l),\mathcal{U}}(z) = \int_{|\xi|>1} e^{i(x_2-gx_2,\xi)} p^z_l(x_1 + x_2, \xi) d\xi dx_2 \]
+ entire function
\[ \simeq \int_1^\infty \int_{|\xi|=r} e^{i(x_2-gx_2,\xi)} p^z_l(x_1 + x_2, \xi) dS^n(r) dx dr. \]

Let $dS^n = dS^n(1) = r^{1-n}dS(r)$ be volume form on the unit sphere in $\mathbb{R}^n$. We would also use the notation $dS^n(\xi)$ to emphasize the variable used in integration. From the homogeneity of $P^z_l$ we obtain
\[ \zeta_{g,\hat{q}(p^z_l),\mathcal{U}}(z) \simeq \int_1^\infty r^{n+l-1} \int_{|\xi|=1} e^{ir(x_2-gx_2,\xi)} p^z_l(x_1 + x_2, \xi) dS^n(\xi) dx dr \]
\[ \simeq \int_1^\infty r^{n+l-1} \int_{|\xi_1|<|\xi_2|} e^{ir(x_2-gx_2,\xi)} p^z_l(x_1 + x_2, \xi) dS^n(\xi) dx_2 dx_1 dr 
+ \int_1^\infty r^{n+l-1} \int_{|\xi_1|\geq|\xi_2|} e^{ir(x_2-gx_2,\xi)} p^z_l(x_1 + x_2, \xi) dS^n(\xi) dx_2 dx_1 dr. \]

Set
\[ \zeta_{g,\hat{q}(p^z_l),\mathcal{U}}(z) \simeq I_1(p^z_l, z) + I_2(p^z_l, z) \]
The first integral $I_1(p^*_i, z)$ above yields an entire function. On the set $|\xi_1| < |\xi_2|$, we have $|\xi_2| > 0$ and thus by $5$ the phase function of the integral

$$I_1(p^*_i, z) = \int_{|\xi_1| < |\xi_2|} e^{ir(x_2 - g(x_2, \xi_2))p^*_i(x_1 + x_2, \xi)dS^n(\xi)dx_2dx_1}$$

will not have any critical points. Hence by Theorem 2.1, it must be rapidly decreasing asymptotically as $r \to \infty$. After integrating in $r$, we have $I_1(p^*_i, z)$ converges for all $z$ and gives an entire function.

However, the phase function $f$ of the integral

$$I_2(p^*_i, z) = \int_{|\xi_1| \geq |\xi_2|} e^{ir(x_2 - g(x_2, \xi_2))p^*_i(x_1 + x_2, \xi)dS^n(\xi)dx_2dx_1}$$

has a stationary phase point whenever $x_2 = 0$ and $\xi_2 = 0$, that is indeed the fixed point manifold $S^*U^g$. For $\xi = (\xi_1, \xi_2)$ on the unit sphere with $|\xi_1| \geq |\xi_2|$, let $\eta_1 = \frac{\xi_1}{|\xi_1|} \in S^{kg}(\xi_1)$ and $\eta_2 = \frac{\xi_2}{|\xi_1|}$. Then rewriting (7) above we have

$$\int \left( \int_{|\eta_2| \leq 1} e^{ir(x_2 - g(x_2, \eta_2))p^*_i(x_1 + x_2, \eta_1 + \eta_2)d\eta_2dx_2} \right) dS^k(\eta_1)dx_1dr.$$

But on each normal fiber $F_{(x_1, \eta_1)}$ of a fixed point $(x_1, \eta_1) \in S^*M^g$, there is only one stationary point of $f$ namely $(x_1, \eta_1, 0, 0)$ and the Hessian $D^2f \neq 0$. Then by the stationary phase method 2.2, for every integer $i \geq 0$ there exist $\mathcal{M}_j(z)(x_1, \eta_1)$ $0 \leq j \leq i$, smooth in $x, \xi \in S^*U$ and holomorphic in $z$, and a constant $c_i > 0$ such that

$$\left| \int_{|\eta_1| \geq |\eta_2|} e^{ir(x_2 - g(x_2, \eta_2))p^*_i(x_1 + x_2, \eta_1 + \eta_2)d\eta_2dx_2} - \sum_j \mathcal{M}_j(z)(x_1, \eta_1)r^{-n+kg-j} \right| < C_i r^{-i} \quad \text{as } r \to \infty.$$

Set

$$M_j(z) = \int \mathcal{M}_j(z)(x_1, \eta_1)dx_1d\eta_1.$$

Therefore, up to a holomorphic function,

$$I_2(p^*_i, z) \approx \int_1^\infty \sum_j M_j(z)r^{kg+l-1-j} dr$$

$$= \sum_j \frac{M_j(z)}{kg + l - j} = \sum_j \frac{M_j(z)}{kg + a - z - j - i}.$$

The last equality is due to the fact that $l$ is the degree of homogeneity, and takes value in the set $a - z - i, (i \in \mathbb{Z})$. Thus $I_2$ and hence $\zeta_{g,A}$ would be meromorphic with at most simple poles as specified.

\[\square\]

Remark 2.5. In the proof of Lemma 2.4 above, the operators $D^{-z}A$ could be replaced by any holomorphic family $R(z)$ with order $R(z) = -z + a$. 
Remark 2.6. For any order $m$ operator positive $D$, the operator $D = D^{\frac{1}{m}}$ is of order one and hence for the operator $D$, its zeta function $\zeta_{g,A}(z) = \text{Tr}(D^{-z}Ag)$ would have a meromorphic extension to the complex numbers with possible poles at $\frac{d}{m}, \frac{d-1}{m}, \ldots$.

Remark 2.7. Nonsimple poles can occur in the zeta function of operators on spaces with singularities [18].

Although the function $\zeta_{g,A}$ does depend on the choice of an invariant operator $D$, the residue of $\zeta_{A,g}$ at $z = 0$ does not. Indeed if $D_1$ is another such operator invariant under $\Gamma$, then let $R(z) = D_1^{-z}D^z$. The family $R(z) \in \Psi^0(M)$ and $R(0) = I$. Thus we define

$$\tau_g(A) := \text{res}_{z=0} \text{Tr}(D^{-z}Ag) = \text{res}_{z=0} \text{Tr}(R(z)D^{-z}Ag) = \text{res}_{z=0} \text{Tr}(D_1^{-z}Ag).$$

It is also immediate from the definition that the residue $\tau_g$ defined in Equation (9) descends to a function on $A(M) = \Psi^\infty(M)/\Psi^{-\infty}(M)$.

In particular, we obtain the proof of Proposition 2.4 that the residue vanishes if $g$ has only isolated fixed points.

Corollary 2.8. If $g$ has no fixed points on the co-sphere bundle $S^*M$, then $\tau_g = 0$.

This can explain in a certain way the result of Atiyah and Bott in [2].

Corollary 2.9. Let $B(M) = A(M) \rtimes \Gamma$. For any conjugacy class $\langle \gamma \rangle \in \langle \Gamma \rangle$, the map $\text{Tr}_R^{\langle \gamma \rangle} : B(M) \mapsto \mathbb{C}$ given by

$$\text{Tr}_R^{\langle \gamma \rangle} \left( \sum_{g \in \Gamma} A_g g \right) := \sum_{g \in \langle \gamma \rangle} \tau_g(A_g)$$

is a trace on $B(M)$.

Proof. By linearity it suffices to check that each $\tau_\gamma$ is a trace on the generators. If $g, h \in \Gamma$ then $gh$ and $hg$ belong to same conjugacy class and thus

$$\tau_{(gh)}[Ag, Bh] = \text{res}_{z=0} \text{Tr}(D^{-z}[Ag, Bh]).$$

For any operators $A, B, C$, such that $ABC, ACB, BAC$ are trace class and $\text{Tr}(ACB) = \text{Tr}(BAC)$ the following obviously holds:

$$\text{Tr}(A[B, C]) = \text{Tr}([A, B]C).$$

Thus for $\text{Re}(z)$ large enough,

$$\text{Tr}(D^{-z}[Ag, Bh]) = \text{Tr}([D^{-z}, Ag]Bh) = \text{Tr}([D^{-z}, A]gBh).$$

Here the equalities holds because $D$ commutes with $g$. But then $[D^{-z}, A] = zR(z)$, for some holomorphic family $R(z)$. Since $\text{Tr}(R(z))$ can at most have a simple pole at $z = 0$, $\text{res}_{z=0}zR(z) = 0$. \hfill \Box
Remark 2.10. If the elements in $\langle \gamma \rangle$ have no fixed points on the cosphere bundle $S^*M$ then $\text{Tr}^{(\gamma)}_R = 0$ as the zeta function of these elements have a holomorphic continuation to the complex plane. But here we remark that by an argument as above the sum of regular values $Ag \to \zeta_{A,g}(0)$ defines a trace on $\Psi^\infty(M) \rtimes \Gamma$. This trace will still be non-trivial if $g$ has isolated fixed points on $M$, but will not descend to be a trace on $\mathcal{A}(M) \rtimes \Gamma$.

For simplicity, let us assume that $S^*M^g$ is connected or empty for the rest of our discussion. Then the Hochschild homology calculation in [8] show that all the traces of the algebra $\mathcal{B}(M)$ are linear combinations of the traces $\text{Tr}^{(\gamma)}_R$.

Lemma 2.11. Let $m = -k_g$, and let $A$ be of order $m$ with principal symbol $a_m = \sigma(A)$. Then there exists a constant $C > 0$ such that $\tau_g(A) = C \int_{S^*M^g} \sigma(A) d\text{vol}$.

In particular, if $S^*M^g \neq \emptyset$ then $\tau(D^{-k}g) \neq 0$ for a positive invariant order one operator $D$.

Proof. Since the principal symbol of an operator is defined independent of coordinates, we can work with normal local coordinates. By Proposition 2.4, the first possible pole of $\zeta_{g,A}(z)$ would be at $z = 0$. In each normal local coordinate, the value of the residue is given by $M_0(0)$, which can be obtained by the Equation (8)

$$M_0(z) = \int M_0(z)(x_1, \eta_1) dS^k(\eta_1) dx_1.$$  

By the stationary phase principal 2.2, we have

$$M_0(0)(x_1, \xi_1) = L_0(p_{-k_g}^0(x_1, \xi_1)) = C \sigma(A)(x_1, \xi_1),$$

where $C = \det(D^2f(x_1, \xi_1)/2\pi)^{-\frac{1}{2}}$ is a constant independent of the fixed point $(x_1, \xi_1)$. Since the principal symbol $\sigma(A)$ is invariantly defined under change of coordinates, the local formula is independent of coordinates and adds up to give the result. □

Related calculations (but not in the equivariant setting) can be found in several other papers, including [15, 16, 17, 19, 20].

3. Local Expression for the traces

We now establish a local formula for the traces $\text{Tr}^{(\gamma)}_R$ for all operators. This in particular would require a generalization of Lemma 2.11 and a more elaborate use of the stationary phase principal. For any operator $A$ of order $m$, the calculation in Proposition 2.4 yields a local formula for the residue of $\zeta_{g,A}$ at $z = 0$. To start, fix a $\Gamma$ invariant metric on $M$. By compactness of $M$ let $r$ be the minimum value of injectivity radius on $M$. Let $U$ be a tubular neighborhood of $M^g$ such that $\text{dist}(y, gy) < \frac{r}{2}$ for any $y \in U$. Denote by $j(x)$
the tangent at $x$ to the unique geodesic $\gamma$ such that $\gamma(0) = x$ and $\gamma(1) = gx$. Then we define a function on $T^*U$ by

$$f(x, \xi) := \langle \xi, j(x) \rangle.$$  

In fact locally in normal coordinates $f(x, \xi) = \langle x - gx, \xi \rangle$. Thus Equation 5 implies that $M^g$ is the critical manifold of $f$ and by Equation 6 $f$ is Morse Bott, the Hessian $f'' =: D^2f$ is a non-degenerate bilinear form on the normal bundle $N(T^*U : T^*M^g) = (1 - g)T^*U_{T^*M^g}$. Therefore the inverse $D^2f^{-1}$ is a nondegenerate bilinear form on the conormal bundle $N^*(T^*U : T^*M^g)$.

Let $U = \{x_1, x_2\}$ be normal local coordinates at a fixed point $x \in M^g$ and choose $\eta_1, \eta_2$ to give a trivialization over $U$ of the cotangent bundle $T^*M^g$ and its orthogonal $(1 - g)T^*M$ in the cotangent bundle. Then by the Equation (8),

$$\text{res}_{z_0} \text{Tr}(\psi^2 D^{-z} A g \psi^2) = \sum_{j=0}^{m+k_y} \int \overline{M}_j(x_1, \eta_1) dS^{k_y}(\eta_1) dx_1.$$  

Let $D$ denote the gradient operator on the normal bundle $(1 - g)T^*M$, that is, $D^\perp = i\partial_{x_2} + i\partial_{\eta_2}$. Again by stationary phase principle, each $M_j$ can be calculated as

$$\overline{M}_j(0)(x_1, \xi_1) = L_j(p^0_{\perp k_y+j}(x_1, \eta_1))$$

$$= C(f^m_{(x_1, \eta_1)} D^\perp, D^\perp) \left( p^0_{\perp k_y+j}(A)(x_1, \eta_1) \right),$$

where the constant $C = \left( \det |\frac{f_{x_2, \eta_2}}{2\pi i}| \right)^{-\frac{1}{2}}$.

Now by using the invariance of the residue under the choice of of operator defining the $\zeta_{g,A}$ one has:

**Proposition 3.1.** Let $U(x_1, x_2)$ be normal coordinates near a point $x \in M^g$ and let $A \in \Psi^m(M)$ be an order $m$ operator given on $U$ by the quantization of the symbol $a(x, \eta) \sim a_m(x, \eta) + a_{m-1}(x, \eta) + \ldots$. Let $dS^{k_y}(\eta)$ denote the volume form on $k_y - 1$ dimensional sphere in $S^*U^g \simeq U \times S^{k_y-1}$.

Let

$$W_{M^g}(A) := \left( \sum_{j=0}^{m} \int_{S^*(\eta_1)} C(f^m_{(x_1, \eta_1)} D^\perp, D^\perp) a_{-k_y+j}(x, \eta) dS^{k_y}(\eta_1) \right) dx_1.$$  

Then $W_{M^g}(A)$ is invariant under $g$-equivariant change of normal coordinates near $M^g$.

**Proof.** Let $\phi : M \to M$ be an $\Gamma$ equivariant isometry. Then $\phi(M^g) = M^g$. Let $A$ be supported in a normal neighborhood $U$ as above and let $\phi(U) = V$. 

Then from the discussion above (applied to $\phi_\ast A$) it follows that:

$$W_{M^g}(A) = \text{res}_{z=0} \ Tr(D^{-z}\phi A \phi^{-1} g)$$

Since $\phi$ is an isometry,

$$\phi^{-1} D^{-z} \phi = \phi^\ast D^{-z} \phi = D_1^{-z},$$

Where $D_1$ is another order 1 positive elliptic operator. Since the residue is independent of choice of such an operator, we have

$$W_{M^g}(\phi_\ast A) = \text{res}_{z=0} \ Tr(D_1^{-z} Ag) = W_{M^g}(A).$$

\[\square\]

### 4. Asymptotics for representations in eigenspaces

Let $\pi \in R(\Gamma)$ be a representation of the group $\Gamma$. We can define a trace on the cross-product algebra $B(M)$ corresponding to $\pi$ as

$$\tau_\pi \left( \sum_{g \in \Gamma} A_g g \right) := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{\pi}(g) \tau_g(A_g g).$$

Since $\chi_\pi$ is class function, this trace is a linear combinations of the traces $Tr_R$ defined before. Let $\epsilon := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$ the idempotent in $\mathbb{C}[\Gamma]$. Then one can likewise obtain traces on the invariant algebra $A(M)^\Gamma$ by $A \to \tau_\pi(A \epsilon)$. Such a trace is the residue of a $\zeta$ function namely the one obtained from the meromorphic extension using Proposition 2.4 of

$$\zeta_{\pi,A}(s) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{\pi}(g) \ Tr(D^{-s} Ag) \quad \text{Re}(s) > n.$$ 

Here $D$ is $\Gamma$ invariant positive order one operator.

We say that action of $\Gamma$ on $M$ is faithful or effective if for any $g \in \Gamma$ the fixed point manifold $M^g = M$ then $g$ must be the identity element.

**Lemma 4.1.** Let $n = \dim(M)$. If the action of $\Gamma$ on $M$ is effective, then $\zeta_{\pi,Id}$ has a pole at $s = n$.

**Proof.** Let $k_g = \dim M^g$ for any group element $g$. If the action of the group $\Gamma$ is effective then $k_g < n$ whenever $g$ is not the identity element $e$. Since $\zeta_{e,Id}(s)$ does have a pole at $s = n$ and all the $\zeta_{g,Id}$ do not, $\zeta_{\pi,Id}$ is holomorphic in the half plane $\text{Re}(s) > n$ and must have a pole at $s = n$. \[\square\]

For the rest of section, we assume that the $\Gamma$ action is effective. Let $\{\lambda_i\}$ be the set of eigenvalues of $D$. Then each eigenspace $V_{\lambda_i} = \ker(D - \lambda_i)$ is
invariant under $\Gamma$, and so acquires a representation $\pi_{V_{\lambda_i}}$ of $\Gamma$. We count the multiplicity of representation $\pi$ asymptotically in the eigenspaces $V_{\lambda_i}$. Let

$$N_{\pi,D}(\lambda) := \sum_{\lambda_i \leq \lambda} \langle V_{\lambda_i}(g), \pi \rangle = \sum_{\lambda_i \leq \lambda} \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_\pi(g) \chi_{V_{\lambda_i}}.$$ 

Let $m_i = \langle V_{\lambda_i}, \pi \rangle$ denote the multiplicity of occurrence of $\pi$ in the $V_{\lambda_i}$ if $\pi$ is irreducible. For the purpose of establishing the asymptotics of the function $N$, we recall the following Tauberian theorem.

**Theorem 4.2** (Tauberian Theorem). Let $v(x)$ be a non-decreasing function with $v(x) = 0$ for all $x < 1$. Suppose the function

$$f(z) = \int_1^\infty x^{-z} dv(x)$$

exists for $\text{Re}(z) > 1$ and is analytic, and suppose that there is a constant $A$ such that the analytic function

$$F(z) = f(z) - \frac{A}{z - 1}$$

extends continuously to the closed half plane $\text{Re}(z) \geq 1$. Then asymptotically,

$$\frac{v(x)}{x} \to A, \text{ as } x \to \infty.$$ 

**Corollary 4.3** (Asymptotic estimates). Let the group action of $\Gamma$ on $M$ be faithful. Then for an irreducible representation $\pi$ and an $\Gamma$ invariant operator $D$, the multiplicity counting function, $N_{\pi,D}(\lambda) \sim C \dim(\pi) \lambda^{n/m}$, where $C > 0$ and $m = \text{ord}(D)$.

**Proof.** First we notice that for sufficiently large values of the complex variable $s$, the zeta functions have a nice expression of the form

$$\zeta_{g,Id}(s) = \text{Tr}(D^{-s} g) = \sum_i \text{Tr}(D^{-s} g|_{V_{\lambda_i}}) \tag{10}$$

$$= \sum_i \chi_{V_{\lambda_i}}(g) \lambda_i^{-s}, \quad \text{Re}(s) > k_g. \tag{11}$$

Then adding them all up, we get

$$\zeta_{\pi,Id}(s) = \frac{1}{|\Gamma|} \sum_{\pi} \chi_\pi(g) \sum_i \chi_{V_{\lambda_i}}(g) \lambda_i^{-s} \quad \text{Re}(s) > k_g \tag{12}$$

$$= \sum_i \lambda_i^{-s} m_i \tag{13}$$

$$= \int_0^\infty \lambda^{-s} dN_{\pi,D}(\lambda). \tag{14}$$

Since by Lemma 4.1 the function $\zeta_{\pi,Id}(s)$ is holomorphic on the half plane $\text{Re}(s) > n$ and has a non-zero residue $K$ at $s = n$, it follows that $\zeta_{\pi,Id}(s) - \frac{K}{s-n}$ has a continuation to the closed half line $\text{Re}(s) \geq n$. Let $C = \text{Tr}_R(D^{-n})$, 

then we note that $K$ is of the form $K = C \chi_{\pi}(e)|_{\Gamma}$. The estimate follows by direct application of Theorem 4.2. □

One could also employ Proposition 2.4 again to obtain relative asymptotic estimates for a pair of irreducible representations $\pi_j$, $j = 1, 2$ of same dimension, and compare their occurrence of in eigenspaces of our invariant operator $D$ assuming that for all but a finitely many eigenvalues $\lambda_i$,

$\langle V_{\lambda_i}, \pi_1 \rangle \geq \langle V_{\lambda_i}, \pi_1 \rangle$.

**Corollary 4.4.** Let $\pi_j$ be irreducible representations as above, and let

$N_{\pi_1, \pi_2, D}(\lambda) := \sum_{\lambda_i \leq \lambda} \langle V_{\lambda_i}, \pi_1 - \pi_2 \rangle$.

Let $k = \max_{g \in \Gamma} \{\dim(M^g)|\chi_{\pi_1}(g) \neq \chi_{\pi_2}(g)\}$. Then asymptotically

$N_{\pi_1, \pi_2, D}(\lambda) \approx C^{\lambda_k}$.

**Remark 4.5.** One application for Weyl’s theorem is to establish the convergences of numerical schemes for solving boundary value problems using spectral decomposition. It can be hoped that the equivariant asymptotic formulas could be applied to establish rates of convergences for numerical solution to the boundary value problems which involve some symmetries but over a domain with not necessarily smooth boundary.

Fix a $\Gamma$-representation $\pi$. Now for an $\Gamma$-invariant compact operator $D$ on $L^2(M)$, let $\{\mu_i\}$ be the decreasing sequence of eigenvalues counted with multiplicity of $\pi$. Then each eigenvalue $\lambda$ is counted the number of times $\pi$ occurs as a subrepresentation in $\ker(D - \lambda I)$. We shall denote this $\pi$-multiplicity of an eigenvalue $\lambda_j$ by $m(\pi)_j$.

**Definition 4.6.** We say that a positive invariant operator $D$ is $\pi$-measurable if the following limit exists and is finite

$Tr^+_{\pi}(D) := \lim_{N \to \infty} \frac{\sum_{i \leq N} \mu_i}{\log N} = \lim_{N \to \infty} \frac{\sum_{\sum m(\pi)_j \leq N} m(\pi)_j \lambda_j}{\log N}$

In particular when $\pi$ is the trivial representation for the trivial group then $Tr^+_{\pi} = Tr^+$ is the Dixmier trace functional which is defined on the measurable elements of the Dixmier ideal $L^{1, \infty}$.

**Proposition 4.7.** With the above notation let $D$ be a positive invariant pseudodifferential operator operator of order $-n$ where $n = \dim(M)$ then $D$ is $\pi$ measurable and,

$Tr^+_{\pi}(D) = Tr_R(D) = Tr^+(D)$.

Here of course the last equality is the Trace formula of Connes[5].

**Proof.** In view of Connes’ theorem it suffices to show $Tr^+_{\pi}(D) = Tr_R(D)$. We need the following lemma
Lemma 4.8. Let $\{a_k\}$ be a decreasing sequence of positive numbers such that $a_k \searrow 0$ and $\sum a_k^s < \infty$ for $s > 1$ and $\sum a_k^s \rightarrow \frac{1}{s-1}$ as $s \searrow 1$. Then $\sum_{k=1}^N a_k \sim \log N$.

We refer the reader to [10] for a proof of the above lemma.

As before we now define

$$f(s) := \frac{1}{|\Gamma|} \sum_{\Gamma} \chi_\pi(g) \text{Tr}(D^s g)$$

$$= \frac{1}{|\Gamma|} \sum_{\Gamma} \chi_\pi(g) \sum_j \chi_{V_{\lambda_j}}(g) \lambda_j^s$$

$$= \sum_j \lambda_j^s m(\pi)_j = \sum_i \mu_i$$

By Proposition 2.4 $f(s)$ is analytic in the half plane $\text{Re}(s) > 1$ and has a simple pole at $s = 1$. Then the proposition follows from the lemma above.

□

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