Holography, unfolding and higher spin theory

M A Vasiliev

I E Tamm Department of Theoretical Physics, Lebedev Physical Institute, Leninsky prospect 53, 119991 Moscow, Russia

E-mail: vasiliev@lpi.ru

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Abstract
Holographic duality is argued to relate classes of models that have equivalent unfolded formulation, hence exhibiting different space-time visualizations for the same theory. This general phenomenon is illustrated by the AdS4 higher spin gauge theory shown to be dual to the theory of 3d conformal currents of all spins interacting with 3d conformal higher spin fields of Chern–Simons type. Generally, the resulting 3d boundary conformal theory is nonlinear, providing an interacting version of the 3d boundary sigma model conjectured by Klebanov and Polyakov to be dual to the AdS4 higher spin theory in the large N limit. Being a gauge theory, it escapes the conditions of the theorem of Maldacena and Zhiboedov, which force a 3d boundary conformal theory to be free. Two reductions of particular higher spin gauge theories where boundary higher spin gauge fields decouple from the currents and which have free-boundary duals are identified. Higher spin holographic duality is also discussed for the cases of AdS3/CFT2 and duality between higher spin theories and nonrelativistic quantum mechanics. In the latter case, it is shown in particular that (dS) AdS geometry in the higher spin setup is dual to the (inverted) harmonic potential in the quantum-mechanical setup.

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1. Introduction

Higher spin (HS) gauge theories describe interactions of massless fields of all spins. The first example of fully nonlinear HS theory was given in the 4d case in [1], while its modern formulation was worked out in [2] (see [3] for a review). A specific property of HS gauge theories is that consistent interactions of propagating massless fields require a curved background which provides a length scale in HS interactions that contain higher derivatives. (A)dS is the most symmetric curved background compatible with HS interactions. The AdS$_4$ HS model is the simplest nontrivial in the sense that $d = 4$ is the lowest dimension where HS massless fields propagate. After the AdS/CFT correspondence conjecture was put forward in [4–6], the fact that HS theories are most naturally formulated in the AdS background was conjectured to play a role in the context of AdS/CFT correspondence [7–11]. This expectation conforms to the fundamental result of Flato and Fronsdal [12] on the relation between tensor products of 3d conformal fields (singletons) and infinite towers of 4d massless fields that appear in HS theories.

In the important work of Klebanov and Polyakov [13], it was argued that the HS gauge theory of [2] should be dual to the 3d $O(N)$ sigma model in the $N \to \infty$ limit. The Klebanov–Polyakov conjecture was checked by Giombi and Yin in [14, 15] where it was shown in particular how the bulk computation in HS gauge theory reproduces at least some of conformal correlators in the free 3d theory. (For related computations in free HS theory see also [16–19].) Recently, Maldacena and Zhiboedov [20] addressed the question on restrictions imposed on a boundary 3d conformal theory by HS conformal symmetries. Assuming very general conditions on conformal theory which included unitarity, locality and conformal operator product algebra, they were able to show that a conformal HS theory, that possesses an HS conserved current, should be free. This conclusion seemingly suggests that any AdS$_4$ HS theory should be equivalent to a free-boundary theory at least in the most symmetric vacuum.
The primary motivation for this work was to analyze directly a 3d dual of the AdS4 HS theory by means of the unfolded dynamics approach which makes this analysis straightforward, describing evolution with respect to different coordinates as independent mutually commuting flows. This property allows us to obtain 3d field equations from 4d HS equations simply by reducing four space-time coordinates of AdS4 to three, relating directly two seemingly different theories in a way anticipated from AdS/CFT correspondence.

An important ingredient of the duality is that unfolded equations for 3d conserved currents result from the 3d reduction of 4d unfolded massless equations. The key observation is based on the interpretation of currents as rank-2 fields within the approach of [21] where it was shown that conserved currents, built from the free fields described by $C(Y|X)$, where $Y$ are auxiliary spinor (twistor) variables while $X$ describe space-time coordinates, are described by the fields $J(Y'|X)$ with the doubled number of spinor variables $Y', i = 1, 2$. In particular, free 3d massless fields are described by functions of two-component spinors $y^a$ and space-time coordinates $x^a = x^{d \alpha}$, where $\alpha, \beta = 1, 2$, while 3d conserved currents $J(y'|x)$ depend on a pair of spinors $y'_i$. On the other hand, a 4d massless field is described by a field $C(Y|X)$, where $Y = (y^a, \bar{y}^\beta)$ are complex two-component spinors and $X = x^{d \alpha}$ encode four real coordinates in the form of $2 \times 2$ Hermitian matrices. It suffices to modify hermiticity conditions for two-component spinors to identify 4d massless fields with 3d conformal conserved currents. The pullback of the known nonlinear 4d massless field equations to a 3d subspace $\Sigma^3 \in \text{AdS}_4$ gives nonlinear 3d equations which describe interactions of conformal current fields with 3d conformal HS gauge fields. Precise identification only requires an appropriate change of the reality conditions and transition to the conformal frame where conformal symmetries are manifest.

In this setup, holography takes place for generic 3d surface $\Sigma$. However, the map from the AdS frame to the conformal frame, where conformal symmetries are properly realized, turns out to be nonlocal for general $\Sigma$. In the unfolded formulation of HS theory, this map has simple meaning in terms of noncommutative twistor variables $Y$, describing the transition from the Weyl star product in the bulk theory to the normal ordered one in the conformal frame. However, in space-time terms, it may look obscure for general $\Sigma$. Remarkably, in the limit where $\Sigma$ is AdS4 infinity, the correspondence between the two frames becomes local and very simple, directly identifying 4d massless fields with (sources for) 3d currents in accordance with the original AdS/CFT prescription of [5, 6].

The conclusion that nonlinear AdS4 HS gauge theory is dual to a nonlinear 3d theory does not contradict the Maldacena–Zhiboedov theorem [20] because the boundary theory turns out to be a gauge theory with currents interacting through 3d Chern–Simons conformal HS gauge fields. As such it escapes at least one of the assumptions of unitarity, locality and/or conformal invariance. Indeed, gauge degrees of freedom correspond to null states, while a gauge fixing procedure breaks covariance and/or locality. There are, however, two special configurations for AdS4 HS fields that correspond to Dirichlet and Neumann boundary conditions in the so-called A and B models, where the 3d superconformal HS gauge fields decouple from currents. These correspond to the free bosonic and fermionic boundary theories in accordance with Klebanov–Polyakov [13] and Sezgin–Sundell [22] conjectures as well as Maldacena–Zhiboedov theorem [20].

The analysis of HS holography within unfolded dynamics shows that the phenomenon of holographic duality is absolutely general, taking place in any AdS theory. To make the correspondence manifest, the theory in question has to be reformulated in the unfolded form. This makes the correspondence to a large extent tautological via reduction of space-time coordinates in the unfolded theory. More precisely, the unfolding procedure effectively reformulates a theory in terms of appropriate (generalized) twistor variables $Y$ rather than
directly in terms of space-time coordinates $X$. In this setup, holography relates theories in different space-times $M$ (coordinates $X$) that have the same description in the twistor space $T$ (coordinates $Y$). Reformulation of a theory in the unfolded form makes it straightforward to identify its holographic duals by choosing different space-times $M$ for the same twistor model.

In the general setup of this paper, the large $N$ parameter does not play any significant role. In HS theories, it can be related to the HS coupling constant so that in the large $N$ limit it brings the boundary theory to the free-field limit. However, interactions with boundary HS gauge fields should be taken into account in the analysis of subleading $1/N$ corrections.

Although in this paper we mainly discuss the correspondence at the level of field equations because the appropriate action principle for HS gauge theories remains unknown (for some interesting conjectures see however [23, 24]), it should be stressed that the general nature of holographic duality extends to the action level. Here, we follow the analogy with the properties of conserved currents discussed in [25] where conserved charges in HS theory were represented in the form

$$q = \int_\sigma \Omega(Y|X),$$

(1.1)

where $\Omega(Y|X)$ is a closed $p$-form in some ‘correspondence’ space $C$ with local coordinates $Y$ (twistor space $T$) and $X$ (space-time $M$). The charge $q$ is independent of local variations of $\sigma$ in $C$, while $\text{dim} C$ may be much larger than $p$. For example, in the case of [25], $q$ could be represented either in the standard form of a space integral or as an integral in $T$.

Since, as shown in [26], in the unfolded dynamics approach the concepts of conserved charge and action are similar, the analogous representation is anticipated to be reachable for HS actions. If so, the generating functional in the bulk theory

$$Z_{\text{bulk}}(\phi_{\text{boun}}) = \int \exp iS(\phi(Y|X))$$

(1.2)

with boundary conditions $\phi_{\text{boun}}$ for the bulk dynamical variables $\phi(Y|X)$ can be represented in many equivalent ways. If the integration is over space-time $M_{\text{bulk}}$, this leads to the AdS/CFT prescription. If it is over some cycle in the twistor space $T$ or in $M_{\text{bound}} \times T$, this provides an independent definition of the boundary theory. (Note that $M_{\text{bound}}$ itself has too small a dimension to support the integral).

An interesting output of the analysis of this paper is that, at least for the HS models in question, it is most convenient to formulate them in the doubled AdS space where the original AdS boundary $z = 0$ is identified with the invariant surface for the reflection automorphism $P$ that maps one copy of AdS to another via reflection $P(z) = -z$. As a result, no boundary conditions at $z = 0$ should be imposed to define the action as the integral over the doubled AdS space-time. In this setup, the holographic duality relates a bulk theory in the doubled AdS space with the ‘boundary theory’ where all possible types of boundary fields $\phi_{\text{boun}}(x)$ contribute. The values of $\phi_{\text{boun}}(x)$ at $z = 0$ determine all fields in the (doubled) bulk and hence the values of the respective action functionals $S(\phi_{\text{boun}})$. We believe that this doubling trick should have a wide area of applicability in HS theories and beyond.

The remaining part of this paper is organized as follows. In section 2, we recall relevant elements of the unfolded dynamics approach. Interpretation of holography in terms of unfolded dynamics is discussed in section 3. Unfolded equations for massless fields in AdS$_4$ are recalled in section 4, while unfolded formulation for 3d conserved currents is summarized in section 5. In section 6, we recall nonlinear HS equations in AdS$_4$ also discussing in section 6.2 their extension with spinor coordinates. In section 7, the holographic duality between HS theory in AdS$_4$ and 3d conformal HS theory is discussed in general terms. Its detailed analysis in terms of familiar Poincaré coordinates is presented in section 8. The general structure of nonlinear
3d conformal HS theory is discussed in section 9. Boundary conditions, the construction of the doubled AdS bulk space and reductions of particular HS theories, associated with free-boundary theories, are considered in section 10. The AdS$_3$/CFT$_2$ HS correspondence is briefly discussed in section 11. The duality between HS gauge theories and nonrelativistic quantum mechanics is considered in section 12. Aspects of the off-shell extension of on-shell unfolded theories are considered in section 13. Conclusions and perspectives are presented in section 14.

2. Unfolded dynamics

2.1. Unfolded equations

Unfolded formulation is a multidimensional coordinate-independent generalization of the first-order formulation

$$\frac{dt}{\partial} q^\alpha = G^\alpha(q), \quad G^\alpha(q) = eF^\alpha(q),$$

available for any system of ordinary differential equations by adding auxiliary variables associated with higher derivatives of the dynamical variables of the original system. Here, $e$ is an einbein 1-form that can be identified with $dt$ because 1d geometry is flat.

Let $M^d$ be a $d$-dimensional space-time manifold with coordinates $x^a$ ($a = 0, 1, \ldots, d - 1$). By unfolded formulation of a linear or a nonlinear system of partial differential equations (PDE) in $M^d$ we mean its reformulation in the first-order form [27]

$$dW^2(x) = G^2(W(x)),$$

where $d = dx^a \frac{\partial}{\partial x^a}$ is the exterior differential in $M^d$, $W^2(x)$ is a set of degree $p_2$ differential forms and $G^2(W)$ is some degree $p_2 + 1$ function of $W^\Lambda$:

$$G^2(W) = \sum_{n=1}^{\infty} f^2_{\Lambda_1\cdots\Lambda_n} W^{\Lambda_1} \wedge \cdots \wedge W^{\Lambda_n},$$

where the coefficients $f^2_{\Lambda_1\cdots\Lambda_n}$ satisfy the (anti)symmetry condition

$$f^2_{\Lambda_1\cdots\Lambda_n} = (-1)^{p_2 + 1 + \Lambda_1 + \cdots + \Lambda_n} f^2_{\Lambda_n\cdots\Lambda_1},$$

(extension to the case with additional boson–fermion grading is straightforward) and $G^2(W)$ satisfies the condition

$$G^\Lambda(W) \wedge \frac{\partial G^2(W)}{\partial W^\Lambda} = 0$$

equivalent to the following generalized Jacobi identity on the structure coefficients:

$$\sum_{n=0}^{m} (n + 1) f^\Phi_{\Lambda_1\cdots\Lambda_{n-1}} f^2_{\Lambda_n\cdots\Lambda_{m+1}} = 0,$$

where the brackets $[\ ]$ denote an appropriate (anti)symmetrization of all indices $\Lambda_i$. Strictly speaking, the generalized Jacobi identities (2.6) have to be satisfied at $p_2 \leq d$ since any $(d + 1)$-form in $M^d$ is zero. Given solution of (2.6), it defines a free differential algebra [28–31]. We call a free differential algebra universal [32, 26] if the generalized Jacobi identity holds independently of a particular value of space-time dimension. All HS free differential algebras relevant to HS theories including those discussed in this paper are universal. For example, the 1d system (2.1) is universal. Here, the condition (2.5) trivializes because $e \wedge e = 0$, i.e. any function $F^\alpha(q)$ is allowed. The generalized Jacobi identity is obeyed for any number of coordinates of the ambient space (i.e. $dx^a$), since $e = dx^a e^a$ carries no fiber indices.
Condition (2.5), which can equivalently be rewritten as
\[ Q^2 = 0, \quad Q = G^\Omega(W) \frac{\partial}{\partial W^\Omega}, \]  
(2.7)
guarantees the formal consistency of the unfolded system (2.2) which can now be put into the form
\[ dF(W(x)) = Q(F(W(x))) \]  
(2.8)
with \( d^2 = 0 \) for all \( F(W) \). Unfolded equations in the form (2.8) are analogous to 1d Hamiltonian equations.

Equation (2.2) is invariant under the gauge transformation
\[ \delta W^\Omega = d \varepsilon + \varepsilon^\Lambda \frac{\partial G^\Omega(W)}{\partial W^\Lambda}, \]  
(2.9)
where the gauge parameter \( \varepsilon^\Omega(x) \) is a \((p_\Omega - 1)\)-form. (0-forms among \( W^\Omega \) do not have associated gauge parameters.)

2.2. Examples

2.2.1. Vacuum. An important example of unfolded equations is provided by Maurer–Cartan equations. Let \( h \) be a Lie algebra with some basis \( \{T_\alpha\} \). Let \( w = w^\alpha T_\alpha \) be an \( h \)-valued 1-form. Setting \( G(w) = -w \wedge w \equiv -\frac{1}{2} w^\alpha \wedge w^\beta [T_\alpha, T_\beta] \), equation (2.2) with \( W = w \) becomes the flatness condition
\[ dw + w \wedge w = 0. \]  
(2.10)
Equation (2.5) amounts to the Jacobi identity for \( h \). Equation (2.9) gives the usual gauge transformation
\[ \delta w = D_0 \varepsilon := d + [w, \varepsilon]. \]  
(2.11)
Usually, the zero-curvature equations (2.10) describe background geometry in a coordinate-independent way. For example, let \( h \) be Poincaré algebra with the gauge fields
\[ w(x) = e^m(x) P_m + \omega^{nm}(x) L_{nm}, \]  
(2.12)
where \( P_m \) and \( L_{nm} \) are generators of translations and Lorentz transformations. The gauge fields \( e^m(x) \) and \( \omega^{nm}(x) \) are identified with the frame 1-form and Lorentz connection, respectively (fiber Lorentz vector indices \( m, n, \ldots \) and base indices \( m, n, \ldots \) run from 0 to \( d - 1 \) and are raised and lowered by the flat Minkowski metric). It is well known that the zero-curvature condition (2.10) for the Poincaré algebra amounts to the zero-torsion condition
\[ R^a = de^a + \omega^a_{\, \mu} \wedge e^\mu = 0, \]  
(2.13)
which expresses \( \omega^{nm}(x) \) in terms of derivatives of the frame field, and the condition that Riemann tensor is zero
\[ R^{mn} = d\omega^{mn} + \omega^m_{\, \nu} \wedge \omega^{\nu n} = 0, \]  
(2.14)
which implies flat Minkowski geometry. As a result, on the condition that the matrix \( e^m_n(x) \) is nondegenerate, the zero-curvature condition (2.10) for the Poincaré algebra gives the coordinate-independent description of Minkowski space-time. Choosing a different Lie algebra \( h \), one can describe different backgrounds like, e.g., (anti-) de Sitter.
2.2.2. Linear fluctuations. If the set $W^\Omega$ contains some $p$-forms denoted by $C^p$ (e.g. 0-forms) and $G'(W)$ is linear both in $w$ and in $C^p$,

\[ G' = -w^\alpha (T_\alpha)^j_i \wedge C^j, \tag{2.15} \]

relation (2.5) implies that $(T_\alpha)^j_i$ form some representation $T$ of $h$, acting in a space $V$ where $C^j$ is valued. The corresponding equation (2.2) is a covariant constancy condition

\[ D_w C^j = 0 \tag{2.16} \]

with $D_w \equiv d + w$ being the covariant derivative in the $h$-module $V$. For $G'(w, C')$ multilinear in the background connections $w$ but still linear in dynamical fields $C'$, unfolded equations can be interpreted in terms of Chevalley–Eilenberg cohomology of $h$ with coefficients in the infinite-dimensional modules carried by differential forms of different degrees among $C'$. [33]

As an illustration, consider the unfolded formulation of a scalar field. Following [34], we introduce the infinite set of 0-forms $C_{m_1 \ldots m_n}(x)$ $(n = 0, 1, 2, \ldots)$, which are totally symmetric tensors $C_{m_1 \ldots m_n} = C_{(m_1 \ldots m_n)}$. The off-shell unfolded equations are

\[ dC_{m_1 \ldots m_n} = \hat{e}^k C_{m_1 \ldots m_n k} \quad (n = 0, 1, \ldots), \tag{2.17} \]

where we use Cartesian coordinates with $D^k = d$. This system is formally consistent because application of $d$ to both sides of (2.17) gives zero by $\hat{e}^k \wedge \hat{e}^l = -\hat{e}^l \wedge \hat{e}^k$. Hence, the space $V$ of 0-forms $C_{m_1 \ldots m_n}$ forms some (infinite-dimensional) $\text{iso}(d - 1, 1)$-module. (Strictly speaking, one has to check that the equation is consistent for any flat $\text{iso}(d - 1, 1)$ connection. It is not hard to see that this is indeed true.)

Let the scalar field $C(x)$ be identified with $C_{m_1 \ldots m_n}(x)$ at $n = 0$. The first two equations of the system (2.17) read

\[ \partial_x C = C_{m_1}, \quad \partial_x C_{m_1} = C_{m_1 m_2}, \tag{2.18} \]

where we have identified the world and tangent indices via $\epsilon^m_n = \delta^m_n$. The first of these equations tells us that $C_{m_1}$ is the first derivative of $C$. The second one identifies $C_{m_1 m_2}$ with the second derivative of $C$. All other equations in (2.17) express highest tensors in terms of the higher order derivatives

\[ C_{m_1 \ldots m_n} = \partial_{m_1} \cdots \partial_{m_n} C. \tag{2.19} \]

From equation (2.19), it is clear that the 0-forms $C_{m_1 \ldots m_n}$ describe all derivatives of the dynamical field $C(x)$ including $C(x)$ itself. The system (2.17) is off-shell: it is equivalent to an infinite set of constraints, imposing no field equations on the dynamical field $C$.

To put the system on shell, we impose an additional condition that all $C_{m_1 \ldots m_n}(x)$ are traceless

\[ C^k_{m_1 \ldots m_n}(x) = 0. \tag{2.20} \]

This condition preserves consistency and puts the system on shell by virtue of equation (2.18).

2.3. Global symmetries

Maximally symmetric vacua of dynamical systems are described by vacuum connections that satisfy the flatness condition (2.10) for some Lie algebra $h$. Choosing some vacuum connection $w_0(x)$, global symmetry transformations that leave $w_0$ invariant are described by the parameters $\hat{e}_\mu(x)$ that satisfy

\[ D_0 \hat{e}_\mu = 0. \tag{2.21} \]

Clearly, in the topologically trivial situation, this equation has dim $h$ independent solutions.
This simple observation immediately uncovers maximal symmetries of the linear unfolded equations of the form (2.16). Indeed, an \( h \)-module \( V \) can be treated as the \( \mathfrak{l}^{\text{max}}(V) \)-module, where \( \mathfrak{l}^{\text{max}}(V) \) is the Lie algebra of commutators of \( \text{End} V \). Indeed, since \( h \in \mathfrak{l}^{\text{max}}(V) \), any flat connection \( \omega_0 \) of \( h \) can be interpreted as a flat connection of \( \mathfrak{l}^{\text{max}}(V) \). Hence, \( \mathfrak{l}^{\text{max}}(V) \) is the maximal symmetry of the linear unfolded equations with dynamical fields valued in \( V \). As a result, via identification of \( V \), unfolding of a dynamical system makes all its symmetries manifest.

Let \( W^\Omega_0 \) be some solution of the unfolded system (2.2), may be containing some nonzero \( p\Omega \)-forms with \( p\Omega \neq 1 \). According to (2.9), symmetries of this solution are described by the symmetry parameters \( \epsilon_{\Omega\lambda}^\Omega(x) \) that satisfy

\[
\frac{d \epsilon_{\Omega\lambda}^\Omega}{d t} + \epsilon_{\Omega\lambda}^\Omega \frac{\partial G^\Omega(W)}{\partial W^\Lambda}|_{W=W_0} = 0. \tag{2.22}
\]

Since equations (2.22) that contain \( d \epsilon_{\Omega\lambda}^\Omega \) are consistent as a consequence of the original unfolded equations, they can be solved locally in terms of \( \epsilon_{\Omega\lambda}^\Omega(x_0) \) at any space-time point \( x_0 \). Naively, it looks like that this gives as many global symmetries as parameters \( \epsilon_{\Omega\lambda}^\Omega \). However, this is not the case because the 0-form part of equation (2.22) may impose constraints on \( \epsilon_{\Omega\lambda}^\Omega(x) \) :

\[
0 \frac{\partial G^\Omega(W)}{\partial W^\Lambda}|_{W=W_0} = 0, \tag{2.23}
\]

where \( pW^\Omega \) denotes \( p \)-forms among \( W^\Omega \). This implies that nontrivial global symmetries should leave invariant vacuum values of 0-forms in the system. This restriction is very strong since, in the unfolded dynamics approach, nontrivial curvatures like Weyl tensor and its HS analogues are described by 0-forms. Most symmetric vacua are associated with those solutions where all 0-forms are zero or central (i.e. singlet with respect to all 1-form connections).

### 2.4. Dynamical content

The general situation can be illustrated by the simple 1d example. First-order ordinary differential equations have the following structure:

\[
\frac{\partial}{\partial t} q^\hat{\lambda} = a^\hat{\lambda} q_{\nu+1} + \cdots, \quad i = 0, 1, 2, \ldots \tag{2.24}
\]

where \( a^\hat{\lambda} \) are some coefficients while an ellipsis denotes nonlinear corrections. If all \( a^\hat{\lambda} \) are nonzero, equations (2.24) treated perturbatively describe an infinite set of constraints that express all \( q_{\nu+1} \) via derivatives of \( q_0 \). If some coefficient \( a^\hat{\lambda} \) vanishes, this implies a nontrivial differential equation on \( q_0 \), which is of order \( j \) at the linearized level. These two options are analogous, respectively, to the off-shell and on-shell cases in the scalar field example of section 2.2.2. Indeed, using that 1d traceless tensors in one dimension are all zero except for \( C \) and \( C_{\nu} \), the on-shell system (2.17), (2.20) at \( d = 1 \) implies \[ a^\hat{\lambda} \] C = 0.

In the first-order formulation (in particular, in the Hamiltonian formalism) the initial data problem is set in terms of values of all variables \( q \) at given time \( t_0 \). In the general case of \( d > 1 \), these properties have clear analogues. Nontrivial dynamical fields (i.e. those that are different from auxiliary fields expressed via derivatives of the dynamical fields), gauge symmetries and true differential field equations are classified in terms of the so-called \( \sigma_- \) cohomology [34] that roughly speaking controls zeros among the coefficients analogous to \( a^\hat{\lambda} \) of the linearized equations. The \( \sigma_- \) cohomology is a perturbative concept that emerges in the linearized analysis with

\[
W^\Omega(x) = W^\Omega_0(x) + W^\Omega_1(x), \tag{2.25}
\]

\[
where \( W_0^\Omega(x) \) is a particular solution of (2.2) and \( W_0^\Omega(x) \) is treated as a perturbation. \( W_0^\Omega(x) \) is nonzero in a field-theoretical system because, as explained in section 2.2.1, it should contain a background gravitational field. Linearized equations (2.2)

\[
\frac{d W_1^\Omega(x)}{d x} = \frac{\delta G^\Omega}{\delta W^\Lambda} \bigg|_{W=W_0}
\]

(2.26) can be rewritten in the form (2.16)

\[
D_0 W_1^\Omega(x) = 0,
\]

(2.27) where \( D_0 \) is some differential that squares to zero to fulfil the consistency condition (2.5):

\[
D_0^2 = 0.
\]

(2.28) Usually, a set of fields \( W_i \) admits a grading \( G \) with the spectrum bounded from below, which typically counts a rank of a tensor. Suppose that

\[
D_0 = D_0 + \sigma_+ + \sigma_-,
\]

(2.29) where

\[
[ G, \sigma_- ] = -\sigma_-,
\]

(2.30) and \( \sigma_+ \) is a sum of operators of positive grade. From (2.28), it follows that

\[
\sigma_-^2 = 0.
\]

(2.31) Provided that \( \sigma_- \) does not differentiate \( x^2 \), the dynamical content of the system in question is determined by cohomology of \( \sigma_- \). Namely, as shown in [34] (see also [32]), for \( p_\Omega \)-forms \( W^\Omega \) valued in a vector space \( V \), \( H^{p+1}(\sigma_-, V) \), \( H^p(\sigma_-, V) \) and \( H^{p-1}(\sigma_-, V) \) describe, respectively, differential equations, dynamical fields and differential gauge symmetries encoded by equation (2.27). The case with \( H^{p+1}(\sigma_-, V) = 0 \) is analogous to that of (2.24) with all coefficients \( a_\alpha^i \) different from zero. Here, no differential equations on the dynamical variables are imposed, i.e. equations (2.2) just encode constraints on auxiliary fields. Equations of this type are referred to as off-shell. (Let us stress that this definition is true both for linear and nonlinear cases: nonlinear equations are off-shell if their linearization is off-shell.) If \( H^{p+1}(\sigma_-, V) \neq 0 \), unfolded equations (2.2) impose some differential equations on the dynamical fields. Such systems are called on-shell.

Degrees of freedom, i.e. variables that determine a (local) solution of equations (2.2) modulo gauge ambiguity, are represented by values of all 0-forms \( C^\Phi(x_0) \) among \( W^\Omega(x_0) \) at any given \( x_0 \) which is analogous to \( t_0 \) of the 1d case. This means in particular that to unfold a field-theoretical system with infinite number of degrees of freedom, an infinite set of 0-forms has to be introduced. In the scalar field example, this is the set of 0-forms \( C_{n_1,...,n_k}(x) \).

### 2.5. Generalized twistor space

The unfolded scalar field system is most conveniently described in terms of generating functions of auxiliary variables \( y^\phi \)

\[
C(y|x) = \sum_{k=0}^{\infty} \frac{1}{k!} y_1^{n_1} \cdots y_k^{n_k} C_{n_1,...,n_k}(x).
\]

(2.32) In the on-shell case, tracelessness of \( C_{n_1,...,n_k}(x) \) is equivalent to the condition that \( C(y|x) \) is harmonic in \( y^n \)

\[
\square C(y|x) = 0
\]

(2.33)
while the unfolded equations (2.17) take the form

\[
(d_t - e^a \frac{\partial}{\partial y^a}) C(y|x) = 0. \tag{2.34}
\]

In these terms,

\[
G = y^a \frac{\partial}{\partial y^a}, \quad D_0 = d, \quad \sigma_+ = 0, \quad \sigma_- = -e^a \frac{\partial}{\partial y^a} \tag{2.35}
\]

In the off-shell case, \(\sigma_-\) acts as exterior differential on polynomials. Hence, the only nontrivial \(\sigma_-\) cohomology is \(H^0(\sigma_-)\). This tells us that scalar \(C(x)\) is the only dynamical field and that it is not restricted by dynamical equations. In the on-shell case, \(H^1(\sigma_-)\) is one dimensional in agreement with the fact that the on-shell unfolded system imposes only one equation on the scalar field, namely the Klein–Gordon equation, which belongs to the following part of the unfolded equations:

\[
em_{m} \frac{\partial^2}{\partial x^m \partial y^n} \left( d_t - e^a \frac{\partial}{\partial y^a} \right) C(y|x) \bigg|_{y=0} = 0, \tag{2.36}
\]

where the vector field \(e^m_{m}\) is the inverse to the (co)vielbein 1-form \(e^m_{a}\).

In field-theoretical systems with infinite degrees of freedom, the label \(\Omega_1\), enumerating differential forms in the unfolded equations (2.2), usually refers to appropriate functional spaces, i.e. \(W^\Omega(X)\) can be represented as a set \(W(Y|X)\) of functions of some auxiliary variables \(Y\) (labels different functions).

The variables \(Y\) are analogous to the twistor coordinates in the twistor theory. Following [35], we interpret \(Y\) as the coordinates of a generalized twistor space T. X are the space-time coordinates of space M. Together, \((Y,X)\) are interpreted as the coordinates of the correspondence space C. More precisely, C is a fiber bundle with M as base manifold and T as fibers.

In this setup, the unfolded equation (2.2) encodes a generalized Penrose transform [37, 36] expressed by the diagram

\[
\begin{array}{c}
\text{C} \\
\eta \\
\text{M} \\
\end{array} \quad \begin{array}{c}
\nu \\
\text{T} \\
\end{array}
\tag{2.37}
\]

Indeed, it reconstructs (locally) the general solution \(W(Y|X)\) of the unfolded equations in terms of arbitrary functions \(W(Y|X_0)\) on the twistor space T. Via restriction of a \(p\)-form \(W(Y|X)\) to dynamical fields associated with \(H^p(\sigma_-)\), this gives a solution of the dynamical equations associated with the \(H^{p+1}(\sigma_-)\) part of the unfolded equations.

As discussed in more detail in section 4, in lower dimensions \(d = 3, 4\), the on-shell condition (2.33) is conveniently resolved in terms of unrestricted functions of spinor (twistor) variables. This greatly simplifies the analysis of the respective unfolded on-shell systems.

As discussed in section 3, the holographic duality relates different looking dynamical systems in different space-times (coordinates \(X)\) that have the same twistor space. From this perspective, unfolded equations perform a generalized Penrose transform from the same twistor space T to one or another space-time M.

1 In HS theories, T may or may not have precise geometric meaning of some twistor space in the twistor theory [36]. Nevertheless, abusing terminology, in this paper, we call it twistor space.
2.6. Properties

The unfolded formulation has a number of useful and important properties discussed in some more detail, e.g., in [33]. In particular, unfolded equations possess manifest gauge and diffeomorphism invariance due to the exterior algebra formalism. As such, they are perfectly suited for the study of gauge-invariant theories in the framework of gravity like, e.g., HS gauge theories. The following properties of unfolded dynamics are most relevant to the analysis of holography.

The method is universal. Any dynamical system can be unfolded. This is analogous to the fact that any system of ordinary differential equations admits a first-order formulation.

Indeed, let \( w = e^a_0 P_a + \frac{1}{2} \epsilon^{ab}_0 M_{ab} \) be a vacuum connection valued in some space-time symmetry algebra \( k \). Let a field \( C^{(0)}(X) \) satisfy some dynamical equations to be unfolded. Consider first the case where \( C^{(0)}(X) \) is a 0-form. One starts by writing the equation \( D_0^C C^{(0)} = e^a_0 C^{(1)}_a \), where \( D_0^C \) is the Lorentz-covariant derivative and the field \( C^{(1)}_a \) is auxiliary. Next, one checks whether the original field equations for \( C^{(0)} \) impose restrictions on the first derivatives of \( C^{(0)} \). A part of \( D_0^C C^{(0)} \), and hence \( C^{(1)} \), may vanish on the mass shell (e.g. for the Dirac equation). The remaining non-zero auxiliary fields \( C^{(1)} \) parameterize on-shell nontrivial components of first derivatives. One proceeds by writing the analogous equation for the first-level auxiliary fields \( D_0^C C^{(1)}_a = e^b_0 C^{(2)}_{a,b} \), where the new fields \( C^{(2)}_{a,b} \) parameterize second derivatives of \( C^{(0)} \). Again, one checks, taking into account Bianchi identities, which components of the second-level fields \( C^{(2)}_{a,b} \) remain nonzero if the original equations of motion are satisfied. This process continues indefinitely, leading to a chain of equations on the chain of fields \( C^{(m)}_{a_1,a_2,...,a_m} (m \in \mathbb{N}) \) parameterizing all on-shell nontrivial derivatives of the original dynamical field. If one starts with some gauge field, the analogous analysis determines a form of shift gauge transformations that subtract extra field components to be introduced to describe a system in terms of differential forms. (For instance, local Lorentz symmetry in the Cartan formulation of gravity appears this way as the shift symmetry that removes extra components of the vielbein 1-form compared to the metric tensor.) By construction, this leads to a particular unfolded system. The correspondence between \( p \geq 1 \) forms and gauge symmetries in the unfolded dynamics approach uncovers patterns of local and global symmetries associated with a given gauge field. In particular, the pattern of the linearized 4d HS algebras was deduced this way in [38]. These results were then used in [39, 40] to find infinite-dimensional non-Abelian HS algebras that underlie the nonlinear 4d HS theories.

In the topologically trivial situation, degrees of freedom are carried by 0-forms at any space-time point \( X = X_0 \). Indeed, by virtue of the Poincaré lemma, unfolded equations express all exterior derivatives in terms of the values of fields themselves modulo exact forms that can be gauged away by the gauge transformation (2.9). What is left is the ‘constant part’ of 0-forms. In terms of functions of twistor variables \( Y \) like \( C(Y|X) \), this means that dynamics is entirely encoded by 0-forms on the twistor space, i.e. \( C(Y|X_0) \) at any \( X_0 \).

3. Unfolding and holographic duality

The unfolded formulation unifies various dual versions of one and the same system. This concerns both the duality between systems in the same space-time and the holographic duality between theories in space-times of different dimensions.

In the former case, the duality results from the ambiguity in which fields are chosen to be dynamical or auxiliary, the nomenclature governed by the choice of the grading \( G \) and \( \sigma \). Different gradings lead to different interpretations of the same unfolded system in terms of different dynamical fields that satisfy seemingly unrelated differential equations. The key
point is that if two dynamical systems give rise to the same unfolded system (more precisely, belonging to the same projective system [33]), they are equivalent.

Holographic duality rests on the striking feature that a universal unfolded system may admit different space-time interpretations. In some sense, the space-time dependence in such systems is auxiliary as was first noted in the context of HS dynamics in [41]. True dynamics is hidden in the twistor sector of the auxiliary variables $Y$.

Indeed, in a universal unfolded system, dynamics is entirely encoded by the function $G^Q(W)$ independently of the original space-time picture. In particular, the unfolded formulation allows one to extend space-time without changing dynamics simply by letting the differential $d$ and differential forms $W^\sigma$ to live in a larger space

$$d = dX^a \frac{\partial}{\partial X^a} \rightarrow \tilde{d} = dX^a \frac{\partial}{\partial X^a} + d\tilde{X}^\tilde{a} \frac{\partial}{\partial \tilde{X}^\tilde{a}} , \quad dX^a W^a \rightarrow dX^\tilde{a} \tilde{W}^\tilde{a}, \quad (3.1)$$

where $\tilde{X}^\tilde{a}$ are some additional coordinates. For a universal unfolded system such substitution neither spoils consistency nor affects local dynamics still determined by the 0-forms at any point of (any) space-time. Indeed, the unfolded system in the $X$ space remains a subsystem of that in the enlarged space while additional equations reconstruct dependence on $\tilde{X}^\tilde{a}$ in terms of solutions of the original system (of course, this consideration is local).

On the other hand, as emphasized in [42], the role of coordinates is that they help to visualize physical local events via a physical process. A particular space-time interpretation of a universal unfolded system, e.g., whether a system is on-shell or off-shell, depends not only on $G^Q(W)$ but, in the first place, on a chosen space-time $M^d$ and vacuum solution $W_0(X)$. The dynamical interpretation may be different for different space-times because $\sigma_-$ cohomology depends on the space-time dimension via the rank of the vielbein 1-form. In particular, the on-shell HS theory in AdS$_4$ will be shown in section 9 to be dual to an off-shell 3d conformal system. This implies that the 3d ‘dynamical’ boundary fields are not restricted by any differential field equations. Among other things, this property makes it possible to identify 3d dynamical fields with unrestricted boundary values of the on-shell bulk fields.

An important point considered in more detail in section 13 is that, in the unfolded formulation, a nontrivial conserved charge or gauge invariant generating functional for correlators is represented as an integral of some $d$-closed form [26]. As a result, correlators in boundary conformal systems are represented by integrals over a space larger than space-time where conformal fields live. It can be either the bulk space as in the standard AdS/CFT treatment or some twistor space. In any case, from this perspective, the nonlinear off-shell system at the boundary is represented by a nonlinear on-shell system in a larger space$^2$.

To summarize, two unfolded systems in different space-times are equivalent (dual) provided that they have the same unfolded form. Generally, a most straightforward way to establish holographic duality between two theories is to unfold both of them to see whether the operators $Q^d(2.7)$ of their unfolded formulations coincide. The other way around, given an unfolded system generates a class of holographically dual theories in different dimensions. It should be stressed that, being simple in terms of unfolded dynamics and the corresponding twistor space $T$, the holographic duality in usual space-time terms may be very complicated.

$^2$ There is a subtlety related to this point to be taken into account in the holography context. Perturbatively, every nonlinear off-shell system is equivalent to some linear system by virtue of a perturbatively local nonlinear field redefinition. Indeed, a system is off-shell in the case where the operator $\sigma_-$ is a kind of nondegenerate. Writing schematically $R_k = \sigma_- w_{k+1} + \cdots$, where ellipsis denote lower grade and/or nonlinear terms, one observes that the latter can all be absorbed into nonlinear field redefinitions of $w_k$ with $k = 1, 2, \ldots$ modulo $\sigma_-$, exact terms that are pure gauge with respect to some shift symmetries. Although such a field redefinition destroys the structure of unfolded equations, the resulting system has a linear form. This should be compared to on-shell nonlinear systems where not all nonlinear terms can be removed by such a field redefinition. In particular, the lhs of field equations, associated with the respective $\sigma_-$ cohomology, cannot be linearized in a general nonlinear on-shell system.
requiring the solution of at least one of the two unfolded systems which is equivalent to a nonlinear integral transform.

4. Free massless fields in AdS$_4$

In this section, we remind the reader about the unfolded formulation of free massless fields of all spins in AdS$_4$ obtained originally in [27]. It is based on the frame-like approach to HS gauge fields [43, 38], where a 4nf spin $s \geq 1$ massless field is described by the set of 1-forms

$$\omega_{a_1 \ldots a_k \bar{a}_1 \ldots \bar{a}_l} = d\varpi^a_{a_2 \ldots a_k \bar{a}_1 \ldots \bar{a}_l}, \quad k + l = 2(s - 1),$$

(4.1)

where $\alpha, \beta, \ldots = 1, 2$ and $\bar{\alpha}, \bar{\beta}, \ldots = 1, 2$ are the two-component spinor indices. The HS gauge fields are totally symmetric with respect to each type of spinor index and obey the reality conditions $\overline{\omega}_{a_1 \ldots a_k \bar{a}_1 \ldots \bar{a}_l} = \omega_{\bar{a}_1 \ldots \bar{a}_l \bar{a}_1 \ldots \bar{a}_l}$. For a given $s$, this set is equivalent to the real 1-form $\omega_{\alpha_1 \ldots \alpha_{2s-1}}$ symmetric in the Majorana indices $A = 1, 2, 3, 4$. As such, it carries an irreducible module of $sp(4, \mathbb{R}) \sim o(3, 2)$.

AdS$_4$ is described by the Lorentz connection $\alpha^{\alpha \beta}$, $\overline{\alpha}^{\alpha \beta}$ and vierbein $e^{a\alpha}$. Altogether they form the $sp(4, \mathbb{R})$ connection $w^{AB} = w^{BA}$ that satisfies the $sp(4, \mathbb{R})$ zero-curvature conditions

$$R^{AB} = 0, \quad R^{AB} = dw^{AB} + w^{AC} \wedge w^B_C,$$

(4.2)

where indices are raised and lowered by an $sp(4, \mathbb{R})$-invariant form $C_{AB} = -C_{BA}$

$$A_B = A^A C_{AB}, \quad A^A = C^{AB} A_B, \quad C_{AC} C^{BC} = \delta^B_A.$$

(4.3)

In terms of Lorentz components $w^{AB} = (\alpha^{\alpha \beta}, \overline{\alpha}^{\alpha \beta}, \lambda e^{a\alpha}, \lambda e^{a\alpha})$, where $\lambda^{-1}$ is the AdS$_4$ radius, the AdS$_4$ equations (4.2) read as

$$R_{\alpha \beta} = 0, \quad \overline{R}_{\bar{a} \bar{b}} = 0, \quad R_{\bar{a} \bar{b}} = 0,$$

(4.4)

where

$$R_{\alpha \beta} = d\omega_{\alpha \beta} + \omega_{\alpha \gamma} \wedge \omega_{\beta \gamma} + \lambda^2 e_a^{\alpha} \wedge e_{\beta b},$$

$$\overline{R}_{\bar{a} \bar{b}} = d\overline{\omega}_{\bar{a} \bar{b}} + \overline{\omega}_{\bar{a} \gamma} \wedge \overline{\omega}_{\bar{b} \gamma} + \lambda^2 e^a_{\bar{\alpha}} \wedge e_{\bar{\beta} b},$$

$$R_{\bar{a} \bar{b}} = d\omega_{\bar{a} \bar{b}} + \omega_{\bar{a} \gamma} \wedge e_{\bar{b} a} + \overline{\omega}_{\bar{b} \gamma} \wedge e_{\bar{a} a},$$

(4.5)

(Two-component indices are raised and lowered as in (4.3) with $e_{\alpha \beta}$ or $e_{\bar{a} \bar{b}}$ instead of $C_{AB}$.)

Unfolded equations of motion of a spin-$s$ massless field are [27]

$$R_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} = \eta_0^{\alpha_1 \alpha_2} H^{\alpha_3 \ldots \alpha_n} \overline{C}_{\bar{a}_1 \ldots \bar{a}_n}, \quad n + m = 2(s - 1),$$

(4.7)

and

$$D^{\bar{w}} C_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} = 0, \quad n - m = 2s, \quad D^{\bar{w}} \overline{C}_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} = 0, \quad m - n = 2s.$$

(4.8)

Here, the HS field strength and adjoint covariant derivative have the form

$$R_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} := D^\gamma \omega_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} + n \lambda e_a^{\alpha_1} \wedge \omega_{\alpha_2 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n + 1} + m \lambda e^{a+1}_{\alpha_1} \wedge \omega_{\alpha_2 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n},$$

$$D^{\bar{w}} C_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} := D^\gamma C_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} - \frac{1}{2} \lambda (e_{\alpha_1})^A_{\alpha_1 \alpha_2} \omega_{\alpha_3 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n} - m e_{\alpha_1} \bar{C}_{\alpha_1 \ldots \alpha_n \bar{a}_1 \ldots \bar{a}_n},$$

(4.9)

where the indices $\alpha$ and $\bar{\alpha}$ are (separately) symmetrized and Lorentz covariant derivative $D^\gamma$ is

$$D^\gamma \psi_\alpha := d\psi_\alpha + \omega_\alpha^\beta \psi_\beta, \quad D^\gamma \overline{\psi}_\bar{\alpha} := d\overline{\psi}_\bar{\alpha} + \overline{\omega}_\bar{\alpha}^\bar{\beta} \overline{\psi}_{\bar{\beta}},$$

(4.10)
\[ H^{\alpha \beta} = C^{\alpha \beta} \] and \[ \bar{H}^{\dot{\alpha} \dot{\beta}} = \bar{C}^{\dot{\alpha} \dot{\beta}} \] are the basis 2-forms

\[ H^{\alpha \beta} = e^{\alpha}_a \wedge e^{\beta}_a, \quad \bar{H}^{\dot{\alpha} \dot{\beta}} = e^{\dot{\alpha}}_a \wedge e^{\dot{\beta}}_a. \tag{4.12} \]

The phase parameters \( \eta \) and \( \bar{\eta} \) (\( \eta \bar{\eta} = 1 \)) are introduced for the future convenience. Although at the linearized level they can be absorbed into redefinition of mutually conjugated \( C \) and \( \bar{C} \), they become nontrivial beyond the linearized approximation [27] where the cases of \( \eta = 1 \) and \( \bar{\eta} = 1 \) correspond to two parity symmetric types of HS theories often referred to as models \( A \) and \( B \), respectively [22].

Formulae are simplified in terms of generating functions

\[ A(y, \bar{y} \mid x) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{a_1} \cdots y_{a_n} \bar{y}_{\dot{b}_1} \cdots \bar{y}_{\dot{b}_m} A^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \tag{4.13} \]

with \( A = \omega, C, \bar{C}, R, \) etc. In particular,

\[ R(y, \bar{y} \mid x) = D^{\alpha \beta} \omega(y, \bar{y} \mid x) = D^{\dot{\alpha} \dot{\beta}} C(y, \bar{y} \mid x) = \lambda e^{\alpha \beta}(y_\alpha \frac{\partial}{\partial y_\beta} + \bar{y}_\dot{\beta} \frac{\partial}{\partial \bar{y}_\dot{\alpha}}) \omega(y, \bar{y} \mid x), \tag{4.14} \]

\[ D^m C(y, \bar{y} \mid x) = D^p C(y, \bar{y} \mid x) + \frac{i}{2} \lambda e^{\alpha \beta} \left( y_\alpha \bar{y}_\dot{\beta} - \frac{\partial^2}{\partial y_\alpha \partial \bar{y}_\dot{\beta}} \right) C(y, \bar{y} \mid x), \tag{4.15} \]

\[ D^q A(y, \bar{y} \mid x) = dA(y, \bar{y} \mid x) = \left( \omega^{\alpha \beta} y_\alpha \frac{\partial}{\partial y_\beta} + \bar{\omega}^{\dot{\alpha} \dot{\beta}} \bar{y}_{\dot{\alpha}} \frac{\partial}{\partial \bar{y}_{\dot{\beta}}} \right) A(y, \bar{y} \mid x). \tag{4.16} \]

As a consequence of the AdS4 zero-curvature equation (4.2), the covariant derivatives \( D^{\alpha \beta} \) and \( D^m \) are flat, i.e.

\[ (D^{\alpha \beta})^2 = (D^m)^2 = 0. \]

These conditions imply the consistency of equations (4.7) and (4.8) (i.e. compatibility with \( d^2 = 0 \) and gauge invariance of the field strength (4.14) (and hence free equations (4.7)) under Abelian HS gauge transformations

\[ \delta \omega(y, \bar{y} \mid x) = D^{\alpha \beta} \epsilon(y, \bar{y} \mid x). \tag{4.17} \]

It is important that the consistency of the equations is not spoiled by the \( C \)-dependent terms in (4.7). As explained in [33], this means that the latter correspond to Chevalley–Eilenberg cohomology of \( sp(4, \mathbb{R}) \) with coefficients in the corresponding infinite-dimensional module.

In equations (4.7) and (4.8), a spin \( s \) field is described by the set of gauge 1-forms \( \omega^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \) with \( n + m = 2(s - 1) \) (for \( s \geq 1 \)) and 0-forms \( C^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \) with \( n - m = 2s \) along with their conjugates \( \bar{C}^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \) with \( m - n = 2s \). Indeed it is easy to see that the field equations (4.7) and (4.8) for such a set of fields form a subsystem for any \( s \).

For example, a spin zero field is described by a set of multispinors \( C^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \) which is equivalent to the set of 4d symmetric tensors \( C_{a_1 \cdots a_n}(x) \) satisfying the tracelessness condition (2.20). This property extends to all spins in the sense that all multispinors \( C^{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \) describe traceless Lorentz tensors. This follows from the Penrose formula that any \( p_{a\dot{a}} = p_a \tilde{p}_{\dot{a}} \) is null [37]. Indeed, unfolded equations just express \( p_{a\dot{a}} \sim \frac{\partial}{\partial x^a} \) via \( p_a \tilde{p}_{\dot{a}} \) with \( p_a \sim \frac{\partial}{\partial y^a} \), \( \tilde{p}_{\dot{a}} \sim \frac{\partial}{\partial \bar{y}^{\dot{a}}} \) (modulo mass-like terms proportional to \( \lambda^2 y^a \bar{y}^{\dot{a}} \) necessary in AdS background).

Dynamical massless fields are
- \( C(x) \) and \( \bar{C}(x) \) for two spin-0 fields,
- \( C_0(x) \) and \( \bar{C}_0(x) \) for a massless spin-1/2 field,
- \( \omega_{a_1 \cdots a_n \dot{b}_1 \cdots \dot{b}_m}(x) \) for an integer spin \( s \geq 1 \) massless field.
• \(\omega_{a_1, \ldots, a_{\nu-2}, \bar{a}_1, \ldots, \bar{a}_{\nu-2}/}(x)\) and its complex conjugate \(\omega_{a_1, \ldots, a_{\nu-2}, \bar{a}_1, \ldots, \bar{a}_{\nu-2}/}(x)\) for a half-integer spin \(s \geq 3/2\) massless field.

All other fields are auxiliary, being expressed via derivatives of the dynamical massless fields by equations (4.7) and (4.8).

The pattern of the unfolded massless field equations is expressed by the so-called central on-shell theorem [27] stating that equations (4.7) and (4.8) express all auxiliary fields in terms of derivatives of the dynamical fields, imposing on the latter massless field equations equivalent to those of Fronsdal [44] and Fang and Fronsdal [45].

In more detail, the meaning of equations (4.7) and (4.8) is as follows. Equations (4.8) are independent of (4.7) for spins \(s = 0\) and \(s = 1\) partially independent for \(s = 1\) but become consequences of (4.7) for \(s > 1\). Equations (4.7) express the holomorphic and antiholomorphic components of spin \(s \geq 1\) 0-forms \(C(y, \bar{y}|x)\) via the derivatives of the gauge 1-forms \(\omega(y, \bar{y}|x)\).

This identifies the spin \(s \geq 1\) holomorphic and antiholomorphic components of the 0-forms \(C(y, \bar{y}|x)\) with the Maxwell tensor, on-shell Rarita–Schwinger curvature, Weyl tensor and their HS counterparts considered already in the seminal works [46, 37]. In addition, equations (4.7) impose usual (second order for bosons and first order for fermions) field equations on the spin \(s > 1\) massless gauge fields so that equations (4.8) become their consequences by virtue of Bianchi identities. Dynamical equations for spins \(s \leq 1\) are contained in equations (4.8).

Although the system (4.7)–(4.8) is consistent at the free-field level, its nonlinear extension requires doubling of fields [27, 1, 2]. This is achieved by introducing the fields

\[
\omega^\mu(y, \bar{y}|x), \quad C^{i\to}(y, \bar{y}|x), \quad i = 0, 1,
\]

such that \(\omega^\mu(y, \bar{y}|x)\) are self-conjugated, while \(C^{0\to}(y, \bar{y}|x)\) and \(C^{1\to}(y, \bar{y}|x)\) are conjugated to one another:

\[
\overline{\omega^\mu(y, \bar{y}|x)} = \omega^\mu(y, \bar{y}|x), \quad \overline{C^{i\to}(y, \bar{y}|x)} = C^{i\to}(y, \bar{y}|x).
\]

The unfolded system for the doubled set of fields is

\[
R^\mu(y, \bar{y}|x) = \eta H^\alpha\beta \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^\beta} C^{i\to}(0, \bar{y}|x) + \bar{\eta} H^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^\beta} C^{i\to}(\bar{y}, 0|x),
\]

\[
D^\mu C^{i\to}(y, \bar{y}|x) = 0.
\]

(4.18)

(4.19)

Note that now all components of the expansions (4.13) of \(C^{i\to}(y, \bar{y}|x)\) contribute to equations (4.18) and (4.19), while in equations (4.7) and (4.8) with the single HS 1-form \(\omega(y, \bar{y})\) only those components of \(C(y, \bar{y})(\bar{C}(y, \bar{y}))\) contributed that carried at least as many \(y\bar{y}\) as \(\bar{y}\bar{y}\).

In the standard formulation of the 4d nonlinear HS gauge theory [2, 3], the field doubling results from the dependence on the Klein operators \(k\) and \(\bar{k}\) that have the properties

\[
k u^\alpha = -u^\alpha k, \quad \bar{k} w^\alpha = w^\alpha \bar{k}, \quad \bar{k} u^\alpha = -\bar{\omega}^\alpha \bar{k}, \quad k^2 = \bar{k}^2 = 1, \quad k\bar{k} = \bar{k}k,
\]

(4.20)

where \(u^\alpha = y^\alpha, \quad \bar{w}^\alpha = \bar{y}^\alpha\). All fields are packed into 1-forms

\[
\omega(y, \bar{y}; k, \bar{k}|x) = \sum_{ij=0,1} (k)^j (\bar{k})^i \omega^{ij}(y, \bar{y}|x)
\]

and 0-forms

\[
C(y, \bar{y}; k, \bar{k}|x) = \sum_{ij=0,1} (k)^j (\bar{k})^i C^{ij}(y, \bar{y}|x).
\]

Now both adjoint and twisted adjoint covariant derivatives result from different sectors of the adjoint covariant derivative in the Weyl algebra extended by the Klein operators.
Massless fields are those with
\[ \omega(y, \bar{y}; -k, -\bar{k} | x) = \omega(y, \bar{y}; k, k | x), \quad C(y, \bar{y}; -k, -\bar{k} | x) = -C(y, \bar{y}; k, k | x). \]

The fields with the opposite oddnesses in the Klein operators are topological, carrying at most a finite number of degrees of freedom per an irreducible subsystem [47].

Truncating out fermions, it is possible to consider a system with bosonic fields in which every integer spin appears once. This is achieved via the projection of the theory with the help of projectors
\[ \Pi_{\pm} = \frac{1}{2} (1 \pm k \bar{k}), \tag{4.21} \]
which are central in the bosonic HS theory. HS gauge fields and Weyl 0-forms of the bosonic theory are \[ \omega_{\pm} = \frac{1}{2} (\omega^{00} \pm \omega^{11}) \text{ and } C_{\pm} = \frac{1}{2} (C^{01} \pm C^{10}). \]

Bosonic HS theories can be further truncated to the minimal system that only contains even spins [3].

5. Conserved currents and massless equations

As observed in [48, 9, 33], conformal invariant massless equations are naturally formulated in the spaces \( M_M \) with matrix coordinates \( X^{AB} = X^{BA} (A, B = 1, \ldots, M) \). More precisely, 3\( d \) Minkowski space-time coincides with \( M_2 \), while 4\( d \) Minkowski space-time is a subspace of the ten-dimensional space \( M_4 \). In both cases, unfolded massless field equations are
\[ dX^{AB} \left( \frac{\partial}{\partial X^{AB}} \pm \frac{\partial^2}{\partial Y^A \partial Y^B} \right) C_{\pm} (Y|X) = 0, \tag{5.1} \]
where \( \pm \) is introduced for future convenience.

In [21], equation (5.1) was extended to the so-called rank-\( r \) unfolded equations
\[ dX^{AB} \left( \frac{\partial}{\partial X^{AB}} \pm \eta^{ij} \frac{\partial^2}{\partial Y^{iA} \partial Y^{jB}} \right) C^r_{\pm} (Y|X) = 0, \tag{5.2} \]
where \( i, j = 1, \ldots, r \) and \( \eta^{ij} = \eta^{ji} \) is some nondegenerate metric. Higher rank systems inherit all symmetries of the underlying lower rank system simply because they correspond to the tensor product of the lower rank representation. In particular, higher rank systems are conformal once the underlying lower rank systems are. In the basis where \( \eta^{ij} \) is diagonal, the higher rank equation (5.2) is satisfied by the product of rank-1 fields \( C^r (Y|X) = C_1 (Y_1|X) C_2 (Y_2|X) \cdots C_r (Y_r|X). \tag{5.3} \)

A rank-\( r \) system in \( M_M \) can be further extended to the rank-1 system (5.1) in the larger space \( M_{rM} \) with the coordinates \( X^{AB}_{ij} \) via reinterpretation of the twistor coordinates
\[ Y^A_i \rightarrow \bar{Y}^A, \quad \bar{A} = 1, \ldots, rM. \tag{5.4} \]

The diagonal embedding of \( M_M \) into \( M_{rM} \) is
\[ X_{11}^{AB} = X_{22}^{AB} = \cdots = X_{rr}^{AB} = X^{AB}. \tag{5.5} \]
In particular, the \( M = 2 \) rank-2 system extends to the \( M = 4 \) rank-1 system. Group theoretically, this provides a realization of the Flato and Fronsdal theorem [12] that relates tensor products of 3\( d \) conformal fields (singletons) to infinite towers of 4\( d \) massless fields of all spins. The key fact underlying the AdS\(_4\)/CFT\(_3\) holographic duality in HS theories is that, as shown in [21], rank-2 systems just describe conserved currents. Direct identification of 3\( d \) conserved conformal currents with 4\( d \) massless fields provides an example of holographic duality via unfolded dynamics.
Let us recall following [25] how rank-2 equations give rise to conserved currents. The rank-2 equation can be rewritten in the form

\[
\frac{\partial}{\partial X^{AB}} - \frac{\partial^2}{\partial Y^\alpha \partial U^B} \bigg|_{\partial X^{AB} = 0} T(U, Y|X) = 0, \tag{5.6}
\]

where \( T(U, Y|X) \) is the generalized stress tensor. In particular, equation (5.6) is obeyed by the bilinear substitution

\[
T(U, Y|X) = \sum_{i=1}^N C_{\pm i}(Y - U|X)C_{-\pm}(U + Y|X), \tag{5.7}
\]

where \( C_{\pm i}(Y|X) \) obey (5.1). Rank-2 fields can be interpreted as bilocal fields in the twistor space. Being seemingly similar to the bilocal space-time formalism of [49–51], the twistorial bilocal formalism is in many respects more efficient.

Since equation (5.6) has the unfolded form, its dynamical pattern can be analyzed with the help of \( \sigma_- \) cohomology techniques with

\[
\sigma_- = -dX^{AB} \frac{\partial^2}{\partial Y^\alpha \partial U^B}. \tag{5.8}
\]

The result is that dynamical currents (primaries), that belong to \( H^0(\sigma_-) \), are [21]

\[
J(U|X) = T(U, 0|X), \quad \tilde{J}(Y|X) = T(0, Y|X), \tag{5.9}
\]

\[
J^{\text{asym}}(U, Y|X) = (U^A Y^B - U^B Y^A) \left( \frac{\partial^2}{\partial U^A \partial Y^B} T(U, Y|X) \bigg|_{U^A = Y^A = 0} \right). \tag{5.10}
\]

In the 3d case of \( M = 2 \) where \( A, B \to \alpha, \beta \), \( J(U|X) \) generates 3d currents of all integer and half-integer spins

\[
J(U|X) = \sum_{\lambda=0}^\infty U^{\alpha_1 \ldots \alpha_{2\lambda}} J_{\lambda-\alpha\ldots\alpha}(X) \tag{5.11}
\]

and

\[
J^{\text{asym}}(U, Y|X) = U_\alpha Y^\alpha J^{\text{asym}}(X). \tag{5.12}
\]

Differential equations, which follow from equation (5.6), are associated with \( H^1(\sigma_-) \) found in [21] for general \( M \). For \( M = 2 \), the structure of \( H^1(\sigma_-) \) is greatly simplified so that the resulting field equations amount solely to the conventional conservation condition

\[
\frac{\partial}{\partial X^{AB}} \frac{\partial^2}{\partial Y^\alpha \partial U^B} J(U|X) = 0, \quad \frac{\partial}{\partial X^{AB}} \frac{\partial^2}{\partial Y^\alpha \partial Y^B} \tilde{J}(Y|X) = 0. \tag{5.13}
\]

To define conserved charges, it is convenient to Fourier transform \( T(U, Y|X) \) to

\[
\tilde{T}(W, Y|X) = (2\pi)^{-M/2} \int_{\mathbb{R}^M} d^M U \exp\left(-iW^C U^C\right) T(U, Y|X), \tag{5.14}
\]

which satisfies the following current equation

\[
\left( \frac{\partial}{\partial X^{AB}} + iW_A \frac{\partial}{\partial Y^B} \right) \tilde{T}(W, Y|X) = 0. \tag{5.15}
\]

The key fact is that a 2M-form

\[
\Omega(T) = (dW_A \wedge (iW_B dX^{AB} - dY^A)) M \tilde{T}(W, Y|X) \tag{5.16}
\]

is closed in \( M_M \times \mathbb{R}^M (W_B) \times C^M (Y^A) \) provided that \( \tilde{T}(W, Y|X) \) obeys (5.15).
Indeed, from (5.15) and (5.16), it follows that
\[
\left(\frac{\partial}{\partial W_A} + dX^{AB} \frac{\partial}{\partial X^{AB}} + dY^A \frac{\partial}{\partial Y^A}\right) \wedge \Omega^{2M}(T(W, Y|X)) = 0.
\]
As a result, the charge
\[
q = q(T) = \int_{\Sigma^{2M}} \Omega^{2M}(T)
\]
is independent of local variations of a 2M-dimensional integration surface \(\Sigma^{2M}\) on solutions of (5.6). In particular, for functions that decrease fast enough at space infinity, it is independent of the time parameter in \(\mathcal{M}_M\), hence being conserved.

A remarkable output of this construction [25] is that it makes it possible to express conserved charges as integrals over the twistor space \(\mathbf{T}\) at any point of space-time.

Since (5.15) is a first-order linear PDE system, its solutions form a commutative algebra \(\mathcal{R}\), i.e. a linear combination of products of any regular solutions of (5.15) is also a solution. \(\mathcal{R}\) is the algebra of functions \(\eta\) of the form
\[
\eta(W, Y|X) = \epsilon(W_A, Y^C - iX^{CB}W_B)
\]
with regular \(\epsilon(W, Y)\). As a result, equation (5.16) generates conserved currents for \(\overline{T}_\eta(W, Y|X)\) of the form
\[
\overline{T}_\eta(W, Y|X) = \eta(W, Y|X) \overline{T}(W, Y|X),
\]
where \(\eta(W, Y|X)\) (5.18) is a polynomial representing a parameter of global HS symmetry. The charges \(q(\overline{T}_\eta)\) with various \(\eta(W, Y|X)\) generate the full set of conformal HS conserved charges. In particular, at \(M = 2\), formula (5.7) generates all conserved charges for free 3d massless fields.

6. Nonlinear higher spin equations in AdS_4

In this section, we first recall the standard formulation of nonlinear 4d HS equations of [2] and then extend it to a larger space with spinor coordinates. In the following, wedge products are implicit.

6.1. Standard formulation

The key element of the construction of [2] is the doubling of auxiliary Majorana spinor variables \(Y_3\) in the HS 1-forms and 0-forms
\[
\omega(Y; K|x) \rightarrow W(Z; Y; K|x), \quad C(Y; K|x) \rightarrow B(Z; Y; K|x)
\]
supplemented with equations which determine the dependence on the additional variables \(Z_3\) in terms of ‘initial data’
\[
\omega(Y; K|x) = W(0; Y; K|x), \quad C(Y; K|x) = B(0; Y; K|x).
\]
To this end, we introduce a spinor field \(S_3(Z; Y; K|x)\) that carries only pure gauge degrees of freedom and plays a role of connection with respect to additional \(Z_3\) directions. It is convenient to introduce anticommuting \(Z\)-differentials \(dZ^A dZ^B = -dZ^B dZ^A\) to interpret \(S_3(Z; Y; K|x)\) as a \(Z-1\)-form,
\[
S = dZ^A S_3(Z; Y; K|x).
\]
The variables $K = (k, \bar{k})$ are Klein operators that satisfy (4.20) with $w^a = (y^a, \bar{z}^a, d\zeta^a)$, $\tilde{w}^a = (\bar{y}^a, z^a, d\bar{z}^a)$. These two cases are distinguished by the property that they respect parity [22].

The nonlinear HS equations are [2]

\begin{align}
\frac{dW}{dt} + W \ast B - B \ast W &= 0, \\
\frac{dB}{dt} + W \ast B - B \ast W &= 0, \\
\frac{dS}{dt} + W \ast S - S \ast W &= 0, \\
S \ast B &= B \ast S, \\
S \ast S &= -i(dZ^A dZ_A + dz^a d\bar{z}^a)F_1(B)k\theta + d\bar{z}^a d\bar{z}_a \bar{F}_1(B)\bar{k}\bar{\theta}) ,
\end{align}

where $F_1(B)$ is some star-product function of the field $B$.

The simplest case of linear functions

\begin{align}
F_1(B) = \eta B, \quad \bar{F}_1(B) = \bar{\eta} B,
\end{align}

where $\eta$ is some phase factor (its absolute value can be absorbed into redefinition of $B$) leads to a class of pairwise nonequivalent nonlinear HS theories. The particular cases of $\eta = 1$ and $\eta = \exp i\pi/4$ are especially interesting, corresponding to the so-called $A$ and $B$ HS models. These two cases are distinguished by the property that they respect parity [22].

The associative star product $\ast$ acts on functions of two spinor variables

$$
(f \ast g)(Z; Y) = \frac{1}{(2\pi)^4} \int d^4U d^4V \exp[iU^AV^B C_{AB}] f(Z + U; Y + U)g(Z - V; Y + V),
$$

(6.10)

where $C_{AB} = (\epsilon_{a\beta}, \bar{\epsilon}_{a\bar{\beta}})$ is the $4d$ charge conjugation matrix and $U^A, V^B$ are real integration variables. It is normalized so that $1$ is a unit element of the star-product algebra, i.e. $f \ast 1 = 1 \ast f = f$. Star product (6.10) provides a particular realization of the Weyl algebra

$$
[Y_A, Y_B] = -[Z_A, Z_B] = 2iC_{AB}, \quad [Y_A, Z_B] = 0 \quad (6.11)
$$

([a, b]_s = a \ast b - b \ast a) resulting from the normal ordering with respect to the elements

\begin{align}
 b_A = \frac{1}{2i}(Y_A - Z_A), \quad a_A = \frac{1}{2}(Y_A + Z_A),
\end{align}

(6.12)

which satisfy

$$
[a_a, b_b]_s = [b_a, b_b]_s = 0, \quad [a_a, b_b]_s = C_{AB}
$$

(6.13)

and can be interpreted as creation and annihilation operators as is most evident from the relations

$$
b_A \ast f(b, a) = b_A f(b, a), \quad f(b, a) \ast a_A = f(b, a) a_A.
$$

(6.14)

From (6.10) it follows that functions $f(Y)$ form a proper subalgebra which is the centralizer of the elements $Z_A$. For $Z$-independent $f(Y)$, the star product (6.10) takes the form of the Weyl star product.

An important property of the star product (6.10) is that it admits the inner Klein operator

$$
\Upsilon = \exp iZ_A Y^A,
$$

(6.15)

which behaves as $(-1)^N$, where $N$ is the spinor number operator. One can easily see that

$$
\Upsilon \ast \Upsilon = 1,
$$

(6.16)

$$
\Upsilon \ast f(Z; Y) = f(-Z; -Y) \ast \Upsilon
$$

(6.17)
and
\[(\Upsilon \ast f)(Z; Y) = \exp iZ Y A f(Y; Z).\] (6.18)

The left and right inner Klein operators
\[\upsilon = \exp i\zeta^a v^a, \quad \bar{\upsilon} = \exp i\bar{z}^a \bar{v}^a,\] (6.19)

which enter equation (6.8), act analogously on undotted and dotted spinors, respectively,
\[(\upsilon \ast f)(z, \bar{z}; y, \bar{y}) = \exp i\zeta^a v^a f(y, \bar{z}; z, \bar{y}), \quad (\bar{\upsilon} \ast f)(z, \bar{z}; y, \bar{y}) = \exp i\bar{z}^a \bar{v}^a f(z, \bar{y}; y, \bar{z}).\] (6.20)

To analyze equations (6.4)–(6.8) perturbatively, one has to linearize them around some vacuum solution. The simplest choice is
\[W_0(Z; Y|x) = W_0(Y|x), \quad S_0(Z; Y|x) = dZ^A Z_A, \quad B_0 = 0,\] (6.23)

where \(W_0(Y|x)\) is some solution of the flatness condition
\[dW_0(Y|x) + W_0(Y|x) * W_0(Y|x) = 0.\] (6.24)

\(W_0(Y|x)\) bilinear in \(Y^A\) describes AdS4. Linearization of the system (6.4)–(6.8) around this vacuum just reproduces the free-field equations (4.7) and (4.8) (for more detail see [2, 3]).

In the purely bosonic case where all fermion fields are zero, the operator \(kk\) remains central in the full nonlinear system. As a result, the bosonic sector of the system (6.4)–(6.8) decomposes into two independent subsectors singled out by the projectors \(\Pi_\pm\) (4.21).

### 6.2. Spinor coordinates

An important feature of the system (6.4)–(6.8) is that equations (6.4)–(6.6), which contain space-time differential \(d\), are flatness conditions. As a result, the flows along space-time coordinates commute to equations (6.7), (6.8). This has two consequences. One is that nontrivial dynamics is hidden entirely in the noncommutative twistor space of Z and Y [41]. Another is that the system remains consistent if original space-time coordinates \(x^a\) are extended to a larger space. Of course, if the differential is extended as in (3.1) with the same vacuum connection, additional equations will simply mean that, up to gauge ambiguity, all fields are independent of the coordinates \(X\). However, the situation becomes more interesting if pullback of a vacuum connection to additional directions is nonzero.

As explained in section 2.2.1, this can be achieved by introducing a connection with respect to some symmetry algebra \(h\) in the system. So far, vacuum connection was introduced for the AdS4 algebra \(s p(4|\mathbb{R})\). We believe that in the context of holographic interpretation of the AdS4 HS theory it may be useful to extend \(s p(4|\mathbb{R})\) to the Lie algebra with the generators
\[T_{AB} = -\frac{1}{2} Y A Y B, \quad t_A = Y_A\] (6.25)

obeying commutation relations
\[[T_{AB}, T_{CD}] = C_{BC} T_{AD} + C_{AC} T_{BD} + C_{BD} T_{AC} + C_{AD} T_{BC},\] (6.26)

\[[T_{AB}, t_C] = C_{BC} t_A + C_{AC} t_B, \quad [t_A, t_B] = 2i C_{AB} -\] (6.27)

(The central element on the rhs of the second relation (6.27), which can be identified with \(h\) in the Heisenberg algebra \(h_4\) spanned by \(T_4\), is set to unity.) Following [35], we call this Lie algebra \(s p h(4|\mathbb{R})\). It should not be confused with the superalgebra \(o s p(1, 4)\) where \(Y_A\)
are treated as supergenerators. Clearly, $sph(4|\mathbb{R}) = sp(4|\mathbb{R}) \oplus h_4$. Note that $sph(4|\mathbb{R})$ is a parabolic subalgebra of $sp(6|\mathbb{R})$.

This generalization is aimed at the extension to the action level of the construction of [25] explained in section 5 where conserved currents were represented by closed forms in the correspondence space unifying space-time and spinor coordinates. As shown in [35], relevant geometry naturally results from the formulation of HS theory in an appropriate coset space of the group $SpH(4|\mathbb{R})$. The idea is to introduce additional commutative coordinates $u^A$ associated with additional generators in $sph(4|\mathbb{R})$ compared to $sp(4|\mathbb{R})$. Namely, we set

$$X = (x^a, u^2), \quad u^A = (u^2, \bar{u}^2).$$

(6.28)

Correspondingly, the space-time HS connection $W_{\alpha}(Z; Y; \mathcal{K}|x) = d_\alpha W_{\alpha}(Z; Y; \mathcal{K}|x)$ is extended to

$$W_X(Z; Y; \mathcal{K}|x) = W_{\alpha}(Z; Y; \mathcal{K}|x), \quad W_{\alpha}(Z; Y; \mathcal{K}|x) = d_\alpha W_{\alpha}(Z; Y; \mathcal{K}|x).$$

(6.29)

A vacuum connection can be chosen in the form

$$W_0(Y|x) = \frac{i}{4} W_{0A}^B(x) Y_A Y_B, \quad W_0 = d_\alpha W_{0A}^B(x) Y_A + idu^B d^B C_{AB}.$$  

(6.31)

where $W_{0A}^B(x)$ is a set of Killing spinors enumerated by the label $A$ that satisfy the covariant constancy condition

$$dW_{0A}^B(x) - W_{0B}^A(x) W_{0A}^B(x) = 0.$$  

(6.32)

As a consequence,

$$C_{AB} = W_{0A}^B(x) W_{0B}^D(x) C_{AB}$$  

(6.33)

is some constant antisymmetric matrix. Requiring

$$W_{0A}^B(x_0) = \delta^B_A$$  

(6.34)

at some point $x_0$, we achieve that $C_{AB}(x) = C_{AB}$ for any $x$.

As explained in section 13, an HS action should be described by some $Q$-closed 4-form where $Q$ is the operator (2.7) associated with the unfolded form of HS equations resulting from the perturbative analysis of equations (6.4)–(6.8). In other words, the action should be $d$-closed by virtue of these equations. Since equations (6.4)–(6.8) are insensitive to particular detail of space-time, this should be true for any space-time coordinates that can be introduced within unfolded formulation. The idea of the spinor extension is to use coordinates $u^A$ instead of $x^2$ in the computations involving bulk-to-boundary propagators, which are expected to be simpler in the spinor space than in $AdS_4$. A toy model for this mechanism is provided by the evaluation of conserved charges along the lines of [25].

7. AdS$_4$ higher spin theory as 3d conformal higher spin theory

As discussed in section 3, an unfolded formulation allows one to choose freely one or another space-time interpretation of the theory. To see HS $AdS_4/CFT_3$ holographic duality, consider pullback of all space-time differential forms (i.e. curvatures and connections) to some 3d surface $\Sigma \in AdS_4$. This gives a subsystem of the original unfolded system in $AdS_4$ that now can be interpreted as a 3d system on $\Sigma$, being by construction equivalent to the original $AdS_4$ system. In the HS model of interest, 0-forms associated with $AdS_4$ massless fields acquire the meaning of 3d conserved currents. Indeed, from the 3d perspective, dotted and undotted indices carry equivalent Lorentz representation. Hence, the 3d pullback of equation (4.19)
gives equation (5.15) for the generalized 3d conformal stress tensors. What is not guaranteed however is that conformal properties are manifest for dynamical variables inherited from the AdS4 HS theory. Let us consider this point in some more detail.

For manifest conformal invariance, it is most convenient to introduce oscillators
\[ y^+_a = \frac{1}{2} (y_a - i\bar{y}_a), \quad y^-_a = \frac{1}{2} (\bar{y}_a - iy_a) \]  
(7.1)
that satisfy
\[ [y^+_a, y^{+\beta}]_s = \delta^\beta_a. \]  
(7.2)
In the AdS4 setup, the reality conditions imply \((y^+_a)^\dagger = iy^-_a\). In the conformal setup, the appropriate reality conditions are
\[ (y^-_a)^\dagger = y^{+a}. \]  
(7.3)

Weyl star-product realization of the 3d conformal algebra \(sp(4; \mathbb{R}) \sim o(3, 2)\) is
\[
L^\alpha_\beta = y^{+\alpha}y^-_\beta - \frac{1}{2}\delta^\alpha_\beta y^{+\gamma}y^-_\gamma, \quad D = \frac{1}{2}y^{+a}y^-_a.
\]  
(7.4)
\[
P_\alpha^\beta = iy^+_\alpha y^-_\beta, \quad K^{\alpha\beta} = -iy^{+\alpha}y^{+\beta}.
\]  
(7.5)

Here, generators of 3d Lorentz transformations \(L^\alpha_\beta\) form \(sp(2; \mathbb{R})\), \(D\) is the dilatation generator. \(P_\alpha^\beta\) and \(K^{\alpha\beta}\) are generators of translations and special conformal transformations, respectively. By (7.2), the conformal dimension of the HS gauge fields counts the difference of the numbers of pluses and minuses
\[
[D, \omega(y^{\pm}|X)] = \frac{1}{2} \left( \frac{\partial}{\partial y^{+\alpha}} - \frac{\partial}{\partial y^{-\alpha}} \right) \omega(y^{\pm}|X). \]  
(7.6)

3d conformal HS algebra coincides with the AdS4 HS algebra rewritten in terms of oscillators \(y^\pm_a\). In this form, it was introduced in [52]. Hence, the pullback \(\tilde{\omega}(y^{\pm}|x)\) of the AdS4 HS gauge fields \(\omega(y^{\pm}|x)\) to \(\Sigma\) just gives the full set of 3d conformal HS gauge fields.

To make contact with the standard approach, it is convenient to foliate AdS4 so that
\[ x^2 = (x^\xi, z), \]  
(7.7)
where \(x^\xi\) are the coordinates of leaves \((q = 0, 1, 2)\) while \(z\) is a foliation parameter. Let \(\tilde{W}(y^{\pm}|x, z) = dx^\xi \tilde{W}_2(y^{\pm}|x, z)\) be pullback of \(W(y^{\pm}|x)\) to a leaf at some \(z\). At every \(z\), the original AdS4 HS theory gives rise to a 3d conformal HS theory with 3d conformal HS connections \(\tilde{W}(y^{\pm}|x, z)\). Similarly, \(W_0(y^{\pm}|x, z)\) inherited from some AdS4 vacuum connection \(W_0(y^{\pm}|x, z)\) provides a flat connection of the 3d conformal algebra \(sp(4)\) on every leaf.

A less trivial part of the 3d reduction is due to the gluing terms in (4.18) and field equations (4.19) on the 0-forms \(C\). First of all, we observe that equations (4.10) and (4.15) give the AdS deformation of equation (5.6). Hence, in agreement with the AdS/CFT correspondence prescription, 0-forms \(C\) describing massless fields in AdS4 should be interpreted as conserved currents in the 3d conformal setup. However, this correspondence is not quite direct because the original 0-forms \(C\) do not transform properly under conformal transformations. Indeed, the dilatation generator \(D\) in the twisted adjoint representation is realized by anticommutator which gives a second-order differential operator
\[
[D, C]_s = \left( y^{+\alpha}y^-_\alpha - \frac{1}{4} \frac{\partial^2}{\partial y^{+\alpha}\partial y^-_\alpha} \right) C \]  
(7.8)
rather than the first-order operator (7.6) in the adjoint representation. This means that the set of fields \(C\) inherited from the AdS4 theory is not manifestly conformal.

To solve this problem, one should change variables from \(C(y^{\pm}|x)\) to \(T(y^{\pm}|x)\) to achieve that \(T(y^{\pm}|x)\) transforms properly under dilatations. In fact, this issue is not specific to the
conformal description, having its direct analogue on the AdS side where the fields $C(y^\pm|x)$ do not exhibit manifest decomposition in terms of eigenfunctions of the energy operator $E$ which is holographically dual to $D$. As shown in [9], the transition from $C(y^\pm|x)$ to the basis which diagonalizes energy is nonlocal, forming a kind of non-unitary Bogolyubov transform. Similar transformation in the conformal setup is achieved by the transition from Weyl to Wick star product with respect to $y^-$ and $y^+$. Let $f(y^\pm)$ be an element of the Weyl star-product algebra. The map from the Weyl star product

\[(f \star g)(y^\pm) = \frac{1}{\pi^4} \int d^4 u^\pm d^4 v^\pm \exp 2(v^- u^+ - u^- v^+) f(y^\pm + u^\pm) g(y^\pm + v^\pm)\]  

(7.9)
to the Wick star product

\[(f_N \star g_N)(y^\pm) = \frac{1}{(2\pi)^2} \int d^4 u^\pm \exp(-u^- u^+ f_N(y^+, y^- + u^-) g_N(y^+ + u^+))\]  

(7.10)
is

\[f_N(y^\pm) = \frac{1}{\pi^4} \int d^4 u^\pm \exp(-2u^- u^+ f(y^\pm + u^\pm))\]  

(7.11)
or, equivalently,

\[f_N(y^\pm) = \exp \left( -\frac{1}{2} \epsilon^{\alpha\beta} \frac{\partial^2}{\partial y^- \partial y^+} \right) f(y^\pm).\]  

(7.12)
Wick star product has the properties

\[f_N(y^+) \star g_N(y^\pm) = f_N(y^+) g_N(y^\pm), \quad f_N(y^\pm) \star g_N(y^-) = f_N(y^+) g_N(y^-).\]  

(7.13)

\[y^- \star = y^- + \frac{\partial}{\partial y^-}, \quad y^+ \star = y^+ + \frac{\partial}{\partial y^+}.\]  

(7.14)
Let us now apply these formulæ to the dilatation operator $D$ (7.4). First of all, we obtain that

\[D_N = \frac{1}{2} (y^- y^+ + 1).\]  

(7.15)
Hence,

\[D_N \star = \frac{1}{2} \left( y^+ \frac{\partial}{\partial y^+} + y^- \frac{\partial}{\partial y^-} + 1 \right), \quad \star D_N = \frac{1}{2} \left( y^- \frac{\partial}{\partial y^-} + y^+ \frac{\partial}{\partial y^+} + 1 \right).\]  

(7.16)
In the twisted adjoint representation, the action of $D_N$ therefore is

\[\{D_N, \ldots \} \star = \frac{1}{2} \left( y^+ \frac{\partial}{\partial y^+} + y^- \frac{\partial}{\partial y^-} + 1 \right) + y^- y^+ + 1.\]  

(7.17)
It remains to introduce

\[T(y^\pm|x) = \exp(-y^- y^+) C_N(y^\pm|x)\]  

(7.18)
to achieve that, in agreement with the interpretation of $T(y^\pm|x)$ as a 3d conformal current,

\[D_N(T(y^\pm)) = \frac{1}{2} \left( y^+ \frac{\partial}{\partial y^+} + y^- \frac{\partial}{\partial y^-} + 1 \right) T(y^\pm).\]  

(7.19)
Let us now look more closely at the action of the translation generator $P_{a\beta}$ (7.5). To this end, we observe that

\[k y^- = \mp i y^+ \mp k, \quad \bar{k} y^+ = \mp i y^- \pm \bar{k}.\]  

(7.20)
Hence

\[P_{a\beta} T(y^\pm)k = \mp \frac{\partial^2}{\partial y^+ \partial y^\mp} T(y^\pm)k, \quad T(y^\pm)k P_{a\beta} = -\frac{\partial^2}{\partial y^- \partial y^-} T(y^\pm)k\]  

(7.21)
and, using 3d Cartesian coordinates along with

\[
\frac{\partial^2}{\partial y^+ \partial y^+} + \frac{\partial^2}{\partial y^- \partial y^-} = 4 \frac{\partial^2}{\partial y^0 \partial y^0},
\]  
(7.22)

for the case where the pullback of the AdS4 connection to \( \Sigma \) has the 3d Cartesian form, the resulting equation on the 3d 0-forms acquires the form of rank-2 equation (5.6)

\[
D_\alpha^\omega T(y, \bar{y}|x) = d_\alpha T(y, \bar{y}|x) + 4dx^\alpha \beta \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} T(y, \bar{y}|x) = 0.
\]  
(7.23)

According to the analysis of \( \sigma \) cohomology of [21] summarized in section 5, equation (7.23) describes two sets of conserved currents of all spins \( s > 0 \) and two spin-zero branches distinguished by their symmetry under \( y \leftrightarrow \bar{y} \). The symmetric branch is generated by \( J^{\text{sym}}(x) = T(0|x) \) that has the conformal dimension \( \Delta = 1 \). The antisymmetric branch is generated by \( J^{\text{asym}}(x) (5.12) \). In accordance with (7.19), \( \Delta(J^{\text{asym}}(x)) = 2 \). These are just correct conformal dimensions for scalar currents associated with the two energy branches of the AdS4 conformal scalar field.

In this setup, the AdS\(_4\)/CFT\(_3\) HS duality takes place on every leaf of the \( z \)-foliation. However, boundary and bulk fields are related by the nonlocal transform from Weyl to Wick star product. Since unfolded equations relate space-time derivatives to those over \( y^\pm \), the nonlocal map in spinor variables is translated to the space-time nonlocality of the map of AdS\(_4\) fields to conformal ones. Without using twistor variables, it may be difficult to establish precise correspondence between the AdS\(_4\) HS theory and its dual on any \( \Sigma \). However, as shown in the next section, the holographic correspondence drastically simplifies if \( \Sigma \) is AdS\(_4\) infinity, just reproducing the standard AdS/CFT recipe [5, 6] where fields at the boundary of AdS\(_{d+1}\) are identified with (sources for) currents in CFT\(_d\).

Opposite \( -+ \) ordering choice leads to equivalent results with the exchange of \( y^+ \) and \( y^- \). This means that \( D \) changes its sign while \( P_{a\beta} \) and \( K^{a\beta} \) should be reinterpreted as generators of special conformal transformations and translations, respectively. With these redefinitions, consideration remains intact.

8. Holographic locality at infinity

8.1. Conformal foliation and Poincaré coordinates

Let \( M^d \) be a \( d \)-dimensional conformally flat space-time with local coordinates \( x \) and \( u_w(x) = u^A_w T_A \) be some flat \( o(d, 2) \) connection\(^3\)

\[
d_\alpha u_w(x) + u_w(x)w_\alpha(x) = 0.
\]  
(8.1)

Let \( D \) be the dilatation generator among \( T_A \) which induces standard \( \mathbb{Z} \) grading on \( o(d, 2) \) so that

\[
[D, T_A] = \Delta(T_A)T_A,
\]  
(8.2)

where \( \Delta(T_A) \) is the conformal dimension of \( T_A \) which takes values \( \pm 1 \) or \( 0 \). Namely,

\[
T_A = (L_{ab}, D, K_a, P_a),
\]  
(8.3)

where the conformal dimensions of generators of Lorentz transformations \( L_{ab} \), dilatations \( D \), special conformal transformations \( K_a \) and translations \( P_a \) are

\[
\Delta L = 0, \quad \Delta(D) = 0, \quad \Delta(K) = 1, \quad \Delta(P) = -1.
\]  
(8.4)

\(^3\) That an \( o(d, 2) \) connection \( u_w(x) \) is flat means that \( M^d \) endowed with the metric resulting from the vielbein associated with the \( P_a \) component of \( u_w(x) \) is (locally) conformally flat (see, e.g., [33]).
A particular flat connection which corresponds to Cartesian coordinates in \( M^d \) is
\[
 u_X(x) = \frac{dx^a P_a}{x},
\]  
(8.5)
Let us now introduce an additional coordinate \( z \) and differential \( dz \) so that \( x = (x, z) \) be the local coordinates of \( AdS_{d+1} \). A conformally foliated connection \( W(x) \) of \( AdS_{d+1} \) can be introduced as follows. The components of \( W(x) \) with differentials \( dx \) are
\[
 W^A_{\xi}(x) T_A = z^{\xi(T_+)} u^A_{\xi}(x) T_A,
\]  
(8.6)
while the only nonzero \( dz \) component of the connection is associated with the dilatation generator \( D \), having the form
\[
 W_z(x) = -z^{-1} dz D. 
\]  
(8.7)
Clearly, so defined connection \( W(x) \) is flat in (a local chart of) \( AdS_{d+1} \). Poincaré coordinates result from this construction applied to the connection \( u_x(x) \). It should be stressed, however, that this construction works for any \( o(d,2) \) flat connection \( u_X(x) \) in \( M^d \). In particular, if \( u_X(x) \) itself is some \( AdS_d \) connection with nonzero components in \( o(d - 1,2) \in o(d,2) \), it can itself be represented in the form of conformal foliation with another foliation parameter \( z_1 \), continuing the process of dimension reduction.

In spinor notation, local coordinates of \( AdS_4 \) are
\[
 x^{a\alpha} = \left( x^{a\alpha}, -\frac{i}{2} e^{a\alpha} z^{-1} \right),
\]  
(8.8)
where the symmetric part of \( 4d \) coordinates \( x^{a\alpha} = x^{a\alpha} \) is identified with coordinates of \( \Sigma \) while \( z \) is the radial coordinate of \( AdS_4 \). The appearance of \( e^{a\alpha} = -e^{a\alpha} \) in the definition of \( z \) breaks \( 4d \) Lorentz symmetry \( sp(2; \mathbb{C}) \) to \( 3d \) Lorentz symmetry \( sp(2; \mathbb{R}) \) that acts on the both types of spinor indices.

Now we are in a position to analyze the dynamical content of \( AdS_4 \) HS equations at \( z \to 0 \). We will work in terms of Weyl star product inherited from \( (6.10) \) in the sector of \( x^\pm \) variables. Using \( (8.5) \) and \( (7.5) \), (4.13), \( AdS_4 \) connection can be chosen in the form
\[
 W = \frac{i}{z} dx^\alpha \bar{y}_\alpha \bar{y}_\beta - \frac{dz}{2z} y_\alpha \bar{y}^\alpha + \frac{1}{4z} (dx^\alpha (y_\alpha y_\beta - \bar{y}_\alpha \bar{y}_\beta) + 2dx^\alpha \bar{y}_\beta y_\alpha + dy_\alpha \bar{y}_\alpha) .
\]  
(8.9)
which is equivalent to
\[
 W = \frac{1}{4z} (dx^\alpha (y_\alpha y_\beta - \bar{y}_\alpha \bar{y}_\beta) + 2dx^\alpha \bar{y}_\beta y_\alpha + dy_\alpha \bar{y}_\alpha) .
\]  
(8.10)
By equation (4.13), \( AdS_4 \) vierbein and Lorentz connection are
\[
 e^{a\alpha} = \frac{1}{2z} dx^a, \quad e^{a\alpha} = -\frac{i}{4z} dx^a, \quad \bar{e}^{a\alpha} = \frac{i}{4z} dx^a.
\]  
(8.11)

### 8.2. 0-forms

The unfolded equation on Weyl tensors in \( AdS_4 \), which is the covariant constancy condition (4.19) in the twisted adjoint representation, decomposes into two equations with respect to the \( 3d \) coordinates \( e^{a\alpha} \) and \( z \), respectively,
\[
 \left[ \frac{dx}{z} + \frac{1}{z} dx^\alpha \left( y_\alpha \frac{\partial}{\partial y^\beta} - \bar{y}_\alpha \frac{\partial}{\partial \bar{y}^\beta} + y_\alpha \bar{y}_\beta - \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \right) \right] C(y, \bar{y}; x, z) = 0, 
\]  
(8.12)
\[
 \left[ \frac{dz}{2z} \left( y_\alpha \bar{y}^\alpha - e^{a\alpha} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \right) \right] C(y, \bar{y}; x, z) = 0. 
\]  
(8.13)
Consider equation (8.12) which should reproduce the rank-2 equation (5.6) at \( M = 2 \). By the substitution
\[
C(y, \tilde{y} | x, z) = \exp(y_\alpha \tilde{y}^\alpha) \tilde{C}(y, \tilde{y} | x, z),
\]
it amounts to
\[
\left[ d_x - \frac{i}{z} dx^\alpha \frac{\partial^2}{\partial y_\alpha \partial \tilde{y}^\beta} \right] \tilde{C}(y, \tilde{y} | x, z) = 0.
\]  
(8.15)
Rescaling the variables \( y^\alpha \) and \( \tilde{y}^\alpha \) via the substitution
\[
C(y, \tilde{y} | x, z) = z \exp(y_0 \tilde{y}^0) T(w, \tilde{w} | x, z),
\]
where the overall factor of \( z \) is introduced for the future convenience and
\[
w^\alpha = z^{1/2} y^\alpha, \quad \tilde{w}^\alpha = z^{1/2} \tilde{y}^\alpha,
\]
we obtain that \( T(w, \tilde{w} | x, z) \) satisfies the conformal-invariant rank-2 unfolded equation
\[
\left[ d_x - i dx^\alpha \frac{\partial^2}{\partial w^\alpha \partial \tilde{w}^\beta} \right] T(w, \tilde{w} | x, z) = 0.
\]  
(8.18)
Substitution of (8.16) into equation (8.13) gives
\[
\left( \frac{\partial}{\partial z} - \frac{1}{2} \epsilon^\alpha_\beta \frac{\partial^2}{\partial w^\alpha \partial \tilde{w}^\beta} \right) T(w, \tilde{w} | x, z) = 0.
\]  
(8.19)
Equations (8.18) and (8.19) are linearized unfolded equations for 0-forms in AdS\(_4\) HS theory in Poincaré coordinates. As anticipated, equation (8.18) describes 3d conserved currents. Equation (8.19) tells us that contractions \( w_\alpha \tilde{w}^\alpha \) in \( T(w, \tilde{w} | x, z) \) should carry appropriate powers of \( z \). This conforms to the fact that, by virtue of the linearized equation (8.18), most such components vanish as a consequence of the conservation equations for the currents \( \tilde{J} \) and \( \tilde{\Delta} \). An important exception is provided by \( \tilde{J} \) (5.12) that describes the scalar mode of conformal dimension \( \Delta = 2 \) properly accounted by equation (8.19) (see also section 8.4).

8.3. 1-forms

Using background connection (8.9) and Weyl star product, we obtain in the sector of HS gauge fields
\[
D_x W(y^\pm | x, z) = \left( d_x + \frac{2i}{z} dx^\alpha \frac{\partial}{\partial y^\alpha} \right) W(y^\pm | x, z),
\]  
(8.20)
\[
D_x W(y^\pm | x, z) = \left( d_x - \frac{dx}{2z} \left( y_+ \frac{\partial}{\partial y_+} - y_- \frac{\partial}{\partial y_-} \right) \right) W(y^\pm | x, z).
\]  
(8.21)
Setting
\[
W(y^\pm | x, z) = \Omega(v^-, w^+ | x, z),
\]
where
\[
v^\pm = z^{-1/2} y^\pm, \quad w^\pm = z^{1/2} y^\pm,
\]
this gives
\[
D_x \Omega(v^-, w^+ | x, z) = \left( d_x + 2i dx^\alpha v_\alpha \frac{\partial}{\partial w^\alpha} \right) \Omega(v^-, w^+ | x, z),
\]  
(8.24)
\[
D_x \Omega(v^-, w^+ | x, z) = d_x \Omega(v^-, w^+ | x, z).
\]  
(8.25)
Now consider equation (4.18) starting from $x$ sector. Its rhs takes the form
\[
- \mathcal{H}^{\alpha \beta} \left( \eta \frac{\partial^2}{\partial \omega^\alpha \partial \omega^\beta} T^{-1 - i}(0, \omega, x, z) + \tilde{\eta} \frac{\partial^2}{\partial \tilde{\omega}^\alpha \partial \tilde{\omega}^\beta} T^{1-i}(w, 0 | x, z) \right),
\]
where
\[
\mathcal{H}^{\alpha \beta} = \frac{1}{4} dx^\gamma \wedge dx^\beta.
\]
Let us stress that explicit dependence on $z$ in equation (8.26) due to derivatives over $y$ and $\tilde{y}$, and the factors of $\tilde{z}$ in the frame field (8.11) and the definition of $T$ (8.16) cancel out. Using
\[
w_\alpha = \omega^\alpha + iz \bar{\omega}^\alpha, \quad \tilde{w}_\alpha = i \omega^\alpha + z \bar{\omega}^\alpha,
\]
equation (4.18) acquires the form
\[
D_x \Omega^i_{ij}(w^-, w^+ | x, z) = \mathcal{H}^{\alpha \beta} \frac{\partial^2}{\partial \omega^\alpha \partial \omega^\beta} (\tilde{\eta} T^{1-i-j}(w^+ + iz \omega^-, 0 | x, z) - \eta T^{1-i-j}(0, iw^+ + z \omega^- | x, z)).
\]
In the limit $z \to 0$, this gives
\[
D_x \Omega^i_{ij}(w^-, w^+ | x, 0) = \mathcal{H}^{\alpha \beta} \frac{\partial^2}{\partial \omega^\alpha \partial \omega^\beta} T^{ii}(w^+, 0 | x, 0),
\]
where
\[
T^{ij}(w^+, w^- | x, z) = \tilde{\eta} T^{1-i-j}(w^+ + iz \omega^-, 0 | x, z) - \eta T^{1-i-j}(0, iw^+ + z \omega^- | x, z)
\]
satisfies the rank-2 equation (8.18)
\[
\left[ d_x - idx^{\alpha \beta} \frac{\partial^2}{\partial \omega^\alpha \partial \omega^\beta} \right] T^{ii}(w^+, w^- | x, z) = 0.
\]
Equations (8.30) and (8.32) are the linearized unfolded equations of free 3d conformal HS theory that describes conserved currents $T^{ij}$ and conformal HS gauge fields $\Omega^{ij}_x$. Being inherited from the nonlinear HS theory in AdS$_3$, the full boundary theory should be a nonlinear conformal HS gauge theory of currents $T^{ij}$ interacting with gauge fields $\Omega^{ij}_x$ of Chern–Simons type. We will come back to this issue in section 9.

In the $x$ sector, equation (4.18) gives
\[
D_x \Omega^i_{ij}(w^-, w^+ | x, z) + D_x \Omega^i_{ij}(w^-, w^+ | x, z) = - \frac{i}{2} dx^{\alpha \beta} dw^{\alpha \beta} \frac{\partial^2}{\partial \omega^\alpha \partial \omega^\beta} \times (\tilde{\eta} T^{1-i-j}(w^+ + iz \omega^-, 0 | x, z) + \eta T^{1-i-j}(0, iw^+ + z \omega^- | x, z)).
\]
Equations (8.19) and (8.33) determine the $z$-evolution of $\Omega^i_{ij}(w^-, w^+ | x, z)$ and $T^{ij}(w^+ | x, z)$. According to [53–55], supplemented with nonlinear corrections, these should be interpreted as renormalization group equations.

### 8.4. Weyl, Wick and Fock

Let us consider in more detail the relation between the Wick star-product formalism of section 7 and the Weyl star-product formalism used in section 8.

Naively, the boundary limit $z \to 0$ of the map (7.12) gives the identity operator since
\[
f_{\lambda}(w^\pm | z) = \exp \left( -2z \epsilon^{\alpha \beta} \frac{\partial^2}{\partial \omega^\alpha \partial \omega^\beta} \right) f(w^\pm | z).
\]
This is however not true because of the exponential factors in (7.18) and (8.16) which are singular in the limit $z \to 0$ at $w$ fixed. Moreover, formulae (7.18) and (8.16) seemingly do not
match each other because the exponential factors look different taking into account that
\[ y^{-\cdot} y^{+\cdot} = - \frac{1}{2} y^{-\cdot} y^{+\cdot}. \]  
(8.35)

This is however just the effect of using different star products in the respective formulae.

Indeed, consider a Weyl star-product element of the form
\[ c(y^\pm) = \exp(-2y^{-\cdot} y^{+\cdot}) t(w^\pm) \]  
(8.36)

which is an analogue of (8.16). Using (7.11), it easy to see that
\[ c_N(y^\pm) = \exp(-y^{-\cdot} y^{+\cdot}) \frac{1}{\pi^2} \int d^4u^\pm \exp(-u^{-\cdot} u^{+\cdot}) t\left(\frac{1}{2} (w^\pm + z^{1/2} u^\pm)\right). \]  
(8.37)

Similarly to (8.34), integration over \( u^\pm \) trivializes at \( z \to 0 \) giving
\[ c_N(y^\pm)|_{z=0} = \exp(-y^{-\cdot} y^{+\cdot}) t\left(\frac{1}{2} w^\pm\right). \]  
(8.38)

The exponential factor in this formula just matches that in (7.18).

In fact, in the both of star products, the exponentials
\[ F_N = \exp -y^{-\cdot} y^{+\cdot}, \quad F = \exp -2y^{-\cdot} y^{+\cdot} \]  
(8.39)

provide the star-product realization of the Fock vacuum that satisfies
\[ y^{-\cdot} F = y^{-\cdot} F_N = 0, \quad F \circ y^{+\cdot} = F_N \circ y^{+\cdot} = 0. \]  
(8.40)

Correspondingly, the substitution of the exponential as in (7.18) maps the Wick star product \( \circ \) to the operation \( \circ \) that describes the action of normal-ordered operators in the Fock bimodule generated from the Fock vacuum (8.39)
\[ f(y^{-\cdot}, y^{+\cdot}) \circ T(y^\pm) = f\left(\frac{\partial}{\partial y^{-\cdot}}, y^{+\cdot}\right) T(y^\pm), \quad T(y^\pm) \circ f(y^{-\cdot}, y^{+\cdot}) = f\left(y^{-\cdot}, -\frac{\partial}{\partial y^{-\cdot}}\right) T(y^\pm), \]  
(8.41)

where derivatives \( \frac{\partial}{\partial y^{-\cdot}} \) act on \( T(y^\pm) \).

Note that, in the conformal setup, the exponential factor in (8.16) trivializes for all primary fields except for \( J_{\text{asym}} \) (5.12) since all other primaries depend either only on \( y^- \) or only on \( y^+ \). In the case of \( J_{\text{asym}} \), the exponential factor accounts properly the asymptotic \( z \)-dependence of \( J_{\text{asym}} \) in accordance with its conformal dimension \( \Delta = 2 \).

Thus, the Wick and Weyl star-product descriptions match at \( \text{AdS}_4 \) infinity. Nonlocality of the holographic conformal map in the bulk trivializes at \( \text{AdS}_4 \) infinity, leading to the standard \( \text{AdS}/\text{CFT} \) prescription where boundary values of the HS gauge fields in the bulk are identified with sources to conformal operators of the boundary theory.

9. Toward nonlinear 3d conformal higher spin theory

Although an unfolded formulation of the nonlinear 3d conformal HS theory is not yet known, it can be systematically reconstructed from the \( \text{AdS}_4 \) HS theory via extension of analysis of section 8 to the nonlinear level. A detailed derivation of the nonlinear 3d conformal HS theory will be presented elsewhere. Here, we only comment on its general structure.

Generally, the holographic image of the \( \text{AdS}_4 \) HS theory should be nonlinear. This is because \( \text{AdS}_4 \) HS gauge connections contain background and fluctuational parts as two pieces of the same field. The same should be true in its conformal version. To be formally consistent, a system of conformal HS equations should be nonlinear with the only exception for the case where rhs of equation (8.30) can be zero in all orders. Let us explain this in some more detail.

To simplify notation, we consider the purely bosonic case with all fields even in spinor variables \( Y^A, Z^A, dZ^A \), where one can discard the doubling of fields in the full nonlinear
system via truncation of the theory by the projectors (4.21). Correspondingly, in this section, we discard the labels \( i, j \) writing \( \Omega_{\alpha} \) and \( T \) instead of \( \Omega_{\alpha i}^{\beta j} \) and \( T^{i} \).

Let us decompose

\[
\Omega_{\alpha}(v^{-}, w^{+})|x\rangle = \sum_{n,m}^{\infty} \Omega_{\alpha}^{n,m}(v^{-}, w^{+}|x), \quad R_{1xx} := D_{x} \Omega_{\alpha} = \sum_{n,m=0}^{\infty} R_{1xx}^{n,m}(v^{-}, w^{+}|x),
\]

(9.1)

\[
A_{\alpha_{1}\ldots \alpha_{n}}^{n,m}(v^{-}, w^{+}|x) = A_{\alpha_{1}\ldots \alpha_{n} \beta_{1}\ldots \beta_{m}}^{n,m}(v^{-}, w^{+}|x)v^{-}_{\alpha_{1}}\ldots v^{-}_{\alpha_{n}}w^{+}_{\beta_{1}}\ldots w^{+}_{\beta_{m}}.
\]

(9.2)

Recall that a spin \( s \) gauge field is described by \( \Omega_{\alpha}^{n,m}(v^{-}, w^{+}|x) \) with \( n + m = 2(s - 1) \). A particularly important role is played by the fields

\[
\Omega_{\alpha}^{-}(v^{-}|x) = \sum_{n}^{\infty} \Omega_{\alpha}^{n,0}(v^{-}, 0|x) = \Omega_{\alpha}^{-}(v^{-}|x).
\]

(9.3)

and curvatures

\[
R_{xx}^{+}(w^{+}|x) = \sum_{m}^{\infty} R_{1xx}^{+}(0, w^{+}|x).
\]

(9.4)

\( \Omega_{\alpha}^{-}(v^{-}|x) \) and \( R_{xx}^{+}(w^{+}|x) \) belong to the subspaces of, respectively, lowest and highest vectors of the 3d conformal subalgebra \( sp(4) \) of the 3d conformal HS algebra.

\( \Omega_{\alpha}^{-}(v^{-}|x) \) is the generating function for dynamical conformal HS gauge fields as can be seen by virtue of the \( \sigma_{-} \) cohomology analysis with

\[
\sigma_{-} = e^{\alpha\beta} P_{\alpha\beta} = 2i d^{\alpha\beta \gamma} v^{-}_{\alpha} \frac{\partial}{\partial w^{+}_{\beta}}.
\]

(9.5)

Dynamical fields associated with \( H^{1}(\sigma_{-}) \) are rank-3 totally symmetric multispinor fields \( \psi_{\alpha_{1}\ldots \alpha_{3}}(x) \) representing \( \Omega_{\alpha}^{-}(v^{-}|x) \) up to Lorentz and dilatation HS gauge shift symmetries

\[
\Omega_{\alpha}^{-}(v^{-}|x) = e^{\alpha_{1}\beta_{1}} v^{-\alpha_{1}} \ldots v^{-\alpha_{3}} \psi_{\alpha_{1}\ldots \alpha_{3}}(x).
\]

(9.6)

\( \psi_{\alpha_{1}\ldots \alpha_{3}}(x) \) provides 3d spinor realization of traceless conformal HS gauge fields introduced by Fradkin and Tseytlin for \( d = 4 \) in [56]. Having the gauge transformation law

\[
\delta \psi_{\alpha_{1}\ldots \alpha_{3}} = \partial_{(\alpha_{1}} \epsilon_{\alpha_{2}\ldots \alpha_{3})},
\]

(9.7)

they are dual to conserved conformal currents, serving as sources for correlators of currents in the holographic interpretation.

At the linearized level, \( R_{xx}^{+}(w^{+}|x) \) is the part of the linearized conformal HS curvature that contains nontrivial gauge-invariant combinations of derivatives of the dynamical fields \( \psi_{\alpha_{1}\ldots \alpha_{3}} \). Namely, the \( \sigma_{-} \) cohomology analysis shows that the conditions

\[
R_{1xx}^{n,m} = 0, \quad n > 0,
\]

(9.8)

give algebraic constraints which express fields \( \Omega_{\alpha}^{n,m} \) via order-\( m \) derivatives of the dynamical fields, imposing no restrictions on the latter. Although most of components of \( R_{1xx}^{n,m}(w^{+}|x) \) vanish by virtue of Bianchi identities applied to (9.8), some may remain nonzero. These are just those parametrized by the 0-forms \( T(w^{+}, 0|x) \) on the rhs of the unfolded equations (8.30) (discarding indices \( \alpha \)). In fact, \( T(w^{+}, 0|x) \) just represents \( H^{2}(\sigma_{-}) \).

An important property of the unfolded system (8.30), (8.32) with currents \( T(w^{+}, w^{-}|x) \) treated as independent 3d fields is that it is off-shell. This means that the system (8.30), (8.32) expresses up to gauge transformations all fields \( \Omega_{\alpha}(v^{-}, w^{+}|x) \) and \( T(w^{+}, w^{-}|x) \) via derivatives of \( \psi_{\alpha_{1}\ldots \alpha_{3}} \) imposing no restrictions on the latter. In particular, this means that equations (8.32) are consequences of equations (8.30) supplemented with constraints which express \( T(w^{+}, w^{-}|x) \) via the derivatives of \( T(w^{+}, 0|x) \). In this setup, the current conservation equation

\[
\frac{\partial}{\partial x^{\mu\nu}} \frac{\partial^{2}}{\partial w_{\nu}^{\alpha} \partial w_{\mu}^{\beta}} T(w^{+}, 0|x) = 0
\]

(9.9)
follows from the expression for $T(w^+, 0|\mathbf{x})$ in terms of derivatives of $\psi_{a_1...a_3}$ by virtue of equation (8.32). The other way around, given conserved current $T(w^+, 0|z)$, equation (8.32) determines dynamical fields $\psi_{a_1...a_3}$ in terms of $T(w^+, 0|\mathbf{x})$ up to a gauge transformation.

The fact that the 3d system (8.30), (8.32) is off-shell means that unrestricted $\psi_{a_1...a_3}$ can be interpreted as arbitrary boundary values of the bulk HS gauge fields. It should be stressed that, being off-shell in $d = 3$, the system (8.30), (8.32) becomes on-shell in a larger space like $\text{AdS}_4$ or a space with additional spinor coordinates of section 6.2. As discussed in section 3, this is crucial for holographic interpretation of the theory.

It should be stressed that unfolded dynamics properly accounts for the asymptotic behavior of relativistic fields in AdS. We have seen this already for a scalar field in AdS4 where unfolded equations reproduce two types of asymptotic behavior with $\Delta = 1$ and $\Delta = 2$. The analysis of this section extends this observation to any spin. Indeed, equation (7.19) shows that conserved currents have the canonical conformal dimension $s + 1$ and hence asymptotic behavior $z^{s+1}$. From equation (8.30), it follows that $\Omega(0, w^+)$ has the asymptotic behavior $z^{-1}$, which is in agreement with (7.6), taking into account the fact (4.1) that the total number of spinor indices carried by a spin-$s$ connection is $2(s - 1)$. From equation (8.23), it follows that $\Omega(v^-, 0)$ has the asymptotic behavior $z^{2-s}$ which is again in agreement with (7.6). Since the background frame field in (9.6) contains the factor of $z^{-1}$, $\psi_{a_1...a_3}$ has the asymptotic behavior $z^{2-s}$ which is just the correct behavior of the boundary source.

Setting to zero $T(w^+, 0|\mathbf{x})$ imposes differential equations on the dynamical fields $\psi_{a_1...a_3}$. Since equation (9.8) at $T(w^+, 0|\mathbf{x}) = 0$ is the linearized flatness condition, in this case, the HS gauge fields become pure gauge. For nonzero $T(w^+, 0|\mathbf{x})$, conformal HS gauge fields are nontrivial. To see whether or not the theory can remain free beyond the linearized approximation, one has to check whether or not the condition $R_{w^+}(w^+, \mathbf{x}) = 0$ is consistent with the full nonlinear HS equations in $\text{AdS}_4$. As discussed in more detail in section 10, this indeed turns out to be possible for two particular truncations of HS models: one for the $A$-model and another for the $B$-model. Correspondingly, the related truncated HS theories turn out to be holographically dual to free-boundary bosonic and fermionic theories in agreement with the Klebanov–Polyakov [13] and the Sezgin–Sundell [22] conjectures. This conclusion is also in agreement with the Maldacena–Zhiboedov theorem [20] because in these cases 3d conformal HS gauge fields decouple from the 3d currents. However, beyond these two cases, the boundary dual of $\text{AdS}_4$ HS theory is nonlinear.

One reason why the boundary theory should be nonlinear is that the conformal HS curvatures inherited from $\text{AdS}_4$ HS theory

$$R_{w^+}(w^+, \mathbf{x}) = d_s \Omega_s(v^-, w^+|\mathbf{x}) + \Omega_s(v^-, w^+|\mathbf{x}) \ast \Omega_s(v^-, w^+|\mathbf{x})$$

(9.10)

are nonlinear. Here, it is crucial that the rescalings (8.17) and (8.23) have opposite scalings in the radial coordinate $z$ so that $v^-$ and $w^+$ obey $z$-independent commutation relations

$$[v^-, w^+], = \delta^\rho_{\sigma}$$

(9.11)

Analogously, the lhs of equation (8.32) deforms to the covariant derivative in the twisted adjoint representation of the non-Abelian 3d conformal HS algebra. Since the substitution (7.18) for the current maps Wick star product to Fock product, the nonlinear extension of equation (8.32) starts from the twisted adjoint covariant derivative

$$\tilde{D}T(w^+|\mathbf{x}) = dT(w^+|\mathbf{x}) + \Omega \left( \frac{\partial}{\partial w^+\sigma} \cdot w^\sigma \right) T(w^+|\mathbf{x}) - T(w^+|\mathbf{x}) \Omega \left( -i \frac{\partial}{\partial w^-\sigma} \cdot -iw^-\sigma \right)$$

(9.12)

where we used (7.20) and that $\Omega(-v^-, -w^+) = \Omega(v^-, w^+)$ for bosons.
It should be stressed that nonlinear terms in equations (9.10) and (9.12) are $z$-independent, hence fully reproducing the non-Abelian structure of 3d conformal HS theory in the $z \to 0$ limit. The nonlinear deformation (9.12) of the twisted adjoint covariant derivative implies a nonlinear deformation of the rank-2 unfolded equation (8.32) which, in turn, implies a nonlinear deformation of the conventional current conservation condition, hence not respecting conditions of Maldacena–Zhiboedov theorem [20].

The nonlinear deformation due to non-Abelian HS algebra is just a first step requiring further $T$-dependent nonlinear deformation of equations (8.30) and (8.32). Similarly to [27] one can search these corrections perturbatively in powers of $T$. However, a straightforward analysis is not simple. It seems more promising to try to guess a closed form of yet unknown nonlinear 3d HS conformal unfolded system as it was guessed for the AdS$_4$ HS system in [1, 2]. We plan to consider this problem elsewhere.

It may look a bit peculiar that the variables $v^+$ and $v^-$ appear asymmetrically in equations (8.30) and (8.32). This happens because of choosing a particular $y^+ y^-$ Wick ordering in the star product (7.10) or, alternatively, a particular form of connection (8.9). Choosing the $y^- y^+$ ordering exchanges the roles of $y^+$ and $y^-$. Full nonlinear conformal HS theory is expected to describe both of these sectors on an equal footing.

Another comment is that in the realization of conformal HS theory described so far the 3d conformal currents appear as independent dynamical objects whose properties are determined by the equations of the 3d conformal HS theory itself. This model does not capture 3d conformal scalar and spinor fields $\Phi'(w^+ | x)$ from which the currents $T(w^+ | x)$ can be built. We believe that this ingredient can also be incorporated into 3d conformal HS theory. Indeed, as shown in [57], 3d conformal scalar and spinor can be described as fields $| \Phi'(w^+ | x) \rangle$ valued in the Fock module of the 3d conformal HS algebra. In these terms, free-field equations for 3d conformal fields have the form

$$(d + \Omega_0(v^-, w^+ | x)) \Phi'(w^+ | x) \Phi' = 0. \quad (9.13)$$

An interesting problem for the future is to find a nonlinear 3d conformal HS theory for the full system of fields $\Omega$, $\mathcal{T}$ and $\Phi$, which relates $\mathcal{T}$ to proper bilinear combinations of $\Phi$. The solution of this problem should clarify the explicit relation of our construction to (generalized) boundary $\sigma$-model constructions of [13, 22].

### 10. Boundary conditions, reductions and AdS doubling

Standard AdS/CFT correspondence assumes certain boundary conditions at infinity. In terms of AdS$_4$ HS Weyl forms, they relate $T^{1-1-j}(w^+, 0 | x, 0)$ and $T^{1-1-j}(0, i w^+ | x, 0)$. Let

$$\mathcal{A}^{i j}(w^+, w^- | x) = T^{1-1-j}(w^+, w^- | x, 0) - T^{1-1-j}(-iw^-, iw^+ | x, 0), \quad (10.1)$$

$$\mathcal{B}^{i j}(w^+, w^- | x) = T^{1-1-j}(w^+, w^- | x, 0) + T^{1-1-j}(-iw^-, iw^+ | x, 0). \quad (10.2)$$

Conditions

$$\mathcal{A}^{i j}(w^+, w^- | x) = 0 \quad (10.3)$$

and

$$\mathcal{B}^{i j}(w^+, w^- | x) = 0 \quad (10.4)$$

will be called $\mathcal{A}$ and $\mathcal{B}$, respectively. We observe that the rhs of equation (8.30) is zero for the $\mathcal{A}$ boundary conditions in the $A$-model ($\eta = 1$) and for the $B$ conditions in the $B$-model ($\eta = i$). On the other hand, the rhs of equation (8.33), that determines $z$-evolution of the HS connection, is zero for the $\mathcal{B}$ conditions in the $A$-model and for the $\mathcal{A}$ conditions in the
B-model. This suggests that the latter boundary conditions correspond to IR fixed points of the model.

\( \mathcal{A}^{\perp} \) and \( \mathcal{B}^{\perp} \) describe independent combinations of \( T^{11−j} \), i.e. each of \( \mathcal{A} \) or \( \mathcal{B} \) conditions leaves some components of \( T^{11−j} \) nonzero. For any other choice of relative coefficients on the rhs of equations (10.1) and (10.2), the corresponding conditions would be too strong, implying \( T^{11−j} = 0 \).

The bosonic model where all fermions are zero decomposes into two independent systems projected out by the projectors \( \Pi_{\pm} \) (4.21). The currents, which contribute to the \( \Pi_{\pm} \) model, are

\[
\mathcal{A}_{\pm}(w^+, w^- | x) = \sum_{j=0,1} (T^{11−j}(w^+, w^- | x, 0) − T^{11−j}(−iw^−, iw^+ | x, 0)),
\]

(10.5)

\[
\mathcal{B}_{\pm}(w^+, w^- | x) = \sum_{j=0,1} (T^{11−j}(w^+, w^- | x, 0) + T^{11−j}(−iw^−, iw^+ | x, 0)).
\]

(10.6)

In this case, the \( \mathcal{A} \) condition implies that the current \( J^{\text{asymp}}(X) \) (5.12) is zero while \( J^{\text{sym}}(0 | X) \) remains free. The \( \mathcal{B} \) condition implies that \( J^{\text{sym}}(0 | X) = 0 \) and \( J^{\text{asymp}}(X) \) remains free. Hence, \( \mathcal{A} \) and \( \mathcal{B} \) boundary conditions just distinguish between two scalar currents that have different conformal dimensions, namely \( \Delta(J^{\text{sym}}) = 1 \) and \( \Delta(J^{\text{asymp}}) = 2 \).

In the \( \Pi_{-} \) model, the \( \mathcal{A} \) and \( \mathcal{B} \) conditions have opposite effect. Namely, the \( \mathcal{A} \) condition implies that the current \( J^{\text{asymp}}(X) \) is zero and \( J^{\text{sym}}(0 | X) \) remains free while the \( \mathcal{B} \) condition implies that \( J^{\text{asymp}}(0 | X) = 0 \) and \( J^{\text{sym}}(X) \) remains free. This conclusion is in agreement with supersymmetry of the model in the presence of fermions: for any type of boundary conditions, the supersymmetric model will contain both types of scalars, one in the \( \Pi_{+} \) sector and another in the \( \Pi_{-} \) sector. \( \mathcal{A} \) and \( \mathcal{B} \) boundary conditions extend two different types of boundary conditions for scalar currents to currents of all spins.

If the rhs of equation (8.30) is zero, it becomes the flatness condition for boundary HS gauge fields. In the gauge where nonzero boundary HS gauge fields belong to the conformal algebra \( sp(4) \), the resulting theory describes free unfolded equations on boundary currents in some conformally flat background. In agreement with the Klebanov–Polyakov [13] and the Sezgin–Sundell [22] conjectures, these two particular cases correspond to the free-boundary models of conformal scalar or spinor in \( \Pi_{+}A \) and \( \Pi_{-}B \) or \( \Pi_{+}B \) and \( \Pi_{-}A \) models, respectively.

In fact, the free-boundary theories are dual to truncations of the full nonlinear HS theories in \( \text{AdS}_4 \) by the parity automorphism \( P \) that exchanges dotted and undotted spinors. Indeed, as observed in [22], the nonlinear HS equations (6.4)–(6.8) are \( P \) invariant provided that \( P(B) = B \) in the \( \text{AdS}_4 \) model or \( P(B) = −B \) in the \( \text{B} \) model.

There is however an interesting and important subtlety in this consideration. Indeed, so defined \( P \) describes the reflection \( z \rightarrow −z \) of the coordinate \( z \) as introduced in (8.8). Hence, to apply \( P \), one Poincaré chart of \( \text{AdS}_4 \) has to be supplemented with another one with negative \( z \) to allow

\[
P(z) = −z.
\]

(10.7)

In fact, in our construction it is important that \( \text{AdS}_4 \) is doubled to contain two Poincaré charts related by \( P \). Although, geometrically, \( P \) leaves \( \text{AdS}_4 \) invariant, the effect of this doubling is nontrivial because the extension of solutions from one chart to another is not necessarily \( P \)-invariant. For example, this is the case in non-\( P \)-invariant HS theories with \( \eta^2 \neq ±1 \). On the other hand, no boundary conditions at \( z = 0 \) should be imposed to define the action as integral over the doubled \( \text{AdS} \) space-time. In this setup, the holographic duality relates a bulk theory in the doubled \( \text{AdS} \) space to the ‘boundary theory’ where all possible types of boundary fields \( \phi_{\text{bound}}(x) \) contribute. In the unfolded dynamics approach, values of \( \phi_{\text{bound}}(x) \) at \( z = 0 \) reconstruct all fields in the (doubled) bulk and hence values of the respective action functionals.
\[ S(\phi_{\text{bound}}). \] Note that in this respect the situation with the surface \( z = 0 \) in the doubled bulk space is analogous to that with a regular 3d surface \( \Sigma \) inside bulk as discussed in section 7.

We believe that the doubled bulk AdS/CFT setup, which follows naturally from unfolded dynamics, has general applicability for HS theories and beyond. The important issue of anomalies also naturally fits this problem setting as we discuss briefly in section 13.

In terms of elementary oscillators of the AdS4 HS theory, \( P \) acts as follows:

\[ P(y_\alpha) = \bar{y}_\alpha, \quad P(\bar{y}_\alpha) = -y_\alpha, \] (10.8)

which is equivalent to

\[ P(y_\alpha^\pm) = \pm iy_\alpha^\pm. \] (10.9)

Being defined in such a way that it maps \( y^+ \) and \( y^- \) to themselves, \( P \) is not involutive.

Namely, \( P^2 = F, F^2 = \text{id} \), where \( F \) is the boson–fermion automorphism that changes a sign of fermions. Although, naively, this property obstructs consistent \( P \)-reduction of the HS theory in the presence of fermions, this is not the case. Remarkably, this is just what doctor ordered to cure additional factors of \( i \) that appear due to the \( z \to -z \) reflection of the factor of \( z^{1/2} \) in the rescaling (8.17), so that (10.8) is replaced by

\[ P(w_\alpha^\pm) = \pm w_\alpha^\pm. \] (10.10)

So defined \( P \) admits two extensions \( P_\pm \) to the full nonlinear HS system

\[ P_\pm(z_\alpha) = -\bar{z}_\alpha, \quad P_\pm(\bar{z}_\alpha) = z_\alpha, \quad P_\pm(dx^\eta) = -d\bar{z}^\eta, \quad P_\pm(d\bar{z}^\eta) = dx^\eta, \] (10.11)

\[ P_\pm(k) = \pm \bar{k}, \quad P_\pm(\bar{k}) = \pm k. \] (10.12)

(Spinor coordinates \( z_\alpha \) and \( \bar{z}_\alpha \) should not be confused with the radial coordinate \( z \) of AdS4.) \( P_\pm \) leave invariant all nonlinear equations except for equation (6.8) which is not invariant for general \( \eta \). However, equation (6.8) is invariant under \( P_+ \) and \( P_- \), in the cases of \( A \) and \( B \) models, respectively. This allows us to truncate nonlinear \( A \) and \( B \) models in AdS4 by the conditions

\[ P_\pm W = W, \quad P_\pm S = S, \quad P_\pm B = B. \] (10.13)

Associated boundary theories are \( A \) and \( B \) models with, respectively, \( A \) and \( B \) boundary conditions at the linearized level. Since in these cases boundary currents decouple from 3d conformal HS gauge fields at the linearized level and since conditions (10.13) are consistent in all orders, the corresponding truncations of the bulk HS theory should correspond to the free-boundary theories in all orders.

Other models and/or boundary conditions do not correspond to any consistent truncation of the full bulk theory. Even if the boundary conditions were imposed in such a way that the rhs of equation (8.30) be zero in the lowest order, it will acquire non-zero higher order corrections from the full nonlinear system. From the perspective of [58], these cases correspond to broken HS symmetry because the current-conservation equations are deformed by nonlinear corrections, i.e. the currents are not conserved in the conventional sense. From the bulk HS theory perspective, this is the effect of a nonlinear deformation of the HS gauge transformation law rather than breaking of HS symmetry.

To summarize, except for the particular cases of \( A \) boundary condition in the \( A \)-model and \( B \) boundary condition in the \( B \)-model, all other possibilities correspond to nonlinear boundary conformal HS theories where boundary conformal HS gauge fields are sourced by boundary currents. This leads to fully nonlinear boundary theories where currents interact via the Chern–Simons-type boundary conformal HS gauge fields. In particular, this happens for all HS theories with \( \eta^2 \neq \pm 1 \).
The holographic duality described in this paper works for any coupling constant in the HS theory, hence not referring to the $N \to \infty$ limit. In this respect, it extends the Klebanov–Polyakov–Sezgin–Sundell conjecture on the critical $O(N)$ and Gross–Neveu models to finite $N$. Beyond the $N \to \infty$ limit, AdS$_3$ HS theory is shown to be dual to a nonlinear theory that describes HS interactions of boundary currents via 3d conformal HS gauge fields. It remains to be seen what is the relation of this boundary HS theory to critical $O(N)$ and Gross–Neveu models as well as to the models with arbitrary $\eta$ discussed in [59, 60].

11. AdS$_3$/CFT$_2$

11.1. AdS$_3$ description

The AdS$_3$/CFT$_2$ correspondence in HS theories has been extensively studied in [61–73]. It is interesting to reconsider this problem along the lines of the AdS$_4$/CFT$_3$ analysis of previous sections. For details of the nonlinear AdS$_3$ HS theory, we refer the reader to [74] (see also [3]). Below we only need the linearized construction.

The AdS$_3$ algebra is semisimple: $\alpha(2, 2) \sim sp(2; R) \oplus sp(2; R)$ with the diagonal subalgebra $sp(2; R) \sim \alpha(2, 1)$ as Lorentz algebra. A particularly useful realization of AdS$_3$ generators is

$$L_{\alpha\beta} = \frac{1}{4i} [\hat{y}_\alpha, \hat{y}_\beta], \quad P_{\alpha\beta} = \frac{1}{4i} [\hat{\psi}_\alpha, \hat{\psi}_\beta] \hat{\psi},$$

(11.1)

with the generating elements $\hat{y}_\alpha$ and $\hat{\psi}$ obeying the relations $[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta\gamma} \hat{y}_\gamma$, $\epsilon_{12} = 1$ and

$$\hat{\psi}^2 = 1, \quad [\hat{\psi}, \hat{y}_\alpha] = 0.$$  (11.2)

$\Pi_{\pm} = \frac{1}{2}(1 \pm \hat{\psi})$ are the projectors to the simple components of $\alpha(2, 2)$.

In [75–79], it was shown that there exists a one-parametric class of infinite-dimensional algebras $hs(2; v)$ ($v$ is an arbitrary real parameter), all containing $sp(2)$ as a subalgebra. This allows one to define a class of HS algebras $g = hs(2; v) \oplus hs(2; v)$ which admit a useful realization in terms of deformed oscillators.

Consider associative algebra $Aq(2; v)$ [78] of elements of the form

$$f(\hat{y}, k) = \sum_{n=0}^{\infty} \sum_{A=0,1} \frac{1}{n!} f^{A_{a_1} \cdots a_n}(k) \hat{y}_{a_1} \cdots \hat{y}_{a_n}$$

(11.3)

under the condition that the coefficients $f^{A_{a_1} \cdots a_n}$ are symmetric with respect to the indices $a_j$ and that the generating elements $\hat{y}_a$ obey

$$[\hat{y}_a, \hat{y}_b] = 2i\epsilon_{a_\beta\gamma} (1 + vk), \quad k\hat{y}_a = -\hat{y}_a k, \quad k^2 = 1,$$

(11.4)

where $v$ is an arbitrary constant (central element). In other words, $Aq(2; v)$ is the enveloping algebra of relations (11.4) often called deformed oscillator algebra.

Its important property is that, for all $v$,

$$T_{\alpha\beta} = \frac{1}{4i} [\hat{y}_\alpha, \hat{y}_\beta]$$

(11.5)

obey $sp(2)$ commutation relations, rotating $\hat{y}_a$ as an $sp(2)$ vector

$$[T_{\alpha\beta}, T_{\gamma\eta}] = \epsilon_{\alpha\gamma} T_{\beta\eta} + \epsilon_{\beta\gamma} T_{\alpha\eta} + \epsilon_{\eta\gamma} T_{\alpha\beta} + \epsilon_{\alpha\eta} T_{\beta\gamma} + \epsilon_{\beta\eta} T_{\alpha\gamma},$$

(11.6)

$$[T_{\alpha\beta}, \hat{\psi}] = \epsilon_{\alpha\gamma} \hat{\psi} + \epsilon_{\beta\gamma} \hat{\psi}. \quad (11.7)$$

Deformed oscillators were originally discovered by Wigner [80] who addressed the question whether it is possible to modify the commutation relations for usual oscillators $a^\pm$ in such a
way that the basic commutation relations \([H, a^\pm] = \pm a^\pm, H = \frac{1}{2}(a^+, a^-)\) remain intact. By analyzing this problem in the Fock-type space, Wigner found a one-parametric deformation of the standard commutation relations which gives a particular representation of the commutation relations (11.4) with the identification \(a^+ = \tilde{y}_1, a^- = \frac{1}{2}\tilde{y}_2, H = T_{12}\) and \(k = (-1)^N\), where \(N\) is the particle number operator. These commutation relations were discussed later on by many authors (see, e.g., [81–83]).

According to (11.5) and (11.7), the \(sp(2)\) generated by \(T_{\alpha\beta}\) extends to \(osp(1, 2)\) via identification of supergenerators with \(\tilde{y}_\alpha\). The quadratic Casimir operator of \(osp(1, 2)\)

\[
C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{4}\tilde{y}_\alpha\tilde{y}^\alpha
\]

is

\[
C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{4}(1 - v^2).
\] (11.8)

Thus, \(Aq(2, v)\) is isomorphic to \(U(osp(1, 2))/I_{C_2+\frac{1}{2}(1-v^2)}\), where the ideal \(I_{C_2+\mu}\) consists of elements proportional to \(C_2 + \mu, \mu \in \mathbb{C}\). This has a number of important consequences. For example, any module of \(osp(1, 2)\) with \(C_2 = -\frac{1}{2}(1 - v^2)\) forms a module of \(Aq(2, v)\) \((v \neq 0)\) and vice versa. In particular, this is the case for finite-dimensional modules corresponding to \(v = 2l + 1, l \in \mathbb{Z}\), with \(C_2 = l(l + 1)\).

The even subalgebra of \(Aq(2, v)\) spanned by \(f(\tilde{y}, k)\) (11.3) obeying \(f(\tilde{y}, k) = f(-\tilde{y}, k)\) decomposes into a direct sum of two subalgebras \(Aq^{\pm}(2, v)\) spanned by the elements \(\Pi_{\pm} f(\tilde{y}, k)\) with \(f(-\tilde{y}, k) = f(\tilde{y}, k)\), \(\Pi_{\pm} = \frac{1}{2}(1 \pm k)\). These algebras are isomorphic to \(U(sp(2))/I_{C_2+\frac{1}{2}l(l+1)}\), where \(C_2 = -\frac{1}{4}T_{\alpha\beta}T^{\alpha\beta}\) is the quadratic Casimir operator of \(sp(2)\), and can be interpreted as (infinite-dimensional) algebras interpolating between ordinary finite-dimensional matrix algebras as discussed in [75, 79].

Algebra \(o(2, 2) \sim sp(2) \oplus sp(2)\) can be spanned by \(\psi\)-dependent bilinears of the oscillators \(\tilde{y}\). Its HS extension results from allowing all powers of \(\tilde{y}\). HS gauge fields are

\[
w(\tilde{y}, \psi, k|x) = \sum_{A, B = 0, 1; n = 0}^{\infty} \frac{1}{n!} w^{AB_{n_1 \ldots n_k}}(x) k^A \psi^{B} \tilde{y}_{n_1} \cdots \tilde{y}_{n_k}.
\] (11.10)

Here \(w^{AB_{n_1 \ldots n_k}}(x)\) describe 3d HS gauge fields of spin \(\frac{1}{2}n\). HS curvatures have the standard form

\[
R(\tilde{y}, \psi, k|x) = dw(\tilde{y}, \psi, k|x) + w(\tilde{y}, \psi, k|x)w(\tilde{y}, \psi, k|x).
\] (11.11)

(This construction for ordinary (i.e. \(v = 0\)) oscillators was suggested in [84].) The labels \(A = 0, 1\) and \(B = 0, 1\) play different roles. \(A\) describes the doubling of all fields as a consequence of \(N = 2\) supersymmetry in the theory. This doubling can be avoided in an appropriately truncated theory [74]. \(B\) distinguishes between the Lorentz-like \((B = 0)\) and frame-like \((B = 1)\) fields.

The 3d linearized system is simpler than the 4d one because, analogously to 3d gravity [86, 87], 3d HS fields do not propagate, being of Chern–Simons type. An equivalent statement is that 3d HS fields admit no HS Weyl tensors. Consequently, the 3d central on-mass-shell theorem has the form

\[
R_1(\tilde{y}, \psi, k|x) := dw(\tilde{y}, \psi, k|x) + w_0(\tilde{y}, \psi, k|x)w(\tilde{y}, \psi, k|x)
+ w(\tilde{y}, \psi, k|x)w_0(\tilde{y}, \psi, k|x) = 0,
\] (11.12)

\[
D_0 C(\tilde{y}, \psi, k|x) := dC(\tilde{y}, \psi, k|x) + w_0(\tilde{y}, \psi, k|x)C(\tilde{y}, \psi, k|x)
- C(\tilde{y}, \psi, k|x)w_0(\tilde{y}, -\psi, k|x) = 0,
\] (11.13)

where \(w_0(\tilde{y}, \psi, k|x)\) is some AdS3 flat connection.
As shown in [85], in the sector of 0-forms, (11.13) describes four massive scalars, $C(\hat{\gamma}, \psi, k|x) = C(-\hat{\gamma}, \psi, k|x)$, and four massive spinors, $C(\hat{\gamma}, \psi, k|x) = -C(-\hat{\gamma}, \psi, k|x)$, arranged into $N = 2$ 3d hypermultiplets. Masses $M$ of matter fields are expressed in terms of $\lambda$ and $\nu$ as follows [85]:

$$M^2_\pm = \lambda^2 \frac{\nu (\nu \mp 2)}{2}$$

for bosons, and

$$M^2_\pm = \lambda^2 \frac{\nu^2}{2}$$

for fermions. Here, $-\lambda^2$ is the cosmological constant of AdS$_3$. The signs ‘$\pm$’ refer to the projections $C^\pm = \Pi_+ C$, $\Pi_\pm = \frac{1}{2}(1 \pm \lambda^2)$. Doubling of fields of the same mass is due to $\psi$ ($\phi^2 = 1$) while that with mass splitting in the bosonic sector is due to $k$. The component form of the covariant constancy conditions (11.13) was originally found in [85] (see also [3]).

11.2. CFT$_2$ description

The analysis of conformal version of AdS$_3$ HS theory is to some extent parallel to the AdS$_4$ case. The radial coordinate $z$ is identified with $z = x^{12}$ while the boundary coordinates are $x = x^{11}$ and $\tilde{x} = x^{22}$:

$$x^{a\beta} = (\tilde{x}^{a\beta}, \sigma_{1a\beta} z), \quad \sigma_{1a\beta} x^{a\beta} = 0, \quad \sigma_1^{12} = \sigma_1^{21} = 1.$$  \hspace{1cm} (11.16)

The 3d conformal algebra $o(2, 2)$ as well as its HS extension decomposes into a direct sum of two subalgebras $o(2, 2) = sp(2) \oplus \tilde{sp}(2)$:

$$sp(2) : T_{a\beta} = \frac{1}{8i} (1 + \psi)(\hat{\gamma}_a \hat{\gamma}_\beta + \hat{\gamma}_\beta \hat{\gamma}_a), \quad \tilde{sp}(2) : \tilde{T}_{a\beta} = \frac{1}{8i} (1 - \bar{\psi})(\bar{\gamma}_a \bar{\gamma}_\beta + \bar{\gamma}_\beta \bar{\gamma}_a).$$  \hspace{1cm} (11.17)

Lorentz and dilatation generators are defined by the relations

$$L - D = \frac{1}{2} \sigma_1^{a\beta} T_{a\beta}, \quad L + D = \frac{1}{2} \sigma_1^{a\beta} \tilde{T}_{a\beta}.$$ \hspace{1cm} (11.18)

From here, it follows that

$$[D, T_{22}] = - T_{22}, \quad [D, T_{11}] = T_{11}, \quad [D, T_{12}] = 0,$$

$$[D, \tilde{T}_{22}] = \tilde{T}_{22}, \quad [D, \tilde{T}_{11}] = - \tilde{T}_{11}, \quad [D, \tilde{T}_{12}] = 0.$$  \hspace{1cm} (11.19)

In accordance with (8.4), we set

$$P = T_{22}, \quad \tilde{P} = \tilde{T}_{11}.$$  \hspace{1cm} (11.20)

Poincaré foliated flat connection (8.6) is

$$W_0 = z^{-1} (d x^P + d \tilde{x}^\tilde{P} - dz D).$$  \hspace{1cm} (11.21)

Manifest conformal invariance is achieved via a transition from the Weyl star product in AdS$_3$ setup to the Fock bimodule realization with the Fock vacua $\mathcal{F}_\pm$ that satisfy

$$y_1 \circ \mathcal{F}_\pm = \mathcal{F}_\pm \circ y_2 = 0, \quad k \mathcal{F}_\pm = \mathcal{F}_\pm k = \pm \mathcal{F}_\pm.$$  \hspace{1cm} (11.22)

An element $F_\pm(y)$ of the Fock bimodule results from the vacuum $\mathcal{F}_\pm$ via action of functions of $y_2$ from the left and functions of $y_1$ from the right. This gives

$$y_1 \circ F_\pm(y) = 2i D_1 F_\pm(y), \quad y_2 \circ F_\pm(y) = y_2 F_\pm(y),$$

$$F_\pm(y) \circ y_1 = F_\pm(y) y_1, \quad F_\pm(y) \circ y_2 = 2i F_\pm(y) \tilde{F}_\pm,$$  \hspace{1cm} (11.23)

$$y_1 \circ F_\pm(y) = 2i D_1 F_\pm(y), \quad y_2 \circ F_\pm(y) = y_2 F_\pm(y),$$

$$F_\pm(y) \circ y_1 = F_\pm(y) y_1, \quad F_\pm(y) \circ y_2 = 2i F_\pm(y) \tilde{F}_\pm,$$  \hspace{1cm} (11.24)

$$F_\pm(y) \circ y_1 = F_\pm(y) y_1, \quad F_\pm(y) \circ y_2 = 2i F_\pm(y) \tilde{F}_\pm,$$  \hspace{1cm} (11.25)
where
\[ D_1 F_\pm(y_1, y_2) = \frac{\partial}{\partial y^2} F_\pm(y_1, y_2) \pm \frac{\nu}{2y_2} (F_\pm(y_1, y_2) - F_\pm(y_1, -y_2)), \]
\[ F_\pm(y_1, y_2) D_2 = \frac{\partial}{\partial y^2} F_\pm(y_1, y_2) \pm \frac{\nu}{2y_1} (F_\pm(y_1, y_2) - F_\pm(-y_1, y_2)). \]

Note that \( D_1 \) and \( D_2 \) are the so-called Dunkl derivatives [88] of the two-body Calogero model.

In this setup, the system becomes manifestly conformal with homogeneous polynomials of \( y_\alpha \) carrying definite conformal dimensions in the adjoint
\[ [D, A_\pm(y)] = \frac{1}{2} \left( y^2 \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^1} \right) A_\pm(y), \]
\[ [D, \tilde{A}_\pm(y)] = \frac{1}{2} \left( y^1 \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2} \right) \tilde{A}_\pm(y) \]

and twisted adjoint representation
\[ D(F_\pm(y)) = \frac{1}{2} \left( \left( y^\alpha \frac{\partial}{\partial y^\alpha} + 2(1 + \nu) \right) F_\pm(y) \mp \nu(F_\pm(-y_1, y_2) + F_\pm(y_1, -y_2)) \right). \]
\[ D(\tilde{F}_\pm(y)) = -\frac{1}{2} \left( \left( y^\alpha \frac{\partial}{\partial y^\alpha} + 2(1 + \nu) \right) \tilde{F}_\pm(y) \mp \nu(\tilde{F}_\pm(-y_1, y_2) + \tilde{F}_\pm(y_1, -y_2)) \right). \]
The Lorentz transformation has the universal form in all cases
\[ LA(y) = \frac{1}{2} \left( y^1 \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2} \right) A(y). \]

The essential difference between AdS_{3/2}/CFT_2 and AdS_{4/3}/CFT_3 dualities is that in the latter case 0-forms \( C \) are glued to the HS curvatures at the linearized level by equation (4.18), which leads to the nontrivial gluing of 3d conformal HS currents to 3d conformal HS gauge fields via (2.27). In the AdS_{3/2} HS gauge theory, no gluing between HS gauge fields and 0-forms \( C \) occurs at the linear level. Moreover, no nontrivial gluing of this type is even possible in AdS_{3/2} because 3d Chern–Simons HS gauge theory admits no Weyl tensor and its HS generalizations. This has a consequence that 2d conformal fields \( J \) associated with \( C \) do not source 2d conformal HS gauge fields at the linearized level which, in fact, allows conformal fields associated with \( C \) to have a continuous conformal dimension parametrized by \( \nu \).

This does not however imply that the 2d conformal fields \( J \) and 2d conformal HS gauge fields are completely independent. AdS_{3/2} HS gauge fields are sourced by the HS currents built from bilinears of the AdS_{3/2} matter fields \( C \) which represent the 3d stress tensor and its HS generalizations. From the CFT_2 dual viewpoint, this means that 2d conformal HS curvatures will receive sources \( T \sim JJ \) starting from the second order in \( J \), where \( T \) is an HS generalization of the stress tensor built from the currents \( J \).

To obtain z-independent 2d equations from the AdS_{3/2} HS theory, one should again properly rescale the oscillators similarly to (8.17) and (8.23). To simplify formulae, we abuse notation denoting the rescaled variables by \( y \). Then, 2d HS field equations have the following structure in the lowest order in which 0-forms contribute to the rhs of the equations for HS gauge fields:
\[ R(y, k, \psi|x, \tilde{x}) = d\mathbf{x} d\tilde{\mathbf{x}} \frac{\partial^2}{\partial y^1 \partial y^2} (T(y_1, y_2, k, \psi|x) + \tilde{T}(y_1, y_2, k, \psi|x)) + \cdots, \]
where the HS curvature is
\[ R(y, k, \psi|x, \tilde{x}) = d\Omega(y, k, \psi|x, \tilde{x}) + \Omega(y, k, \psi|x, \tilde{x}) \ast \Omega(y, k, \psi|x, \tilde{x}) \]
with \(*\) denoting the non-commutative product of deformed oscillators. Equation (11.32) is to some extent analogous to equation (8.33) if \( T \) were treated as an independent field. More
precisely, it is analogous to the 4d equations found in [89] where current interactions of 4d massless fields of all spins were constructed.

It remains to see to what extent the scheme sketched above reproduces the AdS3/CFT2 HS duality conjectures of [63, 67]. The construction of conformal currents \( T \) in terms of bilinears of the 2d fields \( J \) resulting from the 3d fields \( C \) is analogous to Sugawara construction considered in [67]. Correspondingly, the operator product of conformal conserved currents \( T \) is expected to reproduce the \( W_\lambda \) algebra [90, 64] with \( \lambda = \frac{1+\nu}{2} \). Note that at the classical level the construction of the nonlinear \( \nu \)-dependent \( W \) algebra in terms of HS algebras, which is anticipated to be equivalent to the \( W_\lambda \) algebras of [90, 64], was found in [91]. In any case, a conformal theory dual to the AdS3 HS theory should be nonlinear.

12. Higher spin theory and quantum mechanics

Unfolded dynamics provides a powerful direct tool elucidating the duality between theories in various dimensions, sometimes going beyond the conventional framework of AdS\(_{d+1}\)/CFT\(_d\) duality [4–6]. For instance, one can consider a chain of AdS\(_n+1\)/AdS\(_n\) dualities as conjectured in [9] (see also interesting recent work [92]), using the chain of Poincaré foliations (8.7), or, alternatively, by going directly from a higher dimension to the lower one. An intriguing example of the latter option considered in this section is provided by the duality between HS theories in the matrix space \( \mathcal{M}_M \), formulated originally in the unfolded form in [9], and non-relativistic quantum mechanics. This consideration is closely related to the recent analysis of symmetries of quantum mechanical models in [93–95].

Via appropriate rescaling and complexification of variables, the rank-1 equation (5.1) in \( \mathcal{M}_M \) can be rewritten in the form

\[
(i\hbar \frac{\partial}{\partial X^{AB}} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial Y^A \partial Y^B}) \Psi(X) = 0, \quad A, B = 1, \ldots, M. \tag{12.1}
\]

(Note that the factor of \( i \) in this equation naturally appears in the analysis of HS equations in Siegel space [25].) As discussed in section 2.3, maximal symmetries of the free unfolded equations coincide with algebra \( l^{\text{max}}(V) \) (of commutators) of endomorphisms of the space \( V \) where the 0-forms \( \Psi(Y|X) \) at any \( X = X_0 \) are valued. Hence, symmetries of equations (12.1) are generated by various operators in the space \( F \) of functions of \( Y^A \).

Generally, to specify a space of operators in the functional space, one has to specify their properties in some more detail. To respect relativistic symmetries, we should consider the space of differential operators with polynomial coefficients in \( Y^A \) which is equivalent to the Fock space realization with the ‘oscillators’ \( P_A \) and \( Y^B \) that satisfy

\[
[P_A, Y^B] = \delta^B_A, \quad [P_A, P_B] = 0, \quad [Y^A, Y^B] = 0. \tag{12.2}
\]

In these terms, \( F \) is the space of vectors \( f(Y)|0\rangle \) induced from the vacuum \( |0\rangle \) satisfying

\[
P_A|0\rangle = 0. \tag{12.3}
\]

Hence, the symmetry algebra of equation (12.1) is generated by various polynomials of \( P_A \) and \( Y^B \). This is the generalized conformal HS algebra considered in [9]. It contains \( sp(2M) \) generated by

\[
K^{AB} = Y^AY^B, \quad L^A_B = \{ Y^A, P_B \}, \quad P_{AB} = P_AP_B. \tag{12.4}
\]

The \( X \)-dependence of global HS transformations determined by equation (2.21) was found in [9].

In [42], it was shown that time-like directions in \( \mathcal{M}_M \) are associated with positive-definite \( X^{AB} \). In particular, one can set

\[
X^{AB} = t M^{AB}, \tag{12.5}
\]
where $t$ is the time evolution parameter. Restriction of equation (12.1) to $t$ gives the usual $M$-dimensional Schrödinger equation

\[
\left(\frac{i\hbar}{\partial t} + \frac{\hbar^2}{2m} \delta^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B}\right) \Psi(Y|t) = 0,
\]

(12.6)

where $Y^A$ are now interpreted as the coordinates of the Galilean space.

From general properties of unfolded formulation discussed in section 2, it follows that relativistic rank-1 equations in $\mathcal{M}_M$ are equivalent to the nonrelativistic Schrödinger equation in $M$ dimension. The cases of $M = 2$ and $M = 4$ are particularly interesting from the relativistic field theory perspective. Equation (5.1) with $M = 2$ describes the massless scalar ($\Psi(Y|X) = \Psi(-Y|X)$) and spinor ($\Psi(Y|X) = -\Psi(-Y|X)$) in 2+1 dimension. Equation (5.1) with $M = 4$ describes massless particles of all integer ($\Psi(Y|X) = \Psi(-Y|X)$) and half-integer ($\Psi(Y|X) = -\Psi(-Y|X)$) spins in 3+1 dimension [9].

It should be noted that relativistic systems in $\mathcal{M}_M$ are conformal [42, 96]. In particular, $sp(4)$ is just the 3d conformal algebra while $sp(8)$ contains the 4d conformal algebra $su(2, 2)$ as a subalgebra. This immediately implies that these algebras do act on the solutions of the respective non-relativistic field equations as well as the full Weyl algebra of operators built from $P_i$ and $Y^A$. However, this action does not look geometric in terms of twistor variables $Y^A$ interpreted as the space coordinates of nonrelativistic quantum mechanics. (More precisely, beyond the free-field level, these are coordinates $\mu^A$ introduced in section 6.2.) The other way around, nonrelativistic symmetries, which act geometrically in terms of nonrelativistic coordinates $Y^A$, look nongeometric in terms of relativistic coordinates.

This is a manifestation of a very general situation. In the unfolded dynamics approach, it is easy to introduce coordinates in which any symmetry $h$ of a given system acts geometrically by introducing an appropriate non-zero flat connection of $h$. However, different symmetries require different coordinates (spaces) and connections. The description of the same system in different space-times gives holographically dual theories. This is obvious in the unfolded dynamics approach, where it refers to the same twistor space (which is the space of $Y^A$ in the quantum-mechanical model of interest), while in other approaches, the holographic duality may look obscure.

Equation (12.6) is the Schrödinger equation for a free nonrelativistic particle. One may wonder what if the system is deformed by a potential? In the framework of unfolded dynamics, this does not affect the consideration much, at least formally. Indeed, in the presence of potential $U(Y)$, the equation

\[
\left(\frac{i\hbar}{\partial t} + \frac{\hbar^2}{2m} \delta^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B} - U(Y)\right) \Psi(Y|t) = 0
\]

remains linear, hence exhibiting infinite symmetries. In the spirit of unfolded dynamics, it can be interpreted as a flatness condition

\[
D\Psi(Y|t) = 0, \quad D = \frac{\partial}{\partial t} + \Omega,
\]

\[
\Omega = i\hbar^{-1} \frac{\partial}{\partial t} H, \quad H = -\frac{\hbar^2}{2m} \delta^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B} + U(Y). \tag{12.8}
\]

In the one-dimensional case with the single coordinate $t$, any connection is flat, i.e. the compatibility conditions for equation (12.8) are trivially satisfied. Hence, it can be represented in the pure gauge form which is simply

\[
\Omega = \exp(-i\hbar^{-1} H) d \exp(i\hbar^{-1} H). \tag{12.9}
\]

The same similarity transform relates symmetries of the $H = 0$ system to those of $H \neq 0$. 

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The other way around, any HS connection $\Omega(Y|X)$ in $\mathcal{M}_M$ (not necessarily flat) generates a flat connection $\Omega_t$ as its pullback to the time arrow. Hence, any HS geometry is holographically dual to some quantum mechanics. For example, from equation (4.15), we observe that an appropriate $\lambda$-dependent rescaling maps AdS geometry to the harmonic potential

$$U(Y) = \frac{1}{2}m\omega^2 Y^A Y^B \delta_{AB},$$

where the coupling constant is proportional to the cosmological constant $-\Lambda \sim \lambda^2$:

$$\frac{1}{2}m\omega^2 = \lambda^2.$$ (12.10)

On the other hand, dS geometry is holographically dual to the inverted harmonic potential with negative $\omega^2$, i.e. of course not too surprising in the context of inflation.

The correspondence between relativistic systems in higher dimensions and quantum mechanics is not just formal. In particular, these holographically dual systems have the same spectra. Namely, by virtue of unfolded equations, the spectrum of states of free relativistic massless particles of all spins in 3+1 dimensions is identical to that of four quantum harmonic oscillators, while the spectrum of massless particles in 2+1 dimensions is the same as of two harmonic oscillators. The finite-dimensional Schrödinger algebra of nonrelativistic symmetries of the Schrödinger equation (see, e.g., [95] and references therein) forms a subalgebra of the algebra $sph(4|\mathbb{R})$ of section 6.2. In particular, the so-called mass operator $\hat{M}$ is represented by the central element of the Heisenberg subalgebra of $sph(4|\mathbb{R})$.

Let us note that the duality of relativistic and nonrelativistic equations allows a natural interpretation for such a standard tool for the study of relativistic equations as oscillator realization of relativistic symmetries extensively used for the group-theoretic analysis of relativistic theories [97, 98]. From the holographic point of view pursued in this paper, it results from the dual realization of the relativistic system in terms of its nonrelativistic cousin. Moreover, not only are symmetries of holographically dual relativistic and non-relativistic systems the same, but their conserved currents also coincide. This is a simple consequence of the analysis of [25] summarized in section 5. The key fact is that a differential 2$\mathcal{M}$-form $\Omega$ (5.16) is closed in the correspondence space unifying (relativistic) space-time $\mathcal{M}_M$ with coordinates $X^{AB}$ with the twistor space $\mathcal{T}$ (non-relativistic space-time) with coordinates $Y^A$. As a result, a conserved charge can be evaluated both in $\mathcal{M}_M$ and in $\mathcal{T}$. In the first case, it gives a relativistic conserved charge in $\mathcal{M}_M$ while in the second case it appears as a nonrelativistic conserved charge in $\mathcal{T}$. In fact, higher conserved charges of nonrelativistic quantum mechanics constructed recently in [95] just coincide with those resulting from the pullback of $\Omega$ to $\mathcal{T}$.

Surprisingly enough, the equivalence of relativistic and non-relativistic systems described above acquires the interpretation of a Penrose transform induced by the unfolded equations. This should have much in common with the interpretation of non-relativistic physics as the relativistic one in the light-cone higher dimensional system (see [95] and references therein).

13. Toward off-shell formulation

The property underlying holographic duality is that dynamics of universal unfolded systems is characterized entirely by the differential $Q$ (2.7) defined on the ‘target space’ of dynamical variables independently of the original space-time. In particular, invariants like actions and conserved charges are characterized by $Q$ cohomology [26].

First suppose that the system (2.2) is off-shell. As shown in [26], a gauge-invariant action is an integral over a $d$-cycle $M^d$

$$S = \int_{M^d} \mathcal{L}(W)$$ (13.1)
of some $Q$-closed $d$-form Lagrangian function $\mathcal{L}(W)$

$$Q\mathcal{L} = 0 : \quad G^\alpha(W) \frac{\partial}{\partial W^\alpha} \mathcal{L}(W) = 0. \quad (13.2)$$

It is easy to see that, being $Q$-closed, $S$ is invariant under the gauge transformations (2.9). If $\mathcal{L}$ is $Q$-exact, by virtue of (2.8), it is $d$-exact, i.e. nontrivial invariant actions represent the $Q$ cohomology of the system in question.

If the system is on-shell and $\mathcal{L}$ represents $H^0(Q)$, the same formula describes a conserved charge as an integral over a $p$-cycle $\Sigma$

$$q = \int_\Sigma \mathcal{L}(W). \quad (13.3)$$

Examples of application of this construction were given in [26]. Let us stress that the analysis in terms of $Q$ cohomology applies to both linear and nonlinear unfolded systems.

In unfolded dynamics, Noether current interactions are directly related to conserved currents. In the case of interest they result from the expression for the conserved charge (5.19), (5.19). For example, in the case of $M = 2$, since the 4-form $\Omega$ (5.17) is closed for any $T_\eta$ (5.19) with $\eta(W, Y|x)$ that satisfies equation (5.15), the 5-form

$$L = \Omega(W, Y|X) dW_\beta dW^\beta (iW_\beta dX^\alpha - dY^\alpha) (iW^\gamma dX^\gamma - dY_\alpha)$$

$$\times \int dU_\nu dU^\nu \exp -iW_\nu U^\nu J(U, Y|X) \quad (13.4)$$

is closed up to $J^2$ terms by virtue of the unfolded equations (8.30).

Let us look more closely at the relation between on-shell and off-shell systems. Let $W^{\Omega}_\text{on}$ be a set of forms of some on-shell system. Its off-shell extension should contain additional fields $E^\alpha$ that appear on the rhs of the field equations to replace the differential equations by constraints expressing new fields via the lhs of the field equations. Abusing notation we can write

$$L^i(W(x)) = E^i(x), \quad (13.5)$$

where $L^i(W(x))$ describes the lhs of the dynamical equations on $W^{\Omega}_\text{on}$. Note that $E^i$ is a part of the full set of $E^\alpha$ since, in unfolded dynamics, $E^\alpha$ contains $E^i$ along with all their derivatives. We will call $E^i$ primary off-shell fields, saying that they glue the field equations of the on-shell system in question. This is equivalent to the statement that primary off-shell fields match the on-shell $\sigma$-cohomology associated with the lhs of the field equations to enforce the corresponding cohomology of the off-shell system be zero.

The off-shell system with $W^{\Omega}_\text{off} = (W^{\Omega}_\text{on}, E^\alpha)$ is such that $E^\alpha = 0$ puts the off-shell system on shell. This means that the off-shell system is described by such $G^\alpha(W^{\Omega}_\text{off})$ that both

$$Q^{\text{off}} = G^\alpha(W^{\Omega}_\text{off}) \frac{\partial}{\partial W^{\alpha}_{\text{off}}} = G^\alpha(W^{\Omega}_\text{on}, E) \frac{\partial}{\partial W^{\alpha}_{\text{on}}} + G^\alpha(W^{\Omega}_\text{on}, E) \frac{\partial}{\partial E^\alpha} \quad (13.6)$$

and

$$Q^{\text{on}} = G^\alpha(W^{\Omega}_\text{on}, 0) \frac{\partial}{\partial W^{\alpha}_{\text{on}}} \quad (13.7)$$

are nilpotent

$$Q^{\text{off}} Q^{\text{off}} = Q^{\text{on}} Q^{\text{on}} = 0. \quad (13.8)$$

This is a consequence of the property

$$G^\alpha(W^{\Omega}_\text{on}, 0) = 0, \quad (13.9)$$
which should be true for any off-shell extension. Indeed, otherwise, the on-shell fields \( W^\alpha \) would source the fields \( E^\alpha \) not allowing to put the system on shell. Field equations imply \( E^i = 0 \) and hence

\[
E^\alpha = 0. \tag{13.10}
\]

To extend the on-shell analysis of this review to the full quantum level, an off-shell extension of the system has to be considered. This problem has not yet been solved in a fully satisfactory way. An interesting hint from the analysis of [89] is that the system, which describes current interactions of 4d massless fields, can be viewed as the 4d off-shell system with the current fields \( J \) interpreted as off-shell fields \( E^i(x) \). On the other hand, the same fields can be interpreted either as describing two-particle states in the system or as free 6d fields. This suggests the idea that proper account of off-shell quantum effects in terms of unfolded dynamics may result from consideration of the theory in higher and higher dimensions, allowing us to interpret quantum-mechanical effects as classical dynamics in an infinite-dimensional space that has enough room to describe all multiparticle states of the system.

In a more traditional fashion, if an off-shell action is available in the unfolded formulation, it can be used to produce generating functionals in the standard path integral approach. Again, the idea is that using the fact that an action functional is closed in an appropriate correspondence space which extends space-time with some twistor coordinates, the integration can be performed in the twistor space for all holographically dual theories. In that case, various holographic interpretations of the same generating functional will be fully equivalent.

Actions of the form (13.1) are also appropriate for the analysis of anomalies in the formulation in the doubled bulk space suggested in section 10. Although the naïve interpretation of the action (13.1) may be ill-defined because of divergences at \( z = 0 \), it can be regularized via deformation of the integration contour to the complex plane in \( z \), say, via the substitution \( z \to z + i\epsilon \). Anomalous terms will be associated with singularities in \( \epsilon \). In fact, complexification of matrix coordinates \( X^{AB} \) in HS theories has been used in [25] to regularize integrals for HS-conserved charges analogous to the action integral (13.1), where \( X^{AB} \) were complexified to \( Z^{AB} \) from the upper Siegel half-space. In the example of [25], it was shown that the charge integrals are independent of variations of a complex integration contour away from singularity. If the same happens in HS theories, this would imply that the regularized action is independent of \( \epsilon \) hence being anomaly free (though it may be dependent on the contour homotopy class). The same time, the independence of local contour variation implies \( Q \)-closure of the action (13.1) which, in turn, implies its gauge invariance [26]. It would be interesting to see how this scheme works in off-shell HS theories.

Note that for HS theories formulated in matrix space \( M_M \), regularization via deformation to Siegel space has deep meaning in various respects [25]. In particular, solutions analytic in upper and lower Siegel spaces correspond, respectively, to particles and antiparticles. Also solutions of HS equations from the upper Siegel half-space, which are periodic in the \( Y^A \), are closely related to Riemann theta-functions where complexified coordinates \( Z^{AB} \) acquire the meaning of a period matrix.

14. Conclusion

In this review, it is demonstrated how holographic duality results from different interpretations of one and the same theory. This phenomenon is very general and applies to any theory (not necessarily conformal). To establish the holographic duality, it is most useful to reformulate a theory in the unfolded form [27] of coordinate-independent first-order equations formulated in terms of exterior differential as space-time derivative and differential forms as field variables.
Once such a formulation is achieved, one can play freely with space-time dimension, adding
or removing coordinates without changing dynamical content of the theory. This provides
a vast variety of different looking models in space-times of different dimensions which however
are by construction locally equivalent. Since unfolding machinery applies to any theory, every
model belongs to a class of holographically equivalent models.

Since HS theories were originally formulated within the unfolded dynamics approach,
they provide a natural arena illustrating this phenomenon. In this paper, we focused on the
$\text{AdS}_4/CFT_3$ and $\text{AdS}_3/CFT_2$ HS dualities. The latter was put forward in [61–63, 67]. The
former was conjectured by Klebanov and Polyakov [13] to relate the simplest $\text{AdS}_4$ HS theory
to $3d\: O(N)$ sigma-model and was partially proved by Giombi and Yin [14, 15] for correlators
involving any three spins $s_1, s_2, s_3$ that do not respect the triangle inequality.

Recently, Maldacena and Zhiboedov conjectured [20] that $\text{AdS}_4$ HS theory is dual to
the $3d$ free model even beyond the large $N$ limit. The arguments of [20] are very general,
generalizing the Coleman–Mandula theorem [99] to conformal theories. Namely, the authors
of [20] have shown that if a unitary local conformal field theory possesses a conserved HS
current, then it must be a theory of currents of free conformal fields. Since the $\text{AdS}_4$ HS theory
possesses HS symmetries, the conclusion of [20] was that its boundary dual is free.

The analysis of this paper shows however that, except for two particular cases, the
boundary theory dual to $\text{AdS}_4$ HS theory turns out to be nonlinear, escaping some of conditions
of the Maldacena–Zhiboedov theorem. Namely, the boundary theory describes interactions
of conformal currents in the framework of $3d$ conformal HS gauge theory which extends $3d$
(Chern–Simons) conformal gravity to higher spins. Being a gauge theory, it is not unitary,
while a particular gauge choice makes it nonlocal and/or not conformal. Another property
of the boundary dual of the $\text{AdS}_4$ HS theory is that boundary conformal currents associated
with massless fields in $\text{AdS}_4$ are not conserved in the usual sense being instead covariantly
conserved with respect to the $3d$ conformal HS algebra. Analogous phenomena are expected
to take place for higher dimensions $d > 4$, relating nonlinear HS theories in any $d$ [100] to
boundary conformal HS theories in $d − 1$. However, this duality is expected to be far more
complicated because of the complicated structure of the corresponding generalized twistor
space.

We have identified two particular truncations of the bulk HS theories which have free
bosonic and fermionic boundary duals in agreement with the conjectures of Klebanov and
Polyakov [13] and Sezgin and Sundell [22]. In these cases, $3d$ conformal HS gauge fields
decouple from the boundary currents and the corresponding boundary theories indeed turn out
to be free in agreement with the Maldacena–Zhiboedov theorem [20]. Truncations to the free-
boundary theories are based on the parity automorphism $P$ of the $\text{AdS}_4$ system that reflects
the Poincaré coordinate $z$. Its application requires the doubling of the Poincaré chart, identifying
the $\text{AdS}_4$ boundary with the stationary surface of $P$.

In the setup with doubled bulk space, it is not necessary to impose definite boundary
conditions at $z = 0$ since it becomes a regular point in terms of appropriately rescaled twistor
variables. Hence, in our approach, the $3d$ dual of $\text{AdS}_4$ HS gauge theory describes a doubled
number of $3d$ currents which in particular contain two scalar currents of different dimensions.
Generally, all these currents interact via $3d$ conformal HS gauge fields. We believe that the
trick with doubled bulk in $\text{AdS}_4/CFT$, which follows naturally from unfolded dynamics, should
also have interesting applications beyond HS theories.

A new phenomenon found in this review is that both holographically dual theories are
theories of (conformal) gravity. This phenomenon seems to be very general and should take
place beyond the $N \to \infty$ limit for most holographic models of bulk gravity as a consequence
of coordinate independence of the unfolded formulation.
In this paper, we did not check explicitly how our prescription reproduces conformal correlators on the conformal side, leaving discussion of this issue to future work [101].

Taking into account that non-Abelian contributions to conformal HS curvatures exist only for spins $s_1, s_2, s_3$ that respect the triangle inequality, it would be interesting to see whether nonlinear corrections of the boundary theory can help to conform the boundary and bulk calculations of [14] for such spins.

The analysis of this paper is mostly on-shell, operating in terms of field equations rather than action since HS actions are not yet known to all orders on both sides of the HS duality. Once they are available, the analysis can be immediately extended to the action level. Hence, most urgent problems for the future include explicit construction of nonlinear 3d HS conformal gravity and action functionals for both AdS$_4$ HS theory and its 3d dual.

A peculiar feature emerging from the analysis of the particular HS model in this paper is that the infinite boundary limit is not a necessary ingredient of the duality which can formally be established on every co-dimension-1 surface $\Sigma$ in the bulk. However, for general $\Sigma$, the relation between fields and sources in the dual theories, which respects conformal symmetry, is nonlocal while in the infinite boundary limit $z \to 0$ the relation turns out to be local in accordance with the standard prescription of [5, 6]. Being complicated in terms of space-time coordinates, the nonlocal holographic duality map between two theories on general $\Sigma$ acquires natural meaning in terms of non-commutative twistor variables, describing the map between Weyl and Wick star products. It should be noted however that the transition from one ordering prescription to another may, in principle, lead to divergences in the star-product formalism in HS theories because the construction involves nonpolynomial elements like Klein operators (6.19). When this happens, a model exhibits conformal anomaly.

A systematic reformulation of unfolded theories in terms of twistor variables greatly simplifies the analysis of holographic duality making it nearly tautological. Seemingly different theories are described by solutions of the same equations in the generalized twistor space or by the same action-like invariants evaluated as integrals over twistor variables. Two holographically dual models result from different space-time extensions of the same twistor model.

In [9], it was conjectured that massless conformal HS theories may form a chain of dualities between models in space-times of different dimensions. If a boundary theory contains conformal gravity, it can be again put in locally AdS$_d$ background, say, by using the foliation prescription of section 8.1. In the end, one stops at some 2d conformal theory, 1d quantum-mechanical theory or even 0d matrix-like theory, which is nothing but the part of the theory reduced solely to the twistor space (e.g., equations (6.7) and (6.8) in the AdS$_4$ HS system).

The duality between HS theories and nonrelativistic quantum mechanics discussed in section 12 provides an exciting example of 1d dual interpretation. A deep relation between HS theories and quantum mechanics makes it difficult to refrain from speculation that the two systems may be literally equivalent while their different interpretations depend on particular details of physical observation in question. In other words, it is tempting to raise a risky question whether HS theories can tell us what quantum mechanics is. This issue has too many aspects to be discussed in detail in this paper. However, one immediate consequence is that, if true, nonlinear HS theories should imply that the Schrödinger equation has to receive nonlinear corrections of the form prescribed by HS theory. Since the coupling constant inherited from HS theories should be related to the gravitational constant, nonlinear corrections to quantum mechanics should be negligible in the non-relativistic regime. Nevertheless, one can speculate that their appearance may shed some light on such conceptual problems of quantum mechanics as, for instance, momentary wave packet reduction.
Tremendous robustness of the quantum gravity problem suggests that its solution may require modification of both ingredients. HS theory may provide a framework for nontrivial merger of gravity with quantum mechanics, affecting the present-day understanding of both. If so, non-relativistic quantum mechanics may one day provide us with an unexpected tool for the study of quantum gravity in laboratory experiments. At any rate, we believe that HS gauge theory has potential to unify gravity and quantum mechanics in a nontrivial and constructive way.

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