ASYMPTOTIC ANALYSIS OF A 2D OVERHEAD CRANE WITH INPUT DELAYS IN THE BOUNDARY CONTROL

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Abstract. The paper investigates the asymptotic behavior of a 2D overhead crane with input delays in the boundary control. A linear boundary control is proposed. The main feature of such a control lies in the facts that it solely depends on the velocity but under the presence of time-delays. We end-up with a closed-loop system where no displacement term is involved. It is shown that the problem is well-posed in the sense of semigroups theory. LaSalle’s invariance principle is invoked in order to establish the asymptotic convergence for the solutions of the system to a stationary position which depends on the initial data. Using a resolvent method it is proved that the convergence is indeed polynomial.

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1. Introduction

Overhead cranes are extensively utilized in a variety of industrial and construction sites. Usually, it consists of a hoisting mechanism such as a hoisting cable and a hook and a support mechanism like a girder (trolley) [2]. The aim of using such cranes is to horizontally transport point-to-point a suspended mass/load. It is well-known that cables possess the inherent flexibility characteristics and can only develop tension [2]. Such natural features inevitably cause deflection in transversal direction of the cable. Furthermore, the suspended load is always subject to swings due to several reasons. Thereby, the behavior of the overhead crane system with flexible cable can generate complex system dynamics (see [2] for more details).

We shall consider in the present work an overhead crane system which consists of a motorized platform of mass $m$ moving along an horizontal rail. A flexible cable of length $\ell$, holding a load mass $M$, is attached to the platform (see Fig. 1). Furthermore, it is assumed that:

(i) The cable is completely flexible and non-stretching.
(ii) The length of the cable is constant.
(iii) Transversal and angular displacements are small.
(iv) Friction is neglected.

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(v) The masses $m$ and $M$ are point masses.

(vi) The angle of the cable with respect to the vertical $x$-axis is small everywhere.

Under the above assumptions, the overhead crane is modeled by a hybrid PDE-ODE system (see [7] and [27]). For sake of completeness, we shall provide some details about the derivation of such a model (the reader is referred to [7] and [27] for more details).

Let $T$ be the tension of the cable, $\theta(x,t)$ be the angle between $T$ and the $x$-axis, and consider a portion of the cable of length $\Delta x$. Newton’s law leads to

$$\Delta x y_{tt}(x,t) = T(x + \Delta x) \theta(x + \Delta x, t) - T(x) \theta(x,t).$$

We can write $\theta(x,t) \simeq y_x(x,t)$ due to the assumption of smallness of transversal and angular displacements. On the other hand, since the tension of the cable is essentially due to the action on its lower part, we have $|T(x)| = (M + \ell - x)g$, which is the modulus of tension of the cable and will be denoted by $a(x)$. This, together with the above equation imply that

$$y_{tt}(x,t) - (ay_x)_x(x,t) = 0, \quad 0 < x < \ell, \quad t > 0. \tag{1.1}$$

We turn now to the equation of the platform part of the system (see Fig. 2). Taking into account the external controlling force $F(t)$, we have

$$my_{tt}(0,t) = |T(0)|\theta(0,t) + F(t),$$

which can be rewritten

$$my_{tt}(0,t) = a(0)y_x(0,t) + F(t), \quad t > 0, \tag{1.2}$$

as $|T(x)| = a(x)$ and $\theta(0,t) \simeq y_x(0,t)$.

Using similar arguments for the the load mass (see Fig. 3), we have

$$My_{tt}(\ell,t) = -a(\ell)y_x(\ell,t), \quad t > 0. \tag{1.3}$$

Combining (1.1)-(1.3), we have the system

$$\begin{cases}
y_{tt}(x,t) - (ay_x)_x(x,t) = 0, & 0 < x < \ell, \quad t > 0, \\
ym_{tt}(0,t) - (ay_x)(0,t) = F(t), & t > 0, \\
M y_{tt}(\ell,t) + (ay_x)(\ell,t) = 0, & t > 0,
\end{cases} \tag{1.4}$$

where $a(x)$ is supposed to satisfy the following conditions
Figure 2. The platform

Figure 3. The load mass

(1.5) \[
\begin{align*}
\{ & a \in H^1(0, \ell); \\
& \text{there exists a positive constant } a_0 \text{ such that } a(x) \geq a_0 > 0 \text{ for all } x \in [0, \ell].
\end{align*}
\]

For simplicity and without loss of generality, we shall set the length \( \ell = 1 \).

As mentioned above, the objective is to seek a delayed control \( F(t) \) depending solely on the velocity so that the solutions of the closed-loop system asymptotically converge to an equilibrium point in a suitable functional space.

The boundary stabilization of the system (1.4) has been the object of a considerable mathematical research. There are two categories of research articles: in the first category, at least one of the dynamical terms in the boundary conditions is neglected. In other words, either \( my_{tt}(0, t) \) or \( My_{tt}(1, t) \) does not appear in the system or even both terms are not present. For instance, it has been shown in [27] that the feedback law

\[
F(t) = -cy(0, t) - F(y_t(0, t)), \quad c > 0,
\]

exponentially stabilizes the system (1.4) with \( my_{tt}(0, t) = 0 \) under appropriate assumptions on the function \( F \). Another stabilization result for the system (1.4) with \( my_{tt}(0, t) = My_{tt}(1, t) = 0 \) has also been established in [15] via the action of the following feedback:

\[
\begin{align*}
F(t) &= -\alpha y(0, t) - F(y_t(0, t)), \\
U(t) &= -\alpha y(1, t) - F(y_t(1, t)), \quad \alpha > 0,
\end{align*}
\]

where \( U \) is an additional control to be applied on the load mass. In [7], the asymptotic stabilization has been proved as long as a dynamical control is acting on the boundary \( y(1, t) \). We also mention that a stabilization result has been obtained in [12] by proposing the feedback law

\[
F(t) = k_p y(0, t) + k_v y_t(0, t) + \int_0^1 G(x) y(x, t) dx + \frac{k_v}{k_p} \int_0^1 G(x) y_t(x, t) dx,
\]
with $k_p$, $k_v > 0$ and $G$ is a function in $H^1(0, 1)$. Of course, such a result has been established under some conditions on the feedback gains $k_p$, $k_v$ as well as the function $G$. Similar findings have been obtained in \[8\] for other types of controls containing a displacement term. We conclude this discussion about the first category of articles available in the literature by pointing out that it has been noticed in \[11\] that in all references cited above, either the boundary conditions in (1.4) or the stabilizing feedback law $F(t)$ involves the displacement term $y$. This is mainly due to the fact that most of the authors defined the energy-norm of the system by

$$E_0(t) = \frac{1}{2} \int_0^1 (y_x^2 + y_t^2) \, dx.$$  

This observation has motivated the authors in \[11\] to consider a displacement term in the equation and propose a general class of feedback law containing only the velocity. In fact, the closed-loop system in \[11\] has the following form

\[
\begin{align*}
    y_{tt}(x, t) - (ay_x)_x(x, t) + \alpha y_t(x, t) + \beta y(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
    (ay_x)(0, t) &= \epsilon_1 f(y(0, t)), \\
    (ay_x)(1, t) &= \epsilon_2 g(y(1, t)),
\end{align*}
\]

(1.6) in which $f$ and $g$ are two nonlinear functions. The multiplier method has been successfully used in \[11\] to get precise decay rate (polynomial or exponential) estimates of the energy of the system (1.6) according to the type of assumptions on the functions $f$ and $g$. Recently, the back-stepping approach has been successfully applied to a variant of the system (1.4) leading to an exponentially stabilizing boundary feedback controller \[8\]. In the same spirit, the following feedback law

\[
\begin{align*}
    F(t) &= -\alpha_1 y(0, t) - \beta_1 y_t(0, t), \\
    U(t) &= -\alpha_2 y(1, t) - \beta_2 y_t(1, t),
\end{align*}
\]

has been suggested in \[29\] in the case where $\alpha_1 + \alpha_2 \neq 0$, $\beta_1 + \beta_2 \neq 0$ and $my_{tt}(0, t) = My_{tt}(1, t) = 0$ and the Riesz basis property has been shown.

The second category of research papers takes into consideration the dynamics of both the load mass and platform mass. Within this context, it has been proved in \[13\] that the system (1.4) can be strongly (but non-uniformly) stabilized by means of the control

\[
F(t) = -\alpha y(0, t) - f(y(0, t)), \quad \alpha > 0,
\]

where $f$ is a suitable function. This motivated several authors to propose controls of higher orders to reach the uniform exponential stability. Indeed, the uniform stabilization holds if

\[
F(t) = -\alpha y(0, t) - \alpha \beta y_t(0, t) + \beta y_{tt}(0, t), \quad \alpha > 0, \quad \beta > 0.
\]

It turned out that the same result result can be achieved by the control

\[
F(t) = -\alpha y(0, t) - (\beta + \alpha c)y_t(0, t) + cy_{tt}(0, t),
\]

where $\alpha$, $\beta$ and $c$ are positive constants satisfying $\beta c < m$. Motivated by the work of \[11\], a feedback control depending only on the velocity has been proposed in \[14\] for the system (1.4) and an asymptotic convergence result has been established (see also \[11\]).

All the papers mentioned above do not take into consideration time-delay. In turn, it is well-known that delays are inevitable in practice as they naturally arises in most systems due to the time factor needed for the communication among the controllers, the sensors and the actuators of systems or in some cases due to the dependence of the state variables on past states. Furthermore, it has been noticed that the presence of a delay in a system could be a source of poor performance and instability \[17\]-\[19\] (see also \[28\], \[4\], \[5\] and \[6\]).
The present work places primary emphasis on the analysis of the system (1.4) under the action of the following input delay

\begin{equation}
F(t) = -\beta y_t(0, t) + \alpha y_t(0, t - \tau),
\end{equation}

where $\beta > 0$, $\alpha \in \mathbb{R}$ and $\tau > 0$ is the time-delay.

It is worth mentioning that the absence of the displacement term in the closed-loop system prevents the applicability of classical Poincaré inequalities. To overcome this difficulty, an appropriate energy-norm is suggested.

The main contribution of the present work is threefold:

(a) Extend the mathematical findings on the overhead crane available in literature (specially those of [24, 13, 11, 14]), where no delay has been taken into account in the feedback laws.

(b) Show that despite the presence of the delay term in the proposed feedback control law, the closed-loop system possesses the asymptotic convergence property of its solutions to an equilibrium state which depends on the initial conditions.

(c) Provide the rate of convergence of solutions of the closed-loop system to the equilibrium state, in contrast to the work [14] where such a result has not been achieved.

The paper is organized as follows. The next section is devoted to the proof of existence and uniqueness of the solutions to the closed-loop system. Section 3 deals with the asymptotic behavior of solutions via the use of LaSalle’s principle. Section 4 is devoted to the polynomial convergence of solutions. Finally, the paper closes with conclusions and discussions.

2. Well-posedness of the system

With the feedback law in (1.7), we obtain the closed-loop system

\begin{equation}
\begin{aligned}
&y_{tt}(x, t) - (ay_x)_x(x, t) = 0, &0 < x < 1, \ t > 0, \\
&my_{tt}(0, t) - (ay_x)(0, t) = -\beta y_t(0, t) + \alpha y_t(0, t - \tau), &t > 0, \\
&M y_{tt}(1, t) + (ay_x)(1, t) = 0, &t > 0,
\end{aligned}
\end{equation}

where $a$ obeys the condition (1.5), $\alpha \in \mathbb{R}$ and $\beta > 0$.

Our immediate task is to seek an appropriate energy associated to (2.1). To proceed, let

\begin{equation}
E_0(t) = \frac{1}{2} \left\{ \int_0^1 \left( y_{tt}^2(x, t) + a(x)y_x^2(x, t) \right) \, dx + my_{tt}^2(0, t) + My_{tt}^2(1, t) + K\tau \int_0^1 y_t^2(0, t - x\tau) \, dx \right\},
\end{equation}

where $K$ is a positive constant. Using (2.1) and integrating by parts, a formal computation yields

\begin{equation}
E'_0(t) = -\beta y_t^2(0, t) + \alpha y_t(0, t)y_t(0, t - \tau) - \frac{K}{2} \left( y_{tt}^2(0, t - \tau) - y_t^2(0, t) \right).
\end{equation}

Applying Young’s inequality, the latter becomes

\begin{equation}
E'_0(t) \leq \left( \frac{K}{2} + \frac{|\alpha|}{2c} - \beta \right) y_t^2(0, t) + \frac{1}{2} (|\alpha|c - K) y_t^2(0, t - \tau),
\end{equation}

for any positive constant $c$. Subsequently, we introduce the following additional energy functional

\begin{equation}
E_1(t) = \frac{1}{2} \rho^2(t),
\end{equation}

where

\begin{equation}
\rho(t) = \int_0^1 y_t(x, t) \, dx + c_1 y_t(0, t) + c_2 y_t(1, t) + c_3 \int_0^1 y_t(0, t - x\tau) \, dx + c_4 y(0, t),
\end{equation}

with $c_1, c_2, c_3, c_4$ positive constants.
and $c_1, c_2, c_3,$ and $c_4$ are constants to be determined. Following the same arguments as for $E_0(t)$, we get

$$E'_1(t) = \rho(t) \left( (ay_x)(1, t) \left[ 1 - \frac{c_2}{M} \right] + (ay_x)(0, t) \left[ \frac{c_1}{m} - 1 \right] + \beta y_t(0, t) \left[ \frac{c_3}{\tau \beta} + \frac{c_4}{\beta} - \frac{c_1}{m} \right] + \alpha y_t(0, t - \tau) \left[ \frac{c_1}{m} - \frac{c_3}{\tau \alpha} \right] \right).$$

(2.7)

Thereafter, we define the total energy of the system (2.1) as follows

$$E(t) = E_0(t) + E_1(t).$$

This, together with (2.4) and (2.7), imply that

$$E'(t) \leq \left( \frac{K}{2} + \frac{|\alpha|}{2c} - \beta \right) y_t^2(0, t) + \frac{1}{2} (|\alpha| c - K) y_t^2(0, t - \tau) + \rho(t) \left( (ay_x)(1, t) \left[ 1 - \frac{c_2}{M} \right] + (ay_x)(0, t) \left[ \frac{c_1}{m} - 1 \right] + \beta y_t(0, t) \left[ \frac{c_3}{\tau \beta} + \frac{c_4}{\beta} - \frac{c_1}{m} \right] + \alpha y_t(0, t - \tau) \left[ \frac{c_1}{m} - \frac{c_3}{\tau \alpha} \right] \right).$$

(2.9)

In order to make the energy $E(t)$ decreasing, we shall assume that

$$|\alpha| < \beta,$$

and then choose $K$ such that

$$|\alpha| \leq K \leq 2\beta - |\alpha|,$$

whereas the other constants are

$$c = 1, \quad c_1 = m, \quad c_2 = M, \quad c_3 = \tau \alpha, \quad c_4 = \beta - \alpha.$$

In light of (2.9) and (2.10)-(2.12), we deduce that

$$E'(t) \leq \frac{1}{2} \left( (-2\beta + |\alpha| + K) y_t^2(0, t) + (|\alpha| - K) y_t^2(0, t - \tau) \right) \leq 0,$$

and hence the energy $E(t)$ is decreasing.

**Remark 1.** It is clear from the above choices in (2.12), that the additional energy $E_1(t)$ defined by (2.5)-(2.6) is in fact constant.

Here and elsewhere throughout the paper, we shall use the following definitions and notations for the Hilbert space $L^2(0, 1)$ and the Sobolev space $H^m(0, 1)$, more precisely

$$L^2(0, 1) = \left\{ v : (0, 1) \to \mathbb{R} \text{ is measurable and } \int_0^1 |v(x)|^2 \, dx < \infty \right\}$$

equipped with its usual norm

$$\|\varphi\|_{L^2(0, 1)} = \left( \int_0^1 |v(x)|^2 \, dx \right)^{1/2},$$

and

$$H^m(0, 1) = \left\{ g : (0, 1) \to \mathbb{R}; \; g^{(m)} \in L^2(0, 1), \text{ for } m \in \mathbb{N} \right\}.$$
endowed with the standard norm
\[ \|g\|_{H^m(0,1)} = \sum_{i=0}^{\infty} \|g^{(i)}\|_{L^2(0,1)}. \]

Let us return now to our closed-loop system (2.1). Using the well-known change of variables
\[ (\text{2.14}) \quad u(x,t) = y(t_0, t - x\tau), \]
the system (2.1) becomes
\[ (\text{2.15}) \quad \begin{cases} y_t(x,t) - (ay_x)_x(x,t) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \tau u_t(x,t) + u_x(x,t) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ my_t(0,t) - (ay_x)(0,t) = \alpha u(1,t) - \beta u(0,t), & t > 0, \\ M y_t(1,t) + (ay_x)(1,t) = 0, & t > 0, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in (0,1), \\ u(x,0) = y_t(0, -x\tau) = f(-x\tau), & x \in (0,1). \end{cases} \]

Let \( z(\cdot,t) = y_t(\cdot,t), \xi = y_t(0,t), \eta = y_t(1,t) \) and consider the state variable \( \Phi = (y, z, u, \xi, \eta) \). Then, our state space \( \mathcal{X} \) is defined by
\[ \mathcal{X} = H^1(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathbb{R}^2, \]
equipped with the following real inner product (the complex case is similar)
\[ (\text{2.16}) \quad \langle (y,z,u,\xi,\eta),(\tilde{y},\tilde{z},\tilde{u},\tilde{\xi},\tilde{\eta}) \rangle_X = \int_0^1 (ay_x\tilde{y}_x + z\tilde{\eta}) \, dx + K\tau \int_0^1 u\tilde{u} \, dx + m\xi\tilde{\xi} + M\eta\tilde{\eta} + \varpi \left( \int_0^1 zd\xi + m\xi + M\eta + \mu y(0) + \tau\alpha \int_0^1 u \, dx \right) \left( \int_0^1 \tilde{z}d\tilde{\xi} + m\tilde{\xi} + M\tilde{\eta} + \mu\tilde{y}(0) + \tau\alpha \int_0^1 \tilde{u} \, dx \right) \]
in which \( K > 0 \) satisfies the condition (2.11), while \( \mu = \beta - \alpha \) and \( \varpi \) is a positive constant to be determined. Note that \( \mu = \beta - \alpha \) is positive due to (2.10).

The first result is stated below.

**Proposition 1.** Assume that (1.3), (2.10) and (2.11) hold. Then, the state space \( \mathcal{X} \) endowed with the inner product (2.16) is a Hilbert space provided that \( \varpi \) is small enough.

**Proof.** It suffices to show the existence of two positive constants \( A_1 \) and \( A_2 \) such that
\[ (\text{2.17}) \quad A_1 \| (y,z,u,\xi,\eta) \| \leq \| (y,z,u,\xi,\eta) \|_X \leq A_2 \| (y,z,u,\xi,\eta) \|, \]
where \( \| (y,z,u,\xi,\eta) \| \) denotes the usual norm of \( H^1(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathbb{R}^2 \), that is,
\[ \| (y,z,u,\xi,\eta) \|^2 = \int_0^1 \left( y^2 + y_x^2 + z^2 + u^2 \right) \, dx + \xi^2 + \eta^2. \]
The right-hand inequality \( \| (y,z,u,\xi,\eta) \|_X \leq A_2 \| (y,z,u,\xi,\eta) \| \) is straightforward. Indeed, Young’s and Hölder’s inequalities yield
\[ \| (y,z,u,\xi,\eta) \|^2 \leq \int_0^1 (ay_x^2 + z^2) \, dx + K\tau \int_0^1 u^2 \, dx + m\xi^2 + M\eta^2 + 5\varpi \left( \int_0^1 z^2 \, dx + m\xi + M\eta + \mu y(0) + \tau\alpha \int_0^1 u \, dx \right) \left( \int_0^1 \tilde{z}d\tilde{\xi} + m\tilde{\xi} + M\tilde{\eta} + \mu\tilde{y}(0) + \tau\alpha \int_0^1 \tilde{u} \, dx \right). \]
Moreover, by virtue of (1.5) and the well-known trace continuity Theorem [3]

\[ y^2(0) \leq 2 \int_0^1 (y^2 + y_x^2) \, dx, \]

the above inequality leads to the desired result with \( A_2 \) depending on \( m, M, |\alpha|, \tau, \beta \) and \( ||a||_\infty \).

With regard to the other inequality of (2.17), we proceed as follows:

\[
\| (y, z, u, \xi, \eta) \|_X^2 = \int_0^1 (ay_x^2 + z^2) \, dx + K\tau \int_0^1 u^2 \, dx + m\xi^2 + M\eta^2 \\
+ \varpi \left( \int_0^1 z \, dx + \tau \alpha \int_0^1 u \, dx + m\xi + M\eta \right)^2 + \varpi \mu y^2(0) \\
+ 2\varpi \mu y(0) \left[ \int_0^1 z \, dx + \tau \alpha \int_0^1 u \, dx + m\xi + M\eta \right].
\]

(2.18)

It follows from Young’s inequality that for any \( \kappa > 0 \),

\[
2y(0) \left[ \int_0^1 z \, dx + \tau \alpha \int_0^1 u \, dx + m\xi + M\eta \right] \geq \\
- \frac{4}{\kappa} \left( \int_0^1 z^2 \, dx + \tau^2 \alpha^2 \left[ \int_0^1 u^2 \, dx \right]^2 + m^2 \xi^2 + M^2 \eta^2 \right) - \kappa y^2(0).
\]

(2.19)

Combining (2.18) and (2.19), and choosing \( \kappa < \mu = \beta - \alpha \), we obtain

\[
\| (y, z, u, \xi, \eta) \|_X^2 \geq \\
\int_0^1 ay_x^2 \, dx + \left[ 1 + 4\varpi \left( 1 - \frac{\mu}{\kappa} \right) \right] \int_0^1 z^2 \, dx + \tau \left[ K + 4\varpi \tau \alpha^2 \left( 1 - \frac{\mu}{\kappa} \right) \right] \int_0^1 u^2 \, dx \\
+ m \left[ 1 + 4m\varpi \left( 1 - \frac{\mu}{\kappa} \right) \right] \xi^2 + M \left[ 1 + 4M\varpi \left( 1 - \frac{\mu}{\kappa} \right) \right] \eta^2 + \varpi \mu (\mu - \kappa) y^2(0).
\]

(2.20)

A direct computation gives

\[
\int_0^1 y^2 \, dx = y^2(0) + 2 \int_0^1 \int_0^x y_y \, ds \, dx \\
\leq y^2(0) + \varepsilon \int_0^1 y^2 \, dx + \frac{1}{\varepsilon} \int_0^1 y_x^2 \, dx,
\]

(2.21)

for any \( \varepsilon > 0 \). Inserting (2.21) into (2.20) and using (1.5) yields

\[
\| (y, z, u, \xi, \eta) \|_X^2 \geq \left[ a_0 - \varepsilon^{-1} \varpi \mu (\mu - \kappa) \right] \int_0^1 y_x^2 \, dx + \varpi \mu (\mu - \kappa)(1 - \varepsilon) \int_0^1 y^2 \, dx \\
+ \left[ 1 + 4\varpi \left( 1 - \frac{\mu}{\kappa} \right) \right] \int_0^1 z^2 \, dx + \tau \left[ K + 4\varpi \tau \alpha^2 \left( 1 - \frac{\mu}{\kappa} \right) \right] \int_0^1 u^2 \, dx \\
+ m \left[ 1 + 4m\varpi \left( 1 - \frac{\mu}{\kappa} \right) \right] \xi^2 + M \left[ 1 + 4M\varpi \left( 1 - \frac{\mu}{\kappa} \right) \right] \eta^2 + \varpi \mu (\mu - \kappa) y^2(0),
\]

(2.22)

for any \( 0 < \kappa < \mu = \beta - \alpha \) and \( 0 < \varepsilon < 1 \). Finally, we choose \( \varpi \) such that

\[
0 < \varpi < \min \left\{ \frac{\varepsilon a_0}{\mu (\mu - \kappa)}, \delta, \frac{\delta}{m}, \frac{\delta}{M}, \frac{K\delta}{\tau \alpha^2} \right\},
\]

where \( \delta = \frac{\kappa}{4(\mu - \kappa)} > 0 \). Thus, (2.17) holds and the proof of Proposition 1 is achieved. \( \square \)
We are now in a position to set our problem in the state space \( X \). Define a linear operator \( A \) by
\[
\mathcal{D}(A) = \{(y, z, u, \xi, \eta) \in X; y \in H^2(0, 1), \ z, u \in H^1(0, 1), \ \xi = u(0) = z(0), \ \eta = z(1)\},
\]
\[
A(y, z, u, \xi, \eta) = \left(z, (ay_x)_x, -\frac{u_x}{\tau}, \frac{1}{m}[\xi(ay_x)(0) - \beta \xi + \alpha u(0)], -\frac{(ay_x)(1)}{M}\right),
\]
\forall(y, z, u, \xi, \eta) \in \mathcal{D}(A).

The closed-loop system (2.1) can now be formulated in terms of the operator \( A \) by the evolution equation over \( X \)
\[
(2.24) \quad \begin{cases}
\dot{\Phi}(t) = A\Phi(t), \\
\Phi(0) = \Phi_0,
\end{cases}
\]
in which \( \Phi = (y, z, u, \xi, \eta) \) and \( \Phi_0 = (y_0, y_1, f(-\tau \cdot), \xi_0, \eta_0) \).

The well-posedness result is stated below.

**Theorem 1.** Suppose that (1.5), (2.10) and (2.11) are satisfied. Then, we have:
(i) The operator \( A \) defined by (2.23) is densely defined in \( X \) and generates on \( X \) a \( C_0 \)-semigroup of contractions \( e^{tA} \). Moreover, \( \sigma(A) \), the spectrum of \( A \), consists of isolated eigenvalues of finite algebraic multiplicity only.
(ii) For any initial condition \( \Phi_0 \in X \), the system (2.24) has a unique mild solution \( \Phi \in C([0, \infty); X) \). In turn, if \( \Phi_0 \in \mathcal{D}(A) \), then necessarily the solution \( \Phi \) is strong and belongs to \( C([0, \infty); \mathcal{D}(A) \cap C^1([0, \infty); X)) \).

**Proof.** Let \( \Phi = (y, z, u, \xi, \eta) \in \mathcal{D}(A) \). Then, in light of (2.16) and (2.23), a simple integration by parts gives
\[
\langle A\Phi, \Phi \rangle_X = (ay_x)(1)z(1) - (ay_x)(0)z(0) - \frac{K}{2} (u^2(1) - u^2(0)) + \xi(ay_x)(0) - \beta \xi^2 + \alpha \xi u(1)
\]
\[
-\eta(ay_x)(1) + \varpi \left( \int_0^1 z \, dx + \tau \alpha \int_0^1 u \, dx + m \xi + M \eta \right) (\alpha u(0) + \beta \xi + (\beta - \alpha)z(0))
\]
\[
= \alpha \xi u(1) - \frac{K}{2} u^2(1) + \frac{K}{2} u^2(0) - \beta \xi^2
\]
\[
\leq \left( -\beta + \frac{K + |\alpha|}{2} \right) \xi^2 + \frac{|\alpha| - K}{2} u^2(1)
\]
and so the operator \( A \) is dissipative due to the assumption (2.11).

Next, we claim that the operator \( \lambda I - A \) is onto \( X \) for \( \lambda > 0 \) sufficiently large. To ascertain the correctness of this claim, one has to show that given \( (f, g, v, p, q) \in X \), there exists \( (y, z, u, \xi, \eta) \in \mathcal{D}(A) \) for which \( (\lambda I - A)(y, z, u, \xi, \eta) = (f, g, v, p, q) \). Although this can be considered as a classical problem, one can easily verify that the latter is equivalent to solve the following system
\[
(2.26) \quad \begin{cases}
\lambda^2 y - (ay_x)_x = \lambda f + g, \\
u_x + \lambda \tau u = \tau v, \\
\lambda (m\lambda + \beta) y(0) - (ay_x)(0) - \alpha u(0) = mp + (m\lambda + \beta) f(0), \\
\lambda^2 M y(1) + (ay_x)(1) = M q + \lambda M f(1), \\
z = \lambda y - f, \\
\xi = u(0) = z(0) = \xi y(0) - f(0), \\
\eta = z(1) = \lambda y(1) - f(1).
\end{cases}
\]
Solving the equation of \( u \) in the above system, we obtain
\[
(2.27) \quad u(x) = e^{-\tau x}(\lambda y(0) - f(0)) + \tau \int_0^x e^{-\tau(x-s)}v(s) \, ds,
\]
and hence
\[
(2.28) \quad u(1) = e^{-\tau}(\lambda y(0) - f(0)) + \tau \int_0^1 e^{-\tau(1-s)}v(s) \, ds.
\]
This, together with (2.26) and (2.27), imply that one has only to seek \( y \in H^2(0,1) \) satisfying
\[
(2.29) \quad \begin{cases}
\lambda^2 y - (ay_x)_x = \lambda f + g, \\
\lambda \left[ (m\lambda + \beta) - \alpha e^{-\tau \lambda} \right] y(0) - (ay_x)(0) = mp + (m\lambda + \beta - \alpha e^{-\tau \lambda})f(0) \\
+ \tau \alpha \int_0^1 e^{-\tau \lambda(1-s)}v(s) \, ds,
\end{cases}
\]
Multiplying the first equation in (2.29) by \( u \) and hence
\[
(2.27) \quad u(1) = e^{-\tau}(\lambda y(0) - f(0)) + \tau \int_0^1 e^{-\tau(1-s)}v(s) \, ds.
\]
Applying Lax-Milgram Theorem \[10\], one can deduce the existence of a unique solution \( \lambda > 0 \) exists and maps \( X \) into \( \mathbb{X} \) for \( \lambda > 0 \). Thus, according to semigroup theory \[25\], the operator \( A \) is densely defined in \( \mathbb{X} \) and generates on \( \mathbb{X} \) a \( C_0 \)-semigroup of contractions denoted by \( e^{tA} \). As a direct consequence of the fact that, for \( \lambda > 0 \), the range of \( \lambda I - A \) is \( \mathbb{X} \), it follow that \( (\lambda I - A)^{-1} \) exists and maps \( \mathbb{X} \) into \( D(A) \). Finally, using Sobolev embedding \[3\], it follows that \( (\lambda I - A)^{-1} \) is compact and hence the spectrum of \( A \), consists of isolated eigenvalues of finite algebraic multiplicity only \[23\]. This completes the proof of the first assertion (i) in Theorem \[1\].

Concerning the proof of the second assertion, it suffices to use (i) and invoke semigroups theory \[25\].
3. Asymptotic behavior.

We begin this section by recalling the following result.

**Theorem 2.** [22] Let $P$ be the infinitesimal generator of a $C_0$-semigroup $S(t)$ in a Hilbert space $H$ such that $P$ has compact resolvent. Then, $S(t)$ is strongly stable if and only if it is uniformly bounded and Re $\lambda < 0$, for any $\lambda$ in the spectrum of $P$.

It is clear from (2.23) that $\lambda = 0$ is an eigenvalue of $A$ whose eigenfunction is $(c, 0, 0, 0, 0)$, where $c \in \mathbb{R} \setminus \{0\}$. Thus, Theorem 2 implies that the semigroup $e^{tA}$ generated by $A$ is not stable. However, we are able to prove the main result of the paper which is stated next.

**Theorem 3.** Assume that (1.3), (2.10) holds and $K$ satisfies $|\alpha| < K < 2\beta - |\alpha|$. Then, for any initial data $\Phi_0 = (y_0, y_1, f, \zeta_0, \eta_0) \in \mathcal{X}$, the solution $\Phi(t) = \left( y, y_t, y_{tt}, y_{ttt}, y_{ttt} \right)$ of the closed-loop system (2.1) (or equivalently (2.24)) tends in $\mathcal{X}$ to $(\Omega, 0, 0, 0, 0)$ as $t \to +\infty$, where

$$\Omega = \frac{1}{\beta - \alpha} \left[ \int_0^1 y_1 dx + \alpha \int_0^1 f(-\tau x)dx + (\beta - \alpha) y_0(0) + m \zeta_0 + M \eta_0 \right].$$

**Proof.** The proof depends on an essential way on the application of LaSalle’s invariance principle [22]. Using a standard argument of density of $\mathcal{D}(A)$ in $\mathcal{X}$ and the contraction of the semigroup $e^{tA}$, it suffices to prove Theorem 3 for smooth initial data $\Phi_0 = (y_0, y_1, f, \zeta_0, \eta_0) \in \mathcal{X}$. Let $\Phi(t) = (y, y_t, y_{tt}, y_{ttt}, y_{ttt}) = e^{tA}\Phi_0$ be the solution of (2.1). It follows from Theorem 1 that the trajectories set of solutions $\{\Phi(t)\}_{t \geq 0}$ is a bounded for the graph norm and thus precompact by virtue of the compactness of the operator $(I - A)^{-1}$. Invoking LaSalle’s principle, we deduce that $\omega(\Phi_0)$ is non empty, compact, invariant under the semigroup $e^{tA}$ and in addition $e^{tA}\Phi_0 \to \omega(\Phi_0)$ as $t \to \infty$ [22]. Clearly, in order to prove the convergence result, it suffices to show that $\omega(\Phi_0)$ reduces to $(\Omega, 0, 0, 0, 0)$. To this end, let $\tilde{\Phi}_0 = \left( \tilde{y}_0, \tilde{y}_1, f, \tilde{\xi}, \tilde{\eta} \right) \in \omega(\Phi_0) \subset \mathcal{D}(A)$ and consider $\tilde{\Phi}(t) = \left( \tilde{y}(t), \tilde{y}_t(t), \tilde{u}(t), \tilde{\xi}(t), \tilde{\eta}(t) \right) = e^{tA}\tilde{\Phi}_0 \in \mathcal{D}(A)$ as the unique strong solution of (2.24). It is well-known that $\|\tilde{\Phi}(t)\|_{\mathcal{X}}$ is constant [22] and thus $\frac{d}{dt} (\|\tilde{\Phi}(t)\|^2_{\mathcal{X}}) = 0$. This leads to

$$< A\tilde{\Phi}, \tilde{\Phi} >_{\mathcal{X}} = 0$$

which, together with (2.25), imply that $\tilde{\xi} = \tilde{y}_t(0, t) = 0$ and $\tilde{u}(1) = \tilde{y}_t(0, t - \tau) = 0$. Consequently, $\tilde{y}$ is a solution of the system

$$\begin{cases}
\tilde{y}_{ttt} - (a \tilde{y}_{xx})_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\
M \tilde{y}_{ttt}(1, t) + (a \tilde{y}_{xx})(1, t) = 0, & t > 0, \\
\tilde{y}_0(0, t) = \tilde{y}_x(0, t) = 0, & t > 0, \\
\tilde{y}(0) = \tilde{y}_0; \tilde{y}_t(0) = \tilde{y}_1, & x \in (0, 1) \\
\tilde{y} \in H^2(0, 1).
\end{cases}$$

A straightforward computation shows that $\tilde{z} = \tilde{y}_t$ is a solution of

$$\begin{cases}
\tilde{z}_{ttt} - (a \tilde{z}_{xx})_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\
M \tilde{z}_{ttt}(1, t) + (a \tilde{z}_{xx})(1, t) = 0, & t > 0, \\
\tilde{z}(0, t) = \tilde{z}_x(0, t) = 0, & t > 0, \\
\tilde{z}(0) = \tilde{y}_1; \tilde{z}_t(0) = (a \tilde{y}_{0x})_x, & x \in (0, 1).
\end{cases}$$
Theorem 4. The space $X$ is in fact polynomial. The proof of such a desired result is based on applying the following:

\[ \int_0^1 y(t,x) \, dx + my(0,t) + My(1,t) + \alpha \tau \int_0^1 y_t(0,t-x\tau) \, dx + (\beta - \alpha)y(0,t) = \Upsilon, \quad \forall t \geq 0, \]

in the state space $X$. Furthermore, in view to Remark 1, any solution of the closed-loop system (2.24) stemmed from $\Phi_0 = (y_0, y_1, f, \xi_0, \eta_0)$ verifies

\[ \Phi(t_n) = (y(t_n), y_t(t_n), y(t_n), u(t_n), \xi(t_n), \eta(t_n)) = e^{t_nA} \Phi_0 \rightarrow (\Omega, 0, 0, 0, 0) \]

for any $t_n$. To summarize, we have shown that for any $\tilde{\Phi}_0 = \left( \tilde{y}_0, \tilde{y}_1, \tilde{f}, \tilde{\xi}, \tilde{\eta} \right) \in \omega(\Phi_0) \subset D(A)$, the unique solution $\tilde{\Phi}(t) = \left( \tilde{y}(t), \tilde{y}_t(t), \tilde{u}(t), \tilde{\xi}(t), \tilde{\eta}(t) \right) = e^{tA} \tilde{\Phi}_0 \in D(A)$ is actually $(\Omega, 0, 0, 0, 0)$, for any $t \geq 0$, where $\Omega$ is a constant to be determined. This implies that the initial condition $\tilde{\Phi}_0 = \left( \tilde{y}_0, \tilde{y}_1, \tilde{f}, \tilde{\xi}, \tilde{\eta} \right)$ is also equal to $(\Omega, 0, 0, 0, 0)$. Thereby, the $\omega$-limit set $\omega(\Phi_0)$ only consists of constants $(\Omega, 0, 0, 0, 0)$. It remains to provide an explicit expression of the constant $\Omega$ to complete the proof. To do so, let $(\Omega, 0, 0, 0, 0) \in \omega(\Phi_0)$. This implies that there exists $\{t_n\} \rightarrow \infty$, as $n \rightarrow \infty$ such that

\[ \Phi(t_n) = (y(t_n), y_t(t_n), y(t_n), u(t_n), \xi(t_n), \eta(t_n)) = e^{t_nA} \Phi_0 \rightarrow (\Omega, 0, 0, 0, 0) \]

and

\[ \limsup_{n \rightarrow \infty} \|y(t_n)\|_\mathcal{H} < \infty. \]

Lastly, letting $t = t_n$ in (3.6) with $n \rightarrow \infty$ and using (3.5) yield the desired expression of $\Omega$. This achieves the proof of the theorem. \hfill \Box

4. POLYNOMIAL CONVERGENCE

The objective of this section is to show that the convergence result obtained in the previous section is in fact polynomial. The proof of such a desired result is based on applying the following frequency domain theorem for polynomial stability of a $C_0$ semigroup of contractions on a Hilbert space $\mathcal{H}$.

Theorem 4. A $C_0$ semigroup $e^{tL}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies, for all $t > 0$,

\[ \|e^{tL}\|_{\mathcal{L}(D(A), \mathcal{H})} \leq \frac{C}{t^{1/\delta}} \]

for some constant $C, \delta > 0$ if and only if

\[ \rho(\mathcal{L}) \ni \{ i\gamma \mid \gamma \in \mathbb{R} \} \equiv i\mathbb{R}, \]

and

\[ \limsup_{|\gamma| \rightarrow \infty} \|\gamma|^{\delta} (i\gamma I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \]

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$. 
In order to use the above theorem, let us first consider the space
\[ \dot{\mathcal{X}} = \left\{ (y, z, u, \xi, \eta) \in \mathcal{X} ; \int_0^1 z(x)dx + \alpha \tau \int_0^1 u(x)dx + (\beta - \alpha) y(0) + m\xi + M\eta = 0 \right\}. \]
Then, a new operator is defined below
\[ \dot{\mathcal{A}} : D(\dot{\mathcal{A}}) := D(\mathcal{A}) \cap \dot{\mathcal{X}} \subset \dot{\mathcal{X}} \rightarrow \dot{\mathcal{X}}, \]
(4.3)
\[ \dot{\mathcal{A}}(y, z, u, \xi, \eta) = \mathcal{A}(y, z, u, \xi, \eta), \quad \forall (y, z, u, \xi, \eta) \in D(\dot{\mathcal{A}}). \]
Clearly, the operator \( \dot{\mathcal{A}} \) defined by (4.3) generates on \( \dot{\mathcal{X}} \) a \( C^0 \) -semigroup of contractions \( e^{t\dot{\mathcal{A}}} \) provided that the conditions (2.10) and (2.11) are fulfilled. Moreover, \( \sigma(\dot{\mathcal{A}}) \), the spectrum of \( \dot{\mathcal{A}} \), consists of isolated eigenvalues of finite algebraic multiplicity only. In order to achieve the objective of this section, we shall assume that the coefficient \( a \) satisfies stronger conditions than (1.5), namely,
(4.4)
\[ \left\{ \begin{array}{l}
a \in C^1[0, 1]; \\
\text{there exist positive constants } a_0, a_1 \text{ such that } a(x) \geq a_0, \ a'(x) \geq a_1, \text{ for all } x \in [0, \ell].
\end{array} \right. \]
Now, we are ready to state our result which translates the fact that the semigroup operator \( e^{t\dot{\mathcal{A}}} \) is polynomially stable in \( \dot{\mathcal{X}} \).

**Theorem 5.** Assume that (2.10) and (4.4) hold and \( K \) satisfies \( |\alpha| < K < 2\beta - |\alpha| \). Then, there exists \( C > 0 \) such that for all \( t > 0 \) we have
\[ \left\| e^{t\dot{\mathcal{A}}} \right\|_{L(D(\dot{\mathcal{A}}), \dot{\mathcal{X}})} \leq \frac{C}{\sqrt{t}}. \]

**Proof of Theorem 5.** The proof of Theorem 5 is based on the following lemmas.

We first look at the point spectrum.

**Lemma 1.** If \( \gamma \) is a real number, then \( i\gamma \) is not an eigenvalue of \( \dot{\mathcal{A}} \).

**Proof.** We will show that the equation
(4.5) \[ \dot{\mathcal{A}}Z = i\gamma Z \]
with \( Z = (y, z, u, \xi, \eta) \in D(\dot{\mathcal{A}}) \) and \( \gamma \in \mathbb{R} \) has only the trivial solution. Clearly, the system (4.5) writes
(4.6) \[ z = i\gamma y \]
(4.7) \[ (ay_x)_x = i\gamma z \]
(4.8) \[ -\frac{u_x}{\tau} = i\gamma u \]
(4.9) \[ \frac{1}{m} [(ay_x)(0) - \beta \xi + \alpha u(0)] = i\gamma \xi. \]
(4.10) \[ \frac{-(ay_x)(1)}{M} = i\gamma \eta. \]
Let us firstly treat the case where \( \gamma = 0 \). It’s clear that the only solution of (4.5) is the trivial one.
Suppose now that $\gamma \neq 0$. By taking the inner product of (4.5) with $Z$, using the inequality (2.25) we get:

$$\text{Re} \left( < \dot{A}Z, Z >_{X} \right) \leq \frac{1}{2} \left( (-2\beta + |\alpha| + K)|z(0)|^2 + (|\alpha| - K)|u(1)|^2 \right) \leq 0.$$  
(4.11)

Thenceforth, we obtain that $z(0) = 0$ and $u(1) = 0$ and hence $\xi = u(0) = 0$. Lastly, we conclude that the only solution of (4.5) is the trivial one.$\square$

**Lemma 2.** The resolvent operator of $\dot{A}$ obeys the condition (4.2).

**Proof.** Suppose that condition (4.2) is false. By the Banach-Steinhaus Theorem (see [10]), there exist a sequence of real numbers $\gamma_n \to +\infty$ and a sequence of vectors $Z_n = (y_n, z_n, u_n, \xi_n, \eta_n) \in D(\dot{A})$ with $\|Z_n\|_{\dot{X}} = 1$ such that

$$\|\gamma_n^2 (i\gamma_n I - \dot{A}) Z_n\|_{\dot{X}} \to 0 \quad \text{as} \quad n \to \infty,$$
(4.12)

that is, as $n \to \infty$, we have:

$$\gamma_n^2 (i\gamma_n y_n - z_n) \equiv \gamma_n^2 f_n \to 0 \quad \text{in} \quad H^1(0,1),$$
(4.13)

$$\gamma_n^2 (i\gamma_n z_n - (a(y_n)_x)_x) \equiv \gamma_n^2 g_n \to 0 \quad \text{in} \quad L^2(0,1),$$
(4.14)

$$\gamma_n^2 \left( i\gamma_n u_n + \frac{(u_n)_x}{\tau} \right) \equiv \gamma_n^2 v_n \to 0 \quad \text{in} \quad L^2(0,1),$$
(4.15)

$$\gamma_n^2 \left( i\gamma_n \xi_n - \frac{1}{m} [(a(y_n)_x)(0) - \beta \xi_n + \alpha u_n(1)] \right) \equiv \gamma_n^2 p_n \to 0,$$
(4.16)

and

$$\gamma_n^2 \left( i\gamma_n \eta_n + \frac{(a(y_n)_x)(1)}{M} \right) \equiv \gamma_n^2 q_n \to 0.$$  
(4.17)

Our goal is to derive from (4.12) that $\|Z_n\|_{\dot{X}}$ converges to zero, thus there is a contradiction. The proof is divided into three steps

**First step.**

We first notice that we have

$$\|\gamma_n^2 (i\gamma_n I - \dot{A}) Z_n\|_{\dot{X}} \geq \text{Re} \left( \langle \gamma_n^2 (i\gamma_n I - \dot{A}) Z_n, Z_n \rangle_{X} \right).$$
(4.18)

Amalgamating (4.18) with (4.11)-(4.13), it follows that

$$\gamma_n z_n(0) \to 0,$$
(4.19)

and

$$\gamma_n u_n(1) \to 0.$$  

Moreover, since $Z_n \in D(\dot{A})$, we deduce that $\xi_n = u_n(0) = z_n(0)$. Thereby

$$\gamma_n \xi_n = \gamma_n u_n(0) \to 0 \quad \text{and} \quad \xi_n \to 0.$$  
(4.20)

Whereupon, (4.16) gives

$$\gamma_n x(0) \to 0.$$  

Solving (4.15), we have the following identity

$$u_n(x) = u_n(0) e^{-i\gamma_n x} + \tau \int_{0}^{x} e^{-i\gamma_n (x-s)} v_n(s) \, ds,$$
which, together with (4.19), implies that

\[(4.21)\quad u_n \to 0 \text{ in } L^2(0, 1).\]

We have according to (4.12)

\[(4.22)\quad y_n = \frac{1}{i\gamma_n}(z_n + f_n) \to 0 \text{ in } L^2(0, 1).
\]

Invoking (4.13), (4.16) and (4.17), we have

\[
\int_0^1 z_n(x)dx - \frac{1}{i\gamma_n} \int_0^1 g_n(x)dx = \frac{p_n}{i\gamma_n} - m\xi_n - M\eta_n + \frac{g_n}{i\gamma_n} + o(1) = -M\eta_n + o(1).
\]

Then, since \(Z_n \in \mathcal{X}\), we obtain that

\[(4.23)\quad y_n(0) \to 0.
\]

Integrating (4.14) we get

\[
\int_0^1 i\gamma_n z_n(x) a(\overline{y_n})_x(x)dx - \int_0^1 (a(y_n)_x x)(a(x)\overline{y_n})_x(x)dx = \int_0^1 g_n(x)a(x)\overline{y_n}_x(x)dx
\]

and hence

\[
\frac{1}{\gamma_n^2} \left( |(a(1)(y_n)_x)(1)|^2 - |(a(0)(y_n)_x)(0)|^2 \right) = o(1).
\]

Therefore, using (4.16) and (4.17), we have

\[(4.24)\quad M^2\eta_n^2 - m^2\xi_n^2 = o(1) \Rightarrow \eta_n \to 0.
\]

Since \((f_n, g_n, v_n, p_n, q_n) \in \mathcal{X}\), we obtain

\[
\int_0^1 g_n(x)dx + \alpha\tau \int_0^1 v_n(x)dx + (\beta - \alpha) f_n(0) + m\xi_n + M\eta_n = 0, \forall n \in \mathbb{N}
\]

Therefore

\[(4.25)\quad f_n(0) \to 0 \text{ and } f_n(1) = \int_0^1 \overline{(f_n)_x(x)}dx + f_n(0) \to 0.
\]

This also implies, thanks to (4.13), that

\[
i\gamma_n y_n(1) = z_n(1) + f_n(1) \to 0, i\gamma_n y_n(0) = z_n(0) + f_n(0) \to 0.
\]

**Second step.**

We express now \(z_n\) in terms of \(y_n\) from equation (4.13) and substitute it into (4.14) to get

\[(4.26)\quad -\gamma_n^2 y_n - (a(y_n)_x)_x = i\gamma_n f_n + g_n.
\]

Next, we take the inner product of (4.26) with \(b(y_n)_x\) in \(L^2(0, 1)\), where \(b \in C^1([0, 1])\). We obtain

\[
\int_0^1 \left( -\gamma_n^2 y_n(x) - (a(y_n)_x)_x(x) \right) b(x)\overline{(y_n)}_x(x)dx = \int_0^1 (i\gamma_n f_n(x) + g_n(x)) b(x)\overline{y_n}(x)dx =
\]

\[(4.27)\quad - \int_0^1 i\gamma_n f_n(x)b(x)\overline{y_n}(x)dx + \int_0^1 g_n(x)b(x)\overline{y_n}(x)dx + i\gamma_n f_n(1)b(1)\overline{y_n}(1) - i\gamma_n f_n(0)b(0)\overline{y_n}(0).
\]

It is clear that the right-hand side of (4.27) converges to zero since \(f_n, g_n, f_n(0), f_n(1), \gamma_n y_n(0)\) and \(\gamma_n y_n(1)\) converge to zero in \(H^1(0, 1), L^2(0, 1)\) and \(C\), respectively.
On the other hand, a straightforward calculation yields
\[
\Re \left\{ \int_0^1 -\gamma_n^2 y_n b(x) \overline{(y_n)_x(x)} \, dx \right\} = \frac{1}{2} \int_0^1 \gamma_n^2 b'(x) |y_n(x)|^2 \, dx - \frac{1}{2} \left( -\gamma_n^2 b(1) |y_n(1)|^2 + \gamma_n^2 b(0) |y_n(0)|^2 \right)
\]
\[
= \frac{1}{2} \int_0^1 \gamma_n^2 b'(x) |y_n(x)|^2 \, dx + o(1)
\]
and
\[
\Re \left\{ - \int_0^1 (a(y_n)_x(x) b(x) \overline{(y_n)_x(x)} \, dx \right\} = \frac{1}{2} \int_0^1 (ab' - a'b) |(y_n)_x|^2 \, dx
\]
\[
- \frac{1}{2} \left( a(1)b(1) |(y_n)_x(1)|^2 - a(0)b(0) |(y_n)_x(0)|^2 \right)
\]
\[
= \frac{1}{2} \int_0^1 (ab' - a'b) |(y_n)_x|^2 \, dx - \frac{1}{2} a(1)b(1) |(y_n)_x(1)|^2 + o(1).
\]
This leads to
\[
\int_0^1 (ab' - a'b) |(y_n)_x|^2 \, dx + \int_0^1 \gamma_n^2 b'(x) |y_n(x)|^2 \, dx - a(1)b(1) |(y_n)_x(1)|^2 = o(1).
\]
In particular, by taking \( b(x) = x \), for \( x \in [0, 1] \), we get
\[
\int_0^1 (a - xa') |(y_n)_x|^2 \, dx + \int_0^1 \gamma_n^2 |y_n(x)|^2 \, dx - a(1) |(y_n)_x(1)|^2 = o(1),
\]
while for \( b(x) = x - 1, \forall x \in [0, 1] \), we have
\[
\int_0^1 (a - (x - 1)a') |(y_n)_x|^2 \, dx + \int_0^1 \gamma_n^2 |y_n(x)|^2 \, dx = o(1).
\]
Combining (4.29)-(4.30), it follows that
\[
\int_0^1 a' |(y_n)_x|^2 \, dx + a(1) |(y_n)_x(1)|^2 = o(1).
\]
Thus according to (4.4), we obtain
\[
(y_n)_x(1) = o(1),
\]
which, together with (4.22) and (4.31), yields
\[
y_n \to 0 \text{ in } H^1(0, 1).
\]

Third step.

Taking the inner product of (4.14) with \( \overline{\frac{\partial}{\partial n}} x \) in \( L^2(0, 1) \), we have
\[
\left( \frac{\partial}{\partial n} x \right) \left| z_n(x) \right|^2 \, dx + \int_0^1 (a(y_n)_x(x) \overline{i\gamma_n (y_n)_x(x)} - (f_n)_x(x) \, dx
\]
\[
- a(1)(y_n)_x(1) i\gamma_n y_n(1) - f_n(1) + a(0)(y_n)_x(0) i\gamma_n y_n(0) - f_n(0)
\]
\[
= \int_0^1 g_n(x) i\gamma_n y_n(x) - f_n(x) \, dx = o(1).
\]
Hence
\[
\int_0^1 \left| z_n(x) \right|^2 \, dx - \int_0^1 a(x) |(y_n)_x(x)|^2 \, dx = o(1),
\]
which together with (4.33) leads to
\[ z_n \to 0 \text{ in } L^2(0,1). \]
Lastly, the identities (4.19), (4.21), (4.24), (4.33) and (4.34) clearly contradicts the fact that \( \|Z_n\|_{\mathcal{K}} = 1, \forall n \in \mathbb{N}. \)

Thereby, the two assumptions of Theorem 4 are proved and the proof of Theorem 5 is thus completed.

**Remark 2.** Combining Theorem 3 and Theorem 5, one can claim that the solutions of the closed-loop system (2.1) polynomially tend in \( \mathcal{X} \) to \( (\Omega,0,0,0,0) \) as \( t \to +\infty \), where \( \Omega \) is given by (3.1).

5. Conclusions and discussions

To recapitulate, this work dealt with the analysis of overhead system under the presence of a constant time-delay in the boundary velocity control. Assuming that the feedback gain of the delayed term is small, it has been shown that the system is well-posed whose proof is based on the introduction of a suitable energy-norm. Additionally, it has been proved that the solutions of the system asymptotically converge to an equilibrium state which is explicitly given and depends on the initial conditions. The proof of this result utilized the well-known LaSalle principle. More importantly, the polynomial convergence of solutions has been obtained.

We point out that there are many problems which could be treated. For instance, it is quite natural to wonder whether the results obtained in this article could be extended to the case where the control is nonlinear. Moreover, if the delay occurring in the boundary control is time-dependent, then does the convergence result still hold? This will be the focus of our attention in future works.

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