Tegula- exploring a galaxy of two-dimensional periodic tilings

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July 2020

Abstract

Periodic tilings play a role in the decorative arts, in construction and in crystal structures. Combinatorial tiling theory allows the systematic generation, visualization and exploration of such tilings of the plane, sphere and hyperbolic plane, using advanced algorithms and software. Here we present a "galaxy" of tilings that consists of the set of all 2.4 billion different types of periodic tilings that have Dress complexity up to 24. We make these available in a database and provide a new program called Tegula that can be used to search and visualize such tilings.

Availability: All tilings and software are open source and available here: https://ab.inf.uni-tuebingen.de/software/tegula

1 Introduction

Two dimensional periodic tilings play a role in the decorative arts, prominent examples being the euclidean tilings that cover the Alhambra palace in Granada, Spain, and M.C. Escher’s illustrations of euclidean, hyperbolic and spherical tilings involving reptiles, birds and other shapes (Schattschneider 2010). Two dimensional euclidean tilings are used in the construction of floors, walls and roofs. Hyperbolic tilings are used in the analysis of minimal surfaces of three-dimensional crystal structures (Ramsden et al. 2009 Kolbe and Evans 2018).

A two-dimensional periodic tiling \((\mathcal{T}, \Gamma)\) of the euclidean plane, sphere or hyperbolic plane, consists of a set of tiles \(\mathcal{T}\) and a discrete group \(\Gamma\) of symmetries of \(\mathcal{T}\) with compact fundamental domain, see Figure 1. Combinatorial tiling theory, based on the encoding of periodic tilings as “Delaney-Dress symbols” (Dress 1984 1987 Dress and Huson 1987), can be used to systematically enumerate all possible (equivariant) types of two-dimensional tilings by their curvature and increasing number of equivalence classes of tiles (Huson 1993).

In this paper, we introduce the term Dress complexity of a periodic tiling, which is simply the size of the corresponding Delaney-Dress symbol. We discuss how to systematically enumerate all Delaney-Dress symbols up to a given Dress complexity, in the case of two-dimensional periodic tilings. Using this, we have enumerated all two-dimensional periodic tilings of complexity \(\leq 24\). There are 2,395,220,319 such tilings. We refer to this collection as a “galaxy of periodic tilings” in the title of this paper, because, first, the number of tilings is very big (although not as large as the number of stars in a typical galaxy), and second, when viewing these tilings, the impression is that many look very similar to each other, much like stars in the sky.

Each such tiling is represented by its Delaney-Dress symbol and we provide these in a SQLITE database. We provide a new program called Tegula that allows the user to explore and query the database, and to visualize the corresponding tilings in all three geometries. Tegula and the database of periodic tilings are open source and freely available.

2 Conway’s orbifold notation

In this section, we briefly recall results on the classification of surfaces (Seifert and Threlfall 1934 Francis and Week 1999) and their Conway names (Conway and Huson 2002). Any orientable, closed, connected
Figure 1: Periodic tilings of the sphere, plane and hyperbolic plane. Each is labeled by its Dress complexity $\delta = \delta(\mathcal{T}, \Gamma)$ and the orbifold name of the symmetry group $\Gamma$.

A connected surface with boundary is obtained from a closed connected surface by removing $k$ disks from the interior of the surface. In Conway’s notation, a sphere with $b > 0$ boundary components is written as

\[
\ast \ast \ast \ldots \ast
\]

a sphere with $h > 0$ handles and $b > 0$ boundary components is written as

\[
\ast \ast \ldots \ast
\]

and a sphere with $k > 0$ crosscaps and $b > 0$ boundary components is written as

\[
\ast \ast \ldots \ast \ast \ast \ldots \ast
\]

The surface $\ast\ast$ is a Möbius strip.

For the purposes of this paper, a two-dimensional orbifold (Thurston 1980; Conway and Huson 2002) consists of a connected surface $S$, either orientable or non-orientable, with boundary or without, together with a finite set of points $P = \{p_1, \ldots, p_t\}$ in $S$, where each such point $p_i$ is labeled with an integer $v_i \geq 2$ that we call its order. Any such point is called a cone, if it is contained in the interior of $S$, or a corner, if it is contained in the boundary of $S$. For example, in Figure 2 we depict an orbifold obtained by adding three cones and four corners to the surface $\ast\ast\ast$. Note that the set of added points can be empty, so any surface, such as $\ast\ast\ast$, is also an orbifold.
Figure 2: On the left we show the orientable surface \( h** \). On the right we show the orbifold obtained by adding three cones and four corners. (Torus image: Oleg Alexandrov, Wikipedia.)

Conway’s notation, which we introduced above for the naming of surfaces, also covers orbifolds and has this form

\[
\begin{array}{cccccccc}
\circ \circ \ldots \circ & A & B & C & \ldots & a & b & c & \ldots & r & p & q & \ldots & x & x & \ldots & x \\
& h & \text{handles} & \text{cones} & \text{corners} & \text{corners} & \text{corners} & k & \text{crosscaps} \\
& & & & & & & k & \text{boundary components}
\end{array}
\]

Note that the cone degrees \( A, B, C, \ldots \) are unordered, whereas the corner degrees associated with any given boundary component have a cyclic ordering given by the order in which they are encountered along boundary component. One can flip the direction in which corners of an individual boundary component are listed if there are no other boundary components with corners, or if the surface is non-oriented.

3 Equivariant tilings

Through this paper, let \( X \) be one of the three two-dimensional geometries, namely either the sphere \( S^2 \), the euclidean plane \( \mathbb{E}^2 \), or the hyperbolic plane \( \mathbb{H}^2 \).

We use \( (T, \Gamma) \) to denote a equivariant tiling of \( X \), defined in the usual way (Dress and Huson, 1987). That is, any such tiling \( (T, \Gamma) \) consists of a set of tiles \( T \) that is “well-behaved” (i.e. a locally finite tiling whose tiles have simply-connected interiors), and a group \( \Gamma \) of isometries of \( X \) that map the set of tiles \( T \) onto itself. We emphasize that the word equivariant indicates that the symmetry group is prescribed and thus may be only a subgroup of the group of all automorphisms of the tiling.

Such a tiling \( (T, \Gamma) \) is called periodic tiling, if its symmetry group \( \Gamma \) is a discrete group with a compact fundamental domain. Examples of such tilings are shown in Figure 1.

Let \( \Gamma \) be the symmetry group of a two-dimensional periodic tiling. The orbifold \( O(\Gamma) \) of such a group is “the surface divided by the group”, that is, the orbit-manifold given by the quotient topological space whose points are the orbits under the group (Conway and Huson, 2002). Here, reflections are mapped onto boundary segments, rotational centers are mapped onto cones, dihedral centers are mapped onto corners, and glide-reflections give rise to crosscaps. The order of such a point is given by the largest order of any rotation in the symmetry group that fixes the center. This is illustrated in Figure 3 and in Figure 1 we list the orbifold names for the displayed tilings. Given the drawing of a periodic tiling, or other periodic pattern, one can easily determine the orbifold name for corresponding symmetry group, as discussed in (Conway and Huson, 2002).

Do all orbifold names correspond to symmetry groups? Any two-dimensional orbifold with name \( O \), of the form shown above in Equation 1, can be obtained as either

1. \( S^2 \) / an orthogonal group,
2. \( \mathbb{E}^2 \) / a crystallographic group, or
3. \( \mathbb{H}^2 \) / a “non-euclidean crystallographic group”,
except for the “bad orbifolds” with names $p$, $pq$, $\ast p$ and $\ast pq$ with $p, q \geq 2$ and $p \neq q$, see (Conway and Huson 2002).

4 Combinatorial tiling theory

In combinatorial tiling theory, every periodic tiling $(T, \Gamma)$ is represented by a Delaney-Dress symbol $(D, m)$, defined as a finite set $D$, together with the action of a certain free group $\Sigma$, together with with maps $m_{01}, m_{12}, m_{02} : D \to \mathbb{N}$, fulfilling certain conditions, see (Dress 1984; Dress and Huson 1987).

A key result is that the Delaney-Dress symbol describes a periodic tiling up to equivariant equivalence. In more detail, two periodic tilings $(T, \Gamma)$ and $(T', \Gamma')$ are equivariantly equivalent, if and only if their corresponding Delaney-Dress symbols $(D, m)$ and $(D', m')$ are isomorphic (Dress 1984, 1987).

Based on this, all two-dimensional periodic tilings can be systematically enumerated (Huson 1993). Delaney-Dress symbols can be assigned to higher-dimensional tilings, and have been used to address classification problems for three-dimensional euclidean tilings (Molnár et al. 1997; Delgado Friedrichs and Huson 1999; Delgado-Friedrichs and O’Keeffe 2005; Dutour Sikirić et al. 2010) and as a useful data-structure in the context of developing the system of orbifold names for three-dimensional euclidean space groups (Delgado Friedrichs and Huson 1997; Conway et al. 2001).

Rather than repeat the details of the definition of a Delaney-Dress symbol here, we illustrate its construction using an example. Consider the periodic tiling shown in Figure 3(a). To construct its Delaney-Dress symbol, start by triangulating the tiling using a barycentric subdivision, as shown in Figure 4(a). Note that each triangle corresponds to a flag $(v, e, t)$ consisting of a vertex $v$ contained in an edge $e$, which is contained in a tile $t$. Every triangle has exactly three neighbors, which we call its 0-, 1- or 2-neighbor, whose flags differs only in their 0-, 1- or 2-component, respectively.

The second step is to partition the set of triangles into equivalence classes, considering any two triangles to be equivalent, if their exists an symmetry of the tiling that maps the one triangle onto the other. In this example we obtain eight such classes and label them 1–8.

These eight equivalence classes define the Delaney-Dress set $D$, which are represented by nodes in the graph shown in Figure 4(b). Two such nodes are connected by an edge with label $i$, if the $i$-neighbor of any triangle in the one equivalence class is contained in the other equivalence class. For example, nodes 1 and 2
Figure 4: (a) A periodic tiling, triangulated into chambers, all symmetry-equivalent chambers labeled with the same number 1–8. The highlighted fundamental domain contains exactly one chamber for each of the eight numbers. (b) The associated Delaney-Dress symbol has eight corresponding nodes (labeled here 1–8). There are three types of edges, labeled 0–2, indicating neighbor relationships between chambers. Each node is labeled with two bold numbers, indicating the degree of the containing tile and the degree of the incident vertex, respectively.

are connected by an edge labeled 2, because triangles of neighboring triangles in classes 1 and 2 are incident to the same node and edge, but are contained in different tiles.

The final step is to label each node $D$ by two numbers, $p, q$; these record the tile-degree (number of edges of the tile) and vertex-degree associated with the given equivalence class of triangles. More formally, the two numbers are denoted by $m_{01}(D)$ and $m_{12}(D)$. For example, node 1 is labeled 3, 4, because all triangles in equivalence class 1 are contained in a tile of degree 3 and are incident to a vertex of degree 4, whereas node 4 is labeled 4, 6, because the corresponding triangles are contained in tiles of degree 4 and are incident to vertices of degree 6.

So, we can view a two-dimensional Delaney-Dress symbol as a Delaney-Dress graph that is, a connected (multi-) graph in which each node is incident to exactly one edge of each color 0, 1, and 2, together with a labelling of its nodes by two maps $m_{01}$ and $m_{12}$, fulfilling certain conditions.

Let an $i,j$-component $Z$ be set of nodes in $Z \subseteq D$ that is connected by edges of colors $i$ and $j$, with $0 \leq i < j \leq 2$. By the properties of an edge coloring, $Z$ will always be a cycle or a chain, and we define the $i,j$-length of $Z$ to be $|Z|$, in the former case, and $2|Z|$, in the latter. For any node $D \in D$, we define $r_{ij}(D)$ to be the $i,j$-length of the $i,j$-component containing $D$.

A natural interpretation of the Delaney symbol is as a triangulation of the associated orbifold, where each triangle is the image of one equivalence class of triangles of the original tiling. By construction, any cone or corner of the orbifold will lie on a vertex of this triangulation and will be surrounded by triangles that belong to the same $i,j$-component $Z$, for some choice of $0 \leq i < j \leq 2$. We call this a $k$-vertex, with $k$ such that $\{i, j, k\} = \{0, 1, 2\}$. For all nodes $D \in Z$, we define $v_{ij}(D)$ to be the order of the associated cone or corner, that is, the highest order of any rotation in the symmetry group about the vertex. For each $i,j$-orbit $Z$ whose triangles are not incident to a cone or corner, we set $v_{ij}(D) = 1$ for all $D \in Z$.

Note that we have $m_{ij}(D) = v_{ij}(D)r_{ij}(D)$, linking combinatorial features of the tiling, such as vertex degrees, etc, with rotational degrees in the symmetry group. In particular, all equivalence classes of rotational centers and dihedral centers of the symmetry group can be obtained by from the Delaney symbol of a tiling by enumerating all $i,j$-components $Z$ in $D$ for which $v_{ij}(D) = \frac{m_{ij}(D)}{r_{ij}(D)} > 1$ holds for $D \in Z$. 

5
The number of symmetry-equivalence classes of vertices, edges and tiles in a periodic tiling is given by the number of 1, 2-, 0, 2- and 0, 1-components in its Delaney symbol, respectively.

Other properties of a tiling require more involved analysis of the corresponding Delaney symbol, such as the Euler characteristic, curvature, geometry and the corresponding orbifold name (Balke and Huson, 1996). For example, the curvature is given by the following calculation:

\[ K(D, m) = \sum_{D \in \mathcal{D}} \left( \frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} - \frac{1}{2} \right), \]

and this, in turn, defines the geometry associated with the tiling, namely spherical, euclidean or hyperbolic, depending on whether the curvature is positive, 0 or negative, respectively.

A more difficult tiling property to obtain from an analysis of the corresponding Delaney symbol is whether the tiling is pseudo convex, that is, whether the intersection of any two tiles is always either empty or simply connected.

The size of the Delaney-Dress symbol \((D, m)\) is an important invariant for the corresponding tiling \((T, \Gamma)\), albeit the most simplest property to obtain, and we propose to call this the Dress complexity of the tiling, denoted by \(\delta(T, \Gamma)\).

5 Enumeration

A main goal of this paper is to enumerate all periodic tilings of low Dress complexity.

We first start with Dress complexity 1, that is, Delaney-Dress symbols of size one, as displayed in Figure 5. In this case, the curvature is given by

\[ K(D, m) = \frac{1}{p} + \frac{1}{q} - \frac{1}{2}. \]

For \(p = 3, 4, 5, q = 3, 4, 5\) this value is positive, and thus the corresponding tilings are spherical. The same is true for \(p = 3, 4, 5, q = 3\). For \(p = 3, q = 6\), or \(p = 6, q = 3\), or \(p = q = 4\), the curvature is 0 and thus the corresponding tilings are euclidean. In all other cases, for example \(p = 4, q = 5\), the curvature is negative and thus the corresponding tiling is hyperbolic.

If we allow tiles to be digons, that is, to have only two edges, then \(p = 2\) and for any value of \(q \geq 3\) the curvature is positive and so all such tilings of Dress complexity 1 are tilings of the sphere. To reduce the number of resulting classes, in this paper we only enumerate tilings for which all tiles have at least 3 edges, contrast to some of our previous work (Delgado Friedrichs et al., 1992; Huson, 1993).

Already for Dress complexity 1 we encounter infinite families of non-equivalent periodic tilings. To address this, we say that a periodic tiling \((T, \Gamma)\) is geometry minimal, if one of the three cases hold:

1. the tiling is spherical and either the corresponding orbifold is one of 532 and *532, or all rotational degrees are \(\leq 4\),

2. the tiling is euclidean, or

3. the tiling is hyperbolic and one can’t reduce the rotational order of any tile, or vertex, without changing the sign of the curvature of the symmetry group, or without reducing the degree of the tile, or the vertex, respectively, to below 3.

This property is easily inferred from the corresponding Delaney-Dress symbol. For a spherical tiling, determine whether the value of \(v_{ij}\) is \(\leq 5\) on all 0,1- and 1,2-orbits. For a hyperbolic tiling, reduce the value of \(v_{ij}\) on each 0,1- and 1,2-orbit in turn, and check whether the modified Delaney-Dress symbol has negative curvature and that the resulting value for \(m_{ij}\) is \(\geq 3\).

We can now formulate our first result: there exist exactly 12 different equivariant types of geometry minimal, periodic two-dimensional tilings with Dress complexity 1, see Figure 6. Note here we only consider tilings with tiles of degree 3 or more. In addition, there are two types of geometry-minimal tilings with digons, they have parameters \((p, q)\) 2, 3 and 2, 4.
Figure 5: Any Delaney-Dress symbol \((\mathcal{D}, m)\) of Dress complexity 1 consists of a single node \(D\), three self-edges of colors 0, 1, 2, and two number \(p \geq 3\) and \(q \geq 3\).

Figure 6: The 12 geometry-minimal types of periodic two-dimensional tilings with Dress complexity 1. Each labeled with \(p, q\), that is, their tile and vertex degrees, and the orbifold name of their symmetry group.
| $\delta$ | # Spherical | # Euclidean | # Hyperbolic | Total |
|-------|-------------|-------------|-------------|-------|
| 1     | 5           | 3           | 4           | 12    |
| 2     | 13          | 15          | 22          | 50    |
| 3     | 15          | 8           | 13          | 36    |
| 4     | 30          | 37          | 71          | 138   |
| 5     | 26          | 15          | 41          | 82    |
| 6     | 119         | 86          | 221         | 426   |
| 7     | 104         | 64          | 201         | 369   |
| 8     | 252         | 217         | 796         | 1,265 |
| 9     | 296         | 185         | 858         | 1,339 |
| 10    | 697         | 527         | 2,974       | 4,198 |
| 11    | 771         | 506         | 3,993       | 5,270 |
| 12    | 2,014       | 1,573       | 13,987      | 17,574|
| 13    | 2,364       | 1,575       | 22,162      | 26,101|
| 14    | 5,428       | 4,227       | 75,270      | 84,925|
| 15    | 6,627       | 4,528       | 140,024     | 151,179|
| 16    | 15,103      | 12,078      | 475,445     | 502,626|
| 17    | 18,622      | 13,105      | 982,726     | 1,014,453|
| 18    | 42,881      | 34,242      | 3,327,350   | 3,404,473|
| 19    | 53,588      | 38,470      | 7,419,771   | 7,511,829|
| 20    | 120,496     | 98,076      | 25,029,758  | 25,248,330|
| 21    | 151,234     | 111,145     | 58,815,127  | 59,077,506|
| 22    | 340,744     | 280,574     | 197,482,678 | 198,103,996|
| 23    | 428,769     | 322,102     | 482,898,722 | 483,649,593|
| 24    | 965,620     | 805,130     | 1,614,643,799 | 1,616,414,549|
| Total | 2,155,818 | 1,728,488 | 2,391,336,013 | 2,395,220,319 |

Table 1: For Dress-complexity $\delta = 1, \ldots, 24$, we list the number of different geometry-minimal periodic tilings of the sphere, euclidean plane and hyperbolic plane.

An enumeration of all possible equivariant types of geometry minimal, periodic two-dimensional tilings, up to a given maximal Dress complexity $D$, can be obtained by systematically enumerating all non-isomorphic Delaney-Dress symbols of size $\leq D$ that are geometry minimal.

Our second result is: there exist exactly 50 equivariant types of geometry minimal, periodic two-dimensional tilings with Dress complexity 2.

More generally, we have solved this classification up to Dress complexity $D = 24$ and obtain the following main result: There exist exactly $2,395,220,319$ equivariant types of geometry minimal, periodic two-dimensional tilings with Dress complexity $\leq 24$. Summary statistics are provided in Table 1. We have generated and saved all Delaney-Dress symbols for these tilings, and make them available as described below.

6 Visualization

In Table 1 we count billions of Delaney-Dress symbols that correspond to two-dimensional periodic tilings. To enable the exploration of these, we require an algorithm for calculating a drawing of the tiling associated with any given Delaney-Dress symbol $(D, m)$.

Figure 4 illustrates that each node of a Delaney-Dress symbol corresponds to a different equivalence class of triangles in the barycentric subdivision of the corresponding periodic tiling, and that a fundamental domain for the symmetry group can be obtained by selecting a suitable set of representatives of the different classes of triangles.

To construct a tiling associated with a given two-dimensional Delaney-Dress symbol $(D, m)$, we proceed
in three stages:

1. Compute coordinates for a barycentric triangulation of the tiling for a fundamental domain of the symmetry group.

2. Compute a set of isometric transformations that generate the symmetry group.

3. Apply the generators to copies of the triangulation of the fundamental domain so as to cover a desired region of the tiling.

Stage (1) uses an algorithm and code developed by Klaus Westphal ([Westphal 1991](#)). The algorithm assigns a triangle to each node or chamber of the given Delaney-Dress symbol \((D, m)\). Triangles of adjacent nodes are then identified along the corresponding side, in an iterative manner. This is done in such a way that the resulting triangulated region is a topological disc \(B\) and all vertices of the triangulation that correspond to a cone or corner point are located on the boundary. Let \(Z\) be a \(i,j\)-component. We use \(s(Z)\) to denote the number of vertices of the triangulation that are associated with \(Z\). This will be one, if \(Z\) lies in the interior of \(B\). If \(s(Z) > 1\), then the vertices must all lie on the boundary of \(Z\) and we say that \(Z\) is split. For example, in Figure 4, the 1, 2-component containing chambers 1, 2, 3, 5 is split and is represented twice by vertices on the boundary of the fundamental domain, once involving the chambers labeled 1 – 3 and the other time involving 5.

Taking splitting into account, we assign an interior angle to \(Z\) as \(\alpha(Z) = \frac{360^\circ}{s(Z) \times v_{ij}(D)}\), if \(Z\) is a \(i,j\)-cycle, and \(\alpha(Z) = \frac{180^\circ}{s(Z) \times v_{ij}(D)}\), otherwise.

With this, we setup a polygon whose corners are given by the vertices assigned to the boundary of \(B\), using the calculated interior angles. The polygon is heuristically fitted around an incircle and vertices with interior angle \(180^\circ\) are then placed equally-spaced along the sides of the polygon. Then all triangulation vertices that are assigned to the interior of \(B\) are iteratively assigned to the centroid of all adjacent vertices so as to obtain useful coordinates.

To address stage (2), a set of generators for the symmetry group is obtained as follows. For a given chamber \(D\) whose \(i\)-th edge lies on the boundary of \(B\), let \(D'\) be its \(i\)-neighbor. Then a generator of the symmetry group can be obtained by calculating the isometry that maps the 0-, 1- and 2-vertices of the triangle representing \(D\) onto the 0-, 1- and 2-vertices of the triangle representing \(D'\), respectively. This is performed on all boundary chambers.

In stage (3), we repeatedly concatenate generators and keep the transformed copy of the fundamental domain, if it will be visible. The key practical challenge is to avoid placing more than one copy of the fundamental domain at the same location. To address this, we select a reference point within the interior of the fundamental domain and use either a quad-tree (in the case of euclidean tiles), or a oct-tree (for spherical and hyperbolic tilings), to determine whether the current transformation applied to the reference point gives rise to a point that has already been seen.

All hyperbolic calculations are performed using the Minkowski hyperboloid model. Visualization of the Poincare model and Klein model are implemented by observing the hyperboloid model using a perspective camera at locations \((0, 0, −1)\) and \((0, 0, 0)\), respectively.

To speed-up the visualization of translations, copies of the fundamental domain that disappear from view on one side of the tiling are reused and reappear on the other side of the tiling.

### 7 Enumeration and visualization software

We have implemented the enumeration of Delaney-Dress symbols in a program called \(genDSyms\) using the programming language Julia ([Bezanson et al. 2017](#)). For performance purposes, we use the principle of *orderly generation* ([Read 1978](#)) to ensure that every symbol is produced exactly once and no additional effort is required to identify and remove duplicates. The process has two stages.

In the first stage we enumerate all possible Delaney-Dress graphs up to the required size. Note that, for any given Delaney-Dress graph and choice of an initial node, there exists a unique *ordered traversal*, that is, a
breadth-first graph traversal, in which at each node the incident edges are visited in the order of their labels (Delgado-Friedrichs, 2003). We use this to assign numbers to the nodes in the order they are encountered and represent the traversal as a linear string of numbers by listing the 0-, 1- and 2-neighbors of all the vertices in that same order.

As an example, consider the Delaney-Dress graph in Figure 4(b). Beginning at the node on the left labelled 1, we see that it is its own 0- and 1-neighbor. Its 2-neighbor is thus labelled 2, its 1-neighbor in turn 3, and so on. Continuing in this fashion, we see that in fact the vertices are already numbered in accordance with this traversal, which is then represented by the list 1, 1, 2; 2, 3, 1; 4, 2, 5; 3, 6, 7; 7, 5, 3; 6, 4, 8; 5, 8, 4; 8, 7, 6. Of all the possible traversals for this graph, this one turns out to be the lexicographically smallest, because the leftmost node is the only one with both a 0- and a 1-loop, and thus the only one that can give rise to a traversal representation starting with two ones.

To perform an orderly generation of Delaney-Dress graphs up to a given size, we generate all possible ordered traversals and keep only those that are lexicographically smallest for the graph they represent. To speed up the enumeration process, we prune the enumeration tree whenever we identify a partial ordered traversal that cannot be completed to a lexicographically smallest one.

In the second stage of the enumeration, for each Delaney-Dress graph we generate all possible valid definitions of the maps \( m_{01} \) and \( m_{12} \), in particular making use of the restrictions imposed by geometric minimality.

The file containing the complete galaxy of tilings is 322 GB in size, and it is thus impractical to make the file available on a webservice. To provide easy access to much of the classification, we have produced three SQLITE databases of Delaney-Dress symbols. The first, tilings-1-18.tdb, contains all tilings with Dress complexity 1–18. The other two, spherical-1-24.tdb and euclidean-1-24.tdb, contain all spherical and euclidean tilings, respectively of Dress complexity 1–24. Each database contains a table called “tilings” that has the schema shown in Table 2.

We have implemented a new program called Tegula that can be used to explore our galaxy of tilings. Tegula takes as input an SQLITE database of Delaney-Dress symbols and provides drawings of the corresponding tilings in a “collection” tab, on a page-by-page basis. The program provides an interactive dialog for searching for tilings of specific interest. The user can page through all tilings that fulfill a given query.

For example, the query “symmetry.class = ‘Stellate’ and normal = ‘true’ and maximal = ‘true’ and colorable = ‘true’ ” returns 31 tilings of Dress complexity \( \leq 18 \), displayed Figure 7.

Individual tilings can be edited in a number of different ways. Double-clicking on a tiling in a collection tab will open the tiling in a new “editor” tab and there five panels of tools are available to modify the displayed tiling, as illustrated in Figure 8.

Availability

The enumeration program genDSyms is written in Julia. The source is provided here: https://github.com/odf/julia-dsymbols. The visualization and exploration program Tegula is written in Java and uses the OpenJFX library. The source is provided here: https://github.com/husonlab/tegula. Installers for Windows and MacOS are available here: https://software-ab.informatik.uni-tuebingen.de/download/tegula.

Acknowledgements

We thank Klaus Westphal for providing us with his original source code for computing the fundamental domain of a tiling. We thank Julius Vetter and Cornelius Wiehl for programming contributions.
| Column name and type | Explanation |
|----------------------|-------------|
| id INTEGER PRIMARY KEY | number in file |
| symbol TEXT | Delaney-Dress symbol \((\mathcal{D}, m)\) |
| complexity INTEGER | Dress complexity \(\delta(\mathcal{D}, m)\) |
| geometry TEXT | Name of two-dimensional geometry |
| curvature TEXT | Curvature \(K(\mathcal{D}, m)\) |
| euler REAL | Euler characteristic |
| orbifold TEXT | Orbifold name of symmetry group |
| symmetry_class TEXT | Symmetry class of graph, as defined in [Hyde et al., 2014](#) |
| signature TEXT | Signature is expression such as \((3465)\) that indicates the tiling consists of tiles of degree 4 with vertices of degree 3, 4, 6 and 5. |
| tile_deg TEXT | List of tile degrees in ascending order |
| vertex_deg TEXT | List of vertex degrees in ascending order |
| tiles INTEGER | Number of equivalence classes of tiles |
| edges INTEGER | Number of equivalence classes of edges |
| vertices INTEGER | Number of equivalence classes of vertices |
| normal BOOLEAN | Is tiling pseudo-convex? |
| maximal BOOLEAN | Is the symmetry group of the tiling maximal? |
| colorable BOOLEAN | Tiling is colorable, if no two tiles of the same symmetry equivalence class share an edge. |
| orientable BOOLEAN | Does symmetry group only contain orientation-preserving symmetries? |
| fixed_point_free BOOLEAN | Is symmetry group fixed-point free? |
| self_dual BOOLEAN | Is tiling self-dual? |

Table 2: Schema for table “tilings” used for storing Delaney-Dress symbols and some associated properties.
Figure 7: Applying the query \texttt{symmetry\_class = 'Stellate' and normal = 'true' and maximal = 'true' and colorable = 'true'} to the database \texttt{tilings-1-18.tdb} of all Delaney-Dress symbols of Dress complexity $\leq 18$ returns 31 tilings.
Figure 8: Any tiling can be edited using five different panels. The *symmetries* panel allows editing of the rotational orders of the symmetry group; the *hyperbolic model* panel allows selection between the Poincare, Klein and hyperboloid model; the *appearance* panel allows the modification of different aspects of the tiling; the *algorithms* panel provides some transformations of the drawing and the Delaney symbol; and the *fundamental domain* panel allows interactive reshaping of the edges of the tiling.
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