A matched expansion approach to practical self-force calculations

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Abstract
We discuss a practical method of computing the self-force on a particle moving through a curved spacetime. This method involves two expansions to calculate the self-force, one arising from the particle’s immediate past and the other from the more distant past. The expansion in the immediate past is a covariant Taylor series and can be carried out for all geometries. The more distant expansion is a mode sum, and may be carried out in those cases where the wave equation for the field mediating the self-force admits a mode expansion of the solution. In particular, this method can be used to calculate the gravitational self-force for a particle of mass $\mu$ orbiting a black hole of mass $M$ to order $\mu^2$, provided $\mu/M \ll 1$. We discuss how to use these two expansions to construct a full self-force, and in particular investigate criteria for matching the two expansions. As with all methods of computing self-forces for particles moving in black hole spacetimes, one encounters considerable technical difficulty in applying this method; nevertheless, it appears that the convergence of each series is good enough that a practical implementation may be plausible.

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1. Introduction
There is currently considerable interest in calculating the self-force on a charged particle moving in a curved background spacetime. This interest is twofold. On one hand, self-force is an intrinsically interesting phenomenon, apparently both deep and subtle. On the other hand, there is increasing practical interest in understanding the motion of particles with small but non-negligible masses in curved geometries. In particular, there is widespread belief that accurate computation of gravitational self-force is needed to calculate gravitational waveform templates for key data analysis efforts associated with interferometric gravitational wave
detectors. With such templates, the laser interferometer space antenna (LISA) [1] is expected to produce a wealth of information about strong-field gravity [2].

We define self-force to be any force on a particle of quadratic (or higher) order in the charge carried by that particle. For electrically charged particles in flat spacetime, the self-force is given by the Abraham–Lorentz–Dirac formula [3, 4]. DeWitt and Brehme [5] derived the general expression for the self-force on an electrically charged particle in a curved background (however, they missed a term which was later supplied by Hobbs [6]). The gravitational self-force was first calculated almost simultaneously by Mino, Sasaki and Tanaka [7] and by Quinn and Wald [8]. Later, Quinn derived the equivalent formula for a charge coupled to a minimally coupled massless scalar field [9]. These results have resolved many of the issues of principle in computing self-forces in curved spacetime. Poisson [10] has recently written a comprehensive review article in which these expressions and their implications are discussed at length.

The self-force expressions for all fields of physical interest are of similar form. We henceforth restrict our attention, therefore, to the simplest case: a particle with mass $\mu$ and scalar charge $q$ coupled to a minimally coupled massless scalar field in a curved background geometry. Lessons learned here should be extendible to other physical fields without major modification.

The field equation for a minimally coupled massless scalar field $\phi$ is

\[
\Box \phi = -4\pi \rho.
\]

Here, $\Box$ is the D’Alembertian of the curved background and $\rho$ is the charge density. We consider a point particle, in which case

\[
\rho(x) = \int q \delta^4(x, z(\tau)) \, d\tau,
\]

where $\delta^4(\cdot)$ is a generalized Dirac distribution in four dimensions and $z(\tau)$ denotes the worldline of the particle.

For such a particle, Quinn [9] has shown that the self-force is given by

\[
f^\alpha = q^2 \left[ \frac{1}{3}(\dot{a}^\alpha - a^\beta u^\beta) + \frac{1}{12}(2R^\beta \rho_\beta + 2R_{\rho \tau} u^\rho u^\tau u^\alpha - Ru^\alpha) \right. \\
\left. + \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau - \epsilon} \nabla^\alpha G_{\text{ret}}(z(\tau), z(\tau')) \, d\tau' \right].
\]

The quantities in equation (3) are defined as follows: $u^\alpha$ is the 4-velocity of the particle and $a^\alpha = u^\beta \nabla_\beta u^\alpha$ its 4-acceleration. $\dot{a}^\alpha = u^\rho \nabla_\rho a^\alpha$ denotes derivative of the acceleration with respect to its proper time, and $a^2 = a^\alpha a_\alpha$ is the magnitude of the acceleration squared. The quantity $R_{\rho \beta}$ is the Ricci tensor of the background spacetime, and $R$ is its scalar curvature. $\tau$ is the proper time of the particle at its current position, while $\tau'$ denotes the proper time at any other point along the particles worldline. Finally, $G_{\text{ret}}(x, x')$ is the retarded Green function for the scalar field equation (1), which satisfies the Green function equation

\[
\Box G_{\text{ret}}(x, x') = -4\pi \delta^4(x, x'),
\]

and has support only when $x'$ is in the causal past of $x$.

Note that the terms on the right-hand side of (3) can be gathered into three groups. The first group are local terms involving the acceleration of the particle. These give the Abraham–Lorentz–Dirac force on an accelerating particle in flat spacetime [3]. They arise because the accelerating particle radiates, and that radiation produces a ‘radiation reaction’ or recoil force on the particle. It is interesting to note that for a particle in Minkowski spacetime with
constant \(a\) (whose worldline is a hyperbola) that \(\dot{a}^\nu \neq 0\) (because the direction of the 4-vector changes). In fact, \(\dot{a}^\nu = a^2 u^\nu\) in this case, and the first group vanishes. This fact, which might seem surprising at first, can be understood as a consequence of the equivalence principle [11–13]. We further note that for a particle freely falling in curved space (i.e. following a geodesic), \(a^\nu = 0\), and thus these terms will also vanish.

The second group are also local terms, this time involving the background curvature. These terms involve only the Ricci curvature, and thus represent a self-force mediated by the matter content of the background. Interestingly, it is not necessary for the matter to interact directly with the particle for this to be true. We will largely ignore these first two groups of terms for the remainder of our discussion because for a particle following a geodesic in a vacuum background spacetime, which is the case of most practical interest, they vanish. Moreover, even when they do not vanish, they are easily calculated.

In contrast, there is considerable practical difficulty in calculating the single non-local term which constitutes the third group. Interestingly, this non-local self-interaction arises in curved backgrounds, but not in a flat background. This is because of two ways in which the propagation of massless fields differs in curved and flat backgrounds. In Minkowski space, massless fields propagate along null geodesics. Thus, a particle would have to be null separated from some point on its past worldline to affect itself. However, the simple causal structure of Minkowski space does not allow this for a massive particle. The particle is restricted to a time-like geodesic, and no two points in Minkowski space can be connected both by a time-like geodesic and a null geodesic. Thus, any point with which the charged particle can interact, it cannot travel to, and vice versa. This is not true in a curved background, however, where the causal structure can be considerably more complicated. In this case, it is possible for the field, which 'leaves' the particle along a null geodesic, to re-intersect that particle, which follows a time-like geodesic, at a later time.

If this was the only mechanism by which non-local self-interactions arose, then there would be no self-interactions arising from within the normal neighbourhood of the point at which the particle sits. Recall that the normal neighbourhood of a point \(x\) is the set of all other points which are connected to \(x\) via a unique geodesic. Recall also that such a neighbourhood is guaranteed to exist. Thus, if a particle at \(x\) is massive and non-accelerating (i.e. following a time-like geodesic), then any point on the particle’s worldline connected to \(x\) by a null geodesic is connected to \(x\) by (at least) two geodesics. Therefore, by definition, it is not in the normal neighbourhood of \(x\).

However, in general, fields do not propagate only along null geodesics in curved backgrounds. This is apparent from the Hadamard [14] form of the retarded Green function,

\[
G_{\text{ret}}(x, x') = \Theta[t - t'] \left[U(x, x') \delta[\sigma(x, x')] - V(x, x') \Theta[-\sigma(x, x')]\right],
\]

where \(\Theta[\cdot]\) is the Heaviside step function, \(\delta[\cdot]\) is the Dirac delta distribution, \(\sigma(x, x')\) is one half of the square of the geodesic distance between points \(x\) and \(x'\), and \(U(x, x')\) and \(V(x, x')\) are smooth functions which depend on the details of the background and field equation. Note that this expression is only well defined if one can unambiguously define geodesic distance between \(x\) and \(x'\), which implies that \(x'\) is within the normal neighbourhood of \(x\).

Recall that the Green function gives the field at position \(x\) due to a charge at position \(x'\). Now, the first term on the right-hand side of equation (5), which is known as the direct part of the Green function, has support only when the geodesic distance between \(x\) and \(x'\) vanishes, or, in other words, when \(x\) and \(x'\) are separated by a null geodesic. This term is always present, but, as described above, cannot contribute to the self-force. However, the second term, which is known as the tail part of the Green function, does contribute whenever the square of the geodesic distance between \(x\) and \(x'\) is negative, or, in other words, when \(x\)
and \( x' \) are separated by a time-like geodesic\(^1\). Since every point on the particles past worldline is time-like separated from the particle by a time-like geodesic (to wit, the worldline), through this term a self-force can be generated at every time within the normal neighbourhood.

Given that all points on a particles past worldline can interact with the particle, it may seem somewhat arbitrary to have singled out in the above discussion points on the past worldline which are null separated from the particle. Indeed, in the literature, it is often stated that the self-force arises from the tail part of the field (or the Green function) and left at that. This statement can be somewhat confusing, however. Outside of the normal neighbourhood, there is no clear distinction between tail and direct parts—is the field from a point connected by both timelike and null geodesics a direct field, a tail field, or both?

Furthermore, there is reason to believe that these points play a qualitatively different role in the self-interaction. In particular, when doing the integral over the Green function, distributions (i.e. \( \delta \)-functions and step functions) will be encountered when the source point and field point are null separated. In effect, the particle can ‘feel’ its own direct field ‘sent’ from points in the past. This is depicted in figure 1. In the specific case of the \( O[M] \) Green function, such distributions do appear, and their contribution makes up the entire self-force (see equation (24)).

Let us turn now to the matter of calculating the non-local part of the self-force,

\[
\begin{align*}
f^a = q^2 \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(z(\tau), z(\tau')) d\tau'.
\end{align*}
\]  

(6)

The key feature of this formula is that the integral is well behaved over the entire region of integration, i.e. it is integrable throughout its domain. The termination of the integral at \( \tau - \epsilon \) (as opposed to integrating all the way to \( \tau \)) is required for this to be true, however, because the Green function has an unintegrable singularity at the coincidence limit (\( \epsilon = 0 \)). Nonetheless, this contribution is distributional, and the integrand is perfectly well behaved as long as we approach \( \tau' = \tau \) from the past.

At first glance, evaluating equation (6) seems like a relatively straightforward task—an approximate retarded Green function can be calculated, for instance, using mode-sum techniques [16]. However, in practice, we can only sum a finite number of modes and thereby obtain an approximate Green function. Furthermore, as we will demonstrate in section 3, the number of modes needed to obtain a given accuracy grows without bound as epsilon approaches zero. This does not preclude the use of modes to calculate the non-local part of the self-force, and, as can be seen in the pages of this special issue, modes are indeed widely used. The modes must, however, first be regularized by some method.

Nonetheless, it is our purpose in this paper to explore an alternative method which may have some advantages over a regularized mode sum. It does not need to make use of regularized modes, although one might choose to do so. Rather, it is a method of matched expansions, in which one calculates the self-force using the tail within some portion of the normal neighbourhood of the particle and using an unregularized mode expansion for the remainder of the particle’s worldline. More precisely, we propose to express equation (6) as

\[
\begin{align*}
f^a = -q^2 \int_{\tau - \Delta \tau}^{\tau} \nabla^a V(z(\tau), z(\tau')) d\tau' + q^2 \int_{-\infty}^{\tau - \Delta \tau} \nabla^a G_{\text{ret}}(z(\tau), z(\tau')) d\tau,
\end{align*}
\]  

(7)

\(^1\) This fascinating fact was first elaborated upon by Hadamard [14], who described it as a failure of Huygens’ principle to hold in general for hyperbolic partial differential equations—cf the excellent review article in the Paul Günther memorial edition of Zeitschrift für Analysis und ihre Anwendungen [15] addressing the proof of the modified Hadamard conjecture, which states that the only spacetimes for which standard wave equations obey Huygen’s principle are those conformal to Minkowski spacetime and one plane-wave family of spacetimes.
where $\Delta \tau$ is an interval of proper time. We require that $\Delta \tau$ be chosen so that $z(\tau')$ is within the normal neighbourhood of $z(\tau)$ for all $\tau - \Delta \tau < \tau' < \tau$. In other words, $\Delta \tau$ distinguishes the contribution to the self-force coming from the recent history of the particle from the contribution that comes from the more distant past. It is the choice of $\Delta \tau$ and the feasibility of evaluating the two integrals that will occupy us for the remainder of this paper.

There are several notable features of equation (7). First, this expression is exactly equivalent to equation (6), since, as is evident from equation (5), $G_{\text{ret}}(x, x') \equiv -V(x, x')$ provided we restrict ourselves to the interior of the past light cone, as is always the case for this integral. Second, there is no longer a limit needed because $V(x, x')$ is regular everywhere. Finally, note that we now have a new parameter, $\Delta \tau$, which we are free to choose so long as it is not too large.

One might suspect that the contribution to the Green function ‘falls off’ fast enough, in general, so that perhaps only the first integral needs to be evaluated; and, since it is restricted to the normal neighbourhood, only the Hadamard expansion is needed to compute the self-force. Surprisingly, there are simple situations in which this line of reasoning turns out to be false; the second integral gives a significant (perhaps, the dominant) contribution to the total force. We demonstrate this in section 3 using Green’s function for spacetime with a large central mass $M$ where we only keep the terms of leading order in the central mass (the ‘$O[M]$ Green function’). Using the $O[M]$ Green function, none of the force originates from...
the tail contribution in the normal neighbourhood; all the force comes times prior to the light reflection time.

Unfortunately, neither integrand in equation (7) can be calculated exactly. As mentioned above, \( G_{\text{ret}}(x, x') \) can be approximated in the second integrand by a mode sum expansion. The advantage here is that the mode sum does not need to be extended to the limit of the particle’s position. Thus, for any fixed finite precision required, a finite number of modes will be needed for the second integral in equation (7). As noted above, however, that the number will grow as \( \Delta \tau \) decreases.

On the other hand, for \( V(x, x') \), we can take advantage of the fact that we are within the normal neighbourhood, and can therefore define a Riemann normal coordinate system. In such a coordinate system, it is relatively straightforward to expand \( V(x, x') \) in a covariant Taylor series. Such a series will have coefficients constructed of geometric quantities (notably, the particle’s 4-velocity and the curvature of the background) evaluated at the particles position, and will be an expansion in geodesic distance from that position. Again, only a finite number of terms will be available, so only an approximation of \( V(x, x') \) can be calculated. In contrast to the mode sum expansion of \( G_{\text{ret}}(x, x') \), the covariant Taylor series will, in general, require more terms to achieve a given accuracy as \( \Delta \tau \) increases.

The goal, then, is to calculate both series to sufficient accuracy within their applicable domains, and to then integrate and sum. We may take advantage of our parameter, \( \Delta \tau \), to adjust the number of terms required in each series to achieve the required accuracy. However, both series are non-trivial to calculate, and the difficulty of calculating each successive term is greater than for the last. There is, therefore, no guarantee that there is any value of \( \Delta \tau \) such that the number of terms needed in both series can be calculated in practice.

It is the main goal of the remainder of this paper to investigate if it is plausible that there would exist a choice of \( \Delta \tau \) which would make this scheme, first proposed by Poisson and Wiseman [17], viable for practical calculations. Because of the difficulty of performing both expansions, we will exploit existing results when possible. Furthermore, we will use the simplest and most transparent results that still bear upon the problem. We will not explicitly try to calculate the self-force for any specific geometry or particle motion, but will, rather, simply investigate the convergence of both series expansions.

The plan of the remainder of this paper is as follows: we will review and examine the existing results for the expansion of the \( V(x, x') \) in the following section. Following that, in section 3, we will investigate the mode expansion. Finally, we will discuss our conclusions in section 4.

2. The quasi-local part of the self-force

Covariant normal neighbourhood expansions have a venerable history for calculations of both self-forces [5] and quantum field effects in curved backgrounds [18, 19]. There is therefore a plethora of material from which to begin a calculation of the first term in equation (7), which we will call the quasi-local part of the self-force

\[
f_{\text{QL}}^a = -q^2 \int_{\tau - \Delta \tau}^{\tau} \nabla^a V(z(\tau), z(\tau')) \, d\tau'.
\]  

(8)

There are, however, to our knowledge, only three papers in the literature which address the calculation of this part of the self-force. The first was by Roberts, in which he stopped just short of calculating the quasi-local part of the self-force for an electric charge in an arbitrary
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A trivial extension of his paper gives the leading order to the quasi-local part of the self-force in this case,

$$f_{QL}^\mu = - \frac{q^2}{4} C^\sigma_{\beta \gamma \delta \lambda} u^\beta u^\gamma \Delta \tau^2 + O(\Delta \tau^3).$$

(9)

Here, $C_{\alpha \beta \gamma \delta}$ is the Weyl curvature tensor of the background at the particle’s position, $u^\alpha$ is the particle’s 4-velocity and $\Delta \tau$, as previously, is a proper time interval along the particle’s past worldline.

The second result is due to Anderson and Hu [21]. They calculated the tail of the retarded Green function in the normal neighbourhood for a particle with a minimally coupled massless scalar field in a Schwarzschild background geometry. By using a Hadamard–WKB expansion for the Euclidean Green function, they are able to expand $V(\mathbf{x}, \mathbf{x}')$ to sixth order in the geodesic separation. They do not go on to calculate the quasi-local self-force explicitly (although that is clearly the motivation for their paper), but it is a relatively straightforward matter to do so. Because their results are somewhat unwieldy, we refer the reader to their paper for the actual result, although we will be reproducing the self-force derived from it for two simple particle motions below.

Most recently, the quasi-local part of the gravitational self-force for a particle in an arbitrary curved background has been calculated by Anderson, Flanagan and Ottewill [22]. They calculate the first two non-vanishing terms in the Taylor series, and find

$$f_{QL\sigma}(\tau, \Delta \tau) = -\mu^2 \left( \delta^\sigma_{\alpha} u_{\alpha} + u_{\alpha} u^\beta \right) C_{\beta \gamma \delta \lambda} C_{\sigma \rho \tau \delta} u^\beta u^\sigma \Delta \tau^2$$

$$+ \mu^2 \left( \delta^\sigma_{\alpha} u_{\alpha} u^\beta \right) u^\gamma u^\delta \left[ \frac{1}{2} C_{\gamma \mu \nu \lambda} C_{\mu \sigma \tau \beta} u^\nu u^\rho \Delta \tau^2 \right.$$  

$$+ \frac{1}{2} \left( \frac{1}{2} C_{\mu \nu \lambda} C_{\mu \nu \lambda} C_{\sigma \tau \beta} + C_{\mu \nu \lambda} C_{\sigma \tau \beta} C_{\mu \nu \lambda} C_{\sigma \tau \beta} \right) \Delta \tau^3 + O(\Delta \tau^4).$$

(10)

They also provide explicit expressions for some particle motions in Schwarzschild and Kerr geometries.

For the purposes of evaluating the convergence of these expansions, we will focus on the results of Anderson and Hu [21], both because the scalar field case is inherently the simplest and because it provides the highest order expansion with which to work. Unfortunately, Anderson and Hu do not take the derivative of $V(\mathbf{x}, \mathbf{x}')$ necessary to calculate the self-force contribution. Furthermore, their result is expressed in terms of coordinate expansion rather than an expansion in geodesic distance. It is only for relatively simple particle motions that one can easily recast coordinate distance into geodesic distance. Fortunately, it is in the spirit of our explorations here to take such simple cases, and we shall.

The fundamental question we ask in this section, then, is “what is the rate of convergence of the expansion for the quasi-local part the self-force as a function of $\Delta \tau$”? Clearly, as $\Delta \tau \to 0$, the expansion becomes exact at any order. The more interesting limit is that where $\Delta \tau$ approaches the boundary of the normal neighbourhood. It might be reasonable to expect that the expansion ceases to converge at all in that case, since that is the boundary of its domain of validity. Indeed, there is no fundamental reason why the domain of convergence of the series expansion could not be much smaller than the normal neighbourhood. Addressing this question, however, is slightly complicated by the fact that it is not immediately clear what the value of $\Delta \tau$ at the boundary of the normal neighbourhood is.

In the case of a Schwarzschild black hole spacetime, at least, one can get some insight into the normal neighbourhood boundary by studying null geodesics intersecting a particle in
Table 1. Values for a null geodesic leaving a particle in a circular geodesic orbit in Schwarzschild spacetime at radius $R$ and returning to the particle. $R_0$ is the minimal radial coordinate value of the null geodesic (minimum coordinated distance from the centre of the black hole) and $\Delta \tau$ is the proper time the particle has to wait for the null geodesic to return.

| $R/M$ | $R_0/M$ | $\Delta \tau/M$ |
|-------|---------|-----------------|
| 6     | 3.46    | 18.6            |
| 10    | 3.55    | 31.9            |
| 20    | 3.58    | 57.8            |
| 100   | 3.56    | 228             |
| 1000  | 3.54    | 2038            |

A circular geodesic at radius $R$. For such a particle, the angular displacement is related to the particle’s proper time by

$$\Delta \phi_{\text{particle}} = \sqrt{\frac{M}{R - 3M}} \Delta \tau, \quad (11)$$

where $M$ is the mass of the black hole. On the other hand, the angular deflection (as seen at $r = R$) of a null geodesic passing within a nearest distance $R_0 > 3M$ of the black hole is

$$\Delta \phi_{\text{photon}} = 2 \int_{1/R_0}^{1/R} du \sqrt{\frac{R_0^3}{R_0 - 2M - R_0^2u^2(1 - 2Mu)}}, \quad (12)$$

(note that if $R_0 \leq 3M$, there is no turning point for the geodesic and it does not return to larger radii). Finally, the proper time, as measured by the particle, for a photon to descend from a distance $R$ to within a distance $R_0$ of the black hole and return to a distance $R$ is given by

$$\Delta \tau = 2 \left( \frac{R - 2M}{R} \right) \int_{R_0}^{R} dr \frac{r}{r - 2M} \sqrt{\frac{r^3(R_0 - 2M)}{r R_0(r^2 - R_0^2) - 2M(r^3 - R_0^3)}}, \quad (13)$$

What we would like to find is the minimum value of $R_0$ (and hence $\Delta \tau$) for which a null geodesic intersecting the particle’s circular geodesic at a given $R$ can re-intersect it at the same $R$ but a different time. For a particle orbiting at $R \gg 3M$ from the black hole, the condition for this to occur is

$$\Delta \phi_{\text{particle}} + \Delta \phi_{\text{photon}} = 2\pi. \quad (14)$$

Assuming this and substituting equations (11)–(13) into equation (14), we can choose a value of $R$ and solve equation (14) numerically for $R_0$. We give some values obtained in this way in table 1.

There are a number of noteworthy features of table 1. First, at approximately $R_0 = 3.2M$, the angular deflection of the null geodesic is $2\pi$, and this therefore represents the value of $R_0$ corresponding to a particle orbit at $R \rightarrow \infty$. Next, we note that the values of $\Delta \tau$ take values between $\sim 3R$ for the closest orbits and $\sim 2R$ for the most distance orbits, asymptoting to $\Delta \tau = 2R$ in the limit $R \rightarrow \infty$. In other words, for orbits at large distances the time is dominated by the time for the photon to reach the black hole and return.

The $\Delta \tau$ quoted in table 1 represents approximate upper bounds on the extent of the normal neighbourhood along the past worldline of the particle since they demarcate two intersections of the particle’s geodesic with a null geodesic. Thus, for a particle at a fixed radius $R$, we need not worry about the convergence of the expansion of the quasi-local part of the self-force beyond $\sim 2R$ to the past.
Let us now consider two such expansions. First, we consider the expansion for a static particle at radius $R$ coupled to a minimally coupled massless scalar field by scalar charge $q$. We take the expression for $V(x, x')$ given by Anderson and Hu [21] and take partial derivatives with respect to the Schwarzschild coordinates. Next, we convert these coordinates into proper time using the coordinate parametrization

$$\Delta \phi = 0, \quad \Delta \theta = 0, \quad \Delta r = 0, \quad \Delta t = \sqrt{1 - \frac{2M}{r}}(\tau - \tau'),$$

which is appropriate for a static particle. We also, without loss of generality, set $\theta = \pi/2$. We can then integrate the expansion of $\nabla_\alpha V(x, x')$ with respect to proper time $\tau$ to obtain

$$f_{QL} = \frac{9}{2240} \frac{q^2 M^2}{R^{15}} (4R - 11M)(R - 2M)^5 \Delta \tau^5 + \frac{1}{3360} \frac{q^2 M^2}{R^{20}} (20R^3 - 195MR^2 + 598M^2 R - 585M^3)(R - 2M)^6 \Delta \tau^7 + O(\Delta \tau^8),$$

which is the only non-vanishing component of the quasi-local part of the self-force to the order of this expansion.

Denote the fifth and seventh order terms in equation (16) as

$$f_{QL,5} = \frac{9}{2240} \frac{q^2 M^2}{R^{15}} (4R - 11M)(R - 2M)^5 \Delta \tau^5,$$

$$f_{QL,7} = \frac{1}{3360} \frac{q^2 M^2}{R^{20}} (20R^3 - 195MR^2 + 598M^2 R - 585M^3)(R - 2M)^6 \Delta \tau^7.$$

Then the fractional truncation error which is induced by only taking terms in the series expansion of the quasi-local part of the self-force to order $\Delta \tau^7$ can be estimated as

$$\epsilon = \frac{f_{QL,7}}{f_{QL,5} + f_{QL,7}}.$$

This gives an estimate of the upper limit on the local truncation error (the error in truncating the next term in the series) rather than the more desirable global truncation error (the error in truncating all remaining terms in the series). Nonetheless, it is a standard measure of truncation error for this kind of analysis where neither the exact solution nor a form for the general term in the series is known, and is in any case the best estimate of truncation error we have for this series.

As noted previously, the truncation error can be expected to grow with $\Delta \tau$. We have calculated the error as a function of $\Delta \tau$ for particles located at $r = 6M, 10M, 20M$ and $100M$. The results are presented in figure 2. We see that, despite our concern that the quasi-local expansion might fail to converge at the boundary of the normal neighbourhood, it does, in fact, appear to converge everywhere within the normal neighbourhood. Furthermore, it converges quite well for particles at all radii, with estimated fractional error less than 0.1, back to almost $\Delta \tau = 10M$. For $R = 6M$, which is the closest particle that we consider and also the case for which the self-force should be most important in calculating templates for LISA, the fractional error is less than 0.1 more than halfway out to the boundary of the normal neighbourhood.

We can apply exactly the same method to a particle in a circular geodesic orbit at radius $R$ around Schwarzschild. In this case, the coordinates are related to proper time by

$$\Delta \theta = 0, \quad \Delta r = 0, \quad \Delta \phi = \frac{1}{R} \sqrt{\frac{M}{R - 3M}}(\tau - \tau'), \quad \Delta t = \sqrt{\frac{R}{R - 3M}}(\tau - \tau').$$

$$\Delta \phi = 0, \quad \Delta \theta = 0, \quad \Delta r = 0, \quad \Delta t = \sqrt{1 - \frac{2M}{r}}(\tau - \tau'), \quad (15)$$
Again, we have set $\theta = \pi/2$. In this case, there are three non-vanishing components of the quasi-local part of the self-force

$$f_{tQL} = \frac{q^2 M^3}{R^{12}} \sqrt{\frac{R}{R - 3M}} \frac{1}{(R - 3M)^3} \left[ \frac{3}{2240} R^3 (R - 2M) (R - 3M) (5R - 19M) \Delta \tau^4 + \frac{1}{13440} (56R^4 - 728R^3 M + 3461R^2 M^2 - 6990M^3 R + 5073M^4) \Delta \tau^6 + O(\Delta \tau^7) \right]$$

$$f_{rQL} = \frac{q^2 M^3}{R^{14}} \frac{(R - 2M)}{(R - 3M)^3} \left[ \frac{9}{11200} R^3 (R - 3M) (20R^3 - 189M R^2}{47040} (280R^5 - 4690R^4 M + 30780R^3 M^2 - 97302M^3 R^2 + 147777RM^4 - 86481M^5) \Delta \tau^7 + O(\Delta \tau^8) \right]$$

$$f_{\phiQL} = \frac{q^2 M^2}{R^{13}} \sqrt{\frac{M}{R - 3M}} \frac{(R - 4M) (R - 2M)}{(r - 3M)^3} \left[ \frac{9}{2240} R^3 (R - 3M) (3R - 7M) \Delta \tau^4 + \frac{1}{13440} (70R^3 - 685MR^2 + 1968M^2 R - 1731M^3) \Delta \tau^6 + O(\Delta \tau^7) \right]$$

We compute the estimated fractional error for each of these components in the same manner as we did for the static particle. The results are presented in figures 3–5. Again,
Figure 3. Estimated fractional error as a function of $\Delta \tau$ in the time component of the self-force on a charge coupled to a massless minimally coupled scalar field orbiting on a circular geodesic in a Schwarzschild spacetime. We show a particle at four different distances from the black hole. Upper bounds for the range at each distance are again approximately those from table 1.

Figure 4. Estimated fractional error as a function of $\Delta \tau$ in the radial component of the self-force on a charge coupled to a massless minimally coupled scalar field orbiting on a circular geodesic in a Schwarzschild spacetime. We show a particle at four different distances from the black hole.

we see convergence everywhere within the normal neighbourhood for all components of the quasi-local part of the self-force, and good convergence up to fairly high values of $\Delta \tau$. 

3. The contribution to the self-force from the distant past

In the previous section, we examined the contribution to the self-force from the portion of the worldline within the normal neighbourhood of the field point at \( z(\tau) \), i.e. from the first integral in equation (7). In this section, we will examine the contribution from the earlier part of the trajectory. In particular, we will examine the feasibility of computing the second integral in equation (7) using a mode sum expansion for the Green function.

3.1. The \( O[M] \) Green function

In studying the forces on a freely falling electric charge in a curved spacetime, DeWitt and DeWitt [23] developed some clever techniques for finding the Green function for a spherically symmetric spacetime with mass \( M \) at the centre. They give an approximate expression—accurate to leading order in the central mass—for the Green function for an electric charge in this Schwarzschild-like spacetime. This method was later extended by Wiseman [24] and Pfenning and Poisson [25] to the case of a scalar charge. The Green function is found to be

\[
G(x, x') = \frac{\delta[t - t' - |x - x'|]}{|x - x'|} + M \frac{\partial}{\partial t'} \left\{ 2 \delta[t - t' - |x - x'|] \right\} + O[M^2],
\]

where \( \delta \) is the Dirac delta function. The first term is the retarded Green function in Minkowski (flat) spacetime. The logarithmic term is a correction to the retarded Green function in Minkowski (flat) spacetime. The logarithmic term is a correction to the light cones. The final term is the ‘tail’ of the Green function. Note that this

\[ \text{This term reflects the fact that the light cones are bent in curved spacetime. It was not included in the DeWitt–DeWitt [23] calculation.} \]
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term only contributes prior to \( t' = t - (r + r') \), the light reflection time. This is depicted in figure 1. This gives a strong indication that the self-force is dependent on the portion of the worldline that is outside the normal neighbourhood.

Multiplying by spherical harmonics and integrating over the solid angle, we can obtain the angular mode decomposition of this Green function

\[
G(x, x') = \sum_{l=0}^{\infty} \frac{(2l+1)}{2rr'} \left\{ \left[ 1 + 2M \frac{\partial}{\partial t'} \ln \left( \frac{r + r' + (t - t')}{r + r' - (t - t')} \right) \right] \right.
	\times P_l(\xi) \Theta[t - t' - |r - r'|] \Theta[r + r' - (t - t')]
	+ 4M \frac{\partial}{\partial t'} Q_l(\xi) \Theta[t - t' - (r + r')] \right\} P_l[\cos(\gamma)] + O[M^2],
\]

(25)

where \( P_l \) and \( Q_l \) represent Legendre functions and

\[
\xi = \frac{r^2 + r'^2 - (t - t')^2}{2rr'} = \cos \beta
\]

(26)

\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')
\]

(27)

and \((t, r, \theta, \phi)\) and \((t', r', \theta', \phi')\) are the spherical coordinates of the field point and source point, respectively. The angle \( \beta \) is shown in figure 6. In equation (25), it is understood that the first partial derivative not only operates on the natural log, but also on \( P_l(\xi) \) and the \( \Theta \)-functions.

In numerical implementations, we would only be able to include a finite number of multipoles. In this case, we can avoid doing a numerical summation by computing the partial sum analytically, i.e.

\[
G(N; t, r, t', r', \cos \gamma) = \sum_{l=0}^{N} \text{terms in equation (25)}
\]

(28)

As above, the first partial derivative acts on the natural log as well as the Legendre functions and the \( \Theta \)-functions.

This formula can now be used in the second integral in equation (7). As in section 2, we focus our attention on a static source at radius \( R \) and \( \cos \gamma = 1 \) (see figure 6). Note that the final term—the tail term—is a total derivative. When this is substituted into equation (7), we can integrate by parts and see immediately that the tail contribution to the force vanishes. This is just a re-confirmation of the well-known result that there is no self-force on a static scalar charge in Schwarzschild spacetime (cf Wiseman [26]). By the same argument, the natural log term also gives no contribution to the force integral.
world-line of the central mass, $M$

Figure 6. The point charge only has an instant of causal contact with the field point $(t, r)$ when the past light cone intersects the worldline of the charge; however when using multipoles, the point charge is smeared over a sphere and therefore has an extended period of interaction. In equation (29) we have placed the field point on the worldline of the source, i.e. $\gamma = 0$ and $r = b$. In Figure 7 we show how the gradient of the field ‘builds up’ for a partial sum of multipoles as we integrate over $t'$. In computing the numbers in Table 2, we have again placed the field point on the worldline of the source and have integrated from $\beta = \pi$ to $\beta = \pi/2$, i.e. $\cos \beta \in [-1, 0]$.

Substituting the remaining term—the flat-space portion of equation (28)—into equation (7), a straightforward calculation reveals that the first $N$ multipoles of the source give a radial component of the force

$$f_r(\Delta \tau, N) = -\frac{q^2}{R^2 u'} \frac{(N + 1)}{4 \sqrt{2}} \int_{-1}^{\cos \beta_{\text{max}}} \frac{P_{N+1}(\xi) - P_N(\xi)}{(1 - \xi)^{3/2}} \, d\xi,$$

where $u' = 1 - M/b + O(M^2)$ is the leading order contribution to the time component of the 4-velocity of the static charge. The upper limit is defined as

$$\cos \beta_{\text{max}} \equiv 1 - \frac{1}{2} \left( \frac{\Delta \tau}{u' b} \right)^2.$$
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Figure 7. This shows the accumulation of the field gradient as we integrate over $\tau'$. The normalization is $(M^2 u'/q^2) f_r(\Delta\tau, N)$. Here $N$ is chosen to be 50. If we had chosen a larger $N$, the frequency of oscillation in the plot would increase, but the envelope of the oscillation would become smaller. The numbers in table 2 are a measure of how fast the envelope shrinks as the number of modes is increased. For example the entry for 50 modes is just the value of the peak near $\cos \beta_{\text{max}} = 0$.

The only portion of the Green function that contributed to equation (29) was the flat-space term. Since the static point particle would never intersect the past light cone of a field point located on its own worldline, we can ask: why does $f_r(\Delta\tau, N)$ not vanish identically? The answer lies in the fact that we are using only a finite number of modes to describe the field gradient.

The decomposition of the field into angular modes can also be thought of as a mode decomposition of the source. In the case of the static point particle, we are, in effect, replacing the point charge with a sum of spherical shells of charge where each shell has radius $R$ and an angular distribution of charge smeared on it. The point particle is only recovered when a large (infinite) number of shells of charge are summed. As is evident from figure 6, although the point charge does not come into null contact with the source point, the spherical shells of charge do intersect the past light cone of the field point. If we terminate the integral at some $\Delta\tau > 0$, but include an infinite number of terms in the summation (i.e. $N \to \infty$), we will recover the point-particle nature of the source, and therefore there would be no contribution to the force. However, in any practical calculation, we will need to terminate at some finite value of $N$, and thus we will be left with an unwanted, and inescapable, contribution to the force. The key question is can we include enough terms in the summation to squeeze this unwanted contribution to an acceptably low level?

Figure 7 shows how the field gradient accumulates as we integrate over $\tau'$ for a fixed number of modes. The oscillations are artefacts of the finite number of modes summed, and it is the envelope of the function which reflects the level of accuracy that can be achieved. If we are to accurately evaluate the value for the self-force, we must squeeze the envelope of this function below the value of the self-force. For example, on dimensional grounds, one might expect there to be a radial component of the self-force for a static scalar charge in Schwarzschild spacetime of the form

$$f_r = \frac{\lambda q^2 M}{b^3}.$$  (31)
Table 2. Table showing the convergence of the mode sum and the accuracy to which we could use it to compute the gradient appearing in the integrand of equation (7). The first column is the number of modes included in the sum. The integral is terminated at $\Delta \tau$ chosen such that $\beta_{\text{max}} = 0$. The charge is located at $R = 6M$. This value in the second column is the peak nearest $\cos \beta_{\text{max}} = 0$ on plots similar to figure 7. The right column shows the accuracy with which you could constrain the coefficient of a self-force of the form $2\mu^2 M^3/b^3$ with the number of modes used.

| Modes | $(M^2 u'/q^2) f_\tau(\Delta \tau, N)$ | Bound on $\lambda$ |
|-------|------------------------------------|-------------------|
| 10    | 0.0017                             | 0.36              |
| 50    | 0.00078                           | 0.17              |
| 100   | 0.00056                           | 0.12              |
| 200   | 0.00041                           | 0.088             |
| 300   | 0.00036                           | 0.078             |

where $\lambda$ is an unknown coefficient we would be trying to dig out from under the unwanted contribution to the force from our finite mode sum (in the electrostatic case, there is a finite contribution to the self-force of exactly the form shown in equation (31) with $\lambda = 1$). The amplitude of the envelope is therefore an indication of the bound we place on the contribution to the self-force. In the present case, we know the solution vanishes identically, and evaluating equation (16) from section 2 at $R = 6M$, it can be seen that the first integral in equation (7) gives only a tiny contribution to the self-force of a static scalar charge, even when $\Delta \tau$ is extended out to near the edge of the normal neighbourhood. Therefore, in this case, the amplitude of the envelope is nothing but the approximate accuracy to which we have calculated the self-force.

If we choose $\Delta \tau$ such that $\cos \beta_{\text{max}} = 0$, which is about half-way out to our upper bound on the edge of the normal neighbourhood, then the we see from table 2 that using 200 multipoles we can constrain the $\lambda$ to be less than 0.1.

4. Conclusions

Our goal in this paper was to assess the feasibility of the Poisson–Wiseman matched expansion scheme for calculating the self-force. This scheme involves splitting the non-local term in the self-force (which is an integral over the particles worldline $z(\tau)$) at some proper time $\tau - \Delta \tau$. For the $-\infty < \tau' < \tau - \Delta \tau$ (distant) part of the integral, the integrand is expanded as a mode sum. For the $\tau - \Delta \tau < \tau' < \tau$ (quasi-local) part, the integrand is expanded in a covariant Taylor series, which requires $z(\tau - \Delta \tau)$ be in the normal neighbourhood of $z(\tau)$. For the resulting self-force to be accurate, the split must be done in such a way that each expansion reaches sufficient accuracy with a finite and feasible number of terms. Until now, it has not been clear that such a split even existed.

For this preliminary investigation, we have studied the simplest scenarios: minimally coupled massless scalar field, Schwarzschild geometry, static particles and circular geodesic motion. Our results are somewhat surprising. The quasi-local expansion appears to converge well—when the particle is at $6M$ and expanding to order $\Delta \tau^2$, we achieve an estimated truncation error of the order of a few per cent for $\Delta \tau \sim 6M$. This would naively have seemed to us enough of a buffer between the mode-sum integral and the singularity to allow rapid convergence of the mode sum as well.

However, we have found the convergence of the mode sum to be poor. This is because the mode sum smears the charge of the particle over spheres of finite radius which extend outside of the normal neighbourhood. Thus, even the flat-space term in the Green function, which cannot contribute to the self-force for a scalar particle, seems to contribute at every
order. For our simple case of a static particle at $6M$ and $\Delta \tau \sim 6M$, going from 10 modes to 100 increased our accuracy by only a factor of 3.

Clearly, we need a way to speed convergence of mode sum. In the case presented, one could have achieved infinite accuracy at every order by regularizing the modes, which would have removed the flat-space portion of the Green function. It might, therefore, be possible to speed the convergence of the mode sum by regularizing the modes. This would be ironic, since this method is supposed to provide an alternative to mode regularization. Nonetheless, it might be that combining regularization with a two expansion approach will provide much better convergence than either alone. Alternatively, we note in figure 7 that the mode-sum integrand oscillates about its true value. This leads us to speculate that averaging over these oscillations would lead to much quicker convergence of the mode-sum integral.

While we believe our convergence analysis is general, and will apply beyond the simple scenarios we have explored, we would be remiss in not pointing out that there are additional issues for higher spin fields. The foremost of these is the issue of gauge—in order to meaningfully combine the quasi-local and more distant self-force contributions, they will need to be expressed in the same gauge. For instance, the quasi-local contribution for the gravitational field [22] has only be derived in the Lorentz gauge, where no expression for the modes is available. We note, however, that this is a generic difficulty for gravitational self-force calculations because the formal expressions for the self-force are themselves derived in the Lorentz gauge [7, 8], and we are heartened by current progress in understanding gauge transformations in this context [27].

In any case, we feel that these preliminary results are promising enough to warrant further investigation. A good figure of merit for calculational schemes like this is accuracy per floating point operation. As is, this approach would seem to lag behind approaches like mode-sum regularization when measured on this scale. Nonetheless, it might, for instance, provide confirmation for results from other approaches. Further, as mentioned above, it is still possible that this calculational scheme, with refinements, could rival or exceed in computational efficiency other known schemes, especially since the quasi-local expansion, once calculated, can be applied with little further effort to any spacetime or particle motion. In the meantime, this approach remains, in our opinion, in the category of ‘promising but possessing some technical challenges’.

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