Hamiltonian LGT in the complete Fourier analysis basis.

G. Burgio\(^a\), R. De Pietri\(^b\), H. A. Morales-Técoh, L. F. Urrutia\(^d\) and J. D. Vergara\(^d\).

\(^a\)Dipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, Parma, Italy
\(^b\)Centre de Physique Théorique CNRS, Case 907 Campus de Luminy, F-13288 Marseille Cedex 9, France
\(^c\)Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, A. Postal 55-534, 09340 México D.F.
\(^d\)Departamento de Física de Altas Energías, Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, 04510 México D.F.

The main problem in the Hamiltonian formulation of Lattice Gauge Theories is the determination of an appropriate basis avoiding the over-completeness arising from Mandelstam relations. We short-cut this problem using Harmonic analysis on Lie-Groups and intertwining operators formalism to explicitly construct a basis of the Hilbert space. Our analysis is based only on properties of the tensor category of Lie-Group representations. The Hamiltonian of such theories is calculated yielding a sparse matrix whose spectrum and eigenstates could be exactly derived as functions of the coupling \(g^2\).

Preprint UPRF-99-12, September 1999. To appear in Nucl. Phys. Proc. Suppl., LATTICE99.

1. HAMILTONIAN LATTICE GAUGE THEORY

The Hamiltonian formalism for lattice gauge theories (LGT) in \((d+1)\) dimensions is constructed associating gauge field variables \(U_k(x)\) to each link \((x, x + \alpha e_k)\) of a hypercubic periodic lattice of period \(aL\). The corresponding Hilbert space \(\mathcal{H}\) is defined by the gauge invariant square integrable functions \(\psi(U) = \psi(U') = \psi(U_k(x))\) on the tensor product of \(d \cdot L^d\) copies of the gauge group \(G\). Gauge transformations act as \(U_k(x) \mapsto U_k'(x) = \gamma^{-1}(x + \alpha e_k)U_k(x)\gamma(x)\). The variables conjugated to \(U_k(x)\) are the outgoing/ingoing electric fields \(E^\alpha_{\pm k}(x)\) from the lattice point \(x\) in the directions \(e_{\pm k}\).

The standard Hamiltonian operator is

\[
\hat{H} = \frac{g^2}{2a^{d-2}} \sum_{x,k} q_{\alpha\beta} E^\alpha_k(x) E^\beta_k(x) + \sum_P V(U_P),
\]

where \(q_{\alpha\beta}\) is the Cartan metric, the sum over \(P\) ranges over all unoriented plaquettes, \(U_P\) is the plaquette variable and

\[
V(U_P) = \frac{a^{d+4}}{g^2} \left[ 1 - \frac{U_P + U_P^*}{2\text{dim}(U)} \right].
\]

Such choice is not unique since the only condition on the magnetic term potential is \(V(U_P) \simeq \frac{g^2}{2} \text{Tr}[F_P^2]\).

2. THE SPIN NETWORK BASIS

A classical result of representation theory gives a nice way of constructing a basis of LGT Hilbert space. In fact, the set \(R_G = \{ R^j | j \in J[G] \}\) of all the unitary inequivalent representations of a compact group \(G\) is numerable and all the representations \(R^j\) are finite dimensional. Choosing an orthonormal basis for each representation \(R^j\), the matrix elements \(D^{(j)\alpha^j}(U) (\alpha, \beta = 1, \ldots, \text{dim}(R^j))\) of all the representations \(R^j\) are a numerable orthonormal basis of \(L^2(G, dU)\). This result, known as the Peter-Weyl theorem, implies that each vector of \(\mathcal{H}\) can be written as

\[
\psi(U) = \prod_{x=1}^d \sum_{k \in J[G]} \sum_{\alpha^k=1}^{\text{dim}(j^k)} D^{(j^k)\alpha^k}(U) \times c^{(j_{1\alpha^1}\cdots j_{N\alpha^N})\beta_0\cdots\beta_{N\alpha^N}}_{\alpha_0\cdots\alpha_{N\alpha^N}}.
\]
where only gauge invariant combinations should be taken into account.

The implementation of gauge invariance turns into a set of constraints on the coefficients $c$. In particular the $c$'s should factorize in products of group invariant tensors associated to the different lattice sites $\mathbf{x}$.

The concept of invariant tensor is better expressed by the notion of intertwining operators. By definition, an operator $I$ connecting the Hilbert space of two representations, $\mathcal{R}$ and $\mathcal{R}'$, is an intertwining operator if $I \cdot T(U) = T'(U) \cdot I$, for every $U$ in $G$. The set of all intertwining operators $\mathcal{I}(\mathcal{R}, \mathcal{R}')$ is a vector subspace of all the linear operators connecting the Hilbert space of the two representations $\mathcal{R}$ and $\mathcal{R}'$. This concept gives the coordinate free definition of the generalized Clebsh-Gordan coefficients of Yutsis-Levinson-Vanagas which are the matrix elements of these operators on the chosen basis. The integral of the product of $K$ representations decomposes according to

$$
\int dU \prod_{k=1}^{K} D^{\alpha_k}_{\beta_k}(U) = \sum_{\pi} \frac{\prod_{k=1}^{K} \dim(j_k)}{\prod_{k=1}^{K} \dim(j_k)} \prod_{k=1}^{K} \left| \psi_{j_k}(U) \right|^2 \delta_{\alpha_k}^{\beta_k}(U_k) \right]
$$

where the only non diagonal terms are given by the expectation values of the plaquette operator. These are given in equation (26) of Ref.[3] as traces of intertwining operators. In this way the computation amounts to the evaluation of specific Wigner’s $nJ$-symbols, that further reduce to $6J$-symbols only.

An important property of the Hamiltonian matrix in the spin-network basis is that it is sparse.

![Figure 1. A spin network in the case of a 2 dimensional lattice is parametrized by an irreducible representation associated to each link $j^1_2$ and an irreducible representation $\pi^1_k$ parametrizing the irreducible tensor associated to each lattice site.](image)

3. MATRIX ELEMENTS OF THE HAMILTONIAN OPERATOR

Computing the action of the Hamiltonian operator ($\hat{H}$) on the spin-networks basis simply reduces to tracing intertwining operators. In fact, the basis vectors ($\pi$) are eigenstates of the kinetic term, while the potential (magnetic) term is realized as a multiplicative operator. Explicitly

$$
\langle j', \pi' | \hat{H} | j, \pi \rangle = \frac{-a^{d-4}}{2g^2 \dim(U)} \sum_{\gamma} \sum_{r<s=1}^{\gamma} \times

\times \left( \langle j', \pi'| U_{y, r,s} | j, \pi \rangle + \langle j, \pi | U_{y, r,s} | j', \pi' \rangle \right)

+ \left( \frac{g^2}{2a^{d-2}} \sum_{k=1}^{d} \sum_{k'=1}^{d} C^2_{2}[j^2_k] + \frac{a^{d-4}}{g^2 N^2} \right) \delta^2_{j} \delta^2_{\pi},
$$

where the only non diagonal terms are given by the expectation values of the plaquette operator.
un-primed $j^+_1, j^{+2}, \pi^+_1, j^+_2, \pi^+_2, j^+_{1+1}, j^+_{2+1}$ differ by a half integer for $x = y$, being zero otherwise. The explicit expression of the matrix elements of the plaquette operator are:

$$\langle j', \pi' | U_{y,1.2} | j, \pi \rangle = \frac{(-1)^{n} \sum_{i=1}^{n} (|\epsilon_i - \epsilon_{i+1}| + 2)}{\sqrt{\prod_{i=1}^{n} (2X_{y}^i + 1) (2Y_{y}^i + 1)}} \times \prod_{i=1}^{n} R \left[ \begin{array}{c} X_{y}^i, \ Y_{y}^i, \ X_{y}^{i+1}, \ Y_{y}^{i+1}, \ C_{y}^i \end{array} \right]$$

where $\epsilon_i = X_{y}^i - Y_{y}^i = \pm \frac{1}{2}$,

$$X_{y}^i = \frac{j^+_i}{x} \ , \ Y_{y}^i = \frac{j^{+i}}{x} \ , \ C_{y}^i = \pi^i_{x} \ ,$$

$$X_{y}^i = \frac{\pi^i_{x+2}}{x} \ , \ Y_{y}^i = \frac{\pi^{i+2}}{x} \ , \ C_{y}^i = j^i_{x+1} \ ,$$

$$X_{y}^i = \frac{j^i_{x+1}}{x} \ , \ Y_{y}^i = \frac{j^{+i+1}}{x} \ , \ C_{y}^i = \pi^{i+1}_{x+2} \ ,$$

and $R \left[ \begin{array}{c} X_{y}^i, \ Y_{y}^i, \ X_{y}^{i+1}, \ Y_{y}^{i+1}, \ C_{y}^i \end{array} \right]$ is equal to

$$\sqrt{\frac{1 - 2C^i_{y} + X_{y}^i + X_{y}^{i+1} + Y_{y}^i + Y_{y}^{i+1}}{2} + \frac{3 + 2C^i_{y} + X_{y}^i + X_{y}^{i+1} + Y_{y}^i + Y_{y}^{i+1}}{2}}$$

for $|\epsilon_i - \epsilon_{i+1}| = 0$ and

$$\sqrt{\frac{1 - 2C^i_{y} - X_{y}^i - X_{y}^{i+1} + Y_{y}^i - Y_{y}^{i+1}}{2} + \frac{3 - 2C^i_{y} - X_{y}^i - X_{y}^{i+1} - Y_{y}^i - Y_{y}^{i+1}}{2}}$$

for $|\epsilon_i - \epsilon_{i+1}| = 1$

4. CONCLUSIONS

In this work we have shown how harmonic analysis on compact Lie-Groups together with intertwining operators formalism provide a useful basis of the Hilbert space of LDT and an explicit expression for matrix elements of the Hamiltonian operator. Moreover, the latter can be expressed in terms of Wigner’s $nJ$-symbols. Such elements are well known for $SU(2)$ (see for example [3]). Explicit values for the $nJ$-symbol for unitary groups are found in [3].

Our results are derived in terms of the knowledge of: 1) the full set of irreducible representations of the group, 2) a basis on the space of intertwining operators. The generalization to gauge theories coupled to matter is therefore straightforward.

The construction does not depend on the gauge group but only on its tensor category properties.

5. ACKNOWLEDGMENTS

We thank E. Onofri and F. Di Renzo for helpful and enlightening discussions. The work of R.D.P. is supported by a Dalla Riccia Fellowship. Partial support from grants CONACyT No. 3141-P-E9608 and DGAPA-UNAM IN100397 is also acknowledged.

REFERENCES

1. J. Kogut and L. Susskind, Phys. Rev. D 11 (1975) 395.
2. K.G.Wilson, Phys. Rev. D 10 (1974) 2445.
3. A. P. Yutis, I. B. Levinson, and V. V. Vangas, Mathematical Apparatus of the Theory of Angular Momentum, Israel Program for Scientific Translations (Jerusalem, 1962).
4. G. Burgio, R. De Pietri, H. A. Morales-Técotl, L. F. Urrutia, and J. D. Vergara, Nucl. Phys. B (1999), to appear. (hep-lat/9906035).
5. P. Cvitanović, Group Theory, Nordita Lecture Notes in Physics, (Copenhagen, 1994).
6. N. Ja. Vilenkin and A. U. Klimyk, Representation of Lie Groups and Special Functions, volume I, Kweker Academic Publisher (Dordrecht, The Netherland, 1993).
7. D. M. Brink and G. R. Satchler, Angular Momentum, Claredon Press (Oxford, UK, 1968).
8. J. J. De Swart. Rev. Mod. Phys. 35 (1963) 916. L. A. Shelepkin, Z. Eksper. Teor. Fiz. 48 (1965) 360. F. J Archer, Phys. Lett. 295B (1992) 199. N. Ja. Vilenkin and A. U. Klimyk, Representation of Lie Groups and Special Functions, Recent Advances, Kweker Academic Publisher (Dordrecht, The Netherland, 1995).