Minor Invertible Products Assignment and Sparse Hyperdeterminants

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Abstract

We consider an extension of Minor Assignment Problems derived from the determinantal expansion of matrix products, under the condition that the terms of the expansion are units of $\mathbb{C}(t)$. This restriction places constraints on the sparsity and the factorization properties of a family of hyperdeterminants derived from Grassmann-Plücker relations.

We find minimal conditions guaranteeing that allowed assignments returning a determinantal expansion are the trivial ones, i.e., those induced by the action of a diagonal matrix of Laurent monomials on a pair of constant matrices. Counterexamples are provided when such conditions do not hold. Connections with the characterization of forbidden configurations in a different combinatorial context, as well as potential applications to statistical modeling and choice theory, are also discussed.

1 Introduction

1.1 Aim of the paper

The study of relations between minors of matrices belonging to a given class is a problem of both theoretical and applied interest. Algebraic relations between minors have been extensively analyzed [13, 15, 20], and they play a major role in biochemical modeling [10] and algebraic statistics, e.g., partial correlation analysis [18, 24, 4].

The present work extends this investigation, focusing on the assignment of minor products, rather than individual minors, under specific algebraic constraints. This problem arises from the deformation of the terms in the determinantal expansion

$$\det(L \cdot R) = \sum_{\mathcal{I} \in \wp_k[n]} \Delta_L(\mathcal{I}) \cdot \Delta_R(\mathcal{I})$$

where $L, R^T \in \mathbb{C}^{k \times n}$, $k \leq n$, $[n] := \{1, \ldots, n\}$, $\wp_k[n] := \{\mathcal{I} \subseteq [n] : \#\mathcal{I} = k\}$, and $\Delta_L(\mathcal{I})$ (respectively, $\Delta_R(\mathcal{I})$) denotes the maximal minor of $L$ extracted from columns (respectively, rows of $R$) indexed by $\mathcal{I} \subseteq [n]$. Geometric properties of the expansion (1.1) have proved relevant for the study of negative dependence properties [5], determinantal representability in relation to subspace arrangements [9], and subspace learning through Grassmann kernels [14].

Introducing the matroid of non-vanishing maximal minors of $L$ [21] as

$$\mathcal{G}(L) := \{\mathcal{I} \in \wp_k[n] : \Delta_L(\mathcal{I}) \neq 0\}$$
and assuming $\mathcal{G}(R) = \varphi_L[n]$, we consider a deformation of (1.1) as a map

$$I \in \mathcal{G}(L) \mapsto \Delta_L(I) \cdot \Delta_R(I) \cdot c_T \cdot t^{e(I)} \tag{1.3}$$

where $e(I) \in \mathbb{Z}^d$, $d \in \mathbb{N}$, $c_T \in \mathbb{C}^\times$, and $t$ is a $d$-tuple of indeterminates, and $t^e := \prod_{i=1}^d t_i^{e_i}$. Such deformations define a minor assignment in the unit group of the ring $\mathbb{C}[t] := \mathbb{C}[t, t^{-1}]$ of Laurent polynomials in $t$; specifically, they form a group that can be identified with

$$\left(\mathbb{C}(t)^\times\right)^{\#(\mathcal{G}(L))} \cong \left(\mathbb{C}^\times \times \text{Hom } \left((\mathbb{C}^\times)^d, \mathbb{C}^\times\right)\right)^{\#(\mathcal{G}(L))}. \tag{1.4}$$

Then, we look for those deformations that return another determinantal expansion. A trivial attribution of exponents $e(I)$ meeting this requirement follows from the invariance of terms in the expansion (1.3) under

$$(L(t), R(t)) \mapsto (L(t) \cdot D(t)^{-1}, D(t) \cdot R(t)) \tag{1.5}$$

where $D(t)$ is a generalized $n$-dimensional permutation matrix dependent on $t$. Indeed, based on (1.5), we get a determinantal expansion of the type (1.3) choosing

$$L(t) \in \mathbb{C}^{k \times n}, \quad R(t) = \text{diag}(d(t)) \cdot R(1) \tag{1.6}$$

where $d(t)$ is a vector of monic monomials or, equivalently, an element of

$$\left(\text{Hom } \left((\mathbb{C}^\times)^d, \mathbb{C}^\times\right)\right)^n. \tag{1.7}$$

The action of a diagonal matrix $\text{diag}(d(t))$ on the tuple of terms in (1.1) can also be expressed adapting the homomorphism that defines the toric ideal of the matroid $\mathcal{G}(L)$ [17]

$$\varphi_L : \mathbb{C}[y_I : I \in \mathcal{G}(L)] \longrightarrow \mathbb{C}[x_\alpha : \alpha \in [n]], \quad \varphi_L(y_I) := \prod_{\alpha \in I} x_\alpha. \tag{1.8}$$

Indeed, each $n$-tuple of monic monomials $d(t)$ defines a homomorphism

$$\vartheta_d : \mathbb{C}[x_\alpha : \alpha \in [n]] \longrightarrow \mathbb{C}(t), \quad \vartheta_d(x_\alpha) := d(t)_\alpha, \alpha \in [n] \tag{1.9}$$

so we can specify the special deformations (1.6) through the image of the indeterminates $y_I$, $I \in \mathcal{G}(L)$, under the composition $\vartheta_d \circ \varphi_L$ for some $d(t)$. Such deformations can be further combined with a permutation of $[n]$ acting on both the rows of $R(t)$ and the columns of $L(t)$, while preserving the form (1.3).

Our aim is to investigate minimal conditions on the independence structure defined by (1.2) ensuring that such trivial choices, entailing the reduction from (1.4) to (1.7), are the only feasible one.

### 1.2 Motivations and related work

This paper continues the research started in [2] and extended in [3] to explore the combinatorial properties of such deformations and their potential applications, with special regard to algebraic criteria for complexity reduction in different combinatorial systems (sign configurations in [2], permutations of a set system in [3]). These works were originally motivated by the study of combinatorial aspects of Wronskian $\tau$-functions of the bilinear Kadomtsev-Petviashvili II (KP II) equation [16]. These soliton solutions of the bilinear KP II equation can be expanded, using (1.1), as a combination of exponentials, which is central for the analysis of their tropical limit [16, 2]. A logarithmic transformation of variables
converts soliton solutions into Laurent polynomials, where each term in (1.1) is a monomial.

Together with the distinguished role of monomials in algebraic modeling of statistical independence, biochemical reactions [10], and enumerative combinatorics [24], the relations with a special class of soliton solutions bring to the choice (1.3) in our investigation, which also allows us to draw a connection between the Principal Minor Assignment Problem (and its extension, as mentioned in the previous subsection), and the factorization properties of sparse polynomials (we refer to [23] for more details on this topic). Sparse polynomials arise from the combination of Grassmann-Plücker relations for the two matrices \( L(t), R(t) \) due to the choice of monomials derived from (1.4) and (1.7) instead of \( \mathcal{C} (y_L : I \in \mathfrak{G}(L))^\times \) and \( \mathcal{C} (x_\alpha : \alpha \in [n])^\times \), respectively.

This exploration of determinantal expansions, although originates from dynamical systems, concentrates on algebraic conditions to encode information. In particular, [3] introduces a framework to assess the complexity reduction for \( \mathbb{Z}^d \)-valued set functions through the implications of determinantal constraints on the degrees of the terms (1.3); as a special case, when the set function takes values in \{0, 1\}^d and encodes a permutation of subsets of \{1, \ldots, n\} with \( k \) elements, the constraints help check for and identify a permutation of the base index set \{1, \ldots, n\} generating the original set function. This lets us realize an inverse map of (1.3) between monomials and the corresponding subsets of \{1, \ldots, n\} labeling them. Such an approach is suitable for applications involving uncertainty modeling and may be related to recent research on linear regression with shuffled data [22] and unlabeled sensing [25].

1.3 Contribution and organization of the paper

The extension of such a framework discussed here takes into account the effects of the information content encoded in the matroid \( \mathfrak{G}(L) \). The analysis in this work identifies a local and structural property that allows recovering the global integrability, i.e., Assumption 11 stated in Subsection 2.2: this condition is local since it refers to a submatrix of \( L(1) \) and is independent of the full structure \( \mathfrak{G}(L(1)) \); it is structural since it is independent of both the set function \( \Psi \) in (1.10) and the values of non-vanishing entries of \( L(t) \).

Anticipating Remark 10, the role of Assumption 11 in our context is similar to that of graphs \( K_{3,3} \) and \( K_5 \) in Kuratowski’s characterization of planar graphs (see, e.g., [21, Thm. 2.3.8]). This is formalized in Theorem 1 stated below, which will be proved through a more general result (Theorem 27).

**Theorem 1.** Let \( L(t), R(t) \) be two matrices of complex functions of \( d \) indeterminates \( t \) with \( \mathfrak{G}(R(1)) = \psi_k[n] \), and \( \Psi : \mathfrak{G}(L(1)) \rightarrow \mathbb{Z}^d \) be a map satisfying

\[
\Delta_{L(t)}(I) \cdot \Delta_{R(t)}(I) = \Delta_{L(1)}(I) \cdot \Delta_{R(1)}(I) \cdot t^{\Psi(I)}, \quad I \in \mathfrak{G}(L(1)).
\]

(1.10)

Then, Assumption 11 (see below) guarantees the existence of an element \( m_0 \in \mathbb{Z}^d \) and a map \( \psi : [n] \rightarrow \mathbb{Z}^d \) such that

\[
\Delta_{L(t)}(I) \cdot \Delta_{R(t)}(I) = t^{m_0} \cdot \Delta_{L(1)}(I) \cdot \Delta_{R(1)}(I) \cdot \prod_{\alpha \in I} t^{\psi(\alpha)}, \quad I \in \psi_k[n].
\]

(1.11)

This means that, under Assumption 11, the pair \((L(t), R(t))\) induces the same expansion (1.1) as \((L(1), \text{diag}(t^{\psi(\alpha)})_{\alpha \in [n]} \cdot R(1))\), apart from a common unit \( t^{m_0} \) that is irrelevant in terms of Plücker coordinates.

The proofs provided to demonstrate Theorem 1, which rely on basic combinatorial and algebraic arguments, guide us in the construction of counterexamples where the information content provided by \( \mathfrak{G}(L(1)) \) does not suffice to guarantee (1.11), elucidating the minimality of Assumption 11. Here, the notion of minimality is considered with respect to the partial order on the set of matroids of non-vanishing
minors by set-theoretic inclusion.

When specific evaluations $t_0$ of the variables $t$ are used to rank the elements of $\mathcal{G}(L(1))$ based on the norm of the corresponding terms (1.10), these counterexamples and, more generally, the lack of a form (1.11) may generate incompatible rankings for the base set $\{1, \ldots, n\}$. In fact, an allowed configuration of exponents defines, for a given generic value $t_0$, the coordinates of two points in two projective spaces, namely, $\mathbb{CP}^{n-1}$ and the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$. At $\|t_0\| \to \infty$, this lets us explore consistency between two orders on $\{1, \ldots, n\}$ and $\mathcal{G}(L(1))$, respectively, induced by comparison of the coordinate norms. This point, which relates to the inequivalence between tropical structures discussed in [1], is examined in Section 7.1.

The rest of the paper is organized as follows: in Section 2, after setting the main notation adopted throughout the paper, we specify the assumptions of our model. In Section 3, we state preliminary lemmas practical for the subsequent proofs. The effect of the invertibility constraint (1.10) is analyzed in detail in Section 4. In Section 5, the results of this analysis let us identify a basis in $\mathcal{G}(L(1))$ satisfying a precondition, which involves $\mathcal{R}(t)$, that leads to the existence of the representation (1.11): in our notation, we refer to such a base as integrable. In Section 6, we show how the integrability property is transferred among bases under weaker assumptions than those considered in the previous sections. Counterexamples based on the deviation from the model assumptions are presented in Section 7; following the conclusions drawn in Section 8, we present a Mathematica code for symbolic verifications of such examples in Appendix A.

\section{Notation and assumptions}

\subsection{Notation}

We set $\mathcal{G}(L) := \mathcal{G}(L(1))$ and

$$I^\alpha_{\alpha_1, \alpha_2, \ldots} := I \setminus \{i_1, i_2, \ldots\} \cup \{\alpha_1, \alpha_2, \ldots\}, \quad i_1, i_2, \ldots, \in I, \alpha_1, \alpha_2, \ldots \in I^C$$

(2.1)

where $I^C := [n] \setminus I$. The matroid $\mathcal{G}(L)$ satisfies the exchange relation [21]

$$\forall A, B \in \mathcal{G}(L), \alpha \in A \setminus B : \exists \beta \in B \setminus A : A^\alpha_\beta \in \mathcal{G}(L)$$

(2.2)

which is proved equivalent to the following symmetric exchange property [8]

$$\forall A, B \in \mathcal{G}(L), \alpha \in A \setminus B : \exists \beta \in B \setminus A : A^\alpha_\beta, B^\alpha_\beta \in \mathcal{G}(L).$$

(2.3)

We introduce the binary relations $\forall_H \subseteq \forall$ on $[n], H \in \mathcal{G}(L)$, and their union $\forall$:

$$\forall H, \forall \alpha \forall_H \beta \quad \text{def} \quad \forall H \in \mathcal{G}(L) \quad \forall \alpha \forall_H \beta \in \mathcal{G}(L) \quad \forall := \bigcup_{H \in \mathcal{G}(L)} \forall_H.$$ 

(2.4)

We denote as $\mathbb{F}$ the field of fractions of $\mathbb{C}(t)$. For any $P \in \mathbb{C}(t)$, $\text{Supp}(P)$ is the set of monomials composing $P$, i.e., the units $m$ of $\mathbb{C}(t)$ such that the inner product $\langle , \rangle$ between polynomials satisfies $\langle P, m \rangle = \langle m, m \rangle$. Note that the latter condition identifies both the coefficient and exponents of the monomials in $\text{Supp}(P)$; in particular, the associated exponent map is

$$\Psi(P) := \{ e \in \mathbb{Z}^d : \exists c_e \in \mathbb{C}^\times : c_e t^e \in \text{Supp}(P) \}.$$ 

(2.5)
Remark 2. When the polynomial $P$ is a unit in $\mathbb{C}(t)$, i.e., at $\#\Psi(P) = 1$, we explicitly write $\Psi(P) =: \{\Psi^{(1)}(P)\}$.

Note that the symbol $\Psi$ is also used for set functions $\Psi : \mathfrak{S}(\mathbb{L}) \rightarrow \mathbb{Z}^d$ in the statement of Theorem 1: indeed, we can connect the two notions, since the condition (1.3) means that the product

$$h(I) := \Delta_{L(t)}(I) \cdot \Delta_{R(t)}(I), \quad I \in \mathfrak{S}(\mathbb{L})$$

(2.6)

is a unit in $\mathbb{C}(t)$. Therefore, we set $\Psi(I) := \Psi^{(1)}(h(I)), I \in \mathfrak{S}(\mathbb{L})$, to simplify the notation when no ambiguity arises.

With this notation, we give the following:

**Definition 3.** The set

$$\chi(I \mid_{\alpha \beta}^{ij}) := \left\{ h(I) \cdot h(T^{ij}_{\alpha \beta}), h(I) \cdot h(I^0), h(I^0) \right\}$$

(2.7)

is said observable when $\chi(I \mid_{\alpha \beta}^{ij}) \neq \{0\}$; in that case, we use the same attribute for the corresponding index set $c := \{i, j\} \times \{\alpha, \beta\}$. We denote the projections of $c$ on $I$ and $I^0$ as $c_r := \{i, j\}$ and $c_c := \{\alpha, \beta\}$, respectively.

An observable set $\chi(I \mid_{\alpha \beta}^{ij})$, or the underlying index set $\{i, j\} \times \{\alpha, \beta\}$, will be called a local key if $0 \notin \chi(I \mid_{\alpha \beta}^{ij})$, or a weak local key if there are at least three pairs $(l, \gamma) \in \{i, j\} \times \{\alpha, \beta\}$ with $h(T^l) = 0$.

An observable set is said integrable if $\#\Psi(\chi(I \mid_{\alpha \beta}^{ij}) \setminus \{0\}) = 1$, and $I \in \mathfrak{S}(\mathbb{L})$ is integrable if all the observable sets $\chi(I \mid_{\alpha \beta}^{ij})$ are integrable.

The attribute “observable” in Definition 3 refers to non-trivial information contained in $\chi(I \mid_{\alpha \beta}^{ij})$ in terms of Grassmann-Plücker constraints, which, in turn, can be used to extract information on $\Psi(\chi(I \mid_{\alpha \beta}^{ij}) \setminus \{0\})$.

The additional condition on the exponent map $\Psi$ characterizing integrable sets is the restriction of a property entailed by trivial solutions (1.6) to a given set $\chi(I \mid_{\alpha \beta}^{ij})$. The attribute “integrable” also refers to an analogous condition for deformations studied in [2] and related to the bilinear KP II hierarchy.

**Example 4.** In order to provide a simple realization of these notions, we introduce

$$L_{ex} := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 & 4 & 5 & 0 \end{pmatrix}$$

(2.8)

where we make explicit the column labeling, and assume that the associated matrix $R_{ex}$ in (1.10) is generic, i.e., no maximal minor is identically vanishing. We see that $\chi(I \mid_{\alpha \beta}^{ij}) = \{0\}$, so the set $\{i, j\} \times \{\alpha, \beta\}$ is not observable. In contrast, the sets $c_{\beta \gamma} := \{i, j\} \times \{\beta, \gamma\}, c_{\beta \delta} := \{i, j\} \times \{\beta, \delta\}$, and $c_{\gamma \delta} := \{i, j\} \times \{\gamma, \delta\}$ are observable. Both $c_{\beta \delta}$ and $c_{\gamma \delta}$ are local keys, since $0 \notin \chi(I \mid_{\alpha \beta}^{ij}) \cup \chi(I \mid_{\alpha \gamma}^{ij})$. On the other hand, $\chi(I \mid_{\beta \gamma}^{ij})$ is a weak local key, being $\Delta_{L_{ex}}(\{m, \beta, \gamma\}) = 0$. Finally, $\chi(I \mid_{\alpha \beta}^{ij})$ is an observable set that is not a weak local key.

Each $(k \times n)$-dimensional matrix $M$ satisfies the three-term Grassmann-Plücker relations

$$\Delta_M(I) \cdot \Delta_M(T^i_{\alpha \beta}) = c_1 \Delta_M(T^i_{\alpha \beta}) \cdot \Delta_M(T^j_{\alpha \beta}) + c_2 \Delta_M(T^j_{\alpha \beta}) \cdot \Delta_M(T^k_{\alpha \beta})$$

(2.9)

where

$$c_1 := \text{sign } [(i - j)(\alpha - \beta)(i - \beta)(\alpha - j)], \quad c_2 := \text{sign } [(i - j)(\alpha - \beta)(i - \alpha)(j - \beta)].$$

(2.10)
are referred to as $Y$-terms; from (2.9), they transform as follows under changes of bases obtained by a single exchange of indices:

\[ Y(T)^{ij}_{\alpha\beta} := -c_2 \cdot \frac{\Delta_{R(t)}(T^i_{\alpha}) \cdot \Delta_{R(t)}(T^j_{\beta})}{\Delta_{R(t)}(T^i_{\beta}) \cdot \Delta_{R(t)}(T^j_{\alpha})}, \quad i, j \in \mathcal{I}, \quad \alpha, \beta \in \mathcal{I}^c \]  

(2.11)

For each change of basis $\mathcal{I} \mapsto \mathcal{J} := T^c_{\gamma} \in \mathfrak{S}(L)$, where $(l, \gamma) \in (\mathcal{I} \times \mathcal{I}^c)$, we set

\[ c_{\mathcal{I} \mathcal{J}} := (c_i)^{\gamma}_{\gamma} \times (c_j)^{\gamma}_{\gamma}. \]  

(2.14)

Remark 5. Example 4 elucidates the naming “(weak) local key”: the properties of a local key, e.g., $c_{\gamma\delta}$, let us access other bases in $\mathfrak{S}(L)$, starting from $\mathcal{I}$, through a label switching $u \equiv \omega$ at $u \in \{i, j\}$ and $\omega \in \{\gamma, \delta\}$. This accessibility is local, i.e., limited to indices in $c_{\gamma\delta}$, and complete in that it can be iterated for transformed bases (2.14) too. This is not true for the weak local key $c_{\gamma\gamma}$, since $T^c_{\gamma\gamma} \notin \mathfrak{S}(L)$. Finally, (weak) local keys are characterized by the existence of a basis $\mathcal{J}$ obtained from $\mathcal{I}$ through (2.12), (2.13), or the identity map, satisfying

\[ \prod_{(s, r) \in c_{\mathcal{I} \mathcal{J}}} h(T^c_{sr}) \neq 0. \]  

(2.15)

This property, which will be used hereafter, is not met by $c_{\alpha\delta}$ in Example 4.

It is also easy to check

\[ Y^{ij}_{\alpha\beta} Y^{ij}_{\beta\gamma} = -Y^{ij}_{\alpha\gamma}, \quad Y^{im}_{\alpha\beta} \cdot Y^{mj}_{\alpha\beta} = -Y^{ij}_{\alpha\beta}. \]  

(2.16)

Iterating (2.16), for all $i, j, m \in \mathcal{I}$ and $\alpha, \beta, \omega \in \mathcal{I}^c$ we get the decomposition

\[ Y^{ij}_{\alpha\beta} = -Y^{ij}_{\alpha\omega} \cdot Y^{ij}_{\omega\beta} = -Y^{im}_{\alpha\omega} \cdot Y^{mj}_{\omega\beta} \cdot Y^{lm}_{\omega\beta}. \]  

(2.17)

2.2 Assumptions

Non-trivial dependence pattern of $L(t)$

We assume that all the columns of $L(t)$ belong to at least one basis in $\mathfrak{S}(L)$. Dually, each $\mathcal{I} \in \mathfrak{S}(L)$ and $i \in \mathcal{I}$ identify at least one $\alpha \in \mathcal{I}^c$ satisfying $\Delta_{L(t)}(T^i_{\alpha}) \neq 0$. These assumptions entail no loss of generality, since (1.1) is not affected by columns or rows violating them.

A dedicated analysis of configurations with $\mathfrak{S}(L) = \varphi_k[n]$ is carried out in [3], so here we set:

Assumption 6. The matroid $\mathfrak{S}(L)$ is a proper subset of $\varphi_k[n]$.

Generic $R(t)$ and invertible $Y$-terms

The “control” matrix $R(t)$ is assumed to be generic, so the dependence pattern defined by terms $h(\mathcal{I}) = 0$ is fully determined by $\varphi_k[n] \setminus \mathfrak{S}(L)$:

Assumption 7. $\Delta_{R(t)}(\mathcal{I}) \neq 0$ for all $\mathcal{I} \in \varphi_k[n]$ at $t = 1$ and for a generic choice of $t$. Equivalently, each $Y$-term (2.11) is invertible as a function of $t$. 
From (2.12), this assumption gives \( Y_{ij}^{\alpha \beta} \neq -1 \) for all \( I \in \wp_k[n] \), \( i,j \in I \), and \( \alpha, \beta \in T_c \) with \( i \neq j \) and \( \alpha \neq \beta \). We still allow the degenerate cases \( i=j \) or \( \alpha = \beta \), setting \( Y_{ij}^{\alpha \beta} = -1 \) only for these cases, consistently with the definition (2.11).

Assumption 7 is also motivated by an interpretation of matrices \( L(t) \) and \( R(t) \) from a statistical modeling perspective. When \( L = R \) in (1.1), we recover the Gramian of \( L \), which arises in the expression of efficient estimators in linear regression, e.g., using OLS. More general cases where \( L \neq R \) make us assign different roles to the two matrices: the expansion of (1.1) may be suited to the occurrence of coefficient estimators in linear regression, e.g., using OLS. More general cases where \( L \neq R \) make us assign different roles to the two matrices: the expansion of (1.1) may be suited to the occurrence of instrumental variables in statistical models [6, Sect. 1.2], which are widely used for causal modeling (see, e.g., [7], where partial correlations are also discussed in this context). Here, we look at \( L \) as a pattern matrix, whose main role is to define what contributions in the expansion of (1.1) are observable, i.e., \[7\], where partial correlations are also discussed in this context). Here, we look at \( L \) as a pattern matrix, whose main role is to define what contributions in the expansion of (1.1) are observable, i.e., which terms are not vanishing, as encoded in the matroid (1.2). On the other hand, we interpret \( R \) as a generic matrix that is coupled to \( L \) through (1.1): unlike \( L \), which gives structural information, \( R \) carries quantitative information without a priori constraints.

Local integrability condition

Before stating the last condition, we fix the following:

**Definition 8.** Every \( A \subseteq I \), \( I \in \wp_k[n] \), generates a pair of dual sets

\[
N_{I,A} := \{ \gamma \in T_c : \forall i \in A : h(I^\gamma_i) = 0 \},
\]

\[
N^T_{I,H} := \{ m \in I : \forall \alpha \in H : h(I^m_\alpha) = 0 \}.
\]

To simplify the notation, we will omit the subscript \( I \) when no ambiguity arises and define

\[
N(A; H) := \{ (m, \omega) : \omega \in N_{I,A} \text{ or } m \in N^T_{I,H} \}.
\]

Given a local key \( \epsilon = \epsilon_r \times \epsilon_c \subseteq I \times T_c \), we denote the associated set \( N(\epsilon_r; \epsilon_c) \) as \( N(\epsilon) \).

The sets \( N_A \) and \( N_H \) in (2.18) are dual in the sense that they satisfy the adjunction

\[
H \subseteq N_A \Leftrightarrow A \subseteq N_H
\]

so we get consistently say that the \( A \subseteq I \) and \( H \subseteq T_c \) are adjoin if \( H \subseteq N_A \).

**Definition 9.** A local key \( \epsilon \) is said to be planar, or equivocal, if

\[
N(\epsilon_r) = I \setminus \epsilon_r, \quad N(\epsilon_c) = T_c \setminus \epsilon_c.
\]

Otherwise, it is referred to as non-planar:

**Remark 10.** We choose the attribute “non-planar” since there is a correspondence between a non-planar local key and one of the graphs \( K_{4,3} \) or \( K_5 \) characterizing the obstruction to planarity in Kuratowski’s theorem. Fix a non-planar local key \( \epsilon := \{ i_1, i_2 \} \times \{ \alpha_1, \alpha_2 \} \) and take an index, say \( i_3 \in I \), such that \( I_{i_1}^{i_3} \in \mathfrak{S}(L) \). First assume \( J := I_{i_2}^{i_3} \notin \mathfrak{S}(L) \): in this case, we set \( g_r := \{ i_1, i_2, \alpha_2 \}, \ g_c := \{ \alpha_1, i_3 \}, \ g_0 := g_r \cup g_c \cup \{ \epsilon_c \} \), and specify the following correspondence

\[
k_0(u, \omega) := J \setminus \{ u \} \cup \{ \omega \}, \quad k_0(u, g_0) = k_0(g_c, u) := J \setminus g_r \cup \{ \alpha_1, i_3, u \}
\]

with \( u, \omega \in g_r \cup g_c \). For all \( u, w \in g_0, u \neq w \), we say that \( u \) and \( w \) are related if one of the two pairs they generate returns a basis in \( \mathfrak{S}(L) \) through (2.22); then, \( K_{4,3} \) is the graph associated with this relation.
on $g_0$. The same argument holds, by a change of basis, if $0 \in \chi(\mathcal{I}^{|i_i} |_{\alpha_0}^{i_u})$ for some $s, u \in [3], s \neq u$. Otherwise, we state the correspondence with $K_5$ via

$$k_1(\alpha_1, \alpha_2) := I, \quad k_1(i_u, \alpha_w) := T^i_{\alpha_w}, \quad k_1(i_u, i_u) := T^i_{\alpha_1}$$

for all $s, u \in [3], s \neq u$, and $w \in [2]$. This associates each pair in $\mathcal{C} \cup \mathcal{C}$ with a basis in $\mathcal{G}(L)$, so all the indices in $\mathcal{C} \cup \mathcal{C}$ are related in the previous sense.

It is worth looking at Theorem 1 in the light of this observation, since it affirms the obstruction to non-integrable deformations due to the existence of a non-planar local key, just like the existence of a subgraph homeomorphic to $K_{3,3}$ or $K_5$ is an obstruction to planarity.

**Assumption 11.** There exists a basis $\mathcal{I} \in \mathcal{G}(L)$ with a non-planar local key.

### 3 Preliminary lemmas

The following lemmas are stated for future reference: they easily follow from direct computations, some of which are presented in [2, 3] and are summarized here to make this work self-consistent.

**Lemma 12.** Let $\mathcal{H}, \mathcal{K} \in \mathcal{G}(L)$ with $r := \#(\mathcal{H} \setminus \mathcal{K})$. Then, there exists a finite sequence $\mathcal{L}_0 := \mathcal{H}, \mathcal{L}_1, \ldots, \mathcal{L}_r := \mathcal{K}$ of elements of $\mathcal{G}(L)$ such that $\#(\mathcal{L}_{u-1} \Delta \mathcal{L}_u) = 2, u \in [r]$.

**Proof.** This is Lemma 6 in [2] and easily follows from the exchange property of matroids (2.2). \qed

**Lemma 13.** For observable sets $\chi(\mathcal{I}^{i_j} |_{\alpha_{ij}})$, at $h(\mathcal{I}) \cdot h(\mathcal{I}^{i_j} |_{\alpha_{ij}}) = 0$ we find that

$$Y_{\alpha_{ij}}^{ij} = -\frac{h(I^i_j \cdot h(I^{i_j}_{\alpha_{ij}}))}{h(I^i_j, h(I^{i_j}_{\alpha_{ij}}))}$$

is a unit in $\mathcal{C}$, while at $h(I^i_j) \cdot h(I^{i_j}_{\alpha_{ij}}) = 0$ we get

$$Y_{\alpha_{ij}}^{ij} = \frac{h(I) \cdot h(I^{i_j}_{\alpha_{ij}})}{h(I^i_j) \cdot h(I^{i_j}_{\alpha_{ij}})} - 1.$$

**Proof.** Multiplying (2.9) for $L(t)$ and $R(t)$ side by side, we get

$$h(I) \cdot h(I^{i_j}_{\alpha_{ij}}) = h(I^i_j) \cdot h(I^{i_j}_{\alpha_{ij}}) + Y_{\alpha_{ij}}^{ij} \cdot h(I^i_j) \cdot h(I^{i_j}_{\alpha_{ij}}) + \frac{h(I^i_j) \cdot h(I^{i_j}_{\alpha_{ij}})}{Y_{\alpha_{ij}}^{ij}} + h(I^i_j) \cdot h(I^{i_j}_{\alpha_{ij}}).$$

Then, the thesis follows by direct computation. \qed

**Lemma 14.** An observable set $\chi(\mathcal{I}^{i_j}_{\alpha_{ij}})$ is integrable if and only if $Y_{\alpha_{ij}}^{ij} \in \mathcal{C}$.

**Proof.** The thesis follows from (3.1)-(3.2) when $0 \in \chi(\mathcal{I}^{i_j}_{\alpha_{ij}})$. Otherwise, from $Y_{\alpha_{ij}}^{ij} \neq 0$, the left-hand side of (3.3) is a second-degree polynomial in $Y_{\alpha_{ij}}^{ij}$: it is easily checked (see [3]) that, starting from (1.10), the discriminant of this polynomial is a perfect square in $\mathcal{C}(t)$ if and only if $\chi(\mathcal{I}^{i_j}_{\alpha_{ij}})$ is integrable. \qed

**Remark 15.** For every $\mathcal{A} \subseteq \mathcal{I}$ and $i \in \mathcal{A}$, whenever $h(I^i_j) \neq 0$, the set $\mathcal{N}_{\mathcal{I}, \mathcal{A}}$ is invariant under the change of basis $\mathcal{I} \mapsto \mathcal{J} := I^i_j$: indeed, $\alpha \notin \mathcal{N}_{\mathcal{I}, \mathcal{A}}$ and, for all $\beta \in \mathcal{N}_{\mathcal{I}, \mathcal{A}}$, we find $h(\mathcal{J}^\beta_{\alpha_j}) = h(\mathcal{I}^i_j) = 0$ from (2.12)-(2.13). Even for the other indices $j \in \mathcal{A}^i$, we get $h(J^j_{\alpha_j}) = h(I^{i_j}_{\alpha_{ij}}) = 0$, so $\mathcal{N}_{\mathcal{I}, \mathcal{A}} \subseteq \mathcal{N}_{\mathcal{J}, \mathcal{A}_i}^j$: by symmetry under the exchanges $i \mapsto \alpha$ and $\mathcal{I} \mapsto \mathcal{J}$, we infer $\mathcal{N}_{\mathcal{I}, \mathcal{A}} = \mathcal{N}_{\mathcal{J}, \mathcal{A}_i}^j$. Dually, for every $\mathcal{H} \subseteq \mathcal{I}$ and $\alpha \in \mathcal{H}$, the set $\mathcal{N}_{\mathcal{H}}$ is invariant under the change of basis $\mathcal{I} \mapsto I^i_j$ whenever $h(I^i_j) \neq 0$. 

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In particular, transformations (2.14) preserve the set $\mathcal{N}(c)$ and map a (weak) local key into a new (weak) local key, at least one of which satisfies (2.15).

**Lemma 16.** For each weak local key $c = \{i, j\} \times \{\alpha, \beta\}$ satisfying (2.15) and $(m, \omega) \in \mathcal{N}(c)$ with $h(I^m_\omega) \neq 0$, a term $Y_{mi}^\omega \notin \mathcal{C}$ for some $(i, \alpha) \in c$ determines at most two allowed configurations for the set $\{Y_{s}\omega : s \in \mathcal{C}, \delta \in \mathcal{C}\}$.

**Proof.** We define

$$
\mathcal{Y}^+ = \{Y_{s}^\omega, Y_{\alpha^s}^\omega\}, \quad \mathcal{Y}^- = \{Y_{s}^\omega, Y_{\alpha^s}^{\omega^s}\}, \quad \mathcal{Y} = \mathcal{Y}^+ \cup \mathcal{Y}^- \quad (3.4)
$$

and express $\{Y_{s,1}, Y_{s,2}\} := \mathcal{Y}^e$ for each $\sigma \in \{\pm\}$. From (2.17), we find

$$
Y_{\alpha^s \beta} \cdot (Y_{s}^\omega Y_{\alpha^s}^{\omega^s}) = -(Y_{s}^\omega Y_{\alpha^s}^{\omega^s}). \quad (3.5)
$$

The configuration (2.15) assigns to each term in $\mathcal{Y}$, the form (3.2), so $Y_{\alpha^s \beta}^s \in \mathcal{F}$, then $Y_{\alpha^s \beta}^s = g$ is invertible in $\mathcal{C}(\mathcal{F})$ by Lemmas 18-13.

Under the condition $\mathcal{Y}^e \cap \mathcal{C} = \emptyset$, the factors of $Y_{s,1} - Y_{s,2}$ uniquely determine the singletons $\Psi(Y_{s,1} + 1)$ and $\Psi(Y_{s,2} + 1)$ and, hence, the corresponding elements $\Psi(1)(Y_{s,1} + 1)$ and $\Psi(1)(Y_{s,2} + 1)$ as defined in Remark 2. Starting from (3.5) and taking into account Assumption 7, at $\mathcal{Y} \cap \mathcal{C} = \emptyset$ we infer

$$
\{\Psi(1)(Y_{s,1} + 1), \Psi(1)(Y_{s,2} + 1)\} = \{-\Psi(1)(Y_{s,1} + 1), -\Psi(1)(Y_{s,2} + 1)\}. \quad (3.6)
$$

We can extend (3.6) under the condition $\mathcal{Y} \not\subseteq \mathcal{C}$, since it trivially holds at $\mathcal{Y} \subseteq \mathcal{C}$, and it follows from Assumption 7 when $\#(\mathcal{Y} \cap \mathcal{C}) = 2$. In particular, at $\mathcal{Y} \not\subseteq \mathcal{C}$ there exists a unit $\Sigma \in \mathcal{C}(\mathcal{F})$ such that

$$
\mathcal{Y}^+ = \{\tau^{-1} - 1, \tau\theta^{-1} \cdot \mathcal{O}\}, \quad \mathcal{Y}^- = \{\tau - 1, \mathcal{O}\}, \quad \mathcal{O} \in \{0, \theta^{-1} - 1\} \cup \mathcal{C}. \quad (3.7)
$$

At $2 \nmid \#(\mathcal{Y} \cap \mathcal{C})$, which implies $\#(\mathcal{Y} \cap \mathcal{C}) = 3$, there exist a permutation $\sigma$ of $\{\pm\}$ and $Y \in \mathcal{Y}^{e\sigma}$ such that $Y_{\sigma(-)} \in \mathcal{Y}^{e\sigma}$ for all $Y_{\sigma(-)} \in \mathcal{Y}^{e\sigma}$. This condition lets us find $\epsilon_1, \epsilon_2 \in \{1, -1\}$ and $C \in \mathcal{C}$ such that $\sigma = \epsilon_2 \cdot C \cdot \tau^{(1 + \epsilon_1 - 2\epsilon_2)/2}$ and

$$
\mathcal{Y}^{e\sigma(\pm)} = \{\tau^{\epsilon_2} \pm 1, C^{-1}\}, \quad \mathcal{Y}^{e\sigma(-)} = \{\tau - 1, -\epsilon_1 - 1\}. \quad (3.8)
$$

We refer to (3.7) and (3.8) as even- and odd-type configurations, respectively. □

### 4 Integrable sets from non-planar local keys

**Lemma 17.** Let $I \in \mathcal{G}(\mathcal{L})$ be a basis such that $h(I_\omega^s) = 0$ for at least one pair $(s, \omega) \in I \times I^c$, and $c := \{i_1, i_2\} \times \{\alpha_1, \alpha_2\} \subseteq I \times I^c$ be a local key. Then, $Y_{i_1}^{i_2} \in \mathcal{C}$ for any $\gamma_1, \gamma_2 \notin \mathcal{N}(c)$ and $l_1, l_2 \notin \mathcal{N}(c)$.

**Proof.** It is enough to prove $Y_{i_1}^{i_2} \in \mathcal{C}$ for all $\gamma \notin \mathcal{N}(c)$; since this entails $Y_{i_1}^{i_2} = -Y_{i_1}^{i_2} \cdot Y_{i_1}^{i_2} \in \mathcal{C}$; the argument leading to $Y_{i_1}^{i_2} \in \mathcal{C}$, $l_1, l_2 \notin \mathcal{N}(c)$, is analogous.

Observe that, whether $0 \in \chi(I^{(i_1 l_2 \delta)}_{\alpha_1 \alpha_2})$ for some $\delta \in I^c$ and $w \in \{2\}$, at least one pair in $c$, say $(i_2, \alpha_2)$, returns a basis $J := I_{\alpha_2}^{i_2}$ with $h(J_{\delta}) = 0$ for some $j \in (c_\gamma)$. The lack of null columns (Paragraph 2.2) guarantees the existence of $m \in J$ such that $h(J_\delta^m) \neq 0$, and we can take $m \in (c_\gamma)$, at $\delta \notin \mathcal{N}(c_\gamma)$. This gives $Y(J)^{m}_{\gamma} \in \mathcal{F}$ for all $j \in \{i_1, \alpha_2\}$ and $\beta \in \{\alpha_1, i_1\}$, since all of these $Y$-terms assume the value $-1$ (at $m \in (c_\gamma)$) or the form (3.2). By (2.17), $Y(J)^{i_1\alpha_2}_{i_1\alpha_2} \in \mathcal{F}$ and, by the proof of Lemma 14, $Y(J)^{i_1\alpha_2}_{i_1\alpha_2} \in \mathcal{C}$.  

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Setting \( \delta := \gamma \) with \( \gamma \notin \mathcal{N}(e) \) in the previous configuration, the compatibility of \( Y(\mathcal{J})_{\alpha \gamma}^{\omega} \in \mathbb{C} \setminus \{ -1 \} \) (from Assumption 7) with the algebraic forms of \( Y(\mathcal{J})_{\alpha \gamma}^{\omega} \) and \( Y(\mathcal{J})_{\omega}^{\gamma} \) requires \( Y(\mathcal{J})_{\beta_1 \beta_2}^{\gamma} \in \mathbb{C} \) for all \( \beta_1, \beta_2 \in \{ \alpha_1, \alpha_2, \gamma \} \). Moving back to \( \mathcal{J} \), from (2.12)-(2.13) we still have \( Y_{\alpha_1 \gamma}^{i_1}, Y_{\alpha_2 \gamma}^{i_2} \in \mathbb{C} \), and, by (2.16), \( Y_{\alpha_1 \gamma}^{i_1}, Y_{\alpha_2 \gamma}^{i_2} \in \mathbb{C} \). Thus, we may get \( Y_{\alpha_1 \gamma}^{i_1}, Y_{\alpha_2 \gamma}^{i_2} \notin \mathbb{C} \) only if \( \{ i_1, i_2 \} \times \{ \alpha_1, \gamma \} \) is a local key and, iterating the argument in the previous paragraph with \( \gamma \) in place of \( \omega \), \( \{ j_1, j_2 \} \times \{ \alpha_1, \gamma \} \) and \( \{ i_1, i_2 \} \times \{ \beta_1, \beta_2 \} \) are themselves local keys for all \( j_1, j_2 \in \mathcal{J} \) and \( \beta_1, \beta_2 \in \mathcal{J} \). Taking \( (s, \omega) \in \mathcal{J} \times \mathcal{J} \) satisfying \( h(\mathcal{J}_s) = 0 \), which exists by hypothesis, from (3.2) we infer \( Y_{\beta_\omega}^{j_\omega} \in \mathbb{F} \) for all \( j \in \{ i_1, i_2 \} \) and \( \beta \in \{ \alpha_1, \gamma \} \), then \( Y_{\alpha_1 \gamma}^{i_1}, Y_{\alpha_2 \gamma}^{i_2} \in \mathbb{C} \) by (2.17).

\[ \square \]

**Proposition 18.** We have \( Y_{\alpha_1 \gamma}^{i_1}, Y_{\alpha_2 \gamma}^{i_2} \in \mathbb{C} \) for any local key \( e := \{ i_1, i_2 \} \times \{ \alpha_1, \alpha_2 \} \), \( \gamma_1, \gamma_2 \notin \mathcal{N}_e \), \( l_1, l_2 \notin \mathcal{N}^e \).

**Proof.** As in Lemma 17, it suffices to show \( Y_{\alpha_1 \gamma}^{i_1}, Y_{\alpha_2 \gamma}^{i_2} \in \mathbb{C} \) for all \( \gamma \notin \mathcal{N}_e \). We can focus on configurations that do not meet the hypotheses of Lemma 17 and consider the bases \( \mathcal{J} \) with at least one pair \((s, \sigma) \in \mathcal{J} \times \mathcal{J} \) satisfying \( h(\mathcal{J}_s) = 0 \), which exists by Assumption 6. Adopting the same notation as in Lemma 12, we choose any such a basis \( \mathcal{L}(r) \) with minimal distance \( r := \#(\mathcal{I} \Delta \mathcal{L}(r)) \) from \( \mathcal{J} \).

Take any \((m, \omega) \in \mathcal{L}(r-1) \times \mathcal{L}(r-1) \). By construction of \( \mathcal{L}(r-1) \), the assumptions in Paragraph 2.2 guarantee the existence of \((m, \omega) \in \mathcal{L}(r-1) \times \mathcal{L}(r-1) \) such that \( \{ l_r, m \} \times \{ \gamma_r, \omega \} \) is a local key for \( \mathcal{L}(r-1) \), and we choose such \((m, \omega) \in \mathcal{L}(r-1) \times \mathcal{L}(r-1) \) for \( \omega \) if one of these two choices is feasible. In this way, at \( \{ m, \omega \} \cap \{ l_r, m \} \neq \emptyset \), we get a local key \( \{ \gamma_r, \omega \} \in \mathcal{L}(r) \), whose deformations let us apply Lemma 17 to find \( Y(\mathcal{L})(r)_{l_r, m} \in \mathbb{C} \) and, by (2.12), \( Y(\mathcal{L}(r-1))_{l_r, m} \in \mathbb{C} \). Otherwise, we infer

\[ \forall j_1, j_2 \in \mathcal{L}(r-1), \beta_1, \beta_2 \in \mathcal{L}(r-1) : h(\mathcal{L}(r-1))_{j_1, j_2} = h(\mathcal{L}(r-1))_{j_1, j_2} = 0 \]

and find that \( \mathcal{L}(r)_{j_1, j_2} \) is a local key with basis \( \mathcal{L}(r-1) \) for all \( j \in \{ l_r, m \} \) and \( \beta \in \{ \gamma_r, \omega \} \); specifying Lemma 17 at the local keys \( \mathcal{L}(r)_{j_1, j_2} \) for such choices of \( j_1, j_2 \), we get \( Y(\mathcal{L}(r-1))_{j_1, j_2} \in \mathbb{C} \), and, from (2.17), \( Y(\mathcal{L}(r-1))_{l_r, m} \in \mathbb{C} \).

Being \((m, \omega) \) arbitrary, we take \( m_1 \neq \gamma_{r-1}, \omega_1 \neq l_{r-1} \), and consider \( m \in \{ m_1, \gamma_{r-1} \} \) and \( \omega \in \{ \omega_1, l_{r-1} \} \); applying these choices of \( Y(\mathcal{L}(r-1))_{l_r, m} \) to (2.17), we get \( Y(\mathcal{L}(r-1))_{l_r, m} \in \mathbb{C} \), then (2.12) returns \( Y(\mathcal{L}(r-2))_{l_r, m} \in \mathbb{C} \). Iterating this last step, we find that \( \mathcal{L}^{(u-1)} \) is an integrable basis for all \( u \) in \( [r] \), and, in particular, this holds for \( \mathcal{L}^{(0)} \) = \( \mathcal{I} \), which proves the thesis.

\[ \square \]

**Lemma 19.** For \( Y_{\alpha_1 \alpha_3}^{\omega_3}, Y_{\alpha_2 \alpha_3}^{\omega_2}, Y_{\alpha_3 \alpha_2}^{\omega_2} \in \mathbb{F} \), the term \( Y_{\alpha_1 \alpha_2}^{\alpha_3} \) lies in an algebraic extension of \( \mathbb{F} \) of degree at most 2.

**Proof.** Let us introduce

\[ n_{\alpha_1 \alpha_2}^{\alpha_3} := \frac{\Delta_{\mathcal{R}(t)}(\mathcal{I}) \cdot \Delta_{\mathcal{R}(t)}(\mathcal{I})}{\Delta_{\mathcal{R}(t)}(\mathcal{I}) \cdot \Delta_{\mathcal{R}(t)}(\mathcal{I})} = \frac{\Delta_{\mathcal{R}(t)}(\mathcal{I}) \cdot \Delta_{\mathcal{R}(t)}(\mathcal{I})}{\Delta_{\mathcal{R}(t)}(\mathcal{I}) \cdot \Delta_{\mathcal{R}(t)}(\mathcal{I})} \]

where

\[ \varepsilon_{\alpha_1 \alpha_2}^{\alpha_3} := \text{sgn} \left( \prod_{u < w} (a_u - a_w) \cdot \prod_{x < z} (\delta_x - \delta_z) \cdot \prod_{r \neq s} (a_r - a_s) \right) \]

Recalling (2.11), it can be easily verified that the following identity holds:

\[ 1 + Y_{\alpha_2 \alpha_3}^{\alpha_1} + Y_{\alpha_3 \alpha_2}^{\alpha_1} + Y_{\alpha_2 \alpha_3}^{\alpha_1} + Y_{\alpha_3 \alpha_2}^{\alpha_1} = Y_{\alpha_1 \alpha_2}^{\alpha_3} \]

Therefore, \( Y_{\alpha_3 \alpha_2}^{\alpha_1} \) is a root of the quadratic polynomial

\[ \mathcal{P}_{\alpha_1 \alpha_2}^{\alpha_3}(X) := X^2 + \left( 1 + Y_{\alpha_2 \alpha_3}^{\alpha_1} + Y_{\alpha_3 \alpha_2}^{\alpha_1} + Y_{\alpha_2 \alpha_3}^{\alpha_1} - m_{\alpha_1 \alpha_2}^{\alpha_3} \right) \cdot X - Y_{\alpha_1 \alpha_2}^{\alpha_3} \]

where

\[ Y_{\alpha_3 \alpha_2}^{\alpha_1} \]

\[ Y_{\alpha_1 \alpha_2}^{\alpha_3} \]

\[ Y_{\alpha_2 \alpha_3}^{\alpha_1} \]

\[ Y_{\alpha_3 \alpha_2}^{\alpha_1} \]

\[ Y_{\alpha_1 \alpha_2}^{\alpha_3} \]
and the thesis follows.

Remark 20. Note that the discriminant of (4.5)

\[
\Delta_{s_1 s_2 s_3}^{a_1 a_2 a_3} := (1 + Y_{s_2 s_1}^{a_1 a_2} + Y_{s_3 s_1}^{a_1 a_2} + m_{s_1 s_2 s_3}^{a_1 a_2 a_3} - m_{s_2 s_1 s_3}^{a_1 a_2 a_3})^2 + 4 \cdot Y_{s_1 s_2 s_3}^{a_1 a_2 a_3} = 2 \cdot Y_{s_1 s_2 s_3}^{a_1 a_2 a_3}
\]

expressed in terms of \(Y_{s_u a_u}^{w} + 1\), \(1 \leq u < w \leq 3\), is a non-homogeneous analog of the \(2 \times 2 \times 2\) hyperdeterminant \([12, 15]\).

Now, let us use the notation \(c_1^{(s,t)}\) to refer to the sign \(c_1\) in (2.10) with the specification \((i, j, \alpha, \beta) := (a, a, \delta, \delta)\); then, looking at (4.3), we note that

\[
\varepsilon_{s_1 s_2 s_3} = c_1^{(1,2)} c_1^{(1,3)} c_1^{(2,3)}
\]

while from the three-term Grassmann-Plücker relations (2.9) we derive

\[
Y_{\alpha_1 \omega}^{a_1 a_2 a_3} + 1 = c_1^{(s,t)} \frac{\Delta_{R(I)}^{(1)}(T_{\beta}) \cdot \Delta_{R(I)}^{(2)}(T_{\alpha_1 a_1})}{\Delta_{R(I)}^{(1)}(T_{\alpha_1 a_1}) \cdot \Delta_{R(I)}^{(2)}(T_{\beta})}.
\]

From (4.2), (4.7), and (4.8), we verify the identity

\[
m_{s_1 s_2 s_3}^{a_1 a_2 a_3} = (Y(I_{\alpha_1})^{a_1 a_2 a_3} + 1) \cdot (Y(I_{\beta})^{a_1 a_2 a_3} + 1)
\]

which entails that the factor

\[
ge_{s_1 s_2 s_3}^{a_1 a_2 a_3} := \frac{Y(I_{\alpha_1})^{a_1 a_2 a_3} + 1}{Y(I_{\beta})^{a_1 a_2 a_3} + 1}
\]

is invariant under permutations of the three pairs \((a, \delta)\), \(i \in [3]\).

Proposition 21. Let \(c := \{i_1, i_2\} \times \{\alpha_1, \alpha_2\}\) be a non-planar local key. Then, for each \((m, \omega)\) such that \(\chi(I_{\alpha_2})^{m})\) is observable for some \((i, \alpha) \in c\), we have \(Y_{\alpha_2}^{m} = \mathbb{C}\) for all \(u, w \in [2]\).

Proof. The thesis holds for \(I\) if and only if it holds for \(J := I_{\alpha_1}\) for any \((i, \alpha) \in c\) with \(J \in \mathcal{B}(L)\): indeed, the transformation rule (2.13) gives

\[
Y(J)^{m_5}_{\omega} = Y(J)^{m_5}_{\omega} \cdot Y(J)^{m_5}_{\beta} \in \mathbb{C} \Leftrightarrow Y(I)^{m_5}_{\omega}, Y(I)^{m_5}_{\beta}, Y(I)^{m_5}_{\beta} \in \mathbb{C}
\]

for all \(i, j \in c, \alpha, \beta \in c\), and from (2.17) we infer

\[
Y(I)^{m_5}_{\omega} = Y(I)^{m_5}_{\omega} \cdot Y(I)^{m_5}_{\beta} \in \mathbb{C}, \ (j, \beta) \in c.
\]

For each \(\lambda \notin \mathcal{N}_{c_r}\), we use the relations (4.11)-(4.12) to choose an appropriate labeling of \(c_r\) and move to a basis \(H \in \{I, I_{\alpha_1}\}\) where \((i, \alpha) \in c\) satisfies \(h(I_{\alpha_1}) \neq 0\), in order to get a new local key \(c_{\lambda} := (c_{H}) \times \{\alpha_1, \lambda\}\) in addition to \(c_r\).

Being non-planar, there exists an index in \([n]\) that falsifies (2.21), and we can take it as a column index \(\alpha_3 \in \mathcal{T}_c^\perp\), transposing the indices otherwise. Proposition 18 entails the thesis for all \((m, \omega) \notin \mathcal{N}(c) \cup \mathcal{N}(c_3)\): indeed, we can specify \(\lambda := \omega\) in the previous construction of \(c_{\lambda}\) to get a local key \(c_{\lambda} = (c_{H}) \times \{\alpha_1, \omega\}\). From Proposition 18 applied to \(c\) and \(c_{\omega}\), each term on the right-hand side of (4.12) is constant and using (4.11), if necessary, we get the thesis for \(I\).

So we can focus on \((m, \omega) \in \mathcal{N}(c) \cup \mathcal{N}(c_1)\) and choose a basis \(H\) constructed as before for \(\lambda := \alpha_3\), setting \(H = I\) to simplify the notation. Being \(Y_{\alpha_2}^{m_5}_{\omega} \in \mathbb{C}\) for all \(1 \leq u < w \leq 3\), the configurations (3.7) and (3.8) are incompatible, namely, each term \(Y_{\alpha_2}^{m_5}_{\omega} \) derived from (3.7) has roots if and only if it has poles, since \(\tau - 1\) and \(-\theta^{-1} \tau - 1\) are coprime at \(\theta = Y_{\alpha_2}^{m_5}_{\omega} \in \mathbb{C} \setminus \{-1\}\), while \(Y_{\alpha_2}^{m_5}_{\omega} \) from (3.8) has

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either roots or poles, but not both. Furthermore, (3.7) is compatible with Assumption 7 only if it has no constant terms. Thus, each constant $Y$-term can come only from (3.8): counting them for every (weak) local key $\epsilon_{u,w} := \epsilon \times \{u, \omega_u\}$, $1 \leq u < \omega \leq 3$, each of these constants is considered twice. This double-counting returns an even number of odd-type configurations (3.8). In particular, there exists at least one $\epsilon_{u,w}$ that induces (3.7). But, as remarked, (3.7) and (3.8) are not compatible; so we cannot find $u \in [3]$ such that $Y^{i_1 i_2}_{\alpha \omega_1}$ comes from two local keys $\epsilon_{u,w}$ and $\epsilon_{u,m}$ associated with different types (3.7) and (3.8). This means that, for all $u, w, \epsilon_{u,w}$ induces (3.7) when $Y^{i_1 m}_{\alpha \omega_1} \notin \mathbb{C}$. Taking $u = 1$ and setting $Y^{i_1 m}_{\alpha \omega_1} := \xi - 1, \xi$ unit in $\mathcal{C}(t)$, the compatibility of the two choices $w \in \{2, 3\}$ in $\epsilon_{(1,w)}$ subject to Assumption 7, which excludes $Y^{i_1 m}_{\alpha \omega_2} = \xi^{-1} - 1 = Y^{i_1 m}_{\alpha \omega_3}$, returns

$$Y^{i_1 m}_{\alpha \omega_1} = Y^{i_1 m}_{\alpha \omega_2} Y^{i_1 m}_{\omega_1} = \frac{\xi - 1}{\partial_x^2 \xi + 1}, \quad Y^{i_2 m}_{\alpha \omega_2} = \frac{\partial_x^2 \xi}{\partial_x^2 (1 + 1)}$$

where $\partial_x^{-1} := Y^{i_1 i_2}_{\alpha \omega_1}, x \in \{2, 3\}$. Evaluating (4.4) at $(a_1, a_2, a_3) := (i_1, i_2, m)$ and $(\delta_1, \delta_2, \delta_3) := (\alpha_1, \alpha_2, \alpha_3)$, we get $m^{i_1 i_2}_{\alpha_1 \alpha_2} = 0$, at odds with Assumption 7. So $\xi \in \mathbb{C}$ and the thesis holds. \[\square\]

5 Identification of an integrable basis

**Proposition 22.** Let $\epsilon$ denote a non-planar local key. Then, $Y^{m_1 m_2}_{\omega_1 \omega_2} \in \mathbb{C}$ for all $(m_1, \omega_1), (m_2, \omega_2) \in \mathcal{N}(\epsilon)$ with $h(T^{m_1}_{\omega_1}) \neq 0$, $s \in [2]$.

**Proof.** Let $\epsilon := \{i_1, i_2\} \times \{\alpha_1, \alpha_2\}$. For any $(i, \alpha) \in \epsilon$ and $s \in [2], Y^{i m_1}_{\alpha \omega_1} \in \mathbb{C}$ by Proposition 21. This observation can be extended to get the thesis when $h(T^{m_1}_{\omega_1}) \cdot h(T^{m_2}_{\omega_2}) \neq 0$, since Proposition 21 applies to all the observable sets $\chi(T^{m_1}_{\omega_1}) \cdot \epsilon, s \in [2]$, so $Y^{i m_1}_{\alpha \omega_1} \in \mathbb{C}$ and, from (6.3), $Y^{m_1 m_2}_{\omega_1 \omega_2} \in \mathbb{C}$. Then, we assume $h(T^{m_1}_{\omega_1}) \cdot h(T^{m_2}_{\omega_2}) = 0$ (hence, $h(T^{m_1 m_2}_{\omega_1 \omega_2}) \neq 0$) in the rest of the proof.

We consider the change of basis $I \mapsto T^{m_1}_{\omega_1},$ observing that $\epsilon$ remains a non-planar local key being $(m_1, \omega_1) \in \mathcal{N}(\epsilon)$. The set $\chi(T^{m_1}_{\omega_1})$, $(i m_1) \in \mathbb{C}$ is observable since $h(T^{m_1 m_2}_{\omega_1 \omega_2}) \neq 0$. Arguing as before, we get $Y(T^{i m_1}_{\omega_1}) \cdot h(T^{i m_2}_{\omega_2}) \in \mathbb{C}$; specifying (4.9) at $(a_1, a_2, a_3) = (m_1, m_2, 1)$ and $(\delta_1, \delta_2, \delta_3) = (\omega_1, \omega_2, \alpha)$, the conditions $Y(T^{i m_1 m_2}_{\omega_1 \omega_2}) \in \mathbb{C}$ and $Y^{m_1 m_2}_{\omega_1 \omega_2} \in \mathbb{C}$ imply

$$\Psi(m^{i_1 i_2}_{\alpha_1 \alpha_2}) = \Psi(Y^{m_1 m_2}_{\omega_1 \omega_2} + 1). \quad (5.1)$$

The assumption $h(T^{m_1}_{\omega_1}) \cdot h(T^{m_2}_{\omega_2}) = 0$ leads to the form (3.2) for $Y^{m_1 m_2}_{\omega_1 \omega_2} =: \tau - 1, \tau \in \mathcal{C}(t)$ invertible, while (5.1) allows us to introduce the notation $(Y^{i m_1}_{\alpha_1 \omega_1}, Y^{i m_2}_{\alpha_2 \omega_2}) = (c_{u,1} - 1, c_{u,2} - 1) \in \mathbb{C}^2,$ and $m^{i_1 i_2}_{\alpha_1 \alpha_2} = c_{u,3} \cdot \tau$ with $c_{u,3} \in \mathbb{C}, u \in [2]$.

Lemma 19 asserts that $Y^{i m_1}_{\omega_1 \omega_2}$ and $Y^{i m_2}_{\omega_1 \omega_2}$, $u \in [2]$, are algebraic or belong to a quadratic extension of $\mathbb{F}$, depending on the existence of a factor with odd multiplicity of the discriminant (4.6) $\Delta_u := \Delta^{i m_1 m_2}_{\omega_1 \omega_2}$ for both $u \in [1, 2]$. With the notation introduced above, we get

$$\Delta_u = (-c_{u,1} + c_{u,2} + \tau - c_{u,3} \cdot \tau)^2 - 4c_{u,2}(c_{u,1} - 1) \cdot (c_{u,3} \cdot c_{u,2} - 1) \cdot \tau \quad (5.2)$$

and $\sqrt{\Delta_1} \in \mathbb{C}$ if and only if $\sqrt{\Delta_2} \in \mathbb{C},$ finding $Y^{i_1 m_1}_{\alpha_1 \omega_1}, Y^{i_1 m_2}_{\alpha_2 \omega_2} = -Y^{i_1 i_2}_{\alpha_1 \omega_1} \in \mathbb{C}$. Now we exploit the symmetry of $Y$-terms under the simultaneous exchanges $m_1 \equiv m_2$ and $\omega_1 \equiv \omega_2,$ focusing on $Y^{i_1 i_2}_{\alpha_1 \omega_1}$ When these discriminants are not perfect squares, for both $u \in [2]$ the roots

$$\zeta^{(1)} := Y^{m_1 m_2}_{\omega_1 \omega_2}, Y^{i m_1}_{\omega_1 \omega_2}, \quad \zeta^{(2)} := Y^{m_1 m_2}_{\omega_1 \omega_2}, Y^{i m_2}_{\omega_1 \omega_2}$$

of the equation $P^{i_1 i_2}_{\alpha_1 \omega_1}(X) = 0$, where $P^{i_1 i_2}_{\alpha_1 \omega_1}$ is a specification of (4.5), have degree 2 over $\mathbb{F}$. We note that $\zeta^{(1)}$ and $\zeta^{(2)}$ are proportional over $\mathbb{C},$ since $\zeta^{(1)} = -Y^{i_1 i_2}_{\alpha_1 \omega_1}, \zeta^{(2)} = -Y^{i_1 m_2}_{\alpha_1 \omega_1}, Y^{i_1 i_2}_{\alpha_1 \omega_1} \in \mathbb{C}$. 

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Under Assumption 11, we can find an integrable basis. Theorem 23.

Proof. From Lemma 14, the thesis is equivalent to the condition

\[ \{ Y(I)_{i_1,i_2} : \chi(I | i_1,i_2) \text{ is observable} \} \subset \mathbb{C}. \]  

(5.3)

Taking into account Propositions 18 and 21, we only have to check observable sets \( \chi(I | i_1,i_2) \) where at most one (upper or lower) index comes from a non-planar local key \( \epsilon \). We label \( m_1, m_2, \) and \( \omega_1, \omega_2 \) so that \( h(I_{m_1}) \not= 0, u \in [2] \).

We start from observable sets with one index coming from \( c \), say \( m_1 = i_1 \), since an analogous argument holds whether one between \( \omega_1 \) and \( \omega_2 \) lies in \( c \). We easily get \( Y_{\omega_1 \omega_2}^{i_1,m_2} \in \mathbb{C} \) when there exists \( \sigma \in \{ \omega_1, \omega_2, \omega_1, \omega_2 \} \) such that \( h(I_{m_2}) \not= 0 \): in that event, we can pick \( w \in [2] \) such that \( \chi(I | i_1,m_2) \) is observable. Then, noting that \( \chi(I | i_1,m_2) \) is observable as well, Proposition 21 gives \( Y_{\omega_1 \omega_2}^{i_1,m_2} = -Y_{\omega_1 \omega_2}^{i_1,m_2} Y_{\omega_1 \omega_2}^{i_1,m_2} \in \mathbb{C} \).

At \( m_2 \in N(\alpha_1, \alpha_2, \omega_1) \), we specify (4.9)-(4.10) to obtain

\[ \frac{Y(I_{i_1}^{m_1})_{i_2,m_2}}{Y(I)^{i_1,i_2}_{i_1,i_2}} = \frac{Y(I_{i_1}^{m_2})_{i_2,m_2}}{Y(I)^{i_1,i_2}_{i_1,i_2}} \in \mathbb{C} \]  

(5.4)

where the second ratio is constant due to Proposition 21, since \( \epsilon \) remains a non-planar local key under the change \( I \mapsto I_{\omega_1 \omega_2} \) at \( m_2 \in N(\alpha_1, \alpha_2, \omega_1) \). Proposition 21 ensures \( Y(I_{i_1}^{i_2} \omega_1 \omega_2) Y(I_{\alpha_1}^{i_2} \omega_1 \omega_2) \in \mathbb{C} \), which can be combined with (5.4) to get

\[ \frac{Y(I_{i_1}^{m_1})_{i_2,m_2}}{Y(I)^{i_1,i_2}_{i_1,i_2}} \cdot \frac{Y(I_{i_1}^{m_2})_{i_2,m_2}}{Y(I)^{i_1,i_2}_{i_1,i_2}} \in \mathbb{C}. \]  

(5.5)

At \( m_2 \in N(\alpha_1, \alpha_2, \omega_1) \), we also find \( h(H_{m_2}) \not= 0 \) for both \( H \in \{ I, I_{\alpha_1}^{i_1} \} \), while at least one of these two choices, call it \( A \), satisfies \( h(A_{i_1}) \not= 0 \), being \( \chi(I | i_1,i_2) \) a weak local key. Denoting \( \{ i_1 \} := A \cap \{ i_1 \} \), both \( Y(A)^{i_1,m_2}_{i_1,\omega_1} \) and \( Y(A)^{i_1,m_2}_{i_1,\omega_1} \) assume the form (3.2). This is compatible with \( Y(A)^{i_1,m_2}_{i_1,\omega_1} \in \mathbb{C} \) and (5.5) only if \( Y(I)^{m_1}_{i_1,i_2} \in \mathbb{C} \).

Then, we consider all the indices of the observable set outside \( c \) in particular, \( (m_1, \omega_1), (m_2, \omega_2) \in N(c) \) by Proposition 22. So, we can look at \( (m_1, \omega_1) \not\in N(c) \) for some \( s \in [2] \), say \( s = 1 \). Equivalently, this means that there exists \( \{ i_s \} \) and \( \{ i_s \} \) such that \( h(I_{m_1}) h(I_{m_2}) \not= 0 \), which lets us refer to the previous case: the sets \( \chi(I | i_1,m_2) \) and \( \chi(I | i_2,m_2) \) are observable with an index taken from \( c \) (respectively, \( \omega_2 \) and \( i_2 \)), then the previous argument entails \( Y_{\omega_1 \omega_2}^{i_1,m_2} Y_{\omega_1 \omega_2}^{i_1,m_2} \in \mathbb{C} \). On the other hand, we have \( Y_{\omega_1 \omega_2}^{i_1,m_2} Y_{\omega_1 \omega_2}^{i_1,m_2} \in \mathbb{C} \) by Proposition 21, being \( \chi(I | i_1,m_2) \) and \( \chi(I | i_2,m_2) \) observable. In conclusion, we get

\[ Y_{\omega_1 \omega_2}^{i_1,m_2} = -Y_{\omega_1 \omega_2}^{i_1,m_2} Y_{\omega_1 \omega_2}^{i_1,m_2} Y_{\omega_1 \omega_2}^{i_1,m_2} \in \mathbb{C}. \]  

□
6 Propagation of integrability between different bases

To extend the previous result to different bases, we select a weaker condition than Assumption 11, which however suffices to ensure the propagation of the integrability property between bases. In this way, we take advantage of Theorem 23, having a single structural property that guarantees the integrability of all the elements of $\mathcal{G}(L)$. On the other hand, even when Assumption 11 does not hold, but a priori knowledge affirms the existence of an integrable basis, from the Theorem 25 stated below we are able to infer this property for other bases too.

Specifically, this propagation is entailed by the existence of a set $\{g\} \times \{\kappa_1, \kappa_2\} \subseteq I \times I$, such that

$$h(I_{g_1}^2) \cdot h(I_{g_1}^2) \neq 0.$$  \hspace{1cm} (6.1)

Remark 24. The existence of two indices as in (6.1) holds for a set $I$ if and only if it holds for any set $J \in \mathcal{G}(L)$, since the negation of this property entails that there exists a unique map $g : I^c \to I$ returning $h(I_{g(\kappa)}^2) \neq 0$. The possible changes of bases correspond to the substitution of a set $A \subseteq I^c$ with $g(A)$, and vice versa, and this structure is preserved under such exchanges.

Theorem 25. If there exists an integrable basis $I$ and indices $g \in I$, $\kappa_1, \kappa_2 \in I^c$ such that (6.1) holds, then each basis in $\mathcal{G}(L)$ is integrable. In particular, under Assumption 11, each basis in $\mathcal{G}(L)$ is integrable.

Proof. Let $I$ be integrable, and suppose, for the sake of contradiction, that the thesis does not hold for a given basis $J \in \mathcal{G}(L)$. We can construct a finite sequence $\mathcal{L}_0 := I, \mathcal{L}_1, \ldots, \mathcal{L}_u := J$ of elements of $\mathcal{G}(L)$ such that $r = \#(I \Delta J)$ and $\#(L_{u-1} \Delta L_u) = 2, u \in [r]$ as stated in Lemma 12. Say

$$q := \min \{u \in [r] : L_{u-1} \text{ is integrable and } L_u \text{ is non-integrable}\}.$$  \hspace{1cm} (6.2)

To simplify the notation, we denote $A := L_{q-1}, B := L_q$, and indices $v, \omega$ such that $B = A_v^\omega$. Definition (6.2) is equivalent to the existence of an observable set $\chi(B|_{\alpha \beta}^{ij})$ that is not integrable, i.e., $Y(B|_{\alpha \beta}^{ij}) \notin \mathbb{C}$ by Lemma 14. Instantiating (2.17) as

$$Y(B|_{\alpha \beta}^{ij}) = -Y(B)^{\omega \tau \alpha \beta}_s Y(B)^{\omega \tau \alpha \beta}_s Y(B)^{\omega \tau \alpha \beta}_s Y(B)^{\omega \tau \alpha \beta}_s$$  \hspace{1cm} (6.3)

we can move from terms of the form $Y(B|_{\alpha \beta}^{ij})$ to the associated terms $Y(A|_{\alpha \beta}^{ij})$, $s \in \{i, j\}$, and $\tau \in \{\alpha, \beta\}$, through the transformation rules (2.12)-(2.13). From $Y(B|_{\alpha \beta}^{ij}) \notin \mathbb{C}$, the decomposition (6.3), and the transformation rules, we derive the existence of $s \in \{i, j\}$ and $\tau \in \{\alpha, \beta\}$ such that $Y(A|_{\alpha \beta}^{ij}) \notin \mathbb{C}$. By choosing a proper labeling of $\{i, j\}$ and $\{\alpha, \beta\}$, we can write $(s, \tau) = (i, \beta)$. Then, the integrability hypothesis for $A$ allows $Y(A|_{\alpha \beta}^{ij}) \notin \mathbb{C}$ only if $\chi(A|_{\alpha \beta}^{uv})$ is not observable, which entails

$$h(A^i_{\beta}) = 0, \quad h(A^v_{\alpha \beta}) = h(B^i_{\beta}) = 0$$  \hspace{1cm} (6.4)

where the first condition is forced by $h(A^v_{\alpha \beta}) \neq 0$ for the basis $A_v^\omega$. Conversely, the set $\chi(B|_{\alpha \beta}^{ij})$ is assumed to be observable: from the second condition in (6.4), this implies

$$h(B^i_{\beta}) \cdot h(B^i_{\beta}) = h(A^v_{\alpha \beta}) \cdot h(A^v_{\alpha \beta}) \neq 0.$$  \hspace{1cm} (6.5)

So $u \notin \{i, j\}$, $v \notin \{\alpha, \beta\}$, and the sets $\chi(A|_{\alpha \beta}^{uv})$ and $\chi(A|_{\alpha \beta}^{uv})$ are observable too, which entails that each $p \in \{i, j, v\} =: \mathbb{C}_r$ is associated with at least one $\pi \in \{\alpha, \beta, \omega\} =: \mathbb{C}_c$ satisfying $h(A^p_{\alpha \beta}) \neq 0$, and vice versa. When combined with the existence of $Y(A|_{\alpha \beta}^{ij}) \notin \mathbb{C}$, this condition, which requires $h(A^p_{\alpha \beta}) \neq 0$ for at least three pairs $(p, \pi) \in \mathbb{C}_r \times \mathbb{C}_c$, imposes exactly three such pairs, otherwise the observable sets.
\(\chi(A |_{\delta_1^2 \delta_2^2})\) generated by the pairs and the corresponding constant \(Y\)-terms would let us apply (2.17) to find \(Y(A)_{\delta_1^2 \delta_2^2} \in \mathbb{C}\) for all \(a_1, a_2 \in \mathbb{C}\) and \(\delta_1, \delta_2 \in \mathbb{C}\), in particular for \(Y(A)_{\omega_\beta}^{(v)}\). We conclude that there is a map \(g : \mathbb{C}_r \to \mathbb{C}_r\) such that

\[
\forall \delta \in \mathbb{C}_r, a \in \mathbb{C}_r : \ h(A_{\delta}^a) \neq 0 \Leftrightarrow a = g(\delta). \tag{6.6}
\]

Being \(h(A_{\delta}^a) \neq 0\) and \(\chi(A |_{\omega_\beta}) \), \(\chi(A |_{\omega_\beta})\) observable, we can specify \(g(\alpha) : = i\), \(g(\beta) : = j\), and \(g(\omega) : = v\). As a consequence, we get

\[
Y(A)_{\omega_\beta}^{(v)} : Y(A)_{\omega_\alpha}^{(v)}, Y(A)_{\omega_\beta}^{(v)} \in \mathbb{C} \setminus \{0, -1\} \tag{6.7}
\]

since these \(Y\)-terms come from observable sets. Now, we instantiate (4.5), at \((a_1, a_2, a_3) : = (v, i, j)\) and \((\delta_1, \delta_2, \delta_3) : = (\omega, \alpha, \beta)\); then, \(Y_{\omega_\beta}^{(v)}\) satisfies the equation

\[
P_{\omega_\beta}^{(v)}(X) : = P_{\alpha_\beta}^{(v)} (Y_{\omega_\alpha}, X) = 0.
\]

From (6.7) and the condition \(Y(A)_{\omega_\beta}^{(v)} \notin \mathbb{C}\), we infer that the discriminant (4.6) for the set of indices under consideration cannot be a perfect square in \(C(t)\), so the quadratic polynomial \(P_{\omega_\beta}^{(v)}\) is the minimal polynomial of \(Y_{\omega_\beta}^{(v)}\). Furthermore, \(Y(A)_{\omega_\beta}^{(v)} \notin \mathbb{C}\) implies that the coefficient of \(X\) in \(P_{\omega_\beta}^{(v)}(X)\) is not vanishing, so \(Y_{\omega_\alpha}^{(v)}\) is the unique conjugate root of \(Y_{\omega_\beta}^{(v)}\) and \(Y_{\omega_\alpha}^{(v)} \neq Y_{\omega_\beta}^{(v)}\). From the symmetry of (6.6) and \(P_{\alpha_\beta}^{(v)}\) under permutations of \(\{(\rho(\delta), \delta) : \delta \in \mathbb{C}_r\}\), we can summarize this argument and (6.7) as follows:

\[
\delta_1 \neq \delta_2 \neq \delta_3 \neq \delta_1 \Leftrightarrow Y_{c_{\delta_1 \delta_2 \delta_3}^{(v)}(\delta)} \notin \mathbb{C}_r, \ \delta_1, \delta_2, \delta_3 \in \mathbb{C}_r.
\]

Now we invoke the existence of elements \(g \in A\) and \(\kappa_1, \kappa_2 \in A^c\) such that (6.1) holds: elements of this type exist in \(\mathcal{I}\), as follows from the existence of a local key: then, they exist in all the bases in \(\mathfrak{S}(L)\) and, in particular, in \(A\) by Remark 24. Then, we extend the choice map in (6.6), using the same symbol \(g\) with a slight abuse of notation, setting \(g(\kappa_1) = g(\kappa_2) : = g\). For each \(\gamma_1, \gamma_2 \in \{\alpha, \beta, \omega\}\), \(\gamma_1 \neq \gamma_2\), we can adapt the previous argument: when \(Y(A)^{g(\gamma_1)}_{\kappa_1 \gamma_2} \notin \mathbb{C}\), for both \(u, w \in [2]\) the set \(\chi(A |_{\kappa_1 \gamma_2}^{g(\gamma_1)})\) is observable, and the condition (6.6) also holds under the substitution of \(\mathbb{C}_r\) with \(\{\kappa_u, \gamma_1, \gamma_2\}\). Thus, we recover

\[
Y^{g(\gamma_1)}_{\kappa_1 \gamma_2} = Y^{g(\gamma_2)}_{\kappa_2 \gamma_1}
\]

since they coincide with the unique conjugate root of \(Y_{\kappa_2 \gamma_1}^{g(\gamma_1)}\). But this gives \(Y_{\kappa_1 \gamma_2}^{g(\gamma_1)} = -1\), contradicting Assumption 7. Hence, instantiating the discriminant (4.6) to the present set of indices, we find \(\Delta_{\kappa_1 \gamma_2}^{g(\gamma_1)} \in \mathbb{C}\) since it is the only situation where it is a perfect square in \(C(t)\), and the condition

\[
Y(A)^{g(\gamma_1)}_{\kappa_1 \gamma_2}, Y(A)^{g(\gamma_1)}_{\kappa_2 \gamma_1}, Y(A)^{g(\gamma_2)}_{\kappa_2 \gamma_1} \in \mathbb{C}\n\]

holds. In analogy with the proof of Proposition 22, this means

\[
\forall u, s \in [2] : Y_{\kappa_u \gamma_1}^{g(\gamma_1)} \in \mathbb{C}. \tag{6.8}
\]

Finally, the term

\[
Y(A)_{\omega_\beta}^{(v)} = \left(Y(A)_{\omega_\alpha}^{(v)} \cdot Y(A)_{\omega_\beta}^{(v)}\right) \cdot \left(Y(A)_{\omega_\beta}^{(v)} \cdot Y(A)_{\omega_\alpha}^{(v)}\right) \cdot Y(A)_{\omega_\alpha}^{(v)} \tag{6.9}
\]

is constant, since each factor on the right-hand side derives from an observable set or is of the form (6.8). This also includes the cases where \(g \in \{v, i\}\), since this means that some factors are equal to \(-1\) according to the definition (2.11). Having reached a contradiction, the thesis holds and, in particular, it
follows from Assumption 11, since it guarantees the integrability of a basis $I$ by Theorem 23.

Lemma 26. Under the hypothesis of Theorem 25, for all $\alpha \forall \beta$, the function

$$
\psi_2(\alpha; \beta) := \Psi \left( h(J)^{-1} \cdot h(J^p_\beta) \right)
$$

(6.10)
does not depend on the choice of the set $J \in \mathcal{G}(L)$ such that $J^p_\beta \in \mathcal{G}(L)$ too. In particular, this holds if Assumption 11 is verified.

Proof. Take two sets $I := I_1$ and $J := J_1$ in $\mathcal{G}(L)$ such that $I_2 := (I_1)^\alpha$ and $J_2 := (J_1)^\beta$ lie in $\mathcal{G}(L)$ too, then define $\kappa := \#(I \setminus J)$. The proof proceeds by induction on $\kappa$. For the base step $\kappa = 1$, say $I \setminus J := \{j\}$ and $J \setminus I := \{\beta\}$, the condition $I_x, J_x \in \mathcal{G}(L), x \in [2]$, means that $\chi(I \setminus J, \alpha, \beta)$ is observable, therefore the integrability of the bases in $\mathcal{G}(L)$ derived from Theorem 25 entails the thesis.

Then, assume the thesis for all $I, J \in \mathcal{G}(L)$ with $\#(I \setminus J) \leq \kappa$ and all the choices of indices $(i, \alpha) \in (I \setminus J) \times (J \setminus I)$, and look at any 4-tuple of bases $(I, I_\alpha, J, J_\alpha)$ with $\#(I \setminus J) = \kappa + 1$. We can directly infer the thesis when there exists $(j, \beta) \in (I \setminus J) \times (J \setminus I)$ such that $h(I_j) \cdot h(J^j_\beta) \neq 0$ or $h(J_j) \cdot h(J^j_\alpha) \neq 0$: in these cases, we can split the thesis in two claims, i.e., at $(I_j) \cdot h(J^j_\alpha) \neq 0$ we obtain

$$
\Psi \left( h(I_1)^{-1}h(I_2) \right) = \Psi \left( h(I_1)^{j\beta}h(J^j_\beta) \right)
$$

$$
\Psi \left( h(I_1)^{-1}h(I_2) \right) = \Psi \left( h(J^j_\alpha)h(J_2) \right)
$$

(6.11)

The first equation in (6.11) is verified as in the base step, while the second is valid by the induction hypothesis, since $I^j_\beta, J^j_\alpha \in \mathcal{G}(L)$ and $\#(I^j_\beta \setminus J^j_\alpha) = \kappa$. Analogous expressions hold at $h(I_\alpha) \cdot h(J^\alpha_j)$ through the exchange of labels $(I, J) \mapsto (J, I)$ and $(j, \beta) \mapsto (\beta, j)$. Then, we explore some consequences of the lack of such a pair $(j, \beta)$, which is equivalent to the condition

$$
h(I_j) \cdot h(I^j_\alpha) = h(J^j_\beta) \cdot h(J^j_\alpha) = 0, \quad (j, \beta) \in (I \setminus J) \times (J \setminus I).
$$

(6.12)

First, from (2.2) we can find, for each $x \in [2]$ and $j \in I \setminus J$, an element $\beta \in J \setminus I$ with $h(I_x) = 0$: this means that, for each $j$, we can find both indices $\beta_j$, returning $h(I^j_\beta) \neq 0$ (choosing $x = 1$ in the previous statement) and $\beta_j$ with $h(I^j_\beta) = 0$ (choosing $x = 2$ and using (6.12)). The same argument holds for $\beta \in J \setminus I$. On the other hand, we observe that $I^j_\beta, J^j_\alpha \in \mathcal{G}(L)$ holds for all $j \in I \setminus J$ and $\beta \in J \setminus I$: for example, whether $h(I^j_\alpha) = 0$, we can choose $\beta_j \in J \setminus I$ satisfying $h(I^j_\beta) \neq 0$ as before, then $h(I^j_\beta) \neq 0$, which allows us to invoke the inductive hypothesis. Even this argument can be applied starting from $J$ to obtain $J^j_\beta, J^j_\alpha \in \mathcal{G}(L)$.

With these premises, we specify the exchange property (2.2), or its symmetric version (2.3), for the basis $J^j_\alpha$: taking $\pi \in J \setminus I$ and selecting an index $p \in I \setminus J$ such that $h(J^p_\alpha) \neq 0$, from (6.12) we infer $h(J^p_\alpha) = 0$, while the second property discussed above returns $h(I^p_\beta) \cdot h(J^p_\alpha) \neq 0$. In this way, the sets $I^p_\beta, (I^p_\alpha)^{-1}, J^p_\beta,$ and, from $h(J^p_\alpha) \cdot h(J^p_\alpha) \neq 0 = h(J^p_\alpha)$, also $J^p_\alpha$ belong to $\mathcal{G}(L)$; furthermore, $\#(I^p_\beta \setminus J^p_\alpha) = \kappa$, so the inductive hypothesis applies for the pair $(i, p)$, and we obtain

$$
\Psi \left( h(I^p_\alpha) \right) - \Psi \left( h(I^p_\alpha) \right) = \Psi \left( h(J^p_\alpha) \right) - \Psi \left( h(J^p_\alpha) \right)
$$

(6.13)

$$
= \Psi \left( h(J^p_\alpha) \right) - \Psi \left( h(J^p_\alpha) \right)
$$

(6.14)

where (6.13) follows from the induction hypothesis and (6.14) is an instance of the base step. Analogously, for the same index $p$ as above, from (2.3) we can find $\varrho \in J \setminus I$ with $h(I^p_\beta) \cdot h(J^p_\varrho) \neq 0$. Selecting the bases $I^p_\alpha, (I^p_\alpha)^{-1}, J^p_\beta,$ and $(J^p_\alpha)^{-1}, J^p_\varrho$, from $\#(I^p_\beta \setminus J^p_\varrho) = \kappa$ we can apply the induction hypothesis
Fixing an index we get

\[ \Psi(h(\mathcal{I})) - \Psi(h(\mathcal{I}_\alpha)) = \Psi(h(\mathcal{J}_p^0)) - \Psi(h(\mathcal{J}_\alpha^0)) \]  \hspace{1cm} (6.15)

\[ = \Psi(h(\mathcal{J}_p^1)) - \Psi(h(\mathcal{J}_\alpha^1)). \]  \hspace{1cm} (6.16)

Comparing (6.13)-(6.14) and (6.15)-(6.16), we get the thesis.

Theorem 27. Under the hypothesis of Theorem 25, in particular when Assumption 11 is verified, Theorem 1 holds.

Proof. For each \( \mathcal{H} \in \mathfrak{G}(\mathcal{L}) \), denote by \( \bar{\nabla}_\mathcal{H} \) the transitive closure of \( \nabla_\mathcal{H} \). Consider any \( \mathcal{I} \in \mathfrak{G}(\mathcal{L}) \), which is integrable by hypothesis, and focus on \( \bar{\nabla}_\mathcal{I} \). For each \( \alpha \bar{\nabla}_\mathcal{I} \omega \), by definition there exists a finite sequence \( (\delta_1, \ldots, \delta_\kappa) \) with \( \delta_1 = \alpha, \delta_\kappa = \omega \) and \( \delta_i \bar{\nabla}_\mathcal{I} \delta_{i+1} \) for all \( i \in [\kappa - 1] \), so we extend (6.10) introducing

\[ \bar{\psi}_2(\alpha; \omega) := \sum_{i=1}^{\kappa-1} \psi_2(\delta_i; \delta_{i+1}). \]  \hspace{1cm} (6.17)

This definition is consistent, as each pair \( (\delta_i; \delta_{i+1}) \) lies in the domain of \( \psi_2 \). Furthermore, we claim that (6.17) does not depend on the choice of the sequence \( (\delta_1, \ldots, \delta_\kappa) \), but only on its endpoints: due to \( \bar{\psi}_2(\alpha; \beta) = -\bar{\psi}_2(\beta; \alpha) \), it is sufficient to show that \( \bar{\psi}_2 \) vanishes on closed paths, i.e., \( \bar{\psi}(\alpha; \alpha) = 0 \).

From \( \delta_i \in \mathcal{I} \Leftrightarrow \delta_{i+1} \in \mathcal{I}^c \), each closed path contains an odd number of indices including the coinciding endpoints, say \( 2 \cdot p + 1 \) with \( \delta_1 = \delta_{2p+1} \); then we prove \( \psi_2(\alpha; \alpha) = 0 \) for all closed paths by induction on \( p \). Since the case \( p = 1 \) is equivalent to \( \psi_2(\alpha; \beta) = -\psi_2(\beta; \alpha) \), the base step is \( p = 2 \), where the thesis holds due to Theorem 25. Regarding the induction step, we assume that the thesis holds for all \( p \) and consider any closed path \( (\delta_1, \ldots, \delta_{2p+3}) \) with \( \delta_1 = \delta_{2p+3} \) and \( \delta_i \bar{\nabla}_\mathcal{I} \delta_{i+1} \) for all \( i \in [2p + 2] \). To simplify the notation, we consider indices modulo \( 2p + 2 \), that is, \( \delta_{2p+2+1} = \delta_1 \) for all \( l \in [2p + 2] \), and act via a cyclic shift of labels, if necessary, to have \( \delta_1 \in \mathcal{I} \). If we can find \( \delta_u, \delta_w \) such that \( 1 < w - u < 2p + 1 \) and \( \delta_u \bar{\nabla}_\mathcal{I} \delta_w \), then we write

\[ \bar{\psi}_2(\delta_1; \delta_1) = \left( \sum_{i=u+1}^{w-1} \psi_2(\delta_{i-1}; \delta_1) + \psi_2(\delta_u; \delta_u) \right) \]

\[ + \left( \sum_{j=w+1}^{2p+2+u} \psi_2(\delta_{j-1}; \delta_1) + \psi_2(\delta_u; \delta_u) \right). \]

Each of the two sums is over a cycle with length not greater than \( 2p \), then the inductive hypothesis applies, and \( \bar{\psi}_2(\delta_1; \delta_1) = 0 \). Otherwise, \( \delta_u \bar{\nabla}_\mathcal{I} \delta_w \) never holds at \( |u - w| \neq 1 \): in this case, for all \( M \neq 0 \) we get

\[ h(I_{\delta_{2u+1}}^0 \cdot h(I_{\delta_{2u+2M+1}}^0) \neq 0 = h(I_{\delta_{2u+1}}^0 \cdot h(I_{\delta_{2u+2M+1}}^0) \]

then the Plücker relations impose

\[ h(I_{\delta_{2u}^0 \delta_{2u+2M+1}}^0) \neq 0. \]  \hspace{1cm} (6.18)

Fixing an index \( u \), say \( u = 1 \), we obtain

\[ \psi_2(\delta_1; \delta_2) + \psi_2(\delta_2; \delta_3) + \psi_2(\delta_3; \delta_4) \]

\[ = \Psi \left( \frac{h(I_{\delta_2}^0)}{h(I^0_{\delta_2})} \right) + \Psi \left( \frac{h(I)}{h(I_{\delta_2}^0)} \right) + \Psi \left( \frac{h(I_{\delta_3}^0)}{h(I^0_{\delta_2})} \right) \]

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\[
\begin{aligned}
\text{(by (6.18))} & \quad \Psi \left( \frac{h(I_{\delta_2}^{\delta_1})}{h(I_{\delta_2}^{\delta_1})} \right) + \Psi \left( \frac{h(I_{\delta_2}^{\delta_1})}{h(I_{\delta_2}^{\delta_1})} \right) \\
& = \Psi \left( \frac{h(I_{\delta_2}^{\delta_1})}{h(I_{\delta_2}^{\delta_1})} \right) \\
& = \psi_2(\delta_1; \delta_2). & (6.19)
\end{aligned}
\]

Therefore, the sequence \((\delta_1, \delta_1, \ldots, \delta_{2p+2}, \delta_1)\) is valid for the basis \(I_{\delta_2}^{\delta_1} \in \mathfrak{S}(L)\) due to (6.18) and gives rise to the same value for \(\bar{\psi}_2(\delta_1; \delta_1)\) by (6.19) and Lemma 26. The length of the sequence \((\delta_1, \delta_1, \ldots, \delta_{2p+2}, \delta_1)\) is \(2p + 1\), therefore, the inductive hypothesis applies, and the claim is proved.

Finally, we can explicate the function \(\psi\) mentioned in the thesis: fix an arbitrary set \(I \in \mathfrak{S}(L)\), choose a representative \(\bar{t}_e\) for each equivalence class \(c\) of \(\bar{\nu}_I\), and assign a \(d\)-tuple \(\psi(\bar{t}_e) \in \mathbb{Z}^d\) to every chosen representative. Then, for each \(\alpha \in [n]\) belonging to the same class of \(\bar{t}_e\), define
\[
\psi(\alpha) := \psi(\bar{t}_e) + \bar{\psi}_2(\bar{t}_e; \alpha). & (6.20)
\]

The components of each pair \((i_u, \alpha_u)\) such that \(\mathcal{I}_{\alpha_u} \in \mathfrak{S}(L)\) lie in the same equivalence class of \(\bar{\nu}_I\), so we have
\[
\bar{\psi}_2(i_u; \alpha_u) = \bar{\psi}_2(i_u; \bar{t}_e) + \bar{\psi}_2(\bar{t}_e; \alpha_u) & (6.21)
\]

The Grassmann-Plücker relations imply that for any \(\alpha \in \mathcal{J} \setminus I\), there exists \(i \in \mathcal{I} \setminus J\) with \(\mathcal{I}_{\alpha}, \mathcal{J}_i^\alpha \in \mathfrak{S}(L)\). Then, we can choose an ordering \((\alpha_1, \ldots, \alpha_r)\) for the set \(\mathcal{J} \setminus I\) and construct a sequence of sets in \(\mathfrak{S}(L)\) using
\[
\mathcal{T}_0 := \mathcal{J}, \quad \mathcal{T}_u := (\mathcal{T}_{u-1})_{\alpha_u}^\alpha & (6.22)
\]
where each \(i_u\) satisfies \((\mathcal{T}_{u-1})_{\alpha_u}^\alpha, \mathcal{I}_{\alpha_u} \in \mathfrak{S}(L)\). Clearly, \(i_u = i_w\) implies \(u = w\), since \(u < w\) implies \(i_u \in \mathcal{T}_{w-1}\) and \(i_w \notin \mathcal{T}_{w-1}\). In conclusion, we get
\[
\Psi(\mathcal{J}) = \Psi(\mathcal{I}) + \sum_{u=0}^{r-1} \Psi(\mathcal{T}_u) - \Psi((\mathcal{T}_{u+1})_{\alpha_{u+1}}^\alpha) & (6.23)
\]
that is (1.11) with \(m_0 := \psi(\bar{t}_e) - \sum_{i \in \mathcal{I}} \psi(i_u)\).

Remark 28. Lemma 26 involves fixed elements and a transition between bases, while the claim in the proof of Theorem 27 keeps the basis fixed and moves between the elements of \([n]\). Even if the specific properties that characterize each of these two problems are addressed in the corresponding proofs, it should be noted that they can be seen as dual statements in terms of the bases (6.22) obtained from the Plücker relations.
7 Counterexamples

The assumptions leading to the previous results highlight the information content required to recover integrability. In [3], which pertains to cases with generic \( L(t) \), the amount of information is provided by the dimensionality of the matrices. Allowing for larger sparsity of \( L(t) \), as in this work, the information is provided by the existence of a non-planar local key.

When the assumptions underlying the results in Section 6 are not met, we can find counterexamples to integrability: in the following, this is done by relaxing the individual conditions that define a non-planar local key.

Where appropriate, in the rest of this section, we will use subscripts for block matrices to make explicit the dimensions of the constituting blocks.

7.1 Reduction to Principal Minor Assignment

We start with an example that involves a matroid \( \mathcal{G}(L) \) that provides minimal information. Let us consider

\[
L_0 := (I_k \mid 1_k) \in \mathbb{C}^{k \times (2k)}
\]

where \( I_k \) is the \( k \)-dimensional identity matrix, so \( \mathcal{G}(L_0) \) does not satisfy Assumption 11. Setting \( R(t) := (I_k \mid r(t))^T \), this model is equivalent to a Principal Minor Assignment for \( r(t) \) restricted to the set of units in \( C(t) \). Let \( S \in \mathbb{C}^{k \times k} \) be a generic skew-symmetric constant matrix and specify

\[
r_0(\tau) := \tau \cdot 1_k \cdot 1_k^T + S, \quad R_0(\tau) := (1_k \mid r_0)^T
\]

for a non-constant unit \( \tau \in C(t) \), where \( 1 := (1, \ldots, 1) \in \mathbb{C}^k \). These matrices satisfy (1.10), but not (1.11), since the \( Y \)-terms are not constant. This claim is formalized as follows:

**Proposition 29.** For a configuration defined by (7.1)-(7.2) with a generic choice of \( S \) in the space of \((k \times k)\) skew-symmetric complex matrices, we get

\[\Delta_{R_0(\tau)}(\mathcal{I}) \in \mathbb{C} \Leftrightarrow 2 \mid \#([k] \setminus \mathcal{I}).\] (7.3)

In particular, this configuration is not integrable.

**Proof.** According to Remark 24, the evaluation of observable minors for the present configuration is reduced to the evaluation of principal minors of \( r_0(\tau) \). To simplify the notation, we introduce \( \kappa(\mathcal{I}) := [k] \setminus \mathcal{I} \), so that \( \Delta_{R_0(\tau)}(\mathcal{I}) = \det \left( r_0(\tau)_{\kappa(\mathcal{I})} \right) \) for all \( \mathcal{I} \in \mathcal{G}(L_0) \), where \( r_0(\tau)_{\kappa(\mathcal{I})} \) denotes the submatrix of \( r_0(\tau) \) with rows, respectively, columns, indexed by \( A \), respectively, \( B \). For each \( \mathcal{I} \in \mathcal{G}(L_0) \) with \( 2 \mid \#([k] \setminus \mathcal{I}) \), the matrix determinant lemma [11] implies

\[\Delta_{R_0(\tau)}(\mathcal{I}) = \left( 1 + \tau \cdot 1^T \cdot \left( r_0(0)_{\kappa(\mathcal{I})}^{\kappa(\mathcal{I})} \right)^{-1} \cdot 1 \right) \cdot \Delta_{R_0(\tau)}(\mathcal{I}) = \Delta_{R_0(\tau)}(\mathcal{I}) \in \mathbb{C} \] (7.4)

since the skew-symmetric matrix \( \left( r_0(0)_{\kappa(\mathcal{I})}^{\kappa(\mathcal{I})} \right)^{-1} \) verifies \( v^T \cdot \left( r_0(0)_{\kappa(\mathcal{I})}^{\kappa(\mathcal{I})} \right)^{-1} \cdot v = 0 \) for all \( v \in \mathbb{C}^k \), in particular for \( v = 1 \).

For the remaining sets \( \mathcal{I} \in \mathcal{G}(L_0) \), we choose any \( \alpha \in \mathcal{I} \) and introduce

\[r_{(\alpha)}(\tau) := r_0(\tau)_{\kappa(\mathcal{I}) \setminus \{\alpha\}} \cdot \tau^{-1} \cdot \left( r_0(\tau)_{\kappa(\mathcal{I}) \setminus \{\alpha\}} \right)^\top \cdot \left( r_0(\tau)_{\kappa(\mathcal{I}) \setminus \{\alpha\}} \right)^{-1}.\] (7.5)

This is the Schur complement of the invertible matrix \( r_0(\tau)_{\kappa(\mathcal{I}) \setminus \{\alpha\}} = (\tau) \) in \( r_0(\tau)_{\kappa(\mathcal{I})} \). Setting \( s := \]
\( r_0(t) \), with simple algebraic manipulations, we get the equivalent expression
\[
 r_{(\alpha)}(\tau) = \tau^{-1} \cdot s \cdot s^T + r_0(\kappa(\alpha)) + (1 \cdot s^T - s \cdot 1^T). \tag{7.6}
\]
The matrix \( r_0(\kappa(\alpha)) + (1 \cdot s^T - s \cdot 1^T) \) is constant, skew-symmetric, and even-dimensional, so we can invoke the matrix determinant lemma again and find \( \det(r_{(\alpha)}(\tau)) \in \mathbb{C} \). From Schur formula (see, e.g., [11, Sect. 2.5] and [15])
\[
 \Delta_{r_0(\tau)}([k]_\alpha)^{-1} \cdot \Delta_{r_0(\tau)}(I) = \det(r_{(\alpha)}(\tau)) \tag{7.7}
\]
and \( \Delta_{r_0(\tau)}([k]_\alpha) = \tau \), we conclude that \( \Delta_{r_0(\tau)}(I) \) is a non-constant monomial.

In the same way, we can use (7.4) or (7.7) to see that the maximal minors of \( R_0(\tau) \) do not vanish for a generic choice of \( S \).

Specifying this argument for two different indices \( i \neq j \) in \([k]\), and taking \( \alpha, \beta \in [k]^C \) with \( h([k]_\alpha^i) \cdot h([k]_\beta^j) \neq 0 \), we find \( h([k]) \cdot h([k]_\alpha^i, [k]_\beta^j) \in \mathbb{C} \) from (7.4), while \( h([k]_\beta^j) \cdot h([k]_\alpha^i) \notin \mathbb{C} \). So \( Y \in [k]_{\alpha, \beta} \notin \mathbb{C} \), which is not compatible with (1.11) by Lemma 14.

Going back to the concluding remark in the Introduction, we use this counterexample to briefly comment on the incomparability of rankings on \([n]\) for non-integrable configurations, which is of interest in relation to previous work on applications of tropical algebra in statistical physics [1].

Given an injective function \( \psi : [n] \rightarrow \mathbb{Z}^d \), for each permutation \( \sigma \in S_n \) we can consider the domains in \( \mathbb{C}^d \) composed of points \( t_\sigma \) such that \( \| t_\sigma^{(\sigma(\alpha))} \| < \| t_\sigma^{(\sigma(\beta))} \| \) for all \( \alpha \in [n-1] \). Since the union of these domains is dense in \( \mathbb{C}^d \) and their number is bounded by \( n! \) and therefore finite, at least one of them is unbounded, and we will choose an evaluation point \( t_0 \) in this domain.

Each order on \([n]\) that is consistent with this evaluation induces a lexicographic order on \([k]\) and its subsets, including \( \mathfrak{S}(L_0) \). On the other hand, taking \( \tau = t^{(\psi(\sigma^{-1}))} - t^{(\psi(\sigma^1))} \) in (7.2), we see that an incomparability arises: relabeling the indices of columns of \( L_0 \) and rows of \( R_0(\tau) \) through a matrix representation of \( \sigma \) using (1.5), we can assume \( \| t_0^{(\psi(\alpha))} \| < \| t_0^{(\psi(\beta))} \| \) for all \( \alpha < \beta \); then, we introduce
\[
 \tilde{R}_0(t) := \text{diag} \left( t^{(\psi(1))}, \ldots, t^{(\psi(n))} \right) \cdot R_0(\tau)
\]
which still satisfies the invertibility assumption (1.10) with the same structural matrix \( L_0 \). Looking at the expansion generated by \( (L_0, \tilde{R}_0(t)) \), and taking \( I := [k] \) and \( (i, \alpha) \) with \( h(I_\alpha^i) \neq 0 \), we find
\[
 \left\| \Delta_{R_0(t_0)}(I_\alpha^i) \cdot \Delta_{R_0(t_0)}(I) \right\| > 1
\]
which agrees with the chosen ranking \( \| t_0^{(i)} \| < \| t_0^{(\alpha)} \| \). On the other hand, for any \( j \in I, j \notin \{i, n\}, \beta \in I^C \) with \( J := I_\beta^j \neq 0 \), and \( t_0 \) large enough, from Proposition 29 we get
\[
 \left\| \Delta_{R_0(t_0)}(J_\beta^j) \cdot \Delta_{R_0(t_0)}(J) \right\| < 1
\]
since \( 2 \mid \#(J \setminus I) \) and \( 2 \mid \#(J_\beta^j \setminus I) \).

We can say that the comparison of elements \( i \) and \( \alpha \) exhibits contextuality, that is, it depends on the basis used for the comparison. More precisely, we can interpret the norms \( \| t_0^{(\psi(1))} \| \) and \( \| t_0^{(\psi(1))} \| \) (or \( \| \Delta_{R_0(t_0)}(\cdot) \| \) at \( \| t_0 \| \rightarrow \infty \)) as weights that define two rankings on \([n]\) and \( \mathfrak{S}(L) \), respectively. Then, the contextual effect mentioned above or, more generally, a deviation from (1.11), leads to a violation of an Independence of Irrelevant Alternatives property, a fundamental condition in the study of decision-making and choice theory [19, Ch. 1.C].
7.2 Multiple weak local keys

Several weak local keys, none of which is a local key, do not guarantee the integrability of the model. On the basis of the proofs in the previous sections, we can construct a counterexample with

\[
L_w := \begin{pmatrix} 1_k & 1_{2 \times (p+1)} & 0_{2 \times (k-2)} \\ 0_{(k-2) \times (p+1)} & 1_{(k-2) \times (k-2)} \end{pmatrix}
\]  

(7.8)

where \(1_{e \times f}\) and \(0_{e \times f}\) denote the \((e \times f)\)-block matrices with all entries equal to 1 and 0, respectively, and

\[
R_w(\xi) := \begin{pmatrix} 1 & 1_{1 \times p} & -1_{1 \times (k-2)} \\ 1 & d_{1 \times p} & 1_{1 \times (k-2)} \\ f_{(k-2) \times 1} & C_{(k-2) \times p} & S_{(k-2) \times (k-2)} \end{pmatrix}^\tau
\]

(7.9)

where \(\xi \in \mathbb{C}(t)\) is a non-constant unit, \(d \in \mathbb{C}^{1 \times p}\), \(i \in \mathbb{C}^{(k-2) \times p}\), and \(S \in \mathbb{C}^{(k-2) \times (k-2)}\) are generic, \(r_s := \xi \cdot i_s^{-1} - 1\) for all \(s \in [k-2]\), and \(S\) satisfies \(S_{s,u} = i_{s}^{-1}u^\cdot(1 - S_{u,s}) + 1\) and \(S_{s,s} = 1\) for all \(s, u \in [k-2]\). So, we get \(n = 2 \cdot k + p - 1\).

For the same \(L_w\), we can also consider a new matrix obtained from \(R_w(\xi)\) using (1.5) and the invariance of maximal minors of \(R_w(\xi)\) under the action of \(GL_k(\mathbb{C})\) (up to a common factor), that is,

\[
\begin{pmatrix} 1 & 0 & 0_{1 \times (k-2)} & \xi^{-1} & 1_{1 \times p} & -a_{1 \times (k-2)} \\ 0 & 1 & 0_{1 \times (k-2)} & 1 & d_{1 \times p} & 1_{1 \times (k-2)} \\ f_{(k-2) \times 1} & a_{(k-2) \times 1} & c_{(k-2) \times p} & s_{(k-2) \times (k-2)} \end{pmatrix}^\tau
\]

(7.10)

where \(a_u := i_u^{-1}\), \(c_{u,s} := C_{u,s} + 1\), and \(s_{w,u} = -s_{u,w} := a_w \cdot (s_{w,u} - 1)\) for all \(u, w \in [k-2]\), \(u < w\), and \(s \in [p]\). We refer to Appendix A for an example of (7.9) at \((k, p) = (5, 4)\).

We can check that the minors \(\Delta_{R_w(\xi)}(I)\) with \(I \in \Theta(L_w)\) are monomials, but not all of them are constant, while the remaining maximal minors of (7.10) are non-vanishing under the generic assumption for the free constants of the matrix. In particular, \(Y([k]_2^{12})_{(k+1)(k+2)} \notin \mathbb{C}\), which is in contrast to the form (1.11), so integrability is not achieved.

It is worth noting that the pattern provided by (7.8) at \(k = 3, p = 2\) also allows us to present a non-integrable model with an odd-type configuration (3.8), i.e.,

\[
\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -c & c \cdot \xi & c \cdot (\xi - 1) \\ 0 & 1 & 0 & -c \cdot (\xi + 1) & -c \cdot (1 + \xi^{-1}) & -c \end{pmatrix}^\tau, \ c \in \mathbb{C}
\]

(7.11)

7.3 Planar local key

Finally, we find a family of counterexamples that violate the condition (1.11) due to the lack of existence of a non-planar local key. It can be proved that the definition of \(R_p(\zeta)\) shown below arises as a non-integrable configuration whether there exists a local key \(c \subseteq I \times \mathbb{C}\) for a basis \(I \in \Theta(L_p)\) and, for all \(\alpha \in I^2 \setminus c\), there is only one \(g(\alpha) \in I\) such that \(h(I^2_{\alpha}) \neq 0\).

We fix a non-constant monomial \(\xi \in \mathbb{C}(t)\) such that \(Y_{\alpha_1 \alpha_2}^{12} + 1 = c \cdot \zeta\) with \(c \in \mathbb{C}\) and \(w \in (I_{\alpha_1 \alpha_2})^2\), with \(c_{\alpha_2} = 1\). With these data, we can get a non-integrable configuration: applying (1.5), here \(R_p(\zeta)\) takes the block form

\[
\begin{pmatrix} -\tau^{-2} & \tau^{-2} & 0_{1 \times (k-2)} & 0 & -1 & c_r \\ \phi^2 + 1 & 1 & 0_{1 \times (k-2)} & 1 & 0 & 1_{1 \times (n-k-2)} \\ 0_{(k-2) \times 1} & 0_{(k-2) \times 1} & \mathbb{I}_{k-2} & \mathbb{I}_{(k-2) \times 1} & c_c & \mathbb{Z}^{(2,2)} \end{pmatrix}^\tau
\]

(7.12)
where \( \mu_{\{u,w\}} := \Delta_{\alpha_1,\omega_1,\alpha_2,\omega_2} \in \mathbb{C} \) and, for all \( m_u \in \mathcal{I}^{i_1,i_2} \) and \( \omega_w \in (\mathcal{I}_{\alpha_2})^2 \), we specify
\[
(c_r)_{\omega_u} = -(c_r)_{m_u}^{-1} := c_u^2,
\]
\[
(Z^{(2,2)})_{m_u,\omega_w} := 1 + \text{sign}(w-u) \cdot \frac{c_w \mu_{\{u,w\}}}{c_u}.
\]

Similarly to the previous example, we find that (7.12) satisfies (1.10) and the generic condition, but \( Y([k])^{13} \in \mathbb{C} \), in contrast to (1.11).

8 Conclusion

In this work, we focused on the effects of structural information (encoded in the matroid \( \mathcal{G}(L) \) of non-vanishing maximal minors of a matrix \( L \)) on the components of a determinantal expansion under given algebraic conditions (the invertibility of terms (1.3) in the ring \( \mathbb{C}(t) \)).

The discussion in the previous sections suggests different extensions of the present work, starting from the factorization properties of the hyperdeterminants (4.6) in \( \mathbb{C}(t) \). The Principal Minor Assignment Problem, which has already been mentioned as a counterexample in Section 7.1, is of particular interest, since it violates the hypotheses underlying the results of Section 6; furthermore, a detailed study of the Principal Minor Assignment Problem in the unit group \( \mathbb{C}(t)^{\times} \) is needed to characterize non-integrable configurations for a better comprehension of the minimality of Assumption 11, beyond the examples presented in Section 7. This stimulates an in-depth investigation of the relations between \( Y \)-terms, starting from (2.17) and its iterations, and the algebraic extensions of \( F \) where these terms lie. The latter relate to the factor multiplicity for hyperdeterminants of the type (4.6) and their products, which could also support the analysis of algebraic constraints other than (1.3).

The reduction of \( \mathbb{Z}^d \)-valued set functions stated in Theorem 1 when Assumption 11 is satisfied may be used to streamline verification processes involving these set functions, which is the focus of [3]. On the other hand, a deviation from this structural condition could support uncertainty modeling related to the identifiability of individual terms within a set, especially when this uncertainty affects their labeling, and the available information regards subsets. As already mentioned, possible deviations from the integrability condition might be of interest in the comparison of tropical structures borrowed from statistical physics, in continuity with the present work.

Finally, we point out the need to better formalize the potential interpretation of the basic objects of this framework, that is, the two matrices \( L(t) \) and \( R(t) \), in the context of statistical modeling, in particular instrumental variable estimation. The remark in Section 2.2 and the geometric implications of the present model deserve more attention, in line with the properties of the expansion (1.1) mentioned in the Introduction, to better understand the contribution of variable terms (1.10) in this class of regression problems, especially the geometric characteristics of the parametrized families of subspaces associated with \( L(t) \) and \( R(t) \).

A Code for additional verifications

In the following, we report the Wolfram Mathematica commands to check the fulfilment of the invertibility condition (1.10) and the generic property of \( R(t) \) for the configurations presented in Section 7.

First, we introduce two basic functions to select and compute observable sets.

Listing 1: Selection of the matroid indexing the non-vanishing minors of a general matrix \( X \)

```
matro[x_] := Select[Subsets[Range[Length[Transpose[x]]]], (Length[x])], Defined[(All, #)] ||= 0 &
```
Listing 2: Function returning an integer only if P is a monomial in x, in which case the output is the degree

\[
\text{ismon}(p_, x_) \rightarrow \text{FullSimplify}(D(p, x) \times p^\wedge(-1 + 2 \cdot \text{Boole}(\text{SameQ}(p, 0))))
\]

We start analysing the counterexample in Section 7.2. We choose \((k, p) = (5, 4)\), so \(n = 13\), instantiate 11 as (7.8) with these data, and define

\[
\begin{align*}
\mathbf{r}_{11}^T &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 & a_1 & \xi -1 & c_{12} & c_{13} & c_{14} & 1 & aux_1(a_1, a_2, g_{21}) & aux_1(a_1, a_3, g_{21}) \\
0 & 0 & 0 & 1 & 0 & a_2 & \xi -1 & c_{22} & c_{23} & c_{24} & g_{21} & 1 & aux_1(a_2, a_3, g_{21}) & 1 \\
0 & 0 & 0 & 0 & 1 & a_3 & \xi -1 & c_{32} & c_{33} & c_{34} & g_{31} & 1 & aux_1(a_3, a_3, g_{21}) & 1
\end{pmatrix}
\end{align*}
\]

where we have introduced the function

\[
\text{aux}_1(a_1_, a_2_, g_{21}) = a_1/a_2 \cdot (-g_{21} + 1) + 1
\]

The verification of the invertibility condition (1.10) is carried out through the following code:

```
Listing 3: List of non-vanishing terms in the expansion and verification of the invertibility condition

\[
\text{FullSimplify}([\text{Table}([\text{Det}([\{\text{Aux}[i], \text{matro}[l_1]\}_\{i\}], \{i, 1, \text{Length}[\text{matro}[l_1]]\})])]), \{i, 1, \text{Length}[\text{matro}[l_1]]\})
\]

\[
\text{deg}_{1} = \text{FullSimplify}([\text{Table}([\text{ismon}[\text{Det}([\{\text{Aux}[i], \text{matro}[l_1]\}_\{i\}], \{i, 1, \text{Length}[\text{matro}[l_1]]\})]), \{i, 1, \text{Length}[\text{matro}[l_1]]\})])
\]

\[
\text{Apply}[\text{And}, \text{Map}[\text{IntegerQ}, \text{deg}_{1}]]
\]

\[
\text{The output associated with Line 5 is a list of integers, so the associated terms are monomials. In particular, the terms associated with the bases indexed by the following elements of } \varphi_{5}[13]
\]

\[
\begin{align*}
\{1, 3, 4, 5, 6\}, & \quad \{1, 3, 6, 12, 13\}, \quad \{1, 4, 6, 11, 13\}, \\
\{1, 5, 6, 11, 12\}, & \quad \{2, 3, 4, 6, 13\}, \quad \{2, 3, 5, 6, 12\}, \\
\{2, 4, 5, 6, 11\}, & \quad \{2, 6, 11, 12, 13\}
\end{align*}
\]

have degree 1 in \(\xi\), and the remaining bases return constant terms. The fulfilment of the invertibility condition is also summarised by the Boolean output of Line 6, that is \text{True}.

Listing 4: Check that all the maximal minors of \(R_{w}(\xi)\) are non-vanishing

```
\[
\text{sub}_{1} = \text{Subsets}([\{\text{Range}([\text{Length}([\text{Transpose}([1])])], [\text{Length}([1])])])]), [\text{Length}([1])])
\]

\[
\text{MemberQ}([\text{FullSimplify}([\text{Table}([\text{Det}([\{\text{Aux}[i], \text{sub}_{1}[i]\}_\{i\}], \{i, 1, \text{Length}([\text{sub}_{1}[i]])\}], \{i, 1, \text{Length}([\text{sub}_{1}[i]])\})])]), \{i, 1, \text{Length}([\text{sub}_{1}[i]])\}], 0)
\]

The last query returns \text{False}, which means that 0 does not appear in the list of expressions \(\Delta_{r_1}(\xi)\), in line with Assumption 7.

Moving to the second counterexample in Section 7.2 concerning odd-type configurations from the matrix (7.11), we choose \((k, p) = (3, 2)\) and set

\[
\text{oddtype} = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathbf{r}_{12}^T = \begin{pmatrix}
1 & 0 & 0 & 1 & -1 & 1 & \xi & 1 - (\xi - 1) \\
0 & 0 & 1 & 1 - (\xi + 1) & -c(1 + \frac{1}{\xi}) & -c
\end{pmatrix}
\]

The check of the invertibility hypothesis

```
\[
\text{FullSimplify}([\text{Table}([\text{ismon}([\text{Det}([\text{r}_{12}^T\text{oddtype}[[\{\text{Aux}[i], \text{matro}\}[\text{oddtype}\}_\{i\}], \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])\}], \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])])])])]), \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])])])
\]

returns the list of monomials

\[
\{1, -c, -1, c, -c_{2}, -c_{3}, c_{2}^{2}, c_{3}, 1, 1, 1, c + 1, -c_{3}, -c\}
\]

Furthermore, the degrees computed from

```
\[
\text{FullSimplify}([\text{Table}([\text{ismon}([\text{Det}([\text{r}_{12}^T\text{oddtype}[[\{\text{Aux}[i], \text{matro}\}[\text{oddtype}\}_\{i\}], \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])])])]), \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])])])])
\]

returns the list of monomials

\[
\{1, -c, -1, c, -c_{2}, -c_{3}, c_{2}^{2}, c_{3}, 1, 1, 1, c + 1, -c_{3}, -c\}
\]

Furthermore, the degrees computed from

```
\[
\text{FullSimplify}([\text{Table}([\text{ismon}([\text{Det}([\text{r}_{12}^T\text{oddtype}[[\{\text{Aux}[i], \text{matro}\}[\text{oddtype}\}_\{i\}], \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])])])]), \{i, 1, \text{Length}([\text{matro}[\text{oddtype}])])])])
\]

returns the list of monomials

\[
\{1, -c, -1, c, -c_{2}, -c_{3}, c_{2}^{2}, c_{3}, 1, 1, 1, c + 1, -c_{3}, -c\}
\]
are in line with odd-type configurations:

\[ \{0, 0, 0, 0, 0, 0, 0, 1, -1\} \]

Also for the matrix \( r1\text{odd} \), the code

\[
\text{sub1odd} := \text{Subsets[Range[Length[Transpose[oddtype]]], \{Length[oddtype]\}]} \\
\text{MemberQ[FullSimplify[Table[Def[r1\text{odd}(All, sub1odd[i]), i], 1, Length[sub1\text{odd}]]], 0]}
\]

returns \( \text{False} \).

Moving to the counterexample in Section 7.1, we specify (7.1) at \( k = 6 \) and set

\[
r2^T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \tau & \tau + s(1, 2) & \tau + s(1, 3) & \tau + s(1, 4) & \tau + s(1, 5) & \tau + s(1, 6) \\
0 & 1 & 0 & 0 & 0 & \tau - s(1, 2) & \tau & \tau + s(2, 3) & \tau + s(2, 4) & \tau + s(2, 5) & \tau + s(2, 6) \\
0 & 0 & 1 & 0 & 0 & \tau - s(1, 3) & \tau - s(2, 3) & \tau & \tau + s(3, 4) & \tau + s(3, 5) & \tau + s(3, 6) \\
0 & 0 & 0 & 1 & 0 & \tau - s(1, 4) & \tau - s(2, 4) & \tau - s(3, 4) & \tau & \tau + s(4, 5) & \tau + s(4, 6) \\
0 & 0 & 0 & 0 & 1 & \tau - s(1, 5) & \tau - s(2, 5) & \tau - s(3, 5) & \tau - s(4, 5) & \tau & \tau + s(5, 6) \\
0 & 0 & 0 & 0 & 0 & \tau - s(1, 6) & \tau - s(2, 6) & \tau - s(3, 6) & \tau - s(4, 6) & \tau - s(5, 6) & \tau
\end{bmatrix}
\]

As before, the code

\[
pos1 = \text{FullSimplify[Table[If[FullSimplify[r2[[\{All, matro[[2]][[i]\}\}], \{\tau\}], \{i, 1, Length[matro[[2]]]\}]], (I)]]} \\
\text{Apply[And, Map[IntegerQ, pos1]]}
\]

returns \( \text{True} \), so the invertibility condition is satisfied. In addition, we provide the code for a check of the thesis of Proposition 29:

\[
p1 = \text{Flatten[Position[pos1, 0]]} \\
p2 = \text{Complement[Range[Length[pos1]], p1]} \\
\text{Apply[And, Table[Mod[Length[Complement[Range[6], Part[matro[[2]], (p1)]], 2]], 2]} == 0, \{i, 1, Length[matro[[2]]]\}]
\]

Through Lines 15 – 16, we retrieve the positions of the constant and non-constant observable terms. Lines 17 – 18, which together express the equivalence (7.3), both return \( \text{True} \), in agreement with Proposition 29.

Then, we check the generic property of \( r2 \): the corresponding commands

\[
\text{sub2b} := \text{Subsets[Range[Length[Transpose[2]]], \{Length[2]\}]} \\
\text{MemberQ[FullSimplify[Table[Def[r2((\{Range[6], sub2b[i]\}), \{i, 1, Length[sub2b]\})], 0]}
\]

return \( \text{False} \), as required by Assumption 7.

Finally, the existence of a unique, planar local key described in Section 7.3 is addressed specifying, at \( k = 6 \), the matrices

\[
r3^T = \begin{bmatrix}
-\tau^2 & \tau^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Adapting Lines 13-14 and 19-20 to these choices of \( L \) and \( R \), we get analogous validations for \( r3 \) too.
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