On the Stability of Modified Patankar Methods*

Thomas Izgin† and Philipp Öffner‡

Abstract. Patankar schemes have attracted increasing interest in recent years because they preserve the positivity of the analytical solution of a production-destruction system (PDS) irrespective of the chosen time step size. Although they are now of great interest, for a long time it was not clear what stability properties such schemes have. Recently a new stability approach based on Lyapunov stability with an extension of the center manifold theorem has been proposed to study the stability properties of positivity preserving time integrators. In this work, we study the stability properties of the classical modified Patankar–Runge–Kutta schemes (MPRK) and the modified Patankar Deferred Correction (MPDeC) approaches. We prove that most of the considered MPRK schemes are stable for any time step size and compute the stability function of MPDeC. We investigate its properties numerically revealing that also most MPDeC are stable irrespective of the chosen time step size. Finally, we verify our theoretical results with numerical simulations.

AMS subject classifications (2020). 65L05, 65L06, 65L20

Key words. Lyapunov Stability, Conservative, Positivity Preserving, Modified Patankar Schemes, Runge–Kutta, Deferred Correction

1. Introduction. The derivation of structure preserving methods is essentially important and has attracted much attention recently in various different ways, cf. [20, 16, 21, 2]. One specific family of schemes which ensures the positivity and conservation property of the numerical approximations are modified Patankar (MP) schemes. They are constructed for conservative and positive production-destruction systems (PDS) which are used to describe chemical reactions or biological models [4], but can also be found as the semi-discretization of convection equations (with and without) stiff source terms [6, 12]. Such a PDS is defined through the following system of ordinary differential equations (ODEs):

\begin{align}
y'_i &= P_i(y) - D_i(y), \quad i = 1, \ldots, N
\end{align}

with

\begin{align}
P_i(y) &= \sum_{j=1}^{N} p_{ij}(y), \quad D_i(y) = \sum_{j=1}^{N} d_{ij}(y), \quad p_{ij}(y) = d_{ji}(y) \geq 0 \quad \text{and} \quad y = (y_1, \ldots, y_N)^T,
\end{align}

where \( y_i \) denotes the \( i \)-th concentration, e.g. the density of an chemical or biological component. Further, the production rate \( p_{ij} \) denotes the rate of change from the \( j \)-th component to the \( i \)-th component whereas the destruction \( d_{ij} \) tells us the change from the \( i \)-th component to the \( j \)-th component.

The PDS is positive if the componentwise inequality \( y(0) > 0 \) implies \( y(t) > 0 \) for all \( t > 0 \). It is easy to see that due to \( p_{ij}(y) = d_{ji}(y) \), solutions of (1.1) fulfill additionally the
conservation property $\sum_{i=1}^{N} y_i(t) = \text{const}$ for all $t \geq 0$.

To mimic conservation and positivity discretely, modified Patankar schemes have been first developed in [4] and further extended and applied in [16, 17, 19]. They proposed families of second- and third-order accurate modified Patankar–Runge–Kutta (MPRK) methods which are based on the Butcher coefficients whereas Huang and Shu have used as their starting point for the MP developments strong stability preserving (SSP) RK schemes in Shu-Osher form introducing SSPMPRK methods [11, 12]. However, all of those approaches have maximum order three, whereas arbitrary high-order MP schemes have later been proposed in [20]. These schemes are based on the Deferred Correction method [8]. All of those MP schemes are unconditionally positive and conservative for $y(0) > 0$.

Although the benefits of MP schemes have been repeatedly demonstrated numerically since their introduction in 2003, an analytical study of their stability and robustness has been lacking until recently. In [23] some issues of MP schemes have been analyzed whereas in [13, 14, 15], a suitable stability theory for positivity preserving time integration schemes has been proposed and applied to a family of second order MPRK schemes. In [10], this theory has been used to investigate the stability properties of the SSPMPRK schemes. In this paper, we follow the lines of [14, 15] and investigate the stability behavior of several MP schemes. We concentrate on the classical MPRK schemes from [16, 17] and the MPDeC approach up to order 14 [20].

The rest of the paper is organized as follows. In the next section, we shortly repeat the notation, the main definitions and results regarding the stability of fixed points from [15]. Next, in Section 3 and 4, we investigate the stability of MP schemes. First, we focus on a high-efficient MPRK scheme proposed in [23] and explain the general setting. Later, we consider the general families of third order MPRK schemes from [16, 17], denoted by MPRK43($\alpha, \beta$) and MPRK43($\gamma$). In Section 4, we extend our stability investigation to the MPDeC schemes from [20]. In numerical simulations, see Section 5, we verify our theoretical results. Finally, we formulate our conclusions and future research interests in section 6. In the Appendix 7, we give some additional material for completeness.

2. Preliminary Results and Notation. In the following part we shortly introduce the notation and the considered test equation where we also repeat the preliminary stability results for positivity preserving schemes following [13, 14, 15].

2.1. Notation and Test Case. We consider a stable linear positive and conservative PDS of the form

\begin{equation}
    y'(t) = Ay(t)
\end{equation}

with $A \in \mathbb{R}^{N \times N}$ and the initial condition

\begin{equation}
    y(0) = y^0 > 0
\end{equation}

following the notation and investigation in [10, 15]. In case that we have exactly $k > 0$ linear invariants, there exist vectors $n_1, \ldots, n_k$ which form a basis of the $\ker(A^T)$. As a consequence, they fulfill $n_i^T y(t) = n_i^T y^0$ for all $t \geq 0$ and $i = 1, \ldots, k$. Note that for conservative PDS we find $1 \in \ker(A^T)$, i.e. the sum of each column vanishes, that is $\sum_{j=1}^{N} a_{ji} = 0$ for $i = 1, \ldots, N$. 
Due to a classical result from dynamical system theory \cite{18}, we know that (2.1) is positive if and only if $A$ is a Metzler matrix, i.e. the off-diagonal elements are nonnegative. Altogether, (2.1) can be rewritten as a positive and conservative PDS with $p_{ij}(y) = d_{ij}(y) = a_{ij}y_j$ and $p_{ii}(y) = d_{ii}(y) = 0$ for $i \neq j$ and $i, j = 1, \ldots, N$. Moreover,

\begin{equation}
- \sum_{j=1}^{N} d_{ij}(y) = - \sum_{j=1}^{N} a_{ij}y_i = a_{ii}y_i
\end{equation}

holds true, which was already used in \cite{10} to rewrite the SSPMRK schemes in matrix-vector notation when applied to (2.1). To ensure stable steady states $y^* \in \ker(A)$ in the sense of Definition 2.2, the matrix $A$ has a spectrum

$$\sigma(A) \subseteq \mathbb{C}^- = \{ z \in \mathbb{C} \mid \Re(z) \leq 0 \}$$

and the eigenvalues of $A$ with zero real part have to be associated to a Jordan block size of 1, cf. \cite[Theorem 2]{7}. Also, according to the same theorem, no steady state of the test equation (2.1) is asymptotically stable in the sense of Definition 2.2 since $\ker(A) \neq \{0\}$.

To summarize the assumptions on the linear PDS (2.1), we introduce the algebraic multiplicity $\mu_A(\lambda)$ of the eigenvalue $\lambda \in \sigma(A)$ as well as the corresponding geometric multiplicity $\gamma_A(\lambda)$. Altogether we consider in the following systems of the form

\begin{equation}
y' = Ay, \quad A \neq 0, \quad A - \text{diag}(A) \geq 0, \quad \mu_A(0) = \gamma_A(0) = k \geq 1, \quad 1 \in \ker(A^T), \quad \sigma(A) \subseteq \mathbb{C}^-,
\end{equation}

where $\text{diag}(A)$ denotes the diagonal of $A$.

**Remark 2.1.** We want to note, that $1 \in \ker(A^T)$ and $A \neq 0$ imply that the matrix $A$ is even a proper Metzler matrix, which means that in addition to being a Metzler matrix, $A$ possesses at least one negative diagonal element. As a result of \cite[Theorem 10, Corollary 11]{3}, the only eigenvalue $\lambda \in \sigma(A) \subseteq \mathbb{C}^-$ with $\Re(\lambda) = 0$ is $\lambda = 0$. Hence, the only eigenvalue for which we require equality of algebraic and geometric multiplicity in \cite[Theorem 2]{7} is $\lambda = 0$.

As a final notation we want to introduce the diagonal matrix $\text{diag}(y)$ for a given vector $y \in \mathbb{R}^N$ by $(\text{diag}(y))_{ii} = y_i$ for $i = 1, \ldots, N$.

### 2.2. Stability of Positive and Conservative Schemes

Here we recall the main theorem from \cite{15} that we use for the subsequent stability investigation. The theorem provides sufficient conditions for the stability analysis of non-hyperbolic fixed points, i.e. fixed points $y^*$ of a map $g: D \to D$ with $D \subseteq \mathbb{R}^N$, where the Jacobian $Dg(y^*)$ of $g$ at $y^*$ possesses at least one eigenvalue $\lambda$ with $|\lambda| = 1$. We consider the following stability notion of $y^*$.

**Definition 2.2.** Let $y^*$ be a fixed point of an iteration scheme $y^{n+1} = g(y^n)$, that is $y^* = g(y^*)$.

1. $y^*$ is called Lyapunov stable if, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|y^0 - y^*\| < \delta$ implies $\|y^n - y^*\| < \epsilon$ for all $n \geq 0$.

2. If in addition to 1, there exists a constant $c > 0$ such that $\|y^0 - y^*\| < c$ implies $\|y^n - y^*\| \to 0$ for $n \to \infty$, we call $y^*$ asymptotically stable.
3. A fixed point that is not Lyapunov stable is said to be \textit{unstable}.

In the following we will also just speak of stable fixed points. A common result concerning the stability of fixed points is the following.

**Theorem 2.3 ([22, Theorem 1.3.7]).** Let \( y^{n+1} = g(y^n) \) be an iteration scheme with fixed point \( y^* \). Suppose the Jacobian \( Dg(y^*) \) exists and denote its spectral radius by \( \rho(Dg(y^*)) \).

Then

1. \( y^* \) is asymptotically stable if \( \rho(Dg(y^*)) < 1 \).
2. \( y^* \) is unstable if \( \rho(Dg(y^*)) > 1 \).

This theorem does not provide us with a criterion for the analysis of fixed points that are stable but not asymptotically stable, which must belong to the class of non-hyperbolic fixed points. Nevertheless, it gives a sufficient condition for detecting unstable fixed points. Still, the case of \( \rho(Dg(y^*)) = 1 \) is not yet covered by Theorem 2.3, and thus, we use the following theorem from [15]. For a better reading flow we first introduce some notation.

Since we have assumed that the system \( y' = Ay \) has exactly \( k \geq 1 \) linear invariants, there exist \( n_1, \ldots, n_k \) forming a basis of \( \ker(A^T) \). Similar to [15], we introduce

\[
N = \begin{pmatrix} n_1^T \\ \vdots \\ n_k^T \end{pmatrix} \in \mathbb{R}^{k \times N}
\]

as well as the set

\[
(2.5) \quad H = \{ y \in \mathbb{R}^N \mid Ny = Ny^* \},
\]

and repeat the following theorem on the stability of an iteration in its explicit form, i.e. \( y^{n+1} = g(y^n) \).

**Theorem 2.4 ([15]).** Let \( A \in \mathbb{R}^{N \times N} \) such that \( \ker(A) = \text{span}(v_1, \ldots, v_k) \) represents a \( k \)-dimensional subspace of \( \mathbb{R}^N \) with \( k \geq 1 \). Also, let \( y^* \in \ker(A) \) be a fixed point of a map \( g : D \to D \) where \( D \subseteq \mathbb{R}^N \) contains a neighborhood \( D \) of \( y^* \). Moreover, let any element of \( C = \ker(A) \cap D \) be a fixed point of \( g \) and suppose that \( g|_D \in C^1 \) as well as that the first derivatives of \( g \) are Lipschitz continuous on \( D \). Then \( Dg(y^*)v_i = v_i \) for \( i = 1, \ldots, k \) and the following statements hold.

1. If the remaining \( N - k \) eigenvalues of \( Dg(y^*) \) have absolute values smaller than 1, then \( y^* \) is stable.
2. Let \( H \) be defined by (2.5) and \( g \) conserve all linear invariants, which means that \( g(y) \in H \cap D \) for all \( y \in H \cap D \). If additionally the assumption of 1 is satisfied, then there exists a \( \delta > 0 \) such that \( y^0 \in H \cap D \) and \( \|y^0 - y^*\| < \delta \) imply \( y^n \to y^* \) as \( n \to \infty \).

We want to note that according to [10, Remark 2.4], \( g \in C^2(D) \) is enough to conclude that \( g|_D \in C^1 \) has Lipschitz continuous derivatives for an appropriate neighborhood \( D \).

3. **Stability investigation of modified Patankar schemes.** In [14, 15], a novel stability investigation has been developed for conservative and positive time integration schemes. To
that end, Theorem 2.4 was applied to a family of second order MPRK methods and it was shown that all fixed points are unconditional stable, i.e. stable for all $\Delta t > 0$. Recently, also families of SSPMPRK schemes have been investigated inside this framework [10]. In this section, we extend those investigations and consider modified Patankar schemes based on several Runge–Kutta methods. Thereby, the schemes are linearly implicit, so that the function $g$ satisfying $y^{n+1} = g(y^n)$ is in all these cases implicitly given by solving linear systems. By the same reasoning as in [10], one can verify that $g \in C^2$ for all schemes under consideration. The main idea for proving this is to show that the functions $\Phi_i$ defined by the numerical scheme are in $C^2$ and that $g$ is obtained by solving only linear systems with a unique solution.

Similarly, all schemes preserve positive steady states with the same arguments as in [10] or [23, Prop. 2.3]. Thus, in order to apply Theorem 2.3 and part 1 of Theorem 2.4, we need to compute the eigenvalues of the Jacobian of $g$ evaluated at some steady state $y^* \in \ker(A) \cap D^0$. However, to use also part 2 of Theorem 2.4, we need to prove that $g$ additionally conserves all linear invariants.

In order to compute $Dg(y^*)$ we follow the approach as in [15, 10] introducing functions $\Phi_i$ and several Jacobians. The functions $\Phi_i$ arise from rearranging the equations for the $s$ stages of the numerical method leading to

$$0 = \Phi_j(y^n, y^{(1)}(y^n), \ldots, y^{(j)}(y^n)), \quad j = 1, \ldots, s$$

$$0 = \Phi_{n+1}(y^n, y^{(1)}(y^n), \ldots, y^{(s)}(y^n), g(y^n)).$$

Note that $\Phi_j(x_0, \ldots, x_j)$ is a function of $j+1$ vector-valued variables while $\Phi_{n+1}(x_0, \ldots, x_s, y)$ depends on $s+2$ variables. As mentioned above, in all cases we will find $\Phi_{k, \Phi_{n+1}}$ are in $C^2$, so that we may define

$$D_n\Phi_j = \frac{\partial}{\partial x_0} \Phi_j, \quad D_l\Phi_j = \frac{\partial}{\partial x_l} \Phi_j,$$

for $j, l = 1, \ldots, s$ with $l \leq j$, and

$$D_n\Phi_{n+1} = \frac{\partial}{\partial x_0} \Phi_{n+1}, \quad D_l\Phi_{n+1} = \frac{\partial}{\partial x_l} \Phi_{n+1}, \quad D_{n+1}\Phi_{n+1} = \frac{\partial}{\partial y} \Phi_{n+1}.$$

for $l = 1, \ldots, s$. Moreover, the operators $D_i^*$ indicate the evaluation of the corresponding Jacobian at $y^*, y^{(1)}(y^*)$ et cetera, e.g. $D_n^*\Phi_2 = D_n\Phi_2(y^*, y^{(1)}(y^*), y^{(2)}(y^*))$. As we interpret $y^{(k)} = y^{(k)}(y^n)$ we also introduce the Jacobian $D^*y^{(k)} = Dy^{(k)}(y^*)$.

With that we can derive a formula for computing $Dg(y^*)$, where $y^{n+1} = g(y^n)$ is the unique solution to (3.1). The chain rule yields

$$0 = D_n^*\Phi_j + \sum_{l=1}^{j} D_l^*\Phi_j D^*y^{(l)}, \quad j = 1, \ldots, s,$$

$$0 = D_{n+1}^*\Phi_{n+1} + \sum_{l=1}^{s} D_{s+1}^*\Phi_{s+1} D^*y^{(l)} + D_{n+1}^*\Phi_{n+1} Dg(y^*).$$
which can be rewritten to

\[ D^* y^{(j)} = - \left( D_j^* \Phi_j \right)^{-1} \left( D_n^* \Phi_j + \sum_{l=1}^{j-1} D_l^* \Phi_j D^* y^{(l)} \right), \quad j = 1, \ldots, s, \]

(3.4)

\[ D g(y^*) = - \left( D_{n+1}^* \Phi_{n+1} \right)^{-1} \left( D_n^* \Phi_{n+1} + \sum_{l=1}^{s} D_l^* \Phi_{n+1} D^* y^{(l)} \right), \]

if all occurring inverses exist. Also, in order to avoid long formulas in the following, we may omit to write the functions \( \Phi_i \) together with all their arguments.

**Stability of MPRK(3,2).** In [23] a novel three stage, second order MPRK method has been developed and investigated numerically. The scheme has shown favorable properties and is given by

\[ y^{(1)} = y^n + \Delta t \sum_j \left( p_{ij}(y^n) \frac{y_j^{(1)}}{y_j^n} - d_{ij}(y^n) \frac{y_i^{(1)}}{y_i^n} \right), \]

\[ y^{(2)} = y^n + \Delta t \sum_j \left( \frac{p_{ij}(y^n) + p_{ij}(y^{(1)}) y_j^{(2)}}{4} \frac{y_j^{(1)}}{y_j^n} - \frac{d_{ij}(y^n) + d_{ij}(y^{(1)}) y_i^{(2)}}{4} \frac{y_i^{(1)}}{y_i^n} \right), \]

\[ y^{n+1} = y^n + \Delta t \sum_j \left( \frac{p_{ij}(y^n) + p_{ij}(y^{(1)}) + 4p_{ij}(y^{(2)}) y_j^{n+1}}{6} \frac{y_j^{(1)}}{y_j^n} - \frac{d_{ij}(y^n) + d_{ij}(y^{(1)}) + 4d_{ij}(y^{(2)}) y_i^{n+1}}{6} \frac{y_i^{(1)}}{y_i^n} \right). \]

Note that this scheme is steady state preserving as the stages are uniquely determined and \( y^{n+1} = y^{(2)} = y^{(1)} = y^* \) is a valid solution. Hence, writing \( y^{n+1} = g(y^*) \) we obtain \( g(y^*) = y^{(2)}(y^*) = y^{(1)}(y^*) = y^* \).

Next, we apply MPRK(3,2) to (2.4) which, due to (2.3), yields in matrix vector notation the following identity

\[ 0 = (I - \Delta t A)y^{(1)} - y^n = \Phi_1 \]

and we obtain

\[ D_n^* \Phi_1 = -I \quad \text{and} \quad D_1^* \Phi_1 = I - \Delta t A. \]

(3.6)

The second stage can be written as

the linear system

\[ 0 = y^n + \frac{\Delta t}{4} A \text{diag}(y^{(2)})(\text{diag}(y^{(1)}))^{-1}(y^n + y^{(1)}) - y^{(2)} = \Phi_2. \]

(3.7)

This equation can be transformed using the fact that diagonal matrices commute and \( \text{diag}(v^T)w = \text{diag}(w^T)v \) for all \( v, w \in \mathbb{R}^N \), yielding

\[ \Phi_2 = y^n + \frac{\Delta t}{4} A \text{diag}(y^n + y^{(1)})(\text{diag}(y^{(1)}))^{-1}y^{(2)} - y^{(2)}. \]

(3.8)
(3.9) \[ y^n + \frac{\Delta t}{4} A \text{ diag}(y^{(2)}) \left( \text{diag}(y^{(1)})^{-1} + 1 \right) - y^{(2)}, \]

where \( 1 = (1, \ldots, 1)^T \in \mathbb{R}^N \) as well as \( (y^{-1})_i = y_i^{-1} \) for \( y \in \mathbb{R}^N_{>0} \) and \( i = 1, \ldots, N \). Hence, we find

\[
D^*_n \Phi_2 \overset{(3.7)}{=} I + \frac{\Delta t}{4} A, \quad D^*_1 \Phi_2 \overset{(3.9)}{=} -\frac{\Delta t}{4} A, \quad D^*_2 \Phi_2 \overset{(3.8)}{=} \frac{\Delta t}{2} A - I.
\]

With analogous steps as before, the final step reads

\[
0 = y^n + \frac{\Delta t}{6} A \text{ diag}(y^{n+1})(\text{diag}(y^{(1)})^{-1}(y^n + y^{(1)} + 4y^{(2)}) - y^{n+1} = \Phi_{n+1}.
\]

Using again the above mentioned calculation rules, we get

\[
\Phi_{n+1} = y^n + \frac{\Delta t}{6} A \text{ diag}(y^n + y^{(1)} + 4y^{(2)})(\text{diag}(y^{(1)})^{-1}y^{n+1} - y^{n+1})
\]

\[
= y^n + \frac{\Delta t}{6} A \text{ diag}(y^{n+1}) \left( \text{diag}(y^n + 4y^{(2)})(y^{(1)})^{-1} + 1 \right) - y^{n+1}.
\]

Therefore,

\[
D^*_n \Phi_{n+1} \overset{(3.11)}{=} I + \frac{\Delta t}{6} A, \quad D^*_1 \Phi_{n+1} \overset{(3.13)}{=} -\frac{5}{6} \Delta t A,
\]

\[
D^*_2 \Phi_{n+1} \overset{(3.11)}{=} \frac{2}{3} \Delta t A, \quad \Phi_{n+1} \overset{(3.12)}{=} \Delta t A - I.
\]

Since \( \sigma(A) \subseteq \mathbb{C}^- \), we conclude that the inverses of the following matrices exist. We obtain from (3.4):

\[
D^* y^{(1)} = -(D^*_1 \Phi_1)^{-1} D^*_n \Phi_1 = (I - \Delta t A)^{-1},
\]

\[
D^* y^{(2)} = -(D^*_2 \Phi_2)^{-1} \left( D^*_n \Phi_2 + D^*_1 \Phi_2 D^* y^{(1)} \right) = -\left( \frac{\Delta t}{2} A - I \right)^{-1} \left( I + \frac{\Delta t}{4} A - \frac{\Delta t}{4} A(I - \Delta t A)^{-1} \right),
\]

\[
Dg(y^*) = -(D^*_n+1 \Phi_{n+1}^{-1})^{-1}(D^*_n \Phi_{n+1} + D^*_1 \Phi_{n+1} D^* y^{(1)} + D^*_2 \Phi_{n+1} D^* y^{(2)})
\]

\[
= (I - \Delta t A)^{-1} \left( I + \frac{\Delta t}{6} A - \frac{5}{6} \Delta t A(I - \Delta t A)^{-1} \right)
\]

\[
- \frac{2}{3} \Delta t A \left( \frac{\Delta t}{2} A - I \right)^{-1} \left( I + \frac{\Delta t}{4} A - \frac{\Delta t}{4} A(I - \Delta t A)^{-1} \right).
\]

Hence, eigenvalues \( \lambda \) of \( A \) are eigenvalues of the Jacobian \( Dg(y^*) \). By substituting \( z = \lambda \Delta t \), we obtain the stability function \( R(z) \) for MPRK(3,2) with

\[
R(z) = \frac{1}{1 - z} \left( 1 + \frac{z}{6} \frac{5z}{6(1 - z)} - \frac{4z}{3(1 - z)} \left( 1 + \frac{z}{4} \frac{z}{4(1 - z)} \right) \right) = \frac{z^3 + 18z - 12}{6(1 - z)^2(z - 2)},
\]

so that

\[
\sigma(Dg(y^*)) = \{ R(\lambda \Delta t) \mid \lambda \in \sigma(A) \}.
\]
Before we analyze this stability function, we want to point out that, due to (3.12),

\[ \mathbf{n}^T g(y^n) = \mathbf{n}^T y^{n+1} = \mathbf{n}^T y^n \]

holds true for all \( \mathbf{n} \in \ker(\mathbf{A}) \) as \( \mathbf{n}^T \mathbf{A} = \mathbf{0} \). Hence, the map \( g \) generating the iterates of the MPRK(3,2) scheme conserves all linear invariants of the linear test problem. As a consequence, all conditions for applying Theorem 2.3 and Theorem 2.4 are met. It remains to prove that \( |R(z)| < 1 \) holds in order to conclude the stability of the fixed points as well as the local convergence towards the corresponding steady state solution \( y^* \) satisfying \( \mathbf{n}^T y^0 = \mathbf{n}^T y^* \) for all \( \mathbf{n} \in \ker(\mathbf{A}^T) \).

**Proposition 3.1.** The stability function \( R(z) = \frac{z^3 + 18z - 12}{6(z-1)^2(z-2)} \) satisfies \( R(0) = 1 \) and \( |R(z)| < 1 \) for \( z \in \mathbb{C} \setminus \{0\} \).

**Proof.** Obviously, it is \( R(0) = 1 \). Then, we investigate \( |R(z)|^2 \) for \( z = iy \) and \( y \in \mathbb{R} \). Elementary calculations yield for the denominator of \( |R(z)|^2 \):

\[ |6(1 - z)^2(z-2)|^2 = 144 + 324y^2 + 216y^4 + 36y^6, \]

where the numerator gives

\[ |z^3 + 18z - 12|^2 = 144 + 324y^2 - 36y^4 + y^6. \]

Due to the even powers of \( y \), it is obviously that for \( y \neq 0 \), we have \( |R(iy)| < 1 \). Since \( R(z) \) is a rational function with poles\(^1\) at \( z = 1,2 \), we see that \( R(z) \) is holomorphic for all \( z \in \mathbb{C}^- \). We can apply the Phragmén-Lindelöf principle on the union of the origin and the interior of \( \mathbb{C}^- \) and deduce that \( |R(z)| \leq 1 \) for all \( z \in \mathbb{C}^- \). With the maximum principle, we know that \( |R(z)| = 1 \) is only possible on the imaginary axis since \( R \) is not constant. Therefore, it follows \( |R(z)| < 1 \) for all \( z \) with \( \Re(z) < 0 \).

From this result, we can follow as a direct consequence of Theorem 2.4:

**Corollary 3.2.** 1. Let \( y^* \) be a positive steady state of the differential equation (2.1). Then, \( y^* \) is a stable fixed point of the MPRK(3,2) for all \( \Delta t > 0 \).

2. Let \( y^* \) be the unique steady state of the initial value problem (2.1), (2.2). Then, the scheme MPRK(3,2) converges towards \( y^* \) for all \( \Delta t > 0 \), if \( y^0 \) is close enough to \( y^* \).

**Remark 3.3.** We want to mention that as long as the stability function \( R(z) = \frac{N(z)}{D(z)} \) with polynomials \( N, D \) satisfying \( \deg(N) \leq \deg(D) \) and \( D(z) \neq 0 \) for all \( z \in \mathbb{C}^- \), we can conclude \( |R(z)| < 1 \) for \( \Re(z) < 0 \) whenever \( |R(z)| \leq 1 \) holds on the imaginary axis with the same reasoning as in the proof of Proposition 3.1.

Moreover, we point out that the Phragmén-Lindelöf principle can also be applied to different sectors of \( \mathbb{C}^- \). Indeed, according to [3], all eigenvalues of a proper Metzler matrix \( \mathbf{A} \in \mathbb{R}^{N \times N} \) with \( 0 \in \sigma(\mathbf{A}) \subseteq \mathbb{C}^- \) and \( N \leq 4 \) lie in the sector \( S = \left\{ z \in \mathbb{C}^- \mid \arg(z) \in \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right)\right\} \), and

\(^1\)Note that the stability function is always well defined on \( \mathbb{C}^- \) as otherwise a Jacobian of some \( \Phi_i \) must have been singular.
hence, it suffices in this case to investigate the stability function on the boundary of \( S \) rather than the imaginary axis. However, the analysis on \( \partial S \) is more complex.

**Stability of MPRK43\((\alpha, \beta)\).** In the following part, the third order accurate MPRK43-family from \([16, 17]\) will be analyzed, which can be written as

\[
\begin{align*}
y_i^{(1)} &= y_i^n + a_{21} \Delta t \sum_{j=1}^{N} \left( p_{ij}(y^n) \frac{y_j^{(1)}}{y_j^n} - d_{ij}(y^n) \frac{y_i^{(1)}}{y_i^n} \right), \\
y_i^{(2)} &= y_i^n + \Delta t \sum_{j=1}^{N} \left( (a_{31} p_{ij}(y^n) + a_{32} p_{ij}(y^{(1)})) \frac{y_j^{(2)}}{(y_j^n)^{\frac{1}{3}} (y_j^{(1)})^{1-\frac{1}{3}}} - (a_{31} d_{ij}(y^n) + a_{32} d_{ij}(y^{(1)})) \frac{y_i^{(2)}}{(y_i^n)^{\frac{1}{3}} (y_i^{(1)})^{1-\frac{1}{3}}} \right), \\
y_i^{(3)} &= y_i^n + \Delta t \sum_{j=1}^{N} \left( (\beta_1 p_{ij}(y^n) + \beta_2 p_{ij}(y^{(1)})) \frac{y_j^{(3)}}{(y_j^n)^{\frac{1}{4}} (y_j^{(1)})^{1-\frac{1}{4}}} - (\beta_1 d_{ij}(y^n) + \beta_2 d_{ij}(y^{(1)})) \frac{y_i^{(3)}}{(y_i^n)^{\frac{1}{4}} (y_i^{(1)})^{1-\frac{1}{4}}} \right), \\
y_i^{n+1} &= y_i^n + \Delta t \sum_{j=1}^{N} \left( (b_1 p_{ij}(y^n) + b_2 p_{ij}(y^{(1)}) + b_3 p_{ij}(y^{(2)})) \frac{y_j^{n+1}}{y_j^{(3)}} - (b_1 d_{ij}(y^n) + b_2 d_{ij}(y^{(1)}) + b_3 d_{ij}(y^{(2)})) \frac{y_i^{n+1}}{y_i^{(3)}} \right),
\end{align*}
\]

where \( p = 3a_{21} (a_{31} + a_{32}) b_3, \quad q = a_{21}, \quad \beta_2 = \frac{1}{2a_{21}} \) and \( \beta_1 = 1 - \beta_2 \).

We want to note that in \([16]\) the stages are labeled \( y^{(1)}, y^{(2)}, y^{(3)} \) and \( \sigma \), however, the first stage reads \( y^{(1)} = y^n \) and can directly be inserted into the further stages. As a result, the first non-trivial stage is the second one, which we denote by \( y^{(1)} \) here. As a result of this shift in superscript numbering, the stage \( \sigma \) corresponds to \( y^{(3)} \) in our framework. We performed these changes in order to be consistent with our notation and to reduce the number of functions \( \Phi_i \) by one. Also note that by the same reasoning as above, this scheme is also steady state preserving so that \( y^n = y^* > 0 \) yields \( y^{(i)}(y^n) = y^* \) for \( i = 1, 2, 3 \) and \( g(y^n) = y^{n+1}(y^n) = y^* \).

The Butcher tableau with respect to the two parameters is defined by

\[
\begin{array}{c|ccc}
& \alpha & \beta \\
\hline
0 & 0 & \alpha \\
\alpha & 3\alpha^2(1-\alpha) - \beta \alpha & \beta(\beta-\alpha) \\
\beta & \frac{6\alpha^2(\alpha-3\alpha)}{6\alpha(\beta-\alpha)} & \frac{3\beta-2}{6\alpha(\beta-\alpha)} & \frac{2-3\alpha}{6\alpha(\beta-\alpha)} \\
1 + \frac{2-3\alpha(\alpha-\beta)}{6\alpha(\beta-\alpha)} \\
\end{array}
\]

with

\[
\begin{align*}
2/3 \leq \beta &\leq 3\alpha(1-\alpha) & \quad \left\{ \begin{array}{l}
1/3 \leq \alpha < \frac{2}{3}, \\
3\alpha(1-\alpha) \leq \beta \leq 2/3 \quad \text{for} \quad 2/3 \leq \alpha < \alpha_0, \\
\frac{3\alpha-2}{6\alpha-3} \leq \beta \leq 2/3 \quad \text{for} \quad \alpha > \alpha_0,
\end{array} \right.
\end{align*}
\]

and \( \alpha_0 \approx 0.89255 \).

We follow the steps from MPRK32 and obtain

\[0 = (I - a_{21} \Delta t A) y^{(1)} - y^n = \Phi_1.\]
The Jacobians of \( \Phi_1 \) with respect to \( y^n \) and \( y^{(1)} \) are

\[
(3.19) \quad D_n^{*} \Phi_1 = -I \quad \text{and} \quad D_1^{*} \Phi_1 = I - a_{21} \Delta t A.
\]

For the second step, we obtain the linear system

\[
(3.20) \quad 0 = \Delta t A \, \text{diag}(y^{(2)}((\text{diag}(y^{(1)}))^{-1} \frac{1}{p} ((\text{diag}(y^n))^{-1} - \frac{1}{p} (a_{31} y^n + a_{32} y^{(1)})) + y^n - y^{(2)}) = \Phi_2.
\]

In order to compute the Jacobians of \( \Phi_2 \) with respect to \( y^n \) and \( y^{(1)} \), we introduce

\[
f(x, y(x)) = \text{diag}(x^k(a_1 x + a_2 y(x)))
\]

for some \( k, a_1, a_2 \in \mathbb{R} \), where \( y(x^*) = x^* \). We see that

\[
(3.21) \quad \left( \frac{\partial}{\partial x} f(x^*, y(x^*)) \right)_{ij} = \left. \frac{\partial}{\partial x_j} \left( (x_i^*)^k(a_1 x_i + a_2 y_i(x)^*) \right) \right|_{x = x^*}
\]

\[
= \delta_{ij} \left( k(x_i^*)^{k-1}(a_1 + a_2) x_i^* + (x_i^*)^k a_1 \right)
\]

\[
= \left( \text{diag}(x^*)^k \right)_{ij} (k(a_1 + a_2) + a_1).
\]

Hence, the derivatives of \( \Phi_2 \) are given by the following terms, using \( a_{31} + a_{32} = \beta \):

\[
(3.22) \quad D_n^{*} \Phi_2 = I + \Delta t A \left( \frac{1}{p} - 1 \right) \beta + a_{31},
\]

\[
D_1^{*} \Phi_2 = \Delta t A \left( -\frac{\beta}{p} + a_{32} \right),
\]

\[
D_2^{*} \Phi_2 = 0,
\]

\[
D_3^{*} \Phi_2 = -I + \beta \Delta t A.
\]

Next, \( \Phi_3 \) satisfies

\[
(3.23) \quad 0 = \Delta t A \, \text{diag}(y^{(3)}((\text{diag}(y^{(1)}))^{-1} \frac{1}{q} ((\text{diag}(y^n))^{-1} - \frac{1}{q} (\beta_1 y^n + \beta_2 y^{(1)}) + y^n - y^{(3)}) = \Phi_3.
\]

The derivatives are given by the following terms, using \( \beta_1 + \beta_2 = 1 \) and (3.21):

\[
(3.24) \quad D_n^{*} \Phi_3 = I + \Delta t A \left( \frac{1}{q} - 1 + \beta_1 \right),
\]

\[
D_1^{*} \Phi_3 = \Delta t A \left( -\frac{1}{q} + \beta_2 \right),
\]

\[
D_2^{*} \Phi_3 = 0,
\]

\[
D_3^{*} \Phi_3 = -I + \Delta t A.
\]

Finally, the last step can be written as

\[
(3.25) \quad 0 = y^n + \Delta t A \, \text{diag}(y^{n+1}((\text{diag} y^{(3)}))^{-1}) (b_1 y^n + b_2 y^{(1)} + b_3 y^{(2)}) - y^{n+1} = \Phi_{n+1}.
\]
Note that from this equation we can conclude that the generating map \( g \) conserves all linear invariants as \( n^T A = 0 \) implies \( n^T y^n = n^T y^{n+1} \). Furthermore, the derivatives are given by the following terms, using \( b_1 + b_2 + b_3 = 1 \):

\[
\begin{align*}
D_n^* \Phi_{n+1} &= I + b_1 \Delta t A, \\
D_1^* \Phi_{n+1} &= b_2 \Delta t A, \\
D_2^* \Phi_{n+1} &= b_3 \Delta t A, \\
D_3^* \Phi_{n+1} &= -\Delta t A, \\
D_{n+1}^* \Phi_{n+1} &= \Delta t A - I.
\end{align*}
\] (3.26)

With \( \sigma(A) \subseteq \mathbb{C}^- \) the inverses needed for (3.4) exist and we get

\[
\begin{align*}
D^* y^{(1)} &= -(D^*_1 \Phi_1)^{-1} D_n^* \Phi_1 = (I - \alpha \Delta t A)^{-1}, \\
D^* y^{(2)} &= -(D^*_2 \Phi_2)^{-1} \left( D_n^* \Phi_2 + D_1^* \Phi_2 D^* y^{(1)} \right) \\
&= (I - \beta \Delta t A)^{-1} \left( I + \Delta t A \left( \frac{1}{p} - 1 \right) \beta + a_{31} \right) + \Delta t A \left( -\frac{\beta}{p} + a_{32} \right) (I - \alpha \Delta t A)^{-1}, \\
D^* y^{(3)} &= -(D^*_3 \Phi_3)^{-1} \left( D_n^* \Phi_3 + D_1^* \Phi_3 D^* y^{(1)} + D_2^* \Phi_3 D^* y^{(2)} \right) \\
&= (I - \Delta t A)^{-1} \left( I + \Delta t A \left( \frac{1}{q} - 1 + \beta_1 \right) + \Delta t A \left( -\frac{1}{q} + \beta_2 \right) (I - \alpha \Delta t A)^{-1} \right), \\
D_g(y^*) &= -(D_{n+1}^* \Phi_{n+1})^{-1} \left( D_n^* \Phi_{n+1} + \sum_{l=1}^{3} D_l^* \Phi_{n+1} D^* y^{(l)} \right).
\end{align*}
\]

The stability function can be computed similar to the one of MPRK(3,2) and is given by

\[
R(z) = 1 + b_1 z + \frac{b_2 z}{1 - \alpha z} + \frac{b_3 z \left( 1 + z \left( \frac{1}{p} - 1 \right) \beta + a_{31} \right) + \frac{z(-\frac{\beta}{p} + a_{32})}{1 - \alpha z}}{1 - \beta z} - \frac{z \left( 1 + z \left( \frac{1}{q} - 1 + \beta_1 \right) + \frac{z(-\frac{1}{q} + \beta_2)}{1 - \alpha z} \right)}{1 - z}.
\]

This stability function can be rewritten in the form

\[
R(z) = \frac{\sum_{j=0}^{4} a_j z^j}{\sum_{j=0}^{4} \tilde{b}_j z^j},
\] (3.28)

where

\[
\begin{align*}
a_0 &= 1, & \tilde{b}_0 &= 1, \\
a_1 &= -(1 + \alpha + \beta), & \tilde{b}_1 &= -(\alpha + \beta + 2),
\end{align*}
\]
\[ a_2 = \frac{-6(1 + \beta)\alpha^3 + (6\beta^2 + 3)\alpha^2 + (6\beta^2 - 3\beta + 1)\alpha + 2(\beta - 1)}{6\alpha(\beta - \alpha)}, \]
\[ a_3 = \frac{(6\beta - 3)\alpha^3 + (6\beta^2 + 1)\alpha^2 + (3\beta^2 + \beta - 1)\alpha - 2(\beta - 1)}{6\alpha(\beta - \alpha)}, \]
\[ \tilde{b}_2 = (\beta + 2)\alpha + 2\beta + 1, \quad \tilde{b}_3 = -(\alpha + \beta + 2\alpha\beta), \]
\[ a_4 = -\left(\frac{1}{6} + (\beta - \frac{1}{2})\alpha\right), \quad \tilde{b}_4 = \alpha\beta. \]

An analytical investigation of the stability function of MPRK43(\(\alpha, \beta\)) for arbitrary finite sized systems \(N \times N\) is outside the scope of this paper. This is due to the fact \(R\) is a rational map in \(\alpha, \beta\) and \(z \in \mathbb{C}^-\) and by expanding \(R\) such that the coefficients of \(z^j\) are polynomials in \(\alpha\) and \(\beta\), we see that the maximum total degree appearing in the numerator and denominator is \(\text{deg}(\alpha^2\beta^2z^4) = 8\). But it can be seen that \(|R(z)| < 1\) is not true for all \((\alpha, \beta)\) satisfying (3.18), see for instance Figure 1, where the stability region is highlighted in grey for \((\alpha, \beta) = (\frac{1}{2}, \frac{2}{3})\).

![Figure 1: The stability domain of MPRK43(\(\frac{1}{2}, \frac{2}{3}\)) inside \([-0.1, 0] \times [-3, 3]\) is highlighted in grey. The two linear segments are given by \(\Re(z) = -10|\Im(z)|\). The angle between the two linear segments is approximately 168.58°.](image)

Nevertheless, if we restrict ourselves to the case outlined in Remark 3.3, i.e. we investigate \(R\) for systems with \(N \leq 4\), we can prove that MPKR43(\(\alpha, \beta\)) schemes are stable as we will find that \(|R(z)| < 1\) on

\[ S^\circ = \left\{ z \in \mathbb{C}^- \setminus \{0\} \mid \arg(z) \in (\frac{3}{4}\pi, \frac{5}{4}\pi) \right\}. \]
Note that for \((\alpha, \beta) = (\frac{1}{2}, \frac{2}{3})\) the largest cone inside the stability region contains the sector \(S\), see Figure 1. Hence, to see the instability properties of MPRK43\((\frac{1}{2}, \frac{2}{3})\) numerically using a linear positive PDS (2.4), Theorem [3, Theorem 10] and a small calculation reveal that we would have to consider a proper Metzler matrix \(A \in \mathbb{R}^{N \times N}\) with \(N \geq 32\).

**Proposition 3.4.** Let \(R\) be defined by (3.28). Then \(R(0) = 1\), and, if

\[
\begin{align*}
\bar{a}_3 \bar{b}_4 - a_3 a_4 & \leq 0, & a_4^2 - \bar{b}_3^2 & \leq 0, \\
(3.29) & a_1 a_4 - a_2 a_3 - \bar{b}_1 \bar{b}_4 + \bar{b}_2 \bar{b}_3 & \leq 0 \\
& a_2^2 - \bar{b}_2^2 - 2a_4 + 2\bar{b}_4 & \leq 0, & -a_1 a_2 + \bar{b}_1 \bar{b}_2 + a_3 - \bar{b}_3 & \leq 0,
\end{align*}
\]

then \(|R(z)| < 1\) holds true for all \(z \in S^n = \{z \in \mathbb{C}^+ \setminus \{0\} \mid \arg(z) \in (\frac{3}{4}\pi, \frac{5}{4}\pi)\}\).

**Proof.** First, \(a_0 = \bar{b}_0 = 1\) yields \(R(0) = 1\). Next, we make use of Remark 3.3 and investigate \(R\) on \(\partial S\), i.e. we substitute \(z = re^{i\varphi}\) with \(r > 0\) and \(\varphi \in \{\frac{3}{4}\pi, \frac{5}{4}\pi\}\). We obtain

\[
\left| R(re^{i\varphi}) \right|^2 = \frac{\left(\sum_{j=0}^4 r^j a_j \cos(j\varphi)\right)^2 + \left(\sum_{j=0}^4 r^j a_j \sin(j\varphi)\right)^2}{\left(\sum_{j=0}^4 r^j \bar{b}_j \cos(j\varphi)\right)^2 + \left(\sum_{j=0}^4 r^j \bar{b}_j \sin(j\varphi)\right)^2}.
\]

Next, note that \(\left| R(re^{i\varphi}) \right|^2 < 1\) is equivalent to

\[
\begin{align*}
& \left(\sum_{j=0}^4 r^j a_j \cos(j\varphi)\right)^2 + \left(\sum_{j=0}^4 r^j a_j \sin(j\varphi)\right)^2 - \left(\sum_{j=0}^4 r^j \bar{b}_j \cos(j\varphi)\right)^2 - \left(\sum_{j=0}^4 r^j \bar{b}_j \sin(j\varphi)\right)^2 \\
= & (a_4^2 - \bar{b}_4^2)r^8 + (2a_3 a_4 - 2\bar{b}_3 \bar{b}_4) \cos(\varphi)r^7 + (4(a_2 a_4 - \bar{b}_2 \bar{b}_4) \cos(\varphi)r^2 - 2a_2 a_4 + a_3^2 + 2\bar{b}_2 \bar{b}_4 - \bar{b}_3^2)r^6 + (8(a_1 a_4 - \bar{b}_1 \bar{b}_4) \cos(\varphi)^3 + (-6a_1 a_4 + 2a_2 a_3 + 6\bar{b}_1 \bar{b}_4 - 2\bar{b}_2 \bar{b}_3) \cos(\varphi))r^5 + (16(a_4 - \bar{b}_4) \cos(\varphi)^4 + 4(a_1 a_3 - \bar{b}_1 \bar{b}_3 - 4a_4 + 4\bar{b}_4) \cos(\varphi)^2 - 2a_1 a_3 + a_2^2 + 2\bar{b}_1 \bar{b}_3 - \bar{b}_2^2 + 2a_4 - 2\bar{b}_4)r^4 + (8(a_3 - \bar{b}_3) \cos(\varphi)^3 + (2a_1 a_2 - 2\bar{b}_1 \bar{b}_2 - 6a_3 + 6\bar{b}_3) \cos(\varphi))r^3 + (4(a_2 - \bar{b}_2) \cos(\varphi)^2 + a_1^2 - \bar{b}_1^2 - 2a_2 + 2\bar{b}_2)r^2 + (2a_1 - 2\bar{b}_1) \cos(\varphi)r < 0.
\end{align*}
\]

Dividing by \(r > 0\) and exploiting \(\cos(\frac{3}{4}\pi) = \cos(\frac{5}{4}\pi) = -\sqrt{\frac{1}{2}}\) we have to prove that

\[
\begin{align*}
& (a_4^2 - \bar{b}_4^2)r^7 + (-a_3 a_4 + \bar{b}_3 \bar{b}_4) \sqrt{2}r^6 + (a_2^2 - \bar{b}_2^2)r^5 + (a_1 a_4 - a_2 a_3 - \bar{b}_1 \bar{b}_4 + \bar{b}_2 \bar{b}_3) \sqrt{2}r^4 + (a_2^2 - \bar{b}_2^2 - 2a_4 + 2\bar{b}_4)r^3 + (-a_1 a_2 + \bar{b}_1 \bar{b}_2 + a_3 - \bar{b}_3) \sqrt{2}r^2 + (a_1^2 - \bar{b}_1^2)r + (-a_1 + \bar{b}_1) \sqrt{2} < 0.
\end{align*}
\]
The coefficients of $r^j$ can be written in terms of $\alpha$ and $\beta$ resulting in

\[
c_7 = a_4^2 - \tilde{b}_4^2 = \left(\frac{1}{3} \alpha - \alpha^2\right) \beta + \frac{\alpha^2}{4} - \frac{\alpha}{6} + \frac{1}{36}, \quad c_6 = \tilde{b}_3 \tilde{b}_4 - a_3 a_4,
\]

\[
c_5 = a_3^2 - \tilde{b}_3^2, \quad c_4 = a_1 a_4 - a_2 a_3 - \tilde{b}_1 b_4 + \tilde{b}_2 \tilde{b}_3,
\]

\[
c_3 = a_2^2 - \tilde{b}_2^2 - 2a_4 + 2\tilde{b}_4, \quad c_2 = -a_1 a_2 + \tilde{b}_1 b_2 + a_3 - b_3,
\]

\[
c_1 = a_1^2 - \tilde{b}_1^2 = -(2\alpha + 2\beta + 3) < 0, \quad c_0 = \tilde{b}_1 - a_1 = -1 < 0.
\]

Let us consider $c_7$, which is linear in $\beta$. The coefficient of $\beta$ is a parabola in $\alpha$ and vanishes for $\alpha = 0$ or $\alpha = \frac{1}{3}$. Since for $\alpha = \frac{1}{4} \in (0, \frac{1}{3})$, the coefficient reads $\frac{1}{12} - \frac{1}{16} > 0$, we can conclude that the coefficient of $\beta$ is negative for all $\alpha \geq \frac{1}{3}$. Now since $\beta \geq \frac{1}{4}$, the biggest value of $c_7$ can be obtained for $\beta = \frac{1}{4}$, resulting in

\[
c_7 \left(\alpha, \frac{1}{4}\right) = -\frac{\alpha}{12} + \frac{1}{36} \leq -\frac{1}{12} + \frac{1}{36} = 0.
\]

Additionally, we assumed that $c_2, \ldots, c_6 \leq 0$, so that $|R(z)| < 1$ on $S^0$ follows from Remark 3.3.

Remark 3.5. We want to note that the assumption (3.29) of Proposition 3.4 can be easily verified for a given $\alpha$ and $\beta$. Indeed, it can be demonstrated that the assumption is always met for $\alpha, \beta$ satisfying (3.18), however, an analytical proof is outside the scope of this paper. Instead we present 3D plots of (3.29) in the Appendix 7, see Figures 10 - 14.

From this result, we can follow as a direct consequence of Theorem 2.4:

Corollary 3.6. 1. Let $y^\ast$ be a positive steady state of the differential equation (2.1) with $N \leq 4$. Then, $y^\ast$ is a stable fixed point of $\text{MPRK}(\alpha, \beta)$ for all $\Delta t > 0$ and $\alpha, \beta$ satisfying (3.18) and (3.29).

2. If $y^\ast$ is the unique steady state of the initial value problem (2.1), (2.2) with $N \leq 4$ and if $y^0$ is close enough to $y^\ast$, then the $\text{MPRK}(\alpha, \beta)$ iterates converge towards $y^\ast$ for all $\Delta t > 0$ and $\alpha, \beta$ satisfying (3.18) and (3.29).

$\text{MPRK43(}\gamma\text{)}$. Already in [17] another $\text{MPRK43(}\gamma\text{)}$ has been proposed. It is based on

\[
\begin{array}{c|cccc}
0 & 2 & 3 & 1 & \gamma \\
\frac{2}{3} & 2 & \frac{2}{3} & \frac{1}{4} & \frac{1}{17} \\
\frac{2}{3} & 2 & \frac{2}{3} & 3 & \gamma \\
\frac{1}{4} & \frac{1}{17} & 3 & \frac{1}{4} & \gamma
\end{array}
\]
with $\frac{3}{8} \leq \gamma \leq \frac{3}{4}$. The scheme reads

\begin{equation}
(3.31)
\end{equation}

\begin{align*}
y^{(1)}_i &= y^n_i + a_{21} \Delta t \sum_{j=1}^{N} \left( p_{ij}(y^n) \frac{y_j^{(1)}}{y_j^n} - d_{ij}(y^n) \frac{y_j^{(1)}}{y_i^n} \right), \\
y^{(2)}_i &= y^n_i + \Delta t \sum_{j=1}^{N} \left( a_{31} p_{ij}(y^{(1)}) + a_{32} p_{ij}(y^{(1)}) \right) \frac{y_j^{(2)}}{(y_j^n)^{1-\frac{1}{p}}} - \left( a_{31} d_{ij}(y^n) + a_{32} d_{ij}(y^{(1)}) \right) \frac{y_i^{(2)}}{(y_i^n)^{1-\frac{1}{p}}}, \\
y^{(3)}_i &= y^n_i + \Delta t \sum_{j=1}^{N} \left( \beta_{pij}(y^n) + \beta_{2pij}(y^{(1)}) \right) \frac{y_j^{(3)}}{(y_j^n)^{1-\frac{1}{q}}} - \left( \beta_{1dij}(y^n) + \beta_{2dij}(y^{(1)}) \right) \frac{y_i^{(3)}}{(y_i^n)^{1-\frac{1}{q}}}, \\
y^{n+1}_i &= y^n_i + \Delta t \sum_{j=1}^{N} \left( b_{1pij}(y^n) + b_{2pij}(y^{(1)}) + b_{3pij}(y^{(2)}) \right) \frac{y_{j}^{n+1}}{y_j^n} - \left( b_{1dij}(y^n) + b_{2dij}(y^{(1)}) + b_{3dij}(y^{(2)}) \right) \frac{y_{i}^{n+1}}{y_i^n},
\end{align*}

where we again renamed the stages to be consistent with our framework. Moreover, the parameters above are given by

\begin{align*}
p &= 3a_{21}(a_{31} + a_{32})b_3 = \frac{4}{3} \gamma, \quad q = a_{21} = \frac{2}{3}, \\
\beta_1 &= 1 - \frac{1}{2a_{21}} = \frac{1}{4}, \quad \beta_2 = 1 - \beta_1 = \frac{3}{4}.
\end{align*}

Following exactly the steps from before, we obtain $\Phi_i = 0$ for $i \in \{1, 2, 3, n+1\}$ where

\begin{align*}
\Phi_1 &= y^n + a_{21} \Delta t A y^{(1)} - y^{(1)}, \\
\Phi_2 &= y^n + \Delta t A \text{diag}(y^{(2)}) (\text{diag}(y^{(1)}))^{-\frac{1}{p}} (\text{diag}(y^n))^{\frac{1}{p}-1} (a_{31} y^n + a_{32} y^{(1)}) - y^{(2)}, \\
\Phi_3 &= y^n + \Delta t A \text{diag}(y^{(3)}) (\text{diag}(y^{(1)}))^{-\frac{1}{q}} (\text{diag}(y^n))^{\frac{1}{q}-1} (\beta_1 y^n + \beta_2 y^{(1)}) - y^{(3)}, \\
\Phi_{n+1} &= y^n + \Delta t A \text{diag}(y^{n+1}) (\text{diag}(y^{(3)}))^{-1} (b_1 y^n + b_2 y^{(1)} + b_3 y^{(2)}) - y^{n+1}.
\end{align*}

Note that $n^T A = 0$ again leads to $n^T y^n = n^T y^{n+1} = n^T g(y^n)$ proving that the scheme conserves all linear invariants. Furthermore, $y^n = y^{(k)} = y^{n+1} = y^*$ with $k = 1, 2, 3$ solves $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_{n+1} = 0$ so that also this scheme is steady state preserving for positive $y^* \in \ker(A)$. Hence, we need to compute $D g(y^*)$ to apply Theorem 2.4 and Theorem 2.3. We get $D^*_n \Phi_1 = I$ and $D^*_1 \Phi_1 = a_{21} \Delta t A - I$, which leads to

\begin{align*}
D^*_1 y^{(1)} &= (I - a_{21} \Delta t A)^{-1}.
\end{align*}

Next, using (3.21) we have

\begin{align*}
D^*_n \Phi_2 &= I + \Delta t A \left( \frac{1}{p} - 1 \right) (a_{31} + a_{32}) + a_{31}, \\
D^*_1 \Phi_2 &= \Delta t A \left( \frac{1}{p} (a_{31} + a_{32}) + a_{32} \right), \\
D^*_2 \Phi_2 &= \Delta t A (a_{31} + a_{32}) - I,
\end{align*}
which determines $\mathbf{D}^* y^{(2)}$ but we omit to write it down here for a better reading flow. Furthermore,

$$\mathbf{D}^* \Phi_3 = I + \Delta t \mathbf{A} \left( \left( \frac{1}{q} - 1 \right) (\beta_1 + \beta_2) + \beta_1 \right),$$

$$\mathbf{D}^*_1 \Phi_3 = \Delta t \mathbf{A} \left( -\frac{1}{q} (\beta_1 + \beta_2) + \beta_2 \right),$$

$$\mathbf{D}^*_2 \Phi_3 = 0,$$

$$\mathbf{D}^*_3 \Phi_3 = \Delta t \mathbf{A} (\beta_1 + \beta_2) - I.$$

Finally

$$\mathbf{D}^* \Phi_{n+1} = I + b_1 \Delta t \mathbf{A}, \quad \mathbf{D}^*_1 \Phi_{n+1} = b_2 \Delta t \mathbf{A}, \quad \mathbf{D}^*_2 \Phi_{n+1} = b_3 \Delta t \mathbf{A}$$

$$\mathbf{D}^*_3 \Phi_{n+1} = -\Delta t \mathbf{A} (b_1 + b_2 + b_3), \quad \mathbf{D}^*_{n+1} \Phi_{n+1} = \Delta t \mathbf{A} (b_1 + b_2 + b_3) - I.$$

The stability function is thus determined by (3.4) reading (3.32)

$$R(z) = \frac{1}{1 - z} \left( 1 + zb_1 + \frac{zb_2}{1 - za_{21}} + \frac{zb_3 \left( 1 + z \left( \left( \frac{1}{q} - 1 \right) (a_{31} + a_{32}) + a_{31} \right) + \frac{z(-\frac{1}{q} (a_{31} + a_{32}) + a_{32})}{1 - za_{21}} \right)}{1 - z(a_{31} + a_{32})} \right)$$

$$\times \frac{z \left( 1 + z \left( (\frac{1}{q} - 1) (\beta_1 + \beta_2) + \beta_1 \right) + \frac{z(-\frac{1}{q} (\beta_1 + \beta_2) + \beta_2)}{1 - za_{21}} \right)}{1 - z(\beta_1 + \beta_2)}$$

$$= \frac{1 + \frac{1}{4} z + \frac{z(3\gamma - 3\gamma)}{1 - \frac{1}{4} z} + \frac{z(3\gamma - 3\gamma)}{1 - \frac{3}{4} z}}{1 - z} - \frac{z \left( \frac{1 + \frac{3}{4} z + \frac{3}{4} z}{1 - \frac{3}{4} z} \right)}{1 - z} = \frac{-5z^4 + 7z^3 + 23z^2 - 42z + 18}{2(2z - 3)^2(z - 1)^2}.$$

Note that the stability function is independent of the parameter $\gamma$, so that the following investigation is valid for all $\frac{3}{5} \leq \gamma \leq \frac{3}{4}$.

**Proposition 3.7.** Let $R$ be defined by (3.32). Then $|R(z)| < 1$ holds true for all $z \in \mathbb{C}^- \setminus \{0\}$ and $R(0) = 1$.

**Proof.** A straightforward calculation yields

$$R(z) = \frac{-\frac{5}{18} z^4 + \frac{7}{18} z^3 + \frac{23}{18} z^2 - \frac{42}{18} z + 1}{(\frac{3}{5} z - 1)^2(z - 1)^2} = \frac{\sum_{j=0}^{4} a_j z^j}{1 - \sum_{j=0}^{4} b_j z^j},$$

where

$$a_0 = 1, \quad a_1 = -\frac{7}{3}, \quad a_2 = \frac{23}{18}, \quad a_3 = \frac{7}{18}, \quad a_4 = -\frac{5}{18},$$

$$b_0 = 1, \quad b_1 = -\frac{10}{3}, \quad b_2 = \frac{37}{9}, \quad b_3 = -\frac{20}{9}, \quad b_4 = \frac{4}{9}.$$
Hence $R(0) = 1$ and, additionally, we follow the proof of [10, Prop. 3.6], i.e. all we have to prove is that

\[
(a_1^2-b_1^2)y^8 + (-2a_2a_4 + a_3^2 + 2b_2b_4 - b_3^2)y^6 + (2a_4 - 2a_1a_3 + a_2^2 + 2b_1b_3 - b_2^2 - 2b_4)y^4 + (-2a_2 + a_1^2 - b_1^2 + 2b_2)y^2 < 0
\]

for all $y \neq 0$. We denote by $c_k$ the above coefficient of $y^k$. A small calculation reveals

\[
c_8 = -\frac{13}{108} < 0, \quad c_6 = -\frac{137}{324} < 0, \quad c_4 = -\frac{1}{12} < 0, \quad c_2 = 0,
\]

which finishes the proof due to the even powers of $y$.

From this result, we can follow as a direct consequence of Theorem 2.4:

**Corollary 3.8.** 1. Let $y^*$ be a positive steady state of the differential equation (2.1). Then, $y^*$ is a stable fixed point of MPRK($\gamma$) for all $\Delta t > 0$ and $\frac{3}{8} \leq \gamma \leq \frac{3}{4}$.

2. If $y^*$ is the unique steady state of the initial value problem (2.1), (2.2), then, the scheme MPRK($\gamma$) converges towards $y^*$ for all $\Delta t > 0$ and $\frac{3}{8} \leq \gamma \leq \frac{3}{4}$, if $y^0$ is close enough to $y^*$.

4. **Stability Investigation of modified Patankar Deferred Correction schemes.** Arbitrarily high-order conservative and positive modified Patankar Deferred Correction schemes (MPDeC) were introduced in [20] which is based on the Deferred Correction approach developed in [8]. A time step $[t^n, t^{n+1}]$ is divided into $M$ subintervals, where $t^n,0 = t^n$ and $t^n,M = t^{n+1}$. The idea of the scheme is to mimic the Picard–Lindelöf theorem as follows. At each subtimstep $t^n,m$ and each correction step $k$, an approximation $y^{m,(k)}$ is calculated. During each of the $K$ correction steps the approximation of the numerical solution is increased by one order of accuracy. The modified Patankar trick is introduced inside the basic scheme to guarantee positivity and conservation of the intermediate approximations. Using the fact that initial states $y^{0,(k)}_i = y^n_i$ are identical for any correction $k$, the MPDeC correction steps can be rewritten for $k = 1, \ldots, K$, $m = 1, \ldots, M$ and $i = 1, \ldots, N$ as

\[
y^{m,(k)}_i - y^n_i = \sum_{r=0}^{M} \theta^m_r \Delta t \sum_{j=1}^{N} \left( p_{ij}(y^{r,(k-1)}_i, y^{m,(k-1)}_i, y^{m,(k)}_i, y^{m,(k)}_i) \right) = 0,
\]

where $\theta^m_r$ are the correction weights and the index function $\gamma(j, i, \theta^m_r)$ is defined by

\[
\gamma(j, i, \theta^m_r) = \begin{cases} j, & \theta^m_r \geq 0, \\ i, & \theta^m_r < 0. \end{cases}
\]

We also want to note that $y^{s,0} = y^n$ holds true for all $s = 0, \ldots, M$ by definition. The new numerical solution when applied to (2.4) is $y^{n+1} = y^{M,(K)}$.

By means of an affine linear transformation of the interval $[t^n, t^{n+1}]$ to $[0,1]$, the $M$ transformed subintervals are determined by $0 = t^0 < \cdots < t^M = 1$. Then the correction weights can be computed according to the formula

\[
\theta^m_r = \int_0^{t^m} \varphi_r(t) dt,
\]
where \( \varphi_r \) is the \( r \)-th Lagrangian basis polynomials defined by the subtimenodes \( \{t_m^r\}_{m=0}^M \).

Due to the iterative correction process, MPDeC schemes are more complicated than the previous schemes, especially since the index function changes the weights of productive and destructive terms. Since \( \gamma \) depends on the sign of \( \theta_r^m \), we introduce the nonnegative part \( \theta_{m,+} = \min\{0, \theta_r^m\} \) and nonpositive part \( \theta_{m,-} = \max\{0, -\theta_r^m\} \). It is worth mentioning that

\[
\theta_{r,\pm}^m = \frac{\theta_r^m \pm |\theta_r^m|}{2}
\]

and

\[
\theta_r^m = \begin{cases} 
\theta_{r,+}^m, & \theta_r^m < 0, \\
\theta_{r,-}^m, & \theta_r^m \geq 0
\end{cases}
\]

as well as \( \theta_{r,-}^m + \theta_{r,+}^m = \theta_r^m \). With that, we split the sum appearing in (4.1) into two sums containing \( \theta_{r,+}^m \) and \( \theta_{r,-}^m \), respectively. For the separated sums, we know the value of \( \gamma(j, i, \theta_r^m) \) so that we introduce the positive part

\[
p_{r,(k)}(y_{r,(k-1)}, y_{m,(k-1)}, y_{m,(k)}) = A \text{ diag}(y_{m,(k)}) \left( \text{diag}(y_{m,(k-1)}) \right)^{-1} y_{r,(k-1)}
\]

as well as the negative part \( n_{r,(k)} \) given by

\[
n_{i,(k)}(y_{r,(k-1)}, y_{m,(k-1)}, y_{m,(k)}) = \sum_{j=1}^N \left( p_{ij}(y_{r,(k-1)}) \frac{y_{i,m,(k)}}{y_{i,m,(k-1)}} - d_{ij}(y_{r,(k-1)}) \frac{y_{j,m,(k)}}{y_{j,m,(k-1)}} \right)
\]

for \( i = 1, \ldots, N \), \( r = 0, \ldots, M \) and \( k = 1, \ldots, K \). Using \( p_{ij}(y) = d_{ji}(y) = a_{ij}y_j \) for \( i \neq j \) and \( p_{ii}(y) = d_{ii}(y) = 0 \) this can be rewritten as

\[
n_{i,(k)}(y_{r,(k-1)}, y_{m,(k-1)}, y_{m,(k)}) = y_{i,m,(k)} \sum_{j=1}^N \frac{a_{ij}y_j}{y_{i,m,(k)-1}} - y_{i,m,(k-1)} \sum_{j=1}^N \frac{a_{ij}y_j}{y_{j,m,(k)-1}}
\]

Utilizing these vector fields, the iterates from (4.1) satisfy

\[
0 = \Phi_k^m(y^n, y^{1,(k-1)}, \ldots, y^{M,(k-1)}, y_{m,(k)}) = y_{m,(k)} - y^n - \sum_{r=0}^M \theta_{r,\pm}^m \Delta t p_{r,(k)}(y_{r,(k-1)}, y_{m,(k-1)}, y_{m,(k)}) - \sum_{r=0}^M \theta_{r,-}^m \Delta t n_{r,(k)}(y_{r,(k-1)}, y_{m,(k-1)}, y_{m,(k)})
\]

for \( k = 1, \ldots, K \) and \( m = 1, \ldots, M \). Furthermore, analogously to the Jacobians introduced in (3.2) and (3.3), we write \( D_y^* \Phi_k^m \) to represent the Jacobian with respect to the entries of the vector \( y^x \) for some \( x \), evaluated at \( (y^*, \ldots, y_{m,(k)}(y^*)) \). Finally, we introduce similar notations for the Jacobians of \( p_{r,(k)} \) and \( n_{r,(k)} \) with respect to \( y^x \).

Also note that MPDeC schemes are steady state preserving as plugging in \( y_{m,(k)} = y^n = y^* \in \ker(A) \) into (4.1) yields a true statement. Hence, \( y_{m,(k)}(y^*) = y^* \) for all \( k = 1, \ldots, K \) and \( m = 1, \ldots, M \).

With that we are able to compute \( Dg(y^*) \) for MPDeC schemes as the next theorem states.
Theorem 4.1. Let $g: \mathbb{R}_0^N \to \mathbb{R}_0^N$, implicitly given by the solution of (4.6), be the generating map of the MPDeC iterates when applied to (2.4). Furthermore, let $y^* \in \ker(A) \cap \mathbb{R}_0^N$ be a steady state of (2.4).

Then, $g \in C^2$ and the Jacobian of $g$ evaluated at $y^*$ is given by

$$Dg(y^*) = D_n^n y^{M,(k)},$$

(4.7)

$$D^*_n y^{m,(k)} = -(D^*_{m,(k)} \Phi^m)\frac{1}{1-\delta_{k1}} \sum_{r=1}^{M} D^*_r (k-1) \Phi^m D^*_k D^*_n y^{r,(k-1)}$$

for $m = 1, \ldots, M$ and $k = 1, \ldots, K$. Therefore, $\delta_{ij}$ is the Kronecker delta and

$$D^*_n \Phi^m = \begin{cases}
-\theta^m \Delta t A, & k = 1, \\
-I + \theta^m \Delta t A, & k > 1,
\end{cases}$$

(4.8)

as well as

$$D^*_{l,(s)} \Phi^m = \begin{cases}
\sum_{r=0}^{M} \theta^m \Delta t A - \sum_{r=0}^{M} \theta^m \Delta t A (\diag(y^*) A^T (\diag(y^*))^{-1}) - \theta^m \Delta t A, & s = k - 1 > 0, l = m, \\
-I - \sum_{r=0}^{M} \theta^m \Delta t A + \sum_{r=0}^{M} \theta^m \Delta t (\diag(y^*) A^T (\diag(y^*))^{-1}), & s = k, l = m.
\end{cases}$$

(4.9)

Proof. Since the $\theta^m$ are fixed for a given scheme, the functions $\Phi^m_k$ are in $C^2$ and as a consequence of solving only linear systems, the map $g$ is also in $C^2$. Furthermore, the formula (4.7) follows analogously to (3.4), whereby we want to point out that the sum appearing in (4.7) is multiplied with 0 for $k = 1$ since $y^{r,(k-1)} = y^n$ in this case. Hence, we only have to prove the formulas (4.8) and (4.9). For this, we compute the Jacobians of each addend of the sums in (4.6) separately by considering (4.3) and (4.5).

Let us start proving (4.8), first considering $k = 1$. From (4.3) and $y^{s,0} = y^n$ for all $s = 0, \ldots, M$ it follows that

$$p^{s,(1)}(y^n, y^n, y^{m,(1)}) = p^{s,(1)}(y^n, y^{m,(1)}) = Ay^{m,(1)}$$

and hence, $D^*_n p^{s,(1)} = 0$. Moreover, (4.5) for $k = 1$ yields

$$n^{s,(1)}_{i}(y^n, y^n, y^{m,(1)}) = n^{s,(1)}_{i}(y^n, y^{m,(1)}) = \frac{y_i^{m,(1)}}{y_i^n} \sum_{j=1}^{N} a_{ij} y_j^n - \sum_{j=1}^{N} a_{ji} y_j^{m,(1)} y_j^n.$$ 

Hence, using $1 \in \ker(A^T)$, we obtain

$$\frac{\partial}{\partial y_i} n^{s,(1)}_{i}(y^*, y^*) = -\frac{1}{y_i^n} \sum_{j=1}^{N} a_{ij} y_j^n - \sum_{j=1}^{N} a_{ji} = \frac{1}{y_i^*} \left( -\sum_{j=1}^{N} a_{ij} y_j^* + a_{ii} y_i^* \right) = \frac{1}{y_i^*} \left( -\sum_{j=1}^{N} a_{ij} y_j^* + 2a_{ii} y_i^* \right) = 2a_{ii},$$

(4.10)
and for \( q \neq i \) we find

\[
\frac{\partial}{\partial y^*_q} n_i^{r,(1)}(y^*, y^*) = a_{iq} + a_{qi} \frac{y^*_i}{y^*_q}.
\]

Altogether, we obtain

\[
(D^{r,(1)} n^n_r) = A + \begin{pmatrix}
  a_{11} & a_{21} \frac{y^*_1}{y^*_2} & \cdots & a_{N1} \frac{y^*_1}{y^*_N} \\
  a_{12} \frac{y^*_2}{y^*_1} & \ddots & \vdots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  a_{1N} \frac{y^*_N}{y^*_1} & \cdots & \cdots & a_{NN}
\end{pmatrix} = A + \text{diag}(y^*) A^T (\text{diag}(y^*))^{-1},
\]

and thus,

\[
D^*_n \Phi^m_k = - \left( I + (A + \text{diag}(y^*) A^T (\text{diag}(y^*))^{-1}) \Delta t \sum_{r=0}^M \theta^m_{r,-} \right).
\]

Next, for \( k > 1 \) it follows from (4.3) that

\[
D^*_n p^r(k) = \delta_{r0} A.
\]

Similarly, \( D^*_n n^r(k) = 0 \) if \( r \neq 0 \). Furthermore,

\[
n_i^{0,(k)}(y^n, y^{m,(k-1)}, y^{m,(k)}) = \frac{y_i^{m,(k)}}{y_i^{m,(k-1)}} \sum_{j=1}^N a_{ij} y_j^n - \frac{y_i^n}{y_i^{m,(k-1)}} \sum_{j=1}^N a_{ji} y_j^{m,(k)}
\]

yields

\[
\frac{\partial}{\partial y^*_i} n_i^{0,(k)}(y^*, y^*, y^*) = - \sum_{j=1}^N a_{ji} = a_{ii}
\]

\[
\frac{\partial}{\partial y^*_q} n_i^{0,(k)}(y^*, y^*, y^*) = a_{iq}, \quad i \neq q,
\]

so that \( D^*_n n^{r,(k)} = \delta_{r0} A \). This results in

\[
D^*_n \Phi^m_k = - \left( I + (\theta^m_{0,-} + \theta^m_{0,+}) \Delta t A \right) = - \left( I + \theta^m_{0} \Delta t A \right),
\]

proving (4.8).

To derive (4.9) consider first the case \( s = k - 1 > 0 \) and \( 0 < l \neq m \). From (4.3) it follows immediately that

\[
D^*_l n^{r,(k-1)} = \delta_{rl} A.
\]
Moreover, (4.5) yields
\[
\frac{\partial}{\partial y_i^{m(k)}} n_i^{r,(k)}(y^*, y^*, y^*) = -\delta_{rl} \sum_{j=1}^N a_{ji} = \delta_{rl} a_{ii}
\]
\[
\frac{\partial}{\partial y_q^{m(k)}} n_i^{r,(k)}(y^*, y^*, y^*) = \delta_{rl} a_{iq}, \quad i \neq q,
\]
which means that \(D_{l,(k-1)}^* \mathbf{n}^{r,(k)} = \delta_{rl} A\). In total (4.6) gives us
\[
D_{l,(k-1)}^* \Phi^m_k = -\Delta t (\theta_{l,+}^m + \theta_{l,-}^m) A = -\Delta t \theta^m_l A.
\]

Next, we investigate the case of \(s = k - 1 > 0\) and \(l = m\). Using \(\text{diag}(v)w = \text{diag}(w)v\) and (4.3), we receive
\[
D_{m,(k-1)}^* \mathbf{n}^{r,(k)} = D_{m,(k-1)}^* \left( A \text{diag}(y^{m,(k)}) \text{diag} \left( y^{m,(k-1)} \right)^{-1} y^{r,(k-1)} \right) = -(1 - \delta_{rm}) A.
\]
Furthermore, recalling (4.5), i.e.
\[
n_i^{r,(k)}(y^{r,(k-1)}, y^{m,(k-1)}, y^{m,(k)}) = \frac{y_i^{m,(k)}}{y_i^{m,(k-1)}} \sum_{j=1}^N a_{ij} y_j^{r,(k-1)} - y_i^{r,(k-1)} \sum_{j=1}^N a_{ji} y_j^{m,(k-1)},
\]
we also distinguish between \(r = m\) and \(r \neq m\). In the first case we see \(n_i^{m,(k)} = n_i^{m,(k)}(y^{m,(k-1)}, y^{m,(k)})\) and
\[
\frac{\partial}{\partial y_i^{m(k)}} n_i^{m,(k)}(y^*, y^*) = -\frac{1}{y_i^*} \sum_{j=1}^N a_{ij} y_j^* - \sum_{j=1}^N a_{ji} (4.10) = 2a_{ii}
\]
\[
\frac{\partial}{\partial y_q^{m(k)}} n_i^{m,(k)}(y^*, y^*) = a_{iq} + a_{qi} \frac{y_i^*}{y_q^*} (4.11) = (A + \text{diag}(y^*) A^T (\text{diag}(y^*))^{-1})_{iq}, \quad i \neq q,
\]
which means that \(D_{m,(k-1)}^* \mathbf{n}^{m,(k)} = A + \text{diag}(y^*) A^T (\text{diag}(y^*))^{-1}\). Turning to the case \(r \neq m\), we find
\[
\frac{\partial}{\partial y_i^{m(k)}} n_i^{r,(k)}(y^*, y^*, y^*) = -\frac{1}{y_i^*} \sum_{j=1}^N a_{ij} y_j^* (4.10) = a_{ii},
\]
\[
\frac{\partial}{\partial y_q^{m(k)}} n_i^{r,(k)}(y^*, y^*, y^*) = a_{qi} \frac{y_i^*}{y_q^*} (4.11) = (\text{diag}(y^*) A^T (\text{diag}(y^*))^{-1})_{iq}, \quad i \neq q,
\]
resulting in \(D_{m,(k-1)}^* \mathbf{n}^{r,(k)} = \text{diag}(y^*) A^T (\text{diag}(y^*))^{-1}\) for \(r \neq m\). Altogether, we thus end up
with
\[
D_{m,(k-1)}^* \Phi_k^m = \sum_{r=0}^{M} \theta_{r,+}^m \Delta t A - \sum_{r=0}^{M} \theta_{r,-}^m \Delta t (\text{diag}(y^*) A^T (\text{diag}(y^*))^{-1}) - \theta_{m}^m \Delta t A.
\]

Finally, we have to consider the case \( s = k \) and \( l = m \), i.e. we have to compute \( D_{m,(k)}^* \Phi_k^m \).

Using \( \text{diag}(v)w = \text{diag}(w)v \) and (4.3) once again we see that
\[
D_{m,(k)}^* p_{r,(k)} = A.
\]

Furthermore, we obtain
\[
\frac{\partial}{\partial y_i} n_{i,(k)}^{r,(k)}(y^*,y^*,y^*) = \frac{1}{y_i} \sum_{j=1}^{N} a_{ij} y_j^* \quad (4.10)
\]
\[
= -a_{ii},
\]
\[
\frac{\partial}{\partial y_q} n_{i,(k)}^{r,(k)}(y^*,y^*,y^*) = -a_{iq} \frac{y_i^*}{y_q^*} = -(\text{diag}(y^*) A^T (\text{diag}(y^*))^{-1})_{iq}, \quad i \neq q,
\]
resulting in
\[
D_{m,(k)}^* \Phi_k^m = I - \sum_{r=0}^{M} \theta_{r,+}^m \Delta t A + \sum_{r=0}^{M} \theta_{r,-}^m \Delta t (\text{diag}(y^*) A^T (\text{diag}(y^*))^{-1})^{-1}
\]

With this, we have finally proven Theorem 4.1.

The distribution and number \( M \) of the substeps as well as the number of iterations \( K \) determines the order of accuracy of the scheme. In the following, we will use equispaced and Gauss–Lobatto points. To reach order \( p \), we use \( K = p \) corrections and \( M = \max\{K - 1, 1\} \) subintervals\(^2\). We will denote the \( p \)-th order MPDeC method by \( \text{MPDeC}(p) \). Note that \( \text{MPDeC}(1) \) is equivalent to the modified Patankar–Euler scheme and \( \text{MPDeC}(2) \) is equivalent to \( \text{MPRK}(2,2,1) \) investigated in [14]. Due to \( y^{n+1} = y^{M,(K)} \), \( \text{MPDeC} \) conserves all linear invariants, if \( \theta_{r,M} \geq 0 \) for all \( r = 0, \ldots, M \) since in this case the index function yields \( \gamma(j, i, \theta_{r,M}) = j \) and (4.1) can be written as
\[
y^{n+1} - y^n - \sum_{r=0}^{M} \theta_{r}^M \Delta t A \text{diag}(y^{n+1})(\text{diag}(y^{M,(K-1)}))^{-1} y^{r,(K-1)} = 0,
\]
which means that \( n^T y^{n+1} = n^T y^n \) for all \( n \in \ker(A^T) \). Indeed, for equispaced nodes, \( \theta_{r}^M \) with \( r = 0, \ldots, M \) are the weights of the closed Newton–Cotes formulas for integrals over \( I = [0, 1] \). Hence, a negative \( \theta_{r}^M \) occurs for the first time at \( M = 7 \), i.e. with \( \text{MPDeC}(8) \). In

\(^2\)Recent investigations have shown that for Gauss–Lobatto points it suffices to choose \( M = \lceil \frac{K}{2} \rceil \) due to the higher-order quadrature rule resulting in a more accurate approach. However, for consistency reasons we stick with the classical version as introduced and used in [20].
this case, we also have to consider \( n^{r,(K)}(y^{r,(K-1)}, y^{M,(K-1)}, y^{n+1}) \) given in (4.4), resulting in

\[
\begin{align*}
n^T n^{r,(K)}(y^{r,(K-1)}, y^{M,(K-1)}, y^{n+1}) &= \sum_{i,j=1}^{N} n_i \frac{y_i^{n+1}}{y_i^{M,(K-1)}} p_{ij}(y^{r,(K-1)}) - \sum_{i,j=1}^{N} n_i \frac{y_i^{n+1}}{y_j^{M,(K-1)}} d_{ij}(y^{r,(K-1)}) \\
&= \sum_{i,j=1}^{N} n_i \frac{y_i^{n+1}}{y_i^{M,(K-1)}} p_{ij}(y^{r,(K-1)}) - \sum_{i,j=1}^{N} n_j \frac{y_j^{n+1}}{y_i^{M,(K-1)}} p_{ij}(y^{r,(K-1)}),
\end{align*}
\]

where we switched indices and used \( d_{ij} = p_{ji} \) for the last equality. We observe that \( n^T n^{r,(k)} \) does not need to vanish for \( n \notin \text{span}(1) \), so that the preservation of all linear invariants can not be guaranteed anymore for arbitrary systems \((2.4)\) and \(\text{MPDeC}(2.4)\). As a result, \(\text{MPDeC} \) with Gauss–Lobatto nodes conserve all linear invariants when applied to the weights of the corresponding Gauss-Lobatto quadrature, which are always positive. As a result, \(\text{MPDeC} \) with Gauss–Lobatto nodes conserve all linear invariants when applied to \((2.4)\).

Remark 4.2. From Theorem 4.1, we see that the Jacobian in general depends on \( y^* \), if there exist negative correction weights \( \theta_r^m \). For equispaced or Gauss–Lobatto points this is already the case for \( K > 2 \). If \( A \) is normal, then \( A \) and \( A^T \) share the same eigenvectors and the corresponding eigenvalues are the complex conjugate of each other. Since \( 1 \in \ker(A^T) \) this means that even \( 1 \in \ker(A) \). Hence, we may discuss the stability of \( y^* = 1 \). Then, we find

\[
\sigma \left( r_1(A) + r_2 \left( \text{diag}(y^*) A^T (\text{diag}(y^*))^{-1} \right) \right) = \{ r_1(\lambda) + r_2(\bar{\lambda}) | \lambda \in \sigma(A) \}
\]

for any rational maps \( r_1, r_2 \), which means that the spectrum of the Jacobian of the map \( g \) generating the \(\text{MPDeC} \) iterates can be written only in terms of the eigenvalues of \( A \). Using (4.7), (4.8) and (4.9), the stability function \( R_p \) of \(\text{MPDeC}(p) \) for normal matrices \( A \) and \( y^* = 1 \) can be computed by

\begin{align*}
R^{m,(1)}(z) &= \frac{1 + (z + \bar{z}) \sum_{r=0}^{M} \theta_r^m}{1 - \left( z \sum_{r=0}^{M} \theta_r^m - \bar{z} \sum_{r=0}^{M} \theta_r^m \right)}, \\
R^{m,(k)}(z) &= \frac{1 + \theta_r^m z + z \sum_{r=1}^{M} \theta_r^m R^{m,(k-1)}(z) - \left( z \sum_{r=0}^{M} \theta_r^{m+1} - \bar{z} \sum_{r=0}^{M} \theta_r^{m-1} - z \theta_m \right) R^{m,(k)}(z)}{1 - \left( z \sum_{r=0}^{M} \theta_r^m - \bar{z} \sum_{r=0}^{M} \theta_r^m \right)}, \\
R_p(z) &= R^{M,(K)}(z),
\end{align*}

for \( k = 2, \ldots, K \) and \( m = 1, \ldots, M \). Note that if \( A \) is symmetric it is also normal and we
obtain $\sigma(A) \subseteq \mathbb{R}$, so that one can further simplify (4.12) to

$$
R^{m,(1)}(z) = \frac{1 + 2z \sum_{r=0}^{M} \theta_{r}^m}{1 - z \sum_{r=0}^{M} |\theta_{r}^m|},
$$

(4.13)

$$
R^{m,(\tilde{k})}(z) = \frac{1 + \theta_{0}^m z + z \sum_{r=1}^{M} \theta_{r}^m R^{r,(\tilde{k}-1)}(z) - z \left( \sum_{r=0}^{M} |\theta_{r}^m| - \theta_{m}^m \right) R^{m,(\tilde{k}-1)}(z)}{1 - z \sum_{r=0}^{M} |\theta_{r}^m|},
$$

$$
R_{\mu}(z) = R^{M,(K)}(z),
$$

using $\theta_{r,+}^m + \theta_{r,-}^m = \theta_{r}^m$. It is also worth mentioning that for the system matrix

$$
A = \begin{pmatrix}
-a & b \\
-a & -b
\end{pmatrix}
$$

(4.14)

with $a, b > 0$, used in [14, 23], we find that $\ker(A) = \text{span}(y^*)$ with $y_1^* = \frac{b}{a}$, and thus

$$
\text{diag}(y^*)A^T(\text{diag}(y^*))^{-1} = \begin{pmatrix}
-a & \frac{b}{a} \\
\frac{b}{a} & -b
\end{pmatrix} = A,
$$

so that the stability function $R_{\mu}$ in this case is also given by (4.13).

Deferred Correction schemes are described by an iterative process which can be compared with classical RK schemes with more stages [1]. As MPDeC and DeC share the same amount of stages, we thus know that MPDeC(3) contains 5 stages where MPDeC(4) has already 10 stages inside, resulting in rational function with polynomial degree 10 in the numerator and denominator. By going even to arbitrary high-order the stability analysis would even become more complicated. Therefore, we avoid a theoretical analysis of the stability functions and investigate them numerically.

To give a first insight in the stability properties of MPDeC, we analyze the reduced stability function (4.13). In both cases described in Remark 4.2, the eigenvalues of $A$ leading to (4.13) are real, so that it is no loss of generality to restrict ourselves to the $2 \times 2$ case with $A$ given in (4.14).

In Figure 2, we present the absolute value of the stability function over real $z$. To obtain a stable scheme, the absolute value of $R(z)$ has to be always smaller than one (the black line). In 2a, we investigate MPDeC from order 4 to 14 using Gauss–Lobatto points. As can be recognized all MPDeC methods are stable. In 2b, the stability functions of MPDeC schemes from 4th to 14th order are depicted. Here, we recognize that MPDeC(12) and MPDeC(14) are unstable but MPDeC(13) is stable. However, this is not surprising since already in classical DeC using equidistant points has been problematic for high-order methods, cf. [8, 9, 20, 23] and references therein. The reason for this is related with classical interpolation theory where it is known that equidistant points should be avoided. However, we like to point out that our investigation supports as well the numerical investigation in [23] where problems in MPDeC-equidistant have been also recognized.
5. Numerical Experiments. In the following part, we verify the theoretical results of our considered modified Patankar schemes by numerical simulations. For the MPRK schemes we will consider more general systems where for the MPDeC schemes we restrict ourselves as described above to a $2 \times 2$ system.

5.1. Numerical Simulations of MPRK Schemes. Due to our investigations in Section 3, all the considered MPRK schemes are unconditionally stable theoretically and our numerical experience will demonstrate this as well. For comparison, we will focus on the same tests as suggested and investigated in [14, 15, 10]. We consider the following three different settings:

1. Test problem with exclusively real eigenvalues
2. Test problem with complex eigenvalues
3. Test problem with double zero eigenvalues

Test problem with exclusively real eigenvalues. As the first test, we consider the linear initial value problem (IVP)

$$y' = 100 \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & -2 \end{pmatrix} y \text{ with } y(0) = \begin{pmatrix} 1 \\ 9 \\ 5 \end{pmatrix}. \tag{5.1}$$

The IVP (5.1) has only positive off-diagonal elements inside the matrix and is therefore a Metzler matrix. Together with the positive initial values, this ensures that each component of the solution of the IVP is positive for all times. The eigenvalues are given by $\lambda_1 = 0$, $\lambda_2 = -300$ and $\lambda_3 = -500$ and the analytical solution is given by

$$y(t) = c_1 \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-300t} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-500t}.$$
with \((c_1, c_2, c_3) = (1, 4, 6)\). The steady state is given by \(y^* = (5, 3, 7)^T\). The zero eigenvalue is simple and hence we obtain exactly one linear invariant given by \(1^T y^T\). Therefore, the positive system is also conservative, e.g. \(\sum_{i=1}^{3} y_i^n = 15\) for all \(n \in \mathbb{N}_0\). To demonstrate the stability of the MPRK methods, we select three different types focusing on MPRK32, MPRK43(0.5) and MPRK43(0.9,0.6). In Figure 3, the analytical and numerical results are shown. In 3a, we plot it using to a comparably small time step of \(\Delta t = 0.05\), whereas in 3b the time step is \(\Delta t = 25\). In both cases our approximated solution converges towards the fixed point even for a big time step. Note that we still have oscillations except for the MPRK32 scheme, we refer again to [23] for a more detailed investigation on this topic.

Test problem with complex eigenvalues. The second test is the IVP

\[
y' = 100 \begin{pmatrix} -4 & 3 & 1 \\ 2 & -4 & 3 \\ 2 & 1 & -4 \end{pmatrix} y \quad \text{with} \quad y(0) = \begin{pmatrix} 9 \\ 20 \\ 8 \end{pmatrix}.
\]

The system matrix in (5.2) is a Metzler matrix and we have the eigenvalues \(\lambda_1 = 0, \lambda_2 = 100(-6 + i)\) and \(\lambda_3 = \overline{\lambda}\). The analytical solution is given by

\[
y(t) = \begin{pmatrix} 13 \\ 14 \\ 10 \end{pmatrix} - 2e^{-600t} \begin{pmatrix} \cos(100t) & -1 \\ 0 & 1 \end{pmatrix} - 6e^{-600t} \begin{pmatrix} \cos(100t) & 1 \\ -1 & 0 \end{pmatrix} + \sin(100t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
and the steady state is $y^* = (13, 14, 10)^T$. Also in this case a rapid decrease to $y^*$ can be expected through the behaviour of (5.3). In Figure 4, we see again the numerical solution compared to the analytical one. All schemes are stable and converge towards the steady state solution. The conservation property is ensured as well and can be recognized in 4b.

**Test problem with double zero eigenvalues.** The third test is the linear IVP

$$y' = 100 \begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & -4 & 3 & 0 \\ 0 & 4 & -4 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix} y \text{ with } y(0) = \begin{pmatrix} 4 \\ 1 \\ 9 \\ 1 \end{pmatrix}. \tag{5.4}$$

The system matrix in (5.4) is a Metzler matrix with double zero eigenvalues $\lambda_1 = \lambda_2 = 0$. Therefore, we obtain besides $1^T y$ (the conservative property), a second linear invariant $n^T y$ with $n^T = (1, 2, 2, 1)^T$. Using the remaining eigenvalues $\lambda_3 = -300$ and $\lambda_4 = -700$ and corresponding eigenvectors, the analytical solution is given by

$$y(t) = c_1 \begin{pmatrix} 0 \\ 1 \\ 4/3 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + c_3 e^{-700t} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_4 e^{-300t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $(c_1, c_2, c_3, c_4) = (30/7, 5/3, -23/7, 7/3)$. The steady state will be reached quite fast and it is given by $y^* = \frac{1}{7}(7, 90, 120, 70)^T$. In Figures 5 and 6, we see the numerical results for our selected schemes for different $\Delta t$. All schemes converge to the steady state solutions and preserve both linear invariants as presented in the figures.
Figure 5: Approximation of (5.4) with $\Delta t = 0.05$

Figure 6: Approximation of (5.4) with $\Delta t = 25$

5.2. Numerical Simulations of MPDeC Schemes. We consider the initial value problem

\[(5.5)\]

\[y' = \begin{pmatrix} -25 & 25 \\ 25 & -25 \end{pmatrix} y \text{ with } y(0) = \begin{pmatrix} 0.998 \\ 0.002 \end{pmatrix}.\]

The nonzero eigenvalue $\lambda = -50$ and the analytical solutions is given by

\[y(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0.498 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-50t}.\]

The steady state is given by $y^* = (0.5, 0.5)^T$. In Figure 7, the numerical and analytical solutions are plotted for several different MPDeC schemes.

From our investigation in Section 4, we know that using Gauss–Lobatto points leads always stable and conservative approximations as can be also recognized in 7a however for equidistant points, we obtain problems inside the approximation for higher-order approximations. This can also be seen in our numerical approximation for the IVP (5.5) in Figure 7b. As we mentioned earlier this behaviour is not surprising since already the classical DeC method (without using the modified Patankar trick) has problems for high-order approximations, i.e.
Figure 7: Approximation with $\Delta t = 25$

Figure 8: Numerical approximation of (5.5)

for 8-th order or higher, if equidistant points are used. Obviously, MPDeC(14) is unstable and alternate around the steady state solution. Nevertheless, we can obtain a stable discretization for this test case if we decrease the time step. We can estimate the time step using the eigenvalue $\lambda = 50$ and the value $z^* \approx -9.403$ satisfying $|R(z^*)| = 1$ in Figure 2b. From $z = \Delta t \lambda$ we thus find the bound $\Delta t \leq 0.188$. In Figure 8, we see an unstable behaviour for $\Delta t = 0.2$ whereas for $\Delta t \approx 0.17$ also MPDeC(14) converges to the steady state solution. Indeed, increasing $\Delta t$ up to 0.188 to obtain stable approximations, however, the convergence rate will also be quite slow and we need more time to reach the steady state. Note that the presented theory only claims a local convergence of the iterates towards the steady state. Nevertheless, if we violate the stability condition, we can start arbitrary close to the steady
state solution, and still, the iterates will not converge to $y^*$. To clarify this, we change our IVP (5.5) to the following condition

\begin{equation}
(5.6) \quad y' = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} y \quad \text{with } y(0) = y^* + 10^{-6} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{equation}

In Figure 9, we plot the numerical solution $y_2$ calculated with order MPDeC(14) using equispaced points. The time step is selected with respect to the eigenvalue $\lambda_1 = -1$, so that $\Delta t = \frac{1}{\lambda} = |z|$ holds. In 9b, we see a slow convergence against the steady state. In Figures 9c-9d, the conditions for convergence are not fulfilled and we see this also in our figures. A similar behaviour can be seen for $y_1$ analogously.

6. Conclusion. In this paper, we have investigated the stability conditions for several modified Patankar schemes. We were able to demonstrate that the MPRK(3,2) scheme proposed in [23] and the MPRK43($\gamma$) schemes from [16] are stable considering general positive linear $N \times N$ systems. Further, we provided the stability functions for the MPRK43($\alpha, \beta$) schemes and derived several conditions (3.29) on the parameters to obtain a stable scheme. For a fixed method, the conditions (3.29) can easily be checked whereas through a numerical study we have concluded that these conditions should be as well fulfilled for the general parameter selection. However, a formal proof is still missing. This remaining investigation will also be part of future research. We were also able to compute the stability function for MPDeC schemes and investigated the schemes numerically. We have seen that using Gauss–Lobatto points, we always obtain a stable scheme whereas for equispaced points stability issues have been recognized for higher-order methods and large $\Delta t$. This fact is not surprising since already DeC methods show stability issues if equispaced points are used and also the investigation in [23] confirms those problems. We have seen that the $\Delta t$ bound can be estimated from stability functions and eigenvalues.

In the future, we plan to continue our investigation in several directions. First, we consider non-conservative Patankar systems as the ones proposed in [5]. Later, we combine MP schemes with proper selected space discretization and consider hyperbolic conservation/balance laws as it was already successfully done in [6]. However, it is not clear what the stability and convergence properties of such combined schemes are.
7. Appendix. Numerical validation that Assumptions (3.29) are always fulfilled for the general parameter selection.

Acknowledgements. The author Th. Izgin gratefully acknowledges the financial support by the Deutsche Forschungsgemeinschaft (DFG) through grant ME 1889/10-1. P. Öffner was supported by the Gutenberg Research College, JGU Mainz. P. Öffner also wants to thank A. Meister (Kassel) for his invitation to Kassel.

REFERENCES

[1] Rémi Abgrall, Élise Le Mélédo, Philipp Öffner, and Davide Torlo. Relaxation deferred correction methods and their applications to residual distribution schemes. SMAI J. Comput. Math., 8:125–160, 2022.
Figure 12: Plot of $(a_1a_4 - a_2a_3 - b_1b_4 + b_2b_3)(\beta - \alpha)^2$ from (3.29).

Figure 13: Plot of $(a_3^2 - b_3^3)(\beta - \alpha)^2$ from (3.29).

[2] Rémi Abgrall, Philipp Öffner, and Hendrik Ranocha. Reinterpretation and extension of entropy correction terms for residual distribution and discontinuous Galerkin schemes: application to structure preserving discretization. J. Comput. Phys., 453:24, 2022. Id/No 110955.

[3] Luca Benvenuti and Lorenzo Farina. Eigenvalue regions for positive systems. Systems & Control Letters, 51(3):325–330, 2004.

[4] Hans Burchard, Eric Deleersnijder, and Andreas Meister. A high-order conservative Patankar-type discretisation for stiff systems of production–destruction equations. Appl. Numer. Math., 47(1):1–30, 2003.

[5] Alina Chertock, Shumo Cui, Alexander Kurganov, and Tong Wu. Steady state and sign preserving semi-implicit Runge–Kutta methods for ODEs with stiff damping term. SIAM Journal on Numerical Analysis, 53(4):2008–2029, 2015.

[6] M. Cialliella, L. Micalizzi, P. Öffner, and D. Torlo. An arbitrary high order and positivity preserving method for the shallow water equations. Comput. Fluids, 247:21, 2022. Id/No 105630.
(a) $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$, $\frac{2}{3} \leq \beta \leq 3\alpha(1-\alpha)$

(b) $3\alpha(1-\alpha) \leq \beta \leq \frac{2}{3}$, $\alpha < \alpha_0$

(c) $\frac{3\alpha-2}{6\alpha-3} \leq \beta \leq \frac{2}{3}$, $\alpha > \alpha_0$

Figure 14: Plot of $(b_3b_4 - a_3a_4)(\beta - \alpha)^2$ from (3.29).

[7] Peter Deuflhard and Folkmar Bornemann. Scientific computing with ordinary differential equations. Transl. from the German by Werner C. Rheinboldt, volume 42 of Texts Appl. Math. New York, NY: Springer, 2002.

[8] Alok Dutt, Leslie Greengard, and Vladimir Rokhlin. Spectral deferred correction methods for ordinary differential equations. BIT, 40(2):241–266, 2000.

[9] Maria Han Veiga, Philipp Öffner, and Davide Torlo. DeC and ADER: similarities, differences and a unified framework. J. Sci. Comput., 87(1):35, 2021. Id/No 2.

[10] Juntao Huang, Thomas Izgin, Stefan Kopecz, Andreas Meister, and Chi-Wang Shu. On the stability of strong-stability-preserving modified Patankar–Runge–Kutta schemes. arxiv preprint arxiv:2205.01488, 2022.

[11] Juntao Huang and Chi-Wang Shu. Positivity-preserving time discretizations for production-destruction equations with applications to non-equilibrium flows. J. Sci. Comput., 78(3):1811–1839, 2019.

[12] Juntao Huang, Weifeng Zhao, and Chi-Wang Shu. A third-order unconditionally positivity-preserving scheme for production-destruction equations with applications to non-equilibrium flows. J. Sci. Comput., 79(2):1015–1056, 2019.

[13] Thomas Izgin, Stefan Kopecz, and Andreas Meister. Recent developments in the field of modified Patankar Runge-Kutta-methods. PAMM, 21(1):e202100027, 2021.

[14] Thomas Izgin, Stefan Kopecz, and Andreas Meister. On lyapunov stability of positive and conservative time integrators and application to second order modified patankar-runge-kutta schemes. ESAIM: M2AN, 56(3):1053–1080, 2022.

[15] Thomas Izgin, Stefan Kopecz, and Andreas Meister. On the stability of unconditionally positive and linear invariants preserving time integration schemes. SIAM Journal on Numerical Analysis, 60(6):3029–3051, 2022.

[16] Stefan Kopecz and Andreas Meister. Unconditionally positive and conservative third order modified Patankar-Runge-Kutta discretizations of production-destruction systems. BIT, 58(3):691–728, 2018.

[17] Stefan Kopecz and Andreas Meister. On the existence of three-stage third-order modified Patankar-Runge-Kutta schemes. Numer. Algorithms, 81(4):1473–1484, 2019.

[18] David G. Luenberger. Introduction to dynamic systems. Theory, models, and applications. New York etc.: John Wiley & Sons. XIV, 446 p. £ 10.50 (1979)., 1979.

[19] A. Meister and S. Ortleb. On unconditionally positive implicit time integration for the DG scheme applied to shallow water flows. Int. J. Numer. Methods Fluids, 76(2):69–94, 2014.

[20] Philipp Öffner and Davide Torlo. Arbitrary high-order, conservative and positivity preserving Patankar-type deferred correction schemes. Appl. Numer. Math., 153:15–34, 2020.
[21] Hendrik Ranocha, Lajos Lóczy, and David I. Ketcheson. General relaxation methods for initial-value problems with application to multistep schemes. Numer. Math., 146(4):875–906, 2020.

[22] Andrew Stuart and Anthony R Humphries. Dynamical systems and numerical analysis, volume 2. Cambridge University Press, 1998.

[23] Davide Torlo, Philipp Öffner, and Hendrik Ranocha. Issues with positivity-preserving Patankar-type schemes. Appl. Numer. Math., 182:117–147, 2022.