Subgame-perfect equilibrium strategies in state-constrained recursive stochastic control problems

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Abstract. We study time-inconsistent recursive stochastic control problems. Since, for this class of problems, classical optimal controls may fail to exist or to be relevant in practice, we focus on subgame-perfect equilibrium policies. The approach followed in our work relies on the stochastic maximum principle and, indeed, we adapt the classical spike variation technique to obtain a characterization of equilibrium strategies in terms of a generalized Hamiltonian function of second-order defined through a flow of pairs of BSDEs. We deal then time-inconsistent recursive stochastic control problems under a state constraint, defined by means of an additional recursive utility, by adapting Ekeland’s variational principle to this more tricky situation. The theoretical results are applied to the financial field on finite horizon investment-consumption policies with non-exponential actualization. Here the possibility that the constraint is a risk constraint is covered.

Keywords. Time-inconsistency · Equilibrium strategy · Recursive stochastic control problem · Maximum principle · Portfolio management.

Introduction

In this paper we study time-inconsistent recursive stochastic control problems where the notion of optimality is by means of subgame-perfect equilibrium. In a continuous time setting, such kind of controls have been introduced in the papers [20] and [22], later completed in [21], and can be thought of as “infinitesimally optimal via spike variation”: that is, they are optimal with respect to a penalty represented by unilateral deviations during an infinitesimal amount of time. In [22], the authors apply the classical maximum principal theory of [58] to deal with the linear Merton portfolio management problem in the context of pseudo-exponential actualization, introducing the concept of subgame-perfect equilibrium policy as a notion for ensure the time-consistency of the portfolio strategy - possibly not unique. They arrive at equivalent formulations in terms of ODEs and integral equations precisely because of the special form of discounting. See also [5].

What we propose is to do similar operations for a control problem that is not only much more general, but that is also in the context of recursive utilities and furthermore under a state constraint, to then apply everything only at the end to the financial sphere. The theory of recursive optimal control problems in continuous time has become very popular in the recent years. For the time-consistent framework we refer, in particular, to the fundamental works [17] and [26] (see also [25]). For the time-inconsistent setting we mention the series of work carried out by Yong, [55] among the others, whose approach focuses on dynamic programming (Hamilton-Jacobi-Bellman equations). The approach followed in our work, instead, is inspired by [22] and [35] and relies on the stochastic maximum principle; see also [46], [47] and [7]. We adapt the classical spike variation technique to obtain
a characterization of equilibrium strategies in terms of a generalized Hamiltonian function \( H \) defined through a flow of pairs of BSDEs. Our generalized Hamiltonian function, compared to the classical one, contains the driver coefficient of the recursive utility (which has more variables than its analogue of the classical case) and of the presence of a second-order stochastic process. We emphasize that, differently from the classical case, equilibrium strategies are characterized not only by means of a necessary condition, but also through a sufficient condition involving the generalized Hamiltonian even in the absence of extra convexity conditions. The control domain \( U \) does not need to satisfy particular geometric conditions such as convexity or linearity and, even if it were, given our choice made on the use of the concept of equilibrium, we had still to study the variations of the optimal candidate by means of the spike variation technique (and no more familiar ones).

This characterization will be computationally applicable in a smooth way to a portfolio problem that we'll consider at the end. We'd like to observe also that the assumptions we adopt for the coefficients of our stochastic control system - a decoupled FBSDE - are substantially those proposed in [35], although the concept of equilibrium is not used there. See also [1] and [11].

Going further, our analysis extends to treat time-inconsistent recursive stochastic control problems under a state constraint defined by means of an additional recursive utility, under the simplifying assumption that \( U \) is bounded. That constraint refers to an expected value, similarly to what could be seen in [58], so we had to adapt Ekeland’s variational principle (see also [18]) to this more tricky situation. We use a penalization method: that is, we consider equilibria for unconstrained problems, that approximate our constrained problem, to which we then apply the theory developed above on the maximum principle, thus obtaining necessary conditions for them that will be satisfied at the limit by our original equilibrium. The existence of approximating equilibria will be guaranteed precisely by Ekeland’s variational principle. Here we also need a study of the distance function with respect to a closed subset of \( \mathbb{R}^N \). We obtain a transversality condition, losing the sign of the first scalar multiplier, but failing to incorporate the multipliers into the Hamiltonian from within.

We'd like to point out that this procedure seems to be able to manage a wide variety of constraints (quite different from the one just mentioned), such as a constraint on the state process at the terminal instant as the ones considered, e.g., in [39] and [59] (where, however, convexity is assumed and the spike variation technique is not used), or even [30] (local minimizers, Clarke’s tangent cone and adjacent cone of first-order and second-order are used). This will be the subject of future developments, as well as the open issue of \( U \)-unbounded. We suggest to see [57] and [56].

See [49] for a convex and compact constraint defined through a (pseudo) risk measure as VaR - Value at Risk - on a wealth process at a future time instant “very close” to the present (estimations of projected wealth are involved). Here, the market coefficients are random but independent of the Brownian motion driving the stocks. For a generalization, see [44] (CRRA preferences), so [36] and [9] (martingale methods). See also [24], [33] and [34].

Finally, the theoretical results are applied to the financial field of finite horizon investment-consumption policies with non-exponential actualization (e.g. a hyperbolic one). Under appropriate hypotheses, our results cover the case where the constraint is, more specifically, a risk constraint: that is, the additional recursive utility derives from a suitable dynamic risk measure defined by means of a \( g \)-expectation as, e.g., in [51] (that’s a road to investigate better). Of this author see also [13] and [16].

We search here controls in feedback form partially mimicking what’s done in [22] (or [21]), where explicit calculations are feasible. Here the concepts of discounting, utility and time-consistency are well explained. The choice of the shape of the recursive utilities, even those that determine the constraint, follows the classic Uzawa type. There’re countless choices for the wealth equation and the two BSDEs (the recursive utility for the optimization and that one for the constraint). See also [31], [38] and [48].

One of our beliefs is that a study of risk measures turns out to be useful in the sense just explained through some of the following fields: risk measures from BSDEs; law invariance; comonotonicity; less than convexity; links with the underlying scenario space; consistency with respect to appropriate orderings; connections with game theory; set-valued case (all that not just for random variables or portfolio vectors which are bounded and also to get robust representation formulas, hoping finally for the chance of a reasonable limit in the conditional framework).

In this regard, some references about risk measures are [12], [3], [42], [28], [40], [15], [8], [27], [53], [23], [29] and [19] while, about the set-valued case, [2] and [32].

In the bibliography it’s possible to find a remarkable extract also of other fundamental texts which can be reached by deepening the study of all this, such as [54], [50], [10], [45], [52], [14], [43], [4], [6], [37] and [41].
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The paper is organized as follows: in Section 1 we establish the notations; in Section 2 we explain the problem through the notion of equilibrium policy; in Section 3 we give some preliminary results; in Section 4 we retrace the technical steps up to the maximum principle (and corollaries); in Section 5 we deal with the problem under a state constraint; in Section 6 we apply all that to a portfolio management problem.

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1 Notations

Take \( n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\} \) and \( \mathbb{R}^n = \mathbb{R}^n_\mathbb{R} \), \( u = (u_1, \ldots, u_n)^T \), equipped by the Euclidean topology and the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^n) \) with its Lebesgue measure, as will be henceforth for any other Euclidean space, and choose \( U \in \mathcal{B}(\mathbb{R}^n) \setminus \{\emptyset\} \) (not necessarily bounded, for now). Any alphabetic letter appearing as a subscript of a prescribed set, such as \( \mathbf{u} \) for \( \mathbb{R}^n \) and \( U \) here, must be seen as our preference symbol to denote the generic variable element of that domain. Take also a non-empty open interval \( I \subseteq \mathbb{R} = \mathbb{R}_* \).

Set \( T \in [0, \infty] \) as our finite deterministic horizon and let \((\Omega, \mathcal{F}, \mathbf{P})\) be a given complete probability space such that we can define a one-dimensional standard Brownian motion, or Wiener process, \( W = (W(t))_{t \in [0, T]} \) on it and let \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]} \) be the completed filtration generated by \( W \) for which we suppose \( \mathcal{F}_T = \mathcal{F} \) (system noise is the only source of uncertainty in the problem). Thus, our filtered space \((\Omega, \mathcal{F}, \mathbf{P})\) fulfills the usual conditions.

For any non-empty set \( I \) of indices \( i \), we’ll keep implicit the dependence on the sample variable \( \omega \in \Omega \) for each stochastic process on \( I \times \Omega \), as is usually done and as indeed we’ve just done for \( W \).

We want to specify also that any stochastic process on \( I \times \Omega \) must be seen as its equivalence class given by the quotient with respect to the equivalence relation \( \sim \) of indistinguishability: that is, for any process \( X = (X(i))_{i \in I} \), \( \tilde{X} = (\tilde{X}(i))_{i \in I} \) on \( I \times \Omega \), \( X \sim \tilde{X} \) if and only if \( \mathbf{P} \left[ \forall i \in I, X(i) = \tilde{X}(i) \right] = 1 \).

We also introduce the following symbology, rather familiar, where: \( E \in \mathcal{B}(\mathbb{R}) \), \( d \in \mathbb{N}^* \), \( V < \mathbb{R}^d \) (vector subspace), \( t, \tau \in [0, T] \) with \( t \neq T \), and \( p \in [1, \infty] \).

- \( \mathbb{I}_E(\cdot) \). The indicator function of the set \( E \), i.e., for any \( x \in \mathbb{R} \),
  \[
  \mathbb{I}_E(x) := \begin{cases} 
  1, & \text{if } x \in E, \\
  0, & \text{if } x \notin E.
  \end{cases}
  \]

- \( \mathbf{E}[\cdot] \). The expected value w.r.t. \( \mathbf{P} \) of a (\( \mathbf{P} \)-integrable) real-valued \( \mathcal{F}_\tau \)-measurable random variable \( X \) on \( \Omega \), i.e.
  \[
  \mathbf{E}[X] \equiv \mathbf{E}X := \int_{\Omega} X(\omega) \, d\mathbf{P}(\omega) \in \mathbb{R}.
  \]

- \( L^p_p(\Omega; V) \). The (Banach) space of \( V \)-valued \( \mathcal{F}_\tau \)-measurable random variables \( X \) on \( \Omega \) such that
  \[
  \|X\|_p^p := \mathbf{E}|X|^p < \infty.
  \]

- \( L^\infty(\Omega; V) \). The space of (\( \mathbf{P} \)-a.s.) bounded \( V \)-valued \( \mathcal{F}_\tau \)-measurable random variables \( X \) on \( \Omega \), i.e. with
  \[
  \|X\|_\infty := \inf \{ K \in [0, \infty] | \|X\| \leq K \text{ } \mathbf{P}-\text{a.s.} \} < \infty
  \]
  (conventionally, \( \inf \emptyset = \infty \)).

- \( L^p_p(t, T; V) \). The space of \( V \)-valued \((\mathcal{F}_s)_{s \in [t, T]}\)-progressively measurable processes \( X = (X(s))_{s \in [t, T]} \) on \([t, T] \times \Omega \) such that
  \[
  \|X(\cdot)\|_p^p \equiv \|X\|^p_p := \mathbf{E} \int_t^T |X(s)|^p \, ds < \infty.
  \]

Remark. By the classical Jensen’s inequality (w.r.t. the Lebesgue measure on \([t, T]\) - for almost any fixed \( \omega \), if \( X = (X(s))_{s \in [t, T]} \in L^p_p(t, T; V) \), then
  \[
  \mathbf{E} \left( \int_t^T |X(s)|^p \, ds \right) < \infty.
  \]
• \( \mathcal{L}_p^\infty(t, T; V) \). The space of bounded \( V \)-valued \((\mathcal{F}_s)_{s \in [t, T]}\)-progressively measurable processes \( X = (X(s))_{s \in [t, T]} \) on \([t, T] \times \Omega\), i.e. with

\[
\|X(\cdot)\|_\infty \equiv \|X\|_\infty := \inf \left\{ K \in [0, \infty] \mid \sup_{s \in [t, T]}|X(s)| \leq K \text{ P-a.s.} \right\} < \infty.
\]

• \( \mathcal{L}_p^\infty(\Omega; C([t, T]; V)) \). The space of \( V \)-valued \((\mathcal{F}_s)_{s \in [t, T]}\)-adapted (P-a.s.) continuous processes \( X = (X(s))_{s \in [t, T]} \) on \([t, T] \times \Omega\) such that

\[
\|X(\cdot)\|_{p, \infty}^p \equiv \|X\|_{p, \infty}^p := \mathbb{E} \sup_{s \in [t, T]}|X(s)|^p < \infty.
\]

We’d like to remind that a \( \mathbb{R}^d \)-valued measurable \((\mathcal{F}_s)_{s \in [t, T]}\)-adapted process \( X = (X(s))_{s \in [t, T]} \) on \([t, T] \times \Omega\) admits a \((\mathcal{F}_s)_{s \in [t, T]}\)-progressively measurable modification (stochastically equivalent process) and, if \( X \) is also (P-a.s.) left or right continuous as a process, then \( X \) itself is \((\mathcal{F}_s)_{s \in [t, T]}\)-progressively measurable.

2 The problem

For arbitrary fixed initial instant \( t \in [0, T] \) and initial state \( x \in I \), we consider the recursive stochastic optimal control problem on \([t, T] \times \Omega\), in strong formulation, as the ensemble of the controlled system, or state equation,

\[
\begin{align*}
\frac{dX(s)}{ds} &= b(s, X(s), u(s)) ds + \sigma(s, X(s), u(s)) dW(s), \quad s \in [t, T], \\
X(t) &= x
\end{align*}
\]  
(1)

which is a controlled (forward) stochastic differential equation in the Itô’s differential form, or (F)SDE for short, with finite horizon and random coefficients depending on the sample \( \omega \) through the given admissible control \( u(\cdot) \) – decoupled from the stochastic differential (dis)utility system

\[
\begin{align*}
\frac{dY(s; t)}{ds} &= -f(s, X(s), u(s), Y(s; t), Z(s; t); t) ds + Z(s; t) dW(s), \quad s \in [t, T], \\
Y(T; t) &= h(X(T); t)
\end{align*}
\]  
(2)

which is instead a backward stochastic differential equation, still in the Itô’s differential form, or BSDE for short (more general than a stochastic differential (dis)utility - SUD for short - in its original meaning) – with the real-valued generalized Bolza-type cost or disutility functional

\[
J(u(\cdot); t, x) = \mathbb{E} Y(t; t)
\]  
(3)

where \( b, \sigma : [0, T] \times \mathbb{R} \times U_u \rightarrow \mathbb{R}, f : [0, T] \times \mathbb{R} \times U_u \times U_y \times \mathbb{R}_z \times [0, T] \rightarrow \mathbb{R} \) and \( h : \mathbb{R}_x \times [0, T] \rightarrow \mathbb{R} \) are measurable deterministic maps such that, at least, all the above makes sense (see below).

Therefore, by combining (1) and (2), we get the controlled decoupled forward-backward stochastic differential equation (system), or FBSDE for short,

\[
\begin{align*}
\frac{dX(s)}{ds} &= b(s, X(s), u(s)) ds + \sigma(s, X(s), u(s)) dW(s), \quad s \in [t, T], \\
\frac{dY(s; t)}{ds} &= -f(s, X(s), u(s), Y(s; t), Z(s; t); t) ds + Z(s; t) dW(s), \quad s \in [t, T], \\
X(t) &= x, \quad Y(T; t) = h(X(T); t)
\end{align*}
\]  
(4)

where the dependencies between all the elements involved could be clarified by writing, on \([t, T] \times \Omega\),

\[
\begin{align*}
X(\cdot) &= X^{t, x, u(\cdot)} := X(\cdot; t, x, u(\cdot)) \\
Y(\cdot; t) &= Y^{x, u(\cdot); t} := Y(\cdot; t, x, u(\cdot)) \\
Z(\cdot; t) &= Z^{x, u(\cdot); t} := Z(\cdot; t, x, u(\cdot))
\end{align*}
\]  
(5)

and indeed, moreover, \( X(\cdot), Y(\cdot; t) \in \mathcal{L}_p^\infty(\Omega; C([t, T]; \mathbb{R})) \) and \( Z(\cdot; t) \in \mathcal{L}_p^\infty(t, T; \mathbb{R}) \) have their own decomposition in the Itô’s integral form given by, for any \( s \in [t, T] \),

\[
X(s) = x + \int_t^s b(r, X(r), u(r)) dr + \int_t^s \sigma(r, X(r), u(r)) dW(r),
\]  
(6)

\[
Y(s; t) = h(X(T); t) + \int_t^s f(r, X(r), u(r), Y(r; t), Z(r; t); t) dr - \int_t^s Z(r; t) dW(r).
\]  
(7)
In particular, $Y(t; t)$ is actually deterministic since $X(T)$ is measurable w.r.t. also the completed $\sigma$-algebra on $\Omega$ generated by the process $(W(s) - W(t))_{s \in [t, T]}$ and so $Y(t; t)$ is simultaneously measurable w.r.t. $\mathcal{F}_t$ and that one which are mutually independent as filtrations. Consequently, from (3) and (7),

$$J(u(\cdot); t, x) \equiv \mathbb{E} \left[ \int_t^T f(s, X(s), u(s), Y(s; t), Z(s; t); t) \, ds + h(X(T); t) \right] = Y(t; t)$$

where, by the way, the running or intertemporal cost and the terminal cost are explicitly specified.

We assume that $I$ is such that, for any $t \in [0, T]$, and for any $x \in I$ and $s \in [t, T]$, $X(s) \in I$ (P-a.s.) as well - in the “less optimal” case, $I = \mathbb{R}$ - and therefore, if we prefer, we can think that the domain of the coefficients $b, \sigma, f, h$ is restricted. So $I$ could depend on $T$.

We’ll define soon the class $\mathcal{W}([t, T])$ of our admissible controls $u(\cdot)$ on $[t, T] \times \Omega$, which are at least $U$-valued and $(\mathcal{F}_s)_{s \in [t, T]}$-progressively measurable, and some pretty mild conditions for $b, \sigma, f, h$ suitable for our purposes.

For any $u(\cdot) \in \mathcal{W}([t, T])$, we call $u(\cdot)$ an admissible control and, if $X(\cdot) = X^{t, x, u(\cdot)}$ is the solution of (1) corresponding to $u(\cdot)$ (according to (6)) and if $Y(\cdot; t) = Y^{x, u(\cdot); t}, Z(\cdot; t) = Z^{x, u(\cdot); t}$ is the solution of (2) corresponding to them (according to (7)), then we call $X(\cdot)$ an admissible state process, or trajectory, and $(u(\cdot), X(\cdot), Y(\cdot; t), Z(\cdot; t))$ an admissible 4-tuple.

**Definition 1** (spike variation). Let $\bar{u}(\cdot) \in \mathcal{W}([t, T])$. For any other $u(\cdot) \in \mathcal{W}([t, T])$, and for any $\varepsilon \in [0, T - t]$ and $E^\varepsilon_t \in \mathcal{B}([t, T])$ with length $|E^\varepsilon_t| = \varepsilon$, we call the spike variation of $u(\cdot)$ w.r.t. $u(\cdot)$ and $E^\varepsilon_t$ as the function $u^\varepsilon(\cdot) \in \mathcal{W}([t, T])$ defined by setting

$$u^\varepsilon(\cdot) \doteq u + (u - \bar{u})1_{E^\varepsilon_t}$$

that is, for any $s \in [t, T]$,

$$u^\varepsilon(s) = \begin{cases} u(s), & \text{if } s \in [t, T] \setminus E^\varepsilon_t, \\ \bar{u}(s), & \text{if } s \in E^\varepsilon_t. \end{cases}$$

With respect to these symbols and according to (5), we set

$$\begin{align*}
\bar{X}(\cdot) & \doteq X^{t, x, \bar{u}(\cdot)}, \quad X^\varepsilon(\cdot) \doteq X^{t, x, u^\varepsilon(\cdot)} \\
\bar{Y}(\cdot; t) & \doteq Y^{x, \bar{u}(\cdot); t}, \quad Y^\varepsilon(\cdot; t) \doteq Y^{x, u^\varepsilon(\cdot); t} \\
\bar{Z}(\cdot; t) & \doteq Z^{x, \bar{u}(\cdot); t}, \quad Z^\varepsilon(\cdot; t) \doteq Z^{x, u^\varepsilon(\cdot); t}
\end{align*}$$

**Definition 2** (equilibrium policy). We call equilibrium policy for (3) any map $\Pi : [0, T]_s \times J_x \to U$ which is measurable and such that, for any $t \in [0, T]$ and $x \in I$, there exists a (P)-unique $I$-valued Itô process $\bar{X}(\cdot)$ on $[t, T] \times \Omega$ solution of the SDE

$$\begin{cases}
\frac{dX(s)}{ds} = b(s, X(s), \Pi(s, X(s))) \, ds + \sigma(s, X(s), \Pi(s, X(s))) \, dW(s), & s \in [t, T], \\
X(t) = x
\end{cases}$$

such that by denoting, for $s \in [t, T]$ (and P-a.s.),

$$\bar{u}(s) \doteq \Pi(s, \bar{X}(s))$$

then $\bar{u}(\cdot) \in \mathcal{W}([t, T])$ and, for any other $u(\cdot) \in \mathcal{W}([t, T])$,

$$\lim_{\varepsilon \downarrow 0} \frac{J(\bar{u}^\varepsilon(\cdot); t, x) - J(\bar{u}(\cdot); t, x)}{\varepsilon} \geq 0$$

where $u^\varepsilon(\cdot)$ is defined as in (9) or (10).

Our optimal control problem can be stated as follows.

**Problem.** Find an equilibrium policy for (3).
For any $\mathbf{P}$ equilibrium policy, we call $\bar{X}(\cdot)$ an optimal state process, or trajectory, and $(\bar{u}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$ an equilibrium 4-tuple. Such 4-tuples wouldn’t be unique.

It should be clear that that $\liminf_{t \to 0} b(t)$ is morally an actual limit (and for us it really will be).

Now, we need to define a class $\mathcal{W}[t, T]$ of $U$-valued controls $\bar{u}(\cdot)$ on $[t, T] \times \Omega$, which are at least $(F_{s})_{s \in [t, T]}$-progressively measurable, and to impose some mild conditions for $b, \sigma, f$, $h$ such in a way that, for any $\bar{u}(\cdot) \in \mathcal{W}[t, T]$, there exists a $(\mathbf{P})$-unique $I$-valued Itô process on $[t, T] \times \Omega$ (w.r.t. $\mathcal{F}_{s}$) $s \in [t, T]$ and $(W(s))_{s \in [t, T]}$

\[X(\cdot) = X^{t, u(\cdot)}(\cdot) \in \mathcal{F}_{s}(\cdot, t, x, u(\cdot))\]

(13)
such for which: on the one hand, the two processes $b(\cdot, X(\cdot), u(\cdot))$ and $\sigma(\cdot, X(\cdot), u(\cdot))$ on $[t, T] \times \Omega$ turn out to be $(F_{s})_{s \in [t, T]}$-progressively measurable and such that, $\mathcal{P}$-a.s.,

\[\int_{t}^{T} |b(s, X(s), u(s))| + |\sigma(s, X(s), u(s))|^{2} ds < \infty\]

even better if $b(\cdot, X(\cdot), u(\cdot)) \in L_{p}^{2}(t, T; \mathcal{R})$ and $\sigma(\cdot, X(\cdot), u(\cdot)) \in L_{p}^{2}(t, T; \mathcal{R})$; on the other hand, $f(s, X(s), u(s), y, z; t)$ results to be, as a function of $(x, y, z)$, $f(s, X(s), u(s), y, z; t)$ results to be, as a function of $(y, z)$, plus $f(\cdot, X(\cdot), u(\cdot), 0; t) \in L_{p}^{2}(t, T; \mathcal{R})$ and

\[h(X(T); t) \in L_{p}^{2}(\Omega; \mathcal{R})\]

so that the couple $(-f(s, X(s), u(s), y, z; t), h(X(T); t))$ becomes a so-called standard parameter for BSDE.

By this starting point of view, we define, for an appropriate $p \in [2, \infty]$, $\mathcal{W}[t, T] = \{ u(\cdot) \in L_{p}^{2}(t, T; \mathcal{R}^{n}) \mid u(\cdot)$ is $U$-valued \}

and we suppose that $b, \sigma, f, h$ are continuous w.r.t. all their variables and we assume that the following assumption hold.

(A) For any $t \in [0, T]$, there exists $L_{t} \in [0, \infty]$ such that, whatever map $\varphi(s, x, u, y, z; t)$ between $b(s, x, u)$, $\sigma(s, x, u)$, $f(s, x, u, y, z; t)$ and $h(x; t)$ is taken, and for any $s \in [t, T]$, $u \in U$ and $x, y, z \in \mathcal{R}$,

\[|\varphi(s, x, u, y, z; t)| \leq L_{t} (1 + |x| + |u| + |y| + |z|)\]

Next, $b, \sigma, h$ are of (differentiability) class $C^{2}$ in the variable $x \in \mathcal{R}$, $b_{x}, b_{xx}, \sigma_{x}, \sigma_{xx}$ are bounded (on $[t, T]_{s} \times \mathcal{R}_{x} \times U_{a}$) and continuous in $(x, u) \in \mathcal{R} \times U$, $h_{x}, h_{xx}$ are bounded and - continuous - (on $\mathcal{R}_{x}$) and $f(\cdot; t)$ is of class $C^{2}$ w.r.t. $b(\cdot, X(\cdot), u(\cdot), y, z; t)$ and $h(\cdot; t)$. Hence, $f(\cdot; t)$ is of class $C^{2}$ w.r.t. $b(\cdot, X(\cdot), u(\cdot), y, z; t)$ and $h(\cdot; t)$ is taken, for any $s \in [t, T]$, $u \in U$ and $x, \hat{x}, y, z \in \mathcal{R}$,

\[|\varphi(s, x, u, y, z; t) - \varphi(s, \hat{x}, u, \hat{y}, \hat{z}; t)| \leq L_{t} \{ |x - \hat{x}| + |u - \hat{u}| + |y - \hat{y}| + |z - \hat{z}|\}\]

and

\[|\varphi(s, 0, u, 0, 0; t)| \leq L_{t}\]

Observe that all the previous assumed relations, where $\varphi = b, \sigma$, could really depend on $t$ through the fact that $s \in [t, T]$.

Remark (modulus of continuity). Analogues of the previous assumption could work even if, relatively at least to the variable $u \in U$ or $x \in I$ (or both), it uses a more generic modulus of continuity $\bar{\omega}(\cdot)$ than a linear one: that is, a map $\bar{\omega} : [0, \infty] \to [0, \infty]$ which is non-decreasing and with $\lim_{t \to 0} \bar{\omega}(t) = \bar{\omega}(0) = 0$ such that it quantitatively measures the uniform continuity of some (continuous) function between metric spaces $\eta : (V, d_{V}) \to (V, d_{V})$ in the sense that, for any $v, \bar{v} \in V, d_{V}(\eta(v), \eta(\bar{v})) \leq \bar{\omega}(d_{V}(v, \bar{v}))$. The same would also be true by replacing $\varphi$ with $\sigma$. However, the modulus of Lipschitz continuity on the control $u$ is stronger than the one for the state $x$, and the Lipschitz continuity of $b$ on $x$ is stronger than the one for the state $x$. It is important to note that the modulus of the function $\phi$ depends on the modulus of continuity $\bar{\omega}(\cdot)$.
Remark. Observe that $\mathcal{W}[t, T]$ coincides to
\[
\{ u(\cdot) U\text{-valued process on } [t, T] \times \Omega \mid u(\cdot) \text{ is } (\mathcal{F}_s)_{s\in[t, T]}\text{-progressively measurable } \}
\]
if $U$ is bounded.

Therefore, by assumption, for any $t \in [0, T]$, $x \in \mathbb{R}$ and $u(\cdot) \in \mathcal{W}[t, T]$, the FBSDE (4) admits an unique (adapted) solution $(X(\cdot), Y(\cdot; t), Z(\cdot; t)) \equiv (X(\cdot; t, x, u(\cdot)), Y(\cdot; t, x, u(\cdot)), Z(\cdot; t, x, u(\cdot)))$ such that
\[
\mathbb{E} \sup_{s \in [t, T]} |X(s)|^2 + \mathbb{E} \sup_{s \in [t, T]} |Y(s; t)|^2 + \mathbb{E} \int_t^T |Z(s; t)|^2 ds \leq 1 + x^2 + \mathbb{E} \int_t^T |u(s)|^2 ds.
\]

3 Preliminaries

Lemma 1 (continuous dependence of the solution pairs on the assigned data). Take $p \in [1, \infty]$, $\xi_t, \hat{\xi}_t \in L^p_T(\Omega; \mathbb{R})$, and $F, \hat{F} : [0, T] \times \mathbb{R} \times \mathbb{R} \times \Omega \times [0, T] \to \mathbb{R}$ measurable maps which are $s\mathcal{P}$-uniformly Lipschitz w.r.t. $(y, z)$ and such that, for any $y, z \in \mathbb{R}$, $F(\cdot, y, z; t), \hat{F}(\cdot, y, z; t)$ are $\mathcal{F}_s$-valued $(\mathcal{F}_s)_{s\in[t, T]}$-progressively measurable processes with $F(\cdot, 0, 0; t), \hat{F}(\cdot, 0, 0; t) \in L^p_T(t, T; \mathbb{R})$. Consider the two well posed - non linear - BSDEs with parameters $(-F, \xi_t)$ and $(-\hat{F}, \hat{\xi}_t)$ respectively, i.e.
\[
\begin{cases}
    d\bar{X}(s; t) = -F(s, \bar{X}(s; t), \bar{Y}(s; t), \bar{Z}(s; t)) ds + \bar{Y}(s; t) d\mathbb{W}(s), & s \in [t, T], \\
    \bar{X}(T; t) = \xi_t
\end{cases}
\]
and
\[
\begin{cases}
    d\hat{X}(s; t) = -\hat{F}(s, \hat{X}(s; t), \hat{Y}(s; t), \hat{Z}(s; t)) ds + \hat{Y}(s; t) d\mathbb{W}(s), & s \in [t, T], \\
    \hat{X}(T; t) = \hat{\xi}_t
\end{cases}
\]
Then there exists $K_p \in [0, \infty]$ (depending also on $t$, $T$ and the Lipschitz constant, but not on $\xi_t, \hat{\xi}_t, \Xi, \hat{\Xi}$ or $\bar{Y}, \hat{Y})$ such that
\[
\mathbb{E} \sup_{s \in [t, T]} |\Xi(s; t) - \hat{\Xi}(s; t)|^p + \mathbb{E} \left( \int_t^T |\bar{Y}(s; t) - \hat{Y}(s; t)|^2 ds \right)^{p/2} \leq K_p \left[ \mathbb{E} \left| \xi_t - \hat{\xi}_t \right|^p + \mathbb{E} \left( \int_t^T |F(s, \Xi(s; t), \bar{Y}(s; t), \bar{Z}(s; t)) - \hat{F}(s, \Xi(s; t), \hat{Y}(s; t), \hat{Z}(s; t))| ds \right)^p \right].
\]

Remark. The main result underlying the whole theory of BSDEs is the classic representation theorem of integrable square continuous martingales, thus it is crucial that the reference filtration remains the completed filtration generated by $\mathbb{W}$.

Remark. Note the exponentiation - to power $p/2$ or $p$ - of the two deterministic integrals (of which we then calculate the expected value), and not only of the respective integral functions (absolute values of a difference).

Remark (Itô’s integration by parts for a product, and more). Given
\[
\begin{cases}
    d\lambda(s; t) = a(s; t) ds + \theta(s; t) d\mathbb{W}(s), & s \in [t, T], \\
    d\Lambda(s; t) = A(s; t) ds + \Theta(s; t) d\mathbb{W}(s), & s \in [t, T],
\end{cases}
\]
( well-posed) we’ve $d(\lambda(\cdot; t), \Lambda(\cdot; t))(s) = \theta(s; t) \Theta(s; t) ds$ and then
\[
d(\lambda(\cdot; t)\Lambda(\cdot; t))(s) = \lambda(s; t) d\Lambda(s; t) + d\Lambda(s; t) \lambda(s; t) + d(\lambda(\cdot; t), \Lambda(\cdot; t))(s)
\]
\[
= \left[ \lambda(s; t) A(s; t) + a(s; t) \Lambda(s; t) + \theta(s; t) \Theta(s; t) \right] ds + \left[ \lambda(s; t) \Theta(s; t) + \theta(s; t) \lambda(s; t) \right] d\mathbb{W}(s).
\]
In particular,
\[
d(\Lambda^2(\cdot; t))(s) = 2 \left[ A(s; t) \Lambda(s; t) + \frac{1}{2} \Theta^2(s; t) \right] ds + 2 \Theta(s; t) \Lambda(s; t) d\mathbb{W}(s)
\]
or also
\[ E[\lambda(s;t)\Lambda(s;t)] - E[\lambda(t;t)\Lambda(t;t)] = E\left[ \int_t^T \left[ \lambda(r;t)A(r;t) + \alpha(r;t)\Lambda(r;t) + \vartheta(r;t)\Theta(r;t) \right] dr \right]. \]

Moreover, equivalently,
\[ d\left( (\Lambda(\cdot;t) - \lambda(\cdot;t))^2 \right)(s) = 2 \left[ (A(s;t) - \alpha(s;t))\Lambda(s;t) - \lambda(s;t) \right] ds + 2(\Theta(s;t) - \vartheta(s;t))(\Lambda(s;t) - \lambda(s;t))dW(s). \]

**Proposition** (corollary of the Comparison Theorem applied to linear BSDEs). Let's consider \( \beta(\cdot;t), \gamma(\cdot;t) \in \mathcal{L}_D^\infty(t, T; \mathbb{R}) \) and, in their correspondence, \( \eta(\cdot;t) \geq 0 \) such that
\[ \begin{cases} d\eta(s;t) = \eta(s;t)\left[ \beta(s;t)ds + \gamma(s;t)dW(s) \right], & s \in [t, T], \\ \eta(t;t) = 1 \end{cases} \]
i.e.
\[ \eta(s;t) = \exp\left\{ \int_t^s \left[ \beta(r;t) - \frac{\gamma^2(r;t)}{2} \right] dr + \int_s^T \gamma(r;t) dW(r) \right\}. \]

Then, for any \( \alpha(\cdot;t) \in \mathcal{L}_D^2(t, T; \mathbb{R}) \) and \( \xi_t \in \mathcal{L}_F^2(\Omega; \mathbb{R}) \), there exists an unique solution pair \( (\Xi(\cdot;t), O(\cdot;t)) \in \mathcal{L}_D^\infty(\Omega; \mathcal{L}([t, T]; \mathbb{R})) \times \mathcal{L}_D^2(t, T; \mathbb{R}) \) of
\[ \begin{cases} d\Xi(s;t) = -[\alpha(s;t) + \beta(s;t)\Xi(s;t) + \gamma(s;t)O(s;t)]ds + O(s;t)dW(s), & s \in [t, T], \\ \Xi(T;t) = \xi_t \end{cases} \]
and \( \Xi(\cdot;t) \) is given by the conditional expectation
\[ \Xi(s;t) = \eta^{-1}(s;t)E\left[ \eta(T;t)\xi_t + \int_s^T \eta(r;t)\alpha(r;t) dr \bigg| \mathcal{F}_s \right]. \]

**Remark.** In particular, if \( \xi_t \geq 0 \), \( \alpha(\cdot;t) \geq 0 \) (resp. \( \leq 0 \)), then \( \Xi(\cdot;t) \geq 0 \) (resp. \( \leq 0 \)) - resp. with narrow inequalities even if only one of the two inequalities is narrow (for instance, if \( \xi_t > 0 \) and \( \alpha(\cdot;t) \geq 0 \), then \( \Xi(\cdot;t) > 0 \)).

Moreover, in general, for \( \tau \in [s, T] \), \( \eta(\tau;t)\eta^{-1}(s;t) \neq \eta(\tau;s) \) due to the dependence on \( t \) of \( \beta(\cdot;t) \) and \( \gamma(\cdot;t) \), so \( \Xi(s;t) \neq E\left[ \eta(T;t)\xi_t + \int_s^T \eta(r;t)\alpha(r;t) dr \bigg| \mathcal{F}_s \right] \) (term which differs from \( \Xi(s;t) \) through the dependence on \( t \) of \( \alpha(\cdot;t) \) and \( \xi_t \)).

**4 Maximum principle: necessary and sufficient conditions**

W.r.t. \( (\bar{u}(\cdot), \bar{X}(\cdot)) \) and \( u(\cdot) \), for \( \varphi = b, \sigma, b_x, \sigma_x \) etc, we use the notations
\[ \begin{cases} \varphi(s) := \varphi(s, \bar{X}(s), \bar{u}(s)) \\ \delta\varphi(s) := \varphi(s, \bar{X}(s), \bar{u}(s)) - \varphi(s). \end{cases} \]

**Remark.** It is natural to keep each candidate optimal pair as a reference point, so a notation like the following would be unnecessarily burdensome: \( \bar{\varphi}(s) \) and \( \delta\bar{\varphi}(s) \).

We consider approximate variational systems of first order and second order of \((\bar{u}(\cdot), \bar{X}(\cdot))\) w.r.t. \( u(\cdot) \) and \( E^\infty \):
\[ \begin{cases} dX^\infty_1(s;t) = b_x(s)X^\infty_1(s;t)ds + \left[ \sigma_x(s)X^\infty_1(s;t) + \delta\sigma(s)\mathbb{1}_{E^\infty}(s) \right]dW(s), & s \in [t, T], \\ X^\infty_1(t;t) = 0 \end{cases} \]
and
\[ \begin{cases} dX^\infty_2(s;t) = \left[ b_x(s)X^\infty_2(s;t) + \delta b(s)\mathbb{1}_{E^\infty}(s) + \frac{1}{2} b_{xx}(s)X^\infty_1(s;t)^2 \right]ds \\ + \left[ \sigma_x(s)X^\infty_2(s;t) + \delta\sigma_x(s)X^\infty_1(s;t)\mathbb{1}_{E^\infty}(s) + \frac{1}{2} \sigma_{xx}(s)X^\infty_1(s;t)^2 \right]dW(s), & s \in [t, T], \\ X^\infty_2(t;t) = 0. \end{cases} \]
We consider also adjoint processes \((p(\cdot ; t), q(\cdot ; t))\) and \((P(\cdot ; t), Q(\cdot ; t))\) of first order and second order associated to \((\bar{u}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot ; t), \bar{Z}(\cdot ; t))\) (not depending either on \(u(\cdot)\) or \(E_\xi\)), which are linear BSDEs whose coefficients will be explained soon:

\[
\begin{align*}
 dp(s; t) &= -g(s, p(s; t), q(s; t); t)ds + q(s; t)dW(s), \quad s \in [t, T], \\
 p(T; t) &= h_x(\bar{X}(T; t))
\end{align*}
\]

and

\[
\begin{align*}
 dp(s; t) &= -G(s, P(s; t), Q(s; t); t)ds + Q(s; t)dW(s), \quad s \in [t, T], \\
 P(T; t) &= h_{xx}(\bar{X}(T; t)).
\end{align*}
\]

Although it is clear, the process \(p(\cdot; t)\) and the summability exponent \(p\) are not to be confused - also because we can think that it is \(p = 2\).

Why we are interested in such processes derives at first from the next results (where \(O(\cdot) \equiv O_{\varepsilon,0}(\cdot)\) and \(o(\cdot) \equiv o_{\varepsilon,0}(\cdot)\))

**Lemma 2** (Yong-Zhou). For any \(k \in [1, \infty]\),

\[
\begin{align*}
 &\sup_{s \in [t, T]} E \left[ |X^\varepsilon(s) - \bar{X}(s)|^{2k} \right] = O(\varepsilon^k) \\
 &\sup_{s \in [t, T]} E \left[ |X^\varepsilon_1(s; t)|^{2k} \right] = O(\varepsilon^k) \\
 &\sup_{s \in [t, T]} E \left[ |X^\varepsilon - \bar{X}(s) - X^\varepsilon_1(s; t)|^{2k} \right] = O(\varepsilon^{2k}) \\
 &\sup_{s \in [t, T]} E \left[ |X^\varepsilon_2(s; t)|^{2k} \right] = O(\varepsilon^{2k}) \\
 &\sup_{s \in [t, T]} E \left[ |X^\varepsilon(s) - \bar{X}(s) - X^\varepsilon_1(s; t) - X^\varepsilon_2(s; t)|^{2k} \right] = o(\varepsilon^{2k})
\end{align*}
\]

and, moreover,

\[
E \left[ h(X^\varepsilon(T; t)) - h(\bar{X}(T; t)) - h_x(\bar{X}(T; t)) [X^\varepsilon_1(T; t) + X^\varepsilon_2(T; t)] - \frac{1}{2} h_{xx}(\bar{X}(T; t)) X^\varepsilon(T; t)^2 \right] = o(\varepsilon^{2k}).
\]

Thus, what we want to estimate is \(Y^\varepsilon(t; t) - \bar{Y}(t; t)\) but, in the above sense, we know something useful only about \(Y^\varepsilon(T; t) - \bar{Y}(T; t)\). Let’s re-build informations thanks to proper BSDEs and it will turn out to be a good idea to retrace calculations of M. Hu (partly following his notations). So, first of all, to determine \(g\) and \(G\) which work for us and to eliminate the explicit difference \(h(X^\varepsilon(T; t)) - h(\bar{X}(T; t))\), and obviously grouping with respect
to terms \( \mathbb{I}_{E_1}(\cdot) \cdot X_1^2(\cdot; t) \), \( X_2^2(\cdot; t) \), \( X_1^2(\cdot; t)^2 \) and \( X_1(\cdot; t) \mathbb{I}_{E_1}(\cdot) \).  

\[
d\left\{ p(s; t)X_1^2(\cdot; t) + X_2^2(\cdot; t) + \frac{1}{2}P(s; t)X_1^2(\cdot; t)^2 \right\}(s) \
= \left\{ p(s; t)\delta b(s) + q(s; t)\delta \sigma(s) + \frac{1}{2}P(s; t)(\delta \sigma(s))^2 \right\} \mathbb{I}_{E_1}(s) \\
+ \left\{ p(s; t)b_x(s) + q(s; t)\sigma_x(s) + g(s, p(s; t), q(s; t); t) \right\}[X_1^2(\cdot; t) + X_2^2(\cdot; t)] \\
+ \frac{1}{2} \left\{ p(s; t)b_x(s) + q(s; t)\sigma_x(s)^2 + 2P(s; t)b_y(s) + 2Q(s; t)\sigma_x(s) + P(s; t)\sigma_x(s)^2 - G(s, P(s; t), Q(s; t); t) \right\}X_1^2(\cdot; t)^2 \\
+ \left\{ p(s; t)\delta \sigma_x(s) + P(s; t)\sigma_x(s)\delta \sigma(s) + Q(s; t)\delta \sigma(s) \right\} X_1^2(\cdot; t) \mathbb{I}_{E_1}(s) \right\} ds \\
+
\left\{ p(s; t)\delta \sigma(s) \mathbb{I}_{E_1}(s) \\
+ \left\{ p(s; t)\sigma_x(s) + q(s; t) \right\}[X_1^2(\cdot; t) + X_2^2(\cdot; t)] \\
+ \frac{1}{2} \left\{ p(s; t)\sigma_x(s) + 2P(s; t)\sigma_x(s) + Q(s; t) \right\} X_1^2(\cdot; t)^2 \\
+ \left\{ p(s; t)\delta \sigma_x(s) + P(s; t)\delta \sigma(s) \right\} X_1^2(\cdot; t) \mathbb{I}_{E_1}(s) \right\} dW(s).
\]

Observe that \( g \) and \( G \) do not appear in the \( W \)-term. For the sake of brevity, we define

\[
\begin{align*}
A_1(s; t) &:= p(s; t)b(s) + q(s; t)\delta \sigma(s) + \frac{1}{2}P(s; t)(\delta \sigma(s))^2 \\
A_2(s; t) &:= p(s; t)b_x(s) + q(s; t)\sigma_x(s) - g(s, p(s; t), q(s; t); t) \\
A_3(s; t) &:= p(s; t)b_x(s) + q(s; t)\sigma_x(s) + 2P(s; t)b_y(s) + 2Q(s; t)\sigma_x(s) + P(s; t)\sigma_x(s)^2 - G(s, P(s; t), Q(s; t); t) \\
A_4(s; t) &:= q(s; t)\delta \sigma_x(s) + P(s; t)\sigma_x(s)\delta \sigma(s) + Q(s; t)\delta \sigma(s) \\
\Theta_1(s; t) &:= p(s; t)\sigma_x(s) + q(s; t) \\
\Theta_2(s; t) &:= p(s; t)\sigma_x(s) + 2P(s; t)\sigma_x(s) + Q(s; t) \\
\Theta_3(s; t) &:= p(s; t)\delta \sigma_x(s) + P(s; t)\delta \sigma(s)
\end{align*}
\]

while \( p(s; t)\delta \sigma(s) \) is already short enough for the moment. Both the coefficients of \( \mathbb{I}_{E_1}(\cdot) \) will prove to be particularly significant and useful. We point out that \( A_4(\cdot; t) \) will disappear, because \( A_4(s; t)X_1^2(\cdot; t)\mathbb{I}_{E_1}(s) = o(\varepsilon) \) (which not holds for \( \Theta_3(\cdot; t) \)). Now, indeed,

\[
Y^\varepsilon(s; t) = h(X^\varepsilon(T); t) + \int_s^T f(r, X^\varepsilon(r), u^\varepsilon(r), Y^\varepsilon(r; t), Z^\varepsilon(r; t); t) \, dr - \int_s^T Z^\varepsilon(r; t) \, dW(r) \\
= h(X^\varepsilon(T); t) + o(\varepsilon) + p(s; t)\mathbb{I}_{E_1}(s; t) + \frac{1}{2}P(s; t)X_1^2(\cdot; t)^2 \\
+ \int_s^T \left\{ f(r, X^\varepsilon(r), u^\varepsilon(r), Y^\varepsilon(r; t), Z^\varepsilon(r; t); t) + A_1(r; t) \mathbb{I}_{E_1}(r) \\
+ A_2(r; t)X_1^2(\cdot; t) + X_2^2(\cdot; t) \right\} \, dr \\
- \int_s^T \left\{ Z^\varepsilon(r; t) - \left\{ p(r; t)\delta \sigma(r) \mathbb{I}_{E_1}(r) \\
+ \Theta_1(r; t)\mathbb{I}_{E_1}(r; t) + X_2^2(\cdot; t) \right\} + \frac{1}{2}\Theta_2(r; t)X_1^2(\cdot; t)^2 + \Theta_3(r; t)X_1^2(\cdot; t)^2 \mathbb{I}_{E_1}(r) \right\} dW(r).
\]

Thus, defining

\[
\begin{align*}
\tilde{Y}^\varepsilon(s; t) &:= Y^\varepsilon(s; t) - \left\{ p(s; t)\mathbb{I}_{E_1}(s; t) + \frac{1}{2}P(s; t)X_1^2(\cdot; t)^2 \right\} \\
\tilde{Z}^\varepsilon(s; t) &:= Z^\varepsilon(s; t) - \left\{ p(s; t)\delta \sigma(s) \mathbb{I}_{E_1}(s) \\
&+ \Theta_1(s; t)\mathbb{I}_{E_1}(s; t) + X_2^2(\cdot; t) \right\} + \frac{1}{2}\Theta_2(s; t)X_1^2(\cdot; t)^2 + \Theta_3(s; t)X_1^2(\cdot; t)^2 \mathbb{I}_{E_1}(s) \right\}
\end{align*}
\]
we can rewrite
\[ \hat{Y}(s; t) = h(\bar{X}(T); t) + o(\epsilon) \]
\[ + \int_0^T \left[ f(r, X^\epsilon(r), u^\epsilon(r), Y^\epsilon(r; t), Z^\epsilon(r; t); t) + A_1(r; t) \epsilon Y^\epsilon(r) \right. \]
\[ + A_2(r; t) [X_1^\epsilon(r; t) + X_2^\epsilon(r; t)] + \frac{1}{2} A_3(r; t) X_1^2(r; t) \right] dr \]
\[ - \int_0^T \bar{Z}(r; t) dW(r). \]

Now, if
\[
\begin{cases}
Y^\epsilon_1(s; t) & := \hat{Y}(s; t) - \bar{Y}(s; t) \\
Z^\epsilon_1(s; t) & := \bar{Z}(s; t) - \bar{Z}(s; t)
\end{cases}
\]

since
\[ \bar{Y}(s; t) = h(\bar{X}(T); t) + \int_0^T f(r, \bar{X}(r), u(r), \bar{Y}(r; t), \bar{Z}(r; t); t) dr - \int_0^T \bar{Z}(r; t) dW(r) \]

then we obtain
\[ Y^\epsilon_1(s; t) = o(\epsilon) \]
\[ + \int_0^T \left[ f(r, X^\epsilon(r), u^\epsilon(r), Y^\epsilon(r; t), Z^\epsilon(r; t); t) - f(r, \bar{X}(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t); t) + A_1(r; t) \epsilon Y^\epsilon(r) \right. \]
\[ + A_2(r; t) [X_1^\epsilon(r; t) + X_2^\epsilon(r; t)] + \frac{1}{2} A_3(r; t) X_1^2(r; t) \right] dr \]
\[ - \int_0^T Z^\epsilon_1(r; t) dW(r). \]

So, by distinguishing \( r \notin E^\epsilon_{1+} \) and \( r \in E^\epsilon_{1+} \), with the idea to have three pieces \((\bar{u}, u \text{ and } \bar{u})\) and to pass \( \epsilon E^\epsilon_{1+}(\cdot) \) from the inside to the outside,
\[
\begin{align*}
&f(r, \bar{X}(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t); t) = f(r, \bar{X}(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t); t) + \Theta(r; t) \epsilon E^\epsilon_{1+}(r; t) \\
&\quad + [f(r, \bar{X}(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t); t) - f(r, \bar{X}(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t); t) + \Theta(r; t) \epsilon E^\epsilon_{1+}(r; t)] \epsilon E^\epsilon_{1+}(r; t)
\end{align*}
\]

(for arbitrary \( \Theta(\cdot; t) \)) and therefore it can be found out that \( g(\cdot, p(\cdot; t), q(\cdot; t); t) \) and \( G(\cdot, P(\cdot; t), Q(\cdot; t); t) \) must follow the following expressions:
\[
g(s, p, q; t) = [b_x(s) + f_x(s; t) \sigma_x(s) + f_y(s; t)]p + [\sigma_x(s) + f_x(s; t)]q + f_x(s; t)
\]

and
\[
G(s, P, Q; t) = \left[2b_x(s) + \sigma_x(s) + 2f_x(s; t) \sigma_x(s) + f_y(s; t)\right]P + \left[2\sigma_x(s) + f_x(s; t)\right]Q
\]
\[ + b_x(s)p(s; t) + \sigma_x(s)[f_x(s; t)p(s; t) + q(s; t)] \]
\[ + \left(1, p(s; t), \sigma_x(s)p(s; t) + q(s; t)\right) \cdot D^2f(s; t) \cdot (1, p(s; t), \sigma_x(s)p(s; t) + q(s; t))^T
\]

where \( D \equiv D_{x,y,z} \) (all the coefficients are calculated on a candidate optimum).

By assumption, we can be sure that adjoint equations (35) and (36) have unique solutions \((p(\cdot; t), q(\cdot; t))\) and \((P(\cdot; t), Q(\cdot; t))\) respectively such that, for any \( k \geq 1, \)
\[
\mathbf{E} \sup_{s \in [t, T]} \left[ |p(s; t)|^{2k} + |P(s; t)|^{2k} \right] + \mathbf{E} \left( \int_t^T \left[ |q(s; t)|^{2k} + |Q(s; t)|^{2k} \right] ds \right)^k < \infty.
\]
Remark. For the same reasons that \( Y(t; t) \) is deterministic, \( p(t; t) \) and \( P(t; t) \) are also deterministic.

So, the interest was in \( \Theta(s; t) := p(s; t)\delta\sigma(s) \). Now we consider, finally, \( (\Delta Y^\varepsilon(\cdot; t), \Delta Z^\varepsilon(\cdot; t)) \) such that

\[
\Delta Y^\varepsilon(s; t) = \int_s^T \left\{ f_y(r; t)\Delta Y^\varepsilon(r; t) + f_z(r; t)\Delta Z^\varepsilon(r; t) + \left[ p(r; t)\delta b(r) + q(r; t)\delta\sigma(r) + \frac{1}{2}P(r; t)(\delta\sigma(r))^2 \right] \right\} dr
- \int_s^T \Delta Z^\varepsilon(r; t) dW(r)
\]

and in particular \( \Delta Y^\varepsilon(T; t) = 0 \)

**Lemma 3** (Hu). \( Y_1^\varepsilon(s; t) - \Delta Y^\varepsilon(s; t) = o(\varepsilon) \) and \( Z_1^\varepsilon(s; t) - \Delta Z^\varepsilon(s; t) = o(\varepsilon) \).

Thus, by definitions (and \( X_1^\varepsilon(t; t) = X_2^\varepsilon(t; t) = 0 \)),

\[
\begin{align*}
Y^\varepsilon(t; t) &= \tilde{Y}(t; t) + \Delta Y^\varepsilon(t; t) + o(\varepsilon) \\
Z^\varepsilon(t; t) &= \tilde{Z}(t; t) + \Delta Z^\varepsilon(t; t) + p(t; t)\delta\sigma(t) + o(\varepsilon)
\end{align*}
\]

and in particular \( Y^\varepsilon(t; t) - \tilde{Y}(t; t) = \Delta Y^\varepsilon(t; t) + o(\varepsilon) \). Thus we consider \( \kappa(\cdot; t) > 0 \) such that

\[
\begin{align*}
\{ d\kappa(s; t) - \kappa(s; t) f_y(s; t) ds + f_z(s; t) dW(s) \}, \quad s \in [t, T],
\kappa(t; t) = 1
\end{align*}
\]

which can also be interpreted as a change of numéraire (a normalization) relative to the (dis)utility and corresponding to the coefficients \( \{ f_y(\cdot; t), f_z(\cdot; t) \} \), thus a positive constant, so that

\[
d(\kappa(\cdot; t), \Delta Y^\varepsilon(\cdot; t))(s) = \kappa(s; t) f_z(s; t) \Delta Z^\varepsilon(s; t) ds
\]

and in such a way that, due to various cancellations, plus boundary conditions (and \( \Delta Y^\varepsilon(t; t) \) deterministic),

\[
\begin{align*}
\Delta Y^\varepsilon(t; t) &= \mathbb{E} \left[ \int_t^T \kappa(r; t) \left[ p(r; t)\delta b(r) + q(r; t)\delta\sigma(r) + \frac{1}{2}P(r; t)(\delta\sigma(r))^2 \right] \right. \\
&\quad + \left. f(r, \bar{X}(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t) + p(r; t)\delta\sigma(r); t) - f(r, X(r), \bar{u}(r), \bar{Y}(r; t), \bar{Z}(r; t); t) \right] 1_{E_1^\varepsilon}(r) dr
\end{align*}
\]

and thus, finally, for arbitrary \( s \in [t, T] \), selecting

\[
E_1^\varepsilon = [s, s + \varepsilon]
\]

(or even open to the right), or \( [s - \varepsilon, s] \) for \( s = T \) (respectively open to the left), we get, by the classic Lebesgue’s dominated convergence theorem,

\[
\liminf_{\varepsilon \downarrow 0} \frac{J(\tilde{u}^\varepsilon(\cdot); t, x) - J(\tilde{u}(\cdot); t, x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\Delta Y^\varepsilon(t; t)}{\varepsilon} = \mathbb{E} [\kappa(s; t)\delta H(s; t, \bar{u}(s))]
\]

where \( H \) is our generalized Hamiltonian

\[
H(s, x, \bar{u}, y, z, p, q, P; t, \bar{x}, \bar{u}) = pb(s, x, \bar{u}) + q\sigma(s, x, \bar{u}) + \frac{1}{2}P[\sigma(s, x, \bar{u}) - \sigma(s, \bar{x}, \bar{u})]^2 \\
+ f(s, x, \bar{u}, y, z + p[\sigma(s, x, \bar{u}) - \sigma(s, \bar{x}, \bar{u})]; t)
\]
and, thus, where
\[
\begin{align*}
\mathcal{H}(s; t) &\equiv \mathcal{H}(s, \bar{X}(s), \bar{u}(s), \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t), P(s; t); t, \bar{X}(s), \bar{u}(s)) \\
\delta \mathcal{H}(s; t, u) &\equiv \mathcal{H}(s, \bar{X}(s), u, \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t), P(s; t); t, \bar{X}(s), \bar{u}(s)) - \mathcal{H}(s; t)
\end{align*}
\]

(\bar{x} \equiv \bar{X}(s) and \bar{u} \equiv \bar{u}(s)). These are random variables in \(L^2_\mathbb{P} \Omega; \mathbb{R} \) - with more than one variable - and such that \(\delta \mathcal{H}(s; t, \bar{u}(s)) = 0\).

**Theorem 1** (maximum principle). The following three conditions are equivalent for any \(t \in [0, T]\) and \(x \in I\).

1. \((\bar{u}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))\) is an equilibrium \(4\)-tuple.

2. For any \(u \in U\) and \(\mathbb{P}\text{-a.s.}, \delta \mathcal{H}(t; t, u) \geq 0\).

3. For any \(s \in [t, T]\), \(u \in U\) and \(\mathbb{P}\text{-a.s.}, \delta \mathcal{H}(s; t, u) \geq 0\).

**Proof.** 3 \(\Rightarrow\) 2 \(\Rightarrow\) 1: easy (remember that \(\kappa(\cdot; t) > 0\) with \(\kappa(t; t) = 1\)). 1 \(\Rightarrow\) 3: assume, by contradiction, that there exists \(t^* \in [0, T]\), \(s^* \in [t^*, T]\) \(u^* \in U\) and \(\mathcal{N} \in \mathcal{F}\) with \(\mathbb{P}[\mathcal{N}] > 0\) such that \(\delta \mathcal{H}(s^*; t^*, u^*) < 0\) on \(\mathcal{N}\). Then any \(u(\cdot) \in \mathcal{W}[t^*, T]\) such that
\[
u(s^*) = \begin{cases} \bar{u}(s^*), & \text{on } \Omega \setminus \mathcal{N}, \\ u^*, & \text{on } \mathcal{N}, \end{cases}
\]

(for instance, the trivial one) satisfies \(\delta \mathcal{H}(s^*; t^*, u(s^*)) = \delta \mathcal{H}(s^*; t^*, u^*)\mathbb{I}_\mathcal{N}\) and thus \(\mathbb{E}[\kappa(s^*; t^*)\delta \mathcal{H}(s^*; t^*, u(s^*))] < 0\) (\(\mathbb{P}[\mathcal{N}] > 0\)): contradiction w.r.t. the hypotheses and (20).

**Remark.** 2 and 3 mean: there exists \(\Omega' \in \mathcal{F}\) with \(\mathbb{P}[\Omega'] = 1\) such that, for any \(t \in [0, T]\), \(s \in [t, T]\) and \(u \in U\), \(\delta \mathcal{H}(s; t, u) \geq 0\) on \(\Omega'\).

So, for any \(t \in [0, T]\), \(s \in [t, T]\) and \(\mathbb{P}\text{-a.s.}, \)
\[
\bar{u}(s) \in \arg\min_{u \in U} \mathcal{H}(s, \bar{X}(s), u, \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t), P(s; t); t, \bar{X}(s), \bar{u}(s)).
\]

**Corollary 1.** If, for any \(t \in [0, T]\), \(x \in I\) and \(\mathbb{P}\text{-a.s.}, \)
\[
\Pi(t, x) \in \arg\min_{u \in U} \mathcal{H}(t, x, u, \bar{Y}(t; t), \bar{Z}(t; t), p(t; t), q(t; t), P(t; t); t, x, \Pi(t, x))
\]
then \(\Pi\) is an equilibrium policy and, in such a case, for any \(s \in [t, T]\) and \(\mathbb{P}\text{-a.s.}, \)
\[
\Pi(s, \bar{X}(s)) \in \arg\min_{u \in U} \mathcal{H}(s, \bar{X}(s), u, \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t), P(s; t); t, \bar{X}(s), \Pi(s, \bar{X}(s)))
\]
where \(\bar{X}(\cdot)\) is that of definition 2.

Thus, from Comparison Theorem about BSDEs, we obtain the following result, if we define our first-order Hamiltonian \(H\) as
\[
H(s, x, u, y, z, p, q; t, x, \bar{u}) = pb(s, x, u) + q\sigma(s, x, u) + f(s, x, u, y, z) + p[\sigma(s, x, u) - \sigma(s, x, \bar{u})]; t).
\]

**Corollary 2.** If, for any \(t < T\) and \(x \in I\), \(h(\cdot; t)\) is convex, \(G(\cdot, 0; 0; t) \geq 0\) and
\[
\Pi(t, x) \in \arg\min_{u \in U} H(t, x, u, \bar{Y}(t; t), \bar{Z}(t; t), p(t; t), q(t; t); t, x, \Pi(t, x))
\]
then \(\Pi\) is an equilibrium policy and, in such a case, for any \(s \in [t, T]\) and \(\mathbb{P}\text{-a.s.}, \)
\[
\Pi(s, \bar{X}(s)) \in \arg\min_{u \in U} H(s, \bar{X}(s), u, \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t); t, \bar{X}(s), \Pi(s, \bar{X}(s)))
\]
where \(\bar{X}(\cdot)\) is that of definition 2.

Indeed, \(P(\cdot; t) \geq 0\) and, thus, also the term with it in \(\mathcal{H}\) is \(\geq 0\) - so conflicts in calculating the minimum.

**Remark.** When \(f \equiv 0\), everything takes on a much simpler expression.
5 Problems with state constraint: necessary conditions

An equilibrium policy is generally not unique. The idea is to make it unique by means of a constraint (at least in a local sense). Suppose \( U \) is bounded in all this section (in the previous one it might not be). Also, remember that \( p \geq 2 \). We rewrite, for convenience, our FBSDE

\[
\begin{align*}
  dX(s) &= b(s,X(s),u(s)) ds + \sigma(s,X(s),u(s)) dW(s), \quad s \in [t,T], \\
  dY(s;t) &= -f(s,X(s),u(s),Y(s;t),Z(s;t);t) ds + Z(s;t)dW(s), \quad s \in [t,T], \\
  X(t) &= x, \quad Y(T;t) = h(X(T);t)
\end{align*}
\]

and our cost functional

\[
J(u(\cdot);t,x) = E Y(t;t) = E \left[ \int_t^T f(s,X(s),u(s),Y(s;t),Z(s;t);t) ds + h(X(T);t) \right]
\]
to which we put beside a one-dimensional state constraint as

\[
J(u(\cdot);t,x) = E Y(t;t) = E \left[ \int_t^T f(s,X(s),u(s),Y(s;t),Z(s;t);t) ds + h(X(T);t) \right] \in \Gamma_{t,x} \tag{21}
\]

where \( \Gamma_{t,x} \in \mathcal{B}(\mathbb{R}) \setminus \{0\} \) is a non-empty closed interval (possibly a singleton), \( f \) is similar to \( f \) and \( h \) is similar to \( h \) (i.e. they belong to the corresponding vector space of functions) and

\[
\begin{align*}
  dY(s;t) &= -f(s,X(s),u(s),Y(s;t),Z(s;t);t) ds + Z(s;t)dW(s), \quad s \in [t,T], \\
  Y(T;t) &= h(X(T);t).
\end{align*}
\]

\( \textbf{Remark.} \) The use of bold type is preferred for the above mathematical symbols to refer to the idea that this section is generalizable for more generic closed and convex subsets \( \Gamma_{t,x} \) of Euclidean space.

Let \( (\tilde{u}(\cdot),\tilde{X}(\cdot),\tilde{Y}(\cdot;t),\tilde{Z}(\cdot;t)) \) be an equilibrium 4-tuple such that \( (\tilde{u}(\cdot),\tilde{X}(\cdot)) \) satisfies the constraint \( (21) \) (the constraint concerns \( (u(\cdot),X(\cdot)) \) and not the whole 4-tuple \( (u(\cdot),X(\cdot),Y(\cdot;t),Z(\cdot;t)) \)). We assume that

\[
J(\tilde{u}(\cdot);t,x) = 0
\]

without loss of generality. Now we need a couple of classic results.

\textbf{Lemma 4} (Ekeland’s variational principle (corollary)). Let \( (V,d) \) be a complete metric space. Let \( F: V \to \mathbb{R} \) be a lower-semicontinuous function bounded from below. Let \( \tilde{v} \in V \) and \( \varrho > 0 \) be such that

\[
F(\tilde{v}) \leq \inf_{v \in V} F(v) + \varrho.
\]

Then there exists a \( \tilde{v} \in V \) such that

\[
F(\tilde{v}) \leq F(\tilde{v})
\]

\[
d(\tilde{v},\tilde{v}) \leq \varrho
\]

and, for any \( v \in V \),

\[
F(v) - F(\tilde{v}) \geq -\sqrt{\varrho} d(v,\tilde{v}).
\]

Denote as \( m \) the one-dimensional Lebesgue measure on \( \mathbb{R} \) and consider the metric on \( \mathcal{Y} [t,T] \) given by

\[
d(u(\cdot),\tilde{u}(\cdot)) = (m \otimes P)[\{(s,\omega) \in [t,T] \times \Omega \mid u(s,\omega) \neq \tilde{u}(s,\omega)\}] = E \int_t^T \mathbb{I}_{\{u \neq \tilde{u}\}}(s,\omega) ds. \tag{23}
\]

Here we’re taking the quotient space of \( \mathcal{Y} [t,T] \) w.r.t. the following equivalence relation: \( u(\cdot) \sim \tilde{u}(\cdot) \) if, and only if, \( d(u(\cdot),\tilde{u}(\cdot)) = 0 \) - which is weaker, therefore more restrictive, than that of indistinguishability.

Since the space \( \mathcal{Y} [t,T] \) is a \( \mathcal{F}_s \in \mathcal{F}(\mathbb{R},[t,T]) \text{-progressively measurable } \), \( d \) is complete, if \( U \) is bounded then also \( \mathcal{Y} [t,T],d \) becomes complete (because these two spaces coincide).
Remark. \( d(\cdot) \leq T - t. \)

**Lemma 5** (distance function). Fix a non-empty closed interval \( \Gamma \subset \mathbb{R} = \mathbb{R}_v \) and consider the distance function to the set \( \Gamma \), \( d_{\Gamma} : \mathbb{R}_v \to [0, \infty] \), defined, for \( v \in \mathbb{R} \), by

\[
d_{\Gamma}(v) = \inf_{v \in \Gamma} |v - \bar{v}|.
\]

Then this inf is a min, \( d_{\Gamma}(\cdot) \) is a convex and 1-Lipschitz continuous function on \( \mathbb{R} \), such that \( d_{\Gamma}^{-1}(0) = \Gamma \) (fiber) and such that, for any \( v \in \mathbb{R} \), the Clarke’s generalized gradient of \( d_{\Gamma}(\cdot) \) at \( v \), i.e. \( \partial d_{\Gamma}(v) \subset \mathbb{R} \) (nonempty), coincides with its (classical) subgradient at \( v \), that is

\[
\partial d_{\Gamma}(v) = \{ v \in \mathbb{R} | \forall \bar{v} \in \mathbb{R}, \ d_{\Gamma}(\bar{v}) \geq d_{\Gamma}(v) + \nu(\bar{v} - v) \}.
\]

Moreover, for any \( v \notin \Gamma \), \( \partial d_{\Gamma}(v) \) is the singleton or \( \{ -1 \} \) or \( \{ 1 \} \) and so we identify it with its unique element or \(-1 \) or \( 1 \) respectively (a sign). Finally, the square function \( d_{\Gamma}^2(\cdot) \) is \( C^1 \) on \( \mathbb{R} \) with derivative function given, for \( v \in \mathbb{R} \), by

\[
\frac{d}{dv}(d_{\Gamma}^2(v)) = 2d_{\Gamma}(v)\partial d_{\Gamma}(v) := \begin{cases} 0, & \text{if } v \in \Gamma, \\ 2d_{\Gamma}(v)\partial d_{\Gamma}(v), & \text{if } v \notin \Gamma. \end{cases}
\]

In short, we can write \( \frac{d}{dv}(d_{\Gamma}^2(\cdot)) = \pm 2d_{\Gamma}(\cdot). \)

**Remark.** Remember that the Clarke’s generalized gradient is defined at least for any scalar function defined on a region of \( \mathbb{R}^N \) (connected with nonempty interior) which is locally Lipschitz continuous (e.g. convex), and this gradient is a subset of \( \mathbb{R}^N \) which is nonempty, compact. The theory relies on the classical Hahn-Banach theorem of extension with control.

**Remark 1.** For any \( v \in \mathbb{R} \) and any subsequence \( (v_\varepsilon)_\varepsilon \) in \( \mathbb{R} \) such that \( v_\varepsilon \to v \) as \( \varepsilon \downarrow 0 \), it holds that

\[
d_{\Gamma}^2(v_\varepsilon) - d_{\Gamma}^2(v) = [2d_{\Gamma}(v_\varepsilon)\partial d_{\Gamma}(v_\varepsilon) + o_{\varepsilon \downarrow 0}(1)](v_\varepsilon - v)
\]

(in the sense of the previous lemma) and also, by continuity of such derivative function (in \( v \)),

\[
d_{\Gamma}^2(v_\varepsilon) - d_{\Gamma}^2(v) = [2d_{\Gamma}(v_\varepsilon)\partial d_{\Gamma}(v_\varepsilon) + o_{\varepsilon \downarrow 0}(1)](v_\varepsilon - v).
\]

Now, fix \( \rho > 0 \) (small) and define, for \( u(\cdot) \in \mathbb{W}[t,T] \),

\[
J_\rho(u(\cdot);t,x) = \left\{ (J(u(\cdot);t,x) + \rho)^2 + d_{\Gamma}^2_{\varepsilon(x)}(J(u(\cdot);t,x)) \right\}^{1/2}.
\]

\[\text{(24)}\]

Note that, for any \( u(\cdot) \in \mathbb{W}[t,T] \), \( J_\rho(u(\cdot);t,x) \geq 0 \) and

\[
J_\rho(u(\cdot);t,x) = 0 \Leftrightarrow \begin{cases} J(u(\cdot);t,x) = -\rho \\ J(u(\cdot);t,x) \in \Gamma_{t,x} \end{cases}
\]

(where, thus, such a \( u(\cdot) \) would depend on \( \rho \) and \( \Gamma_{t,x} \)). Moreover, \( J_\rho(\bar{u}(\cdot);t,x) = \rho \) and so

\[
J_\rho(\bar{u}(\cdot);t,x) = \rho \leq \inf_{u(\cdot) \in \mathbb{W}[t,T]} J_\rho(u(\cdot);t,x) + \rho.
\]

In particular, if \( J(\cdot;t,x) \geq 0 \) (i.e. \( \bar{u}(\cdot) \) is an actual classical minimum), then \( J_\rho(u(\cdot);t,x) \geq \rho > 0. \)

Now, \( J_\rho(u(\cdot);t,x) : (\mathbb{W}[t,T],d) \to [0,\infty[ \) is continuous because, as we show below, such are \( J \) and \( J \) (w.r.t. \( d \)) using also that \( U \) is bounded.

**Remark (continuity of \( J \)- or, identically, for \( J \)).** For any \( u(\cdot), \bar{u}(\cdot) \in \mathbb{W}[t,T] \) - to which correspond \( X(\cdot), Y(\cdot;t), Z(\cdot;t) \) and \( 
\hat{X}(\cdot), \hat{Y}(\cdot;t), \hat{Z}(\cdot;t) \) respectively - , by definitions (and remarks), lemma 1 (which allows you to make \( Y, Z \) and \( \hat{Y}, \hat{Z} \) disappear), assumptions (up to the Lipschitz property) and Jensen’s inequality (including the discreet form as \( |a + b|^p \leq |a|^p + |b|^p \) we’ve that (attributing to the symbol \( \leq \) the usual meaning of less than or equal to
something, unless there’re positive multiplicative constants - i.e. independent of the variables involved, since \( t \) can be considered fixed

\[
|J(u(\cdot); t, x) - J(\hat{u}(\cdot); t, x)|^p \equiv |Y(t; t) - \hat{Y}(t; t)|^p \\
\leq E \left| h(X(T); t) - h(\hat{X}(T); t) \right|^p \\
+ E \left( \int_t^T \left| f(s, X(s), u(s), Y(s; t), Z(s; t); t) - f(s, \hat{X}(s), \hat{u}(s), Y(s; t), Z(s; t); t) \right| ds \right)^p
\]

\[
\leq E \left| X(T) - \hat{X}(T) \right|^p + E \left( \int_t^T \left( \left| X(s) - \hat{X}(s) \right| + \left| u(s) - \hat{u}(s) \right| \right) ds \right)^p
\]

\[
\leq E \left( \int_t^T \left| X(s) - \hat{X}(s) \right|^p + E \int_t^T \left| u(s) - \hat{u}(s) \right|^p ds \\
\leq E \sup_{s \in [t, T]} \left| X(s) - \hat{X}(s) \right|^p + d(u(\cdot), \hat{u}(\cdot))
\]

using the boundedness of \( U \) for \( |u(s) - \hat{u}(s)|^p \) - and the definition of \( d \), because

\[
E \int_t^T |u(s) - \hat{u}(s)|^p ds = E \int_t^T 1_{u \neq \hat{u}}(s, \omega)|u(s) - \hat{u}(s)|^p ds
\]

(here above, the sup is a max). Now, our claim is that it holds

\[
E \sup_{s \in [t, T]} \left| X(s) - \hat{X}(s) \right|^p \leq E \int_t^T |u(s) - \hat{u}(s)|^p ds
\]

(\( \leq d(u(\cdot), \hat{u}(\cdot)) \) again) which allows us to conclude. But, indeed, for any \( s \in [t, T] \), by the state equations,

\[
\left| X(s) - \hat{X}(s) \right| \leq \int_t^s \left| b(r, X(r), u(r)) - b(r, \hat{X}(r), \hat{u}(r)) \right| dr + \int_t^s \left( \sigma(r, X(r), u(r)) - \sigma(r, \hat{X}(r), \hat{u}(r)) \right) dW(r)
\]

from which, in a similar way to above,

\[
\left| X(s) - \hat{X}(s) \right|^p \leq \int_t^s \left| b(r, X(r), u(r)) - b(r, \hat{X}(r), \hat{u}(r)) \right|^p dr + \int_t^s \left( \sigma(r, X(r), u(r)) - \sigma(r, \hat{X}(r), \hat{u}(r)) \right) dW(r)^p
\]

and so, for any \( r \in [t, T] \), by the classical Burkholder-Davis-Gundy inequality (with times in \([t, T]\) and with exponentiation \( r = p/2\)) after that “\( \sup \Sigma \leq \Sigma \sup \)” and since \( p \geq 2 \) (to use Jensen’s inequality), and by the Lipschitz property,

\[
E \sup_{s \in [t, r]} \left| X(s) - \hat{X}(s) \right|^p \leq E \int_t^r \left| b(r, X(r), u(r)) - b(r, \hat{X}(r), \hat{u}(r)) \right|^p dr \\
+ E \sup_{s \in [t, r]} \left( \int_t^s \left( \sigma(r, X(r), u(r)) - \sigma(r, \hat{X}(r), \hat{u}(r)) \right) dW(r) \right)^p
\]

\[
\leq E \int_t^r \left| b(r, X(r), u(r)) - b(r, \hat{X}(r), \hat{u}(r)) \right|^p dr \\
+ E \left( \int_t^r \left( \sigma(r, X(r), u(r)) - \sigma(r, \hat{X}(r), \hat{u}(r)) \right)^2 \right)^{p/2} dr
\]

\[
\leq E \int_t^r \left| b(r, X(r), u(r)) - b(r, \hat{X}(r), \hat{u}(r)) \right|^p + \left| \sigma(r, X(r), u(r)) - \sigma(r, \hat{X}(r), \hat{u}(r)) \right|^p dr
\]

\[
\leq E \int_t^r \left| X(r) - \hat{X}(r) \right|^p + E \int_t^r \left| u(r) - \hat{u}(r) \right|^p dr
\]

\[
\leq \int_t^r E \sup_{s \in [t, r]} \left| X(s) - \hat{X}(s) \right|^p dr + E \int_t^r \left| u(r) - \hat{u}(r) \right|^p dr
\]
and thus, finally, thanks to the classical Grönwall's inequality in its integral form - applied to the continuous function \( \tau \mapsto E\sup_{s \in [t, \tau]} \left| X(s) - \hat{X}(s) \right|^p \) on \([t, T]\),

\[
E \sup_{s \in [t, \tau]} \left| X(s) - \hat{X}(s) \right|^p \lesssim E \int_t^\tau \left| u(r) - \bar{u}(r) \right|^p dr
\]

- for any \( \tau \in [t, T] \), also for \( \tau = t \) (hence how much we wanted by choosing \( \tau = T \)).

We point out here that, in light of what we have just seen, we understand that we could have done all the calculations with \( p = 2 \) and, explicitly, that \( (a + b)^2 \leq 2(a^2 + b^2) \).

Carrying on, by lemma 4, there exists \( \bar{u}_p(\cdot) \in \mathcal{W}[t, T] \) such that

\[
J_{\bar{v}}(\bar{u}_p(\cdot); t, x) \leq J_{\bar{v}}(\bar{u}_p(\cdot); t, x) = \varnothing
\]

and for any \( v(\cdot) \in \mathcal{W}[t, T] \),

\[
J_{\bar{v}}(v(\cdot); t, x) - J_{\bar{v}}(\bar{u}_p(\cdot); t, x) \geq -\sqrt{\vartheta}\mathbf{d}(v(\cdot), \bar{u}_p(\cdot)).
\]

Fix \( u(\cdot) \in \mathcal{W}[t, T], \varepsilon \in ]0, T - t[\), \( \mathcal{E}_t^\varepsilon \in \mathcal{B}(\mathcal{W}[t, T]) \) with length \( |\mathcal{E}_t^\varepsilon| \equiv m[\mathcal{E}_t^\varepsilon] = \varepsilon \) and consider the spike variation of \( \bar{u}_p(\cdot) \) w.r.t. \( u(\cdot) \) and \( \mathcal{E}_t^\varepsilon \), i.e.,

\[
\bar{u}_p^\varepsilon \doteq \bar{u}_p + (u - \bar{u}_p) \mathbb{1}_{\mathcal{E}_t^\varepsilon}
\]

on \([t, T]\). Since \( \{\bar{u}_p^\varepsilon \neq \bar{u}_p\} \subset \mathcal{E}_t^\varepsilon \times \Omega \), we've \( \mathbf{d}(\bar{u}_p^\varepsilon(\cdot), \bar{u}_p(\cdot)) \leq \varepsilon \) and thus \( \bar{u}_p^\varepsilon(\cdot) \to \bar{u}_p(\cdot) \) w.r.t. \( \mathbf{d} \) as \( \varepsilon \downarrow 0 \).

Thus, from above with the choice \( v(\cdot) = \bar{u}_p^\varepsilon(\cdot) \),

\[
-\sqrt{\vartheta}\varepsilon \leq -\sqrt{\vartheta}\mathbf{d}(\bar{u}_p^\varepsilon(\cdot), \bar{u}_p(\cdot)) \leq J_{\bar{v}}(\bar{u}_p^\varepsilon(\cdot); t, x) - J_{\bar{v}}(\bar{u}_p(\cdot); t, x).
\]

Due to the formula which defines \( J_{\bar{v}}(\cdot; t, x) \), we find it convenient to use the fact that, for any \( a, b \in \mathbb{R} \) with \( a + b \neq 0 \), \( a - b = (a^2 - b^2)/(a + b) \) holds.

**CASE I:** \( J_{\bar{v}}(\bar{u}_p(\cdot); t, x) \neq 0 \), i.e. \( > 0 \), for \( \vartheta \downarrow 0 \) (\( \vartheta \) small enough). We can write

\[
-\sqrt{\vartheta}\varepsilon \leq J_{\bar{v}}(\bar{u}_p^\varepsilon(\cdot); t, x) - J_{\bar{v}}(\bar{u}_p(\cdot); t, x)
\]

\[
= J_{\bar{v}}^2(\bar{u}_p^\varepsilon(\cdot); t, x) - J_{\bar{v}}^2(\bar{u}_p(\cdot); t, x)
\]

\[
= \left( J(\bar{u}_p^\varepsilon(\cdot); t, x) + \varrho \right)^2 - J(\bar{u}_p(\cdot); t, x) + \varrho \right)^2
\]

\[
J_{\bar{v}}(\bar{u}_p^\varepsilon(\cdot); t, x) + J_{\bar{v}}(\bar{u}_p(\cdot); t, x)
\]

\[
+ \frac{\frac{d^2 J_{\bar{v}}}{dr_{\bar{u}_p}^2}(J(\bar{u}_p^\varepsilon(\cdot); t, x)) - J_{\bar{v}}(\bar{u}_p(\cdot); t, x))}{J_{\bar{v}}(\bar{u}_p^\varepsilon(\cdot); t, x) + J_{\bar{v}}(\bar{u}_p(\cdot); t, x)}
\]

\[
=: K_{\varepsilon}^\vartheta + K_{\varepsilon}^\vartheta.
\]

Now, there exist multipliers \( \psi_\varepsilon^\vartheta, \psi_\varepsilon^\vartheta \in \mathbb{R} \) such that

\[
K_{\varepsilon}^\vartheta = \psi_\varepsilon^\vartheta[J(\bar{u}_p^\varepsilon(\cdot); t, x) - J(\bar{u}_p(\cdot); t, x)] \quad \text{and} \quad K_{\varepsilon}^\vartheta = \psi_\varepsilon^\vartheta[J(\bar{u}_p^\varepsilon(\cdot); t, x) - J(\bar{u}_p(\cdot); t, x)]
\]

and they're

\[
\psi_\varepsilon^\vartheta = \frac{J(\bar{u}_p(\cdot); t, x) + \varrho}{J_{\bar{v}}(\bar{u}_p(\cdot); t, x)} + o_{\varepsilon \downarrow 0}(1)
\]

\[
\psi_\varepsilon^\vartheta = \frac{\frac{d}{dr_{\bar{u}_p}}(J(\bar{u}_p(\cdot); t, x))\frac{d^2 J_{\bar{v}}}{dr_{\bar{u}_p}}(J(\bar{u}_p(\cdot); t, x))}{J_{\bar{v}}(\bar{u}_p(\cdot); t, x)} + o_{\varepsilon \downarrow 0}(1)
\]

\[
(25)
\]
Therefore, about subsequences, still denoted by \( o(1) \) (no problem for the continuation - this dependence on \( \varepsilon \) - as we will send \( \varepsilon \) to zero after fixing any sufficiently small \( \varepsilon \)). Indeed, first of all, by continuity w.r.t. \( d \),

\[
J(\tilde{u}_\varepsilon(\cdot);t,x) = J(\tilde{u}_\varepsilon(\cdot);t,x) + o(1) \quad \text{and} \quad J(\tilde{u}_\varepsilon(\cdot);t,x) = J(\tilde{u}_\varepsilon(\cdot);t,x) + o''(1)
\]

as well as \( J(\tilde{u}_\varepsilon(\cdot);t,x) \to J(\tilde{u}_\varepsilon(\cdot);t,x) \) w.r.t. \( d \) as \( \varepsilon \downarrow 0 \), thus also (by remark 1)

\[
d^2_{t,x}(J(\tilde{u}_\varepsilon(\cdot);t,x)) - d^2_{t,x}(J(\tilde{u}_\varepsilon(\cdot);t,x)) = \frac{J(\tilde{u}_\varepsilon(\cdot);t,x) + 2\varepsilon}{J(\tilde{u}_\varepsilon(\cdot);t,x)} - J(\tilde{u}_\varepsilon(\cdot);t,x) = o(1).
\]

Therefore, about \( K^\varepsilon_\psi \):

\[
(J(\tilde{u}_\varepsilon(\cdot);t,x) + \varepsilon)^2 - (J(\tilde{u}_\varepsilon(\cdot);t,x) + \varepsilon)^2 = \frac{J(\tilde{u}_\varepsilon(\cdot);t,x) + 2\varepsilon}{J(\tilde{u}_\varepsilon(\cdot);t,x) + \varepsilon} - \varepsilon = o(1).
\]

and now

\[
\frac{J(\tilde{u}_\varepsilon(\cdot);t,x) + 2\varepsilon}{J(\tilde{u}_\varepsilon(\cdot);t,x) + \varepsilon} - \varepsilon = \frac{J(\tilde{u}_\varepsilon(\cdot);t,x) + \varepsilon}{J(\tilde{u}_\varepsilon(\cdot);t,x) + \varepsilon} + o(1).
\]

Similarly, about \( K^\varepsilon_\psi \), in the light of what has just been observed about the distance function:

\[
\frac{2d_{t,x}(J(\tilde{u}_\varepsilon(\cdot);t,x)) + \varphi}{J(\tilde{u}_\varepsilon(\cdot);t,x) + \varphi} - \varphi = \frac{2d_{t,x}(J(\tilde{u}_\varepsilon(\cdot);t,x)) + \varphi}{J(\tilde{u}_\varepsilon(\cdot);t,x) + \varphi} + o(1).
\]

Convergence of multipliers. We’ve that \( \psi_\varepsilon \) and

\[
\psi_\varepsilon = \begin{cases} \frac{o(1)}{d_{t,x}(J(\tilde{u}_\varepsilon(\cdot);t,x))}, & \text{if } J(\tilde{u}_\varepsilon(\cdot);t,x) \in \Gamma_{t,x}, \\ \pm \frac{d_{t,x}(J(\tilde{u}_\varepsilon(\cdot);t,x))}{\varphi} + o(1), & \text{otherwise,} \end{cases}
\]

could be \( o(1) \) and \( o(1) \) respectively - observe that it could be \( -\varphi > J(\tilde{u}_\varepsilon(\cdot);t,x) \) (while remaining a \( o(1) \)). However, with the squares (from definitions of them and of \( J_\theta \)):

\[
(\psi_\varepsilon^2 + (\psi_\varepsilon^2) = 1 + o(1)
\]

so \( (\psi_\varepsilon^2) \) and \( (\psi_\varepsilon^2) \) are bounded as \( \varepsilon \downarrow 0 \) (in \( \mathbb{R} \)). Consequently, there exist \( \psi_\varepsilon, \psi_\varepsilon \in \mathbb{R} \) and subsequences, still denoted by \( (\psi_\varepsilon^2) \) and \( (\psi_\varepsilon^2) \), such that, as \( \varepsilon \downarrow 0 \),

\[
\psi_\varepsilon \to \psi_\varepsilon, \quad \psi_\varepsilon \to \psi_\varepsilon \quad \text{and} \quad (\psi_\varepsilon^2 + (\psi_\varepsilon^2)^2 = 1.
\]

Now, clearly, \( (\psi_\varepsilon^2) \) and \( (\psi_\varepsilon^2) \) are both bounded as \( \varphi \downarrow 0 \) (among the others), thus again there exist \( \psi, \psi \in \mathbb{R} \) and subsequences, still denoted by \( (\psi_\varepsilon^2) \) and \( (\psi_\varepsilon^2) \), such that, as \( \varphi \downarrow 0 \),

\[
\psi_\varepsilon \to \psi, \quad \psi_\varepsilon \to \psi \quad \text{and} \quad \psi_\varepsilon^2 + \psi_\varepsilon^2 = 1.
\]

In particular, \( \psi, \psi \in [-1, 1] \) (and we’ve the qualification condition when \( \psi \neq 0 \), i.e. \( |\psi| < 1 \)).

Trasversality condition. We show that, for any \( \tilde{v} \in \Gamma_{t,x} \),

\[
\psi [\tilde{v} - J(\tilde{u}(\cdot);t,x)] \leq 0
\]
Subgame-perfect equilibrium strategies in state-constrained recursive stochastic optimal control problems

So, in any case, $\hat{v} \notin \Gamma_{t,x}$, by definition of subgradient we get (using also $\bar{v} \in \Gamma_{t,x}$)

$$
\partial d_{\Gamma_{t,x}}(\bar{J}(\bar{u}_\varphi(\cdot); t, x)) \left[ \bar{v} - J(\bar{u}_\varphi(\cdot); t, x) \right] \geq -d_{\Gamma_{t,x}}(\bar{J}(\bar{u}_\varphi(\cdot); t, x)) \leq 0.
$$

So, in any case, $\psi^\varepsilon_\varphi \left[ \bar{v} - J(\bar{u}_\varphi(\cdot); t, x) \right] \leq \psi^\varepsilon_\varphi \left[ \bar{v} - J(\bar{u}_\varphi(\cdot); t, x) \right] \leq 0$ for $\varepsilon \downarrow 0$ and the thesis for $\bar{v} \downarrow 0$ (remember that $u_{\varphi(\cdot)} \rightarrow \bar{u}_{\cdot}$ w.r.t. $d$ as $\bar{v} \downarrow 0$).

Continuation of (25). Therefore,

$$
-\sqrt{v} \in \psi^\varepsilon_\varphi \left[ J(\bar{u}_\varphi^\varepsilon(\cdot); t, x) - J(\bar{u}_\varphi(\cdot); t, x) \right] + \psi^\varepsilon_\varphi \left[ J(\bar{u}_\varphi(\cdot); t, x) - J(\bar{u}_\varphi(\cdot); t, x) \right]
$$

where, for any $v(\cdot) \in W[t, T]$, $J^\varepsilon(v(\cdot); t, x) \approx \psi^\varepsilon_\varphi J(v(\cdot); t, x)$

and

$$
J^\varepsilon(v(\cdot); t, x) \approx \psi^\varepsilon_\varphi J(v(\cdot); t, x).
$$

Here it happens like if the parameters $(f, h)$ in $J$ become $(\psi^\varepsilon_\varphi f, \psi^\varepsilon_\varphi h)$ in $J^\varepsilon$ while, similarly, the parameters $(f, h)$ in $J$ become $(\psi^\varepsilon_\varphi f, \psi^\varepsilon_\varphi h)$ in $J^\varepsilon$. However, we will proceed on $J$ and $J$ separately as we know how to do it. Indeed, we consider the adjoint processes of first order and second order associated to $(\bar{u}_\varphi(\cdot), \bar{X}_\varphi(\cdot), \bar{Y}_\varphi(\cdot); t, \bar{Z}_\varphi(\cdot); t))$ w.r.t. $J$ (not depending either on $u(\cdot)$ or $E^\varepsilon_\varphi$):

$$
\begin{cases}
\frac{dp_\varphi(s; t)}{ds} = -g_\varphi(s, p_\varphi(s; t), q_\varphi(s; t); t)ds + q_\varphi(s; t)dW(s), & s \in [t, T], \\
p_\varphi(T; t) = h_x(\bar{X}_\varphi(T); t)
\end{cases}
$$

and

$$
\begin{cases}
\frac{dp_\varphi(s; t)}{ds} = -G_\varphi(s, p_\varphi(s; t), Q_\varphi(s; t); t)ds + Q_\varphi(s; t)dW(s), & s \in [t, T], \\
p_\varphi(T; t) = h_x(\bar{X}_\varphi(T); t)
\end{cases}
$$

where in $g_\varphi(s, p_\varphi(s; t), q_\varphi(s; t); t)$ and in $G_\varphi(s, p_\varphi(s; t), Q_\varphi(s; t); t)$ we’ve, of course, $b_\varphi(s) := b(s, \bar{x}_\varphi(s), \bar{u}_\varphi(s)), f_\varphi(s; t) := f_\varphi(s, \bar{X}_\varphi(s), \bar{u}_\varphi(s)), \bar{Y}_\varphi(s; t), \bar{Z}_\varphi(s; t); t)$, etc. Similarly for $(\bar{u}_\varphi(\cdot), \bar{X}_\varphi(\cdot), \bar{Y}_\varphi(\cdot); t, \bar{Z}_\varphi(\cdot); t))$ w.r.t. $J$:

$$
\begin{cases}
\frac{dp_\varphi(s; t)}{ds} = -g_\varphi(s, p_\varphi(s; t), q_\varphi(s; t); t)ds + q_\varphi(s; t)dW(s), & s \in [t, T], \\
p_\varphi(T; t) = h_x(\bar{X}_\varphi(T); t)
\end{cases}
$$

and

$$
\begin{cases}
\frac{dp_\varphi(s; t)}{ds} = -G_\varphi(s, p_\varphi(s; t), Q_\varphi(s; t); t)ds + Q_\varphi(s; t)dW(s), & s \in [t, T], \\
p_\varphi(T; t) = h_x(\bar{X}_\varphi(T; t))
\end{cases}
$$

where in $g_\varphi(s, p_\varphi(s; t), q_\varphi(s; t); t)$ and in $G_\varphi(s, p_\varphi(s; t), Q_\varphi(s; t); t)$ we’ve $b_\varphi(s) := b(s, \bar{x}_\varphi(s), \bar{u}_\varphi(s)), f_\varphi(s; t) := f_\varphi(s, \bar{X}_\varphi(s), \bar{u}_\varphi(s), \bar{Y}_\varphi(s; t), \bar{Z}_\varphi(s; t); t)$, etc. Thus, w.r.t. such notations, on the one hand we consider $\kappa_\varphi(\cdot; t) > 0$ such that

$$
\begin{cases}
\frac{d\kappa_\varphi(s; t)}{ds} = \kappa_\varphi(s; t)\left[ f_\varphi(s; t)ds + f_\varphi(s; t)dW(s) \right], & s \in [t, T], \\
\kappa_\varphi(t; t) = 1
\end{cases}
$$

and $\kappa_\varphi(\cdot; t) > 0$ such that

$$
\begin{cases}
\frac{d\kappa_\varphi(s; t)}{ds} = \kappa_\varphi(s; t)\left[ f_\varphi(s; t)ds + f_\varphi(s; t)dW(s) \right], & s \in [t, T], \\
\kappa_\varphi(t; t) = 1
\end{cases}
$$
so, on the other hand,

\[
\begin{align*}
\mathcal{H}_\varrho(s; t) &= \mathcal{H}(s, \tilde{X}_\varrho(s), \tilde{u}_\varrho(s), \tilde{Y}_\varrho(s; t), \tilde{Z}_\varrho(s; t), p_\varrho(s; t), q_\varrho(s; t), P_\varrho(s; t); t, \tilde{X}_\varrho(s), \tilde{u}_\varrho(s)) \\
\delta \mathcal{H}_\varrho(s; t; u) &= \mathcal{H}(s, \tilde{X}_\varrho(s), u, \tilde{Y}_\varrho(s; t), \tilde{Z}_\varrho(s; t), p_\varrho(s; t), q_\varrho(s; t), P_\varrho(s; t); t, \tilde{X}_\varrho(s), \tilde{u}_\varrho(s)) - \mathcal{H}_\varrho(s; t)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{H}(s, u, y, z, p, q, P; t, \bar{x}, \bar{u}) &= p b(s, x, u) + q \sigma(s, x, u) + \frac{1}{2} P \sigma(s, x, u) - \sigma(s, \bar{x}, \bar{u})^2 \\
&\quad + f(s, x, u, y, z + p \sigma(s, x, u) - \sigma(s, \bar{x}, \bar{u}); t)
\end{align*}
\]

Therefore, as we know,

\[
\liminf_{\varepsilon \downarrow 0} \frac{J(\bar{u}_\varrho^\varepsilon(\cdot); t, x) - J(\bar{u}_\varrho(\cdot); t, x)}{\varepsilon} = \mathbb{E}[\kappa_\varrho(s; t) \delta \mathcal{H}_\varrho(s; t, u(s))]
\]

and

\[
\liminf_{\varepsilon \downarrow 0} \frac{J(u_\varrho^\varepsilon(\cdot); t, x) - J(u_\varrho(\cdot); t, x)}{\varepsilon} = \mathbb{E}[\kappa_\varrho(s; t) \delta \mathcal{H}_\varrho(s; t, u(s))]
\]

so we get

\[
- \sqrt{\varrho} \leq \mathbb{E}[\psi_\varrho \kappa_\varrho(s; t) \delta \mathcal{H}_\varrho(s; t, u(s)) + \psi_\varrho \kappa_\varrho(s; t) \delta \mathcal{H}_\varrho(s; t, u(s))]
\]

and, finally, with the meaning that one expects,

\[
\mathbb{E}[\psi \kappa(s; t) \delta \mathcal{H}(s; t, u(s)) + \psi \kappa(s; t) \delta \mathcal{H}(s; t, u(s))] \geq 0
\]

or, equivalently,

\[
\mathbb{E}[\psi \kappa(s; t) \delta \mathcal{H}(s; t, u(s))] \geq -\psi \mathcal{L}
\]

where

\[
\mathcal{L} := \liminf_{\varepsilon \downarrow 0} \frac{J(u_\varrho^\varepsilon(\cdot); t, x) - J(u_\varrho(\cdot); t, x)}{\varepsilon} \equiv \liminf_{\varepsilon \downarrow 0} \frac{J(u_\varrho^\varepsilon(\cdot); t, x)}{\varepsilon} \geq 0.
\]

Observe here how crucial it is that the lim inf of the equilibrium property is actually a limit (otherwise, we would not have that the lim inf of a sum is less than (or equal to) the sum of the liminf - since the opposite holds true).

**CASE II:** \(J_\varrho(u_\varrho^\varepsilon(\cdot); t, x) = 0\) for infinitely many \(\varrho \downarrow 0\), i.e. \(J(u_\varrho^\varepsilon(\cdot); t, x) = -\varrho\) and \(J(u_\varrho(\cdot); t, x) \in \Gamma_{t,x}\) (situation quite unrealistic, as it is “unlikely” that \(J(u_\varrho^\varepsilon(\cdot); t, x) = -\varrho\) exactly, even countless times - although it remains true that \(J(u_\varrho^\varepsilon(\cdot); t, x) \rightarrow 0\) as \(\varrho \downarrow 0\)). In light of the above, it will be enough to find \(\psi_\varrho^\varepsilon, \psi_\varrho \in \mathbb{R}\) - which at the limits satisfy the three properties we want for them - such that

\[
J_\varrho(u_\varrho^\varepsilon(\cdot); t, x) - J_\varrho(u_\varrho(\cdot); t, x) \equiv J_\varrho(u_\varrho^\varepsilon(\cdot); t, x)
\]

\[
\leq \psi_\varrho^\varepsilon[J(u_\varrho^\varepsilon(\cdot); t, x) - J(u_\varrho(\cdot); t, x)] + \psi_\varrho[J(u_\varrho^\varepsilon(\cdot); t, x) - J(u_\varrho(\cdot); t, x)]
\]

\[
\equiv \psi_\varrho^\varepsilon[J(u_\varrho^\varepsilon(\cdot); t, x) + \varrho] + \psi_\varrho[J(u_\varrho^\varepsilon(\cdot); t, x) - J(u_\varrho(\cdot); t, x)]
\]

(=, for instance).
**Sub-case 1:** $J(\bar{u}_\epsilon(\cdot); t, x) \in \text{int}(\Gamma_{t,x})$ or, more generally, when $\bar{J}(\bar{u}_\epsilon(\cdot); t, x) \in \text{int}(\Gamma_{t,x})$ for infinitely many of such $\epsilon \downarrow 0$ (remember that $\bar{u}_\epsilon \to \bar{u}$ w.r.t. $d$ as $\epsilon \downarrow 0$ and that $\bar{J}$ is continuous w.r.t. $d$) - in case $\text{int}(\Gamma_{t,x}) \neq \emptyset$. That's easy because, in such situation, corresponding to these values of $\epsilon$, also $\bar{J}(\bar{u}_\epsilon^*; t, x) \in \text{int}(\Gamma_{t,x})$ for $\epsilon \downarrow 0$ ($\bar{u}_\epsilon^* \to \bar{u}_\epsilon$ w.r.t. $d$ as $\epsilon \downarrow 0$) and so, since obviously $\text{int}(\Gamma_{t,x}) \subset \Gamma_{t,x}$, we've that $\bar{J}(\bar{u}_\epsilon^*; t, x) = \lim_{\epsilon \to 0} \bar{J}(\bar{u}_\epsilon^*; t, x) + \epsilon$ (for such parameters) and we can take $\psi^*_\epsilon = \text{sgn}(\bar{J}(\bar{u}_\epsilon^*; t, x) + \epsilon)$ and $\psi^*_\epsilon \equiv 0$ - from which $|\psi^*_\epsilon| = 1$ and $\psi^*_\epsilon = 0$ (here we use that $|v| = \text{sgn}(v)v$, or also $\text{sgn}(|v|) = v$, for any $v \in \mathbb{R}$).

**Sub-case 2:** $J(\bar{u}_\epsilon(\cdot); t, x) \in \partial \Gamma_{t,x}$ for infinitely many of such $\epsilon \downarrow 0$ - except at most a finite number - (so also $J(\bar{u}_\epsilon(\cdot); t, x) \in \partial \Gamma_{t,x}$ which certainly happens, for example, when $\Gamma_{t,x}$ is a singleton) and, at the same time, for such $\epsilon$, $J(\bar{u}_\epsilon^*; t, x) \notin \Gamma_{t,x}$ for infinitely many $\epsilon \downarrow 0$ except at most a finite number, i.e. for $\epsilon \downarrow 0$ (indeed, if we had oscillatory convergences, we could proceed as in the previous case unless we pass to subsequences). In such situation, we definitely have that $J(\bar{u}_\epsilon^*; t, x) \notin \Gamma_{t,x}$ (i.e. $> 0$) and indeed, more precisely, by definition of $\bar{J}_\epsilon$ (and it is something already seen at the beginning of CASE I),

$$J(\bar{u}_\epsilon^*(\cdot); t, x) = \frac{(J(\bar{u}_\epsilon^*(\cdot); t, x) + \epsilon)^2}{J(\bar{u}_\epsilon^*(\cdot); t, x)}$$

therefore the idea is to define $\psi^*_\epsilon, \psi^*_\epsilon \in \mathbb{R}$ - which at the limits satisfy the three properties we want for them - such that

$$\frac{(J(\bar{u}_\epsilon^*(\cdot); t, x) + \epsilon)^2}{J(\bar{u}_\epsilon^*(\cdot); t, x)} \leq \psi^*_\epsilon [J(\bar{u}_\epsilon^*(\cdot); t, x) + \epsilon] \quad \text{and} \quad \frac{2d_{t,x}(J(\bar{u}_\epsilon^*(\cdot); t, x))}{J(\bar{u}_\epsilon^*(\cdot); t, x)} \leq \psi^*_\epsilon [J(\bar{u}_\epsilon^*(\cdot); t, x) - J(\bar{u}_\epsilon^*(\cdot); t, x)]$$

(=, for instance). So, about $\psi^*_\epsilon$, we can take

$$\psi^*_\epsilon = \frac{J(\bar{u}_\epsilon^*(\cdot); t, x) + \epsilon}{J(\bar{u}_\epsilon^*(\cdot); t, x)}$$

(not substantially different from CASE I). Now, if we had that, for infinitely many of the $\epsilon \downarrow 0$ of our hypotheses (it won’t be a problem if the following would happen for a finite number of such $\epsilon$) and for infinitely many $\epsilon \downarrow 0$, $J(\bar{u}_\epsilon^*(\cdot); t, x) = -\epsilon$, then we could take $\psi^*_\epsilon \equiv 0$ (corresponding to such parameters), from which $\psi = 0$ at the limit. Otherwise we can assume that all the $\epsilon$ and all the $\epsilon$ are considered are all such that $J(\bar{u}_\epsilon^*(\cdot); t, x) \neq -\epsilon$ (maybe $\epsilon$) and thus we could write ($\epsilon$ fixed, $\epsilon \downarrow 0$)

$$\psi^*_\epsilon = \frac{\text{sgn}(J(\bar{u}_\epsilon^*(\cdot); t, x) + \epsilon)}{\sqrt{1 + \frac{d_{t,x}(J(\bar{u}_\epsilon^*(\cdot); t, x))}{(J(\bar{u}_\epsilon^*(\cdot); t, x) + \epsilon)^2}}}$$

(equality which actually holds in any case in an extended arithmetic sense - $|\text{constant}|/0 = \infty$, constant + $\infty = \infty$, constant/constant = 0), expression that doesn’t allow by itself to establish its behavior at the limit (passing to subsequences) since it depends on both $J$ and $\bar{J}$, also through $\Gamma_{t,x}$.

However, coming to $\psi^*_\epsilon$, we observe that the remark 1 becomes here

$$d_{t,x}^2(J(\bar{u}_\epsilon^*(\cdot); t, x)) = o_{\epsilon < 0}(1) [J(\bar{u}_\epsilon^*(\cdot); t, x) - J(\bar{u}_\epsilon^*(\cdot); t, x)]$$

(there, $o_{\epsilon < 0}(1) \equiv o_{\epsilon < 0}(1)$ and since $d_{t,x}^2(J(\bar{u}_\epsilon^*(\cdot); t, x)) > 0$ ($J(\bar{u}_\epsilon^*(\cdot); t, x) \notin \Gamma_{t,x}$) for $\epsilon \downarrow 0$, it must be $J(\bar{u}_\epsilon^*(\cdot); t, x) \neq J(\bar{u}_\epsilon^*(\cdot); t, x)$ for the same $\epsilon \downarrow 0$ (as well as, for the same reason, the other term $o_{\epsilon < 0}(1)$ next to it isn’t null)), the ratio

$$\frac{d_{t,x}^2(J(\bar{u}_\epsilon^*(\cdot); t, x))}{J(\bar{u}_\epsilon^*(\cdot); t, x) - J(\bar{u}_\epsilon^*(\cdot); t, x)} = o_{\epsilon < 0}(1)$$

is well defined and we can take

$$\psi^*_\epsilon = \frac{d_{t,x}^2(J(\bar{u}_\epsilon^*(\cdot); t, x))}{J(\bar{u}_\epsilon^*(\cdot); t, x)J(\bar{u}_\epsilon^*(\cdot); t, x) - J(\bar{u}_\epsilon^*(\cdot); t, x)} = o_{\epsilon < 0}(1)$$
(basically, we’re dividing by $J(\bar{u}_p^e(\cdot); t, x) - J(\bar{u}_q(\cdot); t, x) \neq 0$). Here, although a relationship between infinitesimals re-emerges quite similar to that which appears in $\psi^e_\theta$ above, we can and must be more explicit: indeed, since $\Gamma_{t,x} \subset \mathbb{R}$ (and by hypotheses),

$$
\frac{d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x))}{J_p(\bar{u}_p^e(\cdot); t, x)} = \left| J(\bar{u}_p^e(\cdot); t, x) - J(\bar{u}_q(\cdot); t, x) \right| \\
\equiv \text{sgn}(J(\bar{u}_p^e(\cdot); t, x) - J(\bar{u}_q(\cdot); t, x)) \left[ J(\bar{u}_p^e(\cdot); t, x) - J(\bar{u}_q(\cdot); t, x) \right] \\
= \partial d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x)) \left[ J(\bar{u}_p^e(\cdot); t, x) - J(\bar{u}_q(\cdot); t, x) \right]
$$

(it is also very easy to see it graphically, or simply from definition of subgradient but with the $\leq$ (see also the lemma 5)) where indeed $\partial d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x)) = \pm 1$ by hypotheses, and therefore

$$
\psi^e_\theta = \frac{d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x))}{J_p(\bar{u}_p^e(\cdot); t, x)} = \pm 1
$$

and we’ve finished from every point of view, because now the sum of squares condition, and the transversality condition - comparing initially $\psi^e_\theta$ with $\psi - J(\bar{u}_p^e(\cdot); t, x)$ for $\psi \in \Gamma_{t,x}$ - work in a completely analogous way to CASE I. We could also rewrite

$$
\psi^e_\theta = \frac{\partial d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x))}{\sqrt{\left( \frac{J(\bar{u}_p^e(\cdot); t, x) + g^2}{d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x))} + 1 \right)}}
$$

If we had that, for infinitely many of the $\theta$ of our hypotheses (it won’t be a problem if the following would happen for a finite number of such $\theta$) and for infinitely many $\varepsilon \downarrow 0$, $J(\bar{u}_p^e(\cdot); t, x) = -\theta$ (i.e. $J_p(\bar{u}_p^e(\cdot); t, x) = d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x)))$, then we know now that $|\psi^e_\theta| \equiv 1$ (corresponding to such parameters), from which $|\psi| = 1$ at the limit. Otherwise we can assume that all the $\theta$ and all the $\varepsilon$ considered are all such that $J(\bar{u}_p^e(\cdot); t, x) \neq -\theta$ and thus the behavior of $\psi^e_\theta$ with respect to that of $\psi^e_\theta$ depends on the order of infinitesimally of $J(\bar{u}_p^e(\cdot); t, x) + \theta$ with respect to that of $d\Gamma_{t,x}(J(\bar{u}_p^e(\cdot); t, x))$ (so, all possible numbers with absolute value less than or equal to 1 are plausible for their limits $\psi$ and $\psi^e_\theta$).

The analogue of such a comment is valid, as it is, for the multipliers of CASE I. Anyway, finally,

**Theorem 2 (multipliers).** For $t \in [0, T]$ and $x \in I$, let $(\bar{u}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$ be an equilibrium 4-tuple such that $(\bar{u}(\cdot), \bar{X}(\cdot))$ satisfies the constraint (21). Assume that $U$ is bounded. Then there exist multipliers $\psi, \psi^e \in [-1, 1]$ satisfying

$$
\psi^2 + \psi^2 = 1
$$

and, for any $\bar{v} \in \Gamma_{t,x}$,

$$
\psi[\bar{v} - J(\bar{u}(\cdot); t, x)] \leq 0
$$

such that, considered $(p(\cdot; t), q(\cdot; t)), (P(\cdot; t), Q(\cdot; t)), \kappa$ and, thus, $H$ and $\delta H$ corresponding to $(\bar{u}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$, and considered $(p(\cdot; t), q(\cdot; t)), (P(\cdot; t), Q(\cdot; t)), \kappa$ and, thus, $H$ and $\delta H$ corresponding to $(\bar{u}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$, then, for any $s \in [t, T]$ and $u \in U$,

$$
\psi^e(s; t)\delta H(s; t, u) + \psi^e(s; t)\delta H(s; t, u) \geq 0
$$

and, in particular,

$$
\psi\delta H(t; t, u) + \psi^e\delta H(t; t, u) \geq 0.
$$

**Proof.** Indeed, $E[\psi^e(s; t)\delta H(s; t, u) + \psi^e(s; t)\delta H(s; t, u)] \geq 0$ and now see the proof by contradiction of the theorem 1.

**Remark.** When $\psi = 1$ and $\psi^e = 0$ we don’t reach anything explicitly new that is different from the hypotheses of the theorem themselves (the part linked to the constraint, which is the most interesting, would not appear at all).
Remark ("structure" of the multiplier $\psi$). Similarly to what has been done above, just a little more explicitly, we’ve that

$$d_{\Gamma_{t,x}}^2 (J(u^*_\theta; \cdot; t, x)) = [2d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) + o_{\varepsilon}](1)$$

($\bar{\theta}_{\varepsilon} = (1)$) still with $d_{\Gamma_{t,x}}^2 (J(u^*_\theta; \cdot; t, x)) > 0 (J(u^*_\theta; \cdot; t, x) \notin \Gamma_{t,x})$ for $\varepsilon \downarrow 0$, so $J(u^*_\theta; \cdot; t, x) \neq J(u^*_\theta; \cdot; t, x)$ for the same $\varepsilon \downarrow 0$ (as well as the other term next to it isn’t null), the ratio

$$\frac{d_{\Gamma_{t,x}}^2 (J(u^*_\theta; \cdot; t, x))}{J(u^*_\theta; \cdot; t, x) - J(u^*_\theta; \cdot; t, x)} = 2d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) + \bar{\theta}_{\varepsilon}(1)$$

is well defined again and

$$\psi^p \equiv \frac{d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x))}{J(u^*_\theta; \cdot; t, x) d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x))} = 2d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) + \bar{\theta}_{\varepsilon}(1)$$

where $\bar{\theta}_{\varepsilon}(1) = \bar{\theta}_{\varepsilon}(1) + \bar{\theta}_{\varepsilon}(1)$. Well, the above is that "we can take" $\bar{\theta}_{\varepsilon}(1) \equiv 0$.

Remark. Understood the procedure, we also understand that we could take any

$$\psi^p \equiv \frac{J(u^*_\theta; \cdot; t, x) + o_{\varepsilon}(1)}{J(u^*_\theta; \cdot; t, x) d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x))}$$

where $o_{\varepsilon}(1) (\equiv o_{\varepsilon}(1))$ and $o_{\varepsilon}(1) (\equiv o_{\varepsilon}(1))$ are such that

$$o_{\varepsilon}(1)[J(u^*_\theta; \cdot; t, x) + o_{\varepsilon}(1)] \geq 0$$

and

$$o_{\varepsilon}(1)[J(u^*_\theta; \cdot; t, x) - J(u^*_\theta; \cdot; t, x)] \geq 0$$

respectively.

Remark (multi-dimensional case). Let $\Gamma_{t,x} \subset \mathbb{R}^N$ be closed and convex. The natural generalization of the lemma 5 holds - where, for instance, for any $z \notin \Gamma$, $\#d_{\Gamma_{t,x}}(z) = 1$ and $|d_{\Gamma_{t,x}}(z)| = 1$ (its Euclidean norm, i.e. length) - and therefore also what we have seen could easily be extended, where however a clarification is necessary: about $\psi^p$, from definition of subgradient (see also the lemma 5 itself),

$$d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) \leq d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) \cdot [J(u^*_\theta; \cdot; t, x) - J(u^*_\theta; \cdot; t, x)]$$

($\equiv [J(u^*_\theta; \cdot; t, x) - J(u^*_\theta; \cdot; t, x)]$), indeed and thus again we can take

$$\psi^p \equiv \frac{d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x)) d_{\Gamma_{t,x}} (J(u^*_\theta; \cdot; t, x))}{J(u^*_\theta; \cdot; t, x)}$$

(now $\psi^p \in \mathbb{R}^N$ - with $|\psi^p| \leq 1$) and it holds

$$\frac{d_{\Gamma_{t,x}}^2 (J(u^*_\theta; \cdot; t, x))}{J(u^*_\theta; \cdot; t, x)} \leq \psi^p \cdot [J(u^*_\theta; \cdot; t, x) - J(u^*_\theta; \cdot; t, x)]$$

which not necessarily is an $\infty$ (except at the limit).
Remark (about two possible assumptions - not required for the proof above).

**Assumption 1**: local uniqueness of an/the equilibrium 4-tuple. The equilibrium 4-tuple \((\bar{u}(\cdot); X(\cdot), Y(\cdot); t), \hat{Z}(\cdot); t)\) (with \(J(\bar{u}(\cdot); t, x) \in \Gamma_{t,x}\)) is such that there exists \(\delta > 0\) small enough such for which, for any other \(\tilde{u}(\cdot) \in \mathcal{W}[t, T] \setminus \{\bar{u}(\cdot)\}\) with \(J(\tilde{u}(\cdot); t, x) \in \Gamma_{t,x}\), if \(d(\tilde{u}(\cdot), \bar{u}(\cdot)) < \delta\) then \(\tilde{u}(\cdot)\) does not “come from” an equilibrium policy (w.r.t. \(J\)) with the following strong meaning: for any \(u(\cdot) \in \mathcal{W}[t, T]\),

\[
\liminf_{\varepsilon \downarrow 0} \frac{J(\tilde{u}^\varepsilon(\cdot); t, x) - J(\bar{u}(\cdot); t, x)}{\varepsilon} < 0
\]

where \(\tilde{u}^\varepsilon(\cdot)\) is the spike variation of \(\tilde{u}(\cdot)\) w.r.t. \(u(\cdot)\) and \(E^\varepsilon_t\). In particular, \(J\) cannot be constant “so near” to \(\tilde{u}(\cdot)\). Therefore, we take \(\varrho \in [0, \delta^2]\) (i.e., \(\sqrt{\varrho} < \delta\).

**Remark**. Under this assumption, it’s easy to verify that at least \(J_\varrho(\tilde{u}_\varrho(\cdot); t, x) + J_\varrho(\bar{u}_\varrho(\cdot); t, x) > 0\) for any \(u(\cdot) \in \mathcal{W}[t, T]\) (these addends are not both null at the same time). Indeed, reasoning by contradiction, if \(J_\varrho(\tilde{u}_\varrho(\cdot); t, x) = J_\varrho(\bar{u}_\varrho(\cdot); t, x) = 0\) for a certain \(u(\cdot) \in \mathcal{W}[t, T]\), then \(J(\tilde{u}_\varrho(\cdot); t, x) = J(\bar{u}_\varrho(\cdot); t, x) = -\varrho\) (in particular, \(\bar{u}_\varrho(\cdot) \neq \tilde{u}(\cdot)\)) and \(J(\bar{u}_\varrho(\cdot); t, x) \in \Gamma_{t,x}\), thus \(\tilde{u}_\varrho(\cdot)\) “come from” an equilibrium policy (w.r.t. \(J\)) with the lim inf equal to zero; but, by lemma 4, \(d(\tilde{u}(\cdot), \bar{u}_\varrho(\cdot)) \leq \sqrt{\varrho} < \delta\): contradiction w.r.t. the assumption (with \(\tilde{u}(\cdot) = \bar{u}_\varrho(\cdot)\)) - don’t forget that \(J(\bar{u}(\cdot); t, x) = 0\).

**Assumption 2**: to have that \(J_\varrho(\bar{u}_\varrho(\cdot); t, x) \neq 0\), i.e., \(\varrho > 0\), precisely for \(\varrho \in [0, \delta^2]\). Under assumption 1 above, that \(\delta > 0\) is such that, for any \(\bar{u}(\cdot) \in \mathcal{W}[t, T]\) with \(J(\bar{u}(\cdot); t, x) \in \Gamma_{t,x}\) and \(d(\bar{u}(\cdot), \tilde{u}(\cdot)) < \delta\) (so \(\bar{u}\) depends also on \(\delta\)), if \(J(\bar{u}(\cdot); t, x) \leq o_\varepsilon(\delta)\) then \(\bar{u}(\cdot) = \tilde{u}(\cdot)\).

**Remark**. The proof is similar to the previous one, still with \(\tilde{u}(\cdot) = \bar{u}_\varrho(\cdot)\).

6 Application to investment-consumption policies

Let’s start with the financial market model, and then we continue with the way of acting in it that interests us and, therefore, with the corresponding wealth process and all that we are interested in saying about it.

**Saving account (bond)**: accrues interest at the constant risk-free interest rate \(r(\cdot) \equiv r > 0\) on \([0, T]\). That is,

\[
dS_0(s) = rS_0(s)ds
\]

(i.e. \(S_0(s) = S_0(0)e^{rs}\), where \(S_0(0) > 0\) is exogenously specified).

**Remark**. Of course, such a \(r\) has nothing to do with the integration variable in intervals of the type \([t, \cdot]\) already indicated with the same symbol somewhere in the previous sections.

**Stock (risky asset)**: the stock price per share follows a geometric Brownian motion

\[
dS(s) = S(s)\left[\varrho ds + \sigma dW(s)\right], \quad s \in [0, T],
\]

(i.e. \(S(s) = S(0)\exp\left\{\left(\varrho - \sigma^2/2\right)s + \sigma W(s)\right\}\), \(S(0) > 0\) exogenously specified as well) where \(\varrho > r\) is the (constant) mean rate of return by the stock and \(\sigma > 0\) the (constant) volatility of the price. We denote

\[
\mu := \varrho - r > 0
\]

the excess return (in mean) by investment. We’ll soon understand that the generalization to the case of multiple stocks is simple.

**Remark**. We do not assume that the market is incomplete, i.e. that there’re more sources of randomness than stocks.

**Investment-consumption policies**: a decision-maker in this market is continuously investing her/his wealth in the bond/stock and is consuming, so an investment-consumption policy (or trading-consumption policy) is determined by the proportion of current wealth she/he invests in the bond/stock and by the consumption rate. Therefore,
Our control (portfolio strategy): \( u(\cdot) = (\zeta(\cdot), c(\cdot)) \in [-1, 1] \times [0, 1] \), where \( c(s) \) is the proportion of wealth consumed at time \( s \) and \( \zeta(s) \) (resp. \( 1 - \zeta(s) \)) is the proportion of wealth invested in the stock (resp. in the bond) at time \( s \). What we’ll research: \( (\zeta(\cdot), c(\cdot)) \) depending, in turn, on \( X(\cdot) \) (feedback form).

Self-financing condition. We assume that the changes in wealth over time are due solely to gains/losses from investing in stock and from consumption, so there’s no cashflow coming in or out.

(Dis)utility functions. The agent is deriving utility from intertemporal (or intermediate) consumption and final wealth, so an agent invests in this market and consumes to maximize resp. minimize her/his expected utility resp. disutility of intertemporal consumption and final wealth. In other words, all decision-makers try to minimize the discounted expectation (see below) of disutility functions \(-v(\cdot), -\hat{v}(\cdot)\), where \( v(\cdot), \hat{v}(\cdot) \) are utility functions of intertemporal amount consumption and of the terminal wealth respectively (which are their von Neumann-Morgenstern common utility). So, \( v, \hat{v} \) satisfy the classical Uzawa-Inada conditions: \( v, \hat{v}: [0, \infty[ \to [0, \infty[ \) are \( C^2 \), strictly increasing and strictly concave - \( v'(x), \hat{v}'(x) > 0 \) and \( v''(x), \hat{v}''(x) < 0 \) for \( x > 0 \), at least - with \( v(0) = \hat{v}(0) = 0 \) and such that \( \lim_{x\downarrow 0} v'(x) = \infty \) and \( \lim_{x\uparrow \infty} v'(x) = 0 \) - and the same for \( \hat{v}' \).

Remark. The \( C^1 \) function \( v'(\cdot): [0, \infty[ \to [0, \infty[ \), i.e. the marginal disutility function (it measures changes in utility), is bijective having inverse function \( (v')^{-1}(\cdot): [0, \infty[ \to [0, \infty[ \) continuous and strictly decreasing which still satisfies the limits as above: \( \lim_{y\downarrow \infty} (v')^{-1}(y) = \infty \) and \( \lim_{y\uparrow \infty} (v')^{-1}(y) = 0 \).

About \(-v(\cdot)\), it holds that \((-v)'(\cdot) \equiv -v'(\cdot): [0, \infty[ \to ]-\infty, 0[ \) and thus \((-v)'^{-1}(\cdot): ]-\infty, 0[ \to [0, \infty[ \).

Example. Consider the CRRA - constant relative risk aversion - utility depending on the parameter \( \lambda \in [0, 1[ \) with crra equal to \( 1 - \lambda = -xv''(x)/v'(x) \), \( x \geq 0 \), i.e. \( v(x) \equiv v_1(x) \equiv x^\lambda/\lambda \). In this case, \( v'(x) = x^{\lambda-1} = 1/(x^{1-\lambda}) \), so \( v''(x) = (\lambda - 1)x^{\lambda-2} \) and \( (v')^{-1}(x) = x^{1/(1-\lambda)} \).

Notation: \( \Upsilon := (-v')^{-1} \).

Discount functions. \( h(\cdot, t), \hat{h}(\cdot, t) \) are discount functions on \([t, T]\) in the sense that \( h(\cdot, t): [t, T] \to [0, \infty[ \) with \( h(t; t) = 1 \) and \( \int_t^T h(s; t) \, ds < \infty \) (for instance, when continuous) - and the same for \( \hat{h} \). An example is when \( h(s; t) \equiv h(s - t) \) where \( h(\cdot): [0, T - t[ \to [0, \infty[ \) with \( h(0) = 1 \) and \( \int_0^{T-t} h(s) \, ds < \infty \), such as an hyperbolic discounting \( h(s) \equiv h_K(s) \equiv 1/(1 + Ks) \), \( K > 0 \).

Remark. wanting to consider such an integrable function on the whole \([0, \infty[ \), we’ve to slightly change it to \( h(s) \equiv 1/(1 + Ks^k) \) with \( k > 1 \).

The corresponding discount rate - or rate of impatience - is the function \(-h'/h\) and we could think about it as a monotone function. That’s the rate of return used to discount future cashflows back to their present value (thus, it’s an interest rate relating to computations of present value).

Time-inconsistency. The key issue - which we manage through the concept of equilibrium policy - is to understand how non-exponential discounting affects an agent’s investment-consumption policies in our problem. Indeed, a decision-maker with high discount rates exhibits more impatience (less care about the future - lower probability/chance of “staying alive”) than one with low discount rates. However, there’s a growing evidence to suggest that in nature discount functions are almost hyperbolic (so that discount rates are hyperbolic as well) - they decrease inversely proportionally, or like a negative power, over time - and therefore the decision-maker becomes time-inconsistent in the sense that a policy, to be implemented after time \( \tau - t > 0 \), which is optimal when discounted at time \( t \in [0, T[ \), no longer is optimal if it is discounted at any later time as \( \tau > t \). Thus, if the decision-maker at time \( t \) can commit her/his successors, she/he can choose the policy which is optimal from her point of view and constraint the others to abide by it, although they don’t see it as optimal for them.

Graphically, the discount mechanism - where \( t < \tau \) and \( X(\cdot) > 0 \):

\[
\begin{array}{ccc}
time t \ (“now”) & \overset{-t}{\longrightarrow} & \text{time } \tau \\
\text{present value} & \downarrow & \text{future utility} \\
h(\tau - t)v(X(\tau)) & \longleftarrow & v(X(\tau))
\end{array}
\]
So, graphically, time-inconsistency - where \( t < s < \tau \) and \( X(\cdot) > 0 \):

\[
\begin{array}{ccc}
\text{time } t & \leftarrow & \tau - t \\
\text{time } s & \leftarrow & \tau - s \\
\text{time } \tau & \leftarrow & \end{array}
\]

\[ h(s - t)v(X(s)) \leftarrow \leftarrow \frac{v(X(s))}{h(\cdot)} \]

\[ h(\tau - s)v(X(\tau)) \leftarrow \leftarrow \frac{v(X(\tau))}{h(\cdot)} \]

from which we recognize that, to have classical time-consistency, it must be (and we can see it in other ways too)

\[
\begin{cases}
    v(X(s)) = h(\tau - s)v(X(\tau)) \\
    h(s - t)v(X(s)) = h(\tau - t)v(X(\tau))
\end{cases}
\]

thus \( h(s - t)h(\tau - s) = h(\tau - t) \) (because \( v(X(\tau)) \neq 0 \)) which is true (if and) only if the discount rate corresponding to \( h \) is constant, i.e. \( h \) is exponential (indeed, deriving w.r.t. \( s \), \( h'(s - t)h(\tau - s) - h(s - t)h'(\tau - s) = 0 \) i.e. \( h'(s - t)/h(s - t) = h'(\tau - s)/h(\tau - t) \), from which the thesis by arbitrariness of \( t, s, \tau \).

Our FBSDE (Merton portfolio management with Uzawa recursive utility):

\[
\begin{cases}
    dX(s) = X(s)\left[(r + \mu \zeta(s) - c(s))\right] ds + \sigma \zeta(s) dW(s), \quad s \in [t, T], \\
    dY(s; t) = -h(s; t)\left[-v(c(s)X(s)) - \beta(s; t)Y(s; t) - \gamma(s; t)Z(s; t)\right] ds + Z(s; t)dW(s), \quad s \in [t, T], \\
    X(t) = x, \quad Y(T; t) = -h(T; t)v(X(T))
\end{cases}
\]

(34)

where: \( x > 0 \) is exogenously specified, \( \zeta(s) = \zeta(s, X(s)), c(s) = c(s, X(s)), \) and \( \beta(\cdot; t), \gamma(\cdot; t) \in L^\infty(t, T; \mathbb{R}) \) and are \( > 0 \). In particular, \( I = [0, \infty[ \).

Cost functional:

\[ J((\zeta(\cdot), c(\cdot)); t, x) \equiv \mathbb{E}Y(t; t). \]

Remark. \( Y(\cdot; t) < 0 \), because \( X(T) > 0 \) and so \( -\hat{v}(X(T)) < 0 \), while \( -v(\cdot) \leq 0 \).

Here, what we might call generators or aggregators are so made due to economic motivations and, substantially, give rise to the so-called Uzawa utility (generalization of the standard additive utility) - which, however, is not the only possible one (see for example the Kreps-Porteus utility).

In order to comply with the assumption (A), we want all the functions involved to be continuous and bounded with their derivatives. Therefore, with slight notational abuses,

\[ b(s, x) \equiv b(s, x, (\zeta, c)) \equiv b(s, x, (\zeta(s, x), c(s, x))) = x[r + \mu \zeta(s, x) - c(s, x)] \]

\[ b_x = r + \mu \zeta(s, x) + x \zeta_x(s, x) - [c(s, x) + xc_x(s, x)] \]

\[ b_{xx} = \mu[2\zeta_x(s, x) + x \zeta_{xx}(s, x)] - [2c_x(s, x) + x c_{xx}(s, x)] \]

\[ \sigma(s, x) \equiv \sigma(s, x, (\zeta, c)) \equiv \sigma(s, x, \zeta(s, x)) = \sigma_x \zeta(s, x) \]

\[ \sigma_x = \sigma[2\zeta_x(s, x) + x \zeta_{xx}(s, x)] \]

\[ f(s, x, y, z; t) \equiv f(s, x, (\zeta, c), y, z; t) \equiv f(s, x, c(s, x), y, z; t) = -h(s; t)[v(xc(s, x)) + \beta(s; t)y + \gamma(s; t)z] \]

\[ \text{D}f = -h(s; t)[v'(xc(s, x))[c(s, x) + xc_x(s, x)], \beta(s; t), \gamma(s; t)]^T \]

\[ f_{xx} = -h(s; t)\left\{v''(xc(s, x))[c(s, x) + xc_x(s, x)]^2 + v'(xc(s, x))[2c_x(s, x) + xc_{xx}(s, x)] \right\} \]
(all the other derivatives of second order of \(f\) are identically zero). Moreover,

\[
h(x; t) = -\dot{h}(T; t)v(x)
\]

and the Hamiltonian \(H\) (the first-order part of the Hamiltonian \(\mathcal{H}\)) - with \(s = t\) and \(\dot{x} = x\) - is

\[
H(t, x, (\zeta, c), y, z, p; q; t, x, (\zeta(t, x), c(t, x))) = pb(t, x, u) + q\sigma(t, x, u) + f(t, x, u, z) + p[\sigma(t, x, u) - \sigma(t, x, u)]; t)
\]

\[
= px[r + \mu c - c] + q\sigma x\zeta - \nu(xc) - \beta(t; t)y - \gamma(t; t)[z + p\sigma x(\zeta - \zeta(t, x))]
\]

(it will be \(P(\cdot; t) \geq 0\) and it has gradient given by

\[
\frac{\partial H}{\partial c} = x\{\mu - \sigma \gamma(t; t)p + \sigma q\}
\]

\[
\frac{\partial H}{\partial q} = -x\{p + \nu'(xc)\}
\]

which we will set \(= 0\) for \(p = p(t; t)\) and \(q = q(t; t)\).\

**Remark.** \(H(t, x, \cdot, Y(t), Z(t), p(t; t), q(t; t); t, x, (\zeta(t, x), c(t, x)))\) admits at most one stationary point in the domain \([-1, 1] \times [0, 1]\) and this must necessarily be the minimum point for it.

Next, adjoint processes \((p(\cdot; t), q(\cdot; t)), (P(\cdot; t), Q(\cdot; t))\) associated to \(((\zeta(\cdot), c(\cdot)), X(\cdot), Y(\cdot; t), Z(\cdot; t))\):

\[
\begin{align*}
dp(s; t) &= -g(s, p(s; t), q(s; t); t)ds + q(s; t)dW(s), \quad s \in [t, T], \\
p(T; t) &= -\dot{h}(T; t)v'(X(T)).
\end{align*}
\]

(35)

and

\[
\begin{align*}
dP(s; t) &= -G(s, P(s; t), Q(s; t); t)ds + Q(s; t)dW(s), \quad s \in [t, T], \\
P(T; t) &= -\dot{h}(T; t)v''(X(T)).
\end{align*}
\]

(36)

where

\[
g(s, p, q; t) = \left[ b_{x}(s) + f_{z}(s; t)\sigma_{x}(s) + f_{y}(s; t)\right]p + \left[\sigma_{x}(s) + f_{z}(s; t)\right]q + f_{z}(s; t)
\]

and

\[
G(s, P, Q; t) = \left[ 2b_{x}(s) + \sigma_{x}(s)^{2} + 2f_{z}(s; t)\sigma_{x}(s) + f_{y}(s; t)\right]P + \left[2\sigma_{x}(s) + f_{z}(s; t)\right]Q + b_{xx}(s)p(s; t) + \sigma_{xx}(s)f_{z}(s; t)p(s; t) + q(s; t) + \left(1, p(s; t), \sigma_{x}(s)p(s; t) + q(s; t)\right)\cdot D^{2}f(s; t) - \left(1, p(s; t), \sigma_{x}(s)p(s; t) + q(s; t)\right)^{T}
\]

so their “homogeneous” term are

\[
g(s, 0, 0; t) = f_{z}(s; t) = -\dot{h}(s; t)v'(\dot{c}(s, X(s))X(s))[\dot{c}(s, X(s)) + \dot{X}(s)\dot{c}_{x}(s, X(s))]
\]

and

\[
G(s, 0, 0; t) = \left[\mu - \sigma \gamma(s; t)p(s; t) + \sigma q(s; t)\right]\left[2\dot{c}_{x}(s, X(s)) + \dot{X}(s)\dot{c}_{xx}(s, X(s))\right] - \left[h(s; t)v''(\dot{c}(s, X(s))\dot{X}(s)) + p(s; t)\right]\left[2\dot{c}_{x}(s, X(s)) + \dot{X}(s)\dot{c}_{xx}(s, X(s))\right] - h(s; t)v''(\dot{c}(s, X(s))\dot{X}(s))[\dot{c}(s, X(s)) + \dot{X}(s)\dot{c}_{x}(s, X(s))^{2}]
\]

Therefore, the sign of \(p(\cdot; t)\) seems unknown for now: indeed, although \(-\dot{v}'(\dot{X}(T)) < 0\) and \(-v' \leq 0\), we cannot know the sign of \(\dot{c}_{x}\) and, thus, of \(\dot{c}(s, X(s)) + \dot{X}(s)\dot{c}_{x}(s, X(s))\).

But, fortunately, the sign of \(P(\cdot; t)\) relates well to the Hamiltonian minimization problem: indeed, since \(-\dot{v}'', -v'' \geq 0\), if we did so that

\[
\begin{align*}
\left\{ \begin{array}{l}
[\mu - \sigma \gamma(s; t)]p(s; t) + \sigma q(s; t) = 0 \\
p(s; t) = -h(s; t)v'(\dot{c}(s, X(s))\dot{X}(s))
\end{array} \right.
\end{align*}
\]

(37)

we obtain \(p(\cdot; t) \leq 0\), so \(\dot{x} < 0\), and

\[
\dot{c}(s, X(s)) = \frac{\chi(p(s; t)/h(s; t))}{\dot{X}(s)}
\]

\[
(\dot{c}(s, x) = \chi(p(s; t)/h(s; t))-x)\) so, consistently, \(\dot{c}(s, X(s)) + \dot{X}(s)\dot{c}_{x}(s, X(s)) = 0\) plus \(2\dot{c}_{x}(s, X(s)) + \dot{X}(s)\dot{c}_{xx}(s, X(s)) = 0\).
Remark. The idea to determine $\zeta(\cdot, X(\cdot))$, instead, is to use the value function of our problem through the Hamilton-Jacobi-Bellman (HJB, for short) equation associated with our problem.

In such a case, in particular $(s = t)$,

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\mu - \sigma \gamma(t; t) p(t; t) + \sigma q(t; t) = 0 \\
p(t; t) = -v'(x \tilde{c}(t, x))
\end{array} \right.
\end{aligned}
\]  

(38)

and all that is consistent with having $P(\cdot; t) > 0 (-v''(X(T)) > 0)$ and minimizing only the first-order part $H$ of the Hamiltonian $\mathcal{H}$, obtaining that

\[
\tilde{c}(t, x) = \frac{\Upsilon(p(t; t))}{x}.
\]

Remark. Let $h(\cdot; t), \hat{h}(\cdot; t)$ be discount functions on $[t, T]$, let $v, \hat{v}$ be utility functions satisfying the Uzawa-Inada conditions and set $\Upsilon := (\hat{v}'(\cdot))^{-1}$, as above. Let $\Pi \equiv (\Pi_1, \Pi_2) : [0, T] \times [0, \infty] \rightarrow [-1, 1] \times [0, 1]$ which is measurable and $C^2$ w.r.t. $x$ with bounded first and second derivatives, even if you multiply them by $x$, on $[0, T] \times [0, \infty]$. For any $t \in [0, T]$ and $x > 0$, $\tilde{X}(\cdot)$ be the solution of

\[
\begin{aligned}
&\left\{ \begin{array}{l}
dx(s) = X(s) \left[ (r + \mu \Pi_1(s, X(s)) - \Pi_2(s, X(s))) ds + \sigma \Pi_1(s, X(s)) dW(s) \right], \quad s \in [t, T], \\
X(t) = x
\end{array} \right.
\end{aligned}
\]

and assume that there exists $a$, the solution pair $(p(\cdot; t), q(\cdot; t))$ - adjoint process of first order - of the BSDE

\[
\begin{aligned}
&dp(s; t) = \left\{ \begin{array}{l}
\left\{ r + \mu [\Pi_1(s, X(s)) + X(s) \frac{\partial}{\partial x} \Pi_1(s, X(s))] - [\Pi_2(s, X(s)) + X(s) \frac{\partial}{\partial x} \Pi_2(s, X(s))] \\
- \sigma h(s; t) \gamma(s; t) [\Pi_1(s, X(s)) + X(s) \frac{\partial}{\partial x} \Pi_1(s, X(s))] - h(s; t) \beta(s; t) \right\} p(s; t) \\
+ \left\{ \sigma [\Pi_1(s, X(s)) + X(s) \frac{\partial}{\partial x} \Pi_1(s, X(s))] - h(s; t) \gamma(s; t) \right\} \hat{q}(s; t) \\
- h(s; t) v'(\Pi_2(s, X(s)) X(s)) [\Pi_2(s, X(s)) + X(s) \frac{\partial}{\partial x} \Pi_2(s, X(s))] \right\} ds \\
+ q(s; t) dW(s), \quad s \in [t, T], \\
p(T; t) = -\hat{h}(T; t) v'(\tilde{X}(T))
\end{array} \right.
\]

which satisfies $(p(\cdot; t) < 0$ and), for any $s \in [t, T]$,

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\mu - \sigma \gamma(s; t) p(s; t) + \sigma q(s; t) = 0 \\
p(s; t) = -h(s; t) v'(\Pi_2(s, X(s)) X(s)).
\end{array} \right.
\end{aligned}
\]

Then $\Pi$ is an equilibrium policy for our problem with, in particular,

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\mu - \sigma \gamma(t; t) p(t; t) + \sigma q(t; t) = 0 \\
\Pi_2(t, x) = \Upsilon(p(t; t))/x.
\end{array} \right.
\end{aligned}
\]

Remark. $(\Pi_1(\cdot, X(\cdot)), \Pi_2(\cdot, X(\cdot))) \in \mathcal{W}[t, T]$.

One-dimensional constraint:

\[
J((\zeta(\cdot), c(\cdot)); t, x) \equiv E Y(t; t) \in \Gamma_{t,x}
\]

where (for instance)

\[
\begin{aligned}
&dY(s; t) = h(s; t) \beta(s; t) Y(s; t) ds + Z(s; t) dW(s), \quad s \in [t, T], \\
Y(T; t) = -\hat{h}(T; t) \bar{b}(X(T)).
\end{aligned}
\]  

(39)

and $\Gamma_{t,x} \in \mathcal{B}(\mathbb{R}) \setminus \{0\}$ is a non-empty closed interval. So,

\[
f(s, x, y; z; t) := f(s, x, (\zeta, c), y, z; t) \equiv f(s, y; t) = -h(s; t) \beta(s; t) y.
\]
Remember that
\[
H(s, x, u, y, z, p, q, P; t, x, \bar{u}) = pb(s, x, u) + q\sigma(s, x, u) + \frac{1}{2}P[\sigma(s, x, u) - \sigma(s, \bar{x}, \bar{u})]^2 \\
+ f(s, x, u, y, z + p[\sigma(s, x, u) - \sigma(s, \bar{x}, \bar{u})]; t)
\]
and
\[
\begin{align*}
\delta H(s; t) &= H(s, \bar{X}(s), \bar{u}(s), \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t), P(s; t); t, \bar{X}(s), \bar{u}(s)) \\
\delta \delta H(s; t, u) &= H(s, \bar{X}(s), u, \bar{Y}(s; t), \bar{Z}(s; t), p(s; t), q(s; t), P(s; t); t, \bar{X}(s), u(s)) - H(s; t)
\end{align*}
\]
so the Hamiltonian $H$ - with $s = t$ and $\bar{x} = x$ - corresponding to $((\bar{\zeta}(\cdot), \bar{c}(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$ is
\[
H(t, x, (\zeta, c), y, z, p, q, P; t, x, (\bar{\zeta}(t, x), \bar{c}(t, x))) = px[r + \mu \zeta - c] + q\sigma x \zeta + \frac{1}{2}P x^2 \sigma^2 (\zeta - \bar{\zeta}(t, x))^2 \\
- v(xc) - \beta(t; t)y - \gamma(t; t)[z + p\sigma(x - \bar{\zeta}(t, x))]
\]
while the Hamiltonian $\mathcal{H}$ - with $s = t$ and $\bar{x} = x$ - corresponding to $((\zeta(\cdot), c(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$ is
\[
\mathcal{H}(t, x, (\zeta, c), y, z, p, q, P; t, x, (\bar{\zeta}(t, x), \bar{c}(t, x))) = px[r + \mu \zeta - c] + q\sigma x \zeta + \frac{1}{2}P x^2 \sigma^2 (\zeta - \bar{\zeta}(t, x))^2 - \beta(t; t)y.
\]
Thus
\[
\mathcal{H}(t; t) = xp(t; t)[r + \mu \bar{\zeta}(t, x) - \bar{c}(t, x)] + x\sigma q(t; t)\bar{\zeta}(t, x) - v(x\bar{c}(t, x)) - \beta(t; t)\bar{Y}(t; t) - \gamma(t; t)\bar{Z}(t; t)
\]
and so, for any $(\zeta, c) \in [-1, 1] \times [0, 1],
\[
\delta \mathcal{H}(t; t, (\zeta, c)) = xp(t; t)[\mu(\zeta - \bar{\zeta}(t, x)) - (c - \bar{c}(t, x))] + x\sigma q(t; t)[\zeta - \bar{\zeta}(t, x)] \\
+ \frac{1}{2}p(t; t) x^2 \sigma^2 [\zeta - \bar{\zeta}(t, x)]^2 - [v(xc) - v(x\bar{c}(t, x))] - x\sigma p(t; t)\gamma(t; t)[\zeta - \bar{\zeta}(t, x)]
\]
and
\[
\delta \delta \mathcal{H}(t; t, (\zeta, c)) = xp(t; t)[\mu(\zeta - \bar{\zeta}(t, x)) - (c - \bar{c}(t, x))] + x\sigma q(t; t)[\zeta - \bar{\zeta}(t, x)] + \frac{1}{2}p(t; t) x^2 \sigma^2 [\zeta - \bar{\zeta}(t, x)]^2.
\]
Therefore, following theorem 2,

**Theorem 4.** For $t \in [0, T]$ and $x \in I$, let $((\bar{\zeta}(\cdot), \bar{c}(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$ be an equilibrium 4-tuple such that $((\zeta(\cdot), c(\cdot)), \bar{X}(\cdot))$ satisfies the constraint above. Then there exist multipliers $\psi, \psi \in [-1, 1]$ satisfying
\[
\psi^2 + \psi^2 = 1
\]
and, for any $\bar{v} \in \Gamma_{t,x},$
\[
\psi[\bar{v} - J((\bar{\zeta}(\cdot), \bar{c}(\cdot)); t, x)] \leq 0
\]
such that, considered $(p(\cdot; t), q(\cdot; t))$, $(P(\cdot; t), Q(\cdot; t))$, $H$ and $\delta H$ corresponding to $((\zeta(\cdot), c(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$, and considered $(p(\cdot; t), q(\cdot; t))$, $(P(\cdot; t), Q(\cdot; t))$, $\mathcal{H}$ and $\delta \mathcal{H}$ corresponding to $((\zeta(\cdot), c(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot; t), \bar{Z}(\cdot; t))$, then, for any $(\zeta, c) \in [-1, 1] \times [0, 1],$
\[
\psi \delta \mathcal{H}(t; t, (\zeta, c)) + \psi \delta \mathcal{H}(t; t, (\zeta, c)) \\
= \left[\psi p(t; t) + \psi p(t; t)[\mu(\zeta - \bar{\zeta}(t, x)) - (c - \bar{c}(t, x))] + \sigma[\psi q(t; t) + \psi q(t; t)][\zeta - \bar{\zeta}(t, x)] \\
+ \frac{1}{2}x^2 \sigma^2 [\psi P(t; t) + \psi P(t; t)][\zeta - \bar{\zeta}(t, x)]^2 - \psi \left\{\frac{1}{2}[v(xc) - v(x\bar{c}(t, x))] + x\sigma p(t; t)\gamma(t; t)[\zeta - \bar{\zeta}(t, x)]}\right\} \geq 0.
\]
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