THE TWISTED SATAKE TRANSFORM AND THE CASSELMAN-SHALIKA FORMULA FOR QUASI-SPLIT GROUPS

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Abstract. We prove the Casselman-Shalika formula for unramified groups over a non-archimedean local field by studying the action of the spherical Hecke algebra on the space of compact spherical Whittaker functions via the twisted Satake transform. This method provides a conceptual explanation of the appearance of characters of a dual group in the Casselman-Shalika formula.

1. Introduction

In this article we provide a conceptual explanation of the appearance of characters of a complex reductive group in the Casselman-Shalika formula of a connected unramified group $G$ over a nonarchimedean local field $F$. This extends the work of the first author in [7], where $G$ was assumed to be split of adjoint type.

For this introduction we assume that $G$ is split unless otherwise specified. Let $T$ be a maximal torus, $K$ a maximal compact subgroup, and $U$ a maximal unipotent subgroup of $G$ such that $G = UT K$ (Iwasawa Decomposition). Let $\Psi : U \to \mathbb{C}^\times$ be a non-degenerate character of conductor $p$. (For definitions see Subsection 2.8 and Section 5. Casselman-Shalika [5] assume $\Psi$ is trivial on $U \cap K$ and nontrivial on any subgroup with a larger abelianization. They call such a $\Psi$ unramified; we will say it has conductor $O$.) Let $\mathcal{H}_K \overset{\text{def}}{=} C_c^\infty(K \backslash G/K)$ be the spherical Hecke algebra. The algebra $\mathcal{H}_K$ acts on $\text{Ind}_G^U(\Psi)^K = \{ f : G \to \mathbb{C} f(ugk) = \Psi(u)f(g), \text{ for all } g \in G, u \in U, k \in K \}$ by right convolution. For $h \in \mathcal{H}_K$ we write $\hat{h}(g) = h(g^{-1})$. A spherical Whittaker function on $G$ with respect to $K, \Psi$, and a character $\chi : \mathcal{H}_K \to \mathbb{C}$ is a function $W_\chi : G \to \mathbb{C}$ that satisfies the following properties.

1. $W_\chi \in \text{Ind}_U^G(\Psi)^K$.
2. $W_\chi * h = \chi(h)W_\chi$, for all $h \in \mathcal{H}_K$.

Since $G = UT K$, item (1) implies that $W_\chi$ is determined by its values on $T/T \cap K$; item (2) in conjunction with the multiplicity one theorem for Whittaker models implies that the associated eigenvalues determine $W_\chi$ up to scaling. The Casselman-Shalika formula ([5], Theorem 5.4) is a formula for the function $W_\chi$ evaluated at the elements of $T/T \cap K$.

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Casselman-Shalika’s original approach [5] is based on decomposing the spherical Whittaker function using Casselman’s basis. One function in this decomposition can be computed directly; the others can then be computed via a symmetry. One can manipulate the resulting formula and apply the Weyl character formula to find a character of a finite dimensional representation of the Langlands dual group of \( G \). When \( G \) is not split one can make a similar statement. In this case the description of the dual group (not necessarily the Langlands dual group) seems to first appear in Tamir [14].

While it is not difficult to identify this character as a byproduct of the method of Casselman-Shalika, Gurevich [7] provided a conceptual explanation of its appearance when \( G \) is split and adjoint. Gurevich’s proof is based on studying the \( \mathcal{H}_K \)-module \( \text{ind}^G_K(\Psi)^K = \{ f \in \text{Ind}^G_K(\Psi) | \text{supp}(f) \text{ is compact mod } U \} \), where \( \Psi \) is the complex conjugate of \( \Psi \). The character naturally appears through the Satake isomorphism.

In addition to providing a conceptual explanation of the appearance of the character, Gurevich’s proof also avoids the multiplicity one theorem for Whittaker functionals, which is invoked by Casselman-Shalika [5]. Thus this method may provide an alternative approach to the Casselman-Shalika formula in the the case of covering groups, first proved by Chinta-Offen [6] and McNamara [10], and \( p \)-adic loop groups, first proved by Patnaik [11].

However, Gurevich’s proof does not determine the normalization factor \( \zeta(\chi) \) (Casselman-Shalika [5], Theorem 5.4), which characterizes the spherical representations that do not possess a Whittaker functional. Nevertheless, almost all spherical representations are principal series, which do possess a Whittaker functional. In these cases the choice of a normalization of the Whittaker functional determines the constant of proportionality in our formula.

The present paper extends Gurevich’s proof of the Casselman-Shalika formula to connected unramified groups, the class of groups considered by Casselman-Shalika [5].

The foundation for our proof is an explicit \( \mathcal{H}_K \)-module isomorphism. To state this precisely we introduce some additional notation. The torus \( T \) has a cocharacter lattice \( X_*(T) \) and Weyl group \( W = N_G(T)/T \). Using the isomorphism \( T/T \cap K \cong X_*(T) \), for any \( \mu \in X_*(T) \) we fix an element \( t_\mu \in T \) representing \( \mu \) in the quotient.

We will write \( \mathbb{C}[X_*(T)] \) for the group ring of \( X_*(T) \) and let \( \{ e^\mu | \mu \in X_*(T) \} \) be the natural basis. The action of \( W \) on \( T \) induces an action on \( \mathbb{C}[X_*(T)] \) and we write \( \mathbb{C}[X_*(T)]^W \) and \( \text{alt}(\mathbb{C}[X_*(T)]) \) for the set of symmetric and alternating elements, respectively.

Let \( B = TU \subset G \) be a Borel subgroup with modular character \( \delta \) and unipotent radical \( U \). Let \( X_*(T)^+ (X_*(T)^{++}) \) be the set of dominant (strictly dominant) cocharacters with respect to \( B \). Let \( \rho' \) be one half the sum of the dominant coroots.

Let \( \hat{G} \) be the Langlands dual group of \( G \) and \( \text{Rep}(\hat{G}) \cong \mathbb{C}[X_*(T)]^W \) its \( \mathbb{C} \)-algebra of finite dimensional characters. The set of characters of highest weight representations \( \{ \text{ch} V_\lambda | \lambda \in X_*(T)^+ \} \) is a basis for \( \text{Rep}(\hat{G}) \). Using the Satake isomorphism \( S : \mathcal{H}_K \to \mathbb{C}[X_*(T)]^W \), for every \( \lambda \in X_*(T)^+ \) we can define \( A_\lambda = S^{-1}(\text{ch} V_\lambda) \in \mathcal{H}_K \).

The space \( \text{ind}^G_U(\Psi)^K \) has a basis \( \{ \phi_\mu | \mu \in X_*(T)^{++} \} \) defined as follows. Consider the \( G \)-invariant pairing \( \langle -,- \rangle_\Psi : \text{ind}^G_U(\Psi) \otimes \text{Ind}^G_U(\Psi) \to \mathbb{C} \) defined by \( \langle h,f \rangle_\Psi = \int_{U \backslash G} h(g)f(g)dg \).
For each \( \mu \in X_*(T)^{++} \) the function \( \phi_\mu \in \text{ind}^{G}_{U}(\Psi)^{K} \) is defined by the property that
\[
delta(t_\mu)^{-1/2} f(t_\mu) = \langle \phi_\mu, f \rangle_{\Psi}, \tag{1}\]
for all \( f \in \text{Ind}^{G}_{P}(\Psi)^{K} \).

Now we can state the main technical theorem.

**Theorem 1.1.** The linear map \( j: \text{ind}^{G}_{U}(\Psi)^{K} \rightarrow \text{alt}(C[X_*(T)]) \) defined by \( j(\phi_\mu) = \text{alt}(e^\mu) \) for \( \mu \in X_*(A)^{++} \) is an \( \mathcal{H}_K \cong C[X_*(T)]^{W} \)-module isomorphism.

When \( G \) is adjoint then \( \rho^\vee \in X_*(T)^{++} \), and Theorem 1.1 quickly yields the Casselman-Shalika formula. Specifically, the Weyl character formula and Theorem 1.1 imply for \( \mu \in X_*(T)^{++} \) and \( \lambda \in X_*(T)^{+} \)
\[
j(\phi_{\rho^\vee} \ast A_\lambda) = j(\phi_{\rho^\vee}) \cdot \text{ch}_{\lambda} = \text{alt}(e^{\rho^\vee}) \cdot \text{ch}_{\lambda} = \text{alt}(e^{\lambda + \rho^\vee}) = j(\phi_{\lambda + \rho^\vee}). \tag{2}\]

Thus \( \phi_{\rho^\vee} \ast A_\lambda = \phi_{\lambda + \rho^\vee} \). Therefore because \( W_\chi \) is an \( \mathcal{H}_K \)-eigenfunction and the pairing is \( G \)-invariant, for any \( \lambda \in X_*(T)^{+} \) we have
\[
\chi(A_\lambda)\delta(t_{\rho^\vee})^{-1/2} W_\chi(t_{\rho^\vee}) = \langle \phi_{\rho^\vee}, W_\chi \ast A_\lambda, W_\chi \rangle_{\Psi} = \langle \phi_{\lambda + \rho^\vee}, W_\chi \rangle_{\Psi} = \delta(t_{\lambda + \rho^\vee})^{-1/2} W_\chi(t_{\lambda + \rho^\vee}). \tag{3}\]

**Theorem 1.2** (Casselman-Shalika Formula for Adjoint Groups). For any \( \lambda \in X_*(T)^{+} \)
\[
W_\chi(t_{\lambda + \rho^\vee}) = \delta^{1/2}(t_\lambda)\chi(S^{-1}(\text{ch}_{\lambda})) W_\chi(t_{\rho^\vee}). \tag{4}\]

Since \( G \) is adjoint Theorem 1.2 directly implies a formula for characters \( \Psi \) of conductor \( \mathcal{O} \), which is important for global applications. For details see Section 7.

If \( G \) is not adjoint, then it may be that \( \rho^\vee \notin X_*(T) \), in which case (2) is meaningless. Nevertheless we prove a substitute (Lemma 6.4). For any \( \lambda \in X_*(T)^{+} \) and \( \mu \in X_*(T)^{++} \) define \( c_{\mu,\lambda}^{\eta} \in \mathbb{C} \) such that \( \text{ch}_{V_{\mu - \rho^\vee}} \cdot \text{ch}_{\lambda} = \sum_{\eta} c_{\mu,\lambda}^{\eta} \text{ch}_{\eta - \rho^\vee} \). (In case \( \rho^\vee \notin X_*(T) \) we view \( \text{ch}_{V_{\mu - \rho^\vee}} \) as a representation of the simply connected cover of \( \hat{G} \).) Then
\[
\phi_\mu \ast A_\lambda = \sum_{\eta \in X_*(A)^{++}} c_{\mu,\lambda}^{\eta} \phi_\eta. \tag{5}\]

Lines (1) and (5) yield a family of recursions for \( W_\chi \). For any \( \lambda \in X_*(T)^{+} \) and \( \mu \in X_*(T)^{++} \)
\[
\delta^{-1/2}(t_\mu)\chi(S^{-1}(\text{ch}_{\lambda})) W_\chi(t_\mu) = \sum_{\eta \in X_*(T)^{++}} c_{\mu,\lambda}^{\eta} \delta^{-1/2}(m_\eta) W_\chi(t_\eta). \tag{6}\]

In Proposition 8.2 we show that these recursions have a solution space of dimension at most one. Moreover, a nonzero solution to these recursions is provided by characters of the simply connected cover of \( \hat{G} \), for almost all \( \chi \). This yields the Casselman-Shalika formula (Theorem 8.3).

So far we have focused on the case where \( G \) is split. This was a matter of convenience and in the main body of this paper we will work with an arbitrary connected unramified group \( G \). The most important new feature required for this more general case is identifying the
correct dual group $L G^\dagger$ (Subsection 2.9). In fact, we work in the context of the universal principal series, which highlights the role played by this dual group.

The above proof of the Casselman-Shalika formula only works for characters of conductor $p$. However, for global applications one must consider characters of conductor $\mathcal{O}$. We include an alternative proof of the Casselman-Shalika formula for characters of conductor $p$ via a reduction to the adjoint case. The advantage of this approach is that it allows us to recover the Casselman-Shalika formula for characters of conductor $\mathcal{O}$.

We conclude this introduction with an outline of this paper. In Section 2 we establish notation and recall relevant theorems. In Section 3 we recall the Bernstein presentation for the Iwahori-Hecke algebra of a connected unramified group (Theorem 3.1). This presentation is used in Section 4 where we generalize Savin's isomorphism [13] to connected unramified groups (Theorem 4.2).

In Section 5 we determine the $H_K$-module structure of $\text{ind}_{L}^{G}(\Psi)^K$ through a study of the twisted Satake transform $S_{\Psi}$. The main technical result is the computation of the kernel of $S_{\Psi}$ (Lemma 5.4). It is in this computation where we use that $\Psi$ has conductor $p$.

In Section 6 we study a spherical Whittaker function of conductor $p$. The main result of this section is a recursive formula for this function (Proposition 6.5). In Section 7 we specialize this to the case where $G$ is adjoint and prove the Casselman-Shalika formula for characters of conductor $p$ (Theorem 7.1) and $\mathcal{O}$ (Proposition 7.3). In Section 8 we we generalize the argument from Section 7 to connected unramified groups. Unfortunately, this does not yield any information about the conductor $\mathcal{O}$ case. Therefore, in Section 9 we prove the Casselman-Shalika formula for connected unramified groups and characters of conductor $p$ via a reduction to the adjoint case. This approach allows us to recover the Casselman-Shalika formula for characters of conductor $\mathcal{O}$, which is described in Section 10.

2. Notation

2.1. Fields. Let $F$ be a nonarchimedean local field with finite residue field. If $L$ is a field extension of $F$, we write $\mathcal{O}_L$ for the ring of integers of $L$ with maximal ideal $p_L$, and $\kappa_L$ for the residue field. If $\kappa_L$ is finite let $q_L = |\kappa_L|$. When $L = F$ we may suppress the subscripts. Let $\text{ord}$ be the discrete valuation of $F$ with value group $\mathbb{Z}$. Let $E$ be a finite unramified extension of $F$. Since $E \supseteq F$ is unramified the canonical extension of $\text{ord}$ from $F$ to $E$ also has value group $\mathbb{Z}$.

2.2. Algebraic Groups. Throughout this paper we use boldface characters for group schemes over $F$, such as $H$, and plain text characters for their group of $F$-points, such as $H$.

Let $H$ be a group scheme defined over $F$. We write $Z(H)$ for the center of $H$ and $\mathcal{D}H$ for the derived subgroup of $H$. For any field $L \supseteq F$, let $X^*(H)_L$ be the group of algebraic characters defined over $L$ and $X_*(H)_L$ the group of algebraic cocharacters defined over $L$. If $L = F_s$, a separable closure of $F$, then we omit the subscript.

Let $G$ be a connected quasi-split reductive group scheme defined over $F$ that is split over $E$. (i.e. $G$ is unramified.) Inside of $G$ we fix a minimal parabolic subgroup $P$ containing a maximal $F$-split torus $A$ with centralizer $M$ and normalizer $N$. Since $G$ is quasi-split
\( \textbf{M} \) is a maximal torus in \( \textbf{G} \). Thus \( X_*(\textbf{M})_F \cong X_*(\textbf{A}) \). The parabolic subgroup \( \textbf{P} \) admits a Levi decomposition \( \textbf{P} = \textbf{M} \textbf{U} \), where \( \textbf{U} \) is the unipotent radical of \( \textbf{P} \). We write \( \overline{\textbf{P}} \) and \( \overline{\textbf{U}} \) for the opposite parabolic and its unipotent radical respectively.

The maximal split torus \( \textbf{A} \) in \( \textbf{G} \) determines a relative root datum \( (X^*(\textbf{A}), \Phi, X_*(\textbf{A}), \Phi^+) \), where \( \Phi = \Phi(\textbf{G}, \textbf{A}) \) is the set of relative roots with respect to \( \textbf{A} \). For each \( \alpha \in \Phi \) we write \( \textbf{U}_\alpha \) for the root subgroup associated to \( \alpha \). (Note that if \( 2\alpha \in \Phi \), then \( \textbf{U}_{2\alpha} \subset \textbf{U}_\alpha \).

Our choice of \( \textbf{P} \) identifies a set of positive roots \( \Phi^+ \), from which we can extract a set of simple roots \( \Delta \). The set of negative roots is \( \Phi^- = -\Phi^+ \). Let \( \langle -, - \rangle : X^*(\textbf{A}) \times X_*(\textbf{A}) \to \mathbb{Z} \) be the pairing defined by \( \lambda \circ \mu(t) = t^{\langle \lambda, \mu \rangle} \). Let \( \Lambda^+ = \{ v \in \text{span}_\mathbb{Z}(\Phi^+) \otimes \mathbb{Q} \mid \langle \alpha, v \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi \} \)

be the coweight lattice in \( \text{span}_\mathbb{Z}(\Phi^+) \otimes \mathbb{Q} \) with respect to \( \Phi \).

Let \( \textbf{W} \) be the relative Weyl group associated to the root system \( \Phi \), which is isomorphic to \( N/M \). The action of \( \textbf{W} \) on \( \textbf{A} \) induces an action on \( X_*(\textbf{A}) \) which extends to an action on \( \mathbb{C}[X_*(\textbf{A})] \). Define \( \text{alt} : \mathbb{C}[X_*(\textbf{A})] \to \mathbb{C}[X_*(\textbf{A})] \) by \( \text{alt}(f) = \sum_{w \in \textbf{W}} \text{sign}(w)w \cdot f \), where \( \text{sign}(w) \) is the sign character of \( \textbf{W} \). Let \( \text{alt}(\mathbb{C}[X_*(\textbf{A})]) \) denote the image of \( \text{alt} \).

The subgroup \( \textbf{P} \) acts on \( \textbf{U} \) by conjugation. The modular character of \( P = \textbf{M} \textbf{U} \) is defined by \( d(pup^{-1}) = \delta_P(p)du \), where \( du \) is a Haar measure of \( \textbf{U} \). Similarly we defined \( \delta_T \) to be the modular character of \( T = \textbf{M}T \). Note that \( \delta_P = \delta_T^{-1} \).

An element \( \lambda \in X_*(\textbf{A}) \) is dominant (strictly dominant) with respect to \( P \) if \( \langle \lambda, \alpha \rangle \geq 0 \) \( \langle \lambda, \alpha \rangle > 0 \) for all \( \alpha \in \Delta \). Let \( X_*(\textbf{A})^\circ \) \( (X_*(\textbf{A})^\circ)^+ \) denote the set of dominant (strictly dominant) elements of \( X_*(\textbf{A}) \) with respect to \( P \).

Later we will have three connected reductive groups \( \textbf{G}, \textbf{G}' \), and \( \textbf{G}'' \) appearing simultaneously. All the notation that we introduced for \( \textbf{G} \) will be carried over to \( \textbf{G}' \) and \( \textbf{G}'' \) and augmented with one or two primes respectively.

2.3. Brouhat-Tits Theory. It will be convenient to establish some terminology from Bruhat-Tits theory. We follow Tits [15]. (For additional details see Bruhat-Tits [2, 3], and Vigneras [16], Section 3.) Let \( \mathcal{B} \) denote the building of \( \textbf{G} \) over \( F \) and let \( \mathcal{A} \subset \mathcal{B} \) be the (enlarged) apartment associated to the maximal \( F \)-split torus \( \textbf{A} \). We fix a hyperspecial vertex \( x_0 \in \mathcal{A} \), which we use to make an identification \( \mathcal{A} \cong X_*(\textbf{A}) \otimes \mathbb{Z} \mathbb{R} \).

The Bruhat-Tits homomorphism \( \nu : M \to X_*(\textbf{M})_F \cong X_*(\textbf{A}) \) is characterized by the equations \( \langle \chi, \nu(m) \rangle = -\text{ord}(\chi(m)) \), for all \( \chi \in X^*(\textbf{M})_F \). Let \( M^0 = \ker \nu \), then we have the exact sequence

\[
1 \to M^0 \to M \xrightarrow{\nu} X_*(\textbf{A}) \to 1.
\] (For surjectivity see Cartier [4], page 135.)

For convenience, we fix a splitting of this sequence \( s : X_*(\textbf{A}) \to M \) (i.e. \( s \) is a group homomorphism such that \( \nu \circ s = \text{id}_{X_*(\textbf{A})} \)), which exists because \( M \) is abelian and \( X_*(\textbf{A}) \) is a free abelian group, and write \( m_\lambda = s(\lambda) \).

Let \( \alpha + k \) be an affine function on \( X_*(\textbf{A}) \otimes \mathbb{R} \), where \( \alpha \in \Phi \) and \( k \in \mathbb{R} \). The group \( U_\alpha \) contains the subgroups \( X_{\alpha + k} \) (Tits [15], Section 1.4). These groups satisfy \( X_{\alpha + k} \subseteq X_{\alpha + \ell} \) if and only if \( k \geq \ell \) and determine a filtration of \( U_\alpha \). This filtration can be used to define the set of affine roots \( \Phi_{\text{aff}} \) as in Tits [15], Section 1.6.
Under the identification $A \cong X_*(A) \otimes \mathbb{R}$, the cone $X_*(A)^+ \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{R}_{\geq 0}$ identifies a unique conical chamber $C$ in $A$ with apex $x_0$. Let $C$ be the unique chamber in $C$ containing $x_0$ and define $I$ to be the Iwahori subgroup associated to $C$ (Tits [15], Section 3.7). Recall that $I$ is contained in the fixator of $C$. The stabilizer of $x_0$ is a maximal compact subgroup $K = \text{Stab}_G(x_0)$.

The next few propositions collect a few facts that will be useful in later computations.

**Proposition 2.1.** Let $\alpha \in \Phi$, $\lambda \in X_*(A)$, and $k \in \mathbb{R}$.

1. $m_\lambda X_{a+k}m_\lambda^{-1} = X_{a+k-\lambda} \cdot \lambda$.
2. Thus if $\lambda$ is dominant with respect to $P$ and $\alpha \in \Phi^-$, then $m_\lambda X_{a+k} m_\lambda^{-1} \subseteq X_{a+k}$.

**Proof:** Item (1) is stated in Tits [15], Section 1.4.2. Item (2) follows from item (1). \(\blacksquare\)

If $\lambda$ is dominant we will informally say that $m_\lambda$ contracts $U$. Furthermore, if $\lambda$ is strictly dominant, then this containment is strict and we will say $m_\lambda$ strictly contracts $U$.

**Proposition 2.2.**

1. $M^\circ = M \cap K$.
2. $P \cap K = (M \cap K)(U \cap K)$.
3. Let $\lambda \in X_*(A)^+$. Then $[\text{Im}_\lambda : I] = \delta_P(m_\lambda)^{-1}$.
4. $G = UMK$. (Iwasawa Decomposition)

**Proof:** For item (1) see Cartier [4], page 145. For item (2) see Cartier [4], page 140. Item (3) can be checked by a direct calculation using equations (3) and (9) on page 145 in Cartier [4]. The Iwasawa decomposition holds because $K$ is the stabilizer of a special vertex. (See Tits [15], 3.3.2.) \(\blacksquare\)

The next proposition states some basic facts about parahoric subgroups. First we introduce some notation. Let $\alpha \in \Delta$. Let $I_\alpha$ be the parahoric subgroup associated to the facet $\mathcal{F}_\alpha$ of $C$ fixed by $s_\alpha$. (See Vigneras [16], Section 3.7.) Note that $I_\alpha$ is contained in the fixator of $\mathcal{F}_\alpha$.

**Proposition 2.3.**

1. The multiplication map induces an isomorphism

$$\left( \prod_{\alpha \in \Phi^+} I \cap U_\alpha \right) \times \left( I \cap M \right) \times \left( \prod_{\alpha \in \Phi^-} I \cap U_\alpha \right) \cong I.$$  

(The factors in the product over $\Phi^\pm$ may be taken in any order.)

Let $\alpha \in \Delta$. Then:

2. $I \cap U_\alpha = I_\alpha \cap U_\alpha = K \cap U_\alpha$;
3. $I \cap U_{-\alpha} \subseteq I_\alpha \cap U_{-\alpha} = K \cap U_{-\alpha}$;
4. $I \cap \overline{U} \subseteq I_\alpha \cap \overline{U}$;
5. $I_\alpha = I \cup Iw_\alpha I$.  

2.4. **Function Spaces.** If $R$ is a $\mathbb{C}$-vector space and $X$ is an $\ell$-space (in the sense of Bernstein-Zelevinski [1]), let $C^\infty(X, R)$ be the set of functions $f : X \to R$ such that $f$ is locally constant and let $C_c^\infty(X, R)$ be the subset of $C^\infty(X, R)$ consisting of compactly supported functions. When $R = \mathbb{C}$ we omit $\mathbb{C}$ from the notation.

Given $f_1 \in C_c^\infty(G)$ and $f_2 \in C_c^\infty(G, R)$ we define $f_2 * f_1(g) \overset{\text{def}}{=} \int_G f_2(h) f_1(h^{-1} g) dg \in C^\infty(G, R)$, where $dg$ is the Haar measure on $G$ such that $\text{meas}(I) = 1$.

Unless otherwise stated, the group $G$ will act on $(g, C^\infty(G, R))$, on the left, via right translation, $(g(h) \cdot f)(g) = f(gh)$. This action induces an action of $C^\infty_c(G)$ on $C^\infty(G, R)$. Let $f_1 \in C^\infty_c(G)$ and $f_2 \in C^\infty(G, R)$, then $(g f_1) \cdot f_2 = f_2 * f_1$, where $f_1(g) = f_1(g^{-1})$.

When $H$ is a subgroup of $G$ and $(\sigma, V)$ is a smooth $H$-representation we define the smooth $G$-representations

$$\text{Ind}_H^G(\sigma) = \{ f \in C^\infty(G, V) | f(h g) = \sigma(h) f(g) \text{ for all } h \in H \},$$

$$\text{ind}_H^G(\sigma) = \{ f \in \text{Ind}_H^G(\sigma) | \text{the support of } f \text{ is compact mod } H \},$$

where $G$ acts by right translation.

The space $C^\infty_c(M/M^0)$ is a $\mathbb{C}$-algebra under convolution. For $\lambda \in X_*(A)$ let $1_{m_\lambda M^0}$ be the characteristic function of the set $m_\lambda M^0$. The set $\{1_{m_\lambda M^0} | \lambda \in X_*(A) \}$ is a basis for $C^\infty_c(M/M^0)$.

For $\lambda \in X_*(A)$ let $e^\lambda \in \mathbb{C}[X_*(A)]$ be the element associated to $\lambda$. The map $1_{m_\lambda M^0} \mapsto e^\lambda$ defines a $\mathbb{C}$-algebra isomorphism $C^\infty_c(M/M^0) \to \mathbb{C}[X_*(A)]$ since $M/M^0 \cong X_*(A)$.

2.5. **Satake Transformation.** Let $S : C^\infty_c(G/K) \to C^\infty_c(M/M^0) \cong \mathbb{C}[X_*(A)]$ be the Satake transform, defined by

$$S(f)(m) = \delta_{\mathcal{T}}(m)^{-1/2} \int_{\mathcal{T}} f(um) du,$$

where the Haar measure of $\mathcal{T}$ is normalized so that $\text{meas}(\mathcal{T} \cap K) = 1$.

The space $\mathcal{H}_K \overset{\text{def}}{=} C^\infty_c(K \backslash G/K)$ has a multiplication defined by $f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1} g) dh$, where the Haar measure is normalized so that $I$ has measure 1.

**Theorem 2.4** (Satake Isomorphism). The map $S : \mathcal{H}_K \to \mathbb{C}[M/M^0]^W$, defined by

$$S(f)(m) = \delta_{\mathcal{T}}(m)^{-1/2} \int_{\mathcal{T}} f(um) du,$$

is an isomorphism of $\mathbb{C}$-algebras.

(For details see Cartier [4] Theorem 4.1, page 147.)

A direct calculation shows that $S : C^\infty_c(G/K) \to C^\infty_c(\mathcal{T} \backslash G/K) \cong \mathbb{C}[M/M^0]$ is an $\mathcal{H}_K$-module homomorphism where $\mathcal{H}_K$ acts on $\mathbb{C}[M/M^0]$ through the Satake isomorphism.
2.6. Universal Principal Series. Our presentation of the Casselman-Shalika formula will utilize the universal principal series, which we now introduce. For additional details see the survey article of Haines-Kottwitz-Prasad [8]. For the remainder of the paper let $R = \mathbb{C}[M/M'] \cong \mathbb{C}[X_s(A)]$ (where the isomorphism is induced via $\nu$) and define $\chi_{\text{univ}} : M \rightarrow R^\times$ to be the tautological character. The universal principal series is defined to be

\[ i_{G,\mathfrak{p}}(\chi_{\text{univ}}^{-1}) = \text{Ind}_{\mathfrak{p}}^{G}((\mathfrak{p}/\mathfrak{p}) \otimes \chi_{\text{univ}}^{-1}) = \{ f \in C^\infty(G, R) | f(mug) = (A_{\mathfrak{p}}^{1/2} \cdot \chi_{\text{univ}})(m)f(g) \}. \tag{11} \]

The space $i_{G,\mathfrak{p}}(\chi_{\text{univ}}^{-1})$ is an $(R, G)$-bimodule, where $r \in R$ acts on $f \in i_{G,\mathfrak{p}}(\chi_{\text{univ}}^{-1})$ via $(r \cdot f)(g) = r \cdot (f(g))$; $G$ acts via right translation.

Let $v_0 \in i_{G,\mathfrak{p}}(\chi_{\text{univ}}^{-1})^K$ be the spherical vector normalized such that $v_0(1) = 1$.

It will be convenient to extend the scalars of the universal principal series. Let $S \supseteq R$ be a commutative $\mathbb{C}$-algebra that is an integral domain. Write $i_s : R \rightarrow S$ for the inclusion of $R$ into $S$.

**Lemma 2.5.** The map $\varpi : S \otimes_R i_{G,\mathfrak{p}}(\chi_{\text{univ}}^{-1}) \rightarrow i_{G,\mathfrak{p}}(i_s \circ \chi_{\text{univ}})$ defined by $s \otimes f(g) \mapsto s i_s(f(g))$ is an isomorphism of $(S, G)$-bimodules.

**Proof:** This follows from smoothness and because $S$ is an integral domain. \hfill \Box

2.7. Relating Universal Principal Series. In this subsection we describe two situations in which we can relate the universal principal series on two groups. First we discuss the case where $G'' = G \times_{\text{Spec}(F)} T$, where $G$ is a connected unramified group and $T$ is a torus. The group scheme $G''$ has a maximal split torus $A'' = A \times_{\text{Spec}(F)} T$ contained in the minimal parabolic subgroup $P'' = P \times_{\text{Spec}(F)} T$ with Levi subgroup $M'' = M \times_{\text{Spec}(F)} T$.

Let $R'' = \mathbb{C}[M''/M'']$ and let $\xi : T \rightarrow \mathbb{C}[T/T^\circ]^\times$ be the tautological character of $T$. Note that the tautological character $\chi_{\text{univ}}'' : M'' \rightarrow R'' \cong R \otimes_{\mathbb{C}} \mathbb{C}[T/T^\circ]$ of the Levi subgroup $M''$ is given by $\chi_{\text{univ}}''((m, t)) = \chi_{\text{univ}}(m) \otimes \xi(t)$.

**Lemma 2.6.** With the notation above, the map $\tau : R'' \otimes_R i_{G,\mathfrak{p}}(\chi_{\text{univ}}^{-1}) \rightarrow i_{G'',\mathfrak{p}''}(\chi_{\text{univ}}'')^{-1}$ defined by $\chi \otimes f \mapsto ((g, t) \mapsto \chi^{-1}(t)f(g))$ is an isomorphism of $(R'', G'')$-bimodules.

**Proof:** This is a consequence of smoothness. \hfill \Box

**Remark:** Recall that the right regular action of $T$ on $\mathbb{C}[T/T^\circ]$ when its elements are viewed as functions corresponds to multiplication by $\xi^{-1}$ when $\mathbb{C}[T/T^\circ]$ is viewed as a group algebra.

Second, we consider the case of two connected unramified groups $G''$ and $G'$ and a map $\pi : G' \rightarrow G''$ that is an open algebraic group homomorphism with finite central kernel such that $P'' \text{Im}(\pi) = G''$. In this case, we take $P' = \pi^{-1}(P'')$ as our minimal parabolic subgroup of $G'$. This implies that $\pi(M') \subseteq M''$, $\pi(A') \subseteq A''$, and $\pi(U') \subseteq U''$.

Let $R'' = \mathbb{C}[X_s(A'')]$ and $R' = \mathbb{C}[X_s(A')]$. Then we have the following maps:

\[ \pi : X_s(A') \rightarrow X_s(A'') \quad \text{(ker } \pi \text{ finite implies injectivity)}, \tag{12} \]

\[ \pi : i_{G'',\mathfrak{p}''}((\chi_{\text{univ}}'')^{-1}) \rightarrow i_{G',\mathfrak{p}'}(\pi \circ (\chi_{\text{univ}}')^{-1}), \tag{13} \]
where $\pi^*(f)(g') = f(\pi(g'))$. The map $\pi^*$ is well-defined because
\[\pi^*(f)(m'g') = (\delta_{\mathcal{T}}^{-1/2} \chi_{\text{univ}}^{-1})(\pi(m'))\pi^*(f)(g'),\]
$$(\chi_{\text{univ}}^{-1}) \circ \pi_* = \pi_\star \circ (\chi_{\text{univ}}^{-1}),$$
and $\delta_{\mathcal{T}}(\pi(m')) = \delta_{\mathcal{T}}(m')$. Furthermore, if $g'' = \pi(g')$, then
$$\pi^*(\varrho(g'')f) = \varrho(g')\pi^*(f). \quad (14)$$

**Lemma 2.7.** The map $\pi^*: \imath_{\mathcal{T}'(\chi_{\text{univ}}^{-1})} \to \imath_{\mathcal{T}'(\chi_{\text{univ}}^{-1})}$ is an isomorphism of $(R'', G')$-bimodules.

**Proof:** We show that the map is a bijection. The rest follows directly from definitions. Since $P''\imath(\pi) = G''$, the map $\pi^*$ is injective.

Now we show that $\pi^*$ is surjective. Let $x' \in G'$ and $K' \subset G'$ be an open compact subgroup such that $K' \cap (x')^{-1} P' x' \subset (x')^{-1} (M')^0 U' x'$. Define $\phi_{x', K'}$ to be the function supported on $P' x' K'$ such that $\phi(m' u' x' k') = \pi_* (\chi_{\text{univ}}^{-1}) (m')$, where $m' \in M'$, $u' \in U'$, $k' \in K'$. The module $i_{\mathcal{T}', \mathcal{T}'}(\pi_* (\chi_{\text{univ}}^{-1}))$ is the $R''$-span of the functions $\phi_{x', K'}$.

Similarly, we can define $\phi_{\pi(x'), \pi(K')} \in i_{\mathcal{T}', \mathcal{T}'}((\chi_{\text{univ}}^{-1}))$. (Since $\pi$ is an open map, $\pi(K')$ is a compact open subgroup of $G''$. By construction $\pi^*(\phi_{\pi(x'), \pi(K')}) = \phi_{x', K'}$. Thus $\pi^*$ is surjective.

We combine Lemmas 2.5 (with $S = R''$) and 2.7 to get the following.

**Lemma 2.8.** The map $\varpi^{-1} \circ \pi^*: i_{\mathcal{T}', \mathcal{T}'}((\chi_{\text{univ}}^{-1})) \to R'' \otimes R' i_{\mathcal{T}', \mathcal{T}'}((\chi_{\text{univ}}^{-1}))$ is an isomorphism of $(R'', G')$-bimodules.

We use the above results in the following setting. Let $G'$ be a connected unramified group. Define the torus $T = G'/\mathcal{D}G'$ and the connected semisimple adjoint group $G = G'/Z(G')$ and let $\pi: G' \to G'' = G \times_{\text{Spec}(F)} T$ be the natural map induced by the quotient maps. The map $\pi$ induces a map on $F$-points $\pi: G' \to G \times T$ such that $\pi$ is an open algebraic group homomorphism with finite central kernel such that $(P \times T)\imath(\pi) = G \times T$. In this case, we will take $S = R''$.

### 2.8. Whittaker Function

Let $\Psi : \mathcal{U} \to \mathbb{C}^\times$ be a smooth character. The character $\Psi$ factors through $\mathcal{U}/[\mathcal{U}, \mathcal{U}] \cong \prod_{\alpha \notin \Delta} U_\alpha / U_{2\alpha}$, where $U_{2\alpha} = \{1\}$ if $2\alpha \notin \Phi$. Thus to define $\Psi$ it suffices to define smooth characters $\Psi_\alpha : U_\alpha \to \mathbb{C}^\times$ for all $\alpha \in -\Delta$ and set $\Psi = \prod_{\alpha \in -\Delta} \Psi_\alpha$. We say that $\Psi$ is nondegenerate if $\Psi_{|U_\alpha}$ is non trivial for all $\alpha \in -\Delta$. We write $\overline{\Psi}$ for the complex conjugate of $\Psi$.

We are interested in $R$-valued Whittaker functionals, but it will be convenient to allow for extension of scalars. Again we let $S \supseteq R$ be a $\mathbb{C}$-algebra that is an integral domain with an inclusion $i_S: R \to S$. For any abelian group $J$, define $J_\Psi$ to be the abelian group $J$ with an action of $\mathcal{U}$ given by $u \cdot j = \Psi(u)j$. Let
$$i_{\mathcal{T}', \mathcal{T}'}(i_S \circ \chi_{\text{univ}})^{-1} | \mathcal{U}/\Psi = i_{\mathcal{T}', \mathcal{T}'}(i_S \circ \chi_{\text{univ}})^{-1} / \text{span}_{S}(u \cdot f - \Psi(u)f | u \in \mathcal{U}, f \in i_{\mathcal{T}', \mathcal{T}'}(i_S \circ \chi_{\text{univ}}))$$
be the $\Psi$-twisted Jacquet module of $i_{G,T}(i_S \circ \chi_{\text{univ}}^{-1})$. An $S$-valued Whittaker functional is an element $\mathcal{W} \in \text{Hom}_{(S,T)}(i_{G,T}(i_S \circ \chi_{\text{univ}}^{-1}), S_\Psi)$. Note that for any $S$-module $S'$ we have
\[ \text{Hom}_{(S,T)}(i_{G,T}(i_S \circ \chi_{\text{univ}}^{-1}), S'_\Psi) \cong \text{Hom}_{S}(i_{G,T}(i_S \circ \chi_{\text{univ}}^{-1}(T,\Psi)), S') \] (15) as $S'$-modules.

**Lemma 2.9.** The natural map
\[ \text{Hom}_{(S,T)}(S \otimes_R i_{G,T}(\chi_{\text{univ}}^{-1}), S_\Psi) \rightarrow \text{Hom}_{(R,T)}(i_{G,T}(\chi_{\text{univ}}^{-1}), S_\Psi) \] defined by $\phi \mapsto \phi\vert_{i_{G,T}(\chi_{\text{univ}}^{-1})}$ is an isomorphism of $S$-modules.

**Proof:** The inverse map is defined by $\phi' \mapsto (s \otimes f \mapsto s\phi'(f))$. \qed

The $S$-valued spherical Whittaker function associated to $\mathcal{W}$ is defined to be $\mathcal{W}(g) = \mathcal{W}(g v_0)$. The Iwasawa decomposition $G = \overline{UMK}$ shows that $\mathcal{W}$ is determined by its values on $M/M^o \cong A/A^o$. Note that $\mathcal{W} \in \text{Ind}_{T}^{G}(S_\Psi)$. The Casselman-Shalika formula is a formula for the function $\mathcal{W}$ evaluated at points in $A/A^o$.

**2.9. Dual Group.** In this subsection we introduce a complex group whose characters will appear in the Casselman-Shalika formula. Let $\mathcal{X} = X_\ast(A) + A^\vee \subseteq X_\ast(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathcal{Y} = \text{Hom}_{\mathbb{Z}}(\mathcal{X}, \mathbb{Z})$. Let $\Phi_{\text{nd}}$ be the set of non-divisible roots in $\Phi$ and let $(\Phi_{\text{nd}})^\vee$ be the set of coroots of $\Phi_{\text{nd}}$.

**Proposition 2.10.** The quadruple $(\mathcal{X}, (\Phi_{\text{nd}})^\vee, \mathcal{Y}, \Phi_{\text{nd}})$ defines a root datum.

**Proof:** The only point that may not be clear is that $\Phi_{\text{nd}} \subseteq \mathcal{Y}$. Let $\alpha \in \Phi_{\text{nd}}$. Then by definition $\alpha \in \text{Hom}_{\mathbb{Z}}(X_\ast(A), \mathbb{Z})$. Thus $\alpha$ extends uniquely to an element of $\text{Hom}_{\mathbb{Q}}(X_\ast(A) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$. Since $\mathcal{X} \subseteq X_\ast(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\alpha \in \text{Hom}_{\mathbb{Z}}(A^\vee, \mathbb{Z})$ it follows that $\alpha \in \text{Hom}_{\mathbb{Z}}(\mathcal{X}, \mathbb{Z})$. \qed

Let $L^G$ be the connected complex reductive group with root datum $(\mathcal{X}, (\Phi_{\text{nd}})^\vee, \mathcal{Y}, \Phi_{\text{nd}})$. Let $\rho^\vee$ be half the sum of the positive roots (with respect to $P$) in $(\Phi_{\text{nd}})^\vee$. Note that $\rho^\vee \in \mathcal{X}$.

**Examples of $L^G$:**

1. If $G$ is a semisimple split group, then $\mathcal{X}$ is the coweight lattice of $G$ and $\Phi_{\text{nd}} = \Phi$. Thus $L^G$ is the simply connected cover of the Langlands dual group of $G$.

2. Let $E/F$ be an unramified quadratic extension. Let $G = SU(2n + 1, E/F)$ be the unramified quasi-split special unitary group over $F$ associated to the Hermitian form $\left( \begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix} \right)$ on the vector space $E^{2n+1}$. In this case, $L^G = \text{Sp}(2n, \mathbb{C})$.

Let $L^A = X_\ast(A) \otimes \mathbb{C}^\times$ and let $L^A$ be a maximal torus of $L^G$. Since $L^A \cong \mathcal{Y} \otimes \mathbb{C}^\times$, the restriction map $\mathcal{Y} \rightarrow X_\ast(A)$ induces an isogeny $L^A \rightarrow L^A$ of tori. In addition we have the inclusion $X_\ast(A) \hookrightarrow \mathcal{X}$. 
For $H$ a complex reductive group, let $\text{Rep}(H)$ denote the $\mathbb{C}$-algebra of characters of finite dimensional algebraic representations of $H$.

**Lemma 2.11.** The inclusion $X^* (L \mathcal{A}) = X^*_s (\mathcal{A}) \xrightarrow{\text{inc}} \mathscr{X}$ induces an inclusion $\mathbb{C} [L \mathcal{A} / W] \hookrightarrow \mathbb{C} [L \mathcal{A}^\dagger / W] \cong \text{Rep}(L \mathcal{G}^\dagger)$.

**Proof:** $L \mathcal{G}^\dagger$ is a connected reductive group over $\mathbb{C}$, thus $\text{Rep}(L \mathcal{G}^\dagger) \cong \mathbb{C} [L \mathcal{A}^\dagger / W]$. ∎

For any $\lambda \in X^*_s (L \mathcal{A})^+$ define $A_\lambda$ to be the element of $\mathcal{H}_K$ that corresponds to $\text{ch} V_{\text{inc}(\lambda)}$ under the inclusion $\mathcal{H}_K \xrightarrow{\text{inc}} \mathbb{C} [L \mathcal{A} / W] \hookrightarrow \text{Rep}(L \mathcal{G}^\dagger)$. Later we will suppress $\text{inc}$ in our notation.

The following coefficients play an important role in our proof of the Casselman-Shalika formula. Given $\lambda \in \mathscr{X}^+$ and $\mu, \eta \in \mathscr{X}^+$ define $c_{\lambda, \mu}^\eta$ so that

$$\text{ch} V_\lambda \cdot \text{ch} V_{\mu - \rho^\vee} = \sum_\eta c_{\lambda, \mu}^\eta \text{ch} V_{\eta - \rho^\vee}.$$  \hfill (17)

Recall that if $c_{\lambda, \mu}^\eta \neq 0$, then $\lambda + \mu - \eta \in \text{span}_{\mathbb{Z}}((\Phi^\text{nd})^\vee)$.

3. Iwahori-Hecke Algebra

In this section we recall the Bernstein presentation of the Iwahori-Hecke algebra $\mathcal{H} \overset{\text{def}}{=} \mathcal{H}_I = C^\infty_c (I \backslash G / I)$ following the presentation of Rostami [12]. We begin with some preliminaries. The space $\mathcal{H}$ has a multiplication defined by $f_1 \ast f_2 (g) = \int_G f_1 (h) f_2 (h^{-1} g) dh$ and the Haar measure is normalized so that $I$ has measure 1. Let $1_X$ be the characteristic function of the set $X \subseteq G$, and for $g \in G$ let $T_g = 1_{1_g} \in \mathcal{H}$. It is convenient to normalize $T_g$, so let $q (g) \overset{\text{def}}{=} [I g I : I]$ and define $T_g \overset{\text{def}}{=} q (g)^{-1 / 2} T_g$.

Next we introduce the elements of Bernstein’s commutative subalgebra. For $\lambda \in X^*_s (\mathcal{A})$ we choose $\lambda_1, \lambda_2 \in X^*_s (\mathcal{A})^+$ such that $\lambda = \lambda_1 - \lambda_2$ and define

$$\theta_\lambda \overset{\text{def}}{=} T_{\lambda_1} \ast T_{\lambda_2}^{-1}.$$  \hfill (18)

The definition of $\theta_\lambda$ depends neither on our choice of the $\lambda_j$ nor the splitting $s$ of the exact sequence on line (7). The invertibility of $T_{\lambda_2}$ follows from the first part of Theorem 3.1; the elements $\theta_\lambda$ do not appear until the second part of that theorem. The elements $\theta_\lambda$ generate a commutative $\mathbb{C}$-algebra $\mathcal{A}$, which is isomorphic to $\mathbb{C} [X^*_s (\mathcal{A})]$.

Let $\tilde{W} = N / M^\circ$ be the extended affine Weyl group. It acts on $\mathscr{A}$ (Tits [15], Section 1.2) and the stabilizer of $x_0$ is isomorphic to $W$, the Weyl group of the relative root system $\Phi$. For each $w \in \tilde{W}$ we will choose a representative $n_w \in N \cap \text{Stab}_G (x_0) = N \cap K$. We write $T_w$ in place of $T_{n_w}$. Note that $T_w$ is independent of the choice of $n_w$, because $M^\circ \subseteq I$.

If $H$ is the group of $F$-points of a connected reductive group scheme, then we let $\Omega_H$ be the image of the Kottwitz homomorphism ([9], Sections 7.1-7.4). Now we can state the Iwahori-Matsumoto presentation and the Bernstein presentation for $\mathcal{H}$. (Note that because $G$ is quasi-split, $M$ is a torus and $\Omega_M = X^*_s (\mathcal{A})$.)
Theorem 3.1 (Rostami [12]; Vigneras [16]). The product of any two basis elements $T_w$ is determined by the relations

- $T_w \cdot T_{w'} = T_{ww'}$, for any $w, w' \in \hat{W}$, such that $\ell(ww') = \ell(w) + \ell(w')$;
- $T_s \cdot T_s = (q(s) - 1)T_s + q(s)$, for any simple affine reflection $s \in \Delta_{\text{aff}}$;
- $T_w \cdot T_{\tau} = T_{ww} \cdot T_{\tau} = T_{\tau} \cdot T_{\tau^{-1}w}$, for any $w \in \hat{W}, \tau \in \Omega_G$.

The set $\{T_w \cdot \theta_\lambda | \lambda \in \Omega_M, w \in \hat{W}\}$ is a $\mathbb{C}$-linear basis for the Iwahori-Hecke algebra $H$, and the product of two basis elements is determined by the relations:

(Add) $\theta_\lambda \cdot \theta_\mu = \theta_{\lambda + \mu}$, for any $\lambda, \mu \in \Omega_M$.

If $\alpha \in \Delta$ and $\lambda \in \Omega_M$, then

\[(BR) \ T_{\alpha} \cdot \theta_\lambda = \begin{cases} 
\theta_{\alpha}(\lambda) \cdot T_{\alpha} + \sum_{j=0}^{<\alpha, \lambda>-1} q_j(\alpha) \theta_{\lambda - j\alpha^\vee} & \text{if } \langle \alpha, \lambda \rangle > 0; \\
\theta_{\alpha}(\lambda) \cdot T_{\alpha} - \sum_{j=0}^{<\alpha, \lambda>-1} q_j(\alpha) \theta_{\lambda - j\alpha^\vee} & \text{if } \langle \alpha, \lambda \rangle < 0.
\end{cases}\]

(We will not need the definition of $q_j(s)$, which can be found in Rostami [12], Section 5.4.)

Remark: The Bernstein relations (BR) above include a minor correction to what appears in Rostami [12]. Namely Rostami’s formula only holds in the case $\langle \alpha, \lambda \rangle \geq 0$. Fortunately, the case $\langle \alpha, \lambda \rangle < 0$ can be deduced directly from the case $\langle \alpha, -\lambda \rangle > 0$.

One consequence of Theorem 3.1 is that the intermediate algebra $H_{IK} \overset{\text{def}}{=} C^\infty_c(\Gamma \backslash G/K)$ has a nice basis. Since $K = \bigcup_{w \in \hat{W}} IwI$, the characteristic function of $K$ can be expressed as $1_K = \sum_{w \in \hat{W}} T_w$. This identity, Theorem 3.1, and the fact that $f(g) \mapsto f(g^{-1})$ defines an anti-isomorphism of $H$ yield the following corollary.

Corollary 3.2. The Hecke algebra $H_{IK}$ has a basis consisting of the elements $\theta^K_\lambda \overset{\text{def}}{=} \theta_\lambda * 1_K$, where $\lambda \in X_s(A)$.

We end this section with an identity that is crucial for the proof of Proposition 5.4. It follows directly from the Bernstein relations.

Corollary 3.3. Let $\alpha \in \Delta$ and let $s$ be the simple reflection associated to $\alpha$. Let $\lambda \in X_s(A)$. Then

$$T_s(\theta_\lambda + \theta_{s(\lambda)}) = (\theta_\lambda + \theta_{s(\lambda)}) T_s. \quad (19)$$

4. Savin’s Isomorphism

In this section we prove Savin’s Isomorphism ([13], Theorem 1) for connected unramified groups, which states that the Satake transform defines an $H_K$-module isomorphism from $H_{IK}$ to $C^\infty_c(M/M^0) \overset{\text{def}}{=} \mathbb{C}[X_s(A)]$. Using the results of Section 3 this amounts to a minor modification of Savin [13]. We begin with some notation and then a few basic results.

Given an equation

$$A = B, \quad (20)$$

let $\text{RHS}(20) = B$ (LHS$(20) = A$) be the right(left)-hand-side of equation $(20)$. 
**Lemma 4.1.** Let $(\pi, V)$ be a smooth $G$-module and $(\pi', V')$ a smooth $\bar{P}$-module with the trivial action of $\bar{T}$. Let $S : V \to V'$ be a map such that $S(\pi(p)v) = \delta_{\bar{P}}^{1/2}(p)\pi'(p)S(v)$ for every $p \in \bar{P}$. Then, for every $\lambda \in X_*(A)$ and $v \in V'$,

$$S(\pi(\theta_\lambda)v) = \pi'(m_\lambda)S(v). \quad (21)$$

**Proof:** To begin, we prove this for $\lambda \in X_*(A)^+$, so $\theta_\lambda = \delta_{\bar{P}}^{1/2}(m_\lambda)1_{Im_\lambda I}$. By definition

$$S(\theta_\lambda v) = \delta_{\bar{P}}^{1/2}(m_\lambda)S\left(\int_{Im_\lambda I} gv\,dg\right). \quad (22)$$

Since $v \in V'$ and $\text{meas}(I) = 1$ we have

$$\text{RHS}(22) = \delta_{\bar{P}}^{1/2}(m_\lambda)S(\sum_{\gamma \in I/Im_\lambda I} \gamma m_\lambda v). \quad (23)$$

By the Iwahori factorization (Proposition 2.3) and the fact that $m_\lambda$ contracts $\bar{T}$ (Proposition 2.1) we can assume that each $\gamma \in I/I \cap m_\lambda Im_\lambda^{-1}$ is represented by an element in $I \cap \bar{T}$. Thus

$$\text{RHS}(23) = \delta_{\bar{P}}^{1/2}(m_\lambda) \sum_{\gamma \in I/Im_\lambda I} m_\lambda S(v) = \delta_{\bar{P}}(m_\lambda)[Im_\lambda : I]m_\lambda S(v). \quad (24)$$

Since $[Im_\lambda : I] = \delta_{\bar{P}}^{-1}(m_\lambda)$ (Proposition 2.2), we get

$$\text{RHS}(24) = m_\lambda S(v). \quad (25)$$

The general result follows from this special case and is left to the reader. \qed

**Remark:** Lemma 4.1 is independent of the choice of the splitting $s$ of sequence (7) because $I \cap M = M^o$ (Cartier [4], page 140).

**Theorem 4.2** (Savin’s Isomorphism). Let $\lambda \in X_*(A)$. The Satake transform $S : C_c^\infty(G/K) \to C_c^\infty(M/M^o) \cong \mathbb{C}[X_*(A)]$ sends the element $\theta^K_\lambda$ to $e^\lambda$. Hence $S$ induces an isomorphism of left $\mathcal{H}_K \cong \mathbb{C}[X_*(A)]^W$-modules

$$\mathcal{H}_{IK} \cong \mathbb{C}[X_*(A)]. \quad (26)$$

**Proof:** First note that $S(1_K) = 1_{M^o} = e^0$. To compute $S(\theta^K_\lambda)$ we apply Lemma 4.1. In particular, $(\pi, V)$ is the $G$-representation where $V = C_c^\infty(G/K)$ and $G$ acts by left translation; $(\pi', V')$ is the $P$-representation where $V' = C_c^\infty(M/M^o) \cong X_*(A)$ and $P$ acts through the quotient $M \cong P/U$ by left translation. Lemma 4.1 states that for any $f \in V' = C_c^\infty(I\backslash G/K) = \mathcal{H}_{IK}$ we have

$$S(\theta_\lambda * f) = \pi'(m_\lambda)S(f). \quad (27)$$

In particular, for $f = 1_K$ we have
\[ S(\theta) = S(\theta \ast 1_K) = \pi'(m_\lambda)S(1_K) = \pi'(m_\lambda)1_{M^\circ} = e^\lambda. \] (28)

\section{Twisted Satake Transform}

In this section we study the structure of \( \text{ind}_{GU}^G(\Psi) \) as an \( \mathcal{H}_K \)-module for a non-degenerate character \( \Psi \) of conductor \( p \). This is accomplished in Proposition 5.6.

When we say that \( \Psi \) has ‘conductor \( p \)’, we mean that for any \( \alpha \in \Delta \) the character \( \Psi|_{U - \alpha} \) is nontrivial on \( U - \alpha \cap I_\alpha = U - \alpha \cap K \) and trivial on \( U - \alpha \cap I \). One can show that such a character exists by reducing to the case where \( G \) is a simply-connected semi-simple unramified group of rank one. (i.e. \( G \) is isomorphic to \( \text{SL}(2, L) \), where \( L \) is an unramified extension of \( F \), or \( \text{SU}(3, E/F) \), where \( E \) is an unramified quadratic extension of \( F \).)

We begin our study of \( \text{ind}_{GU}^G(\Psi) \) by constructing a geometric basis.

**Lemma 5.1.** Suppose that \( f \in \text{ind}_{GU}^G(\Psi) \). Then for any \( \mu \in X^*(A) \cap X^*(A)^{++} \) we have \( f(m_\mu) = 0 \).

**Proof:** The Iwasawa decomposition shows that it suffices to consider \( m_\mu \) such that \( \mu \in X_*(A) \). We will show that if \( \mu \in X_*(A) \) is not strictly dominant, then \( f(m_\mu) = 0 \).

In this case, there exists \( \alpha \in -\Delta \) such that \( m_\mu(U_\alpha \cap K)m_\mu^{-1} \supseteq (U_\alpha \cap K) \). Thus there exists \( u_\alpha \in U_\alpha \cap K \) such that \( \Psi(m_\mu u_\alpha m_\mu^{-1}) \neq 1 \) and
\[
f(m_\mu) = f(m_\mu u_\alpha) = \overline{\Psi}(m_\mu u_\alpha m_\mu^{-1})f(m_\mu).
\]
Thus \( f(m_\mu) = 0 \). \( \square \)

**Remark:** The proof of Lemma 5.1 uses the fact that \( \Psi \) has conductor \( p \).

Now we can construct a geometric basis for \( \text{ind}_{GU}^G(\Psi) \). Let \( \lambda \in X_*(A)^{++} \). Define
\[
\phi_\lambda(g) \overset{\text{def}}{=} \begin{cases} \delta_{\mathcal{F}}(m_\lambda)^{1/2}\overline{\Psi(u)} & \text{if } g = um_\lambda k \in \mathcal{U}m_\lambda K; \\ 0 & \text{otherwise}. \end{cases}
\]
(30)
The function \( \phi_\lambda \) is well-defined because \( \lambda \) is strictly dominant.

**Lemma 5.2.** The set \( \{ \phi_\lambda | \lambda \in X_*(A)^{++} \} \) is a basis for \( \text{ind}_{GU}^G(\Psi) \).

**Proof:** This follows from Lemma 5.1 and the Iwasawa decomposition. \( \square \)

Now we can investigate the \( \mathcal{H}_K \)-module structure of \( \text{ind}_{GU}^G(\Psi) \) by comparing it to \( \mathcal{H}_{IK} \) using the twisted Satake transform. The twisted Satake transform \( S'_\psi : C_c(G/I) \rightarrow \text{ind}_{GU}^G(\Psi) \) is defined by
\[
S'_\psi(f)(m) = \int_{U} f(um)\Psi(u)du,
\]
(31)
where the Haar measure of $\overline{U}$ is normalized so that $\text{meas}(U \cap I) = 1$. (This is not the normalization used to define the Satake transform.) A direct calculation shows that $S'_\Psi$ is a homomorphism of left $\mathcal{H}$-modules. Specifically, $S'_\Psi(\varrho(f_1) \cdot f_2) = \varrho(f_1) \cdot S'_\Psi(f_2)$, where $f_1 \in \mathcal{H}$ and $f_2 \in C_c(G/I)$.

We are primarily interested in $S'_\Psi \overset{\text{def}}{=} S'_\Psi|_{I_1K}$. First, we study the image of $S'_\Psi$.

**Lemma 5.3.** Let $\lambda \in X_s(A)^{++}$. Then $S'_\Psi(\theta^K_\lambda) = \phi_\lambda$. In particular, the twisted Satake transform $S'_\Psi : \mathcal{H}_{1K} \to (\text{ind}_{U}^{G} \Psi)^K$ is surjective.

**Proof:** By Lemma 5.1, it suffices to compute $S'_\Psi(\theta^K_\lambda)(m_\mu)$, where $\mu \in X_s(A)^{++}$. Since $\lambda$ is dominant we have $\theta^K_\lambda = \delta^{1/2}_{\overline{P}}(m_\lambda)1_{I_{m_\lambda}K}$. Thus

$$S'_\Psi(\theta^K_\lambda)(m_\mu) = \delta^{1/2}_{\overline{P}}(m_\lambda) \int_{I_{m_\lambda}K} \Psi(u)du. \quad (32)$$

For $u \in \overline{U}$, we show that $um_\mu \in I_{m_\lambda}K$ implies that $\mu = \lambda$ and $u \in I \cap \overline{U}$.

Since $\lambda$ is dominant the Iwahori factorization implies $I_{m_\lambda}K = (I \cap \overline{U})m_\lambda K$. So, if $um_\mu \in I_{m_\lambda}K \subseteq (I \cap \overline{U})m_\lambda K$, then there exists $u' \in (I \cap \overline{U})$ and $k \in K$ such that $m_\mu = u'^{-1}u'm_\lambda k$. This implies that $(m_\mu^{-1}u'^{-1}u'm_\mu)(m_\mu^{-1}m_\lambda) \in \overline{P} \cap K = (\overline{U} \cap K)(M \cap K)$. Thus $m_\mu^{-1}m_\lambda \in M \cap K$, which implies that $\lambda = \mu$; and $m_\mu^{-1}u'^{-1}u'm_\lambda \in (\overline{U} \cap K)$, which implies that $u'^{-1}u \in m_\lambda(\overline{U} \cap K)m_\lambda^{-1} \subseteq (\overline{U} \cap I)$. The containment is strict because $\lambda$ is strictly dominant. Therefore $u \in I \cap \overline{U}$.

Thus

$$S'_\Psi(\theta^K_\lambda)(m_\lambda) = \delta^{1/2}_{\overline{P}}(m_\lambda) \int_{I \cap \overline{U}} \Psi(u)du = \delta^{1/2}_{\overline{P}}(m_\lambda). \quad (33)$$

Second, we study ker $S'_\Psi$.

**Lemma 5.4.** Let $\alpha \in \Delta$, $s$ the simple reflection corresponding to $\alpha$, and $\iota_\alpha = 1_I + T_s$ (the characteristic function of the parahoric subgroup $I_\alpha$). Then

1. $S'_\Psi(\iota_\alpha)(1) = 0$.
2. $S'_\Psi(\theta^K_\lambda + \theta^K_\lambda s(\lambda)) = 0$, for all $\lambda \in X_s(A)$.

**Proof:** (1) We will prove the following claims.

i) $S'_\Psi(\iota_\alpha)(1) = 0$.

ii) If $\overline{U}_{\lambda,w} \overset{\text{def}}{=} \{u \in \overline{U} | um_\lambda n_w \in I_\alpha \} \neq \emptyset$, then $\lambda = 0$ and $w = 1$ or $s$.

iii) $S'_\Psi(\iota_\alpha)(m_\lambda n_w) = \int_{\overline{U}_{\lambda,w}} \Psi(u)du$.

First we show how these claims imply (1). Note that $S'_\Psi(\iota_\alpha)$ is determined by its values on representatives of the double cosets $\overline{U} \backslash G/I \cong \{m_\lambda, m_\lambda n_w | \lambda \in X_s(A), w \in W\}$. Thus it suffices to compute $S'_\Psi(\iota_\alpha)(m_\lambda n_w)$, where $\lambda \in X_s(A)$ and $w \in W$. By iii), $S'_\Psi(\iota_\alpha)(m_\lambda n_w) = \int_{\overline{U}_{\lambda,w}} \Psi(u)du$. So if $S'_\Psi(\iota_\alpha)(m_\lambda n_w) \neq 0$, then $\overline{U}_{\lambda,w} \neq \emptyset$. In this case, ii) implies that $\lambda = 0$ and $w = 1$ or $s$, so $m_\lambda n_w \in I_\alpha$. Therefore $S'_\Psi(\iota_\alpha)(m_\lambda n_w) = S'_\Psi(\iota_\alpha)(1)$, which is zero by i).
Now we prove the claims. By definition $S_{\Psi}(\iota_\alpha)(1) = \int_{\mathcal{U} \cap I_{\alpha}} \Psi(u)du = 0$, because $\Psi$ is nontrivial on $\mathcal{U} \cap I_{\alpha}$. This proves item i). Item iii) follows from the definition of $\mathcal{U}_{\lambda,w}$.

Finally we will prove item ii). Since $I_{\alpha} \subseteq K$ and $n_w \in K$ we see that if $u \in \mathcal{U}_{\lambda,w}$, then $um_\lambda \in \mathcal{U} \cap K = (\mathcal{U} \cap K)(M \cap K)$. Thus $u \in \mathcal{U} \cap K$ and $m_\lambda \in M \cap K$, which implies that $m_\lambda = 1$. Thus it suffices to determine $\mathcal{U}_{0,w}$.

By Tits [15] Section 3.4 there is a group scheme $\mathcal{G}$ defined over $\mathcal{O}$ such that $\mathcal{G} \times \mathcal{O} F = \mathcal{G}$ and $\mathcal{G}(\mathcal{O}) = K$. Let $\mathcal{G}$ be the group scheme over $\mathcal{O} = \mathcal{O}/p$ defined by reduction mod $p$. Since $u, w \in K = \mathcal{G}(\mathcal{O})$ we apply the canonical homomorphism $\mathcal{O} \rightarrow \mathcal{O}$ and work in the group $\mathcal{G}(\mathcal{O})$. We will suppress the underline to avoid clutter.

Let $P_\alpha(\kappa)$ be the parabolic subgroup of $G(\kappa)$ that is the image of $I_{\alpha}$ under the mod $p$ reduction map. We have $uw \in \mathcal{U}(\kappa)w \cap P_\alpha(\kappa)$. But $\mathcal{U}(\kappa)w \subset \overline{P}_\alpha(\kappa)wU(\kappa)$, and $P_\alpha(\kappa) \subseteq \overline{P}_\alpha(\kappa)U(\kappa)$. ($P(\kappa) = M(\kappa)U(\kappa) \subseteq \overline{P}(\kappa)U(\kappa)$, and $U(\kappa)w, P(\kappa) \subseteq \overline{P}_\alpha(\kappa)U(\kappa)$.) The intersection $[\overline{P}_\alpha(\kappa)wU(\kappa)] \cap \overline{P}_\alpha(\kappa)U(\kappa)$ is nonempty if and only if $w = 1$ or $s$, by the Bruhat decomposition. This completes the proof of item ii).

(2) By definition

$$\theta^K_\lambda + \theta^K_{s(\lambda)} = (\theta_\lambda + \theta_{s(\lambda)}) \ast 1_K = \frac{1}{[I_{\alpha} : I]}(\theta_\lambda + \theta_{s(\lambda)}) \ast \iota_\alpha \ast 1_K. \quad (34)$$

Since $\iota_\alpha = 1_I + T_s$, Corollary 3.3 implies

$$\text{RHS}(34) = \frac{1}{[I_{\alpha} : I]} \iota_\alpha \ast (\theta_\lambda + \theta_{s(\lambda)}) \ast 1_K = \frac{1}{[I_{\alpha} : I]} \iota_\alpha \ast (\theta^K_\lambda + \theta^K_{s(\lambda)}). \quad (35)$$

We apply the $H$-module homomorphism $S_{\Psi}$ to equations (34) and (35) to get

$$S_{\Psi}(\theta^K_\lambda + \theta^K_{s(\lambda)}) = \frac{1}{[I_{\alpha} : I]} S'_{\Psi}(\iota_\alpha) \ast (\theta^K_\lambda + \theta^K_{s(\lambda)}), \quad (36)$$

which is zero by item (1).

Remarks: a) We provide another proof of claim ii) above, when $\lambda = 0$. Specifically, if $u \in \mathcal{U} \cap K$, $w \in W$, and $uw \in I_{\alpha}$, then $w = 1$ or $s$. This approach will utilize the building $\mathcal{B}$. Let $\mathcal{F}_\alpha$ be the facet associated to $I_{\alpha}$. Then we know that $I_{\alpha}$ fixes $\mathcal{F}_\alpha$ pointwise. Since $w^{-1}u^{-1} \in I_{\alpha}$ it must fix $\mathcal{F}_\alpha$. We also know that $w$ maps $\mathcal{A}$ into itself. Thus $u^{-1} \mathcal{F}_\alpha$ must be in $\mathcal{A}$. Since $u^{-1}$ acts via an isometry on $\mathcal{B}$ and $u^{-1}$ fixes $-\mathcal{C}$ we claim that $u^{-1}$ must fix $\mathcal{F}_\alpha$. To see this we will use the following fact. Let $\mathcal{E}$ be a Euclidean space with distance function $d_{\mathcal{E}}$. For any $y \in \mathcal{E}$ and $d \in \mathbb{R}_{\geq 0}$ let $S(y,d) = \{x \in \mathcal{E} | d_{\mathcal{E}}(x,y) = d\}$ be the sphere centered at $y$ or radius $d$. Given $x_0 \in \mathcal{E}$ and $y \in \mathcal{E}$ a set with nonempty interior, we have $\cap_{y \in Y} S(y, d_{\mathcal{E}}(x_0,y)) = \{x_0\}$. The apartment $\mathcal{A}$ is a Euclidean space containing the set $-\mathcal{C}$, which has a nonempty interior. If $x_0 \in \mathcal{F}_\alpha$ then $u^{-1}x_0 \in \cap_{y \in -\mathcal{C}} S(y, d_{\mathcal{E}}(x_0,y)) = \{x_0\}$, since for all $y \in -\mathcal{C}$, we have $u^{-1}y = y$. This implies that $w$ fixes $\mathcal{F}_\alpha$. Thus $w = 1, s$.

b) Our proof of Lemma 5.4, specifically the use of the parahoric subgroup $I_{\alpha}$, forces our choice of a character $\Psi$ of conductor $p$. It seems that a different integral operator in place of $S_{\Psi}$ could be used to circumvent this restriction. Unfortunately this introduces other
complications and will not be pursued here.

Lemmas 5.3 and 5.4 directly imply the following corollary.

**Corollary 5.5.** The kernel of the $H_K$-module homomorphism $S_{\psi} : H_{IK} \to \text{ind}^G_U(\Psi)^K$ is

$$\ker(S_{\psi}) = \text{span}\{\theta^K_{w\lambda} - (-1)^{\ell(w)}\theta^K_{w\lambda}| \lambda \in X_*(A), w \in W\}.$$  \hfill (37)

Now we introduce the fundamental diagram of $H_K$-modules.

$$H_{IK} \xrightarrow{S_{\psi}} \text{ind}^G_U(\Psi)^K \xrightarrow{S} \mathbb{C}[X_*(A)] \xrightarrow{\text{alt}} \text{alt}(\mathbb{C}[X_*(A)])$$  \hfill (38)

In the next proposition we define $j$ and show that it is an isomorphism of right $H_K$-modules.

**Proposition 5.6.** $S(ker(S_{\psi})) = ker(\text{alt})$. Thus we can define the $H_K$-module isomorphism $j : \text{ind}^G_U(\Psi)^K \to \text{alt}(\mathbb{C}[X_*(A)])$ by the formula $j(\phi) = \text{alt}(S(f))$, where $f \in H_{IK}$ such that $S_{\psi}(f) = \phi$. In particular, for $\mu \in X_*(A)^{++}$ we have $j(\phi_\mu) = \text{alt}(e^{\mu})$.

**Proof:** This follows from Theorem 4.2, Lemma 5.3, and Corollary 5.5. \hfill $\square$

### 6. Unramified Whittaker Function

Recall that $R = \mathbb{C}[X_*(A)]$ and $S \supseteq R$ is a $\mathbb{C}$-algebra that is an integral domain. In this section we prove that the $S$-valued Whittaker function $W$ (introduced in Subsection 2.8) satisfies a family of recursions, Proposition 6.5.

Let $\langle -, - \rangle = \langle -, -, - \rangle_{S, \psi} : \text{ind}^G_U(\Psi) \otimes_{\mathbb{C}} \text{Ind}^G_U(S_{\psi}) \to S$ be the bilinear form defined by $\langle \phi_1, \phi_2 \rangle = \int_{U \setminus G} \phi_1(g)\phi_2(g)dg$, where the measure on $U \setminus G$ is induced from the Haar measure on $G$ such that $\text{meas}(K) = 1$. Note that this pairing is $G$-invariant.

Now we establish a few basic properties of $\langle -, W \rangle$.

**Lemma 6.1.** Let $\lambda \in X_*(A)^{++}$ and let $f \in \text{Ind}^G_U(S_{\psi})^K$. Then

$$\langle \phi_\lambda, f \rangle = \delta^{-1/2}(m_\lambda)f(m_\lambda).$$  \hfill (39)

**Proof:** Let $F \in C_c^\infty(G/K, S)$ such that $\int_U F(ug)du = \phi_\lambda(g)f(g)$. Then

$$\langle \phi_\lambda, f \rangle \overset{\text{def}}{=} \int_{U \setminus G} \phi_\lambda(g)f(g)dg = \int_G F(g)dg.$$  \hfill (40)

By the Iwasawa decomposition and the right $K$-invariance of $F$,

$$\text{RHS}(40) = \int_{\mathbb{T}} \int_K F(pk)d\ell_k dp = \int_{\mathbb{T}} F(p)d\ell_p.$$  \hfill (41)

By equation (9) in Cartier [4], page 145,

$$\text{RHS}(41) = \int_M \int_{\mathbb{T}} \delta^{-1}(m)F(um)dudm.$$  \hfill (42)
By integrating over $U$ and applying the definition of $\phi_\lambda$ we have
\begin{equation}
\text{RHS}(42) = \int_M \delta_U(m)^{-1} \phi_\lambda(m)f(m)dm = \delta_U^{-1/2}(m_\lambda)f(m_\lambda).
\end{equation}
\(\blacksquare\)

**Lemma 6.2.** Let $\phi \in \text{ind}_{L \Gamma}^G(\overline{\Psi})^K$ and $f \in \mathcal{H}_K$. Then
\begin{equation}
\langle \phi(f) \cdot \phi, W \rangle = S(\hat{f})\langle \phi, W \rangle.
\end{equation}

**Proof:** The $G$-invariance of the pairing implies that $\langle \phi(f) \cdot \phi, W \rangle = \langle \phi, \phi(f) \cdot W \rangle$. The result follows from the identity $\phi(f) \cdot W = S(\hat{f})W$. \(\blacksquare\)

**Lemma 6.3.** Let $\lambda \in X_s(A)^+$ and let $f \in \mathcal{H}_{IK}$. Then
\begin{equation}
\langle S_\Psi(f \ast A_\lambda), W \rangle = \text{ch}V_\lambda \langle S_\Psi(f), W \rangle.
\end{equation}
In particular, if $\mu \in X_s(A)^{++}$ and $f = \theta^K_\mu$, then
\begin{equation}
\langle S_\Psi(\theta^K_\mu \ast A_\lambda), W \rangle = \delta_U^{-1/2}(m_\mu)\text{ch}V_\lambda W(m_\mu)
\end{equation}
\(\blacksquare\)

Let $\lambda \in X_s(A)^+$ and $\mu \in X_s(A)^{++}$. Now we will describe an explicit formula for $\phi_\mu \ast A_\lambda$ in terms of characters of finite dimensional representations of $L \Gamma$. \(\blacksquare\)

**Lemma 6.4.** Let $\lambda \in X_s(A)^+$ and $\mu \in X_s(A)^{++}$. Let $c_{\mu,\lambda}^\eta \in \mathbb{C}$ be defined as in subsection 2.9. Then
\begin{equation}
\phi_\mu \ast A_\lambda = \sum_{\eta \in X_s(A)^{++}} c_{\mu,\lambda}^\eta \phi_\eta.
\end{equation}

**Proof:** Since $j$ is an isomorphism and $\text{alt} \circ S = j \circ S_\Psi$ it suffices to show that $\text{alt} \circ S(\theta^K_\mu \ast A_\lambda) = \text{alt}(\sum_{\eta} c_{\mu,\lambda}^\eta \theta^K_\eta)$. By Theorem 4.2, we have
\begin{equation}
\text{alt} \circ S(\theta^K_\mu \ast A_\lambda) = \text{ch}V_\lambda \cdot \text{alt}(e^K_\mu).
\end{equation}
We multiply and divide by $\text{alt}(e^{\rho^K_\lambda})$ and apply the Weyl character formula with respect to the group $L \Gamma$, specifically $\text{alt}(e^K_\mu) = \text{ch}V_{\mu - \rho^K} \cdot \text{alt}(e^{\rho^K_\lambda})$, (temporarily working in the field of fractions of $\mathbb{C}[\mathcal{X}]$) to get
\begin{equation}
\text{RHS}(48) = \text{ch}V_{\mu - \rho^K} \cdot \text{ch}V_\lambda \cdot \text{alt}(e^{\rho^K_\lambda})
\end{equation}
Next we apply the identity \( \text{ch} V_{\mu - \rho^\vee} \cdot \text{ch} V_{\lambda} = \sum_{\eta} c^\eta_{\mu,\lambda} \text{ch} V_{\eta - \rho^\vee} \) followed by the Weyl character formula to get

\[
\text{RHS}(49) = \sum_{\eta} c^\eta_{\mu,\lambda} \text{ch} V_{\eta - \rho^\vee} \cdot \text{alt}(e^{\rho^\vee}) = \sum_{\eta} c^\eta_{\mu,\lambda} \text{alt}(e^{\eta}).
\] (50)

Again by Theorem 4.2, we have

\[
\text{RHS}(50) = \text{alt} \circ \mathcal{S} \left( \sum_{\eta} c^\eta_{\mu,\lambda} \theta^K_{\eta} \right).
\] (51)

**Remark:** If \( \rho^\vee \in X_*(\mathbb{A}) \), then Lemma 6.4 implies \( \text{ind}^G_{\mathbb{G}(\Psi)} K \cong \mathcal{H}_K \cdot \phi_{\rho^\vee} \).

We conclude this section with a family of recursions for \( \mathcal{W} \).

**Proposition 6.5.** Let \( \lambda \in X_*(\mathbb{A})^+ \) and \( \mu \in X_*(\mathbb{A})^{++} \). Define \( c^\eta_{\mu,\lambda} \in \mathbb{C} \) be such that

\[
\text{ch} V_{\mu - \rho^\vee} \cdot \text{ch} V_{\lambda} = \sum_{\eta} c^\eta_{\mu,\lambda} \text{ch} V_{\eta - \rho^\vee}. \] (Recall Subsection 2.9.) Then

\[
\delta_{\mathfrak{p}}^{-1/2}(m_{\mu}) \text{ch} V_{\lambda} \cdot \mathcal{W}(m_{\mu}) = \sum_{\eta \in X_*(\mathbb{A})^{++}} c^\eta_{\mu,\lambda} \delta_{\mathfrak{p}}^{-1/2}(m_{\eta}) \mathcal{W}(m_{\eta}). \] (52)

**Proof:** Apply \( \langle \cdot, \mathcal{W} \rangle \) to equation (47) and use Lemmas 6.1 and 6.3. \(\square\)

In the next two sections we prove two special cases of the Casselman-Shalika formula based on Proposition 6.5. In Section 7 we prove the Casselman-Shalika formula where \( \mathbb{G} \) is a semisimple group of adjoint type and \( \Psi \) has conductor \( \mathfrak{p} \) or \( \mathcal{O} \) (defined in Section 7). In Section 8 we prove the Casselman-Shalika formula where \( \mathbb{G} \) is an arbitrary connected unramified group, but \( \Psi \) is of conductor \( \mathfrak{p} \).

**7. Casselman-Shalika Formula for Adjoint Groups**

In this section we suppose that \( \mathbb{G} \) is semisimple of adjoint type and prove the Casselman-Shalika formula for \( R \)-valued spherical Whittaker functions associated to characters of conductor \( \mathfrak{p} \) or \( \mathcal{O} \). Note that \( \mathbb{G} \) adjoint implies that \( \rho^\vee \in X_*(\mathbb{A}) \).

**Theorem 7.1 (Conductor \( \mathfrak{p} \)).** Let \( \lambda \in X_*(\mathbb{A})^+ \). Then

\[
\mathcal{W}(m_{\lambda + \rho^\vee}) = \delta_{\mathfrak{p}}^{1/2}(m_{\lambda}) \text{ch} V_{\lambda} \cdot \mathcal{W}(m_{\rho^\vee}).
\] (53)

**Proof:** Take \( \mu = \rho^\vee \) in Proposition 6.5. Then

\[
\text{ch} V_{\mu - \rho^\vee} \cdot \text{ch} V_{\lambda} = \sum_{\eta} c^\eta_{\mu,\lambda} \text{ch} V_{\eta - \rho^\vee} = \text{ch} V_{\lambda}.
\] (54)

Thus

\[
\delta_{\mathfrak{p}}^{-1/2}(m_{\rho^\vee}) \text{ch} V_{\lambda} \cdot \mathcal{W}(m_{\rho^\vee}) = \delta_{\mathfrak{p}}^{-1/2}(m_{\lambda + \rho^\vee}) \mathcal{W}(m_{\lambda + \rho^\vee}),
\] (55)
from which it follows that
\[ W(m_{\lambda+\rho^\vee}) = \delta^{1/2}(m_\lambda) \cdot \text{ch} V_\lambda \cdot W(m_{\rho^\vee}). \] (56)

\[ \square \]

Remarks: a) The proof in this section is valid for any \( G \) such that \( \rho^\vee \in X_*(A) \).

b) We take this opportunity to correct the formula appearing in Theorem 6.1 in [7]. It is off by a factor of \( \delta^{1/2}(m_{\rho^\vee}) \).

Now we treat the case where \( \Psi \) has conductor \( \mathcal{O} \). We accomplish this by relating the Whittaker function of conductor \( \mathcal{O} \) to a Whittaker function of conductor \( p \).

Lemma 7.2. Suppose that \( \Psi_\mathcal{O} \) is a non-degenerate character of \( \overline{U} \) of conductor \( \mathcal{O} \), \( \mathcal{W}_\mathcal{O} \) is a Whittaker functional with respect to \( \Psi_\mathcal{O} \), and \( \mathcal{W}_\mathcal{O} = \mathcal{W}_\mathcal{O}(gv_0) \). Then

1. \( \Psi_p(u) \overset{\text{def}}{=} \Psi_\mathcal{O}(m_{-\rho^\vee}um_{-1}^{-1}) \) is a non-degenerate character of conductor \( p \);
2. \( \mathcal{W}_p = \mathcal{W}_\mathcal{O} \circ g(m_{-\rho^\vee}) \) is a Whittaker functional with respect to \( \Psi_p \);
3. \( \mathcal{W}_p(g) \overset{\text{def}}{=} \mathcal{W}_\mathcal{O}(m_{-\rho^\vee}g) \) is the spherical Whittaker function associated to \( \mathcal{W}_p \).

Proof: This follows directly from the definitions.

Proposition 7.3 (Conductor \( \mathcal{O} \)). Let \( \mathcal{W}_\mathcal{O}(g) \) be a spherical Whittaker function for the semisimple adjoint group \( G \) of conductor \( \mathcal{O} \). Let \( \lambda \in X_+(A) \). Then

\[ \mathcal{W}_\mathcal{O}(m_\lambda) = \delta^{1/2}(m_\lambda) \cdot \text{ch} V_\lambda \cdot \mathcal{W}_\mathcal{O}(1) \] (57)

Proof: This follows directly from Lemma 7.2 and Theorem 7.1.

Remark: This argument can be applied to arbitrary characters \( \Psi \).

8. Casselman-Shalika Formula Via Recursion

In this section we present a proof of the Casselman-Shalika formula for \( R \)-valued spherical Whittaker functions of conductor \( p \) where \( G \) is a connected unramified group. The main obstruction that prohibits the argument of Section 7 is that \( \rho^\vee \) may not be an element of \( X_+(A) \).

Let \( Q \) be the field of fractions of \( R \) and define the \( Q \)-vector space \( V' = \bigoplus_{\mu \in X_+(A)^{++}} Q e_\mu \) with standard basis elements \( e_\mu \). Recall the coefficients \( c_{\lambda,\mu}^0 \) defined in Subsection 2.9. Let

\[ V'' = \text{span}_Q(\text{ch} V_\lambda e_\mu - \sum c_{\lambda,\mu}^0 e_{\eta} | \lambda \in X_+(A)^+, \mu \in X_+(A)^{++}) \],

and let \( p : V' \to V \overset{\text{def}}{=} V'/V'' \) be the canonical quotient map.

We can define two functionals as follows. Let \( \Psi : V' \to Q \) by \( \Psi(e_\mu) = \delta(m_\mu)^{-1/2} \mathcal{W}(m_\mu) \), and let \( \omega : V' \to Q \) defined by \( e_\mu \mapsto \text{alt}(e^\mu) \cdot \text{ch} V_{\mu-\rho^\vee} = \text{alt}(e^\mu) \) (Weyl character formula).

(Note that \( \text{ch} V_{\mu-\rho^\vee} \) is a character of \( L \mathbb{G}_1 \).)
Lemma 8.1. The functionals $\omega, \mathcal{W}$ factor through $V$. Moreover, $V \neq 0$.

Proof: Proposition 6.5 implies that $\mathcal{W}$ factors through $V$. The Weyl character formula implies that $\omega$ is well-defined and factors through $V$. Note that $V \neq 0$ because $\omega \neq 0$. □

If we can prove that $V \cong Q$ as a $Q$-vector space, then $\omega$ and $\mathcal{W}$ are proportional and the Casselman-Shalika formula follows.

Proposition 8.2. The $Q$-linear map $\omega$ defines an isomorphism $V \cong Q$.

Proof: Let $\mu \in X_*(A)^{++}$. Then $\omega(e_\mu) = \text{alt}(e^\mu) \neq 0$. So it suffices to show that $\ker \omega = V''$.

The multiplication map $R \otimes_C R \to R$ restricts to give the following the exact sequence of $R^W$-modules

$$0 \to \ker(\text{mult}) \to R^W \otimes_C \text{alt}(R) \overset{\text{mult}}{\to} \text{alt}(R) \to 0. \quad (58)$$

Since $\text{alt}(R) = \mathbb{C}[X_*(A) + \rho^\vee] \text{alt}(e^{\rho^\vee})$, the definition of the ring structure of $\text{Rep}(L^G)$ implies that

$$\ker(\text{mult}) = \text{span}_C(\text{chV}_\lambda \otimes \text{chV}_{\mu - \rho^\vee} \text{alt}(e^{\rho^\vee}) - \sum c_{\lambda,\mu}^{\beta}(1 \otimes \text{chV}_{\eta - \rho^\vee} \text{alt}(e^{\rho^\vee}))(\lambda \in X_*(A)^+, \mu \in X_*(A)^{++}). \quad (59)$$

Consider the functor $Q \otimes R^W$. It is exact because it is the composition of the exact functors $Q^W \otimes R^W$ and $Q \otimes Q^W$. The first functor $Q^W \otimes R^W$ is exact because $Q^W$ is the field of fractions of $R^W$; the second functor $Q \otimes Q^W$ is exact because $Q$ is a free $Q^W$-module since $Q^W$ is a field. Thus if we apply $Q \otimes R^W$ to the exact sequence (58) we get the exact sequence

$$0 \to Q \otimes R^W \ker(\text{mult}) \to Q \otimes_C \text{alt}(R) \overset{\text{mult}}{\to} Q \to 0. \quad (60)$$

Let $\Xi : Q \otimes_C \text{alt}(R) \to V'$ be the $Q$-module isomorphism defined by $1 \otimes \text{alt}(e^\mu) \to e_\mu$. By definition we have the following commutative diagram.

$$\begin{array}{ccc}
Q \otimes_C \text{alt}(R) & \overset{\text{mult}}{\to} & Q \\
\downarrow{\Xi} & & \\
V' & \overset{\omega}{\to} & Q
\end{array} \quad (61)$$

Thus $\ker \omega = \Xi(Q \otimes R^W \ker(\text{mult})) = V''$. □

Theorem 8.3. Let $\mu \in X_*(A)^{++}$, then

$$W(m_\mu) = r^{1/2} \prod_{\rho} \mu_\rho \cdot \text{alt}(e^{\rho^\vee}), \quad (62)$$

where $r \in R$ is a normalization factor depending on the choice of Whittaker functional $\mathcal{W}$.
Proof: By Proposition 8.2 we have that $\dim_{Q}(V) = 1$, thus $\text{Hom}_{Q}(V, Q) \cong Q\omega$. Since $\mathcal{W} \in \text{Hom}_{Q}(V, Q)$ there exists $r \in Q$ such that $\mathcal{W} = r\omega$. By the definition of $\mathcal{W}$, equation (62) follows. In fact, $r \in R$ because $\mathcal{W}$ is $R$-valued, $R$ is a unique factorization domain, and $r$ is independent of $\mu$. \hfill \Box

9. The General Case

In this section we deduce the Casselman-Shalika formula for a general connected unramified group $G'$ and $\Psi$ a character of conductor $\mathfrak{p}$. We use the notation of Subsection 2.7. In particular, $G = G'/Z(G')$, $T = G'/DG'$, $G'' = G \times_{\text{Spec}(F)} T$, $R' = \mathbb{C}[X_*(A')]$, and $R'' = \mathbb{C}[X_*(A'')]$.

Let $\mathcal{W} \in \text{Hom}_{(R', \mathcal{U})}(i_{G', \mathcal{P}}((\chi_{\text{univ}}')^{-1}), R'_\Psi)$ be an $R'$-valued Whittaker functional for the group $G'$. This gives rise to the $R'$-valued spherical Whittaker function $\mathcal{W}'(g') = \mathcal{W}(g'v_0')$. We will relate this Whittaker function to a Whittaker function on the semisimple group $G$ of adjoint type. Since $R' \hookrightarrow R''$ (recall line (12)), $\mathcal{W}'$ and $\mathcal{W}$ may also be viewed as $R''$-valued.

Lemma 2.8 and Lemma 2.9 imply that

$$\text{Hom}_{(R', \mathcal{U})}(i_{G', \mathcal{P}}((\chi_{\text{univ}}')^{-1}), R'_\Psi) \cong \text{Hom}_{(R'', \mathcal{U})}(i_{G'', \mathcal{P}}((\chi_{\text{univ}}'')^{-1}), R''_\Psi)$$

as $R''$-modules. Thus $\mathcal{W}'$ corresponds to $\mathcal{W}'' \in \text{Hom}_{(R'', \mathcal{U})}(i_{G'', \mathcal{P}}((\chi_{\text{univ}}'')^{-1}), R''_\Psi)$ such that $\mathcal{W}' = \mathcal{W}''|_{1 \otimes i_{G', \mathcal{P}}((\chi_{\text{univ}}')^{-1})}$, where $1 \otimes i_{G', \mathcal{P}}((\chi_{\text{univ}}')^{-1}) \subseteq i_{G'', \mathcal{P}}((\chi_{\text{univ}}'')^{-1})$ according to Lemma 2.8.

This relationship between Whittaker functionals induces a relationship between spherical Whittaker functions. Let $\mathcal{W}''((g, t)) = \mathcal{W}''((g, t)v_0'')$.

Proposition 9.1. Let $g' \in G'$. Then

$$\mathcal{W}'(g') = \mathcal{W}''(\pi(g')).$$

Proof: Under the isomorphism of Lemma 2.8, $v_0' = 1 \otimes v_0'$. Thus if we write $\pi(g') = (g, t)$ then

$$\mathcal{W}'(g') = \mathcal{W}''((g, t)v_0'') = \mathcal{W}''((g, t)v_0'') = \mathcal{W}''(\pi(g')).$$

With Proposition 9.1 we reduce the computation of a spherical Whittaker function on $G'$ to one on $G''$; next we reduce the computation of a spherical Whittaker function on $G''$ to one on $G$.

We begin with some notation and a relationship between the Whittaker functionals. Let $\mathcal{W}'' \in \text{Hom}_{(R'', \mathcal{U})}(i_{G'', \mathcal{P}}((\chi_{\text{univ}}'')^{-1}), R''_\Psi)$. Lemma 2.6 and Lemma 2.9 imply that

$$\text{Hom}_{(R'', \mathcal{U})}(i_{G'', \mathcal{P}}((\chi_{\text{univ}}'')^{-1}), R''_\Psi) \cong \text{Hom}_{(R', \mathcal{U})}(i_{G', \mathcal{P}}((\chi_{\text{univ}}^{-1})^{-1}), R'_\Psi)$$

as $R''$-modules. Thus $\mathcal{W}''$ corresponds to a $\mathcal{W} \in \text{Hom}_{(R', \mathcal{U})}(i_{G', \mathcal{P}}((\chi_{\text{univ}}^{-1})^{-1}), R'_\Psi)$ such that $\mathcal{W} = \mathcal{W}''|_{1 \otimes i_{G', \mathcal{P}}((\chi_{\text{univ}}^{-1})^{-1})}$. 

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Again the relationship between Whittaker functionals induces a relationship between spherical Whittaker functions. Let $W(g) = \mathfrak{W}(gv_0)$. Recall that $\xi$ is the tautological character of $T$ (Subsection 2.7).

**Proposition 9.2.** Let $(g, t) \in G''$. Then

$$W''((g, t)) = \xi(t)^{-1}W(g). \quad (66)$$

**Proof:** Under the isomorphism of Lemma 2.6, $v''_0 = 1 \otimes v_0$. Thus

$$W''((g, t)) = \xi(t)^{-1}\mathfrak{W}'(1 \otimes gv_0) = \xi(t)^{-1}\mathfrak{W}(gv_0) = \xi(t)^{-1}W(g).$$

□

We combine Proposition 9.1 and Proposition 9.2 to express $W'$ in terms of $W$, a spherical Whittaker function of a semisimple adjoint group $G$.

**Proposition 9.3.** Let $g' \in G'$, $g \in G$, and $t \in T$ such that $\pi(g') = (g, t)$. Then

$$W'(g') = \xi(t)^{-1}W(g). \quad (67)$$

The next theorem is the Casselman-Shalika formula for a general connected reductive group $G'$.

**Theorem 9.4.** Let $\mu' \in X_*(A')^{++}$ and let $\pi_*(\mu') = (\mu, \lambda) \in X_*(A) \times X_*(T) = X_*(A'')$. Then

$$W'(m_{\mu'}) = \xi^{-1}(t_\lambda)A_{1/2}(m_{\mu'})\text{ch}_{V_{\mu' - \rho'}} \cdot W(m_{\rho'}). \quad (68)$$

**Proof:** This is a direct consequence of Proposition 9.3 and Theorem 7.1. □

**Remark:** In Theorem 9.4, $\text{ch}_{V_{\mu' - \rho'}}$ is a character of the Langlands dual group of $G'/Z(G')$ and thus lives in $R''$. We also have $W(\rho') \in R''$ and $\xi^{-1}(t_\lambda) \in R''$. However, since $W'$ is $R'$-valued, RHS(68) is in $R'$.

10. Conductor $\mathcal{O}$

Our focus on Whittaker functions of conductor $p$ is a by-product of our proof technique. However, for global applications one must study Whittaker functions of conductor $\mathcal{O}$. This means that for all $\alpha \in \Delta$, $\Psi_\alpha$ is trivial on $U_{-\alpha} \cap K$ and nontrivial on any larger subgroup of $U_{-\alpha}$. We accomplish this by relating a Whittaker function of conductor $\mathcal{O}$ to a Whittaker function of conductor $p$, for which Theorem 9.4 provides a formula.

We will retain the notation from Section 9 and augment the notation of $\Psi$, $\mathfrak{W}$, and $\mathcal{W}$ to indicate the conductor. For example, $\Psi_\mathcal{O}$ will be a character of $\mathfrak{U}$ of conductor $\mathcal{O}$ and $\Psi_p$ be a character of conductor $p$.

All of the pieces are already in place because the results of Section 9 with the exception of Theorem 9.4 are independent of the conductor of $\Psi$. (The conductor $p$ property is used in our proof of Lemma 5.4, an important ingredient for Theorem 9.4.)
Theorem 10.1. Let $\lambda' \in X_*(A')$ be dominant. Let $\lambda \in X_*(A)$ be dominant, and $\mu \in X_*(T)$ such that $\pi_*(\lambda') = (\lambda, \mu)$. Then
\[
W'_O(m_{\lambda'}) = \xi^{-1}(t_\mu)\frac{1}{\sqrt{2}} (m_\lambda) \text{ch} V_\lambda \cdot W'_O(1). \tag{69}
\]

Proof: Apply Propositions 9.3, 7.3, and Theorem 9.4 and note that $W'_O(1) = W_O(1)$. □

Remarks:
(1) $\xi^{-1}(t_\mu)$ and $\text{ch} V_\lambda$ need not be in the image of $R' \hookrightarrow R'' = R \otimes \mathbb{C}[T/T^0]$ individually, but the product $\xi^{-1}(t_\mu)\text{ch} V_\lambda \cdot W'_O(1)$ is.
(2) One can apply the argument of this section to spherical Whittaker functions associated to characters $\Psi$ with arbitrary conductor.

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