Amplification of explosive width in complex networks

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We present an adaptive coupling strategy to induce hysteresis/explosive synchronization (ES) in complex networks of phase oscillators Sakaguchi-Kuramoto model). The coupling strategy ensures explosive synchronization with significant explosive width enhancement. Results show the robustness of the strategy and the strategy can diminish (by inducing enhanced hysteresis loop) the contrarian impact of phase frustration in the network, irrespective of network structure or frequency distributions. Additionally, we design a set of frequency for the oscillators which eventually ensure complete in-phase synchronization behavior among these oscillators (with enhanced explosive width) in the case of adaptive-coupling scheme. Based on a mean-field analysis, we develop a semi-analytical formalism, which can accurately predict the backward transition of synchronization order parameter.

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The phenomenon of synchronization has fascinated researchers for many years due to its appearance in variety of natural as well as man-made systems. However, the study of synchronization in adaptively coupled complex networks has got less attention. Here, we have presented an adaptive coupling scheme and explored the impact of that in the enhancement of explosive width in Sakaguchi-Kuramoto model on complex networks. Numerical investigation shows that the proposed scheme ensures explosive synchronization (ES) with significant explosive width enhancement for diverse frequency distributions with different network realizations. More importantly, it is found that the proposed coupling scheme can inhibit the contrarian impact of the phase frustration in the network. We also have established a semi analytical treatment for investigating the system using the Ott-Antonsen ansatz. The results obtained from the semi analytical approach are found to match closely with the numerical simulation results. The analysis shows that the adaptive coupling function is robust and can enhance the hysteresis width in different complex networks having variety of natural frequency distributions and for a broad range of frustration parameter.

I. INTRODUCTION

Adaptively coupled oscillators have been found to exhibit a wide range of complex behaviors. For instance, link weights co-evolving with underlying dynamics can enhance the synchronizability in heterogeneous networks and this mechanism can induce spontaneous synchronization in e.g., certain group of neurons of a brain by adjusting the adaptive Hebbian learning process. Recently, it has been reported that the adaptation of the coupling configuration by the local order parameter may result in explosive synchronization, a phenomenon characterized by discontinuous (first-order) phase transitions between incoherent and coherent states accompanied by hysteresis in networks of coupled oscillators.

Of particular interest, the explosive synchronization has been found to emerge among adaptively-coupled oscillators irrespective of the structure of the networks or intrinsic frequency distributions. We raise here the question what happens if we use a modified strategy, i.e incorporating the high power of magnitude of the adapting parameter?

Here, we have used the modified adaptive-coupling scheme proposed by Filatrella et al. on Sakaguchi-Kuramoto model in complex network environment, and results show that the proposed scheme can amplify or increase the hysteresis width (difference between the backward transition and forward transition points of synchronization) for diverse frequency distributions with different network realizations. For demonstration, we start with the classical network-coupled Sakaguchi-Kuramoto phase oscillators of size $N$. Dynamics of each oscillator in the network is governed by the equation

$$\frac{d\theta_i}{dt} = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i - \alpha), \quad (1)$$

where $\omega_i$ is the intrinsic natural frequency of the $i^{th}$ oscillator, $A_{ij}$ is an $ij^{th}$ element of the adjacency matrix.
distribution and set the frustration parameter (we take the natural frequencies from a Lorentzian coherent to the phase synchronized state. In this simulation we considered a heterogeneous Erdős-Rényi (ER) network with nodes. We increase (decrease) the coupling strength $l_i$ adiabatically with an increment (decrement) $\delta \lambda = 0.01$ and compute the stationary value of $R$ for each $\lambda$ during the forward (backward) transition from the incoherent to the phase synchronized state. In this simulation we take the natural frequencies from a Lorentzian distribution and set the frustration parameter $\alpha$ equal to 0.5. In Fig. 1 (a), the order parameter undergoes a continuous transition for the choice of $z = 1$ where the network is a purely diffusive i.e. the coupling function is not controlled by any adaptive coupling. However, setting the adaptive parameter to higher power ($z = 2$), one can generate discontinuous (explosive) transition in the synchronization order parameter (see Fig. 1 (b)). Interestingly, the explosive width is further enhanced if we increase the value of $z$ from 2 to 3 (Fig. 1 (c)). The double headed arrow in Fig. 1 (c) indicates the explosive width for the explosive synchronization transition which is the difference between the backward transition and forward transition points of synchronization. In this paper, we have explored the impact of $z$ on transition to synchronization in different complex networks of Sakaguchi-Kuramoto oscillators for a broad range of frequency distributions, namely Lorentzian or uniform distributions. We have also considered the degree-frequency correlated environment and a special form of degree-frequency correlation which eventually gives perfect synchronization. Our results show that the adaptive coupling function is robust and can induce the enhanced explosive width for a broad range of frustration parameter over diverse network settings. In this work, we validate our detailed numerical results by solving (semi-analytically) order parameter equation on the basis of Ott-Antonsen ansatz. The forward transition curve is perfectly fitted with our semi-analytical findings.

II. ANALYTICAL APPROACH AND NUMERICAL VERIFICATION

We start with the annealed network approximation proposed in Refs. 20-24, which gives critical behavior of phase transitions (including synchronization transition) in complex networks. For the case of a sparse uncorrelated complex network with a degree distribution $P(k)$ in the thermodynamic limit ($N \to \infty$) we may write,

$$\frac{d \theta_i}{dt} = \omega_i + \frac{\lambda R^{-1} k_i}{N} \sum_{j=1}^{N} k_j \sin(\theta_j - \theta_i - \alpha),$$

where $k_i$ is the degree of $i^{th}$ node, and $\langle k \rangle$ is the average degree across nodes. To obtain the explicit set of equations for the time evolution of the complex order parameter $C = Re^{i(\psi-\alpha)}$, we use here the Ott-Antonsen ansatz where the density of the oscillators at time $t$ with phase $\theta$ for a given degree $k$ and frequency $\omega$ be given by the function $f(k, \omega; \theta, t)$, and we normalize it as

$$\int_0^{2\pi} f(k, \omega; \theta, t) d\theta = h(k, \omega) = P(k) g(\omega).$$

To maintain the conservation of the oscillators in a network, the density function $f$ satisfies the continuity equation

$$\frac{\partial}{\partial t} f(k, \omega; \theta, t) + \frac{\partial}{\partial \theta}(v f(k, \omega; \theta, t)) = 0,$$

where $v$ is the velocity field on the circle that drives the dynamics of $f$, which comes from the right hand side of

$$f(k, \omega; \theta, t) = \sum_{j=1}^{N} k_j \delta(\theta_j - \theta - \alpha).$$

Notice that $v = \frac{\partial}{\partial \theta} (\omega + \frac{\lambda R^{-1} k_i}{N} \sum_{j=1}^{N} k_j \sin(\theta_j - \theta_i - \alpha)).$
the Eqn. \ref{eqn:3} as follows

\[ v(k, \omega; \theta, t) = \frac{d\theta}{dt} = \omega + \frac{\lambda k}{2i} R^{-2} \left[ C e^{-i\theta} - C^* e^{i\theta} \right]. \tag{6} \]

Expanding \( f(k, \omega; \theta, t) \) in a Fourier series in \( \theta \), we have

\[ f = \frac{h(k, \omega)}{2\pi} \left( 1 + \sum_{n=1}^{\infty} f_n(k, \omega, t)e^{in\theta} + \text{c.c.} \right), \tag{7} \]

where \( \text{c.c.} \) stands for complex conjugate. We now consider the Ott-Antonsen ansatz

\[ f_n = \beta^n(k, \omega, t) \tag{8} \]

to obtain an equation for the function \( \beta(k, \omega, t) \) and find the following equation

\[ \frac{\partial \beta}{\partial t} + i\omega \beta + \frac{\lambda k}{2} R^{-2} (C\beta^2 - C^*) = 0, \tag{9} \]

where \(|\beta(k, \omega, t)| \leq 1\) to avoid divergence of the series. The above equation is not yet in a closed form. So to make the above equation in a closed form, we get the complex global order parameter as

\[ C(t) = \frac{e^{-i\alpha}}{\langle k \rangle} \int_{k_{\text{min}}}^{\infty} \int_{-\infty}^{\infty} kP(k)g(\omega)\beta^* d\omega dk. \tag{10} \]

Now to study a steady state solution we average \( C(t) = \text{Re} e^{i\psi} - i\alpha + i\Omega t \) for the constant order parameter \( R \), a phase \( \psi \), and a group angular velocity \( \Omega \). By suitably varying the reference frame, \( \omega \rightarrow \omega - \Omega \) and putting \( \psi = 0 \), without loss of generality, we have \( C = \text{Re} e^{-i\alpha} \). Therefore, \( C^* = \text{Re} e^{i\alpha} \) and for stationary points of Eqn. \ref{eqn:3} we find the solution of \( \dot{\beta} = 0 \).

We obtain the solution as follows,

\[ \beta(k, \omega) = \left\{ \begin{array}{ll} -ixe^{i\alpha} + e^{i\alpha} \sqrt{1 - x^2}, & |x| \leq 1, \\ -ixe^{i\alpha} \left[ 1 - \sqrt{1 - \left( \frac{x}{\alpha} \right)^2} \right], & |x| > 1. \end{array} \right. \tag{11} \]

where \( x = \frac{\omega - \Omega}{\langle k \rangle} \) and this solution satisfies the requirement \(|\beta| \leq 1\) for the convergence of the geometrical series (Eqn. \ref{eqn:3}). The first solution corresponds to the synchronous state and the second solution is due to desynchronous state. In this case, the order parameter can be rewritten as

\[ R = \frac{1}{\langle k \rangle} \left[ \int_{k_{\text{min}}}^{\infty} kP(k)g(\omega)\beta^* (k, \omega)H (1 - |x|) dk \omega \\
+ \int_{k_{\text{min}}}^{\infty} kP(k)g(\omega)\beta^* (k, \omega)H (|x| - 1) dk \omega \right]. \tag{12} \]

where \( H \) is Heaviside functions. The first part of right-hand side of Eqn. \ref{eqn:12} encompasses the contribution of locked oscillators and the second part accounts for the contribution of drift oscillators to the order parameter \( R \). Splitting the contributions of real and imaginary parts, we can eventually obtain the coupled self consistent equations of \( R \) and \( \Omega \) (see in the appendix).

To test the impact of \( z \) on transition to synchronization, we use a heterogeneous scale-free network \( \langle N = 2000, \langle k \rangle = 12 \) and \( \gamma = 2.8 \) generated from Barabási-Albert model, the frequencies are drawn from a Lorentzian distribution with mean 0 and standard deviation 1. The bi-stable state (hysteresis regime) is shown by yellow color (regime:II) in the Fig. \ref{fig:2} (a)-(b) for the frustration parameters \( \alpha = 0.0 \) and \( \alpha = 0.5 \) respectively.

For lower \( z \), the hysteresis width is found to be negligibly small and if we increase this adaptive parameter to higher values, the width is significantly enhanced irrespective of the frustration parameter. Here the magenta island (regime: I) shows the de-synchronized regime and cyan island (regime: III) is the locked part of coupled phase oscillators. It is observed from the figure that as \( z \) is increased, the region of the hysteresis is amplified significantly. Interesting to note here that the higher adaptive coupling parameter \( z > 1 \) not only induces explosive synchronization in the network but also enhances the explosive width even for high phase frustration (e.g. \( \alpha = 0.5 \) in Fig. \ref{fig:2}(b)). Typically in this kind of network, higher phase frustration (\( \alpha > 0.3 \)) destroys the hysteresis behaviour in a degree-frequency correlated network when \( z = 1.23 \). However, for higher value of \( z \), it can overcome the situation and explosive synchronization (ES) re-emerges.

The boundary of the yellow and cyan regions which signifies the forward critical point \( (\lambda_f(z)) \) of the hysteresis transition is computed numerically from the model Eqn. \ref{eqn:1}. The backward transition points derived from the
We have also numerically calculated the order parameter and the global frequency $\Omega$ for $z = 2$ and $z = 3$ (see Fig. 3). The order parameter ($R$) and global frequency ($\Omega$) are shown in the Fig. 3(a, c) with blue lines and Fig. 3(b, d) with red lines respectively as a function of $\lambda$. The backward transition points determined from the self-consistent equations are shown with dashed vertical lines in the figure which shows a very close match with the backwards points obtained through numerical simulation of the whole system.

For $z = 2$, the hysteresis width (see inset of Fig. 3(a, b)) occurs due to the presence of adaptive coupling. For $z = 3$ the width is significantly enhanced compared to $z = 2$. Note that, the values of $\Omega$ and $R$ are negligibly small before the forward critical coupling. The reason is as follows: adaptive function $R^{(z-1)}$, acting on each node of the network, decreases the effective coupling ($\Lambda_{\text{eff}} = \lambda \times R^{(z-1)}$) where $0 < R < 1$ and $z > 1$ substantially. Therefore, higher strength is required to establish locked phase. Numerical values of backward $R$ and $\Omega$ closely match with the ones obtained from the semi-analytical expression of $R$ and $\Omega$ derived from the Eqn. 10. We therefore establish the fact, that a tuning parameter associated with adaptive term in power can enhance the width of hysteresis irrespective of the frustration value or network structure. For better realization we also simulate our model Eq. 1 in Fig. 4 with Watts-Strogatz small world network size $N = 5 \times 10^3$, $\langle k \rangle = 4$, and $\beta = 0.5$ for three different values of phase frustration $\alpha = 0, 0.3, 0.6$. In every panel of Fig. 4 we have simulate the model with increasing $z$. We perform the numerical analysis for diverse set of frequency distributions, and we show that the enhancement properties appear for all the cases which only depend on the $z$. The changes in forward and backward points depend on the choices of frequency distributions. Next, we validate our proposition in frustrated networks for a certain choice of frequency distribution directly correlated with its own degree in which a global order parameter achieves perfect synchronization (5) ($R = 1$). This is important, as $z$ mainly contributes in the enhancement of hysteresis width irrespective of $\alpha$. Although $\alpha$ frustrates the system by not allowing the order parameter to reach perfect synchronization. We seek a selection of frequency distribution which can avoid the impact of frustration $\alpha$ by reaching perfect synchronization ($R = 1$) as well as it will create a broader explosive width in presence of higher values of $z$.

### III. Amplification of Hysteresis Widths in Degree-Frequency Environment

We have already shown that our adaptive strategy works efficiently when the frequencies are drawn from Lorentzian distribution. A degree-frequency environment naturally induces ES\textsubscript{L}, however a frustrated network ($\alpha > 0.3$) cannot reveal ES even in the presence of degree-frequency correlated dynamics. Here we seek to understand the impact of such coupling configuration in frustrated dynamics.

We consider a SF network of size $N = 2000$ with $\gamma = 2.8$ and average degree $\langle k \rangle = 14$, where natural frequency of each oscillator scales with its own degree as $\omega_i = \alpha k_i$ where $\alpha$ is a proportionality constant. We set the phase-frustration parameter $\alpha = 1$, and we take $\alpha = \sin \alpha$. In this case order parameter $R$ does not exhibit the first-order transition for non-adaptive environ-

![FIG. 3. Order parameter $R$ and group angular velocity $\Omega$ as a coupling strength $\lambda$ for two values of $z$. Blue curve indicates the synchronization diagram and red curve indicates the group angular velocity calculated from the model Eqn. 1 for SF network of network size $N = 2000$, $\langle k \rangle = 12$ and $\gamma = 2.8$. Black dotted vertical line indicates the backward transition point of $R$ and $\Omega$ calculated from the semi-analytic Eqn. 10 and 14. All the data are simulated with phase frustration $\alpha = 0.5$.](image-url)

![FIG. 4. Realization of the expansion of explosive width with $z$ for three different values of phase frustration $\alpha$. Both the figures are constructed using Watts-Strogatz small world network of size $N = 500$, $\langle k \rangle = 4$, and $\beta = 0.5$. Double headed arrow indicates the explosive width of the hysteresis transition.](image-url)
FIG. 5. Amplification of explosive width in case of degree-frequency correlation. Order parameter $R$ as a function of coupling strength $\lambda$ for a SF network of network size $N = 2000$, $(k) = 14$, and $\gamma = 2.8$ for different values of $z$. Blue line indicates the forward transition and red line indicates backward transition. Magenta dot on the cyan color line indicates our targeted point at (1, 1).

Therefore, we have shown that our adaptive strategy can create ES and enhance the ES width in these types of networks irrespective of the choice of the frequencies. The analytical prediction is confirmed in Fig. 5 assuring perfect synchronization at $\lambda = 1$ which also nullifies the contrarian effect of frustration parameter $\alpha$. The forward line misses the perfect synchronization point (blue line) where as backward line coincides with $R = 1$ at $\lambda = 1$ for each cases of $z$.

IV. CONCLUSIONS

We have investigated the phenomenon of explosive synchronization in complex networks of phase oscillators both numerically and analytically, based on an adaptive coupling strategy. Numerical simulations for different networks both in frustrated and non frustrated environments show that, as a parameter $(z)$ in the coupling function is tuned appropriately, networks undergo abrupt transition to (explosive) synchronization and width of the associated hysteresis loop is greatly enhanced. It is observed that the proposed strategy is quite general and is applicable for any type of networks and frequency distributions for the amplification of hysteresis width. We have confirmed that the strategy can successfully induce the emergence and enhancement of explosive synchronization in diverse frustrated environments. Based on the Ott-Antonsen ansatz, we have derived analytical expressions for the global order parameter and global frequency. The results obtained from the semi-analytical approach based on Ott-Antonsen ansatz match with the backward transition points obtained from the numerical simulation of the entire complex networks.

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VI. APPENDIX: COUPLED EQUATION OF $R$ AND $\Omega$

The contribution of locked oscillators to the order parameter is

$$R_t = \frac{(\cos \alpha - i \sin \alpha)}{(k)} \int_{k_{min}}^{\infty} kP(k)g(\omega) \left[ \sqrt{1 - \left( \frac{\omega - \Omega}{\lambda kR^2} \right)^2} + i \frac{\omega - \Omega}{\lambda kR^2} \right] dk d\omega H \left( 1 - \left| \frac{\omega - \Omega}{\lambda kR^2} \right| \right)$$ (14)
Now the contribution of drift oscillators to the order parameter is given by

\[ R_d = \frac{(\sin \alpha + i \cos \alpha)}{\langle k \rangle} \int_{k_{\text{min}}}^{\infty} \int kP(k)g(\omega) \frac{\omega - \Omega}{\lambda k R^2} \left[ 1 - \sqrt{1 - \frac{(\lambda k R^2)^2}{\omega - \Omega}} \right] dk d\omega H \left( \frac{\omega - \Omega}{\lambda k R^2} - 1 \right) \]  

(15)

Hence we get \( R = R_t + R_d \), where \( R_t \) and \( R_d \) are given by Eqn. (14) and Eqn. (15) respectively.

Now comparing the real and imaginary parts we get

\[ R(\langle \omega \rangle) = \cos \alpha \int_{k_{\text{min}}}^{\infty} \int kP(k)g(\omega) \sqrt{1 - \left( \frac{\omega - \Omega}{\lambda k R^2} \right)} H \left( 1 - \frac{\omega - \Omega}{\lambda k R^2} \right) dk d\omega + \sin \alpha \langle \omega \rangle - \Omega \]

(16)

\[ \langle \omega \rangle - \Omega = \lambda R^2 \tan \alpha \int_{k_{\text{min}}}^{\infty} \int kP(k)g(\omega) \sqrt{1 - \left( \frac{\omega - \Omega}{\lambda k R^2} \right)} H \left( 1 - \frac{\omega - \Omega}{\lambda k R^2} \right) dk d\omega \]

(17)

By solving the above two equations we can get the set of values for order parameter \( R \) corresponding to coupling strength \( \lambda \).

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