Abstract. For the exponential noise-to-state stability system, it’s long known that the probability distribution of the state is controlled by a particular function which is concerned with the maximum noise covariance matrix. In this paper, we show another result for these systems that the distribution of the state points measured in the time horizon also exhibits the same property. It means that the trajectory will fluctuate in a bounded region around the origin with a large proportion of time and, once going out of this region, the state will soon be back to it, which depicts a certain sense of stability for stochastic systems and exposes the meaning of exponential noise-to-state stability in a new perspective. Our analysis is based on detecting the average of up/down cross time of the loops that are constructed for the systems. Finally, A numerical example is provided to present the efficiency of our work.

Key words. Noise-to-stability, stochastic ordinary differential equation, stochastic stability, time average analysis.

AMS subject classifications. 60G52, 60H10, 93E15.

1. Introduction. Since the plants in the real world are always disturbed by all kinds of non-negligible noises, stability analysis for stochastic systems has now attracted more and more researchers’ attentions. This task has never been easy and becomes much more difficult when the systems are equipped with non-vanishing noise at the equilibrium. In this case, almost all classic stability analysis methods fails to be applied to these system, and, even worse, the systems barely admit Lyapunov stability behavior in the stochastic form.

Notwithstanding these difficulties, researchers have done a lot of work intended to deal with stochastic systems with persistent noise in the past few decades. Following the idea of stochastic passivity [4], some efforts are done to weaken the global dissipation conditions in this theory so that it can be applied to the considered systems and some stability behavior is ensured. Among them there’re two major modifications. One is to consider the dissipation of Lyapunov function only outside a neighborhood of the equilibrium ( [10] or [12]) and another is to discuss noise-to-state stability [6]. In fact, according to Proposition 2.5 in [3], noise-to-state stability can be implied by the former one, thus these two modifications mentioned above are actually quite connected. Both of these techniques have been successfully applied to all kinds of practical systems, such as port-Hamiltonian systems [10] and large scale systems [3].

Noise-to-state stability can be seen as a stochastic counterpart of input-to-state stability introduced in [11]. For it, the expectation of increment of Lyapunov function consists of two terms, a dissipative one and an additional one decided by the covariance of noise. With this structure, it’s shown that the state and $p$-th moment of noise-to-state stability system are bounded in probability, which demonstrate a particular sense of stability for the stochastic systems [7].

This result provides a possible solution for the problem, however we are not fully satisfied with it, because the worry about extreme events can’t be eliminated by this
discussion. As a matter of fact, boundedness in probability depicts the distribution of a great many trajectories with the same initial condition, but what we care more is the almost sure long time behavior of a single trial that is robust with the given parameters. Therefore, instead of considering probability distribution, here comes a straightforward idea, discussing the distribution of state points in a single trajectory to be almost surely controlled. Mathematically, it’s to show that the inferior time average of each indicator function \(1_{\{\|x(t)\| \leq r\}}\) is almost surely bounded from below and goes to 1 as the bound, \(r\), going to infinity. In physical sense, it illustrates that the trajectory will fluctuate in a region around the origin with a large proportion of time, and, once going out of this region, the state will soon be back to it. In this paper, we will show such property holding true for exponential noise-to-state stability system by presenting time average analysis of the indicator function.

As matter of fact, this analysis is quite easy to be achieved when the noise of the system is non-singular. In this case, the stochastic process admits Feller and irreducible properties. Therefore, together with the recurrence of noise-to-state stability systems, these facts implies ergodicity [5], which demonstrates the time average of the indicator function equals to the spacial (probabilistic) average. Unfortunately, such non-singular condition is at loss for many stochastic systems, and therefore we have to apply another method to deal with our problem. Borrowing the technique in [5], we construct loops for the process in Definition 2 and, latter, time average analysis is done through detecting the average of time spent in each part of the loop. Although non-singularity no longer presented, its position is taken over by the exponential dissipation condition which makes our analysis possible in this work. Particularly, it should be noted that there’s no any additional condition required in our main result, Theorem 7, showing the lower boundedness of inferior time average of indicator functions for exponential noise-to-state stability systems. This paper demonstrates another sense of stability for noise-to-state stability systems distinct from boundedness in probability, and therefore can be seen as a supplement to noise-to-state stability theory.

In the sequel this paper is organized as follow. In Section 2, we briefly review the basic concept and results of noise-to-state stability and, in Section 3, some strong laws of large numbers are prepared for later time average analysis. The main results are presented in Section 4 which demonstrate the positive lower boundedness of inferior time average of each indicator function. In Section 5, a numerical example is introduced to show the efficiency of our work. Finally, Section 6 concludes this paper.

Here are some notations that the readers will find in the context.

**Notation:**

| Symbol | Description |
|--------|-------------|
| \(\mathcal{K}\) | \(\mathcal{K} = \{\rho(\cdot) \in \mathcal{C}(\mathbb{R}) \mid \rho(\cdot) \text{ is strictly increasing and } \rho(0) = 0\}\) |
| \(\mathcal{K}_\infty\) | \(\mathcal{K}_\infty = \{\rho(\cdot) \in \mathcal{K} \mid \lim_{r \to +\infty} \rho(r) = +\infty\}\) |
| \(\mathbb{1}_{\{\cdot\}}\) | \(\mathbb{1}_{\{\cdot\}}\) is called indicator function which is valued 1 if event \(\{\cdot\}\) happens, otherwise 0. |
| \(W_{-1}\) | Lambert W function in the lower branch. For any real number \(r \geq -e^{-1}\), there’s \(r = W_{-1}(r)e^{W_{-1}(r)}\) and \(W_{-1}(r) \geq -1\). |
| \(U(0, 1)\) | The uniform distribution with lower bound 0 and upper bound 1. |
| \(\exp\{\cdot\}\) | The exponential function. |

2. Noise-to-State stability. In this section, we briefly review the concept of noise-to-state stability and figure out some of its essential results, including the most important one that there exists a probability bound on the system’s state [6].
Usually, a practical system can be modeled by a stochastic differential equation

\begin{equation}
\frac{dx}{dt} = f(x)dt + h(x)\Sigma(t)d\omega
\end{equation}

where $x \in \mathbb{R}^n$ is the state of the system and $\omega \in \mathbb{R}^m$ is a standard Wiener process. For it, $\Sigma(t) \in \mathbb{R}^{n \times m}$ is a positive definite bounded Borel measurable covariance matrix, and $f(x)$, $h(x)$ are locally Lipschitz so that there’s a local solution for the equation. The concept of noise-to-state stability is proposed to analyse the systems with non-vanishing noise and unknown covariance, that is $h(x) \neq 0_{m \times m}$ and $\Sigma(t)$ being unknown. Its specific definition is as

**Definition 1 (Noise-to-state Stability).** For stochastic system (1), suppose there exists a $C^2$ function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\gamma(\cdot)$ such that

\begin{equation}
\alpha_1(||x||_2) \leq V(x) \leq \alpha_2(||x||_2)
\end{equation}

and

\begin{equation}
\mathcal{L}[V(x(t))] \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{tr} \left\{ \Sigma(t)^\top h(x)^\top \frac{\partial^2 V}{\partial x^2} h(x) \Sigma(t) \right\}
\end{equation}

\begin{equation}
\leq -\alpha_3(||x||_2) + \gamma(||\Sigma(t)\Sigma(t)^\top||_F)
\end{equation}

then the system is said to be noise-to-state stability and $V(x)$ is called noise-to-state Lyapunov function. Particularly, if $\alpha_3(||x||) \geq cV(x)$ where $c$ is a positive constant, then the system (1) is said to be exponential noise-to-state stability.

For convenience, we denote $\gamma_{\max} = \gamma(\sup_x ||\Sigma(s)\Sigma(s)^\top||_F)$ in the context.

**Remark 1.** Following the technique in [9], literature [7] notes that for a noise-to-state stability system there’s another Lyapunov function, $\tilde{V}(x)$, together with a quaternary $\mathcal{K}_\infty$ function set $\{\tilde{\alpha}_1(\cdot), \tilde{\alpha}_2(\cdot), \tilde{\alpha}_3(\cdot), \tilde{\gamma}(\cdot)\}$ rendering the system to be exponential noise-to-state stable. Therefore, these two concepts, noise-to-state stability and exponential noise-to-state stability, are actually equivalent. In the sequel, results are all presented in the framework of exponential noise-to-state stability, however they still hold true for the noise-to-state stability systems if the notations $\{V(x), \alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot), \gamma(\cdot)\}$ are substituted by $\{\tilde{V}(x), \tilde{\alpha}_1(\cdot), \tilde{\alpha}_2(\cdot), \tilde{\alpha}_3(\cdot), \tilde{\gamma}(\cdot)\}$.

It’s clear that for a noise-to-state stability system, $V(x)$ is radially unbounded and $LV(x)$ is upper bounded by $\gamma_{\max}$ (a constant). Therefore, according to [8], there’s a global solution for system (1) which ensure the results in the sequel of this paper not to be castles in the air.

As for exponential noise-to-state stability systems, thanks to its exponential dissipation condition, it’s possible for us to construct an essential super-martingale like inequality, shown in the next lemma, which plays a fundamental role in all kinds of analysis. As a matter of fact, our analysis in Section 4 is also heavily based on this inequality.

**Lemma 1.** Consider an exponential noise-to-state stability system (1). For any time $t \geq t_0$ and any stopping time $\tau$, we have

\begin{equation}
E\left\{ [V(x(t \wedge \tau)) - ce^{-\gamma_{\max}}] e^{c(t \wedge \tau - t_0)} \mid x(t_0) \right\} \leq V(x(t_0)) - ce^{-\gamma_{\max}}.
\end{equation}
Proof. Applying Dynkin’s formula and inequality (3), we have

\[
\begin{align*}
&\mathbb{E}\left[V(x(t)) e^{c (t \wedge \tau - t_0)} \mid x(t_0)\right] = V(x(t_0)) + \mathbb{E} \int_{t_0}^{t \wedge \tau} e^{c(s-t_0)} \left(L[V(x(s))] + cV(x(s))\right) \, ds \\
&\quad \leq V(x(t_0)) + \mathbb{E} \int_{t_0}^{t \wedge \tau} e^{c(s-t_0)} \gamma \left(\|\Sigma(s)\Sigma(s)^\top\|_F\right) \, ds \\
&\quad \leq V(x(t_0)) + \mathbb{E} \left[e^{c(t \wedge \tau - t_0)} - 1\right] c^{-1} \gamma_{\max}.
\end{align*}
\]

Hence, the result (4) follows immediately from this inequality, by subtracting the term \(\mathbb{E} \left[e^{c(t \wedge \tau - t_0)}\right] c^{-1} \gamma_{\max}\) at both sides. \(\square\)

Remark 2. For inequality (4), letting \(\tau\) be infinite, then \(t \wedge \tau = t\) and the inequality can be simplified as

\[
E\left[\left| V(x(t)) - c^{-1} \gamma_{\max}\right| e^{ct} \mid x(t_0)\right] \leq \left| V(x(t_0)) - c^{-1} \gamma_{\max}\right| e^{ct_0}
\]

which implies the stochastic process \(\left| V(x(t)) - c^{-1} \gamma_{\max}\right| e^{ct}\) a super-martingale. \(\text{This is the very reason why we call inequality (4) as a super-martingale like inequality in the context.}\)

Applying the super-martingale inequality (5) to the stochastic system (1), we can obtain another important result by setting \(t_0 = 0\), that the expectation of the Lyapunov function is bounded.

Theorem 1 (Adapt from [1]). For exponential noise-to-state stability system (1), there is

\[
\mathbb{E}[V(x(t))] \leq e^{-ct} \left[ V(x(0)) - c^{-1} \gamma_{\max}\right] + c^{-1} \gamma_{\max}.
\]

Now, through this theorem together with Markov inequality \(\text{1}\), we are able to show the boundedness of the state in probability for exponential noise-to-state systems.

Theorem 2 (Adapt from [1]). For exponential noise-to-state stability system (1), \(\forall r > 0\) there is

\[
\mathbb{P}\{\|x(t)\|_2 < r\} \geq 1 - \frac{e^{-ct} \left[ V(x(0)) - c^{-1} \gamma_{\max}\right] + c^{-1} \gamma_{\max}}{\alpha_1(r)}
\]

which implies the state of the system is bounded in probability.

Proof. Applying Markov inequality to (6), it’s straightforward to conclude that for any \(v \geq 0\) there’s

\[
\mathbb{P}\{V(x(t)) < v\} \geq 1 - \frac{e^{-ct} \left[ V(x(0)) - c^{-1} \gamma_{\max}\right] + c^{-1} \gamma_{\max}}{v}.
\]

Besides, there’s the fact \(\{V(x) < \alpha_1(r)\} \subset \{\|x(t)\|_2 < r\}\) suggested by inequality (2). Hence, for any \(r > 0\) we have the result

\[
\mathbb{P}\{\|x(t)\|_2 < r\} \geq \mathbb{P}\{V(x(t)) < \alpha_1(r)\}
\]

\(\text{1}\)A stochastic process \(X(t)\) is called a super-martingale if for any \(t > s\) there’s \(\mathbb{E}[X(t) \mid X(s)] \leq X(s)\).

\(\text{1}\)The Markov inequality states that for any random variable \(X\) and any constant \(a > 0\), \(\mathbb{P}\{|X| \geq a\} \leq \mathbb{E}|X|/a\).
which together with inequality (8) leads immediately to the inequality (7). Since
\( \alpha_1(\cdot) \in K_{\infty} \) presented in the definition, we can observe that the right hand side of
inequality (7) goes to 1 as \( r \) going to infinity, which shows that the state of the system
is bounded in probability.

**Theorem 2** is a beautiful result, revealing the fact that there’s a probability bound
on the system’s state for exponential noise-to-state stability. It illustrates that at a
given time the state will stay in a safe region near the origin with a large probability
(cf. inequality (7)), which can be seen as a certain sense of stability for the stochastic
system.

However, we are not satisfied with it, because this result can not eliminate our
worry about the unexpected extreme event. What we interest more is the almost
surely long time behavior of the trajectory that is robust with given parameters.
Instead of considering the distribution in the probability sense, we come up with
an idea, discussing the distribution of state points measured in time horizon to be
almost surely bounded by a particular function. Mathematically, this distribution,
denoted as \( D(r) \), measures the proportion of time in which the state is bounded by
\( r \), and, therefore, can be calculated by
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{\|x(t)\|_2 < r\}} ds,
\]
which, termed as the
time average of the indicator function \( 1_{\{\|x(t)\|_2 < r\}} \). But such limitation sometimes
may lose its meaning, so in order to overcome this problem we replace the limitation
notation with the inferior limit one, “\( \lim \inf \)”. Hence, our goal becomes showing the
inferior time average of indicator function \( 1_{\{\|x(t)\|_2 < r\}} \) to be bounded from below and
its value converging to 1 as the bound getting to infinity, that is detecting whether
there’s a lower bound function \( b(\cdot) \) s.t.

\[
D(r) = \lim_{T \to \infty} \liminf_{T \to \infty} \int_0^T 1_{\{\|x(t)\|_2 < r\}} ds \geq b(r) \quad \text{a.s.} \quad \text{and} \quad \lim_{r \to \infty} b(r) = 1.
\]

If such \( b(\cdot) \) exists, then we denote \( q_k \) as the \( k \) fractile of distribution \( b(\cdot) \) which means
\( b(q_k) = k \). In the physical sense, if above formulas hold true, then it can be deduced
that the trajectory will fluctuate in a region near the origin (e.g. \( \{x \mid \|x\|_2 < q_{90\%}\} \))
with a large proportion of time (more than 90%), and, once going out, the state will
soon be back to it, which demonstrate another sense of stability distinct from the one
introduced before. In this paper, we will show that this time average analysis can be
achieved for exponential noise-to-state stability system.

### 3. Strong law of large numbers.

Before going to further analysis for noise-
to-state stability systems, it’s necessary for us to prepare some strong law of large
numbers first. The most famous strong law of large numbers is stated that the sample
average of a sequence of independent identical distributed random variables converges
almost surely to the expected value so long as the expected value is finite. Mathematically, it is presented as

**Theorem 3.** Consider a sequence of independent identical distributed random
variables \( \{X_n\} \). If \( EX_1 < \infty \), then

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{X_i}{n} = EX_1.
\]

For the proof of it, readers can refer to [2], which is the most elementary one
according to our knowledge.

In this section, we present some similar results, that the superior (inferior) limit
of sample average is almost surely less (greater) than a particular constant if the
distributions are uniformly controlled. It should be noted that, in these results, we no longer need the i.i.d condition as is presented in Theorem 3.

First of all, we introduce some basic lemmas in the following. In Lemma 2, we provides a method to map arbitrary random variables to the independent uniformly distributed ones. Conversely, it’s also possible for us to generate an arbitrary distributed random variable through a uniformly distributed one, which is shown in Lemma 3 and 4.

**Lemma 2.** Consider a sequence of random variables \( \{X_n\} \) admitting the conditional distribution \( g(\cdot) \) defined as

\[
g(s \mid X_1, \ldots, X_{n-1}) = P \{X_n \leq s \mid X_1, \ldots, X_{n-1}\}
\]

\[
g(s^- \mid X_1, \ldots, X_{n-1}) = \lim_{l \to s^-} g(s \mid X_1, \ldots, X_{n-1})
\]

where \( s \in \mathbb{R} \). Let \( \xi_n \sim U(0, 1) \) be another sequence of independent random variables that are also independent of \( \{X_n\} \). If we define

\[
Y_n = g(X_n^+ \mid X_1, \ldots, X_{n-1}) + \xi_n [g(X_n \mid X_1, \ldots, X_{n-1}) - g(X_n^- \mid X_1, \ldots, X_{n-1})]
\]

then \( \{Y_n\} \) are mutually independent and \( Y_n \sim U(0, 1) \).

**Proof.** According to the basic properties of distribution function, \( g(\cdot) \) is non-decreasing and right continuous respect to the first variable. Then, denoting \( X_n^+ = \max\{l \mid g(l \mid X_1, \ldots, X_{n-1}) < s\} \), we have

\[
g(X_n^+ \mid X_1, \ldots, X_{n-1}) \geq s \quad \text{and} \quad g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right) \leq s.
\]

In the meanwhile, with the help of independence, for any \( s \in (0, 1) \) there's

\[
P \{Y_n < s \mid X_1, \ldots, X_{n-1}, \xi_1, \ldots, \xi_{n-1}\} = P \{g(X_n \mid X_1, \ldots, X_{n-1}) < s \mid X_1, \ldots, X_{n-1}\}
\]

\[
+ P \{g(X_n \mid X_1, \ldots, X_{n-1}) \geq s \text{ and } g(X_n^- \mid X_1, \ldots, X_{n-1}) \leq s \mid X_1, \ldots, X_{n-1}\}
\]

\[
: P \left\{ \xi_n < \frac{s - g(X_n^- \mid X_1, \ldots, X_{n-1})}{g(X_n \mid X_1, \ldots, X_{n-1}) - g(X_n^- \mid X_1, \ldots, X_{n-1})} \mid X_n \right\}
\]

\[
= P \{X_n < X_n^+ \mid X_1, \ldots, X_{n-1}\}
\]

\[
+ P \{X_n = X_n^+ \mid X_1, \ldots, X_{n-1}\} \cdot \frac{s - g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right)}{g(X_n \mid X_1, \ldots, X_{n-1}) - g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right)}
\]

\[
= g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right) + \left[g(X_n \mid X_1, \ldots, X_{n-1}) - g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right)\right]
\]

\[
\cdot \frac{s - g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right)}{g(X_n \mid X_1, \ldots, X_{n-1}) - g\left((X_n^+)^- \mid X_1, \ldots, X_{n-1}\right)}
\]

\[
= s
\]

which shows that \( Y_n \sim U(0, 1) \) and \( Y_n \) is independent of \( X_1, \ldots, X_{n-1} \) and \( \xi_1, \ldots, \xi_{n-1} \). Furthermore, by the fact \( Y_k \) \( (k < n) \) is decided by \( X_1, \ldots, X_k \) and \( \xi_k \), so \( Y_n \) is independent of \( Y_k \). Thus \{\( Y_n \)\} are mutually independent. \( \square \)
Lemma 3. Given a random variable \( Y \sim U(0,1) \) and a distribution function \( F(\cdot) \), the random variable defined as
\[
Z := \inf \{ s \mid F(s) \geq Y \}
\]
is subject to distribution \( F(\cdot) \).

Proof. Since \( F(\cdot) \) is right continuous, there's \( F(Z) = \lim_{s \to Z^+} F(s) \geq Y \). Thus, \( Z \) is the minimum point of the set \( \{ s \mid F(s) \geq Y \} \). Moreover, by monotonicity of \( F(\cdot) \), we have \( \{ s \geq Z \} = \{ F(s) \geq Y \} \). So
\[
\mathbb{P}(Z \leq s) = \mathbb{P}(F(s) \geq Y) = F(s).
\]

Lemma 4. Given a random variable \( Y \sim U(0,1) \) and a distribution function \( F(\cdot) \), the random variable
\[
Z := \sup \{ s \mid F(s) \leq Y \}
\]
are subject to distribution \( F(\cdot) \).

Proof. The proof is similar to the former one. According to the definition of \( Z \) and monotonicity of distribution function, we have \( F(Z^-) \leq Y \), and, furthermore, \( \{ F(s^-) \leq Y \} = \{ Z \geq s \} \). Hence,
\[
\mathbb{P}(Z < s) = 1 - \mathbb{P}(Z \geq s) = 1 - \mathbb{P}(Y > F(s^-)) = F(s^-).
\]

Actually, the random variables defined in Lemma 3 and 4 are almost surely same. In another word, They only differ in zero measure set. Now, with the help of these lemmas, we are able to propose the following strong laws of large numbers.

Theorem 4. Let \( \{X_n\} \) be a sequence of independent random variables and \( X \) be another integrable random variable satisfying \( EX < \infty \). If for all \( n \in \mathbb{N}^* \) and \( s \in \mathbb{R} \)
\[
(11) \quad \mathbb{P}\{X_n \geq s \mid X_1, \ldots, X_{n-1}\} \leq \mathbb{P}\{X > s\}
\]
then
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i \leq EX \quad \text{a.s.}
\]

Proof. Let \( \{\xi_n\} \) be a sequence of independent random variables satisfying \( \xi_n \sim U(0,1) \) and \( \{\xi_n\} \) being independent from \( \{X_n\} \). In the meanwhile, define \( Y_n \) as Eq. (10) and denote
\[
(12) \quad Z_n = \sup \{ s \mid F(s) \leq Y_n \}
\]
where \( F(s) = \mathbb{P}\{X \leq s\} \). According to Lemma 2, \( \{Y_n\} \) are mutually independent and \( Y_n \sim U(0,1) \). Thus, \( \{Z_n\} \) are also mutually independent and, by Lemma 4, \( Z_n \) are subject to \( F(\cdot) \). Hence, by the strong law of large numbers Theorem 3, we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Z_i = EZ_1 = EX \quad \text{a.s.}
\]
Besides, inequality (11) implies \(g(s | X_1, \ldots, X_n) \geq F(s)\). Combing this result with the fact \(Y_n \geq g(X_n^- | X_1, \ldots, X_n)\) (cf. Eq. (10)), we can conclude that

\[ F(X_n^-) \leq g(X_n^- | X_1, \ldots, X_n) \leq Y_n. \]

Thus, by Eq. (12), we know \(X_n \leq Z_n\) for all \(n > 0\). So there’s

\[ \limsup_{n \to \infty} \sum_{i=1}^{n} \frac{X_i}{n} \leq \limsup_{n \to \infty} \sum_{i=1}^{n} \frac{Z_i}{n} = \mathbb{E}X \quad \text{a.s.}. \]

**Theorem 5.** Let \(\{X_n\}\) be a sequence of independent random variables and \(X\) be another integrable random variable satisfying \(\mathbb{E}X \leq \infty\). If for all \(n \in \mathbb{N}^*\) and \(s \in \mathbb{R}\)

\[ P\{X_n > s\} \geq P\{X > s\} \]

then

\[ \liminf_{n \to \infty} \sum_{i=1}^{n} \frac{X_i}{n} \geq \mathbb{E}X \quad \text{a.s.}. \]

**Proof.** The proof is quite similar to the previous one. Let \(\{\xi_n\}\) be a sequence of independent random variables satisfying \(\xi_n \sim U(0,1)\) and \(\{\xi_n\}\) being independent from \(\{X_n\}\). In the meanwhile, define \(Y_n\) as Eq. (10) and denote

\[ Z_n = \inf\{s \mid F(s) \geq Y_n\} \]

where \(F(s) = P\{X \leq s\}\). According to Lemma 2, \(\{Y_n\}\) are mutually independent and \(Y_n \sim U(0,1)\). Thus, \(\{Z_n\}\) are also mutually independent and, by Lemma 3, \(Z_n\) are subject to \(F(\cdot)\). Hence, by the strong law of large numbers Theorem 3, we have

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{Z_i}{n} = \mathbb{E}Z_1 = \mathbb{E}X \quad \text{a.s.}. \]

Besides, inequality (13) implies \(g(s | X_1, \ldots, X_n) \leq F(s)\). Combing this result with the fact \(Y_n \leq g(X_n | X_1, \ldots, X_n)\) (cf. Eq. (10)), we can conclude that

\[ F(X_n^-) \geq g(X_n^- | X_1, \ldots, X_n) \geq Y_n. \]

Thus, by Eq. (14), we know \(X_n \geq Z_n\) for all \(n > 0\). So there’s

\[ \liminf_{n \to \infty} \sum_{i=1}^{n} \frac{X_i}{n} \geq \liminf_{n \to \infty} \sum_{i=1}^{n} \frac{Z_i}{n} = \mathbb{E}X \quad \text{a.s.}. \]

It should be noted that the random variable \(X_n\) considered in Theorem 5 can be valued infinity. In this special case, the limit inferior equals to infinity which for sure consists with the result.

**4. Time average analysis.** In this section, we try to answer the question proposed in Section 2, whether there exists a function \(b(\cdot)\) such that formula (9) is achieved. Of course, our answer is “YES”. The main idea to solve this problem is to construct loops for the system (see the following definition) and analyse inferior time average of indicator functions through detecting the average time spent in each part of the loop.
**Definition 2 (Loops).** For noise-to-state stability system (1), let’s denote $\tau_0 = 0$ and

$$
\tau_{2i+1} = \inf \{ t > \tau_{2i} \mid V(x(t)) \geq V_1 \}
$$

$$
\tau_{2i+2} = \inf \{ t > \tau_{2i+1} \mid V(x(t)) = V_0 \}
$$

where $i = 0, 1, 2, \ldots$, and $V_0$, $V_1$ be some positive constants such that $c^{-1}\gamma_{\max} < V_0 < V_1$. We call the trajectory from time $\tau_{2i}$ to $\tau_{2i+2}$ as the $i$-th loop of the system. Particularly, the time $\tau_{2i+1} - \tau_{2i}$ and $\tau_{2i+2} - \tau_{2i+1}$ are termed as the up cross time and down cross time of the $i$-th loop, respectively.

If the noise of system is nonsingular which implies feller and irreducible conditions, then there’re almost surely infinite many loops and the process $\{x(\tau_{2i})\}$ converges to an ergodic Markov chain [5]. In this case, ergodicity of the system is guaranteed [5] thus the time average of indicator function equals to the spacial (probabilistic) average. However, for system (1), the nonsingular condition is at loss, time average of indicator function can no longer be concluded by ergodicity. In this paper, we borrow some ideas from [5], such as construct the loops in above definition, but follow another method to do time average analysis.

Note that during the up cross period, from $\tau_{2i}$ to $\tau_{2i+1}$, the Lyapunov function is strictly less than $V_1$. Hence, the inferior time average of indicator function $\mathbb{1}_{\{V(x(t)) \leq V_1\}}$ is lower controlled by the average of the up cross time. So is the inferior time average of $\mathbb{1}_{\{\|x(t)\| \leq e^{-\gamma}(V_1)\}}$. Therefore, average analysis for the up cross time and down cross time become the central feature in discussion. As a matter of fact, it’s possible for us to discuss the sample average of the up/down cross time by the strong law of large numbers introduced in previous section so long as the distribution is uniformly controlled. And, fortunately, this task, estimating the distributions of $\tau_{i+1} - \tau_i$, is quite easy to be achieved with the help of super-martingale like inequality (4).

### 4.1. Up cross time estimation

Now, let’s consider the up cross time. In the following, we show that the distributions of the up cross time, $\tau_{2i+1} - \tau_{2i}$ ($i > 0$), are uniformly controlled.

**Lemma 5.** For an exponential noise-to-state stability system (1), for any $s \geq 0$ there’s

$$
\mathbb{P}\{\tau_{2i+1} - \tau_{2i} > s \mid \tau_{2i} < \infty\} \geq \frac{V_1 - V_0}{V_1 - c^{-1}\gamma_{\max} + c^{-1}\gamma_{\max} e^{s}} \quad i \geq 1.
$$

**Proof.** Consider the case that for some $i \geq 1$ there’s $\tau_{2i} < \infty$. By the definition of expectation, for any $t > \tau_{2i}$ there’s

$$
\mathbb{E}\left\{ \left[ V(x(t \wedge \tau_{2i+1})) - c^{-1}\gamma_{\max} \right] e^{c(t \wedge \tau_{2i+1} - \tau_{2i})} \mid x(\tau_{2i}) \right\}
$$

$$
= \mathbb{E}\left[ \left[ V(x(t \wedge \tau_{2i+1})) - c^{-1}\gamma_{\max} \right] e^{c(t \wedge \tau_{2i+1} - \tau_{2i})} \mid \tau_{2i+1} \leq t, x(\tau_{2i}) \right] \mathbb{P}\{\tau_{2i+1} \leq t \mid x(\tau_{2i})\}
$$

$$
+ \mathbb{E}\left[ \left[ V(x(t \wedge \tau_{2i+1})) - c^{-1}\gamma_{\max} \right] e^{c(t \wedge \tau_{2i+1} - \tau_{2i})} \mid \tau_{2i+1} > t, x(\tau_{2i}) \right] \mathbb{P}\{\tau_{2i+1} > t \mid x(\tau_{2i})\}
$$

$$
\geq \mathbb{E}\left[ V(x(t \wedge \tau_{2i+1})) - c^{-1}\gamma_{\max} \mid \tau_{2i+1} \leq t, x(\tau_{2i}) \right] \mathbb{P}\{\tau_{2i+1} \leq t \mid x(\tau_{2i})\}

+ \mathbb{E}\left[ -c^{-1}\gamma_{\max} \cdot e^{c(t \wedge \tau_{2i+1} - \tau_{2i})} \mid \tau_{2i+1} > t, x(\tau_{2i}) \right] \mathbb{P}\{\tau_{2i+1} > t \mid x(\tau_{2i})\}
$$

$$
= \left( V_1 - c^{-1}\gamma_{\max} + c^{-1}\gamma_{\max} \cdot e^{c(t - \tau_{2i})} \right) \mathbb{P}\{\tau_{2i+1} \leq t \mid x(\tau_{2i})\} - c^{-1}\gamma_{\max} \cdot e^{c(t - \tau_{2i})},
$$

$$
\frac{V_1 - V_0}{V_1 - c^{-1}\gamma_{\max} + c^{-1}\gamma_{\max} e^{s}} \geq \mathbb{P}\{\tau_{2i+1} - \tau_{2i} > s \mid \tau_{2i} < \infty\}.
$$
Then applying inequality (4) where \( \tau_{2i} \) is taken as \( t_0 \), we have

\[
\mathcal{P}\{\tau_{2i+1} \leq t \mid x(\tau_{2i})\} \leq \frac{V_0 - e^{-c \gamma_{\max}} + c^{-1} \gamma_{\max} \cdot e^{c(t-\tau_{2i})}}{V_1 - e^{-c \gamma_{\max}} + c^{-1} \gamma_{\max} \cdot e^{c(t-\tau_{2i})}}.
\]

Since \( x(\tau_{2i}) \) is arbitrary, we can substitute it by the notation \( \tau_{2i} \leq \infty \). Therefore, the result follows immediately from above equation by choosing \( s = t - \tau_{2i} \).

Now, we focus on the distribution that controls up cross time. Let’s denote \( \tilde{X} \) as a random variable with the survival function

\[
\mathcal{P}\{\tilde{X} > s\} = \left\{ \begin{array}{ll}
\frac{V_1 - V_0}{V_1 - e^{-c \gamma_{\max}} + c^{-1} \gamma_{\max} \cdot e^{cs}} & s \geq 0 \\
1 & s < 0
\end{array} \right.
\]

Obviously, this survival function decays exponentially as \( s \) getting larger and therefore \( E\tilde{X} \), the integration of survival function, is finite. Moreover, its specific value is calculated as

\[
E\tilde{X} = \int_0^{\infty} \mathcal{P}\{\tilde{X} > s\} ds = \int_0^{\infty} \frac{V_1 - V_0}{V_1 - e^{-c \gamma_{\max}} + c^{-1} \gamma_{\max} \cdot e^{cs}} \cdot \frac{V_1 - e^{-c \gamma_{\max}}}{V_1 - e^{-c \gamma_{\max}} + c^{-1} \gamma_{\max} \cdot e^{cs}} ds
\]

\[
= \frac{V_1 - V_0}{V_1 - e^{-c \gamma_{\max}}} \left[ -e^{-1} \ln (e^{-cs}(V_1 - e^{-c \gamma_{\max}}) + e^{-c \gamma_{\max}}) \right]_0^{\infty}
\]

\[
= c^{-1} \frac{V_1 - V_0}{V_1 - e^{-c \gamma_{\max}}} \ln \frac{V_1}{e^{-c \gamma_{\max}}}
\]

For simplicity, in the rest of this paper we denote it as \( t_{uc} \).

Though we have show that the distributions of up cross time are uniformly controlled by a distribution of a integral random variable, it’s still not enough to apply Theorem 5 to it. Because we can’t figure out whether there’re infinite many loops, under the condition that non-singularity of the noise is lost. So, we need to do some tricks here.

We define a sequence of random variable \( \{\tilde{X}_i\} \ (i \geq 1) \) as

\[
\tilde{X}_i = \left\{ \begin{array}{ll}
\tau_{2i+1} - \tau_{2i} & \tau_{2i} < +\infty \\
+\infty & \tau_{2i} = +\infty
\end{array} \right.
\]

Obviously, the conditional probability, \( \mathcal{P}\{\tilde{X}_i > s \mid \tau_{2i} = +\infty, \tilde{X}_1, ..., \tilde{X}_{i-1}\} \) equals 1 if \( s < +\infty \), otherwise 0. Hence, for any \( s \in \mathbb{R} \) there’s

\[
\mathcal{P}\{\tilde{X}_i > s \mid \tau_{2i} = +\infty, \tilde{X}_1, ..., \tilde{X}_{i-1}\} \geq \mathcal{P}\{\tilde{X} > s\}
\]

Therefore, we can conclude that \( \{\tilde{X}_i\} \) are also controlled by \( \tilde{X} \), because

\[
\mathcal{P}\{\tilde{X}_i > s \mid \tilde{X}_1, ..., \tilde{X}_{i-1}\} = P\{\tilde{X}_i > s \mid \tau_{2i} = \infty, \tilde{X}_1, ..., \tilde{X}_{i-1}\} P\{\tau_{2i} = \infty \mid \tilde{X}_1, ..., \tilde{X}_{i-1}\}
\]

\[
+ P\{\tilde{X}_i > s \mid \tau_{2i} < \infty, \tilde{X}_1, ..., \tilde{X}_{i-1}\} P\{\tau_{2i} < \infty \mid \tilde{X}_1, ..., \tilde{X}_{i-1}\}
\]

\[
\geq \mathcal{P}\{\tilde{X} > s\}
\]

where the last inequality follows directly from (15) and (17). Also, by Theorem 5, \( \lim \inf_{n \to \infty} \sum_{i=1}^{n} \frac{\tilde{X}_i}{n} \geq t_{uc} \). Besides, it’s clear that, when there’re infinite loops, every
\( \hat{X}_i \) equals to \( \tau_{2i+1} - \tau_{2i} \). So, the inferior average of up cross time is almost surely lower bounded by \( t_{uc} \) if the number of loops is infinite.

**Lemma 6.** For an exponential noise-to-state stability system (1), there’s

\[
P \left\{ \text{There’re infinite many loops} \right\} \cap \left\{ \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \tau_{2i+1} - \tau_{2i} < t_{uc} \right\} = 0.
\]

**Proof.** It’s simply because

\[
P \left\{ \text{There’re infinite many loops} \right\} \cap \left\{ \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \tau_{2i+1} - \tau_{2i} < t_{uc} \right\}
= \mathcal{P} \left\{ \text{There’re infinite many loops} \right\} \cap \left\{ \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \hat{X}_i < t_{uc} \right\}
\]

\[
\leq \mathcal{P} \left\{ \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \hat{X}_i < t_{uc} \right\} = 0.
\]

### 4.2. Down cross time estimation.

Similar to the previous subsection, in the following we show that the down cross time is uniformly controlled by a particular distribution function with the help of super-martingale like inequality (4). Then a sequence of random variables \( \{\hat{X}_i\} \), which is quite similar to \( \{X_i\} \), is proposed to help discuss the average of down cross time.

Firstly, it should be noted that, for an exponential noise-to-state stability system, down cross time is almost surely finite.

**Lemma 7.** For an exponential noise-to-state stability system (1), there’s

\[
P \{ \tau_{2i} - \tau_{2i-1} < +\infty \mid \tau_{2i-1} < +\infty \} = 1 \quad \forall i \geq 1.
\]

**Proof.** Since the trajectory of system (1) is continuous, for any \( s \in [t \wedge \tau_{2i}, \tau_{2i-1}] \) \((t > \tau_{2i-1})\) we have \( V(x(s)) > V_0 \). Hence, by Dynkin’s formula, we can conclude

\[
E[V(x(t \wedge \tau_{2i})) \mid \tau_{2i-1} < \infty] = E[V(x(\tau_{2i-1})) \mid \tau_{2i-1} < \infty] + E \int_{\tau_{2i-1}}^{t \wedge \tau_{2i}} LV(x(s))ds
\]

\[
\leq V_1 + (-cV_0 + \gamma_{max})E[t \wedge \tau_{2i} - \tau_{2i-1} \mid \tau_{2i-1} < \infty].
\]

Note that \( V_0 > e^{-1}\gamma_{max} \), shown in Definition 2, thus

\[
E[t \wedge \tau_{2i} - \tau_{2i-1} \mid \tau_{2i-1} < \infty] \leq \frac{V_1 - E[V(x(t \wedge \tau_{2i})) \mid \tau_{2i-1} < \infty]}{cV_0 - \gamma_{max}} \leq \frac{V_1}{cV_0 - \gamma_{max}}.
\]

As \( t \) go to infinity, by monotone convergence theorem, \( E[\tau_{2i} - \tau_{2i-1} \mid \tau_{2i-1} < \infty] \leq \frac{V_1}{cV_0 - \gamma_{max}} < \infty \). Hence, down cross time, \( \tau_{2i} - \tau_{2i-1} \), is almost surely finite under the condition \( \tau_{2i-1} < \infty \).

Therefore, by super-martingale inequality (5), we can figure out that the down cross time is uniformly controlled by an exponentially like distribution function.

**Lemma 8.** For an exponential noise-to-state stability system (1), for any \( s \) bigger than \( \frac{1}{c} \ln \frac{V_1 - e^{-1}\gamma_{max}}{V_0 - e^{-1}\gamma_{max}} \) there’s

\[
P \{ \tau_{2i} - \tau_{2i-1} \geq s \mid \tau_{2i-1} < \infty \} \leq \frac{V_1 - e^{-1}\gamma_{max}}{V_0 - e^{-1}\gamma_{max}} e^{-cs} \quad \forall i \geq 1.
\]
Moreover, combing above inequality (18) and (19), we have
\[
E \left[ (V_0 - c^{-1} \gamma_{\text{max}}) e^{c(\tau_{2i-1} - \tau_{2i-1})} \mid x(\tau_{2i-1}) \right] \leq V_1 - c^{-1} \gamma_{\text{max}} \quad i \geq 1.
\]
Since \( x(\tau_{2i-1}) \) is arbitrary in above inequality, it can be substitute by the notation \( \tau_{2i-1} < \infty \). Furthermore, \( V_0 - c^{-1} \gamma_{\text{max}} \) is a positive constant, so dividing this constant at both sides, we can conclude
\[
E \left[ e^{c(\tau_{2i-1} - \tau_{2i-1})} \mid \tau_{2i-1} < \infty \right] \leq \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} \quad i \geq 1.
\]
Hence, by Markov’s inequality we have
\[
\mathcal{P} \left\{ e^{c(\tau_{2i-1} - \tau_{2i-1})} \geq e^{cs} \mid \tau_{2i-1} < \infty \right\} \leq \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} e^{-cs} \quad i \geq 1.
\]
which implies our result is true. \( \square \)

Similar to the former subsection, though we know that the down cross time is uniformly controlled, it’s still not enough for us to apply strong law of large numbers Theorem 4 to it, because the number of loops may be only finite. We need to do some tricks here to cover this gap.

Following the example of previous subsection, we denote \( \hat{X} \) as a random variable satisfying
\[
\mathcal{P} \left\{ \hat{X} > s \right\} = \begin{cases} \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} e^{cs}, & s \geq \frac{1}{c} \ln \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} \\ 1, & s < \frac{1}{c} \ln \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} \end{cases}
\]
and \( \{\hat{X}_i\} (i \geq 1) \) as a sequence of random variables defined by
\[
\hat{X}_i = \begin{cases} \tau_{2i} - \tau_{2i-1}, & \tau_{2i-1} < +\infty \\ 0, & \tau_{2i-1} = +\infty \end{cases}.
\]
Obviously, the conditional probability \( \mathcal{P} \left\{ \hat{X}_i \geq s \mid \tau_{2i-1} = \infty, \hat{X}_1, ..., \hat{X}_{i-1} \right\} \) equals 0 if \( s > 0 \), otherwise 1. So for any \( s \in \mathbb{R} \) there’s
\[
(19) \quad \mathcal{P} \left\{ \hat{X}_i \geq s \mid \tau_{2i-1} = \infty, \hat{X}_1, ..., \hat{X}_{i-1} \right\} \leq \mathcal{P} \left\{ \hat{X} > s \right\}.
\]
Moreover, combing above inequality (18) and (19), we have
\[
\mathcal{P} \left\{ \hat{X}_i \geq s \mid \hat{X}_1, ..., \hat{X}_{i-1} \right\} \leq \mathcal{P} \left\{ \hat{X} > s \right\}.
\]
Also, it should be note that the expectation of \( \hat{X} \) can be calculated as
\[
\mathbb{E} \hat{X} = \int_0^{\infty} \mathcal{P}(\hat{X} > s) ds = \int_0^{c^{-1} \ln \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}}} 1 ds + \int_{c^{-1} \ln \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}}}^{\infty} \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} e^{-cs} ds
\]
\[
= c^{-1} \left( 1 + \ln \frac{V_1 - c^{-1} \gamma_{\text{max}}}{V_0 - c^{-1} \gamma_{\text{max}}} \right)
\]
which is finite. So, according to Theorem 4, \( \limsup_{n \to \infty} \sum_{i=1}^{n} \frac{\hat{X}_i}{n} \leq E\hat{X} \) almost surely. For simplicity, we denote \( t_{dc} = E\hat{X} \) in the rest of this paper. Besides when there’re infinite many loops, \( \{\hat{X}_i\} \) equal to \( \{\tau_{2i} - \tau_{2i-1}\} \). Hence, superior average of down cross time is also almost surely upper bounded by \( t_{dc} \), if the number of loops is finite.

**Lemma 9.** For an exponential noise-to-state stability system (1), there’s

\[
P \left\{ \left\{ \text{There’re infinite many loops} \right\} \cap \left\{ \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} \tau_{2i} - \tau_{2i-1}}{n} > t_{dc} \right\} \right\} = 0.
\]

**Proof.** It’s simply because

\[
P \left\{ \left\{ \text{There’re infinite many loops} \right\} \cap \left\{ \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} \hat{X}_i}{n} > t_{dc} \right\} \right\}
= P \left\{ \left\{ \text{There’re infinite many loops} \right\} \cap \left\{ \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} \hat{X}_i}{n} > t_{dc} \right\} \right\}
\leq P \left\{ \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} \hat{X}_i}{n} > t_{dc} \right\} = 0.
\]

### 4.3. Main result.

With all these preparation, it’s possible for us to present our main results, here.

**Theorem 6.** For an exponentially noise-to-state stability system (1), we have

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{V(x(s)) < V_1\}} \, ds \geq \frac{t_{uc}}{t_{dc} + d_{uc}}.
\]

**Proof.** We first consider the case that the number of loops is finite. Let’s denote this number as \( i \). Since, down cross time is almost surely finite (Lemma 7), therefore \( \tau_{2i+1} \) should be infinite a.s.. In another word, after time \( \tau_{2i} \) the function \( V(x(t)) \) is strictly less than \( V_1 \), which implies that under this condition

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{V(x(s)) < V_1\}} \, ds = 1 \geq \frac{t_{uc}}{t_{dc} + d_{uc}}.
\]

Hence

(21)
\[
P \left\{ \left\{ \text{There’re finite many loops} \right\} \cap \left\{ \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{V(x(s)) < V_1\}} \, ds < \frac{t_{uc}}{t_{dc} + t_{uc}} \right\} \right\} = 0.
\]

Now, let’s consider the case that there’re infinite many loops. We denote \( i(T) \) as
max \{i \mid \tau_{2i} \leq T\}. According to the definition, we have

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{V(x(s)) < V_1\}} \, ds \geq \liminf_{T \to \infty} \frac{\sum_{i=0}^{n(T)-1} (T_{2i+1} - T_{2i}) + T_{2i}(T)+1 \land T - T_{2i}(T)}{\sum_{i=0}^{n(T)-1} (T_{2i+2} - T_{2i}) + T - T_{2i}(T)}$$

Then combing it with Lemma 6 and 9, we can conclude that

$$(22) \quad \mathcal{P}\left\{ \{\text{There're infinite many loops}\} \cap \left\{ \liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{V(x(s)) < V_1\}} \, ds < \frac{t_{uc}}{t_{dc} + t_{uc}} \right\} \right\} = 0.$$ 

Thus the result immediately follows from equation (21) and (22). \qed

Now we have show the inferior time average of function $1_{\{V(x(t)) < V_1\}}$ is lower bounded by $\frac{t_{uc}}{t_{up} + t_{dc}}$. Note that $t_{uc}$ and $t_{dc}$ are dependent on $V_0$. Hence, a problem arises, what the optimal $V_0$ is to maximize $\frac{t_{uc}}{t_{up} + t_{dc}}$ (or equivalently to minimize $\frac{t_{dc}}{t_{up}}$).

This is an optimization problem. Let’s denote $\beta = \frac{V_0 - c^{-1} \gamma_{max}}{V_1 - c^{-1} \gamma_{max}} (\in (0, 1))$. It’s easy to see that finding the optimal $V_0$ is equivalent to choosing the best $\beta$. Following this notation and Eq. (16) and (20), we can conclude that

$$t_{uc} = (1 - \beta)c^{-1} \ln \frac{V_1}{c^{-1} \gamma_{max}} \quad \text{and} \quad t_{dc} = c^{-1}(1 - \ln \beta).$$

Besides differentiating $t_{dc}/t_{uc}$ with respect to $\beta$, we have

$$\frac{d}{d \beta} \frac{t_{uc}}{t_{dc}} = \frac{2 - \frac{1}{\beta} - \ln \beta}{(1 - \beta)^2} \frac{V_1}{c^{-1} \gamma_{max}} = \ln \left( e^{\frac{1}{\beta}} - 1 \right) \frac{V_1}{c^{-1} \gamma_{max}}.$$

Obviously, when $0 < \beta < \frac{1}{W_{-1}(-e^{-2})}$, the derivative is smaller than 0, and when $\frac{1}{W_{-1}(-e^{-2})} < \beta < 1$, the derivative is larger than 0. Hence, $\frac{t_{uc}}{t_{dc}}$ reaches its minimum value when $\beta = \frac{1}{W_{-1}(-e^{-2})}$, and the optimal value is $-\frac{W_{-1}(-e^{-2})}{\ln \frac{V_1}{c^{-1} \gamma_{max}}}$, which leads to the our main theorem.

**Theorem 7.** For an exponentially noise-to-state stability system (1), we have

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{V(x(s)) < V_1\}} \, ds \geq \frac{\ln \frac{V_1}{c^{-1} \gamma_{max}}}{-W_{-1}(-e^{-2}) + \ln \frac{V_1}{c^{-1} \gamma_{max}}}.$$
and, therefore, for any \( r > \alpha_1^{-1}(c^{-1}\gamma_{\max}) \)

\[
D(r) \triangleq \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{\|z(s)\|_2 < r\}} \, ds \geq \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{V(z(s)) < \alpha_1(r)\}} \, ds
\]

\[
\geq \frac{\ln \alpha_1(r)}{e^{-r\gamma_{\max}}} - W_{-1}(-e^{-2}) + \ln \frac{\alpha_1(r)}{e^{-r\gamma_{\max}}}
\]

(23)

**Remark 3.** Since \( \alpha_1(\cdot) \in K_{\infty} \) shown in Definition 1, it’s easy to check that the lower bound of \( \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{\|z(s)\|_2 < r\}} \, ds \) converges to 1 as bound \( r \) going to infinity. Hence the main result, Theorem 7, has answered our question proposed in Section 2, by providing

\[
b(r) = \frac{\ln \alpha_1(r)}{e^{-r\gamma_{\max}}} - W_{-1}(-e^{-2}) + \ln \frac{\alpha_1(r)}{e^{-r\gamma_{\max}}}
\]

for formula (9). Particularly, in this main result, there’s no any additional condition required for the exponential noise-to-state stability system, in another word it is independent of the given parameters such as initial state and so on. Besides, let’s denote \( q_k \) \((k \in (0, 1))\) as

\[
q_k = \alpha_1^{-1}\left(e^{-1}\gamma_{\max} \cdot \exp\left(-\frac{k}{1-k} W_{-1}\left(-e^{-2}\right)\right)\right).
\]

Then we can observe that \( b(q_k) = k \), which means \( q_k \) is the \( k \) fractile of function \( b(\cdot) \). Hence, there’s no less than \( k \) proportion of the time in which the state is bounded by \( q_k \).

**Remark 4.** Inequality (6) and Eq. (24) state that the lower bound function \( b(\cdot) \) is affected by the covariance matrix \( \Sigma \) in a special way. As the norm of covariance matrix getting smaller, the quantity \( \gamma_{\max} \) will decrease and the lower bound estimation will increases. Particularly, when \( \Sigma = 0 \), the inferior time average of indicator function equals to one, which means the trajectory will maintained in a bounded region forever. This fact consists with our instinct that the behavior of the system is heavily relied on the system’s noise, and exposits the meaning of noise-to-state stability in a new perspective.

5. **A numerical example.** In this section, we present a numerical example to show the efficiency of our result. The considered system is written in the following form

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= -x_1 - x_2
\end{align*}
\]

Obviously, its structure consists with the system (1). Let’s denote a positive definite function \( V(x) = \frac{1}{2} (x_1^2 + x_2^2) \). Hence, following Itô’s formula, we have

\[
\mathcal{L}[V(x)] = - (x_1^2 + x_2^2) + \frac{1}{2} (x_1^2 + x_2^2) + \frac{1}{2} \sqrt{\|\Sigma(t)\Sigma(t)^T\|^2_F - 1}.
\]

If we define \( \alpha_1(\|x\|_2) = \alpha_2(\|x\|_2) = \alpha_3(\|x\|_2) = \frac{1}{2} \|x\|_2^2, c = 1 \) and \( \gamma(\|\Sigma\Sigma\|_F) = \frac{1}{2} \sqrt{\|\Sigma\Sigma^T\|^2_F - 1} \), then combing above inequality we can conclude

\[
\alpha_1(\|x\|_2) = V(x) = \alpha_2(\|x\|_2)
\]
The figures show the evolution of system (26). In figure (a) the distribution function $D(r)$ and lower bound function $b(r)$ are presented by the solid (blue) and dash (red) lines, respectively. In figure (b), we draw the evolution of the state’s norm (solid line), which seldom hit the bound $q_{\frac{1}{3}}$ shown by the dash (red) line. The trajectory of the system is presented in figures (c) and (d).

\[
\mathcal{L}[V(x)] \leq -\alpha_3 (\|x\|_2) + \gamma (\|\Sigma(t)\Sigma(t)^\top \|) = -cV(x) + \gamma (\|\Sigma(t)\Sigma(t)^\top \|)
\]

which are in accordance with conditions (2) and (3) in Definition 1. Therefore system (26) is an exponential noise-to-state stability system with Lyapunov function $V(x)$. Particularly, in this case the coefficient $\gamma_{\text{max}}$ equal to $\frac{1}{2}$.

Note that the noise term in system (26) is singular, which leads to the loss of classic ergodicity analysis for this system. Hence, we can apply an alternative method introduced in this paper to estimate the inferior time average of indicator function $\mathbb{1}_{\{\|x\|_2 \leq r\}}$. According to inequality (23) in Theorem 7 and Eq. (24) in Remark 3, we have that for any $r > \alpha_1^{-1} (e^{-1}\gamma_{\text{max}}) = 1$

\[
D(r) \triangleq \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{\|x(s)\|_2 < r\}} \mathrm{d}s \geq b(r) \triangleq \frac{2 \ln r}{-W_{-1}(-e^{-2}) + 2 \ln r}
\]

which answers our question. Moreover, by Eq. (25), it’s easy to observe $q_{\frac{1}{2}} = \exp \{-0.25W_{-1}(-e^{-2})\}$ approximating 2.2, and, therefore, the state stays in the region $\{x : \|x\|_2 < q_{\frac{1}{2}}\}$ more than one third of the time.

In Fig. 1, we have shown a simulation of the system (26). For this numerical example, we simulate 500 seconds, which, according to our understanding, is large enough to reveal the long time behavior of the system and depict the distribution of state points measured in the time horizon. In accordant with our previous analysis, from figure (a), we can observe that the distribution, $D(r)$ (solid line), is controlled by the function $b(r)$ (dash line). Beside, according to figure (b), there’s more than (actually far more than) one third of the time in which the state is bounded by $q_{\frac{1}{2}}$, and once the state exceeding this boundary it will soon be pulled back. Finally, figures (c) and (d) show that the trajectory of the system fluctuates mostly around the origin.
exhibiting a certain sense of stable behavior. All these facts consist with our results. It shows the efficiency of our work and, again, demonstrate the well-posedness of the concept, noise-to-state stability.

Notwithstanding these achievements, there’s a major shortage here. By observing figure (a), we can see that the distribution function $D(r)$ is far greater than the lower bound function $b(r)$, in another word our estimation is not very precise in this work. It needs our further efforts to improve this result.

6. Conclusion. In this paper, we develop the noise-to-state stability theory by showing an important feature that the distribution of the state points measured in the time horizon is almost surely controlled by $b(r)$. It leads to that the trajectory will fluctuate in a region around the origin with a large proportion of time, and, once going out, the system’s state will soon be back to it, which depicts a certain sense of stability for the systems. The analysis here is mainly based on detecting the average of up/down cross time in the loops that we construct for the systems. Thanks to the super-martingale like inequality (6) for the exponentially noise-to-state stability systems, the probability distribution of up/down cross time are uniformly controlled and our analysis is guaranteed by strong laws of large numbers, presented in Section 3. In Section 5, a numerical example is given to show the efficiency of our work.

In the future, there’s still a lot of work to do to improve the corresponding result. Primarily, we should improve the lower bound function $b(\cdot)$ so that it becomes more accurate to estimate the distribution function $D(\cdot)$.

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