Spanning tree packing, edge-connectivity and eigenvalues of graphs with given girth

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Abstract

Let \( \tau(G) \) and \( \kappa'(G) \) denote the edge-connectivity and the spanning tree packing number of a graph \( G \), respectively. Proving a conjecture initiated by Cioaba and Wong, Liu et al. in 2014 showed that for any simple graph \( G \) with minimum degree \( \delta \geq 2k \geq 4 \), if the second largest adjacency eigenvalue of \( G \) satisfies \( \lambda_2(G) < \delta - \frac{2k}{\delta+1} \), then \( \tau(G) \geq k \). Similar results involving the Laplacian eigenvalues and the signless Laplacian eigenvalues of \( G \) are also obtained. In this paper, we find a function \( f(\delta, k, g) \) such that for every graph \( G \) with minimum degree \( \delta \geq 2k \geq 4 \) and girth \( g \geq 3 \), if its second largest adjacency eigenvalue satisfies \( \lambda_2(G) < f(\delta, k, g) \), then \( \tau(G) \geq k \). As \( f(\delta, k, 3) = \delta - \frac{2k}{\delta+1} \), this extends the above-mentioned result of Liu et al. Related results involving the girth of the graph, Laplacian eigenvalues and the signless Laplacian eigenvalues to describe \( \tau(G) \) and \( \kappa'(G) \) are also obtained.

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Key words: Girth; Edge-connectivity; Edge-disjoint spanning trees; Spanning tree packing number; Eigenvalue; Quotient matrix

1 Introduction

We consider finite and simple graphs and follow [1] for undefined terms and notation. In particular, \( \Delta(G), \delta(G), \kappa'(G) \) and \( \kappa(G) \) denote the maximum degree, the minimum degree, the edge-connectivity and connectivity of a graph \( G \), respectively. The girth of a graph \( G \), is defined as

\[
g(G) = \begin{cases} 
\min\{|E(C)| : C \text{ is a cycle of } G\} & \text{if } G \text{ is not acyclic,} \\
\infty & \text{if } G \text{ is acyclic.}
\end{cases}
\]

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Let $d(G)$ be the average degree of $G$, and $\tau(G)$ be the maximum number of edge-disjoint spanning trees contained in $G$. A literature review on $\tau(G)$ can be found in [17]. As in [1], for a vertex subset $S \subseteq V(G)$, $G[S]$ is the subgraph of $G$ induced by $S$.

Let $G$ be a simple graph of vertex set $\{v_1, \ldots, v_n\}$. The adjacency matrix of $G$ is an $n \times n$ matrix $A(G) = (a_{uv})$, where $u, v \in V(G)$ and $a_{uv}$ is the number of edges joining $u$ and $v$ in $G$. As $G$ is simple, $A(G)$ is symmetric $(0, 1)$-matrix. Eigenvalues of $G$ are the eigenvalues of $A(G)$. We use $\lambda_i(G)$ to denote the $i$th largest eigenvalue of $G$. So $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $D(G)$ be the degree diagonal matrix of $G$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian matrix and the signless Laplacian matrix of $G$, respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the $i$th largest eigenvalue of $L(G)$ and $Q(G)$, respectively.

Fiedler [7] initiated the investigation between graph connectivity and graph eigenvalues. Motivated by Kirchhoff’s matrix tree theorem [11] and by a problem of Seymour (see Reference [19] of [5]), Cioabă and Wong [5] initiated the following conjecture.

**Conjecture 1.1** (Cioabă and Wong [5], Gu et al [8], Li and Shi [13] and Liu et al [14]) Let $k$ be an integer with $k \geq 2$ and $G$ be a graph with minimum degree $\delta \geq 2k$ and maximum degree $\Delta$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.

Several studies made progress towards Conjecture 1.1, as seen in [5, 8, 13, 14, 15]. The conjecture is finally settled in [15].

**Theorem 1.2** (Liu, Hong, Gu and Lai [15]) Let $k \geq 2$ be an integer, and $G$ be a graph with $\delta(G) \geq 2k$ for each of the following holds.

(i) If $\lambda_2(G) < \delta(G) - \frac{2k-1}{\delta(G)+1}$, then $\tau(G) \geq k$.

(ii) If $\mu_{n-1}(G) > \frac{2k-1}{\delta(G)-1}$, then $\tau(G) \geq k$.

(iii) If $q_2(G) < 2\delta(G) - \frac{2k-1}{\delta(G)+1}$, then $\tau(G) \geq k$.

Nash-Williams [16] and Tutte [19] proved a fundamental theorem on spanning tree packing number of a graph $G$.

**Theorem 1.3** (Nash-Williams [16] and Tutte [19]) Let $G$ be a connected graph and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any partition $(V_1, \ldots, V_t)$ of $V(G)$, $\sum_{i=1}^{t} d(V_i) \geq 2k(t-1)$.

As consequences of Theorem 1.3, relationship between $\tau(G)$ and $\kappa'(G)$ has been investigated, as seen in [9] and [12], among others. A characterization is proved in [3].

**Theorem 1.4** (Catlin, Lai and Shao [3]) Let $k \geq 1$ be an integer. Then $\kappa'(G) \geq 2k$ if and only if for any subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.

Cioabă in [4] initiated the investigation on the relationship between graph adjacency eigenvalues and edge-connectivity. A number of results have been obtained.
Theorem 1.5 Let $d$ and $k$ be integers with $d \geq k \geq 2$, and let $G$ be a simple graph on $n$ vertices with $\delta = \delta(G) \geq k$.

(i) (Cioabă [4]) If $G$ is $d$-regular and $\lambda_2(G) \leq d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geq k$.

(ii) (Cioabă [4]) If $G$ is $d$-regular and $\lambda_2(G) < d - \frac{2(k-1)}{d+1}$, then $\kappa'(G) \geq k$.

(iii) (Gu et al [8]) If $\lambda_2(G) < \delta - \frac{2(k-1)}{d+1}$, then $\kappa'(G) \geq k$.

(iv) (Liu et al [14]) If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geq k$.

These motivates the current research. It is natural to understand whether we will have a different range of the eigenvalues to predict the values of $\tau$ or $\kappa'$, when we are restricted to certain graph families such as bipartite graphs. The goal of this study is investigate, when the girth of a graph $G$ is known, the relationship between the eigenvalues of $G$ and $\tau(G)$, as well as $\kappa'(G)$. Motivated by the methods deployed in [15], for any graph $G$ with adjacency matrix $A$ and diagonal degree matrix $D$, we define $\lambda_i(G, a)$ to be the $i$th largest eigenvalues of $aD + A$, where $a \geq -1$ is a real number. For any integers $\delta$ and $g$ with $\delta > 0$ and $g \geq 3$, define $t = \lfloor \frac{\delta - 1}{2} \rfloor$, and $n_1^{*} = n_1^*(\delta, g)$ as follows.

\[
n_1^{*} = \begin{cases} 
1 + \delta + \sum_{i=2}^{t}(\delta - 1)^i, & \text{if } g = 2t + 1; \\
2 + 2(\delta - 1)^t + \sum_{i=1}^{t-1}(\delta - 1)^i, & \text{if } g = 2t + 2.
\end{cases}
\]  \hfill (1)

The main results are the following.

Theorem 1.6 Let $g$ and $k$ be integers with $g \geq 3$ and $k \geq 2$, $a \geq -1$ be a real number, and $G$ be a simple graph of order $n$ with minimum degree $\delta \geq k \geq 2$ and girth $g$. Each of the following holds.

(i) If $\lambda_2(G, a) \leq (a + 1)\delta - \frac{(k-1)n}{n_1^{*}(n-n_1^{*})}$ then $\kappa'(G) \geq k$.

(ii) If $\lambda_2(G, a) < (a + 1)\delta - \frac{2(k-1)}{n_1^{*}}$, then $\kappa'(G) \geq k$.

Theorem 1.7 Let $g$ and $k$ be integers with $g \geq 3$ and $k \geq 2$, $a \geq -1$ be a real number, and $G$ be a simple graph of order $n$ with minimum degree $\delta \geq 2k \geq 4$ and girth $g$. If $\lambda_2(G, a) < (a + 1)\delta - \frac{2k - 1}{n_1^{*}}$, then $\tau(G) \geq k$.

When we choose $a \in \{0, 1, -1\}$, then Theorems 1.6 and 1.7 will lead to results using $\lambda_2(G)$, $\mu_{a-1}(G)$ and $q_2(G)$ to describe $\kappa'(G)$ and $\tau(G)$. In particular, Theorem 1.7 has the following corollary. As $n_1^{*}(\delta, 3) = \delta + 1$, Corollary 1.8 extends Theorem 1.2.

Corollary 1.8 Let $g$ and $k$ be integers with $g \geq 3$ and $k \geq 2$, and $G$ be a simple graph of order $n$ with minimum degree $\delta \geq 2k \geq 4$ and girth $g$. Each of the following holds.

(i) If $\lambda_2(G) < \delta - \frac{2k - 1}{n_1^{*}}$, then $\tau(G) \geq k$.

(ii) If $\mu_{a-1}(G) > \frac{2k}{n_1^{*}}$, then $\tau(G) \geq k$.

(iii) If $q_2(G) < 2\delta - \frac{2k - 1}{n_1^{*}}$, then $\tau(G) \geq k$.

The arguments adopted in this paper are refinements and improvements of those presented in [14] and [15]. In the next section, we present the interlacing technique, a common tool in spectral theory of matrices. The proofs of the main results are in the subsequent sections.
2 Preliminaries

The main tool in our paper is the eigenvalue interlacing technique described below.

Given two non-increasing real sequences \( \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \) and \( \eta_1 \geq \eta_2 \geq \cdots \geq \eta_m \) with \( n > m \), the second sequence is said to *interlace* the first one if \( \theta_i \geq \eta_i \geq \theta_{n-m+i} \) for \( i = 1, 2, \ldots, m \). The interlacing is *tight* if exists an integer \( k \in [0, m] \) such that \( \theta_i = \eta_i \) for \( 1 \leq i \leq k \) and \( \theta_{n-m+i} = \eta_i \) for \( k+1 \leq i \leq m \).

**Lemma 2.1 (Cauchy Interlacing [2])** Let \( A \) be a real symmetric matrix and \( B \) be a principal submatrix of \( A \). Then the eigenvalues of \( B \) interlace the eigenvalues of \( A \).

Consider an \( n \times n \) real symmetric matrix

\[
M = \begin{pmatrix}
M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\
M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
M_{m,1} & M_{m,2} & \cdots & M_{m,m}
\end{pmatrix},
\]

whose rows and columns are partitioned according to a partitioning \( X_1, X_2, \ldots, X_m \) of \( \{1, 2, \ldots, n\} \). The *quotient matrix* \( R \) of the matrix \( M \) is the \( m \times m \) matrix whose entries are the average row sums of the blocks \( M_{i,j} \) of \( M \). The partition is *equitable* if each block \( M_{i,j} \) of \( M \) has constant row (and column) sum.

**Lemma 2.2 (Brouwer and Haemers [2, 10])** Let \( M \) be a real symmetric matrix. Then the eigenvalues of every quotient matrix of \( M \) interlace the ones of \( M \). Furthermore, if the interlacing is tight, then the partition is equitable.

3 Proof of Theorem 1.6

Following [1], for disjoint subsets \( X \) and \( Y \) of \( V(G) \), let \( E(X, Y) \) be the set of edges with one end in \( X \) and the other end in \( Y \), and

\[
e(X, Y) = |E(X, Y)|, \quad \text{and} \quad d(X) = e(X, V(G) - X).
\]

Tutte [18] initiated the cage problem, which seeks, for any given integers \( d \) and \( g \) with \( d \geq 2 \) and \( g \geq 3 \), the smallest possible number \( n(d, g) \) such that there exists a \( d \)-regular simple graph with girth \( g \). A tight lower bound (often referred as the Moore bound) on \( n(d, g) \) can be found in [6].

**Lemma 3.1 (Exoo and Jajcay [6])** For given integers \( d \geq 2 \) and \( g \geq 3 \), let \( t = \lfloor \frac{2d-1}{d-1} \rfloor \). Then

\[
n(d, g) \geq \begin{cases}
1 + d\sum_{i=0}^{t-1}(d-1)^i, & g = 2t + 1; \\
2\sum_{i=0}^{t}(d-1)^i, & g = 2t + 2.
\end{cases}
\]

We start our arguments with a technical lemma. For a subset \( X \subseteq V(G) \), define \( \overline{X} = V(G) - X \), and \( N_G(X) = \{ u \in \overline{X} : \exists v \in X \text{ such that } uv \in E(G) \} \). If \( X = \{v\} \), then we use \( N_G(v) \) for \( N_G(\{v\}) \). When \( G \) is understood from the context, we often omit the subscript \( G \).
Lemma 3.2 Let $G$ be a simple graph with minimum degree $\delta = \delta(G) \geq 2$ and girth $g = g(G) \geq 3$, and $X$ be a vertex subset of $G$. Let $n^*_1 = n_1^*(\delta, g)$ be defined as in (1). If $d(X) < \delta$, then $|X| = n_1 \geq n^*_1$.

Proof. For notational convenience, we use $X$ to denote both a vertex subset of $G$ as well as $G[X]$, the subgraph induced by the vertices of $X$.

Claim 3.3 $X$ contains at least a cycle.

By contradiction, assume that $X$ is acyclic. Then $|E(X)| \leq n_1 - 1$, and so

$$\delta \cdot n_1 = \delta \cdot |X| \leq \sum_{v \in X} d_G(v) = 2|E(X)| + e(X, Y) \leq 2(n_1 - 1) + \delta - 1,$$
leading to a contradiction $n_1 \leq \frac{\delta - 1}{\delta} < 1$. This proves Claim 3.3.

By Claim 3.3, $X$ must contain a cycle with length at least $g$. We shall justify the lemma by making a sequence of claims.

Claim 3.4 Each of the following holds.

(i) If $g \geq 3$, then there exists a vertex $u_0 \in X$ such that $N(u_0) \cap \overline{X} = \emptyset$.

(ii) If $g \geq 3$, then $X$ contains a path $P = u_0u_1u_2 \cdots u_{g-3}$ such that for any $i \in \{0, 1, 2, \ldots, g - 3\}$, $N(u_i) \cap \overline{X} = \emptyset$, for the neighborhood of whose each vertex is contained in $X$.

If (i) does not hold, then for every vertex $v \in X$, we always have $N(v) \cap \overline{X} \neq \emptyset$. Fix a vertex $v_0 \in X$. Then

$$d(X) = |N(v_0) \cap \overline{X}| + |e(X - \{v_0\}, \overline{X})| \geq |N(v_0) \cap \overline{X}| + |X - \{v_0\}|$$

$$\geq |N(v_0) \cap \overline{X}| + |N(v_0) \cap X| = d(v_0) \geq \delta,$$

contrary to the fact $d(X) < \delta$. Hence (i) follows.

We shall prove (ii) by induction on $g$. By (i), (ii) holds if $g = 3$. Assume that $g \geq 4$ and (ii) holds for smaller values of $g$. Thus $X$ contains a path $P' = u_0u_1 \cdots u_{g-4}$ such that for any $i \in \{0, 1, 2, \ldots, g - 4\}$, $N(u_i) \cap \overline{X} = \emptyset$. Let $N' = \{u' \in N(u_0) : N(u') \cap \overline{X} \neq \emptyset\}$ and $N'' = \{u'' \in N(u_{g-4}) : N(u'') \cap \overline{X} \neq \emptyset\}$. Since $g(G) = g$, for any $w \in N(u_0)$, $N(w) \cap V(P') = \{u_0\}$, and for any $w \in N(u_{g-4})$, $N(w) \cap V(P') = \{u_{g-4}\}$. As $u_{g-4} \in X$ and $|N(u_{g-4}) - V(P')| \geq \delta - 1 \geq d(X) \geq |N''|$, either $|N(u_{g-4}) - V(P')| > |N''|$, and so there must be a vertex $u_{g-3} \in N(u_{g-4}) - V(P') \cup N''$; or $|N(u_{g-4}) - V(P')| = |N''|$. If $|N(u_{g-4}) - V(P')| > |N''|$, then a path $P = u_0u_1u_2 \cdots u_{g-3}$ satisfying (ii) is found, and so (ii) holds by induction in this case. Hence we assume that $|N(u_{g-4}) - V(P')| = d(X) = |N''|$. This implies that $N' = \emptyset$ as for any $u' \in N'$, there must be a vertex $w' \in \overline{X}$ such that $u'w' \in E(G)$. Since $d(X) = |N''|$, this forces that $u' \in N''$, and so $E(P') \cup \{u_0u', u'u_{g-4}\}$ is a cycle of length $g - 2$, contrary to the assumption that the girth of $G$ is $g$. Hence if $|N(u_{g-4}) - V(P')| = d(X) = |N''|$, then $N' = \emptyset$, and so there must be a vertex $u_{g-1} \in N(u_0) - V(P')$ such that $N(u_{g-1}) \cap \overline{X} = \emptyset$. This implies that, letting $v_i = u_{i-1}$ for $0 \leq i \leq g - 3$, we obtain a path $P = v_0v_1 \cdots v_{g-3}$ such that for any $i \in \{0, 1, 2, \ldots, g - 3\}$, $N(v_i) \cap \overline{X} = \emptyset$. Hence (ii) is proved by induction. This justifies the claim.
Let $t = \lfloor \frac{n}{g} \rfloor$. By Lemma 3.1 and by Claim 3.4(ii), if $g = 2t + 1$ is odd, then

\[
|X| \geq 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i - d(X) - d(X)(\delta - 1) - \cdots - d(X)(\delta - 1)^{t-2} \geq 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i - \sum_{i=1}^{t-1} (\delta - 1)^i = n_1^*.
\]

By the same reason, if $g = 2t + 2$ is even, then

\[
|X| \geq 2 \sum_{i=0}^{t} (\delta - 1)^i - d(X) - d(X)(\delta - 1) - \cdots - d(X)(\delta - 1)^{t-2} \geq 2 \sum_{i=0}^{t} (\delta - 1)^i - \sum_{i=1}^{t} (\delta - 1)^i = n_1^*.
\]

This completes the proof of the lemma. \Box

3.1 Proof of Theorem 1.6(i)

Suppose that $k$ is an integer with $k \geq 2$. By contradiction, we assume that $\kappa'(G) = r \leq k - 1$. Then there exists a partition $(X, Y)$ with $Y = \overline{X}$ such that $e(X,Y) = r \leq k - 1 \leq \delta - 1$. Let $|X| = n_1, |Y| = n_2$. By Lemma 3.2 and as $n_1 + n_2 = n$, we have $n_1^* \leq \min\{n_1, n_2\} \leq n - n_1^*$. Hence $n_1 n_2 = n_1 (n - n_1) \geq n_1^*(n - n_1^*)$.

Let $\bar{d}_1 = \frac{1}{n_1} \sum_{v \in X} d(v), \bar{d}_2 = \frac{1}{n_2} \sum_{v \in Y} d(v)$. Then $\bar{d}_1, \bar{d}_2 \geq \delta$. Accordingly, the quotient matrix $R(aD + A)$ of $aD + A$ on the partition $(X, Y)$ becomes:

\[
R(aD + A) = \begin{pmatrix}
(a+1)\bar{d}_1 - \frac{r}{n_1} & \frac{r}{n_2} \\
\frac{r}{n_2} & (a+1)\bar{d}_2 - \frac{r}{n_2}
\end{pmatrix}.
\]

As the characteristic polynomial of $R(aD + A)$ is

\[
\lambda^2 - [(a+1)\bar{d}_1 - \frac{r}{n_1} + (a+1)\bar{d}_2 - \frac{r}{n_2}] \lambda + [(a+1)\bar{d}_1 - \frac{r}{n_1}][(a+1)\bar{d}_2 - \frac{r}{n_2}] - \frac{r^2}{n_1 n_2},
\]
we have, by direct computation,

\[
\lambda_2(R) = \frac{1}{2} \{(a + 1)d_1 - \frac{r}{n_1} + (a + 1)d_2 - \frac{r}{n_2}\} - \sqrt{[(a + 1)d_1 - \frac{r}{n_1} + (a + 1)d_2 - \frac{r}{n_2}]^2 - 4[(a + 1)d_1 - \frac{r}{n_1}][(a + 1)d_2 - \frac{r}{n_2} + \frac{4r^2}{n_1n_2}]}
\]

\[
= \frac{1}{2} \{(a + 1)d_1 - \frac{r}{n_1} + (a + 1)d_2 - \frac{r}{n_2}\} - \sqrt{[(a + 1)d_1 - \frac{r}{n_1} + (a + 1)d_2 + \frac{r}{n_2}]^2 + \frac{4r^2}{n_1n_2}}
\]

\[
= \frac{1}{2} \{(a + 1)d_1 - \frac{r}{n_1} + (a + 1)d_2 - \frac{r}{n_2}\} - \sqrt{[(a + 1)(d_1 - d_2) - \frac{r}{n_1} - \frac{r}{n_2}]^2 + \frac{4r^2}{n_1n_2}}
\]

\[
\geq \frac{1}{2} \{(a + 1)(d_1 + d_2) - \frac{r}{n_1} - \frac{r}{n_2}\} - \sqrt{[(a + 1)(d_1 - d_2) - \frac{r}{n_1} + \frac{r}{n_2}]^2 + 2(a + 1)|d_1 - d_2|\frac{r}{n_1} - \frac{r}{n_2})\}
\]

\[
= \frac{1}{2} \{(a + 1)(d_1 + d_2) - \frac{r}{n_1} - \frac{r}{n_2}\} - \sqrt{(a + 1)^2(d_1 - d_2)^2 + (\frac{r}{n_1} + \frac{r}{n_2})^2 + 2(a + 1)|d_1 - d_2|\frac{r}{n_1} - \frac{r}{n_2})\}
\]

\[
g \geq (a + 1)\delta - \frac{(k - 1)n}{n_1(n - n_1)}. \quad (5)
\]

By Lemma 2.2, \(\lambda_2(G, a) \geq \lambda_2(R) \geq (a + 1)\delta - \frac{(k - 1)n}{n_1(n - n_1)}\). By assumption, \(\lambda_2(G, a) \leq (a + 1)\delta - \frac{(k - 1)n}{n_1(n - n_1)}\), and so we must have \(\lambda_2(G, a) = \lambda_2(R) = (a + 1)\delta - \frac{(k - 1)n}{n_1(n - n_1)}\). It follows that all the inequalities in (5) must be equalities. Hence \(r = k - 1\) and \(d_1 = d_2 = \delta\), implying that \(G\) must be a \(\delta\)-regular graph, and so \(\lambda_1(G, a) = (a + 1)\delta\). By algebraic manipulation,
\[ \lambda_1(R) = \frac{1}{2} \left\{ \left[ (a+1)\delta - \frac{r}{n_1} + (a+1)\delta - \frac{r}{n_2} \right] \right. \\
+ \sqrt{\left[ (a+1)\delta - \frac{r}{n_1} + (a+1)\delta - \frac{r}{n_2} \right]^2 - 4\left[ (a+1)\delta - \frac{r}{n_1} + (a+1)\delta - \frac{r}{n_2} \right] + \frac{4r^2}{n_1n_2} } \right\} \\
= \frac{1}{2} \left\{ \left[ 2(a+1)\delta - \frac{r}{n_1} - \frac{r}{n_2} \right] + \sqrt{\left( \frac{r}{n_1} - \frac{r}{n_2} \right)^2 + \frac{4r^2}{n_1n_2} } \right\} \\
= \frac{1}{2} \left\{ \left[ 2(a+1)\delta - \frac{r}{n_1} - \frac{r}{n_2} \right] + \left( \frac{r}{n_1} + \frac{r}{n_2} \right) \right\} \\
= (a+1)\delta. \\
\]

Therefore, the interlacing is tight. By Lemma 2.2, the partition is equitable. This means that every vertex in \( X \) has the same number of neighbors in \( Y \). However, by Claim 3.4(i) of Lemma 3.2, there exists at least one vertex in \( X \) without a neighbor in \( Y \). This implies that \( r = e(X, Y) = k - 1 = 0 \), contrary to the assumption that \( k \geq 2 \). \( \Box \)

### 3.2 Corollaries of Theorem 1.6(i)

Throughout this subsection, \( n_1^* \) is defined as in (1). To see that Theorem 1.6(ii) follows from Theorem 1.6(i), we observe that as \( n_1^* \leq \min\{n_1, n_2\} \leq \frac{n}{2} \leq n - n_1^* \), it follows that

\[ (a+1)\delta - \frac{2(k-1)}{n_1} \leq (a+1)\delta - \frac{(k-1)n}{n_1(n-n_1)}. \]

and so Theorem 1.6(ii) follows from Theorem 1.6(i).

For real numbers \( a \) and \( b \) with \( \frac{n}{2} \geq -1 \), let \( \lambda_i(G, a, b) \) be the \( i \)th largest eigenvalues of the matrix \( aD + bA \). Thus \( \lambda_i(G, a, 1) = \lambda_i(G, a) \).

**Corollary 3.5** Let \( a \) and \( b \) be real numbers with with \( b \neq 0 \) and \( \frac{n}{2} \geq -1, k \) be an integer with \( k \geq 2 \), and \( G \) be a simple graph with \( n = |V(G)| \), \( g = g(G) \) and with minimum degree \( \delta = \delta(G) \geq k \). Then \( \kappa(G) \geq k \) if one of the following holds.

(i) \( b > 0 \) and \( \lambda_2(G, a, b) \leq (a+b)\delta - \frac{b(k-1)n}{n_1(n-n_1)} \)

(ii) \( b < 0 \) and \( \lambda_{n-1}(G, a, b) \geq (a+b)\delta - \frac{b(k-1)n}{n_1(n-n_1)} \).

**Proof.** As \( aD + bA = b(\frac{n}{2}D + A) \), it follows by definition that

\[ \begin{cases} 
\text{if } b > 0, & \text{then } \lambda_i(G, a, b) = b\lambda_i(G, \frac{n}{2}); \\
\text{if } b < 0, & \text{then } \lambda_{n-i+1}(G, a, b) = b\lambda_i(G, \frac{n}{2}).
\end{cases} \]

Hence Corollary 3.5 follows form Theorem 1.6(i). \( \Box \)

Choosing \( a \in \{0, -1, 1\} \) and \( b = 1 \) in Corollary 3.5, we have the following special case.
Corollary 3.6 Let $k$ be an integer with $k \geq 2$, and $G$ be a simple graph with $n = |V(G)|$, $g = g(G)$ and with minimum degree $\delta = \delta(G) \geq k$. Each of the following holds.
(i) If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{n_1(n-1)}$, then $\kappa'(G) \geq k$.
(ii) If $\mu_{n-1}(G) \geq \frac{(k-1)n}{n_1(n-1)}$, then $\kappa'(G) \geq k$.
(iii) If $q_2(G) \leq 2\delta - \frac{(k-1)n}{n_1(n-1)}$, then $\kappa'(G) \geq k$.

As $n_1^*(\delta, 3) = \delta + 1$ and by (6), Theorem 1.5 (iii) and (iv) are consequences of Corollary 3.6. Corollary 3.6 also implies the following result on bipartite graphs by setting $g \geq 4$ in Corollary 3.6.

Corollary 3.7 Let $G$ be a bipartite graph with minimum degree $\delta \geq k \geq 2$. If $\lambda_2(G) < \delta - \frac{k-1}{n_1}$, then $\kappa'(G) \geq k$.

4 Proof of Theorem 1.7 and its Corollaries

Throughout this section, for given integers $\delta$ and $g$, we continue defining $n_1^* = n_1^*(\delta, g)$ as in (1). We utilize the arguments deployed in [15] to prove Theorem 1.7 by imposing the girth requirement. In particular, the following technical lemma will also be used, with an additional condition $a \geq -1$ to justify the algebraic manipulation needed in the proof of the lemma.

Lemma 4.1 (Lemma 3.2 of [15]) Let $a \geq -1$ be a real number and $G$ be a simple graph with minimum degree $\delta = \delta(G)$. For any two disjoint nonempty vertex subsets $X$ and $Y$, if $\lambda_2(G, a) \leq (a + 1)\delta - \max\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$, then

$$[e(X, Y)]^2 \geq [(a + 1)\delta - \frac{d(X)}{|X|} - \lambda_2(G, a)][(a + 1)\delta - \frac{d(Y)}{|Y|} - \lambda_2(G, a)]|X||Y|.$$ (8)

Proof of Theorem 1.7. Let $V_1, \ldots, V_t$ be an arbitrary partition of $V(G)$. Without loss of generality, we assume that $d(V_1) \leq d(V_2) \leq \cdots \leq d(V_t)$. By Theorem 1.3, it suffices to show that $\sum_{i=1}^t d(V_i) \geq 2k(t-1)$. The inequality holds trivially if $t = 1$. Hence we assume that $t \geq 2$. If $d(V_1) \geq 2k$, then $\sum_{i=1}^t d(V_i) \geq t(2k) > 2k(t-1)$. Thus we also assume that $d(V_1) \leq 2k-1$.

Let $s$ be the largest integer such that $d(V_s) \leq 2k-1$. Then as $d(V_s) \leq 2k-1$, $1 \leq s \leq t$, and if $s < t$, then $d(V_{s+1}) \geq 2k$. By Lemma 3.2, $|V_i| \geq n_1^*$ for $1 \leq i \leq s$. It follows that for any $i$ with $i \leq s$,

$$\lambda_2(G, a) \leq (a + 1)\delta - \frac{2k-1}{n_1^*} \leq (a + 1)\delta - \frac{d(V_1)}{|V_1|} \cdot \frac{d(V_i)}{|V_i|}.$$ (8)

By (8) and Lemma 4.1,

$$[e(V_1, V_i)]^2 \geq \left[(a + 1)\delta - \frac{d(V_1)}{|V_1|} - \lambda_2(G, a)\right][a + 1)\delta - \frac{d(V_i)}{|V_i|} - \lambda_2(G, a)]|V_1||V_i|$$

$$\geq [2k - 1 - d(V_1)]|V_1|[2k - 1 - d(V_i)]$$

$$\geq [2k - 1 - d(V_1)]^2.$$
Hence $e(V_1, V_i) > 2k - 1 - d(V_i)$, or $e(V_1, V_i) \geq 2k - d(V_i)$, and so as $d(V_j) \geq 2k$ for all $j \geq s + 1$, we have

$$\sum_{i=1}^{t} d(V_i) = d(V_1) + \sum_{i=2}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \geq 2k(s-1) - \sum_{i=2}^{s} d(V_i) + \sum_{i=2}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \geq 2k(s-1) + 2k(t-s) = 2k(t-1).$$

Hence by Theorem 1.3, $\tau(G) \geq k$, as desired. This completes the proof of Theorem 1.7.

The following seemingly more general result can be derived from Theorem 1.7 by arguing similarly as in [15] and using (7), within certain ranges of the real numbers $a$ and $b$.

**Corollary 4.2** Let $a$ and $b$ be real numbers satisfying $b \neq 0$ and $\frac{a}{b} \geq -1$, $k$ be an integer with $k > 0$ and $G$ be a graph with $n = |V(G)|$, $g = g(G)$ and with minimum degree $\delta = \delta(G) \geq 2k$. Each of the following holds.

(i) If $b > 0$ and $\lambda_2(G, a, b) < (a+b)\delta - \frac{b(2k-1)}{n_1}$, then $\tau(G) \geq k$.

(ii) If $b < 0$ and $\lambda_{n-1}(G, a, b) > (a+b)\delta - \frac{b(2k-1)}{n_1}$, then $\tau(G) \geq k$.

Thus Corollary 1.8 now follows by letting $a \in \{0, 1, -1\}$ and $b = 1$ in Corollary 4.2.

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**References**

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, New York, 2008.

[2] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Springer Universitext, 2012.

[3] P.A. Catlin, H.-J. Lai and Y. Shao, Edge-connectivity and edge-disjoint spanning trees, Discrete Math., 309 (2009), 1033-1040.

[4] S.M. Cioabă, Eigenvalues and edge-connectivity of regular graphs, Linear Algebra Appl. 432 (2010) 458-470.

[5] S.M. Cioabă, W. Wong, Edge-disjoint spanning trees and eigenvalues of regular graphs, Linear Algebra Appl. 437 (2012) 630-647.
[6] G. Exoo and R. Jajcay, Dynamic cage survey, The Electronic Journal of Combinatorics DS16 (2011) 1-54.

[7] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. Journal 23 (1973) 298-305.

[8] X.F. Gu, H.-J. Lai, P. Li, S.M. Yao, Edge-disjoint spanning trees, edge connectivity and eigenvalues in graphs, J. Graph Theory 81 (2016) 16-29.

[9] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inform. Process. Lett. 16 (1983) 87-89.

[10] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 227-228 (1995) 593-616.

[11] G. Kirchhoff, Über die Auflöung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497-508.

[12] S. Kundu, Bounds on the number of disjoint spanning trees, J. Combinatorial Theory, 17 (1974) 199-203.

[13] G. Li and L. Shi, Edge-disjoint spanning trees and eigenvalues of graphs, Linear Algebra Appl. 439 (2013) 2784-2789.

[14] Q.H. Liu, Y.M. Hong, H.-J. Lai, Edge-disjoint spanning trees and eigenvalues, Linear Algebra Appl. 444 (2014) 146-151.

[15] Q.H. Liu, Y.M. Hong, X.F. Gu, H.-J. Lai, Note on edge-disjoint spanning trees and eigenvalues, Linear Algebra Appl. 458 (2014) 128-133.

[16] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445-450.

[17] E.M. Palmer, On the spanning tree packing number of a graph: a survey, Discrete Math. 230 (2001) 13-21.

[18] W.T. Tutte, A family of cubical graphs, Proc. Cambridge Philos. Soc. 43 (1947) 459-474.

[19] W.T. Tutte, On the problem of decomposing a graph into $n$ factors, J. London Math. Soc. 36 (1961) 221-230.