TRANSITION FROM CLUMPY TO SMOOTH ANGULAR DIAMETER DISTANCES

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ABSTRACT

Distance relations in a locally inhomogeneous universe are expected to behave like the Dyer-Roeder solution on small angular scales and the Friedmann-Robertson-Walker solution on large angular scales. Within a simple compact clump model the transition between these asymptotic behaviors is demonstrated and quantified. The redshift-dependent transition scale is of order a few arcseconds; this implies that it should have little influence on large angular scale cosmological tests such as the volume-redshift relation but possibly significant effects on arcsecond angular diameter measurements of radio galaxies and active galactic nuclei. For example, at \( z = 2 \) on arcsecond scales a clumpy flat universe mimics the angular diameter distance of a smooth \( \Omega = 0.27 \) model.

Subject headings: cosmology: theory — distance scale — galaxies: distances and redshifts

1. INTRODUCTION

Light propagation through our universe is generally treated in two separate regimes: either within the smooth Friedmann-Robertson-Walker model or within the gravitational lensing model of density inhomogeneities (see, e.g., Schneider, Ehlers, & Falco 1992). In the actual universe both local inhomogeneities and global homogeneity exist, so each acts as an asymptotic description under the appropriate conditions. Along with image amplifications, distortions, and time delays, the source distance-redshift relation is also affected by inhomogeneities. While many researchers discuss the distance relations in the presence of inhomogeneities (e.g., Futamase & Sasaki 1989; Watanabe & Tomita 1990) and how ignorance of clumpiness can affect determination of cosmological parameters by distance tests (e.g., Linder 1988a; Hadrovic & Binney 1997), the question of the nature of the transition region between the two regimes has not been addressed. It is of importance to understand where this transition occurs so as to use the two limiting cases only where they are valid approximations.

For almost all astrophysical applications of light propagation, geometric optics holds, where the curvature or inhomogeneity scales are much greater than the wavelength of the electromagnetic radiation, so the radiation can be treated as a beam or bundle of light rays. The focusing of the bundle due to the spacetime geometry determines the angular diameter distance \( r \) by

\[
d^2r/d\lambda^2 = - (\mathcal{R} + |\sigma|^2)r, \tag{1}
\]

where \( \lambda \) is the affine parameter measuring the path length, \( \mathcal{R} = \frac{1}{2}R_{\mu\nu}k^\mu k^\nu \) is the Ricci contribution of the gravitational focusing from matter within the beam, and \( \sigma \) is the Weyl contribution of gravitational shear from matter outside the beam. Here \( R_{\mu\nu} \) is the Ricci tensor and \( k^\mu \) the photon four-momentum.

To obtain a universal, i.e., isotropic, distance relation, equation (1) is generally solved under the “average path” assumption (Dyer & Roeder 1973) for an infinitesimal light beam, e.g., a single ray. This posits that the matter density along the line of sight is given by the global average density times a uniform smoothness parameter \( \alpha \), measuring the degree of small-scale inhomogeneity or clumpiness, and that the shear vanishes from global homogeneity. This approximation substitutes for detailed knowledge of the locally inhomogeneous metric, or physical conditions, along the light path, which we generally lack. Given that ignorance, some such effective model must be adopted, with the main caution being insurance that a “typical” path is indeed characteristic of the average.

As the light beam subtends larger and larger solid angles, at some point the inhomogeneities should be smoothed over, and we can legitimately calculate distances within the Friedmann-Robertson-Walker cosmology. The question arises how to connect these two asymptotic behaviors with a reasonable, preferably simple analytic model. (One can of course use numerical ray shooting within a specific pattern of inhomogeneities to compute the distance relations, but they will be relations, not one single universal relation as desired.) Of particular observational interest is how large the transition angle is, as many cosmological tests such as the number-redshift and magnitude-redshift relations are sensitive to the precise distance measure.

2. SMOOTHING

The standard model for density inhomogeneities when calculating cosmological distance relations is the Dyer-Roeder prescription: distribute randomly a fraction \( 1 - \alpha \) of the total matter density in clumps, point masses \( M \) with constant comoving number density \( n_0 \). Light rays passing too close to a clump will be appreciably gravitationally influenced (spatially deflected, cross-sectionally sheared, intensity amplified), and so will be recognized as something special and not treated as being on a typical line of sight. That leaves those paths avoiding clumps, but their neighborhoods only possess a matter density of \( \alpha \rho \), where \( \rho \) is the global average matter density. Thus, an “average” path does not feel the full Friedmann-Robertson-Walker (FRW) density and the calculated angular diameter distance \( r(\alpha, z) \) will differ from the FRW relation \( r_{\text{FRW}} = r(1, z) \). In this case \( \mathcal{R} = 4\pi(1 + z)^2\alpha \rho \), where \( z \) is the source redshift.

One can also generalize the model to redshift-dependent clumpiness \( \alpha(z) \), modeling the effects of evolving inhomogeneity, clump mass, or number density (Linder 1988b). Note that this brings up an interesting point. If we lived in a local inhomogeneity, either a bubble of different density or different clumpiness, then, even though the volume-averaged universe would appear Friedmann, our observed distance-redshift relation would not be the same as in that...
volume-averaged universe. It does not asymptotically approach the Friedmann result even at distances much greater than the bubble size. That is, the observer is in a special position, and volume rather than angle (line-of-sight) averaging may give misleading results; one cannot invoke an ergodic homogeneity because our position as observers selects a unique location. Such an isotropic inhomogeneity would lead to an apparent conflict between cosmological parameters derived from dynamical quantities and those involving the distance-redshift relation (Linder 1998).

Given the Dyer-Roeder Ansatz, one can see that as a light beam subtends a larger solid angle, the probability increases that it will include a clump. This raises the matter density within the beam actually jumps in a step function, a clump angle The lower curves are for a completely clumpy universe, the transverse area on the sky, and 

$\theta_c$ is the global smoothness parameter, i.e., that felt by an infinitesimal ray. So rather than having no clump within the beam and then suddenly all of it, we smoothly add its contribution according to the probability for inclusion $p(\theta)$. This is the main approximation we make in order to talk about a universal distance function. It does not asymptotically approach the need for recursion in determining $a(\theta)$. Because the ratio of $\theta$'s is insensitive to $\alpha$, the approximation is excellent: the fully recursed $x$ differs from $a_{\text{apx}}$ by at most 11% for the extreme case of $\alpha = 0$, $x = 3$ (and is more typically less than 1%)—and this occurs in a region of parameter space such that the distance $r(x)$ is never more than 1% from $r(a_{\text{apx}})$.

The resulting $a(\theta)$ has the pleasing properties of monotonicity, simplicity, and the proper asymptotic behaviors: as $\theta$ increases, the parameter goes from $a(\theta < \theta_c) = \alpha$ to $a(\theta \geq \theta_c) = 1$; i.e., we have effectively introduced an averaging procedure that provides the Dyer-Roeder clumpy universe result at small angles and the FRW result at large angles and defines the transition.

To find the transition angle $\theta_c$, combine equations (2) and (5) to get

$$1 = 2\pi n_0 \int_0^\theta d\theta \int_V \rho_r(1 + z)^3 r^2 [a(\theta)]$$

$$\equiv \pi\theta_c^2 n_0 H_0^{-3} f(z, \alpha)$$

for constant comoving clump density. So

$$\theta_c = \left[ \frac{\pi n_0 H_0^{-3} f(z, \alpha)}{1 - 2} \right]^{1/2}.$$

Taking $n_0 = (3H_0^2/8\pi)\Omega_c M^{-1}$, where $H_0 = 100$ h km s$^{-1}$ Mpc$^{-1}$ is the Hubble constant and $\Omega_c = (1 - \beta)\Omega$ is the clump matter density in units of the FRW critical density,

$$\theta_c = 1.3(M/10^{12})^{-1/2} M_c^{-1/2} h^{1/2}(1 - \beta)^{-1/2} \bar{a}^{-1/2}. \quad (9)$$

At low $z$ or high smoothness, $1 - \beta < 1$, the clump angle formally becomes large, but this is unphysical given our interpretation of $\theta_c$ as a transition angle to FRW behavior.

$\varepsilon_{\alpha}(\theta) = \bar{a} + (1 - \bar{a})[1 - \exp\left(-\frac{(\theta/\theta_c)^2}{2}\right)]$, \hspace{1cm} (6b)

plotted in Figure 1, where $\varepsilon_{\alpha} \equiv a(\theta_c) = 1 - e^{-1}(1 - \bar{a})$. We adopt $\Omega = 1$, which would cause the greatest deviation of the clumpy versus smooth FRW model. Note that the approximation in equation (6b) (solid curve) has removed the need for recursion in determining $a(\theta)$. Because the ratio of $\theta$'s is insensitive to $\alpha$, the approximation is excellent: the fully recursed $a$ differs from $a_{\text{apx}}$ by at most 11% for the extreme case of $\bar{a} = 0$, $x = 3$ (and is more typically less than 1%)—and this occurs in a region of parameter space such that the distance $r(x)$ is never more than 1% from $r(a_{\text{apx}})$.
A more physical transition angle $\Theta$ is defined as that angle where $r(\alpha(\theta))]$ first differs from $r_{\text{FRW}}$ by less than 5%.

Figure 2 plots both $\Theta$ and $\theta_t$ versus $z$. For low $z$ the distances $r$ are insensitive to $z$ (it enters at third order in an expansion of $r$ in $z$; see Linder 1988b), so even for $\theta = 0$ all $r(\alpha)$ are close to $r_{\text{FRW}}$. For high redshifts $\theta_t$ and $\Theta$ formally diverge as $\bar{z} \to 1$, because of the scarcity of clumps for the former and because of the difference in high-$z$ asymptotic behavior between smooth and even slightly clumpy distances for the latter. The growth is extremely slow, however: $\theta_t(z \gg 1) \approx 25^\circ$ for $\bar{z} = 0.999$, and $\Theta(z = 10^3) \approx 7^\circ$ for $\bar{z} = 0.99$. Realistically, $\bar{z}$ evolves because of structure formation such that it approaches unity sufficiently closely (e.g., greater than 0.998 for $z \leq 10^3$) that $\Theta \to 0$ at high redshifts. Thus, in general, distant source observations with fields of view larger than a few arcseconds can legitimately be treated within the FRW model for distances.

As a definite example, consider the volume-redshift cosmological test, which involves the angular diameter distance (note that the luminosity and proper-motion distances are simply related to the angular diameter distance by factors of $1+z$). From equation (5) the volume-redshift relation is

$$V(\theta) = V_{\text{FRW}}(\theta)[g(z, \alpha)/g(z, 1)],$$

(10)

where $x_{\text{FRW}} = 1$. For $z < 1$ the ratio of $g$ factors is close to unity, and for $z > 1$ we observe at $\theta \gg \theta_t$ (larger than arcsecond fields) and so $\alpha(\theta) \approx 1$ and again $V \approx V_{\text{FRW}}$. Thus, clumpiness effects are negligible in this case.

The differential volume test is more sensitive, since it does not possess the integration in equation (5) that includes those very low redshifts where $r$ is nearly independent of $z$ (as discussed above). Then

$$\Delta V(\theta)/V_{\text{FRW}}(\theta) = 2\theta_t^{-2} \int_0^\theta d\theta_0 \theta_0^2 [\theta(\theta_0)]^{-2}.$$

(11)

The ratio is bounded between $(r(\alpha_0)]/r(1)^2$ and $[(\theta_0)/r(1)]^2$ and can approach 1.25 at $z = 1$ as $\theta_0 \to 0$ if $\theta(\theta_0)$ is unreasonably pushed to zero. The deviation from unity is less than 5% for all redshifts and clumpiness factors, however, when $\theta > 12^\circ$. In particular, this assures us that the usual observations can be analyzed properly using the FRW volume element: for example, the Loh & Spiller (1986) fields were $7^\circ \times 10^\circ$ at $z < 0.75$. (See Omote & Yoshida 1990 for how the clumpy luminosity distance affects the analysis of flux-weighted counts.)

Clumpy and transition regimes can be important, however, for cosmological tests involving arcsecond scales. Possible applications include observations of radio galaxy lobes (Kapahi 1989; Guerra & Daly 1996) and milliarcsecond observations of active galactic nuclei (Gurvits 1994) for use in the angular diameter distance-redshift cosmological test. At a redshift of 2, for example, the difference between the clumpy and smooth angular diameter distances for $\Omega = 1$ is $33\%$, and a clumpy flat model gives the same distance as a smooth model with $\Omega = 0.27$.

Figure 3 shows this trade-off between clumpiness in a $\Omega = 1$ model and low density in a smooth model. Thus, a flat but clumpy universe could be misinterpreted through small angular scale observations as a lower density one. In general, one can match a distance in an $(z, \Omega)$ universe with an $(z' > z, \Omega' < \Omega)$ model. The numbers on the plotted curves give the angular scale of the observations in arcseconds and show the transition in clumpiness or, alternatively, density miscalculation from the infinitesimal case (Dyer-Roeder) to the large-scale average (FRW).

3. SHEAR

One point remains about which we must be cautious. The $\mathcal{R}$ term in equation (1) was treated by using an effective
smoothed density, but we neglected the shear term $|\sigma|^2$. On large scales we expect the shear to average to zero as a result of homogeneity, but this is not ensured at smaller scales: if the light passes far from the clump, then the shear on the bundle should be small; but if the light passes near the clump, then the shear on the bundle might be significant.

We can analyze this in terms of an effective clumpiness defined through equation (1) by

$$\alpha_{\text{eff}} = \alpha - (2/3) |\sigma|^2 H_0^{-2} \Omega^{-1}(1 + z)^{-5},$$

where $\Omega$ is the ratio of the total density to the critical density. Then the right-hand side of equation (1) can be written simply as $4\pi(1 + z)^2\alpha_{\text{eff}} \rho r$, and the previous results hold with the substitution of $\alpha_{\text{eff}}$ for $\alpha$. Note that shear always decreases $\alpha_{\text{eff}}$—bringing it further from the FRW value of unity—and so increases the transition angle $\theta_\text{E}$ (more properly, decreases the ratio $\theta_\text{E}/\theta_\text{c}$ for which $\alpha_{\text{eff}}$ is a given value). Also note that $\alpha_{\text{eff}}$ can be negative, which causes no mathematical worries [it merely makes the usual parameter $\beta \equiv (25 - 24\alpha)^{1/2} > 5$].

Alternatively, we can solve the full-beam equation (1). Because the shear squared source term of a point mass dies off as $(\text{distance})^{-4}$, we can approximate its behavior as localized in affine parameter, say, between $\lambda_0$ and $\lambda_0 + \Delta \lambda$.

Dividing the light propagation from a source at $\lambda_0$ to an observer at zero into three regimes, $(0, \lambda_0), (\lambda_0, \lambda_0 + \Delta \lambda)$, and $(\lambda_0 + \Delta \lambda, \lambda_0)$, we match the values of $r$ and its first derivative at the boundaries. For the most extreme case, $\alpha = 0$ and $\Omega = 1$, the solutions to the distance relations are $r_1 = \lambda$, $r_2 = C \sin |\sigma| + D \cos |\sigma|$, and $r_3 = \Delta \lambda + B$. Taylor expanding under the assumption $\Delta \lambda \ll \lambda_0$, $|\sigma|^{-1}$ yields

$$r_3(\lambda) \approx \lambda - |\sigma|^2 \lambda_0^2 \Delta \lambda (\lambda/\lambda_0 - 1)$$

$$\approx r_1(\lambda) - (4/25) |\sigma|^2 \Delta \lambda y_0^5 (1 - y_0^{5/2}) [1 - (y_0/y)^{5/2}],$$

where $y = 1 + z$ and $\lambda = \frac{1}{2} (1 - y^{-5/2})$.

Thus the shear has a significant effect on the angular diameter distance relation only if the second term is non-negligible. The maximum shear on the beam is given in order of magnitude by $|\sigma| \sim M b^{-2}$, where $b$ is the impact parameter from the mass $M$. For $b$ equal to the Einstein radius of the mass, $|\sigma| \sim H_0 r, r_h$, where the $r$'s are respectively measured from observer to source, observer to lens mass, and lens to source. The maximum value of the ratio of the second to first terms in equation (13) is then $(H_0 \lambda_0^2) \Delta \lambda/\lambda \ll 1$, meaning that, when concerned with distances, we can neglect shear for beams passing outside the Einstein ring of the mass.

So we argue that we can neglect the influence of shear on distances overall, because (1) narrow beams ($\theta < \theta_\text{E} = b_\text{E}/r_\text{E}$) would typically miss the Einstein ring because their probability for intersection $(\theta_\text{E}/\theta_\text{E})^2$ is small (e.g., $\theta_\text{E}/\theta_\text{E} \approx \Omega z^2 \ll 1$ for $z \ll 1$), (2) those few appreciably sheared light beams would be recognized as atypical and not used for distance measures, and (3) broad beams would have any shear effects diluted because of those portions of the beam that lie far from the clump and those symmetric (isotropic) about it. Therefore, we do not expect shear to alter significantly the distance-redshift relation for a typical light bundle, except possibly at high redshifts where the transition scale $\theta_\text{E}$ is an order of magnitude below the region of observational interest.

4. Conclusion

Using a simple toy model of an effective density distribution due to inhomogeneities, one can derive a universal angular diameter distance relation applicable over all angular scales. It has the desired asymptotic properties: agreeing for small solid angles with the clumpy universe Dyer-Roeder distance, re-creating for large solid angle observations the Friedmann-Robertson-Walker relation, and interpolating smoothly between them. The transition angle depends on the type of inhomogeneities but is estimated to be unlikely to exceed $10'$ for any cosmological test and should be of order $1'$ for any quantities involving the entire light propagation path between the source and the observer. Still, this can have significant effects on such relations as that between angular diameter and redshift for radio galaxies and active galactic nuclei—for example, a true clumpy flat universe mimicking a smooth open model—and must be looked for in such cosmological tests.

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