Nonlinear conductance of long quantum wires at a conductance plateau transition:
Where does the voltage drop?

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We calculate the linear and nonlinear conductance of spinless fermions in clean, long quantum wires where short-ranged interactions lead locally to equilibration. Close to the quantum phase transition where the conductance jumps from zero to one conductance quantum, the conductance obtains an universal form governed by the ratios of temperature, bias voltage and gate voltage. Asymptotic analytic results are compared to solutions of a Boltzmann equation which includes the effects of three-particle scattering. Surprisingly, we find that for long wires the voltage predominantly drops close to one end of the quantum wire due to a thermoelectric effect.

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Introduction.— A clean quantum wire with adiabatic contacts is characterized by a quantized conductance, \( G = n G_0 \) with \( G_0 = e^2/ \hbar \). The integer \( n \) describes the number of conduction channels (including spin). The conductance quantization is closely related to charge conservation across the wire (i.e. of the electrochemical potential) across the wire while for non-interacting, ballistic quantum wires the voltage drop occurs only close to the two contacts\textsuperscript{22}. In clean interacting quantum wires with low fermion density (and therefore negligible Umklapp scattering) the dc conductivity is infinite for an infinitely long wire due to momentum conservation, \( \sigma(T) = \infty \). The vanishing resistivity strongly suggests that there is again no voltage drop inside the wire.

A recent series of papers\textsuperscript{1,2,24,26,27}, which studied the role of equilibration in long but finite quantum wires of length \( L \), found that in the linear response regime, \( V \to 0 \), there is a linear drop of voltage\textsuperscript{2} along the wire. We resolve this apparent contradiction to \( \sigma(T) = \infty \) by noting that the limits \( V \to 0 \) and \( L \to \infty \) do not commute. The drop of voltage is governed by a new length scale \( \ell_V \) which diverges for \( V \to 0 \). For \( L < \ell_V \) a linear drop of voltage occurs. In the opposite limit, \( L \gg \ell_V \), however, the voltage drops only within a distance of \( \ell_V \) of the contacts. Surprisingly, the voltage drop is not symmetrical and occurs predominantly only at one of the two contacts!

Previous work on equilibrated quantum wires\textsuperscript{1,2,24,26,27} focused on the limit \( T \ll \epsilon_F \), where \( \epsilon_F \) is the Fermi energy. As scattering processes equilibrating left-moving and right-moving fermions involve the bottom of the band\textsuperscript{2,18,26,28}, they are exponentially suppressed and therefore the corresponding equilibration length is exponentially large, \( \ell_{\text{eq}} \sim e^{\Delta / T} \), where \( \Delta \approx \epsilon_F \) for weak interactions (thermal equilibration relevant for heat conductance occurs on shorter length scales\textsuperscript{29}). For \( L \gg \ell_{\text{eq}} \) it was found that the quantized (linear) conductance obtains corrections of order \( (T/\Delta)^2 \). Large effects can therefore be expected close to the conductance plateau transition, where \( \Delta \sim T \), as studied in this paper.

Model.— We consider 1D spin-polarized electrons with quadratic dispersion \( \epsilon_p = p^2 / 2 m \), interacting via a short range potential. Close to the QCP, where filling of the first subband becomes small, interactions are strongly irrelevant in the renormalization group sense\textsuperscript{30,31} and a
single electron description becomes approximately valid. To study equilibration and its effect on transport we may thus use the Boltzmann equation

\[ v_p \partial_x f_{x,p} = -I_{x,p}^{\text{col}}[f], \]

where \( f_{x,p} \) is the quasiclassical distribution function, \( v_p = p/m \) the velocity, and the collision integral \( I_{x,p}^{\text{col}} \) describes collisions. The contacts of the quantum wire to the leads at \( x = \pm \ell/2 \) induce boundary conditions for electrons moving into the quantum wire

\[ f_{x,\pm \ell/2} = \frac{1}{e^{\xi_p/\ell} + 1}, \]

where \( \xi_p = e_p - \mu \mp eV/2 \) with \( \mu = 0 \) at the QCP. Here we assume adiabatic and ballistic contacts, i.e., contacts which are smooth compared to the electronic wavelength but short in comparison to the scattering length.

In 1D systems energy and momentum conservation severely restrict the phase space available for scattering: in a two-particle process, two particles of equal mass can only exchange their momenta \([18, 28]\) which leaves \( f_p \) unchanged. One therefore has to study the effects of three particle collisions \([18, 28]\) described by

\[ I_{x,p_1}^{\text{col}}[f] = \sum_{p_2 \neq p_3 \neq p_1} W_{123}^{123} (1 - f_{x,p_1})(1 - f_{x,p_2})(1 - f_{x,p_3}) - f_{x,p_2} f_{x,p_3} (1 - f_{x,p_1})(1 - f_{x,p_2})(1 - f_{x,p_3}) \]  

where the scattering rate \( W_{123}^{123} \) arises to fourth order in the bare two-particle interactions \([31]\). For low energies and spinless fermions, Pauli principle ensures that it takes the universal form

\[ W_{123}^{123} = W((p_1 - p_2)(p_1 - p_3)(p_2 - p_3) \delta_{p_1(p_1 \mp p_3)(p_2 - p_3)}^2 \delta_{p_2} \delta(E_1 - E_f) \]

where \( P_{i(f)} = p_1 + p_2 + p_3 \) and \( E_{i(f)} = \epsilon_1 + \epsilon_2 + \epsilon_3 \) are the total momentum and energy of the three scattering particles before (after) the collision, respectively. A simple dimensional analysis allows to identify a characteristic length scale of equilibration at the QCP \( (\mu = 0) \) by setting typical momenta to \( \sqrt{2mT} \)

\[ \ell_{\text{eq}} = \frac{2Wm^3L^4}{(2\pi \hbar)^4(2mT)^{13/2}} \]

Measuring all length scales in units of \( \ell_{\text{eq}} \) and all momenta in units of \( \sqrt{2mT} \) allows to scale out the parameters \( W, m \) and \( T \) and the only remaining parameters are \( L/\ell_{\text{eq}}, eV/T \) and \( \mu/T \). We have checked both numerically and analytically that close to the QCP Hartree-Fock potentials (not included in Eq. (1)) can be neglected.

For a numerical solution of the Boltzmann equation (1) it is important to avoid discretization errors leading to a violation of conservation laws. We therefore use a conservative splitting method following Ref. [3], see supplement [31], to solve the time-dependent Boltzmann equation until a steady state has been reached. For the linear response calculation we use a linearized collision integral.

**Conservation laws.**— Three conservation laws govern transport in long quantum wires: charge, energy and momentum conservation. The corresponding currents are the charge current, \( j_c = e \sum_p v_p f_p \), the energy current, \( j_E \approx \sum_p f_p p v_p f_p \), and the momentum current, \( j_p \approx \sum_p f_p p v_p f_p \). The latter can be identified with pressure. For sufficiently long quantum wires and far away from the contacts the system will reach locally equilibrium with the distribution function

\[ f_{x,p}^{eq}(u, \mu, T) = \left[ 1 + e^{-\frac{(e\mu - mu)^2}{2mT}} \right]^{-1} \]  

parametrized by three space-dependent Lagrange parameters.

**FIG. 1:** (Color online) Electron density, \( n(x) = n_0(1 + \delta n(x)) \), at the QCP \( (\mu = 0) \) calculated from a solution of the Boltzmann equation (1), using (a) a linearized collision integral for \( L = 100\ell_{\text{eq}} \) and \( V \to 0 \) and (b) the non-linearized collision integral for \( L = 10\ell_{\text{eq}} \) and larger voltages, \( eV/T = 0.4 \) and \( eV/T = -0.6 \) (inset: \( eV/T = 0.4 \)). The inset shows \( \mu(x), T(x) - T \) and \( u(x) \) (in units of \( T \) and \( \sqrt{2T/m} \)) obtained from fitting the local charge, energy and momentum densities to Eq. (6). While in the linear response regime, \( L \ll |\nu| \), there is a linear voltage drop across the wire, the voltage (and the density) drops predominantly close to one of the two contacts for \( L \gg |\nu| \). Note that due to a finite drift \( u \) the chemical potentials close to the contacts do not match the chemical potential \( \mu \pm eV/2 \) in the leads (shown as separate dots at \( x = \pm L/2 \) in the insets).
eters $\mu(x)$, $T(x)$ and the velocity $u(x)$ reflecting the three conservation laws. For the distribution function \( \delta f \), one can calculate the corresponding equilibrium currents \( j^c_{\alpha} \), \( j^c_E \) and \( j^p_{\alpha} \) as function of \( \mu \), \( T \) and \( u \).

**Voltage drop.**—Fig. 1 shows the density profile and the local chemical potential (insets) of long quantum wires, \( L \gg \ell_{eq} \), obtained from our Boltzmann simulations (to a good approximation, \( \mu(x) \) can be measured by tunneling contacts \( 3D \)). For small \( V \) (Fig. 1a) there is both a linear drop of the chemical potential along the wire and a finite jump directly at the two contacts (the separate points at \( x = \pm L/2 \) show \( \mu \) in the leads) on a length scale set by the geometry of the contact (set to zero within our approach). This jump is also present for larger \( V \) (Fig. 1b), where, however, the linear voltage drop is absent. Surprisingly, there is instead a large asymmetric voltage drop which occurs only close to one of the two contacts. This behavior occurs for sufficiently long wires not only directly at the QCP but also away from it.

The qualitative difference between small and larger voltage can be understood from a simple argument based on matching currents. The steady state for \( V > 0 \) is characterized by the three currents \( j_c \), \( j_E \) and \( j_p \). From the three equations \( j_{\alpha} = j^c_{\alpha} \), \( \alpha = c, E, p \), one can, for sufficiently long wires, determine the three parameters \( \mu \), \( T \) and \( u \) which will be constant along the wire as \( j_{\alpha} = \text{const} \). Therefore, for a sufficiently long wire and finite \( V \), a voltage drop can occur only close to the contacts. In the linear response regime, i.e. for small \( V \), the situation is, however, different. By setting only the three parameters, \( u \), to zero, two currents, \( j^c_{\alpha} \) and \( j^c_E \), vanish in equilibrium. As both \( j^c_E \) and \( j^p_{\alpha} \) are linear in \( u \), their ratio \( j^c_{c} / j^c_E \) is - in the limit of small \( V \) - fixed by the average \( \mu \) and \( T \). This is used below when calculating the linear-response conductance analytically.

To develop an approximate analytical theory valid in both regimes, we consider small, but finite voltages \( V \) and parametrize \( f_{x,p} \) by

\[
f_{x,p} = f^c_{x,p} + \delta f_{x,p},
\]

where \( \delta f_{x,p} \) accounts for deviations from local equilibrium. Here is convenient to determine \( \mu(x) \), \( T(x) \) and \( u(x) \) from the two equations \( j_c = j^c_{c} \) and \( j_p = j^p_{c} \) while the third parameter is fixed by fitting the local density \( n(x) = n^c(x) \).

By linearizing the Boltzmann equation (1) in \( \delta f \), one obtains that \( \delta f \) is proportional to \( \partial n/\partial x \). For the energy current, one therefore obtains

\[
j_E = j^c_E + D^c \frac{\partial n}{\partial x} = \text{const}.
\]

where \( D \) is a certain thermoelectric diffusion constant which for small voltages is approximately constant across the wire. Using Galilei invariance which implies \( u = j^c_{c} / en^c = j_c / en \), we obtain \( j^c_E = \frac{3\beta j_c}{2en} - \frac{uj_c}{c^2n^2} \). For small \( V \) the last term can be neglected and one can linearize the density \( n = n_0 + \delta n \) to obtain

\[
-\frac{3\beta j_c \delta n}{2en^2} + D^c \frac{\partial \delta n}{\partial x} \approx j_E - \frac{3\beta j_c}{2en_0} = \text{const}.
\]

This equation introduces a new length scale

\[
\ell_V = \frac{2D^c en^2}{3\beta j_c}
\]

denotes the track of the change of the charge and energy current along the wire and therefore a linear drop in density and local chemical potential as in our numerical results, Fig. 1b. In the other limit, \( L \gg |\ell_V| \), \( \delta n \) obtains an exponential \( x \) dependence

\[
n(x) = n_L + (n_R - n_L) \exp \left[ \frac{x - L/2}{\ell_V} \right]
\]

with \( n_L/R \approx n(\mp L/2) \). The direction of the current and the sign of \( D \) determines whether the drop of density and voltage occur at the right (\( \ell_V > 0 \)) or left (\( \ell_V < 0 \)) lead, see Fig. 1c. This shows that the strongly asymmetric drop of voltage arises from a thermoelectric effect captured by the simple hydrodynamic equation (9).

**Linear response regime.**—Interestingly, it is possible to calculate in the linear response regime the quantum critical conductance for long quantum wires (\( \ell_{eq} \ll |\ell_V| \)) analytically. We use the approach developed in Ref. 12 (where only \( T \ll \mu \) was considered) and keep track of the change of the charge and energy current carried by right-moving electrons with \( p > 0 \), \( j^R_{c}(x) = c \int_{p>0} v_{kp} f_{x,p} \) and \( j^R_{E}(x) = \int_{p>0} v_{kp} p f_{x,p} \), respectively. We use that far away from the contacts the distribution function obtains local equilibrium, \( f_{x,p} \approx f^c_{x,p} \), described by Eq. (6) with \( T(x) = T + \delta T(x) \), \( \mu(x) = \mu + \delta \mu(x) \) and \( u(x) \). Current conservation implies \( u(x) = \text{const} \). Furthermore, the ratio \( r_1 = j^R_{E}/j^R_{c} \) can easily be calculated as it is independent of \( u \), i.e. just a function of \( V \)-independent parameters. The condition of constant momentum current fixes within linear response another ratio \( r_2 = \partial_x \delta T / \partial_x \delta \mu \). This allows to calculate the ratio \( r_3 = \partial_x j^R_{E} / \partial_x j^R_{c} \) as the remaining unknown \( \partial_x \delta \mu \) drops from this expression. To leading order, \( r_1, r_2 \) and \( r_3 \) are space independent functions of \( \mu \) and \( T \) (calculated in the supplementary material [30]). Finally, one identifies \( \delta f \) the difference in the charge (energy) current of the interacting- and non-interaction system as the total change in the right-moving charge (energy) current along the wire, \( j_c = j^R_{c} + \int v_{kp} j^R_{c} dx \) (\( j^R_{E} = j^R_{E} + \int v_{kp} j^R_{E} dx \)), respectively. If we now assume (as we checked numerically), that these integrals are dominated by their bulk contribution, we obtain the equation

\[
r_3 = \frac{r_1 j_c - j^R_{E}}{j_c - j^R_{c}}
\]
from which one can calculate directly \( j_c \). Combining all results [30], we find for the linear-response conductance up to corrections of \( \mathcal{O}(\ell_{eq}/L) \), i.e. for \( \ell_{eq} \ll L \ll |eV| \)

\[
G(z) = \frac{e^2}{h} \frac{\alpha_0(z)\alpha_2(z) - \alpha_1^2(z)}{\alpha_2(z) + \alpha_0(z)\kappa^2(z) - 2\alpha_1(z)\kappa(z)},
\]

where \( z = \mu/T, \langle ... \rangle = - \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{e^{i\xi\nu}}{\xi} \) with \( f^0_{\xi} = \frac{1}{1+e^\xi} \) and \( \alpha_k = \langle \xi_k \rangle_{z} \), \( \kappa = \frac{\langle \xi \sqrt{\pi/2} \rangle_{z}}{\sqrt{\pi/2} \xi_{z}} \). At the QCP, i.e., for \( \mu = 0 \), this gives

\[
\frac{G_{QCP}}{e^2/h} = \frac{\pi^2}{3} + \frac{9}{8} \zeta(3/2) \log^2 2 \approx 0.420
\]

with \( \zeta(x) \) the Riemann zeta function. \( G_{QCP} \) is about 16% below the non-interacting result \( e^2/2h \).

Fig. 2 displays the linear response conductance as functions of \( \mu/T \) for non-interacting (see below) and fully equilibrated electrons which have a clearly different shape. Our analytical formula (13) fits very well the numerical result (symbols).

Shorter wires. — Upon lowering \( T \), \( \ell_{eq} \) rapidly increases, see Eq. (5). For quantum wires, where \( L/\ell_{eq} \ll 1 \), one can neglect the effects of equilibrating interactions. Half of the voltage drops at the left and right contact, respectively, and there is no voltage drop inside the wire as \( f_{x,p} = f_p \) is independent of \( x \). For \( j_c \), one obtains the well-known non-interacting result \( j^0_c = e^2 F_0 \ln \left( \frac{1 + e_{p}^{2}/2}{1 + e_{p}^{2}/2} \right) \).

The conductance plateau transition in linear response is therefore described by \( G(\mu/T) = G_0/(1 + e^{-\mu/T}) \) while at \( \mu = 0 \) the current is for arbitrary \( eV/T \) given by \( j_0 = G_0V/2 \). In the inset of Fig. 2 we have calculated numerically the crossover from the interacting quantum critical conductance (14) to the non-interacting one, which occurs upon lowering \( T \) when \( \ell_{eq} \sim L \).

Nonlinear response. — Fig. 3 show the nonlinear conductance \( j_c/V \) at the QCP, i.e. for \( \mu = 0 \). It interpolates between the linear-response value (14) and the non-interacting result obtained for \( |eV|/T \to \infty \). For \( |eV| \gg T \) and \( \mu = 0 \) all states with \( p > 0 \) and \( \epsilon_p < |eV|/2 \) are occupied. As this is also an equilibrium distribution function with \( u = \sqrt{|eV|/4m} \) and \( \mu = mu^2/2 \), collisions have no effects in this limit.

For small \( V \) the nonlinear conductance appears to be non-analytic,

\[
j_c(V) = G_{QCP}V + \gamma|V|V + ... \tag{15}
\]

which can be traced back to the asymmetric voltage drop for \( L \gg |eV| \). As \( n_R - n_L \) in Eq. (11) varies linear in \( V \), the density in the center, \( n(0) \approx \max(n_R, n_L) \) according to Eq. (11), obtains for \( L \approx |eV| \) a correction proportional to \( |V| \). As \( j_c \approx eun \) this implies a correction proportional to \( |V|V \) to the current as soon as \( L \gg |eV| \). Due to numerical problems, we were not able to obtain reliable numerical results in the small-\( V \) regime \( L \lesssim |eV| \) where we expect a rounding of the non-analytic correction. Overall, the finite size corrections to the non-linear conductance are smaller than our numerical resolution for \( L \gg \ell_{eq} \), see inset of Fig. 3.

Outlook. — While our results have been derived only for spinless fermions with short ranged interactions, we expect that our main qualitative results are also of direct relevance for quantum wires made of electrons with spin and long-ranged Coulomb interactions. For these systems, the hydrodynamic equation (9) should also be valid (with strongly modified parameters). The highly asymmetric voltage drop predicted by us will probably
be even much easier to observe, as the stronger interactions imply that the regimes $L \gg \ell_{eq}$ and $L \gg |\ell_V|$ are much easier to reach.

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Supplementary material to: “Nonlinear conductance of long quantum wires at a conductance plateau transition: Where does the voltage drop?”

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In this supplementary material we discuss (i) the irrelevance of short range interaction for spinless fermions, (ii) our numerical implementation of the Boltzmann equation, (iii) some more details on the calculation of the linear-response conductance and, finally, (iv) how a voltage drop along the quantum wire can be measured.

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Irrelevance of short ranged interactions at QCP

The imaginary-time action describing spinless fermion with short ranged interaction at the QCP ($\mu = 0$) takes the form

$$ S = \int d\tau \int dx \left\{ \bar{\psi} \left( \partial_\tau - \frac{\hbar^2}{2m} \partial_x^2 \right) \psi + V \bar{\psi} (\partial_x \bar{\psi}) (\partial_x \psi) \right\} $$

(S16)

The derivatives in the interaction term reflect Pauli’s principle, $\bar{\psi} \psi = 0$. From this action one can directly read off the scaling dimensions of the field, $[\psi] \sim L^{-1/2}$, and the (imaginary) time, $[\tau] \sim L^2$, for $[x] \sim L$. This simple power-counting shows that the interaction strength scales to smaller values, $V \rightarrow V/\lambda^2$, as the distance of Fermions is increased, $x \rightarrow \lambda x$, or as temperature is lowered, $T \rightarrow T/\lambda^2$. Therefore the interactions are irrelevant at the QCP. This result also implies that the quasiparticle picture and therefore the Boltzmann equation becomes asymptotically exact upon approaching the QCP.

Conservative splitting method

For the numerical implementation of the Boltzmann equation we follow Ref. \textsuperscript{2} and solve the time-dependent Boltzmann equation, by splitting time evolution into a free flow and a relaxation stage. The advantage of the splitting procedure is that the distribution obtained after a relaxation step can be corrected such that conservation laws are fulfilled in each collision, see Ref. \textsuperscript{3} for details.

Free flow and relaxation steps were implemented by a finite difference scheme, i.e. by discretizing two-dimensional phase-space. We used a first order implicit-explicit upwind scheme to model the free propagation step, and an implicit scheme for the relaxation step. The steady state is reached when currents of the conserved quantities, $j_c$, $j_p$, and $j_E$ are constant along the wire. For the linear response calculation we parametrize $j_{x,p} = f_{x,p}^0 + \delta j_{x,p}$, where $f_{x,p}^0$ is the distribution at zero bias and $\delta j_{x,p}$ is linear in $V$, and linearize the collision integral in $\delta f$. For the numerically more demanding calculations employing the full collision integral we used meshes with 22 and 42 points in momentum space and various different homogeneous and inhomogeneous discretization of space with $\sim 100$ grid-points. We checked that our results are independent of discretization.

Linear conductance in long wires

A method to calculate the linear conductance in long wires $\ell_{eq} \ll L \ll |V|$, based on conservation laws and the distribution of fully equilibrated electrons Eq. (6) in the main text, was originally introduced in Refs. \textsuperscript{1,2}. As described in the main text, we need to calculate the ratios $r_1 = j_E/j_c$, $r_2 = \partial_x \delta T/\partial_x \delta \mu$ and $r_3 = \partial_x j_E^R/\partial_x j_c^R$. The current $j_c$ in response to the applied voltage (to linear order $V$) is then found from

$$ r_3 = \frac{r_1 j_c - j_c^0}{j_c - j_c^0} $$

(S17)

where $j_c^0$ and $j_E^0$ are the (linear response) charge and energy currents of non-interacting electrons which is directly described by Eq. (2) of the main text,

$$ j_c^0 = \frac{e^2 V}{\hbar} \alpha_0, \quad j_E^0 = \frac{e V}{\hbar} (\mu \alpha_0 + T \alpha_1), \quad \alpha_k = \langle \xi^k \rangle_z $$

(S18)

Here and in the following $\langle \ldots \rangle_z = - \int_{-z}^{\infty} d\xi (\ldots) \frac{d f_\xi^0}{d \xi}$, and $z = \mu/T$.

With these definitions we first calculate $r_1$. Using the equilibrium distribution Eq. (6) of the main text we find

$$ r_1 = \frac{j_E}{j_c} = \frac{1}{e} (\mu + T \kappa), \quad \kappa = \frac{\langle \xi \sqrt{1 + \xi/z} \rangle_z}{\langle \sqrt{1 + \xi/z} \rangle_z} $$

(S19)

To calculate $r_2$ we use that momentum conservation implies homogeneity of momentum-current in the steady
state. The latter can again be calculated with help of Eq. (6) given in the main text by expanding $T(x) = T + \delta T(x)$, $\mu(x) = \mu + \delta \mu(x)$ form small $V$. To linear order in $V$, i.e., for small $\delta T$ and $\delta \mu$, we find

$$\text{const.} = j_p(x) = n \delta \mu(x) + n k \delta T(x) + \text{const.} \quad (S20)$$

resulting in

$$r_2 = \frac{\partial_x \delta T}{\partial_x \delta \mu} = -\kappa^{-1} \quad (S21)$$

To obtain $r_3$ we can directly use the definition of $j^R_c$ and $j^R_T$ given in the main text combined with the equilibrium distribution function (6) of the main text

$$\partial_x j^R_c = \frac{e \alpha_0}{\hbar} \partial_x \delta \mu + \frac{e \alpha_1}{\hbar} \partial_x \delta T \quad (S22)$$

and

$$\partial_x j^R_T = \frac{\mu}{e} \partial_x j^R_c + \frac{T}{\hbar} (\alpha_1 \partial_x \delta \mu + \alpha_2 \partial_x \delta T) . \quad (S23)$$

For the ratio $r_2$ we therefore obtain

$$r_3 = \frac{\partial_x j^R_T}{\partial_x j^R_c} = \frac{\mu}{e} + \frac{T}{\hbar} \left( \frac{\alpha_1 \kappa - \alpha_2}{\alpha_0 \kappa - \alpha_1} \right) \quad (S24)$$

Inserting above expressions into (S17) and solving for $j_c$ gives Eq. (13) of the main text.

**Measuring voltage profiles**

To measure the voltage drop across a quantum wire one can, for example, use a weakly coupled tunneling contact realized by the tip of a scanning tunneling microscope. Assuming a constant tunneling matrix element $M$ and a constant density of states $n_0$ of the tunneling tip, the charge current $I_c(x)$ and the energy current $I_E(x)$ through the tip located at position $x$ are given by

$$I_c(x) = \frac{4 \pi e n_0}{\hbar} |M|^2 \int dk (f_{x,p} - f^\text{tip}_{\mu_T}(\epsilon_k)) \quad (S25)$$

$$I_E(x) = \frac{4 \pi e n_0}{\hbar} |M|^2 \int dk \epsilon_k (f_{x,p} - f^\text{tip}_{\mu_T}(\epsilon_k))$$

Here $f_{x,p}$ is the distribution function of the wire and $f^\text{tip}(\epsilon)$ is the Fermi distribution describing the occupation of the states in the tip. The latter is parametrized by the chemical potential $\mu_T$ and the temperature $T_T$.

The local chemical potential $\mu(x)$ of the quantum wire and the local temperature $T(x)$ of the wire are now obtained from the condition that particle and energy currents flow only if there is a difference in the chemical potential and temperature. $T(x)$ and $\mu(x)$ are therefore obtained from the condition

$$T(x) = T, \quad \mu(x) = \mu \quad \text{for} \quad I_c(x) = I_E(x) = 0 \quad (S26)$$

Note that this definition can be used for distribution functions far from equilibrium (and the usual result is obtained in equilibrium). Eq. (S25) implies that the local chemical potential and temperature can directly be obtained from the local charge- and energy density of the system.

In Fig. S4 we show chemical potential and temperature profiles obtained from the definition (S26) for a wire of length $L = 10 \ell_{eq}$ with $eV/T = 0.4$ at the QCP ($\mu = 0$). The separate points at $\pm L/2$ denote the chemical potential in the two leads.

In the insets of Fig. 1 of the main text we also show $\mu(x)$ and $T(x)$ but in this case we use a different definition of these quantities, which is more appropriate to illustrate our analytical arguments. In the main text, we fit the local charge, momentum and energy densities to the equilibrium distribution given by Eq. (6) of the main text which depends not only on $\mu$ and $T$ but also on a local velocity $u$.

For the range of applied voltages discussed in this paper, both methods lead to nearly identical profiles. This is shown in Fig. S4 where temperature and chemical-potential profiles obtained from Eq. (S26) (symbols) are compared to the corresponding curves from the main text (lines).

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