Surfaces in 3-space possessing nontrivial deformations which preserve the shape operator

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Abstract

The class of surfaces in 3-space possessing nontrivial deformations which preserve principal directions and principal curvatures (or, equivalently, the shape operator) was investigated by Finikov and Gambier as far back as in 1933. We review some of the known examples and results, demonstrate the integrability of the corresponding Gauss-Codazzi equations and draw parallels between this geometrical problem and the theory of compatible Poisson brackets of hydrodynamic type. It turns out that coordinate hypersurfaces of the n-orthogonal systems arising in the theory of compatible Poisson brackets of hydrodynamic type must necessarily possess deformations preserving the shape operator.
1 Introduction

In 1865 Bonnet initiated the study of the following two types of deformations of surfaces in the Euclidean 3-space:

1. **Isometric deformations preserving principal directions** \(^3\):
2. **Isometric deformations preserving principal curvatures** \(^4\).

Deformations of the type 1 characterise the so-called moulding surfaces (see \(^5\) for further discussion), while deformations of the type 2 lead to constant mean curvature surfaces and a special class of Weingarten surfaces, which has been recently investigated in \(^2\). In 1933 Finikov and Gambier \(^18\), \(^17\) (see also \(^18\)) introduced deformations of the third type which seem to be a natural counterpart of deformations 1, 2:

3. **Deformations preserving principal directions and principal curvatures or, equivalently, the shape operator (Weingarten operator).**

For short, we will call them S-deformations (notice that S-deformations are no longer isometric). Let \(M^2 \in E^3\) be a surface in the Euclidean space \(E^3\) parametrized by coordinates \(R^1, R^2\) of curvature lines. Let

\[
G_{11}(dR^1)^2 + G_{22}(dR^2)^2
\]

be its third fundamental form (i.e. metric of Gaussian image, which is automatically of constant curvature 1). This condition can be written in the form

\[
(\partial_2 a + a^2)G^{22} + \frac{a}{2}\partial_2 G^{22} + (\partial_1 b + b^2)G^{11} + \frac{b}{2}\partial_1 G^{11} + 1 = 0,
\]

where \(\partial_1 = \partial/\partial R^1\), \(\partial_2 = \partial/\partial R^2\), \(G^{11} = 1/G_{11}\), \(G^{22} = 1/G_{22}\) and the coefficients \(a\) and \(b\) are defined by the formulae

\[
\partial_2 G^{11} = -2aG^{11}, \quad \partial_1 G^{22} = -2bG^{22}.
\]

Notice that \(a\) and \(b\) are the Christoffel symbols of the Levi-Civita connection of the metric \((1)\): \(a = \Gamma^1_{12}, \ b = \Gamma^2_{21}\). We point out that for given \(a\) and \(b\) equations (\(2\), \(3\)) are linear in \(G^{11}, G^{22}\). Let \(k^1, k^2\) be the radii of principal curvature of the surface \(M^2\). In the coordinates of curvature lines Codazzi equations take the form

\[
\frac{\partial_2 k^1}{k^2 - k^1} = a, \quad \frac{\partial_1 k^2}{k^1 - k^2} = b.
\]

Let us assume now that the surface \(M^2\) possesses nontrivial S-deformation. Analytically, this means that for given \(k^1, k^2\) (and hence given \(a, b\)) the linear system (\(2\), \(3\)) is not uniquely solvable for \(G^{11}, G^{22}\). The form of the equations (\(2\)) — (\(4\)) leads to the following simple observations:

1. S-deformations occur in one-parameter (multi-parameter) families. Indeed, if \((G^{11}, G^{22})\) and \((\tilde{G}^{11}, \tilde{G}^{22})\) are two different solutions of (\(2\), \(3\)) (with \(a\) and \(b\) fixed) then \((\lambda G^{11} + (1 - \lambda)\tilde{G}^{11}, \lambda G^{22} + (1 - \lambda)\tilde{G}^{22})\) is a solution as well. This is a consequence of the linearity of equations (\(2\), \(3\)) in \(G^{11}, G^{22}\).
2. The property for a surface $M^2$ to possess nontrivial $S$-deformations entirely depends on geometry of the orthogonal net on the sphere $S^2$ which is the Gauss image of its

curvature lines. Indeed, equations (2), (3) depend on $a$ and $b$ only, so that any solution of the

linear system (4) defines the radii of principal curvature of an $S$-deformable surface. 

Surfaces corresponding to different solutions of (4) have one and the same spherical image of curvature lines and hence one and the same third fundamental form. Thus, they $S$-

deform simultaneously. We refer to [18] for further comments.

In section 2 we list some of the known examples of $S$-deformable surfaces. These, in particular, include arbitrary quadrics, cyclids of Dupin and conformal images of the surfaces of revolution (as well as all other surfaces with the same spherical image of curvature lines). Apart from several degenerate cases, these examples exhaust the list of surfaces which possess 3-parameter families of $S$-deformations (under certain assumptions the number 3 proves to be the maximal possible).

Following [16], in section 3 we demonstrate the integrability of the Gauss-Codazzi equations governing $S$-deformable surfaces by explicitly constructing the Lax pair with a spectral parameter.

Some results on multi-dimensional $S$-deformable hypersurfaces are discussed in section 4. It is pointed out that under certain “genericity” assumptions $n$-dimensional quadrics are the only hypersurfaces which possess $S$-deformations depending on $n + 1$ arbitrary constants.

In sections 5—7 we draw parallels between $S$-deformable hypersurfaces and compatible Poisson brackets of hydrodynamic type. It turns out that coordinate hypersurfaces of the $n$-orthogonal coordinate systems arising in the theory of bi-Hamiltonian systems of hydrodynamic type are necessarily $S$-deformable.

2 Examples

This section contains a list of examples of surfaces possessing nontrivial $S$-deformations. In our discussion we mainly follow [18].

Example 1. Moulding surfaces of Monge correspond to the third fundamental form

$$\frac{(dR_1)^2}{1 + c/cos^2(R_1)} + \frac{\sin^2(R_1)(dR_2)^2}{\psi(R_2)},$$

where $c$ is a constant and $\psi(R_2)$ is an arbitrary function. The metric (5) has constant curvature 1 for any $c, \psi$. The coordinate curves $R_2 = \text{const} \ (R_1 = \text{const})$ represent the meridians (parallels) of the sphere $S^2$, respectively. It can be readily verified that the radii of principal curvature $k_1, k_2$ satisfy the linear system

$$\partial_2 k_1 = 0, \quad \frac{\partial_1 k_2}{k_1 - k_2} = \cot(R_1)$$

which does not explicitly depend on $c, \psi$. Thus, any surface with the third fundamental form (5) possesses $S$-deformations depending on one arbitrary constant and one arbitrary function. Geometrically, the moulding surfaces of Monge are swept by a planar curve while the plane rolls along a curvilinear cylinder.
Example 2. General moulding surfaces are characterised by the third fundamental form

\[(dR^1)^2 + \frac{\sin^2(R^1 + \varphi(R^2))(dR^2)^2}{\psi(R^2)}, \quad (6)\]

where \(\psi(R^2)\) and \(\varphi(R^2)\) are arbitrary functions. The coordinate curves \(R^1 = \text{const}\) represent a one-parameter family of great circles on \(S^2\), the curves \(R^2 = \text{const}\) are their orthogonal trajectories. It can be readily verified that the radii of principal curvature \(k^1, k^2\) satisfy the equations

\[
\partial_2 k^1 = 0, \quad \frac{\partial_1 k^2}{k^1 - k^2} = \cot(R^1 + \varphi(R^2))
\]

which do not explicitly depend on \(\psi\). Thus, any surface with the third fundamental form (6) possesses \(S\)-deformations depending on one arbitrary function of one variable. A general moulding surface is swept by a planar curve while the plane rolls along a developable surface.

If a surface is not the moulding one (that is, both \(\partial_2 k^1\) and \(\partial_1 k^2\) are nonzero), the number of parameters on which \(S\)-deformations depend cannot exceed 3. The proof of this statement can be found in [18] (see also [12]). Examples presented below actually exhaust the list of surfaces which possess exactly 3-parameter families of \(S\)-deformations.

Example 3. Quadrics correspond to the third fundamental form

\[
\frac{R^2 - R^1}{4} \left( \frac{(dR^1)^2}{(R^1)^3 + a(R^1)^2 + bR^1 + c} - \frac{(dR^2)^2}{(R^2)^3 + a(R^2)^2 + bR^2 + c} \right) \quad (7)
\]

where \(a, b, c\) are arbitrary constants. The coordinates \(R^1, R^2\) are known as spherical-conical: the corresponding coordinate curves are intersections of the sphere with a confocal family of quadratic cones. Equations for the radii of principal curvature \(k^1, k^2\) take the form

\[
\frac{\partial_2 k^1}{k^2 - k^1} = \frac{1}{2(R^2 - R^1)}, \quad \frac{\partial_1 k^2}{k^1 - k^2} = \frac{1}{2(R^1 - R^2)}
\]

which does not explicitly depend on \(a, b, c\). Thus, any surface with the third fundamental form (7) possesses \(S\)-deformations depending on three arbitrary constants. The case of quadrics corresponds to the choice

\[
k^1 = \frac{1}{R^1 \sqrt{R^1 R^2}}, \quad k^2 = \frac{1}{R^2 \sqrt{R^1 R^2}}
\]

Example 4. Cyclids of Dupin correspond to the third fundamental form

\[
\frac{1}{(R^1 - R^2)^2} \left( \frac{(dR^1)^2}{a(R^1)^2 + bR^1 + c} - \frac{(dR^2)^2}{a(R^2)^2 + bR^2 + c + 1} \right) \quad (8)
\]

where \(a, b, c\) are arbitrary constants. The coordinate system \(R^1, R^2\) consists of the two orthogonal pencils of circles on \(S^2\). It can be readily verified that the radii of principal curvature \(k^1, k^2\) satisfy the linear system

\[
\frac{\partial_2 k^1}{k^2 - k^1} = \frac{1}{R^1 - R^2}, \quad \frac{\partial_1 k^2}{k^1 - k^2} = \frac{1}{R^2 - R^1}
\]
which does not explicitly depend on \(a, b, c\). Thus, any surface with the third fundamental form (8) possesses \(S\)-deformations depending on three arbitrary constants. The case of cyclids of Dupin corresponds to the choice

\[k^1 = R^2, \quad k^2 = R^1.\]

**Example 5. Conformal transforms of surfaces of revolution** correspond to the third fundamental form

\[
\frac{1}{(R^1 - R^2)^2} \left( \frac{p^2(dR^1)^2}{a(R^1)^2 + bR^1 + c} - \frac{(p + p'(R^1 - R^2))^2(dR^2)^2}{a(R^2)^2 + bR^2 + c + p^2} \right)
\]

where \(a, b, c\) are arbitrary constants and \(p\) is an arbitrary function of \(R^2\). Here the curves \(R^2 = \text{const}\) represent a one-parameter family of circles on \(S^2\). It can be readily verified that the radii of principal curvature \(k^1, k^2\) satisfy the system

\[
\frac{\partial_2 k^1}{k^2 - k^1} = \frac{1}{R^1 - R^2} + \frac{p'}{p}, \quad \frac{\partial_1 k^2}{k^1 - k^2} = \frac{1}{R^2 - R^1} + \frac{p'}{p + p'(R^1 - R^2)}
\]

which does not explicitly depend on \(a, b, c\). Thus, any surface with the third fundamental form (8) possesses \(S\)-deformations depending on three arbitrary constants. The case of conformal images of surfaces of revolution corresponds to the choice

\[k^1 = \frac{1}{p}, \quad k^2 = \frac{1}{p + p'(R^1 - R^2)}\]

(see [12]). The choice \(p = R^2\) and the subsequent change of variables \(R^i \to 1/R^i\) returns to the case of cyclids of Dupin discussed above. Indeed, cyclids of Dupin are known to be conformal images of cylinders, cones and tori of revolution. Notice that equations (10) possess a more general class of solutions

\[k^1 = q, \quad k^2 = q + \frac{qp'(R^1 - R^2)}{p + p'(R^1 - R^2)}\]

where \(q\) is another arbitrary function of \(R^2\). Solution (10) corresponds to the choice \(q = 1/p\).

Examples 3-5 provide a complete list of surfaces possessing 3-parameter families of \(S\)-deformations. We refer to [12] for the proof of this result. Notice that example 5 is missing from the classification proposed in [18] (although examples 3 and 4 are present).

The case of surfaces possessing 2-parameter families of \(S\)-deformations is not understood at the moment well enough. We present here just one example of that type.

**Example 6. Surfaces possessing 2-parameter family of \(S\)-deformations.** Consider the metric

\[
\frac{R^1 - R^2}{(R^1 + R^2)^2} \left( \frac{(dR^1)^2}{a(R^1)^2 - R^1 + c} - \frac{(dR^2)^2}{a(R^2)^2 - R^2 + c} \right)
\]

where \(a, c\) are arbitrary constants. The metric (11) has constant curvature 1 for any values of \(a, c\). The coordinate system \(R^1, R^2\) is the image of the ellipsoidal coordinate system on
the plane $E^2$ under the stereographic projection $E^2 \to S^2$ (see [18]). Equations for the radii of principal curvature $k^1, k^2$

\[
\frac{\partial_2 k^1}{k^2 - k^1} = \frac{1}{2(R^2 - R^1)} - \frac{1}{R^1 + R^2}, \quad \frac{\partial_1 k^2}{k^1 - k^2} = \frac{1}{2(R^1 - R^2)} - \frac{1}{R^1 + R^2},
\]

do not explicitly depend on $a, c$. Thus, any surface with the third fundamental form (11) possesses $S$-deformations depending on two arbitrary constants. The particular case

\[
k^1 = \frac{(R^1 + R^2)^2}{(R^1)^{3/2}(R^2)^{3/2}}, \quad k^2 = \frac{(R^1 + R^2)^2}{(R^1)^{3/2}(R^2)^{3/2}}
\]

was discussed in detail in [18].

**Example 7. Surfaces possessing 1-parameter family of $S$-deformations.** Consider the metric

\[
\frac{2}{\cosh^2(R^1 + R^2)} \left( \frac{(dR^1)^2}{1 + c} + \frac{(dR^2)^2}{1 - c} \right)
\]

(12)

where $c$ is an arbitrary constant. It has constant curvature 1 for any value of $c$. The radii of principal curvature $k^1, k^2$ satisfy the equations

\[
\frac{\partial_2 k^1}{k^2 - k^1} = \frac{\partial_1 k^2}{k^1 - k^2} = -\tanh(R^1 + R^2)
\]

which do not explicitly depend on $c$. Thus, any surface with the third fundamental form (12) possesses $S$-deformations which depend on one arbitrary constant. The choice

\[
k^1 = -k^2 = \cosh^2(R^1 + R^2)
\]

corresponds to a minimal surface. We refer to [18] for the geometry of this particular example. The discussion of surfaces possessing one-parameter families of $S$-deformations will be continued in the next section.

### 3 The Lax pair

In this section we discuss surfaces which possess 1-parameter families of $S$-deformations. Let $M^2 \in E^3$ be a surface parametrized by coordinates $R^1, R^2$ of curvature lines with the third fundamental form (1):

\[
G_{11}(dR^1)^2 + G_{22}(dR^2)^2.
\]

Let $k^1, k^2$ be the radii of principal curvature satisfying the Codazzi equations

\[
\frac{\partial_2 k^1}{k^2 - k^1} = \partial_2 \ln \sqrt{G_{11}}, \quad \frac{\partial_1 k^2}{k^1 - k^2} = \partial_1 \ln \sqrt{G_{22}}.
\]

(13)

Suppose there exists a flat metric

\[
g_{11}(dR^1)^2 + g_{22}(dR^2)^2
\]

(14)
such that
\[ G_{11} = g_{11}/\eta_1, \quad G_{22} = g_{22}/\eta_2, \] (15)
where \( \eta_1, \eta_2 \) are functions of \( R_1, R_2 \), respectively. One can readily verify that under these assumptions the metric
\[ \tilde{G}_{11}(dR_1)^2 + \tilde{G}_{22}(dR_2)^2 = \frac{g_{11}}{\sqrt{\lambda + \eta_1}}(dR_1)^2 + \frac{g_{22}}{\sqrt{\lambda + \eta_2}}(dR_2)^2 \] (16)
has constant curvature 1 for any \( \lambda \). Since equations (13) are still true if we replace \( G_{ii} \) by \( \tilde{G}_{ii} \), we arrive at a 1-parameter family of surfaces \( M^2_\lambda \) with the third fundamental forms (16) (which depend on \( \lambda \)) and the principal curvatures \( k_1, k_2 \) (which are independent of \( \lambda \)). Hence, the shape operators of surfaces \( M^2_\lambda \) coincide. The problem of the classification of surfaces which possess 1-parameter families of \( S \)-deformations is thus reduced to the classification of metrics (16) which have constant Gaussian curvature 1 for any \( \lambda \). Any such metric generates an infinite family of \( S \)-deformable surfaces whose principal curvatures \( k_1, k_2 \) satisfy (13). In terms of the Lame coefficients \( H_1 = \sqrt{g_{11}}, \quad H_2 = \sqrt{g_{22}} \) and the rotation coefficients \( \beta_{12} = \partial_1 H_2/H_1, \quad \beta_{21} = \partial_2 H_1/H_2 \) our problem reduces to the nonlinear system
\[ \partial_1 H_2 = \beta_{12} H_1, \quad \partial_2 H_1 = \beta_{21} H_2, \]
\[ \partial_1 \beta_{12} + \partial_2 \beta_{21} = 0, \]
\[ \eta_1 \partial_1 \beta_{12} + \eta_2 \partial_2 \beta_{21} + \frac{1}{2} \eta_1' \beta_{12} + \frac{1}{2} \eta_2' \beta_{21} + H_1 H_2 = 0, \]
which possesses the Lax pair
\[ \partial_1 \psi = \begin{pmatrix} 0 & -\sqrt{\frac{\lambda + \eta_2}{\lambda + \eta_1}} \beta_{21} & \frac{H_1}{\sqrt{\lambda + \eta_1}} \\ \sqrt{\frac{\lambda + \eta_2}{\lambda + \eta_1}} \beta_{21} & 0 & 0 \\ -\frac{H_1}{\sqrt{\lambda + \eta_1}} & 0 & 0 \end{pmatrix} \psi, \]
\[ \partial_2 \psi = \begin{pmatrix} 0 & \sqrt{\frac{\lambda + \eta_1}{\lambda + \eta_2}} \beta_{12} & 0 \\ -\sqrt{\frac{\lambda + \eta_1}{\lambda + \eta_2}} \beta_{12} & 0 & \frac{H_2}{\sqrt{\lambda + \eta_2}} \\ 0 & -\frac{H_2}{\sqrt{\lambda + \eta_2}} & 0 \end{pmatrix} \psi. \]
Geometrically, this Lax pair governs infinitesimal displacements of the orthonormal frame of the orthogonal coordinate system on the unit sphere \( S^2 \), corresponding to the metric (16). In \( 2 \times 2 \) matrices it takes the form
\[ 2 \sqrt{\lambda + \eta_1} \partial_1 \psi = \begin{pmatrix} \frac{i}{2} \sqrt{\lambda + \eta_2} \beta_{21} & \frac{H_1}{\sqrt{\lambda + \eta_1}} \\ -H_1 & -\frac{i}{2} \sqrt{\lambda + \eta_2} \beta_{21} \end{pmatrix} \psi, \]
\[ 2 \sqrt{\lambda + \eta_2} \partial_2 \psi = i \begin{pmatrix} -\sqrt{\lambda + \eta_1} \beta_{12} & \frac{H_2}{\sqrt{\lambda + \eta_1}} \\ \frac{H_2}{\sqrt{\lambda + \eta_1}} & \sqrt{\lambda + \eta_1} \beta_{12} \end{pmatrix} \psi. \]
Example 8. The case of constant $\eta_i$. Let us assume $\eta_1 = -1/2$, $\eta_2 = 1/2$ in which case equations (17) take the form

$$\begin{align*}
\partial_1 H_2 &= \beta_{12} H_1, \\
\partial_2 H_1 &= \beta_{21} H_2, \\
\partial_1 \beta_{12} &= H_1 H_2, \\
\partial_2 \beta_{21} &= -H_1 H_2.
\end{align*}$$

(18)

It can be readily verified that system (18) possesses 2 integrals

$$\begin{align*}
\partial_1 (\beta_{12}^2 - H_2^2) &= 0, \\
\partial_2 (\beta_{21}^2 + H_1^2) &= 0
\end{align*}$$

implying

$$\begin{align*}
\beta_{12}^2 - H_2^2 &= \mu_2(R^2), \\
\beta_{21}^2 + H_1^2 &= \mu_1(R^1),
\end{align*}$$

where the functions $\mu_1, \mu_2$ can be reduced to $\pm 1$ by virtue of the following symmetry of system (18):

$$\begin{align*}
\tilde{R}^1 &= s_1(R^1), \\
\tilde{H}_1 &= H_1/s_1', \\
\tilde{\beta}_{21} &= \beta_{21}/s_1', \\
\tilde{R}^2 &= s_2(R^2), \\
\tilde{H}_2 &= H_2/s_2', \\
\tilde{\beta}_{12} &= \beta_{12}/s_2'
\end{align*}$$

(here $s_1(R^1), s_2(R^2)$ are arbitrary functions). Let us assume, for instance, that

$$\beta_{12}^2 - H_2^2 = 1, \quad \beta_{21}^2 + H_1^2 = 1.$$

Introducing the parametrisation

$$\begin{align*}
H_1 &= \sin \psi, \\
H_2 &= \sinh \varphi, \\
\beta_{12} &= \cosh \varphi, \\
\beta_{21} &= \cos \psi,
\end{align*}$$

we readily rewrite (18) in the form

$$\begin{align*}
\partial_1 \varphi &= \sin \psi, \\
\partial_2 \psi &= \sinh \varphi,
\end{align*}$$

which implies the following integrable Monge-Ampere equations for $\varphi, \psi$:

$$\partial_1 \partial_2 \varphi = \sinh \varphi \sqrt{1 - (\partial_1 \varphi)^2}, \quad \partial_1 \partial_2 \psi = \sin \psi \sqrt{1 + (\partial_2 \psi)^2}.$$

4 Multidimensional hypersurfaces possessing nontrivial $S$-deformations

Let $M^n$ be a hypersurface in $E^{n+1}$ parametrised by the coordinates $R^i$ of curvature lines. We point out that generic multidimensional hypersurface does not necessarily possess curvature line parametrisation. However, (at least for $n = 3$) such parametrisation appears to be the necessary condition for the existence of nontrivial $S$-deformations. Let $k^i(R)$ be the radii of principal curvature of the hypersurface $M^n$ and $G_{ii}$ $(dR^i)^2$ the third fundamental form (which is automatically of constant curvature 1). These objects satisfy the Codazzi equations

$$\frac{\partial_j k^i}{k^j - k^i} = \partial_j \ln \sqrt{G_{ii}}, \quad i \neq j.$$

We will discuss just one important example.
Example 9. Hyperquadrics are characterized by the third fundamental form

$$\sum_{i=1}^{n} \prod_{k \neq i} (R^k - R^i) \frac{(dR^i)^2}{4P(R^i)},$$

(19)

where $P(R) = \prod_{s=1}^{n+1} (R - a_s)$ is an arbitrary polynomial of the order $n + 1$. The corresponding radii of principal curvature $k^i$ satisfy the linear system

$$\frac{\partial_j k^i}{k^j - k^i} = \frac{1}{2(R^j - R^i)}, \quad i \neq j,$$

which does not explicitly depend on $a_s$. Thus, any hypersurface $M^n$ with the third fundamental form (19) possesses $(n + 1)$-parameter family of $S$-deformations. The case of a hyperquadric corresponds to the choice

$$k^i = \frac{1}{R^i \sqrt{\prod_{j=1}^{n} R^j}}.$$

It can be demonstrated that quadrics are the only $n$-dimensional hypersurfaces which possess $(n + 1)$-parameter families of $S$-deformations and satisfy the “genericity” assumption $\partial_j k^i \neq 0$ for any $i \neq j$ (we recall that for $n = 2$ the condition $\partial_1 k^2 \neq 0$, $\partial_2 k^1 \neq 0$ forbids moulding surfaces). Thus, unlike the case $n = 2$, the class of multidimensional hypersurfaces possessing the “maximal” number of $S$-deformations appears to be very restricted. It is likely that there exist examples of hypersurfaces possessing $k$-parameter families of $S$-deformations for any intermediate $k = 1, 2, \ldots, n + 1$. Numerous examples of that type are provided by coordinate hypersurfaces of the $n$-orthogonal coordinate systems arising in the theory of multi-Hamiltonian equations of hydrodynamic type.

5 Compatible Poisson brackets of hydrodynamic type. Criterion of the compatibility.

In 1983 Dubrovin and Novikov [7] introduced the Poisson brackets of hydrodynamic type

$$\{F, G\} = \int \frac{\delta F}{\delta u^i} A_{ij} \frac{\delta G}{\delta u^j} \, dx$$

(20)

defined by the Hamiltonian operators $A^{ij}$ of the form

$$A^{ij} = g^{ij} \frac{d}{dx} + b_{ij}^k u_k^x, \quad b_{ij}^k = -g^{is} \Gamma_{sk}^j.$$

(21)

They proved that in the nondegenerate case ($\det g^{ij} \neq 0$) the bracket (20), (21) is skew-symmetric and satisfies the Jacobi identities if and only if the metric $g^{ij}$ (with upper indices) is flat, and $\Gamma_{sk}^j$ are the Christoffel symbols of the corresponding Levi-Civita connection.

Let us consider another Poisson bracket of hydrodynamic type defined on the same phase space by the Hamiltonian operator

$$\tilde{A}^{ij} = \tilde{g}^{ij} \frac{d}{dx} + \tilde{b}_{ij}^k u_k^x, \quad \tilde{b}_{ij}^k = -\tilde{g}^{is} \tilde{\Gamma}_{sk}^j,$$

(22)
corresponding to a flat metric $\tilde{g}^{ij}$. Two Poisson brackets (Hamiltonian operators) are called compatible if their linear combinations $\tilde{A}^{ij} + \lambda A^{ij}$ are Hamiltonian as well. This requirement implies, in particular, that the metric $\tilde{g}^{ij} + \lambda g^{ij}$ must be flat for any $\lambda$ (plus certain additional restrictions). The necessary and sufficient conditions of the compatibility were first formulated by Dubrovin [10], [11] (see [24], [25] for further discussion). Below we reformulate these conditions in terms of the operator $r_{ij} = \tilde{g}^{is} g_{sj}$ (Theorem 1) which, in particular, imply the vanishing of the Nijenhuis tensor of the operator $r_j^i$:

$$N_{jk}^i = r_j^i \partial_s r_k^i - r_k^i \partial_s r_j^i - r_s^i(\partial_j r_k^s - \partial_k r_j^s) = 0$$

(see [13], [25]).

Examples of compatible Hamiltonian pairs naturally arise in the theory of Hamiltonian systems of hydrodynamic type — see e.g. [1], [20], [21], [27], [28], [31]. Dubrovin developed a deep theory for the particular class of compatible Poisson brackets arising in the framework of the associativity equations [10], [11]. Compatible Poisson brackets of hydrodynamic type can also be obtained as a result of Whitham averaging (dispersionless limit) from the local compatible Poisson brackets of integrable systems [7], [8], [1], [24], [30], [13], [22]. Some further examples and partial classification results can be found in [12], [15], [27], [29], [24], [19].

If the spectrum of $r_j^i$ is simple, the vanishing of the Nijenhuis tensor implies the existence of a coordinate system where both metrics $g^{ij}$ and $\tilde{g}^{ij}$ become diagonal. In these coordinates the compatibility conditions take the form of an integrable reduction of the Lamé equations. We present the corresponding Lax pairs in section 6. Another approach to the integrability of this system was recently proposed by Mokhov [26] by an appropriate modification of Zakharov’s scheme [35].

Our main observation is the relationship between compatible Poisson brackets of hydrodynamic type and hypersurfaces $M^{n-1} \in E^n$ possessing nontrivial $S$-deformations. In section 7 we demonstrate that the n-orthogonal coordinate systems in $E^n$ corresponding to flat metrics $\tilde{g}^{ij} + \lambda g^{ij}$ (rewritten in the diagonal coordinates) deform with respect to $\lambda$ in such a way that the shape operators of coordinate hypersurfaces are preserved up to constant scaling factors.

Notice that the operator $r_j^i = \tilde{g}^{is} g_{sj}$ is automatically symmetric:

$$r_s^i g^{sj} = r_j^i g^{si}, \tag{23}$$

so that $\tilde{g}^{ij} = r_j^i g^{sj} = r_j^i g^{si} = r^{ij}$. In what follows we use the first metric $g^{ij}$ for raising and lowering the indices.

**Theorem 1** [10]

*Hamiltonian operators* (24), (22) *are compatible if and only if the following conditions are satisfied:*

1. The Nijenhuis tensor of $r_j^i$ vanishes:

$$N_{jk}^i = r_j^i \partial_s r_k^i - r_k^i \partial_s r_j^i - r_s^i(\partial_j r_k^s - \partial_k r_j^s) = 0. \tag{24}$$

2. The metric coefficients $\tilde{g}^{ij} = r^{ij}$ satisfy the equations

$$\nabla^i \nabla^j r^{kl} + \nabla^k \nabla^l r^{ij} = \nabla^i \nabla^k r^{jl} + \nabla^j \nabla^l r^{ik}. \tag{25}$$

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Here $\nabla^i = g^{is} \nabla_s$ is the covariant differentiation corresponding to the metric $g^{ij}$. The vanishing of the Nijenhuis tensor implies the following expression for the coefficients $\tilde{b}^i_k$ in terms of $r^i_j$:

$$2\tilde{b}^{ij}_k = \nabla^i r^j_k - \nabla^j r^i_k + \nabla_k r^{ij} + 2b^j_k r^i_s$$  \hspace{1cm} (26)$$

In a somewhat different form the necessary and sufficient conditions of the compatibility were formulated in [10], [11], [24], [25].

**Remark.** The criterion of the compatibility of Hamiltonian operators of hydrodynamic type resembles that of finite-dimensional Poisson bivectors: two skew-symmetric Poisson bivectors $\omega^{ij}$ and $\tilde{\omega}^{ij}$ are compatible if and only if the Nijenhuis tensor of the corresponding recursion operator $r^i_j = \tilde{\omega}^{is} \omega_{sj}$ vanishes. We emphasize that in our situation operator $r^i_j$ does not coincide with the recursion operator.

**Remark.** If the spectrum of $r^i_j$ is simple the condition (25) is redundant: it is automatically satisfied by virtue of (24) and the flatness of both metrics $g$ and $\tilde{g}$. This was the motivation for me to drop condition (25) in the compatibility criterion formulated in [13]. However, in this general form the criterion proved to be incorrect: recently it was pointed out by Mokhov [25] that in the case when the spectrum of $r^i_j$ is not simple the vanishing of the Nijenhuis tensor is no longer sufficient for the compatibility.

6 Compatibility conditions in the diagonal form: the Lax pairs

If the spectrum of $r^i_j$ is simple, the vanishing of the Nijenhuis tensor implies the existence of the coordinates $R^1, \ldots, R^n$ in which the objects $r^i_j$, $g^{ij}$, $\tilde{g}^{ij}$ become diagonal. Moreover, the $i$-th eigenvalue of $r^i_j$ depends only on the coordinate $R^i$, so that $r^i_j = \text{diag}(\eta_i)$, $g^{ij} = \text{diag}(g^{ii})$, $\tilde{g}^{ij} = \text{diag}(g^{ii} \eta_i)$ where $\eta_i$ is a function of $R^i$. This is a generalization of the analogous observation by Dubrovin [11] in the particular case of compatible Poisson brackets originating from the theory of the associativity equations. Introducing the Lame coefficients $H_i$ and the rotation coefficients $\beta_{ij}$ by the formulae

$$H_i = \sqrt{g_{ii}} = 1/\sqrt{g^{ii}}, \quad \partial_i H_j = \beta_{ij} H_i,$$  \hspace{1cm} (27)$$

we can rewrite the zero curvature conditions for the metric $g$ in the form

$$\partial_k \beta_{ij} = \beta_{ik} \beta_{kj}, \quad (28)$$

$$\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{k \neq i,j} \beta_{ki} \beta_{kj} = 0.$$  \hspace{1cm} (29)$$

The zero curvature condition for the metric $\tilde{g}$ imposes the additional constraint

$$\eta_i \partial_i \beta_{ij} + \eta_j \partial_j \beta_{ji} + \frac{1}{2} \eta_i' \text{et}_{aij} + \frac{1}{2} \eta_j' \beta_{ji} + \sum_{k \neq i,j} \eta_k \beta_{ki} \beta_{kj} = 0,$$  \hspace{1cm} (30)$$

resulting from (29) after the substitution of the rotation coefficients $\tilde{\beta}_{ij} = \beta_{ij} \sqrt{\eta_i/\eta_j}$ of the metric $\tilde{g}$. As can be readily seen, equations (29) and (30) already imply the compatibility,
so that in the diagonalisable case condition (25) of Theorem 1 is indeed superfluous. Solving equations (29), (30) for $\partial_i \beta_{ij}$, we can rewrite (28)–(30) in the form

$$\partial_k \beta_{ij} = \beta_{ik} \beta_{kj},$$  \hspace{1cm} (31)$$

It can be verified by a straightforward calculation that system (31) is compatible for any choice of the functions $\eta_i(R^i)$, and its general solution depends on $n(n-1)$ arbitrary functions of one variable (indeed, one can arbitrarily prescribe the value of $\beta_{ij}$ on the $j$-th coordinate line). Under the additional "Egorov" assumption $\beta_{ij} = \beta_{ji}$, system (31) reduces to the one studied by Dubrovin in [11]. For $n \geq 3$ system (31) is essentially nonlinear. Its integrability follows from the Lax pair [16]

$$\partial_j \psi_i = \beta_{ij} \psi_j, \hspace{0.5cm} \partial_i \psi_i = -\frac{\eta'_i}{2(\lambda + \eta_i)} \psi_i - \sum_{k \neq i} \frac{\lambda + \eta_k}{\lambda + \eta_i} \beta_{ki} \psi_k$$  \hspace{1cm} (32)$$

with a spectral parameter $\lambda$ (another demonstration of the integrability of system (32) has been proposed recently in [26] by an appropriate modification of Zakharov’s approach [35]).

**Remark.** In fact, the Lax pair (32) is gauge-equivalent to the equations

$$(\tilde{g}^{ik} + \lambda g^{ik}) \partial_k \partial_j \psi + (\tilde{b}^{ik} + \lambda b^{ik}) \partial_k \psi = 0$$

for the Casimirs $\int \psi dx$ of the Hamiltonian operator $\tilde{A}^{ij} + \lambda A^{ij}$.

After the gauge transformation $\psi_i = \varphi_i / \sqrt{\lambda + \eta_i}$ the Lax pair (32) assumes the manifestly skew-symmetric form

$$\partial_j \varphi_i = \sqrt{\frac{\lambda + \eta_i}{\lambda + \eta_j}} \beta_{ij} \varphi_j, \hspace{0.5cm} \partial_i \varphi_i = -\sum_{k \neq i} \sqrt{\frac{\lambda + \eta_k}{\lambda + \eta_i}} \beta_{ki} \varphi_k.$$  \hspace{1cm} (33)$$

Thus, we can introduce an orthonormal frame $\varphi_1, ..., \varphi_n$ in the Euclidean space $E^n$ satisfying the equations

$$\partial_j \varphi_i = \sqrt{\frac{\lambda + \eta_i}{\lambda + \eta_j}} \beta_{ij} \varphi_j, \hspace{0.5cm} \partial_i \varphi_i = -\sum_{k \neq i} \sqrt{\frac{\lambda + \eta_k}{\lambda + \eta_i}} \beta_{ki} \varphi_k, \hspace{1cm} (\varphi_i, \varphi_j) = \delta_{ij}.$$  \hspace{1cm} (34)$$

Let us introduce a vector $\vec{r}$ such that

$$\partial_i \vec{r} = \frac{H_i}{\sqrt{\lambda + \eta_i}} \varphi_i$$

(the compatibility of these equations can be readily verified). In view of the formula

$$(\partial_i \vec{r}, \partial_j \vec{r}) = \frac{H_i^2}{\lambda + \eta_i} \delta_{ij}$$
the radius-vector $\vec{r}$ is descriptive of the n-orthogonal coordinate system in $E^n$ corresponding to the flat metric

$$\sum_i \frac{H_i^2}{\lambda + \eta_i} (dR^i)^2.$$ 

Geometrically, $\vec{\varphi}_i$ are the unit vectors along the coordinate lines of this n-orthogonal system.

Let us discuss in some more detail the case $\eta_i = \text{const} = c_i$, in which system (31) takes the form

$$\partial_k \beta_{ij} = \beta_{ik} \beta_{kj},$$

$$\partial_i \beta_{ij} = \sum_{k \neq i,j} \frac{c_k - c_j}{c_j - c_i} \beta_{kj} \beta_{kj}. \quad (35)$$

One can readily verify that the quantity

$$P_i = \sum_{k \neq i} (c_k - c_i) \beta_{ki}^2$$

is an integral of system (33), namely, $\partial_i P_i = 0$ for any $i \neq j$, so that $P_i$ is a function of $R_i$. Utilising the obvious symmetry $R_i \rightarrow s_i(R^i), \ \beta_{ki} \rightarrow \beta_{ki}/s'_i(R^i)$ of system (35), we can reduce $P_i$ to $\pm 1$ (if nonzero). Let us consider the simplest nontrivial case $n = 3$, $P_1 = P_2 = 1, \ P_3 = -1$:

$$P_1 = (c_2 - c_1) \beta_{21}^2 + (c_3 - c_1) \beta_{31}^2 = 1,$$

$$P_2 = (c_1 - c_2) \beta_{12}^2 + (c_3 - c_2) \beta_{32}^2 = 1,$$

$$P_3 = (c_1 - c_3) \beta_{13}^2 + (c_2 - c_3) \beta_{23}^2 = -1.$$ 

Assuming $c_3 > c_2 > c_1$ and introducing the parametrisation

$$\beta_{21} = \sin p/\sqrt{c_2 - c_1}, \ \beta_{31} = \cos p/\sqrt{c_3 - c_1},$$

$$\beta_{12} = \sinh q/\sqrt{c_2 - c_1}, \ \beta_{32} = \cosh q/\sqrt{c_3 - c_2},$$

$$\beta_{13} = \sin r/\sqrt{c_3 - c_1}, \ \beta_{23} = \cos r/\sqrt{c_3 - c_2},$$

we readily rewrite (35) in the form

$$\partial_1 q = \mu_1 \cos p, \ \partial_1 r = -\mu_1 \sin p,$$

$$\partial_2 p = -\mu_2 \cosh q, \ \partial_2 r = \mu_2 \sinh q,$$

$$\partial_3 p = \mu_3 \cos r, \ \partial_3 q = \mu_3 \sin r,$$

where

$$\mu_1 = \sqrt{\frac{c_3 - c_2}{(c_2 - c_1)(c_3 - c_1)}}, \ \mu_2 = \sqrt{\frac{c_3 - c_1}{(c_2 - c_1)(c_3 - c_2)}}, \ \mu_3 = \sqrt{\frac{c_2 - c_1}{(c_3 - c_1)(c_3 - c_2)}}.$$ 

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After rescaling, this system simplifies to
\[ \begin{align*}
\partial_1 q &= \cos p, \quad \partial_1 r = -\sin p, \\
\partial_2 p &= -\cosh q, \quad \partial_2 r = \sinh q, \\
\partial_3 p &= \cos r, \quad \partial_3 q = \sin r.
\end{align*} \tag{36} \]

Expressing \( p \) and \( r \) in the form \( p = \arccos \partial_1 q, \ r = \arcsin \partial_3 q \), we can rewrite (36) as a triple of pairwise commuting Monge-Ampère equations
\[ \begin{align*}
\partial_1 \partial_2 q &= \cosh q \sqrt{1 - \partial_1 q^2}, \\
\partial_1 \partial_3 q &= -\sqrt{1 - \partial_1 q^2} \sqrt{1 - \partial_3 q^2}, \\
\partial_2 \partial_3 q &= \sinh q \sqrt{1 - \partial_3 q^2}.
\end{align*} \]

Similar triples of Monge-Ampère equations were obtained in [14] in the classification of quadruples of \( 3 \times 3 \) hydrodynamic type systems which are closed under the Laplace transformations. However, there is no understanding of this coincidence at the moment. Notice that coordinate surfaces of these 3-orthogonal coordinate systems are of the type discussed in example 8.

7 Deformations of n-orthogonal coordinate systems inducing rescalings of shape operators of the coordinate hypersurfaces

We have demonstrated in sect. 6 that the radius-vector \( \vec{r}(R^1, ..., R^n) \) of the n-orthogonal coordinate system in \( E^n \) corresponding to the flat diagonal metric \( \sum_i H_i^2 (dR^i)^2 \) satisfies the equations
\[ \partial_i \vec{r} = \frac{H_i}{\sqrt{\lambda + \eta_i}} \vec{\varphi}_i, \]
where the infinitesimal displacements of the orthonormal frame \( \vec{\varphi}_i \) are governed by
\[ \begin{align*}
\partial_j \vec{\varphi}_i &= \sqrt{\frac{\lambda + \eta_j}{\lambda + \eta_i}} \beta_{ij} \vec{\varphi}_j, \\
\partial_i \vec{\varphi}_i &= -\sum_{k \neq i} \sqrt{\frac{\lambda + \eta_k}{\lambda + \eta_i}} \beta_{ki} \vec{\varphi}_k.
\end{align*} \]

Since our formulae depend on the spectral parameter, we may speak of the "deformation" of the n-orthogonal coordinate system with respect to \( \lambda \). To investigate this deformation in some more detail, we fix a coordinate hypersurface \( M^{n-1} \subset E^n \) (say, \( R^n = \text{const} \)). Its radius-vector \( \vec{r} \) and the unit normal \( \vec{\varphi}_n \) satisfy the Weingarten equations
\[ \partial_i \vec{\varphi}_n = \frac{\beta_{ni}}{H_i} \sqrt{\lambda + \eta_n} \partial_i \vec{r}, \quad i = 1, ..., n - 1, \]
implying that
\[ k^i = \frac{\beta_{ni}}{H_i} \sqrt{\lambda + \eta_n} \]
are principal curvatures of the hypersurface $M^{n-1}$. Since $\eta_n$ is constant on $M^{n-1}$, our deformation preserves the shape operator of $M^{n-1}$ up to a constant scaling factor $\sqrt{\lambda + \eta_n}$ (we point out that the curvature line parametrisation $R^1, \ldots, R^{n-1}$ is preserved by a construction). Thus, compatible Poisson brackets of hydrodynamic type give rise to deformations of n-orthogonal systems in $E^n$ which, up to scaling factors, preserve shape operators of the coordinate hypersurfaces. If we follow the evolution of a particular coordinate hypersurface $M^{n-1}$, this scaling factor can be eliminated by a homothetic transformation of the ambient space $E^n$, so that we arrive at the nontrivial deformation which preserves the shape operator. However, this scaling factor cannot be eliminated for all coordinate hypersurfaces simultaneously.

8 Concluding remarks

It seems to be an interesting and nontrivial problem to classify hypersurfaces $M^n \in E^{n+1}$ possessing $S$-deformations which depend on $k$ essential parameters where $k$ varies from 2 to $n$ (the limiting cases $k = n + 1$ and $k = 1$ reduce to hyperquadrics and generic $S$-deformable surfaces, respectively). Even the simplest nontrivial case $n = 2, k = 2$ is not understood at the moment.

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