Self-intersection local times for generalized grey Brownian motion in higher dimensions

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May 7, 2018

Abstract

We prove that the self-intersection local times for generalized grey Brownian motion $B^{\beta,\alpha}$ in arbitrary dimension $d$ is a well defined object in a suitable distribution space for $d\alpha < 2$. 
1 Introduction

Intersection local times are an intensively studied object for about 80 years, see e.g. [22]. Heuristically the self-intersection measures the time the process spends on its trajectory, i.e., it serves to count the self-crossings of the trajectory of a random process. In an informal way the self-intersection local time can be expressed by

$$L(Y) \equiv \int d^2t \delta(Y(t_2) - Y(t_1)),$$

where $\delta$ is Donsker’s - $\delta$-function and $Y$ a random process. Indeed the random variable $L$ is intended to sum up the contributions from each pair of "times" $t_1, t_2$ for which the process $Y$ is at the same point. For Gaussian processes the self-intersection local time is defined as a rigorous object, see for the case of Brownian motion e.g. [2, 3, 5, 7, 12, 13, 14, 15, 34, 35, 36, 37, 38, 39] and e.g. [30, 18, 1, 6, 10, 23, 4] for fractional and multifractional Brownian motion. One framework which serves to give a mathematical sound meaning to the object above in the Gaussian setting is White Noise Analysis, see e.g. [17, 29, 21].

For non-Gaussian processes in [9] a similar concept was used to establish the Mittag-Leffler Analysis. The grey noise measure [32, 27] is included as a special case in the class of Mittag-Leffler measures, which offers the possibility to apply the Mittag-Leffler analysis to fractional differential equations, in particular to fractional diffusion equations [31, 32], which carry numerous applications in science, like relaxation type differential equations or viscoelasticity.

The corresponding grey Brownian motion (gBm) was introduced by W. Schneider as a model for slow anomalous diffusions, i.e., the marginal density function of the gBm is the fundamental solution of a time-fractional diffusion equation. This is a class $\{B_\beta(t) : t \geq 0, 0 < \beta \leq 1\}$ of stochastic processes which are self-similar with stationary increments. More recently, this class was extended to the, so called generalized grey Brownian motion (ggBm) to include slow and fast anomalous diffusions which contain either Gaussian or non-Gaussian processes e.g., grey Brownian motion and fractional Brownian motion. In this paper we study the existence of self-intersection local times of ggBm in dimension $d$.

In Section 2 we summarize the construction and basic properties of ggBm in dimension $d$. Section 3 contains the main result on the existence of self-
intersection local times of a $d$-dimensional ggBm as a weak integral in a suitable stochastic distribution space.

2 The Mittag-Leffler Measure

Let $d \in \mathbb{N}$ and $L^2_d$ be the Hilbert space of vector-valued square integrable measurable functions

$$L^2_d := L^2(\mathbb{R}) \otimes \mathbb{R}^d.$$ 

The space $L^2_d$ is unitary isomorphic to a direct sum of $d$ identical copies of $L^2 := L^2(\mathbb{R})$, (i.e., the space of real-valued square integrable measurable functions with Lebesgue measure). Any element $f \in L^2_d$ may be written in the form

$$f = (f_1 \otimes e_1, \ldots, f_d \otimes e_d),$$

where $f_i \in L^2(\mathbb{R})$, $i = 1, \ldots, d$ and $\{e_1, \ldots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. The inner product in $L^2_d$ is given by

$$(f, g)_0 = \sum_{k=1}^d (f_k, g_k)_{L^2} = \sum_{k=1}^d \int_{\mathbb{R}} f_k(x) g_k(x) \, dx,$$

where $g = (g_1 \otimes e_1, \ldots, g_d \otimes e_d)$, $f_k \in L^2$, $k = 1, \ldots, d$, $f$ as given in (1). The corresponding norm in $L^2_d$ is given by

$$|f|_0^2 := \sum_{k=1}^d |f_k|_{L^2}^2 = \sum_{k=1}^d \int_{\mathbb{R}} f_k^2(x) \, dx.$$

As a densely embedded nuclear Fréchet space in $L^2_d$ we choose $S_d := S(\mathbb{R}) \otimes \mathbb{R}^d$, where $S(\mathbb{R})$ is the Schwartz test function space. An element $\varphi \in S_d$ has the form

$$\varphi = (\varphi_1 \otimes e_1, \ldots, \varphi_d \otimes e_d),$$

where $\varphi_i \in S(\mathbb{R})$, $i = 1, \ldots, d$. Together with the dual space $S'_d := S'(\mathbb{R}) \otimes \mathbb{R}^d$ we obtain the basic Gel’fand triple

$$S_d \subset L^2_d \subset S'_d.$$
The dual pairing between $S'_d$ and $S_d$ is given as an extension of the scalar product in $L^2_d$ by
\[ \langle f, \varphi \rangle_0 = \sum_{k=1}^{d} (f_k, \varphi_k)_{L^2}, \]
where $f$ and $\varphi$ as in (1) and (2), respectively. In $S'_d$ we choose the Borel $\sigma$-algebra $\mathcal{B}$ generated by the cylinder sets.

Define the operator $M_{\alpha/2}^{-}$ on $S(\mathbb{R})$ by
\[ M_{\alpha/2}^{-} \varphi := \begin{cases} K_{\alpha/2} D^{- (\alpha-1)/2} \varphi, & \alpha \in (0, 1), \\ \varphi, & \alpha = 1, \\ K_{\alpha/2} I^{- (\alpha-1)/2} \varphi, & \alpha \in (1, 2), \end{cases} \]
where the normalization constant $K_{\alpha/2} := \sqrt{\alpha \sin(\alpha \pi/2) \Gamma(\alpha)}$ and $D^{-}$, $I^{-}$ denote the left-side fractional derivative and fractional integral of order $r$ in the sense of Riemann-Liouville, respectively:
\[ (D^{-}_r f)(x) = -\frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_{x}^{\infty} f(t)(t-x)^{-r} dt \]
\[ (I^{-}_r f)(x) = \frac{1}{\Gamma(r)} \int_{x}^{\infty} f(t)(t-x)^{r-1} dt, \quad x \in \mathbb{R}. \]

We refer to [33] or [19] for the details on these operators. It is possible to obtain a larger domain of the operator $M_{\alpha/2}^{-}$ in order to include the indicator function $\mathbb{1}_{[0,t]}$ such that $M_{\alpha/2}^{-} \mathbb{1}_{[0,t]} \in L^2$, for the details we refer to Appendix A in [8]. We have the following

**Proposition 1 (Corollary 3.5 in [8]).** For all $t,s \geq 0$ and all $0 < \alpha < 2$ it holds that
\[ \left( M_{\alpha/2}^{-} \mathbb{1}_{[0,t]}, M_{\alpha/2}^{-} \mathbb{1}_{[0,s]} \right)_{L^2} = \frac{1}{2} \left( t^{\alpha} + s^{\alpha} - |t-s|^{\alpha} \right). \quad (3) \]

Note that this coincides with the covariance of the fractional Brownian motion with Hurst parameter $H = \frac{\alpha}{2}$.

In order to construct $\mathbb{g} \mathbb{g}$Bm we will use the Mittag-Leffler function which is introduced by G. Mittag-Leffler in a series of papers [24] [25] [26].
Definition 2 (Mittag-Leffler function). For $\beta > 0$ the Mittag-Leffler function $E_{\beta}$ is defined as an entire function by the following series representation

$$E_{\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C},$$

where $\Gamma$ denotes the Euler gamma function.

Note that for $\beta = 1$ the Mittag-Leffler function coincides with the classical exponential function. We also consider the so-called the $M$-Wright function $M_{\beta}$, $0 < \beta \leq 1$ (in one variable) defined by

$$M_{\beta}(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}.$$

For the choice $\beta = 1/2$ the corresponding $M$-Wright function reduces to the Gaussian density

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{z^2}{4} \right).$$

The Mittag-Leffler function $E_{\beta}$ and the $M$-Wright are related through the Laplace transform

$$\int_0^{\infty} e^{-s\tau} M_{\beta}(\tau) \, d\tau = E_{\beta}(-s).$$

The Mittag-Leffler measures $\mu_{\beta}$, $0 < \beta \leq 1$ is a family of probability measures on $S'_d$ whose characteristic functions are given by Mittag-Leffler functions, see Definition 2. Using the Bochner-Minlos theorem, see [11] or [16], we obtain the following definition.

Definition 3 (cf. [9]). For any $\beta \in (0,1]$ the Mittag-Leffler measure is defined as the unique probability measure $\mu_{\beta}$ on $S'_d$ by fixing its characteristic functional

$$\int_{S'_d} e^{i\langle w, \varphi \rangle_0} \, d\mu_{\beta}(w) = E_{\beta} \left( -\frac{1}{2} |\varphi|^2_0 \right), \quad \varphi \in S_d.$$

Remark 4. 1. The measure $\mu_{\beta}$ is also called grey noise (reference) measure, cf. [9] and [8].
2. The range $0 < \beta \leq 1$ ensures the complete monotonicity of $E_\beta(-x)$, see Pollard [28], i.e., $(-1)^n E_\beta^{(n)}(-x) \geq 0$ for all $x \geq 0$ and $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. In other words, it is sufficient to show that

$$S_d \ni \varphi \mapsto E_\beta \left( -\frac{1}{2} |\varphi|^2_0 \right) \in \mathbb{R}$$

is a characteristic function in $S_d$.

By $L^2(\mu_\beta) := L^2(S'_d, \mathcal{B}, \mu_\beta)$ we denote the complex Hilbert space of square integrable measurable functions defined on $S'_d$ with scalar product

$$(F,G)_{L^2(\mu_\beta)} := \int_{S'_d} F(w)\bar{G}(w) \, d\mu_\beta(w), \quad F, G \in L^2(\mu_\beta).$$

The corresponding norm is denoted by $\| \cdot \|_{L^2(\mu_\beta)}$. It follows from (7) that all moments of $\mu_\beta$ exists and we have

**Lemma 5.** For any $\varphi \in S_d$ and $n \in \mathbb{N}_0$ we have

$$\int_{S'_d} \langle w, \varphi \rangle_0^{2n+1} d\mu_\beta(w) = 0,$$

$$\int_{S'_d} \langle w, \varphi \rangle_0^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)} |\varphi|^2_0.$$

In particular, $\| \langle \cdot , \varphi \rangle \|_{L^2(\mu_\beta)}^2 = \frac{1}{\Gamma(\beta + 1)} |\varphi|^2_0$ and by polarization for any $\varphi, \psi \in S_d$ we obtain

$$\int_{S'_d} \langle w, \varphi \rangle_0 \langle w, \psi \rangle_0 d\mu_\beta(w) = \frac{1}{\Gamma(\beta + 1)} \langle \varphi, \psi \rangle_0.$$

3. Generalized grey Brownian motion in dimension $d$

For any test function $\varphi \in S_d$ we define the random variable

$$X^\beta(\varphi) : S'_d \longrightarrow \mathbb{R}^d, \quad w \mapsto X^\beta(\varphi, w) := (\langle w_1, \varphi_1 \rangle, \ldots, \langle w_d, \varphi_d \rangle).$$

The random variable $X^\beta(\varphi)$ has the following properties which are a consequence of Lemma 5 and the characteristic function of $\mu_\beta$ given in (7).
Proposition 6. Let $\varphi, \psi \in S_d$, $k \in \mathbb{R}^d$ be given. Then

1. The characteristic function of $X^\beta(\varphi)$ is given by

$$E(e^{i(k,X^\beta(\varphi))}) = E_\beta \left(-\frac{1}{2} \sum_{j=1}^{d} k_j^2 |\varphi_j|_{L^2}^2 \right).$$

(8)

2. The characteristic function of the random variable $X^\beta(\varphi) - X^\beta(\psi)$ is

$$E(e^{i(k,X^\beta(\varphi) - X^\beta(\psi))}) = E_\beta \left(-\frac{1}{2} \sum_{i=1}^{d} k_i^2 |\varphi_i - \psi_i|_{L^2}^2 \right).$$

(9)

3. The expectation of the $X^\beta(\varphi)$ is zero and

$$\|X^\beta(\varphi)\|_{L^2(\mu_\beta)}^2 = \frac{1}{\Gamma(\beta + 1)} |\varphi|_0^2.$$  

(10)

4. The moments of $X^\beta(\varphi)$ are given by

$$\int_{S_d'} \left|X^\beta(\varphi, w)\right|^{2n+1} d\mu_\beta(w) = 0,$$

$$\int_{S_d'} \left|X^\beta(\varphi, w)\right|^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)} |\varphi|_0^{2n}.$$

Remark 7. 1. The property (10) of $X^\beta(\varphi)$ gives the possibility to extend the definition of $X^\beta$ to any element in $L^2_d$, in fact, if $f \in L^2_d$, then there exists a sequence $(\varphi_k)_{k=1}^\infty \subset S_d$ such that $\varphi_k \to f$, $k \to \infty$ in the norm of $L^2_d$. Hence, the sequence $(X^\beta(\varphi_k))_{k=1}^\infty \subset L^2(\mu_\beta)$ forms a Cauchy sequence which converges to an element denoted by $X^\beta(f) \in L^2(\mu_\beta)$.

2. For $\beta = 1$ property (10) yields the Itô isometry.

We define $\mathbb{I}_{[0,t]} \in L^2_d$, $t \geq 0$, by

$$\mathbb{I}_{[0,t]} := (\mathbb{I}_{[0,t]} \otimes e_1, \ldots, \mathbb{I}_{[0,t]} \otimes e_d)$$

and consider the process $X^\beta(\mathbb{I}_{[0,t]}) \in L^2(\mu_\beta)$ such that the following definition makes sense.
Definition 8. For any $0 < \alpha < 2$ we define the process
\[
S'_d \ni w \mapsto B_{\beta,\alpha}(t, w) := \left( \langle w, (M_{\alpha/2}^- \mathbb{1}_{[0,t])} \otimes e_1 \rangle, \ldots, \langle w, (M_{\alpha/2}^- \mathbb{1}_{[0,t])} \otimes e_d \rangle \right) = \left( \langle w_1, M_{\alpha/2}^- \mathbb{1}_{[0,t])} \rangle, \ldots, \langle w_d, M_{\alpha/2}^- \mathbb{1}_{[0,t])} \rangle \right), \quad t > 0
\]
as an element in $L^2(\mu_\beta)$. This process is called a version of $d$-dimensional generalized grey Brownian motion (ggBm). Its characteristic function has the form
\[
E(e^{i(k, B_{\beta,\alpha}(t))}) = E_\beta \left( -\frac{|k|^2}{2} t^\alpha \right), \quad k \in \mathbb{R}^d.
\]

Remark 9. 1. By Remark 7 the $d$-dimensional ggBm exist as a $L^2(\mu_\beta)$-limit and hence the map $S'_d \ni \omega \mapsto \langle \omega, \mathbb{1}_{[0,t]} \rangle$ yields a version of ggBm, $\mu_\beta$-a.s., but not in the pathwise sense.

2. For a fixed $0 < \alpha < 2$ one can show by the Kolmogorov-Centsov continuity theorem that the paths of the process are $\mu_\beta$-a.s. continuous.

Proposition 10. For any $0 < \alpha < 2$, the process $B_{\beta,\alpha} := \{ B_{\beta,\alpha}(t), t \geq 0 \}$, is $\alpha/2$ self-similar with stationary increments.

Proof. Given $k = (k_1, k_2, \ldots, k_n) \in \mathbb{R}^n$, we have to show that for any $0 < t_1 < t_2 < \ldots < t_n$ and $a > 0$:
\[
E\left( \exp \left( i \left\langle \sum_{j=1}^n k_j M_{\alpha/2}^- \mathbb{1}_{[0,a t_j]} \right\rangle \right) \right) = E\left( \exp \left( ia^{\alpha/2} \left\langle \sum_{j=1}^n k_j M_{\alpha/2}^- \mathbb{1}_{[0,t_j]} \right\rangle \right) \right).
\]
It follows from (13) that eq. (13) is equivalent to
\[
E_\beta \left( -\frac{1}{2} \sum_{j=1}^n |k_j M_{\alpha/2}^- \mathbb{1}_{[0,a t_j]}|^2_{L^2} \right) = E_\beta \left( -\frac{1}{2} a^{\alpha/2} \sum_{j=1}^n |k_j M_{\alpha/2}^- \mathbb{1}_{[0,t_j]}|^2_{L^2} \right).
\]
Because of the complete monotonicity of $E_\beta$, the above equality reduces to
\[
\left| \sum_{j=1}^n k_j M_{\alpha/2}^- \mathbb{1}_{[0,a t_j]} \right|_{L^2}^2 = a^{\alpha} \left| \sum_{j=1}^n k_j M_{\alpha/2}^- \mathbb{1}_{[0,t_j]} \right|_{L^2}^2.
\]
which is easy to show, taking into account (3). A similar procedure may be applied in order to prove the stationarity of the increments. Hence, for any \( h \geq 0 \), we have to show that
\[
E \left( \exp \left( i \sum_{j=1}^{n} k_j (B^\beta,\alpha(t_j + h) - B^\beta,\alpha(h)) \right) \right) = E \left( \exp \left( i \sum_{j=1}^{n} k_j B^\beta,\alpha(t_j) \right) \right).
\]
The above procedure reduces this equality to check the following
\[
\left| \sum_{j=1}^{n} k_j M_{n/2}^{\alpha/2} \mathbb{I}_{[h,t_j+h]} \right|_{L^2}^2 = \left| \sum_{j=1}^{n} k_j M_{n/2}^{\alpha/2} \mathbb{I}_{[0,t_j]} \right|_{L^2}^2
\]
which is verified analogously as for the self-similarity.

**Remark 11.** The family \( \{B^\beta,\alpha(t), t \geq 0, \beta \in (0,1], \alpha \in (0,2)\} \) forms a class of \( \alpha/2 \) self-similar process with stationary increments (\( \alpha/2\)-sssi) which includes:

1. For \( \beta = \alpha = 1 \), the process \( \{B^{1,1}(t), t \geq 0\} \) is a standard \( d \)-dimensional Brownian motion.

2. For \( \beta = 1 \) and \( 0 < \alpha < 2 \), \( \{B^{1,\alpha}(t), t \geq 0\} \) is a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( \alpha/2 \).

3. For \( \alpha = 1 \), \( \{B^{\beta,1}(t), t \geq 0\} \) is \( \alpha/2 \) self-similar non Gaussian process with
\[
E \left( e^{i(k,B^{\beta,1}(t))} \right) = E^\beta \left( -\frac{|k|^2}{2} t \right), \quad k \in \mathbb{R}^d.
\]

4. For \( 0 < \alpha = \beta < 1 \), the process \( \{B^{\beta}(t) := B^{\beta,\beta}(t), t \geq 0\} \) is \( \beta/2 \) self-similar and is called \( d \)-dimensional grey Brownian motion (gBm for short). Its characteristic function is given by
\[
E \left( e^{i(k,B^{\beta}(t))} \right) = E^\beta \left( -\frac{|k|^2}{2} t^\beta \right), \quad k \in \mathbb{R}^d.
\]

For \( d = 1 \), gBm was introduced by W. Schneider in [31, 32].
4 Distributions and characterization theorems

There is a standard way to construct the test and distribution spaces in non Gaussian analysis through Appell systems, the details of this construction can be found in [20], [9], [8] and references therein. In between the many choices of triples which can be constructed we choose the Kondratiev triple

\[(S_d)^1_{\mu,\beta} \subset (H_p)^1_{q,\mu,\beta} \subset L^2(\mu) \subset (H_{-p})^{-1}_{-q,\mu,\beta} \subset (S_d)^{-1}_{\mu,\beta}.
\]

The space \((H_p)^1_{q,\mu,\beta}\) is defined as the completion of the \(\mathcal{P}(S_d')\) (the space of smooth polynomials on \(S_d'\)) w.r.t. the norm \(\| \cdot \|_{p,q,\mu,\beta}\) given by

\[\| \varphi \|^2_{p,q,\mu,\beta} := \sum_{n=0}^{\infty} (n!)^{2nq} |\varphi^{(n)}|^2_p, \quad p, q \in \mathbb{N}_0, \ \varphi \in \mathcal{P}(S_d').\]

The dual space \((H_{-p})^{-1}_{-q,\mu,\beta}\) is a subset of \(\mathcal{P}'(S_d')\) such that if \(\Phi \in (H_{-p})^{-1}_{-q,\mu,\beta}\), then

\[\| \Phi \|^2_{p,-q,\mu,\beta} := \sum_{n=0}^{\infty} 2^{-nq} |\Phi^{(n)}|^2_{-p} < \infty, \quad p, q \in \mathbb{N}_0.
\]

The dual pairing between \((S_d)^{-1}_{\mu,\beta}\) and \((S_d)^1_{\mu,\beta}\), denoted by \(\langle \cdot, \cdot \rangle_{\mu,\beta}\) is a bilinear extension of scalar product in \(L^2(\mu)\). For any \(\varphi \in (S_d)^1_{\mu,\beta}\) and \(\Phi \in (S_d)^{-1}_{\mu,\beta}\) we have

\[\langle \langle \Phi, \varphi \rangle \rangle_{\mu,\beta} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.
\]

The set of \(\mu,\beta\)-exponentials

\[\left\{ e_{\mu,\beta}(\cdot, \varphi) := \frac{e^{\langle \cdot, \varphi \rangle}}{E(e^{\langle \cdot, \varphi \rangle})}, \ \varphi \in S_d, \ |\varphi|_p < 2^{-q} \right\}
\]

forms a total set in \((H_p)^1_{q,\mu,\beta}\) and for any \(\varphi \in S_d\) such that \(|\varphi|_p < 2^{-q}\) we have \(\| e_{\mu,\beta}(\cdot, \varphi) \|_{p,q,\mu,\beta} < \infty\).

Let us introduce an integral transform, the \(S_{\mu,\beta}\)-transform, which is used to characterize the spaces \((S_d)^1_{\mu,\beta}\) and \((S_d)^{-1}_{\mu,\beta}\). For any \(\Phi \in (S_d)^{-1}_{\mu,\beta}\) and \(\varphi \in U \subset S_d\), where \(U\) is a suitable neighborhood of zero, we define

\[S_{\mu,\beta} \Phi(\varphi) := \langle \langle \Phi, e^{\langle \cdot, \varphi \rangle} \rangle \rangle_{\mu,\beta} = \frac{1}{E_{\beta}\left(\frac{1}{2} \langle \varphi, \varphi \rangle\right)} \langle \langle \Phi, e^{\langle \cdot, \varphi \rangle} \rangle \rangle_{\mu,\beta}.
\]
The characterization theorem for the space \((S_d)^{-1}_{\mu_{\beta}}\) via the \(S_{\mu_{\beta}}\)-transform is done using the spaces of holomorphic functions on \(S_{d,\mathbb{C}}\). We denote by \(\text{Hol}_0(S_{d,\mathbb{C}})\) the space of holomorphic functions at zero where we identify two functions which coincides in a neighborhood of zero. The space \(\text{Hol}_0(S_{d,\mathbb{C}})\) is given as the inductive limit of a family of normed spaces, see [20] for the details and the proof of the following characterization theorem.

**Theorem 12** (cf. [20, Theorem 8.34]). The \(S_{\mu_{\beta}}\)-transform is a topological isomorphism from \((S_d)^{-1}_{\mu_{\beta}}\) to \(\text{Hol}_0(S_{d,\mathbb{C}})\).

As a corollary from the characterization theorem the following integration result can be deduced.

**Theorem 13.** Let \((T, \mathcal{B}, \nu)\) be a measure space and \(\Phi_t \in (S_d)^{-1}_{\mu_{\beta}}\) for all \(t \in T\). Let \(U \subset S_{d,\mathbb{C}}\) be an appropriate neighbourhood of zero and \(0 < C < \infty\), such that

1. \(S_{\mu_{\beta}}\Phi_t : T \to \mathbb{C}\) is measurable for all \(t \in U\).
2. \(\int_T |S_{\mu_{\beta}}\Phi_t| d\nu(t) \leq C\) for all \(t \in U\).

Then, there exists \(\Phi \in (S_d)^{-1}_{\mu_{\beta}}\) such that for all \(\xi \in U\)

\[ S_{\mu_{\beta}}\Psi(\xi) = \int_T S_{\mu_{\beta}}\Phi_t(\xi) d\nu(t). \]

We denote \(\Psi\) by \(\int_T \Phi_t d\nu(t)\) and call it the weak integral of \(\Phi\).

In the following we will use the \(T_{\mu_{\beta}}\)-transform which is defined as follows.

**Lemma 14.** Let \(\Phi \in (S_d)^{-1}_{\mu_{\beta}}\) and \(p, q \in \mathbb{N}\) such that \(\Phi \in (H_{-p})^{-1}_{q,\mu_{\beta}}\). Then, the \(T_{\mu_{\beta}}\)-transform given by

\[ T_{\mu_{\beta}}\Phi(\varphi) = \langle \Phi, \exp(i\langle \cdot, \varphi \rangle) \rangle_{\mu_{\beta}} \]

is well-defined for \(\varphi \in U_{p,q}\) and we have

\[ T_{\mu_{\beta}}\Phi(\varphi) = E_\beta \left( \frac{1}{2} \langle \varphi, \varphi \rangle \right) S_{\mu_{\beta}}\Phi(i\varphi). \]

In particular, \(T_{\mu_{\beta}}\Phi \in \text{Hol}_0(S_{d,\mathbb{C}})\) if and only if \(S_{\mu_{\beta}}\Phi \in \text{Hol}_0(S_{d,\mathbb{C}})\). Moreover, **Theorem 13** also holds if the \(S_{\mu_{\beta}}\)-transform is replaced by the \(T_{\mu_{\beta}}\)-transform.

For details and proofs we refer to [9].
5 Self-intersection local times for ggBm in dimension $d$

In this section we consider the self-intersection local times for ggBm which is formally given by

$$L^{\beta,\alpha}(t) := \int_0^t \int_0^t \delta(B^{\beta,\alpha}(s) - B^{\beta,\alpha}(u)) \, du \, ds.$$ 

The (generalized) random variable $L^{\alpha,\beta}(a,t)$ is intended to measure the amount of time in which the sample path of a ggBm spends intersecting itself within the time interval $[0,t]$. A priori the expression above has no mathematical meaning since Lebesgue integration of Dirac delta distribution is not defined. We will prove that we can make sense of this object as a weak integral in Kondratiev distribution space.

**Theorem 15.** Let $0 < \alpha < 1$, $0 < \beta \leq 1$, and $d \in \mathbb{N}$ be such that $d\alpha < 2$.

Then, $L^{\beta,\alpha}(t)$ is a well defined element in $(S_d)^{-1}_{\mu_\beta}$ in the weak sense.

**Proof.** Using the representation

$$\delta(B^{\beta,\alpha}(s) - B^{\beta,\alpha}(u)) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{i\lambda(B^{\beta,\alpha}(s) - B^{\beta,\alpha}(u))} \, d\lambda$$

and denoting $\eta_x := M^{\alpha/2}_\beta I_{[0,x)}$ we compute for any $\varphi \in \mathcal{U}$ the $T_{\mu_\beta}$-transform of $L^{\beta,\alpha}(t)$ to obtain with

$$(T_{\mu_\beta}L^{\beta,\alpha}(t))(\varphi) = \left(\frac{1}{2\pi}\right)^{d/2} \int_0^t \int_0^t T_{\mu_\beta} \delta(B^{\beta,\alpha}(s) - B^{\beta,\alpha}(u))(\varphi) \, du \, ds$$

$$= \left(\frac{1}{2\pi}\right)^{d/2} \int_0^t \int_0^t \prod_{i=1}^d \left[ \int_{\mathbb{R}} E^{\beta}_{\lambda_i} \left( -\frac{1}{2} \lambda_i^2 |\eta_s - \eta_u|^2 - \frac{1}{2} \langle \varphi_i, \varphi_i \rangle - \lambda_i (\varphi_i \eta_t - \eta_u) \right) \, d\lambda_i \right] \, du \, ds.$$ 

By using the Laplace transform of $E^{\beta}_{\lambda_i}$ and computing the Gaussian integral yields

$$\int_{\mathbb{R}} E^{\beta}_{\lambda_i} \left( -\frac{1}{2} \lambda_i^2 |\eta_s - \eta_u|^2 - \frac{1}{2} \langle \varphi_i, \varphi_i \rangle - \lambda_i (\varphi_i \eta_t - \eta_u) \right) \, d\lambda_i$$

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\begin{align*}
= & \int_{\mathbb{R}} \int_{0}^{\infty} M_\beta(\tau) \exp \left( -\tau \left( \frac{1}{2} \lambda_i^2 |\eta_s - \eta_u|^2 + 1/2 \langle \varphi_i, \varphi_i \rangle + \lambda_i \langle \varphi_i, \eta_t - \eta_u \rangle \right) \right) \, d\tau d\lambda_i \\
= & \int_{0}^{\infty} d\tau M_\beta(\tau) \int_{\mathbb{R}} \exp \left( -\tau \left( \frac{1}{2} \lambda_i^2 |\eta_s - \eta_u|^2 + 1/2 \langle \varphi_i, \varphi_i \rangle + \lambda_i \langle \varphi_i, \eta_t - \eta_u \rangle \right) \right) \, d\lambda_i \\
= & \exp \left( 1/2 \langle \varphi_i, \varphi_i \rangle \right) \int_{0}^{\infty} M_\beta(\tau) \sqrt{2\pi/\tau} \exp \left( \tau |\varphi_i|^2/2 \right) \, d\tau.
\end{align*}

Next, by using Lemma A.4 from [9] and applying the Cauchy-Schwarz inequality we obtain

\begin{align*}
\int_{\mathbb{R}} E_\beta \left( -\frac{1}{2} \lambda_i^2 |\eta_s - \eta_u|^2 - 1/2 \langle \varphi_i, \varphi_i \rangle - \lambda_i \langle \varphi_i, \eta_t - \eta_u \rangle \right) \, d\lambda_i \\
= & \left( \frac{2\pi}{|\eta_s - \eta_u|^2} \right) \exp \left( \frac{1}{2} \langle \varphi_i, \varphi_i \rangle \right) \int_{0}^{\infty} \tau^{-1/2} M_\beta(\tau) \exp \left( \frac{\tau \langle \varphi_i, \eta_t - \eta_u \rangle^2}{2|\eta_s - \eta_u|^2} \right) \, d\tau \\
\leq & \left( \frac{2\pi}{|\eta_s - \eta_u|^2} \right) \exp \left( \frac{1}{2} \langle \varphi_i, \varphi_i \rangle \right) \int_{0}^{\infty} \tau^{-1/2} M_\beta(\tau) \exp \left( \frac{\tau |\varphi_i|^2}{2} \right) \, d\tau \\
\leq & K \left( \frac{2\pi}{|\eta_s - \eta_u|^2} \right) \exp \left( \frac{1}{2} \langle \varphi_i, \varphi_i \rangle \right).
\end{align*}

Putting all together gives

\begin{align*}
|(T_{\mu_\beta} L^\beta,\alpha(t))(\varphi)| \leq & \left( \frac{1}{2\pi} \right)^{d/2} \int_{0}^{t} \int_{0}^{t} \prod_{i=1}^{d} K \frac{\sqrt{2\pi}}{|\eta_s - \eta_u|} \exp \left( \frac{1}{2} \langle \varphi_i, \varphi_i \rangle \right) \, duds \\
= & K \exp \left( \frac{1}{2} |\varphi|^2 \right) \int_{0}^{t} \int_{0}^{t} \left( \frac{1}{|\eta_s - \eta_u|^2} \right)^{d/2} \, duds \\
= & 2K \exp \left( \frac{1}{2} |\varphi|^2 \right) \int_{0}^{t} \int_{0}^{s} (s-u)^{-\alpha d/2} \, duds.
\end{align*}

The last integral is finite for \( \alpha d < 2 \). The announced result now follows by an application of Theorem 13.

\section{Conclusion}
In this paper we have studied self-intersection local times of \( g^3Bm \) for the case \( d\alpha < 2 \) and characterized it as a Mittag-Leffler distribution in a suitable
distribution space. The case $d\alpha < 2$ corresponds for the Gaussian case $\beta = 1$ to the case $Hd < 1$. Indeed in this case \cite{18} showed that the self-intersection local time is a square integrable function. For $d\alpha \geq 1$ further renormalizations are needed, like e.g. centering of the random variable. These considerations for the case of ggBm are postponed for a forthcoming paper.

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