Refined functional relations for the elliptic SOS model

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Abstract

In this work we refine the method of [1] and obtain a novel kind of functional equation determining the partition function of the elliptic SOS model with domain wall boundaries. This functional relation arises from the dynamical Yang-Baxter relation and its solution is given in terms of multiple contour integrals.

PACS numbers: 05.50+q, 02.30.IK
Keywords: Dynamical Yang-Baxter equation, Functional relations, Domain wall boundaries

August 2012
1 Introduction

Face models or Solid-on-Solid (SOS) models of statistical mechanics were introduced by Baxter in the process of solving the eight-vertex model with periodic boundary conditions [2]. The Boltzmann weights of Baxter’s eight-vertex model are parameterised by elliptic functions and this feature is intrinsically connected with the requirement that the model statistical weights satisfy the Yang-Baxter equation [3, 4]. The elliptic nature of the eight-vertex model Boltzmann weights is naturally transported to the corresponding SOS model and a new continuous parameter emerges in the course of Baxter’s vertex-face transformation [2]. We shall refer to this new parameter as dynamical parameter [2] and its implications for the analytic theory of the eight-vertex model have been discussed in [5]. Besides the emergence of this new parameter, the resulting statistical weights no longer satisfy the standard Yang-Baxter equation but its dynamical version introduced in [6] and subsequently considered by Felder [7,9] as the quantised form of a modified classical Yang-Baxter equation [10].

In the same fashion as Drinfeld-Jimbo quantum groups [12,15] provide the algebraic structure underlying the solutions of the Yang-Baxter equation, the so called elliptic quantum groups introduced in [7,8] accommodate the solutions of the dynamical Yang-Baxter equation. In this work we shall restrict ourselves to the SOS model built out of the solution of the dynamical Yang-Baxter equation associated with the elliptic quantum group $E_{r,\gamma}[sl_2]$. As far as the boundary conditions are concerned, we shall consider the case of domain wall boundaries firstly introduced by Korepin in the context of vertex models [16] and subsequently extended for SOS models in [1,17,19].
In contrast to the case with periodic boundary conditions, the partition function of vertex and SOS models with domain wall boundaries can be exactly computed without relying on solutions of Bethe ansatz equations. Interestingly enough, the exact solution of the six-vertex model with domain wall boundaries [20] revealed that the model free-energy differs from the case with periodic boundary conditions [21]. This unusual dependence of bulk thermodynamic properties with boundary conditions has also been observed for the elliptic SOS model when the anisotropy parameter assumes a particular value [22]. For general values of the anisotropy parameter this study still poses as an open problem, probably due to the lack of suitable expressions for the partition function allowing to compute physical properties in the thermodynamic limit. In searching for alternative representations for this partition function, which might render the analysis of the thermodynamic limit feasible, we have obtained in [1] a multiple integral formula for the partition function of the trigonometric SOS model with domain wall boundaries. This case consists of a particular limit of a more general elliptic model, the limit where elliptic theta-functions degenerate into trigonometric functions, and here we refine and generalise the method of [1] for the general elliptic case.

This paper is planned as follows. In the Section 2 we give a brief description of SOS models with domain wall boundaries in terms of the generators of Felder’s dynamical Yang-Baxter relations. The conventions employed here are basically the ones already discussed in [1]. In the Section 3 we demonstrate how the dynamical Yang-Baxter relations can be explored in order to obtain a functional equation determining the model partition function. This functional equation is solved in Section 4 and concluding remarks are discussed in Section 5. Technical details required throughout this paper are presented in the Appendices.

2 Operatorial description of the SOS model

Partition functions of two-dimensional lattice models can be described in terms of operators representing the allowed configurations of the lattice. This feature goes back to Kramers and Wannier transfer matrix technique [23,24] and it has found several important generalisations [25]. Remarkably, when the statistical weights of the model satisfy the Yang-Baxter equation [3] or its dynamical counterpart [7–9], we not only have an operatorial description of the model but also an algebra governing its operators for any size of the lattice. In what follows we shall recall the conventions discussed in [1] which consist of an extension of the ones given in [16] for the six-vertex model.

Dynamical Yang-Baxter equation. Following [8,9] we encode the statistical weights of our elliptic SOS model on a matrix $\mathcal{R} \in \text{End}(V \otimes V)$ with $V \cong \mathbb{C}^2$. For variables $\lambda, \gamma, \theta \in \mathbb{C}$, this matrix $\mathcal{R}$ satisfies the dynamical Yang-Baxter equation,

$$\mathcal{R}_{12}(\lambda_1 - \lambda_2, \theta - \gamma \hat{h}_3) \mathcal{R}_{13}(\lambda_1 - \lambda_3, \theta) \mathcal{R}_{23}(\lambda_2 - \lambda_3, \theta - \gamma \hat{h}_1) = \mathcal{R}_{23}(\lambda_2 - \lambda_3, \theta) \mathcal{R}_{13}(\lambda_1 - \lambda_3, \theta - \gamma \hat{h}_2) \mathcal{R}_{12}(\lambda_1 - \lambda_2, \theta),$$

(2.1)
where \( \hat{h} = \text{diag}(1,-1) \). The Eq. (2.1) is defined in \( \text{End}(\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3) \) and the action of \( \mathcal{R}_{12}(\lambda, \theta - \gamma \hat{h}_3) \) on the basis vector \( v_1 \otimes v_2 \otimes v_3 \) is understood as

\[
[\mathcal{R}(\lambda, \theta - \gamma h)v_1 \otimes v_2] \otimes v_3 ,
\]

where \( h \) is a scalar denoting a particular eigenvalue of \( \hat{h} \), i.e. \( \hat{h}_i v_i = hv_i \).

**Definition.** Let \( \tau \) be a complex number such that \( \text{Im}(\tau) > 0 \) and write \( p = e^{i\pi \tau} \) so that \( |p| < 1 \). For \( \lambda \in \mathbb{C} \) we define the elliptic function \( f \) with nome \( p \) as

\[
f(\lambda) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^{n-\frac{1}{2}} p^{(n+\frac{1}{2})^2} e^{-(2n+1)\lambda} .
\]

The function \( f(\lambda) \) corresponds to the Jacobi theta-function \( \Theta_1(i\lambda, \tau)/2 \) \(^{26}\) and in the Appendix \(^3\) we have collected the properties of \( f \) required through this work.

The equation (2.1) has been considered in \(^7\)\(^8\) and its solution reads

\[
\mathcal{R}(\lambda, \theta) = \begin{pmatrix}
a_+(\lambda, \theta) & 0 & 0 & 0 \\
0 & b_+(\lambda, \theta) & c_+(\lambda, \theta) & 0 \\
0 & c_-(\lambda, \theta) & b_-(\lambda, \theta) & 0 \\
0 & 0 & 0 & a_-(\lambda, \theta)
\end{pmatrix}
\]

with non-null entries

\[
a_\pm(\lambda, \theta) = f(\lambda \pm \gamma) \\
b_\pm(\lambda, \theta) = f(\lambda) \frac{f(\theta \mp \gamma)}{f(\theta)} \\
c_\pm(\lambda, \theta) = f(\gamma) \frac{f(\theta \mp \lambda)}{f(\theta)} .
\]

Although we shall not make explicit use of it, we remark here that the algebraic structure underlying (2.4) is the elliptic quantum group \( E_{\tau,\gamma}[\mathfrak{sl}_2] \) \(^7\).

**Dynamical monodromy matrix.** Let \( \hat{\theta}_i \) be the operator valued parameter

\[
\hat{\theta}_i = \theta - \gamma \sum_{k=i+1}^L \hat{h}_k
\]

and consider the following ordered product of dynamical \( \mathcal{R} \)-matrices,

\[
\mathcal{T}_a(\lambda, \theta) = \prod_{1 \leq i \leq L} \mathcal{R}_{ai}(\lambda - \mu_i, \hat{\theta}_i) ,
\]
living in the tensor product space $V_a \otimes V_1 \otimes \cdots \otimes V_L$. We shall refer to $T_a(\lambda, \theta)$ as dynamical monodromy matrix or simply monodromy matrix. Since the dynamical $R$-matrix (2.4) satisfy the weight-zero condition $[R_{ab}(\lambda, \theta), \hat{h}_a + \hat{h}_b] = 0$, one can show that (2.7) obeys the relation

$$R_{ab}(\lambda_1 - \lambda_2, \theta - \gamma H)T_a(\lambda_1, \theta)T_b(\lambda_2, \theta - \gamma \hat{h}_a) = T_b(\lambda_2, \theta)T_a(\lambda_1, \theta - \gamma \hat{h}_b)R_{ab}(\lambda_1 - \lambda_2, \theta)$$

(2.8)

with $H = \sum_{k=1}^{L} \hat{h}_k$. Here we are considering $V \cong \mathbb{C}^2$ and the dynamical monodromy matrix can be recast in the form

$$T_a(\lambda, \theta) = \begin{pmatrix} A(\lambda, \theta) & B(\lambda, \theta) \\ C(\lambda, \theta) & D(\lambda, \theta) \end{pmatrix}$$

(2.9)

whose entries are then defined on $V_1 \otimes \cdots \otimes V_L$. The formula (2.8) encodes commutation relations for the entries of (2.9) which shall be referred to as dynamical Yang-Baxter relations.

**Domain wall boundaries.** The partition function of the elliptic SOS model with domain wall boundaries can be written in terms of entries of (2.9) as described in [1]. More precisely, the elliptic SOS model partition function $Z_\theta$ is given by the expected value

$$Z_\theta = \langle \bar{0} | \prod_{1 \leq j \leq L} B(\lambda_j, \theta + j\gamma) | 0 \rangle$$

(2.10)

where

$$|0\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\bar{0}\rangle = \bigotimes_{i=1}^{L} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

(2.11)

In the next section we shall demonstrate how the dynamical Yang-Baxter relations can be employed to produce a functional equation determining $Z_\theta$.

### 3 Functional relations

The relation (2.8) encodes commutation rules for the operators $A(\lambda, \theta), B(\lambda, \theta), C(\lambda, \theta)$ and $D(\lambda, \theta)$ once the structure (2.9) is considered. Out of the sixteen relations contained in (2.8), we will make use of only two of them in order to derive a functional equation describing the partition function (2.10). More precisely, the required relations are simply:

$$B(\lambda_1, \theta)B(\lambda_2, \theta + \gamma) = B(\lambda_2, \theta)B(\lambda_1, \theta + \gamma)$$

$$A(\lambda_1, \theta + \gamma)B(\lambda_2, \theta) = \frac{f(\lambda_2 - \lambda_1 + \gamma)}{f(\lambda_2 - \lambda_1)} f(\theta + \gamma) \cdot B(\lambda_2, \theta + \gamma)A(\lambda_1, \theta + 2\gamma)$$

$$- \frac{f(\theta + \gamma - \lambda_2 + \lambda_1)}{f(\lambda_2 - \lambda_1)} f(\gamma) \cdot B(\lambda_1, \theta + \gamma)A(\lambda_2, \theta + 2\gamma).$$

(3.1)
In addition to that, the weight-zero condition satisfied by \( (2.4) \) associated with the definition \( (2.7) \) allows us to compute the action of \( A(\lambda, \theta) \) on the states \(|0\rangle\) and \(|\bar{0}\rangle\) defined in \( (2.11) \). The vectors \(|0\rangle\) and \(|\bar{0}\rangle\) are respectively the \( \mathfrak{sl}_2 \) highest and lowest weight states and from \( (2.4) \) and \( (2.7) \) we readily obtain

\[
A(\lambda, \theta) |0\rangle = \prod_{j=1}^{L} f(\lambda - \mu_j + \gamma) |0\rangle
\]

\[
\langle \bar{0} | A(\lambda, \theta) = \frac{f(\theta - \gamma)}{f(\theta + (L - 1)\gamma)} \prod_{j=1}^{L} f(\lambda - \mu_j) \langle \bar{0} | .
\]  

(3.2)

**The framework.** In order to explore the relations \( (3.1) \) and \( (3.2) \) we shall consider the quantity

\[
\langle \bar{0} | A(\lambda_0, \theta + \gamma)Y_{\theta-\gamma}(\lambda_1, \ldots, \lambda_L) |0\rangle ,
\]

where \( Y_{\theta}(\lambda_1, \ldots, \lambda_L) = \prod_{1 \leq j \leq L} B(\lambda_j, \theta + j\gamma) \), computed in two different ways. One of them only makes use of the properties \( (3.2) \) arising from the \( \mathfrak{sl}_2 \) highest weight representation theory, while the second way employ the dynamical Yang-Baxter relations \( (3.1) \) in addition to \( (3.2) \). For instance, we can compute the term \( \langle \bar{0} | A(\lambda_0, \theta + \gamma) \) using solely \( (3.2) \) to find that \( (3.3) \) is proportional to \( Z_{\theta-\gamma}(\lambda_1, \ldots, \lambda_L) \). On the other hand, we could have firstly examined the quantity \( A(\lambda_0, \theta + \gamma)Y_{\theta-\gamma}(\lambda_1, \ldots, \lambda_L) \) \( |0\rangle \). For that we employ the relations \( (3.1) \) to move the operator \( A(\lambda_0, \theta + \gamma) \) through the string of operators \( B(\lambda_j, \theta + (j-1)\gamma) \) and then consider the action of the resulting operator \( A \) on the vector \(|0\rangle\). Exacting this procedure, we repeatedly apply \( (3.1) \) together with the addition rule \( (B.1) \) in order to show that

\[
A(\lambda_0, \theta + \gamma)Y_{\theta-\gamma}(\lambda_1, \ldots, \lambda_L) =
\]

\[
\frac{f(\theta + \gamma)}{f(\theta + (L + 1)\gamma)} \prod_{j=1}^{L} \frac{f(\lambda_j - \lambda_0 + \gamma)}{f(\lambda_j - \lambda_0)} Y_{\theta}(\lambda_1, \ldots, \lambda_L) A(\lambda_0, \theta + (L + 1)\gamma)
\]

\[
- \sum_{i=1}^{L} \frac{f(\theta + \gamma - \lambda_i + \lambda_0)}{f(\theta + (L + 1)\gamma)} \frac{f(\gamma)}{f(\lambda_i - \lambda_0)} \prod_{j=1}^{L} \frac{f(\lambda_j - \lambda_i + \gamma)}{f(\lambda_j - \lambda_i)} \times
\]

\[
Y_{\theta}(\lambda_0, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_L) A(\lambda_i, \theta + (L + 1)\gamma) .
\]  

(3.4)

Then we use \( (3.2) \) to compute the action of the operators \( A(\lambda_i, \theta + (L + 1)\gamma) \) appearing on the RHS of \( (3.4) \) on the vector \(|0\rangle\). Thus the combination of \( (3.4) \) and \( (3.2) \) allows us to write the quantity \( (3.3) \) as a linear combination of terms \( Z_{\theta} \) depending on the set of \( L + 1 \) variables \( \{\lambda_0, \lambda_1, \ldots, \lambda_L\} \) where only \( L \) variables are taken at a time.

**Functional equation.** Taking into account the above discussion, we can see that the consistency between the \( \mathfrak{sl}_2 \) highest weight representation theory, manifested in the
relations (3.2), and the dynamical Yang-Baxter relations implies the functional equation

\[ M_0 Z_{\theta-\gamma}(\lambda_1, \ldots, \lambda_L) + \sum_{i=0}^{L} N_i Z_\theta(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_L) = 0 , \] (3.5)

with coefficients given by

\[ M_0 = \frac{f(\theta)}{f(\theta + L\gamma)} \prod_{j=1}^{L} f(\lambda_0 - \mu_j) \]

\[ N_0 = -\frac{f(\theta + \gamma)}{f(\theta + (L+1)\gamma)} \prod_{j=1}^{L} f(\lambda_0 - \mu_j + \gamma) \prod_{j=1}^{L} f(\lambda_j - \lambda_0 + \gamma) \]

\[ N_i = \frac{f(\theta + \gamma + \lambda_0 - \lambda_i)}{f(\theta + (L+1)\gamma)} \frac{f(\gamma)}{f(\lambda_i - \lambda_0)} \prod_{j=1}^{L} f(\lambda_i - \mu_j + \gamma) \prod_{j \neq i}^{L} f(\lambda_j - \lambda_i) \]

\[ i = 1, \ldots, L . \] (3.6)

Some remarks are in order at this stage. Although the partition function considered here reduces to the one studied in [1] when the elliptic theta-function \( f \) degenerate into a trigonometric function, the functional equation (3.5) still differs significantly from the one obtained in [1]. For instance, (3.5) is a functional equation also over the variable \( \theta \) and even in the limit \( \theta \to \infty \), where \( Z_{\theta-\gamma} \) and \( Z_\theta \) coincide, we still would be left with a functional equation different from the one presented in [27]. This divergence is due to the fact that here we have started our analysis with the quantity (3.3) instead of \( \langle 0 | C(\lambda_0, \theta + \gamma) \prod_{j=1}^{L+1} B(\lambda_j, \theta + (j-1)\gamma) | 0 \rangle \) as employed in the works [1] and [27]. This different starting point allows us to obtain a simpler functional equation whose solution will be discussed in the next section.

4 The partition function

This section is concerned with solving the functional relation (3.5). The method we shall employ is essentially the one described in [1] which exploits special zeroes of \( Z_\theta \) to produce a separation of variables. Some structural properties of (3.5) will be of utility to help us identifying the elements required to solve this functional equation. For instance, the partition function \( Z_\theta \) is a function of two sets of variables, i.e. \( \{\lambda_1, \ldots, \lambda_L\} \) and \( \{\mu_1, \ldots, \mu_L\} \), in addition to the parameters \( \gamma, \theta \) and the elliptic nome \( p \). In our framework, however, the set of variables \( \{\mu_1, \ldots, \mu_L\} \) can also be regarded as parameters while \( \theta \) is promoted to a variable. This follows from the fact that (3.5) is an equation not only over variables \( \lambda_j \) but also \( \theta \).

With this in mind we can see that (3.5) is a homogeneous equation in the sense that \( \alpha Z_\theta \) is a solution if so is \( Z_\theta \) and \( \alpha \) is independent of \( \lambda_j \) and \( \theta \). This property implies that the equation (3.5) will be able to determine the partition function up to an overall
multiplicative factor independent of \( \lambda_j \) and \( \theta \) at most. Thus the complete determination of \( Z_\theta \) will require that we are able to compute it for a particular value of \( \lambda_j \) and \( \theta \) in order to determine this overall factor. Any point on the \((\lambda_j, \theta)\)-plane would serve our need and we can choose it such that the evaluation of \( Z_\theta \) is as simple as possible. As demonstrated in the Appendix A, the evaluation of \( Z_\theta \) in the limit \((\lambda_j, \theta) \to \infty\) can be performed in the same lines of \( [27] \). Moreover, the equation \((3.5)\) is linear which raises the issue of uniqueness of the solution since linear combinations of particular solutions also solve \((3.5)\). Similarly to the case considered in \([1]\), we will see that the location of zeroes of \( Z_\theta \) will select the appropriate solution uniquely.

Asymptotic behaviour. In the limit \((\lambda_j, \theta) \to \infty\) the partition function \((2.10)\) behaves as

\[
Z_\theta(\lambda_1, \ldots, \lambda_L) \sim \frac{f(\gamma)^L}{2L(L-1)} \sum_{n_1 = -\infty}^{\infty} \cdots \sum_{n_{L-1} = -\infty}^{\infty} \sum_{n_L = -\infty}^{\infty} \cdots \sum_{n_{L-1} = -\infty}^{\infty} (-1)^{\sum_{a=1}^{L} n_{a-1}} \prod_{a=1}^{L} \prod_{i=1}^{L-1} p_{n_i(a)} q_{n_i(a)} e^{(\lambda_{a\mu(a)} - \mu_{\sigma(a)})} \sum_{\sigma \in S_L} \prod_{(a,b) \in I_\sigma} (q_{\mu(a)} q_{\mu(b)})^{-1},
\]

where \( e_n = e^{-(2n+1)} \), \( p_n = p^{(n+\frac{3}{2})^2} \), \( q_n = e_n^\gamma \) and \( \mu(a) = \{ \mu_i : i \neq a \} \). Here \( S_L \) denotes the group of permutations of \( L \) objects and \( \sigma = \sigma(1) \ldots \sigma(L) \) stands for a given permutation.

Higher order theta-function. The partition function \( Z_\theta \) is a theta-function of order \( L \) and norm \( t_i \) in each one of its variables \( \lambda_i \) separately. That is to say there exist constants \( C \) and \( \xi^{(i)}_j \) satisfying \( \xi^{(i)}_1 + \cdots + \xi^{(i)}_L = t_i \) such that

\[
Z_\theta = C \prod_{j=1}^{L} f(\lambda_j - \xi^{(i)}_j).
\]

Although an explicit expression for the norm \( t_i \) shall not be required, unveiling special zeroes \( \xi^{(i)}_j \) for a particular specialisation of variables will be an important step for solving \((3.5)\).

Now we shall proceed with the analysis of \((3.5)\) in the lines of \([1]\). For that we look for special values of the variables \( \lambda_j \) such that particular zeroes of \( Z_\theta \) can be identified.

Special zeroes. The coefficients \( M_0 \) and \( N_i \) given in \((3.6)\) exhibit a factorised form and due to that identifying their zeroes is a simple task. For instance, when \( \lambda_0 = \mu_1 \) and
\( \lambda_1 = \mu_1 - \gamma \) we find that \( M_0 = N_0 = N_1 = 0 \). Next we set \( \lambda_j = \lambda_{j+1} - \gamma \) successively for \( j \in [2, L - 1] \) and collect the result at each step. At the last step we find

\[
N_2 \ Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_L - (L - 3)\gamma, \lambda_L - (L - 4)\gamma, \ldots, \lambda_L) = 0 ,
\]

and since \( N_2 \) is different from zero we can conclude that the vanishing of (4.3) is due to \( Z_\theta \).

This result can now be substituted back into the previous steps leading to (4.3). By doing so we find the more general vanishing condition, namely

\[
Z_\theta(\mu_1, \ldots, \mu_1 - \gamma, \lambda_3, \ldots, \lambda_L) = 0 ,
\]

for general values of the variables \( \lambda_j \) with \( j \in [3, L] \). This process can also be performed starting with variables \( \lambda_0 = \mu_1 \) and \( \lambda_j = \mu_1 - \gamma \) for any \( j \in [1, L] \), which allows us to conclude that \( Z_\theta(\mu_1, \ldots, \mu_1 - \gamma, \ldots) = 0 \).

**Building up the solution.** The zeroes of \( Z_\theta \) above unveiled have a special appeal since we are interested in the solution of (3.5) consisting of a higher order theta-function (4.2). Taking that into account, those special zeroes imply that

\[
Z_\theta(\mu_1, \lambda_2, \ldots, \lambda_L) = \prod_{j=2}^{L} f(\lambda_j - \mu_1 + \gamma) \ V_\theta(\lambda_2, \ldots, \lambda_L) ,
\]

where \( V_\theta \) is also a theta-function but of order \( L - 1 \) in each one of its variables. Next we set \( \lambda_0 = \mu_1 \) in the Eq. (3.5) and substitute the expression (4.4) into it. The resulting equation can then be solved for \( Z_\theta(\lambda_1, \ldots, \lambda_L) \) yielding the formula

\[
Z_\theta(\lambda_1, \ldots, \lambda_L) = \sum_{i=1}^{L} m_i \ V_\theta(\ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots) \quad (4.5)
\]

with coefficients

\[
m_i = \frac{f(\theta + \gamma + \mu_1 - \lambda_i)}{f(\theta + \gamma)} \prod_{j=2}^{L} f(\lambda_i - \mu_j + \gamma) \prod_{j=1, j \neq i}^{L} f(\lambda_j - \mu_1 + \gamma) \prod_{j=1}^{L} f(\lambda_j - \lambda_i),
\]

We then substitute the formula (4.5) back into the original equation (3.5) and set \( \lambda_L = \mu_1 \). After eliminating an overall factor we are left with the equation

\[
P_0 \ V_\theta(\ldots, \lambda_{L-1}) + \sum_{i=0}^{L-1} Q_i \ V_\theta(\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{L-1}) = 0 ,
\]

where the coefficients \( P_0 \) and \( Q_i \) correspond respectively to the coefficients \( M_0 \) and \( N_i \) given in (3.6) under the mapping \( L \to L - 1, \ \theta \to \theta + \gamma \) and \( \mu_i \to \mu_{i+1} \). Thus the function \( V_\theta \) obeys essentially the same equation as the partition function \( Z_\theta \) but for a square lattice of dimensions \((L - 1) \times (L - 1)\). Now since \( V_\theta \) is also a theta-function, this procedure can be repeatedly carried out until we reach the equation for \( L = 1 \). The
solution of (3.5) for $L = 1$ can be found in the Appendix and gathering our results we obtain the following solution for general $L$,

$$Z_\theta(\lambda_1, \ldots, \lambda_L) = \sum_{\sigma \in S_L} F_{\sigma(1) \ldots \sigma(L)}$$  (4.8)

where

$$F_{\sigma(1) \ldots \sigma(L)} = \frac{\Omega_L}{\prod_{k=2}^{L} f(\mu_1 - \mu_k + \gamma)} \prod_{n=1}^{L} \frac{f(\theta + n\gamma - \lambda_{\sigma(n)} + \mu_n)}{f(\theta + n\gamma)} \prod_{j>n}^{L} f(\lambda_{\sigma(n)} - \mu_j) \prod_{j<n}^{L} f(\lambda_{\sigma(n)} - \lambda_j)$$  (4.9)

The overall factor $\Omega_L$ arises from the homogeneity of (3.5) as previously discussed, and from (4.1) we obtain $\Omega_L = f(\gamma)^L \prod_{k=2}^{L} f(\mu_1 - \mu_k + \gamma)$. It is important to remark here that this partition function has also been considered in [17–19] where a similar but still different expression for $F_{\sigma(1) \ldots \sigma(L)}$ has been found.

**Multiple integral formula.** The partition function $Z_\theta$ can be represented by a multiple contour integral as follows. The function $V_\theta$ in the formula (4.5) is essentially the partition function for a lattice of size $(L - 1) \times (L - 1)$ and modified parameters. In fact, the decomposition of $Z_\theta$ in terms of $V_\theta$ as described by (4.5) can be thought of as a separation of variables. Moreover, we shall see that the prescription given by (4.5) can be mimicked by the Cauchy like integral

$$Z_\theta(\lambda_1, \ldots, \lambda_L) = \oint \ldots \oint \frac{H(w_1, \ldots, w_L)}{\prod_{i,j=1}^{L} f(w_i - \lambda_j)} \prod_{j=1}^{L} \frac{dw_j}{2i\pi},$$  (4.10)

with integration contours enclosing solely the zeroes of $f$ when $w_i \to \lambda_j$. Also we shall assume that $H(w_1, \ldots, w_L)$ has no poles inside the integration contour. Under those assumptions the formula (4.10) can be for instance, integrated over the variable $w_1$, and by doing so we obtain the relation

$$Z_\theta(\lambda_1, \ldots, \lambda_L) = \sum_{i=1}^{L} f'(0)^{-1} \oint \ldots \oint \frac{H(w_1, \ldots, w_L)_{|w_1=\lambda_i}}{\prod_{j \neq i}^{L} f(\lambda_i - \lambda_j) \prod_{j=2}^{L} f(w_j - \lambda_i)} \times \prod_{k=2}^{L} \prod_{j \neq i}^{L} f(w_k - \lambda_j) \prod_{j=2}^{L} \frac{dw_j}{2i\pi},$$  (4.11)

where $f'(0)$ denotes the derivative of $f(\lambda)$ with respect to $\lambda$ at the point $\lambda = 0$. The expression (4.11) decomposes similarly to (4.5) allowing us to look for a term by term identification. Thus taking into account the explicit form of the factors $m_i$ given in (4.6),
we find the following relation for the function $H$,

$$H(w_1, \ldots, w_L)_{w_i = \lambda_i} = \prod_{j=2}^{L} \frac{f'(0)}{f(\theta + \gamma + \mu_1 - \lambda_i)} \frac{f(\theta + \gamma) \prod_{j=2}^{L} f(\lambda_i - \mu_j + \gamma)}{\prod_{j \neq i}^{L} f(\mu_1 - \lambda_j) \prod_{j \neq i}^{L} f(\lambda_j - \lambda_i + \gamma) \prod_{j=2}^{L} f(w_j - \lambda_i) \bar{H}(w_2, \ldots, w_L)}.$$

(4.12)

The function $\bar{H}$ in (4.12) consists of $H$, up to an overall multiplicative factor independent of $\lambda_j$, $w_j$ and $\theta$, under the mappings $\theta \to \theta + \gamma$ and $\mu_i \to \mu_{i+1}$. Furthermore, the LHS of (4.12) consists of the function $H$ computed at the particular point $w_1 = \lambda_i$, and it would be useful to have a similar relation valid for general values of the variable $w_1$. In order to obtain such relation, we first notice that (4.12) needs to be satisfied for $i \in [1, L]$ and that it is required to hold only when integrated according to (4.11). Thus assuming that $H$ has no poles inside the integration contour, we only need to consider (4.12) under the mappings $\lambda_i \to w_1$ and $\lambda_j \to w_j$ for $j \neq i$ to obtain the relation

$$H(w_1, \ldots, w_L) = \prod_{j=2}^{L} \frac{f'(0)}{f(\theta + \gamma + \mu_1 - w_1)} \frac{f(\theta + \gamma) \prod_{j=2}^{L} f(w_1 - \mu_j + \gamma)}{\prod_{j \neq i}^{L} f(\mu_1 - w_j) \prod_{j \neq i}^{L} f(w_j - w_1 + \gamma) \prod_{j=2}^{L} f(w_j - w_1) \bar{H}(w_2, \ldots, w_L)}.$$

(4.13)

Now the formula (4.13) can be readily iterated once we know $H(w_1)$. For that we consider the results of the Appendix D and from (D.6), we can immediately read

$$H(w_1) = f'(0) f(\gamma) \frac{f(\theta + \gamma - w_1 + \mu_1)}{f(\theta + \gamma)}.$$

(4.14)

Thus the iteration of (4.13) with (4.14) as initial condition yields the formula

$$H(w_1, \ldots, w_L) = [f'(0) f(\gamma)]^L \prod_{j>i}^{L} f(w_j - w_i + \gamma) f(w_j - w_i) \prod_{j=1}^{L} \frac{f(\theta + \gamma - w_j + \mu_j)}{f(\theta + \gamma)} \times \prod_{j<i}^{L} f(\mu_j - w_i) \prod_{j>i}^{L} f(w_i - \mu_j + \gamma).$$

(4.15)

The expression (4.15) already takes into account the asymptotic behaviour (4.1) and though here we have considered a functional equation different from the one obtained in [1], the expression (4.15) indeed reduces to the formula of [1] in the degenerated limit with the conventions properly adjusted. Moreover, it is worth remarking that the homogeneous limit $\lambda_j \to \lambda$ and $\mu_j \to \mu$ can be trivially obtained from the integral formula (4.10) [15].

\footnote{Strictly speaking, the identity (4.12) is only required to hold when integrated as $\oint \cdots \oint dw_2 \cdots dw_L$.}
5 Concluding remarks

In this work the partition function of the elliptic SOS model with domain wall boundaries was studied through a fusion of algebraic and functional techniques. The partition function of the model was shown to obey a functional equation arising from commutation rules encoded in the dynamical Yang-Baxter relation (2.8) which is valid for general values of the model parameters. The solution was then obtained as a multiple contour integral.

The possibility of deriving functional equations for such partition functions from the Yang-Baxter algebra and its dynamical counterpart was firstly demonstrated in [27,1]. Although here we have also employed the dynamical Yang-Baxter relation, the mechanism considered in Section 3 differs from the one used in [27,1], and the resulting functional equation is significantly simpler than the ones previously obtained. Interestingly, solving this new type of functional equation follows the same lines of [1] but each one of the steps required are dramatically simplified.

The elliptic SOS model considered here is also referred to as 8VSOS model in the literature and for the special value of the anisotropy parameter $\gamma = \frac{2\pi}{3}$, it reduces to the so called Three-colouring model [28]. For the case with domain wall boundaries, the partition function of the Three-colouring model was shown to obey a certain functional equation in [29,30] but a possible connection with our results has eluded us so far. In the work [22] this same partition function was studied under the light of the symmetric polynomials theory where a set of two-variables polynomials have been introduced. These polynomials were conjectured in [31] to satisfy a certain partial differential equation and recurrence relation, to which (3.5), (4.10) and (4.15) might shed some light into their proofs.

6 Acknowledgements

The author thanks the Australian Research Council (ARC) and the Centre of Excellence for the Mathematics and Statistics of Complex Systems (MASCOS) for financial support. The author also thanks the anonymous referee for several comments and suggestions which helped to improve this manuscript.

A Asymptotic behaviour

In the limit $\theta \to \infty$ the $\mathcal{R}$-matrix (2.4) resembles the one associated with the six-vertex model, except that the Boltzmann weights (2.5) still consist of elliptic theta-functions. Also, from the definition (2.10) we can readily see that the whole dependence of $Z_\theta$ with a particular variable $\lambda_j$ will be described by the operator $B(\lambda_j, \theta + j\gamma)$, which is very similar to the six-vertex model analogous in the mentioned limit. Moreover, in order to proceed with the analysis of $Z_\theta$ in the full limit $(\lambda_j, \theta) \to \infty$, it will be useful to rewrite (2.9) as

$$\mathcal{T}_a^{(L)}(\lambda, \theta) = \begin{pmatrix} A_L(\lambda, \theta) & B_L(\lambda, \theta) \\ C_L(\lambda, \theta) & D_L(\lambda, \theta) \end{pmatrix}. \quad (A.1)$$
The Eq. (A.1) differs from (2.9) by the index \( K \) where we have introduced the conventions \( R \) that we have inserted in order to emphasise we are considering the ordered product of \( L \) matrices \( R_{n j} \) as given by (2.7). The matrix \( R_{n j}(\lambda, \theta) \) in its turn consists of a \( 2 \times 2 \) matrix in the space \( \mathbb{V}_a \), i.e.

\[
R_{n j}(\lambda, \theta) = \begin{pmatrix} \alpha_j(\lambda, \theta) & \beta_j(\lambda, \theta) \\ \gamma_j(\lambda, \theta) & \delta_j(\lambda, \theta) \end{pmatrix}, \tag{A.2}
\]

whose entries are then matrices acting non-trivially on the \( j \)-th space of the tensor product \( \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L \). More precisely we have

\[
\alpha_j(\lambda, \theta) = \begin{pmatrix} a_+(\lambda, \theta) & 0 \\ 0 & b_+(\lambda, \theta) \end{pmatrix}_j, \quad \beta_j(\lambda, \theta) = \begin{pmatrix} 0 & 0 \\ c_+(\lambda, \theta) & 0 \end{pmatrix}_j, \\
\gamma_j(\lambda, \theta) = \begin{pmatrix} 0 & c_-(\lambda, \theta) \\ 0 & 0 \end{pmatrix}_j, \quad \delta_j(\lambda, \theta) = \begin{pmatrix} b_-(\lambda, \theta) & 0 \\ 0 & a_-(\lambda, \theta) \end{pmatrix}_j. \tag{A.3}
\]

In this way the definition (2.7) can be implemented recursively, i.e.

\[
T_a^{(L+1)}(\lambda, \theta) = T_a^{(L)}(\lambda, \theta) R_{aL+1}(\lambda - \mu_{L+1}, \hat{\theta}_{L+1}), \tag{A.4}
\]

with initial conditions

\[
A_1(\lambda, \theta) = \alpha_1(\lambda - \mu_1, \hat{\theta}_1), \quad B_1(\lambda, \theta) = \beta_1(\lambda - \mu_1, \hat{\theta}_1), \\
C_1(\lambda, \theta) = \gamma_1(\lambda - \mu_1, \hat{\theta}_1), \quad D_1(\lambda, \theta) = \delta_1(\lambda - \mu_1, \hat{\theta}_1). \tag{A.5}
\]

In particular, from (A.4) we can single out the relation

\[
B_{L+1}(\lambda, \theta) = A_L(\lambda, \theta) \beta_{L+1}(\lambda - \mu_{L+1}, \hat{\theta}_{L+1}) + B_L(\lambda, \theta) \delta_{L+1}(\lambda - \mu_{L+1}, \hat{\theta}_{L+1}), \tag{A.6}
\]

which allows us to obtain the behaviour of \( B(\lambda_j, \theta + j \gamma) \) from the analysis of \( \alpha_j, \beta_j, \gamma_j, \delta_j \) in the limit \( (\lambda_j, \theta) \to \infty \). Thus taking into account (2.3), (2.5) and (A.3), in the limit \( (\lambda, \theta) \to \infty \) we find

\[
\alpha_j \sim \frac{1}{2} \sum_{n_j = -\infty}^{+\infty} (-1)^{n_j} q_n \hat{a}_j e_n K_{n_j} \quad \beta_j \sim \frac{1}{2} \sum_{n_j = -\infty}^{+\infty} (-1)^{n_j} p_n q_n X^- \\
\gamma_j \sim \frac{1}{2} \sum_{n_j = -\infty}^{+\infty} (-1)^{n_j} q_n \hat{a}_j X^+ \quad \delta_j \sim \frac{1}{2} \sum_{n_j = -\infty}^{+\infty} (-1)^{n_j} q_n p_n e_n K_{n_j}^{-1}, \tag{A.7}
\]

where we have introduced the conventions \( e_n = e^{-(2n+1)}, \ p_n = p^{(n+\frac{1}{2})} \) and \( q_n = q_n^\gamma \). In their turn the operators \( K_n \) and \( X^\pm \) appearing in (A.7) are given by

\[
K_n = \begin{pmatrix} a_n^{\frac{1}{2}} & 0 \\ 0 & a_n^{-\frac{1}{2}} \end{pmatrix}, \quad X^\pm = \frac{1}{2} \begin{pmatrix} 0 & 1 \pm 1 \\ 1 \mp 1 & 0 \end{pmatrix}. \tag{A.8}
\]
Now the relation (A.6) can be iterated with the help of (A.5) and (A.7). Thus in the limit \((\lambda_j, \theta) \to \infty\) we find the expression

\[
B(\lambda_j, \theta + j\gamma) \sim \frac{f(\gamma)}{2L-1} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_{L-1}=-\infty}^{\infty} (-1)^{\sum_{i=1}^{L-1} n_i} \prod_{i=1}^{L-1} p_{n_i} q_{n_i} e_{n_i}^{\lambda_j} \times 
\sum_{j=1}^{L} e^{-\mu_1} \cdots e^{-\mu_{j-1}} P_{j}^{\bar{n}} e^{-\mu_{j+1}} \cdots e^{-\mu_L},
\]

(A.9)

with \(\bar{n} = (n_1, \ldots, n_{L-1})\) and operators \(P_{j}^{\bar{n}}\) reading

\[
P_{j}^{\bar{n}} = K_{n_1} \otimes \cdots \otimes K_{n_{j-1}} \otimes X^- \otimes K_{n_{j+1}} \otimes \cdots \otimes K_{n_{L-1}}.
\]

(A.10)

The operators \(K_n\) and \(X^\pm\) satisfy the following analogous of the \(q\)-deformed \(su_2\) algebra

\[
K_n X^\pm K_m^{-1} = q_{\frac{n}{2}}^\pm X^\pm
\]

\[
[X^+, X^-] = \frac{K_n K_m - K_{n-1} K_{m-1}}{(q_{n_m} - q_{n_m})},
\]

(A.11)

which allows us to demonstrate the properties

\[
P_{i}^{\bar{n}(a)} P_{j}^{\bar{n}(b)} = q_{n_{j(a)}} q_{n_{j(b)}} P_{j}^{\bar{n}(b)} P_{i}^{\bar{n}(a)} \quad (i < j)
\]

\[
P_{i}^{\bar{n}(a)} P_{i}^{\bar{n}(b)} = 0.
\]

(A.12)

As we shall see, the relations (A.12) will be of utility for the analysis of the asymptotic behaviour of \(Z_\theta\).

Next, in order to analyse the behaviour of \(Z_\theta\) in the proposed limit, we substitute the expansion (A.9) into the definition (2.10) and use the relations (A.12) to reorganise the result properly. By doing so we obtain the expression

\[
Z_\theta(\lambda_1, \ldots, \lambda_L) \sim \frac{f(\gamma)^L}{2L(L-1)} \sum_{n_1^{(1)}=-\infty}^{\infty} \cdots \sum_{n_{L-1}^{(1)}=-\infty}^{\infty} \sum_{n_1^{(L)}=-\infty}^{\infty} \cdots \sum_{n_{L-1}^{(L)}=-\infty}^{\infty} (-1)^{\sum_{i=1}^{L} n_i} \prod_{a=1}^{L} \prod_{i=1}^{L-1} p_{n_i^{(a)}} q_{n_i^{(a)}}^{\lambda_a} \prod_{\sigma \in S_L} (q_{n_{\sigma(b)}} q_{n_{\sigma(b)}})^{1 - \langle 0 | \prod_{a=1}^{L} P_{a}^{\bar{n}(a)} | 0 \rangle},
\]

(A.13)

where \(\mu^{(a)} = \{\mu_i : i \neq a\}\). As usual \(S_L\) denotes the group of permutations of \(L\) objects while \(\sigma(a)\) stands for the permutation of the \(a\)-th object. In order to clarify the meaning of \(I_\sigma\) let us consider the usual two row representation of \(\sigma\). We draw a line starting at the object \(a\) in the top row and ending in the bottom row at the position \(\sigma(a)\) such that
only two lines intersect at any one point. The lines are labelled by their numbers in the top row and the points of intersection are called inversion vertices. In this way an inversion vertex can be labelled by a pair \((a,b)\) with \(a < b\) such that \(a\) and \(b\) label the two intersecting lines originating the inversion vertex. Then denoting \([L] = \{1,\ldots,L\}\), we call \(I_{\sigma} = \{(a,b) \in [L] \times [L] : a < b \text{ and } \sigma(a) > \sigma(b)\}\) the set of inversion vertices labels of a given permutation \(\sigma\).

The next step to obtain an explicit expression for \(\langle 0| \prod_{a=1}^{L} P_{a}^{(a)} |0\rangle\) is to compute the quantity \(\langle 0| \prod_{a=1}^{L} P_{a}^{(a)} |0\rangle\) which can be readily performed since the operators \(P_{a}^{(a)}\) consist of a simple tensor product \((A.10)\). Thus considering \((2.11)\) we obtain

\[
\langle 0| \prod_{a=1}^{L} P_{a}^{(a)} |0\rangle = \prod_{a=1}^{L} \prod_{i=1}^{L-1} q_{n_{i}^{(a)}}^2
\]

which can be substituted in \((A.13)\) yielding the formula

\[
Z_{\theta}(\lambda_{1}, \ldots, \lambda_{L}) \sim \frac{f(\gamma)^{L}}{2^{L(L-1)}} \sum_{n_{1}^{(1)} = -\infty}^{\infty} \cdots \sum_{n_{L}^{(1)} = -\infty}^{\infty} \cdots \sum_{n_{1}^{(L)} = -\infty}^{\infty} \cdots \sum_{n_{L}^{(L)} = -\infty}^{\infty} (-1)^{\sum_{n=1}^{L} \sum_{i=1}^{L-1} n_{i}^{(a)} - \frac{L(L-1)}{2} - \frac{L-1}{2}} \prod_{a=1}^{L} \prod_{i=1}^{L-1} p_{n_{i}^{(a)}} q_{n_{i}^{(a)}} e^{\lambda_{a} - \mu_{i}^{(a)}} \sum_{\sigma \in S_{L}} \prod_{(a,b) \in I_{\sigma}} (q_{n_{b}^{(a)}} q_{n_{a}^{(b)}})^{-1}
\]

in the limit \((\lambda_{j}, \theta) \to \infty\).

### B  Theta-function properties

In this appendix we recall some useful properties of elliptic theta-functions that we have considered through this paper. We remark here that many of these properties have also been discussed in [17]. The function \(f\) defined in Section 2 consists basically of the Jacobi theta-function \(\Theta_{1}\) [26] and in this paper we have omitted the dependence of \(f\) with the elliptic nome \(p\) for brevity. In what follows we summarise some properties of elliptic theta-functions adjusted to our conventions.

**Addition rule.** The function \(f\) satisfy the addition rule

\[
\begin{align*}
&f(\lambda_{1} + \lambda_{2}) f(\lambda_{1} - \lambda_{2}) f(\lambda_{3} + \lambda_{4}) f(\lambda_{3} - \lambda_{4}) = \\
&f(\lambda_{1} + \lambda_{4}) f(\lambda_{1} - \lambda_{4}) f(\lambda_{3} + \lambda_{2}) f(\lambda_{3} - \lambda_{2}) + f(\lambda_{1} + \lambda_{3}) f(\lambda_{1} - \lambda_{3}) f(\lambda_{2} + \lambda_{4}) f(\lambda_{2} - \lambda_{4})
\end{align*}
\]

\[(B.1)\]

**Analyticity and periodicity.** The function \(f\) is an entire function, that is to say all of its singularities are removable, and it has only simple zeroes. It is also an odd function and quasi doubly-periodic, i.e.

\[
\begin{align*}
f(\lambda - i\pi) &= -f(\lambda) & f(\lambda - i\pi \tau) &= -e^{2\lambda - i\pi \tau} f(\lambda)
\end{align*}
\]

\[(B.2)\]
**Trigonometric limit.** In the limit $p \to 0$ the theta-function $f(\lambda)$ degenerate into a trigonometric function. More precisely we have $\lim_{p \to 0} -ip^{-\frac{1}{2}}f(\lambda) = \sinh(\lambda)$, which allows for an easy comparison with previous results in the literature.

**Higher order theta-functions.** For a fixed value of the elliptic nome $\tau$, we call $\mathcal{F}$ a theta-function of order $L$ and norm $t$ if

$$\mathcal{F}(\lambda) = C \prod_{j=1}^{L} f(\lambda - \chi_j) \quad (B.3)$$

for constants $C$ and $\chi_j$ such that $\sum_{j=1}^{L} \chi_j = t$. Moreover, due to $(B.2)$ one can readily show the quasi-periodicity

$$\mathcal{F}(\lambda - i\pi \tau) = (-1)^L e^{2(L\lambda - t) - i\pi L} \mathcal{F}(\lambda) . \quad (B.4)$$

In fact, the factorised form $(B.3)$ and the quasi-periodicity $(B.4)$ for entire functions can be shown to be equivalent properties [32]. This feature allows us to state a more general result. Let $\bar{\mathcal{F}}$ be defined as

$$\bar{\mathcal{F}}(\lambda) = \sum_{i} \bar{C}_i \prod_{j=1}^{L} f(\lambda - \chi^{(i)}_j) , \quad (B.5)$$

with $\bar{C}_i$ being constants and $\sum_{j=1}^{L} \chi^{(i)}_j = t$ for any $i$. The function $\bar{\mathcal{F}}$ is entire and obeys the quasi-periodicity $(B.5)$, thus it can be factored similarly to $(B.3)$.

**$Z_\theta$ as a higher order theta-function.** The partition function $Z_\theta$ defined in $(2.10)$ is written as a product of operators $B(\lambda, \theta)$. As a matter of fact, the whole dependence of $Z_\theta$ with a particular variable $\lambda_j$ is contained in a single operator $B(\lambda_j, \theta + j\gamma)$ since the vectors $|0\rangle$ and $|\bar{0}\rangle$ are constants. Now taking into account $(2.5)$, $(A.3)$ and $(A.5)$, the recurrence relation $(A.6)$ tells us that the entries of $B(\lambda_j, \theta + j\gamma)$ are of the form $(B.3)$. This is because the factors $A_L(\lambda, \theta)\beta_{L+1}(\lambda - \mu_{L+1}, \hat{\theta}_{L+1})$ and $B_L(\lambda, \theta)\delta_{L+1}(\lambda - \mu_{L+1}, \theta_{L+1})$ in $(A.6)$ do not contribute simultaneously to the same entry due to the structure of $(A.3)$. Thus, due to the definition $(2.10)$, we have that the partition function $Z_\theta$ will be of the form $(B.5)$ with respect to a given variable $\lambda_j$. The latter characterises $Z_\theta$ as a higher order theta-function of order $L$ in each one of the variables $\lambda_j$. Although its explicit value shall not be required through this work, we shall use $t_j$ to denote the norm of $Z_\theta$ when factored with respect to the variable $\lambda_j$ as given by $(B.3)$.

**C** **$Z_\theta$ as a symmetric function**

The commutativity of operators $B(\lambda, \theta)$ as described by $(3.1)$, together with the definition $(2.10)$, implies that $Z_\theta$ is a symmetric function. This commutativity has been extensively
employed in the derivation of (3.5) and here we intend to show that this symmetry becomes an inherent property of the solutions of (3.5).

Through the inspection of the coefficients (3.6), we notice that $N_i \leftrightarrow N_j$ under the mapping $\lambda_i \leftrightarrow \lambda_j$ while $M_0 \rightarrow M_0$ and $N_k \rightarrow N_k$ for $k \neq i, j$. Thus performing this mapping on (3.5) and subtracting it from the original equation we obtain the following relation,

$$M_0[Z_{\theta-\gamma}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_L)] + N_0[Z_\theta(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_L) - Z_\theta(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_L)] + \sum_{k=1}^{L} N_k Z_\theta(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_j, \ldots) - \sum_{k=1}^{L} N_k Z_\theta(\lambda_0, \ldots, \lambda_j, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_i, \ldots) = 0.$$  

(C.1)

Next we solve (C.1) for the $l$-th term of the summation over the index $k$ which yields the expression

$$\frac{M_0}{N_l}[Z_{\theta-\gamma}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_L)] + \frac{N_0}{N_l}[Z_\theta(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_L) - Z_\theta(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_L)] + \sum_{k=1}^{L} \frac{N_k}{N_l} Z_\theta(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_j, \ldots) - \sum_{k=1}^{L} \frac{N_k}{N_l} Z_\theta(\lambda_0, \ldots, \lambda_j, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_i, \ldots) = Z_\theta(\lambda_0, \ldots, \lambda_j, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_i, \ldots) - Z_\theta(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_j, \ldots).$$  

(C.2)

The RHS of (C.2) does not depend on $\lambda_l$ so this variable can be chosen such that the LHS of (C.2) vanishes. Thus we can conclude that

$$Z_\theta(\lambda_0, \ldots, \lambda_j, \ldots, \lambda_{l-1}, \lambda_{l+1}, \ldots, \lambda_i, \ldots) = Z_\theta(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_{l-1}, \lambda_{l+1}, \ldots, \lambda_j, \ldots),$$  

(C.3)

and since this is valid for any $i, j$ and $l$, the symmetry property

$$Z_\theta(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) = Z_\theta(\ldots, \lambda_j, \ldots, \lambda_i, \ldots)$$  

(C.4)

immediately follows.

D Solution for $L = 1$

The functional equation (3.5) for $L = 1$ explicitly reads

$$M_0 Z_{\theta-\gamma}(\lambda_1) + N_0 Z_\theta(\lambda_1) + N_1 Z_\theta(\lambda_0) = 0.$$  

(D.1)
with coefficients
\[
M_0 = \frac{f(\theta)}{f(\theta + \gamma)} f(\lambda_0 - \mu_1)
\]
\[
N_0 = -\frac{f(\theta + \gamma)}{f(\theta + 2\gamma)} f(\lambda_0 - \mu_1 + \gamma) \frac{f(\lambda_1 - \lambda_0 + \gamma)}{f(\lambda_1 - \lambda_0)}
\]
\[
N_1 = \frac{f(\theta + \gamma + \lambda_0 - \lambda_1)}{f(\theta + 2\gamma)} \frac{f(\gamma)}{f(\lambda_1 - \lambda_0)} f(\lambda_1 - \mu_1 + \gamma).
\]
(D.2)

By setting \(\lambda_0 = \lambda_1 - \theta - \gamma\), the coefficient \(N_1\) vanishes and (D.1) simplifies to
\[
Z_{\theta}(\lambda_1) \frac{f(\theta + \gamma)}{f(\theta + \gamma + \mu_1 - \lambda_1)} = Z_{\theta-\gamma}(\lambda_1) \frac{f(\theta)}{f(\theta + \mu_1 - \lambda_1)}.
\]
(D.3)

The relation (D.3) is an equation only over the variable \(\theta\) which is readily solved by
\[
Z_{\theta}(\lambda_1) = \frac{f(\theta + \gamma - \lambda_1 + \mu_1)}{f(\theta + \gamma)} F(\lambda_1)
\]
(D.4)

where \(F\) is \(\theta\) independent. After eliminating the dependence with \(\theta\), we can substitute (D.4) back into (D.1). The resulting equation can then be simplified and we obtain the relation
\[
\frac{f(\gamma) f(\theta + \gamma + \mu_1 - \lambda_0) f(\theta + \gamma + \lambda_0 - \lambda_1) f(\lambda_1 - \mu_1 + \gamma)}{f(\theta + \gamma) f(\theta + 2\gamma) f(\lambda_1 - \lambda_0)} (F(\lambda_1) - F(\lambda_0)) = 0.
\]
(D.5)

From (D.5) we can conclude that \(F\) is a constant and it can be fixed by the asymptotic behaviour (A.15). Thus we find for \(L = 1\),
\[
Z(\lambda) = f(\gamma) \frac{f(\theta + \gamma - \lambda + \mu_1)}{f(\theta + \gamma)}.
\]
(D.6)

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