The emergence of charged collective modes from a large $N$ extrapolation of the Hubbard model

D. Foerster

CPMOH, UMR 5798, Université de Bordeaux I
351, cours de la Libération, F - 33405 Talence Cedex

A.A. Ovchinnikov

Max-Planck-Institut für Physik komplexer Systeme
Nöthnitzer Str. 38, D-01187 Dresden, Germany
and
Joint Institute for Chemical Physics of the Russian Academy of Sciences,
117977, Moscow, Russia

We consider a symplectic extrapolation of the Hubbard model of $N$ fold replicated electrons and solve this model exactly in two special cases, at $N = \infty$ in the bosonic sector and for any $N$ on a dimer of two points. At $N = \infty$ we find a multiplet of collective modes that contains neutral spin fluctuations and charged pair fluctuations that are degenerate with each other at zero doping. Our solution of the symplectic model on a dimer of two points for any $N$ interpolates smoothly between $N = 1$ and $N = \infty$ without any visible discontinuity. These results suggest that the inclusion of charged pairing modes in weakly doped antiferromagnets is essential and that an expansion about the $N = \infty$ limit is appropriate in this context.

I. INTRODUCTION AND MOTIVATION

Calculational schemes that are perturbative in the Coulomb interaction are known to be inadequate for describing strongly interacting electrons in systems such as, for example, transition metal oxides, organic conductors and quantum dots and this necessitates the use of non-perturbative methods, such as the renormalization group, dynamical mean field theory and various numerical methods. The $1/N$ expansion for $N$ fold replicated degrees of freedom (with subsequent extrapolation to the physical value of $N = 2$ or $N = 3$) was successfully applied to magnetic impurities and in other areas of physics. In almost all applications to electronic systems, this expansion uses semiclassical saddle point integration and its success in the case of the repulsive Hubbard model is open to question because it is unclear where a suitable saddle point exists. An alternative version of the $1/N$ expansion was discovered by ’t Hooft in the context of $SU(N)$ Quantum Chromodynamics and relies instead on dominance of diagrams of simple topology for $N \to \infty$. Recently it was noticed that ’t Hooft’s topological classification of diagrams can be applied to the Hubbard model, with collective excitations playing the role that gauge bosons play in Quantum Chromodynamics. This raises the challenge of reconstructing the large $N$ extrapolation for electron systems from the very beginning in a way that avoids saddle point integration.

The symplectic extrapolation of the Hubbard model is interesting because, unlike the $SU(N)$ extension, it conserves the notion of two component spins and permits pairing. In the present paper, we solve this model exactly in two special cases (i) in the bosonic sector on an infinite lattice at $N = \infty$ and (ii) on a dimer of two points for any $N$. Our solution at $N = \infty$ shows the existence not only of neutral spin fluctuations but also of charged pair fluctuations that are in the same multiplet and exactly degenerate with the spin fluctuations at zero doping. Our solution of the dimer for any $N$ indicates a smooth interpolation between $N = 1$ and $N = \infty$ without any apparent singularity. Taken together these results indicate that (i) both spin and pair fluctuations must be included from the very beginning in any theory of weakly doped antiferromagnets and (ii) that the expansion in powers of $1/N$ about an appropriate $N = \infty$ limit is a legitimate procedure in this context.

The symplectic extrapolation of Hubbard’s model that we consider here is based on $U(N,q)$, the group of unitary transformations on $N$ quaternions and it leads to the following Hamiltonian:

$$H_{U(N,q)} = \frac{U}{N} \sum_x \left[ \left( \psi_x \bar{\psi}_x \right)^+ \left( \psi_x \bar{\psi}_x \right) \right]_{\text{symm}} + \left( \bar{\psi}_x \psi_x \right)^2 + \sum_{x,y} t_{xy} \bar{\psi}_x \psi_y$$

where

$$\psi_x \bar{\psi}_x = \sum_{i=1\ldots N, \alpha=1,2} \psi_{x\alpha i} (i \sigma_y)_{\alpha\beta} \psi_{x\beta i}, \quad \bar{\psi}_x \psi_x = \sum_{i=1\ldots N, \alpha=1,2} \bar{\psi}_{x\alpha i} \psi_{x\alpha i}$$

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II. EXACT RESULTS FOR LIMITING CASES OF THE SU(N) HUBBARD MODEL

A. The SU(N) Hubbard model for a dimer

For simplicity, we begin with an SU(N) extrapolation \[ \mathcal{H}_{SU(N)} \] of the Hubbard model \[ \mathcal{H}_{\text{Hubb}} \]:

\[
\mathcal{H}_{SU(N)} = \frac{U}{2N} \sum_x n_x^2 + \sum_{x,y} t_{xy} \psi_x^+ \psi_y
\]

which we simplify even further by specializing it to a dimer of only two points:

\[
\mathcal{H}_{SU(N)}^{\text{dimer}} = \frac{U}{2N} \left( n_1^2 + n_2^2 \right) + \frac{1}{2} \sum_{i=1}^{N} \left( \psi_{1i}^+ \psi_{2i} + \psi_{2i}^+ \psi_{1i} \right)
\]

Here \( n_a = \sum_{i=1}^{N} \psi_{ai}^+ \psi_{ai} \), \( a = 1, 2 \) and \( i = 1..N \) enumerates the electron species and we have chosen hopping amplitudes with a (naive) bandwidth of 1. This Hamiltonian describes electrons on a ladder of length \( N \) with hopping along the rungs \( i \) or tunnelling between two mesoscopic quantum dots on a semiconducting substrate, each doped with \( N \) electron donors. The position \( \alpha = 1, 2 \) on the dimer can be viewed as a pseudo spinor index and \( R_{SU(N)}^{\text{dimer}} \) can be expressed entirely in terms of pseudo spin operators as follows:

\[
\mathcal{H}_{SU(N)}^{\text{dimer}} = \frac{U}{N} T_z^2 + T_x + \text{const}
\]

\[
T^z = \frac{1}{2} \sum_{i=1}^{N} \sum_{\alpha=1,2} \psi_{ai}^+ \tau_{\alpha\beta} \psi_{\beta i}
\]

where the constant has physical meaning because it depends on electron filling. Because the Hamiltonian decomposes into blocks of distinct pseudo spins we may diagonalize it separately in each block and determine the optimal value of the pseudo spin at the end (it is impossible to diagonalize the SU(N) dimer for large \( N \) by brute force because the dimension of its Hilbert space explodes like \( 4^N \)). It is useful to diagonalize the hopping term by interchanging the conventional representations of \( T_z \) and \( T_x \):

\[
\mathcal{H}_{SU(N)}^{\text{dimer}} \rightarrow \frac{U}{N} T_x^2 + T_z + \text{const}
\]

Because \( \frac{U}{N} \rightarrow 0 \) as \( N \rightarrow \infty \) a naive first approximation to the ground state is \( |s, s_z = -s > \). To understand the action of the Hamiltonian on states in the vicinity of \( |s, -s > \), we express it in terms of Holstein-Primakoff oscillators:

\[
T_+ = a^+ \sqrt{2s - a^+ a}, \quad T_- = \sqrt{2s - a^+ a} a, \quad T_z = -2s + a^+ a
\]

\[
\mathcal{H}_{SU(N)}^{\text{dimer}} = \frac{1}{2} \left( a^+ a \right) \left( \frac{\mu_s}{N} + 1 \right) \left( \frac{\mu_s}{N} + 1 \right) \left( \frac{a}{a} \right) + \text{const} + O \left( \frac{1}{N} \right)
\]

Here \{sym\} denotes symmetrization and the electrons \( \psi_{x\alpha} \) occur in \( N \) copies with \( i = 1..N \) but we will often suppress this extra index. For our purposes invariance of \( \mathcal{H}_{SU(N)} \) under SU(N) transformations is sufficient and \( \psi_x \psi_x^+ \) in eq(1) are indeed invariant under SU(N) because \( \psi_{xi} \) and \( \psi_{xi}^+ \) transform, by fiat, according to mutually conjugate SU(N) transformations. If we imagine, for the moment, that the interactions in eq(1) come about via exchange of suitable bosons, then both neutral and charged bosons must be exchanged. It will be shown below that this naive expectation is correct and that the symplectic extrapolation of the Hubbard model contains indeed both neutral and charged collective excitations.

This paper is organized as follows. In section 2 we solve the SU(N) Hubbard model first for two points and any \( N \) using SU(2) pseudo spins and we also find the SU(\( \infty \)) Hamiltonian in the bosonic sector for an infinite lattice. In section 3 we determine the \( N \rightarrow \infty \) asymptotics of the symplectic model after mapping it onto a model of the double exchange type and we also solve the symplectic model on a dimer for any \( N \) using the group SP(4) \( \sim SO(5) \) as a classifying symmetry. In section 4 we identify the spectrum of the asymptotic SU(\( \infty \)) and \( U(\infty, q) \) Hamiltonians. Section 5 serves to counterbalance our extensive use of ancient boson operator methods by reformulating our problem in standard path integral language. Our conclusions are given in section 6.
and then diagonalize it with the help of a Bogoljubov transformation:

\[ H = \sqrt{1 + \frac{2Us}{N}} b^+ b + \text{const} + O \left( \frac{1}{N} \right) \]  

(7)

\[ b = a \cosh \theta + a^+ \sinh \theta, \quad \tanh 2\theta = \frac{Us}{1 + \frac{Us}{N}} \]

\[ |G> = \exp(-\frac{\tanh \theta}{2} a^+ a^>)|0> \]

The use of the Holstein-Primakoff expansion for \( N \to \infty \) is justified, a posteriori, because the number of \( a^+ a^+ \) pairs converges in the ground state. The optimal value of the total spin \( s \) is the maximal one that is permitted at a given filling e.g. \( s = N/2 \) at half filling \[^{13}\].

By numerically diagonalizing the SU(\( N \)) dimer for finite \( N \) the convergence to its \( N = \infty \) limit can be checked. We consider the gap \( \Delta_N(U) \) between the ground state and the first excited state and which, according to eq(7) should converge (at half filling) to \( \sqrt{1 + U} \) and our calculation confirms this convergence. Figure 1 displays the gap \( \Delta_N(U) \) for \( N = 2 \ldots 5 \) and shows that the convergence with \( N \) is non uniform in \( U \). This reflects the failure of the limits \( N \to \infty \) and \( U \to \infty \) to commute that follows from eq(2). We therefore expect the extrapolation to the physical value \( N = 2 \) to run into difficulties if the Coulomb interaction is too large compared with the bandwidth.

\[ \text{FIG. 1. Oscillating convergence in } N \text{ of the gaps } \Delta_N(U) \text{ of the SU(N) dimer for } N = 2 \ldots 5 \text{ towards } \Delta(\infty)(U). \text{ The highest curve is for } N = 2, \text{ the lowest for } N = 3 \text{ and the asymptotic one is dotted.} \]

B. Hamiltonian of the SU(\( \infty \)) Hubbard model on any lattice

We now generalize the operator representation of the SU(\( N \)) dimer to an arbitrary lattice. Because we want to use the notion of a Fermi surface, we transform the SU(\( N \)) Hamiltonian of eq(2) to plane waves:

\[ H_{SU(N)} = \frac{U}{2Nvol} \sum_{p-q=r-s} B^*_p B^*_r + \sum e_p B^*_p \]  

(8)

with \( \psi_x = \frac{1}{\sqrt{vol}} \sum e^{ipx} \psi_p, \quad B_{pq} \equiv \psi^+_p \psi_q \quad \text{and} \quad e_p \equiv \sum t_{x0} e^{ipx}. \) To find the \( 1/N \) expansion of the \( B_{pq} \) operators we examine the commutators of their fluctuating components:

\[ [B_{pq}, B_{rs}] = \delta_{qr} B_{ps} - \delta_{ps} B_{qr} \]

\[ = \delta_{qr} (B_{qs} - N\delta_{qs}n_p) - \delta_{ps} (B_{qr} - N\delta_{qr}n_q) + N\delta_{ps}\delta_{qr} (n_p - n_q) \]

\[ n_p = <\psi^+_p \psi_p> = \text{Fermi function} \]
We change notation and use subtracted bilinears $B_{pq} \equiv \psi_p^\dagger \psi_q - N \delta_{pq} n_p$ from now on. The preceding equation shows that the operators $B_{pq}$ for $p$, $q$ on opposite sides of the Fermi surface behave like harmonic oscillators:

$$B_{\mathbf{p},q} \sim \sqrt{N} b_{\mathbf{p},q}^\dagger, \quad B_{p,\mathbf{q}} \sim \sqrt{N} b_{p,\mathbf{q}}$$

with $[b_{\mathbf{p},q}, b_{p,\mathbf{q}}^\dagger] = \delta_{\mathbf{p} \mathbf{q}, \delta_{q,s}}$

with, so far, unknown corrections. Above we denoted a level by $p$ when it is empty and by $\mathbf{p}$ when it is full. Fortunately for us, the expansion, to all orders, of bilinears of fermions in terms of bosons was previously worked out in nuclear physics, see [1] for a review. Although we will not need this representation in its most general form we quote it from ref. [17], according to which the leading density operators are represented by

$$B_{\mathbf{p},q} = \sum_r \left( \sqrt{N - A} \right)_{q,r} b_{\mathbf{p},r}$$

$$[b_{\mathbf{p},p}, b_{p,\mathbf{q}}^\dagger] = \delta_{p,q} \delta_{\mathbf{p},\mathbf{q}}$$

$$A_{pq} = b_{p,\mathbf{q}}^\dagger b_{\mathbf{p},p}$$

and where the square root $\sqrt{N - A}$ is to be interpreted as a series in $1/N$. By contrast, the non leading density operators, with momenta on the same side of the Fermi surface, are simple quadratic forms in the harmonic oscillators:

$$B_{pq} = \sum_r b_{p,r}^\dagger b_{r,q} = A_{pq}$$

$$B_{p,\mathbf{q}} = N - \sum_r b_{q,r}^\dagger b_{\mathbf{p},r}$$

The representation of the algebra [3] via eqs(11,12) constitutes a generalization of the Holstein-Primakoff representation of eq(8), but the proof that it represents the fermion bilinears correctly is much more difficult for this generalization than for the Holstein-Primakoff representation [10]. Here we only use the lowest order terms of this expansion and we retain as dynamical variables the bilinears $B_{\mathbf{p},q}$ and $B_{p,\mathbf{q}}$ with $p$, $q$ on opposite sides of the Fermi surface where they have, from eq(8), harmonic oscillator like commutators

$$n_p + n_q = 1 : [B_{pq}, B_{rs}] = N \delta_{ps} \delta_{qs} (n_p - n_q) + O(1)$$

The non leading $B_{pq}$ for $p$, $q$ on the same side of the Fermi surface are expressed, using eqs(12,11) in terms of the leading operators as follows

$$n_p = n_q :$$

$$B_{pq} = \frac{1}{N} \sum_r (n_r - n_p) B_{pr} B_{rq} + O \left( \frac{1}{N} \right) + \text{const}$$

(our notation no longer distinguishes between filled and empty levels). We may now use eq(14) to express the kinetic energy in eq(8) in terms of leading density operators

$$\sum_p e_p B_{pp} = \frac{1}{2N} \sum_{n(p)+n(q)=1} |e_p - e_q| \{B_{pq} B_{qp}\}_{\text{symm}} + \text{const} + O \left( \frac{1}{N} \right)$$

where we have antisymmetrized $e_p$ with respect to the replacement $p \leftrightarrow q$. Using eq(13) the Hamiltonian in eq(8) reads as follows:

$$H_{SU(N)} = \frac{U}{2N \cdot \text{vol}} \sum_{p-q=-s} B_{pq}^\dagger B_{rs} + \frac{1}{2N} \sum_{n(p)+n(q)=1} |e_p - e_q| \{B_{pq} B_{qp}\}_{\text{symm}} + \text{const} + O \left( \frac{1}{N} \right)$$

As we shall see later in more detail, this describes RPA like particle hole scattering, provided we restrict the summation in the interaction term to $n(p) + n(q) = 1$ and $n(r) + n(s) = 1$ so that $B_{pq}^\dagger B_{rs} \sim N$. The operators $B_{pq}^\dagger B_{rs}$ in eq(16) contain, however, also non leading contributions with either $p, q$ or $r, s$ on the same side of the Fermi surface, or both. By using eq(14) to express these non leading contributions in terms of the leading operators, we find, to next order in $1/\sqrt{N}$, the following three point interaction:
The interaction $H^{(3)}$ is $\sim N^{-1/2}$ because it contains three operators $\sim \sqrt{N}$ each with arguments on mutually opposite sides of the Fermi surface and it gives rise to a correction $\sim \frac{1}{N^3}$ of the energy. In this paper we will not make use of such non leading interactions.

Finally we check whether the RPA like expression of eq(16) reduces to our previous result for the dimer. We choose a naive bandwidth of one so that $|e\pi - e\sigma| = 1$ and take into account $vol = 2$ to obtain:

$$H^{'}_{SU(N)} = \frac{U}{4} (b_{0\pi} + b_{0\sigma})^2 + b_{0\pi}^* b_{0\sigma} + const + O(\frac{1}{N})$$

It is reassuring that this agrees with our previous result (1) that was obtained directly via the Holstein Primakoff representation of pseudo spin operators.

### III. EXACT RESULTS FOR THE SYMPLECTIC HUBBARD MODEL

#### A. Map of the symplectic model onto a double exchange model

The symplectic model of eq(1) is fairly complicated and to make progress we first map it onto a spin model. A hint on the mapping we need is provided by the fact that the operators $\{\psi_x \bar{\psi}_x, (\psi_x \bar{\psi}_x)^+, \psi_x^+ \psi_x\}$ form an SU(2) algebra under commutation. This fact becomes obvious after a particle hole transformation $\psi_{x1} \rightarrow \psi_{x1}, \psi_{x2} \rightarrow \psi_{x2}^+$ on the down spins:

$$\begin{pmatrix} \frac{1}{2} \psi_x^\dagger \bar{\psi}_x^\dagger \\ -\frac{1}{2} \psi_x \bar{\psi}_x \\ \frac{1}{2} \psi_x^\dagger \psi_x \end{pmatrix} \rightarrow \begin{pmatrix} \psi_{x1}^\dagger \psi_{x2} \\ \psi_{x2}^\dagger \psi_{x1} \\ \frac{1}{2} (\psi_{x1}^\dagger \psi_{x1}^\dagger - \psi_{x2}^\dagger \psi_{x2}^\dagger) \end{pmatrix} = \begin{pmatrix} S_x^+ \\ S_x^- \\ S_z \end{pmatrix}$$

We see that this particle hole transformation (17) replaces the difficult charge $\pm 2$ pairing operators by $S_x^\pm$ while density fluctuations get mapped onto $S_z$. Applying this map on the symplectic Hamiltonian of eq(1) we find:

$$H_{U(N,q)} \rightarrow H_{DE} = \frac{4U}{N} \sum_x S_x^+ \cdot S_x^- + \sum_{x,y} t_{xy} \psi_x^\dagger \sigma_3 \psi_y$$

While in the symplectic model of eq(1) up and down electrons transform according to mutually conjugate representations under SU($N$) they transform according to the same SU($N$) representation in the model of eq(20) and therefore all four bilinears $\psi_x^\dagger \psi_\beta^\dagger$ become legitimate SU($N$) invariant operators. The Hamiltonian $H_{DE}$ is of the double exchange type, with only a single band and a Hund coupling of the "wrong sign" that favors antiferromagnetic order and we will refer to it as "double exchange model" for short. The chemical potential $\psi_x^\dagger \sigma_3 \psi_x$ of the symplectic model transforms into $\psi_x^\dagger \sigma_3 \psi_x$ in the double exchange model where it acts like a magnetic field that reduces the symmetry to spin rotations about the z axis (18).

#### B. Asymptotic Hamiltonian of the double exchange model

We first rewrite the double exchange model (20) in terms of plane waves as

$$H_{DE} = \frac{U}{N \text{vol}} \sum_{p+q=r+s} \bar{\sigma}_{pq} \cdot \sigma_{rs} + \sum_p \epsilon_p \rho_{pp}$$

where $\sigma_{\alpha\beta}$ are the Pauli matrices. The fermion bilinears $B_{p\alpha q\beta} = \psi_{p\alpha}^\dagger \psi_{q\beta}$ of the double exchange model now turn into RPA like collective modes in the same way as the spinless $\psi_x^\dagger \psi_x$ pairs did in the SU($N$) model of the last section, except for replacing $p \rightarrow (p, \alpha), q \rightarrow (q, \beta)$ and $n_p \rightarrow n_{p\alpha}$ in eqs(13,14):
oscillators as follows:

\[ n_{p\alpha} + n_{q\beta} = 1 : [\psi^+_{p\alpha}, \psi_{q\beta}, \psi^+_{r\gamma}, \psi_{s\delta}] = N\delta_{pr}\delta_{qs}\delta_{\alpha\delta}\delta_{\beta\gamma} (n_{p\alpha} - n_{q\beta}) + O(1) \]  \hspace{1cm} (22)

\[ n_{p\alpha} = n_{q\beta} : B_{p\alpha,q\beta} = \frac{1}{N} \sum_{r\gamma} (n_{r\gamma} - n_{p\alpha}) B_{p\alpha,r\gamma} B_{r\gamma,q\beta} + O\left(\frac{1}{N}\right) + const \]  \hspace{1cm} (23)

The Fermi factors in the double exchange model depend on the spin direction for non zero doping. This will break its SU(2) \(* U(1)\) symmetry and split the multiplet of modes. To simplify our analysis, we therefore limit ourselves here to zero doping for which the Fermi surfaces in the double exchange model loose their spin dependence. We may expand the collective modes \(\psi^+_{p\alpha}\psi_{q\beta}\) of eq\((22)\) over the Pauli matrices \(\sigma\) and decompose them into scalar \(\rho_{pq}\) and vector \(\sigma_{pq}\) operators according to

\[ \psi^+_{p\alpha}\psi_{q\beta} = \frac{1}{2} \sum_{\mu=0,3} \sigma^\mu_{pq}, \sigma^\mu_{st}, \mu = (1, \sigma) \]  \hspace{1cm} (24)

Using eqs\((22,24)\) the modes \(\rho_{pq}, \sigma_{pq}\) can be shown to be independent degrees of freedom, to leading order in \(N\):

\[ \left[\sigma^\mu_{p,q}, \sigma^\nu_{r,s}\right] = 2N\delta_{\mu\nu}\delta_{ps}\delta_{qr} (n_p - n_q) + O(1) \]  \hspace{1cm} (25)

Use of this representation on eq\((22)\) for \(H_{DE}\) gives us the RPA dynamics of \(\rho_{p,q}\) and \(\sigma_{pq}\) in the undoped DE model:

\[ H_{DE} = \frac{U}{N + \text{vol}} \sum_{p+s=q+t} \sigma_{pq} \cdot \sigma_{st} + \frac{1}{4N} \sum_{p,q} (e_p - e_q) (n_q - n_p) [\rho_{pq}\rho_{qp} + \sigma_{pq} \cdot \sigma_{qp}] \]  \hspace{1cm} (26)

where we antisymmetrized \(e_p\) with respect to interchange \(p \leftrightarrow q\). \(H_{DE}\) contains also nonlinear interactions in \(\sigma_{pq}\cdot\sigma_{st}\) with either \(p, q\) or \(r, s\) on the same side of the Fermi surface, but we will not write them down here. From our previous discussion we know that the operators \(S^\pm, S^3\) correspond, respectively, to chargeland neutral modes and, similarly, \(\sigma_{pq}\) decomposes into charged \((\sigma^\pm_{pq})\) and neutral \((\sigma^3_{pq})\) bosons. The basic reason for the appearance of charged collective modes in the symplectic model of eq\((1)\) is the interaction of pairs in its Hamiltonian.

C. Solution, for arbitrary \(N\), of the symplectic model on a dimer of two points.

We consider the dimer both in its symplectic and its double exchange version by specializing eqs\((1,20)\) to two points:

\[ H_{\text{SY}}^{\text{dimer}} = \frac{U}{N} \sum_{x=1,2} \left\{ (\psi_x e_\psi_x)^+ (\psi_x e_\psi_x) \right\}_{\text{symm}} + (\psi^+_2 \psi_2)^2 \right] + \frac{1}{2} (\psi^+_1 \psi_2 + \psi^+_2 \psi_1) \]  \hspace{1cm} (27)

\[ H_{\text{DE}}^{\text{dimer}} = \frac{4U}{N} \left( \overline{\sigma} \cdot \overline{\sigma} + \overline{S} \cdot \overline{S} \right) + \frac{1}{2} (\psi^+_1 \psi_2 + \psi^+_2 \psi_1) \]  \hspace{1cm} (28)

For a dimer of two points, the Hamiltonian of eq\((26)\) reduces to

\[ H_{DE}^{\text{dimer}} = \frac{U}{2N} (\overline{\sigma}_{0\pi} + \overline{\sigma}_{\pi 0})^2 + \frac{1}{2N} [e^- e_\pi - c_0] [\rho_{0\pi}\rho_{\pi 0} + \overline{\sigma}_{0\pi}\overline{\sigma}_{\pi 0}] + \text{const} + O\left(\frac{1}{N}\right) \]  \hspace{1cm} (28)

According to eq\((23)\) the operators \(\sigma^\mu_{0\pi} = \{\rho_{0\pi}, \overline{\sigma}_{\pi 0}\}\) may be represented as \(\sigma^\mu_{0\pi} = \sqrt{2N} h^\mu\) to leading order and therefore the symplectic dimer with naive bandwidth \(|e^- e_\pi - c_0| = 1\) at \(N = \infty\) is represented in terms of harmonic oscillators as follows:

\[ \lim_{N \to \infty} H_{DE}^{\text{dimer}} = \frac{U}{2} \left( \overline{b}^0 + \overline{b}^+ \right)^2 + (\overline{b}^+ \overline{b}) + \text{const} + O\left(\frac{1}{N}\right) \]  \hspace{1cm} (29)

By referring to eqs\((4, 4)\) for the SU(\(N\)) dimer we can read off the frequencies of the oscillators after Bogoliubov transformation and we find \(\omega_0 = 1, \omega_{1,3} = \sqrt{1 + 2U}\).

Straightforward numerical diagonalization of the dimer is impossible because its number of states grows as \(16^N\) with \(N\). However, according to exact diagonalization for \(N = 1, 2, 3\), the low lying states of the dimer are included.
in a much smaller subspace of $SU(N)$ invariant states. To further simplify the diagonalization within the $SU(N)$ invariant subspace, it is useful to recognize that the building blocks of the symplectic dimer $\psi_a \varepsilon \psi_b$, $\psi_a^+ \psi_b$ for $a, b = 1, 2$ in eq (27) generate the algebra of $SP(4) \sim SO(5)$. This can be seen by choosing two commuting operators from this set such as $\psi_1^+ \psi_1$ and $\psi_2^+ \psi_2$ and by determining the Cartan weights under commutation with the remaining operators and by plotting the two dimensional weight diagram. Opening a textbook on group theory and comparing with the Cartan weight diagrams of the classical groups of low rank [19] one concludes that the algebra is $SP(4) \sim SO(5)$.

The irreducible representations of $SO(5)$ have been classified with respect to their $SU(2) \times SU(2)$ content and we only need the special irreducible representation that contains the vacuum state of the double exchange model and which is characterized by $S_1 = S_2 = k^2$, $k = 0...N$. The kinetic term in this representation can then be determined from the reduced matrix elements given in [20].

![FIG. 2. Monotonic descent and convergence of the gaps $\Delta_N(U)$ of the symplectic dimer for $N=1..5$ towards $\Delta_\infty(U)$. The highest curve is for $N = 1$, the lowest curve represents $N = \infty$.](image)

From numerical diagonalization of the resulting representation of the Hamiltonian that we consider at zero doping we find the first excited state of the dimer to be a degenerate vector triplet. In order for $U$ to have its conventional meaning at $N = 1$ we replaced $U \rightarrow U^6$ in all formulas and plotted the energy $\Delta_N(U)$ of the vector multiplet relative to the ground state in figure 2 for $N = 1..5$ together with its asymptotic limit $\Delta_\infty(U) = \sqrt{1 + U/3}$. Comparing figures 1 and 2 we conclude that convergence of the gaps towards their asymptotic limit is slower for the symplectic extrapolation than it was for $SU(N)$.

**IV. THE SPECTRUM OF THE $SU(\infty)$ AND $U(\infty, Q)$ HAMILTONIANS**

Although eqs (14,26) give the Hamiltonians of the $SU(\infty)$ and $U(\infty, Q)$ models in the sector of even Fermion number in terms of elementary harmonic oscillators the spectra of these Hamiltonians are not obvious at first sight. This is due to a serious overcounting of degrees of freedom, with roughly one boson for any pair of momenta or any pair of points. These essentially non local bosons can describe local physics only by virtue of the interaction $\sim U$ being local.

We wish to determine the spectra of the asymptotic Hamiltonians via the poles of some correlation functions and begin with the correlator $< B_{pq} B_{pq}^+ >$ of the $SU(\infty)$ theory for simplicity. As a further simplification, we adopt the more flexible functional integral representation of the partition function which is fairly obvious here because we are dealing with a collection of independent oscillators:

$$B_{pq} = \sqrt{N} b_{pq}$$

$$n_p + n_q = 1 : [b_{pq}, b_{rs}^+] = \delta_{pr} \delta_{qs} (n_p - n_q) + O(1/N)$$

(we have rescaled the oscillators to have conventional commutation relations). The Lagrangian we need is just a slight generalization of the conventional one.
\[ L = \frac{1}{2} \sum_{p,q} (n_p - n_q) b_{pq}^\dagger \partial_t b_{pq} - h_{SU(N)}(b, b^+) \]  

(31)

\[ Z = \int DBe^{-S} \text{ with } S = \int_0^\beta dt L = b^+ M b \]

Correlators (in imaginary time) of harmonic variables such as \( b \) and \( b^+ \) are well known to be given by the inverse of the quadratic form in the imaginary time action:

\[ < b_{pq}(t)b_{pq}^+(0) > = < p_1 q_1; t | \frac{1}{M} | p_2 q_2; 0 > \]

(32)

At this point we invoke the locality of the interaction \( \sim U \) which, in \( x \) space and in terms of the rescaled variables \( b \), reads:

\[ \int_0^\beta dt h_{\text{int}} = U \int_0^\beta dt \sum_x b_{xx}^+ b_{xx} = U \int_0^\beta dt \sum_x b_{xy}^+ P_{xy,xx'} b_{xx'} \]

(33)

with \( P_{xy,xx'}b_{xx'} = \delta_{xy}b_{xx} \)

The operator \( P \) picks out the coincident piece of the wave function and satisfies the projector relation \( P^2 = P \). We decompose the full action into its "free" and "interacting" part by writing \( b^+ M b = b^+ L b - Ub^+ Pb \) with

\[ b^+ L b = \int dt \left[ \frac{1}{2} \sum_{p,q} (n_p - n_q) b_{pq}^\dagger \partial_t b_{pq} - \frac{1}{2} \sum_{n(p) + n(q) = 1} (n_p - n_q) (\epsilon_p - \epsilon_q) \left\{ b_{pq}^\dagger b_{pq} \right\}_{\text{symm}} \right] \]

(34)

where \( L \) describes free propagation of particle hole pairs with \( U = 0 \). According to general principles, the free propagator \( < b(t)b^+(0) > \) is just the inverse of \( L \):

\[ < b_{pq}(t)b_{pq}^+(0) >_{U=0} = \delta_{pr} \delta_{qs} < p q; t | \frac{1}{L} | r s; 0 > = \delta_{pr} \delta_{qs} \frac{n_p - n_q}{\partial_t - (\epsilon_p - \epsilon_q)} \]

(35)

The last expression resembles the particle hole propagator that enters into the calculation of bubbles and Lindhard’s function and this indicates that we are on the right track. Next we switch on \( U \) and exploit \( P^2 = P \), the locality of the interaction, in a way reminiscent of the treatment of a Friedel resonance:

\[ < b(t)b^+(0) >_{U \neq 0} = \frac{1}{L} \frac{1}{1 - UP} \frac{1}{P} \frac{1}{L} \]

(36)

\[ P \frac{1}{1 - UP} \frac{1}{P} \]

represents propagation and rescattering of particle hole pairs or a sum of bubbles. The preceding argument shows that the spectrum of the \( SU(\infty) \) Hamiltonian corresponds to zeroes of the operator

\[ \left( 1 - UP \frac{1}{L} \right)_{tx,ty} = \delta_{xy} \delta(t) - U < x t | \frac{1}{L} | y 0 > \]

(37)

The locality of the interaction \( \sim U \) leads to a propagator of coincident pairs and to functions of a single label. Fourier transforming we find

\[ < x t | \frac{1}{L} | 0 0 > = < b_{xx}(t)b_{xx}^+(0) > \]

(38)

\[ = \frac{1}{V^2} \sum_{p,q} e^{i(p-q)x} \left\{ \frac{n_p - n_q}{\partial_t - (\epsilon_p - \epsilon_q)} \right\} = \frac{1}{\beta V^2} \sum_{p,q,\omega} e^{i(p-q)x} e^{i\omega t} \frac{n_p - n_q}{i\omega - (\epsilon_p - \epsilon_q)} \]

But the correlation \( < b_{xx}(t)b_{xx}^+(0) > \) for free fields is also calculable via Wicks theorem:

\[ < b_{xx}(t)b_{xx}^+(0) > = O\left( \frac{1}{N} \right) = < \left( \psi_x^+ \psi_x \right) (0) \left( \psi_0^+ \psi_0 \right) > \text{ no internal label} = \chi(x) = -G(x)G(-x) \]

with \( G(x) = - < \psi_x(t)\psi_0^+(0) > = - \frac{1}{\partial_t + h_{\text{hop}}} \frac{1}{\beta V} \sum_{\omega, p} e^{i(p\omega + px)} \frac{i\omega - \epsilon(p)}{\omega - \epsilon(p)} \)

(39)
where $h^{\text{hop}}$ is the hopping matrix. Using conventional Matsubara techniques it is easy to check that

$$\chi(x) = -G(x)G(-x) = \frac{1}{\beta V^2} \sum_Q e^{-iQx} \sum_{p-q=Q} \frac{n_p - n_q}{i\delta_{p0} - (e_p - e_q)}$$

(40)

We conclude that the spectrum of the boson Hamiltonian corresponds to the singularities of

$$\frac{1}{\delta_{x0}\delta(t) - U <xt|\chi|y0>} = \frac{1}{\delta_{x0}\delta(t) - U\chi(\overline{x}, t)}$$

(41)

where $\chi(x) = \langle (\psi^+_x \psi) (\psi^+_0 \psi_0) \rangle$ is a susceptibility. At half filling, in $d=2$ dimensions and at $Q = (\pi, \pi)$ the Fourier transform of $\chi(\overline{x}, t)$ is well known [21] to diverge logarithmically to $+\infty$ because of an incipient antiferromagnetic instability. We conclude that the asymptotic Hamiltonian $H_{SU(\infty)}$ describes bubbles that have a pole near $Q = (\pi, \pi)$ at half filling. This pole in the spin fluctuation channel is also seen in neutron scattering on cuprates [22].

At half filling the spectrum of the asymptotic Hamiltonian $H_{U(\infty,q)}$ on an infinite lattice is very similar to that of the $SU(\infty)$ model. Again the Hamiltonian describes bubbles and again the spectrum is dominated by fluctuations at $Q = (\pi, \pi)$ except that there is now an extra band index. The excitations form a triplet of spin and pair fluctuations that are exactly degenerate at half filling.

V. EXTRAPOLATING FROM $N = \infty$ TO $N = 1$

The aim of the present paper was to establish exact results on the symplectic model in order to see whether the expansion in powers of $1/N$ about $N = \infty$ in this model makes any sense. The boson expansion helped us find the asymptotic Hamiltonian for a finite number of points but as we have seen in the last section this method becomes inconvenient for dealing with infinite lattices where the bosons seriously overcount the degrees of freedom. In this section we return to main stream methods of condensed matter physics and discuss how to extrapolate to from $N = \infty$ to $N = 1$.

From the preceding sections it follows that a treatment of doped antiferromagnets and the Hubbard model at low temperature near half filling requires a self consistent treatment of electron propagation and spin and pair fluctuations. This may be done either using the topological expansion given in [8] or by using the more conventional saddle point method. For simplicity we adopt the saddle point method, although the topological expansion is presumably the more powerful approach. We may set up a $1/N$ expansion of the double exchange model by conventional Hubbard-Stratonovich transformation:

$$\int D\phi \exp \frac{-U}{N} \left( \overrightarrow{\phi}_x - i \overrightarrow{\sigma}_x \right)^2 = \text{const}, \quad \overrightarrow{\sigma}_x = \frac{1}{2} \psi^*_a \sigma_{a\beta} \psi_{\beta i}$$

(42)

$$Z = \int D\phi D\psi \exp -S, \quad S = \int dt L$$

$$L = \sum_x N \overrightarrow{\phi}_x^2 - 2i\sqrt{U} \cdot \overrightarrow{\phi}_x \overrightarrow{\phi} + \psi^* (\partial_t + t^{\text{hop}}) \psi + \mu \psi^* \sigma_3 \psi$$

where $t^{\text{hop}}$ is the hopping amplitude. Like in preceding work on the $SU(N)$ Hubbard model at large $N$ (\cite{8}) the phase factor $"i"$ is essential to get the correct coefficient $+U$ for the coupling in the double exchange model and again the factor $"i"$ causes no difficulty because $\overrightarrow{\phi}$ is purely imaginary at the saddle point. To see this, we integrate over the fermionic degrees of freedom and find the stationary point of the effective action:

$$Z = \int D\overrightarrow{\phi} D\psi \exp -S = \int D\phi \exp -S_{\text{eff}}$$

$$\frac{S_{\text{eff}}(\overrightarrow{\phi})}{N} = \int \sum_x \overrightarrow{\phi}_x^2 dt - \log \det \left( \partial_t + t^{\text{hop}} + \mu \sigma_3 - i\sqrt{U} \cdot \overrightarrow{\phi} \right) + \text{const}$$

$$\delta S_{\text{eff}} = 0 \rightarrow i \overrightarrow{\phi}(x, t) = \frac{1}{2} \sqrt{U} Tr \overrightarrow{\sigma} <xt| \partial_t + t^{\text{hop}} + \mu \sigma_3 - i\sqrt{U} \cdot \overrightarrow{\phi} |xt> = \frac{\sqrt{U}}{N} <\overrightarrow{S}>$$

(there are $N$ species that contribute to $<\overrightarrow{S}>$, hence the factor $1/N$). So the hypothesis that $i \overrightarrow{\phi}$ is real is consistent, but difficulties may be anticipated for $\mu = i\sqrt{U}\overrightarrow{\phi}^3$. Numerical calculations indicate that the spin fluctuations are
quickly killed by doping while the pair fluctuation channel remains intact. The likely explanation is that the spin fluctuations correspond to longitudinal fluctuations of \( \vec{\phi} \) in the 3 direction while the pair fluctuations correspond Goldstone like rotations about the z axis.

One may also expand \( S_{eff} \) to second order in \( \vec{\phi} \) and confirm the existence of degenerate spin and pair fluctuations at half filling and \( N = \infty \) without ever using boson operators (but we do not see how to avoid them in the dimer problem). A detailed study of the extrapolation to \( N = 1 \) via coupled integral equations will be given in a separate paper.

VI. CONCLUSIONS

In this paper we have derived two exact results that serve as first steps towards a \( 1/N \) expansion of the symplectic Hubbard model. Firstly, we diagonalized the symplectic dimer for any \( N \) using an underlying \( SP(4) \sim SO(5) \) classifying symmetry. We found a smooth interpolation, without any singularities, between \( N = 1 \) and \( N = \infty \) which suggests that the expansion, in powers of \( 1/N \), about the point \( N = \infty \) is indeed legitimate.

Secondly, we determined the Hamiltonian of the symplectic model for \( N = \infty \) and found a multiplet of collective modes containing charged pair fluctuations and neutral spin fluctuations that are exactly degenerate with each other at half filling. This tells us that pair fluctuations in doped antiferromagnets are essential degrees of freedom.

On the way to establishing these results we found analogous results for the \( SU(N) \) extrapolation of the Hubbard model and we also indicated how Holstein-Primakoff like boson operators may be replaced by more familiar and more powerful Green’s function techniques. The map of the original symplectic model onto a magnetic model that is reminiscent of a double exchange model proved crucial to our arguments. This map is interesting in its own right because it may provide us with a simple physical picture of the superconducting instability.

The extrapolation from \( N = \infty \) to \( N = 1 \) requires a self consistent treatment of fermions interacting with their own triplet of collective modes and taking into account mutual renormalization of bosonic and fermionic fluctuations in a way similar to the FLEX approach. In the light of our results, the FLEX approach is more closely related to the \( SU(N) \) extrapolation of the Hubbard model than the \( U(N,q) \) one as it ignores the pairing fluctuations of the latter model. The integral equations of the extrapolation to \( N = 1 \) follow essentially from the topological classification given previously in \[ \text{Ref.} \], but the details remain to be worked out.

Several scenarios for the physics of pair fluctuations have been suggested, ranging from preformed pairs \[ \text{[2]} \] to postulating a quantum phase transition at a critical doping \[ \text{[24]} \]. Also, various authors have added pair fluctuations on top of antiferromagnetic ones and obtained interesting results \[ \text{[25]} \]. It is fair to say, however, that the details of the competition between spin and pairing fluctuations still remain to be understood.

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