Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations

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Abstract
We investigate semi-classical generalizations of the Charlier and Meixner polynomials, which are discrete orthogonal polynomials that satisfy three-term recurrence relations. It is shown that the coefficients in these recurrence relations can be expressed in terms of Wronskians of modified Bessel functions and confluent hypergeometric functions, respectively for the generalized Charlier and generalized Meixner polynomials. These Wronskians arise in the description of special function solutions of the third and fifth Painlevé equations.

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1. Introduction

In this paper we are concerned with the coefficients in the three-term recurrence relations for semi-classical orthonormal polynomials, specifically for generalizations of the Charlier and Meixner polynomials which are discrete orthonormal polynomials. It is shown that these recurrence coefficients for the generalized Charlier polynomials and generalized Meixner polynomials can respectively be expressed in terms of Wronskians that arise in the description of special function solutions of the third Painlevé equation (PIII)

\[
\frac{d^2w}{dz^2} = \frac{1}{w} \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{A}{z} + \frac{B}{w} + \frac{C}{w^2} + \frac{D}{w},
\]

(1.1)

where \(A, B, C\) and \(D\) are arbitrary constants, and the fifth Painlevé equation (PV)

\[
\frac{d^2w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left( \frac{Aw + B}{w} \right) + \frac{Cw}{z} + \frac{Dw(w+1)}{w-1}.
\]

(1.2)

Wronskians for special function solutions of PIII are expressed in terms of modified Bessel functions and for PV in terms of confluent hypergeometric functions.
The relationship between semi-classical orthogonal polynomials and integrable equations dates back to the work of Shohat [56] in 1939 and later Freud [30] in 1976. However it was not until the work of Fokas, Its and Kitaev [24, 25] in the early 1990s that these equations were identified as discrete Painlevé equations. The relationship between semi-classical orthogonal polynomials and the (continuous) Painlevé equations was demonstrated by Magnus [41] in 1995. A motivation for this work is that recently it has been shown that recurrence coefficients for several semi-classical orthogonal polynomials can be expressed in terms of solutions of Painlevé equations, see, for example, [2, 3, 5, 6, 8, 10–14, 17, 18, 20–22, 27, 29, 57, 61, 62].

This paper is organized as follows. In section 2 we review properties of PIII (1.1) and PV (1.2), including special function solutions and the Hamiltonian structure of these equations; in section 3 we review properties of orthogonal polynomials and discrete orthogonal polynomials; in section 4 we derive expressions for the recurrence coefficients for the generalized Charlier polynomials in terms of Wronskians that arise in the description of special function solutions of PIII; in section 5 we derive expressions for the recurrence coefficients for the generalized Meixner polynomials in terms of Wronskians that arise in the description of special function solutions of PV; and in section 6 we discuss our results.

2. Painlevé equations

The six Painlevé equations (P_I–P_VI) were first discovered by Painlevé, Gambier and their colleagues in an investigation of which second order ordinary differential equations of the form
\[
\frac{d^2w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right),
\]
where $F$ is rational in $dw/dz$ and $w$ and analytic in $z$, have the property that their solutions have no movable branch points. They showed that there were 50 canonical equations of the form (2.1) with this property, now known as the Painlevé property. Further Painlevé, Gambier and their colleagues showed that of these 50 equations, 44 can be reduced to linear equations, solved in terms of elliptic functions, or are reducible to one of six new nonlinear ordinary differential equations that define new transcendental functions (see Ince [35]). The Painlevé equations can be thought of as nonlinear analogues of the classical special functions [16, 23, 32, 37], and arise in a wide variety of applications, for example random matrices, see [26, 55] and the references therein.

The Painlevé equations P_{III}–P_{VI} possess hierarchies of solutions expressible in terms of classical special functions, cf [16, 32, 44] and the references therein. For P_{III} (1.1) these are expressed in terms of Bessel functions [46, 47, 53], which we discuss in section 2.1, and for P_{V} (1.2) in terms of confluent hypergeometric functions (equivalently, Kummer functions or Whittaker functions) [52, 44, 63], which we discuss in section 2.3.

Each of the Painlevé equations P_{I}–P_{VI} can be written as a (non-autonomous) Hamiltonian system. For P_{III} (1.1) and P_{V} (1.2) these have the form
\[
\frac{dq}{dz} = -\frac{\partial \mathcal{H}_I}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial \mathcal{H}_I}{\partial q}, \quad J = \text{III, V}
\]
for a suitable Hamiltonian function $\mathcal{H}_J(q, p, z)$ [38, 50, 52, 53], which we discuss in sections 2.2 and 2.4. Further, the function $\sigma(z) \equiv \mathcal{H}_J(q, p, z)$ satisfies a second-order, second-degree equation, which is often called the Jimbo–Miwa–Okamoto equation or Painlevé $\sigma$-equation, whose solution is expressible in terms of the solution of the associated Painlevé equation [38, 51–53]. Hence there are special function solutions of these equations, which are also discussed in sections 2.2 and 2.4.
2.1. Special functions solutions of the third Painlevé equation

In the generic case when $CD \neq 0$ in $P_{III}$ (1.1), then we set $C = 1$ and $D = -1$, without loss of generality, so in the following we consider the equation

$$
\frac{d^2 w}{dz^2} = \frac{1}{w} \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{Aw^2 + B}{z} + w^3 - \frac{1}{w}.
$$

(2.2)

Special function solutions of (2.2) are expressed in terms of Bessel functions, see [46, 47, 53].

**Theorem 2.1.** Equation (2.2) has solutions expressible in terms of Bessel functions if and only if

$$
\epsilon_1 A + \epsilon_2 B = 4n + 2,
$$

(2.3)
with $n \in \mathbb{Z}$ and $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ independently.

**Proof.** See Gromak [31], Mansfield and Webster [42] and Umemura and Watanabe [60]; also [32, section 35].

An alternative form of $P_{III}$, due to Okamoto [50, 51, 53], is obtained by making the transformation $w(z) = u(t)/\sqrt{t}$, with $t = \frac{1}{2}z^2$, in (2.2) giving

$$
\frac{d^2 u}{dt^2} = \epsilon_1 u^2 + vu + \epsilon_2 t,
$$

(2.4)
if and only if $A$ and $B$ satisfy (2.3). To solve (2.5), we make the transformation

$$
u(t) = -\epsilon_1 t \frac{d}{dt} \ln \psi_v(t),
$$

then $\psi_v(t)$ satisfies

$$
\frac{d^2 \psi_v}{dt^2} + (1 - v) \frac{d\psi_v}{dt} + \epsilon_1 \epsilon_2 \psi_v = 0,
$$

(2.6)
which has solution

$$
\psi_v(t) = \begin{cases} 
\frac{t^{1/2}}{2} \left( C_1 J_{\nu}(2\sqrt{t}) + C_2 Y_{\nu}(2\sqrt{t}) \right), & \text{if } \epsilon_1 = 1, \epsilon_2 = 1, \\
\frac{t^{-1/2}}{2} \left( C_1 J_{\nu}(2\sqrt{t}) + C_2 Y_{\nu}(2\sqrt{t}) \right), & \text{if } \epsilon_1 = -1, \epsilon_2 = -1, \\
\frac{t^{1/2}}{2} \left( C_1 I_{\nu}(2\sqrt{t}) + C_2 K_{\nu}(2\sqrt{t}) \right), & \text{if } \epsilon_1 = 1, \epsilon_2 = -1, \\
\frac{t^{-1/2}}{2} \left( C_1 I_{\nu}(2\sqrt{t}) + C_2 K_{\nu}(2\sqrt{t}) \right), & \text{if } \epsilon_1 = -1, \epsilon_2 = 1,
\end{cases}
$$

(2.7)
with $C_1$ and $C_2$ arbitrary constants, and where $J_{\nu}(z), Y_{\nu}(z), I_{\nu}(z)$ and $K_{\nu}(z)$ are Bessel functions.

2.2. Hamiltonian structure for the third Painlevé equation

The Hamiltonian associated with $P_{III}$ (2.4) is

$$
\mathcal{H}_{III}(q, p, t) = q^2 p^2 - (q^2 + \vartheta_0 q - t)p + \frac{1}{2}(\vartheta_0 + \vartheta_{\infty})q.
$$

(2.8)
with $\vartheta_0$ and $\vartheta_{\infty}$ parameters, where $p$ and $q$ satisfy

$$
\frac{d}{dt} \frac{dq}{dt} = 2q^2 p - q^2 - \vartheta_0 q + t,
$$

(2.9)
$$
\frac{d}{dt} \frac{dp}{dt} = -2qp^2 + 2qp + \vartheta_0 p - \frac{1}{2}(\vartheta_0 + \vartheta_{\infty}).
$$
see Okamoto [50, 51, 53]. Eliminating \( p \) in (2.9) then \( q = u \) satisfies \( P_{III} \) (2.4) with parameters \((A, B) = (-2\vartheta_\infty, 2(\vartheta_0 + 1))\). Eliminating \( q \) in (2.9) then \( p \) satisfies

\[
\frac{d^2 p}{dt^2} = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{p-1} \right) \left( \frac{dp}{dt} \right)^2 - \frac{1}{t} \frac{dp}{dt} - \frac{2p(p-1)}{t} + \frac{1}{8t^2} \left\{ 4\vartheta_0 \vartheta_\infty - \frac{(\vartheta_0 + \vartheta_\infty)^2}{p} - \frac{(\vartheta_0 - \vartheta_\infty)^2}{p-1} \right\}.
\]

(2.10)

Making the transformation \( p(t) = 1/[1 - w(z)] \), with \( z = t \), in (2.10) yields

\[
\frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + (w-1) \frac{(\vartheta_0 + \vartheta_\infty)^2 w}{8} - (\vartheta_0 - \vartheta_\infty)^2 \frac{w}{8w} = \frac{2w}{z},
\]

(2.11)

which is \( P_{V} \) (1.2) with parameters \((A, B, C, D) = (\frac{3}{2} (\vartheta_0 + \vartheta_\infty)^2, -\frac{1}{2} (\vartheta_0 - \vartheta_\infty)^2, -2, 0)\), illustrating the well-known relationship between \( P_{III} \) and \( P_{V} \) with \( \delta = 0 \), cf [32, section 34].

The second-order, second-degree equation satisfied by the Hamiltonian function is given in the following theorem.

**Theorem 2.2.** The Hamiltonian function

\[ \sigma(t; \vartheta_0, \vartheta_\infty) = \tau_{III}(q, p, t) = \frac{1}{2} t + \frac{1}{2} \vartheta_\infty^2, \]

with \( \tau_{III}(q, p, t) \) given by (2.8), satisfies the second-order, second-degree equation

\[
\left( \frac{d^2 \sigma}{dt^2} \right)^2 + \left( \frac{d \sigma}{dt} \right)^2 - \frac{1}{2} \left( \frac{d^2 \sigma}{dt^2} \right) + \vartheta_0 \vartheta_\infty \frac{d \sigma}{dt} = \frac{1}{2} \left( \vartheta_0^2 + \vartheta_\infty^2 \right).
\]

(2.12)

Conversely, if \( \sigma(t) \) satisfies (2.12) then the solution of the Hamiltonian system (2.9) is given by

\[
q(t) = 2 \frac{t \frac{d \sigma}{dt} - 2 \vartheta_0 \frac{d \sigma}{dt} + \vartheta_\infty}{1 - 4 \left( \frac{d \sigma}{dt} \right)^2}, \quad p(t) = \frac{d \sigma}{dt} + \frac{1}{2}.
\]

(2.13)

**Proof.** See Okamoto [51, 53]; see also Forrester and Witte [28].

The special function solutions of (2.12) are given in the following theorem.

**Theorem 2.3.** Let \( \tau_{n,v}(t) \) be the determinant given by

\[
\tau_{n,v}(t) = \left| \begin{array}{cccc}
\psi_v & \delta_1(\psi_v) & \cdots & \delta_n(\psi_v) \\
\delta_1(\psi_v) & \delta_2(\psi_v) & \cdots & \delta_n(\psi_v) \\
\vdots & \ddots & \ddots & \vdots \\
\delta_n^{-1}(\psi_v) & \delta^n(\psi_v) & \cdots & \delta_{2n-2}(\psi_v) 
\end{array} \right|, \quad \delta_t \equiv \frac{d}{dt},
\]

(2.14)

with \( \psi_v(t) \) a solution of (2.6). Then special function solutions of (2.12) are given by

\[
\sigma(t) = \frac{1}{2} \frac{d}{dt} \left( \ln \tau_{n,v}(t) \right) + \frac{1}{2} \varepsilon_1 \varepsilon_2 t + \frac{1}{4} v^2 + \frac{1}{2} n(1 - \varepsilon_1 v) - \frac{1}{4} n^2,
\]

(2.15)

for the parameters \((\vartheta_0, \vartheta_\infty) = (v + n, \varepsilon_1 \varepsilon_2 (v - n))\).

**Proof.** See Okamoto [53]; see also Forrester and Witte [28].

The determinant \( \tau_{n,v}(t) \) given by (2.14) is often called a ‘\( t \)-function’, see [53].
2.3. Special functions solutions of the fifth Painlevé equation

In the generic case when $D \neq 0$ in $\text{PV} (1.2)$, then we set $D = -\frac{1}{2}$, without loss of generality, so in the following we consider the equation

$$\frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{w} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left( \frac{Aw + B}{w} \right) + \frac{Cw}{z} - \frac{w(w+1)}{2(w-1)}. \tag{2.16}$$

Special function solutions of (2.16) are expressed in terms of confluent hypergeometric functions (equivalently Kummer functions or Whittaker functions), see [52, 44, 63].

**Theorem 2.4.** Equation (2.16) then has solutions expressible in terms of Kummer functions if and only if

$$a + b + \varepsilon_3 C = 2N + 1, \tag{2.17}$$

or

$$(a - N)(b - N) = 0, \tag{2.18}$$

where $N \in \mathbb{N}$, $a = \varepsilon_1 \sqrt{A}$ and $b = \varepsilon_2 \sqrt{-2B}$, with $\varepsilon_j = \pm 1$, $j = 1, 2, 3$, independently.

**Proof.** See Okamoto [52], Masuda [44] and Watanabe [63]; also [32, section 40].

In the case when $N = 0$ in (2.17), then (2.16) has solutions in terms of the associated Riccati equation

$$\frac{dw}{dz} = aw^2 + (b - a + \varepsilon_3 z)w - b. \tag{2.19}$$

If $a \neq 0$, then (2.19) has the solution

$$w(z) = - \frac{z}{a} \frac{d}{dz} \ln \varphi(z),$$

where $\varphi(z)$ satisfies

$$z^2 \frac{d^2 \varphi}{dz^2} + z(a - b + 1 - \varepsilon_3 z) \frac{d\varphi}{dz} - ab\varphi = 0,$$

which has solution

$$\varphi(z) = \begin{cases} z^b [C_1 M(b, 1 + a + b, z) + C_2 U(b, 1 + a + b, z)], & \text{if } \varepsilon_3 = 1, \\ z^b e^{-z} [C_1 M(1 + a, 1 + a + b, z) + C_2 U(1 + a, 1 + a + b, z)], & \text{if } \varepsilon_3 = -1, \end{cases}$$

with $C_1$ and $C_2$ arbitrary constants, and where $M(\alpha, \beta, z)$ and $U(\alpha, \beta, z)$ are Kummer functions.

2.4. Hamiltonian structure for the fifth Painlevé equation

The Hamiltonian associated with (2.16) is

$$H_V(q, p, z) = q(q - 1) \frac{p^2}{2} - ((b + \vartheta)q^2 - (2b + \vartheta - z)q + b)p - \frac{1}{2} [a^2 - (b + \vartheta)^2], \tag{2.20}$$

with $a$, $b$ and $\vartheta$ parameters, where $p$ and $q$ satisfy

$$z \frac{dq}{dz} = 2q(q - 1) \frac{p^2}{2} - (b + \vartheta)q^2 + (2b + \vartheta - z)q - b, \tag{2.21}$$

$$z \frac{dp}{dz} = -(3q - 1) \frac{p^2}{2} + 2(b + \vartheta)qp - (2b + \vartheta - z)p + \frac{1}{2} [a^2 - (b + \vartheta)^2],$$

see Jimbo and Miwa [38] and Okamoto [50–52]. Eliminating $p$ then $q = w$ satisfies (2.16) with parameters $(A, B, C) = (\frac{1}{4} a^2, -\frac{1}{2} b^2, -\vartheta - 1)$.

The second-order, second-degree equation satisfied by the Hamiltonian function is given in the following theorem.
Theorem 2.5. The Hamiltonian function
\[
\sigma (z, a, b, \vartheta) = H_V(q, p, z) + \frac{1}{2} (2b + \vartheta) z = \frac{1}{3} (2b + \vartheta)^2, \tag{2.22}
\]
with \( H_V(q, p, z) \) given by (2.20), satisfies the second-order, second-degrees equation
\[
\left( \frac{d^2 \sigma}{dz^2} \right)^2 - \left( 2 \left( \frac{d \sigma}{dz} \right)^2 - \frac{\sigma}{z} + \sigma + \frac{\sigma'}{\sigma_2} \right)^2 + 4 \prod_{j=1}^n \left( \frac{d \sigma}{dz} + \kappa_j \right) = 0, \tag{2.23}
\]
for the parameters
\[
\kappa_1 = \frac{1}{2} \vartheta + \frac{1}{2} a, \quad \kappa_2 = \frac{1}{2} \vartheta - \frac{1}{2} a, \quad \kappa_3 = -\frac{1}{2} \vartheta - \frac{1}{2} b, \quad \kappa_4 = -\frac{1}{2} \vartheta + \frac{1}{2} b. \tag{2.24}
\]
Conversely, if \( \sigma(z; a, b, \vartheta) \) satisfies (2.23) then the solution of the Hamiltonian system (2.21) is given by
\[
q(z) = \frac{z \sigma'' + 2(\sigma')^2 - z \sigma' + \sigma}{2 (\sigma' + \frac{1}{2} \vartheta - \frac{1}{2} a) (\sigma' + \frac{1}{2} \vartheta + \frac{1}{2} a)}, \quad p(z) = \frac{z \sigma'' - 2(\sigma')^2 + z \sigma' - \sigma}{2 (\sigma' - \frac{1}{2} \vartheta + \frac{1}{2} b)}, \tag{2.25}
\]
where \( \gamma = d/dz \).

\( \square \)

The special function solutions of (2.23) are given in the following theorem.

Theorem 2.6. Let \( W_n(\psi) \) be the determinant given by
\[
W_n(\psi) = \begin{vmatrix}
\psi & \delta_2(\psi) & \cdots & \delta_n^{n-1}(\psi) \\
\delta_1(\psi) & \delta_2^2(\psi) & \cdots & \delta_n^3(\psi) \\
\vdots & \vdots & \ddots & \vdots \\
\delta_1^{n-1}(\psi) & \delta_2^{n-2}(\psi) & \cdots & \delta_n^{2n-2}(\psi)
\end{vmatrix}, \quad \delta_z \equiv z \frac{d}{dz}, \tag{2.26}
\]
and define \( \varphi_{\alpha, \beta}(z) \) by
\[
\varphi_{\alpha, \beta}(z) = C_1 M(\alpha, \beta, z) + C_2 U(\alpha, \beta, z),
\]
with \( C_1 \) and \( C_2 \) arbitrary constants, and where \( M(\alpha, \beta, z) \) and \( U(\alpha, \beta, z) \) are Kummer functions. Then special function solutions of (2.23) are given by
\[
\sigma(z) = z \frac{d}{dz} \left( \ln W_n(\varphi_{\alpha, \beta}) \right) - \frac{1}{4} (3n + 2\alpha - \beta - 1) z
\]
\[
- \frac{5}{8} n^2 - \frac{1}{4} (2\alpha - 3\beta - 1)n - \frac{1}{8} (2\alpha - \beta - 1)^2, \tag{2.27a}
\]
\[
\sigma(z) = z \frac{d}{dz} \left( \ln W_n(z^d \varphi_{\alpha, \beta}) \right) - \frac{1}{4} (3n + 2\alpha - \beta - 1) z
\]
\[
- \frac{5}{8} n^2 - \frac{1}{4} (2\alpha + \beta - 1)n - \frac{1}{8} (2\alpha - \beta - 1)^2, \tag{2.27b}
\]
for the parameters
\[
\kappa_1 = \frac{3}{4} (2\alpha - \beta - n - 1), \quad \kappa_2 = \frac{3}{4} (2\alpha + \beta + n - 3), \quad \kappa_3 = \frac{1}{4} (2\alpha - \beta + 3n - 1), \quad \kappa_4 = \frac{1}{4} (2\alpha - 3\beta + n + 1), \tag{2.27c}
\]
and
\[
\sigma(z) = z \frac{d}{dz} \left( \ln W_n(z^d e^{-z} \varphi_{\alpha, \beta}) \right) + \frac{1}{4} (3n + 2\alpha + \beta - 1) z
\]
\[
- \frac{5}{8} n^2 + \frac{1}{4} (2\alpha - 3\beta + 1)n - \frac{1}{8} (2\alpha - \beta + 1)^2, \tag{2.28a}
\]
for the parameters
\[ \kappa_1 = \frac{1}{2}(2\alpha - \beta + n + 1), \quad \kappa_3 = \frac{1}{2}(2\alpha - \beta - 3n + 1), \]
\[ \kappa_2 = -\frac{1}{2}(2\alpha + \beta - n - 1), \quad \kappa_4 = -\frac{1}{2}(2\alpha - 3\beta - n + 3). \]  
(2.28b)

**Proof.** This result can be inferred from the work of Forrester and Witte [28] and Okamoto [52].

3. Orthonormal polynomials

3.1. Continuous orthonormal polynomials

Let \( p_n(x) \), for \( n \in \mathbb{N} \), be the orthonormal polynomial of degree \( n \) in \( x \) with respect to a positive weight \( \omega(x) \) on \((a, b)\), which a finite or infinite open interval in \( \mathbb{R} \), such that
\[
\int_a^b p_m(x)p_n(x)\omega(x) \, dx = \delta_{m,n},
\]
with \( \delta_{m,n} \) the Kronecker delta. One of the most important properties of orthogonal polynomials is that they satisfy a three-term recurrence relationship of the form
\[
x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),
\]
where the coefficients \( a_n \) and \( b_n \) are given by the integrals
\[
a_n = \int_a^b x p_n(x)p_{n-1}(x)\omega(x) \, dx, \quad b_n = \int_a^b x^2 p_n^2(x)\omega(x) \, dx,
\]
with \( p_{-1}(x) = 0 \). The coefficients in the recurrence relationship (3.1) can also be expressed in terms of determinants whose coefficients are given in terms of the moments associated with the weight \( \omega(x) \). Specifically, the coefficients \( a_n \) and \( b_n \) in the recurrence relation (3.1) are given by
\[
a_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad b_n = \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n},
\]
where the determinants \( \Delta_n \) and \( \tilde{\Delta}_n \) are given by
\[
\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \quad \tilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} \end{vmatrix},
\]
for \( n = 0, 1, 2, \ldots \), with \( \Delta_0 = 1, \Delta_{-1} = 0 \) and \( \tilde{\Delta}_0 = 0 \), and \( \mu_k \), the \( k \)th moment, is given by the integral
\[
\mu_k = \int_a^b x^k \omega(x) \, dx.
\]
A characterization of classical orthogonal polynomials (such as Hermite, Laguerre and Jacobi polynomials), is that their weights satisfy the Pearson equation
\[
\frac{d}{dx} \left[ \sigma(x) \omega(x) \right] = \tau(x) \omega(x),
\]
where \( \sigma(x) \) is a monic polynomial with \( \deg(\sigma) \leq 2 \) and \( \tau(x) \) is a polynomial with \( \deg(\tau) = 1 \), cf [1, 7, 15]. If the weight function \( \omega(x) \) satisfies the Pearson equation (3.5) with either \( \deg(\sigma) > 2 \) or \( \deg(\tau) > 1 \), then the orthogonal polynomial is said to be semi-classical, cf [33, 43].

For further information about orthogonal polynomials see, for example, the books by Chihara [15], Ismail [36] and Szegö [59].
3.2. Discrete orthonormal polynomials

One can also define orthogonal polynomials on an equidistant lattice, rather than an interval. Consider the discrete orthonormal polynomials \( \{p_n(x)\} \), \( n = 0, 1, 2, \ldots \), with respect to a discrete weight \( \omega(k) \) on the lattice \( \mathbb{N} \)

\[
\sum_{k=0}^{\infty} p_n(k) p_m(k) \omega(k) = \delta_{mn},
\]

which also satisfy the recurrence relation (3.1).

The moments \( \mu_n \) of the discrete weight \( \omega(k) \) are given by

\[
\mu_n = \sum_{k=0}^{\infty} k^n \omega(k), \quad n = 0, 1, 2, \ldots,
\]

and, as for the continuous orthonormal polynomials in section 3.1 above, the coefficients in the recurrence relation are given by (3.2), with the determinants \( \Delta_n \) and \( \Delta_n \) given by (3.3).

In the special case when the discrete weight has the special form

\[
\omega(k) = c(k) z^k, \quad z > 0,
\]

which is the case for the Charlier polynomials \( C_n(k; z) \) and the Meixner polynomials \( M_n(k; \alpha, \gamma) \) (see sections 4.1 and 5.1 below, respectively), then

\[
\mu_n(z) = \sum_{k=0}^{\infty} k^n c(k) z^k = \delta^n_z(\mu_0), \quad \delta_z = \frac{d}{dz},
\]

(3.6)

Consequently the determinants \( \Delta_n(z) \) and \( \tilde{\Delta}_n(z) \) given by (3.3) have the form

\[
\Delta_n(z) = \begin{vmatrix}
\mu_0 & \delta_z(\mu_0) & \cdots & \delta_z^{n-1}(\mu_0) \\
\delta_z(\mu_0) & \delta_z^2(\mu_0) & \cdots & \delta_z^n(\mu_0) \\
\vdots & \vdots & \ddots & \vdots \\
\delta_z^{n-1}(\mu_0) & \delta_z^n(\mu_0) & \cdots & \delta_z^{2n-2}(\mu_0)
\end{vmatrix},
\]

\[
\tilde{\Delta}_n(z) = \begin{vmatrix}
\mu_0 & \delta_z(\mu_0) & \cdots & \delta_z^{n-2}(\mu_0) & \delta_z^{n+1}(\mu_0) \\
\delta_z(\mu_0) & \delta_z^2(\mu_0) & \cdots & \delta_z^{n-1}(\mu_0) & \delta_z^{n+2}(\mu_0) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_z^{n-1}(\mu_0) & \delta_z^n(\mu_0) & \cdots & \delta_z^{2n-3}(\mu_0) & \delta_z^{2n-1}(\mu_0)
\end{vmatrix},
\]

respectively. Hence we have the following result.

**Theorem 3.1.** If the moment \( \mu_n(z) \) has the form (3.6), then the determinants \( \Delta_n(z) \) and \( \tilde{\Delta}_n(z) \) can be written in the form

\[
\Delta_n(z) = W_n(\mu_0), \quad \tilde{\Delta}_n(z) = \delta_z W_n(\mu_0),
\]

(3.7)

where \( W_n(\psi) \) is defined by (2.26).

Some properties of the recurrence coefficients \( a_n(z) \) and \( b_n(z) \) are given in the following theorem.

**Theorem 3.2.** If the moment \( \mu_n(z) \) has the form (3.6), then the recurrence coefficients \( a_n(z) \) and \( b_n(z) \) in (3.1) satisfy the Toda system

\[
\delta_z(a_n^2) = a_n^2(b_n - b_{n-1}), \quad \delta_z(b_n) = a_{n+1}^2 - a_n^2.
\]

(3.8)

Further the determinant \( \Delta_n(z) \) satisfies the Toda equation

\[
\delta_z^2(\ln \Delta_n) = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n = 1, 2, \ldots.
\]
Theorem 3.3. The recurrence coefficients $a_n(z)$ and $b_n(z)$ in (3.1) can be expressed in the form

$$a_n^2(z) = \delta^{-1}(\ln W_n(\mu_0)), \quad b_n(z) = \delta(z)\left(\ln \frac{W_{n+1}(\mu_0)}{W_n(\mu_0)}\right).$$

(3.9)

Proof. Applying theorem 3.1 to (3.2) gives the result.

As discussed in section 3.1 above, classical orthogonal polynomials are characterized by the Pearson equation (3.5). Analogously discrete orthogonal polynomials are characterized by the discrete Pearson equation

$$\Delta[\sigma(k)\omega(k)] = \tau(k)\omega(k),$$

(3.10)

where $\Delta$ is the forward difference operator

$$\Delta f(k) = f(k + 1) - f(k),$$

$\sigma(k)$ is a monic polynomial with $\deg(\sigma) \leq 2$ and $\tau(k)$ is a polynomial with $\deg(\tau) = 1$. A discrete Pearson equation can also be defined using the backward difference operator

$$\nabla f(k) = f(k) - f(k - 1).$$

If the discrete weight $\omega(k)$ satisfies (3.10) with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$, then the discrete orthogonal polynomial is said to be semi-classical, cf [19].

For further information about discrete orthogonal polynomials see, for example, the books by Beals and Wong [4, chapter 5], Chihara [15, chapter 6], Ismail [36, chapter 6] and Nikiforov, Suslov and Uvarov [49].

4. Charlier polynomials and generalized Charlier polynomials

4.1. Charlier polynomials

The Charlier polynomials $C_n(k; z)$ are a family of orthogonal polynomials introduced in 1905 by Charlier [9] given by

$$C_n(k; z) = \binom{-n}{k} = (-1)^k n! L_n^{(-1-k)}(-1/z), \quad z > 0,$$

(4.1)

where $\binom{a}{n}$ is the hypergeometric function and $L_n^{(\alpha)}(z)$ is the associated Laguerre polynomial, see [4, 15, 36, 54]. The Charlier polynomials are orthogonal on the lattice $\mathbb{N}$ with respect to the Poisson distribution

$$\omega(k) = \frac{z^k}{k!}, \quad z > 0,$$

(4.2)

and satisfy the orthogonality condition

$$\sum_{k=0}^{\infty} C_m(k; z)C_n(k; z) \frac{z^k}{k!} = n! e^{-z} \delta_{m,n}.$$
Hence from theorem 3.1, the Hankel determinant \( \Delta_n(z) \) is given by

\[
\Delta_n(z) = W_n(\mu_0) = z^{(n-1)/2} e^{\pi i \gamma} \prod_{k=1}^{n-1} (k!),
\]

and so from theorem 3.3 the recurrence coefficients are given by

\[
a_n^2(z) = \delta_n^2 \left( \ln W_n(\mu_0) \right) = nz, \quad b_n(z) = \delta_z \left( \ln \frac{W_{n+1}(\mu_0)}{W_n(\mu_0)} \right) = n + z.
\]

4.2. Generalized Charlier polynomials

Smet and van Assche [57] generalized the Charlier weight (4.2) with one additional parameter through the weight function

\[
\omega(x) = \frac{\Gamma(\beta + 1) x^{\beta}}{\Gamma(\beta + x + 1) \Gamma(x + 1)}, \quad z > 0,
\]

with \( \beta \) a parameter such that \( \beta > -1 \). This gives the discrete weight

\[
\omega(k) = \frac{z^k}{(\beta + 1)_k k!}, \quad z > 0,
\]

where \( (\beta + 1)_k = \Gamma(\beta + 1 + k)/\Gamma(\beta + 1) \) is the Pochhammer symbol, on the lattice \( \mathbb{N} \). The weight (4.3) satisfies the discrete Pearson equation (3.10) with

\[
\sigma(k) = k(k + \beta), \quad \tau(k) = -k^2 - \beta k + z,
\]

and so the generalized Charlier polynomials are semi-classical orthogonal polynomials. The special case \( \beta = 0 \) was first considered by Hounkonnou, Hounga and Ronveaux [34] and later studied by van Assche and Fouquauagnigni [62].

For the generalized Charlier weight (4.3), the orthonormal polynomials \( p_n(k; z) \) satisfy the orthogonality condition

\[
\sum_{k=0}^{\infty} p_n(k; z) p_m(k; z) \frac{z^k}{(\beta + 1)_k k!} = \delta_{m,n},
\]

and the three-term recurrence relation

\[
x p_n(x; z) = a_{n+1}(z) p_{n+1}(x; z) + b_n(z) p_n(x; z) + a_n(z) p_{n-1}(x; z), \tag{4.4}
\]

with \( p_1(x; z) = 0 \) and \( p_0(x; z) = 1 \). Our interest is determining explicit expressions for the coefficients \( a_n(z) \) and \( b_n(z) \) in the recurrence relation (4.4).

Smet and van Assche [57, theorem 2.1] proved the following theorem for recurrence coefficients associated with the generalized Charlier weight (4.3).

**Theorem 4.1.** The recurrence coefficients \( a_n(z) \) and \( b_n(z) \) for orthonormal polynomials associated with the generalized Charlier weight (4.3) on the lattice \( \mathbb{N} \) satisfy the discrete system

\[
\begin{align*}
(a_{n+1}^2 - z)(a_n^2 - z) &= z(b_n - n)(b_n - n + \beta), \\
b_n + b_{n-1} - n + \beta + 1 &= nz/a_n^2, \tag{4.5}
\end{align*}
\]

with initial conditions

\[
a_0^2 = 0, \quad b_0 = \frac{\sqrt{z} I_{\beta+1}(2\sqrt{z})}{I_{\beta}(2\sqrt{z})} = \frac{d}{dz} \ln(z^{-\beta/2} I_{\beta}(2\sqrt{z})), \tag{4.6}
\]

with \( I_{\cdot}(x) \) the modified Bessel function.
Smet and van Assche [57, theorem 2.1] show that the system (4.5) is a limiting case of a discrete Painlevé equation, namely the first dP1V in [61, p 723]. Since the initial conditions (4.6) involve modified Bessel functions, then clearly the solutions of the discrete system (4.5) are expressed in terms modified Bessel functions.

Using the discrete system (4.5) and the Toda system (3.8), Filipuk and van Assche [21] show that the recurrence coefficient \( b_n \) can be expressed in terms of solutions of a special case of PVI (1.2) which can be transformed into PIII. However their proof is rather involved and several details are omitted due to the size of expressions involved. Further Filipuk and van Assche [21] do not obtain explicit expressions for the recurrence coefficients \( a_n \) and \( b_n \).

The relationship between the recurrence coefficients \( a_n(z) \) and \( b_n(z) \) and classical solutions of PIII can be shown in much more straightforward way and also obtain explicit expressions for these coefficients. First we obtain explicit expressions for the moment \( \mu_0(z) \) and the Hankel determinant \( \Delta_n(z) \).

**Theorem 4.2.** For the generalized Charlier weight (4.3) the moment \( \mu_0(z) \) is given by

\[
\mu_0(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\beta + 1)k!} = \frac{\Gamma(\beta + 1)z^{-\beta/2}I_{\beta}(2\sqrt{z})}{\Gamma(\beta + 1)},
\]

(4.7)

with \( I_\nu(x) \) the modified Bessel function, and the Hankel determinant \( \Delta_n(z; \beta) \) is given by

\[
\Delta_n(z; \beta) = \left( \frac{\Gamma(\beta + 1)}{\Delta_n(z; \beta)} \right)^n W_n(z^{-\beta/2}I_{\beta}(2\sqrt{z})).
\]

(4.8)

**Proof.** Since the modified Bessel function \( I_\nu(x) \) has the series expansion [54, section 10.25.2]

\[
I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+k}}{k! \Gamma(\nu+k+1)},
\]

then the expression (4.7) for the moment \( \mu_0(z) \) follows immediately. Then using theorem 3.1, we obtain the expression (4.8) for the Hankel determinant \( \Delta_n(z; \beta) \).

Hence we obtain explicit expressions for the recurrence coefficients \( a_n(z) \) and \( b_n(z) \).

**Corollary 4.3.** The coefficients \( a_n(z) \) and \( b_n(z) \) in the recurrence relation (4.4) have the form

\[
a_n^2(z) = \left( \frac{d}{dz} \ln \Delta_n(z; \beta) \right)^2, \quad b_n(z) = z \frac{d}{dz} \left( \frac{\Delta_{n+1}(z; \beta)}{\Delta_n(z; \beta)} \right),
\]

(4.9)

with \( \Delta_n(z; \beta) \) given by (4.8).

**Proof.** These follow immediately from theorem 3.3.

Finally we relate the Hankel determinant \( \Delta_n(z; \beta) \) to solutions of (2.12), the PIII-\( \sigma \)-equation.

**Theorem 4.4.** The function

\[
S_n(z; \beta) = z \frac{d}{dz} \ln \Delta_n(z; \beta),
\]

(4.10)

with \( \Delta_n(z; \beta) \) given by (4.8), satisfies the second-order, second-degree equation

\[
\left( \frac{d^2 S_n}{dz^2} \right)^2 = \left( n - (n + \beta) \frac{d}{dz} \right) \frac{d^2 S_n}{dz^2} - 4 \frac{d}{dz} \left( \frac{dS_n}{dz} - 1 \right) \left[ z \frac{d}{dz} S_n - S_n + \frac{1}{2} n(n - 1) \right].
\]

(4.11)
Proof. Equation (4.11) is equivalent to (2.12) through the transformation
\[ S_n(z; \beta) = \sigma(z) + \frac{1}{2} z + \frac{1}{4} n^2 - \frac{1}{2} n(\beta + 1) - \frac{1}{2} \beta^2, \]  
(4.12)
with parameters \((\theta_0, \theta_{\infty}) = (n + \beta, n - \beta)\), as is easily verified. Then comparing (4.12), with \(S_n\) given by (4.10), to (2.15), with \(v = \beta, \epsilon_1 = -1\) and \(\epsilon_2 = 1\), gives the result. \(\square\)

Remarks 4.5.

(1) In terms of \(S_n(z; \beta)\) given by (4.10), the coefficients \(a_n(z)\) and \(b_n(z)\) in the recurrence relation (4.4) have the form
\[ a_n^2(z) = -z \frac{d^2}{dz^2} S_n(z; \beta), \quad b_n(z) = S_{n+1}(z; \beta) - S_n(z; \beta). \]

(2) Solving (4.12) for \(\sigma(z)\) and substituting in (2.13) yields the solution of the Hamiltonian system (2.9) given by
\[ q(z) = \frac{z S_n'(z) - 2(n + \beta) S_n(z) + 2n}{4 S_n(z)[1 - S_n'(z)]}, \quad p(z) = S_n(z), \quad z = t, \]
with parameters \((\theta_0, \theta_{\infty}) = (n + \beta, n - \beta)\).

5. Meixner polynomials and generalizations

5.1. Meixner polynomials

The Meixner polynomials \(M_n(k; \alpha, z)\) are a family of discrete orthogonal polynomials introduced in 1934 by Meixner [45] given by
\[ M_n(k; \alpha, z) = _2F_1(-n, -k; -\alpha; 1 - 1/z), \quad 0 < z < 1, \]
(5.1)
with \(\alpha > 0\), where \(_2F_1(a, b; c; z)\) is the hypergeometric function, see [4, 15, 36, 54]. In the case when \(\alpha = -N\), with \(N \in \mathbb{N}\) and \(z = p/(1-p)\), these polynomials are referred to as the Krawtchouk polynomials for \(k \in \{0, 1, \ldots, N\}\)
\[ K(k; p, N) = _2F_1(-n, -k; N; 1/p), \quad 0 < p < 1, \]
(5.2)
which were introduced in 1929 by Krawtchouk [40]. The Meixner polynomials (5.1) are orthogonal with respect to the discrete weight
\[ \omega(k) = \frac{(\alpha)_k z^k}{k!}, \quad 0 < z < 1, \]
(5.3)
and satisfy the orthogonality condition
\[ \sum_{k=0}^{\infty} M_n(k; \alpha, z) M_m(k; \alpha, z) \frac{(\alpha)_k z^k}{k!} = \frac{n! z^{-n}}{(\alpha)_n(1-z)\alpha} \delta_{m,n}. \]
The weight (5.3) satisfies the discrete Pearson equation (3.10) with \(\sigma(k) = k, \quad \tau(k) = (z - 1)k + z\alpha\).

From (5.3), the moment \(\mu_0(z)\) is given by
\[ \mu_0(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{k!} = (1 - z)^{-\alpha}. \]

Hence from theorem 3.1 that the Hankel determinant \(\Delta_n(z)\) is given by
\[ \Delta_n(z) = \mathcal{W}_n(\mu_0) = z^{n(n-1)/2} (1 - z)^{n+\alpha-1} \prod_{k=1}^{n-1} k!(\alpha + k)^{a-k-1}, \]
and so, from theorem 3.3, the recurrence coefficients are given by
\[ a_n^2(z) = \frac{n(n + \alpha - 1)z}{(1 - z)^2}, \quad b_n(z) = \frac{n + (n + \alpha)z}{1 - z}. \]
5.2. Generalized Meixner polynomials

In a similar way to that for the Charlier weight above, Smet and van Assche [57] generalized the Meixner weight (4.2) with one additional parameter through the weight function the weight function

$$\omega(x) = \frac{\Gamma(\alpha + x)\Gamma(\beta)z^x}{\Gamma(\alpha)\Gamma(\beta + x)\Gamma(x + 1)}, \quad z > 0,$$

with $\alpha, \beta > 0$, which gives the weight

$$\omega(k) = \frac{(\alpha)_{k}z^{k}}{(\beta)_{k}k!}, \quad z > 0. \quad (5.4)$$

The weight (5.4) satisfies the discrete Pearson equation (3.10) with

$$\sigma(k) = k(k + \beta - 1), \quad \tau(k) = -k^2 + (z + 1 - \beta)k + z\alpha,$$

and so the generalized Meixner polynomials are semi-classical orthogonal polynomials.

Boelen, Filipuk and van Assche [5] considered the special case of (5.4) when $\beta = 1$ and showed that the recurrence coefficients $a_n$ and $b_n$ satisfy a limiting case of an asymmetric dPIV equation. We note that the special case $\alpha = \beta$ gives the classical Charlier weight (4.2) and the case $\alpha = 1$ corresponds to the classical Charlier weight on the lattice $\mathbb{N} + 1 - \beta$.

For the generalized Meixner weight (5.4), the orthonormal polynomials $p_n(k; z)$ satisfy the orthogonality condition

$$\sum_{k=0}^{\infty} p_n(k; z)p_m(k; z)\frac{(\alpha)_{k}z^{k}}{(\beta)_{k}k!} = \delta_{m,n},$$

and the three-term recurrence relation

$$xp_n(x; z) = a_{n+1}(z)p_{n+1}(x; z) + b_n(z)p_n(x; z) + a_n(z)p_{n-1}(x; z),$$

with $p_1(x; z) = 0$ and $p_0(x; z) = 1$. As for the generalized Charlier weight (4.3), our interest is determining explicit expressions for the coefficients $a_n(z)$ and $b_n(z)$ in the recurrence relation (5.5).

Smet and van Assche [57, theorem 2.1] proved the following theorem for recurrence coefficients associated with the generalized Meixner weight (5.4).

**Theorem 5.1.** The recurrence coefficients $a_n(z)$ and $b_n(z)$ for orthonormal polynomials associated with the generalized Meixner weight (5.4) on the lattice $\mathbb{N}$ satisfy

$$a_n^2 = nz - (\alpha - 1)x_n,$$

$$b_n = n + \alpha - \beta + z - (\alpha - 1)y_n/z,$$

where $x_n$ and $y_n$ satisfy the discrete system

$$\begin{align*}
(x_n + y_n)(x_{n+1} + y_n) &= \frac{\alpha - 1}{z^2}y_n(y_n - z)\left(y_n - z\frac{\alpha - \beta}{\alpha - 1}\right), \\
(x_n + y_n)(x_n + y_{n-1}) &= \frac{\alpha - 1}{(\alpha - 1)x_n - nz}\left(x_n + z\frac{\alpha - \beta}{\alpha - 1}\right),
\end{align*} \quad (5.6)$$

with initial conditions

$$a_0^2 = 0, \quad b_0 = \frac{\alpha z}{\beta} M(\alpha + 1, \beta + 1, z) = z \frac{d}{dz} \ln M(\alpha, \beta, z), \quad (5.7)$$

and $M(\alpha, \beta, z)$ is the Kummer function.
We note that $M(\alpha, \beta, z) = {}_1F_1(\alpha, \beta, z)$, the confluent hypergeometric function [54, section 13]. Smet and van Assche [57] show that the discrete system (5.6) can be identified as a limiting case of an asymmetric dPIV equation. Filipuk and van Assche [20] show that the system (5.6) can be obtained from a of PV (1.2). Since the initial conditions (5.7) involve Kummer functions, then clearly the solutions of the discrete system (5.6) are expressed in terms Kummer functions.

Using the discrete system (5.6) and the Toda system (3.8), Filipuk and van Assche [20] show that the recurrence coefficients $a_n(z)$ and $b_n(z)$ are related to classical solutions of PV (1.2), for the parameters $(A, B, C, D) = \left(\frac{1}{2}(\alpha - 1)^2, -\frac{1}{2}(n + \alpha - \beta)^2, \pm(n + \beta), -\frac{1}{2}\right)$.

However their proof is rather cumbersome and most of the details are omitted due to the size of expressions involved. Further Filipuk and van Assche [20] do not obtain explicit expressions for the recurrence coefficients $a_n$ and $b_n$.

In an analogous way to that for the generalized Charlier polynomials in section 4.2 above, the relationship between the recurrence coefficients $a_n(z)$ and $b_n(z)$ and classical solutions of PV (1.2) can be shown using a much more straightforward way and also obtain explicit expressions for these coefficients. First we obtain explicit expressions for the moment $\mu_0(z)$ and the Hankel determinant $\Delta_1(z; \alpha, \beta)$.

Theorem 5.2. For the generalized Meixner weight (5.4) the moment $\mu_0(z)$ is given by

$$\mu_0(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{(\beta)_k k!} = M(\alpha, \beta, z),$$

with $M(\alpha, \beta, z)$ the Kummer function, and the Hankel determinant $\Delta_n(z)$ given by

$$\Delta_n(z; \alpha, \beta) = W_n(M(\alpha, \beta, z)).$$

Proof. Since the Kummer function $M(\alpha, \beta, z)$ has the series expansion [54, section 13.2.2]

$$M(\alpha, \beta, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{(\beta)_k k!},$$

then the expression (5.8) for the moment $\mu_0(z)$ follows immediately. Then we use theorem 3.1 to obtain the expression (5.9) for the Hankel determinant $\Delta_n(z; \alpha, \beta)$. □

Hence we obtain explicit expressions for the recurrence coefficients $a_n(z)$ and $b_n(z)$.

Theorem 5.3. The coefficients $a_n(z)$ and $b_n(z)$ in the recurrence relation (5.5) have the form

$$a_n(z) = \left(\frac{d}{dz}\right)^2 (\ln \Delta_n(z; \alpha, \beta)), \quad b_n(z) = z \frac{d}{dz} \left(\ln \frac{\Delta_{n+1}(z; \alpha, \beta)}{\Delta_n(z; \alpha, \beta)}\right),$$

with $\Delta_n(z)$ given by (5.9).

Proof. These follow immediately from theorem 3.3. □

Finally we relate the Hankel determinant $\Delta_n(z)$ to solutions of (2.23), the PV $\sigma$-equation.

Theorem 5.4. The function

$$S_n(z; \alpha, \beta) = z \frac{d}{dz} \ln \Delta_n(z; \alpha, \beta),$$

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with $\Delta_n(z; \alpha, \beta)$ given by (5.9), satisfies the second-order, second-degree equation
\[
\left( \frac{d^2 S_n}{dz^2} \right)^2 = \left[ (z + n + \beta - 1) \frac{dS_n}{dz} - S_n - \frac{1}{2} n(n - 1 + 2\alpha) \right]^2 \nonumber \\
-4 \frac{dS_n}{dz} \left( \frac{dS_n}{dz} - n - \alpha + \beta \right) \left[ \frac{dS_n}{dz} - S_n + \frac{1}{2} n(n - 1) \right]. \tag{5.12}
\]

**Proof.** Equation (5.12) is equivalent to (2.23) through the transformation
\[
S_n(z; \alpha, \beta) = \sigma(z) + \frac{1}{4} (2\alpha - \beta + 3n - 1) z \\
+ \frac{5}{8} n^2 + \frac{1}{4} (2\alpha - 3\beta - 1)n + \frac{1}{8} (2\alpha - \beta - 1)^2,
\]
with parameters
\[
\kappa_1 = \frac{1}{4} (2\alpha - \beta - n - 1), \quad \kappa_3 = \frac{1}{4} (2\alpha - \beta + 3n - 1), \nonumber \\
\kappa_2 = -\frac{1}{4} (2\alpha + \beta + n - 3), \quad \kappa_4 = -\frac{1}{4} (2\alpha - 3\beta + n + 1), \tag{5.14}
\]
as is easily verified. Then comparing (5.13), with $S_n$ given by (5.11), to (2.27a) gives the result. $\square$

**Remarks 5.5.**

1. In terms of $S_n(z; \alpha, \beta)$ given by (5.11), the coefficients $a_n(z)$ and $b_n(z)$ in the recurrence relation (5.5) have the form
\[
a'_n(z) = z \frac{d}{dz} S_n(z; \alpha, \beta), \quad b_n(z) = S_{n+1}(z; \alpha, \beta) - S_n(z; \alpha, \beta).
\]
2. Substituting the parameters (5.14) in (2.24) yields
\[
(a, b, \vartheta) = (\alpha - 1, \beta - \alpha - n, 1 - \beta - n), \tag{5.15}
\]
and so (5.12) is equivalent to $P_V$ (1.2) for the parameters
\[
(A, B, C, D) = \left( \frac{1}{4} (\alpha - 1)^2, -\frac{1}{4} (\alpha - \beta + n), \beta + n - 2, -\frac{1}{2} \right).
\]
3. Solving (5.13) for $\sigma(z)$ and substituting in (2.25) yields the solution of the Hamiltonian system (2.21) given by
\[
q(z) = \frac{2zS''_n + 4(S'_n)^2 - 2(z - 2\alpha + \beta - 3n + 1)S'_n + 2S_n + n(n + 2\alpha - 1)}{4(S''_n - \alpha - n + 1)(S'_n - n)}, \nonumber \\
p(z) = \frac{2zS''_n - 4(S'_n)^2 + 2(z - 2\alpha + \beta - 3n + 1)S'_n - 2S_n - n(n + 2\alpha - 1)}{4(S''_n - \alpha + \beta - n)},
\]
for the parameters (5.15).

**6. Discussion**

In this paper we have studied semi-classical generalizations of the Charlier polynomials and the Meixner polynomials. These discrete orthogonal polynomials satisfy three-term recurrence relations whose coefficients depend on a parameter. We have shown that the coefficients in these recurrence relations can be explicitly expressed in terms of Wronskians of modified Bessel functions and Kummer functions, respectively. These Wronskians also arise in the description of special function solutions of the third and fifth Painlevé equations and the second-order, second-degree equations satisfied by the associated Hamiltonian functions. The results in this
paper are more comprehensive than those in [21] for generalized Charlier polynomials and in [5, 20] for generalized Meixner polynomials. The link between the semi-classical discrete orthogonal polynomials and the special function solutions of the Painlevé equations is the moment for the associated weight which enables the Hankel determinant to be written as a Wronskian. In our opinion, this illustrates the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.

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References

[1] Álvarez-Nodarse R 2006 On characterizations of classical polynomials J. Comput. Appl. Math. 196 320–37
[2] Basor E and Chen Y 2009 Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles J. Phys. A: Math. Theor. 42 035203
[3] Basor E, Chen Y and Ehrhardt T 2010 Painlevé V and time-dependent Jacobi polynomials J. Phys. A: Math. Theor. 43 015204
[4] Beals R and Wong R 2010 Special Functions: A Graduate Text (Cambridge Studies in Advanced Mathematics vol 126) (Cambridge: Cambridge University Press)
[5] Boelen L, Filipuk G and van Assche W 2011 Recurrence coefficients of generalized Meixner polynomials and Painlevé equations J. Phys. A: Math. Theor. 44 035202
[6] Boelen L, Filipuk G, Smet C, van Assche W and Zhang L 2013 The generalized Krawtchouk polynomials and the fifth Painlevé equation J. Difference Equ. Appl. at press (doi:10.1080/10236198.2012.755522)
[7] Bochner S 1929 Über Sturm-Liouvillesche Polynomsysteme Math. Z. 29 730–6
[8] Boelen L and van Assche W 2011 Discrete Painlevé equations for recurrence relations of semiclassical Laguerre polynomials Proc. Am. Math. Soc. 138 1317–31
[9] Charlier C V L 1905–6 Über die Darstellung willkürlich Funktionen Ark. Mat. Astr. Fys. 2 1–9
[10] Chen Y and Dai D 2010 Painlevé V and a Pollaczek-Jacobi type orthogonal polynomials J. Approx. Theory 162 2149–67
[11] Chen Y and Feigin M V 2006 Painlevé IV and degenerate Gaussian unitary ensembles J. Phys. A: Math. Gen. 39 12381–93
[12] Chen Y and Its A R 2010 Painlevé III and a singular linear statistics in Hermitian random matrix ensembles: I J. Approx. Theory 162 270–97
[13] Chen Y and Pruessner G 2005 Orthogonal polynomials with discontinuous weights J. Phys. A: Math. Gen. 38 L191–8
[14] Chen Y and Zhang L 2010 Painlevé VI and the unitary Jacobi ensembles Stud. Appl. Math. 125 91–112
[15] Chihara T S 1978 An Introduction to Orthogonal Polynomials (New York: Gordon and Breach) (reprinted by Dover 2011)
[16] Clarkson P A 2006 Painlevé equations—non-linear special functions Orthogonal Polynomials and Special Functions: Computation and Application (Lecture Notes in Mathematics vol 1883) ed F Marcellán and W van Assche (Berlin: Springer) pp 331–411
[17] Clarkson P A and Jordaan K 2013 The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation arXiv:1301.4134 [nlin.SI]
[18] Dai D and Zhang L 2010 Painlevé VI and Hankel determinants for the generalized Jacobi weight J. Phys. A: Math. Theor. 43 055207
[19] Dominici D and Marcellán F 2012 Discrete semiclassical orthogonal polynomials of class one arXiv:1211.2005 [math.CA]
[20] Filipuk G and van Assche W 2011 Recurrence coefficients of a new generalization of the Meixner polynomials SIGMA 7 068
[21] Filipuk G and van Assche W 2013 Recurrence coefficients of generalized Charlier polynomials and the fifth Painlevé equation Proc. Am. Math. Soc. 141 551–62
[22] Filipuk G, van Assche W and Zhang L 2012 The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation J. Phys. A: Math. Theor. 45 205201
[55] Osipov V A and Kanzieper E 2010 Correlations of RMT characteristic polynomials and integrability: Hermitian matrices Ann. Phys. 325 2251–306
[56] Shohat J 1939 A differential equation for orthogonal polynomials Duke Math. J. 5 401–17
[57] Smet C and van Assche W 2012 Orthogonal polynomials on a bi-lattice Constr. Approx. 36 215–42
[58] Sogo K 1993 Time-dependent orthogonal polynomials and theory of soliton—applications to matrix model, vertex model and level statistics J. Phys. Soc. Japan 62 1887–94
[59] Szegő G 1975 Orthogonal Polynomials (AMS Colloquium Publications vol 23) (Providence RI: American Mathematical Society)
[60] Unemura H and Watanabe H 1998 Solutions of the third Painlevé equation I Nagoya Math. J. 151 1–24
[61] van Assche W et al 2007 Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials Difference Equations, Special Functions and Orthogonal Polynomials ed S Elaydi (Singapore: World Scientific) pp 687–725
[62] van Assche W and Fouppouagnigni M 2003 Analysis of non-linear recurrence relations for the recurrence coefficients of generalized Charlier polynomials J. Nonlinear Math. Phys. 10 231–7
[63] Watanabe H 1995 Solutions of the fifth Painlevé equation: 1 Hokkaido Math. J. 24 231–67