A non-singular potential for the Dirac monopole

Ranjan Kumar Ghosh*  
Bidhannagar College, EB-2 Salt Lake City, Calcutta 700064, India

Palash B. Pal†  
Saha Institute of Nuclear Physics, 1/AF Bidhan-Nagar, Calcutta 700064, India

August 2002

Abstract

We propose a new vector potential for the Abelian magnetic monopole. The potential is non-singular in the entire region around the monopole. We argue how the Dirac quantization condition can be derived for any choice of potential.

Magnetic monopoles have not been found experimentally, but there are strong theoretical reasons to believe that they exist. Dirac [1] showed that if a magnetic monopole exists, its magnetic charge $g$ obeys the relation

$$qg = n/2$$

(1)

in natural units, where $q$ is the charge of any particle and $n$ is an integer. This implies quantization of electric charges.

The magnetic field of a monopole is given by

$$B = \frac{g}{r^2} \hat{r},$$

(2)

where $r$ is the radial distance from the monopole and $\hat{r}$ is the unit vector in the radial direction. Thus, the integral of the magnetic flux on a closed surface containing the monopole does not vanish:

$$\oint d\mathbf{S} \cdot \mathbf{B} = 4\pi g.$$  

(3)

Accordingly, one cannot define a vector potential globally by the relation

$$\mathbf{B} = \nabla \times \mathbf{A},$$

(4)
since that would imply vanishing of the divergence of \( B \), and consequently of the surface integral on the left side of Eq. (3). It may not be a problem classically, where one could work entirely in terms of the electric and magnetic fields. But in quantum theory, \( A \) plays a fundamental role, and so it must be defined.

Dirac \[1\] circumvented this problem by hypothesizing a thin string which carries all the flux and defining the divergence-free \( A \) everywhere else. This Dirac string must be unphysical and therefore undetectable. This leads to the Dirac quantization formula \[1, 2, 3, 4\] of Eq. (1). Much later, Wu and Yang \[5, 6\] showed that the Dirac string can be totally avoided in a formulation where one uses two different patches for the vector potential for the space around a monopole:

\[
A_r = A_\theta = 0, \quad A_\phi = \begin{cases} 
\frac{\theta}{r \sin \theta} (1 - \cos \theta) & \text{for } 0 \leq \theta \leq \frac{\pi}{2} + \epsilon, \\
\frac{-g}{r \sin \theta} (1 - \cos \theta) & \text{for } \frac{\pi}{2} - \epsilon \leq \theta \leq \pi,
\end{cases}
\]  

(5)

for any arbitrary \( \epsilon \) in the range \( 0 < \epsilon < \frac{1}{2}\pi \). Notice each patch has a singularity if we try to extend them over the entire region around the monopole as Dirac did, but is regular in its restricted domain of definition shown above. In the overlap region \( \frac{\pi}{2} - \epsilon \leq \theta \leq \frac{\pi}{2} + \epsilon \), the two patches are related by a gauge transformation:

\[
A^+ = A^- + \nabla (2g\phi),
\]  

(6)

where the superscripted plus and minus sign denotes the patch which covers the upper and lower hemisphere respectively. On wave functions (for quantum mechanics) or on other fields (for quantum field theory) in the theory, there will be a corresponding transformation of the form \( \exp(2i\phi g) \). Demanding that this is single valued everywhere, Wu and Yang \[5\] obtained the Dirac quantization condition, Eq. (1).

In the same paper \[5\], Wu and Yang argued that it is impossible to define a singularity-free potential for the space around the monopole. Their argument starts with the assumption that such a potential exists. The loop integral \( \oint d\mathbf{x} \cdot \mathbf{A} \) around any closed curve must then give the magnetic flux through the surface enclosed. If we keep \( r \) and \( \theta \) constant and traverse a loop in the \( \phi \)-direction, the magnetic flux through such a polar cap would be given by

\[
\Phi(r, \theta) = 2\pi g(1 - \cos \theta).
\]  

(7)

This gives \( \Phi(r, \pi) = 4\pi g \). Wu and Yang comment that this is a contradiction, since at \( \theta = \pi \) the curve has shrunk to a point and therefore the flux through it must be zero. This proves, reductio ad absurdum, that a non-singular potential cannot exist, according to Wu and Yang \[5\].

In fact, there is no contradiction here. Any closed curve on the surface of a sphere is the boundary of two regions. For example, the equator can be thought of as the boundary of the northern hemisphere or of the southern hemisphere. Likewise, the closed “curve” at \( \theta = \pi \) is the boundary of either a region of zero area, or of the entire remaining area. Eq. (7) gives the flux through the region bounded by the curve which contains the point \( \theta = 0 \). This is the surface of the entire sphere, so it is no wonder that the flux through that would come out to be \( 4\pi g \).

There is therefore no argument to show that one cannot define a non-singular potential for a monopole. And in fact, non-singular potentials exist. Here we propose such a potential:

\[
A_r = 0, \quad A_\theta = -\frac{g}{r} \phi \sin \theta, \quad A_\phi = 0.
\]  

(8)
It is trivial to check that the curl of this potential gives the magnetic field of Eq. (2). It is also quite obvious that, apart from the essential singularity at \( r = 0 \), there is no other singularity of this potential. The Wu-Yang patches, on the other hand, have singularities outside their domain of definition, which is why no patch can be extended to the entire space. For example, \( A^{(+)} \) is singular on the entire half-line \( \theta = \pi \), whereas \( A^{(-)} \) is singular on the half-line \( \theta = 0 \).

Although our potential is multi-valued because the azimuthal angle \( \phi \) is defined only modulo \( 2\pi \), this does not cause any problem. The reason is that, the value of the potential for two values of \( \phi \) separated by \( 2\pi \) are related by a gauge transformation:

\[
A(r, \theta, \phi + 2\pi) = A(r, \theta, \phi) + \nabla(2\pi g \cos \theta). \tag{9}
\]

Further, our potential is very simply related to the Wu-Yang patches through gauge transformation:

\[
A^{(\pm)} = A + \nabla \left( \left( \pm 1 - \cos \theta \right) \phi \right). \tag{10}
\]

Thus, the two potentials should have identical physical implications, including the quantization condition. However, we should be able to derive the quantization condition without making any reference to the Wu-Yang potential, which is what we argue below.

To begin with, we make an important point about the Wu-Yang derivation of the quantization condition, which we have outlined above. The proof relies on the gauge transformation connecting the different patches. However, it should be realized that such “sewing conditions for patches” cannot guarantee the correct result. To illustrate the point, let us suppose that, instead of the two patches of Eq. (5) used by Wu and Yang, we take three different patches defined by

\[
A_r = A_\theta = 0, \quad A_\phi = \begin{cases} 
\frac{q}{r \sin \theta} (1 - \cos \theta) & \text{for } 0 \leq \theta < \frac{1}{3}\pi, \\
\frac{q}{r \sin \theta} (a - \cos \theta) & \text{for } \frac{1}{4}\pi < \theta < \frac{3}{4}\pi, \\
\frac{q}{r \sin \theta} (-1 - \cos \theta) & \text{for } \frac{2}{3}\pi < \theta \leq \pi,
\end{cases} \tag{11}
\]

where \( a \neq \pm 1 \). In this case, the sewing condition between the two patches in the region \( \frac{1}{4}\pi < \theta < \frac{1}{3}\pi \) will give \( qg = n/(1-a) \), which is not the Dirac quantization condition. One might then make the further stipulation that one must take only the minimum number of patches necessary to cover the space around the monopole. But this only shows that a proper derivation of the quantization condition must somehow take into account the global properties of the space around the monopole, not any local condition.

This can be done, and the quantization condition derived, without making any reference to any specific form of the potential. All one needs is the property, emphasized by Wu and Yang [5], that a complete and unsuperfluous description of electromagnetism is provided by the phase factors

\[
\exp \left( ig \oint dx^\mu A_\mu \right) \tag{12}
\]

around all possible closed loops. For a purely magnetic field, \( A_0 = 0 \), so the integral in the exponent becomes the line integral of the vector potential around a closed loop.
For the space around the monopole, consider such a closed loop. To be specific, one can consider a loop on a sphere of radius $r$, although this restriction is not essential. No matter which vector potential we use, the line integral around a loop $C$ will equal the flux of magnetic field through a surface $S$ whose boundary is $C$.

It is not clear though what we mean by the surface $S$. As pointed out earlier, any loop on a sphere is the boundary of two complementary surfaces. If the outward flux through one of these surfaces is $\Phi$, the outward flux through the other must be $4\pi g - \Phi$. The direction in which the line integral on $C$ is taken determines the direction in which we have to consider the normal to the surface $S$. If the direction of the line integral is such that on the first surface the normals are outward, they will be inward for the second surface. To apply Stokes’ theorem, we must then use the inward flux for the second surface, which is $\Phi - 4\pi g$. Thus, Wu and Yang’s phase factor is given by two expressions, which must give the same result:

$$\exp \left( i g \Phi \right) = \exp \left( i g (\Phi - 4\pi g) \right),$$

(13)

This implies

$$e^{4\pi i g} = 1,$$

(14)

which gives the Dirac quantization condition. As indicated earlier, this makes no reference to the specific form of the vector potential, and in particular applies to our potential as well.

It might naively seem that this argument faces a problem if, for our potential of Eq. (8), we consider a loop with constant $\theta$. Since our potential does not have any component in the $\phi$-direction, it might seem that the line integral on this loop would vanish. The same problem would occur for the Wu-Yang potential if we consider a loop consisting of two semi-circles of constant $\phi$, as shown in Fig. 1a. On the other hand, the magnetic field lines are radial and isotropic. So, through any loop subtending a solid angle $\Omega$ at the center, the magnetic flux should be $g\Omega$. This poses an apparent paradox. Let us first discuss how this paradox is resolved in the Wu-Yang case, before commenting on our potential.

The crucial thing to realize is that the loop in Fig. 1a crosses discontinuities of the Wu-Yang potential. So, one should be careful while taking the line integral. The correct way of doing this is to add two equal and opposite lines in the middle, shown as separate lines for the sake of clarity in Fig. 1b. These lines break the loop into two parts. Each part of the loop entirely lies in a region in which there is a single continuous potential. For each part, the line integral can be calculated in a straightforward way. Then one should add the contributions to obtain the line integral over the entire loop. If the two semicircles of the loop are at $\phi = \phi_1$ and $\phi = \phi_2$, the line integral on the upper part of the loop is given by

$$\Phi_+ = \int_{\phi_1}^{\phi_2} d\phi \, r \sin \theta A^{(+)}_{\phi},$$

(15)

whereas on the lower part it is given by

$$\Phi_- = \int_{\phi_2}^{\phi_1} d\phi \, r \sin \theta A^{(-)}_{\phi}.$$  

(16)

Using the form of the potential from Eq. (5), we then obtain the total flux through the closed loop to be

$$\Phi = 2g(\phi_2 - \phi_1).$$

(17)
Figure 1: Line integral of the Wu-Yang potential is naively zero around the loop on the sphere shown in part (a) of the figure. However, as argued in the text, the loop should be decomposed into two parts as shown in part (b) in order to calculate the line integral properly.

Figure 2: The figure shows a sphere in the polar projection. The center represents the north pole. The radial lines are lines of constant $\phi$, whereas the circular line is a line of constant $\theta$. If one wants to perform the line integral of our vector potential over a loop of constant $\theta$, the loop will have to be deformed as shown in the figure.
This is the correct value from the isotropy argument. When $\phi_2 - \phi_1 = 2\pi$, it gives the total flux to be $4\pi g$.

We now consider a similar situation with our potential. The contentious loops in this case are lines of constant $\theta$. Since $A_\phi$ vanishes for our potential, it would naively seem that the line integral along such a loop vanishes as well. The important thing is to recall that our potential of Eq. (8) is multi-valued. One can circumvent this problem by making the substitution 

$$\phi \rightarrow \phi \mod 2\pi$$

in Eq. (8), which does not affect the curl of the potential. However, the potential is now discontinuous at $\phi = 0$, and the contentious loops certainly cross this discontinuity. As in the Wu-Yang case, we now have to avoid the discontinuity by going around the north pole, as shown in Fig. 2. The entire contribution to the line integral now comes from the added pair of lines, and is given by

$$\int_0^\theta d\theta \, r \left( A_\theta(0+) - A_\theta(0-) \right),$$

where $A_\theta(0+)$, for example, denotes the value of $A_\theta$ for a slightly positive value of $\phi$. Using Eq. (8), we now obtain Eq. (15), which is the correct flux through such a north polar cap. If instead we avoid the discontinuity by going around the south pole, we would obtain the flux through the complementary surface.

We have thus provided a vector potential for the magnetic monopole which is non-singular everywhere in the space around the monopole. It is admittedly multi-valued. It can be made single-valued by restricting the azimuthal angle as in Eq. (18). It then becomes discontinuous, but the whole space can nevertheless be covered with a single patch. Because of this feature, we hope our potential will be easier to work with in practical situations.

**Note added :** After submitting this paper to the hep-th archive, we came to know through Douglas Singleton that Eq. (8) appears in Ref. [7]. The related physics issues have no mention there.

**References**

[1] P. A. M. Dirac, Proc. Roy. Soc. London A133, 60 (1931).

[2] M. N. Saha, Ind. J. Phys. 10, 141 (1936); Phys. Rev. 75, 1968 (1949).

[3] H. A. Wilson, Phys. Rev. 75, 309 (1949).

[4] S. R. Coleman, in “Les Houches 1981, Proceedings, Gauge Theories In High Energy Physics, Part 1”, pp 461-552.

[5] T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3845 (1975).

[6] T. T. Wu and C. N. Yang, Nucl. Phys. B 107, 365 (1976).

[7] G. B. Arfken, H-J. Weber: “Mathematical Methods for Physicists” (Harcourt/Academic Press), 5th Edition, page 130.