1. Introduction

This paper is the fourth one in a series of papers, [24], [34], [35], which aims to a better understanding of the phenomenon of common hypercyclic vectors for uncountable many families of hypercyclic operators of translation type. The notion of hypercyclicity has been studied intensively the last 30 years and there is by now a well developed theory on this subject, see for instance the two important recent books [9], [27] and the survey of Gross-Erdmann [26]. Let us recall the relevant definitions. A sequence \( T_n \) of continuous and linear operators acting on a real or complex topological vector space \( X \) is called hypercyclic, if there exists a vector \( x \in X \) so that the set \( \{ T_n x : n = 1, 2, \ldots \} \) is dense in \( X \); in this case \( x \) is called hypercyclic for \( T_n \) and the symbol \( H.C(T_n) \) stands for the set of all hypercyclic vectors for \( (T_n) \). Let \( T : X \to X \) be a linear and continuous operator.

We define the iterates of \( T \) inductively as follows: \( T^1 := T \) and \( T^{n+1} = T^n o T \), \( n = 1, 2, \ldots \) where with \( T^n o T \) we mean the usual composition of the operators \( T^n \) and \( T \) for \( n = 1, 2, \ldots \). If in the previous definition, the sequence \( (T_n) \) comes from the iterates of a single operator \( T \), i.e. \( T_n = T^n, n = 1, 2, \ldots \), then \( T \) is called hypercyclic, \( x \) is called hypercyclic for \( T \) and \( H.C(T) \) denotes the set of hypercyclic vectors for \( T \). Observe that, in the above situation the topological vector space \( X \) is necessarily separable. For several examples of hypercyclic operators including many classical operators, such as weighted shifts, differential operators, adjoints of multipliers and so on, we refer to [9], [27]. We denote \( C \), the set of complex numbers and \( H.C \) the set of entire functions that is supposed endowed with the topology \( T_u \) of local uniform convergence from now on.

Our interest here lies on a particular operator, the translation operator acting on the space \( H.C \). Let \( a \in C \). For obvious reasons, the operator \( T_a : H.C \to H.C \) defined by \( T_a f(z) := f(z + a) \), for \( f \in H.C \), \( z \in C \), is called translation. A classical result due to Birkhoff [13] says that \( T_i \) is hypercyclic. This means, that there exists an entire function \( f \) whose positive integer translates approximate every entire function, i.e. the set \( \{ f(z + n): n = 1, 2, \ldots \} \) is dense in \( H.C(T_i) \). Actually for every \( a \in C \setminus \{0\} \) the translation operator \( T_a \) is hypercyclic. As an easy application of Baire’s category theorem we have the following dichotomy: if \( X \) is a separable topological vector space and \( T \) is linear and continuous operator on \( X \) then either \( H.C(T) = \emptyset \) or \( H.C(T) = G_3 \) and dense set in \( X \), see [9], [27].

Recall that a subset \( A \) of \( X \) is called \( G_3 \) if it can be written as countable intersection of open sets. Therefore, for every \( a \in C \setminus \{0\} \) the set \( H.C(T_a) \) is \( G_3 \) and dense set in \( H.C(T_a) \) and as an immediate consequence of Baire’s category theorem, we have that the set \( \bigcap_{n \geq 1} H.C(T_{an}) \) is non-empty for every sequence \( (a_n) \) of non-zero complex numbers. Our point of departure is the following extension of Birkhoff’s theorem due to Costakis and Sambarino [23]:
The set \( \cap_{a \in \mathbb{C} \setminus \{0\}} HC(T_a) \) is residual in \((H(C), T_a)\), hence non-empty.

The difficulty of proving such a result is, of course, the uncountable range of \( a \). Subsequently Costakis, in an attempt to generalize the previous result established the following:

**Theorem 1.1.** [20] Let \((\lambda_n)\) be a sequence of non-zero complex numbers with \( |\lambda_n| \to +\infty \) which also satisfies the following condition \((\Sigma)\):

**Condition \((\Sigma)\):** For every \( M > 0 \) there exists a subsequence \((\mu_n)\) of \((\lambda_n)\) such that

(i) \(|\mu_{n+1}| - |\mu_n| > M\) for every \( n = 1, 2, \ldots \) and

(ii) \( \sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty \)

Then the set \( \bigcap_{|a|=1} HC(T_{\lambda_n a}) \) is a \( G_\delta \) and dense set in \((H(C), T_a)\).

The purpose of the present work is to extend the above theorem by allowing a strictly wider class of sequences \((\lambda_n)\). However, it is necessary to impose certain restrictions on \((\lambda_n)\) so that the conclusion of the above theorem holds, as the following result from [24] shows: if \((\lambda_n)\) is a sequence of non-zero complex numbers with \( \liminf_{n \to +\infty} \frac{|\lambda_{n+1}|}{|\lambda_n|} > 2 \) then

\( \bigcap_{|a|=1} HC(T_{\lambda_n a}) = \emptyset \).

Frederic Bayart improved this result by proving that \( \bigcap_{|a|=1} HC(T_{\lambda_n a}) = \emptyset \), when \( \liminf_{n \to +\infty} \frac{|\lambda_{n+1}|}{|\lambda_n|} > 1 \).

We consider a sequence \((\lambda_n)\) of complex numbers that satisfies the following condition which we call it condition \((\Sigma_1)\) from now on.

**Condition \((\Sigma_1)\):** For every \( M > 0 \) there exists a subsequence \((\mu_n)\) of \((\lambda_n)\) such that

(i) \( \mu_1 \neq 0 \)

(ii) \(|\mu_{n+1}| - |\mu_n| > M\) for every \( n = 1, 2, \ldots \) and

(iii) \( \lim_{n \to +\infty} \left( |\mu_n| \cdot \sum_{k=n}^{+\infty} \frac{1}{|\mu_k|} \right) = +\infty \). We denote \( A_1 \) the sequences that satisfy the previous condition \((\Sigma_1)\). At this point it is instructive to observe that the sequences \((n^2), (n^3), \ldots\), satisfy condition \((\Sigma_1)\) but not condition \((\Sigma)\) of Theorem 1.1.

We denote by \( \mathcal{C}_r = \{ z \in \mathbb{C} | |z| = r \} \) the circle with center 0 and radius \( r \), for some \( r > 0 \).

**Theorem 1.2.** Fix a sequence of complex numbers \( \Lambda = (\lambda_n) \) that satisfies the above condition \((\Sigma_1)\). Then for every \( r \in (0, +\infty) \) the set \( \bigcap_{a \in \mathbb{C}_r} HC(T_{\lambda_n a}) \) is a \( G_\delta \) and dense set in \((H(C), T_a)\). In particular,

\[ \bigcap_{a \in \mathbb{C}_r} HC(T_{\lambda_n a}) \neq \emptyset \]

It is clear that property (iii) of condition \((\Sigma_1)\) relaxes the corresponding condition (ii) of \((\Sigma)\).

This differentiate Condition \((\Sigma_1)\) from condition \((\Sigma)\) because for a sequence \(|\mu_n|\) from non-zero complex numbers the series

\[ \sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} \]

has always a unique limit, finite or + infinity. This does not hold for a sequence of the form

\[ |\mu_n| \left( \sum_{k=n}^{+\infty} \frac{1}{|\mu_k|} \right), n = 1, 2, \ldots \]

as someone can see, using elementary calculus because such a sequence can have many limit points.

We note that Frederic Bayart examined in [2] similar problems in \( \mathbb{R}^n \) generally in his significant paper.

One may also wonder if property (iii) of \((\Sigma_1)\) in Theorem 1.2 can be relaxed by replacing the lim with lim sup. We prove now that this has no sense.

More specifically:

We denote \( N \), the set of natural numbers.
We consider the set $A_3$ of sequences $(\lambda_n)$ of complex numbers that satisfy the following condition which we call it Condition $(\Sigma_2)$ from now on.

**Condition $(\Sigma_2)$**. For every $M > 0$ there exists a subsequence $(\mu_n)$ of $(\lambda_n)$ such that:

(i) $\mu_1 \neq 0$

(ii) $|\mu_{n+1}| - |\mu_n| > M$ for every $n = 1, 2, \ldots$

(iii) $\limsup_{n \to +\infty} \left( |\mu_n| \cdot \sum_{k=n+1}^{+\infty} \frac{1}{|\mu_k|} \right) = +\infty$

**Lemma 1.3.** It holds: $A_2 = A_3$.

**Proof.** It is obvious that

$$A_2 \subset A_3 \quad (1.1)$$

We prove now the reverse inclusion.

Let $(\lambda_n) \in A_3$. We fix a positive number $M$. Then, there exists a subsequence $(\mu_n)$ of $(\lambda_n)$ that satisfies the three properties (i), (ii) and (iii) of condition $(\Sigma_2)$.

We distinguish two cases.

If

$$\sum_{n=1}^{+\infty} \frac{1}{\mu_n} = +\infty$$

then the sequence $(\mu_n)$ satisfies of course property (iii) of condition $(\Sigma_i)$.

We suppose that

$$\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = < +\infty$$

We denote $a_n := \sum_{k=n}^{+\infty} \frac{1}{|\mu_k|}$, for $n = 1, 2, \ldots$.

Of course $a_n \to 0$, as $n \to +\infty$.

By property (iii) of $(\Sigma_2)$ there exists $n_1 \in \mathbb{N}$ such that:

$$|\mu_{n_1}| \cdot \sum_{k=n_1}^{+\infty} \frac{1}{|\mu_k|} > 2.$$  

(1.2)

Because $|\mu_{n_1}| \cdot a_n \to 0$, as $n \to +\infty$, there exists $n_2 \in \mathbb{N}$, $n_2 > n_1$: $|\mu_{n_1}| a_{n_2+1} < 1$.  

(1.3)

By (1.2) and (1.3) we get:

$$|\mu_{n_1}| \cdot \sum_{k=n_1+1}^{n_2} \frac{1}{|\mu_k|} > 1.$$  

(1.4)

By property (iii) of $(\Sigma_2)$ there exists $n_3 \in \mathbb{N}$, $n_3 > n_2$ such that:

$$|\mu_{n_3}| \cdot \sum_{k=n_3}^{+\infty} \frac{1}{|\mu_k|} > 3.$$  

(1.5)

Because $|\mu_{n_3}| \cdot a_n \to 0$ as $n \to +\infty$, there exists $n_4 \in \mathbb{N}$, $n_4 > n_3$ such that

$$|\mu_{n_3}| \cdot a_{n_4+1} < 1.$$  

(1.6)

By (1.5) and (1.6) we get:

$$|\mu_{n_3}| \cdot \sum_{k=n_3+1}^{n_4} \frac{1}{|\mu_k|} > 2.$$  

(1.7)

Inductively we construct a subsequence $(\mu_{n_p})$, $p = 1, 2, \ldots$ of $(\mu_n)$ such that:

$$|\mu_{n_{2p-1}}| \cdot \sum_{k=n_{2p-1}}^{n_{2p}} \frac{1}{|\mu_k|} > \rho \quad \text{for every} \quad \rho = 1, 2, \ldots$$

(1.8)

Of course the subsequence $(\mu_{n_p})$, $p = 1, 2, \ldots$ of $(\mu_n)$, $n = 1, 2, \ldots$ satisfies the properties (i), (ii) and (iii) of $(\Sigma_i)$. This shows that $A_3 \subset A_2$, that completes the proof.

It is important to mention that in [35] we obtain a full strength of the conclusion of Theorem 1.1, namely we show that under the assumptions of Theorem 1.1 the set
is $G_δ$ and dense set in $(H(C), T_u)$. On the other hand, relaxing condition (iii) of Theorem 1.1 as above, the price we pay, at least for now, is the “thin”, but still uncountable, range of $a$ in the conclusion of Theorem 1.2. A further connection of the present work with our main result from [34] will be discussed in Section 5.

There are several recent results concerning either the existence or the non-existence of common hypercyclic vectors for uncountable families of operators, see for instance, [1]-[12], [14]-[25], [27]-[29], [31]-[35].

The paper is organized as follows. Sections 2-4 contain the proof of Theorem 1.2. In the last section, Section 5, we connect our work with the main results from [34], [35].

2. Three Basic Lemmas

We fix a positive number $r_0$.

Let us now describe the main steps for the proof of Theorem 1.2. Defining the arcs $A_k:= \{a \in \mathbb{C}| \text{there exists } t \in \left[\frac{k}{4}, \frac{k+1}{4}\right] \text{ such that } a = r_0 e^{2\pi it}, k=0,1,2,3\}$ and using Baire’s category theorem we easily see that Theorem 1.2 reduces to the following

**Proposition 2.1.** Fix a sequence $(\lambda_n)$ of complex numbers that satisfies the above condition $(\Sigma_1)$. Fix three real numbers $r_0$, $\theta_0$, $\theta_1$ such that $r_0 \in (0, +\infty)$,

$0 \leq \theta_0 < \theta_1 \leq 1,$

$\theta_f - \theta_o = \frac{1}{4}$ and consider the arc $A$ defined by

$A = \{a \in \mathbb{C}| \text{there exists } t \in [\theta_0, \theta_1] \text{ such that } a = r_0 e^{2\pi it}\}.$

Then $\bigcap_{a \in A} HC(T_{\lambda_n a})$ is a $G_δ$ and dense subset of $(H(C), T_u)$. For the proof of Proposition 2.1 we introduce some notation which will be carried out throughout this paper. Let $(p_j), j = 1, 2, \ldots$ be a dense sequence of $(H(C), T_u)$, (for instance, all the polynomials in one complex variable with coefficients in $(Q + iQ)$ where $Q$ is the set of rational number. For every $m, j, s, k \in \mathbb{N}$ we consider the set

$E(m, j, s, k) := \{f \in H(C)| \forall a \in A, \exists n \in \mathbb{N}, n \leq m: \sup_{|z| \leq k} |f(z + \lambda_n a) - p_j(z)| < \frac{1}{5}\}$

Clearly, Baire’s category theorem and the three lemmas stated below imply Proposition 2.1.

**Lemma 2.1**

$$\bigcap_{a \in A} HC(T_{\lambda_n a}) = \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} E(m, j, s, k)$$

**Lemma 2.2.** For every $m, j, s, k \in \mathbb{N}$ the set $E(m, j, s, k)$ is open set in $(H(C), T_u)$.

**Lemma 2.3.** For every $j, s, k \in \mathbb{N}$ the set $U_{m=1}^{\infty} E(m, j, s, k)$ is dense set in $(H(C), T_u)$.

The proof of Lemma 2.1 is in [34]. The proof of Lemma 2.2 is similar to that of Lemma 9 in [22] and it is omitted. It remains to prove Lemma 2.3. This will be done in Sections 3 and 4 [23].

3. Construction of the Partition and the Disks

Our main task is to prove Lemma 2.3. This means of course to find a natural number $m$ and an entire function $f$ such that the inequality that is described in the set $E(m, j, s, k)$ is satisfied for every $a \in A$. If the set $A$ is finite this can be done easily if we take different terms from the sequence $(\lambda_n)$ that are far away each-other. However, if $A$ is uncountable as an arc we cannot apply the previous method.

We remark that if the inequality $\sup_{|z| \leq k} |f(z + \lambda_n a) - p_j(z)| < \frac{1}{5}$ is satisfied for some value $a \in A$, then the same inequality is satisfied for values of $a$ that are close to $a$. So the basic idea is to choose a suitable partition from finite many values of $a$ that satisfy the desired inequality and wait that all the others satisfy the some inequality with the same term of the sequence $(\lambda_n)$ of some of the finite many prescribed values.
In a point of the proof it is needed, for technical reasons to have that the
\[ \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \] be large enough for some suitable subsequence \((k_n) \) \( n = 1, 2, \ldots \), from natural numbers.

Of course if we have \( \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} < +\infty \), then we cannot wait to have a progress, for example for the sequence \( \lambda_n = n^2 \), \( n = 1, 2, \ldots \).

For this reason the method in [20] and [23] cannot be applied here. However, if it is permitted to take in the sum \( \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \) some terms many times then there will be the possibility the sum \( \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \) to become large enough. This exactly is the key point of our proof here.

After we choose suitable terms of the sequence \((\lambda_n)\) and some partition from values \( a \in A \) and we apply Runge’s Theorem on convenient discs with center in the point \( \lambda_0 \) and the same radius \( R \) that is assigned by the problem.

More specifically:

Let \( A = (\lambda_n) \) be a sequence of non-zero complex numbers such that satisfies condition \((\Sigma_1)\). Let three real numbers \( \theta_0, \theta_1, r_0 \) be as in Proposition 2.1. For the sequel we fix four positive numbers \( c_1, c_2, c_3, c_4 \) such that \( c_2 > 1 \), \( c_1 > 1 \), \( c_4 > 1 \), where \( c_3 := \frac{c_4}{r_0 c_2} \), \( c_1 := 4(c_3 + 1) \). After the definition of the above numbers we fix a subsequence \((\mu_n)\) of \((\lambda_n)\) such that:

\[ |\mu_n|, |\mu_{n+1}| > c_1 \] for every \( n = 1, 2, \ldots \) and
\[ \lim_{n \to +\infty} \left( |\mu_n| \cdot \sum_{k=n}^{+\infty} \frac{1}{|\mu_k|} \right) = +\infty. \]

The condition \( \lim_{n \to +\infty} \left( |\mu_n| \cdot \sum_{k=n}^{+\infty} \frac{1}{|\mu_k|} \right) = +\infty \) implies that there exists a positive integer \( m_0 \) such that for every \( m \geq m_0 \)
\[ \sum_{k=m}^{+\infty} \frac{1}{|\mu_k|} > c_3 \cdot \frac{1}{|\mu_m|} \]

Throughout Section 3 the positive integer \( m_0 \) will appear frequently and it is fixed from now on.

3.1. Step 1. Partitions of the Interval \([\theta_0, \theta_1]\)

For every sufficiently large positive integer \( m \) (actually \( m \geq m_0 \)) we shall construct a corresponding partition \( \Delta_m \) of \([\theta_0, \theta_1]\). For every \( m \geq m_0 \) let \( m_1(m) \) be the minimum positive integer such that
\[ \sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|} > c_3 \cdot \frac{1}{|\mu_m|}. \] (3.1)

Obviously \( m_1(m) \geq m+1 \) for every \( m=m_0, m_0 + 1, \ldots \), since \( c_3 > 1 \).

Let \( m \) be any positive integer with \( m \geq m_0 \).

We define the numbers \( \theta_0^{(m)} := \theta_0, \theta_1^{(m)} := \theta_0 + \frac{c_2}{|\mu_m|}, \theta_2^{(m)} := \theta_1 + \frac{c_2}{|\mu_{m+1}|}, \ldots, \theta_{m_1(m)-m+1}^{(m)} := \theta_{m_1(m)-m} + \frac{c_2}{|\mu_{m_1(m)}|} \) or in a more compact form
\[ \theta_{n+1}^{(m)} := \theta_n^{(m)} + \frac{c_2}{|\mu_{m+n}|} \] for \( n=0,1, \ldots, m_1(m) - m \). (3.2)

We denote
\[ \sigma_m := \theta_{m_1(m)-m+1}^{(m)} - \theta_0 \]

Now let some positive integer \( \nu > m_1(m) - m + 1, \nu \in \mathbb{N} \). Then there exists a unique pair \((k, j) \in \mathbb{N}^2\), where \( j \in \{0, 1, \ldots, m_1(m) - m\} \) such that:
\[ \nu = k(m_1(m) - m + 1) + j. \]

Define
\[ \theta_{\nu}^{(m)} := \theta_j^{(m)} + k \sigma_m \] for \( \nu > m_1(m) - m + 1 \)

It is obvious that \( \lim_{\nu \to +\infty} \theta_{\nu}^{(m)} = +\infty \) and the sequence \( \left( \theta_{\nu}^{(m)} \right), \nu \in \mathbb{N} \) is strictly increasing.
with respect to \( v \). So there exists the maximum natural number \( v_m \in \mathbb{N} \) such that \( \theta_{v_m}^{(m)} \leq \theta_T \). We set
\[
\Delta_m := \{ \theta_0^{(m)}, \theta_1^{(m)}, \ldots, \theta_{v_m}^{(m)}, \theta_T \} \text{ if } \theta_{v_m}^{(m)} < \theta_T \quad \text{or}
\]
\[
\Delta_m := \{ \theta_0^{(m)}, \theta_1^{(m)}, \ldots, \theta_{v_m}^{(m)} \} \text{ if } \theta_{v_m}^{(m)} = \theta_T
\]

3.2. Step 2. A Lower Bound for the Stopping Time \( v_m \)

**Lemma 3.1.** Let \( m \in \mathbb{N} \) with \( m \geq m_0 \). Then \( \sigma_m := \theta_0^{(m)} - \theta_0 < 1/4 \).

*Proof.* By the definition of the numbers \( \theta_j^{(m)} j = 0, 1, \ldots, m_1(m) - m + 1 \) we have
\[
\theta_{m_1(m) - m + 1}^{(m)} - \theta_0 = c_2 \cdot \frac{\sum_{k=m}^{m_1(m)} 1}{|\mu_k|} \tag{3.3}
\]
In order to bound the right hand term in the equality above, observe that
\[
\sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|} \leq c_3 \cdot \frac{1}{|\mu_m|} + \frac{1}{|\mu_{m+1}|} < (c_3 + 1) \frac{1}{|\mu_m|} \tag{3.4}
\]
which follows from the definition of the number \( m_1(m) \). Since \( c_1 = 4c_3 + 1 \)
\[
c_2 \in (0, 1) \text{ and } |\mu_m| > c_1, \text{ we deduce that}
\]
\[
c_3+1 \frac{1}{|\mu_m|} < \frac{1}{4c_2} \tag{3.5}
\]
Thus, by (3.3), (3.4) and (3.5), \( \sigma_m < \frac{1}{4} \) and the proof is complete.

3.3. Step 3. Partitions of the Arc \( \Phi_{r_0} ([\theta_0, \theta_T]) = A \)

Consider the function \( \phi : [\theta_\sigma, \theta_T] \times (0, +\infty) \to \mathbb{C} \) given by
\[
\phi(t, r) := r e^{2\pi i t}, \quad (t, r) \in [\theta_0, \theta_T] \times (0, +\infty)
\]
and for \( r_0 > 0 \) we define the corresponding curve \( \Phi_{r_0} : [\theta_0, \theta_T] \to \mathbb{C} \) by
\[
\Phi_{r_0}(t) := \phi(t, r_0), \quad t \in [\theta_0, \theta_T]
\]
For any given positive integer \( m \geq m_0 \), \( \Phi_{r_0} (\Delta_m) \) defines a partition of the arc \( \Phi_{r_0} ([\theta_0, \theta_T]) \), where \( m_0 \) is the positive integer defined above in Step 1 and \( \Delta_m \) is the partition of the interval \([\theta_0, \theta_T]\) constructed in Step 1. For every \( m \in \mathbb{N} \) with \( m \geq m_0 \) define
\[
P_m := \Phi_{r_0} (\Delta_m)
\]
which we call partition of the arc \( \Phi_{r_0} ([\theta_0, \theta_T]) = A \).

3.4. Step 4. Construction of the Disks

Our task in this subsection is to assign to each point \( w \) of the partition \( P_m \) for \( m \geq m_0 \) a suitable closed disk with center \( w\mu(w) \) and radius \( c_4 \) (the radius will be the same for every member of the family of the disks), where \( \mu(w) \) will be chosen from the sequence \( (\mu_n) \). We shall see that, the construction of the partition \( P_m \) ensures on the one hand that the points of the partition are close enough to each other on the arc \( A \) and on the other hand that the disks centered at the points \( w\mu(w) \) for \( w \in P_m \) with fixed radius \( c_4 \) are pairwise disjoint.

We set
\[
B := \{ z \in \mathbb{C} | |z| \leq c_4 \}
\]
and fix any positive integer \( m \) with \( m \geq m_0 \). Let \( w \) be an arbitrary point in \( P_m \). There exists unique \( n \in \{ 0, 1, \ldots, v_m \} \) such that \( w = r_0 e^{2\pi i \mu_m} \). Now there exists unique \( k \in \mathbb{N} \cup \{ 0 \} \), and \( j \in \{ 0, 1, \ldots, m_1(m) - m \} \) such that \( n = k (m_1(m) - m + 1) + j \) and define
\[
\mu(w) := \mu_{n+j}.
\]
Thus we assign, in a unique way, a term of the sequence \( (\mu_n) \) to every point of \( P_m \) and more specifically a term of the finite set \( \{ \mu_n, \mu_{n+1}, \ldots, \mu_{m_1(m) - m} \} \) and we introduce the notation
\[
B_w := B + w\mu(w),
\]
to take our construction. The desired disks are the disks \( B \) and \( B_w, w \in P_m \). Denote by
\[
D_m := \{ B \} \cup \{ B_w : w \in P_m \}
\]
the collection of the above disks.

**Remark.** Since \( \mu(w) = \mu_{w,j} \) does not depend on \( K \), this means that several centers \( w\mu(w) \) are on the same disk centered in 0.

### 3.5. Step 5. The Disks are Pairwise Disjoint

**Lemma 3.2.** Let \( m \in \mathbb{N} \) with \( m \geq m_0 \). Then \( B \cap B_w = \emptyset \) for every \( w \in P_m \).

**Proof.** We have \( c_3 = \frac{c_4}{r_0 c_2} > \frac{c_4}{r_0} \) since \( c_2 \in (0, 1) \), and taking into account \( c_1 = 4 (c_3 + 1) > 2c_3 \) we get

\[
c_1 = \frac{2c_4}{r_0} \tag{3.6}
\]

Let \( w \in P_m \). The closed disks \( B, B_w \) are centered at 0, \( w\mu(w) \) respectively and they have the same radius \( c_4 \). Hence, we have to show that \( |w\mu(w)| > 2c_4 \).

Since \( |w| \geq r_0 \), it suffices to prove that \( |\mu(w)| > \frac{2c_4}{r_0} \).

Observe now that, by the definition of \( \mu(w) \) in the previous subsection, \( \mu(w) = \mu_n \) for some positive integer \( n \in \mathbb{N} \). As a consequence of the definition of the sequence \( (\mu_n) \) we have

\[
|\mu_n| > c_1 \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.7}
\]

By (3.6), (3.7) and (3.8) we conclude that \( |\mu(w)| > 2 \frac{c_4}{r_0} \) and this finishes the proof of the lemma.

**Lemma 3.3.** Let \( m \in \mathbb{N} \) with \( m \geq m_0 \) and \( w_1, w_2 \in P_m \) with \( w_1 \neq w_2 \). Then

\[
B_{w_1} \cap B_{w_2} = \emptyset
\]

**Proof.** We distinguish two cases:

(i) \( |\mu(w_1)| < |\mu(w_2)| \).

Our hypothesis implies

\[
|\mu(w_2) - \mu(w_1)| / \geq |\mu(w_2) - |w_2\mu(w_2)| / - |\mu(w_1)| / | = r_0 (|\mu(w_2)| / - |\mu(w_1)| / ) \geq r_0 c_1 > 2c_4,
\]

where the last inequality above follows by (3.6).

Thus \( B_{w_1} \cap B_{w_2} = \emptyset \).

(ii) \( |\mu(w_1)| = |\mu(w_2)| \).

By the definition of the partition \( P_m \) we have \( w_1 = r_0 e^{2\pi i(\theta_1^{(m)})}, w_2 = r_0 e^{2\pi i(\theta_2^{(m)})} \) for some \( n_1, n_2 \in \{0, 1, \ldots, v_m\} \) and \( n_1 \neq n_2 \) because \( w_1 \neq w_2 \). Without loss of generality we suppose that \( n_1 < n_2 \).

Now there exists a unique pair \((k_1, j_1)\), where \( k_1 \in \mathbb{N} \cup \{0\}, j_1 \in \{0, 1, \ldots, m(m) - m\} \) and a unique pair \((k_2, j_2)\) where \( k_2 \in \mathbb{N} \cup \{0\} \) and \( j_2 \in \{0, 1, \ldots, m(m) - m\} \) such that

\[
n_1 = k_1 (m(m) - m + 1) + j_1 \tag{3.9}
\]

and

\[
n_2 = k_2 (m(m) - m + 1) + j_2 \tag{3.10}
\]

By definition we have \( \mu(w_1) = \mu_{m+j_1}, \mu(w_2) = \mu_{m+j_2} \) and the hypothesis yields

\[
|\mu(w_1)| = |\mu(w_2)| \iff \mu(w_1) = \mu(w_2).
\]

So we have \( j_1 = j_2 \). Thus

\[
\theta_{n_1}^{(m)} = \theta_{j_1}^{(m)} + k_1 \sigma_m,
\]

\[
\theta_{n_2}^{(m)} = \theta_{j_1}^{(m)} + k_2 \sigma_m
\]

and

\[
\theta_{n_2}^{(m)} - \theta_{n_1}^{(m)} = (k_2 - k_1) \sigma_m. \tag{3.11}
\]

By (3.9), (3.10) and the fact that \( n_1 < n_2 \) and \( j_1 = j_2 \) we have \( k_1 < k_2 \Rightarrow k_2 \geq k_1 + 1 \).

Using now (3.11) it follows that
\[ \theta^{(m)}_{n_2} - \theta^{(m)}_{n_1} \geq \sigma_m > 0. \]  

(3.12)

A lower bound for the quantity \(|w_2 \mu (w_2) - w_1 \mu (w_1)|\) is:

\[
|w_2 \mu (w_2) - w_1 \mu (w_1)| = |\mu (w_1)||w_1 - w_2| \geq |\mu_m||w_1 - w_2| = r_0 e^{2 \pi i \theta^{(m)}_{n_2}} - r_0 e^{2 \pi i \theta^{(m)}_{n_1}} = r_0 |\mu_m| e^{2 \pi i \theta^{(m)}_{n_2}} - e^{2 \pi i \theta^{(m)}_{n_1}} = r_0 |\mu_m| 2 \sin(\pi(\theta^{(m)}_{n_2} - \theta^{(m)}_{n_1})). \]

(3.13)

We remind that by the suppositions of Proposition 2.1 we have \( \theta_T - \theta_0 = \frac{1}{4} \), so

\[
\theta_0 \leq \theta^{(m)}_{n_1} < \theta^{(m)}_{n_2} \leq \theta_\gamma \Rightarrow \theta^{(m)}_{n_2} - \theta^{(m)}_{n_1} \leq \frac{1}{4}.
\]

The last inequality implies the last equality of (3.13). Consider Jordan’s inequality

\[
\sin x > \frac{2}{\pi} x, \quad x \in (0, \frac{\pi}{2}).
\]

We have

\[
0 < \theta^{(m)}_{n_2} - \theta^{(m)}_{n_1} \leq \frac{1}{4} \Rightarrow 0 < \pi(\theta^{(m)}_{n_2} - \theta^{(m)}_{n_1}) \leq \frac{\pi}{4}.
\]

So, applying Jordan’s inequality for \( x = \pi(\theta^{(m)}_{n_2} - \theta^{(m)}_{n_1}) \) we derive

\[
\sin \left( \pi(\theta^{(m)}_{n_2} - \theta^{(m)}_{n_1}) \right) > 2(\theta^{(m)}_{n_2} - \theta^{(m)}_{n_1}).
\]

(3.14)

Now, inequalities (3.12), (3.13) and (3.14) imply

\[
|w_2 \mu (w_2) - w_1 \mu (w_1)| > 4r_0 |\mu_m| \sigma_m.
\]

(3.15)

By the definition of the number \( \sigma_m \) and relation (3.3) of Lemma 3.1 we obtain

\[
\sigma_m = c_2 \sum_{k=m}^{m_1(m)} \frac{1}{|m_k|}.
\]

The last equality, inequality (3.15) and the definition of the number \( m_1(m) \) give

\[
|w_2 \mu (w_2) - w_1 \mu (w_1)| > 4r_0 |\mu_m| c_2 \sum_{k=m}^{m_1(m)} \frac{1}{|m_k|} > 4r_0 |\mu_m| c_2 \frac{C_3}{|\mu_m|} = 4r_0 c_2 c_3
\]

and the properties of our fixed numbers imply \( 4r_0 c_2 c_3 > 2c_4 \).

It follows that \( B_{w_1} \cap B_{w_2} = \emptyset \) and the proof of this lemma is complete.

By Lemmas 3.2, 3.3 we conclude the following.

**Corollary 3.1.** For every positive integer \( m \) with \( m \geq m_0 \) the family \( D_m := \{ B \} \cup \{ B_w : w \in P_m \} \) consists of pairwise disjoint disks.

### 4. Proof of Lemma 2.3

Let \( j_1, s_1, k_1 \in \mathbb{N} \) be fixed. We will prove that the set \( \bigcup_{m=1}^{\infty} E (m, j_1, s_1, k_1) \) is dense in \((H(C), T_0)\). For simplicity we write \( p_{j_1} = p \). Consider fixed \( g \in H(C) \), a compact set \( K \subseteq C \) and \( \varepsilon_0 > 0 \). We seek \( f \in H(C) \) and a positive integer \( m_1 \) such that

\[
f \in E (m_1, j_1, s_1, k_1)
\]

(4.1)
\[
\sup_{z \in \mathbb{C}} |f(z) - g(z)| < \varepsilon_0
\] 

(4.2)

Fix \( R_1 > 1 \) sufficiently large so that
\[
K \cup \{ z \in \mathbb{C} \mid |z| \leq k_1 \} \subset \{ z \in \mathbb{C} \mid |z| \leq R_1 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)

We set
\[
B := \{ z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0 \}.
\]

and then choose \( 0 < \delta_0 < 2 \) such that
\[
\text{if } |z| \leq R_1 \text{ and } |z \cdot w| < \delta_0, \ w \in \mathbb{C}, \ \text{then } |p(z) - p(w)| < \frac{1}{2S_1}.
\] 

(4.3)
which implies the desired inequality (4.2).

It remains to show (4.1). Let \( \alpha \in A \). There exists a unique \( \theta \in [\theta_0, \theta_T] \) such that \( \alpha = r_0 e^{2 \pi i \theta} \). Now there exists unique \( \rho \in \{0, 1, \ldots, \nu_m - 1\} \) such that:

\[
either \theta^{(m)}_\rho \leq \theta < \theta^{(m)}_{\rho+1} \text{ or } \theta^{(m)}_{\nu_m} \leq \theta \leq \theta_T.
\]

and we then define

\[
\theta_1 := \theta^{(m)}_\rho \text{ and } \theta_2 := \theta^{(m)}_{\rho+1} \text{ if } \theta^{(m)}_\rho \leq \theta < \theta^{(m)}_{\rho+1},
\]

\[
\theta_1 := \theta^{(m)}_{\nu_m} \text{ and } \theta_2 := \theta_T \text{ if } \theta^{(m)}_{\nu_m} \leq \theta \leq \theta_T.
\]

For the above, recall that the definitions of \( \nu_m \) and \( \theta^{(m)}_\rho \), \( \rho \in \{0, 1, \ldots, \nu_m - 1\} \) are defined in Section 3. Set \( w_0 : = r_0 e^{2 \pi i \theta_1} \in P \). We shall prove that for every \( z \in C, |z| \leq R_1 \) we have \( z + \alpha \mu (w_0) \in B_{w_0} \).

Recall that \( B_{w_0} := B + (w_0 \mu (w_0) = B \mu (w_0), R_1 + \delta_0) \). It is suffices to prove that

\[
|z + \alpha \mu (w_0) - w_0 \mu (w_0)| < R_1 + \delta_0 \text{ for } |z| \leq R_1.
\]  

(4.5)

For \( |z| \leq R_1 \) we have:

\[
|z + \alpha \mu (w_0) - w_0 \mu (w_0)| + |\mu (w_0)| |\alpha - w_0| = R_1 + |\mu (w_0)||r_0 e^{2 \pi i \theta} - r_0 e^{2 \pi i \theta_1}|.
\]  

(4.6)

By (4.5) and (4.6) it suffices to prove

\[
|\mu (w_0)||r_0 e^{2 \pi i \theta} - r_0 e^{2 \pi i \theta_1}| < \delta_0.
\]  

(4.7)

We have:

\[
|r_0 e^{2 \pi i \theta} - r_0 e^{2 \pi i \theta_1}| = \left| r_0 e^{2 \pi i \theta} - e^{2 \pi i \theta_1} \right|
\]

\[
= r_0 2 \sin\left( \pi (\theta - \theta_1) \right)
\]

\[
\leq r_0 2 \sin\left( \pi (\theta_2 - \theta_1) \right)
\]

\[
\leq 2 r_0 \pi (\theta_2 - \theta_1)
\]

\[
\leq 2 r_0 \pi \frac{\delta_0}{2 |\mu (w_0)|} \leq \frac{\delta_0}{2 |\mu (w_0)|} < \frac{\delta_0}{2 |\mu (w_0)|},
\]

which implies (4.7). So, for every \( z \in C, |z| \leq R_1 \)

\[
z + \alpha \mu (w_0) \in B_{w_0}\]

(4.8)

The definition of \( h \) and (4.8) give that for every \( z \in C, |z| \leq R_1 \)

\[
|f (z + \alpha \mu (w_0)) - p (z + \mu (w_0)(r_0 e^{2 \pi i \theta} - r_0 e^{2 \pi i \theta_1}))| < \frac{1}{2 S_1}.
\]  

(4.9)

By (4.3) and (4.7) we get: for every \( z \in C, |z| \leq R_1 \)

\[
|p(z + \mu (w_0)(r_0 e^{2 \pi i \theta} - r_0 e^{2 \pi i \theta_1})) - p(z)| < \frac{1}{2 S_1}
\]  

(4.10)

The triangle inequality and the fact that \( k_1 \leq R_1 \) give

\[
\sup_{|z| \leq k_1} |f (z + \alpha \mu (w_0)) - p (z)| < \frac{1}{S_1}
\]  

(4.11)

Setting

\[
m_1 := \max \{n \in \mathbb{N} | \lambda_n = \mu (w), \text{ for some } w \in P \},
\]

observing that the definition of \( m_1 \) is independent of \( \alpha \in A \) and in view of (4.11) we conclude that for every \( \alpha \in A \) there exists some \( n \in \mathbb{N} \) with \( n \leq m_1 \) such that

\[
\sup_{|z| \leq k_1} |f (z + \alpha \lambda_n) - p(z)| < \frac{1}{S_1},
\]

where \( f \in H (C) \), since \( f \) is a polynomial. This implies (4.1) and the proof of the lemma is complete.
5. Examples of Sequences $A := (\lambda_n)$ Satisfying Condition $(\Sigma)$

In this section we show that our main theorem is not covered by our recent results in [34], [35]. Recall that a sequence of non-zero complex numbers $(\lambda_n)$ with $|\lambda_n| \to \infty$ satisfies condition $(\Sigma)$ if:

- for every $M > 0$ there exists a subsequence $(\mu_n)$ of $(\lambda_n)$ such that
  \[
  \sup_{n \in \mathbb{N}} |\lambda_{n+1} - \lambda_n| < +\infty
  \]

(i) $|\mu_{n+1} - \mu_n| > M$ for every $n = 1, 2, \ldots$ and

(ii) $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$.

For instance, sequences of linear growth i.e. $\lambda_n = an + b$, $a, b \in \mathbb{C}$, $a \neq 0$, $n = 1, 2, \ldots$, satisfy condition $(\Sigma)$, or sequences $(\lambda_n)$, such that $\sup_{n \in \mathbb{N}} |\lambda_{n+1} - \lambda_n| < +\infty$.

We introduce the following definitions.

$L := \{A = (\lambda_n) \in \mathbb{C}^N \}$

$L' := \{A \in L \mid \lambda_n \neq 0, \forall n = 1, 2, \ldots, |\lambda_n| \to \infty\}$

$A_1 := \{A = (\lambda_n) \in L' \mid A$ satisfies condition $(\Sigma)$\},

$A_2 := \{A = (\lambda_n) \in L \mid A$ satisfies condition $(\Sigma)$\}.

For $A \in L'$, define

\[ B(A) := \{\alpha \in [0, +\infty] \mid \alpha = \lim_{n \to +\infty} \sup_{n=1}^{+\infty} \frac{|\mu_{n+1}|}{|\mu_n|} \text{ for some subsequence } (\mu_n) \text{ of } A\} \]

and

\[ i(A) := \inf_{B(A)} \]

Clearly,

\[ i(A) \in [1, +\infty] \text{ for every } A \in L'. \]

Our main results in [34], [35] are the following

**Theorem 5.1** ([34]). If $A := (\lambda_n) \in L'$ and $i(A) = 1$ then $\bigcap_{n \in \mathbb{C} \setminus \{0\}} HC(T_{\lambda_n \alpha})$ is a $G_\delta$ and dense subset of $(H(C), T_\alpha)$.

**Theorem 5.2** ([35]). If $A := (\lambda_n) \in A_1$ then $\bigcap_{n \in \mathbb{C} \setminus \{0\}} HC(T_{\lambda_n \alpha})$ is a $G_\delta$ and dense subset of $(H(C), T_\alpha)$.

In order to come closer to a complete picture of our investigations we introduce one last class of sequences. Set

$A_4 := \{A = (\lambda_n) \in L' \mid i(A) = 1\}$.

**Proposition 5.1.** $A_4 \subseteq A_2$

**Proof.** Let $A = (\lambda_n) \in A_4$ and fix a positive number $M$. By Lemma 7.1 in [34] there exists a subsequence $(\mu_n)$ of $(\lambda_n)$ such that $|\mu_{n+1} - \mu_n| > M$ for every $n = 1, 2, \ldots$ and $|\mu_{n+1}|/|\mu_n| \to 1$ as $n \to +\infty$. If we show that $|\mu_n| \sum_{k=1}^{+\infty} |\mu_k|^{-1} \to +\infty$ as $n \to +\infty$ then $A \in A_2$.

To this end, let $A > 0$ and consider any positive number $\varepsilon$ so that $\varepsilon < 1/A$. Since $|\mu_{n+1}|/|\mu_n| \to 1$ as $n \to +\infty$ there exists a positive integer $N$ such that $|\mu_n/\mu_{n+1}| > 1/(1 + \varepsilon)$ for every $n \geq N$.

Repeated using of the previous inequality gives

\[ \left| \frac{\mu_n}{\mu_{n+v}} \right| > \left( \frac{1}{1 + \varepsilon} \right)^v, \quad n \geq N, \quad v = 1, 2, \ldots \]

and upon summation we get

\[ \sum_{v=1}^{+\infty} \left| \frac{\mu_n}{\mu_{n+v}} \right| \geq \frac{1}{\varepsilon} > A, \quad n \geq N \]

and this completes the proof.

From the above we have $A_1 \cup A_4 \subseteq A_2$. 

In view of Theorem 5.1 and in order to completely characterize the sequences \( A := (\lambda_n) \in L \) such that the set \( \cap_{\alpha \in C(\Omega)} HC(T_{\lambda_n}) \) is a \( G_{\delta} \) and dense in \( (H(C), T) \), one has to deal with sequences \( A \in L \) for which \( i(A) > 1 \). This is one of the reasons we introduced the classes \( A_1, A_2 \). Indeed, it is established in [35] that the class \( A_1 \) contains sequences \( A \in L \) with \( i(A) > 1 \). On the other hand, there exist sequences \( A \in L' \) with \( i(A) = 1 \) and \( A \notin A_1 \), see [35]. Since,
\[ A_1 \subset A_2 \]
we conclude that the class \( A_2 \) contains sequences \( A \in L \) with \( i(A) > 1 \). The above inclusion is strict; for instance, the sequence \( \lambda_n = n^2, n = 1, 2, \ldots \), belongs to \( A_2 \) but not in \( A_1 \). However \( i((n^2)) = 1 \), therefore for this sequence the conclusion of Theorem 5.1 holds and of course in this case the conclusion of Theorem 1.2 is covered by the much stronger Theorem 5.1. So the interest here is to show that there exists \( A \in A_2 \setminus A_1 \) with \( i(A) = M \) for some positive real number \( M > 1 \), and this in turn shows that our main result, Theorem 1.2, is not covered by Theorems 5.1, 5.2. This is the content of the following

**Proposition 5.2.** Fix some \( M > 1 \). There exists \( A \in A_2 \setminus A_1 \) such that \( i(A) = M \).

**Proof.** We construct inductively a countable family \( \{ F_n \}, n = 1, 2, \ldots \) of finite sets \( F_n \subset [1, +\infty) \) according to the following rules. Firstly, we fix the sequence \( (\alpha_n), n = 1, 2, \ldots \) inductively as follows: \( \alpha_n = 1 \) and \( \alpha_{n+1} = \sqrt{M}(\alpha_n + [\alpha_n] + 1) \), where with \([\alpha_n]\) we mean the integer part of the real number \( \alpha_n \).

1. \( F_1 = \{1\} \).
2. \( F_n = \{(\alpha_m + v)^2| v = 0, 1, \ldots, [\alpha_n] + 1\} m = 1, 2, \ldots \)
3. \( \min F_{m+1} = \max F_m \) for each \( m = 1, 2, \ldots \).

where \( \alpha_m^2 := \min F_m m = 1, 2, \ldots \) Observe that for every \( m_1, m_2 \in N = \{m \in N | m \neq m_2, F_{m_1} \cap F_{m_2} = 0\} \). Set
\[
\Lambda = \bigcup_{n=1}^{+\infty} F_n.
\]

We define the sequence \( A = (\lambda_n) \) to be the enumeration of \( \Lambda \) by its natural order.

It is obvious that \( \lambda_n \neq 0 \) for every \( n \in N, \lim_{n \to +\infty} |\lambda_n| = +\infty \), and \( \lambda_n \) is a strictly increasing sequence of positive numbers. We divide the proof into several claims.

**Claim 1.** For every subsequence \( (\mu_n) \) of \( A \) we have
\[
\lim_{n \to +\infty} \sup \left| \frac{\mu_{n+1}}{\mu_n} \right| \geq M
\]

**Proof.** Let \( (\mu_n) \) be a fixed subsequence of \( A \). We prove that for every natural number \( n_1 \in N \), there exists \( N \in N \) with \( N \geq n_1 \) such that
\[
\left| \frac{\mu_{N+1}}{\mu_N} \right| \geq M.
\]

So, fix \( n_1 \in N \). Let \( m_2 \) be the unique positive integer such that \( \mu_{n_1} \in F_{m_2} \). We set \( A_{m_2} := \{n \in N | \mu_n \in F_{m_2}\} \). It is obvious that \( A_{m_2} \neq \emptyset \), since \( n_1 \in A_{m_2} \). We set \( m_1 := \max A_{m_2} \). Then \( \mu_{m_1} \in F_{m_2} \) and so \( \mu_{m_1} + 1 \in F_{m_2} \). We have \( \mu_{m_2} \leq \max F_{m_2} \). Thus,
\[
\frac{\mu_{m_3} + 1}{\mu_{m_3}} \geq \frac{\min F_{m_2+1}}{\max F_{m_2}} = M \quad \text{and} \quad m_3 \geq m_2
\]

So we proved that for every \( n \in N \), there exists some \( N \geq n \) such that
\[
\left| \frac{\mu_{N+1}}{\mu_N} \right| \geq M.
\]

This implies \( \lim_{n \to +\infty} \sup \left| \frac{\mu_{n+1}}{\mu_n} \right| \geq M \).

**Claim 2.** \( \lim_{n \to +\infty} \frac{\lambda_{n+1}}{\lambda_n} \geq M \).

**Proof.** It is obvious \( \lim_{n \to +\infty} \alpha_n = +\infty \).
Let \( n \in \mathbb{N} \). If there exists some \( m \in \mathbb{N} \) such that \( \lambda_n \lambda_{n+1} \in F_m \) then by the construction of \( F_m \) we remark by definition of \( F_m \) that \( \lambda_n = (\alpha_m + k)^2, \lambda_{n+1} = (\alpha_m + k + 1)^2 \) for some \( k \in \{0,1,\ldots, [\alpha_m] + 1\} \); thus

\[
\frac{\lambda_{n+1}}{\lambda_n} = \frac{(\alpha_m + k + 1)^2}{(\alpha_m + k)^2} = \left(1 + \frac{1}{\alpha_m + k}\right)^2 \leq \left(1 + \frac{1}{\alpha_m}\right)^2. \tag{5.1}
\]

If there exists no \( m \in \mathbb{N} \) such that \( \lambda_n \lambda_{n+1} \in F_m \) then this happens only if \( \lambda_n = \max F_m \) and \( \lambda_{n+1} = \min F_{m+1} \) for some \( m \in \mathbb{N} \). In this case we have

\[
\frac{\lambda_{n+1}}{\lambda_n} = M. \tag{5.2}
\]

By (5.1), (5.2) and since \( \lim_{n \to +\infty} \alpha_n = +\infty \) the conclusion follows. This completes the proof of Claim 2.

Claims 1 and 2 imply that \( i(A) = M \). We now show the following claim.

**Claim 3.** \( \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} < +\infty \)

**Proof.** Let \( m \in \mathbb{N}, m \geq 2 \). Set \( S_m := \sum_{k \in F_m} \frac{1}{k} \), that is

\[
S_m := \sum_{v=0}^{[\alpha_m]+1} \frac{1}{(\alpha_m + v)^2}.
\]

Since \( \alpha_m > 1 \), we have trivially

\[
S_m \leq \frac{[\alpha_m] + 2}{\alpha_m^2} \leq \frac{3}{\alpha_m}
\]

and by noticing that

\[
\sum_{n=1}^{+\infty} \frac{1}{\lambda_n} = 1 + \sum_{m=2}^{+\infty} S_m \leq 1 + 3 \sum_{m=2}^{+\infty} \frac{1}{\alpha_m}.
\]

Because \( \frac{\alpha_{n+1}^2}{\alpha_n^2} \geq M \) and thus \( \frac{\alpha_{n+1}^2}{\alpha_n} \geq \sqrt{M} > 1 \) the above imply Claim 3.

The above claim shows that \( A \notin A_1 \). We now show our last claim

**Claim 4.** \( A \in A_2 \).

**Proof.** Since previously, we have trivially

\[
S_m \geq \frac{[\alpha_m] + 2}{(\alpha_m + [\alpha_m] + 1)^2} \geq \frac{\alpha_m}{(3\alpha_m)^2} = \frac{1}{9\alpha_m^2}
\]

and using the fact that \( \alpha_m^2 = M(\alpha_m + [\alpha_m] + 1)^2 \) and thus \( 9M\alpha_m^2 > \alpha_{m+1}^2 \) we get if \( \lambda_n \in F_m \),

\[
\lambda_n \sum_{k=n}^{+\infty} \frac{1}{\lambda_k} \geq \alpha_m^2 S_{m+1} \geq \frac{\alpha_m^2}{9\alpha_{m+1}} \geq \frac{\alpha_{m+1}}{81M}.
\]

Finally, since \( \lambda_{n+1} - \lambda_n \to \infty \) we can deduce that the sequence \( (\lambda_n) \) satisfies condition (C). Hence \( A \in A_2 \) and the proof of this proposition is complete.

Proposition 5.2 shows that the previous inclusion \( A_1 \cup A_4 \subset A_2 \) is strict. So, at the level of a circle the main result of this paper is “strictly” stronger than the union of the main results in [34], [35].
ACKNOWLEDGEMENTS

I am grateful to George Costakis for his helpful comments and remarks and for all the help he offered me concerning the presentation of this work.

REFERENCES

[1] E. Abakumov, J. Gordon, Common hypercyclic vectors for multiples of backward shift, J. Funct. Anal. 200 (2003), 494-504.

[2] F. Bayart, Common hypercyclic vectors for high dimensional families of operators arXiv: 1503.08574, 30/03/2015

[3] F. Bayart, Common hypercyclic vectors for composition operators, J. Operator Theory 52 (2004), 353-370.

[4] F. Bayart, Topological and algebraic genericity of divergence and universality, Studia Math. 167 (2005), 161-181.

[5] F. Bayart, Common hypercyclic subspaces, Integral Equations Operator Theory 53 (2005), 467-476.

[6] F. Bayart, Dynamics of holomorphic groups, Semigroup Forum 82 (2011), 229-241.

[7] F. Bayart, G. Costakis, D. Hadjiloucas, Topologically transitive skew-products of operators. Ergodic Theory Dynam. Systems 30 (2010), 33-49.

[8] F. Bayart, S. Grivaux, Frequently hypercyclic operators, Trans. Amer. Math. Soc. 358 (2006), 5083-5117.

[9] F. Bayart and E. Matheron, Dynamics of linear operators, Cambridge Tracts in Math. 179, Cambridge Univ. Press, 2009.

[10] F. Bayart and E. Matheron, How to get common universal vectors, Indiana Univ. Math. J., 56, (2007), 553-580.

[11] F. Bayart, S. Grivaux, R. Mortini, Common bounded universal functions for composition operators, Illinois J. Math. 52 (2008), 995-1006.

[12] L. Bernal-Gonzalez, Common hypercyclic functions for multiples of convolution and non-convolution operators, Proc. Amer. Math. Soc. 137 (2009), 3787-3795.

[13] G. D. Birkhoff, Démonstration d’un théorème élémentaire sur les fonctions entières, C. R. Acad. Sci. Paris, 189 (1929), 473-475.

[14] K. C. Chan, R. Sanders, Common supercyclic vectors for a path of operators, J. Math. Anal. Appl. 337 (2008), 646-658.

[15] K. C. Chan, R. Sanders, Two criteria for a path of operators to have common hypercyclic vectors, J. Operator Theory 61 (2009), 191-223.

[16] K. C. Chan, R. Sanders, Common hypercyclic vectors for the conjugate class of a hyper-cyclic operator, J. Math. Anal. Appl. 375 (2011), 139-148.

[17] K. C. Chan, R. Sanders, An SOT-dense path of chaotic operators with same hypercyclic vectors, J. Operator Theory 66 (2011), 107-124.

[18] K. C. Chan, R. Sanders, Rebecca Common hypercyclic vectors for the unitary orbit of a hypercyclic operator, J. Math. Anal. Appl. 387 (2012), 17-23.

[19] A. Conejero, V. Müller, A. Peris, Hypercyclic behaviour of operators in a hypercyclic C0-semigroup, J. Funct. Anal., 244 (2007), 342-348.

[20] G. Costakis, Approximation by translates of entire functions, Complex and harmonic analysis, 213-219 Destech Publ., Inc., Lancaster, PA, 2007.

[21] G. Costakis, Common Cesaro hypercyclic vectors, Studia Math. 201 (2010), 203-226.

[22] G. Costakis, P. Mavroudis, Common hypercyclic entire functions for multiples of differential operators, Colloq. Math. 111 (2008), 199-203.

[23] G. Costakis and M. Sambarino, Genericity of wild holomorphic functions and common hypercyclic vectors, Adv. Math. 182 (2004), 278-306.

[24] G. Costakis, N. Tsirivas and V. Vlachou, Non-existence of common hypercyclic entire functions for certain families of translations operators, Computational Methods and Function theory, 15 (2015), 393-401.

[25] E. Gallardo-Gutierrez, J. R. Partington, Common hypercyclic vectors for families of operators, Proc. Amer. Math. Soc. 136 (2008), 119-126.

[26] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. (N.S.) 36 (1999), 345-381.

[27] K. Grosse-Erdmann and A. Peris, Linear Chaos, Universitext, 2011, Springer.

[28] F. Leon-Saavedra, Fernando, V. Müller, Rotations of hypercyclic and supercyclic operators, Integral Equations Operator Theory 50 (2004), 385-391.

[29] E. Matheron, Subsemigroups of transitive semigroups, Ergodic Theory Dynam. Systems 32 (2012), 1043-1071.

[30] W. Rudin, Real and Complex Analysis, Mc Graw-Hill, New York, 1966.

[31] R. Sanders, Common hypercyclic vectors and the hypercyclicity criterion, Integral Equations Operator Theory 65 (2009), 131-149.

[32] S. Shkarin, Universal elements for non-linear operators and their applications, J. Math. Anal. Appl. 348 (2008), 193-210.

[33] S. Shkarin, Remarks on common hypercyclic vectors, J. Funct. Anal. 258, (2010), 132-160.

[34] N. Tsirivas, Existence of common hypercyclic vectors for translation operators, arXiv:1411.7815, 1/12/2014.

[35] N. Tsirivas, Common hypercyclic functions for translation operators with large gaps, arXiv: 1412.0827, 3/12/2014.