Short Distance Behavior of (2+1)-dimensional QCD

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Abstract

Within the framework of semiclassical QCD approximations the short distance behavior of two static color charges in (2+1)-dimensional QCD is discussed. A classical linearization of the field equations is exhibited and leads to analytical results producing the static potential. Beyond the dominant classical part proportional to \(\ln \lambda R\), QCD contributions of order \(R^{1/2}\) and \(R\) are found.

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1 Introduction

During the past years many attempts have been made to approximate the theory of quantum chromodynamics (QCD) by abelianized, classical models. Especially Adler and Piran [1,2] managed to analyze the potential of two static, opposite color charges of (3+1)-dimensional QCD (QCD4) with regard to short and long distance behavior using such methods. In particular, they found a confining potential plus correction terms.

The idea consists in describing the vacuum and its properties as a dielectric medium, arising as a consequence of the quantum fluctuations of Yang-Mills fields. One handles the quantum fluctuations with the help of effective Lagrangians whose application leads to nonlinear Maxwell equations of electrostatics.

In most cases, these equations are not exactly soluble, but at least it is possible to obtain the asymptotic behavior of the theory, e.g., corrections to classical field theory.

In this work we study (2+1)-dimensional quantum chromodynamics (QCD3) which has been shown to be a confining theory by Frenkel and Silva Fo. [3].
Unlike the latter, we focus on the short distance behavior of two static QCD color charges, separated by a distance of $R$, and their static potential.

In section 2 we introduce the effective action approach to QCD$_3$. Section 3 answers the question of how the dielectric model is correctly approximated by linear electrostatics employing a formalism developed by Lehmann and Wu [4]. Finally, in section 4 we show how calculation of QCD contributions to the static potential is nevertheless possible, and we obtain the classical part proportional to $\ln \lambda R$ followed by terms of order $R^{1/2}$ and $R$.

2 Effective action in QCD$_3$

In 4-dimensional space the static potential is defined by:

$$V_{\text{static}} = -\text{extremum}_\varphi \left[ \int d^3x (\mathcal{L}(\nabla \varphi) - \varphi j_0) \right] + \Delta V_c$$

(1)

with Lagrangian density $\mathcal{L}$, potential function $\varphi$, current density $j_0$ and Coulomb counter terms $\Delta V_c$ to remove self-energies.

We calculated the 1-loop effective Lagrangian of pure QCD$_3$ by summing over normal modes of a constant color field by means of, for example, the gauge invariant $\zeta$-function regularization method. Our findings agree with Trottier’s result [11]:

$$\mathcal{L}_{\text{eff}}(\nabla \varphi) = \frac{1}{2}(\nabla \varphi)^2 \left[ 1 - \frac{4}{3} \left( \frac{(\nabla \varphi)^2}{\kappa^2} \right)^{-1/4} \right]$$

(2)

where $\kappa^2 > 0$ denotes the minimum of $\mathcal{L}_{\text{eff}}$. For example, in pure QCD$_3$ without fermions:

$$\kappa^{1/2} = g^{3/2} \frac{3}{4\pi} \left[ 1 + \frac{1}{\sqrt{2}} - \frac{\zeta(3/2)}{\sqrt{2} \ 4\pi} \right]$$

(3)

where $\zeta(x)$: Riemannian Zeta-function. Besides, $\kappa$ is not a renormalization group invariant, because there is simply no renormalization group due to the superrenormalizability of the model.

By inserting (2) into (1), we establish quantum fluctuations as a mean field background and define the leading root modell of QCD$_3$ analogous with the leading log modell of QCD$_4$ [5,6].
3 Short distance approximation

It is obvious that the effective Lagrangian (2) is already decomposed into the classical part $\propto (\nabla \varphi)^2$ and a part containing the QCD corrections $\propto [\textstyle{\frac{1}{2}}(\nabla \varphi)^2]^4$. Hence, $L_{\text{eff}}$ naturally reduces to the classical Lagrangian because of increasing field strength while $R$ decreases. In order to treat this statement more quantitatively we have to write down the field equations of our dielectric model. Employing the Euler-Lagrange equations and $L_{\text{eff}}$ (2), yields:

\begin{align*}
\nabla \cdot D &= j_0 \\
D &= \epsilon E \\
\epsilon &= 1 - \sqrt{\frac{\kappa}{E}} \\
E &= |E| \\

j_0 &= Q(\delta^2(|r_1|) - \delta^2(|r_2|)) \\
r_1 &= (x - a, y) \\
r_2 &= (x + a, y) \\
R &= 2a
\end{align*}

The difficulty of imposing boundary conditions on (4) is conveniently handled via introducing a manifestly flux conserving form [8]. In this method, $D$ is expressed as a function of the electric flux through a curve $C$ which intersects the charge axes at a point $x_s > a$ as shown the following figure (fig.1):

![Diagram](image)

fig.1

This has to be done carefully in two dimensions due to the inequality of line and surface integrals although lines and surfaces are similar objects. Explicit calculation with the aid of an arbitrary parametrization of $C$ shows that the flux definition $\Phi = \int_C D \cdot dn$ is always satisfied by the choice:

\begin{align*}
D &= \frac{1}{2} \begin{pmatrix}
\partial_y \Phi \\
-\partial_x \Phi
\end{pmatrix}
\end{align*}

Now, it is possible to incorporate boundary conditions by imposing them on
the flux function \( \Phi(x, y) \):

\[
\Phi(y = 0) = \begin{cases} 
Q & \text{if } |x| < a \\
0 & \text{if } |x| > a 
\end{cases} \tag{9}
\]

\( \Phi \to 0 \) if \( x^2 + y^2 \to \infty \)

The dynamical equation for \( \Phi \) now comes from the second equation of (4):

\( \epsilon^{ij} E_j = 0 \), also expressing \( E \) in terms of \( \Phi \) by inverting (5):

\[
E = f(D) = \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} + \kappa D + D} \tag{10}
\]

where \( D = |D| = \frac{1}{2} \sqrt{\left( \partial_x \Phi \right)^2 + \left( \partial_y \Phi \right)^2} \).

Calculating \( \epsilon^{ij} E_j = 0 \) with the help of (10) we obtain the exact quasilinear, second order differential equation of elliptic type (as Lehmann and Wu found in QCD4):

\[
0 = \left[ \Phi_x^2 + \left( \frac{\partial(D)}{\partial y} + 1 \right) \Phi_y^2 \right] \Phi_{xx} + \left[ \left( \frac{\partial(D)}{\partial x} + 1 \right) \Phi_x^2 + \Phi_y^2 \right] \Phi_{yy} - 2 \frac{\partial(D)}{\partial y} \Phi_x \Phi_y \Phi_{xy} \tag{11}
\]

where \( \Phi_x \) is a short form of \( \partial_x \Phi \) and \( g(D) = \frac{f(D)}{f'(D)} - D \) denotes a coefficient function.

Expanding \( g(D) \) in terms of \( D \) near the sources when \( D \gg \kappa \), yields:

\[
g(D) = \frac{1}{2} \sqrt{\kappa D} + \frac{\kappa}{4} + O(D^{-1/2}) \tag{12}
\]

Thus \( \frac{g(D)}{D} \) is of order \( O(D^{-1/2}) \); therefore we are allowed to omit it for short distances obtaining, not surprisingly, Laplace’s equation as approximation of (11):

\[
0 = (\partial_x^2 + \partial_y^2) \Phi \tag{13}
\]

which has the well-known solution satisfying the boundary conditions (9):

\[
\Phi = \Phi_{cl} = \frac{Q}{\pi} \left[ \arctan \left( \frac{y}{x - a} \right) - \arctan \left( \frac{y}{x + a} \right) \right] \tag{14}
\]

Furthermore we find:

\[
D = D_{cl} = \frac{Q}{2\pi} \left( \frac{\hat{r}_1 - \hat{r}_2}{r_1} \right) \tag{15}
\]
and the potential function:

\[ \varphi = \varphi_{cl} = -\frac{Q}{2\pi} \ln \frac{r_1}{r_2} = -\frac{Q}{2\pi} \ln \frac{\sqrt{(x-a)^2 + y^2}}{\sqrt{(x+a)^2 + y^2}} \]  

(16)

To summarize, we come to the hardly surprising but important conclusion that the classical linearization is appropriate for approximating the short distance behavior of two static color charges.

4 The static potential

In order to finally compute the static potential including QCD\textsubscript{3} contributions we employ another formula of \( V_{\text{static}} \) that is shown to be equal to (1) [1,2]:

\[ V_{\text{static}} = \int d^2 x \int_0^D d\tilde{D} \ f(\tilde{D}) \]  

(17)

where \( f(D) = E(D) \). Inserting (10) and expanding \( (D \gg \kappa) \) leads to:

\[ V_{\text{static}} = \int d^2 x \left[ \frac{\kappa}{2} D + \frac{2}{3\kappa} \sqrt{\frac{\kappa^2}{4} + D^2} - \frac{\kappa^2}{12} + \frac{1}{2} D^2 \right] \]

\[ = \int d^2 x \left[ \frac{1}{2} D^2 + \frac{2}{3} \sqrt{D^3/2} + \frac{\kappa}{2} D + \mathcal{O}(D^{1/2}) \right] \]  

(18)

Of course, \( V_{\text{static}}^{D^2} \) is simply the classical potential and is well known from electrostatics or might be calculated directly by partial integration and use of Poisson’s equation with due consideration of Coulomb counter terms:

\[ V_{\text{static}}^{D^2} = \frac{Q^2}{2\pi} \ln \lambda R \]  

(19)

where \( \lambda \) denotes a mass parameter without physical significance.

In the following it proves to be useful to integrate \( V_{\text{static}}^{D^2} \) first. Note that \( D = \frac{1}{2} \sqrt{(\partial_x \Phi)^2 + (\partial_y \Phi)^2} \) can be written as:

\[ D = \frac{Qa}{\pi} \left[ \left( (x-a)^2 + y^2 \right) \left( (x+a)^2 + y^2 \right) \right]^{-1/2} \]  

(20)
We are able to perform the $y$-integration without restrictions using the symmetry of our system [9]:

$$V_{\text{static}}^D = \frac{\kappa}{2} \int d^2 x \ D = 2\kappa \int_0^\infty dx \int_0^\infty dy \ D$$

$$= \frac{1}{\pi} \kappa Q R \int_0^\infty dk \ K(k), \ k = \frac{x}{a}$$

(21)

where $K(k)$ denotes the complete elliptic integral of the first kind. Now, the $k$-integration does not make sense, because $K(k)$ is well defined for $k^2 < 1$ (or $|x| < a$) only. But there is no reason for concern; as we obtain from the analysis of long distance behavior [3,10], the free boundary of confinement intersects the $x$-axes at $x_b = \pm a$ (in QCD$_3$ as well as in QCD$_4$). Simply because $D$ vanishes outside the boundary the remaining integration interval is part of the defining interval of $K(k)$. We find:

$$V_{\text{static}}^D \rightarrow \frac{1}{\pi} \kappa Q R \int_0^1 dk \ K(k) = \frac{2G}{\pi} \kappa Q R = 0.58 \ldots \kappa Q R$$

(22)

where $G$ denotes Catalan’s constant. It should not come as a surprise that we obtain a term $\propto R$ as in the case of long distances due to the appearance of a term $\propto D$, since it is responsible for the creation of linear confinement both in QCD$_3$ and QCD$_4$. Of course, here it is of less importance. Besides, the limit of integration removes divergent self-energy terms in an elegant way.

The leading order contribution to the classical potential is found by the integration of $V_{\text{static}}^{D^{3/2}}$ which has to be performed with great care. First of all we insert the potential function $\varphi$, integrate by parts and use Poisson’s equation:

$$V_{\text{static}}^{D^{3/2}} = \frac{2}{3} \sqrt{\kappa} \int d^2 x \ D^{3/2} = \frac{2}{3} \sqrt{\kappa} \int d^2 x \ \frac{\nabla \varphi \cdot \nabla \varphi}{\sqrt{D}}$$

$$= \frac{2}{3} \sqrt{\kappa} \left\{ \int d^2 x \ \nabla \cdot \left( \varphi \frac{\nabla \varphi}{\sqrt{D}} \right) + \frac{1}{2} \int d^2 x \ \varphi j_0 \right\}$$

(23)

The second integral completely disappears because the integrands behave as $\lim_{p \rightarrow 0} p \ln p$. For the remaining integral we use the same integration volume as in the case of $V_{\text{static}}^D$ which takes account of the confining boundary and automatically removes divergent self-energies. Next we take care of the specific properties of Gauss’ law in two dimensions and choose the following path of integration with the normal unit vectors pointing outwards (fig.2):
We obtain:

\[
V_{D^{3/2}}^{\text{static}} = \frac{2}{3} \sqrt{\kappa} \left\{ \int_{-a}^{a} \left( \lim_{y \to -\infty} (\hat{y}) \cdot W + \lim_{y \to \infty} (\hat{y}) \cdot W \right) \right. \\
\left. + \int_{-\infty}^{\infty} \left( \lim_{x \to -a} (\hat{x}) \cdot W + \lim_{x \to a} (\hat{x}) \cdot W \right) \right. \\
\left. + \int_{-\infty}^{\infty} \left( \lim_{x \to -a} (\hat{x}) \cdot W + \lim_{x \to a} (\hat{x}) \cdot W \right) \right\} 
\]

(24)

where \( W := \frac{\rho \nabla \rho}{\sqrt{D}} = \frac{1}{4} \sqrt{Q_3^3/\alpha^3} \ln \left( \frac{r_1}{r_2} \right) \left[ \frac{r_1 \sqrt{r_2}}{\sqrt{r_1}} - \frac{r_2 \sqrt{r_1}}{\sqrt{r_2}} \right] \)

The \( x \)-integrals vanish due to \( W \) approaching zero at infinity. There are no divergencies nor are there any analytical problems in the remaining \( y \)-integrations; hence we find \( V_{\text{static}}^{D^{3/2}} \) evaluating to:

\[
V_{\text{static}}^{D^{3/2}} = \frac{\sqrt{2}}{3} \left( \frac{\pi^2}{2} - \frac{\Psi'(1/4)}{4} \right) \sqrt{\frac{\kappa Q^3}{\pi^3}} R^{1/2} 
\]

(25)

where \( \Psi'(x) \) denotes the derivative of the psi function and \( \Psi'(1/4) \simeq 17.1973 \). Therefore the potential of two static color charges \((Q - \bar{Q})\) of QCD\(_5\) in the case of small separations is found to be:

\[
V_{\text{static}} = \frac{Q^2}{2\pi} \ln \lambda R + \frac{\sqrt{2}}{3} \left( \frac{\pi^2}{2} - \frac{\Psi'(1/4)}{4} \right) \sqrt{\frac{\kappa Q^3}{\pi^3}} R^{1/2} + \left( \frac{2G}{\pi} + \ldots \right) \kappa Q R \\
= \frac{Q^2}{2\pi} \ln \lambda R + 0.054..\sqrt{\kappa Q^3} R^{1/2} + (0.583.. + \ldots) \kappa Q R 
\]

(26)

In addition to the dominant classical potential, there are subdominant contributions behaving like \( R^{1/2}, R, \ldots \) vanishing as \( R \) approaches zero. In the last
correction term, it is indicated that the numerical factor will be modified due to quantum corrections of the flux function itself. The magnitude of this contribution lies outside the various approximations applied to reach (26). Detailed analysis shows that the first correction term $\propto R^{1/2}$ is not changed! Equation (26) should be read side by side with Adler’s formula (40) of reference [6].

5 Conclusion

Despite important differences in physical and analytical details, the global structure of QCD$_3$ seems to be comparable to that of QCD$_4$ with reference to quasiclassical approximations, as was pointed out in more detail by Cornwall [7]. Although QCD$_3$ does not provide renormalization group methods, a Callan-Symanzik $\beta$-function or a mass-scale dependent running coupling, it is possible within the framework of Adler and Piran to study the asymptotic freedom of a confining theory.

It is interesting that quantum contributions result from the employment of solutions of purely classical differential equations. Here it proved appropriate not to approximate a solution to an exact differential equation but to approximate the differential equation itself.

In this way the short distance approximation of the static potential was determined including nonclassical contributions of order $R^{1/2}$ and $R$. Even with reservations due to the relevance of low-dimensional field theories the calculations may lend some insight into the mechanisms of classical approximations of real QCD.

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