BTZ black holes and flat space cosmologies in higher derivative theories

Céline Zwikel

Université Libre de Bruxelles, Physique Théorique et Mathématique Campus Plaine
C.P. 231, B-1050 Bruxelles, Belgium

E-mail: czwikel@ulb.ac.be

Received 5 October 2016, revised 2 February 2017
Accepted for publication 14 February 2017
Published 21 March 2017

Abstract
We consider BTZ black holes and flat space cosmologies in generic higher derivative gravity theories in $2 + 1$ dimensions. Our goal is to prove the match between the bulk Iyer–Wald entropy and the field theory entropy for various symmetry algebras ($\text{CFT}_2$, Warped $\text{CFT}_2$, $\text{BMS}_3$). We also discuss phase transitions in higher curvature theories, and argue that, in the flat case, there is strictly speaking no phase transition in the grand canonical ensemble.

Keywords: lower dimensional gravity models, gravitational entropy, higher curvature gravity theories, holographic dualities

1. Introduction

In a 2D conformal field theory (CFT), the Cardy formula counts the asymptotic density of states [1]. Strominger [2], building on Brown–Henneaux’s seminal work [3] relating asymptotically AdS$_3$ spacetimes to a 2D CFT, has shown it exactly accounted for BTZ black holes entropy [4, 5]. This result constitutes one of the milestones of the close relationship between AdS gravity and conformal field theories [6].

In the spirit of a UV complete theory of gravity, higher curvature terms are expected to appear as corrections to its low energy limit, general relativity. Therefore, it is interesting to get to know if the entropy matching still holds for such theories. First, Saida and Soda [7] have proven that the matching is preserved in any theory of gravity without derivative of the Riemann tensor in the Lagrangian. They performed a frame transformation by defining a new metric bringing the original higher curvature frame into an Einstein frame—the new metric obeys the Einstein–Hilbert action—and auxiliary fields, which do not contribute to the gravitational entropy. The new metric is always proportional to the BTZ metric. Consequently, all charges are just multiplied by that proportionality constant. Later, Kraus and Larsen [8] have shown that the matching occurs in any theory of gravity for solutions with a near horizon AdS$_3 \times X$ geometry. Their derivation uses the special asymptotic behaviour of AdS spacetimes...
for Brown–Henneaux boundary conditions. This method is not directly transposable to other cases of interest, in particular the boundary conditions introduced in [9, 10].

One particular term in 3D gravity is the Chern–Simons term, bringing a gravitational anomaly. For BTZ black holes, its effect has already been studied in [11–15] and therefore, it will not be considered in this work.

In this paper, we first recover the results of [7, 8], i.e. the matching between the statistical and gravitational entropy and the expressions of charges for BTZ black holes with Brown–Henneaux boundary conditions [3], using another method: the covariant phase space formalism. We are inspired by [16] where the authors studied the entropy matching for 4D extremal Kerr black holes. We use this formalism throughout this paper.

Secondly, we consider BTZ black holes equipped with the recently new boundary conditions found by Compère, Song and Strominger [9]. The asymptotic symmetry group is generated by a \( U(1) \) Kac–Moody–Virasoro algebra, suggesting a description in terms of a warped conformal field theory [17]. There, the counting of the asymptotic density of states can be made using an analog of the Cardy formula, called the warped Cardy formula. Nevertheless, its form depends of the considered ensemble, closely related to the choice of coordinates. Here, we consider the coordinates of [9] referring to the quadratic ensemble (see sections 4 and 5.3 of [17]) where the warped Cardy formula takes the form of a Cardy formula. In this paper, we show that the gravitational entropy of BTZ can be reproduced by a warped Cardy formula for any gravity theory.

Another interesting class of spacetimes in three dimensions are the flat space cosmologies [18]. These are locally flat, their asymptotic symmetry group is given by BMS\(_3\) [10], and can be obtained as a certain limit of BTZ black holes [18, 19]. Their entropy is also reproduced by the Cardy-like formula for BMS\(_3\) [20, 21]. In this paper, we derive the charges in any diffeomorphism invariant theories and also prove the entropy matching.

Finally, we consider phase transitions. In the case of AdS\(_3\) geometries, it was shown that a phase transition occurs between the thermal AdS\(_3\) spacetime and the BTZ black hole (see [22–24] and references therein). We show that the higher curvature corrections do not modify the phase diagram. For flat space cosmologies, it was argued in [25] that there exists also a phase transition between them and the hot flat spaces in the grand canonical ensemble. Nevertheless, their analysis of local thermodynamic stability needs to be refined and leads to the conclusion that these spacetimes are not locally stable in the grand canonical ensemble for any theories without gravitational anomalies. The study of phase transition at thermodynamic equilibrium requires the comparison between minima of free energies. Therefore, in that context, we exclude the possibility of a phase transition. We also analyse the local stability in the particular cases of topologically massive gravity (TMG) [26, 27] and flat chiral gravity [28], where only the Chern–Simons term subsists in the Lagrangian. Only the latter seems to allow locally stable flat space cosmologies and exhibits a phase transition. But a careful analysis of the consequence of the enhancement of symmetries in flat chiral gravity forbids us to consider the thermodynamics of these solutions and thus the phase transition.

The plan of the paper is the following: in section 2, we define the quantities needed to compute charges. Then, in section 3, we consider the BTZ black holes. Using the symmetries, we compute the charges and show the entropy matching. After, we study the Hawking–Page transition and present examples of theories where we apply that formalism. In the section 4, we prove the entropy matching for the flat space cosmologies and discuss the phase transition. Finally we conclude.

Note that throughout this paper, we set Newton’s constant to one.
2. Method and definitions

In this section, we explicit the expressions of the charges in the covariant phase space formalism without any details. We closely follow [16]. The most general Lagrangian can be written as

\[ L = \ast f(g_{ab}, R_{abcd}, \nabla_v R_{abcd}, \cdots, \nabla_v \cdots \nabla_v R_{abcd}) \]

or equivalently in terms of auxiliary fields

\[ L = \ast f(g_{ab}, R_{abcd}, \cdots, R_{abcd(e_1 \cdots e_k)}) + Z^{abcd}(R_{abcd} - R_{abcd(e_1 \cdots e_k)}) \]

\[ Z^{abcd(e_1 \cdots e_k)}(\nabla_v R_{abcd} - R_{abcd(e_1 \cdots e_k)}) + \cdots + Z^{abcd(e_1 \cdots e_s}(\nabla_v R_{abcd(e_1 \cdots e_{s-1})} - R_{abcd(e_1 \cdots e_s)}) \]

The auxiliary fields \( Z^{abcd(e_1 \cdots e_s}, R_{abcd(e_1 \cdots e_s)} \) for \( 1 \leq s \leq k \) are totally symmetric in the indices \( e_i \) and the symmetrization in \( \nabla_v R_{abcd(e_1 \cdots e_{s-1})} \) is only among the \( e_i \)'s. The equations of motion for \( R_{abcd(e_1 \cdots e_s)} \) and \( Z^{abcd(e_1 \cdots e_s)} \) are

\[ R_{abcd(e_1 \cdots e_s)} = \nabla_v R_{abcd(e_1 \cdots e_{s-1})} \]

\[ Z^{abcd(e_1 \cdots e_s)} = \frac{\partial f}{\partial R_{abcd(e_1 \cdots e_s)}} - \nabla_v Z^{abcd(e_1 \cdots e_{s-1})} \]

where there is no covariant derivative for \( s = 0 \) or \( k \). They can be solved iteratively

\[ R_{abcd(e_1 \cdots e_s)} = \nabla_v R_{abcd(e_1 \cdots e_{s-1})} \]

\[ Z^{abcd} = \delta^{\text{cov}}_{R_{abcd}} - \sum_{i=0}^{k-1} (-1)^i \nabla_{e_i} \cdots \nabla_{e_0} \frac{\partial}{\partial \nabla_{e_i} \cdots \nabla_{e_0} R_{abcd}}. \]

The auxiliary field \( Z^{abcd} \) allows us to write easily the relevant quantities to compute the charges. In 3D, the Noether charge for a Killing vector \( \xi \) is [16]

\[ Q_\xi = ( -Z^{abcd} \nabla_{\xi_d} - 2 \xi \nabla_{\xi_d} Z^{abcd}) + \xi A^{abcd} \epsilon_{abcd} dx^e, \]

with

\[ A^{abcd} = -2(Z^{abcd(e_1 \cdots e_s, \gamma \rho \delta \nu}_{\beta \alpha \delta \nu}) + Z^{abcd(e_1 \cdots e_s, \beta \gamma \delta \nu}_{\alpha \gamma \delta \nu}) + Z^{abcd(e_1 \cdots e_s, \beta \gamma \delta \nu}_{\alpha \gamma \delta \nu}) + \frac{s-1}{2} Z^{abcd(e_1 \cdots e_s, \gamma \rho \delta \nu}_{\alpha \gamma \delta \nu}) \]

while the boundary term in the variation of the action is [16]

\[ \Theta = \frac{1}{2} (-2(Z^{abcd} \nabla_{\xi_d} \delta g_{bc} - (\nabla_{\xi} Z^{abcd} \delta g_{bc})) + \delta g_{ij} R^{ij} - Z^{abcd(e_1 \cdots e_s, \gamma \rho \delta \nu}_{\alpha \gamma \delta \nu}) R_{abcd} dx^b \wedge dx^c. \]
The Iyer–Wald-entropy [29] formula in 3D is

$$S_{\text{Wald}} = -2\pi \int_{\text{horizon}} dA Z^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\mu\nu}$$

(13)

where \( \epsilon_{\mu\nu} \) is the binormal at horizon and \( A_d \) the infinitesimal area element.

We are interested in computing the charges associated to the exact Killing vectors \( \partial_t \) and \( \partial_\phi \). Their expressions are [30]

$$\delta L_0 \equiv -\int_\infty \delta Q_{\partial_\phi},$$

(14)

$$\delta P_0 \equiv \int_\infty \delta Q_{\partial_t} + \int_\infty i_\delta \Theta,$$

(15)

where the integral is taken on the circle at spatial infinity \((r, t = \text{cst})\) and \( \delta \) is the derivation with respect to the parameters of the solution. There is no term with \( \Theta_{\partial_\phi} \) in \( \delta L_0 \) because \( \partial_\phi \) is tangent to this circle.

The asymptotic symmetries we are interested in are of three kinds:

- two Virasoros for BTZ with Brown–Henneaux [3] boundary conditions
  
  \begin{align*}
  i[L_m, L_n] &= (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0} \\
  i[L_m, L_n] &= (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0} \\
  i[L_m, L_n] &= 0,
  \end{align*}

(16)

- Vir \( \ltimes u(1) \) algebra for BTZ with Compère–Song–Strominger [9] boundary conditions, here written in the quadratic ensemble [17]

  \begin{align*}
  i[L_m, L_n] &= (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0} \\
  i[L_m, \hat{P}_n] &= -n \hat{P}_{m+n} + n \hat{P}_0 \delta_{m+n} \\
  i[\hat{P}_m, \hat{P}_n] &= -2m \hat{P}_{m+n},
  \end{align*}

(17)

- BMS\(_3\) [10] for flat space cosmologies

  \begin{align*}
  i[L_m, L_n] &= (m - n) L_{m+n} + c_{LL} m(m^2 - 1) \delta_{m,-n} \\
  i[L_m, M_n] &= (m - n) M_{m+n} + c_{LM} m(m^2 - 1) \delta_{m,-n} \\
  i[M_m, M_n] &= 0.
  \end{align*}

(18)

The central extension of the algebra satisfied by the charges associated to the symmetry generators \( \chi, \xi \) is given through the conserved \( n - 2 \)-form \( k \) (see [31, 32] and references therein)

$$\int_\Sigma k_\Sigma [\delta_{\xi} \Phi, \Phi]$$

(19)

where \( \Phi \) are the fields of the theory and \( \Sigma \) is a Cauchy surface. In the case of gravity, we have \( \delta_{\xi} \Phi = L_{\xi} \Phi \) and \( k \) can be written as [29, 30, 33] (for more details see [16, 34])
\[ \int \Sigma k_\times \{ \xi_\Phi \Phi ; \Phi \} = \int \Sigma (\delta_\xi Q_\times(\Phi) + i_\xi \Theta(\Phi_\xi, \Phi)). \] 

In the literature, another definition for the \( n - 2 \)-form is made by Barnich–Brandt–Compère [31, 32]. Those two differ by the so-called \( E \)-term [16]

\[ E[\xi_\Phi \Phi ; \xi_\Phi g] = \frac{1}{2} \left( -\frac{3}{2} Z^{abcd} \xi_{\xi a} \Phi^e \wedge \xi_{\xi b} \Phi^f \wedge \xi_{\xi c} \Phi^g + 2 Z^{acde} \xi_{\xi a} \Phi^b \wedge \xi_{\xi c} \Phi^d \wedge \xi_{\xi e} \Phi^g \right) f_{abc} \, \text{d}x^g. \]

It turns out that in the considered cases, this term always vanishes and so the two definitions are equivalent.

3. BTZ black holes

3.1. Symmetries

BTZ black holes are locally AdS3. The latter is a maximally symmetric space. One main property of these spacetimes is that all tensors made out of the curvature and its covariant derivatives, like \( \nabla\mu \nabla\nu R_{\mu \nu} \), ... can be expressed in terms of the metric tensor. For example,

\[ \ell^{\mu \nu} = -\frac{2}{\ell^2} g^{\mu \nu}. \]

where \( \ell \) is the AdS radius. Note that there is no way to construct a 3-tensor just from products of the metric.

Another consequence of the symmetries is that the left hand side of the equations of motion \( E_{\mu \nu} \) evaluated on BTZ black holes, can always be rewritten as a constant times the metric

\[ E_{\mu \nu} = K(\lambda_i, \ell) g_{\mu \nu}. \]

where \( K \) is a linear function of the coupling constants of the theory \( \lambda_i \) as well as \( \ell \) (but not on the parameters of the black holes). Solving the equations of motion imposes the constant \( K \) to be zero which is in general solvable.

3.2. Charges and entropy

Now, we can easily write the general form of the key field for the charges computation (7):

\[ Z^{abcd} = \frac{\alpha}{32\pi} (g^{ae} g^{bd} - g^{ad} g^{be}). \]

Indeed, as it depends only on curvature invariants, it can be written in terms of the metric and has the symmetries of the Riemann as it should. The constant \( \alpha \) is a constant depending on \( \ell \) and the coupling constants of the theory\(^1\). Therefore, the Iyer–Wald entropy (13) is proportional to the Bekenstein–Hawking entropy

\[ S^{\text{IW}} = \frac{\pi}{2} r_c \alpha r_c = \alpha S^{\text{BH}} \]

where \( r_c \) is the outer horizon of BTZ.

Moreover, the 3-tensors \( A \) and \( B \) in (9) and (11) vanish. The last term in (11) can easily be shown to be zero as well (it is the variation of products of the metric, so it can be rewritten as a 3-tensor times \( \delta g_{\mu \nu} \)). Thus, the expressions of the Noether charge (9) and the boundary term

\(^1\)The normalization is chosen for later convenience.
are linear functions of the $Z_{abcd}$ field. In addition, the higher derivatives of the Riemann in the Lagrangian do not have an effect on charges for BTZ black holes. Indeed, the terms of the sum with $i > 0$ in (7) are always zero for the charges computation since they are the covariant derivatives of terms proportional to the metric. Also, the higher derivatives might be present in the $i = 0$ term as factor but will not contribute as they vanish on-shell.

Therefore, the exact charges (14) and (15) are proportional to their expressions in general relativity. All the information about the particular theory is encoded only through the proportionality constant $\alpha$ and there is no contribution from higher derivative terms of the Lagrangian, in the sense that they do not modify $\alpha$.

Now, we compute the central charges. For that, we need to specify the boundary conditions. Starting from the Brown–Henneaux boundary conditions [3], we compute the central charges arising in the asymptotic symmetry group, generated by two copies of the Virasoro algebra. We use the following BTZ metric [5]

$$ds^2 = -\frac{\ell^2 r^2 dr^2}{16\ell^2 + 8\ell^2 M^2 r^2 - r^4} + \frac{(8\ell^2 M - r^2)dr^2}{\ell^2} + 8J d\phi + r^2 d\phi^2,$$

where $J \leq \ell M$ and $\phi \sim \phi + 2\pi$. The conserved charges $L_0$ associated to the asymptotic Killing vectors

$$l_0 = \left(\frac{\ell}{2} e^{i(t + s)} + O(r^{-2})\right)\partial_t + \left(\frac{1}{2} i e^{i(t + s)} r r + O(r^{-1})\right)\partial_r + \left(\frac{1}{2} e^{i(t + s)} + O(r^{-2})\right)\partial_\alpha,$$

satisfy one Virasoro of the algebra (16). As the Noether charge (9) and the boundary term (11) are linear functions of $Z_{abcd}$, the central charge given by (20) is just proportional to the general relativity case

$$c = \alpha \left(\frac{3\ell}{2}\right).$$

As argued before, the exact charges are as well multiplied by the constant $\alpha$

$$M^{bc} = \alpha M, \quad J^{bc} = \alpha J.$$

The dual theory is a CFT. The counting of the asymptotic density of states is given by the Cardy formula

$$S^{\text{CFT}} = 2\pi \sqrt{\frac{c}{6} L_0} + 2\pi \sqrt{\frac{c}{6} \bar{L}_0},$$

with $L_0 = \ell M^{bc} \cdot J^{bc}$ and $\bar{L}_0 = \ell M^{bc} \cdot J^{bc}$, so

$$S^{\text{CFT}} = \frac{\pi}{2}\alpha r_+, \quad (30)$$

which is exactly the Iyer–Wald entropy. We therefore have recovered the results of [7, 8].

We continue with the Compère–Song–Strominger boundary conditions [35]. We first rewrite the BTZ metric as [35]:

$$ds^2 = \frac{\ell^2 dr^2}{r^2} + 2(J + \ell M)dr^2 + \frac{(4J^2 - 4\ell^2 M^2 - \ell^2 r^4)dr_+ dt_+}{r^2} - 2(J - \ell M)dr_+^2,$$

Since there is no gravitational anomaly, the two central charges are equal and we only need to consider one of the Virasoro’s.
In these coordinates, the asymptotic Killing vectors are
\[ l_a = e^{\text{int.}} \partial_{x_a} + \frac{r}{2} \text{ine}^{\text{int.}} \partial_{r_a}, \quad t_a = e^{\text{int.}} \partial_{t_a} \] (33)
and their associated conserved charges \( \tilde{L}_a, \tilde{P}_a \) satisfy the algebra (17), i.e. a \( U(1) \)
Kac–Moody–Virasoro algebra in the quadratic ensemble [17]. As argued before, the charges
are proportional to their value in general relativity [35]
\[ c = \alpha \left( \frac{3 \ell}{2} \right), \quad \tilde{P}_0 = \alpha \left( \frac{(J + \ell M)}{2} \right), \quad \tilde{L}_0 = \alpha \left( \frac{(-J + \ell M)}{2} \right). \] (34)
In that particular ensemble, the warped Cardy formula takes the form [17]
\[ S_{\text{WCFT}} = 4 \pi \sqrt{-\tilde{P}_0^{\text{vac}} \tilde{P}_0 + 4 \pi \sqrt{-\tilde{L}_0^{\text{vac}} \tilde{L}_0}} \] (35)
where \( \text{vac} \) refers to the charges of the vacuum. Here, global \( \text{AdS}_3 \) is the vacuum whose charges
are \( M = -1/8 \) and \( J = 0 \). So, we get
\[ S_{\text{WCFT}} = \frac{\pi}{2} \alpha r_r. \] (36)
Therefore, the gravitational entropy is also reproduced by the warped Cardy formula.

### 3.3. Phase transition

In this section, we consider the generalization of the Hawking–Page transition in the grand
canonical ensemble for any theory of gravity where \( \text{BTZ} \) is a solution. The exact charges are
proportional to the general relativity case but the thermodynamic potentials, the temperature \( T \)
and the angular velocity \( \Omega \) remain unchanged because their derivation relies only on the
metric. Thus, the Gibbs free energy is given by
\[ G_{\text{HC}}(T, \Omega) = \alpha \left( \frac{-\ell^2 \pi^2 T^2}{2(1 - \ell^2 \Omega^2)} \right). \] (37)
First, we study the local stability of that phase. In the grand canonical ensemble, the stability
condition is the requirement for a system to have a negative semi-definite Hessian of its
free energy \( G(T, \Omega) \), given by
\[ H = \begin{pmatrix}
\frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Omega} \\
\frac{\partial^2 G}{\partial \Omega \partial T} & \frac{\partial^2 G}{\partial \Omega^2}
\end{pmatrix} = \alpha \begin{pmatrix}
\frac{-\pi^2 \ell^2}{1 - \ell^2 \Omega^2} & -\frac{2\ell^4 \pi^2 T \Omega}{(1 - \ell^2 \Omega^2)^2} \\
-\frac{2\ell^4 \pi^2 T \Omega}{(1 - \ell^2 \Omega^2)^2} & -\frac{\ell^4 \pi^2 T^2(1 + 3\ell^2 \Omega^2)}{(1 - \ell^2 \Omega^2)^3}
\end{pmatrix}. \] (38)
This implies only the following constraint on the coupling constants of the theory
\[ \alpha > 0. \] (39)
Secondly, we consider the global stability. In the classical limit, the dominant phase is the
most probable, i.e. the one that dominates the partition function among the saddle points. Here
the two known phases are the black hole and the vacuum \( \text{AdS}_3 \). So, we compare their free
energies through their difference
\[ \Delta G = \alpha \left( -\frac{1}{8} + \frac{\ell^2 \pi^2 T^2}{2(1 - \ell^2 \Omega^2)} \right). \] (40)
If $\Delta G < 0$, AdS$_3$ dominates and for the opposite sign, BTZ dominates. The constant $\alpha$ factorizes out$^3$. It implies that the phase diagram doesn’t depend on which theory we look at and it is the same as in general relativity.

3.4. A working example: $L = a R - \Lambda + b R^2 + c R_{ab}R^{ab} + d R^3$

We explicit the constant $\alpha$ of an example for which particular cases are general relativity and new massive gravity [36]. The auxiliary field $Z^{abcd}$ is

$$Z^{abcd} = \frac{\delta L}{\delta R_{abcd}} = (a + 2b R + 3d R^2) \frac{\delta R}{\delta R_{abcd}} + 2c R^{ef} \frac{\delta R_{ef}}{\delta R_{abcd}} = \frac{1}{2} (a + 2b R + 3d R^2) (g^{bd} g^{ac} - g^{ad} g^{bc}) + \frac{1}{2} c (g^{bd} R^{ac} - g^{ad} R^{bc} - g^{bc} R^{ad} + g^{ac} R^{bd}).$$

The BTZ solution has $R = -\frac{6}{\ell}$ and $R_{ab} = -\frac{2}{\ell} g_{ab}$. So we get

$$Z^{abcd} = \frac{\alpha}{32 \pi} (g^{bd} g^{ac} - g^{ad} g^{bc}), \text{ with } \alpha = 16 \pi \left( a - \frac{12}{\ell^2} b + \frac{108}{\ell^4} d - \frac{4}{\ell^2} c \right).$$

Examples:

1. The Lagrangian of general relativity is $\frac{1}{16 \pi} R$. So the constant in $Z^{abcd}$ field is

$$\alpha = 1. \quad (43)$$

2. New massive gravity is obtained by taking $a = \frac{1}{16 \pi}$, $b = \frac{-3}{16 \pi 8 m^2}$, $c = \frac{1}{10 \pi m^2}$ and $d = 0$ [36], so

$$\alpha = \left( 1 + \frac{1}{2 \ell^2 m^2} \right). \quad (44)$$

4. Flat space cosmologies

4.1. Charges and entropy

Now we turn to the case of flat space cosmologies (see [18, 19] and references therein)

$$ds^2 = -2 dr du + 8 M du^2 + 8 J du d\phi + \ell^2 d\phi^2$$

with $M > 0$ and $J \neq 0$. The cosmological horizon is located in $r_c = \sqrt{2 J^2 / M}$. The asymptotic Killing vectors associated to the boundary conditions described in [10] are

$$l_a = e^{i m \phi} n_a \partial_\phi - i n_r \partial_r \left( 1 + n_r^2 \frac{R}{r} \right) \partial_\phi, \quad m_a = i e^{im \phi} \partial_\phi.$$

The associated charges $L_a, M_a$ satisfy the BMS$_3$ algebra (18).

$^3$We consider $\alpha > 0$ because the study of the global stability at thermodynamic equilibrium requires to compare locally stable phase.
We show that the charges are always the same regardless of the theory. Indeed, in these solutions, \( R = R_{\mu\nu} = 0 \) and covariant derivatives of the Ricci all vanish, fixing completely \( Z^{abcd} \) (7). Its term \( i = 0 \) takes the derivatives with respect to the Ricci tensors. To have a non zero contribution, we need at least and at most one Ricci tensor, because if it is multiplied by any ingredient of the Lagrangian apart from the metric, it will no contribute on-shell. The terms \( i > 0 \) are always zero by the same argument as for BTZ. So in general, the \( Z^{abcd} \) field is of the form

\[
Z^{abcd} = \frac{1}{32\pi} (g^{bd} g^{ac} - g^{ad} g^{bc}).
\] (47)

For example, if we take the Lagrangian of section 3.4, the steps are identical to BTZ until the equation (41). On-shell, only the term proportional to \( \alpha \) survives. It is the constant of the Einstein–Hilbert term which is usually taken to be \( 1/(16\pi)^2 \).

Therefore, the Iyer–Wald entropy is just the general relativity expression, namely

\[
S_{\text{IW}} = \frac{\pi}{2} \kappa_c.
\] (48)

Furthermore, the corrections tensors \( A(10) \) and \( B(12) \) are proportional to \( R_{abcd}[e_1,...,e_9} \) (6) which is always zero on-shell. Thus, we recover the general relativity result \([19]\)

\[
Q_{\partial_0} = M, \quad Q_{\partial_0} = J
\] (49)

\[
c_{LL} = 0, \quad c_{LM} = \frac{1}{4}.
\] (50)

This result is consistent with the argument following from dimensional analysis. The symmetries of these spacetimes imply that the charges in any higher curvature theory are proportional to their value in general relativity. The proportionality constant is only a linear combination of the coupling constants of the theory whose the coefficients are pure numbers as locally flat spacetimes do not have an intrinsic length. Moreover, this combination should be consistent with dimensional analysis. Thus, the only allowed term is the Einstein–Hilbert term and in the case of theory with diffeomorphism anomaly, also the Chern–Simons term.

One consequence of the non renormalization of the charges is that the results about charges of \([20, 21]\) hold in any higher curvature theories. For example, their entropy is reproduced by the Cardy-like formula for BMS3 \([20, 21]\).

### 4.2. Phase transition

In \([25]\), a phase transition between the flat spaces cosmologies and hot flat space is exhibited. Nevertheless, our local stability analysis differs from theirs and modifies the conclusion.

We consider the flat space cosmologies written for convenience in the following form

\[
d s^2 = R_t^2 dt^2 - 2R_t r_0 d\phi - \frac{1}{R_t^2(1 - r_0^2/r^2)} \, dr^2 + r^2 d\phi^2.
\] (51)

The temperature and angular velocity are \([25]\)

\[
T = \frac{R_t^2}{2 \pi r_0}, \quad \Omega = \frac{R_t}{r_0}.
\] (52)

\(^4\) If we consider a theory with neither an Einstein–Hilbert or Chern–Simons term having flat spaces cosmologies as solutions, their charges will be zero.
We directly consider the case of TMG [26, 27]

\[
S = \frac{1}{16\pi} \int d^3x \sqrt{-g} \left( R - 2\Lambda \right) + \frac{1}{16\pi} \frac{1}{2 \mu} \int d^3x \sqrt{-g} e^{\lambda g_{\mu\nu}} \nabla_\lambda \left( \partial_\mu T^\gamma_{\nu\rho} + \frac{2}{3} T^\gamma_{\mu\nu} \Gamma^\sigma_{\nu\rho} \right) \tag{53}
\]

with \( \mu \) the Chern–Simons coupling taken positive without loss of generality. The conserved charges associated with \( \partial_t \) and \( \partial_\phi \) are [25]

\[
M = \frac{R_i^2}{8}, \quad J = -\frac{R_i r_0}{4} + \frac{R_i^2}{8\mu}, \tag{54}
\]

while the entropy is [25]

\[
S = \frac{\pi r_0}{2} - \frac{\pi R_i}{2 \mu}. \tag{55}
\]

To study the thermodynamic stability, we need to consider the Gibbs free energy \( G \) who can be derived by integrating the first law \( dM = -T dS - \Omega dJ \) [25], or through the on-shell action procedure [37],

\[
G = M + TS + \Omega J \tag{56}
\]

leading to\(^5\)

\[
G^{\text{SCG}}(T, \Omega) = \frac{\pi^2 T^2}{2 \Omega^2} \left( 1 - \frac{\Omega}{\mu} \right). \tag{57}
\]

In the grand canonical ensemble, it is not sufficient to study only the specific heat to determine the local stability as was done in [25]. Instead, the complete requirement is that the Gibbs free energy should be a concave function of the temperature and the angular velocity. In other words, its Hessian \( H \) should be negative semi-definite which is never the case here\(^6\). Indeed, it is given by

\[
H = \begin{pmatrix}
\frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Omega} \\
\frac{\partial^2 G}{\partial \Omega \partial T} & \frac{\partial^2 G}{\partial \Omega^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\pi^2}{\Omega^2} & -\frac{\pi^2 T (2\mu - \Omega)}{\mu \Omega^3} \\
-\frac{\pi^2 T (2\mu - \Omega)}{\mu \Omega^3} & \frac{\pi^2 T^2 (3\mu - \Omega)}{\mu \Omega^4}
\end{pmatrix}. \tag{58}
\]

The determinant of \( H \) is \( -\frac{\pi^2 r_0^2}{\Omega^2} \). It is negative and the two eigenvalues values are non zero (expect in the limit \( \mu \to 0 \) which will be considered below). It implies that the eigenvalues have not the same sign and so the matrix can not be negative semi-definite. So the flat spaces cosmologies are not metastable states and therefore the phase transition can not be studied in the frame of the physics of thermodynamic equilibrium. The extensions for non equilibrium phenomena are far beyond the scope of this paper. It is also the case of pure general relativity \((\mu \to \infty)\) and therefore in any diffeomorphism invariant theories.

\(^5\) The free energy has an opposite sign with respect to [25], because here it is expressed in terms of Lorentzian variables.

\(^6\) It is well known that the flat cosmologies can be obtained as the flat limit of BTZ black holes. Nevertheless, the flat limit of the condition (39) obtained for BTZ does not give to the one for flat space cosmologies. Indeed, it is a consequence of the local stability condition in where \( \alpha \) and the derivatives of the free energy take place. Therefore, we should take the limit of all these ingredients and after discuss. This procedure leads to the result presented in this section.
Only in the particular case of flat chiral gravity [28], where only the Chern–Simons remains, it turns out that the Hessian is negative semi-definite for positive angular velocities

\[ \Omega > 0. \] (59)

If we naively pursue the analysis, we will conclude that the flat space cosmologies are locally stable and therefore we will consider the global stability. However, the limit \( \mu \to 0 \) leads to a theory with an enhancement of symmetry. Indeed, the action is a pure Chern–Simons term who is conformally invariant. Rewriting the flat space cosmologies (51) as

\[
\sqrt{g} \left( R_0^2 dr^2 - 2 R_0 dr d\phi - \frac{1}{R_0^2 (1 - 1/r^2)} dr^2 + r^2 d\phi^2 \right),
\] (60)

it is obvious that the value of \( r_0 \) can always be modified by a conformal transformation. Therefore we can not make sense of the thermodynamics of the flat space cosmologies in flat chiral gravity and obviously it is meaningless to talk about phase transition in that context.

In conclusion, we exclude the possibility of a phase transition between the flat space cosmologies and the hot flat spaces for any theory of gravity, with or without gravitational anomalies, including in the analysis the singular point \( \mu \to 0 \).

5. Conclusion

In this paper, we have analysed in detail the corrections due to higher derivative terms in the action and have shown that they never contribute to the renormalization of the charges. Also, we have proved that the Iyer–Wald entropy of BTZ black holes, equipped with the Compère–Song–Strominger boundary conditions, is reproduced by the warped Cardy formula. As a direct consequence of the charge renormalization of BTZ black holes, the same Hawking–Page transition occurs as in general relativity since the constant of proportionality factorizes out. Nevertheless, the thermodynamic local stability of BTZ black holes imposes an additional constraint on the coupling constants of the theory.

In addition, we have proved that the charges of flat space cosmologies are never modified in any diffeomorphism invariant theory. Therefore, the properties of the charges derived in general relativity in [20, 21], for example the match between the bulk and boundary entropies, holds in general. We have also discussed the phase transition between flat space cosmologies and hot flat spaces where we found a few discrepancies with the work of [25]. We have argued that the local stability requirement excludes the possibility of a phase transition at thermodynamic equilibrium, except potentially in the case of flat chiral gravity. However, we have pointed out that the thermodynamics of flat space cosmologies can not be considered in flat chiral gravity because of its property of conformal invariance.

Acknowledgments

The author is grateful to S Detournay for its comments on the draft and enlightening discussions. She also thanks G Giribet, H González, D Grumiller, V Lekeu, A Marzolla, and G Ng. She is a research fellow of ‘Fonds pour la Formation la Recherche dans l’Industrie et dans l’Agriculture’–FRIA Belgium. This work is partially supported by the ARC grant ‘Holography, Gauge Theories and Quantum Gravity—Building models of quantum black holes’ and by FNRS-Belgium (convention IISN 4.4503.15).
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