Abstract

We construct the non-linear realisation of $E_{11}$ and its first fundamental representation in eleven dimensions at low levels. The fields depend on the usual coordinates of space-time as well as two form and five form coordinates. We derive the terms in the dynamics that contain the three form and six form fields and show that when we restrict their field dependence to be only on the usual space-time we recover the correct self-duality relation. Should this result generalise to the gravity fields then the non-linear realisation is an extension of the maximal supergravity theory, as previously conjectured. We also comment on the connections between the different approaches to generalised geometry.
1. Introduction

It has been realised for more than a quarter of a century that there exists no fundamental theory of strings. The maximal supergravity theories are the complete low energy effective actions for the type II string theories and these have provided a source of certainty. However, the study of these theories has also led to the realisation that a fundamental theory of strings must include branes, but we have really no idea how to describe many of the properties of branes. One way forward may be to try to guess the symmetries of the underlying theory. Some time ago, and with this approach in mind, it was conjectured that a non-linear realisation of the Kac-Moody algebra $E_{11}$ was an extension of the maximal supergravity theories \cite{1}. The different maximal theories emerge from the different possible decompositions of $E_{11}$ into the sub-algebras that arise from deleting the different possible nodes in the $E_{11}$ Dynkin diagram \cite{2,3,4,5,6,7}. In the early papers on $E_{11}$ space-time was introduced into the non-linear realisation by adjoining to the $E_{11}$ algebra the space-time translations. It was understood at the time that this was an adhoc step and in 2003 it was proposed that one should consider the non-linear realisation of the semi-direct product of $E_{11}$ and its first fundamental representation $l_1$ denoted $E_{11} \otimes_s l_1$ \cite{8}. The latter contains the generators $P_a, Z^{ab}, Z^{a_1...a_5}, Z^{a_1...a_7,b}$ as well as an infinite number of other objects. In the non-linear realisation this leads to a generalised space-time with coordinates $x^a, x_{ab}, x_{a_1...a_5}, x_{a_1...a_7,b}, \ldots$ \cite{8}. There is very good evidence that the $l_1$ representation contains all the brane charges \cite{8,9,10,11,3} and so there is a one to one relationship between the coordinates of the generalised space-time and brane charges. To appreciate this proposal one has to understand what is a non-linear realisation which is in this case not the same as what is often called a sigma model. Such non-linear realisations were given in the papers on $E_{11}$, for example \cite{12,1,5}, and an early application was to formulate gravity as a non-linear realisation \cite{14}. The non-linear realisation of $E_{11} \otimes_s l_1$ not only introduces a generalised space-time but also a generalised vielbein and so a corresponding geometry. In particular this non-linear realisation was used to derive almost all of the features of the five dimensional gauged maximal supergravities \cite{5}. However, there has not been a systematic attempt to construct the non-linear realisation of $E_{11} \otimes_s l_1$. In this paper we will construct the dynamics of the $E_{11} \otimes_s l_1$ at lowest level keeping the first few coordinates of the generalised space-time, that is the coordinates $x^a, x_{ab}$ and $x_{a_1...a_5}$ and the three form and six form fields.

2. A review of $E_{11}$ and its $l_1$ representation

In this section we review some of the technical aspects of $E_{11}$ and its first fundamental representation $l_1$ which will be required to construct the non-linear realisation and so the dynamics. The $E_{11}$ algebra, like any Kac-Moody algebra, is just the multiple commutator of the Chevalley generators subject to the Serre relations. The Dynkin diagram of $E_{11}$ is given by

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• 11

• – • – • – • – • – • – •
1 2 7 8 9 10
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The eleven dimensional theory emerges from the $E_{11}$ non-linear realisation if we delete
node eleven and decompose the $E_{11}$ algebra in terms of the remaining $A_{10}$ subalgebra; that is decompose the adjoint representation of $E_{11}$ in terms of representations of $A_{10}$. The generators can be listed according to a level and those of positive level are given by [1,13]

$$K^a_b, R^{a_1 a_2 a_3}, R^{a_1 a_2 \ldots a_6} ~\text{and}~ R^{a_1 a_2 \ldots a_8, b}$$

(2.1)

at levels zero, one, two and three respectively. The generators at level zero are those of $GL(11)$ and are responsible in the non-linear realisation for eleven dimensional gravity. The generator $R^{a_1 a_2 \ldots a_8, b}$ obeys the condition $R^{[a_1 a_2 \ldots a_8, b]} = 0$.

The corresponding negative level generators are given by

$$R_{a_1 a_2 a_3}, R_{a_1 a_2 \ldots a_6} ~\text{and}~ R_{a_1 a_2 \ldots a_8, b},$$

(2.2)

at levels -1, -2, -3 with the last generator satisfying an analogous constraint.

From the mathematical viewpoint the $E_{11}$ algebra is just the multiple commutators of the Chevalley generators subject to the Serre relations. However, it turns out that the Chevalley generators are contained in the generators $K^a_b$, $R^{a_1 a_2 a_3}$ and $R_{a_1 a_2 a_3}$ and so the $E_{11}$ algebra is found by taking the multiple commutators of these generators and at low levels it suffices to just impose the Jacobi identities on the algebra formed from the generators listed above. The commutators must preserve the level and so on the right-hand side of the commutators we can only write all possible terms that preserve the level. We can then implement the Jacobi identities. The level is plus (minus) the number of times the positive (negative) root Chevalley generators associated with node eleven occur in the multiple commutator that creates the generator. However, this is just the same as plus (minus) the number of times the generator $R^{a_1 a_2 a_3}$ ($R_{a_1 a_2 a_3}$) occurs in the multiple commutator.

The generators of $GL(11)$ obey the algebra

$$[K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b,$$

(2.3)

By construction the generators in equations (2.1) and (2.2) are representations of $GL(11)$ and so their commutators with the $K^a_b$ generators are given by

$$[K^a_b, R^{c_1 \ldots c_6}] = \delta^{c_1}_b R^{a c_2 \ldots c_6} + \ldots, \quad [K^a_b, R^{c_1 \ldots c_3}] = \delta^{c_1}_b R^{a c_2 c_3} + \ldots,$$

(2.4)

$$[K^a_b, R^{c_1 \ldots c_8, d}] = (\delta^{c_1}_b R^{a c_2 \ldots c_8, d} + \ldots) + \delta^d_b R^{c_1 \ldots c_8, a},$$

(2.5)

where $+ \ldots$ means the appropriate anti-symmetrisation. The corresponding relations for the negative level generators are

$$[K^a_b, R_{c_1 \ldots c_3}] = -\delta^a_{c_1} R_{b c_2 c_3} - \ldots, \quad [K^a_b, R_{c_1 \ldots c_6}] = -\delta^a_{c_1} R_{b c_2 \ldots c_6} - \ldots,$$

(2.6)

$$[K^a_b, R_{c_1 \ldots c_8, d}] = - (\delta^a_{c_1} R_{b c_2 \ldots c_8, d} + \ldots) - \delta^d_a R_{c_1 \ldots c_8, b},$$

(2.7)

The rest of the $E_{11}$ algebra can be found by remembering that the commutators preserve the level, writing the most general possibility on the right-hand side of the commutator, and enforcing the Jacobi identities. For the positive level generators we find that [1]

$$[R^{c_1 \ldots c_3}, R^{c_4 \ldots c_6}] = 2R^{c_1 \ldots c_6}, \quad [R^{a_1 \ldots a_6}, R^{b_1 \ldots b_3}] = 3R^{a_1 \ldots a_6 [b_1 b_2, b_3]},$$

(2.8)
and for the negative root generators

\[ [R_{c_1...c_3}, R_{c_4...c_6}] = 2R_{c_1...c_6}, \quad [R_{a_1...a_6}, R_{b_1...b_3}] = 3R_{a_1...a_6[b_1b_2b_3]}, \]

Finally, the commutation relations between the positive and negative generators of up to level four are given by [8]

\[ [R^{a_1...a_3}, R_{b_1...b_3}] = 18\delta^{a_1a_2} K^{a_3}_{b_3} - 2\delta^{a_1a_2a_3} D, \quad [R_{b_1...b_3}, R^{a_1...a_6}] = \frac{5!}{2} \delta^{a_1a_2a_3} R^{a_4a_5a_6}_{b_3} \]

\[ [R^{a_1...a_6}, R_{b_1...b_6}] = -5!3.3\delta^{a_1...a_5} K_{b_6}^{a_6} + 5!\delta^{a_1...a_6} D, \]

\[ [R_{a_1...a_4}, R_{b_1...b_8,c}] = 8.7.2(\delta^{b_1b_2b_3} R^{b_4...b_8}|c - \delta^{b_1b_2|c} R^{b_3...b_8}) \]

\[ [R_{a_1...a_6}, R_{b_1...b_8,c}] = \frac{7!2}{3}(\delta^{b_1...b_6} R^{b_7b_8}|c - \delta^{c|b_1...b_5} R^{b_6b_7b_8}) \]

where \( D = \sum_b K^b_b \), \( \delta^{a_1a_2} = \frac{1}{2}(\delta_{b_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_2}^{a_1} \delta_{b_1}^{a_2}) = \delta^{a_1a_2}_{b_2} \) with similar formulae when more indices are involved.

A non-linear realisation is defined by a choice of an algebra together with a subalgebra. For us the subalgebra is chosen to be the one that is invariant under the Cartan involution \( I_c \). This is an involution, that is \( I_c^2 = I_c \) is the identity operator, and an automorphism of the algebra, that is \( I_c(AB) = I_c(A)I_c(B) \), which acts on the generators given above as

\[ I_c(K^a_b) = -\eta^{ac} \eta_{bd} K^d_c, \quad I_c(R^{a_1a_2a_3}) = -\eta^{a_1b_1} \eta^{a_2b_2} \eta^{a_3b_3} R_{b_1b_2b_3}, \]

\[ I_c(R^{a_1...a_6}) = \eta^{a_1b_1} ... \eta^{a_6b_6} R_{b_1...b_6}, \quad I_c(R^{a_1...a_8,c}) = -\eta^{a_1b_1} ... \eta^{a_6b_6} \eta^{cd} R_{b_1...b_8,d} \]

Its more fundamental definition in terms of its action on the Chevalley generators can be found in for example [1]. In fact, we have modified the usual definition to take account of the Minkowski rather than the Euclidean signature. The sub-algebra invariant under the Cartan involution is generated at low levels by

\[ J_{ab} = K^c_b \eta_{ac} - K^c_a \eta_{bc}, \quad S_{a_1a_2a_3} = R^{b_1b_2b_3} \eta_{b_1a_1} \eta_{b_2a_2} \eta_{b_3a_3} - R_{a_1a_2a_3}, \]

\[ S_{a_1...a_6} = R^{b_1...b_6} \eta_{b_1a_1} ... \eta_{b_6a_6} + R_{a_1...a_6} \]

\[ S_{a_1...a_8,c} = R^{b_1...b_8} \eta_{b_1a_1} ... \eta_{b_8a_8} \eta_{bc} - R_{a_1...a_8,c} \]

where \( \eta_{ab} \) is the metric of Minkowski space-time. The generators \( J_{ab} \) are those of the Lorentz algebra \( \text{SO}(1,10) \) and their commutators with the other generators just express the fact that they belong to a representation of the Lorentz algebra. The \( S_{a_1a_2a_3} \) and \( S_{a_1...a_6} \) generators obey the commutators [8]

\[ [S^{a_1a_2a_3}, S_{b_1b_2b_3}] = -18\delta^{a_1a_2} J^{a_3}_{b_3} + 2S^{a_1a_2a_3}_{b_1b_2b_3} \]
\[
[S_{a_1a_2a_3}, S_{b_1...b_6}] = -\frac{5!}{2} \delta_{a_1a_2a_3}^{b_1b_2b_3} S_{b_4b_5b_6} - 3 S_{b_1...b_6}^{a_1a_2a_3}
\] (2.16)

The non-linear realisation of interest to us also includes generators in the fundamental representation of \(E_{11}\), denoted \(l_1\); by definition this representation has highest weight \(\Lambda_1\) which obeys \((\Lambda_1, \alpha_a) = \delta_{a_1}\). Decomposed into representations of \(GL(11)\) the \(l_1\) representation contains, at low levels, the generators \([8,9]\)

\[P_a, Z^{ab}, Z^{a_1...a_5}, Z^{a_1...a_7,b}, Z^{a_1...a_8}, Z^{b_1b_2b_3,a_1...a_8}, \ldots\]
(2.17)

at levels 0,1,2,3,3 and 4 respectively. Here \(P_a\) is the generator of space-time translations and the next two generators can be identified with the central charges in the supersymmetry algebra. The commutators of the low level generators of the \(l_1\) representation with \(R^{a_1a_2a_3}\) are determined up to constants by demanding that the levels match and so we can take \([8]\)

\[[R^{a_1a_2a_3}, P_b] = 3 \delta_b^{[a_1} Z^{a_2a_3]}, \quad [R^{a_1a_2a_3}, Z^{b_1b_2}] = Z^{a_1a_2a_3b_1b_2},
[[R^{a_1a_2a_3}, Z^{b_1...b_5}] = Z^{b_1...b_5}[a_1a_2,a_3] + Z^{b_1...b_5}a_1a_2a_3\]
(2.18)

These equations define the normalisation of these generators of the \(l_1\) representation.

We will be interested in constructing the non-linear realisation of the semi-direct product of the \(E_{11}\) algebra with its \(l_1\) representation, denoted \(E_{11} \otimes l_1\). In this algebra the commutators of the generators of \(E_{11}\) with themselves obey the same algebra as above that is equations (2.4) to (2.10). We will take the generators of the \(l_1\) representation to commute with themselves. The commutators between the generators of \(E_{11}\) and those of the \(l_1\) representation express the fact that they are a representation of \(E_{11}\) and this is enforced by demanding the Jacobi identity involving two \(E_{11}\) generators and one \(l_1\) generator. The construction is essentially the same as that for the Poincare group where the Lorentz group \(L\) plays the role of \(E_{11}\) and the space-time translations \(T\) the role of the \(l_1\) representation; that is we can write the Poincare group as \(L \otimes T\).

As we have decomposed the \(l_1\) representation into representations of \(GL(11)\), the commutators of these generators with those of the \(l_1\) representations are given \([8]\)

\[[K^a_b, P_c] = -\delta^a_c P_b + \frac{1}{2} \delta^a_c P_c, \quad [K^a_b, Z^{c_1c_2}] = 2 \delta_b^{[c_1} Z^{a|c_2]} + \frac{1}{2} \delta^a_b Z^{c_1c_2},
[[K^a_b, Z^{c_1...c_5}] = 5 \delta_b^{[c_1} Z^{a|[c_2...c_5]} + \frac{1}{2} \delta^a_b Z^{c_1...c_5}\]
(2.19)

The term with the factor of \(\frac{1}{2}\) plays an important role in many applications of \(E_{11}\) and it follows from the fact that the \(l_1\) is a highest weight representation of \(E_{11}\) \([8]\). Strictly speaking it is actually a lowest weight representation as usually defined. We also find using the Jacobi identities and equations (2.8) and (2.18) that

\[[R^{a_1...a_6}, P_b] = -3 \delta_b^{[a_1} Z^{...a_6]}, \quad [R^{a_1...a_6}, Z^{b_1b_2}] = -Z^{b_1b_2[a_1...a_5,a_6]} - Z^{b_1b_2a_1...a_6},\]
(2.20)
The commutators with the negative root generators are given by

\[
[R_{a_1a_2a_3}, P_b] = 0, \quad [R_{a_1a_2a_3}, Z^{b_1b_2}] = 6\delta_{[a_1a_2}^{b_1b_2} P_{a_3}], \quad [R_{a_1a_2a_3}, Z^{b_1...b_5}] = \frac{5!}{2} \delta_{a_1a_2a_3}^{b_1b_2b_3} Z^{b_4b_5}
\]

(2.21)

The first equation, just follows from the fact that the \( l_1 \) representation is a highest weight representation, and the subsequent equations follow by using the equation (2.10) and the Jacobi identities.

To conclude this section we now also give the commutation relations between the generators of the Cartan involution invariant subalgebra, given in equations (2.12-2.14), and the generators of the \( l_1 \) representation [8]

\[
[S^{a_1a_2a_3}, P_b] = 6\delta_{[a_1}^{b} Z^{a_2a_3}, \quad [S^{a_1a_2a_3}, Z^{b_1b_2}] = Z_{a_1a_2a_3}^{b_1b_2} - 6\delta_{[a_1a_2}^{b_1b_2} P_{a_3}],
\]

\[
[S^{a_1a_2a_3}, Z^{b_1...b_5}] = Z^{b_1...b_5}_{[a_1a_2a_3]} + Z^{b_1...b_5}_{a_1a_2a_3} - \frac{5!}{2} \delta_{a_1a_2a_3}^{b_1b_2b_3} Z^{b_4b_5}
\]

(2.24)

3. The non-linear realisation of \( E_{11} \otimes_s l_1 \)

It was conjectured [8] that the non-linear realisation of \( E_{11} \otimes_s l_1 \) was an extension of the equations of motion of eleven dimensional supergravity. Put another way, it states that eleven dimensional supergravity was contained in the non-linear realisation of \( E_{11} \otimes_s l_1 \) at low levels. At higher levels one finds not only an infinite number of new fields coming from \( E_{11} \), but also all the fields depend on a generalised space-time encoded in the \( l_1 \) representation.

The non-linear realisation of \( E_{11} \otimes_s l_1 \) is constructed from a group element \( g \in E_{11} \otimes_s l_1 \) which can be written as

\[
g = g_E g_l \quad (3.1)
\]

where

\[
g_E = e^{A_{a_1...a_3} R_{a_1...a_3}} e^{A_{a_1...a_6} R_{a_1...a_6}} e^{h_{a_1...a_8, b} R_{a_1...a_8, b}} \ldots
\]

\[
e^{h_{a b} K_a^b} \ldots e^{h_{a_1...a_8, b} R_{a_1...a_8, b}} e^{A_{a_1...a_6} R_{a_1...a_6}} e^{A_{a_1...a_3} R_{a_1...a_3}}
\]

(3.2)

and

\[
g_l = e^{x^a P_a} e^{x_{ab} Z^{a b}} e^{x_{a_1...a_5} Z^{a_1...a_5}} \ldots = e^{z^A L_A}
\]

(3.3)

where we have denoted the generalised coordinates by \( z^A \) and the generators of the \( l_1 \) representation by \( L_A \). Thus the non-linear realisation introduces a generalised space-time with the coordinates [8]

\[
x^a, x_{ab}, x_{a_1...a_5}, \ldots
\]

(3.4)

The fields that occur in the group element \( g_E \) are taken to depend on the generalised space-time that is the coordinates of equation (3.4). Since the \( l_1 \) representations contains all the brane charges [8,9,3,11] and this was responsible for the generalised space-time there is a one to one relation between the brane charges and the coordinates of the generalised space-time. Furthermore, for every field in \( E_{11} \) there corresponds an element in the \( l_1 \) representation [9]. As such for every field there is an associated coordinate in the generalised
space-time. For example, the metric corresponds to the point particle with charge $P_a$ and coordinate $x^a$, the three form corresponds to the two brane with charge $Z^{a_1a_2}$ and coordinate $x_{a_1a_2}$, the six form corresponds to the five brane with charge $Z^{a_1...a_5}$ and coordinate $x_{a_1...a_5}$ and so on. As the discussion below makes clear, the non-linear realisation $E_{11} \otimes s l_1$ automatically encodes a generalised geometry equipped with a generalised vielbein.

The non-linear realisation is by definition just a set of dynamics that is invariant under the transformations

$$g \to g_0 g, \quad g_0 \in E_{11} \otimes s l_1, \quad \text{as well as} \quad g \to gh, \quad h \in I_c(E_{11}) \quad (3.5)$$

The group element $g_0$ is a rigid transformation, that is a constant, while $h$ is a local transformation, that is it depends on the generalised space-time. As the generators in $g_l$ form a representation of $E_{11}$ the above transformations for $g_0 \in E_{11}$ can be written as

$$g_l \to g_0 g_l g_0^{-1}, g_E \to g_0 g_E \quad \text{and} \quad g_E \to g_E h \quad (3.6)$$

As a consequence the coordinates are inert under the local transformations but transform under the rigid transformations as

$$z^A l_A \to g_0 z^A l_A g_0^{-1} = z^\Pi D(g_0^{-1})_\Pi^A l_A \quad (3.7)$$

Using the local transformation we may bring $g_E$ into the form

$$g_E = e^{h_a b K^{ab} \ldots e^{h_{a_1...a_8,b} R^{a_1...a_8,b}} e^{A_{a_1...a_6} R^{a_1...a_6}} e^{A_{a_1...a_3} R^{a_1...a_3}} (3.8)$$

Thus the theory contains the graviton field $h_{a b}$ associated with the generators $K^{ab}$ of $GL(11)$, as well as the gauge fields $A_{a_1a_2a_3}$ and its dual $A_{a_1...a_6}$ associated with the level one and two generators $R^{a_1a_2a_3}$ and $R^{a_1...a_6}$ respectively. Furthermore, in addition we have a field $h_{a_1...a_8,b}$ corresponding to the generator $R^{a_1...a_8,b}$ which is the dual field of gravity [1]. The parameterisation of the group element differs from that used in some earlier works on $E_{11}$, but this does not affect any physical results.

The dynamics is usually constructed from the Cartan forms $\mathcal{V} = g^{-1} dg$ as these are inert under the $E_{11}$ rigid transformations of equation (3.5) and only transform under the local transformations as

$$\mathcal{V} \to h^{-1} \mathcal{V} h + h^{-1} dh \quad (3.9)$$

Hence if we use the Cartan forms, the problem of finding a set of field equations which are invariant under equation (3.5) reduces to finding a set that is invariant under the local subalgebra $I_c(E_{11})$, that is the transformations of equation (3.9).

The Cartan forms can be written as

$$\mathcal{V} = \mathcal{V}_E + \mathcal{V}_l \quad (3.10)$$

where

$$\mathcal{V}_E = g_E^{-1} dg_E \quad \text{and} \quad \mathcal{V}_l = g_E^{-1}(g_l^{-1} dg_l)g_E \quad (3.11)$$
The first part $\mathcal{V}_E$ is just the Cartan form for $E_{11}$ while $\mathcal{V}_l$ is a sum of generators in the $l_1$ representation. While both $\mathcal{V}_E$ and $\mathcal{V}_l$ are invariant under rigid transformations and under local transformations they change as

$$\mathcal{V}_E \rightarrow h^{-1} \mathcal{V}_E h + h^{-1} dh \quad \text{and} \quad \mathcal{V}_l \rightarrow h^{-1} \mathcal{V}_l h$$  \hfill (3.12)

Let us evaluate the $E_{11}$ part of the Cartan form

$$\mathcal{V}_E = dz^\Pi G_{\Pi,*} R^* = G_a^b K_a^b + G_{c_1\ldots c_3} R^{c_1\ldots c_3} + G_{c_1\ldots c_6} R^{c_1\ldots c_6} + G_{c_1\ldots c_8,b} R^{c_1\ldots c_8,b} + \ldots$$  \hfill (3.13)

where $\star$ denotes the indices on the generators of $E_{11}$. Explicitly one finds that

$$G_a^b = (e^{-1} de)_a^b \quad G_{c_1\ldots c_3} = \tilde{D} A_{c_1\ldots c_3}$$

$$G_{c_1\ldots c_6} = \tilde{D} A_{c_1\ldots c_6} - A_{[c_1\ldots c_3} \tilde{D} A_{c_4\ldots c_6]}$$

$$G_{c_1\ldots c_8,b} = \tilde{D} h_{c_1\ldots c_8,b} - A_{[c_1\ldots c_3} \tilde{D} A_{c_4 c_5 c_6} A_{c_7 c_8]} b + 3 A_{[c_1\ldots c_6} \tilde{D} A_{c_7 c_8]} b$$

$$+ (A_{[c_1\ldots c_3} \tilde{D} A_{c_4 c_5 c_6} A_{c_7 c_8]} b - 3 A_{[c_1\ldots c_6} \tilde{D} A_{c_7 c_8]} b)$$  \hfill (3.14)

where $e^{\mu}_a \equiv (e^h)_{\mu}^a$ and

$$\tilde{D} A_{c_1\ldots c_3} \equiv d A_{c_1 c_2 c_3} + ((e^{-1} de)_{c_1}^b A_{b c_2 c_3} + \ldots$$  \hfill (3.15)

where $+ \ldots$ denotes the action of $(e^{-1} de)$ on the other indices with analogous expressions for other quantities. In the last expression of equation (3.14) we have subtracted the totally anti-symmetric part corresponding to the fact that the generator obeys the condition $R^{[c_1\ldots c_8,b]} = 0$. Evaluating this expression we find that

$$G_{c_1\ldots c_8,b} = \tilde{D} h_{c_1\ldots c_8,b} - A_{[c_1\ldots c_3} \tilde{D} A_{c_4 c_5 c_6} A_{c_7 c_8]} b + 2 A_{[c_1\ldots c_6} \tilde{D} A_{c_7 c_8]} b$$

$$+ 2 \tilde{D} A_{[c_1\ldots c_5} b] A_{c_6 c_7 c_8]}$$  \hfill (3.16)

We note that

$$A_{[c_1\ldots c_3} \tilde{D} A_{c_4 c_5 c_6} A_{c_7 c_8]} b = 0$$  \hfill (3.17)

Let us now evaluate the part of the Cartan form in equation (3.10) containing the generators of the $l_1$ representation; we may write it as

$$\mathcal{V}_l = g^{-1} dg = dz^\Pi E_{\Pi} A_{l} = g^{-1}_E (d x^a P_a + d x_{ab} Z^{ab} + d x_{a_1 \ldots a_5} Z^{a_1 \ldots a_5} + \ldots) g_E$$  \hfill (3.18)

Using equation (2.18-21) we find that $E_{\Pi} A$, viewed as a matrix, is given at low orders by

$$E = (\text{dete})^{-\frac{1}{2}}\begin{pmatrix} e^{\mu}_a & -3 e^{\mu}_c A_{\mu b_1 b_2} & 3 e^{\mu}_c A_{\mu b_1 b_5} + \frac{3}{2} e^{\mu}_c A_{[b_1 b_2 b_3} A_{\mu b_4 b_5]} \\ 0 & (e^{-1})_{b_1}^{\mu_1} (e^{-1})_{b_2}^{\mu_2} & -A_{b_1 b_2 b_3} (e^{-1})_{b_4}^{\mu_1} (e^{-1})_{b_5}^{\mu_2} \\ 0 & 0 & e^{-1}_{b_1}^{\mu_1} \ldots (e^{-1})_{b_5}^{\mu_5} \end{pmatrix}$$  \hfill (3.19)
This illustrates the fact that the generalised space-time leads to a generalised tangent space, which in this case has the usual tangent space, two forms, five forms and higher objects. In general the tangent space can be read off from the \( l_1 \) representation in an obvious way. The \( l_1 \) representation appropriate to ten and \( d \) dimensions is found by decomposing \( E_{11} \) into \( GL(d) \otimes E_{11-d} \) and the results can be found in \([4,9,11,21]\). The tangent space group is \( I_c(E_{11}) \); at lowest level this is \( O(d) \otimes O(d) \) for the IIA theory in ten dimensions while in \( d \) dimensions it is \( SO(d) \otimes I_c(E_{11-d}) \).

Our task is to find a set of dynamics which is invariant under the rigid and local transformations of equation (3.5) and with this in mind we now consider in more detail the transformations of the two parts of the Cartan form beginning with that of \( E_{11} \) part. As noted above the Cartan forms only transform under the local \( I_c(E_{11}) \) transformations. It is useful to introduce the operation \( g^* = (I_c(g))^{-1} \) on the group. While \( I_c \) is an automorphism, i.e. on two group elements \( I_c(g_1 g_2) = I_c(g_1)I_c(g_2) \), the action of \( \ast \) reverses the order, that is \( (g_1 g_2)^* = (g_2)^* (g_1)^* \). The action of \( \ast \) on the algebra is given by \( A^* = -I_c(A) \) and \( (AB)^* = B^* A^* \). A group element belonging to \( I_c(E_{11}) \) obeys \( h^* = h^{-1} \) and the two transformations of equation (3.5) imply that \( g^* \rightarrow h^{-1} g^* (g_0)^* \). We write the Cartan forms \( V_E \) as

\[
\mathcal{V}_E = P + Q, \quad \text{where} \quad P = \frac{1}{2}(\mathcal{V}_E + \mathcal{V}_E^\ast), \quad Q = \frac{1}{2}(\mathcal{V}_E - \mathcal{V}_E^\ast) \quad (3.20)
\]

and then the transformations of equation (3.12) becomes

\[
P \rightarrow h^{-1} Ph, \quad Q \rightarrow h^{-1} Qh + h^{-1} dh \quad (3.21)
\]

Examining equation (3.13) we find that

\[
P = \frac{1}{2} G_a^b (K^a \delta_b + K^b \delta_a) + \frac{1}{2} G_{c_1\ldots c_3} (R^{c_1\ldots c_3} + R_{c_1\ldots c_3}) + \frac{1}{2} G_{c_1\ldots c_6} (R^{c_1\ldots c_6} - R_{c_1\ldots c_6})
\]

\[
+ \frac{1}{2} G_{c_1\ldots c_8, b} (R^{c_1\ldots c_8, b} + R^{c_1\ldots c_8, b}) + \ldots \quad (3.22)
\]

and

\[
Q = \frac{1}{2} G_a^b (K^a \delta_b - K^b \delta_a) + \frac{1}{2} G_{c_1\ldots c_3} (R^{c_1\ldots c_3} - R_{c_1\ldots c_3}) + \frac{1}{2} G_{c_1\ldots c_6} (R^{c_1\ldots c_6} + R_{c_1\ldots c_6})
\]

\[
+ \frac{1}{2} G_{c_1\ldots c_8, b} (R^{c_1\ldots c_8, b} - R^{c_1\ldots c_8, b}) + \ldots \quad (3.23)
\]

We note that the connection \( Q \) contains the same objects as the covariant quantity \( P \).

Taking \( h = 1 - \Lambda_{a_1 a_2 a_3} a^{a_1} a^{a_2} a^{a_3} \), the local transformations of \( P \) of equation (3.21) implies, using equations (2.12-14) and equations (2.18-21) that

\[
\delta G^{ab} = 18 \Lambda_{c_1 c_2 b} (c_1 c_2 a - 2 \delta^{a b} \Lambda_{c_1 c_2 c_3} G_{c_1 c_2 c_3}),
\]

\[
\delta G_{a_1 a_2 a_3} = -\frac{5!}{2} B^{b_1 b_2 b_3 a_1} a_{a_2 a_3} a^{b_1} a^{b_2} a^{b_3} - 6 G^c_{a_1} a_{c a_2 a_3},
\]

9
\[ \delta G_{a_1 \ldots a_6} = 2 \Lambda_{[a_1 a_2 a_3} G_{a_4 a_5 a_6]} - 8.72 G_{b_1 b_2 b_3 [a_1 \ldots a_5, a_6]} \Lambda^{b_1 b_2 b_3} + 8.72 G_{b_1 b_2 [a_1 \ldots a_5 a_6, b_3]} \Lambda^{b_1 b_2 b_3} \]

\[ \delta G_{a_1 \ldots a_8} = -3 G_{[a_1 \ldots a_6} \Lambda_{a_7 a_8]} b + 3 G_{[a_1 \ldots a_6} \Lambda_{a_7 a_8 b]} \]  (3.24)

Let us now turn our attention to the transformation of the part of the Cartan form in the direction of the \( l_1 \) representation, that is \( \mathcal{V}_l \). At lowest level equation (3.12) implies that

\[ E_{\Pi}^A = E_{\Pi}^B D(h)_B^A, \quad \text{and for the inverse} \quad (E^{-1})_A^{\Pi'} = D(h^{-1})_A^B (E^{-1})_B^\Pi \]  (3.25)

if we define \( h^{-1} L_A h = D(h)_A^B L_B \). At lowest levels this implies the local transforms

\[ \delta E_{\Pi}^a = -6 E_{\Pi b_1 b_2} \Lambda^{b_1 b_2 a}, \quad \delta E_{\Pi a_1 a_2} = 3 \Lambda_{a_1 a_2 b} E_{\Pi}^b, \ldots \]

\[ \delta (E^{-1})_a^\Pi = -3 (E^{-1})_{b_1 b_2} \Lambda_{b_1 b_2 a}, \quad \delta (E^{-1})_{a_1 a_2}^\Pi = 6 \Lambda_{a_1 a_2 b} (E^{-1})_b^\Pi, \ldots \]  (3.26)

Even though the Cartan forms are invariant under the rigid transformations, \( E_{\Pi}^A \) and \( G_{\Pi^*} \) are not as the transformation of \( z^\Pi \) of equation (3.7) implies a corresponding inverse transformation acting on the \( \Pi \) index of these two objects. Thus \( E_{\Pi}^A \) transforms under a local transformation on its \( A \) index and by the inverse generalised vielbein on its \( \Pi \) index. As such we can think of it as a generalised vielbein. We can rewrite the Cartan form of \( E_{11} \otimes_s l_1 \) as

\[ \mathcal{V} = g^{-1} dg = dz^\Pi E_{\Pi}^A (L_A + G_{A^*} R^*) \]  (3.27)

where \( G_{A^*} = (E^{-1})_A^\Pi G_{\Pi^*} \). At low levels \( (E^{-1})_A^\Pi \) is the inverse of the matrix of equation (3.19). Clearly \( G_{A^*} \) is inert under rigid transformations, but it transforms under local transformations as in equation (3.24) on its \( * \) index and as the inverse generalised vielbein on its \( A \) index, that is as in equation (3.26).

Thus if we choose to construct the dynamics out of \( G_{A^*} \) we need only worry about the local transformations as the rigid transformations are automatically taken care of. Hence we seek a set of equations that are first order in \( G_{A^*} \) and invariant under \( I_c(E_{11}) \) transformations; thus we are left with a problem in group theory.

We will now focus our attention on finding the terms in such an invariant dynamics that involve the three and six form gauge fields and the coordinates \( x^a, x_{ab} \) and \( x_{a_1 \ldots a_5} \). We can solve this problem using a trick which may not generalise to the full system. In [8] it was shown at the lowest levels that \( I_c(E_{11}) \) is the group \( SL(32) \) and the generators \( P_a, Z^{ab}, Z^{a_1 \ldots a_5} \) can be collected together in the matrix

\[ Z_a^\beta = (\gamma^a P_a + \frac{\gamma_{ab}}{2} Z^{ab} + \frac{\gamma_{a_1 \ldots a_5}}{5!} Z^{a_1 \ldots a_5})_a^\beta \]  (3.28)

where \( \alpha, \beta = 1, \ldots, 32 \) and the \( \gamma^a \) matrices are elements of the eleven dimensional Clifford algebra. In fact these first few components of the \( l_1 \) representation are the charges that occur in the eleven dimensional supersymmetry algebra and in the above equation we recognise the right hand side as the result of the the anti-commutators of two supersymmetry generators; it is the most general symmetric matrix. One can verify that the local
transformations of these generators of the $l_1$ representations, given in equation (2.24), can be written as [8]

\[ [S_{a_1 a_2 a_3}, Z_{\alpha}^\beta] = \begin{cases} \frac{\gamma_{a_1 a_2 a_3}}{2}, & Z \end{cases} \]  

(3.29)

This labeling of the generators as a bispinor implies a corresponding labeling of the coordinates and so the generalised vielbein which we can define as

\[ \mathcal{V}_s = dz^\Pi E_\Pi^A L_A \equiv dz^\Pi E_\Pi^A a Z_{\alpha}^\beta = \frac{1}{32} Tr(E^\Pi Z) \]  

(3.30)

Comparing with the expression of equation (3.18) we find that

\[ E_\Pi^\alpha = (\gamma_a E_\Pi^a - \gamma_{a_1 a_2} E_\Pi a_1 a_2 + \gamma_{a_1 \ldots a_5} E_\Pi a_1 \ldots a_5)^\alpha \]  

(3.31)

we have used that

\[ \frac{1}{32} Tr(\gamma_{a_1 \ldots a_p} \gamma_{b_1 \ldots b_p}) = (-1)^{p(p-1)/2} p! \delta_{a_1 \ldots a_p}^{b_1 \ldots b_p} \equiv (-1)^{p(p-1)/2} p! \delta_{a_1}^{b_1} \ldots \delta_{a_p}^{b_p} \]  

(3.33)

Using equations (3.12) and (3.29) we find that the infinitesimal transformation of the generalised vielbein under a local transformation when written in terms of the bispinor notation is given by

\[ \delta E_\Pi^\alpha = \frac{1}{2} \{ \gamma_{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, E_\Pi^\alpha \} \]  

(3.34)

We can define the inverse generalised vielbein by $E_\Pi^\alpha (E^{-1})^\alpha_{\beta} = \delta^\Lambda_{\Pi}$ and it transforms under a local transformation as

\[ \delta (E^{-1})^\alpha_{\beta} = -\frac{1}{2} \{ \gamma_{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, (E^{-1})^\lambda_{\Lambda} \}^\alpha_{\beta} \]  

(3.35)

Let us also reformulate the transformations of the $E_{11}$ part of the Cartan form which is in the coset, that is the object $P$ contained in equation (3.22), when restricted to contain only the three and six form fields in terms of gamma matrices. Let us define

\[ P_{\Pi, \alpha}^\beta = (\gamma_{a_1 a_2 a_3} 2 G_{\Pi, a_1 a_2 a_3} + \gamma_{a_1 \ldots a_6} 4 G_{\Pi, a_1 \ldots a_6})^\alpha_{\beta} \]  

(3.36)

One can then verify that the transformation

\[ \delta P_{\Pi, \alpha}^\beta = \frac{1}{2} \{ \gamma_{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, P_{\Pi} \}^\alpha_{\beta} \]  

(3.37)

leads to the transformations of equation (3.24) for the parts of the Cartan forms corresponding to the three form and six form fields.

As discussed above we can convert the world index on the Cartan form into a tangent space index using the inverse generalised vielbein; using the bispinor notation we define

\[ P_{\alpha}^{\beta, \gamma, \delta} \equiv (E^{-1})^\alpha_{\beta} P_{\Pi, \gamma}^\delta \]  

(3.38)
Let us define
\[ \mathcal{P}_\alpha^\beta \equiv \mathcal{P}_\alpha^{,\delta} \delta^\beta \] (3.39)
which, using (3.35) and (3.37), we find to transform as
\[ \delta \mathcal{P}_\alpha^\beta = \frac{1}{2} \{ \gamma^{a_1a_2a_3} \Lambda_{a_1a_2a_3}, \mathcal{P} \} \alpha^\beta \] (3.40)

Thus we have found an object which is inert under the rigid transformations and transforms covariantly under the local transformation and as such we have found a candidate for the equation of motion. In fact we have two possible covariant objects as we can symmetrise and anti-symmetrise on the \( \alpha \) and \( \beta \) indices after lowering the latter with the inverse charge conjugation matrix. We note that the eleven dimensional gamma matrices \( \gamma^a C^{-1} \), \( \gamma^{a_1a_2} C^{-1} \) and \( \gamma^{a_1...a_5} C^{-1} \) are symmetric while \( C^{-1} \), \( \gamma^{a_1a_2a_3} C^{-1} \) and \( \gamma^{a_1...a_4} C^{-1} \) are anti-symmetric. Let us consider the anti-symmetric part which we can set to zero to obtain the invariant equation
\[ 0 = \frac{1}{2} (\mathcal{P}_\alpha \delta (C^{-1}) \delta \gamma - \mathcal{P}_\gamma \delta (C^{-1}) \delta \alpha) \]
\[ = (\gamma^{a_1a_2a_3a_4} C^{-1})_\alpha \gamma \mathcal{P}_{a_1a_2a_3a_4} + (\gamma^{a_1a_2a_3} C^{-1})_\alpha \gamma \mathcal{P}_{a_1a_2a_3} + (C^{-1})_\alpha \gamma \mathcal{P} \] (3.41)

Thus we find the equations
\[ 2\mathcal{P}_{a_1a_2a_3a_4} \equiv G_{[a_1,a_2a_3a_4]} - \frac{3.5}{2} G_{b_1b_2, b_1b_2 a_1a_2a_3a_4} - \frac{1}{2.4!} \epsilon_{a_1a_2a_3a_4} b_1...b_7 G_{b_1,b_2...b_7} \]
\[ -\frac{1}{2} G_{b_1b_2[a_1a_2a_3, b_1b_2a_4]} + \frac{5}{4.4!} \epsilon_{a_1a_2a_3a_4} b_1...b_7 G_{c_1c_2b_1b_2b_3, c_1c_2b_4...b_7} = 0 \] (3.42)
\[ 2\mathcal{P}_{a_1a_2a_3} \equiv -6 G_{[a_1|b|, a_2a_3]} + \frac{1}{4} \epsilon_{a_1a_2a_3} b_1...b_8 G_{b_1b_2b_3...b_8} \]
\[ + \frac{1}{3!5!} \epsilon_{a_1a_2a_3} b_1...b_8 G_{b_1b_5b_6...b_8} + \frac{5.3}{2} G_{c_1c_2c_3c_4 [a_1, c_1c_2c_3c_4] a_2a_3} = 0 \] (3.43)
and
\[ 2\mathcal{P} \equiv \frac{1}{2.5!11!} \epsilon^{a_1...a_{11}} G_{a_1...a_{5}a_{6}...a_{11}} = 0 \] (3.44)

In finding these equations use was made of the identity
\[ \gamma^{a_1...a_p} \gamma_{b_1...b_q} = \sum_r (-1)^{\frac{r(r-1)}{2}} (-1)^{(p-r)r} \frac{p! q!}{r!(p-r)!(q-r)!} \epsilon^{[a_1...a_r} \gamma^{a_{r+1}...a_p]} \gamma_{b_1...b_r} \gamma_{b_{r+1}...b_q]} \] (3.45)
and equation (3.31). In considering these equations it is important to recall that we have set to zero all contributions involving the gravity and dual gravity fields.

We note that only the first of these equations involves the derivative with respect to the usual coordinates \( x^a \) of space-time. At the linearised level this equation is given by
\[ \partial_{[a_1} A_{a_2a_3a_4]} - \frac{3.5}{2} \partial_{b_1b_2} A_{b_1b_2, a_1a_2a_3a_4} = \frac{1}{2.4!} \epsilon_{a_1a_2a_3a_4} b_1...b_7 \partial_{[b_1} A_{b_2...b_7]} \]
\[-\frac{1}{2} \partial_{b_1 b_2 [a_1 a_2 a_3} A^{b_1 b_2} A_{a_4]} + \frac{5}{4!} \varepsilon_{a_1 a_2 a_3 a_4} b_1 \ldots b_7 \partial_{c_1 c_2 b_1 b_2 b_3} A^{c_1 c_2} b_4 \ldots b_7 = 0 \] (3.46)

where \( \partial_a = \frac{\partial}{\partial x^a} \), \( \partial^{ab} = \frac{\partial}{\partial x_{ab}} \) and \( \partial^{a_1 \ldots a_5} = \frac{\partial}{\partial x_{a_1 \ldots a_5}} \).

If we were to restrict the dependence of the fields to only be on \( x^a \) then the last equation would be the correct equation of motion for the three and six form fields at linearised level. Indeed at the full non-linear level we find the field equation

\[ F_{c_1 \ldots c_4} = \frac{1}{7.6.2} \varepsilon_{c_1 \ldots c_4} b_1 \ldots b_7 F_{b_1 \ldots b_7} \] (3.47)

where

\[ F_{c_1 \ldots c_4} \equiv 4 \partial_{[c_1 A_{c_2 \ldots c_4]} \] (3.48)

and

\[ F_{c_1 \ldots c_7} \equiv 7(\partial_{[c_1 A_{c_2 \ldots c_7]} - A_{[c_1 c_2 c_3} \partial_{c_4 A_{c_5 c_6 c_7]}}) \] (3.49)

The reader may wish to explicitly vary equation (3.42) using equations (3.24) and (3.26) and show that the resulting terms which contain the usual space-time derivative \( \partial_a \) do actually cancel. A useful intermediate result is that

\[ \delta(G_{[a_1 a_2 a_3 a_4]} - \frac{3.5}{2} G_{b_1 b_2}, b_1 b_2 b_1, a_1 a_2 a_3 a_4) = \frac{5!}{8} G_{[b_1 b_2 b_3 a_1 \ldots a_4]} \lambda^{b_1 b_2 b_3} \] (3.50)

from which we see that although the left-hand side is not a field strength both terms conspire so as to give a variation that is a field strength, as the equation of motion requires.

The most general invariant equation linear in generalised space-time derivatives would be a sum of the symmetric and antisymmetric parts of \( P_\gamma (C^{-1})_\delta \alpha \) with arbitrary coefficients. However, the symmetric part involves terms such as \( G_{b,a_1 a_2 a_3} \eta^{ba_1} \) which are clearly not gauge invariant if one restricts the dependence on the generalised space-time to be only on \( x^a \). The strategy used here is similar to that used in the original paper [14] on gravity, except that they used conformal symmetry to fix the constants, and the approach used in the early \( E_{11} \) papers, such as [1,12]. However, an extension of this procedure was applied in [15,16] where the non-linear realisation of \( E_{11} \otimes_s l_1 \), appropriate to four dimensions and at lowest level, was carried out. This meant keeping only the coordinates of the four dimensional space-time and those in the 56 dimensional representation of \( E_7 \) from the \( l_1 \) representations. An invariant action was then found that contained a number of undetermined constants. The constants were then fixed by demanding general coordinate invariance once the fields had been restricted to depend on the coordinates of the four dimensional space-time and only the usual seven of the 56 other coordinates. It is this strategy we are following here.

4. Discussion

The variation of equation (3.42) under the transformations of equation (3.24) will lead to a duality relation between the derivative of the graviton and that of the dual graviton. It would certainly be interesting to find what these equations are and if they really do describe the correct dynamics for gravity as it appears in the framework of eleven
dimensional supergravity once we neglect the higher $E_{11}$ fields and the higher coordinates. Should this be the case then the $E_{11}$ conjecture [1,8] will be shown to be true, namely that the non-linear realisation of $E_{11} \otimes_{s} l_1$ is an extension of the dynamics of eleven dimensional supergravity. We hope to report on this soon. Although when $E_{11}$ was first proposed the meaning of the higher fields was unknown, we have now come to understand the physical significance for large numbers of the higher level fields. The result in this paper suggests that the additional coordinates will also have a physical meaning. It would be very straightforward to extend the results in this paper to the IIA and IIB theories in ten dimensions and the theories in $d$ dimensions using the techniques previously employed [1,2,3,4,5,6,7].

The non-linear realisation of $E_{11} \otimes_{s} l_1$ has not been systematically computed before. In the early papers on $E_{11}$ only the coordinate $x^a$ was used and the local subalgebra was taken to be just the Lorentz algebra. As a result much of the power of the non-linear realisation was lost. Nonetheless many of the features of the supergravity theories were recovered, for example the fields strengths for all gauged supergravity theories in five dimensions [5]. This paper should open the way to the systematic computation of the $E_{11} \otimes_{s} l_1$ non-linear realisation and so the dynamics it contains.

Above we simply deleted the dependence of the fields on the higher coordinates. However, it remains to understand what physical procedure one should use to reduce the dependence on the fields on the generalised space-time. The work of reference [17] suggested that even though the full theory was $E_{11} \otimes_{s} l_1$ invariant only part of the $l_1$ representation occurred in the second quantised field theory. In particular although the first quantised theory involved all of the coordinates of the $l_1$ representation, the choice of representation of the commutators that takes one from the first to the second quantised theory required one to choose only part of the $l_1$ representation. However, one can make different choices of which part of the $l_1$ representation one takes and these should be equivalent and related by $E_{11}$ transformations. It would be interesting to really understand how this works. However, it is likely that a simple truncation will not be the only allowed possibility; indeed in the construction of all the five dimensional supergravity theories [5] we found a much more subtle procedure involving a slice that included part of $E_{11}$.

To close it could be helpful to discuss the relationship between the various works on generalised geometry. This paper is based on the 2003 proposal to consider the non-linear realisation of $E_{11} \otimes_{s} l_1$ [8], however, there are several other approaches. A generalised space-time appeared in the context of string dynamics where the usual space-time was extended to include an additional coordinate $y_a$. This was done in such a way as to encode the (first quantised) dynamics of the string in an $O(D,D)$ symmetric manner [18,19]; a generalisation to the membrane was also given [20]. In fact the dynamics of strings and membranes can be formulated as a non-linear realisation of $E_{11} \otimes_{s} l_1$ [21]. The difference from the non-linear realisation studied above is that a different choice of local subalgebra is taken and the coordinates associated with the $l_1$ representation become fields. The non-linear realisation can be carried out so as to include the background supergravity fields that belong to the $E_{11}$ part of the non-linear realisation in the same way as above. If one takes the non-linear realisation of $E_{11} \otimes_{s} l_1$ appropriate to ten dimensions and at lowest level one finds the generalised space-time and the string dynamics given in [18,19,20]. Carrying
out this non-linear realisation in \(d\) dimensions at lowest order one finds the coordinates of \(d\) dimensional Minkowski space-time and in addition scalar coordinates belong to the 10, 16, 27, 56 and 248 \(\oplus 1\) of \(SL(5), SO(5,5), E_6, E_7\) and \(E_8\) for \(d\) equal to seven, six, five and four and three dimensions respectively [3, 11, 21, 5]. These are the same coordinates as arises at level zero in the non-linear realisation of \(E_{11} \otimes_s l_1\) used to find the supergravity theories in \(d\) dimensions in the absence of strings and branes.

Another version of generalised geometry was inspired by a version of closed string field theory [22] and the papers of [23]. It goes by the name of doubled field theory as it also doubles the number of coordinates to have a \(x^a\) and \(y_a\) in order to encode an \(O(10,10)\) symmetry [24]. The field theory is defined on this space contains the same fields as in the NS-NS sector of the ten dimensional superstring. An action was constructed and if one restricts the fields to depend on just the usual space-time, that is just on \(x^a\), one finds the well known action for the NS-NS sector [33, 34]. However, doubled field theory is just a sub case of the non-linear realisation of \(E_{11} \otimes_s l_1\). To be precise it is the non-linear realisation of \(E_{11} \otimes_s l_1\) at lowest level in the decomposition appropriate to the IIA theory [25]. This is a very straightforward systematic calculation that requires no guess work and took only six pages in [25] to present in all detail. The advantage of viewing this as a non-linear realisation of \(E_{11} \otimes_s l_1\) is that it is places the construction in a wider conceptual framework in which the true nature of the symmetries is apparent. For example, the presence of the \(GL(1)\) symmetry in addition to \(O(10,10)\) becomes clear and one can construct the extension to the next level [26]. This is also very straightforward and one finds [26] the R-R part of the well known supergravity equations of motion. One could also compute even higher levels involving fields beyond that of the usual maximal supergravity theories and so find new physics.

The approach of [19, 20] just mentioned above had the aim of encoding some duality symmetries in the first quantised dynamics. This work was taken up in [27] which derived a generalised metric from the first quantised theory and used this to construct invariant \(SL(5)\) dynamics for fields living on a space whose coordinates belonged to the ten dimensional representation of \(SL(5)\). This work was then generalised to the duality group \(SO(5,5)\) [28]. This was in agreement with the general framework of [21] and the non-linear realisation of \(E_{11} \otimes_s l_1\) taking into account that the coordinates of the \(l_1\) representations in \(d\) dimensions at level zero are those mentioned just above. Very recently it was shown [29] in detail how these theories [27, 28] were the non-linear realisation of \(E_{11} \otimes_s l_1\) appropriate to seven and six dimensions at lowest level. The work of [29] also contained the generalisation to find the analogues of these results in five and four dimensions and so involving the duality groups \(E_6\) and \(E_7\) respectively. The precise relationship to the work of [15, 16] which earlier computed the non-linear realisation of \(E_{11} \otimes_s l_1\) at lowest level in four dimensions and used the generalised space consisting of the usual coordinates of space-time and the coordinates in the 56 dimensional representation of \(E_7\) has yet to be clarified.

There is yet another approach inspired by the work of Hitchin [30] and Gualtieri [31]. This introduced an extended tangent space, associated with \(O(D,D)\) but does not extend our usual notion of space-time, see for example [32] and references therein. There has not been a study to investigate the connection to the non-linear realisation of \(E_{11} \otimes_s l_1\). However, the generalised tangent space in ten dimensions which encodes \(O(D, D) \otimes GL(1)\)
with tangent group $O(10) \otimes O(10)$ [32], is precisely the same as that which arises in the non-linear realisation of $E_{11} \otimes s l_1$ appropriate to ten dimensions at lowest level [25,26]. As we mentioned below equation (3.19) the generalised tangent space is in general just that given by the $l_1$ representation and the tangent space group is $I_c(E_{11})$. In eleven dimensions this is just the usual tangent space, the space of two forms and five forms and higher objects [8]. While in $d$ dimensions we would find the usual tangent space, scalars belong to the $10$, $16$, $27$, $56$ and $248 \oplus 1$ of SL(5), SO(5,5), $E_6$, $E_7$ and $E_8$ for $d$ equal to seven, six, five, four and three dimensions respectively as well as higher level objects [3,21,5,11]. This leads one to suspect that if one carries out the $E_{11} \otimes s l_1$ non-linear realisation, but at the end sets all the fields to depend on just the usual coordinates $x^a$ then one might obtain this approach.

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In this note added we will derive, from the $E_{11} \otimes s l_1$ non-linear realisation, the equation of motion that relates the usual field of gravity to its dual field. We will work only to the level that the dual graviton occurs and so we will not derive the terms involving the higher level $E_{11}$ fields associated with gravity, nor shall we derive the terms that contain generalised space-time derivatives for coordinates beyond the lowest level, that is, we will only find terms with derivatives with respect to the coordinates of our customary space-time.

The notion of a dual graviton was first introduced by Curtwright [N1]. This work was based on an equation that involved the Riemann tensor. The dual graviton $h_{a_1...a_8,b}$ arises automatically as a field at level three in the $E_{11}$ non-linear realisation and it was proposed that it was related to the vielbein by a duality relation that is first order in derivatives [1]. In reference [1] a non-linear equation, involving the vielbein and a field $Y_{b_1...b_9,b}$, which did correctly describe full gravity was given and it was also shown that at the linearised level could one substitute the field $Y'_{b_1...b_9,b}$ for the derivative of the dual gravity field and find the correct equation for linearised gravity. This left the situation for the full non-linear theory unresolved. However, in reference [13] an equation involving the vielbein, its dual and the field $Y_{b_1...b_9,b}$, which was first order in derivatives, was given and shown the correctly describe gravity at all orders. This result seems to have gone somewhat unnoticed in the subsequent literature.

Although these results were encouraging, there remained the problem of how a duality relation between the vielbein and the dual gravity field would arise in $E_{11}$. This was discussed in reference [N2] and in [N3]; the latter includes considerations based on supersymmetry, but no firm contact with $E_{11}$ was found.

Equation (3.42), which relates the three form gauge field to its dual six form field, was derived by introducing objects with spinor indices in a maneuver that may not generalise to the other fields. As such we begin this section by giving a more conventional derivation of this equation of motion. In particular we will explicitly vary equation (3.42) under the local $I_c(E_{11})$ symmetry variations of equations (3.24) and (3.26). We recall that the way the dynamical equations are constructed ensures that they are automatically invariant under the rigid $E_{11} \otimes s l_1$ transformations. The equation is built from the Cartan forms of equation (3.14-16) whose indices are of the form $G_{\cdot,\cdot}$. The $\cdot$ represent the indices that are contracted with the $E_{11}$ generators that occur in the Cartan form and their variations under $I_c(E_{11})$ are given in equation (3.24). The $\diamond$ indices arise from the forms, when converted to tangent indices with the inverse generalised vielbein, that occur in the Cartan form. These latter indices transform under the local transformations given in the second line of equation (3.26). To give an example, $G_{a_1a_2,b_1b_2b_3}$ occurs with the $E_{11}$ generators $R^{b_1b_2b_3}$ and the two form $dx^\pi E_{\pi}^{a_1a_2}$ in the Cartan form. For simplicity we will often denote a generic Cartan form by just indicating the number of indices, for example we might denote the object just discussed by $G_{2,3}$. We note that this Cartan form contains generalised space-time derivatives that are with respect to $x^{a_1a_2}$ and higher level coordinates, that is it does not contain derivatives with respect to the coordinates $x^a$ of the usual space-time. Similar statements apply to all the other Cartan forms. Of course in principle one should write the variations of the $\cdot$ and $\diamond$ indices in one equation, but it can be convenient to
In deriving this result we have already implemented the cancellations that occur between gravity field and its dual. The terms in the variation that contain the three and six form gauge fields and later collect the terms that contain the Cartan forms associated with generalised space-time derivatives with respect to \(x_{a_1}...x_{a_5}\) and higher level coordinates. We will first focus on terms in the variation that contain the three and six form gauge fields, that is the Cartan formations for both of the two types of variation just mentioned, but keeping terms in the variation that contain Cartan forms \(G\).

We now carry out the variation of the equation (3.42) under the local \(I_c(E_{11})\) transformations for both of the two types of variation just mentioned, but keeping terms in the variation that contain Cartan forms \(G\).\(\ast\), with \(a\) or \(a_1a_2\), that is, we discard Cartan forms associated with generalised space-time derivatives with respect to \(x_{a_1}...x_{a_5}\) and higher level coordinates. We will first focus on terms in the variation that contain the three and six form gauge fields and later collect the terms that contain the gravity field and its dual. The terms in the variation that contain \(G_{2,3}A_3\) are given by

\[
+9G_{[a_1a_2,b_1b_2}a_3A_{a_4]b_1b_2} + 9G_{[a_1a_2,b_1b_2}a_3A_{a_4]b_1b_2} - 18G_{[a_1|b, a_2^cA_{|a_3a_4]} (N.1)
\]

In deriving this result we have already implemented the cancellations that occur between certain terms. While the terms involving \(G_{2,6}A_3\) are given by

\[
+\frac{1}{4.4} \epsilon_{a_1a_2a_3a_4} b_1...b_7 \{ \Lambda_{b_1c_1c_2} G_{c_1c_2} b_2...b_7 + 15 \Lambda_{b_1c_1c_2} G_{b_2b_3} c_1c_2, b_4...b_7 + 5.6 \Lambda_{b_2b_1b_2} G_{b_3c_1, c_1c_2} b_4...b_7 \} (N.2)
\]

where again we have not shown terms that cancel against each other.

The second and third terms of equation (N.1) can be written as

\[
-9G_{[a_1|b, a_2^cA_{|a_3a_4]} (N.3)
\]

where the anti-symmetry on \(a_1, a_2, a_3\) and \(a_4\) in this particular equation is not indicated explicitly but is to be understood to be present.

The first and second terms in equation (N.2) can be written as

\[
+\frac{7}{4} \epsilon_{a_1a_2a_3a_4} b_1...b_7 \{ \Lambda_{b_1c_1c_2} G_{c_1c_2} b_2...b_7 - 6.4 \Lambda_{b_2b_1c_2} G_{b_3c_1, b_3} b_4...b_7 \} (N.4)
\]

To lowest order equation (3.43) can be expressed as

\[
G_{[b_1b_2,b_3...b_8]} = -\frac{6.4}{8!3!} \epsilon_{b_1...b_8} e_1e_2e_3 G_{e_1b, e_2e_3} (N.5)
\]

and substituting this into the first term in equation (N.4) we find that it cancels the terms two and three in equation (N.1).

The net effect of all this is that the local \(I_c(E_{11})\) variation of equation (3.42) which contain only terms that involve the three and six form gauge fields, that is the Cartan forms \(G_{2,3}\) and \(G_{2,6}\), is given by

\[
+9G_{[a_1a_2,b_1b_2}a_3A_{a_4]b_1b_2} + \frac{6}{4.4} \epsilon_{a_1a_2a_3a_4} b_1...b_7 \{ -2 \Lambda_{b_1c_1c_2} G_{b_2c_1,b_3...b_7} + 5 \Lambda_{b_1b_2d} G_{b_3c, cd} b_4...b_7 \} (N.6)
\]

These are the first term in equation (N.1), the second term in equation (N.4) and the third term in equation (N.2). These terms can be canceled by introducing gravity and dual gravity terms into equation (3.42) whose Cartan forms have the local variations of
equations (3.24) and (3.26). Carrying out these modifications equation (3.42) is now given by

\[ 0 = G_{[a_1,a_2,a_3,a_4]} - \frac{3}{2} G_{b_1,b_2,a_1a_2a_3a_4} - \frac{1}{2} \epsilon_{a_1a_2a_3a_4} b_1 \ldots b_7 G_{b_1,b_2 \ldots b_7} \]

\[ -\frac{1}{2} G_{b_1,b_2[a_1a_2a_3,a_4]} b_1 b_2 \frac{5}{4} \epsilon_{a_1a_2a_3a_4} b_1 \ldots b_7 G_{c_1c_2b_1b_2b_3,c_1c_2b_4 \ldots b_7} \]

\[ -\frac{1}{2} G_{[a_1,a_2,a_3,a_4]} - \frac{7}{6} \epsilon_{a_1a_2a_3a_4} b_1 \ldots b_7 G_{c_b,c,b_2 \ldots b_7} \]

\( (N.7) \)

The variation of the second to last and last terms in this equation cancel the first and second variations of equation (N.6) respectively. Thus we have found an equation for the three form and six form gauge fields that is invariant under the local \( I_{c}(E_{11}) \) transformations and so all the transformations of the non-linear realisation if we discard in the variation terms containing gravity and its dual and generalised space-time derivatives with respect to coordinates beyond the two form.

To better understand the above calculation it is useful to represent it in a schematic diagram, which is given in equation (N.8). We searching for an equation with four anti-symmetric indices constructed from the Cartan forms \( G_{\diamond}, \bullet \) and the epsilon symbol, denoted by \( \star \). In the diagram below we list all possible terms in a grid going with increasing level of the fields to the right and increasing level in the generalised space-time derivatives as one goes down. Where there is no term indicated it means that there is no such term that one can write down with the correct indices.

\[ \leftrightarrow \quad G_{1,3} \leftrightarrow \star G_{1,6} \rightarrow \]

\[ \uparrow \quad \downarrow \leftrightarrow \quad \star G_{2,6} \leftrightarrow \star G_{2,8,1} \rightarrow \]

\[ \uparrow \quad \downarrow \leftrightarrow \quad G_{5,1,1} \leftrightarrow \star G_{5,3} \leftrightarrow \star G_{5,6} \leftrightarrow G_{5,8,1} \]

\( (N.8) \)

The arrows indicate what happens when one varies the terms under the local transformations. In particular, the vertical arrows indicate the effect of varying the first indices, that is the \( \diamond \) indices and the horizontal arrows the effect of varying the second indices, that is the \( \bullet \) indices. The terms so obtained contain the Cartan field at the site the arrow point to times the parameter \( \Lambda_3 \). For example, the variation of the first index on \( G_{1,3} \) leads to a term \( \Lambda_3 G_{2,3} \) whose only other sources are given by following the arrows pointing to this site; for example, one such term arises from the variation of the second index of \( G_{2,6} \). We note that just because a Cartan form is absent in the four index equation of motion this does not mean that the same Cartan form does not arise in the local variation of this equation as the variation has in general a different index structure.

Finally, we now compute the local variation of equation (3.42) keeping the remaining terms, that is, those that contain the gravity field or its dual. We will only keep derivatives with respect to the usual coordinates of space-time, that is, those with respect to \( x^a \). We note that the resulting equation will have a different index structure to that for the three form field and as a result the spaces in the diagram where no such contribution can exist are different. Varying equation (N.7) we are interested in the terms correspond to the spaces in the diagram of equation (N.8) that are in the top line at the extreme left and
right hand ends and these can only come from the variation of \( G_{a,b,c} \) \((G_{2,1,1}, G_{a,b_1...b_8,c} \) \((G_{2,8,1})\), by varying their first index, or \( G_{1,3} \) and \( G_{1,6} \) by varying their second index. Since there are no terms in the four index equation that involve the Cartan forms \( G_{a,b,c} \) and \( G_{a,b_1...b_8,c} \) the terms in the variation must either cancel or result in a new equation. In fact this new equation has three indices. The most general terms one can write down with three indices, with no particular symmetry, are given in the diagram of equation (N.9) whose interpretation is analogous to that for the diagram of equation (N.8).

\[
\begin{align*}
G_{1,1,1} & \rightarrow G_{2,3} \quad \leftarrow \star G_{1,8,1} \\
\uparrow & \quad \uparrow \quad \uparrow \\
\downarrow & \quad \downarrow
\end{align*}
\]  
\( (N.9) \)

Setting the variation to zero we find that

\[
3X_{[a_1a_2,[c|\Lambda^e_{a_3a_4}]} + 7\epsilon_{a_1a_2a_3a_4}b_1...b_7 \{G_{b_1,c_1c_2c_3b_2...b_6,b_7}\Lambda^{c_1c_2c_3} - G_{c_1,c_2b_2...b_7}\delta_d^{c_1c_2b_1}\} = 0
\]  
\( (N.10) \)

where

\[
X_{a_1a_2,c} = -G_{[a_1,a_2]c} - G_{[a_1,[c|a_2]} + G_{c,][a_1a_2]}
\]  
\( (N.11) \)

We recall that \( G_{c,a}^b = e_c^\mu(e\delta_\mu e)_a^b \) whereupon we recognise that \( X_{ab,c} = w_{c,ab} \) where \( w_{c,ab} \) is the usual spin connection. Extracting the parameter in equation (N.10) we find the equation

\[
3X_{[a_1a_2,[c|\delta_{a_3a_4}]} + 7\epsilon_{a_1a_2a_3a_4}b_1...b_7 \{G_{b_1,c_1c_2c_3b_2...b_6,b_7} - \delta_{b_1,[c_3}G_{c_1,c_2]}b_2...b_7\delta^{d}\} = 0
\]  
\( (N.12) \)

Setting \( a_3 = c_2 \) and \( a_4 = c_3 \) and summing over these indices we find that

\[
\frac{7.2}{3}w_{c_1,a_1a_2} - \frac{8}{3}w_{d,a_1,\delta^{c_1}_{a_2]} + \epsilon_{a_1a_2}b_1...b_9G_{b_1,b_2...b_9,c_1} + \frac{7}{3}\epsilon_{c_1b_1...b_8G_{b_1,b_2...b_9}}^d \]  
\( (N.13) \)

where we have substituted the spin connection.

We see that the equation has many of the correct features and in particular it relates the derivative of the vielbein, specifically the spin connection, to the derivative of the dual graviton field. Of course the full equation will contain higher level \( E_{11} \) fields and also derivatives with respect to the higher level coordinates of the generalised space-time. We observe that equation (N.13) contains not only the \( P \) part of the Cartan forms of equation (3.22) but also those of equation (3.23), that is, the \( Q \) part, which transform inhomogeneously under the local symmetry. In particular it contains the Cartan form \( G_{c,|[ab]} \) which transforms inhomogeneously under the local Lorentz group, see equation (3.21). The appearance of this Cartan form in the equations of motion is related to the fact that we have not chosen the local Lorentz transformation to set the anti-symmetric part of the graviton to zero, or equivalently, one of the off diagonal parts of the vielbein to zero. It follows that equation (N.13) does not strictly speaking hold as an equality as while the right-hand side transforms covariantly under local Lorentz transformations the left-hand
side transforms like the spin connection and so has an inhomogeneous term of the form $\partial^c \Lambda_{a_1 a_2}$ in its local variation. The $E_{11}$ algebra implies that the dual gravity field satisfies the constraint $h_{[a_1 \ldots a_8 b]} = 0$, but it is easy to see that the effect of allowing equality up to the above term is equivalent to relaxing this constraint and introducing a nine form object into the theory [13].

To find an invariant equation one must form the Riemann tensor $R_{\mu \nu, a}^b$ from the spin connection in the usual way. The corresponding terms involving the dual graviton do not vanish. However they do vanish at least at the linearised level if one then forms the usual contraction to form the Ricci tensor, by setting $\nu = b$ and then summing. Thus at the linearised level one would have the correct equation for gravity if it were it not for the following observation. We note that if we set $c_1$ and $a_1$ and then trace on the resulting index in equation (N.12) then we find that $w_{da, d} = \partial \mu (\text{det} e_a^\mu) = 0$. As has just been remarked equation (N.13) only holds up to an inhomogeneous local Lorentz transformations, but unfortunately $w_{da_1, d}$ is not of this form in general. Thus although equation (N.13) has many of the correct features it does not as it stands describe the correct equation for gravity. One might wonder if there is any term involving the dual gravity field that one could add and that would prevent $w_{da_1, d}$ being zero. However, such a term would have to be constructed from the epsilon symbol, contain one derivative acting on the field $h_{a_1 \ldots a_8, b}$ and have, after the contraction, only one index; one can easily see that this is not possible.

There are several ways out of this dilemma. The above computation could be wrong in its details and in particular if the magnitude of the second term in equation (N.13) was instead given by $-\frac{7}{15}$ then the terms involving the usual gravity field would also be traceless.

Another possibility is that there is some non-trivial dependence on the extra coordinates that leads to a non-trivial trace. However, the most likely possibility is that the trace can be non-zero if one includes the contribution of the higher level $E_{11}$ fields. The gravity fields of the $E_{11} \otimes s L_1$ non-linear realisation occur at level $0, 3, 6, 9, \ldots$ and there is in fact no reason to believe that the full gravity equation arises only at the levels zero and three computed above. In fact at level six we find the fields

$$h_{10,6,2}, h_{9,8,1}, h_{11,4,3}, h_{11,5,1,1}, h_{10,7,1}, h_{11,6,1}, h_{10,8}, h_{11,7} \quad (N.14)$$

To contribute to the gravity equation we need an object with three indices once we have differentiated with respect to the usual space-time. As this leads to an odd number of indices we cannot use a single epsilon. It is very easy to find possible terms that have a trace, for example

$$\partial_a h_{b_1 \ldots b_8}^{a b}, b_1 \ldots b_8, c \quad (N.14)$$

The presence of this term would imply that the duality relation included also higher level Cartan forms on its right-hand side, however, in view of the non-linear nature of gravity this would not be unnatural.

This is the first time that the equation involving the gravity fields has been systematically derived from the $E_{11} \otimes s L_1$ non-linear realisation. To derive this result we have only assumed that the equation is linear in derivatives and we have set to zero one constant; indeed the calculation is just a matter of $E_{11}$ group theory. One cannot help be encouraged
by the very intricate way in which the invariance is achieved and it is a convincing sign, at least to this author, that this equation has almost all the correct features at low levels.

**Additional references given in this note added**

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