I. INTRODUCTION

In the late 70’s and early 80’s, attempts were made to write down field theories that describe scalar mesons in terms of observables like currents and densities rather than the creation and annihilation operators. The motivation for doing this stems from the fact a theory cast directly in terms of observables was more physically intuitive than the more traditional approach based on raising and lowering operators on the Fock space. This attempt however, raised a number of technical questions, among them was how to make sense of the various identities connecting say the kinetic energy density to the currents and particle densities and so on. Elaborate mathematical machinery was erected by the authors who started this line of research to address these issues. However, it seems gaps still remain especially with regard to the crucial question of how one goes about writing down a formula for the annihilation operator (fermi or bose) alone in terms of bilinears like currents and densities. The bilinears in question namely currents and densities satisfy a closed algebra known as the current algebra. This algebra is insensitive to the nature of the statistics of the underlying fields. On the other hand, if one desires information about single-particle properties, it is necessary to relate the annihilation operator (whose commutation rules determine the statistics) to bilinears like currents and densities. That such a correspondence is possible was demonstrated by Goldin, Menikoff and Sharp. However their methods are somewhat abstract and difficult to follow especially for someone trained exclusively in Physics. This is the main motivation for my article which is aimed primarily at Physicists. This allows me the luxury of sacrificing some mathematical rigor and its place make plausibility arguments in the hope of drawing in a wider audience. The attempts made here are partly based on the work of Goldin et.al, Ligouri and Mintchev on generalised statistics and the series by Reed and Simon. This article is organised as follows. In the next couple of paragraphs, we prove a lemma that is going to be used repeatedly in the following sections. In the next section, we write the DPVA ansatz (this was introduced in an earlier preprint and it stands for ”Density Phase Variable Approach”) in the Fock space language. This also means relating the canonical conjugate of the density to observables like currents and densities. In the end we write down the main conjecture which when proven will vindicate the DPVA ansatz. A partial proof(no rigor) of the conjecture is given elsewhere. Now for the Lemma.

**Lemma** Let $F$ be smooth a function from a bounded subset of the real line on to the set of reals. Also let $f$ and $g$ be smooth functions from some bounded subset of $\mathbb{R}^d$ to reals. Let us further assume that the range of these functions are such that it is always possible to find compositions such as $F \circ f$ and they will also be smooth functions with sufficiently big domains. The claim is this, since they possess fourier transforms (they are well-behaved) If,

$$F( f(x) ) = g(x) \quad (1)$$

and,

$$f(x) = \sum_q \hat{f}_q e^{i q \cdot x} \quad (2)$$

$$g(x) = \sum_k \hat{g}_k e^{i k \cdot x} \quad (3)$$

then the following also holds,

$$[F( \sum_q \hat{f}_q T_q(k) )] \delta(k,0) = \hat{g}_k \quad (4)$$
where $T_q(k) = \exp(q, \nabla_k)$. Here the operator $T_q(k)$ acts on the $k$ in the Kronecker delta on the extreme right, and every time it translates the $k$ by an amount $q$.

**Proof**

Proof is by brute force expansion. We know,

$$F(y) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} y^n$$

therefore,

$$F( f(x) ) = F(0) + \sum_{n=1}^{\infty} \frac{F^{(n)}(0)}{n!} \sum_{\{q_i\}} \tilde{f}_{q_1} \tilde{f}_{q_2} \cdots \tilde{f}_{q_n} \exp(i \sum_{i=1}^{n} q_i \cdot x)$$

$$= \sum_k e^{i \cdot k \cdot x} \tilde{g}_k$$

This means (take inverse fourier transform),

$$F(0) \delta_{k,0} + \sum_{n=1}^{\infty} \frac{F^{(n)}(0)}{n!} \sum_{\{q_i\}} \tilde{f}_{q_1} \tilde{f}_{q_2} \cdots \tilde{f}_{q_n} \delta_{k - \sum_{i=1}^{n} q_i} \cdot 0$$

$$= \tilde{g}_k$$

This may also be cleverly rewritten as

$$F(0) \delta_{k,0} + \sum_{n=1}^{\infty} \frac{F^{(n)}(0)}{n!} \sum_{\{q_i\}} \tilde{f}_{q_1} \tilde{f}_{q_2} \cdots \tilde{f}_{q_n} T^{-q_1}(k) T^{-q_2}(k) \cdots T^{-q_n}(k) \delta_{k,0}$$

$$= \tilde{g}_k$$

and therefore,

$$\tilde{g}_k = [F( \sum_q \tilde{f}_q T^{-q}(k) )] \delta_{k,0}$$

and the **Proof** is now complete.

**II. FOCK SPACE REPRESENTATION**

In earlier preprints, we introduced the so-called DPVA ansatz (see Appendix also Ref [4]) that related the annhilation operator to the canonical conjugate of the density distribution. In this section, we try to formulate this correspondence in the Fock space language. We start off with some preliminaries. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert Space. We know from textbooks that such a space possesses a countable orthonormal basis $\mathcal{B} = \{ w_i; i \in \mathbb{Z} \}$. Here, $\mathbb{Z}$ is the set of all integers. Thus $\mathcal{H} = \text{Set of all linear combinations of vectors chosen from } \mathcal{B}$. We construct the tensor product of two such spaces

$$\mathcal{H} \otimes \mathcal{H}$$

This is defined to be the dual space of the space of all bilinear forms on the direct sum. In plain English this means something like this. Let $f \in \mathcal{H}$ and $g \in \mathcal{H}$ define the object $f \otimes g$ to be that object which acts as shown below. Let $<v,w>$ be an element of the Cartesian product $\mathcal{H} \times \mathcal{H}$.

$$f \otimes g <v,w> = (f,v)(g,w)$$
Here, \((f, v)\) stands for the inner product of \(f\) and \(v\). Define also the inner product of two \(f \otimes g\) and \(f' \otimes g'\)

\[
(f \otimes g, f' \otimes g') = (f, f')(g, g')
\]

Construct the space of all finite linear combinations of objects such as \(f \otimes g\) with different choices for \(f\) and \(g\). Lump them all into a set. You get a vector space. It is still not the vector space \(\mathcal{H} \otimes^2\). Because the space of all finite linear combinations of objects such as \(f \otimes g\) is not complete. Not every Cauchy sequence converges. Complete the space by appending the limit points of all Cauchy sequences from the space of all finite linear combinations of vectors of the type \(f \otimes g\). This complete space is the Hilbert space \(\mathcal{H} \otimes^2\). Similarly one can construct \(\mathcal{H} \otimes^n\) for \(n = 0, 1, 2, 3, \ldots\) Where we have set \(\mathcal{H}^0 = \mathbb{C}\) the set of complex numbers. Define the Fock Space over \(\mathcal{H}\) as

\[
\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H} \otimes^n
\]

Physically, each element of it is an ordered collection of wavefunctions with different number of particles

\[(\varphi_0, \varphi_1(x_1), \varphi_2(x_1, x_2), \ldots, \varphi_n, \ldots)\]

is a typical element of \(\mathcal{F}(\mathcal{H})\). This is the Hilbert Space which we shall be working with. Let \(D^n\) be the space of all decomposable vectors.

\[D^n = \{f_1 \otimes \ldots \otimes f_n; f_i \in \mathcal{H}\}\]

For each \(f \in \mathcal{H}\) define

\[
b(f) : D^n \to D^{n-1}, n \geq 1
\]

\[
b^*(f) : D^n \to D^{n+1}, n \geq 0
\]

defined by

\[
b(f) f_1 \otimes \ldots \otimes f_n = \sqrt{n} (f, f_1) f_2 \otimes \ldots \otimes f_n
\]

\[
b^*(f) f_1 \otimes \ldots \otimes f_n = \sqrt{n+1} f \otimes f_1 \otimes f_2 \otimes \ldots \otimes f_n
\]

We also define \(b(f)\mathcal{H}^0 = 0\). By linearity we can extend the definitions to the space of all finite linear combinations of elements of \(D^n\) namely \(L(D^n)\). For any \(\varphi \in L(D^n)\) and \(\psi \in L(D^{n+1})\)

\[
\| b(f)\varphi \| \leq \sqrt{n} \| f \| \| \varphi \|
\]

\[
\| b^*(f)\varphi \| \leq \sqrt{n+1} \| f \| \| \varphi \|
\]

\[(\psi, b^*(f)\varphi) = (b(f)\psi, \varphi)\]

So long as \(\| f \| < \infty\), \(b(f)\) and \(b^*(f)\) are bounded operators. An operator \(\mathcal{O}\) is said to be bounded if

\[
sup_{\| \varphi \| = 1} \| \mathcal{O}\varphi \| < \infty
\]

\(\mathcal{O}\) is unbounded otherwise. The norm of a bounded operator is defined as

\[
\| \mathcal{O} \| = sup_{\| \varphi \| = 1} \| \mathcal{O}\varphi \|
\]

In order to describe fermions, it is necessary to construct orthogonal projectors on \(\mathcal{F}(\mathcal{H})\). In what follows \(c(f)\) will denote a fermi annihilation operator. \(c^*(f)\) will denote a fermi creation operator. Physically, and naively speaking, these are the fermi operators in "momentum space" \(c_k\) and \(c_k^*\). First define \(P_-\) to be the projection operator that projects out only the antisymmetric parts of many body wavefunctions. For example,
We now have
\[ c(f) = P_-(f)P_+ \]
\[ c^*(f) = P_-(b^*(f))P_+ \]

Let us take a more complicated example. Let us find out how \( c^*(f)c(g) \) acts on a vector \( v = f_1 \otimes f_2 \).

\[ c^*(f)c(g) = P_-(b^*(f))P_-(P_-(b(g))P_+) \]
\[ c^*(f)c(g) = P_-(b^*(f))P_-(b(g))P_+ \]
\[ c^*(f)c(g)v = P_-(b^*(f))P_-(b(g))P_+ v \]

\[ P_-v = \frac{1}{2!}(f_1 \otimes f_2 - f_2 \otimes f_1) \]
\[ b(g)P_-v = \frac{1}{2!}\sqrt{2}((g, f_1)f_2 - (g, f_2)f_1) \]
\[ P_-b(g)P_-v = \frac{1}{2!}\sqrt{2}((g, f_1)f_2 - (g, f_2)f_1) \]
\[ b^*(f)P_-b(g)P_-v = \frac{1}{2!}\sqrt{2}^2((g, f_1)f_2 \otimes f_2 - (g, f_2)f_1 \otimes f_1) \]
\[ c^*(f)c(g)v = (\frac{1}{2!})^2\sqrt{2}^2((g, f_1)[f_2 \otimes f_2 - f_2 \otimes f] - (g, f_2)[f_1 \otimes f_1 - f_1 \otimes f]) \]

Having had a feel for how the fermi operators behave, we are now equipped to pose some more pertinent questions. Choose a basis
\[ \mathcal{B} = \{ w_i; i \in \mathbb{Z} \} \]

**A. Definition of the Fermi Density Distribution**

Here we would like to capture the notion of the fermi density operator. Physicists call it \( \rho(x) = \psi^*(x)\psi(x) \). Multiplication of two fermi fields at the same point is a tricky business and we would like to make more sense out of it. For this we have to set our single particle Hilbert Space:
\[ \mathcal{H} = L_p(\mathbb{R}^3) \otimes \mathcal{W} \]
Here, \( L_p(\mathbb{R}^3) \) is the space of all periodic functions with period \( L \) in each space direction. That is if \( u \in L_p(\mathbb{R}^3) \) then
\[ u(x_1 + L, x_2, x_3) = u(x_1, x_2, x_3) \]
\[ u(x_1, x_2 + L, x_3) = u(x_1, x_2, x_3) \]
\( u(x_1, x_2, x_3 + L) = u(x_1, x_2, x_3) \)

\( \mathcal{W} \) is the spin space spanned by two vectors. An orthonormal basis for \( \mathcal{W} \)

\[ \{ \xi_\uparrow, \xi_\downarrow \} \]

A typical element of \( \mathcal{H} \) is given by \( f(x) \otimes \xi_\downarrow \). A basis for \( \mathcal{H} \) is given by

\[ B = \{ \sqrt{\frac{1}{L^3}} \exp (i q_n \cdot x) \otimes \xi_\sigma ; n = (n_1, n_2, n_3) \in \mathbb{Z}^3, s \in \{ \uparrow, \downarrow \}; q_n = \left( \frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L} \right) \} \]

We move on to the definition of the fermi-density distribution. The Hilbert Space \( \mathcal{H} \otimes^n \) is the space of all \( n \)-particle wavefunctions with no symmetry restrictions. From this we may construct orthogonal subspaces

\[ \mathcal{H}_+ \otimes^n = P_+ \mathcal{H} \otimes^n \]

\[ \mathcal{H}_- \otimes^n = P_- \mathcal{H} \otimes^n \]

Tensors from \( \mathcal{H}_+ \otimes^n \) are orthogonal to tensors from \( \mathcal{H}_- \otimes^n \). The only exceptions are when \( n = 0 \) or \( n = 1 \).

\[ \mathcal{H}_+^0 = \mathcal{H}_-^0 = \mathcal{C} \]

\[ \mathcal{H}_+^1 = \mathcal{H}_-^1 = \mathcal{H} \]

The space \( \mathcal{H}_+ \otimes^n \) is the space of bosonic-wavefunctions and the space \( \mathcal{H}_- \otimes^n \) is the space of fermionic wavefunctions.

The definition of the fermi density distribution proceeds as follows. Let \( v \) be written as

\[ v = \sum_{\sigma \in \{ \uparrow, \downarrow \}} a(\sigma) \xi_\sigma \]

The Fermi density distribution is an operator on the Fock Space, given a vector \( f \otimes v \in \mathcal{H} \) in the single particle Hilbert Space, and a tensor \( \varphi \) in the \( n \)-particle subspace of \( \mathcal{F}(\mathcal{H}) \), there exists a corresponding operator \( \rho(f \otimes v) \) that acts as follows:

\[ [\rho(f \otimes v) \varphi]_n(x_1 \sigma_1, x_2 \sigma_2, ..., x_n \sigma_n) = 0 \]

if \( \varphi \in \mathcal{H}_+ \otimes^n \) and

\[ [\rho(f \otimes v) \varphi]_n(x_1 \sigma_1, x_2 \sigma_2, ..., x_n \sigma_n) = \sum_{i=1}^n f(x_i) a(\sigma_i) \varphi_n(x_1 \sigma_1, x_2 \sigma_2, ..., x_n \sigma_n) \]

when \( \varphi \in \mathcal{H}_- \otimes^n \). The physical meaning of this abstract operator will become clear in the next subsection.

**B. Definition of the Canonical Conjugate of the Fermi Density**

We introduce some notation. Let \( g = \exp (i k_m \cdot x) \otimes \xi_r \) (the square root of the volume is not needed in the definition of \( g \) since we want all the operators in the fourier space such as \( \psi(k_m r) \) to be dimensionless.

\[ \psi(k_m r) = c(g) \]
\[ \rho(k_m r) = \rho(g) \]

This \( \rho(k_m r) \) is nothing but the density operator in momentum space, familiar to Physicists

\[ \rho(k_m r) = \sum_{q_n} c_{q_n}^\dagger c_{q_m r} \]

We want to define the canonical conjugate of the density operator as an operator that maps the Fock space(or a subset thereof) on to itself. If \( \varphi \in \mathcal{H}^0 = \mathbb{C} \) then

\[ X_{qm} \varphi = 0 \]

Let \( \varphi \in \mathcal{H}_+^n, n = 2, 3, \ldots \) then

\[ X_{qm} \varphi = 0 \]

The important cases are when \( \varphi \in \mathcal{H}_+^n, n = 2, 3, \ldots \) or if \( \varphi \in \mathcal{H} \). In such a case, we set \( N_s = N_s^0 \neq 0 \) for \( s \in \{\downarrow, \uparrow\} \).

Let us introduce some more notation. \( N_s = \rho_{q_m} \) is the number operator to be distinguished from the c-number \( N_s^0 \). The eigenvalue of \( N_s \) is \( N_s^0 \) when it acts on a state such as \( \varphi \in \mathcal{H}_+^n \). Some more notation.

\[ \delta \psi(k_m s) = \psi(k_m s) - \sqrt{N_s^0} \delta_{km,0} \]

and

\[ \delta \rho(k_m s) = \rho(k_m s) - N_s^0 \delta_{km,0} \]

The Canonical Conjugate of the density distribution in real space denoted by \( \Pi_s(x) \) is defined as follows.

\[ \Pi_s(x + L, x_2, x_3) = \Pi_s(x_1, x_2, x_3) \]

\[ \Pi_s(x_1, x_2 + L, x_3) = \Pi_s(x_1, x_2, x_3) \]

\[ \Pi_s(x_1, x_2, x_3 + L) = \Pi_s(x_1, x_2, x_3) \]

The definition is as follows.

\[ \Pi_s(x) = \sum_{qm} \exp(i q_m x) X_{qm, s} \]

\[ X_{qm, s} = i \ln(1 + \frac{1}{\sqrt{N_s^0}} \sum_{kn} \delta \psi(k_n s) T_{-k_n}(q_m))(1 + \sum_{kn} \delta \rho(k_n s) T_{k_n}(q_m))(1 + \sum_{kn} \phi([\rho], k_n s) T_{k_n}(q_m) T_{-k_n}(q_m)) \delta_{qm,0} \]

\[ \Phi([\rho]; x s) = \sum_{kn} \phi([\rho], k_n s) \exp(-ik_n x) \]

\[ T_{k_n}(q_m) = \exp(k_n \nabla_{q_m}) \]

The translation operator translates the \( q_m \) in the Kronecker delta that appears in the extreme right by \( k_n \) and \( \Phi([\rho]; x s) \) satisfies a recursion explained in detail in the previous manuscript. The logarithm is to be interpreted as an expansion around the leading term which is either \( N_s^0 \) or \( \sqrt{N_s^0} \). The question of existence of \( X_{qm, s} \) now reduces to demonstrating that this operator (possibly unbounded) maps its domain of definition(densely defined in Fock space) on to the Fock space. Defining the limit of the series expansion is likely to be the major bottleneck.
in demonstrating the existence of $X_{\mathbf{q}_m s}$. That this is the canonical conjugate of the density operator is not at all obvious from the above definition. A rigorous proof of that is also likely to be difficult. Considering that we arrived at this formula by first postulating the existence of $\Pi_s(x)$, it is probably safe to just say "it is clear that $\Pi_s(x)$, is in fact the canonical conjugate of $\rho$. The way in which the above formula can be deduced may be motivated as follows:

$$\psi(x_\sigma) = \frac{1}{V^\frac{1}{2}} \sum_k \exp(i\mathbf{k}.x)\psi(k_\sigma)$$

$$= \exp(-i \sum_q \exp(i\mathbf{q}.x)X_{\mathbf{q}_s})\exp(i \sum_q \exp(-i\mathbf{q}.x)\phi(\rho; \mathbf{q}_s))$$

$$(n_\sigma + \frac{1}{V} \sum_{q \neq 0} \rho_{q_\sigma} \exp(-i\mathbf{q}.x))^\frac{1}{2}$$  (11)

In the above Eq.(11) ONLY on the right side make the replacements,

$$\exp(i\mathbf{q}.x) \rightarrow T_{-\mathbf{q}}(\mathbf{k})$$  (12)

$$\exp(-i\mathbf{q}.x) \rightarrow T_{\mathbf{q}}(\mathbf{k})$$  (13)

where

$$T_{\mathbf{q}}(\mathbf{k}) = \exp(\mathbf{q}.\nabla_\mathbf{k})$$  (14)

and append a $\delta_{\mathbf{k},0}$ on the extreme right. Also on the LEFT side of Eqn.(11) make the replacement $\psi(x_\sigma)$ by $\psi(k_\sigma)$. And you get a formula for $\psi(k_\sigma)$. In order to get a formula for $X_{\mathbf{q}_m s}$ we have to invert the relation and obtain,

$$\Pi(x_s) = i \ln(\sqrt{N_0^s} + \sum_{k_n} \exp(i\mathbf{k}_n.x)\delta\psi(k_n s))$$

$$\exp(-i \sum_{k_n} \exp(-i\mathbf{k}_n.x)\phi(\rho; \mathbf{k}_n s))(N_0^s + \sum_{k_n} \delta\rho_{k_n s} \exp(-i\mathbf{k}_n.x))^{-\frac{1}{2}}$$  (15)

Make the replacements on the right side of Eqn.(15)

$$\exp(i\mathbf{k}_n.x) \rightarrow T_{-k_n}(\mathbf{q}_m)$$  (16)

$$\exp(-i\mathbf{k}_n.x) \rightarrow T_{k_n}(\mathbf{q}_m)$$  (17)

where, and append a $\delta_{\mathbf{q}_m,0}$ on the extreme right. Also on the LEFT side of Eqn.(15) replace $\Pi(x_s)$ by $X_{\mathbf{q}_m s}$. This results in formula given in Eqn. (10).

III. THE CANONICAL CONJUGATE IN TERMS OF THE CURRENT

We would like to express the field operator in terms of currents and densities. This is inspired directly by the work of Sharp, Menikoff and Goldin. To this end let us make the following statements. If one insists on having a self-adjoint canonical conjugate of the density then one must sacrifice positivity of the density. On the other hand if one insists on having a positive density, one must sacrifice self-adjointness of the canonical conjugate. This is true even when the underlying single-particle hilbert space is made separable. Let us assume that we have decided one way or the other. Then, we would like to make a statement that gives us a rigorous way of deciding whether it is possible to write down a formula for the field operator in terms of currents and densities. To this end, let us do the following, first define the current operator rigorously. To Physicists, it is,
\[ J(x) = \left( \frac{1}{2i} \right)[\psi^\dagger (\nabla \psi) - (\nabla \psi)^\dagger \psi] \]  

(18)

To the Math-community it is an operator similar to the density, given a typical element \( f \otimes v \) associated with the underlying single-particle hilbert space, there is an operator denoted by \( J_s(f \otimes v) \), ( \( s = 1, 2, 3 \) ) that acts on a typical tensor from the n-particle subspace of the full Fock space as follows,

\[ [J_s(f \otimes v) \varphi]_n(x_1 \sigma_1, x_2 \sigma_2, ..., x_n \sigma_n) = 0 \]  

(19)

\( \omega|6 \) if \( \varphi \in \mathcal{H}_n^+ \) and, and,

\[ [J_s(f \otimes v) \varphi]_n(x_1 \sigma_1, x_2 \sigma_2, ..., x_n \sigma_n) = -i \sum_{k=1}^{n} \{ f(x_k) a(\sigma_k) \nabla_s^b + \frac{1}{2} [\nabla_s^b f(x_k)] a(\sigma_k) \} \varphi_n(x_1 \sigma_1, x_2 \sigma_2, ..., x_n \sigma_n) \]  

(20)

if \( \varphi \in \mathcal{H}_n^+ \). For the bosonic current it is the other way around. Having done all this, we would now like to write the DPVA ansatz more rigorously. Now for some notation. As before, let \( g = \exp(i \ k_m \cdot x) \otimes \xi_r \) (the square root of the volume is not needed as we want all operators in momentum space to be dimensionless). Then as before,

\[ \psi(k_m r) = c(g) \]  

(21)

\[ \rho(k_m r) = \rho(g) \]  

(22)

\[ \delta \rho(k_m r) = \rho(k_m r) - N^0_r \delta_{k_m,0} \]  

(23)

\[ j_s(k_m r) = J_s(g) \]  

(24)

\[ \delta j_s(k_m r) = j_s(k_m r) \]  

(25)

Having done this, we would like to write down another formula for the canonical conjugate. Recall that,

\[ \nabla \Pi(x \sigma) = (-1/\rho(x \sigma)) J(x \sigma) + \nabla \Phi(\rho; x \sigma) - [-i \Phi, \nabla \Pi] \]  

(26)

Then we have (bear in mind here that we have distinguished between the c-number \( N^0_r \) and the operator \( \rho(0r) \) whose expectation value is \( N^0_r \)).

\[ (i \ q_m) X_{q_m r} = -\left[ \frac{1}{N^0_r} \right] \frac{1}{1 + \frac{1}{N^0_r} \sum_{k_n} \delta \rho(k_m r) T_{k_n}(q_m)} \left[ \sum_{p_n} \delta j_l(p_n r) T_{p_n}(q_m) \right] \delta_{q_m,0} + F(\rho; q_m r) \]  

(27)

where,

\[ \sum_{q_m} \exp(i \ q_m \cdot x) F(\rho; q_m r) = \nabla \Phi - [-i \Phi, \nabla \Pi] \]  

(28)

Without loss of generality, we may set \( X_{0_r} = 0 \), as this contributes just a constant phase. For \( q_m \neq 0 \)

\[ X_{q_m r} = \frac{1}{q_m^2} \frac{i}{N^0_r} \frac{1}{1 + \frac{1}{N^0_r} \sum_{k_n} \delta \rho(k_m r) T_{k_n}(q_m)} \left[ \sum_{p_n} \delta j_l(p_n r) T_{p_n}(q_m) \right] \delta_{q_m,0} \]  

(29)

Now define an operator which is defined to be the formal expansion that the formula itself suggests,

\[ \tilde{\psi}(k_m r) = \exp(-i \sum_{q_m} T_{-q_m}(k_m) X_{q_m r}) \exp(i \sum_{q_m} T_{q_m}(k_m) \phi(\rho; q_m r)) \left( N^0_r + \sum_{q_m} \delta \rho(q_m r) T_{q_m}(k_m) \right) \frac{1}{q_m^2} \delta_{q_m,0} \]  

(30)
IV. THE MAIN CONJECTURE

In this section, we would like to write down a statement that would require a proof. This conjecture when proven will vindicate the DPVA ansatz. Let us focus on fermions, the bose case is analogous.

**Conjecture**
There exists a unique functional \( \Phi([\rho];x) \) and a unique odd (for fermions, even for bosons) integer \( m \) such that the following recursion holds,

\[
\Phi([\rho(y_1 \sigma_1) - \delta(y_1 - x') \delta_{\sigma_1 \sigma}];x)
+ \Phi([\rho];x') - \Phi([\rho];x) = m\pi
\]

and has the following additional effects. The domain of definition of \( \tilde{\psi}(k_n r) \) (in which the series expansion converges), is the same as that of \( \psi(k_n r) \) and it acts the same way too. In other words,

\[
\tilde{\psi}(k_n r) = \psi(k_n r)
\]

We know how the ingredients of \( \tilde{\psi}(k_n r) \) namely the current \( j(k_n r) \) and the density \( \delta\rho(q_n r) \) act on typical elements of the Fock space, and we know how \( \psi(k_n r) \) acts on the Fock space, we just have to show that the complicated \( \tilde{\psi}(k_n r) \) acts the same way as the simple \( \psi(k_n r) \). Moreover, this is true for a unique phase functional \( \Phi \).

V. APPENDIX

In this section, we try to recapture in a compact way the results of earlier works. In particular, the DPVA (which stands for the density phase variable approach) ansatz is brought out in a way that is readily appreciated by Physicists. The DPVA ansatz relates the annihilation operator to the canonical conjugate of the relevant density distribution. It may be written as follows,

\[
\psi(x) = \exp(-i \int x d_l (-1/\rho(y))J(y) + \Phi([\rho];x)) \psi(x)
\]

here \( \Phi \) satisfies the recursion explained earlier. \( \Pi \) is canonically conjugate to the density \( \rho = \psi^*\psi \). It satisfies canonical equal-time commutation rules with the density,

\[
[\Pi(x), \rho(y)] = i \delta(x - y)
\]

The formula that relates the canonical conjugate of the density to currents and particle densities may also be written down.

\[
\Pi(x) = \int x d_l (-1/\rho(y))J(y) + \Phi([\rho];x) - \int x d_l [-i \Phi, \nabla \Pi](y)
\]

The formula for the annihilation operator in terms of currents and densities is

\[
\psi(x) = e^{-i \int x d_l (-1/\rho(y))J(y)} e^{i \Phi([\rho];x)} \psi(x) \]

The solution for the phase functional \( \Phi \) has been given in an earlier preprint. In the high density limit, the answer may be written down as follows,

\[
\Phi([\rho];x) = 0
\]

in the bose case and in the fermi case,

\[
\Phi([\rho];x) = \sum_{q\neq 0} \rho_q U_q(x)
\]
and,

\[ U_q(x) = e^{-i q \cdot x} U_0(q) \]  

(39)

where again,

\[ U_0(q) = \frac{1}{N} \left( \frac{\theta(k_f - |q|) - w_1(q)}{w_2(q)} \right)^{\frac{1}{2}} \]  

(40)

and,

\[ w_1(q) = \left( \frac{1}{4 N^2 c_q^2} \right) \sum_k \left( \frac{k \cdot q}{m} \right)^2 (\Lambda_k(-q))^2 \]  

(41)

\[ w_2(q) = \left( \frac{1}{N} \right) \sum_k (\Lambda_k(-q))^2 \]  

(42)

here, \( \Lambda_k(q) = n_F(k + q/2)(1 - n_F(k - q/2)) \)

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