Random Cayley Graphs I: Cutoff and Geometry for Heisenberg Groups

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Abstract

Consider the random Cayley graph of a finite group $G$ with respect to $k$ generators chosen uniformly at random, with $1 \ll \log k \ll \log |G|$. A conjecture of Aldous and Diaconis asserts, for $k \gg \log |G|$, that the random walk on this graph exhibits cutoff at a time which is a function only of $|G|$ and $k$. The conjecture is verified for all Abelian groups, but has neither been verified nor rejected for non-Abelian groups.

We establish cutoff for the Heisenberg group $H_{p,d}$ of $d \times d$ uni-upper triangular matrices with entries modulo $p$, subject to some mild conditions; we allow $1 \ll k \ll \log |H_{p,d}|$ as well as $k \gg \log |H_{p,d}|$. In doing so, we refute the conjecture: the cutoff time is max\{$\log |H_{p,d}|$, $s_0 k$\}, where $s_0$ is the time at which the entropy of Poisson($s_0$) is $(\log |A|)/k$, where $A \cong \mathbb{Z}_p^{d-1}$ is the Abelianisation of $H_{p,d}$; this cannot be written as a function only of $|H_{p,d}|$ and $k$.

We also show that the graph distance from the identity for all but $o(|G|)$ of the elements of $G$ lies in $[M - o(M), M + o(M)]$, where $M$ is the minimal radius of a ball in $\mathbb{Z}_p^k$ of cardinality $p^{d-1}$.

Keywords: cutoff, mixing times, random walk, random Cayley graphs, concentration of measure, entropy, diameter, typical distance

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1 Introduction

Consider a finite group \( G \). Let \( Z \) be a multiset of \( G \), called the generators. We consider the (nearest-neighbour) random walk on the directed Cayley multigraph generated by the multiset \( Z \), denoted \( G^+ (Z) \): the vertex set is \( G \) and edge multiset is

\[
\{(g,g \cdot z) \mid g \in G, z \in Z \}
\]

(of directed edges);

if the walk is at \( g \in G \), then a step involves choosing \( z \in Z \) uniformly at random and moving to \( gz \).

We focus on the case where \( Z = [Z_1, ..., Z_k] \) with \( Z_1, ..., Z_k \sim \text{Unif}(G) \), and denote \( G^+ := G^+ (Z) \).

We say that a statistic is “independent of the algebraic structure of the group” when, up to subleading order terms, it is fully determined by \( n := |G| \) and \( k \), the number of generators used; for example, such a statistic would be the same for both \( G := \mathbb{Z}_n \) and \( G := \mathbb{Z}_n \oplus \mathbb{Z}_\sqrt{n} \).

In this article, our focus is directed towards cutoff and typical distance statistics when the underlying group \( G \) is a Heisenberg matrix group \( H_{p,d} \) (defined precisely later); in a companion article \([32]\), we consider the same statistics for general Abelian groups.

Write \( n := |G| \). Our results are asymptotic as \( n \to \infty \); we require \( 1 \ll \log k \ll \log n \), and emphasise that we do not assume that either of \( p \) or \( d \) is bounded as \( n \to \infty \). We say that an event holds with high probability, abbreviated \( \text{whp} \), if its probability tends to 1 as \( n \to \infty \). See §1.3 for standard usage of asymptotic notation; in particular, “\( \approx \)” means “up to a \( 1 \pm o(1) \) factor” and “\( \asymp \)”, “\( \leq \)”, “\( \geq \)”, and “\( \gg \)” mean, respectively, “\( \leq \)”, “\( = \)”, “\( \geq \)”, and “\( \gg \)”, up to constants”.

1.1 Summarised Statements of Results

We now state our results, in summarised form. More refined statements are given later.

Cutoff Phenomenon

A sequence of Markov chain is said to exhibit cutoff when, in a short time-interval, known as the cutoff window, the total variation (TV) distance of the distribution of the chain from its invariant distribution drops from close to 1 to close to 0. For the random walk on the Cayley graph with generators \( Z \), write \( d_Z(t) \) for this TV distance at time \( t \); this is a random variable (in \( Z \)). We use standard notation and definitions for mixing, cutoff and total variation distance; see, eg, \([37]\).

For the random walk on a random Cayley graph, Aldous and Diaconis \([1]\) conjectured that there is cutoff at a time independent of the algebraic structure of the group; see §1.2 for history.

We now define the non-Abelian group of Heisenberg matrices \( H_{p,d} \): it is all \( d \times d \) upper triangular matrices with 1s on the diagonal and other elements in \( \mathbb{Z}_p \); note that \( |H_{p,d}| = p^{(d-1)/2} \).

For \( N \in \mathbb{N} \), let \( t_0(k,N) \) denote the time at which the entropy of rate-1 Poisson process (abbreviated \( \text{PP} \)) on \( Z \) becomes \( \log N/k \). We call \( t_0 \) an entropic time. The mixing time will be \( \max \{\log_k n, t_0(k,p^{d-1})\} \), which cannot be written as a function only of \( k \) and \( n \); it depends strongly on \( d \) too. This runs counter to the Aldous–Diaconis conjecture. It is the first example of this phenomenon in the regime \( k \gg \log n \); see §1.2.

A more refined statement than the one below is given in Theorem 3.4; see also Hypotheses A.

Theorem A (Cutoff). Let \( G := H_{p,d} \) with \( p \) prime and \( d \geq 3 \). Assume the following: 1 \( \ll \log k \ll d^2 \log p \); if \( k \ll d \log p \), then \( d^3 \ll k \) and \( \log d \leq d \log p/k \); if \( k \gtrsim d \log p \), then \( d \ll \log \log p \).

Whp, the random walk on \( G_k \) exhibits cutoff at time \( t_*(k,p,d) := \max \{t_0(k,p^{d-1}), \log_k |G|\} \).

Moreover, there is a phase transition in the mixing time at \( k = \log(p^{d-1})^{1/2}/(d-2) \):

- for \( k \leq \log(p^{d-1})^{1/2}(d-2) \), we have \( t_*(k,p,d) \approx \log_k (p^{d-1}k^{1/2}G) = k|H_{p,d}|/|H_{p,d}|; H_{p,d}|; H_{p,d}|) \);
- for \( k \geq \log(p^{d-1})^{1/2}(d-2) \), we have \( t_*(k,p,d) \approx \log_k |G| = \left(\frac{d-1}{2}\right) \log p/\log k \).

A description of \( t_0(k,p,d) \), up to subleading order terms, can be found in Proposition 2.2.

The Abelianisation \( H_{p,d}/[H_{p,d}, H_{p,d}] \) is isomorphic to \( \mathbb{Z}_p^{d-1} \); it corresponds to the super-diagonal of the matrices. Our companion article \([32]\) deals with (general) Abelian groups; see specifically \([32, \text{Theorem A}]\). It shows that the mixing time when \( G := \mathbb{Z}_p^{d-1} \) is precisely \( t_0(k,p^{d-1}) \); in general, subject to some conditions, it is \( t_0(k,|G|) \). Roughly, we split the analysis into “the mixing of the
Abelianisation” and “the mixing of the commutator (ie ‘non-Abelian part’)”. The structure of the proof is the same for all $k$, except in bounding one specific (combinatorial) probability.

An diameter-based argument lower bounds the mixing time by $\log k n$.

**Remark A.** Universality of cutoff for this model has been well-studied when the underlying group is Abelian; the vast majority studied $k \gg \log |G|$. The purpose of this article is to establish, for the first time, cutoff when the underlying group is non-Abelian, in the regime $\log k \lesssim \log \log |G|$ (ie $k = (\log |G|)^{O(1)}$), including $k \ll \log |G|$. We use certain Heisenberg matrix groups, which are a canonical class of nilpotent groups; further, our analysis extends somewhat to other nilpotent groups—see §5.2. Hence this is a first step towards establishing universality of cutoff for nilpotent groups. △

**Typical Distance**

Our next result concerns typical distance in the random Cayley graph, for $k \ll \log n / d$: when there are $k$ generators, for $R \geq 0$ and $\beta \in (0,1)$, write

$$B_k(R) := \{ x \in G_k \mid \text{dist}(0, x) \leq R \} \quad \text{and} \quad D_k(\beta) := \min \{ R \geq 0 \mid |B_k(R)| \geq \beta |G| \}.$$  

Informally, we show the mass (in terms of number of vertices) concentrates at a thin ‘slice’, or ‘shell’, consisting of vertices at a distance $M (1 \pm o(1))$ from the origin, where $M \propto k n^{2/(dk)}$.

Investigating this typical distance when $k$ diverges with $n$ was suggested to us by Benjamini [8].

Previous work had concentrated on fixed $k$, ie independent of $n$; see §1.2. As with cutoff, we analyse the Abelianisation and the commutator separately.

A more refined statement than the one below is given in Theorem 4.1; see also Hypotheses B.

**Theorem B (Typical Distance).** Let $G := H_{p,d}$ with $p$ prime and $d \geq 3$. Assume that $1 \ll k \ll d \log p$, that $k \leq \frac{1}{2} d \log p / \log d$ and that $d^3 \ll k$. Write $M := kp^{(d-1)/k} / e$.

Then, for all constants $\beta \in (0,1)$, we have

$$|D(\beta) - M| / M = o(1) \quad \text{whp over } Z.$$  

Alternatively, $M$ can be given by the minimal radius of a ball in $Z^k_n$ with volume at least $p^{d-1}$.

**Remark B.** By a classical result to generate a nilpotent group it is enough that the maps of the generators under $g \mapsto g[G,G]$ generates the Abelianisation $G/[G,G]$; this follows from the fact that for nilpotent groups $[G,G] \leq \Phi(G)$, the Frattini subgroup of non-generators of $G$. We prove a quantitative version of this result, where the typical distance in $G$ is very close to the typical distance in $G/[G,G]$ for the Cayley graph with the generators $[Z_1[G,G], \ldots, Z_k[G,G]]$. △

Interesting is the way we prove Theorem B. It is quite common in mixing proofs to use geometric properties of the graph, such as expansion or distance properties. We, in essence, do the opposite: we adapt the mixing proof to prove this geometric result. (We give a proof-outline in §4.2.) This is in the same spirit as [30]; see §1.2.

**1.2 Historic Overview**

In this section, we give a fairly comprehensive account of previous work on random Cayley graphs, for cutoff and then for typical distance, and compare our results with existing ones. The cutoff phenomenon in particular for this model has received a great deal of attention over the years.

**Cutoff Phenomenon**

Aldous and Diaconis [1] conjectured that there exists a time $t_\ast(k, n)$ so that, for $k \gg \log n$ and any group $G$ with $|G| = n$, for $k$ generators, the random walk exhibits cutoff at time $t_\ast(k, n)$.

There has since been much work on this conjecture and related problems, almost all of which deals with the regime $k \gg \log n$. However, for $\log k \lesssim \log \log n$, the only situations in which cutoff has been established require the underlying group to be Abelian. (We give a detailed account of previous work in this case in our companion article, [32, §1.2].)
An upper bound, valid for any group, has been proved in the regime \( k \gg \log n \): it proves that the mixing time is upper bounded by \( \frac{\log k}{\rho} n \), where \( \rho \) is defined by \( k = (\log n)^\rho \), and \( k \gg \log n \); see Dou and Hildebrand [26, Theorem 1] and Roichman [45, Theorems 1 and 2]. Combined with a basic diameter lower bound of \( \log k n \), this establishes the conjecture when \( \rho \to \infty \), i.e., \( k \gg \log \log n \), as then \( \frac{\log k}{\rho} \to 1 \), with \( t^*_k(n) := \log k n \); see [25, Theorem 4.1]. A matching lower bound, valid only for Abelian groups, again in the regime \( k \gg \log n \), was established by Hildebrand [34, Theorem 3]; see also Hildebrand [35, Theorem 5]. (In \([32, \S 3.3]\), we give a lower bound, valid for any Abelian group and \( 1 \ll k \ll \log n \); further, it is concise.) This established the Aldous–Diaconis conjecture for all Abelian groups when \( k \gg \log n \), with \( t^*_k(n) := \frac{\log k}{\rho} n \).

Aldous and Diaconis [1] made their conjecture in the mid 1980s; the above works were done in the 1990s. It was not until recently, in 2017, that results were established when \( k \ll \log n \); Hough [36, Theorem 1.7] established cutoff when \( G := \mathbb{Z}_p \), for \( p \) prime and \( 1 \ll k \ll p/\log p \). Additionally, in 1997, Wilson [52] established cutoff for the group \( \mathbb{Z}_d^2 \); since \( k \geq d = \log k n \) is needed to generate the group, this implies that \( k \gg \log n \). Wilson’s methods are specialised to the group \( \mathbb{Z}_d^2 \), as are Hough’s to \( \mathbb{Z}_p \).

In our companion article [32], we establish cutoff for a very wide range of Abelian (at a time depending only on \( k \) and \( n \)), for any \( k \) with \( 1 \ll k \ll \log n \); see [32, Theorem A].

Even with all this progress for Abelian groups, no-one had established cutoff for any non-Abelian group, unless \( k \) grew super-polynomially in \( n \); in fact, no lower bound, other than the diameter-based lower bound of \( \log k n \), was known. We improve on this in \([32, \S 3.2]\), giving a lower bound in terms of the size of the commutator \([G, G]\); general groups, see Remark 3.6.

For the Heisenberg matrix group \( H_{p,d} \), with \( p \) prime, we establish cutoff provided \( d \) does not grow too rapidly—in particular, we allow consideration of any constant \( d \).

This cutoff time cannot, however, be written only as a function of \( k \) and \( n \); it depends strongly on \( d \) too. While the aforementioned upper bound, valid for all groups and \( k \gg \log n \), is tight for Abelian groups, it turns out to be off by a factor of \( 2/d \) for \( H_{p,d} \) when \( \log(p^{d-1}) \ll k \ll \log(p^{d-1}) + 2/(d-2) \); in particular, we can allow \( d \to \infty \), sufficiently slowly so that there exists \( k \) in the above range with \( k \gg \log n \), and then the upper bound is not even of the correct order. This shows that the Aldous–Diaconis conjecture does not hold once one allows non-Abelian groups.

Random walks on the Heisenberg group have been the focus of a great deal of attention; focus has primarily been on \( 3 \times 3 \) matrices. (See in particular [15, \S 2.1] and [44, \S 1.1], upon which we have based the description below.) The probabilistic study of random walks on \( H_{p,d} \) was initiated by Zack [53]; she interpreted the walk in terms of random number generation; focus has been on \( d = 3 \). Using a specific generating set of size 4, the correct order of mixing was established by Diaconis and Saloff-Coste [20, 18, 19], in the ‘90s, using geometric theoretical tools: the first proof use Nash inequality; the second ‘moderate growth’, and extends to more general nilpotent groups; the third lifts to a random walk on a free nilpotent group, and uses Harnack inequalities. Work of Alexopoulos [2] should allow similar results for less symmetric walks on quotients of groups with polynomial growth. A fourth proof appears in Diaconis [21] (2010), using conjugacy classes, character theory and comparison theory. Stong, in a series of papers [49, 50, 51] from 1995, proves results which specialise to three yet different proofs. Lastly, another proof, this time using discrete Fourier analysis, was given by Bump et al. [15] (2017).

Moving even further from the realm of Abelian groups, it is natural to consider the set-up where \( p \) is fixed and \( d \) diverges—\( H_{p,d} \) is a step-(\( d-1 \)) nilpotent group; loosely speaking, the larger \( d \) is, the less Abelian the group is. A simple walk on \( H_{p,d} \) was introduced by Coppersmith and Pak [17, 43]; a row is chosen uniformly and added to or subtracted from the row above. The case of growing \( d \) was first studied by Ellenberg [27], then subsequently improved upon by Stong [51] and generalised by Ellenberg and Tyomkzo [28]. It was then studied by Peres and Sly [44], in particular detail when \( p = 2 \), answering a question by Stong [51]. The dependency on \( p \) in the mixing time was analysed recently by Nestoridi [42], using super-character theory and comparison arguments.

In a recent impressive work, Diaconis and Hough [22] introduced a new method for proving a CLT for random walks on nilpotent groups. They illustrate the method on \( H_{o,d} \), obtaining some extremely precise results on the rate at which individual coordinates mix as a function of their distance from the diagonal. They show that the greater the distance from the diagonal is, the faster the mixing time of the coordinate is. In the same spirit, we show that, in many cases, the
bottleneck for mixing is the super-diagonal coordinates, while in the rest of the cases, the cutoff time is given by the diameter-based lower bound $\log_k n$.

Related work includes analysis of the spectrum of a random walk on the Heisenberg group by Bguln, Valette and Zuk [5]. There has also been work on a random walk on a $3 \times 3$ Heisenberg group with entries in $\mathbb{R}$. See, for example, Breuillard [12, 13].

The Cayley graph of the symmetric group $S_n$ has also been studied. Perhaps the most famous is by Diaconis and Shahshahani [23]: they prove cutoff for the Cayley graph generated by all transpositions; it is the first paper in which cutoff is established, for any Markov chain. For arbitrary generating sets, a quasipolynomial bound on the diameter was established by Helfgott and Seress [31]; assuming that the generating set contains an element fixing at least 37% of the points, a polynomial bound is given by Bamberg et al. [4]. See [4, 31] for further background.

We now put our results into a broader context. A common theme in the study of mixing times is that 'generic' instances often exhibit the cutoff phenomenon. In this set-up, a sequence of transition matrices chosen from a certain family of distributions is shown to, whp, give rise to a sequence of Markov chains which exhibits cutoff. A few notable examples include random birth and death chains [24, 48], the simple or non-backtracking random walk on various models of sparse random graphs, including random regular graphs [10], random graphs with given degrees [6, 7, 9, 10], the giant component of the Erdős–Rényi random graph [9] (where the authors consider mixing from a 'typical' starting point) and a large family of sparse Markov chains [10], as well as random walks on a certain generalisation of Ramanujan graphs [11] and random lifts [11, 10].

A recurring idea in the aforementioned works establishing the cutoff phenomenon for certain families of random instances is that the cutoff time can be described in terms of entropy: one can look at some auxiliary random process which up to the cutoff time can be coupled with, or otherwise related to, the original Markov chain—often in the above examples this is the random walk on the corresponding Benjamini–Schramm local limit; this technique has been used recently in [9] and then in [16]. The cutoff time is then shown to be (up to smaller order times) the time at which the entropy of the auxiliary process equals the entropy of the invariant distribution of the original Markov chain. This is the case in the present work also: we use a $k$-dimensional rate-1 Poisson process as our auxiliary random process; more details are given in §2.1.

In all previous examples, the Benjamini–Schramm limit had been a tree, eg a Poisson Galton–Watson tree in [9] and a deterministic period tree in [16]. Ours is the first example where the graphs are not close in the local topology to a tree.

**Typical Distance**

As well as determining cutoff for these random Cayley graph, we study a geometric property of a diameter flavour; recall the concept of typical distance from Theorem B. Previous work (detailed below) had concentrated on the case where the number of generators $k$ is a fixed number, ie one which does not increase as the size $n$ of the group increases. In contrast, our results are in the situation where $k \to \infty$ as $n \to \infty$; this line of enquiry was suggested to use by Benjamini [8].

Amir and Gurel-Gurevich [3] studied the diameter of the random Cayley graph of cyclic groups of prime order. They prove that the diameter is order $n^{1/k}$; see [3, Theorems 1 and 2]. They conjecture that the diameter divided by $n^{1/k}$ converges in distribution to some non-trivial random variable as $n \to \infty$; see [3, Conjecture 3].

Shapira and Zuck [47] verify this conjecture on the diameter; they also consider the girth. Further, they consider an $L_p$-type distance measure; see their “(II)”, [47, Page 2]. The find the limiting distribution for all three of these statistics (for fixed $k$).

Marklof and Strömbergsson [41] consider the diameter of the random Cayley graph of $\mathbb{Z}_n$.

Lubetzky and Peres [39] derive an analogous typical distance result for $n$-vertex, $d$-regular Ramanujan graphs: whp all by $o(n)$ of the vertices lie at a distance $\log_{d-1} n \pm O(\log \log n)$; they establish this by proving cutoff for the non-backtracking random walk at time $\log_{d-1} n$.

Related work on the diameter of random Cayley graphs, including concentration of certain measures, can be found in [38, 46].

The Aldous–Diaconis conjecture for mixing can be transferred naturally to typical distance: the mass should concentrate at a distance $M$, where $M$ can be written as a function only of $k$ and $n$;
ie there is concentration of mass at a distance independent of the algebraic structure of the group.

In our companion article [32, Theorem B] we consider typical distance analogously to this paper, but there the underlying group is a general Abelian group (subject to some mild conditions). The $M$ for these Abelian groups can be written as a function only of $k$ and $n$, while for the Heisenberg groups in Theorem B above this is not the case.

### 1.3 Additional Remarks

#### Asymptotic Results

Our results are asymptotic as the size of the group diverges. More formally, we consider a sequence $(G_N)_{N \in \mathbb{N}}$ of groups, which satisfies $n_N := |G_N| \to \infty$ as $N \to \infty$; also indexed by $N$ is $k_N$. (We require $1 \ll \log k \ll \log n$, which translates to $\lim_N k_N = \infty$ and $\lim_N \log k_N / \log n_N = 0$.)

Instead of writing statements like $G_N = H_{p_N, d_N}$, we drop the $N$ and just write $G = H_{p, d}$; however, the parameters $d$ and $p$ need not be constant, and but diverge as $n = |G|$ increases.

Similarly, we sometimes drop the $t$-dependence from the notation, eg writing $S$ instead of $S(t)$.

#### Invariant Distribution

For statistics regarding the Cayley graph generated by $Z$, we add $Z$ to the notation, eg writing $d_{mix}(\cdot)$ and $\nu_{mix}(\cdot)$: when $Z$ is chosen randomly, these statistics are thus random.

We also denote by $S = (S(t))_{t \geq 0}$ the random walk and by $\pi_G$ the uniform distribution on $G$, which is invariant for $S$; it is the unique invariant distribution when the graph is connected.

#### Additional Notation

Throughout the article, we use the following notation:

$$k = (\log n)^p, \quad \frac{1}{2}d = (\log n)^\nu \quad \text{and} \quad n = |H_{p, d}| = p^{d(d-1)/2}.$$  

We emphasise that we allow $\nu$ (and hence $d$) to depend on $n$—in particular, we may have $\nu \ll 1$, but since $p \geq 2$, this enforces $\limsup \nu \leq \frac{1}{2}$. Precise conditions are given later.

The commutator is $[G, G] := \langle ghg^{-1}h^{-1} \mid g, h \in G \rangle$; the Abelianisation is $G/[G, G]$, and it is the largest Abelian normal subgroup of $G$.

For functions $f$ and $g$, write $f \approx g$ if $f(N)/g(N) \to 1$ as $N \to \infty$; also write $f \ll g$, or $g \gg f$, if $f(N)/g(N) \to 0$ as $N \to \infty$. Write $f \lesssim g$, or $g \gtrsim f$, if there exists a constant $C$ so that $f(N) \leq Cg(N)$ for all $N$; also write $f \asymp g$ if $f \lesssim g \lesssim f$.

Also write $f = O(g)$ if $f \lesssim g$ and $f = o(g)$ if $f \ll g$. Throughout the paper, unless otherwise explicitly mentioned all limits will be as the size of the group diverges.

#### Undirected Cayley Graphs

One can define undirected Cayley graphs analogously to directed ones: the edge set is simply

$$\{(g, g \cdot z) \mid g \in G, z \in Z\} \quad \text{instead of} \quad \{(g, g \cdot z) \mid g \in G, z \in Z\}.$$

The random walk then chooses $z \in Z$ uniformly at random and moves to one of $g z$ or $g z^{-1}$ with equal probability. One can then analyse mixing or typical distance for this undirected graph.

In this article, we stick to the directed case. In our companion article [32], where we study analogous questions for general Abelian groups (instead of Heisenberg groups), we consider both undirected and directed graphs.

#### Simple Graphs

The Cayley graph is simple if and only if no generator is picked twice, ie $Z_i \neq Z_j$ for all $i \neq j$, and, in the undirected case, additionally no generator is the inverse of another, ie $Z_i \neq Z_j^{-1}$ for all $i$ and $j$. Since, by assumption, $k/\sqrt{n} \to 0$ as $n \to \infty$, the probability of this event tends to 1 as $n \to \infty$. Hence our “whp over $Z$” results all also hold when the generators are chosen uniformly at random from $G$ but conditional on giving rise a simple Cayley graph.

6
2 Entropic Method and Times

2.1 Description of Entropic Methodology

We use an ‘entropic method’, as mentioned in §1.2; cf. [9, 10, 11, 16]. The method is fairly general; we now explain the specific application in a little more depth.

For mixing, we define an auxiliary random process \((W(t))_{t \geq 0}\), recording how many times each generator has been used. More precisely, for \(t \geq 0\), for each generator \(i = 1, \ldots, k\), write \(W_i(t)\) for the number of times that it has been picked by time \(t\). By independence, \(W(\cdot)\) forms a rate-1 PP on \(\mathbb{Z}_+^k\); each \(W_i(\cdot)\), for \(i = 1, \ldots, k\), is an independent rate-1/\(k\) PP on \(\mathbb{Z}_+\); in particular, the law of \(W_i(t)\) is Poisson with mean \(t/k\).

Since the underlying group is non-Abelian, \(S(t)\) is not a function of \(W(t)\) alone; rather, the full information \((W(t'))_{W \leq t}\) is needed. Even still, analysis of \(W(t)\) will be integral. A key ingredient is to analyse the commutator \([G, G]\) and Abelianisation \(G/[G, G]\) separately; we elaborate in §3.

Recall that the invariant distribution is uniform on \(G\), giving mass \(1/n\) to each vertex. The proposed mixing time is then the time at which the auxiliary process \(W\) obtains entropy \(\log n\). This time can be calculated fairly precisely in many situations; see Proposition 2.2.

For typical distance, we define a related auxiliary random variable, \(A\), corresponding to the number of times each generator is used: \(A\) is uniformly distributed on a \(k\)-dimensional lattice ball of a certain radius. We apply the chosen generators in a uniformly random order. We do not apply an entropic method here, per se, but the underlying principles of the proof are extremely similar.

2.2 Definition of Entropic Times and Concentration

In this section, we define the notion of entropic times. For \(t \geq 0\), write \(\mu_t\), respectively \(\nu_t\), for the law of \(W(t)\), respectively \(W_i(t)\); so \(\mu_t = \nu_t^{\otimes k}\). Also, for each \(i = 1, \ldots, k\), define

\[
Q_i(t) := -\log \nu_t(W_i(t)), \quad \text{and set} \quad Q(t) := -\log \mu_t(W(t)) = \sum_k Q_i(t).
\]

Definition 2.1. For \(k, N \in \mathbb{N}\), define the entropic time \(t_0(k, N)\) so that \(\mathbb{E}(Q_1(t_0(k, N))) = \log N/k\). We apply this with \(N := p^{d-1} = n^{2/d}\); abbreviate \(t_0 := t_0(k, p^{d-1}) = t_0(k, n^{2/d})\).

Direct calculation, given in §2.3, with the Poisson distribution gives the following relations.

Proposition 2.2. Assume that \(1 \ll \log k \ll \log N\). Write \(\kappa := k/\log N\). For all \(\lambda \in (0, \infty)\), the following relations hold, for some continuous function \(f\):

\[
\begin{align*}
t_0(k, N) &\approx k \cdot N^{2/k}(2\pi e) \quad \text{when} \quad k \ll \log N; \\
t_0(k, N) &\approx k \cdot f(\lambda) \quad \text{when} \quad k \approx \lambda \log N; \\
t_0(k, N) &\approx k \cdot 1/(\kappa \log \kappa) \quad \text{when} \quad k \gg \log N.
\end{align*}
\]

(We apply this with \(n := p^{(d-1)/2}\) and \(N := p^{d-1} = n^{2/d}\); hence \(\log N = \frac{2}{d} \log n\).

By a standard argument considering appropriate subsequences, to cover the general case \(k \approx \log N\), it suffices to assume that \(k/\log N\) actually converges, say to \(\lambda \in (0, \infty)\).

Observe that, since the \(W_i\), and hence the \(Q_i\), are iid, \(Q\) is a sum of \(k\) iid random variables. Also, it turns out that \(\text{Var}(Q(t)) \approx \text{Var}(Q(t_0)) \gg 1\) when \(t \approx t_0\); see the [32, Corollary A.4].

Proposition 2.2 above shows that \(t_0 \approx t_0\) for all \(\alpha \in \mathbb{R}\). This means that, by a simple application of Chebyshev’s inequality, we get the following concentration result.

Proposition 2.3. Assume that \(k\) satisfies \(1 \ll \log k \ll \log N\). Then \(\text{Var}(Q(t_0)) \gg 1\), and further, for \(\xi \in \mathbb{R}\), writing \(\nu := \text{Var}(Q(t_0))\) and \(\omega := \text{Var}(Q(t_0))^{1\/4} = (\nu k)^{1\/4}\), we have

\[
P(Q((1 + \xi)t_0) \geq \log N + \omega) \to 1 (\text{sgn}(\xi) = 1) = \begin{cases} 1 & \text{when} \quad \xi > 0, \\ 0 & \text{when} \quad \xi < 0. \end{cases}
\]

(There is no specific reason for choosing this \(\omega\); we just need some \(\omega\) with \(1 \ll \omega \ll (\nu k)^{1/2}\).)
2.3 Finding the Entropic Times: Proof of Proposition 2.2

This section is devoted to the proof of Proposition 2.2. Write \( s_0 := t_0/k \). For \( s \geq 0 \), write \( H(s) \) for the entropy of the Poisson distribution with mean \( s \). Since \( H \) is strictly increasing, we have

\[ H(s) = s \log(1/s) + s + e^{-s} \sum_{\ell=0}^{\infty} s^{\ell} \log(\ell)! / \ell! \]

\( \) as \( s \to \infty \).

**Proof when \( k \ll \log N \).** It is shown in [29] that

\[ H(s) = \frac{1}{2} \log(2\pi es) - \frac{1}{2} s^{-1} + O(s^{-2}) \]

as \( s \to \infty \). Solving \( H(s_0) = \log N/k \), and using \( t_0 = s_0 k \), gives (2.1a).

**Proof when \( k \ll \log N \).** In this regime, the target entropy \( \log N/k \gg 1 \), and so \( s_0 \ll 1 \). Hence all the random variables in question are order-1 random variables, in the sense that they do not tend to 0 or \( \infty \) in probability as \( n \to \infty \). When \( k \ll \lambda \log N \), all the desired expressions are continuous functions of \( \lambda \). Thus (2.1b) holds, with \( f(\lambda) := H^{-1}(1/\lambda) \).

**Proof when \( k \gg \log n \).** In this regime, the target entropy \( \log N/k \ll 1 \), and so \( s_0 \ll 1 \). It is straightforward to see that the sum \( \sum_{\ell=0}^{\infty} s^{\ell} \log(\ell)! / \ell! = O(s^2) \), as \( s \to 0 \). Hence

\[ H(s) = s \log(1/s) + s + O(s^2) \]

as \( s \to 0 \). Solving \( H(s_0) = \log N/k \), and using \( t_0 = s_0 k \), gives (2.1c).

3 Total Variation Mixing

In this section, we consider mixing for the random walk on the random directed Cayley graph of the Heisenberg group \( H_{p,d} \). We take \( p \) prime and “\( \equiv \)” means “equivalent modulo \( p \”).

It is straightforward to see that \( H_{p,d} \) is a nilpotent group of step \( d-1 \): this means that iteratively quotienting the group by its centre reduces the group to an Abelian group in \( d-2 \) iterations. We call \( [H_{p,d}, H_{p,d}] \) the commutator and \( H_{p,d}/[H_{p,d}, H_{p,d}] \) the Abelianisation (noting that the latter is an Abelian group). Moreover, \( H_{p,d}/[H_{p,d}, H_{p,d}] \approx \mathbb{Z}_p^{d-1} \). Set \( n := [H_{p,d}] = p^{d(d-1)/2} \).

Informally, we show that there is competition between mixing of the Abelianisation and of the commutator. Which part governs the mixing depends on the regime of \( k \): for \( k \ll (\log n)^{1+2/(d-2)} \), it is the Abelianisation, meaning that the overall mixing time is the same as that for \( \mathbb{Z}_p^{d-1} \); for \( k \gg (\log n)^{1+2/(d-2)} \), it is non-Abelian part, and the overall mixing time is given by the standard diameter-based lower bound of \( \log n \); see Definition 3.1 and Theorem 3.4.

Recall that we use the notation \( k = (\log n)^{\epsilon}, \frac{1}{2}d = (\log n)^{\nu} \) and \( n = p^{d(d-1)/2} \).

3.1 Precise Statement and Remarks

In this section, we state the more refined version of Theorem B.

**Definition 3.1.** Define \( t_{\text{diam}}(k,n) := \log_k n \). Define

\[ t_*(k,p,d) := \max\{ t_0(k,p,d-1), t_{\text{diam}}(k,p^{d(d-1)/2}) \} \]

Abbreviate \( t_{\text{diam}} := t_{\text{diam}}(k,p^{d(d-1)/2}) \) and \( t_0 := t_0(k,p,d) \).

The following proposition determines \( t_* \) up to a \( 1 + o(1) \) factor; it follows easily from Proposition 2.2 and Definition 3.1, using \( N := n^{2/d} = p^{d-1} \). Note that \( (\log n)^{1-\nu} = \frac{1}{d} \log n = \log N \).
Proposition 3.2. We have the following approximation to $t_*$:

$$t_* \approx \begin{cases} 
  k \cdot \frac{1}{\alpha^2} n^{4/(dk)} & \text{when} \quad 1 \leq k \leq 2 \log n; \\
  k \cdot f(\lambda) & \text{when} \quad k \approx \frac{\lambda}{\log n}; \\
  \frac{1}{\rho} k^{\log \log n} & \text{when} \quad \frac{2}{\theta} \log n \leq k \leq (\frac{3}{\theta} \log n)^{1+2/(d-2)}; \\
  \frac{1}{\rho} \log n & \text{when} \quad (\frac{4}{\theta} \log n)^{1+2/(d-2)} \leq k, \log k \ll \log n.
\end{cases}$$  \hspace{1cm} (3.1a) (3.1b) (3.1c) (3.1d)

(The third regime is empty if $(\log n)^{1/d} \approx 1$, i.e. $d \geq \log log n$, or equivalently $d \geq \log log p$; in this case, the fourth regime has lower bound $k \gg \frac{1}{\theta} \log n$.)

There are some simple conditions that the parameters must satisfy for our proof to be valid.

**Hypotheses A.** The triple $(k, p, d)$ satisfies Hypotheses A if the following conditions hold:

- $1 \leq \log k \leq d^2 \log p$;
- if $k \leq d^2 \log p$, then $d^2 \leq k$ and $k \leq d \log p / \log d$;
- if $k \geq d \log p$, then $\log d \leq \log \log p$.

(Recall that implicitly we consider sequences $(k_N, p_N, d_N)_{N \in \mathbb{N}}$.)

**Remark.** When $d = 1$, the first condition above implies the second and third. Also, if $k$ is sufficiently small in terms of $p$ (e.g. $k \ll \log p$), then $k \leq d \log p / \log d$ is automatically satisfied. \hspace{1cm} △

While having the conditions only in terms of $(k, p, d)$, as it allows us to be given a group $H_{p,d}$ and then determine for which $k$ our proof is valid, it is also convenient to consider $(k, n, p, d)$ as jointly given and ask for conditions in terms of this quadruple; this latter point of view will turn out to be more helpful for the proof.

**Remark 3.3.** Writing $n := p^{d(d-1)/2}$, the conditions of Hypotheses A imply the following:

- we have $1 \leq \log k \leq \log n$, and in particular $n \gg k \gg 1$;
- if $k \ll \frac{2}{\theta} \log n$, then $d^2 \ll k$ and $k \ll \frac{\log n}{\log d}$;
- if $k \geq \frac{2}{\theta} \log n$, then $\log d \ll \log \log n$ (i.e. $d = (\log n)^{\alpha(1)}$, i.e $\nu \ll 1$).

Again, the first condition implies the latter two whenever $d$ is a constant. \hspace{1cm} △

We now state the main result of this section; it is in essence a restatement of Theorem A.

**Theorem 3.4 (Cutoff).** Let $(k, p, d)$ be integers with $p$ prime and $d \geq 3$, satisfying Hypotheses A. Let $G := H_{p,d}$ be a Heisenberg group. Then the random walk on $G_k$ exhibits cutoff at time $t_*$ w.h.p. over $Z$. Moreover, the implicit lower bound on mixing holds deterministically for all $Z$.

**Remark 3.5 (Window and Shape).** Not only does our proof show cutoff w.h.p., but, in the regime where $k \ll \frac{2}{\theta} \log n$, we find the cutoff window, and even the shape. We explain this in detail in §3.6 below; in summary, for $\alpha \in \mathbb{R}$, we define times $t_\alpha$ and show that they satisfy

$$t_0(k, n^{2/d}) \approx k \cdot \frac{1}{\alpha^2} n^{4/(dk)}, \quad t_\alpha - t_0 \approx \alpha \sqrt{2} t_0 / \sqrt{k} = o(t_0) \quad \text{and} \quad d_Z(t_\alpha) \approx \Psi(\alpha) \text{ w.h.p.}$$ \hspace{1cm} △

**Remark.** For ease of presentation, consider for the moment $d$ independent of $n$. One can see that

$$T(p, N) := \frac{\rho}{p} \log \log n = \frac{\rho}{p} t_{\text{diam}}(k, N)$$

satisfies $T(p, N) \approx t_0((\log N)^p, N)$ when $\rho > 1$ is bounded away from 1. Recall that the Abelianisation $H_{p,d} / [H_{p,d}, H_{p,d}]$ has size $p^{d-1}$. Hence the above says that if $H_{p,d}$ is projected to its Abelianisation, then the random walk would have cutoff at $t_0(k, p^{d-1})$. Our proof shows that cutoff occurs at the maximum of this and the diameter lower bound, $t_{\text{diam}}(k, n)$, with $n = p^{d(d-1)/2}$.

This heuristic is only valid when $\rho > 1$. From our companion article [32, Theorem A], one sees that the walk projected to the Abelianisation has cutoff at $t_0(k, p^{d-1})$ for $\rho \leq 1$ too; in this regime, $t_0(k, p^{d-1}) \gg t_{\text{diam}}(k, n)$. We prove that $t_0(k, p^{d-1})$ is an upper bound on mixing when $\rho \leq 1$. 

\hspace{1cm} 9
Recall that cutoff is already established (for arbitrary groups) when \( k \) grows super-polynomially in \( n \), i.e. \( \log k \gg \log \log n \); below assume that \( \log k \lesssim \log \log n \), i.e. \( k = (\log n)^{\mathcal{O}(1)} \). \( \triangle \)

The fact that the mixing time is a maximum of two quantities suggests some sort of ‘competition’ between the Abelianisation and the rest of the group; this leads to a ‘phase transition’ in the mixing time, which has an interesting consequence for the Aldous–Diaconis conjecture.

**Remark.** Set \( \rho := (1 - \nu)(1 + \frac{1}{d}) \). By choosing \( d \to \infty \) sufficiently slowly, we can ensure that

\[
\log n \ll (\log n)^{(1-\nu)(1+1/d)} \ll (\log n)^{(1-\nu)(1+2/(d-2))}.
\]

According to the Aldous–Diaconis conjecture, there should be cutoff at \( T(\rho, n) \); however, Proposition 3.2 shows that the mixing time \( t_\star \) satisfies \( t_\star \sim T(\rho, n^{2/d}) = \frac{d}{2} T(\rho, n) \) in this regime (provided \( d \) does not grow too quickly). Hence the Aldous–Diaconis conjecture is off by a factor of \( \frac{d}{2} \), and so does not even capture the correct order of the mixing (since we allow \( d \to \infty \)).

Recall that the conjecture has been verified for Abelian groups, in the entire \( k \gg \log n \) regime. These Heisenberg groups give a strong counter-example once one allows non-Abelian groups. \( \triangle \)

### 3.2 Lower Bound

The lower bound is relatively straightforward to prove: we project onto the Abelianisation, then use the lower bound for Abelian groups from our companion article [32, §3.3].

**Proof of Lower Bound in Theorem 3.4.** We assume that \( Z \) is given, and suppress it.

For any \( \varepsilon > 0 \), a lower bound is given by \((1 - \varepsilon) \log \log n\): in \( m \) steps the support of the random walk is (at most) \( k^m \), and hence the walk cannot be mixed in this many steps; cf [39, Fact 2.1].

Write \( A := H_{p,d}/[H_{p,d}, H_{p,d}] \) for the Abelianisation of \( H_{p,d} \), and \( \pi_A : G \to A \) for the projection map. Write \( N := |A| = p^{d-1}. \) Let \( \varepsilon > 0 \) and let \( t := (1 - \varepsilon) t_0(k, N) \). Write

\[
\mathcal{E} := \{ \mu(W(t)) \geq N^{-1} e^\omega \} = \{ Q(t) \leq \log N - \omega \},
\]

with \( \mu, Q \) and \( \omega \gg 1 \) from §2.2. By Proposition 2.3, we have \( P(\mathcal{E}) = 1 - o(1) \). Consider the set

\[
E := \{ x \in A \mid \exists w \in \mathbb{Z}^d \text{ st } \mu_t(w) \geq N^{-1} e^\omega \text{ and } x = \pi_A(w \cdot Z) \}.
\]

Since we use \( W \) to generate \( S \), we have \( P(\pi_A(S(t)) \in E \mid \mathcal{E}) = 1 \). Every element \( x \in E \) satisfies \( x = \pi_A(w_x \cdot Z) \) for some \( w_x \in \mathbb{Z}^d \) with \( \mu_t(w_x) \geq N^{-1} e^\omega \). Hence, for all \( x \in E \), we have

\[
P(\pi_A(S(t)) = x) \geq P(W(t) = w_x) = \mu_t(w_x) \geq N^{-1} e^\omega.
\]

Write \( E' := \pi_A^{-1}(E) \); we have \( \pi_G(E') = \pi_A(E) \cdot |G|/|A| \). From this we deduce that

\[
1 \geq \sum_{x \in E} P(\pi_A(S(t)) = x) \geq |E| \cdot N^{-1} e^\omega, \quad \text{ and hence } |E|/N \leq e^{-\omega} = o(1). \]

Finally we deduce the lower bound from the definition of TV distance:

\[
\| P(S(t) \in \cdot \mid Z) - \pi_G \|_{TV} \geq P(S(t) \in E' - \pi_G(E')) \geq P(\mathcal{E}) - \frac{1}{N} |E| \geq 1 - o(1). \quad \square
\]

**Remark 3.6.** This proof actually generalises somewhat. First, \( t_{\text{diam}} \) is always a lower bound, regardless of the group. For \( t_0(k, p^{d-1}) \), the only property of the Heisenberg group that we used is that the size of its Abelianisation, \([H_{p,d}/[H_{p,d}, H_{p,d}]]\), is \( p^{d-1} \). The same proof holds for an arbitrary group, giving a lower bound on mixing (valid for all \( Z \)) of \( \max \{ t_0(k, |G|/|G|), t_{\text{diam}} \} \). \( \triangle \)

### 3.3 Upper Bound Preliminaries

As mentioned above, the case where \( \log k \gg \log \log n \) (i.e. \( k \) is super-polynomial in \( n \)) is already solved. For concreteness, we assume that \( 1 \ll k \leq (\log n)^{\mathcal{O}(1)} \log n \log n \). We first prove the upper bound for \( d = 3 \): the majority of the ideas are exposed in this case, while the technical
details involved in the general $d$ case somewhat obscure the ideas; the case of general $d$ comes after. Note that the conditions on $d$ are always satisfied when $d = 3$ (or, in fact, any fixed $d$).

Before doing so, we need some preliminary results (for both $d = 3$ and general $d$). First, we need a concept of ‘typicality’ for the auxiliary random variable $W(t)$; later in the proof, we define a set $W \subseteq \mathbb{Z}_+^k$ (dropping the $t$-dependence from the notation) with the property

$$W \subseteq \{ w \in \mathbb{Z}_+^k \mid \mu_i(w) \leq e^{-h}, \max_i w_i < p \};$$

(3.2)

this set will satisfy $\mathbb{P}(W \in W) = 1 - o(1)$, hence the name ‘typical’.

It is often easier to work with $L_2$, rather than TV, distance, since it has a nice explicit representation; on the other hand, with TV one can condition on ‘typical’ events. We combine the two with a ‘modified $L_2$ calculation’: let $S$ and $S'$ be independent copies (given $Z$), and let $W$ and $W'$ be their associated auxiliary random variables; write $\text{typ} := \{ W, W' \in W \}$.

**Lemma 3.7.** Assume that $\mathbb{P}(W((1 + \varepsilon)t_a) \notin W) = o(1)$ for all constants $\varepsilon > 0$. Establishing the upper bound in Theorem 3.4 is equivalent to showing, for all constants $\varepsilon > 0$, that

$$D(t) := n \mathbb{P}(S = S' \mid \text{typ}) - 1 \quad \text{satisfies} \quad D((1 + \varepsilon)t_a) = o(1).$$

**Proof.** Using the triangle inequality and then Cauchy-Schwarz inequality, we obtain the following:

$$\|\mathbb{P}(S \in \cdot \mid Z) - \pi_G\|_{\text{TV}} \leq \|\mathbb{P}(S \in \cdot \mid Z, W \in W) - \pi_G\|_{\text{TV}} + \mathbb{P}(W \notin W);$$

$$2\mathbb{E}(\|\mathbb{P}(S \in \cdot \mid Z, W \in W) - \pi_G\|_{\text{TV}})^2 \leq n \mathbb{P}(S = S' \mid \text{typ}) - 1 = D(t).$$

Combining these with the assumption $\mathbb{P}(W \notin W) = o(1)$, and Markov’s inequality, gives

$$\|\mathbb{P}(S \in \cdot \mid Z) - \pi_G\|_{\text{TV}} = o(1) \quad \text{whp over } Z. \quad \square$$

To upper bound $D := D(t)$, we separate into cases according to whether or not $W = W'$:

$$\mathbb{P}(S = S' \mid \text{typ}) = \mathbb{P}(S = S' \mid W = W', \text{typ})\mathbb{P}(W = W' \mid \text{typ})$$

$$+ \mathbb{P}(S = S' \mid W \neq W', \text{typ})\mathbb{P}(W = W' \mid \text{typ}).$$

Were the underlying group Abelian, $W = W'$ would imply $S = S'$. This is not the case for non-Abelian groups; in fact estimating $\mathbb{P}(S = S' \mid W = W', \text{typ})$ is the main part of the proof.

**Lemma 3.8.** We have

$$\mathbb{P}(W = W' \mid \text{typ}) \leq e^{-h}/\mathbb{P}(\text{typ}).$$

**Proof.** By direct calculation, since $W$ and $W'$ are independent copies, we have

$$\mathbb{P}(W = W', \text{typ}) = \mathbb{P}(W = W', W \in W) = \sum_{w \in W} \mathbb{P}(W = w)\mathbb{P}(W' = w) \leq e^{-h},$$

using the fact that $\sum_{w \in W} \mathbb{P}(W = w) \leq 1$ and $\mathbb{P}(W = w) \leq e^{-h}$ for all $w \in W$. \quad \square

Consideration of $\mathbb{P}(S = S' \mid W \neq W', \text{typ})$ is deferred. The analysis is greatly simplified once we have done a little more preparation; it is proved separately for $d = 3$ and for general $d$. The main ingredient is the following lemma, which follows immediately from the fact that $p$ is prime.

**Lemma 3.9.** Let $\ell \in \mathbb{N}, X_1, \ldots, X_\ell \sim \text{id Unif}(\mathbb{Z}_p)$ and $a_1, \ldots, a_\ell \in [p-1]$. Then $\sum_{i=1}^\ell a_iX_i \sim \text{Unif}(\mathbb{Z}_p)$. We can extend this to general $p \in \mathbb{N}$: we have $\sum_{i=1}^\ell a_iX_i \sim \text{Unif}(g\mathbb{Z}_p)$ where $g := \gcd(a_1, \ldots, a_\ell, p)$ and $g\mathbb{Z}_p = \{ g, 2g, \ldots, p \}$; see [32, Lemma 3.11]. (These statements are in $\mathbb{Z}_p$, i.e. modulo $p$.)

We need to determine the entropy, or at least bounds on it, at the proposed mixing time. Recall that $W(\cdot)$ is a $k$-dimensional PP, that $\mu_\ast$ is the law of $W(t)$, that $Q(t) = -\log \mu_\ast(W(t))$ and that $h(t) = \mathbb{E}(Q(t))$ is the entropy of $W(t)$. Recall that $\log(p^k-1) = \frac{3}{4} \log n$. 

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Lemma 3.10. Let $\xi > 0$. Then, for any $\omega \ll \min\{k, \frac{1}{p} \log n\}$, the following lower bounds hold.

- For $t \geq (1 + \xi)t_0(k, p^{d-1})$, we have $h(t) \geq \log(p^{d-1}) + 2\omega$.
- For $t \geq (1 + \xi)t_{\text{diam}}$, if $t_{\text{diam}}/k \ll 1$, then we have $h(t) \geq (1 - \frac{1}{p}) \log n + 2\omega$.

**Proof.** First we determine what the entropy is without the $(1 + \xi)$ factor. Increasing the time by the $(1 + \xi)$ factor provides the $+2\omega$ additive term; this is proved in Lemma 3.17, given later.

The first bullet follows immediately from the definition of the entropic time $t_0(k, p^{d-1})$. For the second bullet, recall from §2.3, specifically (2.3b), the form of the entropy when $s := t_{\text{diam}}/k \ll 1$:

$$h(t_{\text{diam}}) \approx t_{\text{diam}} \log(k/t_{\text{diam}}) = (1 - \frac{1}{p}) \log n \cdot (1 + \frac{1}{p-1} \log(p \log n)/\log n) \approx (1 - \frac{1}{p}) \log n. \quad \Box$$

Motivated by this lemma, recalling that $t_* = \max\{t_0, t_{\text{diam}}\}$, we make the following definition.

**Definition 3.11.** Define $h_0$ as follows:

$$h_0 := \begin{cases} 
\log(p^{d-1}) = \frac{1}{p} \log n & \text{when } k \leq (\log n)^{1+2/(d-2)}; \\
(1 - \frac{1}{p}) \log n & \text{when } k \geq (\log n)^{1+2/(d-2)}. 
\end{cases}$$

Fix some $\omega$ such that $1 \ll \omega \ll \min\{k, \frac{1}{p} \log n\}$, and set $h := h_0 + \omega$.

Not only does the entropy satisfy this lower bound, but the $Q$ random variable, which is defined, in §2.2, so that $E(Q(t)) = h(t)$, concentrates, giving the following result.

**Lemma 3.12.** Assume that $\omega \ll \min\{k, \frac{1}{p} \log n\}$. Let $\varepsilon > 0$ and $t \geq (1 + 3\varepsilon) \max\{t_0, t_{\text{diam}}\}$. Then

$$\mathbb{P}(Q(t) \geq h) = \mathbb{P}(\mu(W(t)) \leq e^{-h}) = 1 - o(1).$$

**Proof.** Rearrange the inequality $\mu \leq e^{-h}$ into $Q := -\log \mu \geq h$, use Lemma 3.10, the definition of $h$ and $h_0$ from Definition 3.11 and apply the concentration result Proposition 2.3. \(\Box\)

### 3.4 Upper Bound for $3 \times 3$ Heisenberg Matrices

In the $3 \times 3$ case, we only have three terms to play with. For ease of notation, we write

$$M := (M_{1,2}, M_{2,3}, M_{3,1}) \quad \text{for a matrix } M \in H_{p,3}.$$ 

For matrices $M_1, M_2, \ldots \in H_{p,3}$, writing $M_j := (a_j, b_j, c_j)$ for each $j$, we have

$$\Pi^t_1 M_s = \left(\sum_{i=1}^t a_s, \sum_{i=1}^t b_s, \sum_{i=1}^t c_s + f((a_s)_1^t, (b_s)_1^t)\right)$$

where $f((a_j)_1^t, (b_j)_1^t) := \sum_{s=1}^t b_s \sum_{r=1}^{s-1} a_r$.

Note that the first two terms are ‘Abelian’ (and correspond to the Abelianisation): we can reorder the product $M_1 \cdots M_t$ in any way we desire, and the first two terms are unchanged; also, so is the first part of the third term, but the polynomial $f$ is not.

We have $k$ generators $Z = (Z_1, \ldots, Z_k)$; write $Z_i := (A_i, B_i, C_i)$ for each $i$. Recall the definition of $W$ in the directed case, a $k$-dimensional Poisson process. Suppose that $N := N(t)$ steps are taken. Write $(\alpha_1, \beta_1, \gamma_1), \ldots, (\alpha_N, \beta_N, \gamma_N)$ for the steps taken by $S$. Write $G_m$ for the generator index chosen at step $m \in [N]$, ie $G_m = i$ if $(\alpha_m, \beta_m, \gamma_m) = (A_i, B_i, C_i)$. Write $\alpha := (\alpha_m)_1^N$ and $\beta := (\beta_m)_1^N$. Let $S'$ be an independent copy of $S$, and make similar definitions. From (3.3), we have

$$S(t) = \left(\sum_{i=1}^t A_i W_i(t), \sum_{i=1}^t B_i W_i(t), \sum_{i=1}^t C_i W_i(t) + f(\alpha, \beta)\right).$$

**Proof of Theorem 3.4 Given Lemma 3.13 (when $d = 3$).** First, we claim that

$$\mathbb{P}(S = S' \mid W \neq W', \text{typ}) = 1/n = 1/p^3.$$  \(3.5\)
Indeed, for any \( v \in \mathbb{Z}_p^k \setminus \{0\} \), by Lemma 3.9, each of \( \sum_{i=1}^k A_i v_i, \sum_{i=1}^k B_i v_i \) and \( \sum_{i=1}^k C_i v_i \) is an independent \( \text{Unif}(\mathbb{Z}_p) \); also, \( f(\alpha, \beta) \) is independent of \( \sum_{i=1}^k C_i W_i(t) \). Note also that, by typicality, \(|W_i - W'_i| < p\) for all \( i \), and so \( W_i \equiv W'_i \mod p \) if and only if \( W_i = W'_i \). Hence conditioning on \( W \) and \( W' \) and then using (3.4) establishes the claim. Next, recall from Lemma 3.8 that

\[
\mathbb{P}(W = W' \mid \text{typ}) \leq e^{-h/\mathbb{P}(\text{typ})} = e^{-h/\mathbb{P}(\text{typ})}.
\] (3.6)

It remains to consider the case that \( W(t) = W'(t) = w \), for some \( w \in W \). In particular, \( S \) and \( S' \) take the same number of steps: \( N = N' \). Note now, by (3.4), that \( S - S' = (0, 0, f(\alpha, \beta) - f(\alpha', \beta')) \).

Expanding the definition of \( f \) in (3.3), we may write

\[
f(\alpha, \beta) = \sum_{i=1}^k C_{i,j} A_i B_j,
\]

for appropriate \( \{C_{i,j}\}_{i,j=1}^k \); specifically, for \( i, j \in [k] \) with \( i \neq j \), we have \( C_{i,j} = 0 \) and

\[
C_{i,j} = \sum_{t=1}^N 1(G_t = j) \sum_{m=1}^{t-1} 1(G_m = i); \quad \text{write } C := \{C_{i,j} \mid i, j \in [k] \}.
\]

Define \( C_{i,j}' \) and \( C' \) analogously with respect to \( W' \). The body of the proof will be controlling the probability that \( C \equiv C' \) conditional on \( W(t) = W'(t) = w \), for some typical \( w \in W \). Write

\[
\mathcal{E} := \{C \equiv C'\} = \{C_{i,j} \equiv C_{i,j}' \forall i, j \in [k]\}.
\]

We have \( \mathbb{P}(S = S' \mid W = W' = w, \mathcal{E}) = 1 \). We now argue that

\[
\mathbb{P}(S = S' \mid W = W' = w, \mathcal{E}^c) \leq 2/p.
\] (3.10)

Write \( D_{i,j} := C_{i,j} - C_{i,j}' \). On the event \( \mathcal{E}^c \), there exist \( i', j' \in [k] \) with \( D_{i', j'} \neq 0 \). Then

\[
f(\alpha, \beta) - f(\alpha', \beta') = A_{i'}(D_{i', j'} B_{j'} + \sum_{j \neq j'} D_{i,j} B_j) + \sum_{i \neq i'} A_i \sum_j D_{i,j} B_j.
\] (3.11)

We can write this final expression (with the natural association) as

\[
U(V + X) + Y.
\] (3.12)

Since \( D_{i', j'} \neq 0 \) (by choice of \( i' \) and \( j' \)) and \( p \) is prime, \( U, V \sim^\text{id} \text{Unif}(\mathbb{Z}_p) \). Moreover, \( U \) is jointly independent of \( X \) and \( Y \) and \( V \) is independent of \( X \) (but not of \( Y \)); hence \( V + X \sim \text{Unif}(\mathbb{Z}_p) \), independent of \( U \), and so \( U(V + X) \sim \text{Unif}(\mathbb{Z}_p) \) independent of \( Y \) on the event \( \{V + X \neq 0\} \). (These independence statements are all conditional on \( W = W' = w \).) Thus

\[
\mathbb{P}(U(V + X) + Y \equiv 0) \leq \max_{u \equiv 0} \mathbb{P}(U \equiv u) + \mathbb{P}(V + X \equiv 0) = 2/p.
\] (3.13)

This establishes (3.10).

Combining these results, recalling that \( \mathcal{E} = \{C \equiv C'\} \), writing

\[
q(t) := \max_{w \in W} \mathbb{P}(\mathcal{E} \mid W = W' = w),
\]

recalling that \( w \) is an arbitrary (fixed) element of \( W \), we find that

\[
\mathbb{P}(S = S' \mid W = W' = w, \text{typ}) \leq 2/p + q(t).
\] (3.14)

Once we average over \( w \in W \), recalling (3.6), we obtain

\[
\mathbb{P}(S = S', W = W' \mid \text{typ}) \leq 2e^{-h(1/p + q(t))}/\mathbb{P}(\text{typ}).
\] (3.15)

It remains to make an appropriate definition of typicality, i.e. of \( W \): we require that it satisfies (3.2), that \( \mathbb{P}(W \in W) = 1 - o(1) \), and hence \( \mathbb{P}(\text{typ}) = 1 - o(1) \), and that \( e^{-h(2/p + q(t))} = o(1/n) \). This is done in Lemma 3.13 below; it is the main technical part of the proof.

Once this is done, combining (3.5, 3.15) gives

\[
n \mathbb{P}(S = S' \mid \text{typ}) - 1 = o(1).
\]

The upper bound in Theorem 3.4 then follows from Lemma 3.7, modulo Lemma 3.13. \( \square \)
It remains to appropriately upper bound \( q(t) \) so that the right-hand side of (3.14) is \( o(e^h/n) \).

**Lemma 3.13.** Suppose that \( 1 \ll \log k \ll \log n \). There exists a \( W \subseteq \mathbb{Z}_+^k \), satisfying (3.2), so that
\[
\mathbb{P}(W \in W) = 1 - o(1) \quad \text{and} \quad ne^{-h}(1/p + q(t)) = o(1).
\]

**Proof.** Let \( \varepsilon > 0 \), and assume that it is as small as required (but independent of \( n \)). Recall that here \( d = 3 \), so \( \log(n^{2/d}) \ll \log n \). Hence there are three main regimes of interest:
\[
k \ll \log n, \quad \log n \lesssim k \leq (\log n)^3 \quad \text{and} \quad k \geq (\log n)^3.
\]

*Regime \( k \geq (\log n)^3 \).* We have \( t \geq (1 + 3\varepsilon)\text{diam} \). Recall from Definition 3.11 that, in this regime, we take \( h_0 := (1 - \frac{1}{p}) \log n \). Hence \( e^{-h_0} = n^{-1+1/p} \).

Since \( t \ll k \), almost all the generators are picked at most once whp. For \( w \in \mathbb{Z}_+^k \), define
\[
J(w) := \left\{ i \in [k] \mid w_i = 1 \right\} \quad \text{and} \quad J = |J(w)|.
\]

Using this, we make precise our definition of typicality:
\[
W := \left\{ w \in \mathbb{Z}_+^k \mid \mu_t(w) \leq e^{-h}, \quad |J(w) - te^{-t/k}| \leq \frac{k}{t}e^{-t/k}, \quad \max w_i < p \right\},
\]
satisfying (3.2). Using Hypotheses A (using the regime \( k \gtrsim d \log p \approx \frac{d}{4} \log n \)), it is straightforward to see that \( \log p \approx \log n^2/d^2 = (\log n)^{1-o(1)} \), and hence \( p \) grows super-polylogarithmically in \( n \); we are assuming that \( k \) grows at most polylogarithmically in \( n \), and hence \( k^t \ll p \) for any \( t \in \mathbb{N} \). Thus \( \log k \ll \sqrt{p} \) and \( t_0 \lesssim k \ll \sqrt{p} \). Thus the condition \( \{ \max w_i < p \} \) holds with probability \( 1 - o(1) \).

By Binomial concentration and Lemma 3.12, we then have \( \mathbb{P}(W \in W) = 1 - o(1) \).

Let \( w \in W \). We now argue that
\[
\mathbb{P}(\mathcal{E} \mid W = W' = w, \mid J(w) = J) \leq 1/|J|!.
\]

This holds since, conditional on \( W = W' = w \), different (relative) orderings, between \( S \) and \( S' \), of the coordinates chosen once, ie in \( J(w) \), must result in some pair \((i, j)\) such that \( C_{ij} = 1 \) and \( C_{i,j} = 0 \). There are \( |J|! \) different orderings. Applying (3.16) gives
\[
q(t) \leq 1/((1 - \varepsilon)t)!
\]

Applying this with \( t \geq (1 + 3\varepsilon)\text{diam} \), direct calculation using Stirling’s approximation gives
\[
q(t) \leq ((1 + e)\text{diam}/e)^{(1+\varepsilon)\text{diam}} \lesssim n^{-1/p}.
\]

In particular, \( n^{-1/p} \gtrsim n^{-1/\beta} = p^{-1} \) for all \( \rho \geq 3 \). Recalling that \( e^{-h} = e^{-\omega}n^{-1+1/p} \), the proof is completed in the regime \( k \gtrsim (\log n)^3 \):
\[
ne^{-h}(1/p + q(t)) \leq 2n \cdot e^{-\omega}n^{-1+1/p} \cdot n^{-1/p} = 2e^{-\omega} \ll 1.
\]

*Regime \( \log n \lesssim k \lesssim (\log n)^3 \).* We have \( t \geq (1 + 3\varepsilon)t_0 \). Recall from Definition 3.11 that, in this regime, we take \( h_0 := \frac{d}{4} \log n = \frac{d}{4} \log (p^2) \). Hence \( e^{-h_0} = n^{2/\beta} = p^2 \).

Since \( d \gg 1 \), we have \( 1 \ll t \lesssim k \). We use the same definition of typicality here as for \( k \gtrsim (\log n)^3 \).

Since \( 1 \ll t \lesssim k \), we have \( \mathbb{P}(W \in W) = 1 - o(1) \).

Using \( t \geq (1 + 3\varepsilon)t_0 \) in (3.17), direct calculation using (2.1c) Stirling’s approximation gives
\[
q((1 + 3\varepsilon)t_0) \leq ((1 + e)t_0/e)^{(1+\varepsilon)t_0} \lesssim n^{-\frac{1}{(\rho-1)}} \quad \text{when} \quad k \gg \log n.
\]

For \( k \approx \lambda \log n \), with \( \lambda \in (0, \infty) \), we have \( t \approx f(\lambda)k \approx \lambda f(\lambda) \log n \) by (2.1b), and thus
\[
\mathbb{E}(|J|) \approx \lambda f(\lambda)e^{-f(\lambda)} \log n.
\]

Applying (3.16), recalling that \( w \in W \), gives
\[
q(t) \leq 1/((1 - \varepsilon)\lambda f(\lambda)e^{-f(\lambda)} \log n)!
\]

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applying Stirling’s approximation, it is easy to see that this decays super-polynomially, ie
\[
\log(1/q((1 + 3\varepsilon)t_{<})) \gg \log n \quad \text{when} \quad k \approx \lambda \log n,
\]
provided \( \varepsilon \) is sufficiently small. Hence
\[
q(t) \leq n^{-1/3} = 1/p \quad \text{when} \quad \log n \lesssim k \leq (\log n)^3.
\]
Recalling that \( e^{-h} = e^{-\omega p^{-2}} \), the proof is completed in the regime \( \log n \lesssim k \leq (\log n)^3 \):
\[
ne^{-h}(2/p + q(t)) \leq p^3 \cdot e^{-\omega p^{-2}} \cdot 3/p = 3e^{-\omega} \ll 1.
\]

**Regime \( k \ll \log n \).** We have \( t \geq (1 + 3\varepsilon)t_0 \). Recall from Definition 3.11 that, in this regime, we take \( h_0 = \frac{2}{3} \log n = \frac{2}{3} \log n = \log(p^2) \). Hence \( e^{-h} = n^{-2/3} = p^{-2} \).

Since \( \delta \gg 1 \), we have \( t_0 = kp^{1/\delta} = k\sqrt{4/(3\delta)} \gg k \). Hence the same generator is picked lots of times, and so we need a new approach for calculating \( q(t) \). The expected number of times a generator is picked is \( s := t/k \gg 1 \). As part of our typicality requirements, we ask that ‘most’ pairs \((2i,2i-1)\), with \( i \in \{1,\ldots,[k/2]\}\), are picked between \( \eta s \) and \( \eta^{-1}s \), for a small positive constant \( \eta \), to be chosen later; for the moment, let \( \eta \in (0,1) \).

Then, for \( \eta \) sufficiently small (but still a constant), we have
\[
\mathbb{P}(|C(W)| \geq \frac{2}{3}k) = 1 - o(1).
\]
(We could replace \( \frac{2}{3}k \) by any constant less than \( \frac{1}{2} \), at the cost only of making \( \eta \) a smaller constant.)

We use this to make precise our definition of typicality for this regime:
\[
W := \{w \in \mathbb{Z}_+^k \mid \mu_\ell(w) \leq e^{-h}, \ |C(w)| \geq \frac{2}{3}k, \ \max_i w_i < p\}.
\]
Then, like before and additionally using (3.21), we have \( \mathbb{P}(W \in W) = 1 - o(1) \). If \( i \in C(w) \), then \( \max(C_{2i,2i-1},C'_{2i,2i-1}) \leq w_i \leq s^2 \approx p^{1/k} \ll p \), so \( \{C_{2i,2i-1} \equiv C'_{2i,2i-1}\} = \{C_{2i,2i-1} = C'_{2i,2i-1}\} \).

We claim that it is sufficient to fix an arbitrary \( w \in W \) and prove the bound
\[
\max_i q_i \leq p^{-3/k} \quad \text{where} \quad q_i := \max_x \mathbb{P}(C_{2i,2i-1} = x \mid W = w)1(i \in C(w)).
\]
To see this, first make the simple observation that, for any \( \mathcal{I} \subseteq \{1,\ldots,[k/2]\} \), we have
\[
\{C_{i,j}_{i,j \in [k]} = C'_{i,j}\}_{i,j \in [k]} \subseteq \{C_{2i,2i-1} = C'_{2i,2i-1}\}_{i \in \mathcal{I}} \times \{C'_{2i,2i-1} = C_{2i,2i-1}\}_{i \in \mathcal{I}^c}.
\]
Given \( W = W' = w \), the event \( \{C_{i,j} = C'_{i,j}\} \) is determined by the relative order in which the generators \( i \) and \( j \) are chosen, ie \( \{m \mid G_m \in \{i,j\}\} \). Hence, since the pairs \((2i,2i-1)\) are disjoint, the events \( \{C_{2i,2i-1} = C'_{2i,2i-1}\} \) are independent for different \( i \), conditional on \( W = W' = w \). Take \( \mathcal{I} := C(w) \), which has size at least \( \frac{2}{3}k \). By the aforementioned independence, given (3.22), we have
\[
\mathbb{P}(C = C' \mid W = W' = w) \leq (\max_i q_i)^{2k/5} \leq p^{(-3/k)(2k/5)} = p^{-6/5} \ll 1/p,
\]
and hence \( q(t) \ll 1/p \). The proof is then completed as in the regime \( \log n \lesssim k \ll (\log n)^3 \).

It remains to prove (3.22). For simplicity of notation, we assume that \( 1 \in C(w) \) and set \( i := 1 \).

Let \( r := w_1 + w_2 \), and write our random word as \( S = Z_G, \ldots, Z_{G_{2r}} \); here \( N = \sum \frac{1}{2} \) is the number of steps taken and \( G_t \) is the generator index chosen in the \( t \)-th step. Let \( J_1 < \cdots < J_r \) be the (random) indices with \( G_{J_t} \in \{1,2\} \). Now define the vector \( I \in \{1,2\}^r \) by \( I_t := G_{J_t} \). Thus \( I \) encodes the relative order between the different occurrences (with multiplicities) of the generators labelled by \( \{1,2\} \) in the word \( S \). By typicality, \( 2\eta s \leq \tau \leq 2\eta^{-1}s \).

Let \( I' \) be the random vector obtained from \( I \) by picking a 2 uniformly at random and omitting it from \( I \). (Eg, if \( I = (2,2,1,1,2,1) \) and we pick the last 2, then \( I' = (2,2,1,1,1,1) \).) Importantly, we are omitting elements of the relative order of appearances of \( Z_1 \) and \( Z_2 \), not the absolute locations of the corresponding generators.
By the definition of $C_{1,2}$, given in (3.8), given $W = w$, the value of $C_{1,2}$ is a function only of the relative locations $I$. Hence, given $I'$ also, it is a function only of the location of the omitted 2. It is constant on the set of locations which give rise to the same $I$: two different placements of the omitted 2 give rise to the same $I$ if and only if they both lie in the same (possibly empty) interval of consecutive 2s. (Eg, if $I' = (2, 2, 1)$, then there are three locations in which we can insert a 2 to get $I = (2, 2, 1, 2)$, namely the first, second and third positions, and only one to get $I = (2, 2, 1, 2)$, namely the fourth position; the first three give rise to $C_{1,2} = 0$ and the fourth to $C_{1,2} = 1$.)

Hence, writing $L(I')$ for the longest interval of 2s in $I'$, we have

$$ \max_x \mathbb{P}(C_{1,2} = x \mid W = w, I') \leq \frac{(L(I') + 1)}{r}. $$

By Claim 3.14 below, with $m = 2$ which gives $M \approx r \gg 1$, we find that $L(I')/(C \log r) \preceq \text{Geom}(\frac{1}{2})$ for a sufficiently large constant $C$, and so $\mathbb{E}(L(I')) \lesssim \log r$. Hence

$$ \max_x \mathbb{P}(C_{1,2} = x \mid W = w) \leq \mathbb{E}(L(I') + 1 \mid W = w)/r \lesssim (\log r)/r. $$

Since $2\eta s \leq r \leq 2\eta^{-1}s$, as $w \in \mathcal{W}$, and $\eta$ is a (small) constant, this last expression is $o(1/s^{3/4})$.

(In fact, it is $O(\log s)/s$.) Recalling that $s \asymp n^{4/k}$ establishes (3.22). This completes the proof. □

It remains to state and prove the claim regarding $\mathbb{E}(L(I'))$. We actually state and prove a slightly more general claim, that we are then able to use in the analysis of the $d \times d$ matrices.

**Claim 3.14.** Let $m \in \mathbb{N}$ and $\eta \in (0, 1)$. Let $\{w_1, ..., w_m\}$ be arbitrary positive integers satisfying $w_i/w_j \in [\eta^2, \eta^{-2}]$ for all $i, j \in [m]$. For each $k \in \{1, ..., m\}$, let there be $w_k$ balls of colour $k$; write $r := \sum_{k=1}^{m} w_k$ for the total number of balls. Choose a uniform permutation of the balls on positions $\{1, ..., r\}$. For each $k \in \{1, ..., m\}$, let $L_k$ be the longest interval without any balls of colour $k$. Then, for each $k$, we have the stochastic domination

$$ L_k/(\eta^{-2}m \log r) \preceq \text{Geom}(\frac{1}{2}). $$

**Proof.** Without loss of generality, take $k := 1$ and write $L := L_1$. By assumption, $w_i/w_j \in [\eta^2, \eta^{-2}]$ for all $i, j$, and $\eta \in (0, 1)$ is a constant. Hence $r \leq m\eta^{-2}w_1$, and so $w_1 \geq \eta^2 r/m$. Let $\ell \in \mathbb{N}$ to be chosen shortly; write $[1, r] \subseteq [1, \ell] \cup [2, \ell+1] \cup \cdots \cup [r, r+\ell-1]$. By direct calculation,

$$ \mathbb{P}(L > \ell) = r \mathbb{P}(\text{no 1 in the interval } [1, \ell]) = r(1 - \frac{w_1}{r})^{\ell} \leq r \exp(-\ell w_1/r) \leq r \exp(-\ell \eta^2/m), $$

where for the penultimate inequality we used the fact that $\frac{w_1}{r} \leq \frac{\eta^2}{\eta^2}$, which holds since $w_1 \leq r$. Choosing $\ell := (k+1)\eta^{-2}m \log r$ gives

$$ \mathbb{P}(L > (k+1)\eta^{-2}m \log r) \leq r \exp(-(k+1) \log r) = r^{-k}. $$

Thus we may stochastically dominate

$$ L/(\eta^{-2}m \log r) \preceq \text{Geom}(1 - 1/r) \preceq \text{Geom}(\frac{1}{2}). $$

**3.5 Upper Bound for $d \times d$ Heisenberg Matrices**

The high-level ideas of the proof will be similar to the $d = 3$ case, but there are a number of subtleties which need to be navigated. Analogously to Lemma 3.13, there will be a certain probability that requires bounding, and the argument for bounding this will differ depending on $k$; the specific reference will be Lemmas 3.16 and 3.17, and comes at the end of the section.

We also use the same preliminaries (see §3.3), and in particular consider

$$ D(t) = n \mathbb{P}(S = S' \mid \text{typ}) - 1. $$

The analogues of (3.3, 3.4) are different for general $d$ than for $d = 3$: they have the same basic structure, but with the addition of ‘higher order’ terms (given by $g_{a,b}$ in the lemma below).
Lemma 3.15. Let $Z_1, \ldots, Z_k \in H_{p,d}$. Let $\gamma := (\gamma_\ell)_{\ell \in [N]} \in [k]^N$. Let
\[
M := Z_{\gamma_1} \cdots Z_{\gamma_N}.
\]
Then, for all $a \in [d]$, we have $M(a, a) = 1$, $M(a, a + 1) = \sum_{i=0}^{N-1} Z_{\gamma_i}(i, i + 1)$ and, for all $b \geq a + 2$,
\[
M(a, b) = \sum_{i=0}^{N-1} Z_{\gamma_i}(a, b) + \sum_{i,j \in [N]} C_{i,j} (\gamma) Z_i(a, a + 1) Z_j(a + 1, b) + g_{a,b}(\gamma, Z_1, \ldots, Z_k), \quad (3.23)
\]
where $C_{i,j}(\gamma) := \sum_{m < d} 1(\gamma_i = \gamma, \gamma_m = j)$ and where $g_{a,b}(\gamma, Z_1, \ldots, Z_k)$ is a polynomial in $(Z_i(x, y) : i \in [k], x \in [d-1], y > x)$ in which each monomial contains the term $Z_i(a, a + 1)$ either 0 times or exactly once and no monomial contains a term of the form $Z_i(a, a + 1)Z_j(a + 1, b)$ for $i, j \in [k]$.

The proof of this lemma is deferred to the end of this subsection (§3.5).

Proof of Theorem 3.4 Given Lemmas 3.16 and 3.17. When $W(t) \neq W'(t)$, the same argument as for $d = 3$, using Lemma 3.9, applies, replacing (3.4) by (3.23):
\[
\mathbb{P}(S = S' \mid W \neq W', \text{typ}) = 1/n = 1/p^{k(d-1)/2} = p^{-(d-1)(d-2)/2}, \quad (3.24)
\]
This is the analogue of (3.5). Next, recall from Lemma 3.8 that
\[
\mathbb{P}(W = W' \mid \text{typ}) \leq e^{-h}/\mathbb{P}(\text{typ}) = e^{-\omega}e^{-h_0}/\mathbb{P}(\text{typ}). \quad (3.25)
\]

Now suppose that $W(t) = W'(t) = w$, where $w$ is some fixed element of $W$ (yet to be defined fully). Then the ‘Abelian’ parts of $S$ and $S'$ cancel (as was the case when $d = 3$). Write $C := (C_{i,j})$ and $C' := (C'_{i,j})$ for the $C(\gamma)$ in Lemma 3.15 generated by the walks $S$ and $S'$, respectively. Write $E := \{C = C'\}$. On the event $E$, the middle terms of (3.23) now cancel, leaving only the higher-order terms; we upper bound $\mathbb{P}(S = S' \mid W = W', E) \leq 1$.

Now suppose that $E$ does not hold; choose, and fix, $(i', j')$ so that $C_{i',j'} \neq C'_{i',j'}$. By the condition (3.2) which $W$ must satisfy and the definition of $C_{i,j}$, this implies that $C_{i',j'} \neq C'_{i',j'}$.

Analogously to (3.11, 3.12), where $d$ was equal to 3, letting
\[
U_{a,b} := (C_{i',j'} - C'_{i',j'})Z_i(a + 1, b) \quad \text{and} \quad V_{a,b} := Z_{i'}(a, a + 1), \quad (3.26)
\]
we can, for some random variables $X_{a,b}$ and $Y_{a,b}$, write
\[
\sum_{i,j} (C_{i',j'} - C'_{i',j'})Z_i(a + 1, b) \quad \text{naturally as} \quad U_{a,b}(V_{a,b} + X_{a,b}) + Y_{a,b}.
\]
(Here, $Y_{a,b}$ contains terms related to $g_{a,b}$ from (3.23)). For the moment, fix $(a,b)$. Analogously to the $d = 3$ case, i.e. (3.10–3.13), we have that $U_{a,b}, V_{a,b} \sim_{\text{ind}} \text{Unif}(\mathbb{Z}_p)$, that $U_{a,b}$ is jointly independent of $X_{a,b}$ and $Y_{a,b}$, and that $V_{a,b}$ is independent of $X_{a,b}$ (but not of $Y_{a,b}$). Thus $U_{a,b}(V_{a,b} + X_{a,b}) \sim \text{Unif}(\mathbb{Z}_p)$ is independent of $Y$ on the event $V_{a,b} + X_{a,b} \not\equiv 0$. Hence, as for (3.13), we have
\[
\max \mathbb{P}(U_{a,b}(V_{a,b} + X_{a,b}) + Y_{a,b} \equiv r) \leq \max \mathbb{P}(U_{a,b} \equiv u) + \mathbb{P}(V_{a,b} + X_{a,b} \equiv 0) \leq 2/p; \quad (3.27)
\]
Now compare $S_{a,b}$ and $S'_{a,b}$: since $W = W'$, the ‘Abelian’ part cancels and we are left with the $U_{a,b}(V_{a,b} + X_{a,b}) + Y_{a,b}$ part and the higher-order terms, given by the $g_{a,b}$ polynomials in (3.23); these two parts are independent, by the conditions of Lemma 3.15, and hence (3.27) implies that
\[
\mathbb{P}(S_{a,b} = S'_{a,b} \mid W = W' = w, E') \leq 2/p. \quad (3.28)
\]
Now, the random variables $\{X_{a,b}, Y_{a,b}\}_{a,b}$ are not independent. Also, $V_{a,b} = Z_{i,a,a+1}$ does not depend on $b$, and so $\{U_{a,b}, V_{a,b} \mid b \geq a + 2\}_{a,b}$ are not independent either. However, if we fix $b$ then $\{U_{a,b}, V_{a,b} \mid b \geq a + 2\}_a$ is a collection of independent variables. We exploit this.

Partition the $[k]$ generators into $d - 2$ sets $(P_1, \ldots, P_{d})$. For each (fixed) $b \in \{3, \ldots, d\}$, we use generators only from $P_b$; this will give independence when we consider all $b$. (Note that for $b \in \{1, 2\}$ there are no terms above the super-diagonal.) Then for the $(a,b)$-th coordinate we try to get $C_{i,j} \neq C'_{i,j}$ for some $(i,j)$ with $i \in P_b$. Now, for each $b$, using this pair $(i,j)$ in the definition
(3.26) of $U_{a,b}$ and $V_{a,b}$, the random variables $\{U_{a,b}, V_{a,b} \mid b \geq a + 2\}_a$ are independent, since they depend on a disjoint set of generators.

For each $b \in \{3, \ldots, d\}$, write
\[
C_b := (C_{i,j})_{i,j \in P_b}, \quad C'_b := (C'_{i,j})_{i,j \in P_b} \quad \text{and} \quad \mathcal{E}_b := \{C_b = C'_b\}. \tag{3.29}
\]

We wish to get an analogue of (3.14), for general $d$. Write $\tilde{\mathbb{P}}_w(\cdot) := \mathbb{P}(\cdot \mid W = W' = w)$ for $w \in \mathcal{W}$, and $S_{b} := (S_{a,b})_{a = 1, \ldots, b-2}$ for the $b$-th column strictly above the super-diagonal; also, henceforth, in $\sum^d$ and $\prod^d$, the implicit index is always $b$. Then
\[
\tilde{\mathbb{P}}_w(S = S') = \prod^d \tilde{\mathbb{P}}_w(S_{b} = S'_{b}, S_{b'} = S'_{b'}, \forall b' = 3, \ldots, b-1)
\]

Using (3.28), and noting that $S_{b}$ has $b-2$ entries, we obtain
\[
\tilde{\mathbb{P}}_w(S_{b} = S'_{b}, S_{b'} = S'_{b'}, \forall b' = 3, \ldots, b-1) \leq (2/p)^{b-2} + \tilde{\mathbb{P}}_w(\mathcal{E}_b);
\]

this uses the aforementioned independence between columns, guaranteed by the partitioning of the generators. Combining these two equations, we obtain
\[
\tilde{\mathbb{P}}_w(S = S') \leq 2^{d^2/2} \prod^d (1/p^{b-2} + q_b(t)) \quad \text{where} \quad q_b(t) := \max_{w \in \mathcal{W}} \prod^d \tilde{\mathbb{P}}_w(\mathcal{E}_b). \tag{3.30}
\]

It remains to make an appropriate definition of typicality, i.e., of $\mathcal{W}$, and choose the partition $(P_3, \ldots, P_d)$ appropriately. For reasons explained later, we end up choosing $P_b$ so that $R_b := |P_b|/k = (b-2)/(d-1)$, omitting floor/ceiling signs. (Note that $\sum^d R_b = 1$, as required.) We justify the omission of floor/ceiling signs by the fact that $|P_b| = bkd^{-2} \gg 1$ (as $d^2 \ll k$).

This is all done in Lemma 3.16, which gives the following bound:
\[
n \tilde{\mathbb{P}}_w(S = S') \equiv n \mathbb{P}(S = S' \mid W = W' = w) \leq e^{-\log n} 2^{d^2};
\]

recall that if $k \ll d^2 \log n$, then we require $d^3 \ll k$ and $d \log d \leq \log n/k$, as required to apply Lemma 3.16. Combined with (3.24, 3.25) this implies that
\[
n \mathbb{P}(S = S' \mid \text{typ}) - 1 \leq 2 \cdot e^{-\log n} 2^{d^2}, \tag{3.31}
\]

where we shall choose typ so that $\mathbb{P}(\text{typ}) = 1 - o(1)$. If we can show that we can choose $\omega \gg d^2$, then the upper bound in Theorem 3.4 then follows from Lemma 3.7, modulo Lemma 3.16.

It remains to prove that we can choose $\omega \gg d^2$. In Lemma 3.17 below, we show that we can choose any $\omega \ll \min\{k, d^2 \log n\}$. Hence the condition $d^2 \ll \max\{k, d^2 \log n\}$ is sufficient:
\[• \text{when } k \ll \frac{d}{2} \log n, \text{ Hypotheses A (or Remark 3.3) requires that } d^3 \ll k; \]
\[• \text{when } k \geq \frac{d}{4} \log n, \text{ Hypotheses A (or Remark 3.3) requires that } d = (\log n)^o(1). \]

It remains to appropriately bound $q_b(t)$, defined in (3.30). Recall that $1 \ll k \leq (\log n)^{\log \log \log n}$.

**Lemma 3.16.** Let $\varepsilon > 0$ and set $t := (1 + 3\varepsilon) t_\ast$. Assume the conditions of Hypotheses A. Then there exists a $\mathcal{W} \subseteq \mathcal{W}_t$, satisfying (3.2), so that
\[
\mathbb{P}(W \in \mathcal{W}) = 1 - o(1) \quad \text{and} \quad ne^{-\log n} \prod^d (1/p^{b-2} + q_b(t)) \leq 2^{d^2/2}.
\]

Recall the condition on $W$ given by (3.2). Since $t \geq (1 + 3\varepsilon) t_0$, by Lemma 3.12, this condition is satisfied with probability $1 - o(1)$. Hence we need only check that any additional constraints are also satisfied with probability $1 - o(1)$.

**Lemma 3.17.** Let $t_0$ and $t_{2\omega}$ be the entropic times for entropy $\frac{d}{8} \log n$ and $\frac{d}{8} \log n + 2\omega$, respectively. Then we have $t_{2\omega} \approx t_0$ if $\omega \ll \min\{k, d^2 \log n\}$.

**Proof of Lemma 3.16 for $k \geq \frac{d}{4} \log n$.** Let $\varepsilon > 0$ and set $t := (1 + 3\varepsilon) t_\ast$; write $s := t/k$. Typicality. As when $d = 3$, when $k \gg \frac{d}{4} \log n$ almost all the generators are picked at most once; when $k \lessapprox \frac{d}{4} \log n$, a constant proportion are. As part of our typicality requirement (typ),
we ask that at least \((1 - \varepsilon)te^{-t/k}\) generators are picked exactly once—ie at least \((1 - \varepsilon)\) times the expected number. Given this, we can then choose our partition so that, for each \(b \in \{3, ..., d\}\), writing \(R_b := |P_b|/k\), at least \((1 - \varepsilon)te^{-t/k}R_b\) generators from \(P_b\) are picked exactly once.

We can hence use the same definition of typicality, for \(k \geq \frac{2}{3} \log n\), as when \(d = 3\):

\[
\mathcal{W} := \{w \in \mathbb{Z}_+ \mid \mu_d(w) \leq e^{-b}, \ |J(w) - s e^{-s}k| \leq \frac{1}{2}e se^{-s}k, \ \text{max}_i w_i < \sqrt{p}\},
\]

satisfying (3.2). As previously, we have \(P(W \in \mathcal{W}) = 1 - o(1)\).

Analogously to (3.17), when \(k \gg \frac{2}{3} \log n\), we have

\[
q_b(t) \leq 1/(1 - 2\varepsilon)tR_b!,
\]

(3.33)

where we have absorbed the \(e^{-t/k} = 1 - o(1)\) term into the \((1 - 2\varepsilon)\); we consider \(k \approx \frac{2}{3} \log n\) later. Throughout this regime, \(t\) will be an order-1 power of \(\log n\), and \(R_b \gg d^{-2} = (\log n)^{-o(1)}\).

Before continuing, let us note that \((\log n)^{2\nu/(d-2)} \geq (\rho d)^{(2/(d-2)} \rightarrow 1\) as \(d \rightarrow \infty\), and so \((log n)^{(1-\nu)/(2/(d-2))} \approx (log n)^{1+2/(d-2)-\nu}\). This should be kept in mind when we refer to regimes.

**Regime** \(k \geq (\log n)^{1+2/(d-2)-\nu}\). We have \(t \geq (1 + 3\varepsilon)t_{\text{diam}}\). Direct calculation, analogous to (3.18), using (3.14) and (3.33) and Stirling’s approximation gives

\[
q_b(t) \leq 1/(1 - 2\varepsilon) \cdot (1 + 3\varepsilon)t_{\text{diam}} \cdot R_b! \leq \exp(-\frac{1}{p}R_b \log n) = n^{-R_b/p},
\]

(3.34)

using (crucially) the fact that \(d = (\log n)^{o(1)}\) and the fact that \(\varepsilon > 0\) is a constant. If \(d \geq \log \log n \gg \log \log p\), then \((\log n)^{2\nu/(d-2)} \geq (\rho d)^{(2/(d-2))} \rightarrow 1\) as \(d \rightarrow \infty\), and so this regime then covers all of \(k \geq (\log n)^{1-\nu}\).

We subdivide the regime: we first consider \(\rho \geq 1 + \frac{2}{\sigma - 2}\), then \(1 + \frac{2}{\sigma - 2} - \nu \leq \rho \leq 1 + \frac{2}{\sigma - 2}\).

Consider first \(\rho \geq 1 + \frac{2}{\sigma - 2}\). Recall from Definition 3.11 that, in this regime, we take \(h_0 := (1 - \frac{1}{p}) \log n\). Hence \(e^{-ho} = n^{-(1-\nu)/\rho}\). (Note that \(\rho \geq 1\), so \(t_{\text{diam}}/k \ll 1\), as needed in Lemma 3.10.)

In (3.18), we upper bounded \(q(t) \leq n^{-1/\rho}\), and this term was dominant in the sum \(1/p + n^{-1/\rho}\). Here, we compare \(q_b(t) \leq n^{-R_b/p}\) and \(1/p^{1-\nu}\). It is thus natural to choose \(R_b \propto b - 2\), ie \(R_b := (b - 2)/(d - 1)\), for \(b \in \{3, ..., d\}\). Observe that

\[
1/p^{b-2} \leq n^{-R_b/p} \quad \text{if and only if} \quad \rho(b-2)/(d-1) \geq R_b = (b - 2)/(d - 1);
\]

hence we need \(\rho \geq (\frac{d}{d})/(\frac{b}{d}) = 1 + \frac{2}{\sigma - 2}\), which is precisely the regime which we are considering.

Combining the upper bounds just developed, we deduce that

\[
\prod_b (1/p^{b-2} + q_b(t)) \leq 2^d \prod_b q_b(t) \leq 2^d n^{-1/\rho},
\]

since \(\sum_b R_b = 1\). Recalling that \(h_0 = (1 - \frac{1}{p}) \log n\) in this regime, we deduce the desired bound.

Consider now \(\rho \leq 1 + \frac{2}{\sigma - 2}\). Recall from Definition 3.11 that, in this regime, we take \(h_0 := \frac{2}{d} \log n = \log(p^{d-1})\). Hence \(e^{-hop} = p^{-(d-1)}\).

We use the same definition for the partition. Hence, analogously to the above, the dominating term is now \(1/p^{b-2} \geq n^{-R_b/p}\), for this regime of \(\rho\). Hence

\[
\prod_b (1/p^{b-2} + q_b(t)) \leq 2^d p^{-(d-1)} = 2^d n^{-1} p^{-d-1}.
\]

Recalling that \(h_0 = log(p^{d-1})\) in this regime, we deduce the desired bound.

Regime \(\frac{2}{d} \log n \lesssim k \ll (\log n)^{1+2/(d-2)-\nu}\). We have \(t \geq (1 + 3\varepsilon)t_0\). Recall from Definition 3.11 that, in this regime, we take \(h_0 := \log(p^{d-1})\). Hence \(e^{-h_0} = p^{-(d-1)}\). We subdivide the regime: we first consider \(k \gg \frac{2}{3} \log n\), then \(k \approx \frac{2}{3} \log n\).

Consider first \(k \gg \frac{2}{3} \log n\). Using (2.1b), as for Proposition 3.2, in this regime,

\[
t_0 \approx \frac{1}{p + \nu - 1} \cdot \frac{2}{\sigma - 2} \log n.
\]

Direct calculation, analogous to (3.19a), using (3.1c) and (3.33) and Stirling’s approximation, gives

\[
q_b(t) \leq 1/(1 - 2\varepsilon)R_b! \leq \exp(-\frac{2}{\sigma - 2} \cdot \frac{1}{R_b \log n});
\]

(3.35)
again, this crucially uses the fact that \( d = (\log n)^{(\alpha + 1)} \) and \( \varepsilon > 0 \) is a constant.

In (3.19a), we upper bounded \( q(t) \leq n^{-\frac{1}{2d(\nu + 1)}} \), and this term was subdominant in the sum \( 1/p + n^{-\frac{1}{2d(\nu + 1)}} \). Here, we compare \( q(t) \leq n^{-\frac{1}{2d}R_b/(\nu + 1)} + 1/p^b - 2 \). Again, it is thus natural to choose \( R_b = b - 2, i.e. R_b := (b - 2)/(d - 1) \), for \( b \in \{3, \ldots, d\} \). Observe that

\[
1/p^b - 2 \geq \exp\left(-\frac{1}{\alpha + 1} R_b \log n\right)
\]

if and only if \( (\rho + \nu - 1)(b - 2)/(d - 1) \leq R_b = (b - 2)/(d - 1) \);

hence we need \( \rho \leq 1 + \frac{2}{d^2} - \nu \), which is precisely the regime which we are considering.

Combining the upper bounds just developed, we deduce that

\[
\prod_{i,j}(1/p^b - 2 + q_b(t)) \leq 2^{d} \prod_{i,j} 1/p^b - 2 \leq 2^d p^{-(d-1)^2} = 2^d p^{-(d-1)^2}.
\]

Recalling that \( h_0 = \log(p^{d-1}) \) in this regime, we deduce the desired bound.

Consider now \( k = \frac{1}{2} \log n \). Suppose that \( k \approx \lambda \log n \) with \( \lambda \in (0, \infty) \). Direct calculation, analogous to (3.19b), using (3.1b) and (3.33) and Stirling’s approximation gives

\[
q_b(t) \leq \exp(-\lambda f(\lambda) \frac{1}{2} \log n \log \log n) = 1/(p^b - 2)^{2\lambda f(\lambda)} d^{-1} \log \log n.
\]

As noted above, we may assume that \( d \ll \log \log n \), otherwise the previous calculations have already covered this regime. This is crucial, as it allows us to deduce from the above display that \( q_b(t) \ll p^b - 2 \). The proof is then completed in exactly the same way as above.

**Proof of Lemma 3.16 for \( k \ll \frac{1}{2} \log n \).** Set \( t := (1 + 3\varepsilon)t_{\ast} \geq (1 + 3\varepsilon)t_0 \). Recall from Definition 3.11 that, in this regime, we take \( h_0 := \frac{1}{2} \log n = \log(p^{d-1}) \). Hence \( e^{-h_0} = p^{-(d-1)} \). Then \( s := t/k \gg p^{2(d-1)/k} = n^{1/(dk)} \gg 1 \), by Proposition 2.2 and the assumption \( k \ll \frac{1}{2} \log n \).

As noted in the \( d = 3 \) case, neither the actual value of \( t \) nor the fact that \( W \) and \( W' \) are independent PPs is of much consequence. Even the particular form of \( s \) is not important: it can be changed, subject to changing the conditions on \( d \) appropriately. (In fact, we can allow \( k \approx \frac{1}{2} \log n \), which gives \( s \approx 1 \), provided \( k/\log n \) and \( d \) are sufficiently small.)

In the case \( d = 3 \), we looked at (adjacent) pairs of indices \((2i, 2i - 1)\). For general \( d \), this is insufficient; instead, we look at \( m \)-tuples, where \( m \) is a (growing) function of \( d \).

In this regime, the same generator is picked lots of times, with expectation \( s = t/k \gg 1 \). For the moment, let \( \eta \in (0, 1) \). For \( w \in \mathbb{Z}_+^d \), write

\[
C(w) := \{ i \in [k] \mid \eta s \leq w_i \leq \eta^{-1} s \}.
\]

Then, for \( \eta \) sufficiently small (but still a constant), we have

\[
\mathbb{P}(\{C(W)\}/k \geq \frac{1}{2}) = 1 - o(1).
\]

(We could replace \( \frac{1}{2} \) by any constant less than 1.) This will form part of our typicality requirements:

\[
W := \{ w \in \mathbb{Z}_+^k \mid \mu_i(w) \leq e^{-h}, |C(w)| \geq \frac{1}{2}k, \max_i w_i < p \}.
\]

Note that this definition satisfies (3.2). For \( i, j \in C(w) \), as when \( d = 3 \), we have \( \{C_{i,j} = C'_{i,j}\} \), since \( \max\{C_{i,j}, C'_{i,j}\} \leq w_i w_j \leq s^2 = p^{(d-1)/k} \ll p \).

Now recall the partition \((P_3, \ldots, P_d)\) of \( k \), and the definition \( R_b := |P_b|/k = (b - 2)/(d - 1) \). Let \( m \) be an integer (allowed to depend on other parameters) with \( m \ll \text{min}_i|P_i| = k/(d - 1) \ll k/d^2 \). By exchangeability of the generators, for each \( b \in \{3, \ldots, d\} \) assume that the first \( \frac{1}{2} |P_b|/k \) entries \( i \) of \( P_b \) satisfy \( \eta s \leq w_i \leq \eta^{-1} s \).

Our aim is to show that the mode of the vector \( C_m := \{C_{i,j}\}_{i,j \in [m]} \), conditional on \( W = w \), which we denote \( g_m \), is bounded by \( s^f(m) \), for some (suitable) super-linearly growing function \( f \), recalling that \( s \approx p^{2(d-1)/k} \). We prove this in Claim 3.18 below, and in fact show that we can take \( f(m) \approx m^2 \); for now, assume that claim.

Partition \( \{1, \ldots, |P_b|/m\} \) into \( \frac{|P_b|}{m} \geq \frac{3}{4} |P_b|/m \) sequential intervals of length \( m \), say \( I_{1,b}, \ldots, I_{N,b} \). This allows us to decompose

\[
\mathcal{E}_b = \{C_{i,j} = C'_{i,j} \forall i, j \in P_b\} \subseteq \cap_{i=1}^N \{C_{i,j} = C'_{i,j} \forall i, j \in I_{i,b}\}.
\]
Moreover, the events in the intersection are independent. We upper bound each using the mode:

\[ q_b(t) \leq q_m^{N} \leq s^{-f(m)N} \leq p^{-(d-1)k^{-1}f(m)}\frac{(3/4)^{|P_x|/m}}{dR_b} \leq p^{-(b-2)d^{-1}f(m)/m}, \]

as \( s \approx p^{2(d-1)/k} \), and so in particular \( s \geq p^{(d-1)/k} \) (recall that we had said that the exact value of \( s \) would be unimportant); we have \( dR_b = (b-2)d/(d-1) \geq 2(b-2)/d \), and so this becomes

\[ q_b(t) \leq p^{-(b-2)d^{-1}f(m)/m}. \]  \( (3.38) \)

Since \( f(m)/m \approx m \), we can choose a constant \( C \) large enough so that \( m := Cd \) satisfies \( f(m)/m \geq d \), and hence \( q_b(t) \leq 1/p^d \).

We still need \( n \leq k/(d-1) \approx k/d \); since \( m \approx d \), this is equivalent to requiring \( d^3 \ll k \). Finally, to apply Claim 3.18, we need \( m \ll s \); since \( m \approx d \) and \( s \approx n^{4/(dk)} \), this holds by our assumption that \( d \log d \leq 3 \log n/k \).

This establishes the desired bound, as it did for \( \frac{1}{d} \log n \leq k \leq (\log n)^{1+2/(d-2)} \). \( \square \)

**Claim 3.18.** In the notation and under the assumptions of the above proof, there exists an absolute positive constant \( c \) so that, assuming that \( m \ll s \approx p^{2(d-1)/k} \) and \( d \log m \leq n \), we have

\[ q_m \leq s^{-cm^2}. \]

(In fact, we only need \( m \ll c's \) for a sufficiently small, positive constant \( c' \).)

**Proof.** For this proof, we use the following notation: for \( w \in W \), write \( P_{\omega}(\cdot) := \mathbb{P}(\cdot \mid W = W' = w) \) and \( E_{\omega}(\cdot) \) similarly; often we consider events that depend on \( W \) but not on \( W' \), in which case we ignore the conditioning on \( W' \) (noting that \( W \) and \( W' \) are independent). Recall that we write \( G_\ell \in [k] \) for the index of the generators chosen in the \( \ell \)-th step.

Further, we abuse notation and terminology slightly by always assuming that “pairs \( (i, j) \)” have \( i \neq j \), and write \( [m] = \{(i, j) \mid i \neq j\} \), so \( |[m]| = m(m-1) \).

Take an arbitrary ordering of all \( m(m-1) \) distinct pairs \( (i, j) \in [m]^2 \); write \( K := m(m-1) \) and the \( \kappa \)-th term (of the ordering) as \( y_\kappa \), for \( \kappa \in [K] \).

Let \( x_m = (x_{i,j})_{i,j \in [m]^2} \), with \( x_{i,j} \in \mathbb{N}_0 \) for all \( (i, j) \in [m]^2 \), be arbitrary. We are interested in \( P(C_m = x_m \mid W = w) \); cf \((3.22)\). We do this by sequentially estimating the conditional probabilities that \( C_{i,j} = x_{i,j} \). For each \( \kappa \in [K] \), let \( x_\kappa := I(C_{y_\kappa} = x_{y_\kappa}) \), recalling that \( (y_1, ..., y_K) \) is the chosen ordering of all pairs \( (i, j) \in [m]^2 \). Then we want to bound \( E_{\omega}(1 \cdot \cdot \cdot 1) \).

To do this, we use the following general bound, which is an immediate consequence of the tower property for conditional expectation: for random variables \( V_1, ..., V_K \), we have

\[ \mathbb{E}(f_1(V_1, ..., V_K) \cdots f_K(V_1, ..., V_K)) \leq \max_{v_1, ..., v_K} \mathbb{E}(f_1(v_1, v_2, ..., v_K) \cdots f_K(v_1, ..., v_{K-1}, V_K)) \]

\[ = \max_{v_1, ..., v_K} \mathbb{E}\left( \prod_{\kappa=1}^K f_k(v_1, ..., v_{\kappa-1}, V_{\kappa}, v_{\kappa+1}, ..., v_K \mid V_1, ..., V_{\kappa-1}) \right). \]

The following argument is analogous to that given in the \( d = 3 \) case; see the end of the proof of Lemma 3.13. Let \( r = \sum_{i=1}^n w_i \), and write our random word as \( S = Z_{G_1} \cdots Z_{G_N} \); here \( N = \sum_{i=1}^k w_i \) is the number of steps taken and \( G_\ell \) is the generator index chosen in the \( \ell \)-th step. Let \( J_1 < \cdots < J_r \) be the (random) indices with \( G_{J_\ell} \in [m] \). Now define the vector \( I \in \{1, ..., m\}^r \) by \( I_\ell := G_{J_\ell} \).

Thus \( I \) encodes the relative order between the different occurrences (with multiplicities) of the generators labelled by \( [m] \) in the word \( S \). By typicality, \( m \eta s \leq r \approx m \eta^{-1}s \)

Sequentially, and without replacements, for each pair \( (i, j) \), or index \( \kappa \), choose a uniformly random element of \( \{ \ell \mid G_\ell = j \} \); call this \( U_{i,j}, \) or \( U_\kappa \). (This can be done since \( |\{ \ell \mid G_\ell = j \}| \geq s \gg m \) by assumption.) For each pair \( (i, j) \), let \( V_{i,j} \) be the location in \( I \) of the random element \( U_{i,j} \). Now define the vector \( I' \) from \( I \) by omitting the \( K = m(m-1) \) locations \( \{V_{i,j}\}_{i,j \in [m]^2}; \) so \( I' \in \{1, ..., m\}^{r-K} \). Importantly, we are omitting elements of the relative order, not the absolute order.

By definition of \( C_{i,j} \), given in \((3.8)\), given \( W = w \), the value of \( C_{i,j} \) is a function only of the relative locations \( I \). Hence, given \( I' \) also, it is a function only of the location of the omitted \( j \). It is constant on the set of locations which give rise to the same relative order between choices of \( i \) and \( j \); two different placements of the omitted \( j \) give rise to the same relative order between choices of
Recall that the random variables \( \{V_k\} \) are drawn uniformly at random without replacement from \([r]\); hence the distribution of \( V_k \) given \( V_1, \ldots, V_{k-1} \) is uniform on \([r] \setminus \{V_1, \ldots, V_{k-1}\}\). Hence, writing \( L_i := L_i(I') \) for the longest interval in \( I' \) without an \( i \) in it, we obtain

\[
\mathbb{E}_w(\chi_k(v_1, \ldots, v_{k-1}, V_k, v_{k+1}, \ldots, v_K) \mid V_1, \ldots, V_{k-1}, I') \leq (L_k + m(m - 1) + 1)/(r - \kappa + 1).
\]

Hence, applying the above formula and noting that \( m(m - 1) + 1 \leq m^2 \), we obtain

\[
\mathbb{E}_w(\chi_1 \cdots \chi_K \mid I') \leq \frac{\prod_i^m (L_i + m^2)^{m-1}}{r(r-1) \cdots (r - m(m-1) + 1)}.
\]

We now average over \( I' \). To bound this expectation, we use the generalisation of Hölder’s inequality to the product of \( m \) variables: for non-negative random variables \( X_1, \ldots, X_m \), we have

\[
\mathbb{E}(X_1 \cdots X_m) \leq (\mathbb{E}(X_1^m) \cdots \mathbb{E}(X_m^m))^{1/m}.
\]

In our application of this, we take \( X_i = (L_i + m^2)^{m-1} \). This gives

\[
\mathbb{E}(\prod_i^m (L_i + m^2)^{m-1}) \leq \max_i \mathbb{E}((L_i + m^2)^{m(m-1)}).
\]

Since \( r \geq mns \) and \( s \gg m \), we have \( m^2 \leq \frac{1}{2} r \), and so the denominator is at least \( 2^{-m^2 r - m(m-1)} \); recall also that \( r \geq mns \). In summary, we have proved that

\[
\mathbb{E}_w(\chi_1 \cdots \chi_K) \leq 2^m(2\eta^{-1}m^{-1})^{m(m-1)s-m(m-1)} \max_i \mathbb{E}((L_i + m^2)^{m^2}). \quad (3.39)
\]

Recall also that \( s \approx p^{2(d-1)/k} \). It remains to bound this latter expectation.

To do this, recall Claim 3.14, we states that

\[
L_i/(\eta^{-2}m \log r) \leq \text{Geom}(\frac{1}{2}).
\]

Using the the inequality \((a + b)^\ell \leq 2^{\ell-1}(a^\ell + b^\ell)\), valid for \( a, b \geq 0 \) and \( \ell \in \mathbb{N} \), we obtain

\[
(L_i + m^2)^{m_2} \leq (2\eta^{-2}m \log r)^{m^2} (L_i/(\eta^{-2}m \log r))^{m^2} + (2m^2)^{m^2}.
\]

If \( X \sim \text{Geom}(\frac{1}{2}) \), then one can show, for \( \ell \geq 3 \), that \( \mathbb{E}(X^\ell) \leq \ell^\ell \). (This follows by comparison with the exponential-(log 2) distribution.) We apply this with \( X := L_i/(\eta^{-2}m \log r) \) and \( \ell := m^2 \):

\[
\mathbb{E}((L_i + m^2)^{m^2}) \leq (2\eta^{-2}m \log r)^{m^2} \cdot (m^2)^{m^2} + (4m^2)^{m^2} \leq 2(2\eta^{-2}m^3 \log r)^{m^2}.
\]

Plugging this back into (3.39), we obtain

\[
- \log \mathbb{E}_w(\chi_1 \cdots \chi_K) \approx m^2 \log s + m^2 \log m + m^2 \log r.
\]

Recalling that \( p \gg \log s \approx (d/k) \log p \gg (\log n)/(dk) \), we obtain

\[
- \log \mathbb{E}_w(\chi_1 \cdot \chi_K) \approx m^2 \log s + \log m \approx m^2 \cdot (\log n)/(dk) + \log m.
\]

Recall that we desire

\[
\mathbb{E}_w(\chi_1 \cdots \chi_K) \leq s^{-f(m)} \leq p^{-(d-1)f(m)/k},
\]

as \( s \approx p^{2(d-1)/k}/(2\pi e) \geq p^{(d-1)/k} \), where \( f \) is some super-linearly growing function; this holds if

\[
- \log \mathbb{E}(\chi_1 \cdots \chi_K \mid W = w) \geq (d - 1)f(m) \log p/k \geq f(m)(d/k) \log p.
\]

We hence see that this is satisfied, with \( f(m) \approx m^2 \), if

\[
d^{-1} \log n/k \geq \log m, \quad \text{i.e.} \quad d \log m \leq \log n/k.
\]

Recall that we end up taking \( m = d \), and so this condition is equivalent to \( d \log d \leq \log n/k \). \( \square \)
**Proof of Lemma 3.17.** This lemma is a statement simply about the entropic times; it is independent of the group we are using. As in §2.3, write $H(s)$ for the entropy of a single coordinate at time $s := t/k$. Recall that the entropic time $t_0$ is defined so that $H(s_0) = \log(n^{2/d})/k = \frac{2}{d} \log n/k$ where $s_0 := t_0/k$. In §2.3, specifically (2.3a, 2.3b), we established the following relations:

\[
H(s) = \begin{cases} 
\frac{1}{d} \log(2\pi e s) + O(s^{-1}) & \text{when } s \gg 1; \\
\log(1/s) + O(s) & \text{when } s \ll 1; 
\end{cases}
\]

(2.3a)

when $s \approx 1$, we use continuity of $\lambda \mapsto H(1/\lambda)$, as in §2.3. Let $\delta > 0$; assume that $\delta \ll 1$, but more slowly than the error terms in (2.3), recalled above. We need to choose $\delta$ and $\omega$ so that

\[
H(s_0(1 + \delta)) \geq \log n/k + 2\omega/k \quad \text{with} \quad \delta \ll 1 \quad \text{and} \quad \omega \gg 1.
\]

Note that the statement is monotone in $\omega$: if it holds for some $\omega$, then it holds for any $0 \leq \omega \leq \omega'$, since then $t_0 \leq t_{2\omega} \leq t_{2\omega'}$. Hence we may assume lower bounds on $\omega$, if desired.

**Regime** $k \ll \frac{2}{d} \log n$. We have $s_0^{-1/4} \ll n^{-1/(2dk)} \gg 1$. By (2.3a), we have

\[
H(s_0(1 + \delta)) = \frac{1}{d} \log(2\pi e s_0) + \frac{1}{d} \log(1 + \delta) + O(s_0^{-1})
\]

\[= \frac{2}{d} \log n/k + \frac{1}{d} \delta + O(\min\{\delta^2, s_0^{-1}\}).\]

We take $\delta := 5\omega/k$, and so need $n^{-2/(dk)} \ll \omega/k \ll 1$. Hence $\omega \ll k$ suffices.

**Regime** $k \gg \frac{2}{d} \log n$. We have $s_0 \approx 1/(k \log n) \ll 1$ where $k := \frac{1}{d} \log n$. By (2.3b), we have

\[
H(s_0(1 + \delta)) = (1 + \delta) \cdot s_0 \log(1/s_0) - s_0(1 + \delta) \log(1 + \delta) + O(s_0)
\]

\[= \frac{2}{d} \log n/k + \delta s_0 + O(s_0).
\]

We take $\delta := 3\omega d/k$, and so need $1/(\kappa \log n) \ll \omega d/k \ll 1$. Hence $\omega \ll \frac{2}{d} \log n$ suffices.

**Regime** $k \gg \frac{2}{d} \log n$. By continuity and the strict increasing property of the entropy, all we require is that $\frac{2}{d} \log n/k + 2\omega/k = (1 + o(1))\frac{2}{d} \log n/k$, and hence only require $\omega \ll \frac{2}{d} \log n \gg k$. \phantom{.}

The very final thing to do is to prove Lemma 3.15; it uses double induction.

**Proof of Lemma 3.15.** We prove the claim by induction on $b - a$. The cases $b - a \in \{0, 1, 2\}$ are trivial. We now assume the claim is true for all $\ell$ whenever $b - a \leq m$, and prove that it holds also when $b - a = m + 1$. We prove the induction step (in terms of $b - a$) by induction on $\ell$.

Denote $K := Z_{a_1} \cdots Z_{a_{\ell-1}}$. Then, for $(a, b) \in [d]^2$, we can write

\[
M(a, b) = K(a, b) + Z_{a_1}(a, b) + K(a, a + 1)Z_{a_1}(a + 1, b) + \sum_{r=a+2}^{b-1} K(a, r)Z_{a_1}(r, b).
\]

By the induction hypothesis wrt $\ell$, the terms $K(a, b) + Z_{a_1}(a, b) + K(a, a + 1)Z_{a_1}(a + 1, b)$ can be written in the desired form. It thus suffices to show that each term in the sum $\sum_{r=a+2}^{b-1} K(a, r)Z_{a_1}(r, b)$ can be written in the same form as $g_{a,b}(\alpha, Z_1, ..., Z_h)$ from the statement of the lemma.

As the sum is over $r \geq a + 2$, it is in fact enough to show that, for all $r \in [a + 2, b - 1]$, we can write $K(a, r)$ in the same form as $g_{a,b}(\alpha, Z_1, ..., Z_h)$ from the statement of the lemma. Namely, that for all $i, j \in [k]$ and $b' \geq b$, each monomial in $K(a, r)$ contains the term $Z_i(a, b')$ either $0$ times or exactly once, and no monomial in $B(a, r)$ contains a term of the form $Z_i(a, a + 1)Z_j(a, b')$. This follows from the induction hypothesis on $b - a$, using the fact that $s < b$. \phantom{.}

### 3.6 Cutoff Window

In Remark 3.5, we claimed that our analysis actually allows us to find the cutoff window, and even shape, when $k \ll \frac{2}{d} \log n = (\log n)^{1-\nu}$. Key is that, in this regime, $t_s = t_0(k, n^{2/d})$.

Recall that $t_0$ is determined by the entropy of $W$, which was the expectation of the random variable $Q$—see Definition 2.1. For $\alpha \in \mathbb{R}$, we now define $t_\alpha$ according to the variations of $Q$:

\[
E(Q_1(t_\alpha)) = (\log n + \alpha \sqrt{vk})/k \quad \text{where} \quad v := \text{Var}(Q_1(t_0)).
\]
Then, for all $\alpha \in \mathbb{R}$, [32, Proposition 2.2] shows that

$$t_0(k, n^{2/d}) \approx k \cdot n^{4/(dk)}$$

and $t_\alpha - t_0 \approx \alpha \sqrt{2t_0} / \sqrt{k} = o(t_0)$,

and, provided $\omega \ll \sqrt{k}$, [32, Proposition 2.3] shows that

$$\mathbb{P}(Q(t_\alpha) \leq \log n + \omega) \to \Psi(\alpha) \quad \text{as } n \to \infty.$$

For a proof of these claims, see [32, §2 and §A].

Now recall from the analysis of the total variation distance given typicality, from §3.5, in the regime $k \ll \frac{2}{3} \log n$, that the particular value of $t_0$ is unimportant—changing it by a constant would not affect the proof or the result. The above says that $t_\alpha \approx t_0$ for all $\alpha \in \mathbb{R}$. Hence this contribution is $o(1)$ when $t_0$ is replaced by $t_\alpha$, regardless of $\alpha \in \mathbb{R}$.

All that changes is Lemma 3.12 which, by the above, becomes

$$\mathbb{P}(\mu_{t_\alpha}(W(t_\alpha)) \leq e^{-h}) \approx 1 - \Psi(\alpha).$$

Hence, exactly the same argument gives

$$d_Z(t_\alpha) \approx \Psi(\alpha) \quad \text{whp over } Z.$$

When $k$ is large enough so that $t_* = t_{\text{diam}}(k, p^{d(d-1)/2})$, one can look at the relevant part of the proof in §3.5 to get bounds on the window, improving slightly some bounds. It is less easy to analyse the window and shape when $k$ is in the regime so that, in the maximum $t_* = \max\{t_0(k, p^{d-1}), t_{\text{diam}}(k, p^{d(d-1)/2})\}$, the two terms are roughly equal. We do not go into details.

4 Typical Distance

This section focuses on distances from a fixed point in the directed random Cayley graph of a Heisenberg matrix group $H_{p,d}$, with $p$ prime and $d \geq 3$. Recall the definition of typical distance: when $G := H_{p,d}$ and there are $k$ generators, for $R \geq 0$ and $\beta \in (0, 1)$, write

$$B_k(R) := \{x \in G_k \mid \text{dist}(0, x) \leq R\} \quad \text{and} \quad D_k(\beta) := \min\{R \geq 0 \mid |B_k(R)| \geq \beta |G|\},$$

emphasising explicitly the dependence on $d$ for the latter statistic.

4.1 Precise Statement and Remarks

In this section, we state the more refined version of Theorem B. Again, there are some simple conditions that the parameters must satisfy.

**Hypotheses B.** The triple $(k, p, d)$ satisfies Hypotheses B if the following conditions hold:

$$1 \ll k \ll d \log p, \quad \log d \lesssim d \log p/k, \quad d^3 \ll k \quad \text{and} \quad d \ll \max\{\log k, k^{1/2}/p^{(d-1)/(4k)}\}.$$  

(Recall that implicitly we consider sequences $(k_N, p_N, d_N)_{N \in \mathbb{N}^+}$.)

As with Hypotheses A, it is sometimes convenient to write these in a different form.

**Remark.** Writing $n := p^{d(d-1)/2}$, the conditions of Hypotheses B imply the following:

$$1 \ll k \ll \frac{2}{3} \log n, \quad d \log d \lesssim \log n/k, \quad d^3 \ll k \quad \text{and} \quad d \ll \max\{\log k, k^{1/2}/n^{1/(2d)}\}.$$  

Note that these conditions are satisfied by $d$ fixed (order 1) and $k \ll \log n$.  

**Theorem 4.1 (Typical Distance).** Let $(k, p, d)$ be integers with $p$ prime and $d \geq 3$, satisfying Hypotheses B. Let $G := H_{p,d}$ be a Heisenberg group. Write $M^*_k := k p^{(d-1)/k}/e$.

For all constants $\beta \in (0, 1)$, we have

$$|D_k(\beta) - M^*_k|/M^*_k \to 0 \quad \text{whp over } Z \quad (\text{as } N \to \infty).$$
Moreover, the implicit lower bound holds deterministically: for $1 \ll k \ll d \log p$, for all positive constants $\xi$ and $\beta \in (0,1)$ and all multisets $Z$ of size $k$, for $N$ sufficiently large, we have

$$D_k(\beta) \geq M^*_k \cdot (1 - \xi).$$

(Recall that implicitly we consider sequences $(k_N, p_N, d_N)_{N \in \mathbb{N}}$; this is the $N$ above.)

**Remark.** Our results above are for the standard $L_1$ graph distance. It is possible to define a type of $L_q$ graph distance, for general $q \in [1, \infty]$, and prove analogous results, albeit potentially with more restrictive conditions on the triple $(k, p, d)$; see [32, §4], where we do this for general Abelian groups $G$, and in addition study $k \gg \log n$.

As a proxy for the size of the $(L_1)$ balls in the (directed) Cayley graph with $k$ generators, denoted $B_k(\cdot)$, we use the size of discrete, directed $L_1$ balls in dimension $k$, denoted $B_k(\cdot)$: for $R \geq 0$, define $B_k(R) := \{ x \in \mathbb{Z}^d_k | \text{dist}(0, x) \leq R \}$. This is done in Lemma 4.3 below.

Were the underlying group Abelian, we would have the easy inequality $|B_k(R)| \leq |B_k(R)|$. For the Heisenberg group $H_{p,d}$, we develop a similar inequality; roughly, we use the inequality for the Abelianisation, which is of size $p^{d-1}$, and upper bound the number of elements which can be seen by the other vertices by the maximum amount, i.e $n/p^{d-1}$.

In [32, §3], we studied typical distance for general Abelian groups, using the same (overall) method; there, the radius $R$ of the balls in question was chosen so that $|B_k(R)| = |G|$. Since the Abelianisation is of size $p^{d-1}$, here we make the following definition of our candidate radius $M_k$.

**Definition 4.2.** Set $\omega := \max\{(\log k)^2, k/p^{(d-1)/2k}\}$, and choose $M_k$ to be the minimal integer satisfying $|B_k(M_k)| \geq \omega p^{d-1}$. (Note that $1/p^{(d-1)/2k} = 1/n^{k/(dk)} \ll 1$ when $k \ll \frac{1}{n} \log n$.

### 4.2 Outline of Proof

As remarked after the summarised statement (in §1.1), when considering the mixing time on a graph, geometric properties of the graph are often derived and used. In a reversal of this, we use knowledge about the mixing properties of the random walk to derive a geometric result; the style of proof is similar enough that we can quote lemmas from the mixing section.

The main difference between the proofs is the following: previously, $W(\cdot)$ was a PP on $\mathbb{Z}$; we replace this $W(t)$ by $A$ which is uniformly distributed on a $\mathbb{Z}$-ball of radius $R$, for $R$ defined later; this $A$ tells us how many times each generator is used; we apply the sequence of generators, with multiplicities, in an order chosen uniformly at random; call the resulting element $S$.

We choose $M$ so that this ball has size slightly larger than $p^{d-1}$ (the size of the Abelianisation)—recall that this size was used for the entropic time $t_0(k, p^{d-1})$ in the mixing. For a constant $\xi > 0$, if $R := M(1 - \xi)$, then we use a counting argument to show that the ball cannot cover more than a proportion $o(1)$ of the vertices of the graph; hence this gives a deterministic lower bound, valid for all $Z$.

For a constant $\xi > 0$, if $R := M(1 + \xi)$, then we show that not only does the ball cover (almost) all the graph, but the random variable $S$ is well-mixed whp, in the sense that it is very close to the uniform distribution. From this we deduce that, for a proportion $1 - o(1)$ of the vertices, there is a non-zero probability that $S$ is at that vertex, and hence a path to it must exist; furthermore, by choice of $A$, the path must have length at most $R = M(1 + \xi)$. To prove this, we even use an analogous $L_2$ calculation to that used for the mixing, namely Lemma 3.7.

### 4.3 Size of Ball Estimates and Lower Bound

**Lemma 4.3.** For all $R \geq 0$, we have

$$|B_k(R)| = \binom{|R| + k}{k}.$$

**Proof.** Assume that $R \in \mathbb{N}$. It is a standard combinatorial identity that

$$|B_k(R)| = \left| \{ \alpha \in \mathbb{Z}^k_+ \mid \sum^k \alpha_i \leq R \} \right| = \binom{|R| + k}{k}.$$  

$\square$
Recall that $M_k^* = kp^{d-1}/e$. The next lemma shows that the difference between $M_k$ and $M_k^*$ is only in subleading order terms, and so can be absorbed into the error terms.

Lemma 4.4. For $k \ll \frac{2}{\epsilon} \log n = \log(p^{d-1})$, for all constants $\xi \in (0,1)$, we have

$$M_k \leq \left[ M^*_k(1 + \xi) \right] \quad \text{and} \quad |B_k(M^*_k(1 - \xi))| \ll p^{d-1}.$$ 

Proof. Upper bound. Set $M := \lfloor e^k p^{(d-1)/k} / e \rfloor$. By Stirling’s approximation, we have

$$\binom{M+k}{k} \geq M^k / k! \geq k^{-1/2}(eM/k)^k = k^{-1/2}e^{k}p^{d-1}.$$ 

Since $\omega \ll k$, we have $k^{-1/2}e^{k} \gg \epsilon^{\omega}$ and $\binom{M+k}{k} \geq \epsilon^{\omega}p^{d-1}$.

Lower bound. Set $M := \lfloor e^{-k} p^{(d-1)/k} / e \rfloor$. Using the inequality $\binom{N}{k} \leq (eN/k)^k$, we have

$$\binom{M+k}{k} \leq (e(M + k)/k)^k \leq (eM/k)^k \exp(k^2/M) \leq e^{-\omega}p^{d-1} \exp(e \xi k/p^{(d-1)/k}).$$ 

Since $k \ll \frac{2}{\epsilon} \log n$, we have $p^{(d-1)/k} = n^{2/(dk)} \gg 1$. Hence $k/p^{(d-1)/k} \ll \xi k$ and $\binom{M+k}{k} \ll p^{d-1}$. \hfill $\Box$

From these, it is straightforward to deduce the lower bound (for all $Z$) in Theorem 4.1.

Proof of Lower Bound in Theorem 4.1. Were the underlying group Abelian, we would be able to upper bound $|B_k(M)| \leq |B_k(M)|$. However, this does not hold for general groups.

Recall that the Abelianisation of $H_p$ corresponds to ‘modding out all but the super-diagonal’:

$$H_p/[H_p,H_p] \cong \mathbb{Z}_p^{d-1} \quad \text{and} \quad [H_p,H_p] \cong \mathbb{Z}_p^{(d-1)/2} = \mathbb{Z}_p^{(d-1)-(d-2)/2}.$$ 

This establishes, for Heisenberg groups, an inequality similar to $|B_k(M)| \leq |B_k(M)|$ for Abelian groups: for a given number of steps, the number of different elements (of the group, ie matrices) that can be seen is at most $|H_{p,d},H_{p,d}| = p^{(d-1)-(d-2)/2}$ times the number that can be seen in the Abelianisation $H_{p,d}/[H_{p,d},H_{p,d}]$; that is,

$$|B_k(M)| \leq L \cdot |[H_{p,d},H_{p,d}]| \quad \text{where} \quad L := \{|g[H_{p,d},H_{p,d}] \mid g \in B_k(M)\}|.$$ 

We have $|[H_{p,d},H_{p,d}]| = p^{(d-1)-(d-2)/2}$ and $L \leq |B_k(M)|$, so

$$|B_k(M)| \leq p^{(d-1)-(d-2)/2} |B_k(M)|.$$ 

We choose $M$ so that $|B_k(M)| \approx p^{d-1}$, meaning that the right-hand side above is approximately $n$.

By Lemma 4.4, we have $|B_k(M^*_k(1 - \xi))| \ll p^{d-1}$ for any constant $\xi > 0$. Hence

$$|B_k(M^*_k(1 - o(1)))| \ll p^{(d-1)-(d-2)/2 \cdot p^{d-1} = n}.$$ 

This immediately implies that, for any constant $\beta \in (0,1)$, we have $D_k(\beta) \geq M_k^*$. \hfill $\Box$

Remark 4.5. This proof generalises further. Instead of looking at just Heisenberg groups, we can take any group $G$. We then obtain a lower bound analogously, but where now $M_k$ is defined so that $|B_k(M_k)| \geq e^{\omega|G|/|G,G|}$, for some suitable $\omega \gg 1$. (For $H_{p,d}$, this is $e^{\omega p^{d-1}}$.) \hfill $\triangle$

Remark 4.6. The statements are for directed lattice balls, in $\mathbb{Z}_p^d$. Changing to undirected lattice balls, in $\mathbb{Z}^d$, increases the size by a factor at most $2^d$. Since $k \ll \frac{2}{\epsilon} \log n = \log(p^{d-1})$ and we are looking at sizes of the order $p^{d-1}$, analogous statements can easily be proved for directed balls. \hfill $\triangle$

### 4.4 Mixing-Type Results and Upper Bound

As stated in the outline (§4.2), we replace the auxiliary $W$ with $A \sim \text{Unif}(B_k(M_k))$, and then apply the generators in a uniformly chosen order. More precisely, we have the following algorithm.

- First draw $A \sim \text{Unif}(B_k(M_k))$; this tells us how many times we use each of the $k$ generators.
  - Define the vector $g$ by $g_1 = \cdots = g_{A_1} = 1$, $g_{A_1 + 1} = \cdots = g_{A_1 + A_2} = 2$ and so on.
To decide in which order we apply the generators, label the steps 1,...,\( N \), so \( N := \sum_{i} A_{i} \), and then draw a uniform permutation \( \sigma \) on \([N] = \{1,\ldots,N\} \); this will tell us in which order we the generators: \( S := Z_{g_{\sigma(1)}} \cdots Z_{g_{\sigma(N)}} \).

In words, we choose how many times each generator is going to be used by \( A \), and then apply them in a uniformly chosen order.

We now present our ‘mixing-type’ result, showing that \( S \) is close to uniform; cf Proposition 4.7.

**Proposition 4.7.** Assume that the conditions of Theorem 4.1 hold. Then

\[
\mathbb{E}_{Z}(\|P(S = \cdot \mid Z) - \pi_{G}\|_{TV}) = o(1).
\]

**Proof.** For notational ease, write \( M := M_{k} \). Let \( S', A' \) and \( \sigma' \) be independent copies of \( S, A \) and \( \sigma \), respectively. For a set \( A \) (to be defined), the modified \( L_{2} \) calculation used in Lemma 3.7 gives

\[
\mathbb{E}_{Z}(\|P(S = \cdot \mid Z) - \pi_{G}\|_{TV}) \leq n P(S = S' \mid \text{typ}) - 1 + P(A \notin A),
\]

where \( \text{typ} := \{A, A' \in A\} \). Similarly to the mixing case, we separate according to whether or not \( A = A' \). If \( A = A' \), then we do an analysis similar to that of \( W = W' \) from §3. Using Lemma 3.9 in an analogous way as was used to obtain (3.24), we obtain

\[
P(S = S' \mid A \neq A', \text{typ}) = 1/n; \tag{4.3}
\]

recall that the coefficients in Lemma 3.9 (corresponding to the entries of \( A - A' \) here) are deterministic, and hence (4.3) holds regardless of the choice of \( A \). This also uses the fact that \( M \ll p \), and so \( |A_{i} - A'_{i}| \ll p \) for all \( i \); this follows from manipulating the conditions of Hypotheses B and using \( M \approx kp^{(d-1)/k} \). Using the definition of \( M = M_{k} \), it is easy to calculate

\[
P(A = A' \mid \text{typ}) \leq |B_{k}(M)|^{-1} / P(\text{typ}) \leq e^{-d^{2}/2} P(\text{typ}); \tag{4.4}
\]

this replaces the entropic calculation (3.25). Combining (4.3, 4.4) establishes

\[
n P(S = S' \mid \text{typ}) - 1 \leq e^{-d^{2}/2} P(S = S' \mid A = A', \text{typ}) / P(\text{typ}) \leq 1/n.
\]

As stated above, the analysis of \( P(S = S' \mid A = A', \text{typ}) \) is analogous to the \( W = W' \) case from §3. There we stated that it was not important that \( W \) was a PP, and that we would apply the same proof here (§4) for a “different \( W'\)—the \( A \) just defined is this “different \( W'\). So far, we have in essence been following Proof of Theorem 3.4 Given Lemmas 3.16 and 3.17 from the start of §3.5, but with \( W(t) \) replaced by \( A \) and \( W \) replaced by \( A_{i} \); it is not until Lemma 3.16, which upper bounds the analogue of \( P(S = S' \mid A = A', \text{typ}) \), that the choice of typicality (\( W \) there; \( A \) here) is made. Hence (3.30) still holds here: write \( \tilde{P}_{a}(\cdot) := P(\cdot \mid A = A' = a, \text{typ}) \); define \( \tilde{E}_{a} \) as in (3.29):

\[
\tilde{P}_{a}(S = S') \leq 2^{d^{2}/2} \prod_{i} \left(1/p^{d-2} + q_{b}(t)\right) \quad \text{where} \quad q_{b}(t) := \max_{a \in A} \prod_{i} \tilde{P}_{a}(\xi_{b}). \tag{4.5}
\]

In the mixing context, Lemma 3.16 upper bounded this probability by \( 2^{d^{2}} e^{h_{0}+\gamma n^{-1}} \), where \( h_{0} \) was the entropy; for \( k \ll \frac{1}{\gamma} \log n \), we chose \( h_{0} = \log(p^{d-1}) \), so this upper bound became \( 2^{d^{2}/2} n^{-1} p^{d-1} \).

Combined with the entropic calculation (3.25), of which (4.4) is the analogue, established (3.31): \( n \mathbb{P}(S = S' \mid \text{typ}) - 1 \leq 2 e^{-\omega} 2^{d^{2}} \). Conditions on \( d \) ensured that we could choose \( \omega \) to make this \( o(1) \).

We claim that we can copy the proof of Lemma 3.16 to show that \( q_{b}(t) \leq 1/p^{b-2} \). The proof of this claim is deferred to the end of the subsection (§4.4). From this claim, we deduce that

\[
\tilde{P}_{a}(S = S') \leq 2^{d^{2}} \prod_{i} 1/p^{d-2} = 2^{d^{2}} n^{-(d-1)(d-2)/2}. \tag{4.6}
\]

Combining (4.5, 4.6), we obtain

\[
n P(S = S' \mid \text{typ}) - 1 \leq e^{-\omega} 2^{d^{2}} / P(\text{typ}),
\]

where we shall choose \( \text{typ} \) so that \( P(\text{typ}) = 1 - o(1) \); this is analogous to (3.31). We now check that our conditions on \( d \) allow us to choose \( \omega \) so that \( \omega \gg d^{2} \): recall from Definition 4.2 that \( \omega := \max\{\log k\}^{2}, k/p^{(d-1)/(2k)}\) \( d^{2} \ll \omega \) is included in Hypotheses B.
It remains to prove our claim that we can copy of the proof of Lemma 3.16 to prove that \( q_m \leq 1/p^{b-2} \). In said proof, we were particularly interested in the (expected) number of times that an individual generator was picked; this was \( t/k \), and, in the regime \( k \ll d \log p \), satisfied \( s \asymp p^{2(d-1)/k} \). At the start of Proof of Lemma 3.16 for \( k \ll d \log p \), we emphasised that the proof did not rely heavily on the distribution of \( W \), nor did it need \( s \asymp p^{2(d-1)/k} \); we apply the same arguments with \( W \) replaced by \( A \), and in this case the expected number of times that an individual generator is picked, which we still denote \( s \), satisfies \( s \asymp p^{d-1/k} \) since \( A \sim \text{Unif}(B_k(M_k)) \) and, by Lemma 4.4, \( M_k \asymp kp^{d-1/k} \). We elaborate further on how to adapt the proof to this context.

Let \( \eta \in (0, 1) \) be a (small) constant. For \( a \in \mathbb{Z}_+^k \), writing

\[
C(a) := \{ i \in [k] \mid \eta s \leq a_i \leq \eta^{-1} s \},
\]

we have \( \mathbb{P}(|C(A)|/k \geq \frac{2}{3}) = 1 - o(1) \), if \( \eta \) is sufficiently small; this is analogous to (3.36). We use this to define typicality, analogously to (3.37) except recalling that we no longer require the entropic part:

\[
A := \{ a \in \mathbb{Z}_+^k \mid |C(a)| \geq \frac{1}{3} k, \max_i a_i < p \}.
\]

We use exactly the same decomposition of generators; we look at \( m \)-tuples, and require \( m \ll k/(d^2) \). We take \( m \gg d \), and so need \( d^2 \ll k \); this latter condition is included in Hypotheses B. Fixing some \( a \in A \), consider the mode \( q_m \) of the vector \( C_m := (C_{i,j})_{i,j \in [m]} \), conditional on \( A = a \); here \( C_{i,j} \) is defined as in (3.8), but now with \( S \) defined using \( A \) instead of \( W \). Since \( \log s \asymp d \log p/k \), as it did in §3.5, we may apply Claim 3.18, provided its conditions hold: we need \( m \ll p^{(d-1)/k} \) and \( d \log m \lesssim \log n/k \). Since \( m \gg d \), we need \( d \ll p^{(d-1)/k} \) and \( d \log d \ll \log n/k \), ie \( \log d \ll d \log p/k \); Hypotheses B ensures this. We can now apply Lemma 3.16:

\[
q_b(t) \leq p^{-(b-2)}d^{-1}f(m)/m \quad \text{with} \quad f(m) \asymp m^2,
\]

exactly as in (3.38); setting \( m := Cd \) for a sufficiently large constant \( C \) gives \( q_b(t) \leq 1/p^{b-2} \).

5 Concluding Remarks and Open Questions

§5.1 We discuss some statistics in the regime where \( k \) is a fixed constant.

§5.2 We consider how our arguments extend from Heisenberg groups to other nilpotent groups.

§5.3 To conclude, we discuss some questions which remain open and gives some conjectures.

Throughout this section, we only sketch details.

5.1 Mixing, Relaxation and Diameter Bounds for Constant \( k \)

Throughout the paper we have always been assuming that \( k \to \infty \) as \( n \to \infty \). In this section, we derive bounds, asymptotic in \( n \), on the diameter and the mixing and relaxation times for fixed \( k \) (independent of \( n \)). In particular, we show that there is no cutoff in this case.

First, following Diaconis and Saloff-Coste [18], we define moderate growth for Cayley graph. Let \( G \) be a Cayley graph; let \( n := |G| \). Let \( V(R) := |B[R]| \) denote the volume of the ball of radius \( R \) in \( G \). Let \( \Delta := \inf \{ R \mid V(R) \geq n \} \) denote the diameter of \( G \). We say that \( G \) has \((c, a)\)-moderate growth if \( V(R) \geq cn/(R/\Delta)^a \). Breuillard and Tointon [14, Corollary 1.9] proved that if \( G \) is a Cayley graph of fixed degree, then this condition is equivalent in some quantitative sense to the simpler condition that \( n \leq \beta \Delta^a \) for some \( \alpha, \beta > 0 \). Diaconis and Saloff-Coste [18] proved, for Cayley graphs of \((c, a)\)-moderate growth, that

\[
t_{\text{mix}}/k \lesssim_{a,c} \Delta^2 \lesssim_{a,c} t_{\text{rel}} \lesssim_{a,c} t_{\text{mix}}
\]

with implicit constants depending on \( a \) and \( c \), where \( k \) is the degree; see also [14, Corollary 1.10]. (We can replace \( \lesssim_{a,c} \) with \( \lesssim_{a,\beta} \) if the condition of [14] is satisfied.) It follows that for constant \( k \) the product condition \( t_{\text{rel}} \ll t_{\text{mix}} \) fails, and so it is standard that there is no cutoff; see, eg, [37, Proposition 18.4].
Returning to the case $G := H_{p,d}$, it is not difficult to see that $\Delta \gtrsim kp^{(d-1)/k} = kn^{2/(dk)}$. Indeed,

$$|\{g[G,G] \mid \text{dist}(0,g) \leq r\}| \leq |\{x \in \mathbb{Z}_p^k \mid \text{dist}(0,x) \leq r\}| = \left(\frac{r+k}{k}\right)^k \leq (e(r/k + 1))^k;$$

until this latter size is $p^{-d-1} = |G/[G,G]|$, we have $r < \Delta$. Choosing $\alpha := \frac{1}{2}dk$ and $\beta$ sufficiently large, the condition of [14] is satisfied—more precisely, for all $\varepsilon > 0$ there exists a constant $\beta(\varepsilon)$ so that the condition of [14] is satisfied with probability at least $1 - \varepsilon$. Hence there is no cutoff.

The same argument holds for Abelian groups, except that now the Abelianisation is the full group, ie $G = G/[G,G]$; hence $\Delta \gtrsim kn^{1/k}$. Choosing $\alpha := k$ and $\beta$ sufficiently large, we see that the condition of [14] is satisfied (in the above sense). Hence there is no cutoff.

### 5.2 Extending Our Arguments from Heisenberg to Other Nilpotent Groups

In the introduction, in Remark A, we claimed that some of our analysis extends from Heisenberg groups to more general nilpotent groups. We elaborate on this claim here. Most of the following discussion is based on observations made by Péter Varjú during discussions of our work with him; further work on general nilpotent groups in collaboration with him is planned for the near future.

A group is **nilpotent of step at most $\ell$** if all iterated commutators of order at least $\ell + 1$ vanish necessarily. For example, step-1 is Abelian; step-2 has $[[g_1, g_2], g_3] = id$ for all $g_1, g_2$ and $g_3$, ie the commutator subgroup is central. Our analysis has focussed on Heisenberg matrix groups; these are a canonical class of nilpotent groups—$H_{p,d}$ is step-$(d-2)$ nilpotent. However, some of our analysis does extend somewhat to more general nilpotent, as we now explain.

Recall that we wrote $S$ for the location of the walk and $W$ for its auxiliary variable; let $W'$ be an independent copy of $W$, and define $S'$ correspondingly. As previously, we work in the directed regime; so in the word $S$ there are no inverses. Recall the definition of $C_{i,j}$ from (3.8).

**Lemma 5.1.** Up to multiplication by an element of $[G,[G,G]]$, we can express $S$ as

$$S = \left(\prod_{i} Z_i^{W_i}\right) \cdot \left(\prod_{i<j} [Z_i^{-1}, Z_j^{-1}]^{-C_{i,j}}\right)$$

If $G$ is step-2 nilpotent then $[G,[G,G]] = \{id\}$ is the trivial group.

(The second product is unordered, since we are working up to an element of $[G,[G,G]]$, and so we may assume that commutators commute with any element of $G$; the first is ordered $i = 1, \ldots, k$.)

**Sketch of Proof.** Writing a rigorous proof of this lemma is technical, and can obscure what is going on; we use an example to demonstrate how to prove the lemma. In essence, we wish to move all the $Z_1$-s to the left, then all the $Z_2$-s to the left-but-one and so on. To reverse the order terms, we use the fact that $hg = gh^{-1}g^{-1}gh = gh[h^{-1}, g^{-1}]$ and $[h^{-1}, g^{-1}] = g^{-1}, h^{-1}]^{-1}$. For example,

$$ghhg = gh \cdot gh[g^{-1}, h^{-1}]^{-1} = g \cdot gh[g^{-1}, h^{-1}]^{-1} \cdot h[g^{-1}, h^{-1}]^{-1} = g^2 h^2 [g^{-1}, h^{-1}]^{-2}.$$  

To move $Z_i$ past $Z_j$, with $i < j$, for each occurrence of $Z_i$ we need to count the number of times that $Z_j$ appears before it in the word; this is precisely (the definition of) $C_{i,j}$.

Expressing $S^{-1}S'$ as a similar product, it is straightforward to see what we get when $W = W'$. (We actually only need $W_i \equiv W'_i \mod Z_i$ for each $i$, but $W = W'$ is generally easier to analyse.)

**Corollary 5.2.** If $W = W'$, then $S^{-1}S' \in [G,G]/[G,[G,G]]$. The converse holds in the free group, ie when considering $Z_1, \ldots, Z_k$ as formal variables (ie with no relations between them).

If $W = W'$, then, up to multiplication by an element of $[G,[G,G]]$, we can express $S^{-1}S'$ as

$$S^{-1}S' = \prod_{i<j} [Z_i^{-1}, Z_j^{-1}]^{D_{i,j}}$$

where $D_{i,j} := C_{i,j} - C_{i,j}';$ write $D := (D_{i,j})_{i,j}$. In particular, if $C_{i,j} = C_{i,j}'$ for all $i$ and $j$ (which implies that $W_i = W'_i$ for all $i$ by taking $i = j$), then $S^{-1}S' \in [G,[G,G]]; if the group is step-2 nilpotent, then $[G,[G,G]] = \{id\}$, and hence $S = S'$.  

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Consider now step-2 nilpotent groups, of which $H_{p,3}$ is an example. We are interested in analysing $\Pr(S = S' \mid \typ)$; typicality will primarily involve entropic considerations. For ease of presentation, here we drop $\typ$ from the notation. As in §3, we separate this probability as

$$\Pr(S = S') \leq \Pr(S = S' \mid W = W') \Pr(W = W') + \Pr(S = S' \mid W \neq W').$$

Typicality (entropy) will bound $\Pr(W = W') \ll 1/|G/[G,G]|$, as for Heisenberg groups.

Assume that $t \ll k$, and that every generator is picked at most once—eg, this is the case if $k \gg \log n$. The assumption means that some generator is picked once in $S$ and never in $S'$ (or vice versa); this will allow us to deduce that $S^{-1}S' \sim \text{Unif}(G)$, and hence $\Pr(S = S' \mid W \neq W') = 1/n$.

Since $S = S'$ when $D_{i,j} = 0$ for all $i$ and $j$, we have

$$\Pr(S = S' \mid W = W') \leq \Pr(S = S' \mid W = W', D \neq 0) + \Pr(D = 0).$$

When the nilpotent group is of higher step, the bound $\Pr(D = 0) \leq 1$ may be too crude.

We analysed $\Pr(D = 0)$ in §3, obtaining $\Pr(D = 0) \approx 1/t!$. We desire this to be close to $1/|[G,G]|$.

We wish to get $\Pr(S = S' \mid W = W', D \neq 0)$ close to $1/|[G,G]|$. To do this, write

$$S^{-1}S' = \prod_{i<j, D_{i,j} \neq 0} [Z_i^{-1}, Z_j^{-1}]^{D_{i,j}}.$$

While these commutators are neither uniformly random nor independent, we aim to have suitably many $D_{i,j} \neq 0$ so that the commutator product is sufficiently close to uniform (on $[G,G]$).

If “close” can mean “up to a sufficiently small factor”, then combining all these bounds gives $\Pr(S = S', W = W') \ll 1/n$. The modified $L_2$ distance is then given by $n \Pr(S = S') - 1 = o(1)$.

If one desires a regime of $k$ in which $t \gg k$, then another option, instead of restricting to step-2, is to study $p$-groups (which are nilpotent): we studied these in [33, §3]. Replace “=” with “$\equiv \mod p$”.

In [33, §3.5], we proved an upper bound on mixing for arbitrary $p$-groups, around an entropic time for simple random walk on $\mathbb{Z}_p^k$. Key was showing that $S^{-1}S' \sim \text{Unif}(G)$ when $W \neq W'$, even with $t \gg k$. This used conjugation, rather than commutators; see the justification for [33, (3.12)]—there $\{I \neq 0\} = \{W \neq W'\}$. Also, we naively upper bounded $\Pr(S = S' \mid W \equiv W') \leq 1$; see [33, (3.10)]. Instead, we can use the method outlined above to obtain a much better bound.

We can apply the method for nilpotent groups of greater step, by quotienting out $[G,G]$.

However, as the step increases the bounds become more crude: we could have $\Pr(S = S') \ll \Pr(S^{-1}S' \in [G,[G,G]])$, which would be bad for this method; this is in essence what we did for $H_{p,4}$. The analysis also applies to non-nilpotent groups, for which such issues can be even worse.

### 5.3 Open Questions and Conjectures

Below $\text{PP}$ stands for Poisson process and $\text{SRW}$ for simple random walk.

#### 1: Sufficient Conditions for Cutoff for General Groups

Cutoff (whp) has been established at an explicit time for all Abelian groups in the regime $k \gg \log |G|$; this time is $t_0(k, |G|)$. Let $G$ be a group and write $A = G/[G,G]$ for its Abelianisation. Recall from Remark 3.6 that the lower bound of $\max\{t_0(k, |A|), t_{\text{diam}}\}$ is valid for any group.

For nilpotent groups it is known that generating the Abelianisation is equivalent to generating the whole group; see Remark B. So for nilpotent groups, one can extend from considering $k \gg \log |G|$ to any $k \gg \log |A|$, or even $k - \log |A| \gg 1$.

**Question 1.** For all groups $G$, is there cutoff whp at $\max\{t_0(k, |A|), t_{\text{diam}}\}$ in the regime $k \gg \log |G|$? If not, are the relatively mild conditions on $G$ so that the conclusion holds?

For nilpotent groups, does the same claim hold in the larger regime $k \gg \log |A|$?

Do all (sequences of) nilpotent groups exhibit cutoff whp whenever $k - \log |A| \gg 1$? (The time no longer needs be the one above, but can depend more intricately on the group.)
In our companion article [32, Theorem A], we show, in the regime $1 \ll k \lesssim \log n$, that cutoff was universal for Abelian groups satisfying some mild conditions on their algebraic structure, and that the cutoff time was given by the entropic time $t_0(k, |G|)$.

Given that we use the decomposition of $G$ into $G/\langle G, G \rangle$ and $\langle G, G \rangle$ above, perhaps a more tractable starting step is to look at only nilpotent groups with Abelianisation $A$ of ‘significant size’ and a regime of $k$ so that $t_0(k, |A|) \gg t_{\text{diam}}$. One then desires to show that there is cutoff at $t_0(k, |A|)$: this entropic time is precisely the time needed for the random walk on the projection to the Abelianisation to mix (provided $A$ is ‘sufficiently nice’); one then needs to show that ‘the non-Abelian part mixes faster than the Abelian part’.

We also mention work by Gowers [30], on quasirandom groups. He looks for groups whose (non-trivial) irreducible representations all have high dimension; such groups he describes as “very far from being Abelian”. Perhaps a similar criterion would be useful for this question.

2: Cutoff for Heisenberg Group $H_{p,d}$ on an Undirected Graph

Throughout this paper, we always considered directed Cayley graphs. The primary reason for this is that then the value $W_i(t)$ gives the number of times that generator $i$ has been used.

**Conjecture 2.** The behaviour is the same for the undirected graph, at least for the cases which we have studied above, even down to having the same description of the mixing times in terms of the entropic time—now, however, entropic time is taken not with respect to a PP on $\mathbb{Z}_p^k$, but a SRW on $\mathbb{Z}^k$.

In our companion article [32], we consider both directed and undirected Cayley graphs, as noted in §1.3: the PP $W$ on $\mathbb{Z}_p^k$ is replaced by a SRW on $\mathbb{Z}^k$; for $W_i$, a step +1 corresponds to choosing generator $i$ and applying it, while a step −1 corresponds to choosing generator $i$ and applying its inverse. Hence, in the undirected case, we may have $W_i(t) = 0$, yet generator 1 chosen an arbitrary (even) number of times.

The fact that $W_i(t)$ counted the number of times generator $i$ had been used was very helpful: since the group is non-Abelian, applying a generator at some point and its inverse at some other point do not cancel out. One should then introduce another variable which counts the number of times each generator is used.

The entropic time $t_0(k, N)$ for the SRW is the same, up to a $1 \pm o(1)$ factor, as for the PP, except when $k \asymp \log N$ where they differ by an additional constant factor; see [32, Proposition 2.2].

3: Cutoff for Heisenberg Group $H_{p,d}$ with $p$ Small (eg $p = 2$)

When $k \gtrsim d \log p$, we required $d = (\log p)^o(1)$. This appears to be an artefact of the proof—see below. It is natural to expect there to be cutoff even for larger $d$ (ie smaller $p$). The auxiliary variable $W$ should now be a PP on $\mathbb{Z}_p^k$ (ie taken mod $p$), rather than $\mathbb{Z}_p^k$; see below, and cf [33].

Write $\sigma_{\zeta,p}$ for the relative entropy mixing time (defined analogously to the total variation mixing time) with level $\zeta \in (0, \infty)$ for the SRW on $\mathbb{Z}_p$. Set

$$\zeta := \frac{1}{2} \left(1 - \frac{d}{k}\right) \log(p^{d-1}) = (1 - \frac{1}{2})(1 - \frac{d}{k}) \log p = 2d^{-2}(1 - \frac{d}{k})^2 \log n.$$

**Conjecture 3.** Consider $H_{p,d}$ with $p$ prime, $d \geq 3$ and $|H_{p,d}| = p^{d(d-1)/2} \gg 1$. There is cutoff at $k\sigma_{\zeta,p}$ whp over $Z$, under some (relatively mild) condition that $k$ is large enough.

One place in which we loose information is in moving from $3 \times 3$ matrices to $d \times d$ matrices—compare (3.4) with (3.23): in both cases, we used the ‘Abelian’ terms (ie monomials of size 1) and the first ‘non-Abelian’ terms (ie monomials of size 2); for general $d$, there are $d - 1$ terms—coordinates at distance $\ell$ from the diagonal contain monomials of size at most $\ell$. (This corresponds to the fact that $H_{p,3}$ is step-2 nilpotent, while in general $H_{p,d}$ is step-$(d - 1)$ nilpotent; the further from the diagonal a coordinate is, the ‘more non-Abelian’ it is.) Considering only the first two terms meant that the analysis was more analogous to the $3 \times 3$ case, however we threw away a lot of information. In order to study the case where $d$ is very large (compared with $p$, eg $p = 2$ and $d \approx (2 \log_2 n)^{1/2}$), one surely needs to analyse these higher order terms. Cf end of §5.2.
In another companion article [33, Theorem B], we study in detail cutoff for the Abelian group $\mathbb{Z}_p^d$, in particular allowing (prime) $p$ to be fixed; this extends Wilson’s consideration of $\mathbb{Z}_2^d$ (ie $p = 2$). One key difference is instead of letting $W$ be a PP on $\mathbb{Z}_p^d$, we take each coordinate modulo $p$; this leads from entropy to relative entropy considerations. In the current article, $d$ has been small enough so that almost all coordinates of $W$ never reach $p$, and hence the distinction between a PP on $\mathbb{Z}_p^d$ and $\mathbb{Z}_2^d$ is negligible (cf [32] vs [33]); this will not be the case for small $p$.

4: Cutoff for Heisenberg Group $H_{p,d}$ with $p$ Not Prime

Our analysis required that $p$ be prime. This restriction can potentially be removed, however.

**Conjecture 4.** Subject to potentially stronger conditions on $d$, analogous results hold when $p$ is not prime with the same cutoff time.

Perhaps the main obstacle is related to Lemma 3.9, which states that $\sum_{i=1}^{d} a_i X_i \sim \text{id} \text{Unif}[p]$ when $X_i \sim \text{Unif}[p]$ and $a_i \in \{1, \ldots, p\}$ for all $i$. This requires that $p$ be prime, as then $\mathbb{Z}_p \setminus \{0\}$ is a group under multiplication, and so $a_i X_i \sim \text{Unif}[p]$. Otherwise, as noted in the body of the paper, one has $\sum_{i=1}^{d} a_i X_i \sim \text{Unif}\{g, 2g, \ldots, p\}$ where $g := \gcd(a_1, \ldots, a_d, p)$; for a proof of this, see [32, Lemma 3.11]. In particular, the probability that this equals $p$ is then $g/p$, rather than $1/p$. This has knock-on effects, eg to the analysis for (3.10–3.13).

When we studied Abelian groups in our companion article [32], we did not assume that the analogue of $p$ was prime; we did the gcd analysis. It is not unreasonable to imagine that similar techniques applied there—see [32, §3.4]—may well be applicable here too.

5: Spectral Gap for Heisenberg Group $H_{p,d}$ with $\frac{2}{d} \log n \ll k \ll \log n$

We studied typical distance for $k \leq \log |A_{p,d}|$, where $A_{p,d}$ is the Abelianisation and $|A_{p,d}| = p^{d-1} = n^{2/d}$. We show that when $k \asymp \log |A_{p,d}|$ the typical distance is order $k$. Analogously, we study typical distance for Abelian groups in [32, Theorem B], obtaining analogous results—for an Abelian group $G$, its Abelianisation is itself. The regime $k \asymp \log |G|$ is the point at which the Cayley graph of an Abelian group becomes an expander; see [33, Theorem D]. It is natural to conjecture that the analogue holds for Heisenberg groups.

**Conjecture 5.** For $k \gtrsim \log |A_{p,d}| = (d-1) \log p$ and $Z_1, \ldots, Z_k \sim \text{iid Unif}(H_{p,d})$, the (directed or undirected) Cayley graph with generators $Z$ is an expander whp.

This would provide the first example of a group with the property that its Cayley graph with $k$ random generators is an expander whp for some choice of $k$ satisfying $k \ll \log n$.

6: Diameter for Heisenberg Group $H_{p,d}$ for Diverging $k$

We have shown concentration of typical distance, but never considered the diameter. It is trivial that the typical distance is a lower bound on the diameter, and that twice the typical distance is an upper bound. Can more be determined?

**Conjecture 6.** For $G = H_{p,d}$ and $Z_1, \ldots, Z_k \sim \text{iid Unif}(H_{p,d})$, write $\Delta_Z$ for the diameter of the Cayley graph with generators $Z$. Assume that $k$ diverges, sufficiently rapidly in terms of $d$. Does the law of $\Delta_Z$ concentrate?

7: Diameter for Heisenberg Group $H_{p,d}$ for Fixed $k$

Instead of requiring $k \gg 1$ (diverging sufficiently rapidly in terms of $d$), as in the previous question, for fixed $d$ we can ask that $k$ is (at least) a suitable large constant, depending on $d$. Conjecture 6 along with §5.1 suggests that the correct order should be $kp^{(d-1)/k}$. Shapira and Zuck [47] establish convergence in distribution of the normalised diameter for Abelian groups. (Cf Amir and Gurel-Gurevich [3].) Does the same hold for Heisenberg groups?
Conjecture 7. For \( G = H_{p,d} \) and \( Z_1, \ldots, Z_k \sim \text{Unif}(H_{p,d}) \), write \( \Delta_Z \) for the diameter of the Cayley graph with generators \( Z \). Assume that \( d \) and \( k \) are fixed. Under suitable conditions on \( k \) in terms of \( d \) (which do not require \( k \) to diverge), \( \Delta_Z/(kp^{(d-1)/k}) \) converges in distribution to some non-trivial random variable as \( p \to \infty \).

Replacing the Heisenberg Group with a Nilpotent Group

Questions 2–7 for the Heisenberg group can all be extended by replacing \( G = H_{p,d} \) with a general nilpotent group \( G \); one could consider only step-2 nilpotent groups as a first step.

Questions for Typical Distance

Questions for typical distance can be asked analogous to those detailed in Questions 2–4.

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