MEAN CURVATURE FLOW WITH SURGERY OF MEAN CONVEX SURFACES IN $\mathbb{R}^3$

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Abstract. We define a notion of mean curvature flow with surgery for two-dimensional surfaces in $\mathbb{R}^3$ with positive mean curvature. Our construction relies on the earlier work of Huisken and Sinestrari in the higher dimensional case. We also use a new estimate for the inscribed radius under mean curvature flow.

1. Introduction

In [16], Huisken and Sinestrari defined a notion of mean curvature flow with surgery for two-convex hypersurfaces in $\mathbb{R}^{n+1}$, where $n \geq 3$. Our goal in this paper is to extend the results in [16] to the case $n = 2$:

Theorem 1.1. Let $M_0$ be a closed, embedded surface in $\mathbb{R}^3$ with positive mean curvature. Then there exists a mean curvature flow with surgeries starting from $M_0$ which terminates after finitely many steps.

The formation of singularities in geometric flows is a central problem in geometric analysis. In particular, there are related surgery constructions for the Ricci flow due to Hamilton [8], [9] and Perelman [17], [18], [19]. Furthermore, Brian White has obtained several breakthroughs in the analysis of the singularities of mean curvature flow; see [21], [22], [23], [24], [25], [26]. We note that a different approach in the two-dimensional case was suggested by Colding and Kleiner in [4]. Moreover, Wang [20] has obtained a classification of translating solutions to the mean curvature flow in dimension 2. These solutions arise as models for Type II singularities.

Our argument broadly follows the one in [16]. However, there are several major differences. One important difference is that the cylindrical estimate in Section 5 of [16] fails for $n = 2$. To replace the cylindrical estimate, we use an estimate for the inscribed radius established in [3]. Given an oriented surface $M$ and a point $p \in M$, the inscribed radius at $p$ is defined as the radius of the largest open ball in $\mathbb{R}^3$ which is disjoint from $M$ and touches $M$ at $p$ from the inside. Similarly, the outer radius at $p$ is defined as the radius of the largest open ball in $\mathbb{R}^3$ which is disjoint from $M$ and touches $M$ at $p$ from the outside. A mean convex surface $M$ will be called $\alpha$-noncollapsed if the inscribed radius at each point $p \in M$ is bounded from below by $\frac{\alpha}{H}$, where $H$ denotes the mean curvature at the point $p$.

The main theorem in [3] asserts that, for any smooth solution of the mean curvature flow with positive mean curvature, we have a pointwise estimate

$$
\frac{\partial}{\partial t} \frac{1}{\text{dist}(x, \gamma)} \leq C \frac{1}{\text{dist}(x, \gamma)^2} + \frac{1}{\text{dist}(x, \gamma)^3}.
$$
of the form $\mu \leq (1 + \delta) H + C(\delta)$. Here, $\mu$ denotes the reciprocal of the inscribed radius, $\delta$ is a given positive number, and $C(\delta)$ is a constant that depends on $\delta$ and the initial data. We show that this estimate still holds in the presence of surgeries, at least for a suitable choice of surgery parameters. This is a subtle issue, as the ratio $\frac{\mu}{H}$ might deteriorate slightly under surgery. To overcome this obstacle, we show that the ratio $\frac{\mu}{H}$ improves immediately prior to surgery. By choosing the surgery parameters in a suitable way, we can ensure that this improvement in the noncollapsing constant prior to surgery is strong enough to absorb the error terms that arise during each surgery procedure.

Another problem is that the proof of the gradient estimate in Section 6 of [16] does not directly carry over to the case $n = 2$. To get around this issue, we combine the interior gradient estimate established in [10] with a new pseudolocality principle for the mean curvature flow (cf. Theorem 2.2 below).

In Section 2 we collect a number of auxiliary results. These results will be used in Section 3 to establish an analogue of the crucial Neck Continuation Theorem in [16]. Finally, in Sections 4–14 we give the proofs of the auxiliary results stated in Section 2.

It is a pleasure to thank Brian White for discussions concerning the pseudolocality property for the mean curvature flow.

2. Overview of some auxiliary results

We first establish a pseudolocality principle for the mean curvature flow. We begin with a definition.

**Definition 2.1.** Consider a ball $B$ in $\mathbb{R}^3$ and a one-parameter family of smooth surfaces $M_t \subset B$ such that $\partial M_t \subset \partial B$. Moreover, suppose that each surface $M_t$ bounds a domain $\Omega_t \subset B$. We say that the surfaces $M_t$ form a regular mean curvature flow if the surfaces $M_t$ form a smooth solution to mean curvature flow, except at finitely many times where one or more connected components of $\Omega_t$ may be removed.

**Theorem 2.2** (Pseudolocality). There exist positive constants $\beta_0$ and $C$ such that the following holds. Suppose that $M_t$, $t \in [0,T]$, is a regular mean curvature flow in $B_4(0)$ in the sense of Definition 2.1. Moreover, we assume that the initial surface $M_0$ can be expressed as the graph of a (single-valued) function $u$ over a plane. If $\|u\|_{C^4} \leq \beta_0$, then

$$|A| + |\nabla A| + |\nabla^2 A| \leq C$$

for all $t \in [0,\beta_0] \cap [0,T]$ and all $x \in M_t \cap B_1(0)$.

We will also need a variant of Theorem 1.8' in [10]:

**Theorem 2.3** (cf. Haslhofer-Kleiner [10], Theorem 1.8'). Given any $\alpha \in (0, \frac{1}{1000})$, there exists a constant $C(\alpha)$ with the following property. Suppose that $M_t$, $t \in [-1,0]$, is a regular mean curvature flow in the ball $B_4(0)$. 
Moreover, suppose that each surface $M_t$ is outward-minimizing within the ball $B_4(0)$. We further assume that the inscribed radius and the outer radius are at least $\frac{\alpha}{H}$ at each point on $M_t$. Finally, we assume that $M_0$ passes through the origin, and $H(0,0) \leq 1$. Then the surface $M_0$ satisfies $|\nabla A| \leq C(\alpha)$ and $|\nabla^2 A| \leq C(\alpha)$ at the origin.

The difference between Theorem 2.3 and the setting in [10] is that we allow some components of the surface to suddenly disappear. It turns out that this does not affect the argument in [10], as long as the surfaces are outward-minimizing. Therefore, the proof in [10] goes through in our setting.

In the following, we will fix an initial manifold $M_0$. Moreover, let us fix a constant $\alpha \in (0, \frac{1}{1000}]$ such that the inscribed radius and the outer radius of the initial surface $M_0$ are at least $\alpha H$.

We next describe the necks on which we will perform surgery.

**Definition 2.4.** Let $M$ be a mean convex surface in $\mathbb{R}^3$, and let $N$ be a region in $M$. As usual, we denote by $\nu$ the outward pointing unit normal vector field. We say that $N$ is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck of size $r$ if (in a suitable coordinate system in $\mathbb{R}^3$) the following holds:

- There exists a simple closed, convex curve $\Gamma \subset \mathbb{R}^2$ with the property that $\text{dist}_{C_20}(r^{-1} N, \Gamma \times [-L, L]) \leq \varepsilon$.
- At each point on $\Gamma$, the inscribed radius is at least $\frac{1}{1+\hat{\delta}} \kappa$, where $\kappa$ denotes the geodesic curvature of $\Gamma$.
- We have $\sum_{i=1}^{18} |\nabla^i \kappa| \leq \frac{1}{100}$ at each point on $\Gamma$.
- There exists a point on $\Gamma$ where the geodesic curvature $\kappa$ is equal to 1.
- The region $\{x + a \nu(x) : x \in N, a \in (0, 2\hat{\alpha} r)\}$ is disjoint from $M$.

The last assumption is needed to ensure that, immediately after performing surgery, the resulting surface has outer radius at least $\frac{\alpha}{H}$ everywhere (cf. Proposition 5.5). It turns out that the necks obtained via the Neck Detection Lemma satisfy this condition; see Theorem 2.14 below.

Given an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck, we can perform surgery on $N$. The procedure depends on a parameter $\Lambda$. The exact choice of $\Lambda$ will be specified later.

**Theorem 2.5** (Properties of Surgery). Given any number $\hat{\alpha} > \alpha$, there exists a real number $\delta_0$ with the following significance. Suppose that we are given a pair of real numbers $\delta$ and $\hat{\delta}$ such that $\hat{\delta} < \delta < \delta_0$. Then we can find numbers $\hat{\varepsilon}$ and $\Lambda$, depending only on $\delta$ and $\hat{\delta}$, such that the following holds. Suppose that $N$ is an $(\alpha, \delta, \varepsilon, L)$-neck of size $r$ sitting in a mean convex surface in $\mathbb{R}^3$, where $\varepsilon \leq \hat{\varepsilon}$ and $\frac{1}{1000} \geq \Lambda$. If we perform a $\Lambda$-surgery on $N$, then the resulting surface $\tilde{N}$ will be $\frac{\alpha}{H}$-noncollapsed. Furthermore, the outer radius is at least $\frac{\alpha}{H}$ at each point on $\tilde{N}$. Finally, if $\tilde{p} \in \tilde{N} \setminus N$ is a point in the surgically modified region, then either $\lambda_1(\tilde{p}) \geq 0$, or else there exists a point $p \in N$ such that $\lambda_1(\tilde{p}) \geq \lambda_1(p)$ and $H(\tilde{p}) \geq H(p)$. 


A key point is that the deterioration in the noncollapsing constant can be made arbitrarily small by choosing $\varepsilon$ small and $\Lambda$ large.

**Assumption 2.6.** In the following, we will assume that $M_t$ is a solution of the mean curvature flow with surgery. We will assume that this flow satisfies the following assumptions:

- The flow $M_t$ is smooth for $t \in [0, (100 \sup_{M_0} |A|)^{-2}]$.
- Each surgery procedure involves performing an $\Lambda$-surgery on an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck of size $r \in \left[ \frac{1}{2K_*}, \frac{2}{K_*} \right]$, where $\hat{\alpha} > \alpha$, $\hat{\delta} \leq \frac{1}{10}$, $\frac{L}{1000} \geq \Lambda$, and $K_* \geq (\sup_{M_0} |A|)^2$.
- The region enclosed by $M_t$ shrinks under surgery.
- For each $t$, the surface $M_t$ is outward-minimizing within the region enclosed by $M_0$.
- For each $t$, the inscribed radius and the outer radius of $M_t$ are at least $\alpha H$.

The exact values of the parameters $\Lambda, \hat{\alpha}, \hat{\delta}, \varepsilon, L,$ and $K_*$ will be specified later.

In the first step, we want to apply the Pseudolocality Theorem to obtain derivative bounds shortly after a surgery. We begin by showing that surgeries are separated in space:

**Proposition 2.7 (Separation of Surgery Regions).** Let $M_t$ be a mean curvature flow with surgery as above. Suppose that $t_0$ is a surgery time, and $x_0 \in M_{t_0+}$ lies in the surgically modified region at time $t_0+$. Moreover, suppose that $t_1 > t_0$ and $x_1$ is a point on $M_{t_1+}$ satisfying $|x_1 - x_0| \leq \frac{\alpha}{1000} K_*^{-1}$. Then $H(x_1, t_1+) \geq \frac{\alpha}{10} K_*$. Moreover, if $t_1$ is a surgery time, then $x_1$ does not lie in the surgically modified region at time $t_1$.

Thus, if $t_0$ is a surgery time and $x_0$ is a point in the surgically modified region at time $t_0+$, then the flow $M_t \cap B_{\frac{\alpha}{1000} K_*^{-1}}(x_0)$, $t > t_0$, is a regular flow in the sense of Definition 2.1. Using the Pseudolocality Theorem, we can draw the following conclusion:

**Proposition 2.8.** There exist positive constants $\beta_* \in (0, \frac{\alpha}{1000})$ and $C_*$ with the following property. Let $M_t$ be a mean curvature flow with surgery as above. Suppose that $t_0$ is a surgery time and $x_0$ is a point in the surgically modified region at time $t_0+$. Then we have

$$K_*^{-1} |A| + K_*^{-2} |\nabla A| + K_*^{-3} |\nabla^2 A| \leq C_*$$

for all times $t \in (t_0, t_0 + \beta_* K_*^{-2})$ and all points $x \in M_t \cap B_{\beta_* K_*^{-1}}(x_0)$. The constants $\beta_*$ and $C_*$ may depend on the noncollapsing constant $\alpha$, but they do not depend on the surgery parameters $\hat{\alpha}, \hat{\delta}, \varepsilon, L,$ and $K_*$. The exact values of the surgery parameters will depend on the value of the constant in the derivative estimate, which in turn depends on $\beta_*$ and
$C_\ast$. It is therefore critically important that the constants $\beta_\ast$ and $C_\ast$ do not depend on the exact choice of the surgery parameters $\hat{\alpha}$, $\hat{\delta}$, $\varepsilon$, $L$, and $K_\ast$.

Combining Proposition 2.8 with the interior gradient estimate in \[10\], we obtain pointwise bounds for the first and second derivatives of the second fundamental form.

**Proposition 2.9 (Pointwise Derivative Estimate).** There exists a constant $C_\ast$ with the following significance. Suppose that $M_t$ is a mean curvature flow with surgery as above. Then $|\nabla A| \leq C_\ast (H + K_\ast)^2$ and $|\nabla^2 A| \leq C_\ast (H + K_\ast)^3$ for all times $t \geq (1000 \sup_{M_0} |A|)^{-2}$ and all points $x \in M_t$. The constant $C_\ast$ may depend on the initial noncollapsing constant $\alpha$, but is independent of the surgery parameters $\hat{\alpha}$, $\hat{\delta}$, $\varepsilon$, $L$, and $K_\ast$.

Having fixed the constant $C_\ast$ in the derivative estimate, we next define $\Theta = \frac{400}{\alpha}$, $\theta_0 = 10^{-6} \min\{\alpha, \frac{1}{\varepsilon} \}$, and $\hat{\alpha} = \frac{\alpha}{1 - \frac{\Theta}{\Theta}}$. Hence, if we start at a point $(p_0, t_0)$ with $H(p_0, t_0) \geq \frac{K_\ast}{\hat{\alpha}}$ and follow this point back in time, then the mean curvature at the resulting point will be between $\frac{1}{2} H(p_0, t_0)$ and $2 H(p_0, t_0)$, provided that $t \in (t_0 - 2\theta_0 H(p_0, t_0)^{-2}, t_0]$.

We next explain our choice of $\delta$ and $\hat{\delta}$.

**Proposition 2.10.** We can find a real number $\delta > 0$ such that the following holds:

- Suppose that $\Gamma$ is a (possibly non-closed) embedded curve in the plane with the property that $\kappa > 0$, $\left| \frac{d\kappa}{ds} \right| \leq C_\ast (\kappa + 2\Theta)^2$, and $\left| \frac{d^2\kappa}{ds^2} \right| \leq C_\ast (\kappa + 2\Theta)^3$. Moreover, suppose that the inscribed radius is at least $\frac{1}{(1 + \delta)\kappa}$ at each point on $\Gamma$, and the outer radius is at least $\frac{2}{\kappa}$ at each point on $\Gamma$. Finally, we assume that $\kappa(p) = 1$ for some point $p \in \Gamma$. Then $L(\Gamma) \leq 3\pi$ and $\sup_{\Gamma} |\kappa - 1| \leq \frac{1}{100}$.
- Suppose that $\Gamma_t$, $t \in (-2\theta_0, 0]$, is a family of simple closed, convex curves in the plane which evolve by curve shortening flow. Assume that, for each $t \in (-2\theta_0, 0]$, the curve $\Gamma_t$ satisfies the derivative estimates $\left| \frac{d\kappa}{ds} \right| \leq C_\ast (\kappa + 2\Theta)^2$ and $\left| \frac{d^2\kappa}{ds^2} \right| \leq C_\ast (\kappa + 2\Theta)^3$. Moreover, we assume that the inscribed radius is at least $\frac{1}{(1 + \delta)\kappa}$ at each point on $\Gamma_t$, and the outer radius is at least $\frac{2}{\kappa}$ at each point on $\Gamma_t$. Finally, we assume that the geodesic curvature of $\Gamma_0$ is equal to 1 somewhere. Then the curve $\Gamma_0$ satisfies $\sum_{i=1}^{18} |\nabla^i \kappa| \leq \frac{1}{100}$. Moreover, we have $\sup_{\Gamma_0 - \delta_0} \kappa \leq 1 - \frac{\delta_0}{4}$.

We assume that $\delta$ is chosen sufficiently small so that $\delta < \delta_0$, where $\delta_0$ is the constant in Theorem 2.5. In the next step, we choose $\hat{\delta}$ such that the following holds:

**Proposition 2.11.** Given $\theta_0 > 0$ and $\delta > 0$, we can find a real number $\hat{\delta} \in (0, \delta)$ with the following property: Consider a simple closed, convex solution $\Gamma_t$, $t \in (-2\theta_0, 0]$, of the curve shortening flow in the plane which satisfies
the derivative estimates $|\frac{d^2\kappa}{ds^2}| \leq C_\#(\kappa + 2\Theta)^2$ and $|\frac{d\kappa}{ds}| \leq C_\#(\kappa + 2\Theta)^3$. Moreover, we assume that the inscribed radius is at least $\frac{1}{1+\delta} \kappa$ at each point on $\Gamma_t$, and the outer radius is at least $\kappa$ at each point on $\Gamma_t$. Finally, we assume that the geodesic curvature of $\Gamma_0$ is equal to 1 somewhere. Then $\Gamma_0$ is $\frac{1}{1+\delta}$-noncollapsed.

Finally, we choose $\bar{\varepsilon}$ and $\Lambda$ such that the conclusion of Theorem 2.5 holds. We next observe that the convexity estimates of Huisken and Sinestrari (cf. [14], [15]) still hold for mean curvature flow with surgery.

Proposition 2.12 (Huisken-Sinestrari [16], Section 4). The Huisken-Sinestrari convexity estimate holds for mean curvature flow with surgery. More precisely, given any $\eta > 0$, there exists a constant $K(\eta)$ such that $\lambda_1 \geq -\eta H - C_1(\eta)$. The constant $C_1(\eta)$ depends only on $\eta$ and the initial data, but is independent of the surgery parameters $\hat{\alpha}, \hat{\delta}, \bar{\varepsilon}, L, \text{ and } K_\star$.

Theorem 2.5 implies that performing $\Lambda$-surgery on an $(\hat{\alpha}, \hat{\delta}, \bar{\varepsilon}, L)$-neck will produce a surface which is $\frac{1}{1+\delta}$-noncollapsed, provided that $\bar{\varepsilon} \leq \bar{\varepsilon}$ and $L \geq 1000 \Lambda$.

Proposition 2.13 (Cylindrical Estimate). Let $\delta$ and $\hat{\delta}$ be chosen as above. Moreover, suppose that $\bar{\varepsilon}$ and $\Lambda$ are chosen such that the conclusion of Theorem 2.5 holds. Then, if $\mu$ denotes the reciprocal of the inscribed radius, then we have the pointwise estimate $\mu \leq (1 + \delta)H + CH^{1-\sigma}$. Here, $\sigma$ and $C$ may depend on $\delta$ and $\hat{\delta}$, but they are independent of the exact choice of $\varepsilon$ and $L$.

Using the convexity estimate and the cylindrical estimate, we are able to prove an analogue of the Neck Detection Lemma in [16]. In fact, we will need two different versions.

Theorem 2.14 (Neck Detection Lemma, Version A). Let $\delta$ and $\hat{\delta}$ be chosen as above, and let $\bar{\varepsilon}$ and $\Lambda$ be chosen so that the conclusion of Theorem 2.5 holds. Consider a solution of the mean curvature flow with surgery as above. Then, given $\varepsilon_0 > 0$ and $L_0 \geq 100$, we can find $\eta_0 > 0$ and $K_0 > \frac{(1000 \sup_{M_0} |A|)^2}{\inf_{M_0} H}$ with the following significance: Suppose that $t_0$ and $p_0 \in M_{t_0}$ satisfy

- $H(p_0, t_0) \geq \max \{K_0, \frac{K_\star}{\varepsilon_0} \}$, $\frac{\lambda_1(p_0, t_0)}{H(p_0, t_0)} \leq \eta_0$,
- the neighborhood $\hat{P}(p_0, t_0, L_0 + 4.2\theta_0)$ does not contain surgeries.

Then $(p_0, t_0)$ lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)$-neck of size $H(p_0, t_0)^{-1}$. Finally, the constants $\eta_0$ and $K_0$ may depend on $\varepsilon_0, L_0, \delta, \hat{\delta}$, and the initial data, but they are independent of the remaining surgery parameters $\varepsilon, L, \text{ and } K_\star$.

Theorem 2.15 (Neck Detection Lemma, Version B). Let $\delta$ and $\hat{\delta}$ be chosen as above, and let $\bar{\varepsilon}$ and $\Lambda$ be chosen so that the conclusion of Theorem
Let us dilate the surface \( \{ x \in M_{t_0} : d_g(t_0)(p_0, x) \leq L_0 H(p_0, t_0)^{-1} \} \) by the factor \( H(p_0, t_0) \). Then the resulting surface is \( \varepsilon_0 \)-close to a product \( \Gamma \times [-L_0, L_0] \) in the \( C^3 \)-norm. Here, \( \Gamma \) is a closed, convex curve satisfying \( L(\Gamma) \leq 3\pi \) and \( \sup [\kappa - 1] \leq \frac{1}{100} \). The constant \( K_0 \) may depend on \( \theta, \varepsilon_0, L_0, \delta, \hat{\delta} \), and the initial data, but they are independent of the remaining surgery parameters \( \varepsilon, L \), and \( K_\ast \).

Version A requires the assumption that \( \bar{P}(p_0, t_0, L_0 + 4, \theta) \) does not contain surgeries. Version B only requires that the (possibly much smaller) parabolic neighborhood \( \bar{P}(p_0, t_0, L_0 + 4, \theta) \) is free of surgeries. Version B will be needed for the proof of the Neck Continuation Theorem.

The following next result serves as a replacement for Lemma 7.12 in [16]:

**Proposition 2.16** (Replacement for Lemma 7.12 in [16]). Let \( M_t \) be a mean curvature flow with surgery as above. Suppose that \( (p_1, t_1) \) is a point in spacetime with the property that the parabolic neighborhood \( \bar{P}(p_1, t_1, L + 4, 2\theta_0) \) contains at least one point belonging to a surgery region. Then there exists a point \( q_1 \in M_{t_1} \) and an open set \( V_1 \subset M_{t_1} \) such that \( d_{g(t_1)}(p_1, q_1) \leq (L + 4) H(p_1, t_1)^{-1}, \{ x \in M_{t_1} : d_{g(t_1)}(q_1, x) \leq 500 K_\ast^{-1} \} \subset V_1 \), and \( V_1 \) is diffeomorphic to a disk. Moreover, the mean curvature is at most \( 40 K_\ast \) at each point in \( V_1 \).

Since we have a bound for the gradient of the mean curvature, we can apply Theorem 7.14 in [16]. This gives the following result:

**Proposition 2.17** (Huisken-Sinestrari [16], Theorem 7.14). Consider a closed surface in \( \mathbb{R}^3 \) which satisfies the estimate \( |\nabla A| \leq C_{\#} (H + K_\ast)^2 \) for suitable constants \( C_{\#} \) and \( K_\ast \). Then, given any \( \eta > 0 \), we can find \( \rho > 0 \) and \( \gamma_0 > 0 \) (depending only on \( C_{\#} \) and \( \eta \)) with the following significance. Suppose that \( p \) is a point on the surface with \( \lambda_1(p) > \eta H(p) \) and \( H(p) \geq \gamma_0 K_\ast \). Then either \( \lambda_1 > \eta H > 0 \) everywhere on the surface, or else there exists a point \( q \) such that \( \lambda_1(q) \leq \eta H(q) ; d(p, q) \leq \frac{\rho}{H(p)} ; \) and \( H(q') \geq \frac{H(p)}{\gamma_0} \geq K_\ast \) for all points \( q' \) satisfying \( d(p, q') \leq \frac{\rho}{H(p)} \). In particular, \( H(q) \geq \frac{H(p)}{\gamma_0} \geq K_\ast \).

Moreover, using the noncollapsing property we can prove an analogue of Lemma 7.19 in [16]. This result will be needed for the proof of the Neck Continuation Theorem.

\footnote{See [16], pp. 189–190, for the definition of \( \bar{P}(p_0, t_0, L_0 + 4, \theta) \).}
**Proposition 2.18** (Replacement for Lemma 7.19 in [16]). Let \( \Sigma \) be an embedded surface in \( \mathbb{R}^3 \) which is \( \alpha \)-noncollapsed, and let \( y_1 < y_2 \) be two real numbers. We assume that the surface \( \Sigma \) is contained in the cylinder \( \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 100, y_1 \leq x_3 \leq y_2 \} \). Moreover, we assume that \( \partial \Sigma = \Gamma_1 \cup \Gamma_2 \), where \( \Gamma_1 \subset \{ x \in \mathbb{R}^3 : x_3 = y_1 \} \) and \( \Gamma_2 \subset \{ x \in \mathbb{R}^3 : x_3 = y_2 \} \). Then we have \( H(x) \geq \frac{\alpha}{100} \) for all points \( x \in \Sigma \) satisfying \( x_3 \in [y_1 + 1, y_2] \) and \( \langle \nu(x), e_3 \rangle \geq 0 \). Here, \( \nu \) denotes the outward-pointing unit normal to \( \Sigma \) and \( \Theta = \frac{200}{\alpha} \).

### 3. The Neck Continuation Theorem

In this section, we establish an analogue of the Neck Continuation Theorem of Huisken and Sinestrari [16]. We begin by finalizing our choice of the surgery parameters. This step is similar to the discussion on pp. 208–209 in [16]. Recall that the parameters \( \delta, \tilde{\delta}, \tilde{\alpha} \) and the constants \( C\#, \theta_0, \Theta \) have already been chosen at this stage. Moreover, we have chosen \( \tilde{\epsilon} \) and \( \Lambda \) so that the conclusion of Theorem 2.5 holds.

In the next step, we choose numbers \( \epsilon_0 \) and \( L_0 \) so that \( \epsilon_0 < \tilde{\epsilon} \) and \( L_0 > 1000 \Lambda \). In addition, we require that the mean curvature on an \((\tilde{\alpha}, \tilde{\delta}, \epsilon_0, L_0)\)-neck varies by at most a factor of \( 1 + L_0^{-1} \). (This can always be achieved by choosing \( \epsilon_0 \) very small.) We then choose real numbers \( \eta_0 > 0 \) and \( K_0 > 1000 \sup_{M_0} |A| \) so that the conclusion of Version A of the Neck Detection Lemma can be applied for each \( L \in [100, L_0] \). In other words, if \((p_0, t_0)\) satisfies \( H(p_0, t_0) \geq \max\{K_0, \frac{K_0^2}{\eta_0} \} \), \( \lambda_1(p_0, t_0) \leq \eta_0 H(p_0, t_0) \), and if the parabolic neighborhood \( \tilde{\mathcal{P}}(p_0, t_0, L + 4, 2\theta_0) \) is free of surgeries for some \( L \in [100, L_0] \), then \((p_0, t_0)\) lies at the center of a \((\tilde{\alpha}, \tilde{\delta}, \epsilon_0, L)\)-neck in \( M_{t_0} \).

In the next step, we put \( \epsilon_1 = \frac{\eta_0}{100} \). By Version A of the Neck Detection Lemma, we can find constants \( \eta < \eta_0 \) and \( K_1 > K_0 \) such that the following holds: if \((p_0, t_0)\) satisfies \( H(p_0, t_0) \geq \max\{K_1, \frac{K_1^2}{\eta} \} \), \( \lambda_1(p_0, t_0) \leq \eta H(p_0, t_0) \), and if the parabolic neighborhood \( \tilde{\mathcal{P}}(p_0, t_0, 104, 2\theta_0) \) is free of surgeries, then \((p_0, t_0)\) lies at the center of a \((\tilde{\alpha}, \tilde{\delta}, \epsilon_1, 100)\)-neck in \( M_{t_0} \).

Having chosen \( \eta_1 \), we next choose numbers \( \gamma_0 > 0 \) and \( \rho > 0 \) so that the conclusion of Proposition 2.17 holds.

By Version B of the Neck Detection Lemma, we can find a number \( K_2 > K_1 \) such that the following holds: Suppose that \((p_0, t_0)\) satisfies \( H(p_0, t_0) \geq \max\{K_2, \frac{K_2^2}{\rho} \} \) and \( \lambda_1(p_0, t_0) \leq 0 \), and that the parabolic neighborhood \( \tilde{\mathcal{P}}(p_0, t_0, 104, 10^{-6} \Theta^{-2} \tilde{\gamma}_0^{-2}) \) does not contain surgeries. Then, if we dilate the surface \( \{ x \in M_{t_0} : d_{\gamma_0}(p_0, x) \leq 100 H(p_0, t_0)^{-1} \} \) by the factor \( H(p_0, t_0) \), the resulting surface is \( \epsilon_1 \)-close to a product \( \Gamma \times [-100, 100] \) in the \( C^3 \)-norm. Here, \( \Gamma \) is a closed, convex curve satisfying \( L(\Gamma) \leq 3\pi \) and \( \sup_{\Gamma} |\kappa - 1| \leq \frac{1}{100} \).

Finally, we define \( K_* = 1000 \Theta K_2 \). Moreover, we put \( H_1 = K_* \), \( H_2 = 1000 \gamma_0 H_1 \), and \( H_3 = 10 H_2 \).
Before we can prove the Neck Continuation Theorem, we need one additional lemma:

Lemma 3.1. Suppose that $M_t$ is a mean curvature flow with surgeries. Moreover, suppose that $(p_0, t_0)$ satisfies $H(p_0, t_0) \geq 1000 H_1$ and $\lambda_1(p_0, t_0) \leq \eta_0 H(p_0, t_0)$, where $\eta_1$ and $H_1$ are defined as above. Then $p_0$ lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)$-neck.

Proof. We distinguish two cases:

Case 1: Suppose first that the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 2 \theta_0)$ contains a point modified by surgery. By Proposition 2.16 we can find a point $q \in M_{t_0}$ and an open set $V \subset \{ x \in M_{t_0} : H(x, t_0) \leq 40 H_1 \}$ such that $d_{g(t_0)}(p_0, q) \leq 104 H(p_0, t_0)^{-1}$ and $\{ x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500 H_1^{-1} \} \subset V$. Clearly, $p_0 \in V$. Consequently, $H(p_0, t_0) \leq 40 H_1$, contrary to our assumption.

Case 2: We now assume that the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 2 \theta_0)$ is free of surgeries. Let $L \in [100, L_0]$ be the largest number with the property that $\hat{\mathcal{P}}(p_0, t_0, L + 4, 2 \theta_0)$ is free of surgeries. By the Neck Detection Lemma, the point $(p_0, t_0)$ lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon_0, L)$-neck $N$. If $L = L_0$, we are done. Hence, it remains to consider the case when $L < L_0$. In this case, the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, L + 5, 2 \theta_0)$ must contain a point modified by surgery. By Proposition 2.16 we can find a point $q \in M_{t_0}$ and an open set $V \subset \{ x \in M_{t_0} : H(x, t_0) \leq 40 H_1 \}$ such that $d_{g(t_0)}(p_0, q) \leq (L + 5) H(p_0, t_0)^{-1}$ and $\{ x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500 H_1^{-1} \} \subset V$. Since the set $\{ x \in M_{t_0} : d_{g(t_0)}(p_0, x) \leq (L - 1) H(p_0, t_0)^{-1} \}$ is contained in $N$, we conclude that $d_{g(t_0)}(q, N) \leq 6 H(p_0, t_0)^{-1} \leq 6 H_1^{-1}$. Consequently, we have $N \cap V \neq \emptyset$. On the other hand, we have $H \geq \frac{1}{7} H(p_0, t_0)^{-1} \geq 500 H_1$ at each point on $N$ and $H \leq 40 H_1$ at each point on $V$. This is a contradiction. This completes the proof of Lemma 3.1.

The following result is the analogue of the Neck Continuation Theorem in [16]:

Theorem 3.2 (Neck Continuation Theorem). Suppose that $M_t$ is a mean curvature flow with surgeries. Let $t_0$ be a time such that $\sup_{M_{t_0}} H \geq H_3$. Moreover, suppose that $(p_0, t_0)$ satisfies $H(p_0, t_0) \geq 1000 H_1$ and $\lambda_1(p_0, t_0) \leq \eta_1 H(p_0, t_0)$, where $\eta_1$ and $H_1$ are defined as above. Then there exists a finite collection of points $p_1, \ldots, p_l$ with the following properties:

- For each $i = 0, 1, \ldots, l$, the point $p_i$ lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)$-neck $N^{(i)} \subset M_{t_0}$, and we have $H(p_i, t_0) \geq H_1$.
- For each $i = 1, \ldots, l - 1$, the point $p_{i+1}$ lies on the neck $N^{(i)}$, and we have $\text{dist}_{g(t_0)}(p_{i+1}, \partial N^{(i)} \setminus N^{(i-1)}) \in [(L_0 - 100) H(p_i, t_0)^{-1}, (L_0 - 50) H(p_i, t_0)^{-1}]$.
- Finally, one of the following four statements holds: either the set $\mathcal{N} = \bigcup_{i=1}^l N^{(i)}$ covers the entire surface; or $H(p_i, t_0) \in [H_1, 2 H_1]$; or
there exists a closed curve in \( N \cap \{ x \in M_{t_0} : H(x, t_0) \leq 40 H_1 \} \) which is homotopically non-trivial in \( N \) and bounds a disk in \( \{ x \in M_{t_0} : H(x, t_0) \leq 40 H_1 \} \); or the outer boundary \( \partial N^{(k)} \setminus N^{(k-1)} \) bounds a convex cap.

We now describe the proof of the Neck Continuation Theorem. Most of the arguments in [16] carry over to our situation. However, the proof of Lemma 7.19 does not work in our setting. The reason is that the gradient estimate in Proposition 2.18 deteriorates when the curvature is much smaller than \( H_1 \). We will use Proposition 2.15 to overcome this problem.

Proof. By Lemma 3.1, the point \( p_0 \) lies at the center of an \((\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)\)-neck \( N^{(0)} \subset M_{t_0} \). The construction of the points \( p_1, p_2, \ldots \) is by induction. Suppose that we have constructed points \( p_1, \ldots, p_k \) and necks \( N^{(1)}, \ldots, N^{(k)} \) with the following properties:

- For each \( i = 0, 1, \ldots, k \), the point \( p_i \) lies at the center of an \((\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)\)-neck \( N^{(i)} \subset M_{t_0} \), and we have \( H(p_i, t_0) \geq H_1 \).
- For each \( i = 1, \ldots, k - 1 \), the point \( p_{i+1} \) lies on the neck \( N^{(i)} \), and we have dist \( g(t_0)(p_{i+1}, \partial N^{(i)} \setminus N^{(i-1)}) \in [(L_0 - 100) H(p_i, t_0)^{-1}, (L_0 - 50) H(p_i, t_0)^{-1}] \).

If \( H(p_k, t_0) \in [H_1, 2H_1] \), we have achieved our goal and we can stop the process. Hence, it suffices to consider the case that \( H(p_k, t_0) \geq 2H_1 \). Let us distinguish several cases:

Case 1: Suppose that there exists a point \( p \in N^{(k)} \) such that dist \( g(t_0)(p, \partial N^{(k)} \setminus N^{(k-1)}) \in [(L_0 - 100) H(p_k, t_0)^{-1}, (L_0 - 50) H(p_k, t_0)^{-1}] \), and the parabolic neighborhood \( \mathcal{P}(p, t_0, L_0 + 4, 2\theta_0) \) contains a point modified by surgery. In this case, Proposition 2.16 implies that there exists a point \( q \in M_{t_0} \) and an open set \( V \subset \{ x \in M_{t_0} : H(x, t_0) \leq 40 H_1 \} \) such that dist \( g(t_0)(p, q) \leq (L_0 + 4) H(p, t_0)^{-1} \), \( \{ x \in M_{t_0} : d_{g(t_0)}(p, x) \leq 500 H_1^{-1} \} \subset V \), and \( V \) is diffeomorphic to a disk.

By our choice of \( \varepsilon_0 \) and \( L_0 \), the mean curvature on \( N^{(k)} \) varies at most by a factor \( 1 + L_0^{-1} \). Hence, \( H(p_k, t_0) \leq (1 + L_0^{-1}) H(p, t_0) \). Since the set \( \{ x \in M_{t_0} : d_{g(t_0)}(p, x) \leq (L_0 - 100) H(p_k, t_0)^{-1} \} \) is contained in \( N^{(k)} \), we conclude that

\[
\begin{align*}
d_{g(t_0)}(q, N^{(k)}) & \leq (L_0 + 4) H(p, t_0)^{-1} - (L_0 - 100) H(p_k, t_0)^{-1} \\
& \leq (L_0 + 4)(1 + L_0^{-1}) H(p_k, t_0)^{-1} - (L_0 - 100) H(p_k, t_0)^{-1} \\
& \leq 200 H(p_k, t_0)^{-1} \\
& \leq 100 H_1^{-1}.
\end{align*}
\]

Consequently, there exists a closed curve which is contained in \( N^{(k)} \cap V \) and is homotopically non-trivial in \( N^{(k)} \). Since \( V \) is diffeomorphic to a disk, this curve bounds a disk in \( V \). Hence, we can terminate the process at this point.
Case 2: We now assume that the parabolic neighborhood \( \hat{\mathcal{P}}(p, t_0, L_0 + 4, 2\theta_0) \) is free of surgeries for all points \( p \in N(k) \) satisfying \( \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \in [(L_0 - 100)H(p_k, t_0)^{-1}, (L_0 - 50)H(p_k, t_0)^{-1}] \). There are two possibilities now:

Subcase 2.1: Suppose that there exists a point \( p \in N(k) \) with the property that \( \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \in [(L_0 - 100)H(p_k, t_0)^{-1}, (L_0 - 50)H(p_k, t_0)^{-1}] \) and \( \lambda_1(p, t_0) \leq \eta_0 H(p, t_0) \). By Version A of the Neck Detection Lemma, the point \( p \) lies at the center of an \((\hat{\alpha}, \delta, \varepsilon_0, L_0)\)-neck \( N \). Moreover, since \( p \in N(k) \) and \( H(p_k, t_0) \geq 2H_1 \), we have \( H(p, t_0) \geq H_1 \). Hence, we can put \( p^{(k+1)} := p \) and \( N^{(k+1)} := N \) and continue the process.

Subcase 2.2: Suppose that \( \lambda_1(p, t_0) > \eta_0 H(p, t_0) \) for all points \( p \in N(k) \) satisfying \( \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \in [(L_0 - 100)H(p_k, t_0)^{-1}, (L_0 - 50)H(p_k, t_0)^{-1}] \). Let \( N = \bigcup_{i=0}^{k} N(i) \), and let \( A \) be the set of all points \( x \in N \) such that \( \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \geq (L_0 - 50)H(p_k, t_0)^{-1} \) and \( \lambda_1(x, t_0) \leq \eta_1 H(x, t_0) \). By assumption, the initial point \( p_0 \) belongs to \( A \), so \( A \) is non-empty. Let us consider a point \( p^* \) which has maximal distance from \( p_0 \) among all points in \( A \).

Subcase 2.2.1: Suppose that the parabolic neighborhood \( \hat{\mathcal{P}}(p^*, t_0, 104, 2\theta_0) \) contains a point modified by surgery. In this case, Proposition 2.16 implies that there exists a point \( q \in M_{t_0} \) and an open set \( V \subset \{ x \in M_{t_0} : H(x, t_0) \leq 40H_1 \} \) such that \( d_{g(t_0)}(p^*, q) \leq 104H(p^*, t_0)^{-1} \), \( \{ x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500H_1^{-1} \} \subset V \), and \( V \) is diffeomorphic to a disk. Since \( H(p^*, t_0) \geq \frac{H_1}{2} \), this implies

\[
\begin{align*}
\{ x \in M_{t_0} : d_{g(t_0)}(p^*, x) \leq 100H(p^*, t_0)^{-1} \} & \subset \{ x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 204H(p^*, t_0)^{-1} \} \\
& \subset \{ x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500H_1^{-1} \} \\
& \subset V.
\end{align*}
\]

Consequently, there exists a closed curve in \( N \cap V \) which is homotopically non-trivial in \( N \). This curve bounds a disk which is contained in \( V \). Hence, we can again terminate the process.

Subcase 2.2.2: Suppose, finally, that the parabolic neighborhood \( \hat{\mathcal{P}}(p^*, t_0, 104, 2\theta_0) \) is free of surgeries. In this case, the Neck Detection Lemma implies that the point \( p^* \) lies at the center of an \((\hat{\alpha}, \delta, \varepsilon_1, 100)\)-neck \( N^* \). Clearly, \( \lambda_1 \leq \varepsilon_1 H \) at each point on \( N^* \). Consequently, the set \( N^* \) is disjoint from the set \( \{ p \in N^{(k)} : \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \in [(L_0 - 100)H(p_k, t_0)^{-1}, (L_0 - 50)H(p_k, t_0)^{-1}] \} \). Furthermore, since \( p^* \) has maximal distance from \( p_0 \) among all points in \( A \), we conclude that the part of \( N \) that lies between the neck \( N^* \) and the set \( \{ p \in N^{(k)} : \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \in [(L_0 - 100)H(p_k, t_0)^{-1}, (L_0 - 50)H(p_k, t_0)^{-1}] \} \) is strictly convex.

Let \( \omega \) denote the axis of the neck \( N^* \). Arguing as in [19], we can show that \( \langle \nu, \omega \rangle \geq -\varepsilon_1 \) for all points \( p \in N(k) \) satisfying \( \text{dist}_{g(t_0)}(p, \partial N^k \setminus N^{k-1}) \).
For each point on $\Gamma$ the sequence $p$ theorem is proved. On the other hand, if the sequence $p$ family of curves $\Gamma$ hold for all $y$ (isfied for $T$ property follows from work of Head [11]. Therefore, Assumption 2.6 is sat-

$N^{(k-1)} \in [(L_0 - 100) H(p_k, t_0)^{-1}, (L_0 - 50) H(p_k, t_0)^{-1}]$. Moreover, we have $\lambda_1(p, t_0) > \eta_0 H(p, t_0)$ for all points $p \in N^{(k)}$ satisfying $\text{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \in [(L_0 - 100) H(p_k, t_0)^{-1}, (L_0 - 50) H(p_k, t_0)^{-1}]$. Putting these facts together (and using the fact that $\eta_0 \geq 10 \varepsilon_1$), we conclude that $\langle \nu, \omega \rangle \geq 4 \varepsilon_1$ for all points $p \in N^{(k)}$ satisfying $\text{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \leq (L_0 - 100) H(p_k, t_0)^{-1}$.

We claim that the boundary curve $\partial N^{(k)} \setminus N^{(k-1)}$ bounds a convex cap. To prove this, we use the argument on p. 215-216 of [16]. Let us choose a curve $\Gamma_0$ such that $\Gamma_0 \subset \{ p \in N^{(k)} : \text{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \leq (L_0 - 100) H(p_k, t_0)^{-1} \}$ and $\Gamma_0$ is contained in a plane orthogonal to $\omega$. For each point on $\Gamma_0$, we solve the ODE $\dot{\gamma} = \frac{\omega^F(\gamma)}{\omega^T(\gamma)}$, where $\omega^T(\gamma)$ denotes the projection of $\omega$ to the tangent plane to $M_{t_0}$ at the point $\gamma$. This gives a family of curves $\Gamma_y \subset M_{t_0}$, each of which is contained in a plane orthogonal to $\omega$. The curves $\Gamma_y$ are well-defined for $y \in [0, y_{\max})$. Moreover, there exists a point $p \in \Gamma_0$ such that $\nu(\gamma(y, p)) \rightarrow \omega$ as $y \rightarrow y_{\max}$.

Using Proposition 2.18 and the Neck Detection Lemma, we can show that the inequalities

$$\langle \nu, \omega \rangle < 1, \quad \lambda_1 > 0, \quad H > \frac{2H_1}{\Theta}, \quad \langle \nu, \omega \rangle > \varepsilon_1$$

hold for all $y \in [0, y_{\max})$. Therefore, the union of the curves $\Gamma_y$ is a convex cap, and we can terminate the process. This completes the construction of the sequence $p_1, p_2, \ldots$.

If the sequence $p_1, p_2, \ldots$ terminates after finitely many steps, then the theorem is proved. On the other hand, if the sequence $p_1, p_2, \ldots$ never terminates, then the necks $N^{(1)}, N^{(2)}, \ldots$ will eventually cover the entire surface. This completes the proof of the Neck Continuation Theorem.

After these preparations, we can now implement the surgery procedure of Huisken and Sinestrari [16]. More precisely, starting from the given surface $M_0$ we run the mean curvature flow until the maximum of the mean curvature reaches the threshold $H_3$ for the first time. Let us denote this time by $T_1$. It follows from a result of Andrews [1] that Assumption 2.6 is satisfied for $0 \leq t < T_1$. Hence, the Neck Detection Lemma and the Neck Continuation Theorem can be applied, which enables us to perform perform surgery. Immediately after surgery, the maximum of the mean curvature has dropped to a level below $H_2$. We then run the flow again until the maximum of the mean curvature reaches $H_3$ for the second time. Let us denote this time by $T_2$. Again, Assumption 2.6 are satisfied for $T_1 < t < T_2$. Indeed, Theorem 2.5 implies that the inscribed radius and the outer radius of the surface $M_{T_1+}$ are bounded by $\frac{H}{T^2}$, and this property continues to hold for all $t < T_2$ by a result of Andrews [1]. Furthermore, the outward-minimizing property follows from work of Head [11]. Therefore, Assumption 2.6 is satisfied for $T_1 < t < T_2$, and we can again apply the Neck Detection Lemma.
and the Neck Continuation Theorem. Thus, we can again perform surgery to push the maximum of the mean curvature below $H_2$. This process can now be repeated.

4. Proof of Theorem 2.2

We first recall the following analogue of Shi’s local derivative estimate for the Ricci flow. The argument given here is standard and follows the proof in Ecker-Huisken [7]; see also [5], Proposition 3.22.

**Lemma 4.1.** Suppose that $M_t$, $t \in [0, T]$, is a regular mean curvature flow in $B_4(0)$ in the sense of Definition 2.1. Moreover, we assume that $|A| \leq 1$ for all $t \in [0, T]$ and all $x \in M_t \cap B_4(0)$. Finally, we assume that $|\nabla A| \leq 1$ for all $x \in M_0 \cap B_4(0)$. Then $|\nabla A| \leq C$ for all $t \in [0, 1] \cap [0, T]$ and all $x \in M_t \cap B_2(0)$.

**Proof.** Consider the cutoff function $\psi(x) = 1 - \frac{|x|^2}{16}$. A straightforward calculation gives

$$\frac{\partial}{\partial t}(\psi^2 |\nabla A|^2) \leq \Delta(\psi^2 |\nabla A|^2) + C_0 |\nabla A|^2$$

for all $t \in [0, T]$ and all $x \in M_t \cap B_4(0)$. This implies that

$$\frac{\partial}{\partial t}(\psi^2 |\nabla A|^2 + C_0 |A|^2) \leq \Delta(\psi^2 |\nabla A|^2 + C_0 |A|^2) + C_1$$

for all $t \in [0, T]$ and all $x \in M_t \cap B_4(0)$. Applying the maximum principle to the function $\psi^2 |\nabla A|^2 + C_0 |A|^2 - C_1 t$, we obtain

$$\sup_{t \in [0,T]} \sup_{x \in M_t \cap B_4(0)} (\psi^2 |\nabla A|^2 + C_0 |A|^2 - C_1 t)$$

$$\leq \max \left\{ \sup_{x \in M_0 \cap B_4(0)} (\psi^2 |\nabla A|^2 + C_0 |A|^2), \sup_{t \in [0,T]} \sup_{x \in M_t \cap \partial B_4(0)} (C_0 |A|^2 - C_1 t) \right\}$$

$$\leq 1 + C_0.$$

From this, the assertion follows.

A similar estimate holds for the second derivatives of the second fundamental form:

**Lemma 4.2.** Suppose that $M_t$, $t \in [0, T]$, is a regular mean curvature flow in $B_4(0)$ in the sense of Definition 2.1. Moreover, we assume that $|A| \leq 1$ for all $t \in [0, T]$ and all $x \in M_t \cap B_4(0)$. Finally, we assume that $|\nabla A| + |\nabla^2 A| \leq 1$ for all $x \in M_0 \cap B_4(0)$. Then $|\nabla A|^2 + |\nabla^2 A| \leq C$ for all $t \in [0, 1] \cap [0, T]$ and all $x \in M_t \cap B_1(0)$.

**Proof.** By Lemma 4.1 we have $|\nabla A| \leq C$ for all $t \in [0, 1] \cap [0, T]$ and all $x \in M_t \cap B_1(0)$. To get a bound for $|\nabla^2 A|$, we apply the maximum principle to the function $\psi^2 |\nabla^2 A|^2 + C_0 |\nabla A|^2$, where $\psi = 1 - \frac{|x|^2}{16}$ and $C_0$ is a large constant.
Our next result will require the monotonicity formula for mean curvature flow. The monotonicity formula was established in [13]. We will need a local version of this result. Specifically, we consider the modified Gaussian density

\[ \Theta(x_0, t_0; r) = \int_{M_0 - r^2} \frac{1}{4\pi r^2} e^{-\frac{|x-x_0|^2}{4r^2}} (1 - |x-x_0|^2 + 4r^2)^{-\frac{3}{2}}. \]

The local monotonicity formula asserts that the function \( r \mapsto \Theta(x_0, t_0; r) \) is monotone increasing. A proof of this fact can be found in [5], pp. 64–65 (see also [6]).

**Proposition 4.3.** There exist positive constants \( \beta_0 \in (0, 1) \) and \( C \) such that the following holds. Suppose that \( M_0 \), \( t \in [0, T] \), is a regular mean curvature flow in \( B_4(0) \). Moreover, we assume that the initial surface \( M_0 \) can be expressed as the graph of a (single-valued) function \( u \) over a plane. If \( \| u \|_{C^3} \leq \beta_0 \), then \( |A(x, t)| \leq C \) for all \( t \in [0, \beta_0] \cap [0, T] \) and all \( x \in M_t \cap B_1(0) \).

**Proof.** Our argument is inspired in part by the proof of Theorem C.1 in [10]. Suppose that the assertion is false. Then we can find a sequence of regular mean curvature flows \( M_j \) in \( B_4(0) \) with the following properties:

- The initial surface \( M_{0,j} \cap B_4(0) \) is the graph of a (single-valued) function \( u_j \) over a plane, and \( u_j \) satisfies \( \| u_j \|_{C^4} \leq \frac{1}{j} \).
- There exists a sequence of times \( t_j \in [0, \frac{1}{j}] \cap [0, T_j] \) and a sequence of points \( x_j \in M_{t_j,j} \cap B_1(0) \) such that \( |A(x_j, t_j)| \geq j \).

Using a point picking argument as in Appendix C of [10], we can find a pair \((\tilde{x}_j, \tilde{t}_j)\) such that \( \tilde{t}_j \in [0, \frac{1}{j}] \cap [0, T_j] \), \( \tilde{x}_j \in M_{t_j,j} \cap B_2(0) \), \( Q_j := |A(\tilde{x}_j, \tilde{t}_j)| \geq j \), and

\[
\sup_{t \in [0, \tilde{t}_j]} \sup_{x \in M_{t,j} \cap B_2(\tilde{x}_j)} |A(x, t)| \leq 2 Q_j.
\]

At this point, we distinguish two cases:

**Case 1:** Suppose that \( \limsup_{j \to \infty} \tilde{t}_j Q_j^2 = 0 \). By assumption, we have \( |\nabla A| \leq 1 \) and \( |\nabla^2 A| \leq 1 \) on the initial surface \( M_{0,j} \cap B_4(0) \). Hence, it follows from Lemma 4.1 and Lemma 4.2 that

\[
\sup_{t \in [0, \tilde{t}_j]} \sup_{x \in M_{t,j} \cap B_2(\tilde{x}_j)} |\nabla A(x, t)| \leq C_0 Q_j^2
\]

and

\[
\sup_{t \in [0, \tilde{t}_j]} \sup_{x \in M_{t,j} \cap B_2(\tilde{x}_j)} |\nabla^2 A(x, t)| \leq C_0 Q_j^3,
\]

where \( C_0 \) is a uniform constant independent of \( j \). In the next step, we follow the point \( \tilde{x}_j \) back in time. More precisely, we consider a path \( \sigma_j : [0, \tilde{t}_j] \to \mathbb{R}^3 \) such that \( \sigma_j(t) \in M_{t,j} \), \( \sigma_j'(t) \) equals the mean curvature vector of \( M_{t,j} \) at the point \( \sigma_j(t) \), and \( \sigma_j(\tilde{t}_j) = \tilde{x}_j \). Then \( |\sigma_j'(t)| \leq 4 Q_j \) as long as
form a solution of the mean curvature flow in the classical sense. Moreover, the surface for each fine a family of surfaces y for each point (\partial flow in the classical sense. Moreover, we have |\nabla^2 A(\sigma_j(t), t)| \leq C_0 Q_j^3 for all t \in [0, \tilde{t}_j]. This implies
\[ \frac{d}{dt}|A(\sigma_j(t), t)| \leq C_1 Q_j^3 \]
for all t \in [0, \tilde{t}_j], provided that j is sufficiently large. Integrating this inequality from 0 to \tilde{t}_j gives
\[ Q_j = |A(\tilde{x}_j, \tilde{t}_j)| \leq |A(\sigma_j(0), 0)| + C_1 \tilde{t}_j Q_j^3 \leq 1 + C_1 \tilde{t}_j Q_j^3 \]
if j is sufficiently large. Since Q_j \to \infty and \tilde{t}_j Q_j^2 \to 0, we arrive at a contradiction.

Case 2: We now assume that \tau := \limsup_{j \to \infty} \tilde{t}_j Q_j^2 \in (0, \infty]. Let us define a family of surfaces M'_{t,j} \subset M_{t,j} in the following way: The surface M'_{t,j} is defined as the intersection of M_{t,j} with the ball \( B_{\frac{Q_j}{1000}}(\tilde{x}_j) \). Moreover, for each t \in [0, \tilde{t}_j], the surface M'_{t,j} is obtained by following each point on the surface M_{t,j} back in time. It is clear that the surfaces M'_{t,j}, t \in [0, \tilde{t}_j], form a solution of the mean curvature flow in the classical sense. Moreover, we have \( \partial M'_{t,j} \cap B_{\frac{Q_j}{1000}}(\tilde{x}_j) = \emptyset \) for all t \in [0, \tilde{t}_j] and all \tilde{t}_j - \frac{Q_j}{1000} Q_j^{-2}, \tilde{t}_j].

We next consider the rescaled surfaces \( \tilde{M}_{s,j} := Q_j (M'_{t,j+Q_j^{-2}s,j} - \tilde{x}_j), s \in [-\tilde{t}_j Q_j^2, 0] \). These surfaces again form a solution of mean curvature flow in the classical sense. Moreover, we have \( \partial \tilde{M}_{s,j} \cap B_{\frac{1}{2000}}(0) = \emptyset \) for all s \in [-\tilde{t}_j Q_j^2, 0] \setminus [-\frac{Q_j}{1000}, 0]. Finally, the norm of the second fundamental form of \( \tilde{M}_{s,j} \) is bounded from above by 2.

Taking the limit as j \to \infty, we obtain a complete, smooth, non-flat solution to the mean curvature flow which is defined on the time interval (-\tau, 0]. The limiting solution has bounded curvature and nonnegative mean curvature. We claim that the (standard) Gaussian density of the limit flow is at most 1. To see this, let us denote the limit flow by \( M_s, s \in (-\tau, 0] \). Then, for each point (y_0, s_0) \in \mathbb{R}^3 \times (-\tau, 0] and any r \in (0, \sqrt{\tau + s_0}), we have
\[
\int_{M_{y_0+r}} \frac{1}{4\pi r^2} e^{-\frac{|y-y_0|^2}{4r^2}} \leq \lim_{j \to \infty} \Theta_{M_{t,j}}(\tilde{x}_j + Q_j^{-1}y_0, \tilde{t}_j + Q_j^{-2}s_0; Q_j^{-1}r) \\
\leq \lim_{j \to \infty} \Theta_{M_{t,j}}(\tilde{x}_j + Q_j^{-1}y_0, \tilde{t}_j + Q_j^{-2}s_0; \sqrt{\tilde{t}_j + Q_j^{-2}s_0}) \\
\leq 1.
\]
Here, we have used Ecker’s monotonicity formula for the modified Gaussian density $\Theta_{M_j}$. We have also used the fact that $Q_j^{-1} r < \sqrt{\tilde{t}_j + Q_j^{-2} s_0}$ for $j$ large.

Consequently, if we choose $y_0 \in \hat{M}_{s_0}$, then we must have equality in the monotonicity formula; that is,

$$\int_{\hat{M}_{s_0}-r^2} \frac{1}{4\pi r^2} e^{-\frac{|y-y_0|^2}{4r^2}} = 1$$

whenever $s_0 \in (0, \sqrt{\tau + s_0})$. This forces the limiting solution to be a homothetically shrinking solution. Since the base point $y_0 \in \hat{M}_{s_0}$ is arbitrary, the solution must be a flat plane of multiplicity 1, contrary to our assumption. This completes the proof of Proposition 4.3.

Theorem 2.2 follows by combining Proposition 4.3 with Lemma 4.1 and Lemma 4.2.

5. Proof of Theorem 2.5

In this section, we explain our procedure for capping off a neck. We begin by constructing an axially symmetric model surface.

Lemma 5.1. The surface

$$\Sigma = \left\{ \left( \sqrt[3]{\frac{s}{1+s}} \cos(2\pi t), \sqrt[3]{\frac{s}{1+s}} \sin(2\pi t), s \right) : s \in [0, \infty), t \in [0, 1] \right\}$$

closes up smoothly at $s = 0$. Moreover, we have $0 < \lambda_1 < \lambda_2 = \mu$ whenever $s > 0$. Here, $\mu$ denotes the reciprocal of the inscribed radius of $\Sigma$.

Proof. The smoothness of $\Sigma$ is obvious. A straightforward calculation shows that the principal curvatures of $\Sigma$ are given by

$$\lambda_1 = 2 \left( 1 + 4s \left( 1 + s \right)^3 \right)^{-\frac{1}{2}} \left( 1 + s \right)^2 \left( 1 + 4s \right)$$

and

$$\lambda_2 = 2 \left( 1 + 4s \left( 1 + s \right)^3 \right)^{-\frac{1}{2}} \left( 1 + s \right)^2.$$

Clearly, $0 < \lambda_1 < \lambda_2$ for $s > 0$. Hence, it remains to estimate the inscribed radius of $\Sigma$. To that end, let $\Omega = \{ x \in \mathbb{R}^3 : x_3 > 0, x_1^2 + x_2^2 < \frac{x_3}{1+x_3} \}$ be the region enclosed by $\Sigma$. For each $s > 0$, we denote by $W_s$ the open ball of radius $\frac{1}{2} \left( 1 + 4s \left( s+1 \right)^3 \right)^{1/2} \left( 1 + s \right)^{-2}$ centered at the point $(0, 0, s + \frac{1}{2} \left( s+1 \right)^{-2})$.

For each $s > 0$, the circle

$$C_s := \left\{ \left( \sqrt[3]{\frac{s}{1+s}} \cos(2\pi t), \sqrt[3]{\frac{s}{1+s}} \sin(2\pi t), s \right) : t \in [0, 1] \right\}$$

is contained in $\Sigma \cap \partial W_s$. Moreover, the surfaces $\Sigma$ and $\partial W_s$ have the same tangent plane at each point on the circle $C_s$.

It is easy to see that $W_s \subset \Omega$ if $s$ is sufficiently large. We claim that $W_s \subset \Omega$ for all $s > 0$. Suppose this is false. Let $\bar{s} = \sup\{ s > 0 : W_s \not\subset \Omega \}$. 


Then $W_s \subset \Omega$. Moreover, we can find a sequence of numbers $s_j \nearrow \bar{s}$ and a sequence of points $p_j \in W_{s_j} \setminus \Omega$. After passing to a subsequence if necessary, the points $p_j$ converge to some point $p \in W_{\bar{s}} \setminus \Omega$. Since $W_{\bar{s}} \subset \Omega$, we conclude that $p \in \Sigma \cap \partial W_{\bar{s}}$, and the surfaces $\Sigma$ and $\partial W_{\bar{s}}$ have the same tangent plane at the point $p$. On the other hand, since $\lambda_1 < \lambda_2$, we must have $\liminf_{j \to \infty} \text{dist}(p_j, C_{s_j}) > 0$. Consequently, we have $p \in C_{\tilde{s}}$ for some $\tilde{s} \neq \bar{s}$. This implies that $p \in \Sigma \cap \partial W_{\tilde{s}}$, and the surfaces $\Sigma$ and $\partial W_{\tilde{s}}$ have the same tangent plane at the point $p$. Thus, the spheres $\partial W_{\tilde{s}}$ and $\partial W_{\bar{s}}$ touch each other at the point $p$, but this is impossible if $\tilde{s} \neq \bar{s}$. This shows that $W_s \subset \Omega$ for all $s > 0$. Consequently, the inscribed radius is given by $\frac{1}{2} (1 + 4s \lambda^3) (1 + s)^{-2}$. This completes the proof of Lemma 5.1.

In the remainder of this section, we consider an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck $N$ of size 1, which is contained in a closed, embedded, mean convex surface $M \subset \mathbb{R}^3$. It is understood that $\varepsilon$ is much smaller than $\delta$. By definition, we can find a simple closed, convex curve $\Gamma$ with the property that $\text{dist}_{C^{2,0}}(N, \Gamma \times [-L, L]) \leq \varepsilon$. Moreover, the curve $\Gamma$ is $\frac{1 + \delta}{1 + \delta}$ noncollapsed, and the derivatives of the geodesic curvature of $\Gamma$ satisfy $\sum_{t=3}^{18} |\nabla^t \kappa| \leq \frac{1}{100}$ at each point on $\Gamma$. Furthermore, there exists a point on $\Gamma$ where the geodesic curvature $\kappa$ is equal to 1.

Since $\text{dist}_{C^{2,0}}(N, \Gamma \times [-L, L]) \leq \varepsilon$, we can find a collection of curves $\Gamma_s$ such that $$\{ (\gamma_s(t), s) : s \in [-L, L], t \in [0, 1] \} \subset N$$

and

$$\sum_{k+l \leq 2} \left| \frac{\partial^k}{\partial s^k} \frac{\partial^l}{\partial t^l} (\gamma_s(t) - \gamma(t)) \right| \leq O(\varepsilon).$$

Here, we have used the notation $\Gamma = \{ \gamma(t) : t \in [0, 1] \}$ and $\Gamma_s = \{ \gamma_s(t) : t \in [0, 1] \}$.

The following lemma is analogous to Proposition 3.17 in [16]:

**Lemma 5.2** (cf. Huiskens-Sinestrari [16], Proposition 3.17). Consider a bended surface of the form

$$\tilde{N} = \{ (1 - u(s)) \gamma_s(t), s : s \in (0, \Lambda^{1 \over 2}], t \in [0, 1] \},$$

where $|u| + |u'| + |u''| \leq \frac{1}{100}$ everywhere. Then we have the pointwise estimates

$$\tilde{\lambda}_1(s, t) \geq \lambda_1(s, t) + c_0 \lambda^3 - \frac{1}{c_0} (|u(s)| + |u'(s)|)$$

and

$$\tilde{H}(s, t) \geq H(s, t) + c_0 \lambda^2 - \frac{1}{c_0} (|u(s)| + |u'(s)|),$$

where $c_0 > 0$ is a universal constant.
It will be convenient to translate the neck $N$ in space so that the center of mass of $\Gamma$ is at the origin. Using the curve shortening flow, we can construct a homotopy $\tilde{\gamma}(t)$, $(r, t) \in [0, 1] \times [0, 1]$, with the following properties:

- $\tilde{\gamma}(t) = \gamma(t)$ for $r \in [0, \frac{1}{2}]$.
- $\tilde{\gamma}(t) = (\cos(2\pi t), \sin(2\pi t))$ for $r \in [\frac{1}{2}, 1]$.
- For each $r \in [0, 1]$, the curve $\tilde{\gamma}$ is $\frac{1}{1+\delta}$ noncollapsed.
- We have $\sup_{(r, t) \in [0, 1] \times [0, 1]} |\frac{\partial}{\partial r} \tilde{\gamma}(t)| + |\frac{\partial^2}{\partial r \partial t} \tilde{\gamma}(t)| + |\frac{\partial^2}{\partial r^2} \tilde{\gamma}(t)| \leq \omega(\delta)$, where $\omega(\delta) \to 0$ as $\delta \to 0$.

Moreover, let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function such that $\chi = 1$ on $(-\infty, 1]$ and $\chi = 0$ on $[2, \infty)$. We next define a surface $\tilde{F}_\Lambda : [-L, \Lambda] \times [0, 1] \to \mathbb{R}^3$ by

\[
\tilde{F}_\Lambda(s, t) = \begin{cases} 
(\gamma_s(t), s) & \text{for } s \in [-(L-1), 0] \\
((1 - e^{-\frac{4\Lambda}{s}}) \gamma_s(t), s) & \text{for } s \in (0, \Lambda^{\frac{1}{2}}] \\
((1 - e^{-\frac{4\Lambda}{s}}) \chi(s/\Lambda^{\frac{1}{2}}) \gamma_s(t) + (1 - \chi(s/\Lambda^{\frac{1}{2}})) \gamma(t), s) & \text{for } s \in (\Lambda^{\frac{1}{2}}, 2\Lambda^{\frac{1}{2}}] \\
((1 - e^{-\frac{4\Lambda}{s}}) \tilde{\gamma}_s/\Lambda(t), s) & \text{for } s \in (2\Lambda^{\frac{1}{2}}, \Lambda]. 
\end{cases}
\]

It is clear that $\tilde{F}_\Lambda$ is smooth. Moreover, $\tilde{F}_\Lambda$ is axially symmetric for $s \geq \frac{\Lambda}{2}$.

**Lemma 5.3.** We can find real numbers $\delta_1, \Lambda_1$, and a function $E(\delta, \Lambda)$ such that the following statements hold:

- Suppose that $\delta < \delta_1$, $\frac{L}{1000} \geq \Lambda \geq \Lambda_1$, and $\varepsilon \leq E(\delta, \Lambda)$. Then, for each point $(s, t) \in (0, \Lambda^{\frac{1}{2}}] \times [0, 1]$, the mean curvature of $\tilde{F}_\Lambda$ at $(s, t)$ is greater than the mean curvature of the original neck at $(s, t)$, and the smallest curvature eigenvalue of $\tilde{F}_\Lambda$ is greater than the smallest curvature eigenvalue of the original neck at $(s, t)$.

- Suppose that $\delta < \delta_1$, $\frac{L}{1000} \geq \Lambda \geq \Lambda_1$, and $\varepsilon \leq E(\delta, \Lambda)$. Then, for each point $(s, t) \in (\Lambda^{\frac{1}{2}}, 2\Lambda^{\frac{1}{2}}] \times [0, 1]$, the surface $\tilde{F}_\Lambda$ is strictly convex at $(s, t)$.

- Suppose that $\delta < \delta_1$, $\frac{L}{1000} \geq \Lambda \geq \Lambda_1$, and $\varepsilon \leq E(\delta, \Lambda)$. Then, for each point $(s, t) \in (2\Lambda^{\frac{1}{2}}, \Lambda] \times [0, 1]$, the surface $\tilde{F}_\Lambda$ is strictly convex at $(s, t)$.

**Proof.** We begin with the first statement. Let $\tilde{\lambda}_1$ denote the smallest curvature eigenvalue of the bended surface $\tilde{F}_\Lambda$ and let $\lambda_1$ be the smallest curvature eigenvalue of the original neck. Similarly, we denote by $\tilde{H}$ the mean curvature of the bended surface and by $H$ the mean curvature of the original neck. By choosing $\Lambda$ sufficiently large, we can arrange that the function $u(s) = e^{-\frac{4\Lambda}{s}}$ satisfies

\[
u''(s) \geq \frac{1}{C_0} (|u(s)| + |u'(s)|)
\]
for all $s \in (0, \Lambda^{\frac{1}{4}}]$, where $c_0$ is the constant from Lemma 5.2. Using Lemma 5.2, we conclude that $\hat{\lambda}_1 \geq \lambda_1$ and $\hat{H} \geq H$ for all points $(s, t) \in (0, \Lambda^{\frac{1}{4}}] \times [0, 1]$. This proves the first statement.

To verify the second statement, we consider a point $(s, t) \in (\Lambda^{\frac{1}{4}}, 2 \Lambda^{\frac{1}{4}}] \times [0, 1]$. It is easy to see that $\hat{h}_{tt} \geq \frac{1}{2}$ and $\langle (\gamma(t), 0), \nu(s, t) \rangle \geq \frac{1}{2}$. We next compute

$$\frac{\partial^2}{\partial s^2} \tilde{F}_\Lambda(s, t) = -\left(\frac{16\Lambda^2}{s^4} - \frac{8\Lambda}{s^3}\right) e^{-\frac{4\Lambda}{s}} (\gamma(t), 0) + O(\varepsilon)$$

and

$$\frac{\partial^2}{\partial s \partial t} \tilde{F}_\Lambda(s, t) = -\frac{4\Lambda}{s^2} e^{-\frac{4\Lambda}{s}} \left(\chi(s/\Lambda^{\frac{1}{4}}) \frac{\partial}{\partial t} \gamma_s(t) + (1 - \chi(s/\Lambda^{\frac{1}{4}})) \frac{\partial}{\partial t} \gamma(t), 0 \right) + O(\varepsilon)$$

$$= -\frac{4\Lambda}{s^2} e^{-\frac{4\Lambda}{s}} \frac{\partial \tilde{F}_\Lambda}{\partial t}(s, t) + O(\varepsilon)$$

for $(s, t) \in (\Lambda^{\frac{1}{4}}, 2 \Lambda^{\frac{1}{4}}] \times [0, 1]$. From this, we deduce that

$$\hat{h}_{ss} = -\left\langle \frac{\partial^2}{\partial s^2} \tilde{F}_\Lambda(s, t), \nu(s, t) \right\rangle$$

$$= \left(\frac{16\Lambda^2}{s^4} - \frac{8\Lambda}{s^3}\right) e^{-\frac{4\Lambda}{s}} \langle (\gamma(t), 0), \nu(s, t) \rangle + O(\varepsilon)$$

$$\geq \left(\frac{8\Lambda^2}{s^4} - \frac{4\Lambda}{s^3}\right) e^{-\frac{4\Lambda}{s}} + O(\varepsilon)$$

and

$$\hat{h}_{st} = -\left\langle \frac{\partial^2}{\partial s \partial t} \tilde{F}_\Lambda(s, t), \nu(s, t) \right\rangle = O(\varepsilon)$$

for $(s, t) \in (\Lambda^{\frac{1}{4}}, 2 \Lambda^{\frac{1}{4}}] \times [0, 1]$. Hence, if $\varepsilon$ is small enough (depending on $\Lambda$), then $\hat{h}_{ss} \hat{h}_{tt} - \hat{h}_{st}^2 > 0$, and the surface $\tilde{F}_\Lambda$ is strictly convex at $(s, t)$. This completes the proof of the second statement.

To prove the third statement, we consider a point $(s, t) \in (2 \Lambda^{\frac{1}{4}}, \Lambda] \times [0, 1]$. We clearly have $\hat{h}_{tt} \geq \frac{1}{2}$ and $\langle (\gamma_s/\Lambda(t), 0), \nu(s, t) \rangle \geq \frac{1}{2}$. We next compute

$$\frac{\partial^2}{\partial s^2} \tilde{F}_\Lambda(s, t) = -\left(\frac{16\Lambda^2}{s^4} - \frac{8\Lambda}{s^3}\right) e^{-\frac{4\Lambda}{s}} \gamma_s/\Lambda(t), 0 \rangle + O(\Lambda^{-2} \omega(\hat{\delta}) 1_{\{\frac{3}{4} \leq s \leq \frac{5}{4}\}})$$

and

$$\frac{\partial^2}{\partial s \partial t} \tilde{F}_\Lambda(s, t) = -\frac{4\Lambda}{s^2} e^{-\frac{4\Lambda}{s}} \left(\frac{\partial}{\partial \gamma_s/\Lambda(t), 0 \rangle + O(\Lambda^{-1} \omega(\hat{\delta}) 1_{\{\frac{3}{4} \leq s \leq \frac{5}{4}\}})\right)$$

$$= -\frac{4\Lambda}{s^2} e^{-\frac{4\Lambda}{s}} \frac{\partial \tilde{F}_\Lambda}{\partial t}(s, t) + O(\Lambda^{-1} \omega(\hat{\delta}) 1_{\{\frac{3}{4} \leq s \leq \frac{5}{4}\}})$$
for \((s,t) \in (2\Lambda^{\frac{1}{4}}, \Lambda] \times [0,1]\). From this, we deduce that

\[
\hat{h}_{ss} = -\left(\frac{\partial^2}{\partial s^2} \hat{F}_\Lambda(s,t), \hat{v}(s,t)\right) = \left(\frac{16s^2}{s^4} - \frac{8s^3}{s^3}\right)e^{-\frac{4s}{s^2}}((\hat{r}_s/\Lambda(t), 0), \hat{v}(s,t)) + O(\Lambda^{-2}\omega(\hat{\delta})1_{\left[\frac{4}{4\Lambda}, \frac{4}{4\Lambda}\right]}(s,t))
\]

\[
\geq \left(\frac{8s^2}{s^3} - \frac{4s^3}{s^3}\right)e^{-\frac{4s}{s^2}} + O(\Lambda^{-2}\omega(\hat{\delta})1_{\left[\frac{4}{4\Lambda}, \frac{4}{4\Lambda}\right]}(s,t))
\]

and

\[
\hat{h}_{st} = -\left(\frac{\partial^2}{\partial s \partial t} \hat{F}_\Lambda(s,t), \hat{v}(s,t)\right) = O(\Lambda^{-1}\omega(\hat{\delta})1_{\left[\frac{4}{4\Lambda}, \frac{4}{4\Lambda}\right]}(s,t))
\]

for \((s,t) \in (2\Lambda^{\frac{1}{4}}, \Lambda] \times [0,1]\). Hence, if \(\hat{\delta}\) is sufficiently small, then \(\hat{h}_{ss} \hat{h}_{tt} - \hat{h}_{st}^2 > 0\), and the surface \(\hat{F}_\Lambda\) is strictly convex at \((s,t)\). This completes the proof of Lemma 5.3.

Since the surface \(\hat{F}_\Lambda\) is axially symmetric in the region \(\{\frac{4}{4\Lambda} \leq s \leq \Lambda\}\), we want to glue \(\hat{F}_\Lambda\) to a scaled copy of the axially symmetric cap constructed in Lemma 5.1. Let us briefly sketch how this can be done. Let us fix a smooth, convex, even function \(\Phi : \mathbb{R} \rightarrow \mathbb{R}\) such that \(\Phi(z) = |z|\) for \(|z| \geq \frac{1}{100}\). For \(\Lambda\) very large, we define

\[
a = 1 - e^{-4} + \frac{1}{3}(1 - e^{-4})^2\Lambda^{-\frac{1}{4}}
\]

and

\[
v_\Lambda(s) = 1 - e^{-\frac{4s}{s^2}} + a\sqrt{\frac{\Lambda + 2\Lambda^{\frac{1}{4}} - s}{a + \Lambda + 2\Lambda^{\frac{1}{4}} - s}}
\]

\[
- \Lambda^{-\frac{1}{4}}\Phi\left(\Lambda^{\frac{1}{4}}\left(1 - e^{-\frac{4s}{s^2}} - a\sqrt{\frac{\Lambda + 2\Lambda^{\frac{1}{4}} - s}{a + \Lambda + 2\Lambda^{\frac{1}{4}} - s}}\right)\right)
\]

for \(s \in [\Lambda, \Lambda + \Lambda^{\frac{1}{4}}]\). Since the functions \(s \mapsto 1 - e^{-\frac{4s}{s^2}}\) and \(s \mapsto a\sqrt{\frac{\Lambda + 2\Lambda^{\frac{1}{4}} - s}{a + \Lambda + 2\Lambda^{\frac{1}{4}} - s}}\) are concave, the function \(v_\Lambda\) is concave as well. Moreover, we have \(v_\Lambda(s) = 2(1 - e^{-\frac{4s}{s^2}})\) in a neighborhood of the point \(s = \Lambda\), and \(v_\Lambda(s) = 2a\sqrt{\frac{\Lambda + 2\Lambda^{\frac{1}{4}} - s}{a + \Lambda + 2\Lambda^{\frac{1}{4}} - s}}\) in a neighborhood of the point \(s = \Lambda + \Lambda^{\frac{1}{4}}\). We now extend \(\hat{F}_\Lambda\) to the region \(\left[-(L - 1), \Lambda + 2\Lambda^{\frac{1}{4}}\right] \times [0,1]\) by putting

\[
\hat{F}_\Lambda(s,t) = \left(\frac{1}{2}v_\Lambda(s)\cos(2\pi t), \frac{1}{2}v_\Lambda(s)\sin(2\pi t), s\right)
\]

for \(s \in (\Lambda, \Lambda + \Lambda^{\frac{1}{4}}]\) and

\[
\hat{F}_\Lambda(s,t) = \left(a\sqrt{\frac{\Lambda + 2\Lambda^{\frac{1}{4}} - s}{a + \Lambda + 2\Lambda^{\frac{1}{4}} - s}}\cos(2\pi t), a\sqrt{\frac{\Lambda + 2\Lambda^{\frac{1}{4}} - s}{a + \Lambda + 2\Lambda^{\frac{1}{4}} - s}}\sin(2\pi t), s\right)
\]
for \( s \in (\Lambda + \Lambda^{\frac{1}{2}}, \Lambda + 2\Lambda^{\frac{1}{2}}) \). It is straightforward to verify that the resulting surface is smooth and satisfies the curvature bounds \( \frac{1}{10} \leq H \leq 10 \).

**Lemma 5.4.** The surface \( \tilde{F}_\Lambda \) is convex in the region \( (\Lambda, \Lambda + 2\Lambda^{\frac{1}{2}}) \times [0, 1] \).

**Proof.** Since the function \( u_\Lambda \) is concave on the interval \([\Lambda, \Lambda + \Lambda^{\frac{1}{2}}]\), we conclude that \( \tilde{F}_\Lambda \) is convex for \((s, t) \in (\Lambda, \Lambda + \Lambda^{\frac{1}{2}}) \times [0, 1] \). Moreover, it follows from Lemma 5.1 that \( \tilde{F}_\Lambda \) is convex for \((s, t) \in (\Lambda + \Lambda^{\frac{1}{2}}, \Lambda + 2\Lambda^{\frac{1}{2}}) \).

In the next step, we show that in the surgically modified region the inscribed radius is at least \( \frac{1}{(1+\delta)H} \) and the outer radius is at least \( \frac{\alpha}{H} \).

**Proposition 5.5.** Given any number \( \hat{\alpha} > \alpha \), we can find a number \( \delta_2 \) with the following property. Suppose that we are given a pair of real numbers \( \delta \) and \( \hat{\alpha} \) such that \( \delta < \delta < \delta_2 \). Then there exist real numbers \( \bar{\epsilon} \) and \( \Lambda_0 \) such that the surface \( \tilde{F}_\Lambda \) is \( \frac{1}{1+\delta} \)-noncollapsed whenever \( \bar{\epsilon} \leq \bar{\epsilon} \) and \( \frac{L}{1000} \geq \Lambda \geq \Lambda_2 \). Furthermore, if \( \bar{\epsilon} \leq \bar{\epsilon} \) and \( \frac{L}{1000} \geq \Lambda \geq \Lambda_2 \), then the outer radius is at least \( \frac{\alpha}{H} \) at each point on \( \tilde{F}_\Lambda \).

**Proof.** We first establish the bound for the inscribed radius. It follows from Lemma 5.1 that the inscribed radius of \( \tilde{F}_\Lambda \) is at least \( \frac{1}{(1+\delta)H} \) at each point in the region \((\Lambda + \Lambda^{\frac{1}{2}}, \Lambda + 2\Lambda^{\frac{1}{2}}) \). Consider now a point \((s_0, t_0) \in [-L, 0, \Lambda + \Lambda^{\frac{1}{2}}] \times [0, 1] \). If \( \Lambda \) is large, then we can approximate the map \( \tilde{F}_\Lambda \) near the point \((s_0, t_0) \) by a cylinder whose cross-section is a simple closed, convex curve. Furthermore, the noncollapsing constant of the cross section is \( \frac{1}{1+\delta} - O(\epsilon) \) or better. Since \( \delta < \delta \), we conclude that the surface \( \tilde{F}_\Lambda \) is \( \frac{1}{1+\delta} \)-noncollapsed if \( \Lambda \) is sufficiently large and \( \epsilon \) is sufficiently small.

It remains to prove the bound for the outer radius. Let \( \tilde{N} \) denote the image of the map \( \tilde{F}_\Lambda : (-L, \Lambda + 2\Lambda^{\frac{1}{2}}] \times [0, 1] \rightarrow \mathbb{R}^3 \). By assumption, the region \( \{x + a\nu(x) : x \in N, a \in (0, 2\hat{\alpha})\} \) is disjoint from \( M \setminus N \). Since the surface \( \tilde{N} \setminus N \) lies inside the original neck \( N \), it follows that the region \( \{x + a\nu(x) : x \in \tilde{N} \setminus N, a \in (0, 2\hat{\alpha})\} \) is disjoint from \( M \setminus N \). Consequently, for each point \( x \in \tilde{N} \setminus N \), we can find a ball of radius \( \hat{\alpha} \) which touches \( \tilde{N} \) at the point \( x \) from the outside, and which is disjoint from \( M \setminus N \). On the other hand, if \( \delta \) and \( \epsilon \) are sufficiently small and \( \Lambda \) is sufficiently large, then the mean curvature of the surface \( \tilde{N} \setminus N \) is greater than \( \frac{2}{\alpha} \) everywhere. Putting these facts together, we conclude that the outer radius is at least \( \hat{\alpha} > \frac{\alpha}{H} \) at each point in \( \tilde{N} \setminus N \). This completes the proof of Proposition 5.5.

We note that our surgery procedure always produces an embedded surface. Finally, it is clear from the construction that the resulting cap is at least of class \( C^0 \) with uniform bounds independent of the surgery parameters \( \hat{\alpha}, \delta, \epsilon, L, \) and \( \Lambda \).
6. Proof of Proposition 2.7

By assumption, the point $x_0$ lies in the surgically modified region of $M_{t_0+}$. Hence, the surface $M_{t_0-}$ must have contained an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck of size $r \in \left[\frac{1}{2K_\ast}, \frac{2}{K_\ast}\right]$. Let us denote this neck by $N$. At time $t_0$ the neck $N$ is replaced by a capped-off neck $\tilde{N}$. More precisely, suppose that the original neck $N$ satisfies

$$\{ (\gamma_s(t), s) : s \in [-L-1, L-1], t \in [0,1] \} \subset r^{-1} N.$$ 

Then the surface $\tilde{N}$ satisfies

$$N \cap \{ x \in \mathbb{R}^3 : \langle x, e_3 \rangle \leq 0 \} \subset \tilde{N}$$

and

$$\tilde{N} \subset \{ x \in \mathbb{R}^3 : \langle x, e_3 \rangle \leq 4\Lambda K_\ast^{-1} \}.$$ 

Since the point $x_0$ lies in the surgically modified part of $\tilde{N}$, we have $\langle x_0, e_3 \rangle \geq 0$.

By assumption, the outer radius of $M_{t_0+}$ is at least $\frac{1}{\Lambda}$ everywhere. Moreover, we have $H \leq 100 K_\ast$ at each point on $\tilde{N}$. Therefore, for each point $x \in \tilde{N}$, the outer radius is at least $\frac{\alpha}{100} K_\ast^{-1}$. Hence, if we denote by $\nu$ the outward-pointing unit normal vector field to $\tilde{N}$, then the set

$$E = \{ x + a \nu(x) : x \in \tilde{N}, a \in (0, \frac{\alpha}{100} K_\ast^{-1}) \}$$

is disjoint from the region enclosed by $M_{t_0+}$. Since the region enclosed by $M_t$ shrinks as $t$ increases, we conclude that $E$ is disjoint from the region enclosed by $M_{t_1-}$.

We next define a compact region $\Omega \subset \{ x \in \mathbb{R}^3 : \langle x, e_3 \rangle \geq -\frac{L}{4} K_\ast^{-1} \}$ by the condition

$$\partial \Omega \subset \tilde{N} \cup \{ x \in \mathbb{R}^3 : \langle x, e_3 \rangle = -\frac{L}{4} K_\ast^{-1} \}.$$ 

We now consider a time $t_1 > t_0$ and a point $x_1 \in M_{t_1+}$ such that $|x_1 - x_0| \leq \frac{\alpha}{1000} K_\ast^{-1}$. Clearly, $x_1 \notin E$. Since $x_0 \in \tilde{N}$ and $|x_1 - x_0| \leq \frac{\alpha}{1000} K_\ast^{-1}$, it follows that $x_1 \in \Omega$.

We claim that $H(x_1, t_1+) \geq \frac{\alpha}{100} K_\ast$. To prove this, we argue by contradiction. Suppose that $H(x_1, t_1+) \leq \frac{\alpha}{100} K_\ast$. Since the surface $M_{t_1+}$ is $\alpha$-noncollapsed, we can find a ball $B \subset \mathbb{R}^3$ of radius $10 K_\ast^{-1}$ such that $x_1 \in \partial B$ and $B$ is contained in the region enclosed by $M_{t_1+}$. This implies $B \cap E = \emptyset$. Since $x_1 \in \Omega$, it follows that $B \subset \Omega$, which is impossible. Thus, we conclude that $H(x_1, t_1+) \geq \frac{\alpha}{100} K_\ast$.

It remains to show that $x_1$ does not lie in the surgically modified region at time $t_1$. To prove this, we again argue by contradiction. Suppose that $t_1$ is a surgery time and $x_1$ lies in the surgically modified region of $M_{t_1+}$. This region is the result of a surgery that was performed on an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck in $M_{t_1-}$. This neck has length at least $\frac{L}{2} K_\ast^{-1} \geq 50 \Lambda K_\ast^{-1}$. Consequently, we can find two points $y, z \in \mathbb{R}^3$ such that $|y-z| = 40 \Lambda K_\ast^{-1}$, $|\frac{y+z}{2} - x_1| \leq \frac{1}{40}$. Since $x_1$ lies in the surgically modified region of $M_{t_1+}$, the outward-pointing unit normal vector field $\nu$ satisfies $\langle x_1, \nu(x_1) \rangle \geq \frac{\alpha}{100} K_\ast$. Therefore, the region enclosed by $M_{t_1+}$ is disjoint from $E$.

Since the region enclosed by $M_{t_1+}$ is disjoint from $E$, the region enclosed by $M_{t_1-}$ is also disjoint from $E$. This implies that $x_1 \notin \tilde{N}$, which is impossible. Thus, we conclude that $x_1$ does not lie in the surgically modified region at time $t_1$. This completes the proof of Proposition 2.7.
Suppose that \( t \) is contained in the region enclosed by \( M_t \). By Proposition 2.7, the flow times \( t > t \) of a function with \( \alpha \) not on the exact choice of the surgery parameters \( \hat{\delta} \) for all \( t \), the assertion follows.

and confusion (and without any loss of generality), we will assume that \( |x| \) is contained in \( \Omega \). But this is impossible since \( \Omega \subset \{ x \in \mathbb{R}^3 : \langle x, e_3 \rangle \leq 4\Lambda K_*^{-1} \} \) and \( \langle \frac{y}{2}, e_3 \rangle \geq -\frac{\alpha}{1000} K_*^{-1} \). This completes the proof of Proposition 2.7.

7. Proof of Proposition 2.8

Let \( \beta_0 \) be chosen as in Theorem 2.2. Let us fix a positive number \( \beta_1 \in (0, \frac{\alpha}{8000}) \) such that the following holds: Suppose that \( \delta \) is an \( (\hat{\delta}, \delta, \varepsilon, L) \)-neck of size \( r \in [\frac{1}{100r}, \frac{1}{5r}] \). Moreover, let \( \bar{N} \) denote the capped-off neck obtained by performing a \( \Lambda \)-surgery on \( N \). Then, for each point \( x_0 \) in the region enclosed by \( \tilde{M} \), the dilated surface \( (\beta_1^{-1} K_*(\bar{N} - x_0)) \cap \tilde{B}_4(0) \) can be expressed as the graph of function which has \( C^4 \)-norm less than \( \beta_0 \). In view of the construction of the cap in Section 5, we can choose the constant \( \beta_1 \) such a way that \( \beta_1 \) depends only on the noncollapsing constant \( \alpha \), but not on the exact choice of the surgery parameters \( \hat{\alpha}, \hat{\delta}, \varepsilon, L, \) and \( K_* \).

After these preparations, we now complete the proof of Proposition 2.8. Suppose that \( t_0 \) is a surgery time and \( x_0 \) lies in the surgically modified region. By Proposition 2.7 the flow \( M_t \cap B_{4\beta_1 K_*^{-1}}(x_0) \) is smooth for all times \( t > t_0 \). Moreover, the surface \( (\beta_1^{-1} K_*(M_{t_0} + x_0)) \cap B_4(0) \) is a graph of a function with \( C^4 \)-norm less than \( \beta_0 \). Hence, Theorem 2.2 implies that

\[
\beta_1 K_*^{-1} |A| + \beta_1^2 K_*^{-2} |\nabla A| + \beta_1^3 K_*^{-3} |\nabla^2 A| \leq C
\]

for all \( t \in (t_0, t_0 + \beta_0 \beta_1^{-1} K_*^{-2}] \) and all \( x \in M_t \cap B_{\beta_1 K_*^{-1}}(x_0) \). From this, the assertion follows.

8. Proof of Proposition 2.9

Let us consider an arbitrary time \( t_1 \geq (1000 \sup_{M_0} |A|)^{-2} \) and an arbitrary point \( x_1 \in M_{t_1} \) for which we want to verify the estimate. To avoid confusion (and without any loss of generality), we will assume that \( t_1 \) is not a surgery time. There are two cases:

Case 1: There exists a surgery time \( t_0 \) and a point \( x_0 \) such that \( |x_1 - x_0| \leq \beta_* K_*^{-1}, 0 < t_1 - t_0 \leq \beta_* K_*^{-2}, \) and \( x_0 \) lies in the surgically modified region at time \( t_0 + \). Applying Proposition 2.8 we conclude that

\[
K_*^{-1} |A| + K_*^{-2} |\nabla A| + K_*^{-3} |\nabla^2 A| \leq C_*
\]

at the point \( (x_1, t_1) \). Moreover, by Proposition 2.1 we have \( H \geq \frac{\alpha}{10} K_* \) at the point \( (x_1, t_1) \). Putting these facts together, we obtain

\[
|\nabla A| \leq \frac{100 C_*}{\alpha^2} H^2
\]
and

$$|\nabla^2 A| \leq \frac{1000 C_*}{\alpha^3} H^3$$

at the point \((x_1, t_1)\). Hence, if we choose \(C_* \geq \frac{1000 C_*}{\alpha^3}\), then the desired estimate holds in this case.

**Case 2:** There does not exist a surgery time \(t_0\) and a point \(x_0\) such that \(|x_1 - x_0| \leq \beta_* K_*^{-1}\), \(0 < t_1 - t_0 \leq \beta_* K_*^{-2}\), and \(x_0\) lies in the surgically modified region at time \(t_0+\). In this case, the surfaces \(M_t \cap B_{\beta_* K_*^{-1}}(x_1)\), \(t \in (t_1 - \beta_* K_*^{-2}, t_1]\), form a regular mean curvature flow in the sense of Definition 2.1. Note that, since \(t_1 \geq (1000 \sup_{M_0} |A|)^{-2}\), we have \(t_1 - \beta_* K_*^{-2} > 0\). Moreover, the ball \(B_{\beta_* K_*^{-1}}(x_1)\) is contained in the region enclosed by \(M_0\), so the surfaces \(M_t\) are outward-minimizing within the ball \(B_{\beta_* K_*^{-1}}(x_1)\). Hence, Theorem 2.3 implies that \(|\nabla A| \leq B (H + K)^2\) and \(|\nabla^2 A| \leq B (H + K)^3\) at the point \((x_1, t_1)\). Here, \(B\) is a positive constant that depends only on \(\beta_*\) and the noncollapsing constant \(\alpha\). This completes the proof.

#### 9. Proof of Proposition 2.10

We next prove some auxiliary results about curves. In the following, we assume that \(C_*\) is the constant in Proposition 2.9.

**Lemma 9.1.** Let \(\Gamma\) be a (possibly non-closed) embedded curve in the plane of class \(C^3\) with geodesic curvature \(\kappa > 0\). Moreover, suppose that the inscribed radius is at least \(\frac{1}{\kappa}\) at each point on \(\Gamma\). Then \(\kappa\) is constant.

**Proof.** The assumption implies that the function

$$Z(s, t) := \frac{1}{2} \kappa(s) |\gamma(s) - \gamma(t)|^2 - \langle \gamma(s) - \gamma(t), \nu(s) \rangle$$

is nonnegative for all \(s, t\). A straightforward calculation gives

$$\left. \frac{\partial Z}{\partial t} (s, t) \right|_{s=t} = 0, \quad \left. \frac{\partial^2 Z}{\partial t^2} (s, t) \right|_{s=t} = 0, \quad \left. \frac{\partial^3 Z}{\partial t^3} (s, t) \right|_{s=t} = -\frac{d\kappa}{ds}(s).$$

Since \(Z\) is nonnegative everywhere, we conclude that \(\frac{d\kappa}{ds}(s) = 0\) at each point on \(\Gamma\).

**Lemma 9.2.** Let \(\Gamma_j\) be a sequence of (possibly non-closed) embedded curves in the plane with the property that \(\kappa > 0\), \(|\frac{d\kappa}{ds}| \leq C_* (\kappa + 2\Theta)^2\), and \(\frac{d^2 \kappa}{ds^2} \leq C_* (\kappa + 2\Theta)^3\). Moreover, suppose that the inscribed radius is at least \(\frac{1}{(1+\frac{1}{2})\kappa}\) at each point on \(\Gamma_j\). Finally, we assume that \(L(\Gamma_j) \leq 4\pi\) and \(\kappa(p_j) = 1\) for some point \(p_j \in \Gamma_j\). Then \(\sup_{\Gamma_j} \kappa \to 1\) as \(j \to \infty\).

**Proof.** Suppose that there exists a real number \(a > 0\) such that \(\sup_{\Gamma_j} \kappa \geq 1 + 2a\) for \(j\) large. We can find a segment \(\tilde{\Gamma}_j \subset \Gamma_j\) such that the geodesic curvature increases from \(1+a\) to \(1+2a\) along \(\tilde{\Gamma}_j\). Using our assumptions, we obtain \(\lim_{j \to \infty} \sup_{\tilde{\Gamma}_j} \frac{|d\kappa|}{ds} < \infty\) and \(\lim_{j \to \infty} \sup_{\tilde{\Gamma}_j} \frac{|d^2 \kappa|}{ds^2} < \infty\). Since \(\kappa\) varies between \(1+a\) and \(1+2a\) along \(\tilde{\Gamma}_j\), we must have \(\liminf_{j \to \infty} L(\tilde{\Gamma}_j) > 0\).
On the other hand, we have \( L(\hat{\Gamma}_j) \leq L(\Gamma_j) \leq 4\pi \). Hence, after passing to a subsequence, the curves \( \hat{\Gamma}_j \) converge in \( C^3 \) to a curve \( \hat{\Gamma} \). The geodesic curvature of the limiting curve \( \hat{\Gamma} \) increases from \( 1 + a \) and \( 1 + 2a \) as we travel along the curve \( \hat{\Gamma} \). Finally, at each point on \( \hat{\Gamma} \), the inscribed radius is at least \( \frac{1}{\hat{\kappa}} \), where \( \hat{\kappa} \) denotes the geodesic curvature of \( \hat{\Gamma} \). By Lemma 9.1, \( \hat{\kappa} \) is constant. This contradicts the fact that \( \hat{\kappa} \) varies between \( 1 + a \) and \( 1 + 2a \).

**Lemma 9.3.** Let \( \Gamma_j \) be a sequence of (possibly non-closed) embedded curves in the plane with the property that \( \kappa > 0 \), \( |\frac{d\kappa}{ds}| \leq C_\# (\kappa + 2\Theta)^2 \), and \( |\frac{d^2\kappa}{ds^2}| \leq C_\# (\kappa + 2\Theta)^3 \). Moreover, suppose that the inscribed radius is at least \( \frac{1}{(1 + \frac{1}{a})\kappa} \) at each point on \( \Gamma_j \). Finally, we assume that \( L(\Gamma_j) \leq 4\pi \) and \( \kappa(p_j) = 1 \) for some point \( p_j \in \Gamma_j \). Then \( \inf_{\Gamma_j} \kappa \to 1 \) as \( j \to \infty \).

**Proof.** Suppose that there exists a real number \( a > 0 \) such that \( \sup_{\Gamma_j} \kappa \leq 1 - 2a \) for \( j \) large. We can find a segment \( \hat{\Gamma}_j \subset \Gamma_j \) such that the geodesic curvature increases from \( 1 - a \) to \( 1 - 2a \) along \( \hat{\Gamma}_j \). Using our assumptions, we obtain \( \limsup_{j \to \infty} \sup_{\Gamma_j} |\frac{d\kappa}{ds}| < \infty \) and \( \limsup_{j \to \infty} \sup_{\Gamma_j} |\frac{d^2\kappa}{ds^2}| < \infty \). Since \( \kappa \) varies between \( 1 - a \) and \( 1 - 2a \) along \( \hat{\Gamma}_j \), we must have \( \liminf_{j \to \infty} L(\hat{\Gamma}_j) > 0 \). On the other hand, we have \( L(\hat{\Gamma}_j) \leq L(\Gamma_j) \leq 4\pi \). Hence, after passing to a subsequence, the curves \( \hat{\Gamma}_j \) converge in \( C^3 \) to a curve \( \hat{\Gamma} \). The geodesic curvature of the limiting curve \( \hat{\Gamma} \) increases from \( 1 + a \) and \( 1 + 2a \) as we travel along the curve \( \hat{\Gamma} \). Finally, at each point on \( \hat{\Gamma} \), the inscribed radius is at least \( \frac{1}{\hat{\kappa}} \), where \( \hat{\kappa} \) denotes the geodesic curvature of \( \hat{\Gamma} \). By Lemma 9.1, \( \hat{\kappa} \) is constant. This contradicts the fact that \( \hat{\kappa} \) varies between \( 1 + a \) and \( 1 - 2a \).

**Proposition 9.4.** Let \( \Gamma_j \) be a sequence of (possibly non-closed) embedded curves in the plane with the property that \( \kappa > 0 \), \( |\frac{d\kappa}{ds}| \leq C_\# (\kappa + 2\Theta)^2 \), and \( |\frac{d^2\kappa}{ds^2}| \leq C_\# (\kappa + 2\Theta)^3 \). Moreover, suppose that the inscribed radius is at least \( \frac{1}{(1 + \frac{1}{a})\kappa} \) at each point on \( \Gamma_j \), and the outer radius is at least \( \frac{2}{\kappa} \) at each point on \( \Gamma_j \). Finally, we assume that \( \kappa(p_j) = 1 \) for some point \( p_j \in \Gamma_j \). Then \( L(\Gamma_j) \leq 3\pi \) for \( j \) large, and we have \( \lim_{j \to \infty} \sup_{\Gamma_j} |\kappa - 1| = 0 \).

**Proof.** Suppose by contradiction that \( L(\Gamma_j) \geq 3\pi \) for all \( j \). By shortening \( \Gamma_j \) if necessary, we can arrange that \( L(\Gamma_j) = 3\pi \) for all \( j \). Let \( \gamma_j : [0, 3\pi] \to \mathbb{R}^2 \) be a parametrization of \( \Gamma_j \) by arclength. It follows from Lemma 9.2 and Lemma 9.3 that the geodesic curvature of \( \Gamma_j \) is close to 1 when \( j \) is sufficiently large. This implies that \( \gamma_j(2\pi) - \gamma_j(0) \to 0 \) as \( j \to \infty \). Let us pick a sequence of numbers \( s_j \in [0, 3\pi] \) such that \( s_j \to 2\pi \) as \( j \to \infty \) and the function \( s \mapsto |\gamma_j(s) - \gamma_j(0)|^2 \) has a local minimum at \( s_j \). Then the vector \( \gamma_j(s_j) - \gamma_j(0) \) is parallel to \( \nu_j(s_j) \). Consequently, we have

\[
|\langle \gamma_j(s_j) - \gamma_j(0), \nu_j(s_j) \rangle| = |\gamma_j(s_j) - \gamma_j(0)|.
\]
On the other hand, we know that the inscribed radius and the outer radius of $\Gamma_j$ are at least $\frac{\alpha}{\kappa}$. This implies
\[
\frac{1}{2} \kappa_j(s_j) |\gamma_j(s_j) - \gamma_j(0)|^2 \geq \alpha |\langle \gamma_j(s_j) - \gamma_j(0), \nu_j(s_j) \rangle |.
\]
Putting these facts together, we obtain
\[
\frac{1}{2} \kappa_j(s_j) |\gamma_j(s_j) - \gamma_j(0)| \geq \alpha.
\]
But $\kappa_j(s_j) \to 1$ and $|\gamma_j(s_j) - \gamma_j(0)| \to 0$ as $j \to \infty$, so we arrive at a contradiction. Consequently, we must have $L(\Gamma_j) \leq 3\pi$ when $j$ is sufficiently large. Using Lemma 9.2 and Lemma 9.3, we obtain $\lim_{j \to \infty} \sup_{\Gamma_j} \kappa = 1$ and $\lim_{j \to \infty} \inf_{\Gamma_j} \kappa = 1$. This completes the proof.

**Corollary 9.5.** We can find a number $\delta > 0$ with the following property: Suppose that $\Gamma$ is a (possibly non-closed) embedded curve in the plane with the property that $\kappa > 0$, $|d\kappa|/|ds| \leq C \#(\kappa + 2\Theta)^2$, and $|d^2\kappa|/|ds|^2 \leq C \#(\kappa + 2\Theta)^3$. Moreover, suppose that the inscribed radius is at least $\frac{\alpha}{\kappa}$ at each point on $\Gamma$, and the outer radius is at least $\frac{\alpha}{\kappa}$ at each point on $\Gamma$. Finally, we assume that $\kappa = 1$ at some point $p \in \Gamma$. Then $L(\Gamma) \leq 3\pi$ and $\sup_{\Gamma} |\kappa - 1| \leq \frac{1}{100}$.

Note that the constant $\delta$ will depend only on the constants $\alpha$ and $C \#$, which have already been chosen.

In the following, we define $\theta_0 = 10^{-6} \min\{\alpha, \frac{1}{C \# \Theta} \}$.

**Proposition 9.6.** We can choose $\delta$ small enough so that the following holds: Consider a family of simple closed, convex curves $\Gamma_t$, $t \in (-2\theta_0, 0]$, in the plane which evolve by curve shortening flow. Assume that, for each $t \in (-2\theta_0, 0]$, the curve $\Gamma_t$ satisfies the derivative estimates $|d\kappa|/|ds| \leq C \#(\kappa + 2\Theta)^2$ and $|d^2\kappa|/|ds|^2 \leq C \#(\kappa + 2\Theta)^3$. Moreover, we assume that the inscribed radius is at least $\frac{1}{(1+\delta)\kappa}$ at each point on $\Gamma_t$, and the outer radius is at least $\frac{\alpha}{\kappa}$ at each point on $\Gamma_t$. Finally, we assume that the geodesic curvature of $\Gamma_0$ is equal to 1 somewhere. Then the curve $\Gamma_t$ satisfies $\sum_{l=1}^{18} |\nabla^l \kappa| \leq \frac{1}{1000}$. Moreover, we have $\sup_{\Gamma_t} \kappa \leq 1 - \frac{\theta_0}{4}$.

**Proof.** Suppose that the assertion is false, and consider a sequence of counterexamples. These counterexamples converge to a smooth solution of the curve shortening flow which is defined for $t \in (-2\theta_0, 0]$. The limiting solution is a family of homothetically shrinking circles. This gives a contradiction.

Proposition 2.10 follows by combining Corollary 9.4 and Proposition 9.6.

10. **Proof of Proposition 2.11**

We again argue by contradiction. Let us fix $\theta_0$ and $\delta$ as above, and suppose that there is no real number $\tilde{\delta} \in (0, \delta)$ for which the conclusion of Proposition
holds. By taking a sequence of counterexamples and passing to the limit, we obtain a smooth solution \( \Gamma_t, \ t \in (-2\theta_0, 0] \), to the curve shortening flow with the property that \( \sup_{T_t} \frac{\mu}{\kappa} \leq 1 + \delta \) for each \( t \in (-2\theta_0, 0] \) and \( \sup_{T_0} \frac{\mu}{\kappa} = 1 + \delta \). (As usual, \( \mu \) denotes the reciprocal of the inscribed radius and \( \kappa \) denotes the geodesic curvature.) The geodesic curvature satisfies the evolution equation

\[
\frac{\partial}{\partial t} \kappa = \Delta \kappa + \kappa^3.
\]

Moreover, \( \mu \) satisfies the inequality

\[
\frac{\partial}{\partial t} \mu \leq \Delta \mu + \kappa^2 \mu - \frac{2}{\mu - \kappa} |\nabla \mu|^2
\]

on the set \( \{ \mu > \kappa \} \), where \( \Delta \mu \) is interpreted in the sense of distributions.

In particular, the function \( (1 + \delta) \kappa - \mu \) is nonnegative and satisfies the inequality

\[
\frac{\partial}{\partial t} ((1 + \delta) \kappa - \mu) \geq \Delta ((1 + \delta) \kappa - \mu) + \kappa^2 ((1 + \delta) \kappa - \mu) + \frac{2}{\mu - \kappa} |\nabla \mu|^2
\]

on the set \( \{ \mu > \kappa \} \). Since \( \inf_{T_0} ((1 + \delta) \kappa - \mu) = 0 \), the function \( (1 + \delta) \kappa - \mu \) vanishes identically by the strict maximum principle. This implies \( \nabla \mu = 0 \), hence \( \nabla \kappa = 0 \). Therefore, our solution is a family of shrinking circles. In that case, we have \( \mu = \kappa \), which contradicts the fact that \( (1 + \delta) \kappa - \mu = 0 \).

### 11. Proof of Proposition 2.13

Let \( \delta \) and \( \hat{\delta} \) be chosen such that Theorem 2.10 holds. In the following, we put

\[
f_{\delta, \sigma} = H^{\sigma-1} (\mu - (1 + \delta) H) - C_1(\delta),
\]

where \( \mu \) denotes the reciprocal of the inscribed radius and \( C_1(\delta) \) is the constant in the convexity estimate of Huisken and Sinestrari (see Proposition 2.12 above). The following result was established in [3]:

**Proposition 11.1** (cf. [3]). We can find a constant \( c_0 \), depending only on \( \delta \) and the initial data, with the following property: if \( p \geq \frac{1}{c_0} \) and \( \sigma \leq c_0 p^{-\frac{1}{2}} \), then we have

\[
\frac{d}{dt} \left( \int_{M_t} f_{\delta, \sigma} \right) \leq C \sigma p \int_{M_t} f_{\delta, \sigma}^p + \sigma p K_0 \int_{M_t} |A|^2
\]

except if \( t \) is a surgery time. Here, \( C \) and \( K_0 \) depend only on \( \sigma \) and the initial data, but not on \( \sigma \) and \( p \).

We next analyze the behavior of \( f_{\delta, \sigma} \) under surgery.

**Lemma 11.2.** The integral \( \int_{M_t} f_{\delta, \sigma}^p \) does not increase under surgery.

**Proof.** Consider a surgery time \( t_0 \). By assumption, each surgery is being performed on an \((\hat{\alpha}, \hat{\delta}, \hat{\varepsilon}, L)\)-neck with \( \varepsilon \leq \hat{\varepsilon} \) and \( L \geq 1000 \Lambda \). Hence, Theorem 2.5 implies that the inscribed radius of \( M_{t_0+} \) is at least \( \frac{1}{(1+\delta)H} \) in the surgically modified region. In other words, we have \( f_{\delta, \sigma} \leq 0 \) in the surgically
modified region of $M_{t_0+}$. Consequently, we have $\int_{M_{t_0-}} f_{\delta, \alpha, +}^p \leq \int_{M_{t_0-}} f_{\delta, \alpha, +}^p$, as claimed.

Combining Proposition 11.1 and Lemma 11.2 we can draw the following conclusion:

**Proposition 11.3.** We can find a constant $c_0$, depending only on $\delta$ and the initial data, with the following property: if $p \geq \frac{1}{c_0}$ and $\sigma \leq c_0 p^{-\frac{2}{p}}$, then we have

$$\int_{M_t} f_{\delta, \alpha}^p \leq C$$

for all $t$. Here, $C$ is a constant that depends on $\delta$, $\sigma$, $p$, and the initial data.

We can now use Stampacchia iteration to show that $f_{\delta, \sigma} \leq C$, where $\sigma$ and $C$ depend only on $\delta$ and the initial data. This completes the proof of Proposition 2.13.

12. **Proof of the Neck Detection Lemma (Version A)**

The proof is by contradiction. Suppose that the assertion is false. Then there exists a sequence $M_j$ of mean curvature flows with surgery and a sequence of points $(p_j, t_j)$ such that $H_{M_j}(p_j, t_j) \geq \max \{ j, \frac{K_{\alpha, j}}{\Theta}, \frac{\lambda_{\alpha, M_j}(p_j, t_j)}{H_{M_j}(p_j, t_j)} \} \leq \frac{1}{j}$. Moreover, the neighborhood $\hat{P}_{M_j}(p_j, t_j, L_0 + 4, 2\theta_0)$ does not contain surgeries, and $p_j$ does not lie at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)$-neck in the surface $M_{t_j}$.

For each $j$, we put

$$\rho_j = \min \left\{ \inf \{ d_{g(t_j)}(p_j, x) H_{M_j}(p_j, t_j) : x \in M_{t_j}, H_{M_j}(x, t_j) > 4 H_{M_j}(p_j, t_j), L_0 + 2 \} \right\}.$$

Using Proposition 2.9 we obtain

$$\liminf_{j \to \infty} \rho_j > 0.$$

By definition of $\rho_j$, we have

$$H_{M_j}(x, t_j) \leq 4 H_{M_j}(p_j, t_j)$$

for all points $x \in M_{t_j}$ satisfying $d_{g(t_j)}(p_j, x) < \rho_j H_{M_j}(p_j, t_j)^{-1}$. Using Proposition 2.9 we obtain

$$H_{M_j}(x, t) \leq 8 H_{M_j}(p_j, t_j)$$

for all points $(x, t) \in \hat{P}_{M_j}(p_j, t_j, p_j, 2\theta_0)$.

We next consider the restriction of the flow $M_j$ to the parabolic region $\hat{P}_{M_j}(p_j, t_j, p_j, 2\theta_0)$. Let us shift $(p_j, t_j)$ to $(0, 0)$ and dilate the surface by the factor $H_{M_j}(p_j, t_j)$. As a result, we obtain a flow $\tilde{M}_j$ which is defined in the parabolic region $\mathcal{P}_{\tilde{M}_j}(0, 0, \rho_j, 2\theta_0)$ and satisfies $H_{\tilde{M}_j}(0, 0) = 1$ and
Furthermore, the mean curvature of $\hat{M}_j$ is at most 8 everywhere in the parabolic region $P_{\hat{M}_j}(0, 0, \rho_j, 2\theta_0)$.

After passing to a subsequence, the flows $\tilde{M}_j$ converge smoothly to a limit flow $\bar{M}$. The limit flow is defined in a parabolic region $P(0, 0, \rho, 2\theta_0)$, where $\rho = \lim_{j \to \infty} \rho_j > 0$. Moreover, the limit flow $\bar{M}$ satisfies $H(0, 0) = 1$ and $\lambda_1(0, 0) = 0$. Finally, the mean curvature of $\bar{M}$ is at most 8 everywhere in the parabolic region $P(0, 0, \rho, 2\theta_0)$.

By the strict maximum principle, the limit flow $\bar{M}$ splits as a product. In other words, we can find a one-parameter family of curves $\Gamma_t$, $t \in (-2\theta_0, 0)$, such that $\bar{M}_t \subset \Gamma_t \times \mathbb{R}$. We may assume that the curve $\Gamma_t$ coincides with the image of $\tilde{M}_t$ under the projection from $\mathbb{R}^3$ to $\mathbb{R}^2$. Note that the curves $\Gamma_t$ need not be closed.

It follows from the Huisken-Sinestrari convexity estimate that the second fundamental form of the limiting solution $\bar{M}$ is nonnegative. Hence, the curve $\Gamma_t$ has positive geodesic curvature. Since our original flow satisfies the gradient estimate in Proposition 2.9, the curve $\Gamma_t$ satisfies the derivative estimates $|\frac{d\kappa}{dt}| \leq C_\#(\kappa + \Theta)^2$ and $|\frac{d^2\kappa}{dt^2}| \leq C_\#(\kappa + \Theta)^3$. Furthermore, since our original flow $M_j$ satisfies the pointwise estimate $\mu \leq (1 + \delta) H + CH^{1-\sigma}$, the limit flow $\bar{M}_j$ is $\frac{1}{1+\delta}$-noncollapsed. Hence, the inscribed radius of $\Gamma_t$ is at least $\frac{1}{(1+\delta)\kappa}$, where $\kappa$ denotes the geodesic curvature of $\Gamma_t$. Furthermore, the outer radius of $\Gamma_t$ is at least $\frac{2}{\kappa}$ at each point on $\Gamma_t$.

Finally, the curve $\Gamma_0$ passes through the origin, and the geodesic curvature of $\Gamma_0$ is equal to 1 at the origin. Hence, Proposition 2.10 implies that $\Gamma_0$ has length at most $3\pi$, and $\sup_{\Gamma_0} |\kappa - 1| \leq \frac{1}{100}$. Moreover, Proposition 2.9 implies that each curve $\Gamma_t$ contains a point where the geodesic curvature is between $\frac{1}{2}$ and 2. Applying Proposition 2.10 to a scaled copy of $\Gamma_t$, we conclude that each curve $\Gamma_t$ has length at most $6\pi$.

At this point, we distinguish two cases:

**Case 1:** Suppose first that $0 < \rho < L_0 + 2$. Then $\rho_j < L_0 + 2$ for $j$ large. From this, we deduce that $\sup_{\tilde{M}_0} H \geq 4$ for $j$ large. Using this fact and the gradient estimate, we obtain $\sup_{\tilde{M}_0} H \geq 2$. Consequently, $\sup_{\Gamma_0} \kappa \geq 2$, where $\kappa$ denotes the geodesic curvature of $\Gamma_0$. On the other hand, we have established earlier that $\sup_{\Gamma_0} |\kappa - 1| \leq \frac{1}{100}$. This is a contradiction.

**Case 2:** We now assume that $\rho = L_0 + 2 > 100$. Since $\Gamma_t$ has length at most $6\pi$, we conclude that $\Gamma_t$ must be a closed curve. Hence, the curves $\Gamma_t$ are simple closed, convex curves in the plane, which evolve by curve shortening flow.

By Proposition 2.10, the curve $\Gamma_0$ satisfies $\sum_{l=1}^{18} |\nabla^l \kappa| \leq \frac{1}{1000}$. Moreover, we have $\sup_{\Gamma_0} \kappa \leq 1 - \frac{\theta_0}{4}$. Finally, Proposition 2.11 implies that, for each point on $\Gamma_0$, the inscribed radius is at least $\frac{1}{(1+\delta)\kappa}$. 

$\lambda_{1+\delta}(0, 0, 0, 0, 0) \leq \frac{1}{\delta}$. Furthermore, the mean curvature of $\hat{M}_j$ is at most 8 everywhere in the parabolic region $P_{\hat{M}_j}(0, 0, \rho_j, 2\theta_0)$. 

After passing to a subsequence, the flows $\tilde{M}_j$ converge smoothly to a limit flow $\bar{M}$. The limit flow is defined in a parabolic region $P(0, 0, \rho, 2\theta_0)$, where $\rho = \lim_{j \to \infty} \rho_j > 0$. Moreover, the limit flow $\bar{M}$ satisfies $H(0, 0) = 1$ and $\lambda_1(0, 0) = 0$. Finally, the mean curvature of $\bar{M}$ is at most 8 everywhere in the parabolic region $P(0, 0, \rho, 2\theta_0)$.
If \( j \) is sufficiently large, we can find a region \( N_j \subset \{ x \in M_{t,j} : d_{g(t_j)}(p_j, t_j) \leq (L_0 + 1) H(p_j, t_j)^{-1} \} \) such that \( \text{dist}_{C^{2,0}}(H(p_j, t_j) (N_j - p_j), \Gamma_0 \times [-L_0, L_0]) < \varepsilon_0 \). We again divide the discussion into two subcases:

Subcase 2.1: Suppose that, for \( j \) large, the region

\[
\{ x + a \nu(x) : x \in M_{t,j}, d_{g(t_j)}(p_j, x) \leq (L_0 + 1) H(p_j, t_j)^{-1}, a \in (0, 2\hat{\alpha} H(p_j, t_j)^{-1}) \}
\]

is disjoint from \( M_{t,j} \). Consequently, the point \( p_j \) lies at the center of an \((\hat{\alpha}, \hat{\delta}, \varepsilon_0, L_0)\)-neck in \( M_{t,j} \) if \( j \) is sufficiently large. This contradicts our assumption.

Subcase 2.2: Suppose that, for \( j \) large, the region

\[
\{ x + a \nu(x) : x \in M_{t,j}, d_{g(t_j)}(p_j, x) \leq (L_0 + 1) H(p_j, t_j)^{-1}, a \in (0, 2\hat{\alpha} H(p_j, t_j)^{-1}) \}
\]

does intersect \( M_{t,j} \). In this case, we can find a sequence of points \( x_j \in M_{t,j} \) and a sequence of numbers \( a_j \in (0, 2\hat{\alpha} H(p_j, t_j)^{-1}) \) such that \( d_{g(t_j)}(p_j, x_j) \leq (L_0 + 1) H(p_j, t_j)^{-1} \) and \( z_j := x_j + a_j \nu(x_j) \in M_{t,j} \). We next observe that \( H(x_j, t_j) \leq 4 H(p_j, t_j) \) by definition of \( p_j \). Hence, the outer radius of the surface \( M_{t,j} \) at the point \( x_j \) is at least \( \alpha H(x_j, t_j)^{-1} \geq \frac{\alpha}{4} H(p_j, t_j)^{-1} \). Consequently, \( a_j \geq \frac{\alpha}{2} H(p_j, t_j)^{-1} \).

We now let \( \tau_j = t_j - \theta_0 H(p_j, t_j)^{-2} \). If \( j \) is sufficiently large, we may write \( z_j = y_j + b_j \nu(y_j) \), where \( (y_j, \tau_j) \in \mathcal{P}_{M_j}(p_j, t_j, L_0 + 1, 2\theta_0) \) and \( 0 \leq b_j \leq a_j < 2\hat{\alpha} H(p_j, t_j)^{-1} \). On the other hand, since \( \sup_{\Gamma_0} \kappa \leq 1 - \frac{4\alpha}{\theta_0} \), we have \( H(y_j, \tau_j) \leq (1 - \frac{4\alpha}{\theta_0}) H(p_j, t_j) \). This implies that the outer radius of the surface \( M_{t,j} \) at the point \( y_j \) is at least

\[
\alpha H(y_j, \tau_j)^{-1} \geq \frac{\alpha}{1 - \frac{4\alpha}{\theta_0}} H(p_j, t_j)^{-1} = \hat{\alpha} H(p_j, t_j)^{-1} > \frac{b_j}{2}.
\]

Consequently, the point \( z_j = y_j + b_j \nu(y_j) \) does not lie in the region enclosed by \( M_{t,j} \). This contradicts the fact that \( z_j \in M_{t,j} \). This completes the proof.

13. Proof of Proposition 2.16

By assumption, the parabolic neighborhood \( \mathcal{P}(p_1, t_1, L + 4, 2\theta_0) \) contains a point which belongs to a surgery region. Consequently, we can find a surgery time \( t_0 \in [t_1 - 2\theta_0 H(p_1, t_1)^{-2}, t_1) \) and a point \( q_1 \in M_{t_1} \) such that the following holds:

- \( d_{g(t_1)}(p_1, q_1) \leq (L + 4) H(p_1, t_1)^{-1} \).
- If we follow the point \( q_1 \in M_{t_1} \) back in time, then the corresponding point \( q_0 \in M_{t_0} \) lies in the region modified by surgery at time \( t_0 \).

Let us consider the region modified by surgery at time \( t_0 \), and let \( U_0 \) denote the connected component of this set that contains the point \( q_0 \). In other words, \( U_0 \subset M_{t_0} \) is a cap that was inserted at time \( t_0 \). We next define
$V_0 = \{ x \in M_{t_0+} : \text{dist}_{g(t_0+)}(U_0, x) \leq 1000 K_*^{-1} \}$. Clearly, $V_0$ is diffeomorphic to a disk. Let

$$D = \{ y \in \mathbb{R}^3 : \text{there exists a point } x \in V_0 \text{ such that } |y - x| < \frac{\alpha}{1000} K_*^{-1} \}.$$ 

Arguing as in Proposition 2.7 above, we can show that, for every surgery time $t > t_0$, the set $D$ is disjoint from the region modified by surgery at time $t$. Consequently, the surfaces $M_t \cap D$ form a regular mean curvature flow for $t > t_0$. In other words, the surfaces $M_t \cap D$ evolve smoothly for $t > t_0$, but we allow the possibility that some components of $M_t \cap D$ may disappear as a result of surgeries in other regions.

At each point on $V_0 \subset M_{t_0+}$, the mean curvature is at most $20 K_*$. We now follow the surface $V_0 \subset M_{t_0+}$ forward in time. This gives a one-parameter family of surfaces with boundary. It follows from Proposition 2.9 that, for $t \in (t_0, t_0 + 2\theta_0 K_*^{-2}]$, the resulting surfaces remain inside the region $D$ and have mean curvature at most $40 K_*$. Moreover, since $q_1 \in M_{t_1}$, the resulting surfaces cannot disappear before time $t_1$.

Let $V_1 \subset M_{t_1}$ denote the region in $M_{t_1}$ which is obtained by following $V_0 \subset M_{t_0+}$ forward in time. Clearly, $V_1$ is diffeomorphic to a disk, and the mean curvature is at most $40 K_*$ at each point in $V_1$. Since $g_0 \in V_0$, we have $q_1 \in V_1$. Furthermore, since $\text{dist}_{g(t_0+)}(g_0, \partial V_0) \geq 1000 K_*^{-1}$, we obtain $\text{dist}_{g(t_1)}(q_1, \partial V_1) \geq 500 K_*^{-1}$. From this, we deduce that

$$\{ x \in M_{t_1} : d_{g(t_1)}(q_1, x) \leq 500 K_*^{-1} \} \subset V_1,$$

as claimed.

14. Proof of Proposition 2.18

To fix notation, let $\Omega \subset \{ x \in \mathbb{R}^3 : y_1 \leq x_3 \leq y_2 \}$ denote the region enclosed by $\Sigma$. Moreover, let $\nu$ denote the outward-pointing unit normal to $\Omega$.

Suppose by contradiction that there exists a point $\bar{x} \in \Sigma$ such that $\bar{x}_3 \in [y_1 + 1, y_2]$, $\langle \nu(\bar{x}), e_3 \rangle \geq 0$, and $H(\bar{x}) \leq \frac{9}{100}$. The noncollapsing assumption implies that there exists a ball $B \subset \mathbb{R}^3$ of radius 100 such that

$$B \cap \{ x \in \mathbb{R}^3 : y_1 \leq x_3 \leq y_2 \} \subset \Omega.$$

This implies

$$B \cap \{ x \in \mathbb{R}^3 : x_3 = \bar{x}_3 - 1 \} \subset \Omega \cap \{ x \in \mathbb{R}^3 : x_3 = \bar{x}_3 - 1 \}.$$

Since $\langle \nu(\bar{x}), e_3 \rangle \geq 0$, the set

$$B \cap \{ x \in \mathbb{R}^3 : x_3 = \bar{x}_3 - 1 \}$$

is a disk of radius at least $\sqrt{100^2 - 99^2} > 10$. On the other hand, our assumptions imply that the set $\Omega \cap \{ x \in \mathbb{R}^3 : x_3 = \bar{x}_3 - 1 \}$ is contained in a disk of radius 10. This is a contradiction.
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