Computation of gcd chain over the power of an irreducible polynomial

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Abstract. A notion of gcd chain has been introduced by the author at ISSAC 2017 for two univariate monic polynomials with coefficients in a ring $R = k[x_1, \ldots, x_n]/\langle T \rangle$ where $T$ is a primary triangular set of dimension zero. A complete algorithm to compute such a gcd chain remains challenging. This work treats completely the case of a triangular set $T = (T_1(x))$ in one variable, namely a power of an irreducible polynomial. This seemingly “easy” case reveals the main steps necessary for treating the general case, and it allows to isolate the particular one step that does not directly extend and requires more care.

1 Introduction

Computing gcd is without a doubt one of the most fundamental algorithm in computer algebra and computational aspects have been studied extensively, till today. In [4] is introduced the concept of gcd chain to bring a similar notion of the classical gcd of polynomials of one variable over a field, to the case of over a ring $R := k[x_1, \ldots, x_n]/\langle T \rangle$ where $T$ is a primary triangular set of dimension zero. Such a ring has nilpotent elements, and non nilpotent elements are invertible. Some attempts to treat this case in prior [4] have concluded in somewhat unsatisfactory solutions. Indeed, a desirable fundamental property of gcd is an ideal equality $\langle a, b \rangle = \langle g \rangle$. While if $a$ and $b$ have coefficients in such a ring of type $R$, it is well-known that a polynomial $g$ does not exist in general, the outcome of [4] being to present a strategy to circumvent this impediment by “iterating” somehow a Pseudo Remainder Sequence when a nilpotent remainder is met. On the algorithmic side, this raises several challenging questions, even in the seemingly “easy” case of a primary triangular set $T = (T_1(x_1))$ of one variable. As this article shows, this case is already not simple. And it is important since it builds the framework to tackle the case of several variables. In particular we identify that all steps, except one that requires more work, extend to more than one variable.

1.1 Motivation

An early motivation in computing gcds over triangular sets come from the triangular-decomposition algorithm to solve polynomial (commutative or differential) systems [18,2]. This set of computational methods traces back to the early work of Ritt [10], and the major computational advances realized later by Wu-Wen Tsu [19]. This has lead to several new directions of researches, followed by many researchers. In term of algorithms, only pseudo-divisions were initially used. In 1993, in [7] Kalkbrenner introduced a “gcd”-point of view to realize the decomposition, and the elimination (See also the notes [6]). This point of view has later been significantly developed by M. Moreno-Maza et al. in particular with the implementation of the library RegularChains [5] in the software Maple.

However, such a gcd does not handle “faithfully” polynomials having multiplicities; this question was raised as early as 1995 [13] and later studied furthermore in [10], but without a satisfactory general answer. In this regard, the present work situates in the realm of triangular decomposition as initiated by Wu-Wen Tsu.

The gcd chain has the following geometric interpretation. The underlying triangular set can be thought as some algebraic constraints, over which one may want to compute with further polynomials, that is over

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the solutions of the constraints only. It may happen that the solution (a constraint), is multiple. One may think of $t_1(x) = x^3$ for example, in the case where constraints are modeled by a polynomial of one variable like in this work. When computing over $t_1$, this allows to consider Taylor expansions at order 2 (for example to control the first and second order derivative of the solution=constraint being modeled).

In Example 3 below, we want to compute the constraints defined by both polynomials $a$ and $b$ with coefficients in $\mathbb{R}[x]/(t_1) = \mathbb{R}[x]/(x^3)$. We obtain three cases, displayed in Figure 1.1 as computed by the algorithm of this paper.

![Figure 1.1](image)

**Fig. 1.** Precision $x$ (left): Three points intersection of $y = 0, -1, 1$ with $x = 0$. Precision $x^2$ (middle): Two lines, expanded from $y = 0, 1$. Precision $x^3$ (right): One parabola, expanded from $y = 1$

### 1.2 Definitions

A primary triangular set in one variable is just a power of an irreducible polynomial $p$: $T = (T_1(x)) = (p(x)^e)$. The ring $R = k[x]/(T)$ is local of maximal ideal $m = \langle p \rangle$. It is therefore Henselian: monic polynomials admit a unique factorization into coprime factors (not into irreducibles: see [4], § 3.1] for more details). Before giving the technical definitions, a connection with the classical gcd over unique factorization domains will enlighten the differences with the new notion.

Since $R$ is Henselian we can write the unique factorization of the input polynomials $a$ and $b$ into coprime factors as follows:

$$a = a_1 \cdots a_\gamma a_{\gamma+1} \cdots a_\alpha, \quad b = b_1 \cdots b_\gamma b_{\gamma+1} \cdots b_\beta. \quad (1)$$

We have ordered the factors so that $\gcd(a_i \mod \langle p \rangle, b_j \mod \langle p \rangle) = \rho_i^{\nu_i}$ where $\rho_i \in (k[x]/\langle p \rangle)[y]$ is an irreducible polynomial; And $\nu_i = \min(\lambda_i, \delta_i)$ where $a_i = \rho_i^{\lambda_i} + \cdots$ and $b_i = \rho_i^{\lambda_i} + \cdots$ (so that $a_i \equiv \rho_i^{\lambda_i} \mod \langle p \rangle$ and $b_i \equiv \rho_i^{\lambda_i} \mod \langle p \rangle$). If $a$ and $b$ have some factors $a_\ell$ and $b_m$ for $\ell, m > \gamma$ then we assume that $\gcd(a_\ell \mod \langle p \rangle, b_m \mod \langle p \rangle) = 1$.

**Example 1** We reproduce the example of [4, Ex. 3.5, Ex. 5.2]. Consider the polynomials $a$ and $b$ along with their unique factorizations into coprimes modulo $T = x^3$.

$$a = a_1 a_2 a_3 = ((y + 1)^2 + x(2y + 1) + x^2(y + 1))(y + 2x + 3x^2)(y - 1 - x - 2x^2)$$

$$b = b_1 b_2 b_3 = (y + 1 + 2x + x^2)(y + 2x + 4x^2)(y - 1 - x - 2x^2)$$

We have $\gamma = 3$ and in each case $\gcd(a_1 \mod \langle x \rangle, b_1 \mod \langle x \rangle)$ is non trivial.

The informal discussion that follows is rigorously detailed in Sections 4.1-4.2 of [4]. Under this point of view, it is convenient to refer to the more common terminology of unique factorization domains, but there is a caveat: “precision”. Modulo $p$, the gcd of $a$ and $b$ (over the field $k[x]/\langle p \rangle$) is well-defined and is as expected:
g = \gcd(a_i \mod (p), b_j \mod (p)) = \prod_{i=1}^{s} p_i^{c_i}. But there is no isomorphism \langle a, b, T \rangle = \langle g, T \rangle, only the more coarse \langle a, b, p \rangle = \langle g, p \rangle holds. A more refined notion of “common factors at certain precision” allows to obtain an ideal equality \langle a, b, T \rangle.

**Definition 1.** A monic polynomial \( c \) is said to be a common factor of \( a \) and \( b \) at precision \( \ell \) iff the Euclidean divisions of \( a \) and \( b \) by \( c \) have both zero remainder modulo \( p^\ell \), but at least one non-zero modulo \( p^{\ell+1} \).

\[
a = ca' + ra, \quad b = cb' + rb \quad \Rightarrow \quad r_a, r_b \equiv 0 \mod (p^\ell), \quad r_a \text{ or } r_b \not\equiv 0 \mod (p^{\ell+1}).
\]

Let \( I_1 \) be the set of indices of common factors of \( a \) and \( b \) at the smallest precision \( e_1 \). Let \( I_2 \) be the set of indices of common factors \( a \) and \( b \) at the next to smallest precision \( e_2 \), etc. we obtain a partition \((I_1, I_2, \ldots, I_s)\) of the set of common factors of \( a \) and \( b \) and the associated precision exponents \([e_1, \ldots, e_s]\).

**Example 2** (Example continued) We notice that \( e_1 = 1 \) and \( I_1 = \{1\} \). Indeed \( y + 1 \) is the largest common divisor of \( a_1 \) and \( b_1 \), and it is modulo \( x^{e_1} = x \); There is no common divisor of \( a_1 \) and \( b_1 \) modulo \( x^{e_1+1} = x^2 \).

Next we observe that \( e_2 = 2 \) and \( I_2 = \{2\} \): the largest common divisor of \( a_2 \) and \( b_2 \) is \( y + 2x \) and it is modulo \( x^{e_2} = x^2 \). There is no common divisor of \( a_2 \) and \( b_2 \) modulo \( x^{e_2+1} = x^3 \). Last modulo \( x^3 \) the factors \( a_3 \) and \( b_3 \) have a common divisor implying that \( I_3 = \{e_3\} \).

Sections 4.1-4.2 of [4] prove the existence and uniqueness, in the case of a triangular set of one variable, of the tuple of indices \((I_1, I_2, \ldots, I_s)\) and of the sequence of increasing precision powers \([e_1, e_2, \ldots, e_s]\). Before that, with the notations of Eq. (1), define \( G^{(a)} := \prod_{i \in I} a_i \) and \( G^{(b)} := \prod_{i \in I} b_i \), so that:

\[
a = \left( \prod_{i=1}^{s} G^{(a)}_i \right) \cdot a_{\gamma+1} \cdots a_{\alpha_i}, \quad b = \left( \prod_{i=1}^{s} G^{(b)}_i \right) \cdot b_{\gamma+1} \cdots b_{\beta_i}.
\]

(2)

And from the discussion above both \( G^{(a)}_i \) and \( G^{(b)}_i \) have one maximal common factor which is at precision \( p^{e_i} \). Let us write it \( G_i \) (since there it comes alone, it makes no harm to think of \( G_i \) as “the” \( \gcd \) and to write \( G_i \equiv \gcd(G^{(a)}_i, G^{(b)}_i) \mod (p^{e_i}) \). There exist monic polynomials \( c^{(a)}_i \) and \( c^{(b)}_i \) both relatively prime modulo \( p \), such that:

\[
G^{(a)}_i \equiv c^{(a)}_i G_i \mod (p^{e_i}), \quad \text{and} \quad G^{(b)}_i \equiv c^{(b)}_i G_i \mod (p^{e_i}).
\]

(3)

We obtain an equality of ideals: \( \langle a, b, T \rangle = \langle G_1, p^{e_1} \rangle \cdots \langle G_s, p^{e_s} \rangle \). The formal definition of \( \gcd \) chain modulo a triangular set of one variable is as follows:

**Definition 2.** Given two monic polynomials \( a, b \in R[y] \), a \( \gcd \) chain of \( a, b \) is a sequence \((g_i, p^{e_i})_{i=1, \ldots, s}\) such that:

- \( e_1 < \ldots < e_s \leq e \) and \( \deg_y (g_1) > \cdots > \deg_y (g_s) \).
- \( g_i \) is the product of all common factors of \( a \) and \( b \) at precision \( \geq e_i \).
- \( g_{i+1} \) divides \( g_i \) modulo \( p^{e_{i+1}} \). Defining \( G_i := g_i / g_{i+1} \mod p^{e_{i+1}} \) for \( i = 1, \ldots, s - 1 \) and \( G_s := g_s \), the following isomorphism holds:

\[
R[y]/\langle a, b \rangle \cong (k[x]/\langle p^{e_i} \rangle)[y]/\langle G_1 \rangle \times \cdots \times (k[x]/\langle p^{e_s} \rangle)[y]/\langle G_s \rangle
\]

(4)

where the r.h.s is a direct product of rings. We have moreover for all \( i = 1, \ldots, s \):

\[
(k[x]/\langle p^{e_i} \rangle)[y]/\langle a, b \rangle \cong (k[x]/\langle p^{e_i} \rangle)[y]/\langle G_1 \rangle \times \cdots \times (k[x]/\langle p^{e_{i-1}} \rangle)[y]/\langle G_{i-1} \rangle \times (k[x]/\langle p^{e_i} \rangle)[y]/\langle g_i \rangle
\]

(5)

**Example 3** (Example\( II \) continued) Let \( p(x) = x, T = (T_1(x)) = (x^3) = (p^3) \). The two monic polynomials \( a \) and \( b \) when expanded are written:

\[
a = y^4 + (2x^2 + 3x + 1) y^3 + (-x^2 - x - 1) y^2 (-13x^2 - 4x - 1) y - 7x^2 - 2x
\]

\[
b = y^4 + (3x^2 + 3x)y^2 + (-3x^2 - 3x - 1)y - 10x^2 - 2x
\]
According to the discussion made in Example 2, \( e_1 = 1, \ e_2 = 2, \ e_3 = 3 \). The gcd chain is given by:

\[
[(y-1)(y+1), \ x], \ ((y-1-x)(y+2x), \ x^2], \ (y-1-x-2x^2, \ x^3)],
\]

and yields the following isomorphism according to (4).

\[
(k[x]/\langle x^3 \rangle)[y]/\langle a, b \rangle \cong (k[x]/\langle x \rangle)[y]/\langle y+1 \rangle \times (k[x]/\langle x^2 \rangle)[y]/\langle y+2x \rangle \times (k[x]/\langle x^3 \rangle)[y]/\langle y-1-x-2x^2 \rangle \quad (6)
\]

However, Section 5 of [4] dealing with algorithms is more an indication of directions for future work, than a complete and definitive exposition. This is what the present work does, treating the case of one variable completely.

1.3 Related works

First of all, let us clarify what may appear as an elephant in the room:

why not computing the squarefree part of \( T = p^e \) ?

The first reason is that sticking with \( T = p^e \) really allows to discover the gcd-chain. Otherwise starting from the gcd known at precision \( p \) one could perform Hensel lifting and, roughly speaking, see at each step which parts still divide \( a \) and \( b \) etc. But then quadratic convergent lifting may not be convenient enough to recover each precision \( e_i \), in that it may miss some. Some refinements should then be devised... with additional costs. The linearly convergent Hensel lifting is not as efficient. The second reason is for considerations of generalization to a primary triangular set of several variables. Computing the radical is still possible (see e.g. [15,11]) at a reasonable cost, but this time recovering the factors at the required “precision” becomes far more complicated, assuming it is possible. Consequently, for convenience and generalization in mind,

the squarefree part operation is not considered.

Gcds over non-radical triangular sets have been addressed in [13][10]. But none provided a structural isomorphism that mimics the case of gcd of polynomials over a field. In a different direction, some works have focused on representing not only solution points, but their multiplicity as well [3][9][12][1]. The multiplicity is a much coarser information than what provides an ideal isomorphism; Moreover the methods proposed in these works are not simpler than the one proposed in [4] and here. See Section “Previous Work” of [4] for more details. Compared to the algorithmic Section 5 of [4] the present article gives a complete treatment, with Hensel lifting; in particular the over-optimistic over-simplistic Assumption (C) made therein is elucidated with Weierstrass preparation theorem.

When the input irreducible polynomial is \( p(x) = x \), the article [14] which computes a “truncated” resultant of bivariate polynomials \( a \) and \( b \) shares some common features with the present work. However, the gcd-chain carries more information than the classical resultant, and it is not clear wether the quite intricate algorithm that is devised in [14] can be adapted. Therein, a generalized version of the “half-gcd” is proposed, whereas here only classical subresultants are required. Thus it seems possible to use existing “half-gcd” based subresultant computations and reach a similar quasi-linear time complexity. Another significant difference is the use of Weiertrass factorization to cope with nilpotent leading coefficients. This notion is more natural and efficient than the “normalization” lemma 1 of [14]; It does not require Hensel lifting and to know the squarefree part \( p \) of the input \( T = p^e \).

Besides, this study shows that all tasks used in the algorithm extends quite straightforwardly to primary triangular sets of more than one variable, except one: Line 22 which “removes” nilpotent part. See Section “Concluding remarks” for more about this.
1.4 Organization of the paper

Section 2 introduces the core routine “largestFactor” that executes one iteration of the subresultant algorithm. It finds the “last non-nilpotent subresultant”, uses the Weierstrass factorization to make it, together with the first nilpotent subresultant, monic before another iteration. The complete algorithm “gcdChain” that calls “largestFactor” is presented and analyzed in Section 3. The last section 4 details the variation of the Weierstrass preparation theorem that we have used. Some concluding remarks end the article.

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2 A subresultant based main algorithm

The outcome of this section is Algorithm 4 “largestFactor” presented in Subsection 2.2. It introduces Weierstrass factorizations at Lines 23 and 28. In the following subsection, the several subroutines used in this algorithm are introduced first.

2.1 Preliminary routines

The first subroutine catches the last non-nilpotent subresultant in a subresultant sequence.

Input: Subresultant sequence $S = [S_{r_0}, S_{r_1}, \ldots, S_{r_t}]$ and a power of a irreducible polynomial $T = (p(x)^e)$.

Output: index $j$ such that $S_{r_j}$ is not nilpotent, and $S_{r_{j+1}}$ is nilpotent

```
1 j = t
2 while $\text{Res}_x(\text{lc}(S_{r_j}), T) = 0 \text{ in } k$ do // is leading coefficient $\text{lc}(S_{r_j})$ nilpotent mod $p^e$?
3 j = j - 1
4 return $j + 1$
```

Algo 1: index of the last non nilpotent subresultant

Proof (Correctness of Algo 1). It uses the Block structure of a subresultant sequence. In our context, this has been sort out in Lemma 2 of [10]. (see also 2nd paragraph of “Correctness of Algo. 1” in [4, p. 115]). □

Algorithm 2 computes the largest power of $p$ that divides all the coefficients of a polynomial in $(k[x]/(p^e))[y]$.

Input: $T = p^e$, power an irreducible polynomial. Nilpotent polynomial $F = F_0 + F_1 y + \cdots + F_r y^r$ modulo $T$

Output: $P = p^\ell$, where $\ell \leq e$, and $p^{\ell+1} \nmid F$ but $p^{\ell+1} \mid F$

```
5 P ← T, i ← 0
6 while $i < r + 1$ and $\text{deg}(P) > 0$ do
7 P ← $\gcd(F_i, P)$ ; i ← $i + 1$
8 return P
```

Algo 2: nilpotentFactor: find the largest power of $p$ that divides $F$, and $T$

Proof (Correctness of Algo 2). At any value of $i$, one has $P = \gcd(F_0, \ldots, F_i, T)$. Since $T = p^e$, we can write $p^{\ell_i} := \gcd(F_0, \ldots, F_i, T)$. The sequence $\{\ell_i\}$ decreases. Let $\ell$ be the minimal value. By definition $p^\ell \mid F$ since $p^{\ell+1} \nmid F_i$ for all $i$, and $\ell$ is the largest integer having this property. □

The last subroutine is used to “make monic” a non-nilpotent polynomial. It is based on Corollary 2 which details are postponed to Section 4.
Lemma 1 (Correctness of Algo. 4). The output of Algorithm \[4\] verifies: \(\langle f, A, p^{e_1} \rangle = \langle g, p^{e_1} \rangle\).

Proof. If the algorithm exits at Line 13 then \(\langle f, A, T \rangle = \langle f, 0, T \rangle = \langle f, T \rangle\). It then returns \(f, T, \text{“end”}\), hence \(g = f\) and \(p^{e_1} = T\), and \(\langle f, T \rangle = \langle g, p^{e_1} \rangle\). The gcd chain has one block.

If the algorithm exits at Line 11 then \(A\) is monic constant, hence equal to 1; therefore \(\langle f, A, T \rangle = \langle A, T \rangle = \langle T, 1 \rangle = \langle 1 \rangle\). It returns \(A, T, \text{“end”}\), hence \(g = A\) and \(p^{e_1} = T\), thereby \(\langle g, p^{e_1} \rangle = \langle A, T \rangle\). Here too the gcd chain has one block.

Assume now that the algorithm ends at Line 29 or at Line 23. Lines 17–20 computes the subresultant pseudo-remainder sequence modulo \(T\) of \(g\) and \(f\), following the classical Collins-Brown's formula. As it uses
divisions of some leading coefficients, it may raise an error since modulo $T$ elements are not all invertible. This error is caught and the computation of the subresultant is ended. It means that this coefficient is nilpotent which is precisely what we are looking for. That is why no additional treatment is necessary in the “catch” Line 20.

At Line 21 Specification of Algorithm 1 insures that $S_{r_j}$ is the last non nilpotent subresultant, whence $S_{r_{j+1}}$ is nilpotent.

At Line 22 is removed the nilpotent part of $S_{r_{j+1}}$ equal to $p^{e_1}$. If the test of Line 24 is satisfied, then $S_{r_j}$ is not only the last non-nilpotent but also the last non-zero one: classical subresultant theory insures that $(S_{r_j}) = (f, A)$ modulo $T$ (that is $e_1 = e$). From the specifications of the “WeierstrassMonic” algorithm we have $(g) = (S_{r_j})$ and thus $(g, T) = (f, A, T)$ (with $T = p^{e_1}$ since $e = e_1$).

If this test is not satisfied, then we compute $S = S_{r_{j+1}}/p^{e_1}$ at Line 26 which is not nilpotent. We can apply Algorithm 3 “WeierstrassMonic” at Line 23. It outputs a monic polynomial $g$ such that $S_{r_j} = u \cdot g$ where $u$ is a unit in $(k[x]/(p^{e_1}))[y]$. In particular $(g, p^{e_1}) = (S_{r_j}, p^{e_1})$. By the “last non-nilpotent criterion” of subresultant [4 Thm 5.1] $(S_{r_j}, p^{e_1}) = (f, A, p^{e_1})$. Hence $(g, p^{e_1}) = (f, A, p^{e_1})$, as required.

Similarly, Algorithm 3 at Line 28 returns a monic polynomial $B$ such that $S = v \cdot B$, $v$ being a unit modulo $p^e$. Note that $B$ is monic and satisfies $\deg_g(B) \leq \deg_g(g)$, as required. \hfill \Box

**Lemma 2.** If a monic polynomial $c \neq 1$ is a common factor of $f$ and $A$ at precision $r > e_1$ then $c$ is a monic divisor of $B$ at precision $r - e_1$. A monic divisor $c \neq 1$ of $B$ is not a common factor of $f$ and $A$ at precision $< e_2$.

**Proof.** Let $c$ be a factor of $f$ and $A$ at precision $r > e_1$. We can write: $f \equiv cf^r \mod p^{e_1}$ and $A \equiv cA^r \mod p^{e_1}$ (by Definition 1 both equalities do not hold together modulo $p^{e_1+1}$). Consider the Bézout coefficients (a.k.a cofactors) $u_i, v_i$ of the subresultant $S_{r_j}$. From the equality $S_{r_{j+1}} = u_{j+1}f + v_{j+1}A$, one deduce that $S_{r_{j+1}} \equiv c(u_{j+1}f^r + v_{j+1}A^r) \mod p^{e_1}$. We also have $p^{e_1}Bv \equiv c(u_{j+1}f^r + v_{j+1}A^r) \mod p^{e_1}$. Now $c$ being monic $p^{e_1}$ does not divide $c$, and we have: $B \equiv c^{-1}u_{j+1}f^r + c^{-1}v_{j+1}A^r \mod p^{e_1} - e_1$. Since $r > e_1$ by assumption, $B \equiv 0 \mod (p^{e_1} - e_1, c)$ and $c$ is a factor of $B$ at precision at least $r - e_1 - 1$. Moreover, since $p^{e_1}$ is the largest power of $p$ that divides $S_{r_{j+1}}$, the precision is exactly $r - e_1$. This proves the first assertion.

Consider a monic divisor $C$ of $B$ at precision $< e_2$. This means that the remainder of the Euclidean division of $B$ by $C$ is in $(p)$ but not in $(p^{e_1})$. If $C$ were a common factor of $A$ and $f$ at precision $\ell < e_2$, then it is at precision $\ell \leq e_1$ since there is no common factor of $f$ and $A$ at precision $e_1 + 1, \ldots, e_2 - 1$, by definition. We would obtain $p^{e_1}B \equiv v_1^{e_1}u_{j+1}(Cf + v_{j+1}CA^r + pW) \mod (p^{e_1})$ for a non-zero polynomial $W$, and then $B \equiv C^{-e_1}u_{j+1}f^{e_1} + v_{j+1}A^{e_1} + p^{e_1} - e_1W \mod (p^{e_1} - e_1)$. For $C$ to divide $B$ modulo $p^{e_1}$, the term $p^{e_1} - e_1W$ shall be zero at least modulo $p$. This happens only if $\ell > e_1$ since $W \neq 0$. Contradiction with $\ell \leq e_1$. \hfill \Box

## 3 The Gcd-Chain Algorithm

Now that all sub-routines used in Algorithm 1 “largestFactor” are defined, we introduce in this section main Algorithm 5 below. It is made of a while loop that has two main components:

1. a call to Algorithm 1 “largestFactor” (line 30). A loss of precision is entailed in the division at Line 20.
2. recovering this precision loss at each iteration by Hensel lifting (Lines 34 34).

Each iteration computes one “block” of the gcd-chain. The correctness is shown in Theorem 1 with several preliminaries made of Lemmas 3 4, Proposition 1 and its corollary 1. These preliminaries contain a detailed description of each step of the algorithm.

**Remark 1.** In the sequel the triangular sets $T_1$, $T_1'$ are all power of the irreducible polynomial $p$. These exponents denoted $e_1$ or $e_1'$ are introduced only for the analysis and are not needed in the algorithms. This is consistent with the principle made in § 1.3 to not compute squarefree parts.

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With the notation of Lemma 4, we have

\[ (g_1, g_2, \ldots, g_s) \] is the gcd-chain of \((a, b, T)\).

**Lemma 3.** The Hensel lifting at Line 22 of Algorithm 5 returns \(G_i^*, g_i^{e+1}\) verifying: \(G_i^* \equiv G_i \mod (T^i)\), \(g_i^{e+1} \equiv g_i + 1 \mod (T_i^i)\), \(g_i \equiv G_i g_i^{e+1} \mod (T_i^i)\).

**Proof.** This is classical: the algorithm follows exactly the steps presented in Algorithm 15.10 of [17].

The next lemma clarifies the status of the input/output when calling Algorithm “largestFactor” at Line 23 within the several iterations. The notations introduced will be used thereafter.

**Lemma 4.** Write \(T = (T_1, \ldots, T_s)\) with \(T_i = p^{e_i}\), \(C = (g_1, \ldots, g_s)\) and \(D = (G_1, \ldots, G_s)\) the three outputs of Algorithm 3. For all \(i = 0, \ldots, s-1\), consider the output \(g_{i+1}, T_{i+1}, B_{i+1}, S_{i+1}\) of the call to “largestFactor” at Line 23 and as defined at Line 23. They verify:

\[
T_{i+1} = p^{e_{i+1}+e_i}, \quad S_{i+1} = p^{e_{i+1}+1}, \quad T_{i+1} = p^{e_{i+1}} \quad (\text{with } e_0 = 0).
\]

**Proof.** The proof goes by induction on \(i = 0, \ldots, s-1\). According to the specifications of Algorithm 3 “largestFactor”, its input is by induction hypothesis \(S_i = p^{e_i}\). From the Specification of Algorithm 3 the output is \(T_{i+1} = p^{e_{i+1}+e_i}\) because the input \(S_i\) is only at precision \(e - e_i\); And \(S_{i+1} = p^{e_{i+1}+e_i} = p^{e_{i+1}+1}\) as required. The definition of \(T_{i+1}\) at Line 23 gives \(T_{i+1} = T_{i+1}^i T_i = p^{e_{i+1}+e_i} p^{e_{i+1}} = p^{e_{i+1}+e_i}\) as required.

The proposition below connects the different outputs obtained after each iteration, to the initial input \(a, b, T\). It is crucial in the proof of Theorem 1.

**Proposition 1.** With the notation of Lemma 4 we have

\[
\text{for } i \geq 1, \quad p^{e_{i-1}} g_i \in \langle a, b, p^e \rangle, \quad p^{e_i} B_i \in \langle a, b, p^e \rangle.
\]
Corollary 1. Following corollary (of Proposition 1).

Proof. We proceed by induction on $i$, starting with the base case $i = 1$.

By definition $g_1$ is the first output polynomial of the first call to Algorithm 4 “gcdChain”. From the definition of the Weierstrass factorization at Line 23 of Algorithm 4 there is a unit $v_1 \in (k[x]/(p^e))[y]$ such that $v_1 g_1 = s_{r_1}$. Write the Bézout coefficients $u_1(v_1^i)(a + v_1^j(b = s_{r_1}$, so that $g_1 = v_1^{-1}(u_1^i + v_1^j(b)$; this proves that $g_1 \in \langle a, b, T \rangle$ (and that $g_1 \in \langle a, b, p^e \rangle$: indeed $p^e = T$).

$B_1$ is the third output polynomial of the first call to Algorithm 4 “largestFactor” in Algorithm 5 “gcdChain”. From the definition of the Weierstrass factorization, there is a unit $e_1 \in (k[x]/(p^e))[y]$ such that $p^{e_1} e_1 B_1 = s_{r_1+1}$. With the Bézout coefficients written $s_{r_1+1} = u_1^i(v_1^j(b + v_1^j(b)$, we obtain $p^{e_1} B_1 = e_1^{-1}(u_1^i(a + v_1^j(b)) \in \langle a, b, p^e \rangle$.

Next assume that the result holds up to a value $1 \leq i < s$ and let us prove it for $i+1$. By definition $g_i$ (resp. $B_i$) is the first (resp. the third) output polynomial of the $i$-th call to Algorithm 4 “largestFactor” in Algorithm 5 “gcdChain”. From the definition of the Weierstrass factorization at Line 23 (resp. at Line 28) of Algorithm 4 there is a unit $v_{i+1} \in (k[x]/(p^e))[y]$ (resp. $e_{i+1}$) such that $g_{i+1} v_{i+1} = s_{r_{i+1}}$ (resp. $p^{e_{i+1}} e_{i+1} B_{i+1} = s_{r_{i+1}}$). Write the Bézout coefficients as follows:

$$s_{r_{i+1}} = u_{i+1}^i(v_{i+1}^j(b_1, \quad s_{r_{i+1}} = u_{i+1}^i(v_{i+1}^j(b_1).$$

It follows that:

$$p^{e_{i+1}} g_{i+1} \equiv u_{i+1}^{-1}(u_{i+1}^i(p^{e_{i+1}} g_{i+1} + v_{i+1}^j(p^{e_{i+1}} B_1), \quad p^{e_{i+1}} e_{i+1} B_{i+1} = e_{i+1}^{-1}(u_{i+1}^i(v_{i+1}^j(b + v_{i+1}^j(b).$$

The latter equality implies $p^{e_{i+1}} B_{i+1} \equiv e_{i+1}^{-1}(u_{i+1}^i(p^{e_{i+1}} g_{i+1} + v_{i+1}^j(p^{e_{i+1}} B_1).$ By induction hypothesis $p^{e_{i-1}} g_i \in \langle a, b, p^e \rangle$ and $p^{e_{i}} B_i \in \langle a, b, p^e \rangle$. Thus $p^{e_{i}} g_i = p^{e_{i-1}} p^{e_{i}} g_i \in \langle a, b, p^e \rangle$. Consequently, both the r.h.s of Eqs. (**) and (***) are in $\langle a, b, p^e \rangle$, therefore so are $p^{e_{i}} g_{i+1}$ and $p^{e_{i}} B_{i+1}$.

The first iteration of the loop of Line 32 is special since it doesn’t require Hensel Lifting (case $i = 0$). The lemma hereunder makes this base case a black-box, called in the induction proof of Theorem 1.

Lemma 5. The first iteration of the while loop (Line 32 $i = 0$) of Algorithm 5 fills $C$ with $g_1$ and $T$ with $T_1 = p^{e_1}$. We have $\langle g_1, p^{e_1} \rangle = \langle a, b, p^e \rangle$.

Moreover, if there is only one iteration the stronger equality $\langle g_1, p^{e_1} \rangle = \langle a, b, p^e \rangle$ holds.

Proof. From the proof of correctness of Algorithm 4 one has at Line 33 of Algorithm 5 $\langle g_1, T_1 \rangle = \langle g_0, B_0, T_1 \rangle$. But $T_1 = p^{e_1}$, $g_0 = a$ and $B_0 = b$ yielding the first equality.

For the second one, by Proposition 1 we know that $p^{e_0} g_1 = g_1 \in \langle a, b, p^e \rangle$ and that $p^{e_1} B_1 \in \langle a, b, p^e \rangle$. If there is only one iteration of the while loop Line 32 namely for $i = 0$, then there are two sub-cases. First $B_1 \neq \text{"end"}$ and $B_1 = 1$. Then Proposition 1 gives $p^{e_1} B_1 = p^{e_1} \in \langle a, b, p^e \rangle$ which proves $\langle p^{e_1}, g_1 \rangle = \langle a, b, p^e \rangle$. Second, $B_1 = \text{"end"}. This happens when the call to Algorithm 5 “largestFactor” (Line 33) with input $a, b, T$ outputs $B_{i+1} = B_1 = \text{"end"}. According to Lines 14, 15, 23 the output denoted $p^{e_1}$ (in this Lemma) is equal to $T$ (as denoted in Algorithm 4) which is equal to $p^e$. This implies $\langle g_1, p^{e_1} \rangle = \langle g_1, p^e \rangle = \langle a, b, p^e \rangle$.

To prove correctness of the “gcdChain” Algorithm 5 one must consider the isomorphisms 4 and 5. This amounts to prove that a product of ideals of type $\langle G_1, p^{e_1} \rangle \langle G_2, p^{e_2} \rangle \cdots \langle g_i, p^{e_i} \rangle$ is equal to ideals of type $\langle a, b, p^e \rangle$. Thus we must examine the generators of the product of ideals, which is the purpose of the following corollary (of Proposition 1).

Corollary 1. The same notations as in Lemma 4 are used. Let $I$ be a subset of indices in $[1, s - 1]$ and let $\bar{I}$ its complement in $[1, s]$. To show that $I, \bar{I}$ is a partition of $[1, s]$.

Then $g_s \prod_{i \in I} p^{e_i} \prod_{j \in \bar{I}} G_j \in \langle a, b, p^e \rangle$.

Proof. Assume that $I \neq \emptyset$ first and let $\zeta := \max I$. Then $\{\zeta + 1, \ldots, s - 1\} \subset \bar{I}$ (possibly empty). We claim that $p^{e_\zeta} g_s \prod_{j=\zeta+1}^{s-1} G_j \in \langle a, b, p^e \rangle$, from which the conclusion follows since $\langle g_s \prod_{i \in I} p^{e_i} \prod_{j \in \bar{I}} G_j \rangle \subset \langle p^{e_\zeta} g_s \prod_{j=\zeta+1}^{s-1} G_j \rangle$. 

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From the definition of $G_i$ and $g_i$ after Hensel Lifting at Lines 43 and 42 of Algorithm 5 one has: $g_{i-1} \equiv G_{i-1} g_i \pmod{p^{e_i}}$ (see Lemma 3). It implies the following equality where $(\cdots)$ denotes a polynomial:

$\left( \prod_{j=i+1}^{s-1} G_j \right) g_s = \left( \prod_{j=i+1}^{s-2} G_j \right) (g_{s-1} + p^{e_{s-1}} g_s (\cdots)) = \left( \prod_{j=i+1}^{s-3} G_j \right) (g_{s-2} + p^{e_{s-2}} g_{s-1} (\cdots) + p^{e_{s-1}} g_s (\cdots))$

If we multiply the above equation by $p^{e_s}$ then each term is a multiple of $p^{e_s} g_s$ for some $\ell = \xi, \ldots, s-1$. Hence, by Proposition 1 each term is in $(a, b, p^{e_s})$ yielding the conclusion in the case $I \neq \emptyset$.

Otherwise when $I = \emptyset$, the product is equal to $G_1 \cdots G_{s-1} g_s$ and similarly to the above, we get $g_1 + p^{e_1} g_2 (\cdots) + \cdots + p^{e_{s-1}} g_s (\cdots)$, which is in $(a, b, p^{e_s})$ according to Proposition 1.

The proof of Theorem 1 requires a last intermediate result.

**Lemma 6.** With the notations of Lemma 4 assume that $s \geq 2$. Fix an $1 \leq i \leq s-1$ and refer to the notations of Eqs (2)–(3), to define $a' := G_{i+1}^{(a)} \cdots G_s^{(a)}, a_{i+1} \cdots a_\alpha$ and $b' := G_{i+1}^{(b)} \cdots G_s^{(b)} \cdot b_{i+1} \cdots b_\beta$. We have:

1. $(a, b, p^{e+1}) \subset G_1, p^{e_1} \cdots G_i, p^{e_i} (a', b', p^{e+1})$.
2. $\langle a', b', p^{e+1} \rangle = (a_1, B_1, p^{e_1})$.

**Proof.** By Eqs. (2)–(3), we have $a = a' \prod_{j=1}^i G_j^{(a)}$ (similarly $b = b' \prod_{j=1}^i G_j^{(b)}$) with $G_j^{(a)} \equiv c_j^{(a)} G_j \mod{p^{e_j}}$ (respectively $G_j^{(b)} \equiv c_j^{(b)} G_j \mod{p^{e_j}}$). Therefore

$$a = a' \cdot \prod_{j=1}^i (c_j^{(a)} G_j + p^{e_j} (\cdots)), \quad b = b' \cdot \prod_{j=1}^i (c_j^{(b)} G_j + p^{e_j} (\cdots)),$$

where $(\cdots)$ denotes some polynomials. This proves the inclusion 1.

To prove 2 let us treat first the case $i = 1$. Consider the polynomials $g_1, B_1$, which are the output of the first call to "largestFactor" at Line 33 of Algorithm 5. Note that if we apply the lemma 4 with $g_1, B_1, p^{e_1}$ instead of $a, b, T$, we obtain the ideal equality: $\langle g_1, B_1, p^{e_1} \rangle = \langle g_2, p^{e_2} \rangle$; Note the precision loss of $e_1$ entailed at the previous iteration $i = 0$. Let us prove that $(a', b', p^{e+1}) = (g_1, B_1, p^{e_1})$. According to Definition 1 and to the definitions of the polynomials $a'$ and $b'$, the common factors and $a'$ and $b'$ are exactly those of $a$ and $b$ at precision $\geq e_2$. By Lemma 2 such common factors are divisors of $B_1$ at precision $\geq e_2 - e_1$. On the other hand, by Lemma 1 the divisors of $g_1$ (at precision $p$) are exactly the common factors of $a$ and $b$. Therefore, up to precision, the common factors of $g_1$ and $B_1$ are exactly those of $a$ and $b$ at precision $\geq e_2$, hence are the same as those of $a'$ and $b'$. As for precision, the common factors at precision $\geq e_2$ of $a, b$, and those of $g_1, B_1$ (and they are at precision $\geq e_2 - e_1$ due to precision loss). We deduce that $\langle g_2, p^{e_2} \rangle = (a', b', p^{e_2})$. After Hensel lifting at Line 11 we obtain $(g_2, p^{e_2}) = (a', b', p^{e_2})$.

The general case $i > 1$ reduces to the case $i = 1$ by considering instead of $a$ and $b, E := a / G_1^{(a)} \cdots G_{i-1}^{(a)} = a_{i+1} \cdots a_\alpha \cdot \prod_{j=i}^s C_j^{(a)}$ and respectively $F := b / G_1^{(b)} \cdots G_{i-1}^{(b)} = b_{i+1} \cdots b_\beta \cdot \prod_{j=i}^s C_j^{(b)}$; And instead of $a'$ and $b'$ to take $E' := a' / G_1^{(a)} \cdots G_i^{(a)} = a_{i+1} \cdots a_\alpha \cdot \prod_{j=i+1}^s C_j^{(a)}$ and respectively $F' := b' / G_1^{(b)} \cdots G_i^{(b)} = b_{i+1} \cdots b_\beta \cdot \prod_{j=i+1}^s C_j^{(b)}$.\hfill $\square$

**Theorem 1.** At the end of the $i$-th iteration of the while loop (Line 32) of Algorithm 5 the output lists $\mathcal{C}, \mathcal{T}, \mathcal{D}$ are currently filled with:

$$\mathcal{C} = [g_1, \ldots, g_i], \quad \mathcal{T} = [p^{e_1}, \ldots, p^{e_i}], \quad \mathcal{D} = [G_1, \ldots, G_{i-1}],$$

and at the end of the algorithm, after say $s$ iterations, the sequence $[(g_1, p^{e_1}), \ldots, (g_s, p^{e_s})]$ is a gcd chain.
Proof. We must show that Isomorphisms 1 and 5 hold. By the Chinese remaindering theorem it suffices to show that
\[ \langle a, b, p^c \rangle = \langle G_1, p^{c_1} \rangle \cdots \langle G_s-1, p^{c_{s-1}} \rangle \langle g_s, p^{c_s} \rangle. \] (7)
holds, in order to prove 1; and to prove that 5 holds, it suffices to show that:

for all \( s \geq i \geq 1 \), \( \langle a, b, p^c \rangle = \langle G_1, p^{c_1} \rangle \cdots \langle G_{i-1}, p^{c_{i-1}} \rangle \langle g_i, p^{c_i} \rangle \), (8)

Assume first that \( s = 1 \). Then Eq. (8) and Eq. (7) corresponds respectively to the first and to the second equality of Lemma 6.

Assume now that \( s > 1 \). Let us prove Eq. (8) first. The case \( i = 1 \) amounts to \( \langle g_1, p^{c_1} \rangle = \langle a, b, p^c \rangle \) and is provided by Lemma 5. Assuming now \( i > 1 \). Statements 1,2 of Lemma 6 together insure that \( \langle a, b, p^c \rangle \subset \langle G_1, p^{c_1} \rangle \cdots \langle G_{i-1}, p^{c_{i-1}} \rangle \langle g_i, p^{c_i} \rangle \) (†).

On the other hand, Corollary 1 insures of the inclusion:

\[ \langle G_1, p^{c_1} \rangle \cdots \langle G_{i-1}, p^{c_{i-1}} \rangle \langle g_i \rangle \subset \langle a, b, p^c \rangle. \]

Indeed the generators of the product ideals on the l.h.s. are proved in that corollary to belong to the r.h.s. It follows that:

\[ \langle G_1, p^{c_1} \rangle \cdots \langle G_{i-1}, p^{c_{i-1}} \rangle \langle g_i, p^{c_i} \rangle \subset \langle a, b, p^c \rangle. \]

Together with (†), this proves Eq. (8). Let us prove Eq. (7). Eq. (8) with \( i = s \) yields:

\[ \langle G_1, p^{c_1} \rangle \cdots \langle G_{s-1}, p^{c_{s-1}} \rangle \langle g_s, p^{c_s} \rangle = \langle a, b, p^c \rangle. \]

Corollary 1 provides the inclusion:

\[ \langle a, b, p^c \rangle \supset \langle G_1, p^{c_1} \rangle \cdots \langle G_{s-1}, p^{c_{s-1}} \rangle \langle g_s \rangle. \]

Moreover, since by assumption there are \( s \) iterations, the while loop at Line 32 of Algorithm 5 stops at the \( s + 1 \)-th iteration. Either \( B_s \neq \text{end} \) and then \( B_s = 1 \). Proposition 1 then guarantees that \( p^c B_s = p^c \in \langle a, b, p^c \rangle \). We obtain the equality (7) required. Either \( B_s = \text{end} \): the \( s - 1 \)-th call to “largestFactor” at Line 33 with input \( g_{s-1}, B_{s-1}, p^{c_{s-1}} \) exits at one of the Lines 11,15,25. If it is at Line 11 then \( B_{s-1} = 0 \) which happens eventually only if \( b = 0 \), that is if there is only one iteration; Since \( s > 1 \) here, this cannot happen. If it is at Line 15, then \( B_{s-1} = 1 \), and there is no such \( s \)-th iteration since the test at Line 32 is not passed. It remains the case of Line 25. Then the output \( T \) is equal here to \( p^c \) and we have \( e = e_s \). Then Eq. (7) and Eq. (5) coincide.

\[ \square \]

4 A variant of Weierstrass preparation’s theorem

The Weierstrass preparation theorem states that a formal power series \( f = \sum_i a_i X^i \in \mathfrak{a}[X] \) with coefficients in a local complete ring \((\mathfrak{a}, m)\), not all of them lying in \( m \), has a unique factorization \( f = qu \) where \( q = q_0 + \cdots + q_{n-1} X^{n-1} + X^n \) is monic and \( q_i \in m \), and where \( u \in \mathfrak{a}[x]^* \) is an invertible power series.

In our context the local complete ring is \( \mathfrak{a} = k[x]/(p^e) \), \( m = (p) \) (indeed it is equal to \( k[[x]]/(p^e) \)), which is a finite quotient of the local complete ring \( k[[x]] \)). But the factorization supplied by the classical version (e.g. [8] Theorem 9.1) does not fit the needs of this work. The following variant does:

Proposition 2. Let \((\mathfrak{a}, m)\) be a complete local ring and let \( f \in \mathfrak{a}[X] \) a polynomial, say \( f = f_d X^d + \cdots + f_0 \) which has not all of its coefficients \( f_i \) lying in \( m \). Write \( f_k \) the coefficient of highest degree that is not in \( m \). There are unique polynomials \( q \) and \( u \in \mathfrak{a}[X] \) such that \( f = uq \) where:

- \( q \) is monic of degree \( k \).
- \( u = u_0 + u_1 X + \cdots + u_{d-k} X^{d-k} \) where \( u_0 \not\in m \) and \( u_i \in m \) for \( i \geq 1 \) (note that \( u \) is a unit of \( \mathfrak{a}[[X]] \)).
The proof is adapted from [8, Theorem 9.1-9.2].

**Proof.** Write \( \text{rev}_d(f) := X^df \left( \frac{1}{X} \right) \) the reversal polynomial of \( f \) (this is why we need \( f \) to be a polynomial and not a general power series). The term of smallest degree not in \( \mathfrak{m} \) is \( f_k X^{d-k} \). By the standard Weierstrass preparation Theorem 9.2 of [8] \( \text{rev}_d(f) = gs \) where \( s \) is an invertible power series in \( \mathfrak{a}[[X]] \) and \( g = X^{d-k} + g_{d-k-1}X^{d-k-1} + \ldots + g_0 \) is a monic polynomial which satisfies \( g_1 \in \mathfrak{m} \). Since \( \deg(\text{rev}_d(f)) = d \) and that \( g \) is monic, necessarily \( s \in \mathfrak{a}[X] \) and is of degree \( k \). We write \( s = s_0 + \cdots + s_k X^k \). Thus:

\[
\text{rev}_d(\text{rev}_d(f)) = f = \text{rev}_{d-k}(g)\text{rev}_k(s) = (1 + g_{d-k-1}X + \ldots + g_0 X^{d-k})(s_k + \cdots + s_0 X^k)
\]

Now since \( s \) is invertible, \( s_0 \in \mathfrak{a}^* \). Letting \( u = s_0 + \cdots + g_0 s_0 X^{d-k} \) and \( q = s_0^{-1} s_k + \cdots + s_0^{-1} s_1 X^{k-1} + X^k \) provides polynomials satisfying \( f = uq \) as well as the requirement of the proposition. □

The polynomial \( q \) can be computed by an Euclidean division. This possibility is a minor adaptation of [8, Theorem 9.1], that we have reproduced in Appendix for sake of completeness.

**Corollary 2.** With the same notations and hypotheses of Proposition 4 given the inverse coefficient \( f_k^{-1} \), the monic polynomial \( q \) can be computed by mimicking the Euclidean division of \( X^k \) by \( f_k X^k + f_{k-1} X^{k-1} + \ldots \) to obtain \( X^k = g f + r \). The power series \( q \) needs to be computed only up to modulo \( X^{d-k+1} \). Followed by the inversion of the truncated power series \( g \) to get: \( q = (X^k - r)g^{-1} \).

**Proof.** By Lemma 7 there exits a unique invertible power series \( g \in \mathfrak{a}[[X]] \) and a unique polynomial \( r \in \mathfrak{a}[X] \) such that \( X^k = g f + r \). Therefore \( (X^k - r)g^{-1} = f \) is the factorization of Proposition 2 by uniqueness. We deduce that \( g^{-1} \) is a polynomial of degree \( d-k \) that verifies the conditions of the aforementioned proposition. In consequence, the power series \( g \) needs to be known only modulo \( X^{d-k+1} \). □

**Example 4** Let \( \mathfrak{a} = k[[x]]/\langle x^2 \rangle \simeq k[x]/\langle x^2 \rangle \) be the complete local ring of maximal ideal \( \mathfrak{m} = \langle x \rangle \), \( f = xy^2 + y + 1 \). The Weierstrass factorization of Proposition 2 insures the existence of a polynomial \( q = y + \cdots \in \mathfrak{a}[y] \) and \( u = u_0 + u_1 y \) with \( u_0 \in \mathfrak{a}^* \) and \( u_1 \in \mathfrak{m} \), such that \( f = uq \). To compute it, Euclidean division as explained in Corollary 3 works as follows (boxed terms are parts of the remainder).

\[
\begin{array}{c|c}
xy^2 + y + 1 & 1 - xy + x \\
-xy^2 & -y \\
yx & -x
\end{array}
\]

We obtain \( y = (xy^2 + y + 1)(1 - xy + x) - x - 1 \), hence \( qu = (y + x + 1)(1 - xy + x)^{-1} = xy^2 + y + 1 \) is the factorization of Proposition 3. Now \( u = (1 - xy + x)^{-1} = 1 + \sum_{i\geq 1} x^i(y-1)^i = 1 + x(y-1) \) in \( (k[x]/\langle x^2 \rangle)[y] \). We check that \( u_0 = 1 - x \in \mathfrak{a}^* \) and \( u_1 = x \in \mathfrak{m} \).

5 Concluding remarks

**Generalization** All steps of Algorithm 4 “largestFactor” extends to more than one variable, except the management of the first nilpotent subresultant. To illustrate this difficulty, let us consider a primary triangular set \( T = (x^2, y^2) \) of radical \( (x, y) \), and of some input polynomials \( a \) and \( b \) in \( (k[x, y]/\langle T \rangle)[z] \). It may happen that a subresultant is equal to \( xz + y \) (for example \( a = z^2 + 2xz - y \) and \( b = z^2 + xz - 2y \)). It is nilpotent and the iterated resultant criterion allows to detect it. But there is no way to “remove” the nilpotent part as in the case of a polynomial of one variable. To apply Weierstrass preparation theorem, the polynomial must not be nilpotent. A solution consists of “adding” this polynomial to the coefficient ring. How this integration shall be done requires more work and will be investigated in the future.

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**Complexity** The running-time of the algorithm is dominated by that of the subresultant. All other subroutines are indeed based on classical algorithms which have a lower cost. Hensel lifting, inversion of a truncated power series are endowed of fast algorithms; And Weierstrass factorization is reduced to an Euclidean division. While the standard Collins-Brown’s version of the subresultant have a quadratic (operations in $k[x]/(p^e)$) cost in the degrees of the input polynomials, fast “divide and conquer” (a.k.a half-gcd) having a quasi-linear time are well devised. We speculate that a fast version of our algorithm has a quasi-linear cost, but this is not straightforward, and the purpose of this article is really about feasibility.

**Cofactors** It would be interesting to consider cofactors in Isomorphism [4]. We mean polynomials $u, v$ and $\{v_j\}_{1 \leq j \leq s}$, such that:

$$au + bv = v_1(H_1 + p^{e_1}h_1) + \cdots + v_s(H_s + p^{e_s}h_s),$$

where:

$$H_i + p^{e_i}h_i \equiv 1 \mod \langle G_i \rangle \text{ for } i = 1, \ldots, s; \deg_y(v_i) < \deg_y(G_i); \deg_y(H_i) \leq \sum_{j \neq i} \deg_y(G_j).$$

The degree constraints on $u$ and $v$ should be made to insure uniqueness while minimizing the degree.

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Appendix

This is a modification of Lang’s proof [8, Theorem 9.1] up to a minor point indicated below, of the Weierstrass division, which is used in Corollary [2].

Lemma 7. With the same notations and assumptions of Proposition [2], given \( g \in a[[X]] \), we can solve the equation \( g = qf + r \) uniquely with \( q \in a[[X]] \) and \( r \in a[X] \) and \( \deg r < k \).

Proof. Define the linear maps \( \alpha : a[[X]] \to a[X] \), \( \sum a_i X^i \to \sum_{i=0}^{k-1} a_i X^i \) and \( \tau : a[[X]] \to a[[X]] \), \( \sum a_i X^i \to \sum_{i} a_{k+i} X^i \). It is clear that \( b \in a[X] \) has degree \( < k \) if and only if \( \tau(b) = 0 \) or \( \alpha(b) = b \). Therefore if \( g = qf + r \), \( \deg(r) < k \) then \( \tau(g) = \tau(qf) \). Moreover \( \tau(X^k h) = h \) and \( h = \tau(h) X^k + \alpha(h) \) for any \( h \in a[[X]] \). Thus

\[
\tau(g) = \tau(q(\alpha(f) + X^k \tau(f))) = \tau(q \alpha(f)) + \tau(f)q.
\]

Let \( Z = \tau(f)q \). Notice that \( \tau(f) = f_k + X m' \) with \( f_k \in a^* \) and \( m' \in ma[[X]] \) by assumption, hence \( \tau(f) \) is invertible in \( a[[X]] \). The equation above can be rewritten: \( \tau(g) = \tau(Z \frac{\alpha(f)}{\tau(f)}) + Z \). Being able to solve this equation in \( Z \) uniquely gives \( \tau(f)q \), hence \( q \), hence \( r = g - fq \) in a unique way.

To do it, the proof differs slightly from that of [8, Theorem 9.2] to the following point. The reason why the image of the map below

\[
\tau \circ \frac{\alpha(f)}{\tau(f)} : a[[X]] \to ma[[X]]
\]

is \( ma[[X]] \) is because \( \tau(f)^{-1} = f_k^{-1} \sum_{t \geq 0} (-1)^t (mX)^t \) for an \( m \in ma[[X]] \) and hence \( \alpha(f) \tau(f)^{-1} = \alpha(f) f_k^{-1} + \alpha(f) M \) where \( M \in ma[[X]] \). Therefore \( \tau(\alpha(f) \tau(f)^{-1}) \in ma[[X]] \) since \( \deg(\alpha(f) f_{k-1}) = \deg(\alpha(f)) \leq k - 1 \). Thus for any power series \( h \in a[[X]] \),

\[
\left( \tau \circ \frac{\alpha(f)}{\tau(f)} \right)(h) \in ma[[X]].
\]

Now \( a[[X]] \) being a complete ring, \( I + \tau \circ \frac{\alpha(f)}{\tau(f)} \) is invertible and \( Z = (I + \tau \circ \frac{\alpha(f)}{\tau(f)})^{-1}(\tau(g)) \) is determined in a unique way. \( \square \)