Indeterministic Quantum Gravity

III. Gravidynamics versus Geometrodynamics: Revision of the Einstein Equation

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Abstract

This paper is a continuation of the papers [1, 2]. A revision of the Einstein equation shows that its dynamic incompleteness, contrary to a popular opinion, cannot be circumvented by so-called coordinate conditions. Gravidynamics, i.e., dynamics for gravitational potentials $g_{\mu\nu}$, is advanced, which differs from geometrodynamics of general relativity in that the former is based on a projected Einstein equation. Cosmic gravidynamics, due to a global structure of spacetime, is complete. The most important result is a possibility of the closed universe with a density below the critical one.

Keywords: general relativity, cosmic time, cosmic space, Einstein equation, quantum, metric, indeterministic

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Introduction

As is generally known, the Einstein equations $G^{\mu\nu} = T^{\mu\nu}$ have a peculiar feature from the standpoint of the Cauchy problem: It is only the equations $G^{ij} = T^{ij}$ that are dynamic ones, whereas the equations $G^{0\mu} = T^{0\mu}$ should be considered as constraints. The incompleteness of the dynamic equations is circumvented by introducing coordinate conditions.

On the other hand, in indeterministic quantum gravity the equations $G^{ij} = T^{ij}$ are dynamically complete, whereas the equations $G^{0\mu} = T^{0\mu}$ are absent at all. This is due to quantum jumps and a global structure of spacetime connected with them.

Thus, there is an essential divergence of spacetime dynamics in general relativity and indeterministic quantum gravity. The main objective of this paper is to elucidate the cause of the divergence and to advance general spacetime dynamics.

First and foremost, we evidence that the circumvention of the dynamic incompleteness of the Einstein equation by coordinate conditions is a fallacy. The reasoning behind these conditions is as follows. It is possible to pick an arbitrary coordinate system, which involves four arbitrary functions. This compensates for the lack of four dynamic equations. But this construction, giving values for metric components $g^{\mu\nu}$, does not determine to which events, i.e., spacetime points these values correspond. Thus, spacetime dynamics of general relativity, or so-called geometrodynamics [3], which is based on the Einstein equation solely, is dynamically incomplete.

A complete spacetime dynamics should be based on the projected (on a spacelike hypersurface) Einstein equation $G^{ij} = T^{ij}$ and some additional geometric conditions. Such a dynamics is called gravidynamics—dynamics for gravitational potentials $g^{\mu\nu}$.

In local problems, the complete covariant Einstein equation $G^{\mu\nu} = T^{\mu\nu}$ must be fulfilled; so that in local gravidynamics the equations $G^{0\mu} = T^{0\mu}$ should result from the basic ones $G^{ij} = T^{ij}$.

In global problems, the cosmic structure of spacetime suffices for the basic equations to provide complete gravidynamics—cosmic gravidynamics.

As local problems, the Schwarzschild solution and gravitational collapse of a dust ball are considered. For these, the results of general relativity and local gravidynamics coincide.

As a global problem, the Robertson-Walker spacetime and the Friedmann universe are examined. The results of general relativity and cosmodynamics, i.e., dynamics of the universe based on cosmic gravidynamics, differ. The most important conclusion of cosmodynamics is a possibility of the closed universe with a density below the critical one.

1 Unidynamics: Unified dynamics of spacetime and matter
1.1 Spacetime and matter

We assume the universe as a physical system to be a pair \((st, m)\), where \(st\) is spacetime, \(m\) is matter. In keeping with general relativity as close as possible, we describe spacetime classically and set

\[
st = (M, g, \nabla),
\]

where \(M\) is a differentiable 4-dimensional manifold, \(g\) is a metric, and \(\nabla\) is the Levi-Civita connection.

The matter may be assumed to be a family of fields (classical or quantum) on \(M\).

1.2 Spacetime and matter dynamics

Dynamics for the universe or its part is a unified dynamics of spacetime and matter, which will be called unidynamics. Since spacetime is described classically, matter dynamics is that of matter fields on a given spacetime. Because matter dynamics depends substantially on the description of the matter, our main concern will be with spacetime dynamics. Dynamic variables of spacetime are represented by a metric, so that spacetime dynamics is that of the metric for a given time evolution of the matter. In the sense of the aforesaid, spacetime dynamics and that of matter are not entangled with each other.

1.3 Local and global structures and problems

A spacetime manifold \(M\) possesses both local and global structures. The local structure in a neighborhood of a point is determined by the values of the metric \(g\) in the neighborhood. An important element of the global structure is a spacelike hypersurface.

It is also reasonable to distinguish between local and global problems of spacetime and unified dynamics. An example of the local problem is the Schwarzschild solution (albeit it is formally defined on an infinite domain). There is, in fact, the unique global problem—that of the universe and its dynamics.

A solution to a local problem may involve only a local structure of spacetime; a solution to the global problem will involve a global structure as well.

1.4 The principle of covariance and the geometric principle

In the local structures and problems, an essential role is played by the principle of covariance: Equations should be phrased in a covariant tensor form. To include global elements, it is efficient to advance the geometric principle: Spacetime structure and dynamic equations should be phrased in geometric form. The principle of covariance is a local form of the geometric principle.

2 Revision of the Einstein equation
2.1 Geometrodynamics: Spacetime dynamics in general relativity

In general relativity, spacetime dynamics, which may be called geometrodynamics \[3\], is based on the Einstein equation

\[ G = T, \tag{2.1} \]

or, in components,

\[ G^\mu_\nu = T^\mu_\nu, \quad \mu, \nu = 0, 1, 2, 3, \tag{2.2} \]

where \( G \) is the Einstein tensor, \( T \) is the energy-momentum tensor. The Einstein equation is local, so geometrodynamics is also local.

2.2 The structure of the Einstein equation

It is common knowledge that the system of eqs.(2.2) has peculiar features from the standpoint of dynamics, i.e., time evolution: The equations

\[ G^0_\mu = T^0_\mu, \quad \mu = 0, 1, 2, 3, \tag{2.3} \]

do not contain the second time derivatives \( \partial^2 g_{\mu\nu}/\partial t^2 \) \((t = x^0)\); the equations

\[ G^{ij} = T^{ij}, \quad i, j = 1, 2, 3, \tag{2.4} \]

contain the second time derivatives only for the \( g_{ij} \). It follows that there are only six dynamic equations (2.4) for 10 components \( g_{\mu\nu} \), whereas the equations (2.3) are constraints on the initial data. More specifically, eqs.(2.4) determine the six derivatives \( \partial^2 g_{ij}/\partial t^2 \), but leave the remaining four, \( \partial^2 g_{0\mu}/\partial t^2 \), undetermined.

2.3 Involvement of matter dynamics in geometrodynamics

The constraints (2.3) may be used as a part of equations of matter dynamics, which implies an involvement of the latter in geometrodynamics, or an entanglement of spacetime dynamics and matter dynamics with each other. This entanglement may be expressed in terms of the equation

\[ T^{\mu\nu} = 0, \tag{2.5} \]

which follows from eq.(2.2). The entanglement imposes restrictions on matter dynamics.

2.4 Coordinate conditions

The universally accepted way of circumventing the incompleteness of the dynamic equations (2.4) consists in introducing coordinate conditions (see, e.g., \[4\]). The reasoning behind these conditions is as follows. Let \( x \) and \( \bar{x} \) be two coordinate systems:

\[ M \supset N \ni p \leftrightarrow x \leftrightarrow \bar{x}, \tag{2.6} \]

where \( p \) is a point of spacetime, i.e., an event. We have

\[ \bar{g}^{0\mu}(\bar{x}) = \frac{\partial \bar{x}^0}{\partial x^\rho} \frac{\partial \bar{x}^\mu}{\partial x^\sigma} g^{\rho\sigma}(x). \tag{2.7} \]
There are four arbitrary functions
\[ \bar{x}^\mu = \bar{x}^\mu(x), \]  
(2.8)
which makes it possible to prescribe given values for \( \tilde{g}^{0\mu} \). This compensates for the lack of four dynamic equations.

2.5 Incompleteness of geometrodynamics

We affirm that the reasoning outlined above is a fallacy. The problem is to determine \( g_p \)—metric as a function of spacetime point, or event \( p \). (This function may be measured \( \ell \).) It is important that the event is independent of the metric. To emphasize this, we quote [3]: “…nature provides its own way to localize a point in spacetime, as Einstein was the first to emphasize. Characterize the point by what happens there! Give a point in spacetime the name ‘event’… The primitive concept of an event… needs no refinement. The essential property here is identifiability, which is not dependent on the Lorentz metric structure of spacetime.”

Let some values for \( \tilde{g}^{0\mu}(\bar{x}) \) be prescribed. Then the problem is to find the functions \( \bar{x}^\mu(p) \). Let some functions \( x^\rho(p) \) be given. Then the problem reduces to finding the functions \( \bar{x}^\mu(x^\rho) \). But the functions \( g^{\rho\sigma}(x) \) are not known, so we cannot find \( \bar{x}^\mu(x^\rho) \) from (2.7).

Here is a parody of the situation outlined above. Let \( v(x) \) be a contravariant vector field on the real axis \(-\infty < x < \infty\). Let \( \bar{x}(x) \) be a new coordinate, then
\[ \bar{v}(\bar{x}) = \frac{d\bar{x}}{dx}v(x). \]  
(2.9)
For a given \( v(x) \), a prescription for \( \bar{v}(\bar{x}) \) determines a function \( \bar{x}(x) \): (2.9) is an equation for the latter. But if \( v(x) \) is not known, \( \bar{x}(x) \) is also unknown, so \( \bar{v}(\bar{x}) \) gives in fact no information on \( v(x) \).

Thus, so-called coordinate conditions do not circumvent the incompleteness of geometrodynamics.

3 Gravidynamics

3.1 Projection of the Einstein equation

In constructing a complete spacetime dynamics, we first of all disentangle spacetime and matter dynamics from each other. Namely, in the general case, we discard the constraints (2.3). The grounds for this are as follows. In a quantum description of matter, eqs.(2.3) take the form
\[ G^{0\mu} = (\Psi, T^{0\mu}\Psi), \]  
(3.1)
where \( T \) is the operator of the energy-momentum tensor, \( \Psi \) is a state vector for matter. The components \( G^{0\mu} \) do not involve the second time derivatives \( \partial^2 g^{\mu\nu}/\partial t^2 \), so that in indeterministic dynamics eqs.(3.1) are violated at quantum jumps, i.e., jumps of \( \Psi \).

Thus, generally, we retain only the dynamic equations (2.4). These represent a projection of the complete Einstein equation (2.2) on a spacelike hypersurface.
3.2 Gravidynamics: Unconstrained geometrodynamics

As spacetime dynamics we will use unconstrained geometrodynamics, i.e., dynamics based on the projected Einstein equation (2.4). That is dynamics for gravitational potentials $g_{\mu\nu}$, in view of which we call it gravidynamics. Thus gravidynamics is spacetime dynamics based on the projected Einstein equation.

3.3 Problems of completeness and geometric formulation

The projected Einstein equation (2.4) neither provides completeness of spacetime dynamics nor imparts a geometric formulation to it. So a proposal for using gravidynamics as a spacetime dynamics gives rise immediately to problems concerned with completeness and the geometric principle. These problems should be dealt with separately for local and global dynamics.

3.4 Local gravidynamics

Local gravidynamics should generally involve no global structure. This implies that for it the geometric principle boils down to the local form, i.e., the principle of covariance. But a covariant equation corresponding to the projected Einstein equation (2.4) is the complete Einstein equation (2.2). It follows that in local gravidynamics the constraints (2.3) should hold, but they must be a consequence of the dynamic equations (2.4).

As for completeness of local gravidynamics, it should be provided in every concrete case by additional physical and/or geometrical conditions.

3.5 Global structure: Cosmic time and space

Global gravidynamics, in addition to the projected Einstein equation (2.4), should involve a global structure of the spacetime manifold $M$, so that a complete spacetime dynamics would have been obtained. The global structure is suggested by, firstly, the Robertson-Walker spacetime and, secondly, indeterministic quantum gravity, which incorporates quantum jumps giving rise to cosmic time. The structure is as follows. The manifold $M$ is a direct product of two manifolds:

$$M = T \times S, \quad M \ni p = (t,s), \quad t \in T, \quad s \in S. \quad (3.2)$$

The 1-dimensional manifold $T$ is the cosmic time, the 3-dimensional one $S$ is the cosmic space. By eq.(3.2) the tangent space at a point $p \in M$ is

$$M_p = T_p \oplus S_p. \quad (3.3)$$

Next it is assumed that

$$T \perp S. \quad (3.4)$$

It follows for the metric tensor

$$g = g_T + g_S. \quad (3.5)$$

Furthermore,

$$g = dt \otimes dt - \tilde{g}_t, \quad (3.6)$$
which gives a geometric, i.e., coordinateless representation for $g$. A relevant coordinate representation is

$$g = dt^2 - \tilde{g}_{ij} dx^i dx^j, \quad g_{0\mu} = \delta_{0\mu}, \quad g_{ij} = -\tilde{g}_{ij}.$$  \hfill (3.7)

3.6 Cosmic gravidynamics

The equation of motion for the metric is the projected Einstein equation

$$G_S = T_S,$$  \hfill (3.8)

or eq.(2.4). There are six equations for six quantities $\tilde{g}_{ij}$, so that spacetime dynamics is complete. It is natural to call this dynamics cosmic gravidynamics.

Note that the form (3.7) for the metric holds in any Gaussian coordinates, but in cosmic gravidynamics the projected Einstein equation (2.4), or (3.8), is generally fulfilled only in cosmic representation given by eqs.(3.2)-(3.5).

4 On the problem of matter dynamics

In this paper, the main line of investigation concerns spacetime dynamics. As for matter dynamics, we restrict the consideration to a selection of problems.

In local gravidynamics, the complete Einstein equation (2.2) holds. Therefore, local matter dynamics should satisfy the equation (2.5) of a vanishing covariant divergence.

Global matter dynamics may or may not satisfy eq.(2.5). This equation does not imply the complete Einstein equation (2.2): The constraints (2.3) are not generally fulfilled.

5 Cosmodynamics

5.1 Cosmic unidynamics

Dynamics for the universe consists of a global spacetime dynamics and a matter dynamics. As the global spacetime dynamics we assume cosmic gravidynamics, so that the corresponding unidynamics may be called cosmic one, or, for the sake of brevity, cosmodynamics. The problem of constructing the latter is that of matter dynamics.

5.2 Contraposition with general relativity

General relativity per se is a local theory, which involves generally no additional global structure of the spacetime manifold. Spacetime dynamics (geometrodynamics) is based on the complete Einstein equation (2.1). Spacetime and matter dynamics are entangled with each other. Geometrodynamics is incomplete.

Cosmodynamics is a global theory, which involves an additional global structure of the spacetime manifold given by eqs.(3.2)-(3.4). Spacetime dynamics (gravodynamics) is based on the projected Einstein equation (3.8). Spacetime and matter dynamics are disentangled from each other. Cosmic gravidynamics is complete.
5.3 Indeterministic cosmodynamics: Indeterministic quantum gravity as cosmodynamics

Indeterministic quantum gravity, which is being developed in this series of papers, is based on cosmic gravidynamics. Matter dynamics is an indeterministic quantum one. So indeterministic quantum gravity may be called indeterministic cosmodynamics (indeterministic implies quantum).

6 The Schwarzschild solution and gravitational collapse of a dust ball

6.1 Treatment in general relativity

By way of example of a local problem, we consider the Schwarzschild solution and gravitational collapse of a dust ball. Our initial concern will be the treatment in general relativity (see, e.g., [3, 4]), or in geometrodynamics.

In the Schwarzschild solution the metric has the form

\[ g = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]  (6.1)

Nontrivial components of the complete Einstein equation are

\[ G^1_1 = 0 \Rightarrow -e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = 0, \]  (6.2)

\[ G^2_2 = G^3_3 = 0 \Rightarrow \nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} = 0, \]  (6.3)

\[ G^0_0 = 0 \Rightarrow -e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} = 0, \]  (6.4)

prime denotes \( d/dr \).

For the dust ball, the metric is

\[ g = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right]. \]  (6.5)

The complete Einstein equation gives

\[ G_{ij} = 8\pi \kappa T_{ij} \Rightarrow k + 2R \ddot{R} + \dddot{R}^2 = 0, \]  (6.6)

\[ G_{0\mu} = 8\pi \kappa T_{0\mu} \Rightarrow R(\ddot{R}^2 + k) = k, \]  (6.7)

where the gravitational constant \( \kappa \) is introduced, point denotes \( d/dt \),

\[ R(0) = 1, \quad \dot{R}(0) = 0, \]  (6.8)

\[ k = \frac{8\pi \kappa}{3} \rho(0), \]  (6.9)
\[ \rho(t) = \rho(0)/R^3(t) \] (6.10)

is the matter density.

Gluing together the interior and exterior solutions leads to

\[ k = \frac{2M\kappa}{a^3}, \] (6.11)

where \( a \) is the ball radius in the comoving coordinate system,

\[ M = \frac{4\pi}{3} \rho(0)a^3 \] (6.12)

is the ball mass.

### 6.2 Treatment in local gravidynamics

In local gravidynamics, for the Schwarzschild problem eqs.(6.2),(6.3) hold, but eq.(6.4) is not given. Differentiating eq.(6.2) we obtain

\[ \nu'' + \frac{\nu' - \lambda'}{r} = \nu'\lambda', \] (6.13)

from which and eq.(6.3) it follows

\[ \nu'(\nu' + \lambda') = 0. \] (6.14)

The solution

\[ \nu' = 0, \quad \nu = \text{const}, \] (6.15)

leads, by eq.(6.2), to

\[ \lambda = 0, \] (6.16)

so that the metric (6.1) is reduced, by \( e^{\nu}dt^2 \to dt^2 \), to the Minkowski one.

The solution

\[ \nu' + \lambda' = 0 \] (6.17)

to the eq.(6.14) reduces eq.(6.2) to eq.(6.4). Thus the constraint (6.4) is a consequence of the projected Einstein equation (6.2),(6.3), so that the Schwarzschild solution holds in local gravidynamics.

For the dust ball, eq.(6.6) is valid, but eq.(6.7) and the value (6.9) for \( k \) are not given. We obtain from eq.(6.6)

\[ \frac{d}{dR}[R(k + \dot{R}^2)] = 0, \quad R(k + \dot{R}^2) = C. \] (6.18)

From the initial conditions (6.8) it follows \( C = k \), so that eq.(6.18) results in eq.(6.7). Gluing together the interior and exterior solutions leads again to the value (6.11), which, combined with eq.(6.12), reduces to the expression (6.9). Thus, local gravidynamics reproduces the results of geometrodynamics.

### 6.3 Agreement of the results

For the local problems considered, the results given by gravidynamics and geometrodynamics coincide. The essence of the matter is that for these problems, the constraints are consequences of the projected Einstein equation.
7 The Robertson-Walker spacetime and the Friedmann universe

7.1 Treatment in general relativity

As a global problem, we consider the Robertson-Walker spacetime and the Friedmann universe. We begin with the treatment in general relativity (see, e.g., [4]).

The Robertson-Walker metric is of the form
\begin{equation}
    g = dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right\}, \quad k = -1, 0, 1.
\end{equation}

For the Friedmann universe, the complete Einstein equation gives
\begin{equation}
    G_{ij} = 8\pi\kappa T_{ij} \Rightarrow 2\ddot{R}R + \dot{R}^2 + k = -8\pi\kappa p R^2, \quad (7.2)
\end{equation}
\begin{equation}
    G_{00} = 8\pi\kappa T_{00} \Rightarrow \ddot{R}^2 + k = \frac{8\pi\kappa}{3} \rho R^2, \quad (7.3)
\end{equation}
where \( p \) is the pressure.

From eqs. (7.2), (7.3) it follows
\begin{equation}
    \dot{\rho} R^3 + 3(\rho + p)R^2 \dot{R} = 0, \quad (7.4)
\end{equation}
which is equivalent to
\begin{equation}
    T^{0\nu} \, _{;\nu} = 0, \quad (7.5)
\end{equation}
or
\begin{equation}
    dE = -pdV, \quad V \sim R^3. \quad (7.6)
\end{equation}

7.2 Treatment in cosmodynamics

In cosmodynamics, the dynamic equation (7.2) holds, but the constraint (7.3) is not given. Let an equation for matter be (7.5), or (7.6). We obtain from (7.6)
\begin{equation}
    p R^2 = -\frac{1}{3} \frac{d(\rho R^3)}{dR}. \quad (7.7)
\end{equation}
This equation leads to eq. (7.4).

We obtain from (7.2)
\begin{equation}
    \frac{d}{dR} (R \dot{R}^2 + kR) = -8\pi\kappa p R^2, \quad (7.8)
\end{equation}
which, combined with eq. (7.7), results in
\begin{equation}
    \frac{d}{dR} \left( R \dot{R}^2 + kR - \frac{8\pi\kappa}{3} \rho R^3 \right) = 0, \quad (7.9)
\end{equation}
i.e.,
\begin{equation}
    R \dot{R}^2 + kR - \frac{8\pi\kappa}{3} \rho R^3 = L = \text{const.} \quad (7.10)
\end{equation}
7.3 Contraposition

In cosmodynamics, there exists an integral of motion $L$ given by eq.(7.10). In general relativity, the only possible value for this integral is, by eq.(7.3), zero:

$$L = 0.$$ (7.11)

Thus, in cosmodynamics the constraint (7.3) does not hold. The condition $L = \text{const}$ is substantially less restrictive than $L = 0$. So cosmodynamics gives a much more general result for the Friedmann model than the result of general relativity.

7.4 On critical density

The constraint (7.3) may be written in the form

$$\frac{\rho - \rho_c}{\rho_c} = k \frac{\dot{R}}{R^2},$$ (7.12)

where

$$\rho_c = \frac{3}{8\pi\kappa} \left( \frac{\dot{R}}{R} \right)^2 = \frac{3}{8\pi\kappa} H^2$$ (7.13)

is the critical density and $H$ is Hubble’s constant.

In cosmodynamics, we obtain from eq.(7.10), in place of (7.12),

$$\frac{\rho - \rho_c}{\rho_c} = \frac{k R - L}{R \dot{R}^2}.$$ (7.14)

This important result implies, specifically, that there is a possibility of a closed universe ($k = 1$) with a density $\rho$ which is smaller than the critical one, $\rho_c$. This is realized as long as the inequality

$$R < L$$ (7.15)

is fulfilled.

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