Injective Stability for $K_1$ of Classical Modules

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2000 Mathematics Subject Classification: 13C10, 13H05, 15A63, 19B10, 19B14

Key words: regular ring, affine algebra, projective modules, $K_1$, $K_1\text{Sp}$

Abstract: In [13], the second author and W. van der Kallen showed that the injective stabilization bound for $K_1$ of general linear group is $d + 1$ over a regular affine algebra over a perfect $C_1$-field, where $d$ is the Krull dimension of the base ring and it is finite and at least 2. In this article we prove that the injective stabilization bound for $K_1$ of the symplectic group is $d + 1$ over a geometrically regular ring containing a field, where $d$ is the stable dimension of the base ring and it is finite and at least 2. Then using the Local-Global Principle for the transvection subgroup of the automorphism group of projective and symplectic modules we show that the injective stabilization bound is $d + 1$ for $K_1$ of projective and symplectic modules of global rank at least 1 and local rank at least 3 respectively in each of the two cases above.

1 Introduction

In this article we discuss the injective stabilization for the $K_1$ group of projective and symplectic modules.

In the early 1960's Bass-Milnor-Serre began the study of the stabilization for the linear group $\text{GL}_n(R)/\text{E}_n(R)$ for $n \geq 3$, where $R$ is a commutative ring with identity. In [3], they showed that $K_1(R) = \text{GL}_{d+3}(R)/\text{E}_{d+3}(R)$, where $d$ is the dimension of the maximum spectrum. (They also showed that $K_1(R) = \text{GL}_3(R)/\text{E}_3(R)$, when Krull dimension of $R$ is 1.) In [19], L.N. Vaserstein proved their conjectured bound of $(d + 2)$ for an associative ring with identity, where $d$ is the stable dimension of the ring. After that, in [20], he studied the orthogonal and the unitary $K_1$-functors, and obtained stabilization theorems for them. He showed that the natural map

$$
\varphi_{n,n+1} : \frac{S(n,R)}{E(n,R)} \to \frac{S(n+1,R)}{E(n+1,R)} \quad \text{in the linear case}
$$

$$
\varphi_{n,n+2} : \frac{S(n,R)}{E(n,R)} \to \frac{S(n+2,R)}{E(n+2,R)} \quad \text{otherwise}
$$

(where $S(n, R)$ is the group of automorphisms of the projective, symplectic and orthogonal modules of rank $n$ with determinant 1, and $E(n, R)$ is the elementary subgroup in the respective cases) is surjective for $n \geq d + 1$ in the linear case, for $n \geq d$ in

*The first author was partially supported by the T.I.F.R. Endowment Fund.
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the symplectic case, and for $n \geq 2d + 2$ in the orthogonal case, and is injective for $n \geq 2d + 4$ in the symplectic and the orthogonal cases. Soon after, in [22], he studied stabilization for groups of automorphisms of modules over rings and modules with quadratic forms over rings with involution, and obtained similar stabilization results.

In [13], the second author and W. van der Kallen showed that if $A$ is a non-singular affine algebra of dimension $d > 1$ over a perfect $C_1$-field (Definition 4.1), then the natural map

$$\frac{\text{SL}_n(A)}{E_n(A)} \rightarrow \frac{\text{SL}_{n+1}(A)}{E_{n+1}(A)}$$

is injective for $n \geq d + 1$. We generalize this result for the automorphism group of finitely generated projective module of global rank at least 1 and local rank at least 3. (By definition the global rank or simply rank of a finitely generated projective module (resp. symplectic or orthogonal $R$-module) is the largest integer $r$ such that $\oplus R$ (resp. $\downarrow \mathbb{H}(R)$) is a direct summand (resp. orthogonal summand) of the module. $\mathbb{H}(R)$ denotes the hyperbolic plane). More precisely, we prove the following: (We assume that $(H1)$ and $(H2)$ holds, as stated in [24]).

**Theorem 1.** Let $A$ be an affine algebra of dimension $d > 1$ over a perfect $C_1$-field $k$. Assume $(d+1)!A = A$. Let $P$ be a finitely generated projective $A$-module of local rank $n > 1$. If $\gamma \in \text{SL}(P)$ is such that $\gamma \perp \{1\} \in \text{Trans}(P \oplus A)$ and $n \geq d + 1$, then $\gamma$ is isotopic to the identity, i.e. there exists an automorphism $\alpha(X) \in \text{SL}(P[X])$ such that $\alpha(0) = \text{Id}$ and $\alpha(1) = \gamma$. Moreover, if $A$ is non-singular, then $\gamma \in \text{Trans}(P)$. In particular, the map $\rho : \frac{\text{SL}(P)}{\text{Trans}(P)} \rightarrow \frac{\text{SL}(P \oplus A)}{\text{Trans}(P \oplus A)}$ is bijective for $n \geq d + 1$.

**Theorem 2.** Let $R$ be a commutative ring with identity of stable dimension $d > 1$ and $A$ be an associative $R$-algebra such that $A$ is finite as a left $R$-module. Let $(P, \langle . \rangle)$ be a symplectic left $A$-module of even local rank $n \geq \max(3, d + 1)$. If $\gamma \in \text{Sp}(P)$ is such that $\gamma \perp I_2 \in \text{Trans}_{\text{sp}}(P \perp A^2)$, then $\gamma$ is isotopic to the identity, i.e. there exists an automorphism $\alpha(X) \in \text{Sp}(P[X])$ such that $\alpha(0) = \text{Id}$ and $\alpha(1) = \gamma$. Moreover, if $A$ is a geometrically regular ring containing a field $k$, then $\gamma \in \text{Trans}_{\text{sp}}(P)$. In particular, the map $\rho_{\text{sp}} : \frac{\text{Sp}(P)}{\text{Trans}_{\text{sp}}(P)} \rightarrow \frac{\text{Sp}(P \perp A^2)}{\text{Trans}_{\text{sp}}(P \perp A^2)}$ is bijective for $n \geq \max(3, d + 1)$.

However, in a companion article [2] we prove that the injective stabilization bound for $K_1$ of the orthogonal group is not less than $2d + 4$, in general, for an affine algebra over a perfect $C_1$-field.

## 2 Preliminaries

**Definition 2.1** Let $R$ be an associative ring with identity. The following condition was introduced by H. Bass in [4]:

$(R_m)$ for every $(a_1, \ldots, a_{m+1}) \in \text{Um}_{m+1}(R)$, there are $\{x_i\}_{1 \leq i \leq m} \in R$ such that $(a_1 + a_{n+1}x_1)R + \cdots + (a_m + a_{n+1}x_m)R = R$.

The condition $(R_m)$ implies $(R_n)$ with $x_i = 0$ for $i \geq m + 1$.
By stable range for an associative ring $R$ we mean the least $n$ such that $(R_n)$ holds.

Although, it appears that we should have referred to the above condition as $R$ having "right" stable range $n$, it has been shown by L.N. Vaserstein ([21], Theorem 2) that "right stable range $n" and "left stable range $n" are actually equivalent conditions. The integer $n - 1$ is called the stable dimension of $R$ and is denoted by $sdim(R)$.

Lemma 2.2 (cf. [4]) If $R$ is a commutative noetherian ring with identity of Krull dimension $d$, then $sdim(R) \leq d$.

Definition 2.3 Let $R$ be an associative ring with identity. To define other classical modules, we need an involutive antihomomorphism (involution, in short) $*: R \rightarrow R$ (i.e., $(x - y)^* = x^* - y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for any $x, y \in R$. We assume that $1^* = 1$. For any left $R$-module $M$ the involution induces a left module structure to the right $R$-module $M^* = Hom(M, R)$ given by $(xf)v = (fv)x^*$, where $v \in M$, $x \in R$ and $f \in M^*$. In this case if $M$ is a left $R$-module then $O_M(m)$ has a right $R$-module structure. But any right $R$ module can be viewed as a left $R$-module via the convention $ma = a^*m$ for $m \in M$ and $a \in R$.

Blanket Assumption: Let $A$ be an $R$-algebra, where $R$ is a commutative ring with identity, such that $A$ is finite as a left $R$-module. Let $A$ possess an involution $*: r \mapsto \tilde{r}$ for $r \in A$. For a matrix $M = (m_{ij})$ over $A$ we define $M^T = (m^T_{ij})$. Let $\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\psi_n = \psi_{n-1} \perp \psi_1$, for $n > 1$. For a column vector $v \in A^n$ we write $\tilde{v} = \tilde{v}^T$. $\psi_n$ in the symplectic case.

We define a form $\langle , \rangle$ as follows:

$$\langle v, w \rangle = \begin{cases} v^T \cdot w & \text{in the linear case} \\ \tilde{v} \cdot w & \text{in the symplectic case.} \end{cases}$$

(Viewing $M$ as a right $A$-module we can assume the linearity).

Since $R$ is commutative, we can assume that the involution "*" defined on $A$ is trivial over $R$. We shall always assume that 2 is invertible in the ring $R$ while dealing with the symplectic case. For definitions of the automorphism group, the symplectic module, and its transvection and its elementary transvection subgroup, see ([1], §2).

Notation 2.4 In the sequel $P$ will denote either a finitely generated projective left $A$-module of local rank $n$, a symplectic left module of even rank $n = 2r$ with a fixed form $\langle , \rangle$. And $Q$ will denote $P \oplus A$ in the linear case and $P \perp A^2$ in the symplectic case. To denote $(P \oplus A)[X]$ in the linear case and $(P \perp A^2)[X]$ in the symplectic case we will use the notation $Q[X]$. We assume that the local rank of projective module is at least 3 when dealing with the linear case and at least 6 when considering the symplectic case. For a finitely generated projective $A$-module $M$ we use the notation $G(M)$ to denote the automorphism group of the projective module $= Aut(M)$ and the group of isometries of the symplectic module $= Sp(M, \langle , \rangle)$. Let $SL(M)$ denote the automorphism group of the projective module with determinant 1 in the case when $A$ is commutative. We use $S(M)$ to denote $SL(M)$ in the linear case and $Sp(M, \langle , \rangle)$ in the symplectic case. Let $T(M)$ denote the transvection subgroup.
of the automorphism group of the projective module $\text{Trans}(M)$, and the transvection subgroup of the automorphism group of the symplectic module $\text{Trans}_{Sp}(M)$. We write $\text{ET}(M)$ to denote the elementary transvection subgroup of the automorphism group of the projectives module $\text{ETrans}(M)$, and the transvection subgroup of the automorphism group of syoplectic module $\text{ETrans}_{Sp}(M)$. (For details see [1]).

We shall assume

(H1) for every maximal ideal $\mathfrak{m}$ of $A$, $Q_{\mathfrak{m}}$ is isomorphic to $A_{\mathfrak{m}}^{2n+2}$ with the standard bilinear form $\mathbb{H}(A_{\mathfrak{m}}^{n+1})$.

(H2) for every non-nilpotent $s \in A$, if the projective module $Q_s$ is free $A_s$-module, then the symplectic module $Q_s$ is isomorphic to $A_s^{2n+2}$ with the standard bilinear form $\mathbb{H}(A_s^{n+1})$.

Notation 2.5 When $P = A^n$ (n is even is the non-linear cases), we also use the notation $G(n, A)$, $S(n, A)$ and $E(n, A)$ for $G(P)$, $S(P)$ and $T(P)$ respectively. We denote the usual standard elementary generators of $E(n, A)$ by $ge_{ij}(x)$, $x \in A$. $e_i$ will denote the column vector $(0, \ldots, 1, \ldots, 0)^t$ (1 at the i-th position).

Remark 2.6 Let $Q$ be as in 2.4. Note that if $\alpha \in \text{End}(Q)$ then $\alpha$ can be considered as a matrix of the form $\begin{pmatrix} \text{End}(P) & \text{Hom}(P, A) \\ \text{Hom}(A, P) & \text{End}(A) \end{pmatrix}$ in the linear case. In the non-linear cases one has a similar matrix for $\alpha$ of the form $\begin{pmatrix} \text{End}(P) & \text{Hom}(P, A \oplus A) \\ \text{Hom}(A \oplus A, P) & \text{End}(A \oplus A) \end{pmatrix}$.

Definition 2.7 An associative ring $R$ is said to be semilocal if $R/\text{rad}(R)$ is artinian semisimple.

Lemma 2.8 (H. Bass) (cf. [4]) Let $A$ be an associative $R$-algebra such that $A$ is finite as a left $R$-module and $R$ be a commutative local ring with identity. Then $A$ is semilocal.

Proof. Since $R$ is local, $R/\text{rad}(R)$ is a division ring by definition. That implies $A/\text{rad}(A)$ is a finite module over the division ring $R/\text{rad}(R)$ and hence is a finitely generated vector space. Thus $A/\text{rad}(A)$ artinian as $R/\text{rad}(R)$ module and hence $A/\text{rad}(A)$ artinian as $A/\text{rad}(A)$ module, so it is an artinian ring. It is known that a right artin ring is semisimple if its radical is trivial. Now $\text{rad}(A/\text{rad}(A)) = 0$, hence it follows that $A/\text{rad}(A)$ is semisimple. Hence $A/\text{rad}(A)$ artinian semisimple. Therefore, $A$ is semilocal by definition.

We recall the well-known Serre’s unimodular theorem:

Theorem 2.9 (J-P. Serre) (cf. [4]) Let $R$ be a commutative noetherian ring of dimension $d$, and let $P$ be a finitely generated projective $R$-module of local rank $\geq d + 1$. Then $P$ contains a unimodular element.

While dealing with the symplectic case we implicitly use the following well-known fact; which we include for completeness.

Lemma 2.10 Let $R$ be a commutative ring with identity and $(P, \langle \cdot, \cdot \rangle)$ be a symplectic $R$-module. If $P$ contains a unimodular element, then $(P, \langle \cdot, \cdot \rangle)$ contains a hyperbolic plane as a direct summand.
Proof. Let $p \in \text{Um}(P)$ and let $\varphi : P \cong P^*$ be the induced isomorphism. Then there exists $\alpha : P \to R$ such that $\alpha(p) = 1$. Since $\langle p, p \rangle = 0$, it follows that $p \neq \varphi^{-1}(\alpha)$. Hence there exists $f \in P$ such that $f \neq p$ and $\varphi(f) = \alpha$. Now if $x \in Rp \cap Rf$, then $x = tp = sf$, for some $t, s \in R$. Since $\langle x, x \rangle = 0$, it follows that $st = 0$. Hence $sx = 0$. This is a contradiction, as $Rp \cong R$. Hence $Rp \cap Rf = 0$. Also $\langle p, f \rangle = 1$; hence $P$ contains $\mathbb{H}(R)$. We claim that $P$ contains $\mathbb{H}(R)$ as a direct summand. Let

$$Q = \{ q \in P | \langle q, f \rangle = 0, \langle q, p \rangle = 0 \}.$$ 

Again, let $y \in Q \cap (Rp \oplus Rf)$. Then $y = ap + bf$ for some $a, b \in R$. Since $\langle y, p \rangle = \langle y, f \rangle = 0$, it will follow that $y = 0$. Hence $Q \oplus (Rp \oplus Rf) \subseteq P$. Now let $z \in P$ be such that $z \neq p$ and $z \neq f$. Let $z' = z - \langle z, p \rangle f + \langle z, f \rangle p$. Then one checks that $z' \in Q$. Hence $P \cong Q \oplus \mathbb{H}(R)$. (Note: $Q$ inherits a symplectic structure from $P$ given by the restriction $\langle \cdot, \cdot \rangle|_Q : Q \times Q \to R$). Hence the result follows.\hfill $\square$

The following theorem is a well known result:

**Theorem 2.11** Let $R$ be an associative ring of stable dimension $d \geq 1$. Then, for $n \geq d + 2$ in the linear case and for $n \geq 2d + 4$ in the symplectic and the orthogonal cases, $E(n, R)$ acts transitively on $\text{Um}_n(R)$. In other words, any unimodular row of length $n$ over $R$ is complete to an elementary matrix if $n \geq d + 2$ in the linear case and $n \geq 2d + 4$ in the symplectic case.

**Proof.** See ([10], Theorem 7.3', pg. 93) for the linear case and ([20], Theorem 2.7) for the symplectic case. (The key to proving it is Lemma [2.2].)\hfill $\square$

**Definition 2.12** For $\alpha \in M(r, R)$ and $\beta \in M(s, R)$ we have $\alpha \perp \beta$ denotes its embedding $M(r + s, R)$ given by ($r$ and $s$ are even in the non linear cases)

$$\alpha \perp \beta = \left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right).$$

There is an infinite counterpart: Identifying each matrix $\alpha \in \text{GL}_n(R)$ with the large matrix $(\alpha \perp \{1\})$ gives an embedding of $\text{GL}_n(R)$ into $\text{GL}_{n+1}(R)$. Let $\text{GL}(R) = \bigcup_{n=1}^{\infty} \text{GL}_n(R)$, $\text{SL}(R) = \bigcup_{n=1}^{\infty} \text{SL}_n(R)$, and $\text{E}(R) = \bigcup_{n=1}^{\infty} \text{E}_n(R)$ be the corresponding infinite linear groups.

**Definition 2.13** The quotient group

$$K_1(R) = \frac{\text{GL}(R)}{[\text{GL}(R), \text{GL}(R)]} = \frac{\text{GL}(R)}{\text{E}(R)}$$

is called the **Whitehead group** of the ring $R$. For $\alpha \in \text{GL}_n(R)$ let $[\alpha]$ denote its equivalence class in $K_1(R)$. Similarly, one can define the Symplectic Whitehead group $K_1\text{Sp}(R)$.

The following theorem is the key result we use to generalize the results known for free modules to classical modules. Here we state the result. For details see [1].

**Theorem 2.14 (Local-Global Principle) (cf. [2])** Let $A$ be an associative $R$-algebra such that $A$ is finite as a left $R$-module and $R$ be a commutative ring with identity.
Let $P$ and $Q$ be as in \textbf{2.4}. Assume that (H1) holds. Suppose $\sigma(X) \in G(Q[X])$ with $\sigma(0) = \text{Id}$. If
\[
\sigma_p(X) \in \begin{cases} 
E(n + 1, A_p[X]) & \text{in the linear case,} \\
E(2n + 2, A_p[X]) & \text{in the symplectic case,}
\end{cases}
\]
for all $p \in \text{Spec}(R)$, then $\sigma(X) \in ET(Q[X])$.

**Corollary 2.15** Let $A$ be an associative $R$-algebra such that $A$ is finite as a left $R$-module and $R$ be a commutative ring with identity. Let $Q$ be as in \textbf{2.4}. Assume that (H1) holds. Then $T(Q) = ET(Q)$.

## 3 Stabilization Bounds for $K_1$ of Classical Modules

In this section we prove Theorem 1 and Theorem 2 stated in Section 1. We will show that the injective stability estimates for $K_1(R)$ and $K_1\text{Sp}(R)$, stated by L.N. Vaserstein in \textbf{22}, can be improved in the linear and the symplectic cases if $R$ is a regular affine algebra over a perfect $C_1$-field. Recall

**Definition 3.1** Let $A$ be an affine algebra of dimension $d$ over a field $k$ satisfying: For any prime $p \leq d$ one of the following conditions is satisfied: (i) $p \neq \text{char } k$, (ii) $p = \text{char } p$ and $k$ is perfect. In this case we say that $A$ is an affine algebra over a perfect $C_1$-field.

Suslin showed that stably free projective modules of top rank $d$ over an affine algebra over a field $k$, in which $d!$ was invertible, are free if $k$ is algebraically closed in \textbf{14}; and over perfect $C_1$-fields in \textbf{17}. His methods were used to prove their cancellative properties in \textbf{5}; who established the following:

**Theorem 3.2** (S.M. Bhatwadekar) (\textbf{5}, Theorem 4.1) Let $A$ be an affine algebra of Krull dimension $d > 1$ over a perfect $C_1$-field $k$. Assume $d!|A = A$. Let $P$ be a projective $A$-module of local rank $d$. Then for $(p,a) \in \text{Um}(P \oplus A)$ there exists $\tau \in \text{Aut}(P \oplus A)$ such that $(p,a)\tau = (0,1)$. In particular, $P$ is cancellative.

**Lemma 3.3** (cf. \textbf{13}) Let $A$ be as in Theorem \textbf{2.2} and let $P$ be a projective $A$-module of local rank $d + 1$, where $d$ is the stable dimension of $A$ and $d > 1$. Let $v(X) \in \text{Um}((P \oplus A)[X])$ with $v(X) \equiv (0,1)$ modulo $(X^2 - X)$. Then there exists $\sigma(X) \in \text{SL}((P \oplus A)[X])$ with $\sigma(X) \equiv 1 \text{d modulo } (X^2 - X)$ such that $v(X)\sigma(X) = (0,1)$.

**Proof.** Our argument is similar to that in \textbf{13}, Proposition 3.3. Let $Y = X^2 - X$ and $B = A[Y,Z]/(Z^2 - YZ)$. Then $B$ is an affine algebra of dimension $d + 1$ over the field $k$. Let $v(X) = e_{d+2}' + Yv'(X)$ with $v'(X) \in (P \oplus A)[X]$. Let $u(Z) = e_{d+2}' + Zv'(X)$ be its lift in $B$. Then $u(Z) \in \text{Um}((P \oplus B) \oplus B)$ as locally it is unimodular. So $u(Y) = v(X)$ and $u(0) = (0,1)$. By Proposition \textbf{2.2} there exists $\beta(Z) \in \text{SL}((P \oplus B)[Z] \oplus B[Z])$ with $u(Z) = \beta(Z)e_{d+2}'$. Take $\sigma = \beta(0)^{-1}\beta(Y)$. \qed

**Lemma 3.4** (\textbf{3}, Chapter IV, Theorem 3.1) Let $A$ be an associative $R$-algebra such that $A$ is finite as a left $R$-module where $R$ is a commutative ring with identity. Let $A$ have stable dimension $d$, and let $P$ be a projective left $A$-module of rank $n$, where
\[ n \geq d + 1. \] Suppose that \((p, a) \in \text{Um}(P \oplus A)\). Then there exists a homomorphism \(f : A \to P\) such that \(p + f(a) \in \text{Um}(P)\) and there exists \(\tau \in \text{Trans}(P \oplus A)\) such that \(\tau : (p, a) \to (0, 1)\).

**Lemma 3.5** (L.N. Vaserstein, [22]) Let \(A\) be a commutative \(R\)-algebra such that \(A\) is finite as a \(R\)-module and \(R\) is a commutative ring with identity. Let \((P, \langle , \rangle)\) be a symplectic left \(A\)-module of even local rank \(n\), where \(n \geq d\).

Let \((p, b, a) \in \text{Um}(P \oplus A^2)\).

1. There exists \(\tau \in \text{Trans}_{\text{Sp}}(P \perp A^2)\) such that \(\tau : (p, b, a) \to (0, 0, 1)\).

2. If \(I\) is a two sided ideal of \(A\) and \((p, b, a) \equiv (0, 0, 1)\) modulo \(I(P \perp A^2)\), then there exists \(\tau \in \text{Trans}_{\text{Sp}}(P \perp A^2, I)\) such that \(\tau : (p, b, a) \to (0, 0, 1)\).

**Proof.** We prove the result for completeness. We follow the line of proof of R.G. Swan, (see [15, Corollary 9.8]) (Also see [5, Theorem 3.2]).

By Lemma 3.4 there exists \(g \in P\) and \(t \in A\) such that \(O(p + aq, b + at) = A\); i.e. \((p + aq, b + at) \in \text{Um}(P \perp A)\). Hence there exists \(\gamma \in P^*\) and \(g \in A\) such that \(\gamma(p + aq) + g(b + at) = 1\). Let \(\eta = g\Phi(q)\), where \(\Phi : P \cong P^*\) is the induced isomorphism. Then \(\eta(p + aq) = -g(p, q)\). Hence \(\delta(p + aq) + g(b + at + (p, q)) = 1\), where \(\delta = \eta + \gamma\). Now consider the following automorphisms (elementary transvections) of \((P \perp A^2)\):

\[
\theta_{(t, a)} : (p, b, a) \mapsto (p + aq, b + at + (p, q), a),
\]

\[
\tau_{(g, \alpha)} : (p, b, a) \mapsto (p - \beta(b), b, a + gb + \delta(p)),
\]

where \(\beta : A \to \text{P}^*\) with \(\beta^* = \delta \Phi^{-1}\). Let \(\tau'_{(g, \beta)} = (1 - a)\tau_{(g, b)}\), \(\tau = \tau_{(-b, -p_1)}\tau_{(g, \beta)}\theta_{(t, a)}\) for \(b_1 = b + ta + (p, q)\), and \(p_1 = p + aq - \beta(b + ta + (p, q))\). Then \(\tau(p, b, a)^t = (0, 0, 1)^t\); as required.

Next assume \((p, b, a) \equiv (0, 0, 1)\) modulo \(I(P \perp A^2)\). As above we get

\[
\delta(p + aq) + g(b + ta + (p, q)) = 1.
\]

Since \(a \equiv 1\) modulo \(I\), then \(\beta = (\delta \Phi^{-1})^* x = x = \Phi^{-1}(x)\). Then it follows that \(\beta^* \Phi = x\delta\). Let \(x\delta = \xi\) and \(\Delta_1 = \theta_{(0, -a)}\theta_{(f, \xi)}\theta_{(t, a)}\). Then \(\Delta_1 \in \text{Trans}_{\text{Sp}}((P \perp A^2), I)\). Now \(\Delta_1(p, b, a)^t = (p', b', 1)^t\) for some \(p' \in P\) and \(b' \in A\) such that \(p' \equiv 0\) modulo \(I\) and \(b' \equiv 0\) modulo \(I\). So it follows that \(\theta_{(p', b')} \equiv \text{Id}\) modulo \(I\). Let \(\Delta = \theta_{(p', b')}\Delta_1\). Then \(\Delta \in \text{Trans}_{\text{Sp}}((P \perp A^2), I)\) and \(\Delta(p, b, a)^t = (0, 0, 1)^t\); as required.

**Lemma 3.6** Let \(A\) be an associative ring with identity and \(P\) and \(Q\) be as in 2.4. Let \(\Delta\) be a matrix in \(G(Q)\). If for \(m \geq 2\), then \(\Delta e_m = e_m\), then \(\Delta \in T(Q)G(P)\).

**Proof.** If \(\Delta e_m = e_m\), then in the linear case \(\Delta\) is of the form

\[
\begin{pmatrix}
\beta & 0 \\
\gamma & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\gamma \beta^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
\beta & 0 \\
0 & 1
\end{pmatrix},
\]

for some \(\beta \in \text{Aut}(P)\), \(\gamma \in P^*\), and in the symplectic case \(\Delta\) is of the form

\[
\begin{pmatrix}
\beta' & p' & 0 \\
0 & 1 & 0 \\
\gamma' & a' & 1
\end{pmatrix}
\begin{pmatrix}
1 & p' & 0 \\
0 & 1 & 0 \\
\gamma' \beta^{-1} & a' & 1
\end{pmatrix}
\begin{pmatrix}
\beta' & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

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for some \( \beta' \in \text{Sp}(P) \), \( a' \in A \), \( p' \in P \) and \( \beta_1 : A \to P \) such that \( \beta_1(1) = p' \), and \( \gamma' : P \to A \) chosen in a way that \( \gamma' = \beta_1^{-1} \Phi \beta' \).

Clearly, \( \begin{pmatrix} 1 & 0 \\ \gamma \beta^{-1} & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & p' \\ 0 & 1 \end{pmatrix} \) are in \( \text{ET}(Q) \subset \text{T}(Q) \), where \( f : A \to A \) given by \( 1 \mapsto a' \). Hence the result follows.

\[ \square \]

**Definition 3.7** Let \( k \) be a field. A ring \( A \) is said to be essentially of finite type over \( k \) if \( A = S^{-1}C \), with \( S \) is a multiplicative closed subset of \( C \), and \( C = k[X_1, \ldots, X_n]/I \) is a quotient ring of a polynomial ring over \( k \).

In \textbf{[23]}, Theorem 3.3, T. Vorst, following ideas of H. Lindel in \textbf{[21]}, proved the following in the linear case:

Let \( A \) be a regular local ring essentially of finite type over a perfect field \( k \). Then

\[ S_r(A[X]) = E_r(A[X]) \]

for \( r \geq 3 \).

This method of proof also proves the result for the symplectic groups. We revisit this proof below. We treat the linear and the symplectic cases uniformly.

**Theorem 3.8** Let \( A \) be a regular local ring essentially of finite type over a perfect field \( k \). Then

\[ S(r, A[X]) = E(r, A[X]), \]

for \( r \geq 3 \) in the linear case, and \( r \geq 6 \) in the symplectic case.

We sketch a proof of this theorem below. To prove the theorem we need to use the ideas (of A. Suslin and H. Lindel) to establish the statements below in the linear case.

**Lemma 3.9** Let \( A \) be a commutative ring with identity and \( S \subset A \) be a multiplicative closed set. If \( G(r, A[X]) = G(r, A)E(r, A[X]) \), then

\[ G(r, S^{-1}A[X]) = G(r, S^{-1}A)E(r, S^{-1}A[X]). \]

**Proof.** Let \( \alpha(X) \in G(r, S^{-1}A[X]) \). Replacing \( \alpha(X) \) by \( \alpha(X)\alpha(0)^{-1} \), we may assume that \( \alpha(0) = \text{Id} \). Let \( f(X) = \det(\alpha(X)^{-1}) \). Then \( f(0) = 1 \). Therefore, there exists \( s_1 \in S \) such that \( \alpha(s_1X) \) and \( f(s_1X) \) are both defined over \( A[X] \). Let \( \alpha_1(X) \in G(r, A[X]) \) and \( f_1(X) \in A[X] \) with \( \alpha_1(0) = I_r \) and \( f_1(0) = 1 \), localizing into \( \alpha(s_1X) \) and \( f(s_1X) \) respectively. Also \( \det(\alpha(s_1X)).f(s_1X) = 1 \). Thus, there exists \( s_2 \in S \) such that \( \det(\alpha_1(s_2X)).f_1(s_2X) = 1 \). Hence it follows that \( \alpha_1(s_2X) \in G(r, A[X]) \).

Therefore, \( \alpha_1(s_2X) = \gamma \prod_{k=1}^{m} g_{e_{i_k,j_k}}(f_k(X)) \) with \( \gamma \in G(r, A) \) and \( f_k(X) \in A[X] \) for all \( 1 \leq k \leq m \). So,

\[ \alpha(X) = \gamma_S \prod_{k=1}^{m} g_{e_{i_k,j_k}}(f_k(X)/s_1s_2)) \]

We shall assume that \( r \geq 3 \), in the linear case, and that \( r \) is even and \( r \geq 6 \) in the symplectic case.
Proposition 3.10 (A. Suslin) Let $A$ be a commutative ring with identity and $h \in A$ be a non-nilpotent. Let $\delta \in G(r, A_h)$ and $\sigma(X) = \delta g e_{k_1}(X, f) h^{-1}$, where $k \neq 1$ and $f \in A_h[X]$. Then there exists a natural number $m$ and a matrix $\tau \in E(r, A[X], XA[X])$ such that $\tau h = \sigma(h^m X)$.

Proof. For the linear case see (15, Lemma 3.3). For the symplectic case it has been asserted in (9, § 3) that a similar proof works as in the orthogonal case; and for the orthogonal case see (15, Lemma 4.6).

Theorem 3.11 (H. Lindel) (11, Proposition 2 and 3) Let $A$ be a regular local ring essentially of finite type over $k$ with dim $A \geq 1$, where $k$ is perfect. Then there exists a subring $B$ of $A$ with a non-zero divisor $h \in B$ such that

1. $B$ is the localization of a polynomial ring over $k$,
2. $Ah + B = A$ and $Ah \cap B = Bh$.

The following was proved by T. Vorst in the linear case in (23, Lemma 2.4):

Lemma 3.12 Let $A$ be a commutative ring with identity, $B \subset A$, and $h \in B$ be a non-nilpotent.

1. If $Ah + B = A$, then for every $\alpha \in E(r, A_h)$ there exist $\beta \in E(r, B_h)$ and $\gamma \in E(r, A)$ such that $\alpha = \gamma_i \beta$.
2. If moreover $Ah \cap B = Bh$ and $h$ is a non-zero-divisor in $A$, then for every $\alpha \in G(r, A)$ with $\alpha_h \in E(r, A_h)$ there exist $\beta \in G(r, B)$ and $\gamma \in E(r, A)$ such that $\alpha = \gamma \beta$.

Proof. (1): Assume that $\alpha = \prod_{k=1}^{m} g e_{i_k, j_k}(c_k)$ with $c_k \in A_h$. From hypothesis it follows that $Ah^n + B = A$ for all $n$. Hence for all $1 \leq k \leq m$ we can find $a_k \in A$, $b_k \in B$ and a natural number $m_k$ such that

$$c_k = \frac{b_k}{h^{m_k}} + a_k h^s.$$

Let $\sigma_p = \prod_{k=1}^{p} g e_{i_k, j_k}(c_k)$, $(1 \leq p \leq m)$. By Proposition 3.10 there exists a natural number $s$ and $\tau_p(X) \in E(r, A[X], XA[X])$ such that

$$\tau_p(X) = \sigma_p g e_{i_p, j_p}(h^n X) \sigma_p^{-1}.$$ 

So we have

$$\alpha = \prod_{k=1}^{m} g e_{i_k, j_k} \left( \frac{b_k}{h^{m_k}} \right) g e_{i_k, j_k}(a_k h^s) = \prod_{k=1}^{m} \sigma_k g e_{i_k, j_k}(a_k h^s) \sigma_k^{-1} \prod_{k=1}^{m} g e_{i_k, j_k} \left( \frac{b_k}{h^{m_k}} \right).$$

Now let $\gamma = \prod_{k=1}^{m} \tau_k(a_k) \in E(r, A)$ and $\beta = \prod_{k=1}^{m} g e_{i_k, j_k} \left( \frac{b_k}{h^{m_k}} \right)$. Then we are done.

(2): By hypothesis it follows that $Ah^n \cap B = Bh^n$ for all $n$. Hence $B_h \cap A = B$. Using (1) we can write $\alpha_h = \gamma_h \beta$ with $\gamma \in E(r, A)$ and $\beta \in E(r, B_h)$. Now $\gamma^{-1} \alpha = G(r, B)$ and $\beta \in G(r, B_h)$. Moreover $(\gamma^{-1} \alpha)_h = \beta$. But this implies that
\(\gamma^{-1}\alpha \in G(r, B)\). Hence \(\alpha = \gamma(\gamma^{-1}\alpha) \in E(r, A)G(r, B)\).

\[\square\]

**Proof of the Theorem 3.8** We prove the theorem by induction on \(\dim A\). If \(\dim A = 0\) then \(A\) is a field and the result follows. So we assume that \(\dim A \geq 1\).

Let \(\alpha(X) \in S(r, A[X])\). As the hypothesis of Lemma 3.11 is satisfied, we can find a ring \(B\) and can choose \(h \in B\) as in Lemma 3.11. Since \(\dim A_h < \dim A\), by induction hypothesis we have that \(\alpha_h(X) \in E(r, A_h[X])\). Since \(A\) is a regular local ring, we have that \(h\) is a non-zero-divisor in \(A[X]\). Now by applying Lemma 3.12 to \(\alpha(X)\), we get

\[\alpha(X) = \gamma(X)\beta(X)\]

with \(\beta(X) \in G(r, A[X])\) and \(\gamma(X) \in E(r, A[X])\). Hence we have

\[\alpha(X) = \gamma(X)\gamma(0)^{-1}\beta(0)^{-1}\beta(X),\]

where the first two factors are contained in \(E(r, A[X])\). Since the theorem is true for a polynomial ring over a field (proved in [15], Corollary 6.7) by A. Suslin for the linear case (and similarly other cases are also true due to monic inversion) and \(B\) is a localization of a polynomial ring the theorem is also true for \(B\) by Lemma 3.9. Hence

\[\beta(0)^{-1}\beta(X) \in E(r, B[X]) \subset E(r, A[X])\].

\[\square\]

**Theorem 3.13** Let \(A\) be a geometrically regular local ring containing a field \(k\). Then \(S(r, A[X]) = E(r, A[X])\), for \(r \geq 3\), in the linear case, and \(r \geq 6\), in the symplectic case.

**Proof.** If \(\dim(A) = 0\), then \(A\) is a field, and the result follows. Therefore, we assume that \(\dim(A) \geq 1\). In [12], D. Popescu showed that if \(A\) is a geometrically regular local ring, or when the characteristic of the residue field is a regular parameter in \(R\), then it is a filtered inductive limit of regular local rings essentially of finite type over the integers. Hence by Theorem 3.8 it follows that

\[S(r, A[X]) = E(r, A[X])\]

for all \(r \geq 3\) in the linear case and \(r \geq 6\) in the symplectic case.

\[\square\]

**Remark 3.14** Theorem 3.13 is not true for the orthogonal group. It is not true that \(S(r, A) = E(r, A)\), for \(r \geq 4\), for the orthogonal group, in general, even in the case when \(A\) is a field. This is known classically due to results of Dieudonne, since the spinor norm is surjective. In the case when \(A\) is a local ring similar results have been obtained by W. Klingenberg (see [8], [7]), and the references therein for the field case.

**Remark 3.15** The proof of Theorem 3.13 can be used to show that if \(A\) is a geometrically regular local ring containing a field \(k\) then a stably elementary orthogonal matrix \(\sigma(X) \in SO_{2n}(A[X])\), \(n \geq 3\), with \(\sigma(0) = I_{2n}\), is an elementary orthogonal matrix.

**Remark 3.16** Using “deep splitting technique” as in [6], Definition 3.6, Corollary 3.9) one can show that Lemma 3.12 is valid for \(r = 4\). Consequently, Theorem 3.8 and Theorem 3.13 are also valid for \(r = 4\). The above remark is also true when \(n = 2\).

We now establish the main theorems stated in the Introduction.

To have a uniform notation in Theorem 1 we use the notation \(\tilde{S}(P)\) to denote \(SL(P)\) and \(Sp(P)\) and \(\tilde{T}(P)\) to denote \(Trans(P)\) and \(Trans_{0}(P)\).
Proof of Theorem 1 and 2. The homotopy technique used here is as in (13, Proposition 3.4). In view of L.N. Vaserstein’s result in [20], to prove the result it is enough to prove the injectivity for $n = d + 1$. Let $n_1 = n + 1$ and $n + 2$ in the linear and the symplectic cases respectively. Let $\bar{Q}$ denote $P \oplus A$ in the linear case and $P \perp A^2$ in the symplectic case. Consider $\gamma \in \tilde{S}(P)$ such that $\tilde{\gamma} = \gamma \perp \text{Id} \in \tilde{T}(Q)$. Let $\eta(X)$ be the isotopy between $\tilde{\gamma}$ and identity. As before, viewing $\eta(X)$ as a matrix (as in 2.6), it follows that $v(X) \equiv e_{n_1}$ modulo $(X^2 - X)$. Using Lemma 3.3 for Theorem 1 and Lemma 3.5 for Theorem 2 over $A[X]$ it follows that there exists $\sigma(X) \in \tilde{S}(\bar{Q}[X])$ such that $\sigma(X)^t v(X) = e_{n_1}$ and $\sigma(X) \equiv \text{Id}$ modulo $(X^2 - X)$. Therefore, $\sigma(X)^t \eta(X) e_{n_1} = e_{n_1}$. Hence by Lemma 3.6, $\sigma(X)^t \eta(X) = \xi(X) \tilde{\eta}(X)$, where $\xi(X) \in \tilde{E}(\bar{Q}[X])$ and $\tilde{\eta}(X) \in \tilde{S}(P[X])$. Since $\sigma(X) \equiv \text{Id}$ modulo $(X^2 - X)$, $\tilde{\eta}(X)$ is an isotopy between $\gamma$ and the identity.

Now assume $A$ is regular and contains a field $k$. Hence for every prime ideal $p \in \text{Spec}(A)$,

$$\tilde{\eta}_p(X) \in \begin{cases} \tilde{S}(n, A_p[X]) = \tilde{E}(n, A_p[X]) & \text{in the linear case,} \\ \tilde{S}(2n, A_p[X]) = \tilde{E}(2n, A_p[X]) & \text{in the symplectic case.} \end{cases}$$

(by Theorem 3.13). Since $\tilde{\eta}(0) = \text{Id}$, by the L-G Principle (Theorem 2.14) for the tranvection groups we get $\tilde{\eta}(X) \in \tilde{ET}(P[X])$. Whence $\gamma = \tilde{\eta}(1) \in \tilde{ET}(P)$; as required.

Acknowledgement: The authors thank Professor R.G. Swan profusely for his many illuminating comments and corrections, and for his unstinting support and encouragement.

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