SPECTRAL JACOBI-GALERKIN METHODS AND ITERATED METHODS FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND WITH WEAKLY SINGULAR KERNEL

Yin Yang

Hunan Key Laboratory for Computation and Simulation in Science and Engineering
Key Laboratory of Intelligent Computing & Information Processing of Ministry of Education
School of Mathematics and Computational Science, Xiangtan University
Xiangtan 411105, Hunan, China

Yunqing Huang

Hunan Key Laboratory for Computation and Simulation in Science and Engineering
School of Mathematics and Computational Science, Xiangtan University
Xiangtan 411105, Hunan, China

Abstract. We consider spectral and pseudo-spectral Jacobi-Galerkin methods and corresponding iterated methods for Fredholm integral equations of the second kind with weakly singular kernel. The Gauss-Jacobi quadrature formula is used to approximate the integral operator and the inner product based on the Jacobi weight is implemented in the weak formulation in the numerical implementation. We obtain the convergence rates for the approximated solution and iterated solution in weakly singular Fredholm integral equations, which show that the errors of the approximate solution decay exponentially in $L^\infty$-norm and weighted $L^2$-norm. The numerical examples are given to illustrate the theoretical results.

1. Introduction. The integral equations with weakly singular kernels cover many important applications. Integral equations of this kind arise from potential problems, Dirichlet problems, the description of hydrodynamic interaction between elements of a polymer chain in solution, mathematical problems of radiative equilibrium and transport problems.

Let $S$ is a linear operator with weakly singular kernel defined on the Banach space $X = L^2[a, b]$ or $C[a, b]$ to $X$ by

$$Sy(t) = \int_a^b \tilde{K}(t, \tau)y(\tau)d\tau, \quad a \leq t \leq b,$$

where the kernel $\tilde{K}(t, \tau)$ is of the form

$$\tilde{K}(t, \tau) = |t - \tau|^{-\gamma}K(t, \tau), \quad 0 < \gamma < 1,$$

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* Corresponding author: Yin Yang.
and $K(t, \tau) \in C^1([a, b] \times [a, b])$. We are interested for Fredholm integral equations of the second kind,

$$y(t) = \int_a^b \bar{K}(t, \tau)y(\tau)d\tau + g(t), \quad (1)$$

$g$ is a given function, and $y \in X$ is an unknown function to be determined. The equation (1) can be reformulated as

$$(I - S)y = g, \quad (2)$$

where $I$ be the identity operator defined on $X$.

The numerical treatment of (1) is not simple, mainly due to the fact that the solutions of (1) usually have a weak singularity at $t = a$ or $t = b$, even when the inhomogeneous term $g(t)$ is regular, which is discussed in [3]. In the last decade there has been considerable interest in the numerical analysis of solutions of integral equations with weakly singular kernels. Collocation Spectral methods and the corresponding error analysis have been provided recently [18, 20] for for Volterra integral equation without the singular kernel in case of the underlying solutions are smooth. In [26], the authors extended the Legendre-collocation methods to nonlinear Volterra integral equations. Chen and Tang [5, 6, 21, 27, 28] developed a novel spectral Jacobi-collocation method to solve for Volterra integral equation with singular kernel and provided a rigorous error analysis which theoretically justifies the spectral rate of convergence, see also [23] for Jacobi spectral-collocation method for fractional integro-differential equations. Recently, in [20], the authors provided a Legendre spectral Galerkin method for second-kind Volterra integral equations, [22, 19, 24, 25] provide general spectral Jacobi-Petrov-Galerkin approaches for the second kind Volterra integro-differential equations.

In this paper, we apply spectral Jacobi-Galerkin and iterated Galerkin method for weakly singular Fredholm integral equations of the second kind. We use Jacobi polynomials as basis functions to find the approximate solution and iterated solution in weakly singular Fredholm integral equations of the second kind. The purpose of this paper is to obtain the convergence rates for the approximated solution and iterated solution in weakly singular Fredholm integral equations of the second kind using global polynomial bases.

This paper is organized as follows. In Section 2, we demonstrate the implementation of the spectral Jacobi-Galerkin method and corresponding iterated method for Fredholm integral equations with weakly singular kernels. In Section 3, we obtain the convergence results in $L^\infty$ norm and weighted $L^2$-norm. In Section 4, we demonstrate the implementation of the pseudo-spectral Jacobi-Galerkin method and corresponding iterated method for Fredholm integral equations with weakly singular kernels. In Section 5, we obtain the convergence results in $L^\infty$ norm and weighted $L^2$-norm. In Section 6 we present numerical results. Finally, we end with conclusion and future work.

2. Spectral Jacobi-Galerkin methods. Set $\Lambda = [-1, 1]$ and $\omega^{\alpha, \beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$ be a weight function in the usual sense, for $\alpha, \beta > -1$. The set of Jacobi polynomials $\{J_n^{\alpha, \beta}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{\alpha, \beta}}(\Lambda)$-orthogonal system, where $L^2_{\omega^{\alpha, \beta}}(\Lambda)$ is a weighted space defined by

$$L^2_{\omega^{\alpha, \beta}}(\Lambda) = \{v : v \text{ is measurable and } \|v\|_{\omega^{\alpha, \beta}} < \infty\},$$
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equipped with the norm
\[ \| v \|_{\omega^{\alpha, \beta}} = \left( \int_{-1}^{1} |v(x)|^2 \omega^{\alpha, \beta}(x) \,dx \right)^{\frac{1}{2}}, \]
and the inner product
\[ (u, v)_{\omega^{\alpha, \beta}} = \int_{-1}^{1} u(x)v(x) \omega^{\alpha, \beta}(x) \,dx \quad \forall u, v \in L^2_{\omega^{\alpha, \beta}}(\Lambda). \]

Further, define
\[ H^m_{\omega^{\alpha, \beta}}(\Lambda) = \{ u : D^k u \in L^2_{\omega^{\alpha, \beta}}(\Lambda), 0 \leq k \leq m \}, \]
equipped with the norm
\[ \| u \|_{H^m_{\omega^{\alpha, \beta}}} = \left( \sum_{k=0}^{m} \left\| \frac{d^k u}{dx^k} \right\|_{\omega^{\alpha, \beta}}^2 \right)^{\frac{1}{2}}, \]
and seminorm
\[ |u|_{H^m_{\omega^{\alpha, \beta}}} = \left( \sum_{k=\min(m, N+1)}^{m} \left\| \frac{d^k u}{dx^k} \right\|_{\omega^{\alpha, \beta}}^2 \right)^{1/2}. \]

For the sake of applying the theory of orthogonal polynomials, we change the interval \([a, b]\) to the standard interval \(\Lambda = [-1, 1]\), we use the variable transformation
\[ t = \frac{b-a}{2}x + \frac{b+a}{2}, \quad x = \frac{2t}{b-a} - \frac{b+a}{b-a}, \]
\[ \tau = \frac{b-a}{2}s + \frac{b+a}{2}, \quad s = \frac{2\tau}{b-a} - \frac{b+a}{b-a}, \]
and let
\[ u(x) = y \left( \frac{b-a}{2}x + \frac{b+a}{2} \right), \quad f(x) = g \left( \frac{b-a}{2}x + \frac{b+a}{2} \right), \]
the Fredholm integral equations of the second kind with weakly singular kernel in one dimension (2) is of the form
\[ (I - G)u = f, \quad \text{(3)} \]
f is a given function, \(u \in \mathcal{X}\) is an unknown to be determined and \(I\) denote the identity operator from \(\mathcal{X}\) into itself, the integral operator \(G\) defined on the Banach space \(\mathcal{X} = L^2_{\omega^{\alpha, \beta}}[-1, 1]\) or \(C[-1, 1]\) by
\[ Gu = \int_{-1}^{1} \tilde{k}(x, s) u(s) \,ds, \]
where
\[ \tilde{k}(x, s) = |x-s|^{-\gamma}k(x, s), \quad \text{(4)} \]
and
\[ k(x, s) = \left( \frac{b-a}{2} \right)^{1-\gamma} K \left( \frac{b-a}{2} (1 + x), \frac{b-a}{2} (1 + s) \right). \]

\(G\) is a compact linear operator on \(\mathcal{X}\) into \(\mathcal{X}\). Then, the problem (3) reads: find \(u = u(x)\) such that
\[ u(x) = (Gu)(x) + f(x), \quad x \in \Lambda. \quad \text{(5)} \]
Let us demonstrate the numerical implementation of the spectral Jacobi-Galerkin approach first. Denote by $\mathbb{N}$ the set of all nonnegative integers. For any $N \in \mathbb{N}$, $P_N$ denotes the set of all algebraic polynomials of degree at most $N$ in $\Lambda$, $J^\alpha_1(x)$ is the $j$-th Jacobi polynomial corresponding to the weight function $\omega_{\alpha,\beta}(x)$. As a result, span\{$J^\alpha_{1}(x), J^\alpha_{2}(x), \ldots, J^\alpha_{N}(x)$\} as an orthonormal bases for the space $P_N$, and $P_N$ is the subspaces of $L^2_{\omega_{\alpha,\beta}}$.

Our spectral Jacobi-Galerkin approximation of (5) is now defined as: Find $u_N \in P_N$ such that
\[
\langle u_N, v_N \rangle = \langle Gu_N, v_N \rangle + \langle f, v_N \rangle, \quad \forall v_N \in P_N,
\]
where
\[
\langle u, v \rangle := \langle u, v \rangle_{\omega_{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega_{\alpha,\beta}(x)dx,
\]
is the continuous inner product.

Let $\Pi^\alpha_\beta : L^2_{\omega_{\alpha,\beta}} \to P_N$ be the orthogonal projection defined by
\[
\Pi^\alpha_\beta u = \sum_{j=0}^{N} \langle u, J^\alpha_{j}\rangle J^\alpha_{j}, \quad \forall u \in L^2_{\omega_{\alpha,\beta}},
\]

According to (6) and the definition of the projection operator $\Pi^\alpha_\beta$, the spectral Jacobi-Galerkin solution $u_N$ satisfies
\[
u_N = \Pi^\alpha_\beta Gu_N + \Pi^\alpha_\beta f.
\]
Set $u_N(x) = \sum_{j=0}^{N} \xi_j J^\alpha_j(x)$. Substituting it into (6) and taking $v_N = J^\alpha_j(x)$, we obtain
\[
\sum_{j=0}^{N} \xi_j (J^\alpha_j, J^\alpha_j)_{\omega_{\alpha,\beta}} = \sum_{j=0}^{N} \xi_j (J^\alpha_j, GJ^\alpha_j)_{\omega_{\alpha,\beta}} + (J^\alpha_j, f)_{\omega_{\alpha,\beta}},
\]
which leads to an equation of the matrix form
\[
A\xi = B\xi + C
\]
where
\[
\xi = [\xi_0, \xi_1, \ldots, \xi_N]^T, \quad A_{i,j} = (J^\alpha_i, J^\alpha_j)_{\omega_{\alpha,\beta}}, \quad B_{i,j} = (J^\alpha_i, GJ^\alpha_j)_{\omega_{\alpha,\beta}}, \quad C_i = (J^\alpha_i, f)_{\omega_{\alpha,\beta}}.
\]

It is well-known that the iterated solution may improve order of convergence for the original solution of the equations. The iterated spectral Galerkin solution $u^\text{it}_N$ corresponding to spectral Galerkin solution $u_N$ given by (8), is defined as follows:
\[
u^\text{it}_N(x) = \int_{-1}^{1} \tilde{k}(x, s)u^N(s)ds + f(x)
\]
which gives
\[
\Pi^\alpha_\beta u^\text{it}_N = u_N
\]
and
\[
u^\text{it}_N = G\Pi^\alpha_\beta u^\text{it}_N + f.
\]
3. Convergence analysis for spectral Jacobi-Galerkin method. In this section, we discuss the convergence rates for spectral Jacobi-Galerkin approximated solution and iterated spectral Jacobi-Galerkin solution for the Fredholm integral equations of the second kind with weakly singular kernel (4) and kernels of this type is common in integral equations resulting from solving the boundary value problems of partial differential equations. We will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

**Proposition 1.** ([4, 10]) Let $\Pi_N^{\alpha,\beta} : X \to X$ denote the orthogonal projection defined by (7). Then the projection $\Pi_N^{\alpha,\beta}$ satisfies the following conditions:

- (C1) $\{\Pi_N^{\alpha,\beta} : N \in \mathbb{N}\}$ is uniformly bounded in $L^2$-norm, i.e., $\|\Pi_N^{\alpha,\beta}\|_2 \leq M$.
- (C2) There exists a constant $C > 0$ such that for any $N \in \mathbb{N}$ and $u \in X$,
  \[ \|\Pi_N^{\alpha,\beta}u - u\|_2 \leq \inf_{\chi \in X_N} \|u - \chi\|_2. \]

**Definition 3.1.** Let $X$ be a Banach space and $T$ and $T_N$ are bounded linear operators from $X$ into $X$. Then $\{T_N\}$ is said to be $\psi$-convergent to $T$, if
  \[ \|T_N\| \leq C, \quad \|(T_N - T)T\| \to 0, \quad \|(T_N - T)T_N\| \to 0, \quad \text{as} \quad N \to \infty. \]

**Lemma 3.2.** ([4, 10]) Let $\Pi_N^{\alpha,\beta}$ be the orthogonal projection defined by (7), then for any $u \in H^{m,\beta}(\Lambda)$ and $m \geq 1$,
  \[ \|u - \Pi_N^{\alpha,\beta}u\|_{w^{\alpha,\beta}} \leq CN^{-m}\|u\|_{H^{m,\beta}}, \quad (13) \]
  \[ \|u - \Pi_N^{\alpha,\beta}u\|_{\infty} \leq CN^{2-m}\|u\|_{H^{m,\beta}}. \quad (14) \]

**Lemma 3.3.** ([2]) Let $X$ be a Banach space and $S \subset X$ is a relatively compact set. Assume that $T$ and $T_N$ are bounded linear operators from $S \subset X$ into $S \subset X$ satisfying $\|T_N\| \leq C$ for all $N \in \mathbb{N}$, and for each $x \in S$,
  \[ \|T_Nx - Tx\| \to 0, \quad \text{as} \quad N \to \infty \]
  where $C$ is a constant independent of $N$. Then $\|T_Nx - Tx\| \to 0$ uniformly for all $x \in S$.

**Lemma 3.4.** (see [16, 17]) For a nonnegative integer $r$ and $\kappa \in (0, 1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}([-1, 1])$, there exists a polynomial function $K_Nv \in P_N$ such that
  \[ \|v - K_Nv\|_\infty \leq C_{r,\kappa}N^{-(r+\kappa)}\|v\|_{r,\kappa}, \quad (15) \]
  where $\|\cdot\|_{r,\kappa}$ is the standard norm in $C^{r,\kappa}([-1, 1]), K_N$ is a linear operator from $C^{r,\kappa}([-1, 1])$ into $P_N$, as stated in [16, 17].

**Lemma 3.5.** (see [7]) Let $\kappa \in (0, 1)$ and let $G$ be defined by
  \[ (Gv)(x) = \int_{-1}^1 (x - \tau)^{-\kappa} K(x, \tau)v(\tau)d\tau. \]
  Then, for any function $v \in C([-1, 1])$, there exists a positive constant $C$ such that
  \[ \frac{|Gv(x') - Gv(x'')|}{|x' - x''|} \leq C \max_{x \in [-1, 1]} |v(x)|, \]
under the assumption that $0 < \kappa < 1 - \mu$, for any $x', x'' \in [-1, 1]$ and $x' \neq x''$. This implies that
\[
\| Gu \|_{0, \kappa} \leq C \max_{x \in [-1, 1]} |v(x)|, \quad 0 < \kappa < 1 - \mu.
\]

Here and below, $C$ denotes a positive constant which is independent of $N$, and whose particular meaning will become clear by the context in which it arises.

**Theorem 3.6.** Let the kernel $\tilde{k}(x, s)$ satisfies the conditions
\[
\begin{align*}
(A1) & \quad \sup_{x \in [-1, 1]} \int_{-1}^{1} |\tilde{k}(x, s)|ds < \infty, \\
(A2) & \quad \lim_{N \to \infty} \int_{-1}^{1} |\tilde{k}(x, s) - \tilde{k}(a, s)|^2 ds = 0, \quad -1 \leq a \leq 1.
\end{align*}
\]

Let $\Pi_{N}^{\alpha, \beta}$ satisfies (C1) and (C2). Then $\Pi_{N}^{\alpha, \beta}$ is $v$-convergent to $G$ and there is a positive integer $N$ such that for all $n \geq N$, the inverse $(I - \Pi_{N}^{\alpha, \beta})^{-1}$ exists as linear operator defined on $C[-1, 1]$ and there exists positive constants $M$ independent of $N$ such that $\| (I - \Pi_{N}^{\alpha, \beta})^{-1} \|_{\infty} \leq M$.

**Proof.** We need to show that $\Pi_{N}^{\alpha, \beta}$ is $v$-convergent to $G$, i.e. (i) $\| \Pi_{N}^{\alpha, \beta} \| \leq C$, (ii) $\| (\Pi_{N}^{\alpha, \beta} - G)G \| \to 0$ as $N \to \infty$, (iii) $\| (\Pi_{N}^{\alpha, \beta} - G)\Pi_{N}^{\alpha, \beta} \| \to 0$, as $N \to \infty$. To prove (i), consider $u \in C[-1, 1]$, then
\[
\begin{align*}
\| \Pi_{N}^{\alpha, \beta} u(x) - Gu(x) \| = & \left| \int_{-1}^{1} \tilde{k}(x, s) [\Pi_{N}^{\alpha, \beta} u(x) - u(x)] ds \right| \\
= & \left[ \int_{-1}^{1} |\tilde{k}(x, s)|^2 ds \right]^{1/2} \left[ \int_{-1}^{1} |\Pi_{N}^{\alpha, \beta} u(x) - u(x)|^2 ds \right]^{1/2}
\end{align*}
\]

By the condition (A1), it follows that,
\[
\| (\Pi_{N}^{\alpha, \beta} - G)u \|_{\infty} = \sup_{x \in [-1, 1]} |\Pi_{N}^{\alpha, \beta} u(x) - Gu(x)| \leq A \| \Pi_{N}^{\alpha, \beta} u - u \|_{2},
\]
using the estimate (14)
\[
\lim_{N \to \infty} \| \Pi_{N}^{\alpha, \beta} - G \|_{\infty} = 0.
\]

Since $\Pi_{N}^{\alpha, \beta}$ converges to $G$ pointwise for $u$ in $C[-1, 1]$ and $C[-1, 1]$ is complete with respect to $\| \cdot \|_{\infty}$, it follows from Banach-Steinhaus Theorem that
\[
\sup_{N} \| \Pi_{N}^{\alpha, \beta} \|_{\infty} = B < \infty,
\]
which proves (i).

To prove the (ii), let $B$ be a closed unit ball in $C[-1, 1]$. Since $G$ is a compact operator, the set $\{ S = Gx : x \in B \}$ is a relatively compact set in $C[-1, 1]$. Then it follows that
\[
\| (\Pi_{N}^{\alpha, \beta} - G)G \|_{\infty} = \sup \{ \| (\Pi_{N}^{\alpha, \beta} - G)Gu \|_{\infty} : u \in B \}
\]
\[
= \sup \{ \| (\Pi_{N}^{\alpha, \beta} - G)u \|_{\infty} : u \in s \} \to 0, \text{ as } N \to \infty.
\]
This proves (ii).

Now to prove (iii), let the set $S'$ be defined by $S' = \{ \Pi_{N}^{\alpha, \beta} u : N \geq 1, u \in B \}$, where $B$ denotes the closed unit ball in $C[-1, 1]$. To show that $S'$ is relatively compact set, by Arzela-Ascoli Theorem, we need to prove $S'$ is bounded and equicontinuous subset of $C[-1, 1]$. Using the estimate (16), $S'$ is bounded. To prove the
equicontinuity of, let \( \delta > 0 \), and \( x, x' \in C[-1, 1] \), with \( |s-s'| \leq \delta \). For \( u \in C[-1, 1] \),

\[
|G\Pi_N^{\alpha,\beta} u(x) - G\Pi_N^{\alpha,\beta} u(x')| = \left| \int_{-1}^{1} [\tilde{k}(x, s) - \tilde{k}(x', s)] \Pi_N^{\alpha,\beta} u(s) ds \right|
\]

\[
= \left[ \int_{-1}^{1} |\tilde{k}(x, s) - \tilde{k}(x', s)|^2 ds \right]^{1/2} \|\Pi_N^{\alpha,\beta} u\|_2
\]

\[
\leq \sup \left\{ \left( \int_{-1}^{1} |\tilde{k}(t, s) - \tilde{k}(t', s)|^2 ds \right)^{1/2} : \tau, \tau' \in [-1, 1], |\tau - \tau'| \leq \delta \right\}
\]

\[
\times \sup \left\{ \|\Pi_N^{\alpha,\beta} v\|_2 : m \geq 1, v \in B \right\}
\]

The right hand side is independent of \( x, x', N, u \), hence the equicontinuity of \( S' \) will be proved if the right hand side can be shown to converge to zero as \( \delta \to 0 \). The first factor goes to zero from the condition (A2) and second factor is bounded from the estimate (13). This proves that \( S' \) is equicontinuous. Hence, \( S' \) is a relatively compact set in \( C[-1, 1] \). It follows that

\[
\|(G\Pi_N^{\alpha,\beta} - G)G\Pi_N^{\alpha,\beta}\|_\infty = \sup \{ \|(G\Pi_N^{\alpha,\beta} - G)G\Pi_N^{\alpha,\beta} u\|_\infty : u \in B \}
\]

\[
= \sup \{ \|(G\Pi_N^{\alpha,\beta} - G)u\|_\infty : u \in S' \} \to 0, \quad \text{as } N \to \infty.
\]

This completes the proof that \( G\Pi_N^{\alpha,\beta} \) is \( v \)-convergent to \( G \).

Since \( G\Pi_N^{\alpha,\beta} \) is \( v \)-convergent to \( G \), according to Anselone [1], the operator \((I - G\Pi_N^{\alpha,\beta})^{-1}\) exists and are uniformly bounded for all \( N \) sufficiently large, i.e., there exists a positive real number \( M \) and a positive integer \( n \) such that for all \( N \geq n \), we have

\[
\|(I - G\Pi_N^{\alpha,\beta})^{-1}\|_\infty \leq M.
\]

(17)

This completes the proof.

Now, we obtain the convergence rates for the iterated solution in spectral Jacobi-Galerkin method for Fredholm integral equations of the second kind in both weighted \( L^2 \) norm and infinity norm.

**Theorem 3.7.** Let \( u \) and \( u_N^t \) be the solutions (3) and (11), and assume that the hypothesis of last theorem holds and \((I - G)^{-1}\) exists, then we have the following error estimate

\[
\|u - u_N^t\|_\infty \leq C N^{\frac{1}{2} - \gamma} |u|_{H_{\omega, \gamma}^{m, \beta}}, \quad 0 < \kappa < 1 - \gamma,
\]

\[
\|u - u_N^t\|_{\omega^{m, \beta}} \leq C N^{\frac{1}{2} - \kappa} |u|_{H_{\omega, \gamma}^{m, \beta}}, \quad 0 < \kappa < 1 - \gamma,
\]

(18)

where \( C \) is independent of \( N \).

**Proof.** From Equation (3) and (11), we obtain

\[
u - u_N^t = (I - G)^{-1} f - (I - G\Pi_N^{\alpha,\beta})^{-1} f
\]

\[
= (I - G\Pi_N^{\alpha,\beta})^{-1} [I - G\Pi_N^{\alpha,\beta} - I + G](I - G)^{-1} f
\]

\[
= (I - G\Pi_N^{\alpha,\beta})^{-1} [G - G\Pi_N^{\alpha,\beta}] u.
\]
The compactness of $G$ and the pointwise convergence of $\{\Pi_N^{\alpha,\beta} : N > 0\}$ in $L^2_{\omega^{\alpha,\beta}}(-1, 1)$ implies
\[ \|(G - GP_N^{\alpha,\beta})u\|_{\omega^{\alpha,\beta}} \to 0, \quad \text{as} \quad N \to \infty, \]
(cf.[2], Lemma 3.1.2). Then \((I - \Pi_N^{\alpha,\beta}G)^{-1}\) exists and is uniformly bounded on $L^2_{\omega^{\alpha,\beta}}(-1, 1)$ for all large $N$, along with \((I - GP_N^{\alpha,\beta})^{-1}\).

\[ \|u - u_N\|_{\infty} = \|(I - GP_N^{\alpha,\beta})^{-1}\|_{\infty}\|(G - G\Pi_N^{\alpha,\beta})u\|_{\infty} \leq M\|G(I - \Pi_N^{\alpha,\beta})u\|_{\infty}. \]

Since $\Pi_N^{\alpha,\beta}$ is the orthogonal projection from the space $X$ into $P_N$, then we have
\[ \langle \nu, (I - \Pi_N^{\alpha,\beta})u \rangle = 0, \quad \forall \nu \in P_N. \]

Now using Lemma 3.2 and for any $\nu \in P_N$, we have
\[ |G(I - \Pi_N^{\alpha,\beta})u(x)| = \int_{-1}^1 \overline{k}(x, s)(I - \Pi_N^{\alpha,\beta})u(s)\omega^{\alpha,\beta} ds \]
\[ = \left| \langle \overline{k}_x(\cdot), (I - \Pi_N^{\alpha,\beta})u \rangle \right| \]
\[ \leq \|\overline{k}_x(\cdot)\|_{L^1} \|\nu\|_1 \|(I - \Pi_N^{\alpha,\beta})u\|_{\infty} \]
\[ \leq C\|\overline{k}_x(\cdot)\|_{L^1} \|\nu\|_1 N^{2-m}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}} \]

Hence using Lemma 3.4 and Lemma 3.5, we obtain
\[ \|G(I - \Pi_N^{\alpha,\beta})u\|_{\infty} \leq C\|\overline{k}_x(\cdot)\|_{L^1} N^{2-m}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}} \leq CN^{2-m - \kappa}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \quad 0 < \kappa < 1 - \gamma. \]

We obtain the error bounds
\[ \|u - u_N\|_{\infty} \leq M\|G(I - \Pi_N^{\alpha,\beta})u\|_{\infty} \leq CN^{2-m - \kappa}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \quad 0 < \kappa < 1 - \gamma, \]
\[ \|u - u_N\|_{\omega^{\alpha,\beta}} \leq \sqrt{2}\|u - u_N\|_{\infty} \leq CN^{2-m - \kappa}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \quad 0 < \kappa < 1 - \gamma. \]

This completes the proof. \qed

**Theorem 3.8.** Let $u$ and $u_N$ be the solutions of (3) and (8), respectively and the hypotheses of Theorem 3.7 hold, then there exists a positive constant $C$ independent of $N$,
\[ \|u - u_N\|_{\omega^{\alpha,\beta}} \leq \begin{cases} CN^{2-m - \kappa}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}} & \text{for} \quad 0 < \gamma \leq \frac{1}{4}, \\ CN^{-m}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}} & \text{for} \quad \frac{1}{4} \leq \gamma < 1, \end{cases} \]
\[ \|u - u_N\|_{\infty} \leq CN^{2-m}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}}. \]

**Proof.** Using $u_N = \Pi_N^{\alpha,\beta}u^t_N$, we obtain
\[ u - u_N = u - \Pi_N^{\alpha,\beta}u^t_N = u - \Pi_N^{\alpha,\beta}u + \Pi_N^{\alpha,\beta}u - \Pi_N^{\alpha,\beta}u^t_N. \]

Then
\[ \|u - u_N\|_{\omega^{\alpha,\beta}} = \|u - \Pi_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} + \|\Pi_N^{\alpha,\beta}\|_{\omega^{\alpha,\beta}}\|u - u_N^t\|_{\omega^{\alpha,\beta}} \]
\[ \leq CN^{-m}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}} + CN^{2-m - \kappa}\|u\|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \quad 0 < \kappa < 1 - \gamma. \]
Hence
\[
\|u - u_N\|_{\omega^{\alpha,\beta}} \leq \begin{cases} 
CN^{3/4 - \gamma} |u|_{H^{m,N}} & \text{for } 0 < \gamma \leq 1/4, \\
CN^{-m} |u|_{H^{m,N}} & \text{for } 1/4 \leq \gamma < 0.
\end{cases}
\]

Using the fact that \(\|\Pi^{\alpha,\beta}_N\|_{\infty} \leq C \log N\) (cf. [9]), we obtain
\[
\|u - u_N\|_{\infty} \leq \|u - \Pi^{\alpha,\beta}_N u\|_{\infty} + \|\Pi^{\alpha,\beta}_N\|_{\infty} \|u - u_N\|_{\infty}
\leq CN^{3/4 - m} |u|_{H^{m,N}} + C(\log N) N^{3/4 - m - \gamma} |u|_{H^{m,N}}
\leq CN^{3/4 - m} |u|_{H^{m,N}}.
\]

4. Pseudo-spectral Jacobi-Galerkin methods. Now we turn to describe the pseudo-spectral Jacobi-Galerkin method.

\[
Gu(x) = \int_{-1}^1 |(x-s)|^{-\gamma} k(x,s) u(s) ds = \int_{-1}^1 (x-s_1)^{-\gamma} k(x,s_1) u(s_1) ds_1 + \int_{-1}^1 (s_2-x)^{-\gamma} k(x,s_2) u(s_2) ds_2
\]

(19)

Set
\[
s_1(x,\theta_1) = \frac{1+x}{2} \theta + \frac{x-1}{2}, \quad s_2(x,\theta_2) = \frac{1-x}{2} \theta + \frac{x+1}{2}, \quad -1 \leq \theta \leq 1,
\]

it is clear that
\[
Gu(x) = \int_{-1}^1 (1-\theta)^{-\gamma} \tilde{k}_1(x,s_1(x,\theta)) u(s_1(x,\theta)) d\theta
+ \int_{-1}^1 (1+\theta)^{-\gamma} \tilde{k}_2(x,s_2(x,\theta)) u(s_2(x,\theta)) d\theta
\]

(20)

with
\[
\tilde{k}_1(x,s_1(x,\theta_1)) = \left(\frac{1+x}{2}\right)^{1-\gamma} k(x,s_1(x,\theta_1)),
\]
\[
\tilde{k}_2(x,s_2(x,\theta_2)) = \left(\frac{1-x}{2}\right)^{1-\gamma} k(x,s_2(x,\theta_2)).
\]

Using \((N+1)\)-point Gauss quadrature formula to approximate yields
\[
Gu(x) \approx G_N u(x) := \sum_{k=0}^N \tilde{k}_1(x,s_1(x,\theta_k)) u(s_1(x,\theta_k)) \omega_k^{\alpha,\beta,0}
+ \tilde{k}_2(x,s_2(x,\tilde{\theta}_k)) u(s_1(x,\tilde{\theta}_k)) \omega_k^{\beta,0 - \gamma},
\]

(21)

where \(\{\theta_k\}_{k=0}^N\) and \(\{\tilde{\theta}_k\}_{k=0}^N\) are the \((N+1)\)-degree Jacobi-Gauss points corresponding to the weights \(\{\omega_k^{\alpha,\beta,0}\}_{k=0}^N\) and \(\{\omega_k^{\beta,0 - \gamma}\}_{k=0}^N\) corresponding.

On the other hand, instead of the continuous inner product, the discrete inner product will be implemented in (6) and (9), i.e.

\[
(u,v)_{\omega^{\alpha,\beta}} \approx (u,v)_{\omega^{\alpha,\beta,N}} = \sum_{k=0}^N u(x_k) v(x_k) \omega_k^{\alpha,\beta}(x_k),
\]

(22)
with \( \{x_k\}_{k=0}^{N} \) are \( N + 1 \) Jacobi-Gauss points corresponding to the weight function \( \omega^{\alpha,\beta}(x) \). As a result,
\[
(u,v)_{\omega^{\alpha,\beta}} = (u,v)_{\omega^{\alpha,\beta},N}, \quad \text{if} \ uv \in P_{2N}.
\]
Substitute (21) and (22) into (6), the pseudo-spectral Jacobi-Galerkin method is to find
\[
\bar{u}_N(x) = \sum_{j=0}^{N} \bar{\xi}_j \phi_j(x)
\]
such that
\[
(\bar{u}_N,v)_{\omega^{\alpha,\beta}} = (G_N \bar{u}_N,v)_{\omega^{\alpha,\beta}} + (f,v)_{\omega^{\alpha,\beta},N}, \quad \forall v \in P_N,
\]
(23) \( \{\bar{\xi}_j\}_{j=0}^{N} \) are determined by
\[
\sum_{j=0}^{N} \bar{\xi}_j (\phi_i,\phi_j)_{\omega^{\alpha,\beta}} = \sum_{j=0}^{N} \bar{\xi}_j (\phi_i,G_N \phi_j)_{\omega^{\alpha,\beta}} + (\phi_i,f)_{\omega^{\alpha,\beta},N},
\]
(24) the matrix form
\[
\bar{A} \xi = \bar{B} \xi + \bar{C},
\]
(25) where
\[
\bar{\xi} = [\bar{\xi}_0, \bar{\xi}_1, \cdots, \bar{\xi}_N]^T, \quad \bar{A}_{i,j} = (\phi_i,\phi_j)_{\omega^{\alpha,\beta},N};
\]
\[
\bar{B}_{i,j} = (\phi_i,G\phi_j)_{\omega^{\alpha,\beta},N}, \quad \bar{C}_i = (\phi_i,f)_{\omega^{\alpha,\beta},N}.
\]

5. Convergence analysis for pseudo-spectral Jacobi-Galerkin method. In this section, we discuss the convergence rates for pseudo-spectral Jacobi-Galerkin approximated solution for the Fredholm integral equations of the second kind with weakly singular kernel. Now we investigate the estimate of pseudo-spectral Galerkin solution \( \bar{u}_N \) given by (23), is defined as follows:
\[
\bar{u}_N^N(x) = G_N \bar{u}_N(x) + f(x)
\]
(26)

In terms of (23), the pseudo-spectral Galerkin solution \( \bar{u}_N \) satisfies
\[
(\bar{u}_N,v)_{\omega^{\alpha,\beta}} = (I_N^{\alpha,\beta} G_N \bar{u}_N,v)_{\omega^{\alpha,\beta}} + (I_N^{\alpha,\beta} f,v)_{\omega^{\alpha,\beta}}.
\]
(27) Let
\[
I(x) = (G\bar{u}_N - G_N \bar{u}_N)(x)
\]
\[
= \int_{-1}^{1} \tilde{k}(x,s(x,\theta),\bar{u}_N(s(x,\theta))) d\theta - \sum_{k=0}^{N} \tilde{k}(x,s(x,\theta_k),\bar{u}_N(s(x,\theta_k))) \omega_k,
\]
(28)
Combing (27) and (28), yields
\[
\bar{u}_N = (I_N^{\alpha,\beta} \hat{G} u_N - I_N^{\alpha,\beta} I(x), v_N)_{\omega_{\alpha,\beta}} + (I_N^{\alpha,\beta} f, v_N)_{\omega_{\alpha,\beta}},
\]  
(29)
which gives rise to
\[
\bar{u}_N = I_N^{\alpha,\beta} \hat{G} u_N - I_N^{\alpha,\beta} I(x) + I_N^{\alpha,\beta} f.
\]  
(30)

We first consider an auxiliary problem, i.e., we want to find \( \bar{u}_N \in P_N \), such that
\[
(\bar{u}_N, v_N)_{\omega_{\alpha,\beta}, N} = (\hat{G} u_N, v_N)_{\omega_{\alpha,\beta}, N} + (f, v_N)_{\omega_{\alpha,\beta}, N}, \quad \forall v_N \in P_N,
\]  
(31)
in terms of the definition of \( I_N^{\alpha,\beta} \). (31) can be written as
\[
(\bar{u}_N, v_N)_{\omega_{\alpha,\beta}} = (I_N^{\alpha,\beta} \hat{G} u_N, v_N)_{\omega_{\alpha,\beta}} + (I_N^{\alpha,\beta} f, v_N)_{\omega_{\alpha,\beta}}, \quad \forall v_N \in P_N,
\]  
(32)
which is equivalent to
\[
\bar{u}_N = I_N^{\alpha,\beta} \hat{G} u_N + I_N^{\alpha,\beta} f.
\]  
(33)

We quote the following lemma, which helps us to prove Theorems 5.7.

**Lemma 5.1.** ([4, 12]) Assume that an \((N + 1)\)-point Gauss quadrature formula relative to the Jacobi weight is used to integrate the product \( u \varphi \), where \( u \in H^{m,N}_{\omega_{\alpha,\beta}} \) for some \( m \geq 1 \) and \( \varphi \in P_N \). Then there exists a constant \( C \) independent of \( N \) such that
\[
\left| \int_{-1}^{1} u(x) \varphi(x) \omega_{\alpha,\beta}^\prime(x) dx - (u, \varphi)_{\omega_{\alpha,\beta}, N} \right| \leq CN^{-m} \| u \|_{H^{m,N}_{\omega_{\alpha,\beta}}} \| \varphi \|_{\omega_{\alpha,\beta}}.
\]  
(34)

**Lemma 5.2.** ([4, 8]) Assume that \( u \in H^{m,N}_{\omega_{\alpha,\beta}}(\Lambda) \), \( m \geq 1 \) \( I_N^{\alpha,\beta} u \) denotes the interpolation operator of \( u \) based on \((N + 1)\)-degree Jacobi Gauss points corresponding to the weight function \( \omega_{\alpha,\beta}(x) \) with \(-1 < \alpha, \beta < 1\),
\[
\| u - I_N^{\alpha,\beta} u \|_{\omega_{\alpha,\beta}} \leq CN^{-m} \| u \|_{H^{m,N}_{\omega_{\alpha,\beta}}},
\]  
(35a)

\[
\| u - I_N^{\alpha,\beta} u \|_{\infty} \leq \begin{cases} \frac{CN^{\frac{1}{2}-m}}{\log N} \| u \|_{H^{m,N}_{\omega_{\alpha,\beta}}}, & -1 \leq \alpha, \beta < -\frac{1}{2}, \\ CN^{\nu+1-m} \| u \|_{H^{m,N}_{\omega_{\alpha,\beta}}}, & \text{otherwise}, \quad \nu = \max(\alpha, \beta), \end{cases}
\]  
(35b)
where \( \omega_\nu = \omega^{-\frac{1}{2}} - \frac{1}{2} \) denotes the Chebyshev weight function.

**Lemma 5.3.** ([14, 8]) For every bounded function \( u \), there exists a constant \( C \), independent of \( u \) such that
\[
\| I_N^{\alpha,\beta} u(x_j) \|_{\omega_{\alpha,\beta}} \leq C \| u \|_{\infty}.
\]

where \( I_N^{\alpha,\beta} u(x) = \sum_{j=0}^{N} u(x_j) F_j(x) \) is the Lagrange interpolation basis function associated with \((N + 1)\)-degree Jacobi-Gauss points corresponding to the weight function \( \omega_{\alpha,\beta}(x) \).

**Lemma 5.4.** (see [14]) Assume that \( \{F_j(x)\}_{j=0}^{N} \) are the \( N \)-th degree Lagrange basis polynomials associated with the Gauss points of the Jacobi polynomials. Then,
\[
\| I_N^{\alpha,\beta} \|_{L^\infty(I)} \leq \max_{x \in [-1,1]} \sum_{j=0}^{N} F_j(x) = \begin{cases} O(\log N), & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ O(N^{\gamma + \frac{1}{2}}), & \gamma = \max(\alpha, \beta), \text{ otherwise}. \end{cases}
\]  
(36)
Lemma 5.5. (Gronwall inequality [11]) Suppose $L \geq 0$, $0 < \mu < 1$, and $u$ and $v$ are non-negative, locally integrable functions defined on $[-1,1]$ satisfying

$$u(x) \leq v(x) + L \int_{-1}^{x} (x - \tau)^{-\mu} u(\tau) d\tau.$$  

Then there exists a constant $C = C(\mu)$ such that

$$u(x) \leq v(x) + CL \int_{-1}^{x} (x - \tau)^{-\mu} v(\tau) d\tau, \quad \text{for} \quad -1 \leq x \leq 1.$$  

If a nonnegative integrable function $E(x)$ satisfies

$$E(x) \leq L \int_{-1}^{x} E(s) ds + J(x), \quad -1 < x \leq 1,$$

where $J(x)$ is an integrable function, then

$$\|E\|_{L^\infty(-1,1)} \leq C\|J\|_{L^\infty(-1,1)},$$

$$\|E\|_{L^p_{-\nu-\alpha,\beta}(-1,1)} \leq C\|J\|_{L^p_{-\nu-\alpha,\beta}(-1,1)}, \quad q \geq 1.$$  

(37)

Lemma 5.6. Let $\nu = \max(\alpha, \beta)$, $\hat{u}_N$ and $u$ be the solutions of (33) and (3), respectively. If the solution of satisfies $u \in H^{m,N}_{\alpha,\beta}$, we have

$$\|u - \hat{u}_N\|_{\omega,\beta} \leq CN^{-m}|u|_{H^{m,N}_{\alpha,\beta}},$$

$$\|u - \hat{u}_N\|_{\infty} \leq \left\{ \begin{array}{ll}
CN^{\frac{1}{2}-m} \log N |u|_{H^{m,N}_{\omega,\beta}}, & -1 \leq \nu < -\frac{1}{2}, \\
CN^{\nu+1-m}|u|_{H^{m,N}_{\omega,\beta}}, & -\frac{1}{2} \leq \nu < 0.
\end{array} \right.$$  

Proof. Subtracting (33) from (3), yields

$$u - \hat{u}_N = Gu - I_N^{\alpha,\beta} Gu^N + f - I_N^{\alpha,\beta} f.$$  

(38)

Set $\varepsilon = u - \hat{u}_N$, direct computation shows that

$$Gu - I_N^{\alpha,\beta} Gu^N$$

$$= Gu - I_N^{\alpha,\beta} Gu + (Gu - G\hat{u}^N) - \left[ (Gu - G\hat{u}^N) - I_N^{\alpha,\beta} (Gu - G\hat{u}^N) \right]$$

$$= (u - f) - I_N^{\alpha,\beta} (u - f) + G\varepsilon - [G\varepsilon - I_N^{\alpha,\beta} G\varepsilon]$$

$$= I_N^{\alpha,\beta} u - u + I_N^{\alpha,\beta} f - f + G\varepsilon - [G\varepsilon - I_N^{\alpha,\beta} G\varepsilon]$$

(39)

The insertion of (39) into (38) yields

$$\varepsilon = \int_{-1}^{1} |x - s|^{-\gamma} k(x, s) \varepsilon(s) ds + J_1 + J_2,$$  

(40)

where

$$J_1 = u - I_N^{\alpha,\beta} u, \quad J_2 = I_N^{\alpha,\beta} G\varepsilon - G\varepsilon.$$  

It follows from Gronwall inequality that

$$\|\varepsilon(x)\|_{\infty} \leq C \left( \|J_1\|_{\infty} + \|J_2\|_{\infty} \right).$$  

(41)

Due to Lemma 5.2,

$$\|J_1\|_{\infty} = \|u - I_N^{\alpha,\beta} u\|_{\infty} \leq \left\{ \begin{array}{ll}
CN^{\frac{1}{2}-m} \log N |u|_{H^{m,N}_{\omega,\beta}}, & -1 \leq \nu < -\frac{1}{2}, \\
CN^{\nu+1-m}|u|_{H^{m,N}_{\omega,\beta}}, & \text{otherwise}.
\end{array} \right.$$  

(42)
By virtue of Lemma 3.4, Lemma 3.5 and Lemma 5.4, we have
\[
\| J_2 \|_{\infty} = \| (I_N^{\alpha,\beta} - I)G\|_{\infty}
\leq \| (I_N^{\alpha,\beta} - I)(G\| - K_N G\|)_{\infty}
\leq \| I_N^{\alpha,\beta}(G\| - K_N G\|)_{\infty} + \| G\| - K_N G\|_{\infty}
\leq \left\{ \begin{array}{ll}
\mathcal{O}(\log N) \| G\| - K_N G\|_{\infty}, & -1 < \nu \leq -\frac{1}{2}, \\
\mathcal{O}(N^{\nu+\frac{1}{2}}) \| G\| - K_N G\|_{\infty}, & \text{otherwise},
\end{array} \right.
\]
Combining (41), (42) and (43) we obtain, when \( N \) is large enough,
\[
\| u - \bar{u}_N \|_{\infty} \leq \left\{ \begin{array}{ll}
CN^{\frac{1}{2} - m} \log N |u|_{H^m_{\omega,\nu}}, & -1 \leq \nu < -\frac{1}{2}, \\
CN^{\nu+1-m} |u|_{H^m_{\omega,\nu}}, & -\frac{1}{2} \leq \nu < \min(0, \frac{1}{2} - \gamma).
\end{array} \right.
\]
Now we investigate the \( \| \cdot \|_{\omega,\nu} \)-error estimate. It follows from (40) and the Gronwall inequality that
\[
\| \epsilon(x) \|_{\omega,\nu} \leq C (\| J_1 \|_{\omega,\nu} + \| J_2 \|_{\omega,\nu}).
\]
By Lemma 5.2,
\[
\| J_1 \|_{\omega,\nu} = \| u - I_N^{\alpha,\beta} u \|_{\omega,\nu} \leq CN^{m}\| u \|_{H^m_{\omega,\nu}}.
\]
It follows from Lemma 3.5, Lemma 3.4 and Lemma 3.5 that
\[
\| J_2 \|_{\omega,\nu} = \| (I_N^{\alpha,\beta} - I)G\|_{\omega,\nu}
\leq \left\{ \begin{array}{ll}
C \| G\| - K_N G\|_{\infty}, & 0 < \kappa < 1 - \gamma.
\end{array} \right.
\]
Combining (44), (45) and (46) we obtain
\[
\| u - \bar{u}_N \|_{\omega,\nu} \leq \left\{ \begin{array}{ll}
CN^{m}\left( |u|_{H^m_{\omega,\nu}} + \log N N^{\frac{1}{2} - \kappa} |u|_{H^m_{\omega,\nu}} \right), & -1 \leq \nu < -\frac{1}{2}, \\
CN^{m}\left( |u|_{H^m_{\omega,\nu}} + N^{\nu+1-m} |u|_{H^m_{\omega,\nu}} \right), & -\frac{1}{2} \leq \nu < \min(0, \gamma - \frac{1}{2}).
\end{array} \right.
\]
This completes the proof of the lemma.

**Theorem 5.7.** Let \(-1 \leq \nu = \max(\alpha, \beta) \leq \min(0, \frac{1}{2} - \gamma) \) and \( 0 < \kappa < 1 - \gamma \), \( \bar{u}_N \) and \( u \) be the solutions of \((23) \) and \((3) \), respectively. Suppose that \( u \) satisfies
u \in H_{\omega, \beta}^{m,N}(\Lambda), \text{ we have}
\|u - \bar{u}_N\|_{\omega, \beta}
\leq \begin{cases} CN^{-m} \log N \left[ N^{1 \over 2} |u|_{H_{\omega}^{m,N}} + K^* \|u\|_{\infty} \right], & -1 \leq \nu < -{1 \over 2}, \\
CN^{\nu + 1 - m} \left[ N^{1 \over 2} |u|_{H_{\omega}^{m,N}} + K^* \|u\|_{\infty} \right], & -1 \leq \nu \leq \min(0, {1 \over 2} - \gamma). \end{cases}
\leq \begin{cases} CN^{-m} \left[ (1 + \log N N^{-\kappa}) K^* \|u\|_{\infty} + |u|_{H_{\omega}^{m,N}} + \log N N^{1 \over 2 - \kappa} |u|_{H_{\omega}^{m,N}} \right], & -1 \leq \nu < -{1 \over 2}, \\
CN^{-m} \left[ (1 + N^{\nu + 1 - \kappa}) K^* \|u\|_{\infty} + |u|_{H_{\omega}^{m,N}} + N^{\nu + 1 - \kappa} |u|_{H_{\omega}^{m,N}} \right], & -1 \leq \nu \leq \min(0, {1 \over 2} - \gamma). \end{cases}
\leq \begin{cases} CN^{-m} \left[ (1 + \log N N^{-\kappa}) K^* \|u\|_{\infty} + |u|_{H_{\omega}^{m,N}} + \log N N^{1 \over 2 - \kappa} |u|_{H_{\omega}^{m,N}} \right], & -1 \leq \nu < -{1 \over 2}, \\
CN^{-m} \left[ (1 + N^{\nu + 1 - \kappa}) K^* \|u\|_{\infty} + |u|_{H_{\omega}^{m,N}} + N^{\nu + 1 - \kappa} |u|_{H_{\omega}^{m,N}} \right], & -1 \leq \nu \leq \min(0, {1 \over 2} - \gamma). \end{cases}
\leq \begin{cases} CN^{-m} \left[ (1 + \log N N^{-\kappa}) K^* \|u\|_{\infty} + |u|_{H_{\omega}^{m,N}} + \log N N^{1 \over 2 - \kappa} |u|_{H_{\omega}^{m,N}} \right], & -1 \leq \nu < -{1 \over 2}, \\
CN^{-m} \left[ (1 + N^{\nu + 1 - \kappa}) K^* \|u\|_{\infty} + |u|_{H_{\omega}^{m,N}} + N^{\nu + 1 - \kappa} |u|_{H_{\omega}^{m,N}} \right], & -1 \leq \nu \leq \min(0, {1 \over 2} - \gamma). \end{cases}
where K^* = \max_{x \in (-1,1)} (|k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}(\Lambda)} + |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}(\Lambda)}).

Proof. Now subtracting (30) from (33) leads to
\bar{u}_N - \bar{u}_N = I_N^{\alpha,\beta} (G\bar{u}_N - G\bar{u}_N) - I_N^{\alpha,\beta} I(x),
which can be simplified as, by setting E = \bar{u}_N - \bar{u}_N
E = I_N^{\alpha,\beta} GE - I_N^{\alpha,\beta} I(x)
= GE - GE + I_N^{\alpha,\beta} GE - I_N^{\alpha,\beta} I(x)
= GE + Q_1 - I_N^{\alpha,\beta} I(x)
with Q_1 = I_N^{\alpha,\beta} GE - GE. It follows from the Gronwall inequality that
\|E\|_{\infty} \leq C (\|Q_1\|_{\infty} + \|I_N^{\alpha,\beta} I(x)\|_{\infty}).
Similarly to (43), we have
\|Q_1\|_{\infty} = \|I_N^{\alpha,\beta} GE - GE\|_{\infty} \leq \begin{cases} CN^{-\kappa} \log N \|E\|_{\infty}, & -1 \leq \mu < -{1 \over 2}, \\
CN^{\nu + 1 - \kappa} \|E\|_{\infty}, & -{1 \over 2} \leq \nu < \min(0, {1 \over 2} - \gamma). \end{cases}
with 0 < \kappa < 1 - \gamma.
Using Lemma 5.1 and Lemma 5.4, we have
\|I_N^{\alpha,\beta} I(x)\|_{\infty} \leq \begin{cases} C \log N \max_{x \in (-1,1)} I(x), & -1 \leq \nu < -{1 \over 2}, \\
CN^{\nu + 1} \max_{x \in (-1,1)} I(x), & -{1 \over 2} \leq \nu < 0, \end{cases}
\leq \begin{cases} CN^{-m} \max_{x \in (-1,1)} |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}} \|\bar{u}_N\|_{\omega, -\gamma, 0}, & -1 \leq \nu < -{1 \over 2}, \\
CN^{-m} N^{\nu + 1} \max_{x \in (-1,1)} |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}} \|\bar{u}_N\|_{\omega, -\gamma, 0}, & -{1 \over 2} \leq \nu < 0, \end{cases}
\leq \begin{cases} CN^{-m} \max_{x \in (-1,1)} |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}} \|\bar{u}_N\|_{\omega, -\gamma, 0}, & -1 \leq \nu < -{1 \over 2}, \\
CN^{-m} N^{\nu + 1} \max_{x \in (-1,1)} |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}} \|\bar{u}_N\|_{\omega, -\gamma, 0}, & -{1 \over 2} \leq \nu < 0, \end{cases}
\leq \begin{cases} CN^{-m} \max_{x \in (-1,1)} |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}} (\|u\|_{\infty} + \|E\|_{\infty}), & -1 \leq \nu < -{1 \over 2}, \\
CN^{-m} N^{\nu + 1} \max_{x \in (-1,1)} |k(x, s(x, \cdot))|_{H_{\omega,0}^{m,N}} (\|u\|_{\infty} + \|E\|_{\infty}), & -{1 \over 2} \leq \nu < 0. \end{cases}
Set $K^* = \max_{x \in (-1,1)} \left( |k(x, s(x, \cdot))|_{H^{m,N}_{\omega,-\gamma}} + |k(x, \cdot)|_{H^{m,N}_{\omega,-\gamma}} \right)$, we now obtain the estimate $E$ by using (50)
\[ \|E\|_{\infty} \leq \begin{cases} CN^{-m} \log NK^* \|u\|_{\infty}, & -1 \leq \nu < -\frac{1}{2}, \\ CN^{\nu + \frac{1}{2} - \frac{m}{2}}K^* \|u\|_{\infty}, & -\frac{1}{2} \leq \nu < \min(0, \frac{1}{2} - \gamma). \end{cases} \] (51)

Next, we will give the error estimates in $\| \cdot \|_{\omega,\gamma}$. It follows from (49) and the Gronwall inequality that
\[ \|E\|_{\omega,\gamma} = \|Q_1\|_{\omega,\gamma} + \|I_N^{\alpha,\beta}I(x)\|_{\omega,\gamma}. \] (52)

$\|Q_1\|_{\omega,\gamma}$ can be established in a similar way as (46),
\[ \|Q_1\|_{\omega,\gamma} = \|I_N^{\alpha,\beta}GE - GE\|_{\omega,\gamma} \leq CN^{-\kappa}\|E\|_{\infty}, \quad 0 < \kappa < 1 - \gamma. \]

Using Lemma 5.1 and Lemma 5.3, we have
\[ \|I_N^{\alpha,\beta}I(x)\|_{\omega,\gamma} \leq C[I(x)\|_{\omega,\gamma} \leq CN^{-m}K^* \|\bar{u}_N\|_{\omega,-\gamma,\sigma} \leq CN^{-m}K^* (\|u\|_{\infty} + \|E\|_{\infty}). \]

By the convergence result in (51) ($m = 1$), we have
\[ \|E\|_{\infty} \leq \|u\|_{\infty} \]
for sufficiently large $N$. So that
\[ \|I_N^{\alpha,\beta}I(x)\|_{\omega,\gamma} \leq CN^{-m}K^* \|u\|_{\infty}. \]

We obtain, when $N$ is large enough,
\[ \|E\|_{\omega,\gamma} \leq \begin{cases} CN^{-m} (1 + \log NN^{-\kappa})K^* \|u\|_{\infty}, & -1 \leq \nu < -\frac{1}{2}, \\ CN^{-m} (1 + N^{\nu + \frac{1}{2} - \kappa})K^* \|u\|_{\infty}, & -\frac{1}{2} \leq \nu < \min(0, \frac{1}{2} - \gamma). \end{cases} \] (53)

Finally, it follows from triangular inequality,
\[ \|u - \bar{u}_N\|_{\omega,\gamma} \leq \|u - \bar{u}_N\|_{\infty} + \|\bar{u}_N - \bar{u}_N\|_{\omega,\gamma}, \]
\[ \|u - \bar{u}_N\|_{\omega,\gamma} \leq \|u - \bar{u}_N\|_{\omega,\gamma} + \|\bar{u}_N - \bar{u}_N\|_{\omega,\gamma}, \]
as well as Lemma 5.6, (51) and (53), we can obtain the desired estimated (47). $\Box$

6. Numerical experiments. In this section, we present two numerical examples to confirm the theoretical analysis obtained in the previous sections. For different kernels, and for different values of $N$, we compute $u_N$ and $u_N^{\omega}$ in the spectral Jacobi-Galerkin method, $\bar{u}_N$ and $\bar{u}_N^{\omega}$ in the pseudo-spectral Jacobi-Galerkin method, and compare the result with exact solution $u$. To examine the accuracy of the results, $\| \cdot \|_{\infty}$ and $\| \cdot \|_{\omega,\gamma}$ errors are employed to assess the efficiency of the method. All the calculations are supported by the software Matlab.

Example 6.1. Consider the following Fredholm integral equations of the second kind with weakly singular kernel
\[ u(t) - \frac{1}{10} \int_0^1 \frac{u(\tau)}{\sqrt{|t-\tau|}} d\tau = t^2(1-t)^2 - \frac{27}{30800} \left[ 54t^2 - 126t + 77 \right]. \]
and exact solution $u(t) = t^2(1-t)^2$. 

[The rest of the text is not fully readable due to the image quality, but the above content is the correct representation of the text within the image.]
First we implement the spectral Legendre-Galerkin and Chebyshev-Galerkin methods to solve this example. Fig. 1 (left) illustrates the $\| \cdot \|_\infty$ and $\| \cdot \|_{\omega^{\alpha,\beta}}$ errors of the spectral Legendre-Galerkin method ($\alpha = 0, \beta = 0$). Next the $\| \cdot \|_\infty$ and $\| \cdot \|_{\omega^{\alpha,\beta}}$ errors of the spectral Chebyshev-Galerkin method ($\alpha = -\frac{1}{2}, \beta = -\frac{1}{2}$) are demonstrated in Fig. 1 (right). Clearly the desired spectral accuracy is obtained in these approaches.

Example 6.2. Consider the Fredholm integral equation of the second kind with a weakly singular kernel having the form

$$u(t) = \int_0^1 |t - \tau|^{-\frac{1}{2}} u(\tau)ds + f(t).$$

where $f(t) = 1 - \frac{\pi^2}{4} - 2t^{1/2} - 2(1-t)^{1/2} - t \log(1 + (1-t)^{1/2}) - (1-t) \log(1 + t^{1/2}) + \frac{1}{2} t \log(t) + \frac{1}{2} (1-t) \log(1-t)$ and exact solution $u(t) = 1 + t^{1/2} + (1-t)^{1/2}$.

This problem has the property stated at the beginning of this paper, i.e.,

$$u'(t) = -\frac{1}{2} t^{-1/2} + \frac{1}{2} (1-t)^{-1/2}$$

which is singular at $t = 0^+$ and $t = 1^-$.

In Fig. 2 the errors are given for different values of $N$. It can be seen that the errors decay algebraically as the exact solution for this example is not sufficiently smooth.

7. **Concluding remarks.** This work has been concerned with the spectral and pseudo-spectral Jacobi-Galerkin methods and corresponding iterated methods for the Fredholm integral equations with weakly singular kernel. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of spectral approximations decay exponentially in both infinity and weighted norms, which is a desired feature for a spectral method. Although in this work our convergence theory does not cover the nonlinear case, the methods described above remain applicable, it will be possible to extend the results of this paper to nonlinear case which will be the subject of our future work.
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*E-mail address: yangyinxtu@xtu.edu.cn*

*E-mail address: huangyq@xtu.edu.cn*