LOG-HYPERCONVEXITY INDEX AND BERGMAN KERNEL

BO-YONG CHEN AND ZHIYUAN ZHENG

ABSTRACT. We obtain a quantitative estimate of Bergman distance when $\Omega \subset \mathbb{C}^n$ is a bounded domain with log-hyperconvexity index $\alpha(\Omega) > \frac{n-1+\sqrt{(n-1)(n+3)}}{2}$, as well as the $A^2(\log A)^{q}$-integrability of the Bergman kernel $K_\Omega(\cdot, w)$ when $\alpha(\Omega) > 0$.

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1. INTRODUCTION

We say a domain $\Omega \subset \mathbb{C}^n$ is hyperconvex if there exists a continuous negative plurisubharmonic (psh) exhaustion function $\rho$ on $\Omega$, i.e., $\{ \rho < c \} \Subset \Omega$ for every $c < 0$. It is one of the core concepts in the function theory of several complex variables, which can be traced back to 1974, when Stehlé [33] first proposed it in order to study the Serre problem. Clearly, hyperconvexity implies pseudoconvexity, but the converse fails. Thus it is interesting to ask when a pseudoconvex domain is hyperconvex. A large literature of positive results exists (see, e.g., [11, 15, 16, 17, 22, 23, 26, 29]). Among them, the weakest regularity assumption on the boundary is that the boundary can be written locally as the graph of a Hölder continuous function (cf. [11]).

The quantitative characterization of hyperconvexity starts from the seminal work of Diederich-Fornaess [17], that for every bounded pseudoconvex domain $\Omega$ with $C^2$-boundary one has

$$\eta(\Omega) := \sup\{ \eta \geq 0; \exists \rho \in C(\Omega) \cap PSH^-(\Omega) \text{ s.t. } -\rho \asymp \delta^\eta \} > 0,$$

where $\delta$ denotes the boundary distance and $PSH^-(\Omega)$ denotes the set of negative psh functions on $\Omega$. The quantity $\eta(\Omega)$ is also called the Diederich-Fornaess index (D-F index) of $\Omega$, and it has been studied by a number of authors (see, e.g., [3, 17, 19, 20, 22, 27, 29, 31]). In 2017, the first author [12] introduced a related concept, the so-called hyperconvexity index of a bounded domain $\Omega$, defined as

$$\alpha(\Omega) := \sup\{ \alpha \geq 0; \exists \rho \in C(\Omega) \cap PSH^-(\Omega) \text{ s.t. } -\rho \asymp \delta^\alpha \}.$$

Obviously, $\alpha(\Omega) \geq \eta(\Omega)$. These concepts have numerous applications in the study of the Bergman kernel and metric.

In this spirit, we introduce the following
Definition 1.1. For a bounded domain $\Omega \subset \mathbb{C}^n$, we define
\[
\alpha_l(\Omega) := \sup \{ \alpha \geq 0; \exists \rho \in C(\Omega) \cap PSH^-(\Omega) \ s.t. \ -\rho \lesssim (-\log \delta)^{-\alpha} \text{ near } \partial \Omega \}
\]
as the log-hyperconvexity index of $\Omega$.

This concept is motivated by the recent result of the first author [12] that every bounded pseudoconvex Hölder domain locally has a positive log-hyperconvexity index. It remains open whether every bounded pseudoconvex Hölder domain has a (global) positive log-hyperconvexity index (see [13] for some partial results). Note that there are various examples with $\alpha_l(\Omega) = 0$ while $\alpha_l(\Omega) > 0$ (see Appendix).

It is well-known that the growth of negative psh exhaustion functions relates closely to the estimate of the Bergman distance $d_B(z_0, z)$ from a fixed point $z_0$ to $z$. If $\Omega$ is hyperconvex, then $\lim_{z \to \partial \Omega} d_B(z_0, z) = \infty$ (cf. [6] or [24]). Diederich-Ohsawa [18] obtained the quantitative estimate
\[
d_B(z_0, z) \gtrsim |\log \delta(z)|
\]
for all $z$ sufficiently close to $\partial \Omega$ when there exists $\rho \in C(\Omega) \cap PSH^-(\Omega) \ s.t. \ \delta^{\beta} \lesssim -\rho \lesssim \delta^{\alpha}$, where $\beta \geq \alpha > 0$. In case $\alpha = \beta$, i.e., $\eta(\Omega) > 0$, Blocki [5] improved the previous estimate to
\[
d_B(z_0, z) \gtrsim |\log \delta(z)|/|\log |\log \delta(z)||.
\]

In [12], the first author showed that Blocki’s estimate remains valid under the weaker condition $\alpha_l(\Omega) > 0$. Here we shall prove the following

Theorem 1.1. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. If for every $p \in \partial \Omega$ there is an open neighborhood $D$ of $p$ such that
\[
\alpha_l(\Omega \cap D) > \frac{n - 1 + \sqrt{(n - 1)(n + 3)}}{2},
\]
then
\[
d_B(z_0, z) \gtrsim \log \log |\log \delta(z)|
\]
for all $z$ sufficiently close to $\partial \Omega$.

Problem 1. Does the condition $\alpha_l(\Omega) > 0$ imply (1.1)?

Remark. Theorem 1.1 shows that the answer to this problem is affirmative when $n = 1$.

Theorem 1.1 combined with the main result (and its proof) in [11] gives

Corollary 1.2. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $\partial \Omega$ can be written locally as the graph of a Hölder continuous function of order $\gamma \geq \sqrt{(n-1)(n+3)-n+1}/2$. Then (1.1) holds.

Note that positivity of $\alpha(\Omega)$ can be used to obtain the $L^p$-integrability of $K_\Omega(\cdot, w)$ for fixed $w$ when $2 < p < 2 + \frac{2\alpha(\Omega)}{2n-\alpha(\Omega)}$ (cf. [12]). Analogously, we have
Theorem 1.3. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ with $\alpha_l(\Omega) > 0$. For fixed $w \in \Omega$, we have
$$K_{\Omega}(\cdot, w) \in A^2(\log A)^q(\Omega) := \left\{ f \in \mathcal{O}(\Omega); \int_{\Omega} |f|^2(\log + |f|)^q d\lambda < \infty \right\}$$
for all $0 < q < \alpha_l(\Omega)$, where $\log + t = \max\{\log t, 0\}$ and $d\lambda$ denotes the Lebesgue measure.

Note that $A^p(\log A)^q(\Omega)$ is a natural analogue of the $L^\log L$ space, which plays an important role in harmonic analysis and has been studied by many authors (cf. [2, 8, 9, 21, 32] etc).

As an application, we obtain the following

Corollary 1.4. If $\Omega$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\alpha_l(\Omega) > 0$, then $A^2(\log A)^q(\Omega)$ lies dense in $A^2(\Omega)$ for every $0 < q < \alpha_l(\Omega)$.

Barrett [1] found for each $k \in \mathbb{Z}^+$ a smooth bounded (non-pseudoconvex) domain $\Omega_k \subset \mathbb{C}^2$, such that the Bergman space $A^p(\Omega_k)$ is not dense in $A^2(\Omega_k)$ when $p \geq 2 + 1/k$. A mimic of his construction gives

Theorem 1.5. For every $s \in (0, 1)$ and $q \geq 1 - s$, there exists a smooth bounded (non-pseudoconvex) domain $\Omega(s) \subset \mathbb{C}^2$ such that the following properties hold:

1. $A^2(\log A)^q(\Omega(s))$ is not dense in $A^2(\Omega(s))$;
2. $K_{\Omega(s)}(\cdot, w) \notin A^2(\log A)^q(\Omega(s))$ for some $w \in \Omega(s)$.

Remark. Note that if $\Omega \subset \mathbb{C}^n$ is bounded, $p > 2$ and $q > 0$, then
$$A^p(\Omega) \subset A^2(\log A)^q(\Omega) \subset A^2(\Omega).$$

Thus Theorem 1.5 implies that there exists a smooth bounded domain $\Omega \subset \mathbb{C}^2$ such that $\bigcup_{p>2} A^p(\Omega)$ is not dense in $A^2(\Omega)$.

2. The proof of Theorem 1.1

The classical method of estimating the Bergman metric is to use the pluricomplex Green function (see, e.g., [5, 12, 25]). By the well-known localization principle for the Bergman metric, we only need to deal with the case
$$\alpha_l(\Omega) > \frac{n - 1 + \sqrt{(n - 1)(n + 3)}}{2}.$$ 

Let $\varrho = \varrho_{\Omega} := \sup\{u \in PSH^{-}(\Omega); u|_{\partial^{\Omega}} < -1\}$ be the relative extremal function of a fixed closed ball $B \subset \Omega$. By the extremal property of $\varrho$ there is a constant $C_\alpha > 0$ for each $\alpha \in (0, \alpha_l(\Omega))$, such that
$$-\varrho(z) \leq C_\alpha (-\log \delta(z))^{-\alpha}$$
for all $z$ sufficiently close to $\partial^{\Omega}$.

Lemma 2.1. For every $r > 1$ there exists a constant $\varepsilon_r \ll 1$ such that
$$\varrho(z_2) \geq r \varrho(z_1) - C_\alpha (-\log |z_1 - z_2|)^{-\alpha}$$
for all $z_1, z_2 \in \Omega$ with $|z_1 - z_2| \leq \varepsilon_r$. 
Proof. Set \( \varepsilon := |z_1 - z_2|, \Omega' = \Omega - (z_1 - z_2) \) and

\[
u(z) = \begin{cases} \varrho(z) \\ \max\{\varrho(z), r\varrho(z + z_1 - z_2) - C_\alpha(-\log \varepsilon)^{-\alpha}\} \end{cases} \quad z \in \Omega \setminus \Omega'.
\]

For every \( z \in \Omega \cap \partial \Omega' \) we have \( \delta(z) \leq \varepsilon \), so that

\[
\varrho(z) \geq -C_\alpha(-\log \varepsilon)^{-\alpha} \geq r\varrho(z + z_1 - z_2) - C_\alpha(-\log \varepsilon)^{-\alpha}.
\]

Thus \( u \in \text{PSH}^{-}(\Omega) \). On the other hand, for \( \varepsilon \leq \varepsilon, \ll 1 \) we have

\[
\varrho(z + z_1 - z_2) \leq -1/r, \forall z \in \overline{B},
\]

in view of the continuity of \( \varrho \). Thus \( u|_{\overline{B}} \leq -1 \). Since \( z_2 = z_1 - (z_1 - z_2) \in \Omega \cap \Omega' \), we infer from the extremal property of \( \varrho \) that

\[
\varrho(z_2) \geq u(z_2) \geq r\varrho(z_1) - C_\alpha(-\log \varepsilon)^{-\alpha}.
\]

\( \square \)

Let \( g_{\Omega}(z, w) \) be the pluricomplex Green function of \( \Omega \). For \( c > 0 \) we define

\[
A_{\Omega}(w, -c) := \{ z \in \Omega; g_{\Omega}(z, w) \leq -c \}
\]

Lemma 2.2. There exists a constant \( C \gg 1 \) such that for any \( w \in \Omega \)

\[
A_{\Omega}(w, -1) \subset \{ z \in \Omega; \varrho(z) < -C^{-1}\mu(w) \},
\]

where \( \mu(w) := (-\varrho(w))^{1+\frac{1}{\alpha}} \).

Proof. Apply Lemma 2.1 with \( r = \frac{3}{2} \), we conclude that if \( \varrho(z) = \varrho(w)/2 \) and \( |z - w| < \varepsilon_{1/2} \), then for suitable \( C_1 \gg 1 \)

\[
C_1(-\log |z - w|)^{-\alpha} \geq \frac{3}{2}\varrho(z) - \varrho(w) = -\varrho(w)/4.
\]

When \( |z - w| \geq \varepsilon_{1/2} \), it is also easy to see that

\[
C_1(-\log |z - w|)^{-\alpha} \geq -\varrho(w)/4
\]

if \( C_1 \) is large enough. Thus for all \( z \) with \( \varrho(z) = \varrho(w)/2 \) we have

\[
\log \frac{|z - w|}{R} \geq -C_2(-\varrho(w))^{-1/\alpha}
\]

for suitable \( C_2 > 0 \), where \( R \) denotes the diameter of \( \Omega \). It follows that

\[
\psi(z) = \begin{cases} \log\{|z - w|/R\} \\ \max\{\log\{|z - w|/R\}, 2C_2(-\varrho(w))^{-1-1/\alpha}\varrho(z)\} \end{cases} \quad \text{if } \varrho(z) \leq \varrho(w)/2
\]

otherwise

is a negative psh function on \( \Omega \) which has a logarithmic pole at \( w \). Thus for \( \varrho(z) \geq \varrho(w)/2 \) we have

\[
g_{\Omega}(z, w) \geq \psi(z) \geq 2C_2(-\varrho(w))^{-1-1/\alpha}\varrho(z),
\]

so that

\[
A_{\Omega}(w, -1) \cap \{ \varrho \geq \varrho(w)/2 \} \subset \{ \varrho < -C^{-1}\mu(w) \}
\]
whenever $C \gg 1$. Since
\[ \{ \varrho < \varrho(w)/2 \} \subset \{ \varrho < -C^{-1} \mu(w) \} \]
for $C \gg 1$, we conclude the proof. \qed

**Proposition 2.3.** Let $\alpha > \frac{n-1+\sqrt{(n-1)(n+3)}}{2}$ and $n-1 < \beta < \frac{\alpha^2}{\alpha+1}$. Then there exists a constant $C \gg 1$ such that
\[ A_{n}(w,-1) \subset \{ z \in \Omega; -C \nu(w) < \varrho(z) < -C^{-1} \mu(w) \}, \]
where $\nu(w) := \left( -\varrho(w) \right) \left( \frac{1}{\alpha} - \frac{1}{\beta} (1-\frac{1}{\beta}) \right)^{\frac{\alpha}{\beta}}$.

**Remark.** Note that $\alpha > \frac{n-1+\sqrt{(n-1)(n+3)}}{2}$ implies $\frac{\alpha^2}{\alpha+1} > n-1$, and $\beta > n-1$ implies that the exponent in $\nu$ is positive.

The proof of Proposition 2.3 is based on the following

**Lemma 2.4** (cf. [5]). Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Suppose $\zeta, w$ are two points in $\Omega$ such that the closed balls $\overline{B}(\zeta, \varepsilon), \overline{B}(w, \varepsilon) \subset \Omega$ and $\overline{B}(\zeta, \varepsilon) \cap \overline{B}(w, \varepsilon) = \emptyset$. Then there exists $\tilde{\zeta} \in \overline{B}(\zeta, \varepsilon)$ such that
\[ |g_{\zeta}(\tilde{\zeta}, w)|^n \leq n! (\log R/\varepsilon)^{n-1} |g_{\zeta}(w, \zeta)|. \]

**Proof of Proposition 2.3** For fixed $w \in \Omega$ which is sufficiently close to $\partial \Omega$, we set
\[ \varepsilon := \exp \left( -(-\varrho(w))^{-1/\beta} \right), \]
i.e., $-\varrho(w) = \left( -\log \varepsilon \right)^{-\beta}$. Since $-\varrho(w) \leq C_0 \left(-\log \delta(w)\right)^{-\alpha}$ and $\beta < \alpha$, it follows that if $\delta(w)$ is sufficiently small then
\[ 2^\alpha C_0 \left(-\log \varepsilon\right)^{-\alpha} \leq \left(-\log \varepsilon\right)^{-\beta} \leq C_0 \left(-\log \delta(w)\right)^{-\alpha}, \]
so that $\delta(w) \geq \sqrt{\varepsilon}$ and we have $B(w, \varepsilon) \subset \subset \Omega$. If $\delta(z) \leq \varepsilon$, then
\[ -\varrho(z) \leq C_0 \left(-\log \delta(z)\right)^{-\alpha} \leq C_0 \left(-\log \varepsilon\right)^{-\alpha} = C_0 (-\varrho(w))^{\alpha/\beta} \leq -\varrho(w)/2. \]

By the proof of Proposition 2.2, we have
\[ g_{\zeta}(z, w) \geq C_1 (-\varrho(w))^{-1-1/\alpha} \varrho(z) \]
for some constant $C_1 \gg 1$. This combined with (2.2) gives
\[ \sup_{\delta \leq \varepsilon} |g_{\zeta}(\cdot, w)| \leq C_0 C_1 (-\varrho(w))^{\alpha/\beta - 1-1/\alpha}. \]

By Lemma 2.1, we see that if $\varrho(\zeta) \leq 2 \varrho(w)$ then for suitable $C_2 > 0$,
\[ C_2 \left(-\log |\zeta - w|\right)^{-\alpha} \geq \frac{3}{2} \varrho(w) - \varrho(\zeta) \geq -\varrho(w)/2 = \left(-\log \varepsilon\right)^{-\beta}/2 > 2^\alpha C_2 \left(-\log \varepsilon\right)^{-\alpha} \]
whenever $\delta(w) \ll 1$, so that $|\zeta - w| > \sqrt{\varepsilon}$ and
\[ \overline{B}(\zeta, \varepsilon) \cap \overline{B}(w, \varepsilon) = \emptyset. \]
It follows from Lemma 2.4 that there exists \( \tilde{\zeta} \in \overline{B}(\zeta, \varepsilon) \) such that (2.1) holds. We also need the following known inequalities (cf. [4], Proposition 3.3.3)

\[
(2.5) \quad g_\Omega(z, w) \geq (\log R/\varepsilon) \cdot \vartheta_{\Omega(w, \varepsilon)}, \quad \forall z \in \Omega \setminus B(w, \varepsilon)
\]

\[
(2.6) \quad g_\Omega(z, w) \leq (\log \delta(w)/\varepsilon) \cdot \vartheta_{\Omega(w, \varepsilon)}, \quad \forall z \in \Omega.
\]

By (2.4) and (2.6), we have

\[
(2.7) \quad \sup_{\delta \leq \varepsilon} |\vartheta_{\Omega(w, \varepsilon)}| \leq C_0 C_1 (\log \delta(w)/\varepsilon)^{-1} (-\varrho(w))^{\alpha/\beta - 1 - 1/\alpha} =: \tau_\varepsilon(w).
\]

Set \( \tilde{\Omega} := \Omega - (\tilde{\zeta} - \zeta) \) and

\[
v(z) = \begin{cases} 
\vartheta_{\Omega(w, \varepsilon)}(z) & \text{if } z \in \Omega \setminus \tilde{\Omega} \\
\max\{\vartheta_{\Omega(w, \varepsilon)}(z), \vartheta_{\Omega(w, \varepsilon)}(z + \tilde{\zeta} - \zeta - \tau_\varepsilon(w))\} & \text{if } z \in \Omega \cap \tilde{\Omega}.
\end{cases}
\]

By (2.7), we see that \( v \) is a well-defined negative psh function on \( \Omega \). On the other hand, since

\[
\vartheta_{\Omega(w, \varepsilon)}(z) \leq \frac{\log |z-w|/\delta(w)}{\log R/\varepsilon}, \quad z \in \Omega \setminus B(w, \varepsilon),
\]

in view of (2.5), and \( z + \tilde{\zeta} - \zeta \in \overline{B}(w, 2\varepsilon) \) if \( z \in \overline{B}(w, \varepsilon) \), we have

\[
v|_{\overline{B}(w, \varepsilon)} \leq -\frac{\log(\delta(w)/(2\varepsilon))}{\log R/\varepsilon}
\]

so that

\[
\vartheta_{\overline{B}(w, \varepsilon)}(\tilde{\zeta}) - \tau_\varepsilon(w) \leq v(\zeta) \leq -\frac{\log(\delta(w)/(2\varepsilon))}{\log R/\varepsilon} \cdot \vartheta_{\overline{B}(w, \varepsilon)}(\zeta).
\]

This combined with (2.5) and (2.6) gives

\[
\begin{align*}
g_\Omega(\zeta, w) & \geq \frac{(\log R/\varepsilon)^2}{(\log \delta(w)/\varepsilon) \cdot (\log \delta(w)/(2\varepsilon))} \left( g_\Omega(\tilde{\zeta}, w) - C_0 C_1 (-\varrho(w))^{\alpha/\beta - 1 - 1/\alpha} \right) \\
& \geq C_3 \left( g_\Omega(\tilde{\zeta}, w) - C_0 C_1 (-\varrho(w))^{\alpha/\beta - 1 - 1/\alpha} \right) \\
& \geq -C_3(n!)^{1/n} (\log R/\varepsilon)^{-1-1/n} |g_\Omega(w, \zeta)|^{1/n} - C_0 C_1 C_3 (-\varrho(w))^{\alpha/\beta - 1 - 1/\alpha} \\
& \geq -C_4 (-\varrho(w))^{1/\alpha - (1-\frac{n}{\alpha})/\beta} (-\varrho(\zeta))^{-1/\alpha - (1+\frac{n}{\alpha})/\beta} - C_0 C_1 C_3 (-\varrho(w))^{\alpha/\beta - 1 - 1/\alpha}
\end{align*}
\]

(2.8)

for some constants \( C_1 \gg C_3 > 0 \), in view of (2.3). Since \( \alpha/\beta - 1 - 1/\alpha > 0 \), we see that if \( \delta(w) \ll 1 \) then \( C_0 C_1 C_2 (-\varrho(w))^{\alpha/\beta - 1 - 1/\alpha} \leq 1/2 \), so that

\[
A_\Omega(w, -1) \cap \{\varrho \leq 2\varrho(w)\} \subset \{\varrho > -C\nu(\varrho)\}
\]

for some \( C \gg 1 \). On the other hand, we have \( \{\varrho > 2\varrho(w)\} \subset \{\varrho > -C\nu(\varrho)\} \) for \( C \gg 1 \) since the exponent in \( \nu \) is less than one. Thus the proof is complete.

\[ \square \]

**Proof of Theorem 1.1.** Let \( z \in \Omega \) be sufficiently close to \( \partial \Omega \). We may choose a Bergman geodesic jointing \( z_0 \) to \( z \), and a finite number of points \( \{z_k\}_{k=1}^m \) on this geodesic with the following order

\[
\begin{align*}
z_0 & \to z_1 \to z_2 \to \cdots \to z_m \to z
\end{align*}
\]
where
\[ C^{-1} \mu(z_k) = C \nu(z_{k+1}) \quad \text{and} \quad C^{-1} \mu(z_m) \leq -\varrho(z) \leq C \nu(z_m) \]
for some \( C \gg 1 \) so that Proposition \([2,3]\) hold. Thus we have
\[ \{ g \Omega (\cdot, z_k) \leq -1 \} \cap \{ g \Omega (\cdot, z_{k+1}) \leq -1 \} = \emptyset \]
so that \( d_B(z_k, z_{k+1}) \geq c_1 > 0 \) for all \( k \) in view of Theorem 1.1 in \([5]\).

Set \( \gamma := \left( \frac{1}{n} - \frac{1}{\beta} (1 - \frac{1}{n}) \right) \frac{p_0}{1+\alpha} \). Note that
\[
\log(-\varrho(z_0)) = \frac{\gamma \alpha}{1+\alpha} \log(-\varrho(z_1)) + \frac{2\alpha}{\alpha+1} \log C = \cdots
\]
\[ = \left( \frac{\gamma \alpha}{1+\alpha} \right)^m \log(-\varrho(z_m)) + \frac{1 - \left( \frac{\gamma \alpha}{1+\alpha} \right)^m}{1 - \frac{\gamma \alpha}{1+\alpha}} \frac{2\alpha}{\alpha+1} \log C \]
Thus we have
\[ m \asymp \log |\log(-\varrho(z_m))| \asymp \log |\log(-\varrho(z))| \gtrsim \log \log |\log \delta(z)|, \]
so that
\[ d_B(z_0, z) \geq \sum_{k=1}^{m-1} d_B(z_k, z_{k+1}) \geq c_1 (m - 1) \gtrsim \log \log |\log \delta(z)|. \]

\[ \square \]

3. BERGMAN KERNEL AND \( A^p(\log A)^q \)

We first introduce some basic facts about the \( A^p(\log A)^q \) space. For \( p, q > 0 \), let \( L^p(\log L)^q(\Omega) \) be the set of measurable complex-valued functions \( f \) on \( \Omega \) such that
\[ \int_{\Omega} |f|^p (\log^+ |f|)^q d\lambda < \infty. \]
Set
\[ \| f \|_{L^p(\log L)^q(\Omega)} = \inf \left\{ s > 0; \int_{\Omega} \left( \frac{|f|}{s} \right)^p \left( \log^+ \frac{|f|}{s} \right)^q d\lambda \leq 1 \right\} \]
and
\[ A^p(\log A)^q(\Omega) := L^p(\log L)^q(\Omega) \cap \mathcal{O}(\Omega). \]
When \( q = 0 \), \( \| \cdot \|_{L^p(\log L)^q(\Omega)} \) is the usual \( L^p \) norm.

**Proposition 3.1.** If \( \Omega \subset \mathbb{C}^n \) is a bounded domain, then the following properties hold:
\begin{enumerate}
    \item \( L^p(\log L)^q(\Omega) \) is a linear space;
    \item \( L^p(\log L)^q(\Omega) \subset L^p(\Omega) \) and \( \| \cdot \|_{L^p(\Omega)} \lesssim \| \cdot \|_{L^p(\log L)^q(\Omega)} \);
    \item If \( p, q \geq 1 \), then \( L^p(\log L)^q(\Omega) \) is a Banach space with norm \( \| \cdot \|_{L^p(\log L)^q(\Omega)} \);
    \item If \( p, q \geq 1 \), then \( A^p(\log A)^q \) is a closed subspace of \( L^p(\log L)^q(\Omega) \).
\end{enumerate}

The arguments are standard, and we include the proof for the sake of completeness.
Proof. (1) Given \( f, g \in L^p(\log L)^q(\Omega) \) and \( \varepsilon \in \mathbb{C} - \{0\} \), we have

\[
\int_{\Omega} |c|^p (\log^+ |cf|)^q \, d\lambda = \left( \int_{\{ |f| \leq |c| \}} + \int_{\{ |f| > |c| \}} \right) |c|^p (\log^+ |cf|)^q \, d\lambda \\
\leq \text{const.} + \int_{\{ |f| > |c| \}} |c|^p |f|^p (\log^+ |c| + \log^+ |f|)^q \, d\lambda \\
\leq \text{const.} + \int_{\Omega} |c|^p |f|^p (2 \log^+ |f|)^q \, d\lambda \\
= \text{const.} + 2^q |c|^p \int_{\Omega} |f|^p (\log^+ |f|)^q \, d\lambda \\
< +\infty,
\]

and

\[
\int_{\Omega} \left| \frac{f + g}{2} \right|^p \left( \log^+ \left| \frac{f + g}{2} \right| \right)^q \, d\lambda \\
\leq \int_{\Omega} \left( \frac{|f| + |g|}{2} \right)^p \left( \log^+ \left( \frac{|f| + |g|}{2} \right) \right)^q \, d\lambda \\
\leq \left( \int_{\{ |f| \leq |g| \}} + \int_{\{ |f| > |g| \}} \right) \left( \frac{|f| + |g|}{2} \right)^p \left( \log^+ \left( \frac{|f| + |g|}{2} \right) \right)^q \, d\lambda \\
\leq \int_{\Omega} |f|^p (\log^+ |f|)^q \, d\lambda + \int_{\Omega} |g|^p (\log^+ |g|)^q \, d\lambda \\
< \infty.
\]

Hence \( A^p(\log A)^q(\Omega) \) is closed under scalar multiplication and addition, which implies (1).

(2) If \( \|f\|_{L^p(\log L)^q(\Omega)} = 0 \), then for every \( \varepsilon > 0 \) we have

\[
\int_{\Omega} |f|^p \left( \log^+ \left| \frac{f}{\varepsilon} \right| \right)^q \, d\lambda \leq \varepsilon^p,
\]

since \( h(t) = \int_{\Omega} \left( \frac{|f|}{t} \right)^p (\log^+ |f|/t)^q \) is nonincreasing for \( t > 0 \). In particular, we have \( \|f\|_{L^p(\Omega \cap \{ |f| > \varepsilon \})} \leq \varepsilon \). Letting \( \varepsilon \to 0 \), we get \( \|f\|_{L^p(\Omega)} = 0 \). Now suppose

\[
\|f\|_{L^p(\log L)^q(\Omega)} = \tau > 0.
\]

It follows again from the monotonicity of \( h(t) \) that

\[
\int_{\Omega} \left( \frac{|f|}{\tau + \varepsilon} \right)^p \left( \log^+ \frac{|f|}{\tau + \varepsilon} \right)^q \, d\lambda \leq 1, \quad \forall \varepsilon > 0.
\]

Hence

\[
\int_{\Omega} |f|^p \, d\lambda \leq \int_{\{ |f| \leq (\tau + \varepsilon) \}} |f|^p \, d\lambda + \int_{\{ |f| > (\tau + \varepsilon) \}} |f|^p \left( \log^+ \frac{|f|}{\tau + \varepsilon} \right)^q \, d\lambda \\
\leq (|\Omega| e^p + 1) (\tau + \varepsilon)^p, \quad \forall \varepsilon > 0,
\]

from which (2) immediately follows.
(3) The assertion follows directly from (1) and Theorem 10 in Chapter 3 of [30] with \( g(t) = \left| t \right|^p (\log^+ |t|)^q \) as a Young function.

(4) Let \( \{h_n\} \) be a Cauchy sequence in \( A^p(log A)^q(\Omega) \). By (2) and (3) there is an \( h \in L^p(log L)^q(\Omega) \) such that \( h_n \to h \) under \( \| \cdot \|_{L^p(\Omega)} \) and \( \| \cdot \|_{L^p(log L)^q(\Omega)} \). Since \( A^p(\Omega) \) is complete, we have

\[ h \in A^p(\Omega) \cap L^p(log L)^q(\Omega) = A^p(log A)^q(\Omega). \]

Thus \( A^p(log A)^q(\Omega) \) is a closed subspace of \( L^p(log L)^q(\Omega) \). \( \square \)

Remark. Note that for \( p \geq 1, q \geq 0, g(t) = t^p (\log^+ t)^q \) is convex on \( [x_0, +\infty) \) for some \( x_0 > 0 \).

Thus \( h(t) := \max\{0, g(t) - g(t_0)\} \)

is convex on \( \mathbb{R}^+ \) and satisfies

\[ \int_{\Omega} g(|f|)d\lambda < \infty \Leftrightarrow \int_{\Omega} h(|f|)d\lambda < \infty, \]

since \( \Omega \) is bounded. So if we choose \( h \) as a Young function, then \( A^p(log A)^q(\Omega) \) is still a Banach space.

In order to prove Theorem 1.3, we need the following result from [12].

**Proposition 3.2.** Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex domain. Let \( \rho \) be a continuous negative plurisubharmonic function on \( \Omega \). Set

\[ \Omega_t = \{z \in \Omega; -\rho(z) > t\}, \]

where \( t > 0 \). Let \( a > 0 \) be given. For every \( r \in (0, 1) \), there exist constants \( \varepsilon_r, C_r > 0 \), such that

\[ \int_{\{\rho \leq \varepsilon\}} |K_{\Omega}(\cdot, w)|^2d\lambda \leq C_r K_{\Omega_a}(w) \left( \frac{\varepsilon}{a} \right)^r \]

for all \( w \in \Omega_a \) and \( \varepsilon < \varepsilon_r a \).

**Proof of Theorem 1.3.** For every \( \alpha \in (0, \alpha_l(\Omega)) \), there exists a constant \( C_\alpha > 0 \) such that

\[ -\rho \leq C_\alpha (-\log \delta)^{-\alpha}, \]

where \( \rho = \rho_{\overline{B}} \) is the relative extremal function of a fixed closed ball \( \overline{B} \subset \Omega \). From Proposition 3.2, we have

\[ \int_{\{\rho \leq \varepsilon\}} |K_{\Omega}(\cdot, w)|^2d\lambda \leq C\varepsilon^r \]

for all \( 0 < r < 1 \), where \( w \in \Omega \) is fixed. Here and what in follows we use \( C \) to denote all constants depending only on \( \alpha, r, w \) and \( \Omega \). By (3.1), we have

\[ \{(-\log \delta)^{-\alpha} \leq \varepsilon\} \subset \{-\rho \leq C_\alpha \varepsilon\}, \]

so that

\[ \int_{\{(-\log \delta)^{-\alpha} \leq \varepsilon\}} |K_{\Omega}(\cdot, w)|^2d\lambda \leq C\varepsilon^r. \]
Since \(B(z, \delta(z)) \subset \{ \delta \leq 2\delta(z) \} = \{ (\log \delta)^{-\alpha} \leq (\log (2\delta(z)))^{-\alpha} \} \), we infer from the mean value inequality that
\[
|K_\Omega(z, w)|^2 \leq C\delta(z)^{-2n} \int_{\{ (\log \delta)^{-\alpha} \leq (\log (2\delta(z)))^{-\alpha} \}} |K_\Omega(\cdot, w)|^2 d\lambda
\]
(3.3)
\[
\leq C\delta(z)^{-2n} (2 \log(2\delta(z)))^{-\alpha r},
\]
which implies
\[
\log^+ |K_\Omega(\cdot, w)| \leq \max \left\{ 0, C - n \log \delta(\cdot) - \frac{\alpha r}{2} \log(-\log 2\delta(\cdot)) \right\}.
\]
When \(2^{-k-1} \leq (\log \delta(z))^{-\alpha} < 2^{-k} \) (here \(k \geq k_0 \) for some \(k_0 \)), we have
(3.4)
\[
\log^+ |K_\Omega(z, w)| \leq C \cdot 2^{k/\alpha}.
\]
Hence
\[
\int_\Omega |K_\Omega(\cdot, w)|^2 (\log^+ |K_\Omega(\cdot, w)|)^q d\lambda
\]
\[
\leq \left( \int_{\{ (\log \delta)^{-\alpha} > 2^{-k_0} \}} + \sum_{k=k_0}^{\infty} \int_{\{ 2^{-k-1} \leq (\log \delta)^{-\alpha} < 2^{-k} \}} \right) |K_\Omega(\cdot, w)|^2 (\log^+ |K_\Omega(\cdot, w)|)^q d\lambda
\]
\[
\leq \sum_{k=k_0}^{\infty} C \cdot 2^{kq} \int_{\{ (\log \delta)^{-\alpha} < 2^{-k} \}} |K_\Omega(\cdot, w)|^2 d\lambda
\]
\[
\leq \sum_{k=k_0}^{\infty} C \cdot 2^{kq} \cdot 2^{-kr}
\]
\[
\leq (3.2) C + \sum_{k=k_0}^{\infty} C \cdot 2^{(\frac{q}{\alpha} - r)k}.
\]
Thus \(\int_\Omega |K_\Omega(\cdot, w)|^2 (\log^+ |K_\Omega(\cdot, w)|)^q d\lambda < +\infty \) when \(q < \alpha r\). Since \(\alpha < \alpha(\Omega)\) and \(r \in (0, 1)\) can be chosen arbitrarily, we have
\[K_\Omega(\cdot, w) \in A^2(\log A)^q(\Omega), \forall 0 < q < \alpha_l(\Omega).\]

\[\square\]

Proof of Corollary 1.4 We infer from Theorem 1.3 that
\(\Lambda := \{ K_\Omega(\cdot, w); w \in \Omega \} \subset A^2(\log A)^q(\Omega)\).
Since \(A^2(\log A)^q(\Omega)\) is a linear space, we have
\(\text{Span}\{\Lambda\} \subset A^2(\log A)^q(\Omega)\).
If \(f \in A^2(\Omega)\) and \(f \perp \overline{\text{Span}\{\Lambda\}}\), then
\[
f(w) = \int_\Omega f(\zeta)K_\Omega(\zeta, w)d\lambda(\zeta) = 0.
\]
for every \( w \in \Omega \), i.e., \( f \equiv 0 \). In other words, \( \text{Span}\{A\} = A^2(\Omega) \). So \( A^2(\log A)^q(\Omega) \) lies dense in \( A^2(\Omega) \).

4. **Proof of Theorem 1.5**

Set \( \Omega^{(s)} = \{(z, w) \in \mathbb{C}^2; |z| < r_2(|w|), |z - c(|w|)| > r_1(|w|)\} \), where \( r_1, r_2 \) and \( c_s \) are smooth functions on \([1, 6]\) such that

\[
\begin{align*}
  r_1(x) &= \begin{cases} 
    3 - \sqrt{x - 1}, & \text{as } x \to 1^+ \\
    1, & \text{for } x \in [1, 2] \\
    \text{decreasing} & \text{for } x \in [2, 5] \\
    \text{increasing} & \text{for } x \in [5, 6] \\
    3 - \sqrt{6 - x}, & \text{as } x \to 6^-
  \end{cases} \\
  r_2(x) &= \begin{cases} 
    3 + \sqrt{x - 1}, & \text{as } x \to 1^+ \\
    4, & \text{for } x \in [1, 2] \\
    \text{increasing} & \text{for } x \in [2, 5] \\
    \text{decreasing} & \text{for } x \in [5, 6] \\
    3 + \sqrt{6 - x}, & \text{as } x \to 6^-
  \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
c_s(x) &= \begin{cases} 
    0, & \text{for } x \in [1, 2] \\
    \text{decreasing} & \text{for } x \in [2, 3] \\
    \frac{1}{2}e^{-2|x-3|^s} - 1, & \text{for } x \text{ in a small neighborhood of } 3 \\
    \text{increasing} & \text{for } x \in [3, 4] \\
    1 - \frac{1}{2}e^{-2|x-4|^s}, & \text{for } x \text{ in a small neighborhood of } 4 \\
    \text{decreasing} & \text{for } x \in [4, 5] \\
    0, & \text{for } x \in [5, 6].
  \end{cases}
\end{align*}
\]

Clearly, \( g(x) := x^2(\log^+ x)^q \) is increasing on \( \mathbb{R}_+ \) and convex when \( x > x_0 \) for some \( x_0 \in \mathbb{R}_+ \). Define a convex increasing function on \( \mathbb{R}_+ \) as follows

\[
h(x) = \begin{cases} 
  g(x_0), & 0 \leq x \leq x_0 \\
  g(x), & x > x_0.
\end{cases}
\]

**Lemma 4.1.** If \( s \in (0, 1) \) and \( q \geq \frac{1}{s} - 1 \), then \( \frac{1}{z} \in A^2(\Omega^{(s)}) \) while \( \frac{1}{z} \notin A^2(\log A)^q(\Omega^{(s)}) \).

**Proof.** Define \( \Omega_w^{(s)} := \{z \in \mathbb{C}; (z, w) \in \Omega^{(s)}\} \). Clearly, \( \Omega_w^{(s)} = \Omega^{(s)}_{|w|} \).

First of all, we have

\[
\begin{align*}
  \int_{\Omega^{(s)}} \frac{1}{|z|^2} d\lambda(z, w) &= \int_{\{1 \leq |w| \leq 6\}} d\lambda(w) \int_{\Omega_w^{(s)}} \frac{1}{|z|^2} d\lambda(z) \\
  &= 2\pi \int_1^6 t dt \int_{\Omega_w^{(s)}} \frac{1}{|z|^2} d\lambda(z)
\end{align*}
\]
\[ 2\pi \left( \int_{ \{ t \in [1,6], |t-3|>\varepsilon, |t-4|<\varepsilon \} } + \int_{3-\varepsilon}^{3+\varepsilon} + \int_{4-\varepsilon}^{4+\varepsilon} \right) dt \int_{\Omega(t)} \frac{1}{|z|^2} d\lambda(z) \]
\[ \leq C\varepsilon + 2\pi \int_{-\varepsilon}^{\varepsilon} (3 + t) dt \int_{\Omega_{3+t}^{(s)}} \frac{1}{|z|^2} d\lambda + 2\pi \int_{-\varepsilon}^{\varepsilon} (4 + t) dt \int_{\Omega_{4+t}^{(s)}} \frac{1}{|z|^2} d\lambda(z) \]
\[ \leq C\varepsilon + 4\pi \int_{-\varepsilon}^{\varepsilon} (4 + t) dt \int_{\Omega_{3+t}^{(s)} \cup \Omega_{4+t}^{(s)}} \frac{1}{|z|^2} d\lambda(z) \]
\[ \leq C\varepsilon + 4\pi \int_{-\varepsilon}^{\varepsilon} (4 + t) dt \int_{\Omega_{3+t}^{(s)} \cup \Omega_{4+t}^{(s)}} \frac{1}{|z|^2} d\lambda(z) \]
\[ \leq C\varepsilon + 8\pi^2 \int_{-\varepsilon}^{\varepsilon} (4 + t) (\log 8 + 2|t|^{-s}) dt \]
\[ < +\infty. \]

On the other hand, since \( \Omega_{3+t}^{(s)} \cup \Omega_{4+t}^{(s)} \supset \{ z \in \mathbb{C}; e^{-|t|^{-s}} < |z| < 1 \} \), we have
\[ \int_{\Omega(s)} \frac{1}{|z|^2} \left( \log + \frac{1}{|z|} \right)^q d\lambda(z, w) \geq 2\pi \int_{1}^{6} t dt \int_{\Omega(t)} \frac{1}{|z|^2} \left( \log + \frac{1}{|z|} \right)^q d\lambda(z) \]
\[ \geq 2\pi \left( \int_{3-\varepsilon}^{3+\varepsilon} + \int_{4-\varepsilon}^{4+\varepsilon} \right) dt \int_{\Omega(t)} \frac{1}{|z|^2} \left( \log + \frac{1}{|z|} \right)^q d\lambda(z) \]
\[ \geq 2\pi \int_{-\varepsilon}^{\varepsilon} dt \int_{\Omega_{3+t}^{(s)} \cup \Omega_{4+t}^{(s)}} \frac{1}{|z|^2} \left( \log + \frac{1}{|z|} \right)^q d\lambda(z) \]
\[ \geq 2\pi \int_{-\varepsilon}^{\varepsilon} dt \int_{\{ e^{-|t|^{-s}} < |z| < 1 \}} \frac{1}{|z|^2} \left( \log + \frac{1}{|z|} \right)^q d\lambda(z) \]
\[ = 4\pi^2 \int_{-\varepsilon}^{\varepsilon} \frac{1}{q+1} (|t|^{-s})^{q+1} dt \]
\[ = +\infty. \]

\[ \square \]

**Proof of Theorem 1.5.** (1) Suppose on the contrary that \( A^2(\log A)^q(\Omega^{(s)}) \) lies dense in \( A^2(\Omega^{(s)}) \),
then there exists a sequence \( \{ f_n \} \subset A^2(\log A)^q(\Omega^{(s)}) \) such that \( \| f_n - \frac{1}{z} \|_{L^2(\Omega^{(s)})} \to 0 \) \( (n \to \infty) \).
Set
\[ f_n(z, w) = \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{f}_n(z, e^{i\theta} w) d\theta. \]

**Step 1.** We shall verify that \( f_n \in A^2(\log A)^q(\Omega^{(s)}) \) and \( \| f_n - \frac{1}{z} \|_{L^2(\Omega^{(s)})} \to 0 \).
First of all, we have
\[ \int_{\Omega(s)} |f_n|^2 (\log + |f_n|)^q d\lambda(z, w) = \int_{\Omega(s)} g \left( \left\| \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{f}_n(z, e^{i\theta} w) d\theta \right\| \right) d\lambda(z, w) \]
\[ \leq \int_{\Omega(s)} g \left( \left\| \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{f}_n(z, e^{i\theta} w)| d\theta \right\| \right) d\lambda(z, w) \]
where the second inequality follows from Schwarz's inequality. So

\[ \left\| \tilde{f}_n(z, e^{i \theta} w) - \frac{1}{z} \right\|_{L^2(\Omega(t))}^2 \rightarrow 0 \quad (n \to \infty), \]

where the second inequality follows from Schwarz's inequality. So \( f_n \in A^2(\log A)^q(\Omega(t)) \).

Next, we have

\[
\int_{\Omega(t)} \left| f_n - \frac{1}{z} \right|^2 d\lambda(z, w) = \int_{\Omega(t)} \frac{1}{2 \pi} \int_0^{2 \pi} \left( \tilde{f}_n(z, e^{i \theta} w) - \frac{1}{z} \right)^2 d\theta d\lambda(z, w)
\]

\[
\leq \frac{1}{2 \pi} \int_{\Omega(t)} \int_0^{2 \pi} \left| \tilde{f}_n(z, e^{i \theta} w) - \frac{1}{z} \right|^2 d\theta d\lambda(z, w)
\]

\[
= \frac{1}{2 \pi} \int_0^{2 \pi} \left| \tilde{f}_n(z, e^{i \theta} w) - \frac{1}{z} \right|^2 d\lambda(z, w) d\theta
\]

\[
= \frac{1}{2 \pi} \int_0^{2 \pi} \left| \tilde{f}_n(z, e^{i \theta} w) - \frac{1}{z} \right|^2_{L^2(\Omega(t))} d\theta
\]

where the second inequality follows from Schwarz's inequality. So \( f_n \to \frac{1}{z} \) in \( L^2(\Omega(t)) \).

**Step 2.** We claim that each \( f_n \) is independent of \( w \), and can be extended to a holomorphic function on \( \{|z| < 4\} \).

In fact, for every \( z \in \mathbb{C} \) with \( |z| = 3 \), we conclude that

\[
f_n(z, w) = \frac{1}{2 \pi} \int_0^{2 \pi} \tilde{f}_n(z, e^{i \theta} w) d\theta
\]

\[
= \frac{1}{2 \pi} \int_{\{|z| = 1\}} \frac{\tilde{f}_n(z, w \zeta)}{\zeta} d\zeta
\]
is independent of \( w \) when \( 1 < |w| < 6 \), in view of Cauchy’s theorem. Now for any \( w_1, w_2 \in \{ 1 < |w| < 6 \} \), the function \( f_n(\cdot, w_1) - f_n(\cdot, w_2) \) is holomorphic, and vanishes on \( \{|z| = 3\} \), we infer from the identity theorem that \( f_n(\cdot, w_1) - f_n(\cdot, w_2) \equiv 0 \), i.e., \( f_n \) is independent of \( w \), thus we have

\[
 f_n \in \mathcal{O}(\{0 < |z| < 4\}).
\]

Write \( f_n(z) = \sum_{m=-\infty}^{+\infty} a_m^{(n)} z^m \). Since \( f_n \in A^2(\log A)^q(\Omega^{(s)}) \subset A^2(\Omega^{(s)}) \), we have

\[
 +\infty > \int_{\Omega^{(s)}} |f_n(z)|^2 d\lambda(z, w)
 = 2\pi \int_{1}^{6} t dt \int_{\Omega_t^{(s)}} |f_n(z)|^2 d\lambda(z)
 \geq 2\pi \int_{-\varepsilon}^{\varepsilon} dt \int_{\Omega_{4+t}^{(s)} \setminus \Omega_{4-t}^{(s)}} |f_n(z)|^2 d\lambda(z)
 \geq 2\pi \int_{-\varepsilon}^{\varepsilon} dt \int_{|e^{-|t|} - \frac{1}{2} < |z| < 4} \sum_{m=-\infty}^{+\infty} |a_m^{(n)}|^2 |z|^{2m} d\lambda(z)
 = 4\pi^2 \int_{-\varepsilon}^{\varepsilon} dt \int_{e^{-|t|} - \frac{1}{2}}^{4} \sum_{m=-\infty}^{+\infty} |a_m^{(n)}|^2 \cdot |z|^{2m+1} d\lambda(z)
 = 4\pi^2 \sum_{m \neq -1} \int_{-\varepsilon}^{\varepsilon} \frac{|a_m^{(n)}|^2 (4^{2m+2} - e^{-(2m+2)|t| - \frac{1}{2}})}{2m + 2} dt + 4\pi^2 \int_{-\varepsilon}^{\varepsilon} |a_{-1}^{(n)}|^2 (\log 4 + |t|^{-\frac{1}{2}}) dt.
\]

When \( m \leq -2, e^{-(2m+2)|t| - \frac{1}{2}} \) is not integrable, so \( a_m^{(n)} = 0 \). Thus we may write

\[
 f_n(z) = \frac{a_{-1}^{(n)}}{z} + g_n(z),
\]

where \( g_n \) is holomorphic on \( \{|z| < 4\} \). Notice that \( \frac{1}{z} \) is bounded on \( \{|z| \geq \varepsilon\} \), so

\[
 \frac{a_{-1}^{(n)}}{z} \in A^2(\log A)^q(\Omega^{(s)} \cap \{|z| \geq \varepsilon\}).
\]

Also, since \( f_n \in A^2(\log A)^q(\Omega^{(s)}) \) and \( g_n \) is bounded on \( \{|z| < \varepsilon\} \), it follows from Proposition 3.1(1) we have

\[
 \frac{a_{-1}^{(n)}}{z} \in A^2(\log A)^q(\Omega^{(s)} \cap \{|z| < \varepsilon\}).
\]

So \( \frac{a_{-1}^{(n)}}{z} \) lies in \( A^2(\log A)^q(\Omega^{(s)}) \). But we have already known that \( \frac{1}{z} \) is not in \( A^2(\log A)^q(\Omega^{(s)}) \), thus \( a_{-1}^{(n)} \) must be zero, and \( f_n \) is in fact a holomorphic function on \( \{|z| < 4\} \), which concludes the proof of the claim.
Since \( f_n \to \frac{1}{2} \) in \( L^2(\Omega(s)) \), the convergence holds uniformly on \( \{(z, w_0); |z| = 3\} \), where \( w_0 \) is a fixed point in \( \{1 < |w| < 6\} \). So we have

\[
0 = \int_{\{(z, w_0); |z| = 3\}} f_n(z)dz \to \int_{\{(z, w_0); |z| = 3\}} \frac{1}{z}dz = 2\pi i,
\]

which is absurd. Thus \( A^2(\log A)^q(\Omega(s)) \) is not dense in \( A^2(\Omega(s)) \).

(2) Suppose on the contrary that \( K_{\Omega(s)}(\cdot, w) \in A^2(\log A)^q(\Omega(s)) \) for every \( w \in \Omega(s) \). Then an analogous argument as the proof of Corollary 1.4 shows that \( A^2(\log A)^q(\Omega(s)) \) is dense in \( A^2(\Omega(s)) \), a contradiction to (1). \( \square \)

5. Appendix

We shall construct a bounded planar domain \( \Omega \) with \( \alpha(\Omega) = 0 \) while \( \alpha_l(\Omega) > 0 \), using Corollary 3.4 of [14] and Theorem 1.1 of [7]. Set \( D(a, r) = \{|z - a| < r\} \), and \( E_r(a) = \overline{D}(a, r) - \Omega \). We denote by \( C_l(E) \) the logarithmic capacity of a compact set \( E \subset \mathbb{C} \).

**Definition 5.1 ([7]).** For a compact set \( E \subset \mathbb{R}, a \in E \) and \( \varepsilon > 0 \), set

\[
\mathcal{N}_E(a, \varepsilon) = \{n \in \mathbb{N}; C_l(E_{2^{-n}}(a) - \overline{D}(a, 2^{-n-1})) \geq \varepsilon \cdot 2^{-n}\},
\]

and we say that \( \mathcal{N}_E(a, \varepsilon) \) is of positive lower density if

\[
\liminf_{N \to \infty} \frac{\mathcal{N}_E(\varepsilon) \cap \{1, 2, \ldots, N\}}{N + 1} > 0.
\]

**Theorem 5.1 ([7], Theorem 1.1).** For a compact set \( E \subset \mathbb{R} \) the Green's function \( g_{0-E}(\cdot, \infty) \) is Hölder continuous at 0 if and only if \( \mathcal{N}_E(0, \varepsilon) \) is of positive lower density for some \( \varepsilon > 0 \).

**Definition 5.2 ([14]).** For \( \varepsilon > 0, 0 < \lambda < 1 \) and \( \gamma > 1 \) we set

\[
\mathcal{N}_a(\varepsilon, \lambda, \gamma) := \{n \in \mathbb{Z}^+; C_l(E_{\lambda^n}(a)) \geq \varepsilon \lambda^n\}
\]

\[
\mathcal{N}_a^n(\varepsilon, \lambda, \gamma) := \mathcal{N}_a(\varepsilon, \lambda, \gamma) \cap \{1, 2, \ldots, n\}.
\]

We define the \( \gamma \)-capacity density of \( \partial \Omega \) at \( a \) by

\[
\mathcal{D}_a(\varepsilon, \lambda, \gamma) = \liminf_{n \to \infty} \frac{\sum_{k \in \mathcal{N}_a^n(\varepsilon, \lambda, \gamma)} k^{-1}}{\log n},
\]

and the \( \gamma \)-capacity density of \( \partial \Omega \) by

\[
\mathcal{D}_W(\varepsilon, \lambda, \gamma) = \liminf_{n \to \infty} \frac{\inf_{a \in \partial \Omega} \sum_{k \in \mathcal{N}_a^n(\varepsilon, \lambda, \gamma)} k^{-1}}{\log n}.
\]

**Theorem 5.2 ([14], Corollary 3.4).** If \( \mathcal{D}_W(\varepsilon, \lambda, \gamma) > 0 \) for some \( \varepsilon, \lambda, \gamma \), then there exists \( \beta > 0 \) such that

\[
\phi_K(z) \leq (-\log \delta(z))^{-\beta}
\]

for all \( z \) sufficiently close to \( \partial \Omega \), where \( \phi_K \) denotes the capacity potential of a compact subset \( K \) relative to \( \Omega \).

**Example.** Let \( \Omega = D(0, 3) - E \), where \( E = \bigcup_{n=0}^{\infty} [2^{-n}, 2^{-n} + 2^{-2n}] \cup \{0\} \). Then \( \alpha(\Omega) = 0 \) and \( \alpha_l(\Omega) > 0 \).
Proof. We first calculate $N^n_a(\frac{1}{16}, \frac{1}{2}, 2)$:

The case $a = 0$ is simple, for

$$C_l(E_{2^{-n}}(0)) \geq C_l([2^{-n-1}, 2^{-n-1} + 2^{-2(n+1)}]) = \frac{1}{4} \cdot 2^{-2(n+1)} = \frac{1}{16} \cdot 2^{-2n}.$$ 

The case $a \in [2^{-n_0}, 2^{-n_0} + 2^{-2n_0}]$ is divided into three parts:

(i) If $n \leq n_0 - 1$, then $0 \in \overline{D}(a, 2^{-n})$. It is easy to see that $E_{2^{-n}}(0) \subset E_{2^{-n}}(a)$. Thus

$$C_l(E_{2^{-n}}(a)) \geq C_l(E_{2^{-n}}(0)) \geq \frac{1}{16} \cdot 2^{-2n}.$$ 

(ii) If $n_0 \leq n \leq 2n_0$, then $[2^{-n_0}, 2^{-n_0} + 2^{-2n_0}] \subset \overline{D}(a, 2^{-n})$, so

$$C_l(E_{2^{-n}}(a)) \geq \frac{1}{4} \cdot 2^{-2n_0} \geq \frac{1}{4} \cdot 2^{-2n} \geq \frac{1}{16} \cdot 2^{-2n}.$$ 

(iii) If $n \geq 2n_0 + 1$, then $\overline{D}(a, 2^{-n}) \cap [2^{-n_0}, 2^{-n_0} + 2^{-2n_0}]$ contains an interval with length $2^{-n}$, so

$$C_l(E_{2^{-n}}(a)) \geq \frac{1}{4} \cdot 2^{-n} \geq \frac{1}{16} \cdot 2^{-2n}.$$ 

If $a \in \partial D(0, 3)$, then $E_{2^{-n}}(a)$ contains an interval with length $2^{-n+1}$. Thus

$$C_l(E_{2^{-n}}(a)) \geq \frac{1}{4} \cdot 2^{-n+1} \geq \frac{1}{16} \cdot 2^{-2n}.$$ 

Hence $N^n_a(\frac{1}{16}, \frac{1}{2}, 2) = \{1, 2, \ldots, n\}$ for all $a \in \partial \Omega$, which implies $D_W(\frac{1}{16}, \frac{1}{2}, 2) > 0$. By Theorem 5.2, we have $\alpha_l(\Omega) > 0$.

On the other hand, we have

$$C_l(E_{2^{-n}}(0) - \overline{D}(0, 2^{-n-1})) = \frac{1}{4} \cdot 2^{-2(n+1)}.$$ 

Hence for any $\varepsilon > 0$, there exists an integer $n_0$ such that

$$C_l(E_{2^{-n}}(0) - \overline{D}(0, 2^{-n-1})) < \varepsilon \cdot 2^{-n}$$

for every $n > n_0$. From Theorem 5.1, we know that $g_{\overline{D}}(z, \infty)$ is not Hölder continuous at 0. Suppose on the contrary that there exists a function $\phi \in SH^+(\Omega) \cap C(\Omega)$ such that $-\phi \lesssim \delta^\beta$ for some $\beta > 0$. It is easy to see from the maximum principle that

$$g_{\overline{D}}(z, \infty) \lesssim -\phi(z) \lesssim \delta^\beta(z) \leq |z - 0|^\beta$$

in a neighborhood of 0, which is a contradiction. \hfill \Box

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