Multiple chessboard complexes  
and the colored Tverberg problem

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February 23, 2015

Abstract

Following D.B. Karaguezian, V. Reiner, and M.L. Wachs (Matching Complexes, Bounded Degree Graph Complexes, and Weight Spaces of GL-Complexes, Journal of Algebra 2001) we study the connectivity degree and shellability of multiple chessboard complexes. Among applications of the new connectivity bounds is a result of colored Tverberg type where multiple points of the same color are permitted.

1 An overview and motivation

Chessboard complexes and their generalizations belong to the class of most studied graph complexes, with numerous applications in and outside combinatorics [A04, BLV94, BMZ, FH98, G79, J08, KRW, M03, SW07, V94, Z92, Z11, Z04].

The connectivity degree of a simplicial complex was selected in [J08, Chapter 10] as one of the five most important and useful parameters in the study of simplicial complexes of graphs. Following [KRW] we study the connectivity degree of multiple chessboard complexes (Section 1.4). Our central result is Theorem 3.1 which improves the 2-dimensional case of [KRW, Corollary 5.2.] and reduces to the 2-dimensional case of [BLV94, Theorem 3.1.] in the case of standard chessboard complexes.

High connectivity degree is sometimes a consequence of the shellability of the complex (or one of its skeletons), see [Z94] for an early example in the context of chessboard

*Supported by the Ministry for Science and Technology of Serbia, Grant 174034.
†Supported by the Ministry for Science and Technology of Serbia, Grant 174034.
complexes. Theorem 4.3 provides a sufficient condition which guarantees the shellability of the multiple chessboard complexes and yields another proof of Theorem 3.1.

Among applications of the new connectivity bounds established by Theorem 3.1 is a result of colored Tverberg type where multiple points of the same color are permitted (Theorem 5.1 in Section 5).

1.1 Colored Tverberg problems

‘Tverberg problems’ is a common name for a class of theorems and conjectures about finite sets of points (point clouds) in $\mathbb{R}^d$. We start with a brief introduction into this area of topological combinatorics emphasizing, in the spirit of \cite{Z99} and \cite{VZ11}, a graphical or diagrammatic (2)-(6)) presentation of these results. The reader is referred to \cite{Zi11}, \cite{VZ11, Section 14.4}, \cite{Z04}, and \cite{M03} for more complete expositions of these problems and the history of the whole area.

The Tverberg theorem \cite{T66} claims that every set $K \subset \mathbb{R}^d$ with $(d + 1)(q - 1) + 1$ elements can be partitioned $K = K_1 \cup \ldots \cup K_q$ into $q$ nonempty, disjoint subsets $K_1, \ldots, K_q$ such that the corresponding convex hulls have a nonempty intersection:

$$\bigcap_{i=1}^q \text{conv}(K_i) \neq \emptyset. \tag{1}$$

Following \cite{BSS} it can be reformulated as the statement that for each linear (affine) map $f : \Delta^D \xrightarrow{a} \mathbb{R}^d (D = (d + 1)(q - 1))$ there exist $q$ nonempty disjoint faces $\Delta_1, \ldots, \Delta_q$ such that $f(\Delta_1) \cap \ldots \cap f(\Delta_q) \neq \emptyset$. This form of Tverberg’s result can be summarized as follows,

$$(\Delta^{(d+1)(q-1)} \xrightarrow{a} \mathbb{R}^d) \Rightarrow (q - \text{intersection}). \tag{2}$$

Here we tacitly assume that the faces intersecting in the image are always vertex disjoint. The letter “a” over the arrow means that the map is affine and its absence indicates that it can be an arbitrary continuous map.

The following four statements are illustrative for results of ‘colored Tverberg type’.

$$(K_{3,3} \rightarrow \mathbb{R}^2) \Rightarrow (2 - \text{intersection}) \tag{3}$$
$$(K_{3,3,3} \xrightarrow{a} \mathbb{R}^2) \Rightarrow (3 - \text{intersection}) \tag{4}$$
$$(K_{5,5,5} \rightarrow \mathbb{R}^3) \Rightarrow (3 - \text{intersection}) \tag{5}$$
$$(K_{4,4,4,4} \rightarrow \mathbb{R}^3) \Rightarrow (4 - \text{intersection}) \tag{6}$$

$K_{t_1,t_2,\ldots,t_k} = [t_1] * [t_2] * \ldots * [t_k]$ is by definition the complete multipartite simplicial complex obtained as a join of 0-dimensional complexes (finite sets). By definition the vertices of this complex are naturally partitioned into groups of the same ‘color’.

For example $K_{p,q} = [p] * [q]$ is the complete bipartite graph obtained by connecting each of $p$ ‘red vertices’ with each of $q$ ‘blue vertices’. The simplices of $K_{t_1,t_2,\ldots,t_k}$ are called multicolored sets or rainbow simplices and its dimension is $k_- = k - 1$. (We
systematically use the abbreviation $a_- := a - 1$ to emphasize that $a_-$ is the dimension of a non-degenerate simplex with $a$ vertices.)

The implication (3) says that for each continuous map $\phi : K_{3, 3} \to \mathbb{R}^2$ there always exist two vertex-disjoint edges which intersect in the image. This statement is clearly equivalent to the non-planarity of the complete bipartite graph $K_{3, 3}$. The implication (4) is an instance of a result of Bárány and Larman [BL]. It says that each collection of nine points in the plane, evenly colored by three colors, can be partitioned into three multicolored or ‘rainbow triangles’ which have a common point.

Note that a 9-element set $C \subset \mathbb{R}^2$ which is evenly colored by three colors, can be also described by a map $\alpha : [3] \sqcup [3] \sqcup [3] \to \mathbb{R}^2$ from a disjoint sum of three copies of $[3]$. In the same spirit an affine map $\phi : K_{3, 3, 3} \to \mathbb{R}^2$ parameterizes not only the colored set itself but takes into account from the beginning that some simplices (multicolored or rainbow simplices) play a special role.

A similar conclusion have statement (5) which is a formal analogue of the statement (3) in dimension 3. It is an instance of a result of Vrečica and Živaljević [VŽ94], which claims the existence of three intersecting, vertex disjoint rainbow triangles in each constellation of 5 red, 5 blue, and 5 green stars in the 3-space. A consequence of this result is that $K_{5, 5, 5}$ is strongly non-embeddable in $\mathbb{R}^3$ in the sense that there always exists a triple point in the image.

Finally (6) is an instance of the celebrated result of Blagojević, Matschke, and Ziegler [BMZ, Corollary 2.4] saying that 4 intersecting, vertex disjoint rainbow tetrahedra in $\mathbb{R}^3$ will always appear if we are given sixteen points, evenly colored by four colors.

**Remark 1.1** Both statements (6) and (5) are instances of results of colored Tverberg type. There is an important difference between them however, and this is the reason why they are referred to as Type A and Type B colored Tverberg theorems in the Handbook of discrete and computational geometry [Ž04, Chapter 14]. Both results are optimal in the sense that in the cases where they apply they provide the best bounds possible.

### 1.2 General colored Tverberg theorems

From the point of view of results exhibited in Section 1.1 it is quite natural to ask the following general question.

**Problem 1.2** For given integers $r, k, d$ determine the smallest $t = T(r, k, d)$ such that,

\[
(K_{t, t, \ldots, t} \to \mathbb{R}^d) \Rightarrow (r - \text{intersection})
\]

where $K_{t, t, \ldots, t} = K_{t, t, \ldots, t}^{*k}$ is the join of $k + 1$ copies of $[t]$.

The latest developments [Zi11, BMZ] showed the importance of the following even more general, non-homogeneous version of Problem 1.2.
Problem 1.3 For given integers \(r, k, d\) determine when a sequence \(t = (t_0, t_1, \ldots, t_k)\) yields the implication

\[
(K_{t_0,t_1,\ldots,t_k} \longrightarrow \mathbb{R}^d) \Rightarrow (r - \text{intersection})
\]

where \(K_4 = K_{t_0,t_1,\ldots,t_k} = [t_0] \ast [t_1] \ast \cdots \ast [t_k]\).

Historically the first appearance of the colored Tverberg problem is the question of Bárán and Larman [BL]. It corresponds to the case \(k = d\) of Problem 1.2 i.e. to the case when the dimension \(k\) of top-dimensional rainbow simplices is equal to the dimension \(d\) of the ambient Euclidean space. This case is referred to in [ˇZ04] as the Type A of the colored Tverberg problem. The Type B of colored Tverberg problem, corresponding to the case \(k < d\) of Problem 1.2 is introduced in [VˇZ94], see also [ˇZ96, ˇZ98, ˇZ04].

The following two theorems are currently the most general known results about the invariants \(T(r, k, d)\). The first is the recent Type A statement due to Blagojević, Matschke, and Ziegler [BMZ] who improved the original Type A colored Tverberg theorem of Vrečica and Živaljević [VZ92]. The second is a Type B statement proved by Vrečica and Živaljević in [VˇZ94]. Both results are exact in the sense that in the cases where they apply they both evaluate the exact value of the function \(T(r, k, d)\).

**Theorem 1.4** ([BMZ]) If \(r\) is a prime number then \(T(r, d, d) = r\).

**Theorem 1.5** ([VˇZ94]) If \(k \leq d\) and \(r\) is a prime such that \(2 \leq r \leq d/(d-k)\) then

\[
T(r, k, d) = 2r - 1.
\]

The condition \(r \leq d/(d-k)\) in the second statement cannot be removed if \(r \geq 2\). It does not exist in the Type A statement (Theorem 1.4) so it is a characteristic, distinguishing feature of the Type B colored Tverberg theorem (Theorem 1.5).

### 1.3 Chessboard complexes and colored Tverberg problem

Here we briefly outline, following the original sources [ZV92, VZ94] and a more recent exposition given in [VZ11], how the so called chessboard complexes naturally arise in the context of the colored Tverberg problem. For notation and a more systematic exposition of these and related facts the reader is referred to [M03, ˇZ96, ˇZ98, ˇZ04].

Given a map \(f : K \rightarrow \mathbb{R}^d\) (as in examples from Sections 1.1 and 1.2) we want to find \(r\) nonempty, vertex disjoint faces \(\sigma_1, \ldots, \sigma_r\) of \(K\) such that \(f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset\). For this reason we consider the induced map \(F : K_{\Delta}^r \rightarrow (\mathbb{R}^d)^*r\) from the deleted join (see [M03, Sections 5.5 and 6.3]) of \(r\) copies of \(K\) to the \(r\)-fold join of \(\mathbb{R}^d\) and observe that it is sufficient to show that \(\text{Image}(F) \cap D \neq \emptyset\) where \(D \subset (\mathbb{R}^d)^*r\) is the diagonal subspace of the join. Assuming the contrary we obtain a \(S_r\)-equivariant map \(F' : K_{\Delta}^{*r} \rightarrow (\mathbb{R}^d)_\Delta^{*r}\) where \((\mathbb{R}^d)_\Delta^{*r}\) is the \(r\)-fold deleted join of \(\mathbb{R}^d\) [M03, Section 6.3]. It is not difficult to
show that \((\mathbb{R}^d)^r_\Delta\) has the \(S_r\)-homotopy type of the unit sphere \(S(W_r^{\oplus(d+1)})\) where \(W_r\) is the \((r-1)\)-dimensional, standard (real) representation of \(S_r\).

If \(K = K_{t_1,\ldots,t_k} = [t_1] \ast \ldots \ast [t_k]\) (as in examples from Sections 1.1 and 1.2) then,

\[
K_{\Delta}^r \cong ([t_1] \ast \ldots \ast [t_k])_{\Delta}^r \cong [t_1]_{\Delta}^r \ast \ldots \ast [t_k]_{\Delta}^r = \Delta_{r,t_1} \ast \ldots \ast \Delta_{r,t_k}
\]  

(9)

where \(\Delta_{m,n}\) is the so called chessboard complex, defined as the simplicial complex of all non-taking rook placements on an \(m \times n\) ‘chessboard’.

The upshot of this sequence of reductions is that the implication (8) (Problem 1.3) is a consequence of a Borsuk-Ulam type result claiming that here does not exits a \(S_r\)-equivariant map

\[
F : \Delta_{r,t_1} \ast \ldots \ast \Delta_{r,t_k} \to S(W_r^{\oplus(d+1)}).
\]  

(10)

In particular Theorems 1.4 and 1.5 are both consequences of the following generalized Borsuk-Ulam type theorems where the source space is a join of chessboard complexes,

\[
F : (\Delta_{r,r-1})^{s_d} \ast \Delta_{r,1} \to S(W_r^{\oplus d}).
\]  

(11)

\[
F : (\Delta_{r,2r-1})^{s_k} \to S(W_r^{\oplus d}).
\]  

(12)

### 1.4 Multiple chessboard complexes

It is quite natural to apply the scheme outlined in Section 1.3 so some other simplicial complexes aside from \(K_{t_1,\ldots,t_k}\).

Let \([t]^{(p)} = \{A \subset [t] \mid |A| \leq p\}\) be the collection of all subsets of \([t] = \{1, \ldots, t\}\) of size at most \(p\). As a simplicial complex \([t]^{(p)}\) is the \((p-1)\)-skeleton of the simplex spanned by \([t]\). Let \(K_t^p = K_{t_1,\ldots,t_k}^{p_1,\ldots,p_k} = [t_1]^{(p_1)} \ast \ldots \ast [t_k]^{(p_k)}\).

By applying the same procedure as in Section 1.3 to the complex \(K_t^p = K_{t_1,\ldots,t_k}^{p_1,\ldots,p_k}\) we have,

\[
(K_t^p)_\Delta^r \cong ([t_1]^{(p_1)} \ast \ldots \ast [t_k]^{(p_k)})_{\Delta}^r \cong ([t_1]^{(p_1)})_{\Delta}^r \ast \ldots \ast ([t_k]^{(p_k)})_{\Delta}^r = \Delta_{r,t_1}^{1,p_1} \ast \ldots \ast \Delta_{r,t_k}^{1,p_k}
\]  

(13)

where the generalized chessboard complexes \(\Delta_{r,t}^{1,p}\) make their first appearance in this paper.

### 1.5 Bier spheres as multiple chessboard complexes

One of the novelties in the proof of Theorem 1.4 [BMZ], which is particularly visible in the ‘mapping degree’ proof [VZ11] and [BMZ-2], is the use of the (pseudo)manifold structure of the chessboard complex \(\Delta_{r,r-1}\). Here we observe that an important subclass of combinatorial spheres (Bier spheres) arise as multiple chessboard complexes. As shown in Example 2.4 all Bier spheres can be incorporated into this scheme if we allow even more general chessboard complexes.

Recall [M03, Chapter 5] that the Bier sphere \(Bier_m(K)\), associated to a simplicial complex \(K \subset 2^m\), is the ‘disjoint join’ \(K \ast K^\circ\) of \(K\) and its combinatorial Alexander dual \(K^\circ\). The reader is referred to Definitions 2.3-2.6 in Section 2 for the definition of generalized chessboard complexes \(\Delta^{k,l}_{m,2}, \Delta^{m-2,l}_{m,2}(1)\) and their relatives.
Proposition 1.6 Suppose that $K = [m]^{\leq p}$ is the simplicial complex of all subsets of $[m]$ of size at most $p$ and let $\text{Bier}_m(K)$ be the associated Bier sphere. Then

$$S^{m-2} \cong \text{Bier}_m(K) = \text{Bier}_m([m]^{\leq p}) \cong \Delta^k_{m,2}$$

where $k = (p, m - p - 1)$ and $l_1 = \ldots = l_m = 1$, in particular

$$\Delta^{p,1}_{2p+1,2} \cong \text{Bier}_{2p+1}([2p+1]^{\leq p}) \cong S^{2p-1}$$ and

$$\Delta^{m-2,1(1)}_{m,2} \cong \text{Bier}_m([m]) \cong S^{m-2}.$$ 

2 Generalized chessboard complexes

The classical chessboard complex $\Delta_{m,n}$ [BLV94] is often visualized as the simplicial complex of non-taking rook placements on a $(m \times n)$-chessboard. In particular its vertices $\text{Vert}(\Delta_{m,n}) = [m] \times [n]$ are elementary squares in a chessboard which has $n$ rows of size $m$ (here we use Cartesian rather than matrix presentation of the chessboard).

The complex $\Delta_{m,n}$ can be also described as the matching complex of the complete bipartite graph $K_{m,n}$. In this incarnation its vertices correspond to all edges of the graph $K_{m,n}$ and a collection of edges determine a simplex if and only if it is a matching in $K_{m,n}$, see [BLV94] or [J08]. As we have already seen in Section 1.3 the complex $\Delta_{m,n}$ can be also described as the $n$-fold 2-deleted join of the 0-dimensional skeleton of the $(m - 1)$-dimensional simplex

$$(\sigma^{m-1})^{(0)} \Delta(2).$$

Here, the $n$-fold $(q + 1)$-deleted join of the complex $K$, denoted by $K^{n}_{\Delta(q+1)}$, is a subcomplex of $K^{n}$, the $n$-fold join of the complex $K$, consisting of joins of $n$-tuples of simplices from $K$ such that the intersection of any $q + 1$ of them is empty. (In particular the 2-deleted join $K^{n}_{\Delta(2)} = K^{n}$ is the usual deleted join of $K$.)

The 0-dimensional skeleton of the $(m - 1)$-dimensional simplex and the $m$-fold 2-deleted join of a point are both identified as the sets of $m$ points. It follows that,

$$\Delta_{m,n} = ((\sigma^{m-1})^{(0)} \Delta(2))^{\ast n} \cong ([\ast]_{\Delta(2)})^{\ast m} \ast_{\Delta(2)}^{\ast n}.$$ 

This is precisely the description of $\Delta_{m,n}$ that appeared in the original approach to the Colored Tverberg theorem in [ZV92].

In this paper we allow multicolored simplices to have more (say $p$) vertices of the same color, so we consider a generalized chessboard complex which is the $n$-fold $(q + 1)$-deleted join of the $(p - 1)$-dimensional skeleton of the $(m - 1)$-dimensional simplex, i.e. the complex

$$\Delta^{p,q}_{m,n} := ((\sigma^{m-1})^{(p-1)} \Delta(q+1))^{\ast n} = ([\ast]_{\Delta(p+1)})^{\ast m} \ast_{\Delta(q+1)}^{\ast n}$$ (14)

As before, the vertices of this simplicial complex correspond to the squares on the $m \times n$ chessboard and simplices correspond to the collections of vertices so that at most $p$ of them are in the same row, and at most $q$ of them are in the same column. As indicated in (14) we denote this simplicial complex by $\Delta^{p,q}_{m,n}$ so in particular $\Delta_{m,n} = \Delta^{1,1}_{m,n}$. 


Remark 2.1 The meaning of parameters \((m, p; n, q)\) in the complex \(\Delta_{m,n}^{p,q}\) can be memorized as follows. The parameters \(m\) and \(p\) both apply to the rows (\(m\) as the row-length of the chessboard \([m] \times [n]\) and \(p\) as the maximum number of rooks allowed in each row). Similarly, the parameters \(n\) and \(q\) are associated to columns (\([n]\) is the column-height of \([m] \times [n]\) while \(q\) prescribes the largest number of rooks in each of the columns). A similar interpretation can be given in the case of more general chessboard complexes ((15) and (16)).

Remark 2.2 The higher dimensional analogues of complexes \(\Delta_{m,n}^{p,q}\) were introduced and studied in [KRW] and our particular interest in the generalized Tverberg-type problems is the reason why in this paper we focus on the two dimensional case.

2.1 Complexes \(\Delta_{m,n}^{\mathcal{K},\mathcal{L}}\)

Both for heuristic and technical reasons we consider even more general chessboard complexes based on the \((m \times n)\)-chessboard. The following definition provides an ecological niche (and a summary of notation) for all these complexes.

Definition 2.3 Let \(\mathcal{K} = \{K_i\}_{i=1}^n\) and \(\mathcal{L} = \{L_j\}_{j=1}^m\) be two labelled collections of simplicial complexes where \(\text{Vert}(K_i) = [m]\) for each \(i \in [n]\) and \(\text{Vert}(L_j) = [n]\) for each \(j \in [m]\). Define,

\[
\Delta_{m,n}^{\mathcal{K},\mathcal{L}} = \Delta_{m,n}(\mathcal{K}, \mathcal{L})
\]

as the complex of all subsets (rook-placements) \(A \subset [m] \times [n]\) such that \(\{i \in [m] \mid (i, j) \in A\} \in K_j\) for each \(j \in [n]\) and \(\{j \in [n] \mid (i, j) \in A\} \in L_i\) for each \(i \in [m]\).

Example 2.4 Generalizing Proposition 1.6 we observe that the general Bier sphere \(\text{Bier}_m(K)\) arises as the complex \(\Delta_{m,2}^{\mathcal{K},\mathcal{L}}\) where \(\mathcal{K} = (K, K^\circ)\) and \(L_j = \{\emptyset, \{1\}, \{2\}\}\) for each \(j \in [m]\).

Definition 2.3 can be specialized in many ways. Again, we focus on the special cases motivated by intended applications to the generalized Tverberg problem.

Definition 2.5 Suppose that \(k = (k_i)_{i=1}^n\) and \(l = (l_j)_{j=1}^m\) are two sequences of non-negative integers. Then the complex,

\[
\Delta_{m,n}^{k_1,\ldots,k_n; l_1,\ldots,l_m} = \Delta_{m,n}^{k_1,\ldots,k_n; l_1,\ldots,l_m}
\]

arises as the complex of all rook-placements \(A \subset [m] \times [n]\) such that at most \(k_i\) rooks are allowed to be in the \(i\)-th row (for \(i = 1, \ldots, n\)), and at most \(l_j\) rooks are allowed to be in the \(j\)-th column (for \(j = 1, \ldots, m\)).

When \(k_1 = \ldots = k_n = p\) and \(l_1 = \ldots = l_m = q\), we obtain the complex \(\Delta_{m,n}^{p,q}\). For the reasons which will become clear in the final section of the paper, we will be especially interested in the case \(l_1 = \ldots = l_m = 1\), i.e. in the complexes,

\[
\Delta_{m,n}^{k_1,\ldots,k_n; 1} := \Delta_{m,n}^{k_1,\ldots,k_n; 1,\ldots,1}
\]
The inductive argument used in the proof of the main theorem (Theorem 3.1) requires the analysis (Proposition 3.5) of complexes $\Delta_{m,n}^{2,1(j)}$ which arise as follows. We assume that $R \subseteq [n]$ is a $j$-element subset of $[n]$, prescribed in advance, labelling selected $j$ rows in the $(m \times n)$-chessboard.

**Definition 2.6** A rook-placement $A \subseteq [m] \times [n]$ is a simplex in $\Delta_{m,n}^{2,1}$ if and only if at most 2 rooks are allowed in rows indexed by $R$ and at most one in all other rows and columns. Obviously for $j = 0$ we obtain the usual chessboard complex $\Delta_{m,n}^{2,1(0)} = \Delta_{m,n}$, and for $j = n$ the generalized chessboard complex $\Delta_{m,n}^{2,1(n)} = \Delta_{m,n}^{2,1}$. When $n = 2$, aside from these two possibilities, there is only one case remaining, the complex $\Delta_{m,2}^{2,1(1)}$.

**Remark 2.7** The problem of determining the connectivity of generalized chessboard complexes was considered in [KRW], where they proved a result implying that the generalized chessboard complex $\Delta_{m,n}^{k_1,...,k_n;1}$ is $(\mu - 2)$-connected where

$$\mu = \min \{m, \left[ \frac{m + k_1 + \cdots + k_n + 1}{3} \right], k_1 + \cdots + k_n \}.$$  

The inductive argument based on the application of the nerve lemma from [BLVZ94] can be used to provide an alternative proof of their result. Refining this argument, we obtain here (Theorem 3.1) a substantially better estimate in the case $l_1 = \cdots = l_m = 1$. This is exactly the result needed here for a proof of a generalized colored Tverberg theorem (Theorem 5.1) for which the original estimate from [KRW] was not sufficient. We believe and conjecture that the same argument could be used to prove a better estimate in the general (multidimensional) case.

For completeness and the reader’s convenience here we state, following [Bjo95], a version of the Nerve Lemma needed in the proof of the main theorem and other propositions.

**Lemma 2.8 (Nerve Lemma)** Let $\Delta$ be a simplicial complex and $\{L_i\}_{i=1}^k$ a family of subcomplexes such that $\Delta = \cup_{i=1}^k L_i$. Suppose that every intersection $L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_t}$ is $(\mu - t + 1)$-connected for $t \geq 1$. Then $\Delta$ is $\mu$-connected.

### 2.2 Selected examples of complexes $\Delta_{m,n}^{k,L}$

As a preparation for the proof of Theorem 3 and as an illustration of the use and versatility of the Nerve Lemma, here we analyze in some detail the connectivity properties of multiple chessboard complexes for some small values of $m$ and $n$.

**Example 2.9** $\Delta_{m,1}^{2,1} = (\sigma^{m-1})^{(1)}$ is the 1-skeleton of a $(m - 1)$-dimensional simplex, in particular it is connected.

**Example 2.10** $\Delta_{3,2}^{2,1} \approx S^1 \times I$, $\Delta_{4,2}^{2,1}$ has the homology of $S^2$, $\Delta_{5,2}^{2,1} \approx S^3$, and for $m \geq 5$ the complex $\Delta_{m,2}^{2,1}$ is 2-connected.
Proof: The complex $\Delta^{2,1}_{3,3}$ is a triangulation of the surface of a cylinder into 6 triangles. $\Delta^{2,1}_{4,2}$ is a simplicial complex whose simplices are subsets of the $4 \times 2$ chessboard $\{(i, j) \mid 1 \leq i \leq 4, 1 \leq j \leq 2\}$ with at most two vertices in the same row and at most one vertex in each column. This complex is covered by 4 subcomplexes $L_1, L_2, L_3, L_4$ where $L_i$ is the collection of simplices which contain $(i, 1)$ as a vertex, together with their faces. Each $L_i$ is contractible. For $i \neq j$, $L_i \cap L_j$ is a union of a tetrahedron with two intervals joining the vertices of the tetrahedron with two new vertices, hence it is also contractible. The intersection of each three of these subcomplexes is nonempty (the union of three intervals with a common vertex and one additional vertex). Also, the intersection of all 4 subcomplexes is a set of 4 different points. Hence, by the Nerve Lemma, the complex $\Delta^{2,1}_{4,2}$ is 1-connected. Using the Euler-Poincaré formula it is easy to see that this complex has the homology of $S^3$.

We already know (Proposition 1.6) that $\Delta^{2,1}_{5,2}$ is a 3-sphere. However, as in the previous example, this complex can be covered by 5 contractible subcomplexes $L_1, ..., L_5$. The intersection of any two of these subcomplexes is contractible. The intersection of any three of them is also contractible (the union of three triangles with a common edge and two additional edges connecting the vertices of this edge with two additional points). The intersection of any four of these subcomplexes as well as the intersection of all of them is non-empty. Therefore, by Nerve lemma, the complex $\Delta^{2,1}_{5,2}$ is 2-connected.

It is easy to verify that this complex is a simplicial 3-manifold; the link of each vertex is a 2-dimensional sphere, the link of each edge is a circle, and the link of each 2-dimensional simplex is a 0-dimensional sphere. Hence, $\Delta^{2,1}_{5,2} \approx S^3$. $\square$

Example 2.11 $\Delta^{2,1}_{3,3}$ and $\Delta^{2,1}_{4,3}$ are 1-connected, and $\Delta^{2,1}_{5,3}$ and $\Delta^{2,1}_{6,3}$ are 2-connected.

Proof: It is easy to see that each simplex in these complexes is a face of a simplex having a vertex in the first column. As before we apply the Nerve Lemma to the covering by subcomplexes $M_i$, $i \in \{1, 2, 3\}$, where $M_i$ is the collection of simplices having vertex at $(1, i)$, together with their faces. $\square$

Example 2.12 The complex $\Delta^{2,1(1)}_{m,2}$ (Definition 2.6) is connected for $m \geq 2$, and 1-connected for $m \geq 4$.

Proof: The proof goes along the same lines as before and uses the Nerve lemma. As we already know (Proposition 1.6) $\Delta^{2,1(1)}_{3,2}$ is a simplicial surface homeomorphic to $S^2$. This can be proved directly as follows. The link of each vertex is a circle (a triangle or a hexagon), and the link of each edge consists of two points, i.e. it is $S^0$. The Euler characteristic of this complex equals 2, so it must be $S^2$. $\square$

Some of the examples of generalized chessboard complexes are spheres (Proposition 1.6 and Example 2.4). Here we meet some quasi-manifolds.

Example 2.13 $\Delta^{2,1}_{7,3}$ is a 5-dimensional quasi-manifold. More generally $\Delta^{p,1}_{pm+1,n}$ is a $(pn - 1)$-dimensional quasi-manifold for each $p \geq 1$. 

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Proof: It is easy to verify that the link of each 4-dimensional simplex is $S^0$, the link of each 3-dimensional simplex is a combinatorial circle (consisting of either 3 or 6 edges), the link of some 2-dimensional simplices is $S^2$ (triangles with two vertices in the same row), but some other 2-dimensional simplices (whose vertices are in three different rows) have the link homeomorphic to the torus rather than to $S^2$. A similar proof applies in the general case.

Concerning the generalized chessboard complexes $\Delta_{m,n}^{p,q}$ with higher values of $q$, we mention one additional simple example.

**Example 2.14** $\Delta_{m,n}^{m-1,n} \approx S^{(m-1)n-1}$, and $\Delta_{m,n}^{p,n}$ is $(np-2)$-connected.

**Proof:** $\Delta_{m,1}^{m-1,1}$ is the boundary of $(m-1)$-dimensional simplex, and so homeomorphic to $S^{m-2}$. Since $\Delta_{m,n}^{m-1,n}$ is a join of $n$ copies of $\Delta_{m,1}^{m-1,1}$, it is homeomorphic to $S^{(m-1)n-1}$.

Similarly we see that the complex $\Delta_{m,n}^{p,n}$ is a join of $n$ complexes of the type $\Delta_{m,1}^{p,1}$ which is identified as the $(p - 1)$-skeleton of an $(m - 1)$-dimensional simplex. We conclude that this complex is a wedge of $(np - 1)$-dimensional spheres, so must be $(np - 2)$-connected. In particular for $p = 1$ this reduces to the fact that $\Delta_{m,n}^{1,n}$ is a join of $n$ copies of the finite set of $m$ points, and so $(n-2)$-connected.

**Corollary 2.15** The complex $\Delta_{3,2}^{2,2}$ is a 3-sphere, $\Delta_{3,2}^{2,2} \simeq S^3$.

### 3 The main connectivity result

In this section we prove our main result providing an estimate for the connectivity of the generalized chessboard complex $\Delta_{m,n}^{k_1,\ldots,k_n;1}$ (Definition 2.5 and equation (17)).

**Theorem 3.1** The generalized chessboard complex $\Delta_{m,n}^{k_1,\ldots,k_n;1}$ is $(\mu-2)$-connected where,

$$\mu = \min\{m-n+1,k_1+\cdots+k_n\}.$$

In particular if $m \geq k_1+\cdots+k_n+n-1$ then $\Delta_{m,n}^{1,\ldots,1;1}$ is $(k_1+\cdots+k_n-2)$-connected.

**Proof:** We shall carry on the proof by showing that the complex $\Delta_{k_1+\cdots+k_n+n-1,1}^{k_1,\ldots,k_n;1}$ is $(k_1+\cdots+k_n-2)$-connected, and that by reducing the number of columns by 1 the connectivity degree of the complex either reduces by 1 or remains the same.

We proceed by induction. It is easy to check that the statement of the theorem is true for $n = 1$ and every $m$ and $k_1$, and that our estimate is true if $m \leq n$. It also follows directly from the known result from [ˇZV92] when $k_1 = \cdots = k_n = 1$. Let us suppose that the statement is true for the complex $\Delta_{r,s}^{k_1,\ldots,k_s;1}$ when $s < n$ (for every $r$ and $k_1,\ldots,k_s$), and also for $s = n$ if $r < m$.
We now focus attention to the complex $\Delta^{k_1,\ldots,k_n;1}_{m,n}$, taking into the account that the case when all numbers $k_1,\ldots,k_n$ are equal to 1 is already covered.

(i) Let us start with the case $m \geq k_1 + \cdots + k_n + n - 1$. Note that in this case $\mu = k_1 + \cdots + k_n$. Without loss of generality (by permuting the rows if necessary) we can assume that $k_1 \geq \ldots \geq k_n$, in particular we can assume that $k_1 \geq 2$. The complex is covered by the cones $L_i$ having the vertices at the points $(i,1)$, $i = 1,\ldots,m$.

The intersection of any two cones (say the complexes $L_{m-1}$ and $L_m$) can be described as the union of two subcomplexes, one of them, $K$, being the join of the segment (whose vertices are at the vertices of these cones $(m-1,1)$ and $(m,1)$) and the subcomplex $\Delta^{k_1,\ldots,k_n;1}_{m-2,n}$, and the other is a subcomplex $K'$ of $\Delta^{k_1,\ldots,k_n;1}_{m-2,n}$ consisting of simplices having at most $k_1 - 1$ vertices from the first row. The latter subcomplex $K'$ can be presented as the union of complexes $K_t$, each of which being the subcomplex of $K'$ consisting of simplices with the vertices at $(i_1,1),\ldots,(i_{k_1-1},1)$ and all of their faces. The complexes $K$ and $K_t$ are contractible, and we build the complex $L_{m-1} \cap L_m$ by adding the complexes $K_t$ to the complex $K$ one by one. It is important to note that $(K \cup K_1 \cup \ldots \cup K_{j-1}) \cap K_j = K \cap K_j$ is the join of the boundary of the simplex with the vertices $(i_1,1),\ldots,(i_{k_1-1},1)$ (which is homeomorphic to the sphere $S^{k_1-3}$), and the complex $\Delta^{k_2,\ldots,k_n;1}_{m-k_1-1,n-1}$ (which is $(k_2 + \cdots + k_n - 2)$-connected by the induction hypothesis). So, this complex is $(\mu - 4)$-connected. Using the Mayer-Vietoris long exact sequence repeatedly, we see that the complex $L_{m-1} \cap L_m$ is $(\mu - 3)$-connected.

For the readers convenience, we also prove that the intersection of any three of the subcomplexes $L_i$, $i \in \{1,\ldots,m\}$ is $(\mu - 4)$-connected, and proceed to the general case after that. Namely, $L_{m-2} \cap L_{m-1} \cap L_m$ can be built from the complexes $K$, $K'$ and $K''$ (if $k_1 \geq 3$). Here $K$ is a join of the triangle (with vertices $(m-2,1),(m-1,1),(m,1)$) and the complex of the type $\Delta^{k_1,\ldots,k_n;1}_{m-3,n}$, and so contractible. The complex $K'$ is a join of a finite set of 3 points $((m-2,1),(m-1,1),(m,1))$ and the complex of the type $\Delta^{k_1,\ldots,k_n;1}_{m-k_1-1,n-1}$. So, $K'$ could be presented as the union of the complexes which are joins of the three point set with the simplex $\Delta^{k_1-3}$ and with the complex of the type $\Delta^{k_2,\ldots,k_n;1}_{m-k_1-1,n-1}$. These complexes are contractible, and we note that, when we add these complexes to $K$ one by one, the intersection of each of them with the previously built complex is of the type $[3] \ast S^{k_1-4} \ast \Delta^{k_2,\ldots,k_n;1}_{m-k_1-1,n-1}$ which is $(\mu - 4)$-connected. The complex $K''$ is the complex of the type $\Delta^{k_1-1,\ldots,k_n;1}_{m-3,n}$, and could be presented as the union of the complexes which are joins of the simplex $\Delta^{k_1-2}$ with the complex of the type $\Delta^{k_2,\ldots,k_n;1}_{m-k_1-2,n-1}$. These complexes are contractible, and we note that, when adding them to the previously built complex, the intersection is of the type $S^{k_1-3} \ast \Delta^{k_2,\ldots,k_n;1}_{m-k_1-2,n-1}$, and it could be easily checked that it is $(\mu - 3)$-connected.

Using the Mayer-Vietoris long exact sequence repeatedly, we see that the complex $L_{m-2} \cap L_{m-1} \cap L_m$ is $(\mu - 3)$-connected (which is more than necessary).

In the general case, we want to prove that the intersection of any $q$ cones, let us say $L_{m-q+1} \cap L_{m-q+2} \cap \cdots \cap L_m$, is $(\mu - q - 1)$-connected. We will distinguish the cases $q \leq k_1$, and $q > k_1$. (As before we are allowed to assume that $k_1 \geq 2$.)

(a) $q \leq k_1$: This case is treated in exactly the same way as the special cases $q = 2$ and $q = 3$. Our objective is to prove that the intersection $L_{m-q+1} \cap L_{m-q+2} \cap \cdots \cap L_m$
is \((\mu - q - 1)-\)connected. This intersection is conveniently expressed as the union of the complexes \(K_1, \ldots, K_q\) which arise by specifying how many vertices of the type \(((j, 1))\) they have for \(j \geq m - q + 1\). More precisely, \(K_1, K_2, K_3, \ldots, K_q\) are the subcomplexes consisting of simplices having respectively \(q, q - 2, q - 3, \ldots, 0\) vertices in the set \(((m - q + 1, 1)), \ldots, (m, 1))\). The complex \(K_1\) is the join of the \((q - 1)\)-dimensional simplex and the complex of the type \(\Delta_{k_1, k_2, \ldots, k_n+1}^{q}\) (or of the type \(\Delta_{m-q,n-1}^{k_1, q}\) if \(q = k_1\)). So, \(K_1\) is contractible.

The complex \(K_2\) can be presented as the union of complexes which are joins of the \((q - 3)\)-dimensional skeleton of the \((q - 1)\)-dimensional simplex with the \((k_1 - q)\)-dimensional simplex, and with the complex of the type \(\Delta_{k_2, \ldots, k_n+1}^{m-q,n-1}\). These complexes are contractible, and when adding one by one to the complex \(K_1\) we notice that the intersection of each of them with the previously built complex is the complex of the type of the join of the \((q - 3)\)-dimensional skeleton of the \((q - 1)\)-dimensional simplex with the \((k_1 - q)\)-dimensional skeleton of the \((k_1 - q)\)-dimensional simplex, and with the complex of the type \(\Delta_{m-k_1-1,n-1}^{k_2, \ldots, k_n+1}\). This intersection is, by the induction hypothesis, \((\mu - 4)\)-connected. So, the union \(K_1 \cup K_2\) is \((\mu - 3)\)-connected.

We proceed in the same way by adding complexes \(K_j\), one at the time. Finally, \(K_q\) is the subcomplex consisting of simplices having no vertices in the set \(((m - q + 1, 1)), \ldots, (m, 1))\) and so it is of the type \(\Delta_{m-q,n-1}^{k_1, \ldots, k_n+1}\). The complex \(K_q\) could be presented as the union of complexes which are joins of the \((k_1 - 2)\)-dimensional simplex with the complex of the type \(\Delta_{m-q-k_1+1,n-1}^{k_2, \ldots, k_n+1}\). These complexes are contractible, and when adding one by one to the complex \(K_1 \cup \ldots \cup K_{q-1}\) we notice that the intersection of each of them with the previously built complex is the complex of the type of the join of the \((k_1 - 3)\)-dimensional skeleton of the \((k_1 - 2)\)-dimensional simplex with the complex of the type \(\Delta_{m-q-k_1+1,n-1}^{k_2, \ldots, k_n+1}\). This intersection is, by the induction hypothesis, \((\mu - q - 2)\)-connected. So, the union \(K_1 \cup \ldots \cup K_{k_1}\) is \((\mu - q - 1)\)-connected.

(b) \(q > k_1\): Let us prove that the intersection of any \(q\) cones, for example the intersection \(L_{m-q+1} \cap L_{m-q+2} \cap \ldots \cap L_m\), is \((\mu - q - 1)\)-connected in this case as well. As before we express this intersection as the union of complexes \(K_1, \ldots, K_{k_1}\).

Here, \(K_1\) is the subcomplex consisting of simplices having \(k_1 - 1\) vertices in the set \(((m - q + 1, 1)), \ldots, (m, 1))\) and it is the complex of the type of join of the \((k_1 - 2)\)-dimensional skeleton of the \((q - 1)\)-dimensional simplex and the complex of the type \(\Delta_{m-q,n-1}^{k_2, \ldots, k_n+1}\). So, it is \((\mu - q + k_1 - 2)\)-connected by the induction hypothesis. The complex \(K_2\) consists of simplices having \(k_1 - 2\) vertices in the set \(((m - q + 1, 1)), \ldots, (m, 1))\) and one vertex in the remaining vertices of the first row. The type of this complex is the join of the \((k_1 - 3)\)-dimensional skeleton of the \((q - 1)\)-dimensional simplex, and the complex of the type \(\Delta_{m-q,n}^{1,k_2, \ldots, k_n+1}\). The complex \(K_2\) can be presented as the union of complexes which are joins of the \((k_1 - 3)\)-dimensional skeleton of the \((q - 1)\)-dimensional simplex with a point, and with the complex of the type \(\Delta_{m-q-1,n-1}^{k_2, \ldots, k_n+1}\). These complexes are contractible, and when adding one by one to the complex \(K_1\) we notice that the intersection of each of them with the previously built complex is the complex of the type of the join of the \((k_1 - 3)\)-dimensional skeleton of the \((q - 1)\)-dimensional simplex, and the complex of the type \(\Delta_{m-q-1,n-1}^{k_2, \ldots, k_n+1}\). This intersection is, by the induction hypothesis, \((\mu - q + k_1 - 4)\)-connected. So, the union \(K_1 \cup K_2\) is \((\mu - q + k_1 - 3)\)-connected.
We proceed in the same way. Finally, $K_{k_1}$ is the subcomplex consisting of simplices having no vertices in the set $\{(m - q + 1, 1), \ldots, (m, 1)\}$, and so it is of the type $\Delta_{m-q,n}^{k_1-1;\ldots;k_n:1}$. The complex $K_{k_1}$ can be presented as the union of complexes which are joins of the $(k_1 - 2)$-dimensional simplex with the complex of the type $\Delta_{m-q-1,n-1}^{k_1-1;\ldots;k_n:1}$. These complexes are contractible, and when adding one by one to the complex $K_1 \cup \ldots \cup K_{k_1-1}$ we notice that the intersection of each of them with the previously built complex is the complex of the type of the join of the $(k_1 - 3)$-dimensional skeleton of the $(k_1 - 2)$-dimensional simplex with the complex of the type $\Delta_{m-q-1,n-1}^{k_2;\ldots;k_n:1}$. This intersection is, by the induction hypothesis, $(\mu - q - 2)$-connected. So, the union $K_1 \cup \ldots \cup K_{k_1}$ is $(\mu - q - 1)$-connected.

(ii) Consider now the case $k_1 + \cdots + k_n \leq m \leq k_1 + \cdots + k_n + n - 1$ and let us prove that the complex is $(m - n - 1)$-connected. We again cover the complex by the cones $L_i$, $i = 1, \ldots, m_0$, and note that the intersection of any two of them (let us say $L_{m-1}$ and $L_m$) could be built by adding contractible subcomplexes so that the intersection of any of them with previously built complex is of the type $S^{k_1-3} \ast \Delta_{m-k_1-1,n-1}^{k_2;\ldots;k_n:1}$. This complex is $(m - n - 3)$-connected by the induction hypothesis, and so the intersection $L_{m-1} \cap L_m$ is $(m - n - 2)$-connected by the Mayer-Vietoris theorem. In the same way we prove that the intersection of any three cones is $(m - n - 3)$-connected etc.

(iii) In the case $m \leq k_1 + \cdots + k_n - 1$ we cover the complex $\Delta_{m,n}^{k_1;\ldots;k_n:1}$ by the cones $M_i$ with the vertices $(1, i)$. The intersection of any two of them (let us say $M_{n-1}$ and $M_n$) is the complex of the type $\Delta_{m-1,n}^{k_1;\ldots;k_n-1;1}$ (if $k_{n-1}, k_n \geq 2$), or $\Delta_{m-1,n-1}^{k_1;\ldots;k_n-1;1}$ (if exactly one of them, let us say $k_n$, equals 1), or $\Delta_{m-1,n-2}^{k_1;\ldots;k_n-2;1}$ (if $k_{n-1} = k_n = 1$). In any case, this intersection is at least $(m - n - 2)$-connected by the induction hypothesis. In the same way we prove that the intersection of any three cones is $(m - n - 3)$-connected etc.

\[ \square \]

Corollary 3.2 $\Delta_{m,n}^{2,1}$ is $(2n-2)$-connected for $m \geq 3n-1$, $\Delta_{m,n}^{3,1}$ is $(3n-2)$-connected for $m \geq 4n-1$, and generally $\Delta_{m,n}^{p,1}$ is $(pn-2)$-connected for $m \geq (p+1)n-1$.

Notice that the general estimate obtained in [KRW] implies that the complex $\Delta_{m,n}^{p,1}$ is $(pn-2)$-connected for $m \geq 2pn-1$, which is (compared to $m \geq (p+1)n-1$) a weaker estimate (roughly by a factor of 2).

Corollary 3.3 $\Delta_{7,3}^{2,1}$ is 3-connected, but not 4-connected.

Proof: For the proof of the last claim it suffices to compute the Euler characteristic of this complex $\chi(\Delta_{7,3}^{2,1}) = 147$. Since $\Delta_{7,3}^{2,1}$ is 3-connected 5-dimensional quasi-manifold, we have $\beta_5 = 1$ and so $\beta_4 = 147$. \[ \square \]

Remark 3.4 The estimate $m-n-1$ for small values of $m$ in the statement of Theorem 3.1 can be significantly improved. For example, the following result gives the estimate on the connectivity of the generalized chessboard complex $\Delta_{m,n}^{k_1;\ldots;k_n:1}$ in the case $k_1 = \cdots = k_j = 2$ and $k_{j+1} = \cdots = k_n = 1$, i.e. when $j$ of $n$ numbers $k_1, \ldots, k_n$ are equal to 2 and the remaining $n - j$ are equal to 1. We believe that this estimate is close to the
best possible. Recall (Definition 2.6) that this complex is already introduced as the complex \( \Delta_{m,n}^{2,1(j)} \).

**Proposition 3.5** The complex \( \Delta_{m,n}^{2,1(j)} \) (Definition 2.6) is \((\mu - 2)\)-connected where

\[
\mu = \begin{cases} 
  m, & m < \frac{n+j}{2} \\
  \frac{m+n+j+1}{3}, & \frac{n+j}{2} \leq m < n + \frac{j}{2} - 1 \\
  \frac{5m+n+2j+5}{9}, & n + \frac{j}{2} - 1 \leq m < n + 2j \\
  \frac{m+n+2j+1}{3}, & n + 2j \leq m < 2n + j - 1 \\
  n + j, & m \geq 2n + j - 1
\end{cases}
\]

The proof uses exactly the same ideas, so we omit the details.

As a final comment we repeat that, motivated by possible applications to theorems of Tverberg type, we are interested mostly in the values of \( m \) for which the complex \( \Delta_{m,n}^{k_1,\ldots,k_n} \) is \((k_1 + \cdots + k_n - 2)\)-connected. We believe that the assumption \( m \geq k_1 + \cdots + k_n + n - 1 \) is optimal in that respect.

## 4 Shellability of multiple chessboard complexes

Shellability of standard chessboard complexes \( \Delta_{m,n} \) for \( m \geq 2n - 1 \) is established by G. Ziegler in [Zi94]. He established vertex decomposability of these and related complexes, emphasizing that the natural lexicographic order of facets of \( \Delta_{m,n} \) is NOT a shelling order. We demonstrate that a version of ‘cyclic reversed lexicographical order’ is a shelling order both for standard and for generalized chessboard complexes. Before we prove the general case (Theorem 4.3), we outline the main idea by describing a shelling order for standard chessboard complexes \( \Delta_{m,n} \).

**Shelling order for \( \Delta_{m,n} \):** Let \( \Delta_{m,n} \) be a chessboard complex which satisfies the condition \( m \geq 2n - 1 \). If \( A = (a_1, a_2, \ldots, a_n) \) is a sequence of distinct elements of \([m]\) the associated simplex \( \{(a_i,i)\}_{i=1}^n \) in \( \Delta_{m,n} \) is denoted by \( \hat{A} \). Both \( A \) and \( \hat{A} \) are interchangeably referred to as facets of \( \Delta_{m,n} \).

The shelling order \(<\) on \( \Delta_{m,n} \) is introduced by describing a rule (algorithm) which decides for each two distinct facets \( A \) and \( B \) of \( \Delta_{m,n} \) whether \( A < B \) or \( B < A \). We adopt a basic cyclic order \(<\) on \([m]\),

\[
1 < 2 < \ldots < m < 1
\]

which for each \( a \in [m] \) reduces to a genuine linear order \(<_a\) on \([m]\),

\[
a + 1 < a + 2 < \ldots < m < 1 < \ldots < a,
\]

and in particular \(<_m\) is the standard linear order \(<\) on \([m]\).

Suppose that \( A = (a_1, \ldots, a_n) \) and \( B = (b_1, \ldots, b_n) \) are two distinct facets of \( \Delta_{m,n} \). The procedure of comparing \( A \) and \( B \) begins by comparing \( a_1 \) and \( b_1 \). By definition
the relation \( A < B \) is automatically satisfied if \( a_1 < b_1 \). If \( a_1 = b_1 = a \) we use the order \( <_a \) to compare \( A \) and \( B \), first in the column \( a - 1 \) then (if necessary) in column \( a - 2 \), then (if necessary) in column \( a - 3 \), etc. More precisely let us define the ‘comparison interval’ \([a - (p - 1), a - 1]\) by the requirement that,

1. for each \( b \in [a - (p - 1), a - 1] \) both \( \hat{A} \) and \( \hat{B} \) have a rook in the column \( b \);
2. either \( \hat{A} \) or \( \hat{B} \) (or both) have no rooks in the column \( a - p \).

We compare, moving from right to left (descending in the order \( <_a \)), the positions of rooks of facets \( \hat{A} \) and \( \hat{B} \) in the smaller chessboard \([a - (p - 1), a - 1] \times [n]\). If \( \{e\} \times [n] \) is the first column where they disagree, say \( \hat{A} \ni (e, i) \neq (e, j) \in \hat{B} \), then \( A < B \) if \( i > j \) (the rook corresponding to \( (e, i) \) is above the rook associated to \( (e, j) \)).

Alternatively the facets \( \hat{A} \) and \( \hat{B} \) agree on the whole of the smaller chessboard \([a - (p - 1), a - 1] \times [n]\). In this case by definition \( A < B \) if \( \hat{B} \) does have a rook in column \( a - p \) and \( \hat{A} \) does not.

The final possibility is that the facets \( \hat{A} \) and \( \hat{B} \) agree on the comparison interval and neither \( \hat{A} \) nor \( \hat{B} \) have a rook in the column \( a - p \). If this is the case we declare that the first stage of the comparison procedure is over and pass to the second stage.

The second stage of the comparison procedure begins by removing (or simply ignoring) the first row of the chessboard \([m] \times [n]\) and the column \( \{a - p\} \times [n] \), together with the small chessboard \([a - (p - 1), a - 1] \times [n]\) and all the rows of \([m] \times [n]\) associated to the rooks in,

\[
([a - (p - 1), a - 1] \times [n]) \cap \hat{A} = ([a - (p - 1), a - 1] \times [n]) \cap \hat{B}.
\]

This way we obtain a new chessboard \( M \times N \subseteq [m] \times [n] \) which inherits the (cyclic) ‘right to the left’ order in each of the rows so we can continue by applying the first stage of the comparison procedure to the facets \( \hat{A}' = \hat{A} \cap (M \times N) \) and \( \hat{B}' = \hat{B} \cap (M \times N) \).

This process can be continued, stage after stage, and if \( A \neq B \) sooner or later will lead to the decision whether \( A < B \) or \( B < A \).

**Lemma 4.1** The relation \( < \) is a linear order on the set of facets of the chessboard complex \( \Delta_{m,n} \).

**Proof of Lemma 4.1** By construction if \( A < B \) then \( B \not< A \) so it is sufficient to show that the relation \( < \) is transitive. Assume that \( A < B \) and \( B < C \). If both inequalities are decided at the same stage then by inspection of the priorities one easily deduces that \( A < C \). Suppose that the inequalities are decided at different stages, say \( A < B \) is decided at the stage \( i \) and \( B < C \) at the stage \( j \) where (for example) \( i < j \). Since \( B \) and \( C \) are not discernible from each other, up to the stage \( i \), we conclude that the same argument used to decide that \( A < B \) leads to the inequality \( A < C \). \( \square \)

The details of the proof that the described linear order of facets of \( \Delta_{m,n} \) is indeed a shelling are omitted since they will appear in greater generality in the proof of Theorem 1.3. Nonetheless, in the following very simple example we pinpoint the main difference between this linear order and the lexicographic order of facets.
Example 4.2 The lexicographic order of facets in the chessboard complex $\Delta_{m,2}$ (for $m \geq 3$) is not a shelling. Indeed, if $A$ is a predecessor of the simplex $B = \{(2, 1), (1, 2)\}$ in the lexicographic order then $A \cap B = \emptyset$. On the other hand in the shelling order described above each of the simplices $\{ (2, 1), (j, 2) \}$ for $j \geq 3$ is a predecessor of $\sigma$.

Theorem 4.3 For $m \geq k_1 + k_2 + \cdots + k_n + n - 1$ the complex $\Delta_{m,n}^{k_1, \ldots, k_n; 1}$ is shellable.

Proof: Facets of $\Delta_{m,n}^{k_1, \ldots, k_n; 1}$ are encoded as $n$-tuples $A = (A_1, A_2, \ldots, A_n)$ of disjoint subsets of $[m]$ where $|A_i| = k_i$. The elements of $A_i$ represent the positions of rooks in the $i$-th row, so strictly speaking the simplex associated to $A$ is the set $\hat{A} = \{(a, i) \in [m] \times [n] : a \in A_i\}$. Both $A$ and $\hat{A}$ unambiguously refer to the same facet of $\Delta_{m,n}$.

The linear order $<$ of the facets of $\Delta_{m,n}^{k_1, \ldots, k_n; 1}$ is defined by a recursive procedure which generalizes the procedure already described in the case of standard chessboard complex $\Delta_{m,n}$.

If $A_1$ is anti-lexicographically less than $B_1$ in the sense that $\max(A_1 \triangle B_1) \in B_1$ we declare that $A < B$. If $A_1 = B_1 = \{a_1, a_2, \ldots, a_{k_1}\}$ we consider $[m] \setminus A_1 = I_1 \cup I_2 \cup \cdots \cup I_r$ where,

$$I_j = \{x_j, x_j + 1, \cdots, x_j + s_j\} \subset [m]$$

are maximal sets (lacunas) of consecutive integers in $[m] \setminus A_1$. We assume that $\max I_j < \min I_{j+1}$ for all $j = 1, 2, \ldots, r - 1$.

In full agreement with (18) we introduce a priority order $\prec$ of elements contained in the union of all lacunas $I_j$ as follows. The priority order within the lacuna $I_j$ is from ‘right to left’,

$$x_j + s_j \succ x_j + s_j - 1 \succ \ldots \succ x_j + 1 \succ x_j.$$  \hfill (19)

The priority order of lacunas themselves is from ‘left to right’, so summarizing, the elements of $\cup_{j=1}^r I_j$ listed in the priority order $\prec$ from the biggest to the smallest are the following,

$$x_1 + s_1, x_1 + s_1 - 1, \ldots, x_1 + 1, x_1, x_2 + s_2, \ldots, x_2, \ldots, x_r + s_r, \ldots, x_r.$$ \hfill (20)

If $A_1 = B_1$ we define $A < B$ if either of the following conditions is satisfied.

(a) Both facets $A$ and $B$ contain rooks in the first $p - 1$ columns ($p \geq 1$) in the priority order (20) precisely at the same squares (positions). Moreover, the facet $B$ contains a rook in the $p$-th column with respect to this order and $A$ does not have a rook in the $p$-th column.

(b) Both facets $A$ and $B$ contain rooks in the first $p - 1$ columns in the priority order (20) at the same squares, both $A$ and $B$ contain a rook in the $p$-th column and the rook of $A$ is above the rook of $B$. In other words, if the $p$-th column in the order (20) is $x$, then $x \in A_i$, $x \in B_j$ for some $1 < j < i$. 

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(c) Both facets $A$ and $B$ contain rooks in the first $p-1$ columns ($p \geq 1$) in the order $\{\Delta\}$ at the same squares; neither $A$ nor $B$ contains a rook in the $p$-th column and $A' < B'$ where $A'$ and $B'$ are obtained by removing rooks from $A$ and $B$, by the following procedure.

Let $X \subset [m] \times [n]$ be the union of the first $p$ columns in the order $\{\Delta\}$. Define $j_i$ (where $0 \leq j_i \leq k_i$) as the number of rooks in $X \cap \hat{A}$ in the $i$-th row. Remove from the chessboard $[m] \times [n]$ the ‘small chessboard’ $X$, the first row and all rows where $j_i = k_i$. Simplices $A'$ and $B'$ are precisely what is left in $A$ and $B$ respectively after the removal of these rooks. Let $j$ be the number of rows where the equality $j_i = k_i$ is satisfied, that is $j$ is the number of removed rows aside from the first row.

Note that $K \cong \Delta_{m-k_1-p,n-j-1}$ and that the inequality,

$$m - k_1 - p \geq (k_2 - j_2) + \ldots + (k_n - j_n) + n - j - 2$$

is a consequence of the inequality $m \geq k_1 + k_2 + \ldots + k_n + n - 1$ and the relation $p - 1 = j_2 + \ldots + j_n$.

The fact that $<$ is a linear order is established similarly as in the proof of Lemma $4.1$. Let us show that $<$ is indeed a shelling order for the multiple chessboard complex $\Delta_{m,n}^{k_1,\ldots,k_n}$. We are supposed to show that for each pair of facets $A < B$ of $\Delta_{m,n}^{k_1,\ldots,k_n}$ there exists a facet $C < B$ and a vertex $v \in B$ such that

$$A \cap B \subset C \cap B = B \setminus \{v\}.$$

The statement is established by induction on $n$ (the number of rows). The claim is obviously true for $n = 1$. Assume that $<$ is a shelling order of facets of $\Delta_{d,q}^{k_1,\ldots,k_q}$ for all $q < n$ and all $k_1, k_2, \ldots, k_q$ such that $d \geq k_1 + \ldots + k_q + q - 1$.

Given the facets $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ such that $A < B$, let us consider the following cases.

1° Assume that $A_1 = \{a_1, a_2, \ldots, a_{k_1}\} < B_1 = \{b_1, b_2, \ldots, b_{k_1}\}$ and let $b_i = \max(A_1 \cap B_1) \in B_1$. Observe that in this case there exists $a_j \in A_1$, $a_j \notin B_1$, $a_j < b_i$. If there exists a column indexed by $b_0 < b_i$ without a rook from $B$ we let $B'_1 = B_1 \setminus \{b_i\} \cup \{b_0\}$. Define $C = (B'_1, B_2, \ldots, B_n)$ and observe that $C < B$ and

$$A \cap B \subset C \cap B = B \setminus \{(b_i, 1)\}.$$

If all of the columns $1, 2, \ldots, b_i$ contain a rook from $B$, then $a_j \in B_s$ for some $s \in \{2, 3, \ldots, n\}$. Let $b_0$ be the last column in the order $\{\Delta\}$ that does not contain a rook from $B$ and let $B'_s = B_s \setminus \{a_j\} \cup \{b_0\}$. For the facet $B' = (B_1, \ldots, B_{s-1}, B'_s, B_{s+1}, \ldots, B_n)$ we have $B' < B$ and

$$A \cap B \subset B' \cap B = B \setminus \{(a_j, s)\}.$$

2° If $A_1 = B_1$ we have one of the following possibilities:
(a) $A$ and $B$ contain rooks in the first $p - 1$ columns in the order (20) at the same squares; $B$ contains a rook in the $p$-th column in (20) and $A$ does not have a rook in this column.

Let $b$ denote the label of the $p$-th column, so we have that $b \in B_s$ for some $s > 1$ and $b \notin A_i$ for all $i = 1, 2, \ldots, n$. Let $b'$ denote the last column in the order (20) that does not contain a rook from $B$, and let $B'_s = B_s \setminus \{b\} \cup \{b'\}$. For the facet $B' = (B_1, \ldots, B_{s-1}, B'_s, B_{s+1}, \ldots, B_n)$ we have $B' < B$ and

$$A \cap B \subset B' \cap B = B \setminus \{(b, s)\}.$$  

(b) $A$ and $B$ contain rooks in the first $p - 1$ columns in the order (20) at the same squares; both $A$ and $B$ contain a rook in the $p$-th column and the rook in $A$ is above the rook in $B$.

Again, let $b$ denote the label of the $p$-th column. We have that $b \in B_s$ and $b \in A_t$ for some $1 < s < t$. Let $b'$ be the label of the last column in the order (20) that does not contain a rook from $B$, and let $B'_s = B_s \setminus \{b\} \cup \{b'\}$. For the facet $B' = (B_1, \ldots, B_{s-1}, B'_s, B_{s+1}, \ldots, B_n)$ we have $B' < B$ and

$$A \cap B \subset B' \cap B = B \setminus \{(b, s)\}.$$  

(c) Both $A$ and $B$ contain rooks in the first $p - 1$ columns in the order (20) at the same squares; neither $A$ nor $B$ contains a rook in the $p$-th column and $A' < B'$ in $K \cong \Delta^{k_2-j_2,k_3-j_3,\ldots,k_n-j_n:1}_{m-k_1-p,n-j-1}$. By the inductive assumption (taking into account the inequality (21)) we know that $K$ is shellable. Hence, there exists a facet $C' < B'$ of $K$ and a vertex $v \in B'$ such that

$$A' \cap B' \subset C' \cap B' = B' \setminus \{v\}.$$  

If we return the rooks, previously removed from $A$ and $B$ (from the removed rows and columns of the original complex) and add them to $C'$, we obtain a facet $C$ of $\Delta^{k_1,\ldots,k_n:1}_{m,n}$ such that $C < B$ and,

$$A \cap B \subset C \cap B = B \setminus \{v\}.$$  

This observation completes the proof of Theorem 4.3. □

5 An application

The general colored Tverberg problem, as outlined in Sections 1.1 and 1.2, is the question of describing conditions which guarantee the existence of large intersecting families of multicolored (rainbow) simplices. By definition a simplex is multicolored if no two vertices are colored by the same color.
We can modify the problem by allowing multicolored simplices to contain not more than \( p \) points of each color where \( p \geq 1 \) is prescribed in advance. Following the usual scheme (Section 1.3) we arrive at the generalized chessboard complexes \( \Delta_{p,1}^{r,t} \).

This connection was one of the reasons why we were interested in the connectivity properties of multiple chessboard complexes \( \Delta_{p,1}^{r,t} \) and the following statements illustrate some of possible applications.

**Theorem 5.1** Let \( r \) be a prime power. Given \( k \) finite sets of points in \( \mathbb{R}^d \) (called colors), of \( (p+1)r-1 \) points each, so that \( prk \geq (r-1)(d+1) + 1 \), it is possible to divide the points in \( r \) groups with at most \( p \) points of the same color in each group so that their convex hulls intersect.

**Proof:** The multicolored simplices are the simplices of the complex \( ([*]_{\Delta_{p+1}}^{*(p+1)r-1})^k \). Namely, these are the simplices which could have at most \( p \) vertices of any of colors. The configuration space of all \( r \)-tuples of disjoint multicolored simplices is so the simplicial complex

\[
K = \left( ([*]_{\Delta(p+1)}^{*(p+1)r-1})^k \right)_{\Delta(2)}.
\]

Since the join and deleted join commute, this complex equals

\[
K = \left( ([*]_{\Delta(p+1)}^{*(p+1)r-1})^r \right)_{\Delta(2)}^k.
\]

If we suppose, to the contrary, that the intersection of images of any \( r \) disjoint multicolored simplices is empty, the associated mapping \( F : K \rightarrow (\mathbb{R})^r \) would miss the diagonal \( \Delta \). By composing this map with the orthogonal projection to \( \Delta^\perp \), and then with the radial projection to the unit sphere in \( \Delta^\perp \), we obtain a \( \mathbb{Z}/r \)-equivariant mapping

\[
\tilde{F} : K \rightarrow S^{(r-1)(d+1)-1}.
\]

By Corollary 3.2 the complex \( ([*]_{\Delta(p+1)}^{*(p+1)r-1})^r \) \( \Delta(2) \) is \( (pr-2) \)-connected, and so the complex \( K \) is \( (prk-2) \)-connected. By our assumption \( prk-2 \geq (r-1)(d+1) - 1 \), and so such a mapping \( \tilde{F} \) could not exist. \( \square \)

Specializing to the case \( pk = d + 1 \), it is easy to see that we could take \( (p+1)r-1 \) points of each of \( k \) colors in \( \mathbb{R}^{pk-1} \) and obtain the following.

**Corollary 5.2** Let \( r \) be a prime power. Given \( k \) finite sets of points in \( \mathbb{R}^{pk-1} \) (called colors), of \( (p+1)r-1 \) points each, it is possible to divide the points in \( r \) groups with at most \( p \) points of the same color in each group so that their convex hulls intersect.

\[1\] It was kindly pointed by Günter Ziegler that Theorem 5.1 is implicit in [BFZ], see Theorem 4.4 and the remark following the proof of Lemma 4.2.
Of course, the continuous (non-linear) versions of the above results are true as well, and with the same proof.

If $r + 1$ is a prime, there is a simple proof of this corollary using the result of [BMZ] which even obtains better estimate $(pr$ instead of $(p+1)r-1)$ on the number of points of each color. Namely, one could divide each color of $pr$ points in $p$ ”subcolors” of $r$ points each, and obtain the desired division, even with some additional requirements on the points of the same color in each group. However, this argument works only in this case when $r + 1$ is a prime. Also, we stress here the importance of the generalized chessboard complexes and how they naturally appear in combinatorial problems.

**Remark 5.3** A more general result related to Theorem 5.1 can be formulated if we allow each of $r$ multicolored sets to contain $p_1$ points of the first color, $p_2$ points of the second color, etc. $p_k$ points of the $k$-th color. In this case we arrive at the complex $\Delta_{t,r}^{p_1,\ldots,p_k}$ and its connectivity properties established by Theorem 5.1 can be used again. We omit the details since our main goal in this paper was to establish improved bounds on the connectivity of generalized chessboard complexes and Theorem 5.1 was useful to illustrate their importance and to show how they naturally appear in different mathematical contexts.

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