Flip-Moves and Graded Associative Algebras

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Abstract

The relation between discrete topological field theories on triangulations of two-dimensional manifolds and associative algebras was worked out recently. The starting point for this development was the graphical interpretation of the associativity as flip of triangles. We show that there is a more general relation between flip-moves with two \( n \)-gons and \( \mathbb{Z}_{n-2} \)-graded associative algebras. A detailed examination shows that flip-invariant models on a lattice of \( n \)-gons can be constructed from \( \mathbb{Z}_2 \)- or \( \mathbb{Z}_1 \)-graded algebras, reducing in the second case to triangulations of the two-dimensional manifolds. Related problems occur naturally in three-dimensional topological lattice theories.

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Various aspects of topological lattice theories had been considered in the last years. First models had been constructed as discrete analogies of continuous topological field theories. The invariance of the continuous theory under the diffeomorphism group was discretised to the invariance under flip moves of the lattice [1], see fig. 1. The field variables were located on the vertices of the triangulation. Another type of models grow out of matrix models of two-dimensional quantum gravity [2], where one wants to couple a topological action to the model to control the topology-dependence of the series-expansion. These models have the field variables on the edges of the triangles and could be classified by associative algebras [3, 4]. The approach to topological lattice theories from the matrix models poses the problem to handle 'topological' actions coupled to models which not only contain a cubic but higher polynomials in the potential. This was solved in [4] for monomials of degree...
4, leading to quadrangulations of two-dimensional manifolds, and for arbitrary polynoms containing a cubic term, this leads to lattices build out of triangles and higher polygons.

This paper is a part of [3] and treats the remaining models for monomials of arbitrary degree, i.e. for manifolds covered by $n$-gons. This is of interest not only in the two-dimensional case, in a special case of three-dimensional topological lattice-theories [4] one has to deal with polygonals and multivalent hinges. For this the subdivision invariance of the weights must be assumed, a condition which is in our work a consequence of a rather natural condition to the weights.

We construct from the given data, the sets of weights $\Gamma_{i_1...i_n}$ of the $n$-gons and the weights $q^{ij}$ of the edges, an associative graded algebra, which allows for the classification and the computation of the partition function. We recover the topological model on triangulations and the models on chequered graphs already found in [4] and show, that this is a complete classification, all flip-invariant models on polygonizations belong to one of these two types.

First we have to introduce the model. We consider a polygonization of a two-dimensional compact oriented manifold by $n$-gons. On this polygonization we establish a statistical model with variables $i, j, \ldots = 1, \ldots, N$ on the edges of the $n$-gons, weights $\Gamma_{i_1...i_n} \in \mathbb{C}$ on the $n$-gons and $q^{ij} \in \mathbb{C}$ on the edges. The weights $\Gamma$ have to be cyclic, the weights $q$ have to be symmetric.

We assume that the matrix $(q^{ij})$ is regular and the inverse matrix $(q_{ij})$ exists (this condition can always be achieved by a simple transformation and a reduction of the range of the indices, see [4]). The partition function is the sum of the product of all weights over all indices.

For $n = 3$ the model is called topological if the weights are invariant under the moves in fig. [3] these moves are transitive on the set of all two-dimensional simplicial complexes, this was already shown by Alexander in 1930 [7], see also [8] for a discussion.

\[ \Gamma_{ijr}q^{rs}{\Gamma}_{skl} = \Gamma_{jkr}q^{rs}{\Gamma}_{sli} \]

Flip move

\[ \Gamma_{ijk} = \Gamma_{isr}q^{st}{\Gamma}_{jut}q^{uv}{\Gamma}_{kwv}q^{wr} \]

Pyramid move

Figure 1: Moves for $n = 3$
In this case one defines an algebra $\mathcal{A}$ which is a vector space with basis $\{e_1, \ldots, e_N\}$ and multiplication $e_i \cdot e_j = \lambda^k_{ij} e_k$, where the structure constants are formed by

$$\lambda^k_{ij} = \Gamma_{ijr} q^{rk}$$

(Here and in the following, the sum convention is assumed). The elements of the matrix $(q_{ij})$, the inverse matrix of $(q^{ij})$, form the coefficients of a symmetric bilinear form $q$ on $\mathcal{A}$ with $q_{ij} = q(e_i, e_j)$. Due to the cyclicity of the $\Gamma_{ijk}$ this bilinear form is invariant under multiplications in $\mathcal{A}$:

$$q(a \cdot b, c) = q(a, b \cdot c), \quad \forall a, b, c \in \mathcal{A} \quad (1)$$

An algebra together with a metric which fulfills (1) is called *metrised*, see e.g. [9].

As shown in several publications [3, 4], the conditions imposed by the flip and the pyramid move make the algebra associative and semisimple. The flip condition is the cause for associativity, the pyramid condition was thought to be the origin of the semisimplicity, but in [4] it was shown that the “non-semisimple parts” of the algebra (which consists not only of the radical, but also of some Levi subalgebra) give no contributions to the partition function of the statistical models considered here, and can therefore be ignored. What remains is a semisimple algebra. Imposing the pyramid flip is therefore not necessary for the classification of topological models.

The relation between flip moves and associative algebras was (see [4]) extended to the case of flips of two 4-gons, leading to $\mathbb{Z}_2$-graded associative algebras. There occurred the new quality, that some of the models vanish on graphs which can not be chequered.

We now generalize the work in [4] to arbitrary $n$-gons. First we generalize the flip move in fig. 1 for two $n$-gons as shown in fig. 2. Imposing a condition similar to the pyramid move will not be necessary for the classification of the topological models.

The weights invariant under the moves in fig. 2 fulfil the relations

$$\Gamma_{i_1 \ldots i_{n-1} r} q^{rs} \Gamma_{s i_{n-1} i_{2n-2}} = \Gamma_{i_2 \ldots i_n r} q^{rs} \Gamma_{s i_{n+1} \ldots i_{2n-2} i_1} = \ldots$$

Trivial examples of weights invariant under these flips are constructed out of models on triangulations, the weight of the $n$-gon is defined by the fusion of the weights of $n - 2$ triangles. A nontrivial example is the four-vertex model which was discussed in [4]. We will see that all models are analogous to one of these examples.

As in the case $n = 3$ we define a $N$-dimensional complex vector space $\mathcal{A}$ with basis $\{e_1, \ldots, e_N\}$ and a metric $q$ on $\mathcal{A}$ by $q(e_i, e_j) = q_{ij}$. We define a $(n - 1)$-linear map $\Gamma : \mathcal{A} \times \ldots \times \mathcal{A} \mapsto \mathcal{A}$ by

$$\Gamma(e_{i_1}, \ldots, e_{i_{n-1}}) := \Gamma_{i_1 \ldots i_{n-1} r} q^{rs} e_s.$$
Again the metric is invariant with respect to the map $\Gamma$:

$$q(\Gamma(e_{i_1}, \ldots, e_{i_{n-1}}), e_{i_n}) = \Gamma_{i_1 \ldots i_n} = \Gamma_{i_2 \ldots i_{n-1} i_1} = q(\Gamma(e_{i_2}, \ldots, e_{i_n}), e_{i_1})$$

(2)

The flip condition in fig. 2 imposes the following conditions on the map $\Gamma$:

$$\Gamma(\Gamma(a_1, \ldots, a_{n-1}), a_n, \ldots, a_{2n-3}) = \Gamma(a_1, \Gamma(a_2, \ldots, a_n), a_{n+1}, \ldots, a_{2n-3})$$

$$= \ldots = \Gamma(a_1, \ldots, a_{n-2}, \Gamma(a_{n-1}, \ldots, a_{2n-3}))$$

(3)

which are equivalent to

$$\Gamma \circ (\text{id}^r \otimes \Gamma \otimes \text{id}^{n-2-r}) = \Gamma \circ (\text{id}^s \otimes \Gamma \otimes \text{id}^{n-2-s})$$

(4)

for all $r, s = 0, \ldots, n - 2$. This is a generalization of the associativity condition of associative algebras. An easy but time-consuming induction shows, that this general associativity holds for more than two $\Gamma$:

$$\Gamma \circ (\text{id}^{r_1} \otimes \Gamma \otimes \text{id}^{n-2-r_1}) \circ \ldots \circ (\text{id}^{r_k} \otimes \Gamma \otimes \text{id}^{k(n-2)-r_k}) = \Gamma \circ (\text{id}^{s_1} \otimes \Gamma \otimes \text{id}^{n-2-s_1}) \circ \ldots \circ (\text{id}^{s_k} \otimes \Gamma \otimes \text{id}^{k(n-2)-s_k})$$

(5)

for all admissible $r_i, s_i$.

For practical reasons we rename the vector space $\mathcal{A}$ by $\mathcal{A}_1$ and the metric $q$ by $q_1$.

Then we can prove the following main theorem:
Theorem 1 Let $\mathcal{A}_1$ be a $N$-dimensional complex vector space. Let $\Gamma : \mathcal{A}_1^{n-1} \rightarrow \mathcal{A}_1$ be a $C$-multilinear map and $q_1 : \mathcal{A}_1 \times \mathcal{A}_1 \rightarrow C$ a symmetric, non-degenerate metric, which satisfy the invariance condition (3) and the general associativity condition (3).

Then there exists a $Z_{n-2}$-graded, associative, metrised algebra $(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{n-3}, q)$, where $q$ is a non degenerate, symmetric bilinear form on $\mathcal{A}$ with $q|_{\mathcal{A}_1 \times \mathcal{A}_1} = q_1$ and $q|_{\mathcal{A}_i \times \mathcal{A}_j} = 0$ for $i + j \not\equiv 2 \mod (n - 2)$.

The map $\Gamma$ and the algebra multiplication are related by

$$\Gamma(a_1, \ldots, a_{n-1}) = a_1 \cdot \ldots \cdot a_{n-1} \quad \forall a_1, \ldots, a_{n-1} \in \mathcal{A}_1.$$  \hspace{1cm} (6)

Remarks: An algebra $\mathcal{A} = \bigoplus_{i=0}^{m-1} \mathcal{A}_i$ is $Z_m$-graded, if the multiplication fulfills $\mathcal{A}_i \times \mathcal{A}_j \rightarrow \mathcal{A}_{i+j \mod m}$. Hence $a_1 \cdot a_2 \in \mathcal{A}_2$, $a_1 \cdot a_2 \cdot a_3 \in \mathcal{A}_3$, and finally $a_1 \cdot \ldots \cdot a_{n-1} \in \mathcal{A}_1$. We remark, that the algebra is not super-graded, as it is assumed automatically by Lie-algebras.

Let $\{e_1, \ldots, e_{|\mathcal{A}|}\}$ be a ordered basis of $\mathcal{A}$ with respect to the grading. Let $\lambda^k_{ij}$ be the structure constants with respect to this basis. Then we get with (3):

$$\Gamma_{i_1 \ldots i_n} = q_1(\Gamma(e_{i_1}, \ldots, e_{i_{n-1}}), e_{i_n}) = q(e_{i_1} \cdot \ldots \cdot e_{i_{n-1}}, e_{i_n})$$

$$= q_{r_{i_n}} \lambda^1_{i_1 i_2} \lambda^2_{i_2 i_3} \ldots \lambda^r_{n-3 i_{n-1}}$$

$$= \lambda^1_{i_1 i_2} q^1_{i_3} \lambda^2_{i_3 i_4} q^2_{i_4} \ldots q^{r_{n-2 s_{n-2}}} \lambda^s_{n-2 i_{n-1}}.$$ \hspace{1cm} (7)

The inner indices are summed over $1, \ldots, \dim \mathcal{A}$, but due to the grading of the algebra every summation is restricted to the indices belonging to one part of the grading.

Thanks to the associativity we can replace the right hand side of (7) by any evaluation of the associative product in $q(e_{i_1} \cdot \ldots \cdot e_{i_{n-1}}, e_{i_n})$. The graphical interpretation is simple: we can replace the $n$-gon with weight $\Gamma_{i_1 \ldots i_n}$ by a triangulation with $n - 2$ triangles and weights $\lambda_{ijk}$ and sumation over all inner indices, as in fig. 3.

Due to the associativity of the algebra $\mathcal{A}$ is this model flip invariant. We see: flip invariant models on $n$-gonisations of a two-dimensional manifold are equivalent to flip invariant models on triangulations with a greater range of indices and the restriction, that certain indices only take values in the original part. The value $n - 2$ will appear often, therefore we define $p := n - 2$.

To prove the theorem we have to perform the following steps:
1. We define a non-associative algebra structure on the vector space $M = \bigoplus_{k=1}^{p} A_1^{\otimes k}$ by the multiplication

$$a \cdot b = \begin{cases} a_1 \otimes \ldots \otimes a_k \otimes b_1 \otimes \ldots \otimes b_l = a \otimes b & k + l \leq p \\ \Gamma(a_1, \ldots, b_{p+1-k}) \otimes b_{p+2-k} \otimes \ldots \otimes b_l & k + l > p \end{cases}$$

for $a = a_1 \otimes \ldots \otimes a_k$, $b = b_1 \otimes \ldots \otimes b_l$. This algebra is finite dimensional, but non-associative. The properties of $\Gamma$ allow the definition of an ideal $I$, such that $M/I$ is associative. This is not the usual way to construct an associative algebra, which would start with the infinite dimensional universal tensor algebra over $A_1$ and divide out an infinite dimensional ideal to get a finite dimensional associative algebra.

2. We define the subspace $I := \bigoplus_{k=1}^{p} I_k$ of $M$ with $I_1 := \{0\}$ and

$$I_k := \{ a \in A_1^{\otimes k} | a \cdot b = 0 \ \forall \ b \in A_1^{p+1-k} \}$$

for $k = 2, \ldots, p$ and show that $I$ is a two sided ideal of $M$.

3. We can therefore define the algebra

$$\mathcal{A} := M/I = \bigoplus_{k=1}^{p} A_1^{\otimes k}/I_k =: \bigoplus_{k=1}^{p} \mathcal{A}_k$$

which will be shown to be associative and contains the original vector space $A_1$. The relation (9) coincides with the definition of the multiplication.

4. We define a bilinear form $q$ on $M$ for $a = a_1 \otimes \ldots \otimes a_k, b = b_1 \otimes \ldots \otimes b_l \in M$ by

$$q(a, b) := \begin{cases} 0 & k + l \neq 2 \mod p \\ q_1(a_1, \Gamma(a_2, \ldots, b_l)) & k + l = n \\ q_1(a_1, b_1) & k = l = 1 \end{cases}$$

Figure 3: Splitted $n$-gon
and show that the projection of \( q \) on \( \mathcal{A} \), which we will denote also by \( q \), is well-defined and symmetric.

5. We show that \( q \) is non-degenerate on \( \mathcal{A} \).

6. We show that \( q \) is invariant on \( \mathcal{A} \).

Proofs and Remarks:

1) The multiplication (8) is in general not associative, consider e.g.

\[
(a \cdot b) \cdot c = a \cdot (b \cdot c)
\]

But we can show, that the multiplication is associative for factors \( a \in \mathcal{A}_1^{\otimes n_a}, b \in \mathcal{A}_1^{\otimes n_b}, c \in \mathcal{A}_1^{\otimes n_c} \) with \( n_a + n_b + n_c + \ldots \equiv 1 \mod p \), e.g.

\[
(a \cdot b) \cdot c = a \cdot (b \cdot c)
\]

(12)

\[
((a \cdot b) \cdot c) \cdot d = (a \cdot (b \cdot c)) \cdot d
\]

(13)

For this, let \( a = a_1 \otimes \ldots \otimes a_{n_a}, b = b_1 \otimes \ldots \otimes b_{n_b}, c = c_1 \otimes \ldots \otimes c_{n_c}, \ldots \) Since \( n_a + n_b + n_c + \ldots \equiv 1 \mod p \), all the products are of the form

\[
\Gamma \circ (\text{id}^{r_1} \otimes \Gamma \otimes \text{id}^{p-r_1}) \circ (\text{id}^{r_2} \otimes \Gamma \otimes \text{id}^{2p-r_2}) \circ \ldots (a_1, \ldots, a_{n_a}, b_1, \ldots)
\]

and products of the same factors are equivalent by (8).

2) The condition \( a \cdot b = 0 \) for all \( b \in \mathcal{A}_1^{\otimes p+1-k} \) is equivalent to \( b \cdot a = 0 \) for all \( b \in \mathcal{A}_1^{\otimes p+1-k} \) and we can define alternatively

\[
I_k = \{ a \in \mathcal{A}_1^{\otimes k} | b \cdot a = 0 \; \forall \; b \in \mathcal{A}_1^{\otimes p+1-k} \}
\]

(14)

In order to see this, we use the invariance condition (2) and the symmetry of \( q_1 \): For all \( b = b_1 \otimes \ldots \otimes b_{p+1-k} \in \mathcal{A}_1^{\otimes p+1-k} \), for all \( c \in \mathcal{A}_1 \) and for \( a = a^{i_1 \ldots i_k} e_{i_1} \otimes \ldots \otimes e_{i_k} \in I_k \) holds

\[
0 = a \cdot b = a^{i_1 \ldots i_k} \Gamma(e_{i_1}, \ldots, e_{i_k}, b_1, \ldots, b_{p+1-k})
\]

\[
\Leftrightarrow 0 = q_1(a^{i_1 \ldots i_k} \Gamma(e_{i_1}, \ldots, e_{i_k}, b_1, \ldots, b_{p+1-k}), c)
\]

\[
\Leftrightarrow 0 = q_1(a^{i_1 \ldots i_k} \Gamma(b_2, \ldots, b_{p+1-k}, c, e_{i_1}, \ldots, e_{i_k}), b_1)
\]

\[
\Leftrightarrow 0 = a^{i_1 \ldots i_k} \Gamma(b_2, \ldots, b_{p+1-k}, c, e_{i_1}, \ldots, e_{i_k})
\]

\[
\Leftrightarrow 0 = (b_1 \otimes \ldots \otimes b_{p+1-k} \otimes c) \cdot a
\]
Since the elements of the form $b_1 \otimes \ldots \otimes b_{p+1-k} \otimes c$ span $A_1^{\otimes p+1-k}$, the equivalence is proved.

The subspace $I$ of $M$ is an ideal of $M$: Let $a \in I_{n_a}$, then for all $b \in A_1^{\otimes n_b}$, $n_b = 1, \ldots, p$ and all $c \in A_1^{\otimes n_c}$ with $n_c$ such that $n_a + n_b + n_c \equiv 1 \pmod{p}$, we have

$$ (a \cdot b) \cdot c \overset{\text{(12)}}{=} a \cdot (b \cdot c) = 0 \Rightarrow a \cdot b \in I $$

since $b \cdot c \in A_1^{\otimes p+1-k}$ and

$$ (b \cdot a) \cdot c \overset{\text{(12)}}{=} b \cdot (a \cdot c) = 0 $$

by (14), hence also $b \cdot a \in I$ and $I$ is a two sided ideal.

3) The algebra defined in (10) is associative. To prove this, we have to show that for $a, b, c \in M$ holds $(a \cdot b) \cdot c - a \cdot (b \cdot c) \in I$. Again it is sufficient to consider $a \in A_1^{\otimes n_a}$, $b \in A_1^{\otimes n_b}$, $c \in A_1^{\otimes n_c}$. Let $d \in A_1^{\otimes n_d}$ with $n_d$ such that $n_a + n_b + n_c + n_d \equiv 1 \pmod{p}$. Then

$$ ((a \cdot b) \cdot c) \cdot d \overset{\text{3}}{=} (a \cdot (b \cdot c)) \cdot d $$

$$ \Rightarrow \quad ((a \cdot b) \cdot c - a \cdot (b \cdot c)) \cdot d = 0 $$

and since this holds for all $d$, the difference is in $I$ and the algebra $A$ is associative.

We define $A_0 := A_p$. With the multiplication $A_i \cdot A_j \to A_{i+j \mod p}$ becomes $A = A_0 \oplus \ldots \oplus A_{n-3}$ a $Z_p$-graded associative algebra. The condition (3) is satisfied due to the definition (8): Let $a_1, \ldots, a_{p+1} \in A_1$. Then

$$ a_1 \cdot \ldots \cdot a_{p+1} = \Gamma(a_1, \ldots, a_{p+1}) $$

4) The map $q : M \times M \to C^\ast$ defined in (11) is well defined on $A \times A$. To prove this we have to show that $q(a, b)$ is independent of the choice of the representatives of $a$ and $b$, i.e. for all $c_a, c_b \in I$ holds: $q(a + c_a, b + c_b) = q(a, b)$, i.e. $q(a, c_b) = 0 = q(c_a, b) = q(c_a, c_b)$. Due to the block structure of $q$ it is sufficient to consider for $a$, $b$, $c_a$ and $c_b$ only homogeneous elements. Let $a = a_1 \otimes \ldots \otimes a_k \in A_1^{\otimes k}$, $c_b = c_{j_1} \ldots c_{j_l} e_{j_1} \otimes \ldots \otimes e_{j_l} \in I_l$, $k + l \equiv 2 \pmod{p}$. In the case $k = l = 1$ $c_b = 0$ and $q(a, c_b) = 0$, for $k + l = n$ we have

$$ q(a, c_b) = q(a_1 \otimes \ldots \otimes a_k, c_{j_1} \ldots c_{j_l} e_{j_1} \otimes \ldots \otimes e_{j_l}) $$

$$ = q_1(a_1, c_{j_1} \ldots c_{j_l} \Gamma(a_2, \ldots, a_k, e_{j_1}, \ldots, e_{j_l})) $$

$$ = q_1(a_1, (a_2 \otimes \ldots \otimes a_k) \cdot c_b) = 0 \ . $$
analogly $q(c_a, b) = 0$, $q(c_a, c_b) = 0$ is then clear.

$q$ is symmetric: for $a = a_1 \otimes \ldots \otimes a_k$, $b = b_1 \otimes \ldots \otimes b_l$ is

- $k + l \not\equiv 2 \mod p : q(a, b) = 0 = q(b, a)$
- $k = l = 1 : q(a, b) = q_1(a, b) = q_1(b, a) = q(b, a)$
- $k + l = n :$

$$q(a, b) = q(a_1 \otimes \ldots \otimes a_k, b_1 \otimes \ldots \otimes b_l) = q_1(a_1, \Gamma(a_2, \ldots, a_k, b_1, \ldots, b_l))$$

5) $q$ is non-degenerate: Due to the block structure of the metric it is again sufficient to consider homogeneous elements $a \in A_1^\otimes k$ and $c = c_i^{j_1} \otimes \ldots \otimes e_i^{j_l} \in A_1^\otimes l$, $k + l \equiv 2 \mod p$. Let $q(a, c) = 0$ for all $a = a_1 \otimes \ldots \otimes a_k \in A_1^\otimes k$.

$$\iff q(a_1 \otimes \ldots \otimes a_k, c_i^{j_1} \otimes \ldots \otimes e_i^{j_l}) = 0 \ \forall a_1, \ldots a_k \in A_1$$

$$\iff q_1(a_1, c_i^{j_1} \Gamma(a_2, \ldots, a_k, e_i^{j_2}, \ldots, e_i^{j_l})) = 0 \ \forall a_1, \ldots, a_k \in A_1$$

$$\iff c_i^{j_1} \Gamma(a_2, \ldots, a_k, e_i^{j_2}, \ldots, e_i^{j_l}) = (a_2 \otimes \ldots \otimes a_k) \cdot c = 0$$

$$\iff c \in I_1 .$$

6) To prove the invariance of the metric we consider again $a = a_1 \otimes \ldots \otimes a_{n_a}$, $b = b_1 \otimes \ldots \otimes b_{n_b}$, $c = c_1 \otimes \ldots \otimes c_{n_c}$. For $n_a + n_b + n_c \not\equiv 2 \mod p$ we have

$$q(a, b \cdot c) = 0 = q(a \cdot b, c) .$$

For $n_a + n_b + n_c = n$ we have

$$q(a \cdot b, c) = q(c, a \cdot b) = q_1(c_1, \Gamma(c_2, \ldots, c_{n_c}, a_1, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}))$$

$$= q_1(a_1, \Gamma(a_2, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}, c_1, \ldots, c_{n_c})) = q(a, b \cdot c) \quad (16)$$

For $n_a + n_b + n_c \equiv 2 \mod p$ let $a' = a_2 \otimes \ldots \otimes a_k$, i.e. $a = a_1 \cdot a'$. Then

$$q(a \cdot b, c) = q((a_1 \cdot a') \cdot b, c) = q(a_1 \cdot (a' \cdot b), c)$$

$$q(a_1, (a' \cdot b) \cdot c) = q(a_1, a' \cdot (b \cdot c))$$

$$q(a_1 \cdot a', b \cdot c) = q(a, b \cdot c)$$
The metric $q$ has a block structure with respect to the decomposition $\mathcal{A} = \oplus_{k=0}^{n-3} \mathcal{A}_k$, due to $q(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i + j \not\equiv 2 \mod p$ we get for $n > 4$

$$q = (q_{ij}) = \begin{pmatrix}
0 & \square & 0 \\
\square & 0 & \square \\
0 & \cdots & 0
\end{pmatrix}$$

(17)

The $i$-th column and row, respectively, belong to the component $\mathcal{A}_{i-1}$ of $\mathcal{A}$. We use the symbol $q$ for the metric and for the matrix $(q_{ij})$ in a basis. We assume in the following, that we have chosen a basis $\{e_i\}$ which respects the grading of $\mathcal{A}$. It is easy to see, that the inverse matrix $(q_{ij})^{-1}$ has then same structure; matrix elements $q_{ij}$ are only not equal zero, if the basis elements $e_i$ and $e_j$ lie in components $\mathcal{A}_k$ and $\mathcal{A}_l$ with $k + l \equiv 2 \mod p$.

We now use the methods elaborated in [4] to calculate the partition functions of the flip invariant models. We first review a few facts about associative, metrised algebras (see [4] for details).

- Let $\mathcal{A}$ be a complex, associative, metrised algebra. We decompose $\mathcal{A} = \mathcal{B} \oplus L \oplus R$, where $\mathcal{B}$ is the largest semisimple ideal of $\mathcal{A}$, $L$ is a (non-unique) semisimple Levi-subalgebra and $R$ is the radical of $\mathcal{A}$. $\mathcal{B}$ and $L \oplus R$ are orthogonal with respect to $q$, i.e. $L \oplus R = \mathcal{B}^\perp$.

- $q^{ij} \neq 0$ for $e_i \in L$ is only possible if $e_j \in R$

- $\mathcal{B}$ itself is the direct sum of the simple ideals of $\mathcal{A}$, $\mathcal{B} = \oplus_i I_i$, where $I_i$ are the simple ideals of $\mathcal{A}$ and these are all orthogonal: $I_i \perp I_j$ for $i \neq j$.

We now check the relation of the decompositions $\mathcal{A} = \mathcal{B} \oplus L \oplus R$ and $\mathcal{A} = \oplus_k \mathcal{A}_k$. For this end we introduce the grading operator $\theta$ on $\mathcal{A}$ by $\theta(a_k) = \omega^k a_k$ for $a_k \in \mathcal{A}_k$, $\omega = \exp(2\pi i/p)$. $\theta$ is an automorphism of $\mathcal{A}$ since $\mathcal{A}_k \times \mathcal{A}_l \to \mathcal{A}_{k+l \mod p}$.

Every $\theta$-invariant subalgebra $X$ of $\mathcal{A}$ allows for a decomposition $X = \oplus_k X_k$ with $X_k \subset \mathcal{A}_k$. $\mathcal{B}$ is a $\theta$-invariant subalgebra, since the image of a semisimple ideal is a semisimple ideal, therefore $\mathcal{B} = \oplus_k \mathcal{B}_k$. By the theorem 1 in [10] there exists a $\theta$-invariant Levi algebra $L$ which allows for a decomposition $L = \oplus_k L_k$. The image of the radical $R$ is the radical, hence we have also $R = \oplus_k R_k$.

We have therefore a decomposition of each $\mathcal{A}_k = \mathcal{B}_k \oplus L_k \oplus R_k$ and we can choose a basis of $\mathcal{A}$ respecting this decomposition.
Since \( B \) and \( L \oplus R \) are orthogonal the partition function splits into the partition function of a model with the semisimple algebra \( B \) and of the algebra \( L \oplus R \). The latter can be shown to be zero, the arguments are the same as in \([4]\), we will only give a sketch of the discussion.

We consider the splitted graph, let \( i_1 \) be an arbitrary index. Let \( e_{i_1} \in R \), we consider all triangles which contain the vertex opposite to the index \( i_1 \). We label the indices as in fig. 4. The partition function of this part of the graph, summed over all inner indices \( r_1, \ldots, r_N \) and \( s_1, \ldots, s_N \), is given by

\[
Z_{i_1, \ldots, i_N} = \lambda_{r_1 i_1 s_1} q^{s_1 r_2} \lambda_{r_2 i_2 s_2} \cdots \lambda_{r_N i_N s_N} q^{s_N r_1} = \lambda_{r_1 i_1}^{r_2} \lambda_{r_2 i_2}^{r_3} \cdots \lambda_{r_N i_N}^{r_1} = (R_{r_N} \cdot R_{r_{N-1}} \cdot \cdots \cdot R_{r_1} e_{r_1})^{r_1} = \text{tr} R_{e_{i_1} e_{i_2} \cdots e_{i_N}}
\]

where \( R_a \) is the right multiplication in \( A \) considered as an endomorphism: \( R_a b = ba \), \( R_i \) is short for \( R_{e_i} \). If \( e_{i_1} \in R \), then is also \( e_{i_1} e_{i_2} \cdots e_{i_N} \in R \) and the trace vanishes, therefore all configurations with an index in \( R \) give no contribution to the partition function.

Now let \( e_i \in L \), then \( q^{ij} = 0 \) for all \( e_j \not\in R \), but if \( e_j \in R \) then we can repeat the discussion above with the result that also all configurations with an index in \( L \) give no contribution to the partition function.

There remains the discussion of the semisimple algebra \( B \). It is \( \theta \)-invariant and the (orthogonal) direct sum of all simple ideals of \( A \). One might expect that each simple ideal \( I \) is itself \( \theta \)-invariant, but this is not true in general.

\( \theta \) is an automorphism of \( A \), the image of a simple ideal \( I_1 \) is also a simple ideal \( I_2 = \theta(I_1) \) which can be different from \( I_1 \). We get a sequence \( I_1, I_2, \ldots, I_k \) of
disjoint isomorphic simple ideals with \( \theta(I_k) = I_1 \); since \( \theta^p = 1 \) the number \( k \) must be a divisor of \( p \), \( p = kl \). Not each ideal \( I_i \) is \( \theta \)-invariant, but the direct sum \( I = I_1 \oplus \ldots \oplus I_k \) is, and we can decompose it in \( I = I^{(0)} \oplus \ldots \oplus I^{(p-1)} \) with \( I^{(j)} \subset A_j \) and \( \theta(I^{(j)}) = \omega^j I^{(j)} \). The partition function decomposes into several parts belonging to \( \theta \)-invariant semisimple ideals of \( A \).

By the assumptions in theorem 1 \( q_{|I^{(j)}} \) is non degenerate. We will test this condition to gain information about \( k \): let \( a, b \in I^{(1)} \), \( a = a_1 + \ldots + a_k \), \( b = b_1 + \ldots + b_k \), \( a_j, b_j \in I_j \). Since \( \theta(a) = \omega a \) and \( \theta(a_j) \in I_{j+1} \) we get \( \theta(a_j) = \omega a_{j+1} \), \( \theta(a_k) = \omega a_1 \) and therefore

\[
\begin{align*}
  a_j &= \omega^{1-j} \theta^{j-1}(a_1) \\
  b &= \sum_j \omega^{1-j} \theta^{j-1}(b_1)
\end{align*}
\]

\[
\Rightarrow q(a, b) = \sum_{i,j} \omega^{2-i-j} q(\theta^{j-1}(a_1), \theta^{i-1}(b_1)) = \sum_i \omega^{2(1-i)} q(\theta^{i-1}(a_1), \theta^{i-1}(b_1))
\]

\[
= \sum_i \omega^{2(1-i)} \omega^{2(i-1)} q(a_1, b_1) = kq(a_1, b_1)
\]

where we have used \( q(\theta(a), \theta(b)) = \omega^2 q(a, b) \). \( \theta^k \) is an automorphisms of \( I_1 \), which is a simple complex algebra isomorphic to a full complex matrix algebra. By the theorem of Noether-Skolem \( \theta^k \) is an inner automorphism, i.e. there exists an invertible element \( s \in I_1 \) with \( \theta^k(a) = s^{-1}as \) for all \( a \in I_1 \). Then

\[
q(\theta^k(a_1), \theta^k(b_1)) = \omega^{2k} q(a_1, b_1)
\]

\[
= q(s^{-1}a_1s, s^{-1}b_1s) = q(a_1, b_1) \quad \forall a_1, b_1 \in I_1
\]

where we have used the invariance and the symmetry of \( q \). Hence \( \omega^{2k} = 1 \) which is only possible for \( 2k = p \) or \( k = p \). All other cases, e.g. \( k = 1 \) for \( p > 2 \) which corresponds to a \( \theta \)-invariant simple ideal do not occur in the context of flip invariant models.

There remains the discussion of this two cases:

\( k = p \): This is the trivial one. There are \( p \) simple ideals isomorphic to a full complex matrix algebra \( \mathfrak{g}_{r \times r} \). Let \( \{ e_i \} \) be a basis of \( I_1 \), then is \( \{ \tilde{e}_i = e_i + \omega^{-1} \theta(e_i) + \ldots + \omega^{1-p} \theta^{p-1}(e_i) \} \) a basis of \( A_1 \). Denote by \( (a)_1 \) the \( I_1 \) component of \( a \), then we get for the weights

\[
\Gamma_{i_1 \ldots i_n} = q(\Gamma(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{p+1}}), \tilde{e}_{i_n}) = q(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{p+1}}, \tilde{e}_{i_n})
\]

\[
= kq^{(1)}((\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{p+1}})_{1}, (\tilde{e}_{i_n})_{1}) = kq^{(1)}(e_{i_1}, \ldots, e_{i_{p+1}}, e_{i_n})
\]

(18)
This is exactly the weight one would get for a \( n \)-gon glued together out of \( n - 2 \) triangles with a topological weight on the triangles. Therefore this case is called trivial.

For the calculation of the partition function it is convenient to consider the dual graph in the double line representation [4]. We choose in \( I_1 \), which is isomorphic to a full complex matrix algebra, the standard basis \( \{ E_{ij} \} \) of \( r \times r \) matrices with \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \). \( q^{(1)} = q|_{I_1 \times I_1} \) is an invariant metric on \( I_1 \), this is, up to a factor, the trace of the matrices: \( q^{(1)}(a, b) = \beta \text{tr} (ab) \). We get for the weights of the vertices of degree \( n \)

\[
\Gamma_{i_1 i_2 i_3 \ldots i_n} = p\beta \delta_{i_1 i_2} \delta_{i_2 i_3} \ldots \delta_{i_n i_1}
\]  

(19)

and for the weights of the edges (\( q(\tilde{a}, \tilde{b}) = pq^{(1)}(a, b) \))

\[
q^{i_1 i_2 i_3} = (p\beta)^{-1} \delta_{i_1 i_2} \delta_{i_2 i_3}
\]

(20)

Figure 5: Double line representation with equal indices

All indices on a closed line must have the same value as indicated in fig. 5. Each closed line corresponds to a vertex of the original graph. The computation of the partition function is therefore reduced to a counting of factors. We get a factor \( r \) for each vertex of the polygonization, a factor \( (p\beta)^{-1} \) for each edge and a factor \( p\beta \) for each \( n \)-gon. This results in

\[
Z = r^V (p\beta)^{-E} (p\beta)^P = (p\beta)^\chi \frac{r^V}{p\beta^V}
\]

(21)

where \( V \) is the number of vertices of the polygonization, \( E \) ist the number of edges and \( P \) the number of plaquettes, the \( n \)-gons. We get the typical dependence of the partition function on the euler characteristic \( \chi \) of the manifold. If one adjusts the constant \( \beta \), such that \( r = p\beta \), then the partition function will be topological.
The other case $2k = r$ is non trivial and leads to totally new aspects. Let $a = a_1 + \ldots + a_k \in I^{(1)}$, i.e. $\theta(a) = \omega a$. Then $a_i = \omega^{1-i}r^{i-1}(a_1), \theta^k(a_1) = \omega^k a_1 = -a_1$. In this case is $\Theta = \theta^k$ an automorphism from $I_1$ to $I_1$ with $\Theta^2 = 1$. By the theorem of Noether-Skolem it is an inner automorphism, there exist $s \in I_1$ with $\Theta(a) = s^{-1}as$ for all $a \in I_1$. Then $\Theta^2(a) = s^{-2}as^2 = a$ i.e. $[a, s^2] = 0$ for all $a \in I_1$. With Schur’s Lemma we conclude that $s^2 = \lambda_1$, we can set $\lambda_1 = 1$. Then we can choose a basis in $I_1$ such that $s = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ with $M$ times 1 and $N$ times $-1, M + N = r$. $\Theta(a_1) = s^{-1}a_1s = -a_1$ is fulfilled for all matrices $a_1 \in I_1$ which have the off diagonal block form

$$a_1 = \begin{pmatrix} 0 & \square \\ \square & 0 \end{pmatrix}$$

(22)

i.e. $(a_1)_{ij} = 0$ for $i, j \leq M$ or for $i, j > M$. $I_1$ is then a $\mathbb{Z}_2$-graded algebra. A basis of $I^{(1)}$ is given by $\{E_{ij} + \omega^{-1}\theta(E_{ij}) + \ldots + \omega^{1-k}\theta^{k-1}(E_{ij}) | i \leq M < j \text{ or } j \leq M < i\}$. The weights are given by

$$\Gamma_{i_1j_1i_2j_2\ldots i_nj_n} = k_{i_1j_1}^i \delta_{j_1i_2} \delta_{j_2i_3} \ldots \delta_{j_{n-1}i_1}$$

$$q^{i_1j_1i_2j_2} = (k_{i_1j_1})^{-1} \delta_{j_1i_2} \delta_{j_2i_1}$$

(23)

(24)

where the pairs $i_1j_1, i_2j_2, \ldots, i_nj_n$ fulfil alternating the relations $i \leq M < j$ and $j \leq M < i$.

In the double line representation of the dual graph each double line carries both types of indices, the indices of a line must have the same value. See fig. 6 where different linetypes denote different ranges of indices.

Figure 6: Double line representation of a chequered graph

For an arbitrary graph it is not possible to distribute the indices in this manner. In case it is the graph is called chequered [12, 4], i.e. the faces of the dual graph can
be coloured alternating black and white such that nowhere are two black or two white faces are neighboured. If the graph is not chequered the partition function vanishes, otherwise we get

\[ Z = (M^{V_1}N^{V_2} + M^{V_2}N^{V_1})(k\beta)^{P-E} \]  

(25)

where \( V_1 \) and \( V_2 \) are the numbers of vertices whose dual plaquettes carry the same type of index, these are flip invariants of the model.

This models can distinguish smaller classes of graphs, the flip move is therefore not transitive. See [4] for a discussion of the consequences of this fact.

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