Bicategories of operator algebras and Poisson manifolds

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Abstract. It is well known that rings are the objects of a bicategory, whose arrows are bimodules, composed through the bimodule tensor product. We give an analogous bicategorical description of $C^*$-algebras, von Neumann algebras, Lie groupoids, symplectic groupoids, and Poisson manifolds. The upshot is that known definitions of Morita equivalence for any of these cases amount to isomorphism of objects in the pertinent bicategory.

1 Introduction

One of the achievements of Doplicher and Roberts has been to introduce monoidal categories into quantum field theory; indeed, their work provides the most convincing example of categorical thinking in physics by a long shot. Since a monoidal category is nothing but a bicategory with one object, it seems an appropriate tribute to describe a number of more general bicategories with proven relevance to physics.

In a bicategory \([3,18]\) (alternatively called a weak 2-category, which is a special case of a weak \(n\)-category \([1]\)), each space of arrows between two fixed objects is itself a category, so that there is a notion of isomorphisms of such arrows. To distinguish the arrows between objects in a bicategory from the arrows in each category of arrows, the former arrows are called horizontal, the latter vertical. Now, horizontal composition of arrows is merely associative up to isomorphism, and the horizontal composition of an arrow with a local unit arrow is merely isomorphic to the given arrow, rather than equal to it, as in a category.

Our examples of bicategories originated in the theory of constrained quantization \([3,18]\), relating classical and quantum physics in an intriguing way. In the order: Objects, horizontal arrows, composition operation, unit arrows (as in:

\[\begin{array}{ccc}
\text{Objects} & \text{Horizontal arrows} & \text{Composition} \\
\text{Unit arrows} & \text{Unit arrows} & \text{Unit arrows}
\end{array}\]

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\text{Unit arrows} & \text{Unit arrows} & \text{Unit arrows}
\end{array}\]
Rings, bimodules, algebraic bimodule tensor product, canonical bimodule over a ring), these examples are

1. $C^\ast$-algebras, Hilbert ($C^\ast$) bimodules [25], Rieffel’s tensor product [23], canonical Hilbert bimodule;
2. von Neumann algebras, correspondences [6], Connes’s tensor product [6, 26], standard form [13];
3. Lie groupoids [17], regular bibundles [11], Hilsum–Skandalis tensor product [12], canonical bibundle;
4. Symplectic groupoids [8], regular [16] symplectic bibundles [29], Hilsum–Skandalis–Xu tensor product [29], canonical symplectic bibundle;
5. Integrable Poisson manifolds [8], regular [16] symplectic bimodules (dual pairs) [28, 14], Xu’s tensor product [29, 15], s-connected and s-simply connected symplectic groupoid.

In the literature one finds notions of Morita equivalence for each of these cases (see [25, 25, 23, 29, 30], respectively). It is gratifying that all these definitions may be derived from a single notion, namely isomorphism of objects in the pertinent bicategory.

The plan is as follows: we first recall the notion of a bicategory, then briefly discuss rings as a warmup, and subsequently explain each of the above cases in some detail, including the pertinent Morita theory.

Acknowledgements The author is indebted to M. Müger and I. Moerdijk for drawing his attention to bicategories.

2 Bicategories

Our notation for (bi)categories (including groupoids) will be that $C$ denotes a (bi)category as a whole, whose class of objects is $C_0$, and whose class of arrows is $C_1$. For $a, b \in C_0$, the Hom-space $(a, b) \subset C_1$ stands for the collection of arrows from $b$ to $a$, so that composition of arrows is a map from $(a, b) \times (b, c)$ to $(a, c)$. A functor $F : C \to D$ decomposes as $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$, subject to the usual axioms. The unit arrow associated to an object $a \in C_0$ is denoted $1_a \in (a, a)$.

Categories can sometimes be “enriched” so as to become 2-categories. Although we will not encounter this precise structure, 2-categories are a special case of bicategories, which is easier to explain, and may serve as a warmup for the latter. The following material comes straight from [18]; also see [1] for an enthusiastic account, with generalizations to $n$-categories.

**Definition 2.1** A 2-category is a category $C$ for which each class of arrows $(a, b), a, b \in C_0$, is itself a category, such that

1. The map $(a, b) \times (b, c) \to (a, c)$ given by composition of arrows in $C_1$ is the 0-component of a functor (where the left-hand side is the Cartesian product of two categories; Mac Lane [18] then speaks of a bifunctor);
2. For each $a \in C_0$, the map $U_a : 1 \to 1_a$ is the 0-component of a functor from the trivial (or terminal) category $T$ (with one object and one arrow) to $(a, a)$.

These axioms simply mean that the given maps can be extended to well-behaved maps between “arrows between arrows”. The first motivating example of a 2-category is the category of all categories, with functors as arrows, and natural
transformations as arrows between arrows. The second is the category of topological spaces, with continuous maps as arrows, and homotopies as arrows between arrows.

In the context of this paper, the category of rings with homomorphisms as arrows should be seen as a 2-category, with intertwiners as arrows between arrows. C*-algebras and von Neumann algebras with (normal) *-homomorphisms, and Lie groupoids with smooth functors provide further examples.

Bicategories generalize 2-categories, as follows.

**Definition 2.2** A bicategory (or weak 2-category) $C$ consists of
- A class of objects $C_0$;
- A category $(a, b)$ for each pair $(a, b) \in C_0 \times C_0$;
- A (bi)functor $\Phi_{a,b,c} : (a, b) \times (b, c) \to (a, c)$ for each triple $(a, b, c)$;
- A functor $U_a : T \to (a, a)$, $U_a(1) = 1_a$, for each $a \in C_0$,

such that
1. The functors $\Phi_{a,c,d} \Phi_{a,b,c}$ and $\Phi_{a,b,d} \Phi_{b,c,d}$ from $(a, b) \times (b, c) \times (c, d)$ to $(a, d)$ are naturally isomorphic;
2. The functors $f \mapsto f 1_b$ (where $f \in (a, b)$) and $id_{(a,b)}$ from $(a, b)$ to itself are naturally isomorphic;
3. The functors $1_a f \mapsto f$ and $id_{(a,b)}$ from $(a, b)$ to itself are naturally isomorphic,

subject to coherence laws stated on p. 282 of [18] (these laws lead to consistency of various orders of bracketing).

A 2-category is a bicategory in which the natural isomorphisms in the above definition are the identity. Many examples of genuine bicategories will be given in what follows. The bifunctors $\Phi$ are sometimes said to define the “horizontal” composition of arrows, in contradistinction to the “vertical” composition of arrows in each of the categories $(a, b)$.

Since the theory of Morita equivalence will involve isomorphism of objects in a bicategory, we should point out that this notion is weaker than in a category.

**Definition 2.3** Two objects $a, b$ in a bicategory are isomorphic, written $a \cong b$, when there exist arrows $f \in (a, b)$ and $g \in (b, a)$ such that $fg \simeq 1_a$ (isomorphism in the usual sense as objects in the category $(a, a)$) and $gf \simeq 1_b$ in $(b, b)$.

### 3 Rings

The following theorem is already mentioned (as an example) in [3, 18].

**Theorem 3.1** For any two rings $R, S$, let $(R, S)$ be the collection of all bimodules $R \rightarrow \leftarrow S$, seen as the class of objects of a category, whose arrows are $R$-$S$ linear maps.

The collection of all rings as objects, bimodules as arrows, (horizontal) composition $(R, S) \times (S, T) \to (R, T)$ given by $\otimes_S$, and the unit arrow $1_S$ in $(S, S)$ given by the canonical bimodule $S \to S \leftarrow S$, is a bicategory $[\text{Rings}]$.

One looks at bimodules as generalized homomorphisms. The precise connection between bimodules and ordinary homomorphisms is as follows.

**Remark 3.2** Given a (unital) homomorphism $\rho : R \to S$, one constructs a bimodule $R \to S \leftarrow S$ by $r(s) = \rho(r)s$ and $(st) = st$. We write this as $R \overset{\rho}{\to} S \leftarrow S$. 
The identity map on objects and the map

\[ R \xrightarrow{\rho} S \quad \mapsto \quad R \xrightarrow{\rho} S \leftarrow S \]

on arrows is a contravariant functor from the category of rings with homomorphisms as arrows into the bicategory \([\text{Rings}]\).

See [2] for the notion of a functor between two bicategories; we here look at the domain of the embedding functor as a 2-category (cf. the preceding section). The direction of arrows is reversed, since in \([\text{Rings}]\), a bimodule \(R \to M \leftarrow S\) is an arrow from \(S\) to \(R\). This could be remedied by mapping a homomorphism \(\rho : R \to S\) to the bimodule \(S \to S \xleftarrow{\rho} R\), but this construction does not work for \(C^*\)-algebras.

Morita's theorems give a necessary and sufficient condition for the representation categories of two rings to be equivalent.

**Definition 3.3** The representation category \(\text{Rep}(R)\) of a ring \(R\) has left \(R\)-modules as objects, and \(R\)-module maps as arrows (that is, given two \(R\)-modules \(M, N\), elements of \((N, M)\) of arrows are homomorphisms from \(M\) to \(N\) as abelian groups that commute with the \(R\)-action).

In the present language, Morita theory starts as follows.

**Theorem 3.4** Two rings are isomorphic objects in the bicategory \([\text{Rings}]\) iff they have equivalent representation categories (where the equivalence functor is required to be additive).

**Remark 3.5** The first condition is explained in Definition 2.3. To be concrete, it here means that there exists an arrow in \((R, S)\) (that is, a bimodule \(R \to M \leftarrow S\)) that is invertible up to isomorphism. Such an \(M\) is often called an \(R\)-\(S\) equivalence bimodule. In other words, there should, in addition, exist an arrow \(S \to M^{-1} \leftarrow R\) in \((S, R)\), such that

\[
\begin{align*}
S &\to M^{-1} \otimes_R M \leftarrow S \cong S \to S \leftarrow S; \\
R &\to M \otimes_S M^{-1} \leftarrow R \cong R \to R \leftarrow R.
\end{align*}
\]

Here the first \(\cong\) stands for isomorphism in the category \((S, S)\), whereas the second denotes isomorphism in \((R, R)\). The first condition simply means that the composition of the two given bimodules, seen as arrows, is isomorphic to the unit arrow in \((S, S)\), etc.

**Proof** The idea of the proof in the “\(\Rightarrow\)” direction is as follows. One constructs a functor \(G : \text{Rep}(S) \to \text{Rep}(R)\) by taking tensor products: on objects one has \(G_0(L) = M \otimes_S L \in \text{Rep}(R)_0\) for \(L \in \text{Rep}(S)_0\), and on arrows one puts, in obvious notation, \(G_1(f) = \text{id} \otimes_S f\). To go in the opposite direction, one repeats the above procedure, in defining a functor \(F : \text{Rep}(R) \to \text{Rep}(S)\) by means of \(F_0(N) = M^{-1} \otimes_R N\), etc. Using Theorem 3.1 and Remark 3.3, it easily follows that \(FG \cong \text{id}_{\text{Rep}(S)}\) and \(GF \cong \text{id}_{\text{Rep}(R)}\).

In the “\(\Leftarrow\)” direction, one constructs \(M\), given an equivalence functor \(G : \text{Rep}(S) \to \text{Rep}(R)\), by putting \(M = G_0(1_S)\). The left \(S\) action on \(S\) is turned into a left \(R\) action on \(M\) by definition of \(G_0\), and the right \(S\) action on \(S\) is turned into a right \(S\) action on \(M\) through \(G_1\), since \(S^{op} \subseteq (S, S) \subseteq \text{Rep}(S)_1\). Thus \(M \in (R, S)\). Similarly, define \(F_0(1_R) \in (S, R)\). The definition of equivalence of categories then trivially implies that the isomorphisms in Remark 3.3 hold, with \(M^{-1} = F_0(1_R)\). For details, cf. no. 12.13 in [2].
The first part of this proof generalizes to all other classes of mathematical objects we study in this paper. The second part, on the other hand, only generalizes when the analogues of the identity arrows \(1_R\) lie in the representation category under consideration, and when there is enough functoriality around to turn the analogues of \(F_0(1_R)\) into a bimodule of the desired sort. These two conditions are met in the case of von Neumann algebras; in all other cases one has to adapt the setting. In this light, the following comment is relevant.

**Remark 3.6** An equivalence functor \(F : \text{Rep}(R) \to \text{Rep}(S)\) is automatically fibered, in the following sense. For each fixed ring \(T\), the functor \(F\) defines an equivalence \(F_T\) between the categories \((R, T)\) and \((S, T)\), natural in \(T\).

Naturality here means that, for any rings \(T, T'\) and homomorphisms \(\varphi : T \to T'\), one has \(\varphi^* F_{T'} = F_T \varphi^*_R\), where \(\varphi^*_R : (R, T') \to (R, T)\) is the induced functor.

In any case, the connection with the usual Morita theory is

**Proposition 3.7** A bimodule \(M \in (R, S)\) is invertible as an arrow in \([\text{Rings}]\) iff

1. \(R \simeq \text{End}_{S^{op}}(M)\);
2. \(M\) is finitely generated projective as an \(R\)- and as an \(S^{op}\)-module.

In that case, the inverse may be taken as \(M^{-1} = \text{Hom}_{S^{op}}(M, S)\).

**Proof** The “\(\Rightarrow\)” claim is part of “Morita I”, cf. no. 12.10.4 in [10] for 1, and 12.10.2 for 2. The converse follows from nos. 12.8(c) and 4.3(c) in [10]. ■

### 4 \(C^*\)-algebras

The \(C^*\)-algebraic analogue of a bimodule for rings is a Hilbert bimodule. This concept involves the theory of Hilbert \(C^*\)-modules, for which we refer to [24, 17].

**Definition 4.1** An \(\mathfrak{A}-\mathfrak{B}\) Hilbert bimodule, where \(\mathfrak{A}\) and \(\mathfrak{B}\) are \(C^*\)-algebras, is a Hilbert \(C^*\) module \(E\) over \(\mathfrak{B}\), along with a nondegenerate *-homomorphism of \(\mathfrak{A}\) into \(L_{\mathfrak{B}}(E)\). We write \(\mathfrak{A} \to E \rightleftharpoons \mathfrak{B}\).

The following example is the \(C^*\)-algebraic version of the ring bimodule \(R \leftrightarrow R\).

**Example 4.2** A \(C^*\)-algebra \(\mathfrak{B}\) may be seen as a Hilbert bimodule \(\mathfrak{B} \to \mathfrak{B} \rightleftharpoons \mathfrak{B}\) over itself, in which \((A, B)_{\mathfrak{B}} = A^* B\), and the left and right actions are given by left and right multiplication, respectively.

Note that the norm in \(\mathfrak{B}\) as a \(C^*\)-algebra coincides with its norm as a Hilbert module because of the \(C^*\)-axiom \(\|A^* A\| = \|A\|^2\).

The \(C^*\)-algebraic analogue of the bimodule tensor product is the (interior) tensor product \(\hat{\otimes}_\mathfrak{B}\) defined by Rieffel [25]; also see [24, 13]. In complete parallel with ring theory (cf. Theorem 3.1), one now has

**Theorem 4.3** For any two \(C^*\)-algebras, let \((\mathfrak{A}, \mathfrak{B})\) be the collection of all Hilbert bimodules \(\mathfrak{A} \to E \rightleftharpoons \mathfrak{B}\), seen as a category, whose arrows are adjointable \(\mathfrak{A}\) linear maps (such maps are automatically bimodule maps).

With (horizontal) composition \((\mathfrak{A}, \mathfrak{B}) \times (\mathfrak{B}, \mathfrak{C}) \to (\mathfrak{A}, \mathfrak{C})\) given by \(\hat{\otimes}_\mathfrak{B}\), and unit arrow in \((\mathfrak{B}, \mathfrak{B})\) given by \(1_{\mathfrak{B}} = \mathfrak{B} \to \mathfrak{B} \rightleftharpoons \mathfrak{B}\), the collection of all \(C^*\)-algebras as objects, and Hilbert bimodules as arrows, forms a bicategory \([C^*]\).

Along the lines of Remark 3.2, we have
Remark 4.4 An \( \mathfrak{A} \)-\( \mathfrak{B} \) Hilbert bimodule may be seen as a generalization of a nondegenerate \( * \)-homomorphism \( \rho : \mathfrak{A} \to \mathfrak{B} \), for given such a \( \rho \) one constructs a Hilbert bimodule \( \mathfrak{A} \to \mathfrak{B} \rightleftharpoons \mathfrak{B} \) by \( \mathcal{A}(B) = \rho(A)B \), and the other operations as in Example 3.2. We write \( \mathfrak{A} \leftarrow \mathfrak{B} \Rightarrow \mathfrak{B} \).

Thus one obtains a contravariant functor from the category of \( C^* \)-algebras with \( * \)-homomorphism as arrows into the bicategory \([C^*]\).

Rieffel, who launched the theory of Morita equivalence of \( C^* \)-algebras [25], defined the representation category \( \text{Rep}(\mathfrak{A}) \) of a \( C^* \)-algebra \( \mathfrak{A} \) as follows.

Definition 4.5 The representation category \( \text{Rep}(\mathfrak{A}) \) of a \( C^* \)-algebra \( \mathfrak{A} \) has nondegenerate representations of \( \mathfrak{A} \) on a Hilbert space as objects, and bounded linear intertwiners as arrows.

The first attempt to adapt Theorem 3.4 to \( C^* \)-algebras now reads

Theorem 4.6 If two \( C^* \)-algebras are isomorphic objects in the bicategory \([C^*]\), then they have equivalent representation categories (in the above sense), where the equivalence functor is required to be linear and \( * \)-preserving on arrows.

The proof is the same as for rings, the bicategory \([C^*]\) replacing \([\text{Rings}]\). Moreover, one should regard a representation of a \( C^* \)-algebra \( \mathfrak{A} \) on a Hilbert space as an \( \mathfrak{A} \)-C Hilbert bimodule.

The \( C^* \)-algebraic version of Proposition 3.7 is as follows.

Proposition 4.7 A bimodule \( \mathcal{M} \in (\mathfrak{A}, \mathfrak{B}) \) is invertible as an arrow in \([C^*]\), so that \( \mathfrak{A} \) and \( \mathfrak{B} \) are isomorphic in \([C^*]\), iff

1. the linear span of the range of \( (\cdot, \cdot)_{\mathfrak{B}} \) is dense in \( \mathfrak{B} \) (in other words, \( \mathcal{M} \rightleftharpoons \mathfrak{B} \) is full);
2. the \( * \)-homomorphism of \( \mathfrak{A} \) into \( \mathcal{L}_{\mathfrak{B}}(\mathcal{E}) \) of Definition 4.1 is an isomorphism \( \mathfrak{A} \rightleftharpoons \mathcal{K}_{\mathfrak{B}}(\mathcal{M}) \). (If \( \mathfrak{A} \) has a unit, this isomorphism will be \( \mathfrak{A} \rightleftharpoons \mathcal{L}_{\mathfrak{B}}(\mathcal{M}) \).)

If \( \mathfrak{A} \to \mathcal{M} \rightleftharpoons \mathfrak{B} \) is invertible, its inverse (up to isomorphism) is \( \mathfrak{B} \to \overline{\mathcal{M}} \rightleftharpoons \mathfrak{A} \), where \( \overline{\mathcal{M}} \) is the conjugate space of \( \mathcal{M} \), on which \( \mathfrak{B} \) acts from the left by \( B : \Psi \mapsto B^* \Psi \) and \( \mathfrak{A} \) acts from the right by \( A : \Psi \mapsto A^* \Psi \). The \( \mathfrak{A} \)-valued inner product on \( \overline{\mathcal{M}} \) is given by \( \langle \Psi, \Phi \rangle_{\mathfrak{A}} = \varphi^{-1}(T^\mathfrak{A}_{\Psi, \Phi}) \), where \( \varphi : \mathfrak{A} \to \mathcal{K}_{\mathfrak{B}}(\mathcal{M}) \) is the pertinent isomorphism.

Here \( \mathcal{K}_{\mathfrak{B}}(\mathcal{M}) \) is the \( C^* \)-algebra of “compact” operators on \( \mathcal{M} \), seen as a Hilbert \( C^* \)-module over \( \mathfrak{B} \) [23, 24, 15]. Equivalent conditions are given in [24].

Proof This is essentially Prop. 2.3 in [27] (Schweizer works with the category of \( C^* \)-algebras with equivalence classes of Hilbert bimodules as arrows, rather than with the bicategory whose arrows are the Hilbert bimodules themselves, but his proof may trivially be adapted to our situation). We are indebted to Paul Muhly for drawing our attention to this result. Muhly in addition provided a second proof in case that the algebras are unital: combine Thm. 4.3 in [3] with Thm. 6.2 in [6].

It is clear that the ring-theoretic proof of a potential “\( \Leftarrow \)” part of Theorem 4.6 cannot immediately be adapted to the present case, since the bimodule \( 1_{\mathfrak{A}} \) is not itself an element of \( \text{Rep}(\mathfrak{A}) \). To remedy this defect, one should enlarge \( \text{Rep}(\mathfrak{A}) \) so that it contains \( \mathfrak{A} \). This has been done by Blecher [3] in the setting of operator spaces, operator modules, and completely bounded maps, but here we choose a different way out, suggested by Remark 3.6, and by a discussion with J. Mrčun.
Theorem 4.8 Two $C^*$-algebras $\mathfrak{A}, \mathfrak{B}$ are isomorphic in $[C^*]$ iff for any $C^*$-algebra $\mathfrak{C}$ one has a category equivalence $([\mathfrak{A}, \mathfrak{C}] \simeq ([\mathfrak{B}, \mathfrak{C}])$, natural in $\mathfrak{C}$.

The particular enlargement of $\text{Rep}(\mathfrak{A})$ therefore involves all representations of $\mathfrak{A}$ on Hilbert modules over arbitrary $C^*$-algebras $\mathfrak{C}$. This theorem may be derived from the results in [5]; it is an open question whether it can be proved by pure $C^*$-algebraic methods.

5 von Neumann algebras

Although one could adapt the theory of Hilbert bimodules for $C^*$-algebras so as to include normality of the actions, as in [24], there is a much simpler approach to bimodules for von Neumann algebras, initiated by Connes [7].

Definition 5.1 Let $\mathfrak{M}, \mathfrak{N}$ be von Neumann algebras. An $\mathfrak{M}$-$\mathfrak{N}$ correspondence $\mathfrak{M} \rightarrow \mathcal{H} \leftarrow \mathfrak{N}$ is given by a Hilbert space $\mathcal{H}$, a normal unital representation $\pi$ of $\mathfrak{M}$ on $\mathcal{H}$, and a normal unital anti-representation $\varphi$ of $\mathfrak{N}$ on $\mathcal{H}$ (in other words, a normal representation of $\mathfrak{N}^{\text{op}}$), such that $\pi(\mathfrak{M}) \subseteq \varphi(\mathfrak{N})'$ (and hence $\varphi(\mathfrak{N}) \subseteq \pi(\mathfrak{M})'$). The correspondence is called faithful when $\pi$ and $\varphi$ are injective.

Matched correspondences may be composed using Connes’s tensor product $\boxtimes_{\mathfrak{N}}$.

Theorem 5.2 For any two von Neumann algebras, let $(\mathfrak{M}, \mathfrak{N})$ be the collection of all correspondences $\mathfrak{M} \rightarrow \mathcal{H} \leftarrow \mathfrak{N}$, seen as the object space of a category, whose arrows are bounded linear bimodule maps.

The collection of all von Neumann algebras as objects, and correspondences as arrows, forms a bicategory $[[W^*]]$ under (horizontal) composition $\boxtimes_{\mathfrak{N}} : (\mathfrak{M}, \mathfrak{N}) \times (\mathfrak{N}, \mathfrak{P}) \rightarrow (\mathfrak{M}, \mathfrak{P})$, and the unit arrow in $(\mathfrak{N}, \mathfrak{N})$ given by the standard form $\mathfrak{N} \rightarrow L^2(\mathfrak{N}) \leftarrow \mathfrak{N}$.

The von Neumann algebraic version of Remarks 3.2 and 4.4 holds verbatim [7].

The theory of Morita equivalence of von Neumann algebras was initiated by Rieffel [25], whose definition of strong Morita equivalence was directly adapted from the $C^*$-algebraic Proposition 4.7. However, the theory of correspondences enables one to rewrite his theory in a way that practically copies the purely algebraic case of rings. In any case, one puts [25]

Definition 5.3 Let $\mathfrak{M}$ be a von Neumann algebra. The representation category $\text{Rep}(\mathfrak{M})$ has normal unital representations on Hilbert spaces as objects, and bounded linear intertwiners as arrows.

The Morita theorem for von Neumann algebras then reads as follows:

Theorem 5.4 Two von Neumann algebras are isomorphic objects in the bicategory $[[W^*]]$ iff their representation categories are equivalent (and the equivalence functor implementing $\simeq$ is linear and $\text{adj}^*$-preserving on arrows).

The proof is the same as for rings, replacing the bicategory $[[\text{Rings}]]$ by $[[W^*]]$. Von Neumann’s bicommutant theorem provides enough functoriality to make all steps of the ring proof to go through for von Neumann algebras.

Rieffel’s original Morita theorem for von Neumann algebras was based on his own definition of strong Morita equivalence (Def. 7.5 in [25]), which is equivalent to isomorphism in $[[W^*]]$ by the following analogue of Propositions 3.7 and 4.7.
Proposition 5.5 Two von Neumann algebras \( \mathcal{M}, \mathcal{N} \) are isomorphic in \( \mathcal{W}^* \) iff there exists a faithful correspondence \( \mathcal{M} \to \mathcal{H} \leftarrow \mathcal{N} \) for which \( \mathcal{M}' \simeq \mathcal{N}' \) (and hence \( (\mathcal{M}'\mathcal{N})' \simeq \mathcal{M} \)).

In that case, the inverse correspondence is \( \mathcal{N} \to \overline{\mathcal{H}} \leftarrow \mathcal{M} \), where \( \overline{\mathcal{H}} \) is the conjugate of \( \mathcal{H} \), and left and right actions are swapped using the involution on \( \mathcal{N} \) and \( \mathcal{M} \).

**Proof** This follows from a combination of Theorem 4.3 above with Thms. 7.9 and 8.15 in [2] and Theorem 2.2 in [2]. A direct proof would be desirable. □

The von Neumann algebraic version of Remark 3.6 holds verbatim.

6 Lie groupoids

As for general categories, our generic notation for groupoids is that \( G_0 \) is the base space of a groupoid \( G \), with source and target maps \( s, t : G_1 \to G_0 \), multiplication \( m : G_2 \to G_1 \) (where \( G_2 = G_1 \ast G_0 \)), inversion \( i : G_1 \to G_1 \), and object inclusion \( i : G_0 \hookrightarrow G_1 \) (this inclusion map will often be taken for granted, in that \( G_0 \) is seen as a subspace of \( G_1 \)).

A Lie groupoid is a groupoid for which \( G_1 \) and \( G_0 \) are manifolds, \( s \) and \( t \) are surjective submersions, and \( m \) and \( i \) are smooth. It follows that \( i \) is an immersion, that \( I \) is a diffeomorphism, that \( G_2 \) is a closed submanifold of \( G_1 \times G_1 \), and that for each \( q \in G_0 \) the fibers \( s^{-1}(q) \) and \( t^{-1}(q) \) are submanifolds of \( G_1 \). References on Lie groupoids that are relevant to the themes in this paper include [4], [5], [6].

The following notion will be important in what follows: A Lie groupoid (or, more generally, a topological groupoid) is called \( s \)-(simply) connected if the fibers of \( s : G_1 \to G_0 \) are (simply) connected.

We now define actions [17] and bimodules [11, 12, 23, 20, 21, 22] for Lie groupoids.

**Definition 6.1**

1. Let \( G \) be a Lie groupoid and let \( M \overset{s}{\to} G_0 \) be smooth. A left \( G \) action on \( M \) (more precisely, on \( \tau \)) is a smooth map \( (x, m) \mapsto xm \) from \( G_s^1 \ast M \) to \( M \) (i.e., one has \( s(x) = \tau(m) \)), such that \( \tau(xm) = t(x) \), \( xm = m \) for all \( x \in G_0 \), and \( x(ym) = (xy)m \) whenever \( s(y) = \tau(m) \) and \( t(y) = s(x) \).

2. A right action of a Lie groupoid \( H \) on \( M \overset{\sigma}{\to} H_0 \) is a smooth map \( (m, h) \mapsto mh \) from \( M \ast H_0 \) to \( M \) that satisfies \( \sigma(mh) = s(h), mh = m \) for all \( h \in H_0 \), and \( (mh)k = m(hk) \) whenever \( s(m) = t(h) \) and \( t(k) = s(h) \).

3. A \( G-H \) bidebundle \( M \) carries a left \( G \) action as well as a right \( H \) action that commute. That is, one has \( \sigma(mh) = \tau(m), \sigma(xm) = \sigma(m), \) and \( (xm)h = x(mh) \) for all \( (m, h) \in M \ast H \) and \( (x, m) \in G \ast M \). On occasion, we simply write \( G \to M \leftarrow H \).

The maps \( \tau \) and \( \sigma \) will sometimes be called the base maps of the given actions.

In the purely algebraic case, one may form a tensor product between two matched groupoid bidebundles \( G \to M \leftarrow H \) and \( H \to N \leftarrow K \), as follows. The pull-back \( M \ast_H N \) carries a diagonal right \( H \) action, given by \( h : (m, n) \mapsto (mh, h^{-1}n) \) (defined as appropriate). The tensor product over \( H \) is then simply the orbit space \( M \ast_H N = (M \ast_H N)/H \).
seen as a $G$-$K$ bibundle under the obvious maps. We name this tensor product after Hilsum–Skandalis \cite{HilsumSkandalis1998}. In the smooth case, one needs further assumptions for this construction to work.

**Definition 6.2** A (left) $G$ bundle $M$ over a manifold $X$ consists of a (left) $G$ action on $M$ and a smooth map $\pi : M \to X$ that is invariant under the $G$ action. Similarly for right actions.

A (left) $G$ bundle $M$ over $X$ is called principal when $\pi$ is a surjective submersion and the map from $G \rtimes_0 M \to M \ast_X M$ given by $(x, m) \mapsto (xm, m)$ is a diffeomorphism. In other words, the action is free (in that $xm = m$ iff $x \in G_0$) and transitive along the fibers of $\pi$, and one has $G \backslash M \simeq X$ through $\pi$.

A $G$-$H$ bibundle $M$ is called left principal when it is principal for the $G$ action with respect to $X = H_0$ and $\pi = \sigma$. Similarly, it is called right principal when it is principal for the $H$ action with respect to $X = G_0$ and $\pi = \tau$.

A $G$-$H$ bibundle $M$ is called regular when it is left principal and the right $H$ action is proper (in that the map $(m, h) \mapsto (m, mh)$ from $M \ast_{H_0} H$ to $M \times M$ is proper).

The definition of a principal bundle is taken from \cite{Brylinski1993, Mackenzie1987, Moerdijk1995}; it is different from the one in \cite{Moerdijk1995}.

Now, if two Lie groupoid bibundles $G \to M \leftarrow H$ and $H \to N \leftarrow K$ are both regular, then the tensor product $M \otimes_H N$ is a manifold, and is even a $G$-$K$ bibundle. For the surjectivity of $\sigma : M \to H_0$ implies that $M \ast_{H_0} N$ is a submanifold of $M \times N$, and the freeness of the $H$ action on $N$ with the properness of the $H$ action on $M$ implies that the diagonal $H$ action on $M \ast_{H_0} N$ is free and proper, so that the quotient space $M \otimes_H N$ is a manifold.

To explain the unit objects in the bicategory $[LG]$ to be defined shortly, first note the following \cite{Brylinski1993, Mackenzie1987}.

**Remark 6.3** A homomorphism $\Psi : G \to H$ is a smooth functor (in that $\Psi_0$ and $\Psi_1$ are smooth). Such a functor defines a $G$-$H$ bibundle with total space $M = G_0 \ast_{H_0} H$, base maps $pr_1 : M \to G_0$ and $s_2 : M \to H_0$, a left $G$ action inherited from the obvious $G$ action $G \ast_{G_0} G_0 \to G_0$, and the right $H$ action is given by multiplication. Here $pr_1$ is projection onto the first coordinate, and $s_2$ is essentially the source projection of $H$.

A special case is provided by $H = G$ and the identity functor $\Psi = \text{id} : G \to G$, leading to the canonical $G$-$G$ bibundle $G$. This arises from the general case by the isomorphism $G \ast_{G_0} G_0 \simeq G$, $(x, q) \mapsto x$, and may be seen as the groupoid version $1_G$ of the ring bimodule $R \to R \leftarrow R$, or of the standard form of a von Neumann algebra.

Thus one obtains a version of Theorem \cite{HilsumSkandalis1998} for Lie groupoids:

**Theorem 6.4** For any two Lie groupoids, let $(G, H)$ be the collection of all regular $G$-$H$ bibundles, seen as the object space of a category, in which an arrow from $M$ to $N$ is a smooth map that intertwines the maps $M \to G_0$, $M \to H_0$ with the maps $N \to G_0$, $N \to H_0$, and in addition intertwines the $G$ and $H$ actions (the latter condition is well defined because of the former).

Then the collection of all Lie groupoids as objects, regular bibundles as arrows, (horizontal) composition $(G, H) \times (H, K) \to (G, K)$ given by the Hilsum–Skandalis
tensor product \( \otimes \), and the unit arrow \( 1_H \) in \( (H, H) \) as defined above, is a bicategory [\( LG \)]

This theorem was inspired by [21], where the case of topological groupoids is discussed in the setting of ordinary categories, so that one works with equivalence classes of the bimodules in question. Dropping our right properness assumption, such classes are sometimes called Hilsen–Skandalis maps, following their introduction in the special case of groupoids defined by foliations [12]. Also cf. [1, 20, 22].

Remark 6.3 leads to a satisfactory groupoid counterpart of Remarks 3.2 and 3.4 which we leave to the reader to spell out.

We now write down the Lie groupoid counterpart of Definition 3.3.

**Definition 6.5** The objects of the representation category \( \text{Rep}(G) \) of a Lie groupoid \( G \) are left \( G \) actions on smooth maps \( M \twoheadrightarrow G_0 \). The space of arrows \( (N, M) \) in (\( N, M \)) between a representation on \( M \twoheadrightarrow G_0 \) and one on \( N \twoheadrightarrow G_0 \) consists of all smooth maps \( \varphi : M \to N \) that satisfy \( \sigma \varphi = \tau \) and intertwine the \( G \) action.

As in Theorem 3.4, we now have

**Theorem 6.6** If two Lie groupoids are isomorphic objects in the bicategory \([LG]\), then their representation categories are equivalent.

The proof is literally the same as for rings, the bicategory \([LG]\) replacing \([Rings]\).

As pointed out by I. Moerdijk, a “\( \Leftrightarrow \)” Morita theorem is possible along the lines of Remark 3.6. The equivalence functor \( \text{Rep}(G) \to \text{Rep}(H) \) provided by an isomorphism \( G \cong H \) automatically induces equivalence functors \( (G, K) \to (H, K) \), natural in \( K \). Unlike the case of rings and von Neumann algebras, this property is not shared by arbitrary equivalence functors, which is the reason why there is no “\( \Leftrightarrow \)” implication in Theorem 3.4. One then has the following statement: two (Lie) groupoids \( G, H \) are strongly Morita equivalent iff for all \( K \) there exist equivalences \( (G, K) \cong (H, K) \), natural in \( K \). (It is, in fact, only necessary to have such equivalences for a small class of \( G-K \) bibundles, namely those in which \( K \) is the trivial groupoid over itself, acting trivially on the middle space.)

**Proposition 6.7** Two Lie groupoids \( G, H \) are isomorphic objects in the bicategory \([LG] \) iff there exists a \( G-H \) bibundle \( M \) with the additional properties:

1. \( M \) is left and right principal;
2. The \( G \) and \( H \) actions are proper.

In that case, an inverse of \( G \to M \leftarrow H \) is the bibundle \( H \to \overline{M} \leftarrow G \), where \( \overline{M} \) is \( M \) as a manifold, seen as a \( H-G \) bibundle with the same base maps, and left and right actions interchanged using the inverse in \( G \) and \( H \).

This reproduces the definition of Morita equivalence for (Lie) groupoids given in [23]. Other definitions that are in use are equivalent to this one; see, e.g., [1, 20, 22, 23], and refs. therein.

**Proof** The second part of the proposition, which at the same time proves the “\( \Leftarrow \)” claim in the first part, is proved by the argument following Def. 2.1 in [23].

For the “\( \Rightarrow \)” claim in the first part, we are given a regular bibundle \( G \to M \leftarrow H \), with regular inverse \( H \to M^{-1} \leftarrow G \). This leads to two isomorphisms, as displayed in Remark 3.5 for rings. From \( M \otimes_H M^{-1} \simeq G \) as \( G-G \) bibundles we infer that the left \( G \) action on \( M \otimes_H M^{-1} \) is proper (since the canonical left \( G \) action on \( G \) is), which by reductio ad absurdum implies that the left \( G \) action on \( M \) is
proper. Since the target projection \( t : G_1 \to G_0 \) is a surjective submersion, so is the map \( M \otimes_H M \to G_0 \), and therefore \( \tau : M \to G_0 \) must be a surjective submersion as well. In similar vein, the isomorphism \( M^{-1} \otimes_G M \cong H \) as \( H \)-\( H \) bibundles implies that the right \( H \) action on \( M \) is free and transitive on the \( \tau \)-fibers. Together with the assumed regularity of the bibundle \( G \to M \leftarrow H \), we have now proved both conditions in the proposition.

7 Symplectic groupoids

We now specialize to symplectic groupoids, which will be crucial for a bicategorical understanding of Poisson manifolds, and are of interest in themselves. Our basic references are [19, 8].

**Definition 7.1** A symplectic groupoid is a Lie groupoid \( \Gamma \) for which \( \Gamma_1 \) is a symplectic manifold, with the property that the graph of \( \Gamma_2 \subset \Gamma \times \Gamma \) is a Lagrangian submanifold of \( \Gamma \times \Gamma \times \Gamma \).

The notion of a bibundle for symplectic groupoids is an adaptation of Definition 6.1, now also involving the idea of a symplectic groupoid action [19].

**Definition 7.2** An action of a symplectic groupoid \( \Gamma \) on a symplectic manifold \( S \) is called symplectic when the graph of the action in \( \Gamma \times S \times S \) is Lagrangian. Let \( \Gamma, \Sigma \) be symplectic groups. A (regular) symplectic \( \Gamma \)-\( \Sigma \) bibundle consists of a symplectic space \( S \) that is a (regular) bibundle as in Definition 6.4, with the additional requirement that the two groupoid actions be symplectic.

The tensor product of two matched regular bibundles for symplectic groupoids is then defined exactly as in the general (non-symplectic) case. Hence we obtain a symplectic version of Theorem 6.4:

**Theorem 7.3** For any two symplectic groupoids \( \Gamma, \Sigma \), let \( (\Gamma, \Sigma) \) be the collection of all regular symplectic \( \Gamma \)-\( \Sigma \) bibundles, seen as the class of objects of a category, with arrows as described in Proposition 6.4, with the additional requirement that the maps preserve the symplectic form.

Then the collection of all symplectic groupoids as objects, regular symplectic bibundles as arrows, (horizontal) composition \( (\Gamma, \Sigma) \times (\Sigma, \Upsilon) \to (\Gamma, \Upsilon) \) given by \( \otimes \Sigma \), and the unit arrow \( 1_{\Gamma} \) in \( (\Gamma, \Gamma) \) given by the canonical bibundle over \( \Gamma \), is a bicategory \([SG]\).

The discussion of Morita equivalence for symplectic groupoid is an obvious adaptation of the Lie groupoid case.

**Definition 7.4** The objects of the representation category \( \text{Rep}^s(\Gamma) \) of a symplectic groupoid \( \Gamma \) are symplectic left \( \Gamma \) actions on smooth maps \( \tau : S \to \Gamma_0 \), where \( S \) is symplectic. The space of arrows is as in Definition 6.5, with the additional requirement that \( \varphi \) be a complete Poisson map (cf. the next section).

It should be mentioned that \( \tau \) is necessarily a complete Poisson map [19, 8]. As in Theorem 6.6, we again have

**Theorem 7.5** If two symplectic groupoids are isomorphic in \([SG]\), then their representation categories \( \text{Rep}^s(\cdot) \) are equivalent.

Xu’s original Morita theorem for symplectic groupoids is now equivalent to the following statement, which is proved like Proposition 6.7.
Proposition 7.6 Two symplectic groupoids \( \Gamma, \Sigma \) are isomorphic in \([SG]\) iff as Lie groupoids they satisfy the condition stated in Proposition 7.1, with the additional requirement that the middle space \( M = S \) is symplectic, and that the \( \Gamma \) and \( \Sigma \) actions are symplectic (see Definition 7.3).

In that case, an inverse bibundle is \( S^- \), seen as a \( \Sigma, \Gamma \) bibundle by interchanging left and right actions through the groupoid inverse.

8 Poisson manifolds

Poisson algebras are the classical analogues of \( C^* \)-algebras and von Neumann algebras; see, e.g., [13].

Definition 8.1 A Poisson algebra is a commutative associative algebra \( A \) (over \( \mathbb{C} \) or \( \mathbb{R} \)) endowed with a Lie bracket \{, \} such that each \( f \in A \) defines a derivation \( X_f \) on \( A \) (as a commutative algebra) by \( X_f(g) = \{ f, g \} \). In other words, the Leibniz rule \( \{ f, gh \} = \{ f, g \}h + g\{ f, h \} \) holds.

A Poisson manifold is a manifold \( P \) with a Lie bracket on \( C^\infty(P) \) such that the latter becomes a Poisson algebra under pointwise multiplication.

We write \( P^- \) for \( P \) with minus a given Poisson bracket. Not all Poisson algebras are of the form \( A = C^\infty(P) \) (think of singular reduction), but we specialize to this case. The derivation \( X_f \) then corresponds to a vector field on \( P \), called the Hamiltonian vector field of \( f \). If the span of all \( X_f \) (at each point) is \( TP \), then \( P \) is symplectic.

The classical counterpart of a Hilbert bimodule or a correspondence is a symplectic bimodule. First, recall that a Poisson map \( q : S \rightarrow Q \) between Poisson manifolds is a smooth map whose pullback \( q^* : C^\infty(Q) \rightarrow C^\infty(S) \) is a homomorphism of Poisson algebra. A Poisson map \( q \), or rather its pullback \( q^* \), defines a pair of maps. The first of these, \( q_c^* \), defines \( C^\infty(S) \) as a module for \( C^\infty(Q) \) as a commutative algebra through \( q^*_c(f)g = (q^*f)g \). The second, \( q_p^* \), maps \( C^\infty(Q) \) into the Lie algebra \( \text{Der}(C^\infty(S)) \) of derivations of \( C^\infty(S) \) through \( q^*_p(f) = X_{q^*f} \), or, in other words, \( q^*_p(f)g = \{ q^*f, g \} \). This map is a Lie algebra homomorphism by definition of a Poisson map.

A Poisson map \( q : S \rightarrow Q \) is called complete when, for every \( f \in C^\infty(Q) \) with complete Hamiltonian flow, the Hamiltonian flow of \( q^*f \) on \( S \) is complete as well (that is, defined for all times). Requiring a Poisson map to be complete is a classical analogue of the self-adjointness condition on a representation of a \( C^* \)-algebra [13]. The following definition goes back to Weinstein [28] and Karasev [14].

Definition 8.2 A symplectic bimodule \( Q \leftarrow S \rightarrow P \) consists of a symplectic manifold \( S \), Poisson manifolds \( Q \) and \( P \), and complete Poisson maps \( q : S \rightarrow Q \) and \( p : S \rightarrow P^- \), such that \( \{ q^*f, p^*g \} = 0 \) for all \( f \in C^\infty(Q) \) and \( g \in C^\infty(P) \).

In order to give a bicategorical description of Poisson manifolds, we need to explain the connection between Poisson manifolds and symplectic groupoids. The compatibility condition in Definition 7.1 expresses the idea that groupoid multiplication should be a Poisson map. This has the following consequence [3, 13].

Proposition 8.3 Let \( \Gamma \) be a symplectic groupoid. There exists a unique Poisson structure on \( \Gamma_0 \) such that \( t \) is a complete Poisson map and \( s \) is a complete anti-Poisson map. Hence \( \Gamma_0 \leftarrow \Gamma \rightarrow \Gamma_0 \) is a symplectic bimodule.

The objects in the bicategory \([\text{Poisson}]\) will be of the following form [8, 16].
Definition 8.4 A Poisson manifold $P$ is called integrable when there exists an $s$-connected and $s$-simply connected symplectic groupoid $\Gamma(P)$ over $P$ (so that $P$ is isomorphic to $\Gamma(P)_0$ as a Poisson manifold).

It can be shown that $\Gamma(P)$ is unique up to isomorphism (see Thm. 3.21 in [10]).

Running ahead of the Morita theory for Poisson manifolds, we here need

Definition 8.5 The representation category $\text{Rep}(P)$ of a Poisson manifold has complete Poisson maps $q : S \to P$, where $S$ is some symplectic space, as objects, and complete Poisson maps $\varphi : S_1 \to S_2$, where $q_2 \varphi = q_1$, as arrows.

This definition goes back to Weinstein (see, e.g., [28, 8]), although we use a more straightforward choice of arrows. In any case, one now has the following extraordinary relationship between Poisson manifolds and symplectic groupoids.

Theorem 8.6 If $\Gamma(P)$ and $P$ are related as in Definition 8.4, then the categories $\text{Rep}(\Gamma(P))$ and $\text{Rep}(P)$ are equivalent.

This theorem is independently due to Dazord and Xu [9, 30]; key ingredients of the proof already appeared in [8, 19].

Corollary 8.7 1. Let $P$ and $Q$ be integrable Poisson manifolds, with associated $s$-connected and $s$-simply connected symplectic groupoids $\Gamma(P)$ and $\Gamma(Q)$; cf. Definition 8.4.
   There is a natural bijective correspondence between symplectic bimodules $Q \leftarrow S \to P$ and symplectic bibundles $\Gamma(Q) \to S \leftarrow \Gamma(P)$.
   2. Let $R$ be a third integrable Poisson manifold, with associated $s$-connected and $s$-simply connected symplectic groupoid $\Gamma(R)$, and let $Q \leftarrow S_1 \to P$ and $P \leftarrow S_2 \to R$ be symplectic bimodules.
   The Hilsum–Skandalis tensor product $S_1 \circ \Gamma(P) S_2$ of the associated symplectic bibundles $\Gamma(Q) \to S_1 \leftarrow \Gamma(P)$ and $\Gamma(P) \to S_2 \leftarrow \Gamma(R)$ is a $Q$-$R$ symplectic bimodule.

This follows from Theorem 8.6. In the setting of Poisson manifolds and symplectic groupoids, the Hilsum–Skandalis tensor product was introduced by Xu [29]; also cf. [19]. It may alternatively be described without groupoids using the special symplectic reduction procedure of [13], in which case we write it as $S_1 \circ P S_2$ (cf. $S_1 \circ \Gamma(P) S_2$). This remark provides the connection between the mathematical structures in this paper and the physics of constrained dynamical systems.

In the following theorem, given two $Q$-$P$ symplectic bimodules $Q \triangleright S_i \triangleleft P$ for $i = 1, 2$ we say that a smooth map $\varphi : S_1 \to S_2$ is a $Q$-$P$ map when $q_2 \varphi = q_1$ and $p_2 \varphi = p_1$. Also, we call a $Q$-$P$ symplectic bimodule regular when the associated $\Gamma(Q)$-$\Gamma(P)$ symplectic bibundle is regular (cf. Corollary 8.7.1).

Theorem 8.8 For any two integrable Poisson manifolds, let $(P, Q)$ consist of all regular symplectic bimodules $P \leftarrow S \to Q$. The class $(Q, P)$ consists of the objects of a category, whose arrows are complete $Q$-$P$ Poisson maps
   Then the class of all integrable Poisson manifolds as objects, regular symplectic bimodules as arrows, (horizontal) composition of arrows $(Q, P) \times (P, R) \to (Q, R)$ given by the Hilsum–Skandalis–Xu tensor product as in Corollary 8.7, and the unit arrow $1_P \in (P, P)$ given by $\Gamma(P)$, forms a bicategory $\text{Poisson}$.

Proof This follows from Theorem 7.3 and Corollary 8.7.
It would be desirable to translate the stated conditions on the $\Gamma(P)$ and $\Gamma(Q)$ actions into conditions on the maps $p$ and $q$.

The Poisson analogue of Remarks 3.2, 3.4, etc. is as follows:

**Remark 8.9** A complete Poisson map $\rho : P \to Q$ defines a symplectic bimodule $Q \xleftarrow{p} \Gamma(P) \to P$, so that a symplectic bimodule may be seen as a generalized Poisson map. The identity map on objects and the map

$$P \xleftarrow{\rho} Q \quad \mapsto \quad Q \xleftarrow{\rho} \Gamma(P) \to P$$

on arrows is a covariant functor from the category of Poisson manifolds with Poisson maps as arrows into the bicategory $[\text{Poisson}]$.

The theory of Morita equivalence of Poisson manifolds was initiated by Xu [30], and may now be reformulated as follows.

**Theorem 8.10** If two integrable Poisson manifolds are isomorphic objects in the bicategory $[\text{Poisson}]$, then their representation categories are equivalent.

This is proved as for rings. Interestingly, one has [30, 16]

**Proposition 8.11** A Poisson manifold $P$ is integrable iff it is Morita equivalent to itself.

Xu’s original definition of Morita equivalent of Poisson manifolds [30] now becomes an inference.

**Proposition 8.12** The following statements are equivalent:

1. Two integrable Poisson manifolds $P, Q$ are isomorphic objects in $[\text{Poisson}]$.
2. There is a symplectic bimodule $Q \xleftarrow{S} P$ with the following additional properties:
   (a) The maps $p : S \to P$ and $q : S \to Q$ are surjective submersions;
   (b) The level sets of $p$ and $q$ are connected and simply connected;
   (c) The foliations of $S$ defined by the levels of $p$ and $q$ are mutually symplectically orthogonal (in that the tangent bundles to these foliations are each other’s symplectic orthogonal complement).
3. The symplectic groupoids $\Gamma(P)$ and $\Gamma(Q)$ are Morita equivalent.

In that case, an inverse of $Q \xleftarrow{S} P$ is $P \xleftarrow{S} Q$, with the same maps.

**Proof** The equivalence of 2 and 3 is Thm. 3.2 in [30].

By Theorem 7.6, statement 3 is equivalent to $\Gamma(Q) \cong \Gamma(P)$ in $[\text{SG}]$, which by Definition 2.3 means that there is an invertible bibundle $\Gamma(Q) \to S \xleftarrow{} \Gamma(P)$ in $[\text{SG}]$. By Corollary 8.7 this is equivalent to the invertibility of the associated symplectic bimodule $Q \xleftarrow{S} P$ in $[\text{Poisson}]$, which by Definition 2.3 is statement 1. ■

**References**

[1] J.C. Baez, and J. Dolan, *Categorification*, in: Higher category theory (Evanston, IL, 1997), 1–36, Contemp. Math. 230, American Mathematical Society, Providence, RI, 1998.

[2] M. Baillet, Y. Denizeau, and J.-F. Havet, Indice d’une espérance conditionnelle, Compositio Math. 66 (1988), 199–236.

[3] J. Bénabou, Introduction to bicategories, Lecture Notes in Mathematics 47 (1967), 1–77.

[4] D. Blecher, A new approach to Hilbert $C^*$-modules, Math. Ann. 307 (1997), 253–290.

[5] D. Blecher, On Morita’s fundamental theorem for $C^*$-algebras, e-print math.OA/9906082.

[6] D. Blecher, P.S. Muhly, and V.I. Paulsen, Categories of operator modules (Morita equivalence and projective modules), Mem. Amer. Math. Soc. 143, no. 681 (2000).
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[7] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[8] A. Coste, P. Dazord, and A. Weinstein, *Groupoïdes symplectiques*, Publ. Dépt. Math. Univ. C. Bernard–Lyon 12A (1987), 1–62.

[9] P. Dazord, *Groupoïdes symplectiques et troisième théorème de Lie “non linéaire”*, Lecture Notes in Math. 1416 (1990), 39–74.

[10] C. Faith, *Algebra: Rings, Modules and Categories. I*, Springer, New York, 1973.

[11] A. Haefliger, *Groupoïdes d’holonomie et classifiants*, Astérisque 116 (1984), 70–97.

[12] M. Hilsum and G. Skandalis, *Morphismes K-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes)*. Ann. Sci. École Norm. Sup. (4) 20 (1987), 325–390.

[13] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras II. Advanced Theory*, Academic Press, New York, 1986.

[14] M.V. Karasev, *The Maslov quantization conditions in higher cohomology and analogs of notions developed in Lie theory for canonical fibre bundles of symplectic manifolds. I, II.*, Selecta Math. Soviet. 8 (1989), 213–234, 235–258.

[15] N.P. Landsman, *Mathematical Topics Between Classical and Quantum Mechanics*, Springer, New York, 1998.

[16] N.P. Landsman, *Quantized reduction as a tensor product*, in: Deformation quantization of singular Marsden–Weinstein quotients, Proc. Oberwolfach RIW Workshop, April 1999, eds. M. Bordemann, N.P. Landsman, M. Pflaum, and M. Schlichenmaier, to appear.

[17] K.C.H. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, Cambridge, 1987.

[18] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer, New York, 1998.

[19] K. Mikami and A. Weinstein, *Moments and reduction for symplectic groupoids*, Publ. RIMS Kyoto Univ. 24 (1988), 121–140.

[20] I. Moerdijk, *Classifying toposes and foliations*, Ann. Inst. Fourier (Grenoble) 41 (1991), 189–209.

[21] J. Mrčun, *Functoriality of the bimodule associated to a Hilsum–Skandalis map*, K-Theory 18 (1999), 235–253.

[22] J. Mrčun, *Stability and invariants of Hilsum–Skandalis maps*, Ph.D Thesis, Univ. of Utrecht (1996).

[23] P. Muhly, J. Renault, and D. Williams, *Equivalence and isomorphism for groupoid C*-algebras*, J. Operator Th. 17 (1987), 3–22.

[24] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace C*-algebras*, American Mathematical Society, Providence, RI, 1998.

[25] M.A. Rieffel, *Morita equivalence for C*-algebras and W*-algebras*, J. Pure Appl. Alg. 5 (1974), 51–96.

[26] J.-L. Sauvageot, *Sur le produit tensoriel relatif d’espaces de Hilbert*, J. Operator Theory 9 (1983), 237–252.

[27] J. Schweizer, *Crossed products by equivalence bimodules*, Univ. Tübingen preprint (1999).

[28] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom. 18 (1983), 523–557. Err. ibid. 22 (1985), 255.

[29] P. Xu, *Morita equivalent symplectic groupoids*, pp. 291–311 in: P. Dazord and A. Weinstein (eds.), *Symplectic Geometry, Groupoids, and Integrable Systems*, Springer, New York, 1991.

[30] P. Xu, *Morita equivalence of Poisson manifolds*, Commun. Math. Phys. 142 (1991), 493–509.