Normalized solutions for a Schrödinger equation with critical growth in $\mathbb{R}^N$

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Abstract

In this paper we study the existence of normalized solutions to the following nonlinear Schrödinger equation with critical growth

$$
\begin{cases}
-\Delta u + \lambda u = f(u), & \text{in } \mathbb{R}^N, \\
u > 0, & \int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}
$$

where $a > 0$, $\lambda < 0$ and $f$ has an exponential critical growth when $N = 2$, and $f(t) = \mu|u|^{q-2}u + |u|^{2^*-2}u$ with $q \in (2 + \frac{4}{N}, 2^*)$, $\mu > 0$ and $2^* = \frac{2N}{N-2}$ when $N \geq 3$. Our main results complement some recent results for $N \geq 3$ and it is totally new for $N = 2$.

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Keywords: Normalized solutions, Nonlinear Schrödinger equation, Variational methods, Critical exponents.

1 Introduction

This paper concerns the existence of normalized solutions to the following nonlinear Schrödinger equation with critical growth

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The equation (1.1) arises when ones look for the solutions with prescribed mass for the nonlinear Schrödinger equation

\[
i\frac{\partial \psi}{\partial t} + \Delta \psi + g(|\psi|^2)\psi = 0 \quad \text{in } \mathbb{R}^N.
\]

A stationary wave solution is a solution of the form \(\psi(t, x) = e^{i\lambda t}u(x)\), where \(\lambda \in \mathbb{R}\) and \(u : \mathbb{R}^N \to \mathbb{R}\) is a time-independent that must solve the elliptic problem

\[
-\Delta u + \lambda u = g(|u|^2)u, \quad \text{in } \mathbb{R}^N.
\]

(1.2)

For some values of \(\lambda\) the existence of nontrivial solutions for (1.2) are obtained as the critical points of the action functional \(J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}\) given by

\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda |u|^2) \, dx - \int_{\mathbb{R}^N} G(u) \, dx,
\]

where \(G(t) = \int_0^t g(s) \, ds\). In this case the particular attention is devoted to the least action solutions, namely solutions minimizing \(J_\lambda\) among all non-trivial solutions.

Another important way to find the nontrivial solutions for (1.2) is to search for solutions with prescribed mass, and in this case \(\lambda \in \mathbb{R}\) is part of the unknown. This approach seems to be particularly meaningful from the physical point of view, because there is a conservation of mass.

The present paper has been motivated by a seminal paper due to Jeanjean [26] that studied the existence of normalized solutions for a large class of Schrödinger equations of the type

\[
\begin{cases}
-\Delta u + \lambda u = g(u), & \text{in } \mathbb{R}^N, \\
u > 0, & \int_{\mathbb{R}^N} |u|^2 \, dx = a^2,
\end{cases}
\]

(1.3)

with \(N \geq 2\), where function \(g : \mathbb{R} \to \mathbb{R}\) is an odd continuous function with subcritical that satisfies some technical conditions. One of these conditions is the following: \(\exists (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}\) satisfying

\[
\begin{cases}
\frac{2N+4}{N} < \alpha \leq \beta < \frac{2N}{N-2}, & \forall N \geq 3, \\
\frac{2N+4}{N} < \alpha \leq \beta, & \text{for } N = 1, 2,
\end{cases}
\]

such that

\[
\alpha G(s) \leq g(s)s \leq \beta G(s) \quad \text{with } G(s) = \int_0^s g(t) \, dt.
\]

As an example of a function \(g\) that satisfies the above condition is \(g(s) = |s|^{q-2}s\) with \(q \in (2 + \frac{4}{N}, 2^*)\) when \(N \geq 3\) and \(q > 4\) if \(N = 2\). In order to overcome the loss of compactness of the Sobolev embedding in whole \(\mathbb{R}^N\), the author worked on the space \(H^1_{rad}(\mathbb{R}^N)\) to get some compactness. However the most important and interesting point, in our opinion, is the fact that Jeanjean did not work directly with the energy functional \(I : H^1(\mathbb{R}^N) \to \mathbb{R}\) associated with the problem (1.3) given by
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} G(u) dx. \]

In his approach, he considered the functional \( \tilde{I} : H^1(\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R} \) given by
\[
\tilde{I}(u, s) = \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{e^{Ns}} \int_{\mathbb{R}^N} G(e^{Ns}u(x)) dx.
\]

After a careful analysis, it was proved that \( I \) and \( \tilde{I} \) satisfy the mountain pass geometry on the manifold
\[ S(a) = \{ u \in H^{1,2}(\mathbb{R}^N) : |u|_2 = a \}, \]
and their mountain pass levels are equal, which we denote by \( \gamma(a) \). Moreover, using the properties of \( \tilde{I}(u, s) \), it was obtained a \((PS)\) sequence \( (u_n) \) to \( I \) associated with the mountain pass level \( \gamma(a) \) that is bounded in \( H^{1, rad}(\mathbb{R}^N) \). Finally, after some estimates, the author was able to prove that the weak limit of \( (u_n) \), denoted by \( u \), is nontrivial, \( u \in S(a) \) and
\[-\Delta u - g(u) = \lambda_a u \quad \text{in} \quad \mathbb{R}^N,
\]
for some \( \lambda_a < 0 \). An example of a nonlinearity explored in [26] we cite
\[(g_0) \quad g(t) = \mu |t|^{q-2} t \quad t \in \mathbb{R},\]
where \( \mu > 0 \) and \( q \in (2, 2^*) \). We recall that the study of the normalized problem despite being more convenient in the application, this bring some difficulties such as Nehari manifold method can not be applied because the constants \( \mu \) and \( w \) is unknow in the problem; it is necessary to prove that the weak limit belongs to the constrained manifold; and also it brings some difficult to apply some usual approach for get the boundedness of the Palais Smale sequence.

We recall that the number \( \bar{q} := 2 + \frac{4}{N} \) is called in the literature as \( L^2 \)-critical exponent, which come from Gagliardo Nirenberg inequality, (see [15] Theorem 1.3.7, page 9]. If \( g \) is of the form \( g_0 \) with \( q \in (2, \bar{q}) \), we say that the problem is \( L^2 \)-subcritical, while in the case \( q \in (\bar{q}, 2^*) \) the problem is \( L^2 \)-supercritical. Associated with the \( L^2 \)-supercritical, we would like to cite [10], where the authors studied a problem involving vanishing potential. In the purely \( L^2 \)-critical case, that is, \( q = 2 + \frac{4}{N} \), related problems were studied in [16, 32].

In [35], Soave studied the normalized solutions for the nonlinear Schrödinger equation (1.1) with combined power nonlinearities of then type follows
\[(f_0) \quad f(t) = \mu |t|^{q-2} t + |t|^{p-2} t, \quad t \in \mathbb{R},\]
where
\[ 2 < q \leq 2 + \frac{4}{N} \leq p < 2^*, \ p \neq q \quad \text{and} \quad \mu \in \mathbb{R}. \]
He showed that interplay between subcritical, critical and supercritical nonlinearities strongly affects the geometry of the functional and existence and properties of ground states.

Recently some authors have considered the problem (1.1) with \( f \) of the form \( (f_0) \) but with \( p = 2^* \), which implies that \( f \) has a critical growth in the Sobolev sense. In [27], the existence
of a ground state normalized solution is obtained as minimizer of the constrained functional assuming that $q \in (2, 2 + \frac{4}{N})$. While in [29] a multiplicity result is established, where the second solution is not a ground state. For the general case $q \in (2, 2^*)$, we would like to mention Soave [36], where the existence result is obtained by imposing that $\mu a^{(1 - \gamma_p)q} < \alpha$, where $\alpha$ is a specific constant that depends on $N$ and $q$ and $\gamma_p = \frac{N(p - 2)}{2p}$. We have seen that the results in the paper are deeply dependent on the assumptions about $a$ and $\mu$, because by the Pohozaev identity the problem (1.1) does not have any solution if $\lambda = -a \geq 0$ and $\mu > 0$. Still related to the case $q \in (2 + \frac{4}{N}, 2^*)$, we would like to refer [2, 3] where the existence of least action solutions was proved with $\mu > 0$ and $N \geq 4$, and for $N = 3$ smallness of the constants $\lambda$ and $\mu$ are required.

We recall that elliptic problems involving critical Sobolev exponent were studied many researchers after appeared the pioneering paper by Brezis and Nirenberg[13], which have had many progresses in several directions. We would like to mention the excellent book [41], for a review on this subject. In our setting, since if $\lambda = 0$, the problem (1.1) does not have any solution for any $\mu$, then taking $\lambda$ as a Lagrange multiplier, it is able to get solution, by combining the arguments made in [13] with the concentration compactness principle. For the reader interested in normalized solutions for the Schrödinger equations, we would also like to refer [3, 11, 12, 17, 25, 28, 31, 33, 37, 38, 10] and references therein.

Our main result for the Sobolev critical case is the following:

**Theorem 1.1.** Assume that $f$ is of the form $(f_0)$ with $p = 2^*$ and $q \in (2 + \frac{4}{N}, 2^*)$. Then, there exists $\mu^* = \mu^*(a) > 0$ such that the problem (1.1) admits a couple $(u_\alpha, \lambda_\alpha) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions such that $\int_{\mathbb{R}^N} |u|^2 \, dx = a^2$ and $\lambda_\alpha < 0$ for all $\mu \geq \mu^*$.

The above theorem complements the results found in [36] for the $L^2$—supercritical case, because in that paper $\mu \in (0, a^{-(1 - \gamma_p)q} \alpha)$ for some $\alpha > 0$, then $\mu$ cannot be large enough, while in our paper $\mu$ can be large enough, because $\mu \in [\mu^*(a), +\infty)$. Here, we used a different approach from that explored in [36], because we work directly with the mountain pass geometry and concentration-compactness principle due to Lions [30], while in [36], Soave employed minimization technique and properties of the Pohozaev manifold.

Motivated by the research made in the critical Sobolev case, in this paper we also study the exponential critical growth for $N = 2$, which is a novelty for this type of problem. To the best our knowledge we have not found any reference involving normalizing problem with the exponential critical growth. In this case, we assume that $f$ is a continuous function that satisfies the following conditions:

1. \((f_1)\) $\lim_{t \to 0} \frac{|f(t)|}{|t|^\tau} = 0$ as $t \to 0$, for some $\tau > 3$;

2. \((f_2)\) $\lim_{t \to +\infty} \frac{f(t)}{e^{at^2}} = \begin{cases} 0, & \text{for } \alpha > 4\pi, \\ +\infty, & \text{for } 0 < \alpha < 4\pi; \end{cases}$

3. \((f_3)\) there exists a positive constant $\theta > 4$ such that $0 < \theta F(t) \leq tf(t)$, $\forall t \neq 0$, where $F(t) = \int_0^t f(s) \, ds$;
(f_4) the function \( t \to f(t)/t \) is increasing;

(f_5) there exist constants \( p > 4 \) and \( \mu > 0 \) such that

\[
f(t) \geq \mu t^{p-1} \quad \text{for all } t > 0.
\]

Our main result is as follows:

**Theorem 1.2.** Assume that \( f \) satisfies (f_1) - (f_5). If \( a \in (0, 1) \), then there exists \( \mu^* = \mu^*(a) > 0 \) such that the problem (1.1) admits a couple \( (u_a, \lambda_a) \in H^1(\mathbb{R}^2) \times \mathbb{R} \) of weak solutions with \( \int_{\mathbb{R}^2} |u|^2 \, dx = a^2 \) and \( \lambda_a < 0 \) for all \( \mu \geq \mu^* \).

In the proof of Theorem 1.1 and Theorem 1.2 we borrow the ideas developed in Jeanjean [26]. The main difficulty in the proof of these theorems is associated with the fact that we are working with critical nonlinearities in whole \( \mathbb{R}^N \). As above mentioned, in the proof of Theorem 1.1 the concentration-compactness principle due to Lions [30] is crucial in our arguments, while in the proof Theorem 1.2 the Trudinger-Moser inequality developed by Cao [14] plays an important rule in a lot of estimates. Moreover, in the proofs of these theorems we will work on the space \( H^1_{rad}(\mathbb{R}^N) \), because it has very nice compact embeddings.

**Notation:** From now on in this paper, otherwise mentioned, we use the following notations:

- \( B_r(u) \) is an open ball centered at \( u \) with radius \( r > 0 \), \( B_r = B_r(0) \).
- \( C, C_1, C_2, ... \) denote any positive constant, whose value is not relevant.
- \( \| \cdot \|_p \) denotes the usual norm of the Lebesgue space \( L^p(\mathbb{R}^N) \), for \( p \in [1, +\infty] \), \( \| \cdot \| \) denotes the usual norm of the Sobolev space \( H^1(\mathbb{R}^N) \).
- \( o_n(1) \) denotes a real sequence with \( o_n(1) \to 0 \) as \( n \to +\infty \).

## 2 Normalized solutions: The Sobolev critical case for \( N \geq 3 \)

In order to follow the same strategy of [26], we need the following definitions to introduce our variational procedure.

(1) \( S(a) = \{ u \in H^1(\mathbb{R}^N) : |u|_2 = a \} \) is the sphere of radius \( a > 0 \) defined with the norm \( | \cdot |_2 \).

(2) \( J : H^1(\mathbb{R}^N) \to \mathbb{R} \) with

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx,
\]

where

\[
F(t) = \frac{\mu}{q} |t|^q + \frac{1}{2 \ast} |t|^{2 \ast}, \quad t \in \mathbb{R}.
\]
Hereafter, $H = H^1(\mathbb{R}^N) \times \mathbb{R}$ is equipped with the scalar product
\[
\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_{H^1(\mathbb{R}^N)} + \langle \cdot, \cdot \rangle_{\mathbb{R}}
\]
and corresponding norm
\[
\| \cdot \|_H = (\| \cdot \|^2_{H^1(\mathbb{R}^N)} + \| \cdot \|^2_{\mathbb{R}})^{1/2}.
\]
In this section, $f$ denotes the function $f(t) = \mu t^{q-2} t + |t|^{2^* - 2} t$ with $t \in \mathbb{R}$, and so, $F(t) = \int_0^t f(s) \, ds$.

(3) $\mathcal{H} : H^1(\mathbb{R}^N) \times \mathbb{R} \to H^1(\mathbb{R}^N)$ with
\[
\mathcal{H}(u,s)(x) = e^{\frac{Ns}{2}} u(e^s x).
\]

(4) $\tilde{J} : H^1(\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}$ with
\[
\tilde{J}(u,s) = \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{e^{Ns}} \int_{\mathbb{R}^N} F(e^{\frac{Ns}{2}} u(x)) \, dx
\]
or
\[
\tilde{J}(u,s) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} F(v(x)) \, dx = J(v) \quad \text{for } v = \mathcal{H}(u,s)(x).
\]

Throughout this section $S$ denotes the following constant
\[
S = \inf_{\substack{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} \ni u \in L^{2^*}(\mathbb{R}^N)}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{\frac{1}{2^*}}}
\]
where $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and $D^{1,2}(\mathbb{R}^N)$ is the Banach space given by
\[
D^{1,p}(\mathbb{R}^N) = \{ u \in L^{2^*}(\mathbb{R}^N) : |\nabla u|^2 \in L^2(\mathbb{R}^N) \}
\]
endowed with the norm
\[
\| u \|_{D^{1,2}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.
\]
It is well known that the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous.

### 2.1 The minimax approach

We will prove that $\tilde{J}$ on $S(a) \times \mathbb{R}$ possesses a kind of mountain-pass geometrical structure.

**Lemma 2.1.** Let $u \in S(a)$ be arbitrary but fixed. Then we have:

(i) $|\nabla \mathcal{H}(u,s)|_{2} \to 0$ and $\tilde{J}(\mathcal{H}(u,s)) \to 0$ as $s \to -\infty$;

(ii) $|\nabla \mathcal{H}(u,s)|_{2} \to +\infty$ and $\tilde{J}(\mathcal{H}(u,s)) \to -\infty$ as $s \to +\infty$. 

Proof. By a straightforward calculation, it follows that
\[
\int_{\mathbb{R}^N} |\mathcal{H}(u, s)(x)|^2 \, dx = a^2, \quad \int_{\mathbb{R}^N} |\mathcal{H}(u, s)(x)|^\xi \, dx = e^{\frac{(q-2)Ns}{2}} \int_{\mathbb{R}^N} |u(x)|^\xi \, dx, \quad \forall \xi \geq 1, \quad (2.2)
\]
and
\[
\int_{\mathbb{R}^N} |\nabla \mathcal{H}(u, s)(x)|^2 \, dx = e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \quad (2.3)
\]
From the above equalities, fixing $\xi > 2$, we have
\[
|\nabla \mathcal{H}(u, s)|_2 \to 0 \quad \text{and} \quad |\mathcal{H}(u, s)|_\xi \to 0 \quad \text{as} \quad s \to -\infty.
\]
Hence,
\[
\int_{\mathbb{R}^N} |F(\mathcal{H}(u, s))| \, dx \leq \int_{\mathbb{R}^N} C_1|\mathcal{H}(u, s)|^q \, dx + C_2 \int_{\mathbb{R}^N} |\mathcal{H}(u, s)|^{2^*} \, dx \to 0 \quad \text{as} \quad s \to -\infty,
\]
from where it follows that
\[
J(\mathcal{H}(u, s)) \to 0 \quad \text{as} \quad s \to -\infty,
\]
showing (i).

In order to show (ii), note that by (2.3),
\[
|\nabla \mathcal{H}(u, s)|_2 \to +\infty \quad \text{as} \quad s \to +\infty.
\]
On the other hand,
\[
J(\mathcal{H}(u, s)) \leq \frac{1}{2}|\nabla \mathcal{H}(u, s)|_2^2 - \frac{\mu}{p} \int_{\mathbb{R}^N} |\mathcal{H}(u, s)|^p \, dx = e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\mu e^{(q-2)Ns}}{q} \int_{\mathbb{R}^N} |u(x)|^q \, dx.
\]
Since $q > 2 + \frac{4}{N}$, the last inequality yields
\[
J(\mathcal{H}(u, s)) \to -\infty \quad \text{as} \quad s \to +\infty.
\]

Lemma 2.2. There exists $K(a) > 0$ small enough such that
\[
0 < \sup_{u \in A} J(u) < \inf_{u \in B} J(u)
\]
with
\[
A = \left\{ u \in S(a), \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq K(a) \right\} \quad \text{and} \quad B = \left\{ u \in S(a), \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 2K(a) \right\}.
\]
Proof. We will need the following Gagliardo-Sobolev inequality: for any $\xi \geq 2$,

$$|u|_\xi \leq C(\xi, 2)|\nabla u|_2^{|1-\gamma|},$$

where $\gamma = N(\frac{1}{2} - \frac{1}{\xi})$. If we fix $|\nabla u|_2^2 \leq K(a)$ and $|\nabla v|_2^2 = 2K(a)$, we derive that

$$\int_{\mathbb{R}^N} F(u) \, dx \leq C_1 |u|_q^q + C_2 |u|^{2^*}_{2^*}.$$ 

Then, by the Gagliardo-Sobolev inequality,

$$\int_{\mathbb{R}^N} F(v) \, dx \leq C_1 (|\nabla v|_2^2)^{N(\frac{2}{2^*}-\frac{2}{2})} + C_2 (|\nabla v|_2^2)^{N(\frac{2}{2^*}-\frac{2}{2})}.$$ 

Since $F(u) \geq 0$ for any $u \in H^1(\mathbb{R}^N)$, we have

$$J(v) - J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} F(v) \, dx + \int_{\mathbb{R}^N} F(u) \, dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} F(v) \, dx,$$

and so,

$$J(v) - J(u) \geq \frac{1}{2} K(a) - C_3 K(a)^{N(\frac{2}{2^*}-\frac{2}{2})} - C_4 K(a)^{N(\frac{2}{2^*}-\frac{2}{2})}.$$ 

Thereby, fixing $K(a)$ small enough of such way that,

$$\frac{1}{2} K(a) - C_3 K(a)^{N(\frac{2}{2^*}-\frac{2}{2})} - C_4 K(a)^{N(\frac{2}{2^*}-\frac{2}{2})} > 0,$$

we get the desired result. \qed

As a byproduct of the last lemma is the following corollary.

**Corollary 2.1.** There exists $K(a) > 0$ such that if $u \in S(a)$ and $|\nabla u|_2^2 \leq K(a)$, then $J(u) > 0$.

**Proof.** Arguing as in the last lemma,

$$J(u) \geq \frac{1}{2} |\nabla u|_2^2 - C_1 |\nabla u|_2^{N(\frac{2}{2^*}-\frac{2}{2})} - C_2 |\nabla u|_2^{N(\frac{2}{2^*}-\frac{2}{2})} > 0,$$

for $K(a)$ small enough. \qed

In what follows, we fix $u_0 \in S(a)$ and apply Lemma 2.1 to get two numbers $s_1 < 0$ and $s_2 > 0$, of such way that the functions $s_1 = \mathcal{H}(s_1, u_0)$ and $u_2 = \mathcal{H}(s_2, u_0)$ satisfy

$$|\nabla u_1|_2^2 < \frac{K(a)}{2}, |\nabla u_2|_2^2 > 2K(a), J(u_1) > 0 \quad \text{and} \quad J(u_2) < 0.$$ 

Now, following the ideas from Jeanjean [26], we fix the following mountain pass level given by

$$\gamma_\mu(a) = \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t))$$

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where
\[ \Gamma = \{ h \in C([0, 1], S(a)) : h(0) = u_1 \text{ and } h(1) = u_2 \} \].

From Lemma 2.2
\[ \max_{t \in [0, 1]} J(h(t)) > \max \{ J(u_1), J(u_2) \} . \]

**Lemma 2.3.** There holds \( \lim_{\mu \to +\infty} \gamma_\mu(a) = 0 \).

**Proof.** In what follows we set the path \( h_0(t) = H((1 - t)s_1 + ts_2, u_0) \in \Gamma \). Then,
\[ \gamma_\lambda(a) \leq \max_{t \in [0, 1]} J(h_0(t)) \leq \max_{r \geq 0} \left\{ \frac{r^2}{2} |\nabla u|^2 + \frac{\mu}{q} r^{\frac{N(q-2)}{2}} |u|^q \right\}, \]
and so, for some positive constant \( C_2 \),
\[ \gamma_\lambda(a) \leq C_2 \left( \frac{1}{\mu} \right)^{\frac{4}{N(q-2)-2}} \to 0 \text{ as } \mu \to +\infty. \]

Here, we have used the fact that \( q > 2 + \frac{4}{N} \).

In what follows \((u_n)\) denotes the \((PS)\) sequence associated with the level \( \gamma_\mu(a) \), which is obtained by making \( u_n = H(v_n, s_n) \), where \((v_n, s_n)\) is the \((PS)\) sequence for \( J \) obtained by [26, Proposition 2.2], associated with the level \( \gamma_\mu(a) \). More precisely, we have
\[ J(u_n) \to \gamma_\mu(a) \text{ as } n \to +\infty, \quad (2.5) \]
and
\[ \|J'(u_n)\|_{S(a)} \to 0 \text{ as } n \to +\infty. \]

Setting the functional \( \Psi : H^1(\mathbb{R}^N) \to \mathbb{R} \) given by
\[ \Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx, \]
it follows that \( S(a) = \Psi^{-1} \{ a^2/2 \} \). Then, by Willem [41, Proposition 5.12], there exists \((\lambda_n) \subset \mathbb{R}\) such that
\[ \|J'(u_n) - \lambda_n \Psi'(u_n)\|_{H^{-1}} \to 0 \text{ as } n \to +\infty. \]

Hence,
\[ -\Delta u_n - f(u_n) = \lambda_n u_n + o_n(1) \text{ in } (H^1(\mathbb{R}^N))^*. \quad (2.6) \]

Moreover, another important limit involving the sequence \((u_n)\) is
\[ Q(u_n) = \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + N \int_{\mathbb{R}^N} F(u_n) \, dx - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n) u_n \, dx \to 0 \text{ as } n \to +\infty, \quad (2.7) \]
which is obtained using the limit below
\[ \partial_s \tilde{J}(v_n, s_n) \to 0 \text{ as } n \to +\infty. \]
that was also proved in [26].

Arguing as in [26, Lemmas 2.3 and 2.4], we know that \((u_n)\) is a bounded sequence, and so, the number \(\lambda_n\) must satisfy the equality below

\[
\lambda_n = \frac{1}{|u_n|^2} \left\{ |\nabla u_n|^2 - \int_{\mathbb{R}^N} f(u_n) u_n dx \right\} + o_n(1),
\]

or equivalently,

\[
\lambda_n = \frac{1}{a^2} \left\{ |\nabla u_n|^2 - \int_{\mathbb{R}^N} f(u_n) u_n dx \right\} + o_n(1).
\] (2.8)

**Lemma 2.4.** There exists \(C > 0\) such that

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) \, dx \leq C \gamma \mu(a)
\]

and

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n) u_n \, dx \leq C \gamma \mu(a).
\]

**Proof.** From (2.5) and (2.7)

\[
NJ(u_n) + Q(u_n) = N \gamma \mu(a) + o_n(1),
\]

then

\[
\frac{N + 2}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n) u_n dx = N \gamma \mu(a) + o_n(1).
\]

Using again (2.5), we get

\[
\frac{N + 2}{2} \left( 2 \int_{\mathbb{R}^N} F(u_n) dx + 2 \gamma \mu(a) + o_n(1) \right) - \frac{N}{2} \int_{\mathbb{R}^N} f(u_n) u_n dx = N \gamma \mu(a) + o_n(1),
\]

that is,

\[
-(N + 2) \int_{\mathbb{R}^N} F(u_n) dx + \frac{N}{2} \int_{\mathbb{R}^N} f(u_n) u_n dx = 2 \gamma \mu(a) + o_n(1). \tag{2.9}
\]

Since \(q > \frac{2(N+2)}{N}\) and \(F(t) = \frac{q}{q} |t|^q + \frac{1}{2} |t|^{2^*}, \forall t \in \mathbb{R}^N\), we obtain

\[
qF(t) \leq f(t)t, \, t \in \mathbb{R}. \tag{2.10}
\]

This together with (2.9) yields

\[
\left( \frac{qN}{2} - (N+2) \right) \int_{\mathbb{R}^N} F(u_n) dx \leq 2 \gamma \mu(a) + o_n(1),
\]

and so,

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) \, dx \leq C \gamma \mu(a).
\]

This inequality combined again with (2.9) ensures that

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n) u_n \, dx \leq C \gamma \mu(a).
\]
Lemma 2.5. \( \limsup_{n \to +\infty} |\nabla u_n|^2 \leq C \gamma_\mu(a) \).

Proof. First of all, let us recall that

\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = 2\gamma_\mu(a) + \int_{\mathbb{R}^N} F(v_n) \, dx + o_n(1).
\]

Then, from Lemma 2.4,

\[
\limsup_{n \to +\infty} |\nabla u_n|^2 \leq (2 + C) \gamma_\mu(a).
\]

Now, from (2.9), the sequence \( \int_{\mathbb{R}^N} F(u_n) \, dx \) is bounded away from zero, otherwise we would have

\[
\int_{\mathbb{R}^N} F(u_n) \, dx \to 0 \quad \text{as} \quad n \to +\infty,
\]

which leads to

\[
\int_{\mathbb{R}^N} f(u_n) u_n \, dx \to 0 \quad \text{as} \quad n \to +\infty.
\]

These limits combined with (2.9) imply that \( \gamma_\mu(a) = 0 \), which is absurd. From this, in what follows we can assume that

\[
\int_{\mathbb{R}^N} F(u_n) \, dx \to C_1 > 0, \quad \text{as} \quad n \to \infty. \tag{2.11}
\]

Lemma 2.6. The sequence \( (\lambda_n) \) is bounded with

\[
\limsup_{n \to +\infty} \lambda_n \leq -\frac{(N + 2)}{2a^2} \liminf_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) \, dx
\]

and

\[
\limsup_{n \to +\infty} |\lambda_n| \leq \frac{C}{a^2} \gamma_\mu(a),
\]

for some \( C > 0 \).

Proof. The boundedness of \( (u_n) \) yields that \( (\lambda_n) \) is bounded, because

\[
\lambda_n a^2 = \lambda_n |u_n|^2 = |\nabla u_n|^2 - \int_{\mathbb{R}^N} f(u_n) u_n \, dx + o_n(1), \tag{2.12}
\]

and so,

\[
|\lambda_n| \leq \frac{1}{a^2} \left( |\nabla u_n|^2 + \int_{\mathbb{R}^N} f(u_n) u_n \, dx \right) + o_n(1)
\leq \frac{C}{a^2} \gamma_\mu(a) + o_n(1).
\]

This guarantees the boundedness of \( (\lambda_n) \) and the second inequality is proved.
In order to prove the first inequality, we know by (2.7) that
\[
|\nabla u_n|^2 = \frac{N}{2} \int_{\mathbb{R}^N} f(u_n) u_n \, dx - N \int_{\mathbb{R}^N} F(u_n) \, dx + o_n(1).
\]
Inserting this equality in (2.12), we obtain
\[
\lambda_n a^2 = \frac{(N - 2)}{2} \int_{\mathbb{R}^N} f(u_n) u_n \, dx - N \int_{\mathbb{R}^N} F(u_n) \, dx + o_n(1),
\]
showing the first inequality.

In the sequel, we restrict our study to the space \( H^1_{rad}(\mathbb{R}^N) \). Then, it is well known that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^q \, dx = \int_{\mathbb{R}^N} |u|^q \, dx, \tag{2.13}
\]
where \( u_n \to u \) in \( H^1(\mathbb{R}^N) \), because \( q \in (2 + \frac{4}{N}, 2^*) \).

**Lemma 2.7.** There exists \( \mu^* > 0 \) such that \( u \neq 0 \) for all \( \mu \geq \mu^* > 0 \).

**Proof.** Seeking for a contradiction, let us assume that \( u = 0 \). Then,
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^q \, dx = 0, \tag{2.14}
\]
and by Lemma 2.6
\[
\limsup_{n \to +\infty} \lambda_n \leq 0.
\]
The equality
\[
a^2 \lambda_n = |\nabla u_n|^2 - \int_{\mathbb{R}^N} f(u_n) u_n \, dx + o_n(1)
\]
together with (2.14) leads to
\[
a^2 \lambda_n = |\nabla u_n|^2 - |u_n|^{2^*} + o_n(1). \tag{2.15}
\]
In what follows, going to a subsequence, we assume that
\[
|\nabla u_n|^2 - a^2 \lambda_n = L + o_n(1) \quad \text{and} \quad |u_n|^{2^*} = L + o_n(1).
\]
We claim that \( L > 0 \), otherwise if \( L = 0 \), we must have
\[
a^2 \lambda_n = |\nabla u_n|^2 + o_n(1). \tag{2.16}
\]
From this,
\[
0 \geq \limsup_{n \to +\infty} \lambda_n = \limsup_{n \to +\infty} |\nabla u_n|^2 \geq \liminf_{n \to +\infty} |\nabla u_n|^2 \geq 0,
\]
then,
\[
|\nabla u_n|^2 \to 0,
\]
which is absurd, because \( \gamma_\mu(a) > 0 \).
Since $L > 0$, by definition of $S$ in (2.1),
\[ S \leq \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx}{(\int_{\mathbb{R}^N} |u_n|^{2^*} \, dx)^{\frac{2}{2^*}}} = \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx - a^2 \lambda_n}{(\int_{\mathbb{R}^N} |u_n|^{2^*} \, dx)^{\frac{2}{2^*}}} + \frac{a^2 \lambda_n}{(\int_{\mathbb{R}^N} |u_n|^{2^*} \, dx)^{\frac{2}{2^*}}}.
\]
Taking the $\limsup$ as $n \to +\infty$, we obtain
\[ S \leq \frac{L}{L^{\frac{2}{2^*}}}, \]
that is,
\[ L \geq S^{\frac{N}{2^*}}. \]

On the other hand
\[ \omega_n(1) + \gamma_{\mu}(a) - \frac{a^2 \lambda_n}{2} = \frac{1}{2}(|\nabla u_n|^2 - a^2 \lambda_n) - \frac{\mu}{q} |u_n|^q - \frac{1}{2^*} |u_n|^{2^*} = \frac{1}{N} L + o_n(1). \]

Recalling that $\limsup_{n \to +\infty} |\lambda_n| \leq \frac{C}{a^2} \gamma_{\mu}(a)$, it follows that
\[ \frac{1}{N} S^{\frac{N}{2^*}} \leq C \gamma_{\mu}(a). \]

Now, fixing $\mu^*$ large enough of a such way that
\[ C \gamma_{\mu}(a) < \frac{1}{N} S^{\frac{N}{2^*}}, \quad \forall \mu \geq \mu^*, \]
we get a new contradiction. This proves that $u \neq 0$ for $\mu > 0$ large enough.

**Lemma 2.8.** Increasing if necessary $\mu^*$, we have $u_n \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^N)$ for all $\mu \geq \mu^*$.

**Proof.** Using the concentration-compactness principle due to Lions [30], it follows that
(i) $|\nabla u_n|^2 \to \kappa$ weakly-* in the sense of measure
and
(ii) $|u_n|^{2^*} \to \nu$ weakly-* in the sense of measure,

and for a most countable index set $J$, we have
\[
\begin{align*}
(a) & \quad \nu = |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \nu_j \geq 0, \\
(b) & \quad \kappa \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}, \quad \mu_j \geq 0, \\
(c) & \quad S \nu_j^{\frac{2}{2^*}} \leq \kappa_j, \quad \forall j \in J.
\end{align*}
\]

Since
\[-\Delta u_n - f(u_n) = \lambda_n u_n + o_n(1) \quad \text{in} \quad (H^1(\mathbb{R}^N))^*,
\]
we derive that
\[
\int_{\mathbb{R}^N} \nabla u_n \nabla \phi \, dx - \lambda_n \int_{\mathbb{R}^N} u_n \phi \, dx = \mu \int_{\mathbb{R}^N} |u_n|^{q-2} v_n \phi \, dx + \int_{\mathbb{R}^N} |u_n|^{2^*-2} v_n \phi \, dx, \quad \forall \phi \in H^1(\mathbb{R}^N).
\]
Now, arguing as in [24, Lemma 2.3], \( J \) is empty or otherwise \( J \) is nonempty but finite. In the case that \( J \) is nonempty but finite, we must have

\[
\nu_j \geq S^N, \quad \forall j \in J.
\]

However, by Lemma 2.5

\[
\limsup_{n \to +\infty} |\nabla u_n|^2 \leq C_{\gamma\mu}(a).
\]

Then, if \( \mu^* > 0 \) is fixed of such way that

\[
C_{\gamma\mu}(a) < \frac{1}{2} S^N,
\]

we get a contradiction, and so, \( J = \emptyset \). From this,

\[
u_n \to u \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^N).
\]

**Claim 2.1.** For each \( R > 0 \), we have

\[
u_n \to u \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^N \setminus B_R(0)).
\]

Indeed, as \( u_n \in H^1_{rad}(\mathbb{R}^N) \), we know that

\[
|u_n(x)| \leq \frac{\|u_n\|}{|x|^\frac{N}{2}}, \quad \text{a.e. in} \quad \mathbb{R}^N.
\]

Since \((u_n)\) is a bounded sequence in \( H^1(\mathbb{R}^N) \), we obtain

\[
|u_n(x)| \leq \frac{C}{|x|^\frac{N}{2}}, \quad \text{a.e. in} \quad \mathbb{R}^N,
\]

and so,

\[
|u_n(x)|^{2^*} \leq \frac{C_1}{|x|^{N(N-1)\frac{N}{2}}}, \quad \text{a.e. in} \quad \mathbb{R}^N.
\]

Recalling that \( \frac{C_1}{|x|^{N(N-1)\frac{N}{2}}} \in L^1(\mathbb{R}^N \setminus B_R(0)) \) and \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \setminus B_R(0) \), the Lebesgue’s Theorem gives

\[
u_n \to u \quad \text{in} \quad L^2^*(\mathbb{R}^N \setminus B_R(0)),
\]

showing the Claim 2.1. Now, the Claim 2.1 combined with (2.17) ensures that

\[
u_n \to u \quad \text{in} \quad L^2^*(\mathbb{R}^N).
\]
From the previous analysis $u \neq 0$. Therefore, the inequality

$$
\limsup_{n \to +\infty} \lambda_n \leq -C \liminf_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n)\,dx
$$

together with (2.13) and Lemma 2.8 ensures that

$$
\limsup_{n \to +\infty} \lambda_n \leq -C \liminf_{n \to +\infty} \int_{\mathbb{R}^N} F(u)\,dx < 0.
$$

So, according to the Lemma 2.6 we can assume without loss of generality that

$$
\lambda_n \to \lambda_a < 0 \quad \text{as} \quad n \to +\infty.
$$

Now, the equality (2.6) implies that

$$
-\Delta u - f(u) = \lambda_a u, \quad \text{in} \quad \mathbb{R}^N.
$$

(2.18)

Thus,

$$
|\nabla u|^2_2 - \lambda_a |u|^2_2 = \int_{\mathbb{R}^N} f(u)u\,dx.
$$

On the other hand,

$$
|\nabla u_n|^2_2 - \lambda_n |u_n|^2_2 = \int_{\mathbb{R}^N} f(u_n)u_n\,dx + o_n(1),
$$

or yet,

$$
|\nabla u_n|^2_2 - \lambda_a |u_n|^2_2 = \int_{\mathbb{R}^N} f(u_n)u_n\,dx + o_n(1).
$$

Recalling that

$$
u_n \to u \quad \text{in} \quad L^{2^*}(\mathbb{R}^N)
$$

and

$$
u_n \to u \quad \text{in} \quad L^{q}(\mathbb{R}^N),
$$

we obtain

$$
\lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n)u_n\,dx = \int_{\mathbb{R}^N} f(u)u\,dx,
$$

from where it follows that

$$
\lim_{n \to +\infty} (|\nabla u_n|^2_2 - \lambda_a |u_n|^2_2) = |\nabla u|^2_2 - \lambda_a |u|^2_2.
$$

Since $\lambda_a < 0$, the last equality implies that

$$
u_n \to u \quad \text{in} \quad H^1(\mathbb{R}^N),
$$

implying that $|u|^2_2 = a$. This establishes the desired result.

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3 Normalized solutions: The exponential critical growth case for $N = 2$

In this section we will deal with the case $N = 2$, where $f$ has an exponential critical growth and $a \in (0, 1)$. We start our study recalling that by ($f_1$) and ($f_2$), we know that fixed $q \geq 1$, for any $\zeta > 0$ and $\alpha > 4\pi$, there exists a constant $C > 0$, which depends on $q$, $\alpha$, $\zeta$, such that

$$|f(t)| \leq \zeta |t|^\tau + C |t|^{q-1}(e^{\alpha t^2} - 1) \text{ for all } t \in \mathbb{R}$$

(3.1)

and, using ($f_3$), we have

$$|F(t)| \leq \zeta |t|^{	au+1} + C |t|^q(e^{\alpha t^2} - 1) \text{ for all } t \in \mathbb{R}.$$  

(3.2)

Moreover, it is easy to see that, by (3.1),

$$|f(t)t| \leq \zeta |t|^{	au+1} + C |t|^q(e^{\alpha t^2} - 1) \text{ for all } t \in \mathbb{R}.$$  

(3.3)

Finally, let us recall the following version of Trudinger-Moser inequality as stated e.g. in [14].

**Lemma 3.1.** If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (e^{\alpha |u|^2} - 1) dx < +\infty.$$  

Moreover, if $|\nabla u|^2_2 \leq 1$, $|u|^2 \leq M < +\infty$, and $0 < \alpha < 4\pi$, then there exists a positive constant $C(M, \alpha)$, which depends only on $M$ and $\alpha$, such that

$$\int_{\mathbb{R}^2} (e^{\alpha |u|^2} - 1) dx \leq C(M, \alpha).$$

**Lemma 3.2.** Let $(u_n)$ be a sequence in $H^1(\mathbb{R}^2)$ with $u_n \in S(a)$ and

$$\limsup_{n \to +\infty} |\nabla u_n|^2_2 < 1 - a^2.$$  

Then, there exist $t > 1$, $t$ close to 1, and $C > 0$ satisfying

$$\int_{\mathbb{R}^2} \left(e^{4\pi|u_n|^2} - 1\right)^t dx \leq C, \ \forall n \in \mathbb{N}.$$  

**Proof.** As

$$\limsup_{n \to +\infty} |\nabla u_n|^2_2 < 1 \quad \text{and} \quad |u_n|^2_2 = a^2 < 1,$$

there exist $m > 0$ and $n_0 \in \mathbb{N}$ verifying

$$||u_n||^2 < m < 1, \text{ for any } n \geq n_0.$$  

Fix $t > 1$, with $t$ close to 1, and $\beta > t$ satisfying $\beta m < 1$. Then, there exists $C = C(\beta) > 0$ such that

$$\int_{\mathbb{R}^2} \left(e^{4\pi|u_n|^2} - 1\right)^t dx \leq \int_{\mathbb{R}^2} \left(e^{4\beta m \pi(\frac{|u_n|}{||u_n||})^2} - 1\right) dx, \text{ for any } n \geq n_0.$$  

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where we have used the inequality
\[(e^t - 1)^s \leq e^{ts} - 1, \text{ for } s > 1 \text{ and } t \geq 0.\]

Hence, by Lemma 3.1,
\[
\int_{\mathbb{R}^2} \left( e^{4\pi|u_n|^2} - 1 \right)^t \, dx \leq C_1 \quad \forall \, n \geq n_0,
\]
for some positive constant \(C_1\). Now, the lemma follows fixing
\[
C = \max \left\{ C_1, \int_{\mathbb{R}^2} \left( e^{4\pi|u_1|^2} - 1 \right)^t \, dx, \ldots, \int_{\mathbb{R}^2} \left( e^{4\pi|u_{n_0}|^2} - 1 \right)^t \, dx \right\}.
\]

**Corollary 3.1.** Let \((u_n)\) be a sequence in \(H^1(\mathbb{R}^2)\) with \(u_n \in S(a)\) and
\[
\limsup_{n \to +\infty} |\nabla u_n|^2 < 1 - a^2.
\]
If \(u_n \rightharpoonup u\) in \(H^1(\mathbb{R}^2)\) and \(u_n(x) \to u(x)\) a.e in \(\mathbb{R}^2\), then
\[
F(u_n) \to F(u) \quad \text{in} \quad L^1(B_R(0)), \quad \text{for any } R > 0.
\]

**Proof.** By \((f_1) - (f_2)\), for each \(\beta > 1\), there is \(C > 0\) such that
\[
|F(t)| \leq C_1 |t|^\tau + 1 + C(e^{4\beta \pi|t|^2} - 1), \quad \forall \, t \in \mathbb{R},
\]
from where it follows that,
\[
|F(u_n)| \leq C_1 |u_n|^\tau + 1 + C_2 (e^{4\beta \pi|u_n|^2} - 1), \quad \forall n \in \mathbb{N}. \quad (3.4)
\]
Setting
\[
h_n(x) = C(e^{4\beta \pi|u_n(x)|^2} - 1),
\]
we can argue as in the proof of Lemma 3.2 to find two numbers \(\beta, q > 1\), with \(\beta\) and \(q\) close to 1, such that
\[
h_n \in L^q(\mathbb{R}^2) \quad \text{and} \quad \sup_{n \in \mathbb{N}} |h_n|_q < +\infty,
\]
which is an immediate consequence of Lemma 3.2. Therefore, for some subsequence of \((u_n)\), still denoted by itself, we derive that
\[
h_n \to h = C(e^{4\beta \pi|u|^2} - 1), \quad \text{in} \quad L^q(\mathbb{R}^2). \quad (3.5)
\]

**Claim 3.1.** As \(h_n, h \geq 0\), the last limit yields
\[
h_n \to h \quad \text{in} \quad L^1(B_R(0)), \quad \forall \, R > 0.
\]
Indeed, for each \( R > 0 \), we consider the Characteristic function \( \chi_R \) associated with \( B_R(0) \subset \mathbb{R}^2 \), that is,

\[
\chi_R(x) = \begin{cases} 
1, & x \in B_R(0), \\
0, & x \in \mathbb{R}^N \setminus B_R(0),
\end{cases}
\]

which belongs to \( L^q(\mathbb{R}^2) \) with \( \frac{1}{q} + \frac{1}{q'} = 1 \). Thus, by the weak limit (3.5),

\[
\int_{\mathbb{R}^2} h_n \chi_R \, dx \to \int_{\mathbb{R}^2} h \chi_R \, dx,
\]
or equivalently,

\[
\int_{B_R(0)} h_n \, dx \to \int_{B_R(0)} h \, dx.
\]

Once \( h_n, h \geq 0 \), we derive that

\[
|h_n|_{L^1(B_R(0))} \to |h|_{L^1(B_R(0))}.
\]

Moreover, we also have

\[
h_n(x) \to h(x), \quad \text{a.e. in } \mathbb{R}^2.
\]

Now, the Lebesgue’s Theorem combined with the above limits gives

\[
h_n \to h \quad \text{in } L^1(B_R(0)).
\]

Using the last limit, there exists a subsequence of \((h_n)\), still denoted by itself, and a nonnegative function \( g_R \in L^1(B_R(0)) \), such that

\[
|h_n(x)| \leq g_R(x) \quad \text{a.e. in } B_R(0).
\]

Consequently, by (3.4),

\[
|F(u_n)| \leq C_1 |u_n|^{{\tau}+1} + g_R(x) \quad \text{a.e. in } B_R(0).
\]

Since

\[
u_n \to u \quad \text{in } L^{{\tau}+1}(B_R),
\]

there exists a nonnegative function \( Q_R \in L^{{\tau}+1}(B_R) \) such that

\[
|u_n|^{{\tau}+1} \leq Q_R.
\]

Observing that

\[
F(u_n(x)) \to F(u(x)), \quad \text{a.e. in } \mathbb{R}^2,
\]

we can use again the Lebesgue’s Theorem to guarantee that

\[
F(u_n) \to F(u) \quad \text{in } L^1(B_R(0)).
\]
Lemma 3.3. Let \((u_n) \subset H^1_{rad}(\mathbb{R}^2)\) be a sequence with \(u_n \in S(a)\) and
\[
\limsup_{n \to +\infty} |\nabla u_n|^2 < 1 - a^2.
\]
Then, there are \(\beta, q\) close to 1, such that for all \(l > 1,\)
\[
|u_n|^l(e^{4\beta \pi |u_n(x)|^2} - 1) \to |u|^l(e^{4\beta \pi |u(x)|^2} - 1) \text{ in } L^1(\mathbb{R}^N).
\]
Proof. Arguing as in Corollary 3.1, there are \(\beta, q\) close to 1 such that the sequence
\[
h_n(x) = C(e^{4\beta \pi |u_n(x)|^2} - 1),
\]
is a bounded sequence in \(L^q(\mathbb{R}^N)\). Therefore, for some subsequence of \((h_n)\), still denoted by itself, we derive that
\[
h_n \rightharpoonup h = C(e^{4\beta \pi |u|^2} - 1) \text{ in } L^q(\mathbb{R}^2).
\]
For \(q' = \frac{q}{q-1}\), we know that the embedding \(H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^{q'}(\mathbb{R}^N)\) is compact, then
\[
u_n \to u \text{ in } L^{q'}(\mathbb{R}^N),
\]
and so,
\[
|u_n|^l \to |u|^l \text{ in } L^{q'}(\mathbb{R}^N).
\]
Thus,
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} |u_n|^l h_n(x) \, dx = \int_{\mathbb{R}^2} |u|^l h(x) \, dx,
\]
that is,
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} |u_n|^l(e^{4\beta \pi |u_n(x)|^2} - 1) \, dx = \int_{\mathbb{R}^2} |u|^l(e^{4\beta \pi |u(x)|^2} - 1) \, dx.
\]
Since
\[
|u_n|^l(e^{4\beta \pi |u_n(x)|^2} - 1) \geq 0 \quad \text{and} \quad |u|^l(e^{4\beta \pi |u(x)|^2} - 1) \geq 0,
\]
the last limit gives
\[
|u_n|^l(e^{4\beta \pi |u_n(x)|^2} - 1) \to |u|^l(e^{4\beta \pi |u(x)|^2} - 1) \text{ in } L^1(\mathbb{R}^2).
\]
Proof. By \((f_1) - (f_2)\),
\[
|F(t)| \leq C_1 |t|^{r+1} + C_2 |u_n|^l (e^{4\beta \pi |t|^2} - 1) \quad \forall t \in \mathbb{R},
\]
where \(\beta > 1\) is close to 1 and \(l > 1\) as in the last Lemma. Therefore,
\[
|F(u_n)| \leq C_1 |u_n|^{r+1} + C_2 |u_n|^l (e^{4\beta \pi |u|^2} - 1), \quad \forall n \in \mathbb{N}. \tag{3.6}
\]
By Lemma 3.3,
\[
|u_n|^l (e^{4\beta \pi |u_n|} - 1) \to |u|^l (e^{4\beta \pi |u|} - 1) \quad \text{in} \quad L^1(\mathbb{R}^2),
\]
and by the compact embedding \(H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^{r+1}(\mathbb{R}^N),\)
\[
u_n \to u \quad \text{in} \quad L^{r+1}(\mathbb{R}^2).
\]
Now, we can use the Lebesgue’s Theorem to conclude that
\[
F(u_n) \to F(u) \quad \text{in} \quad L^1(\mathbb{R}^2).
\]
A similar argument works to show that
\[
f(u_n)u_n \to f(u)u \quad \text{in} \quad L^1(\mathbb{R}^2).
\]
\[
\square
\]
From now on, we will use the same notations introduced in Section 2 to apply our variational procedure, more precisely

1) \(S(a) = \{ u \in H^1(\mathbb{R}^2) : |u|_2 = a \}\) is the sphere of radius \(a > 0\) defined with the norm \(| |_2\).

2) \(J : H^1(\mathbb{R}^2) \to \mathbb{R}\) with
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \int_{\mathbb{R}^2} F(u) dx.
\]

3) \(\mathcal{H} : H^1(\mathbb{R}^2) \times \mathbb{R} \to H^1(\mathbb{R}^2)\) with
\[
\mathcal{H}(u, s)(x) = e^{s} u(e^{s}x).
\]

4) \(\tilde{J} : H^1(\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}\) with
\[
\tilde{J}(u, s) = \frac{e^{2s}}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{1}{e^{2s}} \int_{\mathbb{R}^2} F(e^{s}u(x)) dx.
\]
3.1 The minimax approach

We will prove that $\tilde{J}$ on $S(a) \times \mathbb{R}$ possesses a kind of mountain-pass geometrical structure.

Lemma 3.4. Assume that $(f_1) - (f_2)$ and $(f_3)$ hold and let $u \in S(a)$ be arbitrary but fixed. Then we have:

(i) $|\nabla \mathcal{H}(u, s)|_2 \to 0$ and $J(\mathcal{H}(u, s)) \to 0$ as $s \to -\infty$;
(ii) $|\nabla \mathcal{H}(u, s)|_2 \to +\infty$ and $J(\mathcal{H}(u, s)) \to -\infty$ as $s \to +\infty$.

Proof. By a straightforward calculation, it follows that

$$
\int_{\mathbb{R}^2} |\mathcal{H}(u, s)(x)|^2 dx = a^2, \quad \int_{\mathbb{R}^2} |\mathcal{H}(u, s)(x)|^\xi dx = e^{(\xi-2)s} \int_{\mathbb{R}^2} |u(x)|^\xi dx, \quad \forall \xi \geq 1, \quad (3.7)
$$

and

$$
\int_{\mathbb{R}^2} |\nabla \mathcal{H}(u, s)(x)|^2 dx = e^{2s} \int_{\mathbb{R}^2} |\nabla u|^2 dx. \quad (3.8)
$$

From the above equalities, fixing $\xi > 2$, we have

$$
|\nabla \mathcal{H}(u, s)|_2 \to 0 \quad \text{and} \quad |\mathcal{H}(u, s)|_\xi \to 0 \quad \text{as} \quad s \to -\infty. \quad (3.9)
$$

Thus, there are $s_1 < 0$ and $m \in (0, 1)$ such that

$$
||\mathcal{H}(u, s)||^2 \leq m, \quad \forall s \in (-\infty, s_1].
$$

By $(f_1) - (f_2)$,

$$
|F(t)| \leq C_1 |t|^\tau + C_2 |u_n|^l (e^{4\beta \pi |t|^2} - 1) \quad \forall t \in \mathbb{R},
$$

where $\beta$ is close to 1 and $l > 1$ as in the last Lemma 3.3. Hence,

$$
|F(\mathcal{H}(u, s))| \leq C_1 |\mathcal{H}(u, s)|^{\tau+1} + C_2 |\mathcal{H}(u, s)|^{l}(e^{4\beta \pi |\mathcal{H}(u, s)|^2} - 1), \quad \forall s \in (-\infty, s_1].
$$

Using the H"older's inequality together with Lemma 3.1, there exists $C = C(u, m) > 0$ such that

$$
\int_{\mathbb{R}^2} |F(\mathcal{H}(u, s))| dx \leq \int_{\mathbb{R}^2} C_1 |\mathcal{H}(u, s)|^{\tau+1} dx + C \int_{\mathbb{R}^2} |\mathcal{H}(u, s)|^{lq'} dx, \quad \forall s \in (-\infty, s_1],
$$

where $q' = \frac{q}{q-1}$, and $q$ is close to 1 of way that $lq' > 2$. Now, by using (3.9),

$$
\int_{\mathbb{R}^2} |F(\mathcal{H}(u, s))| \to 0 \quad \text{as} \quad s \to -\infty,
$$

from where it follows that

$$
J(\mathcal{H}(u, s)) \to 0 \quad \text{as} \quad s \to -\infty,
$$

showing (i).

In order to show (ii), note that by (3.8),

$$
|\nabla \mathcal{H}(u, s)|_2 \to +\infty \quad \text{as} \quad s \to +\infty.
$$
On the other hand, by \((f_5)\),
\[
J(\mathcal{H}(u, s)) \leq \frac{1}{2} |\nabla \mathcal{H}(u, s)|^2 - \frac{\mu}{p} \int_{\mathbb{R}^2} |\mathcal{H}(u, s)|^p dx = e^{2s} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{\mu e^{(p-2)s}}{p} \int_{\mathbb{R}^2} |u(x)|^p dx.
\]
Since \(p > 4\), the last inequality yields
\[
J(\mathcal{H}(u, s)) \to -\infty \text{ as } s \to +\infty.
\]
\(\square\)

**Lemma 3.5.** Assume that \((f_1) - (f_3)\) hold. Then there exists \(K(a) > 0\) small enough such that
\[
0 < \sup_{u \in A} J(u) < \inf_{u \in B} J(u)
\]
with
\[
A = \left\{ u \in S(a), \int_{\mathbb{R}^2} |\nabla u|^2 dx \leq K(a) \right\} \quad \text{and} \quad B = \left\{ u \in S(a), \int_{\mathbb{R}^2} |\nabla u|^2 dx = 2K(a) \right\}.
\]

**Proof.** We will need the following Gagliardo-Sobolev inequality: for any \(\xi \geq 2\),
\[
|u|_{\xi} \leq C(\xi, 2)|\nabla u|_2 |u|_2^{1-\gamma},
\]
where \(\gamma = 2(\frac{1}{2} - \frac{1}{\xi})\). If we fix \(K(a) < \frac{1-\alpha^2}{2}, \ |\nabla u|^2 \leq K(a)\) and \(|\nabla v|^2 = 2K(a)\), the conditions \((f_1) - (f_2)\) combined with Lemma 3.1 ensure that
\[
\int_{\mathbb{R}^2} F(u) dx \leq C_1 |u|_{\tau+1}^{\tau+1} + C_2 |u|_{lq'}^{lq'}
\]
where \(l > 1, q' = \frac{q}{q-1}\) and \(q\) is closed to 1. Then, by the Gagliardo-Sobolev inequality,
\[
\int_{\mathbb{R}^2} F(v) dx \leq C_1 |\nabla v|_2^{\gamma-1} + C_2 |\nabla v|_2^{lq'^{-2}}.
\]
From \((f_3)\), \(F(u) \geq 0\) for any \(u \in H^1(\mathbb{R}^2)\), then
\[
J(v) - J(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \int_{\mathbb{R}^2} F(v) dx + \int_{\mathbb{R}^2} F(u) dx
\geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \int_{\mathbb{R}^2} F(v) dx,
\]
and so,
\[
J(v) - J(u) \geq \frac{1}{2} K(a) - C_3 K(a)^{\frac{\tau-1}{2}} - C_4 K(a)^{\frac{lq'-2}{2}}.
\]
Since \(\tau > 3\) and we can choose \(q\) closed to 1 of way that \(lq' > 4\), decreasing \(K(a)\) if necessary, it follows that
\[
\frac{1}{2} K(a) - C_3 K(a)^{\frac{\tau-1}{2}} - C_4 K(a)^{\frac{lq'-2}{2}} > 0,
\]
showing the desired result. \(\square\)
As a byproduct of the last lemma is the following corollary.

**Corollary 3.3.** There exists $K(a) > 0$ small enough such that if $u \in S(a)$ and $|\nabla u|_2^2 \leq K(a)$, then $J(u) > 0$.

**Proof.** Arguing as in the last lemma,

$$J(u) \geq \frac{1}{2} |\nabla u|_2^2 - C_1 |\nabla u|_2^{p-1} - C_2 |\nabla u|_2^{p' - 2} > 0,$$

for $K(a) > 0$ small enough. \qed

In what follows, we fix $u_0 \in S(a)$ and apply Lemma 3.4 and Corollary 3.3 to get two numbers $s_1 < 0$ and $s_2 > 0$, of such way that the functions $u_1 = H(u_0, s_1)$ and $u_2 = H(u_0, s_2)$ satisfy

$$|\nabla u_1|_2^2 < \frac{K(a)}{2}, \quad |\nabla u_2|_2^2 > 2K(a), \quad J(u_1) > 0 \quad \text{and} \quad J(u_2) < 0.$$

Now, following the ideas from Jeanjean [26], we fix the following mountain pass level given by

$$\gamma_\mu(a) = \inf_{h \in \Gamma} \max_{t \in [0, 1]} J(h(t))$$

where

$$\Gamma = \{ h \in C([0, 1], S(a)) : h(0) = u_1 \quad \text{and} \quad h(1) = u_2 \}.$$

From Lemma 3.5,

$$\max_{t \in [0, 1]} J(h(t)) > \max \{ J(u_1), J(u_2) \}.$$

**Lemma 3.6.** There holds

$$\lim_{\mu \to +\infty} \gamma_\mu(a) = 0.$$

**Proof.** In what follow we set the path $h_0(t) = H(u_0, (1 - t)s_1 + ts_2) \in \Gamma$. Then,

$$\gamma_\mu(a) \leq \max_{t \in [0, 1]} J(h_0(t)) \leq \max_{r \geq 0} \left\{ \frac{r}{2} |\nabla u_0|_2^2 - \frac{\mu r^{p-2}}{p} |u_0|^p \right\}$$

and so,

$$\gamma_\mu(a) \leq C_2 \left( \frac{1}{\mu} \right)^\frac{2}{p-4} \to 0 \quad \text{as} \quad \mu \to +\infty,$$

for some $C_2 > 0$. Here, we have used the fact that $p > 4$. \qed

Arguing as Section 2, in what follows $(u_n)$ denotes the $(PS)$ sequence associated with the level $\gamma_\mu(a)$, which satisfies:

$$J(u_n) \to \gamma_\mu(a), \quad \text{as} \quad n \to +\infty,$$

$$- \Delta u_n - f(u_n) = \lambda_n u_n + o_n(1), \quad \text{in} \quad (H^1(\mathbb{R}^2))^*.$$
for some sequence \((\lambda_n) \subset \mathbb{R}\), and

\[
Q(u_n) = \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + 2 \int_{\mathbb{R}^2} F(u_n) dx - \int_{\mathbb{R}^2} f(u_n)u_n dx \to 0, \text{ as } n \to +\infty. \tag{3.12}
\]

Moreover, \((u_n)\) is a bounded sequence, and so, the number \(\lambda_n\) must satisfy the equality below

\[
\lambda_n = \frac{1}{a^2} \left\{ |\nabla u_n|^2 - \int_{\mathbb{R}^2} f(u_n)u_n dx \right\} + o_n(1). \tag{3.13}
\]

**Lemma 3.7.** There holds

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^2} F(u_n) dx \leq \frac{2}{\theta - 4} \gamma_\mu(a).
\]

**Proof.** Using the fact that \(J(u_n) = \gamma_\mu(a) + o_n(1)\) and \(Q(u_n) = o_n(1)\), it follows that

\[
2J(u_n) + Q(u_n) = 2\gamma_\mu(a) + o_n(1),
\]

and so,

\[
2|\nabla u_n|^2 - \int_{\mathbb{R}^2} f(u_n)u_n dx = 2\gamma_\mu(a) + o_n(1). \tag{3.14}
\]

Using that \(J(u_n) = \gamma_\mu(a) + o_n(1)\), we get

\[
4 \int_{\mathbb{R}^2} F(u_n) dx + 4\gamma_\mu(a) + o_n(1) - \int_{\mathbb{R}^2} f(u_n)u_n dx = 2\gamma_\mu(a) + o_n(1).
\]

Hence,

\[
2\gamma_\mu(a) + o_n(1) = \int_{\mathbb{R}^2} f(u_n)u_n dx - 4 \int_{\mathbb{R}^2} F(u_n) dx \geq (\theta - 4) \int_{\mathbb{R}^2} F(u_n) dx.
\]

By \((f_3)\), we know that \(\theta > 4\), consequently

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^2} F(u_n)u_n dx \leq \frac{2}{\theta - 4} \gamma_\mu(a).
\]

\[
\square
\]

**Lemma 3.8.** The sequence \((u_n)\) satisfies \(\limsup_{n \to +\infty} |\nabla u_n|^2 \leq \frac{2(\theta - 3)}{\theta - 4} \gamma_\mu(a)\). Hence, there exists \(\mu^* > 0\) such that

\[
\limsup_{n \to +\infty} |\nabla u_n|^2 < 1 - a^2, \text{ for any } \mu \geq \mu^*.
\]

**Proof.** Since \(J(u_n) = \gamma_\mu(a) + o_n(1)\), we have

\[
\int_{\mathbb{R}^2} |\nabla u_n|^2 dx = 2\gamma_\mu(a) + \int_{\mathbb{R}^2} F(u_n) dx + o_n(1).
\]

Thereby, by Lemma 3.7

\[
\limsup_{n \to +\infty} |\nabla u_n|^2 \leq \frac{2(\theta - 3)}{\theta - 4} \gamma_\mu(a).
\]

\[
\square
\]
Lemma 3.9. Fix $\mu \geq \mu^*$, where $\mu^*$ is given in Lemma 3.8. Then, $(\lambda_n)$ is a bounded sequence with

$$\limsup_{n \to +\infty} |\lambda_n| \leq \frac{4\theta - 8}{a^2(\theta - 4)} \gamma_\mu(a) \quad \text{and} \quad \limsup_{n \to +\infty} \lambda_n = -\frac{2}{a^2} \liminf_{n \to +\infty} \int_{\mathbb{R}^2} F(u_n) \, dx.$$

Proof. The boundedness of $(u_n)$ yields that $(\lambda_n)$ is bounded, because

$$\lambda_n |u_n|^2 = |\nabla u_n|^2 - \int_{\mathbb{R}^2} f(u_n)u_n \, dx + o_n(1),$$

and as $|u_n|^2 = a^2$, we have

$$\lambda_n a^2 = |\nabla u_n|^2 - \int_{\mathbb{R}^2} f(u_n)u_n \, dx + o_n(1).$$

Hence,

$$|\lambda_n| a^2 \leq |\nabla u_n|^2 + \int_{\mathbb{R}^2} f(u_n)u_n \, dx + o_n(1).$$

The limit (3.12) together with Lemmas 3.7 and 3.8 ensures that $(\int_{\mathbb{R}^2} f(u_n)u_n \, dx)$ is bounded with

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq \frac{2(\theta - 1)}{\theta - 4} \gamma_\mu(a).$$

This is enough to conclude that $(\lambda_n)$ is a bounded sequence with

$$\limsup_{n \to +\infty} |\lambda_n| \leq \frac{4\theta - 8}{a^2(\theta - 4)} \gamma_\mu(a).$$

In order to prove the second inequality, the equality

$$\lambda_n a^2 = |\nabla u_n|^2 - \int_{\mathbb{R}^2} f(u_n)u_n \, dx + o_n(1)$$

together with the limit (3.12) leads to

$$\lambda_n a^2 = -2 \int_{\mathbb{R}^2} F(u_n) \, dx + o_n(1),$$

showing the desired result. \qed

Now, we restrict our study to the space $H^1_{rad}(\mathbb{R}^2)$. For any $\mu \geq \mu^*$, using Lemmas 3.3 and 3.8 it follows that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} f(u_n)u_n \, dx = \int_{\mathbb{R}^2} f(u)u \, dx,$$

and

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} F(u_n) \, dx = \int_{\mathbb{R}^2} F(u) \, dx.$$
where \( u_n \to u \) in \( H^1(\mathbb{R}^2) \). The last limit implies that \( u \neq 0 \), because otherwise, Corollary 3.2 gives
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} F(u_n) \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^2} f(u_n) u \, dx = 0,
\]
and by Lemma 3.9
\[
\limsup_{n \to +\infty} \lambda_n \leq 0.
\]

Since \((u_n)\) is bounded in \( H^1(\mathbb{R}^2) \) and \( \limsup_{n \to +\infty} |\nabla u_n|^2 < 1 - a^2 \) if \( \mu \geq \mu^* \), Corollary 3.2 together with \((f_1) - (f_2)\) and the equality below
\[
\lambda_n |u_n|^2 = |\nabla u_n|^2 - \int_{\mathbb{R}^2} f(u_n) u_n \, dx + o_n(1),
\]
lead to
\[
\lambda_n a^2 = |\nabla u_n|^2 + o_n(1). \tag{3.15}
\]

From this,
\[
0 \geq \limsup_{n \to +\infty} \lambda_n a^2 = \limsup_{n \to +\infty} |\nabla u_n|^2 \geq \liminf_{n \to +\infty} |\nabla u_n|^2 \geq 0,
\]
then
\[
|\nabla u_n|^2 \to 0,
\]
which is absurd, because \( \gamma_\mu(a) > 0 \).

The above analysis ensures that \( u \neq 0 \). Moreover, the equality
\[
\limsup_{n \to +\infty} \lambda_n = -\frac{2}{a^2} \liminf_{n \to +\infty} \int_{\mathbb{R}^2} F(u_n) \, dx
\]
ensures that
\[
\limsup_{n \to +\infty} \lambda_n = -\frac{2}{a^2} \liminf_{n \to +\infty} \int_{\mathbb{R}^2} F(u) \, dx < 0.
\]

From this, for some subsequence, still denoted by \((\lambda_n)\), we can assume that
\[
\lambda_n \to \lambda_a < 0 \quad \text{as} \quad n \to +\infty.
\]

Now, the equality (3.11) implies that
\[
-\Delta u - f(u) = \lambda_a u \quad \text{in} \quad (H^1(\mathbb{R}^2))^* \tag{3.16}
\]
Thus,
\[
|\nabla u|^2 - \lambda_a |u|^2 = \int_{\mathbb{R}^2} f(u) u \, dx.
\]

On the other hand,
\[
|\nabla u_n|^2 - \lambda_n |u_n|^2 = \int_{\mathbb{R}^2} f(u_n) u_n \, dx + o_n(1),
\]
or yet,
\[
|\nabla u_n|^2 - \lambda_a |u_n|^2 = \int_{\mathbb{R}^2} f(u_n) u_n \, dx + o_n(1).
\]
Recalling that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} f(u_n) u_n \, dx = \int_{\mathbb{R}^2} f(u) u \, dx,
\]
we derive that
\[
\lim_{n \to +\infty} (|\nabla u_n|^2 - \lambda_a |u_n|^2) = |\nabla u|^2 - \lambda_a |u|^2.
\]
Since \(\lambda_a < 0\), the last limit implies that
\[
u_n \to u \quad \text{in} \quad H^1(\mathbb{R}^2),
\]
implicating that \(|u|^2_2 = a\). This establishes the desired result.

References

[1] A. Adimurthi. *Existence of Positive solutions of the semilinear Dirichlet problem with critical growth for the N-Laplacian*. Ann. Sc. Norm. Super. Pisa 17 (1990), 393-413.

[2] T. Akahori, S. Ibrahim, H. Kikuchi, and H. Nawa. *Existence of a ground state and scattering for a nonlinear Schr"{o}dinger equation with critical growth*. Selecta Math. (N.S.) 19(2)(2013), 545-609.

[3] T. Akahori, S. Ibrahim, H. Kikuchi, and H. Nawa. *Global dynamics above the ground state energy for the combined power type nonlinear Schrödinger equations with energy critical growth at low frequencies*. arXiv.1510.08034, 2019.

[4] C. O. Alves. *Multiplicity of solutions for a class of elliptic problem in \(\mathbb{R}^2\) with Neumann conditions*. J. Differential Equations 219 (2005), 20–39.

[5] C. O. Alves, J. M. B. do Ó and O. H. Miyagaki. *On nonlinear perturbations of a periodic elliptic problem in \(\mathbb{R}^2\) involving critical growth*, Nonlinear Anal. 56 (2004), 781–791.

[6] C.O. Alves and G.M. Figueiredo. *On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in \(\mathbb{R}^N\)*. J. Differential Equations 219 (2009), 1288-1311.

[7] C.O. Alves and S.H.M. Soares. *Nodal solutions for singularly perturbed equations with critical exponential growth*. J. Differential Equations 234 (2007), 464-484.

[8] C. O. Alves, M. A. S. Souto and M. Montenegro. *Existence of a ground state solution for a nonlinear scalar field equation with critical growth*. Calc. Var. Partial Differential Equations 43(3-4)(2012), 537-554.

[9] T. Bartsch, L. Jeanjean and N. Soave. *Normalized solutions for a system of coupled cubic Schrödinger equations on \(\mathbb{R}^3\)*. J. Math. Pures Appl. (9) 106(4)(2016), 583-614.

[10] T. Bartsch, R. Molle, M. Rizzi and G. Verzini. *Normalized solutions of mass supercritical Schrödinger equations with potential*. arXiv:2008.07431V1, 2020.
[11] T. Bartsch and N. Soave. *Multiple normalized solutions for a competing system of Schrödinger equations*. Calc. Var. Partial Differential Equations 58(1) (2019), art 22, pp.24.

[12] J. Bellazzini, L. Jeanjean and T. Luo. *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations*. Proc. Lond. Math. Soc. (3), 107(2)(2013), 303-339.

[13] H. Brezis and L. Nirenberg. *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437–477.

[14] D. M. Cao. *Nontrivial solution of semilinear elliptic equation with critical exponent in \( \mathbb{R}^2 \)*. Comm. Partial Differential Equation 17 (1992),407–435.

[15] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics 10 (New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003, 323 pp. ISBN: 0-8218-3399-5.

[16] X. Cheng, C.X. Miao and L.F. Zhao. *Global well-posedness and scattering for nonlinear Schrödinger equations with combined nonlinearities in the radial case*. J. Differential Equations 261(6)(2016), 2881-2934.

[17] S. Cingolani and L. Jeanjean. *Stationary waves with prescribed \( L^2 \)-norm for the planar Schrödinger-Poisson system*. SIAM J. Math. Anal. 51(4)(2019), 3533-3568.

[18] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf. *Elliptic equations in \( \mathbb{R}^2 \) with nonlinearities in the critical growth range*. Calc. Var. Partial Differential Equations 3 (1995), 139-153.

[19] D.G. de Figueiredo, João Marcos do Ó and B. Ruf. *On an inequality by N. Trudinger and J. Moser and related elliptic equations*. Comm. Pure Appl. Math. 55 (2002), 1-18.

[20] J. M. B. do Ó and M.A.S. Souto. *On a class of nonlinear Schrödinger equations in \( \mathbb{R}^2 \) involving critical growth*. J. Differential Equations 174 (2001), 289-311.

[21] J. M. B. do Ó and B. Ruf. *On a Schrödinger equation with periodic potential and critical growth in \( \mathbb{R}^2 \)*. Nonlinear Differential Equations Appl. 13 (2006), 167-192.

[22] J. M. B. do Ó. *Quasilinear elliptic equations with exponential nonlinearities*. Comm. Appl. Nonlin. Anal. 2 (1995), 63-72.

[23] J. M. B. do Ó, M. de Souza, E. de Medeiros and U. Severo. *An improvement for the Trudinger-Moser inequality and applications*. J. Differential Equations 256 (2014), 1317-1349.

[24] J. Garcia Azorero and I. Peral Alonso. *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*. Trans. Amer. Math. Soc. 2 (1991), 877-895.
[25] T.X. Gou and L. Jeanjean. Multiple positive normalized solutions for nonlinear Schrödinger systems. Nonlinearity 31(5)(2018), 2319-2345.

[26] L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. 28 (1997), 1633-1659.

[27] L. Jeanjean, J. Jendrej, T. T. Le and N. Visciglia. Orbital stability of ground states for a Sobolev critical Schrödinger equation. arXiv:2008.12084, 2020.

[28] L. Jeanjean and S.S. Lu. Nonradial normalized solutions for nonlinear scalar field equations. Nonlinearity 32(12)(2019), 4942-4966.

[29] L. Jeanjean and T. T. Le. Multiple normalized solutions for a Sobolev critical Schrödinger equations. arXiv:2011.02945v1, 2020.

[30] P.L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case, part 2. Anal. Inst. H. Poincaré, Sect. C 1 (1984), 223-283.

[31] J. Mederski and J. Schino. Least energy solutions to a cooperative system of Schrödinger equations with prescribed $L^2$-bounds: At least $L^2$-critical growth. arXiv:2101.02611v1, 2021

[32] C. X. Miao, G. X. Xu and L. F. Zhao. The dynamics of the 3D radial NLS with the combined terms. Comm. Math. Phys. 318(3)(2013), 767-808.

[33] J. Moser. A sharp form of an inequality by N. Trudinger. Ind. Univ. Math. J. (20) (1971), 1077–1092.

[34] B. Noris, H. Tavares and G. Verzini. Normalized solutions for nonlinear Schrödinger systems on bounded domains. Nonlinearity 32(3)(2019), 10441072.

[35] N. Soave. Normalized ground states for the NLS equation with combined nonlinearities. J. Differential Equations 269(9)(2020), 6941-6987.

[36] N. Soave. Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case. J. Funct. Anal. 279(6)(2020), 108610, 43.

[37] A. Stefanov. On the normalized ground states of second order PDE's with mixed power non-linearities. Comm. Math. Phys. 369(3)(2019), 929-971.

[38] T. Tao, M. Visan and X. Zhang. The nonlinear Schrödinger equation with combined power-type nonlinearities. Comm. Partial Differential Equations 32(7-9)(2007), 1281-1343.

[39] N. S. Trudinger. On imbedding into Orlicz spaces and some application. J. Math Mech. 17 (1967), 473–484.

[40] W. Wang, Q. Li, J. Zhou and Y. Li. Normalized solutions for p-Laplacian equations with a $L^2$-supercritical growth. Ann. Funct. Anal. 12(1) (2020), doi:10.1007/s43034-020-00101-w.
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[41] M. Willem. *Minimax Theorems*, Birkhauser, 1996.