Lie symmetry analysis and exact solutions for a variable coefficient Gardner equation arising in arterial mechanics

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Abstract. In this paper, a variable-coefficient Gardner equation is considered. By using the classical symmetry analysis method symmetries for this equation are obtained. Then, the generalized Jacobi elliptic function expansion method is used to solve the reduced ODE. Some new exact solutions for the considered PDE are obtained.

PACS numbers: 02.30.Ik, 02.30.Jr, 05.45.Yv, 04.20.Jb

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1. Introduction

The investigation of exact solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. The wave phenomena are observed in physics, mechanics, biology, etc. The effort to find these solutions is significant for the understanding of many physical phenomena, thus they may give more insight into the physical aspects of the problems.

This paper is devoted to study the solutions of the variable coefficient Gardner equation which is given by

\[ u_t + \mu_1 uu_x + \mu_2 u^2 u_x + \mu_3 u_{xxx} + h(t)u_x = 0, \]  

(1)

where \( \mu_1, \mu_2 \) and \( \mu_3 \) are constants and \( h(t) \) is a function of \( t \). Equation (1) includes considerably interesting equations, such as KdV equation, when \( \mu_2 \) and \( h(t) \) are equal zero, and mKdV equation, when \( \mu_1 \) and \( h(t) \) are equal zero. In arterial mechanics, some special cases of Equation (1) were considered as follows:

(i) The variable coefficient KdV equation

\[ u_t + \mu_1 uu_x + \mu_3 u_{xxx} + h(t)u_x = 0, \]  

(2)

was considered in [1–7].

(ii) The variable coefficient modified KdV equation

\[ u_t + \mu_2 u^2 u_x + \mu_3 u_{xxx} + h(t)u_x = 0, \]  

(3)

was considered in [6,7].

(iii) The KdV equation

\[ u_t + \mu_1 uu_x + \mu_3 u_{xxx} = 0, \]  

(4)

was considered in [8–19].

(iv) The modified KdV equation

\[ u_t + \mu_2 u^2 u_x + \mu_3 u_{xxx} = 0, \]  

(5)

was considered in [18–20].

Special cases of (1) have been studied in [21–25]. The rest of this paper is arranged as follows. In section 2, we apply the Lie classical symmetry analysis method to (1). In Sec. 3 the mathematical framework of the generalized Jacobi elliptic function expansion method will be provided. In Sec. 4 some new exact solutions of (1) are obtained and application of the obtained solutions in arterial mechanics will be presented in section 5.
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2. Classical symmetries

To apply the Lie classical symmetry analysis method [26, 27] to (1), we consider the one-parameter Lie group of infinitesimal transformations in \( x, t, \) and \( u \) given by

\[
x^* = x + \epsilon \xi(x, t, u) + O(\epsilon^2),
\]

\[
t^* = t + \epsilon \tau(x, t, u) + O(\epsilon^2),
\]

\[
u^* = u + \epsilon \eta(x, t, u) + O(\epsilon^2),
\]

where \( \epsilon \) is the group parameter. We require that the set of solutions of (1) be invariant under this transformation. This yields an overdetermined system of linear equations for the infinitesimals \( \xi(x, t, u), \tau(x, t, u), \) and \( \eta(x, t, u). \) The associated Lie algebra of infinitesimal symmetries is the set of infinitesimal generators of the form

\[
v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}.
\]

Invariance of (1) under a Lie group of point transformations with infinitesimal generator (9) leads to a set of determining equations. Solving this system we will obtain:

\[
\tau = 6c_1 \mu_2 t + c_2,
\]

\[
\xi = c_1 \left( 2\mu_2 x - \mu_1^2 t + 6\mu_2 h(t) - 2\mu_2 \int h(t) dt \right) + c 2h(t) + c_3,
\]

\[
\eta = -c_1 (2\mu_2 u + \mu_1),
\]

where, \( c_1, c_2 \) and \( c_3 \) are arbitrary constants. The associated infinitesimal generators are given by:

\[
v_1 = \left( 2\mu_2 x - \mu_1^2 t + 6\mu_2 h(t) - 2\mu_2 \int h(t) dt \right) \frac{\partial}{\partial x} + 6\mu_2 t \frac{\partial}{\partial t} - (2\mu_2 u + \mu_1) \frac{\partial}{\partial u},
\]

\[
v_2 = \frac{\partial}{\partial x},
\]

\[
v_3 = h(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial t},
\]

When considering the infinitesimal generator \( v_1, \) we will obtain the surface condition

\[
\left( 2\mu_2 x - \mu_1^2 t + 6\mu_2 h(t) - 2\mu_2 \int h(t) dt \right) \frac{\partial u}{\partial x} + 6\mu_2 t \frac{\partial u}{\partial t} + 6\mu_2 \frac{\partial u}{\partial u} = -(2\mu_2 u + \mu_1),
\]

which when solving we will obtain the similarity transformation

\[
u = \frac{-\mu_1}{2\mu_2} + t^{\frac{1}{3}} f(\zeta), \quad \zeta = x t^{\frac{1}{3}} + \frac{\mu_1}{4\mu_2} t^{\frac{2}{3}} - t^{\frac{1}{3}} \int h(t) dt,
\]

when substituting from (17) into (1) we will obtain the following ordinary differential equation

\[- f - \zeta f' + 3\mu_2 f^2 f' + 3\mu_3 f''' = 0.
\]

In general, the exact solution for this nonlinear ODE can’t be obtained by using the elementary functions. The series solution of this equation was obtained in [28].
When considering the infinitesimal generator $v_2 + \alpha v_3$, $\alpha$ is a constant, we will obtain the surface condition
\[ (h(t) + \alpha) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \tag{19} \]
which when solving we will obtain the similarity transformation
\[ u = f(\zeta), \zeta = x - \int h(t) dt - \alpha t, \tag{20} \]
when substituting from (20) into (1) we will obtain the following ordinary differential equation
\[ -\alpha f' + \mu_1 f f' + \mu_2 f^2 f' + \mu_3 f''' = 0, \tag{21} \]
this equation will be solved in the following section.

3. Generalized Jacobi elliptic function expansion method

For solving (21) we will use the generalized Jacobi elliptic function expansion method \cite{29, 30}. It is assumed that (21) has the solutions in the form
\[ u = a_0 + \sum_{i=-n}^{n} a_i \phi^i, \tag{22} \]
where $a_i$ are constants to be determined later and $\phi$ satisfy the following elliptic equations
\[ \phi'' = r + p\phi^2 + q\phi^4, \tag{23} \]
Furthermore we can get
\[ \phi''' = p\phi + q\phi^3, \tag{24} \]
where the primes denotes the derivatives with respect to $\zeta$ and $r$, $p$, $q$ are constants. The solutions of (23) are written in Appendix A \cite{31}. Balancing the highest derivative term $f'''$ with nonlinear term $f f'$ in (21) gives $n = 1$, from which we have
\[ u = a_0 + a_1 \phi + a_{-1} \phi^{-1}. \tag{25} \]
Substituting (25) along with (23) and (24) into (21) and collect all terms with the same powers in $\phi^k \phi^l (k = 0, 1, 2, ..., l = 0, 1, ...)$ and set the coefficients of these terms to zero yields a system of nonlinear algebraic equations for $a_0$, $a_1$ and $a_{-1}$. Solving this system, we find three sets of solutions.

The first set:
\[ a_0 = -\frac{\mu_1}{2\mu_2}, a_1 = 0, a_{-1} = \pm \sqrt{-\frac{6r\mu_3}{\mu_2}}, \alpha = \frac{4\mu_2\mu_3 - \mu_1^2}{4\mu_2}. \tag{26} \]

The second set:
\[ a_0 = -\frac{\mu_1}{2\mu_2}, a_1 = \pm \sqrt{-\frac{6q\mu_3}{\mu_2}}, a_{-1} = 0, \alpha = \frac{4\mu_2\mu_3 - \mu_1^2}{4\mu_2}. \tag{27} \]

The third set:
\[ a_0 = -\frac{\mu_1}{2\mu_2}, a_1 = \pm \sqrt{-\frac{6q\mu_3}{\mu_2}}, a_{-1} = \pm \sqrt{-\frac{6r\mu_3}{\mu_2}}, \alpha = \frac{-4\mu_2\mu_3(\pm\sqrt{rq} - p) - \mu_1^2}{4\mu_2}. \tag{28} \]
4. Exact solutions

In this section we will give some exact solutions for (1). Substituting from (26), (27) and (28) into (25) and using the table in Appendix A we will obtain the following solutions of (1):

Case 1: Soliton and soliton-like solutions

\[ u_1 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{csch} \left( x - \int h(t)dt - \left( \frac{4\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{29}

\[ u_2 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{6\mu_3}{\mu_2}} \text{sech} \left( x - \int h(t)dt - \left( \frac{4\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{30}

\[ u_3 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{tanh} \left( x - \int h(t)dt - \left( \frac{-8\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{31}

\[ u_4 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{coth} \left( x - \int h(t)dt - \left( \frac{-8\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{32}

\[ u_5 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{tanh}(x - \int h(t)dt - \alpha t) \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{coth}(x - \int h(t)dt - \alpha t), \]  \tag{33}

with

\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm 6 + 2) - \mu_1^2}{4\mu_2} \right). \]  \tag{34}

Case 2: Triangular periodic solutions

\[ u_6 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{sec} \left( x - \int h(t)dt - \left( \frac{-4\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{35}

\[ u_7 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{csc} \left( x - \int h(t)dt - \left( \frac{-4\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{36}

\[ u_8 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{tan} \left( x - \int h(t)dt - \left( \frac{8\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{37}

\[ u_9 = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{cot} \left( x - \int h(t)dt - \left( \frac{8\mu_2\mu_3 - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{38}

\[ u_{10} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{tan}(x - \int h(t)dt - \alpha t) \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{cot}(x - \int h(t)dt - \alpha t), \]  \tag{39}

with

\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm 6 - 2) - \mu_1^2}{4\mu_2} \right). \]  \tag{40}

Case 3: Jacobi elliptic function solutions and combined Jacobi elliptic function solutions

\[ u_{11} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \text{ns} \left( x - \int h(t)dt - \left( \frac{-4\mu_2\mu_3(1 + k^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  \tag{41}
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\[ u_{12} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{-4\mu_2\mu_3(1+k^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (42)

\[ u_{13} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(k^2 - k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (43)

\[ u_{14} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (44)

\[ u_{15} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (45)

\[ u_{16} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1-2k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (46)

\[ u_{17} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (47)

\[ u_{18} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (48)

\[ u_{19} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (49)

\[ u_{20} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (50)

\[ u_{21} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (51)

\[ u_{22} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (52)

with

\[ \alpha = \left( \frac{-8\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right). \]  (53)

\[ u_{23} = \frac{-\mu_1}{2\mu_2} \pm \frac{\sqrt{-6\mu_3}}{\mu_2} \left( x - \int h(t) dt - \left( \frac{4\mu_2\mu_3(1+k'^2) - \mu_1^2}{4\mu_2} \right) t \right). \]  (54)
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\begin{align}
\alpha &= \left( \frac{-\mu_2 \mu_3 (1 - 6k + k^2) - \mu_1^2}{4 \mu_2} \right). \\
\left(55\right)
\end{align}

\begin{align}
u_{24} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} \frac{1 - k^2}{4} \frac{2}{1 - k}} \sqrt{\frac{1 + \text{sn}(x - \int h(t) dt - \alpha t)}{1 + \text{sn}(x - \int h(t) dt - \alpha t)}} \\
&\quad \times \frac{(1 + \text{sn}(x - \int h(t) dt - \alpha t))(1 + \text{sn}(x - \int h(t) dt - \alpha t))}{\text{cn}(x - \int h(t) dt - \alpha t)}.
\left(55\right)
\end{align}

\begin{align}
u_{25} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} k \text{sn} \left( x - \int h(t) dt - \left( \frac{-4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}.
\left(57\right)
\end{align}

\begin{align}
u_{26} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} k \text{cd} \left( x - \int h(t) dt - \left( \frac{-4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}.
\left(58\right)
\end{align}

\begin{align}
u_{27} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{6 \mu_3}{\mu_2} k \text{cn} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}.
\left(59\right)
\end{align}

\begin{align}
u_{28} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{6 \mu_3}{\mu_2} k \text{sc} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}.
\left(60\right)
\end{align}

\begin{align}
u_{29} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{6 \mu_3}{\mu_2} \text{dn} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}.
\left(61\right)
\end{align}

\begin{align}
u_{30} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{6 \mu_3}{\mu_2} \text{ds} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 - 2k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}.
\left(62\right)
\end{align}

\begin{align}
u_{31} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} \frac{k^2 - 1}{4} \frac{\text{dn} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}{1 + \text{sn} \left( x - \int h(t) dt - \left( \frac{-4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}}.
\left(63\right)
\end{align}

\begin{align}
u_{32} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} \frac{k^2}{4} \frac{\text{cn} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}{1 + \text{sn} \left( x - \int h(t) dt - \left( \frac{-4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}}.
\left(64\right)
\end{align}

\begin{align}
u_{33} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} \frac{k^2}{4} \frac{\text{ksn} \left( x - \int h(t) dt - \left( \frac{-4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}{1 + \text{dn} \left( x - \int h(t) dt - \left( \frac{-4 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}).
\left(65\right)
\end{align}

\begin{align}
u_{34} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{-6 \mu_3}{\mu_2} \frac{(1 + k')^2}{k'} \frac{\text{dn} \left( x - \int h(t) dt - \left( \frac{-8 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}{k' + \text{dn} \left( x - \int h(t) dt - \left( \frac{-8 \mu_2 \mu_3 (1 + k^2) - \mu_1^2}{4 \mu_2} \right) t \right)}}.
\left(66\right)
\end{align}

\begin{align}
u_{35} &= \frac{-\mu_1}{2 \mu_2} \pm \sqrt{\frac{\pm \frac{6 \mu_3}{\mu_2} (1 + k'^2) 2 \sqrt{k'} \frac{\text{dn} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2 \pm 6k') - \mu_1^2}{4 \mu_2} \right) t \right)}{k' \pm \text{dn} \left( x - \int h(t) dt - \left( \frac{4 \mu_2 \mu_3 (1 + k^2 \pm 6k') - \mu_1^2}{4 \mu_2} \right) t \right)}}.
\left(67\right)
\end{align}
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\( u_{36} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 - k')^2 \cn^2(x - \int h(t) dt - \alpha t) - k' \sn^2(x - \int h(t) dt - \alpha t) + k' \sn^2(x - \int h(t) dt - \alpha t)} \), \hfill (68)

with
\[ \alpha = \left( \frac{-8\mu_2\mu_3(1 + k')^2 - \mu_1^2}{4\mu_2} \right). \hfill (69) \]

\( u_{37} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 + k) \left( 1 \pm \sn \left( x - \int h(t) dt - \left( -\frac{\mu_2\mu_3(1 + 6k + k^2) - \mu_1^2}{4\mu_2} t \right) \right) \right) \frac{1 - k}{2} 2 \left( 1 \pm \ksn \left( x - \int h(t) dt - \left( -\frac{\mu_2\mu_3(1 + 6k + k^2) - \mu_1^2}{4\mu_2} t \right) \right) \right)} \), \hfill (70)

\( u_{38} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 - k)^2} \frac{1 - k}{2} \cn(x - \int h(t) dt - \alpha t) \) \( \frac{1 + \ksn(x - \int h(t) dt - \alpha t)(1 + \sn(x - \int h(t) dt - \alpha t))}{\sqrt{(1 + \ksn(x - \int h(t) dt - \alpha t))(1 + \sn(x - \int h(t) dt - \alpha t))}}, \hfill (71)

with
\[ \alpha = \left( \frac{-\mu_2\mu_3(1 - 6k + k^2) - \mu_1^2}{4\mu_2} \right). \hfill (72) \]

\( u_{39} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} \ns(x - \int h(t) dt - \alpha t) \pm \sqrt{\frac{-6\mu_3}{\mu_2} k^2 \sn(x - \int h(t) dt - \alpha t)}, \hfill (73)

with
\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm 6\sqrt{k^2} + 1 + k^2) - \mu_1^2}{4\mu_2} \right). \hfill (74) \]

\( u_{40} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} \dc(x - \int h(t) dt - \alpha t) \pm \sqrt{\frac{-6\mu_3}{\mu_2} k^2 \cd(x - \int h(t) dt - \alpha t), \hfill (75)

with
\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm 6\sqrt{k^2} + 1 + k^2) - \mu_1^2}{4\mu_2} \right). \hfill (76) \]

\( u_{41} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} k^2 \nc(x - \int h(t) dt - \alpha t) \pm \sqrt{\frac{6\mu_3}{\mu_2} k^2 \cn(x - \int h(t) dt - \alpha t), \hfill (77)

with
\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm 6\sqrt{k^2} + k^2 - k^2) - \mu_1^2}{4\mu_2} \right). \hfill (78) \]

\( u_{42} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} \cs(x - \int h(t) dt - \alpha t) \pm \sqrt{\frac{-6\mu_3}{\mu_2} k^2 \sc(x - \int h(t) dt - \alpha t), \hfill (79)

with
\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm 6\sqrt{k^2} - 1 - k^2) - \mu_1^2}{4\mu_2} \right). \hfill (80) \]
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\[ u_{43} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{6\mu_3}{\mu_2} k'^2}(x - \int h(t) dt - \alpha t) \pm \sqrt{\frac{6\mu_3}{\mu_2} \text{dn}(x - \int h(t) dt - \alpha t),} \]

with

\[ \alpha = \left( \frac{-4\mu_2 \mu_3 (\pm 6\sqrt{k'^2} - 1 - k'^2) - \mu_1^2}{4\mu_2} \right). \]

\[ u_{44} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{6\mu_3}{\mu_2} k'^2} \text{sd}(x - \int h(t) dt - \alpha t) \pm \sqrt{\frac{-6\mu_3}{\mu_2} \text{ds}(x - \int h(t) dt - \alpha t),} \]

with

\[ \alpha = \left( \frac{4\mu_2 \mu_3 (\pm 6\sqrt{-k'^2} - 1 + 2k'^2) - \mu_1^2}{4\mu_2} \right). \]

\[ u_{45} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{-\frac{6\mu_3}{\mu_2} k'^2 - 1 + 11} \text{dn}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} k'^2 - 1} \text{dn}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} k'^2 - 1} \text{ds}(x - \int h(t) dt - \alpha t), \]

with

\[ \alpha = \left( \frac{-4\mu_2 \mu_3 (\pm 3\frac{1}{2} - \frac{1}{2})(k'^2 - 1) - \mu_1^2}{4\mu_2} \right). \]

\[ u_{46} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{-\frac{6\mu_3}{\mu_2} (1 - k'^2) - 1 + 1} \text{cn}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} (1 - k'^2) - 1} \text{cn}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} (1 - k'^2) - 1} \text{sn}(x - \int h(t) dt - \alpha t), \]

with

\[ \alpha = \left( \frac{-4\mu_2 \mu_3 (\frac{3}{2}(1 - k'^2) - \frac{1}{2}(1 + k'^2)) - \mu_1^2}{4\mu_2} \right). \]

\[ u_{47} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{-\frac{6\mu_3}{\mu_2} k'^2 - 1 + 11} \text{dn}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} k'^2 - 1} \text{dn}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} k'^2 - 1} \text{ds}(x - \int h(t) dt - \alpha t), \]

with

\[ \alpha = \left( \frac{-\mu_2 \mu_3 (\pm 3k'^2 + 2(1 + k'^2)) - \mu_1^2}{4\mu_2} \right). \]

\[ u_{48} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{-\frac{6\mu_3}{\mu_2} (1 - k'^2) - 1 + 1} \text{cs}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} (1 - k'^2) - 1} \text{cs}(x - \int h(t) dt - \alpha t) \pm \sqrt{-\frac{6\mu_3}{\mu_2} (1 + k'^2) - 1} \text{dn}(x - \int h(t) dt - \alpha t), \]

with

\[ \alpha = \left( \frac{-\mu_2 \mu_3 (\pm 3k'^2 + 2(1 + k'^2)) - \mu_1^2}{4\mu_2} \right). \]
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with

\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm6\sqrt{(1 - k')^2(1 + k'^2)} + 2(1 + k'^2)) - \mu_1^2}{4\mu_2} \right). \]  (92)

\[ u_{49} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{24\mu_3}{\mu_2} k' \pm \operatorname{dn}^2(x - \int h(t) \, dt - \alpha t)} \]
\[ \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 + k'^2)} \frac{2\sqrt{k'} \operatorname{dn}(x - \int h(t) \, dt - \alpha t)}{k' \pm \operatorname{dn}^2(x - \int h(t) \, dt - \alpha t)}, \]  (93)

with

\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm6\sqrt{(1 + k'^2)^2} - (1 + k'^2) \pm 6k')} {4\mu_2} \right). \]  (94)

\[ u_{50} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 + k'^2) \frac{\operatorname{cn}^2(x - \int h(t) \, dt - \alpha t) + k' \operatorname{sn}^2(x - \int h(t) \, dt - \alpha t)}{\operatorname{cn}^2(x - \int h(t) \, dt - \alpha t) - k' \operatorname{sn}^2(x - \int h(t) \, dt - \alpha t)}} \]
\[ \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 - k'^2) \frac{\operatorname{cn}^2(x - \int h(t) \, dt - \alpha t) - k' \operatorname{sn}^2(x - \int h(t) \, dt - \alpha t)}{\operatorname{cn}^2(x - \int h(t) \, dt - \alpha t) + k' \operatorname{sn}^2(x - \int h(t) \, dt - \alpha t)}}, \]  (95)

with

\[ \alpha = \left( \frac{-4\mu_2\mu_3(\pm6(1 - k'^2) + 2(1 + k'^2)) - \mu_1^2}{4\mu_2} \right). \]  (96)

\[ u_{51} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 + k'^2) \frac{2(1 \pm \operatorname{cn}(x - \int h(t) \, dt - \alpha t))}{4(1 + k)((1 \pm \operatorname{sn}(x - \int h(t) \, dt - \alpha t)))}} \]
\[ \pm \sqrt{\frac{-6\mu_3}{\mu_2} k' \frac{(1 + k)((1 \pm \operatorname{sn}(x - \int h(t) \, dt - \alpha t)))}{2(1 \pm \operatorname{cn}(x - \int h(t) \, dt - \alpha t))}}, \]  (97)

with

\[ \alpha = \left( \frac{-\mu_2\mu_3(\pm12\sqrt{k(1 + k'^2)^2} + 1 + 6k + k'^2)}{4\mu_2} \right). \]  (98)

\[ u_{52} = \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2} \frac{1 - k'^2}{4} \sqrt{\frac{2}{1 - k}} \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 - k'^2)}} \]
\[ \times \sqrt{\frac{(1 \pm \operatorname{sn}(x - \int h(t) \, dt - \alpha t))(1 \pm \operatorname{sn}(x - \int h(t) \, dt - \alpha t))}{\operatorname{cn}(x - \int h(t) \, dt - \alpha t)}}, \]
\[ \pm \sqrt{\frac{-6\mu_3}{\mu_2} (1 - k'^2) \frac{1 - k}{2} \sqrt{\frac{\operatorname{cn}(x - \int h(t) \, dt - \alpha t)}{(1 \pm \operatorname{sn}(x - \int h(t) \, dt - \alpha t))(1 \pm \operatorname{sn}(x - \int h(t) \, dt - \alpha t))}}, \]  (99)

with

\[ \alpha = \left( \frac{-\mu_2\mu_3(\pm6\sqrt{2(1 - k'^2)(1 - k'^2)^2} + 1 - 6k + k'^2) - \mu_1^2}{4\mu_2} \right). \]  (100)
Case 4: rational solution

\[
\begin{aligned}
u_{53} &= \frac{-\mu_1}{2\mu_2} \pm \sqrt{\frac{-6\mu_3}{\mu_2}} \frac{1}{x - \int h(t)dt - \left(\frac{-\mu_1^2}{4\mu_2}\right)t}.
\end{aligned}
\]  

(101)

All the above solutions can be considered only in the case when \(\mu_2 \neq 0\).

5. Applications to arterial mechanics

In arterial mechanics [6], treating the arteries as a thin walled prestressed elastic tube with variable radius (or, with stenosis in [7]) and blood as an inviscid fluid, the governing equation which models the weakly nonlinear waves in such a fluid-filled elastic tubes is the variable-coefficient mKdV equation (3). Also, in [18–20] treating the arteries as a thin walled prestressed elastic tube and blood as an incompressible inviscid fluid, the governing equation which models the weakly nonlinear waves in such a fluid-filled elastic tubes is the mKdV equation (5). We obtained some useful solutions for these two equations such as periodic solution given by (57) and solitary wave solution given by (30) and anti-kink wave solution given by (31) which are shown in, (with \(h(t) = \text{const.}\)), fig 1, fig 2 and fig 3 respectively.

6. Concluding remarks

Using the classical symmetry analysis method we obtained two similarity transformations (17) and (20) that, for arbitrary \(h(t)\), transforms the original nonlinear PDE (1) into two nonlinear ODEs (18) and (21) respectively. Then we used the generalized Jacobi elliptic function expansion method to obtain many solutions for (21). The set of
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Figure 2. Solitary wave solution for the mKdV equation which is given by family2 (with $\mu_2 = 1$, $\mu_3 = 1$ and $h(t) = 1$).

Figure 3. Anti-kink wave solution for the mKdV equation which is given by family5 (with $\mu_2 = 1$, $\mu_3 = -1$ and $h(t) = 1$).

solutions from $u_1$ to $u_4$, from $u_6$ to $u_9$, from $u_{11}$ to $u_{38}$ and $u_{53}$ with $h(t) = 0$, can be found in [21–25]. To our best knowledge, the set of solutions $u_5$, $u_{10}$ and from $u_{39}$ to $u_{52}$ are new solutions of (1) and are not shown in the current literature until now.

The generalized variable-coefficient Gardner equation

$$u_t + a(t)uu_x + b(t)u^2u_x + h_1(t)u_{xxx} + d(t)u_x + f(t)u = 0 \quad (102)$$

was considered in [32] and solved with a new generalized algebraic method. Also, the set of solutions from $u_1$ to $u_4$, from $u_6$ to $u_9$, from $u_{11}$ to $u_{38}$ and $u_{53}$ (with $a(t) = \mu_1$, $b(t) = \mu_2$, $h_1(t) = \mu_3,d(t) = h(t)$ and $f(t) = 0$), are obtained in [32]. But, the set of solutions $u_5$, $u_{10}$ and from $u_{39}$ to $u_{52}$ are not obtained in [32].
Appendix A. Some special solutions of Equation (23).

| r     | p     | q     | φ            |
|-------|-------|-------|--------------|
| 0     | 1     | 0     | $e^{\pm \zeta}$ |
| 1     | 1     | 0     | $\sinh(\zeta)$ |
| -1    | 1     | 0     | $\cosh(\zeta)$ |
| 1     | -1    | 0     | $\sin(\zeta), \cos(\zeta)$ |
| 0     | 1     | -1    | $\frac{1}{\zeta}$ |
| 0     | 1     | 1     | $\text{sech}(\zeta)$ |
| 1     | -2    | 1     | $\tanh(\zeta), \coth(\zeta)$ |
| 0     | 2     | 1     | $\tan(\zeta), \cot(\zeta)$ |
| 0     | -1    | 1     | $\sec(\zeta), \csc(\zeta)$ |
| 1     | $-(1 + k'^2)$ | $k'^2$ | $\text{sn}(\zeta)$ |
| $k'^2$ | $k'^2 - k^2$ | $-k^2$ | $\text{cn}(\zeta)$ |
| -$k'^2$ | $1 + k'^2$ | $k'^2$ | $\text{sc}(\zeta) = \frac{\text{sn}(\zeta)}{\text{cn}(\zeta)}$ |
| -$k'^2k'^2$ | 1 | $-k'^2$ | $\text{dn}(\zeta)$ |
| $\frac{k'^2 - 1}{4}$ | $\frac{1 + k'^2}{2}$ | $\frac{k'^2 - 1}{4}$ | $\frac{(1 + k'^2) + \text{sn}(\zeta)}{\text{dn}(\zeta)}$ |
| $\frac{1 - k'^2}{4}$ | $\frac{1 + k'^2}{2}$ | $\frac{1 - k'^2}{4}$ | $\frac{(1 - k^2) + \text{sn}(\zeta)}{\text{dn}(\zeta)}$ |
| $\frac{k'^2}{4}$ | $\frac{1 + k'^2}{2}$ | $\frac{1 - k'^2}{4}$ | $\frac{k'^2 - \text{dn}(\zeta)}{\text{sn}(\zeta)}$ |
| $(1 - k^2)^2$ | $-2(1 + k'^2)$ | $(1 + k'^2)$ | $(1 + k^2)^2$ |
| $-4k'$ | $(1 + k^2) \pm 6k'$ | $(1 + k'^2)$ | $2\sqrt{(k'^2 + \text{dn}(\zeta))}$ |
| $(1 + k'^2)^2$ | $-2(1 + k'^2)$ | $(1 - k'^2)$ | $2\sqrt{(k'^2 + \text{dn}(\zeta))}$ |
| $\frac{(1 + k^2)^2}{4}$ | $\frac{1 + 6k + k'^2}{4}$ | $\frac{(1 - k)^2}{2}$ | $\frac{(1 + k)(1 \pm \text{sn}(\zeta))}{2(1 \pm \text{sn}(\zeta))}$ |
| $\frac{1 - k^2}{4}$ | $\frac{1 - 6k + k'^2}{4}$ | $\frac{(1 - k)^2}{2}$ | $\sqrt{\frac{1 - k}{2}} \frac{\text{cn}(\zeta)}{\sqrt{(1 + \text{sn}(\zeta))(1 + \text{sn}(\zeta))}}$ |

where $k^2 + k'^2 = 1$.

Acknowledgments

I wish to express my thanks, respectively, to Prof. Yu. Yu. Tarasevich, Prof. A. I. Lobanov and Prof. A. G. Kushner for valuable discussions concerning this work.

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