Lyapunov exponent for Whitney’s problem with random drive

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We consider the statistical properties of a non-falling trajectory in the Whitney problem of an inverted pendulum excited by an external force. In the case when the external force is white noise, we recently found the instantaneous distribution function of the pendulum angle and velocity over an infinite time interval using a transfer-matrix analysis of the supersymmetric field theory. Here, we generalize our approach to the case of finite time intervals and multipoint correlation functions. Using the developed formalism, we calculate the Lyapunov exponent, which determines the decay rate of correlations on a non-falling trajectory.

1. Balancing an inverted pendulum under a given time-dependent horizontal force \( f(t) \) is a famous mathematical problem formulated by Courant and Robbins in their book *What is Mathematics?* (first edition in 1941) \([1]\), where Whitney was credited as the author of the problem. Using fairly general mathematical arguments based on the intermediate value theorem, they showed that for any force \( f(t) \) acting during a finite time interval \([0,T]\), an initial position of the pendulum in the upper half-plane can be chosen such that it will remain in the upper half-plane during the further evolution for all \( t \in [0,T] \). The existence of a non-falling trajectory (non-FT) in the Whitney problem has been the subject of an ongoing debate in the mathematical literature \([2,3]\), resulting in a critical analysis and refinement of the original arguments of Courant and Robbins. Fresh interest in the problem of an inverted pendulum is associated with Arnold, whose view in 2002 was that this problem still awaits a rigorous solution \([4]\). In 2014, Polekhin presented a proof of the existence of the non-FT using the Wazewski topological principle \([5]\). This work provoked several publications generalizing his approach and proposing new topological methods \([6,8]\) (see Refs. \([8,9]\) for good reviews of the history of the Whitney problem).

Recently we developed a theory of the statistical description of a never falling trajectory (NFT) of an inverted pendulum under the action of a random force \([10]\). An NFT can be regarded as the limit of non-FTs in the Whitney problem as the length \( T \) of the time interval tends to infinity. The NFT concept is illustrated in Fig. 1 which shows numerical solutions to the boundary value problem for the pendulum equation (the angle \( \theta \) is measured from the vertical)

\[
\dot{\theta} = \omega^2 \sin \theta + f(t) \cos \theta \tag{1}
\]

with different initial and final values \( \theta (0) = \theta_1 \) and \( \theta (T) = \theta_2 \) and a sufficiently rapidly varying force \( f(t) \). For any \( \theta_{1,2} \) in the strip \( -\pi/2 < \theta_{1,2} < \pi/2 \), a non-falling solution \( -\pi/2 < \theta(T) < \pi/2 \) of this boundary value problem exists and is unique \([10]\). As \( \theta_1 \) and \( \theta_2 \) run through all possible values in the strip, the set of corresponding non-FTs form a bundle, shown in color in Fig. 1. This bundle shrinks as one moves away from the boundary, becoming exponentially thin in the middle of the interval for large \( T \). In the limit \( T \to \infty \), when the pendulum must be balanced on the entire real axis, the non-FT bundle for the Whitney problem on a finite time interval becomes infinitely thin and defines a unique never falling trajectory, which is a functional of the given force \( f(t) \).

In Ref. \([10]\), we studied the statistical properties of an NFT in the case when the driving force is Gaussian white noise with the correlator

\[
\langle f(t)f(t') \rangle = 2\alpha\delta(t-t'), \tag{2}
\]
and calculated the instantaneous distribution function $P(\theta, p)$ of the angle $\theta$ and its velocity $p = \dot{\theta}$. Our approach is based on the supersymmetric field theory formulation of stochastic dynamics proposed by Parisi and Sourlas [11–13], which allows averaging over the random force at the very beginning of the calculations. It is essential that for the considered problem, the Parisi–Sourlas method is free from the problem of the sign of the fermionic determinant because of the uniqueness of the non-FT. Using the idea of reducing the one-dimensional functional integral to an effective quantum mechanics [14], we were able to express the distribution function $P(\theta, p)$ in terms of the zero mode of the transfer-matrix Hamiltonian, which reduces to the Fokker–Planck operator with a special type of boundary conditions ensuring that the trajectories do not leave the strip.

Here, we extend the ideas of Ref. [10] and consider a range of issues related to the Lyapunov exponent for a non-FT. The Lyapunov exponent determines both the statistical properties of a non-FT and the decay of different-time correlators on the NFT. From the technical standpoint, our result consists in describing the entire spectrum of the transfer-matrix Hamiltonian, which reduces to the Fokker–Planck operator with a special type of boundary conditions enabling that the trajectories do not leave the strip.

We show that the Lyapunov exponent in the Whitney problem with white-noise driving [2] can be written as

$$\lambda = \omega g(\alpha/\omega^3),$$

where the function $g(x)$ has the asymptotic behavior

$$g(x) = \begin{cases} 
1 + \frac{3}{8}x - \frac{525}{1024}x^2 + \ldots, & x \ll 1, \\
0.66x^{1/3} + 0.26 + 0.30x^{-1/3} + \ldots, & x \gg 1.
\end{cases}$$

In the absence of driving ($\alpha = 0$), the Lyapunov exponent $\lambda = \omega$ is determined by the exponential instability of the trajectories near the upper pendulum position. For weak driving ($\alpha/\omega^3 \ll 1$), the typical non-FT angle is of the order $\theta \sim (\alpha/\omega^3)^{1/2}$ [10], and the nonlinearity of Eq. (1) leads to an increase in the Lyapunov exponent, which can be expanded in an asymptotic series in powers of the small parameter $\alpha/\omega^3$. Finally, under strong driving ($\alpha/\omega^3 \gg 1$), the Lyapunov exponent becomes independent of $\omega$, reaching the limiting value $\lambda \approx 0.66 \alpha^{1/3}$. We show the numerically found dependence of the Lyapunov exponent on $\alpha/\omega^3$ in Fig. 2.

According to the approach developed in Ref. [10], the statistical properties of a non-FT are expressed in terms of the two-component “wave function” $\Psi(\theta, p) = (\psi, \phi)^T$, whose evolution is governed by the imaginary-time Schrödinger equation with the corresponding transfer-matrix Hamiltonian:

$$\frac{\partial}{\partial t} (\psi, \phi) = -\mathcal{H} (\psi, \phi), \quad \mathcal{H} = \left( L, \frac{-1}{V_2} L \right),$$

where $L$ is the Fokker–Planck operator for the Kramers problem [15],

$$L = p\partial_p + \omega^2 \sin^2 \theta \partial_p - \alpha \cos^2 \theta \partial_p^2,$$

and the potential $V_2$ has the form

$$V_2 = -\omega^2 \cos \theta - \alpha \sin^2 \theta \partial_p.$$

In Ref. [10], we studied the one-point correlation function of the NFT on the infinite time interval, which is determined by the zero mode $\Psi_0$ of the Hamiltonian $\mathcal{H}$. Finding the zero mode is significantly simplified due to the presence of the Becchi-Rouet-Stora-Tyutin (BRST) symmetry of the action in the Parisi–Sourlas representation of stochastic dynamics [13], which allows expressing both components of $\Psi(\theta, p)$ in terms of a scalar superpotential $\psi(\theta, p)$ via

$$\psi = \partial_p \psi, \quad \phi = -\partial_\theta \psi.$$  

The time evolution of $\psi$ is determined by the Fokker–Planck operator $L$:

$$\frac{\partial \psi}{\partial t} = -L \psi.$$  

But reduction (5) works neither for calculating multi-time correlation functions of the NFT nor for describing the statistics of a non-FT on bounded intervals. In the former case, the BRST symmetry is broken by the operators of physically observable quantities acting identically on the wave function components $\psi$ and $\phi$. In the latter case, the BRST symmetry is broken by the
BRST-asymmetric initial condition at the boundary of the interval [see Eq. (10) below]. In both cases, to describe the non-FT statistics, one must work with the two-component wave function \( \Psi, \Phi \) and understand the properties of the Hamiltonian \( H \).

We start with discussing the initial condition for the wave function at the boundary of an interval. To ensure that the non-FT is unique, we must fix the value of \( \theta \) at the boundary. (Generally speaking, one can fix the value of \( \theta \) or even a linear combination of \( \theta \) and \( \dot{\theta} \), but for simplicity, we assume that the angle is given.) By construction, the wave function \( \hat{\Psi} \) is closely related to the partition function of the supersymmetric functional integral [10]. Right at the boundary, it cannot contain Grassmann variables, which leads to the component \( \Phi \) vanishing.

Hence, the wave function at the interval boundary with the fixed value \( \theta = \theta_0 \) has the form

\[
\hat{\Psi}^{(b)}_{\theta_0} = \begin{pmatrix} \delta(\theta-\theta_0) \\ 0 \end{pmatrix}.
\]

Consider the boundary value problem \([T_L, T_R] \) with the boundary conditions \( \theta(t_L) = \theta_1 \) and \( \theta(t_R) = \theta_R \). The essence of the reduction of the Parisi–Sourlas integral to the quantum mechanics (5) is that the correlation function \( \langle O_1(t_1)O_2(t_2) \ldots \rangle \) of physical quantities \( O \) at the instants \( t_i \) \( (t_1 < t_2 < \ldots) \) can be represented as the matrix element

\[
\langle \Psi^{(b)}_{\theta_0} \cdots O_2(t_2)e^{-H(t_2-t_1)}O_1(t_1)e^{-H(t_1-t_0)}|\hat{\Psi}^{(b)}_{\theta_0} \rangle,
\]

where the scalar product of two wave functions is defined as [10]

\[
\langle \Psi|\Psi' \rangle = \int d\theta dp [\Psi(\theta,p)\Phi'(-p) + \Phi(\theta,p)\Psi'(-p)].
\]

In Ref. [10], we studied the instantaneous joint distribution function \( P(\theta,p) \) of the angle and velocity on the NFT corresponding to the operator \( O = \delta(\theta-\theta_0)\delta(p-p_0) \). Replacing \( \Psi_{L,R} \) with the zero mode and using Eq. (5), one can express \( P(\theta,p) \) in terms of the Poisson bracket of the superpotential:

\[
P(\theta,p) = \{\psi(\theta,p),\psi(-p,-\theta)\}_{\theta,p}.
\]

Both the Hamiltonian in Eq. (5) and the Fokker–Planck operator [6] are non-Hermitian. Generally speaking, such operators lack a complete system of eigenfunctions. However it is known that in the presence of friction the Fokker–Planck operator can be diagonalized [15], which makes it possible to construct a system of biorthogonal eigenfunctions and work with them practically as with eigenfunctions of a Hermitian operator [16]. But there is no friction in our case, and we should therefore expect that the operators \( H \) and \( L \) reduce to the Jordan normal form. This results not in a simple exponential decay of correlators as \( t \to \infty \) but in the appearance of additional powers of time [e.g., as can be seen in expression (30)].

3. To illustrate the developed approach, we consider the case of a weak noise \( (\alpha^2/\omega^3 \ll 1) \) in detail, where the Jordan structure of the operators \( H \) and \( L \) can be studied analytically. We start with the Fokker–Planck operator. In the considered limit, the deviation of the pendulum from the vertical is small (\( \theta \ll 1 \)), and the operator [6] can be replaced with

\[
L = p\partial_p + \omega^2 \partial_p \theta - \alpha \partial_p^2.
\]

The zero mode of this operator corresponding to the NFT has the form

\[
\psi_0(\theta,p) = \text{erf}(z)/2,
\]

where we introduce “holomorphic” and “antiholomorphic” coordinates with different signs of the momentum,

\[
z = \kappa (p - \omega \theta), \quad \bar{z} = -\kappa (p + \omega \theta),
\]

where \( \kappa = \sqrt{\omega/2\alpha} \). The spectrum of the operator [14] can be found using the identity \([L, \partial_z] = \omega \partial_z\), which allows generating the eigenfunctions by consecutively differentiating the zero mode with respect to \( z \). We thus find the eigenfunction of the \( n \)th excited state \( (n = 1, 2, 3, \ldots) \) with the energy \( \epsilon_n = n\alpha \omega \)

\[
\psi_n(z) = \frac{1}{\sqrt{\pi}} H_{n-1}(z)e^{-z^2},
\]

where \( H_n(z) = (-1)^n e^{z^2} d^n e^{-z^2}/dz^n \) is the Hermite polynomial (in the physical definition). But the functions \( \psi_n(\theta,p) \) thus constructed depend only on the difference \( p - \omega \theta \) (do not contain \( \bar{z} \)) and therefore do not form a complete basis. This circumstance is related to the fact that the non-Hermitian operator [6] can be brought to the Jordan normal form and in addition to the eigenfunction has several generalized eigenfunction corresponding to the same eigennumber \( \epsilon_n \). It is easy to verify that the eigenfunction \( \psi_n \) has \( n - 1 \) generalized eigenfunctions, which we choose in the form

\[
\psi_{n,k} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} H_k(\bar{z})H_{n-k-1}(z)e^{-z^2},
\]

where the index \( k \) ranges from 1 to \( n - 1 \). Together with \( \psi_{n,0} = \psi_n \), they form the basis of a Jordan block of dimension \( n \) corresponding to the energy \( \epsilon_n = n\alpha \omega \):

\[
L\psi_{n,k} = \epsilon_n \psi_{n,k} + \omega \psi_{n,k-1}
\]

(to truncate the chain at the eigenfunction \( \psi_{n,0} \), we set \( \psi_{n,-1} = 0 \)).

The constructed system of functions is complete. An arbitrary function can be decomposed with respect to the basis \( \psi_{n,k} \) using the orthogonality relation

\[
\langle \psi_{n,k} | \psi_{n',k'} \rangle = (0)_{n+k'+1} \delta_{n,n'} \delta_{k,k'+1,n',}
\]

where the scalar product \( \langle \cdot | \cdot \rangle \) is defined as

\[
\langle \psi | \psi' \rangle = \int dz d\bar{z} \psi(\bar{z}, z)\psi'(z, \bar{z}),
\]
and exchanging the arguments in one of the functions thus agrees with the sign change for $p$ in Eq. (12). We note that the integration measures in Eqs. (12) and (21) are related by $dz \, d\xi = 2 \omega \kappa^2 d\epsilon_i d\rho_j$.

During the evolution of the wave function $\psi_{n,k}$ under the action of the operator $L$, other states of the Jordan block corresponding to the same energy are mixed into it, which leads to the appearance of powers of $t$ on top of the exponential decay:

$$e^{-Lt} \psi_{n,k} = e^{-\omega_{n,m}t} \sum_{m=-\infty}^{\infty} \frac{(-\omega_{n,m})^m}{m!} \psi_{n,k-m}.$$  \hspace{1cm} (22)

We now turn to studying the spectral properties of the Hamiltonian $H$ in Eq. (1). In the considered case of weak noise, Eq. (7) gives $V_2 = -\omega^2$, which partitions the state space of $H$ into even and odd sectors with the wave functions $\psi_{e,o} = (\Psi, \pm \omega \Psi)^T$ evolving independently with the Hamiltonians $H_{e,o} = L \mp \omega$. The system of eigenfunctions and generalized eigenfunctions of the operator $L$ constructed above thus allows completely describing the evolution of the doublet $\Psi$ under the action of the Hamiltonian $H$.

Consider the evolution of the wave function $|10\rangle$ away from the boundary in the limit $\alpha/\omega^3 \ll 1$. Decomposing it into even and odd components, we obtain

$$e^{-Ht} \psi_{10} = \left( \frac{\cosh \omega t}{\omega \sinh \omega t} \right) e^{-Lt} \delta(\theta - \theta_0).$$  \hspace{1cm} (23)

To calculate the evolution of the delta function, we expand it in the basis $\psi_{n,k}$:

$$\delta(\theta - \theta_0) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} c_{n,k} \psi_{n,k}.$$  \hspace{1cm} (24)

The coefficients $c_{n,k}$ can be obtained using orthogonality relations and the properties of Hermite polynomials

$$H_n(x+y) = \sum_{m=0}^{n} \binom{n}{k} (2y)^{n-k} H_k(x)$$  \hspace{1cm} (25)

following from the Taylor expansion, and are given by

$$c_{n,k} = (-1)^{n-k} 2^{n-k} \frac{(\theta_0/2\kappa \omega)^{n-k-1}}{(n-2k-1)!}.$$  \hspace{1cm} (26)

The evolution of the delta function in Eq. (23) follows from expansion and relations (24). The memory of the boundary is lost in the characteristic time $\omega^{-1}$ (the inverse Lyapunov exponent). During this time, the difference between the two components of the wave function $\Psi$ is lost, and they both take the value determined by the state $\psi_{1,0}$ with the minimum energy $\epsilon_1 = \omega$:

$$\lim_{t \to \infty} e^{-Ht} \psi_{10} = \frac{1}{\omega} \kappa \psi_{1,0},$$  \hspace{1cm} (27)

which is just the zero mode of $|5\rangle$ in the limit $\alpha/\omega^3 \ll 1$.

4. We show how the developed spectral theory of the operators $H$ and $L$ allows systematically calculating various correlation functions of the non-FT in the case of weak noise. The results in this section can also be obtained directly by using the explicit expression for the non-FT in terms of $f(t)$ with subsequent averaging over Gaussian white noise [2, 10], but deriving them using the transfer-matrix formalism is important methodologically because it illustrates the general scheme and allows verifying its workability.

We begin by considering the calculation of the pair correlator for the NFT angle on the entire real axis. Substituting the zero mode (27) into the general formula (11) and taking into account that only the even sector of the theory does contribute, we can express the correlator in terms of the scalar product (21) in the $z$-representation as

$$\langle \theta(0) | \theta(t) \rangle = \langle \psi_{1,0} | e^{-(L-\omega)t} \theta | \psi_{1,0} \rangle.$$  \hspace{1cm} (28)

Using Eqs. (10) and (13), we can express $\theta | \psi_{1,0}$ in terms of the functions $\psi_{2,0}$ and $\psi_{2,1}$. Then using the evolution law (22), we obtain

$$e^{-(L-\omega)t} \theta | \psi_{1,0} = e^{-\omega t} \psi_{2,1} - (1/2 + \omega t) \psi_{2,0}.$$  \hspace{1cm} (29)

Calculating the matrix element (28) as the overlap between the states $e^{-(L-\omega)t} \theta | \psi_{1,0}$ and $\theta | \psi_{1,0}$ with the help of relations (20), we find the sought pair correlator:

$$\langle \theta(0) | \theta(t) \rangle = \langle \theta^2 \rangle (1 + \omega t) e^{-\omega t},$$  \hspace{1cm} (30)

where, as obtained in Ref. [10],

$$\langle \theta^2 \rangle = \alpha/2\omega^3.$$  \hspace{1cm} (31)

The appearance on the background $e^{-\omega t}$ of a contribution linearly increasing with time is related to excitation of the exponential decay. But the structure of the correlator (30) is verifying its workability.

In a similar way one can calculate more complicated correlators of the NFT. For example,

$$\langle \theta^2(0) | \theta^2(t) \rangle = \langle \theta^2 \rangle^2 \left[ 1 + 2(1 + \omega t)^2 e^{-2\omega t} \right].$$  \hspace{1cm} (32)

Formally, the operator $\theta^2$ here applied to $\psi_{1,0}$ excites the Jordan triplet $\psi_{3,0}, \psi_{3,1}, \psi_{3,2}$, which leads to the appearance of terms up to $t^2$ on the background of the exponential decay. But the structure of the correlator (30) is related to the Gaussian statistics of $\theta$ on the NFT, which allows expressing it in terms of pair correlator (30) using the Wick theorem. Generalizing the developed formalism to multipoint correlators is also straightforward.

As the next example, we consider the calculation of the average angle $\langle \theta(t) \rangle_{\theta_0}$ for the non-FT on the semi-infinite time interval $t > 0$ with the boundary condition $\theta(0) = \theta_0$. According to Eq. (11), the average angle is given by the matrix element $\langle \theta(t) \theta_0 \rangle = \langle \Psi_0 | e^{-Ht} \theta | \Psi_{10} \rangle$. It is easiest to calculate by convoluting expression (29)
with the wave function $\psi_{1,0}$ at the boundary. Integrating over the momentum, we see that the contribution from $\psi_{2,0} = 2\pi e^{-z^2}/\sqrt{\pi}$ disappears because it is odd in $z$, and we obtain the simple exponential decay

$$
\langle \theta(t) \rangle_{\theta_0} = \theta_0 e^{-\omega t}.
$$

(33)

One can derive the same expression differently by calculating the matrix element $\theta$ between the zero mode $\psi_{1,0}$ and evolved boundary wave function $\psi_{2,1}$. Such matrix elements are nonzero only with the Jordan doublet $\psi_{2,0}$ and $\psi_{2,1}$. But according to $\psi_{2,1}$, $\psi_{2,0}$ is not included in the expansion of the delta function, while $\psi_{2,0}$ is an eigenfunction and does not generate a linear term during evolution. As a result, we again come to expression (33).

A comparison of expressions (30) and (33) shows that despite the presence of the additional factor $\omega t$, we can calculate the first few polynomials $L\psi$ of the correlations at large times, is determined by the kurtosis

$$
\lambda = \lim_{t \to \infty} \frac{\partial \ln(\langle \theta(0)\theta(t) \rangle)}{\partial t} = -\lim_{t \to \infty} \frac{\partial \ln(\langle \theta(t) \rangle_{\theta_0})}{\partial t}.
$$

(34)

5. We now proceed to calculating the Lyapunov exponent for the non-FT algorithm for arbitrary values of the parameter $\alpha/\omega^3$. The Lyapunov exponent, which governs the decay of the correlations at large times, is determined by the finiteness of the interval $(\pi/2, \pi/2)$. Note that the difference from the normal distribution measured by the kurtosis $\langle \theta^4 \rangle / \langle \theta^2 \rangle^2$ occurs only in the first order in $x = \alpha/\omega^3$. A negative value of $\langle \theta^4 \rangle$ is related to suppression of the tails of $P(\theta)$ due to the finiteness of the interval $(-\pi/2, \pi/2)$.

In the case of an arbitrary noise strength, the excited states of operator $[\partial_t]^{\beta}$ can be constructed only numerically. To determine the Lyapunov exponent $\lambda = \epsilon_1$, we must find the first excited state by solving the equation $\psi^{(1)}(\theta, p) = \epsilon_1 \psi(\theta, p)$ with the boundary conditions

$$
\psi(\pi/2, p < 0) = \psi(\theta, -\infty) = 0,
$$

(37a)

$$
\psi(-\pi/2, p > 0) = \psi(\theta, \infty) = 0.
$$

(37b)

These boundary conditions are similar to the boundary conditions for the zero mode of the superpotential derived in Ref. [10], with the only difference that in the part of the boundary where the wave function is specified, its value is zero and not $\pm 1/2$.

In Fig. 3 we show the first excited state determined numerically for various values of the parameter $\alpha/\omega^3$. For small $\alpha/\omega^3$, the function $\psi_1(\theta, p)$ is close to the Gaussian $\psi_{1,0}(z)$, slightly increasing near $\theta = \pm \pi/2$. As $\alpha/\omega^3$ increases, the maximum of $\psi_1(\theta, p)$ near the boundaries of the interval become more pronounced, and at $\alpha/\omega^3 \to \infty$, the first mode has two humps localized near the boundaries. In Fig. 2 we plot the energy of the first mode (which determines the Lyapunov exponent) as a function of the parameter $\alpha/\omega^3$. For small $\alpha/\omega^3$, the numerical calculation agrees with expression (3) obtained using the perturbation theory up to the values $\alpha/\omega^3 \approx 0.25$. For large $\alpha/\omega^3$, the Lyapunov exponent in units of $\omega$ can be expanded in powers of $(\alpha/\omega^3)^{1/3}$ with the leading term $\lambda \approx 0.66 \alpha^{1/3}$.

In conclusion, we note that the developed theory is a generalization of the supersymmetric approach proposed in Ref. [10] to the case of a non-FT on finite time intervals.
and to multipoint correlation functions. The suggested classification of the excited states of the transfer-matrix Hamiltonian completes the construction of the theory of the statistical properties of a non-FT in the Whitney problem with random short-range driving. The developed formalism allows finding any correlation functions on a non-FT by solving partial differential equations of the Fokker–Planck type with specific boundary conditions.

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