UNIQUENESS OF THE STOCHASTIC KELLER–SEGEL MODEL IN ONE DIMENSION

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Abstract. In a recent paper (J. Differential Equations, 310: 506–554, 2022), the authors proved the existence of martingale solutions to a stochastic version of the classical Patlak–Keller–Segel system in 1 dimension (1D), driven by time-homogeneous spatial Wiener processes. The current paper is a continuation and consists of two results about the stochastic Patlak–Keller–Segel system in 1D. First, we establish some additional regularity results of the solutions. The additional regularity is, e.g. important for its numerical modelling. Then, as a second result, we obtain the pathwise uniqueness of the solutions to the stochastic Patlak–Keller–Segel system in 1D. Finally, we conclude the paper with the existence of strong solution to this system in 1D.

Keywords and phrases: Chemotaxis, Keller–Segel model, Stochastic Partial Differential Equations, Stochastic Analysis, Mathematical Biology, pathwise uniqueness.

AMS subject classification (2010): Primary 60H15, 92C17, 35A01; Secondary 35B65, 35K87, 35K51, 35Q92.

1. Introduction

This paper is a continuation of [14], where the authors proved the existence of martingale solutions to a stochastic version of the classical Patlak–Keller–Segel system driven by time-homogeneous spatial Wiener processes. The existence of solutions is shown in the weak probabilistic sense. We start with obtaining some uniform bounds on the higher moments of the solution. Then, we aim to establish the existence of a unique, strong solution of the stochastic version of the classical Patlak–Keller–Segel system driven by time-homogeneous spatial Wiener processes in one dimension.

In the proof of the existence of a solution, compactness arguments are employed. Doing so, the underlying probability space gets lost, and one needs to introduce a concept of probabilistic weak solutions, i.e. martingale solutions (see Definition 2.2 in [14]). This paper aims to give sufficient conditions to ensure pathwise uniqueness of the solutions to the system (1.2). By the Yamada–Watanabe Theorem, it follows that a global solution exists on every stochastic basis and is unique. We proceed first by proving pathwise uniqueness of the solutions, and then we apply a result of Yamada–Watanabe–Kurtz to show that these solutions are strong and unique in law.

The Yamada–Watanabe theory has been well developed for stochastic equations under the influence of Wiener noise, see e.g. [6, 10, 15, 16, 20, 17, 22], even so, only a few results are dealing with the variational setting, see [19].

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Problem description: Let $\mathfrak{F} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual condition. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be some Bessel potential spaces to be specified later. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two cylindrical Wiener processes defined over $\mathfrak{F}$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let us define the Laplacian $\Delta := \frac{\partial^2}{\partial x^2}$ with the Neumann boundary conditions given by

\[
\begin{align*}
D(A) &:= \{ u \in H^2(0,1) : u_x(0) = u_x(1) = 0 \}, \\
Au &:= \Delta u = u_{xx}, \ u \in D(A).
\end{align*}
\]

In this paper, we consider the following equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad du - (r_u \Delta u - \chi \text{div}(u \nabla v)) \ dt = u \circ dW_1, \\
\quad dv - (r_v \Delta v - \alpha v) \ dt = \beta u \ dt + v \circ dW_2,
\end{array} \right.
\end{align*}
\]

where $\mathcal{O} = [0,1]$, $\beta \geq 0$ is the production weight corresponding to $u$. The positive terms $r_u$ and $r_v$ are the diffusivity of the cells and chemotactic sensitivity, respectively. The positive value $\chi$ is the chemotactic sensitivity, $\alpha \geq 0$ is the so-called damping constant, $\beta \geq 0$ is the production weight corresponding to $u$. The initial conditions are given by $u(0) = u_0$ and $v(0) = v_0$. Since we model an intrinsic noise, the stochastic integral above is interpreted in the Stratonovitch sense denoted by $\circ$.

An important limitation of the Stratonovitch stochastic integral is that the Stratonovitch integral is not a martingale and the Burkholder–Davis–Gundy inequality does not hold here. Consequently, it is more appropriate to work the equation in the Itô form. Following the technical analysis involved in the conversion between the Itô and Stratonovitch form, one can see that the system (1.2) is equivalent to the following system:

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad du - (r_u \Delta u - \chi \text{div}(u \nabla v)) \ dt = udW_1 + \gamma u \ dt, \\
\quad dv - (r_v \Delta v - \alpha v) \ dt = \beta u \ dt + vdW_2,
\end{array} \right.
\end{align*}
\]

where $\gamma = \gamma_1 \leq \sum_{k=1}^{\infty} (\lambda_k)^2$ (see Remark 1.3). For further details regarding the form of the correction term and the technical analysis involved in the conversion, we refer to [23], [9, p. 65, Section 4.5.1].

For completeness we add the definition of strong solutions to system (1.2). For a Banach space $E$, the space $C_0^0([0,T]; E)$ is the set of all continuous and bounded functions $u : [0, T] \to E$.

**Definition 1.1.** We call a pair of processes $(u, v)$ a strong solution to system (1.2) if $u : [0, T] \times \Omega \to L^2(\mathcal{O})$ and $v : [0, T] \times \Omega \to H^1(\mathcal{O})$ are $\mathbb{P}$-progressively measurable processes such that $\mathbb{P}$-a.s. $(u, v) \in C_0^0([0,T]; L^2(\mathcal{O})) \times C_0^0([0,T]; H^1(\mathcal{O}))$ and satisfy for all $t \in [0, T]$ and $\mathbb{P}$-a.s. the integral equation

\[
\begin{align*}
\quad u(t) &= e^{t r_u \Delta} u_0 - \chi \int_0^t e^{(t-s) r_u \Delta} \text{div}(u(s) \nabla v(s)) \ ds + \int_0^t e^{(t-s) r_u \Delta} u(s) \circ dW_1(s), \\
\quad v(t) &= e^{t (r_v \Delta - \alpha I)} v_0 + \beta \int_0^t e^{(t-s) (r_v \Delta - \alpha I)} u(s) \ ds + \int_0^t e^{(t-s) (r_v \Delta - \alpha I)} v(s) \circ dW_2(s).
\end{align*}
\]

\[\text{The process } \xi : [0, T] \times \Omega \to X \text{ is said to be progressively measurable over a probability space } (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}) \text{ if, for every time } t \in [0, T], \text{ the map } (s, \omega) \mapsto \xi_s(\omega) \text{ is } B([0,t]) \otimes \mathcal{F}_t\text{-measurable. This implies that } \xi \text{ is } (\mathcal{F}_t)_{t \in [0,T]}\text{-adapted.}\]
Let us consider the complete orthonormal system of the underlying Lebesgue space $L^2(O)$ given by the trigonometric functions (see [13, p. 352])

$$\psi_k(x) = \begin{cases} \sqrt{2} \sin (2\pi k x) & \text{if } k \geq 1, x \in O, \\ 1 & \text{if } k = 0, x \in O, \\ \sqrt{2} \cos (2\pi k x) & \text{if } k \leq -1, x \in O. \end{cases}$$

(1.4)

For the proof of the existence of the solution, the Wiener perturbation needs to satisfy regularity assumptions given in the next hypotheses.

**Assumption 1.2.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Bessel potential spaces such that the embeddings $\iota_1 : \mathcal{H}_1 \hookrightarrow L^2(O)$ and $\iota_2 : \mathcal{H}_2 \hookrightarrow H^1_0(O)$ are Hilbert–Schmidt operators. The Wiener processes $W_1$ and $W_2$ are two cylindrical processes on $\mathcal{H}_1$ and $\mathcal{H}_2$.

**Remark 1.3.** As an example we can take $\mathcal{H}_1 = H^\delta_1(O)$ for $\delta_1 > 1$, and $\mathcal{H}_2 = H^\delta_2(O)$ for $\delta_2 > 2$ where $H^\delta_i(O)$, $i = 1, 2$, is to be defined in Notation 1.1. The Wiener processes $W_1$ and $W_2$ are then given by

$$W_1(t, x) = \sum_{k \in \mathbb{Z}} \psi_k^{(\delta_1)}(x) \beta_k^{(1)}(t) \quad \text{and} \quad W_2(t, x) = \sum_{k \in \mathbb{Z}} \psi_k^{(\delta_2)}(x) \beta_k^{(2)}(t)$$

where $\psi_k^{(\delta_i)} := (1 + (2\pi k)^2)^{-\delta_i/2} \psi_k$, $i = 1, 2$, with $\psi_k$ defined by (1.4). Here $\{\psi_k^{(\delta_i)} : k \in \mathbb{Z}\}$, $i = 1, 2$, are two families of mutually independent identically distributed standard Brownian motions. We note that $\{\psi_k^{(\delta_i)} : k \in \mathbb{Z}\}$ forms an orthonormal system in $H^\delta_i(O)$, $i = 1, 2$. We will also write

$$W_i(t, x) = \sum_{k \in \mathbb{Z}} \lambda_k^{(i)} \psi_k(x) \beta_k^{(i)}(t), \quad i = 1, 2,$$

(1.5)

where, for $i = 1, 2$, the sequence $\{\lambda_k^{(i)} : k \in \mathbb{Z}\}$, $\lambda_k^{(i)} := (1 + \mu_k)^{-\delta_i/2} = (1 + (2\pi k)^2)^{-\delta_i/2}$, is non-negative with $\lambda_k^{(i)} = \lambda_k^{(i)}$ for all $k \in \mathbb{Z}$, where $\mu_k = (2\pi k)^2$ are the corresponding eigenvalues of $-\Delta$ with Neumann boundary conditions. Since we will need it later on, for each $i = 1, 2$, let us define the constant

$$\gamma_i := \sum_{k \in \mathbb{N}} |\lambda_k^{(i)}|^2 \|\psi_k\|_{L^\infty}^2 \leq \sum_{k \in \mathbb{Z}} (\lambda_k^{(i)})^2.$$

Since the solution $(u, v)$ represents the cell density and concentration of the chemical signal, $u$ and $v$ are non-negative. This implies that the initial conditions $u_0$ and $v_0$ are non-negative. Besides, one needs to impose some more regularity assumptions on $u_0$ and $v_0$.

**Assumption 1.4.** Let $u_0 \in L^2(O)$ and $v_0 \in H^1_0(O)$ be two random variables over $\mathfrak{M}$ such that

(a) $u_0 \geq 0$ and $v_0 \geq 0$;

(b) $(u_0, v_0)$ is $\mathcal{F}_0$-measurable;

(c) $\mathbb{E} \left[\|u_0\|_{L^2}^2\right] < \infty$ and $\mathbb{E} \left[\|v_0\|_{L^2}^2\right] < \infty$.

**Notation 1.1.** Let $A_1$ be the positive Laplace operator $-\Delta$ (as an operator defined on $L^2(\mathbb{R})$) restricted to functions defined on $O$, namely,

$$A_1 := -\Delta, \quad D(A_1) := H^2_0(O) \cap H^4_0(O),$$
and let \((\psi_j, \rho_j)\) be the eigenfunctions and eigenvalues of \(A_1\). The Bessel potential space \(H^s(\mathcal{O})\) is defined by

\[
H^s(\mathcal{O}) = \left\{ u = \sum_j a_j \psi_j \in L^2(\mathcal{O}) : \|u\|_{H^s(\mathcal{O})} = \left( \sum_j a_j^2 \rho_j^{2s} \right)^{1/2} < \infty \right\}.
\]

**Notation 1.2.** Let us define by \(L^{\log}\) the Zygmund space (see [4, Definition 6.1, p. 243]) consisting of all Lebesgue-measurable functions \(f : \mathcal{O} \to \mathbb{R}\) for which \(\int_{\mathcal{O}} |f(x)| \log(|f(x)|) \, dx < \infty\). This space is equipped with the norm

\[
|f|_{L^{\log}} := \int_0^1 f^*(t) \log(\frac{1}{t}) \, dt,
\]

where \(f^*\) is defined by

\[
f^*(t) = \inf\{\lambda : \delta_f(\lambda) \geq t\}, \quad t \geq 0.
\]

Here \(\delta_f(\lambda)\) is the Lebesgue measure of the set \(\{x \in \mathcal{O} : |f(x)| > \lambda\}\), \(\lambda \geq 0\). We note that \(f \in L^{\log} \iff \int_{\mathcal{O}} |f(x)| \log(2 + |f(x)|) \, dx < \infty\) (see p. 252 of [4]). Note also that by Theorem 6.5 p. 247 in [4] we have that \(L^{\log}(\mathcal{O}) \hookrightarrow L^1(\mathcal{O})\).

Let us briefly describe the content of this paper. In Section 2 we formulate a proposition to provide some uniform bounds on \(u\) and \(v\) in such a way that we can control the \(L^2(\Omega; L^\infty(0,T; L^2(\mathcal{O})))\) norm of \(u\) and the \(L^2(\Omega; L^\infty(0,T; H^2_2(\mathcal{O})))\) norm of \(v\). We postpone the proof of this Proposition to Section 3. In Section 3 we deduce that if there exist two solutions on the same probability space, then both the solutions are identical. Here, we prove our main result on the existence of a unique strong solution to the system (1.2) which we reformulated as Corollary 3.3. In the Appendix, we collect some elementary results which are needed in the course of analysis.

## 2. Additional regularity

In this section, we formulate the following proposition to obtain some uniform bounds on \(u\) and \(v\) in such a way that we can control the \(L^2(\Omega; L^\infty(0,T; L^2(\mathcal{O})))\) norm of \(u\) and the \(L^2(\Omega; L^\infty(0,T; H^2_2(\mathcal{O})))\) norm of \(v\). While carrying out this formulation, in order to handle the non-linear term \(u \nabla v\), it is essential to obtain the bound for the \(p\)-th moment of \(H^2_2(\mathcal{O})\) norm of \(v\) for \(p > 1\) i.e.,

\[
v \in L^p(\Omega; L^p(0,T; H^2_2(\mathcal{O}))).
\]

To do so, we will apply Theorem 4.5 in [21]. Consequently, we will end up working in the real interpolation space

\[
(H^2_2(\mathcal{O}), H^{-1}_2(\mathcal{O}))_{\frac{1}{1-p}, p} = B^{\frac{1}{p}, \frac{p}{p-1}}_{2, p}(\mathcal{O}),
\]

which is a Besov space; see [21] p. 1406.

**Proposition 2.1.** Let \(T > 0\) and \(\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a probability space satisfying the usual conditions. Also, let \((\mathcal{W}_1, \mathcal{W}_2)\) be a pair of Wiener processes over \(\mathfrak{A}\), and \(H_1\) and \(H_2\) be two Bessel potential spaces which satisfy Assumption 1.2. Let the initial data \((u_0, v_0)\) satisfy Assumption 1.2 with \(\mathbb{E}|u_0|_{L^2}^{12} < \infty\) and \(\mathbb{E}|v_0|_{L^2}^{12} < \infty\). If \((u, v)\) is a martingale solution solution
to the system (1.3), then, there exist positive constants $a_0$ and $a_1$ such that we have the following inequality

$$
\begin{align*}
\frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s)|^2_{L^2} \right] + \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\nabla v(s)|^2_{L^2} \right] + \frac{r_u}{4} \mathbb{E} \left[ \int_0^t |\nabla u(s)|^2_{L^2} \, ds \right] \\
+ \frac{r_v}{2} \mathbb{E} \left[ \int_0^t \int_0^t |\Delta v(s, x)|^2 \, dx \, ds \right] + \alpha \mathbb{E} \left[ \int_0^t \int_0^t |\nabla v(s, x)|^2 \, dx \, ds \right] \\
\leq a_0 \left( \mathbb{E}|u_0|^2_{L^2} + \mathbb{E}|\nabla v_0|^2_{L^2} + T \mathbb{E}|u_0|^8_{L^1} e^{(\gamma^2 + \frac{1}{2})T^2} + T \mathbb{E}|v_0|^1_{B_{\alpha,1}^{\frac{5}{2}}(2)} \right) + T^2 \mathbb{E}|v_0|^{12}_{L^1} e^{(\gamma^2 + \frac{1}{2})T^2} \\
+ T^2 \mathbb{E}|v_0|^{12}_{L^1} e^{(\gamma^2 + \frac{1}{2})T^2} \right) e^{a_1 T}.
\end{align*}
$$

We postpone the proof of this proposition to Section 4.

3. Pathwise uniqueness of the solution

In this section, we prove pathwise uniqueness of the solution to the system (1.3). Thanks to Yamada–Watanabe theorem, (weak) existence and pathwise uniqueness of the solution of a stochastic equation guarantee the existence of a strong solution. Existence of a martingale solution to the given system is obtained in Theorem 2.6 of [14]. Accordingly, our aim is to obtain the pathwise uniqueness to show the existence of a unique, strong solution to the system (1.3) in one dimension. In the following theorem, we will find under which conditions pathwise uniqueness holds. For the sake of the completeness we start with the definition of pathwise uniqueness.

**Definition 3.1.** The equation (1.3) is pathwise unique if, whenever $(\Omega, \mathcal{F}, \mathbb{P}, (u_i, v_i), W_1, W_2)$, $i = 1, 2$, are solutions to (1.3) such that $\mathbb{P}(u_1(0) = u_2(0)) = 1$ and $\mathbb{P}(v_1(0) = v_2(0)) = 1$, then

$$
\mathbb{P}(u_1(t) = u_2(t)) = 1 \quad \text{and} \quad \mathbb{P}(v_1(t) = v_2(t)) = 1, \quad \text{for every } t \in [0, T].
$$

**Theorem 3.2.** Let $T > 0$ and $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a probability space satisfying the usual conditions. Also, let $(W_1, W_2)$ be a pair of Wiener processes over $\mathfrak{A}$, and $H_1$ and $H_2$ be two Bessel potential spaces which satisfy Assumption (1.4). Let the initial data $(u_0, v_0) \in L^2(\Omega) \times H^1_2(\Omega)$ satisfy Assumption (1.4). Let $(u_i, v_i)$, $i = 1, 2$, be two solutions to the system (1.3) such that $\mathbb{P}$–a.s. the following holds

$$
\begin{align*}
\left\{ \begin{array}{l}
u \in X_u \quad \text{where } X_u := L^2(\Omega; C([0, T]; L^2(\Omega))) \cap L^2(\Omega; L^2(0, T; H^1_2(\Omega))), \\
v \in X_v \quad \text{where } X_v := L^2(\Omega; C([0, T]; H^1_2(\Omega))) \cap L^2(\Omega; L^2(0, T; H^2_2(\Omega))).
\end{array} \right.
\end{align*}
$$

Then, the processes $(u_1, v_1)$ and $(u_2, v_2)$ are identical in $X_u \times X_v$.

Since we are interested in the consequences of the pathwise uniqueness, in particular on the existence of a unique strong solution, we will present the following Corollary and postpone the proof of Theorem 3.2.

**Corollary 3.3.** Suppose the conditions in Theorem 3.2 are satisfied. Then, the system (1.3) admits a unique strong solution.

**Proof.** We show that there exists a unique strong solution to the system (1.3) in the framework of the variational approach. To do so, we use Theorem 1.7 of [17]. Define

$$
\mathcal{W} := (W_1, W_2), \quad \mathbb{U} := H_1 \times H_2,
$$

\[ V_1 := H^1_2(O) \times H_2^1(O), \quad H := L^2(O) \times H_2^1(O), \quad V_2 := H^{-1}_2(O) \times L^2(O), \]

where we use the Gelfand triple \( H^1_2(O) \hookrightarrow L^2(O) \hookrightarrow H^{-1}_2(O) \). Let us define the following path space

\[
\mathbb{B} := \left\{ X = (u,v)^T \in C([0,T]; H) : \int_0^T |X(t)|_{V_1} \, dt < \infty \right\}.
\]

Furthermore, let us consider the maps

\[ b : [0,T] \times \mathbb{B} \to V_2 \quad \text{and} \quad \sigma : [0,T] \times \mathbb{B} \to L_2(U, H) \]

defined by

\[ b(t, X) := \left( r_u \Delta u(t) - \chi \text{div}(u(t)\nabla v(t)) + \gamma u(t) \right), \]

and

\[ \sigma(t, X)[h] := \left( \frac{u(t)h_1}{v(t)h_2} \right), \]

for \( X = (u,v)^T \in \mathbb{B}, h = (h_1, h_2)^T \in \mathbb{B} \), and \( t \in [0,T] \).

Let \( \tilde{X} = (\tilde{u}, \tilde{v})^T \) be a martingale solution of (1.33). To show that \( \tilde{X} \) is a strong solution, we need to verify \( \mathbb{P} \)-a.s.

\[ \int_0^T |b(s, \tilde{X}(s))|_{V_2} \, ds + \int_0^T |\sigma(s, \tilde{X}(s))|_{L(U,H)} \, ds < \infty. \]

We first estimate \( |b(s, \tilde{X}(s))|_{V_2} \) as follows:

\[ |b(s, \tilde{X}(s))|_{V_2} = |r_u \Delta \tilde{u}(s) - \chi \text{div}(\tilde{u}(s)\nabla \tilde{v}(s)) - \gamma \tilde{u}(s)|_{H^{-1}_2} + |r_v \Delta \tilde{v}(s) - \alpha \tilde{v}(s) + \beta \tilde{u}(s)|_{L^2} \]

\[ \leq r_u |\Delta \tilde{u}(s)|_{H^{-1}_2} + |\text{div}(\tilde{u}(s)\nabla \tilde{v}(s))|_{H^{-1}_2} + \gamma |\tilde{u}(s)|_{H^{-1}_2} + r_v |\Delta \tilde{v}(s) - \alpha \tilde{v}(s)|_{L^2} + \beta |\tilde{u}(s)|_{L^2} \]

\[ \leq r_u |\tilde{u}(s)|_{H^1_2} + |\text{div}(\tilde{u}(s)\nabla \tilde{v}(s))|_{L^2} + |\gamma (-(\Delta)^{-1} \tilde{u}(s))_{L^2} + (r_v + \alpha) |\tilde{v}(s)|_{H^1_2} + \beta |\tilde{u}(s)|_{L^2} \]

\[ \leq (r_u + c\gamma + c\beta) |\tilde{u}(s)|_{L^2} + c\gamma |\text{div}(\tilde{u}(s)\nabla \tilde{v}(s))|_{L^2} + (r_v + \alpha) |\tilde{v}(s)|_{H^1_2}. \]

To estimate the second term on the right hand side, namely \( |\tilde{u}(s)\nabla \tilde{v}(s)|_{L^2} \), we use the embedding \( H^1_2(O) \hookrightarrow L^\infty(O) \) and the Cauchy–Schwarz inequality to obtain

\[ |\tilde{u}(s)\nabla \tilde{v}(s)|_{L^2} \leq |\tilde{u}(s)|_{L^2} |\nabla \tilde{v}(s)|_{L^\infty} \leq |\tilde{u}(s)|_{L^2} |\tilde{v}(s)|_{H^1_2}. \]

Using (3.4) in (3.3), we have from Proposition 2.1

\[ \int_0^T |b(s, \tilde{X}(s))|_{V_2} \, ds \leq (r_u + c\gamma + c\beta) \int_0^T |\tilde{u}(s)|_{L^2} \, ds + \gamma \int_0^T |\tilde{u}(s)|_{H^1_2} |\tilde{v}(s)|_{H^1_2} \, ds \]

\[ + (r_v + \alpha) \int_0^T |\tilde{v}(s)|_{H^1_2} \, ds \]

\[ \leq (r_u + c\gamma + c\beta) \int_0^T |\tilde{u}(s)|_{L^2} \, ds + \gamma \left\{ \sup_{0 \leq s \leq T} |\tilde{u}(s)|_{L^2} \int_0^T |\tilde{v}(s)|_{H^1_2} \, ds \right\} \]

\[ + (r_v + \alpha) \int_0^T |\tilde{v}(s)|_{H^1_2} \, ds < \infty. \]
Also, using Proposition 2.1 we have $\mathbb{P}$-a.s.
\[
\int_0^T |\sigma(s, \bar{X}(s))|^2_{L_2(U,\mathbb{R})} ds \leq \int_0^T \left[ |\bar{u}(s)|^2_{H^1_2} + |\bar{v}(s)|^2_{H^2_2} \right] ds < \infty.
\]
This yields (3.2). Finally, since $\bar{X} = (\bar{u}, \bar{v})$ is a martingale solution, the solution process $\bar{X}$ with $x = (u_0, v_0) \equiv \mathbb{P}$-a.s. satisfies
\[
\bar{X}(t) = x + \int_0^t b(s, \bar{X}(s)) ds + \int_0^t \sigma(s, \bar{X}(s))d\mathcal{W}(s), \quad t \in [0, T].
\]
In addition, by Theorem 3.2 we have pathwise uniqueness of the solution. Hence, by Theorem 1.7 of [17], we have shown that $\bar{X}(t)$ is the unique strong solution to the system (1.3). This finishes the proof of the corollary.

**Proof of Theorem 3.2** Let us recall, since $(u_1, v_1)$ and $(u_2, v_2)$ are solutions to the system (1.3) with $\mathbb{P}(u_1(0) = u_2(0)) = 1$ and $\mathbb{P}(v_1(0) = v_2(0)) = 1$, we can write
\[
(3.6) \quad du_i(t) = \left( r_u \Delta u_i(t) - \chi \text{div}(u_i(t)\nabla v_i(t)) + \gamma u_i(t) \right) dt + u_i(t) d\mathcal{W}_1(t)
\]
(3.7) \quad dv_i(t) = \left( r_v \Delta v_i(t) + \beta u_i(t) - \alpha v_i(t) \right) dt + v_i(t) d\mathcal{W}_2(t), \quad t \in [0, T], \quad i = 1, 2.
\]

In the first step we will introduce a family of stopping times $\{\tau_N : N \in \mathbb{N}\}$, and show that on the time interval $[0, \tau_N]$ the solutions $u_1$ and $u_2$, respective, $v_1$ and $v_2$, are indistinguishable. In the second step, we will show that $\mathbb{P}(\tau_N < T) \to 0$ for $N \to \infty$. From this follows that $u_1$ and $u_2$ are indistinguishable on the time interval $[0, T]$.

**Step I.** Let us introduce the stopping times $\{\tau_N : N \in \mathbb{N}\}$ as follows: let
\[
\tau_{1,i}^N := \inf\{t \geq 0 : \sup_{s \in [0,t]} |u_i(s)|_{L^2} \geq N\} \land T; \quad i = 1, 2,
\]
\[
\tau_{2,i}^N := \inf\{t \geq 0 : \sup_{r \in [0,t]} |u_i(r)|_{L^1} \geq N\} \land T; \quad i = 1, 2,
\]
and $\tau_N := \min_{i=1,2}(\tau_{1,i}^N, \tau_{2,i}^N)$. The aim is to show that $(u_1, v_1)$ and $(u_2, v_2)$ are indistinguishable on the time interval $[0, \tau_N]$.

Fix $N \in \mathbb{N}$. To get uniqueness on $[0, \tau_N]$ we first stop the original solution processes at time $\tau_N$ and extend the processes $(u_1, v_1)$ and $(u_2, v_2)$ by other processes to the whole interval $[0, T]$. For this purpose, let $(y_1, z_1)$ be solution to
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{dy_1(t)}{dt} &= \Delta y_1(t) dt + y_1(t) d\theta_{\tau_N} \circ \mathcal{W}_1(t), \quad t \geq 0, \\
\frac{dz_1(t)}{dt} &= (\Delta z_1(t) - \alpha z_1(t)) dt + z_1(t) d\theta_{\tau_N} \circ \mathcal{W}_2(t), \quad t \geq 0,
\end{array} \right.
\end{aligned}
\]
with initial data $y_1(0) := u_1(\tau_N)$, $z_1(0) := v_1(\tau_N)$, and let $(y_2, z_2)$ be solutions to
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{dy_2(t)}{dt} &= \Delta y_2(t) dt + y_2(t) d\theta_{\tau_N} \circ \mathcal{W}_1(t), \quad t \geq 0, \\
\frac{dz_2(t)}{dt} &= \Delta z_2(t) - \alpha z_2(t) dt + z_2(t) d\theta_{\tau_N} \circ \mathcal{W}_2(t), \quad t \geq 0,
\end{array} \right.
\end{aligned}
\]
with initial data $y_2(0) := u_2(\tau_N)$ and $z_2(0) := v_2(\tau_N)$. Here, $\theta_{\tau} \circ \mathcal{W}_i(t) = \mathcal{W}_i(t + \tau)$, $i = 1, 2$. Since $(u_1, v_1)$ and $(u_2, v_2)$ are continuous in $L^2(O) \times L^2(O)$, $(u_1(\tau_N), v_1(\tau_N))$ and $(u_2(\tau_N), v_2(\tau_N))$ are well defined and belong $\mathbb{P}$-a.s. to $L^2(O) \times L^2(O)$. Now, we define two new couple of processes in such a way that they coincide with $(u_1, v_1)$ and $(u_2, v_2)$
on the time interval \([0, \tau_N]\) and later on, they represent the processes \((y_1, z_1)\) and \((y_2, z_2)\). In particular, for \(i = 1, 2\), let us describe the pair \((\bar{u}_i, \bar{v}_i)\) such that

\[
\bar{u}_i(t) = \begin{cases} 
  u_i(t) & \text{for } 0 \leq t < \tau_N, \\
  y_i(t - \tau_N) & \text{for } \tau_N \leq t \leq T,
\end{cases}
\quad
\bar{v}_i(t) = \begin{cases} 
  v_i(t) & \text{for } 0 \leq t < \tau_N, \\
  z_i(t - \tau_N) & \text{for } \tau_N \leq t \leq T.
\end{cases}
\]

Note, that \((\bar{u}_1, \bar{v}_1)\) and \((\bar{u}_2, \bar{v}_2)\) solve the truncated equation corresponding to (3.6)–(3.7) in \([0, \tau_N]\) and equation (3.8)–(3.9) in \([\tau_N, T]\).

**Step II.** Our goal is to show that \((u_1, v_1)\) and \((u_2, v_2)\) are identical on the interval \([0, \tau_N]\). To show this, we follow an idea of Gajewski [11]. Let us consider the function \(\phi : \mathbb{R} \to \mathbb{R}\) by

\[
\phi(u) = \begin{cases} 
  u(\ln(u) - 1), & u > 0, \\
  0, & u \leq 0.
\end{cases}
\]

Exploiting the Lemma A.1 in Appendix A (see [11] for more details), we observe that \(\phi\) satisfies

\[
\phi(u_1) - 2\phi\left(\frac{u_1 + u_2}{2}\right) + \phi(u_2) \geq \frac{1}{4}(\sqrt{u_1} - \sqrt{u_2})^2, \quad u_1, u_2 \geq 0.
\]

Let us consider the functional \(\Phi : L^2(\mathcal{O}) \times L^2(\mathcal{O}) \to \mathbb{R}\) by

\[
\Phi(u_1, u_2) := \int_\mathcal{O} \left\{ \phi(u_1(x)) + \phi(u_2(x)) - 2\phi\left(\frac{u_1(x) + u_2(x)}{2}\right) \right\} \, dx.
\]

Using (3.10) and (3.11), one can show the following inequality

\[
\frac{1}{4}(\sqrt{u_1(t)} - \sqrt{u_2(t)})^2_{L^2} \leq \Phi(u_1(t), u_2(t)).
\]

Let us now apply the Itô formula to the functional \(\Phi\) for the process \((u_1(t), u_2(t))\) for \(t \in [0, \tau_N]\). Here, let us first observe that the trace term vanishes. In particular, we have

\[
\left\langle D^2\Phi(u_1, u_2)(s) \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix}, \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \right\rangle_{L^2}
\]

\[
= \int_\mathcal{O} \left\langle \begin{pmatrix} \frac{u_2(x, s)}{u_1(x, s) + u_2(x, s)} \\ \frac{-1}{u_1(x, s) + u_2(x, s)} \end{pmatrix}, \begin{pmatrix} u_1(x, s) \\ u_2(x, s) \end{pmatrix} \right\rangle_{L^2} \, dx
\]

\[
= \int_\mathcal{O} \left\langle 0, \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} \right\rangle_{L^2} \, dx = 0.
\]

Now, by applying the Itô-formula to \(\Phi(u_1(t), u_2(t))\) we can write

\[
\Phi((u_1(t), u_2(t)) - \Phi(u_1(0), u_2(0))
\]

\[
= \int_0^t \left[ \langle \phi'(u_1(s)), du_1(s) \rangle + \langle \phi'(u_2(s)), du_2(s) \rangle 
\right.
\]

\[
\left. - \langle \phi'(u_1(s) + u_2(s) + 2), d(u_1 + u_2(s)) \rangle \right].
\]

More precisely, we have

\[
\Phi((u_1(t), u_2(t)) - \Phi(u_1(0), u_2(0))
\]
\[
\begin{align*}
\Phi((u_1(t), u_2(t)) - \Phi(u_1(0), u_2(0))
&= \int_0^t \int_\Omega \left\{ r_u \Delta u_1(s) - \chi \text{div}(u_1(s)\nabla v_1(s)) \right\} \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) dx ds \\
&\quad + \left\{ r_u \Delta u_2(s) - \chi \text{div}(u_2(s)\nabla v_2(s)) \right\} \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) dx ds \\
&\quad + \gamma \int_0^t \int_\Omega \left[ u_1(s) \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) + u_2(s) \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) \right] dx ds \\
&\quad + \int_0^t \int_\Omega \left[ u_1(s) \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) + u_2(s) \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) \right] dW_1(s).
\end{align*}
\]

Let us split the sum above into the following three terms
\[
S_1 := \int_0^t \int_\Omega \left\{ r_u \Delta u_1(s) - \chi \text{div}(u_1(s)\nabla v_1(s)) \right\} \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) dx ds,
\]
\[
S_2 := \gamma \int_0^t \int_\Omega \left[ u_1(s) \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) + u_2(s) \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) \right] dx ds,
\]
and
\[
S_3 := \int_0^t \int_\Omega \left[ u_1(s) \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) + u_2(s) \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) \right] dW_1(s).
\]

To start with \(S_1\), using integration by parts and rearranging the terms, we obtain
\[
\begin{align*}
\int_\Omega \left\{ r_u \Delta u_1(s) - \chi \text{div}(u_1(s)\nabla v_1(s)) \right\} \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) dx
&\quad + \left\{ r_u \Delta u_2(s) - \chi \text{div}(u_2(s)\nabla v_2(s)) \right\} \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) dx
\end{align*}
\]
\[
\begin{align*}
&= -\int_\Omega \left[ r_u \nabla u_1(s) + \chi u_1(s)\nabla v_1(s) \right] \frac{u_1(s) + u_2(s)}{2u_1(s)} \\
&\quad \times \left( \frac{2(u_1(s) + u_2(s))\nabla u_1(s) - 2u_1(s)(\nabla u_1(s) + \nabla u_2(s))}{(u_1(s) + u_2(s))^2} \right) \\
&\quad + \left[ r_u \nabla u_2(s) + \chi u_2(s)\nabla v_2(s) \right] \frac{u_1(s) + u_2(s)}{2u_2(s)}
\end{align*}
\]
\[
\times \left(\frac{2(u_1(s) + u_2(s))\nabla u_2(s) - 2u_2(s)(\nabla u_1(s) + \nabla u_2(s))}{(u_1(s) + u_2(s))^2}\right)\right] \, dx
\]

\[
e - \int_\Omega \left[ \left( r_u \nabla u_1(s) + \chi u_1(s)\nabla v_1(s) \right) \left( \frac{u_2(s)\nabla u_1(s) - u_1(s)\nabla u_2(s)}{(u_1(s) + u_2(s))u_1(s)} \right) \\
+ \left( r_u \nabla u_2(s) + \chi u_2(s)\nabla v_2(s) \right) \left( \frac{u_1(s)\nabla u_2(s) - u_2(s)\nabla u_1(s)}{(u_1(s) + u_2(s))u_2(s)} \right) \right] \, dx
\]

\[
e - \int_\Omega \left[ \left( \frac{r_u \nabla u_1(s)}{u_1(s)} + \chi \nabla v_1(s) \right) \left( \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \right) \left( \frac{\nabla u_1(s)}{u_1(s)} - \frac{\nabla u_2(s)}{u_2(s)} \right) \\
+ \left( \frac{r_u \nabla u_2(s)}{u_2(s)} + \chi \nabla v_2(s) \right) \left( \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \right) \left( \frac{\nabla u_2(s)}{u_2(s)} - \frac{\nabla u_1(s)}{u_1(s)} \right) \right] \, dx
\]

\[
+ \chi (\nabla v_1(s) - \nabla v_2(s)) \right] \, dx
\]

\[
(3.14) + \chi \nabla \ln \left( \frac{u_1(s)}{u_2(s)} \right) \cdot \nabla \left( v_1(s) - v_2(s) \right) \right] \, dx.
\]

Using the Young inequality we can show for any \( \varepsilon > 0 \), that the left hand side of \( (3.14) \) is dominated by

\[
- r_u(1 - \varepsilon) \int_\Omega \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \left| \nabla \ln \left( \frac{u_1(s)}{u_2(s)} \right) \right|^2 \, dx
\]

\[
+ \frac{\chi^2}{4r_u\varepsilon} \int_\Omega \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \left| \nabla (v_1(s) - v_2(s)) \right|^2 \, dx.
\]

Using \( \frac{u_1u_2}{u_1+u_2} \leq u_2 \), we observe that \( (3.15) \) can be further simplified as

\[
- r_u(1 - \varepsilon) \int_\Omega \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \left| \nabla \ln \left( \frac{u_1(s)}{u_2(s)} \right) \right|^2 \, dx
\]

\[
+ \frac{\chi^2}{4r_u\varepsilon} \int_\Omega \frac{u_2(s)}{u_2(s)} \left| \nabla (v_1(s) - v_2(s)) \right|^2 \, dx.
\]

We now evaluate the second term of \( (3.16) \), i.e. \( \int_\Omega u_2 |\nabla v_1 - \nabla v_2|^2 \, dx \). Applying the Hölder inequality with \( \frac{1}{2} + \frac{1}{2} = 1 \) and using the definition of stopping time, we achieve

\[
(3.17) \int_\Omega u_2 |\nabla v_1 - \nabla v_2|^2 \, dx \leq |\nabla v_1 - \nabla v_2|^2_{L^2} |u_2|_{L^2} \leq N |\nabla v_1 - \nabla v_2|^2_{L^2}.
\]
To evaluate $S_2$, we use Claim A.1. We first note that if $u_1 = u_2 = 0$, then
\[ S_2 = \gamma \int_0^t \int_{\mathcal{O}} \left[ u_1(s) \ln \left( \frac{2u_1(s)}{u_1(s) + u_2(s)} \right) + u_2(s) \ln \left( \frac{2u_2(s)}{u_1(s) + u_2(s)} \right) \right] \, dx \, ds = 0. \]

Otherwise, we note that $S_2$ can be written as
\begin{equation}
S_2 = \begin{cases} 
\gamma \int_0^t \int_{\mathcal{O}} u_1(s) f(u) \, dx \, ds, & \text{if } u_1(s) \neq 0, \ u = \frac{u_2}{u_1}, \\
\gamma \int_0^t \int_{\mathcal{O}} u_2(s) f(u) \, dx \, ds, & \text{if } u_2(s) \neq 0, \ u = \frac{u_1}{u_2},
\end{cases}
\end{equation}

where $f$ is given in Claim A.1. Putting $u = \frac{u_2}{u_1}$ in (A.1) in Claim A.1 we see that
\begin{equation}
S_2 \leq \gamma \int_0^t \int_{\mathcal{O}} |\sqrt{u_1(x,s)} - \sqrt{u_2(x,s)}|^2 \, dx \, ds.
\end{equation}

Furthermore, since $S_3$ is a local martingale, the expectation of this term vanishes. Combining all the estimates above, we obtain from (3.12)
\begin{align*}
\frac{1}{4} |\sqrt{u_1(t)} - \sqrt{u_2(t)}|_{L^2}^2 + r_u(1 - \varepsilon) \int_0^t \int_{\mathcal{O}} \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \left| \nabla \ln \left( \frac{u_1(s)}{u_2(s)} \right) \right|^2 \, dx \, ds \\
\leq \Phi(u_1(0), u_2(0)) + \frac{\chi^2 C_{O,N}}{4r_u\varepsilon} \int_0^t \left| \nabla v_1(s) - \nabla v_2(s) \right|_{L^2}^2 \, ds \\
+ \gamma \int_0^t \int_{\mathcal{O}} |\sqrt{u_1(s)} - \sqrt{u_2(s)}|_{L^2}^2 \, dx \, ds.
\end{align*}

Since $L^1(\mathcal{O}) \hookrightarrow H_{-1}^2(\mathcal{O})$ in one dimension, one can show that there exists some constant $C > 0$ such that
\[ \mathbb{E}\|\nabla v_1 - \nabla v_2\|_{L^2(0,T;L^2)}^2 \leq C \mathbb{E}\|u_1 - u_2\|_{L^2(0,T;H^{-1})}^2. \]

By elementary calculations and using the embedding $L^1(\mathcal{O}) \hookrightarrow H_{-1}^2(\mathcal{O})$, we obtain
\begin{align*}
\mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|_{H_{-1}^2}^2 \, ds \right] &\leq \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|_{L^1}^2 \, ds \right] \\
&\leq \mathbb{E}\left[ \int_0^T \left( \int_{\mathcal{O}} \left| \sqrt{u_1(s)} - \sqrt{u_2(s)} \right| \left| \sqrt{u_1(s)} + \sqrt{u_2(s)} \right| \, dx \right)^2 \, ds \right] \\
&\leq \mathbb{E}\left[ \int_0^T \left( \int_{\mathcal{O}} |\sqrt{u_1(x,s)} - \sqrt{u_2(x,s)}| \left| \sqrt{u_1(x,s)} + \sqrt{u_2(x,s)} \right| \, dx \right)^2 \, ds \right] \\
&\leq C \mathbb{E}\left[ \int_0^T \left( \int_{\mathcal{O}} \left| \sqrt{u_1(x,s)} - \sqrt{u_2(x,s)} \right|^2 \, dx \right) \int_{\mathcal{O}} \left| \sqrt{u_1(x,s)} + \sqrt{u_2(x,s)} \right|^2 \, dx \, ds \right] \\
&\leq \mathbb{E}\left[ \int_0^T \left| \sqrt{u_1(s)} - \sqrt{u_2(s)} \right|_{L^2}^2 \left| u_1(s) + u_2(s) \right|_{L^1} \, ds \right].
\end{align*}

Then, by the definition of the stopping time we know $|u_i|_{L^1} \leq N$ for $i = 1, 2$, and, hence
\[ \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|_{H_{-1}^2}^2 \, ds \right] \leq 2CN \mathbb{E}\left[ \int_0^T \left| \sqrt{u_1(s)} - \sqrt{u_2(s)} \right|_{L^2}^2 \, ds \right]. \]
Substituting above in (3.20), we obtain
\[
\frac{1}{4} \mathbb{E} \left[ \left( \sqrt{u_1(t)} - \sqrt{u_2(t)} \right)^2 \right] + r_u (1 - \varepsilon) \mathbb{E} \left[ \int_0^T \int_{\Omega} \frac{u_1(s)u_2(s)}{u_1(s) + u_2(s)} \left| \nabla \ln \left( \frac{u_1(s)}{u_2(s)} \right) \right|^2 \, dx \, ds \right]
\]
(3.21) \leq \Phi(u_1(0), u_2(0)) + \left( \frac{\chi^2 C C_O N^2}{2 r_u \varepsilon} + \gamma \right) \mathbb{E} \left[ \int_0^T \left( \sqrt{u_1(s)} - \sqrt{u_2(s)} \right)^2 \, ds \right].

Using the fact that for \( u_1(0) = u_2(0) \) we have \( \Phi(u_1(0), u_2(0)) = 0 \), a Gronwall argument gives
\[
\mathbb{E} \left[ \left( \sqrt{u_1(t)} - \sqrt{u_2(t)} \right)^2 \right] \leq e^{2 \left( \frac{\chi^2 C C_O N^2}{2 r_u \varepsilon} + 1 \right) T} \Phi(u_1(0), u_2(0)) = 0.
\]

**Step IV.** We show that \( \mathbb{P} (\tau_N < T) \to 0 \) as \( N \to \infty \).

\[
\{ \tau_N < T \} \subset \left\{ \sup_{s \in [0,T]} |u_1(s)|_{L^2} \geq N \text{ or } \sup_{s \in [0,T]} |u_2(s)|_{L^2} \geq N \text{ or } \sup_{s \in [0,T]} |u_1(s)|_{L^1} \geq N \text{ or } \sup_{s \in [0,T]} |u_2(s)|_{L^1} \geq N \right\}.
\]

Therefore, due to Proposition 2.1 Theorem 2.6 of [14] and, since, \( \text{LlogL(\Omega)} \leftrightarrow L^1(\Omega) \), one can observe using the Chebyscheff inequality
\[
\mathbb{P} (\tau_N < T) \leq \mathbb{P} \left( \sup_{s \in [0,T]} |u_1(s)|_{L^2} \geq N \right) + \mathbb{P} \left( \sup_{s \in [0,T]} |u_2(s)|_{L^2} \geq N \right)
+ \mathbb{P} \left( \sup_{s \in [0,T]} |u_1(s)|_{L^1} \geq N \right) + \mathbb{P} \left( \sup_{s \in [0,T]} |u_2(s)|_{L^1} \geq N \right) \leq \frac{4C}{N}.
\]

It follows
\[
\mathbb{P} (\tau_N \leq T) \to 0 \quad \text{as} \quad N \to \infty.
\]

Hence, both processes \( u_1 \) and \( u_2 \) coincide on \([0,T]\), so are \( v_1 \) and \( v_2 \). This completes the proof of Proposition 3.2.

4. PROOF OF PROPOSITION 2.1

**Proof of Proposition 2.1.** The proof consists of two steps. In the first step, we will estimate the \( p \)-th moment of \( L^1 \) norm of \( u \) for \( p > 1 \). In the next step, we obtain uniform bound for the \( L^2(\Omega; L^\infty([0,T]; L^2(\Omega))) \) norm of \( u \) and the \( L^2(\Omega; L^\infty([0,T]; H^1_2(\Omega))) \) norm of \( v \).

**Step (I)** Let us define the integral operator \( I : L^1(\Omega) \to \mathbb{R} \) by \( I(u) := \int_{\Omega} u(x) \, dx \). Then, \( I \) is linear functional, hence is of class \( C^2 \); (see [11, p. 11 Example 1.3 (b)]). Applying the Itô formula [5, Chapter 4.4]) to the functional \( I(u) = \int_{\Omega} u(x) \, dx \) for the process \( \{u(t)\}_{t \in [0,T]} \), we obtain
\[
I(u(t)) = I(u(0)) + \int_0^t \left< \frac{\partial I}{\partial u}, du_1(s) \right> + \frac{1}{2} \int_0^t \text{Tr} \left[ \left( \frac{\partial^2 I}{\partial u^2} \right)(uu^T) \right] \, ds.
\]

Using \( \frac{\partial^2 I}{\partial u^2} = 0 \), the above equation reduces to
\[
I(u(t)) = I(u(0)) + \int_0^t \int_{\Omega} \left[ r_u \Delta u(s,x) - \chi \text{div}(u(s,x)\nabla v(s,x)) + \gamma u(s,x) \right] \, dx \, ds
+ \int_0^t \int_{\Omega} u(s,x) \, dW_1(s,x).
\]

(4.2)
Now using integration by parts in space variable and by Neumann boundary conditions, we see that the second and third terms in the right hand side of (4.2) vanishes. Indeed, we have the following equality.

\[
\begin{align*}
(4.3) \quad & \quad \frac{r_u}{O} \int \Delta u(s, x) \, dx = \left( \int \frac{\partial u}{\partial v} (s, x) \, dS(x) \right) = 0, \\
(4.4) \quad & \quad \chi \int \frac{\partial u(s, x)}{\partial v(s, x)} \, dx = \left( \int \frac{\partial u}{\partial v} (s, x) \, dS(x) \right) = 0,
\end{align*}
\]

where, \( S \) is the surface measure on \( \partial O \). In this way, substituting (4.3)-(4.4) in (4.2), we obtain

\[
(4.5) \quad \mathcal{I}(u(t)) = \mathcal{I}(u(0)) + \gamma \int_0^t \mathcal{I}(u(s)) \, ds + \int_0^t \int_O u(s, x) \, d\mathcal{W}_1(s, x).
\]

Our aim in this step is to obtain the higher moment of \( \mathcal{I}(u(t)) \) for \( t \in [0, T] \). Using (4.6), and \( \{\psi_k\}_{k \in \mathbb{Z}} \) being orthonormal basis in \( L^2(O) \), it is clear that \( \{\psi_k^{(1)}\}_{k \in \mathbb{Z}} = \{\lambda_k^{(1)} \psi_k\}_{k \in \mathbb{Z}} \) is an orthonormal basis of \( \mathcal{H}_1 \). Therefore, we have

\[
(4.6) \quad \mathcal{W}_1(r, x) = \sum_{k \in \mathbb{Z}} \psi_k^{(1)}(x) \beta_k^{(1)}(r) = \sum_{k \in \mathbb{Z}} \lambda_k^{(1)} \psi_k(x) \beta_k^{(1)}(r).
\]

We define

\[
(4.7) \quad \Phi^u_s : \mathcal{H}_1 \to \mathbb{R} \quad \text{by} \quad \Phi^u_s(\eta) = \int_O u(s, x) \eta(x) \, dx, \quad \text{for all} \ \eta \in \mathcal{H}_1.
\]

Let \( \Phi^u = \left\{ \Phi^u_t, t \in [0, T] \right\} \) be a measurable \( L^2_t \)-valued process. We define the norm of \( \Phi^u \) by

\[
(4.8) \quad \|\Phi^u\|_T := \left[ \mathbb{E} \left( \int_0^T \|\Phi^u_s\|^2_{L^2} \, ds \right) \right]^{\frac{1}{2}}.
\]

We refer to [8] Chapter 4 to define the stochastic integral with respect to \( \mathcal{W} \) of any \( L^2_\mathbb{R} \)-valued predictable process \( \Phi^u \) such that \( \|\Phi^u\|_T < \infty \). Following equation (3.7) in [7] with \( H = \mathbb{R} \) and \( V = \mathcal{H}_1 \), we now compute \( \|\Phi^u\|^2_T \).
Now we use Monotone Convergence Theorem (Theorem 1.2.7 of [18], Theorem 2.15 of [12]). To do so, we take $X = \Omega \times [0, T]$, $f_k(\omega, s) := |\lambda_k^{(1)}|^2 |\psi_k|_L^2 \left( \int_\Omega u(s, x, \omega) \, dx \right)^2$. Then, we have

$$
\int_{\Omega \times [0, T]} \sum_{k \in \mathbb{Z}} f_k(\omega, s) \, ds \, dP(\omega) = \sum_{k \in \mathbb{Z}} \int_{\Omega \times [0, T]} f_k(\omega, s) \, ds \, dP(\omega),
$$

which implies the following equality

$$
\mathbb{E} \left[ \int_0^T \sum_{k \in \mathbb{Z}} |\lambda_k^{(1)}|^2 |\psi_k|_L^2 \left( \int_\Omega u(s, x) \, dx \right)^2 \, ds \right] = \sum_{k \in \mathbb{Z}} |\lambda_k^{(1)}|^2 |\psi_k|_L^2 \mathbb{E} \left[ \int_0^T \left( \int_\Omega u(s, x) \, dx \right)^2 \, ds \right].
$$

Combining (4.10) and (4.11), using Theorem 2.6 of [13] and, since $\log L(\mathcal{O}) \hookrightarrow L^1(\mathcal{O})$, we achieve

$$
\|\Phi^u\|_T^2 \leq \sum_{k \in \mathbb{Z}} |\lambda_k^{(1)}|^2 |\psi_k|_L^2 \mathbb{E} \left[ \int_0^T \left( \int_\Omega u(s, x) \, dx \right)^2 \, ds \right] < \infty.
$$

Now, by Proposition 3.4 of [7] with $H = \mathbb{R}$ and $f_k = 1$, we have

$$
\int_0^T \Phi^u_s \, d\mathcal{W}_1(s) = \sum_{j=1}^{\infty} \int_0^T \Phi^u_s(\psi_j^{(1)}(s)) \, d\beta_j^{(1)}(s) = \sum_{j \in \mathbb{Z}} \lambda_j^{(1)} \int_0^T \left( \int_\Omega u(s, x) \psi_j(x) \, dx \right) \, d\beta_j^{(1)}(s).
$$

Also, using (4.7), we have

$$
\int_0^T \Phi^u_s \, d\mathcal{W}_1(s) = \int_0^T \left( \int_\Omega \sum_{j \in \mathbb{Z}} u(s, x) \lambda_j^{(1)} \psi_j(x) \, dx \right) \, d\beta_j^{(1)}(s).
$$

Using (4.12) and (4.13), we now estimate the following

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \int_\Omega \sum_{k \in \mathbb{Z}} \psi_k^{(1)}(x) u(r, x, \omega) \, dx \, d\beta_k^{(1)}(r) \right| \right]
$$

$$
= \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \sum_{k \in \mathbb{Z}} \int_0^s \left( \int_\Omega u(r, x) \psi_k^{(1)}(x) \, dx \right) \, d\beta_k^{(1)}(r) \right| \right]
$$

$$
\leq \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_\Omega u(r, x) \psi_k^{(1)}(x) \, dx \right) \, d\beta_k^{(1)}(r) \right| \right]
$$

$$
= \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_\Omega u(r, x) \psi_k^{(1)}(x) \, dx \right) \, d\beta_k^{(1)}(r) \right| \right].
$$

Again, we use Monotone Convergence Theorem (Theorem 1.2.7 of [18], Theorem 2.15 of [12]). To do so, we take

$$
X = \Omega, \quad f_k(\omega, t) := \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_\Omega u(r, x) \psi_k^{(1)}(x) \, dx \right) \, d\beta_k^{(1)}(r) \right|.
$$

Then, we have

$$
\int_{\Omega} \sum_{k \in \mathbb{Z}} f_k(\omega, t) \, dP(\omega) = \sum_{k \in \mathbb{Z}} \int_{\Omega} f_k(\omega, t) \, dP(\omega),
$$
which implies the following equality

\[
E\left[ \sum_{k \in \mathbb{Z}} \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_0 \left( u(r, x) \psi_k^{(1)}(x) \, dx \right) d\beta_k^{(1)}(r) \right) \right| \right]
\]

\[
= \sum_{k \in \mathbb{Z}} E\left[ \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_0 \left( u(r, x) \psi_k^{(1)}(x) \, dx \right) d\beta_k^{(1)}(r) \right) \right| \right].
\]

(4.15)

Using the Burkholder-Davis-Gundy inequality for real-valued Brownian motion, we achieve

\[
E\left[ \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_0 \left( u(r, x) \psi_k^{(1)}(x) \, dx \right) d\beta_k^{(1)}(r) \right) \right| \right]
\]

\[
\leq E\left[ \int_0^t \left( \int_0 \left( u(r, x) \psi_k^{(1)}(x) \, dx \right)^2 \, dr \right)^{\frac{p}{2}} \right]
\]

\[
\leq |\psi_k^{(1)}|_{L^\infty}^2 E\left[ \int_0^t \left( \int_0 \left( u(r, x) \, dx \right)^2 \, dr \right)^{\frac{1}{2}} \right].
\]

(4.16)

Thus, summing over all \( k \) and using (4.10) and (4.16) we have

\[
E\left[ \sum_{k \in \mathbb{Z}} \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_0 \left( u(r, x) \psi_k^{(1)}(x) \, dx \right) d\beta_k^{(1)}(r) \right) \right| \right]
\]

\[
= \sum_{k \in \mathbb{Z}} E\left[ \sup_{0 \leq s \leq t} \left| \int_0^s \left( \int_0 \left( u(r, x) \psi_k^{(1)}(x) \, dx \right) d\beta_k^{(1)}(r) \right) \right| \right]
\]

\[
\leq \sum_{k \in \mathbb{Z}} |\psi_k^{(1)}|_{L^\infty}^2 E\left[ \int_0^t \left( \int_0 \left( u(r, x) \, dx \right)^2 \, dr \right)^{\frac{1}{2}} \right].
\]

(4.17)

Using the similar argument for \( p > 1 \), we obtain

\[
E\left[ \sup_{0 \leq s \leq t} \left| \int_0^s \sum_{k \in \mathbb{Z}} \psi_k^{(1)}(x) u(r, x) \, dx d\beta_k^{(1)}(r) \right|^p \right]
\]

\[
\leq C_p \gamma_1^p E\left[ \left( \int_0^t \left( \int_0 \left( u(r, x) \, dx \right)^2 \, dr \right)^{\frac{p}{2}} \right)^{\frac{p}{2}} \right],
\]

(4.18)

where \( \gamma_1 = \sum_{k \in \mathbb{Z}} |\psi_k^{(1)}|_{L^\infty}^2 \leq \sum_{k \in \mathbb{Z}} |\lambda_k^{(1)}| \psi_k|_{L^\infty}^2 \leq \sum_{k \in \mathbb{Z}} |\lambda_k^{(1)}|^2 < \infty. \) Using (4.18) we have

\[
E\left[ \sup_{0 \leq s \leq t} \left| \int_0^s \int_0 u(r, x) \, dw_1(r, x) \right|^p \right] \leq C_p \gamma_1^p E\left[ \left( \int_0^t \left( \int_0 \left( u(s, x) \, dx \right)^2 \, ds \right)^{\frac{p}{2}} \right)^{\frac{p}{2}} \right]
\]

\[
\leq C_p \gamma_1^p E\left[ \left( \sup_{0 \leq s \leq t} \int_0 u(s, x) \, dx \right)^{\frac{p}{2}} \left( \int_0^t \int_0 u(s, x) \, dx \, ds \right)^{\frac{p}{2}} \right]
\]

\[
\leq \frac{1}{2} E\left[ \sup_{0 \leq s \leq t} \left( \int_0 u(s, x) \, dx \right)^p \right] + \frac{1}{2} E\left[ \left( \int_0^t \int_0 u(s, x) \, dx \, ds \right)^p \right].
\]

(4.19)

We now raise power \( p \geq 1 \) on both sides of (4.15), take supremum over \( s \in [0, T] \), and then the expectation. Hence, we obtain

\[
E\left[ \sup_{0 \leq s \leq t} T^p(u(t)) \right] = E[T^p(u(0))] + \gamma p E\left[ \left( \int_0^t I(u(s)) \, ds \right)^p \right] + \frac{1}{2} E\left[ \sup_{0 \leq s \leq t} T^p(u(s)) \right].
\]
To approximate the divergence operator, we now choose a compactly supported function \( \phi \) the unbounded operator \( \Delta \), nevertheless, one can overcome this problem by the following standard procedure: We replace however, would require a regularization of the unbounded operators appearing in equation (1.3).

(4.22) the following inequality holds

\[
|\Delta - \Delta_\varepsilon|_{L(H^{4+\nu},H^{\delta})} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

To approximate the divergence operator, we now choose a compactly supported function \( \varphi \in C_c^2(\mathbb{R}^2) \) such that

\[
\int_{\mathbb{R}^2} \varphi(x)dx = 1, \quad \text{and} \quad \lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = \lim_{\varepsilon \to 0} \varepsilon^{-2}\varphi(x/\varepsilon) = \delta_0 \quad \text{(in distribution)}.
\]

Let \( \text{div}_\varepsilon(w) := \text{div}(\varphi_\varepsilon \ast w) \) where \( \ast \) denotes the convolution. We consider the following system of equations

\[
\begin{align*}
    du_\varepsilon(t) &= \left( r_{\varepsilon} \Delta u_\varepsilon(t) - \chi \text{div}_\varepsilon(u_\varepsilon(t)\nabla v_\varepsilon(t)) \right) dt + u_\varepsilon(t) dW_1(t) + \gamma u_\varepsilon(t) dt \\
    dv_\varepsilon(t) &= \left( r_{\varepsilon} \Delta v_\varepsilon(t) + \beta u_\varepsilon(t) - \alpha v_\varepsilon(t) \right) dt + v_\varepsilon(t) dW_2(t)
\end{align*}
\]

(4.20)

\[ + \frac{1}{2} t^{p-1} C_p \gamma_1^p \mathbb{E}\left[ \int_0^t T^p(u(s)) \, ds \right]. \]

This implies

(4.21)

\[ \frac{1}{2} \mathbb{E}\left[ \sup_{0 \leq s \leq t} T^p(u(s)) \right] \leq \mathbb{E}[T^p(u(0))] + (\gamma p + \frac{C_p \gamma_1^p}{2}) t^{p-1} \mathbb{E}\left[ \int_0^t T^p(u(s)) \, ds \right]. \]

Using the Gronwall inequality we obtain

(4.22)

\[ \mathbb{E}\left[ \sup_{0 \leq s \leq t} T^p(u(s)) \right] \leq \mathbb{E}[T^p(u(0))] e^{(\gamma p + \frac{C_p \gamma_1^p}{2}) T^p}. \]

Since, \( u \), the cell density of the chemotaxis system is non-negative, \( u(x, t) \geq 0 \) for a.e. \( x \in \mathcal{O} \), for all \( t \in [0, T] \), \( \mathbb{P} - a.s. \) therefore, we note that \( I(u) \) coincides with the \( L^1 \) norm of \( u \). Therefore, one can conclude from (4.22) the following inequality holds

(4.23)

\[ \mathbb{E}\left[ \sup_{0 \leq s \leq t} |u(s)|_{L^1}^p \right] \leq \mathbb{E}[|u_0|^p_{L^1}] e^{(\gamma p + \frac{C_p \gamma_1^p}{2}) T^p}. \]

**Step (II)** We will apply the Itô formula to the functions

(4.24) \( \phi(w) = \frac{1}{2} |w|^2_{L^2}, \ w \in L^2(\mathcal{O}) \quad \text{and} \quad \psi(w) = \frac{1}{2} |\nabla w|^2_{L^2}, \ w \in H^1(\mathcal{O}) \)

to the processes \( u(s) \) and \( v(s) \) for \( s \in [0, T] \) respectively. A straight application of the Itô formula, however, would require a regularization of the unbounded operators appearing in equation (1.3).

Nevertheless, one can overcome this problem by the following standard procedure: We replace the unbounded operator \( A := \Delta \) in equation (1.3) by its Yosida approximation \( A_\varepsilon := \frac{1}{\varepsilon}(I - (\varepsilon \Delta - I)^{-1}) \) and the divergence operator by intertwining a mollifier \( \phi_\varepsilon \). Thereupon, we let the limit \( \varepsilon \to 0^+ \) (see e.g. proof of the Lemma 3.2 in [2]). Here, for any \( \nu \geq 2 \) and \( \delta \in \mathbb{R} \), we infer that

\[ |\Delta - \Delta_\varepsilon|_{L(H^{4+\nu},H^{\delta})} \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

Before applying the Itô-formula we see to it that the integrands belong to the appropriate space (see [3] Chapter 4.4). Thereupon, one can directly apply the Itô formula to the functions \( \phi \) and \( \psi \) defined in (4.24) for the processes \( (u_\varepsilon(t), v_\varepsilon(t)) \) for \( t \in [0, T] \). In addition, by the definition of \( \gamma_1 \) we know

\[ \text{Tr}\left[ \phi_{xx}(u)(uQ^\gamma)(uQ^\gamma)^T \right] \leq \gamma_1^2 |u|^2_{L^2}. \]
Again, let us remind the abbreviation (see (1.6)) $\gamma_j := \sum_{k \in \mathbb{N}} |\lambda_k^{(j)}|^2 |\psi_k|^2$; $j = 1, 2$. Let us begin with the following

$$
\frac{1}{2}|u(t)|_{L^2}^2 - \frac{1}{2}|u_0|_{L^2}^2 + r_u \int_0^t |\nabla u(s)|_{L^2}^2 ds
$$

$$
= \chi \int_0^t \langle \nabla u(r), u(r) \nabla v(r) \rangle dr + \int_0^t \langle u(r), u(r) \rangle dW_1(r)
$$

$$
+ \frac{1}{2} \int_0^t \text{Tr}[\phi_{xx}(u(r))(u(r)Q^{\frac{1}{2}})(u(r)Q^{\frac{1}{2}})^T] ds + \int_0^t \langle u(r), \gamma u(r) \rangle dr
$$

$$
\leq \epsilon \int_0^t |\nabla u(r)|_{L^2}^2 dr + \frac{\chi}{4} \int_0^t |u(s)\nabla v(s)|_{L^2}^2 ds + (\gamma + \frac{\epsilon^2}{2}) \int_0^t |u(s)|_{L^2}^2 ds
$$

$$
\int_0^t \sum_{k=1}^{\infty} \lambda_k^{(1)} \langle u(s), u(s)\psi_k \rangle d\beta_k^{(1)}(s),
$$

for $t \in [0, T]$. The Burkholder-Davis-Gundy inequality gives

$$
\mathbb{E} \left[ \int_0^t \sum_{k=1}^{\infty} |\lambda_k^{(1)}|^2 |\langle u(s), u(s)\psi_k \rangle|^2 ds \right] \leq \left[ \int_0^t \sum_{k=1}^{\infty} (\lambda_k^{(1)})^2 |\langle u(s), u(s)\psi_k \rangle|^2 ds \right]^{\frac{1}{2}}
$$

$$
\leq \gamma_1 \mathbb{E} \left[ \int_0^t |u(s)|_{L^2}^4 ds \right]^{\frac{1}{2}} \leq \gamma_1 \mathbb{E} \left[ \left( \sup_{0 \leq s \leq t} |u(s)|_{L^2}^2 \right)^{\frac{3}{2}} \left( \int_0^t |u(s)|_{L^2}^2 ds \right)^{\frac{1}{2}} \right].
$$

The Cauchy-Schwarz and the Young inequality gives that for any $\epsilon_1 > 0$ there exists a constant $C(\epsilon_1) > 0$ such that

$$
\int_0^t \sum_{k=1}^{\infty} \lambda_k^{(1)} \langle u(s), u(s)\psi_k \rangle d\beta_k^{(1)}(s) \leq \epsilon_1 \mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s)|_{L^2}^2 \right] + C(\epsilon_1) \mathbb{E} \left[ \int_0^t |u(s)|_{L^2}^2 ds \right].
$$

Similarly, by applying the Itô-formula to $[0, T] \ni t \mapsto \langle \nabla v(t), \nabla v(t) \rangle$ we obtain

$$
\frac{1}{2} |\nabla v(t)|_{L^2}^2 - \frac{1}{2} |\nabla v_0|_{L^2}^2 \leq -r_v \int_0^t |\Delta v(s)|_{L^2}^2 ds - \alpha \int_0^t |\nabla v(s)|_{L^2}^2 ds
$$

$$
+ \beta \int_0^t \langle \nabla u(s), \nabla v(s) \rangle ds + \int_0^t \sum_{k=1}^{\infty} \lambda_k^{(2)} \langle \nabla v(s), \nabla (v(s)\psi_k) \rangle d\beta_k^{(2)}(s)
$$

$$
+ \frac{\gamma^2}{2} \int_0^t |\nabla v(s)|_{L^2}^2 ds.
$$

Due to the Young inequality, for any $\epsilon_2 > 0$ we have

$$
\frac{1}{2} |\nabla v(t)|_{L^2}^2 + r_v \int_0^t |\Delta v(s)|_{L^2}^2 ds + \alpha \int_0^t |\nabla v(s)|_{L^2}^2 ds
$$

$$
\leq \frac{1}{2} |\nabla v_0|_{L^2}^2 + \frac{\beta^2}{2\epsilon_2} \int_0^t |\nabla v(s)|_{L^2}^2 ds + \epsilon_2 \int_0^t |\nabla u(s)|_{L^2}^2 ds + \frac{\gamma^2}{2} \int_0^t |\nabla v(s)|_{L^2}^2 ds
$$

$$
+ \int_0^t \sum_{k=1}^{\infty} \lambda_k^{(2)} \langle \nabla v(s), \nabla (v(s)\psi_k) \rangle d\beta_k^{(2)}(s).
$$
Using the Burkholder-Davis-Gundy inequality, we obtain
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^t \sum_{k=1}^\infty \lambda_k^{(2)} \langle \nabla v(s), \nabla (v(s) \psi_k) \rangle d\beta_k^{(2)}(s) \right| \right] \leq C \mathbb{E} \left[ \int_0^t \sum_{k=1}^\infty (\lambda_k^{(2)})^2 \left| \langle \nabla v(s), \nabla (v(s) \psi_k) \rangle \right|^2 ds \right]^{\frac{1}{2}}.
\]
The Hölder inequality gives
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^t \sum_{k=1}^\infty \lambda_k^{(2)} \langle \nabla v(s), \nabla (v(s) \psi_k) \rangle d\beta_k^{(2)}(s) \right| \right] \leq C \mathbb{E} \left[ \int_0^t \left( \sum_{k=1}^\infty \lambda_k^{(2)} \right)^2 \left| \nabla v(s) \right|^4 \right]^{\frac{1}{4}}.
\]
Applying the Hölder inequality, and then the Young inequality gives that for all \( \varepsilon_3 > 0 \) there exists a constant \( C(\varepsilon_3) > 0 \) such that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^t \sum_{k=1}^\infty \lambda_k^{(2)} \langle \nabla v(s), \nabla (v(s) \psi_k) \rangle d\beta_k^{(2)}(s) \right| \right] \leq C \gamma_2 \left[ \mathbb{E} \sup_{0 \leq s \leq t} \left| \nabla v(s) \right|^2 \right]^{\frac{1}{2}} \mathbb{E} \left[ \int_0^t \left| \nabla v(s) \right|^2 ds \right]^{\frac{1}{2}} \leq \varepsilon_3 \mathbb{E} \sup_{0 \leq s \leq t} \left| \nabla v(s) \right|^2 + C(\varepsilon_3) C \gamma_2 \mathbb{E} \left[ \int_0^t \left| \nabla v(s) \right|^2 ds \right].
\]
Now, let us consider the term \( |u(s)\nabla v(s)|^2_{L^2} \) in \( [12, 25] \). Using the following interpolation inequality (see \([5, p. 233]\))
\[
| \cdot |_{L^q} \leq C_{p,q} | \cdot |_{H^2}^\theta | \cdot |_{L^p}^{1-\theta} \quad \text{for} \quad 1 \leq p < q \leq \infty, \quad \theta = \frac{1}{q} - \frac{1}{p} = \frac{1}{2},
\]
for once \( q = 4, \ p = 1 \) and once \( q = 4, \ p = 2 \), we achieve
\[
|u(s)\nabla v(s)|^2_{L^2} = |u(s)|^2_{L^4} \left| \nabla v(s) \right|^2_{L^4} \leq C |u(s)|_{H^2} |u(s)|_{L^1} \left| \nabla v(s) \right|^2_{H^2} \left| \nabla v(s) \right|^2_{L^2}.
\]
The Young inequality gives that for any \( \varepsilon_4 > 0 \) there exists constant \( C(\varepsilon_4) > 0 \) such that
\[
\leq \varepsilon_4 \left( |u(s)|^2_{H^2} + |v(s)|^2_{H^2} \right) + C(\varepsilon_4) |u(s)|_{L^1} |v(s)|_{H^2}^{6^2} \left| \nabla v(s) \right|^2_{H^2} \left| \nabla v(s) \right|^2_{L^2}.
\]
The main difficulty is to handle the non-linear term \( u \nabla v \). To estimate the non-linear term which is \( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)\nabla v(s)|^2_{L^2} \right] \), we need to obtain bound for \( \mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s)|_{L^2}^8 \right] \) and
\( \mathbb{E}\left[ \int_0^t |y(s)|_{H_1^2}^{12} ds \right] \). We now apply Theorem 4.5 in [21], for \( p > 1 \) with \( X_0 = (H_1^1(\mathcal{O}))^* \) (the dual space of \( H_1^1(\mathcal{O}) \) corresponding to Neumann boundary conditions), \( X_1 = H_1^2(\mathcal{O}) \), and \((X_0, X_1)_{1-p, p} = B_{2,p} \). Hence, using the embedding \( L^1(\mathcal{O}) \hookrightarrow (H_1^1(\mathcal{O}))^* \) in one dimension, and using (4.12) we obtain

\[
\mathbb{E}\left[ \int_0^t |y(s)|_{H_1^2}^{12} ds \right] \leq C \left( \mathbb{E}\left[ \int_0^t |u(s)|_{H_1^2}^{12} ds \right] \right) \\
\leq C \left( \mathbb{E}\left[ \int_0^t |u(s)|_{H_1^2}^{12} ds \right] \right) \\
\leq C \left( \mathbb{E}\left[ \int_0^t |u(s)|_{H_1^2}^{12} ds \right] \right) \\
\leq C \left( \mathbb{E}\left[ \int_0^t |u(s)|_{H_1^2}^{12} ds \right] \right)
\]

(4.32)

We set \( \varepsilon_1 < \frac{1}{4}, \varepsilon_2 < \frac{2}{7}, \varepsilon_3 < \frac{1}{7}, \varepsilon_4 < \{ \frac{1}{7} \wedge \frac{2}{7} \} \). Taking supremum and then expectation in (4.25), (4.25), and (4.31) and combining (4.26), (4.29), and (4.32) we finally arrive at the following estimate

\[
\frac{1}{4} \mathbb{E}\left[ \sup_{0 \leq s \leq t} |\nabla u(s)|_{L^2}^2 \right] + \frac{1}{4} \mathbb{E}\left[ \sup_{0 \leq s \leq t} |\nabla v(s)|_{L^2}^2 \right] + \frac{r_v}{4} \mathbb{E}\left[ \int_0^t |\nabla u(s)|_{L^2}^4 ds \right] \\
\quad + \frac{r_v}{2} \mathbb{E}\left[ \int_0^t \int_{\Omega} |\nabla u(s, x)|^2 dx ds \right] + \alpha \mathbb{E}\left[ \int_0^t \int_{\Omega} |\nabla v(s, x)|^2 dx ds \right] \\
\leq C \left( \frac{1}{2} \mathbb{E}|u_0|_{L^2}^2 + \frac{1}{4} \mathbb{E}|\nabla v_0|_{L^2}^2 + T \mathbb{E}|u_0|_{L^1}^{8} e^{(s+\frac{1}{2})T^8} + T \mathbb{E}|v_0|_{B_{2,12}}^{12} \right) \\
\quad + T^2 \mathbb{E}|v_0|_{L^1}^{12} e^{(s+\frac{1}{2})T^8} \right) e^{(s+\frac{1}{2})T^8}.
\]

(4.33)

This completes the proof. \( \square \)

**Appendix A. Technical Lemmata**

**Lemma A.1.** \( \phi(x) = x(\ln(x) - 1), x \geq 0 \), satisfies

\[
\phi(x) - 2\phi \left( \frac{x + y}{2} \right) + \phi(y) \geq \frac{1}{4} \left( \sqrt{x} - \sqrt{y} \right)^2, \quad x, y \geq 0.
\]

**Proof.** See e.g. [11] Lemma 5.1 for more details. \( \square \)

To evaluate \( S_2 \) in the proof of Theorem 5.2, we formulate the following Claim.

**Claim A.1.** Let \( f : [1, \infty) \rightarrow \mathbb{R} \) be a function given by

\[
f(u) = \ln \left( \frac{2}{1+u} \right) + u \ln \left( \frac{2u}{1+u} \right) - (\sqrt{u} - 1)^2, \quad u \in [1, \infty).
\]

Then, \( f(u) \leq f(1) = 0 \) for \( u \in [1, \infty) \).

**Proof of Claim A.1.** Let us first evaluate the first two derivatives of \( f \). On differentiating \( f(u) \) and \( f'(u) \) with respect to \( u \), we get

\[
f'(u) = \ln \left( \frac{2u}{1+u} \right) + \frac{1}{\sqrt{u}} - 1, \quad f''(u) = \frac{2\sqrt{u} - (1+u)}{2u^{3/2}(1+u)} = \frac{-(\sqrt{u} - 1)^2}{2u^{3/2}(1+u)} \leq 0, \quad \text{for} \quad u \in [1, \infty).
\]
This implies that $f'$ is a decreasing function on $[1, \infty)$, i.e. $f'(u) \leq f'(1) = 0$ for $u \in [1, \infty)$. This again yields that $f(u) \leq f(1) = 0$ for $u \in [1, \infty)$. This completes the proof of the claim. 

\[\square\]

**References**

[1] A. Ambrosetti, and G. Prodi. *A primer of nonlinear analysis*. Corrected reprint of the 1993 original. Cambridge Studies in Advanced Mathematics, 34. Cambridge University Press, Cambridge, 1995.

[2] V. Barbu, and C. Marinelli. Strong solutions for stochastic porous media equations with jumps. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 12 no. 3, 413–426, 2009.

[3] V. Barbu, G. Da Prato, and M. Röckner. Stochastic porous media equations. Lecture Notes in Mathematics, 2163. Springer, [Cham], 2016. ix+202 pp.

[4] C. Bennett and R. Sharpley. *Interpolation of operators*. Boston, MA etc.: Academic Press, Inc., 1988.

[5] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, xiv+599 pp., 2011.

[6] A.S. Cherny. On the strong and weak solutions of stochastic differential equations governing Bessel processes. *Stochastics Stochastics Rep.*, 70:213–219, 2000.

[7] C.R. Dalang and L.Q. Sardanyons. Stochastic integrals for spde’s: a comparison. *Expo. Math.* 29 no. 1, 67–109, 2011.

[8] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions. 2nd ed.* Cambridge: Cambridge University Press, 2nd ed. edition, 2014.

[9] J. Duan, and W. Wang. Effective dynamics of stochastic partial differential equations. Elsevier Insights. Amsterdam, 2014.

[10] H. Engelbert. On the theorem of T. Yamada and S. Watanabe. *Stochastics Stochastics Rep.*, 36:205–216, 1991.

[11] H. Gajewski. On a variant of monotonicity and its application to differential equations. Nonlinear Anal. 22, no. 1, 73–80, 1994.

[12] G. B. Folland. *Real analysis. Modern techniques and their applications*. Second edition. Pure and Applied Mathematics (New York). 1999.

[13] B. Garrett and R. Gian. *Carlo Ordinary differential equations*. Fourth edition. John Wiley and Sons, Inc., New York, 1989.

[14] E. Hausenblas, D. Mukherjee, and T. Tran. The one-dimensional stochastic keller–segel model with time-homogeneous spatial wiener processes. *J. Differential Equations*, 310, 506–554, 2022.

[15] J. Jacod. Weak and strong solutions of stochastic differential equations. *Stochastics*, 3:171–191, 1980.

[16] M. Ondreját. Uniqueness for stochastic evolution equations in Banach spaces. *Diss. Math.*, 426:1–63, 2004.

[17] H. Qiao. A theorem dual to Yamada-Watanabe theorem for stochastic evolution equations. *Stoch. Dyn.*, 10: 367–374, 2010.

[18] W. Rudin. *Real and complex analysis*. Third edition. McGraw-Hill Book Co., New York, 1987.

[19] M. Röckner, R. Zhu, and X. Zhu. Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions, *Nonlinear Analysis*, 125:358–397, 2015.

[20] J. S. Tappe. The Yamada-Watanabe theorem for mild solutions to stochastic partial differential equations. *Electron. Commun. Probab.*, 18:13, 2013.

[21] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis. Maximal $L^p$-regularity for stochastic evolution equations. *SIAM J. Math. Anal.*, 44(3): 1372–1414, 2012.

[22] T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11:155–167, 1971.

[23] E. Wong and M. Zakai. On the convergence of ordinary integrals to stochastic integrals. *Ann. Math. Stat.*, 36: 1560–1564, 1965.
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