EXACT SOLUTIONS AND CONSERVATION LAWS OF  
A SYSTEM OF 1D PARTIAL DIFFERENTIAL EQUATIONS  
FOR BLOOD FLOW  

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Abstract. Two aspects of a widely used 1D model of blood flow are studied, where the  
variables in the model are the blood pressure and the cross-section area of the blood vessel.  
As one main result, all travelling waves and steady-state solutions are found by explicit  
quadrature of the model. The features, behaviour, and boundary conditions for these solutions  
are discussed. Solutions of interest include shock waves and sharp wave-front pulses  
for the pressure and the blood flow. Another main result is that three new conservation  
laws are derived for inviscid flows. Compared to the well-known conservations laws in 1D  
compressible fluid flow, they describe generalized momentum and generalized axial and vol-  
umetric energies. For viscous flows, these conservation laws get replaced by conservation  
balance equations which contain a dissipative term proportional to the friction coefficient in  
the model.  

1. Introduction  

In recent years, one-dimensional (1D) models of blood flow in human blood vessels have  
been widely used in clinical applications [1, 2]. These models are effective for understanding  
averaged features of blood flow locally, such as velocity, volume flux, and pressure [3, 4].  
They can also be combined with 3D models for detailed simulation of the human cardiovas-  
cular system as a whole [5, 6, 7]. Moreover, 1D models have much less computational cost  
compared to 3D models and can be mathematically analyzed in greater depth.  

A blood vessel in a 1D model is a cylindrical tube whose radius varies as a function of  
time $t$ and axial distance $x$, in which the blood is a compressible fluid governed by the  
Navier-Stokes equations averaged over cross-sections of the tube. The variables consist of  
the cross-section area $A$, the volume flux $Q$ of blood flow, the mean pressure $P$, and the  
mean blood velocity $\bar{u} = Q/A$, while the blood density is taken to be constant. $A$ and $Q$  
satisfy a system of two coupled partial differential equations (PDEs) which are similar in  
form to the Navier-Stokes equations for mass continuity and momentum in fluid mechanics.  
The system is closed by specifying an equation of state which gives the pressure in terms of  
the cross-section area; the simplest widely-used model is that the pressure change across the
vessel wall is proportional to the change in radius of the vessel. There are two important parameters in the resulting closed system: a friction parameter, which is proportional to the viscosity coefficient in the Navier-Stokes equations; and a momentum correction parameter, which arises from how the Navier-Stokes are averaged over a cross-section [8].

In the literature, there is a lot of work on numerical solutions, but very little has been done on exact solutions except for the use of the well-known Riemann method of characteristics [9, 10]. This method, however, can be carried out to obtain explicit solutions only in a special case for the momentum correction parameter [4].

The main purpose of the present paper is to study explicit solutions of the 1D model that describe travelling waves – namely, a wave form that moves with a constant speed and preserves its shape. A complete discussion of all of these solutions and their properties is given.

One type of solution obtained describes a dissipating shock wave in a very long constricting blood vessel with a steady-state near each end; the vessel’s diameter, pressure, and blood flow display a rapid transition in the shock, which moves at a constant speed. A similar shock wave solution is found for a blood vessel of arbitrary length in which the initial state of the blood vessel is close to a steady-state and then rapidly transitions such that the diameter, pressure, and blood flow are increasing. The blood velocity exhibits a shock behaviour between two steady-states.

Another type of solution obtained describes a pulse with a sharp front for the blood vessel’s diameter, pressure, and blood flow; behind the front, these quantities decrease to a steady-state behaviour. Other solutions are obtained that exhibit a similar sharp front, with different behaviours behind the front.

In general, solutions that describe travelling waves on an infinitely long tube can be applied to modelling a very long blood vessel where the morphology at the ends of the vessel is not relevant. When the morphology at the ends is important, an explanation will be given of how the travelling wave solutions are applicable with various boundary conditions satisfied at the end points. More discussion of the applicability of travelling waves will be given at the end of the paper.

A further purpose of the paper is to derive some explicit conservation laws admitted by the 1D model. The only well-known conservation laws to-date have been the total blood volume and the net blood flux. Three new conservation laws are obtained: a generalized momentum and two generalized energies, which hold when the blood flow is modelled as being inviscid. The momentum quantity is a modified form of the well-known momentum in fluid mechanics. The energy quantities represent a volumetric energy and an axial energy, which are similar to the well-known energy in fluid mechanics (for the inviscid Navier-Stokes equations). It is interesting, however, that there are two different conserved energies for the blood flow model. When the blood flow is modelled as being viscous, then these three new quantities are no longer conserved but they satisfy conservation balance equations that contain a dissipative volume term proportional to friction coefficient. Such balance equations are useful in mathematical analysis of the initial-value problem.

In particular, it is known that the Navier-Stokes energy quantity satisfies a balance equation leading to a time-decay inequality [4]. This quantity coincides with the total volumetric energy in the blood flow model in a special case for the momentum correction parameter, but otherwise it is not conserved in the blood flow model even for inviscid flow. Analogous
inequalities can be derived for the viscous blood flow model by use of the volumetric energy and axial energy with no restriction on the momentum correction parameter.

Furthermore, the new conservation laws yield associated boundary conditions for the model such that the flux of the generalized momentum and the generalized energies is zero. These zero-flux boundary conditions can be applied to the steady-state solutions and, when adapted to a moving reference frame, the travelling wave solutions.

All of these results are new. A worthwhile remark is that the viewpoint here is applied mathematics, rather than biological modelling. Nevertheless, the results can be expected to have potential applications, both in biological models and in mathematical analysis. In particular, the travelling waves solutions and the conservation laws can be used for checking the accuracy of numerical schemes.

Section 2 summarizes the blood flow PDEs and equation of state. The kinematical symmetries and the five basic conservation laws of the model are derived.

Section 3 has a general discussion of boundary conditions for the model, including zero-flux boundary conditions coming from the three new conservation equations. In addition, all steady-state solutions of the model, which represent stationary travelling waves, are summarized.

Section 4 starts with deriving the system of ODEs satisfied by travelling wave solutions of the blood flow PDEs. These ODEs have an explicit quadrature which is obtained for the separate cases of inviscid and viscous flow. Spatial domains and boundary conditions for the solutions are then considered.

Section 5 presents all of the exact travelling wave solutions and discusses their basic mathematical and physical features.

Section 6 makes some concluding remarks.

2. Summary and features of the 1D model

The PDE system for the quantities \(A(x,t), Q(x,t), P(x,t)\) describing cross-section area, blood flow, and pressure in a cylindrical blood vessel is given by [8] \[\begin{align*}
A_t + Q_x &= 0, \\
Q_t + \alpha(Q^2/A)_x + \rho_0^{-1}AP_x + kQ/A &= 0
\end{align*}\] where \(\alpha \geq 1\) is a momentum correction coefficient (determined by the axial velocity profile), \(k \geq 0\) is the friction coefficient (proportional to the viscosity). Here \(\rho_0 > 0\) is the blood density, which is constant.

The equation of state for pressure, which closes the system, is taken to be \[P = \beta(\sqrt{A} - \sqrt{A_0}) + P_{\text{ext.}}\] where \(\beta > 0\) is a constant, and \(P_{\text{ext.}}\) is the external pressure caused by the tissue surrounding the blood vessel, which will be assumed to be constant. If there is no change in pressure across the vessel wall, then the blood vessel is assumed to have a constant area \(A = A_0\).

For details of the derivation of this model and further explanation of its biological and physical features, see Ref. [11, 12].
Substitution of the equation of state (3) into the PDEs (1)–(2) yields the closed system

\[ A_t + Q_x = 0, \]
\[ Q_t + \alpha(Q^2/A)_x + \frac{1}{2} \beta_0 \sqrt{AA_x} + kQ/A = 0 \]

where \( \beta_0 = \beta/\rho_0 > 0 \). In terms of \( Q \) and \( A \), the mean blood flow velocity is

\[ \bar{u} = Q/A \]

This system is well known to be hyperbolic when \( A > 0 \), which implies that it possesses two Riemann invariants which propagate with speeds \( c_\pm = \alpha \bar{u} \pm \sqrt{\frac{1}{4} \beta_0^2 A + \alpha (\alpha - 1) \bar{u}^2} \). This means that the system admits nonlinear waves that travel along the paths determined by \( dx/dt = c_\pm \).

The most important parameter in the system (4)–(5) is the friction coefficient \( k \geq 0 \). In applications, there are two main cases of interest.

**Viscous**: \( k > 0 \). In this case, the system (4)–(5) is dissipative. As an illustration, spatially homogeneous solutions \((A(t),Q(t))\) satisfy \( A' = 0 \) and \( Q' = -kQ/A \), which gives \( A = A_s \) and \( Q = Q_0 e^{-A_s^2/k} \), where \( A_s \) is a positive constant and \( Q_0 \) is an arbitrary constant. These solutions describe a blood vessel with a constant radius and a blood flux that exponentially drops to zero on a time scale \( A_s/k \).

**Inviscid**: \( k = 0 \). In this case, the system (4)–(5) is non-dissipative. Spatially homogeneous solutions \((A(t),Q(t))\) simply are constants, \( A = A_s > 0 \) and \( Q = Q_s \), which describes a blood vessel with a constant radius carrying a constant blood flux.

2.1. **Kinematic point symmetries.** The system (4)–(5) has the following kinematic transformation groups of symmetries:

- space reflection \( x \to -x, \ Q \to -Q \)
- time translation \( t \to t + \epsilon \)
- axial translation \( x \to x + \epsilon \)
- scaling \( t \to e^{\epsilon} t, \ x \to e^{\frac{\epsilon}{2}} x, \ A \to e^{\epsilon} A, \ Q \to e^{\frac{\epsilon}{2}} Q \)

where \( \epsilon \) is the parameter in the symmetry group. In the inviscid case, the system has additional kinematic symmetry transformation groups:

- time reversal \( t \to -t, \ Q \to -Q \)
- dilation \( t \to e^{\epsilon} t, \ x \to e^{\epsilon} x \)
- Galilean boost \( x \to x + \epsilon t, \ Q \to Q + \epsilon A, \ \alpha = 1 \)

Note that the Galilean boost corresponds to \( \bar{u} \to \bar{u} + \epsilon \).

These symmetries (7)–(13) are evident by inspection of the form of the system (4)–(5) in comparison to the 1D Navier-Stokes equations for compressible fluids whose Lie point symmetries and discrete symmetries are well known \[13, 14, 15\].

A determination of all point transformation symmetries is fairly complicated and will involve utilizing the form of the Riemann invariants.

For a general discussion of symmetries and their applications to PDEs, see Ref. \[13, 16, 17\].
2.2. Basic conservation laws. Some conservation laws of the system (4)–(5) can be readily found by comparison to the well-known conservation laws of mass, momentum, and energy in the inviscid case, for the 1D fluid dynamics [13, 14, 18]. Mathematically, $A$ is analogous to mass density $\rho$, and $Q$ is analogous to the momentum density $\rho u$, where $\rho$ and $u$ are density and velocity variables in the Navier-Stokes equations. This analogy holds exactly in case $k = 0, \alpha = 1$.

Firstly, the PDE (4) itself is a continuity equation for $A$ viewed as a density. Integration of $A$ over a finite interval $x_1 \leq x \leq x_2$, for instance a portion of a blood vessel, gives the total volume of blood

$$V = \int_{x_1}^{x_2} A \, dx$$

This integral quantity satisfies the conservation law

$$\frac{d}{dt} V = -Q \bigg|_{x_2}^{x_1}$$

stating that the rate of change in blood volume is balanced by the net change in blood flow between the two ends.

Likewise, the PDE (5) in the inviscid case, $k = 0$, is a continuity equation for $Q$ viewed as a density. The integral of $1/L Q$ over $x_1 \leq x \leq x_2$, with $L = x_2 - x_1$, gives the net (mean) blood flux

$$\bar{Q} = \frac{1}{L} \int_{x_1}^{x_2} Q \, dx$$

which satisfies the conservation law

$$\frac{d}{dt} \bar{Q} = -\frac{2}{L} \rho_0^{-1} \bar{P} A \bigg|_{x_1}^{x_2}$$

where $P = \frac{1}{2} \alpha \rho_0 \bar{u}^2 + \frac{1}{6} \bar{P} A$ is the analog of mechanical pressure in inviscid constant-density fluid dynamics [19].

Thus, the rate of change in the net blood flux is proportional to the difference in the mechanical force $\bar{P} A$ on the cross-sections at each end. In the case of viscous blood flow, the conservation law is replaced by a balance equation

$$\frac{d}{dt} \bar{Q} = -\frac{2}{L} \rho_0^{-1} \bar{P} A \bigg|_{x_1}^{x_2} - k \bar{U}$$

where $\bar{U} = \frac{1}{L} \int_{x_1}^{x_2} \bar{u} \, dx$ is mean velocity.

Secondly, balance equations for generalized momentum and generalized energy can be obtained by considering multipliers for the pair of PDEs (4)–(5).

Conservation of momentum for the Navier-Stokes equations arises from multiplying the equations by the respective expressions $(u, \rho)$, called a multiplier. The analogous multiplier for the PDEs here would be $(Q/A, A)$, which will lead to a conservation law in the case $k = 0, \alpha = 1$. In the general case, a local balance equation arises by adjusting the multiplier using a suitable power of $A$. Specifically, the multiplier $\left( (1 - 2\alpha) A^{-2\alpha} Q, A^{1-2\alpha} \right)$ yields

$$\left( Q A^{1-2\alpha} \right)_t + \left( \frac{1}{2} Q^2 A^{-2\alpha} - \frac{1}{4\alpha - 5} \beta_0 A^{\frac{5-2\alpha}{2}} \right)_x = -k QA^{-2\alpha}$$

Integration of the density times $\rho_0$ gives the integral quantity

$$M = \rho_0 \int_{x_1}^{x_2} \frac{Q}{A^{2\alpha-1}} \, dx = \rho_0 \int_{x_1}^{x_2} \bar{u}/A^{2(\alpha-1)} \, dx$$
satisfying the conservation law
\[ \frac{d}{dt} M = -\rho_0 \left( \left( \frac{3}{2} Q^2 - \frac{1}{4(\alpha - 5)} \beta_0 A^{5/2} \right) / A^{2\alpha} \right) |^{x_2}_{x_1} \] (21)
in the inviscid case, \( k = 0 \). This describes conservation of a generalized momentum.

Similarly, conservation of energy for the inviscid Navier-Stokes equations arises from the multiplier \( (\frac{1}{2} u^2, \rho u) \). The analogous multiplier for the PDEs (4)–(5) in the case \( k = 0, \alpha = 1 \) is \( (\frac{1}{2} Q^2 / A^2, Q) \). In the general case, the adjusted multiplier \( (\frac{1}{2} Q^2 A^{1-r}, QA^{-r}) \) yields a local balance equation
\[ \left( \frac{1}{2} Q^2 A^{-r} + \frac{2}{(2r-3)(2r-5)} \beta_0 A^{\frac{5}{2}-r} \right) + \left( \frac{4\alpha-r}{6} Q^3 A^{-r-1} - \frac{1}{2r-3} \beta_0 QA^{\frac{3}{2}-r} \right) = -kQ^2 A^{-r-1} \] (22)
where \( r^2 - (4\alpha - 1)r + 2\alpha = 0 \) whose roots
\[ r_{\pm} = 2\alpha - \frac{1}{2} \pm 2\sqrt{\alpha(\alpha - 1) + \frac{1}{16}} \] (23)
are real since \( \alpha \geq 1 \). In the inviscid case, \( k = 0 \), this yields the conservation laws
\[ \frac{d}{dt} E_{\pm} = -\rho_0 \left( \left( \frac{4\alpha-r}{6} Q^3 - \frac{1}{2r-3} \beta_0 QA^{\frac{3}{2}} \right) / A^{r+1} \right) |^{x_2}_{x_1} \] (24)
for the generalized energies
\[ E_{\pm} = \rho_0 \int_{x_1}^{x_2} \left( \frac{1}{2} Q^2 + \frac{2}{(2r-3)(2r-5)} \beta_0 A^{\frac{5}{2}} \right) / A^{r \pm} dx \] (25)

In the viscous case, the righthand side of these conservation equations (21) and (24) will also contain a dissipative integral term proportional to \( k \).

For \( \alpha = 1 \), note that the generalized momentum integral (20) reduces to \( M = \rho_0 \int_{x_1}^{x_2} u dx \) which is the total momentum of the blood flow. Similarly, the generalized energy integrals (25) reduce to \( E_+ = \rho_0 \int_{x_1}^{x_2} \left( \frac{1}{2} u^2 + \frac{2}{3} \beta_0 \sqrt{A} \right) A dx \) and \( E_- = \rho_0 \int_{x_1}^{x_2} \left( \frac{1}{2} u^2 - 2 \beta_0 \sqrt{A} \right) dx \), where \( r_- = 1, r_+ = 2 \). The quantity \( E_- \) is the total volumetric energy of the blood flow, while the other quantity \( E_+ \) is the total axial energy.

It is interesting that, in contrast to the inviscid Navier-Stokes equations, the blood flow system possesses two conserved energies.

A determination of all low-order conserved integrals and balance equations is, in principle, possible by the method of multipliers. However, similarly to the situation for symmetries, it is fairly complicated and will involve utilizing the form of the Riemann invariants.

For a general discussion of conservation laws of PDEs, see Ref. [16, 17, 20, 21].

3. Boundary conditions and steady-states

As a model for blood flow, the system (4)–(5) must be supplemented by boundary conditions at the ends of blood vessel, \( x = x_1 \) and \( x = x_2 \), with \( x_2 > x_1 \). Various arguments indicate that a single boundary condition can be posed at each end. ** refs The specific type of boundary condition involves the particular biological morphology of the ends of the blood vessel being modelled: an end that is branching; an end that terminates or is blocked; an end that is open; an end that has blood pumped in or out; an end that has a fixed diameter or a fixed pressure; an end at which a pressure wave or blood flow pulse is propagating in or out; an end with a steady-steady pressure. Attention here will be restricted to the latter two cases. Note that, depending on the morphology model, the two ends can have different types of boundary conditions.
Propagation of a pressure wave with speed $c$ along a blood vessel is specified by conditions $cP_x = P_t$ at each end. From the equation of state $\frac{P}{\rho} = \alpha$, this is equivalent to the boundary conditions

$$cA_x(x_1, t) = A_t(x_1, t), \quad cA_x(x_2, t) = A_t(x_2, t), \quad t \geq 0$$

Likewise, boundary conditions specifying a blood flow pulse are given by

$$cQ_x(x_1, t) = Q_t(x_1, t), \quad cQ_x(x_2, t) = Q_t(x_2, t), \quad t \geq 0$$

Steady-state blood pressure as boundary conditions is given by requiring that the gradient of $P$ vanishes at the ends. This is equivalent to the gradient of $A$ vanishing:

$$A_x(x_1, t) = 0, \quad A_x(x_2, t) = 0, \quad t \geq 0$$

For modelling a very long blood vessel, the ends can be regarded as being at $x_1 \to -\infty$ and $x_2 \to \infty$. Boundary conditions are thereby regarded as holding asymptotically. A precise meaning of very long is that the total length $x_2 - x_1$ is much greater than any length scale in the equations (4) and (5) and in the initial conditions $A(x, 0), Q(x, 0)$.

3.1. Zero-flux boundary conditions. Another kind of boundary condition can be obtained by considering the flux terms in the conservation balance equations (21) and (24) for generalized momentum, generalized volumetric energy and axial energy. Setting the respective flux expressions to vanish yields the following conditions:

$$Q(x_1, t)^2 = \frac{2}{4\alpha - 5} \beta_0 A(x_1, t)^3, \quad Q(x_2, t)^2 = \frac{2}{4\alpha - 5} \beta_0 A(x_2, t)^3, \quad \alpha > \frac{5}{4}$$

and

$$\frac{4\alpha - r_+}{6} Q(x_1, t)^2 = \frac{1}{2r_+ - 3} \beta_0 A(x_1, t)^3, \quad \frac{4\alpha - r_+}{6} Q(x_2, t)^2 = \frac{1}{2r_+ - 3} \beta_0 A(x_2, t)^3$$

where $r_+$ is given by expression (23). It is straightforward to show that the coefficients $4\alpha - r_+$ and $2r_+ - 3$ are positive for $\alpha \geq 1$. However, the coefficients with $r_+$ replaced by $r_-$ have opposite signs for $\alpha \geq 1$, and hence the counterpart of condition (30) using $r_-$ would be consistent only for the trivial case $A = Q = 0$.

The meaning of the boundary condition (29) is that the generalized momentum (20) of the blood flow is conserved for a solution $(A(x, t), Q(x, t))$ in the inviscid case. In the viscous case, the meaning is that the generalized momentum for a solution exhibits dissipation with no flux.

Similarly, the meaning of the boundary condition (30) is that the generalized axial energy of the blood flow, given by the integral (25) in the “+” case, is conserved for a solution $(A(x, t), Q(x, t))$ in the case of inviscid flow, while it exhibits dissipation with no flux for a solution in the case of viscous flow.

3.2. Steady-state solutions. A steady-state refers to a time-independent solution $(A(x), Q(x))$ (31)

of the system (4)–(5). Dynamical solutions can be expected to evolve toward a steady-state as $t \to \infty$.

The first equation (4) reduces to $Q_x = 0$, and thus $Q = Q_s$ is arbitrary constant. The solution to the second equation (5) depends on whether the model is viscous or inviscid.
Inviscid: $k = 0$. Equation (5) reduces to $\alpha (Q_s^2/A)_x + \frac{1}{2} \beta_0 \sqrt{A} A_x = 0$. Its general solution is that $A = A_s > 0$ is a positive constant. Thus, steady-states in the inviscid case are spatially constant (homogeneous)

$$Q = Q_s, \quad A = A_s > 0$$

(32)

Since $Q_s$ and $A_s$ can be specified independently, there is a two-parameter family of steady-states. The steady-state blood flow velocity is given by $\bar{u}_s = Q_s/A_s$.

Viscous: $k > 0$. In this case, equation (5) becomes

$$(\beta_0 \sqrt{A^5} - 2\alpha Q_s^2) A_x + 2kQ_s A = 0$$

(33)

which has the general solution

$$\alpha Q_s^2 \ln(A/A_s) - \frac{1}{5} \beta_0 (\sqrt{A^5} - \sqrt{A_s^5}) = kQ_s x$$

(34)

where $A_s$ is an arbitrary positive constant. When $Q_s = 0$, solutions are constant, $A = A_s > 0$. In contrast, when $Q_s \neq 0$, solutions $A = A(x) > 0$ are spatially varying and exhibit two different behaviours, which are parameterized by $Q_s$ and $A_s = A(0)$. Let $A_c = (2\alpha Q_s^2/\beta_0)^{2/5}$.

If $A > A_c$, then $A(x)$ is unbounded as $x \rightarrow -\infty$ and decreases with $x$ to an inverted one-sided cusp at $x = x_c = \alpha k^{-1}Q_s \ln(A_c/A_s) - \beta_0 (5kQ_s)^{-1}(\sqrt{A_c^5} - \sqrt{A_s^5})$ with $A(x_c) = A_c$ and $A_x(x_c) = -\infty$.

If $A < A_c$, then $A(x)$ goes to 0 exponentially as $x \rightarrow -\infty$ and increases with $x$ to $x = x_c$ where $A(x)$ has a one-sided cusp.

In both cases, the solution stops at the one-sided cusp. See Fig. 1.

![Figure 1. area profiles for $A_c = 1.5$ (solid), 2 (dot), 3 (dash).](image)

The domain for $A(x)$ can be taken to be either $x_1 \leq x \leq x_2$ or $-\infty < x \leq x_2$, with $x_2 \leq x_c$.

In the first case, one allowed boundary condition is $A(x_1, t) = A_1 > 0$, $A(x_2, t) = A_2 > 0$, with $A_2 \neq A_1$, or similarly $P(x_1, t) = P_1 > 0$, $P(x_2, t) = P_2 > 0$, with $P_2 \neq P_1$. These conditions can be shown to determine the two parameters $(Q_s, A_s)$ in the solution $A(x)$, which describes a blood vessel that is either narrowing or widening, with a constant blood flow.
Another allowed boundary condition is $A(x_1, t) = A_1 > 0$, $A_x(x_2, t) = \pm \infty$, or similarly $P(x_1, t) = P_1 > 0$, $P_x(x_2, t) = \pm \infty$, which specifies that $x_2 = x_c$ is the location of the one-sided cusp. These conditions likewise determine the solution $A(x)$. The one-sided cusp describes a flaring out or in of the diameter of the blood vessel.

In the second case, allowed boundary conditions are $A \to 0$ as $x \to -\infty$, $A(x_2, t) = A_2 > 0$, or similarly $P \to 0$ as $x \to -\infty$, $P(x_2, t) = P_2 > 0$. These conditions determine a single parameter in the solution $A(x)$. The second parameter is determined if an additional boundary condition is posed at $x = x_2$, such as $Q(x_2, t) = Q_s$. The solution $A(x)$ describes a blood vessel that is constricted as $x \to -\infty$ and widening with $x$, containing a constant blood flow.

A zero-flux boundary condition at $x = x_2$ is also allowed in both cases, as it has the form $Q_s^2 = \lambda A(x_2, t)^{5/2}$ where $\lambda > 0$ is given in terms of the coefficients for either the momentum-flux (29) or the axial energy-flux (30). This directly determines a single parameter in the solution $A(x)$. Note that a different boundary condition will be needed at $x = x_1$ due to $A(x_1, t) \neq A(x_2, t)$.

Extending $A(x) = A_c$ to be constant past the one-sided cusp yields a piecewise solution. Its domain can be $-\infty < x < \infty$, or one of the previous two possibilities with no restriction on $x_2$. In the former case, possible boundary conditions are $A \to 0$ as $x \to -\infty$, $A \to A_c$ and $Q \to Q_s$ as $x \to \infty$. The solution $A(x)$ describes a blood vessel that is constricted as $x \to -\infty$, flares out for $x$ near $x_c$, and has a constant diameter for $x > x_c$, with a constant blood flow throughout.

4. TRAVELLING WAVES

A travelling wave has the form

$$A = A(\xi), \quad Q = Q(\xi), \quad \xi = x - ct$$

(35)

where $c$ is the wave speed. This form arises from group-invariance with respect to the translation symmetry $(t, x) \to (t + \epsilon, x + \epsilon c)$.

If $c = 0$, then a travelling wave reduces to a steady-state solution. Hereafter, $c$ will be taken to be non-zero.

Substitution into the blood flow system (4)–(5) yields the travelling wave ODEs

$$-cA' + Q' = 0, \quad -cQ' + \alpha (Q^2/A)' + \frac{1}{2} \beta_0 \sqrt{A} A' + kQ/A = 0$$

(36)

The first ODE gives $Q$ in terms of $A$, and then the second ODE becomes a nonlinear separable equation for $A$:

$$Q = cA + C_1, \quad (c^2(\alpha - 1) + \frac{1}{2} \beta_0 \sqrt{A} - C_1^2 \alpha A^{-2}) A' + ck + C_1 k A^{-1} = 0$$

(37)

Let

$$C = -C_1/c, \quad \gamma = 2(\alpha - 1)c^2 \geq 0, \quad \sigma = 2C^2 \alpha c^2 > 0, \quad \kappa = 2kc \neq 0$$

(38)

The equations for $Q$ and $A$ now have the simpler form

$$Q = c(A - C)$$

(39)

and

$$A' = \frac{\kappa (A - C) A}{\sigma - \gamma A^2 - \beta_0 A^{5/2}}$$

(40)
Note that the physical parameters are given in terms of $\kappa$, $\sigma$, $\gamma$ by the relations

$$\alpha = \sigma / (\sigma - \gamma C^2), \quad c = \pm \sqrt{\sigma - \gamma C^2} / (\sqrt{2} C), \quad k = \pm \kappa C / (\sqrt{2} \sqrt{\sigma - \gamma C^2})$$  \(41\)

Some general features of solutions in the inviscid and viscous cases will be discussed next.

4.1. **Inviscid flow.** When $k = 0$, equation (40) for $A(\xi)$ reduces to $A' = 0$. Hence, $A$ is constant, and consequently equation (39) shows that $Q$ is also constant. These two constants determine the value of $C = A - Q/c$.

Thus, the general solution is a homogeneous steady state:

$$A = A_s = \text{const.} > 0, \quad Q = Q_s = \text{const.}$$  \(42\)

The mean blood flow velocity is $\bar{u} = \bar{u}_s = Q_s / A_s$, while the pressure is $P = P_s = P_{\text{ext}} + \beta (\sqrt{A_s} - \sqrt{A_0})$.

4.2. **Viscous flow.** For $k > 0$, equation (40) gives a quadrature for $A(\xi)$, which can be evaluated explicitly. Up to a shift in $\xi$, there is a one-parameter family of solutions $A(\xi)$ in terms of the arbitrary constant $C$. The features of the solution family depend strongly on the sign of $C$ and on the value of the positive root of $\gamma A^2 + \beta_0 A^{5/2} = \sigma$. Let

$$A_c > 0, \quad \gamma A_c^2 + \beta_0 A_c^{5/2} = \sigma;$$  \(43\)

An asymptotic expansion of equation (40) for $A$ near $A_c$ shows that $\xi$ is finite and thus it is the location of a one-sided cusp where

$$A(\xi_c) = A_c, \quad A'(\xi_c) = \infty$$  \(44\)

For $A$ near 0, an asymptotic expansion of equation (40) shows that $|\xi| \to \infty$, which thus represents an exponential tail in $A$. Similarly, if $C > 0$, then $A$ near $C$ has an exponential tail.

Attention will be restricted to solutions with positive wave speeds, $c > 0$. Solutions with negative wave speed are given by reflection $\xi \to -\xi$ applied to positive-wave speed solutions.

4.3. **Domain and boundary conditions.** Firstly, consider a travelling wave solution $A(\xi)$ on $-\infty < \xi < \infty$. This corresponds to a solution

$$(A, Q) = (A(x - ct), c(A(x - ct) - C))$$  \(45\)

describing the solution on the spatial domain $-\infty < x < \infty$, where the asymptotic behaviour of $A(\xi)$ determines the type of asymptotic boundary conditions holding for the solution $(A, Q)$.

If $A(\xi)$ asymptotically approaches a steady-state, then $(A, Q)$ will satisfy asymptotic steady-steady boundary conditions

$$A_x \to 0, \quad Q_x \to 0, \quad \text{as} \quad x \to \pm \infty, \quad t \geq 0$$  \(46\)

If $A(\xi)$ has other asymptotic behaviour, then $(A, Q)$ will satisfy asymptotic wave propagation boundary conditions

$$cA_x - A_t \to 0, \quad cQ_x - Q_t \to 0, \quad \text{as} \quad x \to \pm \infty, \quad t \geq 0$$  \(47\)

since travelling waves (45) automatically satisfy such boundary conditions at any point $x$.

Secondly, consider a travelling wave solution $A(\xi)$ on only a finite domain $\xi_1 \leq \xi \leq \xi_2$. This will yield a corresponding solution (45) of the system (4)–(5) on a finite spatial domain $x_1 \leq x \leq x_2$ in a finite time interval $0 \leq t \leq T$ which are given as follows.
Suppose \( c > 0 \). At \( t = 0 \), the front of the wave will define the location of the right end point \( x = x_2 \) via the relation \( \xi_2 = x_2 \). The left end point \( x = x_1 \) will be defined by the location of the back of the wave at \( t = T \) via \( \xi_1 = x_1 - cT \). The size of the domain is thus \( x_2 - x_1 = \xi_2 - \xi_1 - cT \) which requires that \( T < (\xi_2 - \xi_1)/c \). Thus, the point \( \xi = \xi_2 \) on the wave starts at \( x = x_2 \) and moves to the right, out of the spatial domain, while the point \( \xi = \xi_1 \) on the wave starts out of the spatial domain and moves to the right, entering the domain at \( t = T \). A similar discussion applies when \( c < 0 \).

At the end points of the domain \( x_1 \leq x \leq x_2 \), the solution (45) will satisfy wave propagation boundary conditions (26) or (27).

Thirdly, consider a travelling wave solution \( A(\xi) \) on a half-infinite domain \( -\infty < \xi \leq \xi_2 \). The corresponding solution (45) of the system (4)–(5) is defined for \( t \geq 0 \) on a spatial domain that can be either finite, \( x_1 \leq x \leq x_2 \), or half-infinite, \( -\infty < x \leq x_2 \).

Finally, note that a travelling wave solution \( A(\xi) \) on the domain \( -\infty < \xi < \infty \) can be truncated to any interval \( \xi_1 \leq \xi \leq \xi_2 \) to obtain a solution (45) on a finite domain.

Apart from wave propagation boundary conditions, it is possible to consider zero-flux boundary conditions posed on the moving domain with respect to \( \xi \). This will be pursued elsewhere.

5. Exact solutions

All travelling wave solutions (45) will now be presented, starting from the travelling wave ODE (40). Their detailed features and physical interpretation will also be discussed.

5.1. Solutions for \( C = 0 \). In this case, from relations (38), \( \sigma = 0 \) and the quadrature of equation (40) is given by \( \frac{2}{3} \beta_0 A^{3/2} + \gamma A = \kappa (\xi_0 - \xi) \), where \( \xi_0 \) is an integration constant. This is a cubic equation for \( \sqrt{A} \) which can be solved explicitly. Solutions have the behaviour that \( A(\xi) \) is a concave decreasing function of \( \xi \) that reaches zero at \( \xi = \xi_0 \) where \( A'(\xi_0) = -\kappa/\gamma < 0 \) from equation (40). Equation (39) yields \( Q(\xi) = cA(\xi) \), and thus \( \bar{u} = c \) is constant. See Fig. 2.

![Figure 2. C = 0: area profile for \( \gamma/\beta_0 = 0 \) (solid), 1 (dot), 2 (dash).](image-url)
Extending $A(\xi)$ to be a piecewise solution that is 0 past $\xi = \xi_0$, then this describes a blood vessel that is filling behind the front $x = \xi_0 + ct$ of a moving blood flow pulse and that is constricted ahead of the front, with the blood flow velocity being the same as the speed of the front, $\bar{u} = c$.

5.2. Solutions for $C < 0$. In this case, equation (40) has the quadrature

$$2\beta_0 |C|^{3/2} \arctan \left( \sqrt{A}/\sqrt{|C|} \right) - (\sigma/|C|) \ln(A) - (\gamma|C| - \sigma/|C|) \ln(A + |C|)$$

$$+ \frac{2}{3} \beta_0 A^{3/2} + \gamma A - 2\beta_0 |C| \sqrt{A} = \kappa(\xi_0 - \xi)$$

which determines $A(\xi)$. By translation symmetry, it is convenient to put $\xi_0 = 0$, which corresponds to a shift in either the $x$ or $t$ coordinates. Solutions exhibit the following two different behaviours, which are distinguished by whether $A(\xi) \gtrless A_c$.

If $A(\xi) > A_c$, then the solution $A(\xi)$ is a concave decreasing function of $\xi$ that exhibits an inverted one-sided cusp at $\xi = \xi_c$, and the solution does not exist for $\xi > \xi_c$. From equation (39), $Q(\xi) = c(A(\xi) + |C|)$ has a similar behaviour, except for a constant offset. Thus, $\bar{u}(\xi) = c(1 + |C|/A(\xi))$ is a positive, convex increasing function of $\xi$, with a one-sided cusp at $\xi = \xi_c$. See Fig. 3.

By extending $(A(\xi), Q(\xi), \bar{u}(\xi))$ as a piecewise (continuous) solution that is constant past $\xi = \xi_c$, this describes a blood vessel that is expanding behind the sharp front $x = \xi_c + ct$ of a moving blood flow pulse. The blood velocity exceeds the speed $c$ of the pulse ahead of the front, and dips down to the pulse speed $c$ far behind the front, such that rate of change spikes at the front, $\bar{u}_x|_{x=\xi_c+ct} = \infty$.

If $A(\xi) < A_c$, then the solution behaviour is that $A(\xi)$ goes to zero exponentially as $\xi \to -\infty$ and is a convex increasing function of $\xi$ with a one-sided cuspt at $\xi = \xi_c$. $Q(\xi) = c(A(\xi) + |C|)$ again has a similar behaviour with a constant offset, and $\bar{u}(\xi) = c(1 + |C|/A(\xi))$ is a decreasing positive function of $\xi$ with an inverted at $\xi = \xi_c$ where $\bar{u}(\xi_c) = c(1 + |C|/A_c) > 0$. See Fig. 4.

Extension of $(A(\xi), Q(\xi), \bar{u}(\xi))$ as a piecewise (continuous) solution that is constant past $\xi = \xi_c$ describes a blood vessel that is constricting behind the sharp front $x = \xi_c + ct$ of a moving blood flow pulse. The blood velocity $\bar{u}$ exceeds the speed $c$ of the pulse ahead of the front, and rises behind the front, such that rate of change spikes at the front, $\bar{u}_x|_{x=\xi_c+ct} = -\infty$.

5.3. Solutions for $C > 0$. In this case, the quadrature of equation (40) for $A(\xi)$ is given by

$$\beta_0 C^{3/2} \ln \left( |\sqrt{C} - \sqrt{A}|/(\sqrt{C} + \sqrt{A}) \right) + (\sigma/C) \ln(A) + (\gamma C - \sigma/C) \ln(|C| - A)$$

$$+ \frac{2}{3} \beta_0 A^{3/2} + \gamma A + 2\beta_0 C \sqrt{A} = \kappa(\xi_0 - \xi)$$

Again, it is convenient to put $\xi_0 = 0$ by translation symmetry. Solutions exhibit several different behaviours, which are distinguished by whether $A(\xi) \gtrless A_c$ and $A(\xi) \gtrless C$ and also whether $A_c \gtrless C$, as follows.

The solutions in the case $A_c > C$ will be discussed first.

Suppose $A(\xi) > A_c > C$. The qualitative solution behaviour is similar to the corresponding case when $C < 0$: $A(\xi)$ and $Q(\xi) = c(A(\xi) - C)$ are positive concave decreasing functions of $\xi$, which have an inverted one-sided cusp at $\xi = \xi_c$ where the solution stops.
Thus, \( \bar{u}(\xi) = c(1 - C/A(\xi)) \) is a positive convex increasing function that exponentially approaches the value 0 as \( \xi \to -\infty \) and has a one-sided cusp at \( \xi = \xi_c \). See Fig. 5.

By extending \((A(\xi), Q(\xi), \bar{u}(\xi))\) as a piecewise (continuous) solution that is constant past \( \xi = \xi_c \), this describes a blood vessel that is expanding behind the sharp front \( x = \xi_c + ct \) of a moving blood flow pulse. The blood velocity \( \bar{u} \) is less than the speed \( c \) of the pulse ahead of the front, and drops to zero behind the front, such that there is a spike in the rate of change at the front, \( u_x|_{x=\xi_c+ct} = \infty \).

Suppose \( A_c > A(\xi) > C \). The qualitative solution behaviour is again similar to the corresponding case when \( C < 0 \): \( A(\xi), Q(\xi) = c(A(\xi) - C) \), and \( \bar{u}(\xi) = c(1 - C/A(\xi)) \) are positive increasing functions which exhibit a one-sided cusp at \( \xi = \xi_c \), and the solution does not exist for \( \xi > \xi_c \). As \( \xi \to -\infty \), \( A(\xi) \) approaches the value \( C > 0 \) exponentially, while \( Q(\xi) \) and \( \bar{u}(\xi) \) go to zero. See Fig. 6.

Extension of \((A(\xi), Q(\xi), \bar{u}(\xi))\) as a piecewise (continuous) solution that is constant past \( \xi = \xi_c \) describes a blood vessel that contracts sharply inward to a constant diameter behind the front \( x = \xi_c + ct \) of a moving blood pulse at which the rate of decrease in area and blood velocity profiles for \( \gamma/\beta_0 = 0 \) (solid), 1 (dot), 2 (dash).
flow have a spike. The blood velocity \( \bar{u} \) is less than the speed \( c \) of the pulse ahead of the front, and rises slowly to the speed of the pulse behind the front, such that there is a spike in the rate of change at the front, \( \bar{u}_x |_{x=\xi, +ct} = -\infty \).

For \( A_c > C > A(\xi) \), a different behaviour arises. The solution exists for all \( \xi \) and has the asymptotic behaviour that \( A(\xi) \) decreases exponentially to 0 as \( \xi \to \infty \), and exponentially approaches the value \( C \) as \( \xi \to -\infty \). Consequently, \( Q(\xi) = c(A(\xi) - C) \) is a negative function that exponentially approaches the value 0 as \( \xi \to -\infty \) and decreases to the value \( -cC \) as \( \xi \to \infty \). Therefore, \( \bar{u}(\xi) = c(1 - C/A(\xi)) \) is a negative decreasing function that exponentially approaches the value 0 as \( \xi \to -\infty \) but has no lower bound as \( \xi \to \infty \). See Fig. 7.

This describes a blood vessel in which there is a moving compressive pulse with a shock front that causes the blood flow to be in the backward direction. Far ahead of the pulse, the blood vessel is constricted such that the diameter is close to zero, while at the front of the pulse, where convexity of the cross-section area vanishes, the blood flow exhibits a sharp transition from a high flow value to a low flow value.

Next, the solutions in the case \( C > A_c \) will be discussed.
Figure 5. $A(\xi) > A_c > C > 0$: area, blood flow, blood velocity profiles for 
$\gamma/\beta_0 = 0$ (solid), 1 (dot), 2 (dash).

Suppose $A(\xi) > C$. The solution $A(\xi)$ is a positive concave decreasing function of $\xi$ that exponentially approaches the value $C$ as $\xi \to \infty$, while $Q(\xi) = c(A(\xi) - C)$ has a similar behaviour but goes to 0 as $\xi \to \infty$. Thus, $\bar{u}(\xi) = c(1 - C/A(\xi))$ is a positive decreasing function that exponentially goes to 0 as $\xi \to \infty$. See Fig. 8.

This describes a blood vessel whose diameter increases as the blood flows forward along the vessel. Unlike previous cases, the pulse does not have a sharp front, but there is a transition point where the rate of decrease in blood flow and velocity reaches a maximum, with the velocity tapering to zero ahead of this point. The blood velocity resembles a shock whose front, where the convexity vanishes, corresponds to the transition point.

Suppose $C > A(\xi) > A_c$. The solution $A(\xi)$ is a positive concave increasing function of $\xi$ that starts as an inverted one-sided cusp at $\xi = \xi_c$ and exponentially approaches the value $C$ as $\xi \to \infty$. Likewise, $Q(\xi) = c(A(\xi) - C)$ and $\bar{u}(\xi) = c(1 - C/A(\xi))$ start with an inverted negative one-sided cusp at $\xi = \xi_c$ and are increasing functions of $\xi$ that exponentially approach 0 as $\xi \to \infty$. See Fig. 9.
Figure 6. $A_c > A(\xi) > C > 0$: area, blood flow, blood velocity profiles for $\gamma/\beta_0 = 0$ (solid), 1 (dot), 2 (dash).

Extending $(A(\xi), Q(\xi), \bar{u}(\xi))$ as a piecewise (continuous) solution that is constant prior to $\xi = \xi_c$, then this describes a blood vessel in which there is a backward pulse of blood flow with a sharp front $x = \xi_c + ct$ that moves forward at speed $c$. Ahead of the front, the vessel is constricted and the blood velocity $\bar{u}$ is close to zero, while at the front the diameter flares to a constant size $2\sqrt{A_1/\pi}$ and the backward velocity rapidly rises up to the speed $c$ of the pulse at the front.

A different behaviour occurs for $A_c > A(\xi)$. The solution $A(\xi)$ is a positive convex decreasing function of $\xi$ that starts as a one-sided cusp at $\xi = \xi_c$ and exponentially approaches 0 as $\xi \to \infty$. $Q(\xi) = c(A(\xi) - C)$ starts as a negative one-sided cusp at $\xi = \xi_c$ and decreases with $\xi$ such that it exponentially goes to 0 as $\xi \to \infty$. $\bar{u}(\xi) = c(1 - C/A(\xi))$ similarly is a negative decreasing functions of $\xi$ that starts with a one-sided cusp at $\xi = \xi_c$, but it has no lower bound as $\xi \to \infty$. See Fig. 10.

Extension of $(A(\xi), Q(\xi), \bar{u}(\xi))$ as a piecewise (continuous) solution that is constant prior to $\xi = \xi_c$ describes a blood vessel that rapidly collapses to a constant diameter $2\sqrt{A_1/\pi}$
behind the sharp front $x = \xi_c + ct$ of moving pulse with speed $c$. The blood flow and velocity are in the backward direction and their rate of changes spikes at the front, $Q_x|_{x=\xi_c+ct} = -\infty$ and $\bar{u}_x|_{x=\xi_c+ct} = \infty$. Ahead of the front, the blood flow tapers to zero while the blood velocity is rising in magnitude.

Last, consider the case $A_c = C > 0$. The root equation (43) yields $A_c = A_*$ where

$$A_* = \left(\frac{2\rho_0 c^2}{\beta_0}\right)^2, \quad (50)$$

and equation (40) for $A(\xi)$ becomes

$$A' = \frac{\kappa A}{(A + C)(\gamma + \beta_0 A^{3/2})} \quad (51)$$

Its quadrature is given by

$$\gamma A_* \ln(A) + \frac{2}{3} \beta_0 A^{3/2} + \gamma A + 2\beta_0 A_* \sqrt{A} = \kappa (\xi_0 - \xi) \quad (52)$$
Figure 8. $A(\xi) > C > A_c > 0$: area, blood flow, blood velocity profiles for $\gamma/\beta_0 = 0$ (solid), 1 (dot), 2 (dash).

The solution behaviour in this case is similar to the earlier case $V(\xi) > A_c > C$, except that here $A(\xi)$ goes to 0 exponentially, $Q(\xi) = c(A(\xi) - A_*)$ changes sign and goes to $-cA_*$ exponentially, while $\bar{u}(\xi) = c(1 - A_*/A(\xi))$ changes sign and decreases with no lower bound.

A useful final remark is that

$$\text{sgn}(A_c - C) = \text{sgn}(A_* - C)$$

(53)

can be shown to hold from the root equation (43) by the following argument. First, use the relations (38) to write equation (43) as $\tilde{\gamma} A_c^2 + \tilde{\beta}_0 A_c^{5/2} = C^2$ where $\tilde{\gamma} = 1 - \frac{1}{\alpha} \geq 0$ and $\tilde{\beta}_0 = \frac{\beta_0}{2\alpha^2} > 0$ which do not involve $C$. Then, for $A_c = C \neq 0$, $C = (\frac{1-\tilde{\gamma}}{\beta_0})^2 = A_*$. Next, view $A_c - C := f(C)$ as a function of $C$. The root equation shows that the solutions of $f(C) = 0$ are $C = A_*$ and $C = 0$, and that $f'(A_*) = -\frac{1-\tilde{\gamma}}{5-\gamma} = -\frac{1}{4\alpha+1}$ is negative. This implies $f(C) > 0$ for $0 < C < A_*$ and $f(C) < 0$ for $C > A_*$, which establishes the sign relation (53).
6. Concluding remarks

For the simplest widely-used 1D model of blood flow, three new conservation laws have been derived in case of inviscid flow. These conservation laws yield conserved integrals describing generalized momentum and generalized volumetric and axial energies. The generalized momentum differs compared to the momentum in inviscid constant-density 1D fluid dynamics by involving powers of $A$ that depend on $\alpha - 1$, where $\alpha$ is the momentum correction coefficient. Likewise, when $\alpha \neq 1$, both of the generalized energies involve different powers of $A$ compared to the energy in inviscid constant-density fluid dynamics. In the case of viscous blood flow, each conservation law gets replaced by a balance equation containing a dissipative volume term proportional to the friction coefficient in the model.

It is straightforward to derive similar conservation laws and balance equations for more general models, particularly having a general equation of state in which $P$ is an arbitrary function of $A$. 
The main focus of the present work has been to give a complete discussion of all travelling wave solutions of the 1D blood flow model. These solutions are most naturally applicable to the idealized case of a very long blood vessel in which the morphology of ends is not relevant, as the spatial domain of a solution in this situation is unbounded. For a blood vessel whose morphology at the ends is important for understanding the blood flow behaviour, travelling wave solutions are still applicable by considering suitable boundary conditions. Specifically, while travelling waves do not describe common morphologies such as a fixed diameter or pressure, or a fixed blood flow, in a blood vessel, nevertheless they may be relevant if conditions in the vessel wall or surrounding tissue cause a persistent wave pulse to propagate axially with constant speed. They may also be relevant in constructing piecewise solutions for approximating more realistic wave forms \[22\]. Travelling waves also include steady-state (time-independent) solutions as a special case when the wave speed is zero, and these solutions are compatible with all standard morphological boundary conditions.

The travelling wave solutions having a variety of interesting behaviours have been found, including:
• pressure and blood flow shocks
• sharp wave-front pulses in pressure and blood flow

In previous literature, no exact explicit solutions have been derived for this 1D blood flow model.

There are several possible directions for future work: (1) understand piecewise solutions in a framework of weak solutions; (2) derive and apply energy inequalities in the study of the initial-value problem; (3) study similarity solutions using scaling and dilation symmetries; (4) consider improved models, for example, by inclusion of a diffusion term, use of a viscoelastic equation of state, and an improved radial velocity profile.

ACKNOWLEDGMENTS

SCA is supported by an NSERC Discovery Grant. APM and MLG warmly thank the research group FQM-201 from the Andalusian Government for financial support. TMG acknowledges the Plan Propio - UCA 2022-2023.

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