Thom’s Jet Transversality Theorem for Regular Maps

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Abstract
We establish Thom’s jet transversality theorem for regular maps from an affine algebraic manifold to an algebraic manifold satisfying a suitable flexibility condition. It can be considered as the algebraic version of Forstnerič’s jet transversality theorem for holomorphic maps from a Stein manifold to an Oka manifold. Our jet transversality theorem implies genericity theorems for regular maps of maximal ranks. As an application, it follows that every connected compact locally flexible manifold is the image of a holomorphic submersion from an affine space. We also show that every algebraically degenerate subvariety of codimension at least two in a locally flexible manifold has an Oka complement.

Keywords Transversality theorem · Algebraically Oka manifold · Flexibility · Spray

Mathematics Subject Classification Primary 14R20 · 32Q56 · 58A20; Secondary 32E30

1 Introduction
Thom’s jet transversality theorem [21] is one of the most fundamental tools in differential geometry. In 2006, Forstnerič [7] proved the complex analytic version of his theorem for holomorphic maps from a Stein manifold to an Oka manifold (see also [13]). Here, a complex manifold $Y$ is an Oka manifold if for any holomorphic map $f : X \to Y$ from a Stein manifold there exists a holomorphic map $s : X \times \mathbb{C}^N \to Y$ such that $s(x, 0) = f(x)$ and $s(x, \cdot) : \mathbb{C}^N \to Y$ is a submersion at 0 for each $x \in X$ (cf. [8,16,17]).

In the present paper, we establish the complex algebraic version of Thom’s jet transversality theorem. To this end, let us recall the following definition.
Definition 1.1 (cf. [19]) An algebraic manifold $Y$ is an \textit{algebraically Oka manifold} if for any regular map $f: X \to Y$ from an affine algebraic manifold there exists a regular map $s: X \times \mathbb{C}^N \to Y$ such that $s(x, 0) = f(x)$ and $s(x, \cdot): \mathbb{C}^N \to Y$ is a submersion at 0 for each $x \in X$.

The most typical examples of algebraically Oka manifolds are locally flexible manifolds (cf. [1,12]). A quasi-affine algebraic manifold $Y$ is \textit{flexible} if for any $y \in Y$ the tangent space $T_y Y$ is spanned by the tangent vectors to the orbits of one-parameter unipotent subgroups of Aut $Y$ through $y$. An algebraic manifold is \textit{locally flexible} if it is covered by flexible Zariski open subsets. Complex Grassmannians and smooth non-degenerate toric varieties (i.e. smooth toric varieties with no torus factor) are known to be locally flexible [19, Theorem 3] (see [2, §3] for more examples of flexible manifolds). Currently, it is not known whether there exists an algebraically Oka manifold which is not locally flexible.

For regular maps from an affine algebraic manifold to an algebraically Oka manifold, Forstnerič proved the \textit{local} jet transversality theorem [7, Theorem 4.3]. Under an ostensibly stronger assumption on the target (that is ostensibly weaker than local flexibility), we prove the following \textit{global} jet transversality theorem which generalizes [7, Proposition 4.10] (compare with [7, Theorem 4.9] and [13, Corollary 1.3]). Here, $J^k(X, Y)$ denotes the space of $k$-jets of holomorphic maps $X \to Y$, $j^k f: X \to J^k(X, Y)$ denotes the $k$-jet extension of a regular map $f: X \to Y$ and Sing$(s)$ denotes the singular locus of a regular map $s$ (i.e. the set of points where $s$ is not a submersion).

Theorem 1.2 Let $Y$ be an algebraic manifold. Assume that for any $y \in Y$ there exist a Zariski open neighborhood $U \subset Y$ of $y$ and a regular map $s: U \times \mathbb{C}^N \to Y$ such that $\dim \text{Sing}(s) < N$, $s(y, 0) = y$ and $s(y, \cdot): \mathbb{C}^N \to Y$ is a submersion at 0 for each $y \in U$. Then regular maps from affine algebraic manifolds to $Y$ satisfy the following jet transversality theorem:

Let $X$ be an affine algebraic manifold, $X_0 \subset X$ be a closed algebraic subvariety, $k_l$ ($l \in \mathbb{N}$) be nonnegative integers, $A_l \subset X$ and $B_l \subset J^k_l(X, Y)$ ($l \in \mathbb{N}$) be (not necessarily closed or algebraic) complex submanifolds, and $f_0: X \to Y$ be a regular map such that $j_{k_l}^l f_0|_{A_l}$ is transverse to $B_l$ at all points of $X_0 \cap A_l$ for each $l \in \mathbb{N}$. Then, for any integer $k \geq \sup_{l \in \mathbb{N}} k_l$ the regular map $f_0$ can be approximated uniformly on compacts by regular maps $f: X \to Y$ such that

1. $j_k f|_{X_0} = j_k^l f_0|_{X_0}$, and
2. $j_{k_l}^l f|_{A_l}$ is transverse to $B_l$ for each $l \in \mathbb{N}$.

Since $A_l$ and $B_l$ ($l \in \mathbb{N}$) are not assumed to be closed, we can also take stratified complex subvarieties $A_l$ and $B_l$ ($l \in \mathbb{N}$) in Theorem 1.2 (see [7, §4] for the definition of transversality in this setting). The assumption in Theorem 1.2 implies that $Y$ is algebraically Oka (by Lemma 2.2 and [19, Theorem 1]). Conversely, if the conclusion of Theorem 1.2 holds for an algebraically Oka manifold $Y$, it satisfies the assumption in Theorem 1.2 (Remark 3.1). Thus, the assumption in Theorem 1.2 is natural. Since every locally flexible manifold satisfies this assumption (Proposition 3.3), the following corollary holds.

\textsuperscript{1} At present, it is not known whether these implications can be reversed.
Corollary 1.3  Regular maps from any affine algebraic manifold to any locally flexible manifold satisfy the jet transversality theorem.

The proof of Theorem 1.2 is given in Sect. 3. As immediate consequences of Theorem 1.2, we can obtain the following genericity results by the same proof as in [8, Corollary 8.9.3] (see the proof of [9, Theorem 1.6] for (3)). Here, Hom(X, Y) denotes the space of regular maps X → Y equipped with the compact-open topology.

Corollary 1.4  Assume that X is an n-dimensional affine algebraic manifold and Y is an m-dimensional algebraic manifold satisfying the assumption in Theorem 1.2. Then the following hold:

1. If 2n ≤ m, then the set of immersions X → Y is dense in Hom(X, Y).
2. If 2n + 1 ≤ m, then the set of injective immersions X → Y is dense in Hom(X, Y).
3. If n ≥ m, then for any compact subset K ⊂ Y the set of regular maps f : X → Y satisfying K ⊂ f(X) and dim Sing(f) < m is dense in Hom(X, Y).

Using Theorem 1.2, we can also obtain the following approximation theorem for holomorphic maps with a lower bound on the rank which generalizes [8, Theorem 9.12.4] (see the first part of the proof of [8, Theorem 9.12.4]; see also [15]). Note that it contains approximation theorems for immersions (n = r < m) and submersions (n > m = r) as special cases (see [8, Problem 9.14.3]).

Corollary 1.5  Assume that n, m, r are integers such that (n−r+1)(m−r+1) ≥ 2 and Y is an m-dimensional algebraic manifold satisfying the assumption in Theorem 1.2. Then, any holomorphic map f : U → Y from an open neighborhood of a compact convex set K ⊂ C^n satisfying \( \inf_{z \in K} \text{rank} \, df_z \geq r \) can be approximated uniformly on K by holomorphic maps \( \tilde{f} : C^n \to Y \) such that \( \inf_{z \in C^n} \text{rank} \, \tilde{d}f_z \geq r \).

Forstnerič proved that every n-dimensional connected compact algebraically Oka manifold is the image of a strongly dominating regular map from C^n [9, Theorem 1.6]. Recall that every n-dimensional connected complex manifold is the image of a locally biholomorphic map from the unit polydisc in C^n by the result of Fornæss and Stout [5] (see also [6]). This result and Corollary 1.5 imply the following by Rouché’s theorem (cf. [3, p. 110]).

Corollary 1.6  Assume that Y is an n-dimensional connected algebraic manifold satisfying the assumption in Theorem 1.2. Then for any compact subset K ⊂ Y there exists a holomorphic submersion f : C^{n+1} → Y such that K ⊂ f(C^{n+1}). In particular, every n-dimensional connected compact locally flexible manifold is the image of a holomorphic submersion from C^{n+1}.

In Sect. 4, we give another application of Theorem 1.2. In Oka theory, it is important to understand when a closed complex subvariety in an Oka manifold has an Oka complement. By the Kobayashi conjecture [14], it is natural to assume that the codimension of a subvariety is at least two. It is known that there exists a discrete set D ⊂ C^2 such that C^2 \ D is not Oka [20, Theorem 4.5], and that every algebraically degenerate2 closed complex subvariety A ⊂ C^n of codimension at least two has an

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2 A closed complex subvariety in an algebraic manifold Y is algebraically degenerate if it is contained in a proper closed algebraic subvariety of Y.
Oka complement [10, Theorem 1.6]. We obtain the following generalization of the latter.

**Corollary 1.7** Let $Y$ be an algebraic manifold satisfying the assumption in Theorem 1.2 and $A \subset Y$ be a closed complex subvariety of codimension at least two. Assume that there exists a holomorphic automorphism $\varphi$ of $Y$ such that $\varphi(A)$ is algebraically degenerate. Then the complement $Y \setminus A$ is Oka.

Since a connected affine algebraic group without nontrivial characters is flexible [1, Proposition 5.4], Corollary 1.7 generalizes the result of Winkelmann [22, Theorem 2.9] for the complement of a tame discrete set in such an algebraic group (see [22, Theorem 2.1]).

It is known that local flexibility is preserved by removing closed algebraic subvarieties of codimension at least two [4, Theorem 1.1]. Thus, it is reasonable to expect that the algebraic Oka property is also preserved by the same operation (cf. [8, Problem 6.4.3]). At present, however, we can only conclude that a closed algebraic subvariety of codimension at least two in an algebraic manifold satisfying the assumption in Theorem 1.2 has a complement enjoying aCAP (a Runge type approximation property which is implied by the algebraic Oka property; see Definition 4.1 and Remark 4.3).

## 2 Singularities of Algebraic Sprays

In this section, we recall the notion of algebraic sprays and investigate their singularities to prove Theorem 1.2. For a regular map $p : E \to X$, let $E_x$ denote the fiber $p^{-1}(x)$ over a point $x \in X$.

**Definition 2.1** Let $X$ and $Y$ be algebraic manifolds.

1. A (global) algebraic spray over a regular map $f : X \to Y$ is a triple $(E, p, s)$ where $p : E \to X$ is an algebraic vector bundle and $s : E \to Y$ is a regular map such that $s(0_x) = f(x)$ for all $x \in X$. If the algebraic vector bundle $p : E \to X$ is trivial, we simply write $s : E \to Y$ instead of $(E, p, s)$.
2. A family of algebraic sprays $\{(E_j, p_j, s_j)\}_{j=1}^k$ over $f : X \to Y$ is dominating if $\sum_{j=1}^k (ds_j)_0 (E_{j,x}) = T_{f(x)}Y$ holds for each $x \in X$.

Note that the regular map $s : X \times \mathbb{C}^N \to Y$ in Definition 1.1 (resp. $s : U \times \mathbb{C}^N \to Y$ in Theorem 1.2) is nothing but a dominating algebraic spray over $f : X \to Y$ (resp. the inclusion $U \hookrightarrow Y$).

Let us first recall the following lemma which was used in the proof of Gromov’s localization principle for algebraically Oka manifolds [11, 3.5.B] (see also [8, Proposition 6.4.2] and [12, Proposition 1.4]).

**Lemma 2.2** Assume that $Y$ is an algebraic manifold, $D \subset Y$ is an algebraic hypersurface and $(E, p, s)$ is an algebraic spray over the inclusion $Y \setminus D \hookrightarrow Y$. Then there exist an algebraic spray $(\widetilde{E}, \widetilde{p}, \widetilde{s})$ over the identity map $id_Y$ and an isomorphism $\varphi : E \to \widetilde{E}|_{Y \setminus D}$ of algebraic vector bundles such that $\widetilde{s} \circ \varphi = s$ and $\text{Sing}(\widetilde{s}) \cap \widetilde{E}|_D = \emptyset$. 
Theorem 1.2. Then, for any \( y \in Y \) there exists an algebraic spray \((E, p, s)\) over the identity map \( \text{id}_Y \) and an isomorphism \( \varphi : E \to \widetilde{E}|_Y \setminus D \) of algebraic vector bundles such that \( \tilde{s} \circ \varphi = s \). By construction, we may assume that for any \( e \in \widetilde{E}|_D \) there exists a local holomorphic section \( f : U \to \widetilde{E} \) from an open neighborhood \( U \subset Y \) of \( \tilde{p}(e) \) such that \( f(\tilde{p}(e)) = e \) and \( d(\tilde{s} \circ f)_{\tilde{p}(e)} = \text{id}_{\tilde{p}(e)Y} \). This implies \( \text{Sing}(\tilde{s}) \cap \widetilde{E}|_D = \emptyset \). \( \square \)

Using the assumption in Theorem 1.2 and the above lemma, we can construct a dominating family of algebraic sprays over the identity map with small singular loci.

Corollary 2.3 Assume that \( Y \) is an algebraic manifold satisfying the assumption in Theorem 1.2. Then, for any \( y \in Y \) there exists an algebraic spray \((E, p, s)\) over the identity map \( \text{id}_Y \) such that \( \text{dim Sing}(s) < \text{rank } E \) and \( 0 \notin \text{Sing}(p, s) \).

Proof Take a point \( y \in Y \) arbitrarily. By assumption, there exist a Zariski open neighborhood \( U \subset Y \) of \( y \) and a dominating algebraic spray \( s_0 : U \times \mathbb{C}^N \to Y \) over the inclusion \( U \hookrightarrow Y \) satisfying \( \text{dim Sing}(s_0) < N \). After shrinking \( U \ni y \) if necessary, we may assume that \( D = Y \setminus U \) is an algebraic hypersurface. Then Lemma 2.2 implies that the algebraic spray \( s_0 : U \times \mathbb{C}^N \to Y \) extends to an algebraic spray \((E, p, s)\) over the identity map \( \text{id}_Y \) such that \( \text{Sing}(s) \cap E|_D = \emptyset \). This extension clearly satisfies \( \text{dim Sing}(s) < N = \text{rank } E \) and \( 0 \notin \text{Sing}(p, s) \). \( \square \)

In order to prove Theorem 1.2, we need to recall Gromov’s method of composed sprays [11, §1.3] (see also [8, §6.3]).

Definition 2.4 (cf. [8, Definition 6.3.5]) Let \( X, Y \) and \( Z \) be algebraic manifolds.

1. For a family of algebraic sprays \( \{(E_j, p_j, s_j)\}_{j=1}^k \) over the identity map \( \text{id}_Y \), the composed spray \( (E_1 \cdots \cdots E_k, p_1 \cdots \cdots p_k, s_1 \cdots \cdots s_k) \) is defined by

\[
E_1 \cdots \cdots E_k = \left\{ (e_1, \ldots, e_k) \in \prod_{j=1}^k E_j : s_j(e_j) = p_{j+1}(e_{j+1}), j = 1, \ldots, k-1 \right\},
\]

\[ (p_1 \cdots \cdots p_k)(e_1, \ldots, e_k) = p_1(e_1), \quad (s_1 \cdots \cdots s_k)(e_1, \ldots, e_k) = s_k(e_k). \]

2. For an algebraic spray \((E, p, s)\) over the identity map \( \text{id}_Y \) and a natural number \( k \in \mathbb{N} \), the \( k \)-th iterated spray \((E^{(k)}, p^{(k)}, s^{(k)})\) is the composed spray \((E \cdots \cdots E, p \cdots \cdots p, s \cdots \cdots s)\) of \( k \) copies of \((E, p, s)\).

3. For a regular map \( f : X \to Y \) and an algebraic spray \((E, p, s)\) over a regular map \( g : Y \to Z \), the pullback spray \((f^*E, f^*p, f^*s)\) over \( g \circ f : X \to Z \) consists of the pullback bundle \( f^*p : f^*E \to X \) and the composition \( f^*s : f^*E \to Z \) of the induced map \( f^*E \to E \) and \( s : E \to Z \).

Remark 2.5 Note that the projection \( p_1 \cdots \cdots p_k : E_1 \cdots \cdots E_k \to Y \) of a composed spray \((E_1 \cdots \cdots E_k, p_1 \cdots \cdots p_k, s_1 \cdots \cdots s_k)\) does not have any canonical algebraic vector bundle structure. However, it has the zero section \( \{(0, \ldots, 0) \in E_1 \cdots \cdots E_k : y \in Y \} \). The rank of \( E_1 \cdots \cdots E_k \) is defined to be the dimension of the fibers of \( p_1 \cdots \cdots p_k \) and denoted by \( \text{rank}(E_1 \cdots \cdots E_k) \).
Lemma 2.6 Assume that $Y$ is an algebraic manifold and $\{(E_j, p_j, s_j)\}_{j=1}^k$ is a family of algebraic sprays over the identity map $\text{id}_Y$ such that $\dim \text{Sing}(s_j) < \text{rank } E_j$ for each $j$. Set $\Sigma = (E_1 \times \cdots \times E_k) \setminus \prod_{j=1}^k (E_j \setminus \text{Sing}(s_j))$. Then, the following holds:

$$\dim \text{Sing}(s_1 \times \cdots \times s_k) \leq \dim \Sigma < \text{rank}(E_1 \times \cdots \times E_k).$$

Proof By definition, the singular locus of the composed spray satisfies $\text{Sing}(s_1 \times \cdots \times s_k) \subset \Sigma$ and thus $\dim \text{Sing}(s_1 \times \cdots \times s_k) \leq \dim \Sigma$.

Set $\Sigma_l = (E_l \times \cdots \times E_k) \setminus \prod_{j=l}^k (E_j \setminus \text{Sing}(s_j))$ for $l = 1, \ldots, k$. Note that $\Sigma_1 = \Sigma$ and $\Sigma_k = \text{Sing}(s_k)$. For each $l = 1, \ldots, k - 1$, the following holds:

$$\Sigma_l = (E_l \times \cdots \times E_k) \cap ((\text{Sing}(s_l) \times (E_{l+1} \times \cdots \times E_k)) \cup ((E_l \setminus \text{Sing}(s_l)) \times \Sigma_{l+1})).$$

By definition and assumption,

$$\dim((E_l \times \cdots \times E_k) \cap (\text{Sing}(s_l) \times (E_{l+1} \times \cdots \times E_k))) = \dim \text{Sing}(s_l) + \sum_{j=l+1}^k \text{rank } E_j \leq \sum_{j=l}^k \text{rank } E_j = \text{rank}(E_l \times \cdots \times E_k).$$

Since the restriction $s_l : E_l \setminus \text{Sing}(s_l) \to Y$ is a submersion,

$$\dim((E_l \times \cdots \times E_k) \cap (E_l \setminus \text{Sing}(s_l)) \times \Sigma_{l+1})) \leq \dim(E_l \setminus \text{Sing}(s_l)) - \dim Y + \dim \Sigma_{l+1} = \text{rank } E_l + \dim \Sigma_{l+1}.$$

Thus, the desired inequality follows by downward induction on $l$. □

Lemma 2.7 Assume that $Y$ is an algebraic manifold and $\{(E_j, p_j, s_j)\}_{j=1}^k$ is a family of algebraic sprays over the identity map $\text{id}_Y$. Set $\Sigma = (E_1 \times \cdots \times E_k) \setminus \prod_{j=1}^k (E_j \setminus \text{Sing}(s_j))$. Then the following holds:

$$\text{Sing}(p_1 \times \cdots \times p_k, s_1 \times \cdots \times s_k) \setminus \Sigma \subset \prod_{j=1}^k \text{Sing}(p_j, s_j).$$

Proof Take a point $(e_1, \ldots, e_k) \in \text{Sing}(p_1 \times \cdots \times p_k, s_1 \times \cdots \times s_k) \setminus \Sigma$ arbitrarily. To reach a contradiction, assume that $(e_1, \ldots, e_k) \notin \prod_{j=1}^k \text{Sing}(p_j, s_j)$. Then, $e_j \notin \text{Sing}(p_l, s_l)$ for some $l$. This means that $s_l|_{E_l \setminus p_l(e_l)} : E_l \setminus p_l(e_l) \to Y$ is a submersion at $e_l$. Therefore, there exists a holomorphic map $f_l : U_l \to E_l \setminus p_l(e_l)$ from an open neighborhood $U_l \subset Y$ of $s_l(e_l)$ such that $s_l \circ f_l = \text{id}_{U_l}$ and $f_l(s_l(e_l)) = e_l$. Since $e_j \notin \text{Sing}(s_j)$ ($j = l + 1, \ldots, k$), there exist holomorphic maps $f_j : U_j \to E_j$.
(j = l + 1, \ldots, k) from open neighborhoods of U_j \subset Y of s_j(e_j) such that s_j \circ f_j = id_{U_j} and f_j(s_j(e_j)) = e_j. Note that s_j(e_j) = p_{j+1}(e_{j+1}) holds for each j. Thus, after shrinking U_k \ni s_k(p_k) if necessary, we can consider the holomorphic map \( f : U_k \rightarrow (E_1 \ast \ldots \ast E_k)(p_1 \ast \ldots \ast p_k)(e_1, \ldots, e_k) \) defined by

\[
  f(y) = (e_1, \ldots, e_{l-1}, f_l \circ (p_{l+1} \circ f_{l+1}) \circ \cdots \circ (p_k \circ f_k)(y), \ldots, f_k(y)).
\]

By definition, it satisfies \( s_1 \ast \ldots \ast s_k \circ f = id_{U_k} \) and \( f((s_1 \ast \ldots \ast s_k)(e_1, \ldots, e_k)) = (e_1, \ldots, e_k) \). This implies \( (e_1, \ldots, e_k) \notin Sing(p_1 \ast \ldots \ast p_k, s_1 \ast \ldots \ast s_k) \), a contradiction to our assumption. \( \square \)

With the above lemmas and Corollary 2.3, we can construct a dominating composed spray with small relative singular locus under the assumption in Theorem 1.2.

**Corollary 2.8** Assume that Y is an algebraic manifold satisfying the assumption in Theorem 1.2. Then there exists a dominating composed spray \((E, p, s)\) over the identity \(id_Y\) which satisfies \( dim\ Sing(p, s) < rank\ E\).

**Proof** By Corollary 2.3, there exists a dominating family of algebraic sprays \(\{(E_j, p_j, s_j)\}_{j=1}^k\) over the identity map \(id_Y\) such that \( dim\ Sing(s_j) < rank\ E_j \) for each \(j\). Then the composed spray \((\tilde{E}, \tilde{p}, \tilde{s})\) of the family \(\{(E_j, p_j, s_j)\}_{j=1}^k\) is dominating and satisfies \( dim\ Sing(\tilde{s}) < rank\ \tilde{E} \) by Lemma 2.6. Note that the dimensions of the fibers of \(Sing(\tilde{p}, \tilde{s})\) satisfy \( sup_{y \in Y} dim\ Sing(\tilde{p}, \tilde{s})_y \leq rank\ \tilde{E} - 1 \) since the spray \((\tilde{E}, \tilde{p}, \tilde{s})\) is dominating. Let us consider the \(n\)-th iterated spray \((\tilde{E}^{(n)}, \tilde{p}^{(n)}, \tilde{s}^{(n)})\) of \((\tilde{E}, \tilde{p}, \tilde{s})\) for a natural number \(n > dim\ Y\). Set \(Sing(\tilde{p}, \tilde{s})^{(n)} = \tilde{E}^{(n)} \cap (Sing(\tilde{p}, \tilde{s}))^n\) and \(\Sigma = \tilde{E}^{(n)} \setminus (\tilde{E} \setminus Sing(\tilde{s}))^n\). Then the dimensions of the fibers of \(Sing(\tilde{p}, \tilde{s})^{(n)}\) satisfies

\[
  sup_{y \in Y} dim\Sing(\tilde{p}, \tilde{s})^{(n)}_y \leq n sup_{y \in Y} dim\Sing(\tilde{p}, \tilde{s})_y
  \leq n rank\ \tilde{E} - n
  < rank\ \tilde{E}^{(n)} - dim\ Y,
\]

and hence \( dim\ Sing(\tilde{p}, \tilde{s})^{(n)} < rank\ \tilde{E}^{(n)} \). By Lemma 2.6, the closed subvariety \(\Sigma\) also satisfies \( dim\ \Sigma < rank\ \tilde{E}^{(n)} \). Since \(Sing(\tilde{p}^{(n)}, \tilde{s}^{(n)}) \subset Sing(\tilde{p}, \tilde{s})^{(n)} \cup \Sigma\) by Lemma 2.7, it follows that \( dim\ Sing(\tilde{p}^{(n)}, \tilde{s}^{(n)}) < rank\ \tilde{E}^{(n)} \). Thus \((E, p, s) = (\tilde{E}^{(n)}, \tilde{p}^{(n)}, \tilde{s}^{(n)})\) is a desired dominating composed spray. \( \square \)

**3 Proof of Theorem 1.2**

Our proof of Theorem 1.2 is based on the ideas in [7, §4]. Let \( pr_{X_{\lambda_0}} : \prod_{\lambda \in \Lambda} X_{\lambda} \rightarrow X_{\lambda_0} \) denote the projection map to \(X_{\lambda_0} (\lambda_0 \in \Lambda)\).

**Proof of Theorem 1.2** By Corollary 2.8, there exists a dominating composed spray \((E, p, s)\) over the identity \(id_Y\) satisfying \( dim\ Sing(p, s) < rank\ E \). Let us consider
the pullback spray \((f_0^*E, f_0^*p, f_0^*s)\). By the argument in the proof of [7, Lemma 3.6], there exists an algebraic submersion (i.e. a smooth morphism) \(\varphi: X \times \mathbb{C}^N \to f_0^*E\) over \(X\) which preserves the zero section. Take regular functions \(g_1, \ldots, g_L: X \to \mathbb{C}\) which vanish to order \(k + 1\) on the subvariety \(X_0 = \{x \in X: g_j(x) = 0, \ j = 1, \ldots, L\}\). Consider the regular map \(s_{X_0}: X \times (\mathbb{C}^N)^L \to X \times \mathbb{C}^N, (x, (t_1, \ldots, t_L)) \mapsto (x, g_1(x)t_1 + \cdots + g_L(x)t_L)\) over \(X\). Note that \(s_{X_0}(x, \cdot): (\mathbb{C}^N)^L \to \{x\} \times \mathbb{C}^N \cong \mathbb{C}^N\) is a submersion if \(x \in X \setminus X_0\), and it vanishes if \(x \in X_0\). Since \(X\) is affine, we may assume that \(X\) is a closed algebraic submanifold of \(\mathbb{C}^n\) for some \(n \in \mathbb{N}\). Let \(W\) denote the complex vector space of polynomial maps \(\mathbb{C}^n \to \mathbb{C}^{n+L}\) of degree at most \(k\), and \(s_W: X \times W \to X \times \mathbb{C}^{N+L}\) be the algebraic submersion over \(X\) defined by \(s_W(x, P) = (x, P(x))\). We define the algebraic spray \(s_{f_0}: X \times \mathbb{C}^{N+L} \to Y\) over \(f_0\) by \(s_{f_0} = f_0^*s \circ \varphi \circ s_{X_0} \circ s_W\) (we are identifying the zero sections with \(X\) in the following diagram):

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi \circ s_{X_0} \circ s_W} & X \\
\downarrow & & \downarrow \varphi^*f_0 & \downarrow f_0 \\
X \times W & \xrightarrow{s_{f_0}} & f_0^*E & \xrightarrow{p^*f_0} & E & \xrightarrow{s} & Y \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X & \xrightarrow{f_0} & Y & & & & & \\
\end{array}
\]

Note that \(\text{Sing}(\text{pr}_X, s_{f_0}) \setminus (X_0 \times W) = (p^*f_0 \circ \varphi \circ s_{X_0} \circ s_W)^{-1}(\text{Sing}(p, s))) \setminus (X_0 \times W)\) holds by construction. From this and \(\dim \text{Sing}(p, s) < \text{rank} \ E\), it follows that there exists a dense Zariski open subset \(U \subset W\) such that \(\text{Sing}(\text{pr}_X, s_{f_0}) \cap (X \times U) \subset X_0 \times U\).

For each \(l \in \mathbb{N}\), let us consider the map \(\Phi_{k_l}: X \times W \to J^{k_l}(X, Y)\) defined by \(\Phi_{k_l}(x, P) = j^{k_l}(s_{f_0} \circ (\text{id}_X, P))(x)\). Then by the proof of [7, Lemma 4.5], the singular locus of \((\text{pr}_X, \Phi_{k_l})\) satisfies \(\text{Sing}(\text{pr}_X, \Phi_{k_l}) \subset \text{Sing}(\text{pr}_X, s_{f_0})\), and thus \(\text{Sing}(\text{pr}_X, \Phi_{k_l}) \cap (X \times U) \subset X_0 \times U\). Let us consider the complex submanifolds \(A_l \subset X\) and \(B_l \subset J^{k_l}(X, Y)\). For \(P \in U\), we can easily see that the regular map \(\Phi_{k_l}(-, P)|_{A_l \setminus X_0}: A_l \setminus X_0 \to J^{k_l}(X, Y)\) is transverse to \(B_l\) if and only if \(P \notin \text{pr}_U(\text{Sing}(\text{pr}_U|B_l))\) where \(B'_l = (\Phi_{k_l}|_{(A_l \setminus X_0) \times U})^{-1}(B_l)\). By Sard’s theorem, there exists a polynomial map \(P \in U \setminus \bigcup_{l \in \mathbb{N}} \text{pr}_U(\text{Sing}(\text{pr}_U|B_l))\) which is sufficiently close to 0. Then, by construction, the regular map \(f = s_{f_0}(-, P): X \to Y\) satisfies \(j^k f|_{X_0} = j^k f_0|_{X_0}\), and thus \(j^k f|_{A_l}\) is transverse to \(B_l\) for each \(l \in \mathbb{N}\). Furthermore, it also approximates \(f_0\) since \(P\) is close to 0. □

In the rest of this section, we give a few remarks on the assumption in Theorem 1.2.

**Remark 3.1** As we mentioned in the introduction, if the jet transversality theorem holds for regular maps from affine algebraic manifolds to an algebraically Oka manifold \(Y\), then \(Y\) satisfies the assumption in Theorem 1.2. Indeed, for any \(y \in Y\) and any affine Zariski open neighborhood \(U \subset Y\) of \(y\) there exists a dominating algebraic spray \(s_0: U \times \mathbb{C}^N \to Y\) over the inclusion \(U \hookrightarrow Y\) since \(Y\) is algebraically Oka. By the jet transversality theorem and the proof of (3) in Corollary 1.4 (see [8, §8.9]), there
exists a regular map \( s : U \times \mathbb{C}^N \to Y \) such that \( \dim \text{Sing}(s) < \dim Y \leq N \) and \( j^1s|_{U \times \{0\}} = j^1s_0|_{U \times \{0\}} \), and thus it is a dominating algebraic spray over the inclusion \( U \hookrightarrow Y \) with the desired property.

The above remark leads us to the following question.

**Question 3.2** Does every algebraically Oka manifold satisfy the assumption in Theorem 1.2?

Recall that a flexible manifold \( Y \) admits a dominating algebraic spray \( s : Y \times \mathbb{C}^N \to Y \) over the identity map \( \text{id}_Y \) such that \( s(\cdot, w) \in \text{Aut} Y \) for all \( w \in \mathbb{C}^N \) (see the proof of [8, Proposition 5.6.22]). Thus, the following proposition holds.

**Proposition 3.3** Every locally flexible manifold satisfies the assumption in Theorem 1.2.

**4 Proof of Corollary 1.7**

Let us first recall the Convex Approximation Property (CAP) and its algebraic version aCAP.

**Definition 4.1** A complex manifold (resp. an algebraic manifold) \( Y \) enjoys CAP (resp. aCAP) if any holomorphic map from an open neighborhood of a compact convex set \( K \subset \mathbb{C}^n \) (\( n \in \mathbb{N} \)) to \( Y \) can be uniformly approximated on \( K \) by holomorphic maps (resp. regular maps) \( \mathbb{C}^n \to Y \).

In order to prove Corollary 1.7, we also need to recall the following fact.

**Theorem 4.2** (cf. [16, Theorem 1.3], [7, Corollary 1.2 and Proposition 4.6])

1. A complex manifold enjoys CAP if and only if it is Oka.
2. An algebraic manifold enjoys aCAP if it is algebraically Oka.

**Proof of Corollary 1.7** We may assume that \( A \subset Y \) is algebraically degenerate from the beginning. Let us verify that \( Y \setminus A \) enjoys CAP. Take a compact convex set \( K \subset \mathbb{C}^n \) and a holomorphic map \( f : U \to Y \setminus A \) from an open neighborhood of \( K \). Since \( Y \) is algebraically Oka, it enjoys aCAP by Theorem 4.2. Thus, there exists a regular map \( \tilde{f} : \mathbb{C}^n \to Y \) which approximates \( f \) uniformly on \( K \) and satisfies \( \tilde{f}(K) \subset Y \setminus A \). By the jet transversality theorem (Theorem 1.2), we may assume that \( \tilde{f} \) is transverse to \( A \). Then the inverse image \( \tilde{f}^{-1}(A) \subset \mathbb{C}^n \) is an algebraically degenerate closed complex subvariety of codimension at least two. Therefore the complement \( \mathbb{C}^n \setminus \tilde{f}^{-1}(A) \) is Oka by the result of Forstnerič and Prezelj [10, Theorem 1.6], and hence it enjoys CAP by Theorem 4.2. Thus, there exists a holomorphic map \( g : \mathbb{C}^n \to \mathbb{C}^n \setminus \tilde{f}^{-1}(A) \) which approximates the inclusion \( K \hookrightarrow \mathbb{C}^n \setminus \tilde{f}^{-1}(A) \) uniformly. Then the composition \( \tilde{f} \circ g : \mathbb{C}^n \to Y \setminus A \) approximates \( f \) uniformly on \( K \). Therefore \( Y \setminus A \) is Oka by Theorem 4.2. \( \square \)
Remark 4.3  Let $Y$ be an algebraic manifold satisfying the assumption in Theorem 1.2. The above argument also implies that the complement of a closed algebraic subvariety of codimension at least two in $Y$ enjoys aCAP since the complement $\mathbb{C}^n \setminus \tilde{f}^{-1}(A)$ is algebraically Oka in this case (cf. [8, Proposition 5.6.17]). It is not known whether aCAP implies the algebraic Oka property (the converse of (2) in Theorem 4.2).

In the previous paper, we proved that the blowup of an algebraically Oka manifold along an algebraic submanifold is Oka [18, Corollary 1.5]. For an algebraic manifold $Y$ satisfying the assumption in Theorem 1.2, we can reduce the proof of the Oka property of the blowup of $Y$ along an algebraically degenerate complex submanifold to the Euclidean case as in the proof of Corollary 1.7 (see the proof of [18, Corollary 4.3]). Thus, it is natural to ask the following question. It will be studied in future work.

Question 4.4 Is the blowup of $\mathbb{C}^n$ along an algebraically degenerate closed complex submanifold Oka?

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