Scalar curvature of spacelike hypersurfaces and certain class of cosmological models for accelerated expanding universes

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Abstract

We study the scalar curvature of spacelike hypersurfaces in the family of cosmological models known as generalized Robertson-Walker spacetimes, and give several rigidity results under appropriate mathematical and physical assumptions. On the other hand, we show that this family of spacetimes provides suitable models obeying the null convergence condition to explain accelerated expanding universes.

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1 Introduction

In this paper we deal with the class of cosmological models called generalized Robertson-Walker (GRW) spacetimes (see Section 2), which are warped products $I \times_f F$ with base an open interval $(I, -dt^2)$ and fiber a Riemannian manifold $(F, g_F)$ whose sectional curvature is not assumed to be constant. Thus, our ambient spacetimes widely extend to those that are classically called Robertson-Walker (RW) spacetimes. Recall that the class of Robertson-Walker spacetimes includes the usual big-bang cosmological models, the de Sitter spacetime, the steady state spacetime, the Lorentz-Minkowsky spacetime and the Einstein’s static spacetime, among others. Unlikely to these spacetimes, our ambient spacetimes are not necessarily spatially-homogeneous. Note that being spatially-homogeneous, which is reasonable as a first approximation of the large scale structure of the universe, could not be appropriate when we consider a more accurate scale. Thus, a
GRW spacetime could be a suitable spacetime to model a universe with inhomogeneous spacelike geometry [29]. On the other hand, small deformations of the metric on the fiber of classical Robertson-Walker spacetimes fit into the class of GRW spacetimes. Therefore, GRW spacetimes are useful to analyze if a property of a RW spacetime $\mathcal{M}$ is stable, i.e. if it remains true for spacetimes close to $\mathcal{M}$ in a certain topology defined on a suitable family of spacetimes [21]. In fact, a deformation $s \mapsto g(s)$ of the metric of $F$ provides a one parameter family of GRW spacetimes close to $\mathcal{M}$ when $s$ approaches to 0. Note that a conformal change of the metric of a GRW spacetime with a conformal factor which only depends on $t$, produces a new GRW spacetime. Any GRW spacetime has a smooth global time function, and so it is stably causal [5] p. 64]. Moreover, if the fiber is complete then the GRW spacetime is globally hyperbolic [8, Th. 3.66]. On the other hand, if the fiber is compact then it is called spatially closed. In [34] the behaviour of the geodesics of GRW spacetimes is studied.

We will impose the spacetime to obey the null convergence condition (NCC), which says that the Ricci tensor of the spacetime is semi-definite positive on every null (light-like) vector. Recall that the exact solutions to the Einstein equations with cosmological constant, provided that the stress-energy momentum tensor satisfies the weak energy condition, obey the null convergence condition.

On the other hand, the study of spacelike hypersurfaces in General Relativity is relevant for several questions, as foliations of spacetimes, change of expansion or contraction phases, the Cauchy problem for Einstein’s equation, etc. (see, for instance [26], [19]). Moreover, for many problems in General Relativity, including the Positive Mass Theorem and the Penrose Inequality, knowledge of the entire spacetime is not necessary, rather attention may be focused solely on a spacelike hypersurface, playing the scalar curvature of this hypersurface an important role [36]. In addition, the choice of a constant mean curvature (CMC) spacelike hypersurface as initial data have been considered in order to deal with the Cauchy problem for the Einstein’s equation (see [18]).

In the first part of this paper (Section 3) we study the scalar curvature of spacelike hypersurfaces in a GRW spacetime which obeys the NCC (see Lemma 2). Thus, we obtain a general expression for the scalar curvature of an immersed spacelike hypersurface in such an ambient space (7), given several estimations when the spacetime obeys the NCC and characterizing those spacelike hypersurfaces which attain the equality in our estimations (Theorem 3 and Corollary 4). In this setting, we pay a special attention to the important case of maximal hypersurfaces (Corollaries 5 and 6). As a consequence of our results, in the particular case when the spacetime is the de Sitter space we provide a characterization of the totally umbilical spacelike hypersurfaces from a bound of the scalar curvature of the hypersurface (see Theorem 8 and Remark 9). We also particularize our study to the case of compact CMC hypersurfaces in a GRW spacetime which obeys the NCC, so obtaining more strong consequences including a Calabi-Bernstein type result (Theorem 11).

On the other hand, in the second part of the paper (Section 4) we apply our mathematical results to the study of a certain class of cosmological models, specifically GRW spacetimes filled with perfect fluid. In General Relativity one often employs a perfect fluid stress-energy momentum tensor to represent the source of the gravitational field. This fluid description is used where one assumes that the large-scale proprieties of the universe can be studied by assuming a perfect fluid description of the sources. A review of the specific literature shows that, in fact, almost all the cosmological studies use the perfect fluid model. We focus on the case where GRW spacetimes satisfying the NCC constitute perfect fluid models adequate to describe universes at dominant dark energy stage, namely, accelerated expanding universes. We end up particularizing our study
to the family of spatially closed GRW spacetimes. In this setting, we are able to express the total energy on a compact spacelike hypersurface in terms of its scalar and mean curvatures (Theorems 13 and 15). Finally, in the simplest case of a 3-dimensional GRW spacetime, as a consequence of the Gauss-Bonnet theorem we provide a nice expression of the total energy in terms of the Euler characteristic of the surface, its mean curvature and its volume (Theorem 16).

2 Preliminaries

Let \((F, g_F)\) be an \(n(\geq 2)\)-dimensional (connected) Riemannian manifold, \(I\) an open interval in \(\mathbb{R}\) endowed with the metric \(-dt^2\), and \(f\) a positive smooth function defined on \(I\). Then, the product manifold \(I \times F\) endowed with the Lorentzian metric

\[
\bar{g} = -\pi^*_t(dt^2) + f(\pi_i)^2 \pi^*_p(g_F),
\]

where \(\pi_i\) and \(\pi_p\) denote the projections onto \(I\) and \(F\), respectively, is called a Generalized Robertson-Walker (GRW) spacetime with fiber \((F, g_F)\), base \((I, -dt^2)\) and warping function \(f\). Along this paper we will represent this \((n + 1)\)-dimensional Lorentzian manifold by \(M = I \times F\).

The coordinate vector field \(\partial_t := \partial/\partial t\) globally defined on \(M\) is (unitary) timelike, and so \(M\) is time-orientable. We will also consider on \(M\) the conformal closed timelike vector field \(K := f(\pi_i) \partial_t\). From the relationship between the Levi-Civita connections of \(M\) and those of the base and the fiber [27, Cor. 7.35], it follows that

\[
\nabla_X K = f'(\pi_i) X
\]

for any \(X \in \mathfrak{X}(M)\), where \(\nabla\) is the Levi-Civita connection of the Lorentzian metric (1).

We will denote by \(\overline{\text{Ric}}\) and \(\overline{S}\) the Ricci tensor and the scalar curvature of \(\overline{M}\), respectively. It is a straightforward computation (see [27, Cor. 7.43]) to check that

\[
\overline{\text{Ric}}(X, Y) = \text{Ric}^F(X^F, Y^F) + \left( \frac{f''}{f} + (n - 1) \frac{f'^2}{f^2} \right) \overline{g}(X^F, Y^F) - n \frac{f''}{f} \overline{g}(X, \partial_t) \overline{g}(Y, \partial_t)
\]

for \(X, Y \in \mathfrak{X}(\overline{M})\), where \(\text{Ric}^F\) stands for the Ricci tensor of \(F\). Here \(X^F\) denotes the lift of the projection of the vector field \(X\) onto \(F\), that is,

\[
X = X^F - \bar{g}(X, \partial_t)\partial_t.
\]

Recall that a Lorentzian manifold \(\overline{M}\) obeys the Null Convergence Condition (NCC) if its Ricci tensor \(\overline{\text{Ric}}\) satisfies \(\overline{\text{Ric}}(X, X) \geq 0\), for all null vector \(X \in \mathfrak{X}(\overline{M})\). If \(\overline{M} = I \times_f F\), given a null vector field \(X \in \mathfrak{X}(\overline{M})\) we get, by decomposing \(X\) as in (3), that \(\overline{g}(X^F, X^F) = \overline{g}(X, \partial_t)^2 \neq 0\). Then, if we denote by \(\overline{\text{Ric}}\) and \(\text{Ric}^F\) the Ricci curvatures of \(\overline{M}\) and \((F, g_F)\) respectively, we get from (2)

\[
\overline{\text{Ric}}(X) = |X^F|^2_p \left( \text{Ric}^F(X^F) - (n - 1) f^2 (\log f)'' \right)
\]

for all unitary null vector field \(X \in \mathfrak{X}(\overline{M})\), where \(|X^F|^p = g_F(X^F, X^F)^{1/2}\) and
there is a remarkable family of spacelike hypersurfaces, namely its spacelike slices \( \{ t_0 \} \times F, t_0 \in I \). The spacelike slices constitute for each value \( t_0 \) the restspace of the distinguished observers in \( \partial_t \). It can be easily seen that a spacelike hypersurface in \( \overline{M} \) is a (piece of) spacelike slice if and only if the hyperbolic angle \( \varphi \) vanishes.

Therefore, we have:

**Lemma 1** Let \( \overline{M} = I \times_f F \) be a GRW spacetime. Then, \( \overline{M} \) obeys the NCC if and only if

\[
Ric^F(X^F) = \frac{Ric^F(X^F, X^F)}{g^F(X^F, X^F)} = Ric^F \left( \frac{X^F}{|X^F|^F}, \frac{X^F}{|X^F|^F} \right).
\]

Observe that as a consequence of the previous Lemma \( \overline{M} \) obeys the NCC if and only at each point \( p \in \overline{M} \) the Ricci curvature at any direction of \( T_p \overline{M} \) is greater or equal than \( (n-1)f^2(\log f)'' \geq 0 \), for all \( p \in (n-1)f^2(\log f)'' \geq 0 \).

Regarding the scalar curvature \( \overline{S} \) of \( \overline{M} \), we get from \( 2 \) that

\[
\overline{S} = \text{trace}(\overline{Ric}) = \frac{S^F}{f^2} + 2n \frac{f''}{f} + n(n-1) \frac{f^2}{f^2}
\]

where \( S^F \) stands for the scalar curvature of \( F \).

Given an \( n \)-dimensional manifold \( M \), an immersion \( \psi : M \to \overline{M} \) is said to be spacelike if the Lorentzian metric \( \overline{g} \) induces, via \( \psi \), a Riemannian metric \( g \) on \( M \). In this case, \( M \) is called a spacelike hypersurface.

Since \( \overline{M} \) is time-orientable we can take, for each spacelike hypersurface \( M \) in \( \overline{M} \), a unique unitary timelike vector field \( N \in \mathbb{X}^+(M) \) globally defined on \( M \) with the same time-orientation as \( \partial_t \), i.e. such that \( \overline{g}(N, \partial_t) < 0 \). From the wrong-way Cauchy-Schwarz inequality (see [27, Prop. 5.30], for instance), we have \( \overline{g}(N, \partial_t) \leq -1 \), and the equality holds at a point \( p \in M \) if and only if \( N = \partial_t \) at \( p \).

For a spacelike hypersurface \( \psi : M \to \overline{M} \) with Gauss map \( N \), the hyperbolic angle \( \varphi \), at any point of \( M \), between the unit timelike vectors \( N \) and \( \partial_t \), is given by \( \overline{g}(N, \partial_t) = -\cosh \varphi \). By simplicity, throughout this paper we will refer to \( \varphi \) as the hyperbolic angle function on \( M \). In a GRW spacetime \( \overline{M} \) the integral curves of \( \partial_t \) are called comoving observers [33, p. 18]. If \( p \) is a point of a spacelike hypersurface \( M \) in \( \overline{M} \), among the instantaneous observers at \( p \), \( \partial_t(p) \) and \( N_p \) appear naturally. In this sense, observe that the energy \( e(p) \) and the speed \( v(p) \) that \( \partial_t(p) \) measures for \( N_p \) are given, respectively, by \( e(p) = \cosh \theta(p) \) and \( |v(p)|^2 = \tanh^2 \theta(p) \) [33, pp. 45, 67].

We will denote by \( A \) and \( H := -(1/n)\text{trace}(A) \) the shape operator and the mean curvature function associated to \( N \). The mean curvature is zero if and only if the spacelike hypersurface is, locally, a critical point of the \( n \)-dimensional area functional for compactly supported normal variations. A spacelike hypersurface with \( H = 0 \) is called a maximal hypersurface.

In any GRW spacetime \( \overline{M} \) there is a remarkable family of spacelike hypersurfaces, namely its spacelike slices \( \{ t_0 \} \times F, t_0 \in I \). The spacelike slices constitute for each value \( t_0 \) the restspace of the distinguished observers in \( \partial_t \). It can be easily seen that a spacelike hypersurface in \( \overline{M} \) is a (piece of) spacelike slice if and only if the function \( \tau := \pi_I \circ \psi \) is constant. Furthermore, a spacelike hypersurface in \( \overline{M} \) is a (piece of) spacelike slice if and only if the hyperbolic angle \( \varphi \) vanishes.
identically. The shape operator of the spacelike slice $\tau = t_0$ is given by $A = -f'(t_0)/f(t_0)$, where $I$ denotes the identity transformation, and so its (constant) mean curvature is $H = f'(t_0)/f(t_0)$. Thus, a spacelike slice is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic).

3 Spacelike Hypersurfaces in a GRW which obeys the NCC

Let $\psi : M \rightarrow \mathcal{M}$ be a spacelike hypersurface in the GRW spacetime $\mathcal{M} = I \times F$. The curvature tensor $R$ of $M$ can be described in terms of the curvature tensor $\mathcal{R}$ of $\mathcal{M}$ and the shape operator $A$ according the Gauss equation

$$R(X,Y)Z = (\mathcal{R}(X,Y)Z)^T - \mathcal{g}(AX,Z)AY + \mathcal{g}(AY,Z)AX$$

for all tangent vector fields $X, Y, Z \in \mathfrak{X}(M)$, where $(\mathcal{R}(X,Y)Z)^T$ denotes the tangential component of $\mathcal{R}(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$.

From (6) it follows that the Ricci curvature of $M$ is given by

$$\text{Ric}(X,Y) = \mathcal{Ric}(X,Y) + \mathcal{g}(\mathcal{R}(N,N)Y,N) - \text{trace}(A)\mathcal{g}(AX,Y) + \mathcal{g}(AX,AY)$$

for $X, Y \in \mathfrak{X}(M)$. Then, the scalar curvature of $M$ yields

$$S = \text{trace}(\text{Ric}) = \mathcal{S} + 2\mathcal{Ric}(N,N) + \text{trace}(A^2) - n^2H^2.$$  

(7)

If we put $\partial^T_t = \partial_t + \mathcal{g}(\partial_t, N)N$ the tangential part of $\partial_t$ and $N^F = N + \mathcal{g}(N, \partial_t)\partial_t$, it follows from $\mathcal{g}(N,N) = -1 = \mathcal{g}(\partial_t, \partial_t)$ that

$$|\partial^T_t|^2 = \mathcal{g}(\partial^T_t, \partial^T_t) = \mathcal{g}(N^F, N^F) = |X^F|^2 = \sinh^2 \varphi.$$  

(8)

Then, using also (2), we get

$$\mathcal{Ric}(N,N) = \text{Ric}^F(N^F, N^F) - (n - 1) \frac{f''(\tau)}{f(\tau)} |\partial^T_t|^2 + (n - 1) \frac{f'^2(\tau)}{f^2(\tau)} |\partial^F_t|^2 - n \frac{f''(\tau)}{f(\tau)}$$

which jointly with (5) allow to rewrite the scalar curvature (7) as follows

Lemma 2 Let $\psi : M \rightarrow \mathcal{M}$ be a spacelike hypersurface in a GRW spacetime $\mathcal{M} = I \times F$. Then the scalar curvature of $M$ is given by

$$S = \frac{S^F \circ \pi_F}{f^2(\tau)} + 2 \left( \text{Ric}^F(N^F, N^F) - (n - 1)(\log f)''(\tau) |\partial^T_t|^2 \right)$$

$$+ n(n - 1) \left( \frac{f'^2(\tau)}{f^2(\tau)} - H^2 \right) + \text{trace}(A^2) - nH^2.$$  

(9)

As a consequence of Lemmas 1 and 2 and using also (8), we get.
Theorem 3 Let $\psi : M \to \overline{M}$ be a spacelike hypersurface in a GRW spacetime $\overline{M} = I \times f F$ obeying the NCC. Then the scalar curvature of $M$ satisfies

$$S \geq \frac{S^F \circ \pi_F}{f^2(\tau)} + n(n-1) \left( \frac{f'^2(\tau)}{f^2(\tau)} - H^2 \right).$$

Moreover, if the equality holds then the spacelike hypersurface is totally umbilical.

From Lemma 1 it follows that, when the GRW spacetime obeys the NCC, then

$$S^F \circ \pi_F \geq n(n-1) f^2(\tau) \left( \log f \right)''(\tau).$$

Therefore we get the following consequence from Theorem 3

Corollary 4 Let $\psi : M \to \overline{M}$ be a spacelike hypersurface in a GRW spacetime $\overline{M} = I \times f F$ obeying the NCC. Then the scalar curvature of $M$ satisfies

$$S \geq n(n-1) \left( \frac{f''(\tau)}{f(\tau)} - H^2 \right).$$

Moreover, if the equality holds then the spacelike hypersurface is totally umbilical and the scalar curvature of the fiber can be expressed in terms of the warping function $f$ as $S^F \circ \pi_F = n(n - 1) f^2(\tau) \left( \log f \right)''(\tau)$.

For the important particular case of maximal surfaces, Theorem 3 and Corollary 4 can be rewritten as follows

Corollary 5 Let $\psi : M \to \overline{M}$ be a maximal spacelike hypersurface in a GRW spacetime $\overline{M} = I \times f F$ obeying the NCC. Then the scalar curvature of $M$ satisfies

$$S \geq \frac{S^F \circ \pi_F}{f^2(\tau)}.$$ 

Moreover, the equality holds if and only if the spacelike hypersurface is totally geodesic.

Note that if the equality holds in (12) it must be $f'(\tau) \equiv 0$. Hence, in the particular case when the warping function $f$ is non-locally constant, that is, when the GRW spacetime $\overline{M}$ is proper, we can state

Corollary 6 Let $\psi : M \to \overline{M}$ be a maximal spacelike hypersurface in a proper GRW spacetime $\overline{M} = I \times f F$ obeying the NCC. Then the scalar curvature of $M$ satisfies

$$S \geq \frac{S^F \circ \pi_F}{f^2(\tau)}.$$ 

Moreover, the equality holds if and only if the spacelike hypersurface is contained in a totally geodesic slice.
Corollary 4 can be easily adapted for maximal spacelike hypersurfaces by taking \( H = 0 \). Note that under this additional hypothesis if the equality holds in (11) then the spacelike hypersurface is totally geodesic.

**Remark 7** When \( I = \mathbb{R}, \ F = \mathbb{R}^n \) and \( f(t) = e^t \), the GRW spacetime \( \mathcal{N} = \mathbb{R} \times e^t \mathbb{R}^n \) is isometric to a proper open subset of the De Sitter spacetime of sectional curvature 1, which is called the \((n+1)\)-dimensional steady state spacetime. The steady state is the model of the universe proposed by Bondi and Gold [9] and Hoyle [22] when one is looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times [25, Section 14.8]. Since the steady state spacetime is a proper GRW spacetime whose fiber has zero scalar curvature, from Corollary 5 we can state that any maximal spacelike hypersurfaces in \( \mathcal{N} \) has non negative scalar curvature \( S \geq 0 \), and moreover it can not be \( S = 0 \) up to at isolated points.

In the language of General Relativity, the de Sitter spacetime is the maximally symmetric vacuum solution of Einstein’s field equations with a positive (repulsive) cosmological constant corresponding to a positive vacuum energy density and negative pressure. In its intrinsic version, the \((n+1)\)-dimensional De Sitter spacetime \( S^{n+1} \) is given as the Robertson-Walker spacetime \( S^{n+1} = \mathbb{R} \times \cosh t S^n \), where \( S^n \) denotes the \( n \)-dimensional sphere with its usual metric.

Recall that \( S^{n+1} \) has constant sectional curvature 1 and obeys the NCC. Even more, since the fiber \( F = S^n \) has constant Ricci curvature \( \text{Ric}^F = n - 1 \), we have that equality holds in (4). Therefore, using also that \( S^F = n(n - 1) \), we get from Lemma 2 that the scalar curvature of a spacelike hypersurface in \( S^{n+1} \) is given by

\[
S = \frac{n(n-1)}{\cosh^2 \tau} + n(n-1)(\tanh^2 \tau - H^2) + \text{trace}(A^2) - nH^2
\]

Hence, we have

**Theorem 8** Let \( \psi : M \to S^{n+1}_1 \) be a spacelike hypersurface in the de Sitter spacetime. Then the scalar curvature of \( M \) satisfies

\[
S \geq n(n-1)(1 - H^2). \tag{13}
\]

Moreover, the equality holds if and only if the spacelike hypersurface is totally umbilical.

Recall that every totally umbilical spacelike in the De Sitter space has constant mean curvature. In particular, if the equality holds in (13) then the spacelike hypersurface has constant scalar curvature.

**Remark 9** The problem of characterizing the totally umbilical spacelike hypersurfaces in \( S^{n+1}_1 \) under hypotheses relative to the scalar curvature of the hypersurface has received an special attention in the last years [1], [2], [3], [10], [13], [14], [23], [24], [25], [31], [40]. It is worth pointing out that, in these papers, to obtain the rigidity results the completeness (and sometimes even the compactness) of the hypersurface is required. However, in Theorem 8 we do not need to ask the hypersurface to be complete.

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We will say that a spacetime $\mathcal{M}$ verifies the \textit{NCC with strict inequality} if its Ricci tensor $\text{Ric}$ satisfies $\text{Ric}(X,X) > 0$, for all null vector $X \in \mathfrak{X}(\mathcal{M})$. Reasoning as in Lemma 3, a GRW spacetime $\mathcal{M} = I \times_f F$ obeys the NCC with strict inequality if and only if $\text{Ric}^F - (n-1) f^2 (\log f)'' > 0$.

**Corollary 10** Let $\psi : M \to \mathcal{M}$ be a spacelike hypersurface in a GRW spacetime $\mathcal{M} = I \times_f F$ obeying the NCC with strict inequality. Then the scalar curvature of $M$ satisfies

$$S \geq \frac{S^F \circ \pi_F}{f^2(\tau)} + n(n-1) \left( \frac{f^2(\tau)}{f^2(\tau)} - H^2 \right).$$

Moreover, the equality holds if and only if $M$ is contained in a spacelike slice, and as a consequence $S = \frac{S^F \circ \pi_F}{f(\tau)}$.

\textbf{Proof:} It follows from (10) and the fact that $M$ is contained in a spacelike slice if, and only if, $|N^F| = |\partial_t^F| = 0$ on $M$. \hfill $\Box$

If $\mathcal{M} = I \times_f F$ is a spacetime obeying NCC with strict inequality then, reasoning as in (10), we have that

$$S^F \circ \pi_F > n(n-1) f^2(\tau) (\log f)''(\tau).$$

In particular, the scalar curvature of a spacelike hypersurface in $\mathcal{M} = I \times_f F$ can be bounded in terms of the warping function $f$ as

$$S > n(n-1) \left( \frac{f''(\tau)}{f(\tau)} - H^2 \right).$$

Unlike what was happened under the NCC, now the equality cannot be attained.

It is well-known that if a GRW spacetime $\mathcal{M}$ admits a compact spacelike hypersurface, then $\mathcal{M}$ must be \textit{spatially closed}, that is, its fiber $F$ is compact (see [5, Prop.3.2]).

Let $M$ be a compact spacelike hypersurface with constant mean curvature in $\mathcal{M}$. In [3, Sec. 4], the authors showed that the following integral equation holds

$$\int_M \left\{ \overline{\text{Ric}}(K^T, N) + \overline{\mathcal{g}}(K, N)(\text{trace}(A^2) - nH^2) \right\} dV = 0,$$

where $dV$ denotes the Riemannian volume element on $M$. From (2), and provided that $\mathcal{M}$ obeys the NCC, a straightforward computation yields

$$\overline{\text{Ric}}(K^T, N) = \overline{\mathcal{g}}(K, N) \overline{\text{Ric}}(N^F, N^F) - \overline{\mathcal{g}}(K, N) |\partial_t^F|^2 \overline{\text{Ric}}(\partial_t, \partial_t)$$

$$= \overline{\mathcal{g}}(K, N) \left( \text{Ric}^F(N^F, N^F) - (n-1) |N^F|^2 (\log f)''(\tau) \right)$$

$$= \overline{\mathcal{g}}(K, N) |N^F|^2 \left( \text{Ric}^F \left( \frac{N^F}{|N^F|_F} \right) - (n-1) f^2(\tau) (\log f)''(\tau) \right) \leq 0$$

Since also $\overline{\mathcal{g}}(K, N)(\text{trace}(A^2) - nH^2) \leq 0$, from (14) we get that both $\overline{\text{Ric}}(K^T, N)$ and $\text{trace}(A^2) - nH^2$ vanish identically on $M$. Therefore $M$ is totally umbilical and, from [3], its
scalar curvature is given by

\[ S = \frac{S^F \circ \pi_F}{f^2(\tau)} + n(n - 1) \left( \frac{f'^2(\tau)}{f^2(\tau)} - H^2 \right). \]  

Furthermore, if \( M \) obeys NCC with strict inequality, then \( M \) is a spacelike slice. Therefore, the following Calabi-Bernstein type result can be stated:

**Theorem 11** Let \( (F, g_F) \) be an \( n \)-dimensional, \( n \geq 2 \), compact Riemannian manifold and \( f : I \rightarrow (0, \infty) \) a smooth function such that \( \text{Ric}^F - (n - 1) f^2 (\log f)' > 0 \). Then the only entire solutions to the mean curvature surface equation for spacelike hypersurfaces are the constant functions.

The importance in General Relativity of maximal and constant mean curvature spacelike hypersurfaces in spacetimes is well-known; a summary of several reasons justifying it can be found in [26]. In particular, hypersurfaces of non-zero constant mean curvature are particularly suitable for studying the propagation of gravity radiation [37]. Classical papers dealing with uniqueness results for CMC hypersurfaces are [17], [11] and [26], although a previous relevant result in this direction was the proof of the Bernstein-Calabi conjecture [12] for the \( n \)-dimensional Lorentz-Minkowski spacetime given by Cheng and Yau [15]. In [11], Brill and Flaherty replaced the Lorentz-Minkowski spacetime by a spatially closed universe, and proved uniqueness in the large by assuming \( \text{Ric}(z, z) > 0 \) for all timelike vector \( z \). In [26], this energy condition was relaxed by Marsden and Tipler to include, for instance, non-flat vacuum spacetimes. More recently Bartnik proved in [7] very general existence theorems and consequently, he claimed that it would be useful to find new satisfactory uniqueness results. Still more recently, in [5] Alias, Romero and Sánchez proved new uniqueness results in spatially closed GRW spacetimes (which includes the spatially closed Robertson-Walker spacetimes), under the temporal convergence condition. Finally, in [31], Romero, Rubio and Salamanca provided uniqueness results, in the maximal case, for spatially parabolic GRW spacetimes, which are open models whose fiber is a parabolic Riemannian manifold.

4 GRW spacetimes filled with perfect fluid

Currently, the interest of General Relativity in arbitrary dimension is notable for several reasons, as the creation of unified theories and also methodological considerations associated with the possibility of understanding general features for the simpler (2+1)-dimensional models (see [35] and references therein).

Astronomical evidences indicate that the universe can be modeled (in smoothed, averaged form) as a spacetime containing a perfect fluid whose *molecules* are the galaxies. Classically, the dominant contribution to the energy density of the galactic fluid is the mass of the galaxies, with a smaller pressure due mostly to radiations. Nevertheless, over of the 90’s years, evidences for the most striking result in modern cosmology have been steadily growing, namely the existence of a cosmological constant which is driving the current acceleration of the universe as first observed in [28], [30]. Different models for dark energy cosmology and their equivalences can be seen in [6]. Note that a positive vacuum energy density resulting from a cosmological constant implies a negative pressure and vice versa.
Thus, is natural that several exact solutions to the Einstein field equation
\[
\text{Ric} - \frac{1}{2} S g = 8\pi T
\]  
(16)
had been obtained by considering a continuous distribution of matter as a perfect fluid.

Recall that a perfect fluid (see, for example, [27, Def. 12.4]) on a spacetime \(\overline{M}\) is a triple \((U, \rho, p)\) where

1. \(U\) is a timelike future-pointing unit vector field on \(\overline{M}\) called the flow vector field.
2. \(\rho, p \in C^\infty(\mathbb{N})\) are, respectively, the energy density and the pressure functions.
3. The stress-energy momentum tensor is

\[
T = (\rho + p) U^* \otimes U^* + p \overline{g},
\]

where \(\overline{g}\) is the metric of the spacetime \(\overline{M}\).

For an instantaneous observer \(v\), the quantity \(T(v, v)\) is interpreted as the energy density, i.e. the mass-energy per unit of volume, as measured by this observer. For normal matter, this quantity must be non-negative, i.e. the tensor \(T\) must obey the weak energy condition. It is easy to see that an exact solution to (16) for a stress-energy tensor which obeys the weak energy condition must satisfy the null energy condition, that is, \(\text{Ric}(z, z) \geq 0\) for all null vector \(z\). Nevertheless, perfect fluids can also be used to model another scenarios of universes at the dark energy dominated stage (see [20]).

Let us consider a GRW spacetime \((\overline{M}, \overline{g})\) filled with perfect fluid. As is usual in this context, we take the flow vector field \(U = \partial_t\) and therefore \(T = (\rho + p) dt \otimes dt + p \overline{g}\).

Now, taking into account (2), (5) and the equality \(T(\partial_t, \partial_t) = \rho\), we have

\[
8\pi \rho = \frac{1}{2} \left( \frac{S^F}{f^2} + n(n - 1) \frac{f'^2}{f^2} \right). 
\]  
(17)

Analogously, if we consider vector fields \(X, Y \perp \partial_t\), we obtain

\[
8\pi p \overline{g}(X, Y) = \text{Ric}^F(X, Y) + \left( \frac{f''}{f} + (n - 1) \frac{f'^2}{f^2} \right) \overline{g}(X, Y).
\]

As the pressure is a function which depends only of the points of \(\overline{M}\), the Ricci curvature of the fiber must be constant, i.e. the Riemannian manifold \((F, g_\rho)\) must be Einstein. Consequently, we can write

\[
8\pi p = \frac{1}{f^2} \left( \text{Ric}^F - (n - 1) f^2 (\log f)''\right) - \frac{1}{2} \left( \frac{S^F}{f^2} + n(n - 1) \frac{f'^2}{f^2} \right). 
\]  
(18)

If we assume that the spacetime obeys the NCC, then from (17) we have \(p \geq -\rho\). In particular, when \(\rho > 0\) we have

\[
\frac{p}{\rho} \geq -1.
\]
Observe that from the equations \( 17 \) and \( 18 \), the de Sitter spacetime appears as a barotropic perfect fluid solution of the Einstein field equations with constant energy density and pressure functions, and state equation

\[
w = \frac{p}{\rho} = -1.
\]

If for each \( p \in F \) we parametrize \( I \times \{p\} \) by \( \gamma_p(t) = (t, p) \), since \( \partial_t \) is the velocity of each galaxy \( \gamma_p \), they are its integral curves. In particular, the function \( t \) is the common proper time of all galaxies. By taking \( t \) as a constant, we get the hypersurface

\[
M(t) = t \times F = \{(t, p) : p \in F\}.
\]

The distance between two galaxies \( \gamma_p \) and \( \gamma_q \) in \( M(t) \) is \( f(t)d(p, q) \), where \( d \) is the Riemannian distance in the fiber \( F \). In particular, when \( f \) has positive derivative the spaces \( M(t) \) are expanding. Moreover, if \( f'' > 0 \) the GRW spacetimes models universes in accelerated expansion.

Some GRW spacetimes satisfying NCC can be suitable modified models of gravity. For instance, the de Sitter spacetime \( f'(t) = \sinh t \geq 0 \) (and \( f'(t) = 0 \) only for \( t = 0 \)) and so the spaces \( M(t) \) are expanding. Moreover, \( f''(t) = \cosh t > 0 \) and so this expansion is accelerated. That is, the de Sitter spacetime constitutes an accelerated expanding spacetime.

Observe that for GRW spacetimes satisfying the NCC, the inequality \( (\log f)'' > 0 \) implies that the Ricci curvature of the fiber is non-negative, including the Ricci flat case. When \( f' > 0 \), the condition \( (\log f)'' \geq 0 \) assures accelerated expanding models.

On the other hand, when the GRW spacetime obeys NCC, from \( 4 \) we have that the energy density satisfies

\[
\rho \geq \frac{1}{8\pi} \frac{f''}{f}.
\]

Thus, we have

**Theorem 12** Every GRW satisfying the NCC with \( f'' \geq 0 \) and filled with perfect fluid obeys the weak energy condition.

In fact, given \( X \) a timelike vector field we have

\[
T(X, X) = (\rho + p)\mathcal{G}(X, \partial_t)^2 + p\mathcal{G}(X^F, X^F),
\]

being \( \mathcal{G}(X, \partial_t)^2 \geq \mathcal{G}(X^F, X^F), \rho \geq 0 \) and \( \rho \geq |p| \).

### 4.1 Spatially closed GRW spacetimes

Let \( \overline{M} \) be a spatially closed GRW spacetime and \( M \) a compact spacelike hypersurface in \( \overline{M} \). From Lemma 2, we can give the following estimation of the total energy on \( M \)

**Theorem 13** Let \( \psi : M \rightarrow \overline{M} \) be a compact spacelike hypersurface in a GRW spacetime \( \overline{M} = I \times F \) filled with perfect fluid and which is an exact solution of the Einstein field equation. Then the total energy on \( M \) satisfies that

\[
E_M = \int_M \rho \, dV \leq \frac{1}{16\pi} \int_M \left( S + n(n - 1)H^2 \right) \, dV.
\]
If $M$ has constant mean curvature, from (15) we have
\[
\left(\frac{S^F \circ \pi_F}{f^2(\tau)} + n(n-1)\frac{f'^2(\tau)}{f^2(\tau)}\right) = S + n(n-1)H^2.
\]
Hence, we can calculate the total energy on the spacelike hypersurface $M$ as
\[
E_M = \int_M \rho \, dV = \frac{1}{16\pi} \int_M \left( S + n(n-1)H^2 \right) \, dV
\]
\[
= \frac{1}{16\pi} \int_M S \, dV + \frac{n(n-1)}{16\pi} H^2 \text{Vol}(M). \tag{19}
\]
In particular, if $M$ is maximal, then
\[
E_M = \frac{1}{16\pi} \int_M S \, dV.
\]

**Corollary 14** Let $\psi : M \to \overline{M}$ be a compact CMC spacelike hypersurface in a GRW spacetime $\overline{M} = I \times f F$ obeying the NCC, filled with perfect fluid and which is an exact solution of the Einstein field equation. Then $M$ is maximal if and only if the total energy on $M$ coincides with the integral on the hypersurface of its scalar curvature.

On the other hand, observe that if the fiber of $\overline{M}$ has non-negative scalar curvature, then the total energy on every compact maximal hypersurface is non-negative.

Taking into account (19) and [4, Th. 3], we can give the following estimations of the total energy on a CMC compact spacelike hypersurface in terms of the volume of the fiber of the spacetime

**Theorem 15** Let $\psi : M \to \overline{M}$ be a compact CMC spacelike hypersurface in a GRW spacetime $\overline{M} = I \times f F$ obeying the NCC, filled with perfect fluid and which is an exact solution of the Einstein field equation. Then
\[
\frac{1}{16\pi} \int_M S \, dV + \frac{n(n-1)f(\tau_0)^n}{16\pi \cosh \varphi_0} H^2 \text{Vol}(F) \leq E_M \leq \frac{1}{16\pi} \int_M S \, dV + \frac{n(n-1)f(\tau_0)^n}{16\pi \cosh \varphi_0} H^2 \text{Vol}(F),
\]
were, $f(\tau_0)$ (resp. $f(\tau_0)$) denotes the maximum (resp. the minimum) of the warping function on the hypersurface and $\cosh \varphi_0$ (resp. $\cosh \varphi_0$) denotes the maximum (resp. the minimum) on $M$ of the cosine of $\cosh \varphi$.

It is remarkable that in the case (2+1)-dimensional the formulae (19) has an special significance, since $S = 2K$, were $K$ denotes the Gaussian curvature of $M$. Hence, using the Gauss-Bonnet theorem, we get

**Theorem 16** Let $\psi : M \to \overline{M}$ be a compact CMC spacelike hypersurface in a 3-dimensional GRW spacetime $\overline{M} = I \times f F$ obeying the NCC, filled with perfect fluid and which is an exact solution of the Einstein field equation. Then
\[
E_M = \frac{1}{8} \chi(M) + \frac{n(n-1)}{16\pi} H^2 \text{Vol}(M),
\]
where $\chi(M)$ is the Euler characteristic of $M$. 

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Thus, the total energy is explained in terms of topological and extrinsic quantities.

**Remark 17** Observe that taking into account the relation $\rho + p \geq 0$ and the previous considerations, some estimations for the total pressure on a compact spacelike hypersurface can also be given.

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