The Hypertriton in Effective Field Theory

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(Dated: October 15, 2001)

Abstract

Doublet Λd scattering and the hypertriton are studied in the framework of an effective field theory for large scattering lengths. As in the triton case, consistent renormalization requires a one-parameter three-body force at leading order whose renormalization group evolution is governed by a limit cycle. Constraining unknown parameters from symmetry considerations and the measured binding energy of the hypertriton, we calculate the low-energy phase shifts for doublet Λd scattering. For the low-energy parameters, we find $a_{Λd} = (16.8^{+4.4}_{-4.1})$ fm and $r_{Λd} = (2.3 \pm 0.3)$ fm, where the errors are due to the uncertainty in the hypertriton binding energy. Since the hypertriton is extremely shallow, low-energy three-body observables in this channel are very insensitive to the exact values of the ΛN low-energy parameters.

PACS numbers: 21.80.+a, 21.45.+v, 11.10.Ef

Keywords: Effective field theory, hypertriton, three-body forces

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I. INTRODUCTION

In recent years, there has been much interest in applying Effective Field Theory (EFT) methods to nuclear systems with two or more nucleons [1, 2, 3, 4, 5]. EFT’s provide a powerful framework to explore a separation of scales in physical systems in order to perform systematic, model-independent calculations [6]. If, for example, the momenta $k$ of two particles are much smaller than the inverse range of their interaction $1/R$, observables can be expanded in powers of $kR$. All short-distance effects are systematically absorbed into a few low-energy constants using renormalization. The EFT approach allows for accurate calculation of low-energy processes with well-defined error estimates. For low-energy nuclear few-body systems, the long-distance scale is set by the large two-body scattering lengths, while the short distance scale is set by the range of the nuclear force or the inverse pion mass. In an EFT, the long-distance physics is included explicitly, while the corrections from short-distance physics are calculated in an expansion in the ratio of these two scales.

For very low energies and momenta $(p \lesssim m_\pi)$, even pion exchange can be considered “short-distance” physics. In this case, one can use an effective Lagrangian including only contact interactions. The large S-wave scattering lengths require that the leading two-body contact interaction is treated nonperturbatively [7, 8]. In the two-body system, this program has been very successful (see e.g. Refs. [5, 9, 10] and references therein).

In the nuclear three-body system, considerable progress has been made as well [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Most of the work has concentrated on the neutron-deuteron system. In channels where the Pauli principle or centrifugal barrier suppress sensitivity to short-distance physics, the EFT can be extended in a straightforward way [11, 12, 13, 14]. Recently, Coulomb repulsion has been included and the phase shifts for proton-deuteron scattering in the spin quartet ($J = 3/2$) have been calculated [15]. Most interesting, however, is the S-wave in doublet ($J = 1/2$) neutron-deuteron scattering which displays a number of surprising phenomena such as the Thomas and Efimov effects or the Phillips line [21, 22, 23]. In this channel, the renormalization requires a one-parameter three-body force at leading order whose renormalization group evolution is governed by a limit cycle [16, 17]. Variation of this three-body force gives a compelling explanation of the Phillips line [22]. Recently, the effective range corrections to S-wave neutron-deuteron scattering in the doublet channel have been calculated [20]. At this order, good agreement with available phase shift analyses and potential model calculations is obtained.

The purpose of the present paper is to study doublet $\Lambda d$ scattering and the hypertriton using EFT methods. The hypertriton is the lightest hypernucleus and consists of a triton with one neutron replaced by a $\Lambda$. It is the simplest strange three-body halo nucleus with a separation energy into a deuteron and a $\Lambda$ of only $B^{\Lambda} = 0.13 \pm 0.05$ MeV [24, 25]. The total binding energy is $B^{\Lambda}_3 = 2.35$ MeV. The hypertriton has been studied extensively using various hyperon-nucleon potentials (see e.g. Refs. [24, 27, 28, 24, 30, 31] and references therein). The most sophisticated calculations include both tensor forces and $\Lambda \leftrightarrow \Sigma$ conversion effects [27]. Obtaining the correct binding energy for the hypertriton in these potential models is delicate, which has motivated the use of this system to study details of the hyperon-nucleon interaction [30].

The EFT framework offers a different perspective and stresses the universal aspects of the problem. In this paper, we study $\Lambda d$ scattering and the hypertriton in the framework of an EFT for three-body systems with large two-body scattering lengths [16, 17]. The small $N-\Lambda$ mass difference can be exploited by expanding in the parameter $y = (M_\Lambda - M)/(M_\Lambda + M) \approx$
0. After this expansion is performed, the hypertriton emerges naturally as a shallow Efimov state. To leading order, both $NN$ and $ΛN$ S-waves will contribute to the hypertriton. While the $NN$ S-waves have a large scattering length, the $ΛN$ scattering lengths are believed to be natural. Therefore, it is not obvious that an EFT for large two-body scattering lengths is applicable. In the following, we motivate the use of this EFT.

Unfortunately, there are only few $ΛN$ scattering data and these are at relatively high energies. Model independent analyses using effective range theory indicate mainly S-wave scattering, but afford many solutions for the low-energy parameters and are essentially inconclusive [32, 33]. For example in Ref. [33], the extracted one-standard-deviation bounds were $0 > a_1 > -15$ fm, $0 < r_1 < 15$ fm, $-0.6$ fm $> a_3 > -3.2$ fm, and $2.5$ fm $< r_3 < 15$ fm where $a_1$ ($a_3$) and $r_1$ ($r_3$) are the singlet (triplet) scattering length and effective range, respectively. These bounds do not exclude the case of large scattering lengths.

Various sets of low-energy parameters are also known from extrapolations of the few higher energy data using hyperon-nucleon potentials [30, 34, 35, 36, 37]. The extracted $ΛN$ scattering lengths and effective ranges are generally of natural size $[O(1/m_π)]$. Both the $^3S_1$ and $^1S_0$ $ΛN$ partial waves have a virtual bound state. The pole momenta extracted from the low-energy extrapolations are $\sim 60$ MeV for the $^1S_0$ and $\sim 70$ MeV for the $^3S_1$ partial wave. However, it is not clear how large the errors and model-dependence in these low-energy extrapolations are.

An EFT analysis of hyperon nucleon scattering and hyperon mass shifts in the nuclear medium was recently performed in Ref. [38]. The $ΛN$ scattering lengths, however, were used as input in this analysis.

Because the hypertriton is so weakly bound, it should only be sensitive to the extreme long-distance properties of the $ΛN$ interaction. The typical momentum scale of the hypertriton can be estimated from its binding energy via

$$\gamma_3^Λ \sim 2\sqrt{(MB_3^Λ - \gamma_d^2)/3} \approx 0.3\gamma_d,$$

where $M = 938.9$ MeV is the nucleon mass and $\gamma_d = 45.68$ MeV the deuteron pole momentum. Since $\gamma_3^Λ$ is so small, the contribution from the effective ranges will be suppressed even if the $ΛN$ low-energy parameters are $O(1/m_π)$ as extracted from the potential models (see Section [IV] for a more quantitative estimate). Consequently, the use of the EFT for large two-body scattering lengths is justified.

In the next Section, we write down the effective Lagrangian and discuss the power counting of the EFT. We then review the EFT for the $NN$ and $ΛN$ two-body subsystems. In Section [III], we obtain the three-body integral equations for the $ΛNN$ system and in Section [IV] we present our results and conclusions. Some technical details are given in the Appendices.

II. TWO-BODY SYSTEM

The power counting in an EFT is determined by the physical scales in the system under consideration. For the hypertriton, we are only interested in very low-energies, where all physics (even pion exchange) can be considered “short-range”. The two-body subsystems ($NN$ and $ΛN$) are characterized by two scales: a long-distance scale (the large S-wave scattering length) and a short-distance scale that is given by the longest-range physics excluded from the EFT. For an EFT without pions, one expects the short-distance scale $R$ to be of
\( \mathcal{O}(1/m_c) \). However, since the \( \Lambda \) (\( I = 0 \)) and nucleon (\( I = 1/2 \)) cannot exchange a pion with \( I = 1 \) and conserve isospin, the short-distance scale for the hypertriton should be set by the two-pion exchange \([27]\). A power counting to deal with unnaturally large two-body scattering lengths has been proposed in Refs. \([7, 8]\). This counting takes \( p \sim 1/a \sim Q \), where \( p \) is the typical momentum and \( 1/a \) is the inverse scattering length. The pole or binding momentum \( \gamma \) is of \( \mathcal{O}(Q) \) as well since \( \gamma = 1/a + \mathcal{O}(r/a^2) \) with \( r \sim R \) the effective range. The expansion of the EFT is in powers of \( Q R \approx \gamma r \) (see Refs. \([14, 20]\) for more details).

For the study of three-body systems, it is convenient to employ the dibaryon formalism \([39]\), where an auxiliary field is used to describe two baryons interacting in a given partial wave. We write down an effective Lagrangian including nucleons, \( \Lambda \)'s, and dibaryon fields for the deuteron (\( d \)) as well as the \( ^3S_1 \) (\( u^3 \)) and \( ^1S_0 \) (\( u^1 \)) \( \Lambda N \) partial waves,

\[
\mathcal{L} = N^\dagger \left( i\partial_t + \frac{\vec{\nabla}^2}{2M} \right) N + \Lambda^\dagger \left( i\partial_t + \frac{\vec{\nabla}^2}{2M_\Lambda} \right) \Lambda + \Delta_d d^\dagger d - \frac{g_d}{2} \left[ d^\dagger T (i\tau_2) (i\sigma_1 \sigma_2) N + \text{H.c.} \right] + \Delta_3 (u^3_i)^\dagger u^3_i - ig_3 \left[ (u^3_i)^\dagger \Lambda^T (i\sigma_1 \sigma_2) N + \text{H.c.} \right] + \Delta_1 (u^1_i)^\dagger u^1_i - ig_1 \left[ (u^1_i)^\dagger \Lambda^T (i\sigma_2) N + \text{H.c.} \right] + \ldots,
\]

where H.c. denotes the Hermitian conjugate and the dots indicate terms with more derivatives and/or fields. The terms with more derivatives are suppressed at low energies, while four- and higher-body forces do not contribute. Three-body terms will be addressed in the next Section. In Eq. (2), indices occurring repeatedly are summed over and \( \sigma_i (\tau_i) \) are the usual Pauli matrices acting in spin (isospin) space. Spinor and isospinor indices are suppressed. The parameters \( \Delta \) and \( g \) are not independent and only the combination \( \Delta/g^2 \) enters physical observables.

The Lagrangian (2) describes low-energy S-wave scattering of nucleons and \( \Lambda \)'s and reproduces the effective range expansion up to the scattering length term, which is sufficient for our purposes. To go to higher orders in \( QR \), the two-body effective range can be included with a kinetic term for the auxiliary field [39]. Only the partial waves contributing in the hypertriton channel to leading order in \( QR \) are included in the effective Lagrangian (2). The \( ^1S_0 \) \( NN \) amplitude does not contribute because of isospin, while higher partial waves and the tensor force are suppressed by at least two powers of the expansion parameter \( QR \) \([14]\). After integrating out the auxiliary fields \( d, u^3, \) and \( u^1 \), it is straightforward to show that the Lagrangian (2) is equivalent to a Lagrangian with nucleon and \( \Lambda \) fields only (cf. Refs. \([16, 17]\)).

Since the theory is nonrelativistic, all particles propagate forward in time and the nucleon and \( \Lambda \) tadpoles vanish. The propagator for nucleons and \( \Lambda \)'s is simply

\[
iS(p_0, \vec{p}) = \frac{i}{p_0 - \vec{p}^2/(2m) + i\epsilon},
\]

where \( m \) is the mass of the corresponding particle. The dibaryon propagators are more complicated because of the coupling to two-baryon states. The bare dibaryon propagators are constant, \( i/\Delta \), but the full propagators are dressed by baryon loops to all orders, as illustrated for the \( \Lambda N \) state in Fig. [1]. All diagrams in Fig. [1] are of the same order because
FIG. 1: Dressing of the bare ΛN propagator. The dashed line indicates a Λ, the solid line a nucleon, the thick solid line a bare, and the double line a full dibaryon propagator.

\[ 2\pi \Delta/(Mg^2) = 1/a \sim Q. \]  

Summing the resulting geometric series leads to

\[ iD_j(p_0, \vec{p}) = \frac{2\pi}{\mu_Ag_j^2} \left( -i \right) \sqrt{-2\mu_A \left( p_0 - \vec{p}^2 / \left[ 2(M + M_\Lambda) \right] \right) - i\epsilon}, \]  \( (4) \)

where \( \mu_A = MM_\Lambda / (M + M_\Lambda) \) is the reduced mass of the ΛN system, \( j \in \{1, 3\} \), and \( \gamma_j \) is the corresponding pole momentum. Divergent loop integrals are regulated using dimensional regularization. The full propagator for the deuteron is

\[ iD_d(p_0, \vec{p}) = \frac{2\pi}{Mg_d^2} \left( -i \right) \sqrt{-Mp_0 + \vec{p}^2 / 4 - i\epsilon}, \]  \( (5) \)

where \( \gamma_d = 45.68 \text{ MeV} \) is the deuteron pole momentum. The values of \( \gamma_1 \) and \( \gamma_3 \) will be discussed later, together with the three-body results. The scattering amplitudes are obtained by attaching external baryon lines to the full propagators from Eqs. (4,5) \[12, 17\]. All dependence on the bare coupling constants \( g_d, g_1, \) and \( g_3 \) cancels in observable quantities.

III. THREE-BODY SYSTEM

We now apply the Lagrangian (4) to the ΛNN system in the \( J = 1/2 \) channel, which has the hypertriton as a three-body bound state. We do not include explicit Λ ↔ Σ conversion effects [27]. Such effects are accounted for by a ΛNN three-body force to be introduced in the following. Explicit Σ degrees of freedom have been integrated out from the effective Lagrangian. This procedure is justified because the characteristic momentum \( \gamma_3^A \ll \sqrt{M_A(M_\Sigma - M_\Lambda)} \approx 300 \text{ MeV} \)[10].

To leading order in \( QR \), only relative S-waves contribute. A ΛNN state with the quantum numbers of the hypertriton \( (J^P = \frac{1}{2}^+) \) can then be constructed in three different ways: (i) a \( ^3S_1 \) ΛN partial wave plus another nucleon, (ii) a \( ^1S_0 \) ΛN partial wave plus another nucleon, and (iii) a \( ^3S_1 \) NN partial wave (a deuteron) plus a Λ. The \( ^1S_0 \) NN partial wave does not contribute because of isospin. The leading correction comes from the two-body effective ranges which enter at \( O(QR) \). Contributions from higher partial waves and the tensor force are suppressed by at least two powers of \( QR \)[14, 29]; in potential models, their contribution is found to be small as well [30].

As a consequence, we need three three-body amplitudes, \( T_A, T_B, \) and \( T_C \), to describe the hypertriton. The amplitudes satisfy the coupled integral equations shown in Fig. 2. The amplitude \( T_A \) describes \( J = 1/2 \) Λd scattering, while the remaining two amplitudes have a

\[ \text{See Refs. [40, 41] for a discussion of a similar issue in the } N\Delta \text{ system.} \]
\[ T_A(k, p) = -\frac{1}{\pi} \int_0^{\Lambda_c} q^2 \, dq \{ L_B(p, q, E)T_B(k, q) - 3L_C(p, q, E)T_C(k, q) \} \]  
\[ T_B(k, p) = -\frac{4\pi\gamma_d}{M} L_I(p, k, E) - \frac{1}{\pi} \int_0^{\Lambda_c} q^2 \, dq L_A(p, q, E)T_A(k, q) \]  
\[ -\frac{1}{\pi} \int_0^{\Lambda_c} q^2 \, dq \{ L_B(p, q, E)T_B(k, q) + 3L_C(p, q, E)T_C(k, q) \} \]  
\[ T_C(k, p) = \frac{4\pi\gamma_d}{M} L_I(p, k, E) + \frac{1}{\pi} \int_0^{\Lambda_c} q^2 \, dq L_A(p, q, E)T_A(k, q) \]  
\[ -\frac{1}{\pi} \int_0^{\Lambda_c} q^2 \, dq \{ L_B(p, q, E)T_B(k, q) - L_C(p, q, E)T_C(k, q) \} , \]

where \( k \) \( (p) \) denote the incoming (outgoing) momenta in the center-of-mass frame and \( \Lambda_c \) is a momentum cutoff introduced to regulate the integral equations. In Eqs. (6), the outgoing momentum \( p \) is taken off-shell. The total energy is \( E = 3k^2/(4M) - \gamma_d^2/M \).

In the limit \( y = 0 \), the functions \( L_I \) and \( L_A \) are given by

\[ L_I(p, k, E) = \frac{1}{pk} \ln \left( \frac{k^2 + pk + p^2 - ME}{k^2 - pk + p^2 - ME} \right) , \]

\[ L_A(p, k, E) = \frac{1}{pk} \ln \left( \frac{k^2 + pk + p^2 - ME}{k^2 - pk + p^2 - ME} \right) . \]
\[ L_A(p, q, E) = \frac{1}{pq} \ln \left( \frac{q^2 + pq + p^2 - ME}{q^2 - pq + p^2 - ME} \right) \left[ -\gamma_d + \sqrt{\frac{3}{4}q^2 - ME - i\epsilon} \right]^{-1}. \]

\( L_B \) and \( L_C \) are identical to \( L_A \) with \( \gamma_d \) replaced by \( \gamma_3 (\gamma_1) \), respectively. The amplitude \( T_A(p, k) \) is normalized such that

\[ T_A(k, k) = \frac{3\pi}{M k \cot \delta - ik}, \quad (8) \]

with \( \delta \) the elastic \( \Lambda d \) scattering phase shift.

The integral equations (9) show the same feature as in the case of \( J = 1/2 \) neutron-deuteron-scattering or three spinless bosons [13, 17]. Without the cutoff \( \Lambda_c \), their solution would not be unique [42]. The origin of this nonuniqueness can be understood by solving Eq. (8) for asymptotically large off-shell momenta \( \Lambda_c \gg q \gg \gamma_d \sim k \). In the asymptotic limit, the inhomogeneous terms can be neglected and the equations become independent of \( k \),

\[ \begin{align*}
T_A(p) &= -\frac{2}{\sqrt{3\pi}} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) \left\{ \bar{T}_B(q) - 3\bar{T}_C(q) \right\} \\
T_B(p) &= -\frac{2}{\sqrt{3\pi}} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) \left\{ \bar{T}_A(q) + \bar{T}_B(q) + 3\bar{T}_C(q) \right\} \\
T_C(p) &= +\frac{2}{\sqrt{3\pi}} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) \left\{ \bar{T}_A(q) - \bar{T}_B(q) + \bar{T}_C(q) \right\},
\end{align*} \]

where we have defined \( \bar{T}_j(p) = p T_j(k \sim \gamma_d, p) \) for \( j \in \{A, B, C\} \). Note that the asymptotic equations do not determine the overall normalization of the amplitudes, which will be given from matching to the low-energy solution of the full equation. The following discussion, however, is independent of this normalization. We can decouple the equations by defining a new set of amplitudes \( T_1, T_2, \) and \( T_3 \) via

\[ \begin{pmatrix} \bar{T}_A \\ \bar{T}_B \\ \bar{T}_C \end{pmatrix} = \begin{pmatrix} 0 & -1 - \sqrt{5} & -1 + \sqrt{5} \\ 3 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}. \quad (10) \]

The new amplitudes \( T_1, T_2, \) and \( T_3 \) fulfill the equations

\[ \begin{align*}
T_1(p) &= -\frac{4}{\sqrt{3\pi}} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) T_1(q) \\
T_2(p) &= \frac{4}{\sqrt{3\pi}} \frac{1 - \sqrt{5}}{2} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) T_2(q) \\
T_3(p) &= \frac{4}{\sqrt{3\pi}} \frac{1 + \sqrt{5}}{2} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) T_3(q).
\end{align*} \]

All three equations are scale invariant and symmetric under the inversion \( q \rightarrow 1/q \). In Ref. [42], it was shown that an equation of the type

\[ f(p) = \frac{4\lambda}{\sqrt{3\pi}} \int_0^\infty dq \frac{q}{q} \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) f(q) \quad (12) \]
has a unique solution in the form of a power law if \( \lambda < \lambda_c = 3\sqrt{3}/(4\pi) \approx 0.4135 \) (see also Refs. [16, 17]). This condition is clearly satisfied for the first two equations. The equation for \( T_3 \), however, has \( \lambda = (1 + \sqrt{5})/2 \approx 1.62 \). As a consequence, there are two linearly independent complex solutions \( T_3(k, p) = p^{\pm i s_1} \), where \( s_1 = 1.35322 \). The relative phase between these two solutions is not determined by Eq. (11). For a finite cutoff \( \Lambda_c \) this phase is fixed but strongly depends on \( \Lambda_c \). This \( \Lambda_c \) dependence can absorbed by adding a one-parameter three-body force in the equation for \( T_3 \),

\[
T_3(p) = \frac{4}{\sqrt{3\pi}} \frac{1 + \sqrt{5}}{2} \int_0^{\Lambda_c} dq \frac{1}{q} \left\{ \ln \left( \frac{p^2 + pq + q^2}{p^2 - pq + q^2} \right) + 2H(\Lambda_c) \frac{pq}{\Lambda_c^2} \right\} T_3(q),
\]

where \( H(\Lambda_c) \) is a dimensionless function of the cutoff. In Refs. [16, 17], it was shown that if the three-body force runs with cutoff as

\[
H(\Lambda_c) = -\frac{\sin(s_1 \ln(\Lambda_c/\Lambda_\ast) - \arctg(1/s_1))}{\sin(s_1 \ln(\Lambda_c/\Lambda_\ast) + \arctg(1/s_1))},
\]

all low-energy observables are independent of \( \Lambda_c \). The renormalization group evolution of \( H(\Lambda_c) \) is governed by a limit cycle. In particular, if the cutoff is increased by a factor of \( \exp(\pi/s_1) \approx 10.2 \), \( H(\Lambda_c) \) returns to its original value. \( \Lambda_\ast \) is a dimensionful parameter that determines the asymptotic phase of the off-shell amplitude [16, 17]. It has to be fixed from a three-body observable. Once its value is known, all other observables can be predicted.\(^2\)

Note, however, that \( \Lambda_\ast \) is only determined up to factors of \( \exp(\pi/s_1) \).

Formally, the three-body force term is obtained by adding a nonderivative three-body contact term to the effective Lagrangian (2). The form of this term can be determined by starting from a general structure and matching the unknown coefficients to reproduce Eq. (13). The exact expression for this three-body term is given in Appendix B. It is interesting to note that the value \( s_1 \approx 1.35 \) differs from \( s_0 \approx 1.006 \) which characterizes the three-nucleon force in the triton channel [17]. This does not lead to a contradiction because the hypertriton channel (\( I = 0 \)) and the triton channel (\( I = 1/2 \)) have different isospin.

For the numerical studies, it is useful to write a renormalized integral equation by choosing a special cutoff,

\[
\Lambda_n = \Lambda_\ast \exp \left[ \frac{1}{s_1} \left( n\pi + \arctan \left( \frac{1}{s_1} \right) \right) \right],
\]

with \( n > 0 \) an integer, at which the the three-body term proportional to \( H \) in Eq. (13) vanishes [9]. This procedure does not remove the dependence on the three-body parameter \( \Lambda_\ast \) but rather moves its dependence into the cutoff. All observables are independent of \( n \), but it should be chosen such that \( \Lambda_n \gg \gamma_d \) and finite cutoff corrections of order \( \gamma_d/\Lambda_n \) can be neglected. In the following, we will be using Eqs. (6) with \( \Lambda_c = \Lambda_1 \). The \( \Lambda d \) scattering amplitude is obtained by numerically solving Eqs. (6) for the desired value of the total energy. The hypertriton binding energy can be extracted by solving Eqs. (6) for negative energies and locating the position of the bound state pole in \( T_A \).

\(^2\) One can either specify the dimensionless coupling \( H \) at a specific cutoff \( \Lambda_c \) or the dimensionful low-energy parameter \( \Lambda_\ast \). This is similar to dimensional transmutation in QCD.
IV. RESULTS

In order to apply the EFT to the $\Lambda NN$ system, we have to fix the parameters in Eqs. (6). The deuteron binding momentum $\gamma_d = 45.68$ MeV is known very well. As mentioned above, the low-energy parameters for the $\Lambda N$ S-waves cannot be extracted unambiguously from a model-independent analysis of low-energy scattering data \cite{32, 33}. It is known, however, that the $\Lambda N$ system is unbound, which requires $\gamma_1$ and $\gamma_3$ to be negative. The errors in the low-energy parameters extracted from extrapolations using hyperon-nucleon interaction potentials \cite{34, 35, 36, 37} are not well known and can only be estimated from the variation with different potentials. Therefore, we do not take the extracted parameters as input. Rather, we vary the $\Lambda N$ pole momenta over the range of validity of the EFT and study the implications on the hypertriton and $\Lambda d$ scattering.

In principle, we have three unknown parameters: $\gamma_1$, $\gamma_3$, and the three-body parameter $\Lambda_\star$. However, we will first address the simpler scenario $\gamma_1 = \gamma_3$, which follows from the $SU(6)$ spin-flavor symmetry of the quark model \cite{43, 44}. Such an $SU(6)$ symmetry also emerges in the large-$N_c$ limit of QCD \cite{45}. Symmetry violating corrections are suppressed by $1/N_c$ and the strange quark mass. Since both isospin and strangeness of the $^3S_1$ and $^1S_0$ $\Lambda N$ partial waves are the same, the difference between $\gamma_1$ and $\gamma_3$ is a pure spin splitting effect. In most hyperon-nucleon potentials this effect is of the order $30\%$ \cite{34, 35, 36, 37}. Neglecting this splitting leaves us with two parameters: $\gamma_i (= \gamma_1 = \gamma_3)$ and $\Lambda_\star$. In the following, we vary these two parameters and study the constraints that follow from a correct description of the hypertriton. We will also elucidate the consequences for $\Lambda d$ scattering.

First, we study the constraints on the parameters $\gamma_i$ and $\Lambda_\star$ by the requirement that the hypertriton binding energy be reproduced. An interesting question is whether there are certain values of $\gamma_i$ and $\Lambda_\star$ that are incompatible with the hypertriton data. Furthermore, it is possible that the approximation $\gamma_1 = \gamma_3$ proves too restrictive. In Fig. 3, we plot $\Lambda_\star$ as a function of $\gamma_i$ with the requirement that the hypertriton binding energy is reproduced. We vary $\gamma_i$ from 0 to $-2\gamma_d$, which is about where corrections of $O(QR)$ are expected to
become important. Positive values of \( \gamma_i \) are excluded because the \( \Lambda N \) system is not bound. All allowed values fall on a curve with a linear dependence on \( \gamma_i \),

\[
\Lambda_* / \gamma_d = 1.08 - 0.55 \gamma_i / \gamma_d .
\]  

For every value of \( \gamma_i \) in the interval a corresponding value of \( \Lambda_* \) that reproduces the hypertriton can be found. Clearly, the exact value of \( \gamma_i \) and therefore the \( \Lambda N \) scattering lengths cannot be determined from the hypertriton binding energy. This is in contrast to Ref. [30], where it was found that with standard hyperon-nucleon potentials, the correct binding energy for the hypertriton cannot be reproduced if the singlet \( \Lambda N \) scattering length differs more than 10% from 1.85 fm.

In Fig. 4, we compare the scattering phase shifts for \( \Lambda d \) scattering in the hypertriton channel for different points on the curve in Fig. 3. Fig. 4 shows three curves with \( \gamma_i \) chosen to be equal in magnitude to \( \gamma_d \) as well as a factor of two smaller and larger. The phase shifts agree very well for small momenta around threshold and begin to differ slightly close to the deuteron breakup threshold. This implies that low-energy \( \Lambda d \) scattering does not give additional constraints for the values of \( \gamma_i \) and \( \Lambda_* \). Consequently, neither the hypertriton binding energy nor the low-energy \( \Lambda d \) phase shifts can be used to fix the exact value of \( \gamma_i \). Such information can only come from low-energy \( \Lambda N \) scattering. On the other hand, this observation allows for a prediction of the low-energy \( \Lambda d \) phase shifts [given by the curves in Fig. 4] independent of the exact values for \( \gamma_i \) and \( \Lambda_* \). This is a direct consequence of the shallowness of the hypertriton. Extracting the low-energy doublet \( \Lambda d \) scattering parameters from the phase shifts, we find for the scattering length \( a_{\Lambda d} \) and effective range \( r_{\Lambda d} \):

\[
a_{\Lambda d} = (16.8^{+14}_{-2.4}) \text{ fm and } r_{\Lambda d} = (2.3 \pm 0.3) \text{ fm},
\]

where the errors are due to the uncertainty in the hypertriton binding energy. Our value for the scattering length is in good agreement with Ref. [30], while the effective range is about 1 fm smaller.
It is also interesting to look at the “Phillips line”, which describes the correlation between the $\Lambda d$ scattering length $a_{\Lambda d}$ and the hypertriton binding energy $B_3^\Lambda$. The Phillips line was observed in the neutron-deuteron system, where the results of calculations of the doublet $nd$ scattering length and the triton binding energy using different $NN$ potentials fall on a line when plotted in a plane [22]. A similar line exists for the hypertriton (cf. Fig. 7 in Ref. [31]). In the EFT framework, the Phillips line is parametrized by variations in the three-body parameter $\Lambda^*[16, 17]$. In Fig. 5, we show the Phillips line for the three values of $\gamma_i$ from Fig. 4. For small $B_3^\Lambda$, the different Phillips lines coincide exactly (the physical hypertriton corresponds to $MB_3^\Lambda \approx 1.06 \gamma_d^2$) and deviate from each other only at extremely large binding energies. However, the EFT is expected to break down when the three-body binding momentum $\gamma_3^\Lambda$ from Eq. (1) becomes of the order of the pion mass. This is the case for $MB_3^\Lambda \gtrsim 8 \gamma_d^2$. Consequently, the region where the Phillips lines differ is unphysical. For all practical purposes the Phillips line is independent of $\gamma_i$.

The characteristic momentum that sets the scale of corrections to the large-scattering-length approximation is given by the binding momentum of the hypertriton. Using the measured value for $B_3^\Lambda$ and Eq. (4), we obtain $\gamma_3^\Lambda \approx 0.3 \gamma_d$. This momentum can be used to estimate the contribution of the effective range term in the two-body amplitudes which is the leading correction at $O(QR)$. For our approximation to be accurate, we need $|\gamma| \gg |r(\gamma_3^\Lambda)^2/2|$ in each of the two-body partial waves. Since $(\gamma_3^\Lambda)^2/2 \approx 0.05 \gamma_d^2$ is extremely small, the effective range term gives only a small correction even when the scattering length and effective range are of the same order.

We have also studied the more general case $\gamma_1 \neq \gamma_3$. The conclusions in this case are unchanged: requiring the correct binding energy of the hypertriton does not constrain the specific values of $\gamma_1$ and $\gamma_3$ as long as they are within the range of validity of the EFT ($0 > \gamma_1, \gamma_3 \gtrsim -2 \gamma_d$). The results for $\Lambda d$ scattering confirm Fig. 4 and Eq. (17).

The insensitivity to the precise values of $\gamma_1$ and $\gamma_3$ and the small separation energy of

FIG. 5: Phillips line for different values of $\gamma_i$ (all quantities in units of $\gamma_d$).
the hypertriton into a deuteron and a $\Lambda$ ($B^\Lambda = 0.13$ MeV) can be exploited even further: it is possible to write an EFT with only deuterons and $\Lambda$’s as degrees of freedom \cite{46}. Effects from the deuteron structure and $\Lambda N$ interaction can be included perturbatively via local operators.

To summarize, we have discussed the hypertriton and doublet $\Lambda$-d scattering in an EFT for large two-body scattering lengths. Due to the small binding energy of the hypertriton, our analysis is expected to hold even if the $\Lambda N$ low-energy parameters are $O(1/m_\pi)$ as extracted from potential models. As in the triton case, consistent renormalization requires a one-parameter three-body force at leading order whose renormalization group evolution is governed by a limit cycle. The period of the limit cycle $\exp(\pi/s_1) \approx 10^{2.2}$ is a factor two smaller than for the triton. In contrast to standard potential models where the binding energy of the hypertriton is very sensitive to the singlet $\Lambda N$ scattering length \cite{30}, the three-body force can always compensate for changes in the low-energy $\Lambda N$ parameters. While the $\Lambda N$ low-energy parameters cannot be fixed by requiring the correct binding energy for the hypertriton, the shallowness of the hypertriton still allows for a unique prediction of low-energy doublet $\Lambda$-d scattering. Using the experimental binding energy of the hypertriton as input, we find for the scattering length $a_{\Lambda d} = (16.8 \pm 4.4) \text{ fm}$ and for the effective range $r_{\Lambda d} = (2.3 \pm 0.3) \text{ fm}$. In the future, it would be interesting to go to higher orders and apply EFT methods to a wider class of observables and other halo nuclei. Work in this direction is in progress \cite{47}.

Acknowledgements

I would like to thank E. Braaten, R.J. Furnstahl, U. van Kolck, T. Mehen, and A. Parreño for useful discussions. This research was supported by the U.S. National Science Foundation under Grant Nos. PHY-9800964 and PHY-00098645.

APPENDIX A: DERIVATION OF THREE-BODY EQUATION

In this Appendix, we give some details of the derivation of the integral equation (6) for the general case $M_\Lambda \neq M$. From the Lagrangian (2) and the Feynman diagrams in Fig. 2, we obtain

\begin{align}
\tau_{ij}^A(k, p)_{\alpha\beta} &= g_d d_3 \int \frac{d^3q}{(2\pi)^3} t^k_B(k, q)_{\alpha\gamma} \frac{(\sigma_j \sigma_k)_{\gamma\beta}}{E - \frac{q^2}{2M} - \frac{(q+p)^2}{2M}} \frac{D_3(E - \frac{q^2}{2M}, \bar{q})}{i\epsilon} \\
&+ g_d g_1 \int \frac{d^3q}{(2\pi)^3} t^i_C(k, q)_{\alpha\gamma} \frac{(\sigma_j)_{\gamma\beta}}{E - \frac{q^2}{2M} - \frac{q^2}{2M} - \frac{(q+p)^2}{2M} + i\epsilon} \\
\tau_{ij}^B(k, p)_{\alpha\beta} &= -2g_d d_3 \frac{(\sigma_j \sigma_i)_{\alpha\beta}}{E - \frac{k^2}{2M} - \frac{p^2}{2M} - \frac{(k+p)^2}{2M} + i\epsilon} \\
&+ 2g_d g_3 \int \frac{d^3q}{(2\pi)^3} t^i_A(k, q)_{\alpha\gamma} \frac{(\sigma_j \sigma_k)_{\gamma\beta}}{E - \frac{q^2}{2M} - \frac{q^2}{2M} - \frac{(q+p)^2}{2M} + i\epsilon} D_d(E - \frac{q^2}{2M}, \bar{q})
\end{align}

\footnote{See Refs. \cite{14, 17} for details on the evaluation of the Feynman diagrams.}
Defining the deuteron, we finally obtain the integral equations

\[ t^i_C(\vec{k}, \vec{p})_{\alpha \beta} = -2g_4 g_1 \frac{E - k^2/2M_x - p^2/2M_y - (k+\vec{p})^2/2M + i\epsilon}{(k^2/2M_x + p^2/2M_y + (k+\vec{p})^2/2M + i\epsilon)} \]

\[ +2g_4 g_1 \int \frac{d^3q}{(2\pi)^3} t^i_A(\vec{k}, \vec{q})_{\alpha \gamma} \frac{(\sigma_k)_{\gamma \beta} D_d(E - q^2/2M, \vec{q})}{E - q^2/2M - q^2/2M + i\epsilon} \]

\[ -g_4 g_3 \int \frac{d^3q}{(2\pi)^3} t^i_B(\vec{k}, \vec{q})_{\alpha \gamma} \frac{D_1(E - q^2/2M, \vec{q})}{E - p^2/2M - (q+\vec{p})^2/2M + i\epsilon} \]

\[ +g_4^2 \int \frac{d^3q}{(2\pi)^3} t^i_C(\vec{k}, \vec{q})_{\alpha \gamma} \frac{\delta_{\alpha \beta} D_1(E - q^2/2M, \vec{q})}{E - p^2/2M - (q+\vec{p})^2/2M + i\epsilon} \]

where \(i, j, k\) are vector indices and \(\alpha, \beta, \gamma\) are spinor indices in spin space. \(k(p)\) are the incoming (outgoing) momenta in the center-of-mass frame and \(E = k^2/(4M) - \gamma_s^2/M + k^2/(2M_A)\) is the total energy. The momentum \(p\) is taken off-shell. The isospin dependence of the amplitudes \(t_B\) and \(t_C\) has been absorbed into the amplitudes via the definition

\[ t_B^{ij}(\vec{k}, \vec{p})_{\alpha \beta} = t_B^{ij}(\vec{k}, \vec{p})_{\alpha \beta} \delta_{\alpha \beta}(\tau_2)_{be} \]  

(A4)

and similarly for \(t_C\), where \(e, b\) are isospinor indices. Next we project onto total angular momentum \(J = 1/2\) by using the definitions

\[ t_{A/B}(\vec{k}, \vec{p})_{\delta \alpha \beta} = \frac{1}{3} (\sigma_i)_{\alpha \alpha'} t_{A/B}^{ij}(\vec{k}, \vec{p})_{\alpha \beta} (\sigma_j)_{\beta \beta'} \]

(A5)

\[ t_C(\vec{k}, \vec{p})_{\delta \alpha \beta} = \frac{1}{3} (\sigma_i)_{\alpha \alpha'} t_C^{ij}(\vec{k}, \vec{p})_{\alpha \beta} \]

and projecting on relative S-waves. We also have to account for the wave function renormalization of the deuteron,

\[ Z_d^{-1} = i \frac{\partial}{\partial p_0} (iD_d(p_0, \vec{p}))^{-1} \bigg|_{p_0 = -\gamma_s^2/M, \vec{p} = 0} = \frac{M^2 q_d^3}{4\pi \gamma_s^3} \].  

(A6)

Defining

\[ T_A(k, p) = Z_d t_A(k, p) \]

\[ T_B(k, p) = Z_d g_4 t_B(k, p) \]

\[ T_C(k, p) = Z_d g_4 t_C(k, p) \]

we finally obtain the integral equations

\[ T_A(k, p) = -\frac{1}{\pi(1 - y)} \int_0^{\Lambda_c} q^2 dq \left\{ \tilde{L}_B(p, q, E)T_B(k, q) - 3\tilde{L}_C(p, q, E)T_C(k, q) \right\} \]

(A8)
\[ T_B(k, p) = -\frac{4\pi\gamma_d}{\Lambda_c^2} L_I(p, k, E) - \frac{1}{\pi} \int_{0}^{\Lambda_c} q^2 \, dq L_A(p, q, E) T_A(k, q) \]
\[ - \frac{1}{\pi(1 - y)} \int_{0}^{\Lambda_c} q^2 \, dq \{ L_B(p, q, E) T_B(k, q) + 3L_C(p, q, E) T_C(k, q) \} \]
\[ T_C(k, p) = \frac{4\pi\gamma_d}{\Lambda_c^2} L_I(p, k, E) + \frac{1}{\pi} \int_{0}^{\Lambda_c} q^2 \, dq L_A(p, q, E) T_A(k, q) \]
\[ - \frac{1}{\pi(1 - y)} \int_{0}^{\Lambda_c} q^2 \, dq \{ L_B(p, q, E) T_B(k, q) - L_C(p, q, E) T_C(k, q) \} \]

where \( \Lambda_c \) is a momentum cutoff and \( y = (M_\Lambda - M)/(M_\Lambda + M) \approx 0.1 \) characterizes the \( \Lambda - N \) mass difference. The functions \( L_I, L_A, L_B, \) and \( \tilde{L}_B \) are given by

\[ L_I(p, k, E) = \frac{1}{pk} \ln \left( \frac{k^2/(1 + y) + pk + p^2 - ME}{k^2/(1 + y) - pk + p^2 - ME} \right), \quad (A9) \]
\[ L_A(p, q, E) = \frac{1}{pq} \ln \left( \frac{q^2/(1 + y) + pq + p^2 - ME}{q^2/(1 + y) - pq + p^2 - ME} \right) \]
\[ \times \left[ -\gamma_d + \sqrt{q^2(3 - y)/(1 + y)/4 - ME - i\epsilon} \right]^{-1}, \]
\[ L_B(p, q, E) = \frac{1}{pq} \ln \left( \frac{q^2 + p^2 + pq(1 - y) - ME(1 + y)}{q^2 + p^2 - pq(1 - y) - ME(1 + y)} \right) \]
\[ \times \left[ -\gamma_3 + \sqrt{q^2(3 + 2y - y^2)/4 - ME(1 + y) - i\epsilon} \right]^{-1}, \]
\[ \tilde{L}_B(p, q, E) = \frac{1}{pq} \ln \left( \frac{q^2 + pq + p^2/(1 + y) - ME}{q^2 - pq + p^2/(1 + y) - ME} \right) \]
\[ \times \left[ -\gamma_3 + \sqrt{q^2(3 + 2y - y^2)/4 - ME(1 + y) - i\epsilon} \right]^{-1}. \]

\( L_C (L_C) \) is identical to \( L_B (\tilde{L}_B) \) with \( \gamma_3 \) replaced by \( \gamma_1 \).

**APPENDIX B: THREE-BODY FORCE TERM**

The three-body term proportional to \( H(\Lambda_c) \) in Eq. (13) can be obtained by adding a local, nonderivative three-body contact term to the Lagrangian (2). This term is most easily determined by starting from a general structure and matching the coefficients of the individual terms to reproduce the three-body term in Eq. (13). This leads to,

\[ \mathcal{L}_3 = \frac{MH(\Lambda_c)}{20\Lambda_c^2} \left\{ 16\sqrt{5} \, g_3^2 \left[ d_{\gamma}\Lambda_{\alpha}(\sigma_I\sigma_{I'})_{\alpha\beta}\Lambda_{\beta} d_{I'} \right] \right. \]
\[ - (5 + 3\sqrt{5}) \left[ g_3 \left[ (u_{I})_{\alpha}^{\beta} N_{\alpha}(\sigma_I\sigma_{I'})_{\alpha\beta} N_{\beta}^{*}(u_{I'})_{b} \right] - 3g_1 \left[ (u_{I})_{\alpha}^{\beta} N_{\alpha} N_{\beta}^{*}(u_{I})_{b} \right] \right] \]
\[ + 2(5 + \sqrt{5}) g_1 g_3 \left[ d_{\gamma}^* N_{\alpha}(\sigma_I\sigma_{I'})_{\alpha\beta} \Lambda_{\beta}^{*}(u_{I'})_{b} + \text{H.c.} \right] \]
\[ - (5 + 3\sqrt{5}) g_1 g_3 \left[ (u_{I})_{\alpha}^{\beta} N_{\alpha}(\sigma_I\sigma_{I'})_{\alpha\beta} N_{\beta}^{*}(u_{I})_{b} + \text{H.c.} \right] \]
\[ + 2(5 + \sqrt{5}) g_1 g_3 \left[ d_{\gamma}^* N_{\alpha}(\sigma_I\sigma_{I'})_{\alpha\beta} \Lambda_{\beta}^{*}(u_{I'})_{b} + \text{H.c.} \right]. \]
where $\alpha, \beta (a, b)$ are spinor (isospinor) indices and $l, l'$ are vector indices. Eq. (B1) represents a contact three-body force written in terms of dibaryon, nucleon, and $\Lambda$ fields but to leading order is equivalent to a three-baryon contact force. This becomes obvious if the dibaryon fields are integrated out by performing a Gaussian path integral [14, 16, 17].

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