Charging Symmetries and Linearizing Potentials for Gravity Models with Symplectic Symmetry

Oleg Kechkin

Institute of Nuclear Physics,
Moscow State University,
Moscow 119899, RUSSIA,
e-mail: kechkin@monet.npi.msu.su

Abstract

In this paper we continue to study a class of four–dimensional gravity models with \( n \) Abelian vector fields and \( Sp(2n)/U(n) \) coset of scalar fields. This class contains General Relativity \( (n = 0) \) and Einstein–Maxwell dilaton–axion theory \( (n = 1) \), which arises in the low–energy limit of heterotic string theory. We perform reduction of the model with arbitrary \( n \) to three dimensions and study the subgroup of non–gauge symmetries of the resulting theory. First, we find an explicit form these symmetries using Ernst matrix potential formulation. Second, we construct new matrix variable which linearly transforms under the action of the non–gauge transformations. Finally, we establish one general invariant of the non–gauge symmetry subgroup, which allow us to clarify this subgroup structure.
**Introduction**

Four–dimensional gravity models arise in various contexts as some generalizations of the General Relativity. They differs one of another by the form of the Lagrangian of matter fields, or by involving into consideration terms which are non–linear in respect to the curvature tensor. The most theoretically promissing models are obtained in the framework of grand unified theories, as it takes place for the models arising in the low energy limit of (super)string theories compactified to four dimensions [1].

A regular investigation of gravity models is closely related to study of their symmetries. The most progress in the symmetry analysis was achieved for the Einstein and Einstein–Maxwell theories (see [2] for review). These two theories are the simplest four–dimensional gravity models which become *sygma-models with symmetric target space* after reduction to three dimensions. The bosonic sector of heterotic string theory leads to another example of a theory of this class. The complete list of such theories was established in [3].

In the previous work [4] we considered the class of four–dimensional gravity models which become three–dimensional *sigma–models* with the target space possessing a symplectic symmetry. This class contains the Einstein–Maxwell theory with dilaton and axion fields (EMDA), which arises as some truncation of the low–energy heterotic string effective theory. Moreover, a general representative of this class is the natural matrix generalization of the EMDA theory. Actually, the general *symplectic gravity model* (SGM) is described by the action

\[ 4S = \int d^4x \sqrt{|g|} \left\{ -R + \text{Tr} \left[ \frac{1}{2} \left( \partial \mu p^{-1} \right)^2 - p F F^T + \frac{1}{3} (p H)^2 \right] \right\}, \tag{1.1} \]

where \( R \) is the Ricci scalar for the metric \( g_{\mu \nu} \) of the signature \(+,−,−,−;\)

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\[
H_{\mu \nu \lambda} = \partial_\mu B_{\nu \lambda} - \frac{1}{2} (A_{\mu} F_{\nu \lambda}^T + F_{\nu \lambda} A_{\mu}^T) + \text{cyclic}.
\]

Thus, in the special case when all variables are functions one obtains EMDA in the Einstein frame. In this simplest case \( p = e^{-2\phi} \) has the sense of string coupling (here \( \phi \) is the dilaton), and \( B_{\mu \nu} \) is the antisymmetric \((B_{\mu \nu} = -B_{\nu \mu})\) Kalb–Ramond field. In the general SGM case \( p \) and \( B_{\mu \nu} \) obtain two additional indeces (which are hidden in our notations) and become the symmetric matrices of the dimension \( n \), whereas \( A_\mu \) becomes the column of the same dimension. Thus, we consider four–dimensional models with \( n \) Abelian vector fields. The relation of SGM with \( n > 1 \) to any superstring theory is not clear yet, although symplectic symmetry transformations naturally arise in the supersymmetry context.
From the motion equations corresponding to the action (1.1) it follows the possibility to give the SGM an alternative form on–shell. Actually, using the pseudoscalar matrix variable $q$, defining by the relation

$$\nabla_\sigma q = \frac{1}{3} E_{\mu\nu\lambda\sigma} p H^{\mu\nu\lambda\rho} p,$$

one can rewrite the motion equations in the form which corresponds to the action

$$4S = \int d^4x \left | g \right|^\frac{1}{2} \left \{-R + Tr \left[\frac{1}{2} \left( (\nabla p)^2 + (\nabla q)^2\right) - pFF_T - q\tilde{F}F_T\right]\right \}. \quad (1.3)$$

In [4] it was shown that the general SGM allows the $Sp(2n)$ symmetry on shell. It was established that after the reduction to three dimensions this symmetry becomes off shell (a Lagrangian) symmetry. Moreover, the symmetry group enhancement takes place: the complete symmetry group of the resulting three–dimensional gravity model becomes isomorphic to $Sp(2(n+1))$ (we call it ‘U–duality’ because it appears by the same way as the three–dimensional U–duality of the effective superstring theories [5]).

Below we separate U–duality to the gauge and non–gauge sectors. Next, we fix the gauge (the trivial field asymptotics) and construct a representation of the theory which linearizes the non–gauge sector. Transformations of this sector form a charging symmetry (CS) subgroup. They generate charged solutions from neutral ones (see [6] for CS in the heterotic string (HS) theory).

This letter is organized as follows. In Sec. 2 we review the matrix Ernst potential (MEP) formulation for SGM reduced to three dimensions [4]. In Sec. 3 we obtain all the CS transformations in a finite form using the MEP formulation. After that in Sec. 4 we introduce new matrix variable and show that this variable transforms linearly under the action of the all CS transformations. We derive this linearizing potential (LP) for the General Relativity case using Ernst potential formulation (the details can be found in the Appendix A), and directly generalize the result to the general SGM case. After that we construct one charging symmetry invariant (CSI) which allow us to establish the CS group structure (some properties of its algebra are studied in the Appendix B).

We conclude this work with a discussion on the application of CS transformations to the problem of generation of SGM solutions from the GR ones.

**Matrix Ernst Potential**

Matrix Ernst potential contains all information about the dynamical variables of SGM reduced to three dimensions (for definiteness we consider stationary fields). These variables
consist of
a) scalar fields \( f = g_{00}, \ v = \sqrt{2}A_0 \) and \( p; \)
b) pseudoscalar field \( \kappa; \)
c) vector fields \( \omega_i = -f^{-1}g_{0i} \) and \( A_i; \)
d) tensor field (three–metric) \( h_{ij} = -f g_{ij} + f^2 \omega_i \omega_j. \) In three dimensions, both vector fields \( \vec{A} \) and \( \vec{\omega} \) can be dualized on–shell:

\[
\nabla \times \vec{A} = \frac{1}{\sqrt{2}} \left[ f^{-1} p^{-1} (\nabla u - \kappa \nabla v) + \vec{\omega} \times \nabla v \right],
\]

\[
\nabla \times \vec{\omega} = -f^{-2} (\nabla \chi + v^T \nabla u - u^T \nabla v),
\]

(2.1)

The resulting three–dimensional theory describes the scalars \( f, v \) and \( p \) and pseudoscalars \( \kappa, u \) and \( \chi \) coupled to the metric \( h_{ij}. \)

We define the matrix Ernst potential as follows:

\[
\mathcal{E} = \left( \begin{array}{cc} \mathcal{E} & \mathcal{F}^T \\ \mathcal{F} & -z \end{array} \right),
\]

(2.2)

where

\[
z = q + ip, \quad \mathcal{F} = u - zv, \quad \mathcal{E} = if - \chi + v^T \mathcal{F}.
\]

(2.3)

Thus, matrix Ernst potential is a complex symmetric \((n+1) \times (n+1)\) matrix. In \([4]\) it was shown that all the motion equations can be derived from the action

\[
\mathcal{S} = \int d^3x \sqrt{h} \left\{ -\frac{3}{2} R + \mathcal{L}_{SGM} \right\}
\]

\[
= \int d^3x \sqrt{h} \left\{ -\frac{3}{2} R + 2 \text{Tr} \left[ \nabla \mathcal{E} \left( \mathcal{E} - \mathcal{E}^T \right)^{-1} \nabla \mathcal{E} \left( \mathcal{E} - \mathcal{E}^T \right)^{-1} \right] \right\}.
\]

(2.4)

In the case of \( n = 0 \) our theory becomes the standard General Relativity, and Eq. (2.4) reproduces a conventional Ernst formulation of the stationary Einstein gravity \([9]\). If \( n = 1, \) one deals with the Einstein-Maxwell theory with dilaton and axion fields, whose matrix Ernst potential formulation was proposed in \([8]\). We consider these theories as two first representatives of the class of gravity models possessing the symplectic symmetry; all of them allow the matrix Ernst potential formulation.
Charging Symmetries

The complete Lagrangian symmetry group (U–duality) of the symplectic gravity model reduced to three dimensions is \( Sp(2(n + 1)) \). Its action on the matrix Ernst potential \( E \) had been established. There was shown that the discrete symmetry transformation

\[
\mathcal{E} \rightarrow -\mathcal{E}^{-1}.
\]  (3.1)

For the case of General Relativity this transformation was established. In the effective heterotic string theory context (here we have its truncation for \( n = 1 \)) this discrete symmetry is known as strong–weak coupling duality, or S–duality. Below we call it briefly ‘SWCD’. It is easy to prove that SWCD maps the \( E \)–shift symmetry

\[
E \rightarrow E + \lambda
\]  (3.2)

into the Ehlers transformation

\[
E^{-1} \rightarrow E^{-1} + \epsilon
\]  (3.3)

where \( \lambda \) and \( \epsilon \) are the real symmetric matrices (\( \lambda \to \epsilon \)). The remaining symmetry is the scaling transformation

\[
E \rightarrow S^T E S;
\]  (3.4)

it is invariant under the action of (3.1) (with \( S \to (S^T)^{-1} \)). Thus, the SGM U–duality consists of one doublet and one singlet of the strong–weak coupling duality.

Now let us consider the arbitrary constant potential \( E = E_\infty \), which can be interpreted as the asymptotical value of \( E \) near to the spatial infinity. Applying the shift symmetry with \( \lambda = -\text{Re} (E_\infty) \), we remove the real part of \( E \). Next, the scaling with

\[
S = \begin{pmatrix}
    f_\infty^{-\frac{1}{2}} & 0 \\
    -f_\infty^{-\frac{1}{2}} v_\infty & \pi_\infty^{-1}
\end{pmatrix},
\]  (3.5)

where \( p_\infty = \pi_\infty^T \sigma \pi_\infty \), leads \( E \) to its trivial form

\[
\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -\sigma \end{pmatrix}.
\]  (3.6)

Here

\[
\sigma = \begin{pmatrix}
    1_{n-k} & 0 \\
    0 & -1_k
\end{pmatrix}
\]
is the signature matrix for $p_\infty$, whereas $\pi_\infty$ is the corresponding tetrad matrix (for the models with the non–negative energy density $k = 0$). Thus, U–duality contains gauge transformations which can be used for the removing of the all field asymptotics. Conversely, one can apply these ‘dressing’ transformations to obtain arbitrary asymptotics for the originally asymptotically–free field configuration. In the rest part of the letter we fix the gauge and put $E_\infty = i\Sigma$.

Transformations preserving fixed asymptotics form a subgroup, which we call ‘charging’ because these transformations generate charged solutions from neutral ones. Scaling transformation contains a part of charging symmetry (CS) subgroup. Actually, scalings constrained by

$$S^T \Sigma S = \Sigma$$

(3.7)
do not change the chosen $E$–asymptotics. Thus, the group of charging symmetries contains the $SO(n - k, k + 1)$ subgroup of the scaling symmetry. We call the scalings which satisfy Eq. (3.7) normalized scaling transformation (NST).

One can see that the Ehlers transformation with the arbitrary non–trivial parameter $\epsilon$ moves the asymptotical value $E_\infty = i\Sigma$. However, some combination of the Ehlers transformation with special shift and scaling duality belongs to the charging symmetry subgroup. Actually, let us suppose that the Ehlers transformation with the arbitrary antisymmetric parameter $\epsilon$ is applied to the matrix $E_\infty = i\Sigma$. Then $E_\infty$ becomes changed. To remove the real part of new $E_\infty$, we perform the shift transformation with $\lambda = (i\Sigma - \epsilon)^{-1} \epsilon (i\Sigma + \epsilon)^{-1}$. After that, we transform the resulting $E_\infty$–value to $i\Sigma$ using the scaling (3.4) with $\tilde{S}$ satisfying the restriction

$$\tilde{S} \Sigma \tilde{S}^T = \Sigma + \epsilon \Sigma \epsilon.$$  

(3.8)
The resulting normalized Ehlers transformation (NET) has the form

$$E \rightarrow \tilde{S}^T \left[ \left( E^{-1} + \epsilon \right)^{-1} + (i\Sigma - \epsilon)^{-1} \epsilon (i\Sigma + \epsilon)^{-1} \right] \tilde{S}. $$

(3.9)

It is easy to see that NST forms the symmetry group of NET itself, because the condition (3.8) remains unchanged under the action of NST.

The number of dressing symmetries is equal to the number of SGM dynamical variables, i.e. to $(n + 1)(n + 2)$. NST gives $(n + 1)n/2$ independent parameters. Finally, NET is defined by the set of $(n + 1)(n + 2)/2$ parameters (we fix some $\tilde{S}$ satisfying Eq. (3.8)). Thus, all the established transformations from the CS subgroup, being independent, are constructed from $(n + 1)^2$ parameters. Then the common number of dressing and charging
transformations becomes equal to \((n + 1)(2n + 3)\), i.e. to the number of parameters of the whole U–duality group \(Sp(2(n + 1))\). From this it follows that we have found all the gauge (dressing) transformations as well as all the non–gauge (charging) symmetries. Thus, the CS subgroup consists of the normalized scaling (3.7) and Ehlers (3.9) transformations.

In the General Relativity case NST is absent (or coincides with identical one). Next, NET is related with the single parameter \(\epsilon\); from Eqs. (3.8) and (3.9) it follows that

\[
E \rightarrow \frac{E - \epsilon}{1 + \epsilon E},
\]

Thus, the charging symmetry subgroup of the stationary Einstein gravity coincides with the one–parametric normalized Ehlers transformation.

**Linearizing Potential**

One can see that the normalized scaling acts as linear transformation on the matrix Ernst potential \(E\), whereas the normalized Ehlers transformation is some fractional–linear map. In this section we establish new matrix potential \(Z\) which linearly transforms under the action of the all CS transformations, i.e. \(Z\) is a CS linearizing potential. Our plan is following: we calculate \(Z = Z(E)\) for the General Relativity case \((n=0)\) and extend the result to the general SGM case (arbitrary \(n\)).

In the Appendix A one can find the details of the LP derivation for the stationary Einstein gravity. The result is:

\[
Z = 2(E + i)^{-1} + i.
\]

Thus, \(Z_\infty = Z(E_\infty) = Z(i) = 0\), i.e. the near to spatial infinity asymptotics are trivial. The relation (4.1) admits a straightforward generalization to the case of matrix variables. Actually, the simple substitution \(i \rightarrow i\Sigma\) preserves triviality of \(Z_\infty\); using it, we obtain:

\[
Z = 2(E + i\Sigma)^{-1} + i\Sigma.
\]

To verify that the fractional–linear function (4.2) defines the SGM linearizing potential, one must rewrite all the CS transformations in terms of the \(Z\)–representation. For NST one immediately obtains:

\[
Z \rightarrow S^{-1}Z(S^T)^{-1}. \quad \text{(NST)}
\]
After some amount of algebra based on the use of the relation (3.8), one establishes that NET also has a linear form:

$$
Z \rightarrow \tilde{S}^{-1} (1 - i\epsilon \Sigma) Z (1 - i\epsilon\hat{\Sigma}) (\tilde{S}^T)^{-1}. \quad \text{(NET)}
$$

(4.4)

Thus, the introduced matrix $Z$ actually is a linearizing potential of the charging symmetry subgroup of the stationary symplectic gravity model with arbitrary $n$.

To analyse the CS group structure we will need in one general CS invariant. This invariant can be ‘extracted’ from the Lagrangian $L_{SGM}$ (see Eq. (2.4)). To do this, let us consider asymptotically trivial fields with the non–zero Coulomb terms:

$$
Z = \frac{Q}{r} + o\left(\frac{1}{r}\right), \quad \text{(4.5)}
$$

where $Q$ is a charge matrix and $r$ tends to the spatial infinity. Then, from Eq. (4.2) we obtain that

$$
L_{SGM} = \frac{1}{2r^4}\text{Tr}\left\{\overline{Q}\Sigma Q\Sigma\right\} + o\left(\frac{1}{r^4}\right). \quad \text{(4.6)}
$$

The quadratic charge combination

$$
I(Q) = \text{Tr}\left\{\overline{Q}\Sigma Q\Sigma\right\} \quad \text{(4.7)}
$$

is a CS invariant, because $L_{SGM}$ is the CS invariant and all its terms related to the $1/r$ power expansion are also CS invariants. Then, from Eq. (4.5) it follows that the charge and linearizing potential matrices have the same transformation properties. Thus, the function

$$
I(Z) = \text{Tr}\left\{\overline{Z}\Sigma Z\Sigma\right\} \quad \text{(4.8)}
$$

must be a charging symmetry invariant (see [10] for EMDA).

One can see that the charging symmetry transformations are of the form

$$
Z \rightarrow g_i^T Z g_i, \quad \text{(4.9)}
$$

where $i = \text{NST}$ and $\text{NET}$. An explicit form of the matrices $g_{\text{NST}}$ and $g_{\text{NET}}$ can be obtained from Eqs. (4.3) and (4.4). These are:

$$
g_{\text{NST}} = (S^T)^{-1}, \quad g_{\text{NET}} = (1 - i\Sigma\epsilon) (\tilde{S}^T)^{-1},
$$

(4.10)

where $S$ and $\tilde{S}$ satisfy Eqs. (3.7) and (3.8) correspondingly.
To preserve $I(Z)$ both transformations must satisfy the $U(n - k, k + 1)$ group relation

$$G_i^+ \Sigma G_i = \Sigma.$$  \hspace{1cm} (4.11)

This really takes place; thus $G_i \in U(n - k, k + 1)$. Now let us note that the common number of independent parameters of NST and NET is $(n + 1)^2$, i.e. the same one as for the group $U(n - k, k + 1)$. Moreover, if we consider the infinitesimal transformations $\Gamma_i$ ($G_i = e^{\Gamma_i}$ and compute $\Gamma = \sum \Gamma_i$, we obtain

$$\Gamma = -i \Sigma \epsilon - \sigma^T,$$  \hspace{1cm} (4.12)

where $\sigma$ denotes the NST generator ($\mathcal{S} = e^{\sigma}$). This matrix is a general solution of the equation $(\Gamma)^+ = -\Sigma \Gamma^{\text{right}} \Sigma$, which defines the $u(n - k, k + 1)$ algebra (its structure is discussed in the Appendix B). From this we conclude that the general CS transformation matrix is the general matrix of the group $U(n - k, k + 1)$. It can be constructed as the product of NST and NET matrices multiplied in an arbitrary order.

Thus, we have established the following simplest form of the charging symmetry transformations:

$$Z \rightarrow G^T Z G, \quad \text{where} \quad G \in U(n - k, k + 1).$$  \hspace{1cm} (4.13)

It is important to note that the transformation of the charge matrix $Q$ can be obtained from Eq. (4.13) using the replacement $Z \rightarrow Q$.

The strong–weak coupling duality transformation in terms of the linearizing potential $Z$ takes the form:

$$Z \rightarrow -\Sigma Z \Sigma.$$  \hspace{1cm} (4.14)

From Eqs. (3.1) and (4.2) it follows that SWCD acts on $G$ in the following way:

$$G \rightarrow \Sigma G \Sigma \Sigma$$  \hspace{1cm} (4.15)

One can see that this map preserves the group relation (4.11). This means, that the whole CS subgroup is also SWCD invariant. Taking into account that the dressing symmetries do not possess this property we obtain the following alternative definition of the CS subgroup: the charging symmetry subgroup is the maximal subgroup of the $U$–duality which is invariant under the action of the SWCD transformation.
Concluding Remarks

Thus, we have extracted all the charging symmetry transformations from the general Lagrangian symmetry group of the general four–dimensional symplectic gravity model reduced to three dimensions. We have established matrix linearizing potential which undergo linear homogeneous transformations when the charging symmetries act. We have constructed one general invariant of these symmetries, quadratic on the linearizing potentials, and studied the charging symmetry group structure.

Found representation can be applied to the problem of generation of SGM solutions from the GR ones. It is obvious that the LP formalism is the most convenient for the generation of solutions trivial at the spatial infinity (in the three–dimensional sense, see Eq. (4.5); in four dimensions these solutions allow, for example, the NUT charge). Actually, starting from the arbitrary GR solution, rewritten in the LP form, and applying the SGM charging symmetry transformations accordingly Eq. (4.13), one obtains a class of SGM solutions with the manifest $U(n–k,k+1)$ symmetry. Next, the LP formulation gives the most natural tool for the construction of the extremal (with $h_{ij} = \delta_{ij}$) Israel–Wilson–Perj’es–like solutions [11]; these solutions form the CS–invariant class. There are some other directions in applying of the LP formulation; all of them are based on the fact of linearization of the non–gauge symmetries of the stationary symplectic gravity models.

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Appendix A: LP for GR

In this Appendix we derive linearizing potential of the charging symmetry subgroup for the stationary General Relativity.

The generator of the CS subgroup can be obtained from Eq. (3.10); the result is:

$$\mathcal{X} = - \left( E^2 + 1 \right) \partial_E$$

(A.1)

(we write down only the holomorphic part of generators).

The CS transformation (NET) can be realized linearly in the following way. Let $Z$ be the complex variable whose finite transformation has a transparent $U(1)$ form:

$$Z \rightarrow e^{2i\alpha} Z,$$

(A.2)
where \( \alpha \) is a real parameter. The corresponding generator is:

\[
\mathcal{Y} = 2iZ \partial_Z. \tag{A.3}
\]

Now we identify the generators \( \mathcal{X} \) and \( \mathcal{Y} \); they are equal up to a real constant factor:

\[
\mathcal{X} = c \mathcal{Y}. \tag{A.4}
\]

Supposing that the functional relation \( E = E(Z) \) exists, we obtain the differential equation \( 2cZ \dot{E}_Z = i(E^2 + 1) \) of the first order which defines it. Solving this equation, we obtain:

\[
Z = c' \left( \frac{E + i}{E - i} \right)^c, \tag{A.5}
\]

where \( c' \) is an arbitrary complex constant. We choose \( c = -1 \) in order to reach the simplest possible fractional–linear form of the constructed solution (it is important for the matrix generalization of Eq. (A.5)) and the relation \( Z(i) = 0 \). Thus, we choose LP to be trivial at the spatial infinity. Next, the concrete value of \( c' \) is not important, and we put \( c' = i \) for the simplest form of the result. Finally, the relation between the linearizing and Ernst potential takes the form:

\[
Z = \frac{2}{E + i} + i. \tag{A.6}
\]

It is easy to see that if \( Z \) linearizes some transformation, then \( c'Z^{-c} \) also will be a linearizing potential. This explains appearing of two arbitrary constants in Eq. (A.5).

## Appendix B: CS Algebra for SGM

In this Appendix we compute the commutation relations for the charging symmetry algebra of the symplectic gravity model with arbitrary \( n \).

The generators were constructed in the infinitesimal form; this means that \( \epsilon = \xi e \) and \( \sigma = \xi s \), where \( \xi \) is the infinithesimal parameter. Here the matrices \( e \) and \( s \) are finite (\( e^T = e \), \( s^T = -\Sigma s \Sigma \)); they define the finite form of the NST and NET generators:

\[
\Gamma_{\text{NST}}(s) = -s^T, \quad \Gamma_{\text{NET}}(e) = -i \Sigma e. \tag{B.1}
\]

The computation of the commutation relations gives:

\[
\begin{align*}
\left[ \Gamma_{\text{NST}}(s'), \Gamma_{\text{NST}}(s'') \right] &= \Gamma_{\text{NST}} \left( \left[ s', s'' \right] \right), \\
\left[ \Gamma_{\text{NET}}(e'), \Gamma_{\text{NET}}(e'') \right] &= -\Gamma_{\text{NST}} \left( \left[ \Sigma e', \Sigma e'' \right] \right), \\
\left[ \Gamma_{\text{NET}}(e), \Gamma_{\text{NST}}(s) \right] &= -\Gamma_{\text{NET}} \left( s e + e s^T \right),
\end{align*} \tag{B.2}
\]

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One can see, that only NST generators form a subalgebra, and the minimal algebra including NET is equal to the full CS algebra.

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