Quantifying Rational Belief*

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Abstract

Some criticisms that have been raised against the Cox approach to probability theory are addressed. Should we use a single real number to measure a degree of rational belief? Can beliefs be compared? Are the Cox axioms obvious? Are there counterexamples to Cox? Rather than justifying Cox’s choice of axioms we follow a different path and derive the sum and product rules of probability theory as the unique (up to regraduations) consistent representations of the Boolean AND and OR operations.

1 Introduction

The objective of the Cox approach to probability theory is to develop tools for reasoning under conditions of uncertainty [1][2][3]. The method proposed by Cox amounts to ranking statements according to the extent to which one is rationally justified in believing them. The ranking is implemented by associating to each statement a real number meant to represent a degree of rational belief. It is perhaps surprising that a bare minimum of rationality, namely, a requirement of consistency, is sufficient to yield a precise quantitative formalism. Cox’s remarkable theorem states that ranking according to degrees of rational belief is equivalent to following the rules of probability theory.

The importance of Cox’s approach is, first, that it allows one to represent a partial state of knowledge as a consistent web of interconnected beliefs and, second, that it solves the long standing problem of interpretation: degrees of belief are to be manipulated according to the mathematical rules of probability theory and therefore no mistakes will ever be made if we call them “probabilities”. These are not modest claims and it is only appropriate that the Cox approach be subjected to a severe critical scrutiny. The purpose of this paper is to address some of the criticisms that have been raised over the years.

A thoughtful overview and general criticism of induction theories appears in the work of J. D. Norton [4]. He points out that in order to accept the Cox

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argument one must be convinced that beliefs come in numerical degrees, that beliefs can be compared, and furthermore, that they must be transitive (if \( a \) is preferred over \( b \), and \( b \) over \( c \), then \( a \) is preferred over \( c \)). Not obvious at all, he claims, a single number may not be sufficient to capture the richness of our beliefs which could very well be intransitive or even incommensurate. And the doubts do not end there.

Cox assumes as one of his axioms that the degree of belief in a proposition \( a \) assuming that \( b \) is true, which we write as \([a|b]\), is rigidly related to the degree corresponding to its negation, \([-a|b]\), through some definite but initially unspecified function \( f \),

\[
[-a|b] = f ([a|b]) \quad .
\]

To some of us this is intuitively reasonable: the more one believes in \( a|b \), the less one believes in \(-a|b\). Not obvious, writes Norton, “if one does not prejudge what ‘belief’ must be, the assumption of this specific functional dependency is alien and arbitrary”.

A second Cox axiom is that the degree of belief of “\( a \) AND \( b \) assuming \( c \),” written as \([ab|c]\), must depend on \([a|c]\) and \([b|ac]\),

\[
[ab|c] = g ([a|c], [b|ac]) \quad .
\]

This is also very reasonable. When asked to check whether “\( a \) AND \( b \)” is true, we first look at \( a \); if \( a \) turns out to be false the conjunction is false and we need not bother with \( b \); therefore \([ab|c]\) must depend on \([a|c]\). If \( a \) turns out to be true we need to take a further look at \( b \); therefore \([ab|c]\) must also depend on \([b|ac]\). Norton objects that it may be reasonable, but it is also an assumption “likely to be uncontroversial only for someone who already believes that plausibilities are probabilities and has tacitly in mind that we must eventually recover the product rule”.

Strictly \([ab|c]\) could in principle depend on all four quantities \([a|c]\), \([b|c]\), \([a|bc]\) and \([b|ac]\), an objection that has a long history. It was partially addressed by Tribus [5] and then by Smith and Erickson [6] but, unfortunately, as has been convincingly pointed out by Garrett [7], their arguments are not completely satisfactory.

Yet another objection has been raised by Halpern [8]. He shows that in finite domains it is possible to satisfy the consistency constraint that follows from the associativity of the Boolean AND, \((ab)c = a(bc)\), without requiring that the function \( g \) in eq. (2) be itself associative. This allows him to construct counterexamples to Cox’s theorem.

In section 2 we discuss degrees of rational belief and why we are justified in representing them by real numbers. Then to (partially) counter the objection that the Cox axioms are intuitive only to those who are already convinced of the results we reformulate the Cox theory in terms of axioms that differ from the usual ones. The idea is to construct a representation of the Boolean AND and OR by focusing on their associative and distributive properties rather than on the operation of negation. We then argue that it is the nature of our goal — to construct an inductive logic of general applicability — that allows us to escape
Halpern’s criticism, and also to give a proper treatment to the Tribus-Smith-
Erickson objection. In section 3 an associativity constraint is used to derive the
sum rule rather than the product rule as Cox had originally done, and in section
4 we focus on the distributive property of AND over OR to obtain the product
rule. The negation function \( f \) in eq. (1) and the functional equation associated
with it are completely avoided.

Our subject is degrees of rational belief but the algebraic approach followed
here can be pursued in its own right irrespective of any interpretation. It was
used in [9] to derive the manipulation rules for complex numbers interpreted
as quantum mechanical amplitudes. It was also used by K. Knuth [10] in the
purely mathematical problem of assigning real numbers (valuations) on general
distributive lattices.

2 Degrees of rational belief

Different individuals may hold different beliefs and it is certainly important to
figure out what those beliefs might be — perhaps by observing their gambling
behavior — but this is not our present concern. Our objective is neither to assess
nor to describe the subjective beliefs of any particular individual. Instead we
deal with the altogether different but very common problem that arises when
we are confused and we want some guidance about what we are supposed
to believe. Our concern here is not so much with beliefs as they actually are, but
rather, with beliefs as they ought to be.

Rational beliefs are constrained beliefs. Indeed, the essence of rationality
lies precisely in the existence of some constraints. The problem, of course, is
to figure out what those constraints might be. We need to identify normative
criteria of rationality. It must be stressed that the beliefs discussed here are
meant to be those held by an idealized rational individual who is not subject
to practical human limitations. We are concerned with those ideal standards of
rationality that we ought to strive to attain at least when discussing scientific
matters.

Here is our first criterion of rationality: whatever guidelines we pick they
must be of general applicability—otherwise they fail when most needed, namely,
when not much is known about a problem. Different rational individuals can
reason about different topics, or about the same subject but on the basis of
different information, and therefore they could hold different beliefs, but they
must agree to follow the same rules.

As a second criterion of (extremely idealized) rationality we require theories
that allow quantitative reasoning. The obvious question concerns the type of
quantity that will represent the intensity of beliefs. Discrete categorical variables
are not adequate for a theory of general applicability; we need a much more
refined scheme.

Do we believe statement \( a \) more or less than statement \( b \)? Are we even
justified in comparing statements \( a \) and \( b \) ? The problem with statements is not
that they cannot be compared but rather that the comparison can be carried out
in too many different ways. We can classify statements according to the degree we believe they are true, their plausibility; or according to the degree that we desire them to be true, their utility; or according to the degree that they happen to bear on a particular issue at hand, their relevance. We can even compare propositions with respect to the minimal number of bits that are required to state them, the description length. The detailed nature of our relations to statements is too complex to be captured by a single real number. What we claim is that a single real number is sufficient to measure one specific feature, the sheer intensity of rational belief. This should not be too controversial because it amounts to a tautology: an “intensity” is precisely the type of quantity that admits no more qualifications than that of being more intense or less intense; it is captured by a single real number.

However, some preconception about our subject is unavoidable; we need some rough notion that a belief is not the same thing as a desire. But how can we know that we have captured pure belief and not belief contaminated with some hidden desire? Strictly we can’t. We hope that our mathematical description captures a sufficiently purified notion of rational belief, and we can claim success only to the extent that the formalism proves to be useful. Since some preconceived notions are needed, here is one: we take it to be a defining feature of the intensity of rational beliefs that if \( a \) is more believable than \( b \), and \( b \) more than \( c \), then \( a \) is more believable than \( c \). Such transitive rankings can be implemented using real numbers we are again led to claim that degrees of rational belief can be represented by real numbers.

The notation we use is fairly standard: given any two statements \( a \) and \( b \) the disjunction “\( a \) OR \( b \)” and the conjunction “\( a \) AND \( b \)” are denoted respectively by \( a \lor b \) and \( ab \). Typically we want to quantify our beliefs in \( a \lor b \) and in \( ab \) in the context of some background information \( c \), which we write as \( ab|c \). The real number that represents the degree of belief in \( ab|c \) will initially be denoted by \( [ab|c] \). Degrees of rational belief will range from the extreme of total certainty, \( [a|a] = v_T \), to total disbelief, \( [\sim a|a] = v_F \). Note that the transitivity of the ranking scheme implies that there is a single value \( v_F \) and a single \( v_T \).

Here is a second preconceived notion: in order to be rational our beliefs in \( a \lor b \) and \( ab \) must be somehow related to our separate beliefs in \( a \) and \( b \). Since the goal is to design a quantitative theory, we require that these relations be represented by some functions \( F \) and \( G \),

\[
[a \lor b|c] = F([a|c], [b|c], [ab|c], [b|ac]) \quad (3)
\]

and

\[
[ab|c] = G([a|c], [b|c], [ab|c], [b|ac]). \quad (4)
\]

Note the qualitative nature of this assumption: what is being asserted is the existence of some unspecified functions \( F \) and \( G \) and not their specific functional forms. The same \( F \) and \( G \) are meant to apply to all propositions; what is being designed is a single inductive scheme of universal applicability. Note further that unlike eq. (2) the arguments of \( F \) and \( G \) include all four possible degrees of belief in \( a \) and \( b \) in the context of \( c \) and not any potentially questionable subset.
Since it is quite inconceivable that a smooth change in, say, \([a|c]\) could lead to anything but a smooth change in \([a \lor b|c]\) and \([ab|c]\), we will assume that the functions \(F\) and \(G\) are sufficiently smooth and well behaved. Indeed, should it turn out that \(F\) and \(G\) require kinks or discontinuities we would probably feel justified in throwing the whole scheme away. (However, smoothness might not be necessary. See \([11]\).)

Our method is one of eliminative induction: now that we have identified a sufficiently broad class of theories—quantitative theories of universal applicability, with degrees of belief represented by real numbers and the operations of conjunction and disjunction represented by functions—we can start weeding the unacceptable ones out.

3 The sum rule

We start with the function \(F\) that represents \(\lor\). The space of functions of four arguments is very large. To narrow down the field we initially restrict ourselves to propositions \(a\) and \(b\) that are mutually exclusive in the context of \(d\). Thus,

\[
[a \lor b|d] = F([a|d], [b|d], v_F, v_F),
\]

which effectively restricts \(F\) to a function of only two arguments.

The associativity constraint:

We require that the assignment of degrees of belief be consistent—if a degree of belief can be computed in two different ways the two ways must agree—how else could we claim to be rational? All functions \(F\) that fail to satisfy this constraint must be discarded.

Consider any three mutually exclusive statements \(a, b,\) and \(c\) in the context of a fourth \(d\). The consistency constraint that follows from the associativity of the Boolean \(\lor\), \((a \lor b) \lor c = a \lor (b \lor c)\), is remarkably constraining. It essentially determines the function \(F\). Start from

\[
[a \lor b \lor c|d] = F([a \lor b|d], [c|d]) = F([a|d], [b \lor c|d])
\]

and using \(F\) once again for \([a \lor b|d]\) and for \([b \lor c|d]\), we get

\[
F\{F([a|d], [b|d]), [c|d]\} = F\{[a|d], F([b|d], [c|d])\}
\]

If we call \([a|d] = x, [b|d] = y,\) and \([c|d] = z\), then

\[
F\{F(x, y), z\} = F\{x, F(y, z)\}
\]

The function \(F\) must obey \([8]\) for arbitrary choices of the propositions \(a, b, c,\) and \(d\).

Halpern has raised the following objection \([8]\). Suppose we have a belief function \([a|d]\) that associates a real number to each pair of propositions \(a\) and \(d\). He observes that if the total number of such propositions is finite (a discrete universe of discourse) then the triples \((x, y, z)\) to be used in \([8]\) do not form a
dense set and therefore we are not allowed to conclude that the function \( F \) must itself be associative for arbitrary values of \( x, y, \) and \( z \). Thus, in finite universes of discourse it is possible to design models of inference that are consistent without being equivalent to probability theory, and Halpern constructs an explicit example.

The reply to Halpern’s objection is not to be found in any flaw in his mathematics. We must rather focus on the larger project at hand. We are concerned with designing a theory of inference of \textit{universal} applicability, a single scheme to be used by all rational individuals irrespective of their state of knowledge or of subject matter. One individual might assign a plausibility \(|a|d\) = \( x \) while another, who is in possession of different information, might assign \(|a|d\)' = \( x' \), while a third would assign \( x'' \) and so on. Thus the values \( x \) form a dense set, not because the allowed propositions \( a, b, \ldots \) are themselves dense, but rather because the belief functions \(|·|·\), \(|·|·\)'.. are dense. Furthermore, the same general purpose scheme must be applicable to arbitrary subject matter, not just to one particular discrete set, but also to continuous sets of propositions. We conclude that in order to be of universal applicability the function \( F \) must indeed be associative and satisfy (8) for arbitrary values of \((x, y, z)\).

\textbf{The general solution and its regraduation:}

By straightforward substitution one can check that eq.(8) is satisfied if

\[
F (x, y) = \phi^{-1} (\phi (x) + \phi (y)) ,
\]

where \( \phi \) is an arbitrary invertible function. It has been shown that this is also the \textit{general solution} \([11]\). Given \( \phi \) one can calculate \( F \) and, conversely, given \( F \) one can calculate the corresponding \( \phi \). Eq.(9) can be rewritten as

\[
\phi (F (x, y)) = \phi (x) + \phi (y) \quad \text{or} \quad \phi ([a \lor b]\!d) = \phi ([a]d) + \phi ([b]d) .
\]

This last form is the pivotal point of the whole argument: it shows that instead of representing degrees of belief along the scale provided by the numbers \(|a|d|\), we can equally well regraduate to a new scale given by \( \xi (a) = \phi ([a]|d]) \). The original and the regraduated scales are equivalent because being invertible the function \( \phi \) is monotonic and preserves the ranking of propositions. However, the regraduated scale is much more convenient because the OR operation is now represented by a much simpler sum rule,

\[
\xi (a \lor b|d) = \xi (a|d) + \xi (b|d) .
\]

The regraduated \( \xi_F = \phi(\nu_F) \) is easy to evaluate. Setting \( d = \tilde{a} \) in eq.(11) gives \( \xi (a \lor b|\tilde{a}) = \xi (a|\tilde{a}) + \xi (b|\tilde{a}) \). Since \( a \lor b|\tilde{a} \) is true if and only if \( b|\tilde{a} \) is true, the corresponding degrees of belief must coincide, \( \xi (a \lor b|\tilde{a}) = \xi (b|\tilde{a}) \), and therefore \( \xi (a|\tilde{a}) = \xi_F = 0 \).

\textbf{The general sum rule:}

The restriction to mutually exclusive propositions in the sum rule eq.(11) can easily be lifted. Any proposition \( a \) can be written as the disjunction of two
mutually exclusive ones, \(a = (ab) \lor (\bar{a}b)\) and similarly \(b = (ab) \lor (\bar{a}b)\). Therefore for any two arbitrary propositions \(a\) and \(b\) we have

\[
a \lor b = (ab) \lor (\bar{a}b)
\]

(12)

Since each of the terms on the right are mutually exclusive the sum rule (11) applies,

\[
\xi(a \lor b|d) = \xi(ab|d) + \xi(\bar{a}b|d) + [\xi(ab|d) - \xi(ab|d)] \\
= \xi(ab \lor \bar{a}b|d) + \xi(a \lor \bar{a}b|d) - \xi(ab|d)
\]

(13)

which leads to the general sum rule,

\[
\xi(a \lor b|d) = \xi(a|d) + \xi(b|d) - \xi(ab|d)
\]

(14)

4 The product rule

Next we consider the function \(G\) that represents AND. The space of functions of four arguments is very large so we first narrow it down to just two. Then, we impose a consistency constraint that follows from the distributive properties of the Boolean AND and OR. A trivial regradation yields the product rule of probability theory.

From four arguments down to two:

We will separately consider special cases where the function \(G\) depends on only two arguments, then three, and finally all four arguments. Using commutivity, \(ab = ba\), the possibilities are seven:

\[
\begin{align*}
\xi(ab|c) &= G^{(1)}[\xi(a|c), \xi(b|c)] \\
\xi(ab|c) &= G^{(2)}[\xi(a|c), \xi(a|bc)] \\
\xi(ab|c) &= G^{(3)}[\xi(a|c), \xi(b|ac)] \\
\xi(ab|c) &= G^{(4)}[\xi(a|bc), \xi(b|ac)] \\
\xi(ab|c) &= G^{(5)}[\xi(a|c), \xi(b|c), \xi(a|bc)] \\
\xi(ab|c) &= G^{(6)}[\xi(a|c), \xi(a|bc), \xi(b|ac)] \\
\xi(ab|c) &= G^{(7)}[\xi(a|c), \xi(b|c), \xi(a|bc), \xi(b|ac)]
\end{align*}
\]

(15) (16) (17) (18) (19) (20) (21)

Since the method aims at general applicability the arguments of \(G^{(1)} \ldots G^{(7)}\) can be varied independently.

First some notation: complete certainty is denoted \(\xi_T\), while complete disbelief is \(\xi_F = 0\). Derivatives are denoted with a subscript: the derivative of \(G^{(3)}(x, y)\) with respect to its second argument \(y\) is \(G^{(3)}_2(x, y)\).

**Type 1**: \(\xi(ab|c) = G^{(1)}[\xi(a|c), \xi(b|c)]\). The function \(G^{(1)}\) is unsatisfactory because it does not take possible correlations between \(a\) and \(b\) into account. For example, when \(a\) and \(b\) are mutually exclusive — for example \(b = \bar{a}d\), for some arbitrary \(d\) — \(\xi(ab|c) = \xi_F\) but there are no constraints on either \(\xi(a|c) = x\) or
\[ \xi(b|c) = y. \] Thus, in order that \( G^{(1)}(x, y) = \xi_F \) for arbitrary choices of \( x \) and \( y \), \( G^{(1)} \) must be a constant which is unacceptable.

**Type 2:** \( \xi(ab|c) = G^{(4)}[\xi(a|bc), \xi(b|ac)]. \) This function is unsatisfactory because it overlooks the plausibility of \( b|c \). For example: let \( a = \text{"X is big"} \) and \( b = \text{"X is big and green"} \) so that \( ab = b \). Then

\[ \xi(b|c) = G^{(2)}[\xi(a|c), \xi(a|bc)] \quad \text{or} \quad \xi(b|c) = G^{(2)}[\xi(a|c), \xi_T], \] (22)

which is clearly unsatisfactory since “green” does not figure anywhere on the right hand side.

**Type 3:** \( \xi(ab|c) = G^{(3)}[\xi(a|c), \xi(b|ac)]. \) This function turns out to be satisfactory.

**Type 4:** \( \xi(ab|c) = G^{(4)}[\xi(a|bc), \xi(b|ac)]. \) This function strongly violates common sense: when \( a = b \) we have \( \xi(a|c) = G^{(4)}(\xi_T, \xi_T) \), so that \( \xi(a|c) \) takes the same constant value irrespective of what \( a \) might be [0].

**Type 5:** \( \xi(ab|c) = G^{(5)}[\xi(a|c), \xi(b|c), \xi(a|bc)]. \) This function turns out to be equivalent either to \( G^{(1)} \) or to \( G^{(3)} \) and can therefore be ignored. The proof follows from associativity, \( (ab|c)d = a(bc)|d \), which leads to the constraint

\[
G^{(5)} \left[ G^{(5)}[\xi(a|d), \xi(b|d), \xi(a|bd)], \xi(c|d), G^{(5)}[\xi(a|cd), \xi(b|cd), \xi(a|bcd)] \right] = G^{(5)}[\xi(a|d), G^{(5)}[\xi(b|d), \xi(c|d), \xi(b|cd)], \xi(a|bcd)]
\]

and, with the appropriate identifications,

\[
G^{(5)}[G^{(5)}(x, y, z), u, G^{(5)}(v, w, s)] = G^{(5)}[x, G^{(5)}(y, u, w), s]. \] (23)

Since the variables \( x, y \ldots s \) can be varied independently of each other we can take a partial derivative with respect to \( z \),

\[
G^{(5)}_1[G^{(5)}(x, y, z), u, G^{(5)}(v, w, s)]G^{(5)}_3(x, y, z) = 0. \] (24)

Therefore, either

\[
G^{(5)}_3(x, y, z) = 0 \quad \text{or} \quad G^{(5)}_1[G^{(5)}(x, y, z), u, G^{(5)}(v, w, s)] = 0. \] (25)

The first possibility says that \( G^{(5)} \) is independent of its third argument which means that it is of the type \( G^{(1)} \) that has already been ruled out. The second possibility says that \( G^{(5)} \) is independent of its first argument which means that it is already included among the type \( G^{(3)} \).

**Type 6:** \( \xi(ab|c) = G^{(6)}[\xi(a|c), \xi(a|bc), \xi(b|ac)]. \) This function turns out to be equivalent either to \( G^{(3)} \) or to \( G^{(4)} \) and can therefore be ignored. The proof—which we omit because it is identical to the proof above for type 5—also follows from associativity, \( (ab|c)d = a(bc)|d \).

**Type 7:** \( \xi(ab|c) = G^{(7)}[\xi(a|c), \xi(b|c), \xi(a|bc), \xi(b|ac)]. \) This function turns out to be equivalent either to \( G^{(5)} \) or \( G^{(6)} \) and can therefore be ignored. Again the proof which uses associativity, \( (ab|c)d = a(bc)|d \), is omitted because it is identical to type 5.
We conclude that the possible functions $G$ that are viable candidates for a general theory of inductive inference are equivalent to type $G^{(3)}$,

$$\xi(ab|c) = G[\xi(a|c), \xi(b|ac)] \quad .$$

(26)

The distributivity constraint:
Consider three statements $a$, $b$, and $c$, where the last two are mutually exclusive, in the context of a fourth, $d$. Distributivity, $a (b \lor c) = ab \lor ac$, implies that $\xi (a (b \lor c) | d)$ can be computed in two ways, $\xi ((ab|d) \lor (ac|d))$. Using eq. (11) and (26) leads to

$$G \left[ \xi(ab|d), \xi(ac|d) \right] = G[\xi(a|d), \xi(b|ad)] + G[\xi(a|d), \xi(c|ad)] .$$

(27)

Therefore, the requirement of distributivity constrains $G$ to satisfy

$$G(u, v + w) = G(u, v) + G(u, w) ,$$

(28)

where $\xi(a|d) = u$, $\xi(b|ad) = v$, and $\xi(c|ad) = w$. To solve this constraint let $v + w = z$. Differentiating with respect to $v$ and $w$ gives $\partial^2 G(u, z) / \partial z^2 = 0$, so that $G$ is linear in its second argument, $G(u, v) = A(u)v + B(u)$. Substituting back into eq. (28) gives $B(u) = 0$.

To determine the function $A(u)$ we note that the degree to which we believe in $ad|d$ is exactly the degree to which we believe in $a|d$ by itself. Therefore,

$$\xi(a|d) = \xi(ad|d) = G[\xi(a|d), \xi(d|ad)] = G[\xi(a|d), \xi_T] \quad \text{or} \quad u = A(u)\xi_T ,$$

(29)

which means that

$$G(u, v) = uv / \xi_T \quad \text{or} \quad \xi(ab|d) = \xi(a|d) \xi(b|ad) / \xi_T .$$

(30)

The constant $\xi_T$ is easily regraduated away: just normalize $\xi$ to $p = \xi / \xi_T$. In the regraduated scale the AND operation is represented by a simple product rule,

$$p(ab|d) = p(a|d) \ p(b|ad) \quad ,$$

(31)

while the sum rule, eq. (14), remains unaffected,

$$p(a \lor b|d) = p(a|d) + p(b|d) - p(ab|d) .$$

(32)

Degrees of belief $p$ measured in this particularly convenient regraduated scale can be called “probabilities”. The degrees of belief $\xi$ range from total disbelief $\xi_F = 0$ to total certainty $\xi_T$. The corresponding regraduated values are $p_F = 0$ and $p_T = 1$. 

9
5 Conclusion

Probability theory is the unique method of rational, quantitative and consistent inductive inference that can claim to be of general applicability. It focuses on degrees of rational belief and not on other qualities such as simplicity, explanatory power, degree of confirmation, desirability, or amount of information. The reason the method is unique is not because we have succeeded in formulating a precise and rigorous definition of rationality. Rather, the method is unique for the more modest reason that it is the only one left after obvious irrationalities — such as inconsistencies — have been weeded out.

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References

[1] R. T. Cox, Am. J. Phys. 14, 1-13 (1946).
[2] E. T. Jaynes, Probability Theory: The Logic of Science, ed. by L. Bretthorst (Cambridge U.P., 2003).
[3] A. Caticha, Lectures on Probability, Entropy, and Statistical Physics (MaxEn08, São Paulo, 2008) (arXiv.org/abs/0808.0012).
[4] J. D. Norton, Brit. J. Phil. Sci. 58, 141 (2007).
[5] M. Tribus, Rational Descriptions, Decisions and Designs (Pergamon, 1969).
[6] C. R. Smith, G. J. Erickson, p.17 in Maximum Entropy and Bayesian Methods ed. by P. F. Fougère (Kluwer, 1990).
[7] A. Garrett, p.175 in Maximum Entropy and Bayesian Methods ed. by G. R. Heidbreder (Kluwer, 1996).
[8] J. Y. Halpern, Journal of Artificial Intelligence Research 10, 67 (1999).
[9] A. Caticha, Phys. Rev. A57, 1572 (1998) (arXiv.org/abs/quant-ph/9804012).
[10] K. H. Knuth, p. 204 in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by G.J. Erickson and Y. Zhai, AIP Conf. Proc. 707 (2003).
[11] J. Aczél, Lectures on Functional Equations and Their Applications (Academic Press, New York, 1996).
[12] K. Van Horn, Int. J. Approx. Reasoning 34, 3 (2003).

[13] This argument is due to N. Caticha, private communication (2009).