Chern-Simons and Born-Infeld gravity theories
and Maxwell algebras type

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Abstract

Recently was shown that standard odd and even-dimensional General
Relativity can be obtained from a \((2n+1)\)-dimensional Chern-Simons La-
grangian invariant under the \(B_{2n+1}\) algebra and from a \((2n)\)-dimensional
Born-Infeld Lagrangian invariant under a subalgebra \(L_{B_{2n+1}}\) respectively.

Very Recently, it was shown that the generalized Inönü-Wigner con-
traction of the generalized AdS-Maxwell algebras provides Maxwell alge-
bras types \(M_{m}\) which correspond to the so called \(B_{m}\) Lie algebras.

In this article we report on a simple model that suggests a mechanism
by which standard odd-dimensional General Relativity may emerge as
a weak coupling constant limit of a \((2p+1)\)-dimensional Chern-Simons
Lagrangian invariant under the Maxwell algebra type \(M_{2p+1}\), if and only
if \(m \geq p\). Similarly, we show that standard even-dimensional General
Relativity emerges as a weak coupling constant limit of a \((2p)\)-dimensional
Born-Infeld type Lagrangian invariant under a subalgebra \(L_{M_{2m}}\) of the
Maxwell algebra type, if and only if \(m \geq p\). It is shown that when \(m < p\)
this is not possible for a \((2p+1)\)-dimensional Chern-Simons Lagrangian
invariant under the \(M_{2p+1}\) and for a \((2p)\)-dimensional Born-Infeld type
Lagrangian invariant under \(L_{M_{2m}}\) algebra.

1 Introduction

The most general action for the metric satisfying the criteria of general covari-
ce and second-order field equations for \(d > 4\) is a polynomial of degree \([d/2]\)
in the curvature known as the Lanczos-Lovelock gravity theory (LL) [1], [2].
The LL lagrangian in a \(d\)-dimensional Riemannian manifold can be defined as
a linear combination of the dimensional continuation of all the Euler classes of
dimension \(2p < d\) [3], [4]:

\[
S = \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)}
\]  

(1)
where $\alpha_p$ are arbitrary constants and

$$L_p = \varepsilon_{a_1 a_2 \cdots a_d} R^{a_1 a_2} \cdots R^{a_{2p} - 1 a_{2p}} e^{a_{2p} + 1} \cdots e^{a_d}$$

(2)

with $R^{ab} = d \omega^{ab} + \omega^a_c \omega^{cb}$. The expression (1) can be used both for even and for odd dimensions.

The large number of dimensionful constants in the $LL$ theory $\alpha_p$, $p = 0, 1, \cdots, [d/2]$, which are not fixed from first principles, contrast with the two constants of the Einstein-Hilbert action.

In ref. [5] it was found that these parameters can be fixed in terms of the gravitational and the cosmological constants, and that the action in odd dimensions can be formulated as a Chern-Simons theory of the $AdS$ group.

The closest one can get to a Chern-Simons theory in even dimensions is with the so-called Born-Infeld theories [5] [6], [7], [8]. The Born-Infeld lagrangian is obtained by a particular choice of the parameters in the Lovelock series, so that the lagrangian is invariant only under local Lorentz rotations in the same way as is the Einstein-Hilbert action.

If Chern-Simons theory is the appropriate odd-dimensional gauge theory and if Born-Infeld theory is the appropriate even-dimensional theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In Ref. [10] was shown that the standard, odd-dimensional General Relativity (without a cosmological constant) can be obtained from Chern-Simons gravity theory for a certain Lie algebra $\mathfrak{B}$ and recently was found that standard, even-dimensional General Relativity (without a cosmological constant) emerges as a limit of a Born-Infeld theory invariant under a certain subalgebra of the Lie algebra $\mathfrak{B}$ [11].

Very recently was found that the so called $\mathfrak{B}_m$ Lie algebra of Ref. [10] correspond to Maxwell algebras type $\mathcal{M}_m$ [12]. In fact, it was shown that the generalized Inönü-Wigner contraction of the generalized AdS-Maxwell algebras provides maxwell algebras types $\mathcal{M}_m$ which correspond to $\mathfrak{B}_m$ Lie algebra. These Maxwell algebras types $\mathcal{M}_m$ algebras can be obtained by $S$-expansion resonant reduced of the $AdS$ Lie algebra when we use $S_E^{(N)} = \{\lambda_\alpha\}^{N+1}_{\alpha=0}$ as semigroup.

It is the purpose of this paper to show that standard odd General Relativity emerges as a weak coupling constant limit of a $(2p + 1)$-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_{2m+1}$ algebra, if and only if $m \geq p$. Similarly, we show that standard even General Relativity emerges as a weak coupling constant limit of a $(2p)$-dimensional Born-Infeld type Lagrangian invariant under the $L^{M_{2m}}$ algebra, if and only if $m \geq p$. It is shown that when $m < p$ this is not possible for a $(2p + 1)$-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_{2m+1}$ and for a $(2p)$-dimensional Born-Infeld type Lagrangian invariant under $L^{M_{2m}}$.

This paper is organized as follows: In Sec. II we briefly review some aspect of: (i) Lovelock gravity theory, (ii) the construction of the so called $\mathcal{M}_{2n+1}$ algebra and (iii) obtaining odd and even dimensional general relativity from Chern-Simons gravity theory and from Born-Infeld theory respectively.
In Section III it is shown that the odd-dimensional Einstein-Hilbert Lagrangian can be obtained from a Chern-Simons Lagrangian in \((2p+1)\)-dimensions invariant under the algebra \(M_{2m+1}\), if and only if \(m \geq p\). However, this is not possible for Chern-Simons Lagrangian in \((2p+1)\)-dimension invariant under the \(M_{2m+1}\) algebra when \(m < p\).

In Section IV it is shown that the even-dimensional Einstein-Hilbert Lagrangian can be obtained from a Born-Infeld type Lagrangian in \((2p)\)-dimensions invariant under the \(L_{M_{2m}}\) subalgebra of the \(M_{2m}\) algebra, if and only if \(m \geq p\). However, this is not possible for Born-Infeld type Lagrangians in \((2p)\)-dimensions invariant under the \(L_{M_{2m}}\) subalgebra when \(m < p\).

Sec.V concludes the work with a comment about possible developments.

2 The Lovelock action, The \(M_{2n+1}\) algebra and general relativity

In this section we shall review some aspects of higher dimensional gravity, the construction of the so called Maxwell algebra types, and obtaining odd and even dimensional general relativity from Chern-Simons gravity theory and from Born-Infeld theory respectively. The main point of this section is to display the differences between the invariances of Lovelock action when odd and even dimensions are considered.

2.1 The Chern-Simons gravity

The Lovelock action is a polynomial of degree \(\lfloor d/2 \rfloor\) in curvature, which can be written in terms of the Riemann curvature and the vielbein \(e^a\) in the form 1, 2. In first order formalism the Lovelock action is regarded as a functional of the vielbein and spin connection, and the corresponding field equations obtained by varying with respect to \(e^a\) and \(\omega^{ab}\) read 3:

\[
\varepsilon_a = \sum_{p=0}^{\lfloor (d-1)/2 \rfloor} \alpha_p (d - 2p) \varepsilon_p^a = 0; \quad \varepsilon_{ab} = \sum_{p=1}^{\lfloor (d-1)/2 \rfloor} \alpha_p (d - 2p) \varepsilon_p^{ab} = 0 \tag{3}
\]

where

\[
\varepsilon_p^a := \varepsilon_{ab_1 \ldots b_{2p-1}} R^{b_1 b_2} \ldots R^{b_{2p-1} b_{2p}} e^{b_{2p+1}} \ldots e^{b_{2p-1}} \tag{4}
\]

\[
\varepsilon_p^{ab} := \varepsilon_{abc_1 \ldots c_{2p}} R^{a c_1} \ldots R^{a c_{2p-1} c_{2p}} T^{c_{2p+1} e^{c_{2p+2}} \ldots e^{c_{2p}}} \tag{5}
\]

Here \(T^a = de^a + \omega^a_b e^b\) is the torsion 2-form. Using the Bianchi identity one finds 5

\[
D \varepsilon_a = \sum_{p=1}^{\lfloor (d-1)/2 \rfloor} \alpha_{p-1} (d - 2p + 2)(d - 2p + 1) e^b \varepsilon_p^{ab} \tag{6}
\]
Moreover

\[ e^b \epsilon_{ba} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p(d-2p)e^b \epsilon_{ba}, \quad (7) \]

From (6) and (7) one finds for \( d = 2n - 1 \)

\[ \alpha_p = \alpha_0 \frac{(2n-1)(2\gamma)^p}{(2n-2p-1)} \begin{pmatrix} n - 1 \end{pmatrix}, \quad (8) \]

with \( \alpha_0 = \frac{\kappa}{dl}, \gamma = -\text{sign}(\Lambda) \frac{l^2}{d}, \) where for any dimensions \( l \) is a length parameter related to the cosmological constant by \( \Lambda = \pm \frac{(d-1)(d-2)}{2l^2} \).

With these coefficients, the Lovelock action is a Chern-Simons \((2n-1)\)-form invariant not only under standard local Lorentz rotations \( \delta e^a = \kappa_{ab} e^b, \delta \omega_{ab} = -D\kappa_{ab} \), but also under a local AdS boost [5].

2.2 Born-Infeld gravity

For \( d = 2n \) it is necessary to write equation (6) in the form [5]

\[ D\epsilon_a = T^b \sum_{p=1}^{[n-1]} 2\alpha_{p-1}(n-p+1)T^p - \sum_{p=1}^{[n-1]} 4\alpha_{p-1}(n-p+1)(n-p)e^b \epsilon_{ba}, \quad (9) \]

with

\[ T^p = \frac{\delta L}{\delta R^p} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p T^p, \quad (10) \]

where

\[ T^p = \epsilon_{ab_1 \ldots b_d} R^{a_2 \ldots a_{p+1}} \ldots R^{a_{2p-1}} \epsilon_{b_{p+2} \ldots b_d}. \quad (11) \]

The comparison between (7) and (9) leads to [5]

\[ \alpha_p = \alpha_0 (2\gamma)^p \begin{pmatrix} n \end{pmatrix}. \quad (12) \]

With these coefficients the LL lagrangian takes the form [5]

\[ L = \frac{\kappa}{2n} \epsilon_{a_1 \ldots a_d} \bar{R}^{a_1 \ldots a_2} \ldots \bar{R}^{a_{d-1} a_d}, \quad (13) \]

which is the Pfaffian of the 2-form \( \bar{R}^{ab} = R^{ab} + \frac{1}{l^2} \epsilon^a \epsilon^b \) and can be formally written as the Born-Infeld like form [5, 8]. The corresponding action, known as Born-Infeld action is invariant only under local Lorentz rotations.

The corresponding Born-Infeld action is given by [5, 8]

\[ S = \int \sum_{p=0}^{[d/2]} \frac{\kappa}{2n} \begin{pmatrix} n \end{pmatrix}^{2p-d+1} \epsilon_{a_1 \ldots a_d} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} \epsilon^{a_{2p+1} \ldots a_d}. \quad (14) \]
where $e^a$ corresponds to the 1-form vielbein, and $R^{ab} = d\omega^{ab} + \omega^a \epsilon \omega^{cb}$ to the Riemann curvature in the first order formalism.

The action (14) is off-shell invariant under the Lorentz-Lie algebra $SO(2n - 1, 1)$, whose generators $\tilde{J}_{ab}$ of Lorentz transformations satisfy the commutation relationships

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{cb} \tilde{J}_{ad} - \eta_{ca} \tilde{J}_{bd} + \eta_{db} \tilde{J}_{ca} - \eta_{da} \tilde{J}_{cb}.$$  

The Levi-Civita symbol $\epsilon_{a_1 \ldots a_{2n}}$ in (14) should be regarded as the only non-vanishing component of the symmetric, $SO(2n - 1, 1)$, invariant tensor of rank $n$, namely

$$\left( \tilde{J}_{a_1 a_2} \cdots \tilde{J}_{a_{2n-1} a_{2n}} \right) = \frac{2^n}{n} \epsilon_{a_1 \cdots a_{2n}}. \quad (15)$$

In order to interpret the gauge field as the vielbein, one is forced to introduce a length scale $l$ in the theory. To see why this happens, consider the following argument: Given that (i) the exterior derivative operator $d = dx^\mu \partial_\mu$ is dimensionless, and (ii) one always chooses Lie algebra generators $T_A$ to be dimensionless as well, the one-form connection fields $A = A^\mu T_A dx^\mu$ must also be dimensionless. However, the vielbein $e^a = e^a_\mu dx^\mu$ must have dimensions of length if it is to be related to the spacetime metric $g_{\mu\nu}$ through the usual equation $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. This means that the "true" gauge field must be of the form $e^a/l$, with $l$ a length parameter.

Therefore, following Refs. [15], [16], the one-form gauge field $A$ of the Chern–Simons theory is given in this case by

$$A = \frac{1}{l} e^a \tilde{P}_a + \frac{1}{2} \omega^{ab} \tilde{J}_{ab}. \quad (16)$$

It is important to notice that once the length scale $l$ is brought into the Born-Infeld theory, the lagrangian splits into several sectors, each one of them proportional to a different power of $l$, as we can see directly in eq. (14).

### 2.3 The Maxwell algebra type

#### 2.3.1 The S-expansion procedure

In this subsection we shall review the main aspects of the $S$-expansion procedure and their properties introduced in Ref. [13].

Let $S = \{\lambda_a\}$ be an abelian semigroup with 2-selector $K_{\alpha\beta}^\gamma$ defined by

$$K_{\alpha\beta}^\gamma = \begin{cases} 1 & \text{when } \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0 & \text{otherwise}, \end{cases} \quad (17)$$

and $\mathfrak{g}$ a Lie (super)algebra with basis $\{T_A\}$ and structure constant $C_{AB}^C$,

$$[T_A, T_B] = C_{AB}^C T_C. \quad (18)$$

Then it may be shown that the product $\mathcal{G} = S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}^\gamma C_{AB}^C$,

$$[T_{(A,\alpha)}, T_{(B,\beta)}] = C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} T_{(C,\gamma)}. \quad (19)$$
The proof is direct and may be found in Ref. 13.

**Definition 1** Let \( S \) be an abelian semigroup and \( \mathfrak{g} \) a Lie algebra. The Lie algebra \( \mathfrak{g} \) defined by \( \mathfrak{g} = S \times \mathfrak{g} \) is called \( S \)-Expanded algebra of \( \mathfrak{g} \).

When the semigroup has a zero element \( 0_S \in S \), it plays a somewhat peculiar role in the \( S \)-expanded algebra. The above considerations motivate the following definition:

**Definition 2** Let \( S \) be an abelian semigroup with a zero element \( 0_S \in S \), and let \( \mathfrak{g} = S \times \mathfrak{g} \) be an \( S \)-expanded algebra. The algebra obtained by imposing the condition \( 0_S T_A = 0 \) on \( \mathfrak{g} \) (or a subalgebra of it) is called \( 0_S \)-reduced algebra of \( \mathfrak{g} \) (or of the subalgebra).

An \( S \)-expanded algebra has a fairly simple structure. Interestingly, there are at least two ways of extracting smaller algebras from \( S \times \mathfrak{g} \). The first one gives rise to a resonant subalgebra, while the second produces reduced algebras. In particular, a resonant subalgebra can be obtained as follows.

Let \( g = \bigoplus_{p \in I} V_p \) be a decomposition of \( g \) in subspaces \( V_p \), where \( I \) is a set of indices. For each \( p, q \in I \) it is always possible to define \( i_{(p,q)} \subset I \) such that

\[
[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r,
\]

(20)

Now, let \( S = \bigcup_{p \in I} S_p \) be a subset decomposition of the abelian semigroup \( S \) such that

\[
S_p \cdot S_q \subset \bigcup_{r \in i_{(p,q)}} S_p.
\]

(21)

When such subset decomposition \( S = \bigcup_{p \in I} S_p \) exists, then we say that this decomposition is in resonance with the subspace decomposition of \( g \), \( g = \bigoplus_{p \in I} V_p \).

The resonant subset decomposition is crucial in order to systematically extract subalgebras from the \( S \)-expanded algebra \( G = S \times g \), as is proven in the following

Theorem IV.2 of Ref. 13: Let \( g = \bigoplus_{p \in I} V_p \) be a subspace decomposition of \( g \), with a structure described by eq. (20), and let \( S = \bigcup_{p \in I} S_p \) be a resonant subset decomposition of the abelian semigroup \( S \), with the structure given in eq. (21). Define the subspaces of \( G = S \times g \),

\[
W_p = S_p \times V_p, \ p \in I.
\]

(22)

Then,

\[
\mathfrak{g}_R = \bigoplus_{p \in I} W_p
\]

(23)

is a subalgebra of \( G = S \times g \).

Proof: the proof may be found in Ref. 13.

**Definition 3** The algebra \( G_R = \bigoplus_{p \in I} W_p \) obtained is called a Resonant Subalgebra of the \( S \)-expanded algebra \( G = S \times g \).
A useful property of the $S$-expansion procedure is that it provides us with an invariant tensor for the $S$-expanded algebra $\mathfrak{g} = S \times g$ in terms of an invariant tensor for $g$. As shown in Ref. [13] the theorem VII.2 provide a general expression for an invariant tensor for a $0_S$-reduced algebra.

**Theorem VII.2 of Ref. [13]:** Let $S$ be an abelian semigroup with nonzero elements $\lambda_i$, $i = 0, \cdots, N$ and $\lambda_{N+1} = 0_S$. Let $g$ be a Lie (super)algebra of basis $\{T_A\}$, and let $\langle T_{A_n} \cdots T_{A_n} \rangle$ be an invariant tensor for $g$. The expression

$$\langle T_{(A_1, i_1)} \cdots T_{(A_n, i_n)} \rangle = \alpha_j K_{i_1 \cdots i_n} \langle T_{A_1} \cdots T_{A_n} \rangle$$

(24)

where $\alpha_j$ are arbitrary constants, corresponds to an invariant tensor for the $0_S$-reduced algebra obtained from $\mathfrak{g} = S \times g$.

**Proof:** the proof may be found in section 4.5 of Ref. [13].

### 2.3.2 $S$-expansion of $SO(2n, 2)$ algebra

Let us consider the $S$-expansion of the Lie algebra $SO(2n, 2)$ using the Abelian semigroup $S_E^{(2n-1)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \cdots, \lambda_{2n}\}$ defined by the product

$$\lambda_\alpha \lambda_\beta = \begin{cases} 
\lambda_{\alpha + \beta}, & \text{when } \alpha + \beta \leq 2n \\
\lambda_{2n}, & \text{when } \alpha + \beta > 2n
\end{cases}$$

(25)

The $\lambda_\alpha$ elements are dimensionless, and can be represented by the set of $2n \times 2n$ matrices $[\lambda_\alpha]_{ij} = \delta^i_{j+x}$, where $i, j = 1, \cdots, 2n - 1$, $\alpha = 0, \cdots, 2n$, and $\delta$ stands for the Kronecker delta [10].

After extracting a resonant subalgebra and performing its $0_S(= \lambda_{2n})$-reduction, one finds a new Lie algebra the so called Maxwell algebra type $M_{2n+1}$, which in Ref. [10] was called $\mathfrak{B}_{2n+1}$ algebra, whose generators

$$J_{(ab, 2k)} = \lambda_{2k} \otimes \hat{J}_{ab},$$

(26)

$$P_{(a, 2k+1)} = \lambda_{2k+1} \otimes \hat{P}_a,$$

(27)

with $k = 0, \cdots, n - 1$, satisfy the commutation relationships [10]

\[
\begin{align*}
[P_a, P_b] &= Z_a^{(i)} Z_b^{(i)}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b \\
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \\
[J_{ab}, Z_c^{(i)}] &= \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \\
[Z_a^{(i)}, P_c] &= \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \\
[Z_a^{(i)}, Z_c^{(j)}] &= \eta_{bc} Z_a^{(i+j)} - \eta_{ac} Z_b^{(i+j)},
\end{align*}
\]

(28)-(32)

\[
\begin{align*}
[J_{ab}, Z_c^{(i)}] &= \eta_{bc} Z_d^{(i)} - \eta_{ca} Z_d^{(i)} + \eta_{db} Z_c^{(i)} - \eta_{da} Z_b^{(i)} \\
[Z_a^{(i)}, Z_c^{(j)}] &= \eta_{bc} Z_d^{(i+j)} - \eta_{ca} Z_d^{(i+j)} + \eta_{db} Z_c^{(i+j)} - \eta_{da} Z_b^{(i+j)}, \\
[P_a, Z_c^{(i)}] &= Z_a^{(i+1)}, \quad \left[Z_a^{(i)}, Z_c^{(j)}\right] = Z_a^{(i+j)}.
\end{align*}
\]

(33)-(35)
and where we have defined

\[ J_{ab} = J_{(ab,0)} = \lambda_0 \otimes \tilde{J}_{ab}, \]
\[ P_a = P_{(a,1)} = \lambda_1 \otimes \tilde{P}_a, \]
\[ Z_{ab}^{(i)} = J_{(ab,2i)} = \lambda_{2i} \otimes \tilde{J}_{ab}, \]
\[ Z_a^{(i)} = P_{(a,2i+1)} = \lambda_{2i+1} \otimes \tilde{P}_a, \]

with \( i = 1, \ldots, n - 1 \).

We note that commutation relations \((26), (27), (28)\) form a subalgebra of the \(\mathcal{M}_{2n+1}\) algebra which we will denote as \(\mathfrak{L}^{M_{2n+1}}\). This subalgebra can be obtained from \(S\)-expansion of the Lorentz-Lie algebra using as a semigroup the sub-semigroup \(S_0^{(2n-1)} = \{\lambda_0, \lambda_2, \lambda_4, \cdots, \lambda_{2n}\}\) of semigroup \(S_E^{(2n-1)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \cdots, \lambda_{2n}\}\). After extracting a resonant subalgebra and performing its \(S(= \lambda_{2n})\)-reduction, one finds the \(\mathfrak{L}^{M_{2n+1}}\) algebra, which is a subalgebra of the \(\mathcal{M}_{2n+1}\) algebra, whose generators \(J_{ab} = \lambda_0 \tilde{J}_{ab}, Z_{ab}^{(1)} = \lambda_2 \tilde{J}_{ab}, Z_{ab}^{(2)} = \lambda_4 \tilde{J}_{ab}, \cdots, Z_{ab}^{(n)} = \lambda_{2n} \tilde{J}_{ab}\) satisfy the commutation relationships \((26), (27), (28)\).

\[ 2.4 \text{ General Relativity} \]

\[ 2.4.1 \text{ Odd-dimensional general relativity} \]

In Ref. \([10]\), it was shown that the standard, odd-dimensional general relativity (without a cosmological constant) can be obtained from Chern-Simons gravity for the algebra \(\mathcal{M}_{2n+1}\). The Chern-Simons Lagrangian is built from \(\mathcal{M}_{2n+1}\)-valued, one-form gauge connection \(A\) which depends on a scale parameter \(l\) which can be interpreted as a coupling constant that characterizes different regimes within the theory. The field content induced by \(\mathcal{M}_{2n+1}\) includes the vielbein \(e^a\), the spin connection \(\omega^{ab}\), and extra bosonic fields \(\eta^{(i)}\) and \(k^{a(i)}\). The odd-dimensional Chern-Simons Lagrangian invariant under the \(\mathcal{M}_{2n+1}\) algebra is given by \([10]\)

\[
L_{CS}^{(2n+1)} = \sum_{k=1}^{n} \frac{1}{2^{k-2}} c_k \alpha_j \beta_{i_1 + \cdots + i_{n+1}} \epsilon_{p_1 + q_1} \cdots \epsilon_{p_{n-k} + q_{n-k}} \epsilon_{a_{2n+1}} \times R^{c_{12,13}} \cdots R^{c_{a_{2k-1}a_{2k-2},1}} e^{c_{a_{2k+1},p_1}} e^{c_{a_{2k+2},q_1}} \cdots e^{c_{a_{2n-1},p_{n-k}} e^{c_{a_{2n},q_{n-k}}}} e^{c_{a_{2n+1},i_{n+1}}}. \tag{40}
\]

where

\[
c_k = \frac{1}{2(n-k) + 1} \binom{n}{k}
\]
\[
R^{a(b,2k)} = d_\omega^{(ab,2k)} + \eta_{f d_\omega^{(ac,2i)} \omega^{(dh,2j)}} \delta_{i+j}^k.
\]

In the \(l \rightarrow 0\) limit, the only nonzero term in \([10]\) corresponds to the case \(k = 1\), whose only non-vanishing component occurs for \(p = q_1 = \cdots = q_{2n-1} = 0\).
and is proportional to the odd-dimensional Einstein-Hilbert Lagrangian \[10\]

\[
L_{\text{CS}}^{(2n+1)} \bigg|_{l=0} = \frac{n!2n-1}{2^{n-1}} \varepsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} e^{a_3} \cdots e^{a_{2n+1}}
\]

### 2.4.2 Even-dimensional general relativity

In Ref. \[11\], it was recently shown that standard, even-dimensional General Relativity (without a cosmological constant) emerges as a limit of a Born-Infeld theory invariant under the subalgebra $\mathfrak{M}_{2n+1}^+ \subset \mathfrak{M}_{2n+1}$.

The Born-Infeld Lagrangian is built from the two-form curvature $S_{(2n-1)}$ expanded

\[
F = \sum_{k=0}^{n-1} \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)},
\]

where

\[
F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd} \omega^{(ac,2i)} \omega^{(db,2j)} \delta_{i+j}^k + \frac{1}{l^2} \epsilon^{(a,2i+1)} \epsilon^{(b,2j+1)} \delta_{i+j+1}^k
\]

which depends on a scale parameter $l$ which can be interpreted as a coupling constant that characterizes different regimes within the theory. The field content induced by $L_{\text{BI}}^{\mathfrak{M}_{2n+1}}$ includes the vielbein $e^a$, the spin connection $\omega_{ab}$ and extra bosonic fields $h^{a(i)} = e^{(a,2i+1)}$ and $k^{ab(i)} = \omega^{(ab,2i)}$, with $i = 1, \ldots, n-1$. The even-dimensional Born-Infeld Gravity Lagrangian invariant under the $L_{\text{BI}}^{\mathfrak{M}_{2n+1}}$ algebra is given by \[11\]

\[
L_{\text{BI}}^{\mathfrak{M}_{2n+1}} = \sum_{k=1}^{n} l^{2k-2} \frac{1}{2n} \binom{n}{k} \alpha_j \delta_{11+\ldots+q}^j \delta_{p_1+q_1}^{k+1} \cdots \delta_{p_{n-k}+q_{n-k}}^{i_n} e_{a_1 \cdots a_{2n}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2k-1} a_{2k}, i_k)} e^{(a_{2k+1}, p_1)} e^{(a_{2k+2}, q_1)} \cdots e^{(a_{2n-1}, p_{n-k})} e^{(a_{2n}, q_{n-k})},
\]

where we can see that in the limit $l = 0$ the only nonzero term corresponds to the case $k = 1$, whose only nonzero component (corresponding to the case $p = q_1 = \cdots = q_{2n-2} = 0$) \[11\] is proportional to the even-dimensional Einstein-Hilbert Lagrangian

\[
L_{\text{BI}}^{\mathfrak{M}_{2n}} \bigg|_{l=0} = \frac{1}{2} \alpha_{2n-2} e_{a_1 \cdots a_{2n}} R^{(a_1 a_2, 0)} e^{(a_3, 1)} \cdots e^{(a_{2n}, 1)}
\]

\[
= \frac{1}{2} \alpha_{2n-2} e_{a_1 \cdots a_{2n}} R^{a_1 a_2} e^{a_3} \cdots e^{a_{2n}}.
\]
3 Chern-Simons Lagrangians invariant under the Maxwell algebra type

In this section it is shown that the Einstein-Hilbert Lagrangian for odd-dimensions can be obtained from a Chern-Simons Lagrangian in \((2p + 1)\)-dimensions invariant under the \(M_{2m+1}\) algebra, if and only if \(m \geq p\). However, this is not possible when \(m < p\) for Chern-Simons Lagrangians in \((2p + 1)\)-dimensions invariant under the \(M_{2m+1}\) algebra.

The 1-form gauge connection \(A\) \(M_{2n+1}\)-valued, is given by

\[
A = \sum_{k=0}^{n-1} \left[ \frac{1}{2} \omega^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} \epsilon^{(a,2k+1)} P_{(a,2k+1)} \right],
\]

\((46)\)

and the 2-form curvature \(F = dA + A^2\) is

\[
F = \sum_{k=0}^{n-1} \left[ \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} F^{(a,2k+1)} P_{(a,2k+1)} \right],
\]

\((47)\)

where

\[
F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd} \omega^{(ac,2i)} \omega^{(db,2j)} \delta^{k}_{i+j} + \frac{1}{l^2} \epsilon^{(a,2i+1)} \epsilon^{(b,2j+1)} \delta^{k}_{i+j+1},
\]

\((48)\)

\[
F^{(a,2k+1)} = de^{(a,2k+1)} + \eta_{bc} \omega^{(ab,2i)} e^{(c,2j)} \delta^{k}_{i+j}.
\]

\((49)\)

It is interesting to note that the Maxwell algebra type \(M_{2m+1}\) can be used to construct different odd-dimensional Chern-Simons Lagrangians. For example, if we consider a \(S_{E}^{(3)}\)-expansion of the \(AdS\) algebra \(SO(4,2)\) and after extracting a resonant subalgebra and performing its \(0_{S}\)-reduction, one finds \(M_{5}\) algebra in \(D = 5\) dimensions. On the other hand, if we consider a \(S_{E}^{(3)}\)-expansion of the \(AdS\) algebra \(SO(6,2)\) and after extracting a resonant subalgebra and performing its \(0_{S}\)-reduction, one finds \(M_{7}\) algebra in \(D = 7\) dimensions. In this way, the CS Lagrangians \(L_{CS}^{M_{5}}(5)\) and \(L_{CS}^{M_{7}}(7)\) are invariant under the same \(M_{5}\) algebra, however the indices of the generators \(T_{a}\) runs over 5 and 7 values, respectively.

These considerations allow the construction of gravitational theories in every odd-dimension. Nevertheless, as discussed below, only in some dimensions it is possible to obtain General Relativity as a weak coupling constant limit of a Chern-Simons theory.

3.1 \((2 + 1)\)-dimensional Chern-Simons Lagrangians invariant under \(M_{7}\)-algebra

Before considering the Chern-Simons Lagrangian \((2n + 1)\)-dimensional, we study the case of the \(M_{7}\) algebra. The \(M_{7}\)-algebra can be found by \(S\)-expansion of...
the $AdS$ algebra using as semigroup $S_C^{(5)}$. In fact, after extracting a resonant subalgebra and performing the $0s$ reduction, one finds the $\mathcal{M}_7$-algebra whose generators satisfy the following commutation relations

\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \\
[J_{ab}, Z^{(1)}_{c}] &= \eta_{be} Z^{(1)}_a - \eta_{ae} Z^{(1)}_b, \\
[Z^{(1)}_{ab}, Z^{(1)}_c] &= \eta_{be} Z^{(1)}_a - \eta_{ae} Z^{(1)}_b, \\
[Z^{(1)}_{ab}, P_c] &= \eta_{be} Z^{(2)}_a - \eta_{ae} Z^{(2)}_b, \\
[P_a, Z^{(1)}_c] &= Z^{(2)}_{ab}. \\
\end{align*}

Consider the construction of a three-dimensional Chern-Simons Lagrangian invariant under $\mathcal{M}_7$. In fact, Using Theorem VII.2 of Ref. 13, it is possible to show that the only non-vanishing components of a invariant tensor for the $\mathcal{M}_7$ algebra are given by

\begin{align*}
\langle J_{ab} J_{cd} \rangle_{\mathcal{M}_7} &= \alpha_0 (\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}), \\
\langle J_{ab} Z^{(1)}_{cd} \rangle_{\mathcal{M}_7} &= \alpha_2 (\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}), \\
\langle Z^{(1)}_{ab} Z^{(1)}_{cd} \rangle_{\mathcal{M}_7} &= \alpha_4 (\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}), \\
\langle P_a P_c \rangle_{\mathcal{M}_7} &= \alpha_2 \eta_{bc}, \\
\langle P_a Z^{(1)}_c \rangle_{\mathcal{M}_7} &= \alpha_4 \eta_{bc}, \\
\langle J_{ab} P_c \rangle_{\mathcal{M}_7} &= \alpha_3 \epsilon_{abc}, \\
\langle Z^{(1)}_{ab} P_c \rangle_{\mathcal{M}_7} &= \langle J_{ab} Z^{(1)}_c \rangle_{\mathcal{M}_7} = \alpha_3 \epsilon_{abc}, \\
\langle Z^{(1)}_{ab} Z^{(1)}_c \rangle_{\mathcal{M}_7} &= \langle J_{ab} Z^{(2)}_c \rangle_{\mathcal{M}_7} = \alpha_5 \epsilon_{abc}. \\
\end{align*}
where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5$ and $\alpha_5$ are arbitrary independent constant dimensionless. The 1-form gauge connection $A$ is given by,

$$A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} h^{(ab,1)} Z_{ab}^{(1)} + \frac{1}{l} h^{(a,1)} Z_a^{(1)} + \frac{1}{2} h^{(ab,2)} Z_{ab}^{(2)} + \frac{1}{l} h^{(a,2)} Z_a^{(2)},$$

and the 2-form curvature is

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2} \left( D_\omega k^{(ab,1)} + \frac{1}{l^2} e^a e^b \right) Z_{ab}^{(1)} + \frac{1}{l} \left( D_\omega h^{(a,1)} + k^{(1)}_b e^b \right) Z_a^{(1)}$$

$$+ \frac{1}{2} \left( D_\omega k^{(ab,2)} + k^{(1)}_c k^{cb(1)} + \frac{1}{l^2} \left( e^a h^{(b,1)} + h^{(a,1)} e^b \right) \right) Z_{ab}^{(2)}$$

$$+ \frac{1}{l} \left( D_\omega h^{(a,2)} + k^{(2)}_c e^c + k^{(1)}_c h^{(c,1)} \right) Z_a^{(2)}.$$  \hfill (69)

Using the dual procedure of S-expansion, we find that the 3-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_7$-algebra is given by

$$L_{\mathcal{CS}}^{(2+1)} = \frac{\alpha_1}{l} \varepsilon_{abc} \left( R^{ab} e^c - d \left( \frac{1}{2} \omega^{ab} e^c \right) \right)$$

$$+ \frac{\alpha_3}{l} \varepsilon_{abc} \left( R^{ab} h^{(c,1)} + 9 \varepsilon^{(ab,1)} e^c + \frac{1}{3l^2} e^a e^b e^c - \frac{d}{2} \left( \omega^{ab} h^{(c,1)} + k^{(ab,1)} e^c \right) \right)$$

$$+ \frac{\alpha_5}{l} \varepsilon_{abc} \left( R^{ab} h^{(c,2)} + 9 \varepsilon^{(ab,2)} e^c + \frac{1}{l^2} e^a e^b h^{(c,1)} \right)$$

$$- \frac{d}{2} \left( \omega^{ab} h^{(c,2)} + k^{(ab,1)} h^{(c,1)} + k^{(ab,2)} e^c \right)$$

$$+ \frac{\alpha_2}{l} \left( \omega^a d k^b_a (1) + k^{(1)}_b d \omega^b_a + 2 \omega^a \omega^b k^{(1)}_a + \frac{2}{l^2} e^a T^a \right)$$

$$+ \frac{\alpha_4}{l} \left( \omega^a d k^b_a (2) + k^{(2)}_b d \omega^b_a + 2 \omega^a \omega^b k^{(2)}_a + k^{(1)}_b d k^b_a + 2 \omega^a k^{(1)}_b k^{(1)}_a \right)$$

$$+ \frac{2}{l^2} e^a \tau^{(a,1)} + \frac{2}{l^2} h^{(a,1)} T^a.$$  \hfill (70)

where

$$\varepsilon^{(ab,1)} = D_\omega k^{(ab,1)},$$

$$\varepsilon^{(ab,2)} = D_\omega k^{(ab,2)} + k^{(1)}_c k^{cb(1)}$$

$$\tau^{(a,1)} = D_\omega h^{(a,1)} + k^{(1)}_c e^c,$$  \hfill (71)

(72)

(73)

The Lagrangian is split into six independent pieces, each one proportional to $\alpha_1, \alpha_3, \alpha_5, \alpha_0, \alpha_2, \alpha_4$. The term proportional to $\alpha_1$ corresponds to the Chern-Simons Lagrangian for ISO $(2, 1)$ which contains the Einstein-Hilbert term $\varepsilon_{abc} R^{ab} e^c$.  \hfill (70)
Varying the Lagrangian (70) we have

\[
\delta L_{CS}^{M_7 (2+1)} = \frac{1}{l^2} \varepsilon_{abc} \left( \alpha_1 R^{ab} + \frac{\alpha_3}{l^2} e^a e^b + \alpha_5 R^{(ab,1)} + \frac{\alpha_2}{l^2} e^a e^b \right) \delta e^c
\]

\[
+ \frac{1}{l^2} \varepsilon_{abc} \left( \alpha_3 R^{ab} + \alpha_5 R^{(ab,1)} + \frac{\alpha_2}{l^2} e^a e^b \right) \delta h^{(c,1)}
\]

\[
+ \frac{1}{l^2} \varepsilon_{abc} \left( \alpha_5 R^{ab} \right) \delta h^{(c,2)} + \frac{1}{l^2} \varepsilon_{abc} \delta \omega^{ab} \left( \alpha_1 T^c + \alpha_3 D_\omega h^{(c,1)} + \alpha_5 D_\omega h^{(c,2)} \right)
\]

\[
+ \frac{1}{l^2} \varepsilon_{acd} \delta \omega^{ab} \left( \alpha_3 e^a \epsilon_{k(cd,1)} + \alpha_5 h_{b}^{(1)} k^{(cd,1)} + \alpha_5 e^b k^{(cd,2)} \right)
\]

\[
+ \frac{1}{l^2} \varepsilon_{abc} \delta k^{(ab,1)} \left( \alpha_3 T^c + \alpha_5 D_\omega h^{(c,1)} \right) + \frac{1}{l^2} \varepsilon_{acd} \delta k^{(ab,1)} \left( 2 \alpha_5 h_{b}^{(1)} e^c \right)
\]

\[
+ \frac{\alpha_2}{l^2} \left( \delta L_3^{Lorentz} \left( k^{(2)} \right) \right) + \frac{\alpha_4}{l^2} \left( \delta L_3^{Lorentz} \left( k^{(1)} k^{(1)} \right) \right)
\]

\[
+ \delta \epsilon_{a} \left( \frac{\alpha_4}{l^2} \epsilon^{(a,1)} + \frac{2 \alpha_2}{l^2} T^a \right) + \delta \omega^{ab} \left( \frac{\alpha_2}{l^2} e^a e^b + \frac{\alpha_4}{l^2} e^b h_{a}^{(1)} \right)
\]

\[
+ \delta h_{a}^{(1)} \left( \frac{2 \alpha_4}{l^2} T^a \right) + \delta k^{(ab,1)} \left( \frac{\alpha_2}{l^2} e^a e^b \right).
\]

where \( L_3^{Lorentz} = \omega d \omega + \frac{2}{3} \omega^3 \).

If we consider the case where \( k^{(ab,1)} = k^{(ab,2)} = 0, \ h^{(a,1)} = 0 \) and \( h^{(a,2)} = 0 \) with the condition \( \alpha_1 = \alpha_3 = \alpha_5 = 0 \) we have

\[
\delta L_{CS}^{M_7 (2+1)} = \frac{\alpha_0}{2} \left( \delta L_3^{Lorentz} \right) + \frac{\alpha_2}{2 l^2} \delta \omega^{ab} \left( e^a e^b \right) + \frac{\alpha_2}{2 l^2} \delta e^a \left( T_a \right)
\]

\[
= \alpha_0 \delta \omega^{ab} \left( R_{ab} \right) + \alpha_2 \frac{l^2}{2} \delta \omega^{ab} \left( e^a e^b \right) + \frac{\alpha_2}{2 l^2} \delta e^a \left( T_a \right).
\]

Choosing \( \alpha_0 = \alpha_2 \) we have that \( \delta L_{CS}^{M_7 (2+1)} = 0 \), leads to the following equations of motion

\[
R^{ab} + \frac{1}{l^2} e^a e^b = 0, \quad \quad \quad (\text{74})
\]

\[
T_a = 0. \quad \quad \quad (\text{75})
\]

which correspond to the equations of general relativity with cosmological constant in (2 + 1)-dimensions.

### 3.2 (4+1)-dimensional Chern-Simons Lagrangian invariant under \( M_7 \)-algebra

The only non-vanishing components of a invariant tensor for the \( M_7 \) algebra are given by
In $D = 5$, the only non-vanishing components of an invariant tensor for the $\mathcal{M}_7$ algebra are given by

$$\langle J_{ab} J_{cd} P_f \rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_1 \epsilon_{abcd f}.$$  (76)

$$\langle J_{ab} J_{cd} Z_{(1)}^f \rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_3 \epsilon_{abcd f}.$$  (77)

$$\langle J_{ab} Z_{(1)}^f P_f \rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_3 \epsilon_{abcd f}.$$  (78)

$$\langle J_{ab} J_{cd} Z_{(2)}^f \rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_5 \epsilon_{abcd f}.$$  (79)

$$\langle J_{ab} Z_{(1)}^f Z_{(1)}^f \rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_5 \epsilon_{abcd f}.$$  (80)

where $\alpha_1$, $\alpha_3$ and $\alpha_5$ are arbitrary independent constants of dimensions $[\text{length}]^{-3}$.

Using the dual procedure of S-expansion, we find that the 5-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_7$-algebra is given by

$$L^{\mathcal{M}_7}_{(4+1)} = \alpha_1 \epsilon_{abcd f} \left( l^2 R^{ab} R^{cd} e^{f} \right)$$

$$+ \alpha_3 \epsilon_{abcd f} \left( l^2 R^{ab} R^{cd} h^{(f,1)} + 2 l^2 R^{ab} \mathcal{R}^{(cd,1)} e^{f} + \frac{2}{3} R^{ab} e^c e^d e^f \right)$$

$$+ \alpha_5 \epsilon_{abcd f} \left( l^2 R^{ab} R^{cd} h^{(f,2)} + 2 l^2 R^{ab} \mathcal{R}^{(cd,1)} h^{(f,1)} + 2 l^2 R^{ab} \mathcal{R}^{(cd,2)} e^{f}$$

$$+ 2 \mathcal{R}^{(ab,1)} \mathcal{R}^{(cd,1)} e^{f} + 2 R^{ab} e^c e^d h^{(f,1)} + \frac{2}{3} \mathcal{R}^{(ab,1)} e^c e^d e^f + \frac{1}{5 l^2} e^a e^b e^c e^d e^f \right)$$  (81)

Varying the Lagrangian (81) we have
\[
\delta L_{(4+1)}^{M_7} = \varepsilon_{abcd} \left( \alpha_1 l^2 R^{ab} R^{cd} + 2 \alpha_3 l^2 R^{ab} \mathcal{R}^{(cd,1)} + 2 \alpha_5 R^{ab} e^c \epsilon^d + 2 \alpha_5 l^2 R^{ab} \mathcal{R}^{(cd,2)} + 2 \alpha_3 l^2 R^{ab} e^c \epsilon^d + 2 \alpha_5 l^2 R^{ab} \mathcal{R}^{(cd,1)} + 2 \alpha_5 R^{ab} e^c \epsilon^d + \frac{1}{l^2} \alpha_5 e^a \epsilon^b \epsilon^c \epsilon^d \right) \delta e^f \\
+ \varepsilon_{abcd} \left( \alpha_3 l^2 R^{ab} R^{cd} + 2 \alpha_3 l^2 R^{ab} \mathcal{R}^{(cd,1)} + 2 \alpha_5 R^{ab} e^c \epsilon^d \right) \delta h^{(f,1)} \\
+ \varepsilon_{abcd} \alpha_5 l^2 R^{ab} R^{cd} \delta h^{(f,2)} + 2 \varepsilon_{abcd} \alpha_5 l^2 R^{ab} R^{cd} \delta T^{f} \\
+ \varepsilon_{abcd} \delta h^{(a,b)} \left( 2 \alpha_3 l^2 R^{ab} \mathcal{R}^{(cd,1)} + 2 \alpha_5 l^2 R^{ab} \mathcal{R}^{(cd,1)} + 2 \alpha_5 R^{ab} e^c \epsilon^d \right) \delta h^{(f,1)} \\
+ 2 \alpha_5 l^2 D_{\omega} k^{(cd,1)} T^f + \varepsilon_{cd} \delta h \left( 4 \alpha_5 l^2 h_{b}^{(1)} R^{df} \delta g + 2 \alpha_5 l^2 R_{b}^{(df,1)} \delta g \right) \\
+ \varepsilon_{abcd} \delta h^{ab} \left[ 2 \alpha_3 l^2 R^{cd} T^f + 2 \alpha_3 l^2 R^{cd} D_{\omega} h^{(f,1)} + 2 \alpha_3 R^{cd} \mathcal{R}^{(cd,1)} T^{f} \\
- 2 \alpha_3 l^2 R^{cd} \mathcal{R}^{(cd,1)} R^{fg} \delta g + 2 \alpha_3 e^c \delta g T^f + 2 \alpha_3 l^2 R^{cd} D_{\omega} h^{(f,1)} + 2 \alpha_3 \mathcal{R}^{(cd,1)} D_{\omega} h^{(f,1)} \\
- 2 \alpha_3 l^2 \mathcal{R}^{(cd,1)} R^{fg} \delta g + 2 \alpha_3 l^2 D_{\omega} k^{(cd,2)} T^{f} - 2 \alpha_3 l^2 \mathcal{R}^{(cd,1)} R^{fg} \delta g \\
+ 4 \alpha_5 \mathcal{R}^{(cd,1)} T^{f} + 4 \alpha_5 e^c \delta g h^{(f,1)} + 2 \alpha_5 e^c \delta g h^{(f,1)} \right] \\
+ \varepsilon_{abcd} \delta h^{ab} \left[ 2 \alpha_3 l^2 e_b R^{cd} k^{(fg,1)} + 2 \alpha_5 l^2 h_{b}^{(1)} R^{cd} k^{(fg,1)} + 2 \alpha_5 l^2 e_b R^{cd} k^{(fg,1)} \\
- 2 \alpha_3 l^2 |k^{(cd,1)} k^{(fg,1)} + 2 \alpha_5 l^2 |k^{(cd,1)} k^{(fg,1)} + 2 \alpha_5 e^c \delta g h^{(f,1)} \right]
\]

When a solution without matter \((k^{(a,b)} = 0, k^{(a,b,2)} = 0, h^{(a,1)} = 0, h^{(a,2)} = 0)\) is singled out, we are left with

\[
\delta L_{(4+1)}^{M_7} = \varepsilon_{abcd} \left[ \left( \alpha_1 l^2 R^{ab} R^{cd} + 2 \alpha_3 l^2 R^{ab} \mathcal{R}^{(cd,1)} + 2 \alpha_5 l^2 R^{ab} \mathcal{R}^{(cd,2)} + \frac{1}{l^2} \alpha_5 e^a \epsilon^b \epsilon^c \epsilon^d \right) \delta e^f \\
+ \left( \alpha_3 l^2 R^{ab} R^{cd} + 2 \alpha_5 R^{ab} e^c \epsilon^d \right) \delta h^{(f,1)} + 2 \alpha_5 l^2 \mathcal{R}^{(cd,1)} + 2 \alpha_5 R^{ab} R^{cd} \delta h^{(f,2)} \\
+ \delta h^{(a,b)} \left( 2 \alpha_3 l^2 R^{cd} T^{f} + 2 \alpha_3 e^c \delta g T^{f} \right) + \delta h \left( 2 \alpha_3 l^2 R^{cd} T^{f} + 2 \alpha_3 e^c \delta g T^{f} \right) \right]
\]

So when \(\alpha_1\) and \(\alpha_5\) vanish we finally get

\[
\delta L_{(4+1)}^{M_7} = \varepsilon_{abcd} \left( 2 \alpha_3 R^{ab} e^c \epsilon^d \right) \delta e^f + \varepsilon_{abcd} \left( \alpha_3 l^2 R^{ab} R^{cd} \right) \delta h^{(f,1)} \\
+ \varepsilon_{abcd} \delta h^{(a,b)} \left( 2 \alpha_3 l^2 R^{cd} T^{f} \right) + \delta h \left( 2 \alpha_3 l^2 R^{cd} T^{f} \right)
\]

Therefore, if we impose the torsionless condition, we see that the Chern-Simons Lagrangian in \(D = 5\) invariant under \(M_7\) leads to the same equations of motion than the Chern-Simons Lagrangian in \(D = 5\) invariant under \(M_5\) [10]. From (82), like in Ref. (10), we can see that in the limit where \(l = 0\) the extra constraints just vanish, and \(\delta L_{\text{CS}} = 0\) leads us to the Einstein-Hilbert dynamics in vacuum,

\[
\delta L_{(4+1)}^{M_7} = \varepsilon_{abcd} \left( 2 \alpha_3 R^{ab} e^c \epsilon^d \right) \delta e^f + \varepsilon_{abcd} \delta h \left( 2 \alpha_3 e^c \delta g T^{f} \right) \].

(83)
Similarly, when the cosmological constant is not considered and a solution without matter is singled out, the strict limit where the coupling constant \( l \) equals zero yields just to the Einstein-Hilbert term in the Lagrangian

\[
L_{CS}^{(4+1)} = \frac{2}{3} \alpha_3 \varepsilon_{abcd} R^{ab}e^c e^d. \tag{84}
\]

### 3.3 \((6+1)\)-dimensional Chern-Simons Lagrangian invariant under \( \mathcal{M}_5 \)-algebra

Now, consider a Chern-Simons action \((6+1)\)-dimensional invariant under \( \mathcal{M}_5 \)-algebra. The 1-form gauge connection \( A \mathcal{M}_5 \)-valued, is given by

\[
A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a, \tag{85}
\]

and the 2-form curvature is given by

\[
F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2} \left( D \omega^{ab} + \frac{1}{l^2} e^a e^b \right) Z_{ab} + \frac{1}{l} \left( D \omega^a + k^a e^b \right) Z_a. \tag{86}
\]

Using the dual procedure of S-expansion, we find that the 7-dimensional Chern-Simons Lagrangian invariant under the \( \mathcal{M}_5 \)-algebra is given by

\[
L_{\mathcal{M}_5}^{(6+1)} = \frac{\alpha_1}{l} \varepsilon_{abcdefg} \left( R^{ab} R^{cd} R^{ef} e^g \right) \\
+ \frac{\alpha_3}{l} \varepsilon_{abcdefg} \left( R^{ab} R^{cd} R^{ef} h^g + 3 R^{ab} R^{cd} D \omega^{k e f} e^g + \frac{1}{l^2} R^{ab} R^{cd} e^f e^g \right). \tag{87}
\]

where \( \alpha_1 = \lambda_1 \kappa, \alpha_3 = \lambda_3 \kappa \). From here we see that the Einstein-Hilbert term is not present in the Lagrangian. This result holds for all \( D = p \)-dimensional Chern-Simons Lagrangian invariant under an algebra \( \mathcal{M}_m \) if \( p > m \).

Varying the Lagrangian we have

\[
\delta L_{\mathcal{M}_5}^{(6+1)} = \frac{1}{l} \varepsilon_{abcdefg} \left( \alpha_1 R^{ab} R^{cd} R^{ef} + 3 \alpha_3 R^{ab} R^{cd} D \omega^{k e f} + \frac{3}{l^2} \alpha_3 R^{ab} R^{cd} e^f e^g \right) \delta e^g \\
+ \frac{1}{l} \varepsilon_{abcdefg} \left( \alpha_3 R^{ab} R^{cd} R^{ef} \right) \delta h^g \\
+ \frac{1}{l} \varepsilon_{abcdefg} \delta \omega^{ab} \left( 3 \alpha_1 R^{cd} R^{ef} T^g + 3 \alpha_3 R^{cd} R^{ef} D \omega^a + 6 \alpha_3 R^{cd} D \omega^{k e f} T^g \right) \\
+ \frac{6}{l^2} \alpha_3 R^{cd} e^f e^g T^g + \frac{1}{l} \varepsilon_{abcdg} \delta \omega^{ab} \left( 3 e^b R^{cd} R^{ef} k^{gh} \right) \\
+ \frac{1}{l} \varepsilon_{abcdefg} \delta h^{ab} \left( 3 \alpha_3 R^{cd} R^{ef} T^g \right). \tag{88}
\]

from which we can see it is not possible to obtain the Einstein-Hilbert dynamics.
In fact, imposing the torsionless condition and if we consider the case where $k^{ab} = 0$, $h^a = 0$ with $\alpha_1 = 0$ we find

$$\delta L_{(6+1)}^B = \frac{\alpha_3}{l^2} \varepsilon_{abcdefg} R^{ab} R^{cd} e^f e^g \delta e^a + \frac{\alpha_3}{l} \varepsilon_{abcdefg} R^{ab} R^{cd} \delta h^g, \quad (89)$$

which obviously does not correspond to the dynamics of General Relativity.

### 3.4 (6+1)-dimensional Chern-Simons Lagrangian invariant under $M_7$-algebra

Consider the 7-dimensional Chern-Simons Lagrangian invariant under the $M_7$-algebra

$$L_{\text{CS}}^{\text{AdS}}_{(6+1)} = \kappa \left[ \varepsilon_{abcdefg} \left( \frac{1}{l} R^{ab} R^{cd} e^f e^g + \frac{3}{5 l^2} R^{ab} e^d e^e R^{bf} e^g + \frac{1}{l^2} e^a e^b e^c e^d e^e e^f e^g \right) \right]$$

$$+ \beta_{2,2} \left[ R^a_b R^b_a + \frac{2}{l^2} \left( T^a T_a - R^{ab} e_a e_b \right) \right] \left( \omega^c_d d\omega^d_c + \frac{2}{3} \omega^f_d \omega^f_g + \frac{2}{l^2} \omega^e_c T^e \right)$$

$$+ \beta_4 \left[ \omega^a_b \omega^c_d d\omega^d_c + \frac{8}{5} \omega^a_b \omega^c_d d\omega^d_c + \frac{4}{5} \omega^a_b \omega^c_d d\omega^d_c + \frac{4}{5} \omega^a_b \omega^c_d d\omega^d_c \omega^e_f \omega^f_g + \frac{4}{l^2} \omega^a_b \omega^c_d d\omega^d_c \omega^e_f \omega^f_g \right]$$

$$+ \frac{1}{l^2} 4 T^a_b R^a_b R^b_a + \frac{1}{l^4} \left[ 2 \left( R^{ab} e_a e_b + T^a T_a \right) T^c e_c \right].$$

Using the dual procedure of S-expansion, we find that the 7-dimensional Chern-Simons Lagrangian invariant under the $M_7$-algebra is given by
\[ L_{CS}^{(6+1)} \]

\[
= \alpha_1 t^4 \epsilon_{abcdefg} R^{ab} R^{cd} R^{ef} e^g + \alpha_3 \epsilon_{abcdefg} \left( t^4 R^{ab} R^{cd} R^{ef} h^{(g,1)} + 3 t^4 R^{ab} R^{cd} R^{ef} (ef,1) e^g + 1 t^2 R^{ab} R^{cd} e^e e^f e^g \right) \\
+ \alpha_5 \epsilon_{abcdefg} \left( t^4 R^{ab} R^{cd} R^{ef} h^{(g,2)} + 3 t^4 R^{ab} R^{cd} R^{ef} (ef,1) e^g + 3 t^4 R^{ab} R^{cd} R^{ef} (ef,2) e^g \right) \\
+ 3 t^4 R^{ab} R^{cd} R^{ef} \delta^{(ef,1)} h^{(g,1)} + 2 t^2 R^{ab} R^{cd} e^e e^f e^g + 3 t^2 R^{ab} R^{cd} e^e e^f h^{(g,1)} + \frac{3}{5} R^{ab} e^e e^d e^f e^g \\
+ \alpha_{\{2,2\}} t^5 \left[ (R^a_b R^b_a) L_3^{\text{Lorentz}} \right] \\
+ \alpha_{\{2,2\}} t^5 \left[ (R^a_b R^b_a) L_3^{\text{Lorentz}} \left( k^{(1)} \right) + \frac{2}{12} T^c e_T^c \right] + 2 \left( R^a_b R^b_a \right) L_3^{\text{Lorentz}} \\
+ \frac{2}{12} \left( T^a T_a - R^{ab} e_a e_b \right) L_3^{\text{Lorentz}} \\
+ \alpha_{\{2,2\}} t^5 \left[ R^a_b R^b_a \right] L_3^{\text{Lorentz}} + \frac{2}{12} \left( T^a T_a - R^{ab} e_a e_b \right) L_3^{\text{Lorentz}} \left( k^{(1)} \right) + \frac{2}{12} T^c e_T^c \\
+ \frac{2}{12} \left( 2 T^a T^a \left( 1 \right) - 2 R^{ab} e_a e_b \left( 1 \right) - \delta^{ab,1} e_a e_b \right) L_3^{\text{Lorentz}} \\
+ \alpha_{\{4\}} t^5 \left[ L_7^{\text{Lorentz}} + \alpha_{\{4\}} t^5 \left[ L_7^{\text{Lorentz}} \left( k^{(1)} \right) + \frac{1}{12} 4 T_a R^a_b R^b_c e^c \right] \\
+ \alpha_{\{4\}} t^5 \left[ L_7^{\text{Lorentz}} \left( k^{(2)} \right) + L_7^{\text{Lorentz}} \left( k^{(1)} k^{(1)} \right) \right] \\
+ \frac{4}{12} \left( T^a R^a_b R^b_c h^{(c,1)} + \delta^{(1)} R^a_b R^b_c e^c + T_a R^a_b R^b_c e^c + T_a \delta^{(1)} R^b_c e^c \right) \\
+ \frac{1}{12} \left[ 2 \left( R^{ab} e_a e_b + T^a T_a \right) T^c e_c \right]. \tag{90} 
\]

The Lagrangian \((90)\) is split into nine independent pieces, each one proportional to \(\alpha_1, \alpha_3, \alpha_5, \alpha_{\{2,2\}}, \alpha_{\{4\}}, \alpha_{\{4\}}, \alpha_{\{4\}}, \alpha_{\{4\}}\), and \(\alpha_{\{4\}}\). The term proportional to \(\alpha_1\) corresponds to the Chern-Simons Lagrangian for ISO \((6, 1)\) group. The Einstein-Hilbert term \(\epsilon_{abcdefg} R^{ab} e^e d e^f e^g\) appears in the term proportional to \(\alpha_5\).

Varying the Lagrangian \((90)\) for the case \(\alpha_{\{2,2\}} = \alpha_{\{2,2\}} = \alpha_{\{4\}} = \alpha_{\{4\}} = 0\), we have
\[ \delta L_{CS}^{M_7} = \varepsilon_{abcdefg} \left( \alpha_1 R^{ab} R^{cd} R^{ef} + 3 \alpha_3 R^{ab} R^{cd} R^{ef} \alpha^{(ef,1)} + 3 \alpha_5 \right) \delta h^{(g,1)} + \varepsilon_{abcdefg} \left( \alpha_1 R^{ab} R^{cd} R^{ef} + 3 \alpha_3 R^{ab} R^{cd} R^{ef} \right) \delta h^{(g,2)} \]

In the event that (i) \( \alpha_1 \) and \( \alpha_3 \) are zero, (ii) the torsionless condition is imposed and (iii) \( k^{(ab,1)} = 0, k^{(ab,2)} = 0, h^{(a,1)} = 0, h^{(a,2)} = 0 \), is found

\[ \delta L_{CS}^{M_7} = \varepsilon_{abcdefg} \left( 3 \alpha_5 R^{ab} R^{cd} R^{ef} \right) \delta e^g + \varepsilon_{abcdefg} \left( 3 \alpha_5 \right) \delta h^{(g,1)} + \varepsilon_{abcdefg} \left( \alpha_5 \right) \delta h^{(g,2)}. \]  

Where we see that in the limit \( l \to 0 \) we have that the Lagrangian leads to the Einstein Hilbert term

\[ \frac{3}{5} \varepsilon_{abcdefg} R^{ab} R^{cd} R^{ef} R^{cg}. \]  

and the condition \( \delta L_{CS}^{M_7} = 0 \) leads to the Einstein equations,

\[ \delta L_{CS}^{M_7} = \varepsilon_{abcdefg} \left( 3 \alpha_5 R^{ab} R^{cd} R^{ef} \right) \delta e^g + \varepsilon_{abcdefg} \delta \omega^{ab} \left( 3 \alpha_5 e^a e^d e^e e^f T^g \right). \]  

The results show that the \((2p + 1)\)-dimensional Chern-Simons actions invariant under the algebra \( M_{2m+1} \) does not always lead to the action of General Relativity. Indeed, for certain values of \( m \) is impossible to obtain the Einstein-Hilbert term in the \((2p + 1)\)-dimensional Chern-Simons Lagrangian invariant under \( M_{2m+1} \). This is because to obtain the term Einstein-Hilbert, is necessary the presence of the \( \left\langle J_{a_1 a_2} Z_{a_3 a_4} \cdots Z_{a_{2p-1} a_{2p}} P_{2p+1} \right\rangle \) component of the invariant tensor, which is given by
\[ \langle J_{a_1} \cdots a_2 Z_{a_{2p-1}} a_2 P_{a_{2p+1}} \rangle_{\mathcal{M}_{2p+1}} = \begin{cases} t^{2p-1} \alpha_{2p-1} \langle J_{a_1} \cdots a_2 \rangle_{\text{AdS}}, & \text{if } m \geq p \\ 0, & \text{if } m < p. \end{cases} \]  

(94)

This observation leads to state the following theorem:

**Theorem 4** Let \( \mathcal{M}_{2m+1} \) be the Maxwell type algebra, which is obtained from AdS algebra by \( S_{E}^{(2m-1)} \)-expansion resonant reduced. If \( L^{\mathcal{M}_{2m+1}}_{CS} \) is a Chern-Simons Lagrangian \((2p + 1)\)-dimensional invariant under the \( \mathcal{M}_{2m+1} \)-algebra, then the \((2p + 1)\)-dimensional Chern-Simons Lagrangian, lead to the Einstein-Hilbert Lagrangian in a certain limit of the coupling constant \( l \), if and only if \( m \geq p \).

The following table shows a set of Chern-Simons Lagrangian \( L^{\mathcal{M}_{2m+1}}_{CS} \), invariant under the Lie algebra \( \mathcal{M}_{2m+1} \), that flow into the General Relativity Lagrangian in a certain limit:

\[
\begin{array}{|c|c|c|c|}
\hline
\mathcal{M}_3 & L^{\mathcal{M}_3}_{CS} (3) & L^{\mathcal{M}_3}_{CS} (5) & L^{\mathcal{M}_3}_{CS} (7) \\
\hline
\mathcal{M}_5 & L^{\mathcal{M}_5}_{CS} (3) & L^{\mathcal{M}_5}_{CS} (5) & L^{\mathcal{M}_5}_{CS} (7) \\
\hline
\mathcal{M}_7 & L^{\mathcal{M}_7}_{CS} (3) & L^{\mathcal{M}_7}_{CS} (5) & L^{\mathcal{M}_7}_{CS} (7) \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
\mathcal{M}_{2n-1} & L^{\mathcal{M}_{2n-1}}_{CS} (3) & L^{\mathcal{M}_{2n-1}}_{CS} (5) & L^{\mathcal{M}_{2n-1}}_{CS} (7) & \cdots & L^{\mathcal{M}_{2n-1}}_{CS} (2n-1) \\
\hline
\mathcal{M}_{2n+1} & L^{\mathcal{M}_{2n+1}}_{CS} (3) & L^{\mathcal{M}_{2n+1}}_{CS} (5) & L^{\mathcal{M}_{2n+1}}_{CS} (7) & \cdots & L^{\mathcal{M}_{2n+1}}_{CS} (2n+1) \\
\hline
\end{array}
\]

(95)

It is interesting to note that for each dimension \( D \) of space-time, we have the Lagrangian \( L_{CS}^{(D)} \) invariant under the algebra \( \mathcal{M}_{2n+1} \) contains all other \( D \)-dimensional Lagrangian valued in an algebra \( \mathcal{M}_{2m+1} \) with \( m < n \). So it is always possible to obtain an action of a lower algebra off the appropriate fields.

### 4 Born-Infeld Lagrangians invariant under the subalgebra \( \mathfrak{L}^{\mathcal{M}} \)

In this section is shown that the even-dimensional Einstein-Hilbert Lagrangian can be obtained from a Born-Infeld Lagrangian in \((2p)\)-dimensions invariant under the subalgebra \( L^{\mathcal{M}_{2m}} \) of the algebra \( \mathcal{M}_{2m} \), if and only if \( m \geq p \). However, this is not possible when \( m < p \) for Born-Infeld Lagrangian in \((2p)\)-dimensions invariant under the subalgebra \( L^{\mathcal{M}_{2m}} \).

#### 4.1 Born-Infeld lagrangian in \( D = 4 \) invariant under \( \mathfrak{L}^{\mathcal{M}_{5}} \)

Following the definitions of Ref. [13], let us consider the \( S \)-expansion of the Lie algebra \( SO (3, 1) \) using as a semigroup the sub-semigroup \( S^{(3)}_0 = \{ \lambda_0, \lambda_2, \lambda_4 \} \) of
semigroup $S_E^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. After performing its $0_S(= \lambda_4)$-reduction, one finds a new Lie algebra, call it $L^{M_5}$ which is a subalgebra of the so called $M_5$ algebra, whose generators $J_{ab} = \lambda_0 \tilde{J}_{ab}$, $Z_{ab} = \lambda_2 \tilde{J}_{ab}$, satisfy the commutation relationships
\[
\begin{align*}
\{J_{ab}, J_{cd}\} &= \eta_{cb} \tilde{J}_{ad} - \eta_{ca} \tilde{J}_{bd} - \eta_{db} \tilde{J}_{ca} + \eta_{da} \tilde{J}_{cb}, \\
\{J_{ab}, Z_{cd}\} &= \eta_{cb} \tilde{Z}_{ad} - \eta_{ca} \tilde{Z}_{bd} + \eta_{db} \tilde{Z}_{ca} - \eta_{da} \tilde{Z}_{cb}, \\
\{Z_{ab}, Z_{cd}\} &= \eta_{cd} \tilde{Z}_{ab}.
\end{align*}
\]

In order to write down a Born-Infeld, we start from the two-form $L^{M_5}$ curvature $F$
\[
F = \frac{1}{2} R_{ab} \tilde{J}_{ac} + \frac{1}{2} \left( D_\omega k_{ab} + \frac{1}{l^2} \epsilon_ {a} \epsilon_{b} \right) \tilde{Z}_{ab}.
\]

Using Theorem VII.2 of Ref. [13], it is possible to show that the only non-vanishing components of a invariant tensor for the $L^{M_5}$ algebra are given by
\[
\begin{align*}
\langle J_{ab} J_{cd} \rangle_{L^{M_5}} &= \alpha_0 l^2 \epsilon_{abcd}, \\
\langle J_{ab} Z_{cd} \rangle_{L^{M_5}} &= \alpha_2 l^2 \epsilon_{abcd}.
\end{align*}
\]

where $\alpha_0$ and $\alpha_2$ are arbitrary independent constants of dimensions [length]$^{-2}$.

Using the dual procedure of $S$-expansion in terms of the Maurer-Cartan forms [14], we find that the 4-dimensional Born-Infeld Lagrangian invariant under the $L^{M_5}$ algebra is given by [11]
\[
L_{BI}^{L^{M_5}} (4) = \frac{\alpha_0}{4} \epsilon_{abcd} l^2 R_{ab} R_{cd} + \frac{\alpha_2}{2} \epsilon_{abcd} \left( R_{ab} c_{e} e_{d} + l^2 D_\omega k_{ab} R_{cd} \right).
\]

Here we can see that the Lagrangian (100) is split into two independent pieces, one proportional to $\alpha_0$ and the other to $\alpha_2$. The term proportional to $\alpha_0$ corresponds to the Euler invariant. The piece proportional to $\alpha_2$ contains the Einstein-Hilbert term $\epsilon_{abcd} R_{ab} c_{e} e_{d}$ plus a boundary term which contains, besides the usual curvature $R_{ab}$, a bosonic matter field $k_{ab}$.

Unlike the Born-Infeld Lagrangian the coupling constant $l^2$ does not appear explicitly in the Einstein Hilbert term but accompanies the remaining elements of the Lagrangian. This allows recover four dimensional the Einstein-Hilbert Lagrangian in the limit where $l$ equals to zero.

The variation of the Lagrangian, modulo boundary terms, is given by
\[
\delta L_{BI}^{L^{M_5}} (4) = \epsilon_{abcd} \left( \alpha_2 R_{ab} c_{e} e_{d} + \epsilon_{abcd} \epsilon_{abcd} \left( \alpha_2 T_{c} e_{d} + \alpha_2 k_{ab} e_{cd} \right) \right).
\]
from which we see that to recover the field equations of general relativity is not necessary to impose the limit $l = 0$. $\delta L_{BI}^{L^{M_5}} (4) = 0$ leads to the dynamics of Relativity when considering the case of a solution without matter ($k_{ab} = 0$). This is possible only in 4 dimensions. However, to recover the field equations of general relativity in dimensions greater than 4, is necessary to take a limit of the coupling constant $l$. 

21
4.2 Born-Infeld lagrangian in $D = 4$ invariant under $\mathfrak{L}^{M_7}$ algebra

Now, we consider the Born-Infeld lagrangian in $D = 4$ invariant under $\mathfrak{L}^{M_7}$ algebra whose generators satisfy the following commutation relations

\begin{align*}
[J_{ab}, J_{cd}] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \\
[J_{ab}, Z^{(1)}_{cd}] &= \eta_{cb} Z^{(1)}_{ad} - \eta_{ca} Z^{(1)}_{bd} + \eta_{db} Z^{(1)}_{ca} - \eta_{da} Z^{(1)}_{cb} \\
[J_{ab}, Z^{(2)}_{cd}] &= \eta_{cb} Z^{(2)}_{ad} - \eta_{ca} Z^{(2)}_{bd} + \eta_{db} Z^{(2)}_{ca} - \eta_{da} Z^{(2)}_{cb} \\
[Z^{(1)}_{ab}, Z^{(1)}_{cd}] &= 0 = [Z^{(2)}_{ab}, Z^{(2)}_{cd}].
\end{align*}

The two-form curvature $S^{(3)}_0$-expanded reduced is

\begin{align*}
F &= \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l^a} T^a + \frac{1}{2} \left( D_\omega k^{(ab,1)} + \frac{1}{l^a} e^a e^b \right) Z^{(1)}_{ab} \\
&\quad + \frac{1}{2} \left( D_\omega k^{(ab,2)} + k^a (1) k^{cb(1)} + \frac{1}{l^2} \left( e^a h^{(b,1)} + h^{(a,1)} e^b \right) \right) Z^{(2)}_{ab}.
\end{align*}

Using theorem VII.2 of Ref. [13], it is possible to show that the only non-vanishing components of a invariant tensor for the $\mathfrak{L}^{M_7}$ algebra are given by

\begin{align*}
\langle J_{ab} J_{cd} \rangle_{\mathfrak{L}^{M_7}} &= \alpha_0 \varepsilon_{abcd}, \\
\langle J_{ab} Z^{(1)}_{cd} \rangle_{\mathfrak{L}^{M_7}} &= \alpha_2 \varepsilon_{abcd}, \\
\langle J_{ab} Z^{(2)}_{cd} \rangle_{\mathfrak{L}^{M_7}} &= \langle Z^{(1)}_{ab} Z^{(1)}_{cd} \rangle_{\mathfrak{L}^{M_7}} = \alpha_4 \varepsilon_{abcd},
\end{align*}

where $\alpha_0$, $\alpha_2$, and $\alpha_4$ are arbitrary independent constants dimensionless.

Using the dual procedure of $S$-expansion in terms of the Maurer-Cartan forms [14], we find that the 4-dimensional Born-Infeld Lagrangian invariant under the $\mathfrak{L}^{M_7}$ algebra is given by

\begin{align*}
L_{BI}^{M_7} (4) &= \frac{\alpha_0}{4} \varepsilon_{abcd} R^{ab} R^{cd} + \frac{\alpha_2}{2} \varepsilon_{abcd} \left( \mathcal{R}^{(ab,1)} R^{cd} + \frac{1}{l^2} R^{ab} e^c e^d \right) \\
&\quad + \frac{\alpha_4}{4} \varepsilon_{abcd} \left( \mathcal{R}^{(ab,1)} R^{(cd,1)} + \mathcal{R}^{(ab,2)} R^{cd} + \frac{2}{l^2} \mathcal{R}^{(ab,1)} e^c e^d + \frac{4}{l^2} R^{ab} h^{(c,1)} e^d + \frac{1}{l^4} e^a e^b e^c e^d \right),
\end{align*}

where

\begin{align*}
\mathcal{R}^{(ab,1)} &= D_\omega k^{(ab,1)}, \\
\mathcal{R}^{(ab,2)} &= D_\omega k^{(ab,2)} + k^a (1) k^{cb(1)}.
\end{align*}
The Lagrangian \[ (111) \] is split into three independent pieces, each one proportional to \( \alpha_0, \alpha_2, \alpha_4 \) respectively. The term proportional to \( \alpha_0 \) corresponds to the Euler invariant. The piece proportional to \( \alpha_2 \) contains the Einstein-Hilbert term \( \varepsilon_{abcd} R^{ab} e^c e^d \) plus a boundary term which contains, besides the usual curvature \( R^{ab} \), a bosonic matter field \( k^{(ab)} \).

The variation of the Lagrangian, modulo boundary terms, is given by

\[
\delta L_{BI}^{\mathcal{M}_7} = \varepsilon_{abcd} \left( \frac{\alpha_2}{l^2} R^{ab} e^c e^d + \frac{\alpha_4}{l^2} \delta \mathcal{M}_7^{(ab,1)} e^c e^d \right) \delta e^d + \varepsilon_{abcd} \left( \frac{\alpha_4}{l^2} R^{ab} e^c e^d + \frac{\alpha_4}{l^2} \delta \mathcal{M}_7^{(ab,1)} e^c e^d \right) \delta e^d
\]

\[
+ \varepsilon_{abcd} \left( \frac{4}{l^2} F_{ab} \right) \delta h^{(d,1)} + \varepsilon_{abcd} \delta \omega_{ab} \left( \frac{\alpha_2}{l^2} h_{(c,1)} + \frac{\alpha_4}{l^2} \delta \omega_{ab} \right) \delta h^{(d,1)}
\]

where \( \mathcal{M}_7^{(a)} = D \omega h^{(a,1)} + k^{(1)} e^c \). If we consider the case where \( k^{(ab,1)} = h^{(a,1)} = 0 \), we have

\[
\delta L_{BI}^{\mathcal{M}_7} = \varepsilon_{abcd} \left( \frac{\alpha_2}{l^2} R^{ab} e^c e^d + \frac{\alpha_4}{l^2} \delta \mathcal{M}_7^{(ab,1)} e^c e^d \right) \delta e^d + \varepsilon_{abcd} \left( \frac{\alpha_4}{l^2} R^{ab} e^c e^d + \frac{\alpha_4}{l^2} \delta \mathcal{M}_7^{(ab,1)} e^c e^d \right) \delta e^d
\]

from where

\[
\varepsilon_{abcd} R^{ab} e^c = 0, \quad \varepsilon_{abcd} T^{cd} e^d = 0.
\]

That is, we have obtained the Einstein-Hilbert dynamics in a vacuum without any restriction on the coupling constant \( l \).

### 4.3 Born-Infeld Lagrangian in \( D = 4 \) invariant under \( \mathcal{M}_{2n+1} \)

The generators of the \( \mathcal{M}_{2n+1} \) algebra satisfy the commutation relation \[ (28) \] \[ (33) \]. The corresponding 1-form gauge connection \( A \) and the two-form curvature \( \mathcal{M}_{2n+1} \) valued \( F = dA + A^2 \) are given in \[ (16) \] \[ (17) \]. The generators of the \( \mathcal{M}_{2n+1} \) algebra satisfy the following commutation relation

\[
[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{bd} J_{ca} - \eta_{da} J_{cb}
\]

\[
\begin{bmatrix}
J_{ab} Z^{(i)}
\end{bmatrix} = \eta_{bc} Z_{ai} - \eta_{bd} Z_{ca} - \eta_{da} Z_{cb}
\]

\[
\begin{bmatrix}
Z_{ab}^{(i)}
\end{bmatrix} = \eta_{bc} Z_{ai}^{(i+j)} - \eta_{bd} Z_{ca}^{(i+j)} - \eta_{da} Z_{cb}^{(i+j)},
\]

Bearing in mind that the non-zero components tensor invariant, are given by

\[
\langle J_{(ab,2j)} J_{(cd,2j)} \rangle = \alpha_{2j+2j} \varepsilon_{abcd},
\]
where \( \alpha_{2i+2j} \) are arbitrary independent constants dimensionless and where we have defined

\[
J_{ab} = \lambda_0 J_{ab} = J_{(ab,0)} \\
Z^{(i)}_{ab} = \lambda_{2i} J_{(ab,i)}
\]

with \( i = 1, ..., n - 1 \).

Using the same procedure used in the previous cases, we found that the Lagrangian \( L_{BI}^{6n} \) is given by

\[
L_{BI}^{6n+1} = \frac{\alpha_{2i+2j}}{4} \varepsilon_{abcd} F^{(ab,2i)} F^{(cd,2j)}.
\]

Varying the Lagrangian and considering the case without matter, \( \dot{k}^{(ab,i)} = h^{(a,j)} = 0 \), we have

\[
\delta L_{BI}^{6n+1} = \varepsilon_{abcd} \left( \frac{\alpha_2}{l^2} R^{a} e^{c} + \frac{\alpha_4}{l^4} e^{a} e^{b} e^{c} \right) \delta e^{d} + \varepsilon_{abcd} \left( \frac{\alpha_{i+1}}{l^2} R^{a} e^{c} \right) \delta h^{(d,i)}
\]

\[
+ \varepsilon_{abcd} \dot{k}^{ab} \left( \frac{\alpha_2}{l^2} T^{c} e^{d} \right) + \varepsilon_{abcd} \delta k^{(ab,i)} \left( \frac{\alpha_{i+1}}{l^2} T^{c} e^{d} \right).
\]

Equations leading to the field equations of general relativity

\[
\varepsilon_{abcd} R^{a} e^{c} = 0,
\]

\[
\varepsilon_{abcd} T c^{e} d = 0.
\]

### 4.4 Born-Infeld Lagrangian in \( D = 6 \) invariant under \( L^{M_{2n}} \) algebra

It should be noted that the \( L^{M_{2n+1}} \) algebra has the property of being identical to the \( L^{M_{2n}} \) algebra. However, they have different origins: The \( L^{M_{2n+1}} \) algebra corresponds to a reduced \( S_{0}^{(2n-1)} \)-expansion of the Lorentz algebra, as we have seen previously, and the \( L^{M_{2n}} \) algebra corresponds to a reduced \( S_{0}^{(2n-2)} \)-expansion of the Lorentz algebra, where the semigroup \( S_{0}^{(2n-2)} \) is a subsemigroup of the semigroup \( S_{E}^{(2n-2)} = \{ \lambda_{i} \}_{i=0}^{2n-1} \).

It is also interesting to note that the \( L^{M_{2n}} \) algebra can be used to construct different even-dimensional Born-Infeld type lagrangians. For example, if we consider a reduced \( S_{0}^{(4)} \)-expansion of the Lorentz algebra \( SO(3,1) \), the \( L^{M_{6}} \) algebra in \( D = 4 \) dimensions is obtained, and if we consider a reduced \( S_{0}^{(4)} \)-expansion of the Lorentz algebra \( SO(5,1) \), then, we get the \( L^{M_{6}} \) algebra in \( D = 6 \) dimensions. In this way, the lagrangians \( L_{BI}^{6n} \) and \( L_{BI}^{6n+1} \), are invariant under the same algebra \( L^{M_{6}} \), but the indices in the generators \( J_{ab} \), runs over 4 and 6 values, respectively.

These considerations allow the construction of gravitational theories in every even-dimension. However, as discussed below, only in some dimensions it is possible to obtain General Relativity as a weak coupling constant limit of a Born-Infeld theory.
4.4.1 Born-Infeld Lagrangian in $D = 6$ invariant under $\mathfrak{L}_{M^4}$

The Born-Infeld lagrangian invariant under Lorentz algebra is given by

$$L_{BI}^{(6)} = \frac{k}{6} \epsilon_{abcdef} \left( R^{ab} R^{cd} R^{ef} + \frac{3}{l^2} R^{ab} R^{cd} e^e e^f + \frac{3}{l^2} R^{ab} e^e e^d e^f + \frac{1}{l^6} e^a e^b e^c e^d e^e e^f \right).$$

(125)

Following the definitions of Ref. [13], let us consider the $S$-expansion of the Lie algebra $SO(5,1)$ using $S_E^{(2)} = \{\lambda_0, \lambda_2, \lambda_3\}$ as subsemigroup of $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$. After performing its $0_S$-reduction, one finds the $L_{M^4}$ algebra which corresponds to a subalgebra of $M_4$ algebra. The new algebra is generated by $\{J_{ab}, Z_{ab}\}$, where these new generators can be written as

$$\lambda_0 \otimes \tilde{J}_{ab} = J_{ab},$$

$$\lambda_2 \otimes \tilde{J}_{ab} = Z_{ab}.$$  

(126)

(127)

In this case, $\tilde{J}_{ab}$ corresponds to the original generator of $SO(5,1)$ and the $\lambda_\alpha$ belong to a finite abelian semigroup $S^{(2)}_0$. Using the invariant tensors

$$\langle J_{ab} J_{cd} J_{ef} \rangle_{L_{M^4}} = \frac{4}{3} \alpha_0 \epsilon_{abcdef},$$

(128)

$$\langle J_{ab} J_{cd} Z_{ef} \rangle_{L_{M^4}} = \frac{4}{3} \alpha_2 \epsilon_{abcdef},$$

(129)

we find that the six-dimensional Born-Infeld lagrangian invariant under $\mathfrak{L}_{M^4}$ algebra is given by

$$L_{BI- (6)}^\mathfrak{L}_{M^4} = \frac{\alpha_0}{6} \epsilon_{abcdef} R^{ab} R^{cd} R^{ef} + \frac{\alpha_2}{2} \epsilon_{abcdef} \left( \mathfrak{R}^{ab} R^{cd} R^{ef} + \frac{1}{l^2} R^{ab} R^{cd} e^e e^f \right).$$

(130)

where $\mathfrak{R}^{ab} = D_\omega k^{ab}$.

Note that in this case the $S$-expansion procedure caused the Einstein-Hilbert term disappeared. This means that in the case of a six-dimensional Born-Infeld Lagrangian invariant under $\mathfrak{L}_{M^4}$ does not lead to General Relativity in no limit.

4.4.2 Born-Infeld Lagrangian in $D = 6$ invariant under $\mathfrak{L}_{M^6}$ algebra

In this case the 2-form curvature is given by

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2} \left( D_\omega k^{(ab,1)} + \frac{1}{l^2} e^a e^b \right) Z_{ab}^{(1)}$$

$$+ \frac{1}{2} \left( D_\omega k^{(ab,2)} + k^{(1)}_c k^{cb(1)} + \frac{1}{l^2} e^a h^{(b,1)} + h^{(a,1)} e^b \right) Z_{ab}^{(2)}. $$

(131)

Using the invariant tensors

$$\langle J_{ab} J_{cd} J_{ef} \rangle_{\mathfrak{L}_{M^6}} = \frac{4}{3} l^4 \alpha_0 \epsilon_{abcdef},$$

(132)
we find that the six-dimensional Born-Infeld lagrangian invariant under the algebra is given by
\[
L_{BI}^{M_6} = \frac{\alpha_4}{2} \varepsilon_{abcdef} R^{ab} e^d e^c e^f
\]
Varying the Lagrangian and considering the case without matter, \( k^{(ab,1)} = h^{(a,1)} = 0 \), we have
\[
\varepsilon_{abcdef} R^{ab} e^d e^c e^f = 0, \quad (134)
\]
\[
\varepsilon_{abcdef} T^e e^d e^c = 0. \quad (135)
\]
which are the Einstein equations in vacuum. Note that if in the Lagrangian \( L_{BI}^{M_6} \) take the limit \( l = 0 \), we obtain the Einstein Hilbert term.

\section*{4.5 Born-Infeld Lagrangian in \( D = 2n \) invariant under \( \mathfrak{L}^{M_{2n}} \)}

The generators of the algebra \( \mathfrak{L}^{M_{2n}} \) satisfy the following commutation relations
\[
\begin{align*}
\{ J_{ab}, J_{cd} \} &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \\
\{ J_{ab}, Z_{cd}^{(i)} \} &= \eta_{cb} Z_{ad}^{(i)} - \eta_{ca} Z_{bd}^{(i)} + \eta_{db} Z_{ca}^{(i)} - \eta_{da} Z_{cb}^{(i)} \\
\{ Z_{cd}^{(i)}, Z_{cd}^{(j)} \} &= \eta_{cb} Z_{ad}^{(i+j)} - \eta_{ca} Z_{bd}^{(i+j)} + \eta_{db} Z_{ca}^{(i+j)} - \eta_{da} Z_{cb}^{(i+j)},
\end{align*}
\]
where \( j = 0, \ldots, 2n - 2 \) and \( \alpha_j \) are arbitrary independent constants of dimensions \([\text{length}]^{2-2n}\).

\section{Proofs and Theorems}

\textbf{Theorem VII.2} of Ref. [10] allows us to see that the only nonzero components of the tensor invariant are given by
\[
\langle J_{(a_1 a_2 \cdots a_{2n-1}) j} \cdots J_{(a_{2n-i_n})} \rangle = \frac{\alpha_j}{n} \delta_{j_1 \cdots j_n} e_{a_1 \cdots a_{2n}}, \quad (137)
\]
In this case, the 2-form curvature is given by

\[
F = \sum_{k=0}^{n-1} \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)}
\]  

(138)

where

\[
F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd} d\omega^{(ac,2l)} \omega^{(db,2j)} \delta_{l+j} + \frac{1}{12} \epsilon^{(a,2l+1)} \epsilon^{(b,2j+1)} \delta_{l+j+1}.
\]  

(139)

Using the dual procedure of S-expansion in terms of the Maurer-Cartan forms [14], we find that the 2n-dimensional Born-Infeld Lagrangian invariant under the $M^{2n}$ algebra is given by

\[
L_{BI}^{M} (2n) = \sum_{k=1}^{n} f^{2k-2} \left( \frac{n}{2k} \right) \delta_{l+1} \cdots \delta_{p_1+q_1} \cdots \delta_{p_{n-k}+q_{n-k}}
\]

\[
\varepsilon_{a_1} \cdots a_{2n} R^{a_1 a_2 \cdots a_2n} \cdots (a_{2n} a_{2n-2})
\]

(140)

In the $l \rightarrow 0$ limit, the only surviving term in (140) is given by $k = 1$:

\[
L_{BI}^{M} (2n) \bigg|_{l=0} = \frac{1}{2} \delta_{k+1+k_1+\cdots+k_{2n-2}} \varepsilon_{a_1} \cdots a_{2n} R^{a_1 a_2 \cdots a_{2n}} e^{(a_{2n}, k_{2n-2})}
\]

(141)

The only non-vanishing component of this expression occurs for $p = q_1 = \cdots = q_{2n-2} = 0$, namely

\[
L_{BI}^{M} (2n) \bigg|_{l=0} = \frac{1}{2} \alpha_{2n-2} \varepsilon_{a_1} \cdots a_{2n} R^{a_1 a_2 \cdots a_{2n}} e^{a_{2n}, 1} \cdots e^{a_{2n}, 1}
\]

(142)

which is proportional to the Einstein-Hilbert lagrangian.

The results show that the 2p-dimensional Born-Infeld actions invariant under the algebra $M^{2n}$ does not always lead to the action of General Relativity. Indeed, for certain values of $m$ is impossible to obtain the Einstein-Hilbert term in the 2p-dimensional Born-Infeld type Lagrangian invariant under $M^{2n}$. This is because to obtain the term Hilbert-Einstein, is necessary the presence of the $(J_{a_1 a_2 Z_{a_3 a_4} \cdots Z_{a_{2p-1} a_{2p}}})$ component of the invariant tensor, which is given by
\[ \langle J_{a_1a_2} Z_{a_3a_4} \cdots Z_{a_{2p-1}a_{2p}} \rangle_{\mathcal{L}^{M_{2m}}} = \begin{cases} t^{2p-2} \alpha_{2p-2} \langle J_{a_1a_2} \cdots J_{a_{2p-1}a_{2p}} \rangle_{\Sigma}, & \text{if } m \geq p \\ 0, & \text{if } m < p. \end{cases} \] (143)

This observation leads to state the following theorem:

**Theorem 5** Let \( \mathcal{L}^{M_{2m}} \) be the algebra obtained from Lorentz algebra by reduced \( S_0^{(2m-2)} \)-expansion, which corresponds to a subalgebra of the \( M_{2m} \) algebra. If \( L_{BI-2p}^{M_{2m}} \) is a Born-Infeld type Lagrangian \((2p)\)-dimensional built from the 2-form \( \mathcal{L}^{M_{2m}} \) curvature \( F \), then the \( 2p \)-dimensional Lagrangian Born-infeld type leads to the Lagrangian of General Relativity, in a certain limit of the coupling constant \( l \), if and only if \( m \geq p \).

The following table shows a set of Born-Infeld type Lagrangian \( L_{BI-2p}^{M_{2n}} \), invariant under the Lie algebra \( \mathcal{L}^{M_{2n}} \), that flow into the Lagrangian General Relativity in a certain limit:

| \( \mathcal{L}^{M_4} \) | \( L_{BI}^{M_{4}} \) | \( L_{BI}^{M_{6}} \) | \( L_{BI}^{M_{8}} \) |
|---|---|---|---|
| \( \mathcal{L}^{M_6} \) | \( L_{BI}^{M_{6}} \) | \( L_{BI}^{M_{6}} \) | \( L_{BI}^{M_{6}} \) |
| \( \mathcal{L}^{M_8} \) | \( L_{BI}^{M_{8}} \) | \( L_{BI}^{M_{8}} \) | \( L_{BI}^{M_{8}} \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \mathcal{L}^{M_{2n-2}} \) | \( L_{BI}^{M_{2n-2}} \) | \( L_{BI}^{M_{2n-2}} \) | \( L_{BI}^{M_{2n-2}} \) |
| \( \mathcal{L}^{M_{2n}} \) | \( L_{BI}^{M_{2n}} \) | \( L_{BI}^{M_{2n}} \) | \( L_{BI}^{M_{2n}} \) |

(144)

It is interesting to note that for each dimension \( D \) of space-time, we have the Lagrangian \( L_{BI}^{M_{2n}} \) invariant under the \( \mathcal{L}^{M_{2n}} \) algebra contains all other \( D \)-dimensional Lagrangian evaluated in an \( \mathcal{L}^{M_{2m}} \) algebra with \( m < n \). So it is always possible to obtain an action of a lower algebra off the appropriate fields.

It is also of interest to note that it was found that, analogously to what happens in the case of three-dimensiona Chern-Simons gravity, in four dimensions is not necessary to take the limit \( l = 0 \) to result in General Relativity.

### 5 Comments and Possible Developments

In the present work we have shown that:

(i) standard odd-dimensional General Relativity (without a cosmological constant) emerges as a weak coupling constant limit of a \((2p+1)\)-dimensional Chern-Simons Lagrangian invariant under \( M_{2m+1} \) algebra, if and only if \( m \geq p \).

(ii) when \( m < p \), is impossible to obtain odd-dimensional General Relativity from a \((2p+1)\)-dimensional Chern-Simons Lagrangian invariant under the \( M_{2m+1} \) algebra.
(iii) standard even-dimensional General Relativity (without a cosmological constant) emerges as a weak coupling constant limit of a \((2p)\)-dimensional Born-Infeld type Lagrangian invariant under \(\mathcal{L}^{M_{2m}}\) algebra, if and only if \(m \geq p\).

(iv) when \(m < p\), is impossible to obtain even-dimensional General Relativity from a \((2p)\)-dimensional Born-Infeld type Lagrangian invariant under \(\mathcal{L}^{M_{2m}}\) algebra.

The toy model and procedure considered here could play an important role in the context of supergravity in higher dimensions. In fact, it seems likely that it is possible to recover the standard odd and even-dimensional supergravity from a Chern-Simons and Born-Infeld gravity theories, in a way very similar to the one shown here. In this way, the procedure sketched here could provide us with valuable information of what the underlying geometric structure of Supergravity could be (work in progress).

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