ON QUASI-HEREDITARY ALGEBRAS

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Abstract. In this paper we introduce an easily verifiable sufficient condition to determine whether an algebra is quasi-hereditary. In the case of monomial algebras, we give conditions that are both necessary and sufficient to show whether an algebra is quasi-hereditary.

1. Introduction

Quasi-hereditary algebras are an important class of algebras. However, to establish if an algebra is quasi-hereditary or not is, in general, a difficult problem. In this paper we give an explicit criterion to determine whether a monomial algebra is quasi-hereditary. Based on this results, we give a sufficient condition to determine whether any finite dimensional algebra is quasi-hereditary. Although quasi-hereditary algebras originally were defined in Lie theory in connection with Lusztig’s conjecture \[9\], beginning with \[10\] in the early 1990s, a purely algebraic approach to the study of such algebras has emerged. Since \[10\], the algebraic study of quasi-hereditary algebras has evolved as an active area of research with exciting new results appearing in recent years, for example, \[7, 17, 18, 19, 20, 21\].

Quasi-hereditary algebras arose in the context of highest weight categories. Namely, in \[9\] it is shown that every highest weight category with finite weight poset and all objects of finite lengths is the category of finite dimensional modules over a quasi-hereditary algebra. Furthermore, it is shown that a finite dimensional algebra is quasi-hereditary if and only if its module category is a highest weight category.

Highest weight categories play an important role in many areas of mathematics. For example, although only formally defined in \[9\], they connect representation theory with geometry in the work of Beilinson and Bernstein \[4\] and Brylinski and Kashiwara \[6\] linking perverse sheaves and highest weight representation theory in their proof of the Kazhdan-Lusztig conjecture.

Well-known examples of quasi-hereditary algebras are Schur algebras, \(q\)-Schur algebras, hereditary algebras and algebras of global dimension two as well as algebras whose quiver does not have any oriented cycles.

Every finite dimensional algebra over an algebraically closed field is Morita equivalent to a quotient of a path algebra and every quotient of a path algebra has an associated monomial algebra (see Section 2 for details). The properties of the original algebra are closely linked to the properties of the associated monomial algebra, see for example recent work \[8, 15\].

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In this paper, we begin by briefly recalling the definition of quasi-hereditary algebras and the basic facts of non-commutative Gröbner basis theory including the notion of the associated monomial algebra in Section 2. In Section 3 we present an easily verifiable algorithm to determine whether or not a monomial algebra is quasi-hereditary. In Section 4, we prove that an algebra is quasi-hereditary if its associated monomial algebra is quasi-hereditary. More precisely, the main results of this paper are

**Theorem A.** Let $\Gamma = KQ/J$ be a finite dimensional monomial algebra and let $T$ be the minimal generating set of paths of $J$. Then $\Gamma$ is quasi-hereditary if and only if we can order all the vertices $v_1, \ldots, v_n$ in $Q_0$ such that for each $v_i$, there is no $t \in T \setminus (T \cap KQ_{e_{i-1}}KQ)$ such that there exist non-trivial paths $p_1, p_2$ such that $t = p_1v_ip_2$ where $e_{i-1} = v_1 + \cdots + v_{i-1}$.

**Theorem B.** Let $\Lambda = KQ/I$ be a finite dimensional algebra with $I$ admissible in $KQ$ and let $\Lambda_{Mon}$ be the associated monomial algebra of $\Lambda$ with respect to some order on the paths in $Q$. Then $\Lambda$ is quasi-hereditary if $\Lambda_{Mon}$ is quasi-hereditary.

Furthermore, for any algebra $KQ/I$ with $I$ admissible we give a sufficient criterion to determine when an ideal $\Lambda v \Lambda$, for $v \in Q_0$, is hereditary. In the case $KQ/I$ monomial this gives rise to a necessary and sufficient criterion for the ideal to be hereditary.

### 2. Notations and Summary of Known Results

Throughout this paper we assume that $K$ is a field, $Q$ is a quiver, and that, unless otherwise stated, every $K$-algebra of the form $KQ/I$ is such that $I$ is an admissible ideal in $KQ$. For a $K$-algebra $\Lambda$, denote by $J(\Lambda)$ the Jacobson radical of $\Lambda$.

#### 2.1. Quasi-hereditary algebras

We begin by recalling some definitions and results on quasi-hereditary algebras.

A two-sided ideal $L$ in $\Lambda$ is called **heredity** if

1. $L^2 = L$
2. $LJ(\Lambda)L = 0$
3. $L$ is projective as a left or right $\Lambda$-module.

An algebra $\Lambda$ is **quasi-hereditary** if there exists a chain of two-sided ideals

$$(1)\quad 0 = L_0 \subset L_1 \subset \ldots \subset L_{i-1} \subset L_i \subset \ldots \subset L_n = \Lambda$$

such that $L_i/L_{i-1}$ is a heredity ideal in $\Lambda/L_{i-1}$, for all $i$. We call the sequence in (1) a **heredity chain** for $\Lambda$.

Recall that the **trace ideal** of a $\Lambda$-module $M$ in $\Lambda$ is the two-sided ideal generated by the images of homomorphisms of $M$ in $\Lambda$. By [3], a two-sided ideal $a$ in $\Lambda = KQ/I$ is such that $a^2 = a$ if and only if $a$ is the trace ideal of a projective $\Lambda$-module $P$ in $\Lambda$. There exists a set $S$ of distinct vertices in $Q_0$ such that we may assume that $P = \sum_{v \in S} v\Lambda$. Hence the trace of $P$ in $\Lambda$ is the two-sided ideal generated by $S$, that is, it is the ideal $\Lambda e\Lambda$ where $e = \sum_{v \in S} v$. See also [10] Statement (6).

Therefore $\Lambda = KQ/I$ is quasi-hereditary if there exists a sequence of idempotents $e_1, \ldots, e_n$ in $\Lambda$ such that
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\[ 0 \subset \Lambda e_1 \Lambda \subset \ldots \subset \Lambda e_{i-1} \Lambda \subset \Lambda e_i \Lambda \subset \ldots \subset \Lambda e_n \Lambda = \Lambda \]

and such that \( \Lambda e_i \Lambda / \Lambda e_{i-1} \Lambda \) is a hereditary ideal in \( \Lambda / \Lambda e_{i-1} \Lambda \), for all \( i \).

Note that by Statement (7) in [10], for \( \Lambda e \Lambda \) where \( e \) is an idempotent in \( \Lambda \) such that \( eJ(\Lambda)e = 0 \), \( \Lambda e \Lambda \) is projective as a left \( \Lambda \)-module if and only if the map \( \Lambda e \otimes_{\Lambda e} \Lambda \rightarrow \Lambda e \Lambda \) is bijective.

2.2. Non-commutative Gröbner basis theory. We now recall some non-commutative Gröbner basis theory for path algebras needed for our results. Note that in this subsection only, the ideals considered need not be admissible. For more details on non-commutative Gröbner basis theory see [11] or the more detailed summary in [15].

Denote by \( \mathcal{B} \) be the set of finite (directed) paths in a quiver \( Q \). We view the vertices of \( Q \) as paths of length zero and so they are elements in \( \mathcal{B} \). Thus \( \mathcal{B} \) is a \( K \)-basis for \( KQ \).

A nonzero element \( x \in KQ \) is uniform if \( vwx = x \) for vertices \( v, w \in Q_0 \). We set \( v = o(x) \) and call it the origin vertex of \( x \). Similarly, we set \( w = e(x) \), and call it the end vertex of \( x \). Since \( 1 = \sum_{v \in Q_0} v \), every nonzero element of \( KQ \) is a sum of uniform elements; namely \( x = \sum_{u,v \in Q_0} uxv \).

To define a Gröbner basis theory, we need the notion of an admissible order on \( \mathcal{B} \).

Definition 2.1. An admissible order \( \succ \) on \( \mathcal{B} \) is a well-order on \( \mathcal{B} \) that preserves non-zero left and right multiplication and such that if \( p = rqs \), for \( p, q, r, s \in \mathcal{B} \) then \( p \succeq q \).

Recall that a well-order is a total order such that every non-empty subset has a minimal element.

An example of an admissible order is the left or right length-lexicographical order.

We now fix an admissible order \( \succ \) on \( \mathcal{B} \). The order \( \succ \) enables us find a largest path occurring in an element in \( KQ \). We also can find the largest path that occur in elements of a subset of \( KQ \).

Definition 2.2. If \( x = \sum_{p \in \mathcal{B}} \alpha_p p \), with \( \alpha_p \in K \), almost all \( \alpha_p = 0 \), and \( x \neq 0 \) then define the tip of \( x \) to be

\[ \text{tip}(x) = p \text{ if } \alpha_p \neq 0 \text{ and } p \succeq q \text{ for all } q \text{ with } \alpha_q \neq 0. \]

If \( X \subseteq KQ \) then

\[ \text{tip}(X) = \{ \text{tip}(x) | x \in X \setminus \{0\} \}. \]

For \( p, q \in \mathcal{B} \), we say that \( p \) is a subpath of \( q \) and write \( p \mid q \), if there exist \( r, s \in \mathcal{B} \) such that \( q = rps \). For \( A \subseteq KQ \), we denote by \( \langle A \rangle \) the ideal generated by \( A \).

Definition 2.3. A subset \( \mathcal{G} \) of uniform elements in \( I \) is a Gröbner basis for \( I \) (with respect to \( \succ \)) if

\[ \langle \text{tip}(I) \rangle = \langle \text{tip}(\mathcal{G}) \rangle. \]
Equivalently, a subset $\mathcal{G}$ of $I$ consisting of uniform elements, is a Gröbner basis for $I$ with respect to $\succ$ if, for every $x \in I$, $x \neq 0$, there exists a $g \in \mathcal{G}$ such that $\text{tip}(g) \mid \text{tip}(x)$.

It can be shown that

**Proposition 2.4.** \[\Pi\] If $\mathcal{G}$ is a Gröbner basis for $I$, then $\mathcal{G}$ is a generating set for $I$; that is,

$$\langle \mathcal{G} \rangle = I.$$

An ideal in $KQ$ is called *monomial* if it can be generated by monomials. Recall that the *monomial* elements in $KQ$ are the elements of $\mathcal{B}$. If $\Lambda = KQ/I$ and $I$ is a monomial ideal, then we say that $\Lambda$ is a *monomial algebra*. We recall the following well-known facts about monomial ideals.

**Proposition 2.5.** \[\Pi\] Let $I$ be a monomial ideal in $KQ$. Then

1. there is a unique minimal set $\mathcal{T}$ of monomial generators for $I$ and $\mathcal{T}$ is a Gröbner basis for $I$ for any admissible order on $\mathcal{B}$.
2. if $x = \sum_{p \in \mathcal{B}} \alpha_p p$ then $x \in I$ if and only if $p \in I$ for all $p$ such that $\alpha_p \neq 0$.

Set $\mathcal{N} = \mathcal{B} \setminus \text{tip}(I)$. Note that it follows that we also have $\mathcal{N} = \mathcal{B} \setminus \langle \text{tip}(I) \rangle$: Clearly $\mathcal{B} \setminus \langle \text{tip}(I) \rangle$ is a subset of $\mathcal{B} \setminus \text{tip}(I)$. Now let $p$ be in $\mathcal{B} \setminus \text{tip}(I)$. Suppose $p \in \langle \text{tip}(I) \rangle$. Then there are paths $s_i, u_i \in \mathcal{B}$ and $t_i \in \text{tip}(I)$ such that $p = \text{tip}(\sum_i s_i t_i u_i)$. Hence $p = s_i t_i u_i$ for some $i$. There exists $x \in I$ such that $\text{tip}(x) = t_i$. Hence $s_i x u_i \in I$ and $\text{tip}(s_i x u_i) = \text{tip}(s_i t_i u_i) = p$. Thus $p \in \text{tip}(I)$, a contradiction. In the following let $\mathcal{T}$ be the minimal set of monomials that generates the monomial ideal $\langle \text{tip}(I) \rangle$. It follows from the above that we have

$$\mathcal{N} = \{ p \in \mathcal{B} \mid t \text{ is not a subpath of } p \text{ for all } t \in \mathcal{T} \}.$$

The following result is of central importance.

**Lemma 2.6. (Fundamental Lemma)** \[\Pi\] There is a $K$-vector space isomorphism

$$KQ \simeq I \oplus \text{Span}_K \mathcal{N}.$$

It is an immediate consequence of the Fundamental Lemma that if $x \in KQ \setminus \{ 0 \}$ then $x = i_x + n_x$ for a unique $i_x \in I$ and a unique $n_x \in \text{Span}_K \mathcal{N}$.

Let $\pi: KQ \to KQ/I$ be the canonical surjection. Then the map $\sigma: KQ/I \to KQ$ given by $\sigma \pi(x) = n_x$ for $x \in KQ$ is a $K$-vector space splitting of $\pi$. We see that $\sigma$ is well-defined since if $x, y \in KQ$ are such that $\pi(x) = \pi(y)$, then $x - y \in I$. Hence $n_{x-y} = n_x - n_y = 0$ and we conclude that $n_x = n_y$. Thus, restricting to $\text{Span}_K(\mathcal{N})$, we have inverse $K$-isomorphisms $\pi: \text{Span}_K(\mathcal{N}) \to KQ/I$ and $\sigma: KQ/I \to \text{Span}_K(\mathcal{N})$. Therefore, as vector spaces, we can identify $KQ/I$ with $\text{Span}_K \mathcal{N}$. We note that for $x, y \in \text{Span}_K \mathcal{N}$, the multiplication of $x$ and $y$ in $KQ/I$ equals $n_{x\cdot y}$ where $x \cdot y$ is the usual multiplication in $KQ$.

Summarising, we have the following useful characterisation of a basis of $KQ/I$. 
Proposition 2.7. As K-vector spaces, $\text{Span}_K(N)$ is isomorphic to $KQ/I$ and hence $N$ can be identified with a $K$-basis of $KQ/I$.

Definition 2.8. We call $I_{\text{Mon}} = \langle \text{tip}(I) \rangle$ the ideal in $KQ$ generated by $\text{tip}(I)$ and define the associated monomial algebra of $\Lambda = KQ/I$ to be $\Lambda_{\text{Mon}} = KQ/I_{\text{Mon}}$.

Recall that $I_{\text{Mon}}$ is a monomial ideal, and that by Proposition 2.5(1), there is a unique minimal set $T$ of paths that generate $I_{\text{Mon}}$. By the Fundamental Lemma there exist unique elements $g_t \in I$ and $n_t \in \text{Span}_K(N)$, such that $t = g_t + n_t$, for $t \in T$. Since $t$ is uniform, $g_t$ and $n_t$ are uniform. Furthermore, since $n_t \in \text{Span}_K(N)$, we have that $\text{tip}(g_t) = t$. We now set $\mathcal{G} = \{g_t \mid t \in T\} \subset I$. Then $\text{tip}(\mathcal{G}) = T$ and hence $\mathcal{G}$ is a Gröbner basis for $I$ (with respect to any admissible order) since $\langle \text{tip}(\mathcal{G}) \rangle = \langle T \rangle = \langle I_{\text{Mon}} \rangle = \langle \text{tip}(I) \rangle$.

Definition 2.9. The set $\mathcal{G} = \{g_t \mid t \in T\} \subset I$ defined above is called the reduced Gröbner basis for $I$ (with respect to $>$).

The next result lists some facts about reduced Gröbner bases and the associated monomial algebras, which will be useful for the proofs later in the paper.

Proposition 2.10. Let $I$ be an ideal in $KQ$ and let $\Lambda = KQ/I$. Let $T$ be the unique minimal set of monomials generating $\langle \text{tip}(I) \rangle$ and let $\mathcal{G}$ be the reduced Gröbner basis for $I$. Then the following hold.

1. The reduced Gröbner basis for $I_{\text{Mon}}$ is $T$.
2. $\dim_K(\Lambda) = \dim_K(\Lambda_{\text{Mon}}) = |N|$ where $|N|$ denotes the cardinality of the set $N$.

Keeping the notation above, we write elements of both $\Lambda$ and $\Lambda_{\text{Mon}}$ as $K$-linear combinations of elements in $N$. The difference is how these elements multiply in $\Lambda$ and $\Lambda_{\text{Mon}}$. Next assume that $I$ is an admissible ideal. Notice that $I$ admissible implies that $I_{\text{Mon}}$ is admissible. The converse is false in general. If $I$ is admissible, then the set of vertices and arrows are always in $N$ and both $T$ and $N$ are finite sets.

3. Quasi-hereditary monomial algebras

In this section we give necessary and sufficient conditions for a monomial algebra to be quasi-hereditary. We also describe the structure of an algebra of the form $\Lambda/\Lambda e \Lambda$.

We fix the following notation for the remainder of this paper: $K$ will denote a field, $Q$ a quiver, $B$ the set of paths in $KQ$, $>$ an admissible order on $B$, $v_1, \ldots, v_r$ a set of distinct vertices in $Q$, and $e = \sum_{i=1}^r v_i$. We fix an admissible ideal $I$ in $KQ$, let $\Lambda = KQ/I$, and $J(\Lambda)$ be the Jacobson radical of $\Lambda$.

Definition 3.1. We say that a vertex $v$ is properly internal to a path $p \in B$, if there exist $p_1, p_2 \in B$ both of length greater than or equal to 1 and $p = p_1vp_2$.

If $T$ is a set of paths, then a vertex $v$ is not properly internal to $T$ if, for each $t \in T$, $v$ can only occur in $t$ as either the start or end vertex of $t$.
The next result shows the importance of this definition.

**Proposition 3.2.** Let \( \Lambda = KQ/I \) be a finite dimensional monomial algebra and let \( \mathcal{T} \) be a minimal set of generators of paths of \( I \). For \( v \in Q_0 \), the ideal \( \Lambda v \Lambda \) is heredity if and only if \( v \) is not properly internal to \( \mathcal{T} \).

**Proof.** First assume \( v \) is internal to some \( t \in \mathcal{T} \). If \( vJ(\Lambda)v \neq 0 \) then \( \Lambda v \Lambda \) is not a heredity ideal. Now assume that \( vJ(\Lambda)v = 0 \). Since \( v \) is internal to \( t \), \( t = t_1vt_2 \) with the length of \( t_1 \) and \( t_2 \) being at least 1 and \( t_1 \) and \( t_2 \) are not in \( I \). But \( t_1v \otimes vt_2 \) is nonzero (since \( t_1v \in vN \) and \( vt_2 \in vN \)) and maps to \( \mu(t_1v \otimes vt_2) = 0 \) since \( t \in I \). Therefore \( \mu \) is not injective. Hence \( \Lambda v \Lambda \) is not a heredity ideal in \( \Lambda \).

Now assume that \( v \) is not properly internal to any path in \( \mathcal{T} \). We begin by showing that \( vJ(\Lambda)v = 0 \). Suppose that \( vJ(\Lambda)v \neq 0 \). Then there exists a path \( p \) in \( J(\Lambda) \) such that \( vp \notin I \). Since \( \Lambda \) is finite dimensional, \( (vpv)^n \in I \) for sufficiently large \( n \). Since \( \mathcal{T} \) is a Gröbner basis for \( I \), there is some \( t \in \mathcal{T} \) such that \( t \) is a subpath of \( (vpv)^n \). But \( t \) is not a subpath of \( p \) and hence \( v \) must be internal to \( t \) which is a contradiction. Now consider the multiplication map \( \mu : \Lambda v \otimes v_\Lambda v \Lambda \rightarrow \Lambda v \Lambda \). It suffices to show that this map is bijective. The map is clearly onto. Since \( vJ(\Lambda)v = 0 \), we have that \( n \otimes n' \), for \( n \in NV \) and \( n' \in vN \) form a basis of \( \Lambda v \otimes v_\Lambda v \Lambda \). Suppose that \( \sum_{n_i \in NV, n'_i \in vN} n_i v n'_i \in I \). Then there exists \( t \in \mathcal{T} \) such that \( t \) is a subpath of \( n_i v n'_i \) for some \( i \). But \( t \) is not a subpath of \( n_i \) or \( n'_i \) therefore \( v \) is internal to \( t \) which is a contradiction. Hence \( \mu \) is bijective and \( \Lambda v \Lambda \) is a heredity ideal in \( \Lambda \). \( \square \)

Note that the second part of the proof of Proposition 3.2 gives a criterion for \( \Lambda v \Lambda \) to be a heredity ideal for not only monomial algebras but for any \( \Lambda = KQ/I \).

**Corollary 3.3.** Let \( \Lambda = KQ/I \), \( \mathcal{G} \) be the reduced Gröbner basis for \( I \) and let \( \mathcal{T} = \text{tip}(\mathcal{G}) \). Then \( \Lambda v \Lambda \) is a heredity ideal in \( \Lambda \) if \( v \) is not properly internal to any path in \( \mathcal{T} \).

For \( v = v_1 + \cdots + v_r \), we define \( Q_\varepsilon \) to be the subquiver of \( Q \) obtained by removing the vertices \( v_1, \ldots, v_r \) and all arrows entering or leaving any of the \( v_i \)'s. Let \( B_\varepsilon \) be the set of paths in \( KQ_\varepsilon \). Note that the admissible order \( \succ \) restricts to an admissible order on \( B_\varepsilon \). We leave the proof of the next result to the reader.

**Proposition 3.4.** As \( K \)-vector spaces,

\[
KQ = KQ_\varepsilon \oplus KQ e KQ.
\]

\( \square \)

If \( x \in KQ \), we write \( x = x_\varepsilon + x_e \) with \( x_\varepsilon \in KQ_\varepsilon \) and \( x_e \in KQ e KQ \), and if \( X \subseteq KQ \), then define \( X_\varepsilon = \{ x_\varepsilon \mid x \in X \} \).

We list some of the basic results that relate these definitions and leave the proofs to the reader.
Lemma 3.9. Recall the following well-known Lemma which follows directly from [3], which shows that in our context it is enough to consider idempotents of the form $KQ_i$ ideal in $I$.

Remark 3.8. Corollary 3.7. There is natural isomorphism of algebras

Hence $ι: KQ_i \to KQ$ be the inclusion map and $π: KQ \to Λ$ and $ρ: Λ \to Λ/ΛeΛ$ be the canonical surjections. Because of its usefulness, we include a proof of the following well-known result.

Lemma 3.6. Let $Λ = KQ/I$ be finite dimensional and for $v_1, \ldots, v_r \in Q_0$, set $e = v_1 + \cdots v_r$. Let

$σ: KQ_i \to KQ \xrightarrow{π} Λ \xrightarrow{ρ} Λ/ΛeΛ$.

Then $σ$ is surjective and $κer(σ) = I_\varepsilon$.

Proof. We begin by showing $σ$ is surjective. Let $\bar{λ} \in Λ/ΛeΛ$ and $λ \in Λ$ such that $ρ(λ) = \bar{λ}$. Let $r \in KQ$ map to $λ$ under the canonical surjection $π$. Then $r = r_\varepsilon + r_e$ and $σ(r_\varepsilon) = ρπ(r_\varepsilon) = ρ(π(r_\varepsilon + r_e))$ since $ρ(π(r_e)) = 0$. We have $σ(r_\varepsilon) = ρ(π(r)) = \bar{λ}$ and we conclude that $σ$ is surjective.

Now we prove that $κer(σ) = I_\varepsilon$. Let $x \in I$. We have that $x = x_\varepsilon + x_e \in I$ and hence $x_\varepsilon \in KQ_i$ such that there exists $x' \in KQeKQ$ with $x_\varepsilon + x' \in I$. By Proposition $3.5$, $x_\varepsilon \in I_\varepsilon$. On the other hand, let $x \in κer(σ)$. Then $π(i(x)) \in ΛeΛ$ and therefore $π(i(x)) = \sum_i ρ(y_i)eπ(z_i)$ for some $y_i, z_i \in KQ$. Set $-x' = \sum_i y_i ez_i$. Then $x + x' \in I$. Hence $x \in I_\varepsilon$.

As an immediate consequence we have the following corollary.

Corollary 3.7. There is natural isomorphism of algebras

$Λ/ΛeΛ \simeq KQ_i/I_\varepsilon$.

Remark 3.8. If $I$ is a monomial ideal with minimal generating set of paths $T$, then $I_\varepsilon = \langle T \rangle_\varepsilon = \langle T_\varepsilon \rangle$. In particular, if $I$ is a monomial ideal in $KQ$, then $I_\varepsilon$ is a monomial ideal in $KQ_i$.

Recall the following well-known Lemma which follows directly from [3], which shows that in our context it is enough to consider idempotents of the form $\sum v_i$, where the $v_i$ are some vertices in $Q_0$.

Lemma 3.9. Let $e$ be an idempotent in $Λ = KQ/I$, $I$ admissible. Then $ΛeΛ = Λ\sum_i v_i Λ$ for $b$ some $v_i \in Q_0$. 

□
Combining Remark 3.3 and the proof of Proposition 4.4 we obtain the following necessary and sufficient conditions for a monomial algebra to be quasi-hereditary.

**Theorem 3.10.** Let $\Lambda = KQ/I$ be a finite dimensional monomial algebra, where $I$ admissible and let $T$ be the minimal set of generators of paths of $I$. Then $\Lambda$ is quasi-hereditary if and only if we can order all the vertices $v_1, \ldots, v_n$ in $Q_0$ such that for each $i$, the vertex $v_i$ is not properly internal to $T_{v_1 + \cdots + v_{i-1}}$.

**Proof.** First assume that we can order all the vertices $v_1, \ldots, v_n$ in $Q_0$ such that for each $i$, the vertex $v_i$ is not properly internal to $T_{v_1 + \cdots + v_{i-1}}$, and for all $i$, set $e_i = v_1 + \cdots + v_i$. It suffices to show that $\Lambda e_i \Lambda / \Lambda e_{i-1} \Lambda$ is a heredity ideal in $\Lambda / \Lambda e_{i-1} \Lambda$ for all $i$. Since $v_1$ is not properly internal in $T$, by Corollary 3.3 we have that $\Lambda v_1 \Lambda$ is heredity in $\Lambda$. It follows from Corollary 3.7 that $\Lambda / \Lambda v_1 \Lambda \simeq KQ_{\hat{e}_1} / \langle \hat{T}_{\hat{e}_1} \rangle$ since $\langle \hat{T}_{\hat{e}_1} \rangle = \langle T_{\hat{e}_1} \rangle$. We proceed by induction on $i$. Assume that we have shown the result for $i$. That is $\Lambda e_i \Lambda / \Lambda e_{i-1} \Lambda$ is a heredity ideal in $\Lambda / \Lambda e_{i-1} \Lambda$. We wish to show that $\Lambda e_{i+1} \Lambda / \Lambda e_i \Lambda$ is a heredity ideal in $\Lambda / \Lambda e_i \Lambda$. By Corollary 3.7 we have that for all $i$,

$$\Lambda / \Lambda e_i \Lambda \simeq KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle$$

Consider the following exact commutative diagram:

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\begin{array}{cccccc}
& & 0 & & 0 & \\
& & & \downarrow{f} & & \\
0 & \rightarrow & \Lambda e_i \Lambda & \rightarrow & \Lambda / \Lambda e_i \Lambda & \rightarrow 0 \\
& \downarrow{=} & & \downarrow{=} & & \\
0 & \rightarrow & \Lambda e_{i+1} \Lambda & \rightarrow & \Lambda / \Lambda e_{i+1} \Lambda & \rightarrow 0 \\
& & \downarrow{=} & & \downarrow{=} & \\
\Lambda e_{i+1} \Lambda / \Lambda e_i \Lambda & & 0 & & 0 & \\
& & \downarrow{=} & & \downarrow{=} & \\
0 & & & & & \\
\end{array}
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By the diagram above, showing that $\Lambda e_{i+1} \Lambda / \Lambda e_i \Lambda$ is a heredity ideal in $\Lambda / \Lambda e_i \Lambda$ is equivalent to showing that $\ker(f)$ is a heredity ideal in $KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle$. The reader may check that

$$KQ_{\hat{e}_{i+1}} / \langle \hat{T}_{\hat{e}_{i+1}} \rangle \text{ is isomorphic to } KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle / (KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle v_{i+1} KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle).$$

Hence $\ker(f) \simeq KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle v_{i+1} KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle$. By hypothesis, $v_{i+1}$ is not properly internal to $T_{\hat{e}_i}$, and so by Corollary 3.3, $\ker(f)$ is a heredity ideal in $KQ_{\hat{e}_i} / \langle \hat{T}_{\hat{e}_i} \rangle$. We conclude that $\Lambda$ is quasi-hereditary.

Suppose now that $\Lambda$ is quasi-hereditary and let

$$0 \subset \Lambda e_1 \Lambda \subset \cdots \subset \Lambda e_n \Lambda = \Lambda$$

be a heredity chain for $\Lambda$. We proceed by induction on $n$. We begin by showing that if $e_1 = v_1 + \cdots + v_m$ then each $v_i$ is not properly internal to $T$. Without loss of generality
we do this for $v_1$. We have that $v_1J(\Lambda)v_1 = 0$ since $eJ(\Lambda)e = 0$. Consider the following diagram

$$
\begin{array}{ccc}
\Lambda v_1 \otimes_{v_1,\Lambda} v_1\Lambda & \xrightarrow{f} & \Lambda v_1\Lambda \\
\downarrow{h} & & \downarrow{h} \\
\Lambda e_1 \otimes_{e_1,\Lambda} e_1\Lambda & \xrightarrow{g} & \Lambda e_1\Lambda
\end{array}
$$

where $g$ is a bijection. The map $f$ is clearly onto and $h$ is injective. That implies that $f$ injective and hence bijective. Hence $\Lambda v_1\Lambda$ is heredity in $\Lambda = KQ/\langle T \rangle$ and by Proposition 3.2 $v_1$ is not properly internal to $T$. If $n = 1$ this proves the result. Assume that the result holds for $i = n - 1$. By the induction hypothesis $\Lambda/\Lambda e_i\Lambda$ is quasi-hereditary with heredity chain

$$0 \subset \Lambda e_{i+1}\Lambda/\Lambda e_i\Lambda \subset \cdots \subset \Lambda e_n\Lambda/\Lambda e_i\Lambda = \Lambda/\Lambda e_i\Lambda.$$ 

By Corollary 3.7 together with Remark 3.8 $\Lambda/\Lambda e_i\Lambda = KQ_{v_1 + \cdots + v_i}/T_{v_1 + \cdots + v_i}$. Since $\Lambda/\Lambda e_i\Lambda$ has a heredity chain of length $n - i - 1$ and $T_{v_1 + \cdots + v_i} \subset T$, we see by induction that the result holds.

\[ \square \]

4. **Quasi-hereditary algebras**

In this Section we show that if an algebra $\Lambda$ is such that $\Lambda_{Mon}$ is quasi-hereditary then $\Lambda$ is quasi-hereditary. More precisely, we show

**Theorem 4.1.** Let $\Lambda = KQ/I$ with $I$ admissible. If $\Lambda_{Mon}$ is quasi-hereditary then $\Lambda$ is quasi-hereditary.

Before proving Theorem 4.1 we begin with some preliminary results.

**Lemma 4.2.** Let $G$ be the reduced Gröbner basis for an ideal $I$ in $KQ$ and $T = \text{tip}(G)$. Assume that $v_i$, for $1 \leq i \leq r$, is not properly internal to $T$ and set $e = v_1 + \cdots + v_r$. Then for any $g \in G$, either $g = g_e \in KQeKQ$, or $\text{tip}(g) = \text{tip}(g_e)$.

**Proof.** Let $g \in G$ and let $t = \text{tip}(g) \in T$. Suppose that $t \in KQeKQ$. There is some $i$ such that $v_i$ occurs in $t$. Since $v_i$ is not properly internal to $t$, $v_i$ is either the start vertex or end vertex of $t$. Since $G$ is the reduced Gröbner basis for $I$, $g$ is uniform. Hence, $g \in KQeKQ$ and we are done. \[ \square \]

**Lemma 4.3.** Let $G$ be the reduced Gröbner basis for an ideal $I$ in $KQ$ and let $T = \text{tip}(G)$. Assume that $v_i$, for $1 \leq i \leq r$, is not properly internal to $T$ and set $e = v_1 + \cdots + v_r$. Then $G_e$ is a Gröbner basis for $I_e$ in $KQ_e$.

**Proof.** Assume that $v_i$, for $1 \leq i \leq r$, is not properly internal to $T$. We need to show that $\langle \text{tip}(G_e) \rangle = \langle \text{tip}(I_e) \rangle$. For this it is enough to show that for any $x \neq 0 \in I_e$ there is $g \in G$ and paths $p$ and $q$ in $B_e$ such that $\text{tip}(x) = p \text{tip}(g)q$. By Lemma 1.2 it then follows that $\text{tip}(g) = \text{tip}(g_e)$, showing that $G_e$ is a Gröbner basis for $I_e$. \[ \square \]
Suppose not; that is, suppose that \( \text{tip}(x) \) is a heredity chain for \( \Lambda \). We have that

\[
\text{tip}(x) = \text{tip}(x^*) \subset \text{tip}(x^*) \subset \cdots \subset \text{tip}(x) = \text{tip}(x^*)
\]

Since \( x + x^* \in I \), there is some \( g \in G \) such that \( \text{tip}(g) = \text{tip}(x^*) \). This finishes the proof that \( \text{tip}(g) = \text{tip}(x^*) \).

Since \( x + y^* \in I \), there is some \( g \in G \) such that \( \text{tip}(g) = \text{tip}(x^*) \). This finishes the proof that \( \text{tip}(g) = \text{tip}(x^*) \).

**Proposition 4.4.** Let \( G \) be the reduced Gröbner basis of \( I \) and let \( \mathcal{T} = \text{tip}(G) \). Assume that \( v_i \), for \( 1 \leq i \leq r \), is not properly internal to \( \mathcal{T} \) and set \( e = v_1 + \cdots + v_r \). Then

\[
(\Lambda/\lambda \Lambda)_{\mathcal{M}o}n \simeq (\Lambda_{\mathcal{M}o}n/\Lambda_{\mathcal{M}o}n \varepsilon \Lambda_{\mathcal{M}o}n).
\]

**Proof.** We have that \( I_{\mathcal{M}o}n = \langle \mathcal{T} \rangle \) and \( \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_c \) where \( \mathcal{T}_c = \mathcal{T} \cap KQeKQ \). Furthermore, by Corollary 3.7, we have \( \Lambda_{\mathcal{M}o}n/\Lambda_{\mathcal{M}o}n \varepsilon \Lambda_{\mathcal{M}o}n \simeq KQeKQ/\langle \mathcal{T} \rangle \). By Lemma 4.3, the set \( \mathcal{G}_e \) is a Gröbner basis of \( I_e \) and hence \( \text{tip}(\mathcal{G}_e) = \mathcal{T}_e \) is a Gröbner basis of \( (I_e)_{\mathcal{M}o}n \) and the result follows.

Since it is a crucial point in the proof of Theorem 4.1, we recall the following well-known result.

**Lemma 4.5.** Let \( \Lambda = KQ/I \) and suppose that

\[
(\star) \quad (0) \subset L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = \Lambda
\]

is a chain of ideals in \( \Lambda \). Then \( \star \) is a heredity chain for \( \Lambda \) if and only if \( L_1 \) is a heredity ideal in \( \Lambda \) and

\[
(0) \subset L_2/\lambda L_1 \subset L_3/\lambda L_1 \subset \cdots \subset L_{k-1}/\lambda L_1 \subset L_k/\lambda L_1 = \lambda/\lambda L_1
\]

is a heredity chain for \( \lambda/\lambda L_1 \).

**Proof of Theorem 4.1.** Assume \( \lambda_{\mathcal{M}o}n \) is quasi-hereditary and that

\[
0 \subset \lambda_{\mathcal{M}o}n e_1 \lambda_{\mathcal{M}o}n \subset \lambda_{\mathcal{M}o}n e_2 \lambda_{\mathcal{M}o}n \subset \cdots \subset \lambda_{\mathcal{M}o}n
\]

is a heredity chain for \( \lambda_{\mathcal{M}o}n \). By Theorem 3.10, we can assume that \( e_i = e_{i-1} + v \) for some vertex \( v \in Q_0 \) and in particular, \( e_1 = v_1 \). We show that

\[
0 \subset \lambda e_1 \lambda \subset \lambda e_2 \lambda \subset \cdots \subset \lambda
\]

is a heredity chain for \( \Lambda \). We proceed by induction on the number of vertices in \( Q \). If \( Q \) has one vertex, the result is vacuously true. Assume the result is true for quivers with \( k - 1 \) vertices and that \( Q \) has \( k \) vertices and \( \lambda_{\mathcal{M}o}n = KQ/(I_{\mathcal{M}o}n) \) is quasi-hereditary.
Since $\Lambda_{Mon}v_1\Lambda_{Mon}$ is a heredity ideal in $\Lambda_{Mon}$, $v_1$ is not properly internal to $T = \text{tip}(\mathcal{G})$ where $\mathcal{G}$ is the reduced Gröbner basis for $I$. By Corollary 5.3, the ideal $\Lambda v_1 \Lambda$ is a heredity ideal in $\Lambda$.

We have that $\Lambda_{Mon}$ quasi-hereditary implies $\Lambda_{Mon}/\Lambda_{Mon}v_1\Lambda_{Mon}$ quasi-hereditary and by Proposition 4.4, we have that $\Lambda_{Mon}/\Lambda_{Mon}v_1\Lambda_{Mon} \simeq (\Lambda/\Lambda v_1 \Lambda)_{Mon}$. By Lemma 4.5

$$0 \subset \Lambda_{Mon}e_2\Lambda_{Mon}/\Lambda_{Mon}v_1\Lambda_{Mon} \subset \cdots \subset \Lambda_{Mon}/\Lambda_{Mon}v_1\Lambda_{Mon}$$

is a heredity chain for $\Lambda_{Mon}/\Lambda_{Mon}v_1\Lambda_{Mon}$.

Since $\Lambda/\Lambda v_1 \Lambda$ has strictly fewer vertices than $\Lambda$, by induction on the number of vertices $\Lambda/\Lambda v_1 \Lambda$ is quasi-hereditary with heredity chain

$$\Lambda e_2 \Lambda/\Lambda v_1 \Lambda \subset \cdots \subset \Lambda/\Lambda v_1 \Lambda.$$

Applying Lemma 4.6 again, we conclude that $\Lambda$ is quasi-hereditary. □

The following example illustrates how the above results can be applied.

**Example 4.6.** Let $Q$ be the quiver

![Quiver Diagram]

Let $\succ$ be the length-(left)lexicographic order with $a \succ b \succ c \succ \cdots \succ f \succ g$. Let $T = \{ab, be, de, eh, hc\}$. First we show that the monomial algebra $\Lambda = KQ/\langle T \rangle$ is quasi-hereditary.

The vertices $v_1, v_2, v_4$ and $v_6$ are properly internal to $T$ and $v_3$ and $v_5$ are not properly internal to any path in $T$. We choose as first vertex in our vertex ordering $v_3$. Hence, $\Lambda v_3 \Lambda$ is a heredity ideal in $\Lambda$. By our earlier results $\Lambda/\Lambda v_3 \Lambda$ is isomorphic to $KQ_{\bar{v}_3}/T_{\bar{v}_3}$ where $Q_{\bar{v}_3}$ is

![Revised Quiver Diagram]

and $T_{\bar{v}_3} = \{ab, be, eh\}$. Now, for example, vertex $v_1$ is not properly internal to $T_{\bar{v}_3}$. Thus $(\Lambda/\Lambda v_3 \Lambda)v_1(\Lambda/\Lambda v_3 \Lambda)$ is a heredity ideal in $(\Lambda/\Lambda v_3 \Lambda)$. Continuing one obtains a heredity chain

$$0 \subset \Lambda v_3 \Lambda \subset \Lambda v_3 + v_1 \Lambda \subset \Lambda v_3 + v_1 + v_2 + v_4 \Lambda \subset \Lambda v_3 + v_1 + v_2 + v_4 + v_5 \Lambda \subset \Lambda v_3 + v_1 + v_2 + v_4 + v_5 + v_6 \Lambda = \Lambda$$

It follows from Theorem 4.4 that if $\Gamma = KQ/J$ with $J$ admissible and such that $\Gamma_{Mon} = \Lambda$ then $\Gamma$ is quasi-hereditary. Using the results in [15], we see that $\Gamma$ necessarily is of the form $\Gamma = KQ/(g_1, g_2, \ldots, g_5)$ with $g_1 = ab - Xcd, g_2 = be, g_3 = de - Yfg, g_4 = eh, g_5 = hc$, and $X, Y \in K$ are arbitrary.
We now present an example of a quasi-hereditary algebra $\Lambda$ such that $\Lambda_{\text{Mon}}$ is not quasi-hereditary.

**Example 4.7.** Let $Q$ be the quiver

\[
\begin{array}{c}
v_1 \xrightarrow{a} v_2 \\
\downarrow c & \searrow d \\
v_3 & \downarrow b \\
\nw_3 \xleftarrow{e} w_4
\end{array}
\]

Let $I = \langle ab - cd, be, ea \rangle$. Let $\Lambda = KQ/I$. Choose the length-lexicographic order on $B$ with $a \succ b \succ c \succ d \succ e$. Then $\Lambda_{\text{Mon}} = KQ/\langle ab, be, ea, cde, ecd \rangle$, by [14] the global dimension of $\Lambda_{\text{Mon}}$ is infinite and thus $\Lambda_{\text{Mon}}$ cannot be quasi-hereditary.

On the other hand, if we change the order to length-lexicographic with $e \succ d \succ c \succ b \succ a$ then $\Lambda_{\text{Mon}} = KQ/\langle cd, ea, be \rangle$. We now construct a heredity chain for $\Lambda_{\text{Mon}}$ for this order. The only vertex not properly internal to any element in $T$ is the vertex $v_2$. So $\Lambda_{\text{Mon}}/\Lambda_{\text{Mon}}v_2\Lambda_{\text{Mon}} = KQv_2/\langle cd \rangle$. The vertex $v_1$ is not properly internal to $cd$ and therefore by Theorem 3.10

\[
0 \subset \Lambda_{\text{Mon}}v_2\Lambda_{\text{Mon}} \subset \Lambda_{\text{Mon}}(v_1 + v_2)\Lambda_{\text{Mon}} \subset \Lambda_{\text{Mon}}(v_1 + v_2 + v_3)\Lambda_{\text{Mon}} \subset \Lambda_{\text{Mon}}
\]

is a heredity chain for $\Lambda_{\text{Mon}}$. This implies that $\Lambda_{\text{Mon}}$ is quasi-hereditary and therefore $\Lambda$ is quasi-hereditary.

**References**

[1] Achar, Pramod N. Perverse coherent sheaves on the nilpotent cone in good characteristic. Recent developments in Lie algebras, groups and representation theory, 1–23, Proc. Sympos. Pure Math., 86, Amer. Math. Soc., Providence, RI, 2012.

[2] Anick, David J.; Green, Edward L. On the homology of quotients of path algebras. Comm. Algebra 15 (1987), no. 1-2, 309–341.

[3] Auslander, Maurice; Platzeck, Maria Ines; Todorov, Gordana. Homological theory of idempotent ideals. Trans. Amer. Math. Soc. 332 (1992), 667–692.

[4] Belinson, A.; Bernstein, J.. Localisation de g-modules, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 15–18.

[5] Bergman, George M. The diamond lemma for ring theory. Adv. in Math. 29 (1978), no. 2, 178–218.

[6] Brylinski, J.-L.; Kashiwara, M. Kazhdan-Lusztig conjecture and holonomic systems, Invent. Math. 64 (1981), no. 3, 387–410.

[7] Buchweitz, Ragnar-Olaf; Leuschke, Graham J.; Van den Bergh, Michel. On the derived category of Grassmannians in arbitrary characteristic. Compos. Math. 151 (2015), no. 7, 1242–1264.

[8] Chouhy, Sergio; Solotar, Andrea. Projective resolutions of associative algebras and ambiguities. J. Algebra 432 (2015), 22–61.

[9] Cline, E.; Parshall, B.; Scott, L. Finite-dimensional algebras and highest weight categories. J. Reine Angew. Math. 391 (1988), 85–99.

[10] Dlab, Vlastimil; Ringel, Claus Michael. Quasi-hereditary algebras. Illinois J. Math. 33 (1989), no. 2, 280–291.

[11] Green, Edward L. Noncommutative Gröbner bases, and projective resolutions. Computational methods for representations of groups and algebras (Essen, 1997), 29–60, Progr. Math., 173, Birkhäuser, Basel, 1999.

[12] Green, Edward L. Multiplicative bases, Gröbner bases, and right Gröbner bases. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). J. Symbolic Comput. 29 (2000), no. 4-5, 601–623.

[13] Green, Edward; Huang, Rosa Q. Projective resolutions of straightening closed algebras generated by minors. Adv. Math. 110 (1995), no. 2, 314–333.

[14] Green, Edward L.; Happel, Dieter.; Zacharia, Dan. Projective resolutions over Artin algebras with zero relations. Illinois J. Math. 29 (1985), no. 1, 180–90.

[15] Green, Edward L.; Hille, Lutz; Schroll, Sibylle. Algebras and varieties. arXiv:1707.07877.
[16] Green, Edward L.; Solberg, Øyvind. An algorithmic approach to resolutions. J. Symbolic Comput. 42 (2007), no. 11-12, 1012–1033.
[17] Hille, Lutz; Perling, Markus. Tilting bundles on rational surfaces and quasi-hereditary algebras. Ann. Inst. Fourier (Grenoble) 64 (2014), no. 2, 625–644.
[18] Iyama, Osamu. Finiteness of representation dimension. Proc. Amer. Math. Soc. 131 (2003), no. 4, 1011–1014.
[19] Iyama, Osamu; Reiten, Idun. 2-Auslander algebras associated with reduced words in Coxeter groups. Int. Math. Res. Not. IMRN 2011, no. 8, 1782–1803.
[20] Raedschelders, Theo; Van den Bergh, Michel. The Manin Hopf algebra of a Koszul Artin-Schelter regular algebra is quasi-hereditary. Adv. Math. 305 (2017), 601–660.
[21] Ringel, Claus Michael. Iyama’s finiteness theorem via strongly quasi-hereditary algebras. J. Pure Appl. Algebra 214 (2010), no. 9, 1687–1692.
[22] Wilson, George V. The Cartan map on categories of graded modules. J. Algebra 85 (1983), no. 2, 390–398.

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