Heinz mean curvature estimates in warped product spaces $M \times e^\psi N$

Isabel M.C. Salavessa

Center of Physics and Engineering of Advanced Materials (CeFEMA), Instituto Superior Técnico, University of Lisbon, Edifício Ciência, Piso 3, Av. Rovisco Pais, 1049-001 Lisboa, Portugal; e-mail: isabel.salavessa@ist.utl.pt

In grateful memory of James Eells

Abstract: If a graph submanifold $(x, f(x))$ of a Riemannian warped product space $(M^m \times e^\psi N^n, \tilde{g} = g + e^{2\psi}h)$ is immersed with parallel mean curvature $H$, then we obtain a Heinz type estimation of the mean curvature. Namely, on each compact domain $D$ of $M$, $m\|H\| \leq \frac{A_\psi(\partial D)}{V_\psi(D)}$ holds, where $A_\psi(\partial D)$ and $V_\psi(D)$ are the $\psi$-weighted area and volume, respectively. In particular, $H = 0$ if $(M, g)$ has zero weighted Cheeger constant, a concept recently introduced by D. Impera et al. (13). This generalizes the known cases $n = 1$ or $\psi = 0$. We also conclude minimality using a closed calibration, assuming $(M, g_*)$ is complete where $g_* = g + e^{2\psi}f^*h$, and for some constants $\alpha \geq \delta \geq 0$, $C_1 > 0$ and $\beta \in [0, 1)$, $\|\nabla^* \psi\|_{g_*}^2 \leq \delta$, Ricci$_{\psi, g_*} \geq \alpha$, and $\det g(\psi) \leq C_1 r^{2\beta}$ holds when $r \to +\infty$, where $r(x)$ is the distance function on $(M, g_*)$ from some fixed point. Both results rely on expressing the squared norm of the mean curvature as a weighted divergence of a suitable vector field.

1 Introduction

In 1955, E. Heinz [11] obtained an estimative of the mean curvature of a piece of surface of $\mathbb{R}^3$ described by a graph of a function in terms of an isoperimetric inequality. Using

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some integral formulas, he proved that, if \( f(x, y) \) is a function defined on the disc \( x^2 + y^2 < R^2 \) and the mean curvature of the surface \( (x, y, f(x, y)) \) satisfies \( \|H\| \geq c > 0 \), where \( c \) is a constant, then \( R \leq \frac{1}{c} \). Thus, if \( f \) is defined in all \( \mathbb{R}^2 \) and \( \|H\| \) is constant, then \( H = 0 \). Ten years later this problem was extended and solved for the case of a function \( f : \mathbb{R}^m \to \mathbb{R} \), by Chern [7], and independently by Flanders [10]. In 1985, James Eells proposed us to find out if such isoperimetric estimations on the mean curvature could be extended in higher codimensions. Given a map \( f : M \to N \) between two Riemannian manifolds \((M, g)\) and \((N, h)\), of dimensions \( m \) and \( n \), respectively, assuming the graph submanifold \( \Gamma_f := \{(x, f(x)) : x \in M\} \) of the Riemannian product \( M = (M \times N, g \times h) \) has parallel mean curvature \( H \), we proved the following inequality ([19], [20])

\[
\|H\| \leq \frac{1}{m} \frac{A(\partial D)}{V(D)},
\]

holds on each compact domain \( D \subset M \), where \( A(\partial D) \) and \( V(D) \) are respectively the area of \( \partial D \) and the volume of \( D \) with respect to the metric \( g \). Since \( \|H\| \) is constant, we have

\[
\|H\| \leq \frac{1}{m} h(M, g),
\]

where

\[
h(M, g) = \inf_D \frac{A(\partial D)}{V(D)},
\]

is the Cheeger constant of \((M, g)\), with \( D \) ranging over all open domains of \( M \) with compact closure in \( M \) and smooth boundary (see e.g. [5]). This constant is zero, if, for example, \( M \) is a closed manifold or it is a simple noncompact Riemannian manifold, i.e., there exists a diffeomorphism \( \phi : (M, g) \to (\mathbb{R}^m, <, >) \) onto \( \mathbb{R}^m \) such that \( \lambda g \leq \phi^* <, > \leq \mu g \) for some positive constants \( \lambda, \mu \). Another large class of spaces with zero Cheeger constant are the complete Riemannian manifolds with nonnegative Ricci tensor (cf. [3]). Inequality (1.2) is sharp. In [19, 20, 21, 14] it is given examples of graphic hypersurfaces in \( \mathbb{H}^m \times \mathbb{R} \), with constant mean curvature \( c \) any real in \([0, m - 1]\), where \( \mathbb{H}^m \) is the hyperbolic space, a space with Cheeger constant equal to \((m - 1)\). These examples are of the form \( (x, f(r(x))) \) where \( r(x) \) is the distance function to a point of \( \mathbb{H}^m \).

This problem was generalized by Guanghan Li and the author in [14], in the context of calibrated manifolds. Given a \( m \)-form defining a calibration \( \Omega \), not necessarily closed, on a Riemannian manifold \((\overline{M}, \overline{g})\) of dimension \( m + n \), and \( F : M \to \overline{M} \) an oriented immersed submanifold of dimension \( m \), the \( \Omega \)-angle of \( M \) is the function \( \cos \theta : M \to [-1, 1] \) given by

\[
\cos \theta = \Omega(X_1, \ldots, X_m),
\]

where \( X_i \) is a direct o.n. frame of \( T_p M \). The \( \Omega \)-calibrated submanifolds (i.e. they satisfy \( \cos \theta = 1 \)) are minimal if \( \Omega \) is closed (see [12]). In [14] we have got the following \( \Omega \)-integral
isoperimetric inequality on a domain $D$ of $M$, that we display now in the most general case of $\Omega$, not necessarily closed, by following the same proof,

$$\left| \int_D (m \cos \theta \|H\|^2 - \langle \nabla^H H, \Phi \rangle) dV^* - \int_D (\nabla^H \Omega - H\omega d\Omega) \right| \leq \int_{\partial D} \sin \theta \|H\|dA^*. \quad (1.4)$$

In this inequality, $dV^*$ and $dA^*$ are the volume and area forms for the induced metric $\bar{g} = F^* \bar{g}$ on $M$, $\nabla$ is and Levi Civita connection on $(\bar{M}, \bar{g})$, and $\nabla^H \Omega - H\omega d\Omega$ is restricted to $M$ via pullback by $F$. The morphism $\Phi : TM \to N M$, with values on the normal bundle of $M$, and appearing in the inequality, is defined by

$$\bar{g}(\Phi(X), U) = \Omega(U, *X), \quad (1.5)$$

for any $X \in T_p M$ and $U \in N M_p$, where $*: TM \to \wedge^{(m-1)} TM$ is the star operator. If $\Omega$ is parallel, $\cos \theta > 0$ on $\bar{D}$, and $F$ has parallel mean curvature, from the above inequality we obtain

$$\|H\| \leq \frac{1}{m} \left( \sup_{\partial D} \sin \theta \right) \frac{A^*(\partial D)}{V^*(D)}. \quad (1.6)$$

In particular, if $\mathfrak{h}(M, g_\star) = 0$ and $\cos \theta > \epsilon > 0$ on $M$, then $H = 0$. We may relax the later assumption on $\cos \theta$ giving conditions at infinity. Assuming $(M, g_\star)$ is complete with nonnegative Ricci tensor, then for some constant $C_2 > 0$ we have $\mathfrak{h}(B_r, g_\star) \leq C_2/r$, $\forall r > 0$, where $B_r$ is the ball of radius $r$ centred at a fixed point of $(M, g_\star)$ (cf. [3]), with corresponding distance function $r(x)$. If in addition we assume for some constant $\beta \in [0, 1)$, $\cos \theta(x) \geq C_1 r^{-\beta}(x)$, these two conditions are sufficient to obtain $\|H\| \leq C r^{-\beta-1}(x)$ (see proof of Theorem 1.4 of [14]), and conclude $H = 0$. Here $C$ and $C_1$ are positive constants.

A Riemannian product manifold $\bar{M} = M \times N$ has a natural calibration defined by the volume of the projection onto the first component

$$\Omega((X_1, Y_1), \ldots, (X_m, Y_m)) = Vol_M(X_1, \ldots, X_m).$$

In the case of a graph submanifold, $\Gamma_f : M \to M \times N$, $\Gamma_f(x) = (x, f(x))$, if $(X_1^*, df(X_1^*))$ is a direct o.n. frame of $\Gamma_f$, then

$$\cos \theta = \Omega((X_1^*, df(X_1^*)), \ldots, (X_m^*, df(X_m^*)) = \frac{1}{\sqrt{\det \bar{g}(g + f^* h)}} > 0,$$

where the determinant of the graph metric $g_\star = \Gamma_f^* \bar{g} = g + f^* h$ is taken with respect to a diagonalizing $g$-o.n. frame of $f^* h$. The condition $\cos \theta \geq \epsilon > 0$ is equivalent to $\|df\|^2$ to be bounded, and consequently the metrics $g_\star$ and $g$ are equivalent. In this case, $(M, g_\star)$ has zero Cheeger constant if and only if $(M, g)$ has so. In ([14]) we obtain the conclusion $H = 0$ for graph submanifolds with parallel mean curvature using the isoperimetric inequality with
the $\Omega$-angle, but we need an extra condition on $\cos \theta$ at infinity as explained above. Hence, this approach applied to graphs seems weaker than the one in [19], [20], but the curvature conditions may be different.

In this work we generalize the isoperimetric inequality (1.1) for a graph submanifold $\Gamma_f$ in an ambient space a warped product of two Riemannian manifolds, $(M^m, g)$ and $(N^n, h)$, defined by a warping function $\rho : M \to (0, +\infty)$. We denote this space by $(\tilde{M}, \tilde{g})$, where $\tilde{M} = M \times_\rho N$ is the product space $M \times N$ with the warped metric

$$\tilde{g} = g + \rho^2 h,$$

where $\rho = e^\psi$.

If $n = 1$, some Bernstein type results for a graph hypersurface in a warped product have been obtained in [1, 2, 3, 4, 8, 16, 18]. In this paper we work in any codimension, and consider the Heinz estimation type problem for the mean curvature. This problem has been studied in [13] for $n = 1$, in the context of weighted manifolds. We will extend Theorem 1 of [13] to higher codimensions, using their concept of weighted Cheeger constant.

A graph submanifold is defined as an immersion $\Gamma_f : M \to \tilde{M}$, $\Gamma_f(x) = (x, f(x))$, for a given map $f : M \to N$. It defines on $M$ a induced metric, the graph metric

$$g_* (X, Y) = \Gamma^*_f \tilde{g} (X, Y) = g(x) (X, Y) + \tilde{h}(x) (df(X), df(Y)), \tag{1.7}$$

where $\tilde{h}(x) = \rho(x) h(f(x))$ is a Riemannian metric on the pullback tangent bundle $f^{-1}TN$. We denote by $df^* : f^{-1}TN \to TM$, the adjoint morphism of $df : TM \to f^{-1}TN$ when we consider on $TM$ the graph metric $g_*$, and on $f^{-1}TN$ the metric $\tilde{h}$. The endomorphisms, $Id - dfdf^* : f^{-1}TN \to f^{-1}TN$, and $Id - df^*df : TM \to TM$, symmetric for the metrics $\tilde{h}$ and $g_*$, respectively, are both positive diffeomorphisms, with the same set of eigenvalues. These are useful diffeomorphisms. For example, for any function $\Theta : M \to \mathbb{R}$, the following equality holds

$$(Id - df^*df)(\nabla^M \Theta) = \nabla^* \Theta, \tag{1.8}$$

where $\nabla^M \Theta$ and $\nabla^* \Theta$ are the gradients with respect to $g$ and $g_*$, respectively. We denote by $\nabla df$ the Hessian of $f : (M, g) \to (N, h)$, and consider the following vector fields $\Psi_*$, $W, W_1$ of $f^{-1}TN$ and $Z_1$ of $TM$, given by

$$\Psi_* = df (\|df\|_g^2 \nabla^M \psi + 2 \nabla^* \psi), \tag{1.9}$$

$$W = \text{trace}_{g_*} \nabla df \tag{1.10}$$

$$Z_1 = df^* (W + \Psi_*) \tag{1.11}$$

$$W_1 = (Id - df^*df) (W + \Psi_*). \tag{1.12}$$

We will prove in Lemma 2.3 that the mean curvature of $\Gamma_f$, $H = (H_M, H_N)$, is given by

$$mH = m(H_M, H_N) = (-Z_1, W_1) = (0, W + \Psi_*)^\perp, \tag{1.13}$$
where $\cdot ^\perp$ denotes the orthogonal projection onto the normal bundle, and a minimal graph on $\tilde{M}$ is defined by the equality
\[ W_* := W + \Psi_* = 0. \tag{1.14} \]

We state our first main Theorem 1.1 for graph submanifolds of $\tilde{M}$ that generalizes the case $\rho = 1$ of [20], and the case $n = 1$ of [13]. We have the following relations between the angle of $\nabla^M \psi$ with the $M$ and $N$ components of the mean curvature
\[ g(H_M, \nabla^M \psi) = -\tilde{h}(H_N, df(\nabla^M \psi)). \tag{1.15} \]
Then we define the following set, that is empty if $f$ is constant along integral curves of $\nabla^M \psi$,
\[ M^- = \{ x \in M : g(H_M, \nabla^M \psi) < 0 \} = \{ x \in M : \tilde{h}(H_N, df(\nabla^M \psi)) > 0 \}. \tag{1.16} \]

The proof of Theorem 1.1 relies on a key formula that expresses $\|H\|^2$ as a weighted divergence of a suitable vector. We will show the following equality holds
\[ e^{-\psi} \text{div}_g(e^\psi H_M) = \text{div}_g(H_M) + g(H_M, \nabla^M \psi) = -m\|H\|^2. \tag{1.17} \]
The weighted Cheeger constant, introduced in [13], is given by
\[ h(M, g, \psi) := \inf_D \frac{A_\psi(\partial D)}{V_\psi(D)}, \tag{1.18} \]
where the $\psi$-weighted area and volume are considered on compact domains $\bar{D} = D \cup \partial D$ of $(M, g)$ with smooth boundary,
\[ A_\psi(\partial D) = \int_{\partial D} e^{\psi} dA, \quad V_\psi(D) = \int_D e^{\psi} dM. \tag{1.19} \]
Clearly, if $h(M, g) = 0$, and $\psi$ is bounded, then $h(M, g, \psi) = 0$. The $\psi$-weighted Cheeger constant has the following spectral property (see Section 3), for $M$ compact with non-empty smooth boundary,
\[ \lambda_{\psi,1}(M) \geq \frac{1}{4} (h(M, g, \psi))^2, \tag{1.20} \]
where $\lambda_{\psi,1}(M)$ is the lowest eigenvalue of the spectrum of the drift $\psi$-Laplacian, $-\Delta_\psi u = -\Delta u - g(\nabla \psi, \nabla u)$, with Dirichlet boundary condition, $u = 0$ on $\partial M$.

**Theorem 1.1.** If $f : M \to N$ defines a graph submanifold $\Gamma_f : M \to M \times_\rho N$, $\Gamma_f(x) = (x, f(x))$, with parallel mean curvature $H = (H_M, H_N)$, then the following estimates hold
\[ m\|H\| \leq h(M, g, \psi), \tag{1.21} \]
\[ m\|H\| \leq h(M, g) + \sup_{M^-} \|\nabla^M \psi\|, \tag{1.22} \]
where $M^-$ is defined in (1.16). In particular, if $h(M, g, \psi) = 0$, or if $h(M, g) = 0$ and either $g(H_M, \nabla^M \psi) \geq 0$ for all $x \in M$ or $\psi$ is bounded, then $H = 0$. 
Corollary 1.1. If \((M, g)\) is a closed Riemannian manifold then any graph submanifold \(\Gamma_f\) of \(M \times_{\rho} N\) with parallel mean curvature is minimal.

We consider the particular case \(M\) is a complete non-compact spherically symmetric Riemannian manifold, \(M = M_\tau := [0, +\infty) \times_\tau S^{m-1}\) with metric \(g = g_\tau := dt^2 + \tau^2(t)\sigma^2\), where \(\tau \in C^2([0, +\infty))\) satisfies \(\tau(0) = \tau''(0) = 0\), \(\tau'(0) = 1\), \(\tau(t) > 0\) for all \(t > 0\), and \(d\sigma^2\) is the Euclidean metric of the unit \((m - 1)\)-sphere. For \(x = (t, \xi)\), let \(r(x)\) be the distance function to the origin \(o\) of \(M_\tau\), the point that is identified with the class of all elements \((0, \xi)\), with \(\xi \in S^{m-1}\). In next proposition we build entire graph hypersurfaces with nonzero constant mean curvature in \(M \times_{\rho} \mathbb{R}\) under suitable conditions on the warping functions \(\tau\) and \(\rho = e^\psi\), following a similar construction as in [19, 20, 21]. Some details on spherically symmetric spaces with a density can be seen in [9]. Assume \(\psi\) is a radial function on \(M_\tau\), that is, \(\psi(x) = \Psi(r(x))\) where \(\Psi : [0, +\infty) \rightarrow [0, +\infty)\) satisfies \(\Psi'(0) = 0\) (to make \(\psi\) of class \(C^1\) at \(x = o\)). Consider the function \(X : [0, +\infty) \rightarrow [0, +\infty)\) given by

\[
X(t) = e^{\Psi(t)}(\tau(t))^{(m-1)}.
\]

This function is positive for \(t > 0\) and has a zero of finite order \((m - 1)\) at \(t = 0\). The function \(\phi : [0, +\infty) \rightarrow [0, +\infty)\),

\[
\phi(t) = \frac{\int_0^t X(s)ds}{X(t)},
\]

satisfies the equation \(\phi'(t) = 1 - \frac{X'(t)}{X(t)}\phi(t)\), with initial conditions \(\phi(0) = 0\) and \(\phi'(0) = 1\).

Proposition 1.1. Assume the infimum \(C_0 := \inf_{t \geq 0} \frac{1}{\phi(t)}\) is positive. For each pair of constants \(|c| < C_0\), and \(d \in \mathbb{R}\), consider the functions, \(\phi_c : [0, +\infty) \rightarrow (-1, 1)\) defined by \(\phi_c(t) = c\phi(t)\), and \(F = F_{c,d} : [0, +\infty) \rightarrow \mathbb{R}\) by

\[
F(t) = \int_0^t \left( e^{-\Psi(s)} \frac{\phi_c(s)}{\sqrt{1 - \phi_c(s)^2}} \right) ds + d. \tag{1.23}
\]

Then \(f(x) = F(r(x))\) defines a graph hypersurface \(\Gamma_f(x) = (x, f(x))\) that has constant mean curvature \(\frac{c}{m}\) in \(\tilde{M} = M \times_{e^\psi} \mathbb{R}\). In particular, \(\mathfrak{h}(M_\tau, g_\tau, \psi) \geq C_0\).

If the infimum \(C_0\) is taken at a point \(t_0 \in (0, +\infty)\), then \(C_0\) is positive. Since \(\lim_{t \to 0} \frac{1}{\phi(t)} = \lim_{t \to 0} \Psi'(t) + (m - 1)\frac{\phi'(t)}{\phi(t)} = +\infty\), in order to have \(C_0 > 0\) we only need to assume \(\phi(t)\) bounded. This is the case when, \(M_\tau\) is the \(m\)-dimensional Hyperbolic space where \(\tau(t) = \sinh(t)\) and \(\psi = 0\), giving \(C_0 = (m - 1) = \mathfrak{h}(\mathbb{H}^m)\).
The slices $M \times \{q\}$ are totally geodesic submanifolds of $\tilde{M}$, but the $m$-form on $\tilde{M}$, 
\[ \Omega((X_1, U_1), \ldots, (X_m, U_m)) = Vol_g(X_1, \ldots, X_m), \]
is not a parallel $m$-form if $\psi$ is not constant. However, we will show in Lemma 2.5 that $\Omega$ is a closed calibration that calibrates the slices. Now we state our second main Theorem 1.2 that generalizes Theorem 1.4 of [14], for graph submanifolds of $M = M \times \rho N$ with parallel mean curvature. The key of the proof is the following divergence-type formula
\[
e^{-\psi} \text{div}_{g^*}(e^\psi \cos \theta H_M) = \text{div}_{g^*}(\cos \theta H_M) + g^* (\nabla^* \psi)
\]
\[
= \text{div}_{g^*}(\cos \theta H_M) - (\tilde{\nabla}_H \Omega)(X_1^*, \ldots, X_m^*)
\]
\[
= -m \cos \theta \parallel H \parallel^2,
\]
where $X_i^*$ is a $g_*$-o.n. frame of $M$. We are assuming $M$ is oriented. If $(M, g)$ is complete, then so is $(M, g^*)$.

**Theorem 1.2.** Assume $\Gamma_f$ has parallel mean curvature on $M \times \rho N$, and $(M, g_*)$ is complete. Moreover, assume that for some constants $\beta \in [0, 1)$ and $C_1 > 0$, the $\Omega$-angle of $\Gamma_f$, 
\[
\cos \theta := \Omega(d\Gamma_f(X_1^*), \ldots, d\Gamma_f(X_m^*)) = \frac{1}{\sqrt{\det g_*}},
\]
satisfies $\cos \theta \geq C_1 r^{-\beta}$, when $r \to +\infty$, where $r(x)$ denotes the distance function on $(M, g_*)$ from a fixed point $x_0$. Furthermore, assume for some nonnegative constants $\alpha \geq \delta$, the $\psi$-Ricci tensor of $(M, g_*)$ is bounded from below by $\alpha$, and $\|\nabla^* \psi\| \leq \delta^{1/2}$. Then $\Gamma_f$ is a minimal submanifold.

We will see in Corollary 3.1 of Section 3, under the above boundedness conditions on $\|\nabla^* \psi\|$ and on the $\psi$-Ricci tensor, that $h(M, g_*, \psi) = 0$.

## 2 Graphs in warped products

We consider two Riemannian manifolds $(M^m, g)$ and $(N^n, h)$, and a function $\psi : M \to \mathbb{R}$, defining a Riemannian space $(\tilde{M}, \tilde{g})$, where $\tilde{M} = M \times N$ is endowed with the warped metric $\tilde{g} = g + e^{2\psi} h$. Let $\nabla^M$, $\nabla^N$ and $\tilde{\nabla}$ denote the Levi-Civita connections of $(M, g)$, $(N, h)$ and $(\tilde{M}, \tilde{g})$ respectively. We recall the following properties of $\tilde{\nabla}$ (cf. [17]). If $X, Y$ are vector fields of $M$ and $U, W$ of $N$ then
\[
\tilde{\nabla}_X Y = \nabla^M_X Y, \quad \tilde{\nabla}_X U = \tilde{\nabla}_U X = d\psi(X)U,
\]
\[
\tilde{\nabla}_U W = (\tilde{\nabla}_U W)^{\text{tan}} + (\tilde{\nabla}_U W)^{\text{nor}} = \nabla^N_U W - \tilde{g}(U, W)\nabla^M \psi,
\]
Lemma 2.1. If denote by \(\Gamma_f\) form of \(\Gamma\), \(h\) is a Riemannian metric on the pullback tangent bundle \(f^{-1}TN\). We denote by \(\nabla^{f^{-1}}\) the pullback connection of \(\nabla^N\) by \(f\) on \(f^{-1}TN\).

**Lemma 2.1.** If \(X, Y\) are smooth vector fields on \(M\), and \(U\) is a section of \(f^{-1}TN\), then \((Y, U)\) is a section of \(\Gamma_{f^{-1}}\)

\[
\nabla_{X}^{f^{-1}} (Y, U) = (\nabla_{X}M, \nabla_{X}^{f^{-1}} U) + (-\tilde{h}(df(X), U))\nabla_{X}^{M} \psi, \ d\psi(X)U + d\psi(Y)df(X)).
\]

**Proof.** We take \(U_i\) a local o.n. frame of \(TN\), and write \(U(x) = \sum_i \lambda_i(x, f(x))U_i(f(x))\), where \(\lambda_i(x, q) = \lambda_i(x)\). Hence, \((Y, U) = (Y, 0) + \sum_i \lambda_i(0, U_i)\), and we have

\[
\nabla_{X}^{f^{-1}} (Y, U) = \nabla_{(X, df(X))}(Y, 0) + \sum_i \lambda_i(0, U_i) \nabla_{X}^{(0, df(X))}(Y, 0) + \sum_i \lambda_i \nabla_{(0, df(X))}(Y, 0) + \sum_i \lambda_i \nabla_{(0, df(X))}(Y, 0) + \sum_i \lambda_i \nabla_{(0, df(X))}(Y, 0).
\]

Applying the above rules of the \(\nabla\)-connection, we get the final expression. \(\square\)

Let \(\nabla^*\) be the Levi Civita connection of \(M\) for the graph metric \(g_*\). The second fundamental form of \(\Gamma_f\) takes values on the normal bundle \(N\Gamma_f \subset \Gamma_{f^{-1}}\)

\[
\nabla^* d\Gamma_f (X, Y) = \nabla_{X}^{f^{-1}} (d\Gamma_f (Y)) - d\Gamma_f (\nabla^{*} Y).
\]

We denote by \((Y, U)^\perp\) and \((Y, U)^\perp\) the \(\tilde{g}\)-orthogonal projections onto \(TT\Gamma_f\) and \(N\Gamma_f\) respectively, and the Hessian of \(f\) for the Levi-Civita connections \(\nabla^M\) and \(\nabla^N\) by

\[
\nabla df(X, Y) = \nabla_{X}^{f^{-1}} (df(Y)) - df(\nabla_{X}^{M} Y) = \nabla_{df(X)}^{N} (df(Y)) - df(\nabla_{df(X)}^{N} Y).
\]

**Lemma 2.2.** (1) \((Y, U)^\perp = (0, U - df(Y))^\perp\). Hence, \((Y, U)^\perp = (0, df(Z))^\perp\) if and only if \((0, U)^\perp = (0, df(Z + Y))^\perp\); (2) \((X, 0) \in N\Gamma_f\) if and only if \(X = 0\); (3) \((0, U)^\perp = 0\) if and only if \(U = 0\).
\textbf{Proof.} (1) \((Y, U) \perp = (Y, U) \perp \) - \((Y, df(Y)) \perp\). (2) If \((X, 0) \in N\Gamma_f\), then \(0 = \tilde{g}((X, 0), (X, df(X))) = g(X, X)\). (3) If \((0, U) \in T\Gamma_f\) then \((0, U) = (X, df(X))\) for some \(X\), and so \(X = 0\). \(\Box\)

We define a \(TM\)-valued symmetric bilinear form, \(\Xi : T_xM \times T_xM \to T_xM\),

\[
\Xi(X, Y) = \tilde{h}(df(X), df(Y))\nabla^M \psi + g(\nabla^M \psi, X)Y + g(\nabla^M \psi, Y)X,
\]

(2.25)

and consider the adjoint linear morphism \(df^* : f^{-1}TN \to TM\) of \(df : TM \to f^{-1}TN\), considering \(TM\) with the metric \(g_*\) and \(f^{-1}TN\) with \(\tilde{h}\), that is, it is defined by

\[
g_*(df^*(U), X) = \tilde{h}(df(X), U).
\]

We recall the vector fields \(\Psi_*, W, Z_1\) and \(W_1\) given in (1.9), (1.10), (1.11), and (1.12), and set

\[
W_* = W + \Psi_*, \quad \Xi_* = \text{trace}_{g_*}\Xi.
\]

(2.26)

(2.27)

We have

\[
\Psi_* = df(\Xi_*), \quad \text{and} \quad Z_1 = df^*(W_*).
\]

(2.28)

Let \(X_i\) be a local o.n. frame of \(M\) with respect to \(g\), and set

\[
g_{*ij} := g_*(X_i, X_j) = \delta_{ij} + \tilde{h}(x)(df(X_i), df(X_j)) = \tilde{g}(d\Gamma_f(X_i), d\Gamma_f(X_j)) = \bar{g}_{ij}.
\]

Note that,

\[
\sum_k g_{kr}^* \tilde{h}(W_*, df(X_k)) = \sum_k g_{kr}^* g_*(df^*(W_*), X_k)
\]

\[
= \sum_{ks} g_{kr}^* g(Z_1, X_s)g_{sk} = g(Z_1, X_r).
\]

(2.29)

\textbf{Lemma 2.3.} We have

\[
\nabla^* d\Gamma_f(X, Y) = (0, \nabla df(X, Y) + df(\Xi(X, Y))) \perp
\]

(2.30)

\[
mH = (0, W_*) \perp = (-Z_1, W_1).
\]

(2.31)

In particular,

\[
m\|H\| \geq \|Z_1\|_g.
\]

(2.32)

Furthermore, \(H = 0\) if and only if \(W = -\Psi_*\).
Proof. Applying Lemma 2.1 to \( d\Gamma_f(Y) = (Y, df(Y)) \), and using the fact that the second fundamental form takes values on the normal bundle \( \mathcal{N} \), we get

\[
\nabla^* d\Gamma_f(X,Y) = d\Gamma_f(\nabla^N Y - \nabla_X Y) + (0, \nabla df(X,Y)) + (0, \nabla df(X,Y)) + \tilde{h}(df(X), df(Y))\nabla^N \psi , \ d\psi(X)df(Y) + d\psi(Y)df(X) \\
= (0, \nabla df(X,Y))^\perp + (0, \nabla df(X,Y))^\perp,
\]

where in the last equality we used Lemma 2.2(1). Taking the \( g^* \)-trace of \( \nabla^* d\Gamma_f \), we get the first expression for the mean curvature in (2.31). From (1.11), (1.12), (2.26), (2.27) and (2.29), we have

\[
mH = (0, W_*)^\perp = (0, W_*) - (0, W_*)^\perp
\]

\[
= (0, W_*) - \sum_{kr} g^{kr}_* \tilde{g}((0, W_*), (X_k, df(X_k)))(X_r, df(X_r))
\]

\[
= (0, W_*) - \sum_{kr} g^{kr}_* \tilde{h}(W_*, df(X_k))(X_r, df(X_r))
\]

\[
= (0, W_*) - \sum_r g(\lambda_1, X_r)(X_r, df(X_r))
\]

\[
= (0, W_*) - (\lambda_1, df(Z_1)) = (-\lambda_1, W_1).
\]

Consequently, \( m^2\|H\|^2 = \|Z_1\|_{g}^2 + \tilde{h}(W_1, W_1) \geq \|Z_1\|_{g}^2 \). Therefore, \( H = 0 \) if and only if \( Z_1 = W_1 = 0 \). By Lemma 2.2(3), the later is equivalent to \( W_* = 0 \).

We now choose \( X_i \) that is a diagonalizing \( g^* \)-o.n. basis of \( \rho^2 f^* h \) at a given point \( x \), that is,

\[
\rho^2(x) h(f(x))(df(X_i), df(X_j)) = \lambda_i^2 \delta_{ij}, \quad (2.33)
\]

with \( \lambda_1^2 \geq \lambda_2^2 \geq \ldots \lambda_k^2 \geq \lambda_{k+1}^2 \geq \ldots \lambda_{m}^2 \geq 0 \), where \( k \) is defined by \( \lambda_k > 0 \) and \( \lambda_{k+1} = 0 \) if \( df(x) \neq 0 \), otherwise we set \( k = 0 \). Thus, \( m - k \) is the dimension of the kernel of \( df \), and taking

\[
X_i^* := \frac{X_i}{\sqrt{1 + \lambda_i^2}}, \quad \text{for} \quad i = 1, \ldots, m, \quad (2.34)
\]

we obtain a \( g^* \)-o.n. frame. Hence, \( \{X_1, \ldots, X_k\} \) span \( df^* (f^{-1} T\mathcal{N}) = (\text{Kern } df)^\perp \), and \( \{X_{k+1}, \ldots, X_m\} \) span \( \text{Kern } df \). The vector fields of \( f^{-1} T\mathcal{N} \),

\[
U_i := \lambda_i^{-1} df(X_i), \quad \text{for} \quad i \leq k, \quad (2.35)
\]

span \( df(TM) \), a subspace which \( \tilde{h} \)-orthogonal complement is the kernel of \( df^* \). We extend this \( \tilde{h} \)-o.n. system to give a \( \tilde{h} \)-o.n. basis of \( f^{-1} T\mathcal{N} \), defining the subspace \( \text{Kern } df^* = \)
$(df(TM))^\perp = \text{span}\{U_{k+1}, \ldots, U_n\}$. Recall that we are considering $\lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_{\max\{m,n\}} = 0$, where $k \leq \min\{m,n\}$. Then we have,

$$
\begin{cases}
    df(X_i) = \frac{\lambda_i}{\sqrt{1+\lambda_i^2}} U_i, & df^*\psi(X_i) = \frac{\lambda_i^2}{1+\lambda_i^2} X_i, & (Id - df^*df)(X_i) = \frac{1}{1+\lambda_i^2} X_i, & i = 1, \ldots, m, \\
    df^*(U_i) = \frac{\lambda_i}{\sqrt{1+\lambda_i^2}} X_i^*, & df^*d\psi(U_i) = \frac{\lambda_i^2}{1+\lambda_i^2} U_i, & (Id - df^*df)(U_i) = \frac{1}{1+\lambda_i^2} U_i, & i = 1, \ldots, n.
\end{cases}
$$

(2.38)

It follows that, $||df||_g^2 = \sum_i \lambda_i^2$, and $||df||_*^2 = ||df^*||_*^2 = \text{trace}(df^*df) = \text{trace}(df^*df) = \sum_i \frac{\lambda_i^2}{1+\lambda_i^2}$. From the above identities (2.38) we also derive the formula on the gradients (1.18).

Now we consider a morphism, $Q_\psi : f^{-1}TN \to \mathbb{R}_M$, where for $U \in T_{f(x)}N$ it is given by

$$Q_\psi(U) = \tilde{h}(U, df(\nabla^*\psi)).$$

(2.37)

The morphism $Q_\psi$ is null if $f$ is constant along the integral curves of $\nabla^*\psi$, or of $\nabla^M\psi$, as a consequence of the identity (2.38) in next Lemma 2.4. Using the eigenvectors $X_i, X_i^*, U_i$ we see that $Q_\psi(\Psi_s) = \sum_j \frac{\lambda_j^2}{1+\lambda_j^2} \left(||df||^2 + \frac{2}{1+\lambda_j^2}\right) g(\nabla^M\psi, X_j)^2 \geq 0$.

**Lemma 2.4.** The following equalities hold

$$Q_\psi(U) = g(df^*(U), \nabla^M\psi) = \tilde{h}\left((Id - df^*df)(U), df(\nabla^M\psi)\right).$$

(2.38)

Furthermore,

$$Q_\psi(W_*) = g(Z_1, \nabla^M\psi) = \tilde{h}(W_1, df(\nabla^M\psi)).$$

(2.39)

Hence, $M^- = \{x : Q_\psi(W_*) > 0\}$, where $M^-$ is defined in (1.16). Then $M^- = \emptyset$ if and only if $Q_\psi(W) \leq -Q_\psi(\Psi_s)$ everywhere.

**Proof.** The first equality of (2.38) follows from the identities

$$Q_\psi(U) = \tilde{h}\left(U, df(\nabla^*\psi)\right) = g_s(df^*(U), \nabla^*\psi) = df(\psi(df^*(U))) = g(df^*(U), \nabla^M\psi).$$

Now we show the second.

$$\tilde{h}\left((Id - df^*df)(U), df(\nabla^M\psi)\right) = \sum_{i,j} \tilde{h}(U, U_i) d\psi(X_j) \tilde{h}\left((Id - df^*df)(U_i), df(X_j)\right)$$

$$= \sum_{i \leq n, j \leq k} \tilde{h}(U, U_i) d\psi(X_j) \frac{\lambda_j}{1+\lambda_i^2} \tilde{h}(U_i, U_j) = \sum_{i \leq k} \tilde{h}(U, U_i) d\psi(X_i) \frac{\lambda_i}{1+\lambda_i^2}$$

$$= \sum_i \tilde{h}(U, df(X_i^*)) d\psi(X_i^*) = \tilde{h}(U, df(\nabla^*\psi)) = Q_\psi(U).$$

Then, $Q_\psi(W_*) = g_s(df^*(W_*), \nabla^*\psi) = g_s(Z_1, \nabla^*\psi) = g(Z_1, \nabla^M\psi)$. The proof of last equality in (2.39) is a direct consequence of $H$ to take values on the normal bundle and Lemma 2.4 and so $\bar{g}((-Z_1, W_1), (\nabla^M\psi, df(\nabla^M\psi))) = 0$. \qed
If $X^*_i$ is a g*-o.n. basis of $T_x M$, then
\[
\langle d\Gamma_f(\cdot), \tilde{\nabla}^f_{\cdot} H \rangle = \sum_i \tilde{h}(d\Gamma_f(X^*_i), \tilde{\nabla}^f_{X^*_i} H) = - \sum_i \tilde{h}(\nabla^*_X d\Gamma_f(X^*_i), H)
\]
Hence,
\[
\langle d\Gamma_f(\cdot), \tilde{\nabla} H \rangle = -m\|H\|^2.
\] (2.40)

Now we prove main Theorem 1.1. We denote by $\nabla^\perp$ the covariant derivative on $NT_f$.

**Proof of Theorem 1.1** From equalities (2.26), (2.27) and (2.28), $W_1 = W + df(\Xi_* - Z_1)$, and from Lemma 2.3 $mH = (-Z_1, W_1)$. Thus, by Lemma 2.1

\[
m\nabla^\Gamma_f H = \]
\[
\begin{align*}
&= (-\nabla^M_X Z_1, \nabla^f_{\cdot} W_1) + \left( -\tilde{h}(df(X), W_1)\nabla^M \psi , d\psi(X)W_1 + d\psi(-Z_1)df(X) \right) \\
&= -\left( \nabla_X^M Z_1, df(\nabla^M_X Z_1) \right) + \left( 0 , \nabla^f_{\cdot} W + \nabla_X df(\Xi_* - Z_1) + df(\nabla^M_X \Xi_*) \right) \\
&+ \left( -\tilde{h}(df(X), W_1)\nabla^M \psi , d\psi(X)W_1 + d\psi(-Z_1)df(X) \right).
\end{align*}
\]

By Lemma 2.2 (1), if $\nabla^\perp H = 0$ then
\[
\left( 0 , \nabla^f_{\cdot} W + \nabla_X df(\Xi_* - Z_1) + df(\nabla^M_X \Xi_*) \right)^\perp
\]
\[
+ \left( 0 , d\psi(X)W_1 + d\psi(-Z_1)df(X) + \tilde{h}(df(X), W_1)df(\nabla^M_X \psi) \right)^\perp = 0,
\]
and so, by Lemma 2.2 (3)
\[
\nabla^f_{\cdot} W + \nabla_X df(\Xi_* - Z_1) + df(\nabla^M_X \Xi_*) = \\
= -d\psi(X)W_1 + d\psi(Z_1)df(X) - \tilde{h}(df(X), W_1)df(\nabla^M_X \psi).
\]

Thus,
\[
m\nabla^\Gamma_f H = -(\nabla^M_X Z_1, df(\nabla^M_X Z_1)) + \left( -\tilde{h}(df(X), W_1)\nabla^M \psi , -\tilde{h}(df(X), W_1)df(\nabla^M_X \psi) \right) \\
= d\Gamma_f \left( -\nabla^M_X Z_1 - \tilde{h}(df(X), W_1)\nabla^M \psi \right).
\]

Now we fix a point $x_0$ and take $X_i$ s.t. $\nabla^M X_i(x_0) = 0$. Then, by (2.29) at $x_0$
\[
\nabla^M_X Z_1 = \sum_{kr} d \left( g^{kr}_* \tilde{h}(W_*, df(X_k)) \right)(X_i) X_r,
\]
and so, for each $i,j$
\[
\tilde{g} \left( (\nabla^M_X Z_1, df(\nabla^M_X Z_1)) , (X_j, df(X_j)) \right) = \sum_{kr} d \left( g^{kr}_* \tilde{h}(W_*, df(X_k)) \right)(X_i) \tilde{g}_{ij}.
\]
In particular, at \( x_0 \)
\[
div_g(Z_1) = \sum_r g(\nabla^M_{X_r}Z_1, X_r) = \sum_{kr} d\left( g^{kr}_{*} \tilde{h}(W_*, df(X_k)) \right)(X_r).
\]
Therefore, at \( x_0 \),
\[
-m^2 \| H \|^2 = m(\nabla^{\Gamma g^{-1}} H, d\Gamma_f) = \sum_{ij} g^{ij}_{*} \tilde{g}(\nabla^{\Gamma g^{-1}} H, d\Gamma_f(X_j))
\]
\[
= \sum_{ij} g^{ij}_{*} \tilde{g} \left( -d\Gamma_f(\nabla^M_{X_i}Z_1 + \tilde{h}(df(X_i), W_1)\nabla^M \psi), d\Gamma_f(X_j) \right)
\]
\[
= \sum_{ij} g^{ij}_{*} \tilde{g} \left( -\nabla^M_{X_i}Z_1, df(-\nabla^M_{X_i}Z_1), (X_j, df(X_j)) \right)
\]
\[
+ \sum_{ij} g^{ij}_{*} \left( -\tilde{h}(df(X_i), W_1)\tilde{g}(d\Gamma_f(\nabla^M \psi), d\Gamma_f(X_j)) \right)
\]
\[
= -\sum_{ijklr} g^{ij}_{*} d \left( g^{kr}_{*} \tilde{h}(W_*, df(X_k)) \right)(X_i) \tilde{g}_{rj}
\]
\[
- \sum_{ijl} g^{ij}_{*} \tilde{g}(d\Gamma_f(df^*(W_1)), d\Gamma_f(X_i)) \tilde{g}(d\Gamma_f(\nabla^M \psi), d\Gamma_f(X_j))
\]
\[
= -\div_g(Z_1) - \tilde{g}(d\Gamma_f(df^*(W_1)), d\Gamma_f(\nabla^M \psi))
\]
\[
= -\div_g(Z_1) - g_*(df^*(W_1), \nabla^M \psi) = -\div_g(Z_1) - \tilde{h}(W_1, df(\nabla^M \psi)).
\]
Hence, by Lemma 2.4
\[
m^2 \| H \|^2 = \div_g(Z_1) + g(Z_1, \nabla^M \psi) = e^{-\psi} \div_g(e^\psi Z_1), \tag{2.41}
\]
where \( \| H \| \) is constant by assumption on parallel mean curvature. Weighted integration over \( D \) of \( e^{-\psi} \div_g(e^\psi Z_1) \), using Stokes’, Schwartz inequality and applying Lemma 2.3 gives
\[
m^2 \| H \|^2 V_\psi(D) = \int_D e^{-\psi} \div_g(e^\psi Z_1) e^\psi dM = \int_D \div_g(e^\psi Z_1) dM
\]
\[
= \int_{\partial D} e^\psi g(Z_1, \nu) dA \leq \int_{\partial D} \| Z_1 \| e^\psi dA \leq m \| H \| A_\psi(\partial D),
\]
where \( \nu \) is the unit outward of \( (\partial D, g) \). Hence
\[
m \| H \| \leq \frac{A_\psi(\partial D)}{V_\psi(D)},
\]
and (1.21) of the Theorem is proved. Usual integration of \( \div_g(Z_1) + g(Z_1, \nabla^M \psi) \), gives
\[
m^2 \| H \|^2 V(D) \leq m \| H \| A(\partial D) + \int_{D \cap M^-} g(Z_1, \nabla^M \psi) dM
\]
\[
\leq m \| H \| A(\partial D) + m \| H \| \sup_{M^-} \| \nabla^M \psi \| V(D).
\]
Hence \( m\|H\| \leq \frac{A(\partial D)}{V(D)} + \sup_{M^-}\|\nabla^M \psi\| \), and (1.22) is proved. \( \square \)

**Proof of Proposition 1.1.** For any Riemannian manifold \((M, g)\), if \(N = \mathbb{R}\) and \(f : M \to \mathbb{R}\), the unit normal of \(\Gamma_f\) in \(\tilde{M} = M \times_{\rho} \mathbb{R}\) is given by

\[
N = \frac{(-\nabla f, e^{-2\psi})}{\sqrt{e^{-2\psi} + \|\nabla f\|^2}}.
\]

where \(\nabla = \nabla^M\), and the mean curvature \(H\) satisfies the equation

\[
m \tilde{g}(H, N) = \text{div}_\psi \left( \frac{\nabla f}{\sqrt{e^{-2\psi} + \|\nabla f\|^2}} \right) := e^{-\psi} \text{div}_g \left( e^{\psi} \frac{\nabla f}{\sqrt{e^{-2\psi} + \|\nabla f\|^2}} \right).
\]

Hence \(m \tilde{g}(H, N) = c\) if and only if

\[
\text{div}_g \left( \frac{e^\psi \nabla f}{\sqrt{e^{-2\psi} + \|\nabla f\|^2}} \right) = c e^{\psi}.
\]

Now we are assuming \((M, g) = (M', g')\), and so \(r(t, \xi) = t\), and \(\nabla r(t, \xi) = \partial_t(t, \xi)\). Moreover, for any radial function \(A(r)\) we have

\[
\text{div}_g (A(r) \nabla^M r) = A'(r) + A(r)(m - 1)(\log \tau)' = A'(t) + A(t)(m - 1) \frac{\tau'(t)}{\tau(t)}.
\]

We have \(f(x) = F(r(x)), \psi(x) = \Psi(r(x)), \) and \(X(t) = e^{\Psi(t)} \tau(t)^m\). Then we consider the following function

\[
\xi(t) = \frac{F'(t)}{\sqrt{e^{-2\Psi(t)} + (F'(t))^2}} \in (-1, 1).
\]

It follows that, \(m \tilde{g}(H, N) = c\) if and only if \(c = \xi' + (\Psi' + (m - 1)(\ln \tau)')\xi\), that is

\[
\xi' = c - (\ln X)'\xi.
\]

Solutions \(\xi\) of this equation are of the form

\[
\xi = \phi_c(t) = c \int_0^t X(s) ds / X(t).
\]

Since this function must take values in \((-1, 1)\), we conclude \(|c| < C_0\), and if \(c \neq 0, c\phi_c(t) > 0\) for \(t > 0\). Furthermore, \(\phi_c(0) = c d^{m-1}X \big/ dt^{m-1}(0)/d^mX(0)(0) = 0\) and \(\phi_c'(0) = c\). \(\square\)

Theorem 1.2 is a generalization of Theorem 1.4 of [14], so we will give a sketch of the proof, detailed in general, except on some statements that are easy to follow from the references that will be indicated along the proof. First we prove some properties of the calibration \(\Omega\).
Lemma 2.5. The m-form Ω is a closed calibration of \( \tilde{M} \) that calibrates the slices. It is parallel if and only if \( \psi \) is constant. Moreover, if \( X^*_i \) is a \( g_* \)-o.n. frame of \( M \), we have

\[
(\tilde{\nabla}_H \Omega)(d\Gamma_f(X^*_1), \ldots, d\Gamma_f(X^*_m)) = \cos \theta \tilde{h}(H_N, df(\nabla^M \psi))
\]

\[
= -\cos \theta g(H_M, \nabla^M \psi) = -\cos \theta g_*(H_M, \nabla^* \psi). \quad (2.42)
\]

Proof. To see that \( \Omega \) is a closed m-form we use a \( \tilde{g} \)-o.n frame of the form \((X_i, 0), (0, W_\alpha)\) on \( \tilde{M} \), being \( X_i \) a direct o.n. frame of \((M, g)\), with \( \nabla^M X_i(x_0) = 0 \) at a given point \( x_0 \in M \). Using the properties of \( \tilde{\nabla} \), we see that the covariant derivatives of \( \Omega \) in the directions of all these vector fields vanish except for,

\[
(\tilde{\nabla}_{(0, W_\alpha)} \Omega)((X_1, 0), \ldots, (X_{i-1}, 0), (0, W_\beta), (X_{i+1}, 0), \ldots, (X_m, 0)) = d\psi(X_i) \tilde{h}(W_\alpha, W_\beta) = \delta_{\alpha\beta} g(\nabla^M \psi, X_i).
\]

Thus, \( \Omega \) is parallel only if \( \psi \) is constant. To conclude \( d\Omega = 0 \) we only need to check the following components of \( d\Omega \),

\[
\pm d\Omega((X_1, 0), \ldots, (X_{m-1}, 0), (0, W_\alpha), (0, W_\beta)) =
\]

\[
= (\tilde{\nabla}_{(0, W_\alpha)} \Omega)((X_1, 0), \ldots, (X_{m-1}, 0), (0, W_\beta)) - (\tilde{\nabla}_{(0, W_\beta)} \Omega)((X_1, 0), \ldots, (X_{m-1}, 0), (0, W_\alpha)) = (d\psi(X_m) - d\psi(X_1)) \delta_{\alpha\beta} = 0.
\]

Note that \( \tilde{\nabla}_H \Omega(d\Gamma_f(X^*_1), \ldots, d\Gamma_f(X^*_m)) \) is independent of the \( g_* \)-o.n. basis \( X^*_i \) we take. We may assume that \( X^*_i \) is defined by (2.34), where \( X_i \) is a diagonalizing \( g \)-o.n. basis of \( T_{x_0}M \) of \( \rho^2 f^* h \), and we chose \( W_\alpha \) a local frame of \( N \) that at \( f(x_0) \) is given by \( U_\alpha \) defined as in (2.35) with respect to \( X_i \). Then, writing \( H = (H_M, H_N) = \sum_i g(H_M, X_i)X_i + \sum_\alpha \tilde{h}(H_N, U_\alpha)U_\alpha \), we see that

\[
(\tilde{\nabla}_H \Omega)(d\Gamma_f(X^*_1), \ldots, d\Gamma_f(X^*_m)) =
\]

\[
= (\tilde{\nabla}_H \Omega)((X_1, df(X_1)), \ldots, (X_m, df(X_m))) \frac{1}{\sqrt{\Pi_i(1 + \lambda_i^2)}}
\]

\[
= \sum_{\alpha, i} \tilde{h}(H_N, U_\alpha)(\tilde{\nabla}_U_\alpha \Omega)((X_1, 0), \ldots, (X_{i-1}, 0), (0, df(X_i)), (X_{i+1}, 0), \ldots, (X_m, 0)) \cos \theta
\]

\[
= \sum_{i, \alpha} \tilde{h}(H_N, U_\alpha)\tilde{h}(df(X_i), U_\alpha) d\psi(X_i) \cos \theta = \tilde{h}(\cos \theta H_N, df(\nabla^M \psi)).
\]

Lemma 2.4 proves the other equalities in (2.42). \( \square \)

Proof of Theorem 1.2. Following the proofs of Theorems 1.2 and 1.3 of [14], we consider the morphism \( \Phi : TM \rightarrow \mathcal{N} f, \) defined in (1.5). By Lemma 2.1 of [14], \( \|\Phi(X)\|^2 \leq \sin^2 \theta \|X\|^2_{g_*}. \) We define a vector field \( Z \) on \( M \) by the equality,

\[
g_*(Z, X) = \tilde{g}(\Phi(X), H).
\]
Then, \( \|Z\|_{g_*} \leq \sin \theta \|H\| \). Moreover, since \(*X_i^* = (-1)^{i-1} X_1^* \wedge X_{i-1}^* \wedge \ldots X_m^*, then\):

\[
g_*(Z, X_i^*) = \frac{1}{\Pi_{s \neq i} \sqrt{1 + \lambda_s^2}} \Omega((X_1, 0), \ldots, (H, 0), \ldots, (X_m, 0)) = g(H, X_i) \cos \theta \sqrt{1 + \lambda_i^2},
\]

that is

\[
Z = \sum_i g_*(Z, X_i^*) X_i^* = g_*(\cos \theta H, X_i) X_i = \cos \theta H.
\]

Furthermore, as in Theorems 1.2 and 1.3 of [14],

\[
div g_*(Z) = -\tilde{g}(\delta \Phi, H) + \sum_i \tilde{g}(\Phi(X_i^*), \tilde{\nabla}_{X_i^*} H).
\]

If \( N_\alpha \) is an \( \tilde{g} \)-o.n. frame of \( NG_f \), and identifying \( X_i^* \) with \( d\Gamma_f(X_i^*) \) then, as in the proof of Lemma 2.2 of [14], we have

\[
\delta \Phi = -\tilde{\nabla}_{X_i^*} \Phi(X_i^*) = \sum_{\alpha} d\Omega(N_\alpha, X_1^*, \ldots, X_m^*) N_\alpha - (\tilde{\nabla}_{N_\alpha} \Omega)(X_1^*, \ldots, X_m^*) N_\alpha + m \cos \theta H.
\]

Hence, for \( \Gamma_f \) with parallel mean curvature,

\[
div g_*(Z) = -m \cos \theta \|H\|^2 + \tilde{\nabla}_H \Omega(X_1^*, \ldots, X_m^*).
\]

Therefore, by Lemma 2.5

\[
div g_*(\cos \theta H) = -m \cos \theta \|H\|^2 - g_*(\cos \theta H, \nabla^* \psi),
\]

and we have proved equation (1.24). Weighted integration of (1.24) on a compact domain \( D \) of \((M, g_*)\), gives

\[
m \|H\|^2 \int_D \cos \theta e^\psi dM^* = -\int_{\partial D} g_*(\cos \theta H, \nu)e^\psi dA^*,
\]

where \( dM^* \) is the volume element of \((M, g_*)\), and \( dA^* \) of \( \partial D \), for the induced metric, and \( \nu \) the outward unit normal to \( \partial D \). Thus,

\[
m \|H\|^2 (\inf_D \cos \theta) V^*_\psi(D) \leq \|H\|^2 A^*_\psi(\partial D).
\]

If we take \( D \) a domain of geodesic ball \( B_r \) of \((M, g_*)\) of radius \( r \), with \( \bar{D} \subset B_r \), we have obtained

\[
m \|H\| \inf_{B_r} \cos \theta \leq \frac{A^*_\psi(\partial D)}{V^*_\psi(D)}.
\]
Hence,

\[ m\| H \| \inf_{B_r} \cos \theta \leq h(B_r, g_*, \psi). \]

Using the assumption on \( \cos \theta \), we have

\[ m\| H \| \leq C_1^{-1} r^\beta h(B_r, g_*, \psi). \]

Under the boundedness conditions on Ricci_\psi, g_*, and \( \| \nabla^* \psi \|_g \) we have from Corollary 3.1 of Section 3, \( h(B_r, g_*, \psi) \leq C'/r \). Hence

\[ \| H \| \leq Cr^{\beta-1}. \]

Making \( r \to +\infty \) we get \( H = 0 \). 

\[ \square \]

3  Weighted Cheeger inequality

We prove the weighted Cheeger inequality (1.20) that generalizes the well known Cheeger inequality (cf. [5], Section IV, Theorem 3). Recall that the eigenvalue problem of the \( \psi \)-Laplacian, \( -\Delta_\psi u = -\Delta u - g(\nabla u, \nabla \psi) \), on a smooth compact Riemannian manifold \( M \) with boundary, with Dirichlet boundary condition, \( u = 0 \) on \( \partial M \), consists on a discrete sequence

\[ 0 < \lambda_{\psi, 1} < \lambda_{\psi, 2} \leq \lambda_{\psi, 3} \leq \ldots \to +\infty \]

and each eigenvalue has a variational characterization of Rayleight type (cf. [15]). We have for the principal eigenvalue \( \lambda_{\psi, 1} \),

\[ \lambda_{\psi, 1}(M) = \sup_{u \in C^\infty_0(M)} \frac{\int_M \| \nabla u \|^2 e^\psi dM}{\int_M u^2 e^\psi dM}, \]

and the infimum is attained at a principal eigenfunction \( u \).

Theorem 3.1.

\[ \lambda_{\psi, 1}(M) \geq \frac{1}{4} (h(M, g, \psi))^2. \]

Proof. We follow the proof of Theorem 3 of [5], Section IV, for the case \( \psi = 0 \). Let \( u \) be a principal eigenfunction, normalized s.t. \( \int_M u^2 e^\psi dM = 1 \). Then \( u > 0 \) on \( M \) and vanishes on \( \partial M \). Set \( M(t) = \{ x \in M : u^2(x) > t \} \), \( \Sigma(t) = \{ x \in M : u^2(x) = t \} \), and

\[ V_\psi(t) = V_\psi(M(t)) = \int_{M(t)} e^\psi dM, \quad A_\psi(t) = A_\psi(\Sigma(t)) = \int_{\Sigma(t)} e^\psi dA. \]
We first recall the co-area formula (cf. [5], Sec IV., Theorem 1) for \( f = u^2 \) and any function \( h \in L^1(M) \),

\[
\int_M h \| \nabla u^2 \| dM = \int_0^{+\infty} ds \int_{\Sigma(s)} h dA.
\]

Considering \( h = e^\psi \chi_{\mathcal{M}(t)} \| \nabla u^2 \|^{-1} \), where \( \chi \) is the characteristic function, we get

\[
V_\psi(t) = \int_M \chi_{\mathcal{M}(t)} e^\psi dM = \int_0^{+\infty} ds \int_{\Sigma(s)} \chi_{\mathcal{M}(t)} \| \nabla u^2 \|^{-1} e^\psi dA = \int_t^{+\infty} ds \int_{\Sigma(s)} \| \nabla u^2 \|^{-1} e^\psi dA
\]

Hence \( V_\psi'(t) = -\int_{\Sigma(t)} \| \nabla u^2 \|^{-1} e^\psi dA \). Again, by the co-area formula with \( h = e^\psi \),

\[
\int_M \| \nabla u^2 \| e^\psi dM = \int_0^{+\infty} A_\psi(s) ds.
\]

Since \( L^2(M) \) is an Hilbert space, we have

\[
\left( \int_M \| \nabla u^2 \| e^\psi dM \right)^2 = 4 \left( \int_M u \| \nabla u \| e^\psi dM \right)^2 \leq 4 \left( \int_M \| u \|^2 e^\psi dM \right) \left( \int_M u^2 e^\psi dM \right) = 4 \int_M \| \nabla u \|^2 e^\psi dM.
\]

Hence,

\[
\lambda_{\psi,1} = \int_M \| \nabla u \|^2 e^\psi dM \geq \frac{1}{4} \left( \int_M \| \nabla u^2 \| e^\psi dM \right)^2
\]

Now, using that \( tV_\psi(t) \) vanish at \( t = 0 \) and \( t = +\infty \), and the co-area formula with \( h = u^2 \| \nabla u^2 \|^{-1} e^\psi \), we have

\[
\int_M \| \nabla u^2 \| e^\psi dM = \int_0^{+\infty} A_\psi(s) ds \geq \int_0^{+\infty} \mathfrak{h}(M(s), g, \psi) V_\psi(s) ds
\]

\[
\geq \mathfrak{h}(M, g, \psi) \int_0^{+\infty} V_\psi(s) ds = -\mathfrak{h}(M, g, \psi) \int_0^{+\infty} sV_\psi'(s) ds
\]

\[
= \mathfrak{h}(M, g, \psi) \int_0^{+\infty} sds \int_{\Sigma(s)} \| \nabla u^2 \|^{-1} e^\psi dA
\]

\[
= \mathfrak{h}(M, g, \psi) \int_M u^2 e^\psi dM = \mathfrak{h}(M, g, \psi).
\]

Consequently \( \lambda_{\psi,1} \geq \frac{1}{4} \mathfrak{h}(M, g, \psi)^2 \), and the theorem is proved.

On a weighted manifold \( (M, g, e^\psi) \), it is defined the \( \psi \)-Ricci tensor, \( \text{Ricci}_{\psi,g}(X,Y) := \text{Ricci}_g - \text{Hess}(\psi) \). We recall the following comparison result due to Setti ([22], Theorem 4.2). In our
notation Ricci\(_{\psi,g} = S_\omega\) given in (2.2) of [22], where \(\omega = e^\psi\), and \(R_\omega\) defined in [22] corresponds to \(\text{Ricci}_{\psi,g} - d\psi \otimes d\psi\). The conditions on \(\text{Ricci}_{\psi,g}\) and \(\|\nabla \psi\|\) stated in the next theorem implies \(R_\omega \geq (\alpha - \delta)\).

**Theorem 3.2.** If \((M^m, g, e^\psi)\) is a complete weighted manifold, and \(\text{Ricci}_{\psi,g} \geq \alpha\), and \(\|\nabla \psi\| \leq \delta^{1/2}\), where \(\alpha, \delta\) are constants, then, the first eigenvalue for the \(\psi\)-Laplacian on a geodesic ball of radius \(r\) on \(M, B_r\), satisfies

\[
\lambda_{\psi,1}(B_r) \leq \lambda_1(B_r^0),
\]

where \(\lambda_1(B_r^0)\) is the first eigenvalue for the \(0\)-Laplacian of a geodesic ball of radius \(r\) with Dirichlet boundary condition on a \((m + 1)\)-dimensional space form of constant sectional curvature \((\alpha - \delta)/m\).

Obviously we are assuming \(\delta \geq 0\). If \(\alpha - \delta \geq 0\) then for some constant \(C' > 0\), \(\lambda_1(B_r^0) \leq \frac{C'}{r^2}\) (cf. [3] or proof of Proposition 4.2 in [14]), as a consequence of Cheng’s comparison result [6]). Therefore, we have the following conclusion.

**Corollary 3.1.** In the previous theorem, if \(\alpha \geq \delta\), then, for each ball of radius \(r\), we have \(h(B_r, g, \psi) \leq C/r\). In particular, \(h(M, g, \psi) = 0\).

**Proof.** By previous theorem \(\lambda_{\psi,1}(B_r) \leq \lambda_1(B_r^0) \leq C'/r^2\), Then, applying Theorem 3.1 \(h(M, g, \psi) \leq h(B_r, p, \psi) \leq C/r \rightarrow 0\), when \(r \rightarrow +\infty\).

**References**

[1] L.J. Alias and M. Dajczer, Constant mean curvature hypersurfaces in warped product spaces Proc. Edinb. Math. Soc.(2) **50** (2007) no 3 ; 511–526.

[2] L.J. Alías, M. Dajczer, Constant mean curvature graphs in a class of warped product spaces, Geom. Dedicata **131** (2008), 173–179.

[3] L. Alías, M. Dajczer, and J. Ripoll J., A Bernstein-type theorem for Riemannian manifolds with a Killing field, Ann. Glob. Anal. Geom. **31**(2007), 363–373.

[4] S. Brendle, Constant mean curvature surfaces in warped product manifolds, Publications mathematiques de l’IHES, June 2013, Volume **117**, Issue 1, 247–269

[5] I. Chavel, Eigenvalues in Riemannian geometry, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.

[6] S.Y. Cheng, Eigenvalue comparison theorems and its applications, Math. Z., **143** (1975), 289–297
[7] S.S. Chern, *On the curvatures of a piece of hypersurface in Euclidean space*, Abh. Math. Sem. Univ. Hamburg 29 (1965), 77–91.

[8] M. Dajczer, J.H. de Lira, * Entire unbounded constant mean curvature Killing Graphs*, Bull. Braz. Math. Soc. (NS), (2016) on-line

[9] A.C. Ferreira, I. Salavessa *Dirichlet principal eigenvalue comparison theorems in geometry with torsion*, J. Math. Anal. and App., in Press, online 20 April (2017), https://doi.org/10.1016/j.jmaa.2017.04.030

[10] H. Flanders, *Remark on mean curvature*, J. London Math. Soc. 41 (1966), 364–366.

[11] E. Heinz, *¨Uber Fl¨achen mit eineindeutiger Projektion auf eine Ebene, deren Kr¨ ummungen durch Ungleichungen eingeschr¨ ankt sind.*, Math. Ann. 129 (1955), 451–454.

[12] R. Harvey & H.B. Lawson, Jr. *Calibrated geometries*, Acta Math. 148 (1982), 47–157.

[13] D. Impera, J.H. de Lira, S. Pigola and A. G. Setti, *Height Estimates for Killing graphs*, arXiv:1612.01257.

[14] G. Li and I. Salavessa, *Bernstein-Heinz-Chern results in calibrated manifolds*, Rev. Mat. Iberoam., 26 (2) (2010), 651-692.

[15] Z. Lu, J. Rowlett, *Eigenvalues of collapsing domains and drift Laplacians*, Mat. Res. Lett. 19, no.3 (2012), 627-648.

[16] H.F. Lima, J.R. Lima, and M.A.L. Velasquez, *Entire conformal Killing graphs in foliated Riemannian spaces*, J. Geom. Anal. 25, (2015), 171-188.

[17] B. O’Neil, *Semi Riemannian Geometry (with applications to Relativity)*, Academic Press (1983)

[18] S. Rafalski, *A relative isoperimetric inequality for certain warped product spaces*, Adv. Geom. 12 (2012), no. 4, 639 –646.

[19] I.M.C. Salavessa, *Graphs with parallel mean curvature and a variational problem in conformal geometry*, Ph.D. Thesis, University of Warwick, 1987.

[20] I.M.C. Salavessa, *Graphs with parallel mean curvature*, Proc. Amer. Math. Soc. 107 (1989), no. 2, 449–458.

[21] I.M.C. Salavessa, *Spacelike graphs with parallel mean curvature*, Bull. Belgian Math. Soc. Simon Stevin 15 (1) (2008), 65-76.

[22] A.G. Setti, *Eigenvalue estimates for the weighted Laplacian on a Riemannian manifold*, Rend. Sem. Mat. Padova 100 (1998), 27-55.