The problem of $\Pi^2$-cut-introduction

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Abstract

We describe an algorithmic method of proof compression based on the introduction of $\Pi^2$-cuts into a cut-free LK-proof. This method extends previous research on the introduction of $\Pi_1$-cuts and is based on a connection between proof theory and formal language theory. Given is a cut-free proof $\pi$ of a sequent $S$ and a so called schematic $\Pi^2$-grammar $\mathcal{G}$, a grammar formalizing the substitutions of quantifiers in the elimination of $\Pi^2$-cuts and describing the instantiations for the generation of a Herbrand sequent of $\pi$. An algorithm is developed to automatically construct a $\Pi^2$-cut $A$ and a proof $\pi'$ of $S$ with one cut on $A$. Basically, the method inverts Gentzen’s method of cut-elimination. It is shown that the algorithm can achieve an exponential compression of proof length.

Keywords: Proof Theory, Automated Deduction, Proof Grammar, Sequent Calculus, Lemma Generation

1. Introduction

Cut-elimination, introduced by Gentzen [4], is the most prominent form of proof transformation in logic and plays an important role in automating the analysis of mathematical proofs. The removal of cuts corresponds to the elimination of intermediate statements (lemmas), resulting in a proof which is analytic in the sense that all statements in the proof are subformulas of the result. Therefore, the proof of a combinatorial statement is converted into a purely combinatorial proof. Cut-elimination is therefore an essential tool for the analysis of proofs, especially to make implicit parameters explicit.

The method of Gentzen is based on reductions of cut-derivations (subproofs ending in a cut), transforming them into simpler ones; basically the cut is replaced by one or more cuts with lower logical complexity. In the construction of a Herbrand sequent $S'$ corresponding to a cut-free proof $\varphi'$ (see e.g. [2]) obtained by cut-elimination on a proof $\varphi$ of a sequent $S$ with cuts, only the substitutions generated by cut-elimination of quantified cuts are relevant. In fact,
it is shown in \[5\] that, for proofs with $\Sigma_1$ and $\Pi_1$-cuts only, $S'$ can be obtained just by computing the substitutions defined by cut-elimination without applying Gentzen’s procedure as a whole. More recently this result was generalized to proofs with $\Pi_2$-cuts (see \[1\]). Via the cuts in the proof $\varphi$, one can define a tree grammar generating a language which contains exactly the terms (to be instantiated for quantified variables in $S$) for obtaining a Herbrand sequent $S'$ of $S$. Hence, generating a tree grammar $G$ from a set of Herbrand terms $T$ (s.t. $G$ generates $T$) corresponds to an inversion of the quantifier part of Gentzen’s procedure. The computation of such an inversion forms the basis of a method of cut-introduction for $\Pi_1$-cuts into cut-free proofs presented in \[8\][7][6]. The key concept is that of a schematic extended Herbrand sequent $S$ corresponding to a grammar generating the Herbrand term set; $S$ contains second-order variables for the unknown cut-formulas. It is shown that the schematic extended Herbrand sequent is always solvable (i.e. the second-order variables can be replaced by $\lambda$-terms defining cut-formulas). The method of cut-introduction for $\Pi_1$-cuts also obtains the best possible proof compression, which is exponential.

In this paper we generalize the existing methods to the introduction of a (single) $\Pi_2$-cut. There are various benefits of $\Pi_2$-cut introduction: first of all, the introduction of just a single $\Pi_2$-cut can lead to an exponential proof compression (while a single $\Pi_1$-cut yields at most a quadratic compression). Moreover, interesting mathematical lemmas are frequently in $\Pi_2$-form and thus an algorithmic method of cut-introduction for $\Pi_2$-cuts may yield interesting mathematical results.

The paper is organized as follows: in Section 4 we describe the proof-theoretic infrastructure and repeat the most important results from \[7\]. Then we extend the former terminology to $\Pi_2$-cut introduction (in particular we adapt the concepts of extended Herbrand sequents and schematic extended Herbrand sequents). In Section 5 we define schematic $\Pi_2$-grammars, a simplified version of the grammars defined in \[1\]. A characterization of the solvability of the schematic extended Herbrand sequent, which is the key step in our method of cut-introduction, is given in Section 6. This characterization admits an algorithmic generation of $\Pi_2$-cuts (in case that a $\Pi_2$-cut can be introduced at all). We also show that there are schematic $\Pi_2$-grammars which do not yield $\Pi_2$-cuts at all, in contrast to the $\Pi_1$ case where every $\Pi_1$-grammar leads to a solution. Given a so called starting set of atoms, the solvability of the $\Pi_2$-cut-introduction problem is shown to be decidable. However, the decidability of the general problem is yet unknown. The characterization of solvability described above yields a rather inefficient algorithm for cut-introduction. In Section 7 we define a method of constructing $\Pi_2$-cut formulas which is based on a unification procedure ($G^*$-unifiability); this method is more efficient and works in case so called balanced solutions of the problem exist. Finally, in section 8 we construct an infinite sequence of proofs for which the method of $G^*$-unifiability can achieve an exponential proof compression by the introduction of $\Pi_2$-cuts.

This work can be seen as a first step in the algorithmic introduction of cuts beyond $\Pi_1$. A full characterization of $\Pi_2$-cut introduction for a single cut and
the generalization to the introduction of several $\Pi_2$-cuts are left to future work. The computation of maximally compressing schematic $\Pi_2$-grammars and an implementation of the $G^*$-unifiability method is planned for the near future.

2. Preliminaries and notation

Most of the symbols in the following paper are explained when they are introduced the first time. But due to the high number of different symbols used in this paper we describe below those which are most frequently used.

The capital letters $F,G$ are quantifier-free formulas which occur in the end-sequents. While $L$ is denoting a literal, $P$ denotes an atom and $Q$ might be a literal or an atom (if a second literal or atom is needed). Small letters, such as $f$ and $g$, are function symbols, $r,s,t$ are usually representations of terms, and the characters $r_1,\ldots,r_m$ and $t_1,\ldots,t_p$ are the designated terms of the observed schematic $\Pi_2$-grammar $G$ (see definition 9) which are substitutions of the designated variables (if not defined otherwise) $x,y$ of the binary cut formula $C$. The small letters $m,p$ are the corresponding natural numbers. For running indices we use $i,j,k,l$, or $q$ and for fixed natural numbers we tend to use $n$.

By an over-line $\cdot$ we either denote the dual of a literal $L$, atom $P$, or sequent $S$ (see definition 17) or we denote a tuple of terms $\bar{t}$ or a tuple of variables $\bar{x}$. The Greek characters $\alpha,\beta_1,\ldots,\beta_m$ denote the eigenvariables of the binary cut formula in a proof $\pi$. An arbitrary sequent is denoted by $S$, while we are denoting special sequents by $H$ (Herbrand sequent), $EH_A$ (extended Herbrand sequent with the cut-formula $A$), $S(X)$ (schematic extended Herbrand sequent with the formula variable $X$), and $R$ (reduced representation). If we consider a non-tautological leaf of a proof, we use the symbols $S,S'$ or $J$.

Furthermore, we will abbreviate the sets $\{1,\ldots,n\}$ by $\mathbb{N}_n$. For terms $t$ or formulas $F$ we denote by $V(t)$ or $V(F)$ the number of variables in $t$ and $F$ respectively. The transformation of sets of clauses $C$ into formulas in disjunctive normal form will be denoted by $\text{DNF}(C)$. For a given tuple of variables $\bar{x} = (x_1,\ldots,x_n)$ and a tuple of terms $\bar{t} = (t_1,\ldots,t_n)$ we denote by $F[\bar{x}\backslash\bar{t}]$ the substitution of all occurrences of $x_i$ with $i \in \mathbb{N}_n$ in a given formula (or term) $F$ by $t_i$. For abbreviation we write instead of $F[\bar{x}\backslash\bar{t}_1],\ldots,F[\bar{x}\backslash\bar{t}_k]$ simply $F[\bar{x}\backslash\bar{T}]$ with $T = \{\bar{t}_1,\ldots,\bar{t}_k\}$ and $k$ being an arbitrary natural number.

3. A motivating example

To illustrate the effect of $\Pi_2$-cuts, we present the following sequence of sequents. Let $n \geq 2$ be a natural number and

$A_n := \forall x.(P(x,f_1x) \vee \ldots \vee P(x,f_nx)),$
$B := \forall x,y.(P(x,y) \rightarrow P(x,fy)),$
$Z_n := P(x_1,fx_2) \land P(fx_2,fx_3) \land \ldots \land P(fx_{n-1},fx_n),$  
$C_n := \forall x_1,\ldots,x_n.(Z_n \rightarrow P(x_1,fx_n)),$
and $D := \exists u,v.P(u,gv)$
then the sequents $S_n := A_n, B, C_n \vdash D$ are provable. All cut-free LK-proofs of the sequents $S_n$ require more than $n^n$ quantifier inferences of $A_n, B, C_n$ and $D$. But note that there are proofs of the sequents $S_n$ by a linear number of instances; this compression can be achieved by the $\Pi_2$-cut-formula $\forall x \exists y. P(x, fy)$ (see section 8).

4. Proof-theoretic infrastructure

A sequent $S$ is an ordered pair of sets of formulas, written as $\Delta \vdash \Gamma$; we call $\Delta$ the antecedent and $\Gamma$ the succedent of $S$. The sequent calculus we use is $G3c$ together with the cut-rule, which we call $G3c^+$; note that $G3c$ is an invertible version of cut-free LK (see [10]). In this paper we only consider proofs of prenex skolemized end-sequents (the antecedents are only universal, the succedents existential); note that this restriction is not essential as every sequent is provability-equivalent to such a normal form. Every cut-free proof $\phi$ of a prenex end-sequent $S$ can be transformed into a cut-free proof $\psi$ of $S$ (without increase of proof length) s.t. $\psi$ contains a so-called midsequent $S^*$ s.t. all quantifier inferences in $\psi$ are below $S^*$ and all propositional ones above [4].

We also call $S^*$ a Herbrand sequent of $S$. Herbrand sequents of cut-free proofs will play a crucial role in our cut-introduction method.

Let $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{t} = (t_1, \ldots, t_n)$ are tuples of terms; then $[\bar{x}\backslash\bar{t}]$ denotes the substitution $[x_1\backslash t_1, \ldots, x_n\backslash t_n]$. Furthermore, we denote by $|_i$ for $i \in \mathbb{N}_n$ the projection that gives the $i$-th element of a tuple $\bar{t}_i := t_i$. For the complexity measurement of Herbrand sequents we have to count different terms in tuples of terms.

**Definition 1.** Let $T$ be a set of tuples of terms with arity $n$ and $\bar{t} = (t_1, \ldots, t_n)$, $\bar{r} = (r_1, \ldots, r_n)$ be tuples of terms $t_1, \ldots, t_n, r_1, \ldots, r_n$. We define the number of different terms $\sharp T$ in a set of term-tuples $T$ by induction:

$$
\sharp \emptyset := 0 \\
\sharp (T \cup \{\bar{t}\}) := \sharp T + k
$$

such that $k$ is the maximal number where

$$
\exists s_1, \ldots, s_k \in \mathbb{N}_n \forall \bar{r} \in T \forall i \in \mathbb{N}_k. \bar{r}|_{s_i} \neq \bar{r}|_{s_i}
$$

holds.

**Definition 2** (Herbrand sequent). Let $S : \forall \bar{x}. F \vdash \exists \bar{y}. G$ be a given sequent, where $\bar{x} = (x_1, \ldots, x_m)$, $\bar{y} = (y_1, \ldots, y_l)$ and

$$
H := F[\bar{x}\backslash \bar{t}_1], \ldots, F[\bar{x}\backslash \bar{t}_k] \vdash G[\bar{y}\backslash \bar{t}_{k+1}], \ldots, G[\bar{y}\backslash \bar{t}_n]
$$

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be a valid sequent where $F[x_i]$ for $i \in \mathbb{N}_k$ are instances of $F$ and $G[y_j]$ for $j \in \{k+1, \ldots, n\}$ are instances of $G$. Then we call $H$ a Herbrand sequent of $S$. The complexity of $H$ is defined as

$$|H| := \sharp\{\bar{t}_1, \ldots, \bar{t}_k\} + \sharp\{\bar{t}_{k+1}, \ldots, \bar{t}_n\}.$$

We could further simplify the sequents in definition 2 to sequents with an empty succedent, since every sequent is provability-equivalent to such a normal form as well. But for the sake of readability of the definitions in the following chapters we will always consider sequents of this form. Otherwise, we would have to include a discussion about when two subformulas are separable within a sequent. To give an notion of separable subformulas assume a formula $A$ with subformulas $B$ and $C$ within a sequent $S$. We would call $B$ and $C$ separable if there are formulas $A_1$ and $A_2$ such that $B$ is a subformula of $A_1$, $C$ is a subformula of $A_2$ and there is a unary rule in G3c where $A_1$ and $A_2$ are the active parts of the premise and $A$ is the active part of the conclusion. In the sequent $\vdash B \lor (C \land D)$ are $B$ and $C$ separable but not $C$ and $D$.

**Definition 3.** $\forall$-left inferences $\forall: l$ and $\exists$-right inferences $\exists: r$ in a proof are called weak quantifier inferences, $\forall$-right inferences $\forall: r$ and $\exists$-left inferences $\exists: l$ are called strong.

We measure the quantifier-complexity of a proof $\varphi$ by the number of weak quantifier-inferences in $\varphi$. Note that all quantifier inferences on ancestors of the end-sequent are weak, and multiple uses of quantified formulas in cuts is necessary only for formulas starting with weak quantifiers. Hence in any such proof the number of strong quantifier-inferences is less or equal to the number of weak quantifier-inferences (see [9]).

**Definition 4.** Let $\pi$ be a proof in G3c$^+$; then the quantifier-complexity of $\pi$ is defined as the number of weak quantifier-inferences in $\pi$. We write $|\pi|_q = n$ if $\pi$ has quantifier-complexity $n$.

**Theorem 1.** Assume a sequent $S : \forall \bar{x}.F \vdash \exists \bar{y}.G$. There is a Herbrand-sequent $H$ of $S$ with $|H| = n$ iff there exists a minimal cut-free proof $\pi$ of $S$ such that $|\pi|_q = n$.

**Proof.** A Herbrand sequent describes exactly the terms we have to introduce by weak quantifier inferences. Let $H$ be a Herbrand sequent of $S$ with $|H| = n$. Then a cut-free proof $\pi$ with $|\pi|_q = n$ can be constructed in the following way: apply first all propositional inferences and afterwards all quantifier rules. Let $\pi$ be a minimal cut-free proof of $S$. Then different terms for a given position of an atom can only be produced by weak quantifier inferences. Hence, the number of weak quantifier inferences in $\pi$ is equal to the number of different terms obtained by substitution, and therefore $|\pi|_q = |H|$ for $H$ being the Herbrand sequent obtained from $\pi$.

We define the notion of an extended Herbrand sequent as in [7]; for simplicity we do not consider blocks of quantifiers in the cuts, but only formulas of the
form $\forall x \exists y. A$ where $A$ is quantifier-free, $V(A) \subseteq \{x, y\}$, and $V(A)$ denotes the variables in $A$. As in the case of $\forall$-cuts, extended Herbrand sequents represent proofs with cuts by encoding the cuts by implication formulas. As we consider only the introduction of a single $\Pi_2$ cut, we need only one formula for coding the cut.

If $U$ is a set of term tuples $\{\bar{t}_1, \ldots, \bar{t}_m\}$ and $F$ is a formula $F[\bar{x}\setminus U]$ stands for the set of instances $\{F[\bar{x}\setminus \bar{t}_1], \ldots, F[\bar{x}\setminus \bar{t}_m]\}$.

**Definition 5** (Extended Herbrand-sequent). Let $S$ be a sequent of the form $\forall \bar{x} F \vdash \exists \bar{y} G$ (with $\bar{x} = (x_1, \ldots, x_k)$ and $\bar{y} = (y_1, \ldots, y_l)$ and $A$ be a quantifier-free formula with $V(A) \subseteq \{x, y\}$. Let $U_1 := \{\bar{u}_1, \ldots, \bar{u}_N\}$ be a set of term tuples of the length $k$, $U_2 := \{\bar{v}_1, \ldots, \bar{v}_M\}$ be a set term tuples of the length $l$. Let $\beta_1, \ldots, \beta_m, \alpha$ be variables and $t_i$ for $i \in \mathbb{N}_p$, $r_j$ for $j \in \mathbb{N}_m$ be terms s.t.

$$V(U_1) \subseteq \{\alpha\}, \; V(U_2) \subseteq \{\beta_1, \ldots, \beta_m\}, \; V(t_i) \subseteq \{\alpha\} \text{ for all } i, \; V(r_j) \subseteq \{\beta_1, \ldots, \beta_{j-1}\} \text{ for } j \geq 2, \; \text{and } \; V(r_1) = \emptyset.$$

Then the sequent

$$EH_A := F[\bar{x}\setminus U_1], \; \bigwedge_{i=1}^{p} A[x\setminus \alpha, y\setminus t_i] \rightarrow \bigwedge_{j=1}^{m} A[x\setminus r_j, y\setminus \beta_j] \vdash G[\bar{y}\setminus U_2]$$

is called an extended Herbrand-sequent of $S$ if $EH_A$ is a tautology.

The complexity of an extended Herbrand sequent $EH_A$ is defined as $|EH_A| = k \cdot N + l \cdot M + p + m$.

**Example 1.** Consider the following proof with a single $\Pi_2$ cut.

\[
\begin{array}{c}
\text{cut} \quad \pi_1 \\
\hline \forall x. P(x, t_1 x) \lor P(x, t_2 x) \vdash \exists y, z. P(r_1, y) \land P(r_2 y, z)
\end{array}
\]

with $\pi_1 :=$

\[
\forall l. \; \frac{P(\alpha, t_1 \alpha) \vdash P(\alpha, t_1 \alpha), P(\alpha, t_2 \alpha) \vdash P(\alpha, t_2 \alpha) \vdash P(\alpha, t_1 \alpha), P(\alpha, t_2 \alpha)}{P(\alpha, t_1 \alpha) \lor P(\alpha, t_2 \alpha) \vdash P(\alpha, t_1 \alpha), P(\alpha, t_2 \alpha)}
\]

and $\pi_r :=$

\[
\forall l. \; \frac{P(r_1, \beta_1), P(r_2 \beta_1, \beta_2) \vdash P(r_1, \beta_1), P(r_2 \beta_1, \beta_2) \vdash P(r_2 \beta_1, \beta_2)}{P(r_1, \beta_1), P(r_2 \beta_1, \beta_2) \vdash P(r_1, \beta_1) \land P(r_2 \beta_1, \beta_2)}
\]
Then the extended Herbrand sequent $EH_P$ is given by

$$P(\alpha,t_1\alpha) \lor P(\alpha,t_2\alpha),$$

$$(P(\alpha,t_1\alpha) \lor P(\alpha,t_2\alpha)) \rightarrow (P(r_1,\beta_1) \land P(r_2\beta_1,\beta_2)) \vdash P(r_1,\beta_1) \land P(r_2\beta_1,\beta_2)$$

where $U_1 = \{\alpha\}$ and $U_2 = \{\beta_1,\beta_2\}$. We will not give a cut-free proof because of its size. Instead, we define a Herbrand sequent $H$, which provides the quantifier information of a corresponding cut-free proof.

$$P(r_1,t_1r_1) \lor P(r_1,t_2r_1)$$

$$P(r_2\alpha r_1, t_1r_2t_1r_1) \lor P(r_2\alpha r_1, t_1r_2t_2t_1r_1)$$

$$P(r_2\alpha r_1, t_1r_2t_2r_1) \lor P(r_2\alpha r_1, t_1r_2t_2r_2r_1)$$

$$P(r_2\alpha r_1, t_1r_2t_2r_1) \lor P(r_2\alpha r_1, t_1r_2t_2r_1)$$

Both sequents, $EH_P$ and $H$, are tautological as one can easily verify.

We obtain a result analogous to that in $\Pi_1$-cut-introduction:

**Theorem 2.** The sequent $\forall \bar{x}F \vdash \exists yG$ has a proof $\pi$ with a single $\Pi_2$-cut $\forall x \exists y.A$ such that $|\pi|_{\eta} = n$ iff it has an extended Herbrand-sequent $EH_A$ with $|EH_A| = n$.

**Proof.** For the left-to-right direction we pass through the proof $\pi$ and read off the instances of quantified formulas (of both the end-formula and the cut). We obtain an extended Herbrand sequent $EH$ with $|EH| \leq |\pi|_{\eta}$ (which can be padded with dummy instances if necessary in order to obtain $|EH| = |\pi|_{\eta}$). Let

$$F := F[\bar{\alpha}',U_1],$$

$$G := G[\bar{\beta}',U_2],$$

$$A := \forall \bar{x}.F,$$

$$\mathcal{B} := \exists \bar{y}.G.$$  

For the right-to-left direction we conclude from

$$\forall r \vdash A[x\alpha, \bar{y}]_{i=1}^p, \exists r \vdash A[x\beta, \bar{y}]_{i=1}^m \vdash G$$

$$\vdash A[x\alpha, \bar{y}]_{i=1}^p, \exists r \vdash A[x\beta, \bar{y}]_{i=1}^m \vdash G$$

$$\vdash A[x\alpha, \bar{y}]_{i=1}^p, \exists r \vdash A[x\beta, \bar{y}]_{i=1}^m \vdash G$$

that

$$\vdash A[x\alpha, \bar{y}]_{i=1}^p, \exists r \vdash A[x\beta, \bar{y}]_{i=1}^m \vdash G$$

$$\vdash A[x\alpha, \bar{y}]_{i=1}^p, \exists r \vdash A[x\beta, \bar{y}]_{i=1}^m \vdash G$$

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is provable and obtain a proof with a single $\Pi_2$-cut. The notation $A[x, y,t_1^{p}]$ is an abbreviation for $A[x, y,t_1], \ldots, A[x, y,t_p]$. The labels $\forall: r$, $\rightarrow : l$, and $\land : l$ denote the used rule, i.e. the right-disjunction rule, the left-implication rule, and the left-conjunction rule. In the latter variant the dots represent a multiple application of $\forall : r$ and $\exists : r$. In the particular case between the sequents $A, A[x, y,t_1^{m}] \vdash B$ and $A, \forall x \exists y.A, A[x, r_1] \vdash B$ the dots denote an alternating application of $\forall : l$ and $\exists : l (m-1)$ times.

Given that every term of $EH_A$ is used exactly once in a quantifier rule the quantifier complexity is equal to $|EH_A|$. 

\[ \exists : r \quad \forall : l \quad L \vdash F \\quad \exists : r \quad \forall : l \quad L \vdash F \]

5. Grammars

The way variables are replaced in the procedure of cut-elimination can be defined by grammars modeling substitutions of terms. A characterization of the substitutions defining the Herbrand instances of a proof after cut-elimination of $\Pi_1$-cuts can be found in [3]. Below we give some necessary definitions.

**Definition 6** (Regular tree grammar). A regular tree grammar $G$ is a tuple $\langle \tau, N, \Sigma, Pr \rangle$ where $N$ is a finite set of non-terminal symbols with arity 0 such that $\tau \in N$. Furthermore, $\Sigma$ is a finite set of function symbols of arbitrary arities, i.e. a term signature, satisfying $N \cap \Sigma = \emptyset$. The productions $Pr$ are a finite set of rules of the form $\gamma \rightarrow t$ where $\gamma \in N$ and $t \in T(\Sigma \cup N)$, where $T(\Sigma \cup N)$ denotes the set of all terms definable from symbols in $\Sigma \cup N$. As usual $L(G)$, the language defined by $G$, is the set of all terminal strings (ground terms) derivable in $G$.

The languages of grammars specifying Herbrand instances are finite (see [7]) and therefore their productions must be acyclic.

**Definition 7** (Acyclic tree grammar). We call a regular tree grammar acyclic if there is a total order $<$ on the non-terminals $N$ such that for each rule $\gamma \rightarrow t$ in $Pr$, only non-terminals smaller than $\gamma$ occur in $t$.

We are interested in grammars specifying substitutions. As substitutions are homomorphic mappings on terms, variables have to be replaced only by single terms within a derivation. Therefore we need so-called rigid derivations.
**Definition 8** (Rigid derivation). We call a derivation *rigid* with respect to a non-terminal $\gamma$ if only a single rule for $\gamma$ is allowed to occur in the derivation.

The following type of grammar describes the substitutions generated in the elimination of a $\Pi_2$-cut; this type grammar is a special case of more general grammars defined in [1].

**Definition 9** (Schematic $\Pi_2$-grammar). Let $G = \langle \tau, N, \Sigma, Pr \rangle$ be an acyclic tree grammar and $N = \{\tau, \alpha, \beta_1, \ldots, \beta_m\}$. Let the variables be ordered according to $\beta_1 < \ldots < \beta_m < \alpha < \tau$. We call $G$ a *schematic $\Pi_2$-grammar* if the production rules for given $m$ and $p$ are of the following form:

$$
\begin{align*}
\tau &\rightarrow s_1, \ldots, s_N \mid w_1, \ldots, w_M &\text{ with } V(s_i) \subseteq \{\alpha\} \text{ for } 1 \leq i \leq N \text{ and } \\
V(w_j) &\subseteq \{\beta_1, \ldots, \beta_m\} \text{ for } 1 \leq j \leq M \\
\alpha &\rightarrow r_1 \mid \ldots \mid r_m \text{ with } V(r_j) \subseteq \{\beta_1, \ldots, \beta_{j-1}\} \text{ for } 2 \leq j \leq m \text{ and } V(r_1) = \emptyset \\
\beta_j &\rightarrow t_1(r_j) \mid \ldots \mid t_p(r_j) \text{ for } 1 \leq j \leq m
\end{align*}
$$

where $t(s)$ is an abbreviation for the $\beta$-normal form of $(\lambda \alpha.t)s$. We call $m$ the $\forall$-multiplicity and $p$ the $\exists$-multiplicity.

Let $EH$ be an extended Herbrand sequent as defined in Definition [5] every such $EH$ defines a schematic $\Pi_2$-grammar. As the sequent $S: \forall \vec{x} F \vdash \exists \vec{y} G$ contains blocks of quantifiers and we want to use an ordinary term grammar, we generate function symbols $h_F, h_G$ where $h_F$ is of the arity of the length of $\vec{x}$, $h_G$ of the arity of the length of $\vec{y}$. So every term tuple $\vec{u}_i \in U_1$ is represented by $h_F(\vec{u}_i)$, and every term tuple $\vec{v}_j \in U_2$ by $h_G(\vec{v}_j)$.

**Definition 10.** Let $EH_A$ as in Definition [5]. We define $G(EH_A) = \langle \tau, N, \Sigma, Pr \rangle$, the schematic $\Pi_2$-grammar corresponding to $EH_A$, where $N = \{\tau, \alpha, \beta_1, \ldots, \beta_m\}$ and the variables are ordered as in Definition [9] the production rules are as in Definition [9] except for the variable $\tau$ where we have

$$
\tau \rightarrow h_F(\vec{u}_1), \ldots, h_F(\vec{u}_N) \mid h_G(\vec{v}_1), \ldots, h_G(\vec{v}_M).
$$

We call the production rules $\tau \rightarrow h_F(\vec{u}_1), \ldots, h_F(\vec{u}_N)$ $F$-productions and the production rules $\tau \rightarrow h_G(\vec{v}_1), \ldots, h_G(\vec{v}_M)$ $G$-productions.

At this point it becomes apparent why we have chosen the form $\forall \vec{x}. F \vdash \exists \vec{y}. G$ as end sequent. In a schematic $\Pi_2$-grammar we have terms depending on $\alpha$ and terms depending on some $\beta_i$ with $i \in \mathbb{N}_m$. These terms correspond to the function symbols $h_F$ and $h_G$, i.e. we implicitly ask for formulas that can be separated within one sequent (by a comma on the right side, a comma on the left side, or the sequent symbol $\vdash$). This separated formulas depend either on $\alpha$ or on some $\beta_i$ with $i \in \mathbb{N}_m$. Hence, there are no atoms that depend on both, $\alpha$ and $\beta_i$ for $i \in \mathbb{N}_m$.

We consider Example [1] to give an example of a schematic $\Pi_2$-grammar.
Example 2. Let $F := P(x, t_1x) \lor P(x, t_2x)$ and $G := P(r_1, y) \land P(r_2y, z)$. Then the only $F$-production is $\tau \rightarrow h_F(\alpha)$ and the only $G$-production is $\tau \rightarrow h_G(\beta_1, \beta_2)$. The other production rules are given by
\begin{align*}
\beta_2 & \rightarrow t_1\beta_1 | t_2\beta_1, \\
\beta_1 & \rightarrow t_1r_1 | t_2r_1, \quad \text{and} \\
\alpha & \rightarrow r_1 | r_2t_1r_1 | r_2t_2r_1.
\end{align*}

The set $\{\tau, \alpha, \beta_1, \beta_2\}$ contains all non-terminals and the order is $\beta_1 < \beta_2 < \alpha < \tau$.

Note that the rigidity condition is important. Without this condition the language would also contain expressions as $h_G(t_1r_1, t_1r_2t_2r_1)$; to generate this term we first replace $\beta_1$ by $t_1r_1$ and then by $t_2r_1$.

6. Cut-Introduction

We have shown that from any proof with a $\Pi_2$-cut we can extract a schematic $\Pi_2$-grammar. The language of this grammar covers the so-called Herbrand term set, a representation of the instantiations defining a Herbrand sequent. Now the question arises, whether we can invert this step, i.e. to construct a proof with a $\Pi_2$-cut from a cut-free proof $\varphi$ and a given schematic $\Pi_2$-grammar specifying the Herbrand instances of $\varphi$.

Definition 11 (Herbrand term set). Let $S: \forall \bar{x}F \vdash \exists \bar{y}G$ be a sequent and $H$ be a Herbrand sequent of $S$ of the form

\[ H := F[\bar{x}\setminus \bar{t}_1], \ldots, F[\bar{x}\setminus \bar{t}_k] \vdash G[\bar{y}\setminus \bar{t}_{k+1}], \ldots, G[\bar{y}\setminus \bar{t}_n] \]

as in Definition 2 and $h_F, h_G$ function symbols as defined above. Then the set

\[ H_*(S) := \{ h_F(t_1), \ldots, h_F(t_k), h_G(t_{k+1}), \ldots, h_G(t_n) \} \]

is called a Herbrand term set of $S$.

While Herbrand sequents represent cut-free proofs extended Herbrand sequents represent proofs with cuts. To introduce (yet unknown) cut-formulas we consider the Herbrand sequent of a cut-free proof and specify the Herbrand term set by a schematic $\Pi_2$-grammar. The unknown cut formula is represented by a second-order variable $X$.

Definition 12 (Schematic extended Herbrand sequent). Let $S: \forall \bar{x}F \vdash \exists \bar{y}G$ be a provable sequent and $H_*(S)$ be a Herbrand term set of $S$. Let $G: (\tau, N, \Sigma, Pr)$ be a schematic $\Pi_2$-grammar with $N = \{ \tau, \alpha, \beta_1, \ldots, \beta_m \}$, $\beta_1 < \ldots < \beta_m < \alpha < \tau$, and the production rules
\begin{align*}
\tau & \rightarrow h_F(\bar{u}_1), \ldots, h_F(\bar{u}_N) | h_G(\bar{v}_1), \ldots, h_G(\bar{v}_M) \\
\text{with } V(\bar{u}_i) & \subseteq \{ \alpha \} \text{ for } 1 \leq i \leq N \\
\text{and } V(\bar{v}_j) & \subseteq \{ \beta_1, \ldots, \beta_m \} \text{ for } 1 \leq j \leq M \\
\alpha & \rightarrow r_1 | \ldots | r_m \text{ with } V(r_j) \subseteq \{ \beta_1, \ldots, \beta_{j-1} \} \text{ for } 2 \leq j \leq m \text{ and } V(r_1) = \emptyset \\
\beta_j & \rightarrow t_1(r_j) | \ldots | t_p(r_j) \text{ for } 1 \leq j \leq m.
\end{align*}
Let $L(G)$ be the language of $G$ generated only by rigid derivations with respect to all non-terminals, and $H_s(S) \subseteq L(G)$. Let $U_1 := \{\bar{u}_1, \ldots, \bar{u}_N\}$ and $U_2 := \{\bar{v}_1, \ldots, \bar{v}_M\}$ then we call the sequent

$$S(X) : F[\bar{x}\backslash U_1], \bigvee_{i=1}^{p} X\alpha t_1 \rightarrow \bigwedge_{j=1}^{m} X r_j \beta_j \vdash G[\bar{y}\backslash U_2],$$

where $X$ is a two-place predicate variable, a schematic extended Herbrand sequent corresponding to $G$ and $S$ (in the following abbreviated by SEHS).

Furthermore we call $F[\bar{x}\backslash U_1] \vdash G[\bar{y}\backslash U_2]$ the reduced representation of $S(X)$.

Note that we did not require $L(G) = H_s(S)$; indeed if we generate a proper superset of $H_s(S)$ we still obtain a Herbrand sequent of $S$ (but not a minimal one). Generating supersets can be beneficial to the construction of cut-formulas.

A solution of an SEHS gives us a cut formula for a proof with a $\Pi_2$-cut.

**Definition 13.** Let $S$ be a provable sequent, $G$ a schematic $\Pi_2$-grammar, and $S(X)$ the corresponding SEHS. We call $A$ a solution of the SEHS if $S(A)$, i.e. the SEHS where $X$ is replaced by $A$, is a tautology.

In the following we will think of proofs as trees. This will facilitate the description of our approaches to find a solution for the SEHS. Hence, the leaves of a proof represent tautological or non-tautological axioms.

**Definition 14.** Let $S$ be a sequent. We call an arbitrary tree a $G3c$-derivation if it has only sequents as nodes, has $S$ as lowest element such that each node is a conclusion of a rule of $G3c$, and the immediate successors are the premise of that rule.

**Definition 15.** Let $S$ be a quantifier-free sequent. We call a $G3c$-derivation of $S$ maximal if the leaves of the tree cannot be conclusions of rules.

But before we discuss a characterization of the solvability of the SEHS we show that in general it is not solvable.

**Lemma 1.** Let $F := P(x, t_1 x) \land Q(x, t_2 x)$ and $G := P(r_1, x) \land Q(r_2, y)$. Assume the sequent $\forall x. F \vdash \exists x, y. G$, the SEHS $S(X)$

$$P(\alpha, t_1 \alpha) \land Q(\alpha, t_2 \alpha),$$

$$X(\alpha, t_1 \alpha) \lor X(\alpha, t_2 \alpha) \rightarrow X(r_1, \beta_1) \land X(r_2, \beta_2) \vdash P(r_1, \beta_1) \land Q(r_2, \beta_2),$$

and the schematic $\Pi_2$-grammar $G = (\tau, N, \Sigma, \Pr)$ where $N = \{\tau, \alpha, \beta_1, \beta_2\}$ and

$$\Pr = \{\tau \rightarrow h_F \alpha, \tau \rightarrow h_G \beta_1 \beta_2, \alpha \rightarrow r_1 | r_2, \beta_2 \rightarrow t_1 r_2 | t_2 r_2, \beta_1 \rightarrow t_1 r_1 | t_2 r_1\}.$$ 

Then the SEHS does not have a solution, i.e. there is no formula $C$ such that $S(C)$ is a tautology.
To prove the given lemma we have to make a detour. At first we prove a simpler case.

Lemma 2. Let $F := P(x, t_1 x) \land Q(x, t_2 x)$ and $G := P(r, x) \land Q(r, y)$. Assume the sequent $\forall x. F \vdash \exists x, y, G$, the SEHS $S(X)$

$$P(\alpha, t_1 \alpha) \land Q(\alpha, t_2 \alpha),$$
$$X(\alpha, t_3 \alpha) \lor X(\alpha, t_2 \alpha) \rightarrow X(r, \beta_1) \land X(r, \beta_2) \vdash P(r, \beta_1) \land Q(r, \beta_2)$$

and the schematic $\Pi_2$-grammar $\mathcal{G} = (\tau, N, \Sigma, \text{Pr})$ where $N = \{\tau, \alpha, \beta_1, \beta_2\}$ and

$$\text{Pr} = \{\tau \rightarrow h_F \alpha, \tau \rightarrow h_G \beta_1 \beta_2, \alpha \rightarrow r | r, \beta_2 \rightarrow t_1 r, t_2 r, \beta_1 \rightarrow t_1 r | t_2 r\}.$$

Then the SEHS does not have a solution.

Proof. We prove the lemma by contradiction. Let us assume a valid cut-formula $E$ that corresponds to the grammar $\mathcal{G}$. A maximal $\text{G3c}$-derivation $\psi$ of the reduced representation produces the leaves

$$\{P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash P(r, \beta_1); P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash Q(r, \beta_2)\}.$$

Given that $E$ is a valid cut-formula the following sequents have to be tautologies

$$\{P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash P(r, \beta_1), E(\alpha, t_1 \alpha), E(\alpha, t_2 \alpha); P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash Q(r, \beta_2), E(\alpha, t_1 \alpha), E(\alpha, t_2 \alpha); E(r, \beta_1), E(r, \beta_2), P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash P(r, \beta_1); E(r, \beta_1), E(r, \beta_2), P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash Q(r, \beta_2)\}$$

and, hence, also the following sequents

$$\mathcal{B} := \{P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha) \vdash E(\alpha, t_1 \alpha), E(\alpha, t_2 \alpha); E(\alpha, t_1 \alpha) \vdash P(r, \beta_1); E(\beta_1) \vdash Q(r, \beta_2)\}.$$

That we can drop $P(\alpha, t_1 \alpha)$ and $Q(r, \beta_2)$ in the first two lines and $P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha)$ in the last two lines is obvious (Neither $E(\alpha, t_1 \alpha), E(\alpha, t_2 \alpha)$ can contain an atom depending on $\beta_1$ or $\beta_2$ nor $E(r, \beta_1)$ and $E(r, \beta_2)$ can contain an atom depending on $\alpha$). To prove that we can also ignore $E(r, \beta_2)$ in the third line we assume that $T := E(r, \beta_1) \vdash P(r, \beta_1)$ is not provable. Hence, there is a non-tautological branch $\Lambda_1 \vdash \Theta_1, P(r, \beta_1)$ in every maximal $\text{G3c}$-derivation $\psi$ of $T$. Given that $E(\beta_1)$ has the same logical structure as $E(r, \beta_1)$ we can apply the same $\text{G3c}$-rules of $\psi$ to $E(r, \beta_2)$ and get the sequent $\Lambda_2 \vdash \Theta_2$. The atoms of the sets $\Lambda_1$ and $\Theta_1$ are the same as the atoms in $\Lambda_2$ and $\Theta_2$ except for those which depend on the second argument of $E$, i.e. they contain $\beta_1$ or $\beta_2$. Thus, the sequent $\Lambda_1, \Lambda_2 \vdash \Theta_1, \Theta_2$ is not a tautology and also the atom $P(r, \beta_1)$ is not
an element of $\Lambda_1 \cup \Lambda_2$. Then also $S := \Lambda_1, \Lambda_2 \vdash P(r, \beta_1), \Theta_1, \Theta_2$ is not a tautology. But $S$ is a leaf of every proof tree of $E(r, \beta_1), E(r, \beta_2) \vdash P(r, \beta_1)$. This is a contradiction and, therefore, $T$ has to be a tautology. Analogously we can prove that $E(r, \beta_2) \vdash Q(r, \beta_2)$ has to be a tautology if $E(r, \beta_1), E(r, \beta_2) \vdash P(r, \beta_1)$ is a tautology.

If the sequents in $B$ are provable then we can replace in their proofs $\alpha$ with $r, \beta_1$ with $t_2r$, and $\beta_2$ with $t_1r$ to get the provable sequents

\[
\{ P(r, t_1r), Q(r, t_2r) \vdash E(r, t_1r), E(r, t_2r); \quad E(r, t_2r) \vdash P(r, t_2r);
\quad E(r, t_1r) \vdash Q(r, t_1r) \}. \]

Now we can apply two times the cut-rule

\[
\frac{\pi_t \quad E(r, t_2r) \vdash P(r, t_2r)}{P(r, t_1r), Q(r, t_2r) \vdash Q(r, t_1r), P(r, t_2r)}
\]

with $\pi_t :=$

\[
\frac{P(r, t_2r), Q(r, t_2r) \vdash E(r, t_2r), E(r, t_2r) \quad E(r, t_1r) \vdash Q(r, t_1r)}{P(r, t_1r), Q(r, t_2r) \vdash Q(r, t_1r), E(r, t_2r)}
\]

and derive the sequent $P(r, t_3r), Q(r, t_2r) \vdash Q(r, t_3r), P(r, t_2r)$. But this sequent is not valid and, by contradiction, there is no cut-formula.

In general this example suffices to show that there is not always a solution for an SEHS. But at this point one can argue that we have to refine the definition of schematic $\Pi_2$-grammars. If production rules have to be unique then the given example would be inappropriate. The more complex example in lemma 1 would be still appropriate and now we are able to prove this lemma.

Proof of lemma 1

To prove the lemma we give a model in which $r_1$ and $r_2$ are equal because for this case lemma 2 shows the non-existence of a cut-formula.

Assume the natural numbers modulo 2. We interpret $r_1$ as 0, $r_2$ as 2, $\lambda x.t_1$ as the successor function $\lambda x.ssx$, and $\lambda x.t_2$ as $\lambda x.ssx$. In this model $r_1$ is equal to $r_2$ and, hence, there cannot be a cut-formula.

Remark 1. If we take a sequent calculus with equality and add the formula $\neg r_1 = r_2$ to the left of the end-sequent, i.e. an additional assumption, then $\forall x \exists y.(x = r_1 \rightarrow P(x, y)) \land (\neg x = r_1 \rightarrow Q(x, y))$ is a valid cut-formula that corresponds to the given schematic $\Pi_2$-grammar.

Both examples show that, in general, we cannot expect to find a solution for an SEHS. Moreover, it is difficult to give an easy restriction to the grammar such that the solvability is guaranteed.

We start now to characterize some conditions for the introduction of $\Pi_2$ cuts. We begin with $a$, so called, starting set. It may contain a set of clauses that is interpreted as a formula in DNF which is a solution for the SEHS, i.e. the SEHS where $X$ is replaced with this formula is a tautology. Later, we will
define starting sets that always contain a solution as a subset for certain classes of solutions.

Definition 16. We call a finite set of finite sets of literals $C$ s.t. $V(C) \subseteq \{x, y\}$ for designated variables $x, y$ a starting set. The variables $\beta_1, \ldots, \beta_m$, and $\alpha$ may not occur in $C$.

Now we define a normal form for the representation of the leaves of a reduced representation. Therefore we need the following proposition.

Proposition 3. Let $R$ be a reduced representation of an SEHS as in Definition 12 and $\psi$ be a maximal $G3c$-derivation of $R$. Let $NTA(\psi)$ be the set of non-tautological axioms of $\psi$. Let $S \in NTA(\psi)$. Then $S$ is of the form $A(S) \circ B(S) \circ N(S)$ where $A(S)$ is the set of all atoms in $S$ containing $\alpha$, $B(S)$ the set of all atoms in $S$ containing a non-empty subset of the variables $\{\beta_1, \ldots, \beta_m\}$, and $N(S)$ ($N$ stands for “neutral”) the set of all atoms in $S$ neither containing $\alpha$ nor $\beta_i$-s.

The proposition gives us a representation of the leaves, but in this form we are not able to distinguish between atoms occurring on the left hand-side of a sequent and atoms occurring on the right hand-side of the sequent.

Definition 17. Let $S := P_1, \ldots, P_i \vdash Q_1, \ldots, Q_j$ be a sequent containing only atoms. Then we define the literal normal form $D(S)$ of the sequent $S$ as $\neg Q_1, \ldots, \neg Q_j, P_1, \ldots, P_i \vdash$.

Now each literal carries the information on which side of the sequent it occurs. If it is an atom it occurs on the left hand-side. If it is a negated atom it occurs on the right hand-side. Hence, we can define a normal form of the sequents.

Definition 18. Let $NTA(\psi)$ be the set of non-tautological axioms of a maximal $G3c$-derivation of a reduced representation $R$. We define the set of non-tautological axioms in literal normal form $DNTA(\psi) := \{D(S) \mid S \in NTA(\psi)\}$. Let $S \in DNTA(\psi)$. Then $S$ is also of the form $A(S) \circ B(S) \circ N(S)$ where $A(S)$ is the set of all literals in $S$ containing $\alpha$, $B(S)$ the set of all literals in $S$ containing a non-empty subset of the variables $\{\beta_1, \ldots, \beta_m\}$, and $N(S)$ the set of all literals in $S$ neither containing $\alpha$ nor $\beta_i$-s.

Let $Lit(A)$ be true iff $A$ is an atom or a negated atom. For all literals $L \in A(S)$ let $\xi(L) := \{A \mid Lit(A) \text{ and } V(A) \subseteq \{x, y\} \text{ and } i \in \mathbb{N}_p \text{ and } A[x \backslash \alpha, y \backslash t_i] = L\}$ then

$$A'(S) := \bigcup_{L \in A(S)} \xi(L)$$

denotes the set of all literals that can be mapped to an element of $A(S)$. 

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Now we reconsider the main problem of $\Pi_2$-cut introduction and reformulate the necessary conditions. Instead of finding a substitution for $X$ such that the SEHS

$$F[x\backslash U_1], \bigvee_{i=1}^{p} X\alpha t_i \rightarrow \bigwedge_{j=1}^{m} Xr_j\beta_j \vdash G[\bar{y}\backslash U_2]$$

is a tautology we have to find for each leaf $S \in \text{DNTA}(\psi)$ of the reduced representation $R$ a substitution such that

$$A(S), B(S), N(S), \bigvee_{i=1}^{p} X\alpha t_i \rightarrow \bigwedge_{j=1}^{m} Xr_j\beta_j \vdash$$

is a tautology. Hence, we can divide it into two problems

$$A(S), B(S), N(S), Xr_1\beta_1, \ldots, Xr_m\beta_m \vdash \quad (1)$$

$$A(S), B(S), N(S) \vdash X\alpha t_1, \ldots, X\alpha t_p \quad (2)$$

and say that $V$ is a solution of the SEHS if the sequents (1) and (2), where $X$ is replaced by $V$, are tautologies.

Now we want to find formulas in disjunctive normal form that are solutions. Therefore we assume an arbitrary starting set $\mathcal{A}$ and consider all formulas in DNF that are subsets of $\mathcal{A}$. We formulate the restrictions given by the (1) and (2) and eliminate all subsets of $\mathcal{A}$ that are not solutions. In definition 19 we check if the sequent (1) is a tautology. If we substitute a possible solution in DNF for $X$ then the sequent branches into all possible sequents with one clause for each $Xr_1\beta_1, \ldots, Xr_m\beta_m$ on the left hand-side of the sequent. In definition 19 the choice of these $m$ arbitrary clauses is represented by the $m$-tuples $(C_1, \ldots, C_m)$ where $C_i$ is instantiated with $r_i$ and $\beta_i$ for $i \in \mathbb{N}_m$. For each choice we guarantee the provability by demanding an axiomatic constant $(T_1)$, an axiomatic literal $(T_2)$, or an interactive literal $(T_3)$. These literals cover every possible case in which there is a literal and its dual on the left hand-side of the sequent. Then we can shift the negated literal to the right and receive a tautological axiom.

**Definition 19** (Set of possible sets of clauses). Let $R$ be a given reduced representation of an SEHS $S(X)$ and $\psi$ be a maximal $\textbf{G3c}$-derivation of $R$. Let $S \in \text{DNTA}(\psi)$, $m$ be the $\forall$-multiplicity, $\mathcal{C}$ be a set of clauses. Let $\vec{C}_m$ be the set of all $m$-tuples $(C_1, \ldots, C_m)$ where $C_i \in \mathcal{C}$ for $i \in \mathbb{N}_m$. If $\vec{C} \in \vec{C}_m$, $\vec{C} = (C_1, \ldots, C_m)$, and $i \in \mathbb{N}_m$ we write $\vec{C}(i)$ for $C_i$. Furthermore, let $\mathcal{A}$ be a starting set.

We define the three conditions - $(T_1)$ axiomatic constant, $(T_2)$ axiomatic literal,
Furthermore we assume the schematic $\Pi_2$ is the set of possible sets of clauses.

Then

$$CI(A) := \{ C \subseteq A \mid \forall \tilde{C} \in \hat{C}_n \forall S \in \text{DNTA}(\psi).T(\tilde{C}, S) \}$$

is the set of possible sets of clauses.

In the next step we want to guarantee that the sequent \(2\) becomes a tautological axiom. Consider the following example.

**Example 3.** Let

$$\forall \vdash r \quad P(r, t_1 r), Q(r, t_2 r) \vdash P(r, t_1 r), Q(r, t_1 r)$$

$$\exists \vdash r \quad P(r, t_1 r), Q(r, t_2 r) \vdash P(r, t_1 r) \lor Q(r, t_1 r)$$

$$\land \vdash 1 \quad P(r, t_1 r) \land Q(r, t_2 r) \vdash \exists x. P(r, x) \lor Q(r, x)$$

$$\forall \vdash 1 \quad (P(r, t_1 r) \land Q(r, t_2 r)) \lor (P(r, t_2 r) \land Q(r, t_1 r)) \vdash \exists x. P(r, x) \lor Q(r, x)$$

with $\pi_r :=$

$$\forall \vdash r \quad P(r, t_2 r), Q(r, t_1 r) \vdash P(r, t_2 r), Q(r, t_2 r)$$

$$\exists \vdash r \quad P(r, t_2 r), Q(r, t_1 r) \vdash P(r, t_2 r) \lor Q(r, t_2 r)$$

$$\land \vdash 1 \quad P(r, t_2 r) \land Q(r, t_1 r) \vdash \exists x. P(r, x) \lor Q(r, x)$$

$$\forall \vdash 1 \quad (P(x, t_1 x) \land Q(x, t_2 x)) \lor (P(x, t_2 x) \land Q(x, t_1 x)) \vdash \exists x. P(r, x) \lor Q(r, x).$$

Furthermore we assume the schematic $\Pi_2$-grammar $G = (\tau, N, \Sigma, Pr)$ with $N = \{ \tau, \alpha, \beta \}$ and $Pr = \{ \tau \rightarrow P\alpha, \tau \rightarrow Q\beta, \alpha \rightarrow r, \beta \rightarrow t_2 r \}$ where $P$ is the expression $\lambda x.(P(x, t_1 x) \land Q(x, t_2 x)) \lor (P(x, t_2 x) \land Q(x, t_1 x))$ and $Q$ is the expression $\lambda x.P(r, x) \lor Q(r, x)$. Hence, the reduced representation of the SEHS is given by

$$P(\alpha, t_1 \alpha) \lor Q(\alpha, t_2 \alpha) \lor (P(\alpha, t_2 \alpha) \land Q(\alpha, t_1 \alpha)) \vdash P(r, \beta) \lor Q(r, \beta).$$
A maximal $G_{3c}$-derivation $\psi$ gives us the set of non-tautological axioms

$$DNTA(\psi) = \{ P(\alpha, t_1\alpha), Q(\alpha, t_2\alpha), \neg P(r, \beta), \neg Q(r, \beta) \vdash; \\
P(\alpha, t_2\alpha), Q(\alpha, t_1\alpha), \neg P(r, \beta), \neg Q(r, \beta) \vdash \}.$$ 

Now we consider the starting set $A = \{ (P(x, y), Q(x, y)) \}$ and compute $Cl(A)$. The only relevant subset of $A$ is $A$ itself. In $A$ we find for each $S \in DNTA(\psi)$ an axiomatic literal. Thus, $Cl(A) = \{ A \}$. But the SEHS where $X$ is replaced with $\lambda xy. P(x, y) \land Q(x, y)$ is not a tautology. A maximal $G_{3c}$-derivation of

$$(P(\alpha, t_1\alpha) \land Q(\alpha, t_2\alpha)) \lor (P(\alpha, t_2\alpha) \land Q(\alpha, t_1\alpha)), \\
(P(\alpha, t_1\alpha) \land Q(\alpha, t_1\alpha)) \lor (P(\alpha, t_2\alpha) \land Q(\alpha, t_2\alpha))$$

$$\rightarrow P(r, \beta) \land Q(r, \beta) \vdash P(r, \beta), Q(r, \beta)$$

gives us the the non-tautological leaves

$$\{ P(\alpha, t_1\alpha), Q(\alpha, t_2\alpha) \vdash P(r, \beta), Q(r, \beta), Q(\alpha, t_1\alpha), P(\alpha, t_2\alpha); \\
P(\alpha, t_2\alpha), Q(\alpha, t_1\alpha) \vdash P(r, \beta), Q(r, \beta), P(\alpha, t_1\alpha), Q(\alpha, t_2\alpha) \}.$$ 

This is due to the existence of a leaf $S$ in $DNTA(\psi)$ that fulfills the following property: we find for each term $t_1\alpha$ and $t_2\alpha$ an atom $P(\alpha, t_1\alpha)$ or $Q(\alpha, t_2\alpha)$ that does not appear in the leaf $S$.

In definition 20 we generalize this property and define a set $I(S)$ for each leaf $S$ that contains only allowed clauses. Clauses as $\{ P(x, y), Q(x, y) \}$ in the previous example are excluded.

**Definition 20.** Let $R$ be a given reduced representation of an SEHS $S(X)$, $\psi$ be a maximal $G_{3c}$-derivation of $R$, and $S \in DNTA(\psi)$. $A'(S)$ is defined as in definition 18. Let $M(k) \subseteq A'(S)$ such that $|M(k)| = k$. Then

$$I(S) := \bigcup_{k \leq |A'(S)|} \{ M(k) \mid \exists i \in \{1, \ldots, p\} \forall \ell \in M(k).L[x, y, t_i] \in A(S) \}.$$ 

is the set of allowed clauses.

A useful tool for the application of the set of allowed clauses in practice can be obtained from the following proposition.

**Proposition 4.** Let $R$ be a given reduced representation of an SEHS $S(X)$, $\psi$ be a maximal $G_{3c}$-derivation of $R$, $S \in DNTA(\psi)$ and $I(S)$ the set of allowed clauses. If $I$ is an element of $I(S)$ and $J$ is a non-empty subset of $I$ then $J$ is an element of $I(S)$.

Now we can formulate the conditions that guarantee the provability of sequent [2]. Again we need for each non-tautological leaf an axiomatic constant, an axiomatic literal, or an interactive literal. The differences to definition [19] are due to the different behavior of formulas in disjunctive normal form on different sides of a sequent in a proof in sequent calculus.
Definition 21 (Set of solution candidates). Let $R$ be a given reduced representation of an SEHS $S(X)$ and $\psi$ be a maximal G3c-derivation of $R$. Let $S \in \text{DNTA}(\psi)$, $p$ be the $\exists$-multiplicity, and $C$ be a set of clauses. Let $\vec{L}_p(C)$ be the set of all $p$-tuples $(L_1, \ldots, L_p)$ where $L_i \in C$ for $i \in \mathbb{N}_p$ and $C \in \mathcal{C}$. If $\vec{L} \in \vec{L}_p(C)$, $\vec{L} = (L_1, \ldots, L_p)$, and $i \in \mathbb{N}_p$ we write $\vec{L}(i)$ for $L_i$. Let $\vec{C} = \prod_{C \in \mathcal{C}} \vec{L}_p(C)$ be the Cartesian product of the subspaces $\vec{L}_p(C)$ where $C \in \mathcal{C}$.

We define the three conditions - $(T_1')$ axiomatic constant, $(T_2')$ axiomatic literal, $(T_3')$ interactive literal -

$$T_1'(C, \vec{C}, S) := \exists C \in \mathcal{C} \exists i \in \mathbb{N}_p, \vec{C}(C, i)[y \setminus t_i] \in N(S),$$

$$T_2'(C, \vec{C}, S) := \exists C \in \mathcal{C} \exists I \in I(S) \forall i \in \mathbb{N}_p, \vec{C}(C, i) \in I,$$

$$T_3'(C, \vec{C}) := \exists C, D \in \mathcal{C}, i, j \in \mathbb{N}_p, \vec{C}(C, i)[x \setminus \alpha, y \setminus t_i] = C(D, j)[x \setminus \alpha, y \setminus t_j],$$

and

$$T'(C, \vec{C}, S) := T_1'(C, \vec{C}, S) \text{ or } T_2'(C, \vec{C}, S) \text{ or } T_3'(C, \vec{C}).$$

Then

$$\text{Sol}(A) := \{C \in Cl(A) \mid \forall \vec{C} \in \vec{C} \forall S \in \text{DNTA}(\psi), T'(C, \vec{C}, S)\}$$

is the set of solution candidates for a given starting set and a given SEHS in DNF.

Example 4. If we consider example 3 again and compute $\text{Sol}(A)$ we will get the empty set. To prove that the set is empty we compute $I(S)$ first. By proposition 4 we do not have to compute the whole set. We only have to take those clauses into consideration, that can be build by the literals occurring in $A$, i.e. $P(x, y)$ and $Q(x, y)$. No matter which of the two non-tautological leaves we choose, the only sets in $I(S)$ that contain just $P(x, y)$ or $Q(x, y)$ are $\{P(x, y)\}$ and $\{Q(x, y)\}$. Hence, also all other sets that contain one of the two literals contain at most one of them.

Let us discuss $\text{Sol}(A) = \text{Sol} \{\{P(x, y), Q(x, y)\}\}$. We want to show that it is empty. Therefore, we choose the non-tautological leaf

$$P(\alpha, t_1 \alpha), Q(\alpha, t_2 \alpha), \neg P(r, \beta), \neg Q(r, \beta) \vdash$$

and the 1-tuple of 2-tuples

$$((P(x, y), Q(x, y))) \in \prod_{C \in \{P(x, y), Q(x, y)\}} \vec{L}_p(C).$$

The only element of $Cl(A)$ is $A$ and $A$ contains only a single clause, such that $C$ and $D$ in $T_1', T_2'$, or $T_3'$ are always $\{P(x, y), Q(x, y)\}$. The disjunct $T_1'$ is never
true because \( N(S) \) is in our case empty. In the discussion before we saw that all sets in \( I(S) \) contain at most one literal of \( \{ P(x, y), Q(x, y) \} \). Hence, there is no set \( I \) in \( I(S) \) such that all elements of \( I \) are elements of \( I \) and, thus, \( T_2' \) is not true. \( T_2' \) is not true because \( P(\alpha, t_2 \alpha) \) is, in general, not the contrary to \( Q(\alpha, t_1 \alpha) \). Given that \( T_1', T_2', \) and \( T_3' \) are not true for all elements in \( Cl(A) \) the set of solution candidates is empty. The interested reader may check the procedure with the starting set \( \{ \{ P(x, y) \} \} \).

We can show that each solution candidate is actually a solution.

**Theorem 5 (Soundness).** Let

\[
F[\bar{x}\setminus U_1], \bigvee_{i=1}^{p} X \alpha t_i \rightarrow \bigwedge_{j=1}^{m} X r_j \beta_j \vdash G[\bar{y}\setminus U_2]
\]

be an SEHS, \( Sol(A) \neq \emptyset \) be defined as in definition 22 for a given starting set \( A \), and \( C \in Sol(A) \). Let \( E = DNF(C) \) be the formula in DNF corresponding to \( C \) and \( \hat{E} = \lambda xy. E \). Then

\[
F[\bar{x}\setminus U_1], \bigvee_{i=1}^{p} \hat{E} \alpha t_i \rightarrow \bigwedge_{j=1}^{m} \hat{E} r_j \beta_j \vdash G[\bar{y}\setminus U_2]
\]

is a tautology, i.e. a solution candidate is a solution.

**Proof.** If we want to prove that

\[
F[\bar{x}\setminus U_1], \bigvee_{i=1}^{p} \hat{E} \alpha t_i \rightarrow \bigwedge_{j=1}^{m} \hat{E} r_j \beta_j \vdash G[\bar{y}\setminus U_2]
\]

is a tautology we have to prove the sequents

\[
F[\bar{x}\setminus U_1], \hat{E} r_1 \beta_1, \ldots, \hat{E} r_m \beta_m \vdash G[\bar{y}\setminus U_2] \quad \text{(3)}
\]

\[
F[\bar{x}\setminus U_1] \vdash \hat{E} \alpha t_1, \ldots, \hat{E} \alpha t_p, G[\bar{y}\setminus U_2]. \quad \text{(4)}
\]

Both sequents can be proved by contradiction, i.e. we assume that the sequents are not provable and derive a contradiction.

First we will show that the sequent (3) is a tautology. Assume it is not provable. The formulas \( \hat{E} r_j \beta_j \) for \( j \in \mathbb{N}_m \) are formulas in DNF which can be interpreted as sets of sets of literals. In \( G3c \) a disjunction on the left

\[
\Delta, \bigvee_{i \in I} A_i \vdash \Gamma
\]

is considered to be true if the sequents \( \Delta, A_i \vdash \Gamma \) are true for all \( i \in I \). Hence, if the sequent (3) is not provable then there are clauses \( C_1, \ldots, C_m \) in \( C \) such that, for \( E_i = \lambda xy. DNF(\{ C_1 \}) \),

\[
F[\bar{x}\setminus U_1], E_1 r_1 \beta_1, \ldots, E_m r_m \beta_m \vdash G[\bar{y}\setminus U_2] \quad \text{(5)}
\]

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is not provable.

Now we apply the rules of a maximal $\text{G}3\text{c}$-derivation $\psi$ of $R$ and let the instantiations $E_1 r_1 \beta_1, \ldots, E_m r_m \beta_m$ be untouched. The non-tautological axioms of $R$ can be represented by $\text{DNTA}(\psi)$ where $A(S) \circ B(S) \circ N(S)$ for $S \in \text{DNTA}(\psi)$ is defined as in definition 18. Hence, we can add the literals of the clauses $E_1 r_1 \beta_1, \ldots, E_m r_m \beta_m$ to $B(S)$ and $N(S)$ to get a representation of the non-tautological axioms of $\psi$.

The part of literals that has been added to $B(S)$ will be denoted by $B$ and the part that has been added to $N(S)$ will be denoted by $N$. If the sequent (5) is not provable there has to be a non-tautological axiom $S'$, i.e.

$$\forall L, Q \in A(S') \cup B(S') \cup B \cup N(S') \cup N.L \neq Q.$$ 

But this implies that there is no axiomatic constant $(T_1')$, axiomatic literal $(T_2')$, or interactive literal $(T_3')$. Thus it contradicts definition 21 and the sequent (5) is provable.

Now we have to prove that the sequent (4) is a tautology. We will again assume that it is not a tautology and derive a contradiction. Let us assume there are $k$ clauses $C_1, \ldots, C_k$ in $C$. Thus, the sequent

$$F[\bar{x} \backslash U_1] \vdash E_1 \alpha t_1, \ldots, E_k \alpha t_p, G[\bar{y} \backslash U_2]$$

where $E_i = \lambda xy. \text{DNF}(\{C_i\})$ is also not a tautology. Now we apply again the rules of a maximal $\text{G}3\text{c}$-derivation $\psi$ of $R$ and let the clauses be untouched. Given that the sequent above is not a tautology, there is also a leaf $S'$ in the derivation that is not a tautology. We find in each $E_i \alpha t_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, p\}$ a literal $L_i$ with $l = (i - 1) \cdot p + j$ such that

$$S' := A(S), B(S), N(S) \vdash L_1, \ldots, L_k$$

is not a tautology. But this implies that there is neither an axiomatic constant $(T_1')$, nor an axiomatic literal $(T_2')$, nor an interactive literal $(T_3')$ and contradicts definition 21. Hence, the sequent (4) is a tautology.

Furthermore we can show that the definitions 19 and 21 do not eliminate solutions, i.e. if there is a subset in the starting set $A$ that is a solution then this set will also be an element of $\text{Sol}(A)$.

**Theorem 6 (Partial completeness).** Let

$$F[\bar{x} \backslash U_1], \bigvee_{i=1}^p X \alpha t_i \rightarrow \bigwedge_{j=1}^m X r_j \beta_j \vdash G[\bar{y} \backslash U_2]$$

be an SEHS, $A$ be a starting set, and $C \subseteq A$. Let $E = \text{DNF}(C)$ be the formula in DNF corresponding to $C$ and $F = \lambda xy. E$. If

$$F[\bar{x} \backslash U_1], \bigvee_{i=1}^p \hat{E} \alpha t_i \rightarrow \bigwedge_{j=1}^m \hat{E} r_j \beta_j \vdash G[\bar{y} \backslash U_2]$$

is a tautology then $C \in \text{Sol}(A)$. 

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Proof. At first we assume that there is a solution $C$ for the SEHS that is a subset of the starting set $A$ but $C$ is not an element of $Cl(A)$ of definition $[19]$ Let $\psi$ be a maximal $G3c$-derivation. If $\vec{C}$ is not an element of $Cl(A)$ but $\vec{C} \subseteq A$ then

$$\exists \vec{C} \in \vec{C}_m \exists S \in DNTA(\psi). T(\vec{C}, S)$$

with

$$T(\vec{C}, S) := T_1(\vec{C}, S) \text{ and } T_2(\vec{C}, S) \text{ and } T_3(\vec{C}),$$

$$T_1(\vec{C}, S) := \forall i \in \mathbb{N}_m \forall L \in \vec{C}(i), L[x \backslash r] \notin N(S)$$

where $N(S)$ denotes the dualized set $N(S)$,

$$T_2(\vec{C}, S) := \forall i \in \mathbb{N}_m \forall L \in \vec{C}(i). L[x \backslash r_1, y \backslash \beta_i] \notin B(S),$$

and

$$T_3(\vec{C}) := \forall i, j \in \mathbb{N}_m \forall L \in \vec{C}(i) \forall Q \in \vec{C}(j). L[x \backslash r_i, y \backslash \beta_i] \neq \overline{Q}[x \backslash r_j, y \backslash \beta_j]$$

where $\vec{C}_m$ is defined as in definition $[19]$ Let $S$ be an element of $DNTA(\psi)$ of the form $A(S) \circ B(S) \circ N(S)$. There is a $m$-tuple of clauses $(C_1, \ldots, C_m)$ with $C_i \in C$ for $i \in \mathbb{N}_m$ fulfilling the following property. Let $E_k := DNF(\{C_k\})$ and $\hat{E}_k := \lambda xy.E_k$ for $k \in \mathbb{N}_m$ then

$$A(S), B(S), N(S), \hat{E}_1 r_1 \beta_1, \ldots, \hat{E}_m r_m \beta_m \vdash$$

is not a tautology. But then also

$$F[\vec{x} \backslash U_1], \bigwedge_{j=1}^{m} \hat{E} r_j \beta_j \vdash G[\vec{y} \backslash U_2]$$

is not a tautology, i.e. $C$ is not a solution and by contradiction $C \notin Cl(A)$.

Now we assume $C \notin Sol(A)$. Given that $C \in Cl(A)$ we find an element $\vec{C} \in \vec{C}$ and a leaf $S \in DNTA(\psi)$ such that

$$\forall C \in C \forall i \in \mathbb{N}_p, \vec{C}(C, i)[y \backslash t_i] \notin N(S)$$

and

$$\forall C \in C \forall i \in I(S) \exists i \in \mathbb{N}_p. \vec{C}(C, i) \notin I$$

and

$$\forall C, D \in C \forall i, j \in \mathbb{N}_p, \vec{C}(C, i)[x \backslash \alpha, y \backslash t_i] \neq \overline{C(D, j)[x \backslash \alpha, y \backslash t_j]}$$

where $\vec{C}$ is defined as in definition $[21]$ and $I(S)$ is defined as in definition $[20]$. Let $k$ be the number of clauses in $C$ then we find for all of them $p$ literals

$$\vec{C}(C_1, 1), \ldots, \vec{C}(C_1, p), \ldots, \vec{C}(C_k, 1), \ldots, \vec{C}(C_k, p)$$

where $C_1, \ldots, C_k$ are the $k$ clauses such that the sequent

$$A(S), B(S), N(S) \vdash \hat{L}_1 \alpha t_1, \ldots, \hat{L}_p \alpha t_p, \ldots, \hat{L}_{(k-1) \cdot p+1} \alpha t_1, \ldots, \hat{L}_{k \cdot p} \alpha t_p$$

with $\hat{L}_q := \lambda xy.DNF(\{\vec{C}(C_i, j)\})$ for $q \in \mathbb{N}_{k \cdot p}$, $q = (i-1) \cdot p + j$, $i \in \mathbb{N}_k$, and $j \in \mathbb{N}_p$ does not contain an axiomatic constant $(T_q')$, an axiomatic literal $(T_2)$, or
an interactive literal ($T_3'$). Furthermore $A(S), B(S), N(S) \vdash$ is not a tautology and the literals

$$\hat{L}_1\alpha t_1, \ldots, \hat{L}_p\alpha t_p, \ldots, \hat{L}_{(k-1)p+1}\alpha t_1, \ldots, \hat{L}_{kp}\alpha t_p$$

do not contain the eigenvariables $\beta_1, \ldots, \beta_m$. Hence, none of the literals occurs in $A(S), B(S)$, or $N(S)$ and the found sequent is not a tautology. This contradicts the assumption that $C$ is a solution and is not an element of $Sol(A)$. Thus, $C \in Sol(A)$.

To prove full completeness we need a starting set for every possible reduced representation. In chapter 7 we show that we can define starting sets, provided a balanced solution of the SEHS exists. The general case is not treated in this paper.

**Theorem 7.** Let

$$F[\bar{x} \setminus U_1], \bigvee_{i=1}^p X\alpha t_i \rightarrow \bigwedge_{j=1}^m Xr_j\beta_j \vdash G[\bar{y} \setminus U_2]$$

be an SEHS corresponding to a Herbrand sequent of a cut-free proof of $S$ and a grammar $G$ covering the Herbrand term set of $S$. Let $Sol(A) \neq \emptyset$ be defined as in definition 21 for a given starting set $A$, and $C \in Sol(A)$. Let $E = DNF(C)$ be the formula in DNF corresponding to $C$ and $V(E) = \{x, y\}$. Then there exists a proof of $S$ with one cut and the cut formula $\forall x\exists y.E$

**Proof.** If there is an element $C$ in $Sol(A)$ for a given starting set $A$ and a given SEHS, we are able to construct a proof with a $\Pi_2$-cut. Let

$$\mathbb{F} := F[\bar{x} \setminus U_1], \quad \mathbb{G} := G[\bar{y} \setminus U_2], \quad \mathbb{A} := \forall \bar{x}.F, \quad \mathbb{B} := \exists \bar{y}.G.$$ 

Assume an SEHS

$$\mathbb{F}, \bigvee_{i=1}^p X[x, y\setminus t_i] \rightarrow \bigwedge_{j=1}^m X[x, r_j, y\setminus \beta_j] \vdash \mathbb{G}$$

and the clause set $C \in Sol(A)$ for the starting set $A$. Then there are maximal $G3c$-derivations $\pi_l$ and $\pi_r$ with axioms as leaves for the sequents

$$\mathbb{F} \vdash \bigvee_{i=1}^p \lambda x y, (DNF(C))\alpha t_i, \mathbb{G}$$

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and
\[
F, \bigwedge_{j=1}^{m} \lambda xy. (DNF(C))r_j \beta_j \vdash G,
\]
respectively. Furthermore, the following proof is valid and contains a single \( \Pi_2 \)-cut:

\[
\begin{array}{c}
\pi_l \\
F \vdash \bigvee_{i=1}^{p} \lambda xy. (DNF(C)) \alpha t_i, G \ \\
\vdots \\
A \vdash \forall x \exists y. DNF(C), B \\
\end{array}
\quad
\begin{array}{c}
\pi_r \\
F, \bigwedge_{j=1}^{m} \lambda xy. (DNF(C))r_j \beta_j \vdash G \\
\vdots \\
A, \forall x \exists y. DNF(C) \vdash B \\
\end{array}
\]

This is guaranteed by the theorems 5 and 2 and, hence, solves the main problem of our paper.

\[
\square
\]

7. \( G^* \)-unifiability

The previous section gives us a tool to check whether a given starting set contains a solution for an SEHS. Now we discuss a method that produces a starting set for a given reduced representation of an SEHS. This starting set will contain a solution if there is a so-called balanced solution.

But at first we present the construction of the starting set. To understand the starting set we look at the leaves \( \text{DNTA}(\psi) \) of a maximal \( \text{G3c} \)-derivation \( \psi \) of a given reduced representation \( R \). A solution of the corresponding SEHS contains for each leaf \( S \) in \( \text{DNTA}(\psi) \) at least one literal \( L \) with \( V(L) \subseteq \{x, y\} \), that is an element of \( A(S), B(S), \) or \( N(S) \) with the correct substitutions for \( x \) and \( y \). Hence, the first approach is to collect all literals that can be substituted such that they become at least one element of \( A(S), B(S), \) and \( N(S) \). Then we consider all possible sets containing a subset of these literals (see the naive starting set in definition 38).

Let us assume that a literal \( L \) of the solution interacts twice with a literal in \( A(S), B(S), \) or \( N(S) \), once when we replace \( x \) and \( y \) with \( \alpha \) and \( t_i \), and once when we replace \( x \) and \( y \) with \( r_j \) and \( \beta_j \) for \( i \in \mathbb{N}_p \) and \( j \in \mathbb{N}_m \). Then the two literals that interact with \( L \) are called interacting literals and fulfill a property we describe in the following. A second approach is to search all possible literals that have interacting literals and construct a starting set with them. To find them we define a unification method. In the end we prove that the first approach only finds solutions if the second approach does.

First we collect for each leaf \( S \) the pairs of literals that are candidates for interacting literals.
Definition 22 (Unification candidates). Assume an SEHS with the corresponding reduced representation $R$. Let $S,S' \in \text{DTNA}(\psi)$ for a maximal $\text{G3c}$-derivation $\psi$ of $R$. Then

$$UC(S,S') = \{(L,Q) \mid L \in A(S) \cup N(S) \text{ and } Q \in B(S') \cup N(S')\}$$

is the set of unification candidates for the leaves $S$ and $S'$.

To be able to unify them we introduce a specific type-0-grammar $\tilde{G}$.

Definition 23. Let $G = \langle \tau, N, \Sigma, \text{Pr} \rangle$ be a schematic $\Pi_2$-grammar with the non-terminals $\tau, \alpha, \beta_1, \ldots, \beta_m$. We define the type-0-grammar $G^* = \langle \tau, N, \Sigma^*, \text{Pr}^* \rangle$ by

$$\Sigma^* = \Sigma \cup \{x, y\} \text{ and } \text{Pr}^* = \tau_1 \cup \tau_2 \cup \tau_3$$

where

$$\tau_1 = \{q \mid q \text{ is a } F\text{-production or a } G\text{-production}\}$$

$$\tau_2 = \{\alpha \rightarrow x, r_1 \rightarrow x, \ldots, r_m \rightarrow x\}, \text{ and}$$

$$\tau_3 = \{t_1(\alpha) \rightarrow y, \ldots, t_p(\alpha) \rightarrow y, \beta_1 \rightarrow y, \ldots, \beta_m \rightarrow y\}.$$

For the definition of the unification method we need a notion of a derivation applied to a literal. A derivation $d$ is a finite number of positions $p_1, \ldots, p_n$ and production rules $\theta_1 \rightarrow s_1, \ldots, \theta_n \rightarrow s_n$. If we apply $d$ to a literal $L$, i.e. $L \vdash d$ then we replace gradually the term $\theta_i$ with $s_i$ at position $p_i$ for $i = 1$ until $i = n$.

Definition 24 ($G^*$-unifiability). Assume an SEHS with the corresponding reduced representation $R$ and schematic $\Pi_2$-grammar $G$. Let $S, S' \in \text{DTNA}(\psi)$ for a maximal $\text{G3c}$-derivation $\psi$ of $R$, $(L,Q) \in UC(S,S')$, and $G^* = \langle \tau, N, \Sigma^*, \text{Pr}^* \rangle$ defined as in definition 23. We say $(L,Q)$ is $G^*$-unifiable if there are derivations $d$ and $b$ in $G^*$ such that $L[d = Q \vdash b]$ and $V(L[d]) \subseteq \{x, y\}$. Furthermore we call $L[d]$ the $G^*$-unified literal. We call $R G^*$-unifiable if we find for every $S \in \text{DTNA}(\psi)$ a $S' \in \text{DTNA}(\psi)$ such that there is a $G^*$-unifiable unification candidate in $UC(S,S')$.

If a reduced representation is $G^*$-unifiable we find for each non-tautological leaf interacting literals. To find a cut formula we collect all $G^*$-unified literals that correspond to the interacting literals.

Definition 25. Let $R$ be a reduced representation with the corresponding grammar $G$ and $\psi$ be a maximal $\text{G3c}$-derivation of $R$. Then we define for all $S, S' \in \text{DTNA}(\psi)$ the maximal set of $G^*$-unified literals

$$\text{MGUL}(S,S') := \{O \mid O \text{ is a } G^*\text{-unified term of } (L,Q) \in UC(S,S')\}.$$  

Now we can define the starting set for $G^*$-unifiable sequents. To this aim we take all possible clauses that consist of $G^*$-unified literals.

Definition 26 (Starting set for $G^*$-unifiable sequents). Let $R := F[\bar{x}\setminus U_1] \vdash G[\bar{x}\setminus U_2]$ be a $G^*$-unifiable reduced representation $R$ of an SEHS with a corresponding schematic $\Pi_2$-grammar $G$. Let $\psi$ be a fixed maximal $\text{G3c}$-derivation.
We get for each pair of leaves \( S, S' \in \text{DNTA}(\psi) \) the maximal set of \( G^* \)-unifiable terms \( \text{MGUL}(S, S') \). Then the starting set for the \( G^* \)-unifiable reduced representation \( R \) is defined as

\[
\mathcal{U} := \{ U \mid U \subseteq \bigcup_{S, S' \in \text{DNTA}(\psi)} \text{MGUL}(S, S') \}.
\]

The starting set for \( G^* \)-unifiable sequents suffices to find so-called balanced solutions.

**Definition 27** (Balanced solution). Let

\[
S(X) := F[\bar{x}\backslash U_1], \bigvee_{i=1}^p X\alpha t_i \rightarrow X r_j \beta_j \vdash G[\bar{y}\backslash U_2]
\]

be an SEHS, \( C \) a finite set of sets of literals, and \( \hat{E} := \lambda xy.DNF(C) \) such that

\[
S(\hat{E}) := F[\bar{x}\backslash U_1], \bigvee_{i=1}^p \hat{E}\alpha t_i \rightarrow \bigwedge_{j=1}^m \hat{E} r_j \beta_j \vdash G[\bar{y}\backslash U_2]
\]

is a tautology. Let \( \psi \) be a maximal \( G_3c \)-derivation of \( S(\hat{E}) \). We say \( C \) is a balanced solution if in all axioms of \( S(\hat{E}) \) at least one of the active formulas is not an ancestor of \( \hat{E} \) in \( \psi \).

A balanced solution does not contain interactive literals (not to be confused with interacting literals) as described in definition 19 by \( T_3 \) and in definition 21 by \( T'_3 \).

**Theorem 8.** Let \( S \) be \( \forall \bar{x}.F \vdash \exists \bar{y}.G \), \( G \) be a schematic \( \Pi_2 \)-grammar, and

\[
S(X) := F[\bar{x}\backslash U_1], \bigvee_{i=1}^p X\alpha t_i \rightarrow X r_j \beta_j \vdash G[\bar{y}\backslash U_2]
\]

be an SEHS for \( S \) and \( G \). Assume that \( S(X) \) has a balanced solution \( C \). Then the set of solution candidates \( \text{Sol}(\mathcal{U}) \) (defined as in definition 21) is not empty where \( \mathcal{U} \) is the starting set for the \( G^* \)-unifiable sequent \( R \) as in definition 26.

To prove the theorem we show the same result for the naive starting set instead of the starting set for \( G^* \)-unifiable sequents \( \mathcal{U} \) and conclude that \( \text{Sol}(\mathcal{U}) \) is also not empty.

**Definition 28** (Naive starting set). Let \( R \) be a reduced representation and \( \psi \) a maximal \( G_3c \)-derivation of \( R \). We define for each leaf \( S \in \text{DNTA}(\psi) \) of the form \( A(S) \circ B(S) \circ N(S) \) the sets

\[
\text{NA}(S) = \{ L \mid \exists j \in N_p.(\lambda xy.L)\alpha t_j \in A(S) \cup N(S), V(L) \subseteq \{ x, y \} \}.
\]

\[
\text{NB}(S) = \{ L \mid \exists j \in N_m.(\lambda xy.L) r_j \beta_j \in B(S) \cup N(S), V(L) \subseteq \{ x, y \} \}.
\]
\[
\mathcal{N} := \{ N \mid N \subseteq \bigcup_{S \in \text{DNTA}(\psi)} NA(S) \cup NB(S) \}
\]
is then called the naive starting set.

**Corollary 1.** Let \( S \) be \( \forall \bar{x}.F \vdash \exists \bar{y}.G \), \( \mathcal{G} \) be a schematic \( \Pi_2 \)-grammar, and

\[
S(X) := F[\bar{x}\setminus U_1], \sqrt[p]{Xa t_i} \rightarrow \bigwedge_{j=1}^{m} Xr_j \beta_j \vdash G[\bar{y}\setminus U_2]
\]

be an SEHS for \( S \) and \( \mathcal{G} \). Assume there is a balanced solution \( C \). Then \( C \in \text{Sol}(\mathcal{N}) \) where \( \mathcal{N} \) is the naive starting set and \( \text{Sol}(\cdot) \) is defined as in definition 21.

**Proof.** The definition 27 of a balanced solution implies that every literal \( L \) of the balanced solution \( C \) is either an element of \( N(S) \cup \overline{N(S)} \) for a leaf \( S \in \text{DNTA}(\psi) \) of the maximal \( G_3c \)-derivation \( \psi \) of the SEHS or it is an element of the sets \( NA(S) \) and \( NB(S) \). In the first case we can define \( \lambda x, y. L \) even though \( L \) is a constant. This is again an element of \( N(S) \) or \( \overline{N(S)} \). By theorem 6, \( C \in \text{Sol}(\mathcal{N}) \).

Now we construct from a given solution, that is a subset of the naive starting set, a solution that is a subset of the starting set for \( G^* \)-unifiable sequents.

**Lemma 3.** Let \( \text{Sol}(\mathcal{N}) \) contain a balanced solution for a given SEHS, for a maximal \( G_3c \)-derivation \( \psi \) of its reduced representation \( R \), and for the naive starting set \( \mathcal{N} \). Let \( \mathcal{G} \) be the corresponding schematic \( \Pi_2 \)-grammar. Then \( \text{Sol}(\mathcal{U}) \neq \emptyset \) for the starting set for \( G^* \)-unifiable sequents \( \mathcal{U} \).

**Proof.** Let \( C \in \text{Sol}(\mathcal{N}) \) be a balanced solution. We choose an arbitrary literal \( L \) of \( C \) that is not an element of any set of literals in \( \mathcal{U} \). If there are none, all literals of \( C \) occur in \( \mathcal{U} \). Given that we consider in \( \mathcal{U} \) all possible sets constructed by a finite number of literals, \( C \) is an element of \( \mathcal{U} \), \( \text{Sol}(\mathcal{U}) \neq \emptyset \), and we are done. Otherwise we distinguish between two cases

\[
\begin{align*}
L &\in \bigcup_{S \in \text{DNTA}(\psi)} NA(S) \text{ and } \{ L \} \notin \mathcal{U} \\
L &\in \bigcup_{S \in \text{DNTA}(\psi)} NB(S) \text{ and } \{ L \} \notin \mathcal{U}.
\end{align*}
\]

Let us assume case (6). Then there is a leaf \( S \in \text{DNTA}(\psi) \) and there is a \( j \in \{1, \ldots, p\} \) such that

\[
(\lambda x y. L) a t_j \in A(S) \cup N(S).
\]

Given that \( \{ L \} \notin \mathcal{U} \) there is no leaf \( S' \) such that \( Q \in B(S') \cup N(S) \) where \((\lambda x y. L) a t_j, Q) \) is \( G^* \)-unifiable with the \( G^* \)-unifiable term \( L \). If \( C = \{ L \} \) is a unit clause then the sequent

\[
(\lambda x y. L)r_1 \beta_1, \ldots, (\lambda x y. L)r_m \beta_m, A(S), B(S), N(S) \vdash
\]

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is not a tautology and \( C \) is not a solution. We define the new clause \( C' = C\setminus\{L\} \) that is \( C \) without the literal \( L \). Given that \( C \) is not a unit clause \( C' \) is not empty. A maximal \textbf{G3c}-derivation of the sequent \( A(J), B(J), N(J) \vdash C' \) where \( C' \) is obtained by substituting \( C' \) for \( C \) in \( C \) and \( J \) is an arbitrary element of \( \text{DNTA}(\psi) \) contains only axioms that also appear in \( A(J), B(J), N(J) \vdash C \). Hence, the new sequent is a tautology, too.

Now we consider the sequent \( C', A(J), B(J), N(J) \vdash \) for an arbitrary \( J \in \text{DNTA}(\psi) \). If it were not a tautology there would be a leaf \( S' \in \text{DNTA}(\psi) \) and an \( i \in \mathbb{N}_m \) such that

\[
(\lambda xy. L)_{r_i\beta_i} \in B(S') \cup N(S').
\]

(Note that we need here, that the given solution is a balanced solution. Otherwise we would have to consider the case that \((\lambda xy. L)_{r_i\beta_i} \) appears in \( C' \), too).

But then there exists the \( G^* \)-unifiable pair

\[
((\lambda xy. L)\alpha t_j, (\lambda xy. L)_{r_i\beta_i}).
\]

By contradiction \( C', A(J), B(J), N(J) \vdash \) is a tautology.

With this procedure we can erase all literals of \( C \) that are elements of

\[
\bigcup_{S \in \text{DNTA}(\psi)} NA(S)
\]

and do not appear in a clause of \( U \).

Let us assume case (7). Then there is a leaf \( S \in \text{DNTA}(\psi) \) and there is a \( j \in \mathbb{N}_m \) such that \((\lambda xy. L)_{r_j\beta_j} \in B(S) \cup N(S)\). Given that \( \{L\} \notin U \) there is no leaf \( S' \) such that \( Q \in A(S') \cup N(S') \) where \( Q, (\lambda xy. L)_{r_j\beta_j} \) is \( G^* \)-unifiable with the \( G^* \)-unifiable term \( L \). Let \( C \) be the clause in which \( L \) appears. Assume \( C \) is the only clause then \( C \) is not a solution because

\[
A(S), B(S), N(S) \vdash (\lambda xy. L)\alpha t_1, \ldots, (\lambda xy. L)\alpha t_p
\]

is not a tautology. We define the new clause set \( C' := C\setminus\{C\} \), i.e. \( C \) without the clause \( C \). Given that \( C \) contains more than one clause \( C' \) is not empty. A maximal \textbf{G3c}-derivation of the sequent \( A(J), B(J), N(J), C' \vdash \) where \( J \) is an arbitrary element of \( \text{DNTA}(\psi) \) contains only axioms that also appear in \( A(J), B(J), N(J), C \vdash \). Hence, the new sequent is a tautology, too.

Now we consider the sequent \( A(J), B(J), N(J) \vdash C' \) for an arbitrary \( J \in \text{DNTA}(\psi) \). If it would not be a tautology there would be a leaf \( S' \in \text{DNTA}(\psi) \) and an \( i \in \mathbb{N}_p \) such that \((\lambda xy. L)\alpha t_i \in A(S') \cup N(S')\). But then exists the \( G^* \)-unifiable pair

\[
((\lambda xy. L)\alpha t_j, (\lambda xy. L)_{r_i\beta_i}).
\]

By contradiction, \( A(J), B(J), N(J) \vdash C' \) is a tautology.

With this procedure we can erase all literals of \( C \) that are elements of

\[
\bigcup_{S \in \text{DNTA}(\psi)} NB(S)
\]
and do not appear in a clause of $U$.

By an exhaustive application of these two methods we get a solution that is a subset of $U$.

\textit{Proof of theorem 8}. The proof can be obtained by combining corollary 1 and lemma 3.

\section*{8. Proof compression}

In Section 7 we have defined a method to find so-called balanced solutions for SEHS. Here we demonstrate their potential of proof compression via $\Pi_2$ cuts. Again we consider the example from Section 3. At the beginning we prove each sequent of the sequence by constructing a Herbrand sequent. Afterwards we measure the complexity in three different ways.

We either count the number of weak quantifier inferences (\textit{quantifier complexity}), the number of inferences (\textit{logical complexity}), or the number of symbols (\textit{symbol complexity}). We know that for instance a compression in terms of weak quantifier inferences can be easily achieved by increasing the logical complexity or the symbol complexity of the cut-formula. By measuring all of them we ensure that the compression we achieve is largely independent of the measurement. In the end we will see that by the method of $G^*$-unifiability we find for all sequents of the sequence $S_n$ defined in Section 3 proofs $\psi_n$ with the cut-formula $\forall x \exists y. P(x, fy)$ which are polynomially bounded in $n$. We also show that \textit{all} sequences of cut-free proofs of $S_n$ grow exponentially in $n$, which yields an exponential compression of proof complexity.

We already defined the quantifier complexity (see Definition 4).

\begin{definition}[Logical complexity] Let $\pi$ be a given LK proof. If $\pi$ is of the form

\begin{center}
Axiom $\Delta \vdash \Gamma$
\end{center}

then the \textit{logical complexity} $|\pi|_l$ is defined to be 0. If $\pi$ is of the form

\begin{center}
Binary rule $\frac{\pi_l}{\Delta \vdash \Gamma}$ $\frac{\pi_r}{\Delta \vdash \Gamma}$
\end{center}

with an arbitrary binary rule of LK subproofs $\pi_l$ and $\pi_r$, then $|\pi|_l := |\pi_l|_l + |\pi_r|_l + 1$. If $\pi$ is of the form

\begin{center}
Unary rule $\frac{\pi'}{\Delta \vdash \Gamma}$
\end{center}

with the subproof $\pi'$ and an arbitrary unary rule of LK then $|\pi|_l := |\pi'|_l + 1$.

The symbol complexity counts the number of symbols in each sequent of the proof and the number of rules that connect these sequents with each other. Therefore, it can be defined with the help of the logical complexity which represents the number of LK-rules.
Definition 30 (Symbol complexity). Let $\pi$ be a given LK proof and $\Sigma$ the corresponding signature. Let $S_1, \ldots, S_n$ be the sequents occurring in $\pi$. The symbol complexity $|S_i|$ of a sequent $S_i$ for $i \in \mathbb{N}_n$ is equal to the number of occurrences of the symbols of the set $\Sigma \cup \{\lor, \land, \to, \neg, \exists, \forall, \cdot, \triangleright\}$ and of variables occurring in $S_i$. The symbol complexity $|\pi|$ of the proof is defined as

$$|\pi|_s := |\pi|_l + \sum_{i \in \mathbb{N}_n} S_i.$$  

It is easy to see that the different measurements follow an order. While the quantifier complexity is the most coarse one, the symbol complexity is the finest.

Proposition 9. Let $\pi$ be a given LK proof. Then the following inequalities hold:

$$|\pi|_q \leq |\pi|_l \leq |\pi|_s.$$  

Proof. Obvious. \qed

Before we start to compute the different complexities of our example we adjust the form of the end-sequents $A_n, B, C_n \vdash D$. In the presented method we require a sequent of the form $\forall \bar{x} F \vdash \exists \bar{y} G$. Let $A'_n, B', C'_n$, and $D'$ be the quantifier free part of $A_n, B, C_n$, and $D$ (we rename the variables)

$$A'_n := P(x_1, f_1 x_1) \lor \ldots \lor P(x_1, f_n x_1),$$
$$B' := P(x_2, x_3) \to P(x_2, f x_3),$$
$$C'_n := P(y_1, f y_2) \land P(f y_2, f y_3) \land \ldots \land P(f y_{n-1}, f y_n) \to P(y_1, g y_n),$$
$$D' := P(y_{n+1}, g y_{n+2}).$$

Furthermore, let $\bar{x} = (x_1, x_2, x_3)$ be the tuple of the 3 variables occurring in $A'_n \land B'$, and $\bar{y} = (y_1, \ldots, y_{n+2})$ be tuples of the $n + 2$ variables occurring in $C'_n \lor D'$; let $\overline{C'_n}$ be the negation of $C'_n$

$$\overline{C'_n} := P(y_1, f y_2) \land P(f y_2, f y_3) \land \ldots \land P(f y_{n-1}, f y_n) \land \neg P(y_1, g y_n).$$

Then we can define equivalent sequents

$$S'_n := \forall \bar{x}. A'_n \land B' \vdash \exists \bar{y}. \overline{C'_n} \lor D'.$$

From now on $S'_n$ will always refer to the rewritten sequence of sequents that is in the correct form for the presented cut-introduction method. $S_n$ will refer to the original version.
8.1. Minimal cut-free proofs

In this section we consider cut-free proofs of \( S_n \) for a fixed natural number \( n \). For convenience we will compute lower bounds on the different complexities of minimal proofs of \( S'_n \) in terms of the respective complexity measurement instead of computing the exact complexity. Moreover, we will show that minimal proofs of \( S_n \) always have a smaller complexity than minimal proofs of \( S'_n \) no matter which complexity measurement we choose.

Lemma 4. Let \( \pi \) be a minimal proof of \( S_n \) in terms of quantifier, logical, or symbol complexity and \( \pi' \) a minimal proof of \( S'_n \) in terms of quantifier, logical, and symbol complexity, respectively then

\[
|\pi|_\diamond \leq |\pi'|_\diamond
\]

where \( \diamond \in \{q, l, s\} \).

Proof sketch. Each minimal proof of \( S_n \) can be transformed into a minimal proof of \( S'_n \). This transformation will at most add inferences and, therefore, the respective complexity can only increase.

Before we compute the complexities of minimal proofs of \( S_n \) we have to show some properties of a potential minimal proof. In a first step we show that in a minimal proof (with respect to an arbitrary complexity measurement) all atoms that appear in an instantiation of \( A_n, B, C_n \), or \( D \) are active in an axiom.

Lemma 5. Let \( \pi \) be a minimal proof in terms of quantifier, logical, or symbol complexity of \( S_n \) and

\[
P(a, f_1a) \lor \ldots \lor P(a, f_na),
\]

\[
P(b_1, b_2) \rightarrow P(b_1, f_2b_2),
\]

\[
(P(c_1, f_2c_2) \land P(f_2c_2, f_3c_3) \land \ldots \land P(f_{n-1}c_n, fc_n)) \rightarrow P(c_1, gc_n), \text{ and}
\]

\[
P(d_1, gd_2)
\]

be instantiations of \( A_n, B, C_n \), and \( D \) for some proof-specific terms \( a, b_1, b_2, c_1, \ldots, c_n, d_1, \) and \( d_2 \) then there are axioms for each atom

\[
P(a, f_1a), \ldots, P(a, f_na), P(b_1, b_2), P(b_1, f_2b_2),
\]

\[
P(c_1, f_2c_2), P(f_2c_2, f_3c_3), \ldots, P(f_{n-1}c_n, fc_n), P(c_1, gc_n), \text{ and}
\]

\[
P(d_1, gd_2)
\]

in which the respective atom is active.

Proof. The proof works for all four formulas in a similar way. We will only consider the formula

\[
A_n^{\alpha} := P(a, f_1a) \lor \ldots \lor P(a, f_na).
\]
Assume there is an $i \in \mathbb{N}_n$ such that $P(a, f_i a)$ is not active in any axiom. Then we can order $\pi$ such that the $\lor$ $l$-rules that apply to $A^{i,a}_n$ are the rules in the new minimal proof $\pi'$ that appear at the top of the corresponding proof tree. Let $S := A^{i,a}_n, \Delta \vdash \Gamma$ be a sequent in which $A^{i,a}_n$ appears. The provability implies that also $P(a, f_i a), \Delta \vdash \Gamma$ is a tautological axiom. Hence, $\Delta \vdash \Gamma$ is already tautological and we can drop all the $\lor$ $l$-rules applied to $A^{i,a}_n$ (and even the instantiation rules). Thus, there is a proof with smaller quantifier, logical, and symbol complexity which contradicts the assumption that $\pi$ was already minimal in these terms. Hence, there is no such instantiation.

The next property guarantees that $A_n, B_n, C_n,$ and $D$ have to be instantiated at least once.

**Lemma 6.** Let $\pi$ be a proof of $S_n$ then the formulas

\[
P(a, f_1 a) \lor \ldots \lor P(a, f_n a),
\]

\[
P(b_1, b_2) \rightarrow P(b_1, f b_2), \quad \text{and}
\]

\[
(P(c_1, f c_2) \land P(f c_2, f c_3) \land \ldots \land P(f c_{n-1}, f c_n)) \rightarrow P(c_1, g c_n)
\]

with some proof-specific terms $a, b_1, b_2, c_1, \ldots, c_n$ appear on the left side of some sequents in $\pi$ and the formula

\[
P(d_1, g d_2)
\]

with proof-specific terms $d_1, d_2$ appears on the right side of some sequent in $\pi$.

**Proof.** First of all at least one formula has to be instantiated. Otherwise, there cannot be a valid proof. By showing that an instantiation of an arbitrary formula enforces all other formulas to be instantiated at least once we will complete the proof. This can easily be seen by lemma 5 and the facts that all potential atoms of $A_n$ can only build valid axioms with potential atoms of $B_n$ ($P(a, f_i a), \Delta \vdash \Gamma, P(b_1, b_2)$ with $a = b_1$ and $f_i a = b_2$), all potential atoms of $B$ has to build axioms with $A_n$ and $C_n$, and so on. In the end we have to instantiate $A_n, B, C_n, \text{and} D$.

Now we can describe sets of instantiations that belong to a minimal proof of $S_n$. We will not write down the whole proof because of its large size. But by proving the minimality of this instantiations we will implicitly give a sketch of the proof and show its validity.

**Theorem 10.** Let $n$ be a fixed natural number and $S_n = A_n, B, C_n \vdash D$ be
given. Then the sets

$$A_n^1 := \{c\}, A_n^2 := \{fh_1c \mid h_1 \in \{f_1, \ldots, f_n\}\},$$
$$A_n^i := \{fh_{i-1} \ldots fh_1c \mid h_1, \ldots, h_{i-1} \in \{f_1, \ldots, f_n\}\} \text{ for } i \in \{3, \ldots, n-1\},$$
$$A'_n := \bigcup_{i=1}^{n-1} A_n^i, B' := \{(t, ft) \mid t \in A'_n \text{ and } i \in \{1, \ldots, n\}\},$$
$$C_n := \{(t_1, \ldots, t_{n+2}) \mid t_1 = c \text{ and } t_2 = h_1 t_1 \text{ and } t_3 = h_2 f t_2 \text{ and } \ldots \text{ and } t_n = h_{n-1} f t_{n-1} \text{ and } t_{n+1} = t_1 \text{ and } t_{n+2} = t_n \text{ and } h_1, \ldots, h_{n-1} \in \{f_1, \ldots, f_n\}\},$$
$$D' := \{(c, t) \mid t = t_n \text{ and } \exists t_1, \ldots, t_{n-1}. (t_1, \ldots, t_n) \in C'_n\}.$$

are instantiations of the formulas $A_n, B, C_n,$ and $D$ such that the corresponding fully instantiated sequent $S_n^\dagger$ is tautological and there is a minimal (in terms of quantifier, logical, or symbol complexity) proof $\pi$ of $S_n$ with the midsequent $S_n^\dagger$.

**Proof.** By lemma 5 we can assume an instantiation $(t_1, \ldots, t_n)$ of $C_n$. Let $c := t_1$. Given that atomic subformulas of an instantiated formula in a minimal proof have to be active (see lemma 5) we know that $P(c, ft_2)$ of

$$P(c, ft_2) \land P(ft_2, ft_3) \land \ldots \land P(ft_{n-1}, t_n) \rightarrow P(t_1, gt_n)$$

has to be active in an axiom. In an axiom $P(c, ft_2)$ appears on the right side of the sequent and, hence, the only formula that can become $P(c, ft_2)$ on the left side of the sequent is $P(b_1, fb_2)$ of

$$B = P(b_1, b_2) \rightarrow P(b_1, fb_2).$$

Then $b_1$ has to be equal to $c$ and $b_2$ has to be equal to $t_2$. By applying lemma 5 again we have to find the counterpart of $P(b_1, b_2) = P(c, t_2)$. Hence, there has to be an instantiation of $A_n$, i.e.

$$P(c, f_1c) \lor \ldots \lor P(c, f_nc).$$

Given that this is the only possibility we can conclude that there have to be instantiations of $B$ and $C'_n$ where $t_2$ is equal to $f_1c, \ldots, f_{n-1}c,$ and $f_nc$. So far we described $A_n^1$, the parts of $B'$ where $A_n^{i-1}$ is replaced with $A_n^1$, the first two elements of the tuples in $C'_n$, and the first element of the tuples in $D'$. With the second elements $f_1c, \ldots, f_n c$ of the tuples in $C'_n$ we have to go through the same procedure as we did with $c$. That is, we will get new instantiations of $A_n$, i.e. $A_n^2$, a new part of $B'$ and the third elements of tuples in $C'_n$. After $n$ applications of this procedure we would have constructed the sets of the theorem such that each atom has exactly one necessary counterpart, i.e. all atoms appear as an active formula in an axiom and we cannot drop a single atom without making the proof invalid. Hence, the instantiation correspond to a minimal proof of $S_n$ in terms of quantifier complexity. Given that all proofs contain at least as many instantiations as the given one there also has to be a corresponding minimal proof in terms of logical and symbol complexity.  \[\Box\]
Now we can compute the quantifier complexity of $S_n$. Let $A'_n, B', C'_n, D'$, and $S_n^\downarrow$ be defined as in theorem 10 then

$$|S_n^\downarrow|_q = (|A'_n|) + (|B'| + \sum_{i=1}^{n} n^{i-1}) + (|C'_n| + \sum_{i=1}^{n} n^{i-1}) + (|D'| + 1).$$

The additional instantiations, besides the ones covered by $|A'_n| + |B'| + |C'_n| + |D'|$, derive from the number of variables in each formula $A_n, B, C_n, \text{and } D$. Given that

$$|A'_n| = \sum_{i=1}^{n} n^{i-1},$$

$$|B'| = n \cdot |A'_n| = n \cdot \sum_{i=1}^{n} n^{i-1},$$

$$|C'_n| = n^{n-1}, \text{ and }$$

$$|D'| = |C'_n| = n^{n-1}$$

the quantifier complexity sums up to

$$|S_n^\downarrow|_q = n^n + 6 \cdot n^{n-1} + 4 \cdot n^{n-2} + \ldots + 4 \cdot n + 5 > n^n$$

for $n \geq 3$ and

$$|S_n^\downarrow|_q = n^n + 6 \cdot n^{n-1} + 5 > n^n$$

for $n = 2$.

By lemma 4 we can give a lower bound for the quantifier complexity of $S_n$. Moreover, the quantifier complexity is a lower bound for the logical complexity and the symbol complexity (see proposition 9). To summarize: the various complexities of minimal proofs of $S_n$ are bigger than $n^n$.

### 8.2. A proof scheme with a Pi-2-cut

Now we want to apply our method. We consider the scheme of schematic $\Pi_2$-grammars $G_n$. $G_n$ is defined by the starting symbol $\tau$, the non-terminals $\tau, \beta_1, \ldots, \beta_{n-1}, \alpha$, and the production rules

$$\tau \to h_{F_n}(\alpha, \alpha, f_1 \alpha) | \ldots | h_{F_n}(\alpha, \alpha, f_n \alpha) \mid h_{G_n}(c, \beta_1, \ldots, \beta_{n-1}, c, \beta_{n-1}),$$

$$\alpha \to f_{\beta_{n-1}} | \ldots | f_{\beta_1} | c,$$

$$\beta_{n-1} \to f_1 f_{\beta_{n-2}} | \ldots | f_n f_{\beta_{n-2}},$$

$$\vdots$$

$$\beta_2 \to f_1 f_{\beta_1} | \ldots | f_n f_{\beta_1}, \text{ and }$$

$$\beta_1 \to f_1 c | \ldots | f_n c$$

where $h_{F_n}$ and $h_{G_n}$ are function symbols that correspond to the $\lambda$-terms $\gamma_n = \lambda \bar{x}.A'_n \land B'$ and $\delta_n = \lambda \bar{y}.C'_n \lor D'$. Note that the language $L(G_n)$ of $G_n$ covers...
the Herbrand term set that can be derived from the instantiations of chapter 8.4. The leaves of a maximal $G_3c$-derivation $\psi_n$ of the reduced representation

$$(\lambda x. A'_n \land B') \alpha \alpha(f_1 \alpha), \ldots, (\lambda x. A'_n \land B') \alpha \alpha(f_n \alpha)$$

$$\vdash (\lambda y. C'_n \lor D') c \beta_1 \ldots \beta_{n-1} c \beta_{n-1}$$

can be represented in the normal form of definition 17, i.e.

$$\{ P(\alpha, h \alpha),$$

$$\{ \neg P(\alpha, f_i \alpha) \mid i \in I_1 \}, \{ P(\alpha, f_i \alpha) \mid i \in I \setminus I_1 \},$$

$$\{ \neg P(c, f \beta_1) \mid j = 1 \},$$

$$\{ \neg P(f \beta_1, f \beta_2) \mid j = 2 \}, \ldots, \{ \neg P(f \beta_{n-2}, f \beta_{n-1}) \mid j = n - 1 \},$$

$$\{ P(c, g \beta_{n-1}) \mid j = n \},$$

$$\neg P(c, g \beta_{n-1}) \vdash$$

$$\mid h \in \{ f_1, \ldots, f_n \}, I = N_n, I_1 \subseteq I, j \in I \}$$

and the non-tautological leaves are

$$DNTA(\psi_n) = \{ P(\alpha, f_i \alpha),$$

$$\{ \neg P(\alpha, f_i \alpha) \mid i \in I_1 \}, P(\alpha, f_i \alpha), \{ P(\alpha, f_k \alpha) \mid k \in I \setminus I_1 \},$$

$$\neg P(f \beta_{j-1}, f \beta_j),$$

$$\neg P(c, g \beta_{n-1}) \vdash$$

$$\mid i \in N_n, j \in N_{n-1}, I_1 \subseteq I = N_n \setminus i, f \beta_0 := c \}.$$
Then we find the correct cut-formula and the proof $\pi_n$ with cut can be sketched by

\[
\begin{array}{c}
\forall x.A'_n \land B'_n \vdash [P(\alpha, \beta_1) \ldots, P(\beta_{n-2}, \beta_{n-1})]_{i=1}^n \\
\vdots \\
\forall x.A'_n \land B'_n \vdash \neg \exists y.P(\alpha, f) \\
\forall x.A'_n \land B'_n \vdash \forall x \exists y.P(x, f) \\
\forall x.A'_n \land B'_n \vdash \exists y.\neg \exists y.P(\alpha, f) \\
\forall x.\exists y.\neg \exists y.P(x, f) \\
\forall x.\exists y.\neg \exists y.P(x, f) \\
\end{array}
\]

where $A'_n, B'_n, C'_n$, and $D_n$ are instantiations of $\forall x.\lambda x.\exists y.\neg \exists y.P(x, f)$ and $\exists y.\neg \exists y.P(x, f)$. To check the correctness of the proofs we look at the instantiations and the leaves of a maximal $\textbf{G3e}$-derivation. Let $\beta_1, \ldots, \beta_{n-1}$, and $\alpha$ be the derivations of the cut-formula then

\[
A' := \{\alpha\}, \quad B'_n := \{(\alpha, f_i) \mid i \in \{1, \ldots, n\}\}, \\
C'_n := \{(c, \beta_1, \ldots, \beta_{n-1})\}, \quad D'_n := \{(c, \beta_{n-1})\}
\]

are the sets of instantiations. We can define the substituted formulas as follows

\[
A'_n := \{\lambda x.A' \mid t \in A'\}, \quad B'_n := \{\lambda x.B' \mid \bar{t} \in B'_n\}, \\
C'_n := \{\lambda x.C' \mid \bar{t} \in C'_n\}, \quad C'_n := \{P \mid \bar{T} \in C'_n\}, \quad D'_n := \{\lambda x.D' \mid \bar{t} \in D'_n\}.
\]

The leaves of the left branch $\mathbb{L}_l$ in literal normal form (see definition 17) are

\[
\begin{align*}
& \{ P(\alpha, h) \mid i \in I_1 \}, \quad \{ P(\alpha, f_i) \mid i \in I \setminus I_1 \}, \\
& \{ \neg P(\alpha, f_i) \mid i \in I, \quad h \in \{f_1, \ldots, f_n\}, \quad I = \mathbb{N}_n, \quad I \subseteq I \}
\end{align*}
\]

and the leaves of the right branch $\mathbb{L}_r$ in literal normal form are

\[
\begin{align*}
& \{ \neg P(c, f) \mid j = 1 \}, \\
& \{ \neg P(f, f) \mid j = 2 \}, \ldots, \{ \neg P(\beta_{n-2}, f) \mid j = n-1 \}, \\
& \{ P(c, g) \mid j = n \}, \\
& \neg P(c, g) \mid i \in \mathbb{N}_n, \quad \text{and} \quad f-beta := c \} \vdash \\
& \{ h \in \{f_1, \ldots, f_n\}, \quad I = \mathbb{N}_n, \quad I \subseteq I, \quad j \in I \}.
\end{align*}
\]

Let us assume a leaf $L$ of the set $\mathbb{L}_d$. It contains an atom $P(\alpha, f_k)$ for a given $k \in \mathbb{N}_n$. If $k \in I_1$ then $L$ contains also $\neg P(\alpha, f_k)$ and is therefore a tautology. Let us assume $k \in I \setminus I_1$. Then $P(\alpha, f_k)$ is an element of $L$. But each leaf contains the set $\{ \neg P(\alpha, f_i) \mid i \in I \}$, i.e. $L$ contains also $\neg P(\alpha, f_k)$ and is a tautology.
Let us assume a leaf $L$ of the set $\mathbb{L}_r$. Then it contains the set $\{P(r_i, f \beta_i) \mid i \in \mathbb{N}_{n-1}\}$. If $j \in \mathbb{N}_{n-1}$ we get the dual of an element of $\{P(r_i, f \beta_i) \mid i \in \mathbb{N}_{n-1}\}$. If $j = n$ the leaf contains $P(c, g \beta_{n-1})$ and $\neg P(c, g \beta_{n-1})$. Hence, all leaves in $\mathbb{L}_r$ are tautologies and thus, the proof scheme is a correct.

Now we compute the quantifier complexity of the proof sequence. Let $|A_n \land B_n|$ denote the number of instantiations of $\forall \bar{x}. A_n' \land B'$ and $|C_n \lor D_n|$ denote the number of instantiations of $\exists \bar{y}. C_n' \lor D'$. The number of instantiations of the end-sequent is

$$|A_n \land B_n| = n + 2, \quad |C_n \lor D_n| = n + 2, \quad |A_n \land B_n| + |C_n \lor D_n| = 2 \cdot n + 4$$

and the number of instantiations of the cut-formula is $2 \cdot n - 1$, and therefore the quantifier complexity of the proof is

$$|\pi_n|_q = |A_n \land B_n| + |C_n \lor D_n| + (2 \cdot n - 1) = 4 \cdot n + 3 \in O(n).$$

The logical complexity can easily be verified by counting. Hence, the number of inferences is

$$|\pi_n|_l = (n \cdot n^1) \cdot n \cdot n + n + 1 + 1 = n^4 + n + 2.$$ 

To give an upper bound on the symbol complexity we have to compute the maximal symbol complexity of the sequents appearing in the proofs. This depends heavily on the used sequent calculus and the order of the proofs. Therefore, we will assume a polynomial function $P(\cdot)$ that maps from natural numbers to natural numbers such that the maximal size of each sequent in the proofs is smaller than $P(n)$. The interested reader is invited to prove the existence of such a function. Given $P$ we can define the upper polynomial bound

$$|\pi_n|_s \leq 2 \cdot P(n) \cdot |\pi_n|_q + |\pi_n|_q.$$ 

While the complexity in terms of logical inferences, in terms of weak quantifier inferences, or in terms of symbol complexity is bigger than $n^n$ for the cut-free proofs the introduction of the $\Pi_2$-cut decreases the complexity by an exponential factor.

9. Conclusion

In this paper we extended the current range of algorithmic cut introduction from $\Pi_1$-cuts to $\Pi_2$-cuts. While any $\Pi_1$-grammar specifying the set of Herbrand instances of a cut-free proof yields a solution of the corresponding $\Pi_1$-cut-introduction problem (the so-called canonical solution) this does not hold for schematic $\Pi_2$-grammars and $\Pi_2$-cuts; In Section [6] we have presented a schematic $\Pi_2$-grammar specifying a set of Herbrand instances which is not solvable in the sense that the corresponding schematic extended Herbrand sequent (representing the cut-introduction problem) does not have a solution. As for the
Π₂ case canonical solutions do not exist in general we have chosen a different approach to compute Π₂-cuts corresponding to given schematic Π₂-grammars by characterizing the validity of cut formulas. Given a so-called starting set (a set of sets of literals with two designated free variables x, y of the intended Π₂-cut formula ∀x∃y.A) we have developed a method to decide whether this starting set contains a logical equivalent version of such a formula A. However, the general problem to decide whether such a starting set exists at all remains unsolved. But in case balanced solutions exist appropriate starting sets can be defined. However, the straightforward method to construct so-called naive starting sets for balanced problems is computationally inefficient. To improve the resulting cut-introduction method we developed a unification method which yields much smaller sufficient starting sets for the computation of the cut-formulas. Finally we have shown that our method of introducing (single) Π₂-cuts is capable of achieving an exponential proof compression: there exists a sequence of sequents having only cut-free proofs of at least exponential size for which our method of G*-unification efficiently generates a sequence of proofs of polynomial size with Π₂ cuts.

Concerning future work we plan to implement the G*-unification method developed in this paper and to test it on proof data bases. For practical applications the presented method should be modified: for instance it is not necessary to compute the starting set as a whole already in the beginning. Moreover the algorithm should be enriched by use of effective heuristics. There are also several open theoretical questions: is it possible to construct starting sets whenever there is a solution to the cut-introduction problem and to decide whether a problem is solvable at all? A positive answer would yield a decision procedure for the Π₂ cut-introduction problem and (in case of solvability) a complete method to construct proofs with Π₂-cuts. So far our method can only deal with cut-formulas of the form ∀x∃y.A(x, y). In a next step blocks of quantifiers of the form ∀x₁...∀xₙ∃y₁...∃yₘ should be considered. An extension of the method to the introduction of several Π₂-cuts promises the same compression as can be obtained by a single Π₃-cut, i.e. a super-exponential one. Finally, a method to introduce Πₙ-cuts could be capable of a nonelementary proof compression and would represent a long range goal of this research.

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