We recall the Einstein velocity addition on the open unit ball $B$ of $\mathbb{R}^3$ and its algebraic structure, called the Einstein gyrogroup. We establish an isomorphism between the Einstein gyrogroup on $B$ and the set of all qubit density matrices representing mixed states endowed with an appropriate addition. Our main result establishes a relation between the trace metric for the qubit density matrices and the rapidity metric on the Einstein gyrogroup $B$.

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1. Introduction

Einstein addition is the standard velocity addition of relativistically admissible vectors that Einstein introduced in his 1905 paper that founded the special theory of relativity. In his book [3] the presentation of Einstein’s special theory of relativity is based on Einstein velocity addition law. It also allows the reader to take the properties of the underlying hyperbolic geometry.

A. A. Ungar has introduced and studied in [5] gyrogroups that are algebraic settings of hyperbolic geometry, and suggested three examples of gyrogroups corresponding to three models of hyperbolic geometry. It has been known that gyrogroups correspond equivalently to loop structures; see [4]. In Section 2 we review the Einstein gyrogroup on the open unit ball of the Euclidean three-dimensional space $\mathbb{R}^3$ corresponding to the Beltrami-Klein ball model of hyperbolic geometry.

Bloch vectors in the unit ball of $\mathbb{R}^3$ are well-known in quantum information and computation theory. A qubit density matrix, a 2-by-2 positive semidefinite Hermitian matrix with trace 1, is completely described by the Bloch vectors. Chen and Ungar have computed the Bures fidelity of qubit density matrices generated by two Bloch vectors and showed some equivalent formulas in their papers [6] and [7]. The Bures fidelity that plays an important role for the geometry of quantum state space has been manipulated into a form possessing distance properties, called the Bures metric. On the other hand, we apply the trace metric to the space of all invertible qubit density matrices, investigate its properties, and show in
Section 5 the relation between the rapidity metric of the Einstein gyrogroup on $B$ and the trace metric of the qubit density matrices.

2. Einstein addition and gyrogroups

The velocities in Einstein’s theory of special relativity are three-dimensional vectors with magnitude bounded by the speed of light. We assume the speed of light is normalized by the value 1, and call such velocities admissible vectors. The relativistic sum of two admissible vectors $u$ and $v$ of norm less than 1 is given by

$$u \oplus v = \frac{1}{1 + u^T v} \left\{ u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1 + \gamma_u (u^T v)} u \right\},$$

(2.1)

where $\gamma_u$ is the well-known Lorentz factor

$$\gamma_u = \frac{1}{\sqrt{1 - \|u\|^2}}.$$  

(2.2)

Note that $u^T v$ is just the Euclidean inner product of $u$ and $v$ written in matrix form.

Definition 2.1. The formula (2.1) defines a binary operation called Einstein addition on the open unit ball $B = \{ u : \|u\| < 1 \}$ of $\mathbb{R}^3$.

The Einstein addition can be naturally defined on the open unit ball $B$ of $n$-dimensional space $\mathbb{R}^n$. See [2, Chapter 1] for a derivation of the Einstein addition law.

Remark 2.2. The Einstein vector addition on the open unit ball $B$ of $\mathbb{R}^n$ can also be defined by applying the Lorentz boost

$$B(u) = \begin{pmatrix} \gamma_u & \frac{\gamma_u u^T}{\gamma_u^2} \\ \gamma_u u & I + \frac{\gamma_u^2}{1 - \gamma_u^2} uu^T \end{pmatrix}$$



to \(\begin{pmatrix} 1 \\ v \end{pmatrix}\) and obtaining

$$B(u) \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} t \\ t(u \oplus v) \end{pmatrix},$$

where $t = \gamma_u (1 + u^T v)$.

If two vectors $u$ and $v$ are parallel, that is, there exists a nonzero constant $\lambda$ such that $v = \lambda u$, then

$$u \oplus v = \frac{u + v}{1 + u^T v}.$$
Definition 2.3. A triple \((G, \oplus, 0)\) is a gyrogroup if the following axioms are satisfied for all \(a, b, c \in G\).

\((G1)\) \(0 \oplus a = a \oplus 0 = a\) (existence of identity);
\((G2)\) \(a \oplus (-a) = (-a) \oplus a = 0\) (existence of inverses);
\((G3)\) There is an automorphism \(\text{gyr}[a, b] : G \to G\) for each \(a, b \in G\) such that
\[ a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c \] (gyroassociativity);
\((G4)\) \(\text{gyr}[0, a] = \text{id}\);
\((G5)\) \(\text{gyr}[a \oplus b, b] = \text{gyr}[a, b]\) (loop property).

A gyrogroup \((G, \oplus)\) is gyrocommutative if it satisfies
\[ a \oplus b = \text{gyr}[a, b](b \oplus a) \] (gyrocommutativity).

A gyrogroup is uniquely 2-divisible if for every \(b \in G\), there exists a unique \(a \in G\) such that \(a \oplus a = b\).

We call \(\text{gyr}[a, b]\) the gyroautomorphism or Thomas gyration generated by \(a\) and \(b\).

Remark 2.4. It has been shown that gyrocommutative gyrogroups are equivalent to Bruck loops [4], i.e., a loop is a Bruck loop if and only if it is a gyrocommutative gyrogroup with respect to the same operation. It follows that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to \(B\)-loops, uniquely 2-divisible Bruck loops.

A. A. Ungar has shown in [5, Chapter 3] by computer algebra that Einstein addition on the ball \(B\) is a gyrocommutative gyrogroup operation, and the gyroautomorphisms are orthogonal transformations preserving the Euclidean inner product and the inherited norm. We call \((B, \oplus)\) the Einstein (gyrocommutative) gyrogroup, where \(\oplus\) is defined by (2.1).

In references [5], [6], and [7] the Einstein vector addition is mostly defined on the open unit ball \(\mathbb{B}\) of \(\mathbb{R}^n\). It turns our interest to extending the Einstein addition on the closed unit ball \(\overline{\mathbb{B}}\). Substituting \(\alpha_u = \frac{1}{\gamma_u}\) in the definition (2.1) we have an alternative expression of the Einstein vector addition
\[ u \oplus v = \frac{1}{1 + u^T v} \left\{ u + \alpha_u v + \frac{1}{1 + \alpha_u} (u^T v) u \right\}. \]

This is well defined for all \((u, v) \in \overline{\mathbb{B}} \times \overline{\mathbb{B}}\) excluding the antipodal vectors. Since \(u \oplus v = u\) for any \(v \in B\) if \(\|u\| = 1\), we are able to define the Einstein addition naturally for the
antipodal vectors by continuity such that

\[ u \oplus v = u \]

for any \( v \in \mathcal{B} \) whenever \( \|u\| = 1 \). Hence, the extended Einstein addition of \( u \) and \( v \) in \( \mathcal{B} \) can be defined as

\[
\begin{cases} 
\mathcal{B} & \text{if } u \in \mathcal{B} \\
 u & \text{if } \|u\| = 1 
\end{cases}
\]

**Remark 2.5.** The closed unit ball \( \mathcal{B} \) with respect to the extended Einstein addition is a binary system, but not a gyrogroup since it has no unique inverse.

### 3. Bloch vectors and density matrices

A qubit is a member of a 2-dimensional Hilbert space, or a two-state quantum system. A qubit density matrix is a 2-by-2 positive semidefinite Hermitian matrix with trace 1. Indeed, any 2-by-2 Hermitian matrix of trace 1 must have a parametrization of the form

\[
\rho_v = \frac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix},
\]

where

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3.
\]

So the qubit density matrix can be described as \( \rho_v \) for some \( v \in \mathbb{R}^3 \) such that \( \|v\| \leq 1 \). In this case the vector \( v \) is known as the Bloch vector or Bloch vector representation of the state \( \rho_v \).

**Remark 3.1.** Via the characteristic polynomial of the qubit \( \rho_v \), we obtain that its eigenvalues are

\[
\frac{1 + \|v\|}{2}, \quad \frac{1 - \|v\|}{2},
\]

and its determinant is

\[
\det \rho_v = \frac{1 - \|v\|^2}{4}.
\]

So the mixed states are parameterized by the open unit ball \( \mathcal{B} \) in \( \mathbb{R}^3 \).

The Pauli matrices are given by

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where $i = \sqrt{-1}$. The parameterization of qubit density matrices can be written in an alternative method using the Pauli matrices:

$$
\rho_v = \frac{1}{2}(I + v^T \sigma),
$$

where $v^T \sigma = v_1 \sigma_x + v_2 \sigma_y + v_3 \sigma_z$ for

$$
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}.
$$

Let $D = \{\rho_u : u \in B\}$ be the set of all qubit density matrices representing the mixed states. We define a binary map $\odot : D \times D \rightarrow D$ as

$$
\rho_u \odot \rho_v = \frac{1}{\text{tr} (\rho_u \ast \rho_v)} \rho_u \ast \rho_v
$$

where $\rho_u \ast \rho_v = \rho_u^{1/2} \rho_v \rho_u^{1/2}$.

**Remark 3.2.** From the fact that $\text{tr} AB = \text{tr} BA$ for any matrices $A$ and $B$, we have [5 Eq. (9.67)]

$$
\text{tr} (\rho_u \ast \rho_v) = \text{tr} \rho_u \rho_v = \frac{1 + u^T v}{2}.
$$

This gives us that the binary map $\odot$ is well defined since $1 + u^T v \neq 0$ whenever $u, v \in B$.

Since every element in $D$ is a Hermitian positive definite matrix, it has a unique square root. We have the explicit form and can prove it via the direct matrix computation of squaring the form (3.3).

**Lemma 3.3.** For any $v \in B$ the square root of the qubit density matrix $\rho_v$ is uniquely given by

$$
\rho_v^{1/2} = \sqrt{\frac{\gamma_v}{1 + \gamma_v}} \left( \rho_v + \frac{1}{2 \gamma_v} I \right).
$$

(3.3)

We now see an isomorphism between the open unit ball $B$ with the Einstein velocity addition $\oplus$ and the binary system $(D, \odot)$.

**Theorem 3.4.** The map $\rho : (B, \oplus) \rightarrow (D, \odot)$, $v \mapsto \rho_v$ is an isomorphism.

**Proof.** Obviously the map $\rho$ is a bijection. We need to show that $\rho_{u \oplus v} = \rho_u \odot \rho_v$ for any

$$
u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in B.$$
We set $x := \frac{\gamma_u}{1 + \gamma_u}$. By Lemma 3.3 we have

$$\rho_u^{1/2} = \frac{1}{2} \sqrt{x} \begin{pmatrix} u_3 + \frac{1}{x} & u_1 - iv_2 \\ u_1 + iv_2 & -u_3 + \frac{1}{x} \end{pmatrix}.$$ 

It is enough to check the (1,1) and (1,2) entries of $\rho_u \odot \rho_v$ since $\rho_u \odot \rho_v$ is Hermitian.

Let us first compute the (1,1) entry of $\rho_u \star \rho_v$. Then

$$\frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (1 + v_3) + (u_1 - iv_2)(v_1 + iv_2) \right\} \left( u_3 + \frac{1}{x} \right) + \frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (v_1 - iv_2) + (u_1 - iv_2)(1 - v_3) \right\} (u_1 + iv_2) = \frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right)^2 (1 + v_3) + 2 \left( u_3 + \frac{1}{x} \right) (u_1v_1 + u_2v_2) + (u_1^2 + u_2^2)(1 - v_3) \right\}$$

$$= \frac{x}{8} \left\{ \left( \|u\|^2 + 2(\mathbf{u}^T \mathbf{v})u_3 - \|u\|^2v_3 + \frac{2}{x}u_3 + \frac{2}{x} (\mathbf{u}^T \mathbf{v}) + \frac{1}{x^2}(1 + v_3) \right) \right\}$$

$$= \frac{x}{8} \left\{ \|u\|^2 (1 - v_3) + \frac{x}{4} (\mathbf{u}^T \mathbf{v})u_3 + \frac{1}{4} u_3 + \frac{1}{4} (\mathbf{u}^T \mathbf{v}) + \frac{1}{8x} (1 + v_3) \right\}$$

$$= \gamma_u \frac{1}{8\gamma_u} (1 - v_3) + \gamma_u \frac{1}{4(1 + \gamma_u)} (\mathbf{u}^T \mathbf{v})u_3 + \frac{1}{4} u_3 + \frac{1}{4} (\mathbf{u}^T \mathbf{v}) + \frac{1 + \gamma_u}{8\gamma_u} (1 + v_3)$$

$$= \frac{1}{4} \left\{ (1 + \mathbf{u}^T \mathbf{v}) + u_3 + \frac{1}{\gamma_u} v_3 + \frac{\gamma_u}{1 + \gamma_u} (\mathbf{u}^T \mathbf{v})u_3 \right\}.$$

By Remark 3.2 the (1,1) entry of $\rho_u \odot \rho_v$ is

$$\left( \frac{2}{1 + \mathbf{u}^T \mathbf{v}} \right) \frac{1}{4} \left\{ (1 + \mathbf{u}^T \mathbf{v}) + u_3 + \frac{1}{\gamma_u} v_3 + \frac{\gamma_u}{1 + \gamma_u} (\mathbf{u}^T \mathbf{v})u_3 \right\}$$

$$= \frac{1}{2} \left\{ 1 + \frac{1}{1 + \mathbf{u}^T \mathbf{v}} \left( u_3 + \frac{1}{\gamma_u} v_3 + \frac{\gamma_u}{1 + \gamma_u} (\mathbf{u}^T \mathbf{v})u_3 \right) \right\}$$

$$= \frac{1}{2} (1 + (\mathbf{u} \oplus \mathbf{v})v_3),$$

where $(\mathbf{u} \oplus \mathbf{v})_j$ represents the $j$th coordinate of Einstein vector addition $\mathbf{u} \oplus \mathbf{v}$.

Similarly, let us compute the (1,2) entry of $\rho_u \star \rho_v$. Then we have

$$\frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (1 + v_3) + (u_1 - iv_2)(v_1 + iv_2) \right\} (u_1 - iv_2) + \frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (v_1 - iv_2) + (u_1 - iv_2)(1 - v_3) \right\} \left( -u_3 + \frac{1}{x} \right)$$
The real part of the (1,2) entry is

$$\frac{x}{8} \{ (u_3 + \frac{1}{x})(1 + v_3)u_1 + (u_1^2 - u_2^2)v_1 + 2u_1u_2v_2 \}
$$

$$+ \frac{x}{8} \{ (u_3 + \frac{1}{x})v_1(-u_3 + \frac{1}{x}) + u_1(1 - v_3)(-u_3 + \frac{1}{x}) \}
$$

$$= \frac{x}{8} \{ 2u_1(u_3v_3 + \frac{1}{x}) + (u_1^2 - u_2^2 - u_3^2)v_1 + 2u_1u_2v_2 + \frac{1}{x^2}v_1 \}
$$

$$= \frac{x}{8} \{ \frac{2}{x}u_1 + 2(u^TV)u_1 - \|u\|^2v_1 + \frac{1}{x^2}v_1 \}
$$

$$= \frac{1}{4} \left( u_1 + \frac{1}{\gamma_u}v_1 + \frac{\gamma_u}{1 + \gamma_u}(u^TV)u_1 \right) ,
$$

and the imaginary part is

$$\frac{x}{8} \{ - (u_3 + \frac{1}{x})(1 + v_3)u_2 + (u_1^2 - u_2^2)v_2 - 2u_1u_2v_2 \}
$$

$$- \frac{x}{8} \{ (u_3 + \frac{1}{x})v_2(-u_3 + \frac{1}{x}) - u_2(1 - v_3)(-u_3 + \frac{1}{x}) \}
$$

$$= \frac{x}{8} \{ -2u_2(u_3v_3 + \frac{1}{x}) + (u_1^2 - u_2^2 - u_3^2)v_2 - 2u_1u_2v_2 - \frac{1}{x^2}v_2 \}
$$

$$= \frac{x}{8} \{ \frac{2}{x}u_2 - 2(u^TV)u_2 + \|u\|^2v_2 - \frac{1}{x^2}v_2 \}
$$

$$= \frac{1}{4} \left( u_2 + \frac{1}{\gamma_u}v_2 + \frac{\gamma_u}{1 + \gamma_u}(u^TV)u_2 \right) .
$$

By Remark 3.2 the (1,2) entry of $\rho_u \odot \rho_v$ is

$$\left( \frac{2}{1 + u^T v} \right) \frac{1 + u^T v}{4} ((u \oplus v)_1 - i(u \oplus v)_2) = \frac{1}{2} ((u \oplus v)_1 - i(u \oplus v)_2) .
$$

Therefore, we conclude that $\mathbb{D}$ Eq. (9.23)],

$$\rho_u \odot \rho_v = \frac{1}{2} \left( \frac{1 + (u \oplus v)_3}{(u \oplus v)_1 + i(u \oplus v)_2} \right) = \rho_{u \oplus v}
$$

for any $u, v \in B$. This means the map $\rho$ is a homomorphism. □

**Remark 3.5.** From Theorem 3.4 we obtain that $\mathbb{D}$ is a gyrocommutative gyrogroup with respect to the operation $\odot$ defined by

$$\rho_u \odot \rho_v = \frac{1}{\text{tr} \left( \rho_u^{1/2} \rho_v^{1/2} \rho_u^{1/2} \rho_v^{1/2} \right)} \rho_u^{1/2} \rho_v^{1/2} \rho_u^{1/2} .$$
On $D$, moreover, the identity is $(1/2)I$ and the inverse for $\rho_u$ is
\[ \rho^{-1}_u = \frac{1}{4\gamma_u} \rho_u^{-1}. \] (3.4)

4. The Rapidity Metric

The Einstein gyrogroup on the open unit ball $B$ admits an analytic scalar multiplication given by
\[ (t, u) \mapsto t \cdot u = \tanh(t \tanh^{-1}(\|u\|))(u/\|u\|) \] (4.5)
for $u \neq 0$, and $0$ for $t = 0$ or $u = 0$ (see [5, Chapter 6]). Expressing the magnitude of the velocity parameter $u$ by the hyperbolic parameter $\phi_u$ called the rapidity of $u \in B$,
\[ \phi_u = \tanh^{-1}(\|u\|), \]
we have $t \cdot u = \tanh(t\phi_u)(u/\|u\|)$, or $\phi_{t \cdot u} = t\phi_u$.

For the Einstein gyrogroup $(B, \oplus, 0)$, A. A. Ungar considers what we call the Ungar gyrometric defined by $\varrho(u, v) = \| - u \oplus v \|$. He also defines what we call the rapidity metric by $d(u, v) = \tanh^{-1} \varrho(u, v)$. It is known as the Cayley-Klein metric on the Beltrami-Klein model of hyperbolic geometry (see [2]), or the Bergman metric on the symmetric structure $B$ with symmetries $S_w(v) = u \oplus (-v)$ for some $u = 2 \cdot w$ (see [5]). For real numbers $s, t$, we define
\[ s \oplus t = \frac{s + t}{1 + st}, \]
the restricted Einstein addition analogous to the Einstein sum of parallel vectors. We see some properties of rapidity metric on $B$.

Lemma 4.1. The following properties hold for all $u, v, w \in B$.
(i) $0 \leq \varrho(u, v), d(u, v)$
(ii) $\varrho(u, v) = 0 \iff d(u, v) = 0 \iff u = v$
(iii) $\varrho(u, v) = \varrho(v, u)$, $d(u, v) = d(v, u)$
(iv) $\|u \oplus v\| \leq \|u\| + \|v\| \iff \varrho(u, w) \leq \varrho(u, v) + \varrho(v, w) \iff d(u, w) \leq d(u, v) + d(v, w)$
(v) $\varrho(u \oplus v, u \oplus w) = \varrho(v, w)$ and $d(u \oplus v, u \oplus w) = d(v, w)$
(vi) $d(0, r \cdot w) = |r|d(0, w)$.

We establish the relation of metrics on gyrocommutative gyrogroups under an injective homomorphism.

Lemma 4.2. Let $(G_1, \oplus, 0)$ and $(G_2, \oplus, 0)$ be gyrocommutative gyrogroups each equipped with a metric invariant under left translations. If $\psi : G_1 \to G_2$ is an injective gyrogroup
homomorphism and if there exists $\kappa > 0$ such that $d(\mathbf{0}, \psi(u)) \geq \kappa d(\mathbf{0}, u)$ for each $u \in G_1$, then $d(\psi(u), \psi(v)) \geq \kappa d(u, v)$ for all $u, v \in G_1$.

Proof. Let $u, v \in G_1$ and set $w = -u \oplus v \in G_1$. By hypothesis $d(\mathbf{0}, \psi(w)) \geq \kappa d(\mathbf{0}, w)$. By invariance of the metrics under left translation $d(\mathbf{0}, w) = d(\mathbf{0}, \mathbf{u} \oplus \mathbf{0}, \mathbf{u} \oplus (-u \oplus v)) = d(u, v)$ and similarly

$d(\psi(0), \psi(w)) = d(\psi(u), \psi(u) \oplus \psi(-u \oplus v)) = d(\psi(u), \psi(v))$.

We conclude that $d(\psi(u), \psi(v)) \geq \kappa d(u, v)$. $\square$

5. The trace metric for qubit density matrices

Here we review the definition of the trace metric on the open convex cone $\Omega$ of (Hermitian) positive definite matrices. The trace metric on $\Omega$ is determined locally at the point $A$ by the differential

$$ds = \| A^{-1/2} dA A^{-1/2} \|_F,$$

where $\| \cdot \|_F$ means the Frobenius or Hilbert-Schmidt norm. This relation is a mnemonic for computing the length of a differentiable path $\gamma : [a, b] \rightarrow \Omega$

$$L(\gamma) = \int_a^b \| \gamma^{-1/2}(t) \gamma'(t) \gamma^{-1/2}(t) \|_F dt.$$ 

Based on the above notion of length, we define the trace metric $\delta$ between two points $A$ and $B$ in $\Omega$ as the infimum of lengths of curves connecting them. That is,

$$\delta(A, B) := \inf \{ L(\gamma) : \gamma \text{ is a differentiable path from } A \text{ to } B \}.$$ 

R. Bhatia and J. Holbrook have established some basic properties of the trace metric on $\Omega$ in [1].

Lemma 5.1. For any $A, B \in \Omega$ and any invertible matrix $X$,

(i) $\delta(\Gamma_X(A), \Gamma_X(B)) = \delta(A, B)$, where $\Gamma_X$ is a congruence transformation by $X$.

(ii) $\delta(A, B) = \| \log A^{-1/2} BA^{-1/2} \|_F = \| \log(A^{-1} B) \|_F$.

(iii) $\delta(A^{1/2}, B^{1/2}) \leq \frac{1}{2} \delta(A, B)$.

We now consider the trace metric on the set $\mathbb{D}$ of all invertible qubit density matrices, and then see the connection with the rapidity metric on the open unit ball $B$.

Proposition 5.2. For any $u, v \in B$,

$$\delta(\rho_u, \rho_v) \leq \left\{ \ln^2 \left( \frac{x}{a} \right) + \ln^2 \left( \frac{x}{b} \right) \right\}^{1/2},$$
where $x = \frac{1 - u^T v}{2}$, $a = \frac{1}{4 \gamma_u^2}$, and $b = \frac{1}{4 \gamma_v^2}$.

Proof. By the equation (3.4) and Theorem 3.4 we have

$$\rho^{-1/2}u \rho^{-1/2}v = (2 \gamma_u^2)^{1/2} \rho_+ \rho_+^{-1/2}.$$ 

We set $K = 4 \gamma_u^2 \left( \frac{1 - u^T v}{2} \right)$ and $p = \frac{1}{2} (1 + \| - u \oplus v \|)$, where $p$ and $1 - p$ are the eigenvalues of $\rho_+ \rho_+$. Then

$$\delta(\rho_+, \rho_v)^2 = \| \log \rho_+^{-1/2} \rho_v \rho_+^{-1/2} \|^2_F$$

$$= \ln^2(Kp) + \ln^2(K(1 - p))$$

$$= 2 \ln^2 K + 2 \ln K \{ \ln p + \ln (1 - p) \} + \ln^2 p + \ln^2 (1 - p)$$

$$\leq 2 \ln^2 K + 2 \ln K \ln (p(1 - p)) + \ln^2 (p(1 - p))$$

$$= \ln^2 K + \{ \ln K + \ln (p(1 - p)) \}^2$$

$$= \ln^2 K + \ln^2 (Kp(1 - p))$$

$$= \ln^2 \left( \frac{x}{a} \right) + \ln^2 \left( \frac{x}{b} \right).$$

The inequality holds since $\ln p < 0$ whenever $0 < p < 1$. The last equality follows from the property $\gamma_u \gamma_v = \gamma_u \gamma_v (1 + u^T v)$. Indeed,

$$Kp(1 - p) = 4 \gamma_u^2 \left( \frac{1 - u^T v}{2} \right) \left( \frac{1 - \| - u \oplus v \|^2}{4} \right)$$

$$= 4 \gamma_u^2 \left( \frac{1 - u^T v}{2} \right) \frac{1}{4 \gamma_u^2 \gamma_v}$$

$$= \frac{1}{2 \gamma_v^2 (1 - u^T v)}$$

$$= \frac{x}{b}.$$

Remark 5.3. From Remark 3.2 and Remark 3.1, we see that

$$x = \frac{1 - u^T v}{2} = \text{tr} (\rho_+ \rho_+$$

$$a = \frac{1}{4 \gamma_u^2} = \det \rho_u$$

and $b = \frac{1}{4 \gamma_v^2} = \det \rho_v$. 

\[\square\]
Theorem 5.4. For any $u, v \in B$, 
\[ d(u, v) \leq \frac{1}{\sqrt{2}} \delta(\rho_u, \rho_v). \]

Proof. We have seen in Remark 3.5 that $(\mathbb{D}, \circ)$ is a gyrocommutative gyrogroup via an isomorphism 
\[ \rho : (B, \oplus) \rightarrow (\mathbb{D}, \circ). \]

By Lemma 4.1(v) and Lemma 5.1(i) we have the rapidity metric on $B$ and the trace metric on $\mathbb{D}$ are both invariant under left translations. By Lemma 5.1(ii) and Remark 3.1 
\[ \delta((1/2)I, \rho_u)^2 = \| \log (2\rho_u) \|^2_F = \ln^2 (1 + \|u\|) + \ln^2 (1 - \|u\|) \geq \frac{1}{2} \ln^2 \frac{1 + \|u\|}{1 - \|u\|} = \frac{1}{2} (2 \tanh^{-1} \|u\|)^2 = 2d(0, u)^2. \]

So $\delta((1/2)I, \rho_u) \geq \sqrt{2}d(0, u)$. Therefore, by Lemma 4.2 we conclude 
\[ \delta(\rho_u, \rho_v) \geq \sqrt{2}d(u, v). \]

\[ \square \]

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