System Size Stochastic Resonance from the Viewpoint of the
Nonequilibrium Potential

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Abstract

We analyze the phenomenon of system size stochastic resonance in a simple spatially extended system by exploiting the knowledge of the nonequilibrium potential. We show that through the analysis of that potential, and particularly its “symmetry”, we can obtain a clear physical interpretation of this phenomenon in a wide class of extended systems, and also analyze, for the same simple model, the effect of a general class of boundary conditions (albedo) on this kind of phenomena.

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Among the many noise induced phenomena extensively studied during the last few decades like: stochastic resonance \[1\], noise-induced transitions \[2\], noise-induced phase transitions \[3\], noise-induced transport \[4, 5\], noise-induced limit cycles \[6\], stochastic resonance (SR) detaches as one that attracted –and still attracts– the attention of a large number of researchers, probably due to its interest from a technological as well as a biological points of view. There is a broad range of phenomena for which this mechanism can offer an explanation as has been put in evidence by many reviews and conference proceedings \[1\].

The fingerprint of the SR phenomenon is the enhancement of the output signal-to-noise ratio (SNR) caused by the injection of an optimal amount of noise into a periodically –or even aperiodically– driven nonlinear system. Such enhancement is the result of a cooperative effect arising from the interplay between deterministic and random dynamics in a nonlinear system. In almost all the studies of SR, the relevant control variable of the phenomenon was the noise intensity, while the system’s size didn’t play any relevant role. However, some recent studies on biological models of the Hodgkin-Huxley type \[7, 8\] have shown that ion concentrations along biological cell membranes presents intrinsic SR-like phenomena as the number of ion channels is varied. A related result \[9\] shows that even in the absence of external forcing, the regularity of the collective firing of a set of coupled excitable FitzHugh-Nagumo units results optimal for a given value of the number of elements. From a physical system point of view, the same phenomenon –that has been called system size stochastic resonance (SSSR)– has also been found in an Ising model as well as in a set of globally coupled units described by a $\phi^4$ theory \[10\]. It was even shown to arise in opinion formation models \[11\].

In a recent series of papers \[12, 13, 14, 15, 16\] the phenomenon of SR in extended systems was studied exploiting the concept of nonequilibrium potential (NEP). This potential is a special Lyapunov functional of the associated deterministic system which, for nonequilibrium systems, plays a role similar to that played by a thermodynamic potential in equilibrium thermodynamics \[17\]. Such a NEP, closely related to the solution of the time independent Fokker-Planck equation of the system, characterizes the global properties of the dynamics, that is: attractors, relative (or nonlinear) stability of these attractors, height of the barriers separating attraction basins, allowing to evaluate the transition rates among the different attractors. In \[18, 19\], this approach was applied to the global stability analysis of some reaction-diffusion systems. A kind of “mini-review” on the application of this approach to
the study of SR in one, two, and three species reaction-diffusion systems could be found in [16]. However, in spite of its potentiality, this kind of approach has not been exploited for the study of SSSR.

In this paper we present an analysis of the SSSR phenomenon in a simple spatially extended system, exploiting previous results obtained using the notion of NEP within the context of a simple reaction–diffusion model. The specific model we shall focus on, with a known form of the Lyapunov function, corresponds to a one–dimensional, one–component model [20, 21] that, with a piecewise linear form, mimics general bistable reaction–diffusion models [20]. In particular we will exploit some of the results for the influence of general boundary conditions (called “albedo”) found in [22] as well as previous studies of the NEP [18, 19] and of SR [12, 13].

The particular non-dimensional form of the model that we work with is [12, 13, 22]
\[
\partial_t \phi = \partial_{yy} \phi - \phi + \phi_h \theta(\phi - \phi_c).
\]

We consider here a class of stationary structures \(\phi(y)\) in the bounded domain \(y \in [-y_L, y_L]\) with albedo boundary conditions at both ends, \(\frac{d\phi}{dy}\big|_{y=\pm y_L} = \mp k \phi(\pm y_L)\), where \(k > 0\) is the albedo parameter. These are the spatially symmetric solutions to Eq. (1) already studied in [22]. The explicit forms of these stationary patterns –not shown here– are given by Eq. (9) of [22] (see also Fig. 4 in [22]).

The double-valued coordinate \(y_c\), at which \(\phi = \phi_c\), is given by
\[
y_c^{\pm} = \frac{1}{2} y_L - \frac{1}{2} \ln \left[ \frac{z \gamma(k, y_L) \pm \sqrt{z^2 \gamma(k, y_L)^2 + 1 - k^2}}{1 + k} \right],
\]
with \(\gamma(k, y) = \sinh(y) + k \cosh(y)\), and \(z = 1 - 2\phi_c/\phi_h\) (\(-1 < z < 1\)).

When \(y_c^{\pm}\) exists and \(y_c^{\pm} < y_L\), this pair of solutions represents a structure with a central “excited” zone \((\phi > \phi_c)\) and two lateral “resting” regions \((\phi < \phi_c)\). For each parameter set, there are two stationary solutions, given by the two values of \(y_c\). Figure 5 in [22] depicts the curves corresponding to the relation \(y_c/y_L\) vs. \(k\), for various values of \(\phi_c/\phi_h\).

It has been shown [22] that the structure with the smallest “excited” region (with \(y_c = y_c^{+}\), denoted by \(\phi_u(y)\)) is unstable, whereas the other one (with \(y_c = y_c^{-}\), denoted by \(\phi_1(y)\)) is linearly stable. The trivial homogeneous solution \(\phi_0(y) = 0\) (denoted by \(\phi_0\)) exists for any parameter set and is always linearly stable. These two linearly stable solutions are the only stable stationary structures under the given albedo boundary conditions. We will
concentrate on the region of values of $z$, $y_L$ and $k$, where both nonhomogeneous structures exist.

For the system with the albedo b.c. that we are considering here, the NEP reads

$$
\mathcal{F}[\phi, k, y_L] = \int_{-y_L}^{y_L} \left\{ -\phi' + \phi_h \theta(\phi' - \phi_c) \right\} d\phi' + \frac{1}{2} (\partial_y \phi(y, t))^2 \right|_{\pm y_L} + \frac{k}{2} \phi(y, t)^2.
$$

(3)

Replacing the explicit forms of the stationary nonhomogeneous solutions (see for instance Eq.(9) in [22]), we obtain the explicit expression

$$
\mathcal{F}^\pm = \mathcal{F}[\phi_{u,1}, k, y_L] = -\phi_h^2 y_c^\pm z + \phi_h^2 \sinh(y_c^\pm) \frac{\gamma(k, y_L - y_c^\pm)}{\gamma(k, y_L)}.
$$

(4)

while for the homogeneous trivial solution $\phi_0 = 0$, we have instead $\mathcal{F}[\phi_0, k, y_L] = \mathcal{F}^0 = 0$.

Figure 1, part (a) depicts $\mathcal{F}[\phi, k, y_L]$ as a function of the system size $y_L$, for a fixed albedo parameter $k$, and a fixed value of the ratio $\phi_c/\phi_h$ (i.e. fixed value of $z$). The curves correspond to the nonhomogeneous structures, $\mathcal{F}^\pm$, whereas the horizontal line stands for $\mathcal{F}^0$, the NEP of the trivial solution. We have focused on the bistable zone, the upper branch being the NEP of the unstable structure, where $\mathcal{F}$ attains a maximum, while in the lower branch (for $\phi = \phi_0$ or $\phi = \phi_1$), the NEP has local minima. We see that when $y_L$ becomes small, the difference between the NEP for the states $\phi_u(y)$ and $\phi_1(y)$ reduces until, for $y_L \approx 0.72$, they coalesce and, for even lower values of $y_L$, disappear. For completeness and latter use, in part (b) of Fig. 1 we show $\mathcal{F}[\phi, k, y_L]$ but now as a function of $k$, for a fixed value of $y_L$ and the same value of $z$. Here we see that the initial large difference between the NEP for the states $\phi_u(y)$ and $\phi_1(y)$ reduces for increasing $k$ until, for $k \to \infty$, the values for Dirichlet b.c. are asymptotically reached.

It is important to note that, since the NEP for the unstable solution $\phi_u$ is always positive and, for the stable nonhomogeneous structure $\phi_1$, $\mathcal{F} < 0$ for $y_L$ large enough, and $\mathcal{F} > 0$ for small values of $y_L$, the NEP for this structure vanishes for an intermediate value $y_L = y_L^*$ of the system size. At that point, the stable nonhomogeneous structure $\phi_1(y)$ and the trivial solution $\phi_0(y)$ exchange their relative stability.

In order to account for the effect of fluctuations, we include in the time–evolution equation of our model (Eq.(1)) a fluctuation term, that we model as an additive noise source

$$
\partial_t \phi(y, t) = \partial^2_{yy} \phi - \phi + \phi_h \theta(\phi - \phi_c) + \xi(y, t).
$$

(5)
We make the simplest assumptions about the fluctuation term $\xi(y, t)$, i.e. that it is a Gaussian white noise with zero mean and a correlation function given by: 
$$
\langle \xi(y, t) \xi(y', t') \rangle = 2 \gamma \delta(t - t') \delta(y - y'),
$$
where $\gamma$ denotes the noise strength.

As was discussed in [12, 13, 14, 15, 16], using known results for activation processes in multidimensional systems [24], we can estimate the activation rate according to the following Kramers’ like result for $\langle \tau \rangle$, the first-passage-time for the transitions between attractors,

$$
\langle \tau_i \rangle = \tau_0 \exp \left\{ \frac{\Delta F_i[\phi, k]}{\gamma} \right\}, \quad (6)
$$

where $\Delta F_i[\phi, k, y_L] = F[\phi_u(y), k, y_L] - F[\phi_i(y), k, y_L]$ ($i = 0, 1$). The pre-factor $\tau_0$ is usually determined by the curvature of $F[\phi, k, y_L]$ at its extreme (minima) and typically is, in one hand, several orders of magnitude smaller than the average time $\langle \tau \rangle$, while on the other does not change significatively when changing the system’s parameters. Hence, in order to simplify the analysis, we assume here that $\tau_0$ is constant and scale it out of our results. The behavior of $\langle \tau \rangle$ as a function of the different parameters $(k, \phi_c)$ was shown in [12, 13, 14, 15, 16, 18, 19].

We assume now that the system is subject (adiabatically) to an external harmonic variation of the parameter $\phi_c$: $\phi_c(t) = \phi_c + \delta \phi_c \cos(\omega t)$ [13, 16], and exploit the “two-state approximation” [1] as in [13, 14, 15, 16]. For all details on the general two-state approximation we refer to [15].

Up to first-order in the amplitude $\delta \phi_c$ (assumed to be small in order to have a sub-threshold periodic input) the transition rates $W_i$ take the form

$$
W_i = \tau_0^{-1} \exp \left\{ -\frac{\Delta F[i][\phi, k, y_L]}{\gamma} \right\}, \quad (7)
$$
where
\[ \Delta F^i[\phi, k, y_L] = \Delta F^i[\phi, k, y_L] + \delta \phi_c \left( \frac{\partial \Delta F^i[\phi, k, y_L]}{\partial \phi_c} \right)_{\phi_c = \phi_c^*} \cos(\omega t). \] 

(8)

This yields for the transition probabilities
\[ W_i \approx \frac{1}{2} \left( \mu_i \mp \alpha_i \frac{\delta \phi_c}{\gamma} \cos(\omega t) \right), \]

(9)

where \( \mu_i \approx \exp(-\Delta F^i[\phi, k, y_L]) \) and \( \alpha_i \approx \pm \mu_i \frac{\partial \Delta F^i}{\partial \phi_c} |_{\phi_c} (i = 1, 2) \). Using Eq. (4), \( \frac{\partial \Delta F^i}{\partial \phi_c} |_{\phi_c} \) can be obtained analytically.

These results allow us to calculate the autocorrelation function, the power spectrum and finally the SNR, that we indicate by \( R \). The details of the calculation were shown in [15] and will not be repeated here. For \( R \), and up to the relevant (second) order in the signal amplitude \( \delta \phi_c \), we obtain [15]
\[ R = \frac{\pi}{4 \mu_1 \mu_2} \left( \frac{\alpha_2 \mu_1 + \alpha_1 \mu_2}{\mu_1 + \mu_2} \right)^2. \]

(10)

We have now all the elements required to analyze the problem of SSSR.

Figure 2 shows the typical behavior of SR, but now –in the horizontal axis– the noise intensity is replaced by the the system length \( y_L \), for fixed values of \( k, \gamma \) (the noise intensity) and \( \phi_c/\phi_h \). Such a response is the expected one for a system exhibiting SSSR. Within the context of NEP, it results clear that, in this kind of systems, the phenomenon arises due to the breaking of the NEP’s potential symmetry. That is: both attractors change their relative stability due to the variation of \( y_L \) as shown in Fig. 1.a. Hence, within this framework, SSSR arises as a particular case of the general discussion in [15].

Let us now change the point of view. In Fig. 3 we show the curves of the SNR as a function of \( k \), while keeping fixed values of \( y_L \), and \( z \). When \( k \) is not too large, indicating a high degree of reflectiveness at the boundary (or a reduced exchange with the environment), we see that the SNR changes for \( k \) varying from low to larger values. Remember that a large value of \( k \) indicates that the system boundaries become absorbent. Also, the robustness of the systems’ response when changing \( k \), a parameter that somehow indicates the degree of coupling with the environment, is apparent. According to the previous argument –about the breaking of NEP’s symmetry– from Fig. 1.b this is the expected result.

As a final remark, let us consider one of the models discussed in [10] from the point of view of the above indicated approach. The model we refer to is described by a set of coupled
nonlinear bistable oscillators

\[
\dot{x}_j = x_j - x_j^3 + \frac{\varepsilon}{N} \sum_{k=1}^{N} (x_k - x_j) + \sqrt{2} \gamma \xi_j(t) + f_j(t),
\]
\[
\dot{x}_j = -\frac{\partial}{\partial x_j} U(\{x\}, t) + \sqrt{2} \gamma \xi_j(t),
\]

(11)

with \( f_j(t) = A \cos(\omega t) \), \( \{x\} = (x_1, x_2, \ldots, x_N) \), and

\[
U(\{x\}, t) = \sum_{j=1}^{N} u_0(x_j) - A \cos(\omega t) \sum_{j=1}^{N} x_j
\]
\[ U_0(\{x\}) - A \cos(\omega t) \sum_{j=1}^{N} x_j \]
\[ = \sum_{j=1}^{N} \left( \frac{x_j^4}{4} - \frac{x_j^2}{2} \right) + \varepsilon \sum_{j=1}^{N} \sum_{k=1}^{N} (x_k - x_j)^2 - A \cos(\omega t) \sum_{j=1}^{N} x_j. \]  

(12)

Due to the structure of Eq. (11) it is clear that the potential function in Eq. (12) is the NEP of this problem and –for \( A = 0 \)– that the stationary distribution of the multidimensional Fokker-Planck equation associated to Eq. (11) is

\[ P_{\text{stat}}(\{x\}) \approx \exp \left( -\frac{U_0(\{x\})}{\gamma} \right). \]  

(13)

Clearly, this potential has two attractors corresponding to \( x_1 = x_2 = ... = x_N = \pm 1 \), and a barrier separating them at \( x_1 = x_2 = ... = x_N = 0 \).

Now, exploiting the same scheme as before, but reduced to the symmetric case (as both attractors have the same “energy”), we get

\[ SNR \approx \exp \left( -\frac{\Delta U_0(\{x\})}{\gamma} \right) \approx \frac{N}{\gamma} \exp \left( -\frac{N \Delta u_0(X)}{\gamma} \right), \]  

(14)

where \( X \) is a “collective coordinate” (that can be approximately interpreted as \( X \approx \frac{1}{N} \sum_{j=1}^{N} x_j \)), and \( \Delta u_0(X) = u_0(x = \pm 1) - u_0(x = 0) \). This SNR clearly shows similar SSSR characteristics as those described in [10]. However, as in this situation the NEP’s symmetry is retained when varying \( N \), and we could speak of a genuine SSSR.

The above results clearly show that the “nonequilibrium potential”, even if not known in detail [25], offers a very adequate framework to analyze a wide spectrum of noise induced phenomena in spatially extended or coupled systems. Within this framework the phenomenon of SSSR looks, as in other aspects of SR in extended systems [15], as a natural consequence of the breaking of the symmetry of the NEP. In addition, we have seen that through the variation of its coupling with the surroundings, a system can increase the robustness of its response to an external signal, opening new possibilities of analyzing and interpreting the behavior of biological systems.

An important conclusion to be drawn from the identification of the Lyapunov functional with a “thermodynamical-like potential”, is that for a wide range of parameters where the system is essentially bistable (with both attractors not necessarily having the same “energy”), the problem admits a one–dimensional analogue [16,18,19]. This feature is in contrast with the infinite dimensional character of the whole function space, and has been
exploited to strongly simplify the analysis.

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