GRADIENT REGULARITY IN MIXED LOCAL AND NONLOCAL PROBLEMS

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Abstract. Minimizers of functionals of the type

$$w \mapsto \int_{\Omega} |Du|^p - fw \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+2s}} \, dx \, dy$$

with $p, \gamma > 1 > s > 0$ and $p > s\gamma$, are locally $C^{1,\alpha}$-regular in $\Omega$.

1. Introduction

Mixed local and nonlocal problems are a subject of recent, emerging interest and intensive investigation. Essentially, the main object in question is an elliptic operator that combines two different orders of differentiation, the simplest model case being $-\Delta + (-\Delta)^s$, for $s \in (0,1)$. Here, the simultaneous presence of a leading local operator, and a lower order fractional one, constitutes the essence of the matter. In this special case, from a variational viewpoint, one is considering energies of the type

$$w \mapsto \int_{\Omega} |Du|^p - fw \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+2s}} \, dx \, dy,$$

Here, as in all the rest of the paper, $\Omega \subset \mathbb{R}^n$ denotes a bounded, Lipschitz regular domain and $n \geq 2$. First results in this direction have been obtained in [17,19], via probabilistic methods. More recently, in a series of interesting papers, Biagi, Dipierro, Valdinoci, and Vecchi [5–8] have started a systematic investigation of problems involving mixed operators, proving a number of results concerning regularity and qualitative behaviour for solutions, maximum principles, and related variational principles. Up to now, the literature is mainly devoted to the study of linear operators. As for nonlinear cases, for instance those arising from functionals as

$$(1.1) \quad w \mapsto \int_{\Omega} |Du|^p - fw \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+2s}} \, dx \, dy,$$

the study of regularity of solutions has been confined to $L^\infty_{\text{loc}}(\Omega)$ and $C^{0,\alpha}_{\text{loc}}(\Omega)$ estimates (for small $\alpha$), that is, the classical De Giorgi-Nash-Moser theory. In this paper, our aim is to propose a different approach, aimed at proving maximal regularity of solutions to variational mixed problems in nonlinear, possibly degenerate cases as in (1.1). Specifically, we shall prove the local Hölder continuity of the gradient of minimizers. A sample of our results is indeed

Theorem 1. Let $u \in W^{1,p}_0(\Omega) \cap W^{s,\gamma}(\mathbb{R}^n)$ be a minimizer of $1.1$, with $p, \gamma > 1 > s > 0$ and $p > s\gamma$, and such that $u \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. If $\partial \Omega \in C^{\alpha_0}$ for some $\alpha_0 \in (0,1)$, and $f \in L^d(\Omega)$ for some $d > n$, then $Du$ is locally Hölder continuous in $\Omega$ and $u \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for every $\alpha < 1$.

Our approach is flexible and allows us to consider general functionals of the type

$$(1.2) \quad \mathcal{J}(u) := \int_{\Omega} F(Du) - fw \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(w(x) - w(y))K(x,y) \, dx \, dy$$

modelled on the one in [11], i.e. $F(Du) \approx |Du|^p$ in the $C^2$-sense, $\Phi(t) \approx t^\gamma$ in the $C^1$-sense and $K(x,y) \approx |x - y|^{-n-s\gamma}$. Note that, although we specialize to the variational setting, the regularity estimates we are presenting here actually work for general mixed equations almost verbatim, as our

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analysis is essentially based on the use of the Euler-Lagrange equation of functionals as in [12]; for this, see Section 1.2. For the correct notion of minimality, and the related functional setting, as well as for results in full generality, see Section 1.1. Theorem 1 achieves the maximal regularity of minimas, namely, the local Hölder continuity of the gradient of minimizers in \( \Omega \). This is the best possible result already in the purely local case given by the \( p \)-Laplacean equation \( -\Delta_p u = 0 \), which is covered by Uraltseva-Uhlenbeck theory and related counterexamples [35, 51, 52, 49]. In addition, the case \( p \neq \gamma \) is here considered for the first time, thereby allowing a full mixing between local and nonlocal terms. In this respect, the central assumption is

\[
(1.3) \quad p > s\gamma,
\]

that says, roughly speaking, that the fractional \( W^{s,\gamma} \)-capacity generated by the nonlocal term in (1.1) can be controlled by the \( W^{1,p} \)-capacity (the standard \( p \)-capacity) generated by \( w \mapsto \int \left| Dw \right|^p \, dx \). This is exactly the point ensuring that the nonlocal term in (1.1) has less regularizing effects that the local one, as it happens in the basic case \( -\Delta + (-\Delta)^\gamma \), when \( p = \gamma = 2 \), and also in the nonlinear models of the type \( -\Delta_p + (-\Delta)^\gamma_p \), where the fractional \( p \)-Laplacean operator appears [22, 33, 45, 41, 44]. We also note that, as far as we known, allowing the condition \( p \neq \gamma \) is a new, non-trivial feature already when \( p = 2 \) and that even the basic De Giorgi-Nash-Moser theory is not available when \( p \neq \gamma \). As a matter of fact, all our estimates simplify in the case \( p = \gamma \).

We have reported Theorem 1 for the sake of exposition but it is actually a very special case of more general results, i.e., Theorems 2-4, whose statements are necessarily more involved due to their greater generality. Before stating the precise assumptions and the results in full generality, we spend a few words about the techniques we are going to use, and on some relevant connections. Up to now, the methods proposed in the literature to deal with mixed operators are, in a sense, direct. More precisely, both the local terms and the nonlocal ones stemming from the equations interact simultaneously via energy methods. These techniques ultimately rely on those used in the nonlocal case [9, 11, 22, 23, 24, 11, 44] for purely nonlocal operators. This approach does not allow to prove regularity of solutions beyond that allowed by nonlocal operators techniques, which is not the best one can hope for, as, in mixed operators, the leading regularizing term is the local one. In this paper we reverse the approach, relying more on the methods, and, especially, on the estimates available in regularity theory of local operators. In a sense, we separate the local and nonlocal part combining energy estimates of Caccioppoli type with a perturbative like approach. The crucial point is to fit the terms stemming from the nonlocal term in the iteration procedures that would naturally come up from considering the local part only. For this we have to consider a complex scheme of quantities, interacting with each other, and controlling simultaneously both the oscillations of the solution on small balls, and those averaging the oscillations over their complement (such quantities are detailed in Section 3). This first leads to Hölder regularity of solutions with every exponent (Theorem 2) and then to the same kind of estimates globally (Theorem 3). Combining these ingredients with a priori regularity estimates from the classical local theory, leads to Theorem 4. We mention that, due to the assumption \( p \neq \gamma \), functionals as in (1.1) - (1.2) connect to a large family of problems featuring anisotropic operators and integrands with so-called nonstandard growth conditions [21, 24, 32, 10, 19], and to some other classes of anisotropic nonlocal problems [13, 16, 26, 50]. We mention that a further connection has been established in [24], where a class of mixed functionals has been used to approximate local functionals with \( (p, q) \)-growth in order to prove higher integrability of minimizers. Further approximations via mixed operators occur in the interesting paper [56].

### 1.1. Assumptions and results

When considering the functional \( I \) in (1.2), the integrand \( F : \mathbb{R}^n \to \mathbb{R} \) is assumed to be \( C^2(\mathbb{R}^n \setminus \{0\}) \cap C^1(\mathbb{R}^n) \)-regular and to satisfy the following standard \( p \)-growth and coercivity assumptions (see [37, 51, 52])

\[
\begin{align*}
\Lambda^{-1}(|z|^2 + \mu^2)^{p/2} \leq F(z) & \leq \Lambda(|z|^2 + \mu^2)^{p/2} \\
|\partial_z F(z)| + (|z|^2 + \mu^2)^{1/2} |\partial_{zz} F(z)| & \leq \Lambda(|z|^2 + \mu^2)^{(p-1)/2} \\
\Lambda^{-1}(|z|^2 + \mu^2)^{(p-2)/2} |\xi|^2 & \leq \partial_{z\xi} F(z) \xi \cdot \xi
\end{align*}
\]

(1.4)
for all $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, where $\mu \in [0, 1]$ and $\Lambda \geq 1$ are fixed constants. The function $\Phi : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy

\begin{equation}
\begin{aligned}
\Phi(t) \in C^1(\mathbb{R}), & \quad t \mapsto \Phi(t) \text{ is convex} \\
\Lambda^{-1}|t|^{\gamma} \leq \Phi(t) \leq \Lambda|t|^{\gamma}, & \quad \Lambda^{-1}|t|^{\gamma} \leq \Phi'(t)t \leq \Lambda|t|^{\gamma}
\end{aligned}
\end{equation}

for all $t \in \mathbb{R}$. The kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies

\begin{equation}
1 \leq \frac{1}{|x - y|^{n+\gamma}} \leq K(x, y) \leq \Lambda \frac{1}{|x - y|^{n+\gamma}}
\end{equation}

for all $x, y \in \mathbb{R}^n$, $x \neq y$. As already mentioned, unless otherwise stated, $p, s, \gamma$ are such that $p, \gamma > 1 > s > 0$, with $p > s\gamma$. In order to get gradient continuity of minimizers, we consider the following requirements on $f$, $g$ and the boundary $\partial \Omega$:

\begin{equation}
\begin{aligned}
\partial \Omega & \in C^{2,\alpha}, \quad \alpha \in (0, 1) \\
f & \in L^d(\Omega), \quad d > n \\
g & \in W^{1,p}(\Omega) \cap W^{s,\gamma}(\mathbb{R}^n) \cap W^{n,\chi}(\mathbb{R}^n) \\
q & > p, \quad a > s, \quad \chi > \gamma, \quad \kappa := \min\{1 - n/q, a - n/\chi\} > 0
\end{aligned}
\end{equation}

In particular, this implies that $q, a\chi > n$. Interior Hölder estimates need less; for this, we shall replace (1.7) by the weaker

\begin{equation}
\begin{aligned}
\partial \Omega & \in C^{0,1}, \quad f \in L^n(\Omega) \\
g & \in W^{1,p}(\Omega) \cap W^{s,\gamma}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)
\end{aligned}
\end{equation}

Note that $W^{n,\chi}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ holds provided $a - n/\chi > 0$. Moreover, in the following, letting $f \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, we can always take $f \in L^d(\mathbb{R}^n)$ and $f \in L^n(\mathbb{R}^n)$ in (1.7) and (1.8), respectively. Conditions (1.4)-(1.8) lead to consider the following natural functional setting:

\begin{equation}
\begin{aligned}
X_g(\Omega) & := \left\{ w \in g + W^{1,p}(\Omega) \cap W^{s,\gamma}(\mathbb{R}^n) : w \equiv g \text{ in } \mathbb{R}^n \setminus \Omega \right\} \\
X_0(\Omega) & := \left\{ w \in W^{1,p}(\Omega) \cap W^{s,\gamma}(\mathbb{R}^n) : w \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\}
\end{aligned}
\end{equation}

Note that some ambiguity arises in the definition of $X_g$; in fact, this is actually meant as the subspace of functions $w \in W^{s,\gamma}(\mathbb{R}^n)$ whose restriction on $\Omega$ belongs to $g + W^{1,p}(\Omega)$. Compare for instance with the discussion made in [5, 8], where related functional settings are considered. Under assumptions (1.4)-(1.6) and (1.8), there exists a unique solution $u \in X_g(\Omega)$ to

\begin{equation}
X_g(\Omega) \ni u \mapsto \min_{w \in X_g(\Omega)} J(w).
\end{equation}

Moreover

\begin{equation}
\int_\Omega \left[ \partial_t F(Du) \cdot D\varphi - f\varphi \right] \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi'(u(x) - u(y))(\varphi(x) - \varphi(y)) K(x, y) \, dx \, dy = 0
\end{equation}

holds for every $\varphi \in X_0(\Omega)$. The proof of these facts is quite standard, and relies on the application of Direct Methods of the Calculus of Variations. The details can be found for instance in [23] Sections 3.3-3.5, where actually a more delicate case of mixed operators is considered. As for the derivation of the Euler-Lagrange equation, this is standard once (1.4)-(1.6) are assumed, and, for the nonlocal part, proceeds as in [23][25]. Assumptions (1.4)-(1.8) come along with two different lists of parameters (the data of the problem) that we shall use to simplify the dependence on the various constants. These are

\begin{equation}
\begin{aligned}
data_{ab} & := (n, p, s, \gamma, \Lambda, \|f\|_{L^p(\Omega)}, \|g\|_{W^{1,p}(\Omega)}, \|g\|_{W^{s,\gamma}(\mathbb{R}^n)}, \|g\|_{L^\infty(\Omega)}, \Omega) \\
data_{ab} & := (n, p, s, \gamma, \Lambda) \text{ if } \gamma \leq p \quad \text{and } data_{ab} := data_{ab} \text{ if } \gamma > p \\
data & := (n, p, s, \gamma, \|f\|_{L^p(\Omega)}, \|g\|_{W^{1,p}(\Omega)}, \|g\|_{W^{s,\gamma}(\mathbb{R}^n)}, \|g\|_{W^{n,\chi}(\mathbb{R}^n)}, \Omega)
\end{aligned}
\end{equation}

For the sake of brevity we shall sometimes indicate a dependence of a constant $c$ on one of the lists in (1.11), also when it will actually occur on a subset of the parameters involved. For example, a constant $c$ depending only on $n, p, s, \gamma$ might be still indicated as $c \equiv c(data_{ab})$.

**Theorem 2 (Almost Lipschitz local continuity).** Under assumptions (1.4)-(1.6) and (1.8), let $u \in X_g(\Omega)$ be as in (1.9). Then $u \in C_{loc}^{\alpha,\alpha}(\Omega)$ for every $\alpha \in (0, 1)$ and, for every open subset $\Omega_0 \ni 0$, $|u|_{\alpha,\alpha,\Omega_0} \leq c$ holds with $c \equiv c(data_{ab}, \alpha, \text{dist}(\Omega_0, \partial \Omega))$. 
Theorem 3 (Global Hölder continuity). Under assumptions (1.4)–(1.7), let \( u \in X_\delta(\Omega) \) be as in (1.9). Then \( u \in C^{0,\alpha}(\mathbb{R}^n) \) for every \( \alpha < \kappa \) and \( |u|_{C^{0,\alpha};\mathbb{R}^n} \leq c(\text{data}) \). In particular, if in addition \( g \in W^{1,\infty}(\mathbb{R}^n) \), then \( u \in C^{0,\alpha}(\mathbb{R}^n) \) for every \( \alpha < 1 \).

Theorem 4 (Gradient local Hölder continuity). Under assumptions (1.4)–(1.7) with \( \kappa \geq s \), let \( u \in X_\delta(\Omega) \) be as in (1.9). Then there exists \( \alpha \equiv \alpha(n,p,s,\gamma,d,\Lambda) \in (0,1) \), such that \( Du \in C^{0,\alpha}_{\text{loc}}(\Omega;\mathbb{R}^n) \) and, for every open subset \( \Omega_0 \Subset \Omega \), \( |Du|_{C^{0,\alpha}\Omega_0} \leq c \) holds with \( c \equiv c(\text{data}, dist(\Omega_0,\partial\Omega)) \).

Let us briefly comment on the assumptions considered in Theorems 2–4. These are essentially sharp. For instance, the statement of Theorem 2 does not hold when only assuming that \( f \in L^1 \), for any \( t < n \). As for Theorem 4, one cannot obtain in general the gradient Hölder continuity only assuming that \( W^{n,\infty} \subset C^{0,\alpha} \) (see [26, Theorem 8.2]), which is essentially known for the classical Sobolev space. In fact, in this setting one can only expect regularity of solutions obtained matches with the one of the boundary data. Note that, accordingly, Morrey–Sobolev embedding gives \( W^{1,\infty} \subset C^{0,\alpha} \) in other words, assumption (1.7) encodes the necessary Hölder continuity of the boundary data \( g \) both with respect to the Sobolev space related to the local part of the functional in (1.2), and with respect to the nonlocal one.

Theorems (1.4) come along with explicit a priori estimates. These can be directly inferred from the proofs and whose shape reflects the optimal approach used here. For brevity we confine ourselves to report the a priori estimate related to Theorem 4. This is in the next

Theorem 5 (Campanato type estimate for Theorem 4). Under assumptions (1.4)–(1.6) and (1.8), let \( u \in X_\delta(\Omega) \) be as in (1.9). For every \( \alpha < 1 \) there exists \( r_\alpha \equiv r_\alpha(\text{data}_\alpha,\alpha) > 0 \) such that

\[
\int_{B_{r_\alpha}} |u-(u)_{B_{r_\alpha}}|^p \, dx \leq c \left( \frac{p}{r} \right) \alpha p \left[ \int_{B_{r_\alpha}} |u-(u)_{B_{r_\alpha}}|^p \, dx + r^{\alpha p} \|f\|_{L^p(B_{r_\alpha})} \right]
+ c \left( \frac{p}{r} \right) \alpha p \int_{\mathbb{R}^n \setminus B_{r_\alpha}} |u-(u)_{B_{r_\alpha}}|^\gamma \, d\lambda_{x_0}, \quad d\lambda_{x_0}(x) := \frac{dx}{|x-x_0|^{n+s+\gamma}}
\]

(1.12)

holds whenever \( B_{r_\alpha} = B_{r_\alpha}(x_0) \subset B_r(x_0) \equiv B_r \subset \Omega \) are concentric balls with \( r \leq r_\alpha \), where \( c \equiv c(\text{data}_\alpha,\alpha) \).

We note that, when \( \gamma > p \) (keep (1.1) in mind), the constant \( c \) in (1.12) only depends on \( n,p,s,\gamma,\Lambda \). Therefore in this case (1.12), if reduced to the content of the first line, gives back the classical Campanato type decay estimate for solutions to local non-homogeneous equations (see for instance [27, Theorem 7.7]). As it is well-known, such decay estimates on the integral average of \( u-(u)_{B_r} \) imply the local \( C^{0,\alpha} \)-regularity of solutions. Instead, the second line of (1.12) encodes the long-range interactions due to the presence of the nonlocal term in the functional. In this respect, the average \( u-(u)_{B_r} \), is performed with respect to a suitable measure on the complement of \( B_r \); the resulting term is often called \( \text{snail} \), it is essentially the nonlocal counterpart of the integrals appearing in the first line and some variations of it are of common use in nonlocal problems (see Section 3 for more). In the range \( \gamma > p \), the nonlocal term exhibits a growth larger than the local one, and a careful analysis of the proofs, actually reveals that the constant \( c \) appearing in (1.12), depends on \( n,p,s,\gamma,\Lambda \) and \( \|u\|_{L^\infty(\mathbb{R}^n)} \) (see Remark 11 for details). This typically happens in all those situations when anisotropic operators are considered, especially in the setting of nonuniformly elliptic problems (see for instance the a priori estimates in [21,23,24]). Apart from this unavoidable detail, the shape of (1.12) still neatly reproduces the one known for the classical local case. We note that estimate (1.12) can be further improved including the decay rate of the last term appearing in (1.12); this follows from the estimate on certain (fractional) sharp maximal operators considered in Section 4.2, estimate (4.11), eventually implying (1.12).

1.2. Possible extensions. The results in this paper can be extended in several directions. For instance, one can consider more general functionals of the type

\[
w \mapsto \int_\Omega F(x,Dw) - fw \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(w(x) - w(y)) K(x,y) \, dx \, dy,
\]

where this time we assume that \( z \mapsto F(x,z) \) satisfies (1.4) uniformly with respect to \( x \in \Omega \). The assumption regulating coefficients is

\[
|\partial_x F(x,z) - \partial_y F(y,z)| \leq \Lambda \omega(\|x-y\|(|z|^2 + \mu^2)^{(p-1)/2}),
\]

(1.13)
to hold for every choice \( x, y \in \Omega \) and \( z \in \mathbb{R}^n \). Here \( \omega : [0, \infty) \to [0, 1) \) is a modulus of continuity, that is, a continuous and non-decreasing function, such that \( \omega(0) = 0 \). Under assumption (1.13), it is then easy to see that Theorems 2-3 continue to hold. In order to get an analog of Theorem 3 we assume in addition that \( \omega(t) \leq t^\alpha \) holds for some \( \sigma \in (0, 1) \), this condition being necessary; then the Hölder exponent of \( Du \) does not exceed \( \sigma \). We note the proof of these assertions is in fact implicit in the proof of boundary regularity provided in Proposition 5.1 below.

Another extension, already mentioned above, is about general solutions to nonlinear integrodifferential operators, not necessarily coming from integral functionals. Moreover, a purely local regularity provided in Proposition 5.1 below.

(1.14)

\[
\begin{align*}
\left| A(\zeta) + (|\zeta|^2 + \mu^2)^{1/2} \partial_\zeta A(\zeta) \right| & \leq A(|\zeta|^2 + \mu^2)^{(p-1)/2} \\
\Lambda^{-1}(|\zeta|^2 + \mu^2)^{(p-2)/2}\xi^2 & \leq \partial_\zeta A(\xi) \cdot \xi \\
\Lambda^{-1}|t|^\gamma & \leq \Psi(t) t \leq \Lambda|t|^\gamma
\end{align*}
\]

with the same meaning of (1.13)-(1.15). Note that the classical \( p \)-Laplacean operator given by \( A(\zeta) \equiv |\zeta|^{p-2}\zeta \) is covered by (1.14). We consider functions \( u \in W^{1, p}(\Omega) \cap W^{s, \gamma}(\mathbb{R}^n) \), where \( \Omega \subset \mathbb{R}^n \) is as usual a bounded and Lipschitz-regular domain, such that

\[
\int_\Omega A(Du) \cdot D\varphi - f \varphi \, dx + \int_{\partial \Omega} \int_{\mathbb{R}^n} \Psi(u(x) - u(y))(\varphi(x) - \varphi(y)) K(x, y) \, dx \, dy = 0
\]

holds for every \( \varphi \in \mathcal{X}_0(\Omega) \). Note that here no boundary datum \( g \) appears. The definition of solution is instead purely local. In this case we have

**Theorem 6.** Under assumptions (1.10) and (1.14), let \( u \in W^{1, p}(\Omega) \cap W^{s, \gamma}(\mathbb{R}^n) \) be a solution to (1.15).

- If \( u \in L^\infty(\mathbb{R}^n) \) and \( f \in L^n(\Omega) \), then \( u \in C^{0, \alpha}(\Omega) \) for every \( \alpha \in (0, 1) \).
- If \( u \in \mathcal{X}_0^0(\Omega) \) and conditions (1.14) hold, then \( u \in C^{0, \alpha}(\mathbb{R}^n) \) for every \( \alpha < \kappa \).
- If \( u \in \mathcal{X}_0^0(\Omega) \) and conditions (1.14) hold with \( \kappa \geq s \), then \( Du \) is locally Hölder continuous in \( \Omega \).

The proof of Theorem 6 follows verbatim the ones for Theorems 1-4. The assumption \( u \in L^\infty(\mathbb{R}^n) \) in the first bullet point is put here in order to recover Theorem 6 from the proof of Theorems 1-4 and make it self-contained with respect to the content of this paper. In turn, this assumption can be dropped in several cases by adapting a localized De Giorgi fractional iteration of the type reported in Section 2.4 for the global case (see for instance [28, 33]).

2. Preliminaries

2.1. Notation. Unless otherwise specified, we denote by \( c \) a general constant larger or equal to 1. Different occurrences from line to line will be still denoted by \( c \). Special occurrences will be denoted by \( c_* \) or \( c_1 \). Relevant dependencies on parameters will be as usual emphasized by putting them in parentheses. In the following, given \( a \in \mathbb{R} \), we denote \( a_+ := \max\{a, 0\} \). We denote by \( B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\} \) the open ball with center \( x_0 \) and radius \( r > 0 \); we omit denoting the center when it is not necessary, i.e., \( B \equiv B_r \equiv B_r(x_0) \); this especially happens when various balls in the same context will share the same center. With \( B_r^+(x_0) \) we mean the upper half ball \( B_r(x_0) \cap \{x \in \mathbb{R}^n : x_n > 0\} \); in connection, we denote \( \Gamma_r(x_0) := B_r^+(x_0) \cap \{x_n = 0\} \), whenever \( x_0 \in \{x_n = 0\} \). Moreover, given a domain \( \Omega \subset \mathbb{R}^n \), with \( \mathcal{B} \subset \mathbb{R}^n \) being a measurable subset with respect to a Borel (non-negative) measure \( \lambda_0 \) in \( \mathbb{R}^n \), with bounded positive measure \( 0 < \lambda_0(\mathcal{B}) < \infty \), and with \( b : \mathcal{B} \to \mathbb{R}^k \), \( k \geq 1 \), being a measurable map, we denote

\[
(b)_\mathcal{B} \equiv \int_{\mathcal{B}} b(x) \, d\lambda_0(x) := \frac{1}{\lambda_0(\mathcal{B})} \int_{\mathcal{B}} b(x) \, d\lambda_0(x)
\]

According to the standard notation, given \( b : \mathcal{B} \to \mathbb{R}^k \), we denote

\[
|b|_{0, \alpha, \mathcal{B}} := \sup_{x, y \in \mathcal{B}, x \neq y} \frac{|b(x) - b(y)|}{|x - y|^\alpha}, \quad \text{osc } b := \sup_{x \in \mathcal{B}} |b(x) - b(y)|
\]

for \( 0 < \alpha \leq 1 \) and \( \mathcal{B} \subset \mathbb{R}^n \) being a set.
2.2. Fractional spaces. For $\gamma \geq 1$ and $s \in (0,1)$, the space $W^{s,\gamma}(\mathbb{R}^n)$ is defined via

$$W^{s,\gamma}(\mathbb{R}^n) := \left\{ w \in L^\gamma(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+\gamma}} \, dx \, dy < \infty \right\},$$

and it is endowed with the norm

$$\|w\|_{W^{s,\gamma}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |w|^\gamma \, dx \right)^{1/\gamma} + \left( \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^{\gamma}}{|x-y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma}.\]$$

With $w \in W^{s,\gamma}(\mathbb{R}^n)$, we also denote

$$[w]_{s,\gamma;A} := \left( \int_A \int_A \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma}$$

whenever $A \subset \mathbb{R}^n$ is measurable. In a similar way, by replacing $\mathbb{R}^n$ by $\Omega$ in the domain of integration, it is possible to define the fractional Sobolev space $W^{s,\gamma}(\Omega)$ in an open domain $\Omega \subset \mathbb{R}^n$. Good general references for fractional Sobolev spaces are [1, 30]. For the next result, see also [2] and related references.

Lemma 2.1 (Fractional Poincaré). Let $\gamma \in [1, \infty)$, $s \in (0,1)$, $B_0 \subset \mathbb{R}^n$ be a ball. If $w \in W^{s,\gamma}(B_0)$, then

$$\left( \int_{B_0} |w - (w)_{B_0}|^\gamma \, dx \right)^{1/\gamma} \leq c g^s \left( \int_{B_0} \int_{B_0} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma},$$

holds with $c \equiv c(n,s,\gamma)$.

Proof. By standard rescaling - i.e., passing to $B_0 \ni x \mapsto w(x_0 + gx)$, with $x_0$ being the center of $B_0$ - we can reduce to the case $B_0 \equiv B_1(0)$. The assertion then follows by [30] Proposition 2.2 and standard Poincaré’s inequality, as $w \in W^{1,\infty}(B_1)$. \qed

Lemma 2.2 (Embedding). Let $1 \leq \gamma \leq p < \infty$, $s \in (0,1)$ and $B_0 \subset \mathbb{R}^n$ be a ball. If $w \in W^{1,p}_0(B_0)$, then $w \in W^{s,\gamma}(B_0)$ and

$$\left( \int_{B_0} \int_{B_0} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma} \leq c g^{1-s} \left( \int_{B_0} |Dw|^p \, dx \right)^{1/p},$$

holds with $c \equiv c(n,p,s,\gamma)$.

Proof. Note that, on the contrary to the rest of the paper, here we are allowing $p = s\gamma$; this is not really needed in what follows, but we include this case for completeness. Again we can assume that $B_0(x_0) \equiv B_1(0)$, and, letting $w \equiv 0$ outside $B_1(0)$, we can assume $w \in W^{1,p}_0(\mathbb{R}^n) \cap L^\infty(B_0)$, then $w \in W^{s,\gamma}(B_0)$ and

$$\left( \int_{B_0} \int_{B_0} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma} \leq c \|w\|_{L^\infty(B_0)} \left( \int_{B_0} |Dw|^p \, dx \right)^{s/p},$$

holds with $c \equiv c(n,p,s,\gamma)$.

Proof. Note that, on the contrary to the rest of the paper, here we are allowing $p = s\gamma$; this is not really needed in what follows, but we include this case for completeness. Again we can assume that $B_0(x_0) \equiv B_1(0)$, and, letting $w \equiv 0$ outside $B_1(0)$, we can assume $w \in W^{1,p}_0(\mathbb{R}^n) \cap L^\infty(B_0)$, then $w \in W^{s,\gamma}(B_0)$ and

$$\left( \int_{B_0} \int_{B_0} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma} \leq c \|w\|_{L^\infty(B_0)} \left( \int_{B_0} |Dw|^p \, dx \right)^{s/p},$$

holds with $c \equiv c(n,p,s,\gamma)$.
with \( c \equiv c(n,p,s,\gamma) \), that is (2.2) when \( s\gamma < p \). On the other hand, if \( s\gamma = p \), we use (12) Corollary 3.2, (c), that is \( \|w\|_{L^\infty(B_\theta)} \leq c\|w\|_{L^\infty(\mathbb{R}^n)}^{1-s} \), that holds whenever \( \theta \in (0,1) \), where \( c \equiv c(n,\sigma,\lambda,\theta) \). We use this with \( \sigma = 1, \lambda = p, \theta = s \) and get

\[
[w]_{L^\infty(B_\theta)} \leq c\|w\|_{L^\infty(B_\theta)} \leq c\|w\|_{L^\infty(\mathbb{R}^n)}^{1-s} \parallel Dw\parallel_{L^p(B_\theta)},
\]

with \( c \equiv c(n,p,s) \), and the proof is complete.

We find it useful to have a unified reformulation of Lemmas 2.2, 2.3. For this, we introduce, with reference to the exponents \( p, s, \gamma \) considered in Theorems 1.1 the following quantities:

\[
\vartheta := \begin{cases} 
  s & \text{if } \gamma > p, \\
  1 & \text{if } \gamma \leq p,
\end{cases}
\]

and we set

\[
A_\gamma := 1 & \text{if } \gamma > p \text{ and } 0 \text{ otherwise},
\]

\[
B_\gamma := 1 & \text{if } \gamma < p \text{ and } 0 \text{ otherwise},
\]

\[
C_\gamma := 1 & \text{if } \gamma = p \text{ and } 0 \text{ otherwise}.
\]

Note that \( A_\gamma + B_\gamma + C_\gamma = 1 \). With this definition we note that (1.3) translates into

\[
\vartheta \neq \gamma \implies p > \vartheta \gamma \quad \text{and} \quad p \geq \vartheta \gamma.
\]

We can now summarize the parts we need of Lemmas 2.2 and 2.3 in the following:

**Lemma 2.4.** Let \( p, \gamma > 1, s \in (0,1) \) be such that \( s\gamma \leq p \). If \( w \in W^{1,p}_0(B_\theta) \cap L^\infty(B_\phi) \), then \( w \in W^{s,\gamma}(B_\phi) \) and

\[
\left( \int_{B_\phi} \int_{B_\phi} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+s\gamma}} \, dx \, dy \right)^{1/\gamma} \leq c\|w\|_{L^\infty(B_\phi)} \vartheta^{s-s} \left( \int_{B_\phi} | Dw |^p \, dx \right)^{\vartheta/p},
\]

holds with \( c \equiv c(n,p,s,\gamma) \).

### 2.3. Miscellaneous

We shall often use the auxiliary vector field \( V_\mu : \mathbb{R}^n \to \mathbb{R}^n \), defined by

\[
V_\mu(z) := (|z|^2 + \mu^2)^{(p-2)/2} z
\]

whenever \( z \in \mathbb{R}^n \), where \( p \in (1, \infty) \) and \( \mu \in [0,1] \) are as in (1.1). It follows that

\[
|V_\mu(z_1) - V_\mu(z_2)| \approx (|z_1|^2 + |z_2|^2 + \mu^2)^{(p-2)/2} |z_1 - z_2|,
\]

where the equivalence holds up to constants depending only on \( n, p \). A standard consequence of (1.1) is the following strict monotonicity inequality:

\[
|V_\mu(z_1) - V_\mu(z_2)|^2 \leq c(\partial_\nu F(z_2) - \partial_\nu F(z_1)) : (z_2 - z_1)
\]

holds whenever \( z_1, z_2 \in \mathbb{R}^n \), where \( c \equiv c(n,p,\Lambda) \). The two inequalities in the last two displays are in turn based on the following one

\[
\int_0^1 (|z_1 + \lambda(z_2 - z_1)|^2 + \mu^2)^{1/2} \, d\lambda \approx_{n,t} (|z_1|^2 + |z_2|^2 + \mu^2)^{1/2}
\]

that holds whenever \( t > -1 \) and \( z_1, z_2 \in \mathbb{R}^n \) are such that \( |z_1| + |z_2| + \mu > 0 \). As a consequence of (2.9) and (2.10), it also follows that

\[
|z|^p \leq c(\partial_\nu F(z)) \cdot z + c\mu^p
\]

holds for every \( z \in \mathbb{R}^n \), where, again, it is \( c \equiv c(n,p,\Lambda) \) for the facts in the last four displays see for instance [2,23,28] and related references. Finally, three classical iteration lemmas. The first one can be obtained by [37] Lemma 6.1 after a straightforward adaptation. Lemma 2.4 comes via a reading of the proof of (the very similar) [36] Lemma 2.2. Finally, Lemma 2.7 is nothing but De Giorgi’s geometric convergence lemma [37] Lemma 7.1.

**Lemma 2.5.** Let \( h : [0, q_1) \to \mathbb{R} \) be a non-negative and bounded function, and let \( \theta \in (0,1) \), \( a_i, \gamma_i, b \geq 0 \) be numbers, \( i \leq k \in \mathbb{N} \). Assume that

\[
h(t) \leq \theta h(s) + \sum_{i=1}^k \frac{a_i}{(s - t)\gamma_i} + b
\]

holds whenever \( q_0 \leq t < s \leq q_1 \). Then

\[
h(q_0) \leq c\sum_{i=1}^k \frac{a_i}{(q_1 - q_0)\gamma_i} + cb
\]
holds too, where $c \equiv c(\theta, \gamma_i)$.

Lemma 2.6. Let $h : [0, r_0] \to \mathbb{R}$ be a non-negative and non-decreasing function such that the inequality $h(t) \leq a(t \gamma_0)^\beta + c h(t) + a_0^\beta$ holds whenever $0 \leq t \leq \gamma_0$, where $a > 0$ and $0 < \beta < n$. For every positive $\beta < n$, there exists $\varepsilon_0 \equiv \varepsilon_0(a, n, \beta, \beta)$ such that, if $\varepsilon \leq \varepsilon_0$, then $h(t) \leq c(t \gamma_0)^\beta h(t) + c\varepsilon$ holds too, whenever $0 \leq t \leq \gamma_0$, where $c \equiv c(a, n, \beta, \beta)$. 

Lemma 2.7. Let $t > 0$ and $\{\tilde{v}_i\}_{i \in \mathbb{N}} \subset [0, \infty)$ be such that $\tilde{v}_{i+1} \leq c_a \tilde{v}_i^{1+\varepsilon}$ holds for every $i \geq 0$, with $c_a > 0$, $a \geq 1$ and $t > 0$. If $\tilde{v}_0 \leq c_a^{-1/t} \tilde{v}_1^{-1/\gamma_0}$, then $\tilde{v}_i \leq c_a^{-i/t} \tilde{v}_0$ holds for every $i \geq 0$ and hence $\tilde{v}_i \to 0$.

2.4. Global boundedness. Instrumental to the proof of Theorems 4.1 is the boundedness of minimizers. This proceeds via a variation of the classical De Giorgi’s iteration scheme (see for instance [5, Theorem 4.7], [37, Chapter 7]), and we report the full details for completeness in the subsequent

Proposition 2.1. Under assumptions [1.4]–[1.6] and [1.8], let $u \in \mathcal{X}_a(\Omega)$ be as in [1.9]. There exists a constant $c \equiv c(data_u)$ such that $\|u\|_{W^{1, p}(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^n)} \leq c$.

Proof. We denote the Sobolev conjugate exponent $p^* = np/(n - p)$ when $p < n$, and $p^* = np/(n - 1)$ when $p \geq n$. By the minimality of $u$, Sobolev and Young’s inequalities, we get, after a few standard manipulations involving in particular [1.4], [1.5] and [1.6]

$$
\int_\Omega |Du|^p \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^\gamma}{|x - y|^{n + \gamma}} \, dx \, dy
\leq c \int_{\mathbb{R}^n} |Du|^p \, dx + c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x) - g(y)|^\gamma}{|x - y|^{n + \gamma}} \, dx \, dy + c \left( \int_\Omega |f(x)|^p \, dx \right)^{\frac{n}{n-p}},
$$

c \equiv c(n, p, \gamma, \Lambda, \Omega).$$

Note that this still holds for critical points, i.e., solutions to [1.10], and therefore connects to the setting of Theorem 6 this goes via the use of [2.12]. Using Sobolev inequality of the left-hand side of the inequality in the above display yields

$$
\|u\|_{L^{p^*}(\Omega)} \leq c\|f\|_{L^{p/(p-1)}(\Omega)} + c(data_u) : = M \equiv M(data_u),
$$

with $c \equiv c(n, p, \gamma, \Lambda, \Omega)$. This implies the bound $\|u\|_{W^{1, p}(\Omega)} \leq c(data_u)$. It remains to prove a similar bound for $\|u\|_{L^\infty(\mathbb{R}^n)}$. We start taking $m$ large enough to have

$$
\|u\|_{L^\infty(\mathbb{R}^n)} \leq M^{1/p} + 1.
$$

Eventually, we shall further enlarge the above lower bound on $m$. For $i \in \mathbb{N}_0$, define the increasing sequence $\{\kappa_i\}_{i \in \mathbb{N}_0} := \{2m(1 - 2^{-i-1})\}_{i \in \mathbb{N}_0}$ so that $2m \geq \kappa_i \geq m$ holds for all $i \in \mathbb{N}_0$. By [2.13] and $u \in \mathcal{X}_a(\Omega)$, we see that $v_i := \{u - \kappa_i\}_+ \in \mathcal{X}_a(\Omega)$ for all $i \in \mathbb{N}_0$. Testing [1.10] against $v_{i+1}$ we have

$$
0 = \int_\Omega \left[ \partial_i F(Du) \cdot Du_{i+1} - f_{n+1} \right] \, dx
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi'(u(x) - u(y))(v_{i+1}(x) - v_{i+1}(y)) K(x, y) \, dx \, dy =: (I) + (II)
$$

for every $i \geq 0$. Using [2.12], Sobolev embedding and H"older’s inequalities yield

$$
(I) \geq \frac{1}{c} \|v_{i+1}\|_{L^{p^*}(\Omega)}^p - c\|f\|_{L^n(\Omega)} \|v_{i+1}\|_{L^{p^*}(\Omega)} |\Omega \cap \{v_{i+1} > 0\}|^{(1/p^*)-1/p^*}
$$

(2.15)

for $c \equiv c(n, p, \Lambda)$.

To estimate term (II), first consider the case $u(x) > \kappa_{i+1}$ and $u(y) > \kappa_{i+1}$, when we have, via [1.16]

$$
\Phi'(u(x) - u(y))(v_{i+1}(x) - v_{i+1}(y)) = \Phi'(v_{i+1}(x) - v_{i+1}(y))(v_{i+1}(x) - v_{i+1}(y)) \geq \Lambda^{-1} |v_{i+1}(x) - v_{i+1}(y)|^\gamma.
$$

On the other hand, when $u(x) > \kappa_{i+1}$ and $u(y) \leq \kappa_{i+1}$, by [1.15] it is

$$
\Phi'(u(x) - u(y))(v_{i+1}(x) - v_{i+1}(y)) = \Phi'((u(x) - \kappa_{i+1})_+ + (\kappa_{i+1} - u(y))_+)(u(x) - \kappa_{i+1})_+ + (\kappa_{i+1} - u(y))_+ v_{i+1}(x)
$$

$$
\geq \Lambda^{-1} |v_{i+1}(x) + (\kappa_{i+1} - u(y))_+ v_{i+1}(x)|^\gamma \geq \Lambda^{-1} |v_{i+1}(x) - v_{i+1}(y)|^\gamma.
$$

where $\kappa_{i+1} = \kappa_i + 1$.

On the other hand, when $u(x) > \kappa_{i+1}$ and $u(y) \leq \kappa_{i+1}$, by [1.15] it is

$$
\Phi'(u(x) - u(y))(v_{i+1}(x) - v_{i+1}(y)) = \Phi'((u(x) - \kappa_{i+1})_+ + (\kappa_{i+1} - u(y))_+)(u(x) - \kappa_{i+1})_+ + (\kappa_{i+1} - u(y))_+ v_{i+1}(x)
$$

$$
\geq \Lambda^{-1} |v_{i+1}(x) + (\kappa_{i+1} - u(y))_+ v_{i+1}(x)|^\gamma \geq \Lambda^{-1} |v_{i+1}(x) - v_{i+1}(y)|^\gamma.
$$

where $\kappa_{i+1} = \kappa_i + 1$.
In the opposite situation, i.e. when \( u(x) \leq \kappa_{i+1} \) and \( u(y) > \kappa_{i+1} \), again by (1.5) we have

\[
\Phi'(u(x) - u(y))(v_{i+1}(x) - v_{i+1}(y)) = -\Phi'(u(y) - \kappa_{i+1}) + (\kappa_{i+1} - u(x))_+ v_{i+1}(y)
\]

where \( \Phi'(z) = \frac{\Phi(z)}{z} \). Integrating from \( a \) to \( b \) and using (2.18), we obtain

\[
\Phi(v_{i+1}(x) + (\kappa_{i+1} - u(x))_+) - \Phi(v_{i+1}(y) + (\kappa_{i+1} - u(x))_+) \leq \int_a^b \frac{\Phi'(u(z) - u(y))(v_{i+1}(z) - v_{i+1}(y))}{|z - y|^{n-\gamma}} \, dz
\]

Finally, when \( u(x) \leq \kappa_{i+1} \) and \( u(y) \leq \kappa_{i+1} \), it is \( \Phi'(u(x) - u(y))(v_{i+1}(x) - v_{i+1}(y)) = 0 \). Collecting all the above cases and recalling (1.6), (2.19) and (2.20), it then follows that

\[
\|v_{i+1}\|_{L^{p'}(\Omega)}^p \leq \|u\|_{L^{p'}(\Omega)}^p \|\nabla u\|_{L^{p'}(\Omega)} + (\kappa_{i+1} - u(x))_+ v_{i+1}(y)
\]

with \( c \equiv c(\text{data}) \). Setting \( \tilde{v}_i := m^{-p}\|v_{i+1}\|_{L^{p'}(\Omega)} \), (2.13) and (2.14) imply \( \tilde{v}_i \leq 1 \), and (2.17) reads as

\[
\tilde{v}_{i+1} \leq c\tilde{v}_i \frac{(i+1)(p'-1)\gamma}{p'} \frac{m^{-p}\|v_i\|_{L^{p'}(\Omega)}}{m^{-p}\|v_i\|_{L^{p'}(\Omega)}} + c\frac{2(i+1)p^{-p'}}{p'} \leq c\tilde{v}_i
\]

for \( c \equiv c(\text{data}) \) and \( t := \frac{p'}{p} - 1 > 0 \). In addition to (2.14), we increase \( m \) in such a way that \( m \geq c_1 \frac{2^{i/2}}{2^{i/2}} M \) that implies, via (2.13), \( \tilde{v}_0 \leq c_1^{-1/2} \frac{2^{i/2}}{2^{i/2}} \). Lemma 2.1 now applies and gives

\[
0 = \lim_{i \to \infty} \tilde{v}_i = \lim_{i \to \infty} m^{-p} \left( \int_\Omega \tilde{v}_i \, dx \right)^{p'/p} = m^{-p} \left( \int_\Omega (u - 2m) \, dx \right)^{p'/p}
\]

so that \( |\Omega \cap \{u > 2m\}| = 0 \), and therefore \( u \leq 2m \) holds a.e. in \( \Omega \). For a lower bound, set \( \tilde{g} := -g \in X(g; \Omega) \), \( \tilde{f} := -f \in L^p(\Omega) \) and consider functional \( X_0(\Omega) \ni w \to \tilde{F}(w) \), where

\[
\tilde{F}(z) := F(-z), \quad \tilde{\Phi}(z) := \Phi(-z), \quad \tilde{\Phi}'(z) := \tilde{\Phi}'(-z)
\]

and \( \tilde{\Phi}(\cdot) \) and \( \tilde{\Phi}_i(\cdot) \) satisfy (1.3) and (1.5) and \( \tilde{u} := -u \) is the unique minimizer of \( \tilde{F}(\cdot) \) in \( X_0(\Omega) \). The above argument apply to \( \tilde{u} \) and leads to \( \tilde{u} \leq 2m \) a.e. in \( \Omega \). All in all we have that \( |u| \leq 2m \) a.e. in \( \Omega \) and the proof is complete recalling the way \( m \) has been determined.

\[\square\]

**Remark 2.1.** In the proof of Proposition 2.1 we do not need to assume that \( p > s\gamma \); any choice of \( p, \gamma > 1 \) and \( s \in (0,1) \) suffices. Moreover, the assumption \( f \in L^p(\Omega) \) can be relaxed in \( f \in L^q(\Omega) \), where \( q > n/p \), if \( p < n \) and \( q > 1 \) otherwise, in that case we take \( p' = p(n-1)/n = n' \) when \( p \geq n \). This is in accordance with the classical results for the local equation \(-\Delta_p u = -\div(Du^{p-2}Du) = f \).

### 2.5. Rewriting the Euler-Lagrange equation

Following Section 1.5, let us set

\[
K'(x, y) := \frac{\Phi'(u(x) - u(y))(K(x, y) - u(x))}{|x - y|^{n-\gamma}} \quad \text{if} \quad x \neq y, \quad u(x) \neq u(y)
\]

\[
K'(x, y) := \frac{\Phi'(u(x) - u(y))(K(x, y) - u(x))}{|x - y|^{n-\gamma}} \quad \text{if} \quad x \neq y, \quad u(x) = u(y),
\]

By (1.5) and (2.18), (2.19), it then follows that

\[
K'(x, y) = K'(y, x)
\]

for every \( x, y \in \mathbb{R}^n \) provided \( x \neq y \). Then, changing variables, (1.10) can be rewritten as

\[
\int_\Omega \left[ \partial_\nu \tilde{F}(Du) \cdot D\varphi - f\varphi \right] \, dx
\]

\[
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{-2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x, y) \, dx \, dy = 0
\]

that holds for every \( \varphi \in X_0(\Omega) \). From now on, we shall use (2.21) instead of (1.10).
3. Integral quantities measuring oscillations

In this section we fix two generic functions \( w \) and \( f \), such that, unless otherwise specified, \( w \in W^{1,p}(\Omega) \cap W^{s,\gamma}(\mathbb{R}^n) \) and \( f \in L^p(\mathbb{R}^n) \), and an arbitrary ball \( B_R(x_0) \subset \mathbb{R}^n \). We are going to list a number of basic quantities that will play an important role in this paper. A fundamental tool in the regularity theory of fractional problems is the nonlocal tail, first introduced in [28], which, in some sense, keeps track of long range interactions. In [9], a related nonlocal quantity, called snail, was considered, namely

\[
(3.1) \quad \left( \varphi^\gamma \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|w(x)|^\gamma}{|x-x_0|^{n+\gamma}} \, dx \right)^{1/\gamma}.
\]

The snail can be essentially seen as the \( L^{\gamma} \)-average of \( |w| \) on \( \mathbb{R}^n \setminus B_R(x_0) \) with respect to the measure defined by \( d\lambda_{x_0} := |x-x_0|^{-n-\gamma} \, dx \). We refer to [9, 11, 28, 42, 43, 55] for extra details on this matter. In this paper we use a Campanato-type variation of (3.1), that is

\[
(3.2) \quad \text{snails}(\varrho) \equiv \text{snails}(w, B_R(x_0)) := \left( \varphi^\delta \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|w(y) - (w)_{B_R(x_0)}|^\gamma}{|x-x_0|^{n+\gamma}} \, dy \right)^{1/\gamma}, \quad \delta \geq s\gamma.
\]

Note that \( \text{snails}(w, B_R(x_0)) \leq c(n, s, \gamma) \varphi^\delta \|w\|_{L^\infty(\mathbb{R}^n)} \quad \forall \varrho \leq r < \infty \).

This clearly involves the oscillations of \( u \) and it is a nonlocal version of the more classical object

\[
(3.4) \quad \text{exs}_\lambda(\varrho) \equiv \text{exs}_\lambda(w, B_R(x_0)) := \text{av}_\lambda(w, B_R(x_0)) + \left[ \text{snails}(w, B_R(x_0)) \right]^{\gamma/p}.
\]

With \( \theta \in (0, 1) \) and \( \delta \geq s\gamma \), we further define

\[
(3.5) \quad |\text{rhs}_\lambda(\varrho)|^p \equiv |\text{rhs}_\lambda(B_R(x_0))|^p := \varphi^{-\theta} \left( \|f\|_{L^p(B_R(x_0))} + 1 \right)^p
\]

\[
(3.6) \quad \text{ccp}_{\lambda, s}(\varrho) \equiv \text{ccp}_{\lambda, s}(w, B_R(x_0)) := \varphi^{-p}[\text{av}_\lambda(w, B_R(x_0))]^p + \varphi^{-\gamma}[\text{av}_\gamma(w, B_R(x_0))]^\gamma
\]

\[
(3.7) \quad \text{ccp}_{\lambda}(\varrho) \equiv \text{ccp}_{\lambda}(w, B_R(x_0)) := \varphi^{-p}[\text{av}_\lambda(\varrho)]^p + \varphi^{-\gamma}[\text{snails}(\varrho)]^\gamma + \|f\|_{L^p(B_R(x_0))}^p + 1
\]

\[
(3.8) \quad \left[ \text{gl}_{\lambda, s}(w, B_R(x_0)) \right]^p := |\text{exs}_\lambda(w, B_R(x_0))|^p + |\text{rhs}_\lambda(B_R(x_0))|^p.
\]

Note that

\[
(3.9) \quad p \geq \delta, \varrho \leq 1 \implies \varphi^p \text{ccp}(\varrho) \leq \left[ \text{gl}_{\lambda, s}(\varrho) \right]^p.
\]

Abbreviations above such as \( \text{av}_\lambda(\varrho) \equiv \text{av}_\lambda(w, B_R(x_0)) \), \( \text{ccp}_{\lambda, s}(\varrho) \equiv \text{ccp}_{\lambda, s}(w, B_R(x_0)) \), and the like, will be made in the following whenever there will be no ambiguity on what \( w \) and \( B_R(x_0) \) are. Of course all the quantities defined above also depend on \( f \), but this dependence will be omitted as it will be clear from the context. The motivation for the notation above is that terms of the type \( \text{rhs}_{\lambda}(\varrho) \) appear as right-sides quantities of certain inequalities related to equations as in [1, 10]. Terms of the type \( \text{ccp}_{\lambda}(\varrho) \) will instead occur in certain Caccioppoli type inequalities.

**Lemma 3.1.** Let \( B_t(x_0) \subset B_R(x_0) \) be two concentric balls, \( \gamma \geq 1, \delta \geq s\gamma \) and \( w \in W^{s,\gamma}(\mathbb{R}^n) \).

- Whenever \( 0 < t < \varrho \leq 1 \), it holds that

\[
(3.10) \quad \text{snails}(w, B_t(x_0)) \leq c \left( \frac{t}{\varrho} \right)^{\delta/s} \text{snails}(w, B_R(x_0)) + ct^{\delta/s - \gamma} \int_t^{\varrho} \left( \frac{t}{\nu} \right)^{\delta/s} \text{av}_\nu(w, B_t(x_0)) \, d\nu,
\]

with \( c \equiv c(n, s, \gamma) \).
• With \( q \geq 1 \), if \( \nu > 0 \) and \( \theta \in (0, 1) \) are such that \( \theta \nu \leq \nu \leq \theta \), then

\[
\text{AV}_q(w, B_\nu(x_0)) \leq 2\theta^{-n/q} \text{AV}_q(w, B_\nu(x_0)).
\]

Proof. In the following all the balls will be centred at \( x_0 \). Let us first recall the standard property

\[
\left(\int_{B_\nu} |w - (w)_{B_\nu}|^q dx\right)^{1/q} \leq 2 \left(\int_{B_\nu} |w - w|^q dx\right)^{1/q}
\]

that holds whenever \( w \in \mathbb{R} \) and \( q \geq 1 \); from this follows immediately. For the proof of (3.10), we shall use a few arguments developed in [33]. Let \( B_t \subset B_{\nu} \), we then split

\[
\text{snail}_t(t) \leq c \left(\frac{t}{\theta}\right)^{\delta/\gamma} \text{snail}_\theta(g) + ct^{\delta / \gamma - s} \left(\frac{t}{\theta}\right)^s |(w)_{B_t} - (w)_{B_{\nu}}|.
\]

and Hölder’s inequalities, we estimate, using (3.11)-(3.12) repeatedly

\[
1 \leq ct^{\delta / \gamma - s} \left(\frac{t}{\theta}\right)^s \text{snail}_\theta(g) + ct_1 + ct_2,
\]

where \( c \equiv c(n, s, \gamma) \). We have used

\[
d\lambda_\nu(R^n \setminus B_t) = ct^{-\gamma}, \quad d\lambda_\nu(x) := \frac{dx}{|x - x_0|^{n+\gamma}},
\]

where \( c \equiv c(n, s, \gamma) \). If \( \theta / t < 1 \), also using this last identity, standard manipulations based on (3.11) ensure that \( T_1 + T_2 \leq \text{ct}^{\delta / \gamma - s} (t / \theta)^s \text{AV}_\nu(g) \) holds with \( c \equiv c(n, s, \gamma) \). We can therefore assume that \( t < \theta / t \). This means that there exists \( \lambda \in (1, 1/2) \) and \( \kappa \in \mathbb{N}, \kappa \geq 2 \) so that \( t = \lambda^{-\gamma} \). Using triangle and Hölder’s inequalities, we estimate, using (3.11)-(3.12) repeatedly

\[
T_1 \leq \text{ct}^{\delta / \gamma - s} \left(\frac{t}{\theta}\right)^s |(w)_{B_{\lambda^{-\gamma}}} - (w)_{B_{\nu}}| + \text{ct}^{\delta / \gamma - s} \left(\frac{t}{\theta}\right)^s |(w)_{B_{\lambda^{-\gamma}}} - (w)_{B_{\lambda^{-\gamma}}}|.
\]

with \( c \equiv c(n, s, \gamma) \). For \( T_2 \), we rewrite \( \theta = \lambda^{-\gamma} t \) and estimate, by telescoping and Jensen’s inequality

\[
\left(\int_{B_{\lambda^{-\gamma}}} |w(x) - (w)_{B_{\lambda^{-\gamma}}}|^\gamma dx\right)^{1/\gamma} \leq \text{ct}^{\delta / \gamma - s} \sum_{m=0}^{\infty} \text{AV}_\nu(\lambda^{-m} t),
\]

for \( 0 \leq i \leq k \). Then, via (3.12), (3.15) and the discrete Fubini theorem, we obtain

\[
T_2 \leq \text{ct}^{\delta / \gamma - s} \sum_{i=0}^{\infty} \lambda \sum_{m=0}^{\infty} \text{AV}_\nu(\lambda^{-m} t).
\]
Step 1: Basic Caccioppoli inequality.

4.1. that \( \eta \)

\[
\text{Proof. By (3.14) note that}
\]

\[
\text{permanently assume (1.4)-(1.6) and (1.8) and}
\]

\[
\text{and apply Jensen’s inequality with respect to the concave function}
\]

\[
\text{for } c \equiv c(n,s,\gamma). \text{ Merging the estimates found for } T_1 \text{ and } T_2 \text{ to (3.13), we obtain (3.11).} \]

Lemma 3.2. Let \( w \in L^\infty(\mathbb{R}^n) \) and \( B_t(x_0) \subset \mathbb{R}^n \) be a ball. Then

\[
(3.16) \quad \int_{\mathbb{R}^n \setminus B_t} \frac{|w(y)|^{\gamma-1}}{|y-x_0|^{n+\gamma}} \, dy \leq \frac{c}{t^s} \left( \int_{\mathbb{R}^n \setminus B_t} \frac{|w(y)|^\gamma}{|y-x_0|^{n+s\gamma}} \, dy \right)^{1-1/\gamma},
\]

where \( c \equiv c(n,s,\gamma) \).

Proof. By (3.13) note that

\[
\int_{\mathbb{R}^n \setminus B_t} \frac{|w(y)|^{\gamma-1}}{|y-x_0|^{n+\gamma}} \, dy = \frac{c}{t^s} \int_{\mathbb{R}^n \setminus B_t} |w(y)|^{\gamma-1} \, d\lambda_{n}(y)
\]

and apply Jensen’s inequality with respect to the concave function \( \tau \mapsto \tau^{1-1/\gamma} \).

4. Proof of Theorems 2 and 5

The main steps of the proofs of Theorems 2 and 5 are contained in Sections 4.1-4.3 below, where we permanently assume (1.4)-(1.6) and (1.8) and \( u \) is as in (1.9). Any ball \( B_\varphi \equiv B_\varphi(x_0) \subset \Omega \) will be such that \( \varphi \leq 1 \).

4.1. Step 1: Basic Caccioppoli inequality. This is in the following:

Lemma 4.1. The inequality

\[
(4.1) \quad \int_{B_{\varphi/2}(x_0)} (|D\eta|^2 + \mu^2)n/2 \, dx + \int_{B_{\varphi/2}(x_0)} \int_{B_{\varphi/2}(x_0)} \frac{|u(x) - u(y)|^\gamma}{|x-y|^{n+s\gamma}} \, dy \, dx \leq c \, cc \, p \equiv c(u, B_\varphi(x_0))
\]

holds whenever \( B_\varphi \equiv B_\varphi(x_0) \subset \Omega \) with \( \varphi \in (0,1) \), where \( c \equiv c(n,p,s,\gamma,\lambda) \) and \( \delta \geq s\gamma. \)

Proof. All the balls will be centred at \( x_0 \). We denote \( u_a := u - (u)_{B_a} \), fix \( \eta \in C_0^1(B_a) \) such that \( \mathbf{1}_{B_{\varphi/2}} \leq \eta \leq \mathbf{1}_{B_{\varphi/4}} \) and \( |D\eta| \leq 1/\varphi \), and set \( m := \max\{p,\gamma\} \). Note that \( \varphi := \eta^m u_a \in \mathcal{X}_0(\Omega) \), so that it can be used in (3.11); this yields

\[
0 = \int_{B_a} \left[ \partial_i F(D(u) : D(\eta^m u_a)) - \eta^m f u_a \right] \, dx
\]

\[
+ |B_a|^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{\gamma-2} \, (u(x) - u(y)) \, (\eta^m(x) u_a(x) - \eta^m(y) u_a(y)) \, K_s(x,y) \, dx \, dy
\]

\[
= (I) + (II).
\]

The estimation of (I) goes via (2.12) and Young and Sobolev inequalities as follows:

\[
(I) \geq c \int_{B_a} \eta^m(|D\eta|^2 + \mu^2)n/2 \, dx - c\eta^{-p} \int_{B_a} |u_a|^p \, dx - c - \left( \int_{B_a} |f|^m \, dx \right)^{1/n} \left( \int_{B_a} |\eta^m u_a|^p \, dx \right)^{1/p^*}
\]

\[
\geq c \int_{B_a} \eta^m(|D\eta|^2 + \mu^2)n/2 \, dx - c\eta^{-p} \int_{B_a} |u_a|^p \, dx - c\|f\|_{L^n(B_a)} \left( \int_{B_a} |D(\eta^m u_a)|^p \, dx \right)^{1/p} - c
\]

\[
\geq c \int_{B_a} \eta^m(|D\eta|^2 + \mu^2)n/2 \, dx - c\eta^{-p} |av_p(\eta)| - c\|f\|_{L^n(B_a)} - c.
\]
with \(c \equiv c(n, p, \Lambda)\). Here \(p^*\) is the Sobolev conjugate exponent as described at the beginning of the proof of Proposition \ref{prop:1}. Using \ref{eq:220} we find

\[
\begin{align*}
(\Pi) &= \int_{B_\rho} \int_{B_\rho} |u_m(x) - u_m(y)|^{-2} (u_m(x) - u_m(y))(\eta^m(x)u_m(x) - \eta^m(y)u_m(y))K_s(x, y) \, dx \, dy \\
&\quad + 2 \int_{\mathbb{R}^n \setminus B_\rho} \int_{B_\rho} |u_m(x) - u_m(y)|^{-2} (u_m(x) - u_m(y))\eta^m(x)u_m(x)K_s(x, y) \, dx \, dy \\
&=: (\Pi)_1 + (\Pi)_2.
\end{align*}
\]

We now observe that

\[
(\Pi)_1 \geq \frac{1}{c} \int_{B_\rho} \int_{B_\rho} \frac{|\eta^m(x)u_m(x) - \eta^m(y)u_m(y)|}{|x - y|^{n+\gamma}} \, dx \, dy
\]

\[
- c \int_{B_\rho} \int_{B_\rho} \max\{|u_m(x)|, |u_m(y)|\} \frac{|\eta^m(x) - \eta^m(y)|}{|x - y|^{n+\gamma}} \, dx \, dy
\]

\[
\geq \frac{1}{c} \int_{B_\rho} \int_{B_\rho} \frac{|\eta^m(x)u_m(x) - \eta^m(y)u_m(y)|}{|x - y|^{n+\gamma}} \, dx \, dy - c \theta^{-\gamma}|av_s(\theta)|\gamma
\]

\[
\geq \frac{1}{c} \int_{B_\rho/2} \int_{B_\rho/2} \frac{|u(x) - u(y)|}{|x - y|^{n+\gamma}} \, dx \, dy - c \frac{\theta^{-\gamma}|a\eta(\theta)|}{\gamma}
\]

for \(c \equiv c(n, p, \gamma, \Lambda)\). For \((\Pi)_2\), note that

\[
x \in B_{3\rho/4}, \ y \in \mathbb{R}^n \setminus B_\rho \implies 1 \leq \frac{|y - x_0|}{|x - y|} \leq 4
\]

and then, recalling that \(\eta\) is supported in \(B_{3\rho/4}\), we have

\[
\begin{align*}
|\Pi_2| &\leq c \int_{\mathbb{R}^n \setminus B_\rho} \int_{B_\rho} \frac{|u_m(x) - u_m(y)|^{-1}|u_m(x)|\eta^m(x)}{|x - y|^{n+\gamma}} \, dx \, dy \\
&\leq c \int_{\mathbb{R}^n \setminus B_\rho} \int_{B_\rho} \max\{|u_m(x)|, |u_m(y)|\}^{-1} |u_m(x)| \, dx \, dy \\
&\leq c \theta^{-\gamma} \int_{B_\rho} |u_m|^\gamma \, dx + c \int_{\mathbb{R}^n \setminus B_\rho} \frac{|u_m(y)|^{-1}}{|y - x_0|^{n+\gamma}} \left( \int_{B_\rho} |u_m| \, dx \right)^{1/\gamma} \\
&\leq c \theta^{-\gamma} |a\eta(\theta)| \gamma + c \left( \int_{\mathbb{R}^n \setminus B_\rho} \frac{|u_m(y)|^{-1}}{|y - x_0|^{n+\gamma}} \, dy \right)^{1-1/\gamma} \theta^{-\gamma} |a\eta(\theta)| \gamma
\]

whenever \(\delta \geq s\gamma\), and where \(c \equiv c(n, s, \gamma, \Lambda)\). Combining the estimates for the terms (I)-(II), and recalling that \(\eta \equiv 1\) on \(B_{\rho/2}\), we arrive at \ref{eq:4.1}. \qed
4.2. **Step 2: Localization.** We define $h \in u + W^{1,p}_0(B_{\rho/4}(x_0))$ as the (unique) solution to
\begin{equation}
 h \mapsto \min_{w \in u + W^{1,p}_0(B_{\rho/4}(x_0))} \int_{B_{\rho/4}(x_0)} F(Dw) \, dx.
\end{equation}
The function $h$ solves the Euler–Lagrange equation
\begin{equation}
 \int_{B_{\rho/4}(x_0)} \partial_r F(Dh) \cdot D\varphi \, dx = 0 \quad \text{for every } \varphi \in W^{1,p}_0(B_{\rho/4}).
\end{equation}
Moreover, by minimality of $h$, (1.4) and (4.1) we gain
\begin{equation}
 \int_{B_{\rho/4}(x_0)} (|Dh|^2 + \mu^2)^{p/2} \, dx \leq \Lambda^2 \int_{B_{\rho/4}} (|Du|^2 + \mu^2)^{p/2} \, dx \leq c \text{ccp}(\rho) \tag{4.7}
\end{equation}
with $c \equiv c(n, p, s, \gamma, \Lambda)$. The standard Maximum Principle and Proposition 2.1 yield
\begin{equation}
 \|h\|_{L^\infty(B_{\rho/4})} \leq \|u\|_{L^\infty(B_{\rho/4})}. \tag{4.8}
\end{equation}
Finally, we recall the $L^\infty$ inequality for $p$-harmonic type functions (see [51, 52])
\begin{equation}
 \|Dh\|_{L^\infty(B_{\rho/4})} \leq c \int_{B_{\rho/4}} (|Dh|^2 + \mu^2)^{p/2} \, dx \leq c \text{ccp}(\rho) \tag{4.9}
\end{equation}
that holds with $c \equiv c(n, p, s, \gamma, \Lambda)$.

**Lemma 4.2.** Let $h \in u + W^{1,p}_0(B_{\rho/4}(x_0))$ be as in (1.5). There exists $\sigma \equiv \sigma(p, s, \gamma) \in (0, 1)$ such that
\begin{equation}
 \int_{B_{\rho/4}(x_0)} |u - h|^p \, dx \leq c\sigma^p |\mathbf{1}_{B_{\rho/4}(u, B_\rho(x_0))}|^p \tag{4.10}
\end{equation}
for every $\theta \in (0, 1)$, where $c \equiv c(\text{data}_u)$.

**Proof.** We are going to use Lemma 4.1 with
\begin{equation}
 \delta \in (s\gamma, p) \tag{4.11}
\end{equation}
in (3.2), which makes sense by $p > s\gamma$. We keep this choice until the end of the proof of Theorem 2 later on, in Step 3, we shall choose $\delta$ suitably close to $p$. We preliminary observe that
\begin{equation}
 \text{ccp}(\rho) \leq c \text{ccp}(\rho) \tag{4.12}
\end{equation}
holds with $c \equiv c(\text{data}_u)$. Indeed, recalling (3.6)–(3.7), it is sufficient to estimate the term $\varphi^{-s\gamma} |\text{av}_t(\rho)|^\gamma$ appearing in the definition of $\text{ccp}(\rho)$; for this, still denoting $\text{av}_t(u) \equiv \text{av}_t(u, B_t(x_0))$ for every $q > 0$ and $t \leq \rho$, observe that
\begin{equation}
 \varphi^{-s\gamma} |\text{av}_t(\rho)|^\gamma \leq c \|u\|_{L^\infty(B_{\rho/4})}^{-s\gamma} |\text{av}_t(\rho)|^\gamma \leq c \|u\|_{L^\infty(B_{\rho/4})}^{-s\gamma} |\text{av}_t(\rho)|^\gamma \leq c \text{ccp}(\rho) \tag{4.13}
\end{equation}
with $c \equiv c(\text{data}_u)$, from which (1.12) follows, with the required dependence of the constants; we have used (2.4) and that $\text{ccp}(\rho) \geq 1 \geq \rho$. We now extend $u \equiv u$ outside $B_{\rho/4}$, thereby getting, in particular, that $h \in W^{1,p}(\Omega) \cap L^\infty(\mathbb{R}^n)$ by Proposition 4.1. If we set $w := u - h$, then $w \in W^{1,p}_0(B_{\rho}) \cap L^\infty(B_{\rho})$. Lemmas 2.2 and 2.3 imply $w \in W^{s\gamma}(B_{\rho})$ and, since $w \equiv 0$ in $B_\rho \setminus B_{\rho/4}$, by 3.4 Lemma 5.1 it follows that $w \in W^{s\gamma}(\mathbb{R}^n)$. In this way $w \in \mathcal{C}_0(\Omega)$ and can be used as a test function both in (2.14) and (4.6).

Setting $V^2 := |V_\rho(Du) - V_\rho(Dh)|^2$, with $V_\rho(\cdot)$ being defined in (2.8), we have
\begin{align}
 \int_{B_{\rho/4}} V^2 \, dx & \leq c \int_{B_{\rho/4}} (\partial_r F(Du) - \partial_r F(Dh)) \cdot Dw \, dx \leq c \int_{B_{\rho/4}} \partial_r F(Du) \cdot Dw \, dx 
 & \equiv c \int_{B_{\rho/4}} f w \, dx - c \int_{B_{\rho/2} \setminus B_{\rho/4}} \int_{B_{\rho/2}} |u(x) - u(y)|^{-2}(u(x) - u(y))(w(x) - w(y))K_s(x, y) \, dx \, dy \nonumber \\
 & \quad - 2c \int_{\mathbb{R}^n \setminus B_{\rho/2}} \int_{B_{\rho/2}} |u(x) - u(y)|^{-2}(u(x) - u(y))w(x)K_s(x, y) \, dx \, dy 
 & \quad =: (I) + (II) + (III) \tag{14.14}.
\end{align}
where \( c \equiv c(n, p, \Lambda) \). Hölder and Sobolev inequalities (as in Lemma 4.1) yield

\[
|\|I\| | \leq \|f\|_{L^\infty(B_{\varrho/4})} \left( \int_{B_{\varrho/4}} |Dw|^p \, dx \right)^{1/p} \tag{4.15}
\]

\[
\leq c \|f\|_{L^\infty(B_{\varrho/4})} |ccp_{\delta, \star}(\varrho)|^{1/p} \leq c |\|f\|_{L^\infty(B_{\varrho/4})} |ccp_{\delta}(\varrho)|^{1/p} ,
\]

with \( c \equiv c(\text{data}_0) \). Again by Hölder’s inequality, it is

\[
|\|II\| | \leq c \left( \int_{B_{\varrho/2}} \int_{B_{\varrho/2}} \frac{|u(x) - u(y)|^\gamma}{|x - y|^{n+\gamma}} \, dy \, dx \right)^{1-1/\gamma} \left( \int_{B_{\varrho/4}} \int_{B_{\varrho/4}} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+\gamma}} \, dy \, dx \right)^{1/\gamma} \tag{4.16}
\]

\[
\leq c |ccp_{\delta}(\varrho)|^{1-1/\gamma} \left( \int_{B_{\varrho/4}} \int_{B_{\varrho/4}} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+\gamma}} \, dy \, dx \right)^{1/\gamma} \leq c |ccp_{\delta}(\varrho)|^{1-1/\gamma} \|w\|_{L^\infty(\varrho/4)}^{\delta - \gamma} \left( \int_{B_{\varrho/4}} |Dw|^p \, dx \right)^{\delta/p} ,
\]

\[
\leq c |ccp_{\delta}(\varrho)|^{1-1/\gamma} \|w\|_{L^\infty(\varrho/4)}^{\delta - \gamma} \left( \int_{B_{\varrho/4}} |Dw|^p \, dx + |Dh|^p \right)^{\delta/p} \tag{4.17}
\]

with \( c \equiv c(\text{data}_0) \). Note that in the last line we have also used the content of Proposition 2.1. For (III) we note that we can replace \( u \) by \( u - (u)_{B_{\varrho/2}} \) and use that \( x \in B_{\varrho/4}, y \in \mathbb{R}^n \setminus B_{\varrho/2} \) imply \( |y - x_0|/|x - y| \leq 2 \). Recalling that \( w \) is supported in \( B_{\varrho/4} \), we then have

\[
|\|III\| | \leq c \int_{\mathbb{R}^n \setminus B_{\varrho/2}} \int_{B_{\varrho/2}} \frac{|u(x) - (u)_{B_{\varrho/2}}| |u(y) - (u)_{B_{\varrho/2}}|^\gamma - 1 |w(x)|}{|x - y|^{n+\gamma}} \, dx \, dy \tag{4.18}
\]

\[
\leq c \int_{\mathbb{R}^n \setminus B_{\varrho/2}} \int_{B_{\varrho/2}} \frac{|u(x) - (u)_{B_{\varrho/2}}| |u(y) - (u)_{B_{\varrho/2}}|^\gamma - 1 |w(x)|}{|y - x_0|^{n+\gamma}} \, dy \, dx \leq c |ccp_{\delta, \star}(\varrho)|^{1-1/\gamma} \int_{B_{\varrho/4}} |w|^\gamma \, dx \leq c |ccp_{\delta}(\varrho)|^{1-1/\gamma} \int_{B_{\varrho/4}} |w|^\gamma \, dx \leq c |ccp_{\delta}(\varrho)|^{1-1/\gamma} \left( \int_{B_{\varrho/4}} |w|^\gamma \, dx \right)^{1/\gamma} ,
\]

for \( c \equiv c(\text{data}_0) \); we have used (3.10)–(3.11) in the third-last line. Similarly to (4.13), we have

\[
\int_{B_{\varrho/4}} |w|^\gamma \, dx \leq c \left( \|u\|_{L^\infty(B_{\varrho/4})} + \|h\|_{L^\infty(B_{\varrho/4})} \right)^{(1-\gamma)\varrho} \left( \int_{B_{\varrho/4}} |w|^\gamma \, dx \right)^{\varrho/\gamma} \leq c |ccp_{\delta}(\varrho)|^{1-1/\gamma} \left( \int_{B_{\varrho/4}} |w|^\gamma \, dx \right)^{1/\gamma} 
\]
Combining the content of the last displays we conclude with
\[
|\text{(III)}| \leq c\phi^{p-\varepsilon}|\cC^p(\varepsilon)|^{1-1/\gamma + \theta/p}.
\]
Using this last estimate with (4.15)–(4.16) in (4.14) we conclude that
\[
\int_{B_\rho/4} v^2 \, dx \leq c\|f\|_{L^\gamma(B_\rho)}|\cC^p(\varepsilon)|^{1/p} + c\phi^{p-\varepsilon}|\cC^p(\varepsilon)|^{1-1/\gamma + \theta/p},
\]
holds with \(c \equiv c(\text{data}_0)\). For the specific dependence of the constant on \(\text{data}_0\), see also Remark 4.1. To proceed, for the moment we consider the case \(p \neq \gamma\), when \(|p(\gamma - 1) + \theta\gamma|/(p\gamma) < 1\) and \([2\gamma - (p - \theta\gamma)]/(2\gamma) < 1\) are true by (2.6); these facts will be used in the cases \(p \geq 2\) and \(1 < p < 2\), respectively. Now, if \(p \geq 2\), we take \(\theta \in (0, 1)\) as in (3.5) and estimate, via Poincaré and Young’s inequality
\[
\int_{B_\rho/4} |u - h|^p \, dx \leq c\phi^p \int_{B_\rho/4} |Du - Dh|^p \, dx \leq c\phi^p \int_{B_\rho/4} v^2 \, dx
\]
\[
\leq c\phi^{p-1} \|f\|_{L^\gamma(B_\rho)} \left( \phi^{p+\theta(p-1)/2}|\cC^p(\varepsilon)|^{1/p} + c\phi^{p-\varepsilon}|\cC^p(\varepsilon)|^{1-1/\gamma + \theta/p} \right)
\]
\[
\leq c \left( \phi^{\rho(p-1)/2} + \phi^{\theta(p-\theta\gamma)/(2\gamma)} \right) \phi^\theta |\cC^p(\varepsilon)| + c\phi^{p-\theta/2} \left( \|f\|_{L^\gamma(B_\rho)} + \phi^{p(p-\theta\gamma)/(2\gamma)} \right),
\]
\[
(4.20)
\]
where \(\sigma := \frac{\rho-\theta\gamma}{\rho + \theta\gamma} > 0\) and \(c \equiv c(\text{data}_0)\). When \(p < 2\), we instead estimate
\[
\int_{B_\rho/4} |u - h|^p \, dx \leq c\phi^p \int_{B_\rho/4} |Du - Dh|^p \, dx
\]
\[
\leq c\phi^p \left( \int_{B_\rho/4} v^2 \, dx \right)^{p/2} \left( \int_{B_\rho/4} (|Du|^2 + |Dh|^2 + \mu^2)^{p/2} \, dx \right)^{1-p/2}
\]
\[
\leq c\phi^p \|f\|_{L^\gamma(B_\rho)}^{p/2} |\cC^p(\varepsilon)|^{1/p} + \phi^{p-\varepsilon}|\cC^p(\varepsilon)|^{1-1/\gamma + \theta/p}\left( |\cC^p(\varepsilon)|^{1-p/2} \right)
\]
\[
\leq c \left( \phi^{\rho(p-1)/p} + \phi^{\theta(p-\theta\gamma)/(2\gamma)} \right) \phi^\theta |\cC^p(\varepsilon)| + c\phi^{p-\theta/2} \left( \|f\|_{L^\gamma(B_\rho)} + \phi^{p(p-\theta\gamma)/(2\gamma)} \right),
\]
\[
(4.21)
\]
where \(\sigma := \frac{\rho-\theta\gamma}{\rho + \theta\gamma} > 0\) and \(c \equiv c(\text{data}_0)\). We have so far proved (4.10) in the case \(p \neq \gamma\). When \(p = \gamma\) we partially proceed as in (4.20)–(4.21). When \(p \geq 2\), from (4.19) we directly gain
\[
\int_{B_\rho/4} |u - h|^p \, dx \leq c \left( \phi^{\rho(p-1)/2} + \phi^{1-\varepsilon} \right) \phi^\theta |\cC^p(\varepsilon)| + c\phi^{p-\theta/2} \|f\|_{L^\gamma(B_\rho)}^{p/2}
\]
with \(c \equiv c(\text{data}_0)\), so that (4.10) follows via (5.9), with \(\sigma := (1-s)/2\). If \(p < 2\), we have
\[
\int_{B_\rho/4} |u - h|^p \, dx \leq c \left( \phi^{\rho(p-1)/p} + \phi^{p(p-\theta\gamma)/(2\gamma)} \right) \phi^\theta |\cC^p(\varepsilon)| + c\phi^{p-\theta/2} \|f\|_{L^\gamma(B_\rho)}^{p/2},
\]
where \(c \equiv c(\text{data}_0)\), so that (4.10) follows with \(\sigma := \frac{1}{2} \min \left\{ \frac{p-1}{p}, p(1-s) \right\} \).
\(\square\)
4.3. Step 3: Hölder integral decay and conclusion. With \( t \leq \varrho/8 \), we bound
\[
\text{av}_p(t) \leq c \left( \int_{B_t} |h - (h)_{B_t}|^p \, dx \right)^{1/p} + c \left( \frac{\varrho}{t} \right)^{\alpha/p} \left( \int_{B_{t/4}} |u - h|^p \, dx \right)^{1/p}
\]
and Poincaré
\[
\leq ct \left( \int_{B_t} |Dh|^p \, dx \right)^{1/p} + c \left( \frac{\varrho}{t} \right)^{\alpha/p} \left( \int_{B_{t/4}} |u - h|^p \, dx \right)^{1/p}
\]
with \( c \equiv c(\text{data}_0) \); the same inequality holds in the case \( \varrho/8 \leq t \leq \varrho \) by (4.11). It follows
\[
\left\{ \begin{array}{l}
\text{av}_s(t) \leq 2\|u\|_{L^\infty(B_t)}^2 \|\text{av}_p(t)\|^p \\
\text{av}_r(t) \leq c\left((t/\varrho)^{\alpha/p} + \varrho^{\alpha/p}(\varrho/t)^{\alpha/p}\right)\|\text{g}_{1,\varrho}(\varrho)\|^{\alpha/p}
\end{array} \right. \quad \forall t \leq \varrho,
\]
for \( c \equiv c(\text{data}_0) \). Indeed, (4.15) follows as in (4.14), while (4.25) follows from (4.24) and (4.25). Taking \( t \equiv \varrho \) in (4.24) with \( \tau \in (0,1/8) \), we find, in particular
\[
\text{av}_s(\tau \varrho) \leq c(\tau + \varrho^{\alpha/p}(\varrho/\tau)^{\alpha/p})\|\text{g}_{1,\varrho}(\varrho)\|^{\alpha/p}
\]
with \( c \equiv c(\text{data}_0) \). In order to get a full decay estimate for \( \|\text{g}_{1,\varrho}(\cdot)\|^{\alpha/p} \) from (4.26), we need to evaluate the snail and the rhs terms. For this we use (4.11), that yields
\[
\begin{align*}
\text{sail}_s(\tau \varrho) \geq & \ c \tau^\delta |\text{sail}_s(\varrho)|^\gamma + c\tau^\delta |\text{av}_r(\varrho)|^\gamma + c\tau^\delta \varrho^\gamma |\text{av}_s(\varrho)|^\gamma \\
= & \ : S_1 + S_2 + S_3.
\end{align*}
\]
We have \( S_1 \leq c\tau^\delta \|\text{g}_{1,\varrho}(\varrho)\|^{\alpha/p} \) by (3.3) and (3.8). By (4.25a) and Young’s inequality (recall (2.6)), we have
\[
S_2 \leq c\tau^\delta \varrho^{\delta - \gamma} \left( \int_{\tau \varrho} \frac{d\nu}{\nu^{1+\delta - \varrho}} \right)^{\gamma} |\text{g}_{1,\varrho}(\varrho)|^\gamma + c\tau^\delta \varrho^{\delta + (\delta + \gamma)\varrho} \left( \int_{\tau \varrho} \frac{d\nu}{\nu^{1+\delta + \gamma + \gamma\varrho}} \right)^{\gamma} |\text{g}_{1,\varrho}(\varrho)|^{\alpha\gamma}
\leq c\tau^\delta \varrho^{\delta - \gamma} \log \left( \frac{1}{\tau} \right)^{\gamma} |\text{g}_{1,\varrho}(\varrho)|^\gamma + c\tau^\delta \varrho^{\delta - \gamma - \gamma\varrho} \varrho^{\delta + \gamma + \gamma\varrho} |\text{g}_{1,\varrho}(\varrho)|^{\alpha\gamma}
\leq c \left[ \tau^\delta \varrho^{\delta - \gamma}\log \left( \frac{1}{\tau} \right) + \varrho^{\delta - \gamma - \gamma\varrho} \right] |\text{g}_{1,\varrho}(\varrho)|^\gamma + c(\Lambda_\gamma + B_\gamma) \tau^\delta \varrho^{\frac{\delta - \gamma}{\varrho + \gamma}}.
\]
where \( c \equiv c(\text{data}_0) \) and \( \Lambda_\gamma, B_\gamma, C_\gamma \) are defined in (2.5). Using again Young’s inequality, we have
\[
S_3 \leq c\tau^\delta \varrho^{\delta - \gamma} |\text{av}_r(\varrho)|^{\alpha\gamma} \leq c\tau^\delta |\text{g}_{1,\varrho}(\varrho)|^{\alpha\gamma} + c(\Lambda_\gamma + B_\gamma) \tau^\delta \varrho^{\frac{\delta - \gamma}{\varrho + \gamma}}.
\]
Connecting the above inequalities for \( S_1, S_2, S_3 \), and gathering terms, leads to
\[
|\text{sail}_s(\tau \varrho)|^\gamma \leq c \left[ \tau^\delta \varrho^{\delta - \gamma}\log \left( \frac{1}{\tau} \right) + \varrho^{\delta - \gamma - \gamma\varrho} \right] |\text{g}_{1,\varrho}(\varrho)|^\gamma + c(\Lambda_\gamma + B_\gamma) \tau^\delta \varrho^{\frac{\delta - \gamma}{\varrho + \gamma}}.
\]
Noting that \( |\text{rhs}_\varrho(\tau \varrho)|^\gamma \leq \tau^\gamma |\text{rhs}_\varrho(\varrho)|^\gamma \), recalling (3.8), and connecting (4.20) and (4.28), leads to
\[
|\text{g}_{1,\varrho}(\tau \varrho)| \leq c \left[ \frac{\delta}{\varrho}\log \left( \frac{1}{\tau} \right) + \tau^{1-\delta/p} + \varrho^{\delta - \gamma - \gamma\varrho} \right] |\text{g}_{1,\varrho}(\varrho)|^\gamma + c(\Lambda_\gamma + B_\gamma) \tau^\delta \varrho^{\frac{\delta - \gamma}{\varrho + \gamma}}.
\]
and \( c \equiv c(\text{data}_0) \). From now on we consider balls \( B_\varrho \equiv B_\varrho(x_0) \subset B_s(x_0) \equiv B_{\varrho s} \subset \Omega \) with \( r \leq r^* \leq 1 \); further restrictions on \( r^* \) will be put in a few lines. We now fix \( \alpha \) such that \( 0 < \alpha < 1 \) and set \( \alpha_1 := (1 + \alpha)/2 \). We then find \( \delta \equiv \delta(p,\alpha) \in (0,1) \) sufficiently small and then \( \delta \equiv \delta(p,s,\gamma,\alpha) \in (s\gamma, p) \) sufficiently close to \( p \), such that
\[
\alpha_1 < 1 - \frac{\theta}{p}, \quad \alpha_1 < \frac{\delta}{p} < \frac{\delta - s\gamma}{p - \varrho\gamma}.
\]
Proposition 4.1. \( \alpha \in \text{in connection with the way data} \)

Remark 4.1. A careful inspection of the proofs in Sections 4.1-4.3 reveals that the constant \( c \) appearing in (1.4) imply

\[
(A_\gamma + B_\gamma)\frac{\beta}{\delta - \alpha} \leq \gamma \leq \text{rhs}(\varepsilon).
\]

Using this inequality in (4.29), and recalling the definitions in Section 3 yields

\[
g^{1,\alpha} = \tau^{\alpha/p} \log^{1/\alpha} \left( \frac{1}{\gamma} \right) \left( r \right) + \tau^{1-\alpha/p} + \gamma^{\alpha/p}\tau^{\alpha/n} + \tau^{1/\alpha} \\text{data} \equiv c_1 \left( \text{data} \right).
\]

We eventually determine \( \alpha \equiv \tau \left( \text{data} \right) \alpha \leq 1/8 \) such that \( 3c_1 \tau^{1/p} \log^{1/\alpha} \left( 1/\tau \right) \leq 1 \), \( 3c_1 \tau^{1-\alpha} \leq 1 \) and \( \tau^{1/\alpha} \leq 1/2 \). Once \( \tau \) has been determined, we can find \( r_* \equiv r_* \left( \text{data}_n, \alpha \right) \) such that if \( \rho \leq r \leq r_* \), then \( 3c_1 \gamma^{1/p} \tau^{1/\alpha} \leq 1 \). With such choices (4.30) becomes

\[
g^{1,\alpha}(\tau) \leq \tau^{\alpha} g^{1,\alpha}(\varepsilon),
\]

that now holds whenever \( \varepsilon \leq r \leq r_* \). We now introduce the sharp fractional maximal type operator

\[
M \left( x_0, \varepsilon \right) := \sup_{\varepsilon \leq r \leq \rho} \frac{\nu}{\varepsilon} \text{gl}_{\alpha}(u, B_r(x_0))
\]

and its truncated version

\[
M_\varepsilon \left( x_0, \varepsilon \right) := \sup_{\varepsilon \leq r \leq \rho} \frac{\nu}{\varepsilon} \text{gl}_{\alpha}(u, B_r(x_0)), \quad 0 \leq \varepsilon \leq \tau.
\]

Multiplying both sides of (4.32) by \( \tau^{\alpha} \), taking the sum with respect to \( \rho \in (\varepsilon, r) \), we arrive at

\[
M \left( x_0, \tau \right) \leq \sup_{\varepsilon \leq r \leq \rho} \frac{\nu}{\varepsilon} \text{gl}_{\alpha}(u, B_r(x_0)) \leq \tau^{1/\alpha}(\text{data}) \text{gl}_{\alpha}(u, B_r(x_0)).
\]

that in turn implies, reabsorbing terms (note that \( M_\varepsilon \) is always finite), and recalling that \( \tau \equiv \tau \left( \text{data}_n, \alpha \right) \)

\[
M_\varepsilon \left( x_0, \tau \right) \leq \frac{c}{\varepsilon} \sup_{\varepsilon \leq r \leq \rho} \text{gl}_{\alpha}(u, B_r(x_0)).
\]

Letting \( \varepsilon \to 0 \) yields

\[
M \left( x_0, r \right) \leq \frac{c}{\rho^p} \sup_{\varepsilon \leq r \leq \rho} \text{gl}_{\alpha}(u, B_r(x_0)),
\]

with again \( c \equiv c \left( \text{data}_n, \alpha \right) \). In order to estimate the right hand side we use (3.10)-(3.11), that yields

\[
M \left( x_0, r \right) \leq \frac{c}{\rho^p} \left( \text{av} \left( r \right) + \gamma^{1/p} \text{snail}_\gamma \left( r \right) \right)^{1/p} + \text{rhs}(\varepsilon)
\]

(4.35)

\[
\leq \frac{c}{\rho^p} \left( \text{av} \left( r \right) + \gamma^{1/p} \text{snail}_\gamma \left( r \right) \right)^{1/p} + \gamma^{1/\alpha} \left( \rho^p \right)^{1/\alpha} + \gamma^{1/\alpha} \left( \rho^p \right)^{1/\alpha},
\]

where \( c \equiv c \left( \text{data}_n, \alpha \right) \). From (4.35), recalling the definition in (4.33), estimate (4.12) and Theorem 3 follow via elementary manipulations. Moreover, using Proposition 2.4 and (4.33) to estimate \( \text{av}_f \left( r \right) + \text{snail}_\gamma \left( r \right) \leq c \left( \text{data}_n, \alpha \right) \), we have proved the following:

**Proposition 4.1.** Under assumptions (1.1) and (1.2), let \( u \in X_\alpha \left( \Omega \right) \) be as in (1.9). For every \( \alpha \in (0,1) \) there exist \( r_* \equiv r_* \left( \text{data}_n, \alpha \right) \in (0,1) \) and \( c \equiv c \left( \text{data}_n, \alpha \right) \geq 1 \), such that

\[
\int_{B_{r*}} \left| \frac{u - u_{B_{r*}}} {\rho} \right|^p \, dx \leq c \left( \frac{\rho}{r_*} \right)^{\alpha p}
\]

holds whenever \( B_\rho \Subset \Omega \) and \( \rho \leq r_* \).

Theorem 2 now follows from (4.36) and the classical Campanato-Meyers integral characterization of Hölder continuity (via a standard covering argument); see for instance (3.7) and Remark 1.1.
Remark 4.2. When neglecting the presence of the \( a_{\mu\nu}\) and \( b_{\mu\nu}\) in the definition of \( g_{\Theta,\beta}\) in (4.2), that is, when considering the purely local, homogenous setting, we have that (4.33) turns into
\[
M(x_0, \varrho) = \sup_{\nu \leq \varrho} \left( \int_{B_\nu(x_0)} |u - (u)_{B_\nu(x_0)}|^p \, dx \right)^{1/p}.
\]
This is nothing but the classical local and fractional variant of the Fefferman-Stein Sharp Maximal Operator widely used in [27]. Moreover, note that a bound of the type in (4.35) immediately implies the local H"{o}lder continuity of \( u \) as
\[
|u(x) - u(y)| \leq \frac{c}{\alpha} [M(x, \varrho) + M(y, \varrho)] |x - y|^{\alpha}
\]
holds whenever \( x, y \in B_{\rho/4}, \) for every ball \( B_{\rho} \subset \mathbb{R}^n \) (see [27]).

5. Proof of Theorem 3

In this section we permanently work under the assumptions of Theorem 3 that is (4.3)–(1.10) and (1.1). The proof goes in seven different steps.

5.1. Step 1: Flattening of the boundary and global diffeomorphisms. The classical flattening-of-the-boundary procedure needs to be upgraded here, as we are in a nonlocal setting. We first recall the standard local procedure, as for instance described in [27,15,16], and summarize its main points. Let us consider \( x_0 \in \partial \Omega \); without loss of generality (by translation) we can assume that \( x_0 \in [x_0 = 0] \) and that \( \Omega \) touches \( \{x_0 = 0\} \) tangentially, so that its normal at \( x_0 \) is \( e_n \), where \( \{e_i\}_{i \in \mathbb{N}} \) is the standard basis of \( \mathbb{R}^n \). By the assumption \( \partial \Omega \subset C^{1,\alpha} \), there exists a radius \( r_0 \equiv r_{\alpha}\), depending on \( x_0 \), and a \( C^{1,\alpha}\)-regular diffeomorphism \( \mathcal{F} \equiv \mathcal{F}_{x_0} : B_{4r_0}(x_0) \to \mathbb{R}^n \) such that \( \mathcal{F}(x_0) = x_0, B^+_{2r_0}(x_0) \subset \mathcal{F}(\Omega_{2r_0}(x_0)) \subset B^+_{4r_0}(x_0), \) \( \Gamma_{2r}(x_0) \subset \mathcal{F}(\partial \Omega \cap B_{2r}(x_0)) \subset \Gamma_{3r}(x_0) \) and \( |\mathcal{F}(z) - e_n| \leq c|z|, x \in B_{4r_0}(x_0), \) where \( c \in (1,10/9) \) can be chosen close to 1 at will taking a smaller \( r_0 \). Moreover, it is
\[
\begin{aligned}
\|\mathcal{F}\|_{C^{1,\alpha}(B_{4r_0}(x_0))} &+ \|\mathcal{F}^{-1}\|_{C^{1,\alpha}(B_{4r_0}(x_0))} < \infty, \\
\|J_{\mathcal{F}}\|_{L^{\infty}(B_{4r_0}(x_0))} &+ \|J_{\mathcal{F}}^{-1}\|_{L^{\infty}(B_{4r_0}(x_0))} < \infty,
\end{aligned}
\]
where \( J_{\mathcal{F}} \) and \( J_{{\mathcal{F}}^{-1}} \) denote the Jacobian determinants of \( \mathcal{F} \) and \( \mathcal{F}^{-1} \), respectively. We refer for instance to [3, Section 3.2] and [4, pages 306 and 318] for the full details and for the explicit expression of the map \( \mathcal{F} \) considered here; see also [15,16]. We next extend \( \mathcal{F} \) to a \( C^1\)-regular global diffeomorphism of \( \mathbb{R}^n \) into itself, following a discussion we found in math.stackexchange. With \( \eta \in C^{0,\alpha}_c(B_{4r_0}(x_0)) \) being such that \( \mathbf{1}_{B_{2r_0}} \leq \eta \leq \mathbf{1}_{B_{4r_0}} \) and \( |D\eta| \leq 1/r_0 \), we define
\[
\begin{aligned}
\tilde{\mathcal{F}}_{x_0}(x) &:= \mathcal{F}(x_0) + D\mathcal{F}(x_0) \cdot (x - x_0), \\
\tilde{\mathcal{F}}_{x_0}(x) &:= (1 - \eta(x_0))\mathcal{F}_{x_0}(x) + \eta(x_0)\mathcal{F}(x).
\end{aligned}
\]
It follows that \( \tilde{\mathcal{F}}_{x_0} \) is \( C^{1,\alpha}\)-regular and, being \( D\mathcal{F}(x_0) \) invertible, that \( \tilde{\mathcal{F}}_{x_0} \) is a smooth global diffeomorphism of \( \mathbb{R}^n \). We now use that the set of \( C^1\)-diffeomorphisms of \( \mathbb{R}^n \) (into itself) is open in the (strong) topology of \( C^1(\mathbb{R}^n, \mathbb{R}^n) \) (see [39, Chapter 2, Theorem 1.6], also for the relevant definitions). For this, we take \( r_{\alpha} > 0 \), such that if \( \mathcal{H} \in C^1(\mathbb{R}^n, \mathbb{R}^n) \) and \( \|\mathcal{H} - \mathcal{F}_{x_0}\|_{C^1(\mathbb{R}^n, \mathbb{R}^n)} < r_{\alpha} \), then \( \mathcal{H} \) is a \( C^1\)-regular diffeomorphism. By using (5.4) and mean value theorem, it now easily follows that
\[
\|\tilde{\mathcal{F}}_{x_0} - \mathcal{F}_{x_0}\|_{C^1(\mathbb{R}^n)} \leq c\|\mathcal{F}\|_{C^{1,\alpha}(B_{4r_0}(x_0))} r_{\alpha} \equiv cr_{\alpha}^\alpha,
\]
with \( c \) depending again on \( x_0 \), so that, by taking \( r_0 \) such that \( cr_{\alpha}^\alpha < r_{\alpha} \), we obtain that \( \tilde{\mathcal{F}}_{x_0} \) (from now on also denoted by \( \mathcal{F} \)) is a \( C^1\)-regular global diffeomorphism. Summarizing, and recalling the explicit expression of \( \tilde{\mathcal{F}}_{x_0} \) in (5.2), we have that for every \( x_0 \in \partial \Omega \), there exists a \( C^1\)-regular diffeomorphism \( \mathcal{F} \equiv \tilde{\mathcal{F}}_{x_0} \) such that
\[
\begin{aligned}
\|D\tilde{\mathcal{F}}\|_{L^{\infty}(\mathbb{R}^n)} &+ \|D\tilde{\mathcal{F}}^{-1}\|_{L^{\infty}(\mathbb{R}^n)} \leq c_0 < \infty, \\
\|J_{\tilde{\mathcal{F}}}\|_{L^{\infty}(\mathbb{R}^n)} &+ \|J_{\tilde{\mathcal{F}}^{-1}}\|_{L^{\infty}(\mathbb{R}^n)} \leq c_0 < \infty.
\end{aligned}
\]
(Here we are further enlarging \( c_0 \) and which is \( C^{1,\alpha}\)-regular diffeomorphism on \( B_{2r_0} \). A comment needs perhaps to be made here, on the inequalities in (5.4). Since \( \tilde{\mathcal{F}}_{x_0} \) is a \( C^1\)-regular diffeomorphism, then

https://math.stackexchange.com/questions/148808/the-extension-of-diffeomorphism
From now on, any dependence of the various constants from $T_0$, the data $r_0$ appearing in (5.1), is still depending on the point $x_0$ via the diffeomorphism $T$. As we are going to flatten the entire boundary $\partial\Omega$ with maps as $T$, by compactness we can assume that $r_0$ and $c_0$ are independent of $x_0 \in \partial\Omega$; see also Remark 5.2 below for more on this aspect.

5.2. Step 2: The flattened functional around a point $x_0 \in \partial\Omega$. We set $\Omega := T(\Omega)$, so that $\Omega := T^{-1}(\Omega)$, and also set $\tilde{g} := g \circ T^{-1}$. Note that if $w \in \mathcal{X}_g(\Omega)$, then $\tilde{w} := w \circ T^{-1} \in \mathcal{X}_g(\Omega)$; on the other hand, any $\tilde{w} \in \mathcal{X}_g(\Omega)$ can be written as $\tilde{w} = w \circ T^{-1}$ where $w \in \mathcal{X}_g(\Omega)$ is simply defined by $w := \tilde{w} \circ T$. By \ref{eq:5.8} and (5.3) it follows
\begin{equation}
\left\{ \begin{array}{l}
\tilde{g} \in W^{1,2}(\tilde{\Omega}) \cap W^{1,2}(\mathbb{R}^n) \\
\|\tilde{g}\|_{W^{1,2}(\tilde{\Omega})} + \|\tilde{g}\|_{W^{1,2}(\mathbb{R}^n)} \leq c(\text{data}) \cdot \|\mathcal{T}\|
\end{array} \right. \cdot
\end{equation}

We then define the (flattened) functional
\begin{equation}
\mathcal{J}_g(\tilde{\Omega}) \ni \tilde{u} \mapsto \mathcal{F}(\tilde{u}) := \int_{\tilde{\Omega}} c(x)|\mathcal{F}(x, D\tilde{u}) - \tilde{f}\tilde{u}| \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(\tilde{u}(x) - \tilde{u}(y))K(x, y) \, dx \, dy
\end{equation}
where
\begin{equation}
\mathcal{F}(x, z) := F(zD\mathcal{T}(T^{-1}(x))), \quad c(x) := |f_{T^{-1}}(x)|, \\
\tilde{f}(x) := f(T^{-1}(x)), \quad K(x, y) := c(x)c(y)K(T^{-1}(x), T^{-1}(y)).
\end{equation}

Defining $\tilde{u} := u \circ T^{-1}$, by \ref{eq:1.9} we have
\begin{equation}
\mathcal{X}_g(\tilde{\Omega}) \ni \tilde{u} \mapsto \min_{\tilde{u} \in \mathcal{X}_g(\tilde{\Omega})} \mathcal{F}(\tilde{u})
\end{equation}

By the very definition of $\tilde{u}$, Proposition \ref{prop:2.1} and directly from \ref{eq:5.5}, we also find
\begin{equation}
\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{g}\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{f}\|_{L^4(\mathbb{R}^n)} \leq c(\text{data}).
\end{equation}
From now on, any dependence of the various constants from $\mathcal{T}$, that is $\|\mathcal{T}\|_{C^1,q}(B_{r_0}(x_0)) \cdot \|\mathcal{T}\|_{C^1(\mathbb{R}^n)}$ and the like, will be incorporated in the dependence on $\Omega$, and therefore on $\text{data}$ (compare with \ref{eq:11}). It follows from the very definitions given, \ref{eq:1.6} and \ref{eq:5.5} that $c(\cdot)$ is continuous and
\begin{equation}
\begin{array}{l}
\{c(x) - c(y)\} \leq \tilde{\Lambda}'|x - y|^\alpha_q, \quad \forall \ x, y \in B_{r_0}(x_0) \\
0 < \tilde{\Lambda}^{-1} \leq c(x) \leq \tilde{\Lambda}, \quad \forall \ x \in \mathbb{R}^n \\
\tilde{\Lambda}^{-1}|x - y|^{-n + \gamma_q} \leq K(x, y) \leq \tilde{\Lambda}|x - y|^{-n + \gamma_q}, \quad \forall \ x, y \in \mathbb{R}^n, \ x \neq y \\
\tilde{f} \in L^n(\mathbb{R}^n)
\end{array}
\end{equation}
Again by \ref{eq:1.4} and \ref{eq:5.5}, as for the new integrand $\mathcal{F}(\cdot)$, we have
\begin{equation}
\begin{array}{l}
z \mapsto \tilde{F}(x, z) \in C^2(\mathbb{R}^n \backslash \{0\}) \cap C^1(\mathbb{R}^n) \\
\tilde{\Lambda}^{-1}(\sqrt{z^2 + \mu^2})^{p/2} \leq \tilde{F}(x, z) \leq \tilde{\Lambda}(\sqrt{z^2 + \mu^2})^{p/2} \\
\{(\sqrt{z^2 + \mu^2})^{1/2} \partial_z \tilde{F}(x, z) + ((\sqrt{z^2 + \mu^2})^{1/2} \partial_z \tilde{F}(x, z)) - \tilde{\Lambda}(\sqrt{z^2 + \mu^2})^{p/2} \\
\tilde{\Lambda}^{-1}((\sqrt{z^2 + \mu^2})^{1/2} - (\sqrt{z^2 + \mu^2})^{(p-2)/2}) |\xi|^2 \leq \partial_z \tilde{F}(x, z) - \partial_z \tilde{F}(y, z) \leq \tilde{\Lambda}|x - y|\alpha_q(\sqrt{z^2 + \mu^2})^{(p-1)/2},
\end{array}
\end{equation}
for all $\xi \in \mathbb{R}^n$, $z \in \mathbb{R}^n \backslash \{0\}$, $x, y \in B_{r_0}(x_0)$. In \ref{eq:5.6} and \ref{eq:15.1} it is $\tilde{\Lambda} \equiv \tilde{\Lambda}(\text{data}) \geq 1$. The Euler-Lagrange equation corresponding to \ref{eq:5.10} is now
\begin{equation}
\begin{split}
\int_\Omega c(x) \left[ \partial_z \tilde{F}(x, D\tilde{u}) \cdot D\tilde{\varphi} - \tilde{f}\tilde{\varphi} \right] \, dx \\
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(\tilde{u}(x) - \tilde{u}(y))(\tilde{\varphi}(x) - \tilde{\varphi}(y))K(x, y) \, dx \, dy = 0,
\end{split}
\end{equation}
and holds for all $\tilde{\varphi} \in \mathcal{X}_0(\tilde{\Omega})$. Performing the same transformation described in Section 2.4, we can use
\begin{equation}
\begin{split}
\int_\Omega c(x) \left[ \partial_z \tilde{F}(x, D\tilde{u}) \cdot D\tilde{\varphi} - \tilde{f}\tilde{\varphi} \right] \, dx \\
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\tilde{u}(x) - \tilde{u}(y)|^{\gamma_q - 2}(\tilde{u}(x) - \tilde{u}(y))(\tilde{\varphi}(x) - \tilde{\varphi}(y))K_\varsigma(x, y) \, dx \, dy = 0,
\end{split}
\end{equation}
with the new kernel $\tilde{K}_s(\cdot)$ that can be obtained by $\tilde{K}(\cdot)$ as explained in (5.12) and satisfies
\begin{equation}
\tilde{K}_s(x, y) = \tilde{K}_s(y, x) \quad \text{and} \quad \tilde{K}_s(x, y) \approx \frac{1}{|x - y|^{n + 2\gamma}}
\end{equation}
for every $x, y \in \mathbb{R}^n$, $x \neq y$.

Remark 5.1. The various constants generically appealed to as $\tilde{\Lambda}$, $c_0$ and $c \equiv c(\text{data})$ from Sections 5.1 and 5.2 actually depend on the point $x_0$ via the features of the map $\mathcal{T}$ considered; this dependence has been omitted above, and we will continue to do so. Indeed, by a standard compactness argument, we can cover and flatten the whole boundary $\partial \Omega$ by using a finite number of such diffeomorphisms $\{\mathcal{T}_i\}_{i \leq k}$ (and points $\{x_i\}_{i \leq k}$), generating the corresponding constants in the estimates. Eventually, we take the largest constant/lowest and make all the resulting constants independent of the specific point $x_i$ considered.

We note that all such dependences will be incorporated in $\text{data}$, since this last one also depends on $\Omega$. Similarly, we can assume that the size of the radius $r_0$, that can be decreased at will, is independent of the point $x_0$; we remark that such reasoning is standard [3, 4, 15, 16].

5.3. Step 3: Localized Regularity. In order to prove Theorem 3 it is now sufficient to show that $u \in C^{0, \alpha}(\Omega)$ holds for every $\alpha < \kappa$, with $[u]_{0, \alpha, \Omega} \leq c(\text{data}, \alpha)$, and where $\kappa$ is defined in (1.7). This follows from the fact that $u \in W^1_0(\Omega)$ and $g \in W^{\alpha, \chi}(\mathbb{R}^n)$, and therefore $g \in C^{0, \alpha-n/\chi}(\mathbb{R}^n)$, as $W^{\alpha, \chi}(\mathbb{R}^n) \subset C^{0, \alpha-n/\chi}(\mathbb{R}^n)$ with $\|g\|_{C^{\alpha-n/\chi}(\mathbb{R}^n)} \leq \|g\|_{W^{\alpha, \chi}(\mathbb{R}^n)}$. This is implied by (1.7) and [20].

The last two estimates also give $[u]_{0, \alpha, \mathbb{R}^n} \leq c(\text{data}, \alpha)$ as claimed in Theorem 3. Finally, to get that $u \in C^{0, \alpha}(\Omega)$ for every $\alpha < 1$ when $g \in W^{1, \infty}(\mathbb{R}^n)$, it is then sufficient to note that a careful reading of the (forthcoming) proof of Theorem 3 reveals that Theorem 3 continuous to hold when replacing the assumption $g \in W^{\alpha, \chi}(\mathbb{R}^n)$ by $g \in W^{1, \infty}(\mathbb{R}^n)$ and $g \in C^{0, \alpha}(\mathbb{R}^n)$ (or even by taking $g \in W^{\alpha, \chi}(\Omega')$ with $\Omega \subset \Omega'$). If $g \in W^{1, \infty}(\mathbb{R}^n)$, then these new conditions are obviously satisfied. Also taking Remark 5.1 into account, via a standard covering argument, we are left to prove the following fact, from which Theorem 3 follows:

Proposition 5.1. Let $\tilde{u} \in X_2(\Omega)$ be the solution to (5.10). Then $\tilde{u} \in C^{0, \alpha}(\bar{B}_{r_0/2}(x_0))$ for every $\alpha < \kappa$. Moreover, there exists a constant $c \equiv c(\text{data}, \alpha)$ such that $[\tilde{u}]_{0, \alpha, \bar{B}_{r_0/2}(x_0)} \leq c$.

For the proof of Proposition 5.1 from now on we shall consider points $\tilde{x}_0 \in \Gamma_{r_0/2}(x_0)$, radii $\varrho \leq r_0/4 \leq 1/4$, and upper balls $B_{\varrho} \equiv B_{\varrho}(\tilde{x}_0) \subset B_{r_0}(x_0)$. Unless otherwise stated, all the upper balls will be centred at $\tilde{x}_0$, and $\tilde{x}_0$ will be a fixed, but generic point as just specified. In analogy to the interior case, with $\delta$ being such that $s\gamma < \delta < p$ (such a choice is allowed by (13)), we define the boundary analog of the quantities introduced in Section 3 as follows:

\begin{equation}
\text{exs}^+_{\varrho, \delta}(\tilde{u}, B_{\varrho}(\tilde{x}_0)) := \left( \int_{B_{\varrho}^+(\tilde{x}_0)} |\tilde{u} - \tilde{g}|^p \, dx \right)^{1/p} + |\text{snail}^+_{\delta}(\tilde{u}, B_{\varrho}(\tilde{x}_0))|^p,
\end{equation}

\begin{equation}
\text{rhs}^+_{\varrho, \delta}(\tilde{u}, B_{\varrho}(\tilde{x}_0))^p := \varrho^{-\theta} \left( \|\tilde{f}\|^p_{L^n(B_{\varrho}^+(\tilde{x}_0))} + 1 \right) + \varrho^\theta \int_{B_{\varrho}^+(\tilde{x}_0)} |\tilde{g}(x) - \tilde{g}(\tilde{y})|^\theta \, dx \, dy \right)^p,
\end{equation}

where $\theta$ has been defined in (2.3)

\begin{equation}
\text{ccp}^+_{\varrho, \delta}(\tilde{u}, B_{\varrho}(\tilde{x}_0)) := \varrho^{-\varrho} \int_{B_{\varrho}^+(\tilde{x}_0)} |\tilde{u} - \tilde{g}|^p \, dx + \varrho^{\varrho} |\text{snail}^+_{\delta}(\tilde{u}, B_{\varrho}(\tilde{x}_0))|^p
\end{equation}

\begin{equation}
\left( \|\tilde{f}\|^p_{L^n(B_{\varrho}^+(\tilde{x}_0))} + 1 \right) + \int_{B_{\varrho}^+(\tilde{x}_0)} |\tilde{g}(x) - \tilde{g}(\tilde{y})|^p \, dx \right)^{p/\varrho}.
\end{equation}

(5.15)

and, finally

\begin{equation}
|\text{gl}^+_{\varrho, \delta}(\tilde{u}, B_{\varrho}(\tilde{x}_0))|^p
\end{equation}
The term (III) holds with
\[ (5.18) \]
with \( \rho \) and \( \theta \), we find \( \rho \tau \) and \( \theta \). The above definitions, and the content of the last display, yield
\[ (5.17) \]
with \( c \equiv c(s, \gamma, p) \). We shall often use the inequality
\[ (5.18) \]
that follows by a simple application of Hölder’s inequality.

### 5.4. Step 4: Boundary Caccioppoli type inequality.
We begin the proof of Proposition [5.1] with

**Lemma 5.1.** The inequality
\[ (5.19) \]
holds with \( c \equiv c(\text{data}) \).

**Proof.** Fix parameters \( q/2 \leq \tau_1 \leq \tau_2 \leq q \), a function \( \eta \in C^1_0(B_{\tau_2}) \) such that \( \mathbb{1}_{B_{\tau_1}} \leq \eta \leq \mathbb{1}_{B_{(3\tau_2+\tau_1)/4}} \) and \( |D\eta| \leq 1/(\tau_2 - \tau_1) \). With \( m := \max\{\gamma, p\} \), set \( \tilde{u}_a := \tilde{u} - (\tilde{u} - \bar{u})B_{\tau_2} \), \( \tilde{g}_a := \tilde{g} - (\tilde{g} - \bar{g})B_{\tau_2} \), \( \tilde{u}_a := \tilde{u}_a - \bar{u}_a = \tilde{u} - \tilde{g} \) and consider \( \varphi := \eta^m \tilde{u}_a \). By its very definition, \( \varphi \) vanishes outside \( B_{\tau_2}^+ \subset B_{\tau_1}^- \subset B_{\tau_1}^0(x_0) \), so that \( \tilde{\varphi} \) implies \( \varphi \in C^0(B_{\tau_2}^+ \). Testing (5.11) with \( \varphi \) we find
\[ 0 = \int_{B_{\tau_2}^+} \eta^m c(x) \left[ \tilde{\varphi}_x \tilde{\tilde{F}}(x, D\tilde{u}) \cdot D\tilde{u}_a - \tilde{f} \tilde{\varphi}_a \right] dx + m \int_{B_{\tau_2}^+} \eta^m \tilde{u}_a(x) \tilde{\varphi}_x \tilde{\tilde{F}}(x, D\tilde{u}) \cdot D\tilde{u} dx \]
\[ + \int_{B_{\tau_2}^+} \int_{\mathbb{R}^n} |\tilde{\varphi}(x) - \tilde{\varphi}(y)|^{\gamma-2} (\tilde{\varphi}(x) - \tilde{\varphi}(y)) (\eta^m(x) \tilde{u}_a(x) - \eta^m(y) \tilde{u}_a(y)) \tilde{K}_a(x, y) \, dx \, dy \]
\[ + 2 \int_{B_{\tau_1}^+ \setminus B_{\tau_2}^+} \int_{\mathbb{R}^n} |\tilde{\varphi}(x) - \tilde{\varphi}(y)|^{\gamma-2} (\tilde{\varphi}(x) - \tilde{\varphi}(y)) (\eta^m(x) \tilde{u}_a(x) - \eta^m(y) \tilde{u}_a(y)) \tilde{K}_a(x, y) \, dx \, dy \]
\[ =: (I) + (II) + (III) + (IV). \]

Via \( \Box \), \( \Box \), \( \Box \), and Sobolev, Poincaré and Young’s inequalities (as in Lemma [4.1]), we obtain
\[ I + II \geq \frac{1}{c} \int_{B_{\tau_2}^+} \eta^m (|D\tilde{u}|^2 + \mu^2)^{p/2} dx - c |B_{\tau_2}^+| \left( \int_{B_{\tau_2}^+} |D\tilde{u}|^p dx \right)^{p/q} \]
\[ - \frac{c}{(\tau_2 - \tau_1)^p} \int_{B_{\tau_2}^+} |\tilde{\varphi} - \tilde{\varphi}|^p dx - c |B_{\tau_2}^+| \left( \|f\|_{L^p(B_{\tau_2}^+)}^{p/(p-1)} + 1 \right), \]
where \( c \equiv c(\text{data}) \). We then write (III) as
\[ (III) = \int_{B_{\tau_2}^+} \int_{B_{\tau_2}^+} |\tilde{u}_a(x) - \tilde{u}_a(y)|^{\gamma-2} (\tilde{u}_a(x) - \tilde{u}_a(y)) (\eta^m(x) \tilde{u}_a(x) - \eta^m(y) \tilde{u}_a(y)) \tilde{K}_a(x, y) \, dx \, dy \]
\[ - \int_{B_{\tau_2}^+} \int_{B_{\tau_2}^+} |\tilde{u}_a(x) - \tilde{u}_a(y)|^{\gamma-2} (\tilde{u}_a(x) - \tilde{u}_a(y)) (\eta^m(x) \tilde{u}_a(x) - \eta^m(y) \tilde{u}_a(y)) \tilde{K}_a(x, y) \, dx \, dy \]
\[ =: (III)_1 + (III)_2. \]
The term (III)_1 can be estimated similarly to \( \Box \), \( \Box \), \( \Box \), i.e.:
\[ (III)_1 \geq \frac{1}{c} \int_{B_{\tau_2}^+} \int_{B_{\tau_2}^+} \left| \eta^m(x) \tilde{u}_a(x) - \eta^m(y) \tilde{u}_a(y) \right|^{\gamma} |x - y|^{n+\gamma} \, dx \, dy \]
\[ - c \int_{B_{\tau_2}^+} \int_{B_{\tau_2}^+} \left( \max \{ \tilde{u}_a(x), \tilde{u}_a(y) \} \right)^\gamma \left| \eta^m(x) - \eta^m(y) \right|^{\gamma} |x - y|^{n+\gamma} \, dx \, dy \]
By further using (3.10) and (3.11), we find
\[
\tau + \int_{B_2} |\bar{u} - \tilde{u}| \, dx \geq \frac{1}{c_0} \int_{B_2} |\bar{u} - \tilde{u}| \, dx,
\]
for \(c, c_0 \equiv c, c_0\) (data). As for (III), we have
\[
\int_{(\tau^2 - \tau^2)} \left[ \frac{1}{2c_0} |\bar{u}|^\gamma \right] \int_{B_{2\tau}} \frac{|\eta(x) - \eta(y)|^\gamma}{|x - y|} \, dx \, dy \leq \frac{1}{2c_0} \int_{B_{2\tau}} \frac{|\eta(x) - \eta(y)|^\gamma}{|x - y|} \, dx \, dy.
\]
We can assume that the constant \(c_0\) appearing in the last two displays is the same. Note that in the last line we have also used (3.11) and (3.13). In order to estimate (IV), we note
\[
x \in B_{(3\tau^2 + \tau^2)}/4, \quad y \in \mathbb{R}^n \setminus B_{2\tau} \quad \Rightarrow \quad 1 \leq \frac{|x - \tilde{x}|}{|y - \tilde{y}|} \leq 1 + \frac{3\tau^2 + \tau^2}{\tau^2 - \tau^2} = \frac{4\tau^2}{\tau^2 - \tau^2}.
\]
Recalling that \(\eta\) is supported in \(B_{(3\tau^2 + \tau^2)/4}\), and using (3.11) and (3.12), we get
\[
\int_{IV} \leq \frac{c^\gamma_0}{(\tau^2 - \tau^2)^{n+\gamma}} \int_{B_{2\tau}} \int_{B_{2\tau}} |\bar{u}(x) - \bar{u}(y)|^\gamma - |\bar{u}(x)|^\gamma \, dx \, dy
\]
for \(c \equiv c_0\). Merging the estimates for terms (I)-(IV), and again using (3.11), yields
\[
\int_{B_2^+} |D\tilde{u}|^2 + \tilde{u}^{p/2} \, dx + \int_{B_{2\tau}} |\bar{u}|^\gamma \, dx \leq \frac{1}{2} \int_{B_{2\tau}} |\bar{u}|^\gamma \, dx + \frac{c}{(\tau^2 - \tau^2)^{n+\gamma}} \int_{B_{2\tau}} |\bar{u} - \tilde{u}|^\gamma \, dx
\]
On the other hand, proceeding as in the proof of (4.18), we obtain
\[ (5.26) \]
with \( c \equiv c(data) \). Applying Lemma 2.5 with the choice
\[ (5.26) \]
now yields, after a few manipulations, and recalling his definition in (5.16)
\[ (5.22) \]
Then we have
\[ (5.23) \]
On the other hand, proceeding as in the proof of (5.18), we obtain
\[ (5.24) \]
Using the last two inequalities in (5.22) finally leads to (5.19).

5.5. Step 5: Boundary \( p \)-harmonic functions. Here we have

**Lemma 5.2.** Let \( \bar{h} \in \bar{u} + W_{0}^{1,p}(B_{\theta/4}(\bar{x}_0)) \) be the solution to
\[ (5.25) \]
Then
\[ (5.26) \]
holds for any \( \theta \in (0,1) \), where \( c \equiv c(data) \). Here \( \bar{\sigma} \equiv \bar{\sigma}(p,s,\gamma,\alpha_\theta) \in (0,1) \) is given by \( \bar{\sigma} := \min\{\sigma,\alpha_\theta,p\alpha_\theta/2\} \) and \( \sigma \) comes from (4.10).
Proof. We shall abbreviate, as usual, \( B^+_1 \equiv B^+_1(\tilde{x}_0) \). From (5.29) it follows that

\[
\int_{B^+_1/4} c(\tilde{x}_0) \partial_\nu F(\tilde{x}_0, D\tilde{h}) \cdot D\varphi \, dx = 0 \quad \text{for all } \varphi \in W^{1,\infty}_0(B^+_1/4) \tag{5.27}
\]

and, as for (4.7) - (4.8)

\[
\int_{B^+_1/4} (|D\tilde{h}|^2 + \mu^2)^{p/2} \, dx \leq \lambda^2 \int_{B^+_1/4} (|D\tilde{u}|^2 + \mu^2)^{p/2} \, dx, \quad \|\tilde{h}\|_{L^\infty(B^+_1/4)} \leq \|\tilde{u}\|_{L^\infty(B^+_1/4)} \tag{5.28}
\]

hold. As \( \tilde{h} = \tilde{u} \) on \( \partial B^+_1/4 \) (in the sense of traces), we define \( \tilde{w} := \tilde{u} - \tilde{h} \in W^{1,\infty}_0(B^+_1/4) \) and extend it to the whole \( \mathbb{R}^n \) by setting \( \tilde{w} \equiv 0 \) in \( \mathbb{R}^n \setminus B^+_1/4 \). This implies \( \tilde{w} \in \mathcal{X}_0(\Omega) \), so that \( \tilde{w} \) is an admissible test function for both (5.11) and (5.27). Indeed, note that \( \tilde{w} \in W^{1,\infty}(B^1/2) \cap L^\infty(\mathbb{R}^n) \) and therefore by Lemmas 2.2, 2.3 it follows that \( \tilde{w} \in W^{s,\gamma}(\mathbb{R}^n) \). As \( \tilde{w} \equiv 0 \) outside \( B^1/4 \), it follows that \( \tilde{w} \in W^{s,\gamma}(\mathbb{R}^n) \) by [30] Lemma 5.1, and therefore \( \tilde{w} \in \mathcal{X}_0(\Omega) \). This means that \( \tilde{w} \) can be used as a test function both in (5.11) and in (5.27). Moreover, by (5.10) and (5.28), it follows that

\[
\int_{B^+_1/4} (|D\tilde{w}|^2 + \mu^2)^{p/2} \, dx \leq c \int_{B^+_1/4} (|D\tilde{u}|^2 + \mu^2)^{p/2} \, dx \leq c \text{ccp}_s^+(\gamma) \tag{5.29}
\]

With \( \tilde{0}^2 := |V_\nu(D\tilde{u}) - V_\nu(D\tilde{h})|^2 \), we estimate (via inequality (5.10) applied to \( \partial_\nu \tilde{F} \), as allowed by (5.29))

\[
\frac{1}{c} \int_{B^+_1/4} \tilde{w}^2 \, dx \leq \int_{B^+_1/4} c(x)(\partial_\nu \tilde{F}(\tilde{x}_0, D\tilde{u}) - \partial_\nu \tilde{F}(\tilde{x}_0, D\tilde{h})) \cdot D\tilde{w} \, dx \tag{5.10}
\]

\[
= \int_{B^+_1/4} \frac{c(\tilde{x}_0) - c(x)}{c(x)} \partial_\nu \tilde{F}(\tilde{x}_0, D\tilde{u}) \cdot D\tilde{w} \, dx \tag{5.11}
\]

\[
+ \int_{B^+_1/4} c(x)(\partial_\nu \tilde{F}(\tilde{x}_0, D\tilde{u}) - \partial_\nu \tilde{F}(x, D\tilde{u})) \cdot D\tilde{w} \, dx + \int_{B^+_1/4} c(x)\tilde{f}\tilde{w} \, dx
\]

\[
- \int_{B^+_1/4} \int_{\mathbb{R}^n} |\tilde{u}(x) - \tilde{u}(y)|^{\gamma - 2} (\tilde{u}(x) - \tilde{u}(y))(\tilde{w}(x) - \tilde{w}(y))K_s(x, y) \, dx \, dy \tag{5.12}
\]

\[
\leq c_\delta \int_{B^+_1/4} (|D\tilde{u}|^2 + \mu^2)^{(p-1)/2} |D\tilde{w}| \, dx + c \int_{B^+_1/4} |\tilde{f}| \, dx \tag{5.13}
\]

\[
+ c \int_{B^1/2} \int_{B^1/2} |\tilde{u}(x) - \tilde{u}(y)|^{\gamma - 1} |\tilde{w}(x) - \tilde{w}(y)| \frac{dx \, dy}{|x - y|^{n+\gamma}}
\]

\[
+ c \int_{\mathbb{R}^n \setminus B^1/2} \int_{B^1/2} |\tilde{u}(x) - \tilde{u}(y)|^{\gamma - 1} |\tilde{w}(x)| \frac{dx}{|x - y|^{n+\gamma}} \tag{5.14}
\]

\[
=: (O) + (I) + (II) + (III),
\]

where \( c \equiv c(n, p, \delta) \); we have also used (5.8). The first two terms can be controlled via Sobolev inequality

\[
(O) + (I) \leq c \left[ c^2 \text{ccp}_s^+(\gamma)^{1-1/p} + \|\tilde{f}\|_{L^\infty(B^+_1/4)} \right] \left( \int_{B^+_1/4} |D\tilde{w}|^p \, dx \right)^{1/p} \tag{5.15}
\]

\[
\leq c \text{ccp}_s^+(\gamma) + c \|\tilde{f}\|_{L^\infty(B^+_1/4)} \text{ccp}_s^+(\gamma)^{1/p}, \tag{5.16}
\]

with \( c \equiv c(\text{data}) \) (also recall (4.13)). The term (II) can be estimated as the homonym term in Lemma 1.3 but this time using (5.7) and (5.28); this yields

\[
(II) \leq c \left( \int_{B^+_1/4} \int_{B^+_1/4} |\tilde{w}(x) - \tilde{w}(y)|^\gamma \frac{dx \, dy}{|x - y|^{n+\gamma}} \right)^{1/\gamma} \leq c \text{ccp}_s^+(\gamma)^{1-1/\gamma + \delta/\gamma}
\]

\[
\leq c \text{ccp}_s^+(\gamma)^{1-1/\gamma + \delta/\gamma} \tag{5.17}
\]
where $\vartheta$ is in $\mathcal{O}_2(n)$ and $c \equiv c(\text{data})$. Now, similarly to (5.30), but using $\mathcal{O}_2(n)$ and $\mathcal{O}_3(n)$, we find

$$
(5.31) \quad \left( \int_{B_{c \varrho}^+} |\tilde{u}|^\gamma \, dx \right)^{1/\gamma} \leq c \left( \int_{B_{c \varrho}^+} |\tilde{u}|^{\vartheta} \, dx \right)^{\vartheta/p} \leq c \left( \int_{B_{c \varrho}^+} |D\tilde{u}|^p \, dx \right)^{1/p} \leq c g^{\vartheta} [ccp_+^s (p)]^{\vartheta/p}.
$$

We then have

$$
(III) \quad \leq e \left( \int_{\mathbb{R}^n \setminus B_{c \varrho}^+/2} \frac{\max \{ |\tilde{u}(x) - (\tilde{u})_{B_{c \varrho}^+/2} |, |(\tilde{u})_{B_{c \varrho}^+/2} - \tilde{u}(x) | \}^{\gamma-1} |\tilde{u}(x)| \right) dx dy
$$

$$
\leq e g^{-\gamma} \left( \int_{B_{c \varrho}^+/2} |\tilde{u}(x) - (\tilde{u})_{B_{c \varrho}^+/2}|^\gamma dx \right)^{1-\gamma/\gamma} \left( \int_{B_{c \varrho}^+/4} |\tilde{u}|^\gamma dx \right)^{1/\gamma} + c \left( \int_{\mathbb{R}^n \setminus B_{c \varrho}^+/2} \frac{|(\tilde{u})_{B_{c \varrho}^+/2} - \tilde{u}(x)|}{|y - x_0|^{n+s+\gamma}} dy \right) \left( \int_{B_{c \varrho}^+/4} |\tilde{u}|^\gamma dx \right)^{1/\gamma}
$$

$$
\leq e g^{-\gamma} \left[ ccp_+^s (\vartheta) \right]^{1-1/\gamma+\vartheta/p} + c \left[ \frac{\vartheta}{\gamma} |\mathcal{O}(\vartheta)| \right]^{1-1/\gamma+\vartheta/p}
$$

$$
\leq e g^{-\gamma} [ccp_+^s (\vartheta)]^{1/\gamma+\vartheta/p} + c [\mathcal{O}(\vartheta)]^{1/\gamma+\vartheta/p},
$$

with $c \equiv c(\text{data})$. Combining the estimates for the terms (O), (I), (II) and (III) with (5.30), we obtain

$$
(5.32) \quad \left( \int_{B_{c \varrho}^+/4} |\tilde{u} - h|^p \, dx \right)^{1/p} \leq c g^{\vartheta} [ccp_+^s (\vartheta)]^{1/\gamma+\vartheta/p} + c [\mathcal{O}(\vartheta)]^{1/\gamma+\vartheta/p},
$$

for $c \equiv c(\text{data})$. This is the boundary analog of (4.19). We can then proceed as in (4.20)-(4.23), but using (5.32) instead of (4.19), and (5.17) instead of (5.30), to obtain

$$
\left( \int_{B_{c \varrho}^+/4} |\tilde{u} - h|^p \, dx \right)^{1/p} \leq c g^{\vartheta} [ccp_+^s (\vartheta)]^{1/\gamma+\vartheta/p} + c [\mathcal{O}(\vartheta)]^{1/\gamma+\vartheta/p},
$$

where $\sigma$ is as in Lemma 1.2 and from which (5.27) follows again using (5.17).}$"
By using (5.17) and recalling (5.34), we conclude with
\[
\text{where (5.41)}
\]
\[
\text{c} = \cdots
\]
\[
\text{with (5.37)}
\]
\[
\text{c} \equiv c(\text{data}).
\]
\[
\text{Next observe that, using (5.18) and recalling the definitions in Section (5.22), we find}
\]
\[
\left( \int_{B_t} |\bar{u} - \bar{\psi}(y)|^{1/n} \right)^{1/n} \leq c \left( \frac{t}{\vartheta} \right)^{1-n/q} \| \phi \|_p^{n/p} \| \psi \|_q^{n/q} \text{ } \vartheta \equiv \cdots
\]
\[
\text{with (5.35), we have}
\]
\[
\text{av}_{\tau}(t) \leq c \left( \frac{t}{\vartheta} \right)^{1-n/q} \| \phi \|_p^{n/p} \| \psi \|_q^{n/q} \vartheta \equiv \cdots
\]
\[
\text{with (5.36),}
\]
\[
\text{with (5.37),}
\]
\[
\text{with (5.38),}
\]
\[
\text{with (5.39),}
\]
\[
\text{with (5.30) and (5.36) gives}
\]
\[
\text{av}_{\tau}(t) \leq c \left( \frac{t}{\vartheta} \right)^{1-n/q} \| \phi \|_p^{n/p} \| \psi \|_q^{n/q} \vartheta \equiv \cdots
\]
\[
\text{holds and we can conclude that (5.38) takes place in the full range 0 < t \leq \vartheta. Taking t = \tau \vartheta in (5.35), with 0 < \tau \leq 1/8, yields}
\]
\[
\int_{B_{\tau \vartheta}^*} |\bar{u} - \bar{\psi}(y)|^{1/n} \leq c \left( \tau^{1-n/q} + \| \phi \|_p^{n/p} \tau^{-n/p} \right) \| \psi \|_q^{n/q} \vartheta \equiv \cdots
\]
\[
\text{for c \equiv c(\text{data}). As for the snail, we have}
\]
\[
\text{[snails(\tau \vartheta)]} \gamma \leq c \tau^\delta \text{[snails(\vartheta)]} \gamma + c(\tau \vartheta)^\delta \left( \int_{\tau \vartheta} \text{av}_{\tau \vartheta}(\nu) \frac{d\nu}{\nu} \right)^\gamma + c \tau^\delta \vartheta^{-\gamma} \text{[av}_{\tau \vartheta}(\vartheta)] \gamma
\]
\[
\text{= S_5 + S_6 + S_7.}
\]
We have $S_6 \leq c^{\rho} |g_{\theta,\delta}(\phi)|^p$ by (5.10). For $S_6$, we use (5.32) to estimate $a_{\nu}(\nu)$ inside the integral, and in turn estimate separately the resulting three pieces $S_{6,1}, S_{6,2}$ and $S_{6,3}$ generated by the terms appearing in the right-hand side of (5.32). To estimate $S_{6,1}$ we first consider the case $s \leq 1 - n/q$; we have

$$S_{6,1} \leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(1-n/q)} \left( \int_{\tau \phi}^{\tau^{\rho}(1-n/q)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq cA_\tau \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}.$$ 

The other case is when $s > 1 - n/q$, and we have similarly

$$S_{6,1} \leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(1-n/q)} \left( \int_{\tau \phi}^{\tau^{\rho}(1-n/q)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq cA_\tau \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} |g_{\theta,\delta}^+(\phi)|^{\gamma}.$$ 

Note that here we have used $\delta - s \gamma > \delta > \delta - s \gamma + (1-n/q)$, implied by $s < 1$. Moreover,

$$S_{6,2} \leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(1-n/q)} \left( \int_{\tau \phi}^{\tau^{\rho}(1-n/q)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} |g_{\theta,\delta}^+(\phi)|^{\gamma}.$$ 

For $S_{6,3}$ we first consider the case $a - \chi/n \geq s$, and we have

$$S_{6,3} \leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \left( \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}.$$ 

When $a - \chi/n < s$, using also that $\delta(a - n/x) < \delta > \delta - s \gamma + (a - n/x)$, we instead have

$$S_{6,3} \leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \left( \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \log \left( \frac{1}{\tau} \right) \left( \tau^{s} \right)^{\frac{s-n}{q}} |g_{\theta,\delta}^+(\phi)|^{\gamma} + c(\phi + C_\nu) \tau^{\rho} |g_{\theta,\delta}^+(\phi)|^{\gamma}.$$ 

The last term is dealt with as

$$S_7 \leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \left( \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}$$

$$\leq c^{\rho} \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \left( \int_{\tau \phi}^{\tau^{\rho}(a-n/x)} \frac{d\nu}{\nu^{1-s}(1-s)} \right)^{\gamma} |g_{\theta,\delta}^+(\phi)|^{\gamma}.$$ 

Connecting the estimates found for $S_6, S_7$ and $S_7$ to (5.11) we obtain

$$\left( \tau^{\delta - n/p} \right)^{\gamma} \leq c \left( \tau^{\delta - n/p} + \tau^{\delta - n/p} \right)^{\gamma} \left( \tau^{\delta - n/p} \right)^{\gamma}.$$
On the other hand, by the very definition in (6.13), we trivially have
\begin{equation}
\text{r}\text{h}\text{se}_{\theta}^+ (\tau \theta) \leq \left( \tau^{1 - \theta/p} + \tau^{1 - n/q} + \tau^{n - \theta/\nu} \right) \text{g}^1_{\nu, \tilde{a}}(\theta).
\end{equation}
Connecting (5.30), (5.42) and (5.43), and yet keeping (5.34) in mind, we arrive at
\begin{equation}
\text{g}^1_{\nu, \tilde{a}}(\tau \theta) \leq c_1 \left( \tau^{1 - \theta/p} + \tau^{\theta/p - n/q} + \tau^{\theta(a - n/\nu)/p} + \theta^\theta \tau^{\theta/\nu} \right) \text{g}^1_{\nu, \tilde{a}}(\theta),
\end{equation}
where $c_1 \equiv c_1(\text{data})$. With $\kappa > 0$ being defined in (1.4), we select a positive $\alpha < \kappa$ and then set $\alpha_1 := (\alpha + \kappa)/2$, so that $\alpha < \alpha_1 < \kappa$. We can find $\delta := \delta(n, p, q, \alpha, \chi, \lambda)$ (close enough to $p$) and $\theta := \theta(n, p, q, a, \chi, \lambda)$ (close enough to zero), such that $\min\{1 - \theta/p, \delta/p - n/q, \delta(a - n/\nu)/p\} > \alpha_1$. Then we take $\gamma \equiv \gamma(\text{data}, \alpha)$ small enough to have
\begin{equation}
c_1 \left( \tau^{1 - \theta/p - \alpha_1} + \tau^{\theta/p - n/q - \alpha_1} + \tau^{\theta(a - n/\nu)/p - \alpha_1} \right) \leq \frac{1}{2} \quad \text{and} \quad \tau^{(n-\alpha)/2} \leq \frac{1}{2}.
\end{equation}
With $\gamma$ being determined, we now select a positive radius $r_* \equiv r_*(\text{data}, \alpha) \leq r_0/4$ such that $\tau \leq r_*$ implies $c_1 \theta^{\theta/\nu} \tau^{\theta/p - n/q - \alpha_1} - \tau^{(n-\alpha)/2} \leq 1/2$. Using this last inequality, and the one in the last display, in (5.31), implies $\text{g}^1_{\nu, \tilde{a}}(\tau \theta) \leq \tau^{\gamma(\text{data}, \alpha)}$, which is the boundary analog of (4.32). This leads to consider the maximal operators
\begin{equation}
M^+(\tilde{x}_0, \theta) := \sup_{\varepsilon \leq \theta} \nu^{-\alpha} \text{g}^1_{\nu, \tilde{a}}(u, B_r(\tilde{x}_0)), \quad M^+_{c}(\tilde{x}_0, \theta) := \sup_{\varepsilon \leq \theta} \nu^{-\alpha} \text{g}^1_{\nu, \tilde{a}}(u, B_r(\tilde{x}_0))
\end{equation}
for $\varepsilon \leq \tau$. Proceeding as after (1.3), and taking into account (5.3) and (5.7), we arrive a $\mathcal{M}(x_0, r) \leq c(\text{data})$. From this and the fact that the chosen point $\tilde{x}_0$ is arbitrary, we conclude with
\begin{equation}
\sup_{\tilde{x}_0 \in \Gamma_{r_0/2}} \sup_{\varepsilon \leq r_*} \int_{B^+_{2\varepsilon}(\tilde{x}_0)} |\tilde{u} - \tilde{g}|^p dx \leq c \varepsilon^\alpha.
\end{equation}
Here recall that $r_* \equiv r_*(\text{data}, \alpha)$. Using Sobolev-Morrey embedding theorem, we find
\begin{equation}
\int_{B^+_{2\varepsilon}(\tilde{x}_0)} |\tilde{g} - \tilde{g}|_{B^+_{2\varepsilon}(\tilde{x}_0)}^p dx \leq \left( \text{osc}_{B^+_{2\varepsilon}(\tilde{x}_0)} \tilde{g} \right)^p \leq c \varepsilon^{(1-n)/p} \|D\tilde{g}\|_{L^p(B^+_{2\varepsilon}(\tilde{x}_0))} \leq c \varepsilon^\alpha \leq c \varepsilon^\alpha,
\end{equation}
where $c \equiv c(\text{data})$. Combining the two inequalities above, and yet using (5.11), we finally get that
\begin{equation}
\sup_{\tilde{x}_0 \in \Gamma_{r_0/2}} \sup_{\varepsilon \leq r_*} \int_{B^+_{2\varepsilon}(\tilde{x}_0)} |\tilde{u} - \tilde{g}|_{B^+_{2\varepsilon}(\tilde{x}_0)}^p dx \leq c \varepsilon^\alpha
\end{equation}
holds whenever $\varepsilon \leq r_*$, where $c \equiv c(\text{data}, \alpha)$. On the other hand, by Proposition 5.3, there exists $c \equiv c(\text{data}) \geq 1$ and another positive radius $r_* \equiv r_*(\text{data}, \alpha) \leq r_0/4$, such that
\begin{equation}
\int_{B^+_{r_0}(\tilde{y})} |\tilde{u} - \tilde{g}|_{B^+_{r_0}(\tilde{y})}^p dx \leq c \varepsilon^\alpha
\end{equation}
holds whenever $\varepsilon \leq r_*$ and $B_{r_0}(\tilde{y}) \subset B^+_{r_0}(\tilde{x}_0)$. Combining the information in the last two displays in a standard way yields that now (5.46) holds not only when $\tilde{x}_0$ belongs to $\Gamma_{r_0/2}$ as in (5.45), but whenever $\tilde{x}_0 \in B^+_{r_0}(\tilde{x})$ and $\tilde{g} \leq \min\{r_*, r_*\}/8 \leq r_0/4$. This implies the validity of Proposition 5.3 via Campanato-Meyers integral characterization of Hölder continuity.

5.7. Step 7: Estimate (5.33). Estimates like (5.33) can be found in various places in the literature under additional structure conditions and assumptions. We did not find and explicit reference for it and therefore we offer a rapid derivation here for the sake of completeness. We denote $F_0(z) := c(\tilde{x}_0) \tilde{F}(\tilde{x}_0, z)$, using the same notation of Section 5.3. Note that $\tilde{w} = \tilde{h} - \tilde{g}$ solves
\begin{equation}
\begin{cases}
-\text{div} \partial_t F_0 (D\tilde{g} + D\tilde{w}) = 0 & \text{in } B^+_{r_0/4} \\
\tilde{w} \equiv 0 & \text{on } \Gamma_{r_0/4}.
\end{cases}
\end{equation}
We denote by $\tilde{v} \in \tilde{w} + W^{1,p}_0(B^+_{r_0/4})$ as the solution to
\begin{equation}
\begin{cases}
-\text{div} \partial_t F_0 (D\tilde{v}) = 0 & \text{in } B^+_{r_0/4} \\
\tilde{v} \equiv \tilde{w} & \text{on } \partial B^+_{r_0/4}.
\end{cases}
\end{equation}
By [21] Theorem 2.2] we obtain that

\[(5.49) \quad \|D\tilde{u}\|_{L^\infty(B_{1/4}^+)} \leq c \int_{B_{1/4}^+} (|D\tilde{v}|^2 + \mu^2)^{p/2} \, dx \leq c \int_{B_{1/4}^+} (|D\tilde{w}|^2 + \mu^2)^{p/2} \, dx \]

with \(c \equiv c(n, p, \Lambda)\) (note that [21] Theorem 2.2] is stated for the degenerate case \(\mu = 0\), but the proof applies verbatim in the non-degenerate case \(\mu > 0\), which is actually simpler). The former inequality in (5.49) follows from a delicate barrier argument, and the latter is a consequence of minimality of \(\tilde{v}\) (it solves an Euler-Lagrange equation). In turn, also using the minimality of \(\tilde{h}\) in (5.39), we find

\[
\int_{B_{1/4}^+} (|D\tilde{w}|^2 + \mu^2)^{p/2} \, dx \leq c \int_{B_{1/4}^+} (|D\tilde{h}|^2 + |D\tilde{g}|^2 + \mu^2)^{p/2} \, dx \leq c \int_{B_{1/4}^+} (|D\tilde{u}|^2 + |D\tilde{g}|^2 + \mu^2)^{p/2} \, dx
\]

with \(c \equiv c(n, p, \Lambda)\). On the other hand, this time being \(\tilde{v}^2 := |V_n(D\tilde{v}) - V_n(D\tilde{w})|^2\), we have

\[
\int_{B_{1/4}^+} \tilde{v}^2 \, dx \leq c \int_{B_{1/4}^+} (\partial_s F_0(D\tilde{v}) - \partial_s F_0(D\tilde{w})) \cdot (D\tilde{v} - D\tilde{w}) \, dx
\]

\[
\leq c \int_{B_{1/4}^+} (\partial_s F_0(D\tilde{g} + D\tilde{w}) - \partial_s F_0(D\tilde{w})) \cdot (D\tilde{v} - D\tilde{w}) \, dx
\]

\[\leq c \int_{B_{1/4}^+} (|D\tilde{g}|^2 + |D\tilde{w}|^2 + \mu^2)^{(p-2)/2} (D\tilde{g})|D\tilde{v} - D\tilde{w}| \, dx.
\]

In the case \(p \geq 2\), (5.44) implies \(|D\tilde{v} - D\tilde{w}|^p \leq c \tilde{v}^2\) and, by repeated use of Young’s inequality, and reabsorbing terms, we find

\[(5.50) \quad \int_{B_{1/4}^+} |D\tilde{v} - D\tilde{w}|^p \, dx \leq \varepsilon \int_{B_{1/4}^+} (|D\tilde{v}|^2 + \mu^2)^{p/2} \, dx + c_\varepsilon \int_{B_{1/4}^+} |D\tilde{w}|^p \, dx
\]

for every \(\varepsilon \in (0, 1)\), where \(c_\varepsilon\) depends on \(n, p, \Lambda, \varepsilon\). In the case \(1 < p < 2\), as in (4.21), we instead find

\[
\int_{B_{1/4}^+} |D\tilde{v} - D\tilde{w}|^p \, dx \leq c \left( \int_{B_{1/4}^+} \tilde{v}^2 \, dx \right)^{p/2} \left( \int_{B_{1/4}^+} (|D\tilde{v}|^p + |D\tilde{w}|^p) \, dx \right)^{1-p/2}
\]

\[
\leq c \left( \int_{B_{1/4}^+} |D\tilde{g}|^{p-1} |D\tilde{v} - D\tilde{w}| \, dx \right)^{p/2} \left( \int_{B_{1/4}^+} |D\tilde{w}|^p \, dx \right)^{1-p/2}
\]

\[
\leq c \left( \int_{B_{1/4}^+} |D\tilde{v} - D\tilde{w}|^p \, dx \right)^{1/2} \left( \int_{B_{1/4}^+} |D\tilde{g}|^p \, dx \right)^{p/2} \left( \int_{B_{1/4}^+} |D\tilde{w}|^p \, dx \right)^{2-p/2},
\]

from which (5.50) follows again via Young’s inequality with conjugate exponents \((1/(p-1), 1/(2-p))\).

Combining (5.49) with (5.50) in a standard way, we arrive at

\[
\int_{B_1^+} (|D\tilde{w}|^2 + \mu^2)^{p/2} \, dx
\]

\[
\leq c \left( \frac{t^n}{\varrho^n} + \varepsilon \right) \int_{B_{1/4}^+} (|D\tilde{w}|^2 + \mu^2)^{p/2} \, dx + c \int_{B_{1/4}^+} |D\tilde{g}|^q \, dx \right)^{p/q} \varrho^{n(1-p/q)},
\]

for all \(t \leq \varrho/4\), where \(c \equiv c(n, p, \Lambda)\). By recalling the definition of \(\tilde{w}\), the above inequality holds with \(\tilde{w}\) replaced by \(w\), so that (5.44) follows applying Lemma 2.6 with the choice \(h(t) := ||(D\tilde{w}|^2 + \mu^2)^{p/2}||_{L^1(B_t)}\).

6. PROOF OF THEOREM 8

Here we work under the assumptions of Theorem 3 that is (3.1)–(3.7) with \(\kappa \geq s\) and \(u\) denotes the minimizer considered there. In the following, \(B_\varrho \equiv B_\varrho(x_0) \subset \Omega\) shall denote a generic ball such that \(0 < \varrho \leq 1\) and all the balls will be centred at \(x_0\). Moreover, \(\beta_0, \beta_0 < 1\) and \(\lambda > 0\); their precise value will depend on the context they are going to be employed in. We start with two preliminary lemmas.

**Lemma 6.1.** Under the assumptions on Theorem 8
If $s < \beta_1 < 1$, then
\begin{equation}
\int_{B_{r/2}} \int_{B_{r/2}} \frac{|u(x) - u(y)|^\gamma}{|x - y|^{n+\gamma}} \, dx \, dy \leq c \rho^{(\beta_1 - s)\gamma}
\end{equation}
holds with $c \equiv c(\text{data}_u, \text{dist}(B_p, \partial \Omega), \beta_1)$.

If $\beta_2 < s$, then
\begin{equation}
t^{-\delta}[\text{snail}_d(t)]^\gamma \equiv t^{-\delta}[\text{snail}_d(u, B_t(x_0))]^\gamma \leq ct^\gamma(\beta_2 - s)
\end{equation}
holds whenever $0 < t \leq \theta$, where $c \equiv c(\text{data}, \beta_2)$.

If $\lambda > 0$, then
\begin{equation}
\int_{B_{r/2}} (|Du|^2 + \mu^2)^{p/2} \, dx \leq c \rho^{-p\lambda}
\end{equation}
holds with $c \equiv c(\text{data}, \text{dist}(B_p, \partial \Omega), \lambda)$.

**Proof.** Estimate in (6.1) is a direct consequence of Theorem 2 with $\alpha \equiv \beta_1$. To prove (6.2) we use Theorem 3 with $\alpha \equiv \beta_2$ (recall that in Theorem 4 we are assuming $\kappa \geq s$, so that we can choose any positive $\beta_2 < s$), and estimate as follows:
\begin{equation}
t^{-\delta}[\text{snail}_d(t)]^\gamma \leq c \int_{\mathbb{R} \setminus B_t} \frac{|u(y) - u(x_0)|^\gamma}{|y - x_0|^{n+\gamma}} \, dy + ct^{-\gamma} |u(x_0) - (u)_{B_t(x_0)}|^\gamma \leq c \int_{\mathbb{R} \setminus B_t} \frac{|y - x_0|^{-n}(s - \beta_2)^\gamma}{|y - x_0|^{n+\gamma}} \, dy + ct^\gamma(\beta_2 - s) \leq ct^\gamma(\beta_2 - s),
\end{equation}
where $c \equiv c(\text{data}, \beta_2)$, that is (6.2). Finally, to prove (6.3) we use (6.1) and estimate the various terms stemming from $c_{cp,b}(\rho) \equiv c_{cp,b}(u, B_p(x_0))$, whose definition is in (3.6). Again by Theorem 2 we have that $\varphi^{-\rho}[\text{av}_r(\rho)]^p + \varphi^{-s\gamma}[\text{av}_r(\rho)]^\gamma \leq c \rho^{p(\beta_1 - 1)} + c \rho^s(\beta_1 - s) \leq c \rho^{p(\beta_1 - 1)}$ holds with $c \equiv c(\text{data}_u, \text{dist}(B_p, \partial \Omega), \beta_1)$ (for the precise dependence of the constant also recall Proposition 2.1). By (6.2) we get the last bound $\varphi^{-\delta}[\text{snail}_d(\rho)]^\gamma \leq c \rho^{\gamma(\beta_2 - s)}$. Finally, $\|f\|_{L^\infty(B_4)}^p + 1 \leq c \rho^{\gamma(\beta_2 - s)}$ holds trivially. Choosing $\beta_1$ such that $1 - \beta_1 \leq \lambda$ and $\beta_2$ such that $\gamma(s - \beta_2) \leq 4\lambda$, we arrive at (6.3). \qed

**Lemma 6.2.** If $h \in u + W^{1,p}_d(B_{\rho/4}(x_0))$ as in (1.10), then
\begin{equation}
\int_{B_{\rho/4}(x_0)} |Du - Dh|^p \, dx \leq c \rho^{sp^2p}
\end{equation}
holds where $\sigma_2 \equiv \sigma_2(n, p, s, \gamma, d) \in (0, 1)$, and $c \equiv c(\text{data}, \text{dist}(B_p, \partial \Omega), \lambda)$.

**Proof.** We go back to Lemma 4.2, estimate (4.14), and, adopting the notation introduced there, we improve the estimates for the terms (I)-(III). As in (4.15) and in Lemma 4.1 we find
\begin{equation}
|\text{III}| \leq c \|f\|_{L^\infty(B_{\rho/4})} \left( \int_{B_{\rho/4}} (|Du|^2 + \mu^2)^{p/2} \, dx \right)^{1/p} \leq c \|f\|_{L^\infty(B_{\rho/4})} \theta^{1-n/d-\lambda},
\end{equation}
for every $\lambda > 0$, where $c \equiv c(\text{data}, \text{dist}(B_p, \partial \Omega), \lambda)$. In order to estimate terms (II) and (III), we recall that a basic consequence of the maximum is
\begin{equation}
\text{osc}_{B_{\rho/4}} h \leq \text{osc}_{B_{\rho/4}} u.
\end{equation}
Recall also that $u$ is Hölder continuous; by the Maz'ya-Wiener boundary regularity theory, $h$ is continuous on $B_{\rho/4}$ and therefore
\begin{equation}
\|u\|_{L^\infty(B_{\rho/4})} = \|u - h\|_{L^\infty(B_{\rho/4})} \leq 2 \text{osc}_{B_{\rho/4}} u \leq 4 \|u\|_{L^\infty(B_{\rho/4})} \theta^{\delta_1} \leq c \theta^{\delta_1},
\end{equation}
where $c \equiv c(\text{data}, \text{dist}(B_p, \partial \Omega), \beta_1)$. For (II), as in (4.10), we have, with $w = u - h$ (defined and extended as in Lemma 4.2) so that $w \equiv 0$ outside $B_{\rho/4}$
\begin{equation}
|\text{II}| \leq c \theta^{\delta_1(\gamma - 1)} \left( \int_{B_{\rho/4}} \int_{B_{\rho/4}} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+\gamma}} \, dx \, dy \right)^{1/\gamma}
\end{equation}
Whenever \( s < \beta_1 < 1 \) and \( \lambda > 0 \), where \( c \equiv c(\text{data}, \text{dist}(B_\rho, \partial \Omega), \beta_1, \lambda) \). To estimate (III) we restart from the fifth line of (11.7), and using also (6.2) and (6.7), we easily find

\[
\frac{1}{2} < s < \beta_2 < \frac{1}{4(\gamma - 1)}
\]

and plug (6.5), (6.8) and (6.9) into (11.4), to obtain

\[
\int_{B_{\rho/4}} |V_\mu(Du) - V_\mu(Dh)|^2 \, dx \leq c \sigma_1^{p/4}, \quad \sigma_1 := \frac{1}{p} \min\left\{ \frac{1}{\beta}, \frac{1 - s}{4}, \frac{1 - s}{4}, \frac{1 - s}{4(\gamma - 1)} \right\} > 0
\]

for \( c \equiv c(\text{data}, \text{dist}(B_\rho, \partial \Omega)) \). Now, we want to prove that

\[
\int_{B_{\rho/4}} |Du - Dh|^p \, dx \leq c \sigma_2^{p/4},
\]

where \( \sigma_2 = \sigma_1 \) if \( p \geq 2 \) and \( \sigma_2 := \sigma_1 p/4 \) if \( 1 < p < 2 \). Indeed, If \( p \geq 2 \), then (11.11) follows thanks to (2.9) and (6.10). When \( p \in (1, 2) \), as in (11.2) and (11.7), we have

\[
\int_{B_{\rho/4}} |Du - Dh|^p \, dx \leq c \sigma_1^{p/2 - \lambda(1 - p/2)} p.
\]

By choosing \( \lambda \) such that \( \sigma_1 p/2 - \lambda(1 - p/2) > \sigma_1 p/4 \), we finally conclude with (11.11). \( \square \)

Once (11.11) is established, we can conclude with the local Hölder continuity of \( Du \) by means of a by now classical comparison argument (see for instance [52]). We briefly report it here for the sake of completeness. We select an open subset \( B_\rho \Subset \Omega \), let \( r_* := \text{dist}(B_\rho, \partial \Omega)/4 \), and fix a generic point \( x_0 \in \Omega \); from now we consider balls of the type \( B_{2r} \equiv B_{2r}(x_0) \) with \( r_* \leq r \), and all the balls will be centred in \( x_0 \).

Next, we recall the following classical decay estimate, which is satisfied by \( h \)

\[
\text{osc}_{B_{\rho/4}} Dh \leq c \left( \frac{1}{\rho} \right)^{\alpha_0} \left( \int_{B_{\rho/4}} (|Dh|^2 + \mu^2)^{p/2} \right)^{1/p},
\]

that holds whenever \( 0 < t \leq \rho/8 \), where \( c \equiv c(n, \mu, \Lambda) \geq 1 \) and \( \alpha_0 \equiv \alpha_0(n, \mu, \Lambda) \in (0, 1) \); see [51,52]. We estimate, also using (11.11)

\[
\int_{B_{\rho/4}} |Du - (Du)_{B_{\rho/4}}|^p \, dx \leq c \left( \frac{1}{\rho} \right)^{\alpha_0} \left( \int_{B_{\rho/4}} (|Dh|^2 + \mu^2)^{p/2} \right)^{1/p} \leq c \left( \frac{1}{\rho} \right)^{\alpha_0} \int_{B_{\rho/4}} (|Dh|^2 + \mu^2)^{p/2} \, dx + c \left( \frac{1}{\rho} \right)^n \theta^{\sigma_2 p},
\]

(11.13)
with $c \equiv c(\text{data}, r_*, \lambda)$. In the above inequality, we take $t = t^{1+\sigma_3/(2n)}/8$ and choose $\lambda := \sigma_2 p_0/(4n)$ in (6.3). We conclude with
\[
\int_{B_1} |Du - (Du)_{B_1}|^p \, dx \leq \alpha^{p^n}, \quad \alpha := \frac{\sigma_2 p_0}{2\sigma_2 p + 4n},
\]
where $c \equiv c(\text{data}, r_*, \lambda)$. This holds whenever $B_1 \subseteq \Omega$ is a ball centred in $\Omega_0$, with $t \leq t^{1+\sigma_3/(2n)}/8$. As the $\Omega_0 \supseteq \Omega$ is arbitrary, this implies the local $C^{1,\alpha}$-regularity of $Du$ in $\Omega$, via the classical Campanato’s integral characterization of $Du$ together with the estimate for $|Du|_{0,\alpha,\Omega_0}$, and the proof is complete.

References

[1] R.A. Adams, J. F. Fournier, Sobolev spaces. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003. xiv+305.
[2] B. Avellan, T. Kuusi, G. Mingione, Nonlinear Calderón-Zygmund theory in the limiting case. Arch. Ration. Mech. Anal. 227, 663–714, (2018).
[3] L. Beck, Boundary regularity results for weak solutions of subquadratic elliptic systems. Ph. D. Thesis, Erlangen, (2008).
[4] L. Beck, Boundary regularity for elliptic problems with continuous coefficients. J. Convex Anal. 16, 287–320, (2009).
[5] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, Mixed local and nonlocal elliptic operators: regularity and maximum principles. Commun. PDE (2021). [https://doi.org/10.1080/03605302.2021.1998599]
[6] S. Biagi, S. Dipierro, E. Valdinoci & E. Vecchi, Semilinear elliptic equations involving mixed local and nonlocal operators. Proc. Royal Soc. Edinburgh A: Mathematics 151, 1611–1641, (2021).
[7] S. Biagi, S. Dipierro, E. Valdinoci & E. Vecchi, A Faber-Krahn inequality for mixed local and nonlocal operators. arXiv:2104.00810
[8] S. Biagi, S. Dipierro, E. Valdinoci & E. Vecchi, A Hong-Krahn-Szego inequality for mixed local and nonlocal operators. arXiv:2110.07129
[9] L. Brasco, E. Lindgren, Higher Sobolev regularity for the fractional $p$-Laplace equation in the supercritical case. Adv. in Math. 304, 300–354, (2017).
[10] L. Brasco, E. Lindgren, A. Schikorra, Higher Hölder regularity for the fractional $p$-Laplacian in the supercritical case. Adv. in Math. 358, 782–846, (2018).
[11] L. Brasco, E. Parini, The second eigenvalue of the fractional Laplacian. Adv. Calc. Var. 9, 323–355, (2016).
[12] H. Brezis, P. Mironescu, Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces. J. Evol. Equ. 1, 387–404, (2001)
[13] S.S. Byun, H. Kim, J. Ok, Local Hölder continuity for fractional nonlinear equations with general growth. arXiv:2112.13995
[14] S.S. Byun, J. Ok, K. Song, Hölder regularity for weak solutions to nonlocal double phase problems. arXiv:2108.05623
[15] J. Chaker, M. Kassmann, Nonlocal operators with singular anisotropic kernels. J. Convex Anal. 16, 287–320, (2009).
[16] J. Chaker, M. Kim, M. Weidner, Regularity for nonlocal problems with non-standard growth. arXiv:2111.09182
[17] L. Brasco, E. Lindgren, Higher Sobolev regularity for the fractional $p$-Laplace equation in the supercritical case. Adv. in Math. 304, 300–354, (2017).
[18] Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček, Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets and their applications. Illinois J. Math. 54 981–1024, (2010).
[19] Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček, Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$. Trans. Amer. Math. Soc. 364, 4169–4205, (2012).
[20] Z.-Q. Chen, T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps. Rev. Mat. Iberoam. 26, 551-589, (2010).
[21] A. Cianchi, Maximizing the $L^\infty$ norm of the gradient of solutions to the Poisson equation. J. Geom. Anal. 2, 499–515, (1992).
[22] M. Colombo, G. Mingione, Calderón-Zygmund estimates and non-uniformly elliptic operators. J. Funct. Anal. 270, 1446–1478, (2016).
[23] J. Da Silva, A. M. Salort, A limiting problem for local/non-local $p$-Laplacians with concave-convex nonlinearities. Z. Ang. Math. Physik 71(6), 191, (2020).
[24] C. De Filippis, G. Mingione, Interpolative gap bounds for nonuniformly integrable. Anal. Math. Physics 11, 117, (2021).
[25] C. De Filippis, G. Mingione, Lipschitz bounds and nonuniformly integrals. Arch. Ration. Mech. Anal. 242, 973–1057, (2021).
[26] C. De Filippis, G. Mingione, On the regularity of minima of non-autonomous functionals. J. Geom. Anal. 30:1584–1626, (2020).
[27] C. De Filippis, G. Palatucci, Hölder regularity for nonlocal double phase equations. J. Diff. Equ. 267, 547–586, (2019).
[28] R. DeVore, R.C. Sharpley, Maximal functions measuring smoothness. Mem. Amer. Math. Soc. 47 no. 293, vii+115 pp. (1984).
[29] A. Di Castro, T. Kuusi, G. Palatucci, Local behavior of fractional $p$-minimizers. Ann. IHP–AN 33, 1279–1299, (2016).
[30] A. Di Castro, T. Kuusi, G. Palatucci, Nonlocal Harnack inequalities. J. Funct. Anal. 267, 1807–1836, (2014).
[31] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 136, 5, 521–573, (2012).
[32] S. Dipierro, E. Prieto Lippi, E. Valdinoci, (Non)local logistic equations with Neumann conditions arXiv:2101.02315
[33] L. Esposito, F. Leonetti, G. Mingione, Sharp regularity for functionals with $(p, q)$ growth. J. Diff. Equ. 204, 5–55, (2004).
[34] Y. Fang, B. Shang, C. Zhang, Regularity theory for mixed local and nonlocal parabolic $p$-Laplace equations. J. Geom. Anal. 32, 22, (2022).
[35] Y. Fang, C. Zhang, On Weak and viscosity solutions of nonlocal double phase equations. Int. Math. Res. Not. https://doi.org/10.1093/imrn/rnab351
[36] P. Garain, J. Kinnunen, On the regularity theory for mixed local and nonlocal quasilinear elliptic equations. Trans. Amer. Math. Soc. [https://doi.org/10.1090/tran/8622]
[37] M. Giona, D. Guidetti, Differentiability of minima of non-differentiable functionals. Invent. Math. 72, 285–298 (1983).
[37] E. Giusti, *Direct Methods in the Calculus of Variations*. World Scientific Publishing Co., Inc., River Edge (2003).
[38] C. Hamburger, Regularity of differential forms minimizing degenerate elliptic functionals. *J. Reine Angew. Math. (Crelles J.)* 431, 7–64, (1992).
[39] M.W. Hirsch, *Differential topology*. GMT, No. 33. Springer-Verlag, New York-Heidelberg, 1976. x+221 pp.
[40] L. Koch, Global higher integrability for minimisers of convex obstacle problems with \((p, q)\)-growth, *Calc. Var. & PDE* 61:88, (2022).
[41] J. Korvenpää, T. Kuusi, E. Lindgren. Equivalence of solutions to fractional \(p\)-Laplace type equations. *J. Math. Pures Appl.* (9) 132, 1–26, (2019).
[42] J. Korvenpää, T. Kuusi, G. Palatucci, The obstacle problem for nonlinear integro-differential operators. *Calc. Var. & PDE* 55:63, (2016).
[43] T. Kuusi, G. Mingione, Y. Sire, Nonlocal equations with measure data. *Comm. Math. Phys.* 337, 1317–1368, (2015).
[44] T. Kuusi, G. Mingione, Y. Sire, Nonlocal self-improving properties. *Anal. PDE* 8, 57–114, (2015).
[45] J. Kristensen, C. Melcher, Regularity in oscillatory nonlinear elliptic systems. *Math. Z.* 260, 813–847, (2008).
[46] M. Kronz, Boundary regularity for almost minimizers of quasiconvex variational problems. *NoDEA* 12, 351–382, (2005).
[47] O. A. Ladyzhenskaya, N. N. Ural’tseva, *Linear and quasilinear elliptic equations*. Academic Press Inc. (1968).
[48] J. L. Lewis, Smoothness of certain degenerate elliptic equations. *Proc. Amer. Math. Soc.* 80, 259–265, (1980).
[49] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations. *Comm. PDE* 16, 311–361, (1991).
[50] E. Lindgren, Hölder estimates for viscosity solutions of equations of fractional \(p\)-Laplace type. *NoDEA* 23, nr. 5, 55, (2016).
[51] J.J. Manfredi, *Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations*. Ph.D. Thesis, University of Washington, St. Louis (1986).
[52] J.J. Manfredi, Regularity for minima of functionals with \(p\)-growth. *J. Diff. Equ.* 76, 203–212, (1989).
[53] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Arch. Ration. Mech. Anal.* 105, 267–284, (1989).
[54] J. M. Scott, T. Mengesha. Self-improving inequalities for bounded weak solutions to nonlocal double phase equations. *Commun. Pure Appl. Anal.* 21, 183–212, (2022).
[55] G. Palatucci, The Dirichlet problem for the \(p\)-fractional Laplace equation. *Nonlinear Anal.* 177, 699–732, (2018).
[56] N. Soave, H. Tavares, S. Terracini, A. Zilio, Variational Problems with Long-Range Interaction. *Arch. Rat. Mech. Anal.* 228, 743–772, (2018).
[57] H. Triebel, *Theory of function spaces*. Birkhäuser, Basel and Boston, (1983).
[58] E. Valdinoci, Regularity for a class of non-linear elliptic systems. *Acta Math.* 138, 219–240, (1977).
[59] N.N. Ural’tseva, Degenerate quasilinear elliptic systems. *Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 7, 184–222, (1968).

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