Supervariable approach to nilpotent symmetries of a couple of \( \mathcal{N} = 2 \) supersymmetric quantum mechanical models

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Abstract: We derive the on-shell as well as off-shell nilpotent supersymmetric (SUSY) symmetry transformations for the \( \mathcal{N} = 2 \) SUSY quantum mechanical model of a one (0+1)-dimensional (1D) free SUSY particle by exploiting the SUSY invariant restrictions (SUSYIRs) on the (anti-)chiral supervariables of the SUSY theory that is defined on a (1, 2)-dimensional supermanifold (parametrized by a bosonic variable \( t \) and a pair of Grassmannian variables \( \theta \) and \( \bar{\theta} \) with \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \)). Within the framework of our novel approach, we express the Lagrangian and conserved SUSY charges in terms of the (anti-)chiral supervariables to demonstrate the SUSY invariance of the Lagrangian as well as the nilpotency of the SUSY conserved charges in a simple manner. Our approach has the potential to be generalized to the description of other \( \mathcal{N} = 2 \) SUSY quantum mechanical systems with physically interesting potential functions. To corroborate the above assertion, we apply our method to derive the \( \mathcal{N} = 2 \) continuous and nilpotent SUSY transformations for one of the simplest interacting SUSY system of a 1D harmonic oscillator.

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1 Introduction

One of the most elegant, intuitive and geometrically rich approaches to derive the “quantum” symmetries corresponding to the “classical” gauge and/or reparameterization symmetries is the superfield formalism [1-8]. In particular, Bonora-Tonin (BT) superfield approach [4,5] is very suitable for the derivation of the off-shell nilpotent and absolutely anticommuting Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetries for a given $p$-form ($p=1,2,3,...$) (non-)Abelian gauge theory in any arbitrary dimension of spacetime where the celebrated horizontality condition (HC) plays a very decisive role. The HC leads to the derivation of “quantum” gauge [i.e. (anti-)BRST] symmetries for the bosonic gauge and corresponding fermionic (anti-)ghost fields of the (anti-)BRST invariant gauge theory. This statement is true in any arbitrary dimension of spacetime.

In a set of papers (see, e.g. [9-12]), the augmented version of BT-formalism has been developed where, in addition to the HC, the gauge invariant restrictions (GIRs) on the superfields have been also imposed for the derivation of the “quantum” gauge [i.e. (anti-)BRST] symmetries for the matter fields, in addition to the (anti-)BRST symmetries for the gauge and corresponding (anti-)ghost fields (which emerge from the HC) for a given $p$-form interacting (non-)Abelian gauge theory. As it has turned out, the nilpotent (anti-)BRST symmetries and their geometrical interpretations derived from the HC and GIRs, have been found to be consistent with one-another thereby leading to the derivation of full set of proper (i.e. nilpotent and absolutely anticommuting) (anti-)BRST symmetries for a given $p$-form interacting gauge theory.

The above superfield formalisms [1-12] have, however, not yet been exploited in the context of supersymmetric (SUSY) theories where the nilpotent symmetries are also found to exist. The purpose of our present investigation is to exploit the key ideas of the augmented version of superfield formalism [9-12] (without any use of the HC) to derive the on-shell nilpotent SUSY symmetries of one of the simplest $\mathcal{N}=2$ SUSY quantum mechanical models of the free SUSY particle\footnote{We also briefly mention about the derivation of $\mathcal{N}=2$ SUSY symmetries for an interacting 1D model of SUSY harmonic oscillator in our Appendix A and derive the off-shell nilpotent $\mathcal{N}=2$ SUSY symmetries for the free SUSY particle in our Appendix B.}. We accomplish our goals by taking the help of chiral and anti-chiral supervariables and imposing SUSY invariant restrictions (SUSYIRs) on them. Our method of derivation of SUSY transformations, to the best of our knowledge, has never been exploited in the context of $\mathcal{N}=2$ SUSY quantum mechanical (and/or field theoretic) examples as far as the derivation of symmetries is concerned.

The main motivating factors behind our present investigation are as follows. First, it is very challenging to find out an alternative to the usual mathematical method of deriving the SUSY symmetries for a SUSY quantum mechanical model. In our present investigation, we accomplish this goal by exploiting the SUSY invariance. Second, our approach is physically more appealing because we exploit the SUSY invariance to put restrictions on the (anti-)chiral supervariables. Third, to prove the generality of our approach, we discuss one of the simplest interacting $\mathcal{N}=2$ SUSY system of a harmonic oscillator in our Appendix A. Finally, our present attempt is our modest first step in the direction of developing a general rule to derive the SUSY symmetry transformations for the $\mathcal{N}=2$ SUSY systems of physical interest.
The contents of our present investigation are organized as follows. In our Sec. 2, we exploit the bare essentials of $\mathcal{N} = 2$ superspace formulation to derive the action (and/or Lagrangian) for our present model of $\mathcal{N} = 2$ SUSY free particle. Our Sec. 3 is devoted to the derivation of SUSY transformations generated by one of the two SUSY charges by using the anti-chiral supervariables. We derive the other SUSY transformations in Sec. 4 by exploiting the chiral supervariables. Our Sec. 5 contains the derivation of standard $\mathcal{N} = 2$ SUSY algebra and we provide its interpretation in the language of cohomological operators of differential geometry. Finally, we make some concluding remarks in Sec. 6 and say a few words about the reasons behind our choice of (anti-)chiral supervariables in the context of our present discussion.

In our Appendix A, we apply our method to derive the off-shell as well as on-shell $\mathcal{N} = 2$ SUSY continuous symmetry transformations for one of the simplest interacting SUSY system of a one $(0 + 1)$-dimensional harmonic oscillator. Our Appendix B is devoted to the derivation of $\mathcal{N} = 2$ off-shell nilpotent ($s_1^2 = s_2^2 = 0$) SUSY symmetry transformations for the free SUSY particle.

General notations and conventions: Throughout the whole body of text, we shall denote the fermionic ($s_1^2 = s_2^2 = 0$) $\mathcal{N} = 2$ SUSY transformations by $s_1$ and $s_2$ and shall adopt the convention of the left derivative w.r.t. the fermionic variables. We shall use the notations $(x(t), \psi(t), \bar{\psi}(t))$ for the bosonic variable $(x)$ and a pair of fermionic variables $\psi$ and $\bar{\psi}$ for the $\mathcal{N} = 2$ SUSY QM theory. The corresponding supervariables would be denoted by $X, \Psi, \bar{\Psi}$ which would be defined on the (anti-)chiral super-submanifolds. The auxiliary (super)variables would be represented by $(\bar{A})A$.

2 Preliminaries: $\mathcal{N} = 2$ superspace approach to derive the action integral and $\mathcal{N} = 2$ SUSY symmetries for the free SUSY particle

We begin with the supervariable $[X(Z) \equiv X(t, \theta, \bar{\theta})]$ where the $\mathcal{N} = 2$ superspace coordinates $Z^M = (t, \theta, \bar{\theta})$ are parameterized by the bosonic evolution parameter $t$ and a pair of Grassmannian variables $\theta$ and $\bar{\theta}$ (with $\bar{\theta}^2 = \theta^2 = 0, \theta \bar{\theta} = 0$). The above supervariable can be expanded along the Grassmannian directions $\theta$ and $\bar{\theta}$ of the $(1, 2)$-dimensional supermanifold (on which our present theory is considered) as follows (see, e.g. [13,14])

$$X(t, \theta, \bar{\theta}) = x(t) + i \theta \bar{\psi}(t) + i \bar{\theta} \psi(t) + \theta \bar{\theta} A(t),$$

(1)

where, on the r.h.s., the component variables $(x, A)$ are bosonic and $(\psi, \bar{\psi})$ are fermionic ($\psi^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0$) at the classical level. The basic dynamical variables of the $\mathcal{N} = 2$ SUSY free particle are $x(t), \psi(t)$ and $\bar{\psi}(t)$, in terms of which, the Lagrangian function is defined. The latter includes the corresponding generalized “velocities” (i.e. $\dot{x}(t), \dot{\psi}(t), \dot{\bar{\psi}}(t)$), too. The auxiliary variable $A(t)$ does not play any role in the description of the free $\mathcal{N} = 2$ SUSY particle because it is connected with the potential function of a given physical system. Thus, for our further discussions (on the free SUSY particle within the supervariable approach), we set $A = 0$, right from the beginning, in the above expansion (1) for the supervariable $X(t, \theta, \bar{\theta})$. 

3
The action integral for the $\mathcal{N} = 2$ SUSY free particle can be written as [13-15]

$$S = \int dt \int d\theta \int d\bar{\theta} \mathcal{D}X(t, \theta, \bar{\theta}) \mathcal{D}X(t, \theta, \bar{\theta}),$$

where $\mathcal{D}$ and $\mathcal{D}$ are the super covariant derivatives

$$\mathcal{D} = \frac{\partial}{\partial \theta} - i \theta \frac{\partial}{\partial \bar{\theta}} \equiv \partial_{\theta} - i \theta \partial_{\bar{\theta}}, \quad \mathcal{D} = \frac{\partial}{\partial \bar{\theta}} - i \bar{\theta} \frac{\partial}{\partial \theta} \equiv \partial_{\bar{\theta}} - i \bar{\theta} \partial_{\theta}. \quad (3)$$

The superspace derivatives $\partial_M = \partial/\partial Z^M \equiv (\partial_t, \partial_\theta, \partial_{\bar{\theta}})$ are the generators of the shift transformations along the superspace coordinates as:

$$t \rightarrow t' = t + i(\varepsilon \bar{\theta} + \varepsilon \theta), \quad \theta \rightarrow \theta' = \theta + \varepsilon, \quad \bar{\theta} \rightarrow \bar{\theta}' = \bar{\theta} + \bar{\varepsilon}, \quad (4)$$

where $\varepsilon$ and $\bar{\varepsilon}$ are the infinitesimal time-independent shift parameters along the Grassmannian directions $\theta$ and $\bar{\theta}$ of the $(1, 2)$-dimensional supermanifold. As a consequence, these parameters are also fermionic in nature ($\varepsilon^2 = \bar{\varepsilon}^2 = 0, \varepsilon \bar{\varepsilon} + \bar{\varepsilon} \varepsilon = 0$).

Substitutions of $X(t, \theta, \bar{\theta})$ from (1) (with $A = 0$) and the operations of the super covariant derivatives (3) on them, finally, lead to the derivation of the action integral ($S = \int dt \mathcal{L}_0$) for the $\mathcal{N} = 2$ SUSY free particle as follows:

$$S = \int dt \left[ \frac{1}{2} \dot{x}^2 - \frac{i}{2} (\dot{\psi} \psi - \bar{\psi} \bar{\psi}) \right] \equiv \int dt \mathcal{L}_0, \quad (5)$$

where we have already performed the Grassmannian integrations and, for the sake of brevity, we have chosen the mass ($m$) of the $\mathcal{N} = 2$ SUSY free particle to be one (i.e. $m = 1$). Ultimately, we obtain the Lagrangian ($\mathcal{L}_0$) for the free particle as

$$\mathcal{L}_0 = \frac{1}{2} \dot{x}^2 + i \bar{\psi} \dot{\psi}, \quad (m = 1) \quad (6)$$

where $\dot{x} = (dx/dt), \dot{\psi} = (d\psi/dt)$ and we have dropped a total time derivative term. It is elementary to check that the Euler-Lagrange (EL) equations of motion are: $\ddot{x} = 0, \dot{\psi} = 0, \bar{\psi} = 0$. These EL equations of motion ensure that there is no potential function (and/or force) for the description of our present model (which is nothing but the $\mathcal{N} = 2$ SUSY free particle). The analogue of the Lagrangian (6) can be derived from the general $\mathcal{N} = 2$ superspace approach for any arbitrary potential function (see, e.g. [15]). For instance, the Lagrangian for the $\mathcal{N} = 2$ SUSY harmonic oscillator (see Appendix A) is a special case of $\mathcal{N} = 2$ SUSY theory for the general potential function (see, e.g. [13-15] for details) where $A(t)$ is chosen in a particular fashion (i.e. $A = \omega x$).

Two SUSY transformations ($\delta_1$ and $\delta_2$) can be computed from the standard superspace formula for the $\mathcal{N} = 2$ SUSY quantum mechanical theory. These transformations are

$$(\delta_1 + \delta_2) X(t, \theta, \bar{\theta}) = (\delta_1 + \delta_2) x(t) + i \theta (\delta_1 + \delta_2) \bar{\psi}(t) + i \bar{\theta} (\delta_1 + \delta_2) \psi(t)$$

$$+ \theta \bar{\theta} (\delta_1 + \delta_2) A(t) \equiv (\varepsilon \bar{Q} + \bar{\varepsilon} Q) X(t, \theta, \bar{\theta}), \quad (7)$$

where $Q$ and $\bar{Q}$ are the $\mathcal{N} = 2$ SUSY (fermionic) charges (with $Q^2 = \bar{Q}^2 = 0$) that are defined, in their operator form, as follows:

$$Q = \partial_{\bar{\theta}} + i \theta \partial_{\theta}, \quad \bar{Q} = \partial_{\theta} + i \bar{\theta} \partial_{\bar{\theta}}. \quad (8)$$
Application of (8) in (7) (with $A = 0$) leads to the following transformations:

\[
\begin{align*}
\delta_1 x &= i \bar{\varepsilon} \psi, & \delta_1 \psi &= 0, & \delta_1 \bar{\psi} &= -\bar{\varepsilon} \dot{x}, \\
\delta_2 x &= i \varepsilon \psi, & \delta_2 \psi &= 0, & \delta_2 \bar{\psi} &= -\varepsilon \dot{x}.
\end{align*}
\]

(9)

Infinitesimal versions of the fermionic ($s_1^2 = s_2^2 = 0$) transformations ($s_1, s_2$) can be derived from the above bosonic infinitesimal transformations by defining $\delta_1 = \bar{\varepsilon} s_1$, $\delta_2 = \varepsilon s_2$. These fermionic ($s_1^2 = s_2^2 = 0$) transformations, for the Lagrangian (6), are

\[
\begin{align*}
s_1 x &= i \psi, & s_1 \psi &= 0, & s_1 \bar{\psi} &= -\dot{x}, \\
s_2 x &= i \bar{\psi}, & s_2 \bar{\psi} &= 0, & s_2 \psi &= -\dot{x}.
\end{align*}
\]

(10)

It is elementary to check that the action integral $S = \int dt L_0$ remains invariant under the above SUSY transformations $s_1$ and $s_2$ because we have the following

\[
s_1 L_0 = 0, \quad s_2 L_0 = \frac{d}{dt}(i \dot{x} \bar{\psi}).
\]

(11)

Using Noether’s theorem, it is straightforward to derive the conserved ($\dot{Q} = 0$, $\dot{\bar{Q}} = 0$) charges $Q = (i \dot{x} \psi)$ and $\bar{Q} = (i \dot{x} \bar{\psi})$ which are nilpotent of order two (i.e. $Q^2 = \bar{Q}^2 = 0$). In the forthcoming sections, we shall derive the transformations (10) by our novel supervariable approach.

3 On-shell nilpotent SUSY symmetry transformations generated by $Q$: Anti-chiral supervariables

The central aim of our present section and the forthcoming section is to capture the SUSY transformations $s_1$ and $s_2$ [cf. (10)] in the language of Grassmannian derivatives ($\partial_\theta$, $\partial_{\bar{\theta}}$) defined on the $(1, 2)$-dimensional supermanifold. Towards this goal in mind, we note that the fermionic ($s_1^2 = s_2^2 = 0$) transformations (10) are not absolutely anticommuting (i.e. $\{s_1, s_2\} \neq 0$). As a consequence, we cannot have expansions like (1) to capture these symmetry transformations in the language of ordinary super derivatives ($\partial_\theta$, $\partial_{\bar{\theta}}$) w.r.t. the Grassmannian variables $\theta$ and $\bar{\theta}$. In other words, we have to truncate the expansion (1) to derive the fermionic ($s_1^2 = s_2^2 = 0$) transformations $s_1$ and $s_2$ independently from the $\mathcal{N} = 2$ SUSY invariant restrictions (SUSYIRs) on the (anti-)chiral supervariables.

To accomplish the above goals, first of all, we focus on the derivation of $s_1$ in the language of the Grassmannian derivative $\partial_\theta$ which is defined on the anti-chiral $(1, 1)$-dimensional super-submanifold (parametrized only by $t$ and $\theta$) of the general $(1, 2)$-dimensional supermanifold. As a first step, we generalize the basic dynamical variables $x(t), \psi(t), \bar{\psi}(t)$ onto

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1Basically, the transformations in (9) (and (10)) are global SUSY transformations because the parameters ($\varepsilon, \bar{\varepsilon}$) are time-independent for the 1D system of a free SUSY particle. One of the key characteristic features of a $\mathcal{N} = 2$ SUSY theory is the existence of these nilpotent (but not absolutely anticommuting) symmetries. (see, also, Sec. V for more discussions).
the (1, 1)-dimensional anti-chiral super-submanifold as:

\[ x(t) \rightarrow X(t, \theta, \bar{\theta}) \big|_{\theta=0} = X(t, \bar{\theta}), \]
\[ X(t, \bar{\theta}) = x(t) + \bar{\theta} f(t), \]
\[ \psi(t) \rightarrow \Psi(t, \theta, \bar{\theta}) \big|_{\theta=0} = \Psi(t, \bar{\theta}), \]
\[ \Psi(t, \bar{\theta}) = \psi(t) + i \bar{\theta} b_1(t), \]
\[ \bar{\psi}(t) \rightarrow \bar{\Psi}(t, \theta, \bar{\theta}) \big|_{\theta=0} = \bar{\Psi}(t, \bar{\theta}), \]
\[ \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) + i \bar{\theta} b_2(t), \] (12)

where the expansions of the supervariables \((X, \Psi, \bar{\Psi})\) are along \(\bar{\theta}\)-direction only and \((b_1, b_2)\) are the bosonic secondary variables and \(f(t)\) is a fermionic secondary variable that would be determined by exploiting the SUSY invariant restrictions on the anti-chiral supervariables in a specific fashion. This approach would be physically more appealing than the standard mathematical method used in Sec. 2 because we shall deal with the SUSY invariance of our present theory.

We observe that the variable \(\bar{\psi}(t)\) is an invariant quantity under the transformation \(s_1\) [cf. (10)]. We demand that such kind of SUSY invariant quantities should not depend on the Grassmannian variable \(\bar{\theta}\) of the (1, 1)-dimensional anti-chiral super-submanifold. Thus, we impose the following restriction on the supervariable:

\[ \Psi(t, \bar{\theta}) = \psi(t) \implies b_1(t) = 0. \] (13)

We also note that \(s_1(x \psi) = 0\) due to the fermionic \((\bar{\psi}^2 = 0)\) nature of the variable \(\psi(t)\). Thus, we put the SUSYIR on the composite anti-chiral supervariables as

\[ X(t, \bar{\theta}) \Psi(t, \bar{\theta}) = x(t) \psi(t). \] (14)

Using (13), we obtain the following relationship:

\[ f(t) \psi(t) = 0. \] (15)

Similarly, we observe that \(s_1(x \dot{\psi}) = 0\) which, finally, implies that

\[ \dot{X}(t, \bar{\theta}) \bar{\Psi}(t, \bar{\theta}) = \dot{x}(t) \bar{\psi}(t) \implies \dot{f}(t) \bar{\psi}(t) = 0. \] (16)

The non-trivial solution of (15) and (16) is \(f(t) \propto \psi(t)\). For algebraic convenience, however, we choose here the fermionic secondary variable \(f(t) = i \psi(t)\).

Now, we focus on (11) and draw the most important conclusion for our purpose that \(s_1 L_0 = 0\). Thus, the Lagrangian function \(L_0\) is itself an invariant quantity. Accordingly, it should remain independent of the Grassmannian variable \(\bar{\theta}\). As a consequence, we have:

\[ \frac{1}{2} \dot{X}^2(t, \bar{\theta}) + i \bar{\Psi}(t, \bar{\theta}) \dot{\Psi}(t, \bar{\theta}) = \frac{1}{2} \dot{x}^2(t) + i \bar{\psi}(t) \dot{\psi}(t). \] (17)

Substitutions from (12) and (13) into the above equation imply the following:

\[ \dot{x}(t) \dot{f}(t) = b_2(t) \dot{\psi}(t). \] (18)
If we take \( f(t) = i \psi(t) \), it is evident that \( b_2(t) = i \dot{x}(t) \). Thus, it is obvious that SUSY invariant restrictions \((13), (14), (16)\) and \((17)\) lead to the determination of secondary variables of expansion \((12)\), in terms of the basic variables\(^1\), as: \( b_1(t) = 0, f(t) = i \psi(t), b_2(t) = i \dot{x}(t) \).

The substitution of the above secondary variables into the expansion \((12)\) of the supervariables implies the following explicit expansions for the supervariables

\[
\begin{align*}
X^{(1)}(t, \bar{\theta}) &= x(t) + \bar{\theta} (i \psi) \equiv x(t) + \bar{\theta} (s_1 x), \\
\Psi^{(1)}(t, \bar{\theta}) &= \psi(t) + \bar{\theta} (0) \equiv \psi(t) + \bar{\theta} (s_1 \psi), \\
\bar{\Psi}^{(1)}(t, \bar{\theta}) &= \bar{\psi}(t) + \bar{\theta} (-\dot{x}) \equiv \bar{\psi}(t) + \bar{\theta} (s_1 \bar{\psi}),
\end{align*}
\]

where the superscript \((1)\) denotes the expansion of the supervariables after the SUSY invariant restrictions that lead to the derivation of the SUSY transformations \(s_1\) under which the Lagrangian \(L_0\) for the \(N = 2\) SUSY free particle remains invariant [cf. \((11)\)]. A close look at the above expansions \((19)\) leads to the following mapping:

\[
\frac{\partial}{\partial \bar{\theta}} \Omega^{(1)}(t, \theta, \bar{\theta})|_{\bar{\theta}=0} = s_1 \omega(t) \implies s_1 \leftrightarrow \partial_{\bar{\theta}},
\]

where, in the above, the anti-chiral generic supervariable \(\Omega^{(1)}(t, \theta, \bar{\theta})|_{\bar{\theta}=0}\) stands for \(X^{(1)}(t, \bar{\theta}), \Psi^{(1)}(t, \bar{\theta})\) and \(\tilde{\Psi}^{(1)}(t, \bar{\theta})\) and generic variable \(\omega(t)\) corresponds to the basic variables \(x(t), \psi(t)\) and \(\bar{\psi}(t)\) of the Lagrangian \(L_0\) defined on the one \((0 + 1)\)-dimensional ordinary manifold. It is evident that \(\partial^2_{\bar{\theta}} = 0\). This property obviously implies the nilpotency \((s_1^2 = 0)\) of the transformations \(s_1\).

The starting Lagrangian \(L_0\) can be generalized onto the \((1, 1)\)-dimensional anti-chiral super-submanifold (of the general \((1, 2)\)-dimensional supermanifold) as follows

\[
L_0 \implies \tilde{L}^{(ac)}_0 = \frac{1}{2} X^{(1)}(t, \bar{\theta}) X^{(1)}(t, \bar{\theta}) + i \Psi^{(1)}(t, \bar{\theta}) \tilde{\Psi}^{(1)}(t, \bar{\theta}),
\]

where the supervariables \(X^{(1)}(t, \bar{\theta}), \Psi^{(1)}(t, \bar{\theta}), \tilde{\Psi}^{(1)}(t, \bar{\theta})\) are defined in \((19)\). Furthermore, the SUSY invariance of \(L_0\) (i.e. \(s_1 L_0 = 0\)) can be captured in the following manner

\[
\frac{\partial}{\partial \bar{\theta}} \tilde{L}^{(ac)}_0 = 0 \iff s_1 L_0 = 0.
\]

Geometrically, this observation shows that the SUSY invariance of the Lagrangian \((L_0)\), under symmetry transformation \(s_1\), is equivalent to the translation of the composite supervariables of the super anti-chiral Lagrangian \((\tilde{L}^{(ac)}_0)\) along \(\bar{\theta}\)-direction of the anti-chiral super-submanifold such that the result is zero. Here the superscript \((ac)\) on the super Lagrangian stands for the anti-chiral behavior of the super Lagrangian \((21)\).

By exploiting the Noether theorem, it is clear that the conserved \((\dot{Q} = 0)\) charge \(Q\) for the nilpotent \((s_1^2 = 0)\) transformation \(s_1\) is \(Q = (i \dot{x} \psi)\) which turns out to be the generator of the transformations \(s_1\) as is evident from the following:

\[
s_1 \phi = -i [\phi, Q]|_{\pm}, \quad \phi = x, \psi, \bar{\psi}, \quad \phi = x, \psi, \bar{\psi}, \quad (23)
\]

\(^1\)We have the freedom to choose \(f(t) = \pm i \psi(t)\) and \(b_2(t) = \pm i \dot{x}(t)\) (modulo some constant multiplicative factors). However, we have chosen \(f(t) = + i \psi(t), b_2(t) = + i \dot{x}(t)\) to be consistent with the transformations \((10)\) which have been derived from the mathematical method of superspace formalism.
where the (±) signs, as the subscripts on the square bracket, stand for the (anti)commutator for the generic variable φ being (fermionic) bosonic in nature. This charge can be expressed in terms of the supervariables (19) in two different ways:

\[ Q = \frac{\partial}{\partial \bar{\theta}} [ -i \bar{\Psi}^{(1)}(t, \bar{\theta}) \Psi^{(1)}(t, \bar{\theta}) ] \equiv \int d\bar{\theta} \left[ -i \bar{\Psi}^{(1)}(t, \bar{\theta}) \Psi^{(1)}(t, \bar{\theta}) \right], \]

\[ Q = \frac{\partial}{\partial \bar{\theta}} [ \dot{x}(t) X^{(1)}(t, \bar{\theta}) ] \equiv \int d\bar{\theta} \left[ \dot{x}(t) X^{(1)}(t, \bar{\theta}) \right]. \]

(24)

In view of the mapping (20), it is pretty obvious that (24) can be also expressed as

\[ Q = s_1 \left( -i \bar{\psi}(t) \psi(t) \right), \quad Q = s_1 \left( \dot{x}(t) x(t) \right), \]

(25)

where the on-shell conditions (\(\ddot{x} = 0, \dot{\psi} = 0, \dot{\bar{\psi}} = 0\)) are to be used. Exploiting (23), it is now clear that \(s_1 Q = -i \{ Q, Q \} = 0\) due to the nilpotency of the transformations \(s_1\) where \(s_1^2 = 0\). In the language of Grassmannian derivative \(\partial_{\bar{\theta}}\), it is obvious that \(\partial_{\bar{\theta}} Q = 0\) due to \((\partial_{\bar{\theta}})^2 = 0\). Thus, the nilpotency of \(s_1, \partial_{\bar{\theta}}\) and \(Q\) are inter-related very beautifully.

4 On-shell nilpotent SUSY symmetry transformations generated by \(\bar{Q}\): Chiral supervariables

In this section, we concentrate on the derivation of \(s_2\) by exploiting the SUSY invariant restrictions on the chiral supervariables which are the generalizations of the basic ordinary dynamical variables \(x(t), \psi(t)\) and \(\bar{\psi}(t)\) onto the \((1, 1)\)-dimensional chiral super-submanifold of the general \((1, 2)\)-dimensional supermanifold. The generalizations and their expansions along \(\theta\)-direction of the \((1, 1)\)-dimensional super-submanifold are

\[ x(t) \rightarrow X(t, \theta, \bar{\theta})|_{\bar{\theta}=0} = X(t, \theta), \]

\[ X(t, \theta) = x(t) + \theta \bar{f}(t), \]

\[ \psi(t) \rightarrow \Psi(t, \theta, \bar{\theta})|_{\bar{\theta}=0} = \Psi(t, \theta), \]

\[ \Psi(t, \theta) = \psi(t) + i \theta \bar{b}_1(t), \]

\[ \bar{\psi}(t) \rightarrow \bar{\Psi}(t, \theta, \bar{\theta})|_{\bar{\theta}=0} = \bar{\Psi}(t, \theta), \]

\[ \bar{\Psi}(t, \theta) = \bar{\psi}(t) + i \theta \bar{b}_2(t), \]

(26)

where \((\bar{b}_1, \bar{b}_2)\) are the bosonic variables and \(\bar{f}\) is a fermionic secondary variable on the r.h.s. of expansion (26). These secondary variables would be expressed in terms of the basic variables (and derivatives on them) by exploiting theoretically important restrictions on the chiral-supervariables defined on the \((1, 1)\)-dimensional chiral super-submanifold.

It is elementary to note that \(s_2 \bar{\psi} = 0\). This implies that \(\bar{\psi}(t)\) is an invariant quantity under the transformation \(s_2\). Thus, we demand the \(\theta\)-independence of the supervariables \(\bar{\Psi}(t, \theta)\) which can be mathematically expressed as:

\[ \Psi(t, \theta) = \bar{\psi}(t) \quad \Rightarrow \quad \bar{b}_2(t) = 0. \]

(27)
Similarly, we note that $s_2(x \dot{\psi}) = 0$ and $s_2(\dot{x} \dot{\psi}) = 0$ when we exploit the fermionic property of $\dot{\psi}(t)$ variable (which satisfies $\dot{\psi}^2 = 0$). The above observations, together with (27), imply the following SUSYIRs on the composite chiral supervariables:

$$X(t, \theta) \dot{\Psi}(t, \theta) = x(t) \dot{\psi}(t), \quad \dot{X}(t, \theta) \dot{\Psi}(t, \theta) = \dot{x}(t) \dot{\psi}(t). \quad (28)$$

The above restrictions imply $\dot{f}(t) \dot{\psi}(t) = 0$, $\dot{\phi}(t) \dot{\psi}(t) = 0$. The non-trivial solution for these restrictions is $\dot{f}(t) \propto \dot{\psi}(t)$. For the algebraic convenience, however, we choose $\dot{f}(t) = i \dot{\psi}(t)$. Now, we take note of the SUSY invariance of the following quantity:

$$s_2 \left[ \frac{1}{2} \dot{x}^2(t) - i \dot{\psi}(t) \psi(t) \right] = 0. \quad (29)$$

As a consequence, we demand that the SUSY invariant quantity [contained within the square bracket of (29)], should remain independent of the Grassmannian variable $\theta$ when generalized onto the $(1, 1)$-dimensional chiral super-submanifold. In other words, we have the following equality in the language of mathematical equation:

$$\frac{1}{2} \dot{X}^2(t, \theta) - i \dot{\Psi}(t, \theta) \Psi(t, \theta) = \frac{1}{2} \dot{x}^2(t) - i \dot{\psi}(t) \psi(t). \quad (30)$$

The substitutions of (26), along with the relationship (27), yield the following:

$$\dot{f}(t) \dot{x}(t) = \bar{b}_1(t) \dot{\psi}(t). \quad (31)$$

Plugging in the value $\dot{f}(t) = i \dot{\psi}(t)$, we obtain $\bar{b}_1(t) = i \dot{x}(t)$. Finally, we obtain the following expansions in their full blaze of glory, namely:

$$X^{(2)}(t, \theta) = x(t) + \theta (i \dot{\psi}) \equiv x(t) + \theta (s_2 x),$$
$$\Psi^{(2)}(t, \theta) = \psi(t) + \theta (\dot{x}) \equiv \psi(t) + \theta (s_2 \psi),$$
$$\dot{\Psi}^{(2)}(t, \theta) = \dot{\psi}(t) + \theta (0) \equiv \dot{\psi}(t) + \theta (s_2 \dot{\psi}). \quad (32)$$

where we have inserted the values $\bar{b}_1(t) = i \dot{x}(t), \bar{f}(t) = i \dot{\psi}(t), \bar{b}_2(t) = 0$. The superscript (2) on the supervariables in (32) denotes the chiral supervariables that have been obtained after the SUSY invariant restrictions (27), (28) and (30) have been imposed.

A close look at (32) establishes the fact that we have already derived the transformations $s_2$ of equation (10). Furthermore, we have obtained the following mapping:

$$\frac{\partial}{\partial \theta} \Sigma^{(2)}(t, \theta, \bar{\theta}) \big|_{\bar{\theta} = 0} = s_2 \sigma(t) \quad \Rightarrow \quad s_2 \leftrightarrow \partial_\theta, \quad (33)$$

where $\Sigma^{(2)}(t, \theta, \bar{\theta})|_{\bar{\theta} = 0}$ is the generic chiral supervariable [e.g. $X^{(2)}(t, \theta), \Psi^{(2)}(t, \theta)$ and $\dot{\Psi}^{(2)}(t, \theta)$ of (32)] and $\sigma(t)$ stands for the basic variables $x(t), \psi(t)$ and $\dot{\psi}(t)$ of the Lagrangian (6) of our present theory. We also note that the nilpotency of $\partial_\theta$ (i.e. $\partial_\theta^2 = 0$) implies $s_2^2 = 0$. Geometrically, the relation (33) shows that the SUSY transformations $s_2$ on a 1D ordinary generic variable $\sigma(t)$ is equivalent to the translation of the corresponding generic supervariable $\Sigma^{(2)}(t, \theta, \bar{\theta})|_{\bar{\theta} = 0}$ along $\theta$-direction of the $(1, 1)$-dimensional chiral super-submanifold.
The Lagrangian $L_0$ can be generalized to the super chiral Lagrangian $\tilde{L}_0^{(c)}$ which can be expressed in terms of the chiral supervariables (32) in the following fashion

$$L_0 \implies \tilde{L}_0^{(c)} = \frac{1}{2} X^{(2)}(t, \theta) X^{(2)}(t, \theta) + i \bar{\Psi}^{(2)}(t, \theta) \dot{\Psi}^{(2)}(t, \theta), \quad (34)$$

The invariance of the Lagrangian $L_0$ under the transformations $s_2$ can be captured in the language of the supervariables and Grassmannian derivative ($\partial_s$) as follows:

$$\frac{\partial}{\partial \theta} \left[ \tilde{L}_0^{(c)} \right] = \frac{d}{dt} \left( i \dot{\bar{\psi}} \right) \iff s_2 L_0 = \frac{d}{dt} \left( i \dot{x} \right). \quad (35)$$

The above expression shows the invariance of action integral $S = \int dt L_0$ under the nilpotent transformations $s_2$ which can also be expressed in terms of $\tilde{L}_0^{(c)}$ and $\partial_s$. The geometrical interpretation of the relationship in (35) for $\tilde{L}_0^{(c)}$ can also be provided analogous to $\tilde{L}_0^{(ac)}$, as we have elaborated on, after the equation (22).

The conserved ($\bar{Q} = 0$) and nilpotent ($\bar{Q}^2 = 0$) SUSY charge $\bar{Q} = (i \dot{x} \bar{\psi})$ can be expressed in terms of the supervariables [cf. expansions in (32)] and super derivative ($\partial_s$) in the following two different forms:

$$\bar{Q} = \frac{\partial}{\partial \theta} \left[ i \bar{\Psi}^{(2)}(t, \theta) \Psi^{(2)}(t, \theta) \right] = \int d\theta \left[ i \bar{\Psi}^{(2)}(t, \theta) \Psi^{(2)}(t, \theta) \right],$$

$$\bar{Q} = \frac{\partial}{\partial \theta} \left[ \dot{x}(t) X^{(2)}(t, \theta) \right] = \int d\theta \left[ \dot{x}(t) X^{(2)}(t, \theta) \right], \quad (36)$$

which can be re-expressed in terms of the basic variables $[x(t), \psi(t), \bar{\psi}(t)]$ and the transformations $s_2$ in the following manner:

$$\bar{Q} = s_2 \left( i \bar{\psi}(t) \psi(t) \right), \quad \bar{Q} = s_2 \left( \dot{x}(t) x(t) \right), \quad (37)$$

where we have to use the on-shell condition $\dot{\bar{\psi}} = 0$ for the validity of the second expression for $\bar{Q}$ in (37). The expressions in (37) demonstrate that $s_2 \bar{Q} = -i \{ \bar{Q}, \bar{Q} \} = 0$ due to the nilpotency of $s_2$ (i.e. $s_2^2 = 0$). This observation, in turn, establishes the nilpotency of $\bar{Q}$ (i.e. $\bar{Q}^2 = 0$) which can also be expressed in the language of $\partial_s$ as it is clear that $\partial_s \bar{Q} = 0$ due to $\partial_s^2 = 0$. Thus, we note that the nilpotency of $s_2$, $\bar{Q}$ and $\partial_s$ are inter-related in a beautiful fashion within the framework of our novel approach.

5 Specific $\mathcal{N} = 2$ SUSY algebra and its interpretation

To derive the specific $\mathcal{N} = 2$ SUSY algebra, generated by the charges $Q, \bar{Q}$ and the Hamiltonian ($H_0$) of the theory, we modify the transformations $s_1$ and $s_2$ [cf. (10)] by a constant multiplicative factor in the following manner [see, e.g., footnote after (18)]:

$$s_1 x = \frac{i \psi}{\sqrt{2}}, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = -\frac{\dot{x}}{\sqrt{2}},$$

$$s_2 x = \frac{i \bar{\psi}}{\sqrt{2}}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = -\frac{\dot{x}}{\sqrt{2}}, \quad (38)$$
which lead to the derivation of the conserved \((\dot{Q} = \tilde{Q} = 0)\) and nilpotent \((Q^2 = \tilde{Q}^2 = 0)\)
SUSY charges as
\[
Q = \frac{i \dot{x} \psi}{\sqrt{2}}, \quad \tilde{Q} = \frac{i \dot{\psi}}{\sqrt{2}}.
\] (39)
The canonical Hamiltonian is \(H_0 = \dot{x}p + \dot{\psi}\Pi_{\psi} - L_0 = p^2/2\) where \(p = \dot{x}, \Pi_{\psi} = -i\tilde{\psi}\) are the canonical momenta w.r.t. \(x\) and \(\psi\) from the Lagrangian (6).

Using the basic canonical quantum (anti)commutators \(\{\psi, \tilde{\psi}\} = -1\) and \([x, p] = i\) (in natural units \(\hbar = c = 1\)), we observe that the operators \((Q, \tilde{Q}, H_0)\) satisfy one of the simplest form of the \(\mathcal{N} = 2\) SUSY quantum mechanical \(sl(1|1)\) algebra (without any central extension):
\[
Q^2 = \tilde{Q}^2 = 0, \quad \{Q, \tilde{Q}\} = H_0, \quad [H_0, Q] = [H_0, \tilde{Q}] = 0,
\] (40)
which is identical to the algebra satisfied by the celebrated de Rham cohomological operators \((d, \delta, \Delta)\) of differential geometry [17-19], namely;
\[
d^2 = \delta^2 = 0, \quad \{d, \delta\} = \Delta, \quad [\Delta, d] = [\Delta, \delta] = 0,
\] (41)
where \((\delta)d\) are the (co-)exterior derivatives and \(\Delta = (d + \delta)^2 \equiv \{d, \delta\}\) is the Laplacian operator. We note that \(\Delta\) and \(H_0\) are the Casimir operators for the algebras (41) and (40), respectively, because both of them commute with all the rest of the operators.

The well-known relationship \(\delta = \pm \ast d \ast\) can also be captured in the language of symmetry properties of the Lagrangian \((L_0)\) of equation (6). For instance, it can be seen that \(L_0\) remains invariant under the following discrete symmetry transformations
\[
x \rightarrow -x, \quad t \rightarrow -t, \quad \psi \rightarrow + \psi, \quad \tilde{\psi} \rightarrow -\tilde{\psi}.
\] (42)
This symmetry turns out to be the analogue of the Hodge duality \((\ast)\) operation of differential geometry because we observe that the following interesting relationships:
\[
s_1 \phi = \pm \ast s_2 \ast \phi, \quad \ast \phi = x, \psi, \tilde{\psi},
\] (43)
are true for the generic variable \(\phi = x, \psi, \tilde{\psi}\) of the theory where the analogue of \((\ast)\) operation is nothing but the discrete symmetry transformations (42) and \((s_1, s_2)\) are the continuous symmetry transformations (38) for the Lagrangian \((L_0)\).

For a duality invariant theory (see, e.g. [20]), the \((\pm)\) signs on the r.h.s. of (43) are determined by two successive operations of the discrete symmetry transformations on a specific variable of our theory. In this context, we observe the following:
\[
\ast(\ast x) = x, \quad \ast(\ast \psi) = -\psi, \quad \ast(\ast \tilde{\psi}) = -\tilde{\psi}.
\] (44)

---

3If we modify (8) [i.e. \(Q = \partial_\theta + (i/2) \theta \bar{\partial}_{\bar{\theta}}, \quad \tilde{Q} = \bar{\partial}_{\bar{\theta}} + (i/2) \theta \partial_\theta\)], these operators, too, satisfy the algebra (40) (with \(H_0 = i \partial_\theta\)). Similarly, if we modify (3) [i.e. \(\bar{D} = \partial_\theta - (i/2) \theta \partial_{\bar{\theta}}, \quad \bar{D} = \partial_{\bar{\theta}} - (i/2) \bar{\theta} \partial_\theta\)], the set \((\bar{D}, \bar{D}, H_0)\) satisfies the algebra (40) except \(\{\bar{D}, \bar{D}\} = -H_0\). The operator \(H_0\) is the Casimir operator in the sets \((Q, \tilde{Q}, H_0)\) and \((\bar{D}, \bar{D}, H_0)\) because it commutes with all the other operators.

4On a compact manifold without a boundary, a set of three operators \((d, \delta, \Delta)\) is called as the de Rham cohomological operators of differential geometry where \(d\) is the exterior derivative, \(\delta\) is the co-exterior derivative and \(\Delta\) is the Laplacian operator. The operators \(d\) and \(\delta\) are connected with each-other by the relation \(\delta = \pm \ast d \ast\) where \((\ast)\) is the Hodge duality operation on the above compact manifold [17-19].
Thus, it can be readily checked, from (43), that we have the following relationships

\[ s_1 x = + * s_2 * x, \quad s_1 \psi = - * s_2 * \psi, \quad s_1 \bar{\psi} = - * s_2 * \bar{\psi}. \quad (45) \]

It is the dimensionality of our 1D system that allows us to have a reverse relationship amongst the continuous \((s_1, s_2)\) and discrete symmetry \((*)\) transformations, as

\[ s_2 x = - * s_1 * x, \quad s_2 \psi = + * s_1 * \psi, \quad s_2 \bar{\psi} = + * s_1 * \bar{\psi}. \quad (46) \]

Thus, we have provided the physical realizations of the relationship \(\delta = \pm * d *\) in the language of the interplay between the continuous and discrete symmetries of our theory of \(\mathcal{N} = 2\) SUSY quantum mechanical model.

We wrap up this section with the remark that, under the discrete transformations (42), the conserved charges \((Q, \bar{Q})\) and the Hamiltonian \((H_0)\) transform as

\[ * Q = - \bar{Q}, \quad * \bar{Q} = - Q, \quad * H_0 = + H_0, \quad *(* Q) = + Q, \quad *(* \bar{Q}) = + \bar{Q}, \quad *(* H_0) = + H_0. \quad (47) \]

The above observations establish that the specific \(\mathcal{N} = 2\) SUSY quantum mechanical algebra (40) remains duality invariant as it does not change its form under any arbitrary number of operations of the discrete symmetry \((*)\) transformations (42). The detailed discussions about the proof of a \(\mathcal{N} = 2\) SUSY quantum mechanical model to be a physical example of Hodge theory have been performed in our earlier work (see, e.g. [15] for details).

6 Conclusions

In our present endeavor, we have taken the simplest \(\mathcal{N} = 2\) SUSY quantum mechanical model of a free SUSY particle to demonstrate that the SUSY symmetries of this theory can be derived from the physical arguments where we demand that the SUSY invariant quantities, generalized onto the (anti-)chiral super-submanifolds, should remain independent of the Grassmannian variables. The latter are physically not realized by experiments. In the old literature (see, e.g. [16]), the Grassmannian variables have been christened as the “soul” coordinates because they do not physically manifest themselves in nature whereas the spacetime coordinates have been called as the “body” coordinates because they can be realized physically and can be measured by appropriate physical instruments in a precise manner. Hence, a physical quantity should remain independent of the “soul” coordinates.

We have provided the geometrical meaning to the nilpotent symmetry transformations [cf. (10)] in the language of the nilpotency of the Grassmannian derivatives \(\partial_\theta\) and \(\partial_{\bar{\theta}}\). It is interesting to point out that, within the framework of our novel approach, we have established the inter-relationships amongst the SUSY transformations, SUSY charges and the Grassmannian derivatives of the (anti-)chiral super-submanifolds, on which, the SUSY continuous (and discrete) symmetries are realized and interpreted geometrically. In particular, the nilpotency property of

(i) the SUSY transformations (10),

(ii) the translation generators along the Grassmannian directions, and
(iii) the SUSY conserved charges is very deeply intertwined.

We know that $\mathcal{N} = 2$ SUSY symmetry transformations are nilpotent of order two but they are not absolutely anticommuting. To avoid the latter property, we have been theoretically compelled to choose the (anti-)chiral supervariables in our present endeavor. In the context of gauge theories, we have to have full expansions of the superfields [like (1)] because (anti-)BRST symmetry transformations (corresponding to a given local gauge symmetry) are nilpotent as well as absolutely anticommuting. These properties are encoded in the similar properties (i.e. $\partial_\theta^2 = \bar{\partial}_\theta^2 = 0, \theta \bar{\partial} + \bar{\partial} \theta = 0$) obeyed by the translational generators ($\partial_\theta, \bar{\partial}_\theta$) along the ($\theta, \bar{\theta}$)-directions of supermanifold on which the full expansions are taken into account. Thus, we have made an intelligent choice of the (anti-)chiral supervariables so that we could avoid the anticommutativity property ($\partial_\theta \partial_\bar{\theta} + \partial_\bar{\theta} \partial_\theta = 0$) for the $\mathcal{N} = 2$ SUSY symmetries.

In our Appendix A, we have shown the generalization of our method in deriving the $\mathcal{N} = 2$ SUSY symmetry transformations for the system of a SUSY harmonic oscillator. We plan to discuss this system, in great detail, in our future endeavor where we shall try to study the phenomenological implications of our results. In our Appendix B, for the sake of completeness, we have derived the off-shell nilpotent symmetries for the 1D free SUSY particle within the framework of our supervariable approach. The on-shell nilpotent symmetries of this theory have been discussed in the main body of our text.

It would be a very nice future endeavor to extend our present ideas in the description of the $\mathcal{N} = 2$ SUSY quantum mechanical models of physical interest [15,21,22] which have been recently shown by us to be the models for the Hodge theory. Finally, we make a passing comment that our supervariable approach can not be applied to $\mathcal{N} = 1$ SUSY quantum mechanical model where the symmetry and the corresponding charge are not nilpotent but the charge obeys the algebra.

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Appendix A: On $\mathcal{N} = 2$ continuous symmetry transformations for the interacting system of a SUSY harmonic oscillator

We apply our method of derivation to one of the simplest interacting $\mathcal{N} = 2$ SUSY system of a 1D harmonic oscillator which is described by the following Lagrangian (with mass $m = 1$ and natural frequency $\omega$) [see, e.g. [22] for details]

$$L^{(0)}_H = \frac{1}{2} \dot{x}^2 + i \bar{\psi} \dot{\psi} - \frac{1}{2} \omega^2 x^2 - \omega \bar{\psi} \psi, \quad (A1)$$

where $\dot{x} = (dx/dt), \dot{\psi} = (d\psi/dt)$ are the generalized “velocities” and $x(t)$ is the bosonic variable and its $\mathcal{N} = 2$ SUSY counterparts are fermionic variables $\psi(t)$ and $\bar{\psi}(t)$ (with $\psi^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0$) which are function of the evolution parameter $t$. The above
Lagrangian respects the following two continuous SUSY symmetry transformations
\[ s_1 x = i \psi, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = -(\dot{x} + i \omega x), \]
\[ s_2 x = i \bar{\psi}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = -i(\dot{x} - i \omega x), \tag{A2} \]
where the infinitesimal transformations \( s_1 \) and \( s_2 \) are on-shell \((\dot{\psi} + i \omega \psi = 0, \dot{\bar{\psi}} - i \omega \bar{\psi} = 0)\) nilpotent \((s_1^2 = s_2^2 = 0)\) of order two. It can be checked that the anticommutator of \( s_1 \) and \( s_2 \) generates the time translation which is one of the key requirements of the general \( \mathcal{N} = 2 \) SUSY theory [defined on a 1D spacetime manifold] (see, e.g. [22]).

One can obtain the off-shell nilpotent \((s_1^2 = s_2^2 = 0)\) continuous \( \mathcal{N} = 2 \) SUSY transformations by linearizing the potential (i.e. \( A^2/2 - \omega x A = -\omega^2 x^2/2 \)) by introducing an auxiliary variable (i.e. Lagrange multiplier) \( A(t) \) as follows
\[ L_H^{(1)} = \frac{1}{2} \dot{x}^2 + i x \dot{\bar{\psi}} - \omega x A + \frac{1}{2} A^2 - \omega \bar{\psi} \psi. \tag{A3} \]

The following \( \mathcal{N} = 2 \) continuous SUSY transformations
\[ s_1 x = i \psi, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = -(\dot{x} + i A), \quad s_1 A = -\dot{\psi}, \]
\[ s_2 x = i \bar{\psi}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = -i(\dot{x} - i A), \quad s_2 A = +\dot{\psi}, \tag{A4} \]
are the symmetry transformations for the Lagrangian (A3) because
\[ s_1 L_H^{(1)} = \frac{d}{dt} \left[ -\omega x \psi \right], \quad s_2 L_H^{(1)} = \frac{d}{dt} \left[ \bar{\psi} (i \dot{x} + A - \omega x) \right]. \tag{A5} \]

It is evident that the action integral \( S = \int dt L_H^{(1)} \) remains invariant under the above \( \mathcal{N} = 2 \) SUSY transformations (A4). We can check explicitly that \( s_1^2 = s_2^2 = 0 \) without any help from the equations of motion \((\ddot{x} + \omega^2 x = 0, \dot{\psi} + i \omega \psi = 0, \dot{\bar{\psi}} - i \omega x = 0, A = \omega x)\).

At this juncture, we shall, first of all, derive the transformations (A4) by exploiting the SUSY invariant restrictions (SUSYIRs) on the (anti-)chiral super variables that have been defined in (12) and (26), respectively. In addition to these, we shall include the following generalizations of the auxiliary variable \( A(t) \), namely;
\[ A(t) \rightarrow \tilde{A}(t, \theta, \bar{\theta})|_{\theta = 0} = A(t, \bar{\theta}) \]
\[ A(t, \bar{\theta}) = A(t) + \bar{\theta} f_1(t), \]
\[ A(t) \rightarrow \tilde{A}(t, \theta, \bar{\theta})|_{\bar{\theta} = 0} = A(t, \theta) \]
\[ A(t, \theta) = A(t) + \theta \bar{f}_1(t), \tag{A6} \]
on to its (anti-)chiral SUSY counterparts \( \tilde{A}(t, \bar{\theta}) \) and \( \tilde{A}(t, \theta) \) in the equations (12) and (26), respectively. It is evident that, in the above expansions (A6), we have the fermionic secondary variables \( f_1(t) \) and \( \bar{f}_1(t) \) because \( A(t) \) is bosonic variable and the pair \((\theta, \bar{\theta})\) is fermionic in nature. The derivation of the off-shell nilpotent symmetry transformation (A4) would, finally, enable us to derive the on-shell nilpotent symmetry (A2), too, by the substitution \( A = \omega x \) which emerges as the equation of motion from (A3).
We focus first on the derivation of the transformations \( s_1 \) of (A4) by applying SUSYIRs (13), (14) and (15) which lead to the derivation of the secondary variables \( f(t) \) and \( b_1(t) \) in terms of the basic variable as: \( f(t) = i \psi(t), b_1(t) = 0 \). Next, it is clear that \( s_1(x + i A) = 0 \) due to the off-shell nilpotency of \( s_1 \) because \( s_1^2 \Psi = 0 \). Thus, we have the following SUSYIR on the (super)variables:

\[
\tilde{X}(t, \bar{\theta}) + i \tilde{A}(t, \bar{\theta}) = \dot{x}(t) + i A(t).
\] (A7)

Plugging in \( f(t) = i \psi(t) \) in the expansion of \( X(t, \bar{\theta}) \), we obtain explicitly \( f_1(t) = -\dot{\psi}(t) \) if we use the expansion from (A6) for \( \tilde{A}(t, \bar{\theta}) \). Finally, we note that we have the following SUSY IR on the (super)variables:

\[
s_1 \left[ \frac{1}{2} \ddot{x} + i \dot{\psi} + \frac{1}{2} A^2 \right] = 0.
\] (A8)

Thus, we have the following SUSYIR on the anti-chiral (super)variables:

\[
\frac{1}{2} \tilde{X}(t, \bar{\theta}) \tilde{X}(t, \bar{\theta}) + i \tilde{\Psi}(t, \bar{\theta}) \Psi(t, \bar{\theta}) + \frac{1}{2} \tilde{A}(t, \bar{\theta}) \tilde{A}(t, \bar{\theta}) = \frac{1}{2} x^2(t) + i \dot{\psi}(t) \psi(t) + \frac{1}{2} A^2(t).
\] (A9)

Substitution of our earlier results \( b_1(t) = 0, f(t) = i \psi(t) \) and \( f_1(t) = -\dot{\psi}(t) \) in the expansions of the supervariables \( \Psi(t, \bar{\theta}), X(t, \bar{\theta}) \) and \( \tilde{A}(t, \bar{\theta}) \), respectively, leads to the determination of \( b_2(t) \) in terms of the dynamical and auxiliary variables of the Lagrangian (A3) as given below:

\[
b_2(t) = i [\dot{x}(t) + i A(t)].
\] (A10)

Thus, ultimately, we have the following expansions for (12) and (A6):

\[
\begin{align*}
X^{(h1)}(t, \bar{\theta}) &= x(t) + \bar{\theta} (i \psi) \equiv x(t) + \bar{\theta} [s_1 x(t)], \\
\Psi^{(h1)}(t, \bar{\theta}) &= \psi(t) + \bar{\theta} (0) \equiv \psi(t) + \bar{\theta} [s_1 \psi(t)], \\
\tilde{\Psi}^{(h1)}(t, \bar{\theta}) &= \tilde{\psi}(t) + \bar{\theta} [-(\dot{x} + i A)] \equiv \tilde{\psi}(t) + \bar{\theta} [s_1 \tilde{\psi}(t)], \\
\tilde{A}^{(h1)}(t, \bar{\theta}) &= A(t) + \bar{\theta} \equiv A(t) + \bar{\theta} [s_1 A(t)],
\end{align*}
\] (A11)

where the superscript \((h1)\) denotes the expansions of the supervariables after the SUSYIRs (in connection with the description of the SUSY harmonic oscillator). A close look at (A11) demonstrates that we have already derived the SUSY transformations \( s_1 \) of equation (A4) in a subtle manner and there exists an explicit mapping \( s_1 \leftrightarrow \partial \).

Let us now concentrate on the derivation of \( s_2 \) by exploiting the SUSYIRs on the chiral supervariables defined in (26) and (A6). Using (27) and (29), it is evident that we obtain:

\[
b_2(t) = 0, \tilde{f}(t) = i \dot{\psi}(t).
\]

The off-shell nilpotency of the transformations \( s_2 \) ensures that \( s_2 [\dot{x}(t) - i A(t)] = 0 \). Thus, we have the following SUSYIR on the (super)variables:

\[
\dot{X}(t, \theta) - i \dot{\tilde{A}}(t, \theta) = \dot{x}(t) - i A(t),
\] (A12)

which leads to the determination of \( \tilde{f}_1(t) = \dot{\psi}(t) \). Finally, we observe that a modified part of Lagrangian \( L_{H}^{(1)} \) [cf. (A3)] remains invariant under \( s_2 \) because we have:

\[
s_2 \left[ \frac{1}{2} \ddot{x} - i \dot{\psi} \psi + \frac{1}{2} A^2 \right] = 0.
\] (A13)
Thus, we have the following SUSYIR on the chiral (super)variables:

\[
\frac{1}{2} \dot{X}(t, \theta) X(t, \theta) - i \dot{\Psi}(t, \theta) \Psi(t, \theta) + \frac{1}{2} \dot{A}(t, \theta) A(t, \theta) = \frac{1}{2} \dot{x}^2(t) - i \dot{\psi}(t) \psi(t) + \frac{1}{2} A^2(t), \tag{A14}
\]

which leads to the determination of \( \bar{b}_1(t) = i [\dot{x}(t) - i A(t)] \) in terms of the dynamical and auxiliary variables of the Lagrangian (A3). Finally, the substitution of the values:

\[
\bar{b}_2(t) = 0, \quad \bar{f}_1 = \dot{\psi}(t), \quad \bar{f}(t) = i \dot{\psi}(t), \quad \bar{b}_1(t) = i [\dot{x}(t) - i A(t)], \tag{A15}
\]

leads to the following expansions of the chiral supervariables (26) and (A6), namely;

\[
X^{(h2)}(t, \theta) = x(t) + \theta (i \bar{\psi}) \equiv x(t) + \theta [s_2 x(t)],
\]

\[
\Psi^{(h2)}(t, \theta) = \psi(t) + \theta [- (\dot{x} - iA)] \equiv \psi(t) + \theta [s_2 \psi(t)],
\]

\[
\bar{\Psi}^{(h2)}(t, \theta) = \bar{\psi}(t) + \theta (0) \equiv \bar{\psi}(t) + \theta [s_2 \bar{\psi}(t)],
\]

\[
\bar{A}^{(h2)}(t, \theta) = A(t) + \theta (\dot{\psi}) \equiv A(t) + \theta [s_2 A(t)], \tag{A16}
\]

where the superscript \((h2)\), in the above, denotes the supervariables obtained after the application of SUSYIRs. It is clear, from the above expansions (A16), that we have already derived the off-shell nilpotent SUSY transformations \((s_2)\) of (A4).

We wrap up this Appendix with the remarks that the supercharges \(Q\) and \(\bar{Q}\) can be computed by exploiting Noether’s theorem and these can be expressed in terms of the supervariables obtained after SUSYIRs analogous to (24) and (36). Similarly, the Lagrangian (A3) can be expressed in terms of (anti-)chiral supervariables (A11) and (A16) and the geometrical basis for the SUSY invariance of the Lagrangian as well as the nilpotency of \(Q\) and \(\bar{Q}\) could be provided within the framework of supervariable approach. Finally, the on-shell nilpotent symmetries (A2) can be obtained from (A11) and (A16) if we substitute \(A = \omega x\) which emerges from the Lagrangian (A3) due to the Euler-Lagrange equation of motion w.r.t. the auxiliary variable \(A(t)\).

**Appendix B: On the derivation of \(N = 2\) off-shell nilpotent SUSY transformations for the 1D free SUSY particle**

In the main body of our present paper, we have discussed only the derivation of \(N = 2\) on-shell nilpotent SUSY symmetries for the free SUSY particle. We can also derive the off-shell nilpotent SUSY symmetries for the same system. Towards this goal, we note that the modified form of the Lagrangian (6), with an auxiliary variable \(A(t)\), namely;

\[
L_0^{(m)} = A(t) \dot{x}(t) - \frac{1}{2} A^2(t) + i \bar{\psi}(t) \dot{\psi}(t), \tag{B1}
\]
respects the following off-shell nilpotent $\mathcal{N} = 2$ SUSY symmetries:

\[
\begin{align*}
  s_1 x &= i \psi, \quad s_1 \psi = 0, \quad s_1 \bar{\psi} = -A, \quad s_1 A = 0, \\
  s_2 x &= i \bar{\psi}, \quad s_2 \bar{\psi} = 0, \quad s_2 \psi = -A, \quad s_2 A = 0,
\end{align*}
\]  

(B2)

because the Lagrangian (B1) transforms as follows

\[
\begin{align*}
  s_1 L_0^{(m)} &= 0, \quad s_2 L_0^{(m)} = \frac{d}{dt} (i A \bar{\psi}).
\end{align*}
\]  

(B3)

It is elementary to note that we get back $\mathcal{N} = 2$ on-shell nilpotent SUSY transformation (10) from (B2) by the substitution $A = \dot{x}$ which is an Euler-Lagrange equation of motion from (B1).

Taking the help of expansions in (12), (26) and (A6), we can derive the SUSY transformations (B2) by our supervariable approach. Let us first focus on the derivation of $s_1$. The SUSYIRs (13), (14) and (16) lead to the derivation of $b_1(t) = 0$ and $f(t) = i \psi(t)$. Furthermore, we observe that $s_1 A(t) = 0$ which shows that $A(t)$ is a SUSY invariant quantity. Thus, we have the following SUSYIR [cf. (A6)]

\[
\tilde{A}(t, \bar{\theta}) = A(t) \implies f_1(t) = 0.
\]  

(B4)

The above result implies that we have already obtained the exact expressions for three secondary variables in the expansions (12) and (A6). These are as follows:

\[
\begin{align*}
  b_1(t) &= 0, \quad f(t) = i \psi(t), \quad f_1(t) = 0.
\end{align*}
\]  

(B5)

Finally, we note that $s_1 [A(t) \dot{x}(t) + i \bar{\psi}(t) \dot{\psi}(t)] = 0$ which lead to the following SUSYIR on the composite supervariables, namely;

\[
\tilde{A}(t, \bar{\theta}) \tilde{X}(t, \bar{\theta}) + i \tilde{\Psi}(t, \bar{\theta}) \dot{\Psi}(t, \bar{\theta}) = A(t) \dot{x}(t) + i \bar{\psi}(t) \dot{\psi}(t).
\]  

(B6)

Plugging in the values from (B5) in the expansions for $\Psi(t, \bar{\theta})$, $X(t, \bar{\theta})$ and $\tilde{A}(t, \bar{\theta})$ [cf. (12), (A6)], we get the expression for the secondary variable $b_2(t)$, namely;

\[
b_2(t) = i A(t).
\]  

(B7)

Thus, ultimately, we obtain the expansions for the appropriate super expansions (12) and (A6) as follows:

\[
\begin{align*}
  X^{(m_1)}(t, \bar{\theta}) &= x(t) + \bar{\theta} [i \psi(t)] \equiv x(t) + \bar{\theta} [s_1 x(t)], \\
  \Psi^{(m_1)}(t, \bar{\theta}) &= \psi(t) + \bar{\theta} (0) \equiv \psi(t) + \bar{\theta} [s_1 \psi(t)], \\
  \bar{\Psi}^{(m_1)}(t, \bar{\theta}) &= \bar{\psi}(t) + \bar{\theta} [-A(t)] \equiv \bar{\psi}(t) + \bar{\theta} [s_1 \bar{\psi}(t)], \\
  \tilde{A}^{(m_1)}(t, \bar{\theta}) &= A(t) + \bar{\theta} (0) \equiv A(t) + \bar{\theta} [s_1 A(t)],
\end{align*}
\]  

(B8)

where the superscript $(m_1)$ denotes the expansions of the supervariables after the application of the appropriate SUSYIRs. In a subtle way, we have already derived the off-shell nilpotent transformations $(s_1)$ of (B2).
Now we focus on the derivation of the off-shell nilpotent \((s_2^2 = 0)\) symmetry transformations \(s_2\). In this connection, we observe that the SUSYIRs (27) and (28) lead to the determination of \(\bar{b}_2(t) = 0, \tilde{f}(t) = i \bar{\psi}(t)\) in the expansions (26). Furthermore, the SUSY invariance \(s_2 A = 0\) leads to the following SUSYIR [cf. (A6)]

\[
\tilde{A}(t, \theta) = A(t) \implies \tilde{f}_1(t) = 0.
\] (B9)

We note that the following SUSY invariance, under the off-shell nilpotent transformations \((s_2)\)

\[
s_2 [A(t) \dot{x}(t) - i \dot{\psi}(t) \psi(t)] = 0,
\] (B10)

is true. Thus, we have the following SUSYIR on the composite (super)variables:

\[
\tilde{A}(t, \theta) \dot{X}(t, \theta) - i \dot{\bar{\psi}}(t, \theta) \bar{\psi}(t, \theta) = A(t) \dot{x}(t) - i \dot{\psi}(t) \psi(t),
\] (B11)

which leads to the determination of \(\bar{b}_1(t) = i A(t)\).

Finally, we have the following super expansions of the appropriate supervariables in (26) and (A6) in the language of the transformations \((s_2)\), namely;

\[
X^{(m2)}(t, \theta) = x(t) + \theta [i \dot{\psi}(t)] \equiv x(t) + \theta [s_2 x(t)],
\]

\[
\Psi^{(m2)}(t, \theta) = \psi(t) + \theta [-A(t)] \equiv \psi(t) + \theta [s_2 \psi(t)],
\]

\[
\bar{\Psi}^{(m2)}(t, \theta) = \bar{\psi}(t) + \theta (0) \equiv \bar{\psi}(t) + \theta [s_2 \bar{\psi}(t)],
\]

\[
\tilde{A}^{(m2)}(t, \theta) = A(t) + \theta (0) \equiv A(t) + \theta [s_2 A(t)],
\] (B12)

where the superscript \((m2)\), on the supervariables, denotes the expansions obtained after the application of the appropriate SUSYIRs. A close look at the expansions (B8) and (B12) demonstrates that there exists a connection between the off-shell nilpotent symmetries \(s_1\) and the translation generator \(\partial_\theta\) along the \(\theta\)-direction of the anti-chiral super-submanifold [of the general \((1, 2)\)-dimensional supermanifold]. In exactly similar fashion, we have the mapping: \(s_2 \leftrightarrow \partial_\theta\) which demonstrates the connection between the nilpotent symmetry transformations \(s_2\) and the translational generator \(\partial_\theta\) along the \(\theta\)-direction of the \((1, 1)\)-dimensional chiral super-submanifold. We conclude that the nilpotency \((s_1^2 = s_2^2 = 0)\) property of the \(\mathcal{N} = 2\) transformations (B2) has its origin in the nilpotency \((\partial_\theta^2 = \partial_\theta^2 = 0)\) property of the (anti-)chiral Grassmannian translational generators \(\partial_\theta\) and \(\partial_\theta\), respectively.

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