Nanopteron solution of the Korteweg-de Vries equation

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Abstract – The nanopteron, which is a permanent but weakly nonlocal soliton, has been an interesting topic in numerical studies for many decades. However, the analytical solution of such a special soliton is rarely considered. In this letter, we study the explicit nanopteron solution of the Korteweg-de Vries (KdV) equation. Starting from the soliton-cnoidal wave solution of the KdV equation, the nanopteron structure is shown to exist. It is found that for the suitable choice of the wave parameters, the soliton core of the soliton-cnoidal wave trends to be a classical soliton of the KdV equation and the surrounded cnoidal periodic wave appears as small amplitude sinusoidal variations on both sides of the main core. Some interesting features of the wave propagation are revealed. In addition to the elastic interaction, it is surprising that the phase shift of the cnoidal periodic wave after the interaction with the soliton core is always half its wavelength, and this conclusion is universal to soliton-cnoidal wave interactions.

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Introduction. – The Korteweg-de Vries (KdV) equation
\[ u_t + Auu_x + Bu_{xxx} = 0, \] (1)
which was originally derived to describe the propagation of gravity waves in shallow water [1], is now regarded as one of the most important systems in soliton theory. It arises as a fundamental model in diverse branches of physics, such as nonlinear optics, Bose-Einstein condensates and hydrodynamics [2]. In particular, the KdV equation plays a significant role in the study of small but finite amplitude ion acoustic waves, magnetoacoustic waves, Alfvén waves in plasma physics [3]. It has been reported in an experimental observation that dynamical properties of dust acoustic waves are found to agree quite well, particularly at low amplitudes and low Mach numbers, with the classical soliton solution of the KdV equation [4].

Since the dramatic discovery of the particle-like behavior of the localized waves by Zabusky and Kruskal in 1965 [5], there has been an unprecedented burst of research activities on solitons. Several effective methods, such as the inverse scattering transformation method [6], the Hirota bilinear formalism [7], the Darboux transformation (DT) [8], the Bäcklund transformation (BT) [9], etc., have been developed to find the multiple-soliton solutions of the KdV equation and other integrable systems. Besides multiple-soliton solutions, interactions between solitons and other types of nonlinear waves are another topic of great interest [10–14]. Recently, by combining the symmetry reduction method with the DT or BT related nonlocal symmetries, researchers have established the interaction solutions between solitons and cnoidal periodic waves of the KdV equation [11] as well as the nonlinear Schrödinger equation [12]. Meanwhile, hinted by these results, two equivalent simple direct methods, the truncated Painlevé and the generalized tanh function expansion approach, are developed to find interaction solutions between solitons and other types of nonlinear waves, such as cnoidal waves, Painlevé waves, Airy waves and Bessel waves [13,14].

In this letter, we report a new analytical solution of a special weakly nonlocal soliton of the KdV equation explicitly. Such a particular solution is called nanopteron.

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The concept of nanopteron was originally introduced by Boyd when he was studying a weakly nonlocal soliton in the φ^4 model numerically. It is a quasisoliton which almost satisfies the classical soliton, but fails because of small amplitude oscillatory tails extending to infinity in space [15]. During the past decades, the nanopteron structure in both continuous and discrete systems has been studied extensively [15–29]. For instance, Hunter and Scheurer have shown asymptotically that capillary-gravity water waves can be consistently modeled by a singularly perturbed KdV equation and solutions of this wave equation are of nanopteron type when the Bond number is less than one-third [18]. Actually, the investigation of the interaction between a topological soliton and a background small amplitude wave has been an important topic in condensed-matter physics for more than three decades [19]. In particular, it was shown that a small amplitude oscillatory wave can propagate transparently through a standing topological soliton with a phase shift. In addition, the nanopteron structure has also been investigated in plasma physics [30,31]. Keane et al. studied the Alfvéen solitons in a fermionic quantum plasma numerically [31].

Starting from the governing equations for Hall magnetohydrodynamics including quantum corrections, a coupled Zakharov-type system was derived and numerically solved for both time-independent and dependent cases. The time-independent Alfvéen density soliton shares a form similar to that of a nanopteron structure as an approximate Gaussian peak surrounded by smaller sinusoidal variations. Obviously, some of the above results suggest that the interaction between a soliton and a small amplitude background wave is elastic, otherwise the moving waves will degenerate during their propagations. Consequently, it is rather meaningful and significant to obtain an analytical solution describing such types of waves.

**Soliton-cnoidal wave solution of the KdV equation.**—Balancing the highest nonlinearity and dispersive terms in the KdV equation (1), we assume its solution in the following generalized truncated tanh expansion:

\[ u = u_0 + u_1 \tanh(w) + u_2 \tanh(w)^2, \]

where \(u_0, u_1, u_2\) and \(w\) are functions of \((x,t)\) to be determined later.

Substituting eq. (2) into eq. (1) and vanishing all the coefficients of the different powers of \(\tanh(w)\), we obtain the system of six overdetermined equations that \(u_0, u_1, u_2\) and \(w\) need to satisfy. It is fortunate to find that three of these overdetermined equations are consistent. From the coefficients of \(\tanh(w)^5\), \(\tanh(w)^4\) and \(\tanh(w)^3\), we find that \(u_2, u_1\) and \(u_0\) can be solved as

\[ u_2 = -\frac{12Bw_u^2}{A}, \]  
\[ u_1 = \frac{12Bw_{uu}}{A}, \]  
\[ u_0 = \frac{B}{A} \left( \frac{3w_{xx}}{w_x} - \frac{4w_{xxx}}{w_x} + 8w_x^2 \right) - \frac{w_1}{Aw_x} \]

Consequently, the solution (2) can be reformed in terms of \(w\)

\[ u = \frac{B}{A} \left[ 12(w_{xx} \tanh(w) + w_x^2 \tanh^2(w)) - \frac{w_1}{Bw_x} + 4w_{xxx} \frac{3w_{xx}}{w_x^2} - 2w_x^2 \right]. \]

Then, from the coefficient of \(\tanh(w)^2\), we obtain the associated compatibility condition of \(w\)

\[ \frac{w_t}{w_x} + B \left( \frac{w_{xxx}}{w_x} - \frac{3w_{xx}^2}{2w_x^2} - 2w_x^2 \right) + \lambda = 0, \]

where \(\lambda\) is a constant of integration. Finally, one can verify that the remaining two overdetermined equations obtained from vanishing the coefficients of \(\tanh(w)^3\) and \(\tanh(w)^0\) are identically satisfied by using eqs. (3), (4), (5) and (7).

In order to find the interaction solution between a soliton and a cnoidal wave, we make the following ansatz for the solution of eq. (7):

\[ w = \xi + c_1 \text{arctanh}(c_2 \text{sn}(\eta, m)), \]
\[ \xi = \frac{x - V_1 t}{W_1}, \quad \eta = \frac{x - V_2 t}{W_2}, \]

where \(\text{sn}\) is the usual Jacobi elliptic sine function and the parameter \(m\) is known as its modulus. \(V_1\) and \(V_2\) are velocities of the soliton and its surrounded cnoidal wave, respectively. \(W_1\) and \(W_2\) are quantities related to the soliton width and the cnoidal wavelength, respectively.

Substituting the ansatz (8) back into eqs. (6) and (7) and setting as zero the coefficients of the different powers of Jacobi elliptic functions, we obtain a group of overdetermined equations of the wave parameters \(\{m, V_1, V_2, W_1, W_2, c_1, c_2, \lambda\}\). When solving these overdetermined equations, if we take the elliptic modulus \(m\), velocities \(V_1\) and \(V_2\) as arbitrary, a nontrivial solution of the other five wave parameters \(\{W_1, W_2, c_1, c_2, \lambda\}\) can be determined as

\[ W_1 = \sqrt{\frac{8B(1-m^2)}{V_1 - V_2}}, \quad W_2 = \sqrt{\frac{2B(1-m^2)}{V_1 - V_2}}, \]
\[ \lambda = \frac{(m^2 - 5)V_1 + (3m^2 + 1)V_2}{4(\delta^2 - 1)}, \]
\[ c_1 = \delta, \quad c_2 = m, \quad \delta^2 = 1. \]
By combining eqs. (6), (8) and (9), the explicit soliton-cnoidal wave solution of the KdV equation can be obtained as

\[
u = \frac{3(V_1 - V_2)}{2AG^2} \left[ \frac{(m^2 - 1 + 2G)^2}{(m^2 - 1)} \tanh(w)^2 \right. \\
-2\delta m S(m^2 - 1 + 2G) \tanh(w) + m^2 - 1 \left. \right] - \frac{(m^2 + 7)V_1 - (3m^2 + 5)V_2}{2A(m^2 - 1)}
\]

(10)

with

\[
G = 1 - m^2 S^2 + \delta m CD, \\
S \equiv sn(\eta, m), \quad C \equiv cn(\eta, m), \quad D \equiv dn(\eta, m).
\]

As pointed out in our previous paper \cite{12}, the soliton-cnoidal wave can be viewed as a dressed soliton, namely, a soliton is dressed by a cnoidal periodic wave. Consequently, the soliton-cnoidal wave can be divided into two parts. By taking the limit \( \tanh(w) = \pm 1 \) in eq. (10), we obtain the cnoidal periodic wave part of the dressed soliton,

\[
C_L = \frac{3(V_1 - V_2)(1 + \delta m S)(m^2 - 1 + 2G)}{AG^2} \\
+ \frac{(5 - m^2)V_1 + (3m^2 - 7)V_2}{2A(m^2 - 1)} \quad x - V_1 t < 0,
\]

(11)

and

\[
C_R = \frac{3(V_1 - V_2)(1 - \delta m S)(m^2 - 1 + 2G)}{AG^2} \\
+ \frac{(5 - m^2)V_1 + (3m^2 - 7)V_2}{2A(m^2 - 1)} \quad x - V_1 t > 0.
\]

(12)

Correspondingly, the soliton part of the wave is

\[
S_L = \nu - C_L, \quad x - V_1 t < 0,
\]

(13)

and

\[
S_R = \nu - C_R, \quad x - V_1 t > 0.
\]

(14)

To illustrate the dressed structure more clearly, let us look at some figures. Figure 1(a) exhibits the soliton-cnoidal wave structure of \( \nu \) determined by eq. (10) at \( t = 0 \). Figure 1(b) and fig. 1(c) reveal the related structures of the cnoidal periodic wave and the soliton core of \( \nu \), respectively. Obviously, the superposition of fig. 1(b) and fig. 1(c) is just fig. 1(a). It is observed from fig. 1(b) that apart from the soliton center, the solution rapidly tends to a cnoidal periodic wave. It is clear from fig. 1(c) that after removing the periodic wave background from \( \nu \), the left is just a soliton structure given by eqs. (13) and (14). Figure 1(d) shows an elastic overtaking collision process between a soliton and a cnoidal wave where both are right-going and the soliton is traveling faster. It can be concluded from fig. 1(d) that despite the cnoidal periodic wave is a delocalized structure, in the space-time evolution of the soliton-cnoidal wave, every peak of the cnoidal periodic wave elastically interacts with the soliton core except for a phase shift. To plot fig. 1, the selection of the nonlinearity coefficient \( A \) and the dispersion coefficient \( B \) are given by

\[
A = \frac{48619\sqrt{1973}\sqrt{1980}}{315778650} \approx 0.304,
\]
which is derived from the KdV equation describing the propagation of ion acoustic waves.

The dressed structure enables us to compute the collision-induced phase shift of the cnoidal periodic wave. For instance, now we consider an overtaking collision process as depicted in fig. 1. At time \( t = 0 \), the cnoidal periodic wave peaks on the left side of the soliton core have interacted with the soliton core, while not the cnoidal periodic peaks on the right side. So there is a phase shift between them. Obviously, the cnoidal periodic wave peaks on the left side of the soliton core have interacted with the soliton core, while not the cnoidal periodic wave peaks on the right side. So there is a phase shift.

\[ \Delta_{cn} = 2W_2K(m) = \frac{\lambda_c}{2}, \tag{16} \]

where \( \lambda_c \) is the wavelength of the cnoidal periodic wave, \( K(m) \) is the first kind of complete elliptic integral. This result can be easily verified. By substituting \( \eta = \eta + 2K(m) \) into (11) and using the Jacobi elliptic identities \( sn(2K(m), m) = 0, cn(2K(m), m) = -1 \), and \( dn(2K(m), m) = 1 \), we can directly demonstrate that \( C_L(\eta + 2K(m)) = C_R(\eta) \). Similarly, by substituting \( \eta = \eta + 4K(m) \) into eqs. (11), (12) and using the Jacobi elliptic identities \( sn(4K(m), m) = 0, cn(4K(m), m) = 1 \), and \( dn(4K(m), m) = 1 \), we can also demonstrate that \( C_L(\eta + 4K(m)) = C_L(\eta) \), and \( C_R(\eta + 4K(m)) = C_R(\eta) \). So, \( C_L \) and \( C_R \) are functions of the period \( 4K(m) \). Incidentally, the periods of the \( sn \) and \( cn \) functions are also \( 4K(m) \), while the period of the \( dn \) function is \( 2K(m) \). It is noted that the phase shift of the interaction wave in fig. 1(b) can be computed from eq. (16) as 24.95, which coincides with the figure.

The phase shift formula (16) tells that the phase shift of a cnoidal periodic wave after its interaction with a soliton is always half its wavelength. More significantly, this phase shift formula is universal to all the soliton-cnoidal wave solutions obtained in refs. [11–13]. Unfortunately, it is still difficult to calculate the phase shift of the soliton because of the mixture of the tanh and Jacobi elliptic functions. The existence of the Jacobi elliptic functions prevents us from calculating the difference of the phases at two different time limits, approaching negative and positive infinities, respectively. Therefore, an alternative method should be designed to overcome this difficulty.

Due to the fact that the parameter \( m \) appears as not only the modulus of the Jacobi elliptic function but also its coefficient since \( c_2 = m \), the amplitude of the cnoidal periodic wave trends to thrive with \( m \) increasing. From fig. 2(a), which is plotted to illustrate this phenomenon at \( t = 0 \), it can be observed that as soon as the parameter \( m \) approaches 0.99, the amplitude of the cnoidal wave becomes comparable to the soliton core. Figure 2(b) shows that the solution (10) exponentially approaches the cnoidal wave as \( x \to \pm \infty \). We also notice from fig. 2(c) that after the periodic wave peaks \( C_L \) and \( C_R \) are taken away from the exact solution \( u \), only a tall and slim soliton structure is revived. Figure 2(d) reveals that the soliton core and every peak of the cnoidal periodic wave can pass through each other transparently with a phase shift.

**Quasisoliton behavior as a nanopteron.** – Before we proceed further, let us first review the classical soliton solution of the KdV equation. By using the usual tanh expansion method, the single-soliton solution of the KdV
$\delta = 1$, $V$ is a small amplitude sinusoidal wave oscillating around zero. This wave profile, in the conoidal periodic wave becomes a small amplitude sinusoidal oscillation. The solution (10) reduces to the single-soliton solution (17) obtained by the usual tanh function expansion method. If we take $\omega = (x-Vt)/W$, the solution (6) reduces to the usual tanh function expansion method.

Now, let us consider the asymptotic behavior of the soliton-cnoidal wave solution (10). Under the ultra limit condition $m = 0$ ($G = 1$), $V_1 = V$ and $V_2 = -V$, the wave parameters (9) degenerate to

$W_1 = \sqrt{4B/V}$, $W_2 = \sqrt{B/V}$,

$\lambda = 3V/2$, $c_1 = \delta/2$, $c_2 = 0$, (20)

and the soliton-cnoidal wave solution (10) reduces to the classical soliton solution (19). From eq. (20), it is interesting to notice that the substitution of the straight line solution $w = (x-Vt)/W$ into the compatibility condition (7), the width of the soliton can also be determined as

$u = 3V/A \text{sech}^2 \left( \frac{x-Vt}{W} \right)$, $W = \sqrt{4B/V}$. (19)

Under the limit $m \to 0$, the function $K(m)$ tends to $\pi/2$. Thus, the collision-induced phase shift of the small amplitude background wave can be approximately taken as

$\Delta_{\text{syn}} = 2W_2K(m) \approx W_2\pi$. (21)

A comparison of the classical soliton to the nanopteron for $m = 0.001$ at $t = 0$ is given in fig. 3, which demonstrates that the curves of two solutions coincide exactly with each other at a large space scale. However, the inset on the right side of fig. 3 shows that the oscillating tail is nonvanishing despite of a tiny amplitude. When
m becomes a little larger, the nanopteron tail grows up conspicuously. Figure 4(a) presents a comparison of classical soliton to the nanopteron structures for $m = 0.02$ and $\delta = \pm 1$ at $t = 0$. It is observed that the soliton and the surrounded cnoidal periodic wave appears as small amplitude sinusoidal variations on both sides of the main core. From fig. 4(b), the shift of the cnoidal periodic wave is always half its wavelength during the collision with the cnoidal wave oscillating around zero. From eq. (21), the phase dance with fig. 4(b). Figure 4(c) reveals that only a soliton core preserves its shape and velocity during the collision with the cnoidal wave. Second, from the dressed structure of the solution, it is found that the collision-induced phase shift of the cnoidal periodic wave is always half its wavelength, which is believed to be universal to all the soliton-cnoidal interactions. Third, the nanopteron structure is realized as a special limit case. It is found that for the suitable choice of the wave parameters, the soliton core of the soliton-cnoidal wave trends to be the classical KdV soliton and the surrounded cnoidal periodic wave appears as small amplitude sinusoidal variations on both sides of the main core.

The explicit solution obtained in this letter can be applied in many physical scenarios. For instance, the nanopteron structure can be viewed as a perturbed classical soliton, and it may provide some correction to the classical soliton in both theoretical and experimental studies.

Summary and discussion. – In this letter, we present a new soliton-cnoidal wave solution of the KdV equation, which we also name as a nanopteron solution. Based on this solution, some interesting features are revealed. First, it has been observed that the soliton core preserves its shape and velocity during the collision with the cnoidal periodic wave peaks. Second, from the dressed structure of the solution, it is found that the collision-induced phase shift of the cnoidal periodic wave is always half its wavelength, which is believed to be universal to all the soliton-cnoidal interactions. Third, the nanopteron structure is realized as a special limit case. It is found that for the suitable choice of the wave parameters, the soliton core of the soliton-cnoidal wave trends to be the classical KdV soliton and the surrounded cnoidal periodic wave appears as small amplitude sinusoidal variations on both sides of the main core.

The explicit solution obtained in this letter can be applied in many physical scenarios. For instance, the nanopteron structure can be viewed as a perturbed classical soliton, and it may provide some correction to the classical soliton in both theoretical and experimental studies.

REFERENCES

[1] Korteweg D. J. and de Vries H., Philos. Mag., 39 (1895) 422.

[2] Dauxois T. and Peyrard M., Physics of Solitons (Cambridge University Press, Cambridge, England) 2006.

[3] Jeffrey A. and Kakutani T., SIAM Rev., 14 (1972) 582.

[4] Bandyopadhyay P., Prasad G., Sen A. and Kaw P. K., Phys. Rev. Lett., 101 (2008) 065006.

[5] Zabusky N. J. and Kruskal M. D., Phys. Rev. Lett., 15 (1965) 240.

[6] Gardner C. S., Greene J. M., Kruskal M. D. and Miura R. M., Phys. Rev. Lett., 19 (1967) 1095.

[7] Hirota R., Phys. Rev. Lett., 27 (1971) 1192.

[8] Gu C. H., Hu H. S. and Zhou Z. X., Darboux Transformations in Integrable Systems Theory and their Applications to Geometry, Mathematical Physics Studies Series, Vol. 26 (Springer, Dordrecht) 2005.

[9] Wahlquist H. D. and Estabrook F. B., Phys. Rev. Lett., 31 (1973) 1386.

[10] Shen H. J., Phys. Rev. E, 71 (2005) 036628.

[11] Hu X. R., Lou S. Y. and Chen Y., Phys. Rev. E, 85 (2012) 056607.

[12] Cheng X. P., Lou S. Y., Chen C. L. and Tang X. Y., Phys. Rev. E, 89 (2014) 043202.

[13] Gao X. N., Lou S. Y. and Tang X. Y., JHEP, 05 (2013) 029.

[14] Chen C. L. and Lou S. Y., Chin. Phys. Lett., 30 (2013) 110202.

[15] Boyd J. P., Nonlinearity, 3 (1990) 177.

[16] Boyd J. P., Weakly Nonlocal Solitary Waves and Beyond-All-Orders Asymptotics (Kluwer, Dodrecht, Boston, London) 1998.

[17] Boyd J. P., Physica D, 48 (1991) 129.

[18] Hunter J. K. and Scheurle J., Physica D, 32 (1988) 253.

[19] Bishop A. R., Krüthjens J. A. and Trullinger S. E., Physica D, 1 (1980) 1.

[20] Flach S. and Willis C. R., Phys. Rep., 295 (1998) 181.

[21] Sánchez-Brey B. and Johansson M., Phys. Rev. E, 71 (2005) 036627.

[22] Alfimov G. L., Eleonskii V. M., Kulagin N. E. and Mitskevich N. V., Chaos, 3 (1993) 405.

[23] Alfimov G. L. and Medvedeva E. V., Phys. Rev. E, 84 (2011) 056606.

[24] Fodor G., Forgács P., Grandclément P. and Rácz I., Phys. Rev. D, 74 (2006) 124003.

[25] Dash R. K. and Daripa P., Appl. Math. Comput., 126 (2002) 1.

[26] Speight J. M. and Zolotaryuk Y., Nonlinearity, 19 (2006) 1365.

[27] Duncan D. B., Eilbeck J. C., Feddersen H. and Wattis J. A. D., Physica D, 68 (1993) 1.

[28] Yang J., Malomed B. A. and Kaup D. J., Phys. Rev. Lett., 83 (1999) 1958.

[29] Savin A. V., Zolotaryuk Y. and Eilbeck J. C., Physica D, 138 (2000) 267.

[30] Deeskov P., Schamel H., Rao N. N., Yu M. Y., Varma R. K. and Shukla P. K., Phys. Fluids, 30 (1987) 2703.

[31] Keane A. J., Mushtaq A. and Wheatland M. S., Phys. Rev. E, 83 (2011) 066407.