Experimental demonstration of the Brachistochrone property of the cycloid

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Abstract. In the present work, the Brachistochrone property of the cycloid is tested experimentally. For this purpose, comparative tracks are built, sensors are placed at the end of the track to measure the travel time of the test object in each track. Mathematically, it is reaffirmed that the fastest curve an object can travel when falling by its own weight from the highest point to the lowest located not on the same vertical line of the first is a cycloid. Although it would be thought that the fastest curve is a straight line between the two points, in the experiment the object takes less time to move in the cycloid due to the property of the Brachistochrone. This curve is subject only to the gravitational force in which friction does not act. A cycloid is only an approximation of the optimal trajectory if there are forces other than gravity and the support reaction. The Brachistochrone theory was experimentally demonstrated with three types of curves and three types of objects. The constructed model can be useful for educational purposes.

1. Introduction

The problem of the brachistocrone, or the fastest descent curve, is one of the oldest problems in the history of calculating variations. The first solution was given by Johann Bernoulli in 1696, other mathematicians and physicists gave solutions among them, Jacob Bernoulli, Leibniz and Newton. In 1969, Johann Bernoulli published a letter addressed to the mathematicians of that time, proposing a problem about the fastest sliding lines [1, 2].

Galileo was the first to discover this property of the cycloid when considering that an object would fall faster if a wire was bent, but this was demonstrated by Johann Bernoulli giving rise to the brachistochronic curve. Christian Huygens discovered that two objects placed in different positions of a cycloid will arrive at the same time giving origin to the tautochronous curve, the cycloid is the only curve that possesses this property [3]. Bernoulli indicated that at any point of the brachistocrone, the ratio between the sine of the angle formed by the tangent and the vertical and the square root of the height (distances from the point of the curve to the upper horizontal) will be constant. It was precisely the development of the ideas contained in this solution that led in the 18th century to the creation of variational calculus [4].

In addition to the use of mathematical methods of explanation, many took the idea of interpreting the results using the hint system. One of the best known is that of the experimental physics of the French Sigaud De Lafond [5], being the base of the system of 3 tracks for the verification of the brachistochrone property of the curve.
Although currently several have replicated the model of Lafond, for checking the fastest curve, Da Silviera G. in his university project, through tests carried out on two models with 3 tracks each, both a portable with graphics on a plane, and a small scale structure that shows the trajectory of small "cat's eye" marbles running through them, explains that it is more efficient to build 3 types of track "Straight vs Hyperbole vs Cycloid" in order to test which is the fastest track this in order to give more viable results than in models of type "Straight vs Cycloid [6].

Gustavo Andrés Cortés Pulgarín [7], a graduate of the National Pedagogical University of Bogotá, built his own runway system. He exposes the brachistochrone as a discrepant experiment and instead of focusing on the pure mathematical field he focuses more on the demonstrative character towards the user with questions before and after an experience test, in which the user in spite of having observed the clues still has the doubt of which is the fastest, since the perspective of the human eye is not exact.

Currently, the brachistochronic property of the cycloid is used in architecture for the design of buildings, in pendulums, in slides, as well as in roller-coaster tracks and skate-skating rinks. It serves to improve the user's experience since, as the rink goes faster, it helps to take a better route towards another type of straight, increasing the intensity of the adrenaline that can be acquired in this one. This curve helps to increase the speed in each application and decrease the time [8, 9].

In [10] the authors explain the Brachistochrone problem in a resistant medium, the particle is moving in the vertical plane under influence of gravity and viscous drag that proportional to n-th degree of the velocity. The articles [11, 12] study the brachistochrone problem in a resisting medium and for a solid.

The objective of this article is to demonstrate the brachistochronic property and for this we use the experimental method by means of a model in which we carry out several tests with different objects, in order to demonstrate that a cycloid was the fastest curve possible. The model was made in wood with a dimension of one-meter radius, to make the test more noticeable. In addition, we have included the electronic part to improve the experience of the user, since many in spite of seeing the experiment do not manage to capture with their glance the fastest object. The experiment includes time sensors.

2. Mathematical development
The answer to the question about the fastest curve for the sliding of an object, is based on velocity, potential energy and kinetics in the following way:

Let $S$ be the length of the arc joining $A$ with $P$ (Figure 1). The variable $x$ represents the height of the point $P$ and the variable $y$ the horizontal displacement. The velocity of the particle in $P(x,y)$ would be $v=ds/dr$, clearing $dt$ would be: $dt = ds / v$. The time that the particle takes to move from $A$ to $P$ is obtained by integrating: $t=\int_0^A \frac{ds}{v}$. The variables $v$ and $ds$ must be a function of $x$ and $y$ [13].

![Figure 1. Graph for deduction.](image1.png)  
![Figure 2. Analysis for arc variation.](image2.png)
2.1. Velocity
Suppose the particle part of the repose. As the conditions indicate that there is no friction, then the Principle of Energy is fulfilled. This means that when the particle moves from a high point to a low point, the potential energy will become kinetic. The energy potency at a point at height \( x \) equals \( a: E_p = m \ g \ x \), where \( m \) is the mass of the particle and \( g \) is the acceleration of gravity. On the other hand, the kinetic energy at a point is \( E_c = \frac{1}{2} m \ v^2 \). The kinetic energy is zero at the starting point of motion A. The kinetic energy at point P is equal to the variation of the potential energy between A and P, that is:

\[
E_c = E_p \\
\frac{1}{2} m \ v^2 = m \ g \ x \\
v = \sqrt{2 \ g \ x}
\]

2.2. Differential arc length
Let points P(\(x,y)\) and Q(\(x+\Delta x,y+\Delta y)\) be part of the curve. The variation in the length of the curve arc when going from P to Q is \( \Delta s \). If \( \Delta x \) is small, then from Figure 2:

\[
\Delta s = \sqrt{\Delta x^2 + \Delta y^2} \\
\frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \\
\frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
\]

If you take the limit when \( \Delta x \) tends to zero, then:

\[
\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\
ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx
\]

Replacing \( v \) and \( ds \):

\[
t = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx \cdot \frac{1}{\sqrt{2} \ g \ x} \\
t = \frac{1}{\sqrt{2} \ g} \int_0^a \frac{1 \ + \ y'^2}{x} \ dx \ \text{depend of} \ x
\]

If the Euler condition [6] is applied to the integral, then:

\[
F(x, y(x), y'(x)) = \frac{1 + y'^2}{x} \\
F(x, y(x), y'(x)) = \frac{1}{\sqrt{x}} \sqrt{1 + y'^2}
\]

Deriving F with respect to y: \( F_y = 0 \).

Deriving \( F_y \) with respect to x:

\[
\frac{d}{dx} F_y = 0 \\
\int d F_y = \int 0 \ dx \\
F_y = \int 0 \ dx = C
\]
\[ F_x = \frac{1}{\sqrt{x}} \left( \frac{1}{2} \right) \frac{1}{\sqrt{1 + y'^2}} (2)(y') \]

Matching the last two expressions:

\[ y' \sqrt{x} \sqrt{1 + y'^2} = C \]

\[ \frac{y'^2}{x (1 + y'^2)} = C^2 \]

Making a change to the constant to facilitate the calculation:

\[ \frac{y'^2}{x (1 + y'^2)} = \frac{1}{C_1^2} \]

\[ C_1^2 y'^2 = x (1 + y'^2) \]

\[ (C_1^2 - x) y'^2 = x \]

\[ y' = \frac{x}{\sqrt{C_1^2 - x}} \]

Introducing a variable for parameterization and using double angle equality:

\[ x(t) = \frac{C_1^2}{4} (1 - \cos t) \]

In Equation 9 for the denominator of the root we have:

\[ C_1^2 - x = \frac{C_1^2}{4} (1 - \cos t) \]

\[ C_1^2 - x = \frac{C_1^2}{2} (1 + \cos t) \]

Replacing 11 in 9:

\[ y' = \frac{x}{\sqrt{\frac{C_1^2}{2} (1 - \cos t)} \sqrt{\frac{C_1^2}{2} (1 + \cos t)}} \]

\[ y' = \frac{1 - \cos t}{\sqrt{1 + \cos t}} \]

If the chain rule is used: \( \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \):

\[ \frac{dy}{dt} = \frac{\sqrt{1 - \cos t}}{\sqrt{1 + \cos t}} \left( \frac{C_1^2}{2} \sin t \right) \]

\[ \frac{dy}{dt} = \frac{C_1^2 \sqrt{1 - \cos t}}{2 \sqrt{1 + \cos t}} \left( \sqrt{1 - \cos^2 t} \right) \]

\[ \frac{dy}{dt} = \frac{C_1^2}{2} \left( \frac{1 - \cos t}{1 + \cos t} \right) \]

\[ \frac{dy}{dt} = \frac{C_1^2}{2} (1 - \cos t) \]

Integrating:

\[ y = \int \frac{C_1^2}{2} (1 - \cos t) dt \]

\[ y = \frac{C_1^2}{2} (t - \sin t) + C_2 \]
If the starting point is A(0,0):

\[ 0 = \frac{c_1^2}{2} (0 - \sin 0) + c_2 \]

(15)

Replacing \( c_2 \) and if we call R to \( \frac{c_1^2}{2} \):

\[ y(t) = R(t - \sin t) \]

(16)

\[ x(t) = R(1 - \cos t) \]

(17)

The Equations 16 and 17 represent the parametric shape of a cycloid, which confirms that the brachistochrone property is typical of this curve.

3. Test and results

Once exposed the mathematical development of the verification of the equations of the brachistochrone curve, as is the cycloid, and with the antecedents of the previous tracks already constructed by De Lafond S., Da Silviera G., and Cortés G., an own model was constructed that demonstrates it with the characteristics in Table 1:

| Track      | Equation | Length (m) |
|------------|----------|------------|
| Line       | \( y = mx + b \) | 1.71       |
| Hyperbole  | \( x^2 - \frac{y^2}{a^2} = 1 \) | 2.15       |
| Cycloid    | \( x = r(\theta + \sin \theta); y = r(1 - \cos \theta) \) | 1.92       |

The lateral view in one of the tests to the system of tracks is indicated in Figure 3.

![Figure 3. Test tracks.](image)

The description of the test objects is given in Table 2.

| Objects         | Dimensions/Diameter (mm) |
|-----------------|--------------------------|
| Toy cars        | 50 x 20                  |
| Small marble    | 10                       |
| Medium marble   | 30                       |
The appearance of the objects can be seen in Figure 4.

![Test objects](image1)

**Figure 4.** Test objects.

A group of sensors known as race finals are placed at the beginning and end of each track. The sensors are connected to an Arduino UNO card. The travel time is measured by the sensors and interpreted by the card. A display shows the calculated travel time. The first object to pass in front of the sensor turns on a led (Figure 5).

![Results of the first test](image2)

**Figure 5.** Results of the first test with the toy cars.

Six tests were conducted on three objects of the same type, each on a different track. The travel times obtained with the toy carts are shown in Table 3.

| Test | Line  | Hyperbole | Cycloid |
|------|-------|-----------|---------|
| 1    | 0.80  | 1.06      | 0.77    |
| 2    | 0.84  | 1.00      | 0.72    |
| 3    | 0.83  | 1.01      | 0.75    |
| 4    | 0.87  | 0.98      | 0.70    |
| 5    | 0.90  | 1.05      | 0.69    |
| 6    | 0.88  | 0.99      | 0.74    |
The particle traversed the cycloid curve faster, the second fastest was the line and lastly the hyperbole in all measurements. Also, in the second test with the 10 mm diameter marble, the particle traversed the cycloid curve faster, the straight being the second fastest and last, the hyperbole (Table 4).

**Table 4. Comparative table of times in seconds for particle 2 (10 mm diameter marble).**

| Test | Line | Hyperbole | Cycloid |
|------|------|-----------|---------|
| 1    | 0.85 | 0.99      | 0.76    |
| 2    | 0.84 | 1.07      | 0.85    |
| 3    | 0.93 | 1.05      | 0.80    |
| 4    | 0.81 | 0.96      | 0.73    |
| 5    | 0.83 | 1.02      | 0.90    |
| 6    | 0.96 | 1.10      | 0.79    |

In the third test as well, the marble of 30 mm in diameter went through the Cycloid curve faster, being the second fastest the Line, and lastly, is the Hyperbole in all measurements (Table 5).

**Table 5. Comparative table of times in seconds for particle 3 (30 mm diameter marble).**

| Test | Line | Hyperbole | Cycloid |
|------|------|-----------|---------|
| 1    | 0.85 | 0.81      | 0.71    |
| 2    | 0.81 | 0.78      | 0.65    |
| 3    | 0.84 | 0.73      | 0.61    |
| 4    | 0.90 | 0.82      | 0.62    |
| 5    | 0.92 | 0.83      | 0.68    |
| 6    | 0.82 | 0.75      | 0.60    |

4. Conclusions

In each one of the tests, independently of which object is put to cross the track, the times verify that the cycloid is the curve the fast in being crossed, therefore, it is the one that possesses the property brachistocrona.

Our model uses balls and solids, for which the moment of inertia should be taken into account. In addition, they are additionally affected by air resistance. Thus, a cycloid is only an approximation of the optimal trajectory.

The electronic part implemented, with the Arduino card, the sensors, the led and the display, improves the observation of results and reinforces the reliability of the brachistocrona property on the curve.

The three test curves Line, Hyperbole and Cycloid helped to maximize the comparison and avoid doubts on the part of the observer, at the time of testing the tracks.

The results of experiments illustrate mathematical calculations. The results obtained are in agreement with modern concepts of movement along the classical brachistochrone.

References

[1] Gutiérrez J de J, Quintas I I 2016 El cálculo de variaciones aplicado a problemas de las ciencias sociales *Política y Cultura* no 46 pp 211-230

[2] Argáez-Mendoza S, Oliva A I 2011 El principio de Fermat, la braquistócrona y la curvatura de la luz? *Ingeniería* vol 15 no 1

[3] Da Siveira Batista G 2006 Experiências com a braquistócrona. Universidade Federal do Ceará, Fortaleza, págs 59-60

[4] Haro J, Pérez M J 2012 El problema de la Braquistócrona y otros problemas de la física como introducción al cálculo de variaciones *SUMA* vol 70 pp 43-63

[5] Pyenson L, Gauvin J-F 2002 *The art of teaching physics: the eighteenth-century demonstration apparatus of Jean Antoine Nollet* Les éditions du Septentrion
[6] El problema de la Braquistócrona. Available on: https://www.researchgate.net/profile/Jonathan_Estevez-Fernandez/publication/305903033_Calculo_variacional_del_problema_de_la_braquistocrona_y_la_tautocrona/links/57a4aaf808ae455e853980bb/Calculo_variacional_del_problema_de_la_braquistocrona_y_la_tautocrona.pdf

[7] Cortés P, Gustavo A 2015 La braquistócrona un experimento discrepante útil para la enseñanza del principio de conservación de la energía mecánica. Available on: http://repositorio.pedagogica.edu.co/handle/20.500.12209/2103

[8] Sirvent J B 1996 La cicloide y su propiedad de braquistocrona: una actividad interdisciplinar Epsilon: Revista de la Sociedad Andaluza de Educación Matemática "Thales" no 36 pp 407-416

[9] Blánquez M H 1985 El problema de la braquistocrona En Homenaje al profesor D. Rafael Rodríguez Vidal Universidad de Zaragoza pp 221-238

[10] Zarodnyuk A V, Cherkasov O Y 2017 On the Maximization of the Horizontal Range and the Brachistochrone with an Accelerating Force and Viscous Friction Journal of Computer and Systems Sciences International 56(4) pp 553-560

[11] Čović V, Vesković M 2009 Brachistochronic motion of a multibody system with Coulomb friction European Journal of Mechanics-A/Solids vol 28 no 4 pp 882-890

[12] Akulenko L D 2008 An Analog of the classical brachistochrone for a disk Dokl. Physics 53 156-159

[13] The Tautochrone Problem. Available on: http://www.math.montana.edu/malo/172f16/Tautochrone.pdf