KKP CONJECTURE FOR MINIMAL ADJOINT ORBITS

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Abstract. We prove that LG models for minimal semisimple adjoint orbits satisfy the Katzarkov–Kontsevich–Pantev conjecture about new Hodge theoretical invariants.

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1. KKP CONJECTURE AND OUR RESULT

The Katzarkov–Kontsevich–Pantev conjecture is a numerical prediction expected to follow from the Homological Mirror Symmetry conjecture of Kontsevich, a master conjecture predicting a wide range of categorical equivalences which so far have been established in only a few cases.

In [KKP] three types of new Hodge theoretical invariants were defined:

$$f^{p,q}(Y, w), \ h^{p,q}(Y, w), \ i^{p,q}(Y, w),$$

for tamely compactifiable Landau–Ginzburg (LG) models $w: Y \to \mathbb{C}$; we recall the definitions in 2.3, 5, 7, 9. [KKP] proved that these numbers satisfy the identities

$$\dim H^m(Y, Y_b; \mathbb{C}) = \sum_{p+q=m} i^{p,q}(Y, w) = \sum_{p+q=m} h^{p,q}(Y, w) = \sum_{p+q=m} f^{p,q}(Y, w),$$

(1)

where $Y_b$ is a smooth fiber of $w$, and conjectured the equality of the 3 invariants.
Conjecture. [KKP, LP] Assume that \((Y, w)\) is a Landau–Ginzburg model of Fano type. Then for every \(p, q\) there are equalities

\[ h^{p,q}(Y, w) = f^{p,q}(Y, w) = i^{p,q}(Y, w). \]

For \(Y\) a specific rational surface with a map \(w: Y \to \mathbb{C}\) such that the generic fiber is an elliptic curve [LP] Lunts and Przyjalkowski proved the equality \(f^{p,q}(Y, w) = h^{p,q}(Y, w)\) and gave an example where \(i^{p,q}(Y, w) \neq h^{p,q}(Y, w)\). In [Sh] Shamoto gave sufficient conditions for a tamely compactifiable LG models to satisfy \(f^{p,q}(Y, w) = h^{p,q}(Y, w)\). In [CP] Cheltsov and Przyjalkowski proved the conjecture for Fano threefolds. There remains open the question of what varieties satisfy the KKP conjecture.

The goal of this paper is to provide examples of LG models coming from Lie theory that do satisfy the KKP conjecture. Our examples will use some of the symplectic Lefschetz fibrations constructed in [GGSM1], which we now recall.

Let \(g\) be a complex semisimple Lie algebra with Lie group \(G\), and \(\mathfrak{h}\) the Cartan subalgebra. Consider the adjoint orbit \(O(H_0)\) of an element \(H_0 \in \mathfrak{h}\), that is,

\[ O(H_0) := \{ \text{Ad}(g)H_0, g \in G \}. \]

Let \(H \in \mathfrak{h} \mathbb{R}\) be a regular element, and \(\langle \cdot, \cdot \rangle\) the Cartan–Killing form. Then [GGSM1, Thm. 2.2] shows that the height function

\[ f_H: O(H_0) \to \mathbb{C} \]

\[ X \mapsto \langle H, X \rangle \]

gives the orbit the structure of a symplectic Lefschetz fibration.

Here we consider the case of \(G = \text{SL}(n + 1, \mathbb{C})\) and focus on the adjoint orbit passing through \(H_0 = \text{Diag}(n, -1, \ldots, -1)\). The diffeomorphism type is then \(O(H_0) \simeq T^n \mathbb{P}^n\). Among all choices of elements \(H_0 \in \mathfrak{h} \subset \mathfrak{sl}(n + 1, \mathbb{C})\), the choice \(H_0 = \text{Diag}(n, -1, \ldots, -1)\) produces the homogeneous manifold of smallest dimension, for this reason we set the following terminology:

**Definition 1.** Let \(O_n\) denote the adjoint orbit of \(H_0 = \text{Diag}(n, -1, \ldots, -1)\) in \(\mathfrak{sl}(n + 1, \mathbb{C})\), we call it the **minimal orbit**.

For every \(n \geq 2\) and for every choice of regular element \(H \in \mathfrak{h}\), we prove:

**Theorem.** The LG model \((O_n, f_H)\) admits a tame compactification and satisfies the KKP conjecture.

2. **Landau–Ginzburg Hodge numbers**

This section is just a summary of parts of the nicely written text of [LP].

**Definition 2.** A Landau–Ginzburg model is a pair \((Y, w)\), where

1. \(Y\) is a smooth complex quasi-projective variety with trivial canonical bundle \(K_Y\);
2. \(w: Y \to \mathbb{C}\) is a morphism with a compact critical locus \(\text{crit}(w) \subset Y\).

**Definition 3.** [KKP] A tame compactified Landau–Ginzburg model is the data \(((Z, f), D_Z)\), where

1. \(Z\) is a smooth projective variety and \(f: Z \to \mathbb{P}^1\) is a flat morphism.
(2) $D_Z = (\cup_i \mathcal{D}_i^h) \cup (\cup_j \mathcal{D}_j^v)$ is a reduced normal crossings divisor such that
\begin{enumerate}
\item $D^v = \cup_j \mathcal{D}_j^v$ is a scheme theoretical pole divisor of $f$, i.e. $f^{-1}(\infty) = D^v$. In particular $\text{ord}_{D_i^v}(f) = -1$ for all $j$;
\item each component $D_i^h$ of $D^h = \cup_i \mathcal{D}_i^h$ is smooth and horizontal for $f$, i.e. $f|_{D_i^h}$ is a flat morphism;
\item The critical locus $\text{crit}(f) \subset Z$ does not intersect $D^h$.
\end{enumerate}

(3) $D_Z$ is an anticanonical divisor on $Z$.

One says that $((Z, f), D_Z)$ is a compactification of the Landau–Ginzburg model $(Y, w)$ if in addition the following holds:

(4) $Y = Z \setminus D_Z$, $f|_Y = w$.

Assume that we are given a Landau–Ginzburg model $(Y, w)$ with a tame compactification $((Z, f), D_Z)$ as above. We denote by $n = \dim Y = \dim Z$ the (complex) dimension of $Y$ and $Z$. Choose a point $b \in \mathbb{C}$ which is near $\infty$ and such that $\text{the fiber } Y_b = w^{-1}(b) \subset Y$ is smooth. In [KKP] the authors define geometrically the numbers $f^{p,q}(Y, w)$, $h^{p,q}(Y, w)$, $f^{p,q}(Y, w)$. Let us recall the definitions.

2.1. $f^{p,q}(Y, w)$. Recall the logarithmic de Rham complex $\Omega^*_Z(\log D_Z)$. Namely, $\Omega^0_Z(\log D_Z) = \wedge^q \Omega^1_Z(\log D_Z)$ and $\Omega^1_Z(\log D_Z)$ is a locally free $\mathcal{O}_Z$-module generated locally by
$$\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$$
if $z_1 \cdot \ldots \cdot z_k = 0$ is a local equation of the divisor $D_Z$. Hence in particular $\Omega^0_Z(\log D_Z) = \mathcal{O}_Z$.

The numbers $f^{p,q}(Y, w)$ are defined using the subcomplex $\Omega^*_Z(\log D_Z, f) \subset \Omega^*_Z(\log D_Z)$ of $f$-adapted forms, which we recall next.

**Definition 4.** For each $a \geq 0$ define a sheaf $\Omega^a_Z(\log D_Z, f)$ of $f$-adapted logarithmic forms as a subsheaf of $\Omega^a_Z(\log D_Z)$ consisting of forms which stay logarithmic after multiplication by $df$. Thus
$$\Omega^a_Z(\log D_Z, f) = \{ \alpha \in \Omega^a_Z(\log D_Z) \mid df \wedge \alpha \in \Omega^{a+1}_Z(\log D_Z) \},$$
where one considers $f$ as a meromorphic function on $Z$ and $df$ is viewed as a meromorphic 1-form.

**Definition 5.** The Landau–Ginzburg Hodge numbers $f^{p,q}(Y, w)$ are defined as follows:
$$f^{p,q}(Y, w) = \dim H^p(Z, \Omega^q_Z(\log D_Z, f)).$$

2.2. $h^{p,q}(Y, w)$. Let $N : V \to V$ be a nilpotent operator on a finite dimensional vector space $V$ such that $N^{m+1} = 0$. Such data defines a canonical (monodromy) weight filtration centered at $m$, $W = W^*(N, m)$ of $V$

$$0 \subset W_0(N, m) \subset W_1(N, m) \subset \ldots \subset W_{m-1}(N, m) \subset W_m(N, m) = V$$
with the properties
\begin{enumerate}
\item $N(W_i) \subset W_{i-2}$,
\item the map $N^l : gr^{W_m}_V \to gr^{W_m}_V$ is an isomorphism for all $l \geq 0$.
\end{enumerate}
Let $S^1 \simeq C \subset \mathbb{P}^1$ be a loop passing through the point $b$ that goes once around $\infty$ in the counter clockwise direction in such a way that there are no singular points of $w$ on or inside $C$. It gives the monodromy transformation

$$T: H^\bullet(Y_b) \to H^\bullet(Y_b)$$

and also the corresponding monodromy transformation on the relative cohomology

$$T: H^\bullet(Y,Y_b) \to H^\bullet(Y,Y_b).$$

in such a way that the sequence

$$\ldots \to H^m(Y,Y_b) \to H^m(Y) \to H^m(Y_b) \to H^{m+1}(Y,Y_b) \to \ldots$$

is $T$-equivariant, where $T$ acts trivially on $H^\bullet(Y)$. Since we assume that the infinite fiber $f^{-1}(\infty) \subset Z$ is a reduced divisor with normal crossings, by Griffiths–Landman–Grothendieck Theorem see [Kna] the operator $T: H^m(Y_b) \to H^m(Y_b)$ is unipotent and $(T - \text{id})^{m+1} = 0$. It follows that the transformation (3) is also unipotent. Denote by $N$ the logarithm of the transformation (3), which is therefore a nilpotent operator on $H^\bullet(Y,Y_b)$. One has $N^{m+1} = 0$.

**Definition 6.** We say that the Landau–Ginzburg model $(Y, w)$ is of Fano type if the operator $N$ on the relative cohomology $H^{p+a}(Y,Y_b)$ has the following properties:

1. $N^{n-|a|} \neq 0$,
2. $N^{n-|a|+1} = 0$.

The above definition is motivated by the expectation that the Landau–Ginzburg model of Fano type usually appears as a mirror of a projective Fano manifold $X$.

**Definition 7.** [LP, Def. 8] Assume that $(Y, w)$ is a Landau–Ginzburg model of Fano type. Consider the relative cohomology $H^\bullet(Y,Y_b)$ with the nilpotent operator $N$ and the induced canonical filtration $W$. The Landau–Ginzburg numbers $h^{p,q}(Y, w)$ are defined as follows:

$$h^{p,n-q}(Y, w) = \dim \text{gr}^{W,n-a}_{2(n-p)} H^{n+p-q}(Y,Y_b) \text{ if } a = p - q \geq 0,$$

$$h^{p,n-q}(Y, w) = \dim \text{gr}^{W,n-a}_{2(n-q)} H^{n+p-q}(Y,Y_b) \text{ if } a = p - q < 0.$$

**Remark 8.** Definition 7 differs from [KKP, Definition 3.2]

$$h^{p,q}(Y, w) = \dim \text{gr}^W_{p} H^{p+q}(Y,Y_b)$$

by the indices of the grading.

2.3. $i^{p,q}(Y, w)$. For each $\lambda \in \mathbb{C}$ one has the corresponding sheaf $\phi_{w-\lambda} \mathcal{C}_Y$ of vanishing cycles for the fiber $Y_\lambda$. The sheaf $\phi_{w-\lambda} \mathcal{C}_Y$ is supported on the fiber $Y_\lambda$ and is equal to zero if $\lambda$ is not a critical value of $w$. From the works of Schmid, Steenbrink, and Saito it is classically known that the constructible complex $\phi_{w-\lambda} \mathcal{C}_Y$ carries a structure of a mixed Hodge module and so its hypercohomology inherits a mixed Hodge structure. For a mixed Hodge module $S$ we will denote by $i^{p,q}S$ the $(p,q)$ Hodge numbers of the $p + q$ weight graded piece $\text{gr}^W_{p+q} S$.

**Definition 9.** Assume that the horizontal divisor $D^h \subset Z$ is empty, i.e. assume that the map $w: Y \to \mathbb{C}$ is proper. Then
Let $\mathfrak{g}$ be a noncompact semisimple Lie algebra (not necessarily complex) with group $G$. A compactification of $\mathcal{O}(H_0)$ to a product of flags $F_\Theta \times F_\Theta^*$ is described in [GGSM2, Sec. 3]. We now describe the orbits of the diagonal action of $G$ in this product. For the case considered here, the one of minimal orbits, we will have $F_\Theta = \mathbb{P}^n \cong Gr(n, n+1) = F_{\Theta^*}$, see example 10.

Let $\Sigma$ be a system of simple roots of $(\mathfrak{g}, \mathfrak{a})$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ is an Iwasawa decomposition) and $\Theta \subset \Sigma$ a subset of roots, cf. example 11. Choose $H_\Theta$ defined by $\Theta = \{\alpha \in \Sigma : \alpha (H_\Theta) = 0\}$, then set

$$n_{H_\Theta}^+ = \sum_{\alpha(H_\Theta) > 0} \mathfrak{g}_\alpha, \quad n_{H_\Theta}^- = \sum_{\alpha(H_\Theta) < 0} \mathfrak{g}_\alpha$$

and take the parabolic subalgebra

$$\mathfrak{p}_\Theta = \sum_{\lambda \geq 0} \mathfrak{g}_\lambda = \mathfrak{z}_\Theta \oplus n_{H_\Theta}^+$$

where $\lambda$ varies over the eigenvalues of $ad (H_\Theta)$ and $\mathfrak{z}_\Theta$ is the centralizer of $H_\Theta$. The dual of $\Theta$ is by definition

$$\Theta^* := -w_0 (\Theta) \subset \Sigma$$

where $w_0$ is the main involution of the Weyl group $W$. Set

$$\mathfrak{q}_{\Theta^*} = \sum_{\lambda \leq 0} \mathfrak{g}_\lambda = \mathfrak{z}_{\Theta^*} \oplus n_{H_\Theta}^-,$$

the parabolic subalgebra of $\mathfrak{g}$ conjugate to $\mathfrak{p}_{\Theta^*}$. In fact, $\mathfrak{q}_{\Theta^*} = Ad (w_0) (\mathfrak{p}_{\Theta^*})$ where $w_0$ is a representative of the main involution $w_0$ in $\text{Norm}_G (\mathfrak{a})$, and this is precisely the reason to consider here the dual flag $F_{\Theta^*}$.

The parabolic subgroups $P_\Theta$ and $P_{\Theta^*}$ are the normalizers of $\mathfrak{p}_\Theta$ and $\mathfrak{p}_{\Theta^*}$ respectively. Their flags are $F_\Theta = G/P_\Theta$ and $F_{\Theta^*} = G/P_{\Theta^*}$.

Denote by $b_\Theta = 1 : P_\Theta$ the origin of $F_\Theta = G/P_\Theta$ and by $b_{\Theta^*} = 1 : P_{\Theta^*}$ the origin of $F_{\Theta^*} = G/P_{\Theta^*}$. If $w \in W$ then $wb_\Theta$ (respectively $wb_{\Theta^*}$) denotes the image of $b_\Theta$ by $w$ (actually, the image $wb_\Theta$ of any representative $w \in \text{Norm}_G (\mathfrak{a})$ of $w$).

The diagonal action is given by $g (x, y) = (gx, gy)$, $g \in G$, $x \in F_\Theta$ and $y \in F_{\Theta^*}$. The following statements describe the diagonal action and its properties.

1. Orbits of the diagonal action have the form $G \cdot (b_\Theta, wb_{\Theta^*})$ with $w \in \mathcal{W}$.

   In fact, given $(x, y) \in F_\Theta \times F_{\Theta^*}$ there exists $g \in G$ such that $x = gb_\Theta$. Therefore, $(x, y)$ is in the orbit of $(b_\Theta, z)$ for some $z \in F_{\Theta^*}$.
On the other hand $F_{\Theta^*}$ is the union of orbits $N^+ \cdot \wb_{\Theta^*}$, $w \in \mathcal{W}$. Thus, $z \in N^+ \cdot \wb_{\Theta^*} \subset P_{\Theta} \cdot \wb_{\Theta^*}$ for some $w \in \mathcal{W}$. This shows that any $(x, y)$ belongs to an orbit $G \cdot (b_{\Theta}, \wb_{\Theta^*})$ for some $w \in \mathcal{W}$.

Note that for different $w \in \mathcal{W}$ it might happen that the orbits $G \cdot (b_{\Theta}, \wb_{\Theta^*})$ coincide.

(2) Dualizing, it follows that orbits of the diagonal action are of the form $G \cdot (\wb_{\Theta}, b_{\Theta^*})$ with $w \in \mathcal{W}$. The two descriptions are equivalent, since $(\wb_{\Theta}, b_{\Theta^*})$ and $(b_{\Theta}, w^{-1} \wb_{\Theta^*})$ belong to the same orbit.

(3) The two previous items show that the diagonal action has only a finite number of orbits.

Actually, the orbits $G \cdot (b_{\Theta}, \wb_{\Theta^*})$ are in bijection with the orbits $P_{\Theta} \cdot \wb_{\Theta^*}$, which are all the orbits of $P_{\Theta}$ in $F_{\Theta^*}$. They are as well in bijection with the orbits $P_{\Theta^*} \cdot \wb_{\Theta}$ which are the orbits of $P_{\Theta^*}$ in $F_{\Theta}$. In fact, if $(b_{\Theta}, \wb_{\Theta^*})$ and $(b_{\Theta}, w_1 \wb_{\Theta^*})$ belong to the same orbit, then there exists $g \in G$ such that $g (b_{\Theta}, \wb_{\Theta^*}) = (b_{\Theta}, w_1 \wb_{\Theta^*})$. This means that $gb_{\Theta^*} = b_{\Theta^*}$, that is, $g \in P_{\Theta^*}$. Consequently $\wb_{\Theta^*} = gw_1 \wb_{\Theta^*}$ with $g \in P_{\Theta}$, that is, $\wb_{\Theta^*}$ and $w_1 \wb_{\Theta^*}$ belong to the same orbit of $P_{\Theta^*}$.

Reciprocally, if $\wb_{\Theta^*}$ and $w_1 \wb_{\Theta^*}$ are in the same orbit of $P_{\Theta}$ then $(b_{\Theta}, \wb_{\Theta^*})$ and $(b_{\Theta}, w_1 \wb_{\Theta^*})$ belong to the same orbit of $G$.

**Example 10.** If $F_{\Theta}$ is a projective space (real or complex) $\mathbb{P}^n$, then $F_{\Theta^*}$ is the Grassmannian $\Gr(n, n+1)$. In the language of roots, $\Theta$ is the complement of $\{\alpha_{12}\}$ and $\Theta^*$ is the complement of $\{\alpha_{n,n+1}\}$.

Taking the basis $\{e_1, \ldots, e_{n+1}\}$, in the canonical realization, $b_{\Theta} = [e_1]$ whereas $b_{\Theta^*} = [e_1, \ldots, e_n]$. An element $w \in \mathcal{W}$ is a permutation, so that $\wb_{\Theta}$ (respectively $\wb_{\Theta^*}$) is obtained from $b_{\Theta}$ (respectively $b_{\Theta^*}$) by permutation of the indices. For instance, $w_0 [e_1] = [e_{n+1}]$ and $w_0 b_{\Theta^*} = [e_2, \ldots, e_{n+1}]$ since $w_0$ inverts the order of the indices.

In this case $P_{\Theta}$ is the group of $(n+1) \times (n+1)$ matrices of type

$$
\begin{pmatrix}
*_{1 \times 1} & * \\
0 & *_{n \times n}
\end{pmatrix}
$$

It has two orbits in $\Gr(n, n+1)$. They are:

1. The hyperplanes containing $[e_1]$, that is, the orbit of $b_{\Theta^*} = [e_1, \ldots, e_n]$. In fact, such a hyperplane is determined by its intersection with $[e_2, \ldots, e_{n+1}]$ and the subgroup of matrices

$$
\begin{pmatrix}
*_{1 \times 1} & 0 \\
0 & *_{n \times n}
\end{pmatrix}
$$

is transitive already in the Grassmannian of subspaces of dim $n-1$ in $[e_2, \ldots, e_{n+1}]$.

2. The hyperplanes transversal to $[e_1]$, that is, the orbit of $w_0 b_{\Theta^*} = [e_2, \ldots, e_{n+1}]$. In fact, if $V$ is a hyperplane transversal to $[e_1]$ then the matrix $g \in P_{\Theta}$ whose columns from 2 to $n+1$ are the coordinates of a basis $\{v_2, \ldots, v_{n+1}\}$ of $V$ satisfies $g [e_2, \ldots, e_{n+1}] = V$.

In conclusion, the diagonal action of $\SL(n+1, *)$ in $\mathbb{P}^n \times \Gr(n, n+1)$ has two orbits, an open one and a closed one. The open orbit is isomorphic to the adjoint orbit $\Ad{(G)} H_{\Theta}$ with $H_{\Theta} = \Diag(n, -1, \ldots, -1)$ and is formed by the pairs of transversal elements in $\mathbb{P}^n \times \Gr(n, n+1)$. On the other
hand, the closed orbit is isomorphic to the flag $F_{\Theta \cap \Theta^*}$. Since $\Theta \cap \Theta^*$ is the complement of $\{\alpha_{12}, \alpha_{n,n+1}\}$ it follows that $F_{\Theta \cap \Theta^*} = \mathbb{F}(1,n)$.

**Example 11.** $F_{\Theta} = \text{Gr}(k,n+1)$ and $F_{\Theta^*} = \text{Gr}(n+1-k,n+1)$, real or complex (it is preferable to assume $k < (n+1)/2$). In terms of roots, $\Theta$ is the complement of $\{\alpha_{k,k+1}, \ldots, \alpha_{n,n+1}\}$. In $\text{Gr}(k,n+1) \times \text{Gr}(n+1-k,n+1)$ there exist $k+1$ orbits of the diagonal action determined by the pairs $(V,W) \in \text{Gr}(k,n+1) \times \text{Gr}(n+1-k,n+1)$ such that $\dim(V \cap W) = 0$. The orbit determined by $\dim(V \cap W) = 0$ is the open orbit (transversal pairs) whereas the closed orbit is given by $\dim(V \cap W) = k$, that is $V \subset W$. This closed orbit is the flag $F_{\Theta}$ with $\Theta$ the complement of $\{\alpha_{k,k+1}, \alpha_{n+1-k,n+1-k+1}\}$.

**Example 12.** $F_{\Theta} = F$ the maximal flag is self-dual (for any group). The orbits of $P_{\Theta} = P$ are the same as the orbit of $N^+$ which give the Bruhat decomposition. In this case the closed orbit is $F$ itself.

4. Partial extension of the potential

Next we describe how get our LG-models. Let $H = \text{Diag}(\lambda_1, \ldots, \lambda_n) \in \mathfrak{sl}(n,\mathbb{C})$ be a regular element so that all $\lambda_i$ are distinct. So we consider the Landau–Ginzburg models $(\mathcal{O}_n, f_H)$ where $\mathcal{O}_n$ is the adjoint orbit of $H$ and $f_H$ is the height function described in [GGSM1].

**Notation 13.** We denote by $\text{LG}_n$ the Landau–Ginzburg model $(\mathcal{O}_n, f_H)$.

We are looking for a tame compactification $\overline{\text{LG}}_n = (Z, w)$ such that $Z \setminus D = \mathcal{O}_n$ for some divisor $D$ and such that $w$ is a holomorphic extension of $f_H$.

In this section we accomplish an intermediate step of the construction, namely, that of describing an extension of $f_H$ to a rational map $R_H$ defined in codimension 2 on the compactification $\mathbb{P}^n \times \mathbb{P}^n$ from example 10. We also verify that the critical points of $R_H$ coincide with those of $f_H$ outside of the indeterminacy locus $\mathcal{I}$ (Def. 17), once this is done we can then obtain a holomorphic extension after blowing up $\mathcal{I}$, which we will do in section 5.

4.1. $\text{LG}_2$. Let $H \in \mathfrak{sl}(2,\mathbb{C})$ be the diagonal matrix

$$H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and $\mathcal{O}_2$ its adjoint orbit. As we said at the beginning of this section, the adjoint orbit is promoted to a Landau–Ginzburg model by adding the potential $f_H$. Accordingly, if $A = \begin{pmatrix} x_1 & y_2 \\ x_2 & y_1 \end{pmatrix}$ (the awkward notation chosen so that the answer matches the expression appearing in 4.2), the potential then reads:

$$f_H(A) = \lambda_1 x_1 + \lambda_2 x_2.$$

The case $\lambda_1 = 1$, $\lambda_2 = -1$ was studied extensively in [BBGGSM] and the absence of projective mirrors for $\text{LG}_2$ and for its compactification $\overline{\text{LG}}_2$ was shown.
We modify the notation so that generalizations to higher dimensions become apparent. Hence, we set $H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

The compactification $\overline{LG_2}$ is obtained by taking the manifold $\mathbb{P}^1 \times \mathbb{P}^1$ together with the potential

$$R_H([x_1 : x_2], [y_1 : y_2]) = [\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 : x_1 y_1 - x_2 y_2].$$

that coincides with $f_H$ on the orbit, that is, $R_H|_{O_2}$ reads

$$([x_1 : x_2], [y_1 : y_2]) \mapsto [f_H : 1].$$

Outside the orbit, $R_H$ is defined as

$$([x_1 : x_2], [y_1 : y_2]) \mapsto [2x_1 y_1 : 0],$$

except at the points of the indeterminacy locus

$$\mathcal{I} = \{P_1, P_2\},$$

where $P_1 = ([1 : 0], [1 : 0])$ and $P_2 = ([0 : 1], [0 : 1])$, are the coordinate points, where the map is ill defined because

$$R_H(P_1) = R_H(P_2) = [0 : 0] \notin \mathbb{P}^1.$$

The tame compactification is then obtained by setting $\overline{LG_2} = (Z, w)$ where $Z$ is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $P_1$ and $P_2$. We take coordinates $[r : s]$ on the target of

$$\pi_3 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

and consider the graph $\Gamma$ of $R_H$ inside the product. We denote by $Z := \overline{\Gamma}$ the closure of $\Gamma$ in the product, hence $Z$ is the surface cut out inside $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$s(xw + yz) = r(xw - yz)$$

and

$$w := \pi_3|_Z.$$

4.2. $LG_3$. As described in example 10, the adjoint orbit $O_3$ compactifies to the product $\mathbb{P}^2 \times \mathbb{P}^2$ as the open orbit of the diagonal action of $\text{SL}(3, \mathbb{C})$. This action has as closed orbit the divisor $D = F(1, 2)$ and we have $\mathbb{P}^2 \times \mathbb{P}^2 \setminus F(1, 2) \simeq O_3$.

If $H = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$, the rational map $R_H$ extending $f_H$ to $\mathbb{P}^2 \times \mathbb{P}^2$ is described in [BGGS] as

$$R_H(v \otimes \varepsilon) = \frac{\text{tr}((v \otimes \varepsilon)\rho(H))}{\text{tr}(v \otimes \varepsilon)} =$$

$$= \frac{\lambda_1 a_{11} (a_{33} a_{22} - a_{23} a_{32}) + \lambda_2 a_{21} (a_{13} a_{32} - a_{33} a_{12}) + \lambda_3 a_{31} (a_{23} a_{12} - a_{13} a_{22})}{a_{11} (a_{33} a_{22} - a_{23} a_{32}) + a_{21} (a_{13} a_{32} - a_{33} a_{12}) + a_{31} (a_{23} a_{12} - a_{13} a_{22})}.$$
So that we can reinterpret \( R_H \) as a rational map defined over \( \mathbb{P}^2 \times \mathbb{P}^2 \) with homogeneous coordinates \([x_1 : x_2 : x_3], (y_1 : y_2 : y_3)\], given by

\[
R_H[(x_1 : x_2 : x_3), (y_1 : y_2 : y_3)] = \frac{\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \lambda_3 x_3 y_3}{x_1 y_1 + x_2 y_2 + x_3 y_3}.
\]

Here the denominator vanishes precisely over the flag manifold, that is

\[
F(1, 2) = \{(x_1 : x_2 : x_3), (y_1 : y_2 : y_3) \in \mathbb{P}^2 \times \mathbb{P}^2; x_1 y_1 + x_2 y_2 + x_3 y_3 = 0\}.
\]

The indeterminacy locus \( \mathcal{I} \) of \( R_H \), is the divisor in \( F(1, 2) \) given by

\[
\mathcal{I} = \{(x_1 : x_2 : x_3), (y_1 : y_2 : y_3) \in F(1, 2); \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \lambda_3 x_3 y_3 = 0\}.
\]

To describe the geometry of \( \mathcal{I} \) we consider the Jacobian matrix \( J \) of the polynomials that define it. This gives:

\[
J = \begin{pmatrix}
\lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\
y_1 & y_2 & y_3 \\
x_1 & x_2 & x_3
\end{pmatrix}.
\]

Lemma 14. The Jacobian matrix \( J \) has rank 2 everywhere on \( \mathcal{I} \), therefore \( \mathcal{I} \) is smooth.

Proof. Consider the \( 2 \times 2 \) determinants that contain the coordinate \( y_1 \). These are:

\[
\begin{align*}
(\lambda_1 - \lambda_2)y_1 y_2 \\
(\lambda_1 - \lambda_3)y_1 y_3 \\
(\lambda_1 - \lambda_2)y_1 x_2 \\
(\lambda_1 - \lambda_3)y_1 x_3.
\end{align*}
\]

For \( J \) to have rank lower than 2 these all must vanish. Because we have chosen the \( \lambda_i \) all distinct, if \( y_1 \neq 0 \) we then obtain the unique solution \([1 : 0 : 0], (1 : 0 : 0)]\) which does not belong to \( \mathcal{I} \). So, we conclude that we must have \( y_1 = 0 \). But the equations are symmetric and we can repeat the reasoning with the other variables. We conclude that \( J \) has full rank at all points of \( \mathcal{I} \), which is therefore smooth.

We now discuss the critical points of \( R_H \) at the points where it is defined.

Lemma 15. The rational map \( R_H \) has no critical points on \( F(1, 2) \setminus \mathcal{I} \).

Proof. We will work with Lagrange multipliers in \( \mathbb{C}^3 \times \mathbb{C}^3 \) and then apply the result to the product \( \mathbb{P}^2 \times \mathbb{P}^2 \). We are looking for critical points of the potential \( f_H = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \lambda_3 x_3 y_3 \) assuming \( \lambda_i \)s are all different such that \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \) subject to the condition \( x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \). Call

\[
g = x_1 y_1 + x_2 y_2 + x_3 y_3
\]

our constraint function. So that, writing the parameter as a new variable \( t \) we need to find critical points of

\[
F = f_H - tg = (\lambda_1 - t)x_1 y_1 + (\lambda_2 - t)x_2 y_2 + (\lambda_3 - t)x_3 y_3
\]
which is easy to solve. By the method of multipliers we need now to find critical points of $F$ by solving the system:

$$\begin{cases}
\frac{\partial F}{\partial x_1} = (\lambda_1 - t)y_1 &= 0 \\
\frac{\partial F}{\partial x_2} = (\lambda_2 - t)y_2 &= 0 \\
\frac{\partial F}{\partial x_3} = (\lambda_3 - t)y_3 &= 0 \\
\frac{\partial F}{\partial t} = (\lambda_1 - t)x_1 &= 0 \\
\frac{\partial F}{\partial t} = (\lambda_2 - t)x_2 &= 0 \\
\frac{\partial F}{\partial t} = (\lambda_3 - t)x_3 &= 0 \\
\end{cases}$$

$$x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

A nontrivial solution of the system can only occur if $t = \lambda_i$ for some $i = 1, 2, 3$, else we would have $[(x_1 : x_2 : x_3), (y_1 : y_2 : y_3)] = [(0 : 0 : 0), (0 : 0 : 0)]$. Suppose $t = \lambda_1$, then we still must have that $x_2 = x_3 = y_2 = y_3 = 0$, which in homogeneous coordinates implies

$$[(x_1 : x_2 : x_3), (y_1 : y_2 : y_3)] = [(1 : 0 : 0), (1 : 0 : 0)],$$

but then the last equation is not satisfied. Similarly for $t = \lambda_2$ and $t = \lambda_3$. We conclude that $F$ has precisely 3 critical points, which are

$$[(1 : 0 : 0), (1 : 0 : 0)], [(0 : 1 : 0), (0 : 1 : 0)], [(0 : 0 : 1), (0 : 0 : 1)],$$

which lie outside $F(1, 2)$, and they correspond in fact to the 3 critical points of $f_H$ on $O_3$. The lemma does not verify points of $I$ because $R_H$ is not defined there. \hfill \qed

4.3. $L_{G_n}$. Our rational map is described in [GGSM2, Sec. 4.2] as

$$R_H : \mathbb{P}^n \times Gr(n, n + 1) \to \mathbb{P}^1,$$

$$R_H([v], [\varepsilon]) = \frac{\text{tr}((v \otimes \varepsilon)\rho(H))}{\text{tr}(v \otimes \varepsilon)} = \frac{\sum_{i=1}^{n+1} \lambda_ia_{i1}(\text{adj } g)_{1i}}{\sum_{i=1}^{n+1} a_{i1}(\text{adj } g)_{1i}}. \quad (5)$$

On $\mathbb{P}^n \times \mathbb{P}^n$ using bihomogeneous coordinates $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}$ we have:

$$R_H([x_1, \ldots, x_{n+1}], [y_1, \ldots, y_{n+1}]) = \left[\sum_{i=1}^{n+1} \lambda_i x_i y_i, \sum_{i=1}^{n+1} x_i y_i\right].$$

For the general case, we have the following definitions:

**Definition 16.** The flag variety $F(1, n)$ is the hyperplane section of $\mathbb{P}^n \times \mathbb{P}^n$ defined by:

$$x_1y_1 + \cdots + x_{n+1}y_{n+1} = 0.$$

**Definition 17.** The indeterminacy locus $I$ is given by:

$$I := F(1, n) \cap \{\lambda_1x_1y_1 + \cdots + \lambda_{n+1}x_{n+1}y_{n+1} = 0\}.$$

We then write a Jacobian matrix

$$J = \begin{pmatrix}
\lambda_1y_1 & \ldots & \lambda_{n+1}y_{n+1} & \lambda_1x_1 & \ldots & \lambda_{n+1}x_{n+1} \\
y_1 & \ldots & y_{n+1} & x_1 & \ldots & x_{n+1}
\end{pmatrix},$$

and direct generalization of lemmas 14 and 15 then prove:

**Lemma 18.** $I$ is smooth and $R_H$ has no critical points on $F(1, n) \setminus I$. 


Furthermore, the generalization of the proof of lemma 15 shows that $R_H$ has critical points on the coordinate points $(e_i, e_i)$, that is, those points $[x_1, \ldots, x_{n+1}, [y_1, \ldots, y_{n+1}]$ with only 2 nonzero coordinates $x_i, y_j$ with $i = j$ and these lie outside $F(1, n)$ and correspond to the critical points of $f_H$. The lemma does not verify points of $I$ where the map is ill defined.

**Lemma 19.** $\omega_I \cong O_I(-n + 1, -n + 1)$.

**Proof.** $I$ is given as the intersection of 2 divisors of type $(1, 1)$ in $\mathbb{P}^n \times \mathbb{P}^n$, namely, $x_1 y_1 + \ldots + x_{n+1} y_{n+1} = 0$ and $\lambda_1 x_1 y_1 + \ldots + \lambda_{n+1} x_{n+1} y_{n+1} = 0$, thus adjunction formula gives $\omega_I \cong O_I(-n + 1, -n + 1)$. \(\square\)

5. **Holomorphic extension of the potential**

The blowing up $Z$ of $\mathbb{P}^n \times \mathbb{P}^n$ along $I$ is obtained as follows:

Take $\mathbb{P}^1$ with homogeneous coordinates $[t : s]$. The pencil $\{t g + s f_H\}_{t,s \in \mathbb{C}}$ induces a rational map $\mathbb{P}^n \times \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ with $I$ as its indeterminacy locus. Call $Z$ the closure of the graph of this map. So we get a map $w : Z \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$,

$$(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}, t, s) \mapsto [t : s].$$

Note that if $s \neq 0$ then $[t : s] = \left[\frac{t}{s} : 1\right] = \left[\frac{f_H}{g} : 1\right] = [f_H : g]$ where the middle equality holds since $tg = sf_H$, and $g = 1$ over the orbit $O_n$ because in fact $g = \det(A_{ij})$ with $A \in SL(n, \mathbb{C})$. Thus, we obtain:

**Lemma 20.** The map $w : Z \rightarrow \mathbb{P}^1$ is a holomorphic extension of $f_H$. The critical points of $w$ coincide with the critical points of $f_H$.

$$
\begin{array}{c}
Z \\
\leftarrow \pi
\end{array}
\begin{array}{c}
\mathcal{O}_n \\
\rightarrow \mathbb{P}^n \times \mathbb{P}^n
\end{array}
\begin{array}{c}
f_H \\
\rightarrow \mathbb{P}^1
\end{array}
\begin{array}{c}
R_H
\end{array}
$$

**Proof.** We know by Lemma 18 that the indeterminacy locus $I$ is smooth. We want to show that $w$ has no critical points over $E$. Since surjectivity of the derivative is a local question, it is enough to analyze an open neighborhood of the point in question (the analytic topology is sufficient, though the Zariski topology will work as well).

As an example, we take a point $p$ in $E$ mapping to the coordinate point $P = ([1 : 0 : \ldots : 0], [0 : 1 : \ldots : 0]) \in I$ by the blow down map, hence $p = ([1 : 0 : \ldots : 0], [0 : 1 : \ldots : 0][t_0 : s_0])$. We take the open neighborhood $U_{12} = \{x_1 \neq 0, y_2 \neq 0\} \subset \mathbb{P}^n \times \mathbb{P}^n$ of the point $P$. In this neighborhood the defining equations for $S = I \cap U_{12}$ become

$$
\begin{cases}
 f = \lambda_1 Y_1 + \lambda_2 X_2 + \lambda_3 X_3 Y_3 + \cdots + \lambda_{n+1} X_{n+1} Y_{n+1} = 0 \\
g = Y_1 + X_2 + X_3 Y_3 + \cdots + X_{n+1} Y_{n+1} = 0,
\end{cases}
$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are constants.
where \( X_i = x_i / x_1, Y_i = y_i / y_2 \), and hence the local expressions for \( f \) and \( g \) are nonsingular. Since \( S \) is a smooth submanifold of \( U_{12} \cong \mathbb{C}^{2n} \) of codimension 2, we can change coordinates to \( \Delta = \mathbb{C}[Z_1, \ldots, Z_{2n}] \) so that \( Z_{2n-1} = f, \ Z_{2n} = g \) and therefore \( S \) is cut out inside \( \Delta \) as the linear submanifold \( Z_{2n} - 1 = f, Z_{2n} = g \). Now, following \([Gh, P. 603]\), we know that an extension of \( w \) is defined over \( \tilde{\Delta} = \mathbb{C}[Z_1, \ldots, Z_{2n}] \) so that \( Z_{2n} - 1 = f, Z_{2n} = g \) and therefore \( S \) is cut out inside \( \tilde{\Delta} \) as the linear submanifold \( Z_{2n} - 1 = Z_{2n} = 0 \). Now, following \([Gh, P. 603]\), we know that an extension of \( w \) is defined over \( \tilde{\Delta} \) by \((w(z), l')\) where

\[
l' = \left( \frac{\partial w}{\partial Z_{2n-1}} t, \frac{\partial w}{\partial Z_{2n}} s \right).
\]

Now consider the chart on the target \( \mathbb{P}^1 \) where \( s \neq 0 \), then on \( \tilde{\Delta} \setminus E \) we have \( w = f/g = t/s \), with \( s \neq 0 \) hence \( g \neq 0 \) thus

\[
l' = \left( \frac{\partial f / g}{\partial f} t_0, \frac{\partial f / g}{\partial g} s_0 \right) = (t_0 / g, -s_0 f / g^2) \).
\]

So, that \( w|_S \) can be extended over \( E \) to \( w|_S(z, l') = (w(z), (t_0 / g, -s_0 f / g^2)) \) without critical points. We conclude that \( P \) is not a critical point of \( w \).

Now, generalization to other coordinate points is evident, and generalizing to more general points of \( E \) come automatically because every point on \( E \) belongs to an open neighborhood of some coordinate point, and calculations will get us to points \((t_0, s_0)\) with neither of the coordinates vanishing. In all cases we conclude surjectivity of the derivative, hence \( w \) has no critical points on \( E \). \( \square \)

We then conclude that \( w: Z \to \mathbb{P}^1 \) is the desired tame compactification of our LG model.

6. Hodge structures for the adjoint orbits

The following results will be used to show that the adjoint orbits have pure Hodge structures.

**Theorem 21.** [D, Thm.1] Let \( X \) and \( Y \) be smooth complex projective varieties, and suppose that \( X = X_1 \cup \cdots \cup X_n \) and \( Y = Y_1 \cup \cdots \cup Y_n \) are disjoint unions of quasiprojective subvarieties. Suppose that \( X_i \) is algebraically isomorphic to \( Y_i \) for all \( i \). Then the Betti numbers of \( X \) and \( Y \) are equal and in fact their Hodge numbers are equal.

**Theorem 22.** [D, Thm.2] Let \( X \) be a complex quasiprojective variety. Suppose that \( X \) is a finite disjoint union \( X = X_1 \cup \cdots \cup X_n \) where the \( X_i \) are quasiprojective subvarieties. Then

\[
\chi^{p,q}(X) = \sum_i \chi^{p,q}(X_i).
\]

Consider \( X \) smooth projective, and \( Y \) smooth projective of codimension 1 in \( X \). We are interested in the mixed Hodge structure of \( U = X \setminus Y \). We consider the Gysin map

\[
H^{k-2}(Y) \to H^k(X, U) \to H^k(X).
\]
It turns out that to make this a map of Hodge structures, it is sufficient to shift the weights up by (1,1) \[ Ho, \text{Sec.2.1}, \] then we use:

**Theorem 23.** \[ Ho, \text{Thm. 10} \] The Gysin map \( \delta_k : H^{k-2}(Y) \otimes \mathbb{Q}(-1) \rightarrow H^k(X) \) is a map of Hodge structures of weight \( k \) on \( \ker \delta_k \) and \( \coker \delta_k \). Furthermore, the weight filtrations on \( \ker \delta_{k+1} \) and \( \coker \delta_k \) are the same as those induced by the Hodge filtration on \( H^k(U) \) via the short exact sequence (arising from the Gysin sequence)

\[
0 \rightarrow \coker \delta_k \rightarrow H^k(U) \rightarrow \ker \delta_{k+1} \rightarrow 0.
\]

**Corollary 24.** \[ Ho, \text{Cor. 11} \] \( H^n(U) \) admits a natural mixed Hodge structure with weight filtration

\[
W^k H^n(U) = \begin{cases} 
0 & k < n \\
\im H^n(X) & k = n \\
H^n(U) & k > n
\end{cases}
\]

and Hodge filtration \( F^p H^n(U) \) given by classes represented by \( \geq p \)-holomorphic logarithmic differential forms such that

\[
Gr^k H^n(U) = \begin{cases} 
0 & k < n, k > n + 1 \\
\coker \delta_n & k = n \\
\ker \delta_{n+1} & k = n + 1
\end{cases}
\]

where the kernel and cokernel of \( \delta_k \) are given by their natural Hodge structures (\( \delta_k \) is a map of Hodge structures \( H^{k-2}(Y) \otimes \mathbb{Q}(-1) \rightarrow H^k(X) \)).

Let \( W \) be a smooth projective variety and \( U \subset W \) a non-empty open subset of \( W \). Let \( c \) be the maximal dimension of an irreducible component of \( W \setminus U \). Since \( \Omega^p_W \) is locally free for each \( p = 0, \ldots, \dim W \), the restriction map

\[
H^i(W, \Omega^p_W) \rightarrow H^i(U, \Omega^p_U)
\]

is bijective if \( i \leq \dim(W) - c - 2 \) and injective if \( i = \dim(W) - c - 1 \).

Recall that given a complex algebraic variety \( X \) with a weight filtration \( W \) and a Hodge filtration \( F \) of its cohomology \( H^*(X) \) (or of its cohomology with compact support \( H^*_c(X) \)), the graded Euler characteristics of \( X \) are defined as

\[
\chi(X) = \sum (-1)^k \dim H^k(X)
\]

\[
\chi_m(X) = \sum (-1)^k \dim Gr^W_m H^k(X)
\]

\[
\chi^{p,q}(X) = \sum (-1)^k \dim Gr^W_p Gr^W_{p+q} H^k(X),
\]

and similarly \( \chi_c^c(X), \chi_m^c(X) \) and \( \chi^{c}_{pq}(X) \) with \( H^*_c(X) \) in place of \( H^*(X) \) for cohomology with compact support. These satisfy:

\[
\chi(X) = \sum_m \chi_m(X)
\]

\[
\chi_m(X) = \sum_{p+q=m} \chi^{p,q}(X).
\]

For \( X \) smooth of dimension \( n \), \[ D, \text{p. 100} \] shows that Poincaré duality implies

\[
\chi^{p,q}_c(X) = \chi^{n-p,-n+q}(X).
\]

(7)

If \( X \) is smooth projective, then

\[
\chi(X) = \sum_m (-1)^m \dim H^m(X).
\]

(8)
If in addition all the odd Betti numbers of $X$ are zero (e.g. if $X = F_\Theta \times F_{\Theta^*}$ or $X = F(1, n)$), then
\[ \chi(X) = \sum_m \dim H^m(X). \] (9)

6.0.1. The Hodge structure of $O_n$. We consider now the adjoint orbit $O_n$ as defined in 1, so that $U = O(H_0) \cong T^*\mathbb{P}^n$. In this case the compactification is $W = \mathbb{P}^n \times \mathbb{P}^{n*}$ and $W \setminus U = F(1, n)$ is a flag variety. Here $c = \dim(W \setminus U) = \dim F(1, n) = 4n - 2$ (over $\mathbb{R}$). We identify $\mathbb{P}^n$ and $\mathbb{P}^{n*}$ using the Riemannian metric. Projection onto the second coordinate $\pi_1 : F(1, n) \to \mathbb{P}^n$, endows $F(1, n)$ with the structure of a locally trivial $\mathbb{P}^{n-1}$ bundle over $\mathbb{P}^n$. Even though this bundle is not trivial, its cohomology is the same as the one of the product. Therefore, another application of the Künneth formula gives:
\[ h^{u,v}(F(1, n)) = \sum_{p+r=u, q+s=v} h^{p,q}(\mathbb{P}^n) h^{r,s}(\mathbb{P}^{n-1}). \]

Thus,
\[ h^{u,v}(F(1, n)) = h^{u,v}(\mathbb{P}^n \times \mathbb{P}^{n-1}) = \begin{cases} 0 & \text{if } u \neq v \text{ or } u + v > 2n - 1 \\ u + 1 & \text{if } 0 \leq u = v \leq n - 1 \\ 2n - u & \text{if } n \leq u = v \leq 2n - 1 \end{cases}. \] (10)

Example 25. For our first example we consider the case $n = 1$. Here $H_0 = \text{Diag}(1, -1)$ and $F(1, 1) = \mathbb{P}^1$. An application of theorem 22 gives:
\[ \chi_c^{p,q}(\mathbb{P}^1 \times \mathbb{P}_{1*}) = \chi_c^{p,q}(O(H_0)) + \chi_c^{p,q}(\mathbb{P}^1), \]
and we obtain
\[ \chi_c^{p,q}(O(H_0)) = \begin{cases} 1 & \text{if } p = q = 1 \text{ or } p = q = 2 \\ 0 & \text{otherwise} \end{cases}. \]

Thus, duality 7 gives
\[ \chi_c^{p,q}(O(H_0)) = \begin{cases} 1 & \text{if } p = q = 0 \text{ or } p = q = 1 \\ 0 & \text{otherwise} \end{cases}. \]

Example 26. We now generalize to higher $n$. Here $H_0 = \text{Diag}(n, -1, \ldots, -1)$. Applying theorem 22 to $W = \mathbb{P}^n \times \mathbb{P}^{n*}$, $U = O(H_0)$ and $W \setminus U = F(1, n)$, for all $p, q$ we have
\[ \chi_c^{p,q}(W) = \chi_c^{p,q}(O(H_0)) + \chi_c^{p,q}(F(1, n)). \] (11)

These are all zero for $p \neq q$. Using formula 10, we obtain the following numbers:

| $0 \leq p \leq n - 1$ | $p = n$ | $n + 1 \leq p \leq 2n - 1$ | $p = 2n$ |
|------------------------|--------|--------------------------|--------|
| $0 \leq p \leq n - 1$ | $\chi_c^{p,p}(W)$ | $\chi_c^{p,p}(F(1, n))$ | $\chi_c^{p,p}(O(H_0))$ |
| $p = n$ | $p + 1$ | $p + 1$ | $0$ |
| $n + 1 \leq p \leq 2n - 1$ | $p + 1$ | $p$ | 1 |
| $p = 2n$ | $2n - p + 1$ | $2n - p$ | 1 |
| 1 | 0 | 1 |

Thus
\[ \chi_c^{p,q}(O(H_0)) = \begin{cases} 1 & \text{if } n \leq p = q \leq 2n \\ 0 & \text{otherwise} \end{cases}. \]
Since $\mathcal{O}(X_0)$ is smooth, connected and quasi-projective of dimension $2n$, an application of Poincaré duality formula 7 gives

$$\chi^{p,q}_c(\mathcal{O}(H_0)) = \chi^{2n-p,2n-q}(\mathcal{O}(H_0)).$$

So,

$$\chi^{p,q}(\mathcal{O}(H_0)) = \begin{cases} 1 & \text{if } 0 \leq p = q \leq n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

We also know the cohomologies with integer coefficients, in fact, since there is a real diffeomorphism $\mathcal{O}(H_0) \simeq T^\ast \mathbb{P}^n$, the homotopy type of $\mathcal{O}(H_0)$ is that of $\mathbb{P}^n$. Here odd Betti numbers are zero and we tabulate the even Betti numbers:

\[
\begin{array}{c|c|c|c}
0 \leq p \leq n - 1 & h^{2p}(W) & h^{2p}(F(1,n)) & h^{2p}(\mathcal{O}(H_0)) \\
p = n & p + 1 & p + 1 & 1 \\
n + 1 \leq p \leq 2n - 1 & 2n - p + 1 & 2n - p & 0 \\
p = 2n & 1 & 0 & 0
\end{array}
\]

Following [Ho], we use the Gysin sequence

$$\rightarrow H^{k-2}(Y) \rightarrow H^k(W) \rightarrow H^k(\mathcal{O}(H_0)) \rightarrow H^{k-1}(Y) \rightarrow$$

and the induced maps of Hodge structures

$$\delta_k: H^{k-2}(Y) \otimes \mathbb{Q}(-1) \rightarrow H^k(X)$$

to find the Hodge structure of $\mathcal{O}(H_0)$.

**Example 27.** For $n = 1$ we have that the homotopy type of $\mathcal{O}(H_0)$ is that of $\mathbb{P}^1$, so the only relevant cohomology is the second one. Using 12 with $k = 2$ we get

$$H^1(\mathcal{O}(H_0)) \rightarrow H^0(\mathbb{P}^1) \rightarrow H^2(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow H^2(\mathcal{O}(H_0)) \rightarrow H^1(\mathbb{P}^1) \rightarrow .$$

Thus,

$$0 \rightarrow H^0(\mathbb{P}^1) = \mathbb{Z} \rightarrow H^2(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z} \rightarrow H^2(\mathcal{O}(H_0)) = \mathbb{Z} \rightarrow 0 \rightarrow ,$$

so the map of Hodge structures

$$\delta_2: H^0(\mathbb{P}^1) \otimes \mathbb{Q}(-1) \rightarrow H^2(\mathbb{P}^1 \times \mathbb{P}^1)$$

is injective, and Corollary 24 gives that

$$W^k H^2(\mathcal{O}(H_0)) = \begin{cases} 0 & k < 2 \\ H^2(\mathcal{O}(H_0)) & k \geq 2 \end{cases}$$

and Hodge filtration $F^p H^2(\mathcal{O}(H_0))$ given by classes represented by $\geq p$-holomorphic logarithmic differential forms such that

$$Gr^k H^2(\mathcal{O}(H_0)) = \begin{cases} 0 & k \neq 2 \\ H^2(\mathcal{O}(H_0)) & k = 2 \end{cases}.$$

**Example 28.** For $n = 2$ we have that the homotopy type of $\mathcal{O}(H_0)$ is that of $\mathbb{P}^2$, so the relevant cohomologies are 2 and 4. In this case $\mathcal{O}(H_0)$ has the diffeomorphism type of $T^\ast \mathbb{P}^2$, $W = \mathbb{P}^2 \times \mathbb{P}^2$, and $F(1,2)$ has the cohomology of $\mathbb{P}^2 \times \mathbb{P}^1$. Using 12 with $k = 2$ we get

$$H^1(\mathcal{O}(H_0)) \rightarrow H^0(\mathbb{F}(1,2)) \rightarrow H^2(\mathbb{P}^2 \times \mathbb{P}^2) \rightarrow H^2(\mathcal{O}(H_0)) \rightarrow H^1(\mathbb{F}(1,2)) \rightarrow .$$
Thus, 
\[ 0 \to H^0(F(1, 2)) = \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^2(O(H_0)) = \mathbb{Z} \to 0, \]
so the map of Hodge structures
\[ \delta_2 : H^0(F(1, 2)) \otimes \mathbb{Q}(-1) \to H^2(\mathbb{P}^2 \times \mathbb{P}^2) \]
is injective and and Corollary 12 gives that
\[ W^k H^2(O(H_0)) = \begin{cases} 0 & k < 2 \\ \mathbb{H}^2(O(H_0)) & k \geq 2 \end{cases} \]
and Hodge filtration \( F^p H^2(O(H_0)) \) given by classes represented by \( \geq p \)-holomorphic logarithmic differential forms such that
\[ \text{Gr}_k H^n(U) = \begin{cases} 0 & k \neq 2 \\ \mathbb{H}^n(O(H_0)) & k = 2 \end{cases}. \]

Next, we use 12 with \( k = 4 \) and get
\[ H^3(O(H_0)) \to H^2(F(1, 2)) \to H^4(\mathbb{P}^2 \times \mathbb{P}^2) \to H^4(O(H_0)) \to H^0(F(1, 2)), \]
so the map of Hodge structures
\[ \delta_4 : H^0(F(1, 2)) \otimes \mathbb{Q}(-1) \to H^2(\mathbb{P}^2 \times \mathbb{P}^2) \]
is injective, and Corollary 12 gives that
\[ W^k H^4(O(H_0)) = \begin{cases} 0 & k \neq 4 \\ \mathbb{H}^4(O(H_0)) & k = 4 \end{cases} \]
and Hodge filtration \( F^p H^4(O(H_0)) \) given by classes represented by \( \geq p \)-holomorphic logarithmic differential forms such that
\[ \text{Gr}_k H^4(O(H_0)) = \begin{cases} 0 & k \neq 4 \\ \mathbb{H}^4(O(H_0)) & k = 4 \end{cases}. \]

Generalization to higher dimensions is clear, and we have always a trivial weight filtration. Hence, we obtain in general that the Hodge structures on
the adjoint orbits are pure.

6.0.1. \((p, p)\) cohomology only. Let \( Z = \text{Bl}_I \mathbb{P}^n \times \mathbb{P}^n \) be and let \( E \) be the exceptional divisor. So we have that \( Z \setminus E = \mathbb{P}^n \times \mathbb{P}^n \setminus I \). We compute the Hodge numbers of \( E \), and show:

**Lemma 29.** The exceptional divisor \( E \) has only \((p, p)\) cohomology.

**Proof.** Firstly, observe that \( F(1, n) \) is a \( \mathbb{P}^1 \)-bundle over \( I \). Secondly, observe that \( E \) is a \( \mathbb{P}^1 \)-bundle over \( I \). So we have the Hodge polynomials
\[ \alpha(E) = \alpha(F(1, n)). \]

On the other hand \( F(1, n) \) fibres over \( \mathbb{P}^n \) with fibre \( \mathbb{P}^{n-1} \). Projective space has the Hodge polynomial \( \alpha(\mathbb{P}^n) = 1 + uv + \cdots + u^n v^n \). Thus,
\[ \alpha(F(1, n)) = (1 + uv + \cdots + u^{n-1} v^{n-1})(1 + uv + \cdots + u^n v^n) = \alpha(E). \]

\( \square \)
Remark 30. We observe that $O_n$ has trivial canonical bundle, or equivalently, $\mathbb{P}^n \times \mathbb{P}^n \setminus F(1, n)$ has a trivial canonical bundle. We show there exists a nowhere vanishing section of $\omega_{\mathbb{P}^n \times \mathbb{P}^n \setminus F(1, n)}$. This is true for the following reason. It is sufficient to find a nowhere vanishing section of $\omega_{\mathbb{P}^n \times \mathbb{P}^n \setminus F(1, n)}$. We have $\omega_{\mathbb{P}^n \times \mathbb{P}^n} \cong O_{\mathbb{P}^n \times \mathbb{P}^n}(-n - 1, -n - 1)$ and $F(1, n) \in |O_{\mathbb{P}^n \times \mathbb{P}^n}(1, 1)|$. Thus $(n + 1)F(1, n)$ induces an element of $\omega_{\mathbb{P}^n \times \mathbb{P}^n}$ vanishing only at $F(1, n)$.

7. KKP Numbers for Minimal Adjoint Orbits

We now calculate the KKP numbers for the Landau–Ginzburg model $\text{LG}_n$ (see 13). We write the Hodge diamonds of $O_n$, of its partial compactification $\overline{O}_n$ and the divisor $D_n$ at infinity, that is, $\overline{O}_n = O_n \cup D_n$.

7.0.1. Classical Hodge numbers of $\text{LG}_2$. Hodge diamonds corresponding to $\overline{O}_2 = O_2 \cup D_2$.

\[
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
\infty & 0 & 0 & 0 \\
\infty & 0 & 1 \\
O_2 & D_2
\end{array}
\]

Figure 1. Hodge diamonds for $O_2$ and $D_2$, followed by $\overline{O}_2$.

7.0.2. KKP numbers of $\text{LG}_2$. Now, for $\overline{\text{LG}_2}$ we want to compute the KKP numbers $f^{p,q}(Y, w)$, $h^{p,q}(Y, w)$, $i^{p,q}(Y, w)$ defined in section 2. These Hodge theoretical invariants are for the tamely compactified Landau–Ginzburg model $w: Z \to \mathbb{C}$ as defined in lemma 20. The invariants satisfy the equalities

$$\dim H^m(Y, Y_b; \mathbb{C}) = \sum_{p+q=m} f^{p,q}(Y, w) = \sum_{p+q=m} h^{p,q}(Y, w) = \sum_{p+q=m} i^{p,q}(Y, w)$$

where $Y_b$ is a smooth fibre of $w$. Since here, by sections 6.0.1 and 6.0.2, our spaces have only $(p, p)$ cohomologies each of the sums reduce to a single term, in particular the KKP conjecture is satisfied for $\text{LG}_2$. To find the invariants we only need to calculate the relative cohomology, which is invariant by homotopy. For $\text{LG}_2$, using [GGSM2, Thm. 2.1], we have that $Y = O_2 \sim \mathbb{P}^1$ ($\sim$ denotes homotopy equivalence) and using [GGSM1, Cor. 3.4] and the fact
that \( f_H \) has 2 critical points, we get that \( Y_b \sim \mathbb{P}^1 \setminus \{N, S\} \sim S^1 \). Now we consider the exact sequence of the pair

\[
\ldots \to H^i(Y, Y_b; \mathbb{C}) \to H^i(Y; \mathbb{C}) \to H^i(Y_b; \mathbb{C}) \to H^{i+1}(Y, Y_b; \mathbb{C}) \to \ldots
\]

Putting \( Y = S^2 \) and \( Y_b = S^1 \), the sequence becomes

\[
0 \to H^0(S^2, S^1; \mathbb{C}) \to H^0(S^2; \mathbb{C}) \to H^0(S^1; \mathbb{C}) \to H^1(S^2, S^1; \mathbb{C}) \to H^1(S^2; \mathbb{C})
\]

and we obtain

\[
0 \to H^0(S^2, S^1; \mathbb{C}) \to \mathbb{C} \xrightarrow{\sim} \mathbb{C} \to H^1(S^2, S^1; \mathbb{C}) \to 0 \to \mathbb{C} \to H^2(S^2, S^1; \mathbb{C}) \to 0.
\]

Therefore,

\[
H^0(S^2, S^1; \mathbb{C}) = 0, \quad H^1(S^2, S^1; \mathbb{C}) = 0, \quad H^2(S^2, S^1; \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}.
\]

We conclude that

\[
f^{1,1}(Y, w) = h^{1,1}(Y, w) = i^{1,1}(Y, w) = h^2(Y, Y_b) = 2
\]

and vanish for \( (p, q) \neq (1, 1) \). Observe also that \( h^{1,1}(O_2, f_H) = 2 = h^{1,1}(\overline{O_2}) \).

Therefore, the KKP diamond for \( LG_2 \) is:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

7.0.3. Hodge numbers of \( LG_3 \). In this case we have that \( O_3 \) has the homotopy type of \( T^*\mathbb{P}^2 \), but it is an affine variety, whereas \( D_3 = F(1, 2) \) and \( \overline{O_3} = \mathbb{P}^2 \times \mathbb{P}^2 \).

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\infty & 0 & 0 & 0 & 0 & 2 \\
\infty & 0 & 0 & 0 & 0 & 0 \\
\infty & 0 & 0 & 0 & 0 & 0 \\
\infty & 0 & 0 & 0 & 0 & 0 \\
\infty & 0 & 0 & 0 & 0 & 0 \\
\infty & 0 & 0 & 0 & 0 & 2 \\
\end{array}
\]

\( O_3 \) \hspace{1cm} \( D_3 \)

\textbf{Figure 2.} Hodge diamonds for \( O_3 \) and \( D_3 \), followed by \( \overline{O_3} \)
7.0.4. **KKP numbers of \( LG_3 \).** Calculations go along the same lines as those in 7.0.2, but are geometrically more intricate. First observe that \( Y = T^*\mathbb{P}^2 \sim \mathbb{P}^2 \), and that the regular fibre \( Y_b \) corresponds to \( \mathbb{P}^2 \setminus \{P_1, P_2, P_3\} \) where \( P_1, P_2, P_3 \) are three points in \( \mathbb{P}^2 \). Denote by \( D_i \) a small open disc around \( P_i \), hence \( D_i \sim S^3 \) is a 4-disc. Then consider the following decomposition \( \mathbb{P}^2 = A \cup B \) where:

- \( A = D_1 \cup D_2 \cup D_3 \)
- \( B = \mathbb{P}^2 \setminus \{P_1, P_2, P_3\} \).

Mayer-Vietoris gives:

\[
\begin{align*}
\rightarrow H^i(\mathbb{P}^2; \mathbb{C}) & \rightarrow H^i(A; \mathbb{C}) \oplus H^i(B; \mathbb{C}) \rightarrow H^i(A \cap B; \mathbb{C}) \rightarrow H^{i+1}(\mathbb{P}^2; \mathbb{C}) \rightarrow
\end{align*}
\]

Noting that \( A \cap B = D_1 \setminus \{P_1\} \cup D_2 \setminus \{P_2\} \cup D_3 \setminus \{P_3\} \) and \( D_i \setminus \{P_i\} \sim S^3 \) the above sequence gives:

\[
\begin{align*}
0 \rightarrow & \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus H^0(B; \mathbb{C}) \rightarrow \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \rightarrow \\
0 \rightarrow & 0 \oplus H^1(B; \mathbb{C}) \rightarrow 0 \rightarrow \\
\mathbb{C} \rightarrow & 0 \oplus H^2(B; \mathbb{C}) \rightarrow 0 \rightarrow \\
0 \rightarrow & 0 \oplus H^3(B; \mathbb{C}) \rightarrow \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \rightarrow \mathbb{C} \rightarrow 0.
\end{align*}
\]

Here, we have that \( Y_b = B \) and we get that:

\[
H^i(Y_b; \mathbb{C}) = \begin{cases} 
\mathbb{C} & \text{if } i = 0, 2 \\
\mathbb{C} \oplus \mathbb{C} & \text{if } i = 3 \\
0 & \text{otherwise.}
\end{cases} \tag{13}
\]

Now consider the long exact sequence of the pair:

\[
0 \rightarrow \ldots \rightarrow H^n(Y, Y_b; \mathbb{C}) \overset{\delta}{\longrightarrow} H^n(Y; \mathbb{C}) \overset{i^*}{\longrightarrow} H^n(Y_b; \mathbb{C}) \overset{\delta}{\longrightarrow} H^{n+1}(Y, Y_b; \mathbb{C}) \rightarrow \ldots
\]

Using 13, the above sequence reduces to:

\[
\begin{align*}
0 \rightarrow & H^0(Y, Y_b; \mathbb{C}) \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow H^1(Y, Y_b; \mathbb{C}) \rightarrow 0 \rightarrow 0 \rightarrow \\
H^2(Y, Y_b; \mathbb{C}) \rightarrow & \mathbb{C} \rightarrow \mathbb{C} \rightarrow H^3(Y, Y_b; \mathbb{C}) \rightarrow 0 \rightarrow \mathbb{C} \oplus \mathbb{C} \rightarrow H^4(Y, Y_b; \mathbb{C}) \rightarrow \mathbb{C} \rightarrow 0.
\end{align*}
\]

Thus we conclude that

\[
H^i(Y, Y_b; \mathbb{C}) = \begin{cases} 
\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} & \text{if } i = 4, \\
0 & \text{otherwise.}
\end{cases}
\]

It follows that the KKP diamond for \( LG_3 \) is:

\[
\begin{array}{cccccc}
& 1 & 0 & 0 & \\
0 & 2 & 0 & 0 & \\
0 & 0 & 3 & 0 & 0 & \\
0 & 0 & 0 & 0 & \\
0 & 2 & 0 & \\
0 & 0 & \\
1 & \\
\end{array}
\]

\[
\mathcal{O}_3
\]
7.0.5. **KKP numbers for** $LG_n$. **We now consider the general case of the minimal adjoint orbit** $O_n$ of $\mathfrak{sl}(n+1, \mathbb{C})$. **We denote by** $Y$ **the adjoint orbit** $O_n$ **of the element** $\text{Diag}(n, -1, \ldots, -1)$ **and by** $Y_b$ **a regular fibre of the superpotential** $f_H$.

**Proposition 31.** $h^{2k}(Y, Y_b; \mathbb{C}) = \begin{cases} k + 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$

**Proof.** Using [GGSM2, Thm. 2.1], we have that $Y = O_n \sim \mathbb{P}^n$ ($\sim$ denotes homotopy equivalence) and using [GGSM1, Cor. 3.4] and the fact that $f_H$ has $n+1$ critical points, we get that $Y_b \sim \mathbb{P}^n \setminus \{P_1, \ldots, P_{n+1}\}$ where $P_1, \ldots, P_{n+1}$ are points in $\mathbb{P}^n$. **We begin considering the decomposition** $\mathbb{P}^n = A \cup B$ **where:**

- $A = D_1 \cup \ldots \cup D_{n+1}$ **with** $D_i$ **an open disc around** $P_i$ **and**
- $B = \mathbb{P}^k \setminus \{P_1, \ldots, P_{n+1}\} \sim Y_b$.

**Mayer-Vietoris gives:**

$$
\cdots \to H^{2k}(\mathbb{P}^n; \mathbb{C}) \to H^{2k}(A; \mathbb{C}) \oplus H^{2k}(B; \mathbb{C}) \to H^{2k}(A \cap B; \mathbb{C}) \to H^{2k+1}(\mathbb{P}^n; \mathbb{C}) \to \cdots
$$

Here $A \cap B$ is the union of $n+1$ punctured discs, therefore has the cohomology of $\bigcup_{i=1}^{n+1} S^{2n-1}$. **Thus, the Mayer–Vietoris sequence becomes:**

$$
0 \to \mathbb{C} \to \mathbb{C}^{\oplus n+1} \oplus H^0(B; \mathbb{C}) \to \mathbb{C}^{\oplus n+1} \to \\
0 \to 0 \oplus H^1(B; \mathbb{C}) \to 0 \\
\vdots \quad \vdots \\
\to \mathbb{C} \to 0 \oplus H^{2k-2}(B; \mathbb{C}) \to 0 \\
0 \to 0 \oplus H^{2k-1}(B; \mathbb{C}) \to \mathbb{C}^{\oplus n+1} \to \\
\mathbb{C} \to \mathbb{C}^{\oplus n+1} \oplus H^{2k}(B; \mathbb{C}) \to 0 \to 0
$$

and we conclude that:

$$
H^i(Y_b; \mathbb{C}) = \begin{cases} 
\mathbb{C} & i \text{ even, } i < 2n - 1 \\
\mathbb{C}^{\oplus n+1} & i = 2n - 1 \\
0 & \text{otherwise}.
\end{cases}
$$

To compute the relative cohomology $H^*(Y, Y_b; \mathbb{C})$ we consider the long exact sequence of the pair:

$$
\cdots \to H^n(Y, Y_b; \mathbb{C}) \xrightarrow{\partial^*} H^n(Y; \mathbb{C}) \xrightarrow{\delta} H^n(Y_b; \mathbb{C}) \xrightarrow{\delta} H^{n+1}(Y, Y_b; \mathbb{C}) \to \cdots
$$
Note that the cohomology of $Y \sim \mathbb{P}^n$ is the same as that of $Y_b$ from 0 to $2n - 2$, therefore we conclude that:

$$H^i(Y, Y_b; \mathbb{C}) = \begin{cases} \mathbb{C}^{\oplus n+1} & i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

□

Thus, the KKP diamond of the $(Y, Y_b)$ is the following:

\[ \begin{array}{ccccccc} & & & & & & \\
& & 0 & & 0 & & 0 \\
& & 0 & & 0 & & 0 \\
& 0 & & 0 & & 0 & \\
& 0 & 0 & 0 & 0 & 0 & \\
& 0 & 0 & 0 & 0 & 0 & 0 \\
& & \vdots & & & & \\
0 & 0 & 0 & \cdots & n + 1 & \cdots & 0 & 0 & 0 \\
& & \vdots & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 \\
& & 0 & & 0 & & \\
\end{array} \]

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