Spectrum of $D = 6$, $N = 4b$ Supergravity on $AdS_3 \times S^3$

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ABSTRACT

The complete spectrum of $D = 6$, $N = 4b$ supergravity with $n$ tensor multiplets compactified on $AdS_3 \times S^3$ is determined. The $D = 6$ theory obtained from the $K_3$ compactification of Type IIB string requires that $n = 21$, but we let $n$ be arbitrary. The superalgebra that underlies the symmetry of the resulting supergravity theory in $AdS_3$ coupled to matter is $SU(1,1|2)_L \times SU(1,1|2)_R$. The theory also has an unbroken global $SO(4)_R \times SO(n)$ symmetry inherited from $D = 6$. The spectrum of states arranges itself into a tower of spin 2 supermultiplets, a tower of spin 1, $SO(n)$ singlet supermultiplets, a tower of spin 1 supermultiplets in the vector representation of $SO(n)$ and a special spin $\frac{1}{2}$ supermultiplet also in the vector representation of $SO(n)$. The $SU(2)_L \times SU(2)_R$ Yang-Mills states reside in the second level of the spin 2 tower and the lowest level of the spin 1, $SO(n)$ singlet tower and the associated field theory exhibits interesting properties.

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1 Introduction

Compactifications of supergravity on anti de Sitter spacetimes have been studied extensively in the past (see [1] for a survey). In particular, the $AdS_4 \times S^7$ compactification of $D = 11$ supergravity has been a subject of much study (see [2] for a review). AdS spacetimes have also played role in the study of supermembrane compactifications [3]. More recently, AdS backgrounds have made a remarkable appearance in the context of M-theory, thanks to the Maldacena conjecture [4] according to which, the low energy dynamics of M-theory on AdS is captured by the dynamics of a conformal field theory on the boundary of AdS. The calculational framework for testing this conjecture has been spelled out in [5, 6].

Much of the work done so far in this subject has focused on the $AdS_5 \times S^5$ compactification of type IIB supergravity. While this is a very interesting model to explore, the fact that the boundary theory is $N = 4$ super Yang-Mills theory in 3+1 dimensions makes some of the relevant calculations rather difficult to carry out. On the other hand, $AdS_3$ compactifications of M-theory involve boundary conformal field theories in 1 + 1 dimensions which are far more amenable to exact calculations. Furthermore, $AdS_3$ has been shown to arise in the horizon geometry limit [7] of the self-dual string [8], the horizon geometry limit of black holes in Type II string theory [9] and intersecting branes [10, 11]. More recently, $AdS_3$ bulk/boundary correspondence has been studied in [12, 13, 14, 15] with interesting results. In particular, the $AdS_3 \times S^3 \times K_3$ compactification of type IIB supergravity was considered in [4, 12]. The $K_3$ compactification gives rise to $D = 6, N = 4b$ supergravity coupled to 21 tensor multiplets. The covariant field equations for this model were constructed by Romans [16]. (The quartic fermion terms for this model were recently obtained in [17]). A subsequent compactification on $AdS^3 \times S^3$ yields a spectrum of states some of which have already been found and the corresponding operators in the $AdS_3$ boundary conformal field theory have been identified in [12].

The purpose of this paper is to determine the complete spectrum of $D = 6, N = 4b$ supergravity coupled to $n$ tensor multiplets. For anomaly-freedom one sets $n = 21$ but the results are valid for any $n$. The complete spectrum is given in Tables 3, 4 and 5 and the supermultiplet structure is shown in Figures 1, 2 and 3, with the notation for representations defined in (110). These results are expected to shed further light on the nature of the boundary/bulk duality in the context of $AdS_3$.

The organization of the paper is as follows. In section 2 we describe the model and linearize the field equations. In section 3 the bosonic sector of the model is studied; harmonic expansions on $S^3$ are performed, following the techniques of [18], in the de Donder-Lorentz gauge and the linearized field equations are diagonalized. In section 4, the fermionic sector is studied by taking similar steps. In section 5, $AdS_3$ is analytically continued to a sphere $S^3_E$ on which harmonic expansions are carried out, following the procedures described in [19, 20]. Continuing back to $AdS_3$ we then determine the complete representation content of all the physical states. In section 6, the supermultiplet structure of the complete spectrum is determined. We summarize our results in the concluding section where we also comment on the nature of the sector involving the Yang-Mills vector fields originating from the $S^3$ compactification. Notations and conventions are provided in the Appendix.
2 The $D = 6, N = 4b$ Supergravity Plus $n$ Tensor Multiplets

2.1 Field Content

The equations of motion for $D = 6, N = 4b$ supergravity coupled to $n$ tensor multiplets were constructed in [16]. The theory consists of a pure supergravity multiplet containing a graviton, 4 gravitinos and 5 self-dual tensor fields coupled to $n$ tensor multiplets containing an anti-self-dual tensor field, 4 fermions and 5 scalars. The field content is summarized in the following Table.

| Field | $g_{MN}$ | $\psi_M$ | $B^i_{MN}$ | $B^r_{MN}$ | $\chi^r$ | $\phi^{ir}$ |
|-------|----------|---------|------------|------------|--------|-----------|
| $SO(5)$ | 1 | 4 | 5 | 1 | 4 | 5 |
| $SO(n)$ | 1 | 1 | 1 | $n$ | $n$ | $n$ |

where $M, N, \ldots = 1, \ldots, 6$ are curved 6D indices, $i, j, \ldots = 1, \ldots, 5$ are $SO(5)$ vector indices and $r, s, \ldots = 1, \ldots n$ are $SO(n)$ vector indices. The $SO(5)$ spinor indices carried by $\psi_M$ and $\chi^r$ are suppressed.

In the coupled theory the scalar sector constitute a sigma-model over the coset space

$$\frac{SO(5,n)}{SO(5) \times SO(n)}.$$  

The scalars parametrize a vielbein $(V^i_I, V^r_I)$ obeying the $SO(5,n)$ relation

$$V^i_I V^j_I - V^r_I V^r_I = \eta_{IJ},$$  

where the index $I = 1, \ldots, 5 + n$ transforms under global $SO(5,n)$ transformations and is raised and lowered with the metric $\eta_{IJ} = \text{diag} (++++\cdots \cdots - \cdots \cdots)$. The indices $(i, r)$ are transforming under composite local $SO(5) \times SO(n)$ transformations. Defining

$$dVV^{-1} = \left( \begin{array}{cc} Q^{ij} & \sqrt{2} P^{is} \\ \sqrt{2} P^{jr} & Q^{rs} \end{array} \right),$$

$Q^{ij}$ and $Q^{rs}$ are the composite $SO(5)$ and $SO(n)$ connections solvable in terms of the $5n$ physical scalars $\phi^{ir}$ via the Cartan-Maurer equation. The covariant derivatives in the scalar and the fermionic sectors are given by

$$D_M P^i_N = \nabla_M P^i_N - Q^i_{MN} \Gamma^i_M \chi_N - Q^{rs}_{MN} P^{irs}_N,$$

$$D_M \chi^r = \nabla_M \chi^r - \frac{1}{4} Q^i_{MN} \Gamma^i_M \chi^r - Q^{rs}_{MN} \chi^s,$$

$$D_M \psi_N = \nabla_M \psi_N - \frac{1}{4} Q^i_{MN} \Gamma^i_M \psi_N,$$

$$D_M \phi^{ir} = \nabla_M \phi^{ir} - \frac{1}{4} Q^i_{MN} \Gamma^i_M \phi^{ir}.$$
where $\Gamma^i$ are the $SO(5)$ $\Gamma$-matrices and $\nabla_M$ denotes the Lorentz covariant derivative. In the spin-1 sector the supersymmetric couplings are effected via the modified 3-form field strengths

$$H^i = G^I_{MPQ} V^i_{PQ}, \quad H^r = G^I_{MPQ} V^r_{PQ},$$

where the elementary field strengths $G^I_{MNP}$ obey the globally $SO(5, n)$ invariant Bianchi identity

$$dG^I = 0, \quad G^I = dB^I.$$  \hspace{1cm} (6)

### 2.2 Field Equations and Supersymmetry Transformations

In the bosonic sector, the field equations are the Einstein equation

$$R_{MN} = H^i_{MPQ} H^i_{N PQ} + H^r_{MPQ} H^r_{N PQ} + 2 P^i_{M} P^r_{N},$$

the scalar equation

$$D^M P^i_{M} - \frac{\sqrt{2}}{3} H^i_{MNP} H^r_{MNP} = 0,$$  \hspace{1cm} (8)

and the following $SO(5) \times SO(n)$ invariant Hodge-duality conditions on the 3-form field strengths

$$H^i_{MNP} = \frac{1}{3!} e_{MNPQRS} H^{QRS}, \quad H^r_{MNP} = -\frac{1}{3!} e_{MNPQRS} H^{QRS}.$$  \hspace{1cm} (9)

In the fermionic sector the field equations are

$$\Gamma^{MNP} D_N \psi_P - H^i_{MN} \Gamma^i \psi_P + \frac{1}{2} H^r_{MN} \Gamma^r \chi^r - \frac{1}{\sqrt{2}} P^i_{M} \Gamma^i \chi^r = 0,$$  \hspace{1cm} (10)

$$\Gamma^{M} D_M \chi^r + \frac{1}{12} \Gamma^{MNP} H^i_{MNP} \Gamma^i \psi_P - \frac{1}{\sqrt{2}} P^i_{M} \Gamma^i \psi_M = 0.$$

The supersymmetry transformations acting covariantly on the field equations are

$$\delta e^A_M = -\bar{\epsilon} \Gamma^A \psi_M,$$

$$\delta \psi_M = D_M \epsilon - \frac{1}{2} H^i_{MNP} \Gamma^i \epsilon,$$

$$\delta B^i_{MN} = -V^i_{M} \bar{\epsilon} \Gamma^i \psi_N + \frac{1}{2} V^r_{M} \bar{\epsilon} \Gamma^r \chi^r,$$

$$\delta \chi^r = \frac{1}{\sqrt{2}} \Gamma^M P^i_{M} \Gamma^i \epsilon + \frac{1}{12} \Gamma^{MNP} H^r_{MNP} \epsilon,$$

$$\delta V^i_{M} = \bar{\epsilon} \Gamma^i \psi^r V^r_{M},$$

$$\delta V^r = \bar{\epsilon} \Gamma^i \psi^r V^i_{M}.$$

where $A = 1, \ldots, 6$ is an $SO(5, 1)$ tangentspace index and $e^A_M$ is the sechsbein. The superalgebra closes on general coordinate, local Lorentz, tensor gauge and composite local $SO(5) \times SO(n)$ transformations.
2.3 The Vacuum Solution

We shall study the compactification on the maximally supersymmetric vacuum solution with the geometry of $\text{AdS}_3 \times \text{S}^3$, with curvature tensors given by

\begin{align}
R_{\mu \nu \rho \sigma} &= -m^2 (g_{\mu \rho} g_{\nu \sigma} - g_{\nu \rho} g_{\mu \sigma}), \\
R_{abcd} &= m^2 (g_{ac} g_{bd} - g_{bc} g_{ad}),
\end{align}

(12)

where $\mu, \nu, \ldots = 1, 2, 3$ are curved $\text{AdS}_3$ indices, $a, b, \ldots = 1, 2, 3$ are curved $\text{S}^3$ indices and $g_{\mu \nu}$ and $g_{ab}$ are the metrics of $\text{AdS}_3$ and $\text{S}^3$ with radius $m^{-1}$. One of the components of the self-dual field strength is singled out and put equal to the Levi-Cevita tensors, while the remaining field strengths vanish

\begin{align}
H_{\mu \rho}^i &= m \epsilon_{\mu \rho \delta} \delta_5^i, \\
H_{abc}^i &= m \epsilon_{abc} \delta_5^i, \\
H_{MNP}^r &= 0.
\end{align}

(13)

This solution uses a relative orientation between the six-dimensional space-time and the two three-dimensional factors where $e_{\mu \rho \nu \delta} = e_{\mu \rho} \epsilon_{\nu \delta}$. The $SO(5, n)$ vielbein is taken to be constant. By a global $SO(5, n)$ rotation it can be set equal to unity

\begin{align}
V_I^i &= \delta_I^i, \\
V_I^r &= \delta_I^r.
\end{align}

(14)

Finally we set the fermions equal to zero

\begin{align}
\chi_r &= 0, \\
\psi_M &= 0.
\end{align}

(15)

Supersymmetry of the solution requires that $\delta \psi_M = 0$ and $\delta \chi = 0$. The latter is trivially satisfied, while the first condition yields the Killing spinor equations

\begin{align}
D_\mu \epsilon + \frac{1}{2} m \gamma_\mu \Gamma^5 \epsilon &= 0, \\
D_a \epsilon - \frac{i}{2} m \gamma_a \Gamma^5 \epsilon &= 0,
\end{align}

(16)

where the spinor depend on both $\text{AdS}_3 \times \text{S}^3$ coordinates and carries indices of $SO(2, 1) \times SO(3)$ that have been supressed (see the appendix for full details). The integrability of these equations follows from (12).

2.4 Linearized Field Equations

We parametrize the linearized fluctuations around a background with non-vanishing metric $\bar{g}_{MN}$, three-form field strength $\bar{H}_{MNP}^i$ and constant vielbein as follows
g_{MN} = \bar{g}_{MN} + h_{MN},

G_{MNP}^I = \bar{G}_{MNP}^I + g_{MNP}^I, \quad (17)

V_I^i = \delta_i^I + \phi^{ir} \delta_i^r,

V_I^r = \delta_i^I + \phi^{ir} \delta_i^r,

P_{ir}^M = \frac{1}{\sqrt{2}} \partial_M \phi^{ir}. \quad (18)

The composite $SO(5) \times SO(n)$ connection $Q$ is then quadratic in scalar fluctuations. Note that in the background the flat and curved $SO(5, n)$ indices are identified. Taking the background to obey the classical field equations to zeroth order, the resulting linearized field equations for the bosonic fluctuations are

\begin{align*}
-\frac{1}{2} \bar{\nabla}^2 h_{MN} + R_{(M}^P h_{N)P} + R_{P_{MNP}Q} h^{PQ} + \bar{\nabla}_{(M} \bar{\nabla}^P h_{N)P} - \frac{1}{2} \bar{\nabla}_M \bar{\nabla}_N \left( \bar{g}^{PQ} h_{PQ} \right) \\
= 2 \bar{H}_i^{PQ} g_{iNP}^{Q} - 2 \bar{H}_i^{NP} \bar{H}^i_{N} \bar{H}_P R_{QR}, \quad (19)
\end{align*}

\begin{align*}
\frac{1}{6} \bar{e}_{MNP}^{QRS} g_{iPQRS}^{i} = g_{iMNP}^{i} - 3 h_{[M}^{Q} \bar{H}^{i}_{NP]Q} + \frac{1}{2} h_{Q} \bar{H}^{i}_{MNP}, \quad (20)
\end{align*}

\begin{align*}
\frac{1}{6} \bar{e}_{MNP}^{QRS} g_{iPQRS}^{i} = -g_{iMNP}^{i} - 2 \phi^{ir} \bar{H}^{i}_{MNP}, \quad (21)
\end{align*}

\begin{align*}
\bar{\nabla}^2 \phi^{ir} = \frac{2}{3} g^{rMNP} \bar{H}^{i}_{MNP}. \quad (22)
\end{align*}

and

\begin{align*}
\Gamma^{MNP}_N \psi_P - \bar{H}^{iMNP}_i \Gamma^i_N \psi_P = 0, \\
\Gamma^M_D \chi^r + \frac{1}{12} \Gamma^{MNP}_i \bar{H}^{i}_{MP} \Gamma^i_M \chi^r = 0. \quad (23)
\end{align*}

for the fermionic fluctuations. The linearized field equations are invariant under the linearized reparametrizations and tensor gauge transformations

\begin{align*}
\delta h_{MN} &= \bar{\nabla}_M \xi_N + \bar{\nabla}_N \xi_M, \\
\delta B_{MN}^I &= 2 \partial_{[M} \eta_{N]}^I + \delta^I_i \xi^P \bar{H}^{i}_{MNP}, \quad (24)
\end{align*}

\begin{align*}
\delta \phi^{ir} &= 0.
\end{align*}

The general coordinate transformation part of (24) is the Lie derivative of $B_{MN}^I$ combined with a tensor gauge transformation with the parameter $-\xi^Q B_{QP}^I$.

## 3 The Bosonic Sector

### 3.1 Harmonic Expansion on $S^3$ and Gauge Fixing

Around the $AdS_3 \times S^3$ background we parametrize the metric fluctuations as
\[ h_{\mu\nu} = H_{\mu\nu} + \bar{g}_{\mu\nu} M, \quad \bar{g}^{\mu\nu} H_{\mu\nu} = 0, \]
\[ h_{\mu a} = K_{\mu a}, \]
\[ h_{ab} = L_{ab} + \bar{g}_{ab} N, \quad \bar{g}^{ab} L_{ab} = 0, \]  
(25)

and anti-symmetric tensor fluctuations as

\[ g^{I}_{MNP} = 3 \partial [M b^{I}_{NP}], \]
\[ b^{I}_{\mu\nu} = e_{\mu\nu\rho} X^{I\rho}, \quad b^{I}_{ab} = e_{abc} U^{Ic}, \quad b^{I}_{\mu a} = Z^{I}_{\mu a}, \]  
(26)

where the metric and the Levi-Cevita tensors are understood to be in the background geometry. From here on we shall suppress the bar-notation on any quantity defined with respect to the background geometry.

Expanding the fluctuations in harmonic functions on \( S^3 \) one has

\[
\begin{align*}
H_{\mu\nu}(x, y) &= \sum H^{(\ell_0)}_{\mu\nu}(x) Y^{(\ell_0)}(y), \\
M(x, y) &= \sum M^{(\ell_0)}(x) Y^{(\ell_0)}(y), \\
K_{\mu a}(x, y) &= \sum \left[ K^{(\ell, \pm 1)}_{\mu}(x) Y^{(\ell, \pm 1)}(y) + K^{(\ell_0)}_{\mu}(x) \partial_{\mu} Y^{(\ell_0)}(y) \right], \\
L_{ab}(x, y) &= \sum \left[ L^{(\ell, \pm 2)}(x) Y^{(\ell, \pm 2)}(y) + L^{(\ell, \pm 1)}(x) \nabla_{[a} Y^{(\ell, \pm 1)}_{b]}(y) + L^{(\ell_0)}(x) \nabla_{[a} \nabla_{b]} Y^{(\ell_0)}(y) \right], \\
N(x, y) &= \sum N^{(\ell_0)}(x) Y^{(\ell_0)}(y), \\
X^{I}_{\mu}(x, y) &= \sum X^{I(\ell_0)}_{\mu}(x) Y^{(\ell_0)}(y), \\
Z^{I}_{\mu a}(x, y) &= \sum \left[ Z^{I(\ell, \pm 1)}_{\mu}(x) Y^{(\ell, \pm 1)}(y) + Z^{I(\ell_0)}_{\mu}(x) \partial_{\mu} Y^{(\ell_0)}(y) \right], \\
U^{I}_{a}(x, y) &= \sum \left[ U^{I(\ell, \pm 1)}_{a}(x) Y^{(\ell, \pm 1)}(y) + U^{I(\ell_0)}(x) \partial_{a} Y^{(\ell_0)}(y) \right], \\
\phi^{ir}(x, y) &= \sum \phi^{ir(\ell_0)}(x) Y^{(\ell_0)}(y),
\end{align*}
\]  
(27)

where \((x, y)\) are the coordinates of \( AdS_3 \times S^3 \) and \( \{ab\} \) denotes the traceless symmetric part. The harmonic functions \( Y^{(\ell_1, \ell_2)}_{(s)} \), where \((s)\) denotes the \( SO(3) \) content, are obtained from the Wigner functions of \( SO(4) \) in the irreducible representation labelled by the highest weight vector \((\ell_1 \ell_2)\) satisfying the restriction

\[ Y^{(\ell_1 \ell_2)}_{(s)} : \quad \ell_1 \geq |\ell_2|. \]  
(28)

The dimension of the representation with highest weight \((\ell_1 \ell_2)\) is

\[ d_{(\ell_1 \ell_2)} = (\ell_1 + 1)^2 - \ell_2^2. \]  
(29)
The action of the d’Alembertian on the Wigner functions is determined by group theoretical means to be [21]

\[ \nabla_y^2 Y_{(s)}^{(\ell_1 \ell_2)} = - \left[ \ell_1 (\ell_1 + 2) + \ell_2^2 - s(s + 1) \right] Y_{(s)}^{(\ell_1 \ell_2)} , \tag{30} \]

where we have set the \( S^3 \) radius equal to one. The right hand side is simply the difference between the second order Casimir of \( SO(4) \) in the \((\ell_1, \ell_2)\) representation and that of \( SO(3) \) in the spin \( s \) representation. Applying this formula to the different cases yields

\[ \begin{align*}
\nabla_y^2 Y_{(\ell, \pm 1)}^{(0)} &= \left[ 1 - (\ell + 1)^2 \right] Y_{(\ell, \pm 1)}^{(0)}, \\
\nabla_y^2 Y_{(\ell, \pm 2)}^{(0)} &= \left[ 2 - (\ell + 1)^2 \right] Y_{(\ell, \pm 2)}^{(0)}, \\
\nabla_y^2 Y_{(\ell, \pm 1, a)}^{(0)} &= \left[ 3 - (\ell + 1)^2 \right] Y_{(\ell, \pm 1, a)}^{(0)}, \\
\n\end{align*} \tag{31} \]

The harmonic one-forms \( Y_{(\ell, \pm 1)}^{(0)} \) are transverse, and together with the longitudinal modes \( \partial_a Y_{(\ell, \pm 1)}^{(0)} \) they form complete set of one-forms \( S^3 \). Similarly the transverse \( Y_{(\ell, \pm 2)}^{(0)} \) modes and the longitudinal modes \( \nabla_{(a} Y_{b)}^{(\ell, \pm 1)} \) and \( \nabla_{(a} \nabla_{b)} Y_{(\ell, \pm 0)} \) span the space of traceless and symmetric tensors on \( S^3 \). Note the zero modes in the form of the single constant mode \( Y_{(00)}^{(0)} \), the six Killing vectors \( Y_{(1, \pm 1)}^{(0)} \) in the adjoint representation of \( SO(4) \) and the four conformal Killing vectors \( \partial_a Y_{(10)}^{(0)} \). These obey

\[ \begin{align*}
\partial_a Y_{(00)}^{(0)} &= 0, \\
\nabla_{(a} Y_{b)}^{(10)} &= 0, \\
\nabla_{(a} \nabla_{b)} Y_{(1, \pm 1)}^{(0)} &= 0. \tag{32} \end{align*} \]

The Killing vectors and the conformal Killing vectors together generate the conformal group \( SO(4,1) \) of the internal \( S^3 \).

By a field dependent gauge transformation generated by the harmonic non-zero modes, we can remove all longitudinal gauge modes:

\[ \begin{align*}
\ell \geq 1 & : \quad K_{\mu}^{(\ell)} = Z_{\mu}^{(\ell)} = U^I(\ell, \pm 1) = 0, \\
\ell \geq 2 & : \quad L^{(\ell, \pm 1)} = L^{(\ell, 0)} = 0. \tag{33} \end{align*} \]

This shows the admissibility of the so called de Donder - Lorentz gauge

\[ \begin{align*}
\nabla^a h_{(ab)} &= 0, \\
\nabla^a h_{a\mu} &= 0, \\
\nabla^a b_{aM} &= 0, \tag{34} \end{align*} \]

where the harmonic expansions (27) thus read
\[ H_{\mu\nu} = \sum H^{(\ell)}_{\mu\nu} Y^{(\ell)} \quad , \quad K_{\mu a} = \sum K^{(\ell, \pm 1)}_{\mu a} Y^{(\ell, \pm 1)} , \]
\[ M = \sum M^{(\ell)} Y^{(\ell)} , \quad X^I_{\mu} = \sum X^{I(\ell)}_{\mu} Y^{(\ell)} , \]
\[ L_{ab} = \sum L^{(\ell, \pm 2)}_{ab} Y^{(\ell, \pm 2)} , \quad Z^I_{\mu a} = \sum Z^{I(\ell, \pm 1)}_{\mu a} Y^{(\ell, \pm 1)} , \]
\[ N = \sum N^{(\ell)} Y^{(\ell)} , \quad U^I_{\mu a} = \sum U^{I(\ell)}_{\mu a} \partial a Y^{(\ell)} . \]

(35)

### 3.2 AdS$_3$ Gauge Symmetries

The de Donder - Lorentz gauge completely fix the gauge symmetries that are spontaneously broken by the compactification. These symmetries are generated by harmonic non-zero modes and they transform states of different energy into each other. The de Donder - Lorentz gauge does not touch the remaining unbroken gauge symmetries that are generated by the harmonic zero modes and that correspond to local gauge symmetries in AdS$_3$. The gauge fixing of these symmetries involves fixing of residual gauge transformations leading to decoupling of longitudinal modes from the physical spectrum and they therefore need special treatment. The gauge symmetries are:

- The Stueckelberg shift symmetries

\[
\delta H^{(10)}_{\mu\nu}(x) = -2\nabla_{\{\mu} \nabla_{\nu\}} \lambda^{(10)}(x) , \\
\delta M^{(10)}(x) = -\frac{2}{3} \nabla^2 \lambda^{(10)}(x) , \\
\delta X^I_{\mu(10)}(x) = -\delta^{I5} \partial_{\mu} \lambda^{(10)}(x) , \\
\delta N^{(10)}(x) = -2\lambda^{(10)}(x) , \\
\delta U^I_{\mu(10)}(x) = \delta^{I5} \lambda^{(10)}(x) .
\]

(36)

- The AdS$_3$ reparametrizations and tensor gauge transformations

\[
\delta M^{(00)}(x) = \frac{2}{3} \nabla^\mu \xi_{\mu}(x) , \\
\delta H^{(00)}_{\mu\nu}(x) = 2\nabla_{\{\mu} \xi_{\nu\}}(x) , \\
\delta X^I_{\mu(00)}(x) = -\epsilon^{\mu\nu\rho \eta_{\rho}} \partial_{\nu} \eta_{\mu}(x) + \delta^{I5} \xi_{\mu}(x) , \\
\delta N^{(00)}(x) = 0 .
\]

(37)

- The SO(4) Yang-Mills symmetries

\[
\delta K^{(1, \pm 1)}_{\mu}(x) = \partial_{\mu} \Lambda^{(1, \pm 1)}(x) , \\
\delta Z^I_{\mu(1, \pm 1)}(x) = \mp \delta^{I5} \frac{1}{2} \partial_{\mu} \Lambda^{(1, \pm 1)}(x) .
\]

(38)

The Lorentz gauge (34) for the anti-symmetric tensor fluctuations allows no residual gauge transformations, since there are no harmonic one-forms on $S^3$. For the same reason, the Yang-Mills transformation (38) contains a compensating $y$-dependent tensor gauge transformation of $b^5_{ab}$ in order to preserve the Lorentz gauge condition (that is why $\delta Z^{5(1, \pm 1)}_{\mu}$ is non-zero in (38)).
Table 1: The $SO(4)$ representations arising in the harmonic expansion of the bosonic fields in the de Donder - Lorentz gauge. The fields in each row are described by a coupled system of equations which must be diagonalized to find the spectrum. The residual gauge symmetries are indicated by their parameters. The index $i$ labels the vector representation of the unbroken $SO(4)_R \subset SO(5)$.

### 3.3 The Linearized Field Equations in the de Donder - Lorentz Gauge

In the de Donder - Lorentz gauge the linearized Einstein equations take the form

\[-(\nabla_x^2 + \nabla_y^2 + 2) H_{\mu\nu} + 2\nabla_{(\mu} \nabla^\rho H_{\nu)} - \nabla_{(\mu} \nabla_\nu) (M + 3N) = 0 , \quad (39)\]

\[-(\nabla_x^2 + \frac{3}{4} \nabla_y^2 + 6) M + \frac{1}{2} \nabla^\mu \nabla^\nu H_{\mu\nu} - \frac{3}{4} \nabla_x^2 N + 6 \nabla^\mu X_5^\mu = 0 , \quad (40)\]

\[-(\nabla_x^2 + \nabla_y^2) K_{\mu a} + \nabla_\mu \nabla^\nu K_{\nu a} + \nabla_a \nabla^\nu H_{\mu\nu} - 2 \nabla_\mu \nabla_a (M + N) + 4 \nabla_a X_5^\mu - 4 \nabla_\mu U_5^a - 4 e_\mu^{\rho\nu} \nabla_\rho Z_5^\nu + 4 e_a^{\mu\nu} \nabla_b Z_5^\mu = 0 , \quad (41)\]

\[-(\nabla_x^2 + \nabla_y^2 - 2) L_{ab} + 2 \nabla_{(a} \nabla^\mu K_{\mu b)} - \nabla_{(a} \nabla_b) (3M + N) = 0 , \quad (42)\]

\[-(\nabla_x^2 + \frac{4}{3} \nabla_y^2 - 8) N - \nabla_y^2 M - 8 \nabla^a U_5^a = 0 , \quad (43)\]

where $\nabla_x^2 = \nabla^\mu \nabla_\mu$ and $\nabla_y^2 = \nabla^a \nabla_a$. Each one of the linearized Hodge-duality conditions (20) and (21) splits into two pairs of equations, for the indices $(\mu\nu\rho, abc)$ and $(\mu\nu c, \mu bc)$, where the two equations in each pair turn out to be equivalent. The result is
\[ \nabla^\mu X_i^\mu \left[ \nabla^a U_i^a + \frac{3}{2} \delta^{i5} (N - M) \right] = 0 , \quad (44) \]
\[ \nabla_\mu X_i^\mu + \nabla_\mu U_i^a - e_\mu^{\nu p} \nabla_\nu Z_i^{\nu a} - e_a^{bc} \nabla_\nu Z_i^{\mu c} - \delta^{i5} K_{a \mu} = 0 , \quad (45) \]
\[ \nabla^\mu X_i^r + \nabla^a U_i^r + 2 \phi^5 = 0 , \quad (46) \]
\[ \nabla_\mu X_i^r - \nabla_\mu U_i^r - e_\mu^{\nu p} \nabla_\nu Z_i^r - e_a^{bc} \nabla_\nu Z_i^{\mu c} = 0 . \quad (47) \]

Finally, the scalar equation (22) in the AdS_3 \times S^3 vacuum reads
\[ -(\nabla^2_{x} + \nabla^2_{y}) \phi^{ir} - 4 \delta^{i5} \left( \nabla^\mu X_i^\mu - \nabla^a U_i^a \right) = 0 . \quad (48) \]

We next insert the harmonic expansions into the above equations. This leads to the following set of irreducible equations (see Table 1 for the grouping of the fields according to their SO(4) content):

- **The J = 2 sector**

\((\ell, 0), \ell \geq 2: H_{\mu \nu}^{(\ell, 0)}\) gives rise to a massive spin-2 mode described by the transverse and symmetric spin-2 tensor

\[ S_{\mu \nu}^{(\ell, 0)} = H_{\mu \nu}^{(\ell, 0)} - \frac{2}{(\ell + 1)^2} \nabla_{\{ \mu} \nabla_{\nu\}} \left( N^{(\ell, 0)} - 2 U^{(\ell, 0)} \right) , \quad (49) \]

satisfying the equation

\[ \left[ \nabla^2_{x} + 3 - (\ell + 1)^2 \right] S_{\mu \nu}^{(\ell, 0)} = 0 . \quad (50) \]

In obtaining this equation, we have used

\[ \nabla^\nu H_{\mu \nu}^{(\ell, 0)} - 2 \partial_\mu M^{(\ell, 0)} - 2 \partial_\mu N^{(\ell, 0)} + 4 X_5^{(\ell, 0)} - 4 \partial_\mu U_5^{(\ell, 0)} = 0 , \quad \ell \geq 1 , \quad (51) \]

which follows from (41). We have also used (58) and

\[ 3 M^{(\ell, 0)} + N^{(\ell, 0)} = 0 , \quad \ell \geq 2 . \quad (52) \]

- **The J = 1 sector**

The relevant vectors are contained in the \((\ell, \pm 1)\) sectors, which we group as follows:
i) \((\ell, \pm 1)\). From (41) and (45) we find that the two vectors \(\ell \geq 2\): \(K^{(\ell, \pm 1)}_\mu\) and \(Z^{5(\ell, \pm 1)}_\mu\) are described by the coupled system

\[
\left( \nabla^2 + \left[ 2 - \frac{\ell + 1}{2} \right] \right) K^{(\ell, \pm 1)}_\mu - \nabla_\mu \nabla^\nu K^{(\ell, \pm 1)}_\nu
\]

\[+ 4e^{\mu \nu \rho} \partial_\nu Z^{5(\ell, \pm 1)}_\rho \mp 4(\ell + 1)Z^{5(\ell, \pm 1)}_\mu = 0 , \quad (53)\]

\[e^{\mu \nu \rho} \partial_\nu Z^{5(\ell, \pm 1)}_\rho \pm (\ell + 1)Z^{5(\ell, \pm 1)}_\mu + K^{(1, \pm 1)}_\mu = 0 , \quad (54)\]

subject to the condition

\[\nabla^\mu K^{(\ell, \pm 1)}_\mu = 0 , \quad \ell \geq 2 , \quad (55)\]

which follows from (42).

ii) \((\ell, \pm 1), \ell \geq 1\): From (45) we find that the vectors \(Z^{i(\ell, \pm 1)}_\mu\) are described by the equations

\[e^{\mu \nu \rho} \partial_\nu Z^{i(\ell, \pm 1)}_\rho \pm (\ell + 1)Z^{i(\ell, \pm 1)}_\mu = 0 , \quad (56)\]

where we have introduced the index \(i = 1, ..., 4\), which labels the vector representation of the unbroken \(SO(4)_R \subset SO(5)\).

iii) \((\ell, \pm 1), \ell \geq 1\): From (47) we find that the vectors \(Z^{r(\ell, \pm 1)}_\mu\) are described by the equations

\[e^{\mu \nu \rho} \partial_\nu Z^{r(\ell, \pm 1)}_\rho \mp (\ell + 1)Z^{r(\ell, \pm 1)}_\mu = 0 . \quad (57)\]

The first order equations automatically imply the Lorentz condition and when squared they yield the usual second order Proca equations for massive vector fields. The first order nature of the equations however implies that only half the number of helicity states are found in the spectrum compared to the case of a second order Proca equation. The vector spectrum will be analyzed in more detail in section 5 using harmonic expansion on \(AdS_3\).

**The \(J = 0\) sector**

Using the algebraic equation (52) and the \(S^3\) harmonic expansion formulae, yields four sets of coupled equations (see Table 1) that are diagonalized as follows:

i) \((\ell 0), \ell \geq 2\): \(M^{(\ell 0)}\), \(N^{(\ell 0)}\), \(U^{5(\ell 0)}\) and \(X^{5(\ell 0)}\) yields two scalar modes and a massive spin 2 mode. Analyzing (40), (43) and (52) we find that the two scalar modes are the two eigenmodes of

\[
\nabla^2_{x} \begin{pmatrix} U^{5(\ell 0)} \\ N^{(\ell 0)} \end{pmatrix} - \begin{bmatrix} \ell(\ell + 2) & 2 \\ 8\ell(\ell + 2) & \ell(\ell + 2) + 8 \end{bmatrix} \begin{pmatrix} U^{5(\ell 0)} \\ N^{(\ell 0)} \end{pmatrix} = 0 . \quad (58)\]

This equation system is diagonalized as
\[
\left[ \nabla_x^2 - \ell(\ell + 2) - 4 \pm 4(\ell + 1) \right] N^{(\ell)_{\pm}} = 0 .
\] (59)

The vector \( X_{\mu}^{(\ell)_{5}} \) is not an independent field and from (45) we find that it is given by
\[
X_{\mu}^{(\ell)_{5}} = -\partial_{\mu} U^{(\ell)_{5}} .
\] (60)

ii) \((\ell, \pm 2), \ell \geq 2: \) From (42) we find that \( L^{(\ell, \pm 2)} \) describes the scalar modes
\[
(\nabla_x^2 + 1 - (\ell + 1)^2) L^{(\ell, \pm 2)} = 0
\] (61)

ii) \((\ell 0), \ell \geq 1: \) From (44) and (45) follows that \( X_{\mu}^{(\ell 0)} \) and \( U_{\mu}^{(\ell 0)} \) yield four scalar modes described by the eigenmodes of
\[
\left[ \nabla_x^2 + 1 - (\ell + 1)^2 \right] U_{\mu}^{(\ell 0)} = 0 , \quad X_{\mu}^{(\ell 0)} = -\partial_{\mu} U^{(\ell 0)},
\] (62)

iv) \((\ell 0), \ell \geq 1: \) From (46) and (48) follows that \( X_{\mu}^{(\ell 0)} \), \( U_{r}^{(\ell 0)} \) and \( \phi_{5r}^{(\ell 0)} \) yield 2\(n\) scalar modes described by the eigenmodes of
\[
\nabla_x^2 \left( \frac{U_{r}^{(\ell 0)}}{\phi_{5r}^{(\ell 0)}} \right) - \left[ \frac{\ell(\ell + 2)}{-8\ell(\ell + 2)} - \frac{2}{\ell(\ell + 2) + 8} \right] \left( \frac{U_{r}^{(\ell 0)}}{\phi_{5r}^{(\ell 0)}} \right) = 0 ,
\] (63)

which is diagonalized exactly as in (59):
\[
\left[ \nabla_x^2 + \ell(\ell + 2) - 4 \pm 4(\ell + 1) \right] \phi_{\pm}^{(\ell 0)} = 0 .
\] (64)

From (47) follows that the vector \( X_{\mu}^{(\ell 0)} \) is given by
\[
X_{\mu}^{(\ell 0)} = \partial_{\mu} U_{r}^{(\ell 0)}. \] (65)

v) \((\ell 0), \ell \geq 0: \) From (48) follows that \( \phi_{5r}^{(\ell 0)} \) yields 4\(n\) scalar modes which from (48)
\[
\left[ \nabla_x^2 + 1 - (\ell + 1)^2 \right] \phi_{5r}^{(\ell 0)} = 0 .
\] (66)

There remains the analysis of the zero mode sectors transforming under the \( AdS_3 \) gauge symmetries discussed in section 3.2 (see Table 1). These sectors are:

- **The \((10)\) Sector**

The Stueckelberg shift symmetry (36) acts on the spin 2 mode \( H_{\mu\nu}^{(10)} \) and the four scalar modes \( M^{(10)}, N^{(10)}, X_{\mu}^{(5(10))} \) and \( U_{\mu}^{(5(10))} \). For \( \ell = 1 \) the algebraic condition (52) does not follow from Einstein equations of motion due to the condition on the conformal Killing vectors \( \partial_{a} Y_{(10)} \) given in (32). However, using a gauge transformation (36) with shift parameter \( \lambda^{(10)} \) obeying
\[
(\nabla_x^2 + 1) \lambda^{(10)} = \frac{1}{2} \left( 3 M^{(10)} + N^{(10)} \right) ,
\] (67)

we may instead impose (52) as the gauge condition
\[ 3M^{(10)} + N^{(10)} = 0 . \] (68)

This gauge choice implies (58), where \( \ell \) is now to be set equal to 1, and a massive spin 2 mode described by (49) for \( \ell = 1 \), that is

\[
S^{(10)}_{\mu\nu} = H^{(10)}_{\mu\nu} - \frac{1}{2} \nabla\{\mu \nabla\nu\} \left( N^{(10)} - 2U^{(10)} \right) , \quad \nabla^\mu S^{(10)}_{\mu\nu} = 0 , \quad \left( \nabla^2_x - 1 \right) S^{(10)}_{\mu\nu} = 0 . \] (69)

We have to remember, though, that the gauge choice (68) allows residual shift transformations with parameter \( \lambda^{(10)} \) satisfying

\[
(\nabla^2_x + 1)\lambda^{(10)} = 0 . \] (70)

This symmetry can be used to gauge away the scalar mode \( N^{(10)}_+ \). We are thus left with the scalar mode

\[
(\nabla^2_x - 15)N^{(10)}_- = 0 . \] (71)

Finally, the vector is given by

\[
X^{5(10)}_\mu = -\partial_\mu U^{5(10)} . \] (72)

**The (00) Sector**

This sector can be grouped as follows (see Table 1):

\( i \) The only contribution to the spectrum from the fields \( H^{(00)}_{\mu\nu} \), \( M^{(00)} \), \( N^{(00)} \) and \( X^{5(00)}_\mu \) is the single massive mode described by (43), that is

\[
(\nabla^2_x - 8)N^{(00)} = 0 . \] (73)

To obtain this result, we first use the \( AdS_3 \) reparametrization invariance (37) to fix the gauge

\[
\nabla^\mu H^{(00)}_{\mu\nu} = \frac{15}{4} \nabla_\nu N^{(00)} . \] (74)

We then obtain from (40) the equation

\[
(\nabla^2_x - 3)M^{(00)} = 0 . \] (75)

In the gauge (74), the residual reparametrizations obey

\[
(\nabla^2_x - 3)\nabla^\mu \xi_\mu = 0 , \quad (\nabla^2_x + 2)\nabla\{\mu \xi_\nu\} + \frac{1}{3} \nabla\{\mu \nabla\nu\} \nabla^\rho \xi_\rho = 0 . \] (76)
Using the trace part $\nabla^\mu \xi_\mu$, we set
\[ M^{(00)} = \frac{9}{2} n^{(00)}. \] (77)

Consequently (39) takes the form
\[ (\nabla^2_x + 2) H^{(00)}_{\mu\nu} = 0. \] (78)

The residual gauge transformations now obey
\[ (\nabla^2_x + 2) \nabla^\mu (\xi^\mu) = 0, \quad \nabla^\mu \xi^\mu = 0, \] (79)
and they allow us to impose the gauge
\[ H^{(00)}_{\mu\nu} = 0, \] (80)
with residual global reparametrizations generated by the Killing vectors of $AdS_3$. Finally, the equation of motion (44) determines $X^{(00)}_5$ in terms of $N^{(00)}$ up to a homogeneous solution described by a pure gauge degree of freedom which is removed completely from the spectrum by the tensor gauge transformation (37) generated by $\eta^\Delta$

ii) From (44) it follows that $X^{(00)}_\mu$ describes a pure gauge field which is gauged away completely by the tensor gauge transformations (37) generated by $\eta^\Delta$.

iii) From (46) and (48) it follows that $\phi^{5r(00)}$ and $X^{(00)}_\mu$ modulo the tensor gauge transformations (37) generated by $\eta^r$ give rise to one scalar mode
\[ (\nabla^2_x - 8) \phi^{5r(00)} = 0. \] (81)

**The (1, ±1) Sector**
The two vector modes $K^{(1, \pm 1)}_\mu$ and $Z^{(1, \pm 1)}_\mu$ transform under the the Yang-Mills symmetry (38). The Lorentz gauge condition does not follow from the Einstein equations of motion in this sector due to the Killing equation (32). The gauge invariant equations of motion read
\[ \left( \nabla^2_x - 2 \right) K^{(1, \pm 1)}_\mu - \nabla^\nu K^{(1, \pm 1)}_\nu + 4 e^{\mu \nu \rho} \partial_\nu Z^{5(1, \pm 1)}_\rho \mp 8 Z^{5(1, \pm 1)}_\mu = 0, \] (82)
\[ e^{\mu \nu \rho} \partial_\nu Z^{5(1, \pm 1)}_\rho \pm 2 Z^{5(1, \pm 1)}_\mu + K^{(1, \pm 1)}_\mu = 0. \] (83)

We use (38) to impose the Lorentz gauge
\[ \nabla^\mu K^{(1, \pm 1)}_\mu = 0, \] (84)
with residual gauge symmetries generated by parameters obeying the massless vector field equation
\[ (\nabla^2_x + 2) \partial_\mu \Lambda^{(1, \pm 1)} = 0. \] (85)
As a consequence the massless modes in this sector turn out to be pure gauge, leaving only massive modes in the spectrum, as will be detailed in section 5 using $AdS_3$ harmonic expansion. In the concluding section we shall discuss further the gauge fields arising from the isometries of $S^3$ and the generation of their masses.

4 Fermionic Sector

4.1 Harmonic Expansion on $S^3$ and Gauge Fixing

We expand the fermionic fluctuations in harmonic spinors on $S^3$

$$\psi_\mu(x, y) = \sum \psi^{(\ell, \pm 1/2)}_\mu(x) Y^{(\ell, \pm 1/2)}(y),$$
$$\psi_a(x, y) = \sum \left[ \chi^{(\ell, \pm 3/2)}(x) Y_a^{(\ell, \pm 3/2)} + \left\{ \zeta_1^{(\ell, \pm 1/2)}(x) \right\} Y^{(\ell, \pm 1/2)}(y) \right],$$
$$\chi^r(x, y) = \sum \chi^{r(\ell, \pm 1/2)}(x) Y^{(\ell, \pm 1/2)}(y),$$
$$\nabla_{\{a\}} \equiv \nabla_a - \frac{1}{2} \gamma_a \gamma^b \nabla_b. \quad (86)$$

The spinor harmonics are Wigner functions in the coset $SO(4)/SO(3)$. They carry a suppressed row index transforming as a spinor of $SO(3)$. The representation label $\ell$ is a positive half-integer subject to the condition (28). The spinor harmonics obey

$$\nabla_y^2 Y^{(\ell, \pm 1/2)} = \left[ \frac{3}{2} - (\ell + 1)^2 \right] Y^{(\ell, \pm 1/2)}, \quad \nabla Y^{(\ell, \pm 1/2)} = \pm i (\ell + 1) Y^{(\ell, \pm 1/2)},$$
$$\nabla_a^2 Y_a^{(\ell, \pm 3/2)} = \left[ \frac{5}{2} - (\ell + 1)^2 \right] Y_a^{(\ell, \pm 3/2)}, \quad \nabla_a Y_a^{(\ell, \pm 3/2)} = \pm i (\ell + 1) Y_a^{(\ell, \pm 3/2)},$$
$$\gamma^a Y_a^{(\ell, \pm 3/2)} = 0. \quad (87)$$

The spinor harmonics $Y^{(\ell, \pm 1/2)}$ form a complete set of spinors on $S^3$. A complete set of vector-spinors on $S^3$ consists of the transverse, $\gamma$-traceless vector-spinor modes $Y_a^{(\ell, \pm 3/2)}$, the longitudinal, $\gamma$-traceless modes $\nabla_{\{a\}} Y^{(\ell, \pm 1/2)}$ and $\gamma_a Y^{(\ell, \pm 1/2)}$. Note the zero modes

$$\nabla_{\{a\}} Y^{(1/2, \pm 1/2)} = 0. \quad (88)$$

The spinors $\psi_M$ and $\chi^r$ split into left- and right-handed doublets of the unbroken $SO(4)_R \subset SO(5)$, labelled by the eigenvalue of $\Gamma^5$ as follows

$$\Gamma^5 \psi_M^{(\alpha)} = \alpha \psi_M^{(k)}, \quad \Gamma^5 \chi_r^{(\alpha)} = \alpha \chi_r^{(k)}, \quad k = \pm 1. \quad (89)$$

Since there is no mixing between the left- and the right-handed $SO(4)_R$ spinors in the equations of motion or transformation rules, we will study the sectors with a fixed $\Gamma^5$ eigenvalue separately and suppress the label $\alpha$ on the spinors. The zero modes $Y^{(1/2, \alpha/2)}$ are identified with the $S^3$ component of the Killing spinor (16) in the sector with internal $SO(5)$ eigenvalue $\alpha$. 

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The linearized local supersymmetry transformations read

\[
\begin{align*}
\delta \psi_a &= (\nabla_a - i \frac{\alpha}{2} \gamma_a) \epsilon, \\
\delta \psi_\mu &= (\nabla_\mu + \frac{\alpha}{2} \gamma_\mu) \epsilon, \\
\delta \chi^r &= 0.
\end{align*}
\]

(90)

By a field dependent supersymmetry transformation generated by the harmonic non-zero modes and the zero mode \(Y^{(1/2, -\alpha/2)}\) (which contribute to \(\psi^{(\alpha)}_a\)), we can remove all the terms in the expansion for the internal gravitino proportional to \(\gamma_a\) except for the zero-mode \(\gamma_a Y^{(1/2, \alpha/2)}\). Thus we set to zero the following gauge modes:

\[
\ell \geq \frac{3}{2} : \quad \zeta_{(\ell, \pm 1/2)}^{(1/2, -\alpha/2)} = 0, \\
\zeta_{(1/2, -\alpha/2)}^{(1/2, -\alpha/2)} = 0.
\]

(91)

(92)

This shows the admissibility of the gauge condition

\[
\psi_a = \psi_{(a)} + \zeta^{(1/2, \alpha/2)} \gamma_a Y^{(1/2, \alpha/2)}, \quad \gamma_a \psi_{(a)} \equiv 0,
\]

(93)

where \(\psi_{(a)}\) has an expansion in terms of \(\gamma\)-traceless non-zero modes

\[
\psi_{(a)} = \sum \left[ \lambda^{(\ell, \pm 3/2)} Y_{(a)}^{(\ell, \pm 3/2)} + \zeta^{(\ell, \pm 1/2)} \nabla_{(a)} Y^{(\ell, \pm 1/2)} \right],
\]

(94)

and we have set

\[
\ell \geq \frac{3}{2} : \quad \zeta_{(\ell, \pm 1/2)}^{(1/2, \alpha/2)} \equiv \zeta^{(\ell, 1/2)}, \\
\zeta_{(1/2, \alpha/2)}^{(1/2, \alpha/2)} \equiv \zeta^{(\ell, \alpha/2)}.
\]

(95)

The gauge (93) completely fixes the local supersymmetries that are spontaneously broken by the compactification, that is the symmetries generated by the spinor harmonic non-zero modes \(Y^{(\ell, \pm 1/2)} \quad (\ell \geq \frac{3}{2})\) and the zero mode \(Y^{(1/2, -\alpha/2)}\). The unfixed zero modes \(Y^{(1/2, \alpha/2)}\) generate the unbroken local \(AdS_3\) supersymmetries acting on \(\psi_{(1/2, \alpha/2)}(x)\) as

\[
\delta \psi_{(1/2, \alpha/2)}(x) = (\nabla_\mu + \frac{\alpha}{2} \gamma_\mu) \epsilon^{(1/2, \alpha/2)}(x).
\]

(96)

The gauge fixing of these local supersymmetries involves fixing of residual gauge transformations that completely gauge away the massless gravitino from the spectrum. After the gravitino has been gauged away there is a remaining global supersymmetry that arranges the \(AdS_3\) spectrum into supermultiplets. The nature of the residual supersymmetry algebra and the supermultiplet structure of the spectrum will be explained in section 6.
The linearized gravitino and fermion equations read

\begin{align}
\gamma^\mu\nu\nabla_\nu\psi_\rho + i \gamma^\mu\nabla_\nu \gamma^a \psi_a + \alpha \gamma^\mu\nabla_\nu \gamma^a \psi_a + \gamma^\mu \nabla_\nu \gamma^a \psi_a &= 0 , \\
(i\nabla_x + \nabla_y)\psi^a + \gamma^a \left( \gamma^\mu\nabla_\mu \psi_\mu + \nabla_y(i \gamma^\mu \psi_\mu + \gamma^b \psi_b) - \nabla^b \psi_b - \gamma^a \gamma^b \psi_b - i\alpha \gamma^b \psi_b \right) - \nabla^a \left( i \gamma^b \psi_b + \gamma^b \psi_b \right) + i\alpha \psi^a &= 0 , \\
(\nabla_x + i \nabla_y - \alpha) \chi^r &= 0 ,
\end{align}

where we have used the decomposition of $\Gamma^M$ under $SO(1,5) \rightarrow SO(1,2) \times SO(3)$ given in Appendix A. Note that the last two terms in the gravitino equations only contribute for the zero-modes in (93). Inserting the harmonic expansions for $\psi_\mu$ and $\chi^r$ given in (86) and the expansion for $\psi_{\{a\}}$ given in (94) into the linearized equations (97) - (99) and using

\begin{align}
\nabla^a \nabla_{\{a\}} Y^{(\ell, \pm 1/2)} &= -\frac{2}{3} \left( \ell + 1 \right)^2 - \left( \frac{3}{2} \right)^2 Y^{(\ell, \pm 1/2)} , \\
\nabla_y \nabla_{\{a\}} Y^{(\ell, \pm 1/2)} &= \pm \frac{i(\ell+1)}{3} \nabla_a Y^{(\ell, \pm 1/2)} - \frac{4}{3} \left( \ell + 1 \right)^2 - \left( \frac{3}{2} \right)^2 \gamma_a Y^{(\ell, \pm 1/2)} ,
\end{align}

we find the following set of irreducible equations (see Table 2 for the $SO(4)$ content of the fermionic modes):

1. The $J = \frac{3}{2}$ Sector
\((\ell, \pm \frac{3}{2}), \ell \geq \frac{3}{2}\) and \((\frac{3}{2}, -\frac{3}{2})\): From (97) and (98) follows that \(\psi^{(\ell, \pm \frac{1}{2})}_{\mu}\) gives rise to a massive spin \(\frac{3}{2}\) field described by a transverse and \(\gamma\)-traceless vector-spinor

\[
\begin{align*}
\psi^{(\ell, \pm \frac{1}{2})}_{\mu} &= \psi^{(\ell, \pm \frac{1}{2})}_{\mu} - \frac{1}{3} \ell + 1 \pm \frac{3\alpha}{2} \nabla_{\{\mu\}} \zeta^{(\ell, \pm \frac{1}{2})}, \quad \ell \geq \frac{3}{2} \\
\psi^{(1/2, -\alpha/2)}_{\mu} &= \psi^{(1/2, -\alpha/2)}_{\mu}
\end{align*}
\]

satisfying

\[
\begin{align*}
[\nabla_{\pm (\ell + 1) - \alpha}] \psi^{(\ell, \pm \frac{1}{2})}_{\mu} &= 0, \quad \ell \geq \frac{3}{2}, \\
\left(\nabla_{\pm} - \frac{\alpha}{2}\right) \psi^{(1/2, -\alpha/2)}_{\mu} &= 0.
\end{align*}
\]

**The \(J = \frac{1}{2}\) Sector**

The diagonalized spin \(\frac{1}{2}\) eigenmodes are

\(i\) \((\ell, \pm \frac{3}{2}), \ell \geq \frac{3}{2}\): \(\lambda^{(\ell, \pm \frac{3}{2})}\) which from (98) can be seen to obey

\[
[\nabla_{\pm (\ell + 1) + \alpha}] \lambda^{(\ell, \pm \frac{3}{2})} = 0.
\]

\(ii\) \((\ell, \pm \frac{1}{2}), \ell \geq \frac{3}{2}\): \(\zeta^{(\ell, \pm \frac{1}{2})}\) which from (97) and (98) can be seen to obey

\[
[\nabla_{\pm (\ell + 1) - \alpha}] \zeta^{(\ell, \pm \frac{1}{2})} = 0.
\]

\(iii\) \((\ell, \pm \frac{1}{2}), \ell \geq \frac{1}{2}\): \(\chi^{r(\ell, \pm \frac{1}{2})}\) which from (99) can be seen to obey

\[
[\nabla_{\pm (\ell + 1) - \alpha}] \chi^{r(\ell, \pm \frac{1}{2})} = 0.
\]

**The \((\frac{3}{2}, \frac{3}{2})\) Sector**

This sector contains the zero modes \(\zeta^{(1/2, \alpha/2)}\) and the gravitino modes \(\psi^{(1/2, \alpha/2)}_{\mu}\) and from (97) and (98) follows that they satisfy

\[
\begin{align*}
\gamma^{\mu \rho} (\nabla_{\nu} + \frac{1}{2} \alpha \gamma_{\nu}) \psi^{(1/2, \alpha/2)}_{\rho} - 3i \gamma^{\mu \nu} \nabla_{\nu} \zeta^{(1/2, \alpha/2)} + 3i \alpha \gamma^{\mu} \zeta^{(1/2, \alpha/2)} &= 0, \\
\gamma^{\mu \nu} \nabla_{\mu} \psi^{(1/2, \alpha/2)}_{\nu} - \alpha \gamma^{\mu} \psi^{(1/2, \alpha/2)}_{\mu} - 2i \nabla_{\nu} \zeta^{(1/2, \alpha/2)} - i \alpha \zeta^{(1/2, \alpha/2)} &= 0
\end{align*}
\]

which are invariant under the local supersymmetry transformations (96). Combining (107) and (108) one finds that the zero modes obey

\[
(\nabla_{x} - \frac{\alpha}{2}) \zeta^{(1/2, \alpha/2)} = 0.
\]

The rest of the sector consists of pure gauge degrees of freedom.
5 Harmonic Expansion on AdS and the Full Spectrum

The matrix elements of the AdS group $SO(2, 2)$ can be labelled by using the maximal compact subgroup $O(2)_{I} \times O(2)_{II}$ as a basis. In this basis, a unitary irreducible representation of $SO(2, 2)$ is labelled by $(E_{0}, s_{0})$ where $E_{0}$ is the lowest eigenvalue of the energy operator $M_{03}$ that generates $O(2)_{I}$ and $s_{0}$ is the of $M_{12}$ (helicity) of the state with lowest energy.

The other symmetries of the $AdS_{3} \times S^{3}$ compactified theory are:

- the $SO(4)$ isometry group of the three sphere
- the $SO(4)_{R}$ subgroup of the original $SO(5)_{R}$-symmetry group in $D = 6$
- the $SO(n)$ symmetry group inherited from $D = 6$

Thus, we shall denote a given state in the spectrum by

$$ D^{(\ell_{1}, \ell_{2})} (E_{0}, s_{0}) (R \times S) \tag{110} $$

where $(\ell_{1}, \ell_{2})$ label the $S^{3}$ isometry group $SO(4)$; $(E_{0}, s_{0})$ label the representation of the AdS group $SO(2, 2)$; $R$ denotes the representation of $SO(4)_{R}$ and $S$ denotes a representation of $SO(n)$.

The internal $SO(4)$ is isomorphic to $SU(2)_{L} \times SU(2)_{R}$ with isospins $(j, \bar{j})$ related to the $SO(4)$ highest weight labels $(\ell_{1}, \ell_{2})$ as

$$ j = \frac{1}{2}(\ell_{1} + \ell_{2}) , \quad \bar{j} = \frac{1}{2}(\ell_{1} - \ell_{2}) . \tag{111} $$

Similarly $SO(2, 2) \approx SU(1, 1)_{L} \times SU(1, 1)_{R}$ with conformal weights $(h, \bar{h})$ given by

$$ h = \frac{1}{2}(E_{0} + s_{0}) , \quad \bar{h} = \frac{1}{2}(E_{0} - s_{0}) . \tag{112} $$

To determine the $SO(2, 2)$ content of the spectrum, we shall follow the technique used in [19, 20] which is based on the analytic continuation of $AdS_{3}$ to a three-sphere $S^{3}_{E}$, and consequently the group $SO(2, 2)$ to $SO(4)_{E}$. The Casimir eigenvalues for an $SO(2, 2)$ representation $D(E_{0}, s_{0})$ and an $SO(4)_{E}$ representation with highest weight $(n_{1}, n_{2})$ are

$$ SO(2, 2) : \quad C_{2} = E_{0}(E_{0} - 2) + s_{0}^{2} $$

$$ SO(4)_{E} : \quad C_{2} = n_{1}(n_{1} + 2) + n_{2}^{2} . \tag{113} $$

This suggests that in continuing from $S^{3}_{E}$ back to $AdS_{3}$ one makes the identifications

$$ n_{1} = -E_{0} , \quad n_{2} = s_{0} , \tag{114} $$

where, without loss of generality, we have chosen the plus sign in the last relation. There are a number of subtleties involved in the analytical continuation of $AdS_{3}$ to $S^{3}$. One particular
consequence is that one performs the change of coordinates \((x^1, x^2) \rightarrow (ix^1, ix^2)\), and therefore the metric on \(S_E^3\) has signature \((-\ldots-\ldots-\ldots-)\) and the replacement

\[

g_{\mu\nu} \rightarrow -\bar{g}_{\mu\nu}^E, \quad \nabla^2_{x} \rightarrow -\nabla^2_{E}
\]  

(115)

needs to be made. A number of other issues involved here are direct analogs of those discussed in detail in [19, 20] for the case of \(AdS_4\) continued to \(S^4\). We shall not repeat those discussions here but we shall outline the salient features of the harmonic expansions on \(AdS_3\) which lead to the determination of the complete spectrum.

We need to expand scalars, vectors and symmetric traceless tensors on \(AdS_3\) in terms of suitable harmonic functions. Upon analytical continuation to \(S_E^3\) these expansions become exactly like those listed in (27) where the the AdS vector indices \((\mu, \nu, \ldots)\) are to be treated like the \(S^3\) vector indices \((a, b, \ldots)\). In other words, one makes the replacements

\[
Y_{(s)}^{(\ell_1, \ell_2)} \rightarrow Y_{(s_0)}^{(n_1, n_2)} , \quad n_1 \geq |n_2| .
\]  

(116)

Formulae such as (31) and (87) can of course be used with suitable renaming of the labels. Indeed, the fact that the harmonic expansions are being performed in \(S_E^3 \times S^3\) simplifies the calculations considerably since the same formula can be used for both of the three spheres.

Next, we substitute the \(AdS_3 \rightarrow S_E^3\) harmonic expansions in all the diagonalized field equations listed in sections 3.3 and 4.2. Let us label the \(SO(4)_E\) representations involved by \((nn_2)\). In fact, only the \(0, \pm 1, \pm 2\) values of \(n_2\) occur in the harmonic expansions. We then determine the critical values of \(n\) for which the linearized wave operators vanish. Continuing back to \(AdS_3\), we make the identification \(n = -E_0\), and interpret the value of \(E_0 \geq 1\) as the lowest weight of the \(AdS_3\) representation described by the field. The significance of the value \(E_0 = 1\) is due to the fact that it is the fixed point of the interchange \(E_0 \rightarrow 2 - E_0\) which leaves the wave equations invariant. Therefore the solutions of the field equations for \(E_0 < 1\) can be obtained from those with \(E_0 > 1\) by this interchange and the two solutions together characterize the representation that we label with \((E_0, s_0)\). For first order equations, such as Dirac equations and the Hodge duality conditions, there is an additional flip of helicity \(s_0 \rightarrow -s_0\) to be taken into account as well. This procedure will become transparent as we present its application to various sectors.

- **The \(J = 2\) Sector**

  From (50) and (69), using the formula (30) and the replacement (116), we obtain

\[
\left[ n(n+2) + 1 - (\ell + 1)^2 \right] S^{(\ell_0)(n, \pm 2)} = 0 , \quad n \geq 2 , \quad \ell \geq 1 ,
\]  

(117)

where \(S^{(\ell_0)(n, \pm 2)}\) is the coefficient of the \(S_E^3\) harmonic function \(Y^{(n, \pm 2)}(xE)\). The critical values of \(n\) are \(n_{\pm} = -1 \pm (\ell + 1)\). Continuing back to \(AdS_3\) by letting \(n \rightarrow -E_0\), we find that \(E_0 = \ell + 2\) is the lowest energy of the \(AdS_3\) representation \(D(E_0 = \ell + 2, s_0 = \pm 2)\). Using the notation (110), we then fully characterize the \(J = 2\) tower of states as

\[
D^{(\ell+1, 0)}(\ell + 3, \pm 2) (0, 0) , \quad \ell \geq 0 .
\]  

(118)

Note that we have shifted \(\ell\) by one so that its minimum value is zero. Subsequently we shall do so when necessary, so that \(\ell\) will serve as the level number. It is also worth noting
that the lowest member of the spin 2 tower exhibited in (118) is a massive vector of $SO(4)$ internal symmetry group. The massless graviton does not arise in the physical spectrum, as expected.

- **The $J = \frac{3}{2}$ Sector**
  Starting from (87) and making the replacement (116) we obtain
  \[
  \nabla_E Y^{(n_1, \pm 3/2)}_\nu(x_E) = \pm i (n_1 + 1) Y^{(n_1, \pm 3/2)}_\mu(x_E), \quad n_1 \geq 3/2.
  \]
  Continuing the gravitino equations (101) from $AdS_3$ to $S^3_E$ using technique spelled out above, and the replacement
  \[
  \frac{\gamma^\mu_E}{i} \rightarrow i \gamma^\mu,
  \]
  we find critical values of the energy and the helicity implying that the vector-spinor $\varphi_\mu$ contains the representations
  \[
  D^{(\ell + 3/2, \pm 1/2)}(\ell + \frac{5}{2}, \pm \frac{3}{2})(0, 0), \quad \ell \geq 0,
  \]
  \[
  D^{(\ell + 1/2, \pm 1/2)}(\ell + \frac{7}{2}, \pm \frac{3}{2})(0, 0), \quad \ell \geq 0.
  \]
  The correlation between the $AdS_3$ helicity $s_0$ and the $SO(4)$ helicity $l_2$ is a consequence of choosing the positive energy root.

- **The $J = 1$ Sector**
  Starting from the last equation in (31) and making the replacement (116) we obtain
  \[
  e^{E\nu}_\mu \partial_\nu Y^{(n_1, \pm 1)}_\rho(x_E) = \pm (n_1 + 1) Y^{(n_1, \pm 1)}_\nu(x_E), \quad n_1 \geq 1.
  \]
  Continuing back to $AdS_3$ and using the replacement
  \[
  e^{E\nu}_\mu \rightarrow -e^{\nu\rho}_\mu,
  \]
  we find that the resulting mode functions $Y^{(E_0, s_0)}_\mu(x)$ with energy $E_0$ and helicity $s_0$ obey the relation
  \[
  \epsilon^{\mu\nu\rho} \partial_\nu Y^{(E_0, \pm 1)}_\rho = \mp (E_0 - 1) Y^{(E_0, \pm 1)}_\mu, \quad E_0 \geq 1.
  \]
  From the equations (53), (54) (55) for $K^{(\ell, \pm 1)}_\mu$ and $Z^{(5, \ell, \pm 1)}_\mu$ ($\ell \geq 2$) and from the equations (82), (83) and (84) for $K^{(1, \pm 1)}_\mu$ and $Z^{(5, 1, \pm 1)}_\mu$, and using (124) and the last equation in (31), we find the critical $E_0$ values which correspond to the representations
  \[
  D^{(\ell + 2, \pm 1)}(\ell + 2, \pm 1)(0, 0), \quad \ell \geq 0,
  \]
  \[
  D^{(\ell + 1, \pm 1)}(\ell + 3, \mp 1)(0, 0), \quad \ell \geq 0,
  \]
  \[
  D^{(\ell + 1, \pm 1)}(\ell + 5, \pm 1)(0, 0), \quad \ell \geq 0.
  \]
For each label \((\ell, \pm 1)\) \((\ell \geq 2)\) the system thus yields three degrees of freedom. This can be understood as follows. A single vector field obeying a second order Proca equation together with a Lorentz gauge condition gives rise to two degrees of freedom; one with helicity \(s_0 = 1\) and one with helicity \(s_0 = -1\). On the other hand, a single vector \(V_\mu\) obeying the first order equation \(e^{\mu \nu \rho} \partial_\nu V_\rho + M V_\mu = 0\), where \(M\) is a mass parameter, gives rise to one degree of freedom with energy \(E_0 = 1 + |M|\) and helicity \(s_0 = M/|M|\).

In our case there is an additional contribution to the the second order equation (53) in the form of a first order derivative term, which splits the energy levels of the two critical solutions. The direction of the energy shift is determined by the sign of the internal \(SO(4)\) helicity label \(\ell_2\). Also the sign of the mass parameter \(M\) of the first order equation introduced in the previous paragraph is determined by the sign of \(\ell_2\), as can be seen from (54). As a result, in all the three towers of representations in (125), (126) and (127), the helicity \(s_0\) of a positive energy solution is correlated with the \(SO(4)\) helicity label \(\ell_2\).

For \((1, \pm 1)\), \(K_\mu^{(1, \pm 1)}\) and \(Z_\mu^{c(1, \pm 1)}\) gives rise to four critical solutions; \((E_0 = 1, s_0 = \ell_2)\), \((E_0 = 1, s_0 = -\ell_2)\) \((E_0 = 3, s_0 = -\ell_2)\) and \((E_0 = 5, s_0 = \ell_2)\). The equation (124) implies that the \(E_0 = 1\) modes are total derivatives, which can be gauged away using the residual gauge parameter that satisfy (85). As a result the tower (125) starts with a \((2, \pm)\) representation of \(SO(4)\), and the two physical Yang-Mills states in the adjoint representations \((1, \pm 1)\) of \(SU(2)_{L,R}\) sit at the bottom of the two towers (126) and (127).\(^1\)

From the first order equation (56) we find that the representation content of \(Z_\mu^{c(\ell, \pm 1)}\) is

\[
D^{(\ell + 1, \pm 1)} (\ell + 3, \pm 1) (4, 0) , \quad \ell \geq 0 .
\]

Finally, the representation content of \(Z_\mu^{r(\ell, \pm 1)}\) is found from the first order equation (57) to be

\[
D^{(\ell + 1, \pm 1)} (\ell + 3, \mp 1) (0, n) , \quad \ell \geq 0 .
\]

**The \(J = \frac{1}{2}\) Sector**

From (104) and the relation

\[
\nabla_\mu Y^{(E_0, \pm 1/2)}(x_E) = \mp (E_0 - 1) Y^{(E_0, \pm 1/2)}(x_E) ,
\]

we find that the spinor \(\lambda\) contains the representations

\[
D^{(\ell + 3/2, \pm 3/2)} (\ell + \frac{9}{2}, \pm \frac{1}{2}) (2, 0) , \quad \ell \geq 0 ,
\]

\[
D^{(\ell + 3/2, \pm 3/2)} (\ell + \frac{5}{2}, \pm \frac{1}{2}) (2, 0) , \quad \ell \geq 0 .
\]

\(^1\)In 3D, the Yang-Mills system \(\nabla_\mu F^{\mu \nu} + M e^{\mu \rho \sigma} F_{\nu \rho \sigma} = 0\), where \(F^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) and \(M \neq 0\) is a topological mass parameter, has three critical solutions; \((E_0, s_0) = (1, \pm 1)\) and \((E_0, s_0) = (1 + 2|M|, -M/|M|)\). The \(E_0 = 1\) modes are pure gauge and the system thus has one physical state with topological mass. The compactified theory happens to fix the value of \(M\) in terms of the cosmological constant.
\[ D^{(\ell+1/2, \pm 1/2)}(\ell + \frac{7}{2}, \mp \frac{1}{2})(2_{\pm}, 0), \quad \ell \geq 0, \]
\[ D^{(\ell+3/2, \pm 1/2)}(\ell + \frac{5}{2}, \mp \frac{1}{2})(2_{\mp}, 0), \quad \ell \geq 0. \]  

(132)

From (106) we find that the spinor \( \chi^r \) contains the representations

\[ D^{(\ell+1/2, \pm 1/2)}(\ell + \frac{7}{2}, \mp \frac{1}{2})(2_{\mp}, n), \quad \ell \geq 0, \]  
\[ D^{(\ell+3/2, \pm 1/2)}(\ell + \frac{5}{2}, \mp \frac{1}{2})(2_{\pm}, n), \quad \ell \geq 0. \]  

(133)

Note that for \( \ell = 0 \) (134) actually has a solution with positive energy \( E_0 = \frac{1}{2} \), but as explained earlier this solution only characterize the field configurations obtained by the interchange \( E_0 \rightarrow 2 - E_0, s_0 \rightarrow -s_0 \), and do not represent new states in the spectrum.

• The \( J = 0 \) Sector

From (59), (71) and (73) we find that \( N^{(\ell)}_-(\ell \geq 1) \) and \( N^{(00)} \) together form the representation tower

\[ D^{(\ell, 0)}(\ell + 4, 0)(0, 0), \quad \ell \geq 0 \]  

(135)

Similarly, from (59) it follows that the field \( N^{(\ell)}_+(\ell \geq 2) \) describes the representations

\[ D^{(\ell+2, 0)}(\ell + 2, 0)(0, 0), \quad \ell \geq 0. \]  

(136)

It follows from (61) that \( L^{(\ell, \pm 2)} \) has the representation content

\[ D^{(\ell+2, \pm 2)}(\ell + 4, 0)(0, 0), \quad \ell \geq 0. \]  

(137)

Next, from (62) we find the representation content of the field \( U^{(\ell)} \) \( (\ell \geq 1) \) to be

\[ D^{(\ell+1, 0)}(\ell + 3, 0)(4, 0), \quad \ell \geq 0. \]  

(138)

From (64) and (81) follows the two towers of representations obtained from \( U^{(\ell)} \) \( (\ell \geq 1) \) and \( \phi^{5r(\ell)} \) \( (\ell \geq 0) \):

\[ D^{(\ell, 0)}(\ell + 4, 0)(0, n), \quad \ell \geq 0, \]  

(139)

\[ D^{(\ell+1, 0)}(\ell + 1, 0)(0, n), \quad \ell \geq 0. \]  

(140)

Finally, from (66) we find that \( \phi^{4r(\ell)} \) has the representation content

\[ D^{(\ell, 0)}(\ell + 2, 0)(4, n), \quad \ell \geq 0 \]  

(141)
6 The Supermultiplet Structure of the Spectrum

Applying harmonic analysis to the Killing equation (16) we find that the Killing spinors, associated with the supercharges \( Q_+ \) and \( \overline{Q}_- \), contain the representations

\[
Q_+ : \quad D^{(1/2, 1/2)}(-\tfrac12, \tfrac12)(2_+, 0), \quad \overline{Q}_- : \quad D^{(1/2, -1/2)}(-\tfrac12, \tfrac12)(2_-, 0).
\]

These supercharges obey

\[
\begin{align*}
[E, Q_+] &= -\tfrac12 Q_+ , & [E, \overline{Q}_-] &= -\tfrac12 \overline{Q}_- , \\
[J, Q_+] &= -\tfrac12 Q_+ , & [J, \overline{Q}_-] &= \tfrac12 \overline{Q}_-, \quad (143) \\
\Gamma^5 Q_+ &= Q_+ , & \overline{Q}_- \Gamma^5 &= -\overline{Q}_- ,
\end{align*}
\]

where \( E \) and \( J \) are the hermitian energy and the helicity operators of \( SO(2,2) \). Taking the hermitian conjugates of these equations we find that the supercharges \( \overline{Q}_+ \) and \( Q_- \) carry the representations

\[
\begin{align*}
\overline{Q}_+ : \quad D^{(1/2, 1/2)}(\tfrac12, \tfrac12)(2_+, 0) , \\
Q_- : \quad D^{(1/2, -1/2)}(\tfrac12, -\tfrac12)(2_-, 0) , \quad (144)
\end{align*}
\]

and obey the commutation rules

\[
\begin{align*}
[E, \overline{Q}_+] &= \tfrac12 \overline{Q}_+ , & [E, Q_-] &= \tfrac12 Q_- , \\
[J, \overline{Q}_+] &= \tfrac12 \overline{Q}_+ , & [J, Q_-] &= -\tfrac12 Q_- , \quad (145) \\
\overline{Q}_+ \Gamma^5 &= Q_+ , & \Gamma^5 Q_- &= -\overline{Q}_- ,
\end{align*}
\]

The full algebra satisfied by the supercharges \( Q_\pm \) and \( \overline{Q}_\pm \) is

\[
SU(1,1|2)_L \oplus SU(1,1|2)_R , \quad (146)
\]

where the bosonic subalgebras are \( (SU(1,1) \oplus SU(2))_{LR} \) (see the Appendix where the algebra is given). The \( SU(2)_L \times SU(2)_R \) is the isometry group of the internal \( S^3 \). Representing the superalgebra in this way requires that the supercharges \( Q_L \) are Dirac and that they carry the doublet indices of \( SU(1,1)_L \times SU(2)_L \). Hence there are four complex supercharges in the left-handed sector. One can go over to a real basis by introducing a doublet index which transforms under a global \( SU(2)_+ \) group that commutes with the superalgebra. A similar step in the right-handed sector yields an \( SU(2)_- \). Together \( SU(2)_+ \times SU(2)_- \) can be interpreted as a the
global $SO(4)_R$ inherited from the $R$ symmetry group $SO(5)$ in $D = 6$. We emphasize that the generators of $SO(4)_R$ do not arise on the right hand side of the superalgebra.

To find the supermultiplets structure of the spectrum we begin by acting with the *helicity lowering* supercharges $Q_\pm$ on the tower of highest helicity representations in the physical spectrum, that is the representations

$$D^{(\ell+1,0)}(\ell + 3, 2)(0,0)$$

found in (118)$^2$. Using the $SO(4)$ tensor product rule

$$(\ell_1, \ell_2) \otimes (\frac{1}{2}, \pm \frac{1}{2}) = (\ell_1 + \frac{1}{2}, \ell_2 \pm \frac{1}{2}) \oplus (\ell_1 - \frac{1}{2}, \ell_2 \mp \frac{1}{2}) \text{,}$$

and discarding representations not in the fully antisymmetrized product, we can combine the representations (147), (121), (125), (127), (128), (131) and (137) into the tower of spin 2 supermultiplets given in Table 3 and the supersymmetry transformation rules are shown in Figure 1. This tower contains all $|s_0| > 1$ representations and all the $s_0 = 1$ representations except the $SO(4) \times SO(n)$ singlets in (126), that is the representations

$$D^{(\ell+1,-1)}(\ell + 3, 1)(0,0) \text{,}$$

and the $SO(4) \times SO(n) (0, n)$-plets in (129), that is the representations

$$D^{(\ell, \pm 1)}(\ell + 3, \mp 1)(0, n) \text{.}$$

Acting with the helicity lowering supercharges $Q_\pm$ on (149) we find that (150) together with the representations (132), (136), (138) and (135) fit into the two tower of supermultiplets given in Table 4, with supersymmetry transformation rules in Figure 2. Similarly (150), (134), (140), (141) and (139) constitute the tower 5, with supersymmetry transformation rules in Figure 3.

Note that by repeated action of $Q_\pm$ starting from the helicity +2 graviton state one can only reach the helicities $s_0 \geq 0$. The reason for this is that the tower of “massive” spin 2 multiplets originates from the shortened, massless $D = 6$ gauge multiplets of the infinitely many gauge symmetries that are spontaneously broken by the $AdS_3 \times S^3$ vacuum [2]. Technically speaking, the momenta of the internal $S^3$ that are picked up by the $SO(4)$ generators in the right hand side of the superalgebra implies that the set of fermionic creation operators formed out of the supercharges $Q_\pm$ can be applied at most twice to physical states, as can be seen from Figures 1, 2 and 3.

Thus, the two spin 1 towers consist of self-conjugate spin 1 supermultiplets. In the case of the spin 2 tower, the full multiplet structure is obtained by adding the conjugate tower of multiplets with the replacements $\ell_2 \rightarrow -\ell_2, s_0 \rightarrow -s_0, 2_\pm \rightarrow 2_\mp$ made. This conjugate tower of supermultiplets can be obtained by repeated action of $\overline{Q}_\pm$ starting from the helicity $-2$ graviton state.

The spin 2 tower of physical states shown in Figure 1 are labelled by a level number $\ell$ which starts from zero. If one extrapolates to $\ell = -1$, one finds the nonpropagating supergravity

$^2$Alternatively, the same supermultiplet structure can be found by first identifying the ground state of minimal energy $E_0$ and then act on the ground state with the *energy raising* supercharges $\overline{Q}_+$ and $Q_-$ given in (144).
multplet in the upper left diagonal and its conjugate image consisting of a graviton, gravitini in the \((2_L, 2_+)^{(4)}\) and \((2_R, 2_-)^{(4)}\) representations of \(SO(4) \times SO(4)_R\) and \(SO(4)\) vector fields.

For \(\ell = 0\), the states on the lower right diagonal are absent because of the group theoretical restriction \(\ell_1 \geq |\ell_2|\), on the \(SO(4)\) representation labels \((\ell_1, \ell_2)\). Thus one is left with a spin 2 multiplet with 48 Bose and 48 Fermi degrees of freedom and lowest spin \(\frac{1}{2}\).

For \(\ell \geq 1\) the supermultiplet structure is generic, and has a total of \(16(\ell + 1)(\ell + 3)\) Bose and \(16(\ell + 1)(\ell + 3)\) Fermi degrees of freedom. Note however, that the \(SO(4) \approx SU(2)_L \times SU(2)_R\) Yang-Mills fields originating from the isometries of the internal \(S^3\) reside at level \(\ell = 1\) multiplet with 128 Bose and 128 Fermi degrees of freedom. Furthermore, the helicity +1 Yang-Mills states are in the adjoint representation of \(SU(2)_L\) while the helicity –1 Yang-Mills states carry the adjoint representation of \(SU(2)_R\). The complementary helicity states are found in the spin 1 tower of \(SO(n)\) singlets.

The \(D = 6\) origin of various members of the spin 2 tower is as follows. The spin 2 and \(\frac{3}{2}\) members, of course, come from the metric \(g_{\mu \nu}\) and the gravitino \(\psi_\mu\). The singlet spin 1 states involve a mixing of the Kaluza-Klein vectors \(g_\mu a\) and the vectors \(B_{\mu a}^5\) originating from the self-dual tensors. The spin 1 states in the 4 of \(SO(4)_R\) come from the vectors \(B_{\mu a}^5\). The spin \(\frac{1}{2}\) states come from the internal traceless and transverse components of the internal gravitino field \(\psi_a\). Finally, the scalars originate from the internal metric \(g_{ab}\).

The spin 1 tower of \(SO(n)\) singlet states in figure 2 starts at level \(\ell = 0\). The level 0 multiplet which has 32 Bose and 32 Fermi degrees of freedom is special since it contains the the Yang-Mills states in the adjoint of \(SU(2)_R\) with helicity +1 and in the adjoint of \(SU(2)_L\) with helicity –1 that are complementary to those in the spin 2 tower.

At level \(l \geq 1\) the supermultiplet is generic and it has a total of \(8(\ell + 2)^2\) Bose and \(8(\ell + 2)^2\) Fermi degrees of freedom.

The \(D = 6\) origin of the states is as follows. The spin 1 states come from the mixture between the Kaluza-Klein vector \(g_{\mu a}\) and the selfdual tensor component \(B_{\mu a}^5\). The spin \(\frac{1}{2}\) states originate from the longitudinal part of the internal graviton \(\psi_a\). The scalar \(SO(4)_R\) 4-plet come from the internal components \(B_{\mu a}^5\) of the selfdual tensor and the two scalar \(SO(4)_R\) singlets come from the mixture between \(g_{\mu a}^\rho, g_{\mu a}^\rho\) and \(B_{\mu a}^5\).

The spin \(\frac{1}{2}\) multiplet residing at the left diamond for \(\ell = -1\) does not occur in the spectrum; the scalar \(SO(4)_R\) singlet in the \((10)\) of \(SO(4)\) are the scalar states gauged away by the residual Stueckelberg shift transformations; the scalar \(SO(4)_R\) 4-plet in the singlet representation of \(SO(4)\) never occured in the harmonic expansion of \(B_{\mu a}^5\) (see (27)); and finally the \(SO(4)_R\) doublet of fermions in the \((\frac{1}{2}, \pm \frac{1}{2})\) of \(SO(4)\) is put equal to zero in the gauge (93) for the local supersymmetry.

The spin 1 tower of \(SO(n)\) singlet states in figure 2 is a replica of the singlet spin 1 tower except that this tower starts at level \(\ell = -1\). At this level the spin \(\frac{1}{2}\) multiplet residing in the left diamond survives. This multiplet contains 8 Bose and 8 Fermi degrees of freedom.

The \(D = 6\) origin of the states is as follows. The spin 1 states come from the anti-selfdual tensor \(B_{\mu a}^r\). The spin \(\frac{1}{2}\) states come from the \(SO(n)\) n-plet \(\chi^r\) of chiral fermions. The scalars in the \((4, n)\) of \(SO(4)_R \times SO(n)\) come from the coset scalars \(\phi^a\). The remaining scalars come from the mixture between \(\phi^5\) and the internal components \(B_{\mu a}^5\) of the anti-selfdual tensor.
7 Discussion

The complete spectrum of $AdS_3 \times S^3$ compactified $D = 6$, $N = 4b$ supergravity which we have determined in this paper shares many similarities to other known AdS compactified supergravity theories in the literature. A new feature is that the spectrum contains more than one tower of supermultiplets. This is not too surprising since the underlying supersymmetry contains only 16 real supercharges, which is half of the maximal number of supersymmetries that arise in other AdS compactified supergravity theories studied in the past.

As expected, the massless supergravity sector of the compactified theory is non-propagating. Other gauge modes one normally expects in $AdS$ compactifications of supergravity theories are the singletons/doubletons. For example, in $AdS_4 \times S^7$ compactification of $D = 11$ supergravity singletons arise in the $\ell = -1$ level of the massive spin 2 tower [19, 20]. In our case, the supergroup $SU(1,1|2)_L \times SU(1,1|2)_R$ does not have singletons [22]. However, the massless representations of the group can be interpreted as doubletons. The self-conjugate such multiplet can be found in [23]. The representation with lowest energy in this multiplet is $D^{(0,0)}(1,0)(0,0)$.

The Yang-Mills sector of the theory is rather interesting. They gauge the $SO(4) \approx SU(2)_L \times SU(2)_R$ isometry group of the full $SO(4) \times SO(4)_R$ R-symmetry group of the superalgebra (see discussion on p. 25). The equations of motion (82) and (83) prior to gauge fixing can be written as

\[
\nabla F^{\mu\nu} + 2\epsilon^{\mu\nu\rho} G_{\nu\rho} = 0 , \tag{151}
\]

\[
e^{\mu\nu\rho} G_{\nu\rho} + 4(A^\mu + B^\mu) = 0 , \tag{152}
\]

where we have used the field strengths

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu , \tag{153}
\]

where we have set $A_\mu \equiv K_\mu^{(1,1)}$, $B_\mu \equiv 2Z_\mu^{5(1,1)}$ and we have suppressed the adjoint $SU(2)_L$ indices. The gauge transformations (38) leaving (151) and (152) invariant are

\[
\delta A_\mu = \partial_\mu \Lambda , \quad \delta B_\mu = -\partial_\mu \Lambda . \tag{154}
\]

The equations (151) and (152) can be derived from the Lagrangian

\[
e^{-1}L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \epsilon^{\mu\nu\rho} G_{\mu\nu} B_\rho - 4(A_\mu + B_\mu)(A^\mu + B^\mu) . \tag{155}
\]

The non-abelian version is obtained using the field strengths $F = dA + A \wedge A$ and $G = dB - B \wedge B$ (the sign difference leads to that the mass term remains invariant under the non-abelian gauge transformations). The essential property of this action is that a single gauge symmetry is associated with two gauge fields (note that if one uses the gauge symmetry (154) to impose the Lorentz gauge condition on $A_\mu$ or $B_\mu$, say $\nabla_\mu A_\mu = 0$, then the condition $\nabla_\mu B_\mu = 0$ follows from the first order equation (152)). Consequently this system describes only two degrees of freedom.
on-shell (one from each vector), as shown in detail in the treatment of the $J = 1$ sector section 5.

Note that one can take the curl of the first order equation (152) and diagonalize the resulting system of coupled second order equations by going over to the field strengths $F_1 = F + 2G$ and $F_2 = \frac{1}{3}(F - G)$. The resulting equations can be derived from an action which is the sum of two actions each one containing a Chern-Simons term and a kinetic term (causing propagation in the bulk):

$$e^{-1}L' = -\frac{1}{4}F_{1\mu\nu}F_{1}^{\mu\nu} + \epsilon^{\mu\nu\rho}F_{1\mu\nu}A_{1\rho} - \frac{1}{4}F_{2\mu\nu}F_{2}^{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu\rho}F_{2\mu\nu}A_{2\rho}.$$  

(156)

This action yields the same on-shell content as (155) since the action (156) has two independent gauge symmetries (see footnote on page 23)

$$\delta A_{1\mu} = \partial_{\mu}A_{1}, \quad \delta A_{2\mu} = \partial_{\mu}A_{2}.$$  

(157)

This would however imply the enlargement of the gauge group from $SU(2)_L$ to $SO(4)_L$ which the non-linear theory does not possess. Thus, while the actions (155) and (156) describe the same physical degrees of freedom, the theory chooses the Lagrangian $L$ given in (155).

Finally, we turn to the issue of the expected duality between the $AdS_3$ supergravity theories and two-dimensional conformal field theories [12, 13, 14, 15]. In [12] the authors find the states of the conformal field theory corresponding to the representations (140) and (139) originating from the coset scalars:

$$D^{\ell+2,0}(\ell + 2, 0) \quad (\ell \geq -1)(0, 21) \quad \text{and} \quad D^{\ell,0}(\ell + 4, 0)(0, 21) \quad (\ell \geq 0)$$  

(158)

where we used labelling of Figure 3. The first representation turns out to be a chiral primary with the left and right moving weights $h = \bar{h} = (\ell + 2)/2 (\ell \geq -1)$ and the second representation a descendant of a chiral primary with $h = \bar{h} = (\ell + 4)/2 (\ell \geq 0)$; they arise in Type IIB string on $AdS_3 \times M_4$ where $M_4$ is a k-fold product of $K_3$ orbifolded with the permutation group $S_k$ [12]. The resulting $N = (4, 4)$ SCFT has level $k$ and the unitarity bound leads to a cutoff in the CFT spectrum at $\ell = \frac{1}{2}k$ not seen in the perturbation theory around $AdS_3$. The authors of [12] relates this to a stringy exclusion principle for $AdS_3$ black holes. We expect this map to account for all the sectors of the $AdS_3 \times S^3$ spectrum presented here.

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Appendix

Starting from the conventions of [16], in which the signature of space-time is mostly negative, and making the replacements

\[
\begin{align*}
g_{MN} &\rightarrow -g_{MN}, & \psi_M &\rightarrow \psi_M, & \epsilon_{M_1...M_6} &\rightarrow -\epsilon_{M_1...M_6}, \\
\Gamma^M &\rightarrow i\Gamma^M, & \chi &\rightarrow i\chi, \\
\Gamma^7 &\rightarrow \Gamma^7, & \bar{\theta} &\rightarrow i\bar{\theta}\Gamma^7,
\end{align*}
\]

we end up with a space-time with signature \((-++++)\) and the spinor conventions

\[
\begin{align*}
\left\{ \Gamma^M, \Gamma^N \right\} &= 2g^{MN}, & \bar{\theta} &= i(\theta)\Gamma^0, \\
\Gamma^{M_1...M_6} &= \epsilon_{M_1...M_6}\Gamma^7, & \theta_a &= i\Gamma^0(\theta_{\bar{a}})^*\Omega_{ba}, \\
\frac{1}{2}(1 + \Gamma^7)\psi_M &= \psi_M, & \frac{1}{2}(1 - \Gamma^7)\chi &= \chi,
\end{align*}
\]

where \(\Omega\) is the anti-symmetric charge conjugation matrix of \(SO(5)\), whose \(\Gamma^i\)-matrices are anti-symmetric. With this redefinition both \(\psi_M\) and \(\chi\) remain symplectic Majorana-Weyl (this spinor type is allowed in both signatures) and \(C\) remains symmetric and \(\Gamma^M\) remain anti-symmetric.

As a check one can verify the supertransformations (11) have the correct hermicity properties (without any factors of \(i\)). Both \(SO(1,5)\) and \(SO(5)\) spinor indices are raised and lowered using north-east-south-west contraction. We split the \(\Gamma^M\)-matrices under \(SO(1,5) \rightarrow SO(2,1) \times SO(3)\) as follows

\[
\begin{align*}
\Gamma^\mu &= \gamma^\mu \times 1 \times \sigma_1, & C &= \epsilon \times \eta \times \sigma^1 \\
\Gamma^a &= 1 \times \gamma^a \times \sigma^2, & \Gamma^7 &= 1 \times 1 \times (-\sigma_3),
\end{align*}
\]

where \(\epsilon\) and \(\eta\) are the charge conjugation matrices. The resulting two-component \(SO(1,2)\) spinors are Majorana and the two-component \(SO(3)\) spinors are pseudo-symplectic Majorana.

In our conventions \(D_M\epsilon = (\partial_M + \frac{1}{4}\omega_M^{AB}C_{AB})\epsilon\) and \([\nabla_M, \nabla_N]\epsilon = \frac{1}{4}R_{MAB}\Gamma^{AB}\epsilon\).

The non-trivial part of the superalgebra \(SU(1,1|2)_L\) is

\[
\left\{ Q_{\alpha' \alpha''}, Q_{\beta' \beta''} \right\} = \left( \epsilon_{\alpha' \beta'} + \epsilon_{\alpha'' \beta''} T_{\alpha' \alpha''} \right) \epsilon_{\alpha' \beta'} \epsilon_{\alpha'' \beta''},
\]

where \(\alpha, \alpha', \alpha''\) label the chiral spinors of \(SO(2,2), SO(4)\) and \(SO(4)_R\) respectively, \(M_{\alpha\beta} = M_{\beta\alpha}\) are the \(SU(1,1)_L \subset SO(2,2)\) generators and \(T_{\alpha' \alpha''} = T_{\beta' \beta''}\) are the \(SU(2)_L \subset SO(4)\) generators. The superalgebra \(SU(1,1|2)_R\) has identical structure and commutes with \(SU(1,1|2)_L\).
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| Helicity $s_0$ | $SO(4)$ content $\ell \pm n \pm 1$ | $SO(4)_R \times SO(n)$ content $(\ell, 0)$ | Lowest energy $E_0 \ell + 2$ | Conformal weights $(h, \bar{h})$ $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | Isospins $(j, \bar{j})$ $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
|---|---|---|---|---|---|
| $\pm 2$ | $(\ell + 1, 0)$ | $(0, 0)$ | $\ell + 3$ | $(\ell + \frac{3 + 2}{2}, \ell + \frac{3 + 2}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 3/2$ | $(\ell + \frac{3}{2}, \pm \frac{1}{2})$ | $(2_{\pm}, 0)$ | $\ell + \frac{3}{2}$ | $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 3/2$ | $(\ell + \frac{3}{2}, \pm \frac{1}{2})$ | $(2_{\mp}, 0)$ | $\ell + \frac{3}{2}$ | $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 1$ | $(\ell + 2, \pm 1)$ | $(0, 0)$ | $\ell + 2$ | $(\ell + \frac{2 + 1}{2}, \ell + \frac{2 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 1$ | $(\ell + 1, \pm 1)$ | $(4, 0)$ | $\ell + 3$ | $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 1$ | $(\ell, \pm 1)$ | $(0, 0)$ | $\ell + 4$ | $(\ell + \frac{4 + 1}{2}, \ell + \frac{4 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 1/2$ | $(\ell + \frac{1}{2}, \pm \frac{3}{2})$ | $(2_{\mp}, 0)$ | $\ell + \frac{5}{2}$ | $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $\pm 1/2$ | $(\ell + \frac{1}{2}, \pm \frac{3}{2})$ | $(2_{\mp}, 0)$ | $\ell + \frac{5}{2}$ | $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |
| $0$ | $(\ell + 1, \pm 2)$ | $(0, 0)$ | $\ell + 3$ | $(\ell + \frac{3 + 1}{2}, \ell + \frac{3 + 1}{2})$ | $(\ell + \frac{1}{2}, \ell + \frac{1}{2})$ |

Table 3: The spin 2 tower of supermultiplets for $\ell \geq 0$. The $SO(4)$ content is characterized by the highest weight labels $(\ell_1, \ell_2)$. The $SO(4)_R \times SO(n)$ content is given by the dimensions of the irreducible representations. The internal $SO(4)$ is isomorphic to $SU(2)_L \times SU(2)_R$ with isospins $(j, \bar{j})$ related to the $SO(4)$ highest weight labels as $j = \frac{1}{2}(\ell_1 + \ell_2)$ and $\bar{j} = \frac{1}{2}(\ell_1 - \ell_2)$. Similarly $SO(2, 2) \approx SU(1, 1)_L \times SU(1, 1)_R$ with conformal weights $(h, \bar{h})$ given by $h = \frac{1}{2}(E_0 + s_0)$ and $\bar{h} = \frac{1}{2}(E_0 - s_0)$. At level $\ell$ the number of bosonic and fermionic degrees of freedom match and separately equal to $16(\ell + 1)(\ell + 3)$. For the supersymmetry transformations rules see Figure 1.
Table 4: The spin 1, $SO(n)$ singlet tower of supermultiplets for $\ell \geq 0$. At level $\ell$ the number of bosonic and fermionic degrees of freedom match and separately equal to $8(\ell + 2)^2$. For the supersymmetry transformation rules see Figure 2 and for further notation see the caption of Table 3.

| Helicity $s_0$ | $SO(4)$ Content | $SO(4)_R \times SO(n)$ Content | Lowest Energy $E_0$ | Conformal weights $(\bar{h}, \bar{h})$ | Isospins $(j, \bar{j})$ |
|----------------|-----------------|-------------------------------|-------------------|---------------------------------|------------------|
| $\pm 1$        | $(\ell + 1, \mp 1)$ | $(0, 0)$                     | $\ell + 3$       | $\left(\frac{\ell + 3 + 1}{2}, \frac{\ell + 3 + 1}{2}\right)$ | $\left(\frac{\ell + 1 + 1}{2}, \frac{\ell + 1 + 1}{2}\right)$ |
| $\pm 1/2$      | $(\ell + \frac{3}{2}, \mp \frac{1}{2})$ | $(2, 0)$                   | $\ell + \frac{3}{2}$ | $\left(\frac{2\ell + 5 + 1}{4}, \frac{2\ell + 5 + 1}{4}\right)$ | $\left(\frac{2\ell + 3 + 1}{4}, \frac{2\ell + 3 + 1}{4}\right)$ |
| $\pm 1/2$      | $(\ell + \frac{1}{2}, \mp \frac{1}{2})$ | $(2, 0)$                   | $\ell + \frac{1}{2}$ | $\left(\frac{2\ell + 7 + 1}{4}, \frac{2\ell + 7 + 1}{4}\right)$ | $\left(\frac{2\ell + 1 + 1}{4}, \frac{2\ell + 1 + 1}{4}\right)$ |
| $0$            | $(\ell + 2, 0)$  | $(0, 0)$                     | $\ell + 2$       | $\left(\frac{\ell + 2}{2}, \frac{\ell + 2}{2}\right)$ | $\left(\frac{\ell + 2}{2}, \frac{\ell + 2}{2}\right)$ |
| $0$            | $(\ell + 1, 0)$  | $(4, 0)$                     | $\ell + 3$       | $\left(\frac{\ell + 3}{2}, \frac{\ell + 3}{2}\right)$ | $\left(\frac{\ell + 1}{2}, \frac{\ell + 1}{2}\right)$ |
| $0$            | $(\ell, 0)$      | $(0, 0)$                     | $\ell + 4$       | $\left(\frac{\ell + 4}{2}, \frac{\ell + 4}{2}\right)$ | $\left(\frac{\ell}{2}, \frac{\ell}{2}\right)$ |

Table 5: The spin 1 tower of supermultiplets in the vector representation of $SO(n)$ for $\ell \geq -1$. At level $\ell$ the number of bosonic and fermionic degrees of freedom match and separately equal to $8n(\ell + 2)^2$. For the supermultiplet transformation rules see Figure 3 and for further notation see the caption of Table 3.
Figure 1: The spin 2 supermultiplet structure for $\ell \geq 0$. A given representation is denoted by $D^{(\ell_1, \ell_2)}(E_0, s_0)$ ($R \times S$) where $(\ell_1, \ell_2)$ label the $S^3$ isometry group $SO(4)$; $(E_0, s_0)$ label the representation of the AdS group $SO(2, 2)$; $R$ denotes the representation of $SO(4)_R$ and $S$ denotes a representation of $SO(n)$. The supercharge $Q^{(1/2, 1/2)}_+(-\frac{1}{2}, -\frac{1}{2})(2_+, 0)$ acts to the southwest and the supercharge $Q^{(-1/2, -1/2)}_-(\frac{1}{2}, \frac{1}{2})(2_-, 0)$ acts to the southeast. The full multiplet structure is obtained by adding the conjugate tower of multiplets in which the replacements $\ell_2 \to -\ell_2$, $s_0 \to -s_0$, $2_{\pm} \to 2_{\mp}$ are made. The multiplet at level $\ell$ thus contains $16(\ell+1)(\ell+3)$ Bose and that many Fermi states. At level $\ell = -1$ there are $SU(2)_L$ Yang-Mills states with $(E_0, s_0) = (1, 1)$ and $SU(2)_R$ Yang-Mills states with $(E_0, s_0) = (1, -1)$ which are pure gauge. At level $-1$ one also finds the non-propagating graviton and gravitini. At level $\ell = 0$, the states on the lower right diagonal are absent and the resulting special spin 2 multiplet consists of 48 Bose and 48 Fermi degrees of freedom. At level $\ell = 1$ there are physical $SU(2)_L$ Yang-Mills states with $(E_0, s_0) = (5, 1)$ and physical $SU(2)_R$ Yang-Mills states with $(E_0, s_0) = (5, -1)$. For $\ell \geq 1$ the structure of the multiplets is generic, and no other Yang-Mills states arise.
Figure 2: The spin 1, $SO(n)$ singlet supermultiplet structure for $\ell \geq 0$. The multiplet is self-conjugate and contains $8(\ell + 2)^2$ Bose and that many Fermi states at level $\ell$. At level $\ell = -1$ there is an unphysical spin $1/2$ multiplet residing at the left diamond. At level $\ell = 0$ there is triplet of $SU(2)_R$ Yang-Mills states with $(E_0, s_0) = (3, 1)$ and a triplet of $SU(2)_L$ Yang-Mills states with $(E_0, s_0) = (3, -1)$. For $\ell \geq 0$ the structure of the multiplets is generic and no other Yang-Mills states arise. See caption of Figure 1 for further notations.
Figure 3: The structure of the spin 1 supermultiplet in the vector representation of $SO(n)$ for $\ell \geq -1$. The multiplet is self-conjugate and contains $8n(\ell + 2)^2$ Bose and that many Fermi states at level $\ell$. For the special value $\ell = -1$ one finds a spin $\frac{1}{2}$ multiplet consisting of $8n$ Bose and $8n$ Fermi states residing at the left diamond. For $\ell \geq 0$ the structure of the multiplets is generic. These are matter spin 1 multiplets. See the caption of Figure 1 for further notations.