THE MONOIDAL CENTER AND THE CHARACTER ALGEBRA

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Abstract. For a pivotal finite tensor category \( C \) over an algebraically closed field \( k \), we define the adjoint object \( A \in C \), the algebra \( \text{CF}(C) = \text{Hom}_C(A, \mathbb{1}) \) of class functions, and the internal character \( \text{ch}(X) : A \rightarrow \mathbb{1} \) in \( C \) for an object \( X \in C \). Let \( \text{Gr}_k(C) \) be the Grothendieck algebra of \( C \) defined over \( k \). We show that the map \( \text{ch} : \text{Gr}_k(C) \rightarrow \text{CF}(C) \) given by \( [X] \mapsto \text{ch}(X) \) is a well-defined injective algebra map. We also show that the algebra \( \text{CF}(C) \) is isomorphic to the endomorphism algebra of \( R(\mathbb{1}) \), where \( R \) is a right adjoint of the forgetful functor from the monoidal center. As an application, we show that \( \text{Gr}_k(C) \) is a semisimple algebra if \( C \) is a non-degenerate spherical fusion category over \( k \).

1. Introduction

To begin with, let \( G \) be a finite group. For every finite-dimensional representation \( V \) of \( G \) over a field, say \( k \), the character \( \chi_V : kG \rightarrow k \) is defined. As is well-known, the character is a class function on \( G \). This means that \( \chi_V \) is a \( G \)-equivariant map if we view \( A = kG \) and \( \mathbb{1} = k \) as the adjoint representation and the trivial representation, respectively. Thus, in this way, we can deal with characters inside the representation category of \( G \). The aim of this paper is to develop such a method in the setting of finite tensor categories \([EO04]\), a class of tensor categories including the representation category of a finite-dimensional Hopf algebra.

For a finite tensor category \( C \) over an algebraically closed field \( k \), we denote by \( Z(C) \) the monoidal center of \( C \). It is known that the forgetful functor \( U : Z(C) \rightarrow C \) has a right adjoint \( R \). The adjoint object \([Shi15]\) is defined by \( A = UR(\mathbb{1}) \), where \( \mathbb{1} \in C \) is the unit object. If \( C \) has a pivotal structure, then we can define the internal character \( \text{ch}(X) : A \rightarrow \mathbb{1} \) for each \( X \in C \). We call \( \text{CF}(C) := \text{Hom}_C(A, \mathbb{1}) \) the space of class functions in \( C \) and endow it with a structure of an algebra. Let \( \text{Gr}(C) \) be the Grothendieck ring of \( C \). One of our main results is that the map

\[
(1.1) \quad \text{ch} : \text{Gr}_k(C) := k \otimes \mathbb{Z} \text{Gr}(C) \rightarrow \text{Hom}_C(A, \mathbb{1}), \quad [X] \mapsto \text{ch}(X)
\]

is a well-defined injective algebra map. Moreover, there is an isomorphism

\[
\text{Hom}_C(A, \mathbb{1}) \cong \mathcal{E} := \text{End}_{\mathcal{Z}(C)}(R(\mathbb{1}))
\]

of algebras. As a consequence, we can embed the algebra \( \text{Gr}_k(C) \) into \( \mathcal{E} \). Furthermore, if \( C \) is semisimple (i.e., a fusion category), then the embedding is surjective and thus we have an isomorphism \( \text{Gr}_k(C) \cong \mathcal{E} \) of algebras. As an application of this result, we show that the Grothendieck algebra \( \text{Gr}_k(C) \) of a spherical fusion category \( C \) is semisimple if \( C \) is non-degenerate in the sense of \([ENO07]\), as conjectured by Ostrik in \([Ost15]\).

This paper is organized as follows: In Section 2 we recall some tools from the category theory. In Section 3 we first recall the construction of the central Hopf
\textit{monad} on a rigid monoidal category. Using the central Hopf monad, we define the adjoint object \( A \in \mathbb{C} \), the space \( \text{CF}(\mathbb{C}) \) of class functions, and the internal character of an object. We also endow \( \text{CF}(\mathbb{C}) \) with a product and show that it is isomorphic to \( \text{End}_{\mathbb{Z}(\mathbb{C})}(R(1)) \) (Theorems 3.3 and 3.4). Finally, we observe that, if \( \mathbb{C} = \text{H-mod} \) is the category of representations of a finite-dimensional pivotal Hopf algebra over \( k \) with antipode \( S \) and pivotal element \( g \), then:

- the adjoint object is the adjoint representation of \( H \),
- the space of class functions is the set of linear maps \( f : H \to k \) such that \( f(ab) = f(S^2(b)a) \) for all \( a, b \in H \),
- the product of class functions is given by the convolution product, and
- the internal character of \( X \in \text{H-mod} \) is the linear map \( a \mapsto \text{Tr}_X( ga) \), where \( \text{Tr}_X(h) \) is the trace of the action of \( h \in H \) on \( X \).

In Section 4, we show that the map (1.1) is injective if \( \mathbb{C} \) is a finite tensor category over \( k \). The injectivity is obvious if \( \mathbb{C} \) is semisimple or if \( \mathbb{C} = \text{H-mod} \) as above (Examples 4.3 and 4.4). The proof in the general case is technical: We first introduce a quotient \( A^{ss} \in \mathbb{C} \) of the adjoint object, which corresponds to the quotient of the adjoint representation by the Jacobson radical if \( \mathbb{C} = \text{H-mod} \). It turns out that every internal character factors through the quotient morphism \( A \to A^{ss} \), and the linear independence of “irreducible characters” is proved in a similar way as the case where \( \mathbb{C} = \text{H-mod} \).

In Section 5, we apply our results to fusion categories. We show that the Grothendieck algebra \( \text{Gr}_k(\mathbb{C}) \) of a spherical fusion category \( \mathbb{C} \) is non-degenerate (Theorem 5.1). We also show that, under the non-degeneracy assumption, the object \( R(1) \in Z(\mathbb{C}) \) is multiplicity-free if and only if \( \text{Gr}_k(\mathbb{C}) \) is commutative (Theorem 5.2). This result can be considered as a generalization of a result of Andruskiewitsch and Natale \([AN00]\) on Gelfand pairs of Hopf algebras.

2. Preliminaries

2.1. Monoidal categories. For the basic theory of monoidal categories, we refer the reader to [ML98] and [Kas95]. In this paper, we assume that all monoidal categories are strict. Thus, a monoidal category is a triple \( \mathbb{C} = (\mathbb{C}, \otimes, 1) \) consisting of a category \( \mathbb{C} \), a functor \( \otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) (called the tensor product), and an object \( 1 \in \mathbb{C} \) (called the unit object) such that

\[
(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \quad \text{and} \quad 1 \otimes X = X = X \otimes 1
\]

for all \( X, Y, Z \in \mathbb{C} \). For a monoidal category \( \mathbb{C} = (\mathbb{C}, \otimes, 1) \), we set

\[
\mathbb{C}^{op} = (\mathbb{C}^{op}, \otimes, 1) \quad \text{and} \quad \mathbb{C}^{rev} = (\mathbb{C}, \otimes^{rev}, 1),
\]

where \( \otimes^{rev} \) is the reversed tensor product given by \( X \otimes^{rev} Y = Y \otimes X \).

A\ monoidal functor \([ML98, \text{XI.2}]\) is a functor \( F : \mathbb{C} \to \mathbb{D} \) between monoidal categories \( \mathbb{C} \) and \( \mathbb{D} \) endowed with a natural transformation

\[
F_2(X, Y) : F(X) \otimes F(Y) \to F(X \otimes Y) \quad (X, Y \in \mathbb{C})
\]

and a morphism \( F_0 : 1 \to F(1) \) satisfying a certain coherence condition. We say that a monoidal functor \( (F, F_0, F_2) \) is strong if \( F_0 \) and \( F_2 \) are invertible, and strict if they are identities. A\monoidal functor from \( \mathbb{C} \) to \( \mathbb{D} \) is the same thing as a monoidal functor from \( \mathbb{C}^{op} \) to \( \mathbb{D}^{op} \).
A left dual object of \( X \in \mathcal{C} \) is a triple \((X', e, d)\) consisting of an object \( X' \in \mathcal{C} \) and morphisms \( e : X' \otimes X \to \mathbb{1} \) and \( d : \mathbb{1} \to X \otimes X' \) such that
\[
(e \otimes \text{id}_X) \circ (\text{id}_X \otimes d) = \text{id}_X \quad \text{and} \quad (\text{id}_{X'} \otimes e) \circ (d \otimes \text{id}_{X'}) = \text{id}_{X'}.
\]
Suppose that every object of \( \mathcal{C} \) has a left dual object (we say that \( \mathcal{C} \) is left rigid if this is the case). For each \( X \in \mathcal{C} \), we choose a left dual object
\[
(X^*, \text{ev}_X : X^* \otimes X \to \mathbb{1}, \text{coev}_X : \mathbb{1} \to X \otimes X^*)
\]
of \( X \in \mathcal{C} \). Then the assignment \( X \mapsto X^* \) gives rise to a strong monoidal functor \((-)^* : \mathcal{C} \to \mathcal{C}^{\text{op}, \text{rev}}\), which we call the left duality functor.

A rigid monoidal category is a monoidal category \( \mathcal{C} \) such that both \( \mathcal{C} \) and \( \mathcal{C}^{\text{rev}} \) are left rigid. If \( \mathcal{C} \) is a rigid monoidal category, then the duality functor \((-)^*\) is an equivalence. A quasi-inverse functor of \((-)^*\) is denoted by \(*(-)\). As explained in [Shi15, Lemma 5.4], we may assume that \((-)^* \text{ and } *(-) \) are strict monoidal and mutually inverse to each other.

### 2.2. Pivotal monoidal category

A pivotal monoidal category is a rigid monoidal category \( \mathcal{C} \) endowed with a natural isomorphism \( j_V : V \to V^{**} \) \((V \in \mathcal{C})\) of monoidal functors. The isomorphism \( j \) is often referred to as a pivotal structure.

Let \( \mathcal{C} \) be a pivotal monoidal category with pivotal structure \( j \). We use the following notation:
\[
(\mathbf{1})
\]
Let \( A, B, X \in \mathcal{C} \) be objects. For a morphism \( f : A \otimes X \to B \otimes X \), the (right) partial pivotal trace \( \text{tr}^X_{A,B}(f) \) of \( f \) is defined by
\[
\text{tr}^X_{A,B}(f) = (\text{id}_B \otimes \text{ev}_{X^*}) \circ (f \otimes \text{id}_{X^*}) \circ (\text{id}_A \otimes \text{coev}_X).
\]

If \( A = B = \mathbb{1} \), then \( \text{tr}^X_{\mathbb{1},A}(f) \) is written simple as \( \text{tr}(f) \) and is called the pivotal trace of \( f \). We say that \( \mathcal{C} \) is spherical if \( \text{tr}(f) = \text{tr}(f^\ast) \) for all endomorphism \( f \) in \( \mathcal{C} \).

### 2.3. The monoidal center

A braided monoidal category [Kas95, XIII.1] is a monoidal category \( \mathcal{B} \) endowed with a natural isomorphism \( \sigma : \otimes \to \otimes^{\text{rev}} \), called the braiding, satisfying the so-called hexagon axiom. Given a braided monoidal category \( \mathcal{B} = (\mathcal{B}, \otimes, \mathbb{1}, \sigma) \), we set
\[
\mathcal{B}^{\text{rev}} = (\mathcal{B}, \otimes^{\text{rev}}, \mathbb{1}, \sigma^{\text{rev}}) \quad \text{and} \quad \mathcal{B}^{\text{mir}} = (\mathcal{B}, \otimes, \mathbb{1}, \sigma^{\text{mir}}),
\]
where \( \sigma^{\text{rev}}_{X,Y} = \sigma_{Y,X} \) and \( \sigma^{\text{mir}}_{X,Y} = \sigma_{Y,X}^{-1} \) for \( X, Y \in \mathcal{B} \).

Now let \( \mathcal{C} \) be a monoidal category. A left half-braiding of \( \mathcal{C} \) is a pair \((V, \sigma)\) consisting of an object \( X \in \mathcal{C} \) and a natural isomorphism
\[
\sigma_X : V \otimes X \to X \otimes V \quad (X \in \mathcal{C})
\]
such that \( \sigma_{X\otimes Y} = (\text{id}_X \otimes \sigma_Y) \circ (\sigma_X \otimes \text{id}_Y) \) for all \( X, Y \in \mathcal{C} \). Left half-braidings of \( \mathcal{C} \) form a category, denoted by \( \mathcal{Z}(\mathcal{C}) \) and called the monoidal center. The category \( \mathcal{Z}(\mathcal{C}) \) is in fact a braided monoidal category such that the forgetful functor
\[
U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}, \quad (V, \sigma) \mapsto V
\]
is a strict monoidal functor; see, e.g., [Kas95, XIII.3] for details.
Remark 2.1. A right half-braiding of \( \mathcal{C} \) is a pair \((V, \sigma')\) consisting of an object \( V \in \mathcal{C} \) and a natural isomorphism \( \sigma' : (-) \otimes V \to V \otimes (-) \) satisfying an analogous axiom. The braided monoidal category \( Z_r(\mathcal{C}) \) of right half-braiding is defined in an analogous way. The monoidal center introduced in [Kas95, XIII.3] is in fact \( Z_r(\mathcal{C}) \), and our \( Z(\mathcal{C}) \) coincides with \( Z_r(\mathcal{C}_{rev})^{rev, mir} \).

2.4. Hopf monads. Let \( \mathcal{C} \) be a monoidal category. A bimonad \([BV07]\) on \( \mathcal{C} \) is a monad \((T, \mu, \eta)\) on \( \mathcal{C} \) endowed with a comonoidal structure

\[
T_0 : T(1) \to 1 \quad \text{and} \quad T_2(V, W) : T(V \otimes W) \to T(V) \otimes T(W) \quad (V, W \in \mathcal{C})
\]

such the multiplication \( \mu : T^2 \to T \) and the unit \( \eta : id_\mathcal{C} \to T \) of the monad \( T \) are comonoidal natural transformations. If \( T \) is a bimonad on \( \mathcal{C} \), then the category \( \tau \mathcal{C} \) of \( T \)-modules (= the Eilenberg-Moore category of \( T \)) is a monoidal category in such way that the forgetful functor \( U : \tau \mathcal{C} \to \mathcal{C} \) is a strong monoidal functor. More precisely, given \( T \)-modules \((M, a)\) and \((N, b)\), their tensor product is defined by

\[
(M, a) \otimes (N, b) = \left( M \otimes N, T(M \otimes N) \xrightarrow{T_2} T(M) \otimes T(N) \xrightarrow{a \otimes b} M \otimes N \right).
\]

Now suppose that \( \mathcal{C} \) is a rigid monoidal category. Then a Hopf monad \([BV07]\) on \( \mathcal{C} \) is a bimonad \( T \) on \( \mathcal{C} \) endowed with natural transformations

\[
S_V^{(l)} : T(T(V)^*) \to V^* \quad \text{and} \quad S_V^{(r)} : T(*T(V)) \to *V \quad (V \in \mathcal{C}),
\]

called the left antipode and the right antipode, respectively, satisfying certain conditions. If \( T \) is a Hopf monad on \( \mathcal{C} \), then the monoidal category \( \tau \mathcal{C} \) is rigid.

2.5. Ends and coends. Let \( \mathcal{C} \) and \( \mathcal{V} \) be categories, and let \( S, T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V} \) be functors. A dinatural transformation \( \xi : S \to T \) is a family

\[
\xi = \{ \xi_X : S(X, X) \to T(X, X) \}_{X \in \mathcal{C}}
\]

of morphisms in \( \mathcal{V} \) such that

\[
T(id_X, f) \circ \xi_X \circ S(f, id_X) = T(f, id_Y) \circ \xi_Y \circ S(id_Y, f)
\]

for all morphism \( f : X \to Y \) in \( \mathcal{C} \). An end of \( S \) is a pair \((E, p)\) consisting of an object \( E \in \mathcal{V} \) (regarded as a constant functor from \( \mathcal{C}^{op} \times \mathcal{C} \) to \( \mathcal{V} \)) and a dinatural transformation \( p : E \Rightarrow S \) that enjoys the following universal property: For any pair \((E', p')\) consisting of an object \( E' \in \mathcal{V} \) and a dinatural transformation \( p : E \Rightarrow S \), there exists a unique morphism \( \phi : E \to E' \) such that \( p'_X = \phi \circ p_X \) for all \( X \in \mathcal{C} \). A coend of \( T \) is a pair \((C, i)\) consisting of an object \( C \in \mathcal{V} \) and a dinatural transformation \( i : T \Rightarrow C \) satisfying a similar universal property. The end of \( S \) and the coend of \( T \) are expressed as

\[
\int_{X \in \mathcal{C}} S(X, X) \quad \text{and} \quad \int_{X \in \mathcal{C}} T(X, X),
\]

respectively; see [ML98] for more details.

Lemma 2.2. Let \( \mathcal{B}, \mathcal{C} \) and \( \mathcal{V} \) be categories, and let \( F : \mathcal{B} \to \mathcal{C} \) be a functor that has a right adjoint \( G \). Then, for any functor \( \Phi : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V} \), we have

\[
\int_{X \in \mathcal{C}} \Phi(X, FG(X)) \cong \int_{V \in \mathcal{B}} \Phi(F(V), F(V)),
\]

meaning that if either one of coends exists, then the other one exists and there is a canonical isomorphism between them.
Proof. This is a special case of [BV12, Lemma 3.9]. □

Let \(C_1\) and \(C_2\) denote the left hand side and the right hand side of (2.2), respectively, and let \(\phi_{12} : C_1 \to C_2\) be the isomorphism given by the above lemma. By the proof of [BV12, Lemma 3.9], we see that \(\phi_{12}\) is characterized by
\[
\phi_{12} \circ \iota^{(1)}_X = \iota^{(2)}_{G(X)} \circ \Phi(\varepsilon_X, \text{id}_{FG(X)})
\]
for all \(X \in \mathcal{C}\), where \(\iota^{(a)} (a = 1, 2)\) is the universal dinatural transformation to \(C_a\) and \(\varepsilon : FG \to \text{id}_\mathcal{C}\) is the counit of the adjunction.

We now assume that the coend \(C_3 = \int^{X \in \mathcal{C}} \Phi(X, X)\) also exists. By the universal property, there exists a unique morphism \(\phi_{23} : C_2 \to C_3\) such that
\[
\phi_{23} \circ \iota^{(2)}_V = \iota^{(3)}_{F(G(V))}
\]
for all \(V \in \mathcal{B}\), where \(\iota^{(3)}\) is the universal dinatural transformation to \(C_3\). Thus
\[
\phi_{23} \circ \phi_{12} \circ \iota^{(1)}_X = \phi_{23} \circ \iota^{(2)}_{G(X)} \circ \Phi(\varepsilon_X, \text{id}_{FG(X)}) = \iota^{(3)}_{F(G(X))} \circ \Phi(\varepsilon_X, \text{id}_{FG(X)}) = \iota^{(3)}_X \circ \Phi(\text{id}_X, \varepsilon_X)
\]
for all \(X \in \mathcal{C}\), where the third equality follows from the dinaturality of \(\iota^{(3)}\). Summarizing the above argument, we have the following conclusion:

**Lemma 2.3.** The canonical morphism
\[
\int^{V \in \mathcal{B}} \Phi(F(V), F(V)) \to \int^{X \in \mathcal{C}} \Phi(X, X)
\]
obtained by the universal property corresponds to the morphism
\[
\int^{X \in \mathcal{C}} \Phi(\text{id}_X, \varepsilon_X) : \int^{X \in \mathcal{C}} \Phi(X, FG(X)) \to \int^{X \in \mathcal{C}} \Phi(X, X)
\]
via the isomorphism given in Lemma 2.2.

## 2.6. Conjugation by the duality.

For a functor \(T : \mathcal{B} \to \mathcal{C}\) between rigid monoidal categories, we define \(T^! : \mathcal{B} \to \mathcal{C}\) by
\[
T^!(X) = T(\ast V)^* \quad (V \in \mathcal{B}).
\]
A natural transformation \(\alpha : S \to T (S, T : \mathcal{B} \to \mathcal{C})\) induces
\[
\alpha^! : S^! \to T^!, \quad \alpha^!_V : S^!(V) = S(\ast V)^* \xrightarrow{(\alpha_V)^*} T(\ast V)^* = T^!(V).
\]
The operation \((-)^!\) preserves the composition of functors between rigid monoidal categories and the horizontal composition of natural transformations, but reverses the vertical composition of natural transformations. Moreover, this operation is invertible: The inverse is given by
\[
T \mapsto \mathcal{T}, \quad \mathcal{T}(V) = \ast T(V^*).
\]
Now let \(\mathcal{B}\) and \(\mathcal{C}\) be rigid monoidal categories. By the above observation, one easily proves:
- \(T : \mathcal{C} \to \mathcal{C}\) is a monad if and only if \(T^!\) is a comonad.
- \(F : \mathcal{B} \to \mathcal{C}\) is a monoidal functor if and only if \(F^!\) is a comonoidal functor.
Let $F : B \to C$ be a functor. Then $F \dashv G$ (i.e., $F$ is left adjoint to $G : C \to B$) if and only if $G^\dagger \dashv F^\dagger$.

Suppose that $F$ is a strong monoidal functor. Since $F^\dagger \cong F$, 

\[ L \dashv F \iff F \dashv L^\dagger \quad \text{and} \quad F \dashv R \iff R^\dagger \dashv F. \]

3. Internal characters

3.1. The central Hopf monad. Let $C$ be a rigid monoidal category such that

\[ Z(V) = \int^X \in C \ X^* \otimes V \otimes X \]

exists for all $V \in C$. Day and Street [DS07] showed that the functor $V \mapsto Z(V)$ has a structure of a monad such that the category $\mathbb{Z}C$ of $Z$-modules can be identified with $Z(C)$. By the Tannaka reconstruction, the monad $Z$ has a structure of a quasitriangular Hopf monad in the sense of [BV07, BV12] such that the isomorphism $ZC \cong Z(C)$ is in fact an isomorphism of braided monoidal categories. In this section, we first recall the definition of the Hopf monad $Z$, which we call the central Hopf monad.

Let $i_{V,X} : X^* \otimes V \otimes X \to Z(V)$ be the universal dinatural transformation of the coend (3.1). The comonoidal structure

\[ Z_0 : Z(1) \to 1 \quad \text{and} \quad Z_2(V,W) : Z(V \otimes W) \to Z(V) \otimes Z(W) \quad (V,W \in C) \]

is determined by $Z_0 \circ i_{1,X} = \text{ev}_X$ and

\[ Z_2(V,W) \circ i_{X;V \otimes W} = (i_{X,V} \otimes i_X;W) \circ (\text{id}_{X^*} \otimes \text{id}_V \otimes \text{coev}_X \otimes \text{id}_W \otimes \text{id}_X) \]

for all $V,W,X \in C$. To define the multiplication of $Z$, we note that, by the Fubini theorem for coends, $Z^2(V)$ is a coend of the functor

\[ (C \times C)^{op} \times (C \times C) \to C, \quad (X_1,Y_1,X_2,Y_2) \mapsto Y_1^* \otimes X_1^* \otimes V \otimes X_2 \otimes Y_2 \]

with universal dinatural transformation

\[ i_{V;X,Y}^{(2)} = i_{Z(V);Y} \circ (\text{id}_{Y^*} \otimes i_{V,X} \otimes \text{id}_Y) \quad (X,Y \in C). \]

By using the universal property, the multiplication $\mu : Z^2 \to Z$ is defined by

\[ \mu_V \circ i_{V;X,Y}^{(2)} = i_{V;X \otimes Y} \]

for all $V,X,Y \in C$. Finally, we define the unit $\eta : \text{id}_C \to Z$ of $Z$ by $\eta_V = i_{V;1}$ for $V \in C$. We omit the description of antipodes and the universal R-matrix since we will not use them in this paper. See [BV12] for the details, where, more generally, the quantum double of a Hopf monad is considered (the Hopf monad $Z$ introduced here is in fact the quantum double of $\text{id}_C$).

**Definition 3.1.** We call $Z$ the central Hopf monad on $C$.

To establish an isomorphism $\mathbb{Z}C \cong Z(C)$, we define $\partial_{V,X}$ to be the morphism corresponding to $i_{V,X}$ under the canonical isomorphism

\[ \text{Hom}_C(X^* \otimes V \otimes X, Z(V)) \cong \text{Hom}_C(V \otimes X, X \otimes Z(V)). \]

Given a $Z$-module $(M, a)$, we define $\sigma : M \otimes (-) \to (-) \otimes M$ by

\[ \sigma_X : M \otimes X \xrightarrow{\delta_{M,X}} X \otimes Z(M) \xrightarrow{\text{id}_X \otimes a} X \otimes M \quad (X \in C). \]

The assignment $(M, a) \mapsto (M, \sigma)$ gives rise to an isomorphism $\mathbb{Z}C \cong Z(C)$ of braided monoidal categories.
3.2. The adjoint object. Let $\mathcal{C}$ be a rigid monoidal category such that the forgetful functor $U : Z(\mathcal{C}) \to \mathcal{C}$ has a right adjoint functor $R$. We now introduce the following terminologies:

**Definition 3.2.** We call $A = UR(\mathbb{1})$ the adjoint object. An element of the set $\text{CF}(\mathcal{C}) = \text{Hom}_\mathcal{C}(A, \mathbb{1})$ is referred to as a class function in $\mathcal{C}$.

We note that $R : \mathcal{C} \to Z(\mathcal{C})$ is a monoidal functor as a right adjoint of a strong monoidal functor. Thus the adjoint object is an algebra in $\mathcal{C}$ as the image of the trivial algebra $\mathbb{1} \in \mathcal{C}$ under a monoidal functor.

For class functions $f$ and $g$ in $\mathcal{C}$, we define $f \star g \in \text{CF}(\mathcal{C})$ by

$$f \star g = f \circ E(g) \circ \delta,$$

where $E = UR$ is the comonad associated with $U \dashv R$ and $\delta : E \to E^2$ is the comultiplication of $E$. This operation is nothing but the composition of morphisms in the co-Kleisli category of $E$.

Thus we have:

**Theorem 3.3.** $\text{CF}(\mathcal{C})$ is a monoid with respect to $\star$.

The category of $E$-comodules can be identified with the category of $Z$-comodules, and hence with $Z(\mathcal{C})$. Since the co-Kleisli category of $E$ is equivalent to the category of free $E$-comodules, we have the following theorem:

**Theorem 3.4.** The adjunction isomorphism

$$\text{CF}(\mathcal{C}) = \text{Hom}_\mathcal{C}(UR(\mathbb{1}), \mathbb{1}) \cong \text{End}_{Z(\mathcal{C})}(R(\mathbb{1}))$$

is an isomorphism of monoids.

If $\mathcal{C}$ is a linear monoidal category over a field $k$, then the isomorphism of this theorem is in fact an isomorphism of algebras over $k$.

3.3. The adjoint object as an end. Let $\mathcal{C}$ be a rigid monoidal category such that the coend (3.1) exists for all $V \in \mathcal{C}$. Then, as we have seen, $Z(\mathcal{C})$ can be identified with the category of modules over the central Hopf monad $Z$ on $\mathcal{C}$. Under the identification, the free $Z$-module functor $L : \mathcal{C} \to Z(\mathcal{C})$, $V \mapsto (Z(V), \mu_V)$ $(V \in \mathcal{C})$ is a left adjoint of the forgetful functor $U : Z(\mathcal{C}) \to \mathcal{C}$.

By the results of §2.6, the functor $R := L^!$ is right adjoint to $U$. The universal dinatural transformation $i_{V;X}$ for the coend $Z(V)$ induces

$$\pi_{V;X} : Z^! V \to \int_X X \otimes V \otimes X^*$$

that is natural in $V$ and dinatural in $X$. Since the duality is an anti-equivalence, the functor $Z^! = UR$ and the adjoint object $A = UR(\mathbb{1})$ are expressed as

$$Z^! V = \int_{X \in \mathcal{C}} X \otimes V \otimes X^* \quad \text{and} \quad A = \int_{X \in \mathcal{C}} X \otimes X^*.$$

By a Hopf comonad on $\mathcal{C}$, we mean a comonad $T$ on $\mathcal{C}$ endowed with a monoidal structure such that, in a word, $T^\text{op} : \mathcal{C}^\text{op} \to \mathcal{C}^\text{op}$ is a Hopf monad on $\mathcal{C}^\text{op}$. Again by the argument of §2.6 the functor $Z^!$ is a Hopf comonad on $\mathcal{C}$. For later use, we
express the structure of $Z^!$ in terms of $\pi$. By (3.3), we see that the comultiplication $\delta$ of the comonad $Z^!$ is the unique morphism such that

$$(3.8) \quad (\text{id}_X \otimes \pi_{V,Y} \otimes \text{id}_X) \circ \pi_{Z^!(V),X} \circ \delta_V = \pi_V(X \otimes Y)$$

for all $V, X, Y \in \mathcal{C}$. The counit is given by $\varepsilon_V = \pi_V(\mathbb{1})$. We omit to describe the monoidal structure, but note that the multiplication and the unit of the algebra $A$ (that are defined by the monoidal structure of $Z^!$) are determined by

$$(3.9) \quad \pi_{1;X} \circ u = \text{coev}_X,$$

$$(3.10) \quad \pi_{1;X} \circ m = (\text{id}_X \otimes \text{ev}_X \otimes \text{id}_X) \circ (\pi_{1;X} \otimes \pi_{1;X})$$

for all $X \in \mathcal{C}$, respectively.

**Definition 3.5.** For $X \in \mathcal{C}$, we define the *canonical action* of $A$ on $X$ by

$$(3.11) \quad \rho_X : A \otimes X \xrightarrow{\pi_{1;X} \otimes \text{id}_X} X \otimes X^* \otimes X \xrightarrow{\text{id}_X \otimes \text{ev}_X} X.$$

By (3.9) and (3.10), we see that $(X, \rho_X)$ is in fact a left $A$-module in $\mathcal{C}$.

**Remark 3.6.** The dinaturality of $\pi_{1;X}$ in $X$ implies that $\rho_X$ is natural in $X$. The universal property of $\pi_1(X)$ is translated as follows: For every pair $(A', \rho')$ consisting of an object $A' \in \mathcal{C}$ and a natural transformation $\rho' : A' \otimes X \to X$ ($X \in \mathcal{C}$), there exists a unique morphism $f : A' \to A$ in $\mathcal{C}$ such that $\rho'_X = \rho_X \circ (f \otimes \text{id}_X)$ for all objects $X \in \mathcal{C}$.

3.4. **Internal characters.** Let $\mathcal{C}$ be as in the previous subsection, and let $A \in \mathcal{C}$ denote the adjoint object. We suppose that $\mathcal{C}$ has a pivotal structure $j$. We now introduce a category-theoretical analogue of the notion of a character:

**Definition 3.7.** The *internal character* of $X \in \mathcal{C}$ is a class function given by

$$(3.12) \quad \text{ch}(X) = \text{tr}^X_{A,1}(\rho_X),$$

where $\rho_X$ is the canonical action given by (3.11).

The internal character is multiplicative with respect to the tensor product:

**Theorem 3.8.** $\text{ch}(X \otimes Y) = \text{ch}(X) \star \text{ch}(Y)$ for all $X, Y \in \mathcal{C}$.

**Proof.** To compute the product of internal characters, it is convenient to identify $Z(\mathcal{C})$ as the category $\mathcal{Z}\mathcal{C}$ of modules over the central Hopf monad $Z$. We define $L$ by (3.5), and then set $R = L'$. As noted before, $L$ and $R$ are a left adjoint and a right adjoint of the forgetful functor $U : Z(\mathcal{C}) \to \mathcal{C}$, respectively.

By the definition of the canonical action of $A$ on $X$, we have

$$(3.12) \quad \text{ch}(X) = \text{ev}_X \circ \pi_{1;X},$$

where $\text{ev}_X$ is the morphism defined by (2.4) and $\pi_{1;X}$ is the universal dinatural transformation for the end $Z^!(V)$ given by (3.6). We also have

$$(3.13) \quad \text{ev}_X \otimes Y = \text{ev}_X \circ (\text{id}_X \otimes \text{ev}_Y \otimes \text{id}_X).$$
since \( j_{X \otimes Y} = j_X \otimes j_Y \) for all \( X, Y \in \mathcal{C} \). Now we compute
\[
\text{ch}(X) \star \text{ch}(Y) = \tilde{\ev}_X \circ \pi_{1:1} \circ Z^!(\text{ch}(Y)) \circ \delta_1
\]
by (3.11) and (3.12)
\[
= \tilde{\ev}_X \circ (\text{id}_X \otimes \text{ch}(Y) \otimes \text{id}_X^*) \circ \pi_{Z!(1):X} \circ \delta_1
\]
(the naturality of \( \pi \))
\[
= \tilde{\ev}_X \circ (\text{id}_X \otimes \tilde{\ev}_Y \otimes \text{id}_X^*) \circ (\text{id}_X \otimes \pi_{1:Y} \otimes \text{id}_X^*) \circ \pi_{Z!(1):X} \circ \delta_1
\]
by (3.12)
\[
= \tilde{\ev}_{X \otimes Y} \circ \pi_{1:1;X \otimes Y}
\]
by (3.8) and (3.13)
\[
= \text{ch}(X \otimes Y)
\]
for all \( X, Y \in \mathcal{C} \) (the graphical calculus for the central Hopf monad \( Z \), explained in [Shi15], would be helpful to understand this computation).

3.5. **The case of Hopf algebras.** We now explain what our results say to the representation theory of Hopf algebras. Let \( H \) be a finite-dimensional Hopf algebra over a field \( k \) with comultiplication \( \Delta \), counit \( \varepsilon \) and antipode \( S \). We freely use the Sweedler notation such as
\[
\Delta(h) = h_{(1)} \otimes h_{(2)} \quad \text{and} \quad \Delta(h_{(1)}) \otimes h_{(2)} = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = h_{(1)} \otimes \Delta(h_{(2)})
\]
for \( h \in H \). Recall that a (left-left-) Yetter-Drinfeld module over \( H \) is a left \( H \)-module endowed with a left \( H \)-comodule structure such that

\[
(h \cdot v)_{-1} \otimes (h \cdot v)_{0} = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}v
\]
for all \( h \in H \) and \( v \in V \), where \( x \mapsto x_{-1} \otimes x_{0} \) is the coaction of \( H \). It is well-known that the monoidal center of the category \( \mathcal{C} = H\text{-mod} \) of finite-dimensional left \( H \)-modules can be identified with the category \( H^!YD \) of finite-dimensional Yetter-Drinfeld modules over \( H \).

Under the identification \( Z(\mathcal{C}) \cong H^!YD \), the forgetful functor \( U : Z(\mathcal{C}) \to \mathcal{C} \) corresponds to the functor forgeting the \( H \)-comodule structure. A right adjoint \( R \) of \( U \) is given as follows: As a vector space, \( R(V) = H \otimes_k V \) for \( V \in \mathcal{C} \). The action and the coaction of \( H \) on \( R(V) \) are given by
\[
h \cdot (a \otimes v) = h_{(1)}aS(h_{(3)}) \otimes h_{(2)}v \quad \text{and} \quad a \otimes v \mapsto a_{(1)} \otimes a_{(2)} \otimes v
\]
for \( a, h \in H \) and \( v \in V \). Thus \( A = UR(k) \), where \( k \) is the trivial \( H \)-module, can be identified with \( H \) as a vector space via the canonical isomorphism \( H \otimes_k k = H \) of vector spaces. The action of \( H \) on \( A \) is the adjoint action \( \triangleright \) given by
\[
h \triangleright a = h_{(1)}aS(h_{(2)})
\]
for \( h \in H \) and \( a \in A \). The algebra structure of \( A \) is the same as that of \( H \). See, e.g., [Shi15] and [Shi14] for details.

By definition, a class function in \( \mathcal{C} \) is an \( H \)-linear map \( A \to k \). Thus, by easy computation, we see that the space of class functions in \( \mathcal{C} \) is
\[
\text{CF}(\mathcal{C}) = \{ f \in H^* \mid f(ab) = f(S^2(b)a) \text{ for all } a, b \in H \}.
\]
The product of \( \text{CF}(\mathcal{C}) \) is in fact given by the convolution product. To see this, we note that the comultiplication of the comonad \( E = UR \) is given by
\[
\delta_V : E(V) \to E^2(V), \quad a \otimes v \mapsto a_{(1)} \otimes a_{(2)} \otimes v \quad (a \in H, v \in V)
\]
by the definition of \( \star \), we have
\[
\langle f \star g, a \rangle = \langle f \circ E(g), a_{(1)} \otimes a_{(2)} \rangle = \langle f, a_{(1)} \rangle \langle g, a_{(2)} \rangle
\]
for all \( f, g \in \text{CF}(\mathcal{C}) \) and \( a \in A \). Namely, the product \( * \) coincides with the convolution product of \( H^* \).

For \( X \in \mathcal{C} \), the canonical action \( \rho_X : A \otimes_k X \to X \) is given by the linear map induced by the action of \( H \) on \( X \). Indeed, it is obvious that \( \rho_X \) is a morphism in \( \mathcal{C} \) that is natural in \( X \in \mathcal{C} \). If \( A' \in \mathcal{C} \) is an object and \( \rho_X' : A' \otimes X \to X \) is a natural transformation, then we have

\[
\rho_X'(a' \otimes x) = \rho_H'(a' \otimes 1)x \quad (a' \in A', x \in X)
\]

by the naturality. From this, we see that \( f(a') = \rho_H'(a' \otimes 1) \) is a unique morphism in \( \mathcal{C} \) such that \( \rho_X' = \rho_X \circ (f \otimes \text{id}_X) \) for all \( X \in \mathcal{C} \). Thus, by Remark 3.6, \( \rho_X \) is the canonical action of \( A \) on \( X \).

Now we suppose that \( H \) has a pivotal element \( g \in H \), i.e., an invertible element such that \( \Delta(g) = g \otimes g \) and \( S^2(h) = ghg^{-1} \) for all \( h \in H \). Then \( \mathcal{C} \) is a pivotal monoidal category with pivotal structure

\[
j_X : X \to X^{**} \quad (X \in \mathcal{C}), \quad \langle j_X(x), p \rangle = \langle p, gx \rangle \quad (x \in X, p \in X^*)
\]

The internal character of \( X \in \mathcal{C} \) is given by

\[
\langle \text{ch}(X), a \rangle = \text{Tr}_X(ga) \quad (a \in A),
\]

where \( \text{Tr}_X(h) \) is the trace of the action of \( h \in H \) on \( X \).

**Remark 3.9.** The coend \( F = \int^{X \in \mathcal{C}} X^* \otimes X \) has a structure of a coalgebra in \( \mathcal{C} \) and every object \( V \in \mathcal{C} \) has a natural structure of a right \( F \)-comodule, which we call the canonical coaction [KL01, §5.2.2]. Taking the left partial pivotal trace of the canonical coaction of \( F \) on \( X \), we obtain a morphism \( \text{ch}'(X) : 1 \to F \). If we realize the adjoint object by \( A = F^* \) (as in [33], then we have

\[
\text{ch}'(X) = \ast \text{ch}(X^*)
\]

for all \( X \in \mathcal{C} \). The morphism \( \text{ch}'(X) \) has been appeared in the study of conformal field theories [FSS13a, FSS13b]. We prefer to deal with \( \text{ch}(X) \) since it is more convenient for applications to the representation theory of Hopf algebras.

4. LINEAR INDEPENDENCE OF IRREDUCIBLE CHARACTERS

4.1. **Finite tensor categories.** In this section, we work over an algebraically closed field \( k \). A finite abelian category is a \( k \)-linear category that is equivalent to \( A\text{-mod} \) for some finite-dimensional algebra \( A \) over \( k \). A finite tensor category [EO04] is a rigid monoidal category \( \mathcal{C} \) such that

1. \( \mathcal{C} \) is a finite abelian category,
2. the tensor product of \( \mathcal{C} \) is \( k \)-linear in each variable, and
3. \( \text{End}_\mathcal{C}(1) \cong k \).

For a finite abelian category \( \mathcal{A} \), we denote by \( \text{Gr}(\mathcal{A}) \) its Grothendieck group and set \( \text{Gr}_k(\mathcal{A}) = k \otimes \mathbb{Z} \text{Gr}(\mathcal{A}) \). It is well-known that the isomorphism classes of simple objects of \( \mathcal{A} \) is a basis of \( \text{Gr}_k(\mathcal{A}) \). If \( \mathcal{C} \) is a finite tensor category, then \( \text{Gr}_k(\mathcal{C}) \) is in fact a \( k \)-algebra with respect to the multiplication \( [V] \cdot [W] = [V \otimes W] \).

Now let \( \mathcal{C} \) be a pivotal finite tensor category over an algebraically closed field \( k \) with pivotal structure \( j : \text{id}_C \to \ast \ast \). Following [KL01, §5], the functor (3.1) has a coend for all \( V \in \mathcal{C} \). Thus we can define the adjoint object \( A \in \mathcal{C} \), its canonical action \( \rho_X : A \otimes X \to X \) on an object \( X \in \mathcal{C} \), and the internal character \( \text{ch}(X) \) of \( X \in \mathcal{C} \) in the way of the previous section.
Recall that the internal character $\text{ch}(X)$ is defined to be the partial trace of $\rho_X$. Since the partial trace is additive in exact sequences, the map 

$$\text{ch} : \text{Gr}_k(C) \to \text{CF}(C), \quad [X] \mapsto \text{ch}(X) \quad (X \in C)$$

is well-defined. Moreover, Theorem 3.8 implies that this map is a homomorphism of algebras. The main result of this section is stated as follows:

**Theorem 4.1.** The map $\text{ch} : \text{Gr}_k(C) \to \text{CF}(C)$ is injective.

Let $R$ be a right adjoint of the forgetful functor $U : Z(C) \to C$. By the result of Subsection 3.2, we have the following corollary:

**Corollary 4.2.** The algebra $\text{Gr}_k(C)$ can be embedded into $\text{End}_{Z(C)}(R(1))$.

Let $\{V_1, \ldots, V_m\}$ be a complete set of representatives of isomorphism classes of simple objects of $C$. Theorem 4.1 is equivalent to that

$$\{\text{ch}(V_i)\}_{i=1}^m$$

is linearly independent in $\text{CF}(C)$.

For applications discussed in this paper, we only need the case where $C$ is semisimple. The assertion (4.1) is easily proved in such a case (see Example 4.3 below). However, since Theorem 4.1 seems to be of independent interest, we give a proof in the general setting.

**Example 4.3.** If $C$ in the above is semisimple, then $A = \bigoplus_{i=1}^m V_i \otimes V_i^*$ as an object in $C$ and the universal dinatural transformation $\pi_X : A \to X \otimes X^*$ is just the projection if $X$ is one of $V_1, \ldots, V_m$ [KL01, §5.1.3]. Hence, by (3.12), the internal character of $V_i$ coincides with the composition

$$A \xrightarrow{\text{projection}} V_i \otimes V_i^* \xrightarrow{\tilde{e}_{V_i}} \mathbb{1}.$$

In other words, $\text{ch}(V_i)$ corresponds to $\tilde{e}_{V_i} \in \text{Hom}_C(V_i \otimes V_i^*, \mathbb{1})$ under

$$\text{CF}(C) = \text{Hom}_C(A, \mathbb{1}) \cong \bigoplus_{i=1}^m \text{Hom}_C(V_i \otimes V_i^*, \mathbb{1}).$$

It is obvious that $\text{Hom}_C(V_i \otimes V_i^*, \mathbb{1})$ is a one-dimensional vector space spanned by the morphism $\tilde{e}_{V_i}$. Thus the set $\{\text{ch}(V_i)\}_{i=1}^m$ is a basis of $\text{CF}(C)$.

**Example 4.4.** Let $H$ be a finite-dimensional Hopf algebra over $k$ with pivotal element $g$, and let $V_1, \ldots, V_m$ be the complete set of representatives of isomorphism classes of simple left $H$-modules. As explained in [KL01] then $C = H\text{-mod}$ is a pivotal finite tensor category. Since $g$ is invertible, Theorem 4.1 is equivalent to that

$$\{\text{Tr}_{V_i}\}_{i=1}^m$$

in this particular case.

The assertion (4.1) is well-known to be true and can be proved as follows: We set $H^{ss} = H/J(H)$, where $J(H)$ is the Jacobson radical. By the Artin-Wedderburn theorem, there is an isomorphism

$$H^{ss} \cong \text{End}_k(V_1) \oplus \cdots \oplus \text{End}_k(V_m)$$

of algebras. Since $\text{Tr}_{V_i}$ coincides with the composition

$$H \xrightarrow{\text{quotient}} H^{ss} \xrightarrow{\text{projection}} \text{End}_k(V_i) \xrightarrow{\text{trace}} k,$$

the assertion (4.2) follows.
Our proof of Theorem 4.1 basically follows this scheme. Namely, to prove the theorem, we show that the adjoint object $A$ in a finite tensor category $C$ has a canonical quotient $A^{ss}$. It turns out that $A^{ss}$ is of the form like $4.3$, and every internal character factors through $A \to A^{ss}$ like $4.4$.

4.2. A categorical analogue of $H/J(H)$. Let $C$ be as above, and let $S$ be the full subcategory of $C$ consisting of all semisimple objects. As a categorical analogue of $H^{ss} = H/J(H)$ in Example 4.4 we propose to consider the following object:

**Definition 4.5.** $A^{ss} := \int_{X \in S} X \otimes X^\ast$.

Namely, $A^{ss}$ is an end of $S^{ss} \otimes S \to C$, $(X, Y) \mapsto Y \otimes X^\ast$. The end $A^{ss}$ indeed exists by the argument of $[KL01]$, §5.1.3. Recall that the adjoint object $A$ is an end of the functor $C^{op} \times C \to C$ defined by the same expression. Hence we have a morphism $q : A \to A^{ss}$ by the universal property of $A^{ss}$ as an end. We now state the following key lemma:

**Lemma 4.6.** The morphism $q : A \to A^{ss}$ is epic.

In the setting of Example 4.4, $A^{ss}$ is obtained as the quotient of the adjoint representation $A$ by the Jacobson radical (which is indeed an $H$-submodule of $A$ since it is a two-sided ideal). The map $q : A \to A^{ss}$ is just the quotient map in this case, and thus it is obviously epic in $H$-$mod$. In the general setting, Lemma 4.6 does not seem to be obvious. To prove the lemma, we go back to the proof of the existence of certain coends including (3.1).

4.3. Existence of certain coends. For an object $X \in C$ and a finite-dimensional vector space $K \subset k$-$mod$, their tensor product $K \cdot X \in C$ (also called the *copower*) is defined by

$$\text{Hom}_C(K \cdot X, Y) \cong \text{Hom}_k(K, \text{Hom}_C(X, Y))$$  \hspace{1cm} (4.5)

for $Y \in C$. Let $\text{Lex}(C)$ be the category of $k$-linear left exact endofunctors on $C$. As explained in $[Shi14]$, the Eilenberg-Watts theorem implies that the functor

$$\Phi : C^{op} \boxtimes C \to \text{Lex}(C), \quad X \boxtimes Y \mapsto \text{Hom}_C(X, -) \cdot Y \quad (X, Y \in C)$$  \hspace{1cm} (4.6)

is an equivalence of $k$-linear categories, where $\boxtimes$ means the Deligne tensor product of $k$-linear abelian categories $[Del90]$, $[5]$. Now let $\Phi$ denote an inverse functor of $\Phi$, and let $F \in \text{Lex}(C)$. As shown in $[Shi14]$, the object $\Phi(F)$ represents

$$C^{op} \boxtimes C \to \text{Vec}, \quad M \mapsto \int_{X \in C} \text{Hom}_{C^{op} \boxtimes C}(X \boxtimes F(X), M) \quad (M \in C^{op} \boxtimes C),$$

where $\text{Vec}$ is the category of all vector spaces over $k$. Thus,

$$F \cong \Phi(\Phi(F)) \cong \int_{X \in C} \Phi(X \boxtimes F(X)) = \int_{X \in C} \text{Hom}_C(X, -) \cdot F(X).$$  \hspace{1cm} (4.7)

For $X \in C$, we denote by $\text{soc}(X)$ the socle of $X$, i.e., the maximal semisimple subobject of $X$. Obviously, $X \mapsto \text{soc}(X)$ is a functor $\text{soc} : C \to S$. Let $i_S : S \hookrightarrow C$ denote the inclusion functor. By abuse of notation, we write $i_S \circ \text{soc}$ simply as $\text{soc}$.

For $X \in S$ and $Y \in C$, we have $\text{Hom}_C(X, Y) = \text{Hom}_S(X, \text{soc}(Y))$. Namely, the socle functor $\text{soc} : C \to S$ is right adjoint to $i_S$. Hence, by Lemma 2.2

$$\text{soc} \cong \int_{X \in S} \text{Hom}_C(X, -) \cdot \text{soc}(X) \cong \int_{X \in S} \text{Hom}_C(X, -) \cdot X.$$  \hspace{1cm} (4.8)
Lemma 4.7. There is a commutative diagram

\[
\begin{array}{c}
soc \\
\downarrow \text{inclusion} \\
id_C
\end{array}
\quad \xymatrix{
\int_{X \in C} \Hom_C(X, -) \cdot X \ar[r]^{\Phi} & \int_{X \in C} \Hom_C(X, -) \cdot X \\
\downarrow_{\phi} & \\
\int_{X \in \mathcal{C}} \Hom_C(X, -) \cdot X
\end{array}
\]

in $\text{Lex}(\mathcal{C})$, where $\phi$ is the canonical morphism obtained by the universal property of the coend.

Proof. By Lemma 2.3, the morphism $\phi$ corresponds to $\int_{X \in C} \Hom_C(\text{id}_X, -) \cdot \varepsilon_X$ via the second isomorphism in (4.8), where $\varepsilon$ is the counit of $\lambda^{-1} \circ \text{soc}$. Observe that $\varepsilon$ is the inclusion $\text{soc} \hookrightarrow \text{id}_C$. Now the result follows from the naturality of (4.7). □

We define a right action $\diamond$ of $C^{op} \otimes C$ on $C$ by $V \diamond (X \boxtimes Y) = X^* \otimes V \otimes Y$. Since the base field $k$ is assumed to be algebraically closed, the action $\diamond$ is exact in each variable by [Del01, Proposition 5.13]. Hence, in particular, $V \diamond (-)$ preserves coends for all $V \in C$. So we define

\[
Z(V) = V \diamond \Phi(\text{id}_C) \quad \text{and} \quad Z^s(V) = V \diamond \Phi(\text{soc})
\]

for $V \in C$. By (4.7) and (4.8), we have, symbolically,

\[
Z(V) = \int_{X \in \mathcal{C}} X^* \otimes V \otimes X \quad \text{and} \quad Z^s(V) = \int_{X \in \mathcal{S}} X^* \otimes V \otimes X.
\]

The above argument not only proves that the above coends exist, but also clarifies where they come from. Here is an application: For each $V \in C$, there is a canonical morphism $\phi_V : Z^s(V) \to Z(V)$ obtained by the universal property of $Z^s(V)$ as a coend. By Lemma 4.7, we have

\[
\phi_V = \text{id}_V \diamond \Phi(i),
\]

where $i : \text{soc} \hookrightarrow \text{id}_C$ is the inclusion. Obviously, $i$ is a monic. Since $V \diamond (-)$ is exact, the morphism $\phi_V$ is also monic.

Proof of Lemma 4.8. If we realize ends $A$ and $A^{ss}$ by $A = Z(1)^*$ and $A^{ss} = Z^s(1)^*$, respectively, then we have $q = \phi_1^*$. Thus $q$ is epic as a dual of a monomorphism. □

4.4. Proof of Theorem 4.1. Let $A$ and $A^{ss}$ be as in Lemma 4.6 and let

\[
\begin{array}{ll}
\pi_X : A \to X \otimes X^* \quad (X \in \mathcal{C}) & \quad \text{and} \quad \pi_X^{ss} : A^{ss} \to X \otimes X^* \quad (X \in \mathcal{S})
\end{array}
\]

be the universal dinatural transformations. By definition, $q : A \to A^{ss}$ is the unique morphism in $\mathcal{C}$ such that $\pi_X^{ss} \circ q = \pi_X$ for all $X \in \mathcal{S}$.

Now let $\{V_1, \ldots, V_m\}$ be a complete set of representatives of isomorphism classes of simple objects. Then there exists an isomorphism $r : A^{ss} \cong \bigoplus_{i=1}^m V_i \otimes V_i^*$ of objects such that $\pi_X^{ss} : A^{ss} \to X \otimes X^*$ corresponds to the projection via $r$ if $X$ is one of $V_1, \ldots, V_m$ [KL01, §5.1.3]. Thus the internal character of $V_i$ is given by the composition

\[
\begin{array}{cccc}
\text{ch}(V_i) : A & \xrightarrow{q} & A^{ss} & \xrightarrow{r} \bigoplus_{i=1}^m V_i \otimes V_i^* & \xrightarrow{p_i} V_i \otimes V_i^* & \xrightarrow{\varepsilon_{V_i}} 1
\end{array}
\]

where $p_i$ is the projection. It is easy to see that the set $\{\varepsilon_{V_i} \circ p_i\}_{i=1}^m$ is linearly independent. Since $q$ is an epimorphism, the assertion (4.1) follows.
Remark 4.8. The character of the coend $F := \int^{X \in \mathcal{C}} X^* \boxtimes X \in \mathcal{C} \boxtimes \mathcal{C}$ has been considered in [FSS13b, FSS13a]. Our argument in this section is useful to compute the character of $F$. Let $\{V_1, \ldots, V_m\}$ be a complete set of representatives of isomorphism classes of simple objects of $\mathcal{C}$. Since the base field $k$ is algebraically closed, the set $\{V_i \boxtimes V_j\}_{i,j=1,\ldots,m}$ is a complete set of representatives of isomorphism classes of simple objects of $\mathcal{C} \boxtimes \mathcal{C}$. Hence we have an isomorphism

$$\text{Gr}_k(\mathcal{C}) \otimes_k \text{Gr}(\mathcal{C}) \cong \text{Gr}_k(\mathcal{C} \boxtimes \mathcal{C}), \quad [V] \otimes [W] \leftrightarrow [V \boxtimes W].$$

Let $P_i$ be a projective cover of $V_i$. Then we have

$$F = \sum_{i,j=1}^m N_{ij} [V_i] \otimes [V_j]$$

in $\text{Gr}_k(\mathcal{C})^{\otimes 2}$, where $N_{ij} = \dim_k \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}}(P_i \boxtimes P_j, F)$. To determine $N_{ij}$’s, we consider the functor

$$\Psi : \mathcal{C} \boxtimes \mathcal{C} \xrightarrow{(-) \boxtimes \text{id}_\mathcal{C}} \mathcal{C}^{\text{op}} \boxtimes \mathcal{C} \xrightarrow{\Phi} \text{LEX}(\mathcal{C}),$$

where $\Phi$ is the equivalence used in §4.3. For $V, W \in \mathcal{C}$, we compute

$$\text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}}(V \boxtimes W, F) \cong \text{Nat}(\Psi(V \boxtimes W), \Psi(F))$$

$$\cong \text{Nat}(\text{Hom}_\mathcal{C}(*V, -) \cdot W, \text{id}_\mathcal{C})$$

$$\cong \int_{X \in \mathcal{C}} \text{Hom}_\mathcal{C}(*V, X) \cdot W, X)$$

$$\cong \int_{X \in \mathcal{C}} \text{Hom}_k(\text{Hom}(*V, X), \text{Hom}(W, X))$$

$$\cong \text{Hom}_k(W, *V) \quad \text{(by the Yoneda lemma)}$$

$$\cong \text{Hom}_\mathcal{C}(V, W^*),$$

where we have used the equation

$$\text{Nat}(F, G) = \int_{X \in A} \text{Hom}_B(F(X), G(X))$$

for functors $F, G : A \to B$. Hence, $N_{ij} = \dim_k \text{Hom}_\mathcal{C}(P_i, P_j^*)$.

5. Applications to fusion categories

5.1. Fusion categories. Throughout this section, we work over an algebraically closed field $k$. A fusion category [ENO05] is a semisimple finite tensor category. We give some applications of our results to the theory of fusion categories.

5.2. The commutativity of the Grothendieck algebra. Let $\mathcal{C}$ be a spherical fusion category, and let $R$ be a right adjoint of the forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. By the results of Sections 3 and 4, we have isomorphisms

$$\text{Gr}_k(\mathcal{C}) \cong \text{CF}(\mathcal{C}) \cong \text{End}_{\mathcal{Z}(\mathcal{C})}(R(\mathbb{I}))$$

of algebras.

Now let $\{V_1, \ldots, V_m\}$ be a complete set of representatives of isomorphism classes of simple objects of $\mathcal{C}$. The dimension of $\mathcal{C}$ is defined by

$$\dim(\mathcal{C}) = \text{tr}(\text{id}_{V_1})^2 + \cdots + \text{tr}(\text{id}_{V_m})^2,$$

where $\text{tr}$ is the pivotal trace.
Theorem 5.1. The following assertions are equivalent:

1. $C$ is non-degenerate, i.e., $\dim(C) \neq 0$.
2. The Grothendieck algebra $\text{Gr}_k(C)$ is semisimple.

Proof. The implication (2) $\Rightarrow$ (1) is proved in [Ost15, Proposition 2.9]. The converse is conjectured in [Ost15, Remark 2.10] and is proved as follows: Suppose that $C$ is non-degenerate. Since $\dim(C) \neq 0$, $\mathcal{Z}(C)$ is semisimple [BV13]. Hence, in particular, the algebra $\text{End}_{\mathcal{Z}(C)}(R(1))$ is semisimple. Now the result follows from (5.1). $\square$

5.3. Gelfand pairs. Andruskiewitsch and Natale [AN00] introduced the Hecke algebra $H(A, B)$ for a pair $(A, B)$ of a semisimple Hopf algebra $A$ and its Hopf subalgebra $B$. The pair $(A, B)$ is called a Gelfand pair if the algebra $H(A, B)$ is commutative. As shown in [AN00], the pair $(A, B)$ is a Gelfand pair if and only if the left $A$-module $A \otimes_B 1_B$ is multiplicity-free, where $1_B$ is the trivial $B$-module.

For a semisimple Hopf algebra $A$, they also showed that the pair $(D(A), A)$ is a Gelfand pair if and only if $A$ is quasi-cocommutative, i.e., the Grothendieck ring of $A$-mod is commutative. Here, $D(A)$ is the Drinfeld double. The following result can be thought of as a category-theoretical generalization of their result:

Theorem 5.2. Let $C$ be a spherical fusion category, and let $R$ be a right adjoint of the forgetful functor $U : \mathcal{Z}(C) \to C$. Then the following assertions are equivalent:

1. $R(1)$ is multiplicity-free.
2. $\text{Gr}_k(C)$ is commutative.

Proof. As we have seen in the proof of Theorem 5.1, $\mathcal{Z}(C)$ is semisimple if this is the case. Thus $\text{End}_{\mathcal{Z}(C)}(R(1))$ is commutative if and only if $R(1)$ is multiplicity-free. The result follows from (5.1). $\square$

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