We study finite group actions that are parametrised by coloured graphs, as the basis of the graph calculus. In this setting, a derivative with respect to a certain graph yields its respective group action. The graph calculus is built on a suitable quotient of the monoid algebra $A[G]$ corresponding to a certain function space $A$ and to the free monoid $G$ in finitely many graph variables. The largest section is dedicated solely to these algebraic structures, which, although motivated by Tensor Field Theory (TFT), are introduced and dealt with without reference to it. These abstract results are subsequently applied to a TFT problem:

Tensor field theory focuses on quantum field theory aspects of random tensor models, a quantum-gravity-motivated generalisation of random matrix models. The correlation functions of complex tensor models have a rich combinatorial structure: they are classified by boundary graphs that describe the geometry of the boundary states. These graphs can be disconnected, although the correlation functions are themselves connected. In a recent work, the Schwinger-Dyson equations for an arbitrary albeit connected boundary were obtained. Here, we use a graph calculus—where derivatives of graphs yield group actions by their coloured automorphism—in order to report on the missing equations for correlation functions with disconnected boundary, thus completing the Schwinger-Dyson pyramid for quartic melonic ('pillow'-vertices) in arbitrary rank. We hope that the present result sheds light on the non-perturbative large-$N$ limit of tensor field theories. Moreover, we presume that it can be interesting if one addresses the solvability of the theory by using methods that generalise the topological recursion to the tensor field theory setting.

1. Introduction and motivation

The quest for laws of physics near the Planck scale leads some quantum gravitologist and quantum cosmologists to replace the smooth space-time paradigm with new geometrical structures that are suitable for said high-energy scale. Those new structures include discretisation of space-time (e.g. causal dynamical triangulations [AGJL13]), the algebraisation of space-time (e.g. noncommutative geometry [CC97; Mar18]), just to name some. Already the sole description of a space-time by a single mathematical object is expected to require therefore novel geometrical ideas.

However, if one adopts a path-integral approach, the exploration the quantum theory of space-time requires additionally a multi-geometry description, to which 'off-shell' geometries also contribute. Each of these geometries $\xi$ is weighted via $\exp(iS(\xi)/\hbar)d\xi$ by a 'classical' action $S(\xi)$, bounded to resemble the Einstein-Hilbert action in the classical limit in which $\xi$ starts looking like a Riemannian or Lorentzian manifold.

Random tensors [ADJ91; GR12; Riv16] offers precisely a built-in description of both random and discrete geometry in arbitrary dimensions, and is thus a tool to test models of background independent quantum gravity (e.g. [EKLP18]). Interest in the study of Euclidean quantum field theory (QFT) aspects of random tensors leads to tensor field theory (TFT) [BGS13; BGR13; SV14]. Usually TFT fits in a 'QFT + $\epsilon$' framework, that is to say a conservative modification of QFT, which one can pursue in the perturbative or non-perturbative approaches.
Non-perturbative TFT deals with the geometry of boundary states. Single geometries in TFT are represented by certain decorated graphs called \textit{coloured graphs}; this decoration is precisely the information that allows the construction of PL-manifolds from graphs. Bulk-geometry graphs —Feynman diagrams of TFT— have a colour more than the graphs that triangulate the boundary geometries, which are called therefore \textit{boundary graphs}. The two-fold purpose of this article is to define in abstract way a calculus with coloured boundary-graph variables, and shortly afterwards, to apply this construct to a particular problem in non-perturbative TFT.

The single-variable graph calculus has been used there as a toolkit for non-perturbative field tensor theory, leading to the Ward-Takahashi identity [Pér18]. The single-variable graph calculus allows to define each correlation function as a graph derivative of the free energy. Boundary graphs turn out to classify the correlation functions of tensor field theories; these obeys an analytic\textsuperscript{2} Schwinger-Dyson Equation (SDE).

Each and every analytic SDE for a \textit{connected} correlation function corresponding to \textit{connected}, but otherwise arbitrary, boundary graph was presented in [PPW17] in terms of a general formula that relates a given correlation function with its neighbourings (relative to the number of points) in terms of simple graph operations. In order to obtain these results, the single-variable graph calculus was enough. However, the derivation of SDE for connected correlation functions with disconnected boundary needs a multivariable graph calculus, the variables being the different boundary components.

We introduce graph-group actions as the basis of the multivariable calculus, and study their generating functionals. Concretely, we obtain formulae for the graph derivative of products of functionals, i.e. the corresponding Leibniz rule that generalises
\[
\partial^\alpha (F \cdot G) = \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} \partial^\gamma F \cdot \partial^{\alpha - \gamma} G \quad \text{(in multi-index notation)},
\]
when one replaces usual partial derivatives $\partial_\mu$ (say, with $x^\mu$ a coordinate of $\mathbb{R}^n$, and $F, G$ real-valued smooth functions there) by graph derivatives $\partial_g = \partial/\partial g$, for $g$ a graph. The formula for graphs takes a different form, but reduces to (1.1) when one replaces functionals with functions and simultaneously considers trivial group actions, as it should.

We prove that this abstract structure underlies tensor models functionals and use it to find a general formula for the SDE of the quartic ‘pillow’-model, for the \textit{connected} correlation functions with arbitrary \textit{disconnected} boundary. This is the missing piece that complements the connected-boundary SDE-pyramid obtained in [PPW17]. To have it complete is important for the analysis of the non-perturbative large-$N$ limit of tensor field theories. Moreover, although it is not clear which recursion should generalise the topological recursion [Eyn14], it is clear that the disconnected-boundary correlation functions play an important role\textsuperscript{3}.

This article is organised in six sections. The next one introduces the abstract concepts; in the author’s viewpoint, the results are more transparently proven in that framework than by directly working with tensor models structures. We explain in which sense and when, for two coloured graphs, $g$ and $h$,
\[
\frac{\partial g}{\partial h} = \delta(h, g) \cdot \text{group action of } G(g). \tag{1.2}
\]

\textsuperscript{2}We write ‘analytic’ as opposed to algebraic SDE for expectation values. We conceive tensor field theory as a discretisation (therefore, 0-dimensional) of a $D$-dimensional quantum field theory. The $2k$-point correlation functions are thus functions of $\mathbb{Z}^{k \times D} \rightarrow \mathbb{C}$ which in the continuum limit pass to functions $\mathbb{R}^{k \times D} \rightarrow \mathbb{C}$, and render the SDE integro-differential equations.

\textsuperscript{3}For instance, higher dimensional analogue of the ‘pair of pants’ being represented by a correlation function with three melonic boundary components.
Here $\delta(g, h) = 1$ if the graphs $g$ and $h$ are isomorphic, and $\delta(g, h)$ vanishes otherwise; moreover $G(g)$ is a group determined by $g$. In Section 3 we make the connection between the two first sections and see that indeed in tensor models graph-generated functionals of group actions appear. In Section 4, the model is detailed and the results of the previous sections are applied to the main problem, namely to find the SDE for connected correlation functions with arbitrary disconnected boundary with arbitrary number of connected components. Section 5 gives explicitly some of the SDE for 4, 6-point functions for rank-3 theories, before concluding and presenting an outlook (Sec. 6). The useful coefficients that encode the insertion of the 4-point functions into the 2-point function and the 6-point functions into the 4-point functions are given in the Appendix A.

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2. Graph calculus

In this section we explain what we mean by graph calculus. For practical reasons$^4$ we consider the empty graph $\emptyset$ as coloured and add it in the set of (possibly) disconnected, closed, regularly edge-$D$-coloured, vertex-bipartite graphs (‘$D$-coloured graphs’) $\text{Grph}^{\text{H,cl}}_D$, to form $G_D = \{\emptyset\} \cup \text{Grph}^{\text{H,cl}}_D$. Henceforth, all graphs are coloured, but other types of graphs (even non-graph objects) could be used for the next constructs.

$^4$It is not the first time one has to do so. For instance, see [KT16].
The nature of these graphs is not important at this point and examples will be presented in later sections. As the motivation—and last reference to TFT in this section—graphs treated here are not Feynman graphs, but boundary graphs of these in a rank-$D$ TFT.

2.1. Single variable graph calculus. We regard $G_D$ with a monoidal structure, the product and the unit being given by

$$c_1c_2 = c_1 \amalg c_2 \quad \text{and} \quad c \amalg \emptyset = c = \emptyset \amalg c,$$

respectively, for all $c, c_1, c_2 \in G_D$. We choose to remember the order of the factors, so this product is generally non-commutative.

**Definition 2.1** (System of graph-group actions). For a finite collection $H \subset G_D$, consider the following structures:

- for each connected graph $c \in H$:
  - a set $\mathcal{V}(c)$; a singleton is associated to the empty graph, $\mathcal{V}(\emptyset) = \{\ast\}$
  - a finite group $G(c)$ and
  - a group action $G(c) \curvearrowright \mathcal{V}(c)$ of $G(c)$ on $\mathcal{V}(c)$
- if $g = c_1c_2 \cdots c_n$ is a factorisation in connected components $c_i$, then $\mathcal{V}(g)$ satisfies\(^5\)

$$\mathcal{V}(g) \subset \mathcal{V}(c_1) \amalg \cdots \amalg \mathcal{V}(c_n)$$

The collection $\{\mathcal{V}(g), G(g)\}_{g \in H}$ is a system of graph-group actions. Moreover, if

- for each graph in $g \in H$, one has functions

$$u_g : \mathcal{V}(g) \rightarrow \mathbb{C} \text{ (or } \mathbb{R}),$$

additionally, one says that $\{u_g, \mathcal{V}(g), G(g)\}_{g \in H}$, or more succinctly, $\{u_g\}_{g \in H}$ is a family of functions supported on the system of graph-group actions $\{\mathcal{V}(g), G(g)\}_{g \in H}$.

We are interested in triples $\{u_g, \mathcal{V}(g), G(g)\}_{g \in H}$ and formal sums of the type $U = \sum_{h \in H} u_h h$ which we refer to as their generating functionals. In this case we say that $H$ spans $U$. Why these are functionals instead of functions will become apparent by addressing the applications. At this point also the following terminology, inherited from the physical significance, might seem mysterious: we call the elements of $\mathcal{V}(c_i)$ the momenta of the graph $c_i$, (notion extended to the elements $\mathcal{V}(g)$ too). Notice that $u_\emptyset$ is a constant.

We are motivated by a tensor model context, in which functions $u_g$ are unknown\(^6\), and one derives equations that they should satisfy. Only after knowing solutions we would be able to fix a function space $u_g$ should belong to, which is for now unspecified. We vaguely refer then to them as ‘functions’.

Next, some words on notation. For $g = c_1 \cdots c_n$ a factorisation in connected graphs $c_j$, we let $g/c_i$ be the graph with the $i$-th connected component deleted,

$$g/c_i := c_1 \cdots \hat{c}_i \cdots c_n = c_1 \cdots c_{i-1}c_{i+1} \cdots c_n.$$  

Notice that this deletion does not care about the graph-class, but only about its spot in the factorisation, which we can keep track of thanks to the monoidal structure of $G_D$.

Let $Y \in \mathcal{V}(c_r)$, $1 \leq r \leq n$. We define the insertion $\iota_Y$ as the evaluation of the $r$-th argument of $v_g : \mathcal{V}(g) \rightarrow \mathbb{C}$ at $Y$, that is

$$(\iota_Y v_g)(X_1, \ldots, X_{r-1}, X_{r+1}, \ldots, X_n) = v_g(X_1, \ldots, X_{r-1}, Y, X_{r+1}, \ldots, X_n),$$

where $X_i \in \mathcal{V}(g) \cap \mathcal{V}(c_i)$ for $i \neq r$.

\(^5\)One could strengthen this condition so that $\mathcal{V}(g) = \mathcal{V}(c_1) \amalg \cdots \amalg \mathcal{V}(c_n)$ for sake of naturality. However, this would rule out an important application.

\(^6\)For instance, $u_g$ can be the correlation functions.
Definition 2.2. Let $H \subseteq G_D$ span the functional $U = \sum g u_g g$. Given any connected graph $h \in G_D$, $h \neq \emptyset$, and an arbitrary graph $g = c_1 \cdots c_n$ factorised in connected components $c_i$, we define $I(g, h)$ as the set of spots in $g$ at which $h$ occurs, i.e. $I(g, h) = \{ i \in [1, n] | h = c_i \}$. For $r \in I(g, h)$, we label by $h^{(r)}$ the $r$-th appearance of $h$ in the factorisation $g = c_1 \cdots c_n$. We define the functional graph derivative with respect to $h$ (evaluated at $X$) as the functional

\[
\frac{\delta U}{\delta h(X)} = \sum_{g \in H} \sum_{r \in I(g, h)} \sum_{\sigma \in G(h)} t^r_{\sigma(X)} u_g (g/h^{(r)}), \quad X \in \mathcal{V}(h).
\]

If $h$ does not occur as a factor in $g$, the sum is empty, and thus $\delta U/\delta h \equiv 0$. The derivative with respect to $\emptyset$ is the coefficient of that graph, $\delta U/\delta \emptyset = u_\emptyset \in \mathbb{C}$.

To clarify these elements, consider a monomial functional, $U = (u_h^n)(h^n)$, for some integer $n \geq 1$ and a connected graph $h$. Then one has

\[
\frac{\delta U}{\delta h(X)} = \sum_{r=1}^n \sum_{\sigma \in G(h)} (t^r_{\sigma(X)} u_{h^n}) h^{n-1},
\]

which one can rethink as

\[
\frac{\delta (h^n)}{\delta h} = \left( \frac{\delta h}{\delta h} h^{n-1} + h \frac{\delta h}{\delta h} h^{n-2} + \ldots + h^{n-1} \frac{\delta h}{\delta h} \right) = \left( G(h) h^{n-1} + h G(h) h^{n-2} + \ldots + h^{n-1} G(h) \right),
\]

if

\[
\left( \frac{\delta h}{\delta h} u_h \right)(X) = \sum_{\sigma \in G(h)} (\sigma \cdot u_h)(X) = \sum_{\sigma \in G(h)} u_h(\sigma X).
\]

If none of the $c_j$ is isomorphic to $h$, then

\[
\left( \frac{\delta h}{\delta h} u_{c_1 \cdots c_{r-1} h c_{r+1} \cdots} \right)(X) = \sum_{\sigma \in G(h)} t^r_{\sigma(X)} u_{c_1 \cdots c_{r-1} h c_{r+1} \cdots} .
\]

On this account, the useful symbolism to keep in mind is that the derivative of a graph with respect to itself is the group action of $G(h)$ on $\mathcal{V}(h)$,

\[
\frac{\delta h}{\delta h} = G(h) \lhd \{ \text{functions } \mathcal{V}(h) \to \mathbb{C} \}.
\]

In the sequel, we will often abuse on notation and write this equality without the curved action-arrow, as we already did above in eq. (2.1). One can calculate iterated derivatives, too. That is, for $n \geq 2$, it is clear that the iteration of $n$ graph derivatives applied to $h^n$ yields

\[
\frac{\delta^n (h^n)}{\delta h \delta h \cdots \delta h} (X_1, \ldots, X_n) = \sum_{\mu \in \mathfrak{S}(n)} \sum_{\{\sigma_1, \ldots, \sigma_n\} \in G(h)^n} t^1_{\sigma_1(X_1)} \cdots t^n_{\sigma_n(X_n)},
\]

If $G_i(h)$ is the $i$-th factor of the group $G(h)^n$, a more transparent notation of last equation is

\[
\frac{\delta^n (h^n)}{\delta h \delta h \cdots \delta h} = \sum_{\mu \in \mathfrak{S}(n)} G_{\mu(1)}(h) G_{\mu(2)}(h) \cdots G_{\mu(n)}(h),
\]

where the group $G_{\mu(i)}(h)$ acts on the $i$-th factor of the set $\mathcal{V}(h) \sqcup \cdots \sqcup \mathcal{V}(h)$. The group corresponding to the $n$-th derivative of the $n$-th power of a graph $h$ with respect to itself is

\[
\frac{\delta^n (h^n)}{\delta h \delta h \cdots \delta h} = G(h) \lhd \mathfrak{S}(n).
\]

In this case, the wreath product $G(h) \lhd \mathfrak{S}(n)$ is the semi-direct product $G(h)^n \rtimes_{\psi} \mathfrak{S}(n)$, with the obvious action $\psi$ of $\mathfrak{S}(n)$ on the $n$ copies of $G(h)$.
To give further detail, given a generating system of graph-group actions \{\mathcal{V}(g), G(g)\}_{g \in H} and \( h \in H \), consider a function \( F : \mathcal{V}(h) \amalg \amalg \mathcal{V}(h) \to \mathbb{C} \). An element \( \Omega = (\sigma; \mu) = (\sigma_1, \ldots, \sigma_n; \mu) \) of the group in eq. (2.6) acts as follows:

\[
(\Omega \cdot F)(X_1, \ldots, X_n) = F(\sigma_1(X_{\mu(1)}), \ldots, \sigma_n(X_{\mu(n)})). \tag{2.8}
\]

By departing from eq. (2.8), the composition with another element \( \Xi = (\tau, \nu) \) in the group (2.6) is easily proven to yield \( \Xi \circ \Omega = (\tau \psi_{\nu}(\sigma); \nu \mu) \) where

\[
\psi : \mathfrak{S}(n) \to \text{Aut}(G(h)^n) \quad \mu \mapsto [\psi_{\mu} : (\sigma_i)_{i=1}^n \mapsto (\sigma_{\mu(i)})_{i=1}^n],
\]

which is the product of \( G(h)^n \times_{\psi} \mathfrak{S}(n) \), as claimed.

### 2.2. Examples of graph-group action systems.

Roughly stated, a multivariable graph calculus (of graph variables) consists of generating functionals of systems of graph-group actions \{u_g, \mathcal{V}(g), G(g)\}_{g \in H} that are spanned by finite subsets \( H \) of the free monoid \( \text{FM}([h_1, \ldots, h_n]) \) generated\(^7\) by \( n \) non-isomorphic graphs \( \mathfrak{g} = \{h_1, \ldots, h_n\} \). For a multivariable graph calculus the key property is that the graph-group actions \( G(h_i) \) are pairwise independent, that is for each \( i, j = 1, \ldots, n \),

\[
\frac{\delta h_i}{\delta h_j} = \delta_j^i G(h_i), \quad h_i, h_j \in \mathfrak{g}. \tag{2.9}
\]

For the special element \( g = h_1^{a_1} \cdots h_n^{a_n} \) the restriction imposed by eq. (2.9) implies

\[
\frac{\delta g}{\delta g} = G(g) = G(h_1) \wr \mathfrak{S}(a_1) \times G(h_2) \wr \mathfrak{S}(a_2) \times \cdots \times G(h_n) \wr \mathfrak{S}(a_n). \tag{2.10}
\]

Before formally defining multivariable graph calculus, the next examples are just meant to illustrate last action (2.10), rather than the role of the graphs in graph-generated actions.

**Example 2.3.** Let \( \zeta \neq 1 \) denote a \( n \)-th root of unit \( (n \geq 2) \), and consider the system of graph-group actions with a single graph \( g \). Let \( \zeta_n \) span the group \( G(g) \) which we let act on \( \mathcal{V}(g) = \mathbb{C} \) by multiplication. Then the functional graph derivative of \( g \) with respect to itself on the identity \( \text{id}_\mathbb{C} \) vanishes identically:

\[
\left( \frac{\delta g}{\delta g} \right) \text{id}_\mathbb{C} = 0.
\]

The \( G(g) \)-orbit of the function \( f_n : z \mapsto z^n \) yields

\[
\left( \frac{\delta g}{\delta g(z)} \right) (f_n) = n \cdot z^n.
\]

**Example 2.4.** Consider a finite set \( H \subset \text{FM}(\mathfrak{g}) \) and the following graph-group actions system

\[
G(g) = \mathfrak{S}(|g^0|), \quad \mathcal{V}(g) = M_{|g^0| \times |g^0|}(\mathbb{R}).
\]

Here \( g^0 \) is the vertex set of \( g \). The action of the symmetric group on the matrices permutes columns (or rows). Then the orbit of the determinant \( \det : \mathcal{V}(g) \to \mathbb{R} \) vanishes identically.

This follows from considering, for an arbitrary matrix \( X = (X_{ab}) \in \mathcal{V}(g) \),

\[
\left( \frac{\delta g}{\delta g(X)} \right) \det(\bullet) = \sum_{\sigma \in \mathfrak{S}(|g^0|)} \det(X_{\sigma(a)b})
= \sum_{\sigma \in A} \det(X_{\sigma(a)b}) + \sum_{\sigma \in \mathfrak{S}(|g^0|) - A} \det(X_{\sigma(a)b})
= \sum_{\sigma \in A} \det(X_{ab}) - \sum_{\sigma \in \mathfrak{S}(|g^0|) - A} \det(X_{ab}) = 0,
\]

\(^7\)We recall that the free monoid generated by \( \mathfrak{g} = \{h_1, \ldots, h_n\} \) is in this case the following set \( \text{FM}(\mathfrak{g}) = \{l_1 \cdots l_m : m \in \mathbb{Z}_{\geq 0} \text{ and } l_i \in \mathfrak{g}\} \) endowed with the concatenation operation; containing the empty graph, i.e. the empty word.
where $A$ is the alternating subgroup; its complement in the symmetric group consists of odd-degree permutations, whence the common minus sign in the last line. Both have the same order, which explains why the sum vanishes independent from $X$.

**Example 2.5.** Let $K$ be a finite group that accepts faithful irreducible representations. Consider $n$ of them $\pi^i : K \to \text{End}(W_i)$, and set $G(h_i) = K$ for each $i = 1, \ldots, n$. Define $\mathcal{V}(h_i) = \pi^i(K)$ as the set of matrices that correspond to of $K$. The group $K$ acts on $\mathcal{V}(h_i)$ by

$$\mathcal{V}(h_i) \ni \pi^i(m) \mapsto \pi^i(u)\pi^i(m) = \pi^i(um), \quad (m, u \in K).$$

Consider the following function defined in terms of the characters $\chi^i(u) = \text{Tr}(\pi^i(u))$, $u \in K$,

$$f^i : \mathcal{V}(h_i) \to \mathbb{C}, \quad f^i(X_i) = \chi^i(m)^*\chi^i(m), \quad (X_i = \pi^i(m)).$$

Then for $X_i = \pi^i(m)$, the following holds:

$$\left( \frac{\delta h_i}{\delta h_i(X)} \right) f^i = \sum_{u \in K} (u \cdot f^i)(\pi^i(m)) = \sum_{u \in K} \chi^i(um)^*\chi^i(um) = \sum_{u \in K} \chi^i(u)^*\chi^i(u) = |K|.$$

Fix any $g \in \mathfrak{h}$ and let $\pi$ be the associated representation. Define for any $h_i, h_j \in \mathfrak{h}$, and $X = \pi(m)$,

$$F^{ij} : \mathcal{V}(g) \to \mathbb{C}, \quad F^{ij}(X) = \chi^i(m)^*\chi^j(m).$$

For $X = \pi(m)$,

$$\left( \frac{\delta g}{\delta g(X)} \right) F^{ij} = \sum_{u \in K} \chi^i(um)^*\chi^j(um) = \sum_{u' \in K} \chi^i(u')^*\chi^j(u') = \delta_{ij}|K|.$$

Last equality is due to Schur orthogonality.

**Example 2.6.** Let $D \in \mathbb{Z}_{\geq 1}$. Let two non-isomorphic graphs $H = \{g, h\} \subset G_D$ parametrise the system of graph-group actions given by $\{\mathcal{V}(l), G(l)\}_{l \in H}$, being

$$\begin{align*}
\mu &\in G(g) = \mathfrak{S}(D) \quad Z = (z_i), i \in \mathcal{V}(g) = \mathbb{C}^D \quad \mu : (z_i) \mapsto (z_{\mu(i)}) \\
\tau &\in G(h) = \mathbb{Z}_2 \quad \varepsilon \in \mathcal{V}(h) = \mathbb{Z}_2 \quad \varepsilon : s \mapsto \varepsilon s
\end{align*}$$

with $\mathbb{Z}_2$ written multiplicatively $\{-1, 1\}$. Let $F : \mathcal{V}(g) \amalg \mathcal{V}(g) \amalg \mathcal{V}(h) \to \mathbb{C}$ be given by, say,

$$F(Z, W, \varepsilon) = \frac{\varepsilon}{2(D)^2} e^{-|W|+|Z|} =: c(D) \cdot \varepsilon \cdot e^{-|W|+|Z|}.$$

Then the functional graph derivative of $g^2f$ with respect to itself yields the following group-orbit, when applied to $F$:

$$\frac{\delta^3(g^2f)}{\delta g(Z_1) \delta g(Z_2) \delta h(\varepsilon)}$$

$$= \left( \frac{\delta^3 ggh}{\delta g \delta g \delta h} \right) (Z_1, Z_2, \varepsilon)$$

$$= \left( \left[ G(g) i \mathfrak{S}(2) \times G(h) \right] \cdot F \right) (Z_1, Z_2, \varepsilon)$$

$$= \sum_{\mu \in \mathfrak{S}(2)} \sum_{\tau \in \mathbb{Z}_2} \sum_{\sigma \in \mathfrak{S}(D)} \sum_{\rho \in \mathfrak{S}(D)} F(\sigma(Z_{\mu(1)}), \rho(Z_{\mu(2)}), \tau \varepsilon)$$

$$= \sum_{\tau \in \mathbb{Z}_2} \sum_{\sigma \in \mathfrak{S}(D)} \sum_{\rho \in \mathfrak{S}(D)} \left\{ F(\sigma(Z_1), \rho(Z_2), \tau \varepsilon) + F(\sigma(Z_2), \rho(Z_1), \tau \varepsilon) \right\}$$

$$= 2c(D) \cdot (D!)^2 \cdot \varepsilon \cdot (z_1|Z_1| - |Z_2| + z_1|Z_2| - |Z_1|)$$

$$+ e^{-|Z_1| - |Z_2|} - e^{-|Z_2| - |Z_1|}$$

$$= \varepsilon (e^{-|Z_1|} \sinh |Z_2| + e^{-|Z_2|} \sinh |Z_1|).$$
We have used the invariance under the action of two copies $\mathcal{G}(D)$, which contributed a factor $(D!)^2$.

2.3. Multivariable graph calculus. Let $\mathfrak{h} = \{h_1, \ldots, h_n\} \subset \mathcal{G}_D$ be a set of non-isomorphic graphs. For the basis of the multivariable calculus the free monoid $\mathcal{F}M(\{h_1, \ldots, h_n\})$ is too 'verbose', and not each of its elements has the ordered form $h_1^{n_1} \cdots h_n^{n_n}$. This could in principle be solved by taking the free commutative monoid $\mathcal{F}M^{ab}(\mathfrak{h})$ instead, which, however, turns out to be overly restrictive (for our aims). A mild compromise between these two alternatives — the free monoid and its abelianisation — is to allow to permute letters in an arbitrary word, as to make use of the action (2.10), and then in some sense undo the changes. Next definition introduces precisely such reordering.

**Definition 2.7.** Given a finite set of graphs $\mathfrak{h} = \{h_1, \ldots, h_n\}$, the degree $|g|$ of an element $g$ in $\mathcal{F}M(\mathfrak{h})$ is the number of factors of $g$, i.e. the number of connected components $g$ consists of. We let $\mathcal{G}(|g|)$ act by permuting the factors of $g$, $g \mapsto \sigma(g)$; notice that $\mathcal{G}(|g|)$ left-acts naturally as $\sigma \cdot f = f \circ \sigma^{-1}$ on functions $f : \mathcal{V}(g) \to \mathbb{C}$. Given a family of functions $\{u_g\}_{g \in H}$ supported on a system of graph-group actions $\{\mathcal{V}(g), G(h)\}_{g \in H \subset \mathcal{F}M(\mathfrak{h})}$, and given a $g \in H$, we declare the pairs $(u_g, g) \sim (\sigma \cdot u_g, \sigma(g))$ equivalent for each $\sigma \in \mathcal{G}(|g|)$. The notation we choose for this equivalence, called *reordering*, is

$$u_g g \sim u_h h \quad \text{if and only if} \quad u_h = \sigma \cdot u_g \text{ and } h = \sigma(g) \text{ for certain } \sigma \in \mathcal{G}(|g|).$$  

(2.11)

**Definition 2.8.** Given a finite set of graphs $\mathfrak{h} = \{h_1, \ldots, h_n\} \subset \mathcal{G}_D$, a system of graph-group actions $\mathcal{S} = \{\mathcal{V}(h), G(h)\}_{h \in \mathfrak{h}}$ is said to be independent if eq. (2.9) holds. When the context is clear, we just say that ‘$\mathfrak{h}$ is independent’, or that $\mathcal{S}$ is.

**Definition 2.9.** A multivariable graph calculus $\mathcal{C}(\mathfrak{h})$ or a graph calculus with variables $\mathfrak{h} = \{h_1, \ldots, h_n\} \subset \mathcal{G}_D$ consists in two objects:

- the choice of an independent system of graph-group actions $\{\mathcal{V}(h), G(h)\}$ for $\mathfrak{h}$ and
- the set of finite formal sums in elements of $g \in \mathcal{F}M(\mathfrak{h})$ having each of these as coefficient a function of the form $v_g : \mathcal{V}(g) \to \mathbb{C}$, modulo reordering. That is,

$$\mathcal{C}(\mathfrak{h}) = \{V = \sum_g v_g g \mid v_g \equiv 0 \text{ for almost all } g \in \mathcal{F}M(\mathfrak{h})\} \sim$$

(2.12)

where $\sim$ is the linear extension of relation (2.11), abusing on the same symbol.

2.4. Algebraic structure. We now explore the structure of a graph calculus $\mathcal{C}(\mathfrak{h})$ with variables $\mathfrak{h} = \{h_1, \ldots, h_n\}$. The elements of $\mathcal{C}(\mathfrak{h})$, called also functionals, have a non-unique representation, since $\sum_g v_g g = \sum_{\tilde{g}} v_{\tilde{g}} \tilde{g}$ where $\tilde{g} = \tau_g(g)$ and $v_{\tilde{g}} = \tau_g(v_g)$ for an arbitrary $\tau_g \in \mathcal{G}(|g|)$. For sake of computability, it will be helpful to be able to fix representing elements $g$ that span a functional, and subordinate the order of the arguments of the coefficient-functions to that choice.

We write $g \sim h$ for any $g, h \in \mathcal{F}M(\mathfrak{h})$ if $g = h$ in the free commutative monoid $\mathcal{F}M^{ab}(\mathfrak{h})$ spanned by $\mathfrak{h}$. In other words, $g \sim h$ if and only if $h$ and $g$ match in $\mathcal{F}M(\mathfrak{h})$ up to a rearranging $\sigma \in \mathcal{G}(|g|)$.

**Definition 2.10.** Given $\{v_l, \mathcal{V}(l), G(l)\}_{l \in H}$, a system of graph-group actions with $g, h \in H \subset \mathcal{F}M(\mathfrak{h})$, suppose $h \sim g$. We define for a function $v_h : \mathcal{V}(h) \to \mathbb{C}$

$$\langle v_h \rangle_g = \sigma \cdot v_g, \text{ if } \sigma(g) = h \text{ as elements of } \mathcal{F}M(\mathfrak{h}),$$

being $\sigma$ the rearranging element $\sigma \in \mathcal{G}(|g|) = \mathcal{G}(|h|)$.

We shall drop the subindex $g$ in $\langle \cdot \rangle_g$ when the context is clear. If one factors $g$ as $g_1 g_2$ with respect to a ‘abelianised’ product, an element $\sigma \in \mathcal{G}(|g|)$ serves as correction, so that
σ(g_1g_2) = g. Their rearranging yields \( \langle u_{g_1} t_{g_2} \rangle_g = \sigma \cdot (u_{g_1} t_{g_2}) \) for suitably chosen functions \( u_{g_1} \) and \( t_{g_2} \). In general, if a collection of graphs \( g_1, \ldots, g_r \) are not required to be connected. If the context is clear, we pick this rearranging element in a smaller group \( \sigma \in \mathfrak{S}(r) \) that only permutes the arguments \( \nu \ast(g_i) \).

**Definition 2.11.** Denote by \( H_1H_2 \) the subset \( \{g_1g_2 \mid g_a \in H_a \} \) in the free commutative monoid \( \text{FM}^{ab}(h) \) spanned by an independent set of graphs \( h \). Given two functionals in \( \mathcal{C}(h) \), \( U = \sum_{h \in H_1} u_h h \) and \( T = \sum_{h \in H_2} t_h h \), we define their product \( V = (U \cdot T) \) as the functional

\[
V = U \cdot T = \sum_{g \in H_1H_2 \subset \text{FM}^{ab}(h)} v_g g,
\]

whose coefficients \( v_g \) are given by the ‘ordered convolution’

\[
v_g = \sum_{(g_1, g_2) \in \cup H_1 \cap H_2 \setminus \cup \gamma g} \langle u_{g_1} t_{g_2} \rangle_g.
\]

(2.13)

**Lemma 2.12.** This product on \( \mathcal{C}(h) \) is commutative.

**Proof.** Let \( U = \sum_{j \in J} u_j j \) and \( T = \sum_{l \in L} t_l l \) be in \( \mathcal{C}(h) \). Given \( g \in \text{FM}(h) \) with \( g \sim jl \) for some \( j \in J \) and some \( l \in L \) we show that the components \( v_g \) of \( V = U \cdot T \) and \( \tilde{v}_g \) of \( \tilde{V} = T \cdot U \) satisfy \( v_g g = \tilde{v}_g g \) in \( \mathcal{C}(h) \). In order to show that \( \langle u_j t_l \rangle_g g = \langle t_l u_j \rangle_g \nu(g) \), it suffices to exhibit an element \( \nu \in \mathfrak{S}(|g|) \) that satisfies \( \langle u_j t_l \rangle_g = (t_l u_j) \nu(g) \). This \( \nu \) will be next constructed.

We have the freedom to assume that \( g = h_1^{\alpha_1} \cdots h_n^{\alpha_n} \). Since \( j, l \in \text{FM}(h) \),

\[
j \sim h_1^{\theta_1} h_2^{\theta_2} \cdots h_n^{\theta_n} \quad \text{and} \quad l \sim h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n}
\]

(2.14)

for some \( 0 \leq \lambda_i, \theta_i \leq \alpha_i \) that satisfy \( \alpha_i = \theta_i + \lambda_i, \ i = 1, \ldots, n \). In the notation introduced above, \( I(j, h_i) = \theta_i \) and \( I(l, h_i) = \lambda_i \). We begin assuming that the relations above are equalities,

\[
j = h_1^{\theta_1} h_2^{\theta_2} \cdots h_n^{\theta_n} \quad \text{and} \quad l = h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n},
\]

(2.14’)

and restore towards the end the more general form (2.14). Let \( |j| = \theta_1 + \ldots + \theta_n \) and \( |l| = \lambda_1 + \ldots + \lambda_n \) be the orders of \( j \) and \( l \). We define first \( \sigma \in \mathfrak{S}(|g|) \) as the \((|j|, |l|)\)-shuffle determined by

\[
\sigma = h_1^{\theta_1} h_2^{\theta_2} \cdots h_n^{\theta_n} \quad \text{and} \quad l = h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n},
\]

and \( \in \mathfrak{S}(|g|) \)

Each \( h_i \) before the bars is a factor of \( j \); after the double bar in the first row, the \( h_i \)'s represent the factors of \( l \). The lower row are the factors of \( g \). Thus the diagram states that \( \sigma(jl) = g \). Analogously, we can define a \((|l|, |j|)\)-shuffle \( \rho \) that satisfies \( \rho(lj) = g \). This is depicted in the following diagram, in which we represent \( l \) to the left the double bar and \( j \) to the right.
One has $\rho(lj) = g = \sigma(jl)$ but this still does not guarantee that $\rho^{-1}\sigma(vjl) = \hat{v}_{lj}$. In order to correct this, we define certain permutations that are constant everywhere except in the elements pertaining a particular $h_i$ for fixed $i$. This embeds $\mathcal{G}(|I(g; h_i)|) \subset \mathcal{G}(|g|)$. Notice first that for each such element $\mu \in \mathcal{G}(|I(g; h_i)|)$

$$w_\gamma g = (\mu \cdot w_\gamma) \mu(g) = (\mu \cdot w_\gamma) g$$

holds for any $w_\gamma : \mathcal{Y}(g) \to \mathbb{C}$. For each $i = 1, \ldots, n$, define $\tau_i$ as certain permutation $\beta_i$ (given below) in the the range $[\alpha_{i-1} + 1, \alpha_i]$ and constant outside it:

$$\tau_i(x) = \begin{cases} \beta_i(x) & 0 < x - s_i < \alpha_i \\ x & \text{otherwise} \end{cases},$$

where $s_i := (\alpha_1 + \ldots + \alpha_{i-1})$

$$\beta_i(x) = \begin{cases} x + \lambda_i & s_i < x \leq s_i + \theta_i \\ x - \theta_i & \theta_i < x \leq \theta_i + \lambda_i = \alpha_i. \end{cases}$$

Then the sought-after $\nu$ is

$$\nu = \rho^{-1} \circ (\tau_1 \circ \cdots \circ \tau_n) \circ \sigma,$$

which by construction satisfies

$$(\nu \cdot v_{jl}) \nu(jl) = \hat{v}_{lj} l j.$$

(2.15)

The map $\nu$ is thus given by

We now come back to the general case in which $l$ and $j$ have the weaker form (2.14), instead of (2.14'). This means that there are permutations $\gamma \in \mathcal{G}(|j|) \subset \mathcal{G}(|j| + |l|)$ and $\delta \in \mathcal{G}(|l|) \subset \mathcal{G}(|j| + |l|)$ with

$$\gamma(j) = h_1^{\theta_1} h_2^{\theta_2} \cdots h_n^{\theta_n} \quad \text{and} \quad \delta(l) = h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n}$$

(2.16)
Then we correct \( \nu \) by these two elements:

\[
\nu = \rho^{-1} \circ (\tau_1 \circ \cdots \circ \tau_n) \circ \sigma \circ (\gamma, \delta)
\]

which satisfies, in the most general case, eq. (2.15). The statement follows by linear extension of it. \( \square \)

**Example 2.13.** To illustrate this notation, consider the sets \( K = \{fg, f^2\} \) and \( H = \{fg, g^2\} \) of coloured graphs and let \( U = \sum_{e \in K} u_e e \) and \( T = \sum_{e \in H} u_e e \) be the corresponding generating functionals. If \( X_A \) are momenta of \( f \) and \( Z_A \) of \( g \) (for \( A = 1, 2 \)), picking a particular graph \( l = f^2g^2 \), and defining the following permutations in \( S(4) \)

\[
\sigma = (13)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \tau = (1)(23)(4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}
\]

the coefficient \( v_l \) is given by

\[
v_l(X_1, X_2, Z_1, Z_2) = \left((u_{fg}t_{fg})_l + (u_{g^2t_{f^2}})_l\right)(X_1, X_2, Z_1, Z_2)
\]

\[
= (\tau \cdot (u_{fg}t_{fg}))(X_1, X_2, Z_1, Z_2)
\]

\[
+ (\sigma \cdot (u_{g^2t_{f^2}}))(X_1, X_2, Z_1, Z_2)
\]

\[
= u_{fg}t_{fg}(X_1, Z_1, X_2, Z_2) + u_{g^2t_{f^2}}(Z_1, X_2, X_1, Z_2).
\]

Structures appearing in the graph calculus resemble the monoid ring. Given a commutative unit ring \( R \) and a monoid \( M \), the monoid ring [Lan05, Ch. II] is formed by formal finite sums in \( M \) with coefficients in \( R \), and endowed with the convolution product. The structure of the graph calculus generated by \( n \) variables \( \mathbf{h} = \{h_1, \ldots, h_n\} \) requires to define, instead of the ring \( R \), the commutative algebra \( \mathcal{A} \) of functions

\[
\mathcal{A} = \left\{ \prod_{c \in \text{FM}(\mathbf{h})} ^{\forall} (c) \to \mathbb{C} \right\}
\]

and then to consider the following restricted version of the monoid algebra

\[
\mathcal{A}[\text{FM}(\mathbf{h})] := \left\{ \sum_{g \in \text{FM}(\mathbf{h})} u_g g \mid u_g \in \mathcal{A}_g \text{ with } u_h = 0 \text{ for almost all } h \in \text{FM}(\mathbf{h}) \right\},
\]

being \( \mathcal{A}_g = \{f : \forall(g) \to \mathbb{C}\} \). Then we see that

\[
\mathcal{E}(\mathbf{h}) = \mathcal{A}[\text{FM}(\mathbf{h})] / \sim.
\]

Given functionals \( U = \sum_{h \in H} u_h h \), \( T = \sum_{t \in L} t_l l \) in \( \mathcal{E}(\mathbf{h}) \) one defines their sum by

\[
U + T = \sum_{m \in H \cup L} \left( \sum_{h \in H} (u_h)_m + \sum_{l \in L} (t_l)_m \right) m;
\]

for a scalar \( x \in \mathbb{C} \), the functional \( xU \) is defined by componentwise multiplication by \( x \),

\[
xU = \sum_{h \in H} (x \cdot u_h) h.
\]

The in-depth study of the structure of \( \mathcal{E}(\mathbf{h}) \) is beyond the scope of this paper. Here, we will prove the properties that are useful to us in later sections.

**Definition 2.14** (Coloured Borel transformation). Let \( V = \sum_{h \in H} v_h g \) be a generating functional of graph-group actions. Then define the coloured Borel transformation by

\[
B_c(V) := \sum_{g \in H \subset G_D} \frac{1}{|G(g)|} v_g g.
\]
If \( H \subset G_D \ni h \), the following set

\[ H_h := \{ g \in G_D \mid hg \not\sim j \text{ for some element } j \text{ of } H \}, \]

will be relevant for the next lemma.

**Lemma 2.15.** Consider the generating functional of a system of graph-group actions \( V = \sum_{g \in H} v_g g \) belonging to a graph calculus in variables \( h \subset G_D \). Suppose that the coefficients \( v_g \) satisfy

\[ v_{c_1 \cdots c_p} = (ab)^* v_{c_{p-1} \cdots c_0} \]

for each \( c_a, c_b \in \mathfrak{h} \), where \( a \) and \( b \) denote the number of factor (connected component) where \( c_a \) and \( c_b \) are located. Moreover, assume that \( v_{c_1 \cdots c_p} \) is invariant under \( G(c) \) for each factor \( c \) in \( c_1 \cdots c_p \). Then, one has

\[ \frac{\delta B_c(V)}{\delta h} = \sum_{g \in H_h} \frac{1}{|G(g)|} v_{hg} g, \]

which means that for each \( X \in \mathcal{V}(h) \),

\[ \frac{\delta B_c(V)}{\delta h(X)} = \sum_{g \in H_h} \frac{1}{|G(g)|} (i_X v_{hg}) g = \sum_{g \in H_h} \frac{1}{|G(g)|} v_{hg}(X, \cdot) g, \]

being \( i_X \) evaluation at momentum \( X \) of \( h \) in the first argument of the function \( v_{hg} \).

**Proof.** By assumption, one can bring then every element \( g \in H \) to the form \( g = h_1^{\alpha_1} \cdots h_n^{\alpha_n} \), and change the coefficients \( v_g \) accordingly without alteration. Notice that each \( g \in H \) can be further factorised as \( h^\ell f \), where \( \delta f / \delta h \equiv 0 \), and \( \ell \) depends on \( g \). Through direct computation,

\[ \frac{\delta B_c(V)}{\delta h(X)} = \sum_{g=h^\ell f \in H} \sum_{r=1, \ldots, \ell} \sum_{\sigma \in G(h)} \frac{1}{|G(h^\ell f)|} (i_{\sigma}(X) v_{h^\ell f}) h^{\ell-1} f \]

(2.18)

\[ = \sum_{g=h^\ell f \in H} \sum_{\sigma \in G(h)} \frac{\ell}{|G(h^\ell f)|} (i_{\sigma}(X) v_{h^\ell f}) h^{\ell-1} f \]

First, we used the invariance \( i_{\omega}^R v_{h^\ell f} = i_{\omega}^P v_{h^\ell f} \) for each \( 1 \leq p, m \leq \ell \); then, the invariance under \( G(h) \). Throughout, we can assume \( \ell \geq 1 \), since this is required for a summand in the first equality of eq. (2.18) not to vanish. Also, since

\[ G(h^\ell f) = G(h^\ell) \circ \mathcal{S}(\ell) \times G(f), \]

the orders of the groups should satisfy

\[ |G(h^\ell f)| = \ell ! \cdot |G(h)|^\ell \cdot |G(f)| = \ell |G(h)| \cdot |G(h^{\ell-1} f)|. \]

Hence, after cancellation one gets

\[ \frac{\delta B_c(V)}{\delta h(X)} = \sum_{g \in H_h} \frac{1}{|G(g)|} (i_X v_{hg}) g. \]

**Definition 2.16.** The graph derivative \( \partial U / \partial h \) of a generating functional of group actions \( U \) is given by the coefficient \( v_\emptyset \) of the empty graph \( \emptyset \) of the functional derivative of \( U \) with respect to \( h \), \( \partial U / \partial h =: \sum_{g} v_g g \), to wit

\[ \frac{\partial U}{\partial h} = \left( \frac{\partial U}{\partial h} \right)_\emptyset. \]

Parenthetically, the difference in notations for ‘partial derivative’ and ‘functional derivative’ does not intend to mirror any difference between ordinary multivariable and functional calculi.

The next result is simple and useful at the same time:
Lemma 2.17 (Graph calculus Leibniz product rule). Consider a graph calculus $\mathcal{C}(\mathfrak{h})$ and let $J, L \subset \text{FM}(\mathfrak{h})$ span functionals $U$ and $T$ in $\mathcal{C}(\mathfrak{h})$,

$$
U = \sum_{j \in J} u_j \quad , \quad T = \sum_{l \in L} t_l .
$$

Then the graph derivative of the product $V = U \cdot T = \sum_g v_g g$ is

$$
\frac{\partial (U \cdot T)}{\partial g} = \sum_{\Omega \in G(g)} \Omega \cdot v_g = \sum_{\Omega \in G(g)} \sum_{(j,l) \in J U L} \Omega \cdot \langle u_j t_l \rangle_g .
$$

Proof. We compute directly the derivative of the product $U \cdot T$ with respect to $g$ from the $\emptyset$-coefficient of $\delta (U \cdot T)/\delta g$:

$$
\frac{\partial (U \cdot T)}{\partial g} = \left( \frac{\delta (U \cdot T)}{\delta g} \right)_\emptyset
= \left( \frac{\delta}{\delta g} \sum_{f \in J L \subset \text{FM}^{ab}(\mathfrak{h})} \sum_{(j,l) \in J U L} \langle u_j t_l \rangle_f f \right)_\emptyset
= \sum_{f \in J L \subset \text{FM}^{ab}(\mathfrak{h})} \frac{\delta f}{\delta g} \sum_{(j,l) \in J U L} \langle u_j t_l \rangle_f
= G(g) \cap \left[ \sum_{(j,l) \in J U L} \langle u_j t_l \rangle_g \right]
= \sum_{\Omega \in G(g)} \sum_{(j,l) \in J U L} \Omega \cdot \langle u_j t_l \rangle_g .
$$

For the second equality we inserted the coefficients explicitly, according to the definition of the product. The fourth equality holds by graph independence, eq. (2.9), which holds for a graph calculus.

From now on let $2k(g)$ denote the number of vertices of $g \in G_D$. Our predominant examples of system of graph-group actions have the form

$$
\{ u_g, \mathcal{V} \!(g) = M_{D \times k(g)}(\mathbb{Z}), G(g) = \text{Aut}_c(g) \} \}_{g \in H}
$$

or

$$
\{ u_g, \mathcal{V} \!(g) = \mathcal{F}_{D,k}(g), G(g) = \text{Aut}_c(g) \} \}_{g \in H} .
$$

Here $\mathcal{F}_{D,k}(g)$ is the momentum subspace of $\mathbb{Z}^{D \times k(g)} \cong M_{D \times k(g)}(\mathbb{Z})$ consisting of points outside all the coloured diagonals, i.e.

$$
\mathcal{F}_{D,k} := \{ (y^1, \ldots, y^k) \in M_{D \times k}(\mathbb{Z}) \mid y^c_\alpha \neq y^c_\nu \text{ for all } c = 1, \ldots, D \text{ and } \alpha, \nu = 1, \ldots, k, \alpha \neq \nu \} .
$$

This space is essential for us. The action of the coloured automorphisms $^8 \sigma \in \text{Aut}_c(g) \subset \mathfrak{G}(k(g))$ on $\mathcal{F}_{D,k}$ is by permutation of the matrix columns, $y^i \mapsto y^{\sigma(i)}$, $i = 1, \ldots, k$. As a notation remark, we will often write $k$ instead of $k(g)$, as we just did, if the context is clear.

---

8There are more than one definition of ‘automorphism of a coloured graph’. The one used here is introduced in [Pé18]. In this setting, an automorphism of a coloured graph is a graph-morphism that preserves the colouring of the edges and bipartiteness of the vertex-set in a strict way (not up to a permutation of colours as in [CT16]). That is, edges of colour $a$ have to be mapped to edges of colour $a$; black (resp. white) vertices to black (resp. white) vertices.
2.5. **Three limit cases and examples.** The previous lemma implies the Leibniz multivari-
rule. Before elaborating on it for generating functionals of automorphism groups, it will be
helpful to exhibit this group action on a function $v_g$ in three limit cases, according to the
graph type of $g$.

Consider a set $\{h_1, \ldots, h_n\} \subset G_D$ of pairwise non-isomorphic graphs. Let $g = h_1^{\alpha_1} \cdots h_n^{\alpha_n}$ and let $U$ and $T$ graph-generated functionals by $J$ and $L$, respectively, being these subsets of the the monoid generated by $\{h_1, \ldots, h_n\}$. Then according to Lemma 2.17,

$$\frac{\partial (U \cdot T)}{\partial g} = \sum_{\Omega \in \text{Aut}_c(g)} \sum_{(j, t) \in J \subset \text{FM}^{\alpha_n}(h)} \Omega \cdot (u_j t_l) g.$$

(2.21)

Recalling that $\text{Aut}_c(g) = \text{Aut}_c(h_1^{\alpha_1} \cdots h_n^{\alpha_n}) = \prod_{i=1}^n \text{Aut}_c(h_i) \cdot \mathcal{S}(\alpha_i)$, first we elaborate on three simple cases:

- **Case I:** if $n = 1$. Then $g = h^\alpha$ and any $\Omega \in \text{Aut}_c(g) = \text{Aut}(h) \cdot \mathcal{S}(\alpha)$ is given by $\sigma = (\sigma_1, \ldots, \sigma_n) \in \text{Aut}_c(h)^\alpha$ and $\mu \in \mathcal{S}(\alpha)$, giving for eq. (2.21)

$$\Omega^{-1} \cdot (v_g(X_1, \ldots, X_n)) = v_g(\sigma X_{\mu(1)}, \ldots, \sigma X_{\mu(n)}),$$

where each $X_A = (x_A^{(1)}, \ldots, x_A^{(k)}) \in F_D(k,h)$ and the action of the automorphism group $\tau \in \text{Aut}_c(h)$ is given by $\tau(X_A) = (x_A^{(1)}, \ldots, x_A^{(k)}).

- **Case II:** if $n \neq 1$ but $\alpha_A = 1$ for all $A = 1, \ldots, n$. In this case, $g = h_1 h_2 \cdots h_n$. Then $\text{Aut}_c(g) = \prod_A \text{Aut}_c(h_A) \ni \Omega = (\sigma_1, \ldots, \sigma_n)$, which acts like

$$\Omega^{-1} \cdot (v_g(X_1, \ldots, X_n)) = v_g(\sigma_1(X_1), \ldots, \sigma_n(X_n)).$$

- **Case III:** If $g = h_1^{\alpha_1} \cdots h_n^{\alpha_n}$ but all automorphisms $\text{Aut}_c(h_i)$ are trivial. Then

$$\text{Aut}_c(g) = \prod_i \text{Aut}_c(h_i) \cdot \mathcal{S}(\alpha) = \mathcal{S}(\alpha_1) \times \cdots \times \mathcal{S}(\alpha_n) \ni (\mu_1, \ldots, \mu_n),$$

We use now multi-index notation for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ and abbreviate $u_{\gamma_1,\ldots,\gamma_n} = u_{h_1^{\gamma_1} \cdots h_n^{\gamma_n}}$, following a similar notation for $t_{\gamma_1,\ldots,\gamma_n}$. One has then

$$\frac{\partial (U \cdot T)}{\partial g} = \sum_{\mu_1 \in \mathcal{S}(\alpha_1)} \cdots \sum_{\mu_n \in \mathcal{S}(\alpha_n)} \sum_{(\gamma_1, \ldots, \gamma_n) \geq 0} (\mu_1, \ldots, \mu_n)^* (u_{\gamma_1, \ldots, \gamma_n} t_{v_1, \ldots, v_n}) g,$$

$$= \sum_{\mu_1 \in \mathcal{S}(\alpha_1)} \cdots \sum_{\mu_n \in \mathcal{S}(\alpha_n)} \sum_{0 \leq i = \gamma_i} (\mu_1, \ldots, \mu_n)^* (u_{\gamma_1, \ldots, \gamma_n} t_{\alpha_1 - \gamma_1, \ldots, \alpha_n - \gamma_n}) g$$

which one can rewrite in multi-index notation as

$$\frac{\partial (U \cdot T)}{\partial g} = \sum_{\mu \in \prod_i \mathcal{S}(\alpha_i)} \sum_{\gamma \leq \alpha} (\mu)^* (u_{-\gamma} t_{\alpha - \gamma}) g.$$

For constant functions $u_{\gamma_1, \ldots, \gamma_n}, t_{\gamma_1, \ldots, \gamma_n}$, this should reduce to the multivariable product formula. Indeed,

$$\frac{\partial (U \cdot T)}{\partial g} = \sum_{\gamma \leq \alpha} (\gamma_1) \cdots (\gamma_n) u_{\gamma} t_{\alpha - \gamma}$$

which is just the Leibniz rule eq. (1.1).

---

9 Of course, one could just take the union of the both spanning sets of graphs, if they do not a priori coincide.
Conveniently, lower-case (super)indices \((i = 1, \ldots, n)\) label the graph type, whereas upper-case (sub)indices indicate the number of copy \((A = 1, \ldots, \alpha_i)\) of the \(i\)-th graph type. We describe now the action of \(\text{Aut}_c(g)\) on a general function \(v_g : \mathcal{F}_{D,k(g)} \rightarrow \mathbb{C}\). Let \(X = ((X_1^1, \ldots, X_{\alpha_1}^1), \ldots, (X_1^n, \ldots, X_{\alpha_n}^n)) \in \mathcal{F}_{D,k(g)}\), being \(X_A^i \in \mathcal{F}_{D,k(h_i)}\) the momentum of the \(A\)-th copy of \(h_i\), for \(A = 1, \ldots, \alpha_i\). Picking an element \(\Omega = (\sigma, \mu) \in \text{Aut}_c(g)\), with

\[
(\sigma_1, \ldots, \sigma_n) \in \prod_{i=1}^n \text{Aut}(h_i)^{\alpha_i}, \quad \sigma_i = (\sigma_1^i, \ldots, \sigma_{\alpha_i}^i),
\]

(2.22)

\[
\mu = (\mu_1, \ldots, \mu_n) \in \prod_{i=1}^n \mathcal{G}(\alpha_i),
\]

(2.23)

the following holds:

\[
(\Omega^{-1} \cdot v_g)(X_A^i) = v_g((\sigma^A_i X_{\mu_i(A)}^i)_A^i),
\]

(2.24)

which is short-hand notation for

\[
\Omega^{-1} \cdot v_g(X) = v_g((\sigma^1_1 X_{\mu_1(1)}^1), \sigma^1_2 X_{\mu_1(2)}^1, \ldots, \sigma^1_{\alpha_1} X_{\mu_1(\alpha_1)}^1, \ldots, (\sigma^n_1 X_{\mu_n(1)}^n), \ldots, (\sigma^n_{\alpha_n} X_{\mu_n(\alpha_n)}^n)).
\]

Usually, it is summed all over \(\Omega \in \text{Aut}_c(g)\), which being a group, allows us to choose where we put the inverse in (2.24).

**Example 2.18.** For \(e = \varnothing, f = \{1\}\) and \(g = \varnothing\), let the subsets

\[
H = \{\varnothing, e^2, f^2\} \quad \text{and} \quad L = \{e^3 f, e^2 g, e^2 f^2 g\}
\]

span the functionals \(U = \sum_{h \in H} u_h h\) and \(T = \sum_{l \in L} t_l l\), and consider their product \(V = U \cdot T\). According to the action (2.24), one has the following formulae:

- For \(b = e^3 f\),
  \[
  \frac{\partial V}{\partial b} = \sum_{\Omega \in \text{Aut}_c(b)} \Omega \cdot \langle u_h t_l \rangle_b
  = \sum_{\Omega \in \text{Aut}_c(e^3 f)} \Omega \cdot \langle u_e^2 t_e^3 f \rangle_b
  = \sum_{(\sigma, e) \in \mathcal{G}(5) \times 2^2} (\sigma, e) \cdot (u_e^2 t_e^3 f),
  \]
  (2.25)

  since \(\text{Aut}_c(e^3 f) = \text{Aut}_c(\varnothing) \times \mathcal{G}(5) \times \text{Aut}_c(\{1\}) \times \mathcal{G}(1) = \{1\} \times \mathcal{G}(5) \times 2^2 \times \{1\}\). Also the ordering \(\langle \cdot \rangle_b\) is trivial. This in turn means that for the five momenta \(X_A \in \mathcal{F}_{1, D = 3} = \mathbb{Z}^3\) of \(e^3\) and the momentum \(Z = (z^1, z^2) \in \mathcal{F}_{2, 3}\),

\[
\frac{\partial V}{\partial e^3 f}(X_A, Z) = \sum_{(\sigma, e) \in \mathcal{G}(5) \times 2^2} (u_e^2 t_e^3 f)(X_{\sigma(A)}, z^{(1)}, z^{(2)})
\]

in abstract notation, or displaying the graphs as follows:

\[
\frac{\partial V}{\partial e^3 f}(X_A, Z) = \sum_{(\sigma, e) \in \mathcal{G}(5) \times 2^2} u_{\varnothing \mid \mathcal{G}(1)}(X_{\sigma(1)}, X_{\sigma(2)})
\]

\[
\times t_{\varnothing \mid \mathcal{G}(1)}(X_{\sigma(3)}, X_{\sigma(4)}, X_{\sigma(5)}, z^{(1)}, z^{(2)}).
\]

- For \(b' = e^2 f^2 g\),
  \[
  \frac{\partial V}{\partial b'} = \sum_{\Omega \in \text{Aut}_c(e^2 f^2 g)} \Omega \cdot (u_{e^2 t_e f^2 g}) + \Omega \cdot (u_{f^2 t_f e^2 g}).
  \]

For \(X_A\) momenta of \(e^2\), \(Z_A\) momenta of \(f^2\) and total momentum \(X = (X_1, X_2, Z_1, Z_2, W)\), one sums over elements \(\Omega = (\epsilon, (\sigma^1, \sigma^2; \mu), \tau) \in \text{Aut}_c(e^2 f^2 g) = \mathcal{G}(2) \times (\mathbb{Z}_2 \times \mathcal{G}(2)) \times \mathbb{Z}_3\), which yields for \((\partial V/\partial b')(X) = (\partial V/\partial (\epsilon^2 \mid \{1, \{\}\}) (X))\) the expression

\[
\sum_{\Omega} [\Omega \cdot (u_{e^2 t_e f^2 g}) + \Omega \cdot ([\{13\}(24)]^* u_{f^2 t_f e^2 g}^*) (X)]
\]

\[
= u_{\varnothing} \cdot \left\{ \sum_{(\epsilon, (\sigma^1, \sigma^2; \mu), \tau)} [t_e^2 f^2 g(X_{\epsilon(1)}, X_{\epsilon(2)}), (\sigma^1 Z_{\mu(1)}, \sigma^2 Z_{\mu(2)}, \tau(W))] \right\}
\]

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formed by \((\sigma^1 Z_{\mu(1)}, \sigma^2 Z_{\mu(2)}))\)\(t_{\epsilon^2 g}(X_{\epsilon(1)}, X_{\epsilon(2)}, \tau(W)))\)

\[ u_\mathfrak{B} \cdot \left\{ \sum_{(\epsilon, (\sigma^1, \sigma^2; \mu), \tau)} [t_{\mathfrak{D}^2}] |_{\mathfrak{D}^2}^{\mathfrak{D}_0^2} (X_{\epsilon(1)}, X_{\epsilon(2)}), (\sigma^1 Z_{\mu(1)}, \sigma^2 Z_{\mu(2)}), \tau(W)] \right\} \]

\[ + \sum_{(\epsilon, (\sigma^1, \sigma^2; \mu), \tau)} u_\mathfrak{D} t_{\mathfrak{D}^2} (\sigma^1 Z_{\mu(1)}, \sigma^2 Z_{\mu(2)}) \times t_{\mathfrak{D}^2} (X_{\epsilon(1)}, X_{\epsilon(2)}, \tau(W)). \]

One should still insert the explicit momenta \(Z_A = (z_A^1, z_A^2) \in \mathcal{F}_{2,3}, W = (w^1, w^2, w^3) \in \mathcal{F}_{3,3}, \tau(W) = (w^{\tau(1)}, w^{\tau(2)}, w^{\tau(3)}),\) and \(\sigma^A Z_{\mu(A)} = (\sigma^{A(1)}_{\mu(A)}, \sigma^{A(2)}_{\mu(A)}).

3. Tensor models

In this section, we implement the graph calculus for TFT.

3.1. Tensor Field Theory. The idea to use a the Ward-identity [DGMR07] to decouple the Schwinger-Dyson equations (at a planar sector) and obtain a master integral equation for the 2-point functions for matrix models [GW14]; some progress along these lines has been made for complex tensor field theory [Pér18; PPW17; PPTW18], leading lately to the large-\(N\) limit of the connected-boundary SDE. We treat a TFT as inspired by group field theory [SV14; BG12; BGS13; COR14].

Unlike matrix models, where there is a canonical choice for forming a scalar, for tensor models a specific trace \(\text{Tr}_B\), indicating how to contract the indices, should be specified. These traces are indexed by \(D\)-coloured graphs, where \(D\) is the rank of the tensors \(\varphi_{x_1...x_D}, \tilde{\varphi}_{x_1...x_D} \).

The graphical representation of these traces derives from to the independence of the imposed transformation rules under the action of \(U(N)\) on the spaces corresponding to each index \(x_a\) of \(\varphi_{x_1...x_D}, \tilde{\varphi}_{x_1...x_D},\) for \(a = 1, \ldots, D\), deemed colouring. That is to say, to form invariants only indices of identical colour can be contracted.

Therefore, a trace corresponding to a quartic interaction would be, say, \(\text{Tr}[\mathfrak{D}]:\) formed by colour-wise contracting with deltas the indices, as follows:

\[ \text{Tr}_\mathfrak{D} (\varphi, \tilde{\varphi}) = \sum_{x,y} \varphi_{x_1y_2x_3} \tilde{\varphi}_{x_1y_2y_3} \varphi_{y_1y_2y_3} \tilde{\varphi}_{y_1x_2x_3}. \]

The actual index-set of the tensors is \(\{1, \ldots, N\}\), but thinking of \(N\) as a large integer, we typically write these sums over \(\mathbb{N}\) or \(\mathbb{Z}\), or omit them in the sums. Although orthogonal groups [CT16] and mixed symmetries [Tan15] could be used to define other classes of tensor models, we restrict our discussion to the \(U(N)\)-tensor models we just introduced.

A tensor model is thus determined by a dimension \(D\) (the rank of the tensors) and an action \(S[\varphi, \tilde{\varphi}]\) given by a finite sum of traces indexed by connected \(D\)-coloured graphs. The partition function is given by

\[ Z[J, \tilde{J}] = Z[0,0] \frac{\int \mathcal{D}[\varphi, \tilde{\varphi}] e^{\text{Tr}(J\varphi) + \text{Tr}(\tilde{\varphi} \tilde{J}) - N^{D-1} S[\varphi, \tilde{\varphi}]}}{\int \mathcal{D}[\varphi, \tilde{\varphi}] e^{-N^{D-1} S[\varphi, \tilde{\varphi}]}} \]

\[ \mathcal{D}[\varphi, \tilde{\varphi}] := \prod_x N^{D-1} \frac{d^D \varphi_x d^D \tilde{\varphi}_x}{2\pi i}. \quad (3.1) \]

Its logarithm, \(W[J, \tilde{J}] = \log Z[J, \tilde{J}]\) is the free energy and generates the connected correlation functions, which as pointed out before, are classified by boundary (\(D\)-coloured possibly disconnected) graphs.

3.2. From graph calculus to tensor models. For a deeper exposition and motivation of the terminology and proofs of the results exposed in this section, we refer to [Pér18].

Many objects of interest in tensor models are functionals generated by graphs (e.g. the free energy). By this we mean that graphs have functions (or distributions) as coefficients of
For a graph $g \in H \subset G_D$ we denote by $2k(g)$ its number of vertices. One is interested in collections of functions

$$\{u_g : \mathbb{Z}^{D \times k(g)} \to \mathbb{C}\}_{g \in H}$$

and in their generating functionals

$$U[J, \bar{J}] = \sum_{g \in H \subset G_D} u_g \ast \mathbb{I}(g), \quad \text{where } u_g \ast \mathbb{I}(g) := \sum_{X \in \mathbb{Z}^{D \times k}} (u_g(X))\mathbb{I}(g)(X),$$

Here $\mathbb{I}(g)(X) = \prod_{\alpha=1}^{k(g)} J_{x^\alpha} J_{y^\alpha}$, where $\{y^\alpha\}_\alpha$ are determined by $g$ and $X$ through $g_\ast(X) = (y^1, \ldots, y^{k(g)})$. The induced map $g_\ast$ is defined as follows. The momentum $x^\alpha$ (resp. the $y^\alpha$) indexes white (resp. black) vertices in a graph. Then $g_\ast : M_{D \times k(g)}(\mathbb{Z}) \to M_{D \times k(g)}(\mathbb{Z})$ is given by $X = (x^1, \ldots, x^{k(g)}) \mapsto g_\ast(X) = (y^1, \ldots, y^{k(g)})$, where $y^\alpha_\sigma = x^\alpha_{g_\ast(\sigma)}$ (for $\alpha = 1, \ldots, k$) if and only in the graph $g$ there exists a $c$-coloured edge starting at $x^\alpha$ and ending at $y^\nu$. Each one of these arguments $x^\alpha$, but often $X$ itself too, is referred to as (entering) momentum of the white vertex $J_{x^\alpha}$. Similarly $y^\nu$ is the (outgoing) momentum at the black vertex $J_{y^\nu}$, because they are determined as in Figure 1.

For a connected graph $g$, due to the rigidity of a coloured graph, the elements of $\text{Aut}_c(g)$ are a lift $\tilde{\sigma}$ of an element $\sigma$ of the symmetric group $\Sigma(k(g))$. Defining

$$\frac{\partial U[J, \bar{J}]}{\partial g}(X) = \prod_{\alpha=1}^{k(g)} \frac{\delta}{\delta J_{x^\alpha}} \frac{\delta}{\delta J_{y^\alpha}} U[J, \bar{J}] \bigg|_{J = \bar{J} = 0}$$

the meaning of eq. (1.2) is a permutation of the arguments of $u_g$, i.e.

$$\frac{\partial U[J, \bar{J}]}{\partial g}(X) = \sum_{\tilde{\sigma} \in \text{Aut}_c(g)} (\sigma \cdot u_g)(X), \quad \text{where}$$

$$\quad (\sigma \cdot u_g)(x^1, \ldots, x^{k(g)}) := u_g(x^{\sigma^{-1}(1)}, \ldots, x^{\sigma^{-1}(k(g))}),$$

for all $X = (x^1, \ldots, x^{k(g)}) \in \mathcal{F}_{D,k(g)}$. 

Figure 1. Left. On the interpretation of the induced map $g_\ast$. One takes any representing graph $G$ such that $\partial G = g$. For the quartic melodic model (pillow-interactions) we can due so because it has been proven in [Pér18] that the spectrum of boundary states is full, which is to say, the boundary graphs are all of $G_D$. Assuming an ordering on the white and on the black vertices, the components $y^\alpha$ of $g_\ast(x^1, \ldots, x^k) = (y^1, \ldots, y^k)$ are determined by this picture, seeing the $x^\alpha$’s as independent momenta entering the graph, with output $g_\ast(x^1, \ldots, x^k)$. Right. If $g = \tilde{\sigma}$, and we choose the numbering $x^\alpha_1 = y^\alpha_0$ for $\alpha = 1, 2, 3$; there that $G$ satisfying $\partial G = \tilde{\sigma}$ is shown. The map $(\tilde{\sigma})_\star(x^1, x^2, x^3) = (y^1, y^2, y^3) = ((x^1, x^2, x^3), (x^1, x^2, x^3), (x^1, x^2, x^3))$ is determined by following the $0\alpha$ momenta lines for each colour $a$. 

...
**Remark 3.1.** Notice that in eq. (3.2) it is summed over the group $\text{Aut}_c(g)$. We are therefore entitled to drop the inverse in the RHS in (3.3). However, if the sum is not over all the group, we will keep the right ‘orientation’ of the action, for the convention in single terms $(\sigma \cdot u_g)$ is important in that case.

For graphs $h, g$, functions $u_g$ and functionals $U[J, \bar{J}]$ we abbreviate the usual notation:

$$U = U[J, \bar{J}], \quad u_g g := u_g \star (g) \quad \text{and} \quad g h := (g)(h),$$

and treat the latter as a product of graphs. This product is not considered commutative, since the star $\star$ implies the arguments of a function $u_{gh}$, which needs not satisfy $u_{gh} = u_{hg}$. Now we exhibit the relation of to the generating functional of group actions. With the product $G^\bigcirc B$ defined above, consider the functional

$$V^n = \sum_{B \in G^n_B} G_B B.$$

In tensor models, the subindices of the functions corresponding to the graph $A \times B$ (otherwise written as juxtaposition $AB$), are rather denoted by $G_{A|B}$. The functions $G_D$ satisfy invariance under each of the $\text{Aut}_c$-groups of the connected components of $D$. Also, $G_{A|B|C|D} = G_{A|B|C|D}_\gamma$ for any connected components $A$ and $B$ of $D$. Here $n$ is a large integer, and $G^n_B$ is a finite set of coloured graphs whose elements satisfy

$$\#(\text{vertices of } D) \leq 2n.$$

This truncation would declare vanishing all the floors about the $n$-th floor of the SDE-tower, but we can increase $n$ at desired accuracy. It limits the number of connected components to $n$ precisely. In rank-3, since the canonical (optimal in number of vertices, 3-coloured) graph of genus $g$ has $2(2g + 1)$ vertices, this truncation bounds the genus through $(2g + 1) \leq n$, making higher-genera contributions zero. We write the infinite sums keeping in mind that we meant their $n \to \infty$ limit. The coloured Borel transform of $V^n = \lim_n V^n$ is called the **free energy**:

$$W = B_c(V^n) = \sum_{D \in G_D} \frac{1}{|\text{Aut}_c(D)|} G_D D.$$  \hfill (3.5)

This equation holds in ‘$N = 1$-units’ and this assumption is innocuous within the scope of this article. However, if one plans to proceed perturbatively in $1/N$, the realistic case that drops this simplification ought to be addressed. Adding the power counting conjectured in [PPTW18] that scales $G_D \to \gamma(D) G_D$, where $\gamma(D)$ is certain factor already determined for the $2$-pt and $4$-pt functions, would help analyse convergence of $W$ (see Sec. 6).

This functional (but not this) corresponds to a system of graph-group actions. This is given by the next elements:

- For each graph $A$, $\mathcal{V}(A) = \mathcal{F}_{D,k(A)}$.
- For a disconnected graph $D = \sqcup_p A_p$ one has $\mathcal{V}(D) = \mathcal{F}_{D,k(D)} \subset \prod_p \mathcal{V}(A_p)$.
- There is an action of each $\text{Aut}_c(A_p)$ on $\mathcal{F}_{D,k(A_p)}$. Since $\sigma \in \text{Aut}_c(A_p) \subset \mathcal{G}(k(A))$, pre-composition by a function by $\sigma$ (permuting the columns of $\mathcal{F}_{D,k(A_p)} \subset M_{D \times k(A_p)(\mathbb{C})}$) gives this action.
- For connected graphs $B_i$ and $B_j$, the following holds [Pér18, Lemma 4]:

$$\frac{\delta B_i}{\delta B_j} = \delta_{ij} \text{Aut}_c(B_i),$$

acting on $\mathcal{F}_{D,k(B_i)}$, so for general disconnected graphs $D = B_1^\alpha_1 \sqcup \cdots \sqcup B_m^\alpha_m$

$$\frac{\delta D}{\delta D} = \text{Aut}_c(D) = \text{Aut}_c(B_1) \cdot \mathcal{G}(\alpha_1) \times \cdots \times \text{Aut}_c(B_n) \cdot \mathcal{G}(\alpha_m).$$  \hfill (3.6)
One can then operate with functionals, now without evaluating the sources at 0. That is
\[
\frac{\delta U[J,J]}{\delta g(X)} = \prod_{a=1}^{k(g)} \delta \frac{\delta}{\delta J^{x_a}} U.
\]
This object is, unlike the function \(\delta U/\delta g\), a functional and can be graph-derived again (without getting something a trivial result). We now derive equations for the free energy coefficients.

Another important functional in the next derivation is the so-called \(Y\)-term that emerged in the derivation of the Ward-Takahashi identity \([\text{Pér18}]\) and which encodes all the pertinent insertions of \(2p\) point functions into \(2p - 2\) point functions.

The expression to order six is given in \([\text{PPW17}, \text{Lemma 4.1}]\), but this article only will evoke the \(Y\)-term up to order four, located in Appendix A. For this paper, it is sufficient to additionally know the expression
\[
Y^{(c)}_x = \sum_{B \in G_D} d^{(c)}_{B,x} B,
\]
where \(c \in \{1, \ldots, D\}\) is a colour and \(x \in \mathbb{Z}\). It is important to notice that unlike \(W\) (for which we set \(G_\partial = 0\)), there is a non-vanishing constant term \(d^{(c)}_{\phi,x}\) in \(Y^{(c)}_x\).

4. Disconnected-boundary Schwinger-Dyson equations

The next section exposes the model, before proving the main result in Section 4.2.

4.1. The quartic melonic tensor field theory. The \(\phi^4_{D,m}\)-theory has quartic interaction vertices \(V[\varphi, \bar{\varphi}] = \lambda \sum_{a=1}^{D} \text{Tr}_{\phi_a}(\varphi, \bar{\varphi})\). These vertices are sometimes called pillows, since the graphs they correspond to have that appearance:
\[
V_a = \begin{pmatrix} a \\ b \\ c \\ d \\ \end{pmatrix}
\]
for \(a = 1, \ldots, D\). We analyse this theory with an abstract Laplacian \(E : \mathbb{Z}^D \to \mathbb{R}_{\geq 0}\) as propagator, \(S_0[\varphi, \bar{\varphi}] = \text{Tr}_{\phi}(\bar{\varphi}, E\varphi) = \sum_x \varphi_x E_x \bar{\varphi}_x\). A technical assumption is that the difference
\[
E(t, s) := E_{p_1 \ldots t_\rho \ldots p_D} - E_{p_1 \ldots s_\rho \ldots p_D}
\]
is independent of the fixed momenta in colours different from \(c\).

4.2. Main result. We prepare\(^\text{10}\) now some notions and notations needed for the formulation of the main result. Let \(\mathcal{R}\) be a connected graph, \(2r\) its number of vertices, and \(X \in \mathcal{F}_{r,D}\). Given a colour \(c\) and a numeration \(w^1, \ldots, w^r\) of the black vertices of \(\mathcal{R}\), we set, for any \(1 \leq \rho \leq r\),
\[
\text{Br}(\mathcal{R}, \rho, c) := \{\tau \in \{1, \ldots, r\} | \varsigma_{\tau}(\mathcal{R}, w^\rho, w^\tau) \text{ is disconnected}\}.
\]
Thus, \(\tau\) belonging to this set indices the vertices \(w^\tau\) at which a swap of the colour \(c\) edge at \(w^\tau\) and \(w^\rho\) disconnects \(\mathcal{R}\).

Example 4.1. The complete coloured graph \(\Box\) satisfies \(\text{Br}(\Box, \rho, c) = \emptyset\) for any colour \(c\) and vertex \(\rho\) (‘no edge-swap separates it’). On the other hand, if \(\{\rho, \mu\} = \{1, 2\}\) label the two black vertices of the the pillow \(\Box_{ij}\), then \(\text{Br}(\Box_{ij}, \rho, c = 1) = \{\mu\}\) and \(\text{Br}(\Box_{ij}, \rho, c = 2, 3) = \emptyset\).

Notation of the theorem. Let \(\mathcal{D}\) be a disconnected boundary graph of the \(\phi^4_{D,m}\)-theory having \(2d\) vertices. Given any \(X \in \mathcal{F}_{d,D}\), we select a momentum \(s = y^\beta\) listed in
\[
(y^1, \ldots, y^\beta, \ldots, y^d) = \mathcal{D}_s(X) \in \mathcal{F}_{d,D}.
\]
\(^\text{10}\)We come back to the usual notation for graphs or ‘bubbles’. 19
This $s$ determines both a connected graph component $\mathcal{R}$ of $\mathcal{D}$, and an $r$-tuple $X_0 \in \mathcal{F}_{r,D}$ of momenta, being $2r$ the number of vertices in $\mathcal{R}$, by asking that $s$ appears listed in the $r$-tuple $\mathcal{R}_*(X_0)$, particularly.

For the rest of components of $\mathcal{D}$ we write $Q (\mathcal{D} = \mathcal{R} \amalg \mathcal{Q})$. We factorise $Q$ in $\alpha_u$ copies of pairwse non-isomorphic connected graphs $\{Q_u\}_{u=1,...,n}$:

$$Q = Q_1^{\alpha_1} \amalg ... \amalg Q_n^{\alpha_n}, \quad (Q_A^{\alpha_A} = \bigcup_{i=1}^{\alpha_A} Q_A).$$

We split the $d$-tuple into momenta $X_0$ of $\mathcal{R}$ and momenta $\mathcal{Q}$ of $\mathcal{Q}$, so that, $(X_0, \mathcal{Q}) = \mathcal{X}$ up to reordering. For $\tau \in \text{Br}(\mathcal{R}, \beta, c)$ we can thus write $\zeta_c(\mathcal{R}; \beta, \tau) = \mathcal{R}' \amalg \mathcal{R}''$, and split accordingly the momentum $X_0$ in momenta $X_0'$ of $\mathcal{R}'$ and $X_0''$ of $\mathcal{R}''$, $X_0 = (X_0', X_0'')$. Furthermore, for any factorising pair of graphs

$$\mathcal{C}, \mathcal{B} \in \mathcal{G}_D \text{ with } \mathcal{C} \amalg \mathcal{B} \sim \mathcal{Q},$$

we define the two functions $H^{(c,\tau)}_{\mathcal{C}, \mathcal{B}}$ and $I^{(c)}_{\mathcal{C}, \mathcal{B}}$ by the following products:

$$H^{(c,\tau)}_{\mathcal{C}, \mathcal{B}} = \langle G_{\mathcal{R}}|_{\mathcal{C}} (X'_0; \cdot) \times G_{\mathcal{R}''}|_{\mathcal{B}} (X''_0; \cdot) \rangle_Q, \quad (\tau \in \text{Br}(\mathcal{R}, \beta, c))$$

$$I^{(c)}_{\mathcal{C}, \mathcal{B}} = \frac{1}{|	ext{Aut}_c(\mathcal{B})|} \langle I^{(c)}_{\mathcal{C}, \mathcal{B}} \times G_{\mathcal{R}''}|_{\mathcal{B}} (X''_0; \cdot) \rangle_Q,$$

where the reorderings refer only to the graph components of the graphs in the pair $(\mathcal{C}, \mathcal{B})$. The pivotal term is

$$I^{(c)}_{\mathcal{B}, \mathcal{C}} = Y^{(c)}(0, 0) = \sum_{q \in \mathcal{Q}} G_{(2)}^{(2)}(\mathcal{C}, \mathcal{B}).$$

**Theorem 4.2.** The Schwinger-Dyson equations for the disconnected graph $\mathcal{D}$ read, for each vertex choice $s = \mathbf{y}^\beta$, as follows:

$$G^{(2k)}_D (\mathcal{X}) = \frac{(-2\lambda)}{E_s^l} \sum_{c=1}^{D} \left\{ \sum_{\sigma \in \text{Aut}_c(\mathcal{D})} \sigma \cdot I^{(c)}_{\mathcal{D}, \mathcal{X}}(\mathcal{X}) \right\}$$

$$+ \sum_{\rho \neq \beta} \frac{1}{E(y^\rho_c, s_c)} \left[ \frac{\partial W[J, J]}{\partial \rho_c(\mathcal{D}; \beta, \rho)} (\mathcal{X}) \right]$$

$$- \sum_{b_c} \frac{1}{E(s_c, b_c)} \left[ G^{(2k)}_{\mathcal{D}} (\mathcal{X}) - G^{(2k)}_{\mathcal{D}} (\mathcal{X}|_{s_c \rightarrow b_c}) \right]$$

$$+ \sum_{\tau \in \text{Br}(\mathcal{R}; \beta, c)} \sum_{(\mathcal{C}, \mathcal{B}) \in \mathcal{G}_D \cup \mathcal{G}_D} \sum_{\Omega \in \text{Aut}_c(\mathcal{Q})} \left( \frac{\Omega \cdot H_{(\mathcal{C}, \mathcal{B})}^{(c, \tau)}(\mathcal{X}) - (\Omega \cdot H_{(\mathcal{B}, \mathcal{C})}^{(c, \tau)}(\mathcal{X})|_{s_c \rightarrow y_c^\tau}}{E(y_c^\tau, s_c)} \right)$$

$$+ \sum_{(\mathcal{C}, \mathcal{B}) \in \mathcal{G}_D \cup \mathcal{G}_D} \sum_{\Omega \in \text{Aut}_c(\mathcal{Q})} \left( \Omega \cdot I_{(\mathcal{B}, \mathcal{C})}^{(c)}(\mathcal{X}) \right).$$

**Proof.** The partition function $Z[J, J] = \exp(W[J, J])$ has been shown [Pér18] to satisfy

$$\frac{\delta W}{\delta J_s} = \frac{1}{E_s} \left\{ J_s - e^{-W} \left( \frac{\partial V(\varphi, \bar{\varphi})}{\partial \bar{\varphi}_s} \right) \right\} \delta \varphi \rightarrow \delta J_s \delta \bar{\varphi} \rightarrow \delta J_s e^{+W}, \quad (4.4)$$

The colour-$c$-WTI lead to

$$\left( \frac{\partial V(\varphi, \bar{\varphi})}{\partial \bar{\varphi}_s} \right) \delta \varphi \rightarrow \delta J_s \delta \bar{\varphi} \rightarrow \delta J_s Z[J, J] = 2\lambda \sum_{c=1}^{D} (A_c(s) - B_c(s) + C_c(s) + D_c(s) + E_c(s)), \quad (4.5)$$

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where each of the summands is given by

\[
A_c(s) = Y^{(c)}_{s_c} [J, J] \cdot \frac{\delta Z[J, J]}{\delta J_s},
\]

\[
B_c(s) = \sum_b \frac{J_{b,s_c}}{E(b_c, s_c)} \frac{\delta^2 Z[J, J]}{\delta J_{b,b,c}} \cdot \frac{\delta W[J, J]}{\delta J_b},
\]

\[
C_c(s) = \sum_{b_c} \frac{1}{E(b_c, s_c)} \frac{\delta Z[J, J]}{\delta J_s},
\]

\[
D_c(s) = \sum_b \frac{\bar{J}_b}{E(b_c, s_c)} \frac{\delta^2 Z[J, J]}{\delta J_{b,b,c}} \cdot \frac{\delta W[J, J]}{\delta J_{b,b,c}},
\]

\[
F_c(s) = \frac{\delta Y^{(c)}_{s_c} [J, J]}{\delta J_s} \cdot Z[J, J],
\]

any \( c = 1, \ldots, D \). In order to derive the SDE for connected boundary graphs these expressions were enough to start. In this new derivation, it is convenient to work with

\[
e^{-W} A_c(s) = Y^{(c)}_{s_c} [J, J] \cdot \frac{\delta W[J, J]}{\delta J_s}, \tag{4.6}
\]

\[
e^{-W} B_c(s) = \sum_b \frac{J_{b,s_c}}{E(b_c, s_c)} \left[ \frac{\delta^2 W[J, J]}{\delta J_{b,b,c}} \cdot \frac{\delta W[J, J]}{\delta J_b} + \frac{\delta W[J, J]}{\delta J_{b,b,c}} \cdot \frac{\delta W[J, J]}{\delta J_{b,b,c}} \right], \tag{4.7}
\]

\[
e^{-W} C_c(s) = \sum_{b_c} \frac{1}{E(b_c, s_c)} \frac{\delta W[J, J]}{\delta J_s}, \tag{4.8}
\]

\[
e^{-W} D_c(s) = \sum_b \frac{\bar{J}_b}{E(b_c, s_c)} \left[ \frac{\delta^2 W[J, J]}{\delta J_{b,b,c}} \cdot \frac{\delta W[J, J]}{\delta J_{b,b,c}} + \frac{\delta W[J, J]}{\delta J_{b,b,c}} \cdot \frac{\delta W[J, J]}{\delta J_{b,b,c}} \right], \tag{4.9}
\]

\[
e^{-W} F_c(s) = \frac{\delta Y^{(c)}_{s_c} [J, J]}{\delta J_s}. \tag{4.10}
\]

Notice the presence of the product of derivatives in the \( B_c \) and \( D_c \) terms. These are called \( B^{(\text{prod})}_c \) and \( D^{(\text{prod})}_c \), respectively. The other two terms (that already appeared on the SDEs for connected boundaries) containing a double derivative are denoted by \( B^{(\text{dd})}_c \) and \( D^{(\text{dd})}_c \), respectively.

By definition,

\[
G_D = \frac{\partial W}{\partial D} = \frac{\delta}{\delta Q} \left( \frac{\delta W}{\delta R} \right) \bigg|_{J=\bar{J}=0}.
\]

Spelled out this means that

\[
G_D(X) = \left\{ \left[ \prod_{i=1}^n \frac{\delta^{a_i}}{\delta Q^{a_i}_1} \frac{\delta^{a_i}}{\delta Q^{a_i}_2} \cdots \frac{\delta^{a_i}}{\delta Q^{a_i}_{r_i}} \{-\frac{\delta}{\delta J_{x^\alpha_0}} \left\{ \prod_{\alpha=1}^{r} \frac{\delta}{\delta J_{x^\alpha_0}} \left\{ \frac{\delta}{\delta J_{y_0^\nu}} \left\{ \frac{\delta}{\delta J_{y_0^\nu}} \right\} \right\} \right\} \right] \bigg|_{J=\bar{J}=0} \right\}. \tag{4.11}
\]

We have the freedom to numerate momenta starting with the component \( R \), thus

\[
X_0 = (x_0^1, \ldots, x_0^n) \in F_{D,r}, \quad R_+(X_0) = (y_0^1, \ldots, y_0^r),
\]

and \( s = y^\alpha \). For each item in the list

\[
(m, M) \in \{(a, A), (b, B), (c, C), (d, D), (f, F)\}.
\]

we define the following functions:

\[
m_c(X; s; D) := \frac{\delta^{a}}{\delta Q^{a}(X)} \prod_{\alpha \neq \beta} \frac{\delta^{a}}{\delta J_{y_0^\nu}} \prod_{\nu=1, \ldots, r}^{r_i} \frac{\delta^{a}}{\delta J_{x^\alpha_0}^{a_i}} \left[ e^{-W[J,J]} M_c(s) \right] \bigg|_{J=\bar{J}=0}. \tag{4.12}
\]
The splitting of $b$ and $\mathcal{d}$ into terms $b^{(dd)}$, $b^{(prod)}$, $\mathcal{d}^{(dd)}$, $\mathcal{d}^{(prod)}$, respectively, still makes sense. We now determine all coefficients, beginning with the easiest.

The $c_c$ and $f_c$ terms are readily computed:

$$c_c(X; s; D) = \sum_{\beta} \frac{1}{E(b_c, s_c)} G_D^{(2k)}(X), \quad f_c(X; s; D) = \sum_{\pi \in \text{Aut}_c(D)} \pi^* j^{(c)}_D(X). \quad (4.13)$$

The term $c_c$ itself is not finite, but a term arising from one of the three functions will render it finite. The three remaining $m_c$-functions involve derivatives of products of functionals. We first observe that the $2r - 1$ derivatives in the sources complete the graph derivative $\delta W/\delta \mathcal{R}$, so the $a_c$ yields

$$a_c(X; s; D) = \frac{\delta^\alpha}{\delta Q^\alpha(X)} \prod_{\substack{\alpha \neq \beta, \nu \leq \rho \leq \tau \nu = 1, \ldots, r}} \frac{\delta}{\delta J_{x_0}^\nu} \left( \frac{\delta W[J, J]}{\delta J_s} \right) \bigg|_{J = \bar{J} = 0} \quad (4.14)$$

$$= \frac{\delta^\alpha}{\delta Q^\alpha(X)} \left( \frac{\delta W[J, J]}{\delta \mathcal{R}(X_0)} \right) \bigg|_{J = \bar{J} = 0}. \quad (4.15)$$

Notice that, after evaluation in the sources at 0,

$$\frac{\partial}{\partial A} \cdots \frac{\partial}{\partial A'} \frac{\partial}{\partial \mathcal{R}(X_0)} W[J, \bar{J}] = G_{\mathcal{R}|A| \cdots |A'}(X_0, \cdot)$$

for any boundary graphs $A, \ldots, A'$. Using the formula for the graph derivative of products (Lem. 2.17) and subsequently Lemma 2.15 one deduces

$$a_c(X; s; D) = \sum_{B, \mathcal{C}} \sum_{\Omega \in \text{Aut}_c(Q)} \sum_{\sigma \in \text{Aut}_c(\mathcal{R})} \frac{1}{|\text{Aut}_c(\mathcal{R} \Pi B)|} \Omega \cdot \langle G_{\mathcal{R}|B}(\sigma(X_0); \cdot) \times j^{(c)}_{C, s_c} \rangle Q$$

$$= \sum_{B, \mathcal{C}} \sum_{\Omega \in \text{Aut}_c(Q)} \frac{1}{|\text{Aut}_c(B)|} \Omega \cdot \langle G_{\mathcal{R}|B}(X_0; \cdot) \times j^{(c)}_{C, s_c} \rangle Q$$

$$= \sum_{B, \mathcal{C}} \sum_{\Omega \in \text{Aut}_c(Q)} \Omega \cdot I^{(c)}_{\mathcal{C}, B}(X). \quad (4.16)$$

We compute now the term $\mathcal{d}_c$. We can split $D_c$ into the double derivative contribution to the the double derivative contribution $D^{(dd)}_c$ and the product of single derivatives $D^{(prod)}_c$. If we decompose $\delta^\alpha/\delta Q^\alpha$ and the rest of derivatives implied in $\mathcal{R}$ into single functional derivatives,

$$\mathcal{O}(J) + \prod_{\nu = 1, \ldots, d} \frac{\delta}{\delta J_{x_0}^\nu} \left( e^{-W} D^{(dd)}_c(s) \right)$$

$$= \prod_{\alpha \neq \beta, \rho} \frac{\delta}{\delta J_{y_0}^\nu} \frac{\delta}{\delta J_{x_0}^\nu} \left[ \sum_{b} \frac{1}{E(b_c, s_c)} J_b \delta W[J, \bar{J}] \right] \quad (4.16)$$

$$= \prod_{\rho = 1}^d \prod_{\beta \neq \alpha, \rho} \frac{\delta}{\delta J_{y_0}^\nu} \frac{\delta}{\delta J_{x_0}^\nu} \left[ \sum_{b} \frac{\delta^b_{y_0}}{E(b_c, s_c)} \delta J_{y_0}^b \delta W[J, \bar{J}] \right]$$

The part of the calculation of the contributions of the double derivatives $\delta^2 W/\delta J_0 \delta J$ or $\delta^2 W/\delta J_0 \delta \mathcal{R}$ to $b_c$ and $\mathcal{d}_c$ is shortened, due to the very similar derivation provided already in [PPW17].
\[ s = s_c s_{\hat{c}} \]
\[ y^\rho = (y^c_\rho y^\rho_\hat{c}) = (x^c_\rho y^\rho_\hat{c}) \]
\[ x^\gamma = (x^\gamma_c x^\gamma_{\hat{c}}) \]

**Figure 2.** The swapping operation \( \varsigma_c \)

\[
= \sum_{\rho=1}^{d} \prod_{\alpha(\beta \neq \alpha, \beta) \neq \beta} \frac{\delta}{\delta y^\alpha_\nu} \frac{\delta}{\delta y^\beta_\nu} \left[ \frac{1}{E(y^c_\nu, s_c)} \frac{\delta^2 W[J, \bar{J}]]}{\delta J^c_{\alpha \nu} \delta J^c_{\beta \nu} \delta J^c_{\gamma c} \delta J^c_{\gamma \hat{c}}} \right].
\]

This is, after evaluation at \( \bar{J} = J = 0 \), all the (coloured) graphs obtained from \( D \) (also implying the other connected components) after the colour-c swapping at \( s \): for \( \rho \neq \beta \), i.e. running over black vertices skipping \( \bar{J}_s = \bar{J}_{\gamma \beta} \). Thus the derivatives on \( D^c_{\text{conn}} \) contribute

\[
\sum_{\rho \neq \beta, \text{any black vertex}} \frac{1}{E(y^c_\nu, s_c)} \frac{\partial W[J, J]}{\partial y^\rho_{\nu}(D; 1, \rho)(X)} \text{ for } \rho \neq \beta.
\]

To fully compute \( D_c \), we add now the product of derivatives, \( D^\text{prod} \). Unlike \( D^\text{dd} \), with \( D^\text{prod} \) one can first see the effect of applying the rest of the derivatives of the graph \( R \), i.e. those with momenta \( (x^c_\nu)_{\alpha} \) and \( (y^\rho_\nu)_{\alpha \neq \beta} \). This is due to the product of derivatives of \( W \), which forces both factors to be derived with respect to momenta in the same graph in order not to vanish. Thus,

\[
\prod_{\alpha \neq \beta, \nu = 1, \ldots, r} \frac{\delta}{\delta y^\alpha_\nu} \frac{\delta}{\delta y^\beta_\nu} \left( e^{-W D^\text{prod}_c(s)} \right) + \mathcal{O}(\bar{J})
\]

\[
= \prod_{\alpha \neq \beta, \nu = 1, \ldots, r} \frac{\delta}{\delta y^\alpha_\nu} \frac{\delta}{\delta y^\beta_\nu} \left[ \sum_{b} \frac{\delta}{\delta J^c_{\alpha \nu}} \frac{\delta}{\delta J^c_{\beta \nu}} \left( \frac{\delta}{\delta b} E(b_s, s_c) \right) \frac{\delta}{\delta J^c_{\gamma c}} \frac{\delta}{\delta J^c_{\gamma \hat{c}}}ight]
\]

\[
= \sum_{\rho \neq \beta, \alpha \neq \beta(\alpha \neq \rho)} \prod_{\nu = 1, \ldots, r} \frac{\delta}{\delta y^\rho_\nu} \frac{\delta}{\delta y^\beta_\nu} \left[ \sum_{b} \frac{\delta}{\delta b} E(b_s, s_c) \frac{\delta}{\delta J^c_{\alpha \nu}} \frac{\delta}{\delta J^c_{\beta \nu}} \frac{\delta}{\delta J^c_{\gamma c}} \frac{\delta}{\delta J^c_{\gamma \hat{c}}}ight],
\]

where in the last step we only used that \( s = y^\beta = y^\beta_0 \). It is evident that these two derivatives in the square bracket form of a colour-c edge swap at \( y^\beta_0 \) and \( y^\rho_0 \), but in order not to lead to a vanishing term, they also have to lie on a different component of a graph (as otherwise a graph derivative would be incomplete). It is therefore additionally required that the graph after the swapping at \( y^\beta_0 \) and \( y^\rho_0 \), is disconnected, that is, there are connected graphs \( R_1 \) and \( R_2 \), such that

\[ s_c(R; \beta, \rho) = R_1 \sqcup R_2 \quad \text{if and only if} \quad \rho \in \text{Br}(R; \beta, c). \]
Therefore
\[
\frac{\delta}{\delta Q^\alpha(X)} \prod_{\alpha \neq \beta, \nu = 1, \ldots, r} \frac{\delta}{\delta J_{\gamma\beta}^\nu} \frac{\delta}{\delta J_{\gamma\nu}^\alpha} (e^{-W D_c(\text{prod})}(s)) + \mathcal{O}(J)
\]
\[
= \sum_{\tau \in \text{Br} (\mathbb{R}, \beta, c)} \frac{1}{E(y_c^\tau, s_c)} \frac{\delta}{\delta Q^\alpha(X)} \prod_{\alpha \neq \beta, \nu = 1, \ldots, r} \frac{\delta}{\delta J_{\gamma\beta}^\nu} \frac{\delta}{\delta J_{\gamma\nu}^\alpha} \left[ \frac{\delta W}{\delta J_{\gamma\beta}^\nu} y_c^\tau y_c^\nu - \frac{\delta W}{\delta J_{\gamma\nu}^\alpha} y_c^\tau y_c^\nu \right].
\]
\[
= \sum_{\tau \in \text{Br} (\mathbb{R}, \beta, c)} \frac{1}{E(y_c^\tau, s_c)} \frac{\delta}{\delta Q^\alpha(X)} \left( \frac{\delta W}{\delta J_{\gamma\beta}^\nu} \frac{\delta W}{\delta J_{\gamma\nu}^\alpha} \right).
\]

Here is where one uses the multi-graph calculus Leibniz rule (Lemma 2.17), namely
\[
\frac{\delta}{\delta Q^\alpha(X)} \prod_{\alpha \neq \beta, \nu = 1, \ldots, r} \frac{\delta}{\delta J_{\gamma\beta}^\nu} \frac{\delta}{\delta J_{\gamma\nu}^\alpha} (e^{-W D_c(\text{prod})}(s) ) \bigg|_{J = J = 0}
\]
\[
= \sum_{\tau \in \text{Br} (\mathbb{R}, \beta, c)} \frac{1}{E(y_c^\tau, s_c)} \sum_{B \in \mathcal{C} \in Q^\alpha} \sum_{\Omega \in \text{Aut}_c(Q^\alpha)} \Omega \cdot \left[(G_{\mathcal{R}_1}(X'_0, \cdots X'_0) \times G_{\mathcal{R}_1}(X''_0, \cdots X''_0)) Q^\alpha \right](X)
\]
\[
= \sum_{\tau \in \text{Br} (\mathbb{R}, \beta, c)} \frac{1}{E(y_c^\tau, s_c)} \sum_{B \in \mathcal{C} \in Q^\alpha} \sum_{\Omega \in \text{Aut}_c(Q^\alpha)} (\Omega \cdot H_{B, C}^{\alpha, \tau})(X).
\]

In summary, the $\mathfrak{d}_c$-term is given by
\[
\mathfrak{d}_c(X; s; D) = \sum_{\substack{\nu \neq \beta \\ \rho, \text{any black vertex}}} \frac{1}{E(y_c^\tau, s_c)} \frac{\partial W[J, J]}{\partial c_d(D; 1, \rho)}(X)
\]
\[
+ \sum_{\tau \in \text{Br} (\mathbb{R}, \beta, c)} \frac{1}{E(y_c^\tau, s_c)} \sum_{B \in \mathcal{C} \in Q^\alpha} \sum_{\Omega \in \text{Aut}_c(Q^\alpha)} (\Omega \cdot H_{B, C}(X)).
\]

As for the derivatives on $B_c(s)$, we divide the derivation in two parts. One concerns the double derivative, $B_c^{(\text{dd})}$:
\[
\mathcal{O}(J) + \prod_{\alpha \neq \beta} \frac{\delta}{\delta J_{\gamma\beta}^\nu} B_c^{(\text{dd})}(s) = \prod_{\alpha \neq \beta} \frac{\delta}{\delta J_{\gamma\beta}^\nu} \frac{\delta}{\delta J_{\gamma\nu}^\alpha} \left[ \sum_b \frac{1}{E(b_c, s_c)} J_{b_c s_c} \frac{\delta^2 W[J, J]}{\delta J_{b_c s_c} \delta J_{b_c s_c}} \right]
\]
\[
= \sum_{\theta = 1}^d \prod_{\alpha \neq \beta, \nu \neq \theta} \frac{\delta}{\delta J_{\gamma\beta}^\nu} \frac{\delta}{\delta J_{\gamma\nu}^\alpha} \left[ \sum_b \frac{1}{E(b_c, s_c)} \frac{\delta^2 W[J, J]}{\delta J_{b_c s_c} \delta J_{b_c s_c}} \right].
\]

(4.21)

since there is a single vertex $x^\gamma$, $\gamma = \gamma(c)$, with $x^\gamma = s_c$, so $\delta^2_{x^\gamma} = \delta^2_{x^\gamma}$. The term $\delta^2 W[J, J]/\delta J_{b_c s_c} \delta J_{x^\gamma b_c}$ is selected by $\delta^2_{x^\gamma}$, and $\Omega$. If $x^\gamma$ is selected by (the running) $b_c$. But when
\[
b_c \in \{y_1^c, y_2^c, \ldots, y_l^c \},
\]
one does not have exactly a ‘graph derivative’, since we are evaluating it not in $\mathcal{F}_D, d$, but in one of its diagonals of colour $c$. The direct computation yields then a second contribution to
\(b_c^{(dd)}(X; s; D)\) (the second line below):

\[
b_c^{(dd)}(X; s; D) = \sum_{b_c} \frac{1}{E(b_c, x_c)} G_D^{(2k)}(x_1, \ldots, x_\gamma^{-1}, (x_1^\gamma, \ldots, x_{a_1}^\gamma, b_c, x_{a_1+1}^\gamma, \ldots, x_D^\gamma), x_1^{\gamma+1}, \ldots, x^d) + \sum_{\rho>1} \frac{1}{E(x_c^{(\rho)}, x_c)} \frac{\partial W[J, \tilde{J}]}{\partial s_c(D; 1, \rho)}(x_1, \ldots, (x_1^\gamma, \ldots, x_{a_1}^\gamma, x_1^{\gamma+1}, \ldots, x^d).\]

(4.22)

where \(\kappa(\rho)\) is defined in the Figure 2 (i.e. \(x_c^{(\rho)} = y_c^\rho\)).

The last computation is \(b_c^{(prod)}\),

\[
b_c^{(prod)}(X; s; D) = \frac{\delta^\alpha}{\delta Q^\alpha(X)} \prod_{\alpha \neq \beta} \frac{\delta}{\delta J_\beta} \left[ \sum_{b_c} \frac{J_{b_c, s_c}}{E(b_c, s_c)} \frac{\delta W[J, \tilde{J}]}{\delta J_b} \frac{\delta W[J, \tilde{J}]}{\delta J_{b_c, b_c}} \right]_{J=J=0}.\]

(4.23)

Because of the repetion of \(b_c\) in both factors, it seems that the term vanishes after deriving it. However, \(b_c\) runs, and it does so also through the particular \(c\)-coloured entries of the black vertices of \(\mathcal{R}\). \(b_c \in \{y_0^1, y_0^2, \ldots, y_0^d\}\). Only if we also require that \(b_c \in \{y_0^\tau \mid \tau \in \text{Br}(\mathcal{R}, \beta, c)\}\), we guarantee that each one of those factors forms a graph derivative. However, notice that momentum in the white vertex \(\mathbf{x}^\gamma = s_c \mathbf{x}_c^\gamma\) has changed to \(y_c^\gamma \mathbf{x}_c^\gamma\). Thus, one changes \(X_0\) into \(X_0\big|_{s_c \rightarrow y_c^\gamma}\). Therefore

\[
b_c^{(prod)}(X; s; D) = \sum_{\tau \in \text{Br}(\mathcal{R}; \beta, c)} \sum_{(\mathcal{C}, \mathcal{B}) \in \Omega_0 \Omega D} \sum_{\Omega \in \text{Aut}_c(Q)} \frac{(\Omega \cdot H_{B,c})(X)|_{s_c \rightarrow y_c^\gamma}}{E(y_c^\gamma, s_c)}.\]

We apply again Lemma 2.17 and find that

\[
b_c^{(prod)}(X; s; D) = \sum_{\tau \in \text{Br}(\mathcal{R}; \beta, c)} \sum_{(\mathcal{C}, \mathcal{B}) \in \Omega_0 \Omega D} \sum_{\Omega \in \text{Aut}_c(Q)} \frac{(\Omega \cdot H_{B,c})(X)|_{s_c \rightarrow y_c^\gamma}}{E(y_c^\gamma, s_c)}.\]

Due to eqs. (4.5) one has

\[
\frac{\partial W[J, \tilde{J}]}{\partial D(X)} = \prod_i \frac{\delta^\alpha_i}{\delta Q_i(X^\gamma_i)} \cdot Q_i(X^\gamma_i)
\]

\[
\left\{ \prod_{\alpha \neq \beta} \frac{\delta}{\delta J_\beta} \left[ (-2\lambda E_{\gamma-1}) \sum_{c=1}^D c^{-W(A_c(s))} + C_c(s) + D_c(s) + F_c(s) - B_c(s)) \right] \right\}_{J=0} = \frac{(-2\lambda)}{E_s} \sum_c \left( a_c(X; s; D) + c_c(X; s; D) + d_c^{(dd)}(X; s; D) + d_c^{(prod)}(X; s; D) \right)
\]

This is precisely \(G_D(X)\) and the result follows.

\[\square\]
Correlation functions

| Order | Graph notation | Simplified notation |
|-------|----------------|---------------------|
| 2-pt function | $G^{(2)}_{\varnothing}$ | $G^{(2)}_m$ or $G^{(2)}$ |
| 4-pt functions | $G^{(4)}_{\varnothing \varnothing}$ | $G^{(4)}_{V_a}$ |
| 6-pt functions | $G^{(6)}_{\varnothing \varnothing \varnothing}$ | $G^{(6)}_{Q_a}$ |

Table 1. Two notations for the correlation functions. Here $a, b, c$ are colours bound to satisfy $\{a, b, c\} = \{1, 2, 3\}$. The subindex $m$ originates from ‘melon’.

5. Four and six point SDE with disconnected boundary

Concrete SDEs for the rank-3 $\varphi^4_3$-theory are presented next. Recall, the interaction in this case is $\lambda (\varnothing \varnothing \varnothing + 4\varnothing \varnothing + 3\varnothing \varnothing \varnothing)$. The display the first equation in traditional notation, which allows to see immediately the

5.1. Schwinger-Dyson equations for $G^{(6)}_{\varnothing \varnothing \varnothing \varnothing \varnothing}$. We single out the terms in the derivation of the SDE for this case, which is the most complicated presented here. The rest of the results are obtained in a similar and more direct way.

There are two equations, depending on whether one chooses $s$ (cf. Theorem 4.2 above) as a component of the outgoing momentum in $\varnothing$ or in $\varnothing \varnothing$. We choose this last vertex to be $V_1 = \varnothing \varnothing \varnothing$, for sake of clarity (since the model is colour-invariant, the other three equations are readily obtained).

- If $s = x$ is outgoing momentum of the graph $\varnothing$. In the notation of the theorem, here $D = m|V_1$ being $R = m$, since $y^1 = s = X_0$ is the momentum of the black vertex of $m$. Therefore $Q = V_1$. The remaining momenta $X$ equal $(y, z)$. The $I^{(c)}_{C,B}$-terms are then computed as follows, from any of the factorisations $C, B = (\varnothing, V_1)$ or $(V_1, \varnothing)$ and read

$$I^{(c)}_{C,B}(X) = \frac{1}{|\text{Aut}_c(B)|} : f^{(c)}_{C,s} \times G_{m|V_1}(X; \cdot : V_1(X)) : (5.1)$$

$$= \begin{cases} f^{(c)}_{V_1}(y, z) G^{(2)}_{m|V_1}(x) & (C, B) = (V_1, \varnothing), \\ \frac{1}{2} f^{(c)}_{\varnothing; s} C^{(6)}_{m|V_1}(x, y, z) & (C, B) = (\varnothing, V_1). \end{cases} \quad (5.2)$$

On these functions $\hat{\sigma} \in \text{Aut}_c(V_1)$ acts exchanging $y$ with $z$, just as on the terms coming from the derivative of the $Y$-term with respect to $(m|V_1)$:

$$\sum_{\hat{\sigma} \in \mathbb{Z}_2} \sigma \cdot f^{(c)}_{m|V_1:x}(x, y, z) = f^{(c)}_{m|V_1:x}(x, y, z) + f^{(c)}_{m|V_1:x}(y, x, z). \quad (5.3)$$
Next, we obtain the second line in eq. (4.3) (the ‘swap-term’). We have chosen $y^1 = s$ ($\beta = 1$), so

$$\frac{\partial W[J, J]}{\partial s_{c-1}(m | V_1; \beta = 1, \rho = 2, 3)}(x, y, z) = G_{Q_1}^{(6)}(x, y, z)$$

(5.4a)

$$\frac{\partial W[J, J]}{\partial s_{c-2}(m | V_1; \beta = 1, \rho = 2)}(x, y, z) = G_{F_2}^{(6)}(x, z, y)$$

(5.4b)

$$\frac{\partial W[J, J]}{\partial s_{c-1}(m | V_1; \beta = 1, \rho = 3)}(x, y, z) = G_{F_2}^{(6)}(x, y, z)$$

(5.4c)

$$\frac{\partial W[J, J]}{\partial s_{c-2}(m | V_1; \beta = 1, \rho = 3)}(x, y, z) = G_{F_2}^{(6)}(x, z, y)$$

(5.4d)

In this case the set $\text{Br}(\varnothing, \rho, c)$ are empty, for any values of $\rho$ and $c$. Therefore, the sum over the $H$-terms vanishes. For each $(x, y, z) \in \mathcal{F}_{3,3}$,

$$G_{m|V_1}^{(6)}(x, y, z) = \left(\frac{-2\lambda}{E_x}\right) \times \left\{ \sum_{c=1}^{3} \left[ I_{m|V_1|x_c}^{(c)}(x, y, z) + \right. \right.$$  

$$+ \left. I_{m|V_1|x_c}^{(c)}(y, x, z) \right\} \times \left\{ \sum_{c=1}^{3} \left[ \sum_{b_c} \frac{1}{E(x_c, b_c)} \left[ G_{m|V_1}^{(6)}(x, y, z) - G_{m|V_1}^{(6)}(x, b_c, y, z) \right] \right. \right.$$  

$$+ \left. \left[ I_{V_1|x_c}^{(c)}(z, y) + I_{V_1|x_c}^{(c)}(y, z) \right] \cdot G^{(2)}(x) + \right. \right.$$  

$$\left. \left[ I_{\varnothing|x_c}^{(c)} G_{m|V_1}^{(6)}(x, y, z) \right] \right\} \right\}.$$  

*If $s = x$ is outgoing momentum of the boundary graph $\emptyset_{j_{[1]}}$. For $s = (x_1, y_2, y_3)$ an outgoing momentum of $V_1$ we derive now the SDE for $G_{m|V_1}^{(6)}$. Then $\mathcal{R} = V_1$, $Q = m$, by definition. The $I_{c, \mathcal{B}}^{(c)}$-coefficients are given by

$$I_{m, \varnothing}^{(c)}(x, y, z) = I_{m|x_c}^{(c)}(z) \cdot G_{V_1}(x, y), \quad I_{\varnothing, m}^{(c)}(x, y, z) = I_{\varnothing|x_c}^{(c)} \cdot G_{V_1|m}(x, y, z).$$

The $H_{c, \mathcal{B}}^{(c, \tau)}$-terms are computed from the set

$$\text{Br}(\emptyset_{j_{[1]}}, \beta = 1, c) = \begin{cases} \{2\} & c = 1, \\ \emptyset & \text{otherwise}, \end{cases}$$

since only the colour-1 swap at the vertex $s$ with the vertex with outgoing momentum $y^2$ (also in $V_1$) separates $\emptyset_{j_{[1]}}$. The only contributions are therefore

$$H_{\varnothing, m}^{(c=1, \tau=2)}(X) = G^{(2)}(x) \cdot G_{m|V_1}^{(4)}(y, z), \quad H_{m, \varnothing}^{(c=1, \tau=2)}(X) = G_{m|V_1}^{(4)}(x, z) \cdot G^{(2)}(y).$$
Thus \( G_{V_1|m}^{(6)} \) satisfies, for all \((x, y, z) \in F_{3,3}\),

\[
G_{V_1|m}^{(6)}(x, y, z) = \left( \frac{-2\lambda}{E_x} \right) \times \left\{ \sum_{c=1}^{3} f_{m|V_1:s_c}^{(c)}(z, x, y) + f_{m|V_1:s_c}^{(c)}(z, y, x) \right\} \quad (5.6)
\]

\[
+ \frac{1}{E(y_1, x_1)} \left[ G_{m|m|m}^{(6)}(x, y, z) - G_{m|m|m}^{(6)}(y_1, x_2, x_3, y, z) \right] 
+ \frac{1}{E(z_1, x_1)} \left[ G_{Q_1}^{(6)}(x, y, z) - G_{Q_1}^{(6)}(z_1, x_2, x_3; y, z) \right] 
+ \frac{1}{E(x_2, y_2)} \left[ G_{V_2|m}^{(6)}(x, y, z) - G_{V_2|m}^{(6)}(x_1, y_2, x_3; y, z) \right] 
+ \frac{1}{E(x_2, y_2)} \left[ G_{F_{3,12}}^{(6)}(x, y, z) - G_{F_{3,12}}^{(6)}(x_1, y_2, y_3; z) \right] 
+ \frac{1}{E(x_3, y_3)} \left[ G_{V_3|m}^{(6)}(x, y, z) - G_{V_3|m}^{(6)}(x_1, y_2, x_3; z) \right] 
+ \frac{1}{E(x_3, y_3)} \left[ G_{F_{2,13}}^{(6)}(x, y, z) - G_{F_{2,13}}^{(6)}(x_1, y_2, y_3; z) \right] 

- \sum_{c=1}^{3} \left[ \sum_{b_c} \frac{1}{E(s_c, b_c)} \left[ G_{V_1|m}^{(6)}(x, y, z) - G_{V_1|m}^{(6)}(x_c, y_c, z_c) \right] 
+ \left[ f_{V_1:s_c}^{(c)}(z, y) + f_{V_1:s_c}^{(c)}(y, z) \right] \right],
\]

5.2. Schwinger-Dyson equation for \( G_{\Theta|\Theta|\Theta}^{(4)} \). There is only one SDE for the ‘disconnected-boundary’ 4-point function. For every \((x, y, z) \in F_{3,3}\),

\[
G_{\Theta|\Theta|\Theta}^{(4)}(x, y) = \left( \frac{-2\lambda}{E_x} \right) \times \sum_{c=1}^{3} \left( \sum_{q_c} G_{\Theta}^{(2)}(x_c, q_c) \cdot G_{\Theta|\Theta}(x, y) + G_{\Theta}^{(2)}(x) f_{\Theta|\Theta}(y) \right) 
+ \sum_{b_c} \frac{1}{E(b_c, x_c)} \left[ G_{\Theta|\Theta}^{(4)}(x, y) - G_{\Theta|\Theta}^{(4)}(b_c x_c, y) \right] 
+ \frac{1}{E(y_c, x_c)} \left[ G_{\Theta|\Theta}^{(4)}(x, y) - G_{\Theta|\Theta}^{(4)}(b_c x_c, y) \right] 
+ f_{x_c, \Theta|\Theta}(x, y) + f_{x_c, \Theta|\Theta}(y, x) \right). \quad (5.7)
\]

5.3. Schwinger-Dyson equation for \( G_{\Theta|\Theta|\Theta}^{(6)} \). Similarly, since one can permute the arguments of \( G_{m|m|m}^{(6)} \), it satisfies only one SDE:

\[
\left( 1 + \frac{2\lambda}{E_x} \sum_{c=1}^{3} \sum_{q_c} G_{\Theta}^{(2)}(x_c, q_c) \right) \times G_{\Theta|\Theta|\Theta}^{(6)}(x, y, Z) 
= \left( \frac{-2\lambda}{E_x} \right) \sum_{c=1}^{3} \left( f_{\Theta|\Theta}(y) G_{\Theta|\Theta}(x, z) + f_{\Theta|\Theta}(z) G_{\Theta|\Theta}(x, y) \right) + G_{\Theta|\Theta}(x) \cdot f_{\Theta|\Theta}(y, z) 
- \sum_{b_c} \frac{1}{E(x_c, b_c)} \left[ G_{\Theta|\Theta}^{(6)}(x, y, z) - G_{\Theta|\Theta}^{(6)}(b_c x_c, y, z) \right] 
+ \frac{1}{E(y_c, x_c)} \left[ G_{\Theta|\Theta}^{(6)}(z, x, y) - G_{\Theta|\Theta}^{(6)}(y, x_c, y) \right] \right). \quad (5.8)
\]
\[ + \frac{1}{E(x, y, z)} \left[ G^{(6)}_{\varnothing} (y, x, z) - G^{(6)}_{\varnothing} (y, x, z) \right] \]
\[ + \sum_{\sigma \in \mathcal{S}(3)} \sigma \cdot f^{(c)}_{\varnothing \varnothing \varnothing} (x, y, z) \].

We kept the graph-notation in order to ease the reading of the graph-movements. Equivalently,
\[ \left( 1 + \frac{2\lambda}{E_x} \sum_{c=1}^{3} \sum_{q_c} G^{(2)} (x, q_c) \right) \times G^{(6)}_{m|m|m} (x, y, z) \]
\[ = \left( \frac{-2\lambda}{E_x} \right) \sum_{c=1}^{3} \left( f^{(c)}_{m;x_c} (y) G^{(4)}_{m|m} (x, z) + f^{(c)}_{m;x_c} (z) G^{(4)}_{m|m} (x, y) \right. \]
\[ + G^{(2)} (x) \cdot f^{(c)}_{m|m} (y, z) \]
\[ - \sum_{b_c} \frac{1}{E(x, b_c)} \left[ G^{(6)}_{m|m|m} (x, y, z) - G^{(6)}_{m|m|m} (b_c x_c, y, z) \right] \]
\[ + \frac{1}{E(y_c, x_c)} \left[ G^{(6)}_{m|V_x} (z, x, y) - G^{(6)}_{m|V_x} (z, y_c x_c, y) \right] \]
\[ + \frac{1}{E(z_c, x_c)} \left[ G^{(6)}_{m|V_x} (y, y, z) - G^{(6)}_{m|V_x} (y, y_c x_c, z) \right] \]
\[ + \sum_{\sigma \in \mathcal{S}(3)} \sigma \cdot f^{(c)}_{m|m|m} (x, y, z) \].

6. Outlook

The large-\( N \) limit of the disconnected-boundary SDE should be analysed in order to access their physical significance. At leading order, the melonic approximation [OPVW15] of these equations, is expected to yield closed equations. A significant progress in this direction has been undertaken in [PPTW18], whose techniques will be applied to the present disconnected-boundary SDE in a next project [Pas18]. The main question is the nature of the recursion for the disconnected correlation functions; this is based in the idea of the ubiquitous ‘topological recursion’ (TR) [Eyn14; Bor17; Sul18; ACP+13; EO07] (and references therein) — which for tensor field theory would be rather a geometrical recursion. This is justified by the expected emergence of Gurău’s degree [GR12], which replaces the genus. The former integer encodes geometrical information [BGRR11]. The blobbed TR for quartic tensor models has already been obtained [BD18]—yet it would be interesting to look for a TR for tensor field theory\(^\text{12}\).

On the purely mathematical side, systems of graph-group actions can be extended to Lie groups actions and to calculi in infinitely many graph-variables by using rigorous analytic tools. It would be also interesting to consider the coefficient functions \( u_g \) directly in certain algebra (of functions) and one could dispense with the functions \( \mathcal{V}(g) \to \mathbb{C} \) by using instead directly algebras.

Finally, the (symmetric) monoidal structure on the set of boundary graphs emerges in a natural way. This guides us towards the language of Topological Quantum Field Theories (TQFT) [Ati89]. Since these boundary graphs triangulate boundary states, an interesting program would be to obtain discrete TQFT from matrix models and TQFT with observables

\(^{12}\) The difference between tensor models and tensor field theory is here substantial. The former usually focus on numerical observables \( \langle \text{Tr} (\phi, \bar{\phi}) \rangle \in \mathbb{C} \) and the latter on functions (or distributions) \( G_B : \mathcal{V}(B) \to \mathbb{C} \). In contrast, the loop equations, Ward Identities [IMM17] and SDE [Gur12] are algebraic in the for tensor models, whereas for tensor field theory ‘loop equations’ [Pér18; PPW17] are integro-differential, as shown also in [PPTW18] explicitly.
from tensor models, or enframe these in Oeckl’s positive boundary formalism (op. cit.), which also facilitates the ‘gluing-boundary’ procedure that TQFT provides. In the tensor and matrix models case, the gluing of boundaries should be implemented as an operation \( \wedge \) on two correlation functions sharing a boundary state \( \mathcal{A} \). \( G_{\mathcal{A}[B|\cdots|C']} \wedge \mathcal{A} G_{\mathcal{A}[B'|\cdots|C']} \), which should be related to \( G_{B[B'|\cdots|C'|C} \) due to their geometrical interpretation.

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APPENDIX A. THE FIRST COEFFICIENTS OF THE Y-TERM

For completeness, we quote the first coefficients of the Y-term, keeping in mind the notation simplification (Table 1). The computation of these functions is presented in detail in [Pér18]. As before, the set equality \( \{a, b, c\} = \{1, 2, 3\} \) holds.

\[
\hat{f}_{(a)}^{(a)}_{m; s_a}(x) = G_{(a)}^{(4)}(x, s_a, x_b, x_c) + \sum_{c \neq a, q_b \in \mathbb{Z}} G_{(a)}^{(4)}(x; s_a, q_b, x_c) + \sum_{q_b, q_c} G_{(a)}^{(4)}(x; s_a, q_b, q_c),
\]

\[
\hat{f}_{(a)}^{(a)}_{V_6; s_a}(x, y) = \frac{1}{3} \left( G_{(a)}^{(6)}(s_a, x, x_b, x_c, x, y) + \text{cyclic perm. in } (s_a, x_b, x_c, x) \text{ and } y \right) + \frac{1}{3} \left( G_{(a)}^{(6)}(s_a, x_b, y_c; x, y) + \text{cyclic perm.} \right)
\]

\[
+ \sum_{q_b} G_{(a)}^{(6)}(x; y; s_a, q_b, y_c) + \sum_{q_c} G_{(a)}^{(6)}(x; y; s_a, q_c, y_b)
\]

\[
+ \frac{1}{2} \sum_{q_b, q_c} G_{(a)}^{(6)}(s_a, q_b, q_c; x, y),
\]

\[
\hat{f}_{(a)}^{(a)}_{V_6; s_a}(x, y) = \frac{1}{3} \left( \sum_{q_c} G_{(a)}^{(6)}(s_a, q_b, q_c; x, y) + \text{cyclic perm.} \right) + G_{(a)}^{(6)}(s_a, y_b, x_c; x, y)
\]

\[
+ G_{(a)}^{(6)}(x; s_a, x_b, x_c; x, y) + \sum_{q_b} G_{(a)}^{(6)}(x; y; s_a, q_b, y_c)
\]

\[
- \frac{1}{2} \sum_{q_b, q_c} G_{(a)}^{(6)}(s_a, q_b, q_c; x, y),
\]

\[
\hat{f}_{(a)}^{(a)}_{m; s_a}(x, y) = \left( \sum_{q_b, q_c} G_{(a)}^{(6)}(s_a, q_b, q_c; x, y) + \text{cyclic perm.} \right) + G_{(a)}^{(6)}(x; s_a, x_b, y_c, y)
\]

\[
+ G_{(a)}^{(6)}(x; s_a, y_b, y_c, y) + \sum_{q_c} G_{(a)}^{(6)}(x, s_a, y_b, q_c, y)
\]

\[
+ \sum_{q_b} G_{(a)}^{(6)}(x, s_a, q_b, y_c, y) + \sum_{q_c} G_{(a)}^{(6)}(x, y, s_a, y_b, q_c)
\]

\[
+ \sum_{q_b} G_{(a)}^{(6)}(x, y, s_a, q_b, y_c) + G_{(a)}^{(6)}(x, y, s_a, y_b, y_c),
\]
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