**SOME BOUNDS ON THE SIZE OF MAXIMUM G-FREE SETS IN GRAPHS**

YASER ROWSHAN

**Abstract.** For given graph $H$, the independence number $\alpha(H)$ of $H$, is the size of the maximum independent set of $V(H)$. Finding the maximum independent set in a graph is a NP-hard problem. Another version of the independence number is defined as the size of the maximum induced forest of $H$, and called the forest number of $H$, and denoted by $f(H)$. Finding $f(H)$ is also a NP-hard problem. Suppose that $H = (V(H), E(H))$ be a graph, and $G$ be a family of graphs, a graph $H$ has a $G$-free $k$-coloring if there exists a decomposition of $V(H)$ into sets $V_i$, $i = 1, 2, \ldots , k$, so that $G \not\subseteq H[V_i]$ for each $i$, and $G \in G$. $S \subseteq V(H)$ is $G$-free, where the subgraph of $H$ induced by $S$, be $G$-free, i.e. it contains no copy of $G$. Finding a maximum subset of $H$, so that $H[S]$ be a $G$-free graph is a very hard problem as well. In this paper, we study the generalized version of the independence number of a graph. Also giving some bounds about the size of the maximum $G$-free subset of graphs is another purpose of this article.

1. Introduction

All graphs considered here are undirected, simple, and finite graphs. For given graph $H = (V(H), E(H))$, its maximum degree and minimum degree are denoted by $\Delta(H)$ and $\delta(H)$, respectively. The degree and neighbors of $v$ in $H$, denoted by $deg_H(v)$ (deg($v$)) and $N_H(v)$, respectively. Suppose that $H$ be a graph, and let $V$ and $V'$ be two disjoint subsets of $V(H)$. Suppose that $W$ is any subset of $V(H)$, the induced subgraph $H[W]$ is the graph whose vertex set is $W$ and whose edge set consists of all of the edges in $E(H)$ that have both endpoints in $W$. The set $E(V, V')$ is the set of all the edges $vv'$, which $v \in V$ and $v' \in V'$. Recall that an independent set is a set of vertices in a graph, no two of which are adjacent. A maximum independent set in a graph is an independent set in which the graph contains no larger independent set. The independence number of a graph $H$ is the cardinality of a maximum independent set, and denoted by $\alpha(H)$. This problem was solved by Erdős, and after that by Moon and Moser in [13]. There are very few works about counting the number of maximum independent sets, see [8, 10, 12] and [10]. Finding a maximum independent set in a graph is a NP-hard problem.

Another version of the independence number is the forest number of a graph. Let $H$ be a graph, and $S \subseteq V(H)$, if $H[S]$ is acyclic, then $S$ is called the induced forest of $H$. The forest number of a graph $H$ is the size of a maximum induced forest of $H$, and is denoted by $f(H)$. The decycling number $\phi(H)$ of a graph $H$ is the smallest number of vertices which can be removed from $H$ so that the resultant graph contains no cycle. Thus, for a graph $H$ of order $n$, $\phi(H) + f(H) = n$. The decycling number was first proposed by Beineke and Vandell [3]. There is a fairly large literature of papers dealing with the forest number of a graph. See for example [12, 14], and [10].

The first item of the next results is attributed to P.K. Kwok and has come as an exercise in [15], and the second item discussed in [4]. Suppose that $|V(H)| = n$, $\Delta(H) = \Delta$ and $|E(H)| = e$, then:

- (Kwok Bound): $\alpha(H) \leq n - \frac{\Delta}{2}$,
- (Borg Bound): $\alpha(H) \leq n - \left\lceil \frac{n-1}{\Delta} \right\rceil$.

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The Borg Bound is an efficiently countable upper bound for the $\alpha(H)$. However, generally gives an estimation, greater than or equal to the Kwok Bound.

**Theorem 1.** $[7]$ Let $H$ is a graph with $n$ member, and $p$ is an integer, so that (A) holds, then: thus:

$$\alpha(H) \geq \frac{2n}{p}. \tag{A}$$

For any clique $K$ in $H$ there exists a member of $V(K)$ say $v$, so that $\deg(v) \leq p - |V(K)| - 1$.

1.1. $G$-free coloring.

The conditional chromatic number $\chi(H, P)$ of $H$, is the smallest integer $k$, for which there exists a decomposition of $V(H)$ into sets $V_i$, $i = 1, 2, \ldots, k$, so that for each $i$, $H[V_i]$ satisfies the property $P$, where $P$ is a graphical property and $H[V_i]$ is the induced subgraph on $V_i$. Harary in 1985 presented this extension of graph coloring $[8]$. Suppose that $G$ is a family of graphs, when $P$ is the feature that a subgraph induced by each color class does not contain any copy of members of $G$, we write $\chi_G(H)$ instead of $\chi(H, P)$. A graph $H$ has a $G$-free $k$-coloring if there exists a decomposition of $V(H)$ into sets $V_i$, $i = 1, 2, \ldots, k$, so that for each $i$, $H[V_i]$ does not include any copy of the members of $G$. For simplicity of notation, if $G = \{G\}$, then we write $\chi_G(H)$ instead of $\chi_G(H)$. An ordinary $k$-coloring of $H$ can be viewed as $G$-free $k$-coloring of a graph $H$ by taking $G = \{K_2\}$.

For any two graphs $H$ and $G$, recall that $\chi_G(H)$ is the $G$-free chromatic number of the graph $H$, now suppose that $S$ is a maximum subset of $V(H)$, so that $H[S]$ is $G$-free, therefore it is easy to say that $|S| \geq \frac{n[H]}{\chi_G(H)}$. By considering $H = K_6$ and $G = K_3$, one can check that $\chi_G(H) = 3$, that is $|S| = 2 = \frac{n[H]}{\chi_G(H)}$, which means that this bond is sharp. Set $G = \{C_n, n \geq 3\}$ and let $H$ be a graph, therefore one can say that $|S| = f(H)$, where $H[S]$ is $G$-free and $S$ has the maximum size possible. In this article, we prove results as follow:

**Theorem 2.** Let $H$ and $G$ are two graphs, where $|V(H)| = n, \Delta(H) = \Delta, |E(H)| = e_H, |E(G)| = e_G$ and $\delta(G) = \delta$. Then:

- $|S| \geq n + \frac{e_G + e_H - \Delta}{\delta}$
- $|S| \geq n - \frac{\delta n_3(S) + (\delta + 1)n_{\delta + 1}(S) + \ldots + \Delta n_{\Delta}(S)}{\Delta}$
- $|S| \leq n - \frac{\delta n_3(S) + (\delta + 1)n_{\delta + 1}(S) + \ldots + \Delta n_{\Delta}(S)}{\Delta}$

**Theorem 3** (Main theorem). Let $H$ and $G$ are two graphs, where $|V(H)| = n$ and $\delta(G) = \delta$. Suppose that $P$ is a positive integer, which for each connected component $X \in R(H)$, there exists a vertex of $X$ say $x$, so that $\deg_H(x) \leq P - |X| - \delta$. Then:

$$|S| \geq \frac{(\delta + 1)n}{P}$$

Where $S$ has the maximum size possible and $H[S]$ is $G$-free.

2. Main results

Let $H$ and $G$ are two graphs, in this section, we give some upper and lower bounds on the size of the maximum $G$-free subset of $H$. Next results offers further investigations to get some good bounds on the size of the maximum $G$-free subset of $H$ and exact results, when feasible. The next results are examples of two graphs $H$ and $G$, in which the maximum $G$-free subsets of $H$ easily obtained.

- If $|V(H)| = n$, then $|S| = n$ if and only if (iff) $H$ is $G$-free.
- If $|V(H)| = n$, then $|S| = n - 1$ iff either $H \cong G$ or $|V(G)| = n$ and $G \subseteq H$.
- If $H \cong K_n$, and $G$ has $m$ members, where $m \leq n$, then $|S| = m - 1$.
- If $|V(H)| = n$ and $G$ has $n - 1$ members, then $|S| = n - 2$ iff $G \subseteq H \setminus \{v\}$ for each $v \in V(H)$. 


In the next two theorems, we give lower and upper bounds on the size of the maximum $G$-free subset of $H$, in terms of the number of vertices and edges, maximum degree, and minimum degree of $H$ and $G$.

**Theorem 4.** Suppose that $H$ and $G$ are two graphs, where $|V(H)| = n, \Delta(H) = \Delta, |E(H)| = e_H, |E(G)| = e_G$, and $\delta(G) = \delta$. Suppose that $S \subseteq V(H)$ is maximum $G$-free. Then:

$$|S| \geq n + \frac{e_G + e_{H'} - e_H - \Delta}{\delta}.$$  

Where $e_{H'} = |E(H[V(H) \setminus S])|$. 

**Proof.** Suppose that $S \subseteq V(H)$, where $H[S]$ is $G$-free and $S$ has maximum size as possible, and $|S| = m$. As $S$ is maximum, for each $v \in V(H) \setminus S$ so $H[S \cup \{v\}]$ contains at least one copy of $G$. Therefore, since $\deg_H(v) \leq \Delta$, thus $E(H[S]) \geq e_G - \Delta$. As each vertex of $V(H) \setminus S$ has at least $\delta$ neighbors in $S$, so we have:

$$e_H \geq (n - m)\delta + e_G - \Delta + e_{H'}.$$  

Thus, it can be checked that:

$$m\delta \geq n\delta + e_G + e_{H'} - e_H - \Delta.$$  

Hence, $m \geq n + \frac{e_G + e_{H'} - e_H - \Delta}{\delta}$. Which means that the proof is complete.

**Theorem 5.** Suppose that $H$ and $G$ are two graphs, where $H$ has $n$ members, $\Delta(H) = \Delta$. Let $S$ be a maximum $G$-free subset of $V(H)$. Then:

$$n - \frac{\delta n_\delta(S) + (\delta + 1)n_{\delta+1}(S) + \ldots + \Delta n_\Delta(S)}{\delta} \leq |S| \leq n - \frac{\delta n_\delta(S) + (\delta + 1)n_{\delta+1}(S) + \ldots + \Delta n_\Delta(S)}{\Delta}.$$  

**Proof.** Assume that $S \subseteq V(H)$, where $H[S]$ is $G$-free and $S$ has maximum size as possible, and $|S| = m$. Suppose that $n_i(S)$ be the vertices of $V(H) \setminus S$, which has exactly $i$ neighbors in $S$. Now by maximality of $S$, it is easy to check that $n_i(S) = 0$ for $i = 0, 1, 2, \ldots, \delta - 1$. So:

$$n - m = n_\delta(S) + n_{\delta+1}(S) + \ldots + n_m(S).$$  

As each vertex of $V(H) \setminus S$ has at least $\delta$ and at most $\Delta$ neighbors in $S$, then:

$$\delta(n - m) \leq \delta n_\delta(S) + (\delta + 1)n_{\delta+1}(S) + \ldots + \Delta n_\Delta(S) = \sum_{v \in V(H) \setminus S} \deg(v) \leq (n - m)\Delta.$$  

Therefore by Equation 2 it can be checked that:

- $m \geq n - \frac{\delta n_\Delta(S) + (\delta + 1)n_{\delta+1}(S) + \ldots + \Delta n_\Delta(S)}{\delta}$
- $m \leq n - \frac{\delta n_\Delta(S) + (\delta + 1)n_{\delta+1}(S) + \ldots + \Delta n_\Delta(S)}{\Delta}$

Which means that the proof is complete.

Combining Theorems 4 and 5, The correctness of Theorem 2 is obtained. In the next theorem, we generalize Theorem 4 to specify an appropriate lower bound for the size of the maximum $G$-free subgraphs of graph $H$. We need to determine a series of special subgraphs of $H$, which are expressed in the following definition:

**Definition 6.** Let $H$ and $G$ be two graphs, where $|V(H)| = n$ and $\delta(G) = \delta$. Suppose that $S$ be the maximum subset of $V(H)$ so that $H[S]$ is $G$-free. Therefore as $S$ has maximum size and $H[S]$ is $G$-free, then for each $v \in V(H) \setminus S$, it can be say that $H[S \cup \{v\}]$, contain at least one copy of $G$. In other word for each $v \in V(H) \setminus S$, there is at least one copy of $G - v$ in $H[S]$. Now, for each $v \in V(H) \setminus S$ define $A_v$ as follows:

$$A_v = \{G_v, G_v \cong G - v \subseteq H[S]\}.$$
Assume that $N^i_v = N(v) \cap G^i_v$, for each $i \in \{1, \ldots, |A_v|\}$. Now for each $i \in \{1, \ldots, |A_v|\}$, define $M^i_v$ as follow:

$$M^i_v = \{u \in V(H) \setminus S, \ N(u) \cap S = N^i_v\}.$$

Therefore, define $R(H)$ as follow:

$$R(H) = \{\text{Clique of } H[M^i_v], \text{ for each } i \in [A_v], \text{ and each } v \in V(H) \setminus S\}.$$

Where $[A_v] = \{1, 2, \ldots, |A_v|\}$.

To prove the next results, we present an argument that is similar to the proof of Theorem 11 in [7]. However, in our proof, we carefully choose a maximum $G$-free set $S$ in the graph $H$, so that $|E(S, V(H) \setminus S)|$ is minimize and $H[S]$ is $G$-free. With this choice of $S$, we establish a property on $H$ by considering the operation of replacing a vertex in $S$ with $V(H) \setminus S$, to get a smaller number of edges between $V(H) \setminus S$ and $S$.

**Theorem 7.** Let $H$ and $G$ are two graphs, where $|V(H)| = n$ and $\delta(G) = \delta$. Suppose that $P$ is a positive integer, where for each $X \in R(H)$, there exists a vertex of $X$ say $x$, so that $\deg_H(x) \leq P - |X| - \delta$. Then:

$$|S| \geq \frac{(\delta + 1)n}{P}$$

Where $S \subseteq V(H)$, and $S$ has the maximum size possible, so that $H[S]$ is $G$-free.

**Proof.** Suppose that $m$ is the size of the maximum $G$-free subset of $V(H)$. Now set $A$ as follow:

$$A = \{S \subseteq V(H), G \not\subseteq H[S], |S| = m\}.$$

Therefore, for any member of $A$, say $S$, we define $\beta(S)$ as follow:

$$\beta(S) = \sum_{y \in V(H) \setminus S} |N(y) \cap S| = \sum_{x \in S} |N(x) \cap (V(H) \setminus S)|.$$

In other word, $\beta(S) = |E(S, V(H) \setminus S)|$. Now we define $B$ as follow:

$$B = \{\beta(S), S \in A\}.$$

Let $\beta$ be a minimal members of $B$, and without loss of generality suppose that $\beta = \beta(S^*)$, that is $|E(S^*, V(H) \setminus S^*)|$ is minimize. Assume that $\gamma_i(S^*)$ be the vertices of $V(H) \setminus S^*$, such that its vertices have exactly $i$ neighbors in $S^*$. Since $S^* \in A$, it is easy to check that $\gamma_i(S^*) = 0$ for $i = 0, 1, 2, \ldots, \delta - 1$. Hence, one can say that:

$$|V(H) \setminus S| = n - m = \gamma_\delta(S^*) + \gamma_{\delta+1}(S^*) + \ldots + \gamma_m(S^*).$$

Furthermore, by considering $\beta(S^*) = |E(S^*, V(H)) \setminus S^*|$, and by Equation 8 and 9 it is easy to say that:

$$\sum_{y \in S^*} |N(y) \cap (V(H) \setminus S^*)| = \delta \gamma_\delta(S^*) + (\delta + 1)\gamma_{\delta+1}(S^*) + \ldots + m\gamma_m(S^*) = \sum_{i=\delta}^m i\gamma_i(S^*).$$

Multiplying Equation 9 by $\delta + 1$, and subtracting Equation 8 we acquire the next:

$$(\delta + 1)(n - m) - \sum_{y \in S^*} |N(y) \cap (V(H) \setminus S^*)|$$

$$= (\delta + 1)(\gamma_\delta(S^*) + \gamma_{\delta+1}(S^*) + \ldots + \gamma_m(S^*)) - \sum_{y \in S^*} |N(y) \cap (V(H) \setminus S^*)|$$

$$= \gamma_\delta(S^*) - \gamma_{\delta+2}(S^*) - \ldots - (m - (\delta + 1))\gamma_m(S^*) \leq \gamma_\delta(S^*).$$

As $S^* \in A$, therefore by maximality of $S^*$, for each vertex of $V(H) \setminus S$ say $v$, one can say that $H[S \cup \{v\}]$ contains at least one copy of $G$, namely $G_v$. Suppose that $v'$ be a vertex of $V(G_v)$ with
minimum degree in $G_w$. Let $N(v') \cap V(G') = X_\delta$. Hence $X_\delta$ is a fixed subset of $S^*$ where $|X_\delta| = \delta$.

Now we define $Y_{X_\delta}$ as follow:

(7) 
$$Y_{X_\delta} = \{w \in V(H) \setminus S^*, \forall w \in S^* = X_\delta\}.$$ 

In other word, assume that $Y_{X_\delta}$ is the set of all vertices in $V(H) \setminus S^*$, so that adjacent to each vertex of $X_\delta$ but no other vertices of $S^* \setminus X_\delta$, so every vertex in $Y_{X_\delta}$ has no neighbor in $S^* \setminus X_\delta$. Therefore, we have the next claim.

**Claim 8.** $H[Y_{X_\delta}]$ is a clique in $H$.

**Proof.** By contradiction, suppose that there exist at least two vertices of $Y_{X_\delta}$, say $x, x'$ so that $xx' \notin E(H)$. Therefore, since $S^* \in A$, one can check that $H[S^* \cup \{x\}]$ and $H[S^* \cup \{x'\}]$ contains at least one copy of $G$, say $G_x$ and $G_{x'}$, respectively. Now, suppose that $x'' \in X_\delta$, and set $S'' = S^* \setminus \{x''\} \cup \{x, x'\}$, hence it is easy to see that:

$$|S'| = |S^*| + 1.$$

And $|N(x) \cap S'| = |N(x') \cap S'| = \delta - 1$, that is $H[S']$ is $G$-free, a contradiction to maximality of $S^*$. So $xx' \in E(H)$ for each $x, x' \in Y_{X_\delta}$, that is $H[Y_{X_\delta}]$ is a clique in $H$. ■

Therefore, by Claim 8 it can be checked that $Y_{X_\delta} \in R(H)$. Now for a fixed vertex of $X_\delta$ say $x$, assume that:

(8) 
$$|N(x) \cap (V(H) \setminus S^*)| + |Y_{X_\delta}| + \delta \geq P.$$

So, by considering the vertices of $Y_{X_\delta}$, we have the following claim:

**Claim 9.** For each $w \in Y_{X_\delta}$, we have $|N(w) \cap (V(H) \setminus S^*)| \geq |N_H(x)| - |N(x) \cap S^*|.$

**Proof.** By contradiction, suppose that there exists a vertex of $Y_{X_\delta}$ say $w$ so that:

$$|N(w) \cap (V(H) \setminus S^*)| \leq |N_H(x)| - |N(x) \cap S^*| - 1.$$

Hence, as $w \in Y_{X_\delta}$, so $|N(w) \cap S^*| = \delta$. Therefore, it can be checked that $|N_H(w)| \leq |N_H(x)| - |N(x) \cap S^*| - \delta - 1$. Then set $S'' = S^* \setminus \{x\} \cup \{w\}$, hence seeing $|S''| = |S^*|$ is obvious. Also as $w \in Y_{X_\delta}$, $wx \in E(H)$, and $x \in S^*$, it can be checked that $H[S'']$ is $G$-free. Now by considering $\beta(S'')$ we have the following fact:

**Fact 9.1.** $\beta(S'') \leq \beta(S^*) - 1$.

**proof of the fact:** As $S'' = S^* \setminus \{x\} \cup \{w\}$, one can say that $\beta(S'') = \beta(S^*) + |N(x) \cap S^*| - |N(x) \cap (V(H) \setminus S^*)| + |N(w) \cap (V(H) \setminus S^*)| - |N(w) \cap S^*|$. As $|N(w) \cap S^*| = \delta$, and $|N(w) \cap (V(H) \setminus S^*)| \leq |N_H(x)| - |N(x) \cap S^*| - 1$, one can check that $|N(x) \cap S^*| - |N(x) \cap (V(H) \setminus S^*)| + |N(w) \cap (V(H) \setminus S^*)| - |N(w) \cap S^*| \leq -\delta$, that is:

(9)  
$$\sum_{x \in S''} |N(x) \cap (V(H) \setminus S^*)| = \beta(S'') \leq \sum_{x \in S^*} |N(x) \cap (V(H) \setminus S^*)| - \delta = \beta(S^*) - \delta.$$

Therefore by Fact 9.1 $\beta(S'') \leq \beta(S^*) - 1$, where $S'' \in A$. A contradiction to minimality of $\beta(S^*)$. Hence for each $w \in Y_{X_\delta}$, we have $|N(w) \cap (V(H) \setminus S^*)| \geq |N_H(x)| - |N(x) \cap S^*|$. Which means that the proof of the claim is complete. ■

Therefore, by Claim 9.1 for each $w \in Y_{X_\delta}$ and each $x \in X_\delta$, we have the following equation:

$$|N(w) \cap (V(H) \setminus S^*)| \geq |N_H(x)| - |N(x) \cap S^*| = |N(x) \cap (V(H) \setminus S^*)|.$$

Therefore:

(10)  
$$|N_H(w)| = |N(w) \cap (V(H) \setminus S^*)| + |N(w) \cap S^*| \geq |N(x) \cap (V(H) \setminus S^*)| + 1.$$
Therefore, as $Y_{X_\delta} \in R(H)$ and by Equation 11, $\deg(y) \geq P + 1 - |Y_{X_\delta}| - \delta$, for each $y \in Y_{X_\delta}$, which is a contradiction to assumption. Hence $|N(x) \cap (V(H) \setminus S^*)| \leq P - |Y_{X_\delta}| - \delta - 1$ for each $x \in X_\delta$. As $X_\delta \subseteq S^*$, so by Equation 6,

\[(\delta + 1)n \leq \gamma(S^*) + \sum_{y \in S^*} |N(y) \cap (V(H) \setminus S^*)| + (\delta + 1)m\]

\[\leq \sum_{y \in S^*} (|Y_{X_\delta}| + |N(y) \cap (V(H) \setminus S^*)| + \delta + 1) \leq |S^*|.P = mp.\]

Therefore, $(\delta + 1)n \leq m.P$, thus:

\[|S^*| = m \geq \frac{(\delta + 1)n}{P}\]

Which means that the proof is complete.

In Theorem 7 if we take $G = K_2$, then we get Theorem 1. By setting $\mathcal{G} = \{C_n, n \geq 3\}$ and any arbitrary graph for $H$, it is easy to say that $|S| = f(H)$, where $S$ is the maximum subset of $V(H)$, so that $H[S]$ is $\mathcal{G}$-free and $f(H)$ is the forest number of $H$. In particular, by setting $C$ as 2-regular connected graph in Theorem 7 we can show that the following result is true:

**Theorem 10.** Let $H$ and $G$ are two graphs, where $|V(H)| = n$ and $G \in \mathcal{C}$, that is $G$ is a 2-regular connected graph (cycle). Suppose that $P$ is a positive integer, where for each $X \in R(H)$ there exists a vertex of $X$ say $x$, such that $\deg_H(x) \leq P - |X| - 2$. Then:

\[f(H) \geq \frac{3n}{P}\]

Where $f(H)$ is the forest number of $H$ and $R(H)$ defined in 6.

Suppose that $\mathcal{G}_k = \{G, \omega(G) = k\}$, and $H$ be any such graph. Also, assume that $I$ is the maximum independent set in $H$. In the following theorem, a suitable lower bound for the size of the maximum $\mathcal{G}_k$-free subgraph of the graph $H$ is determined.

**Theorem 11.** Suppose that $H = H_1$ is a graph, and $I_1$ is the maximum independent set in $H_1$ where $|I_1| = i_1$. For each $2 \leq j$, set $H_j = H_{j-1} \setminus I_{j-1}$, and set $I_j$ as the maximum independent set in $H_j$, where $|I_j| = i_j$ for each $1 \leq j$. Then:

\[|S| \geq \sum_{j=1}^{k-1} i_j\]

Where $S$ has the maximum size possible, and $H[S]$ is $\mathcal{G}_k$-free.

**Proof.** Since, for each $j$, $I_j$ is the maximum independent set in $H_j$, then for each $n$, one can check that $|\omega(H[\cup_{j=1}^n I_j])| \leq n$. Therefore, for $n = k - 1$, we have $|\omega(H[\cup_{j=1}^{k-1} I_j])| \leq k - 1$. Now, as $\mathcal{G}_k = \{G, \omega(G) = k\}$, so $U$ is a $G$-free subset of $H$ for each $G \in \mathcal{G}_k$, where $U = \cup_{j=1}^{k-1} I_j$. Hence, it is clear to see that $U$ is a $\mathcal{G}_k$-free subset of $H$. As $|U| = \sum_{j=1}^{k-1} i_j$, so:

\[|S| \geq |U| = \sum_{j=1}^{k-1} i_j\]

Which means that the proof is complete.
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1Y. Rowshan, Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

Email address: y.rowshan@iasbs.ac.ir