On semiclassical approximation
and spinning string vertex operators in $AdS_5 \times S^5$

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Abstract

Following earlier work by Polyakov and Gubser, Klebanov and Polyakov, we attempt to clarify the structure of vertex operators representing particular string states which have large ("semiclassical") values of AdS energy or 4-d dimension $E = \Delta$ and angular momentum $J$ in $S^5$ or spin $S$ in $AdS_5$. We comment on the meaning of semiclassical limit in the context of $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ perturbative expansion for the 2-d anomalous dimensions of the corresponding vertex operators. We consider in detail the leading-order 1-loop renormalization of these operators in $AdS_5 \times S^5$ sigma model (ignoring fermionic contributions). We find more examples of operators (in addition to the one in [hep-th/0110196]) for which the 1-loop anomalous dimension can be made small by tuning quantum numbers. We also comment on a possibility of deriving the semiclassical relation between $\Delta$ and $J$ or $S$ from the marginality condition for the vertex operators, using a stationary phase approximation in the path integral expression for their 2-point correlator on a complex plane.

April 2003

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1. Introduction

One of the important issues in gauge theory – string theory duality is to understand the precise formulation of the relation between observables on the two sides of the duality beyond the supergravity (chiral primary) mode sector. Progress in that direction was initiated in [1,2,3]. A natural expectation [1] is that each on-shell string state or marginal string vertex operator in \( AdS_5 \times S^5 \) string theory should be associated with a local gauge-invariant operator in \( N = 4 \) SYM theory with definite conformal dimension \( \Delta = \Delta(Q, \lambda) \) (\( Q \) are global charges and \( \lambda \) is the 't Hooft coupling). Then the original suggestion of [9,10] implies that one is supposed to relate correlators of local gauge-invariant conformal operators on the gauge theory side with elements of string “S-matrix” in AdS, i.e. with correlators of the appropriate local string vertex operators (having the same quantum numbers and corresponding to a specific choice of boundary conditions).

Our aim here is to try to shed light on the structure of string vertex operators corresponding to particular string states in \( AdS_5 \times S^5 \) with large values of \( AdS_5 \) energy \( E \) in global coordinates (equal to 4-d dimension \( \Delta \)) and global charges \( Q \) (e.g., angular momentum \( J \) in \( S^5 \) or spin \( S \) in \( AdS_5 \)). We shall be guided by the previous work in this direction in [1,11] (see also [12]) and by the predictions [3] for \( \Delta = \Delta(Q) \) for the associated semiclassical string states.

In [3] it was suggested that a spinning string state represented by the classical solution for a long rotating string in AdS [13] may be related, in the limit of large spin \( S \), to the minimal twist operators on the gauge theory side. An evidence in favour of this proposal is the scaling of the dimension as

\[
\Delta = S + c_1 \ln S + ... \tag{1.1}
\]

for large \( S \) in the two limits (\( \lambda \gg 1 \) and \( \lambda \ll 1 \)) of the duality, as well as the absence [14,15] of higher-order (\( \ln S \))^n corrections on the string side to all orders in the inverse string tension (\( \frac{1}{\sqrt{\lambda}} = \frac{\alpha'}{R^2} \)). Similar behaviour \( \Delta = J + c_2 + ... \) was found [3] also (in the limit of large \( J \)) for a stretched string rotating in \( S^5 \).

In flat space one may represent a spinning string state on the leading Regge trajectory either by a local vertex operator \( e^{-iEt}(\partial X \partial X)^{S/2} \), \( X = x_1 + ix_2 \) (or as a Fock space state \( (a_1^\dagger a_{-1}^\dagger)^{S/2}\!|0, E > \)) or, if the spin \( S \) is of order of string tension, by a semiclassical state

\footnote{A related formulation or manifestation of duality is the equality of the gauge theory and string theory expectation values for a Wilson loop [3,12].}
described by a “solitonic” rotating string solution (in Fock space this is a coherent state \( \exp(\sqrt{S}a_1^{+} + \sqrt{S}a_{-1}^{+})|0, E > \) with respect to the elementary oscillator vacuum).

By analogy, one should expect that the same \( \Delta = \Delta(S) \) relation as found for the “solitonic” rotating string solution in AdS should be found also for an “elementary” string state created by a local vertex operator \( V_S \) with the same energy \( E = \Delta \) and spin \( S \) (and having the correct flat-space limit representing state on a leading Regge trajectory). The solutions in [3] should be associated with closed string states on the 2-d cylinder. Via the standard conformal mapping, they may be expected to be in correspondence with states created by the two vertex operators inserted on a complex plane with the “mass-shell” condition \( \Delta = \Delta(S) \) following from the marginality condition of the vertex operators.

Identifying string vertex operators in AdS\(_5 \times S^5\) superstring theory is, in general, a complicated task. First, they may have a non-trivial dependence on fermions. In addition, vertex operators in curved space may in general depend on \( \alpha' \). However, one may hope that in the semiclassical approximation considered in [3], i.e. \( (E = \Delta) \)

\[
\frac{\Delta}{\sqrt{\lambda}} = \kappa = \text{fixed} , \quad \frac{Q}{\sqrt{\lambda}} = q = \text{fixed} , \quad \frac{1}{\sqrt{\lambda}} \ll 1 , \quad Q = J \text{ or } S , \quad (1.2)
\]

where \( \sqrt{\lambda} \) is related to the effective string tension \( T = \frac{R^2}{2\pi \alpha'} = \frac{\sqrt{\lambda}}{2\pi} \), there may be some simplifications. In particular, given that the spinning classical solution in [3] involved only the bosonic AdS part of the AdS\(_5 \times S^5\) string theory, the result for \( \Delta = \Delta(S) \) may be universal, i.e. it may not be sensitive to the fermionic part of the vertex operator and details of \( S^5 \) factor. With this expectation we shall concentrate below only on bosonic parts of the corresponding operators.

Instead of computing the form of the marginality condition for a spinning string vertex operator directly in perturbation theory in \( \frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}} \) one may hope that in the limit of large \( \Delta \) and \( S \) the relation \( \Delta(S) \) may be derived by using semiclassical approximation in the string path integral on the 2-sphere (complex plane) with two insertions of the vertex operators \( V_S \). This idea works indeed in the case of a point-like string mode (supergravity state) with large \( S^5 \) orbital momentum \( J \) [3,11]. One could then try to follow the same logic in the spinning case, i.e. start with the 2-point function of unintegrated operators on a 2-sphere, compute the dependence on the 2-d distance \( |\xi_1 - \xi_2| \) in the semiclassical approximation and show that insisting that the 2-point correlator scales as \( |\xi_1 - \xi_2|^{-4} \) implies the same relation \( \Delta = \Delta(S) \) as found in [3]. Such computation can indeed be done for a spinning string state in flat space where, as we shall explain below, it leads indeed to the Regge trajectory relation \( \Delta \equiv E = \sqrt{\frac{2}{\alpha'}} S \).
Indeed, since in the semiclassical limit (1.2) $S$ scales as string tension, one expects the 2-point correlator to be saturated by a classical trajectory which should be closely related to the rotating string solution in $\mathbb{R}^8$. Presumably, the role of the two vertex operators is only to insert proper boundary conditions in mapping the 2-sphere back to the cylinder, while their detailed pre-exponential form should not be that important. Unfortunately, we were not be able to find a precise implementation of this idea in the $AdS_5$ (or $S^5$) spinning string case.

One may wonder how the expression for $\Delta(S)$ found in semiclassical approximation may be related to the direct computation of the 2-d anomalous dimension of the corresponding vertex operator in sigma model perturbation theory in $\frac{1}{\sqrt{\lambda}}$. In general, an eigenvalue of the anomalous dimension matrix for the operator at level (i.e. with number of 2-d derivatives) equal to $Q$ and with quantum numbers $\Delta$ and $Q$ is expected to have the structure

$$\gamma = 2 - Q + \frac{1}{\sqrt{\lambda}}(a_1\Delta^2 + a_2Q^2 + a_3\Delta + a_4Q)$$

$$+ \frac{1}{(\sqrt{\lambda})^2}(b_1\Delta^3 + b_2Q^3 + b_3\Delta^2 + b_4Q^2 + b_5\Delta + b_6Q) + \ldots + \frac{1}{(\sqrt{\lambda})^n}(c_1\Delta^{n+1} + c_2Q^{n+1} + \ldots) + \ldots .$$

(1.3)

The marginality condition

$$\gamma = 0$$

(1.4)

should then produce a complicated relation $\Delta = \Delta(Q, \sqrt{\lambda})$. Equivalent relation should follow from solving the generalized Klein-Gordon type equation $\hat{\gamma}f = (2 - Q + \frac{1}{2}\alpha'\nabla^2 + \ldots)f = 0$ for the corresponding wave function $f$, with the operator $\hat{\gamma}$ representing the functional form of the anomalous dimension operator $[17,18]$.\footnote{The discussion in $[16]$ applied to string theory in Minkowski-signature AdS space defined on a 2-d Minkowski-signature cylinder. This describes propagation of a particular closed string mode in real time. The vertex-operator 2-point computation done on Euclidean 2-sphere may be mapped onto Euclidean 2-cylinder with the vertex operators specifying a particular string state propagating on the cylinder. It is natural to expect that the relevant semiclassical trajectory should then be a (complex and conformally transformed) analog of the one in $[16]$. As already apparent in the $\Delta = J$ case $[11,12]$, one should not attribute a special meaning to the complex nature of the semiclassical trajectory saturating path integral (cf. $[16]$). An imaginary nature of it may be related to external sources one puts in to specify the required boundary/initial conditions. Like in the case of a euclidean gaussian path integral with imaginary sources the result is an analytic function of $J$ so that one can make analytic continuations.}
While determining the detailed form of higher order corrections in (1.3) is in general a very complicated problem depending on details of both the superstring action and particular vertex operator, a semiclassical path integral computation of the anomalous dimension would bypass this problem, effectively summing up leading terms in (1.3) from each order in $\frac{1}{\sqrt{\lambda}}$. That all orders in $\alpha'$ should be contributing is obvious from the form of the semiclassical relation found in [3] in the limit (1.2): $\Delta = \sqrt{\lambda} \kappa(q) = \sqrt{\lambda} \kappa(\frac{Q}{\sqrt{\lambda}})$. Indeed, assuming that $\Delta$ and $Q$ scale with $\lambda$ as in (1.2) and thus keeping only the leading terms in $\Delta$ and $Q$ at each order in $\frac{1}{\sqrt{\lambda}}$ one finds from (1.3)

$$\gamma = 2 - \sqrt{\lambda} f(\kappa, q) + h(\kappa, q) + O(\frac{1}{\sqrt{\lambda}}) , \tag{1.5}$$

$$f = q + a_1 \kappa^2 + a_2 q^2 + b_1 \kappa^3 + b_2 q^3 + ... \ , \quad h = a_3 \kappa + a_4 q + b_3 \kappa^2 + b_4 q^2 + ... ,$$

so that the leading-order solution of $\gamma = 0$ condition at large $\lambda$ should be determined by solving $f(\kappa, q) = 0$, which gives $\kappa = \kappa_0(q) + O(\frac{1}{\sqrt{\lambda}})$. Remarkably, the semiclassical argument should thus be effectively determining the whole function $f(\kappa, q)$ which contains information coming from all higher-order sigma model loop corrections to the anomalous dimension.

One may expect that some features of the semiclassical relation (1.1) for $\Delta(S)$ can be seen already in the perturbative expansion (1.3). For example, one would be able to deduce that $\Delta = S + ...$ at large $S$ provided each of the expansion terms in (1.3) would have the form $\Delta^{n+1} - S^{n+1} + ...$. In particular, the one-loop correction would then to start with $\Delta^2 - S^2 + ...$, similarly to what happens in the example (corresponding to a scalar string state) pointed out in [1]. As we shall discuss below, such “difference of squares” form of the leading 1-loop correction to the 2-d anomalous dimension is indeed characteristic also to vertex operators representing spinning string states. However, the relative coefficient of the two terms in the difference does not turn out to be exactly one. One possibility is that the precise value of this 1-loop coefficient depends in fact on the ignored contribution of the fermionic terms (both in the string sigma model action and the vertex operator). An alternative explanation is that one is first to sum all orders in $\frac{1}{\sqrt{\lambda}}$ to determine the function $f(\frac{\Delta}{\sqrt{\lambda}}, \frac{S}{\sqrt{\lambda}})$ in (1.3) and then find $\Delta = S + ...$ only after solving $f = 0$.

We shall start in section 2 with a review of the simple $S^5$ orbital rotation case [2,11]. We shall then consider in section 3 the semiclassical path integral derivation of the $E \sim \sqrt{S}$ relation for spinning string vertex operator in flat space.
In section 4 we shall describe the expected structure of vertex operators corresponding to string rotating in $S^5$ and to string rotating in $AdS_5$. We shall then sketch a possible generalization of the semiclassical argument in flat space to the case of the string spinning in $S^5$.

With the motivation to find the precise structure of the spinning string vertex operators in $AdS_5 \times S^5$ in section 5 we shall study in detail the 1-loop renormalisation of composite operators in $O(N)$ invariant bosonic sigma model. We shall follow [1] and apply the method of [19]. We shall find that in general the vertex operators are given by mixtures of various possible operators of the same canonical dimension and spin, with 1-loop corrected eigenvalues of the anomalous dimension matrix having a “difference of squares” structure as in the example considered in [1]. Assuming that higher loop corrections will have similar structure, this seems to be in qualitative agreement with the semiclassical results of [3] ($\Delta \approx S$ and $\Delta \approx J$) for the corresponding string states on a 2-d cylinder.

2. Case of orbital angular momentum in $S^5$: review

2.1. Comments on scalar vertex operators in $AdS$

Let us briefly review some relevant points regarding the general structure of integrated and unintegrated vertex operators in AdS space [1]. In any curved background we can define string vertex operators as perturbations of string sigma model action that solve the conformal invariance (marginality) conditions. The latter are analogs of curved space Laplace equation with various $\alpha'$ corrections. These equations are known explicitly in very few cases (gauged WZW models, etc). As discussed above, for certain types of string modes the semiclassical approach may allow one to fix implicitly the form of the leading terms in these equations, i.e. to determine their solutions with proper boundary conditions and thus determine the space-time dimensions of the associated CFT operators.

Ignoring $\alpha'$-corrections, a scalar mode like dilaton should satisfy the KG equation $(-\nabla^2 + m^2)T = 0$ in $AdS_5 \times S^5$. To define vertex operators leading to an analog of “S-matrix” we need to specify appropriate boundary conditions [9,10]. In flat space the relevant solutions are plane waves with fixed momentum, i.e. vertex operators contain factors of $e^{ipx}$. In AdS, while a 2-d conformal vertex is any $\hat{V} = \int d^2 \xi \, T(x(\xi))$ where $T$ solves the KG equation, we need to choose a specific solution. The required one is ($k = 1, \ldots, d = 4$)

$$T(\hat{x}) = \int d^4x' \, K(\hat{x}, x')T_0(x'), \quad \hat{x} = (z, x), \quad ds^2 = \frac{dz^2 + dx_\mu dx_\mu}{z^2}. \quad (2.1)$$
Here $T_0(x)$ is any “source” function at the boundary of AdS, $K$ is the Dirichlet bulk to boundary propagator,

$$K = c(\Delta)[\frac{z}{z^2 + (x - x')^2}]^\Delta, \quad K_{z \to 0} \to \delta^{(4)}(x - x'),$$

and $\Delta$ is determined from the condition $0 = \gamma = -\frac{1}{2} \alpha' m^2 + \frac{1}{2\sqrt{\lambda}} \Delta(\Delta - 4)$. The vertex operator that enters the expressions for string correlators is the “unintegrated one” – the one integrated over the 2-d world sheet but depending (instead of the usual momentum) on a point at the boundary of AdS:

$$\hat{V}(x) = \int d^2 \xi V(\xi), \quad V = K(\hat{x}(\xi), x), \quad \hat{V}(T_0) = \int d^4 x \hat{V}(x) T_0(x).$$

The AdS/CFT conjecture \[9,10\] can then be formulated as the statement that the string generating functional as functional of $T_0(x)$ is equal to the corresponding gauge-theory generating functional, $< e^\int d^4 x T_0(x) O(x) >$, where $O(x)$ is a local gauge-invariant operator with appropriate quantum numbers and dimension.\(^3\) Choosing $T_0$ to be a delta-function gives us simply the correlators of unintegrated vertex operators each localised at a given boundary point and having specific dependence on radial direction $z$.

This (euclidean) AdS/CFT duality prescription of the equality of the two generating functionals \[9,10\] was not yet tested at the full string-theory (as opposed to supergravity) level. One motivation to study the structure of string-level vertex operators and of semiclassical approach in the string path integral is that this may help to clarify the duality relation.

\[2.2.\] **Operators with orbital momentum $J$ and semiclassical approximation**

Let us now review the case of the simplest vertex operator carrying large orbital momentum $J$ in $S^5$ and the semiclassical derivation of its dimension.

Consider the dilaton-type vertex operator corresponding to the ground-state (supergravity) level state. According to (2.3), the unintegrated operator localised at point 0 at the boundary can be written as can be written as (see \[1,11\])

$$V_J(\xi) \equiv V_J(x(\xi), z(\xi), \varphi(\xi)) = C(N_+)^{-\Delta} (n_x)^J V^{(2)},$$

\(^3\) It does not seem to be known how to directly compute the string 2-point function in this set-up. In compact 2-d CFT case the result for the 2-point correlator $< VV >$ is zero due to division by the Mobius group volume. In the non-compact AdS case we are supposed to get a finite result; this should be a consequence, in particular, of the special role of the boundary terms in AdS.
where \( V^{(2)} = -\partial N_M \bar{\partial} N_M + \partial n_k \bar{\partial} n_k + \ldots \) is a scalar operator that has the same form as the \( AdS_5 \times S^5 \) sigma model Lagrangian and is contributing to the 2-d dimension to make a \((1,1)\) operator.\(^4\) In particular, the Euclidean \( AdS \) space and \( S^5 \) are described by the two 6-vectors \((\mu = 0, 1, 2, 3; \ k = 1, \ldots, 6)\)

\[
N_M N_M \equiv N_5^2 - N_\mu^2 - N_4^2 = 1, \quad n_k n_k = 1.
\]  

(2.5)

The linear combinations appearing in (2.4) are\(^5\)

\[
N_+ \equiv X_4 + X_5 = \frac{1}{z} (z^2 + x_{\mu} x_{\mu}) , \quad N_- = \frac{1}{z} ,
\]  

(2.6)

and

\[
n_x \equiv n_1 + in_2 = \frac{z_1 + iz_2}{z} = e^{i\phi} ,
\]  

(2.7)

which represents rotation in \((z_1, z_2)\) plane in \(R^6\) or along the big circle of \(S^5\).

To localise such operator at the boundary we need to form a linear superposition integrating over \(\xi\)

\[
\hat{V}_J(x) = \int d^2\xi \ V_J(x(\xi) - x, z(\xi), \varphi(\xi)) .
\]  

(2.8)

This operator will have zero world-sheet dimension and the right space-time dimension to be dual to the corresponding gauge theory operator with dimension \(\Delta\). Then for \(z \to 0\) (cf. (2.2))

\[
\hat{V}_J(x) \to \int d^2\xi \ \delta^{(4)}(x(\xi) - x) \ (n_z(\xi))^J \ V^{(2)}(\xi) ,
\]  

(2.9)

i.e. we get indeed an operator localised at one point at the boundary.\(^6\) The correlators of \(\hat{V}_J\) computed using \(AdS_5 \times S^5\) string sigma model on the 2-sphere should give correlators

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\(^4\) Dots stand for the fermionic contributions (\(\alpha'\)-corrections are expected to be absent for a vertex operator representing a supergravity state which is dual to a chiral primary operator, see also below). For the dilaton vertex operator in flat \(D\) dimensions defined on 2-sphere one is to add also the term \(aR^{(2)}\) , \(a = \frac{1}{4}\alpha'(D - 2)\).

\(^5\) Note that \(N_+\) scales under the dilation isometry of AdS in the same way as the boundary coordinate \(x_\mu\) \((z \to \lambda z, \ x_\mu \to \lambda x_\mu)\). Let us also recall the relation between \(N_M\), global \(AdS_5\) coordinates \((t, \rho, \) and \(S^3\) angles with metric \(ds^2 = -\cosh^2 \rho \ dt^2 + d\rho^2 + \sinh^2 \rho \ d\Omega_3\) and the Poincare coordinates: \(N_0 = \frac{x_0}{x} = \cosh \rho \ \sin t\) , \(N_5 = \frac{1}{x^2} (1 + 2z - x_0^2 + x_1^2) = \cosh \rho \ \cos t\) , \(N_i = \frac{x_i}{x} = \tilde{n}_i \sinh \rho\) , \(N_4 = \frac{1}{x^2} (-1 + 2z^2 - x_0^2 + x_1^2) = \tilde{n}_4 \sinh \rho\) , so that \(t = \frac{2x_0}{1 + 2z - x_0^2 + x_1^2}\) and \(z^{-1} = \cosh \rho \ \cos t - \tilde{n}_4 \sinh \rho\). Here the unit vector \(\tilde{n}_k\) \((k = 1, 2, 3, 4)\) \(\tilde{n}_i^2 + \tilde{n}_4^2 = 1\) parametrizes the 3-sphere: \(d\tilde{n}_k d\tilde{n}_k = d\Omega_3\)

\(^6\) It has an arbitrary 4-momentum if we do the Fourier transform in \(x\), i.e. it is “on-shell” in 5-d but “off-shell” in 4-d sense.
of the dual gauge theory at the boundary, with \( \hat{V}_J(x) \leftrightarrow O_J(x) \). On the basis of conformal symmetry we expect
\[
< \hat{V}_J(x) \hat{V}_{-J}(x') > \sim |x - x'|^{-\Delta}.
\] (2.10)

To determine \( \Delta = \Delta(J) \) one may compute the anomalous dimension of (2.4) using sigma-model perturbation theory in \( \frac{1}{\sqrt{\lambda}} \). One can show that at least in the 1-loop approximation (and ignoring all fermionic contributions)\(^7\) the operator (2.4) is an eigen-operator of the anomalous dimension matrix and so the resulting marginality condition takes the form (1.3) (see [1,3] and below)
\[
0 = \gamma = 2 - 2 + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - J(J + 4)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right).
\] (2.11)

This implies \( \Delta = J + ... \) for large \( J \).

In the case of the large “classical” values of \( \Delta \) and \( J \) in (1.2) the same relation can be obtained using a semiclassical approximation in the path integral for the 2-point function of the unintegrated vertex operators [11]. As was mentioned above in (1.3),(1.5), this semiclassical approach in general goes beyond the perturbative 1-loop result.

Using the global AdS coordinates in the string action, rotating the global AdS time \( t \) to the Euclidean one \( t_e = it \) (so that \( N_+ \sim e^{t_e} \)) and expanding near point where \( \rho = 0 \) with all angles except \( \varphi \) in (2.7) being trivial, we get for the relevant part of the Euclidean string action on the 2-sphere
\[
I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\xi \left[ (\partial_a t_e)^2(1 + ...) + (\partial_a \varphi)^2(1 + ...) + ... \right].
\] (2.12)

The aim is to show that demanding that the two-point function has canonical 2-d dimension
\[
< V_J(\xi_1)V_{-J}(\xi_2) > = \int [dt_e \, d\varphi \, ...] \, e^{-I[t_e,\varphi, ...]} \, V_J(\xi_1)V_{-J}(\xi_2) \sim |\xi_1 - \xi_2|^{-4}
\] (2.13)
implies that \( \Delta = J \). The insertions of the vertex operators provide effective source terms to the above action:
\[
e^{-\Delta[t_e(\xi_1) - t_e(\xi_2)] + iJ[\varphi(\xi_1) - \varphi(\xi_2)]}. \] (2.14)

\(^7\) Contrary to naive expectation, fermions may in principle contribute to renormalization of composite operators even in the 1-loop approximation: the superstring action contains the RR 5-form coupling term [20] \( \sim \theta \bar{\theta} \partial x \), and the fermions in it may be “paired” with the fermions in the \( \partial x + \bar{\theta} \theta \) factors in the vertex operator.
When $\Delta$ and $J$ have large “classical” values, the action and the “source” terms are of the same order, and the path integral is saturated by the stationary point trajectory. Then (as in the usual case of a gaussian integral) we get a complex semiclassical trajectory for the angle $\phi$ ($\phi = \sqrt[\lambda]{\ln |\xi - \xi_1| - \ln |\xi - \xi_2|}$). The final semiclassical result for the correlator (2.13) is then proportional to

$$< V^{+}_J(\xi_1) V^{-}_J(\xi_2) > \sim |\xi_1 - \xi_2|^{2\gamma - 4}, \quad \gamma = \frac{1}{2\sqrt{\lambda}}(\Delta^2 - J^2).$$

Here -4 comes (as in the flat space case) from the contribution of $< V^{(2)} V^{(2)} >$. Demanding the marginality of the vertex operators, i.e. $\gamma = 0$, we are thus led to the relation

$$\Delta = J.$$

To relate this derivation to the classical solution of string theory defined on a 2-d cylinder one may take $\xi_1 = 0$, map the point $\xi_2$ to $\infty$ and then map the complex $z$-plane with two punctures into the Euclidean cylinder by setting (see also section 3)

$$z = e^w, \quad w = \tau e + i\sigma, \quad \tau = i\tau.$$

After we rotate back to the Minkowski signature the semiclassical trajectory will become the same as the geodesic in $\mathbb{P}$ (i.e. $t = \kappa \tau$, $\varphi = \kappa \tau$, $E = \Delta = \sqrt{\lambda} \kappa$).

### 3. Semiclassical derivation of dimension of spinning string vertex operator in flat space

We would like in principle to apply a similar semiclassical argument to the case of the spinning string states. Though this turns out to be rather hard to achieve in practice in $AdS_5 \times S^5$ case, as we will describe in this section the semiclassical derivation of the relation between energy and spin works indeed in the flat space case.

The vertex operator that describes a (bosonic) string state on the leading Regge trajectory with spin $S$ and energy $E$ in flat space can be represented as follows

$$V_S(\xi) = e^{-iEt} (\partial X \bar{\partial} X)^{S/2},$$

$$X = x_1 + ix_2, \quad \bar{X} = x_1 - ix_2.$$
For any $S$ the marginality condition implies that

$$\gamma = 2 - S - \frac{1}{2} \alpha' E^2 = 0 , \quad \text{i.e.} \quad \alpha' E^2 = 2(S - 2) . \quad (3.3)$$

In flat space the spin factor provides only a contribution to canonical dimension, while the anomalous dimension comes from the 1-loop renormalisation of the exponential (energy) factor.

Let us show that for large (classical) value of $S$ the flat Regge trajectory relation (3.3) can be found by a semiclassical path integral argument similar to the one in the previous section. The aim is to compute

$$A_2(\xi_1 - \xi_2) \equiv < V_S(\xi_1) V_{-S}(\xi_2) > , \quad V_{-S} \equiv V_S(X \rightarrow \bar{X}) \quad (3.4)$$

in the semiclassical approximation, and to show that (cf. (2.15))

$$A_2(\xi) \sim |\xi_1 - \xi_2|^{2\gamma - 4} , \quad \gamma = \gamma(E, S) , \quad (3.5)$$

where $\gamma$ is approximately the same as in (3.3), i.e. $\gamma \approx -S - \frac{1}{2} \alpha' E^2$ so that demanding the 2-d conformal invariance implies the leading Regge trajectory relation, $E = \sqrt{\frac{2}{\alpha'} S}$.

The idea is again that in the limit when both $E$ and $S$ are of order of the string tension the path integral representation for $A_2$ in (3.4) should be dominated by a (complex) classical trajectory with the source terms (or boundary conditions) determined by the vertex operator insertions.

Note that the Regge trajectory relation may be rewritten as $\alpha' E = \sqrt{2 \alpha' S}$ and thus remains the same in the limit when both $E$ and $S$ are of the order of string tension $\frac{1}{2\pi \alpha'}$. Indeed, as is well known, the same relation is obtained for a classical string solution describing folded closed string rotating about its center of mass. One finds from the closed string equations defined on a 2-cylinder ($\tau, \sigma \equiv \sigma + 2\pi$) with Minkowski signature in both target space and world sheet

$$t = \kappa \tau , \quad X = x_1 + ix_2 = r(\sigma) e^{i\phi(\tau)} , \quad \bar{X} = x_1 - ix_2 = r(\sigma) e^{-i\phi(\tau)} , \quad (3.6)$$

$$\phi(\tau) = n\tau , \quad r(\sigma) = \omega \sin(n\sigma) , \quad \omega = \frac{\kappa}{n} , \quad n = 1, 2, \ldots . \quad (3.7)$$

The relation between $\omega$ and $\kappa$ follows from the conformal gauge constraint. The energy and the spin are

$$E = \frac{1}{2\pi \alpha'} \int_0^{2\pi} d\sigma \ i = \frac{\kappa}{\alpha'} , \quad S = \frac{i}{4\pi \alpha'} \int_0^{2\pi} d\sigma \ (X \dot{X} - \bar{X} \dot{\bar{X}}) = \frac{n\omega^2}{2\alpha'} , \quad (3.8)$$
so that for the leading Regge trajectory state (having minimal energy for a given spin) one has \( n = 1 \), i.e. a single fold of the closed string. Then \( E = \sqrt{2\alpha'} S \).

The explicit form of the correlator (3.4) computed in conformal gauge on a 2-sphere or complex plane is (we rotate to the Euclidean time \( t_e = it \))

\[
A_2(\xi_1 - \xi_2) = \int [dt_e dX d\bar{X} \ldots] \ e^{-\frac{1}{\pi\alpha'} \int d^2\xi (\partial t_e \partial t_e + \partial X \partial \bar{X} \ldots)} \times [e^{-E_{t_e} (\partial X \partial \bar{X})^{S/2}}(\xi_1) \ e^{E_{t_e} (\partial \bar{X} \partial \bar{X})^{S/2}}](\xi_2). \tag{3.9}
\]

This is a gaussian integral that of course can be easily computed exactly, giving (3.5) with the coefficient \( \gamma \) in (3.3).

In the case of the “classical” values of \( E \) and \( S \) we may instead apply the semiclassical method to compute \( \gamma \). We start by rewriting the vertex operator insertions in (3.9) in the exponential form as

\[
\exp \left( -E [t_e(\xi_1) - t_e(\xi_2)] \right) \exp \left[ \frac{1}{2} S \left( \ln \left[ \partial X \partial \bar{X}(\xi_1) \right] + \ln \left[ \partial \bar{X} \partial \bar{X}(\xi_2) \right] \right) \right], \tag{3.10}
\]

and then look for a stationary point of the total expression in the exponent in (3.9), i.e. of the sum of the classical action with the terms in the (3.10), i.e. \( O\left(\frac{1}{\alpha'}\right) + O(E) + O(S) \).

While the contribution of the time coordinate is the obvious one (i.e. the same as in the case discussed in section 2.2), the solution for \( X, \bar{X} \) may seem to be less trivial to find, given a complicated non-linear form of the terms coming from the vertex operator insertions. However, one may expect that details of these insertions should not be too important: the they should just select the relevant solution of the free Laplace equation for \( X, \bar{X} \) which effectively describes a classical trajectory associated with a spin \( S \) configuration. If the role of the “sources” is only to specify appropriate boundary conditions (though different from simply \( \delta \)-functions appearing in the case of the external momentum insertions like \( e^{-Et} \)

---

8 Another classical solution describing oscillating closed string is formally obtained from (3.6),(3.7) by interchanging \( \tau \) and \( \sigma \) in the expression for \( X \), i.e. \( \phi = n\sigma, \ r = \omega \sin n\tau \). In this case the spin is equal to zero, while we still get \( E = \kappa/\alpha' \) and \( \omega = \kappa/n \). The vertex operator representing such state is a scalar operator from level \( n \): \( e^{-iEt} \partial^n \bar{X} \partial^n \bar{X} \). Similar solutions in AdS case were constructed in [21].

9 In our notation \( \xi = \xi^1 + i\xi^2, \ \bar{\xi} = \xi^1 - i\xi^2, \ d^2\xi = d\xi^1 d\xi^2 \), and \( \partial = \frac{1}{2}(\partial_1 - i\partial_2), \ \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \). The Green’s function of the Laplacian \( \partial^a \partial_a = 4\partial \bar{\partial} \) on the complex plane is \( G(\xi - \xi') = -\frac{1}{4\pi} \ln |\xi - \xi'|^2 \), i.e. \( -4\partial \bar{\partial} G(\xi) = \delta(\xi) \). Here \( \delta(\xi) \) is the 2-d delta-function normalized in such a way that \( \int d^2\xi \ \delta(\xi) = 1 \), so that \( \partial_\xi \frac{1}{\xi} = \pi \delta(\xi) \), etc.
one) a natural guess is that the associated stationary point solution should be a complex analog of the real Minkowski-time rotating folded string solution (3.6) with spin $S$.

Using translational invariance we may set $\xi_1 = 0$. Then the equations obtained by varying $t$, $X$ and $\bar{X}$ in the effective action in (3.9), (3.10) are

$$\partial \bar{t} = \frac{1}{2} \pi \alpha' E[\delta(\xi) - \delta(\xi - \xi_2)], \quad (3.11)$$

$$\partial \bar{X} = \frac{1}{2} \pi \alpha' S(\partial[ \frac{1}{\partial X} \delta(\xi)] + \bar{\partial}[ \frac{1}{\partial X} \delta(\bar{\xi})]), \quad (3.12)$$

$$\partial \bar{\bar{X}} = \frac{1}{2} \pi \alpha' S(\partial[ \frac{1}{\partial \bar{X}} \delta(\xi - \xi_2)] + \bar{\partial}[ \frac{1}{\partial \bar{X}} \delta(\bar{\xi} - \xi_2)]). \quad (3.13)$$

The relevant solution of this system can be constructed by starting from the known classical solution (3.6), (3.7) on the 2-cylinder. We shall first note that the conformal transformation of the complex plane $\xi \to z$ defined by

$$z^{-1} = \xi^{-1} - \xi_2^{-1} \quad (3.14)$$

maps the points $\xi = \xi_1 = 0$ and $\xi = \xi_2$ to points $z = 0$ and $z = \infty$, respectively. Then the complex $z$-plane with punctures at 0 and $\infty$ can be mapped (as in the context of the standard operator – state correspondence) to a Euclidean cylinder $(\tau_e, \sigma)$ by the transformation (2.16), i.e.

$$z = e^{\tau e + i\sigma}, \quad \tau_e = i\tau. \quad (3.15)$$

The classical solution (3.6) on the Minkowski 2-cylinder can be rotated to the Euclidean solution ($t \to -it_e$, $\tau \to -i\tau_e$) which takes the following form in terms of $z$:

$$t_e = \frac{1}{2} \kappa \ln |z|^2, \quad X = \frac{\omega}{2i}(z - \bar{z}), \quad \bar{X} = \frac{\omega}{2i}(\bar{z}^{-1} - z^{-1}). \quad (3.16)$$

We have introduced a parameter $\omega$ which is equal to $\kappa$ on the classical single-fold solution (3.7) satisfying the conformal constraint.\[11\]

\[10\] The argument that follows was worked out in collaboration with S. Frolov. A related derivation of the marginality condition using a single vertex operator insertion and imposing the conformal gauge constraint was given K. Zarembo.

\[11\] Note that $X$ and $\bar{X}$ in the Euclidean solution (3.16) are no longer related by a complex conjugation. Instead, they are related by the transformation $z \to \frac{1}{z}$ or $\tau_e \to -\tau_e$, which, as expected, interchanges the two insertion points, or reverses the time axis on the cylinder.
Transformed to the z-plane the system (3.11), (3.12), (3.13) has the same form with
$\xi \to z$ and $\xi_2 \to \infty$. Then $t_e, X$ and $\bar{X}$ in (3.16) solve the free Laplace equation on the
z-plane apart from the points $z = 0$ and $z = \infty$. At these special points they are indeed
supported by the source terms in the right hand side of (3.11), (3.12), (3.13) provided $\kappa$ and
$\omega$ in (3.16) are related to the “source” coefficients $\alpha' E$ and $\alpha' S$ in (3.11), (3.12), (3.13) by
\[ \alpha' E = \kappa, \quad \omega^2 = 2\alpha' S. \] (3.17)
These are the same relations as the ones (3.8) for the classical solution (3.7), but here obtained in a very different way.

The solution of the original system (3.11), (3.12), (3.13) is then given by (3.16), (3.17)
with $z$ replaced by $\xi$ according to (3.14). The final step is to evaluate the effective action
in the path integral (3.9), (3.10) on this Euclidean solution, i.e. find the coefficient $\gamma(E, S)$
in (3.5) and thus determine $E = E(S)$ from the conformal invariance condition $\gamma = 0$.
Following the discussion in section 2, the time coordinate contribution is given, as in
the standard gaussian integral, by the 1/2 of the external point contribution. One finds
that the $(X, \bar{X})$ part of the classical string action vanishes on the solution so that the
semiclassical contribution to the $(X, \bar{X})$ path integral comes only from the $S$-dependent
vertex operator insertion terms in (3.10) evaluated on the solution. The latter is essentially
determined by the canonical dimension of the vertex operators. Differentiating $X, \bar{X}$ in
(3.9) or (3.10) over $\xi$ in (3.14), we get (regularising and omitting some obvious singular
terms) the factor $|\xi_2|^{-2S}$, which, combined with the $t$-contribution, then reproduces (3.3)
with $\gamma = \frac{1}{2}\alpha'E^2 - S$, i.e. gives the leading Regge trajectory relation.

To summarize:

(i) The role of matching onto the source terms in the classical equations (3.11)–(3.13)
is to relate the parameters $\kappa, \omega$ of the semiclassical trajectory to the quantum numbers
$E, S$ of the vertex operators. In the classical rotating string solution (3.6) this was done
by fixing the values (3.8) of the conserved quantities on the solution.

(ii) The role of the marginality condition $\gamma(E, S) = 0$ is to relate $E$ to $S$. In the
classical solution this followed from the conformal constraint that related the parameter $\kappa$
determining the energy $E$ and the parameter $\omega$ determining the spin $S$.

---

12 As always for string excited states in flat space, the mass shell condition expresses the bal-
ance between the anomalous dimension of the $e^{ipx}$ operator and canonical dimension of the pre-
exponential $\partial X$ factors. The spin $S$ term in (3.3) represents the canonical dimension.
4. Vertex operators for spinning string states in $AdS_5$ and $S^5$

One would like to try to give a similar semiclassical path integral derivation of dimensions of string states with large spin $S$ in $AdS_5$ or intrinsic angular momentum $J$ in $S^5$ with for which the classical solutions of [3] predict that

$$\Delta(S) \frac{1}{\sqrt{\lambda}} \gg 1 = S + \frac{1}{\pi} \sqrt{\lambda} \ln S + \ldots , \quad \Delta(J) \frac{1}{\sqrt{\lambda}} \gg 1 = J + \frac{2}{\pi} \sqrt{\lambda} + \ldots . \quad (4.1)$$

The first step should be to identify the corresponding vertex operators that should be analogs of the operator with orbital momentum (2.4). One could then attempt to compute the 2-point functions like (3.4),(3.5) in semiclassical approximation to determine the anomalous power $\gamma = \gamma(\Delta, S)$ or $\gamma = \gamma(\Delta, J)$, which, when set to zero, should be reproducing hopefully the results (4.1).

Here we will not be able to complete the second part of this program (though we will make some comments on it in section 4.2), and will most concentrate (in sections 4.1 and 5) on the first part, i.e. determining the structure of vertex operators, which already turns out to be quite non-trivial.

4.1. General structure of vertex operators

The natural conditions on such vertex operators are: (i) they should carry the required quantum numbers $\Delta$ and $S$ or $J$, i.e. should have appropriate transformation properties under $SO(1,5)$ and $SO(6)$; (ii) in the flat-space limit they should reduce to the leading Regge trajectory vertex operator (3.1). The latter condition implies that their level number or canonical 2-d dimension (equal to the number of 2-d derivatives) should be $S$ (or $J$).

An obvious candidate for a vertex operator representing a string state with angular momentum $J$ in $S^5$ is then (cf. (2.4),(3.1))

$$V_J(\xi) = c(N_+)^{-\Delta} (\partial n_x \bar{\partial} n_x)^{J/2} , \quad n_x \equiv n_1 + in_2 . \quad (4.2)$$

---

13 As above, we shall be ignoring terms depending on the fermionic string coordinates (as well as possible $\alpha'$-corrections). A possible excuse is that one expects that they should not be important in the leading semiclassical approximation (fermions were of course set to zero in the classical solutions of [3]).

14 In flat space limit $n_6 \to 1, n_1 \to \theta \cos \varphi, n_2 \to \theta \sin \varphi$, etc., with $\theta$ playing the role of a radial coordinate.
The analogous operator for a spinning string in $AdS_5$ is
\[ V_S(\xi) = c(N_+)^{-\Delta} (\partial N_x \bar{\partial} N_x)^{S/2} , \quad N_x \equiv N_1 + i N_2 . \quad (4.3) \]

It may be useful to record also the explicit form of $V_S$ in terms of independent coordinates on $AdS_5$. The Euclidean $AdS_5$ sigma model Lagrangian
\[ L = -\partial N_+ \bar{\partial} N_- + \partial N_\mu \bar{\partial} N_\mu , \quad N_+ N_- - N_\mu N_\mu = 1 \]
can be written as
\[ L = \partial T \bar{\partial} T + \partial N_\mu \bar{\partial} N_\mu + (\partial \bar{\partial} T + \partial T \bar{\partial} T) N_\mu N_\mu , \quad T \equiv \ln N_+ , \quad (4.4) \]

or as
\[ L = (1 + N_\mu N_\mu) \partial \bar{\partial} t - \frac{\partial(N_\mu N_\mu) \bar{\partial}(N_\nu N_\nu)}{4(1 + N_\lambda N_\lambda)} + \partial N_\mu \bar{\partial} N_\mu \]
\[ = (1 + N_x N_x) \partial \bar{\partial} t - \frac{\partial(N_x \bar{N}_x) \bar{\partial}(N_x \bar{N}_x)}{4(1 + N_x N_x)} + \partial N_x \bar{\partial} N_x + ... , \quad (4.5) \]

where dots stand for the terms depending on the remaining coordinates $N_3, N_4$. Here $t$ is the global “Euclidean AdS time” variable\[.\]
\[ t \equiv \ln N_+ - \frac{1}{2} \ln (1 + N_\mu N_\mu) = T - \frac{1}{2} \ln (1 + N_\mu N_\mu) , \quad e^{2t} = \frac{N_+}{N_-} . \quad (4.6) \]

In these coordinates \[4.3\] becomes
\[ V_S(\xi) = c e^{-\Delta T} (\partial N_x \bar{\partial} N_x)^{S/2} = c e^{-\Delta t} (1 + N_\mu N_\mu)^{-\Delta/2} (\partial N_x \bar{\partial} N_x)^{S/2} . \quad (4.7) \]

Written in terms of the Poincare coordinates this vertex operator takes the form (cf. \[3.1\])
\[ V_S(\xi) = c \left( \frac{z^2 + x_\mu x_\mu}{z} \right) ^{-\Delta} \left[ \partial \left( \frac{X}{z} \right) \bar{\partial} \left( \frac{X}{z} \right) \right]^{S/2} , \quad X = x_1 + i x_2 . \quad (4.8) \]

While the vertex operators \[4.2\] and \[4.3\] appear to be natural choices, there remains a question then if they are indeed the correct ones at the quantum $AdS_5 \times S^5$ string level,

\[15\] Note that this $t$ is not the same as Euclidean rotation of $t$ used in a footnote below \[2.6\]: the Euclidean global coordinates here are related to the direct Euclidean analog of the coordinates there by a global $O(1,5)$ transformation which effectively interchanges $N_0$ and $N_4$ (so that now $N_4 = \cosh \rho \sin t, \ N_0 = \hat{n}_4 \sinh \rho$). The relation to the Poincare coordinates is however is the same.
in particular, whether they mix with other similar operators. That would mean that they represent eigen-operators of the anomalous dimension matrix, or, equivalently, that the associated wave functions solve the corresponding ($\alpha'$-corrected) KG equation.

It turns out that, in contrast to what was in the case of the spinning state operator (3.1) in flat space, the operators (4.2) and (4.3) do mix with other similar operators already in the 1-loop $O(\sqrt{\lambda})$ approximation for the $AdS_5 \times S^5$ sigma model. At the same time, the operator (2.4) representing a point-like string state with an orbital momentum $J$ is indeed an eigen-operator of the anomalous dimension matrix.

As we shall explain in section 5, for any value of $J$ the operator $(\partial n_x \bar{\partial} n_x)^{J/2}$ of (bosonic) sigma model on a sphere $S^{N-1}$ mixes under the 1-loop renormalization with the following operators (all of which have the same canonical dimension equal to the angular momentum $J$)

$$(n_x)^{2p+2q}(\partial n_x)^{J/2-2p}(\bar{\partial} n_x)^{J/2-2q}(\partial n_m \partial n_m)^p(\bar{\partial} n_k \bar{\partial} n_k)^q, \quad p, q = 0, \ldots, J/4,$$  \hspace{1cm} (4.9)

where there are sums over the indices $m$ and $k$ with range 1, ..., $N$ (in the 5-d case of our interest $N = 6$). A similar but somewhat more complicated mixing takes place for the spin $S$ operator (4.3) in the bosonic $AdS_{N-1}$ sigma model – the operator (4.3) may mix with

$$N^+_{-\Delta-\frac{p-2}{2}}(\partial N_\pm)^p(\bar{\partial} N_\pm)^{S/2-\frac{q}{2}}(\partial N_\pm)^q(\bar{\partial} N_\pm)^{S/2-\frac{q}{2}} + \ldots, \quad p, q = 0, \ldots, S/4,$$  \hspace{1cm} (4.10)

where dots stand for terms involving powers of scalar operators $\partial N_M \bar{\partial} N_M$ and $\bar{\partial} N_K \bar{\partial} N_K$ similar to the ones in (4.3). Again, these operators have the same $SO(1,5)$ quantum numbers $(\Delta, S)$ as (4.3).

This obviously raises the question of whether this mixing is important to take into account in an attempt of a semiclassical derivation of the corresponding anomalous dimensions for large values of quantum numbers. The linear combinations of the above operators that represent the true spinning string vertex operators may simplify in the limit of large quantum numbers. We are unable to answer this question here, and in the next subsection will only sketch a strategy of possible semiclassical argument in the $S^5$ rotation case starting simply with the “naive” form of the corresponding vertex operator (4.2). The $S^5$ rotation case is more transparent than the spinning $AdS_5$ case but is already quite non-trivial to illustrate the main points.

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16 Note that we do not include extra real factors like $(\partial n_m \bar{\partial} n_m)^r$ or $(\partial N_M \bar{\partial} N_M)^r$. Their maximal power $r$ plays the role of an additional quantum number, and their (1-loop) renormalization always decreases $r$.  

16
4.2. Towards a semiclassical derivation of anomalous dimension of vertex operator with $S^5$ angular momentum

In the semiclassical approximation the $AdS_5$ and $S^5$ factors $N_+^\Delta$ and $(\partial n_x \partial n_x)^J/2$ in (4.2) can be treated independently. As in the $S^5$ orbital momentum case of section 2.2 the $AdS_5$ factor contributes to the 2-point correlator the same term as in (2.15) with $\gamma(AdS_5) = \frac{1}{2\sqrt{\lambda}} \Delta^2$. For $\xi_1 = 0$ the stationary point is as in (3.16), i.e.

$$t_c = \frac{1}{2} \kappa \ln |z|^2, \quad \kappa = \frac{\Delta}{\sqrt{\lambda}}, \quad (4.11)$$

where $z$ is given by (3.14). The main complication lies in the $S^5$ sector. Since the vertex operator depends only on $n_x$ components of 6-vector, we may assume that $n_4 = n_5 = n_6 = 0$ and consider the $S^2$ subspace of $S^5$ parametrized by $n_x, \bar{n}_x$ and $n_3 = \sqrt{1 - n_x \bar{n}_x}$. Then the relevant part of the $AdS_5 \times S^5$ string action will be the $S^2$ sigma model (cf. (4.5))

$$I = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi \left[ \partial n_x \partial \bar{n}_x + \frac{\partial(n_x \bar{n}_x)\bar{\partial}(n_x \bar{n}_x)}{4(1-n_x \bar{n}_x)} \right]. \quad (4.12)$$

Adding to this action the (minus) logarithm of the product of the two operators $V_J$ and $V_{-J} = V_J(n_x \rightarrow \bar{n}_x)$ at points $\xi_1 = 0$ and $\xi_2$, i.e.

$$-\frac{1}{2}J \ln[(\partial n_x \partial n_x)(0)(\partial \bar{n}_x \partial \bar{n}_x)(\xi_2)] \quad (4.13)$$

and assuming that the angular momentum $J$ is of order of the string tension $\sqrt{\lambda}$, we can then try to find a (complex) semiclassical trajectory $n_x(\xi)$ that saturates this path integral in the semiclassical approximation. This would then determine the $S^5$-factor contribution to $\gamma$ in the analog of (2.15).

In view of the discussion of the flat space case in section 3 it is natural to anticipate that the stationary point solution will be related by an analytic continuation and the “cylinder $\rightarrow$ complex plane” map to the classical solution in [3] describing folded closed string rotating at the north pole of $S^5$. In Minkowski notation this solution has the following form on the 2-d cylinder (here the metric of $S^5$ taken to be $ds^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 + \cos^2 \theta \, d\Omega_3^2$)

$$\varphi = \omega \tau, \quad \theta = \theta(\sigma), \quad \theta'' + \omega^2 \sin \theta \cos \theta = 0. \quad (4.14)$$

We should not impose the conformal gauge constraint that in [3] related $\omega$ to $\kappa$ in $t = \kappa \tau$ and thus $J$ to $\Delta$: here this relation should follow from the marginality condition of the
vertex operators, i.e. from the vanishing of the coefficient $\gamma$ in the 2-point correlator (2.13). Using the parametrization

$$n_x = \sin \theta \ e^{i\varphi}, \quad \bar{n}_x = \sin \theta \ e^{-i\varphi},$$  \hfill (4.15)

rotating the solution to the euclidean one, $\tau \to -i\tau_e$, and mapping the cylinder onto the complex plane according to (3.14), (3.15) we end up with the following complex solution for the independent functions $n_x$ and $\bar{n}_x$ on the z-plane.\textsuperscript{17}

$$n_x = f\left(\frac{z}{\bar{z}}\right) |z|^\omega, \quad \bar{n}_x = f\left(\frac{z}{\bar{z}}\right) |z|^{-\omega},$$  \hfill (4.16)

where $f$ satisfies the equation (the transformed form of the equation for $\theta$ in (4.14))

$$(1 - f^2)f'' + ff'^2 + \omega^2 f(1 - f^2)^2 = 0,$$  \hfill (4.17)

$$f' = \frac{df}{d\sigma}, \quad \sigma = -\frac{i}{2} \ln \frac{z}{\bar{z}}.$$ 

Eqs. (4.16), (4.17) represent a particular vacuum solution of the $S^2$ sigma model (4.12).\textsuperscript{13} The next and main issue is then to show that matching it onto the source terms coming from (4.13) we get the required relation between the angular momentum $J$ and the parameter $\omega$ of the solution.\textsuperscript{13} The explicit form of this relation is in general quite complicated \textsuperscript{[13]} but it should at least simplify in the limit of large $J$. Unfortunately, the non-linear form of the equation (4.17) makes matching onto $\delta$-function source terms (cf. (3.12), (3.13)) hard to implement explicitly. To relate $J$ to $\omega$ in the limit $J/\sqrt{\lambda} \gg 1$ it may be sufficient to use only an asymptotic form of the solution (4.16), with introduction of a 2-d cutoff near $\xi = \xi_1$ or $\xi_2$.

As in section 3, the final step will be to evaluate the sigma model action (plus source terms) on the solution (4.16) to find the $J$-dependent contribution to the coefficient $\gamma$ in the analog of (2.13), and then hopefully reproduce $\Delta = J + ...$ in (4.1) from the conformal invariance requirement that $\gamma = 0$.

\textsuperscript{17} Note that $n_x \to \bar{n}_x$ under $z \to \bar{z}^{-1}$ in agreement with the exchange symmetry of the two vertex operators or time reversal on the cylinder.

\textsuperscript{18} The solution in the case of the spinning string state in $AdS_5$ (i.e. the one that should be relevant for the semiclassical computation of the 2-point function of the operators (4.13)) has a similar structure: $N_x = F(\tilde{\xi}) |z|^\omega$, $\bar{N}_x = F(\tilde{\xi}) |z|^{-\omega}$, where $F = \sinh \rho(\sigma)$, with $\rho'' = (\kappa^2 - \omega^2) \sinh \rho \cosh \rho$.

\textsuperscript{19} For this it is sufficient to consider just a single vertex operator insertion.
One may object the utility of this method, given that it seems to be much more complicated and indirect as compared to the original (“semiclassical string states on a cylinder”) approach of [3]. Still, we hope that further understanding of this method may shed more light on details of the “string state – vertex operator” correspondence in the nonlinear sigma model cases like the present one.

5. Renormalization of composite operators in $S^{N-1}$ and $AdS_{N-1}$ sigma model

Let us now return to the issue of the mixing of the rotation vertex operators, i.e. (4.2) mixing with (4.9) and (4.3) with (4.10).

Having in mind applications to the $AdS_{5} \times S^{5}$ string theory, one may expect that in the 1-loop approximation one may ignore the fermions and thus, in particular, treat renormalizations in the $AdS_{5}$ and $S^{5}$ sectors as independent. However, as already mentioned above, this is not a priori obvious. Indeed, the superstring action [20] contains the R-R 5-form coupling term which has structure $\theta \theta (\partial n + \partial N)$. String vertex operators may also contain fermionic terms like $(\partial n + \theta \theta) (...)$ and $(\partial N + \theta \theta) (...)$. Pairing the fermionic terms in the action and in the vertex operators one may thus get additional fermionic contributions to the renormalization of the vertex operators. One may also get, via such fermionic contributions, extra mixings between the operator factors from the $AdS_{5}$ and from the $S^{5}$. Here we shall not attempt to take such fermionic contributions into the account, hoping that they would not qualitatively change our main conclusions.

Extending the discussion in [1], in this section we shall study mixing and anomalous dimensions for vertex-type operators in the $O(N)$ invariant 2-d sigma model, applying the general method described in [19] (see also [22]). As in [19], we will consider explicitly only the 1-loop approximation in bosonic sigma model, but we expect that these 1-loop results may clarify some important features of the general case. The discussion of the $S^{N-1}$ sigma model case can be generalized also to the $AdS_{N-1}$ case using a simple continuation rule as in [1].

Let us start with some general remarks. Given a sigma model with the action

$$S = \frac{1}{\pi \alpha'} \int d^{2} \xi \ G_{m n}(x) \partial x^{m} \bar{\partial} x^{n}$$

one may perturb it by an arbitrary higher-dimensional

\[20\] In particular, it seems unlikely that one will be able to derive the full relation $\Delta(J)$ in [3] for any value of $J$ in the present vertex operator approach.

\[21\] In general, the sigma model should be defined on a curved 2-d background; one should include all possible couplings to the curvature of the 2-d metric and instead of usual scale invariance demand the vanishing of the Weyl anomaly coefficients. In addition to Klein-Gordon-type equation for the tensor function, that leads, as in flat space [23], also to divergence-free and tracelessness conditions (see, e.g., [24] and refs. there).
local operator of the type (with total number of 2-d derivatives or level number $Q$)

$$V(f) = f_{m_1...m_j}(x)\partial^{k_1}x^{m_1}...\bar{\partial}^{k_h}x^{m_j}$$

and compute the renormalization of the coupling tensor $f_{m_1...m_j}$. Demanding the vanishing of the corresponding beta-function to linear order in $f_{m_1...m_j}$ (but to all orders in $\alpha'G^{mn}$) determines the corresponding marginal string vertex operator [18]. The resulting equation has symbolic form

$$\hat{\gamma}f = \left[2 - Q + \frac{1}{2}\alpha'\nabla^2 + \alpha' R_{...} + \sum \alpha'^k R_{...}R\nabla...\nabla\right]f = 0,$$

where $\hat{\gamma}$ is the “anomalous dimension” operator (in which we included also the canonical dimension term). For example, a 2-nd rank tensor $f$ (massive spin 2) satisfies a Lichnerowicz-type equation [24]. Solving such equations for $f$ is equivalent to finding the the eigenvalues and eigen-vectors of the anomalous dimension operator. Unfortunately, the general form of $\hat{\gamma}$ operator for generic tensor $f$ and metric $G$ is not known even to the leading (1-loop) order in $\alpha'$. For this reason, in most cases (with possible exceptions being WZW models or some plane-wave type models) one is not able to use a universal expression for $\hat{\gamma}$ and needs to calculate the anomalous dimensions of specific higher-derivative operators starting from “first principles”.

When the sigma model metric $G_{mn}$ has a large global symmetry (e.g., corresponds to a symmetric space) it is natural to construct the operator $V(f)$ so that it is covariant under the global symmetry. In particular, instead of using local coordinates $x^m$, we may write $V(f)$ in terms some global coordinates on which the group acts linearly. This is what we will do in the discussion below.

### 5.1. Rules of 1-loop renormalization in $S^{N-1}$ sigma model

To set up the notation, let us start with the action of the $S^{N-1}$ sigma-model action defined on a complex plane

$$S = \frac{R^2}{4\pi\alpha'} \int d^2 \xi \left(\partial_a n_m\right)^2 = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi \partial n_m \bar{\partial} n_m , \quad n_m n_m = 1 , \quad (5.1)$$

where $m = 1, ..., N$ and $g \equiv \frac{1}{\sqrt{\lambda}} = \frac{\alpha'}{\pi}$ plays the role of the standard dimensionless sigma-model coupling constant, which runs according to the well-known RG equation\footnote{The full $AdS_5 \times S^5$ string sigma-model is of course conformally invariant due to additional fermionic contributions to beta-functions of each of the $AdS_5$ and $S^5$ coupling constants coming from the R-R 5-form coupling term.}

$$\dot{g} = -\epsilon g + (N - 2)g^2 + (N - 2)g^3 + ... \quad g \equiv \frac{1}{\sqrt{\lambda}} , \quad \epsilon = d - 2 . \quad (5.2)$$
In view of the discussion in the previous section we shall be interested in renormalization of the following class of $O(N)$ covariant operators:

$$O_{\ell,s} = f_{k_1...k_l m_1...m_s} n_{k_1}...n_{k_l} \partial n_{m_1}...\partial n_{m_2}...\partial n_{m_{2s-1}} \partial n_{m_{2s}} .$$  \hspace{1cm} (5.3)

These form a special class of composite operators considered in \cite{19} (see sects. 8,9 there). We shall assume that $N$ is even and thus we can split the components of $n_k$ into complex pairs, i.e.

$$n_x = n_1 + in_2 , \quad \bar{n}_x = n_1 - in_2 , \quad n_y = n_3 + in_4 , \quad ... \hspace{1cm} (5.4)$$

In the case of Euclidean $AdS_{N-1}$ sigma-model the corresponding split of the $SO(1,N-1)$ unit vector $N_M$ is into $N_+, N_-, N_x, ...$, where $N_\pm$ are real, i.e. $N_+ N_- - N_x N_\bar{x} - ... = 1.$

To compute the 1-loop renormalisation of the operators (5.3) one may follow the general method of \cite{21} and use the following results for “pairing” of different building blocks of composite operators:

$$< n_k > = -\frac{1}{2} (N-1) I n_k , \quad < n_k, n_l > = -I(n_k n_l - \delta_{kl}) , \quad I = -\frac{1}{2\pi \epsilon} \to \infty , \hspace{1cm} (5.5)$$

$$\bar{\partial} n_k = -I \partial n_k , \quad < n_k, \bar{\partial} n_l > = -I \partial n_k n_l , \hspace{1cm} (5.6)$$

$$< \partial n_k, \partial n_l > = I n_k n_l \partial n_m \partial n_m , \quad < \bar{\partial} n_k, \bar{\partial} n_l > = I n_k n_l \bar{\partial} n_m \bar{\partial} n_m , \hspace{1cm} (5.7)$$

$$< \partial n_k, \bar{\partial} n_l > = -I (\bar{\partial} n_k \partial n_l - \delta_{kl} \partial n_m \bar{\partial} n_m) . \hspace{1cm} (5.8)$$

Contracting indices $k, l$ in (5.8) we get $< \partial n_k, \bar{\partial} n_k > = I (N-1) \partial n_k \bar{\partial} n_k$. Then using $< n_k >= -\frac{1}{2} (N-1) I n_k$ we conclude that

$$< \partial n_k \bar{\partial} n_k >= \partial < n_k > \bar{\partial} n_k + \partial n_k \bar{\partial} < n_k > + < \partial n_k, \bar{\partial} n_k > = 0 . \hspace{1cm} (5.9)$$

\footnote{23 Since eventually we will be interested in operators corresponding to states on leading Regge trajectory we do not include terms with multiple derivatives.}

\footnote{24 We keep only the singular term proportional to $I$ and use the notation of \cite{19}: $< AB > = < A > B + A < B > + < A, B >$, where $A$ and $B$ are some operators; if they depend on $n_k(x)$ only then $< A, B > = \int d^2 \xi d^2 \xi' < n_k(\xi), n_m(\xi') > \frac{\delta A}{\delta n_k(\xi)} \frac{\delta B}{\delta n_m(\xi')}$, etc. Also, $< A(n) >= \frac{1}{2} \int d^2 \xi d^2 \xi' < n_k(\xi), n_m(\xi') > \frac{\delta^2 A}{\delta n_k(\xi)^2 \delta n_m(\xi')}$. These pairing rules can be understood also in the framework of the background field method \cite{22}. Note that in the process of renormalization one is allowed to ignore redundant operators proportional to equations of motion, e.g., $[\partial \partial n_k - n_k(\partial n_m \partial n_m)] C_k(n)$, where $C_k$ is any local operator.}
For other scalar contractions one finds \[19\]
\[
< (\partial n_k \partial n_k) > = -(N - 2) I \partial n_k \partial n_k , \quad < (\bar{\partial} n_k \bar{\partial} n_k) > = -(N - 2) I \bar{\partial} n_k \bar{\partial} n_k , \quad (5.10)
\]
\[
< (\partial n_k \bar{\partial} n_k), (\partial m \bar{\partial} m) > = 2I[(\partial n_k \bar{\partial} n_k)(\partial m \bar{\partial} m) - (\partial n_k \partial n_k)(\bar{\partial} m \bar{\partial} m)] , \quad (5.11)
\]
\[
< (\partial n_k \partial n_k), (\partial_a n_m \bar{\partial}_b n_m) > = 0 , \quad < (\partial n_k \bar{\partial} n_k), (\partial_a n_m \bar{\partial}_b n_m) > = 0 , \quad (5.12)
\]
\[
< n_l, (\partial_a n_k \partial n_k) > = 0 , \quad < \partial n_l, (\partial_a n_k \partial n_k) > = 0 , \quad < \partial n_l, (\partial n_k \bar{\partial} n_k) > = 0 , \quad (5.13)
\]
\[
< \partial n_l, (\partial n_k \bar{\partial} n_k) > = I[\partial n_l(\partial n_k \bar{\partial} n_k) - \bar{\partial} n_l(\partial n_k \bar{\partial} n_k)] . \quad (5.14)
\]

Here \(\partial_a = (\partial, \bar{\partial})\), and similar relations are valid for complex conjugate combinations. Some of the above relations simplify when written in terms of the complex combinations \(n_x\) in \((5.4)\) (note that \(\delta_{xx} = 0\))
\[
< n_x, n_x > = -I n_x n_x , \quad < \partial n_x, \bar{\partial} n_x > = -I \partial n_x \bar{\partial} n_x , \quad < (\partial n_x \bar{\partial} n_x) > = -NI \partial n_x \bar{\partial} n_x , \quad (5.15)
\]
\[
< \partial n_x, \partial n_x > = In_x n_x \partial n_m \bar{\partial} n_m , \quad < \bar{\partial} n_x, \bar{\partial} n_x > = In_x n_x \bar{\partial} n_m \partial n_m .
\]

5.2. Anomalous dimensions of some simple operators

We are now ready to discuss some examples of operators \((5.3)\). The simplest “lowest level” scalar operator \(f_{k_1...k_{\ell}} n_{k_1}...n_{k_{\ell}}\) with traceless \(f_{k_1...k_{\ell}}\) is transformed under renormalization into itself (see \((5.5)\)). It has the same anomalous dimension \(\gamma\) as its simplest (highest-weight) representative:
\[
A_{\ell} = (n_x)^{\ell} : \quad \gamma = 2 - \frac{1}{2}[\ell(N-1)+\ell(\ell-1)]g+O(g^2) = -\frac{1}{2} \ell(\ell+N-2)g+O(g^2) . \quad (5.16)
\]

The two \(O(g)\) terms came from the two terms in \((5.5)\) (note that for \(n_x^{\ell}\) the number of 2-point pairings is \(\frac{1}{2} l(l-1)\)). The total coefficient \(\ell(\ell+N-2)\) is of course the value of the corresponding quadratic Casimir (this is a particular spherical harmonic, i.e. it solves the Laplace equation on the sphere).

As follows from \((5.10)\), the Lagrangian operator \(L = \partial n_m \bar{\partial} n_m\) has zero dimension. Also, eq. \((5.13)\) implies that it does not mix with powers of \(n_k\), i.e. the dimension of \(n_x^{\ell} \partial n_m \bar{\partial} n_m\) is given simply by the sum of the corresponding dimensions, \(\gamma = -\frac{1}{2} \ell(\ell+N-2)g+O(g^2)\).

\[25\] If \(f_\ell\) is the coupling constant that multiplies a composite operator when it is added to the sigma-model action, the anomalous dimension is defined by \(f_\ell^{-1} f_\ell = \gamma(\ell)(g) = 2 + O(g)\).
Similar results for operators of $AdS_{N-1}$ sigma-model are found by replacing $n^\ell$ and $\partial n \bar{\partial} m$ by $N^\ell$ and $\partial N \bar{\partial} N$, taking $\ell = -\Delta$ and changing the sign of the coupling $g$, i.e. of the sigma-model coupling. That implies that the dimension of $N^{-\Delta} \partial N \bar{\partial} N$ is $\gamma = \frac{1}{2} \Delta (\Delta - N + 2) g + O(g^2)$. Applying these remarks to the case of the “dilaton” operator (2.4) we verify the expression (2.11) for its perturbative anomalous dimension ($\ell = J$ for the sphere factor and $N = 6$).

Note that according to (5.8) $\partial n \bar{\partial} m < \partial n \bar{\partial} k > 0$. Thus the left-right factorized operators $(\partial n \partial m)^p (\partial n \bar{\partial} n)^q$ do not mix with operators containing $\partial n \bar{\partial} n$ factors. In particular, the number of $\partial n \bar{\partial} n$ factors never increases, so it can be used as a quantum number to characterise the leading term in the corresponding operator [19]. An example of such an operator considered in [25,19,22] is

$$(\partial n \bar{\partial} n)^r + \ldots : \gamma = 2 - 2r + r(r-1)g + \frac{2}{3} r(r-1)(r-\frac{7}{2}) + r(N-2)g^2 + O(g^3). \quad (5.17)$$

Here the 2-loop correction was computed in [22]. Dots following the operator $(\partial n \bar{\partial} n)^r$ stand for other operators it mixes with, and $\gamma$ is the corresponding diagonal element of the anomalous dimension matrix. In more detail, the above pairing rules imply that a renormalization-invariant linear combination of the operators of the same canonical dimension $2 - 2r$ is given by [19] (we assume that $r$ is even integer)

$$O_r = \sum_{p=0}^{r/2} c_p (\partial n \bar{\partial} n)^{r-2p} (\partial n \partial m)^p (\partial n \bar{\partial} n)^q . \quad (5.18)$$

The 1-loop renormalization matrix has only the diagonal and the next off-diagonal line of entries, with the diagonal ones being proportional to [19]

$$a_{1+p,1+p} = (r - 2p)(r - 1 - 2p) - 2(N-2)p , \quad p = 0, 1, 2, \ldots, r/2 . \quad (5.19)$$

The diagonal values are thus the eigenvalues, with largest one for large $r$ being $a_{11} = r(r-1)$ (corresponding to (5.17)). The associated eigen-operator (5.18) has all terms in the linear

26 Note that the term $\partial N \bar{\partial} N$ (with signature $+-+-+-$) enters the action with the opposite sign compared to the term $\partial n \bar{\partial} n$; this reflects the opposite signs of the curvatures of $S^{N-1}$ and $AdS_{N-1}$.

27 Here (as in (5.16)) one is to account for the number of 2-point pairings $\frac{1}{2} r(r-1)$, but there is extra -2 in (5.11) as compared to (5.5). The dimension indeed vanishes for $r = 1$ in agreement with the above discussion of the dilaton operator.

28 When $r$ is odd one is to consider $c_0 (\partial n \bar{\partial} n)^r + \ldots + c_{[r/2]} (\partial n \bar{\partial} n)(\partial n \bar{\partial} n)^{[r/2]}$. 

23
combination being present (i.e. all \( c_p \) are non-vanishing). At the same time, the “last” operator is an eigen-operator by itself,

\[
(\partial n_k \partial n_k \bar{n}_m \bar{n}_m)^{r/2} : \quad \gamma = 2 - 2r - r(N - 2)g + O(g^2). \tag{5.20}
\]

Since \( n^\ell_x \) does mix with the scalar operators like \((\partial n_k \bar{n}_k)^r\) or \((5.18)\) (cf. \((5.13)\)), it can be directly combined with them; using \((5.16),(5.17)\) we get

\[
n^\ell_x[(\partial n_k \bar{n}_k)^r + \ldots] : \quad \gamma = 2 - 2r - \frac{1}{2}[\ell(\ell + N - 2) - 2r(r - 1)]g + O(g^2). \tag{5.21}
\]

One finds that due to the opposite signs of the two contributions the total 1-loop correction to the anomalous dimension can be made small \([1]\) for large \( r \) by choosing \( \ell \approx \sqrt{2}r \). The analogous observation can be made \([1]\) for the corresponding \( AdS_{N-1} \) operator

\[
N^{-\Delta}[(\partial N_M \partial N_M)^r + \ldots] : \quad \gamma = 2 - 2r + \frac{1}{2}[\Delta(\Delta - N + 2) - 2r(r - 1)]g + O(g^2). \tag{5.22}
\]

Provided a similar pattern persists at higher loop orders (i.e. the 2-loop correction is proportional to \( \Delta^3 - cr^3 + \ldots \) at large \( \Delta \) and \( r \), cf. \((1.3),(5.17)\)) one may conjecture \([1]\) that the dimension \( \Delta(r) \) of the corresponding high-level string vertex operators should not blow up in the limit \( \sqrt{\lambda} \to 0 \). In the \( AdS_5 \times S^5 \) string context such operator may represent a particular high-level scalar string mode with no extra global charges like spin or \( S^5 \) angular momentum.\(^{29}\)

The 1-loop correction in \((5.21)\) has a very similar structure to the one of the dimension of the “dilaton” operator with \( S^5 \) orbital momentum \( J \) \((2.4)\) in \((2.11)\). As we shall see below, an analogous structure of the 1-loop anomalous dimension is characteristic also to the vertex operators \((1.9),(1.10)\) representing spinning string states on leading Regge trajectory.

5.3. Operators with angular momentum in \( S^5 \)

We would like now to discuss renormalization of the operator \((\partial n_x \bar{n}_x)^{J/2}\), which is expected to be a part of the vertex operator representing string state with angular momentum \( J \) in the 5-sphere \((1.2)\). It is clear from the above rules \((5.7)\), etc., that this operator may mix with \( \partial n_k \partial n_k \) and \( \bar{n}_k \bar{n}_k \), but not with \( \partial n_k \bar{n}_k \). On symmetry grounds,

\(^{29}\) Such operators may be dual to singlet scalar SYM operators of high dimension like the ones considered in \([24]\).
the general operator of the same angular momentum \( J \) and dimension \( 2 - J \) (not containing \( \partial n_k \bar{\partial} n_k \) factors and higher derivatives) which is stable under the renormalization is given by a linear combination of the following operators (4.9) (we shall assume that \( s = J/2 \) is even):

\[
O_s = \sum_{p,q=0}^{s/2} c_{pq} M_{pq}, \quad M_{pq} = V_{pq} + V_{qp}, \quad s = J/2 ,
\]

\[
V_{pq} = (n_x)^{2p+2q} (\partial n_x)^{s-2p} (\bar{\partial} n_x)^{s-2q} (\partial n_m \bar{\partial} n_m)^p (\bar{\partial} n_k \bar{\partial} n_k)^q .
\]

We have symmetrised the operators over \( p, q \) so that there is the invariance under \( \partial \rightarrow \bar{\partial} \) (or “2-d reality” property). The left and right sectors in \( V_{pq} \) look completely independent so can be analysed separately.

The analysis of renormalisation of this operator is similar to the case of (5.18). We find that under the 1-loop renormalization the operators \( V_{pq} \) transform as follows (here there is no summation under \( p, q \))

\[
V_{pq} \rightarrow V_{pq} + g I (a_{pq} V_{pq} + f_p V_{p+1,q} + f_q V_{p,q+1}) ,
\]

where

\[
a_{pq} = -s(N - 1) - (p + q)(2p + 2q - 1) - 4(p + q)(s - p - q) - (N - 2)(p + q) - (s - 2p)(s - 2q)
\]

\[
= 2p^2 + 2q^2 - (p + q)(2s + N - 3) - s(s + N - 1) ,
\]

\[
f_p = \frac{1}{2} (s - 2p)(s - 2p - 1) , \quad f_q = \frac{1}{2} (s - 2q)(s - 2q - 1) .
\]

Here the first \( s(N - 1) \) term came from the \( < n_k > \) in (5.3) and the last one – from (5.8). The “2-d real” combination \( M_{pq} = V_{pq} + V_{qp} \) transforms into a “2-d real” combination:

\[
M_{pq} \rightarrow M_{pq} + g I (a_{pq} M_{pq} + f_p M_{p+1,q} + f_q M_{p,q+1}) .
\]

The anomalous dimension matrix is then upper-triangular, and so its eigenvalues are equal to the diagonal entries \( a_{pq} \).

The minimal value of \( |a_{pq}| \) is at \( p = q = 0 \), i.e.

\[
a_{00} = -s(s + N - 1) ,
\]

\[30\] This would no longer be true for the operators with the same angular momentum and dimension but containing extra factors of \( \partial n_m \bar{\partial} n_m \). In the flat space limit, such operators would represent states on a subleading Regge trajectory.
while the maximal value is formally reached at the extremum of the quadratic polynomial (5.26), \( p = q = \frac{1}{2}s + \frac{1}{4}(N - 3) \), i.e. at \( p = q = \frac{1}{2}s \) in the present case,

\[
a_{\frac{s}{2}, \frac{s}{2}} = -2s(s + N - 2) .
\]  

(5.30)

The associated eigen-operators are particular linear combinations (5.23). The eigen-operator for the the eigenvalue \( a_{pp} \) starts with the term in (5.23) which has the coefficient \( c_{pp} \). In particular, the operator \((\partial n_x \bar{\partial} n_x)^s\) is present only in the combination corresponding to the minimal eigenvalue (5.29), i.e. in this case the eigen-operator contains all terms in (5.23)

\[
O_s = c_{00}(\partial n_x \bar{\partial} n_x)^s + \ldots + c_{\frac{s}{2}, \frac{s}{2}}(n_x)^{2s}(\partial n_k \bar{\partial} n_m \bar{\partial} n_m)^{s/2} .
\]  

(5.31)

Remarkably, in contrast to what happens in flat space, in \( S^{N-1} \) (or \( AdS_{N-1} \)) space an operator with a factor \((\partial n_x \bar{\partial} n_x)^{J/2}\) representing a string state with an “intrinsic” angular momentum may mix with an operator with a factor \( n_x^J \) which may be interpreted as representing a state with “orbital” component of the angular momentum. At the same time, the “last” operator in the sum (5.23) or (5.31) does not mix with others – it is by itself the eigen-operator for the eigenvalue (5.30).

To construct a string vertex operator of canonical dimension \( 2 - J \) corresponding to a string state with angular momentum \( J \) in \( S^5 \) and energy \( E = \Delta \) in \( AdS_5 \) we need to multiply the above operators by \( N_+^{-\Delta} \). This way we get for the case (5.29) (\( N = 6, s = J/2 \))

\[
N_+^{-\Delta}[(\partial n_x \bar{\partial} n_x)^{J/2} + \ldots] : \quad \gamma(\Delta, J) = 2 - J + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}J(J + 10)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right) ,
\]  

(5.32)

and for the case (5.30)

\[
N_+^{-\Delta}n_x^J(\partial n_k \bar{\partial} n_m \bar{\partial} n_m)^{J/4} : \quad \gamma(\Delta, J) = 2 - J + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - J(J + 8)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right) .
\]  

(5.34)

Note that the coefficient \( J(J + 8) \) in (5.33) is the sum of the 1-loop contribution \( J(J + 4) \) of \( n_x^J \) in (5.16),(2.11) and 4\( J \) of the operator (5.20) with \( r = J/2 \).

\[31\] In general, the anomalous dimensions of particular operators should be related to eigenvalues of generalized massive Laplace equation on a sphere. In particular, 1-loop dimensions in (5.23),(5.33) should correspond to special symmetric tensor harmonics on \( S^{N-1} \) (cf. [27]).
The 1-loop correction in (5.33) has the same structure as in the dimension (2.11) of the operator (2.4) from the “massless” level carrying orbital momentum \( J \), i.e. \( \gamma(\Delta, J) = \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - J(J + 4)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right) \). At the same time, (5.33) has the same form as the dimension of the scalar \( AdS_{N-1} \) operator in (5.22), which for the same level number \( 2r = J \) as in (5.33),(5.35) and \( N = 6 \) is given by

\[
\gamma(\Delta, J) = 2 - J + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}J(J - 2)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right).
\]

All of these operators share with (5.21) the property [1] that the 1-loop anomalous dimension can be made small by balancing the \( \Delta \)-contribution against the \( J \)-contribution. Thus imposing marginality for large “classical” values of \( J \) and \( \Delta \) one finds that \( \Delta = J + ... \) for (2.11),(5.35) and

\[
\Delta = \frac{1}{\sqrt{2}}J + ...
\]

(5.36) for (5.33),(5.22) (assuming that higher order corrections do not spoil this conclusion, cf. (1.3),(1.5)).

Expecting that the operator in (5.32) should be representing a folded rotating string state in \( S^5 \) it may seem strange to find the proportionality coefficient in (5.36) to be \( \frac{1}{\sqrt{2}} \) instead of 1, as implied (4.1) by the corresponding classical solution in [3]. There may be several possible explanations for this discrepancy. First, it may be that higher-order corrections in (5.33) (that are important for “classical” values of \( J \) and \( \Delta \)) first need to be summed up as in (1.5) and only then one finds \( \Delta = J + ... \) in the limit \( q = \frac{1}{\sqrt{\lambda}} \gg 1 \). Another possibility is that additional fermionic contributions that we ignored above change the coefficient \( \frac{1}{2} \) in front of \( J^2 \) term (5.33) into 1, and thus change \( \frac{1}{\sqrt{2}} \) in (5.36) into 1. Then assuming that all higher order corrections (1.3) will also have the form \( \Delta^{n+1} - J^{n+1} \), we would then get the agreement with the semiclassical prediction \( \Delta = J + ... \) in (4.1).

Finally, it could be that in spite of the fact that the operator (5.32) has the required angular momentum and flat-space limit it is not the right vertex operator to be associated with the semiclassical rotating string state in \( S^5 \). An indication of which vertex operator is an adequate one could come from the discussion in section 4.2 of an attempt to reproduce the relation between \( \Delta \) and \( J \) using semiclassical path integral approach. For string rotating around north pole of \( S^5 \) one has [3]:

\[
ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\Omega_3^2,
\]

Finding subleading correction in \( \Delta(J) \) in (4.1) would in any case require summing contributions from all orders in \( \frac{1}{\sqrt{\lambda}} \) expansion.
\( t = \kappa \tau, \; \phi = \omega \tau, \; \theta = \theta(\sigma), \) and \( \theta'^2 = \kappa^2 - \omega^2 \sin^2 \theta. \) For example, while the operator in (5.34) has the factor \( n_x^J \) that would provide a natural source for the angle \( \phi \) (cf. the discussion in section 2.2), the derivative part of it is completely \( SO(6) \) invariant (i.e. does not select a particular rotation plane) and thus it is unlikely to provide the required source for the angle \( \theta. \)

To attempt to carry out the semiclassical argument sketched in section 4.2 we would need to decide on specific form of the operator in (5.31), (5.32). In particular, one could try to use just the leading term in it given explicitly in (5.32). One could try to justify this provided the coefficients in the linear combination (5.31) were small in the limit of large \( s = J/2. \) However, this is not the case according to (5.25), (5.27): the mixing is not suppressed in the large \( s \) limit. Indeed, fixing the coefficient of the first term in (5.31) to be 1, the coefficients of other terms are proportional to products of off-diagonal values divided over products of differences \( a_{pp} - a_{00} = 2p^2 + q^2 - (p + q)(2s + N - 1). \)

5.4. Operators with spin in AdS

The analog of the operator (5.32) that carries spin in \( AdS_5 \) would be the one in (4.3). Computing its dimension in a naive way, i.e. ignoring its mixing with other operators we would get

\[
N_+^\Delta \left[ (\partial N_x \bar{\partial} N_x)^{S/2} + \ldots \right] : (5.37)
\]

\[
\gamma_0(\Delta, J) = 2 - S + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) + \frac{1}{2} S(S + 10)] + O\left( \frac{1}{(\sqrt{\lambda})^2} \right), \quad (5.38)
\]

where we used that in the AdS case one is to reverse the sign of coupling compared to (5.33). In contrast to (5.33), here both contributions to the 1-loop anomalous dimension term have the same sign and thus can not cancel each other.\(^{33}\)

However, as already mentioned above, the operator (4.3) mixes with other operators (4.10) under the renormalization, and so one should first diagonalize the corresponding matrix of anomalous dimensions.

Instead of addressing the question of renormalisation of operators (5.37) directly in the \( AdS_{N-1} \) context, it is useful to follow the same notation as in the previous subsection

\(^{33}\) At small \( S \) and large \( \Delta \) we would still get from \( \gamma = 0 \) the usual linear Regge trajectory relation, \( \Delta^2 \approx 2\sqrt{\lambda}(S - 2). \) To interpolate to the expected large \( S \) behaviour (4.1) it may seem that we need first to sum up all higher-loop contributions to \( \gamma. \)
and solve the equivalent problem in the $S^{N-1}$ sigma model. Namely, we shall consider the renormalization of the $S^{N-1}$ operators of the type

$$n_y^\ell (\partial n_x \bar{\partial} n_x)^s + \ldots , \quad n_x = n_1 + in_2 , \quad n_y = n_3 + in_4 , \quad (5.39)$$

which are the direct analogs of (5.37). The relevant $AdS_5$ case will be obtained by setting

$$N = 6 , \quad \ell = -\Delta , \quad s = S/2 , \quad g \to -\frac{1}{\sqrt{\lambda}} . \quad (5.40)$$

There will be two different types of mixings: (i) in view of (5.36), i.e. $< n_y, \partial n_x > = -In_x \partial n_y$, we will need to include operators with $n_x$ factors with no derivatives and with factors where derivatives act on $n_y$; (ii) pairings of $\partial n_x$ with $\partial n_x$ or $\partial n_y$, etc., will generate as in (5.24) terms with extra scalar factors $\partial n_k \partial n_k$ and $\bar{\partial} n_m \bar{\partial} n_m$. As a result, a generic operator which has the same quantum numbers and canonical dimension as (5.39) and which is an eigen-vector of the corresponding anomalous dimension matrix will have the form (we shall assume that $\ell \geq 2s$)

$$O_{\ell,s} = \sum_{u,v=0}^{s} \sum_{p=0}^{\frac{s-u}{2}} \sum_{q=0}^{\frac{s-v}{2}} c_{uvpq} M_{uvpq} , \quad (5.41)$$

$$M_{uvpq} \equiv n_y^{\ell-u} n_x^{u+v} (\partial n_y)^u (\partial n_x)^{s-u-2p} (\bar{\partial} n_y)^v (\bar{\partial} n_x)^{s-v-2q} (\partial n_k \partial n_k)^p (\bar{\partial} n_m \bar{\partial} n_m)^q . \quad (5.42)$$

The “last” term in this sum is the direct analog of (5.34) and is an eigen-operator by itself:

$$n_y^\ell n_x^{2s} (\partial n_k \partial n_k \partial n_m \bar{\partial} n_m)^{s/2} : \quad (5.43)$$

$$\gamma(\ell,s) = 2 - 2s - \frac{1}{2} g[(\ell + 2s)^2 + (\ell + 4s)(N - 2)] + O(g^2) . \quad (5.44)$$

To compute the dimension (5.44) we noted that according to (5.33) pairings in $n_y^\ell n_x^{2s}$ give the contribution $-\frac{1}{2}(\ell(\ell - 1) + \frac{1}{2} 2s(2s - 1) + 2s\ell) = -\frac{1}{2}(\ell + 2s)(\ell + 2s - 1)$. Adding the contribution of $< n_a >$ and of the scalar operator (5.20) we end up with (5.44). Note that for the corresponding $AdS_5$ operator one is to use (5.40), and so for large $S$ and $\Delta$ the leading term in the 1-loop correction vanishes if $\Delta = S$.

To simplify the discussion let us now ignore all mixings with the scalar operators $(\partial n_k \partial n_k)^p (\bar{\partial} n_m \bar{\partial} n_m)^q$. They can be readily included as in section 5.3 and that leads only
to upper off-diagonal terms in the anomalous dimension matrix, i.e. should not change the eigenvalues of the anomalous dimension matrix. The $p = q = 0$ term in (5.41) is

$$O_{\ell,s} = \sum_{u,v=0}^{s} c_{uv} M_{uv},$$

(5.45)

$$M_{uv} \equiv n_{y}^{\ell-u-v} n_{x}^{u+v} (\partial n_{y})^{u}(\partial n_{x})^{s-u}(\bar{\partial} n_{y})^{v}(\bar{\partial} n_{x})^{s-v}.$$  

(5.46)

Using the rules of section 5.1 we conclude that under the 1-loop renormalization

$$M_{uv} \rightarrow (1 + g I a_{uv}) M_{uv}$$

$$- g I \left( (\ell - u - v) [(s - u) M_{u+1,v} + (s - v) M_{u,v+1}] + (u + v) [u M_{u-1,v} + v M_{u,v-1}] ight. + (s - u) v M_{u+1,v-1} + u(s - v) M_{u-1,v+1} \right),$$  

(5.47)

where $u, v = 0, \ldots, s$ and the diagonal renormalization coefficients are

$$a_{uv} = -\frac{1}{2} [\ell(\ell + N - 2) + 2s(N - 1)] - (\ell - u - v)(u + v) - (u + v)(2s - u - v) - uv - (s - u)(s - v)$$

$$= -\frac{1}{2} [\ell(\ell + N - 2) + 2s(s + N - 1)] - (u + v)(\ell + s - 2u - 2v) - 2uv .$$

(5.48)

A naive guess is that for large $\ell, s$ the “upper off-diagonal” term $(\ell - u - v) [(s - u) M_{u+1,v} + (s - v) M_{u,v+1}]$ in (5.47) will be dominating over the other off-diagonal terms, and then the eigenvalues of anomalous dimension matrix will be given by the diagonal values (5.48).

For $u, v = 0$ (5.48) becomes

$$a_{00} = -\frac{1}{2} \ell(\ell + N - 2) - s(s + N - 1) ,$$

(5.49)

which is related to the 1-loop coefficient in (5.38) in the special case (5.40). For $u = v = s$ we get

$$a_{ss} = -\frac{1}{2} \ell(\ell + N - 2) - s(s + N - 1) - 2s\ell + 4s^{2} ,$$

(5.50)

so that $|a_{00}| < |a_{ss}|$ for $\ell > 2s$.

It is possible to find the eigenvalues of the anomalous dimension matrix in (5.47) following the method described in section 9 of [19].\footnote{I am grateful to F. Wegner for an important clarification of this method.} In general, the eigen-operators can be characterised by three Young tableaux: $Y_\partial$ and $Y_{\bar{\partial}}$ that describe linear combinations of...
permutations of the $2s$ derivatives $\partial$ and $2s$ derivatives $\bar{\partial}$ in (5.46), respectively, as well as by $Y_n$ that describes permutations of $\ell + 2s$ factors $n_x$ and $n_y$. Then the coefficients in eigenvalues of the anomalous dimension matrix normalised as in (5.48) are given by

$$a = -\frac{1}{2}(\ell + 2s)(N - 1) + a_{\text{exchange}},$$

(5.51)

where the first term comes from the renormalization (cf. (5.5)) of the $n_x$ and $n_y$ factors and

$$a_{\text{exchange}} = \xi(Y_\partial) + \xi(Y_{\bar{\partial}}) - \xi(Y_n).$$

(5.52)

Here $\xi(Y)$ is an eigenvalue of the sum of transpositions

$$\xi(Y) = \frac{1}{2}\sum_{i=1,2,...} k_i(k_i - 2i + 1), \quad Y = (k_1, k_2, ...),$$

(5.53)

where $k_i$ are the numbers of boxes in the rows of a Young tableau $Y$. Since (5.46) contains only first powers of derivatives, we get

$$Y_\partial = (s, 0, 0, ...), \quad Y_{\bar{\partial}} = (s, 0, 0, ...), \quad \xi(Y_\partial) = \xi(Y_{\bar{\partial}}) = \frac{1}{2}s(s - 1).$$

(5.54)

Given that we have only two types of $n_k$-components, i.e. $n_x$ and $n_y$, $Y_n$ will have only two rows at most (we assume that $\ell \geq 2s$)

$$Y_n^{(k)} = (\ell + 2s - k, k, 0, 0, ...), \quad k = 0, 1, ..., 2s,$$

(5.55)

$$\xi(Y_n^{(k)}) = \frac{1}{2}(\ell + 2s - k)(\ell + 2s - k - 1) + \frac{1}{2}k(k - 3).$$

(5.56)

Thus the eigenvalue coefficients (5.51) are given by

$$a^{(k)} = -\frac{1}{2}(\ell + 2s)(N - 1) + s(s - 1) - \frac{1}{2}(\ell + 2s - k)(\ell + 2s - k - 1) - \frac{1}{2}k(k - 3), \quad k = 0, 1, ..., 2s.$$  

(5.57)

$a^{(k)}$ has a minimum at $k = 0$ and a maximum at $k = 2s$ with values (cf. (5.49),(5.50))

$$a^{(0)} = -\frac{1}{2}\ell(\ell + N - 2) - s(s + N - 1) - 2\ell s,$$

(5.58)

$$a^{(2s)} = -\frac{1}{2}\ell(\ell + N - 2) - s(s + N - 3).$$

(5.59)

Written in $\text{AdS}_5$ notation (5.40) these expressions become (cf. (5.38))

$$a^{(0)} = -\frac{1}{2}[\Delta(\Delta - 4) + \frac{1}{2}S(S + 10) - 2\Delta S],$$

(5.60)
Interestingly, in contrast to the naive dimension (5.38) where the 1-loop term was positive for large \( \Delta \) and \( S \), here \( a^{(0)} \) can be made to vanish. The corresponding anomalous dimension

\[
\gamma(\Delta, J) = 2 - S + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) + \frac{1}{2}S(S + 10) - 2\Delta S] + O\left(\frac{1}{\sqrt{\lambda}}\right) \quad (5.62)
\]

can thus be made small for large \( \Delta \) and \( S \) provided \((\Delta, S \gg \sqrt{\lambda} \gg 1)\)

\[
\Delta(S) \gg \frac{1}{\sqrt{\lambda}} \approx (1 \pm \frac{1}{\sqrt{2}})S + \sqrt{2}\sqrt{\lambda} + O(1) \quad (5.63)
\]

As a result, for large \( \Delta, S \) one is able to construct a marginal vertex operator with the required quantum numbers. This is, at least qualitatively, consistent with the semiclassical prediction (4.1). To reproduce the same proportionality coefficient (i.e. 1) between \( \Delta \) and \( S \) as in (4.1) one probably needs to sum up all relevant higher-order \( \frac{1}{\sqrt{\lambda}} \) corrections in (5.62) (cf. (1.5)).

Our main conclusion is that, as in the \( S^5 \) rotation case in section 5.3, the vertex operators that correspond to string states with spin in \( AdS_5 \) are given by linear combinations of local operators like (5.45). That seems to complicate possible rederivation of the semiclassical prediction (4.1) for \( \Delta(S) \) from the string path integral expression for the 2-point correlator of the two vertex operators.

**Acknowledgments**

We are grateful for S. Frolov for a collaboration at an initial stage of this project. We would like also to thank K. Zarembo for many important discussions. We also acknowledge I. Klebanov, R. Metsaev and F. Wegner for discussions and very useful explanations. This work of was supported in part by the grants DOE DE-FG02-91ER40690, PPARC SPG 00613, INTAS 99-1590, and by the Royal Society Wolfson Award.
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