Isometries between non-commutative symmetric spaces associated with semi-finite von Neumann algebras

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Abstract
We show that positive surjective isometries between symmetric spaces associated with semi-finite von Neumann algebras are projection disjointness preserving if they are finiteness preserving. This is subsequently used to obtain a structural description of such isometries. Furthermore, it is shown that if the initial symmetric space is a strongly symmetric space with absolutely continuous norm, then a similar structural description can be obtained without requiring positivity of the isometry.

Keywords Isometries · Disjointness preserving · Non-commutative symmetric spaces · Jordan homomorphisms

Mathematics Subject Classification Primary 47B38; Secondary 46L52

1 Introduction
The form of isometries between $L^p$-spaces was first described by Banach (in the case of finite measure spaces \cite{1}) and Lamperti (for $\sigma$-finite measure spaces \cite{22}). In the proofs of these results essential use is made of the fact that isometries map functions with disjoint support to functions with disjoint support. This will form the basis of the techniques to be employed in this paper. Representations of isometries between more general symmetric function spaces were obtained by Zaidenberg \cite{28}. We will define

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symmetric spaces below, but mention that well-known examples of such spaces include
the $L^p$, Orlicz and Lorentz function spaces. A detailed account of results on isometries
in the commutative settings and the techniques used in the proofs can be found in [12].

Non-commutative symmetric spaces are Banach spaces of closed, densely-defined
operators affiliated with a von Neumann algebra. In the special case where the under-
lying von Neumann algebra is commutative, and hence isometrically isomorphic to an
$L^\infty$-space over some localizable measure space, we obtain the commutative (classical)
symmetric function spaces. In the more general non-commutative (quantum) setting,
isometries of $L^p$-spaces associated with a semi-finite von Neumann algebra equipped
with a faithful, normal semi-finite trace have been characterized by Yeadon [27], but
the description of isometries between more general symmetric spaces have typically
been limited to the finite trace setting or particular examples of semi-finite von Neu-
mann algebras. In particular, structural descriptions for surjective isometries between
Lorentz spaces [3], positive surjective isometries between a symmetric space and a
fully symmetric space [3], and positive (not necessarily surjective) isometries between
a symmetric space and a fully symmetric space with $K$-strictly monotone norm [25]
have been obtained in the setting where the von Neumann algebra is equipped with a
finite trace. Furthermore, surjective isometries on a separable symmetric space have
been characterized [24] under the assumption that the underlying von Neumann alge-
bra is an AFD (almost finite-dimensional) factor of type $II_1$ or $II_\infty$. In this paper
we complement these results by considering surjective isometries between (general)
symmetric spaces associated with (general) semi-finite von Neumann algebras.

The technique we will employ is to analyze and utilize disjointness preserving
properties of isometries. The motivation is as follows. Every von Neumann algebra is
generated by its lattice of projections and therefore it is unsurprising that any isometric
isomorphism between von Neumann algebras has to be implemented by a map that
preserves this lattice structure, namely a Jordan *-isomorphism, possibly multiplied
by a unitary operator [17]. Furthermore, one would anticipate that there would be a
relationship between the isometries of symmetric spaces associated with semi-finite
von Neumann algebras and the isometries of the underlying von Neumann algebras. In
describing the structure of an isometry between symmetric spaces it is therefore natural
to use the isometry to initially define a map on projections. In order to ensure that this
map preserves the projection lattice structure and can be extended in a well-defined
and linear manner, this map should preserve orthogonality of projections. In the setting
of commutative and non-commutative $L^p$-spaces, for example, this can be achieved
by showing that the isometry is disjointness preserving ([22,27], respectively). More
recently it has been shown [25] that a positive isometry $T : E \to F$ between symmetric
spaces associated with semi-finite von Neumann algebras is disjointness preserving
provided $F$ is contained in $L_0(\tau)$ and $F$ has $K$-strictly monotone norm (definitions
to follow). This result is then used to describe the structure of a positive isometry
$T : E \to F$, where $E$ is a symmetric space on a trace-finite von Neumann algebra
and $F$ is a fully symmetric space with $K$-strictly monotone norm on a trace-finite von
Neumann algebra. In this paper we define a weaker notion of projection disjointness
preserving maps, identify positive isometries satisfying this condition and show that
even in the semi-finite setting, this weaker notion is sufficient to describe the structure
of such isometries.
The structure of the paper is as follows. In Sect. 3 we obtain a local representation of positive surjective isometries, which enables us to show that these isometries are projection disjointness preserving. We then investigate projection disjointness preserving isometries in Sect. 4 and show that even if these are not necessarily positive nor surjective we can describe their structure on an ideal contained in the intersection of the von Neumann algebra and the symmetric space. In order to obtain a global representation we consider isometries with more structure for the remainder of Sect. 4. In Sect. 5 we show that we can also obtain a global representation of projection disjointness preserving isometries with fewer assumptions on their structure if the initial symmetric space has slightly more structure.

Most of results in this paper will be proved under the assumption that the isometry under consideration is what we will call finiteness preserving. We show in [7] that surjective isometries between Lorentz spaces associated with semi-finite von Neumann algebras satisfy this condition (and are also projection disjointness preserving). Furthermore, this condition is trivially satisfied if the final von Neumann algebra is equipped with a finite trace.

We are grateful to the reviewer for pointing out the preprint [15], which contains some complementary results on the properties and structure of positive isometries and disjointness preserving maps in the more general setting of symmetrically $\Delta$-normed spaces. Furthermore, one of our results, namely Theorem 4.11, follows partially from [15, Theorem 3.6], but was independently obtained using different methods.

2 Preliminaries

Throughout this paper, unless indicated otherwise, we will use $\mathcal{A} \subseteq B(H)$ and $\mathcal{B} \subseteq B(K)$ to denote semi-finite von Neumann algebras, where $B(H)$ and $B(K)$ are the spaces of all bounded linear operators on Hilbert spaces $H$ and $K$, respectively. Let $\tau$ and $\nu$ denote distinguished faithful normal semi-finite traces on $\mathcal{A}$ and $\mathcal{B}$, respectively. The lattice of all projections in $\mathcal{A}$ will be denoted $\mathcal{P}(\mathcal{A})$ and the sublattice of projections with finite trace will be denoted $\mathcal{P}(\mathcal{A})_f$. We will use $1_{\mathcal{A}}$ (or just $1$ if no confusion is possible) to denote the identity of $\mathcal{A}$. The set of all finite linear combinations of mutually orthogonal projections in $\mathcal{P}(\mathcal{A})_f$ (respectively $\mathcal{G}(\mathcal{A})_f$). Convergence in $\mathcal{A}$ with respect to the operator norm topology, the strong operator topology (SOT) and the weak operator topology (WOT) will be denoted by respectively $\mathcal{A} \rightarrow$, $\text{SOT} \rightarrow$ and $\text{WOT} \rightarrow$. A linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is called a Jordan homomorphism if $\Phi(\text{yx} + \text{xy}) = \Phi(y)\Phi(x) + \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{A}$. If, in addition, $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$, then $\Phi$ is called a Jordan $\ast$-homomorphism.

Further details regarding von Neumann algebras and Jordan homomorphisms may be found in [18].

A closed operator $x$ with domain $\mathcal{D}(x)$ dense in $H$ is affiliated with $\mathcal{A}$ if $u^*xu = x$ for all unitary operators $u$ in the commutant $\mathcal{A}'$ of $\mathcal{A}$. A closed densely defined self-adjoint operator $x$ with spectral measure $e^x$ is affiliated to $\mathcal{A}$ iff $e^x(B) \in \mathcal{P}(\mathcal{A})$ for every Borel subset $B$ of $\mathbb{R}$. For such an operator we will write $x = \int_{-\infty}^{\infty} \lambda de^x_\lambda$ if $\{e^x_\lambda\}_\lambda$ is the unique resolution of the identity such that $x\eta = \int_{-\infty}^{\infty} \lambda de^x_\lambda \eta$ for each $\eta \in f_n(H)$ and
all $n$, and $\bigcup_{n=1}^{\infty} f_n(H)$ is a core for $x$, where $f_n := e_n^x - e_{-n}^x$ (see [18, Theorem 5.6.12]). If $x : D(x) \to H$ is a closed and densely defined operator, then the projection onto the kernel of $x$ will be denoted by $n(x)$, the projection onto closure of the range of $x$ by $r(x)$, and the support projection $1 - n(x)$ by $s(x)$. It follows that $x = r(x)x = xs(x)$, and if $x = x^*$, then $r(x) = s(x)$ and $x = s(x)x = xs(x)$. If $x$ is affiliated with $A$, all these three projections are in $A$. A closed, densely defined operator $x$ affiliated to $A$ is called $\tau$-measurable if there is a sequence $(p_n)$ in $\mathcal{P}(A)$ such that $p_n \uparrow 1$, $p_n(H) \subseteq D(x)$ and $1 - p_n \in \mathcal{P}(A)_f$ for every $n$. It is known that if $x = u|x|$ is the polar decomposition of $x$, then $x$ is $\tau$-measurable if and only if it is affiliated to $A$ and there is a $\lambda > 0$ such that $\tau(e^{ix\lambda}(\lambda, \infty)) < \infty$. A vector subspace $D \subseteq H$ is called $\tau$-dense if there exists a sequence $(p_n)$ in $\mathcal{P}(A)$ such that $p_n(H) \subseteq D$ for all $n$, $p_n \uparrow 1$ and $\tau(1 - p_n) < \infty$ for all $n$. Clearly a closed densely defined operator $x$ affiliated to $A$ is $\tau$-measurable if and only if its domain $D(x)$ is $\tau$-dense. The set of all $\tau$-measurable operators affiliated with $A$ will be denoted $S(A, \tau)$ or $S(A)$. It becomes a $*$-algebra when sums and products are defined as the closures of respectively the algebraic sum and algebraic product. For $x \in S(A, \tau)$ we write $x \geq 0$ if $(\langle x\xi, \xi \rangle) \geq 0$ for all $\xi$ in the domain of $x$ (where $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$), and we put $S(A, \tau)^+ = \{x \in S(A, \tau) : x \geq 0\}$. The cone $S(A, \tau)^+$ defines a partial order on the self-adjoint elements of $S(A, \tau)$. If $H$ is any collection of $\tau$-measurable operators, then we will write $H^{\text{sa}} = \{x \in H : x = x^*\}$ and $H^+ = \{x \in H : x \geq 0\}$. Note that $A$ is an absolutely solid subspace of $S(A, \tau)$, i.e. if $x \in S(A, \tau)$ and $y \in A$ with $|x| \leq |y|$, then $x \in A$.

For $\epsilon, \delta > 0$, define $N(\epsilon, \delta) := \{x \in S(A, \tau) : \tau(e^{ix\lambda}(\epsilon, \infty)) \leq \delta\}$. The collection $\{N(\epsilon, \delta) : \epsilon, \delta > 0\}$ defines a neighbourhood base for a vector space topology $T_m$ on $S(A, \tau)$. This topology is called the measure topology and with respect to this topology $S(A, \tau)$ is a complete metrisable topological $*$-algebra. We will repeatedly use the fact that multiplication is jointly continuous in the measure topology. Another important vector space topology on $S(A, \tau)$ is the local measure topology, denoted $T_{lm}$, which has a neighbourhood base consisting of the collection of sets of the form $N(\epsilon, \delta, p) := \{x \in S(A, \tau) : pxp \in N(\epsilon, \delta)\}$, where $\epsilon, \delta > 0$ and $p \in \mathcal{P}(A)_f$.

Multiplication is separately, but not jointly continuous with respect to the local measure topology, that is $\lim_{\lambda \to 0} T_{lm}(\lambda) x'y \not\to xy$ and $\lim_{\lambda \to 0} T_{lm}(\lambda) xy' \not\to yx$ whenever $y \in S(A, \tau)$ and $\{x_\alpha\}$ is a net in $S(A, \tau)$ with $\lim_{\lambda \to 0} T_{lm}(\lambda) x_\alpha \to x \in S(A, \tau)$.

If $(x_\lambda)_{\lambda \in \Lambda}$ is an increasing net in $S(A, \tau)^{sa}$ and $x = \sup\{x_\lambda : \lambda \in \Lambda\} \in S(A, \tau)^{sa}$, we write $x_\lambda \uparrow x$. In the case of a decreasing net $(x_\lambda)_{\lambda \in \Lambda}$ with infimum $0$ we write $x_\lambda \downarrow 0$. If $H \subseteq S(A, \tau)$ and $T : H \to S(\mathcal{B}, \nu)$ is a linear map such that $T(x_\lambda) \uparrow T(x)$ whenever $(x_\lambda)_{\lambda \in \Lambda}$ is a net in $H^{sa}$ such that $x_\lambda \uparrow x \in H^{sa}$, then $T$ will be called normal (on $H$). If $E$ is a linear subspace of $S(A, \tau)$, a linear map $T : E \to S(\mathcal{B}, \nu)$ will be called finiteness preserving if $\nu(s(T(p))) < \infty$ whenever $p \in \mathcal{P}(A)_f$. For background and further details regarding trace-measurable operators the interested reader is referred to [11,26].

For $x \in S(A, \tau)$, the distribution function of $|x|$ is defined as $d(|x|)(s) := \tau(e^{ixs}(s, \infty))$, for $s \geq 0$. The singular value function of $x$, denoted $\mu_x$, is defined to be the right continuous inverse of the distribution function of $|x|$, namely
\[ \mu_x(t) = \inf \{ s \geq 0 : d(\{x\})(s) \leq t \} \quad t \geq 0. \]

If \( x, y \in S(\mathcal{A}, \tau) \), then we will say that \( x \) is submajorized by \( y \) and write \( x \ll y \) if \( \int_0^t \mu_x(s)ds \leq \int_0^t \mu_y(s)ds \) for all \( t > 0 \). Let \( S_0(\mathcal{A}, \tau) \) denote the ideal of \( \tau \)-compact operators, which is defined as the set of all \( \tau \)-measurable operators \( x \) for which \( \lim_{t \to \infty} \mu_x(t) = 0 \).

A linear subspace \( E \subseteq S(\mathcal{A}, \tau) \), equipped with a norm \( \| \cdot \|_E \), is called a symmetric space if \( E \) is a Banach space and \( x \in E \) with \( \| x \|_E \leq \| y \|_E \), whenever \( y \in E \) and \( x \in S(\mathcal{A}, \tau) \) with \( \mu_x \leq \mu_y \). In this case we also have that \( \| uv \|_E \leq \| u \|_E \| v \|_E \) for all \( x \in E, u, v \in \mathcal{A} \). Furthermore, \( \| x \|_E = \| x^* \|_E = \| x \|_F \) for all \( x \in E \), and \( \| x \|_E \leq \| y \|_E \) whenever \( x, y \in E \) with \( |x| \leq |y| \). A symmetric space is an absolutely solid subspace of \( S(\mathcal{A}, \tau) \). A symmetric space \( E \subseteq S(\mathcal{A}, \tau) \) is called strongly symmetric if its norm has the additional property that \( \| x \|_E \leq \| y \|_E \), whenever \( x, y \in E \) satisfy \( x \ll y \). If \( E \) is a symmetric space and it follows from \( x \in S(\mathcal{A}, \tau), y \in E \) and \( x \ll y \) that \( x \in E \) and \( \| x \|_E \leq \| y \|_E \), then \( E \) is called a fully symmetric space. Let \( E \subseteq S(\mathcal{A}, \tau) \) be a symmetric space. Convergence in \( E \) with respect to the norm of \( E \) will be denoted by \( \rightarrow \). The carrier projection \( c_E \) of \( E \) is defined to be the supremum of all projections in \( \mathcal{A} \) that are also in \( E \). If \( c_E = 1 \), then \( E \) is continuously embedded in \( S(\mathcal{A}, \tau) \) equipped with the measure topology \( T_m \). We will assume throughout this paper that \( c_E = 1 \). The norm \( \| \cdot \|_E \) on a symmetric space \( E \) is called order continuous if \( \| x_\lambda \| \downarrow 0 \) whenever \( x_\lambda \downarrow 0 \) in \( E \). If this is the case, \( \mathcal{F}(\tau) := \{ x \in \mathcal{A} : s(x) \in \mathcal{P}(\mathcal{A}) \} \) is norm dense in \( E \), and it can be shown, using the spectral theorem, that for every \( x \in \mathcal{A}^\sigma \), there is a sequence \( (x_n)_{n=1}^\infty \) in \( \mathcal{G}(\mathcal{A}) \) such that \( x_n \overset{E}{\rightarrow} x \). If \( E \) is a strongly symmetric space, then it can be shown [10, Proposition 6.12] that \( E \) has order continuous norm if and only if it has absolutely continuous norm, that is \( \| p_n xp_n \|_E \to 0 \) for every sequence \( (p_n)_{n=1}^\infty \) in \( \mathcal{P}(\mathcal{A}) \) satisfying \( p_n \downarrow 0 \) and every \( x \in E \).

If \( \mathcal{A} = L^\infty(0, \infty) \) is the abelian semi-finite von Neumann algebra of all essentially bounded Lebesgue measurable functions on \((0, \infty)\) and the trace \( \tau \) is given by integration with respect to Lebesgue measure, then \( S(\mathcal{A}, \tau) = S(0, \infty) \) is the space of all Lebesgue measurable functions on \((0, \infty)\) that are bounded except possibly on a set of finite measure. In this case the singular value function \( \mu_x \) corresponds to the decreasing rearrangement \( f^* \) of a measurable function \( f \). It follows from [9, Corollaries 2.6 and 2.7] that if \( (\mathcal{A}, \tau) \) is a semi-finite von Neumann algebra and \( E(0, \infty) \subseteq S(0, \infty) \) is a fully symmetric space, then the set \( E(\mathcal{A}) := \{ x \in S(\mathcal{A}, \tau) : \mu_x \in E(0, \infty) \} \) is a fully symmetric space, when equipped with the norm \( \| x \|_{E(\mathcal{A})} = \| \mu_x \|_{E(0, \infty)} \) for \( x \in E(\mathcal{A}) \). Furthermore, similar results hold for symmetric spaces and strongly symmetric spaces (see [11,20]). Additional information about symmetric spaces may be found in [8,11].

The following easily verifiable result will be used repeatedly and details conditions under which convergence in a von Neumann algebra yields convergence in an associated symmetric space.
Proposition 2.1 Suppose $E \subseteq S(\mathcal{A}, \tau)$ is a symmetric space. If $(x_n)_{n=1}^{\infty}$ is a sequence in $E \cap A$ such that $x_n \xrightarrow{A} x \in E \cap A$ and either $s(x), s(x_n) \leq p$ or $r(x), r(x_n) \leq p$ for all $n \in \mathbb{N}^+$ and for some $p \in \mathcal{P}(A)_f$, then $x_n \xrightarrow{F} x$.

Since any symmetric space $E \subseteq S(\mathcal{A}, \tau)$ is continuously embedded in $S(\mathcal{A}, \tau)$ equipped with the measure topology [11, Proposition 20], we obtain the following corollary.

Corollary 2.2 Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras and $E \subseteq S(\mathcal{A}, \tau)$ and $F \subseteq S(\mathcal{B}, \nu)$ are symmetric spaces. If $U : E \rightarrow F$ is a continuous map with respect to the norms on $E$ and $F$, then $U(x_n) \xrightarrow{T_{F}} U(x)$, whenever $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{F}(\tau)$ such that $x_n \xrightarrow{A} x \in \mathcal{F}(\tau)$ and $s(x_n) \leq s(x)$ or $r(x_n) \leq r(x)$ for all $n \in \mathbb{N}^+$.

In [25], a positive linear map $U : E \rightarrow F$ between symmetric spaces is called disjointness preserving if $U(x)U(y) = 0$ whenever $x, y \in E^+$ with $xy = 0$. For the purposes of this paper we introduce a slightly weaker notion. We will call a linear map $U : E \subseteq S(\mathcal{A}, \tau) \rightarrow S(\mathcal{B}, \nu)$ projection disjointness preserving if $U(p)^*U(q) = U(p)U(q)^* = 0$, whenever $p, q \in \mathcal{P}(A)_f$ with $pq = 0$. It is clear that a positive map will be projection disjointness preserving whenever it is disjointness preserving. We provide sufficient conditions for the converse to hold.

Proposition 2.3 Suppose $E \subseteq S(\mathcal{A}, \tau)$ and $F \subseteq S(\mathcal{B}, \nu)$ are symmetric spaces and $U : E \rightarrow F$ is a bounded linear projection disjointness preserving map. If $E$ is strongly symmetric with absolutely continuous norm, or $F \subseteq S_0(\mathcal{B}, \nu)$ and $U$ is normal, then $U$ is disjointness preserving.

Proof Suppose $x = \sum_{i=1}^{n} \alpha_i p_i, y = \sum_{j=1}^{m} \beta_j q_j \in \mathcal{G}(A)^+_f$ with $xy = 0$, where $\alpha_i$ $(i = 1, \ldots, n)$ and $\beta_j$ $(j = 1, \ldots, m)$ are positive numbers and $\{p_i\}_{i=1}^{n}$ and $\{q_j\}_{j=1}^{m}$ are two systems of mutually orthogonal $\tau$-finite projections. Then it is easily checked that $s(x)s(y) = 0$, and for every $i, j$ we have that $p_i q_j = 0$, since $p_i \leq s(x)$ and $q_j \leq s(y)$. Using the linearity and projection disjointness preserving nature of $U$ we therefore have that $U(x)U(y) = 0$.

If $x, y \in \mathcal{F}(\tau)^+$, then it follows from the Spectral Theorem that there exist $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subseteq \mathcal{G}(A)^+_f$ such that $x_n \xrightarrow{A} x, y_n \xrightarrow{A} y, s(x_n) \leq s(x)$ and $s(y_n) \leq s(y)$ for every $n$. By Proposition 2.1, $x_n \xrightarrow{E} x$ and $y_n \xrightarrow{E} y$. Therefore $U(x_n) \xrightarrow{F} U(x)$ and $U(y_n) \xrightarrow{F} U(y)$. By [11, Proposition 20], this implies that $U(x_n) \xrightarrow{T_{F}} U(x)$ and $U(y_n) \xrightarrow{T_{F}} U(y)$ and so $U(x_n)U(y_n) \xrightarrow{T_{F}} U(x)U(y)$, since multiplication is jointly continuous in the measure topology [11, p. 210]. Furthermore, $s(x_n)s(y_n) = 0$ for every $n$ and thus $x_n y_n = 0$ for every $n$. It follows that $U(x_n)U(y_n) = 0$ for every $n$ and so $U(x)U(y) = 0$.

Finally, if $x, y \in E^+$, then there exist $(x_\lambda)_{\lambda \in \Lambda}, (y_\alpha)_{\alpha \in \Lambda} \subseteq \mathcal{F}(\tau)^+$ such that $x_\lambda \uparrow x$ and $y_\alpha \uparrow y$ (see [11, Proposition 1(viii)]). Since $s(x_\lambda) \leq s(x)$ and $s(y_\alpha) \leq s(y)$ for each $\lambda$ and $\alpha$, we have that $U(x_\lambda)U(y_\alpha) = 0$ for each $\lambda$ and $\alpha$. If $E$ is strongly
symmetric with absolutely continuous norm, then we have that \( x_\alpha \xrightarrow{E} x \) and \( y_\alpha \xrightarrow{E} y \) and we can show, as in the previous case, that \( U(x)U(y) = 0 \). If \( F \subseteq S_0(B, \nu) \) and \( U \) is normal, then we have that \( U(x_\alpha) \uparrow U(x) \) and therefore \( U(x_\alpha) \xrightarrow{\tau} U(x) \), by [11, Proposition 2(iv)] (since \( F \subseteq S_0(B, \nu) \)). Similarly, \( U(y_\alpha) \xrightarrow{\tau} U(y) \) and so \( U(x)U(y) = 0 \) as before.

In the preprint [14] a projection disjointness preserving operator is also called a **support separating operator** and corresponds to the notion of a **Lampert operator** in the setting of \( L^p \)-spaces (see [21], for example). Furthermore, in the paper [14] those bounded linear maps on non-commutative \( L^p \)-spaces that are support separating are characterized using an adaptation of the proof of Yeadon’s characterization of \( L^p \)-isometries (see [27, Theorem 2]). We also mention that in the preprint [23] a notion of \( \ell^1 \)-contractiveness is defined and it is shown that an isometry \( T : L^2(A) \to L^2(B) \) is (support) separating if and only if it is \( \ell^1 \)-contractive and this is equivalent to it admitting a Yeadon-type factorization (see [23, Propositions 3.11 and 4.2]).

### 3 The projection disjointness preserving property of positive surjective isometries

In order to describe the structure of positive surjective isometries we will start by showing that under certain conditions such isometries are projection disjointness preserving. It is shown in [25, Corollary 5] that if \( T : E \to F \) is a positive isometry, where \( E \subseteq S(A, \tau) \) is a symmetric space and \( F \subseteq L_0(B, \nu) := S_0(B, \nu) \cap L^1 + L^\infty(B) \) is a symmetric space with \( K \)-strictly monotone norm, then \( T \) is disjointness preserving. In this section we complement this result by showing that a finiteness preserving positive surjective isometry between arbitrary symmetric spaces is projection disjointness preserving. Suppose \( E \subseteq S(A, \tau) \) and \( F \subseteq S(B, \nu) \) are symmetric spaces. We will start by showing that if \( U : E \to F \) is a positive surjective isometry, then \( U \) is an order isomorphism and for each \( p \in \mathcal{P}(A) \), \( U \) maps \( pE \) into \( s(U(p))Fs(U(p)) \). Since we were not able to show that \( U \) in fact maps \( pE \) onto \( s(U(p))Fs(U(p)) \) and we are not assuming full symmetry of \( F \), we do not have access to [3, Theorem 3.1], which would have enabled us to describe the structure of \( U \) under the additional assumption that \( U \) is finiteness preserving. Nevertheless, under this assumption we are able to adapt the technique employed in the proof of [3, Theorem 3.1] to prove a local representation of such isometries in the sense that for each \( p \in \mathcal{P}(A) \) we will show that there exists a Jordan *-isomorphism \( \Phi_p \) from \( pAp \) onto \( s(U(p))Bs(U(p)) \) such that \( U(x) = U(p)\Phi_p(x) \) for all \( x \in pAp \). The projection disjointness preserving property of positive surjective isometries will then follow from this.

**Lemma 3.1** Suppose \( E \subseteq S(A, \tau) \) and \( F \subseteq S(B, \nu) \) are symmetric spaces. If \( U : E \to F \) is a positive isometry, then \( z \geq 0 \), whenever \( z \in E \) and \( U(z) \geq 0 \). If in addition, \( U \) is surjective, then \( U \) is an order isomorphism and hence also normal.

**Proof** The proof of the corresponding result in the setting where \( F \) is a fully symmetric space and \( \tau(1), \nu(1) < \infty \) [3, Lemma 3.2] requires only one significant adjustment to
be generalized to spaces associated with arbitrary semi-finite von Neumann algebras. This proof uses the fact that if \( v(1) < \infty \), then \( x - y \leq x + y \) whenever \( x, y \in L^1(\mathcal{B}, v) \) (see [3, Lemma 2.1]). The full symmetry of \( F \) is then used to show that 
\[
\|x - y\|_F \leq \|x + y\|_F \text{, if in addition } x, y \in F^+.
\]

To extend [3, Lemma 3.2] to the general semi-finite setting we note that it has recently been shown in [2, Corollary 4] that, even in this more general setting, 
\[
\|x - y\|_F \leq \|x + y\|_F \text{ whenever } x, y \in F^+ \text{ and } F \text{ is a normed solid space.}
\]

Since symmetric spaces are normed solid spaces, we do not require the full symmetry assumption. Finally, it is easily checked that an order isomorphism is necessarily normal. \( \Box \)

**Remark 3.2** In the preprint [15] the proof of [3, Lemma 3.2] is adapted in a slightly different way to show that the same result holds in the more general setting of symmetrically \( \Delta \)-normed spaces, but under the additional assumption that the norm of \( F \) is log-monotone (see [15, Lemma 5.1]).

The following lemma will play an important role in obtaining a local representation of positive surjective isometries.

**Lemma 3.3** Suppose \( E \subseteq S(\mathcal{A}, \tau) \) and \( F \subseteq S(\mathcal{B}, v) \) are symmetric spaces and \( U : E \to F \) is a positive surjective isometry. If \( p \in \mathcal{P}(\mathcal{A}), \) then \( U(pEp) \subseteq s(U(p))Fs(U(p)) \).

**Proof** Since \( U \) is positive we have that \( s(U(p)) = s(U(p)^*) = r(U(p)) \). This implies that \( s(U(p))U(p)s(U(p)) = U(p) \) and hence \( U(p) \in s(U(p))Fs(U(p)) \).

If \( q \in \mathcal{P}(\mathcal{A}), \) then \( 0 \leq pqp \leq p1p \), by [11, Proposition 1(iii)] and so \( 0 \leq U(pqp) \leq U(p) \). This implies that \( U(pqp) \in s(U(p))Fs(U(p)) \). It follows that \( U(pG(\mathcal{A})p) \subseteq s(U(p))Fs(U(p)) \). If \( x \in pEp \cap \mathcal{A} \subseteq p\mathcal{F}(\tau)^+p \), then using the spectral theorem there exists \((x_n)_{n=1}^\infty \subseteq G(\mathcal{A})^+ \) such that \( x_n \xrightarrow{A} x \) and \( r(x_n) = s(x_n) \leq s(x) \leq p \) for each \( n \in \mathbb{N}^+ \). Then \( x_n \in pG(\mathcal{A})p \) for each \( n \) and \( U(x_n) \xrightarrow{F} U(x) \). Since \( U(x_n) \in s(U(p))Fs(U(p)) \) for each \( n \) and it is easily checked that \( s(U(p))Fs(U(p)) \) is closed in \( F \), we have that \( U(x) \in s(U(p))Fs(U(p)) \).

Finally, if \( x \in pE^+p \), then by [11, Proposition 1(vii)] there exists \( \{x_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}(\tau)^+ \) such that \( x_\lambda \uparrow x \). Then \( px_\lambda p \uparrow pxp = x \). It follows by Lemma 3.1 that \( U \) is normal and therefore \( U(px_\lambda p) \uparrow U(x) \). It follows that \( U(px_\lambda p) \xrightarrow{T_{im}} U(x) \), by [11, Proposition 2(v)]. Therefore \( s(U(p))U(px_\lambda p) \xrightarrow{T_{im}} s(U(p))U(x) \) and hence (see [11, p. 211])

\[
[s(U(p))U(px_\lambda p)]s(U(p)) \xrightarrow{T_{im}} [s(U(p))U(x)]s(U(p)).
\]

Since \( U(x) \in s(U(p))Fs(U(p)) \), and hence \( s(U(p))U(px_\lambda p)s(U(p)) = U(px_\lambda p) \) for each \( \lambda \), it follows that \( U(x) = s(U(p))U(x)s(U(p)) \). \( \Box \)

Next we show how the techniques of [3, Sect. 3] may be adapted to obtain a local representation of positive surjective isometries. To facilitate this we mention a few aspects of reduced spaces (see [11, p. 211, 212 and 215]). For \( p \in \mathcal{P}(\mathcal{A}) \) and \( x \in S(\mathcal{A}, \tau), \) let \( x(p) := (pxp) \uparrow_{p(H)} \), where \( H \) denotes the Hilbert space.
on which $\mathcal{A}$ acts. It can be shown that $\{x(p) : x \in S(\mathcal{A}, \tau)\} = S(\mathcal{A}_p, \tau_p)$, where $\mathcal{A}_p := \{x(p) : x \in \mathcal{A}\}$ and $\tau_p(x(p)) := \tau(px)$ for every $x \in \mathcal{A}$. Let $\phi_p$ denote the canonical map $x \mapsto x(p)$ from $pS(\mathcal{A}, \tau)p$ onto $S(\mathcal{A}_p, \tau_p)$. Note that $\phi_p$ is a $*$-isomorphism, $E_p$ is a symmetric space if $E$ is a symmetric space, and that the restrictions of $\phi_p$ to $p\mathcal{A}p$ and $pEp$ respectively are isometries onto the reduced spaces $\mathcal{A}_p = \{x(p) : x \in \mathcal{A}\}$ and $E_p = \{x(p) : x \in E\}$. Let $\psi_p$ denote the canonical map from $s(U(p))S(\mathcal{B}, v)s(U(p))$ onto $S(B_s(U(p)), v_s(U(p)))$. We will make use of the fact that if $x \in pS(\mathcal{A}, \tau)p$ and $f$ is a Borel measurable function on $\mathbb{R}$ that is bounded on compact sets, then $f(\phi_p(x)) = \phi_p(f(x))$ and a similar relationship holds for elements in $s(U(p))S(\mathcal{B}, v)s(U(p))$.

**Proposition 3.4** Suppose $U : E \to F$ is a positive surjective isometry. If $U$ is finiteness preserving, then for each $p \in \mathcal{P}(\mathcal{A})_f$, there exists a Jordan $*$-isomorphism $\Phi_p$ from $p\mathcal{A}p$ onto $s(U(p))\mathcal{B}_s(U(p))$ such that $U(x) = U(p)\Phi_p(x)$ for every $x \in p\mathcal{A}p$. Furthermore, $a_p := U(p)$ commutes with every element in $s(U(p))S(\mathcal{B}, v)s(U(p))$.

**Proof** Let the reduced space corresponding to $s(U(p))\mathcal{B}_s(U(p))$ be denoted $B_s(U(p))$. If we identify $a_p$ with the corresponding element in the reduced space $S(B_s(U(p))) \cong s(U(p))S(\mathcal{B})s(U(p))$, then we have that $a_p$ is invertible in $S(B_s(U(p)))$ (this follows from the functional calculus for $a_p$ and noting that $s(a_p) = s(U(p))$ is the identity of $B_s(U(p))$ and has finite trace). We will use $a_p^{-1}$ to denote the inverse of $a_p$ in $S(B_s(U(p)))$ (bearing in mind that $a_p$ need not be invertible in $S(\mathcal{B})$). Working in these reduced spaces and using these identifications, we have that $a_p^{-1} \geq 0$ and $a_p^{-1/2} = (a_p^{-1})^{1/2} = (a_p^{1/2})^{-1}$. In this setting, we let

$$\Phi_p(x) = a_p^{-1/2}U(x)a_p^{-1/2} \quad x \in \mathcal{A}_p.$$ 

Note that since $\mathcal{A}_p$ is trace-finite, $\mathcal{A}_p \subseteq E_p \cong pEp$ and so $U$ is defined on all of $\mathcal{A}_p$. It is easily checked that $\Phi_p$ is a positive unital map. To show that $\Phi_p$ maps $\mathcal{A}_p$ into $B_s(U(p))$ note that if $y \in \mathcal{A}_p^+$, then $0 \leq y \leq \|y\|_{\mathcal{A}_p}p$, by [18, Proposition 4.2.3]. This implies that $0 \leq \Phi_p(y) \leq \|y\|_{\mathcal{A}_p} \Phi_p(p) = \|y\|_{\mathcal{A}_p} s(U(p))$ since $\Phi_p$ is positive, linear and unital. It follows that $\Phi_p(y) \in B_s(U(p))$, since $\|y\|_{\mathcal{A}_p} s(U(p)) \in B_s(U(p))$ and $B_s(U(p))$ is an absolutely solid subspace of $S(B_s(U(p)))$. Since any element of $\mathcal{A}_p$ can be written as a linear combination of positive elements, we have that $\Phi_p(\mathcal{A}_p) \subseteq B_s(U(p))$.

Next we show that $\Phi_p$ is surjective. Let $b \in B_s^+(U(p))$ and define $c = a_p^{1/2}b a_p^{1/2}$. Then

$$0 \leq c = a_p^{1/2}b a_p^{1/2} \leq a_p^{1/2} \|b\|_{B_s(U(p))} s(U(p)) a_p^{1/2} = \|b\|_{B_s(U(p))} a_p^{1/2}.$$ 

(3.1)

Since $F$ is symmetric, $F_s(U(p))$ is also symmetric and hence absolutely solid. This, combined with (3.1), implies that $c \in F_s(U(p))$, since $\|b\|_{B_s(U(p))} a_p \in F_s(U(p))$. By Lemma 3.1, $U^{-1}$ is positive and therefore $0 \leq U^{-1}(c) \leq \|b\|_{B_s(U(p))} p$, using (3.1). It follows that $U^{-1}(c) \in p\mathcal{A}p$. Furthermore, it is easily checked that $\Phi_p(U^{-1}(c)) = b$. It follows that $\Phi_p$ is surjective and for $y \in B_s(U(p))$, $\Phi_p^{-1}(y) = U^{-1}(a_p^{1/2} y a_p^{1/2})$.

Using this formula for the inverse of $\Phi_p$, [11, Proposition 1(iii)] and the positivity
of $U^{-1}$, we see that $\Phi_p^{-1}$ is positive. We have shown that $\Phi_p$ is a unital order isomorphism of $A_p$ onto $B_s(U(p))$ and therefore $\Phi_p$ is a Jordan *-isomorphism, by [19, Exercise 10.5.32].

By definition of $\Phi_p$, we have that $\Phi_p(x) = a_p^{-1/2}U(x)a_p^{-1/2}$ and therefore $U(x) = a_p^{1/2}\Phi_p(x)a_p^{1/2}$. Essentially the same technique as the one employed in the proof of [3, Lemma 3.5] can be used to show that $a_p \in S(Z(B_s(U(p))))$, where $Z(B_s(U(p)))$ denotes the center of the von Neumann algebra $B_s(U(p))$ (the interested reader is directed to the first author’s doctoral thesis [5, Lemma 5.1.3] for a detailed verification of this claim).

It now follows that $a_pb \mapsto ba_p$ for every $b \in S(U(p))S(B)s(U(p))$ and therefore $U(x) = a_p\Phi_p(x)$ for every $x \in pAp$.

**Remark 3.5** This result complements [15, Theorem 3.4], which holds in the more general setting of symmetrically $\Delta$-normed spaces, but under the additional assumption that the maps are disjointness preserving.

**Corollary 3.6** Let $E \subseteq S(A, \tau)$ and $F \subseteq S(B, \nu)$ be symmetric spaces and $U : E \to F$ a positive surjective isometry. If $U$ is finiteness preserving (in particular if $v(1) < \infty$), then $U$ is projection disjointness preserving.

**Proof** It follows from the previous result that if $p, q \in \mathcal{P}(A)_f$ with $pq = 0$, then $p + q \in \mathcal{P}(A)_f$ and $U(p)U(q) = a_{p+q}^2\Phi_{p+q}(p)\Phi_{p+q}(q) = 0$ (see [19, Exercise 10.5.22(viii)])

**Remark 3.7** In the preprint [15] it is shown that a similar result holds even for positive isometries which are not necessarily surjective and in the more general setting of symmetrically $\Delta$-normed spaces, but under the additional assumption that the norm of $F$ is strictly log-monotone (see [15, Proposition 4.8]).

### 4 The structure of positive surjective isometries

Our aim in this section is to describe the structure of positive surjective isometries. We saw in the previous section that if, in addition, such an isometry is finiteness preserving, then there exists a family of Jordan *-isomorphisms $\Phi_p : pAp \to s(U(p))B_s(U(p))$ ($p \in \mathcal{P}(A)_f$) that can be used to describe the isometry locally. It is natural to consider if it is not possible to combine these maps to construct a Jordan *-isomorphism on the whole von Neumann algebra (this is, in fact, the strategy employed in the first author’s doctoral thesis—see [5, Sect. 5]). It is, however, beneficial to rather develop a technique for disjointness-preserving isometries, since this will also be useful for the description of certain isometries which are not necessarily positive (see Sect. 5) and also for the description of isometries between Lorentz spaces (see [7]). As such we start by considering projection disjointness preserving isometries that are not necessarily positive nor surjective. We show that the ideas of Yeadon’s Theorem and the extension procedures developed in [6] can be used to describe such isometries on $\mathcal{F}(\tau)$. More specifically we will show that if $V$ is a projection disjointness preserving isometry between symmetric spaces $E \subseteq S(A, \tau)$ and $F \subseteq S(B, \nu)$, then letting $\Psi(p) = s(V(p))$ for $p \in \mathcal{P}(A)_f$ yields a projection mapping which can be extended to a positive linear
map (still denoted $\Psi$) on $\mathcal{F}(\tau)$, which preserves squares of self-adjoint elements and therefore has many Jordan $^*$-homomorphism-like properties (see [6, Proposition 2.3]). Furthermore, we will show that $V(x) = V(p)\Psi(x) = v_p b_p \Psi(x)$ for any $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(\mathcal{A})_f$ with $p \geq s(x)$, where $v_p$ and $b_p$ are respectively the partial isometry and positive operator occurring in the polar decomposition $V(p) = v_p b_p$. Attempts to extend $\Psi$ to all of $\mathcal{A}$ and use the $v_p$’s and $b_p$’s to construct single elements which can be used in a global representation of $\mathcal{V}$ have proven to be problematic without further conditions on the symmetric spaces $\mathcal{E}$ and $\mathcal{F}$ or the isometry $\mathcal{V}$. In this section we will show that the extension and representation can be achieved in the general setting of symmetric spaces if the isometry has more structure, and in the following section we will show how the extension and representation can be achieved if the isometry does not necessarily have all of this additional structure, provided the symmetric spaces have more structure.

We will need the following extension result.

**Theorem 4.1** [6, Theorems 3.7 and 5.1] Suppose $\Phi : \mathcal{P}(\mathcal{A})_f \to \mathcal{P}(\mathcal{B})$ is a map such that $\Phi(p + q) = \Phi(p) + \Phi(q)$ whenever $p, q \in \mathcal{P}(\mathcal{A})_f$ with $pq = 0$. If there exists a linear map $U$ from $\mathcal{F}(\tau)$ into $\mathcal{S}(\mathcal{B}, v)$ such that $\Phi(p) = s(U(p))$ for all $p \in \mathcal{P}(\mathcal{A})_f$, and which has the property that $U(x_n) \xrightarrow{\text{Top}} U(x)$ whenever $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{F}(\tau)$ such that $x_n \xrightarrow{\mathcal{A}} x \in \mathcal{F}(\tau)$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$, then $\Phi$ can be extended to a positive linear map (still denoted by $\Phi$) from $\mathcal{F}(\tau)$ into $\mathcal{B}$ such that $\|\Phi(x)\|_{\mathcal{B}} \leq \|x\|_{\mathcal{A}}$ and $\Phi(x^2) = \Phi(x)^2$ for all $x \in \mathcal{F}(\tau)^{sa}$. Suppose, in addition, that $U$ is positive and normal.

1. If $x \in \mathcal{F}(\tau)^{sa}$ and $p \in \mathcal{P}(\mathcal{A})_f$ with $p \geq s(x)$, then $\Phi(x)U(p) = U(x) = U(p)\Phi(x) = U(p)^{1/2}\Phi(x)U(p)^{1/2}$.
2. If $x \in \mathcal{F}(\tau)^{sa}$ and $p \in \mathcal{P}(\mathcal{A})_f$ with $p \geq s(x)$, then there exists a $w_p \in \mathcal{S}(\mathcal{B}, v)$ such that $U(p)^{1/2}w_p = \Phi(p) = w_p U(p)^{1/2}$ and $\Phi(x) = w_p U(x)w_p$.
3. $\Phi$ can be extended to a normal Jordan $^*$-homomorphism (still denoted by $\Phi$) from $\mathcal{A}$ into $\mathcal{B}$. Furthermore, in this case, $\Phi(x)$ is the SOT-limit of $[\Phi(px)p]_{p \in \mathcal{P}(\mathcal{A})_f}$ for any $x \in \mathcal{A}$, and $\|\Phi(x)\|_{\mathcal{B}} \leq \|x\|_{\mathcal{A}}$ for all $x \in \mathcal{A}^{sa}$.

Using this result we provide a preliminary structural description of projection disjointness preserving isometries.

**Theorem 4.2** Suppose $\mathcal{E} \subset \mathcal{S}(\mathcal{A}, \tau)$ and $\mathcal{F} \subset \mathcal{S}(\mathcal{B}, v)$ are symmetric spaces. If $\mathcal{V} : \mathcal{E} \to \mathcal{F}$ is a projection disjointness preserving isometry, then letting $\Psi(p) := s(V(p))$ for $p \in \mathcal{P}(\mathcal{A})_f$, yields a projection mapping that can be extended to a positive linear map (also denoted by $\Psi$) from $\mathcal{F}(\tau)$ into $\mathcal{B}$ such that $\|\Psi(x)\|_{\mathcal{B}} = \|x\|_{\mathcal{A}}$ and $\Psi(x^2) = \Psi(x)^2$ for all $x \in \mathcal{F}(\tau)^{sa}$. Furthermore, for any $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(\mathcal{A})_f$ with $p \geq s(x) \lor r(x)$, we have

1. $V(x) = V(p)\Psi(x)$
2. $b_p \Psi(x) = \Psi(x)b_p$, where $V(p) = v_p b_p$ is the polar decomposition of $V(p)$ into a partial isometry $v_p$ and positive operator $b_p = |V(p)|$. 

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Proof For $p \in \mathcal{P}(A)_f$, let $\Psi(p) = s(V(p)) = v^*_p v_p$. If $p, q \in \mathcal{P}(A)_f$ with $pq = 0$, then $V(p)^* V(q) = 0 = V(p) V(q)^*$ and so, as in the proof of Yeadon’s Theorem [27, Theorem 2], we have that $v^*_p v_q = 0 = v_p v^*_q$. Furthermore, $v_p + v_q$ is a partial isometry, $|V(p) + V(q)| = b_p + b_q$ and $V(p) + V(q) = (v_p + v_q)(b_p + b_q)$ is the polar decomposition of $V(p + q) = V(p) + V(q)$. Therefore $v_p + v_q = v_{p+q}$ and $b_p + b_q = b_{p+q}$. It follows that

$$
\Psi(p + q) = v^*_{p+q} v_{p+q} = (v_p + v_q)^* (v_p + v_q) = v^*_p v_p + v^*_q v_q = \Psi(p) + \Psi(q).
$$

Using [4, Exercise 2.3.4] we have that $\Psi(p) \Psi(q) = 0$. Furthermore, if $0 \neq p \in \mathcal{P}(A)_f$, then $V(p) \neq 0$, since $V$ is injective. It follows that $\Psi(p) = s(V(p)) \neq 0$.

Furthermore, by Corollary 2.2, $V$ has the property that $V(x_n) \xrightarrow{T_m} V(x)$ whenever $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{F}(\tau)$ such that $x_n \xrightarrow{A} x \in \mathcal{F}(\tau)$ and $s(x_n) \leq s(x)$ for all $n \in \mathbb{N}^+$. By Theorem 4.1, $\Psi$ can therefore be extended to a positive linear map (also denoted by $\Psi$) from $\mathcal{F}(\tau)$ into $\mathcal{B}$ with the desired properties.

Next we prove 1. Since $\Psi(p) = s(b_p) = r(b_p) = s(v_p)$, we have that

$$
\Psi(p) b_p = b_p = b_p \Psi(p) \quad \text{and} \quad v_p \Psi(p) = v_p. \quad (4.1)
$$

Suppose $x = q \in \mathcal{P}(A)_f$ and $p \in \mathcal{P}(A)_f$ with $p \geq q$. Then $p - q \in \mathcal{P}(A)_f$ and $q(p - q) = 0$. Note that $b_{p-q} \Psi(q) = (b_{p-q} \Psi(p - q)) \Psi(q) = 0$, using (4.1) and the fact that $q(p - q) = 0$ implies that $\Psi(q) \Psi(p - q) = 0$. Similarly, we have that $v_{p-q} b_q = v_{p-q} \Psi(p - q) \Psi(q) b_q = 0$. Therefore,

$$
V(p) \Psi(q) = v_{q+(p-q)} b_{q+(p-q)} \Psi(q) = (v_q + v_{p-q}) (b_q + b_{p-q}) \Psi(q) = v_q b_q \Psi(q) = V(q).
$$

Using the linearity of $V$ and $\Psi$, we therefore have that $V(x) = V(p) \Psi(x)$ for any $x \in \mathcal{G}(A)_f$ and $p \in \mathcal{P}(A)_f$ with $p \geq s(x)$. Suppose $x \in \mathcal{F}(\tau)^A$ and $p \in \mathcal{P}(A)_f$ with $p \geq s(x)$. As a consequence of the Spectral Theorem, we can find a sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{G}(A)^A_f$ such that $x_n \xrightarrow{A} x$ and $s(x_n) \leq s(x) \leq p$ for all $n \in \mathbb{N}^+$. By Proposition 2.1, this implies that $x_n \xrightarrow{F} x$. Therefore $V(x_n) \xrightarrow{F} V(x)$ and $\Psi(x_n) \xrightarrow{B} \Psi(x)$, since $V$ is an isometry and $\Psi$ is linear, and isometric on self-adjoint elements in $\mathcal{F}(\tau)$. Furthermore, since $F$ is a normed $\mathcal{B}$-bimodule,

$$
\| V(p)(\Psi(x_n) - \Psi(x)) \|_F \leq \| V(p) \|_F \| \Psi(x_n) - \Psi(x) \|_\mathcal{B} \to 0
$$

and so $V(p) \Psi(x_n) \xrightarrow{F} V(p) \Psi(x)$. However, $V(p) \Psi(x_n) = V(x_n) \xrightarrow{F} V(x)$. It follows that $V(x) = V(p) \Psi(x)$. Finally, if $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(A)_f$ with $p \geq s(x) \vee r(x)$, then $p \geq s(\text{Re} x), s(\text{Im} x)$ and so $V(x) = V(p) \Psi(x)$ using the linearity of $V$ and $\Psi$. 

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To prove 2, suppose $x = q$ and $p \in \mathcal{P}(\mathcal{A})_f$ with $p \geq q$. Then $b_{p-q}\Psi(q) = b_{p-q}\Psi(p - q)\Psi(q) = 0$ and $\Psi(q)b_{p-q} = \Psi(q)\Psi(p - q)b_{p-q} = 0$. We therefore have that

$$b_p\Psi(q) = b_{q+(p-q)}\Psi(q) = (b_q + b_{p-q})\Psi(q) = b_q\Psi(q) = \Psi(q)b_q = \Psi(q)(b_q + b_{p-q}) = \Psi(q)b_p.$$  

Noting that for any $p \in \mathcal{P}(\mathcal{A})_f$, $b_p = v_p^*V(p) \in F$ (since $V(p) \in F$, $v_p^* \in \mathcal{B}$ and $F$ is a bimodule), we can employ a similar strategy to the one used in 1 to complete the proof.

The previous result allows us to completely describe the structure of projection disjointness preserving isometries in the setting where the initial von Neumann algebra is equipped with a finite trace.

**Corollary 4.3** Suppose $E \subseteq S(\mathcal{A}, \tau)$ and $F \subseteq S(\mathcal{B}, \nu)$ are symmetric spaces, and that $\tau(1) < \infty$. If $V : E \to F$ is a projection disjointness preserving isometry, then there exists a Jordan $\ast$-homomorphism $\Psi$ from $\mathcal{A}$ into $\mathcal{B}$ such that $V(x) = V(1)\Psi(x)$ for every $x \in \mathcal{A}$.

**Remark 4.4** In the preprint [15] it is shown that the same result holds for a linear disjointness preserving map (which is not necessarily isometric) between symmetrically $\Delta$-normed spaces, but under the additional assumption that the map is positive (see [15, Theorem 3.1]).

For the remainder of this section we will suppose that $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are arbitrary semi-finite von Neumann algebras, $E \subseteq S(\mathcal{A}, \tau)$ and $F \subseteq S(\mathcal{B}, \nu)$ are symmetric spaces and $U : E \to F$ is a finiteness preserving positive surjective isometry. By Corollary 3.6, $U$ is projection disjointness preserving and therefore, by Theorem 4.2, letting $\Phi(p) := s(U(p))$ for $p \in \mathcal{P}(\mathcal{A})_f$ yields a projection mapping which can be extended to a positive linear map (still denoted by $\Phi$) from $\mathcal{F}(\tau)$, which preserves squares of self-adjoint elements and for which $U(x) = U(p)\Phi(x)$ for any $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(\mathcal{A})_f$ with $p \geq s(x) \lor r(x)$. Furthermore, it follows from Lemma 3.1 that $U$ is normal and therefore Theorem 4.1 can be used to extend $\Phi$ further to a normal Jordan $\ast$-homomorphism (still denoted by $\Phi$) from $\mathcal{A}$ into $\mathcal{B}$. We need to show that $\Phi$ is surjective and define the element $a$ to be used in the representation of $U$. The following lemma will play an important role in both. For $p \in \mathcal{P}(\mathcal{A})_f$, we will let $a_p := U(p)$.

**Lemma 4.5** For any $p \in \mathcal{P}(\mathcal{A})_f$, $\Phi(pAp) = \Phi(p)B\Phi(p)$.

**Proof** Since $\Phi(px_p) = \Phi(p)\Phi(x)\Phi(p)$ for any $x \in \mathcal{A}$ (see [19, Exercise 10.5.21]), we have that $\Phi(pAp) \subseteq \Phi(p)B\Phi(p)$. Let $y \in \Phi(p)B\Phi(p)^+$ and define $c = a_p^{1/2}ya_p^{1/2}$. Then, since $0 \leq y \leq \|y\|_B\Phi(p)$, repeated application of [11, Proposition 1(iii)] yields

$$0 \leq c \leq a_p^{1/2}\|y\|_B\Phi(p)a_p^{1/2} = \|y\|_B a_p,$$

(4.2)
using the fact that $\Phi(p) = s(a_p) = s(a_p^{1/2})$. Since $F$ is symmetric (and hence absolutely solid) and $\|y\|_{a_p} = \|y\|_{U(p)} \in F$, it follows that $c \in F$. By Lemma 3.1, $U^{-1}$ is positive and therefore $0 \leq U^{-1}(c) \leq \|p\|$. It follows that $U^{-1}(c) \in pA_p$.

By Theorem 4.1, there exists a $w_p \in S(B,v)$ such that $U(p)^{1/2}w_p = \Phi(p) = w_pU(p)^{1/2}$ and $\Phi(x) = w_pU(x)w_p$. Since $a_p = U(p)$, it follows that

$$\Phi(U^{-1}(c)) = w_pU(U^{-1}(c))w_p = w_p(a_p^{1/2}ya_p^{1/2})w_p = \Phi(p)y\Phi(p) = y.$$

Since elements in $\Phi(p)B\Phi(p)$ can be written as finite linear combinations of elements in $\Phi(p)B\Phi(p)^+$, we have that $\Phi(p)B\Phi(p) \subseteq \Phi(pA_p)$.

Next we define $a$. Let $a_p = \int_0^\infty \lambda de^\lambda_p$ denote the spectral representation of $a_p$.

We start by showing that for a fixed $\lambda \geq 0$, $\{e^\lambda_p(\lambda, \infty)\}_{p \in P(A)}$ is an increasing net, where $e^\lambda_p(\lambda, \infty) = 1 - e^\lambda_p$. Suppose $q \in P(A_\lambda)$ with $q \geq p$. Note that $e^\lambda_p(\lambda, \infty) \leq s(a_p) = \Phi(p) = \Phi(q)$ and so, by Lemma 4.5, there exists an $x \in AQ$ such that $e^\lambda_p(\lambda, \infty) = \Phi(x)$. It follows by Theorem 4.2(2) that $aqe^\lambda_p(q, \infty) = e^\lambda_p(\lambda, \infty)aq$ and therefore $e^\lambda_p(\lambda, \infty)e^\lambda_p(\lambda, \infty) = e^\lambda_p(\lambda, \infty)e^\lambda_p(\lambda, \infty)$. Since $U$ is positive, we also have that $a_p = U(p) \leq U(q) = a_q$. Therefore $e^\lambda_p(\lambda, \infty) \leq e^\lambda_q(\lambda, \infty)$ for all $\lambda \geq 0$. By [18, Proposition 2.5.6], $\{e^\lambda_p(\lambda, \infty)\}_{p \in P(A)}$ converges in the strong operator topology. Define $e^\lambda(\lambda, \infty) := \text{SOT lim}_{p \in P(A)} e^\lambda_p(\lambda, \infty)$ and $e^\lambda = 1 - e^\lambda(\lambda, \infty)$. One can show that $\{e^\lambda\}_{\lambda \geq 0}$ is a resolution of the identity and, by [18, Lemma 5.6.9], letting

$$a = \int_0^\infty \lambda de^\lambda$$

yields a closed and densely defined positive operator. Furthermore $a_p = U(p) \in F \subseteq S(B,v)$ and so $e^\lambda_p \in B$ for each $\lambda \geq 0$. Since $B$ is closed in the strong operator topology, it follows that $e^\lambda \in B$ for each $\lambda \geq 0$ and therefore $a$ is affiliated with $B$.

Before discussing the relationship between $a$ and $\Phi$, which will enable us to show that $a \in S(B,v)$, we include a result that we will need. It is likely that this is a known result, but since the authors were unable to find an appropriate reference we also include a short proof. Recall that if $x \in S(A, \tau)$ and $p \in P(A)$, then $x(p) := pxp \upharpoonright p(H)$.

**Proposition 4.6** Let $x$ be a closed, densely defined self-adjoint operator on $H$ with spectral representation $x = \int_{-\infty}^\infty \lambda d(e^\lambda_x)$. If $p$ is a projection such that $px = xp$, then $x(p) = \int_{-\infty}^\infty \lambda d((e^\lambda_x)_p)$, i.e. $\{(e^\lambda_x)_p\}_{\lambda \in \mathbb{R}}$ is the spectral resolution for $x(p)$.

**Proof** Let $\{e^\lambda_x\}_{\lambda \in \mathbb{R}}$ denote the spectral resolution for $x$. For each $n \in \mathbb{N}$, put $f^\lambda_n = e^\lambda_n - e^\lambda_{-n}$. Then for each $n$ and each $\xi \in f^\lambda_n(H)$, $x\xi = \int_{-n}^n \lambda d(e^\lambda_x)\xi$ [18, Lemma 5.6.7]. Since $px = xp$, $p$ commutes with $e^\lambda_x$ for each $\lambda \in \mathbb{R}$ (see [18, Sect. 5.6]) and so $(e^\lambda_x)_p = pe^\lambda_xp \upharpoonright p(H)$ is a projection on the Hilbert space $p(H)$ for each $\lambda$. It is easily checked that $\{(e^\lambda_x)_p\}_{\lambda \in \mathbb{R}}$ is a resolution of the identity on $p(H)$. It follows, using the fact that the integral is a limit of linear combinations of disjoint spectral projections commuting with $p$, that $x(p)\xi = p(\int_{-n}^n \lambda d(e^\lambda_x)_p)\xi = \int_{-n}^n \lambda d((e^\lambda_x)_p)\xi$ for $n \in \mathbb{N}$ and $\xi \in (f^\lambda_n)_p(H) \subseteq p(H)$. Since $\bigcup_{n=1}^\infty (f^\lambda_n)_p(H)$ is a core for $x(p)$, the result follows by [18, Theorem 5.6.12]. \qed

\[ Springer\]
We return now to discussing the relationship between $a$ and $\Phi$.

**Lemma 4.7** If $p \in \mathcal{P}(\mathcal{A})_f$, then $a\Phi(p) = a_p$.

**Proof** We start by showing that $e^a_{\lambda} \Phi(p) = e^{a_p}_{\lambda} = \Phi(p)e^a_{\lambda}$ for $\lambda > 0$ and $p \in \mathcal{P}(\mathcal{A})_f$. Let $q \in \mathcal{P}(\mathcal{A})_f$ with $q \geq p$. Then, using the definition of $a_p$ and applying Theorem 4.2, we obtain

$$a_p = U(p) = U(q)\Phi(p) = a_q \Phi(p) \tag{4.3}$$

Furthermore, $\Phi(p)$ is a projection and $a_q \Phi(p) = \Phi(p)a_q$, by Theorem 4.2(2). Using Proposition 4.6 and (4.3) it follows that $\{(e^{a_q}_{\lambda})_{\Phi(p)}\}_{\lambda}$ is the spectral resolution for $(a_q)(\Phi(p)) = (a_p)(\Phi(p))$, and so $(e^{a_p}_{\lambda})(\Phi(p)) = (e^{a_q}_{\lambda}(\Phi(p))(\Phi(p))$ for every $\lambda \geq 0$. Recalling that $\Phi(p) = s(U(p)) = s(a_p)$, we have that $(e^{a_p}_{\lambda}) \mid_{\Phi(p)\downarrow(K)} = 0 = (e^{a_q}_{\lambda}(\Phi(p)) \mid_{\Phi(p)\downarrow(K)}$ and so $e^{a_p}_{\lambda} = e^{a_q}_{\lambda} \Phi(p)$. Furthermore, $e^{a_q}_{\lambda} \Phi(p) \xrightarrow{SOT} e^{a_q}_{\lambda} \Phi(p)$ as $q \uparrow 1$, and therefore $e^{a_p}_{\lambda} = e^{a_q}_{\lambda} \Phi(p)$. Since $a_q \Phi(p) = \Phi(p)a_q$, we have that $e^{a_q}_{\lambda} \Phi(p) = \Phi(p)e^{a_q}_{\lambda}$ and therefore appropriate adjustments to the last few lines yields $e^{a_p}_{\lambda} = \Phi(p)e^{a_q}_{\lambda}$. Combining this with what was shown earlier we obtain $e^{a}_{\lambda} \Phi(p) = e^{a_q}_{\lambda} \Phi(p)$. Therefore, using a similar argument to the one employed at the end of Proposition 4.6, we obtain

$$a_p = \int_0^{\infty} \lambda d(e^{a}_{\lambda} \Phi(p)) = \int_0^{\infty} \lambda d(e^{a_q}_{\lambda} \Phi(p)) = \left(\int_0^{\infty} \lambda d(e^{a_q}_{\lambda})\right) \Phi(p) = a \Phi(p).$$

Strictly speaking these equalities hold modulo the restriction to $p(H)$, but since $(a_p) \mid_{\Phi(p)\downarrow(K)} = 0 = a \Phi(p) \mid_{\Phi(p)\downarrow(K)}$, we obtain $a_p = a \Phi(p)$ globally. \qed

Since $\Phi(p)$ is defined everywhere, we have that $\mathcal{D}(\Phi(p)a) = \{ \eta \in \mathcal{D}(a) : a\eta \in \mathcal{D}(\Phi(p))\} = \mathcal{D}(a)$ and so $\Phi(p)a$ is densely defined. Furthermore, $\Phi(p)$ commutes with every spectral projection of $a$ and therefore $\Phi(p)a \subseteq a \Phi(p)$ (see [18, Sect. 5.6]). Applying [13, Proposition 1.1] we obtain $\mathcal{F}(\Phi(p)a) = [a \Phi(p)] = [a_p] = a_p$, where $[x]$ denotes the closure of an operator. Therefore $\Phi(p)a$ is $\nu$-premeasurable and hence $\mathcal{D}(\Phi(p)a) = \mathcal{D}(a)$ is $\nu$-dense. Thus $a \in \mathcal{S}(\mathcal{B}, \nu)$, since we have already shown that $a$ is a closed densely defined operator affiliated with $\mathcal{B}$.

**Remark 4.8** It is interesting to note that in a number of structural descriptions of isometries it is not claimed that the operator $a$ is $\nu$-measurable (see [27, Theorem 2] and [15, Theorem 3.6]), for example). We used the positivity and surjectivity of the isometry $U$ to define the element $a$, but the claim that $a$ is $\nu$-measurable (as shown above) follows from the facts that $\Phi(p)$ commutes with every spectral projection of $a$ and that $a_p = a \Phi(p)$. Since these facts are also shown in the proofs of [27, Theorem 2] and [15, Theorem 3.6], the elements they obtain are, in fact, $\nu$-measurable.

**Lemma 4.9** If $x \in A \cap E$, then $U(x) = a \Phi(x)$. 

\[ Springer \]
Proof Suppose $x \in \mathcal{F}(\tau)^{sa}$ and let $p = s(x)$. Then $r(\Phi(x)) \leq \Phi(p)$, by [6, Lemma 3.5]. Using Theorem 4.2 and Lemma 4.7, we therefore have $U(x) = a_p \Phi(x) = a \Phi(p) \Phi(x) = a \Phi(x)$. Next, suppose that $x \in \mathcal{A}^+ \cap E$. By [11, Proposition 1(vii)] there exists an increasing net $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(\tau)^+$ such that $x_\lambda \uparrow x$. Then using the normality of $U$ and $\Phi$ we have that $U(x_\lambda) \uparrow U(x)$ and $\Phi(x_\lambda) \uparrow \Phi(x)$. Therefore $U(x_\lambda) \overset{T_{lm}}{\rightarrow} U(x)$ and $\Phi(x_\lambda) \overset{T_{lm}}{\rightarrow} \Phi(x)$, by [11, Proposition 2(v)]. It follows that $\Phi(x_\lambda) \overset{T_{lm}}{\rightarrow} a \Phi(x)$ (see [11, p. 211]). Since $\Phi(x_\lambda) = U(x_\lambda)$ for each $\lambda$ and the local measure topology is Hausdorff, we have that $U(x) = a \Phi(x)$. \hfill \Box

Lemma 4.10 $\Phi$ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$

Proof Assume that $1 - \Phi(1) \neq 0$. Since $(\mathcal{B}, \nu)$ is semi-finite, there exists a $q \in \mathcal{P}(\mathcal{B})$ such that $0 < q \leq 1 - \Phi(1)$ and $\nu(q) < \infty$. This implies that $q \in F$ and hence there exists an $x \in E^+$ such that $U(x) = q$, using the surjectivity of $U$ and Lemma 3.1. By [11, Proposition 1(vii)], there exists $\{x_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{F}(\tau)$ such that $x_\lambda \uparrow x$. Then, using Lemma 4.9 and the normality of $U$, we obtain $a \Phi(x_\lambda) = U(x_\lambda) \uparrow U(x) = q$. Therefore $a \Phi(x_\lambda) \overset{T_{lm}}{\rightarrow} q$. However, we also have that $a \Phi(x_\lambda) = a \Phi(\lambda) \Phi(1) \overset{T_{lm}}{\rightarrow} q \Phi(1)$, by [19, Exercise 10.5.22], Lemma 4.9 and [11, p. 211]. It follows that $q = q \Phi(1)$. However, since $q \leq 1 - \Phi(1)$, we have that $q(1 - \Phi(1)) = q$, and so $q = q \Phi(1) = (q(1 - \Phi(1))) \Phi(1) = 0$. This is a contradiction and so $\Phi$ is unital.

Noting that [6, Theorem 4.5] is employed in the proof of [6, Theorem 5.1] and considering [6, Remark 4.6], it follows that $\Phi$ is isometric on $\mathcal{A}^{sa}$, since $\Phi(p) = s(U(p)) = 0$ if and only if $p = 0$. By Lemma 4.5, $\Phi(p) \mathcal{B} \Phi(p) = \Phi(p \mathcal{A} p) \subseteq \Phi(\mathcal{A})$ for every $p \in \mathcal{P}(\mathcal{A})_f$ and therefore $\Phi$ is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, by [6, Proposition 6.2]. \hfill \Box

We have therefore obtained the following result.

Theorem 4.11 Suppose $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \nu)$ are semi-finite von Neumann algebras, $E \subseteq S(\mathcal{A})$, $F \subseteq S(\mathcal{B})$ are symmetric spaces and $U : E \rightarrow F$ is a positive surjective isometry. If $U$ is finiteness preserving (in particular if $\nu(1) < \infty$), then there exists a positive operator $a \in S(\mathcal{B})$ and a Jordan $*$-isomorphism $\Phi$ of $\mathcal{A}$ onto $\mathcal{B}$ such that $U(x) = a \Phi(x)$ for all $x \in \mathcal{A} \cap E$.

Remark 4.12 In the preprint [15] it is shown (see [15, Theorem 3.6]) that if $E \subseteq S(\mathcal{A})$ and $F \subseteq S(\mathcal{B})$ are symmetrically $\Delta$-normed spaces and $U : E \rightarrow F$ is a normal positive disjointness preserving operator (not necessarily surjective nor isometric), then there exist a positive operator $a$ affiliated with $\mathcal{B}$ and a normal Jordan $*$-monomorphism (injective Jordan $*$-homomorphism) $\Phi$ onto a weakly closed $*$-subalgebra of $\mathcal{B}$ such that $U(x) = a \Phi(x)$ for all $x \in E \cap \mathcal{A}$. Since symmetrically $\Delta$-normed spaces are generalizations of symmetric spaces and the positive surjective isometries we are considering are automatically normal (see Lemma 3.1) and projection disjointness preserving (see Corollary 3.6), the greater part of Theorem 4.11 follows from [15, Theorem 3.6]. Strictly speaking we require the isometry to be disjointness preserving to apply this result, but by considering the proof of this result it is easily checked that the projection disjointness preserving property suffices. It is worth...
noting, however, that having assumed more, we are able conclude more, namely that the Jordan *-monomorphism is a bijection onto $B$ and that the positive operator $a$ is, in fact, $v$-measurable. Furthermore, there are significant differences in the techniques employed in the proofs of these results. In particular, the Jordan *-mono/isomorphism is constructed differently and since we are able to show that $a$ is $v$-measurable, the claim that $U(x) = a\Phi(x)$ for every $x \in A \cap E$ follows more easily.

**Remark 4.13** A number of structural descriptions of surjective isometries exclude the case where the norm of $E$ is proportional to the norm of $L^2(A)$ (see [28, Theorem 1], [27, Theorem 2] and [24, Theorem 4.1], for example). It follows from Theorem 4.11 that if $v(I) < \infty$ and $U : L^2(A) \rightarrow L^2(B)$ is a positive surjective isometry, then there exists a positive operator $a \in S(B, v)$ and a Jordan *-isomorphism $\Phi$ of $A$ onto $B$ such that $U(x) = a\Phi(x)$ for all $x \in A \cap L^2(A)$. In the preprint [23] it is, in fact, shown that such a Yeadon-type factorization holds even if $v(I) = \infty$ and even if the positive isometry is not surjective (see [23, Corollary 4.5]).

# 5 The structure of projection disjointness preserving isometries

In the previous section we showed that under certain conditions the structure of a positive surjective isometry can be described in terms of a positive operator and Jordan *-isomorphism. We will use this result to show that we can obtain a similar representation for a surjective isometry, which is not necessarily positive, if it is projection disjointness preserving and if the initial symmetric space has more structure. Throughout this section we will suppose that $E \subseteq S(A, \tau)$ is a strongly symmetric space with absolutely continuous norm, $F \subseteq S(B, \nu)$ is a symmetric space and $V : E \rightarrow F$ is a projection disjointness and finiteness preserving surjective isometry. The idea of the proof, inspired by [3, Sect. 5], is to use the isometry $V$ to construct a unitary operator $v$ such that $v^*V(\cdot)$ yields a positive surjective isometry and whose structure can therefore be described by the results of the previous section.

By Theorem 4.2, letting $\Psi(p) := s(V(p))$ for $p \in \mathcal{P}(A)_f$, yields a projection mapping that can be extended to a positive linear map (also denoted by $\Psi$) from $\mathcal{F}(\tau)$ into $B$ with Jordan *-homomorphism-like properties (i.e. $\Psi$ is positive, $\|\Psi(x)\|_B = \|x\|_A$ and $\Psi(x^2) = \Psi(x)^2$ for all $x \in \mathcal{F}(\tau)^sa$). Furthermore, for any $x \in \mathcal{F}(\tau)$ and $p \in \mathcal{P}(A)_f$ with $p \geq s(x) \lor r(x)$, we have $V(x) = V(p)\Psi(x)$. As in Theorem 4.2, we will, for each $p \in \mathcal{P}(A)_f$, write $V(p) = v_p b_p$ for the the polar decomposition of $V(p)$.

**Lemma 5.1** \{$(v_p)_{p \in \mathcal{P}(A)_f}$\} converges in the strong operator topology to a unitary operator $v \in B$ and $v\Psi(p) = v_p$ for all $p \in \mathcal{P}(A)_f$.

**Proof** We start by noting that if $p, q \in \mathcal{P}(A)_f$ are such that $0 < q \leq p$, then

$$v_q = v_p \Psi(q). \quad (5.1)$$

To show this, note that if $p = q$, then (5.1) holds using (4.1). If $p > q$, then $0 \neq p - q \in \mathcal{P}(A)_f$ and $q(p - q) = 0$. Therefore, $v_q + v_{p-q} = v_p$ (see [5, Proposition B.1.32(5)]).
It follows that \( v_p \Psi(q) = (v_q + v_{p-q})\Psi(q) = v_q + v_{p-q}\Psi(p-q)\Psi(q) = v_q \), using (4.1) and the fact that \((p-q)q = 0\) implies that \(\Psi(p-q)\Psi(q) = 0\).

Next, we show that \( \{v(p)\}_{p \in \mathcal{P}(A)_f} \) is SOT-convergent to a partial isometry. Let \( \eta \in K \) (where \( B \subseteq B(K) \)) and suppose \( \epsilon > 0 \). Since \( \{\Psi(p)\}_{p \in \mathcal{P}(A)_f} \) is an increasing net of projections, it converges in the strong operator topology to a projection \( y \in \mathcal{P}(B) \). It follows that there exists a \( p_\epsilon \in \mathcal{P}(A)_f \) such that \( p, q \in \mathcal{P}(A)_f \) with \( p, q \geq p_\epsilon \) implies that \( \|\Psi(p) - \Psi(q)\eta\| < \epsilon \). Let \( p, q \in \mathcal{P}(A)_f \) with \( p, q \geq p_\epsilon \). Since \( \mathcal{P}(A)_f \) is a directed set, there exists an \( r \in \mathcal{P}(A)_f \) with \( r \geq p, q \). Using (5.1), we then have

\[
\| (v_p - v_q)\eta | = \| v_r(\Psi(p) - \Psi(q))\eta | \leq \| v_r \|_B \| (\Psi(p) - \Psi(q))\eta | < \epsilon.
\]

Therefore \( \{v(p)\}_{p \in \mathcal{P}(A)_f} \) is Cauchy in \( K \). Since this holds for every \( \eta \in K \), we have that \( \{v_p\}_{p \in \mathcal{P}(A)_f} \) is SOT-Cauchy. Furthermore, \( \{v_p\}_{p \in \mathcal{P}(A)_f} \) is contained in the unit ball of \( B(K) \) and so \( v_p \xrightarrow{SOT} v \) for some \( v \in B(K) \), since norm-closed balls in \( B(K) \) are SOT-complete by [18, Proposition 2.5.11]. Since \( B \) is SOT-closed, \( v \in B \).

Furthermore, for any \( q \in \mathcal{P}(A)_f \) with \( q \geq p \), we have \( v_p = v_q \Psi(p) \xrightarrow{SOT} v\Psi(p) \) as \( q \uparrow q \in \mathcal{P}(A)_f \) by using (5.1) and the fact that multiplication is separately continuous in the strong operator topology. It follows that \( v_p = v\Psi(p) \). We show that \( v \) is a partial isometry and \( s(v) = y \). Note that \( v_p \xrightarrow{SOT} v \) implies that \( v_p \xrightarrow{WOT} v \) since the WOT is coarser than the SOT. Therefore \( v^*_p \xrightarrow{WOT} v^*(y) \) (see [18, Exercise 5.7.1]) and \( v^*_p v_p \xrightarrow{WOT} v^*v \). Furthermore, \( v^*_p v_p = |v_p| = (\Psi(p) \xrightarrow{SOT} y) \) and so \( v^*_p v_p \xrightarrow{WOT} y \).

It follows from the uniqueness of weak operator topology limits, this implies that \( y = v^*v \). Therefore \( v \) is a partial isometry (see [19, Proposition 6.1.1]) and \( s(v) = y \).

We show that \( \psi = 1 \). Suppose \( x \in \mathcal{F}(\tau) \). For \( p \in \mathcal{P}(A)_f \) with \( p \geq s(x) \lor r(x) \), we have that \( px = xp \) and hence \( \Psi(x) = \Psi(xp) = \Psi(x)\Psi(p) \). Therefore,

\[
\Psi(x)y = \Psi(x)SOT \lim_{p \in \mathcal{P}(A)_f} \Psi(p) = SOT \lim_{p \in \mathcal{P}(A)_f : p \geq s(x) \lor r(x)} [\Psi(x)\Psi(p)] = \Psi(x).
\]

It follows that if \( p \geq s(x) \lor r(x) \), then \( V(x) = b_p v_p \Psi(x) = b_p v_p \Psi(x)y = V(x)y \), using Theorem 4.2. Assume that \( 1 - y \neq 0 \). Since \((B, v)\) is semi-finite, there exists a \( q \in \mathcal{P}(B) \) such that \( 0 < q \leq 1 - y \) and \( v(q) < \infty \). This implies that \( q \in F \) and hence there exists an \( x \in E \) such that \( V(x) = q \), since \( V \) is surjective. \( E \) has absolutely continuous norm and therefore \( \mathcal{F}(\tau) \) is dense in \( E \) (see [11, p. 241]). Let \( (x_n)_{n=1}^{\infty} \) be a sequence in \( \mathcal{F}(\tau) \) such that \( x_n \xrightarrow{E} x \). Then \( V(x_n) \xrightarrow{F} V(x) = q \). However \( V(x_n) = V(x_n)y \xrightarrow{F} V(x)y = qy \) and so \( q = qy = 0 \), since \( q \leq 1 - y \). This is a contradiction and so \( v^*v = y = 1 \). By repeating this process with the projection disjointness preserving and finiteness preserving surjective isometry \( \tilde{V} \), defined by \( \tilde{V}(x) := \tilde{V}(x)^* (x \in E) \), we can similarly show that \( vv^* = 1 \). \[ \square \]
Lemma 5.2 The map \( U : E \to F \) defined by \( U(x) = v^*V(x) \) is a positive surjective isometry.

**Proof** Since \( v^* \) is a unitary operator, it is easily checked that \( U \) is a surjective isometry. To see that \( U \) is positive note that if \( x \in \mathcal{F}(\tau)^+ \) and \( p = s(x) \), then \( p \in \mathcal{P}(\mathcal{A}_f) \) and \( V(x) = v_p b_p \Psi(x) = v \Psi(p) b_p \Psi(x) = v b_p \Psi(x) \), by Theorem 4.2, Lemma 5.1 and (4.1). It follows that \( v^*V(x) = b_p \Psi(x) = b_p^{1/2} \Psi(x) b_p^{1/2} \geq 0 \) using Theorem 4.2, [11, Proposition 1(iii)] and the fact that \( \Psi \) is positive. Suppose \( x \in E^+ \). Since \( E \) has absolutely continuous norm, there exists a sequence \( (x_n)_{n=1}^{\infty} \) in \( \mathcal{F}(\tau)^+ \) such that \( x_n \to x \). As \( U \) is an isometry, \( v^*V(x_n) = U(x_n) \to U(x) \). We have that \( v^*V(x_n) \geq 0 \) for all \( n \in \mathbb{N}^+ \) and therefore \( U(x) \geq 0 \), since \( F^+ \) is closed by [11, Corollary 12(i)].

\( \square \)

Theorem 5.3 Suppose \( E \) is a strongly symmetric space with absolutely continuous norm and \( F \) is a symmetric space. If \( V : E \to F \) is a projection disjointness and finiteness preserving surjective isometry, then there exists a unitary operator \( v \), a positive operator \( a \in S(\mathcal{B}, v) \) and a Jordan *-isomorphism \( \Phi \) from \( \mathcal{A} \) onto \( \mathcal{B} \) such that \( V(x) = va \Phi(x) \) for all \( x \in \mathcal{A} \cap E \).

**Proof** In order to apply Theorem 4.11 to describe the structure of \( U \) as defined by the previous lemma we need to show that \( U \) is finiteness preserving. To this end, suppose that \( p \in \mathcal{P}(\mathcal{A}_f) \). Then

\[
U(p) = v^*V(p) = v^* v_p b_p = v^* v \Psi(p) b_p = b_p ,
\]

by Theorem 4.2, Lemma 5.1 and (4.1). It follows from the above and the finiteness preserving assumption on \( V \) that \( v(s(U(p))) = v(s(b_p)) = v(s(V(p))) < \infty \). By Theorem 4.11, there exists a positive operator \( a \in S(\mathcal{B}, v) \) and a Jordan *-isomorphism \( \Phi \) from \( \mathcal{A} \) onto \( \mathcal{B} \) such that \( U(x) = a \Phi(x) \) for all \( x \in \mathcal{A} \cap E \) and so \( V(x) = va \Phi(x) \) for all \( x \in \mathcal{A} \cap E \).

\( \square \)

Remark 5.4 We demonstrate briefly that \( \Phi \) (obtained in the theorem above) is the unique normal extension of \( \Psi : \mathcal{F}(\tau) \to \mathcal{B} \) (as obtained earlier in this section by extending the map \( \Psi(p) := s(V(p)) \) for \( p \in \mathcal{P}(\mathcal{A}_f) \) and that \( b_p = a_p \) for every \( p \in \mathcal{P}(\mathcal{A}_f) \), where the \( a_p \)'s are the positive operators used to construct \( a \) as in Sect. 4. Recall that \( a_p = U(p) = \int_{\text{s}}^{\infty} \lambda d e_{\lambda}^{a_p} , e_{\lambda}^{a_p} = \text{SOT lim}_{p \in \mathcal{P}(\mathcal{A}_f)} e_{\lambda}^{a_p} \) and \( a = \int_{\text{s}}^{\infty} \lambda d e^{a}(\lambda) \). However, \( b_p = v^*V(p) = U(p) \) and so \( b_p = a_p \) for every \( p \in \mathcal{P}(\mathcal{A}_f) \). To demonstrate the relationship between \( \Phi \) and \( \Psi \), recall that \( \Phi \) is obtained using Theorem 4.1 and as such \( \Phi(p) = s(U(p)) = s(v^*V(p)) = s(V(p)) = \Psi(p) \) for every \( p \in \mathcal{P}(\mathcal{A}_f) \), since \( v^* \) is unitary.

Remark 5.5 We present a brief argument to indicate why the techniques presented in this section hold in the more general setting of symmetrically \( \Delta \)-normed spaces. Suppose \( E \subseteq S(\mathcal{A}, \tau) \) and \( F \subseteq S(\mathcal{B}, v) \) are symmetrically \( \Delta \)-normed spaces with \( E \) having order continuous norm. Care is required since symmetrically \( \Delta \)-normed spaces are not necessarily normed \( \mathcal{A} \)-bimodules (see [16, p.4]). However,
\[\|xy_n\|_E, \|ynx\|_E \to 0\] as \(n \to \infty\) whenever \(x \in E\) and \(\{yn\}_{n=1}^{\infty} \subseteq \mathcal{A}\) with \(yn \xrightarrow{\mathcal{A}} 0\) (see [15, p.8]). Furthermore, \(E\) is continuously embedded in \(S(\mathcal{A}, \tau)\) equipped with the measure topology (see [16, Theorem 4.2]). One is therefore able to obtain analogs of Proposition 2.1, Corollary 2.2 and hence also Theorem 4.2. These results combined with the facts that \(\mathcal{F}^+\) is closed [16, Corollary 4.3] and that \(\mathcal{F}(\tau)\) is dense in \(E\) if \(E\) has order continuous norm (see [15, Remark 2.9]) suffice to ensure that one obtains analogs of Lemmas 5.1, 5.2 and hence Theorem 5.3.

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