NEW APPROXIMATION PROPERTIES OF THE BERNSTEIN MAX-MIN OPERATORS AND BERNSTEIN MAX-PRODUCT OPERATORS

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Abstract. In this paper we put in evidence localization results for the so-called Bernstein max-min operators and a property of translation for the Bernstein max-product operators.

1. Introduction. As an alternative to the well-known linear approximation operators, recently, various types of nonlinear approximation operators have been introduced. First, we mention the so called max-product type operators, a class of subadditive and positively homogeneous operators, used as alternatives for the linear counterparts in many areas such as, approximation operators (see, e. g., [2], [3], [20]), interpolation operators (see, e. g., [5], [9]), sampling operators (see, e. g., [6], [8]), neural network operators (see, e. g., [1], [13], [14]) and others (see, e. g. [18], [17], [19], [21]). A detailed account on the theory of max-product type operators can be found in the monograph [4]. Note that in some remarkable cases the max-product operators have better approximation properties than their linear counterparts. While, in general the rate of uniform convergence in the approximation of continuous functions is the same as in the case of the linear counterparts, for the Bernstein operator of max-product kind we obtained Jackson type estimations in the approximation of continuous and concave functions (see [2]) and in the approximation of Lipschitz functions (see [10]). In general, the connections between the rates of uniform convergence for the linear operators and their max-product counterparts are given in the very recent paper [7].

Another remarkable property satisfied by the Bernstein max-product operators is a strong localization result. More exactly, if \( f = g \) on \([a, b]\) with \(0 < a < b < 1\) and \(0 < a < c < d < b < 1\) are arbitrary, then there exists \(\tilde{n}\) such that \(B_n^{[M]}(f) - B_n^{[M]}(g) = 0\) on \([c, d]\) for all \(n \geq \tilde{n}\) (see, Theorem 2.1 in [11], or, e.g., Theorem 2.4.1, p. 76 in [4]). For the classical Bernstein polynomials we have the weaker result \(B_n(f) - B_n(g) = o(1/n)\) on \([c, d]\), for all \(n \geq \tilde{n}\). Also note that the Bernstein max-product operators conserve the monotonicity and more generally, the quasi-convexity and the quasi-concavity.

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Another type of nonlinear operators that were introduced recently in paper [16], are the so called max-min operators. In this paper their convergence is proved and also shape preserving properties are obtained.

In this note we propose two new theoretical results. First, in Section 3 we prove that abstraction making by a translation with a constant value, we get a Jackson type rate of uniform convergence for the approximation of strictly positive continuous functions by the Bernstein operators of max-product kind. The constant is actually equal to \( n \), that is the order of the operator. Therefore, the construction is really simple. Note that in the case of the linear counterpart, such result is impossible because the operator preserves the constant functions and due to its linearity the order of uniform convergence does not depend on translations with constant values. Actually, it does not depend on translations with affine functions since such functions too are preserved by the linear Bernstein operator. The second theoretical result is given in Section 4 and it is an analogue of the localization property obtained for the Bernstein max-product operators, this time for the max-min version. The property is exactly the same as described just above for the Bernstein max-product operator. Besides the theoretical results in Section 3 and 4, in Section 2 we present the basic notions considering the Bernstein max-product and max-min operators and moreover, we make a comparative discussion concerning the max-product Bernstein operators and max-min Bernstein operators.

2. Definitions of the Bernstein max-product and max-min operators. For an arbitrary bounded function \( f : [0, 1] \rightarrow \mathbb{R} \), the Bernstein polynomial of order \( n \) applied to \( f \) is given by

\[
B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right), \quad x \in [0, 1],
\]

where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) for all \( x \in [0, 1] \) and \( k \in \{0, ..., n\} \). The corresponding Bernstein operator of max-product kind denoted with \( B_n^{[M]}(f) \), is given by (see, e. g. [2])

\[
B_n^{[M]}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, \quad x \in [0, 1].
\] (1)

(here, “\( \bigvee \)” means maximum). Since \( \sum_{k=0}^{n} p_{n,k}(x) = 1 \) for all \( x \in [0, 1] \), we can write

\[
B_n(f)(x) = \frac{\sum_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right)}{\sum_{k=0}^{n} p_{n,k}(x)}, \quad x \in [0, 1].
\]

Now replacing the sum operator with the maximum operator denoted with \( \bigvee \), again, we obtain formula (1) for \( B_n^{[M]}(f)(x) \). This construction was given in [15].

We can easily extend formula (1) to the case when \( f \) is defined on some compact interval. This generalization can be found in [12].
The Bernstein max-min operators are given by the formula (see [16])

$$B_n^{[m]}(f)(x) = \bigvee_{k=0}^{n} K_{n,k}(x) \land f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

(2)

where

$$K_{n,k}(x) = \frac{p_{n,k}(x)}{\bigvee_{i=0}^{n} p_{n,i}(x)}, \quad x \in [0,1], \quad k = 0, \ldots, n$$

(here, “∧” means minimum). Note that $B_n^{[m]}(f)$ is well-defined even if $f$ is unbounded. Then, if in (2) we replace “∧” with multiplication then we obtain the formula of the Bernstein max-product operator.

Some comparisons between the max-product Bernstein operators and max-min Bernstein operators can be described by the following.

**Remarks.** 1) Since for all $x \in [0,1]$ and $n \in \mathbb{N}$, we have

$$K_{n,k}(x) \land f\left(\frac{k}{n}\right) \leq K_{n,k}(x) \leq 1,$$

passing to maximum after k, it follows that for any $f : [0,1] \rightarrow [0,\infty)$, we get

$$0 \leq B_n^{[m]}(f)(x) \leq 1,$$

for all $x \in [0,1], n \in \mathbb{N}$.

This fact immediately implies that if $f(x_0) > 1$ then $B_n^{[m]}(f)(x_0)$ cannot converge to $f(x_0)$ as $n \rightarrow \infty$. In particular, it easily follows that if $f(x) > 1$ for all $x \in [0,1]$, then $B_n^{[m]}(f)(x) = 1$, for all $x \in [0,1], n \in \mathbb{N}$.

Therefore, over the max-min Bernstein operators, the max-product Bernstein operators present the advantage that they can approximate functions with values in the whole interval $[0,\infty)$.

2) When $f : [0,1] \rightarrow [0,1]$, both kinds of max-min and max-product Bernstein operators converge uniformly to $f$ on $[0,1]$ (for the max-min case see [16], while for the max-product case see, e.g., [2] or [4], p. 30, Theorem 2.1.5 However, if, for example $f$ is a Lipschitz $\alpha$ function on $[0,1]$, then the degree of approximation given by the max-product Bernstein operators is $O(1/n^{\alpha/2})$, and the degree of approximation given by the max-min Bernstein operators, $O(1/n^{\alpha/(2\alpha+1)})$, (see [16]). Therefore, when $\alpha \in (1/2,1]$ the degree of approximation given by the Bernstein max-product operator is essentially better that the one given by the Bernstein max-min operator. On the other hand, when $\alpha \in (0,1/2)$ the degree of approximation given by the Bernstein max-min operator is essentially better that the one given by the Bernstein max-product operator.

3) Since for any numbers $a, b \in [0,1]$ we have the inequality $a \cdot b \leq \min\{a, b\}$, choosing $a = K_{n,k}(x)$ and $b = f(k/n)$, it immediately follows that for any $f : [0,1] \rightarrow [0,1]$ we have $B_n^{[M]}(f)(x) \leq B_n^{[m]}(f)(x)$, for all $x \in [0,1], n \in \mathbb{N}$.

4) The max-product Bernstein operators are sublinear (that is subadditive and positive homogeneous), while the max-min Bernstein operators are not neither subadditive nor positive homogeneous. However, according to [16], the max-min Bernstein operators are pseudo-linear.

3. **A translation property for the Bernstein max-product operators.** Although in general, the approximation order by the Bernstein max-product operators is $\omega(f, 1/\sqrt{n})$, in this section we show that by a certain translation, equal with the
order of the operator, we can obtain the better order $\omega(f, 1/n)$. In order to achieve this result we need the following auxiliary result.

**Lemma 3.1.** (see Lemma 3.4 in [2]) One has

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), \ (\forall) \ x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], \ j = 0, \ldots, n.$$  

From the above lemma it easily results that for any bounded function $f : [0, 1] \to \mathbb{R}$, $j \in \{0, 1, \ldots, n\}$ and $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$, one has

$$B_{n}^{[M]}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x),$$

where

$$f_{k,n,j}(x) = \left(\begin{array}{c} n \\ j \end{array}\right) x^{j}(1-x)^{n-k} \cdot f\left(\frac{k}{n}\right)$$

$$= \left(\begin{array}{c} n \\ j \end{array}\right) \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f\left(\frac{k}{n}\right).$$

We can now present the main result of this section.

**Theorem 3.2.** Let $f : [0, 1] \to [0, \infty)$ be a continuous function. Then, for sufficiently large $n$ we have

$$\left|B_{n}^{[M]}(f_{n})(x) - f_{n}(x)\right| \leq 2\omega(f, 1/n), x \in [0, 1]$$

where $f_{n}(x) = f(x) + n$ for all $x \in [0, 1]$. Here $\omega(f; \delta) = \sup\{|f(x) - f(y)|; |x-y| \leq \delta\}$ denotes the usual modulus of continuity.

**Proof.** Since $f$ is uniformly continuous on $[0, 1]$, it is well known that $\lim_{\delta \searrow 0} \omega(f, \delta) = 0$. Therefore, there exists $n_{0} \in \mathbb{N}$ such that $\omega(f, 1/n) \leq 1$ for all $n \geq n_{0}$. This implies $|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)| \leq 1$ and $|f\left(\frac{k+1}{n}\right) - f\left(\frac{j}{n}\right)| \leq 1$ for all $n \geq n_{0}$ and $k \in \{1, 2, \ldots, n-1\}$. Now, let us fix $x \in [0, 1]$ and for $n \geq n_{0}$ let $j \in \{0, 1, \ldots, n\}$ be such that $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$. For $k \geq j + 1$ we have

$$\frac{(f_{n})_{k+1,n,j}(x)}{(f_{n})_{k,n,j}(x)}$$

$$= \frac{n-k}{k+1} \cdot \frac{x}{1-x} \cdot \frac{f_{n}\left(\frac{k+1}{n}\right)}{f_{n}\left(\frac{j}{n}\right)} = \frac{n-k}{k+1} \cdot \frac{x}{1-x} \cdot \frac{f\left(\frac{k+1}{n}\right) + n}{f\left(\frac{j}{n}\right) + n}$$

$$\leq \frac{n-k}{k+1} \cdot \frac{\frac{j+1}{n+1}}{1 - \frac{j+1}{n+1}} \cdot \frac{f\left(\frac{j}{n}\right) + 1 + n}{f\left(\frac{j}{n}\right) + n} \leq \frac{n-k}{k+1} \cdot \frac{j+1}{n-j} \cdot \frac{n+1}{n}$$

$$\leq \frac{n-k}{k+1} \cdot \frac{j+1}{n-j} \cdot \frac{k+1}{n} = \frac{n-k}{n-j} \cdot \frac{j+1}{k} \leq 1.$$
On the other hand, for \( k \leq j - 1 \) we have
\[
\frac{(f_n)_{k,n,j}(x)}{(f_n)_{k-1,n,j}(x)} = \frac{n - k + 1}{k} \cdot \frac{x}{1 - x} \cdot \frac{f(\frac{k}{n} + n)}{f(\frac{k-1}{n} + n)} \geq \frac{n - k + 1}{k} \cdot \frac{j}{n+1} \cdot \frac{f(\frac{k}{n} + n)}{f(\frac{k}{n}) + 1 + n} \\
\geq \frac{n - k + 1}{k} \cdot \frac{j}{n + 1 - j} \cdot \frac{n - k}{n + 1 - j} = \frac{j}{k} \cdot \frac{n - k}{n + 1 - j} \geq 1.
\]

The above inequality combined with the previous one gives
\[
B_{n}^{[M]}(f_n)(x) = \max\{(f_n)_{j-1,n,j}(x), (f_n)_{j,n,j}(x), (f_n)_{j+1,n,j}(x)\}
\]
which implies (see Lemma 4.4 in \cite{2})
\[
|B_{n}^{[M]}(f_n)(x) - f_n(x)| \leq 2\omega(f_n, 1/n) = 2\omega(f, 1/n)
\]
and this proves the theorem. \(\square\)

**Remark 1.** The same result is valid for the truncated max-product Favard-Szász and truncated max-product Baskakov operators.

**Remark 2.** It is obvious that Theorem 3.2 remains valid if instead of \( f + n \) we take \((f_n,a)(x) = f(x) + an\) with arbitrary fixed \( a > 0 \) independent of \( n \).

**Remark 3.** If \( \omega(f, 1/n) \leq O(n^{-\alpha}) \) with \( \alpha \in (0, 1] \), then Theorem 3.2 remains valid if instead of \( F_n(x) = f(x) + n \), we take \( f_n(x) = f(x) + n^{1-\alpha} \).

**Remark 4.** It is known that if \( f \) is a strictly positive Lipschitz function on \([0,1]\) (see Corollary 4.7 in \cite{10}) then for all \( x \in [0,1] \) and \( n \in \mathbb{N} \), we have
\[
|B_{n}^{[M]}(f)(x) - f(x)| \leq C/n, x \in [0,1],
\]
where \( C > 0 \) depends on \( f \) but it is independent of \( n \). Therefore, even if we define \( F(x) = f(x) + M \) with arbitrary constant \( M > 0 \), it is clear that \( B_{n}^{[M]}(F)(x) \) does not estimate \( F(x) \) with a better order of approximation.

4. **Localization results for Bernstein max-min operators.** In this section we present strong localization results of approximation satisfied by the Bernstein max-min operators similar to those satisfied by the Bernstein max-product operators. Also, as a consequence, we obtain a localization result concerning the preservation of the monotonicity.

We start with the strong localization result given in the next theorem.

**Theorem 4.1.** Let us consider the functions \( f, g : [0,1] \to [0,\infty) \) and suppose that there exists \( a, b \in [0,1] \), \( 0 < a < b < 1 \) such that \( f(x) = g(x) \) for all \( x \in [a, b] \) and such that \( \min\{ f(x) : x \in [a, b] \} > 0 \). Then for all \( c, d \in [a, b] \) satisfying \( a < c < d < b \) there exists \( \bar{n} \in \mathbb{N} \) depending only on \( f, g, a, b, c, d \) such that \( B_{n}^{[m]}(f)(x) = B_{n}^{[m]}(g)(x) \) for all \( x \in [c, d] \) and \( n \in \mathbb{N} \) with \( n \geq \bar{n} \).

**Proof.** The proof has many common points with the proof of Theorem 2.1 in \cite{11}. To make the paper more readable we will insist on some details that can be found in the proof of the afore mentioned theorem too.
Let us choose arbitrary \( x \in [c, d] \) and for each \( n \in \mathbb{N} \) let \( j_x \in \{0, 1, ..., n\} \) (\( j_x \) depends on \( n \) too, but there is no need at all to complicate on the notations) be such that \( x \in [j_x/(n+1), (j_x+1)/(n+1)]. \) Since \( x \in [c, d] \cap [j_x/(n+1), (j_x+1)/(n+1)] \) and since \( a < c < d < b \) it is immediate that for \( n \geq n_0 \) where \( n_0 \) is chosen such that \( 1/n_0 < \min\{c-a, b-d\} \), we obtain \( a < j_x/(n+1) < j_x/n < b \) (see also the proof of Theorem 2.1 in [11]). This implies that if \( n \geq n_0 \) then for any \( x \in [c, d] \) there exists \( \alpha_x \in [a, b] \) such that \( j_x = n\alpha_x. \)

Now, by Lemma 3.1 it follows that

\[
K_{n,k}(x) = \frac{p_{n,k}(x)}{p_{n,j_x}(x)}, \quad k = 0, ..., n
\]

and in particular this implies

\[
K_{n,j_x}(x) \wedge f \left( \frac{j_x}{n} \right) = \min \left\{ 1, f \left( \frac{j_x}{n} \right) \right\}.
\]

Let us denote

\[
m_f(a, b) = \min\{f(x) : x \in [a, b]\},
\]

where the hypotheses imply that \( m_f(a, b) > 0. \) It also means that \( f \left( \frac{j_x}{n} \right) \geq m_f(a, b). \)

Let us denote \( \overline{m} = \min\{1, m_f(a, b)\}, \) where obviously we have \( \overline{m} > 0. \) Thus, we have

\[
K_{n,j_x}(x) \wedge f \left( \frac{j_x}{n} \right) \geq \overline{m}.
\]

We need to introduce the sequence \( (a_n)_{n \geq 1}, \) \( a_n = \sqrt[2]{n^2}. \) For this sequence there exists \( n_1 \in \mathbb{N} \) such that \( na_n > 0 \) for all \( n \geq n_1. \) We will prove that there exists an absolute constant \( N_0 \in \mathbb{N} \) which does not depend on \( x \in [c, d], \) such that for any \( n \geq N_0 \) and \( x \in [c, d] \) we have \( B_0^{m}(f)(x) = \sqrt[n]{K_{n,k}(x) \wedge f \left( \frac{j_x}{n} \right)}, \)

where

\[
I_{n,x} = \{k \in \{0, 1, ..., n\} : j_x - a_n \leq k \leq j_x + a_n\}.
\]

In order to obtain this conclusion, for \( n \geq \max\{n_0, n_1\} \) let us choose \( k \in \{0, 1, ..., n\} \setminus I_{n,x}. \) We have two cases: i) \( k + a_n < j_x, \) and ii) \( j_x + a_n < k. \)

Case i) Since \( x \in [j_x/(n+1), (j_x+1)/(n+1)], \) we observe that \( \frac{1-x}{x} \leq \frac{1-j_x/(n+1)}{j_x} = \frac{n+1-j_x}{j_x} \) and noting that \( j_x = n\alpha_x, \) after some simple calculations we obtain

\[
K_{n,k}(x) = \frac{p_{n,k}(x)}{p_{n,j_x}(x)} = \left( \frac{n}{n_{j_x}} \right) \left( 1 - \frac{x}{x} \right)^{j_x - k} = \frac{(k+1) \cdot (k+2) \cdots n\alpha_x}{(n-n\alpha_x+1) \cdot (n-n\alpha_x+2) \cdots (n-k)} \cdot \left( \frac{1-x}{x} \right)^{n\alpha_x - k} \leq \frac{(k+1) \cdot (k+2) \cdots n\alpha_x}{(n-n\alpha_x+1) \cdot (n-n\alpha_x+2) \cdots (n-k)} \cdot \left( \frac{n+1-j_x}{j_x} \right)^{n\alpha_x - k}.
\]

We have two subcases: i_1) \( n - n\alpha_x + 1 \leq k + 1 \) and ii_1) \( n - n\alpha_x + 1 > k + 1. \)

Case i_1) It is clear that \( 0 < \frac{n\alpha_x}{n-1} \leq \frac{n\alpha_x-1}{n-2} \leq \cdots \leq \frac{k+1}{n-n\alpha_x+1}, \) which implies

\[
K_{n,k}(x) \leq \left( \frac{k+1}{n-n\alpha_x+1} \right)^{n\alpha_x - k} \cdot \left( \frac{n+1-n\alpha_x}{n\alpha_x} \right)^{n\alpha_x - k} = \left( \frac{k+1}{n\alpha_x} \right)^{n\alpha_x - k}.
\]
Since \( k < n\alpha_x - a_n \) it easily results that
\[
K_{n,k}(x) \leq \left( \frac{n\alpha_x - a_n}{na_x} \right)^{a_n}.
\]

Then, since \( 0 < \frac{n\alpha_x - a_n}{na_x} \leq \frac{n\alpha_x - a_n}{nb} \), it follows that
\[
K_{n,k}(x) \leq \left( \frac{nb - a_n}{nb} \right)^{a_n} = \left( 1 - \frac{a_n}{nb} \right)^{a_n}
\]

We observe that \( \lim_{n \to \infty} \left( 1 - \frac{a_n}{nb} \right)^{a_n} = e^{\lim_{n \to \infty} -\frac{a_n^2}{nb}} = 0 \). It follows that there exists \( n_2 \in \mathbb{N}, n_2 \geq \max\{n_0, n_1\} \) such that \( K_{n,k}(x) < \overline{m} \) for all \( x \in [c,d], n \geq n_2 \) and \( k \in \{0, 1, ..., n\}, k < j_x - a_n \). In addition, we notice that \( n_2 \) does not depend on \( x \in [c,d] \) but it may depend on \( f \).

Case ii) In this case we have \( \frac{k+1}{n-\alpha_x+1} \leq \frac{k+2}{n-\alpha_x+2} \leq \frac{n\alpha_x}{n-k} \), and by similar reasoning to those from the previous case, we get
\[
K_{n,k}(x) \leq \left( \frac{n\alpha_x}{n-k} \right)^{j_x-k} \cdot \left( \frac{n+1-n\alpha_x}{n\alpha_x} \right)^{j_x-k} = \left( \frac{n+1-n\alpha_x}{n-k} \right)^{j_x-k}
\]

Since \( n - k > n - \alpha_x + a_n \geq n + 1 - \alpha_x \) and since \( n\alpha_x - k > a_n \), we get
\[
K_{n,k}(x) \leq \left( \frac{n+1-n\alpha_x}{n-\alpha_x+a_n} \right)^{a_n}
\]

Reasoning as in the previous case we will obtain that there exists an absolute constant \( n_3 \in \mathbb{N}, n_3 \geq \max\{n_0, n_1\} \) such that \( K_{n,k}(x) < \overline{m} \), for all \( x \in [c,d], n \geq n_3 \) and \( k \in \{0, 1, ..., n\}, k < j_x - a_n \).

Summarizing the case (i), we conclude that there exists a constant \( N_1 = \max\{n_2, n_3\} \) (depending only on \( f, a, b, c, d \)), such that \( K_{n,k}(x) < \overline{m} \), for all \( x \in [c,d], n \geq N_1 \) and \( k \in \{0, 1, ..., n\}, k < j_x - a_n \).

Case ii) This case is easily reduced to case i) noticing that in general we have \( K_{n,k}(x) = K_{n,n-k}(1-x) \), for all \( x \in [0,1] \) and \( k \in \{0,1, ..., n\} \). See also the discussion for case ii) in the proof of Theorem 2.1 in [11]. Therefore, there exists \( N_2 \in \mathbb{N} \), depending only on \( f, a, b, c, d \), such that \( K_{n,k}(x) < \overline{m} \), for all \( x \in [c,d], n \geq N_1 \) and \( k \in \{0, 1, ..., n\}, j_x + a_n < k \).

Analyzing the results obtained in cases i)-ii), it results that for all \( x \in [c,d], n \geq N_0, N_0 = \max\{N_1, N_2\} \) and \( k \in \{0,1, ..., n\} \), with \( k < j_x - a_n \) or \( k > j_x + a_n \), we have \( K_{n,k}(x) < \overline{m} \). Combining this fact with relation (3), it follows that
\[
B^{m}\left[ f \right](x) = \bigvee_{k \in I_{n,x}} K_{n,k}(x) \wedge f \left( \frac{k}{n} \right), \quad x \in [c,d], n \geq N_0,
\]

where \( I_{n,x} = \{ k \in \{0,1, ..., n\} : j_x - a_n \leq k \leq j_x + a_n \} \).

Next, let us choose arbitrary \( x \in [c,d] \) and \( n \in \mathbb{N} \) so that \( n \geq N_0 \). If there exists \( k \in I_{n,x} \) such that \( k/n \notin [c,d] \) then we distinguish two cases. Either \( k/n < c \) or \( k/n > d \). In the first case we observe that
\[
0 < \frac{k}{n} \leq x - \frac{k}{n} \leq \frac{j_x + 1}{n + 1} - \frac{k}{n} \leq \frac{j_x + 1}{n} - \frac{k}{n} = \frac{a_n + 1}{n}.
\]
Since \( \lim_{n \to \infty} \frac{a_n + 1}{n} = 0 \), it results that for sufficiently large \( n \) we necessarily have \( \frac{a_n + 1}{n} < c - a \) which clearly implies that \( k/n \in [a, c] \). In the same manner, when \( k/n > d \), for sufficiently large \( n \) we necessarily have \( k/n \in [d, b] \). Summarizing there exists a constant \( \bar{N}_1 \in \mathbb{N} \) independent of any \( x \in [c, d] \) such that

\[
B_{n}^{[m]}(f)(x) = \bigvee_{k \in I_{n,x}} K_{n,k}(x) \wedge f \left( \frac{k}{n} \right), \quad x \in [c, d], \quad n \geq \bar{N}_1
\]

and in addition for any \( x \in [c, d], \ n \geq \bar{N}_1 \) and \( k \in I_{n,x} \), we have \( k/n \in [a, b] \). Also, it is easy to check that \( \bar{N}_1 \) depends only on \( a, b, c, d, f \).

Reasoning for the function \( g \) exactly as in the case of the function \( f \), it follows that there exists \( \bar{N}_2 \in \mathbb{N} \) which depends only on \( a, b, c, d, f, g \) such that

\[
B_{n}^{[m]}(g)(x) = \bigvee_{k \in I_{n,x}} K_{n,k}(x) \wedge g \left( \frac{k}{n} \right), \quad x \in [c, d], \quad n \geq \bar{N}_2
\]

and in addition for any \( x \in [c, d], \ n \geq \bar{N}_2 \) and \( k \in I_{n,x} \), we have \( k/n \in [a, b] \). Taking \( \bar{n} = \max \{ \bar{N}_1, \bar{N}_2 \} \) we easily obtain the desired conclusion.

Let us now discuss the special case when \( f : [0, 1] \to [0, \infty) \) is constant with constant value \( \alpha \in (0, 1) \). Since \( K_{n,k}(x) \wedge f \left( \frac{k}{n} \right) \leq \alpha \), for all \( x \in [0, 1] \) and \( k \in \{0, ..., n\} \), and \( K_{n,j_x}(x) \wedge f \left( \frac{j_x}{n} \right) = f \left( \frac{j_x}{n} \right) = \alpha \), it follows that \( B_{n}^{[m]}(f) = f \). More generally, we have the following property.

**Corollary 1.** Let \( f : [0, 1] \to [0, \infty) \) and suppose that there exists \( a, b \in [0, 1], \ 0 < a < b < 1 \), such that \( f \) is constant on \( [a, b] \) with the constant value \( \alpha \in (0, 1) \). Then for any \( c, d \in [a, b] \) with \( a < c < d < b \), there exists \( \bar{N} \in \mathbb{N} \) depending only on \( a, b, c, d \) and \( f \) , such that \( B_{n}^{[m]}(f)(x) = \alpha \) for all \( x \in [c, d] \) and \( n \in \mathbb{N} \) with \( n \geq \bar{N} \).

**Proof.** Reasoning as in the proof of Theorem 4.1, it follows that there exists \( \bar{N} \in \mathbb{N} \) which depends only on \( a, b, c, d \) and \( f \), such that

\[
B_{n}^{[m]}(f)(x) = \bigvee_{k \in I_{n,x}} K_{n,k}(x) \wedge f \left( \frac{k}{n} \right), \quad x \in [c, d], \quad n \geq \bar{N}
\]

and in addition for any \( x \in [c, d], \ n \geq \bar{N} \) and \( k \in I_{n,x} \), we have \( k/n \in [a, b] \). On one hand this implies that \( B_{n}^{[m]}(f)(x) \leq \alpha \) and on the other hand, since \( j_x \in I_{n,x} \), it follows that

\[
K_{n,j_x}(x) \wedge f \left( \frac{j_x}{n} \right) = f \left( \frac{j_x}{n} \right) = \alpha.
\]

It means that \( B_{n}^{[m]}(f)(x) = \alpha \), for all \( x \in [c, d], \ n \geq \bar{N} \). \( \square \)

Finally, we present a result on local preservation of monotonicity. First we need the following general result on the preservation of monotonicity.

**Theorem 4.2.** (see Corollary 4.1 in [16]). If \( f : [0, 1] \to [0, \infty) \) is continuous and monotone then \( B_{n}^{[m]}(f) \) preserves the monotonicity of \( f \).

Note that in [16] the function \( f \) is assumed to take values in \([0, 1]\). However, this hypothesis is not used in the proof of Corollary 4.1 of paper [16]. Therefore, the statement holds in the setting given in the previous theorem. Combining the previous theorem with Theorem 4.1, we obtain the following property on local preservation of monotonicity.
Corollary 2. Let \( f : [0, 1] \to [0, \infty) \) be continuous and suppose that there exists \( a, b \in [0, 1] \), \( 0 < a < b < 1 \), such that \( f \) is monotone on \([a, b]\) and such that \( f(a) > 0 \). Then for any \( c, d \in [a, b] \) with \( a < c < d < b \), there exists \( \tilde{n} \in \mathbb{N} \) depending only on \( a, b, c, d \) and \( f \), such that \( B_n^{[m]}(f) \) preserves the monotonicity of \( f \) on \([c, d]\), for all \( n \in \mathbb{N} \) with \( n \geq \tilde{n} \).

Proof. Take \( g : [0, 1] \to [0, \infty) \),
\[
g(x) = \begin{cases} f(a) & \text{if } x \in [0, a], \\ f(x) & \text{if } x \in [a, b], \\ f(b) & \text{if } x \in [b, 1]. \end{cases}
\]

Obviously, \( g \) is continuous and monotone on \([0, 1]\). By Theorem 4.1, there exists \( \tilde{n} \in \mathbb{N} \) depending only on \( f, a, b, c, d \) such that \( B_n^{[m]}(f)(x) = B_n^{[m]}(g)(x) \) for all \( x \in [c, d] \) and \( n \in \mathbb{N} \) with \( n \geq \tilde{n} \). Since \( g \) is monotone and since by Theorem 4.2 \( B_n^{[m]}(g) \) preserves the monotonicity of \( g \), it follows that \( B_n^{[m]}(f) \) preserves the monotonicity of \( f \) on \([c, d]\), for all \( n \in \mathbb{N} \) with \( n \geq \tilde{n} \).

5. Conclusions. In this paper, first we proved that it is enough to translate a strictly positive continuous function with the constant value given by the order of the Bernstein operator of max-product kind in order to obtain a Jackson type rate of uniform convergence in the approximation of the translated function by the Bernstein operator of max-product kind. Then, similarly to the case of Bernstein operators of max-product kind we obtained a strong localization property for the max-min type Bernstein operators. Considering the first theoretical result, an open question is whether the results remains valid even for functions that are not strictly positive. The constant does not need to be exactly the order of the operator but maybe it can be this order multiplied by an absolute constant value. The present proof does not work in such cases. Then, another important topic that worth further investigation is whether the present result or some of its more general formulations could be extended for other types of max-product type operators. It seems that this property depends essentially on the kernels at least from the perspective of the proof used in this note. Considering the max-min operators, it is of interest to consider for these operators, analogous studies with those made for max-product operators, as, for example: iterations and fixed points for max-min Bernstein operators, approximation by other max-min operators (of Favard-Szász-Mirakjan kind, of Baskakov kind, of Bleimann-Butzer-Hahn kind, of Meyer-König-Zeller kind), approximation by max-min interpolation kind operators on various knots, approximation by max-min sampling operators, approximation by bivariate variants of the previous mentioned max-min operators, global smoothness preservation by max-min Bernstein operators and by max-min Hermite-Fejér operators and properties of the max-min Weierstrass kind functions.

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