Revisiting the Dolinar Receiver through
Multiple–Copy State Discrimination Theory

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I. INTRODUCTION

Discrimination between two non–orthogonal quantum states is a fundamental issue in quantum mechanics and, in particular, in quantum communications. From a theoretical point of view, the problem was completely analyzed and solved by Helstrom [1], which found the optimal measurement operators, and the corresponding correct detection probability (Helstrom bound), for both pure and mixed quantum states. Unfortunately, also for pure states, often the optimal measurements do not correspond to quantum observables that are easily measurable, so that the experimental implementation of the optimal discrimination is a very difficult task.

For the case of two coherent states of a traveling single radiation mode, in 1973 Dolinar [2] proposed an adaptive measurement scheme, based on a combination of photon counting and feedback control, that precisely achieves the Helstrom bound (see also [3]). However, since the scheme requires a very precise control of an optical–electrical loop, only recently the Dolinar’s idea has obtained a satisfactory practical implementation [4].

Recent years have seen an increasing interest for adaptive measurements from both a theoretical and an experimental point of view, also for optical phase measurements and estimation (see [5] and references therein). A notably interesting theoretical result has been obtained by Acin et al. [6] for discrimination between pure quantum states, when multiple identical copies of a quantum state are available. They proved (see also [7]) that in this case the Helstrom bound can be achieved by local adaptive measurements applied to single copies. The result is particularly attractive in that it offers a useful insight into the Dolinar’s approach for discrimination between coherent states. In this paper we discuss the strict connection, already recognized in [6], between the ideas underlying measurements of multiple copies of pure states [6] and the Dolinar receiver [2].

Acin et al. [6] considered the discrimination between pure quantum states \( |\gamma_0\rangle \) and \( |\gamma_1\rangle \), when \( n \) identical copies of an unknown state are given. Formally, the problem consists in discriminating between the pure states

\[
|\alpha_0\rangle = |\gamma_0\rangle \otimes \cdots \otimes |\gamma_0\rangle \\
|\alpha_1\rangle = |\gamma_1\rangle \otimes \cdots \otimes |\gamma_1\rangle
\]

in the tensorial product Hilbert space \( \mathcal{H}^\otimes n \), where \( \mathcal{H} \) is the Hilbert space spanned by the single copies \( |\gamma_0\rangle \) and \( |\gamma_1\rangle \). Of course, also in this case one could apply the Helstrom theory and find an optimal collective measurement in \( \mathcal{H}^\otimes n \) achieving the Helstrom bound. Unfortunately, collective measurements are difficult to realize experimentally. With the adaptive approach suggested in [6], the experimenter performs on each copy a local measurement which is optimized on the basis of the results of the measurements on the previous copies. The surprising enough conclusion is that the optimized local measurements achieve the Helstrom bound, exactly the same as the optimal collective measurement.

The discrimination between two coherent states of a single–mode harmonic oscillator (without loss of generality \( |\gamma\rangle \) and \( |−\gamma\rangle \)) presents a difficulty similar to that of collective measurements on multiple copies. Namely, the Helstrom theory gives optimum measurement vectors that are linear superposition of \( |\gamma\rangle \) and \( |−\gamma\rangle \) and do not correspond to any measurable observable. On the other hand, owing to their peculiar properties, the coherent states \( |\gamma\rangle \) and \( |−\gamma\rangle \) of duration \( T \) can be thought as sequences of shorter and weaker modes of duration \( T/n \), namely,

\[
|\gamma\rangle = \frac{|\gamma\rangle}{\sqrt{n}} \otimes \cdots \otimes \frac{|\gamma\rangle}{\sqrt{n}} \\
|−\gamma\rangle = \frac{|−\gamma\rangle}{\sqrt{n}} \otimes \cdots \otimes \frac{|−\gamma\rangle}{\sqrt{n}}
\]

Moreover, as \( n \) increases, and the average number of photons per copy goes to zero, the optimal Helstrom measurement on each copy may be conveniently approximated by a displacement followed by a photon detection. Then, in principle, we may think to apply the
multiple–copy adaptive measurement to the segmented quantum states \(2\). As \(n\) goes to infinity, it appears natural the transition to the Dolinar scheme \(2\), with a continuous time–varying displacement controlled by the photon counting results.

This paper is organized as follows. In Section II we illustrate the optimal multiple–copy measurement paradigm from a novel point of view leading in a natural way to the continuous–time extension as in the Dolinar receiver. In Section III the feasibility of near–optimal discrimination of weak coherent states is discussed. In Section IV we extend the optimal multiple–copy adaptive measurements to coherent states. In particular, we derive the theory of the Dolinar receiver in a simple way, avoiding the cumbersome machinery of dynamic programming. Finally, in Section V we propose a suboptimal simplified version of the Dolinar’s scheme.

II. MULTIPLE COPY ADAPTIVE MEASUREMENT

Measurement strategies for discrimination between multiple copies of two pure quantum states were discussed by Acin et al. \(9\) (see also \(7\)). Alice, according to the binary input symbol \(a \in \{0, 1\}\), chooses between two pure states \(|\gamma_0\rangle\) and \(|\gamma_1\rangle\) in the Hilbert space \(\mathcal{H}\) with probability \(q_0\) and \(q_1 = 1 - q_0\), respectively. (Without loss of generality we assume \(q_0 \geq q_1\).) Then, Alice sends Bob \(n\) identical copies of the chosen state, corresponding to the states \(\mathcal{H}^\otimes n\), which are pure states in the tensorial product Hilbert space \(\mathcal{H}^\otimes n\). Bob performs a measurement on the system, trying to guess the original state with maximum correct detection probability.

The optimal result is given by the well–known Helstrom bound \(11\)

\[
P_c = \frac{1}{2} \left[ 1 + \sqrt{1 - 4q_0q_1X^2} \right]
\]

\[
= \frac{1}{2} \left[ 1 + \sqrt{1 - 4q_0q_1\chi^{2n}} \right]
\]

where

\[
X = \langle \alpha_0 | \alpha_1 \rangle = \langle \gamma_0 | \gamma_1 \rangle^n = \chi^n
\]

\[
\chi = |\langle \gamma_0 | \gamma_1 \rangle| = |\langle \gamma_0 | \gamma_1 \rangle|^n = \chi^n\]

and \(\chi = |\langle \gamma_0 | \gamma_1 \rangle|\) is the overlap coefficient of the single copies. Bob may achieve this bound by a global measurement using suitable von Neumann projectors \(\Pi_0 = |\beta_0\rangle \langle \beta_0|\) and \(\Pi_1 = |\beta_1\rangle \langle \beta_1|\) over the product space \(\mathcal{H}^\otimes n\). Unfortunately, the optimum measurement vectors \(|\beta_0\rangle\) and \(|\beta_1\rangle\) turn out to be non separable linear superposition of the pure states \(|\alpha_0\rangle\) and \(|\alpha_1\rangle\), i.e., an entangling measurement which is hard to implement experimentally.

The problem of optimizing local adaptive measurements had been tackled by Acin et al. \(9\) and it may be formalized in the following way. Let us assume, without loss of generality, that the single copies are given by

\[
|\gamma_0\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle \]

\[
|\gamma_1\rangle = \cos \theta |x\rangle - \sin \theta |y\rangle
\]

where \(|x\rangle\) and \(|y\rangle\) form an orthonormal basis of the Hilbert space \(\mathcal{F}\) spanned by the states \(|\gamma_0\rangle\) and \(|\gamma_1\rangle\) and the overlap coefficient is given by

\[
\chi = |\langle \gamma_0 | \gamma_1 \rangle| = \cos 2\theta\]

Assume that the local measurement orthonormal vectors on the \(k\)-th copy, \(k = 1, \ldots, n\),

\[
|\mu_{k0}\rangle = \cos \phi_k |x\rangle + \sin \phi_k |y\rangle
\]

\[
|\mu_{k1}\rangle = \sin \phi_k |x\rangle - \cos \phi_k |y\rangle
\]

are completely specified by the measurement angle \(\phi_k\). Let \(z_k \in \{0, 1\}\) be the outcome of the \(k\)-th measurement, which is also assumed as the result of a provisional decision. The adaptive optimization problem consists in finding a starting measurement angle \(\phi_1\) and a recursive rule

\[
\phi_k = f_k(z_1, \ldots, z_{k-1})
\]

in such a way that the final outcome \(z_n\) gives the correct detection with maximum probability. In the recursion \(8\) the information gained by the previous \(k - 1\) measurements is exploited in order to optimize the choice of the next measurement angle.

This appears to be a dynamic programming problem \(8\) and, as such, it had been dealt with and solved in \(6\).

The main results, surprisingly simple, are summarized as follows: i) the optimal solution of the dynamic programming approach gives correct detection probability coinciding with the Helstrom bound \(4\), so that the global optimal measurement may be replaced by more easily implementable local measurements; ii) the problem reduces to a bayesian updating problem (see also \(7\)) with recursive relation \(\phi_k = f_k(z_{k-1})\) so that the new optimal measurement angle depends only on the outcome of the last measurement; iii) if the \((k-1)\)-th result is \(z_{k-1} = i\), the optimal measurement angle \(\phi_k\) is the solution of the Helstrom optimization problem obtained replacing the a priori probabilities \(q_0\) and \(q_1\) with the a posteriori probabilities \(P[a=0|z_{k-1}=i]\) and \(P[a=1|z_{k-1}=i]\), respectively.

In particular, after the measurement on the \((k-1)\)-th copy, the provisional correct detection probability coincides with the Helstrom bound on \(k-1\) copies, namely,

\[
P_c^{(k-1)} = P[z_{k-1} = a] = \frac{1}{2} \left[ 1 + \sqrt{1 - 4q_0q_1\chi^{2(k-1)}} \right].
\]

The next measurement angle \(\phi_k\) for the \(k\)-th copy is chosen to maximize the probability of correct detection under the assumption that the a priori probabilities \(q_0\) and \(q_1\) are replaced by the corresponding a posteriori probabilities of the input simbol, given the last result \(z_{k-1}\). These turn out to be \(P[a=i|z_{k-1}=i]\) \(= P_c^{(k-1)}\), \(i = 0, 1\). Finally, the measurement angles are given by

\[
\phi_k = \frac{1}{2} \arctan \frac{1}{\sqrt{1 - 4q_0q_1\chi^{2(k-1)}}} \tan 2\theta\]

\[
k = 1, \ldots, n
\]
if $z_{k-1} = 0$ and $\pi/2 - \phi_k$ if $z_{k-1} = 1$. A simple proof of these results is given in the Appendix.\(^1\)

The optimum local adaptive measurement can be summarized by the following step-by-step procedure.

1. From the overlap coefficient $\chi$ and the input probabilities $q_0$ and $q_1$ compute the two sequences of measurement angles

\[
\begin{align*}
\phi_1 & = \frac{\phi}{2} - \phi_1, \\
\phi_2 & = \frac{\phi}{2} - \phi_2, \\
& \ldots \\
\phi_n & = \frac{\phi}{2} - \phi_n
\end{align*}
\] (11)

2. Start with angle $\phi_1$ if $q_0 \geq 1/2$ (and $\pi/2 - \phi_1$ otherwise).

3. Use the angles of the first sequence (11) as long as the measurement result is 0.

4. Change angle sequence every time the result changes and accept $z_n$ as the global result.

We will show in the sequel that this paradigm is mimicked in a continuous time version by the Dolinar receiver.

As a further comment, we note that the multiple-copy optimization requires the discrimination between two hypotheses in the $2^n$-dimensional Hilbert space $\mathcal{H}^{\otimes n}$, so that the optimal solution is not uniquely defined. On the contrary, the Helstrom solution discriminates between the hypotheses in the restricted subspace $\mathcal{H}_0$ spanned by $| \alpha_0 \rangle$ and $| \alpha_1 \rangle$. Of course, each optimal measurement in $\mathcal{H}^{\otimes n}$, once projected in $\mathcal{H}_0$, returns the Helstrom projectors. In particular, in the above procedure, to each sequence $z_1, \ldots, z_n$ of results it corresponds a measurement vector $| \mu_1 \rangle \otimes \ldots \otimes | \mu_n \rangle$ in $\mathcal{H}^{\otimes n}$ with measurement angles chosen in the sequences (11). It can be easily verified that the $2^n$ measurement vectors in $\mathcal{H}^{\otimes n}$ are orthonormal and they globally give a von Neumann projective measure.

### III. COHERENT SINGLE-COPY MEASUREMENT

Now we consider the possibility of applying the multiple-copy adaptive approach to the discrimination between two coherent states segmented like in (2). Let us suppose that Alice prepares a single copy of binary coherent states. Without loss of generality, we can assume $| \alpha_0 \rangle = | \gamma \rangle$ and $| \alpha_1 \rangle = | - \gamma \rangle$, with $\gamma$ real, as in the Binary Phase Shift Keying (BPSK) modulation scheme, with overlap coefficient

\[
X = |\langle \alpha_0 | \alpha_1 \rangle| = e^{-2\gamma^2},
\] (12)

where $\gamma^2$ represents the average number of photons in each state. A straightforward application of the Helstrom theory leads to the Helstrom bound

\[
P_c = \frac{1}{2} \left[ 1 + \sqrt{1 - 4q_0 q_1 e^{-4\gamma^2}} \right],
\] (13)

but the corresponding measurement vectors are difficult to implement. Then, several near-optimal, but simpler to implement, detection schemes had been devised in the past.

For the photon counting detection corresponds to measuring $| \alpha_1 \rangle \otimes | \alpha_1 \rangle$ and tests the resulting state with a photon counter. The photon counting detection corresponds to measurement projectors $\Pi_0 = |0\rangle \langle 0 |$ and $\Pi_1 = I - \Pi_0$. Decision $a = 1$ is accepted if the photon counter clicks, otherwise $a = 0$ is chosen. The correct detection probability reads

\[
P_K = q_0 (| \alpha'_0 \rangle \langle \alpha'_0 | + | \alpha'_1 \rangle \langle \alpha'_1 |) = q_0 + q_1 (1 - e^{-4\gamma^2}).
\] (14)

An improved version (11) of the Kennedy receiver employs a displacement $D(-\beta)$ to be optimized, so that the correct detection probability becomes

\[
P_{IK} = q_0 (| \gamma - \beta | \gamma - \beta | + q_1 (| \gamma + \beta | \gamma + \beta |) = q_0 e^{-(\gamma - \beta)^2} + q_1 \left( 1 - e^{-(\gamma + \beta)^2} \right).
\] (15)

By mulling the derivative with respect to $\beta$, we find that the displacement quantity $\beta$ maximizing (15) satisfies the transcendental equation

\[
\frac{q_0}{q_1} = \frac{\beta}{\beta - \gamma} e^{-4\beta \gamma},
\] (16)

that can be numerically solved, and the corresponding value of $P_{IK}$ evaluated.

In Fig. 1 the performance of the Kennedy receiver and of the improved Kennedy receiver are compared with the Helstrom bound. (The figure also includes the simplified Dolinar receiver that will be introduced in Section IV). For large values of $\gamma$ the improvement obtained by optimizing the displacement $\beta$ is negligible. On the other hand, as $\gamma$ goes to 0, both the Helstrom bound and $P_{IK}$ approach similar values. Indeed, as shown in the figure, the performance of the improved Kennedy receiver strictly approximates the Helstrom bound for weak coherent states.

### IV. A SIMPLE APPROACH TO THE DOLINAR RECEIVER

The above considerations suggest that, for $n$ large enough, such that copies of weak enough coherent states are obtained, the optimum measurements on the segmented states (2) can be well approximated by suitable
displacements and photon counting. In other words, the sequences of measurement angles may be interpreted as sequences of displacements. Then, the transition to the continuous time–scheme depicted in Fig. 2 appears to be natural.

The input field \( \psi(t) \), \( 0 < t < T \), corresponding to the coherent state \( |\pm \gamma \rangle \), is represented by

\[
\psi(t) = \pm \psi e^{i 2\pi f_0 t},
\]
where \( f_0 \) is the optical frequency and \( T \) is the pulse duration. The mean number of photons arriving to the detector is given by

\[
\gamma^2 = \int_0^T |\psi(t)|^2 dt = \psi^2 T.
\]

The detector subtracts from the input field a time–varying field generated by a local laser with envelope chosen between either \( u_0(t) \) or \( u_1(t) \), accordingly to the value of \( z(t) \), a binary signal with possible values 0 and 1, giving the provisional decision at time \( t \). By mimicking the behavior of the optimal multiple–copy detection, we assume that the decision signal \( z(t) \) changes at any photon arrival at the counter. Then, the optical signal at the photon–counter has envelope either \( \pm \psi - u_0(t) \) or \( \pm \psi - u_1(t) \), depending on the value of \( z(t) \). Moreover \( z(T) \) is assumed to be the final decision.

The mathematical problem is to choose the functions \( u_0(t) \) and \( u_1(t) \) that maximize the correct detection probability

\[
P_e = P \left[z(T) = a \right].
\]

Figure 2: Block diagram of the Dolinar receiver: The received signal with envelope \( \pm \psi \) is displaced by a quantity \(-u_z(t)\), where \( z(t) \in \{0, 1\} \) is the temporary estimation at time \( t \). The value of \( z(t) \) alternately changes from 0 to 1 at each single photon detection of the displaced signal \( \pm \psi - u_z(t) \). The shape of the feedback signal \( u_z(t) \) is also changed accordingly.

Let us assume that \( a = 0 \), so that \( \psi(t) = \psi e^{i 2\pi f_0 t} \). Then, the process \( z(t) \) can be interpreted as a telegraph process \( [\xi] \) alternately driven by non–homogeneous Poisson processes with rates

\[
\lambda(t) = |\psi - u_0(t)|^2 \quad \text{and} \quad \mu(t) = |\psi - u_1(t)|^2.
\]

A simple application of the properties of Poisson processes gives the evolution of the conditional correct detection probability \( p_0(t) = P( |z(t) = 0| a = 0) \). In fact, let \( N(t, t + \Delta t) \) denote the number of photon arrivals at the counter in the interval \( (t, t + \Delta t) \), therefore

\[
p_0(t + \Delta t) = P( z(t) = 0, N(t, t + \Delta t) = 0 | a = 0) + p_0(t) + P( N(t, t + \Delta t) = 1 | z(t) = 0) [1 - p_0(t)] + o(\Delta t).
\]

Hence, the differential equation

\[
p_0'(t) = \frac{\delta p_0(t)}{\delta t} = \mu(t) - [\lambda(t) + \mu(t)] p_0(t)
\]

follows. In a similar way for \( p_1(t) = P( |z(t) = 1| a = 1) \) we get

\[
p_1'(t) = \frac{\delta p_1(t)}{\delta t} = \tilde{\mu}(t) - [\tilde{\lambda}(t) + \tilde{\mu}(t)] p_1(t)
\]

with

\[
\tilde{\lambda}(t) = | - \psi - u_1(t)|^2 \quad \text{and} \quad \tilde{\mu}(t) = | - \psi - u_0(t)|^2.
\]

If our search is confined to symmetric solutions, namely, \( u_1(t) = -u_0(t) \) we get \( \tilde{\lambda}(t) = \lambda(t) \) and \( \tilde{\mu}(t) = \mu(t) \), and the correct detection probability satisfies the differential equation

\[
P_e'(t) = \frac{\delta P_e(t)}{\delta t} = q_0 p_0'(t) + q_1 p_1'(t) = \lambda(t) - [\lambda(t) + \mu(t)] P_e(t).
\]

On the basis of the results on multiple–copy measurements we expect that, for some choice \( u(t) \) of the envelope of the feedback signal \( u_0(t) \), the provisional correct
detection probability $P_c(t)$ is exactly equal to the Helstrom bound applied to the interval $(0,t)$, namely,

$$P_c(t) = \frac{1}{2} \left[ 1 + \sqrt{1 - 4q_0q_1e^{-4\psi^2t}} \right].$$

(26)

By substituting the above expression in (25), and defined $R(t) = \sqrt{1 - 4q_0q_1e^{-4\psi^2t}}$, we get

$$\psi^2 \frac{1 - R^2(t)}{R(t)} = \psi^2 + u^2(t) + 2\psi u(t) - \left[ \psi^2 + u^2(t) \right] (1 + R(t))$$

(27)

and after some algebra

$$u(t) = \frac{\psi}{R(t)} = \frac{\psi}{\sqrt{1 - 4q_0q_1e^{-4\psi^2t}}}$$

(28)

coinciding indeed with the Dolinar’s solution (see also [13]).

V. A SUBOPTIMAL RECEIVER

The Dolinar receiver requires a time-varying feedback signal $u_0(t)$, whereas both the Kennedy receiver [11] and its improved version [10] make use of a constant fixed displacement $\beta$ leading to a much simpler implementation. Therefore, it is worthwhile to consider a simplified version of the Dolinar receiver where the feedback signal is constrained to have a constant fixed envelope $u_0(t) = \beta$. Such a setting would mean that only phase modulation, and specifically phase inversion, is required, whereas the optimal receiver has to also employ an amplitude modulator capable of generating an optical signal with shape defined by (23), which decays with $t$ but it is divergent about $t = 0$ for the important case of equiprobable states $q_0 = q_1 = 1/2$.

Hence, by substitution in (24), i.e., setting $\lambda(t) = \lambda = |\psi - \beta|^2$ and $\mu(t) = \mu = |\psi + \beta|^2$, with the initial condition $P_c(0) = q_0$, we get the following final probability of correct decision

$$P_c(T) = \frac{1}{2} + \frac{\psi \beta}{\psi^2 + \beta^2} + \left[ q_0 - \frac{1}{2} - \frac{\psi \beta}{\psi^2 + \beta^2} \right] e^{-2(\psi^2 + \beta^2)T}.$$  

(29)

The optimized value of $\beta$ can be found by numerically solving the following transcendental equation, which is obtained by nulling the derivative of $P_c(T)$ made with respect to $\beta$

$$\beta T (\psi^2 + \beta^2) [2q_0 - 1] (\psi^2 + \beta^2) - 2\psi \beta ] e^{-(\psi^2 + \beta^2)T} = \psi (\psi^2 - \beta^2) \sinh((\psi^2 + \beta^2)T).$$

(30)

From Figure 1 we note that the simplified Dolinar receiver slightly outperforms the improved Kennedy receiver. Therefore, its experimental demonstration, and

a study of its robustness to the presence of possible impairments, is an interesting task for future contributions.

In Figure 3 the intensity $|\beta|^2$ of the fixed displacement $D(-\beta)$ for the considered receivers; setting $q_0 = q_1 = 1/2$, $T = 1$.

VI. CONCLUSIONS

A whole single coherent quantum state can be interpreted as a succession of many (possibly infinite) weaker copies of the same state, so that a different interpretation of the quantum discrimination task can be given. Such a view may provide further insights to well-established problems and give rise to novel detection solutions.

We have proposed an analysis of the behavior of the Dolinar receiver based on recent findings in the field of multiple-copy state discrimination. With such an approach it has been possible to provide a quite intuitive explanation, and simple mathematical derivation, of the Dolinar receiver. We also proposed a suboptimal simplified detection scheme that employs a photon counter, a phase inverter and, contrary to the Dolinar’s solution, a constant-envelope displacement.

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Appendix: Optimality of the multiple-copy adaptive measurement

The proof is done by induction. Using measurement vectors (7) with angle $\phi_1$, the correct detection probability after the first measurement turns out to be

$$P_c^{(1)} = q_0 |\langle \mu_{10} | \gamma_0 \rangle|^2 + q_1 |\langle \mu_{11} | \gamma_1 \rangle|^2$$

and, as it can be easily verified [6], it is maximized by $\phi_1$ given by (10) for $k = 1$, with provisional correct detection probability given by

$$P_c^{(1)} = \frac{1}{2} \left[ 1 + \sqrt{1 - 4q_0q_1\chi^2} \right].$$  \hspace{1cm} (A.1)

Then, the result is proven for $k = 1$. In particular, simple computations give the a posteriori probabilities

$$P \left[ a = i \mid z_1 = i \right] = P_c^{(1)}.$$  \hspace{1cm} (A.2)

Now, suppose that the result holds true for $k − 1$. From the inductive hypothesis, the provisional correct detection coincides with the Helstrom bound, the adaptive measurement up to the $(k − 1)$-th copy coincides with the optimal global measurement and the a posteriori probabilities are

$$P \left[ a = i \mid z_{k-1} = i \right] = P_c^{(k-1)}.$$  \hspace{1cm} (A.3)

If these probabilities replace $q_0$ and $q_1$ in the expression (10) of the angle $\phi_k$ and in (A.2), one gets

$$P_c^{(k)} = \frac{1}{2} \left[ 1 + \sqrt{1 - 4P_c^{(k-1)} \left[ 1 - P_c^{(k-1)} \right] \chi^2} \right]$$

and the proof is complete.

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