Spectral asymptotics of Kreĭn-Feller operators for weak Gibbs measures on self-conformal fractals with overlaps

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Abstract

We study the spectral dimensions and spectral asymptotics of Kreĭn-Feller operators for weak Gibbs measures on self-conformal fractals with or without overlaps. We show that, restricted to the unit interval, the $L^q$-spectrum for every weak Gibbs measure $\varrho$ with respect to a $C^1$-IFS exists as a limit. Building on recent results of the authors, we can deduce that the spectral dimension with respect to a weak Gibbs measure exists and equals the fixed point of its $L^q$-spectrum. For an IFS satisfying the open set condition, it turns out that the spectral dimension equals the unique zero of the associated pressure function. Moreover, for a Gibbs measure with respect to a $C^{1+\gamma}$-IFS under OSC, we are able to determine the asymptotics of the eigenvalue counting function.

Keywords: Kreĭn-Feller operator, spectral dimension, $L^q$-spectrum, spectral asymptotics, (weak) Gibbs measure

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1. Introduction and statement of main results

We investigate the spectral properties under Dirichlet boundary conditions of the classical Krein–Feller operator $\Delta_\varrho$ for weak Gibbs measures $\varrho$ with respect to a conformal iterated function systems on $[0, 1]$ with or without overlaps (see Section 4).

Spectral properties of the operator $\Delta_\varrho$ have attracted much attention in the last century, beginning with Feller [9], Kac [15], Hong and Uno [39], McKean and Ray [28], Kotani and Watanabe [26], Fujita [13], Solomyak and Verbitsky [37] and more recently by Vladimirov and Shepaku [40], Faggionato [8], Arzt [1, 2], Ngai [30], Ngai, Tang and Xie [31, 32], Freiberg, Minorics [29, 12], and by the authors in [21, 22, 20].

In the framework of the weak approach starting from the Dirichlet form $E_\varrho$ as defined in Section 2 it is well known that there exists an orthonormal system of eigenfunctions of $\Delta_\varrho$ with non-negative eigenvalues $(\lambda_n^\varrho)_{n \in \mathbb{N}}$ in increasing order tending to $\infty$ whenever the support $\text{supp}(\varrho)$ of $\varrho$ is not a finite set. We denote the number of eigenvalues of $\Delta_\varrho$ not exceeding $x \geq 0$ by $N_\varrho(x)$ and refer to $N_\varrho$ as the eigenvalue counting function. We define the upper and lower exponent of divergence by

$$s_\varrho^\text{up} := \liminf_{x \to \infty} \frac{\log(N_\varrho(x))}{\log(x)} \quad \text{and} \quad s_\varrho^\text{low} := \limsup_{x \to \infty} \frac{\log(N_\varrho(x))}{\log(x)},$$

and refer to these numbers as the upper, resp. lower, spectral dimension. If the two values coincide we denote the common value by $s_\varrho$ and call it the spectral dimension $\Delta_\varrho$, or of $E_\varrho$, respectively. Note that, we always have $s_\varrho^\text{low} \leq 1/2$, which has been shown in [4, 5]. If $\varrho$ has a non-trivial absolutely continuous part $\sigma \Lambda|_{[0,1]}$ and a singular part $\eta$ with $\eta([0, 1]) = 0$, then we have

$$\lim_{x \to \infty} \frac{N_{\varrho+\sigma\Lambda|_{[0,1]}}(x)}{x^{1/2}} = \frac{1}{\pi} \int_{[0,1]} \sqrt{\sigma} \, d\Lambda,$$

where $\Lambda$ denotes the Lebesgue measure on $\mathbb{R}$ (see [28, 6]). In particular, the spectral dimension equals $s_{\varrho+\sigma\Lambda} = 1/2$. A first account for smooth densities and no singular part is contained in the famous work [42] of Weyl.

In the case of self-similar measures $\varrho$ under the open set condition (OSC) with contraction rates $r_1, \ldots, r_n \in (-1, 1)$ and probability weights $p_1, \ldots, p_n \in (0, 1), n \geq 2$, it has been shown in [39, 13, 36] that the spectral dimension $s_\varrho$ is given by the unique $q > 0$ such that

$$\sum_{i=1}^{n} (p_i |r_i|)^q = 1. \quad (1.1)$$

We will generalize this result in three ways.

- We provide a first contribution to the nonlinear setting in a broad sense. More specifically, we consider weak Gibbs measures on fractals which are generated by
non-trivial $C^1$ iterated function systems ($C^1$-IFS) under the OSC. It turns out that
the spectral dimension is given by the zero of the associated pressure function
(see (4.2)) which is a natural generalization of (1.1).

- As a second novelty, we drop the assumption of the OSC and allow overlaps. In
this situation the computation of the spectral dimension is much more involved
compared to (1.1). However, building on ideas developed in [21] combined with
results from [33, 10], we are able to specify the spectral dimension as the fixed
point of the associated $L^q$-spectrum as defined below in (1.2). Finally, using a
recent result of [3], for the self-similar case with possible overlaps and some
additional assumption, we can express the spectral dimension in terms of $\tau$ which
is implicitly given by $\sum_{i=1}^{n} p_i |r_i|^q = 1$.

- Our final contribution to the nonlinear setting concerns Gibbs measures on fractals
generated by $C^1$-IFS’s under the OSC. For this class, we are able to prove
spectral asymptotics using renewal theory developed for a dynamical context, as
in [25, 19].

Investigating the spectral dimension of $\Delta_\varrho$, the authors have recently shown in [21] that
the $L^q$-spectrum $\beta_{\varrho}^q$ of $\varrho$ carries the crucial information. For $q \geq 0$, it is given by

$$
\beta_{\varrho}^q (q) := \limsup_{n \to \infty} \beta_n^q (q) \quad \text{with} \quad \beta_n^q (q) := \frac{1}{\log 2^n} \log \sum_{C \in D_n} \varrho (C)^q , \hspace{1cm} (1.2)
$$

where $D_n := \{A_k^n : k \in \mathbb{Z}, \varrho \left( A_k^n \right) > 0 \}$ and $A_k^n := ((k-1) 2^{-n}, k 2^{-n}]$. Each $\beta_n^q$ defines a
non-increasing, differentiable and convex function with unique fixed point $q_n^\varrho \in (0, 1)$, i.e. $\beta_n^q (q_n^\varrho) = q_n^\varrho$. We have $\beta_n^1 (1) = \beta_{\varrho}^1 (1) = 0$, $n \in \mathbb{N}$, and $\beta_{\varrho}^1 (0)$ is equal to the
upper Minkowski dimension $\dim_M (\text{supp} (\varrho))$ of the support $\text{supp} (\varrho)$ of $\varrho$. The following quantity

$$
q_{\varrho} := \limsup_{n \to \infty} q_n^\varrho
$$

has been introduced by the authors in [21] and plays a central role for the spectral
problem. In fact, by extending and combining ideas from there and [3, 33], we can
prove that for weak Gibbs measures $\varrho$, as defined in Section 4, the spectral dimension
$s_{\varrho(\text{b} \text{a} \text{r} \text{y} \text{r} \text{e} \text{i} \text{a} \text{t} \text{i} \text{v} \text{a} \text{i} \text{c} \text{e} \text{n} \text{p} \text{o} \text{i} \text{t} \text{p} \text{o} \text{i} \text{n} \text{t} \text{a} \text{l})}$ always exists and is equal to $q_{\varrho}$, which generalizes previous results for linear IFS
under the OSC in [39, 37]. The restriction of $\varrho$ to the open unit interval guarantees
that there are no atoms at the boundary points, which on the one hand allows the weak
Dirichlet approach, while on the other hand the $L^q$-spectrum on $[0, 1]$ and the value of
$q_{\varrho}$ are not affected by this restriction.

**Theorem 1.1.** Let $\varrho$ be a weak Gibbs measure on $[0, 1]$ with respect to a non-trivial
$C^1$-IFS (with or without overlap). Then the spectral dimension $s_{\varrho(\text{b} \text{a} \text{r} \text{y} \text{r} \text{e} \text{i} \text{a} \text{t} \text{i} \text{v} \text{a} \text{i} \text{c} \text{e} \text{n} \text{p} \text{o} \text{i} \text{t} \text{p} \text{o} \text{i} \text{n} \text{t} \text{a} \text{l})}$ exists and equals
$q_{\varrho}$. If, additionally, the OSC is fulfilled, then $q_{\varrho}$ coincides with the unique zero $z_{\varrho}$ of the
pressure function as defined in (4.2).

**Corollary 1.2.** Let $\varrho$ be a weak Gibbs measure with respect to a $C^1$-IFS (with or without
overlap). If $q_{\varrho} < 1/2$, then $\varrho$ is singular with respect to $\Lambda$. 

As a by-product, using ideas of Riedi [35, 34], we can show that the Minkowski dimension of the self-conformal set generated by a $C^1$-IFS with overlaps always exists (Proposition 5.11)—a fact we could not find in the literature.

In the special case of dimensionally regular linear IFS (cf. Definition 1.3) we can apply a recent result by Barral and Feng [3] to compute the spectral dimension more explicitly. We consider an IFS given by contracting similarities $\Phi = (T_i : [0, 1] \rightarrow [0, 1] : i = 1, \ldots, n)$ where $T_i(x) = r_i x + b_i$, $x \in \mathbb{R}$ with $b_i \in \mathbb{R}$, $|r_i| < 1$, $i = 1, \ldots, n$. For a given probability vector $(p_1, \ldots, p_n) \in (0, 1)^n$, we call the unique Borel probability measure $\varrho$ satisfying

$$\varrho(A) = \sum_{i=1}^{n} p_i \cdot \varrho \circ T_i^{-1}(A), \ A \in \mathcal{B}([0, 1])$$

(1.3)

the self-similar measure of $\Phi$ with probability vector $(p_1, \ldots, p_n)$, where $\mathcal{B}([0, 1])$ denotes the Borel $\sigma$-algebra of $[0, 1]$. For every $q \in \mathbb{R}$ let $\tau(q)$ be the unique solution of

$$\sum_{i=1}^{n} p_i^q |r_i|^\tau(q) = 1,$$

(1.4)

which defines an analytic function $q \mapsto \tau(q)$. With the help of $\tau$, the similarity dimension of its attractor $\text{supp} \varrho$ is set to be

$$\dim_s(\text{supp} \varrho) := \tau(0)$$

and we define the similarity dimension of the measure $\varrho$ to be

$$\dim_s(\varrho) := -\tau'(1) = \frac{\sum_{i=1}^{n} \log(p_i) p_i}{\sum_{i=1}^{n} \log(|r_i|) p_i}.$$  

Definition 1.3. An IFS $\Phi := (\varphi_i)_{i=1,\ldots,n}$ of similarities on $\mathbb{R}$ with contraction rates $(r_1, \ldots, r_n)$ is said to be dimensionally regular, if every self-similar measure $\varrho$ of $\Phi$ with probability vector $(p_1, \ldots, p_n) \in (0, 1)^n$ has Hausdorff dimension

$$\dim_H(\varrho) = \min\{1, \dim_s(\varrho)\}.$$  

Remark 1.4. From Hochman [14, Theorem 1.1], it follows that if the similarities $(T_i)_{i=1,\ldots,n}$ satisfy the exponential separation condition (ESC) (see e.g. [3, Definition 2.2]), then $\varrho$ is dimensionally regular.

Theorem 1.5. Assume that the IFS $\Phi := (\varphi_i)_{i=1,\ldots,n}$ of similarities is dimensionally regular and let $\varrho$ be the self-similar measure of $\Phi$ with probability vector $(p_1, \ldots, p_n) \in (0, 1)^n$. With $\zeta$ uniquely determined by $\tau(\zeta) = \zeta$ and

$${\bar{q}} := \inf \{q > 0 : -\tau'(q) q + \tau(q) \leq 1 \} \cup \{1\},$$

the spectral dimension of $\Delta_\varrho$ is given by

$$s_\varrho = \begin{cases} 
\zeta & \text{if } \zeta \geq \bar{q}, \\
\bar{q}/(1 - \tau(\bar{q}) + \bar{q}) & \text{if } \zeta < \bar{q}.
\end{cases}$$
Remark 1.6. We remark in the case $\dim S(\varrho) \geq 1$ we have $s_\varrho = 1/2$, and if $\dim S(\varrho) < 1$ and $\dim S(\text{supp } \varrho) = \tau(0) \leq 1$ then $s_\varrho = \zeta$. Only in the remaining case $s_\varrho$ depends on $\tilde{q}$; such a case is illustrated in Fig. 2.1 on page 5.

Finally, for the more restricted class of $\psi$-Gibbs measures with Hölder continuous potential $\psi$ and with respect to a $C^{1+\gamma}$-IFS under the OSC, which includes self-similar measures under the OSC as a special case, we can show that the eigenvalue counting function obeys a power law with exponent $z_\varrho$. For this we use the following notation: For $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we write $f \ll g$ if there exists a positive constant $c$ such that $f(x) \leq cg(x)$ for all $x$ large and we write $f \asymp g$, if $f \ll g$ and $g \ll f$.

**Theorem 1.7** (Spectral asymptotics). If $\varrho$ is a $\psi$-Gibbs measure for some Hölder continuous potential $\psi$ and with respect to a $C^{1+\gamma}$-IFS satisfying the OSC, then

\[ N_\varrho(x) \asymp x^{z_\varrho}, \]

where $z_\varrho$ is the unique zero of the pressure function as defined in (4.2).

2. Dirichlet forms for generalized Kreǐn–Feller operators

![Graph](image)

Figure 2.1: The graph of $\beta_\varrho$ (solid line) for a dimensionally regular IFS with four contraction ratios equal to 1/2 and an associated self-similar measure $\varrho$ with probability vector (0.001, 0.001, 0.05, 0.948). The graph of $\beta_\varrho$ coincides on $[\tilde{q}, 1]$ with $\tau$ (dotted line) as defined in (1.4) and we have $\tau(0) = 2$. The linear part of $\beta_\varrho$ is determined by the tangent to the graph of $\tau$ over the positive $x$-axis through the point (0, 1). The intersection with the dashed line with slope 1 gives the value for the spectral dimension $s_\varrho$.

In this section we will define the classical and generalized Kreĩn–Feller operator. The spectral properties for the generalized case were studied in [27], [41] and [11]. The connection between the generalized and the classical Kreǐn–Feller operator has been elaborated in [23]. In there, it has been shown that the spectral behavior can be reduced to the classical Kreǐn–Feller operator by a straightforward transformation of measure spaces. In the context of this paper the generalized Kreǐn–Feller operator will be an
important tool in the proofs of our main results (see e.g. Lemma 5.1). Therefore, in this section we present a short proof of this fact, which only refers to the Dirichlet form approach.

Our framework closely follows [16, 8, 21]. Throughout this section, let $\mu$ and $\varrho$ be finite Borel measures on $[a, b]$ such that $\text{supp}(\mu) = [a, b]$, $\mu$ is atomless and $\varrho([a, b]) = 0$. Let $L^2_\varrho = L^2_\varrho([a, b])$ denote the Hilbert space of square integrable functions with respect to $\varrho$, $F_\mu$ the (strictly increasing and continuous) distribution function of $\mu$ and set

\[
H^1_\varrho([a, b]) := \left\{ f : [a, b] \to \mathbb{R} : \exists \nabla_\mu f \in L^2_\mu : f(x) = f(a) + \int_{[a,e]} \nabla_\mu f \, d\mu, \ x \in [a, b] \right\},
\]

\[
H^0_\mu([a, b]) := \left\{ f \in H^1_\mu([a, b]) : f(a) = f(b) = 0 \right\}
\]

as well as

\[
C_{\varrho,\mu}([a, b]) := \left\{ f \in C([a, b]) : f \text{ is aff. lin. in } F_\mu \text{ on each comp. of } [a, b] \setminus \text{supp}(g) \right\}.
\]

We say that $f$ is affine in the linear $F_\mu$ on the interval $I$ if, restricted to $I$, it can be written as $x \mapsto a + b F_\mu(x)$ for some $a, b \in \mathbb{R}$. Note that $\nabla_\mu f$ is unique as an element of $L^2_\mu$ (see [1, Proposition 2.1.3] for a detailed proof). In the case of the Lebesgue measure $\mu = \Lambda$ we write $H^1_0([a, b]) := H^1_{0,\mu}([a, b])$ and $C_{\varrho,\mu}([a, b]) := C_{\varrho,\Lambda}([a, b])$.

Restricted to the (Dirichlet) domain $\text{dom}(\mathcal{E}_{\varrho,\mu}) \times \text{dom}(\mathcal{E}_{\varrho,\mu})$ with $\text{dom}(\mathcal{E}_{\varrho,\mu}) := H^1_{0,\mu}([a, b]) \cap C_{\varrho,\mu}([a, b])$ we define the form

\[
\mathcal{E}_{\varrho,\mu}(f, g) := \mathcal{E}_{\varrho,\mu,[a,b]}(f, g) := \int_{(a,b)} \nabla_\mu f \nabla_\mu g \, d\mu, \ f, g \in \text{dom}(\mathcal{E}_{\varrho,\mu}).
\]

Again, for the Lebesgue case, we write $\mathcal{E}_{\varrho}(f, g) := \mathcal{E}_{\varrho,[a,b]}(f, g) := \mathcal{E}_{\varrho,\Lambda,[a,b]}(f, g)$. We define the linear map

\[
t_\mu : \mathbb{R}[F_\mu(a), F_\mu(b)] \to \mathbb{R}[a,b], \ f \mapsto f \circ F_\mu,
\]

which is injective as a consequence of $F_\mu \circ F_\mu^{-1} = \text{id}_{[F_\mu(a), F_\mu(b)]}$. The inverse on its image

\[
t_\mu^{-1} : t_\mu \left( \mathbb{R}[F_\mu(a), F_\mu(b)] \right) \to \mathbb{R}[F_\mu(a), F_\mu(b)], \ f \mapsto f \circ F_\mu^{-1}
\]

is therefore bijective and linear.

**Lemma 2.1.** For linear subspaces $A_i \subset \mathbb{R}[F_\mu(a), F_\mu(b)]$, $i = 1, 2, 3$, with inner product $(\cdot, \cdot)_{A_i}$ we consider the restrictions

\[
t_\mu^{-1} : t_\mu A_i \to A_i
\]

and equip $t_\mu A_i$ with the pull-back inner product $(f, g)_{\mu,A_i} := (t_\mu^{-1}f, t_\mu^{-1}g)_{A_i}$. This gives rise to the following list

| $i$ | $A_i$ | $(\cdot, \cdot)_{A_i}$ | $t_\mu A_i$ | $(\cdot, \cdot)_{\mu,A_i}$ |
|-----|-------|----------------------|------------|-------------------------|
| 1   | $L^2_{\varrho,F_\mu^{-1}}([F_\mu(a), F_\mu(b)])$ | $(\cdot, \cdot)_{\varrho,F_\mu^{-1}}$ | $L^2_{\varrho}([a,b])$ | $(\cdot, \cdot)_{\varrho}$ |
| 2   | $H^0_\mu([F_\mu(a), F_\mu(b)])$ | $\mathcal{E}_{\varrho,F_\mu^{-1},[F_\mu(a), F_\mu(b)]}$ | $H^0_\mu([a,b])$ | $\mathcal{E}_{\varrho,\mu,[a,b]}$ |
| 3   | $\text{dom}(\mathcal{E}_{\varrho,F_\mu^{-1},[F_\mu(a), F_\mu(b)]})$ | $\mathcal{E}_{\varrho,F_\mu^{-1},[F_\mu(a), F_\mu(b)]}$ | $\text{dom}(\mathcal{E}_{\varrho,\mu,[a,b]})$ | $\mathcal{E}_{\varrho,\mu,[a,b]}$ |
and in all cases \( \psi^{-1}_i \) defines an isometric isomorphism of Hilbert spaces. Moreover, for \( f \in H^1_0\left( [F_{\mu}(a), F_{\mu}(b)] \right) \),

\[
\nabla_{\mu} \left( \psi f \right) = \psi \left( \nabla_{N_{[F_{\mu}(a), F_{\mu}(b)]}} f \right)
\]

and for \( f \in H^1_0([a, b]) \),

\[
\nabla_{N_{[F_{\mu}(a), F_{\mu}(b)]}} \left( \psi^{-1}_i f \right) = \psi^{-1}_i \left( \nabla_{\mu} f \right).
\]

**Proof.** First we show that \( \psi_i A_i \) is equal to the claimed spaces. The case \( i = 1 \) is clear. In treating the cases \( i = 2, 3 \) we will also show the two identities regarding the derivatives. Indeed, for \( f \in H^1_0\left( [F_{\mu}(a), F_{\mu}(b)] \right) \) and all \( x \in [a, b] \), we have

\[
f \left( F_{\mu}(x) \right) = f \left( F_{\mu}(a) \right) + \int_{[F_{\mu}(a), F_{\mu}(x)]} \nabla_{N_{[F_{\mu}(a), F_{\mu}(x)]}} f \, d\lambda
\]

\[
= f \left( F_{\mu}(a) \right) + \int_{[F_{\mu}(a), F_{\mu}(x)]} \nabla_{N_{[F_{\mu}(a), F_{\mu}(x)]}} f \, d\mu \circ F_{\mu}^{-1}
\]

\[
= f \left( F_{\mu}(a) \right) + \int_{[a, x]} \left( \nabla_{N_{[F_{\mu}(a), F_{\mu}(x)]}} f \right) \circ F_{\mu} \, d\mu.
\]

Hence, \( f \circ F_{\mu} \in H^1_0([a, b]) \) and \( \nabla_{\mu} \left( f \circ F_{\mu} \right) = \left( \nabla_{N_{[F_{\mu}(a), F_{\mu}(x)]}} f \right) \circ F_{\mu} \). Using the fact that \( f \in \text{dom} \left( E_{\psi \circ F_{\mu}^{-1}, [F_{\mu}(a), F_{\mu}(b)]} \right) \) is affine linear on the connected components of \( [F_{\mu}(a), F_{\mu}(b)] \setminus \text{supp} \left( \rho \circ F_{\mu}^{-1} \right) \), we deduce that \( f \circ F_{\mu} \) is affine linear in \( F_{\mu} \) on the components of \( [a, b] \setminus \text{supp}(\rho) \). Consequently, we have \( f \circ F_{\mu} \in \text{dom} \left( E_{\psi \circ F_{\mu}^{-1}, [a, b]} \right) \). To see the reverse inclusion, note that for \( f \in H^1_0([a, b]) \) and \( x \in [F_{\mu}(a), F_{\mu}(b)] \),

\[
f \circ F_{\mu}^{-1}(x) = f(a) + \int_{[a, F_{\mu}^{-1}(x)]} \nabla_{\mu} f \, d\mu
\]

\[
= f \circ F_{\mu}^{-1}(a) + \int_{[a, F_{\mu}^{-1}(x)]} \left( \nabla_{\mu} f \right) \circ F_{\mu}^{-1} \circ F_{\mu} \, d\mu
\]

\[
= f \circ F_{\mu}^{-1}(0) + \int_{[0, x]} \left( \nabla_{\mu} f \right) \circ F_{\mu}^{-1} \, d\lambda,
\]

where we used \( F_{\mu}^{-1} \circ F_{\mu} = \text{id}_{[a, b]} \). Since \( a, b \in \text{supp}(\mu) \), it follows \( f \circ F_{\mu}^{-1} \in H^1_0\left( [F_{\mu}(a), F_{\mu}(b)] \right) \) and \( \nabla_{N_{[F_{\mu}(a), F_{\mu}(x)]}} \left( f \circ F_{\mu}^{-1} \right) = \left( \nabla_{\mu} f \right) \circ F_{\mu}^{-1} \). As above using the fact that \( f \in \text{dom} \left( E_{\psi \circ F_{\mu}^{-1}, [a, b]} \right) \) is affine linear in \( F_{\mu} \) on the connected components of \( [a, b] \setminus \text{supp}(\rho) \), we deduce that \( f \circ F_{\mu}^{-1} \) is affine linear on the components of \( [F_{\mu}(a), F_{\mu}(b)] \setminus \text{supp}(\rho \circ F_{\mu}^{-1}) \). Consequently, we have \( f \circ F_{\mu}^{-1} \in \text{dom} \left( E_{\psi \circ F_{\mu}^{-1}, [F_{\mu}(a), F_{\mu}(b)]} \right) \).

To see that the pull-back inner products are as claimed, we note that the case \( i = 1 \) is again obvious. For \( i = 2, 3 \), we obtain by the above identities for the derivatives that
we find that

Remark 2.3

If the measure $\mu$ is bijective and $A_i$ is a Hilbert space with respect to $(\cdot, \cdot)_i$ for each for $i = 1, 2, 3$, we find that $t_\mu^1$ restricted to $t_\mu A_i$ defines an isometric isomorphism of Hilbert spaces in all three cases. □

Proposition 2.2. The set dom$(E_{\varrho,\mu})$ is dense in $L_2^\varrho$ and equipped with the inner product

$$(f, g)_{\varrho,\mu} := (f, g)_\varrho + E_{\varrho,\mu}(f, g),$$

defines a Hilbert space, i.e. $E_{\varrho,\mu}$ is closed with respect to $L_2^\varrho$.

Proof. For the classical case with $\mu = \Lambda$ this follows from [21, Proposition 2.2, Proposition 2.3]. The general case follows from Lemma 2.1 and the fact that for all $f \in \text{dom}(E_{\varrho,\mu})$ we have $(f, g)_\varrho \leq \sqrt{\mu([a, b])} g([a, b]) E_{\varrho,\mu}(f, g)$. □

Remark 2.3. Since

$$(f, g)_\varrho \leq \sqrt{\mu([a, b])} g([a, b]) E_{\varrho,\mu}(f, g)$$

both bilinear forms $(\cdot, \cdot)_{\varrho,\mu}$ and $E_{\varrho,\mu}(\cdot, \cdot)$ give rise to equivalent induced norms.

Using Proposition 2.2, we can define a non-negative, self-adjoint, unbounded operator. Namely, we say $f \in \text{dom}(E_{\varrho,\mu})$ lies in the domain $\mathcal{D}(\Delta_{\varrho,\mu, [a, b]})$ of the generalized Krein–Feller operator $\Delta_{\varrho,\mu, [a, b]} = \Delta_{\varrho,\mu}$ if and only if $g \mapsto E_{\varrho,\mu}(g, f)$ extends continuously to a linear form on $L_2^\varrho$ and then $\Delta_{\varrho,\mu} f$ is uniquely determined by the identity

$$E_{\varrho,\mu}(g, f) = \langle g, \Delta_{\varrho,\mu} f \rangle_{\varrho,\mu} \quad \text{for all } g \in \text{dom}(E_{\varrho,\mu}).$$

If the measure $\mu$ is equal to the Lebesgue measure restricted to $[a, b]$, we call the associated Laplacian $\Delta_\varrho := \Delta_{\varrho, [a, b]} := \Delta_{\varrho,\Lambda, [a, b]}$ the classical Krein–Feller operator.

An element $f \in \text{dom}(E_{\varrho,\mu}) \setminus \{0\}$ is called eigenfunction for $E_{\varrho,\mu}$ with eigenvalue $\lambda$ if for all $g \in \text{dom}(E_{\varrho,\mu})$, we have

$$E_{\varrho,\mu}(f, g) = \lambda \cdot (f, g)_{\varrho,\mu}.$$
Proof. If \( f \) be an eigenfunction of \( \Delta_{\varrho,\mu} \) with eigenvalue \( \lambda \), then for all \( g \in \text{dom}(\mathcal{E}_{\varrho,\mu}) \), by Lemma 2.1 and \( F^{-1}_\mu \circ F_\mu = \text{id}_{[a,b]} \), we have
\[
\int_{[F_\mu(a),F_\mu(b)]} \nabla_\Lambda \left( f \circ F^{-1}_\mu \right) \nabla_\Lambda \left( g \circ F^{-1}_\mu \right) \ d\Lambda = \int_{[F_\mu(a),F_\mu(b)]} \nabla_\mu f \circ F^{-1}_\mu \cdot \nabla_\mu g \circ F^{-1}_\mu \ d\mu \circ F^{-1}_\mu
\]
\[
= \int_{[a,b]} \nabla_\mu f \nabla_\mu g \ d\mu = \lambda \int_{[a,b]} g f \ d\varrho = \lambda \int_{[a,b]} g f \ d\varrho = \lambda \int_{[F_\mu(a),F_\mu(b)]} g \circ F^{-1}_\mu \cdot f \circ F^{-1}_\mu \ d\mu \circ F^{-1}_\mu,
\]
which shows that \( f \circ F^{-1}_\mu \) is an eigenfunction of \( \Delta_{\varrho,\mu} [F_\mu(a),F_\mu(b)] \) with eigenvalue \( \lambda \). The reverse implication is similar. \( \square \)

Remark 2.5. Recall from [21] that the inclusion from the Hilbert space \( \left( \text{dom}(\mathcal{E}_{\varrho,\Lambda}), \mathcal{E}_{\varrho,\Lambda} \right) \) into \( L^2 \) is compact. Hence, we conclude that there exists an orthonormal system of eigenfunctions of \( \Delta_{\varrho,\mu} \) of \( L^2 \) with non-negative eigenvalues \( \left( \lambda_n^{\mu} \right)_{n \in \mathbb{N}} \) in increasing order tending to \( \infty \) given \( \text{supp}(\varrho) \) is not finite (see e.g. [38, Theorem 4.5.1 and p. 258]), we write \( \lambda_0^\mu := \lambda_0^\mu(\varrho,\mu) \). By Lemma 2.4 the same holds true for \( \Delta_{\varrho,\mu,\mu} \) with eigenvalues \( \left( \lambda_n^{\mu,\mu} \right)_{n \in \mathbb{N}} \).

We denote the number of eigenvalues of \( \mathcal{E}_{\varrho,\mu} \) not exceeding \( x \) by \( N_{\varrho,\mu,\mu}(a,b)(x) \) and refer to \( N_{\varrho,\mu,\mu}(a,b) \) as the eigenvalue counting function. In the case \( \mu = \mu \) we write \( N_{\varrho,\mu,\mu}(a,b) = N_{\varrho,\mu}(a,b) \). The remaining two observations in this section will play a central role in the proofs of our main results.

Lemma 2.6. For all \( i \in \mathbb{N} \), we have
\[
\lambda_i^{\mu} = \inf \left\{ \sup \left\{ \mathcal{E}_{\varrho,\mu}(\psi,\psi) \over (\psi,\psi)_\varrho : \psi \in G \setminus \{0\} \right\} : G \text{ i-dim. subspace of } \left( \text{dom}(\mathcal{E}_{\varrho,\mu}), \mathcal{E}_{\varrho,\mu} \right) \right\}.
\]
\[
= \inf \left\{ \sup \left\{ \mathcal{E}_{\varrho,\mu}(\psi,\psi) \over (\psi,\psi)_\mu : \psi \in G \setminus \{0\} \right\} : G \text{ i-dim. subspace of } \left( H^1_{\varrho,\mu}(a,b), \mathcal{E}_{\varrho,\mu} \right) \right\}.
\]

Proof. This follows from [21, Lemma 2.7] in tandem with Lemma 2.4 and Lemma 2.1. \( \square \)

Theorem 2.7. Let \( (a_i)_{i=0,...,n+1} \) be a subdivision vector of \( [a,b] \) such that
\[
a = a_0 < a_1 < \cdots < a_{n+1} = b
\]
and \( \varrho ((a_i)) = 0 \). Then, for all \( x \geq 0 \), we have
\[
\sum_{i=0}^n N_{\varrho,\mu,\mu}(a_i,a_{i+1})(x) \leq N_{\varrho,\mu}(x) \leq \sum_{i=0}^n N_{\varrho,\mu,\mu}(a_i,a_{i+1})(x) + n.
\]

Proof. This follows from [21, Proposition 2.16] and Lemma 2.4. \( \square \)
3. The $L^q$-spectrum

Let us recall some basic facts on the $L^q$-spectrum as presented in [21]. In this section, let $\varrho$ be any given finite Borel measures on $[0, 1]$ with $\text{card} (\text{supp} (\varrho)) = \infty$. Note that $\lambda$ is eigenvalue of $\Delta \varrho$ if, and only if $\lambda/\varrho([0, 1])$ is eigenvalue of $\Delta_{\varrho/\varrho([0, 1])}$. Hence, without loss of generality we assume that $\varrho$ is a probability measure. We begin this section with some additional properties of the $L^q$-spectrum of $\varrho$ as given in (1.2). The function $\beta_{\varrho}$ will not alter when we take $d$-adic intervals instead of dyadic ones. (see e.g. [35, Proposition 2 and Remarks, p. 466] or [34, Proposition 1.6]) and note that the definition in [34, Proposition 1.6] coincides with our definition for $q \geq 0$. More precisely, for fixed $\delta > 0$, let us define

$$G_{\delta} := \{(l\delta, (l + 1)\delta) : l \in \mathbb{Z}, \varrho((l\delta, (l + 1)\delta)) > 0\}$$

and let $(\delta_n) \in (0, 1)^{\mathbb{N}}$ be with $\delta_n \to 0$ an admissible sequence i.e. there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have $C\delta_n \leq \delta_{n+1} \leq \delta_n$. Then, for $q \geq 0$,

$$\limsup_{\delta \downarrow 0} \frac{1}{-\log(\delta)} \log \sum_{C \in G_{\delta}} \varrho(C)^q = \limsup_{m \to \infty} \frac{1}{-\log(\delta_m)} \log \sum_{C \in G_{\delta_m}} \varrho(C)^q.$$

In particular, for $\delta_m = 2^{-m}$ we obtain

$$\limsup_{\delta \downarrow 0} \frac{1}{-\log(\delta)} \log \sum_{C \in G_{\delta}} \varrho(C)^q = \beta_{\varrho}(q).$$

The function $\beta_{\varrho}$ is as a pointwise limit superior of convex function again convex and we have

$$\beta_{\varrho}(0) = \overline{\dim}_M (\text{supp} (\varrho)) \quad \text{and} \quad \beta_{\varrho}(1) = 0.$$

Hence, the Legendre transform is given by

$$\widehat{\beta}_{\varrho}(\alpha) := \liminf_q \beta_{\varrho}(q) + q \alpha.$$

The critical exponent $q_{\varrho}$ defined in the introduction can also be characterized as follows (see [21, Fact 4.8]).

$$q_{\varrho} = \inf \left\{ q > 0 : \limsup_{n \to \infty} \frac{1}{n} \log \sum_{C \in D_n} (\varrho(C) \Lambda_n(C))^q \leq 0 \right\}$$

$$= \inf \left\{ q > 0 : \sum_{C \in D} (\varrho(C) \Lambda(C))^q < \infty \right\} = \sup_{\alpha \geq 0} \frac{\widehat{\beta}_{\varrho}(\alpha)}{1 + \alpha}.$$

4. Iterated function systems and the thermodynamic formalism

In the following we consider the special case of weak $\varphi$-Gibbs measure with respect to not necessary linear iterated function systems. For fixed $n \in \mathbb{N}$ we call the family $\Phi := \{T_i : [0, 1] \to [0, 1] : i = 1, \ldots, n\}$ a $C^1$-iterated function systems ($C^1$-IFS) if its members are $C^1$-maps such that
1. we have uniform contraction, i.e. for all \( j \in I \) we have \( \sup_{x \in [0,1]} |T'_j(x)| < 1 \).

2. the derivatives \( T'_1, \ldots, T'_n \) are bounded away from zero, i.e. for all \( i \in I \) we have \( 0 < \inf_{x \in [0,1]} |T'_i(x)| \).

3. \( \Phi \) is non-trivial, i.e. there is more than one contraction and the \( T'_i \)'s do not share a common fixed point.

If additionally the \( T_1, \ldots, T_n \) are \( C^{1+\gamma} \)-maps with \( \gamma \in (0,1) \), we call the system a \( C^{1+\gamma} \) iterated function systems (\( C^{1+\gamma} \)-IFS). Here \( C^{1+\gamma} \) denotes the set of differentiable maps with \( \gamma \)-Hölder continuous derivative. We call the unique nonempty compact invariant set \( K \subset [0,1] \) of a \( C^{1} \)-IFS \( \Phi \) the self-conformal set associated to \( \Phi \).

Let \( I := \{1, \ldots, n\} \) denote the alphabet and \( I^m \) the set of words of length \( m \in \mathbb{N} \) over \( I \) and by \( I^* = \bigcup_{m \in \mathbb{N}} I^m \cup \{\varnothing\} \) we refer to the set of all words with finite length including the empty word \( \varnothing \). Furthermore, the set of words with infinite length will be denoted by \( I^\infty \) equipped with the metric \( d(x,y) := 2^{-\sup\{\|x\|,\|y\|\}} \) and let \( B(I^\infty) \) denote the Borel \( \sigma \)-algebra of \( I^\infty \). The length of a finite word \( \omega \in I^* \) will be denoted by \( |\omega| \) and for the concatenation of \( \omega, \omega' \) we define \( \omega \omega' \in I^\infty \). Note that \( B(I^\infty) \) is generated by the set of cylinders sets of arbitrary lengths. The set of \( \sigma \)-invariant probability measures on \( B(I^\infty) \) is denoted by \( M_\nu(I^\infty) \), where the measure \( \nu \) is called \( \sigma \)-invariant if \( \nu = \nu \circ \sigma^{-1} \). Further, for \( u = u_1 \cdots u_n \in I^n, n \in \mathbb{N} \), we set \( u^{-1} = u_n \cdots u_1 \). We say \( P \subset I^* \) is a partition of \( I^\infty \) if

\[
\bigcup_{\omega \in P}[\omega] = I^\infty \text{ and } [\omega] \cap [\omega'] = \varnothing, \text{ for all } \omega, \omega' \in P \text{ with } \omega \neq \omega'.
\]

Now, we are able to give a coding of the self-conformal set in terms of \( I^\infty \). For \( \omega \in I^* \) we put \( T_\omega := T_{\omega_1} \circ \cdots \circ T_{\omega_n} \) and define \( T_\emptyset := \text{id}_{[0,1]} \) to be the identity map on \([0,1]\). For \( \omega_1, \omega_2, \ldots, \omega_m \in I^\infty \) and \( m \in \mathbb{N} \) we define the initial word by \( \omega_m := \omega_1 \cdots \omega_m \). For every \( \omega \in I^\infty \) the intersection \( \bigcap_{m \in \mathbb{N}} T_{\omega_m}([0,1]) \) contains exactly one point \( x_\omega \in K \) and gives rise to a surjection \( \pi : I^\infty \to K, \omega \mapsto x_\omega \), which we call the natural coding map. Let \( C(I^\infty) \) denote the space of continuous real valued functions on \( I^\infty \). Fix \( \psi \in C(I^\infty) \) (sometimes called potential function). For \( f \in C(I^\infty) \) we define the Perron-Frobenius operator (with respect to \( \psi \)) via \( L_\psi f(x) := \Sigma_{y \in \mathbb{N}} e^{\psi(y)} f(y), x \in I^\infty \).

**Definition 4.1.** For \( f \in C(I^\infty) \), \( \alpha \in (0,1) \) and \( n \in \mathbb{N}_0 \) define

\[
\var_{\alpha}(f) := \sup \{ |f(\omega) - f(u)| : \omega, u \in I^\infty \text{ and } \omega_i = u_i \text{ for all } i \in \{1, \ldots, n\} \},
\]

\[
|f|_\alpha := \sup_{n \geq 0} \frac{\var_{\alpha}(f)}{\alpha^n} \text{ and } \mathcal{T}_\alpha := \{ f \in C(I^\infty) : |f|_\alpha < \infty \}.
\]

Elements of \( \mathcal{T}_\alpha \) are called \( \alpha \)-Hölder continuous functions on \( I^\infty \). Furthermore, the Birkhoff sum of \( f \) is defined by \( S_n f(x) := \Sigma_{k=0}^{n-1} f \circ \sigma^k(x), x \in I^\infty, n \in \mathbb{N} \) and \( S_0 f = 0 \).
For \( \psi \in C\left(\mathbb{R}^\ell\right) \) with \( L_\psi \mathbb{1} = \mathbb{1} \) let \( \nu \in \mathcal{M}_\nu \left(\mathbb{R}^\ell\right) \) denote a fixed point probability measure of \( L_\psi \), that is \( L_\psi^* \nu = \nu \) where \( L_\psi^* \) denotes the dual operator of \( L_\psi \) acting on the set of Borel probability measures supported on \( \mathbb{R}^\ell \). Such a fixed point always exists by Schauder-Tychonov fixed point theorem (see also [17]) and the \( \sigma \)-invariance of \( \nu \) follows for \( E \in \mathcal{B}\left(\mathbb{R}^\ell\right) \), by

\[
\nu(\sigma^{-1}(E)) = \int \sum_{j \in I} e^{S_{\psi}(jy)} \mathbb{1}_{\sigma^{-1}(E)}(jy) \, d\nu(y) = \int \sum_{j \in I} e^{S_{\psi}(jy)} \mathbb{1}_E(y) \, d\nu(y) = \nu(E).
\]

We call \( \nu \) a weak \( \psi \)-Gibbs measure and \( \varrho := \nu \circ \pi^{-1} \) a weak \( \psi \)-Gibbs measure with respect to the IFS \( \Phi \). For \( \omega \in \Gamma \), we define the measure \( \varrho_\omega \) and \( \Lambda_\omega \) by \( d\varrho_\omega := g_\omega \, d\varrho \) with \( g_\omega := e^{\psi_{\omega \circ \pi^{-1}} \circ \sigma_\omega} \) and \( d\Lambda_\omega := \left|T_\omega\right| \, d\Lambda\mid_{[0,1]} \).

**Remark 4.2.** The following list of comments proves useful in our context.

1. \( \nu \) is always a weak Gibbs measure in the sense of [17, Proposition 1], in particular, for all \( u \in \mathbb{R}^\ell \) and \( n \in \mathbb{N} \), we have

\[
e^{-\sum_{i=0}^{n-1} \text{var}_i(\psi)} \leq \frac{\nu([u_\omega])}{e^{S_{\psi}(u_0)}} \leq e^{\sum_{i=0}^{n-1} \text{var}_i(\psi)}. \tag{4.1}
\]

2. The measure \( \nu \) has no atoms, since \( \sum_{i=0}^{n} \text{var}_i(\psi) = o(n) \) and \( S_{\psi} \leq n \max \psi \), where \( o \) denotes the usual Landau symbol, i.e. \( a_n = o(n) \) if \( a_n/n \to 0 \) for \( n \to \infty \).

3. The topological support \( \text{supp}(\varrho) \) of \( \varrho \) is equal to \( K \). To see this, note that \( K \) is covered by the sets \( \bigcup_{\omega \in \Gamma} T_\omega ([0,1]) \), \( n \in \mathbb{N} \), and by (4.1) each \( T_\omega ([0,1]) \) has positive \( \varrho \)-measure \( \varrho \left(T_\omega ([0,1])\right) \geq \exp \left(-\sum_{i=0}^{n-1} \text{var}_i(\psi)\right) \nu ([\omega]) \).

4. If \( \psi \) is additionally Hölder continuous, then \( \nu \) is the unique invariant ergodic \( \psi \)-Gibbs measure and the bounds in the above inequality (4.1) can be chosen to be positive constants.

5. For an arbitrary Hölder continuous function \( \psi : \mathbb{R}^\ell \to \mathbb{R} \) (without assuming \( L_\psi \mathbb{1} = \mathbb{1} \)) there always exists a \( \sigma \)-invariant \( \psi \)-Gibbs measure \( \nu \) on the symbolic space as a consequence of the general thermodynamic formalism and the Perron-Frobenius theorem for Hölder potentials (see e.g. [7]). Let \( h \) denote the only eigenfunction of the Perron-Frobenius operator for the maximal eigenvalue \( \lambda > 0 \), which is positive and in the same Hölder class. Then \( \psi_1 := \psi - \log \lambda + \log h - \log h \circ \sigma \) defines another Hölder continuous function for which \( L_{\psi_1} \mathbb{1} = 1 \) and for which \( \nu \) is the (unique) \( \psi_1 \)-Gibbs measure, as defined here.

6. If \( \psi \) depends only on the first coordinate and is normalized such that \( p_i := \exp \psi (i, \ldots) \), \( i \in I \), defines a probability vector, then \( \nu \) is in fact a Bernoulli measure and the bounding constants in the above inequalities ((4.1)) can be chosen to be 1. If additionally the \( (T_i) \) are contracting similarities, then \( \varrho \) coincides with the self-similar measure as defined in (1.3).
Let us define the geometric potential function
\[ \varphi(\omega_1 \omega_2 \cdots) := \log \left( |T_{\omega_1}^{\prime}(\pi(\omega_2 \omega_3 \cdots))| \right). \]

We will make use of the following relation between \( \varphi \) with \( T_{\omega}^{\prime} \) with \( \omega = \omega_1 \cdots \omega_n \in P^n, n \in \mathbb{N} \). For any \( x \in K \) there exists \( \alpha \in I^{[1]} \) such that \( \pi(\alpha, x) = x \). Hence,
\[ |T_{\alpha}^{\prime}(x)| = e^{\sum_{i=1}^{n} \log |T_{\omega_i}^{\prime}(\pi(\alpha_i, x))|} = e^{\sum_{i=1}^{n} \log |T_{\omega_i}^{\prime}(\pi(\alpha_i, x))|} = e^{S_{\varphi}(\omega \alpha \omega_1 \cdots \omega_n)}. \]

Note that \( \varphi \) is Hölder continuous if the underlying IFS is a \( C^{1+\gamma} \)-IFS. Moreover, if all the \( T_i \) are affine, then \( \varphi \) depends only on the first coordinate.

The pressure of a continuous function \( f : I^{[1]} \to \mathbb{R} \) is defined by
\[ P(f) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \exp(S_{\omega f}(\omega)), \]
with \( S_{\omega f} := \sup_{x \in [0,1]} S_{|\omega|f}(x) \).

For \( f : I^{[1]} \to \mathbb{R} \) and every \( m \in \mathbb{N} \), we have
\[ P(f) = P(\bar{\sigma}^m f) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \exp(\bar{S}_{|\omega|f}(\omega)) \]
with \( \bar{S}_{|\omega|f}(\omega) := \sum_{i=0}^{m-1} f^{m}(\sigma^i(\omega)) \).

Lemma 4.3. For \( f \in C(I^{[1]}) \) and every \( m \in \mathbb{N} \), we have
\[ mP(f) = P(\bar{\sigma}^m f) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \exp(\bar{S}_{|\omega|f}(\omega)). \]

Proof. The assertion follows immediately from the identity, for \( n \in \mathbb{N} \),
\[ \frac{1}{n} \log \sum_{\omega \in I^n} \exp(\bar{S}_{|\omega|f}(\omega)) = \frac{1}{n} \log \sum_{\omega \in I^n} \exp(\sup_{x \in [0,1]} \bar{S}_{|\omega|f}(x)) = m \frac{1}{mn} \log \sum_{\omega \in I^n} \exp(\sup_{x \in [0,1]} S_{|\omega|f}(x)). \]

In the following we show that the weak bounded distortion property (wBDP) holds true for the IFS \( \Phi = (T_1, \ldots, T_n) \).
**Lemma 4.4** (Weak Bounded Distortion Property). There exists a sequence of non-negative numbers \( (b_m)_{m \in \mathbb{N}} \) with \( b_m = o(m) \) such that for all \( \omega \in I^* \) and \( x, y \in [0, 1] \)

\[
e^{-b_m} \leq \frac{T_\omega'(x)}{T_\omega'(y)} \leq e^{b_m}.
\]

**Proof.** Here, we follow the arguments in [18, Lemma 3.4]. For \( \omega := \omega_1 \cdots \omega_l \in I^* \), we have for all \( x, y \in [0, 1] \),

\[
\frac{T_\omega'(x)}{T_\omega'(y)} \leq \exp \left( \sum_{k=1}^{l} \max_{x, y \in [0, 1]} \max_{i=1,...,n} \left| \log \left( \left| T'_{\omega_k} \left( T_{\sigma^i \omega} (x) \right) \right| \right) - \log \left( \left| T'_{\omega_k} \left( T_{\sigma^i \omega} (y) \right) \right| \right) \right) \right).
\]

Let \( 0 < R < 1 \) be a common bound for the contraction ratios of the maps \( T_1, \ldots, T_n \). Then we have

\[
|T_{\sigma^i \omega} (x) - T_{\sigma^i \omega} (y)| \leq R^{l-k} |x - y| \leq R^{l-k}.
\]

Hence, we conclude

\[
A_{l-k} \leq \max_{a, b \in [0, 1], \omega, \eta \in I^*} \max_{i=1,...,n} \left| \log \left( \left| T'_{i} (a) \right| \right) - \log \left( \left| T'_{i} (b) \right| \right) \right| =: B_{l-k}
\]

Using that each \( T'_1, \ldots, T'_n \) is bounded away from zero and continuous, we obtain \( B_k \to 0 \) for \( k \to \infty \). With \( b_m := \sum_{k=1}^{m-1} B_k \) we have \( \lim_{m} b_m/m \) equals \( \lim_{k} B_k = 0 \) as a Cesàro limit and the second inequality holds. The first inequality follows by interchanging the roles of \( x \) and \( y \). \( \square \)

5. Spectral dimensions and asymptotics

In this last part we give the proofs of all three main theorems.

5.1. Weak Gibbs measures under the OSC

Let \( \varrho \) and \( \nu \) be defined as in Section 4. In this section we assume the open set condition (OSC) with feasible open set \((0, 1)\), i.e. \( T_i((0, 1)) \cap T_j((0, 1)) = \emptyset \) for all \( i \neq j, i, j \in I \). Note that in this case \( \varrho \) has no atoms. We start with some basic observations.

**Lemma 5.1.** For fixed \( \omega, \eta \in I^* \) we have that \( f \) is an eigenfunction with eigenvalue \( \lambda \) of \( \Delta_{\varrho, \Lambda_\omega, \Lambda_\eta} \) with \( I_{\omega \eta} := T_{\omega \eta} ((0, 1)) \) if and only if if \( f \circ T_\omega \) is an eigenfunction with eigenvalue \( \lambda \) of \( \Delta_{\varrho, \Lambda_\omega, \Lambda_\eta} \).
5.1 Weak Gibbs measures under the OSC

**Proof.** Clearly, by a change of variables

\[
\int_{I_\eta} (\nabla_{A_{\eta\omega}} f)^2 \, d\Lambda = \int_{I_\eta} ((\nabla_{A_{\eta\omega}} f) \circ T_\omega)^2 |T_\omega| \, d\Lambda
\]

\[
= \int_{I_\eta} \nabla_{A_{\eta\omega}} (f \circ T_\omega)^2 \, d\Lambda = \int_{I_\eta} (\nabla_{A_{\eta\omega\omega}} (f \circ T_\omega))^2 |T_\omega| \, d\Lambda,
\]

where we used \((\nabla_{A_{\eta\omega}} f) \circ T_\omega |T_\omega| = \nabla_{A_{\eta\omega}} (f \circ T_\omega)\) and \(\nabla_{A_{\eta\omega}} (f \circ T_\omega) = \nabla_{A_{\eta\omega\omega}} (f \circ T_\omega) |T_\omega|\). For the right hand side of the defining equality of the eigenfunction we have

\[
\int_{I_\omega} f^2 \, d\varrho = \int_{I_\omega} f^2 \circ \pi \, dv = \int_{I_\omega} L_{\nu}^{(\omega)} (I_{[\omega]}(x)) f^2(\pi(x)) \, dv(x)
\]

\[
= \int \sum_{j \in F_i} e^{S_{\omega \varrho}(i,j)} I_{[\omega]}(j) f^2(\pi(j)) \, dv(x)
\]

\[
= \int e^{S_{\omega \varrho}(\pi(\omega))} I_{[\eta]}(x) f^2(\pi(\omega x)) \, dv(x) = \int_{I_\eta} (f \circ T_\omega)^2 e^{S_{\omega \varrho}(-1)T_\omega} \, d\varrho,
\]

where we used the fact that \(\pi(\omega x) = T_\omega(\pi(x))\). \(\square\)

Set \(S_{\omega,\varrho} f := \sup_{x \in [\omega]} S_{\omega,\varrho} f(x)\) and \(s_{\omega,\eta,\omega} \varrho := \inf_{x \in [\eta]} S_{\omega,\varrho} f(x)\). If \(\eta\) is the empty word, then \(S_{\omega,\varrho} f = S_{\omega,\varrho,\omega} f\) as defined above and we set \(s_{\omega,\varrho} f := s_{\omega,\varrho,\omega} f\).

**Lemma 5.2.** For all \(i \in \mathbb{N}\) and \(\omega, \eta \in I^*\), we have

\[
\frac{\lambda_i^\eta}{e^{S_{\omega \varrho}\bar{\eta} + S_{\omega \varrho} \bar{\eta}}} \leq \lambda_i^\eta \leq \lambda_i^{\omega,\omega,\omega} \leq \frac{\lambda_i^{\eta,\eta,\omega\omega}}{e^{S_{\omega \varrho}\bar{\eta} + S_{\omega \varrho} \bar{\eta}}}
\]

where \((b_m)_{m \in \mathbb{N}}\) is the sequence defined in Lemma 4.4 with \(b_n = o(n)\).

**Proof.** Note that the equality is a direct consequence of Lemma 5.1. For every \(f \in H_0^1(I_\eta)\) we have

\[
\int_{I_\eta} (\nabla_{A_{\eta\omega}} f)^2 \, d\Lambda_{\omega} = \int_{I_\eta} (\nabla_{A_{\eta\omega}} f)^2 |T_\omega|^{-1} \, d\Lambda = \int_{I_\eta} (\nabla_{A_{\eta}} f)^2 |T_\omega|^{-1} \, d\Lambda = \int_{I_\eta} f^2 e^{S_{\omega \varrho} \pi^{-1}T_\omega} \, d\varrho
\]

and hence using the wBDP stated Lemma 4.4 gives

\[
\frac{1}{e^{S_{\omega \varrho}\bar{\eta} + S_{\omega \varrho} \bar{\eta}}} \leq \frac{\int_{I_\eta} f^2 \, d\varrho}{\int_{I_\eta} (\nabla_{A_{\eta\omega}} f)^2 \, d\Lambda_{\omega}} \leq \frac{\int_{I_\eta} (\nabla_{A_{\eta\omega}} f)^2 \, d\Lambda_{\omega}}{\int_{I_\eta} f^2 \, d\varrho_{\omega}} \leq \frac{1}{e^{S_{\omega \varrho}\bar{\eta} + S_{\omega \varrho} \bar{\eta}}}.
\]

Using the fact that \(H_0^1(I_n) = H_{A_{\eta\omega}}^1(I_n)\) and \(\nabla_{A_{\eta\omega}} f = \nabla_{A_{\omega\omega}} f \, |T_\omega|\), the claim follows as a consequence of Lemma 2.6. \(\square\)

**Corollary 5.3.** For \(m \in \mathbb{N}\) large enough, for all \(x > \lambda_i^{\eta,\omega}/r_{m,\min}\), we have

\[
\left(\frac{x r_{m,\min}}{\lambda_i^{\eta,\omega}}\right)^{\tilde{\omega}_n} \leq N_{\omega,\lambda}(x) \leq 2 \left(\frac{\lambda_i^{\eta,\omega}}{\lambda_i^{\eta,\omega,\omega}}\right)^{\tilde{\omega}_n} + 1
\]
where, for \( \omega \in \mathbb{F}^m \), set \( r_\omega := \exp(s_\omega \varphi - b_m + s_\omega \psi) \), \( R_\omega := \exp(S_\omega \varphi + b_m + S_\omega \psi) \), \( r_{m,\text{min}} := \min_{i \in \mathbb{N}} r_i \), \( R_{m,\text{min}} := \min_{i \in \mathbb{N}} R_i \) and let \( u_m, t_m \in \mathbb{R}_{>0} \) be the unique solutions of
\[
\sum_{\omega \in \mathbb{F}^m} e^{u_m (s_\omega \varphi + b_m)} = \sum_{\omega \in \mathbb{F}^m} e^{t_m (s_\omega \psi + b_m)} = 1.
\]

**Proof.** This proof follows the arguments used in [24, Lemma 2.7]. First, note that for \( m \in \mathbb{N} \) sufficiently large for all \( \omega \in \mathbb{F}^m \) we have \( S_\omega \varphi + S_\omega \psi + b_m < 0 \) where we used \( b_m = o(m) \) and \( S_\omega \psi + S_\omega \varphi \leq m(\max \psi + \max \varphi) \). Therefore there exists \( t_m \in \mathbb{R}_{>0} \) such that
\[
\sum_{\omega \in \mathbb{F}^m} R_m^\omega = 1.
\]
Moreover, iterating Lemma 5.2 for \( \omega := \omega_1 \cdots \omega_n \in (\mathbb{F}^m)^n \), \( n \in \mathbb{N} \), gives
\[
\frac{\lambda^{1 \cdot \Lambda}}{R_\omega} \leq \frac{\lambda^{1 \cdot \Lambda}}{R_{\omega}} \leq \frac{\lambda^{1 \cdot \Lambda}}{R_{\omega}}
\]
with \( R_\omega := \prod_{i=1}^{\lambda \cdot n} R_{\omega_i} \) and \( r_\omega := \prod_{i=1}^{\lambda \cdot n} r_{\omega_i} \). Let \( x > \lambda^{1 \cdot \Lambda} \) be and define for \( m \in \mathbb{N} \) the following partition of \((\mathbb{F}^m)^\mathbb{N}\)
\[
P_{m,x} := \left\{ \omega \in (\mathbb{F}^m)^\mathbb{N} : R_\omega < \frac{x}{x} \leq R_{\omega} \right\},
\]
with \( R_{\omega} := \prod_{i=1}^{\lambda \cdot n} R_{\omega_i} \). Considering the Bernoulli measure on \((\mathbb{F}^m)^\mathbb{N}\) given by the probability vector \((R_{\omega})^\mathbb{N}\) and using the fact that \( P_{m,x} \) defines a partition of \((\mathbb{F}^m)^\mathbb{N}\) we obtain \( \sum_{\omega \in P_{m,x}} R_\omega^m = 1 \), which leads to \( \text{card} (P_{m,x}) \leq x^{\pi_n} / (\lambda R_{m,\text{min}})^{\pi_n} \). Since for all \( \omega \in P_{m,x} \),
\[
x < \frac{\lambda}{R_{\omega}} \leq \lambda^{1 \cdot \Lambda} L_i,
\]
we conclude from Theorem 2.7
\[
N_{\lambda \cdot \Lambda} (x) \leq \sum_{\omega \in P_{m,x}} N_{\lambda \cdot \Lambda} (x) + 2 \text{card} (P_{m,x}) + 1 = 2 \text{card} (P_{m,x}) + 1
\]
\[
\leq 2 \frac{x^{\pi_n}}{(\lambda R_{m,\text{min}})^{\pi_n}} + 1.
\]
For the estimate from below we define for \( x > \frac{\lambda}{r_{m,\text{min}}} \) the following partition of \((\mathbb{F}^m)^\mathbb{N}\)
\[
\Xi_{m,x} := \left\{ \omega \in (\mathbb{F}^m)^\mathbb{N} : r_\omega < \frac{\lambda}{x r_{m,\text{min}}} \leq R_{\omega} \right\},
\]
with \( r_{\omega} := \prod_{i=1}^{\lambda \cdot n} r_{\omega_i} \). Hence, for all \( \omega \in \Xi_{m,x} \), we have by (5.1)
\[
\lambda^{1 \cdot \Lambda} L_i \leq \frac{\lambda}{r_{\omega}} \leq \frac{\lambda}{r_{m,\text{min}} r_{\omega}} \leq x.
\]
Again, there exists \( \overline{\nu}_m \in \mathbb{R}_{>0} \) such that \( \sum_{\omega \in \mathcal{P}_n} \frac{\overline{u}_m^\omega}{u^\omega} = 1 \) and we obtain \( \sum_{\omega \in \mathbb{E}_{n, r}} \frac{\overline{u}_m^\omega}{u^\omega} = 1 \). This implies
\[
1 = \sum_{\omega \in \mathbb{E}_{n, r}} \frac{u^\omega}{u^\omega} \leq \left( \frac{\lambda}{x_{m, \min}} \right)^{\nu_m} \operatorname{card}(\mathbb{E}_{m, r}),
\]
and we conclude from Theorem 2.7
\[
\left( \frac{x_{m, \min}}{A} \right)^{\nu_m} \leq \operatorname{card}(\mathbb{E}_{m, r}) \leq \sum_{\omega \in \mathbb{E}_{n, r}} N_{\nu, \Lambda, I_m}(x) \leq N_{\nu, \Lambda}(x).
\]

In the case of self-similar measures, we obtain the following classical result of [13].

**Corollary 5.4.** Assume \( 0 < T'_j \equiv \sigma_j < 1 \) and \( \psi(\omega) = \log(p_{\omega j}) \), for \( \omega = (\omega_1, \omega_2, \ldots) \in I^n \), where \( (p_j) \in (0, 1)^n \) is a given probability vector. Then, for all \( i, m \in \mathbb{N} \) and \( \omega = (\omega_1 \cdots \omega_m) \in I^m \), we have
\[
\lambda_{\nu, \Lambda}^1 = \prod_{j=1}^m \sigma_{\omega_j} \omega_j \Lambda_{\nu, \Lambda}^i,
\]
and, for all \( x > \lambda_{\nu, \Lambda}^1 (\min p_i |\sigma_i|)^{-1} \), we have
\[
x^n \left( \frac{\min p_i |\sigma_i|}{\lambda_{\nu, \Lambda}^1} \right)^{u} \leq N_{\nu, \Lambda}(x) \leq \frac{2x^n}{(\lambda_{\nu, \Lambda}^1 \min p_i |\sigma_i|)^{u}} + 1,
\]
where \( u \) is the unique solution of \( \sum_{i=1}^m (\sigma_i)^y = 1 \).

The following lemma is elementary and we give its short proof for completeness.

**Lemma 5.5.** For \( a, b \in \mathbb{R} \) with \( a < b \), let \((f_n : [a, b] \to \mathbb{R})_{n \in \mathbb{N}}\) be a sequence of decreasing functions converging pointwise to a function \( f \). We assume that \( f_n \) has a unique zero in \( x_n \), for \( n \in \mathbb{N} \) and \( f \) has a unique zero in \( x \). Then \( x = \lim_{n \to \infty} x_n \).

**Proof.** Assume that \( \lim_{n} x_n \neq x \). Then there exists a subsequence \( n_k \) such that \( x_{n_k} \to x^* \neq x \) and for all \( k \in \mathbb{N} \) we have \( |x^* - x|/2 < |x_{n_k} - x| \) and \( |x_{n_k} - x| < |x - x^*|/2 \). Without loss of generality we assume \( x^* < x \). Then \( x_{n_k} \leq (x^* + x)/2 \) and for each \( y \in ((x^* + x)/2, x) \), we have
\[
0 = f_{n_k}(x_{n_k}) > f_{n_k}(y) \geq f_{n_k}(x) \to f(x) = 0, \text{ for } k \to \infty.
\]
Consequently, \( f(y) = 0 \) for all \( y \in ((x^* + x)/2, x) \), contradicting the uniqueness of the zero of \( f \).

**Lemma 5.6.** For fixed \( m \in \mathbb{N} \) large enough and \( u_m, \overline{u}_m \in \mathbb{R}_{>0} \) denoting the unique solutions of
\[
\sum_{\omega \in \mathcal{P}_n} e^{u_m(S(\omega) \varphi + b(\omega) + S(\omega) \varphi)} = \sum_{\omega \in \mathcal{P}_n} e^{\overline{u}_m(S(\omega) \varphi - b(\omega) + S(\omega) \varphi)} = 1,
\]
then we have \( \lim_{m \to \infty} \overline{u}_m = \lim_{m \to \infty} u_m = z_\varphi \).
5.2 Weak Gibbs measures with overlap

Proof. Define for $m \in \mathbb{N}$ and $t \geq 0$

$$P_m(t) := \frac{1}{m} \log \sum_{\omega \in F^m} \exp(t(s_\omega \varphi - b_m + s_\omega \psi)),$$

$$\bar{P}_m(t) := \frac{1}{m} \log \sum_{\omega \in F^m} \exp(t(S_\omega \varphi + b_m + S_\omega \psi)),$$

$$P_m(t) := \frac{1}{m} \log \sum_{\omega \in F^m} \exp(tS_\omega \xi).$$

We obtain

$$P_m(t) \leq \bar{P}_m(t) \leq P_m(t) - \frac{b_m}{m}$$

$$= \frac{1}{m} \log \sum_{\omega \in F^m} \exp(t(s_\omega \varphi + s_\omega \psi + S_\omega \varphi - s_\omega \varphi + S_\omega \psi - s_\omega \psi)) - \frac{b_m}{m}$$

$$\leq \frac{1}{m} \log \sum_{\omega \in F^m} \exp \left( t(s_\omega \varphi + s_\omega \psi) + t \left( \sum_{j=0}^{m-1} \text{var}_j \varphi + \sum_{j=0}^{m-1} \text{var}_j \psi \right) \right) - \frac{b_m}{m}$$

$$\leq P_m(t) + \frac{t}{m} \left( \sum_{j=0}^{m-1} \text{var}_j \varphi + \sum_{j=0}^{m-1} \text{var}_j \psi - b_m \right).$$

Using the continuity of $\varphi, \psi$ and $\lim_{m \to \infty} b_m/m = 0$, we deduce $\lim_{m \to \infty} \bar{P}_m(t) = \lim_{m \to \infty} P_m(t) = P(t\xi)$. Furthermore, for all $t \geq 0$, we have

$$P_m(t) \leq \bar{P}_m(t) \leq t \frac{b_m}{m} + \frac{1}{m} \log \sum_{\omega \in F^m} \exp(tm(\max \psi + \max \varphi))$$

$$= \log(n) + t \left( \frac{b_m}{m} + (\max \psi + \max \varphi) \right).$$

Observe that for $m$ so large that $b_m/m \leq -\max \psi/2$, each map $t \mapsto \bar{P}_m(t), t \mapsto P_m(t)$ and $t \mapsto P(t)$ is decreasing and has a unique zero in $[0, -\log(n)/(\max \psi/2 + \max \varphi)]$. Hence the statement follows from Lemma 5.5.

Now, we can give the proof of Theorem 1.1 under the OSC.

Proof of Theorem 1.1 under the OSC. The proof Theorem 1.1 assuming the OSC is now an immediate consequence of Corollary 5.3 and Lemma 5.6.

5.2 Weak Gibbs measures with overlap

This section relies on results from [33, 10, 3] on the $L^q$-spectrum together with the recent results in [21]. Let $\nu$ and $\varrho$ be defined as in Section 4 and recall that $\Phi$ is non-trivial, i.e. there is more than one contraction and the $T_i$’s do not share a common fixed point. It is easy to see that self-similar measures with or without OSC are atomless.
as long as $\Phi$ is non-trivial (see [21]). It is an open question under which condition the same applies to weak Gibbs measures without OSC. For our purposes it is enough to observe that the non-triviality of $\Phi$ implies $\text{card}(K) = \infty$ and since $\text{supp}(\varrho) = K$, we infer the important observation $\varrho((0, 1)) > 0$. Also note that for every $\varepsilon > 0$ we can extend each $T_i$ to an injective contracting $C^1$-map $T_i : (-\varepsilon, 1 + \varepsilon) \rightarrow (-\varepsilon, 1 + \varepsilon)$. Hence, the results of [33, 10] are valid in our setting.

First, we will prove that the $L^q$-spectrum of $\varrho$ exists in $(0, 1]$. Combining this with [21, Theorem 1.1, Theorem 1.2] we conclude that the spectral dimension exists and is given by $q_{\varrho}$. To this end we need the following lemmata.

**Lemma 5.7.** We have for any $G \subset I^*$ with $\bigcup_{u \in G} \{u\} = I^*$ and $E \in \mathfrak{B}([0, 1])$ that

$$\varrho(E) \geq \sum_{u \in G} c_u \nu\left(\{u\}\right) \varrho(T_i^{-1}(E))$$

with $c_u := e^{-\sum_{i \in [0, 1]} \text{var}_i(\psi)}$ (and therefore $\log(c_u) = o(n)$).

**Proof.** For all $E \in \mathfrak{B}([0, 1])$ and $u \in I^*$, we have

$$\nu(\pi^{-1}(E) \cap \{u\}) = \int_{\{u\}} 1_E \circ \pi \, d\nu = \int \left(\int_{\{j\}} L_u^{|}\{j\} \chi_{E}(\pi(x)) \, d\nu(x) \right) \, dx$$

$$= \int \sum_{j \in F^u} e^{S_u(\psi(jx))} 1_{\{j\}} \chi_{E}(\pi(jx)) \, d\nu(x)$$

$$= \int e^{S_u(\psi(ux))} \chi_{E}(\pi(ux)) \, d\nu(x) \geq e^{-\sum_{i \in [0, 1]} \text{var}_i(\psi)} \varrho(T_i^{-1}(E)) \nu(\{u\}).$$

Setting $c_u := e^{-\sum_{i \in [0, 1]} \text{var}_i(\psi)}$ and summing over $u \in G$ we obtain

$$\varrho(E) = \sum_{u \in G} \nu(\pi^{-1}(E) \cap \{u\}) \geq \sum_{u \in G} c_u \nu(\{u\}) \varrho(T_i^{-1}(E)).$$

Also, the continuity of the potential $\psi$ implies $\log(c_u) = o(n)$. □

For $u \in I^*$ let us define $K_u := T_u(K)$. Then, for $n \geq 2$ the set

$$W_n := \{u \in I^* : \text{diam}(K_u) \leq 2^{-n} < \text{diam}(K_u^{-1})\},$$

defines a partition of $I^*$.

**Lemma 5.8.** For any $0 < q < 1$ there exists a sequence $(s_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}_{>0}$ with $\log s_n = o(n)$ such that for every $n, m \in \mathbb{N}$ and $Q \in \mathcal{D}_n$

$$\sum_{B \in \mathcal{D}_m, B - \overline{Q} \neq \emptyset} \sum_{Q \in \mathcal{D}_n, Q \subset B} \varrho(Q)^q \geq s_n \varrho(\overline{Q})^q \min_{u \in W_n} \sum_{Q \in \mathcal{D}_{n+1}} \varrho(T_i^{-1}(Q))^q$$

where $B \sim \overline{Q}$ means that the closures of $B$ and $\overline{Q}$ intersect.
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Proof. As in [33] for $n, m \in \mathbb{N}$, $u \in W_n$ and $A \in \mathcal{D}_n$, let us define

$$w(u, A) := \sum_{Q \in \mathcal{D}_{n+m} : Q \subset A} \varrho \left( T^{-1}_u (Q) \right)^q.$$ 

The interval $A \in \mathcal{D}_n$ on which $w(u, A)$ attains its maximum will be called $q$-heavy for $u \in W_n$. We will denote the $q$-heavy box by $H(u)$ (if there are more than one interval which maximizes $w(u, \cdot)$ we choose one of them arbitrarily). Note that every $K_u$ with $u \in W_n$ intersects at most 3 intervals in $\mathcal{D}_n$. Hence, we obtain for $u \in W_n$,

$$\sum_{Q \in \mathcal{D}_{n+m}} \varrho \left( T^{-1}_u (Q) \right)^q = \sum_{B' \in \mathcal{D}_n} \sum_{Q \in \mathcal{D}_{n+m} : Q \subset B'} \varrho \left( T^{-1}_u (Q) \right)^q \leq 3 \sum_{Q \in \mathcal{D}_{n+m} : Q \subset H(u)} \varrho \left( T^{-1}_u (Q) \right)^q.$$ 

This leads to

$$\sum_{Q \in \mathcal{D}_{n+m} : Q \subset H(u)} \varrho \left( T^{-1}_u (Q) \right)^q \geq \frac{1}{3} \sum_{Q \in \mathcal{D}_{n+m}} \varrho \left( T^{-1}_u (Q) \right)^q \geq \frac{1}{3} \min_{u \in W_n} \sum_{Q \in \mathcal{D}_{n+m}} \varrho \left( T^{-1}_u (Q) \right)^q. \quad (5.2)$$

Further, for every $Q \in \mathcal{D}_{n+m}$ and $B \in \mathcal{D}_n$, by Lemma 5.7, we have

$$\varrho(Q) \geq \sum_{u \in W_n} c_{\nu(u)} \nu(\{u\}) \varrho \left( T^{-1}_u (Q) \right) \geq \sum_{u \in W_n, B = H(u)} c_{\nu(u)} \nu(\{u\}) \varrho \left( T^{-1}_u (Q) \right) \geq \left( \min_{u \in W_n} c_{\nu(u)} \right) \sum_{u \in W_n, B = H(u)} \nu(\{u\}) \varrho \left( T^{-1}_u (Q) \right).$$

Setting

$$p_{-}(B) := \sum_{u \in W_n, B = H(u)} \nu(\{u\})$$

and, if $p_{-}(B) > 0$, using the concavity of the function $x \mapsto x^q$ for $0 < q < 1$, we obtain

$$\varrho(Q)^q \geq p_{-}(B)^q \left( \min_{u \in W_n} c_{\nu(u)} \right)^q \left( \sum_{u \in W_n, B = H(u)} \nu(\{u\}) \varrho \left( T^{-1}_u (Q) \right) \right)^q \geq p_{-}(B)^q \left( \min_{u \in W_n} c_{\nu(u)} \right)^q \sum_{u \in W_n, B = H(u)} \nu(\{u\}) \varrho \left( T^{-1}_u (Q) \right)^q.$$

Summing over $Q \in \mathcal{D}_{n+m}$ with $Q \subset B$, and using (5.2), we infer

$$\sum_{Q \in B, Q \in \mathcal{D}_{n+m}} \varrho(Q)^q \geq p_{-}(B)^q \left( \min_{u \in W_n} c_{\nu(u)} \right)^q \sum_{u \in W_n, B = H(u)} \nu(\{u\}) \sum_{Q \in B, Q \in \mathcal{D}_{n+m}} \varrho \left( T^{-1}_u (Q) \right)^q \geq p_{-}(B)^q \left( \min_{u \in W_n} c_{\nu(u)} \right)^q \min_{u \in W_n} \sum_{Q \in \mathcal{D}_{n+m}} \varrho \left( T^{-1}_u (Q) \right)^q,$$
which is also valid in the case \( p_-(B) = 0 \). For \( \bar{Q} \in \mathcal{D}_n \) and \( u \in W_n \) with \( K_u \cap \bar{Q} \neq \emptyset \) we have \( K_u \subset \bigcup_{B \sim \bar{Q}, B \in \mathcal{D}_n} B \), as a consequence of \( \text{diam}(K_u) \leq 2^{-u} \). In particular, every \( K_u \) that intersects \( \bar{Q} \) must have an interval \( B \in \mathcal{D}_n \) with \( B \sim \bar{Q} \) which is \( q \)-heavy for \( u \). Hence, we obtain

\[
q(\bar{Q}) \leq \sum_{u \in W_n: K_u \cap \bar{Q} \neq \emptyset} \nu([u]) \leq \sum_{B \sim \bar{Q}, B \in \mathcal{D}_n} \nu([u]) = \sum_{B \sim \bar{Q}, B \in \mathcal{D}_n} p_-(B).
\]

Using \( 0 < q < 1 \), we conclude

\[
q(\bar{Q})^q \leq \left( \sum_{B \sim \bar{Q}, B \in \mathcal{D}_n} p_-(B)^q \right)^{1/q} \leq \sum_{B \sim \bar{Q}, B \in \mathcal{D}_n} p_-(B)^q.
\]

Summing over all \( B \in \mathcal{D}_n \) with \( B \sim \bar{Q} \) gives

\[
\sum_{B \sim \bar{Q}, B \in \mathcal{D}_n} \sum_{Q \subset B, Q \in \mathcal{D}_{n_{\text{max}}}} q(\bar{Q})^q \
\geq \sum_{B \sim \bar{Q}, B \in \mathcal{D}_n} \frac{p_-(B)^q}{3} \left( \min_{u \in W_n} c_{[u]} \right)^q \min_{v \in W_v} \sum_{Q \in \mathcal{D}_{n_{\text{max}}}} q(T_v^{-1}(Q))^q.
\]

Note that for every \( u \in W_n \), by the definition of \( W_n \), we have

\[
|u| < n \frac{\log(2) - \log(\alpha_{\text{max}})}{-\log(\alpha_{\text{max}})},
\]

with \( \alpha_{\text{max}} := \max_{i=1,\ldots,n} \max_{x \in [0,1]} |T'_i(x)| \). Thus, setting \( s_n := 3^{-1} \min_{u \in W_n} c_{[u]}^q \) we have \( \lim_{n \to \infty} n \log s_n = 0 \), where we used the elementary fact that for any two sequences \( (x_n)_{n \in \mathbb{N}} \in \mathbb{R}_{>0}^\mathbb{N} \) and \( (y_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \) with \( x_n = o(n), y_n \ll n \), we have \( x_n = o(n) \). \( \Box \)

Proposition 5.9. The \( L^q \)-spectrum \( \beta_q \) of \( \nu \) exists on \( (0,1] \) as a limit.

Proof. Let \( 0 < q < 1 \). From [10, Proposition 3.3] (which holds true for all Borel probability measures with support \( K \), see remark after Proposition 3.3 in [10]) it follows that there exists a sequence \( \left\{ b_{q,n} \right\}_{n \in \mathbb{N}} \) of positive numbers with \( \log(b_{q,n}) = o(n) \), such that for all \( m, n \in \mathbb{N} \) and \( u \in W_m \)

\[
b_{q,n} \sum_{Q \in \mathcal{D}_m} q(T_u^{-1}(Q))^q \leq \sum_{Q \in \mathcal{D}_{n_{\text{max}}}} q(T_u^{-1}(Q))^q.
\]

In tandem with Lemma 5.8 we obtain for every \( \bar{Q} \in \mathcal{D}_n \),

\[
\sum_{B \in \mathcal{D}_n, B \sim \bar{Q}} \sum_{Q \subset B, Q \in \mathcal{D}_{n_{\text{max}}}} q(\bar{Q})^q \geq s_n q(\bar{Q})^q \min_{u \in W_n} \sum_{Q \in \mathcal{D}_{n_{\text{max}}}} q(T_u^{-1}(Q))^q
\geq \left( b_{q,n} s_n \right) q(\bar{Q})^q \sum_{Q \in \mathcal{D}_n} q(\bar{Q})^q.
\]

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Clearly, \( \log \left( b_{q,n,s_n} \right) = o(n) \). Hence, we can apply \([10, \text{Proposition 4.4}]\), which shows that \( \beta_q \) exists as a limit on \((0,1)\).

With this knowledge, we can prove the remaining parts of Theorem 1.1.

**Proof of Theorem 1.1 with overlaps.** The proof follows from Proposition 5.9 and \([21, \text{Theorem 1.1, Theorem 1.2}]\) and using \( \varrho((0,1)) > 0 \), the fact that the \( L^q \)-spectrum of \( \varrho \) and \( \varrho((0,1)) \) coincide on \([0,1] \) as well as \( \beta_{\varrho((0,1))} \) exists as limit on \((0,1)\).

The following lemma is needed in the proof of the existence of the Minkowski dimension for weak Gibbs measures without assuming any separation conditions.

**Lemma 5.10.** If \( \varrho \) is a weak Gibbs measure for a \( C^1 \)-IFS, then

\[
M_n := \max \left( -\log \varrho ((C)_1) : C \in \mathcal{D}_n \right) < n,
\]

where for \( C = (2^{-nk},2^{-n(k+1)}], k \in \{0,\ldots,2^n-1\} \), we define the centered interval with triple size as \( (C)_1 := (2^{-n}(k-1),2^{-n}(k+2)] \).

**Proof.** Fix \( C \in \mathcal{D}_n \) that maximizes \(-\log \varrho ((C)_1)\). Since \( \varrho (C) > 0 \) there exists \( u \in W_\varrho \) such that \( K_u \cap C \neq \emptyset \). Since \( \text{diam}(K_u) \leq 2^{-n} \) we have \( K_u \subset (C)_1 \) and for arbitrary \( x \in I_n^\mathbb{N} \), the weak Gibbs property gives \( \varrho (K_u) \geq \nu (|u|) \geq c_{|u|} \exp (S_u \psi (ux)) \) with \( c_{|u|} := e^{-\sum_{i=0}^{\infty} \lim \varrho_i \phi} \). Since \( |u| \leq (\log (\alpha_{\text{max}}) - n \log(2)) / \log (\alpha_{\text{max}}) \) with \( \alpha_{\text{max}} := \max_{i=1,\ldots,n} \max_{x \in [0,1]} |T_i^\mathbb{N}_u(x)| \) and \( S_u \psi (ux) \geq |u| \min \psi \) we get

\[
\varrho ((C)_1) \geq \varrho (K_u) \geq c_{|u|}^{-1} \exp (|u| \min \psi)
\]
and further

\[
\limsup_{n \to \infty} \frac{-\log \varrho ((C)_1)}{n} \leq \limsup_{n \to \infty} \frac{\log c_{|u|} - \min \psi |u|}{n} \leq \limsup_{n \to \infty} \frac{\log c_{|u|} + \min \psi (n \log(2) - \log (\alpha_{\text{max}})) / \log (\alpha_{\text{max}})}{n} \leq \frac{\min \psi \log(2)}{\log (\alpha_{\text{max}})} < \infty,
\]

where we used as above the fact that \( \log c_n = o(n) \) and \( \min \psi < 0 \).

**Proposition 5.11.** If \( \varrho \) is a weak Gibbs measure for an \( C^1 \)-IFS, then the upper and lower Minkowski dimension of \( \text{supp} (\varrho) \) exists, i.e.

\[
\dim_m (\text{supp} (\varrho)) = \overline{\dim}_M (\text{supp} (\varrho)).
\]

In particular, the \( L^q \)-spectrum \( \beta_q \) exists as a limit on the closed unit interval.

**Proof.** We will make use of an observation from \([35, \text{Proposition 2}]\) that: If we replace \( \varrho (C) \) with \( \varrho ((C)_1) \) for \( C \in \mathcal{D}_n, n \in \mathbb{N} \), in the definition of \( \beta_q \), its value does not change
for $q \geq 0$. In this way we can extend $\beta_\varrho$ to the negative half-line and denote this extension, now defined on $\mathbb{R}$, by $\tilde{\beta}_\varrho$. On the one hand, by (4.1), for all $q \in (0, 1)$,

$$
\tilde{\beta}_\varrho (q) = \beta_\varrho (q) = \liminf_{n \to \infty} \frac{1}{2^n \log n} \sum_{C \in D_n} \varrho (C)^q \leq \liminf_{n \to \infty} \frac{\log \text{card} (D_n)}{2^n \log n} = \dim_M (\text{supp} (\varrho)).
$$

Hence, $\lim_{q \downarrow 0} \tilde{\beta}_\varrho (q) \leq \dim_M (\text{supp} (\varrho))$. On the other hand, for $q < 0$, our assumption gives

$$
0 \leq \tilde{\beta}_\varrho (q) = \limsup_{n \to \infty} \frac{1}{2^n \log n} \sum_{C \in D_n} \varrho ((C)_1)^q \leq \limsup_{n \to \infty} \frac{\log (\max_{C \in D_n} \varrho ((C)_1)^q + \log \text{card} (D_n))}{2^n} \leq \limsup_{n \to \infty} \frac{-q \max_{C \in D_n} \varrho ((C)_1) - \log \varrho ((C)_1)}{2^n} \leq -q \limsup_{n \to \infty} \frac{M_n}{n \log 2} + \dim_M (\text{supp} (\varrho)) < \infty.
$$

Hence, $\tilde{\beta}_\varrho$ is finite in a neighborhood of 0 and in particular continuous in 0. Consequently, we have

$$
\tilde{\beta}_\varrho (0) = \lim_{q \downarrow 0} \tilde{\beta}_\varrho (q) \leq \dim_M (\text{supp} (\varrho)) \leq \dim_M (\text{supp} (\varrho)) = \tilde{\beta}_\varrho (0). \quad \square
$$

We finish this section with the proof for the self-similar case without OSC.

**Proof of Theorem 1.5.** In the following we use the results of [3] to prove Theorem 1.5. We consider contracting similarities that is, for every $i = 1, \ldots, n, T_i : [0, 1) \to [0, 1], T_i(x) = r_i x + b_i, x \in \mathbb{R}$ with $b_i \in \mathbb{R}$ and $|r_i| < 1$. For given probability vector $(p_1, \ldots, p_n) \in (0, 1)^n$ let $\tau$ be the analytic function defined in (1.4) and set, as before,

$$
\tilde{q} := \inf \{ q \in (0, 1) : \tau'(q) q - \tau(q) \geq -1 \} \cup \{ 1 \}.
$$

Let $\varrho$ be the unique Borel probability measure defined in (1.3). By the result from [3, Theorem 1.2] the $L^q$-spectrum exists as a limit on $[0, 1]$ with

$$
\beta_\varrho (q) = \begin{cases} 
1 + \frac{q (\tau(q) - 1)}{\tilde{q}} & q \in [0, \tilde{q}), \\
\tau(q) & q \in [\tilde{q}, 1].
\end{cases}
$$

Fig. 2.1 on page 5 illustrates how the spectral dimension depends on the position of $\tilde{q}$ in $[0, 1]$. Now, Theorem 1.5 follows from this observation combined with [21, Theorem 1.1, Theorem 1.2]. \quad \square
5.3 Gibbs measure for $C^{1+\gamma}$-IFS under the OSC

Let $\varphi$ and $v$ be defined as in Section 4. In the following we assume that $\psi$ in the definition of the Gibbs measure $v$ is Hölder continuous and the underlying IFS $\{T_1, \ldots, T_m\}$ is $C^{1+\gamma}$, which implies $\varphi$ is Hölder continuous, in which case the following refined bounded distortion property holds (see [18, Lemma 3.4]).

**Lemma 5.12** (Strong Bounded Distortion Property). Assume $T_1, \ldots, T_n$ are $C^{1+\gamma}$-IFS then we have the following strong bounded distortion property (sBDP). There exists a sequence of positive numbers $(a_n)_{n \in \mathbb{N}}$ converging to 1 such that for $\omega, \eta \in I^*$ and $x, y \in T_\omega([0, 1])$ we have

$$a_n^{-1} \leq \frac{T^{(n)}_\omega(x)}{T^{(n)}_\omega(y)} \leq a_n.$$

Using the sBDP, we can improve Lemma 5.2 in the following way.

**Lemma 5.13.** For all $i \in \mathbb{N}$, $\omega \in I^*$ and $x, y \in I^n$, we have

$$\frac{\lambda_i^{(n)}[\omega, [0, 1]]}{e^{S_{\omega}(\psi(x)) + S_{\omega}(\psi(\omega x)) + d_0}} \leq \lambda_i^{(n)} = \lambda_i^{(n)}[\omega, [0, 1]] \leq \frac{\lambda_i^{(n)}[\omega, [0, 1]]}{e^{S_{\omega}(\psi(y)) + S_{\omega}(\psi(\omega y)) - d_0}},$$

where $d_0 := \log(a_0) + \sum_{k=0}^{\infty} \text{var}_k(\psi)$ and $a_0$ is defined in Lemma 5.12.

**Proof.** For all $\omega \in I^*$ and $x, z \in I^n$, we have

$$|S_{[\omega]}(\psi(\omega x)) - S_{[\omega]}(\psi(\omega z))| \leq \sum_{k=0}^{\infty} \text{var}_k(\psi)$$

and for all $y, v \in [0, 1]$ by Lemma 5.12, we obtain

$$|\log \left(\left|T^{(n)}_\omega(y)\right|\right) - \log \left(\left|T^{(n)}_\omega(v)\right|\right)| \leq \log(a_0).$$

Since there exists $y \in K$ such that $\pi(x) = y$, we obtain $\log \left(\left|T^{(n)}_\omega(y)\right|\right) = S_{[\omega]}(\varphi(\omega x))$. Thus, we infer

$$\frac{1}{e^{S_{\omega}(\psi(y)) + S_{\omega}(\psi(\omega y)) - d_0}} \leq \frac{1}{e^{S_{\omega}(\psi(y)) + S_{\omega}(\psi(\omega y)) + d_0}} \leq \int_{[0, 1]} (\nabla f)^2 \left|T^{(n)}_\omega\right|^{-1} d\Lambda \leq \int_{I^n} e^{S_{\omega}(\psi(y)) + S_{\omega}(\psi(\omega y)) - d_0} d\omega.$$

To complete the proof, we can argue in the same way as in the proof of Lemma 5.12. □

**Lemma 5.14.** For every $t > c > 0$, we have

$$\Gamma_t := \{\omega \in I^* : S_{[\omega]} \xi < \log(c/t) \leq S_{[\omega]} \xi\}$$

is a partition of $I^n$. In particular, for every $\omega \in \Gamma_t$ and $x \in I^n$, we have

$$\log(Me^{d_0 t/c}) \geq -S_{[\omega]}(\xi)$$

with $M := \exp\left(\max(-\xi)\right)$ and $d_0 := \log(a_0) + \sum_{k=0}^{\infty} \text{var}_k(\psi)$ with $a_0$ defined in Lemma 5.12.
First note, that two cylinder sets are either disjoint or one is contained in the other. From $\omega \in \Gamma$, and all $\eta \in \Gamma^*$, we have

$$S_{\omega \eta}^\xi \leq \sup_{x \in F^t} S_{\omega \xi}(\omega x) \leq \sup_{x \in F^t} S_{\omega \xi}(\omega x) = S_{\omega \xi} < \log(c/t),$$

where we used $\max \xi < 0$, which shows for $\eta \neq \emptyset$ that $\omega \eta \notin \Gamma$. Moreover, since $\min \xi < 0$, it follows that $S_{\omega \xi}$ converge to $-\infty$ for $|\omega| \to \infty$. Consequently, the set $\Gamma_t$ is finite. In particular, for every $\omega \in \Gamma$ we have $S_{\omega \xi} \xrightarrow{n \to \infty} -\infty$ as $n$ tends to infinity. Therefore, there exists $N \in \mathbb{N}$ such that $S_{\omega \eta}^\xi < \log(c/t) \leq S_{\omega \xi}^\xi$ and the first statement follows. For the second claim fix $\omega \in \Gamma$, then

$$\log(t/c) \geq -S_{\omega \xi}(\omega x) - (S_{\omega \xi} - S_{\omega_{|-1}}(\omega x)) \geq S_{\omega_{|-1}}(\omega x) - d_0 \geq -S_{\omega_{|-1}}(\omega x) - \xi (\varphi_{|0}^{-1}(\omega)x) + \xi (\varphi_{|0}^{-1}(\omega)x) - d_0 \geq -S_{\omega \xi}(\omega x) - \log(M - d_0),$$

and hence we obtain $\log(Mt/c) \geq -S_{\omega \xi}(\omega x)$. □

Recall for $m \in \mathbb{N}$ and $x \in (F^m)^*$

$$\xi_m(x) = \sum_{i=0}^{m-1} \xi (\varphi_i(x)).$$

**Lemma 5.15.** Set $d_0 := \log(a_0) + \sum_{i=0}^{\infty} \text{var}_x \psi$ where $a_0$ is defined in Lemma 5.12. Then for $t > c > 0$, $m \in \mathbb{N}$ such that $-m \max \xi - d_0 > 0$, $x \in (F^m)^*$, we have that

$$\Gamma_{t,m}^L := \{ \omega \in (F^m)^* : -S_{\omega_{|0}^m}(\omega x) \leq \log(t/c) < \min_{x \in F^m} -S_{\omega_{|0}^m}(\omega (\omega x)) \}$$

defines a disjoint family, meaning $\omega \neq \omega'$ implies $[\omega] \cap [\omega'] = \emptyset$. With $k_m := \exp(-m \max \xi)$ and for every $\omega \in (F^m)^*$

$$\log(\varphi_t^{k_m}/(k_m c)) \leq -S_{\omega_{|0}^m}(\omega x) \leq \log(t/c),$$

we have $\omega \in \Gamma_{t,m}^L$.

**Proof.** For every $\omega \in \Gamma_{t,m}^L$ and every $\nu \in F^m$ we have $\log(t/c) < -S_{\omega_{|0}^m}(\omega (\omega x))$ implying $\omega \nu \notin \Gamma_{t,m}$. Further, using $-m \max \xi - d_0 > 0$ and the BDP, for every $\eta \in (F^m)^* \setminus \{\emptyset\}$ we have

$$\log(c/t) > \overline{S}_{\omega_{|0}^m}(\omega (\omega x))$$

$$\geq S_{\omega_{|0}^m}(\omega (\omega x)) - d_0$$

$$= S_{\omega_{|0}^m}(\omega (\omega x)) + \sum_{i=0}^{k_m^{-1}} \left( S_{\omega_{|0}^m}(\omega (\omega x)) + \sum_{i=0}^{k_m^{-1}} \left( S_{\omega_{|0}^m}(\omega (\omega x)) - d_0 \right) \right)$$

$$\geq S_{\omega_{|0}^m}(\omega (\omega x)) - m \cdot |\eta|_{\omega m} \max \xi - d_0$$

$$\geq S_{\omega_{|0}^m}(\omega (\omega x)) - m \cdot \max \xi - d_0$$

$$> \overline{S}_{\omega_{|0}^m}(\omega (\omega x)).$$
Thus, for every $\omega \in \Gamma^1_{\text{m}}$ and $\eta' \in (I^m)^* \setminus \{\varnothing\}$ it follows $\omega \eta' \notin \Gamma^1_{\text{m}}$.

For second assertion fix $x \in (I^m)^*$, $\omega \in (I^m)^*$ and assume
\[
\log\left(\exp(\bar{d}_0/(k_m c))\right) < -\bar{S}_{\text{locm}}\varepsilon^m(\omega x) \leq \log(t/c).
\]
Using the BDP, we obtain for all $v \in I^m$, $\omega \in (I^m)^*$,
\[
|\bar{S}_{\text{locm}}\varepsilon^m(\omega x) - \bar{S}_{\text{locm}}\varepsilon^m(\omega vx)| \leq d_0,
\]
and consequently,
\[
\log(t/c) < \log(k_m) - d_0 - \bar{S}_{\text{locm}}\varepsilon^m(\omega x)
\leq \log(k_m) + \varepsilon^m(vx) - \varepsilon^m(\omega vx) - \bar{S}_{\text{locm}}\varepsilon^m(\omega vx)
\leq -\bar{S}_{\text{locm}}\varepsilon^m(\omega vx).
\]
Since $-\bar{S}_{\text{locm}}\varepsilon^m(\omega x) \leq \log(t/c)$, we conclude $\omega \in \Gamma^1_{\text{m}}$.

Now we are in the position to give the proof of our last main theorem.

**Proof of Theorem 1.7.** Let $\lambda := \lambda_{1,\text{m}}^1$ be the smallest positive eigenvalue of $\Delta_{\text{m}}$. Then by Lemma 5.13 for $\omega \in I^*$ and $t > 0$ with $t < \lambda/\exp(S_{\omega}^{\xi} + d_0) \leq \lambda_{1,\text{m}}^1$, we have
\[
N_{\nu_{\xi}}(t) = N_{\nu_{\xi_{\text{m}}}}(t) = 0.
\]
Now, by Theorem 2.7, for $t > c_R := \lambda e^{-d_0}$, we conclude
\[
N_{\bar{\mu}}(t) \leq \sum_{\omega \in \Gamma^\beta} N_{\nu_{\xi_{\text{m}}}}(t) + 2|\Gamma^\beta| + 1 = 2|\Gamma^\beta| + 1,
\]
where
\[
\Gamma^\beta = \{\omega \in I^* : S_{\omega}^{\xi} < \log(c_R/t) \leq S_{\omega_{\text{m}}}^{\xi}\},
\]
which is a partition of $I^\beta$ by Lemma 5.14 for $t > c_R$. Hence, for the upper bound, we are left to show that $|\Gamma^\beta| \ll t^\nu$. For this we use [25, Theorem 3.2] adapted to our situation, i.e.
\[
Z(x, t) := \sum_{n=0}^{\infty} \sum_{y \in x} 1_{\{S_{x,y}(t) \leq \log t\}} \sim G(x, \log(t)) t^\nu,
\]
where $(x, s) \mapsto G(x, s)$, defined on $I^\beta \times \mathbb{R}_{>0}$, is bounded from above by inspecting the corresponding function $G$ in [25, Theorem 3.2], and $s \mapsto G(x, s)$ is a constant function in the aperiodic case and a periodic function in the periodic case. Therefore, by Lemma 5.14, for $y \in I^\beta$,
\[
\text{card}(\Gamma^\beta) \leq Z\left(y, e^{\max(-\varepsilon^m, \min(\xi_{\text{m}}))}t/\lambda \right) \ll t^\nu.
\]
For the lower estimate we use an approximation argument involving the strong bounded distortion property. Applying Lemma 5.15 for $x \in (I^m)^\beta$ and $m \in \mathbb{N}$ such $\log(k_m) - d_0 > 0$ with $k_m := \exp(- \varepsilon^m)$, we have
\[
\{\omega \in (I^m)^* : \log(\exp(\bar{d}_0/(k_m c))) < -\bar{S}_{\text{locm}}\varepsilon^m(\omega x) \leq \log(t/c)\} \subseteq \Gamma^1_{\text{m}}.
\]
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and

$$\Gamma^I_{r,m} = \left\{ \omega \in (I^m)^*: -\bar{S}_{|\omega|}\xi^m(\omega x) \leq \log(t/c) < \min_{v \in I^m} -\bar{S}_{|\omega|}\xi^m(\omega v x) \right\}$$

with $c := e^{\delta_0}\lambda$. By Lemma 5.13, for $\omega \in \Gamma^L_{r,m}$, we have

$$\lambda^{1}_{\epsilon,\Lambda,I,\omega} \leq \frac{\lambda}{e^{\xi e^{\epsilon}(\omega x) - d_0}} = \frac{c}{e^{\xi e^{\epsilon}(\omega x) - d_0}} \leq t.$$ 

For $t > c$, this leads to

$$N^I_{\epsilon}(t) \geq \sum_{\omega \in \Gamma^L_{r,m}} N^I_{\epsilon,I,\omega}(t) \geq \text{card}\left( \Gamma^I_{r,m} \right) \geq \text{card}\left( \{ \omega \in (I^m)^*: \log(te^{\delta_0}/(k_m c)) < -\bar{S}_{|\omega|}\xi^m(\omega x) \leq \log(t/c) \} \right).$$

We conclude

$$N^I_{\epsilon}(t) \geq \sum_{n=0}^{\infty} \sum_{\omega \in (I^m)^p} 1_{\{ -\bar{S}_{|\omega|}\xi^m(\omega x) \leq \log(t/c) \}} - \sum_{n=0}^{\infty} \sum_{\omega \in (I^m)^p} 1_{\{ -\bar{S}_{|\omega|}\xi^m(\omega x) \leq \log(te^{\delta_0}/(k_m c)) \}}.$$

Moreover, by Lemma 4.3 we have $0 = P(z_{\epsilon}e) = P(z_{\epsilon}\xi^m)$ as defined in Lemma 4.3. Again, [25, Theorem 3.2] applied to $\xi^m$ gives that there exists a function $(x, s) \mapsto \bar{G}(x, s)$ defined on $(I^m)^2 \times \mathbb{R}_{>0}$, which is bounded away from zero by inspecting the corresponding function $G$ in [25, Theorem 3.2], such that

$$\bar{Z}(x, t) := \sum_{n=0}^{\infty} \sum_{\omega \in (I^m)^p} 1_{\{ -\bar{S}_{|\omega|}\xi^m(\omega x) \leq \log(t) \}} \sim \bar{G}(x, \log(t))t^{\epsilon}.$$

In the aperiodic case $s \mapsto \bar{G}(x, s)$ is a constant function and hence in this case we immediately get $t^{\epsilon} \ll N^I_{\epsilon,\Lambda}(t)$. In the periodic case, $s \mapsto \bar{G}(x, s)$ is periodic with
minimal period \( a > 0 \). For \( \ell := \lceil a / (\log(k_m) - d_0) \rceil \), we finally have

\[
N_{\varphi}(t) \geq \frac{1}{\ell} \sum_{i=0}^{\ell-1} \left( \left\lfloor \frac{t}{e^{d_0/k_m}} \right\rfloor^{\eta} - \left\lfloor \frac{t}{e^{d_0/k_m}} \right\rfloor^{\eta+1} \right)
\geq \frac{1}{\ell} \sum_{i=0}^{\ell-1} \text{card} \left( \Gamma_i^{\ell} \cap (e^{d_0/k_m})_m \right)
\geq \frac{1}{\ell} \left( \sum_{i=0}^{\ell-1} \tilde{Z}(x, (e^{d_0/k_m})^i (t/c)) - \tilde{Z}(x, (e^{d_0/k_m})^i+1 (t/c)) \right)
\geq \frac{1}{\ell} \left( \tilde{Z}(x, t/c) - \tilde{Z}(x, (e^{d_0/k_m})^{a/(\log(k_m) - d_0)} (t/c)) \right)
\geq \frac{1}{\ell} \left( \tilde{G}(x, \log(t/c)) \left( \frac{t}{c} \right)^{\gamma_\varphi} - \tilde{G}(x, \log(t/c)) \left( \frac{t}{c} \right)^{\gamma_\varphi} \left( (e^{d_0/k_m})^{a/(\log(k_m) - d_0)} \right)^{\gamma_\varphi} \right)
= \frac{1}{\ell} \left( \frac{t}{c} \right)^{\gamma_\varphi} \left( 1 - (e^{d_0/k_m})^{a/(\log(k_m) - d_0)} \right)
\gg \frac{t}{c},
\]

where we used \( \log(t/c) - \log \left( (e^{d_0/k_m})^{a/(\log(k_m) - d_0)} \right) = a. \)

\( \square \)

**Example 5.16.** A natural ‘geometric’ choice for the potential \( \psi \) is given by \( \delta \varphi \), where \( \delta \geq 0 \) fulfills \( P(\delta \varphi) = 0 \) and, by Bowen’s formula, is equal to the Hausdorff dimension of the self-conformal set \( K \). We then have \( \xi = (1 + \delta) \varphi, P(\delta/(\delta + 1) \xi) = 0 \) and consequently, \( s_\varphi = \delta/(\delta + 1) \).

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