On the Definition of Nondeterministic Mechanisms

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We present here three different approaches to the problem of modeling mathematically the concept of a nondeterministic mechanism. Each of these three approaches leads to a mathematical definition. We then show that all the three mathematical concepts are equivalent to one another. This insight gives us the option of approaching the \textit{wp} formalism of Dijkstra from a different viewpoint that is easier to understand and to teach.

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1. INTRODUCTION

In his well-known book [Dijkstra 1976] Dijkstra speaks of his intention to present “a number of beautiful algorithms in such a way that the reader (can) appreciate their beauty” and do so “by describing the ... design process that would each time lead to the program concerned”.

He then introduces the \textit{wp} formalism. In his hands this becomes a powerful tool to carry out his agenda. Surely this methodology should be more widely taught and learned. Not only that, it is necessary to examine if it can be extended to cover the present programming paradigms. However, the \textit{wp} formalism is hard to learn and use. One is therefore interested in exploring alternative approaches to the formalism that make it simpler to understand and easier to practise. In this article we show that the backward mapping predicate transformers that Dijkstra uses may be effectively replaced by forward mapping state choice maps. It becomes possible then to use the alternative approach suggested by the results of this paper to carry out his agenda in a different and perhaps more transparent manner.
2. THREE DEFINITIONS

The mapcode approach to the understanding of computing concepts in the deterministic case has been elaborated in [Viswanath 2006; 2008] and in the references cited there. This approach models a program as the repeated application of a self-map on a set, following [D.E.Knuth 2002], page 7. It has been shown [Viswanath 2008] that this generic model is sufficient to convey an understanding of many concepts ranging from machine language to neural networks. At the same time it is sufficiently practical to formulate many standard programs rigorously and prove their total correctness.

It is necessary to extend this approach to the study of parallelism and concurrency. To this end it is necessary to first choose a mathematical model for nondeterministic programs. The mapcode philosophy suggests that we set aside for the time being the formal language problems of how to get a machine to do what we want it to do, and strive for clarity in the language of sets and maps as to what exactly we want the machine to do and why.

We start with the concept of a state space $X$. This is the space of variables on which the computation takes place. As in the deterministic case we take the point of view that a generic nondeterministic program consists of repeatedly invoking a nondeterministic mechanism till a stopping condition is met. Thus the focus shifts to the modeling of a nondeterministic mechanism. Looking at the question ab initio we show that there are three natural viewpoints. Fortunately all three turn out to lead to equivalent mathematical structures. We are thus enabled to proceed with the theory of nondeterministic computation in subsequent articles basing ourselves on any one of the definitions studied here.

The first approach is the simplest and the most natural. Given $x \in X$, let $\Delta(x)$ denote a subset of $X$. We can model nondeterminism by requiring that if the current state is $x$, then the mechanism when invoked presents us with one of the states $y$ in $\Delta(x)$ in a finite amount of time. How exactly the state $y$ is produced is hidden from us. It is observed in [Walicki and Meldal 1997] that this is the most common approach.

Our second approach is the one suggested by Dijkstra [Dijkstra 1976]. In this approach the focus shifts from individual states to sets of states and from initial states to final outcomes. We ask the question: given a set $A \subseteq X$ what is the set of all states $\mu(A)$ for which if the initial state $x \in \mu(A)$, then the mechanism when invoked returns with certainty an outcome that is in $A$? If we knew $\mu(A)$ for every $A$, then it is reasonable to feel that we have understood the mechanism well.\footnote{The symbol $\mu$ has been chosen to represent a multiplicative map. Later on, we use the symbol $\alpha$ to denote an additive map.}

Because the map $\mu : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ has been derived by a particular line of reasoning, it is automatically endowed with certain properties. For example, if $A \subseteq B$, then we should have $\mu(A) \subseteq \mu(B)$. After all, if starting in $\mu(A)$ guarantees that we will move into $A$, it should also guarantee that we will move into $B$, because $A \subseteq B$. It is also reasonable to require that $\mu$ should carry the empty set to the empty set.

Let $\{A_j \mid j \in J\}$ be any collection of sets. If starting in $\mu(A_j)$ guarantees the outcome to be in $A_j$, then starting in $\cap \mu(A_j)$ should guarantee the outcome to be in $\cap A_j$. So it is necessary that $\cap \mu(A_j) \subseteq \mu(\cap A_j)$. Because $\cap A_j \subseteq A_j$ for all $j$, by the monotonicity property of $\mu$ just observed, the reverse inequality also holds. So we must have $\mu(\cap A_j) = \cap \mu(A_j)$.
In the case of unions the monotonicity property implies that \( \cup \mu(A_j) \subseteq \mu(\cup A_j) \). A little reflection shows that the reverse inequality need not hold. Given a state \( x \) one may be able to guarantee that the outcome \( y \) is in the union, though \( y \) may not be uniquely determined by \( x \). It is possible that \( y \) could belong to one \( A_j \) on one invocation of the mechanism and in another \( A_j \) for another invocation. So we may not be in a position to say that the outcome \( y \) will definitely be in one of them.

In the discussion above, we used the monotonicity property to establish the intersection preserving property. It is possible to show, and we shall do so later, that the intersection preserving property implies the monotonicity property. So let us choose the defining properties of \( \mu \) to be \( \mu(\emptyset) = \emptyset \) and \( \mu(\cap A_j) = \cap \mu(A_j) \). \( \mu \) is our second definition for a nondeterministic mechanism.

The third approach is similar to the second. Now we ask: given a set \( A \) what is the set \( \alpha(A) \) of all states with the property that we can guarantee that at least one of the outcomes will be in \( A \)? (Earlier we wanted every possible outcome to be in \( A \), now we only ask for at least one outcome in \( A \).) Repeating the thought processes that led us to derive the properties of \( \mu \) it is not difficult to conclude that \( \alpha \) should carry the empty set to itself and preserve arbitrary unions.

In the rest of the article we shall study these three definitions mathematically and show how they relate to one another and to Dijkstra’s theory.

### 3. CHOICE SET MAPS

In what follows \( X \) denotes a non-empty set called the state space. \( \mathcal{P}(X) \) is the powerset of \( X \). \( \emptyset \) denotes the empty set. \( A \) will denote an arbitrary subset of \( X \) and \( \{A_j\} \) will be an arbitrary collection of subsets of \( X \). The symbol \( \doteq \) may be read as ‘is defined to be’.

**Definition 3.1.** A map \( \Delta : X \to \mathcal{P}(X) \) is called a choice set map on \( X \). \( \Delta(x) \) is called the choice set at \( x \). The pair \( (X, \Delta) \) is called a choice structure.

Suppose \( (X, \Delta) \) is a choice structure and \( A \subseteq X \).

**Definitions 3.2.**
1. \( x \in X \) is called a dynamic element of \( \Delta \) if \( \Delta(x) \neq \emptyset \); otherwise it is called a static element. The set of all dynamic elements of \( \Delta \) is denoted by \( \text{dyn}(\Delta) \).
2. \( \Delta^{-1}(A) \doteq \{ x \mid \emptyset \neq \Delta(x) \subseteq A \} \). \( \Delta^{-1}(A) \) is called the inverse image of \( A \) under \( \Delta \). \( \Delta^{-1}(y) \doteq \Delta^{-1}(\{y\}) \). Note that \( \Delta^{-1} : \mathcal{P}(X) \to \mathcal{P}(X) \).
3. \( \Delta_{\text{w}}^{-1}(A) \doteq \{ x \mid \Delta(x) \cap A \neq \emptyset \} \). \( \Delta_{\text{w}}^{-1}(A) \) is called the weak inverse image of \( A \) under \( \Delta \). \( \Delta_{\text{w}}^{-1}(y) \doteq \Delta_{\text{w}}^{-1}(\{y\}) \). Note that \( \Delta_{\text{w}}^{-1} : \mathcal{P}(X) \to \mathcal{P}(X) \).

The examples in Section 6 may be studied in conjunction with the theory being developed here to help understanding.

**Remarks 3.3.** Given \( X, \Delta, A \) and \( \{A_j\} \) we have:

1. \( \Delta^{-1}(\emptyset) = \emptyset = \Delta_{\text{w}}^{-1}(\emptyset) \).
2. \( \Delta^{-1}(A) \subseteq \Delta_{\text{w}}^{-1}(A) \subseteq \text{dyn}(\Delta) \).
\( \Delta^{-1}(\bigcap_j A_j) = \bigcap_j \Delta^{-1}(A_j) \) and \( \Delta^{-1}_w(\bigcup_j A_j) = \bigcup_j \Delta^{-1}_w(A_j) \).

(4) \( \Delta^{-1}_w(A) = \text{dyn}(\Delta) \setminus \Delta^{-1}(A^c) \) and \( \Delta^{-1}(A) = \text{dyn}(\Delta) \setminus \Delta^{-1}_w(A^c) \)

4. MULTIPlicative AND ADDITIVE MAPS

**Definition 4.1.** Suppose \( \mu : \mathcal{P}(X) \to \mathcal{P}(X) \) is a map such that \( \mu(\emptyset) = \emptyset \) and for any \( \{A_j\} \), \( \mu(\bigcap_j A_j) = \bigcap_j \mu(A_j) \). Then \( \mu \) is said to be a multiplicative map. \( \mu(x) \equiv \mu(\{x\}) \).

**Remark 4.2.** If \( \Delta \) is a choice set map on \( X \) then \( \Delta^{-1} \) is a multiplicative map.

**Theorem 4.3.** Suppose \( \mu \) is a multiplicative map.

(1) If \( A_1 \subseteq A_2 \) then \( \mu(A_1) \subseteq \mu(A_2) \).

(2) \( \bigcup \mu(A_j) \subseteq \mu(\bigcup_j A_j) \) for all \( \{A_j\} \). Equality need not hold.

(3) There exists a unique choice set map \( \Delta \) on \( X \) such that \( \mu = \Delta^{-1} \).

**Proof**

(1) The monotonicity property holds because \( A_1 \subseteq A_2 \Rightarrow A_1 = A_1 \cap A_2 \Rightarrow \mu(A_1) = \mu(A_1) \cap \mu(A_2) \Rightarrow \mu(A_1) \subseteq \mu(A_2) \).

(2) The inclusion relation follows from the monotonicity property above. To see that equality need not hold, let \( X = \mathbb{Z} \), the set of integers, and suppose \( \Delta(x) = \{ \pm x \} \) for all \( x \in \mathbb{Z} \). We then have \( \Delta^{-1}(\{0 : \infty\}) = \{0\} \) and similarly \( \Delta^{-1}((\infty : 0]) = \{0\} \), but \( \Delta^{-1}(\mathbb{Z}) = \mathbb{Z} \).

(3) Suppose \( \mu(X) = B \). By the monotonicity property \( A \subseteq X \Rightarrow \mu(A) \subseteq B \). Also, for any \( x \in B \), there is at least one \( A \subseteq X \) such that \( x \in \mu(A) \), namely \( A = X \). Suppose

\[
\Delta(x) = \begin{cases} 
\cap \{A \mid x \in \mu(A)\}, & \text{if } x \in B; \\
\emptyset, & \text{if } x \notin B.
\end{cases}
\]

Let \( x \in B \). We show that then \( x \in \mu(\Delta(x)) \). This proves incidentally that \( \Delta(x) \neq \emptyset \) if and only if \( x \in B \) so that \( \text{dyn}(\Delta) = B \). By the multiplicative property \( \mu(\Delta(x)) = \mu(\cap \{A \mid x \in \mu(A)\}) = \cap \{\mu(A) \mid x \in \mu(A)\} \). Clearly \( x \) is in the set on the right hand side of the above equality. So \( x \in \mu(\Delta(x)) \).

Let \( A \subseteq X \). By definition of \( \Delta(x) \), \( x \in \mu(A) \Rightarrow \emptyset \neq \Delta(x) \subseteq A \Rightarrow x \in \Delta^{-1}(A) \). So \( \mu(A) \subseteq \Delta^{-1}(A) \)

Suppose next that \( x \in \Delta^{-1}(A) \). Then \( \emptyset \neq \Delta(x) \subseteq A \). By the monotonicity property of \( \mu \) this implies that \( \mu(\Delta(x)) \subseteq \mu(A) \). We have already seen that \( x \in \mu(\Delta(x)) \). So \( x \in \mu(A) \). This shows that \( \Delta^{-1}(A) \subseteq \mu(A) \).

Combining the last two observations above we see that \( \mu = \Delta^{-1} \).

To prove the uniqueness of \( \Delta \) suppose there are two choice set maps \( \Delta_1 \) and \( \Delta_2 \) such that \( \Delta_1^{-1} = \mu = \Delta_2^{-1} \). We have then \( \text{dyn}(\Delta_1) = \Delta_1^{-1}(X) = \Delta_2^{-1}(X) = \text{dyn}(\Delta_2) = B \), say.

If \( x \notin B \), \( \Delta_1(x) = \emptyset = \Delta_2(x) \). Suppose \( x \in B \). Let \( \Delta_1(x) = A_1, \Delta_2(x) = A_2 \). Then \( x \in \Delta_1^{-1}(A_1) \Rightarrow x \in \Delta_2^{-1}(A_1) \). So \( \Delta_2(x) \not\subseteq A_1 \), or \( A_2 \not\subseteq A_1 \). By symmetry \( A_1 \not\subseteq A_2 \). Hence \( \Delta_1(x) = \Delta_2(x) \) for all \( x \in B \) also, so that \( \Delta_1 = \Delta_2 \).

One can have a characterization of the \( \Delta^{-1}_w \) map as of the \( \Delta^{-1} \) map by introducing the notion of an additive map as below.

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Definition 4.4. Suppose $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a map such that $\alpha(\emptyset) = \emptyset$ and for any $\{A_j\}$, $\alpha(\cup_j A_j) = \cup_j \alpha(A_j)$. Then $\alpha$ is called an additive map. $\alpha(x) = \alpha(\{x\})$. \hfill $\square$

The proof of the next theorem is left to the reader.

Theorem 4.5. (1) Suppose $\mu$ is a multiplicative map, $\mu(X) = B$, and for any $A$, $\alpha_\mu(A) = B \setminus \mu(A^c)$. Then $\alpha_\mu$ is an additive map.

(2) Suppose $\alpha$ is an additive map, $\alpha(X) = C$, and for any $A$, $\mu_\alpha(A) = C \setminus \alpha(A^c)$. Then $\mu_\alpha$ is a multiplicative map.

(3) $\mu_{\alpha_\mu} = \mu$ and $\alpha_{\mu_\alpha} = \alpha$.

(4) If $\mu = \Delta^{-1}$ then $\alpha_\mu = \Delta_w^{-1}$. If $\alpha = \Delta_w^{-1}$ then $\mu_\alpha = \Delta^{-1}$. \hfill $\square$

The next theorem gives the properties of additive maps.

Theorem 4.6. Suppose $\alpha$ is an additive map.

(1) If $A_1 \subseteq A_2$ then $\alpha(A_1) \subseteq \alpha(A_2)$.

(2) $\alpha(\cap_j A_j) \subseteq \cap_j \alpha(A_j)$ for all $\{A_j\}$. Equality need not hold.

(3) There exists a unique choice set map $\Delta$ on $X$ such that $\alpha = \Delta_w^{-1}$.

Proof

(1) The monotonicity property holds because $A_1 \subseteq A_2 \Rightarrow A_2 = A_1 \cup A_2 \Rightarrow \alpha(A_2) = \alpha(A_1) \cup \alpha(A_2) \Rightarrow \alpha(A_1) \subseteq \alpha(A_2)$.

(2) The inclusion relation follows from the monotonicity property above. To see that equality need not hold, let $X = \mathbb{Z}$, the set of integers, and suppose $\Delta(x) = \{\pm x\}$ for all $x \in \mathbb{Z}$. We then have $\Delta_w^{-1}([0: \infty)) = \mathbb{Z}$ and similarly $\Delta_w^{-1}((-\infty: 0]) = \mathbb{Z}$, but $\Delta_w^{-1}(\{0\}) = \{0\}$.

(3) Let $C = \alpha(X)$. By the monotonicity property $A \subseteq X \Rightarrow \alpha(A) \subseteq C$. Suppose

$$\Delta(x) = \begin{cases} \{y \mid x \in \alpha(y)\}, & \text{if } x \in C; \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Notice that $C = \alpha(X) = \cup_{y \in X} \alpha(y)$. So if $x \in C$ there is at least one $y \in X$ such that $x \in \alpha(y)$. This proves that $\Delta(x) \neq \emptyset$ if and only if $x \in C$ so that $\text{dyn}(\Delta) = C$.

Let $A \subseteq X$. Then $x \in \alpha(A) \Leftrightarrow$ there exists $y \in A$ such that $x \in \alpha(y) \Leftrightarrow$ there exists $y \in \Delta(x) \cap A \Leftrightarrow \Delta(x) \cap A \neq \emptyset \Leftrightarrow x \in \Delta_w^{-1}(A)$. So $\alpha(A) = \Delta_w^{-1}(A)$.

To prove uniqueness, suppose there are two choice set maps $\Delta_1$ and $\Delta_2$ such that $(\Delta_1)_w^{-1} = \alpha = (\Delta_2)_w^{-1}$. In particular then $\text{dyn}(\Delta_1) = (\Delta_1)_w^{-1}(X) = (\Delta_2)_w^{-1}(X) = \text{dyn}(\Delta_2) = C$, say.

If $x \notin C$, $\Delta_1(x) = \emptyset = \Delta_2(x)$. Suppose $x \in C$. Then $y \in \Delta_1(x) \Rightarrow x \in (\Delta_1)_w^{-1}(y) \Rightarrow x \in (\Delta_2)_w^{-1}(y) \Rightarrow \Delta_2(x) \cap \{y\} \neq \emptyset \Rightarrow y \in \Delta_2(x)$. So $\Delta_1(x) \subseteq \Delta_2(x)$. By symmetry $\Delta_2(x) \subseteq \Delta_1(x)$. Hence $\Delta_1(x) = \Delta_2(x)$ for all $x \in C$, so that $\Delta_1 = \Delta_2$. \hfill $\square$

The results proved so far show that

(1) There is a one-to-one correspondence between choice set maps and multiplicative maps.

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(2) There is a one-to-one correspondence between multiplicative maps and additive maps.
(3) There is a one-to-one correspondence between additive maps and choice set maps.
(4) The three correspondences commute.

Suppose \((A_n)\) is a sequence of subsets of \(X\). Recall that \(\lim sup A_n = \cap_n(\cup_{k\geq n}A_k)\) and \(\lim inf A_n = \cup_n(\cap_{k\geq n}A_k)\) and that if both are equal the common value is called \(\lim A_n\). If \((A_n \uparrow)\) denotes a monotone increasing sequence of subsets of \(X\) then \(\lim A_n = \cup A_n\). If \((A_n \downarrow)\) denotes a monotone decreasing sequence of subsets of \(X\) then \(\lim A_n = \cap A_n\). \((A_n)\) is said to be a convergent sequence if \(\lim A_n\) exists.

**Definition 4.7.** A map \(\sigma : \mathcal{P}(X) \to \mathcal{P}(X)\) is said to be continuous if \(\sigma(\lim A_n) = \lim \sigma(A_n)\) for all convergent sequences \((A_n)\).

**Remarks 4.8.** (1) \(\sigma : \mathcal{P}(X) \to \mathcal{P}(X)\) is continuous if and only if \(\sigma(\lim A_n) = \lim \sigma(A_n)\) for all monotone sequences \((A_n)\).
(2) The continuity spoken of here is continuity in the space of sets \(\mathcal{P}(X)\). It needs to be studied how this is related to the concept of continuity in denotational semantics [D.A. Schmidt 1986].

**Theorem 4.9.** Suppose \(\mu, \alpha, \text{ and } \Delta\) correspond to one another. The following are equivalent.

(1) \(\mu\) is continuous.
(2) \(\alpha\) is continuous.
(3) \(\Delta(x)\) is finite for all \(x \in X\).

**Proof:** We have seen earlier that \(\mu(X) = \alpha(X) = \text{dyn}(\Delta)\). Let this set be denoted by \(B\). We first show that statements (1) and (2) are equivalent.

Since \(\mu\) is multiplicative it preserves limits of monotone decreasing sequences. So \(\mu\) is continuous if and only if it preserves limits of monotone increasing sequences. The situation is just the other way round for \(\alpha\) because \(\mu\) and \(\alpha\) are related by the equality \(\alpha(A^c) = B \setminus \mu(A)\).

So \(\mu\) is continuous \(\iff\) \(\mu(\cup A_n) = \cup \mu(A_n), \forall (A_n \uparrow) \iff B \setminus \mu(\cup A_n) = B \setminus \cup \mu(A_n), \forall (A_n \uparrow) \iff \alpha(\cap A_n^c) = \cap \alpha(A_n^c), \forall (A_n \downarrow) \iff \alpha(\cap B_n) = \cap \alpha(B_n), \forall (B_n \downarrow) \iff \alpha\) is continuous.

We next show that (1) and (3) are equivalent. For this it is enough to show that \(\Delta(x)\) is finite for all \(x\) if and only if \(\Delta^{-1}\) preserves limits of increasing sequences of sets.

Suppose \(\Delta(x)\) is finite for all \(x \in X\) and let \((A_n \uparrow)\). By Theorem 4.3 we have \(\cup(\Delta^{-1}(A_n)) \subseteq \Delta^{-1}(\cup A_n)\). To prove the opposite inequality suppose \(x \in \Delta^{-1}(\cup A_n)\). Then \(\Delta(x) \subseteq \cup A_n\). Since \(\Delta(x)\) is finite and \(A_n\)'s are monotone, there exists \(m\) such that \(\Delta(x) \subseteq A_m\). So \(x \in \Delta^{-1}(A_m) \subseteq \cup(\Delta^{-1}(A_n))\).

To prove the converse, suppose \(\Delta(x)\) is infinite for some \(x\), say \(\Delta(x) = \{y_1, y_2, \ldots\}\). Let \(A_n = \{y_1, y_2, \ldots, y_n\}\). Then, whatever be \(n\), \(\Delta(x) \not\subseteq A_n\) so that \(x \notin \Delta^{-1}(A_n)\) and hence \(x \notin \cup \Delta^{-1}(A_n)\). But \(x \in \Delta^{-1}(\cup A_n)\). So \(\Delta^{-1}\) is not continuous.

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5. CONVERGENCE

Definitions 5.1. Suppose $\Delta, \Delta_1,$ and $\Delta_2$ are choice set maps on $X$.

1. For $A \subseteq X$, $\Delta(A) \triangleq \cup \{ \Delta(x) \mid x \in A \}$, if $A \neq \emptyset$; and $\Delta(\emptyset) \triangleq \emptyset$.
2. For $x \in X$, $(\Delta_2 \circ \Delta_1)(x) \triangleq \Delta_2(\Delta_1(x))$. $\Delta_2 \circ \Delta_1$ is a choice set map on $X$ called the composition of $\Delta_2$ with $\Delta_1$. $\Delta^0(x) \triangleq \{x\}$ so that $\Delta^0(A) = A$ for any $A \subseteq X$, and recursively for $k \geq 1$, $\Delta^k \triangleq \Delta \circ \Delta^{k-1} \triangleq \Delta^{k-1} \circ \Delta$.

Definitions 5.2. Let $\Delta$ be a choice set map on $X$.

1. $\text{fix}(\Delta) = \{ x \mid \Delta(x) = \{x\} \}$ is called the set of fixed points of $\Delta$.
2. $\text{stab}(\Delta) = \{ x \mid \Delta^n(x) \subseteq \text{dyn}(\Delta) \text{ for all } n \geq 0 \}$ is called the set of stable points of $\Delta$.
3. $\text{con}^\Delta = \{ x \mid x \in \text{stab}(\Delta), \Delta^k(x) \subseteq \text{fix}(\Delta) \exists k \geq 0 \}$ is called the set of convergent points of $\Delta$.
4. $\text{con}^\Delta_w = \{ x \mid \Delta^k(x) \cap \text{fix}(F) \neq \emptyset \exists k \geq 0 \}$ is said to be the set of weakly convergent points of $\Delta$.

Remarks 5.3. (1) $\text{fix}(\Delta) \subseteq \text{con}^\Delta \subseteq \text{con}^\Delta_w \cap \text{stab}(\Delta)$.
2. $\Delta^k(x) \cap \text{fix}(F) \subseteq \Delta^{k+1}(x) \cap \text{fix}(F)$ for $k \geq 1$.
3. $\Delta^{-1}(\text{stab}(\Delta)) \subseteq \text{stab}(\Delta)$.

The definitions of convergence and weak convergence given above are conceptually easy to understand but verifying convergence using these definitions is not convenient in practice. So we give below a more practical characterization of convergence.

Definitions 5.4. (1) If $y \in \Delta(x)$ we write $x \mapsto y$ and say that $x$ maps to $y$. $\mapsto$ defines a binary relation on $X$.
2. For $n \geq 1$, a finite sequence $(x_0, x_1, \ldots, x_n)$ of elements of $X$ is called a run of length $n$ starting at $x_0$ and ending at $x_n$ if $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n$. In this case $x_1 \in \Delta(x_0)$, $x_2 \in \Delta(x_1)$, $\cdots$, $x_n \in \Delta(x_{n-1})$. Also $x_n \in \Delta^n(x_0)$.
3. If $(x_0, x_1, \ldots, x_n)$ is a run we write $x_0 \mapsto^* x_n$ and say that $x_n$ is reachable from $x_0$. It may be observed that $\mapsto^*$ is the transitive closure of $\mapsto$.
4. If $(x_0, x_1, \ldots, x_n)$ is a run and $0 < m < n$ then $(x_0, x_1, \ldots, x_m)$ is also a run. In such a case, we say that $(x_0, x_1, \ldots, x_m)$ is an extension of $(x_0, x_1, \ldots, x_m)$.
5. A run is said to be aborted if it ends in a state that is not in $\text{dyn}(\Delta)$; that is, if it has no extension.
6. A run is said to be terminal if it ends in a fixed point of $\Delta$. If $(x_0, x_1, \ldots, x_n)$ is a terminal run and $m$ is the least positive integer such that $x_m \in \text{fix}(F)$, then $x_m = x_{m+1} = \cdots = x_n$.

Theorem 5.5. Suppose $\Delta$ is a choice set map on $X$ and $x \in X$.

1. $x \in \text{con}(\Delta)$ if and only if
   (a) there are runs at $x$;
   (b) every run at $x$ can be extended;

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(c) there exists $k \geq 1$ such that every run starting at $x$ of length $k$ or more is a terminal run.

(2) $x \in \text{con}_w(\Delta)$ if and only if there exists a run starting at $x$ that is terminal.

**Proof**

(1) Suppose that $x \in \text{con}(\Delta)$ so that $\Delta^n(x) \subseteq \text{dyn}(\Delta)$ for all $n \geq 0$, and $\Delta^k(x) \subseteq \text{fix}(F)$ for some $k \geq 1$. We need to prove (a), (b), and (c).

Since $x \in \text{dyn}(\Delta)$, $\Delta(x)$ is not empty. Let $y \in \Delta(x)$. Then $(x, y)$ is a run. So there are runs at $x$. This argument can be repeated with the last element of the run replacing $x$ above. This shows that any run at $x$ can be extended.

Consider any run $(x, x_1, x_2, \ldots, x_k)$ of length $k$. Then $x_k \in \Delta^k(x) \subseteq \text{fix}(F)$. Hence it is a terminal run.

Conversely assume that $x$ satisfies (a), (b) and (c). Since there are runs at $x$, $x \in \text{dyn}(\Delta)$. Since every run at $x$ can be extended it follows that $\Delta^n(x) \subseteq \text{dyn}(\Delta)$ for every $n \geq 0$. So $x$ is stable. Let $k$ be given by (c). We need to prove that $\Delta^k(x) \subseteq \text{fix}(F)$. Let $x_k \in \Delta^k(x)$. Since $k > 1$, $\Delta^k(x) = \Delta(\Delta^{k-1}(x))$. So there exists $x_{k-1} \in \Delta^{k-1}(x)$ such that $x_k \in \Delta(x_{k-1})$. Continuing in this way we can construct a run $(x, x_1, \ldots, x_k)$. Since this run has length $k$ it is terminal. So $x_k \in \text{fix}(F)$.

Since $x_k$ was chosen to be an arbitrary element in $\Delta^k(x)$ it follows that $\Delta^k(x) \subseteq \text{fix}(F)$.

(2) Suppose $x \in \text{con}_w(\Delta)$. Then there exists $k \geq 1$ such that $\Delta^k(x) \cap \text{fix}(F) \neq \emptyset$. So there exists an element $x_k \in \Delta^k(x) \cap \text{fix}(F)$. Since $x_k \in \Delta^k(x) = \Delta(\Delta^{k-1}(x))$ and $k > 1$, there exists $x_{k-1} \in \Delta^{k-1}(x)$ such that $x_k \in \Delta(x_{k-1})$. Continuing inductively we get a sequence $x_i, 1 \leq i \leq k$ such that $x \mapsto x_1 \mapsto \cdots \mapsto x_k \in \text{fix}(F)$. Its length is $k$.

Conversely assume that there exists a terminal run starting at $x$ of length $k$. Then there exist $x_i, 1 \leq i \leq k$, such that $x \mapsto x_1 \mapsto \cdots \mapsto x_k \in \text{fix}(F)$. Then $x_k \in \Delta^k(x)$. Hence $\Delta^k(x) \cap \text{fix}(F) \neq \emptyset$. □

Let $\Delta$ be a choice set map on $X$. We had observed in Remarks 5.3 that the sequence of sets $\Delta^k(x) \cap \text{fix}(F)$ is monotonically increasing.

**Definition 5.6.** Let $(X, \Delta)$ be a choice structure. For any $x \in X$ define $\Delta^\infty(x) = \cup (\Delta^k(x) \cap \text{fix}(F))$. The choice set map $\Delta^\infty$ is called the limit map of $\Delta$. Elements of $\Delta^\infty(x)$ are called the limit points of $\Delta$ at $x$. □

**Remarks 5.7.** (1) $\text{fix}(\Delta^\infty) = \text{fix}(\Delta)$.

(2) $\Delta^\infty(x) = \{ y : x \mapsto y, y \in \text{fix}(F) \}$. These are the points of $\text{fix}(F)$ that are reachable from $x$.

(3) $x \in \text{con}(\Delta)$ if and only if every run at $x$ when sufficiently extended ends up in $\text{fix}(\Delta)$. The set of all such reachable points of $\text{fix}(\Delta)$ is precisely $\Delta^\infty(x)$.

(4) $x \in \text{con}_w(\Delta)$ if and only if $\Delta^\infty(x) \neq \emptyset$ so that $\text{dyn}(\Delta^\infty) = \text{con}_w(\Delta)$.

(5) Suppose $\Delta(x)$ is finite for all $x$. If $x \in \text{con}(\Delta)$ then there exists $k \geq 1$ such that $\Delta^k(x) \subseteq \text{fix}(F)$. In such a case $\Delta^k(x) = \Delta^\infty(x)$. In particular $\Delta^\infty(x)$ is finite. So it is impossible to have a convergent choice structure with $\Delta(x)$ finite and $\Delta^\infty(x)$ infinite for $x \in X$. It is this fact that Dijkstra is
pointing out when he says [Dijkstra 1976] that there can not exist a program that says “set \( x \) to any positive integer”. Example 6.3 illustrates this point further.

**Definition 5.8.** For any \( A \subseteq X \) the set \( \text{bas}(\Delta, A) = \{ x \in \text{con}(\Delta) \mid \Delta^{\infty}(x) \subseteq A \} \) is called the basin of \( A \) with respect to \( \Delta \).

**Remarks 5.9.** (1) Recalling Definition 3.2 we see that \( \text{bas}(\Delta, A) = \text{con}(\Delta) \cap (\Delta^{\infty})^{-1}(A) \) for all \( A \subseteq X \).

(2) It is not true that \( \text{bas}(\Delta, A) = (\Delta^{\infty})^{-1}(A) \) for all \( A \subseteq X \) if \( \text{con}(\Delta) \neq \text{con}_{w}(\Delta) \). For let \( x \in \text{con}_{w}(\Delta) \setminus \text{con}(\Delta) \) and take \( A = \text{fix}(\Delta) \). Then \( \Delta^{\infty}(x) \subseteq A \) so that \( x \in (\Delta^{\infty})^{-1}(A) \) but \( x \notin \text{bas}(\Delta, A) \).

6. **Examples**

**Example 6.1.** Suppose \( X \) is any set and \( \Delta(x) = \emptyset \) for all \( x \). Then \( \text{dyn}(\Delta) = \text{fix}(\Delta) = \emptyset \), \( \Delta_{w}^{-1}(A) = \Delta^{-1}(A) = \emptyset \) for all \( A \subseteq X \), and \( \Delta^{k}(x) = \emptyset \) for all \( x \in X \). No state maps to any state nor yields any state. There are no runs, no stable points, no convergent points and no weakly convergent points. So \( \text{stab}(\Delta) = \text{con}(\Delta) = \emptyset \). Also \( \Delta^{\infty}(x) = \emptyset \), \( x \in X \) and \( \text{bas}(\Delta, A) = \emptyset \) for every \( A \subseteq X \). \( \Delta \) may be identified with the *abort* command.

**Example 6.2.** Suppose \( X \) is any set and \( \Delta(x) = \{ x \} \) for all \( x \in X \). Then \( \text{dyn}(\Delta) = \text{fix}(\Delta) = X \), \( \Delta_{w}^{-1}(A) = \Delta^{-1}(A) = A \) for all \( A \in \mathcal{P}(X) \), and \( \Delta^{k}(x) = \{ x \} \) for all \( x \). Every element maps to itself and yields only itself. Every run is of the form \( (x, x, \cdots, x) \) and is terminal. Every element yields only itself. So \( \text{stab}(\Delta) = \text{con}(\Delta) = \text{con}_{w}(\Delta) = X \). \( \Delta^{\infty}(x) = \{ x \} \) for all \( x \in X \). \( \text{bas}(\Delta, A) = A \) for all \( A \in \mathcal{P}(X) \). This structure may be identified with *skip*, because leaves everything unchanged.

**Example 6.3.** Suppose \( X \) is any infinite set and \( \Delta(x) = X \) for all \( x \). Then \( \text{dyn}(\Delta) = \text{stab}(\Delta) = X \), \( \text{fix}(\Delta) = \emptyset \), \( \Delta^{-1}(X) = X \) and \( \Delta^{-1}(A) = \emptyset \) if \( A \neq X \). \( \Delta_{w}^{-1}(A) = X \) if \( \emptyset \neq A \subseteq X \). \( \Delta^{k}(x) = X \) for all \( x \in X \) and \( k \geq 1 \). Every element maps to every other element and yields every other element. Any finite sequence of elements of \( X \) is a run and no run is terminal. There are no convergent points or weakly convergent points, so that \( \text{con}(\Delta) = \emptyset = \text{con}_{w}(\Delta) \). \( \Delta^{\infty}(x) = \emptyset \) for all \( x \in X \). \( \text{bas}(\Delta, A) = \emptyset \) for every \( A \in \mathcal{P}(X) \).

**Example 6.4.** Suppose \( F : X \to X \) is a map and \( \Delta(x) = \{ F(x) \} \) for all \( x \in X \). We call \( \Delta \) a deterministic map. In this case \( \text{dyn}(\Delta) = X \), \( \text{stab}(\Delta) = X \), \( \text{fix}(\Delta) = \text{fix}(F) \), and all the definitions we have given above reduce to the corresponding definitions for the deterministic flow \( (X, F) \) as given in [Viswanath 2008]. We have \( \Delta^{k}(x) = \{ F^{k}(x) \} \) for all \( x \) and \( k \). \( \text{con}(\Delta) = \text{con}_{w}(\Delta) = \text{con}(F) \). \( \Delta_{w}^{-1}(A) = \Delta^{-1}(A) = F^{-1}(A) \) for all \( A \in \mathcal{P}(X) \). Further \( \Delta^{\infty}(x) = \{ F^{\infty}(x) \} \) for all \( x \in \text{con}(F) \), and \( \text{bas}(\Delta, A) = \cup_{k \geq 0} F^{-k}(A \cap \text{fix}(F)) \).

**Example 6.5.** Let \( X = \mathbb{N} \) and suppose \( \Delta \) is a choice set map on \( \mathbb{N} \) defined by

\[
\Delta(x) = \begin{cases} 
\{0\}, & \text{if } x = 0; \\
\{x - 1, x + 1\}, & \text{if } x > 0.
\end{cases}
\]

Then \( \text{dyn}(\Delta) = \text{stab}(\Delta) = \mathbb{N} \) and \( \text{fix}(\Delta) = \{0\} \). If \( x > 0 \), \( (x, x - 1, x - 2, \cdots, 0) \) is a terminal run of length \( x \). It follows that every state is weakly convergent.
It may be noted that for any $x$ and $n > 0$, a run of the form $(x, x + 1, x + 2, \ldots, x + n, x + n - 1, \ldots, x, x - 1, x - 2, \ldots, 0)$ is also terminal with length $x + 2n$. So there exist arbitrarily long terminal runs at any $x$. At the same time for any $x$, $(x, x + 1, x + 2, \ldots, x + n)$ is a nonterminal run for every $n$. So there also exist arbitrarily long nonterminal runs starting at every $x > 0$. It follows that no state is convergent except 0.

For this example $\text{con}(\Delta) = \text{fix}(\Delta) = \{0\}$ and $\text{con}_w(\Delta) = X$. $\Delta^\infty(x) = \{0\}$ for all $x \in N$. $\text{bas}(\Delta, A) = \{0\} \Leftrightarrow 0 \in A$. □

**Example 6.6.** In this example we show that the sets $\text{dyn}(\Delta), \text{stab}(\Delta), \text{con}(\Delta), \text{con}_w(\Delta)$ can all be different. Let $X = \mathbb{Z}$ and suppose $\Delta$ is defined by

$$
\Delta(x) = \begin{cases}
\{x - 2, x + 2\}, & \text{if } x \geq 0, x \neq 2; \\
\{2\}, & \text{if } x = 2; \\
\emptyset, & \text{if } x < 0.
\end{cases}
$$

Then $\text{dyn}(\Delta) = N$, $\text{stab}(\Delta) = 2N \setminus \{0\}$, and $\text{fix}(\Delta) = \{2\}$. If $x > 0$ is an odd number, $\Delta^k(x)$ contains only odd numbers and hence $\Delta^k(x) \cap \text{fix}(\Delta) = \emptyset$ for every $k \geq 0$.

If $x = 0$, $(0, 2)$ is a terminal run of length 1. If $x > 0$ is even, $(x, x - 2, x - 4, \ldots, 0)$ is a terminal run of length $x/2$. It may be noted that $(0, -2)$ is an aborted run. For any $x > 2$, $x$ even, and $n > 0$, a run of the form $(x, x + 2, x + 4, \ldots, x + 2n, x + 2n - 2, \ldots, x, x - 2, x - 4, \ldots, 2)$ is a terminal with length $2n + (x/2) - 1$. So there exist arbitrarily long terminal runs at any even $x, x > 2$. At the same time for any such $x$, $(x, x + 2, \ldots, x + 2n)$ is a nonterminal run for every $n$. So there also exist arbitrarily long nonterminal runs starting at every even $x > 2$.

It follows that $\text{con}_w(\Delta) = 2N$ and $\text{con}(\Delta) = \{2\}$. $\Delta^\infty(x) = \{2\}$ for all $x \in 2N$. $\text{bas}(\Delta, A) = \{2\} \Leftrightarrow 2 \in A$ and $\text{bas}(\Delta, A) = \emptyset \Leftrightarrow 2 \notin A$. □

**Example 6.7.** In the above example we saw that there exist $x \in X$ such that there could be terminal runs of arbitrary length starting at $x$. However all of the runs end up in the same final state. The present example [Dijkstra 1976] is one where there are terminal runs of arbitrary length that start at the same state but end up at different states.

Let $X = N \times \{0, 1\}$. Define $\Delta$ by

$$
\Delta(x, y) = \begin{cases}
\{(x, 0), (x + 1, 1)\}, & \text{if } y = 1; \\
\{(x, 0)\}, & \text{if } y = 0.
\end{cases}
$$

It is left to the reader to check that $\text{fix}(\Delta) = \text{con}(\Delta) = N \times \{0\}$, $\text{stab}(\Delta) = \text{con}_w(\Delta) = X$, $\Delta^\infty(x, 0) = \{(x, 0)\}$ and $\Delta^\infty(x, 1) = \{(x + n, 0) \mid n \geq 0\}$. □

7. **DIJKSTRA’S IF AND DO CONSTRUCTS**

After describing the concept of a state and introducing the state space (which we have called $X$) Dijkstra [Dijkstra 1976](p.13) introduces the notion of a nondeterministic machine. He says that “activation (of such a machine) in a given initial state will give rise to one out of a class of possible happenings, the initial state only fixing the class as a whole”. We have interpreted this statement to mean that for every...
We are given a set \( \Delta(x) \subseteq X \) such that if \( x \) is the initial state, then \( \Delta(x) \) is the set of all possible happenings when the nondeterministic machine is invoked once. Thus a choice structure \( (X, \Delta) \) is our model for a nondeterministic machine.

However, even after almost defining a choice structure, Dijkstra does not formalize nondeterminism in this way. He says that “the design of such a system is a goal-directed activity, in other words that we want to achieve something with that system.” What we want to achieve is a “post-condition”. That is to say after the machine is invoked we want to insure that the resulting state belongs to a certain set \( A \subseteq X \). He then says that “we should like to know .... the set of (all) initial states such that activation will certainly result in a properly terminating happening leaving the system in the final state satisfying the post-condition”. In our notation this is \( \Delta^{-1}(A) \). This set he calls the “weakest pre-condition” and denotes it by \( wp(S, A) \), where \( S \) is his notation for the mechanism. Without giving a definition of \( S \) directly he wants to characterize it by the map \( A \mapsto wp(S, A) \). He shows that, as we have done in Section 2, that this map is multiplicative in \( A \). So, for Dijkstra, every nondeterministic mechanism is given by a multiplicative map. In our notation we shall henceforth take \( wp(\Delta, A) \) to be the same as \( \Delta^{-1}(A) \).

We need now to connect the theory of nondeterminism developed so far using \( \Delta \) to the theory that may be developed using \( \Delta^{-1} \). Before doing that, we shall define the structures \( IF \) and \( DO \) directly in terms of choice set maps and derive the two main theorems about them to show how simple the definitions and proofs are in our approach.

A patch on \( X \) is a pair \((D, F)\) where \( D \subseteq X \) and \( F : D \rightarrow X \) [Viswanath 2008]. A patch \((D, F)\) can be interpreted to be a guarded command. Its action is first to check if a given state \( x \) is in \( D \). If it is, \( x \) is changed to \( F(x) \). If it is not, then no action is taken.\(^2\)

**Definitions 7.1.** 
1. A quilt \( Q \) is a collection of patches: \( Q = \{(D_1, F_1), (D_2, F_2), \ldots, (D_k, F_k)\} \).
2. Given a quilt \( Q \) let \( D = \bigcup_{1 \leq i \leq k} D_i \) and define the choice set map \( \Delta_Q \) by
   \[
   \Delta_Q(x) = \begin{cases} 
   \{F_i(x) \mid x \in D_i \text{ for some } i\} & \text{if } x \in D; \\
   \{x\} & \text{if } x \notin D
   \end{cases}
   \]

   By definition, \( \Delta_Q(x) \neq \emptyset \) for all \( x \in X \). So \( \text{dyn}(\Delta_Q) = X \). What about \( \text{fix}(\Delta_Q) \)? Clearly \( D^c \subseteq \text{fix}(\Delta_Q) \). There could be points of \( D \) also in \( \text{fix}(\Delta_Q) \). Let \( E = \{x \in D \mid x \in D_i \Rightarrow F_i(x) = x\} \). Then \( E \subseteq \text{fix}(\Delta_Q) \) and in fact \( \text{fix}(\Delta_Q) = D^c \cup E \). It is to be noted that the set \( E \) is not mentioned explicitly by Dijkstra.

**Definition 7.2.** Let \( Q \) be a quilt as above let \( D = \bigcup_{1 \leq i \leq k} D_i \). The choice structure \( \Delta_{IF} \) is defined by

\[
\Delta_{IF}(x) = \begin{cases} 
\Delta_Q(x) & \text{if } x \in D; \\
\emptyset & \text{if } x \notin D
\end{cases}
\]

\(^2\)In Dijkstra’s definition of a guarded command \((D, F)\) the map \( F \) is taken to be a global map, for certain technical reasons which do not concern us here.
We then have $\text{dyn}(\Delta_{IF}) = D$ and $\text{fix}(\Delta_{IF}) = E$.

The “basic theorem for the alternative construct” takes the following form. The proof follows immediately from the definitions of $\Delta_Q$ and $\Delta_{IF}$.

**Theorem 7.3.** Let $A, B \subseteq X$ be such that $A \subseteq D$, and $F_j(A \cap D_j) \subseteq B$ for all $j$. Then $\Delta_{IF}(A) \subseteq B$.

Next, let us consider the repetitive construct $DO$. It seems natural to define it by either $\Delta_0^\infty_Q$ or $\Delta_1^\infty_{IF}$. However, Dijkstra does neither, for two reasons. The first is that he does not want the points that are weakly convergent for $\Delta_Q$, but not convergent, in the domain of $DO$. Secondly, he does not consider the computation terminated unless the state enters $D^c$. This means that if the state finds itself in the set $E$ then, even though it is a fixed point, for both $\Delta_Q$ and $\Delta_{IF}$, the computation is not considered to terminate: The points of $E$ should be considered to be the points where the computation “hangs”.

To construct a properly terminating program guaranteeing an outcome in $D^c$ we need therefore to take away from $\text{dyn}(\Delta_Q)$ all the points that are weakly convergent but not convergent, and also all those points that end up in $E$. This means that we need to restrict ourselves to the set $\text{bas}(\Delta_Q, D^c)$. By Remark 5.9 this is the set $\text{con}(\Delta_Q) \cap (\Delta_0^\infty_Q)^{-1}(D^c)$. We have then the following definition.

**Definition 7.4.** Let a quilt $Q$ be given as above. Then the choice structure $\Delta_{DO}$ is defined by

$$\Delta_{DO}(x) = \begin{cases} \Delta_0^\infty_Q(x), & \text{if } x \in \text{bas}(\Delta_Q, D^c), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly $\text{dyn}(\Delta_{DO}) = \text{bas}(\Delta_Q, D^c)$ and $\text{fix}(\Delta_{DO}) = D^c$. Also $\Delta_{DO}(X) \subseteq D^c$.

The “fundamental invariance theorem for loops” takes the following form.

**Theorem 7.5.** Let $V \subseteq X$ be such that $\Delta_{IF}(V \cap D) \subseteq V$. Then $\Delta_{DO}(V \cap \text{con}(\Delta_Q)) \subseteq V \cap D^c$.

**Proof** Since $\Delta_{IF} = \Delta_Q$ on $D$ we are given that $\Delta_Q(V \cap D) \subseteq V$.

Let $x \in V \cap \text{con}(\Delta_Q)$. If $x \in D^c$ there is nothing to prove. So let $x \in D$. Since $x \in \text{con}(\Delta_Q)$ there exists $k \geq 0$ such that $\Delta^k(x) = \Delta_0^\infty_Q(x)$. Let $y \in \Delta^k(x)$. Then $y \in D^c$ and there exists a run $x = x_0, x_1, \ldots, x_k = y$. Let $j$ be the least integer such that $x_j \in D^c$. Then $x_j = y$ and $x_0, x_1, \ldots, x_{j-1} \in D$. So we have successively

$$x_0 \in V \cap D \Rightarrow x_1 = \Delta_Q(x_0) \in V \cap D \Rightarrow x_2 = \Delta_Q(x_1) \in V \cap D \Rightarrow \cdots \Rightarrow x_{j-1} = \Delta_Q(x_{j-2}) \in V \cap D \Rightarrow x_j = \Delta_Q(x_{j-1}) \in V \cap D^c.$$  

The theorem is proved. \hfill \Box

We have thus seen that using the formalism of choice set maps it is very easy to understand the structures $IF$ and $DO$. We now need to prove that our definitions coincide with Dijkstra’s.
Let us consider $IF$ first, and let us consider the special case when there is only one patch $(D, F)$. In this case $wp\ (F, A)$ is described on p.17 of [Dijkstra 1976] by the following sentence (notation changed): “If the initial state satisfies $wp\ (F, A)$, the mechanism is certain to establish eventually the truth of $A$”. This means that if $x \in wp\ (F, A)$ then $F(x) \in A$. Or $wp\ (F, A) = F^{-1}(A)$.

Consider next the case of a general quilt $Q$ as above. Then
\[
\Delta_{IF}^{-1}(A) = \{x \mid \emptyset \neq \Delta_{IF}(x) \subseteq A\}
\]
\[
= \{x \in D \mid \Delta_{IF}(x) \subseteq A\}
\]
\[
= \{x \in D \mid x \in D_i \Rightarrow F_i(x) \in A\}
\]
\[
= D \cap \{x \mid x \in D_i \Rightarrow x \in F_i^{-1}(A)\}
\]
\[
= D \cap \{x \mid x \in D_i \Rightarrow x \in wp\ (F, A)\}
\]

But this is exactly the definition of $wp\ (IF, A)$ on p.34 of [Dijkstra 1976]. So $wp\ (IF, A) = \Delta_{IF}^{-1}(A)$ for all $A \subseteq X$.

For the $DO$ construct also we need to show that $\Delta_{DO}^{-1}(A) = wp\ (DO, A)$ for all $A \subseteq X$. This takes some hard work. By Definition 7.4 we see that $\Delta_{DO}^{-1}(A) = {\text{bas}}(Q, A \cap D^c)$. So we need to show that $\{\text{bas}(Q, A \cap D^c) = wp\ (DO, A)\}$. For this purpose, we need to first characterize the set $\{\text{bas}(\Delta, A)\}$ in terms of iterates of $\Delta^{-1}$ for any choice set map $\Delta$ and for any $A \subseteq X$.

**Definition 7.6.** Given a choice set map $\Delta$ and $A \subseteq X$, $\Delta^{-k}(A) = (\Delta^{-1})^k(A)$ for $k \geq 1$ and $(\Delta^{-1})^0 = \Delta^0$.

It is natural to ask ourselves at this stage how $(\Delta^{-1})^k$ is related to $(\Delta^k)^{-1}$. First of all we note that they need not be equal.

**Example 7.7.** Let $X = \{a, b, c\}$, and let $\Delta(a) = \{b, c\}$, $\Delta(b) = \emptyset$, $\Delta(c) = \{c\}$. Then $\Delta^2(a) = \{c\}$, so that $a \in (\Delta^2)^{-1}(c)$. But $(\Delta^{-1})^2(c) = \Delta^{-1}(c) = \{c\}$.

We have the following result.

**Lemma 7.8.** Let $\Delta$ be a choice set map on $X$, $A \subseteq X$, and $k > 0$. Then

1. $(\Delta^{-1})^k(A) \subseteq (\Delta^k)^{-1}(A)$;
2. $(\Delta^k)^{-1}(A) \cap \text{stab}(\Delta) \subseteq (\Delta^{-1})^k(A)$.

**Proof** We prove the theorem by induction on $k$.

1. For $k = 1$ equality holds. Assume the result for $k$.

   $x \in (\Delta^{-1})^{k+1}(A) \Rightarrow \emptyset \neq \Delta(x) \subseteq (\Delta^{-1})^{k}(A)$
   \[
   \Rightarrow \emptyset \neq \Delta(x) \subseteq (\Delta^k)^{-1}(A)
   \]
   \[
   \Rightarrow \emptyset \neq (\Delta^k)^{-1}(x) \subseteq A
   \]
\[ \Rightarrow x \in (\Delta^{k+1})^{-1}(A) \]

Hence \((\Delta^{-1})^{k+1}(A) \subseteq (\Delta^{k+1})^{-1}(A)\).

(2) For \(k = 1\) the relation holds. Assume that \((\Delta^{-k})(A) \cap \text{stab}(\Delta) \subseteq (\Delta^{-1})^{k}(A)\). Then

\[
x \in (\Delta^{k+1})^{-1}(A) \cap \text{stab}(\Delta) \Rightarrow x \in \text{stab}(\Delta) \text{ and } \emptyset \neq \Delta^{k}(\Delta(x)) = \Delta^{k+1}(x) \subseteq A,
\]

\[
\Rightarrow x \in \text{stab}(\Delta) \text{ and } \emptyset \neq \Delta^{k}(y) \subseteq A \text{ for every } y \in \Delta(x)
\]

\[
\Rightarrow x \in \text{stab}(\Delta) \text{ and } y \in (\Delta^{k})^{-1}(A) \text{ for every } y \in \Delta(x)
\]

\[
\Rightarrow x \in \text{stab}(\Delta) \text{ and } y \in (\Delta^{k})^{-1}(A) \cap \text{stab}(\Delta) \text{ for every } y \in \Delta(x)
\]

\[
\Rightarrow x \in (\Delta^{-1})^{k+1}(A)
\]

This proves that \((\Delta^{k+1})^{-1}(A) \cap \text{stab}(\Delta) \subseteq (\Delta^{-1})^{k+1}(A)\). \qed

**Remark 7.9.** It follows from Remark 5.3 that \(\Delta^{-k}(\text{stab}(\Delta)) \subseteq \text{stab}(\Delta)\), for \(k \geq 0\). In particular \(\Delta^{-k}(\text{fix}(\Delta)) \subseteq \text{stab}(\Delta)\), for \(k \geq 0\). \(\Box\)

**Theorem 7.10.** For any \(A \subseteq X\), \(\text{bas}(\Delta, A) = \bigcup_{k \geq 0} \Delta^{-k}(A \cap \text{fix}(\Delta))\).

**Proof** It is enough to consider the case \(A \cap \text{fix}(\Delta) \neq \emptyset\).

Suppose \(x \in \text{bas}(\Delta, A)\). Then \(x \in \text{con}(\Delta) \subseteq \text{stab}(\Delta)\), and there exists \(k \geq 0\) such that \(\emptyset \neq \Delta^{\infty}(x) = \Delta^{k}(x) \subseteq A \cap \text{fix}(\Delta)\). This implies that \(x \in (\Delta^{k})^{-1}(A \cap \text{fix}(\Delta)) \cap \text{stab}(\Delta) \subseteq (\Delta^{-1})^{k}(A \cap \text{fix}(\Delta))\) and hence \(x \in \Delta^{-k}(A \cap \text{fix}(\Delta))\).

So \(\text{bas}(\Delta, A) \subseteq \bigcup_{k \geq 0} \Delta^{-k}(A \cap \text{fix}(\Delta))\).

Conversely, suppose \(x \in \Delta^{-k}(A \cap \text{fix}(\Delta))\) for some \(k \geq 0\). By the remark 7.9, \(x \in \text{stab}(\Delta)\) also. Since \((\Delta^{-1})^{k}(A \cap \text{fix}(\Delta)) \subseteq (\Delta^{-1})^{k}(A \cap \text{fix}(\Delta))\), we have \(\Delta^{k}(x) \subseteq A \cap \text{fix}(\Delta)\). Then \(x \in \text{con}(\Delta)\) and \(\Delta^{\infty}(x) = \Delta^{k}(x) \subseteq A\) so that \(x \in \text{bas}(\Delta, A)\). So \(\bigcup_{k \geq 0} \Delta^{-k}(A \cap \text{fix}(\Delta)) \subseteq \text{bas}(\Delta, A)\).

This proves that \(\text{bas}(\Delta, A) = \bigcup_{k \geq 0} \Delta^{-k}(A \cap \text{fix}(\Delta))\). \(\Box\)

To complete the connection to Dijkstra’s \(wp\) formalism we need to connect the map \(\Delta^{-1}_{DO}\) with the iterates of \(\Delta_{IF}^{-1}\).

**Lemma 7.11.** Let \(A \subseteq X\). Define \(H_0(A) = A \cap D^c\), and for \(k > 0\), \(H_{k+1}(A) = \text{wp}(IF, H_{k}(A)) \cup H_0(A)\). Then \(\Delta^{-k}_{Q}(A \cap D^c) = H_{k}(A)\) for all \(k \geq 0\). \(\Box\)

**Proof** Note first that if \(x \in H_{0}(A)\), then \(\Delta_{Q}(x) = \{x\} \subseteq A \cap D^c\) so that \(H_{0}(A) \subseteq \Delta^{-1}_{Q}(A \cap D^c)\). By the monotonicity property of multiplicative maps we have \(H_{0}(A) \subseteq \Delta^{-k}_{Q}(A \cap D^c)\) for all \(k > 0\).

For \(k = 0\) we have \(H_{0}(A) = A \cap D^c = (\Delta_{IF}^{-1})^{0}(A \cap D^c)\). Assume that \(\Delta^{-k}_{Q}(A \cap D^c) = H_{k}(A)\) for some
Then

\[ x \in H_{k+1}(A) \iff x \in \text{wp}(IF, H_k(A)) \text{ or } x \in H_0(A) \]
\[ \iff x \in (\Delta IF)^{-1}(H_k(A)) \text{ or } x \in H_0(A) \]
\[ \iff x \in D \text{ and } \Delta IF(x) \subseteq H_k(A) \text{ or } x \in H_0(A) \]
\[ \iff x \in D \text{ and } \Delta_Q(x) \subseteq \Delta^{-k}_Q(A \cap D^c) \text{ or } x \in H_0(A) \]
\[ \iff x \in D \text{ and } x \in \Delta^{-1}_Q(\Delta^{-k}_Q(A \cap D^c)) \text{ or } x \in H_0(A) \]
\[ \iff x \in \Delta^{-k-1}(A \cap D^c). \]

This proves the lemma. \(\Box\)

**Theorem 7.12.** \(\Delta^{-1}_{DO}(A) = \text{wp}(DO, A)\) for all \(A \subseteq X\).

**Proof** For the proof we only need to collect our earlier results and see the definition of \(\text{wp}(DO, A)\) on p.35 of [Dijkstra 1976].

\[
\Delta^{-1}_{DO}(A) = \text{bas}(\Delta_Q, A \cap D^c)
= \cup_{k \geq 0}^{\Delta^{-k}_Q(A \cap D^c)}
= \cup_{k \geq 0} H_k(A)
= \text{wp}(DO, A)
\]

This proves the theorem. \(\Box\)

8. **CONCLUDING REMARKS**

Dijkstra [Dijkstra 1976] introduces the notion of a nondeterministic mechanism acting on a state space \(X\) but does not define the notion. Rather he says that such a mechanism induces a set action that we have denoted by \(\mu\) and that the action characterizes the mechanism. We have shown that \(\mu\) is determined by a choice set map and that the backward acting \(\mu\) is equivalent to the forward acting \(\Delta\). Thus this article presents an alternative approach to the understanding of Dijkstra’s formalism. We have also shown that there is a third way and equivalent way of defining nondeterminism that is dual to that of Dijkstra, in terms of additive maps.

Our approach also suggests there is a weak convergence related to additive maps that could operate in nondeterministic mechanisms. In subsequent articles we shall choose the choice set map as our primary way of modeling nondeterminism and present an exposition of the design of algorithms as suggested by Dijkstra, and also the standard concepts of computability, complexity, witness certificates and other such ideas studied in a standard course in the theory of computation [Lewis and Papadimitriou 2005]. It turns out that weak inverses of choice set maps have an important role to play.

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