Lax Pairs for Integrable Lattice Systems.

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Abstract. This paper studies the structure of Lax pairs associated with integrable lattice systems (where space is a one-dimensional lattice, and time is continuous). It describes a procedure for generating examples of such systems, and emphasizes the features that are needed to obtain equations which are local on the spatial lattice.

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I. Introduction.

There has long been interest in integrable differential-difference equations (integrable lattice systems), especially since the discovery of the Toda lattice. Such systems can have direct applications, for example in condensed-matter physics; and also occur as spatially-discrete versions of integrable partial differential equations. Associated with each integrable lattice is a Lax pair, involving a matrix $L$ which “steps” along the lattice, together with another matrix $V$ which generates the time evolution. The purpose of this paper is to investigate the structure of this Lax pair, and how it affects the nature of the associated integrable systems.

Throughout the paper, we work with functions $\varphi_n(t)$ which depend on time $t$, and on an integer variable $n$. Such a function will be written simply as $\varphi$, its dependence on $t$ and $n$ being understood; then $\varphi_+$ denotes $\varphi_{n+1}(t)$, and $\varphi_-$ denotes $\varphi_{n-1}(t)$. The symbol $\Delta$ denotes the forward-difference operator, i.e. $\Delta \varphi = \varphi_+ - \varphi_-$. For example, the Toda lattice equation (in first-order form) is

$$\frac{d}{dt} \varphi = \psi, \quad \frac{d}{dt} \psi = \exp \Delta \varphi - \exp \Delta \varphi_-. \quad (1)$$

We shall take the integer $n$ to be unrestricted (i.e. the lattice is infinite). Our primary interest is in systems which are local, in the sense that the time-derivative of a variable at site $n$ equals some expression involving the variables at sites $n - 1$, $n$ and $n + 1$: i.e. nearest neighbours only.

The point of view adopted here is that a lattice equation is integrable if it can be written as the consistency condition for a linear system (Lax pair) of a suitable type. This involves two $q \times q$ matrices $L$ and $V$, the entries of which are functions of a “spectral parameter” $\lambda$, as well as of $t$ and $n$. In what follows, we shall, for the sake of simplicity, restrict to the case $q = 2$ (i.e. $2 \times 2$ matrices). The linear system is

$$\Psi_+ = L \Psi, \quad \frac{d}{dt} \Psi = V \Psi, \quad (2)$$

where $\Psi$ is a column 2-vector (depending on $\lambda$, $t$ and $n$). The consistency condition for (2) is

$$\frac{d}{dt} L = V_+ L - L V. \quad (3)$$
The crucial feature of (3) is that it specifies the evolution only of \( L \), and not of \( V \); so in order to get a meaningful equation, \( V \) has to be determined in terms of \( L \). In the next section, we shall see how this happens. The subsequent sections illustrate how this structure can be used to generate integrable lattice systems. We shall see how known examples fit into this framework; and as a new example, construct a system which couples lattice versions of the Heisenberg ferromagnet and the derivative nonlinear Schrödinger equation.

II. HOW \( L(\lambda) \) DETERMINES \( V(\lambda) \).

In order to analyse the structure of \( L(\lambda) \) and \( V(\lambda) \), one needs to impose some requirements on the way that they depend on \( \lambda \). Let us assume that \( L, L^{-1} \) and \( V \) are rational functions of \( \lambda \), with poles at constant values of \( \lambda \) (that is, the location of the poles does not depend on \( t \) or \( n \)). This is not the only possibility: for example, there is the well-known case of the lattice Landau-Lifshitz equation\(^3\), which involves elliptic functions of \( \lambda \). But we shall restrict to the rational case.

By making a Möbius transformation on \( \lambda \), we may ensure that \( V(\lambda) \) is finite at \( \lambda = \infty \), \textit{i.e.} that its poles occur at finite values of \( \lambda \). Furthermore, since (3) is homogeneous in \( L \), we have the freedom to multiply \( L \) by a scalar function of \( \lambda \) (not depending on \( t \) or \( n \)). We can use this freedom to ensure that \( L \) is a (matrix) polynomial in \( \lambda \) which is nonzero at each of the poles of \( V \). Let \( p \) denote the degree of \( L \) as a polynomial in \( \lambda \).

Equation (3) determines the evolution of each matrix coefficient in the polynomial

\[
L(\lambda) = A\lambda^p + B\lambda^{p-1} + \ldots + D;
\]

so at this stage it is a set of coupled equations for \( q^2(p+1) \) functions (with \( q = 2 \) in what follows). As was emphasized above, the matrix \( V \) has to be determined in terms of \( L \), since (3) does not specify its evolution: let us now examine how this happens.

Assume for the time being that the poles of \( V \) are all simple. So \( V \) has the form

\[
V(\lambda) = \sum_{\alpha=1}^{N} V^{(\alpha)}(\lambda - \lambda_{\alpha})^{-1} + V^{(0)},
\]

where \( V^{(0)}, V^{(1)}, \ldots, V^{(N)} \) are matrices independent of \( \lambda \). The general idea is that \( V^{(0)} \) is determined by a choice of gauge, whereas each \( V^{(\alpha)} \) for \( 1 \leq \alpha \leq N \) is determined by the residue of (3) at the pole \( \lambda = \lambda_{\alpha} \). Note that equations (2) and (3) are invariant under the
gauge transformations
\[ \Psi \mapsto \Lambda \Psi, \]
\[ L \mapsto \Lambda L \Lambda^{-1}, \]
\[ V \mapsto \Lambda V \Lambda^{-1} + \left( \frac{d}{dt} \Lambda \right) \Lambda^{-1}, \]
(5)
where \( \Lambda \) is a nonsingular \( 2 \times 2 \) matrix depending on \( t \) and \( n \) (but not on \( \lambda \)). A choice of gauge involves the following steps:

(i) choose a form for \( L(\lambda) \) (an algebraic condition on the entries in the matrix \( L \)), such that a necessary condition for this form to be preserved under gauge transformations (5) is that \( \Lambda_+ = \Lambda \);

(ii) then choose any \( V^{(0)} \) which is consistent with (i) and the evolution equation (3).

As an example to illustrate how this works, choose the coefficient of \( \lambda^p \) in \( L(\lambda) \) (i.e. the leading term) to be the identity matrix. Then the remaining gauge freedom is (5), with \( \Lambda \) independent of \( n \) as required. And the leading term in (3) gives \( V^{(0)}_+ = V^{(0)} \), so any choice \( V^{(0)} = V^{(0)}(t) \) then fixes the gauge completely; to obtain an autonomous system of equations, one chooses \( V^{(0)} \) to be a constant matrix. This is the most straightforward choice of gauge; for other gauges, \( V^{(0)} \) will depend on the functions appearing in \( L(\lambda) \), and this dependence is in general nonlocal as we shall see below.

Consider, next, the pole at \( \lambda = \lambda_\alpha \). Clearly the residue of the right-hand side of (3) at this pole must vanish, i.e.
\[ V^{(\alpha)}_+ L^{(\alpha)} - L^{(\alpha)} V^{(\alpha)} = 0, \]
(6)
where \( L^{(\alpha)} = L(\lambda_\alpha) \). The idea is that this constraint determines \( V^{(\alpha)} \). Since \( L^{(\alpha)} \) is a non-zero \( 2 \times 2 \) matrix, there are two gauge-invariant possibilities: the rank of \( L^{(\alpha)} \) could be either 2 or 1. The next section deals with the rank 2 case; thereafter, we concentrate on the rank 1 case.

III. RANK 2 CASE: NONLOCAL SYSTEMS.

If \( L^{(\alpha)} \) is invertible, then (6) is a difference equation
\[ V^{(\alpha)}_+ = L^{(\alpha)} V^{(\alpha)} L^{(\alpha)-1}, \]
which determines \( V^{(\alpha)}_n \) in terms of (say) \( L^{(\alpha)}_{n-1}, L^{(\alpha)}_{n-2}, \ldots \) and \( V^{(\alpha)}_\infty \). In other words, \( V^{(\alpha)} \) is a nonlocal function of the entries in \( L^{(\alpha)} \).
To obtain a simple example which illustrates this case, take $p = N = 1$. Without loss of generality, we may set $\lambda_1 = 0$. Write $L = A\lambda + B$, where $B$ is invertible; and choose a gauge by specifying

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $V^{(0)} = 0$. So $V(\lambda) = V^{(1)}/\lambda$, where $V^{(1)}$ is the solution of the difference equation

$$V^{(1)}_+ = BV^{(1)}B^{-1}. \quad (7)$$

In general, the $2 \times 2$ matrix $B$ contains four functions; let us effect a reduction to one function $y_n(t)$ by taking $B$ to have the form

$$B = \begin{pmatrix} e^y & 0 \\ 0 & e^{-y} \end{pmatrix}. \quad (8)$$

In order for the reduction to be consistent, we need $V^{(1)}_{-\infty}$ to have the form

$$V^{(1)}_{-\infty} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

(plus a multiple of the identity, which has no eventual effect). The resulting system of equations for $y$ is

$$\frac{d}{dt}y = b\exp(Y) - c\exp(-Y), \quad (9)$$

where $Y_n(t)$ is given by the non-local expression

$$Y_n = y_n + 2 \sum_{k=-\infty}^{n-1} y_k.$$

This equation can be transformed to a form which looks local, by writing $y_n = \phi_n - \phi_{n+1}$. In terms of $\phi_n(t)$, eqn (9) becomes

$$\frac{d}{dt}(\phi_+ - \phi) = c\exp(\phi_+ + \phi) - b\exp(-\phi_+ - \phi). \quad (10)$$

If (for example) $b = 0$, then (10) is a differential-difference version of the Liouville equation

$$\phi_{tx} = \exp(2\phi). \quad (11)$$

To get (11) from (10), we interpret $c$ as the lattice spacing, put $x = nc$, and take the continuum limit $c \to 0$. Similarly, putting $b = c$ in (10) gives a differential-difference
version of the sine-Gordon equation\(^4\). But for these lattice Liouville and sine-Gordon systems, \(x\) and \(t\) (or \(n\) and \(t\)) do not represent space and time; rather they are characteristic (null) coordinates. In particular, one cannot specify arbitrary initial data at \(t = 0\). In these lattice equations, one of the characteristic coordinates has become discrete, while the other (namely \(t\)) remains continuous.

This example can be generalized in several directions, as follows. If one does not make the reduction (8), then one obtains an equation for the matrix \(B\). Choosing the slightly different gauge \(A = I\) (the identity matrix) and \(V^{(0)} = 0\), leads to

\[
\frac{d}{dt} B - \Delta V^{(1)} = 0
\]

(12)
together with (7). Now (7) and (12) ensure that there exists a matrix \(R_n(t)\) with

\[
B = 1 + R_+ R^{-1}, \quad V^{(1)} = -\left(\frac{d}{dt} R\right) R^{-1}.
\]

Then (12) becomes

\[
\frac{d}{dt} (R_+ R^{-1}) + \Delta \left[\left(\frac{d}{dt} R\right) R^{-1}\right] = 0.
\]

(13)

This is a differential-difference version of the principal chiral equation

\[
(R_x R^{-1})_t + (R_t R^{-1})_x = 0
\]

(14)
in which, as before, one of the characteristic coordinates has become discrete.

This chiral equation generalizes, of course, to larger matrices \((q > 2)\). Similarly, the Liouville and sine-Gordon examples generalize to differential-difference versions of other Toda field equations.

Finally, it might be noted that there are difference-difference versions of the principal chiral\(^5\) and Toda field\(^6\) equations in which both characteristic coordinates (here \(x\) and \(t\)) become discrete. Another, very general, example of this type is the Hirota bilinear difference equation\(^7\).

So systems with \(\det L^{(\alpha)} \neq 0\) may be thought of as time-evolution equations which are nonlocal on the spatial lattice, or as equations where a characteristic coordinate (neither space nor time) has become discrete. To get local evolution equations, it is necessary for each \(L^{(\alpha)}\) to have rank 1. From now on, we shall concentrate on this case.

IV. RANK 1 CASE: LOCAL SYSTEMS.
Given that \( L^{(\alpha)} \) has rank 1, the constraint (6) may be solved as follows. Write \( K^{(\alpha)} = L^{(\alpha)} L^{(\alpha)} \). Assuming that \( \text{tr} K^{(\alpha)} \) is non-zero, the general solution of (6) is

\[
V^{(\alpha)} = \frac{1}{\text{tr} K^{(\alpha)}} [f^{(\alpha)} K^{(\alpha)} + g^{(\alpha)} \text{adj} K^{(\alpha)}],
\]

where \( \text{adj} K^{(\alpha)} \) denotes the adjoint matrix of \( K^{(\alpha)} \), and where \( f^{(\alpha)} \) and \( g^{(\alpha)} \) are scalar functions, with \( f^{(\alpha)}_+ = f^{(\alpha)} \). So the constraint (6) does not determine \( V^{(1)}, \ldots, V^{(N)} \) uniquely: in particular one has the arbitrary functions \( g^{(\alpha)} \). However, there is a further constraint, namely that the condition \( \det L^{(\alpha)} = 0 \) has to be preserved by the evolution (3). This gives equations on the \( g^{(\alpha)} \), which are precisely that they are constant on the lattice, just as the \( f^{(\alpha)} \) are: \( g^{(\alpha)}_+ = g^{(\alpha)} \). Then (15) can be re-written as

\[
V^{(\alpha)} = \frac{c^{(\alpha)}}{\text{tr} K^{(\alpha)}} K^{(\alpha)} + d^{(\alpha)} I,
\]

where \( c^{(\alpha)} \) and \( d^{(\alpha)} \) are functions of \( t \) only. It is clear that the \( d^{(\alpha)} \) term will not contribute in the evolution equations, and so only the \( c^{(\alpha)} \) remain; we may as well set \( d^{(\alpha)} = 0 \), and take

\[
V^{(\alpha)} = \frac{c^{(\alpha)}}{\text{tr} K^{(\alpha)}} K^{(\alpha)}.
\]

At this stage, the \( c^{(\alpha)} \) could still be functions of time \( t \); for simplicity, let us take them to be constants. One point to note about (16) is that \( V^{(\alpha)} \) is local: \( V^{(\alpha)}_n \) is expressed in terms of \( L_n \) and \( L_{n-1} \).

So to obtain local evolution equations with \( 2 \times 2 \) matrix Lax pairs, one first specifies the integer \( p \) (the degree of \( L(\lambda) \)); the integer \( N \) appearing in (4) equals \( 2p \), since the \( \lambda_\alpha \) all have to be roots of \( \det L(\lambda) \). The matrices \( V^{(\alpha)} \) for \( 1 \leq \alpha \leq 2p \) are given by (16), and involve the \( 2p \) constants \( c^{(\alpha)} \). Finally, there is the choice of gauge, which determines \( V^{(0)} \). In general, \( V^{(0)} \) turns out to be nonlocal, and special gauge choices are needed to ensure that it is local.

One can relate all this to the \( r \)-matrix description (see, for example, reference 3; and also reference 8 which addresses the construction of an \( r \)-matrix from a given Lax pair). Suppose one has an \( L(\lambda) \), a Poisson bracket and an \( r \)-matrix such that the Fundamental Poisson Bracket Relations are satisfied. Suppose also that there exist \( \lambda_1, \ldots, \lambda_N \) such that \( \det L(\lambda_\alpha) = 0 \) for each \( \alpha \). Let \( \tau(\lambda) \) be the trace of the monodromy matrix \( \ldots L_2(\lambda) L_1(\lambda) L_0(\lambda) L_{-1}(\lambda) \ldots \) (which propagates from \( n = -\infty \) to \( n = +\infty \)). Then

\[
H = \sum_{\alpha=1}^{N} c^{(\alpha)} \log \tau(\lambda_\alpha)
\]
is a local Hamiltonian, and the corresponding Hamiltonian equations are just (3); the constants \( c^{(α)} \) are the same as those appearing in (16). The problem from this point of view is to choose \( L(λ) \), in a suitable gauge, such that a compatible \( r \)-matrix structure exists.

More generally, one wants \( τ(λ) \) to be conserved in time, for all \( λ \) — this then gives infinitely many conserved quantities. If one has a Lax pair (2) and boundary conditions which imply that \( V_+∞ = V_-∞ \), then \( τ(λ) \) is indeed conserved. When \( V(λ) \) depends locally on the fields, then then it is easy to ensure that this condition is met; if, on the other hand, \( V \) is nonlocal, then conservation of \( τ(λ) \) is not guaranteed. This is one reason why locality is desirable, in the present context.

If \( p = 1 \) (in other words, \( L(λ) \) is linear in \( λ \)), then \( \det L(λ) \) is a quadratic polynomial in \( λ \), the roots of which are \( λ_1 \) and \( λ_2 \). For the time being, let us assume that these roots are distinct; and by translating \( λ \) set \( λ_1 = 1 \) and \( λ_2 = −1 \). It follows that \( L(λ) \) has the form

\[
L(λ) = \frac{1}{2}(λ + 1)L^{(1)} − \frac{1}{2}(λ − 1)L^{(2)},
\]

(17)

where \( L^{(1)} \) and \( L^{(2)} \) are 2 × 2 matrices each having zero determinant. So the entries in \( L^{(α)} \) involve six independent functions of \( t \) and \( n \) (in effect, the requirement that \( λ_1 \) and \( λ_2 \) be constant has reduced the number of functions in \( L(λ) \) from eight to six). The two \( L^{(α)} \) satisfy evolution equations obtained by expanding (3); these are

\[
\frac{d}{dt}L^{(α)} = V_+^{(0)}L^{(α)} − L^{(α)}V^{(0)} − \frac{i}{2}Ξ,
\]

(18)

where \( Ξ = V_+^{(1)}L^{(2)} − L^{(2)}V^{(1)} − V_+^{(2)}L^{(1)} + L^{(1)}V^{(2)} \). The gauge choice reduces the number of functions by four (since \( Λ \) contains four entries), and we end up with a system involving two functions. A number of examples of this type are described in the following section.

V. SOME \( p = 1 \) EXAMPLES.

This section exhibits some systems of the type described in the previous section. Such examples are simple to generate; but before doing so, we should ask when two lattice equations are to be regarded as being “the same”. More specifically, is there an appropriate equivalence relation on the set of all such systems? Certainly such an equivalence would include gauge transformations in which \( Λ \) was constant; and strictly-local redefinitions of the functions appearing in \( L \) (i.e. the new functions at lattice-site \( n \) depend on the old
functions at site \( n \) only). However, it is customary to allow more general transformations than just these. A well-known case is that of the Toda lattice (1). If one replaces \( \varphi \) by

\[
    r = \Delta \varphi_- = \varphi_- - \varphi_+ ,
\]

then (1) becomes

\[
    \frac{d^2}{dt^2} r = e^r+ - 2e^r + e^r- ;
\]

and this is regarded as simply another form of the Toda lattice equation.

But any equivalence relation which admitted (1) and (20) to the same class, would also have to allow the transformations \( \varphi \mapsto \Delta^k \varphi \) for all integers \( k \) (negative as well as positive). If one allows such highly-nonlocal transformations, then ones ends up with rather few equivalence classes; in fact, one might as well transform to action-angle variables, and say that the “only” integrable lattice is linear. Clearly this is inappropriate.

The point of this argument is to conclude that there is no useful equivalence relation on integrable systems of the type that we are considering (unless we insist that (1) and (20) are to be regarded as distinct). This means that the task of listing such systems in a systematic way is not really well-defined. The best that one can do is to exhibit examples, and indicate how they are related to one another.

**Example (i).** Choose a gauge such that \( L^{(1)} - L^{(2)} = 2\mathbf{I} \). This is the gauge which was mentioned as an example in section II. As was remarked there, the gauge is then fixed completely by specifying some \( V^{(0)}(t) \). The simplest choice is to set \( V^{(0)} = 0 \). Note that \( L^{(1)} \) and \( L^{(2)} \) must have the form

\[
    L^{(1)} = \mathbf{I} + M, \quad L^{(2)} = -\mathbf{I} + M ,
\]

where \( M \) is trace-free and \( \det M = -1 \). So we may write \( M = \mathbf{f} \cdot \sigma \), where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the Pauli matrices, and \( \mathbf{f} = \mathbf{f}_n(t) \) is a unit 3-vector. The dot denotes the usual 3-dimensional Euclidean scalar product (and \( \wedge \) below will denote the vector product). The evolution equation for \( \mathbf{f} \), derived from (18), is then

\[
    \frac{d}{dt} \mathbf{f} = \Delta \left[ (1 + \mathbf{f}_- \cdot \mathbf{f})^{-1} (\mu \mathbf{f}_- + \mu \mathbf{f} + \nu \mathbf{f}_- \wedge \mathbf{f}) \right] ,
\]

where \( \mu = \tfrac{1}{2}(c^{(1)} - c^{(2)}) \) and \( \nu = \tfrac{1}{2}i(c^{(1)} + c^{(2)}) \) are constants. Equation (21), then, is an integrable equation for the unit-vector function \( \mathbf{f} = \mathbf{f}_n(t) \). If the parameter \( \nu \) is non-zero, it
can be set to unity by scaling \(t\); so the system effectively depends on the single parameter \(\mu\). The case \(\mu = 0\) is the “Lattice Heisenberg Model”\(^3,9\), so-called because it has the equation of the Heisenberg ferromagnet as a continuum limit. Indeed, if we set \(\nu = 2/h^2\), \(\mu = \hat{\mu}/h\), and let \(h \to 0\), then (21) becomes

\[
\mathbf{f}_t = \hat{\mu} \mathbf{f}_x + \mathbf{f} \wedge \mathbf{f}_{xx};
\]  

(22)

the Heisenberg model corresponds to \(\hat{\mu} = 0\). A slightly different choice of gauge, namely one in which \(V(0)\) is a nonzero constant matrix, yields a lattice nonlinear Schrödinger equation (different from the one in the next example). This is the lattice counterpart of the well-known gauge equivalence of the nonlinear Schrödinger and Heisenberg systems.

\textit{Example (ii).} Choose a gauge such that

\[
L^{(1)} = \begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix}, \quad L^{(2)} = -\begin{pmatrix} 0 & v \\ 0 & 1 \end{pmatrix},
\]  

(23)

where \(u\) and \(v\) are functions of \(t\) and \(n\). The remaining gauge freedom has \(\Lambda\) being a diagonal matrix function of \(t\) only. Substituting (23) into the evolution equation (18) determines \(V^{(0)}\), as

\[
V^{(0)} = \frac{1}{2} \begin{pmatrix} -c^{(1)} v u_- & c^{(1)} v + c^{(2)} v_- \\ c^{(2)} u + c^{(1)} u_- & -c^{(2)} u v_- \end{pmatrix}
\]

plus a diagonal matrix function of \(t\), which by the residual gauge freedom may be set to zero. In addition, (18) gives the equations for \(u\) and \(v\), namely

\[
\frac{d}{dt} u = \frac{1}{2} (c^{(2)} \Delta u + c^{(1)} \Delta u_- + c^{(1)} v u u_- - c^{(2)} v u u_+),
\]

\[
\frac{d}{dt} v = \frac{1}{2} (c^{(1)} \Delta v + c^{(2)} \Delta v_- + c^{(2)} u v v_- - c^{(1)} u v v_+).
\]  

(24)

This is exactly the Ablowitz-Ladik system\(^2\); their \(L\)-operator is slightly different from the one presented here (it is, in effect, quadratic rather than linear in \(\lambda\)); but it is easily seen to be equivalent. In particular, if we choose \(c^{(1)} = 2i = -c^{(2)}\), and impose the (consistent) reduction \(v = \pm u^*\), then (24) reduces to a lattice nonlinear Schrödinger equation

\[
i \frac{d}{dt} u = u_+ - 2u + u_- \mp uu^*(u_+ + u_-).
\]
Example (iii). Here we choose a gauge such that $L^{(1)}$ is constant. Without loss of generality, we may take $L^{(1)}$ to be

$$L^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (25)$$

The most general form for the matrix $L^{(2)}$ is

$$L^{(2)} = \begin{pmatrix} uw & uw \\ v & w \end{pmatrix}, \quad (26)$$

where $u$, $v$ and $w$ are functions of $n$ and $t$. The evolution equation (18) for $L^{(1)}$ implies that $V^{(0)}$ must have the form

$$V^{(0)} = -\frac{c^{(1)}}{2} \begin{pmatrix} 0 & uw \\ v_- & 0 \end{pmatrix} - \frac{c^{(2)}}{2(vu_- + w)} \begin{pmatrix} vu_- & wu_- \\ v & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad (27)$$

where $\Delta A = 0$ (i.e. $A$ depends only on $t$). The residual gauge freedom, i.e. that which preserves (25), is (5) with

$$\Lambda = \begin{pmatrix} f(t) & 0 \\ 0 & g(n, t) \end{pmatrix};$$

this has to be used to determine $A$ and $D$, and to eliminate one of the three functions $u$, $v$ or $w$. In fact, the role of $f(t)$ is simply to fix $A$: let us choose $A = \frac{1}{2} c^{(1)}$. The remaining freedom now is

$$u \mapsto g_-^{-1} u, \quad v \mapsto g_+ v, \quad w \mapsto g_-^{-1} g_+ w. \quad (28)$$

Equation (18) for $L^{(2)}$ gives equations for $u$, $v$ and $w$, one form of which is

$$\frac{d}{dt} \log v = D_+ + \frac{1}{2} c^{(1)} (wu_- / v - uv) + \frac{1}{2} c^{(2)} \frac{v_+ + vw_+}{v(wu_+ + w_+)};$$

$$\frac{d}{dt} \log w = \Delta \left( D - \frac{1}{2} c^{(2)} \frac{vu_-}{vu_- + w} \right);$$

$$\frac{d}{dt}(uv) = \Delta \left( -\frac{1}{2} c^{(1)} uv_+ + \frac{1}{2} c^{(2)} \frac{vu_-}{vu_- + w} \right);$$

$$\frac{d}{dt} \log (uv) = -D - \frac{1}{2} c^{(1)} (uw_+ / u - uv) - \frac{1}{2} c^{(2)} \frac{uw + u_-}{u(vu_- + w)}. \quad (29)$$

(Any three of these equations implies the fourth.) We see from (28) that in order to remove the remaining gauge freedom, i.e. fix $g$ (at least up to a function of $t$), there are
three possibilities. Namely, we can specify either \( w \), or \( uw \), or \( v \) as a function of the gauge-invariant combination \( uv \). This in turn will determine \( D \), and hence \( V^{(0)} \), on eliminating the relevant variable from (29). For example, specifying \( uw \) as a function of \( uv \) will give a local formula for \( D \). Similarly, \( v \) can be specified as any function of \( uv \). But if we impose \( w = F(uv) \), then we need \( F \) to be an exponential in order to get a local expression for \( D \). This illustrates the way in which some choices of gauge lead to a nonlocal expression for \( V^{(0)} \).

As an example, let us take the gauge \( w = \text{constant} \). Choose \( w = -1 \), and write \( u = -e^x, uv = y \). Then (29) reduces to the system

\[
\frac{dy}{dt} = \frac{1}{2} c^{(1)} \Delta (y - e^{\Delta x}) + \frac{1}{2} c^{(2)} \Delta \left( \frac{y}{y - \exp \Delta x} \right), \\
\frac{dx}{dt} = \frac{1}{2} c^{(1)} (y + e^{\Delta x}) - \frac{c^{(2)}}{2(y - \exp \Delta x)}.
\]  

(30)

This is a version of the relativistic Toda lattice\(^{10-14} \).

**Example (iv).** The gauge choice

\[
L^{(1)} = \begin{pmatrix} -1 & e^x \\ 0 & 0 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} e^y & 0 \\ -ke^{y-x} & 0 \end{pmatrix},
\]

where \( k \) is a constant, gives

\[
V^{(0)} = \frac{c^{(1)}}{2} \begin{pmatrix} 1 - ke^{y+x} & -e^x \\ -ke^{y-x} & 0 \end{pmatrix} + \frac{c^{(2)}}{2} \begin{pmatrix} 0 & e^{x-y} \\ ke^{-y} & 1 - ke^{-y+\Delta x} \end{pmatrix},
\]

and again leads to the relativistic Toda system\(^{10,14,15} \), this time in the form

\[
\frac{dy}{dt} = -\frac{1}{2} c^{(1)} k \Delta (e^{y+\Delta x}) - \frac{1}{2} c^{(2)} k \Delta (e^{-y+\Delta x}), \\
\frac{dx}{dt} = -\frac{1}{2} c^{(1)} e^y (1 + ke^{\Delta x}) + \frac{1}{2} c^{(2)} e^{-y} (1 + ke^{\Delta x}).
\]  

(31)

The limit \( k \to 0 \) gives the Toda system (1); it is worth examining this in more detail. In order to get (1), one may replace the variables \( x \) and \( y \) by \( \varphi \) and \( \psi \), where

\[
e^x = -\sqrt{k} e^\varphi, \quad e^y = -1 + \sqrt{k} \psi,
\]

and set \( c^{(1)} = c^{(2)} = -1/\sqrt{k} \). Then the \( k \to 0 \) limit of (31) is indeed (1). But since \( c^{(\alpha)} \to \infty \) in this limit, we need to re-interpret the associated Lax pair. The way to
get a well-behaved limit is to replace $\lambda$ by $2\lambda/\sqrt{k}$. The roots of $\det L(\lambda)$ now occur at $\lambda = \pm \frac{1}{2} \sqrt{k}$, and so in the $k \to 0$ limit they coincide. In fact, when $k = 0$ we have

$$L(\lambda) = -\begin{pmatrix} 1 + \psi \lambda & e^{\varphi} \lambda \\ -e^{-\varphi} \lambda & 0 \end{pmatrix},$$

and $\det L(\lambda)$ has a double zero at $\lambda = 0$ (cf. ref 3). The corresponding expression for $V(\lambda)$ is obtained by taking the $k \to 0$ limit after first subtracting a constant multiple of the identity matrix: this yields

$$V(\lambda) = \begin{pmatrix} 0 & -e^{\varphi} \\ e^{-\varphi} & \lambda^{-1} \end{pmatrix}.$$

VI. Some $p = 2$ Examples.

In the $p = 2$ case, $L(\lambda)$ has the form $L(\lambda) = A\lambda^2 + B\lambda + C$, where $A$, $B$ and $C$ are $2 \times 2$ matrices; so to begin with, one has twelve functions of $n$ and $t$. The requirement that the zeros $\lambda_\alpha$ of the quartic polynomial $\det L(\lambda)$ be constant imposes four equations on these functions, and choice of gauge imposes a further four, so one is left with four independent functions. In other words, the generic system in this $p = 2$, $q = 2$ case is a system of coupled evolution equations for four lattice variables.

Reproductions of such systems, so that fewer functions are involved, are of course possible. One example that has been known for some time is a lattice version of the sine-Gordon equation in which space is discrete and time continuous (by contrast with the version mentioned in section III). Here the $L$-operator has the form

$$L = \begin{pmatrix} \lambda f(\varphi)e^{i\eta} & \frac{1}{4}h(e^{-i\varphi/2} - \lambda^2 e^{i\varphi/2}) \\ \frac{1}{4}h(\lambda^2 e^{-i\varphi/2} - e^{i\varphi/2}) & \lambda f(\varphi)e^{-i\eta} \end{pmatrix},$$

where $\varphi$ and $\eta$ are functions of $n$ and $t$, $h$ is a constant corresponding to the lattice spacing, and $f(\varphi) = (1 + \frac{1}{2} h^2 \cos \varphi)^{1/2}$. The resulting integrable lattice has sine-Gordon in “laboratory coordinates” as a continuum limit: if we replace $n$ by $x = nh$ and let $h \to 0$, then $\varphi$ satisfies $\varphi_{tt} - \varphi_{xx} + \sin \varphi = 0$.

In order to obtain another example, let us choose a different gauge, namely $A = I$, $V^{(0)} = 0$. On $B$ and $C$ we impose the four constraints $\text{tr} B = 0$, $\det C = l$ constant, $\text{tr}(BC) = 0$, and $\text{tr} C + \det B = -2k$ constant. It then follows that $\det L(\lambda) = \lambda^4 - 2k\lambda^2 + l$ has constant zeros. The matrices $B$ and $C$ now involve four independent functions, and
their evolution is given by
\[
\frac{d}{dt} B = \Delta Q, \quad \frac{d}{dt} C = R + C - CR,
\]
where \( Q = \sum_\alpha V^{(\alpha)}, \ R = -\sum_\alpha \lambda_\alpha^{-1} V^{(\alpha)}, \) and the \( V^{(\alpha)} \) are constructed as in (16). There are four parameters, namely the \( c^{(\alpha)} \).

We can get an idea of what this system represents by looking at a continuum limit. To keep things simple, we assign particular values to the parameters, and this leads to the following continuum integrable system.

Let \( B \) and \( C \) be 3-vectors, functions of \( x \) and \( t \), satisfying the constraints \( B \cdot C = 0 \) and \( C \cdot C = 1 \). Their time-evolution is given by
\[
\begin{align*}
B_t &= (C \wedge B_x)_x - \{(C \wedge C_x \cdot B)C\}_x + \frac{1}{2}((B \cdot B)B)_x, \\
C_t &= (C \wedge C_x)_x + (C \wedge C_x \cdot B)B \wedge C + \frac{1}{2}(B \cdot B)C_x.
\end{align*}
\]

This is integrable: it has a Lax pair of the form \( \Psi_x = U \Psi, \Psi_t = V \Psi \), where \( U = -\frac{1}{2}i(B\lambda^{-1} + C\lambda^{-2}) \cdot \sigma \) corresponds to \( L(\lambda) \), and \( V = \sum_{k=1}^{4} V_k \lambda^{-k} \) is a limiting version of the \( V(\lambda) \) of the lattice system.

The equations (33) have two obvious reductions: if \( B = 0 \), then we are left with the Heisenberg ferromagnet equation for \( C \); while if \( C \) is a constant unit vector, then \( B \) satisfies the derivative nonlinear Schrödinger equation. So (33) may be viewed as a coupled Heisenberg-DNLS system; and (32) is a spatially-discrete version of this coupled system.

VII. Concluding Remarks.

It is clear that the examples given above provide only a very small sample of integrable lattice systems. One may envisage a classification involving the three integers \( q \) (the size of the matrices \( L \) and \( V \)), \( p \) (the degree of \( L(\lambda) \)) and \( r \) (the maximum order of the poles of \( V(\lambda) \)). But in view of the remarks at the beginning of section V, a complete classification would require a way of dealing with the problem of equivalence.

We conclude with a remark on higher values of \( r \). One can take a given \( L(\lambda) \) (thereby fixing \( q \) and \( p \)), and allow \( r > 1 \): in other words, higher-order poles in \( V(\lambda) \). This leads to hierarchies of lattice systems, of which the \( r = 1 \) cases are the first members. So hierarches also fit naturally into this framework.
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