Moduli Spaces of Bundles over Riemann Surfaces and the Yang–Mills Stratification Revisited

Frances Kirwan

Abstract. Refinements of the Yang–Mills stratifications of spaces of connections over a compact Riemann surface $\Sigma$ are investigated. The motivation for this study is the search for a complete set of relations between the standard generators for the cohomology of the moduli spaces $\mathcal{M}(n, d)$ of stable holomorphic bundles of rank $n$ and degree $d$ when $n$ and $d$ are coprime and $n > 2$.

The moduli space $\mathcal{M}(n, d)$ of semistable holomorphic bundles of rank $n$ and degree $d$ over a Riemann surface $\Sigma$ of genus $g \geq 2$ can be constructed as a quotient of an infinite dimensional affine space of connections $\mathcal{C}(n, d)$ by a complexified gauge group $\mathcal{G}_c(n, d)$, in an infinite-dimensional version of the construction of quotients in Mumford’s geometric invariant theory [30]. When $n$ and $d$ are coprime, $\mathcal{M}(n, d)$ is the topological quotient of the semistable subset $\mathcal{C}(n, d)^{ss}$ of $\mathcal{C}(n, d)$ by the action of $\mathcal{G}_c(n, d)$. Any nonsingular complex projective variety on which a complex reductive group $G$ acts linearly has a $G$-equivariantly perfect stratification by locally closed nonsingular $G$-invariant subvarieties with its set of semistable points $X^{ss}$ as an open stratum. This stratification can be obtained as the Morse stratification for the normsquare of a moment map on $X$ [22]; in the case of the moduli space $\mathcal{M}(n, d)$ the role of the normsquare of the moment map is played by the Yang-Mills functional. In [28] this Morse stratification of $X$ is refined to obtain a stratification of $X$ by locally closed nonsingular $G$-invariant subvarieties with the set of stable points $X^s$ of stable points of $X$ as an open stratum. The other strata can be defined inductively in terms of the sets of stable points of closed nonsingular subvarieties of $X$, acted on by reductive subgroups of $G$, and their projectivised normal bundles.

In their fundamental paper [1], Atiyah and Bott studied a stratification of $\mathcal{C}(n, d)$ defined using the Harder-Narasimhan type of a holomorphic bundle over $\Sigma$, which they expected to be the Morse stratification of the Yang-Mills functional (this was later shown to be the case [5]). The aim of this paper is to apply the methods of [28] to the Yang-Mills stratification to obtain refined stratifications of

1991 Mathematics Subject Classification. Primary 14D20, 32G13.

Key words and phrases. Moduli spaces of vector bundles, Yang-Mills stratification.

The author is a member of VBAC (Vector Bundles on Algebraic Curves), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099), and acknowledges with gratitude the hospitality of the University of Melbourne and the Australian National University during the writing of part of this paper.
\( \mathcal{C}(n, d) \), and to relate these stratifications to natural refinements of the notion of the Harder-Narasimhan type of a holomorphic bundle over \( \Sigma \).

The motivation for this study was the search for a complete set of relations among the standard generators for the cohomology of the moduli spaces \( \mathcal{M}(n, d) \) when the rank \( n \) and degree \( d \) are coprime and \( n > 2 \) [11]. The cohomology rings of the moduli spaces \( \mathcal{M}(n, d) \) have been the subject of much interest over many years; see for example [1, 3, 5, 14, 16, 17, 20, 21, 29, 31, 32, 38, 40, 41] among many other pieces of work. In the case when \( n = 2 \) we now have a very thorough understanding of the structure of the cohomology ring [2, 21, 37, 41]. For arbitrary \( n \) it is known that the cohomology has no torsion and formulas for computing the Betti numbers have been obtained, as well as a set of generators for the cohomology ring [1, 6, 7, 16, 42]. When \( n = 2 \) the relations between these generators can be explicitly described and in particular a conjecture of Mumford, that a certain set of relations is a complete set, was proved some years ago [2, 21, 27, 37, 41]. However less is known about the relations between the generators when \( n > 2 \), and the most obvious generalisation of Mumford’s conjecture to the cases when \( n > 2 \) is false, although a modified version of the conjecture (using ‘dual Mumford relations’ together with the original Mumford relations) is true for \( n = 3 \) [9]. There is, however, a further generalisation of Mumford’s relations, and the attempt by Richard Earl and the author to prove that this set is indeed complete was the original stimulus for studying refinements of the Yang-Mills stratification, although a different application has now appeared [11, 19].

The layout of this paper is as follows. §1 recalls background material on moduli spaces of bundles and different versions of Mumford’s conjecture, while §2 reviews the Morse stratification of the normsquare of a moment map and some refinements of this stratification. §3 studies the structure of subbundles of semistable bundles over \( \Sigma \) which are direct sums of stable bundles all of the same slope, and this is used in §4 to define two canonical refinements of the Harder-Narasimhan filtration of a holomorphic bundle over \( \Sigma \), and thus to construct two refinements of the Yang-Mills stratification. In the next two sections the stratification defined in §2 is applied to holomorphic bundles over \( \Sigma \); its indexing set is studied in §5 and the associated strata are investigated in §6. This stratification corresponds to a third refinement of the Harder-Narasimhan filtration whose subquotients are all direct sums of stable bundles of the same slope. The relationship between these three filtrations is considered in §7, and §8 provides a brief conclusion.

1. Background material on moduli spaces of bundles

When \( n \) and \( d \) are coprime, the generators for the rational cohomology\(^1\) of the moduli space \( \mathcal{M}(n, d) \) given by Atiyah and Bott in [1] are obtained from a (normalised) universal bundle \( V \) over \( \mathcal{M}(n, d) \times \Sigma \). With respect to the Künneth decomposition of \( H^*(\mathcal{M}(n, d) \times \Sigma) \) the \( r \)th Chern class \( c_r(V) \) of \( V \) can be written as

\[
c_r(V) = a_r \otimes 1 + \sum_{j=1}^{2g} b^j_r \otimes \alpha_j + f_r \otimes \omega
\]

\(^1\)In this paper all cohomology will have rational coefficients.
where \( \{1\}, \{\alpha_j : 1 \leq j \leq 2g\} \), and \( \{\omega\} \) are standard bases for \( H^0(\Sigma), H^1(\Sigma) \) and \( H^2(\Sigma) \), and

\[
1.1 \quad a_r \in H^{2r}(\mathcal{M}(n,d)), \quad b_r^j \in H^{2r-2}(\mathcal{M}(n,d)), \quad f_r \in H^{2r-2}(\mathcal{M}(n,d)),
\]

for \( 1 \leq r \leq n \) and \( 1 \leq j \leq 2g \). It was shown by Atiyah and Bott \[^{[1]}\] Prop. 2.20 and p.580 that \( \{a_r, b_r^j, f_r\} \) (for \( 2 \leq r \leq n \)) and \( b_r^j \) (for \( 1 \leq r \leq n \) and \( 1 \leq j \leq 2g \)) generate the rational cohomology ring of \( \mathcal{M}(n,\hat{d}) \).

Since tensoring by a fixed holomorphic line bundle of degree \( e \) gives an isomorphism between the moduli spaces \( \mathcal{M}(n,d) \) and \( \mathcal{M}(n,d + ne) \), we may assume without loss of generality that

\[
(2g - 2)n < d < (2g - 1)n.
\]

This implies that \( H^1(\Sigma, E) = 0 \) for any stable bundle of rank \( n \) and degree \( d \) \[^{[32]}\] Lemma 5.2], and hence that \( \pi V \) is a bundle of rank \( d - n(g - 1) \) over \( \mathcal{M}(n,d) \), where

\[
\pi : \mathcal{M}(n,d) \times \Sigma \to \mathcal{M}(n,d)
\]

is the projection onto the first component and \( \pi_1 \) is the K-theoretic direct image map. It follows that

\[
c_r(\pi_! V) = 0
\]

for \( r > d - n(g - 1) \). Via the Grothendieck-Riemann-Roch theorem we can express the Chern classes of \( \pi V \) as polynomials in the generators \( a_r, b_r^j, f_r \) described above, and hence their vanishing gives us relations between these generators. These are Mumford’s relations, and they give us a complete set of relations when \( n = 2 \). We can generalise them for \( n > 2 \) as follows.

Suppose that \( 0 \leq \hat{n} < n \), and that \( \hat{d} \) is coprime to \( \hat{n} \). Then we have a universal bundle \( \hat{V} \) over \( \mathcal{M}(\hat{n}, \hat{d}) \times \Sigma \), and both \( V \) and \( \hat{V} \) can be pulled back to \( \mathcal{M}(\hat{n}, \hat{d}) \times \mathcal{M}(n,d) \times \Sigma \). If

\[
\hat{d} > \frac{d}{n}
\]

then there are no non-zero holomorphic bundle maps from a stable bundle of rank \( \hat{n} \) and degree \( \hat{d} \) to a stable bundle of rank \( n \) and degree \( d \), and hence, if

\[
\pi : \mathcal{M}(\hat{n}, \hat{d}) \times \mathcal{M}(n,d) \times \Sigma \to \mathcal{M}(\hat{n}, \hat{d}) \times \mathcal{M}(n,d),
\]

is the projection onto the first two components, it follows that \( -\pi(\hat{V}^* \otimes V) \) is a bundle of rank \( n\hat{n}(g - 1) - d\hat{n} + d\hat{n} \) over \( \mathcal{M}(\hat{n}, \hat{d}) \times \mathcal{M}(n,d) \). Thus

\[
0 = c_r(-\pi(\hat{V}^* \otimes V)) \in H^*(\mathcal{M}(\hat{n}, \hat{d}) \times \mathcal{M}(n,d))
\]

if \( r > n\hat{n}(g - 1) - d\hat{n} + d\hat{n} \) and hence the slant product

\[
c_r(-\pi(\hat{V}^* \otimes V)) \gamma \in H^*(\mathcal{M}(n,d))
\]

of \( c_r(-\pi(\hat{V}^* \otimes V)) \) with any homology class \( \gamma \in H_*(\mathcal{M}(\hat{n}, \hat{d})) \) vanishes when

\[
r > n\hat{n}(g - 1) - d\hat{n} + d\hat{n}.
\]

The relations between the generators \( a_r, b_r^j, f_r \) obtained in this way for \( 0 < \hat{n} < n \) and

\[
\frac{d}{n} + 1 > \frac{\hat{d}}{\hat{n}} > \frac{d}{n}
\]

and

\[
n\hat{n}(g - 1) - d\hat{n} + d\hat{n} < r < n\hat{n}(g + 1) - d\hat{n} + d\hat{n}
\]
(with a little more care taken when \( \hat{n} \) and \( \hat{d} \) are not coprime) are the ones we consider. They are essentially Mumford’s relations when \( n = 2 \). To show that these form a complete set of relations, a natural strategy is to consider the Yang-Mills stratification which was used by Atiyah and Bott to obtain their generators for the cohomology ring.

Recall that a holomorphic vector bundle \( E \) over \( \Sigma \) is called semistable (respectively stable) if every holomorphic subbundle \( D \) of \( E \) satisfies
\[
\mu(D) \leq \mu(E), \quad \text{(respectively } \mu(D) < \mu(E)),
\]
where \( \mu(D) = \text{degree}(D)/\text{rk}(D) \) is the slope of \( D \). Bundles which are not semistable are said to be unstable. Note that semistable bundles of coprime rank and degree are stable.

Let \( \mathcal{E} \) be a fixed \( C^\infty \) complex vector bundle of rank \( n \) and degree \( d \) over \( \Sigma \). Let \( \mathcal{C} \) be the space of all holomorphic structures on \( \mathcal{E} \) and let \( \mathcal{G}_c \) denote the group of all \( C^\infty \) complex automorphisms of \( \mathcal{E} \). Atiyah and Bott identify the moduli space \( \mathcal{M}(n, d) \) with the quotient \( \mathcal{C}^\ast/\mathcal{G}_c \), where \( \mathcal{C}^\ast \) is the open subset of \( \mathcal{C} \) consisting of all semistable holomorphic structures on \( \mathcal{E} \). The group \( \mathcal{G}_c \) is the complexification of the gauge group \( \mathcal{G} \) which consists of all smooth automorphisms of \( \mathcal{E} \) which are unitary with respect to a fixed Hermitian structure on \( \mathcal{E} \). We shall write \( \mathcal{G}^\ast \) for the quotient of \( \mathcal{G} \) by its \( U(1) \)-centre and \( \mathcal{G}^\ast_c \) for the quotient of \( \mathcal{G}_c \) by its \( \mathcal{C}^\ast \)-centre.

There are natural isomorphisms
\[
H^i(\mathcal{M}(n, d)) \cong H^i_c(\mathcal{C}^\ast/\mathcal{G}_c) = H^i_c(\mathcal{C}^\ast/\mathcal{G}^\ast_c) \cong H^i_c(\mathcal{C}^\ast/\mathcal{G}^\ast) \cong H^i_c(\mathcal{C}^\ast)
\]
between the cohomology of the moduli space and the \( \mathcal{G}^\ast \)-equivariant cohomology of \( \mathcal{C}^\ast \), since the \( \mathcal{C}^\ast \)-centre of \( \mathcal{G}_c \) acts trivially on \( \mathcal{C}^\ast \), while \( \mathcal{G}^\ast_c \) acts freely on \( \mathcal{C}^\ast \) and \( \mathcal{G}^\ast \) is the complexification of \( \mathcal{G} \).

In order to show that the restriction map \( H^\ast_c(\mathcal{E}) \rightarrow H^\ast_c(\mathcal{C}^\ast) \) is surjective, Atiyah and Bott consider the Yang–Mills (or Atiyah–Bott–Shatz) stratification of \( \mathcal{C} \). This stratification \( \{ \mathcal{C}_\mu : \mu \in \mathcal{M} \} \) is the Morse stratification for the Yang–Mills functional on \( \mathcal{C} \), but it also has a more explicit description. It is indexed by the partially ordered set \( \mathcal{M} \) consisting of all the Harder–Narasimhan types of holomorphic bundles of rank \( n \) and degree \( d \), defined as follows. Any holomorphic bundle \( E \) over \( \mathcal{M} \) of rank \( n \) and degree \( d \) has a filtration
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_p = E
\]
of subbundles such that the subquotients \( Q_p = E_p/E_{p-1} \) are semi-stable for \( 1 \leq p \leq P \) and satisfy
\[
\mu(Q_p) = \frac{d_p}{n_p} > \frac{d_{p+1}}{n_{p+1}} = \mu(Q_{p+1})
\]
where \( d_p \) and \( n_p \) are the degree and rank of \( Q_p \) and \( \mu(Q_p) \) is its slope. This filtration is canonically associated to \( E \) and is called the Harder-Narasimhan filtration of \( E \).

We define the type of \( E \) to be
\[
\mu = (\mu(Q_1), \ldots, \mu(Q_P)) \in \mathbb{Q}^n
\]
where the entry \( \mu(Q_p) \) is repeated \( n_p \) times. The semistable bundles have type \( \mu_0 = (d/n, \ldots, d/n) \) and form the unique open stratum. The set \( \mathcal{M} \) of all possible types of holomorphic vector bundles over \( \Sigma \) provides our indexing set, and if \( \mu \in \mathcal{M} \) the subset \( \mathcal{C}_\mu \subseteq \mathcal{C} \) is defined to be the set of all holomorphic vector bundles over \( \Sigma \) of type \( \mu \). A partial order on \( \mathcal{M} \) with the property that the closure of the stratum
indexed by $\mu$ is contained in the union of all strata indexed by $\mu' \geq \mu$ is defined as follows. Let $\sigma = (\sigma_1, ..., \sigma_n)$ and $\tau = (\tau_1, ..., \tau_n)$ be two types; then
\begin{equation}
\sigma \geq \tau \text{ if and only if } \sum_{j \leq i} \sigma_j \geq \sum_{j \leq i} \tau_j \text{ for } 1 \leq i \leq n - 1.
\end{equation}

The gauge group $G$ acts on $C$ preserving the stratification which is equivariantly perfect with respect to this action, which means that its equivariant Thom-Gysin sequences
\[ \cdots \rightarrow H^{j-2d_\mu}_{G}(C_\mu) \rightarrow H^j_{G}(U_\mu) \rightarrow H^j_{G}(U_\mu - C_\mu) \rightarrow \cdots \]
break up into short exact sequences
\[ 0 \rightarrow H^{j-2d_\mu}_{G}(C_\mu) \rightarrow H^j_{G}(U_\mu) \rightarrow H^j_{G}(U_\mu - C_\mu) \rightarrow 0. \]
Here
\begin{equation}
d_\mu = \sum_{i > j} (n_i d_j - n_j d_i + n_i n_j (g - 1)),
\end{equation}
is the complex codimension of $C_\mu$ in $C$ and $U_\mu$ is the open subset of $C$ which is the union of all those strata labelled by $\mu' \leq \mu$; we also have
\[ H^*_{G}(C_\mu) \cong \bigotimes_{j=1}^p H^*_{G}(n_j, d_j)(C(n_j, d_j)^{ss}). \]

Atiyah and Bott show that the stratification is equivariantly perfect by considering the composition of the Thom-Gysin map $H^{j-2d_\mu}_{G}(C_\mu) \rightarrow H^j_{G}(U_\mu)$ with restriction to $C_\mu$, which is multiplication by the equivariant Euler class $e_\mu$ of the normal bundle to $C_\mu$ in $C$. They show that $e_\mu$ is not a zero-divisor in $H^*_{G}(C_\mu)$ and deduce that the Thom-Gysin maps $H^{j-2d_\mu}_{G}(C_\mu) \rightarrow H^j_{G}(U_\mu)$ are all injective.

So putting this all together Atiyah and Bott obtain inductive formulas for the $G$-equivariant Betti numbers of $C^{ss}$, and they also conclude that there is a natural surjection
\begin{equation}
H^*(BG) \cong H^*_G(C) \rightarrow H^*_G(C^{ss}) \cong H^*(\mathcal{M}(n, d)).
\end{equation}

Thus generators of the cohomology ring $H^*(BG)$ give generators of the cohomology ring $\mathcal{M}(n, d)$.

The classifying space $BG$ can be identified with the space $\text{Map}_d(\Sigma, BU(n))$ of all smooth maps $f : \Sigma \rightarrow BU(n)$ such that the pullback to $\Sigma$ of the universal vector bundle over $BU(n)$ has degree $d$. If we pull back this universal bundle using the evaluation map
\[ \text{Map}_d(\Sigma, BU(n)) \times \Sigma \rightarrow BU(n) : (f, m) \mapsto f(m) \]
then we obtain a rank $n$ vector bundle $V$ over $BG \times \Sigma$. If further we restrict the pullback bundle induced by the maps
\[ C^{ss} \times EG \times \Sigma \rightarrow C \times EG \times \Sigma \rightarrow C \times G \times \Sigma \times E \rightarrow BG \times \Sigma \]
to $C^{ss} \times \{e\} \times \Sigma$ for some $e \in EG$ then we obtain a $G$-equivariant holomorphic bundle on $C^{ss} \times \Sigma$. The $C^*$-centre of $G$ acts as scalar multiplication on the fibres, and the associated projective bundle descends to a holomorphic projective bundle over $\mathcal{M}(n, d) \times \Sigma$. In fact this is the projective bundle of a holomorphic vector bundle $V$ over $\mathcal{M}(n, d) \times \Sigma$ which has the universal property that, for any $[E] \in \mathcal{M}(n, d)$ representing a bundle $E$ over $\Sigma$, the restriction of $V$ to $\{[E]\} \times \Sigma$ is isomorphic to
By a slight abuse of notation we define elements \( a_r, b_r, f_r \) in \( H^*(BG; \mathbb{Q}) \) by writing
\[
c_r(V) = a_r \otimes 1 + \sum_{j=1}^{2g} b_j \otimes \alpha_j + f_r \otimes \omega \quad 1 \leq r \leq n.
\]
where, as before, \( \omega \) is the standard generator of \( H^2(\Sigma) \) and \( \alpha_1, \ldots, \alpha_{2g} \) form a fixed canonical cohomology basis for \( H^1(\Sigma) \). In fact the ring \( H^*(BG) \) is freely generated as a graded super-commutative algebra over \( \mathbb{Q} \) by the elements
\[
\{ a_r : 1 \leq r \leq n \} \cup \{ b_j : 1 \leq r \leq n, 1 \leq j \leq 2g \} \cup \{ f_r : 2 \leq r \leq n \}
\]
and if we omit \( a_1 \) we get \( H^*(BG) \). These generators restrict to the generators \( a_r, b_r, f_r \) given at (1.1) for \( H^*(\mathcal{M}(n,d)) \) under the surjection (1.1).

The relations among these generators for \( H^*(\mathcal{M}(n,d); \mathbb{Q}) \) are then given by the kernel of the restriction map (1.1) which is in turn determined by the map
(1.5) \( H^*_G(\mathcal{C}) \cong H^*_G(\mathcal{C}) \otimes H^*(BU(1)) \to H^*_G(\mathcal{C}^{ss}) \otimes H^*(BU(1)) \cong H^*_G(\mathcal{C}^{ss}) \),
and the proof that the Yang–Mills stratification is equivariantly perfect leads to completeness criteria for a set of relations to be complete. Let \( R \) be a subset of the kernel of the restriction map \( H^*_G(\mathcal{C}) \to H^*_G(\mathcal{C}^{ss}) \). Suppose that for each unstable type \( \mu \neq \mu_0 \) there is a subset \( R_\mu \) of the ideal generated by \( R \) in \( H^*_G(\mathcal{C}) \) such that the image of \( R_\mu \) under the restriction map
\[
H^*_G(\mathcal{C}) \to H^*_G(\mathcal{C}_\nu)
\]
is zero unless \( \nu \geq \mu \) and when \( \nu = \mu \) contains the ideal of \( H^*_G(\mathcal{C}_\mu) \) generated by the equivariant Euler class \( e_\mu \) of the normal bundle to the stratum \( \mathcal{C}_\mu \) in \( \mathcal{C} \). Then \( R \) generates the kernel of the restriction map \( H^*_G(\mathcal{C}) \to H^*_G(\mathcal{C}^{ss}) \) as an ideal of \( H^*_G(\mathcal{C}) \).

In fact Atiyah and Bott could have replaced the Yang–Mills stratification with a coarser stratification of \( \mathcal{C} \) and obtained equivalent results. For any integers \( n_1 \) and \( d_1 \) let \( S_{n_1,d_1} \) be the subset of \( \mathcal{C} \) consisting of all those holomorphic structures with Harder-Narasimhan filtration \( 0 = E_0 \subset E_1 \subset \cdots \subset E_s = E \) where \( E_1 \) has rank \( n_1 \) and degree \( d_1 \). We shall say that such a holomorphic structure has coarse type \( (n_1, d_1) \). Then \( S_{n_1,d_1} \) is locally a submanifold of finite codimension
\[
\delta_{n_1,d_1} = nd_1 - n_1d + n_1(n - n_1)(g - 1)
\]
in \( \mathcal{C} \) and
(1.6) \( H^*_G(S_{n_1,d_1}) \cong H^*_G(S_{n_1,d_1})(\mathcal{C}(n_1, d_1)^{ss}) \otimes H^*_G(\mathcal{C}(n_1, d_1)) (U(n_1, d_1)) \)
where
\[
U(n_1, d_1) = \bigcup_{\frac{d_1}{n_1} < \frac{d}{n}} S(n - n_1, d - d_1)_{n_2,d_2}
\]
is an open subset of \( \mathcal{C}(n - n_1, d - d_1) \). Moreover the equivariant Euler class \( e_{n_1,d_1} \) of the normal to \( S_{n_1,d_1} \) in \( \mathcal{C} \) is not a zero divisor in \( H^*_G(S_{n_1,d_1}) \), so the stratification of \( \mathcal{C} \) by coarse type
\[
\left\{ S_{n_1,d_1} : 0 < \frac{d_1}{n_1} \right\} \bigcup \{ S_{n,d} \}
\]
is equivariantly perfect. This means that we can modify our completeness criteria, so that for each pair of positive integers \( (\hat{n}, \hat{d}) \) with \( 0 < \hat{n} < n \) and \( \frac{\hat{d}}{\hat{n}} > \frac{d}{n} \) we
require a set of relations whose restriction in $H^*_G(S_{n_1, d_1})$ is zero when $d_1/n_1 < \hat{d}/\hat{n}$ or $d_1/n_1 = \hat{d}/\hat{n}$ and $n_1 < \hat{n}$, and when $(n_1, d_1) = (\hat{n}, \hat{d})$ equals the ideal of $H^*_G(S_{\hat{n}, \hat{d}})$ generated by the equivariant Euler class $e_{\hat{n}, \hat{d}}$ of the normal to $S_{\hat{n}, \hat{d}}$ in $C$.

It is easy enough to prove that if $\gamma \in H^*_G(\hat{C}(\hat{n}, \hat{d}))$ such that $\hat{d}/\hat{n} > d/n$ and if $r > n\hat{n}(g-1) + \hat{n}d - nd$ then the image of the slant product $c_r(-\pi(\mathcal{V}^* \otimes \mathcal{V}))\gamma$ under the restriction map

$$H^*_G(C) \to H^*_G(S_{n_1, d_1}) \cong H^*_G(\hat{C}(n_1, d_1)) \otimes H^*_G(\hat{C}(n-n_1, d-d_1))(U(n_1, d_1))$$

is zero when $d_1/n_1 < \hat{d}/\hat{n}$, and also when $d_1/n_1 = \hat{d}/\hat{n}$ and $n_1 < \hat{n}$.

By Lefschetz duality, since $\hat{C}(\hat{n}, \hat{d})/\hat{G}(\hat{n}, \hat{d}) = M^*(\hat{n}, \hat{d})$ is a manifold of dimension

$$D(\hat{n}, \hat{d}) = 2(\hat{n}^2 - 1)(g-1) + g = 2(n^2 - \hat{n}^2 + 1)$$

we have a natural map

$$LD : H^*_G(\hat{C}(\hat{n}, \hat{d})) \cong H_*(M^*(\hat{n}, \hat{d})) \to H^*_G(\hat{C}(\hat{n}, \hat{d}))$$

such that if $\gamma \in H^*_G(\hat{C}(\hat{n}, \hat{d}))$ then $LD(\gamma)$ lies in the dual of $H^*_G(\hat{C}(\hat{n}, \hat{d}))$ and takes a $\hat{G}(\hat{n}, \hat{d})$-equivariant cycle on $\hat{C}(\hat{n}, \hat{d})$ to its intersection, modulo $\hat{G}(\hat{n}, \hat{d})$, with $\gamma$. When $\hat{n}$ and $\hat{d}$ are coprime then $C(\hat{n}, \hat{d})$ equals $C(\hat{n}, \hat{d})$ and its quotient by $\hat{G}(\hat{n}, \hat{d})$, namely $M(\hat{n}, \hat{d})$, is a compact manifold. In this case Lefschetz duality is essentially Poincaré duality and the map $LD$ is an isomorphism.

We need to consider the restriction of a relation of the form $c_r(-\pi(\mathcal{V}^* \otimes \mathcal{V}))\gamma$ to $H^*_G(S_{\hat{n}, \hat{d}})$. If $\gamma \in H^*_G(\hat{C}(\hat{n}, \hat{d}))$ where $\hat{d}/\hat{n} > d/n$ and if $r = n\hat{n}(g-1) + \hat{n}d - nd + 1 + j$, then it turns out that the image of the slant product $c_r(-\pi(\mathcal{V}^* \otimes \mathcal{V}))\gamma$ under the restriction map

$$H^*_G(C) \to H^*_G(S_{\hat{n}, \hat{d}}) \cong H^*_G(\hat{C}(\hat{n}, \hat{d})) \otimes H^*_G(\hat{C}(n-n_1, d-d_1))(U(\hat{n}, \hat{d}))$$

is the product $(-a_1^{(1)})^j LD(\gamma)e_{\hat{n}, \hat{d}}$ of the equivariant Euler class $e_{\hat{n}, \hat{d}}$ of the normal bundle to $S_{\hat{n}, \hat{d}}$ in $C$ with the image of $\gamma$ under the Lefschetz duality map $LD$ and $j$ copies of minus the generator $a_1^{(1)} \in H^*_G(\hat{C}(\hat{n}, \hat{d}))$. This comes from the copy of the polynomial ring $H^*(BU(1))$ which is the product of the Atiyah and Bott classes $a_r, b_r, f_r$. Recall that $\mathcal{V}$ is a line bundle on $\Delta^*$ and $U(\hat{n}, \hat{d})$ in $C(\hat{n}, \hat{d})$.

The proof of this is based on Porteous’s Formula (as in Beauville’s alternative proof [2] of the theorem of Atiyah and Bott that the classes $a_r, b_r, f_r$ generate $H^*(M(n, d))$; cf. [39] and [12]), which allows us to deduce that the Poincaré dual of $\Delta^* \times U(\hat{n}, \hat{d})$ in $H^*_G(\hat{C}(\hat{n}, \hat{d}) \times C(\hat{n}, \hat{d})) \otimes H^*_G(\hat{C}(n-n_1, d-d_1))(U(\hat{n}, \hat{d}))$

is $c_{\hat{n}^2(g-1)+1}(-\pi(\mathcal{V}^* \otimes \mathcal{V}_1))$. In other words the restriction of $c_{\hat{n}^2(g-1)+1}(-\pi(\mathcal{V}^* \otimes \mathcal{V}_1))$ to $C(\hat{n}, \hat{d}) \times C(\hat{n}, \hat{d}) \times U(\hat{n}, \hat{d})$ is the image of $1$ under the $\hat{G}(\hat{n}, \hat{d}) \times \hat{G}(\hat{n}, \hat{d}) \times \hat{G}(n-n_1, d-d_1)$-equivariant Thom-Gysin map associated to the inclusion of $\Delta^* \times U(\hat{n}, \hat{d})$ in $C(\hat{n}, \hat{d}) \times C(\hat{n}, \hat{d}) \times U(\hat{n}, \hat{d})$. We can express the higher Chern classes of $-\pi(\mathcal{V}^* \otimes \mathcal{V}_1)$ in a similar way [11] by using Fulton’s Excess Porteous Formula [13].
We have found such a relation when \( \eta \) lies in the image of the Lefschetz duality map \( LD \) which maps \( H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^s) \) to \( H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^{ss}) \) and thus into

\[
H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^{ss}) = H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^{ss}) \otimes H^*(BU(1)),
\]

and more generally when \( \eta \) has the form \( \eta = (-a_{\hat{n}}^{(1)})^j LD(\gamma) \), for some element \( \gamma \) of \( H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^s) \). When \( \hat{n} \) and \( \hat{d} \) are coprime this gives us all \( \eta \in H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}^{ss}) \).

Moreover

\[
H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^{ss}) \otimes H^*_{\mathcal{G}(\hat{n} - \hat{n}, \hat{d} - \hat{d})}(\mathcal{C}(\hat{n} - \hat{n}, \hat{d} - \hat{d}))
\]

is generated as a module over \( H^*_{\mathcal{G}(\mathcal{C})} \) by \( H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^{ss}) \), so when \( \hat{n} \) and \( \hat{d} \) are coprime, we have now obtained the relations we need from the slant products

\[
\{c_r((-\pi_{1}(\hat{\mathcal{V}}^s \otimes \mathcal{V}))\gamma : r \geq n\hat{n}(g - 1) + \hat{n}\hat{d} - n\hat{d} + 1, \gamma \in H^*_{\mathcal{G}(\hat{n}, \hat{d})}(\mathcal{C}(\hat{n}, \hat{d})^s)\},
\]

and a little more work reduces the range of \( r \) and \( \hat{d} \) (see [11] for more details).

This deals with the case when \( \hat{n} \) and \( \hat{d} \) are coprime, but the completeness criteria have not yet been shown to hold for pairs \( \hat{n} \) and \( \hat{d} \) with common factors. This was the original motivation for considering further modifications to the Yang–Mills stratification. The difficulty with using the Yang–Mills stratification itself, or the stratification of \( \mathcal{C} \) by coarse type, is that in each case, although \( n \) and \( d \) are chosen to be coprime so that semistability and stability coincide for \( n \) and \( d \), in the construction of the stratification other \( \hat{n} \) and \( \hat{d} \) appear for which semistability and stability do not coincide.

### 2. Stratifying a set of semistable points

In this section we shall describe briefly how to stratify the set \( X^{ss} \) of semistable points of a complex projective variety \( X \) equipped with a linear action of a complex reductive group \( G \), so that the set \( X^s \) of stable points of \( X \) is an open stratum (see [15, 22, 30, 34] for background and [28] for more details).

We assume that \( X \) has some stable points but also has semistable points which are not stable. In [28, 25] it is described how one can blow \( X \) up along a sequence of nonsingular \( G \)-invariant subvarieties to obtain a \( G \)-invariant morphism \( \hat{X} \to X \) where \( \hat{X} \) is a complex projective variety acted on linearly by \( G \) such that \( \hat{X}^{ss} = \hat{X}^s \).

The set \( \hat{X}^{ss} \) can be obtained from \( X^{ss} \) as follows. Let \( r > 0 \) be the maximal dimension of a reductive subgroup of \( G \) fixing a point of \( X^{ss} \), and let \( R(r) \) be a set of representatives of conjugacy classes of all connected reductive subgroups \( R \) of dimension \( r \) in \( G \) such that

\[
Z^{ss}_R = \{ x \in X^{ss} : R \text{ fixes } x \}
\]

is non-empty. Then \( \bigcup_{R \in R(r)} GZ^{ss}_R \) is a disjoint union of nonsingular closed subvarieties of \( X^{ss} \). The action of \( G \) on \( X^{ss} \) lifts to an action on the blow-up of \( X^{ss} \) along \( \bigcup_{R \in R(r)} GZ^{ss}_R \), which can be linearised so that the complement of the set of semistable points in the blow-up is the proper transform of the subset \( \phi^{-1}(\phi(GZ^{ss}_R)) \) of \( X^{ss} \) where \( \phi : X^{ss} \to X/G \) is the quotient map (see [23, 7.17]). Moreover no semistable point in the blow-up is fixed by a reductive subgroup of \( G \) of dimension at least \( r \), and a semistable point in the blow-up is fixed by a reductive subgroup
R of dimension less than r in G if and only if it belongs to the proper transform of the subvariety $Z^s_R$ of $X^s$.

If we repeat this process enough times, we obtain $\pi : \tilde{X}^s \to X^s$ such that $X^s = \tilde{X}^s$. Equivalently we can construct a sequence

$$X^s_{(R_0)} = X^s_{(R_1)} \cdots , X^s_{(R_\tau)} = \tilde{X}^s$$

where $R_1, \ldots, R_\tau$ are connected reductive subgroups of G with

$$r = \dim R_1 \geq \dim R_2 \geq \cdots \dim R_\tau \geq 1,$$

and if $1 \leq l \leq \tau$ then $X_{(R_l)}$ is the blow up of $X^s_{(R_{l-1})}$ along its closed nonsingular subvariety $GZ^s_{R_l} \cong G \times N_l Z^s_{R_l}$, where $N_l$ is the normaliser of $R_l$ in G. Similarly, $\tilde{X} / G = \tilde{X}^s / G$ can be obtained from $X / G$ by blowing up along the proper transforms of the images $Z_R / N$ in $X / G$ of the subvarieties $GZ^s_R$ of $X^s$ in decreasing order of $\dim R$.

If $1 \leq l \leq \tau$ then we have a $G$-equivariant stratification

$$\{ S_{\beta,l} : (\beta, l) \in B_l \times \{ 1 \} \}$$

of $X_{(R_l)}$ by nonsingular $G$-invariant locally closed subvarieties such that one of the strata, indexed by $(0, l) \in B_l \times \{ l \}$, coincides with the open subset $X^s_{(R_l)}$ of $X_{(R_l)}$. Here $B_l$ is a finite subset of a fixed positive Weyl chamber $t_+$ in the Lie algebra $t$ of a maximal compact torus $T$ of $G$. In fact $\beta \in t_+$ lies in $B_l$ if and only if $\beta$ is the closest point to 0 in the convex hull in $t$ of some nonempty subset of the set of weights $\{ \alpha_0, \ldots, \alpha_n \}$ for the linear action of $T$ on $X_{(R_l)}$.

There is a partial ordering on $B_l$ given by $\gamma \succ \beta$ if $|\gamma| > |\beta|$, with 0 as its minimal element, such that if $\beta \in B_l$ then the closure in $X_l$ of the stratum $S_{\beta,l}$ satisfies

$$S_{\beta,l} \subseteq \bigcup_{\gamma \in B_l, \gamma \succ \beta} S_{\gamma,l}.$$  

If $\beta \in B_l$ and $\beta \neq 0$ then the stratum $S_{\beta,l}$ retracts $G$-equivariantly onto its (transverse) intersection with the exceptional divisor $E_l$ for the blow-up $X_{(R_l)} \to X^s_{(R_{l-1})}$. This exceptional divisor is isomorphic to the projective bundle $\mathbb{P}(N_l)$ over $GZ^s_{R_l}$, where $Z^s_{R_l}$ is the proper transform of $Z^s_{R_l}$ in $X^s_{(R_{l-1})}$ and $N_l$ is the normal bundle to $GZ^s_{R_l}$ in $X^s_{(R_{l-1})}$. The stratification $\{ S_{\beta,l} : \beta \in B_l \}$ is determined by the action of $R_l$ on the fibres of $N_l$ over $Z^s_{R_l}$ (see [23, §7]).

There is thus a stratification $\{ \Sigma_{\gamma} : \gamma \in \Gamma \}$ of $X^s$ indexed by

$$\Gamma = \{ R_1 \} \sqcup \{ B_1 \setminus \{ 0 \} \} \times \{ 1 \} \sqcup \cdots \sqcup \{ R_\tau \} \sqcup \{ B_\tau \setminus \{ 0 \} \} \times \{ \tau \} \sqcup \{ 0 \}$$

defined as follows. We take as the highest stratum $\Sigma_{R_1}$ the nonsingular closed subvariety $GZ^s_{R_1}$ whose complement in $X^s$ can be naturally identified with the complement $X_{(R_1)} \setminus E_1$ of the exceptional divisor $E_1$ in $X_{(R_1)}$. We have $GZ^s_{R_1} \cong G \times N_1 Z^s_{R_1}$, where $N_1$ is the normaliser of $R_1$ in $G$, and $Z^s_{R_1}$ is equal to the set of semistable points for the action of $N_1$, or equivalently for the induced action of $N_1 / R_1$, on $Z_{R_1}$, which is a union of connected components of the fixed point set of $R_1$ in $X$. Moreover since $R_1$ has maximal dimension among those reductive subgroups of $G$ with fixed points in $X^s$, we have $Z^s_{R_1} = Z^s_{R_\tau}$ where $Z^s_{R_\tau}$ denotes the set of stable points for the action of $N_1 / R_\tau$ on $Z_{R_\tau}$ for $1 \leq l \leq \tau$. 
Next we take as strata the nonsingular locally closed subvarieties
\[ \Sigma_{\beta,1} = S_{\beta,1}\backslash E_1 \text{ for } \beta \in \mathcal{B}_1 \text{ with } \beta \neq 0 \]
of \( X_{(R_1)}\backslash E_1 = X^{ss}\backslash GZ_{R_1}^{ss} \), whose complement in \( X_{(R_1)}\backslash E_1 \) is just \( X^{ss}_{(R_1)}\backslash E_1 = X_{(R_1)}^{ss}\backslash E_1 \) where \( E_1^{ss} = X^{ss}_{(R_1)} \cap E_1 \), and then we take the intersection of \( X^{ss}_{(R_1)}\backslash E_1 \) with \( GZ_{R_2}^{ss} \). This intersection is \( GZ_{R_2}^{ss} \) where \( Z_{R_2}^{ss} \) is the set of stable points for the action of \( N_2/R_2 \) on \( Z_{R_2} \), and its complement in \( X_{(R_1)}^{ss}\backslash E_1 \) can be naturally identified with the complement in \( X_{(R_2)} \) of the union of \( E_2 \) and the proper transform \( \hat{E}_1 \) of \( E_1 \).

The next strata are the nonsingular locally closed subvarieties
\[ \Sigma_{\beta,2} = S_{\beta,2}\backslash (E_2 \cup \hat{E}_1) \text{ for } \beta \in \mathcal{B}_2 \text{ with } \beta \neq 0 \]
of \( X_{(R_2)}\backslash (E_2 \cup \hat{E}_1) \), whose complement in \( X_{(R_2)}\backslash (E_2 \cup \hat{E}_1) \) is \( X_{(R_2)}^{ss}\backslash (E_2 \cup \hat{E}_1) \), and the stratum after these is \( GZ_{R_1}^s \). We repeat this process for \( 1 \leq l \leq \tau \) and take \( X^s \) as our final stratum indexed by 0.

The given partial orderings on \( \mathcal{B}_1, \ldots, \mathcal{B}_\tau \) together with the ordering in the expression (2.2) above for \( \Gamma \) induce a partial ordering on \( \Gamma \), with \( R_1 \) as the maximal element and 0 as the minimal element, such that the closure in \( X^{ss} \) of the stratum \( \Sigma_\gamma \) indexed by \( \gamma \in \Gamma \) satisfies
\[
\bigcup_{\gamma \geq \gamma} \Sigma_\gamma \subseteq \bigcup \Sigma_{\gamma \prime}.
\]

It is possible to describe the strata \( \Sigma_\gamma \) in more detail. Either \( \Sigma_\gamma \) is \( GZ_{R_1}^s \), for some \( l \), or else it is
\[
S_{\beta,l}\backslash (E_l \cup \hat{E}_{l-1} \cup \ldots \cup \hat{E}_1)
\]
for some \( l \) and \( \beta \in \mathcal{B}_1\backslash \{0\} \), in which case by [22] §5 we have
\[
S_{\beta,l} = GY_{\beta,l}^{ss} \cong G \times p_\beta (Y_{\beta,l}^{ss})
\]
where \( Y_{\beta,l}^{ss} \) fibres over \( Z_{\beta,l}^{ss} \) with fibre \( \mathcal{C}^{m_{\beta,l}} \) for some \( m_{\beta,l} > 0 \), and \( p_\beta \) is a parabolic subgroup of \( G \) with the stabiliser \( \text{Stab}(\beta) \) of \( \beta \) under the adjoint action of \( G \) as its maximal reductive subgroup. Here the fibration \( p_\beta : Y_{\beta,l}^{ss} \rightarrow Z_{\beta,l}^{ss} \) sends \( y \) to a limit point of its orbit under the complex one-parameter subgroup of \( R_\beta \) generated by \( \beta \). Moreover
\[
S_{\beta,l} \cap E_l = G (Y_{\beta,l}^{ss} \cap E_l) \cong G \times p_\beta (Y_{\beta,l}^{ss} \cap E_l)
\]
where \( Y_{\beta,l}^{ss} \cap E_l \) fibres over \( Z_{\beta,l}^{ss} \) with fibre \( \mathcal{C}^{m_{\beta,l}-1} \) (see [23] Lemmas 7.6 and 7.11). Thus
\[
S_{\beta,l} \backslash E_l \cong G \times p_\beta (Y_{\beta,l}^{ss} \backslash E_l)
\]
where \( Y_{\beta,l}^{ss} \backslash E_l \) fibres over \( Z_{\beta,l}^{ss} \) with fibre \( \mathcal{C}^{m_{\beta,l}-1} \times (\mathcal{C} \backslash \{0\}) \). Furthermore if \( \pi_l : E_l \cong \mathbb{P}(\mathcal{N}_l) \rightarrow GZ_{R_1}^s \) is the projection then Lemma 7.9 of [23] tells us that when \( x \in Z_{R_1}^s \) the intersection of \( S_{\beta,l} \) with the fibre \( \pi_l^{-1}(x) = \mathbb{P}(\mathcal{N}_{l,x}) \) of \( \pi_l \) at \( x \) is the union of those strata indexed by points in the adjoint orbit \( Ad(G)\beta \) in the stratification of \( \mathbb{P}(\mathcal{N}_{l,x}) \) induced by the representation \( \rho_l \) of \( R_l \) on the normal \( \mathcal{N}_{l,x} \) to \( GZ_{R_1}^s \) at \( x \). Careful analysis [28] shows that we can, if we wish, replace the indexing set \( \mathcal{B}_1\backslash \{0\} \), whose elements correspond to the \( G \)-adjoint orbits \( Ad(G)\beta \) of elements of the indexing set for the stratification of \( \mathbb{P}(\mathcal{N}_{l,x}) \) induced by the representation \( \rho_l \),
the tautological bundle $T$. Set properties (see \[ (2.7) \; \Sigma = \Sigma_{\beta, l} = S_{\beta, l}(E_l \cup \hat{E}_l \cup \ldots \cup \hat{E}_1) \cong G \times_{Q_{\beta, l}} Y_{\beta, l}^{\lambda E} \]
where $Q_{\beta, l} = q_{\beta}^{-1}(N_l \cap \text{Stab}(\beta))$ and

$$Y_{\beta, l}^{\lambda E} = Y_{\beta, l}^{ss}(E_l \cup \hat{E}_l \cup \ldots \cup \hat{E}_1) \cap p_{\beta}^{-1}\left(Z_{\beta, l}^{ss} \cap \pi^{-1}_l(\hat{Z}_{R_l}^s)\right),$$

and $p_{\beta} : Y_{\beta, l}^{\lambda E} \rightarrow Z_{\beta, l}^{ss} \cap \pi^{-1}_l(\hat{Z}_{R_l}^s)$ is a fibration with fibre $\mathbb{C}^{m_{\beta, l} - 1} \times (\mathbb{C} \setminus \{0\})$.

This process gives us a stratification $\{\Sigma_\gamma : \gamma \in \Gamma\}$ of $X^{ss}$ such that the stratum indexed by the minimal element $0$ of $\Gamma$ coincides with the open subset $X^s$ of $X^{ss}$. We shall apply this construction to obtain a stratification of $C^{ss}$, and thus inductively to refine the Yang-Mills stratification $\{C_\mu : \mu \in M\}$ of $C$ by Harder–Narasimhan type.

### 3. Direct sums of stable bundles of equal slope

In the good case when $n$ and $d$ are coprime, then $C^{ss} = C^s$ and $M(n, d) = C^{ss}/G_c$ is a nonsingular projective variety. When $n$ and $d$ are not coprime, we can use the description of $M(n, d)$ as the geometric invariant theoretic quotient $C/G_c$ to construct a partial desingularisation $\tilde{M}(n, d)$ of $M(n, d)$. From this construction we can use the methods described in §2 to obtain a stratification of $C^{ss}$ with $C^s$ as an open stratum, and thus (by considering the subquotients of the Harder–Narasimhan filtration) obtain a stratification of $C$ refining the stratification $\{C_\mu : \mu \in M\}$. To understand this refined stratification we need to use the description in [26] of the partial desingularisation $\tilde{M}(n, d)$.

In fact in [26] $\tilde{M}(n, d)$ is not constructed using the representation of $M(n, d) as the geometric invariant theoretic quotient of $C$ by $G_c$, although (as is noted at [26], p.246) this representation of $M(n, d)$ would lead to the same partial desingularisation. Instead in [26] $\tilde{M}(n, d)$ is represented as a geometric invariant theoretic quotient of a finite-dimensional nonsingular quasi-projective variety $R(\hat{n}, \hat{d})$ by a linear action of $SL(p, \mathbb{C})$ where $p = d + \hat{n}(1 - g)$ with $\hat{d} >> 0$. We may assume that $\hat{d} >> 0$, since tensoring by a line bundle of degree $l$ gives an isomorphism of $\tilde{M}(n, d)$ with $M(\hat{n}, \hat{d} + \hat{n}l)$ for any $l \in \mathbb{Z}$. By [31] Lemma 5.2 if $E$ is a semistable bundle over $\Sigma$ of rank $\hat{n}$ and degree $\hat{d} > \hat{n}(2g - 1)$ where $g$ is the genus of $\Sigma$, then $E$ is generated by its sections and $H^1(\Sigma, E) = 0$. If $p = d + \hat{n}(1 - g)$, this implies that $\dim H^0(\Sigma, E) = p$ and that there is a holomorphic map $h$ from $\Sigma$ to the Grassmannian $G(\hat{n}, p)$ of $\hat{n}$-dimensional quotients of $\mathbb{C}^p$ such that the pullback $E(h) = h^*T$ of the tautological bundle $T$ on $G(\hat{n}, p)$ is isomorphic to $E$.

Let $R(\hat{n}, \hat{d})$ be the set of all holomorphic maps $h : \Sigma \rightarrow G(\hat{n}, p)$ such that $E(h) = h^*T$ has degree $d$ and the map on sections $\mathbb{C}^p \rightarrow H^0(\Sigma, E(h))$ induced from the quotient bundle map $\mathbb{C}^p \times \Sigma \rightarrow E(h)$ is an isomorphism. For $\hat{d} >> 0$ this set $R(\hat{n}, \hat{d})$ has the structure of a nonsingular quasi-projective variety and there is a quotient $\mathcal{E}$ of the trivial bundle of rank $p$ over $R(\hat{n}, \hat{d}) \times \Sigma$ satisfying the following properties (see [34] §5).

(i) If $h \in R(\hat{n}, \hat{d})$ then the restriction of $E$ to $\{h\} \times \Sigma$ is the pullback $E(h)$ of the tautological bundle $T$ on $G(\hat{n}, p)$ along the map $h : \Sigma \rightarrow G(\hat{n}, p)$.

(ii) If $h_1$ and $h_2$ lie in $R(\hat{n}, \hat{d})$ then $E(h_1)$ and $E(h_2)$ are isomorphic as bundles
over $\Sigma$ if and only if $h_1$ and $h_2$ lie in the same orbit of the natural action of $GL(p; \mathbb{C})$ on $R(\hat{n}, \hat{d})$.

(iii) If $h \in R(\hat{n}, \hat{d})$ then the stabiliser of $h$ in $GL(p; \mathbb{C})$ is isomorphic to the group $\text{Aut}(E(h))$ of complex analytic automorphisms of $E(h)$.

If $N >> 0$ then $R(\hat{n}, \hat{d})$ can be embedded as a quasi-projective subvariety of the product $(G(\hat{n}, p))^{\mathbb{N}}$ by a map of the form $h \mapsto (h(x_1), \ldots, h(x_N))$ where $x_1, \ldots, x_N$ are points of $\Sigma$. This embedding gives a linearisation of the action of $GL(p; \mathbb{C})$ on $R(\hat{n}, \hat{d})$. If $N >> 0$ and $d >> 0$ then we also have the following:

(iv) The point $h \in R(\hat{n}, \hat{d})$ is semistable in the sense of geometric invariant theory for the linear action of $SL(p; \mathbb{C})$ on $R(\hat{n}, \hat{d})$ if and only if $E(h)$ is a semistable bundle. If $h_1$ and $h_2$ lie in $R(\hat{n}, \hat{d})^{ss}$ then they represent the same point of $R(\hat{n}, \hat{d})/SL(p; \mathbb{C})$ if and only if $\text{gr}(E(h_1)) \cong \text{gr}(E(h_2))$, and thus there is a natural identification of $\mathcal{M}(n, d)$ with $R(\hat{n}, \hat{d})/SL(p; \mathbb{C})$ (see for example 34 §5).

It is shown in 24 that the Atiyah–Bott formulas for the equivariant Betti numbers of $C^{ss}$ can be obtained by stratifying $R(n, d)$ instead of $C(n, d)$, and in fact throughout this paper we could work with either $R(n, d)$ or $C(n, d)$. In particular properties (i) to (iv) above imply that the analysis in 26 of the construction of the partial desingularisation of $\hat{\mathcal{M}}(n, \hat{d})$ as $R(\hat{n}, \hat{d})/SL(p; \mathbb{C})$ applies equally well if we work with $\mathcal{G}/G_c$.

To describe the construction of $\hat{\mathcal{M}}(n, d)$, first of all we need to find a set $\mathcal{R}$ of representatives of the conjugacy classes of reductive subgroups of $SL(p; \mathbb{C})$ which occur as the connected components of stabilisers of semistable points of $R(n, d)$. In fact it is slightly simpler to describe the corresponding subgroups of $\hat{G} = GL(p; \mathbb{C})$, and since the central one-parameter subgroup of $GL(p; \mathbb{C})$ consisting of nonzero scalar multiples of the identity acts trivially on $R(n, d)$, finding stabilisers in $GL(p; \mathbb{C})$ is essentially equivalent to finding stabilisers in $SL(p; \mathbb{C})$. By 26 pp. 248-9 such conjugacy classes in $GL(p; \mathbb{C})$ correspond to unordered sequences $(m_1, n_1), \ldots, (m_q, n_q)$ of pairs of positive integers such that $m_1 n_1 + \ldots + m_q n_q = n$ and $n$ divides $n_i d$ for each $i$. A representative $R$ of the corresponding conjugacy class is given by

$$R = GL(m_1; \mathbb{C}) \times \ldots \times GL(m_q; \mathbb{C})$$

embedded in $GL(p; \mathbb{C})$ using a fixed isomorphism of $\mathbb{C}^p$ with $\bigoplus_{i=1}^q (\mathbb{C}^{m_i} \otimes \mathbb{C}^{p_i})$ where $d_i = n_i d / n$ and $p_i = d_i + n_i (1 - g) = n_i p / n$. Then $GZ_R^{ss}$ consists of all those holomorphic structures $E$ with

$$E \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \ldots \oplus (\mathbb{C}^{m_p} \otimes D_q)$$

where $D_1, \ldots, D_q$ are all semistable and $D_i$ has rank $n_i$ and degree $d_i$, while $GZ_R^s$ consists of all those holomorphic structures $E$ with

$$E \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \ldots \oplus (\mathbb{C}^{m_p} \otimes D_q)$$

where $D_1, \ldots, D_q$ are all stable and not isomorphic to one another, and $D_i$ has rank $n_i$ and rank $d_i$. Moreover the normaliser $N$ of $R$ in $GL(p; \mathbb{C})$ has connected component

$$N_0 \cong \prod_{1 \leq i \leq q} (GL(m_i; \mathbb{C}) \times GL(p_i; \mathbb{C}))/\mathbb{C}^*$$
where $\mathbb{C}^*$ is the diagonal central one-parameter subgroup of $GL(m_i; \mathbb{C}) \times GL(p_i; \mathbb{C})$, and $\pi_0(N) = N/N_0$ is the product
\begin{equation}
\pi_0(N) = \prod_{j \geq 0, k \geq 0} \text{Sym}(\#\{i : m_i = j \text{ and } n_i = k\})
\end{equation}
where $\text{Sym}(b)$ denotes the symmetric group of permutations of a set with $b$ elements. Furthermore in terms of the notation of §2, if $R = R_1$ then a holomorphic structure belongs to one of the strata $\Sigma_{\beta,l}$ with $\beta \in B_1 \setminus \{0\}$ if and only if it has a filtration
\[0 = E_0 \subset E_1 \subset \ldots \subset E_s = E\]
such that $E$ is not isomorphic to $\bigoplus_{1 \leq k \leq s} E_k/E_{k-1}$ but
\[\bigoplus_{1 \leq k \leq s} E_k/E_{k-1} \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_r} \otimes D_q)\]
where $D_1, \ldots, D_q$ are all stable and not isomorphic to one another, and $D_i$ has rank $n_i$ and rank $d_i$. Thus to understand the refined Yang-Mills stratification of $\mathcal{C}$, we need to study refinements
\[0 = E_0 \subset E_1 \subset \ldots \subset E_s = E\]
of the Harder-Narasimhan filtration of a holomorphic bundle $E$, such that each subquotient $E_j/E_{j-1}$ is a direct sum of stable bundles all of the same slope. For this recall the following standard result (cf. the proof of [26, Lemma 3.2] and [35]).

**Proposition 3.1.** Any semistable bundle $E$ has a canonical subbundle of the form
\[(\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_s} \otimes D_q)\]
with $D_1, \ldots, D_q$ not isomorphic to each other and all stable of the same slope as $E$, such that any other subbundle of the same form
\[(\mathbb{C}^{m_1'} \otimes D_1') \oplus \cdots \oplus (\mathbb{C}^{m_r'} \otimes D_r')\]
with $D_1', \ldots, D_r'$ not isomorphic to each other and all stable of the same slope as $E$, satisfies $r \leq q$ and (after permuting the order of $D_1, \ldots, D_q$ suitably) $D_j \cong D_j'$ for all $1 \leq j \leq r$ and the inclusion
\[(\mathbb{C}^{m_j'} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_r'} \otimes D_r) \to E\]
factors through the inclusion $(\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q) \to E$ via injections $\mathbb{C}^{m_j'} \to \mathbb{C}^{m_j}$ for $1 \leq j \leq r$.

If we choose a subbundle of $E$ of maximal rank among those of the required form, this follows immediately from

**Lemma 3.2.** Let $E, D_1, \ldots, D_q, D_1', \ldots, D_r'$ be bundles over $\Sigma$ all of the same slope, with $E$ semistable, $D_1, \ldots, D_q$ and $D_1', \ldots, D_r'$ all stable and $D_j = D_j'$ for $1 \leq j \leq k$ for some $0 \leq k \leq \min\{q, r\}$, but with no other isomorphisms between the bundles $D_1, \ldots, D_q$ and $D_1', \ldots, D_r'$. If
\[\alpha : (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q) \to E\]
and
\[\beta : (\mathbb{C}^{m_1'} \otimes D_1') \oplus \cdots \oplus (\mathbb{C}^{m_r'} \otimes D_r') \to E\]
are injective bundle homomorphisms, then there exist nonnegative integers $n_1, \ldots, n_k$ and linear injections $i_j : \mathbb{C}^{m_j} \to \mathbb{C}^{n_j}$ and $i'_j : \mathbb{C}^{m_j'} \to \mathbb{C}^{n_j}$ for $1 \leq j \leq k$ and an injective bundle homomorphism $\gamma$ from
\[(\mathbb{C}^{n_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{n_k} \otimes D_k) \oplus (\mathbb{C}^{m_{k+1}} \otimes D_{k+1}) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)\]
\( \oplus(C^{m_i+1} \otimes D'_{j+1}) \oplus \cdots \oplus (C^{m_r} \otimes D'_r) \)

to \( E \) such that \( \alpha \) and \( \beta \) both factorise through \( \gamma \) via the injections \( i_j \) and \( i'_j \) for \( 1 \leq j \leq k \) in the obvious way.

**Proof:** Consider the kernel of
\[
\begin{aligned}
\alpha \oplus \beta : (C^{m_1+m_1} \otimes D_1) \oplus \cdots \oplus (C^{m_k+m_k} \otimes D_k) & \oplus \\
(C^{m_{r+1}} \otimes D_{k+1}) \oplus \cdots \oplus (C^{m_r} \otimes D_q) & \oplus \\
(C^{m_{k+1}+1} \otimes D'_{k+1}) \oplus \cdots \oplus (C^{m_r} \otimes D'_r) & \rightarrow \ E.
\end{aligned}
\]

The proof of [34] Lemma 5.1 (iii) shows that this kernel is a subsheaf of the domain of \( \alpha \oplus \beta \) which has the same slope as \( E, D_1, \ldots, D_q, \) and \( D'_1, \ldots, D'_r \). Induction on \( m_1 + \ldots + m_q + m'_1 + \ldots + m'_r \) using [34] Lemma 5.1] and the obvious projection from \( C^{m_i} \) onto \( C^{m_i-1} \) shows that such a subsheaf is in fact of the form
\[
(U_1 \otimes D_1) \oplus \cdots \oplus (U_q \otimes D_q) \oplus (U_{q+1} \otimes D'_{k+1}) \oplus \cdots \oplus (U_{q-k+r} \otimes D'_r)
\]
for some linear subspaces \( U_j \) of \( C^{m_1+m_1}, \ldots, C^{m_{k+1}}, \ldots, C^{m_r} \). Then since \( \alpha \) and \( \beta \) are injective, we must have \( \ker(\alpha \oplus \beta) = (U_1 \otimes D_1) \oplus \cdots \oplus (U_k \otimes D_k) \) for linear subspaces \( U_j \) of \( C^{m_j}, C^{m_j'} \) satisfying \( U_j \cap C^{m_j} = \{0\} = U_j \cap C^{m_j'} \) for \( 1 \leq j \leq k \). The result follows easily.

A similar argument gives us

**Corollary 3.3.** Let \( E \) be semistable and \( D'_1, \ldots, D'_r \) all stable of the same slope as \( E \) and not isomorphic to each other. Then
\[
H^0(\Sigma, ((C^{m_1} \otimes D'_1) \oplus \cdots \oplus (C^{m_r} \otimes D'_r))^* \otimes E) \cong \bigoplus_{j=1}^k ((C^{m_j'})^* \otimes C^{m_j})
\]
where \( (C^{m_1} \otimes D_1) \oplus \cdots \oplus (C^{m_q} \otimes D_q) \) is the canonical subbundle of this form associated to \( E \) as in Proposition 3.1, and without loss of generality we assume \( D_j \cong D'_j \) for \( 1 \leq j \leq k \) for some \( 0 \leq k \leq \min\{q, r\} \) and that there are no other isomorphisms between the bundles \( D_1, \ldots, D_q \) and \( D'_1, \ldots, D'_r \).

**Definition 3.4.** With the notation above we set
\[
\text{gr}(E) = \bigoplus_{i=1}^q \bigoplus_{j=1}^r (C^{m_{ij}} \otimes D_i).
\]

for any semistable bundle \( E \).

**Remark 3.5.** Of course, by the Jordan–Hölder theorem, given any filtration \( 0 = D_0 \subset D_1 \subset \ldots \subset D_t = E \) of \( E \) such that \( D_j/D_{j-1} \) is a direct sum of stable bundles of the same slope as \( E \), we have \( \bigoplus_{j=1}^t D_j/D_{j-1} \cong \text{gr}(E) \).

**4. Maximal and minimal Jordan–Hölder filtrations**

Recall that the Harder-Narasimhan filtration of a holomorphic bundle \( E \) over \( \Sigma \) is a canonical filtration
\[
0 = F_0 \subset F_1 \subset \ldots \subset F_s = E
\]
of $E$ such that $F_j/F_{j-1}$ is semistable and slope($F_j/F_{j-1}$) $>$ slope($F_{j+1}/F_j$) for $0 < j < s$. In the last section we saw that any semistable bundle $E$ has a canonical maximal subbundle of the form

$$(C^{m_1} \otimes D_1) \oplus \cdots \oplus (C^{m_q} \otimes D_q)$$

where $D_1, \ldots, D_q$ are not isomorphic to each other and are all stable of the same slope as $E$. This subbundle is nonzero if $E \neq 0$, since any nonzero semistable bundle is either stable itself or it has a proper stable subbundle of the same slope.

Therefore any semistable bundle $E$ has a canonical filtration

$$(4.1) \quad 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E$$

whose subquotients are direct sums of stable bundles, which is defined inductively so that

$$(4.2) \quad E_j/E_{j-1} \cong (C^{m_{ij}} \otimes D_1) \oplus \cdots \oplus (C^{m_{ij}} \otimes D_q)$$

where $D_1, \ldots, D_q$ are stable nonisomorphic bundles all of the same slope as $E$ with nonnegative integers $m_{ij}$ for $1 \leq i \leq q$ and $1 \leq j \leq r$, and $E_j/E_{j-1}$ is the maximal subbundle of $E/E_{j-1}$ of this form. If, moreover, we assume that

$$\sum_{j=1}^r m_{ij} > 0 \text{ for all } 1 \leq i \leq q$$

then the filtration $(4.1)$, the bundles $D_i$ and integers $m_{ij}$ (for $1 \leq i \leq q$ and $1 \leq j \leq r$) and the decompositions $(4.2)$ are canonically associated to $E$ up to isomorphism of the bundles $D_i$, the usual action of $GL(m_{ij}; \mathbb{C})$ on $C^{m_{ij}}$ and the obvious action of the permutation group $Sym(q)$ on this data. We can generalise the definition to the case when $E$ is not necessarily semistable, by applying this construction to the subquotients of the Harder-Narasimhan filtration of $E$. This gives us a canonical refinement

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

of the Harder-Narasimhan filtration such that each subquotient $E_j/E_{j-1}$ is the maximal subbundle of $E/E_{j-1}$ which is a direct sum of stable bundles all having maximal slope among the nonzero subbundles of $E/E_{j-1}$. We shall call this refinement of the Harder-Narasimhan filtration the maximal Jordan–Hölder filtration of $E$.

**Definition 4.1.** Let $s$, $q_1$, ..., $q_s$ and $r_1$, ..., $r_s$ be positive integers and let $d_k$, $n_{ik}$ and $m_{ijk}$ (for $1 \leq i \leq q_k$, $1 \leq j \leq r_k$ and $1 \leq k \leq s$) be integers satisfying

$$n_{ik} > 0, \quad m_{ijk} \geq 0, \quad n_k > 0$$

and

$$n = \sum_{k=1}^s n_k, \quad d = \sum_{k=1}^s d_k, \quad \frac{d_k n_{ik}}{n_k} = d_{ik} \in \mathbb{Z}, \quad \frac{d_1}{n_1} > \frac{d_2}{n_2} > \ldots > \frac{d_s}{n_s}$$

where

$$n_k = \sum_{j=1}^{q_k} \sum_{i=1}^{r_k} n_{ik} m_{ijk}.$$ 

Denote by $[d, n, m] = [(d_k)_{k=1}^s, (n_{ik})_{i=1,k=1}^{q_k,r_k}, (m_{ijk})_{i=1,j=1,k=1}^{q_k,r_k}]$ the orbit of

$$(d, n, m) = ((d_k)_{k=1}^s, (n_{ik})_{i=1,k=1}^{q_k,r_k}, (m_{ijk})_{i=1,j=1,k=1}^{q_k,r_k})$$
under the action of the product of symmetric groups $\Sigma_{q_1} \times \ldots \times \Sigma_{q_k}$ on the set of such sequences, and let $\mathcal{I} = \mathcal{I}(\hat{n}, \hat{d})$ denote the set of all such orbits, for fixed $\hat{n}$ and $\hat{d}$. Given $[d, n, m] \in \mathcal{I}(\hat{n}, \hat{d})$ let $s([d, n, m]) = s$ and let $S_{[d, n, m]}^{\text{maxJH}}$ denote the subset of $\mathcal{C}$ consisting of those holomorphic structures on our fixed smooth bundle of rank $\hat{n}$ and degree $\hat{d}$ whose maximal Jordan–Hölder filtration

$$0 = E_{0,1} \subset E_{1,1} \subset \cdots \subset E_{r_1,1} = E_{0,2} \subset E_{1,2} \subset \cdots$$

$$\cdots \subset E_{r_s,1} = E_{r_s,0} \subset \cdots \subset E_{r_s,s} = E$$

satisfies

$$E_{j,k}/E_{j-1,k} \cong (\mathbb{C}^{m_{1j}} \otimes D_{1j}) \oplus \cdots \oplus (\mathbb{C}^{m_{qj}} \otimes D_{qj})$$

for $1 \leq k \leq s$ and $1 \leq j \leq q$, where $D_{1j}, \ldots, D_{qj}$ are nonisomorphic stable bundles with

$$\operatorname{rank}(D_{ij}) = n_{ij} \quad \text{and} \quad \deg(D_{ij}) = d_{ij}$$

and $E_{j,k}/E_{j-1,k}$ is the maximal subbundle of $E/E_{j-1,k}$ isomorphic to a direct sum of stable bundles of slope $d_{ij}/n_{ij}$. To make the notation easier on the eye, $S_{[d, n, m]}^{\text{maxJH}}$ will often be denoted by $S_{[d, n, m]}$.

Let $\mathcal{I}^{ss}$ denote the subset of $\mathcal{I}$ consisting of all orbits $[d, n, m]$ for which $s([d, n, m]) = 1$. For simplicity we shall write $[n, m]$ for $[d, n, m]$ when $s([d, n, m]) = 1$ (which means that $d = (d)$).

We have now proved

**Lemma 4.2.** $\mathcal{C}$ is the disjoint union of the subsets

$$\{S_{[d, n, m]} : [d, n, m] \in \mathcal{I}\}.$$

**Remark 4.3.** Note that when $s([d, n, m]) = 1$ and $q_1 = r_1 = 1$ and $m_{111} = \hat{n}$ and $m_{111} = \hat{n}$ then we get $S_{[n, (1)]} = \mathcal{C}^s$, and moreover

$$\mathcal{C}(n, d)^{ss} = \bigcup_{[d, n, m] \in \mathcal{I}, s([d, n, m]) = 1} S_{[d, n, m]} = \bigcup_{[n, m] \in \mathcal{I}^{ss}} S_{[n, m]}.$$

**Remark 4.4.** Let $E$ be a semistable holomorphic structure on $\mathcal{E}$. If $E$ represents an element of the closure of $S_{[n, m]}$ in $\mathcal{C}(n, d)^{ss}$ for some

$$[n, m] = [(n_i)_{i=1}^q, (m_{ij})_{i=1,j=1}^r] \in \mathcal{I}^{ss},$$

then $E$ has a filtration $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E$ such that if $1 \leq j \leq r$ then

$$E_j/E_{j-1} \cong (\mathbb{C}^{m_{ij}} \otimes D_{1j}) \oplus \cdots \oplus (\mathbb{C}^{m_{ij}} \otimes D_{qj})$$

where $D_1, \ldots, D_q$ are semistable bundles all having the same slope as $E$, but this filtration is not necessarily the maximal Jordan–Hölder filtration of $E$. If the bundles $D_1, \ldots, D_q$ are not all stable or if two of them are isomorphic to each other, then

$$\dim \operatorname{Aut}(\operatorname{gr}(E)) > \sum_{i=1}^q \left( \sum_{j=1}^r m_{ij} \right)^2$$

and so $E$ lies in $S_{[n', m']}$ where $m' = (m'_{ij})_{i=1,j=1}^{q', r'}$ satisfies

$$\sum_{i=1}^{q'} \left( \sum_{j=1}^{r'} m'_{ij} \right)^2 > \sum_{i=1}^q \left( \sum_{j=1}^r m_{ij} \right)^2.$$
If, on the other hand, $D_1, \ldots, D_q$ are all stable and not isomorphic to each other, then the maximal Jordan–Hölder filtration of $E$ is of the form

$$0 = E'_0 \subset E'_1 \subset E'_2 \subset \cdots \subset E'_r = E$$

with

$$E'_j / E'_{j-1} \cong (\mathbb{C}^{m'_{i_1} \oplus D_1} \oplus \cdots \oplus (\mathbb{C}^{m'_{i_q} \oplus D_q})$$

for $1 \leq j \leq r'$, where $1 \leq r' \leq r$ and

$$m'_{i_1} + \ldots + m'_{i_r} = m_{i_1} + \ldots + m_{i_r}$$

and $m'_{i_1} + \ldots + m'_{i_q} \geq m_{i_1} + \ldots + m_{i_j}$ for $1 \leq i \leq q$ and $1 \leq j \leq r'$. Thus we can define a partial order $\geq$ on $\mathcal{I}^{ss}$ such that $[n', m'] \geq [n, m]$ if and only if either

$$\sum_{i=1}^{q'} \sum_{j=1}^{r'} (m'_{ij})^2 > \sum_{i=1}^{q} \sum_{j=1}^{r} (m_{ij})^2$$

or $n' = n$ and $1 \leq r' \leq r$ and

$$m'_{i_1} + \ldots + m'_{i_r} = m_{i_1} + \ldots + m_{i_r}$$

and $m'_{i_1} + \ldots + m'_{i_q} \geq m_{i_1} + \ldots + m_{i_j}$ for $1 \leq i \leq q$ and $1 \leq j \leq r'$, and then the closure of $S_{[n, m]}$ in $\mathcal{C}(n, d)^{ss}$ is contained in

$$\bigcup_{[n', m'] \geq [n, m]} S_{[n', m']}:$$

Using \cite{12}, we can then extend this partial order to $\mathcal{I}$ so that

$$\overline{S_{[d, n, m]}} \subseteq \bigcup_{[d', n', m'] \geq [d, n, m]} S_{[d', n', m']}.$$

**Proposition 4.5.** Let $[n, m] = [(n_i)_{i=1}^{q}, (m_{ij})_{i=1}^{q}, j=1] \in \mathcal{I}^{ss}$. Then $S_{[n, m]}$ is a locally closed complex submanifold of $\mathcal{C}^{ss}$ of finite codimension

$$\sum_{i=1}^{q} \sum_{j=1}^{r} m_{ij}m_{ij+1} + (g-1) \left( \sum_{i=1}^{q} \sum_{j=1}^{r} m_{ij}m_{ij+1} \right) = \sum_{i=1}^{q} \sum_{j=1}^{r} (m_{ij})^2.$$

**Proof:** (cf. \cite{11} \S 7) The rank and degree are the only $C^\infty$ invariants of a vector bundle over $\Sigma$. Thus we may choose a $C^\infty$ isomorphism of our fixed $C^\infty$ bundle $\mathcal{E}$ over $\Sigma$ with a bundle of the form

$$\bigoplus_{i=1}^{q} \bigoplus_{j=1}^{r} (\mathbb{C}^{m_{ij}} \otimes D_i)$$

where $D_i$ is a fixed $C^\infty$ bundle over $\Sigma$ of rank $n_i$ and degree $d_i = n_i d / n$ for $1 \leq i \leq q$.

Let $\mathcal{Y}_{[n, m]}$ be the subset of $\mathcal{C}^{ss}$ consisting of all semistable holomorphic structures $E$ on $\mathcal{E}$ for which the subbundles

$$E_{ij} = \bigoplus_{k=1}^{q} (\mathbb{C}^{m_{ik}} \otimes D_k)$$

are holomorphic for $1 \leq j \leq r$, and for which there are nonisomorphic stable holomorphic structures $D_1, \ldots, D_q$ on the $C^\infty$ bundles $D_1, \ldots, D_q$ such that the natural identification of $E_{ij} / E_{j-1}$ with $\bigoplus_{i=1}^{q} C^{m_{ij}} \otimes D_i$ becomes an isomorphism of $E_{ij} / E_{j-1}$ with $\bigoplus_{i=1}^{q} C^{m_{ij}} \otimes D_i$ for $1 \leq j \leq r$, and finally for each $1 \leq j \leq r$ the
quotient $E_j/E_{j-1}$ is the maximal subbundle of $E/E_{j-1}$ isomorphic to a direct sum of stable bundles of the same slope as $E/E_{j-1}$. Let $\mathcal{G}_c[n,m]$ be the subgroup of the complexified gauge group $\mathcal{G}_c$ consisting of all $C^\infty$ complex automorphisms of $\mathcal{E}$ which preserve the filtration of $\mathcal{E}$ by the subbundles $\bigoplus_{i=1}^q \bigoplus_{k=1}^{l} (\mathbb{C}^{m_{ik}} \otimes D_i)$ and the decomposition of $E_j/E_{j-1}$ as $\bigoplus_{i=1}^q (\mathbb{C}^{m_{ij}} \otimes D_i)$ up to the action of the general linear groups $GL(m_{ij}; \mathbb{C})$ and the permutation groups

$$Sym(\#\{i : n_i = k \text{ and } m_{ij} = l_j, j = 1, ..., r\})$$

for all nonnegative integers $k$ and $l_1, ..., l_r$. Since the filtration (4.1) and decompositions of $E_1/E_0, ..., E_r/E_{r-1}$ are canonically associated to $E$ up to the actions of these general linear and permutation groups, we have

$$S_{[n,m]} = \mathcal{G}_c \mathcal{Y}_{[n,m]} \cong \mathcal{G}_c \times \mathcal{G}_c [n,m] \mathcal{Y}_{[n,m]}.$$ 

As in [11] §7 we have that $\mathcal{Y}_{[n,m]}$ is an open subset of an affine subspace of the infinite-dimensional affine space $\mathcal{C}$ and the injection

(4.3) $$\mathcal{G}_c \times \mathcal{G}_c [n,m] \mathcal{Y}_{[n,m]} \to \mathcal{C}$$

is holomorphic with image $S_{[n,m]}$.

If $E \in \mathcal{Y}_{[n,m]}$, let $\text{End}'E$ be the subbundle of $\text{End}E$ consisting of holomorphic endomorphisms of $E$ preserving the maximal Jordan–Hölder filtration (4.1) and decomposition (4.2) up to isomorphism of the bundles $D_i$ and the vector spaces $\mathbb{C}^{m_{ij}}$. Let $\text{End}''E$ be the quotient of $\text{End}E$ by $\text{End}'E$. The normal to the $\mathcal{G}_c$-orbit of $E$ in $\mathcal{C}$ can be canonically identified with $H^1(\Sigma, \text{End}E)$ (see [11] §7) and the image of $T_E \mathcal{Y}_{[n,m]}$ in this can be canonically identified with the image of the natural map

$$H^1(\Sigma, \text{End}'E) \to H^1(\Sigma, \text{End}E)$$

which fits into the long exact sequence of cohomology induced by the short exact sequence of bundles

$$0 \to \text{End}'E \to \text{End}E \to \text{End}''E \to 0.$$ 

Thus we get an isomorphism

$$T_E \mathcal{C}/(T_E \mathcal{Y}_{[n,m]} + T_E \mathcal{O}) \cong H^1(\Sigma, \text{End}''E)$$

where $\mathcal{O}$ is the $\mathcal{G}_c$-orbit of $E$ in $\mathcal{C}$.

We have short exact sequences

$$0 \to (E/E_1)^* \otimes E \to \text{End}E \to E_1^* \otimes E \to 0$$

and

$$0 \to E_1^* \otimes E_1 \to E_1^* \otimes E \to E_1^* \otimes (E/E_1) \to 0.$$ 

Let $K$ be the kernel of the composition of surjections

$$\text{End}E \to E_1^* \otimes E \to E_1^* \otimes (E/E_1).$$

Then we have short exact sequences

$$0 \to K \to \text{End}E \to E_1^* \otimes (E/E_1) \to 0$$

and

$$0 \to (E/E_1)^* \otimes E \to K \to E_1^* \otimes E_1 \to 0.$$ 

Now $\text{End}'E \subseteq K$ and the image of $\text{End}'E$ in $E_1^* \otimes E_1$ is

$$\bigoplus_{i=1}^q \text{End}(\mathbb{C}^{m_{i1}}) \otimes \text{End}(D_i),$$
so since $\text{End}'' E = \text{End} E / \text{End}' E$ we get short exact sequences

\begin{equation}
0 \rightarrow \frac{K}{\text{End}' E} \rightarrow \text{End}'' E \rightarrow E_1^* \otimes \left( \frac{E}{E_1} \right) \rightarrow 0
\end{equation}

and

\begin{equation}
0 \rightarrow \frac{\text{End}' E + ((E/E_1)^* \otimes E)}{\text{End}' E} \rightarrow \frac{K}{\text{End}' E} \rightarrow \bigoplus_{i=1}^q \text{End}(C^{m_i}) \otimes \text{End}(D_i) \rightarrow 0.
\end{equation}

Since

\[
\text{End}' E + ((E/E_1)^* \otimes E)
\]

the short exact sequence \((4.5)\) becomes

\begin{equation}
0 \rightarrow \text{End}'' \left( \frac{E}{E_1} \right) \rightarrow \frac{K}{\text{End}' E} \rightarrow E_1^* \otimes E_1 \rightarrow \bigoplus_{i=1}^q \text{End}(C^{m_i}) \otimes D_i^* \otimes D_i \rightarrow 0.
\end{equation}

From the sequences \((4.4)\) and \((4.6)\) it follows that the rank of $\text{End}'' E$ is

\[
\text{rank}(\text{End}'' E) = \text{rank}(K/\text{End}' E) + \text{rank}(E_1^* \otimes (E/E_1))
\]

\[
= \text{rank}(\text{End}'' (E/E_1)) + \text{rank}(E_1^* \otimes E_1) - \sum_{i=1}^q (m_{i_1})^2 \text{rank}(D_i^* \otimes D_i)
\]

\[
+ \text{rank}(E_1^* \otimes (E/E_1))
\]

\[
= \text{rank}(\text{End}'' (E/E_1)) + \text{rank}(E_1^* \otimes E) - \sum_{i=1}^q (m_{i_1})^2 (n_i)^2.
\]

Thus by induction on $r$ we have

\[
\text{rank}(\text{End}'' E) = \sum_{i_1, i_2=1}^q \sum_{1 \leq j_1 \leq j_2 \leq r} m_{i_1, j_1} m_{i_2, j_2} n_{i_1} n_{i_2} - \sum_{i=1}^q \sum_{j=1}^r (n_i)^2.
\]

Since $D_1, \ldots, D_q$ all have the same slope $\hat{d}/\hat{n}$ as $E$ we have $\deg(\text{End}'' E) = 0$. Therefore by Riemann-Roch

\[
\dim H^1(\Sigma, \text{End}'' E) = \dim H^0(\Sigma, \text{End}'' E)
\]

\[
+ (g - 1) \left( \sum_{i_1, i_2=1}^q \sum_{1 \leq j_1 \leq j_2 \leq r} m_{i_1, j_1} m_{i_2, j_2} n_{i_1} n_{i_2} - \sum_{i=1}^q \sum_{j=1}^r (n_i)^2 \right).
\]

Moreover the short exact sequences \((4.4)\) and \((4.6)\) give us long exact sequences of cohomology

\[
0 \rightarrow H^0(\Sigma, K/\text{End}' E) \rightarrow H^0(\Sigma, \text{End}'' E) \rightarrow H^0(\Sigma, E_1^* \otimes (E/E_1)) \rightarrow \cdots
\]

and

\[
0 \rightarrow H^0(\Sigma, \text{End}'' (E/E_1)) \rightarrow H^0(\Sigma, K/\text{End}' E)
\]
→ \( H^0(\Sigma, E_1^* \otimes E_1 / \bigoplus_{i=1}^q \text{End}(\mathbb{C}^{m_{1i}}) \otimes \text{End}(D_i)) \rightarrow \cdots \)

Now \( E_1 \cong \bigoplus_{i=1}^q \mathbb{C}^{m_{1i}} \otimes D_i \) where \( D_1, \ldots, D_q \) are nonisomorphic stable bundles all of the same slope as \( E_1 \), and

\[
E_2/E_1 \cong \bigoplus_{i=1}^q \mathbb{C}^{m_{2i}} \otimes D_i
\]

is the maximal subbundle of \( E/E_1 \) which is a direct sum of stable bundles all of the same slope as \( E/E_1 \). Since \( E_1 \) and \( E/E_1 \) have the same slope, it follows from Corollary 3.3 that

\[
H^0(\Sigma, E_1^* \otimes (E/E_1)) = H^0(\Sigma, E_1^* \otimes (E_2/E_1)) \cong \bigoplus_{i=1}^q (\mathbb{C}^{m_{1i}})^* \otimes \mathbb{C}^{m_{2i}}.
\]

By choosing an open cover \( \mathcal{U} \) of \( \Sigma \) such that the filtration \( 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E \) is trivial over each \( U \in \mathcal{U} \), and describing \( E \) in terms of upper triangular transition functions on \( E_1 \otimes (E_2/E_1) \otimes \cdots \otimes (E/E_{r-1}) \mid U \cap V \) for \( U, V \in \mathcal{U} \) which induce the identity on \( E_j/E_{j-1} \) for \( 1 \leq j \leq r \), we see that the natural map

\[
H^0(\Sigma, \text{End}'''E) \rightarrow H^0(\Sigma, E_1^* \otimes (E/E_1)) = H^0(\Sigma, E_1^* \otimes (E_2/E_1))
\]

is surjective. Also

\[
H^0(\Sigma, E_1^* \otimes E_1 / \bigoplus_{i=1}^q \text{End}(\mathbb{C}^{m_{1i}} \otimes \text{End}(D_i)) = \bigoplus_{i \neq j} (\mathbb{C}^{m_{1i}})^* \otimes \mathbb{C}^{m_{1j}} \otimes H^0(\Sigma, D_1^* \otimes D_j) = 0,
\]

so

\[
\dim H^0(\Sigma, \text{End}'''E) = \sum_{i=1}^q m_{1i} m_{12} + \dim H^0(\Sigma, \text{End}''(E/E_1))
\]

and thus by induction on \( r \) we have

\[
\dim H^0(\Sigma, \text{End}'''E) = \sum_{i=1}^q \sum_{j=1}^{r-1} m_{ij} m_{ij+1}.
\]

Therefore \( \dim H^1(\Sigma, \text{End}'''E) \) is equal to

\[
\sum_{i=1}^q \sum_{j=1}^{r-1} m_{ij} m_{ij+1} + (g-1) \left( \sum_{i_1,i_2=1}^q \sum_{1 \leq i_1 \leq i_2 \leq r} m_{1i_1} m_{2i_2} n_{i_1} n_{i_2} - \sum_{i=1}^r \sum_{j=1}^q (m_{ij} n_j)^2 \right).
\]

In particular this tells us that the image in \( \mathcal{C} \) of the derivative of the injection \( \text{(4.3)} \) has constant codimension, and it follows as in \( \text{(11, 87)} \) (see also \( \text{(11, 8814 and 15)} \)) that the subset \( S_{[n,m]} \) is locally a complex submanifold of \( \mathcal{C} \) of finite codimension given by \( \text{(14)} \).

**Corollary 4.6.** If \( [d,n,m] \in \mathcal{I} \) then \( S_{[d,n,m]} \) is a locally closed complex submanifold of \( \mathcal{C} \) of codimension

\[
\sum_{1 \leq k_2 < k_1 \leq s} (n_{k_2} d_{k_2} - n_{k_1} d_{k_1} + n_{k_1} n_{k_2} (g-1)) + \sum_{k=1}^s \sum_{i=1}^q \sum_{j=1}^{r-1} m_{ijk} m_{ij+1k}
\]
filtration of a bundle $E$

Proof: This follows immediately from \[(13), \text{Lemma 4.5 and the definition of } S_{[d,n,m]} \text{ (Definition 4.1).}

Remark 4.7. Let $0 = E_0 \subset E_1 \subset ... \subset E_t = E$ be the maximal Jordan–Hölder filtration of a bundle $E$. Then the kernels of the duals of the inclusions $E_j \to E$ give us a filtration

$$0 = F_0 \subset F_1 \subset ... \subset F_t = E'$$

of the dual $E'$ of $E$, such that if $1 \leq j \leq t$ then

$$F_j/F_{j-1} \cong (E_{t-j+1}/E_{t-j})'.$$

Thus $F_j/F_{j-1}$ is a direct sum of stable bundles all of the same slope, and moreover $F_{j-1}$ is the minimal subbundle of $F_j$ such that $F_j/F_{j-1}$ is a direct sum of stable bundles all of which have minimal slope among quotients of $F_j$.

Applying this construction with $E$ replaced by $E'$, we find that every holomorphic bundle $E$ over $\Sigma$ has a canonical filtration

$$0 = F_0 \subset F_1 \subset ... \subset F_t = E,$$

which we will call the minimal Jordan–Hölder filtration of $E$, with the property that if $1 \leq j \leq t$ then $F_{j-1}$ is the minimal subbundle of $F_j$ such that $F_j/F_{j-1}$ is a direct sum of stable bundles all of which have minimal slope among quotients of $F_j$.

The minimal and maximal Jordan–Hölder filtrations of a bundle do not necessarily coincide. For example, consider the direct sum $E \oplus F$ of two semistable bundles with maximal Jordan–Hölder filtrations $0 = E_0 \subset E_1 \subset ... \subset E_t = E$ and $0 = F_0 \subset F_1 \subset ... \subset F_s = F$ where without loss of generality we may assume that $s \leq t$. If $E$ and $F$ have the same slope, then it is easy to check that the maximal Jordan–Hölder filtration of $E \oplus F$ is

$$0 = E_0 \oplus F_0 \subset E_1 \oplus F_1 \subset ... \subset E_t \oplus F_t \subset E \oplus F,$$

that is, it is the direct sum of the maximal Jordan–Hölder filtrations of $E$ and $F$ with the shorter one extended trivially at the top. Similarly the minimal Jordan–Hölder filtration of $E \oplus F$ is the direct sum of the minimal Jordan–Hölder filtrations of $E$ and $F$ with the shorter one extended trivially at the bottom. Thus if the minimal and maximal Jordan–Hölder filtrations of $E$ and $F$ coincide (which will be the case if, for example, each of the subquotients $E_j/E_{j-1}$ and $F_j/F_{j-1}$ are stable) but these filtrations are not of the same length, then the minimal Jordan–Hölder filtration

$$0 = E_0 \oplus F_0 \subset E_1 \oplus F_0 \subset ... \subset E_t \oplus F_0 \subset E \oplus F$$

of $E \oplus F$ will be different from its maximal Jordan–Hölder filtration.

Definition 4.8. Given $[d,n,m] \in I$, let $S^{\min \text{JH}}_{[d,n,m]}$ denote the subset of $C$ consisting of those holomorphic structures on our fixed smooth bundle of rank $\hat{n}$ and degree $\hat{d}$ whose minimal Jordan–Hölder filtration is of the form

$$0 = E_{0,1} \subset E_{1,1} \subset ... \subset E_{r_1,1} = E_{0,2} \subset E_{1,2} \subset ... \subset E_{r_s,1},$$

... $E_{r_{s-1},s-1} = E_{r_s,0} \subset ... \subset E_{r_s,s} = E$.
with
\[ E_{j,k} / E_{j-1,k} \cong (\mathbb{C}^{m_{1,k}} \otimes D_{1,k}) \oplus \cdots \oplus (\mathbb{C}^{m_{q,k}} \otimes D_{q,k}) \]
for \( 1 \leq k \leq s \) and \( 1 \leq j \leq q_k \), where \( D_{1,k}, \ldots, D_{q,k} \) are nonisomorphic stable bundles with
\[ \text{rank}(D_{ik}) = n_{ik} \text{ and } \deg(D_{ik}) = d_{ik} \]
and \( E_{j-1,k} \) is the minimal subbundle of \( E_{j,k} \) such that \( E_{j,k} / E_{j-1,k} \) is a direct sum of stable bundles of slope \( d_k / n_k \).

5. More indexing sets

In this section we will consider the indexing set \( \Gamma \) for the stratification \( \{ \Sigma_{\gamma} : \gamma \in \Gamma \} \) of \( \mathcal{C}^{\infty} \) defined as in \( \S 2 \).

If \( \gamma \in \Gamma \) then by (2.2) either \( \gamma = 0 \) or \( \gamma = R_l \) or \( \gamma \in \mathcal{B}_l \setminus \{ 0 \} \times \{ \} \) for some \( 1 \leq l \leq \tau \). If \( \gamma = 0 \) then \( \Sigma_{\gamma} = \mathcal{C}^{\infty} \), while by [26] pp.
248-9 if \( \gamma = R_l \) then there exists \( [n, m] = [(m_i)_{i=1}^q, (m_{ij})_{i=1,j=1}^{q,r}] \in \mathcal{T}^{\infty} \) with \( r = 1 \) and \( q = q_1 \), such that

\[ R_l = \prod_{i=1}^q GL(m_i; \mathbb{C}) \]

where \( m_i = m_{i1} \), and \( \Sigma_{R_l} \) consists of all those holomorphic structures \( E \) with

\[ E \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q) \]

where \( D_1, \ldots, D_q \) are all stable of slope \( d / \hat{n} \) and not isomorphic to one another.

In order to describe the strata \( \{ \Sigma_{\beta, l} : \beta \in \mathcal{B}_l \setminus \{ 0 \} \} \) more explicitly, we need to look at the action of \( R_l \) on the normal \( \mathcal{N}_{R_l} \) to \( GZ_{R_l}^{\infty} \) at a point represented by a holomorphic structure \( E \) of the form (5.2), and to understand the stratification on \( \mathbb{P}(\mathcal{N}_{R_l}) \) induced by this action of \( R_l \). If we choose a \( C^\infty \) isomorphism of our fixed \( C^\infty \) bundle \( E \) with \( (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q) \), then we can identify \( \mathcal{C} \) with the infinite-dimensional vector space

\[ \Omega^{0,1}(\text{End}([(\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)])) \]

and the normal to the \( \mathcal{G}_c \)-orbit at \( E \) can be identified with \( H^1(\Sigma, \text{End}E) \), where \( \text{End}E \) is the bundle of holomorphic endomorphisms of \( E \) \([1] \S 7 \). If \( \delta_1^j \) denotes the Kronecker delta then the normal to \( GZ_{R_l}^{\infty} \) can be identified with

\[ H^1(\Sigma, \text{End}_\beta^E) \cong \bigoplus_{i_1,i_2=1}^q \mathbb{C}^{m_{i1}, m_{i2} - \delta_1^j} \otimes H^1(\Sigma, D_{i1}^* \otimes D_{i2}) \]

where \( \text{End}_\beta^E \) is the quotient of the bundle \( \text{End}E \) of holomorphic endomorphisms of \( E \) by the subbundle \( \text{End}_\beta^E \) consisting of those endomorphisms which preserve the decomposition \([22] \). The action of \( R_l = \prod_{i=1}^q GL(m_i; \mathbb{C}) \) on this is given by the natural action on \( \mathbb{C}^{m_{i1}, m_{i2} - \delta_1^j} \) identified with the set of \( m_{i1} \times m_{i2} \) matrices if \( i_1 \neq i_2 \) and the set of trace-free \( m_{i1} \times m_{i1} \) matrices if \( i_1 = i_2 \); its weights \( \alpha \) are therefore of the form \( \alpha = (\xi - \xi') \) where \( \xi \) and \( \xi' \) are weights of the standard representation of \( R_l \) on \( \bigoplus_{i=1}^q \mathbb{C}^{m_i} \) (see [26] pp.
251-2 noting the error immediately before (3.18)).

Any element \( \beta \) of the indexing set \( \mathcal{B}_l \) is represented by the closest point to 0 of the convex hull of some nonempty set of these weights, and two such closest points can be taken to represent the same element of \( \mathcal{B}_l \) if and only if they lie in the same \( Ad(N_l) \)-orbit, where \( N_l \) is the normaliser of \( R_l \) (see [22] or [23]). By [23]...
the orbit of $\beta$ under the adjoint action of the connected component of $N_i$ is just its $\text{Ad}(R_i)$-orbit, and so by \cite{E} the $\text{Ad}(N_i)$-orbit of $\beta$ is the union
\[ \bigcup_{w \in \pi_0(N_i)} w.\text{Ad}(N_i)(\beta) \]
where $\pi_0(N_i)$ is the product of permutation groups
\[ \pi_0(N_i) = \prod_{j \geq 0, k \geq 0} \text{Sym}(\# \{i : m_i = j \text{ and } n_i = k\}). \]

We can describe this indexing set $\mathcal{B}_l$ more explicitly as follows. Let us take our maximal compact torus $T_i$ in $R_i$ to be the product of the standard maximal tori of the unitary groups $U(m_1), \ldots, U(m_q)$ consisting of the diagonal matrices, and let $t_l$ be its Lie algebra. Let
\[ M = m_1 + \ldots + m_q \]
and let $e_1, \ldots, e_M$ be the weights of the standard representation of $T_l$ on $\mathbb{C}^{m_1} \oplus \ldots \oplus \mathbb{C}^{m_q}$. We take the usual invariant inner product on the Lie algebra $\mathfrak{u}(p)$ of $U(p)$ given by $\langle A, B \rangle = -\text{tr}AB^t$ and restrict it to $T_l$. Since $R_l$ is embedded in $\text{GL}(p; \mathbb{C})$ by identifying $\oplus_{i=1}^q (\mathbb{C}^{m_i} \otimes \mathbb{C}^p)$ with $\mathbb{C}^p$, it follows that $e_1, \ldots, e_M$ are mutually orthogonal and $\|e_j\|^2 = 1/p_i$ if $m_1 + \ldots + m_i < j \leq m_1 + \ldots + m_i$.

**Proposition 5.1.** Let $\beta$ be any nonzero element of the Lie algebra $t_l$ of the maximal compact torus $T_l$ of $R_l$. Then $\beta$ represents an element of $\mathcal{B}_l \setminus \{0\}$ if and only if there is a partition
\[ \{\Delta_{h,m} : (h, m) \in J\} \]
of $\{1, \ldots, M\}$, indexed by a subset $J$ of $\mathbb{Z} \times \mathbb{Z}$ of the form
\[ J = \{(h, m) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq h \leq L, l_1(h) \leq m \leq l_2(h)\} \]
for some positive integer $L$ and functions $l_1$ and $l_2 : \{1, \ldots, L\} \to \mathbb{Z}$ such that $l_1(h) \leq l_2(h)$ for all $h \in \{1, \ldots, L\}$, with the following properties. If
\[ r_{h,m} = \sum_{j \in \Delta_{h,m}} \|e_j\|^{-2} \]
then the function $\epsilon : \{1, \ldots, L\} \to \mathbb{Q}$ defined by
\[ \epsilon(h) = \left( \sum_{m=l_1(h)}^{l_2(h)} m r_{h,m} \right)^{-1} \left( \sum_{m=l_1(h)}^{l_2(h)} r_{h,m} \right)^{-1} \]
satisfies $-1/2 \leq \epsilon(h) < 1/2$ and $\epsilon(1) > \epsilon(2) > \ldots > \epsilon(L)$, and
\[ \frac{\beta}{\|\beta\|^2} = \sum_{h=1}^{L} \sum_{m=l_1(h)}^{l_2(h)} \sum_{j \in \Delta_{h,m}} (\epsilon(h) - m) \frac{e_j}{\|e_j\|^2}. \]

**Remark 5.2.** Note that because of the conditions on the function $\epsilon$, the partition $\{\Delta_{h,m} : (h, m) \in J\}$ and its indexing can be recovered from the coefficients of $\beta$ with respect to the basis $e_1/\|e_1\|^2, \ldots, e_M/\|e_M\|^2$ of $t_l$.

**Proof of Proposition 5.1** $\beta \in t_l \setminus \{0\}$ represents an element of $\mathcal{B}_l \setminus \{0\}$ if and only if it is the closest point to 0 of the convex hull of
\[ \{e_i - e_j : (i, j) \in S\} \]
for some nonempty subset $S$ of $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq i, j \leq M\}$. Then $\beta$ can be expressed in the form

$$\beta = \sum_{(i,j) \in S} \lambda_{ij}^\beta (e_i - e_j)$$

for some $\lambda_{ij}^\beta \in \mathbb{R}$ for $(i,j) \in S$ such that $\lambda_{ij}^\beta \geq 0$ and $\sum_{(i,j) \in S} \lambda_{ij}^\beta = 1$. Replacing $S$ with its subset $\{(i,j) \in S : \lambda_{ij}^\beta > 0\}$ we may assume that $\lambda_{ij}^\beta > 0$ for all $(i,j) \in S$. Moreover clearly if $\lambda$ without loss of generality that $\lambda_{ij}^\beta = 1$ then take $J = \{(1,0),(1,1)\}$ with $\Delta_{(1,0)} = \{j\}$ and $\Delta_{(1,1)} = \{i\}$, so we can assume without loss of generality that $\lambda_{ij}^\beta < 1$ for all $(i,j) \in S$. Since $\beta \neq 0$ we can also assume that the convex hull of $\{e_i - e_j : (i,j) \in S\}$ does not contain 0.

In order to find the closest point to 0 of the convex hull of $\{e_i - e_j : (i,j) \in S\}$ we minimise

$$\| \sum_{(i,j) \in S} \lambda_{ij}(e_i - e_j) \|^2$$

subject to the constraints that $\lambda_{ij} \geq 0$ for all $(i,j) \in S$ and $\sum_{(i,j) \in S} \lambda_{ij} = 1$. Since the weights $e_1, ..., e_M$ are mutually orthogonal, we have

$$\| \sum_{(i,j) \in S} \lambda_{ij}(e_i - e_j) \|^2 = \|\sum_{i=1}^M (\sum_{j:(j,i) \in S} \lambda_{ij} - \sum_{j:(j,i) \in S} \lambda_{ji}) e_i \|^2$$

$$= \sum_{i=1}^M (\sum_{j:(j,i) \in S} \lambda_{ij} - \sum_{j:(j,i) \in S} \lambda_{ji})^2\|e_i\|^2.$$

Using the method of Lagrange multipliers, we consider

$$\sum_{i=1}^M (\sum_{j:(j,i) \in S} \lambda_{ij} - \sum_{j:(j,i) \in S} \lambda_{ji})^2\|e_i\|^2 - \lambda (\sum_{(i,j) \in S} \lambda_{ij} - 1).$$

If $(i,j) \in S$ then $i \neq j$ and $(j,i) \not\in S$ since the convex hull of $\{e_i - e_j : (i,j) \in S\}$ does not contain 0, so

$$\frac{\partial}{\partial \lambda_{ij}} \sum_{i=1}^M (\sum_{j:(j,i) \in S} \lambda_{ij} - \sum_{j:(j,i) \in S} \lambda_{ji})^2\|e_i\|^2 - \lambda (\sum_{(i,j) \in S} \lambda_{ij} - 1))$$

is equal to

$$2(\sum_{k:(i,k) \in S} \lambda_{ik} - \sum_{k:(i,k) \in S} \lambda_{ik})\|e_i\|^2 - 2(\sum_{k:(j,k) \in S} \lambda_{jk} - \sum_{k:(j,k) \in S} \lambda_{kj})\|e_j\|^2 - \lambda.$$

Thus $\beta = \sum_{(i,j) \in S} \lambda_{ij}^\beta (e_i - e_j)$ where for each $(i,j) \in S$ we have either $\lambda_{ij}^\beta = 0$ or $\lambda_{ij}^\beta = 1$ (both of which are ruled out by the assumptions on $S$) or

$$(5.3) \quad (\sum_{k:(i,k) \in S} \lambda_{ik} - \sum_{k:(i,k) \in S} \lambda_{ik})\|e_i\|^2 - (\sum_{k:(j,k) \in S} \lambda_{jk} - \sum_{k:(j,k) \in S} \lambda_{kj})\|e_j\|^2 = \lambda/2$$

where $\lambda$ is independent of $(i,j) \in S$.

From $S$ we can construct a directed graph $G(S)$ with vertices $1, ..., M$ and directed edges from $i$ to $j$ whenever $(i,j) \in S$. Let $\Delta_1, ..., \Delta_N$ be the connected components of this graph. Then $\{e_i - e_j : (i,j) \in S\}$ is the disjoint union of its subsets $\{e_i - e_j : (i,j) \in S, i, j \in \Delta_h\}$ for $1 \leq h \leq N$, and $\{e_i - e_j : (i,j) \in S, i, j \in \Delta_h\}$ is contained in the vector subspace of $t_i$ spanned by the basis vectors
\{e_k : k \in \Delta_h \}. Since these subspaces are mutually orthogonal for \(1 \leq h \leq N\), it follows that

\[
(5.4) \quad \beta = \left( \sum_{h=1}^{L} \frac{1}{\| \beta_h \|^2} \right)^{-1} \sum_{h=1}^{L} \frac{\beta_h}{\| \beta_h \|^2}
\]

where

\[
\beta_h = \sum_{(i,j) \in S, i,j \in \Delta_h} \lambda_{ij}^{\beta_h} (e_i - e_j)
\]

is the closest point to 0 of the convex hull of \(\{e_i - e_j : (i, j) \in S, i, j \in \Delta_h\}\) for \(1 \leq h \leq N\), and without loss of generality we assume that \(\beta_h\) is nonzero when \(1 \leq h \leq L\) and zero when \(L < h \leq N\). Note that then

\[
(5.5) \quad \| \beta \|^2 = \left( \sum_{h=1}^{L} \frac{1}{\| \beta_h \|^2} \right)^{-2} \sum_{h=1}^{L} \frac{\| \beta_h \|^2}{\| \beta_h \|^4} = \left( \sum_{h=1}^{L} \frac{1}{\| \beta_h \|^2} \right)^{-1}
\]

so that

\[
(5.6) \quad \frac{\beta}{\| \beta \|^2} = \sum_{h=1}^{L} \frac{\beta_h}{\| \beta_h \|^2}
\]

and

\[
\lambda_{ij}^{\beta_h} = (\| \beta_h \|^2 / \| \beta \|^2) \lambda_{ij}^{\beta}
\]

if \(i, j \in \Delta_h\). For \(1 \leq h \leq L\) let \(\kappa_h\) be defined by

\[
\kappa_h = \max \{ ( \sum_{k:(i,k) \in S} \lambda_i^\beta - \sum_{k:(k,i) \in S} \lambda_k^\beta) |e_i|^2 : i \in \Delta_h \}
\]

where a sum over the empty set is interpreted as 0. Then by (5.3) for \(1 \leq h \leq L\) we can express \(\Delta_h\) as a disjoint union

\[
\Delta_h = \hat{\Delta}_{h,0} \cup \hat{\Delta}_{h,1} \cup \ldots \cup \hat{\Delta}_{h,l_h}
\]

where

\[
(5.7) \quad \hat{\Delta}_{h,m} = \{ i \in \Delta_h : ( \sum_{j:(i,j) \in S} \lambda_{ij}^\beta - \sum_{j:(j,i) \in S} \lambda_{ji}^\beta ) |e_i|^2 = \kappa_h - m |\lambda|/2 \}. \]

Let us assume that \(\lambda \leq 0\); the argument is similar if \(\lambda \geq 0\). Then (5.3) tells us that if \((i,j) \in S\) then there exist \(h \) and \(m\) such that \(i \in \hat{\Delta}_{h,m}\) and \(j \in \hat{\Delta}_{h,m+1}\). Note that if \(\hat{\Delta}_{h,m_1}\) and \(\hat{\Delta}_{h,m_2}\) are nonempty then so is \(\hat{\Delta}_{h,m}\) whenever \(m_1 < m < m_2\), so without loss of generality we may assume that \(\hat{\Delta}_{h,m}\) is nonempty when \(1 \leq h \leq L\) and \(0 \leq m \leq l_h\). Thus if \(1 \leq h \leq L\) we have

\[
\beta_h = (\| \beta_h \|^2 / \| \beta \|^2) \sum_{i \in \Delta_h} ( \sum_{j \in \hat{\Delta}_{h,(i,j)} \in S} \lambda_{ij}^\beta - \sum_{j \in \hat{\Delta}_{h,(j,i)} \in S} \lambda_{ji}^\beta ) e_i
\]

\[
= (\| \beta_h \|^2 / \| \beta \|^2) \sum_{m=1}^{l_h} \sum_{i \in \hat{\Delta}_{h,m}} (\kappa_h - m |\lambda|/2) \frac{e_i}{|e_i|^2}.
\]

For \(1 \leq h \leq L\) and \(0 \leq m \leq l_h\) let \(r_{h,m} = \sum_{j \in \hat{\Delta}_{h,m}} |e_j|^{-2}\); then by (5.1) we have

\[
\sum_{i \in \hat{\Delta}_{h,m}} ( \sum_{j:(i,j) \in S} \lambda_{ij}^\beta - \sum_{j:(j,i) \in S} \lambda_{ji}^\beta ) = r_{h,m} (\kappa_h - m |\lambda|/2). \]
Recall that if \((i, j) \in S\) then \(i \in \hat{\Delta}_{h,m}\) if and only if \(j \in \hat{\Delta}_{h,m+1}\), so we get
\[
\sum_{i \in \hat{\Delta}_{h,0}} \sum_{j \in (i, j) \in S} \lambda^\beta_{ij} = r_{h,0} \kappa_h,
\]
and hence
\[
\sum_{i \in \hat{\Delta}_{h,1}} \sum_{j \in (i, j) \in S} \lambda^\beta_{ij} = \sum_{i \in \hat{\Delta}_{h,1}} \left( \sum_{j \in (i, j) \in S} \lambda^\beta_{ij} - \sum_{j \in (i, j) \in S} \lambda^\beta_{ji} \right) + \sum_{j \in \hat{\Delta}_{h,0}} \sum_{i \in (j, i) \in S} \lambda^\beta_{ji}
\]
\[
= r_{h,1} (\kappa_h - |\lambda|/2) + r_{h,0} \kappa_h,
\]
and similarly
\[
\sum_{i \in \hat{\Delta}_{h,m}} \sum_{j \in (i, j) \in S} \lambda^\beta_{ij} = (r_{h,0} + r_{h,1} + ... + r_{h,m}) \kappa_h - (r_{h,1} + 2r_{h,2} + ... + mr_{h,m}) \frac{|\lambda|}{2}
\]
if \(1 \leq m \leq l_h\). Since \(\sum_{(i, j) \in S, i \in \Delta_h} \lambda^\beta_{ij} = 1\), it follows that
\[
|\beta|^2 / |\beta| = ((l_h + 1)r_{h,0} + l_hr_{h,1} + ... + r_{h,l_h}) \kappa_h
\]
\[-l_hr_{h,1} + 2(l_h - 1)r_{h,2} + 3(l_h - 2)r_{h,3} + ... + l_hr_{h,l_h}) |\lambda|/2,
\]
and since \(\sum_{i \in S, j \in \Delta_h} \lambda^\beta_{ij} - \sum_{i \in S, j \in \Delta_h} \lambda^\beta_{ji} = 0\) we have
\[
0 = (r_{h,0} + r_{h,1} + ... + r_{h,l_h}) \kappa_h - (r_{h,1} + 2r_{h,2} + ... + l_hr_{h,l_h}) |\lambda|/2.
\]
Thus
\[
|\lambda|/2 = (|\beta|^2 / |\beta|) (r_{h,0} + r_{h,1} + ... + r_{h,l_h}) / \mu_h
\]
and
\[
\kappa_h = (|\beta|^2 / |\beta|) (r_{h,1} + 2r_{h,2} + ... + l_hr_{h,l_h}) / \mu_h
\]
where
\[
\mu_h = \sum_{i, j=0}^{l_h} (((l_h - i + 1)j - j(l_h - j + 1))r_{h,i}r_{h,j}
\]
\[
= \sum_{0 \leq i < j \leq l_h} ((j - i)i + (i - j)i)r_{h,i}r_{h,j} = \sum_{0 \leq i < j \leq l_h} (j - i)^2 r_{h,i}r_{h,j}.
\]
Therefore
\[
\beta_h = \sum_{m=0}^{l_h} \sum_{j=0}^{l_h} (j - m)r_{h,j} \mu_h \sum_{i \in \Delta_{h,m}} \frac{e_i}{\|e_i\|^2}
\]
and
\[
|\beta_h|^2 = \sum_{h=1}^{L} \sum_{m=1}^{l_{h(h)}} \frac{e_i}{\|e_i\|^2}
\]
By defining \(\Delta_{h,m} = \Delta_{h,m-1(h)}\) for an appropriate integer \(l_{1(h)}\), we can arrange that the function \(\epsilon\) defined in the statement of the proposition takes values in the interval \([-1/2, 1/2]\), and then by amalgamating those \(\Delta_h\) for which \(\epsilon(h)\) takes the same value and rearranging them so that \(\epsilon\) is a strictly decreasing function, we can assume that the required conditions on \(\epsilon\) are satisfied, and we have
\[
\frac{\beta}{|\beta|} = \sum_{h=1}^{L} \frac{l_{h(h)}}{m_{h(h)}} \sum_{i \in \Delta_{h,m}} (\epsilon(h) - m) \frac{e_i}{\|e_i\|^2}.
\]
This gives us all the required properties if \(L = N\); that is, if \(\bigcup_{(h,m) \in J} \Delta_{h,m}\) is equal to \(\{1, ..., M\}\). Otherwise we amalgamate \(\{1, ..., M\} \setminus \bigcup_{(h,m) \in J} \Delta_{h,m}\) with \(\Delta_{h_0, m_0}\) where \((h_0, m_0)\) is the unique element of \(J\) such that \(\epsilon(h_0) = 0 = m_0\) if such an
element exists, and if there is no such element of \( J \) then we adjoin \((L + 1, 0)\) to \( J \) and define

\[
\Delta_{L+1,0} = \{1, \ldots, M\} \setminus \bigcup_{h=1}^{L} \bigcup_{m=l_1(h)}^{l_2(h)} \Delta_{h,m}.
\]

Conversely, suppose that we are given any partition \( \{\Delta_{h,m} : (h,m) \in J\} \) of \( \{1, \ldots, M\} \) indexed by

\[
J = \{(h,m) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq h \leq L, l_1(h) \leq m \leq l_2(h)\}
\]

for some positive integer \( L \) and functions \( l_1 \) and \( l_2 : \{1, \ldots, L\} \rightarrow \mathbb{Z} \) with \( l_1 \leq l_2 \), satisfying \( \epsilon(h) \in [-1/2, 1/2] \) and \( \epsilon(1) > \epsilon(2) > \ldots > \epsilon(L) \) where

\[
\epsilon(h) = \left( \frac{l_2(h)}{m=r_{h,m}} m \right) \left( \frac{l_2(h)}{m=l_1(h)} r_{h,m} \right)^{-1}
\]

for \( r_{h,m} = \sum_{i \in \Delta_{h,m}} |e_i|^{-2} \). Suppose also that

\[
\beta = \frac{\hat{\beta}}{\|\beta\|^2}
\]

(or equivalently \( \hat{\beta} = \beta/\|\beta\|^2 \)) where

\[
\hat{\beta} = \sum_{h=1}^{L} \sum_{m=l_1(h)}^{l_2(h)} \sum_{j \in \Delta_{h,m}} (\epsilon(h) - m) \frac{e_j}{\|e_j\|^2}.
\]

It suffices to show that \( \beta \) is the closest point to 0 of the convex hull of \( \{e_i - e_j : (i,j) \in S\} \), where \( S \) is the set of ordered pairs \((i,j)\) with \( i,j \in \{1, \ldots, M\} \) such that \( e_i \in \Delta_{h,m} \) and \( e_j \in \Delta_{h,m+1} \) for some \((h,m) \in J\) such that \((h,m+1) \in J\). For this, it is enough to prove firstly that \( \beta \) lies in the convex hull of \( \{e_i - e_j : (i,j) \in S\} \) and secondly that \( (e_i - e_j) \beta = \|\beta\|^2 \) (or equivalently that \( (e_i - e_j) \hat{\beta} = 1 \)) for all \((i,j) \in S\). The latter follows easily from the choice of \( S \): if \((i,j) \in S\) then there exists \((h,m) \in J\) such that \((h,m+1) \in J\) and \( e_i \in \Delta_{h,m} \) and \( e_j \in \Delta_{h,m+1} \), so

\[
(e_i - e_j) \hat{\beta} = \epsilon(h) - m - \epsilon(h) + m + 1 = 1.
\]

To show that \( \beta \) lies in the convex hull of \( \{e_i - e_j : (i,j) \in S\} \), we note that

\[
\sum_{m=l_1(h)}^{l_2(h)} \sum_{j \in \Delta_{h,m}} \sum_{k=l_1(h)}^{m} \frac{\epsilon(h) r_{h,k} - kr_{h,k}}{r_{h,m} r_{h,m+1} \|e_i\|^2 \|e_j\|^2} (e_i - e_j)
\]

\[
= \sum_{m=l_1(h)}^{l_2(h)} \sum_{j \in \Delta_{h,m}} (\epsilon(h) - m) \frac{e_j}{\|e_j\|^2}.
\]

This means that

\[
\beta = \sum_{(i,j) \in S} \lambda_{ij}^\beta (e_i - e_j)
\]

where

\[
\frac{\lambda_{ij}^\beta}{\|\beta\|^2} = \sum_{k=l_1(h)}^{m} \frac{\epsilon(h) r_{h,k} - kr_{h,k}}{r_{h,m} r_{h,m+1} \|e_i\|^2 \|e_j\|^2}
\]

\[(5.8)\]
if $i \in \Delta_{h,m}$ and $j \in \Delta_{h,m+1}$ for some $(h,m) \in J$ such that $(h,m+1) \in J$. Then

$$\sum_{(i,j) \in S} \lambda^\beta_{ij} \frac{1}{\|\beta\|^2} = \sum_{h=1}^L \sum_{m=l_i(h)}^{l_h(h)-1} \sum_{j \in \Delta_{h,m}} \sum_{k \in \Delta_{h,m+1}} \frac{\epsilon(h) r_{h,k} - k r_{h,k}}{r_{h,m} r_{h,m+1} \| e_i \|^2 \| e_j \|^2} = \sum_{h=1}^L \sum_{m=l_i(h)}^{l_h(h)-1} \sum_{j \in \Delta_{h,m}} \frac{\epsilon(h) r_{h,k} - k r_{h,k}}{r_{h,m} r_{h,m+1} \| e_i \|^2 \| e_j \|^2} \sum_{h=1}^L \sum_{m=l_i(h)}^{l_h(h)-1} \sum_{j \in \Delta_{h,m}} \frac{\epsilon(h) r_{h,k} - k r_{h,k}}{r_{h,m} r_{h,m+1} \| e_i \|^2 \| e_j \|^2}$$

Expanding the brackets, using the definition of $\epsilon$ and replacing the index $k$ by $m$ shows that this equals

$$\sum_{h=1}^L \sum_{m=l_i(h)}^{l_h(h)-1} \sum_{j \in \Delta_{h,m}} \frac{\epsilon(h) r_{h,k} - k r_{h,k}}{r_{h,m} r_{h,m+1} \| e_i \|^2 \| e_j \|^2} = \sum_{h=1}^L \sum_{m=l_i(h)}^{l_h(h)-1} \sum_{j \in \Delta_{h,m}} \frac{\epsilon(h) r_{h,k} - k r_{h,k}}{r_{h,m} r_{h,m+1} \| e_i \|^2 \| e_j \|^2}$$

and so $\sum_{(i,j) \in S} \lambda^\beta_{ij} = 1$. Finally note that

$$(\sum_{k_1=l_i(h)}^{l_h(h)} (\epsilon(h) - k_1) r_{h,k_1})(\sum_{k_2=l_1(h)}^{l_h(h)} r_{h,k_2}) = \sum_{k_1=l_i(h)}^{l_h(h)} \sum_{k_2=l_1(h)}^{l_h(h)} (k_2 r_{h,k_2} - k_1 r_{h,k_2}) r_{h,k_1}.$$ 

This sum is positive because if $k_2 > m$ then the contribution of the pair $(k_1, k_2)$ to the sum is $(k_2 - k_1) r_{h,k_1} r_{h,k_2} > 0$, whereas if $k_2 \leq m$ then the total contribution of the pairs $(k_1, k_2)$ and $(k_2, k_1)$ is zero. Thus by (5.8) we have $\lambda^\beta_{ij} \geq 0$ for all $(i,j) \in S$, and hence $\beta$ lies in the convex hull of $\{e_i - e_j : (i,j) \in S\}$ as required.

**Lemma 5.3.** If $\beta$ satisfies the conditions of Proposition 5.7 and if $i \in \Delta_{h,m}$ and $j \in \Delta_{h',m'}$, then

$$\beta.(e_i - e_j) = \|\beta\|^2$$

if and only if $h' = h$ and $m' = m + 1$, and

$$\beta.(e_i - e_j) \geq \|\beta\|^2$$

if and only if either $m' \geq m + 2$ or $m' = m + 1$ and $h' \geq h$.

**Proof:** This follows immediately from the formula for $\beta$ and conditions on the function $\epsilon$ in the statement of Proposition 5.7.

**Remark 5.4.** Let $\beta$ be as in Proposition 5.1. Then there is a unique bijection $\phi : J \rightarrow \{1, ..., t\}$ from the indexing set $J$ of the partition $\{\Delta_{h,m} : (h,m) \in J\}$ of $\{1, ..., M\}$ to the set of positive integers $\{1, ..., t\}$, where $t$ is the size of $J$, which takes the Hebrew lexicographic ordering on $J$ to the standard ordering on integers; that is, $\phi(h,m) \leq \phi(h',m')$ if and only if either $m < m'$ or $m = m'$ and $h \leq h'$. We can define an increasing function

$$\delta : \{1, ..., t\} \rightarrow \{1, ..., t\}$$

such that $\delta(\phi(h,m))$ is the number of elements $(h',m') \in J$ such that either $m' < m + 1$ or $m' = m + 1$ and $h' < h$. Then $\delta(k) \geq k$ for all $k \in \{1, ..., t\}$, and if $(h,m)$ and $(h,m+1)$ both belong to $J$ then $\delta(\phi(h,m)) = \phi(h,m+1) - 1$ and $\delta(\phi(h,m)) < \delta(\phi(h,m+1))$. Conversely if

$$k - 1 = \delta(k_1) < \delta(k_1 + 1)$$
then there exists \((h, m) \in J\) with \((h, m + 1) \in J\) such that \(k_1 = \phi(h, m)\) and \(k_2 = \phi(h, m + 1)\).

When it is helpful to make the dependence on \(\beta\) explicit, we shall write \(\delta_{\beta} : \{1, \ldots, t_{\beta}\} \to \{1, \ldots, t_{\beta}\}\) and \(\delta_{h, m}(\beta) : (h, m) \in J_{\beta}\).

Lemma 5.3 tells us that if \(i \in \Delta_{h, m}\) and \(j \in \Delta_{h', m'}\), then \(\beta(e_i - e_j) \geq |\beta|^2\) if and only if \(\phi(h', m') > \phi(h, m))\).

**Definition 5.5.** Recall that if \(1 \leq i \leq q\) then \(e_{m_1 + \ldots + m_{i-1} + 1, \ldots, m_1 + \ldots + m_i}\) are the weights of the standard representation on \(\mathbb{C}^{m_i}\) of the component \(GL(m_i; \mathbb{C})\) of \(R_t = \prod_{i=1}^q GL(m_i; \mathbb{C})\). If \(\phi : J \to \{1, \ldots, t\}\) and \(1 \leq i \leq q\) and \(1 \leq k \leq t\), then set

\[
\Delta^k = \Delta_{\phi^{-1}(k)}, \quad \Delta^k_i = \Delta_{\phi^{-1}(k)} \cap \{m_1 + \ldots + m_{i-1} + 1, \ldots, m_1 + \ldots + m_i\}
\]

and let \(m^k_i\) denote the size of \(\Delta^k_i\), so that \(m^1_i + \ldots + m^k_i = m_i\).

**Remark 5.6.** By Remark 5.3, the partition \(\{\Delta^k(\beta) : 1 \leq k \leq t_{\beta}\}\) of \(\{1, \ldots, M\}\) and the function \(\delta_{\beta} : \{1, \ldots, t_{\beta}\} \to \{1, \ldots, t_{\beta}\}\) are determined by \(\beta\). Conversely, from the partition \(\{\Delta^k(\beta) : 1 \leq k \leq t_{\beta}\}\) of \(\{1, \ldots, M\}\) and the function \(\delta_{\beta} : \{1, \ldots, t_{\beta}\} \to \{1, \ldots, t_{\beta}\}\) we can recover \(\beta\) as the closest point to 0 of the convex hull of

\[
\{e_i - e_j : i \in \Delta^{k_1}(\beta)\text{ and } j \in \Delta^{k_2}(\beta)\} \quad \text{where } k_2 > \delta_{\beta}(k_1)\}.
\]

Note, however, that although given any partition \(\{\Delta^k : 1 \leq k \leq t\}\) of \(\{1, \ldots, M\}\) and increasing function \(\delta : \{1, \ldots, t\} \to \{1, \ldots, t\}\) satisfying \(\delta(k) \geq k\) for \(1 \leq k \leq t\), we can consider the closest point \(\beta\) to 0 of the convex hull of

\[
\{e_i - e_j : i \in \Delta^{k_1}\text{ and } j \in \Delta^{k_2}\text{ where } k_2 > \delta(k_1)\}\}
\]

it is not necessarily the case that the associated partition \(\{\Delta^k(\beta) : 1 \leq k \leq t_{\beta}\}\) of \(\{1, \ldots, M\}\) and function \(\delta_{\beta} : \{1, \ldots, t_{\beta}\} \to \{1, \ldots, t_{\beta}\}\) coincide with the given partition \(\{\Delta^k : 1 \leq k \leq t\}\) of \(\{1, \ldots, M\}\) and function \(\delta : \{1, \ldots, t\} \to \{1, \ldots, t\}\). For example, some amalgamation and rearrangement may be needed as in the proof of Proposition 9.1.

### 6. Balanced \(\delta\)-filtrations

The last section studied the indexing set \(\Gamma\) for the stratification \(\{\Sigma_{\gamma} : \gamma \in \Gamma\}\) of \(\mathcal{C}^{ss}\) defined as in §2. In this section we will consider what it means for a semistable holomorphic bundle over the Riemann surface \(\Sigma\) to belong to a stratum \(\Sigma_{\gamma} = \Sigma_{\beta, I}\), where \(\beta\) is as in Proposition 5.3.

**Definition 6.1.** We shall say that a semistable bundle \(E\) has a \(\delta\)-filtration

\[
0 = E_0 \subset E_1 \subset \ldots \subset E_t = E
\]

with associated function \(\delta : \{1, \ldots, t\} \to \{1, \ldots, t\}\) if \(\delta\) is an increasing function such that if \(1 \leq k \leq t\) then \(\delta(k) \geq k\) and the induced filtration

\[
0 \subset \frac{E_k}{E_{k-1}} \subset \frac{E_{k+1}}{E_{k-1}} \subset \ldots \subset \frac{E_{\delta(k)}}{E_{k-1}}
\]

is trivial.

Let \(G(\delta)\) be the graph with vertices \(1, \ldots, t\) and edges joining \(i\) to \(j\) if \(j - 1 = \delta(i) < \delta(i + 1)\). Then the connected components of \(G(\delta)\) are of the form

\[
\{i_{t_1(h)}, \ldots, i_{t_2(h)}\}
\]
for $1 \leq h \leq L$, where $i_j^h - 1 = \delta(i_j^{h-1}) < \delta(i_j^{h-1} + 1)$ if $1 < j \leq s_h$, and $i_j^h - 1$ is not in the image of $\delta$ and either $\delta(i_{s_h}^h) = u$ or $\delta(i_{s_h}^h) = \delta(i_{s_h}^h + 1)$. Moreover $l_1(h) \leq l_2(h)$ can be chosen so that

$$\epsilon(h) = \left( \sum_{m=l_1(h)}^{l_2(h)} m\tilde{r}_{h,m} \right) \left( \sum_{m=l_1(h)}^{l_2(h)} \tilde{r}_{h,m} \right)^{-1},$$

where $\tilde{r}_{h,m} = \text{rank}(E^h_{i_m^h} / E^{h-1}_{i_m^h})$, then $-1/2 \leq \epsilon(h) < 1/2$, and the ordering of the components of $G(\delta)$ can be chosen so that

$$\epsilon(1) \geq \epsilon(2) \geq ... \geq \epsilon(L).$$

We shall say that the $\delta$-filtration is balanced if the inequalities in (6.1) are all strict and if

(6.2) $i_{m_1}^{h_1} \leq i_{m_2}^{h_2}$ if and only if $m_1 < m_2$ or $m_1 = m_2$ and $h_1 \leq h_2$;

that is, if the usual ordering on $\{1, ..., t\}$ is the same as the Hebrew lexicographic ordering via the pairs $(h, m)$.

**Remark 6.2.** If $\beta$ is as in Proposition 6.1 then the proof of that proposition shows that

$$\left( \sum_{h=1}^{L} \frac{l_2(h) \tilde{r}_{h,m}}{l_1(h) \tilde{r}_{h,i}} \right) = \sum_{h=1}^{L} \sum_{m=l_1(h)}^{l_2(h)} (m - \epsilon(h))^2 \tilde{r}_{h,m}$$

where

$$r_{h,m} = \sum_{j \in \Delta_{h,m}} \|e_j\|^2 = \sum_{j \in \Delta_{h,m}} \sum_{i=1}^{q} \|e_j\|^2$$

$$= \sum_{j \in \Delta_{h,m}} p_i = \sum_{j \in \Delta_{h,m}} m_i^{\phi(h,m)} p_i = (1 - g + d/n) \sum_{i=1}^{q} m_i^{\phi(h,m)} n_i.$$

Thus

$$\left( \sum_{h=1}^{L} \frac{l_2(h) \tilde{r}_{h,m}}{l_1(h) \tilde{r}_{h,i}} \right) = (1 - g + d/n) \sum_{h=1}^{L} \sum_{m=l_1(h)}^{l_2(h)} (m - \epsilon(h))^2 \tilde{r}_{h,m}$$

where

$$\tilde{r}_{h,m} = \sum_{i=1}^{q} m_i^{\phi(h,m)} n_i$$

and $\epsilon(h)$ is given by

$$\left( \sum_{m=l_1(h)}^{l_2(h)} m\tilde{r}_{h,m} \right) \left( \sum_{m=l_1(h)}^{l_2(h)} \tilde{r}_{h,m} \right)^{-1} = \left( \sum_{m=l_1(h)}^{l_2(h)} m\tilde{r}_{h,m} \right) \left( \sum_{m=l_1(h)}^{l_2(h)} \tilde{r}_{h,m} \right)^{-1}.$$

Note that if $0 = E_0 \subset E_1 \subset ... \subset E_t = E$ is a filtration such that

$$E_k / E_{k-1} \cong \bigoplus_{i=1}^{q} C^{m_i^k} \otimes D_i$$
if $1 \leq k \leq t$, where $t$ and $m_i^k$ for $1 \leq i \leq q$ and $1 \leq k \leq t$ are as in Definition $5.6$ and $D_1, ..., D_q$ are nonisomorphic stable bundles of ranks $n_1, ..., n_q$ and all of the same slope $d/n$, then

$$\hat{r}_{n,m} = \text{rank}(E_{\phi(h,m)}/E_{\phi(h,m)-1}).$$

**Proposition 6.3.** Let $\beta$ be as in Proposition $5.7$ let $\delta$ be as in Remark $5.6$ and let $E$ be a semistable holomorphic structure on $\mathcal{E}$.

(i) If $E$ represents an element of the stratum $\Sigma_{\beta,l}$ then $E$ has a unique balanced $\delta$-filtration $0 = E_0 \subset E_1 \subset ... \subset E_t = E$ such that

$$E_k/E_{k-1} \cong \bigoplus_{i=1}^{q} \mathbb{C}^{m_i^k} \otimes D_i$$

and hence

$$E_{\delta(k)}/E_{k-1} \cong \bigoplus_{i=1}^{q} \mathbb{C}^{m_i^k + ... + m_i^{k(k)}} \otimes D_i$$

if $1 \leq k \leq t$, where $t$ and $m_i^k$ for $1 \leq i \leq q$ and $1 \leq k \leq t$ are as in Definition $5.6$ and $D_1, ..., D_q$ are nonisomorphic stable bundles of ranks $n_1, ..., n_q$ and all of the same slope $d/n$.

(ii) Conversely, if $E$ has a balanced $\delta$-filtration $0 = E_0 \subset E_1 \subset ... \subset E_t = E$ as in (i) then $E$ represents an element of the stratum $\Sigma_{\beta,l}$ if and only if $E$ has no filtration with the corresponding properties for any $\beta'$ satisfying $|\beta'| > |\beta|$.

**Proof:** Recall from (2.7) that

$$(6.5) \quad \Sigma_{\beta,l} = \mathcal{G}_c Y^E_{\beta,l} \cong \mathcal{G}_c \times \mathcal{Q}_{\beta,l} Y^E_{\beta,l}$$

and that if $E$ represents an element of $Y^E_{\beta,l}$ then its orbit under the complex one-parameter subgroup of $R_t$ generated by $\beta$ has a limit point in $Z_{R_t}$. This limit point is represented by the bundle $\text{gr}(E)$ which is of the form

$$\text{gr}(E) \cong \bigoplus_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i$$

where $D_1, ..., D_q$ are nonisomorphic stable bundles of ranks $n_1, ..., n_q$ and all of the same slope $d/n$. Recall also from (11 § 7) that $\mathcal{C}$ is an infinite dimensional affine space, and if we fix a $C^\infty$ identification of the fixed $C^\infty$ hermitian bundle $\mathcal{E}$ with $\bigoplus_{i=1}^{t} \mathbb{C}^{m_i} \otimes D_i$ then we can identify $\mathcal{C}$ with the infinite dimensional vector space

$$\Omega^{0,1}(\text{End}(\bigoplus_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i))$$

in such a way that the zero element of $\Omega^{0,1}(\text{End}(\bigoplus_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i))$ corresponds to the given holomorphic structure on $\bigoplus_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i$. With respect to this identification, the action of $R_t = \prod_{i=1}^{t} GL(m_i; \mathbb{C})$ on $\mathcal{C}$ is the action induced by the obvious action of $R_t$ on $\bigoplus_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i$. The one-parameter subgroup of $R_t$ generated by $\beta$ acts diagonally on $\mathbb{C}^{m_1} \oplus ... \oplus \mathbb{C}^{m_q}$ with weights $\beta.e_j$ for $j \in \{1, ..., M\}$ where $M = m_1 + ... + m_q$, and so it acts on

$$\Omega^{0,1}(\text{End}(\bigoplus_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i)) = \bigoplus_{i_1,i_2=1}^{q} \Omega^{0,1}(\mathbb{C}^{m_{i_1}} \otimes (\mathbb{C}^{m_{i_2}})^* \otimes D_{i_1} \otimes D_{i_2}^*)$$
with weights \(\beta(e_i - e_j)\) for \(i, j \in \{1, ..., M\}\). If \(E \in \Sigma_\beta\) then \(E\) lies in the \(G_c\)-orbit of an element of the sum of those weight spaces for which the weight \(\beta(e_i - e_j)\) satisfies \(\beta(e_i - e_j) \geq \|\beta\|^2\). By Remark 5.4 and Definition 5.5, we have a partition \(\{\Delta^1, ..., \Delta^t\}\) of \(\{1, ..., M\}\) such that \(\beta(e_i - e_j) \geq \|\beta\|^2\) if and only if \(i \in \Delta^{k_1}\) and \(j \in \Delta^{k_2}\) where \(k_2 > \delta(k_1)\). So if we make identifications

\[
\mathbb{C}^M = \bigoplus_{i=1}^q \mathbb{C}^{m_i} = \bigoplus_{i=1}^q \bigoplus_{k=1}^t \mathbb{C}^{m^k_i}
\]

using the induced partition \(\{\Delta^i_k : 1 \leq i \leq t, 1 \leq k \leq q\}\) of \(\{1, ..., M\}\) as in Definition 6.3, then any \(E \in \Sigma_\beta\) lies in the \(G_c\)-orbit of an element of

\[
\bigoplus_{i_1, i_2=1}^q \bigoplus_{k_1=1}^t \bigoplus_{k_2=1}^t \Omega^{0,1}(\mathbb{C}^{m_{i_1}^{k_1}} \otimes (\mathbb{C}^{m_{i_2}^{k_2}})^* \otimes D_{i_1} \otimes D_{i_2}^*)
\]

This completes the proof of (i), as such an element of \(\Omega^{0,1}(\mathbb{C}^{m_{i_1}^{k_1}} \otimes D_{i_1})\) represents a holomorphic structure \(E\) on \(\mathcal{E}\) with a filtration of the required form (uniqueness follows from 6.3, and the fact that \(Q_{\beta, \delta}\) preserves the filtration), and (ii) is now a consequence of (6.1).

**Corollary 6.4.** Let \(\beta\) be as in Proposition 5.4, let \(\delta\) be as in Remark 5.4, and let \(E\) be a semistable holomorphic structure on \(\mathcal{E}\) with a balanced \(\delta\)-filtration

\[
0 = E_0 \subset E_1 \subset ... \subset E_t = E
\]

whose subquotients satisfy

\[
E_k / E_{k-1} \cong \bigoplus_{i=1}^q \mathbb{C}^{m^k_i} \otimes D_i,
\]

where \(t\) and \(m^k_i\) for \(1 \leq i \leq q\) and \(1 \leq k \leq t\) are as in Definition 6.3, and \(D_1, ..., D_q\) are nonisomorphic stable bundles of ranks \(n_1, ..., n_q\) and all of the same slope \(d/n\). Then \(E\) represents an element of the stratum \(\Sigma_{\beta, \delta}\) if and only if, in the notation of Definition 6.4, there is no \(h \in \{1, ..., L\}\) and refinement

\[
0 = E_0 \subset ... \subset E_{i_1(h)-1} \subset F_{i_1(h)} \subset E_{i_1(h)} \subset ... \subset E_{i_m(h)-1} \subset F_m \subset E_{i_m(h)} \subset ... \subset E_t = E
\]

of (6.6) with

\[
\sum_{m=i_1(h)}^{i_2(h)} \frac{m \text{ rank}(F_m/E_{i_m(h)-1})}{\text{ rank}(F_m/E_{i_m(h)-1})} < \sum_{m=i_1(h)}^{i_2(h)} \frac{m \text{ rank}(E_{i_m(h)}/E_{i_{m-1}(h)-1})}{\text{ rank}(E_{i_m(h)}/E_{i_{m-1}(h)-1})} = \epsilon(h),
\]

such that the induced filtrations

\[
0 \subset \frac{E_{i_1(h)-1}}{E_{i_{m_1(h)-1}}} \subset \frac{F_m}{E_{i_{m_1(h)-1}}}
\]

with

\[
m_1 - \epsilon(h_1) \leq \sum_{m=i_1(h)}^{i_2(h)} \frac{m \text{ rank}(F_m/E_{i_m(h)-1})}{\text{ rank}(F_m/E_{i_m(h)-1})}
\]

and

\[
0 \subset \frac{E_{i_m(h)-1}}{F_m} \subset \frac{E_{i_{m+1}(h)-1}}{F_m}
\]
with
\[ m_2 - \epsilon(h_2) \geq m - \frac{\sum_{m=1_1(h)}^{l_2(h)} m \ \text{rank}(F_m/E_{i(h) - 1})}{\sum_{m=1_1(h)}^{l_2(h)} \ \text{rank}(F_m/E_{i(h) - 1})} \]
are all trivial.

**Remark 6.5.** If \( 0 = E_0 \subset E_1 \subset \ldots \subset E_{j-1} \subset F \subset E_j \subset \ldots \subset E_t = E \) is a refinement of the filtration (6.6) of \( E \) such that the induced filtration
\[ 0 \leq \frac{E_{j-1}}{E_i} \subset \frac{F}{E_{i(h)}} \]
is trivial for some \( i < j - 1 \), then \( F/E_i \) is isomorphic to
\[ \frac{E_{j-1}}{E_i} \oplus \frac{F}{E_{j-1}} \]
and so \( E \) can be given a filtration of the form
\[ 0 = E_0 \subset E_1 \subset \ldots \subset E_i \subset E_{i+1} \subset E_{i+2} \subset \ldots \subset E_{j-1} \subset F \subset E_j \subset \ldots \subset E_t = E \]
where \( E_{i+1}/E_i \cong F/E_{i(h)} \), \( E_{k+1}/E_k \cong E_k/E_{k-1} \) for \( i < k < j \) and \( F/E_{j-1} \cong E_{j-1}/E_{j-2} \). A similar result is true if the induced filtration
\[ 0 \subset \frac{F}{E_{j-1}} \subset \frac{E_i}{E_{j-1}} \]
is trivial for some \( i > j \).

**Proof of Corollary 6.4.** This follows from [28] and the proof of Proposition [21] which tells us that if \( \beta' \neq \beta \) is the closest point to 0 of the convex hull of \( \{ e_i - e_j : (i, j) \in S' \} \) where \( S' \) is a subset of \( S \), then \( S' \) can be chosen so that the connected components of the graph \( G(S') \) give us a refinement of the partition \( \{ \Delta_{h,m} : (h, m) \in J \} \) of \( \{ 1, \ldots, M \} \) associated to \( \beta \), which in turn gives us a refinement of the filtration (6.6) with the required properties.

**Remark 6.6.** Recall from Proposition [6.3] that a semistable bundle \( E \) represents an element of the stratum \( \Sigma_{\beta, t} \) if and only if it has a balanced \( \delta \)-filtration
\[ (6.7) \quad 0 = E_0 \subset E_1 \subset \ldots \subset E_t = E \]
such that if \( 1 \leq k \leq t \) then \( E_k/E_{k-1} \) is of the form \( \bigotimes_{i=1}^{q} \mathbb{C}^{m_i} \otimes D_i \) where \( D_1, \ldots, D_q \) are nonisomorphic stable bundles of ranks \( n_1, \ldots, n_q \) and all of the same slope \( d/n \), and moreover \( E \) has no balanced \( \delta \)-filtration with the corresponding properties when \( \beta \) is replaced with \( \beta' \) satisfying \( \| \beta' \| > \| \beta \| \). From (6.4) we have
\[ (6.8) \quad 1/\| \beta \|^2 = (1 - g + d/n) \sum_{h=1}^{L} \left( \frac{\sum_{l_1(h) \leq i < j \leq l_2(h)} (j - i)^2 \hat{r}_{h,i} \hat{r}_{h,j}}{\sum_{l_1(h) \leq i < j \leq l_2(h)} \hat{r}_{h,i}} \right), \]
where
\[ \hat{r}_{h,m} = \text{rank}(E_{\phi(h,m)}/E_{\phi(h,m)-1}). \]
This gives us some sort of measure of the triviality of the balanced \( \delta \)-filtration (6.7); very roughly speaking, the more trivial this filtration, the smaller the size of \( 1/\| \beta \|^2 \) and hence the larger \( \| \beta \| \) becomes.
Let us therefore define the \textit{triviality} of the balanced \(\delta\)-filtration 6.7 with associated function \(\delta\) to be

\[
\left( \frac{\sum_{h=1}^{L} \left( \sum_{l_{1}(h) \leq i < j \leq l_{2}(h)} (j-i)^{2} \tilde{r}_{h,i}\tilde{r}_{h,j} \right)}{\sum_{l_{1}(h) \leq i \leq l_{2}(h)} \tilde{r}_{h,i}} \right)^{-1/2}
\]

(6.9)

\[
\left( \frac{L}{\sum_{h=1}^{L} \sum_{m=l_{1}(h)}^{l_{2}(h)} (m-\epsilon(h))^{2} \tilde{r}_{h,m}} \right)^{-1/2} ;
\]

Remarks 5.2 and 5.6 tell us that this is well defined. Thus if \(E_{\delta}\) (6.9) = \(\delta\) maximal triviality (according to this measure) among the balanced \(E_{\delta}\) associated function \(\delta\).

\[\sum_{m} E_{\delta}\]

Let us therefore define the \textit{triviality} of the balanced \(\delta\)-filtration 6.7 with associated function \(\delta\) to be

\[
\left( \frac{\sum_{h=1}^{L} \left( \sum_{l_{1}(h) \leq i < j \leq l_{2}(h)} (j-i)^{2} \tilde{r}_{h,i}\tilde{r}_{h,j} \right)}{\sum_{l_{1}(h) \leq i \leq l_{2}(h)} \tilde{r}_{h,i}} \right)^{-1/2}
\]

(6.9)

\[
\left( \frac{L}{\sum_{h=1}^{L} \sum_{m=l_{1}(h)}^{l_{2}(h)} (m-\epsilon(h))^{2} \tilde{r}_{h,m}} \right)^{-1/2} ;
\]

Remarks 5.2 and 5.6 tell us that this is well defined. Thus if \(E \in \Sigma_{\beta,l}\) the balanced \(\delta\)-filtration 6.7 associated to \(E\) by Proposition 10.1 can be thought of as having maximal triviality (according to this measure) among the balanced \(\delta\)-filtrations of \(E\).

**Remark 6.7.** Let \(\beta\) be as in Proposition 5.1 let \(\delta\) be as in Remark 5.4 and let \(E\) be a semistable holomorphic structure on \(E\) with a \(\delta\)-filtration

\[0 = E_{0} \subset E_{1} \subset ... \subset E_{r} = E\]

such that if \(1 \leq k \leq t\) then \(E_{k}/E_{k-1} \cong \bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}} \otimes D_{i}\) where \(D_{1}, ..., D_{q}\) are nonisomorphic stable bundles of ranks \(n_{1}, ..., n_{q}\) and all of the same slope \(d/n\). Then the proof of Proposition 6.3 and (2.3) shows that \(E_{\delta}\) satisfies (6.10) that

\[
\sum_{ij} \bigoplus_{1 \leq i < j \leq r} \mathbb{C}^{m_{i}m_{j}} \cong \mathbb{C}^{m_{i}m_{j}}\]

given by identifying \(\bigoplus_{j=1}^{r} \mathbb{C}^{m_{i}m_{j}}\) with \(\mathbb{C}^{m_{i}}\) for \(1 \leq i \leq q\). Then we have from Remark 6.7 that

\[
S_{\beta[n,m]} \subseteq \bigcup_{\gamma \geq \beta[n,m]} \Sigma_{\gamma},
\]

and from Remark 3.3 and Proposition 6.3 that

\[
\Sigma_{\beta[n,m]} \subseteq \bigcup_{\gamma' \geq \beta[n,m]} S_{\gamma'[n',m']}
\]

where \(\geq\) denotes in (6.10) the partial order on \(\Gamma\) used in Remark 6.7 whereas in (6.11) it denotes the partial order on \(I^{ss}\) described in Remark 1.4.
Remark 6.9. It follows from Proposition 6.3 that if $R_l$ is as at 6.1 then a holomorphic structure belongs to

$\bigcup_{\beta \in B_1 \setminus \{0\}} \Sigma_{\beta, l}$

if and only if $E \not\cong \text{gr}(E) \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)$ where $D_1, \ldots, D_q$ are all stable of slope $d/n$ and ranks $n_1, \ldots, n_q$ and are not isomorphic to one another.

7. Pivotal filtrations

Let us now consider the relationship between the balanced $\delta$-filtration associated to a semistable bundle $E$ as in Proposition 6.3 and the maximal and minimal Jordan–Hölder filtrations defined in §8.

Indeed, motivated by Proposition 6.3 we can try to carry our analysis of the maximal Jordan–Hölder filtration

(7.1) $0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$

of a bundle $E$ a bit further. Recall that if $1 \leq j \leq t$ then the subquotient $E_j/E_{j-1}$ is the maximal subbundle of $E/E_{j-1}$ which is a direct sum of stable bundles all having maximal slope among the nonzero subbundles of $E/E_{j-1}$. We can ask whether it is true for every subbundle $F$ of $E$ satisfying $E_{j-1} \subset F \subset E_j$ and slope($E_j/F$) = slope($E_j/E_{j-1}$) that $E_j/F$ is the maximal subbundle of $E/F$ which is a direct sum of stable bundles all having maximal slope among the nonzero subbundles of $E/F$. Of course if $E_j/E_{j-1}$ is itself stable there are no such intermediate subbundles $F$, so this is trivially true, but it is not always the case (as Example 7.1 below shows).

If there does exist such an intermediate subbundle $F$, then by Lemma 6.2 both $F/E_{j-1}$ and $E_j/F$ are of the form

(7.2) $(\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)$

where $D_1, \ldots, D_q$ are all stable of slope $d/n$ and ranks $n_1, \ldots, n_q$ and are not isomorphic to one another. So we can then ask whether it is possible to find a canonical refinement

(7.3) $0 = F_0 \subset F_1 \subset \ldots \subset F_u = E$

of the maximal Jordan–Hölder filtration 7.1 of $E$ with an increasing function $\delta : \{1, \ldots, u\} \rightarrow \{1, \ldots, t\}$ such that $\delta(j) \geq j$ if $1 \leq j \leq u$, each subquotient $F_{\delta(j)}/F_{j-1}$ is of the form 7.2 and moreover for every subbundle $F$ of $E$ satisfying $F_{j-1} \subset F \subset F_j$ and slope($F_j/F$) = slope($F/F_{j-1}$), the quotient $F_{\delta(j)}/F$ is the maximal subbundle of $E/F$ which is a direct sum of stable bundles all having maximal slope among the nonzero subbundles of $E/F$. The following example shows that even this cannot be achieved in a canonical way.

Example 7.1. Let $E_1$ and $E_2$ be semistable bundles over $\Sigma$ such that slope($E_1$) equals slope($E_2$), with maximal Jordan–Hölder filtrations

$0 \subset D_1 \subset E_1$ and $0 \subset D_2 \subset E_2$

where $D_1, D_2, E_1/D_1$ and $E_2/D_2$ are nonisomorphic stable bundles all of the same slope as $E_1$ and $E_2$, and let $E = E_1 \oplus E_2$.

We observed in Remark 6.7 that the maximal Jordan–Hölder filtration of a direct sum of semistable bundles of the same slope is the direct sum of their maximal
Jordan–Hölder filtrations (with the shorter one extended trivially at the top if they are not of the same length). Thus the maximal Jordan–Hölder filtration of $E$ is

\[(7.4) \quad 0 \subset D_1 \oplus D_2 \subset E_1 \oplus E_2 = E.\]

By Lemma 3.2 there are precisely two proper subbundles $F$ of $D_1 \oplus D_2$ with $\text{slope}(F) = \text{slope}(D_1 \oplus D_2/F) = \text{slope}(D_1 \oplus D_2)$, namely $D_1$ and $D_2$. The maximal Jordan–Hölder filtration of $E/D_1 = (E_1/D_1) \oplus E_2$ is

\[(7.5) \quad 0 \subset (E_1/D_1) \oplus D_2 \subset (E_1/D_1) \oplus E_2,\]

so we can refine the filtration \((7.4)\) of $E$ to get

\[(7.6) \quad 0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = E\]

where $F_1 = D_1$, $F_2 = D_1 \oplus D_2$ and $F_3 = E_1 \oplus D_2$, and we can define $\delta : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ by $\delta(1) = 2$, $\delta(2) = 3$ and $\delta(3) = \delta(4) = 4$. If $1 \leq j \leq 4$ then $F_\delta(j)/F_{j-1}$ is the maximal subbundle of $E/F_{j-1}$ which is a direct sum of stable subbundles all having maximal slope among the nonzero subbundles of $E/F_{j-1}$. Moreover this is trivially still true if we replace $F_{j-1}$ by any subbundle $F$ of $E$ satisfying $F_{j-1} \subset F \subset F_j$ and $\text{slope}(F_j/F) = \text{slope}(F_j/F_{j-1})$, since the only such $F$ is $F_{j-1}$ itself. We can of course reverse the rôles of $E_1$ and $E_2$ in this construction, to get another refinement

\[(7.7) \quad 0 \subset D_2 \subset D_1 \oplus D_2 \subset D_1 \oplus E_2 \subset E_1 \oplus E_2 = E\]

of \((7.6)\). Thus there are precisely two refinements of the maximal Jordan–Hölder filtration of $E_1 \oplus E_2$ with the required properties, and if $E_1$ has the same rank as $E_2$ and $D_1$ has the same rank as $D_2$ then by symmetry there can be no canonical choice.

Notice that if $\text{rank}(D_1)/\text{rank}(E_1) = \text{rank}(D_2)/\text{rank}(E_2)$ then neither of the $\delta$-filtrations \((7.5)\) and \((7.6)\) is balanced since the inequalities \((6.1)\) are not strict; however the $\delta$-filtration \((7.4)\) is balanced and has maximal triviality, in the sense of \((6.9)\), among balanced $\delta$-filtrations of $E_1 \oplus E_2$. If on the other hand $\text{rank}(D_1)/\text{rank}(E_1) \neq \text{rank}(D_2)/\text{rank}(E_2)$ then precisely one of the $\delta$-filtrations \((7.5)\) and \((7.6)\) is balanced and it has maximal triviality, in the sense of \((6.9)\), among balanced $\delta$-filtrations of $E_1 \oplus E_2$. This filtration then determines the stratum $\Sigma_\gamma$ to which $E$ belongs, and in this case $E$ represents an element of the open subset $\Sigma_\delta^\gamma$ of $\Sigma_\gamma$.

**Lemma 7.2.** Let $E$ be a bundle over $\Sigma$ with a filtration

\[0 = F_0 \subset F_1 \subset \ldots \subset F_u = E\]

and let $\{\Delta_1, \ldots, \Delta_L\}$ be a partition of $\{1, \ldots, u\}$ such that if $1 \leq h \leq L$ and $\Delta_h = \{i_1^h, \ldots, i_u^h\}$ where $i_1^h < i_2^h < \ldots < i_u^h$, then the induced extension

\[(7.7) \quad 0 \to F_j^h/F_{j-1}^h \to F_{j-1}^h/F_{j-2}^h \to \cdots \to F_1^h/F_0^h \to 0\]

is trivial. Then we can associate to this filtration of $E$, partition $\{\Delta_1, \ldots, \Delta_L\}$ of $\{1, \ldots, u\}$ and trivialisations of the induced extensions \((7.7)\) a sequence of elements of

\[H^1(\Sigma, (F_j^h/F_{j-1}^h) \otimes (F_{j+1}^h/F_{j+2}^h)^*)\]

or equivalently of extensions

\[0 \to F_j^h/F_{j-1}^h \to E_j \to F_{j+1}^h/F_{j+2}^h \to 0\]

for $1 \leq h \leq L$ and $1 \leq j \leq s_h - 1$. 
Proof: This lemma follows immediately from the well known bijective correspondence between holomorphic extensions of a holomorphic bundle $D_1$ over $\Sigma$ by another holomorphic bundle $D_2$ and elements of $H^1(\Sigma, D_1 \otimes D_2)$. The extension

$$0 \to F_{h+1}^h / F_{h+1}^h \to F_{h+1}^h / F_{h+1}^h \to F_{h+1}^h / F_{h+1}^h \to 0$$

induced by the given filtration gives us an element of

$$H^1(\Sigma, (F_{h+1}^h / F_{h+1}^h) \otimes (F_{h+1}^h / F_{h+1}^h))^*)$$

and the given trivialisation of the extension gives us a decomposition of this as

$$H^1(\Sigma, (F_{h+1}^h / F_{h+1}^h) \otimes (F_{h+1}^h / F_{h+1}^h))^*) \oplus H^1(\Sigma, (F_{h+1}^h / F_{h+1}^h) \otimes (F_{h+1}^h / F_{h+1}^h))^*)$$

Projection onto the first summand gives us an extension

$$0 \to F_{h+1}^h / F_{h+1}^h \to E_{h+1}^h / F_{h+1}^h \to 0$$

as required.

Remark 7.3. Lemma 5.3 and Remark 5.4 tell us that if $i \in \Delta_{i_1}^k$ and $j \in \Delta_{i_2}^k$ where $k_2 > \delta(k_1)$ then $(e_i - e_j, \beta) \geq \|\beta\|^2$, and equality occurs if and only if there exists $(h, m) \in J$ such that $k_1 = \phi(h, m)$ and $k_2 = \phi(h, m+1)$. Thus, retaining the notation of the proof of Proposition 5.3 we observe that if $E$ is represented by an element of

$$\bigoplus_{i_1, i_2=1}^q \bigoplus_{k_1=1}^t \bigoplus_{k_2=1}^t \bigotimes_{i_1, i_2=1}^{\delta(k_1)+1} \Omega^{0,1}(\mathbb{C}^{m_{i_1}} \otimes (\mathbb{C}^{m_{i_2}})^* \otimes D_{i_1} \otimes D_{i_2}^*)$$

then the limit in $\mathcal{C}$ as $t \to \infty$ of $\exp(-it\beta)E$ is the bundle

$$\text{gr}(E) \cong \bigoplus_{i=1}^q \mathbb{C}^{m_i} \otimes D_i$$

which is represented by the zero vector in

$$\bigoplus_{i_1, i_2=1}^q \bigoplus_{k_1=1}^t \bigoplus_{k_2=1}^t \bigotimes_{i_1, i_2=1}^{\delta(k_1)+1} \Omega^{0,1}(\mathbb{C}^{m_{i_1}} \otimes (\mathbb{C}^{m_{i_2}})^* \otimes D_{i_1} \otimes D_{i_2}^*),$$

and so the limit $p_\beta(E)$ of $\exp(-it\beta)E$ in the blow-up of $\mathcal{C}$ along $G, Z_{R_i}^n$ is an element of the fibre

$$\mathbb{P}(\mathcal{N}_{i,\text{gr}E}) = \mathbb{P}(H^1(\Sigma, \bigoplus_{i_1, i_2=1}^q \mathbb{C}^{m_{i_1} - \delta_{i_1}^2} \otimes D_{i_1} \otimes D_{i_2}^*))$$

of the exceptional divisor over $\text{gr}E$. Indeed $p_\beta(E)$ is the element of this fibre represented by the sum in

$$\bigotimes_{h=1}^L \bigotimes_{m=\tau_1(h)}^{\tau_2(h)-1} H^1(\Sigma, \bigoplus_{i_1}^q \mathbb{C}^{m_{i_1}} \otimes D_{i_1}) \otimes (\bigoplus_{i_2}^q \mathbb{C}^{m_{i_2}} \otimes D_{i_2})^*)$$

of the elements of

$$H^1(\Sigma, \bigoplus_{i_1}^q \mathbb{C}^{m_{i_1}} \otimes D_{i_1}) \otimes (\bigoplus_{i_2}^q \mathbb{C}^{m_{i_2}} \otimes D_{i_2})^*)$$
for $1 \leq h \leq L$ and $l_1(h) \leq m \leq l_2(h)$ corresponding to the extensions

$$0 \to \bigoplus_{i \leq 1} \mathbb{C}^{m^{2(h,m)}} \otimes D_i \to E^h_m \to \bigoplus_{i \leq 2} \mathbb{C}^{m^{2(h,m+1)}} \otimes D_i \to 0$$

associated as in Lemma 162 to the $\delta$-filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$ of Proposition 6.3.

**Proposition 7.4.** Let $\beta$ be as in Proposition 6.3 and let $E$ be a semistable bundle representing an element of $\Sigma_{\beta,1}$ with a balanced $\delta$-filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$$

such that if $1 \leq k \leq t$ then $E_k/E_{k-1} \cong \bigoplus_{i=1}^q \mathbb{C}^{m^{2_i}} \otimes D_i$ as in Proposition 6.3. Suppose also that $k_1 \in \{1, \ldots, t\}$ is such that

$$\beta.e_j < 0 \text{ whenever } j \in \Delta^{k_2} \text{ with } k_2 > \delta(k_1).$$

Then whenever $F$ is a subbundle of $E$ with slope($F$) = slope($E$) and such that $E_{k_1-1} \subseteq F \subseteq E_{k_1}$, the subquotient $E_{\delta(k_1)}/F$ is the maximal subbundle of $E/F$ which is a direct sum of stable bundles all having the same slope as $E/F$.

**Proof:** Since $F/E_{k_1-1}$ is a subbundle of $E_{k_1}/E_{k_1-1} \cong \bigoplus_{i=1}^q \mathbb{C}^{m^{2_i}} \otimes D_i$ having the same slope as $D_1, \ldots, D_q$, it follows from Lemma 8.2 that

$$F/E_{k_1-1} = \bigoplus_{i=1}^q U_i \otimes D_i$$

where $U_i$ is a linear subspace of $\mathbb{C}^{m^{2_i}}$ for $1 \leq i \leq q$, and so

$$E_{\delta(k_1)}/F \cong \bigoplus_{i=1}^q ((\mathbb{C}^{m^{2_i}}/U_i) \oplus \mathbb{C}^{m^{2_i+1} + \ldots + m^{2_{(k_1)}}}) \otimes D_i$$

is a sum of stable bundles all having the same slope (which is equal to slope($E$) and slope($E/F$)). Let us suppose for a contradiction that $E'/F$ has a subbundle $E'/E$ which is not contained in $E_{\delta(k_1)}/F$ and which is of the required form. Then we can choose $k_2 > \delta(k_1)$ such that $E' \subseteq E_{k_2}$ but $E'$ is not contained in $E_{k_2-1}$, and then the inclusion of $E'$ in $E_{k_2}$ induces a nonzero map

$$\theta : E'/F \to E_{k_2}/E_{k_2-1} \cong \bigoplus_{i=1}^q \mathbb{C}^{m^{2_i}} \otimes D_i.$$

Since nonzero bundle maps between stable bundles of the same slope are always isomorphisms, by replacing $E'$ by a suitable subbundle we can assume that $E'/F \cong D_{i_0}$ for some $i_0 \in \{1, \ldots, q\}$, and that we can decompose $\mathbb{C}^{m^{2_{i_0}}} \otimes \mathbb{C}^{m^{2_{i_0}-1}}$ in such a way that the projection $\theta_0 : E'/F \to D_{i_0}$ of $\theta$ onto the corresponding component $D_{i_0}$ of $E_{k_2}/E_{k_2-1}$ is an isomorphism. Then

$$\theta_0^{-1} : D_{i_0} \to E'/F \subseteq E_{k_2}/F$$

gives us a trivialisation of the extension of $E_{k_2-1}/E_{k_1}$ by this component $D_{i_0}$ of $E_{k_2}/E_{k_2-1}$. By the definition of $\Sigma_{\beta,t}$ the limit $p_{\beta}(E) \in \mathbb{P}(N,E)$ of $\exp(-it\beta)E$ as $t \to \infty$ is semistable for the induced action of $\text{Stab}(\beta)/T_{\beta}^\delta$ where $\text{Stab}(\beta)$ is the stabiliser of $\beta$ under the coadjoint action of $R_t$ and $T_{\beta}^\delta$ is the complex subtorus.
generated by $\beta$ (see [28]), and by Remark 7.2, $p_\beta(E)$ is represented by the sum over $h \in \{1, \ldots, L\}$ and $m \in \{l_1(h), \ldots, l_2(h)\}$ of the elements of

$$H^1(\Sigma, (\bigoplus_{i_1}^{q} \mathbb{C}^{m_{i_1}^{(h,m)}} \otimes D_{i_1}) \otimes (\bigoplus_{i_2}^{q} \mathbb{C}^{m_{i_2}^{(h,m+1)}} \otimes D_{i_2}))$$

corresponding to the extensions

$$0 \to \bigoplus_{i_1}^{q} \mathbb{C}^{m_{i_1}^{(h,m)}} \otimes D_{i_1} \to E^0_m \to \bigoplus_{i_2}^{q} \mathbb{C}^{m_{i_2}^{(h,m+1)}} \otimes D_{i_2} \to 0$$

associated by Lemma 7.2 to the filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$.

Let $S_0$ be the set of ordered pairs $(i, j)$ with $i, j \in \{1, \ldots, M\}$ such that the component of $p_\beta(E)$ in the weight space corresponding to the weight $e_i - e_j$ for the action of the maximal torus $T_i$ of $R_t = \prod_{i=1}^{t} GL(m_i; \mathbb{C})$ on

$$\bigoplus_{h=1}^{L} \bigoplus_{m=1(h)}^{i_2(h)-1} H^1(\Sigma, (\bigoplus_{i_1}^{q} \mathbb{C}^{m_{i_1}^{(h,m)}} \otimes D_{i_1}) \otimes (\bigoplus_{i_2}^{q} \mathbb{C}^{m_{i_2}^{(h,m+1)}} \otimes D_{i_2}))$$

is nonzero. Since $p_\beta(E)$ is semistable for the action of $\text{Stab}(\beta)/T^*_J$, it follows that $\beta$ is the closest point to 0 in the convex hull of $\{e_i - e_j : (i, j) \in S_0\}$. We may assume that $T_i$ acts diagonally with respect to the decomposition of $\mathbb{C}^{m_{a_2}^{k_2}}$ as $\mathbb{C} \oplus \mathbb{C}^{m_{a_0}^{k_2}}$; let $e_{j_0}$ be the weight of the action of $T_i$ on the component $\mathbb{C}$ of $\mathbb{C}^{m_{a_0}^{k_2}}$ with respect to this decomposition. Since $k_2 > \delta(k_1)$ and the extension of $E_{k_2-1}/E_{k_1}$ by the component of $E_{k_2}/E_{k_2-1} \cong \bigoplus_{i=1}^{q} \mathbb{C}^{m_{a_2}^{k_2}} \otimes D_{i}$ corresponding to the weight $e_{j_0}$ is trivial, it follows that if $(i, j) \in S_0$ then $j \neq j_0$. Since $\beta$ lies in the convex hull of $\{e_i - e_j : (i, j) \in S_0\}$ and $e_1, \ldots, e_M$ are mutually orthogonal, this means that

$$\beta.e_{j_0} \geq 0,$$

and as $j_0 \in \Delta^{k_2}$ and $k_2 > \delta(k_1)$, this gives us the required contradiction.

 Remark 7.5. Dual to the definition of $\delta$ in Remark 5.4, we can define an increasing function $\delta' : \{1, \ldots, t\} \to \{1, \ldots, t\}$ such that $\delta'(\phi(h, m)) - 1$ is the number of elements $(h', m') \in J$ such that either $m' < m - 1$ or $m' = m - 1$ and $h' \leq h$. Then $\delta'(k) \leq k$ for all $k \in \{1, \ldots, t\}$, and if $(h, m)$ and $(h, m - 1)$ both belong to $J$ then $\delta'(\phi(h, m)) = \phi(h, m - 1) + 1$. Also $k_1 < \delta'(k_2)$ if and only if $k_2 > \delta(k_1)$, and Lemma 5.3 tells us that if $i \in \Delta^{k_1}$ and $j \in \Delta^{k_2}$ then $\beta.(e_i - e_j) \geq |\beta|^2$ if and only if $k_1 < \delta'(k_2)$. The dual version of Proposition 6.3 tells us that if $1 \leq k \leq t$ then $E_{k}/E_{k'(k)-1}$ is a direct sum of stable bundles all of the same slope, and using Remark 7.2 we obtain the following dual version of Proposition 7.4.

 Proposition 7.6. Let $\beta$ be as in Proposition 7.2 and let $E$ be a semistable bundle representing an element of $\Sigma_{\beta,1}$ with $\delta$-filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$$

such that if $1 \leq k \leq t$ then $E_{k}/E_{k-1} \cong \bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}^{k}} \otimes D_{i}$ as in Proposition 6.3. Suppose also that $k_1 \in \{1, \ldots, t\}$ is such that

$$\beta.e_{j} > 0 \text{ whenever } j \in \Delta^{k_2} \text{ with } k_2 < \delta'(k_1).$$

Then whenever $F$ is a subbundle of $E$ with slope$(F) = \text{slope}(E)$ and such that $E_{k_1-1} \subset F \subset E_{k_1}$, the subbundle $E_{\delta'(k_1)-1}$ is the minimal subbundle of $F$ such that $F/E_{\delta'(k_1)-1}$ is a direct sum of stable bundles all with the same slope as $F$.\]
REMARK 7.7. It follows from the definition of $\Delta^1, \ldots, \Delta^t$ (Definition 5.10) that if $j_1 \in \Delta^{k_1}$ and $j_2 \in \Delta^{k_2}$ then $\beta. e_{j_1} < \beta. e_{j_2}$ if and only if $k_1 > k_2$, so we can choose $k_-$ and $k_+$ such that $\beta. e_j < 0$ (respectively $\beta. e_j > 0$) if and only if $j \in \Delta^k$ with $k > k_-$ (respectively $k < k_+$). Then $k_- = k_+$ or $k_- = k_+ - 1$, depending on whether there exists $j$ with $\beta. e_j = 0$. Propositions 7.4 and 7.6 tell us that if $E$ is a semistable bundle representing an element of $\Sigma_\beta$ with filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$$

as in Proposition 7.6 then

$$0 \subset E_{\delta(j)+1}/E_{j-1} \subset E_{\delta(j)+1}/E_{j-1} \subset \ldots \subset E/E_j$$

is the maximal Jordan–Hölder filtration of $E/E_j$ if $j = \delta(k_-)$, and

$$0 \subset \ldots \subset E_{\delta(j)+1}/E_{j-1} \subset E_{\delta(j)-1} \subset E_j$$

is the minimal Jordan–Hölder filtration of $E_j$ if $j = \delta(k_+)$. Note also that $\delta'(k_-) \leq k_- \leq \delta(k_+)$, so there are values of $j$ satisfying both $j \geq \delta'(k_-)$ and $j \leq \delta(k_+)$. There is a converse to Propositions 7.4 and 7.6.

**PROPOSITION 7.8.** Let $\beta$ be as in Proposition 5.1 and let $E$ be a semistable bundle with $\delta$-filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_t = E$$

such that if $1 \leq k \leq t$ then $E_k/E_{k-1} \cong \bigoplus_{i=1}^t \mathbb{C}^{m_i} \otimes D_i$ as in Proposition 7.6. Suppose that every subbundle $F$ of $E$ with the same slope as $E$ satisfies the following two properties:

(i) if $E_{k_1-1} \subset F \subset E_{k_1}$ for some $k_1 \in \{1, \ldots, t\}$ such that $\beta. e_j < 0$ whenever $j \in \Delta^{k_2}$ with $k_2 > \delta(k_1)$, then the subquotient $E_{\delta(k_1)}/E$ is the maximal subbundle of $E/F$ which is a direct sum of stable bundles all with the same slope as $E/F$;

(ii) if $E_{k_1-1} \subset F \subset E_{k_1}$ for some $k_1 \in \{1, \ldots, t\}$ such that $\beta. e_j > 0$ whenever $j \in \Delta^{k_2}$ with $k_2 < \delta(k_1)$, then the subbundle $E_{\delta(k_1)-1}$ is the minimal subbundle of $F$ such that $F/E_{\delta(k_1)-1}$ is a direct sum of stable bundles all with the same slope as $F$.

Then $E$ represents an element of the stratum $\Sigma_{\beta, t}$.

**Proof:** Suppose for a contradiction that $E$ does not represent an element of $\Sigma_{\beta, t}$. Then (cf. 28) after applying a change of coordinates to $\mathbb{C}^{m_i}$ for $1 \leq i \leq q$ and $1 \leq k \leq t$, we can assume that $\beta$ is not equal to the closest point to 0 in the convex hull of $\{e_i - e_j : (i, j) \in S_0\}$ where $S_0$ is as in the proof of Proposition 8.4. Moreover

$$S_0 \subseteq \{(i, j) : \beta_0(e_i - e_j) \geq \|\beta\|^2\}.$$ 

Thus $\beta$ does not lie in the convex hull of

$$\{e_i - e_j : (i, j) \in S_0 \cap S\}$$

where $S = \{(i, j) : \beta_0(e_i - e_j) = \|\beta\|^2\}$. From Lemma 5.3 and Remark 5.4 we know that if $i \in \Delta_{k, m}$ and $j \in \Delta_{k', m}$ then $\beta_0(e_i - e_j) \geq \|\beta\|^2$ if and only if $m' \geq m + 2$ or $m' = m + 1$ and $h' \geq h$, and this happens if and only if $\phi(h', m') > \phi(h, m)$, while $\beta_0(e_i - e_j) = \|\beta\|^2$ if and only if $m' = m + 1$ and $h' = h$. By Remark 5.4 the hypothesis (i) on subbundles $F$ of $E$ tells us that if $E_{k_1-1} \subset F \subset E_{k_2}$ where $k_1 \geq \delta(k_-)$, then the subquotient $E_{\delta(k_1)}/E$ is the maximal subbundle of $E/F$ which is a direct sum of stable bundles all with the same slope as $E/F$. This implies that
if \( (h, m) \) and \( (h, m + 1) \) both lie in \( J \) and \( \phi(h, m) \geq k_\ast \) then every pair \( (i, j) \) with \( i \in \Delta_{h,m} \) and \( j \in \Delta_{h,m+1} \) lies in \( S_0 \). Similarly the hypothesis (ii) tells us that if \( (h, m) \) and \( (h, m - 1) \) both lie in \( J \) and \( \phi(h, m) \leq k_+ \) then every pair \( (i, j) \) with \( i \in \Delta_{h,m} \) and \( j \in \Delta_{h,m-1} \) lies in \( S_0 \). Since \( k_\ast \leq k_+ \) this means that \( S_0 \cap S = S \).

This contradicts the fact that \( \beta \) is the closest point to 0 in the convex hull of \( \{ e_i - e_j : (i, j) \in S \} \) but does not lie in the convex hull of \( \{ e_i - e_j : (i, j) \in S_0 \cap S \} \), and thus completes the proof.

**Remark 7.9.** Suppose that \( \beta \) corresponds to a partition \( \{ \Delta_{h,m} : (h, m) \in J \} \) of \( \{1, \ldots, M\} \), indexed by

\[
J = \{(h, m) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq h \leq L, l_1(h) \leq m \leq l_2(h)\}
\]

where \( l_1 \) and \( l_2 : \{1, \ldots, L\} \to \mathbb{Z} \) satisfy \( l_1(h) \leq l_2(h) \) for all \( h \in \{1, \ldots, L\} \), as in Proposition 7.8. Let \( E \) be a semistable bundle representing an element of \( \Sigma_\beta \) with \( \delta \)-filtration

\[
0 = E_0 \subset E_1 \subset \ldots \subset E_i = E
\]

as in Proposition 7.8. If

\[
l_1(h_1) - \epsilon(h_1) < l_1(h_2) - \epsilon(h_2) + 1
\]

for all \( h_1, h_2 \in \{1, \ldots, L\} \), or equivalently if

\[
\phi(h, l_1(h)) \leq \delta(1) + 1
\]

for all \( h \in \{1, \ldots, L\} \), then the proof of Proposition 7.8 shows that

\[
(7.8) \quad 0 \subset E_{\delta(1)+1} \subset E_{\delta(\delta(1)+1)+1} \subset \ldots \subset E
\]

is the maximal Jordan–Hölder filtration of \( E \). Thus for such a \( \beta \) the stratum \( \Sigma_\beta \) is contained in the subset \( S^{\max/JH}_{n_\beta, m_\beta} \) of \( \mathcal{C}^{ss} \) defined at Definition 4.1 where \( n_\beta \) and \( m_\beta \) are determined by the filtration (7.8). If, on the other hand, there exists \( h_0 \in \{1, \ldots, L\} \) with

\[
\phi(h_0, l_1(h_0)) > \delta(1) + 1,
\]

then \( E_{\delta(1)+1} \) may not be the maximal subbundle of \( E \) which is a direct sum of stable bundles all with the same slope as \( E \); there may be a subbundle of \( E_{\phi(h_0, l_1(h_0))}/E_{\phi(h_0, l_1(h_0))}^{-1} \) which provides a trivial extension of \( E_{\delta(1)+1} \) by a direct sum of stable bundles all with the same slope as \( E \) (see Example 8.1 below). However, even in this case a careful analysis of the proof of Proposition 4.3 reveals that it can be modified to show that the intersection \( \Sigma_\beta \cap S^{\max/JH}_{n, m} \) is a locally closed complex submanifold of \( \mathcal{C}^{ss} \) of finite codimension for each \( \beta \in \Gamma \) and \( [n, m] \in \mathcal{I}^{ss} \).

**Definition 7.10.** We shall call a filtration \( 0 \subset P_1 \subset \ldots \subset P_\tau \subset E \) of a semistable bundle \( E \) a pivot if each subbundle \( P_j \) has the same slope as \( E \) and \( P_1 \) is the minimal subbundle of \( P_\tau \) such that \( P_\tau/P_1 \) is a direct sum of stable bundles of the same slope, while \( P_\tau/P_1 \) is the maximal subbundle of \( E/P_1 \) which is a direct sum of stable bundles of the same slope. Any pivot determines a filtration

\[
0 \subset \ldots \subset P_{\tau-2} = P_{\tau-2}^{-1} \subset \ldots \subset P_{\tau-1}^{-1} \subset \ldots \subset P_{\tau-1} = P_\tau = P_\tau^{-1} = P_1 \subset \ldots
\]

\[
\ldots \subset P_\tau = P_\tau^0 = P_\tau^0 \subset P_{\tau+1}^{-1} \subset \ldots \subset P_{\tau+1} = P_{\tau+1}^+ \subset \ldots \subset E
\]

of \( E \) where

\[
0 \subset P_j^+ = P_j^+ \subset P_j^+ \subset \ldots \subset E/P_j
\]
is the maximal Jordan–Hölder filtration of $E/P_j$, and

$$0 \subset \cdots \subset P_j^{-2} \subset P_j^{-1} \subset P_j$$

is the minimal Jordan–Hölder filtration of $P_j$. A filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_t = E$ of this form (once repetitions have been omitted) for some pivot $0 \subset P_1 \subset \cdots \subset P_\tau \subset E$ will be called a pivotal filtration. It will be called a strongly pivotal filtration if every subbundle $F$ of $E$ such that $P_{m j} \subset F \subset P_{m+1 j}$ for some $m \geq 0$ has

$$0 \subset \cdots \subset P_{m-1 j} \subset P_m j \subset F$$

as its minimal Jordan–Hölder filtration.

Note that a pivotal filtration is a $\delta$-filtration where $\delta(k_1)$ is the number of $k_2 \in \{1, ..., t\}$ for which it is not the case that $E_{k_1} = P_{j_1}^{m_1}$ and $E_{k_2} = P_{j_2}^{m_2}$ with $m_1 \geq m_2$ or $m_1 = m_2 - 1$ and $j_1 \geq j_2$; if the associated $\delta$-filtration is balanced then we will call the pivotal filtration balanced.

**Theorem 7.11.** Let $\beta$ be as in Proposition 5.1 and let $E$ be a semistable bundle representing an element of $\Sigma_{\beta, t}$ with $\delta$-filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_t = E$$

such that if $1 \leq k \leq t$ then $E_k/E_{k-1} \cong \bigoplus_{i=1}^q \mathbb{C}^{m_i k} \otimes D_i$ as in Proposition 6.3. Then (7.9) is a balanced strongly pivotal $\delta$-filtration with pivot

$$0 \subset E_{k_1-1} \subset E_{k_1} \subset \cdots \subset E_{\delta(k_1)} \subset E$$

for any $k_1$ satisfying $\delta'(k_-) \leq k_1 \leq k_+$.  

**Proof:** This is an immediate consequence of Propositions 7.4 and 7.6 as in Remark 7.7.

8. Refinements of the Yang-Mills stratification

We thus have three refinements of the Harder-Narasimhan filtration of a holomorphic bundle $E$ over $\Sigma$: the maximal Jordan–Hölder filtration, the minimal Jordan–Hölder filtration and the balanced $\delta$-filtration of maximal triviality obtained by applying Proposition 6.3 to the subquotients of the Harder-Narasimhan filtration. Associated to these we have refinements of the Yang-Mills stratification of $\mathcal{C}$, each of which is a stratification of $\mathcal{C}$ by locally closed complex submanifolds of finite codimension, and has the set $\mathcal{C}^s$ of stable holomorphic structures on $\mathcal{E}$ as its open stratum. The first of these refined stratifications is the stratification

$$\{ S_{\text{max}JH}^{[d,n,m]} : [d,n,m] \in \mathcal{I} \}$$

of $\mathcal{C}$ defined in Definition 8.1, and another is the stratification

$$\{ S_{\text{min}JH}^{[d,n,m]} : [d,n,m] \in \mathcal{I} \}$$
defined dually using minimal Jordan–Hölder filtrations as in Remark \[6.7\] and Definition \[4.8\]. A third refinement is the stratification obtained by applying the stratification \(\{\Sigma_{\gamma} : \gamma \in \Gamma\}\) of \(\mathcal{C}^{ss}\), whose indexing set was determined in \(\S 5\) and whose strata were described in terms of balanced \(\delta\)-filtrations in \(\S 6\), to the \(\mathcal{C}(n', d')^{ss}\) which appear inductively in the description of the Yang–Mills stratification.

**Example 8.1.** Recall from Remark \[6.7\] that the maximal Jordan–Hölder filtration of the direct sum \(E \oplus F\) of two semistable bundles of the same slope is the direct sum of the maximal Jordan–Hölder filtrations of \(E\) and \(F\) with the shorter one extended trivially at the top, while the minimal Jordan–Hölder filtration of \(E \oplus F\) is the direct sum of their minimal Jordan–Hölder filtrations with the shorter one extended trivially at the bottom. Suppose now that \(E\) and \(F\) have balanced \(\delta\)-filtrations of maximal triviality given by

\[
0 \subset E_{l_1(1)} \subset E_{l_1(1)+1} \subset \ldots \subset E_{l_1(1)+l_1(1)} = E
\]

and

\[
0 \subset F_{l_2(2)} \subset F_{l_2(2)+1} \subset \ldots \subset F_{l_2(2)+l_2(2)} = F
\]

where the indices \(l_1(1), ..., l_2(2) \in \mathbb{Z}\) and \(l_1(2), ..., l_2(2) \in \mathbb{Z}\) have been chosen so that

\[
\epsilon(1) = \sum_{m=l_1(1)}^{l_2(1)} m \ \text{rank}(E_m/E_{m-1})
\]

and

\[
\epsilon(2) = \sum_{m=l_1(2)}^{l_2(2)} m \ \text{rank}(F_m/F_{m-1})
\]

lie in the interval \([-1/2, 1/2]\). To simplify the notation let us assume that \(l_1(1) \leq l_1(2) \leq l_2(2) \leq l_2(1)\). If \(\epsilon(1) > \epsilon(2)\) then \(E \oplus F\) has a balanced \(\delta\)-filtration given by

\[
0 \subset E_{l_1(1)} \oplus 0 \subset \ldots \subset E_{l_1(2)} \oplus 0 \subset E_{l_1(2)} \oplus F_{l_2(2)} \subset \ldots \\
\ldots \subset E_{l_1(2)} \oplus F_{l_2(2)} \subset E_{l_1(2)+1} \oplus F_{l_2(2)} \subset \ldots \subset E_{l_2(1)} + F_{l_2(2)} = E \oplus F.
\]

If we assume that \(E_i/E_{i-1}\) and \(F_j/F_{j-1}\) are stable for \(l_1(1) \leq i \leq l_2(1)\) and \(l_1(2) \leq j \leq l_2(2)\), then this filtration has no proper refinements with subquotients of the same slope as \(E \oplus F\), so by Corollary \[6.4\] it is a balanced \(\delta\)-filtration of \(E \oplus F\) with maximal triviality (and in fact it is not hard to check that this is still true without the simplifying assumption). If \(\epsilon(2) > \epsilon(1)\) then we replace the filtration above with the balanced \(\delta\)-filtration

\[
0 \subset E_{l_1(1)} \oplus 0 \subset \ldots \subset E_{l_1(2)-1} \oplus 0 \subset E_{l_1(2)-1} \oplus F_{l_2(2)} \subset \ldots \\
\ldots \subset E_{l_1(2)-1} \oplus F_{l_2(2)} \subset E_{l_1(2)+1} \oplus F_{l_2(2)} \subset \ldots \subset E_{l_2(1)} \oplus F_{l_2(2)} = E \oplus F.
\]

Thus we see that the the maximal Jordan–Hölder filtration, the minimal Jordan–Hölder filtration and the balanced \(\delta\)-filtration of maximal triviality of a bundle \(E\) can all be different from one another, and that none of them is necessarily a refinement of the other two. Nonetheless, the concepts of maximal Jordan–Hölder filtration, minimal Jordan–Hölder filtration and balanced \(\delta\)-filtration of maximal triviality on a bundle \(E\) are related by Theorem \[7.4\] via the notion of a pivotal filtration (see also Remark \[6.8\] Propositions \[7.3\] \[7.6\] and \[7.8\] and Remark \[7.9\]).
References

[1] M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces Philos. Trans. Roy. Soc. London Ser. A 308 (1982), 523-615.
[2] V. Baranovsky, Cohomology ring of the moduli space of stable vector bundles with odd determinant, Izv. Russ. Acad. Nauk. 58 n4 (1994), 204-210.
[3] A. Beauville, Sur la cohomologie de certains espaces de modules de fibrés vectoriels in Geometry and Physics, (Bombay 1992), Tata Inst. Fund. Res. (1995), 37-40.
[4] S. Cappell, R. Lee and E. Miller, The action of the Torelli group on the homology of representation spaces is non-trivial, Topology 39 (2000), 851-871.
[5] G. Daskalopoulos, The topology of the space of stable bundles on a Riemann surface, J. Differential Geometry 36 (1992), 699-746.
[6] S. del Baño, On the Chow motive of some moduli spaces, J. Reine Angew. Math. 532 (2001), 105-132.
[7] U.V. Desale and S. Ramanan, Poincaré polynomials of the variety of stable bundles, Math. Ann. 216 (1975), 233-244.
[8] S.K. Donaldson, Gomu techniques in the cohomology of moduli spaces, in Topological methods in modern mathematics (Proceedings of 1991 Stony Brook conference in honour of the sixtieth birthday of J.Milnor), Publish or Perish (1993), 137-170.
[9] R. Earl, The Mumford relations and the moduli of rank three stable bundles, Compositio Math. 109 (1997), 13-48.
[10] R. Earl and F. Kirwan, The Pontryagin rings of moduli spaces of arbitrary rank holomorphic bundles over a Riemann surface, J. London Math. Soc. (2) 60 (1999), 835-846.
[11] , Complete sets of relations in the cohomology rings of moduli spaces of arbitrary rank holomorphic bundles over a Riemann surface, 2003 preprint.
[12] G. Ellingsrud and S. Stromme, On the Chow ring of a geometric quotient, Ann. Math. 130 (1989), 130-159.
[13] W. Fulton, Intersection Theory (Second Edition) Springer (1998).
[14] A. Gieseker, A Degeneration of the Moduli Spaces of Stable Bundles, J. Differential Geom. 19 (1984), 173-206.
[15] , Geometric Invariant Theory and applications to moduli problems, Proc. Int. Cong. Math. (Helsinki, 1978) Academia Scientarium Fennica (Helsinki, 1980).
[16] G. Harder and M.S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles over curves, Math. Ann. 212 (1975) 215-248.
[17] R. Herrera and S. Salamon, Intersection numbers on moduli spaces and symmetries of a Verlinde formula, Comm. Math. Phys. 188 (1997), 521-534.
[18] L. Jeffrey, Y-H. Kiem, F. Kirwan and J. Woolf, Cohomology pairings on singular quotients in geometric invariant theory, to appear in Transformation Groups.
[19] , Intersection pairings on singular moduli spaces of bundles over a Riemann surface, in preparation.
[20] L. Jeffrey and F. Kirwan, Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface, Ann. of Math. 148 (1998), 109-196.
[21] A. King and P.E. Newstead, On the cohomology of the moduli space of rank 2 vector bundles on a curve, Topology 37 (1998), 407-418.
[22] F. Kirwan, Cohomology of quotients in symplectic and algebraic geometry, Math. Notes vol. 31 Princeton Univ. Press, Princeton, NJ 1985.
[23] , Partial desingularisations of quotients of non-singular varieties and their Betti numbers, Ann. Math. 122 (1985), 41-85.
[24] , On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles Ark. Math. 24 (1986), 221-275.
[25] , Rational intersection cohomology of quotient varieties, Invent. Math. 86 (1986), 471-505.
[26] , On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface, Proc. London Math. Soc. 53 (1986), 237-266.
[27] , Cohomology rings of moduli spaces of bundles over Riemann surfaces, J. Amer. Math. Soc. 5 (1992), 853-906.
[28] , Refinements of the Morse stratification of the norm-square of the moment map, 2003 preprint.
[29] J. Le Potier, *Lectures on Vector Bundles*, Cambridge Studies in Advanced Mathematics 54 CUP (1997).

[30] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, Third Edition, Ser. Modern Surveys Math. 34, Springer (1994).

[31] M.S. Narasimhan and S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. Math. 101 (1975), 391-417.

[32] M.S. Narasimhan and C.S. Seshadri, *Stable and unitary vector bundles over an algebraic curve*, Ann. Math. 82 (1965), 540-567.

[33] P.E. Newstead, *Characteristic classes of stable bundles of rank 2 over an algebraic curve*, Trans. Amer. Math. Soc. 169 (1972), 337-345.

[34] C.S. Seshadri, *Introduction to moduli problems and orbit spaces*, Tata Inst. Lect. 51 (1978).

[35] M.S. Narasimhan and C.S. Seshadri, *Stable and unitary vector bundles over an algebraic curve*, Ann. Math. 82 (1965), 540-567.

[36] P.E. Newstead, *Introduction to the topology of the moduli space of stable bundles on a Riemann surface*, in *Geometry and Physics*, Lecture Notes in Pure and Applied Mathematics 184 (1997), 71-100.

[37] S.S. Shatz, *The decomposition and specialization of algebraic families of vector bundles*, Compositio Math. 35 (1977), 163-187.

[38] M. Thaddeus, *Conformal field theory and the cohomology of the moduli space of stable bundles*, J. Differential Geom. 35 (1992), 131-149.

[39] B. Siebert and G. Tian, *Recursive relations for the cohomology ring of moduli spaces of stable bundles*, Tr. J. of Math. 19 (1995), 131-144.

[40] E. Witten, *Two dimensional gauge theories revisited*, J. Geom. Phys. 9 (1992), 303-368.

[41] D. Zagier, *On the cohomology of moduli spaces of rank 2 vector bundles over curves*, Progress in Mathematics 129 The Moduli space of Curves (1995), 533-563.

[42] E. Witten, *Elementary aspects of the Verlinde formula and of the Harder–Narasimhan–Atiyah–Bott formula*, in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, Israel Math. Conf. Proc. 9 (1996), 445-462.