SYMMETRIZED $p$-CONVEXITY AND RELATED SOME INTEGRAL INEQUALITIES

İMDAT İŞCAN

Abstract. In this paper, the author introduces the concept of the symmetrized $p$-convex function, gives Hermite-Hadamard type inequalities for symmetrized $p$-convex functions.

1. Introduction

Let real function $f$ be defined on some nonempty interval $I$ of real line $\mathbb{R}$. The function $f$ is said to be convex on $I$ if inequality

\begin{equation}
\label{eq:1.1}
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\end{equation}

holds for all $x, y \in I$ and $t \in [0,1]$.

In [3], the author gave definition Harmonically convex and concave functions as follow.

Definition 1. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

\begin{equation}
\label{eq:1.2}
f \left( \frac{xy}{tx+(1-t)y} \right) \leq tf(y) + (1-t)f(x)
\end{equation}

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds for harmonically convex functions.

Theorem 1 ([3]). Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold

\begin{equation}
\label{eq:1.3}
f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.
\end{equation}

The above inequalities are sharp.

In [4], the author gave the definition of $p$-convex function as follow:

Definition 2 ([4]). Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said to be a $p$-convex function, if

\begin{equation}
\label{eq:1.4}
f \left( \left[ tx^p + (1-t)y^p \right]^{1/p} \right) \leq tf(x) + (1-t)f(y)
\end{equation}

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for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.4) is reversed, then $f$ is said to be $p$-concave.

According to Definition 2, it can be easily seen that for $p = 1$ and $p = -1$, $p$-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subseteq (0, \infty)$, respectively.

Since the condition (1.4) can be written as
\[(f \circ g) (tx^p + (1 - t) y^p) \leq t (f \circ g) (x^p) + (1 - t) (f \circ g) (y^p), \quad g(x) = x^{1/p},\]
then we observe that $f : I \subseteq (0, \infty) \to \mathbb{R}$ is $p$-convex on $I$ if and only if $f \circ g$ is convex on $g^{-1}(I) := \{ z^p, z \in I \}$.

**Example 1.** Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^p, p \neq 0$, and $g : (0, \infty) \to \mathbb{R}$, $g(x) = c, \quad c \in \mathbb{R}$, then $f$ and $g$ are both $p$-convex and $p$-concave functions.

In [5], Kunt and İshan gave Hermite-Hadamard-Fejér type inequalities for $p$-convex functions as follow:

**Theorem 2.** Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a $p$-convex function, $p \in \mathbb{R} \setminus \{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then we have
\[
(1.5) \quad f \left( \frac{[ap + bp]}{2} \right)^{1/p} \leq \frac{p}{bp - ap} \int_a^b f(x) \frac{dx}{x^{1-p}} \leq \frac{f(a) + f(b)}{2}.
\]

**Definition 3.** Let $p \in \mathbb{R} \setminus \{0\}$. A function $w : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is said to be $p$-symmetric with respect to $\frac{[ap + bp]}{2}^{1/p}$ if
\[
w(x) = w \left( \frac{[ap + bp - x^p]}{x^{1/p}} \right)
\]
holds for all $x \in [a, b]$.

In [5], Kunt and İshan gave Hermite-Hadamard-Fejér type inequalities for $p$-convex functions as follow:

**Theorem 3.** Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a $p$-convex function, $p \in \mathbb{R} \setminus \{0\}$, $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable and $p$-symmetric with respect to $\frac{[ap + bp]}{2}^{1/p}$, then the following inequalities hold:
\[
f \left( \frac{[ap + bp]}{2} \right)^{1/p} \int_a^b \frac{w(x)}{x^{1-p}} \frac{dx}{x^{1-p}} \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{w(x)}{x^{1-p}} \frac{dx}{x^{1-p}}.
\]

**Definition 4.** Let $f \in L[a, b]$. The left-sided and right-sided Hadamard fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by
\[
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, \quad a < x < b
\]
and
functions in fractional integral forms as follows:

\[ J_{a}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t - x)^{\alpha - 1} f(t) dt, \quad a < x < b \]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt \) (see [7]).

In [6], the authors presented Hermite–Hadamard-Fejer inequalities for p-convex functions in fractional integral forms as follows:

**Theorem 4.** Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a p-convex function, \( p \in \mathbb{R} \setminus \{0\} \), \( \alpha > 0 \) and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) and \( w : [a, b] \rightarrow \mathbb{R} \) is nonnegative, integrable and p-symmetric with respect to \( \frac{a p + b p}{2} \), then the following inequalities for fractional integrals hold:

i.) If \( p > 0 \)

\[ f \left( \frac{a p + b p}{2} \right)^{1/p} \left[ J_{a\alpha}^{\alpha} (w \circ g) (b p) + J_{b\alpha}^{\alpha} (w \circ g) (a p) \right] \]

\[ \leq \left[ J_{a\alpha}^{\alpha} (f w \circ g) (b p) + J_{b\alpha}^{\alpha} (f w \circ g) (a p) \right] \]

\[ \leq \frac{f(a) + f(b)}{2} \left[ J_{a\alpha}^{\alpha} (w \circ g) (a p) + J_{b\alpha}^{\alpha} (w \circ g) (b p) \right]. \]

with \( g(x) = x^{1/p}, x \in [a p, b p] \).

ii.) If \( p > 0 \)

\[ f \left( \frac{a p + b p}{2} \right)^{1/p} \left[ J_{b\alpha}^{\alpha} (w \circ g) (a p) + J_{a\alpha}^{\alpha} (w \circ g) (b p) \right] \]

\[ \leq \left[ J_{b\alpha}^{\alpha} (f w \circ g) (a p) + J_{a\alpha}^{\alpha} (f w \circ g) (b p) \right] \]

\[ \leq \frac{f(a) + f(b)}{2} \left[ J_{b\alpha}^{\alpha} (w \circ g) (a p) + J_{a\alpha}^{\alpha} (w \circ g) (b p) \right]. \]

with \( g(x) = x^{1/p}, x \in [b p, a p] \).

For a function \( f : [a, b] \rightarrow \mathbb{R} \) we consider the symmetrical transform of \( f \) on the interval \( [a, b] \), denoted by \( \tilde{f}_{[a, b]} \) or simply \( \tilde{f} \), when the interval \( [a, b] \) is implicit, which is defined by

\[ \tilde{f}(x) := \frac{1}{2} \left[ f(x) + f(a + b - x) \right], \quad x \in [a, b]. \]

The anti symmetrical transform of \( f \) on the interval \( [a, b] \) is denoted by \( \tilde{f}_{[a, b]} \) or simply \( \tilde{f} \) as defined by

\[ \tilde{f}(x) := \frac{1}{2} \left[ f(x) - f(a + b - x) \right], \quad x \in [a, b]. \]

It is obvious that for any function \( f \) we have \( \tilde{f} + \tilde{f} = f \).

If \( f \) is convex on \( [a, b] \), then \( \tilde{f} \) is also convex on \( [a, b] \). But, when \( \tilde{f} \) is convex on \( [a, b] \), \( f \) may not be convex on \( [a, b] \) (H).

In [1], Dragomir introduced symmetrized convexity concept as follow:
Definition 5. A function \( f : [a, b] \to \mathbb{R} \) is said to be symmetrized convex (concave) on \([a, b]\) if symmetrical transform \( \overline{f} \) is convex (concave) on \([a, b]\).

Dragomir extends the Hermite-Hadamard inequality to the class of symmetrized convex functions as follow:

Theorem 5 \([\text{I}]\). Assume that \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\), then we have the Hermite-Hadamard inequalities

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

Theorem 6 \([\text{II}]\). Assume that \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\). Then for any \( x \in [a, b] \) we have the bounds

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{2} \left[ f(x) + f(a + b - x) \right] \leq \frac{f(a) + f(b)}{2}.
\]

Corollary 1. If \( f : [a, b] \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\) and \( w : [a, b] \to [0, \infty) \) is integrable on \([a, b]\), then

\[
f \left( \frac{a+b}{2} \right) \int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b w(x)dx.
\]

Theorem 7 \([\text{III}]\). Assume that \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized convex on the interval \([a, b]\). Then for any \( x, y \in [a, b] \) with \( x \neq y \) we have the Hermite-Hadamard inequalities

\[
\frac{1}{2} \left[ f \left( \frac{x+y}{2} \right) + f \left( a + b - \frac{x+y}{2} \right) \right]
\leq \frac{1}{2(y-x)} \left[ \int_x^y f(t)dt + \int_{a+b-y}^{a+b-x} f(t)dt \right]
\leq \frac{1}{4} \left[ f(x) + f(y) + f(a + b - x) + f(a + b - y) \right].
\]

For a function \( f : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{C} \), we consider the symmetrical transform of \( f \) on the interval \( I \), denoted by \( Hf_{[a,b]} \) or simply \( Hf \), when the interval \([a, b]\) as defined by

\[
Hf(x) := \frac{1}{2} \left[ f(x) + f \left( \frac{abx}{(a+b)x-ab} \right) \right], \quad x \in [a, b].
\]

Definition 6 \([\text{IV}]\). A function \( f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is said to be symmetrized harmonic convex (concave) on \([a, b]\) if \( Hf \) is harmonic convex (concave) on \( I \).

The similars of above results given for the class of symmetrized convex functions, in \([\text{V}]\) it has been obtained by Wu et al. for the class of symmetrized harmonic convex functions as follow:

Theorem 8 \([\text{VI}]\). Assume that \( f : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is symmetrized harmonic convex and integrable on the interval \([a, b]\). Then we have the Hermite-Hadamard type \( \mathcal{I} \) can inequalities

\[
f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b f(x)\frac{x}{x^2}dx \leq \frac{f(a) + f(b)}{2}.
\]
Theorem 9 (Symmetrized p-Convexity). Assume that \( f : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is symmetrized harmonic convex on the interval \([a, b]\). Then for any \( x \in [a, b] \) we have the bounds

\[
(1.12) \quad f \left( \frac{2ab}{a + b} \right) \leq \bar{H} f(x) = \frac{1}{2} \left[ f(x) + f \left( \frac{abx}{(a + b)x - ab} \right) \right] \leq \frac{f(a) + f(b)}{2}
\]

Theorem 10 (Symmetrized p-Convexity). Assume that \( f : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is symmetrized harmonic convex on the interval \([a, b]\). Then for any \( x, y \in [a, b] \) with \( x \neq y \) we have the Hermite-Hadamard inequalities

\[
(1.13) \quad \frac{1}{2} \left[ f \left( \frac{2xy}{x + y} \right) + f \left( \frac{2abxy}{2xy(a + b) - ab(x + y)} \right) \right] \leq \frac{xy}{2(y - x)} \left[ \int_x^y f(t) \frac{dt}{t^2} + \int_y^x f(t) \frac{dt}{t^2} \right] \leq \frac{1}{4} \left[ f(x) + f(y) + f \left( \frac{abx}{(a + b)x - ab} \right) + f \left( \frac{aby}{(a + b)y - ab} \right) \right].
\]

Motivated by the above results, in this paper we introduces the concept of the symmetrized p-convex function and establish some Hermite-Hadamard type inequalities. Some examples of interest are provided as well.

2. Symmetrized p-Convexity

For a function \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) we consider the \( p \)-symmetrical transform of \( f \) on the interval \([a, b]\), denoted by \( P_{(f,p),[a,b]} \) or simply \( P_{(f,p)} \), when the interval \([a, b]\) is implicit, which is defined by

\[
P_{(f,p)}(x) := \frac{1}{2} \left[ f(x) + f \left( \left[ a^p + b^p - x^p \right]^{1/p} \right) \right], \quad x \in [a, b].
\]

The anti-\( p \)-symmetrical transform of \( f \) on the interval \([a, b]\) is denoted by \( AP_{(f,p),[a,b]} \) or simply \( AP_{(f,p)} \) as defined by

\[
AP_{(f,p)}(x) := \frac{1}{2} \left[ f(x) - f \left( \left[ a^p + b^p - x^p \right]^{1/p} \right) \right], \quad x \in [a, b].
\]

It is obvious that for any function \( f \) we have \( P_{(f,p)} + AP_{(f,p)} = f \). Also it is seen that \( P_{(f,1)}(x) = \frac{1}{2} \left[ f(x) + f(a + b - x) \right] = \bar{H} f(x) \) and \( P_{(f,-1)}(x) = \frac{1}{2} \left[ f(x) + f \left( \frac{abx}{(a+b)x-ab} \right) \right] = \bar{H} f(x) \).

If \( f \) is \( p \)-convex on \([a, b]\), then \( P_{(f,p)} \) is also \( p \)-convex on \([a, b]\). Indeed, for any \( x, y \in [a, b] \) and \( t \in [0, 1] \) we have

\[
P_{(f,p)}(tx^p + (1-t)y^p)^{1/p}
= \frac{1}{2} \left[ f(tx^p + (1-t)y^p)^{1/p} + f \left( (a^p + b^p - tx^p - (1-t)y^p)^{1/p} \right) \right]
= \frac{1}{2} \left[ f(tx^p + (1-t)y^p)^{1/p} + f \left( t(a^p + b^p - x^p) + (1-t) \left( a^p + b^p - y^p \right) \right)^{1/p} \right]
\leq \frac{1}{2} \left[ f(x) + f \left( (a^p + b^p - x^p)^{1/p} \right) \right] + (1-t) \frac{1}{2} \left[ f(y) + f \left( (a^p + b^p - y^p)^{1/p} \right) \right]
= tP_{(f,p)}(x) + (1-t)P_{(f,p)}(y).
\]
Remark 1. If $P_{(f,p)}$ is $p$-convex on $[a,b]$ for a function $f : [a,b] \subseteq (0,\infty) \rightarrow \mathbb{R}$, then the function $f$ is nor necessary $p$-convex on $[a,b]$. For example, let $p = -1$. Consider the function $f(x) = -\ln x, x \in (0,\infty)$. The function $f$ is not $-1$-convex (or harmonically convex), but $P_{(f,-1)}$ is $-1$-convex [8].

Definition 7. A function $f : [a,b] \subseteq (0,\infty) \rightarrow \mathbb{R}$ is said to be symmetrized $p$-convex (p-concave) on $[a,b]$ if $p$-symmetrical transform $P_{(f,p)}$ is $p$-convex (p-concave) on $[a,b]$.

Example 2. Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $\alpha \geq 2$. Then the function $f : [a,b] \rightarrow \mathbb{R}$, $f(x) = (x^p - a^p)^{\alpha-1}$, $p > 0$, (or $f(x) = (a^p - x^p)^{\alpha-1}$, $p < 0$) is $p$-convex on $[a,b]$. Indeed, for any $u,v,\alpha \in [a,b]$ and $t \in [0,1]$ by convexity of the function $g(s) = s^{\alpha-1}, \alpha \geq 0$, we have
\[
\begin{align*}
f((tu^p + (1-t)v^p)^{1/p}) &= (tu^p + (1-t)v^p - a^p)^{\alpha-1} \\
&= (t[u^p - a^p] + (1-t)[v^p - a^p])^{\alpha-1} \\
&\leq t(u^p - a^p)^{\alpha-1} + (1-t)(v^p - a^p)^{\alpha-1} \\
&= tf(u) + (1-t)f(v).
\end{align*}
\]
Thus $P_{(f,p)}$ is also $p$-convex on $[a,b]$. Therefore $f$ is symmetrized $p$-convex function.

Example 3. Let $\alpha \geq 2$. Then the function $f : [a,b] \subseteq (0,\infty) \rightarrow \mathbb{R}$, $f(x) = (b^p - x^p)^{\alpha-1}$, $p > 0$, (or $f(x) = (x^p - b^p)^{\alpha-1}$, $p < 0$) is $p$-convex on $[a,b]$. Therefore $f$ is symmetrized $p$-convex function.

Example 4. Let $\alpha \geq 2$. Then the function $f : [a,b] \subseteq (0,\infty) \rightarrow \mathbb{R}$, $f(x) = (x^p - a^p)^{\alpha-1} + (b^p - x^p)^{\alpha-1}$, $p > 0$, (or $f(x) = (a^p - x^p)^{\alpha-1} + (x^p - b^p)^{\alpha-1}$, $p < 0$) is symmetrized $p$-convex function.

Now if $P'C[a,b]$ is the class of $p$-convex functions defined on I and $SPC[a,b]$ is the class of symmetrized $p$-convex functions on $[a,b]$ then

$$P'C[a,b] \subseteq SPC[a,b].$$

Also, if $[c,d] \subseteq [a,b]$ and $f \in SPC[a,b]$, then this does not imply in general that $f \in SPC[c,d]$.

Proposition 1. Let $f : [a,b] \subseteq (0,\infty) \rightarrow \mathbb{R}$ be a function and $g(x) = x^{1/p}, x > 0, p \neq 0$. $f$ is symmetrized $p$-convex on the interval $[a,b]$ if and only if $f \circ g$ is symmetrized convex on the interval $g^{-1}([a,b]) = [a^p, b^p]$ (or $[b^p, a^p]$).

Proof. Let $f$ be a symmetrized $p$-convex function on the interval $[a,b]$. If we take arbitrary $x,y \in g^{-1}([a,b])$, then there exist $u,v \in [a,b]$ such that $x = u^p$ and $y = g(v) = v^p$.

\begin{align*}
(f \circ g)(tx + (1-t)y) &= \frac{1}{2} [f \circ g](tx + (1-t)y) + (f \circ g)(a^p + b^p - tx - (1-t)y) \\
&= \frac{1}{2} [(f \circ g)(tu^p + (1-t)v^p) + (f \circ g)(a^p + b^p - [tu^p + (1-t)v^p])] \\
&= P_{(f,p)}((tu^p + (1-t)v^p)^{1/p}).
\end{align*}

Since $f$ is a symmetrized $p$-convex function on the interval $[a,b]$, we have

\begin{align*}
P_{(f,p)}((tu^p + (1-t)v^p)^{1/p}) &\leq tP_{(f,p)}(u) + (1-t)P_{(f,p)}(v)
\end{align*}
have

\[ \text{Theorem 11. If } f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \text{ is symmetrized } p\text{-convex on the interval } [a, b], \text{ then we have the Hermite-Hadamard inequalities} \]

\begin{equation}
  f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{p}{b^p - a^p} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\end{equation}

Proof. Since \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p\)-convex on the interval \([a, b]\), then by writing the Hermite-Hadamard inequality for the function \( P_{(f,p)}(x) \) we have

\begin{equation}
  P_{(f,p)} \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{p}{b^p - a^p} \int_a^b P_{(f,p)}(x) \, dx \leq \frac{P_{(f,p)}(a) + P_{(f,p)}(b)}{2},
\end{equation}

where, it is easily seen that

\[ P_{(f,p)} \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) = f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right), \quad \frac{P_{(f,p)}(a) + P_{(f,p)}(b)}{2} = \frac{f(a) + f(b)}{2}, \]

and

\[ \frac{p}{b^p - a^p} \int_a^b P_{(f,p)}(x) \, dx = \frac{p}{b^p - a^p} \int_a^b f(x) \, dx \]

Then by (2.4) we get required inequalities. "}

**Remark 2.** In Theorem 11

i.) if we choose \( p = 1 \), then the inequalities (2.5) reduces to the inequalities (1.8) in Theorem 5.

ii.) if we choose \( p = -1 \), then the inequalities (2.5) reduces to the inequalities (1.11) in Theorem 5.

**Remark 3.** By helping Theorem 5 and Proposition 1 the proof of Theorem 11 can also be given as follows:

Since \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p\)-convex on the interval \([a, b]\), \( f \circ g \) is symmetrized convex on the interval \([a^p, b^p]\) (or \([b^p, a^p]\)) with \( g(x) = x^{1/p}, x > 0 \), \( p \neq 0 \). So, by Theorem 5 we have

\[ (f \circ g) \left( \frac{a^p + b^p}{2} \right) \leq \frac{1}{b^p - a^p} \int_a^{b^p} (f \circ g)(x) \, dx \leq \frac{(f \circ g)(a^p) + (f \circ g)(b^p)}{2}, \]

i.e.

\[ f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{p}{b^p - a^p} \int_a^{b^p} f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \]
\textbf{Theorem 12.} If \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p \)-convex on the interval \([a, b]\). Then for any \( x \in [a, b] \) we have the bounds

\begin{equation}
(2.5) \\
f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq P_{(f,p)}(x) \leq \frac{f(a) + f(b)}{2}.
\end{equation}

\textbf{Proof.} Since \( P_{(f,p)} \) is \( p \)-convex on \([a, b]\) then for any \( x \in [a, b] \) we have

\[ f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) = P_{(f,p)} \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{P_{(f,p)}(x) + P_{(f,p)}([a^p + b^p - x^p]^{1/p})}{2} = P_{(f,p)}(x). \]

This gives us the first inequality in (2.5).

Also, for any \( x \in [a, b] \) there exist a number \( t_0 \in [0, 1] \) such that \( x = [t_0a^p + (1 - t_0)b^p]^{1/p} \).

By the \( p \)-convexity of \( P_{(f,p)} \) we have

\[ P_{(f,p)}(x) \leq t_0 P_{(f,p)}(a) + (1 - t_0) P_{(f,p)}(b) = \frac{P_{(f,p)}(a) + P_{(f,p)}(b)}{2} \]

which gives the second inequality in (2.5). \( \square \)

\textbf{Remark 4.} In Theorem 12

i.) if we choose \( p = 1 \), then the inequalities (2.5) reduces to the inequalities (1.9) in Theorem 6.

ii.) if we choose \( p = -1 \), then the inequalities (2.5) reduces to the inequalities (1.12) in Theorem 4.

\textbf{Remark 5.} By helping Theorem 6 and Proposition 4 the proof of Theorem 12 can also be given as follows:

Since \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p \)-convex on the interval \([a, b]\), \( f \circ g \) is symmetrized convex on the interval \([a^p, b^p]\) with \( g(x) = x^{1/p}, x > 0, p \neq 0 \). So, by Theorem 6 we have

\[ (f \circ g) \left( \frac{a^p + b^p}{2} \right) \leq (f \circ g)(x^p) \leq \frac{(f \circ g)(a^p) + (f \circ g)(b^p)}{2}, \]

i.e.

\[ f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq P_{(f,p)}(x) \leq \frac{f(a) + f(b)}{2} \]

for any \( x \in [a, b] \).

\textbf{Remark 6.} If \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p \)-convex on the interval \([a, b]\), then we have the bounds

\[ \inf_{x \in [a, b]} P_{(f,p)}(x) = f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \]

and

\[ \sup_{x \in [a, b]} P_{(f,p)}(x) = \frac{f(a) + f(b)}{2}. \]
Corollary 2. If \( f: [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p \)-convex on the interval \([a, b]\) and \( w: [a, b] \to [0, \infty) \) is integrable on \([a, b]\), then
\[
(2.6) \quad f \left( \frac{[a^p + b^p]^{1/p}}{2} \right) \int_a^b \frac{w(x)}{x^{1-p}} dx \leq \int_a^b \frac{w(x) P_{f,p}(x)}{x^{1-p}} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{w(x)}{x^{1-p}} dx.
\]
Moreover, if \( w \) is \( p \)-symmetric with respect to \( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \) on \([a, b]\), i.e. \( w(x) = w([a^p + b^p - x^p]) \) for all \( x \in [a, b] \), then
\[
(2.7) \quad f \left( \frac{[a^p + b^p]^{1/p}}{2} \right) \int_a^b \frac{w(x)}{x^{1-p}} dx \leq \int_a^b \frac{w(x)f(x)}{x^{1-p}} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{w(x)}{x^{1-p}} dx.
\]

Proof: The inequality (2.6) follows by (2.5) multiplying by \( w(x)/x^{1-p} \geq 0 \) and integrating over \( x \) on \([a, b]\).

By changing the variable, we have
\[
\int_a^b \frac{w(x)f \left( \frac{[a^p + b^p - x^p]^{1/p}}{2} \right)}{x^{1-p}} dx = \int_a^b \frac{w([a^p + b^p - x^p])f(x)}{x^{1-p}} dx.
\]
Since \( w \) is \( p \)-symmetric with respect to \( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \), then
\[
\int_a^b \frac{w([a^p + b^p - x^p])f(x)}{x^{1-p}} dx = \int_a^b \frac{w(x)f(x)}{x^{1-p}} dx.
\]
Thus
\[
\int_a^b \frac{w(x) P_{f,p}(x)}{x^{1-p}} dx = \frac{1}{2} \left[ \int_a^b \frac{w(x)f(x)}{x^{1-p}} dx + \int_a^b \frac{w(x)f \left( \frac{[a^p + b^p - x^p]^{1/p}}{2} \right)}{x^{1-p}} dx \right]
\]
and by (2.6) we get (2.7). \( \square \)

Remark 7. The inequality (2.7) is known as weighted generalization of Hermite-Hadamard inequality for \( p \)-convex functions (it is also given in Theorem 3). It has been shown now that this inequality remains valid for the larger class of symmetrized \( p \)-convex functions \( f \) on the interval \([a, b]\).

Remark 8. We note that by helping Corollary 1 and Proposition 1 the proof of Corollary 2 can also be given. The details are omitted.

Remark 9. Let \( a, b, \alpha \in \mathbb{R} \) with \( 0 < a < b \) and \( \alpha \geq 2 \). Then the function \( f: [a, b] \to \mathbb{R} \), \( f(x) = (x^p - a^p)^{\alpha-1} \), \( p > 0 \), is symmetrized \( p \)-convex on \([a, b]\) i.) If we consider the function
\[ f(x) = (x^p - a^p)^{\alpha-1} \]
which is symmetrized \( p \)-convex on \([a, b]\) in the inequality (2.6), then we have
\[
\frac{1}{2\alpha-1} \int_a^b \frac{w(x)}{x^{1-p}} dx \leq \frac{\Gamma(\alpha)}{2p(b^p - a^p)^{\alpha-1}} \left[ J_{a^p}^{b^p} (w \circ g)(b^p) + J_{b^p}^{a^p} (w \circ g)(a^p) \right] \leq \frac{1}{2} \int_a^b \frac{w(x)}{x^{1-p}} dx
\]
for any \( w: [a, b] \to [0, \infty) \) is integrable on \([a, b]\) with \( g(x) = x^{1/p}, x \in [a^p, b^p] \).
ii.) If we consider the function

\[ w(x) = (x^p - a^p)^{α-1} + (b^p - x^p)^{α-1} \]

which is \( p \)-symmetric with respect to \( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \) in the inequality (2.7), then we have the following inequalities

\[
f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{Γ(α + 1)}{2(b^p - a^p)^α} \left[ J^α_{a^p+} (f \circ g) (b^p) + J^α_{b^p-} (f \circ g) (a^p) \right] \leq \frac{f(a) + f(b)}{2},
\]

where \( g(x) = x^{1/p}, x \in [a^p, b^p] \).

iii.) Let \( ϕ \) be \( p \)-symmetric with respect to \( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \). If we consider the function

\[ w(x) = \left( (x^p - a^p)^{α-1} + (b^p - x^p)^{α-1} \right) \varphi(x) \]

which is \( p \)-symmetric with respect to \( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \) in the inequality (2.7), then we have the following inequalities

\[
f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \left[ J^α_{a^p+} (ϕ \circ g) (b^p) + J^α_{b^p-} (ϕ \circ g) (a^p) \right] \leq \frac{f(a) + f(b)}{2} \left[ J^α_{a^p+} (ϕ \circ g) (b^p) + J^α_{b^p-} (ϕ \circ g) (a^p) \right]
\]

which are the same of inequalities in (1.7). Where \( g(x) = x^{1/p}, x \in [a^p, b^p] \).

**Theorem 13.** Assume that \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is symmetrized \( p \)-convex on the interval \([a, b]\) with \( p \in \mathbb{R} \setminus \{0\} \). Then for any \( x, y \in [a, b] \) with \( x ≠ y \) we have the Hermite-Hadamard inequalities

\[
\frac{1}{2} \left[ f \left( \left[ \frac{x^p + y^p}{2} \right]^{1/p} \right) + f \left( \left[ a^p + b^p - \frac{x^p + y^p}{2} \right]^{1/p} \right) \right] \leq \frac{p}{2(y^p - x^p)} \left[ \int_x^y \frac{f(t)}{t^{1-p}} dt + \int_{[a^p + b^p - x^p]^{1/p}} \frac{f(t)}{t^{1-p}} dt \right] \leq \frac{1}{4} \left[ f(x) + f(y) + f \left( [a^p + b^p - x^p]^{1/p} \right) + f \left( [a^p + b^p - y^p]^{1/p} \right) \right].
\]

**Proof.** Since \( P_{(f,p),[a,b]} \) is \( p \)-convex on \([a, b]\), then \( P_{(f,p),[a,b]} \) is also \( p \)-convex on any subinterval \([x, y]\) (or \([y, x]\)) where \( x, y \in [a, b] \).

By Hermite-Hadamard inequalities for convex functions we have

\[
P_{(f,p),[a,b]} \left( \left[ \frac{x^p + y^p}{2} \right]^{1/p} \right) \leq \frac{p}{y^p - x^p} \int_x^y P_{(f,p),[a,b]} (t) \frac{f(t)}{t^{1-p}} dt \leq \frac{P_{(f,p),[a,b]}(x) + P_{(f,p),[a,b]}(y)}{2}
\]

for any \( x, y \in [a, b] \) with \( x ≠ y \).

By definition of \( P_{(f,p)} \), we have

\[
P_{(f,p),[a,b]} \left( \left[ \frac{x^p + y^p}{2} \right]^{1/p} \right) = \frac{1}{2} \left[ f \left( \left[ \frac{x^p + y^p}{2} \right]^{1/p} \right) + f \left( \left[ a^p + b^p - \frac{x^p + y^p}{2} \right]^{1/p} \right) \right].
\]
\[
\int_x^y \frac{P_{f,p;a,b}(t)}{t^{1-p}} \, dt = \frac{1}{2} \int_x^y \frac{1}{t^{1-p}} \left[ f(t) + f \left( \left( \frac{a^p + b^p}{2} \right) \right) \right] \, dt \\
= \frac{1}{2} \int_x^y f(t) \, dt + \frac{1}{2} \int_x^y f \left( \left( \frac{a^p + b^p}{2} \right) \right) \, dt \\
= \frac{1}{2} \int_x^y f(t) \, dt + \frac{1}{2} \int_{[a^p + b^p - x^p]^{1/p}} f(t) \, dt \\
\]

and

\[
P_{f,p;a,b}(x) + P_{f,p;a,b}(y) \\
= \frac{1}{4} \left[ f(x) + f(y) + f \left( \left( \frac{a^p + b^p - x^p}{2} \right) \right) + f \left( \left( \frac{a^p + b^p - y^p}{2} \right) \right) \right].
\]

Thus by (2.9) we obtain the desired result (2.8). □

**Remark 10.** We note that by helping Theorem 7 and Proposition 1, the proof of Theorem 13 can also be given. The details is omitted.

**Remark 11.** If we take \(x = a\) and \(y = b\) in (2.8), then we get (2.3). If, for a given \(x \in [a, b]\), we take \(y = \left( \frac{a^p + b^p - x^p}{2} \right)^{1/p}\), then from (2.9) we get

\[
f \left( \left( \frac{a^p + b^p}{2} \right) \right) \leq \frac{p}{a^p + b^p - 2x^p} \int_x^b \frac{f(t) \, dt}{t^{1-p}} \leq \frac{1}{2} \left[ f(x) + f \left( \left( \frac{a^p + b^p - x^p}{2} \right) \right) \right],
\]

where \(x \neq \left( \frac{a^p + b^p}{2} \right)^{1/p}\), provided that \(f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}\) is symmetrized \(p\)-convex on the interval \([a, b]\).

Multiplying the inequalities (2.10) by \(\frac{1}{x^{1-p}}\), then integrating the resulting inequality over \(x\) we get the following refinement of the first part of (2.9)

\[
f \left( \left( \frac{a^p + b^p}{2} \right) \right) \\
\leq \frac{p^2}{(b^p - a^p)} \int_a^b \frac{1}{x^{1-p} (a^p + b^p - 2x^p)} \int_x^b \frac{f(t) \, dt}{t^{1-p}} \, dx \\
\leq \frac{p}{b^p - a^p} \int_a^b \frac{f(x) \, dx}{x^{1-p}}
\]

provided that \(f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}\) is symmetrized \(p\)-convex on the interval \([a, b]\).

When the function is \(p\)-convex, we have the following inequalities as well:
Remark 12. If \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) is \( p \)-convex, then from (2.8) we have the inequalities

\[
(2.11) \quad f \left( \frac{a^p + b^p}{2} \right)^{1/p} \leq \frac{1}{2} \left[ f \left( \frac{x^p + y^p}{2} \right)^{1/p} \right] + f \left( \frac{a^p + b^p - x^p + y^p}{2} \right)^{1/p} \leq \frac{p}{2(y^p - x^p)} \int_x^y f(t) \frac{dt}{t^{1-p}} + \int_{[a^p + b^p - x^p]^1/p} \frac{f(t)}{t^{1-p}} dt \leq \frac{1}{4} \left[ f(x) + f(y) + f \left( [a^p + b^p - x^p]^{1/p} \right) + f \left( [a^p + b^p - y^p]^{1/p} \right) \right],
\]

for any \( x, y \in [a, b] \) with \( x \neq y \).

If we multiply the inequalities (2.11) by \( \frac{1}{(xy)^{1/p}} \) and integrate (2.11) over \((x, y)\) on the square \([a, b]^2\) and divide by \( \frac{p^2}{(b^p - a^p)^2} \), then we get the following refinement of the first Hermite-Hadamard inequality for \( p \)-convex functions

\[
f \left( \frac{a^p + b^p}{2} \right)^{1/p} \leq \frac{p^2}{2(b^p - a^p)^2} \int_a^b \int_a^b f \left( \frac{x^p + y^p}{2} \right)^{1/p} \frac{dx}{(xy)^{1-p}} + \int_a^b \int_a^b f \left( \frac{a^p + b^p - x^p + y^p}{2} \right)^{1/p} \frac{dx}{(xy)^{1-p}} \leq \frac{p^2}{2(b^p - a^p)^2} \int_a^b \int_a^b \frac{1}{(xy)^{1-p}} (y^p - x^p) \left[ \int_x^y \frac{f(t)}{t^{1-p}} dt + \int_{[a^p + b^p - x^p]^{1/p}} \frac{f(t)}{t^{1-p}} dt \right] dx dy \leq \frac{p}{(b^p - a^p)} \int_a^b \frac{f(x)}{x^{1-p}} dx.
\]

Remark 13. In Theorem 1.13

i.) if we choose \( p = 1 \), then the inequalities (2.8) reduces to the inequalities (1.10) in Theorem 1.10.

ii.) if we choose \( p = -1 \), then the inequalities (2.8) reduces to the inequalities (1.13) in Theorem 1.10.

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Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28200, Giresun, Turkey

E-mail address: imdat.iscan@giresun.edu.tr, imdati@yahoo.com