Körpinar, Talat
New characterization of b-m2 developable surfaces
Acta Scientiarum. Technology, vol. 37, núm. 2, abril-junio, 2015, pp. 245-250
Universidade Estadual de Maringá
Maringá, Brasil

Available in: http://www.redalyc.org/articulo.oa?id=303238471007
New characterization of b-m₂ developable surfaces

Talat Körpınar

Department of Mathematics, Muş Alparslan University, 49250, Muş, Turkey. Author for correspondence. E-mail: talatkorpinar@gmail.com

ABSTRACT. In this paper, b-m₂ developable surfaces of biharmonic b-slant helices in the special three-dimensional φ − Ricci symmetric para-Sasakian manifold P is studied. Explicit parametric equations of b-m₂ developable surfaces of biharmonic b-slant helices in the special three-dimensional φ − Ricci symmetric para-Sasakian manifold P are characterized.

Keywords: bienergy, biharmonic curve, para-Sasakian manifold, Bishop frame, developable surface.

Novas caracterizações das superfícies desenvolvimentáveis de b – m₂

RESUMO. Analisam-se as superfícies desenvolvimentáveis b – m₂ de hélices bi-harmônicas com inclinação b em φ − tri-dimencional especial de múltiplo P para-Sasakiano simétrico. São caracterizados as equações paramétricas das superfícies desenvolvimentáveis b – m₂ de hélices bi-harmônicas com inclinação b em φ − tri-dimencional especial de múltiplo P para-Sasakiano simétrico.

Palavras-chave: bi-energia; curva bi-harmônica; múltiplo para-Sasakiano; superfície desenvolvimentável; esquema de Bishop

Introduction

In differential geometry (CADDÉO; MONTALDO, 2001), (DIMITRIC, 1992), (LOUBEAU; ONICIUC, 2007), (O’NEILL, 1983) that under the assumption of sufficient differentiability, a developable surface is either a plane, conical surface, cylindrical surface or tangent surface of a curve or a composition of these types. Thus a developable surface is a ruled surface, where all points of the same generator line share a common tangent plane. The rulings are principal curvature lines with vanishing normal curvature and the Gaussian curvature vanishes at all surface points. Therefore developable surfaces are also called single-curved surfaces, as opposed to double-curved surfaces (CARMO, 1976).

On the other hand, a smooth map φ: N → M is said to be biharmonic if it is a critical point of the bienergy functional:

\[ E_2(\phi) = \int_N \frac{1}{2} T(\phi)^2 d\nu, \]

where:

\[ T(\phi) := \text{tr} \nabla^2 d\phi \]

is the tension field of \( \phi \).

The Euler–Lagrange equation of the bienergy is given by \( T_2(\phi) = 0 \). Here the section \( T_2(\phi) \) is defined by \( T_2(\phi) = -\Delta_T(\phi) + \text{tr} R(\phi, d\phi) d\phi \), and called the bitension field of \( \phi \) (ARSLAN et al., 2005), (EELLS; LEMAIRE, 1978), (EELLS; SAMPSON, 1964). Non-harmonic biharmonic maps are called proper biharmonic maps.

New methods for constructing a canal surface surrounding a biharmonic curve in the Lorentzian Heisenberg group Heis³ were given, (KORPINAR; TURHAN, 2011; 2012), (TURHAN; KORPINAR, 2011). Also, in (KORPINAR; TURHAN, 2010), (TURHAN; KORPINAR, 2010) they characterized biharmonic curves in terms of their curvature and torsion. Also, by using timelike biharmonic curves, they give explicit parametrizations of canal surfaces in the Lorentzian Heisenberg group Heis³.

In this paper, we study b-m₂ developable surfaces of biharmonic b-slant helices in the special three-dimensional φ − Ricci symmetric para-Sasakian manifold P. We characterize the biharmonic curves in terms of their curvature and torsion in the special three-dimensional φ − Ricci symmetric para-Sasakian manifold P. Finally, we find out explicit parametric equations of b-m₂ developable surfaces of biharmonic b-slant helices in the special three-dimensional φ − Ricci symmetric para-Sasakian manifold P. Additionally, we illustrate our results in Figure 1, 2 and 3.
Preliminaries

An $n$-dimensional differentiable manifold $M$ is said to admit an almost para-contact Riemannian structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $M$ such that

\[
\phi\xi = 0, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X),
\]

\[
\phi^2(X) = X - \eta(\xi)\xi,
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

for any vector fields $X, Y$ on $M$ (BLAIR, 1975).

In addition, if $(\phi, \xi, \eta, g)$ satisfy the equations

\[
\nabla_X g(\eta, \xi) = 0, \quad \nabla_X \phi = 0,
\]

then $M$ is called a para-Sasakian manifold or, briefly a $P$-Sasakian manifold.

Definition 1: A para-Sasakian manifold $M$ is said to be locally $\phi$-symmetric if

\[
\phi^2((\nabla_w R)(X, Y)Z) = 0,
\]

where:

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

This notion was introduced by (SATO, 1976) for a Sasakian manifold.

Definition 2: A para-Sasakian manifold $M$ is said to be $\phi$-symmetric if

\[
\phi^2((\nabla_w R)(X, Y)Z) = 0,
\]

where:

for all vector fields $X, Y, Z, W$ on $M$ (BLAIR, 1976).

Definition 3: A para-Sasakian manifold $M$ is said to be $\phi$-Ricci symmetric if the Ricci operator satisfies

\[
\phi^2((\nabla_X Q)(Y)) = 0,
\]

where:

for all vector fields $X$ and $Y$ on $M$ and $Q(X, Y) = g(QX, Y)$. (TAKASHI, 1977).

If $X, Y$ are orthogonal to $\xi$, then the manifold is said to be locally $\phi$-Ricci symmetric, (SATO, 1976).

The three-dimensional manifold $P = \{(x, y, z) \in R^3 : (x, y, z) \neq (0, 0, 0)\}$ where:

$(x, y, z)$ are the standard coordinates in $R^3$. We choose the vector fields

\[
e_1 = e^x \frac{\partial}{\partial y}, \quad e_2 = e^z \frac{\partial}{\partial z}, \quad e_3 = -\frac{\partial}{\partial x}
\]

are linearly independent at each point of $P$.

Let $\eta$ be the 1-form defined by

$\eta(Z) = g(Z, e_1)$ for any $Z \in \chi(P)$.

Let $\phi$ be the $(1,1)$ tensor field defined by

$\phi(e_1) = e_2, \phi(e_2) = e_1, \phi(e_3) = 0$.

Then using the linearity of $\phi$ and $g$ we have

$\eta(e_1) = 1, \quad \phi^2(Z) = Z - \eta(Z)e_1, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$

where:

for any $Z, W \in \chi(P)$. Thus for $e_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost para-contact metric structure on $P$.

Let $\nabla$ be the Levi-Civita connection with respect to $g$. Then, we have

$[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2$.

Taking $e_3 = \xi$, and using the Koszul’s formula, we obtain

$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_i$.

$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2.$

$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$

Moreover we put

$R_{ijk} = R(e_i, e_j)e_k, \quad R_{ijk} = R(e_i, e_j, e_k, e_l),$

where:

the indices $i, j, k$ and $l$ take the values 1, 2 and 3.

$R_{122} = -e_1, \quad R_{133} = -e_1, \quad R_{233} = -e_2.$
and
\[ R_{1212} = R_{1313} = R_{2323} = 1. \] (8)

Biharmonic b-slant helices in the special three-dimensional \( \phi \)-Ricci symmetric para-Sasakian manifold \( P \)

Let us consider biharmonicity of curves in the special three-dimensional \( \phi \)-Ricci Symmetric para-Sasakian manifold \( P \). Let \( \{ t, n, b \} \) be the Frenet frame field along \( \gamma \). Then, the Frenet frame satisfies the following Frenet--Serret equations:

\[
\nabla_t t = \kappa n, \quad \nabla_t n = -\kappa t + \tau b, \quad \nabla_t b = -\sigma n,
\] (9)

where:
\[ \kappa = |T(\gamma)| = |\nabla_t t| \] is the curvature of \( \gamma \) and \( \tau \) its torsion and
\[
g(t, t) = 1, \quad g(n, n) = 1, \quad g(b, b) = 1, \\
g(t, n) = g(t, b) = g(n, b) = 0.
\]

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, (BISHOP, 1975). The Bishop frame is expressed as

\[
\nabla_t t = k_1 m_1 + k_2 m_2, \\
\nabla_t m_1 = -k_1 t, \\
\nabla_t m_2 = -k_2 t,
\] (10)

where:
\[
g(t, t) = 1, \quad g(m_1, m_1) = 1, \quad g(m_2, m_2) = 1, \\
g(t, m_1) = g(t, m_2) = g(m_1, m_2) = 0.
\]

Here, we shall call the set \( \{ t, m_1, m_2 \} \) as Bishop trihedra, \( k_1 \) and \( k_2 \) as Bishop curvatures and \( \zeta(s) = \arctan \frac{k_2}{k_1} \), \( \tau(s) = \xi'(s) \) and \( \kappa(s) = \sqrt{k_1^2 + k_2^2} \).

Bishop curvatures are defined by
\[
k_1 = \kappa(s) \cos \zeta(s), \\
k_2 = \kappa(s) \sin \zeta(s).
\]

The relation matrix may be expressed as
\[
t = t, \\
n = \cos \zeta(s) m_1 + \sin \zeta(s) m_2, \\
b = -\sin \zeta(s) m_1 + \cos \zeta(s) m_2.
\]

On the other hand, using above equation we have
\[
t = t, \\
m_1 = \cos \zeta(s) n - \sin \zeta(s) b, \\
m_2 = \sin \zeta(s) n + \cos \zeta(s) b.
\]

With respect to the orthonormal basis \( \{ e_1, e_2, e_3 \} \) we can write
\[
t = t^1 e_1 + t^2 e_2 + t^3 e_3, \\
m_1 = m_1^1 e_1 + m_1^2 e_2 + m_1^3 e_3, \\
m_2 = m_2^1 e_1 + m_2^2 e_2 + m_2^3 e_3.
\] (11)

Theorem 1: \( \gamma: I \rightarrow P \) is a biharmonic curve with Bishop frame if and only if
\[ k_1^2 + k_2^2 = \text{constant} \neq 0, \]

Definition 4: A regular curve \( \gamma: I \rightarrow P \) is called a slant helix provided the unit vector \( m_1 \) of the curve \( \gamma \) has constant angle \( \varepsilon \) with unit vector field \( u \) along \( \gamma \), that is
\[
g(m_1(s), u) = \cos \varepsilon \text{ for all } s \in I.
\] (12)

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on Bishop curvatures.

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as b-slant helix.

Lemma 1: Let \( \gamma: I \rightarrow P \) be a unit speed non-geodesic biharmonic b-slant helix. Then, the parametric equations of \( \gamma \) are

\[
k_1 = \kappa(s) \cos \zeta(s), \\
k_2 = \kappa(s) \sin \zeta(s).
\]
\[ x(s) = \sin E s + A_1, \]
\[ y(s) = \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} (\sin E - A_0)\cos E \cos [A_0 s + A] \] (13)
\[ + \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} (\sin E + A_0)\cos E \sin [A_0 s + A]) + A_2, \]
\[ z(s) = \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} A_0 \cos E \cos [A_0 s + A] \]
\[ - \frac{e^{\sin E s + A_1}}{A_0^2 + \sin^2 E} \sin E \cos E \sin [A_0 s + A]) + A_3, \]

where:
\[ A, A_1, A_2, A_3 \] are constants of integration and
\[ A_0 = \left( k_0^2 + k_2^2 - \frac{\cos^2 E}{\cos E} \right)^{\frac{1}{2}}. \]

Now, we illustrate our result in Figure 1.

![Figure 1](image)

Figure 1. For \( A = A_1 = A_2 = A_3 = 0. \)

**b-m\text{2}** developable surfaces of biharmonic b-slant helices in the special three-dimensional \( \phi^- \) ricci symmetric para-sasakian manifold \( P \)

The **b-m\text{2}** developable of \( \gamma \) is a ruled surface

\[ R(s, u) = \gamma(s) + u m_2(s). \] (14)

Firstly, we need following Lemma.

**Lemma 2:** Let \( \gamma : I \to P \) be a unit speed non-geodesic biharmonic b-slant helix. Then, the parametric equations of \( \gamma \) are

\[ \gamma(s) = \left[ \frac{1}{A_0^2 + \sin^2 E} (\sin E - A_0) \cos E \cos [A_0 s + A] \right] \]
\[ \left[ \frac{1}{A_0^2 + \sin^2 E} (\sin E + A_0) \cos E \sin [A_0 s + A] \right] + e^{-\sin E} A_2 \]
\[ \left[ \frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos [A_0 s + A] \right] \]
\[ - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin [A_0 s + A]) + e^{-\sin E} A_3 \] e_1 \]
\[ \left[ \frac{1}{A_0^2 + \sin^2 E} A_0 \cos E \cos [A_0 s + A] \right] - u \cos [A_0 s + A] \]
\[ - \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin [A_0 s + A]) + e^{-\sin E} A_3 \] e_2 \]
\[ - \sin E s + A_1] e_3, \]

where:
\[ A, A_1, A_2, A_3 \] are constants of integration and
\[ A_0 = \left( k_0^2 + k_2^2 - \frac{\cos^2 E}{\cos E} \right)^{\frac{1}{2}}. \]
Proof. By the Bishop (1975) formula, we have the following equation

\[ \mathbf{m}_2 = \sin[A_0 s + A] \mathbf{k}_1 - \cos[A_0 s + A] \mathbf{k}_2, \quad (17) \]

where:
- \( A \) is constants of integration and

\[ A_0 = \left( \frac{k_2^2 + k_3^2 - \cos^2 E}{\cos^2 E} \right)^{1/2}. \]

Consequently, the equation of \( R \) can be found from (14), (17). This concludes the proof of Theorem.

We can prove the following interesting main result.

Theorem 3. Let \( R \) be \( b\text{-}\mathbf{m}_2 \) developable of a unit speed non-geodesic biharmonic \( b\)-slant helix in \( P \). Then the parametric equations of \( b\text{-}\mathbf{m}_2 \) developable are given by

\[
\begin{align*}
    x_R(s, u) &= \sin E s + A_1,
    \\
y_R(s, u) &= e^{\sin E A_1} \left[ \frac{1}{A_0 + \sin^2 E} (\sin E - A_0) \cos E \cos[A_0 s + A] \right. \\
    &+ \frac{1}{A_0^2 + \sin^2 E} (\sin E + A_0) \cos E \sin[A_0 s + A] + e^{-\sin E A_1} A_2 \\
    &+ \frac{1}{A_0 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] + u \sin[A_0 s + A] \\
    &- \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + e^{-\sin E A_1} A_3 \right],
    \\
z_R(s, u) &= e^{\sin E A_1} \left[ \frac{1}{A_0 + \sin^2 E} A_0 \cos E \cos[A_0 s + A] - u \cos[A_0 s + A] \\
    &- \frac{1}{A_0^2 + \sin^2 E} \sin E \cos E \sin[A_0 s + A] + e^{-\sin E A_1} A_3 \right],
\end{align*}
\]

where:
- \( A, A_1, A_2, A_3 \) are constants of integration.

Proof. We assume that \( \gamma \) is a unit speed \( b\)-slant helix.

Substituting (4) to (16), we have (18). Thus, the proof is completed.

Thus, we proved the following:

Corollary 1: Let \( R \) be \( b\text{-}\mathbf{m}_2 \) developable of a unit speed non-geodesic biharmonic \( b\)-slant helix in \( P \). Then, normal of \( b\text{-}\mathbf{m}_2 \) developable of \( \gamma \) is

\[ \mathbf{n}_R = \sin E \cos[\tan E s + A] \mathbf{k}_1 + \sin E \sin[\tan E s + A] \mathbf{k}_2 + \cos E \mathbf{k}_3, \]

where:
- \( A \) is constants of integration.

Proof. It is obvious from Theorem 15 Hence the proof is completed.

Finally, the obtained parametric equations are illustrated in Figure 2 and 3:

**Figure 2.** For \( A = -A_1 = A_2 = A_3 = 1 \).

**Figure 3.** For \( -A_1 = A_2 = A_3 = -1 \).

**Conclusion**

In the last decade there has been a growing interest in the theory of biharmonic maps which can
be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations.

In this paper, \( b-m_2 \) developable surfaces of biharmonic \( b \)-slant helices in the special three-dimensional \( \phi \)-Ricci symmetric \( \phi \)-Sasakian manifold \( P \) is studied. Explicit parametric equations of \( b-m_2 \) developable surfaces of biharmonic \( b \)-slant helices in the special three-dimensional \( \phi \)-Ricci symmetric \( \phi \)-Sasakian manifold \( P \) are characterized.

Acknowledgements

The author would like to thank the referees for valuable suggestions which improved the paper considerably.

References

ARSLAN, K.; EZENTAS, R.; MURATHAN, C.; SASAHARA, T. Biharmonic submanifolds 3-dimensional \((\kappa, \mu)\)-manifolds. *Internat International Journal of Mathematics and Mathematical Sciences*, v. 22, p. 3575-3586, 2005.

BISHOP, L. R. There is more than one way to frame a curve. *American Mathematical Monthly*, v. 82, n. 3, p. 246-251, 1975.

BLAIR, D. E. *Contact manifolds in riemannian geometry*. Berlin and New York: Lecture Notes in Mathematics, Springer-Verlag, 1976.

CADDEO, R.; MONTALDO, S. Biharmonic submanifolds of \( S^3 \). *International Journal of Mathematics*, v. 12, n. 8, p. 867-876, 2001.

CARMO, M. D. *Differential geometry of curves and surfaces*. New Jersey: Prentice Hall, 1976.

DIMITRICO, I. Submanifolds of \( \mathbb{E}^m \) with harmonic mean curvature vector. *Bulletin of the Institute of Mathematics. Academia Sinica*, v. 20, p. 53-65, 1992.

EELLS, J.; LEMAIRE, L. A report on harmonic maps. *Bulletin of the London Mathematical Society*, v. 10, n. 1, p. 1-68, 1978.

EELLS, J.; SAMPSON, J. H. Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics*, v. 86, n. 1, p. 109-160, 1964.

KORPINAR, T.; TURHAN, E. On characterization of \( B \)-canal surfaces in terms of biharmonic \( B \)-slant helices according to Bishop frame in Heisenberg group \( Heis^3 \). *Journal of Mathematical Analysis and Applications*, v. 382, n. 1, p. 57-65, 2011.

KORPINAR, T.; TURHAN, E. On horizontal biharmonic curves in the heisenberg group \( Heis^3 \). *Arabian Journal for Science and Engineering Section A*, v. 35, n. 1, p. 79-85, 2010.

KORPINAR, T.; TURHAN, E. Tubular surfaces around timelike biharmonic curves in lorentzian heisenberg group \( Heis^3 \). *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica*, v. 20, n. 1, p. 431-445, 2012.

LOUBEAU, E.; ONICIUC, C. On the biharmonic and harmonic indices of the Hopf map. *Transactions of the American Mathematical Society*, v. 359, n. 11, p. 5239-5256, 2007.

O’NEILL, B. *Semi-Riemannian geometry*. New York: Academic Press, 1983.

SATO, I. On a structure similar to the almost contact structure. *Tensor (N.S.)*, v. 30, n. 3, p. 219-224, 1976.

TAKASHI, T. Sasakian \( \phi \)-symmetric spaces. *Tohoku Mathematical Journal*, v. 29, n. 1, p. 91-113, 1977.

TURHAN, E.; KORPINAR, T. On characterization canal surfaces around timelike horizontal biharmonic curves in lorentzian heisenberg group \( Heis^3 \). *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, v. 66, n. a, p. 441-449, 2011.

TURHAN, E.; KORPINAR, T. On characterization of timelike horizontal biharmonic curves in the lorentzian heisenberg group \( Heis^3 \). *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, v. 65, n. a, p. 641-648, 2010.

Received on May 25, 2012.

Accepted on June 29, 2012.

License information: This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.