Establishing a non-Fermi liquid theory for disordered metals near two dimensions

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(March 22, 2022)

We consider the Finkelstein action describing a system of spin polarized or spinless electrons in 2 + 2ε dimensions, in the presence of disorder as well as the Coulomb interactions. We extend the renormalization group analysis of our previous work and evaluate the metal-insulator transition of the electron gas to second order in an ε expansion. We obtain the complete scaling behavior of physical observables like the conductivity and the specific heat with varying frequency, temperature and/or electron density.

We extend the results for the interacting electron gas in 2 + 2ε dimensions to include the quantum critical behavior of the plateau transitions in the quantum Hall regime. Although these transitions have a very different microscopic origin and are controlled by a topological term in the action (θ term), the quantum critical behavior is in many ways the same in both cases. We show that the two independent critical exponents of the quantum Hall plateau transitions, previously denoted as ν and p, control not only the scaling behavior of the conductances σ_{xx} and σ_{xy} at finite temperatures T, but also the non-Fermi liquid behavior of the specific heat (c_v ∝ T^n). To extract the numerical values of ν and p it is necessary to extend the experiments on transport to include the specific heat of the electron gas.

I. INTRODUCTION

The integral quantum Hall regime has traditionally been viewed as a (nearly) free particle localization problem with interactions playing only a minor role. Although it is well known that many features of the experimental data, taken from low mobility heterostructures, can be explained as the behavior of free particles, a much sharper formulation of the problem is obtained by considering the quantum Hall plateaus. Following the experimental work by H.P. Wei et al., these transitions behave in all respects like a disorder driven metal-insulator transition that is characterized by two independent critical indices, i.e., a localization length exponent ν and a phase breaking length exponent p. Whereas transport measurements usually provide an experimental value of only the ratio κ = p/2ν, it is generally not known how the values of ν and p can be extracted separately.

Inspite of the fact that one can not proceed without having a microscopic theory of electron-electron interaction effects, there is nevertheless a strong empirical belief in the literature, which says that the zero temperature localization length exponent ν is given precisely by the free electron value ν = 2.3 as obtained from numerical simulations. The experimental situation has not been sufficiently well understood, however, to justify the bold assumption of Fermi liquid behavior. In fact, the progress that has been made over the last few years in the theory of localization and interaction effects clearly indicates that Fermi liquid principles do not exist in general. The Coulomb interaction problem lies in a different universality class of transport phenomena with a previously unrecognized symmetry, called F invariance. The theory relies in many ways on the approach as initiated by Finkelstein and adapted to the case of the spin polarized or spinless electrons. By reconciling the Finkelstein theory with the topological concept of an instanton vacuum and the Chern Simons statistical gauge fields, the foundations have been laid for a complete renormalization theory that unifies the quantum theory of metals with that of the abelian quantum Hall states.

A. A historical problem

The unification of the integral and fractional quantum Hall regimes is based on the assumption that Finkelstein approaches is renormalizable and generates a strong coupling, insulating phase with a massgap. However, the traditional analyses of the Finkelstein theory have actually not provided any guarantee that this is indeed so.

Inspite of Finkelstein’s pioneering and deep contributions to the field, it is well known that the conventional momentum shell renormalization schemes do not facilitate any computations of the quantum theory beyond one loop order. At the same time, application of the more advanced technique of dimensional regularization has led to conceptual difficulties with such aspects like dynamical scaling. One can therefore not rule out the possibility that there are complications, either in the idea of renormalizability, or in other aspects of the theory such as the Matsubara frequency technique.
Nothing much has been clarified, however, by repeating similar kinds of analyses in a different formalism, like the Keldish technique [17]. What has been lacking all along is the understanding of a fundamental principle that has prevented the Finkelstein approach from becoming a fully fledged field theory for localization and interaction effects.

B. $\mathcal{F}$ invariance

In our previous work [18] we have shown that the Finkelstein action has an exact symmetry ($\mathcal{F}$ invariance) that is intimately related to the electrodynamic $U(1)$ gauge invariance of the theory. $\mathcal{F}$ invariance is the basic mechanism that protects the renormalization of the problem with infinitely ranged interaction potentials such as the Coulomb potential. Moreover, it has turned out that the infrared behavior of physical observables can only be extracted from $\mathcal{F}$ invariant quantities and correlations, and these include the linear response to external potentials. Arbitrary renormalization group schemes break the $\mathcal{F}$ invariance of the action and this generally complicates the attempt to obtain the temperature and/or frequency dependence of physical quantities such as the conductivity and specific heat.

Quantum Hall physics is in many ways a unique laboratory for investigating and exploring the various different consequences of $\mathcal{F}$ invariance. For example, one of the longstanding questions in the field is whether and how the theory dynamically generates the exact quantization of the Hall conductance. Important progress has been made recently by demonstrating that the instanton vacuum, on the strong coupling side of the problem, generally displays massless excitations at the edge of the system. These massless edge excitations are identical to those described by the more familiar theory of chiral edge bosons. Our theory of massless edge excitations implies that the concept of $\mathcal{F}$ invariance retains its fundamental significance all the way down to the regime of strong coupling.

C. Outline of this paper

In this paper we put the concept of $\mathcal{F}$ invariance at work and evaluate the renormalization behavior of the Finkelstein theory at a two loop level. As shown in our previous papers [19] the technique of dimensional regularization is a unique procedure, not only for the computation of critical indices, but also for extracting the dynamical scaling functions. In fact, the metal-insulator transition in $2 + 2\epsilon$ spatial dimensions is the only place in the theory where the temperature and/or frequency dependence of physical observables can be obtained explicitly. This motivates us to further investigate the problem in $2 + 2\epsilon$ dimensions and use it as a stage setting for the much more complex problem of the quantum Hall plateau transitions.

The final results of this paper are remarkably similar to those of the more familiar classical Heisenberg ferromagnet [20]. For example, unlike the free electron gas, the Coulomb interaction problem displays a conventional phase transition (metal-insulator transition) in $2 + 2\epsilon$ dimensions with an ordinary order parameter. The theory is therefore quite different from that of free electrons which has a different dimensionality and displays, as is well known, anomalies or multifractal density fluctuations near criticality [21].

It is important to bear in mind, however, that the analogy with the Heisenberg model is rather formal and it fails on many other fronts. For example, the classification of critical operators is very different from what one is used to, in ordinary sigma models. Moreover, the Feynman diagrams of the Finkelstein theory are more complex, involving internal frequency sums which indicate that the theory effectively exists in $2 + 1$ space-time dimensions, rather than in two spatial dimensions alone. The complexity of $\mathcal{F}$ invariant systems is furthermore illustrated by the lack of such principles like Griffith analyticity that facilitates a discussion of the symmetric phase in conventional sigma models [22]. In a subsequent paper we shall address the strong coupling insulating phase of the electron gas and show that the dynamics is distinctly different from that of the Goldstone (metallic) phase and controlled by different operators in the theory [23].

This paper is organized as follows. After introducing the formalism (Section II) we embark on the details of the two loop contributions to the conductivity in Section III. As in our earlier work, we employ an $\mathcal{F}$-invariance-breaking parameter $\alpha$ to regularize the infinite sums over frequency. This methodology actually provides numerous self consistency checks and a major part of the computation consists of finding the ways in which the various singular contributions in $\alpha$ cancel each other. The actual computation of the diagrams is described in the Appendices which contain the list of the momentum and frequency integrals that are used in the text. In tables I and II we summarize how the different singular contributions in $\alpha$ cancel each other. Table III lists the various finite contributions to the pole term in $\epsilon$. The final result for the $\beta$ function is given by Eqs. (15)–(18).

In Section IV we summarize the consequences for scaling. We extend the discussion to include the plateau transitions in the quantum Hall regime in Section IVC. We briefly address several new advancements, both from theoretical and experimental sides, that seem to have general consequences for the quantum theory of conductances. Finally, we show how the results of this paper can be used in the problem of critical exponents $p$ and $\nu$.

We end this paper with a conclusion (Section V).
II. EFFECTIVE PARAMETERS

A. Introduction

The theory for spinless electrons involves unitary matrix field variables $Q^\alpha_{nm}$ where the superscripts $\alpha \beta$ are the replica indices, the subscripts $n, m$ denote the Matsubara frequency indices. The $Q$ fields obey the nonlinear constraint $Q^2 = 1$ and we are interested in the following action

$$S[Q, A] = -\frac{\sigma_0}{8} \int_x \left( \text{tr}\left[ \hat{D}, Q \right]^2 + 2h_0^2 \text{tr} \Lambda Q \right) + z_0 \pi T \int_x \left( \sum_{\alpha n} c_0 \text{tr} I^\alpha_n Q \text{tr} I^\alpha_{m n} Q + 4 \eta \eta Q - 6 \eta \text{tr} \Lambda \right).$$

(1)

The explanation of the symbols is as follows. The parameter $\sigma_0$ plays the role of conductivity of the electron gas, $z_0$ is the so-called singlet interaction amplitude and $T$ stands for the temperature. The parameter $c_0 = 1 - \alpha$ is such that the theory interpolates between the Coulomb case ($\alpha = 0$) and the free particle case ($\alpha = 1$). Here, the quantity $\alpha$ breaks the $\mathcal{F}$ invariance of the theory and we shall eventually be interested in the limit where $\alpha$ goes to zero. For a detailed exposure to the meaning of $\mathcal{F}$ invariance we refer the reader to the original papers.

We generally need the definition of two more diagonal matrices $\Lambda$ and $\eta$, and one more off-diagonal matrix $I^\alpha_n$. They are given by

$$\Lambda^\alpha_{nm} = \text{sign}(n) \delta^{\alpha \beta} \delta_{nm},$$

$$\eta^\alpha_{nm} = n \delta^{\alpha \beta} \delta_{nm},$$

$$(I^\alpha_n)_{kl} = \delta^{\alpha \beta} \delta_{n,k-l}.$$ 

Here, $\eta$, being multiplied by $2 \pi T$, represents the Matsubara frequencies in matrix language. The $I^\alpha_n$ are shifted diagonals in frequency space and they generally represent the generators of the $U(1)$ gauge transformations.

The term proportional to $h_0^2$ is not a part of the theory but we shall use it later on as a convenient infrared regulator of the theory. Finally, the $\hat{D}$ are covariant derivatives

$$\hat{D}_a = \nabla_a - i \hat{A}_a,$$

where

$$\hat{A}_a = \sum_{\alpha, n} (A^\alpha_a)_{n} I_n^\alpha,$$

and $(A^\alpha_a)_n$ is the Fourier transform of the homogeneous external vector potential $A^\alpha_a(\tau)$: $A^\alpha_a(\tau) = \sum_n (A^\alpha_a)_n \exp(-i \omega_n \tau)$, $\omega_n = 2 \pi T n$ is the Matsubara frequency.

B. Linear response

The "effective" action for the external vector potential is defined according to

$$\exp S_{eff}[A] = \int DQ \exp S[Q, A].$$

(2)

The quadratic part can generally be written as

$$S_{eff}[A] = \sum_x \sigma'(n)n(A^\alpha_n)_{n} (A^\alpha_{-n}).$$

(3)

The quantity $\sigma'(n)$ is the true conductivity of the electron gas, and in terms of the $Q$ matrix fields the following Kubo like expression can be obtained

$$\sigma'(n) = \langle O_1 \rangle + \langle O_2 \rangle,$$

(4)

where

$$O_1 = -\frac{\sigma_0}{4n} \text{tr} [I^\alpha_n, Q(x)] [I^\alpha_{-n}, Q(x)]$$

and

$$O_2 = \frac{\sigma_0^2}{16 \pi n^2} \int_{x \neq x'} \text{tr} [I^\alpha_n, Q(x)] \nabla Q(x) \text{tr} [I^\alpha_{-n}, Q(x')] \nabla Q(x').$$

Here the expectations are with respect to the theory without the vector potentials.

C. The $h_0$ field

Although we are interested, strictly speaking, in evaluating $\sigma(n)$ with varying values of external frequencies $\omega_n$ and temperature, the computation simplifies dramatically if we put these parameters equal to zero in the end and work with a finite value of the $h_0$ field instead. This procedure has been analyzed in exhaustive detail in our previous work and, in what follows, we shall greatly benefit from the technical advantages that make the two-loop analysis of the conductivity possible. We shall return to finite frequency and temperature problem in the end of this paper (Sections IV).

The infrared regularization by the $h_0$ field relies on the following statement

$$\sigma_0 h_0^2<(Q(\vec{x})) = \sigma' h^2 \Lambda,$$

(7)

which says that there is an effective mass $h'$ in the problem that is being induced by the presence of the $h_0$ field. It is very well known that, since the quantity $\langle Q(\vec{x}) \rangle$ is not a gauge invariant object, the definition of the $h'$ field is singular as $\alpha$ goes to zero and the theory is generally not renormalizable. However, the effective parameter $\sigma'$ is truly defined in terms of the effective mass $h'$
rather than the bare parameter $h_0$. Hence, all the non-renormalizable singularities are removed from the theory, provided we express $\sigma'$ in terms of the $h'$ rather than the $h_0$. We shall show that the ultraviolet singularities of the theory can be extracted directly from the final result for $\sigma'(h')$. On the other hand, we can make use of our previous results and express the final answer in terms of frequencies and temperature, rather than the mass $h'$.

III. COMPUTATION OF CONDUCTIVITY IN 2 + 2\epsilon DIMENSIONS

A. Introduction

To define a theory for perturbative expansions we use the following parametrization

$$Q = \left( \frac{\sqrt{1 - q q^+}}{q} - \sqrt{1 - q^+ q} \right),$$

(8)

The action can be written as an infinite series in the independent fields $g_{n_1 n_2}^{\alpha \beta}$ and $[q^+]_{n_1 n_2}^{\alpha \beta}$. We use the convention that Matsibara indices with odd subscripts: $n_1, n_3, ...$, run over non-negative integers, whereas those with even subscripts: $n_2, n_4, ...$, run over negative integers. The propagators can be written in the form

$$(g_{n_1 n_2}^{\alpha \beta}(q)[q^+]_{n_1 n_2}^{\alpha \beta}(-p)) = \frac{4}{\sigma_0} \delta^{\alpha \delta} \delta^{\beta \gamma} \delta_{n_1 n_2 n_3 n_4} D_\alpha(q)$$

(9)

where

$$[D_\alpha(q)]^{-1} = p^2 + h_0^2 + \kappa^2 n_1 n_2,$$

(10)

$$[D_{\alpha}^{\dagger}(q)]^{-1} = p^2 + h_0^2 + \kappa^2 n_1 n_2,$$

(11)

$$\kappa^2 = \frac{8\pi T}{\sigma_0}.$$  

(12)

Here we use the notation $n_{12} = n_1 - n_2$.

The expression for the DC conductivity is known to one loop order

$$\sigma'_{\text{one}} = \sigma_0 + \frac{4\Omega_d h_0^2}{\epsilon}, \quad \Omega_d = \frac{S_d}{2(2\pi)^d},$$

(13)

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of a $d$ dimensional unit sphere.

B. The two-loop theory

To proceed we need the action terms obtained by expanding the action (2.1) in terms of $q$ and $q^\dagger$ fields:

$$S^{(1)}_{\text{int}} = -\frac{a\sigma_0}{8} \int \sum_{\beta, m \geq 0} \{ \text{tr} I_{m\beta} I_{\beta m}^\dagger [q, q^\dagger] + \text{tr} I_{m\beta} I_{\beta m} [q, q^\dagger] \},$$

(14)

$$S^{(2)}_{\text{int}} = \frac{a\sigma_0}{16} \int \sum_{\beta, m \geq 0} \{ \text{tr} I_{m\beta} I_{\beta m} [q, q^\dagger] \}^2,$$

(15)

$$S^{(3)}_{\text{int}} = \frac{a\sigma_0}{32} \int \delta(p_1 + p_2 + p_3 + p_4) \sigma^\dagger \sigma \sum_{n_1 n_2 n_3}^4 h^\dagger \sum_{n_1 n_2 n_3 n_4}^4,$$

(16)

where we define $a = \kappa^2 z_{00}$.

In addition, we need the following terms obtained by expanding the expression for the conductivity, Eq. [2],

$$O_1^{(2)} = -\frac{\sigma_0}{2} \text{tr} \{ I_n^\dagger I_{-n} I_n^\dagger I_{-n} I_n q - 2(I_n^\dagger I_{-n} I_n^\dagger I_n^\dagger I_{-n} I_n^\dagger) [q, q^\dagger] \},$$

(17)

$$O_1^{(3)} = \frac{\sigma_0}{4} \text{tr} \{ I_n^\dagger (q + q^\dagger) I_n^\dagger I_n [q, q^\dagger] - I_n^\dagger I_n [q, q^\dagger] I_n^\dagger I_n \}$,

(18)

$$O_1^{(4)} = \frac{\sigma_0}{16} \text{tr} \{ I_n^\dagger I_n^\dagger I_n I_n^\dagger I_n^\dagger [q, q^\dagger] - 2I_n^\dagger I_n I_n^\dagger I_n [q, q^\dagger] \},$$

(19)

$$O_2^{(4)} = \frac{\sigma_0^2}{32} \int_{x = x'} \text{tr} I_{n}^\dagger (q \nabla q^\dagger + q^\dagger \nabla q) I_n^\dagger (q \nabla q^\dagger + q^\dagger \nabla q),$$

(20)

$$O_2^{(5)} = \frac{\sigma_0^2}{8d} \int_{x = x'} \{ \text{tr} I_n^\dagger (q \nabla q^\dagger + q^\dagger \nabla q) I_n q (\nabla q^\dagger q) + \text{tr} I_n^\dagger q (\nabla q^\dagger q) \},$$

(21)

$$O_2^{(6)} = \frac{\sigma_0^2}{16d} \int_{x = x'} \{ \text{tr} I_n^\dagger q (\nabla q^\dagger q) I_n q (\nabla q^\dagger q) \},$$

(22)
Next we give the complete list of two loop contributions to the conductivity as follows

\[ \sigma_{\text{two}}(n) = \left\langle O_4^{(4)} + O_4^{(3)} S_{\text{int}}^{(3)} + O_1^{(2)} (S_{\text{int}}^{(4)} + S_0^{(4)} + \frac{1}{2} (c_{\text{int}})^2) \right\rangle \]

\[ + O_2^{(6)} + O_2^{(5)} S_{\text{int}}^{(3)} + O_1^{(4)} (S_{\text{int}}^{(4)} + S_0^{(4)} + \frac{1}{2} (c_{\text{int}})^2) \right\rangle. \] (23)

The computations of the terms in Eq. (23) are straightforward but lengthy and tedious. In what follows we present the expressions in terms of the momentum integrals, frequency sums and propagators \( D, D^c \) for each term in Eq. (23) separately, along with the final answer. In the Appendices we give the complete list of integrals and symbols that we shall make use of here.

C. Computation of contractions

1. \( \left\langle O_4^{(4)} \right\rangle \)

\[ \frac{2}{\sigma_0} \left( \int D_p(0) \right)^2 + \frac{2a^2}{\sigma_0} \left( \sum_{m>0} \int D^c_p(m) \right)^2 \]

\[ = \frac{Q_{\text{int}}^4}{\sigma_0 e^2} (2 + 2 \ln^2 \alpha) \] (24)

with \( D^c_p(m) \equiv D_q(m)D^c_q(m) \).

2. \( \left\langle O_3^{(3)} S_{\text{int}}^{(3)} \right\rangle \)

\[ - \frac{8a}{\sigma_0} \left\{ \sum_{k>0} D^c_p(k)D_q(k)D_p(k) \right\} \]

\[ + a \sum_{k,m>0} D^c_p(m)D^c_q(k)D_{p+q}(k + m) \}

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} (4S_0 + 4A_{00}) \]

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} \left[ -4 - 4 \ln^2 \alpha + \epsilon(8 + 4\zeta(3)) \right], \] (25)

where \( \zeta(z) \) is the Riemann zeta-function.

3. \( \left\langle O_2^{(2)} (S_{\text{int}}^{(4)} + S_0^{(4)} + \frac{1}{2} (S_{\text{int}}^{(3)})^2) \right\rangle \)

\[ \frac{4a}{\sigma_0} \left\{ \sum_{k>0} D^c_p(k)D_q(k) \right\} \]

\[ + a \sum_{k,m>0} D^c_p(m)D^c_q(k)D^2_{p+q}(k + m) \]

\[ + a \sum_{k,m>0} (1 + amD^c_p(m))DD^c_q(k)D^2_{p+q}(k + m) \}

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} \left[ -2S_0 - 2D_1 - 2T_{01} - 4A_{10} \right] \]

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} \left[ 2 + 2 \ln^2 \alpha - \epsilon(4 + 2\zeta(3) + 2\pi^2/3) \right]. \] (26)

4. \( \left\langle O_2^{(6)} \right\rangle \)

\[ - \frac{4}{\sigma_0 e^2} \int p^2 \left\{ D_p(0)D_q(0)D_{p+q}(0) \right\} \]

\[ - 4a^2 \sum_{k,m>0} D^2D^c_p(m)\hat{S}_mDD^c_q(k) \]

\[ - a^2 \sum_{k,m>0} [ D_p(k + m)DD^c_q(m)D^c_p(k + m) ] \}

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} \left[ -16 \ln \alpha - 2 + \epsilon(-4 \ln \alpha - \frac{\pi^2}{3} + \frac{\pi^2}{2} \ln 2 \right) \]

\[ + \frac{\pi^4}{12} + \frac{11\zeta(3)}{2} + \frac{\pi^2}{3} \ln^2 2 - \frac{1}{3} \ln^4 2 - 7\zeta(3) \ln 2 \]

\[ - 8Li_4\left( \frac{1}{2} \right) \right]. \] (27)

Here \( D^n D^c_q(m) \equiv D^n_q(m)D^c_q(m) \) and

\[ Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{kn} \] (28)

is the polylogarithmic function \( Li_4(1/2) = 0.517\ldots \), and we have introduced an operator \( \hat{S}_m \) which acts only on frequency \( k \) according to the following rule \( \hat{S}_m f(k) = f(k) + f(k + m) \).

5. \( \left\langle O_2^{(5)} S_{\text{int}}^{(3)} \right\rangle \)

\[ \frac{16a^2}{\sigma_0 e^2} \int p \cdot (p - q) \sum_{k>0} D^c_{p+q}(0)D^2_q(k)D_q(k) \]

\[ + \frac{16a^2}{\sigma_0 e^2} \left\{ \sum_{k,m>0} D^c_p(m)D^2_q(k)D^c_q(k) \right\} \]

\[ + D^2D^c_q(k + m)D_q(k) \}

\[ - \frac{16a^2}{\sigma_0 e^2} \int (p \cdot q) \left\{ \sum_{k,m>0} [ D^2D^c_p(m)D^c_q(p + k + m) ] \right\} \]

\[ + D^c_p(k + m)D^2D^c_q(k)D_q(k + m) \}

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} \left[ -8S_0 - 4A_{01}^1 - 4H_0 - 4C_0 - 4A_0 \right] \]

\[ = \frac{2Q_{\text{int}}^4}{\sigma_0 e^2} \left[ -8 \ln \alpha + 4 + \epsilon(4 \ln^2 \alpha + 20 \ln \alpha \right) \]

\[ - 12 + 4\zeta(3) + 4\pi^2/3 - 4A_0 + 4C_0^1 \right]. \] (29)

Where we have introduced yet another operator \( \hat{T}_m \) which acts only on frequency \( k \) but now according to the rule \( \hat{T}_m f(k) = f(k) - f(k + m) \).
D. Results of the computations

We proceed by presenting the final answer for all the pole terms in $\epsilon$. By putting the external frequency equal to zero and in the limit $\alpha \to 0$ we obtain

$$
\sigma'_{\text{two}}(0) = \frac{\Omega_d^2 h_0^4}{\sigma_0^2} (A - 8(2 + \ln \alpha)) .
$$

Here, the $A$ stands for all the terms that are finite in $\alpha$. The complete list is as follows

$$
A = 50 + \frac{1}{6} - 3\pi^2 + \frac{19}{2} \zeta(3) + 16 \ln^2 2
$$
$$
- 44 \ln 2 + \frac{\pi^2}{2} \ln 2 + 16G + \frac{\pi^4}{12} + \frac{\pi^2}{3} \ln^2 2
$$
$$
- \frac{1}{3} \ln^4 2 - 7\zeta(3) \ln 2 - 8L_i(\frac{1}{2})
$$
$$
\approx 1.64 .
$$

Before Eq. (33) is obtained, one has to deal with a host of other contributions that are more singular in $\alpha$ and/or $\epsilon$. These more singular contributions all cancel one another in the end, however. There are in total six different types of contributions that are more singular than the simple pole term $1/\epsilon$. In Tables I and II we list these terms, show where they come from and how they sum up to zero. There is one exception, namely the terms proportional to $\ln(\alpha)/\epsilon$, and their contribution is written in Eq. (33). However, these terms are absorbed in the definition of an “effective” $h$’ field. More specifically, from the two-loop computation of the singlet amplitude $z$ we know that the effective $h$, field is given by

$$
h_0^2 \to h^2 = h_0^2 \left( 1 - \frac{2 + \ln \alpha h_0^2 t_0}{2 \epsilon} \right) .
$$

Using this result, as well as Eqs. (13) and (33), we can write the total answer for the conductivity as follows

$$
\sigma' = \sigma_0 \left( 1 + \frac{h^4 t_0}{2 \epsilon} + A \frac{h^4 t_0^2}{\epsilon} \right) .
$$

Here we have written $t_0 = 4\Omega_d/\sigma_0$. Eq. (34) no longer contains $\alpha$ and is therefore the desired result.

E. $\beta$ and $\gamma$ functions

Recall that $h$ is just the effective mass in the problem and we can replace it by the effective mass that is being induced by working with finite external frequencies, or finite temperatures. However, we can use Eq. (34) directly for extracting the renormalization constant $Z_1$ for the $t$ field. Introducing the renormalized fields $t$ and $z$ as usual
\[ t_0 = \mu^{-2t} Z_1(t), \quad z_0 = z Z_2(t), \]  
then, following the scheme of minimal subtraction, we obtain

\[ Z_1 = 1 + \frac{t}{\epsilon} + \frac{t^2}{\epsilon^2} (1 + \epsilon A) \]  
\[ Z_2 = 1 - \frac{t}{2\epsilon} - \frac{t^2}{4\epsilon^2} \left( \frac{1}{2} + \epsilon \left( \frac{\pi^2}{6} + 2 \right) \right). \]

Here, we have listed also the result for \( Z_2 \) that was obtained in Ref. 10. The \( \beta \) and \( \gamma \) functions are defined by

\[ \beta = \frac{dt}{d\ln \mu} = \frac{2\epsilon t}{1 + td\ln Z_1/dt}, \]  
\[ \gamma = -\frac{d\ln z}{d\ln \mu} = \beta \frac{d\ln Z_2}{dt}, \]

and the final answer can be written as

\[ \beta = 2\epsilon t - 2t^2 - 4At^3, \quad \gamma = -t - \left( \frac{\pi^2}{6} + 3 \right) t^2. \]

| Contractions | Diagrams                          | \( \frac{1}{\epsilon^2} \) | \( \log^2 \alpha \) | \( \log \alpha \) | \( \frac{\log^2 \alpha}{\epsilon} \) | \( \frac{\log \alpha}{\epsilon} \) | \( \frac{1}{\epsilon^2} \) |
|--------------|----------------------------------|-----------------------------|----------------------|------------------|--------------------------------|------------------|------------------------|
| \( \langle O_1^{(4)} \rangle \) | ![Diagram](image)               | 2                           |                      |                  |                                |                  |                        |
| \( \langle O_1^{(3)} S_{int}^{(3)} \rangle \) | ![Diagram](image)               | -4                          |                      |                  |                                |                  |                        |
| \( \langle O_1^{(2)} (S_{int}^{(4)} + s_0^{(4)} + \frac{1}{2} (S_{int}^{(3)})^2) \rangle \) | ![Diagram](image)               | 2                           |                      |                  |                                |                  |                        |
| Total        |                                 | 0                           | 0                     | 0                 | 0                              | 0                | 0                      |

TABLE I. The second-loop contributions to the \( O_1 \) term in the effective conductivity. The \( \alpha \)-dependent and \( 1/\epsilon^2 \) contributions. A black solid dot denotes the vertex in \( O_1 \) term, a white solid dot denotes the vertex in \( S \) terms, and
| Contractions | Diagrams | $\frac{1}{\epsilon^3}$ | $\frac{\log^2 \epsilon}{\epsilon^2}$ | $\frac{\log \alpha}{\epsilon}$ | $\frac{\log^2 \alpha}{\epsilon}$ | $\frac{\log \alpha}{\epsilon}$ | $\frac{1}{\epsilon^2}$ |
|--------------|----------|------------------------|---------------------------------------|-------------------------------|--------------------------------|-------------------------------|------------------------|
| $\langle O_2^{(6)} \rangle$ | ![Diagram](image1) and ![Diagram](image2) | 16 | -4 | -2 | | | |
| $\langle O_2^{(5)} S_{int}^{(3)} \rangle^*$ | ![Diagram](image3) | -8 | 4 | 20 | 4 | | |
| $\langle O_2^{(4)} S_0^{(4)} \rangle$ | ![Diagram](image4) | 2 | 4 | -8 | -2 | 1 | 2 |
| $\langle O_2^{(2)} (S_{int}^{(4)} + \frac{1}{2} (S_{int}^{(3)})^2) \rangle^*$ | ![Diagram](image5) | -2 | -4 | -2 | -25 | -4 | |
| Total | | 0 | 0 | 0 | 0 | -8 | 0 |

TABLE II. The second-loop contributions to the $O_2$ term in the effective conductivity. The $\alpha$-dependent and $1/\epsilon^3$ contributions. The symbol $^*$ denotes that we exclude integrals $A_0$ and $C_0'$ which cancel in the sum of the two terms. A black solid triangle denotes the current vertex in $O_2$ term, a white solid dot denotes the vertex in $S$ terms, and
TABLE III. The second-loop contributions to the $O_2$ term in the effective conductivity. The $1/\epsilon$ contributions. The symbol $^*$ denotes that we exclude integrals $A_0$ and $C_0'$ which cancel in the sum of the two terms. A black solid dot denotes the vertex in $O_1$ term, a black solid triangle denotes the current vertex in $O_2$ term, a white solid dot denotes the vertex in $S$ terms, and

IV. DYNAMICAL SCALING

A. Relation between $h'$ and $\omega_s$

In the Section we combine the two loop computations of this paper with those of the amplitude $z_0$ and establish the connection between the effective mass $h'$ and the frequency $\omega_s$. For this purpose, recall that the renormalization of the $z$ field was obtained from the derivative of the free energy $F$ (or rather, the grand canonical potential) with respect to $\ln T$. The result of the computation was as follows
where

\[ M_b(t_0, h_s^2) = 1 + \frac{h_s^2 t_0}{2 \epsilon} + \frac{h_s^2 t_0^2}{\epsilon^2} \left( -\frac{1}{8} + \epsilon \left( \frac{1}{4} + \frac{\pi^2}{24} \right) \right). \]  

(44)

Here, the frequency enters through the quantity \( h_s^2 = \kappa^2 \lambda_0 = \frac{2 \pi^2 \lambda_0 \omega_s t_0}{\epsilon} \) which has the dimension of mass squared. The frequency dependence in \( \sigma'(s) \) is restored by writing

\[ \sigma'(s) = \frac{4 \Omega_d}{t_0} R_b(t_0, h_s^2) \]  

(45)

with

\[ R_b(t_0, h_s^2) = 1 + \frac{h_s^2 t_0}{\epsilon} + (A - 1/2) \frac{h_s^2 t_0^2}{\epsilon^2} \]  

(46)

One can easily verify that Eqs. (4.1–4.4) lead to the same results as those of the previous Section. Eq. (4.4) is therefore the correct result.

The relation between \( h \) and \( \omega_s \) can now be made more explicit by writing

\[ h^2 = h_s^2 M_b(t_0, h_s^2) / R_b(t_0, h_s^2). \]  

(47)

Here, \( h^2 \) is the effective mass that is induced by the frequency \( \omega_s \) and the result is consistent with all previous statements and explicit computations.

B. The Goldstone phase

1. Specific heat and AC conductivity

The zero of the \( \beta \) function, Eq. (42), determines a critical point \( t_c = O(\epsilon) \) that separates the Goldstone or metallic phase (\( t < t_c \)) from an insulating phase (\( t > t_c \)). To second order in \( \epsilon \) we have

\[ t_c = \epsilon - 2 A \epsilon^2 \approx \epsilon - 3.28 \epsilon^2. \]  

(48)

We see that the \( \epsilon^2 \) contribution is rather large and the expansion can clearly not be trusted for \( \epsilon = 1/2 \) or three spatial dimensions. This is a well-known drawback of asymptotic expansions and the two-loop theory is otherwise necessary to completely establish the scaling behavior of the electron gas in \( 2 + 2 \epsilon \) spatial dimensions. To discuss this scaling behaviour, we proceed and express Eqs. (43) and (45) in terms of the renormalized parameters \( t \) and \( z \). The results can be written in the following general form

\[ \frac{dF}{d\ln T} = 2 \sum_{s>0} \omega_s z_0 M_b(t_0, h_s^2), \]  

(43)

\[ \frac{dF}{d\ln T} = 2 \sum_{s>0} \mu^2 \omega_s z M(t, \omega_s z), \]  

(49)

\[ \sigma'(s) = \frac{4 \Omega_d}{t} R(t, \omega_s z). \]  

(50)

The expressions are now finite in \( \epsilon \). The AC conductivity is obtained from \( \sigma'(s) \) by replacing the imaginary frequencies \( i \omega_s \) by real ones \( \omega \). On the other hand, the specific heat of the electron gas can be expressed as

\[ c_v = \int_0^\infty d\omega \frac{\partial f_{BE}}{\partial T} \omega \rho_{qp}(\omega), \]  

(51)

where

\[ f_{BE} = \frac{1}{e^{\omega / T} - 1} \]  

(52)

and

\[ \rho_{qp}(\omega) = \frac{1}{\pi} \int M(t, i \omega z) + M(t, -i \omega z) \]  

(53)

is the density of states of bosonic quasiparticles indicating that the Coulomb system is unstable with respect to the formation of particle-hole bound states.

2. Scaling results

Next, from the method of characteristics we can obtain the general scaling behavior of the quantities \( M \) and \( R \) as usual:

\[ M(t, \omega_s z) = M_0(t) G(\omega_s z \xi^d M_0(t)), \]  

\[ R(t, \omega_s z) = R_0(t) H(\omega_s z \xi^d R_0(t)). \]  

(54)

Here \( G \) and \( H \) are unspecified functions, whereas \( \xi, R_0 \) and \( M_0 \) each have a clear physical significance and are identified as the correlation length, the DC conductivity and \( \rho_{qp}(0) \) respectively. They obey the following equations

\[ (\mu \partial_t + \beta \partial_\xi) \xi(t) = 0, \]  

\[ (\beta \partial_t - 2 \epsilon - \beta / t) R_0(t) = 0, \]  

\[ (\beta \partial_t + \gamma) M_0(t) = 0. \]  

(55)

In the metallic phase (\( t < t_c \)) the solutions can be written as follows

\[ R_0(t) = (1 - t / t_c)^{2\nu}, \]  

\[ M_0(t) = (1 - t / t_c)^{\beta_0}, \]  

(56)

\[ \xi = \mu^{-1} t^{1/2\nu} (1 - t / t_c)^{-\nu}, \]  

(57)

where the critical exponents \( \nu \) and \( \beta_0 \) are obtained as

\[ \nu^{-1} = \beta_0 (t_c), \]  

\[ \beta_0 = -\nu \gamma(t_c). \]  

(58)

To second order in \( \epsilon \) the results are
\[ \nu^{-1} = 2\epsilon(1 + 2A\epsilon) \approx 2\epsilon + 6.56\epsilon^2 \]
\[ \beta_0 = \left(1 + (\pi^2/6 + 3 - 4A)\epsilon\right)/2 \approx 0.50 - 0.96\epsilon. \] (59)

Both the DC conductivity \( R_0 \) and the quantity \( M_0 \) vanish as one approaches the metal-insulator transition at \( t_c \). The results are quite familiar from the Heisenberg ferromagnet where \( M_0 \) stands for the spontaneous magnetization. Unlike the free electron gas\[\text{[54x-744]}\] however, the interacting system with Coulomb interactions has a true order parameter, \( M_0 \), which is associated with a non-Fermi liquid behavior of the specific heat.

3. Equations of state

The explicit results of Section A can be used to completely determine the quantities \( M \) and \( R \) in the Goldstone phase. They take the form of an "equation of state"\[\text{[54x-744]}\]
\[ \frac{\omega_{st} t}{M^\delta} = \left(\frac{t_c}{t}\right)^{1/\epsilon} \left(1 + (2\epsilon t - 1 - t/t_c) - 2\epsilon t_0 \Gamma(t/t_c)\right)^{1/\epsilon}. \] (60)

\[ \frac{\omega_{st} t}{R^\kappa} = \left(\frac{t_c}{t}\right)^{1/\epsilon} \left(1 - 1 - t/t_c\right)^{1/\epsilon}. \] (61)

Here, the exponents \( \delta \) and \( \kappa \) can be obtained from the values of \( \nu \) and \( \beta_0 \) following the relations
\[ d\nu = \beta_0(\delta + 1), \quad 2\epsilon \nu \kappa = \beta_0\delta. \] (62)

The universal features of the "equations of state" are the Goldstone singularities at \( t = 0 \) and the critical singularities near \( t_c \). As for the specific heat, we find the usual behavior \( c_v = \gamma_0 T \) at \( t = 0 \) but at criticality the following algebraic behavior is found \( c_v = \gamma_1 T^{1+\delta}/\delta \).

It is important to remark that the expression for the conductivity \( R \) can also be used in the case of finite temperatures and we may, on simple dimensional grounds, substitute \( T \) for \( \omega_{st} \). The results, however, strictly hold for the Goldstone and critical phases only. The "equations of state" cannot be analytically continued and used to obtain information on the insulating phase. As we already mentioned in the introduction, the strong coupling phase is controlled by different operators in the theory and has a distinctly different frequency and temperature dependence.\[\text{[54x-744]}\]

C. Plateau transitions in the quantum Hall regime

1. Introduction

In this Section we briefly describe how the results of this paper are extended to include the plateau transitions in the quantum Hall regime. For this purpose recall that the theory in two spatial dimensions and strong magnetic fields is given by
\[ S[Q, A] \rightarrow S[Q, A] + \frac{\sigma_{xy}^{\theta}}{8} \int_x \text{tr} \epsilon_{ij} Q[D_i, Q][D_j, Q]. \] (63)

The theory depends on the \( \theta \) term, or \( \sigma_{xy} \) term, in a non-perturbative manner and the general form of the renormalization group equations can now be written as
\[ \frac{d\sigma_{xx}}{d\ln \mu} = \beta_{xx}(\sigma_{xx}, \sigma_{xy}), \]
\[ \frac{d\sigma_{xy}}{d\ln \mu} = \beta_{xx}(\sigma_{xx}, \sigma_{xy}), \] (64)
\[ \frac{d\ln \gamma}{d\ln \mu} = \gamma(\sigma_{xx}, \sigma_{xy}). \]

The interesting physics actually occurs in the strong coupling phase \( (\sigma_{xx} < 1) \) where the crossover takes place from the perturbative regime of quantum interference effects, as studied in this paper, to the quantum Hall regime that generally appears in the limit of much larger distances only (Fig. 1).

As an important general remark we can say that the quantum Hall effect is a universal, strong coupling feature of the \( \theta \) term, or instanton vacuum, and fundamental aspects of the problem have not been recognized until recently. We mention in particular the fact that the theory displays massless excitations that always exist at the edge of the system.\[\text{[54x-744]}\] This new ingredient turns out to have fundamental consequences for longstanding problems such as the quantization of topological charge, the general meaning of instantons etc. Moreover, the concept of massless chiral edge modes can be used to unravel some of the outstanding strong coupling aspects of the theory such as the exact quantization of the Hall conductance.
which is represented by the infrared stable fixed points at $\sigma_{xx} = 0$ and $\sigma_{xy} = k$ in the scaling diagram of Fig. 1.

Perhaps more surprisingly, a gapless phase seems to always exist in the theory at $\theta = \pi$ or $\sigma_{xy}^* = k$ equal to an half-integer. This fundamental aspect of the quantum Hall effect is displayed even by the $CP^{N-1}$ theory with large values of $N$.\cite{3} These results indicate that the quantum Hall effect is a generic feature of the $\theta$ term in asymptotically free field theory and, contrary to the previous believes, the number of field components plays a secondary role only. Recall that the free electron theory, the Finkelstein approach and the $CP^{N-1}$ model with large $N$ are all topologically equivalent. They have important features in common such as asymptotic freedom and instantons. They are only different in the manner the number of field components in the theory is being handled. This does not affect the fundamentals of the quantum Hall effect, however, but only the critical singularities at $\theta = \pi$ which are different in each case.

2. Scaling of conductances

We next focus on the consequences of the unstable fixed points in Fig. 1, located at $\sigma_{xy} = k + \frac{1}{2}$ and $\sigma_{xx} = \sigma_{xx}^*$ which is of order unity. These fixed points describe the critical singularities of the quantum Hall plateau transitions.\cite{1} A finite value of $\sigma_{xx}^*$ indicates that we are dealing with a critical metallic state which is much the same phenomenon as the metal-insulator transition can be explored and investigated in detail.

Let us first recall the results for the conductances $\sigma_{xx}'$ and $\sigma_{xy}'$ as obtained in Ref. 8

$$\sigma_{xx}' = f_{xx}'[(zT)^{-\kappa}(\nu_B - \nu_B^*)],$$

$$\sigma_{xy}' = f_{xy}'[(zT)^{-\kappa}(\nu_B - \nu_B^*)].$$

Here, the functions $f_{xx}(X)$ and $f_{xy}(X)$ are regular (differentiable) functions for small $X$, $\nu_B = \sigma_{xx}^0 \propto 1/B$ is the filling fraction of the Landau levels and $\nu_B^* = k + 1/2$ is the critical value, corresponding to the center of the Landau band. The exponent $\kappa = p/2\nu \approx 0.42$ has been extracted from the experimental transport data taken from low mobility heterostructures in the quantum Hall regime.\cite{8}

Notice that the scaling variable $X$ can be expressed as $(h \xi)^{-1/\nu}$ where $h$ is the mass that is induced by finite temperatures (or frequency)

$$h' = (zT)^{\nu/2}, \quad (z\omega)^{\nu/2}. \quad (66)$$

The $\xi$ is the diverging correlation length at the center of the Landau band

$$\xi \propto |\nu_B - \nu_B^*|^{-\nu} = |\sigma_{xy} - k - \frac{1}{2}|^{-\nu}. \quad (67)$$

The critical exponent $\nu$ has the same meaning as before whereas $p$ was originally introduced as the inelastic scattering time exponent.\cite{3} Both are defined formally by the $\beta_{xy}$ and $\gamma$ functions according to

$$\nu^{-1} = \frac{\partial \beta_{xy}^*}{\partial \sigma_{xy}},$$

$$p = 1 + \frac{1}{\delta} = \frac{1}{1 + \gamma^* / 2},$$

(68)

where $\beta_{xy}^* = \beta_{xy}(\sigma_{xx}^*, k + 1/2)$ and $\gamma^* = \gamma(\sigma_{xx}^*, k + 1/2)$.\cite{3}

3. Particle-hole symmetry, duality

Generally speaking, one expects the functions $f_{xx}(X)$ and $f_{xy}(X)$ to be universal scaling functions, describing the points on the renormalization group trajectory that connects the unstable fixed points with the stable ones (Fig. 1).\cite{3} There is, however, interesting physics associated with this statement of universality and the subject is an extremely important objective for experimental research.

The problem with the plateau transitions is that although the macroscopic conductances $\sigma_{xx}'$ and $\sigma_{xy}'$ are well defined and sharply distributed at finite $T$, this is not the case for the mesoscopic conductances which are defined for finite lengthscales, of the order of the phase breaking length $1/h'$. The mesoscopic conductances are, in fact, broadly distributed and the size of the fluctuations is comparable or larger than the mean value. Since the $1/h'$ is the only length scale in the problem with Coulomb interactions and at finite $T$, it directly follows that the relation between the mesoscopic conductance distributions and the measured or macroscopic conductance must be non-trivial in general.\cite{3} For example, it is necessary to construct block models that describe the electron transport process in terms of a classical network of (mesoscopic) conductances that are randomly distributed over the different areas (blocks) in the system of size $1/h'$.

The concept of block models complicates such aspects like particle-hole symmetry that is displayed by the physical observables of the electron gas. Particle-hole symmetry, just like the quantization of the Hall conductance, is a direct consequence of one of the most fundamental aspects of the instanton vacuum, namely quantization of topological charge. It can be expressed as follows

$$f_{xx}(X) = f_{xx}(-X),$$

$$f_{xy}(X) = 2k + 1 - f_{xy}(-X) \quad (69)$$

More generally, one can show that particle-hole symmetry is displayed by the entire distribution functions of the mesoscopic conductances, rather than by the macroscopic quantities or averaged quantities alone.
It is clear that the theory of block models is particularly sensitive with regard to the many controversial issues that presently span the subject of mesoscopic fluctuations. It is important to keep in mind that the quantum Hall plateau transitions take place in precisely the regime ($\sigma_{xx} < 1$) where not only the conductance fluctuation are uncontrolled, but also the infinite set of higher dimensional operators that enters in the definition of the higher order moments of the distribution functions. Obviously, for the more difficult problems like quantum criticality in the presence of the Coulomb interactions, one can not just assume that the theory automatically takes care of itself in each and every fronts.

Following Kivelson et al., however, one can proceed in a pragmatic fashion and employ the Chern Simons mapping of abelian quantum Hall states to show that the system has a dual symmetry. Provided one works at finite $T$ and with system sizes that are much larger than $1/h'$, the mapping of conductances is not affected by the fluctuations that occur at mesoscopic length scales. By making furthermore use of particle-hole symmetry and by identifying the functions $f_{xx}(X)$ and $f_{xy}(X)$ as the subspace of conductances that is dual under the Chern Simons mapping, one arrives at the following result

$$f_{xx}(X) = \frac{g(X)}{1 + g^2(X)}$$

$$f_{xy}(X) = k + \frac{1}{1 + g^2(X)}$$ \hspace{1cm} (70)

where the function $g(X) = e^{a_1 X + a_3 X^3 + ...}$ obeys the general contraint

$$g(X) = g^{-1}(-X)$$ \hspace{1cm} (71)

These results imply that the sequence of plateau transitions in the quantum Hall regime end up at $k = 0$ in a so-called quantum Hall insulating phase which means that the Hall resistance $\rho_{xy}$ remains quantized throughout the lowest Landau level.

It is important to remark that the statement of duality has been carried out in a manner which is consistent with the gradient expansion that generally defines the effective action or sigma model approach. If, on the other hand, the effective action procedure were to fail and, say, terms of higher dimension would generally become important, then the statements made by Eqs. (70) and (71) would clearly have no meaning and the theory of quantum transport must be largely reconsidered.

With regard to the universality of the functions $f_{xx}(X)$ and $f_{xy}(X)$, the experimental situation has remained unresolved for a long time. However, recent experiments have clearly demonstrated that Eqs. (70) and (71) are valid, at least for the lowest Landau level. The transport data were taken from a low mobility InGaAs/InP heterostructure in strong magnetic fields and at low temperatures.

The new results indicate that the lack of universality, that was previously found, is merely the consequence of sample inhomogeneities. This means that there is little room left for the type of complications that arose in the perturbative theory of mesoscopic fluctuations. The experiments are in favor of duality as a fundamental symmetry of the electron gas with Coulomb interactions. As shown by Eqs. (70) and (71), this symmetry provides fundamental support for the results of the renormalization theory.

It should be mentioned that the Chern Simons mapping of conductances can be carried out for almost any type of disorder and duality by itself does therefore not provide any guarantee that the system is actually in a quantum critical state. For example, it is well known that complications arise in systems with long ranged potential fluctuations and the matter has been extensive addressed in Ref. 11.

4. Specific heat

As we have mentioned earlier, it is necessary to identify other physical observables in the problem that can in principle be measured and used to extract the value of $p$ and $\nu$ separately. The microscopic theory of the electron gas in $2 + 2\epsilon$ dimensions tells us that the natural quantity to consider is the specific heat, Eq. (51). Moreover, we have shown in Ref. 8 that this quantity is unchanged under the Chern Simons mapping.

By using our general knowledge on the renormalization group functions $\beta_{xx}$, $\beta_{xy}$ and $\gamma$ one can derive, in the standard manner, the scaling form of the quantity $M(\sigma_{xx}, \sigma_{xy}, \omega, \infty)$ in the quantum Hall regime. This leads to the same expression as in Eq. (24) with $M_0(t)$ now replaced by $M_0(\nu_B) = |\nu_B - \nu_B^*|^{\beta_0}$ and $\xi$ given as in Eq. (25). At the quantum critical point ($\nu_B = \nu_B^*$) we obtain the same non-Fermi liquid expression as before

$$c_v = \gamma_1 T^{\beta_0}.$$ \hspace{1cm} (72)

In different words, the physical observable, associated with the "inelastic scattering" exponent $p$ in quantum Hall systems, is none other than the specific heat of the electron gas. A measurement of $c_v$ should therefore provide the ultimate test on the consistency of the theory. This information is not present as of yet.

V. CONCLUSION

In this paper we have completed the two-loop analysis of the Finkelstein theory with the singlet interaction term. We have reported the detailed computations of the conductivity which is technically the most difficult part of the analysis. We have benifitted from the regularization procedure involving the $h_0$ field, which has substantially simplified the two-loop computations. Moreover, we have obtained a general relation between the effective masses
that are being induced by the $h_0$ field on the one hand, and the frequency $\omega_n$ on the other. This enables one to re-express the final answer in terms of finite frequencies and/or temperature, simply by a substitution of the $h_0$ regulating field.

By combining the concept of $F$ invariance with technique of dimensional regularization, we have extracted new physical information on the disordered electron gas with Coulomb interactions in low dimensions. In particular, we now have a non-Fermi liquid theory for the specific heat and dynamical scaling.

The metal-insulator transition in $2+2$ dimensions sets the stage for the plateau transitions in the quantum Hall regime. We have identified the specific heat $c_v$ as the physical observable that determines the exponent $p$, previously introduced as the exponent for "inelastic scattering."

As a final remark we can say that our knowledge of the theory is limited only by the accuracy with which one can give a numerical estimate of the critical exponents $\nu$ and $\eta$. Except for the fact that $p$ is bounded by $1 < p < 2\frac{d}{d-1}$, the detailed values of $\nu$ and $p$ can only be obtained by performing the renormalization group numerically. Notice that the situation is somewhat similar for the metal-insulator transition in $2+2\epsilon$ dimensions. In that case, the limitations of the expansion prevent us from having accurate exponents for the electron gas in three spatial dimensions.

VI. ACKNOWLEDGEMENT

We are indebted to E. Brézin and A. Finkelstein for numerous conversations. One of us (I.B.) is gratefully to M. Feigel’man, M. Lashkevich, D. Podolsky and P. Ostrovsky for stimulating discussions. The research has been supported in part by the Dutch Science Foundation FOM and by INTAS (Grant 99-1070).

VII. APPENDIX A

In this Appendix we present the final results for the various integrals listed in Eqs. (23)-(22). We shall follow the same methodology as used in the two-loop computation of Ref. 10 and employ the standard representation for the momentum and frequency integrals in terms of the Feynman variables $x_1, x_2$ and $x_3$. We classify the different contributions in Eqs. (23)-(22) in different categories, labeled $A$-integrals, $B$-integrals etc. In total we have seven different categories, i.e. $A, B, C, D$, $H, S$ and $T$ respectively, which are discussed separately in Sections A - G of this Appendix. The last Section, $H$, contains a list of abbreviations and a list of symbols for those integrals that need not be computed explicitly because their various contributions sum up to zero in the final answer.

In Appendix B we present the main computational steps for a specific example, the so-called $A_{10}$-integral. We show how the integral representation of hypergeometric functions can be used to define both the $\epsilon$ expansion and the limit where $\alpha \to 0$.

A. The $A$ - integrals

1. Definition

To set the notation, we consider the integral

$$X_{\nu,\eta}^{\mu} = -\frac{2\nu+1}{2\nu} \frac{a^{2+\mu}}{\sigma_0 d\nu} \int p^{2\nu} \sum_{k,m > 0} m^{\mu}$$

$$D_{\nu+\eta}(m) D_{\eta}^{q}(k) D_{\frac{1}{q} \nu+\eta}(k + m). \quad (A.1)$$

Here, the three indices $\mu, \nu$ and $\eta$ generally take on the values 0, 1. We shall only need those quantities $X_{\nu,\eta}^{\mu}$ which have $\eta = \nu$, however.

Using the Feynman trick, one can write (for the notation, see Section $H$)

$$X_{\nu,\eta}^{\mu} = -\frac{2\nu+1}{2\nu} \frac{a^{2+\mu}}{\sigma_0 d\nu} \int p^{2\nu} \int dm m^{\mu} \int dk$$

$$\Gamma(\mu + \eta + 1) \int \frac{1}{dz} \int x_2 x_3^{\mu+\eta}$$

$$\left[ h_0^2 + q^2 x_2 x_2 + p^2 x_2 x_3 + 2q \cdot q \right]^{\mu+\eta}$$

$$(A.2)$$

Next, by shifting $q \to q - px_1/x_2$, we can decouple the vector variables $p$ and $q$ in the denominator. The integration over $k, m, p$ and $q$ then leads to an expression that only involves the integral over $z$ and the Feynman variables $x_1, x_2$ and $x_3$. Write

$$X_{\nu,\eta}^{\mu} = \frac{\Omega h_0^{d-4}}{\sigma_0 \epsilon} A_{\mu,\eta}^{\nu} \quad (A.3)$$

then

$$A_{\mu,\eta}^{\nu} \frac{1}{dz} \int x_2 x_3^{1+\mu+\eta}(x_1 + x_2)^{\nu}(x_1 + x_2)^{-1-\nu-\epsilon} \frac{1}{(z x_2 + x_3)(x_3 x_3)^{1+\nu}}. \quad (A.4)$$

To complete the list of $A$-integrals, we next define quantities that carry either two indices $\mu, \nu$ or only a single index $\mu$. Like $A_{\mu,\eta}^{\nu}$, they all describe contractions that contain both momentum and frequency integrals. The results are all expressed in terms of integrals over $z, x_1, x_2$ and $x_3$.

$$A_{\mu}^{\nu} \frac{1}{dz} (z - \alpha)^{1+\nu-\alpha}$$

$$\times \int x_2 x_3^{2+\nu-\mu} x_3^{1-\mu}(x_1 + x_2)^{-2-\epsilon}$$

$$((\alpha x_2 + x_3)^{1+\nu}(z x_2 + x_3)^{1+\nu}), \quad (A.4)$$
\[ A_0 = \int_\alpha dz(z-\alpha) \frac{x_2^2 x_1(x_1 x_2)^{-2-\epsilon}}{(x_1 + x_2)(x_2 + \alpha x_1 + 2 x_3)}, \quad (A.5) \]

\[ A_1 = \int_\alpha dz(z-\alpha)^2 \frac{x_2^2(x_1 + x_3)(x_2 + x_4)}{(x_2 + x_3)^2} \times (x_1 x_2)^{-2-\epsilon} (\frac{1}{(\alpha x_1 + x_3)^2} - \frac{1}{(x_2 + \alpha x_1 + 2 x_3)^2}), \quad (A.6) \]

\[ A_2 = \int_\alpha dz(z-\alpha)(1-z) \frac{x_2^2(x_1 + x_3)(x_1 x_2)^{-2-\epsilon}}{(x_1 + x_2)(x_2 + \alpha x_1 + 2 x_3)}, \quad (A.7) \]

\[ A_3 = \int_\alpha dz(z-\alpha) \frac{x_2^2(x_1 + x_3)(x_1 x_2)^{-2-\epsilon}}{(x_2 + \alpha x_1 + 2 x_3)(x_2 + x_3)}. \]

2. \(\epsilon\) expansion

The calculation of integrals is straightforward but tedious and lengthy. Here we only present the final results of those quantities that are needed. The list does not contain the final answer for the \(A_0\)-integral because the various contributions to \(A_0\) sum up to zero in the final answer. These same holds for some other integrals that are defined in Section H and that we do not specify any further.

\[ A_{00}^0 = -\frac{\ln^2 \alpha}{\epsilon} + \zeta(3), \]

\[ A_{01}^0 = -\frac{\ln^2 \alpha + \ln \alpha}{\epsilon} - \frac{\ln^2 \alpha}{2} + \frac{\pi^2}{6} + \zeta(3), \]

\[ A_{01}^1 = \frac{\ln \alpha}{\epsilon} - \frac{\ln^2 \alpha}{2} - \frac{2 \ln \alpha - \pi^2}{3} + 1, \]

\[ A_{11}^1 = \frac{\ln \alpha}{\epsilon} - \frac{\ln^2 \alpha}{2} - \frac{2 \ln \alpha - \pi^2}{3}, \]

\[ A_{00} = \frac{\ln \alpha}{\epsilon} + \frac{\ln^2 \alpha}{2} + \frac{2 \ln \alpha + \pi^2}{3} - 1, \]

\[ A_{10} = -\frac{1}{\alpha} - \frac{2 \ln \alpha + 3}{\epsilon} - \ln^2 \alpha - 5 \ln \alpha - \frac{2 \pi^2}{3} + 3, \]

\[ A_{01} = -\ln \alpha - \frac{\pi^2}{6} + 1, \]

\[ A_{11} = \frac{\ln \alpha + 2}{\epsilon} + \frac{\ln^2 \alpha}{2} + 3 \ln \alpha + \frac{\pi^2}{2}, \quad (A.8) \]

\[ A_1 = -\frac{2}{\alpha} + \frac{2 \ln^2 \alpha + 4 \ln \alpha}{\epsilon} - 3 \ln^2 \alpha \]

\[ + 8 \ln 2 \ln \alpha - \frac{17}{2} \ln \alpha + 4 K_1(\alpha) + 8 J'_1(\alpha) \]

\[ - \pi^2 - 2 \zeta(3) - 6 \ln^2 2 + 10 \ln 2 - \frac{1}{2}. \quad (A.9) \]

\[ A_2 = -\frac{\ln^2 \alpha + 2 \ln \alpha}{\epsilon} - 2 \ln \alpha - 3 \ln 2 \ln \alpha \]

\[ - J_1(\alpha) - K_1(\alpha) - 2 J'_1(\alpha) + A_0 - \frac{\pi^2}{6} \]

\[ + 1 + \zeta(3) + 3 \ln^2 2 - 3 \ln 2 - 3 \zeta(2)/2, \quad (A.10) \]

\[ A_3 = A_0 - 2 L_2(\frac{1}{2}) + \frac{\pi^2}{6}. \quad (A.11) \]

**B. The \(B\)-integrals**

1. \(B\)-definition

The \(B\)-integrals are similarly defined in terms of the variables \(z, x_1, x_2\) and \(x_3\). However, they describe only those contractions that contain frequency sums and no momentum integrals.

\[ B_\mu = \int_\alpha \frac{dz}{z^\mu} \frac{x_1^{\mu-1}x_2x_3^{\mu-\epsilon}(x_1 + x_2)^{-\mu-\epsilon}}{(ax_2 + zzx_3 + x_1)}, \quad (A.12) \]

2. \(\epsilon\) expansion

\[ B_1 = \frac{\ln \alpha}{\epsilon} + \frac{\ln^2 \alpha}{2} + \ln \alpha, \]

\[ B_2 = -\frac{1}{\alpha} + \frac{\ln^2 \alpha}{\epsilon} + \frac{2 \ln \alpha}{\epsilon} - 2 \ln \alpha - 2. \quad (A.13) \]

**C. The \(C\)-integrals**

1. \(C\)-definition

The \(C\)-integrals contain one additional integration over \(y\), besides the ones over \(z\) and the Feynman variables \(x_1, x_2\) and \(x_3\). They originate from expressions involving integrations over both frequencies and momenta.

We distinguish between quantities with two indices \(\mu\) and \(\nu\)

\[ C_{\mu\nu} = \int dzdy \frac{x_1^\mu x_2x_3(x_2 + x_3)^{1-\mu}(x_1 x_2)^{-2-\epsilon}}{(zx_3 + x_1)yx_2 + x_1)^{\nu}(zx_3 + yx_2)^{1-\nu}} \quad (A.14) \]
and those that carry only a single index $\nu$

$$C_\nu = \int_\alpha^1 dz (1-z)^\nu \int_\alpha^1 dy \times \int \left[ \frac{x_2^{2-\nu} x_3^{1+\nu} (x_1 + x_2)^\nu (x_1 x_3)^{-2-\epsilon}}{(z x_3 + x_1)(y x_2 + x_1)^\nu (z x_3 + y x_2 + 2 x_1)} \right]. \quad (A.15)$$

2. $\epsilon$ expansion

$$C_{00} = \frac{\ln \alpha}{\epsilon} - \frac{\ln^2 \alpha}{2} - 2 \ln \alpha + \frac{\pi^2}{4} \ln 2 - \frac{\pi^2}{6} + \frac{15}{4} \zeta(3)$$

$$- \frac{\pi^4}{24} - \frac{\pi^2}{6} \ln 2 + \frac{1}{6} \ln^4 2 + \frac{7}{2} \zeta(3) \ln 2 + 4Li_4(\frac{1}{2}),$$

$$C_{01} = \frac{2 \ln \alpha}{\epsilon} - \ln^2 \alpha - 4 \ln \alpha - 2 - \zeta(3),$$

$$C_{11} = \zeta(3),$$

$$C_0 = \frac{\ln \alpha}{\epsilon} - \frac{\ln^2 \alpha}{2} - 2 \ln \alpha - 1 - \zeta(3) - C_0' - 1,$$

$$C_1 = 4 \ln 2 \ln \alpha + 2J_1(\alpha) - C_1' - 2 - \frac{\zeta(3)}{2},$$

$$- 4 \ln 2 - \frac{\pi^2}{6} + 4G, \quad (A.16)$$

where the Catalan constant $G = 0.517\ldots$ appears as the integral

$$G = - \int_0^1 du \frac{\ln u}{1+u^2}. \quad (A.17)$$

D. The D-integrals

1. Definition

These are integrals over the Feynman variables only. They originate from the contractions which contain sums over both momenta and frequencies.

$$D_\nu = \int \left[ \frac{x_2^{\mu} (x_1 + x_2)^{\nu-1} (x_1 x_3)^{-\nu-\epsilon}}{(\alpha x_1 + x_3)(\alpha x_2 + x_3)} \right]. \quad (A.18)$$

2. $\epsilon$ expansion

$$D_1 = - \ln^2 \alpha - \frac{\pi^2}{6},$$

$$D_2 = - 2 \ln \alpha. \quad (A.19)$$

E. The H - integrals

1. Definition

The $H$-integrals involve the variable $z$ and the Feynman variables. All of them originate from contractions with sums over both momenta and frequencies.

$$H_\nu = \int_\alpha^1 dz (z - \alpha)^{2\nu} \int \left[ \frac{x_2^{2+\nu} (x_1 + x_3) (x_1 x_3)^{-2-\epsilon}}{(\alpha x_1 + z x_2)(\alpha x_2 + x_3)} \right]. \quad (A.19)$$

2. $\epsilon$ expansion

$$H_0 = - \ln \alpha + 1,$$

$$H_1 = - \ln \alpha. \quad (A.20)$$

F. The S - integrals

1. Definition

These are integrals over the Feynman variables only and they do not not contain the parameter $\alpha$. All of them originate from the expressions with sums over both momenta and frequencies.

$$S_{\mu\nu} = \int \left[ \frac{x_2^{\mu} x_3^{1+\nu} ((2 - \nu - \mu)x_1 + x_3)(x_1 x_3)^{-2-\epsilon}}{(x_2 + x_3)^{1+\nu}} \right]. \quad (A.21)$$

$$S_\nu = \int \left[ \frac{(x_1 + x_2)^{-1+2\nu} (x_1 x_3)^{-1-\nu-\epsilon}}{(x_1 + x_2)^{-1+2\nu}} \right]. \quad (A.22)$$

2. $\epsilon$ expansion

$$S_{00} = - \frac{1}{\epsilon} + 2,$$

$$S_{01} = - \frac{1}{3\epsilon} + \frac{8}{9},$$

$$S_{11} = - \frac{1}{6\epsilon} + \frac{1}{9},$$

$$S_0 = - \frac{1}{\epsilon} + 2,$$

$$S_1 = - \frac{2}{\epsilon} + 2. \quad (A.23)$$
G. The T-integrals

1. Definition

The integrals are over the Feynman variables only. They come from the expressions which only contain sums over frequency.

\[ T_{\mu\nu}^0 = \frac{(1 - \alpha)^2}{\alpha^3} \int \frac{x_1^{2-\eta_s}\eta^\nu_x - 1}{x_1 + x_2 + x_3 + x_4} \] (A.24)

\[ T_{\mu\nu} = \int \frac{x_1^{2-2\nu-2} + x_2^{2-2\nu-2} - x_1 - x_2 - x_3 + (\alpha + \mu)x_2}{x_3^2 + (1 + \mu)x_3 + x_1} \] (A.25)

2. \( \epsilon \) expansion

\[ T_{10}^0 = -\frac{1}{\alpha} + 1, \]
\[ T_{11}^0 = -\frac{1}{\alpha} + \frac{1}{\epsilon} + \ln \alpha + 1, \]
\[ T_{20}^0 = \frac{1}{6\alpha^2} - \frac{3\alpha}{\epsilon} - \ln \alpha - \frac{11}{12\epsilon} \]
\[ T_{21}^0 = \frac{1}{6\alpha^2} + \frac{2\ln \alpha + 5/2}{\epsilon} + \frac{\ln^2 \alpha}{2} + 4\ln \alpha + \frac{17}{12\epsilon} \]
\[ T_{10}^1 = -\frac{1}{\alpha} - 2\ln \alpha - 2, \]
\[ T_{01} = \frac{\ln \alpha}{\epsilon} - \frac{\ln^2 \alpha}{2}, \]
\[ T_{02} = \frac{1}{\epsilon}, \]
\[ T_{12} = -\frac{3\ln \alpha + 11/2}{\epsilon} + \frac{3\ln^2 \alpha}{2} + \frac{9\ln \alpha}{2} - 4\ln 2\ln \alpha + \frac{\pi^2}{6} - 4\ln^2 \frac{1}{2} - 12\ln 2 + \frac{27}{4}. \] (A.26)

H. List of symbols and abbreviations

\[ \int_{x_i}^{x_j} = \int_{x_i}^{x_j} \int_{x_2}^{x_j} \int_{x_3}^{x_4} \delta(x_1 + x_2 + x_3 + 1), \]
\[ x_i = x_i + x_j, \]
\[ x_i = x_i + x_j + x_2x_3 + x_3x_1. \] (A.27)

\[ K_1(\alpha) = \frac{1}{\alpha} \int \frac{dx}{\int_{x_i}^{x_j} x_3(x_1 + x_3)(x_i x_j - 2 - \epsilon)} \]
\[ \int \frac{1}{\int_{x_i}^{x_j} (x_2 + x_3)(x_1 + x_2 + x_3 + 1)} \] (A.28)

VIII. APPENDIX B

In this appendix we present the calculation of the integral \( A_{10} \) as a typical example. We start with the integral

\[ X_{10} = -\frac{32\alpha^3}{\epsilon} \int_{p_0}^{\infty} \int_{m_0}^{\infty} \int_{k_0}^{\infty} \int_{q_0}^{\infty} \frac{m}{p_0 + m_0 + k_0 + q_0} D(p_0 + m_0 + k_0 + q_0). \] (B.1)

Using the Feynman trick, one can write

\[ X_{10} = -\frac{16\alpha^3}{\epsilon} \int_{p_0}^{\infty} \int_{m_0}^{\infty} \int_{k_0}^{\infty} \int_{q_0}^{\infty} \frac{m}{p_0 + m_0 + k_0 + q_0} \Gamma(6) \int dz \frac{(z - \alpha)^2}{\alpha} \int \int \left[ h_0^2 + q^2 x_1 + p^2 x_2 + 2p \cdot q x_1 + an x_1 x_3 + ak x_2 x_3 \right] \] (B.2)

Shifting \( q \rightarrow q - px_1/x_3 \), we can decouple \( p \) and \( q \) in the denominator. We are then able to perform the integration over \( k, p, m, q, \) and \( \alpha \), resulting in

\[ X_{10} = \frac{4\alpha^3}{\epsilon} A_{10}, \]

where

\[ A_{10} = \frac{1}{\alpha} \int dz (z - \alpha)^2 \int \frac{1}{\alpha} \frac{x_3(x_1 + x_3)(x_1 x_2 - 2 - \epsilon)}{(x_2 + x_3)(x_1 + x_3)^2} \] (B.3)

Next we write the integral as a sum of four terms

\[ A_{10} = \frac{1}{\alpha} \int dz (z - \alpha)^2 \int \frac{1}{\alpha} \frac{x_3(x_1 + x_3)(x_1 x_2 - 2 - \epsilon)}{(x_2 + x_3)(x_1 + x_3)^2} \]
\[ \times \left[ 1 - x_1 x_3 x_2 (x_i x_j) - x_3(x_1 + x_3)(x_i x_j)^{-1} + x_3^2(x_1 + x_3)(x_i x_j)^{-1} \right] = I_0 - I_1 - I_2 + I_3. \] (B.3)

In what follows we retain the full \( \epsilon \) dependence in the \( I_0 \), \( I_1 \) and \( I_2 \) and it suffices to put \( \epsilon = 0 \) in the fourth piece \( I_3 \). Introducing a change of variables
\[
x_1 = \frac{u}{s + 1}; \quad x_2 = \frac{s}{s + 1}; \quad x_3 = \frac{1 - u}{s + 1},
\]
where \(0 < s \ll 0 < u < 1\), then the four different pieces can be written as follows

\[
I_0 = \left(\frac{1}{2} - 2\alpha\right) \int_0^1 \frac{du}{\left(\alpha u + 1 - u\right)^2} \int_0^{\infty} ds \frac{s(s + 1)^{2\epsilon}}{(s + u(1 - u))^{1+\epsilon}},
\]
\[
I_1 = \frac{1}{2} \int_0^1 \frac{u(1 - u)}{(\alpha u + 1 - u)^2} \int_0^{\infty} ds \frac{s(s + 1)^{2\epsilon}}{(s + u(1 - u))^{2+\epsilon}},
\]
\[
I_2 = \frac{1}{2} \int_0^1 \frac{(1 - u)}{(\alpha u + 1 - u)^2} \int_0^{\infty} ds \frac{s(s + 1)^{2\epsilon}}{(s + u(1 - u))^{2+\epsilon}},
\]
\[
I_3 = \int_0^{\infty} \frac{dz}{\left(\frac{z - \alpha}{z}\right)^2} \int_0^{\infty} du \frac{u(1 - u)^2}{(\alpha u + 1 - u)^2} \int_0^{\infty} ds \frac{(s + 1 - u)}{(s + u(1 - u))^{2(\epsilon s + 1 - u)^2}}.
\]  

The integrals over \(s\) in Eq. \(\text{[B.4]}\) can now be recognized as integral representations of the hypergeometric function \(\text{}_2F_1\). Write

\[
I_0 = \left(\frac{1}{2} - 2\alpha\right) \int_0^1 \frac{du}{\left(\alpha u + 1 - u\right)^2} \left[\frac{1}{1 + \epsilon} G_0(u(1 - u)) + \frac{1}{\epsilon} G_1(u(1 - u))\right],
\]
\[
I_1 = \frac{1}{2} \int_0^1 \frac{u(1 - u)}{(\alpha u + 1 - u)^2} \left[\frac{1}{1 + \epsilon} G_1(u(1 - u)) - \frac{1}{1 - \epsilon} G_2(u(1 - u))\right],
\]
\[
I_2 = \frac{1}{2} \int_0^1 \frac{(1 - u)}{(\alpha u + 1 - u)^2} \left[\frac{1}{1 + \epsilon} G_1(u(1 - u)) - \frac{1}{1 - \epsilon} G_2(u(1 - u))\right],
\]
\[
I_3 = \int_\alpha^{\infty} \frac{dz}{z} \left(\frac{z - \alpha}{z}\right)^2 \int_0^{\infty} du \frac{u}{2u + 1 - u} \left[\frac{1}{2} H_3(1 - \alpha u) + \frac{(1 - u)\Gamma(3)}{u\Gamma(4)} H_4(1 - \alpha u)\right],
\]
then, in the limit where \(\epsilon \rightarrow 0\), we can identify the functions \(G_i\) and \(H\) as follows

\[
G_0(1 - z) = \text{}_2F_1(1, -2\epsilon, -\epsilon; z) = \frac{1 + z}{1 - z},
\]
\[
G_1(1 - z) = \text{}_2F_1(1, -2\epsilon, 1 - \epsilon; z) = 1 + 2\epsilon \ln(1 - z),
\]
\[
G_2(1 - z) = \text{}_2F_1(1, -2\epsilon, 2 - \epsilon; z) = \frac{2}{\epsilon} \ln(1 - z) + z - z^2/2.
\]

Using these results we obtain

\[
I_0 = \frac{1}{\epsilon} - \frac{\ln \alpha + 2}{\epsilon} - \frac{2\ln \alpha - \pi^2}{3},
\]
\[
I_1 = \frac{\ln \alpha + 2}{\epsilon} + \frac{\ln^2 \alpha}{2} + 2\ln \alpha + \frac{\pi^2}{3},
\]
\[
I_2 = \frac{\ln \alpha + 1}{\epsilon} + \frac{\ln^2 \alpha}{2} + 2\ln \alpha + \frac{\pi^2}{3} + 1,
\]
\[
I_3 = -\ln \alpha.
\]

The final answer is therefore

\[
A_{10} = -\frac{1}{\alpha} - \frac{2\ln \alpha + 3}{\epsilon} - \frac{\ln^2 \alpha - 5\ln \alpha}{\epsilon} - \frac{2\pi^2}{3} + 3.
\]
11 A.M.M. Pruisken, B. Škorić, and M.A. Baranov, Phys. Rev. B 60, 16838 (1999).
12 A.M.M. Pruisken, M.A. Baranov, and I.S. Burmistrov, e-print cond-mat/0104387.
13 A.M. Finkelstein, Pis'ma Zh. Eksp. Teor. Fiz. 37, 436 (1983) [JETP Lett. 37, 517 (1983)]; Zh. Eksp. Teor. Fiz. 86, 367 (1984) [Sov. Phys. JETP 59, 212 (1984)]; Physica B 197, 636 (1994).
14 A.M.M. Pruisken, Nucl. Phys. B 235, 277 (1984); 285, 719 (1987); 290, 61 (1987); Phys. Rev. B 31, 416 (1985); H. Levine, S. Libby, and A.M.M. Pruisken, Phys. Rev. Lett. 51, 20 (1983); Nucl. Phys. B 240 [FS12], 30, 49, 71 (1984).
15 See o.g. O. Heinonen, ed., Composite fermions (World Scientific) 1998.
16 T.R. Kirkpatrick and D. Belitz, Phys. Rev. B 41, 11082 (1990).
17 A. Kamenev and A. Andreev, Phys.Rev. B 60, 2218 (1999).
18 C. Chamon, A.W.W. Ludwig, and C. Nayak, Phys. Rev. B 60, 2239 (1999).
19 E. Brézin, S. Hikami, and J. Zinn-Justin, Nucl. Phys. B 165, 528 (1980).
20 F. Wegner, Nucl. Phys B 280, 210 (1987).
21 E. Brézin, J. Zinn-Justin, and J.C. Le Guillon, Phys. Rev. D14, 2615 (1976); E. Brézin, J. Zinn-Justin, Phys. Rev. B14, 3110 (1976).
22 A.M.M. Pruisken and M.A. Baranov, unpublished.
23 M.H. Cohen and A.M.M. Pruisken, Phys. Rev. B 49, 4593 (1994).
24 A.M.M. Pruisken, M.A. Baranov, and M. Voropaev, e-print cond-mat/0101003.
25 A.L. Efros and B.I. Shklovskii, J. Phys. C 8, L49 (1975).
26 A.M.M. Pruisken and Z. Wang, Nucl. Phys. B 322, 721 (1989).
27 B.L. Altshuler, V.E. Kravtsov, and I.V. Lerner, in Mesoscopic Phenomena in Solids, ed. B.L. Altshuler, P.A. Lee, and R.A. Webb, North Holland, New York, p. 449 (1991); F. Wegner, Nucl. Phys B 354, 441 (1991).
28 S. Kivelson, D.H. Lee, and S.C. Zhang, Phys. Rev. B 46, 2223 (1992).