The intersection number of two curves and their subsurface projection distance

Yohsuke Watanabe

Abstract

Suppose $\xi(S) \geq 1$. Let $x, y \in C(S)$, we recall the following classical inequality which goes back to the work of Lickorish [Lic62] and stated by Hempel [Hem01] later on,

$$d_S(x, y) \prec \log i(x, y).$$

In this paper, we improve on the inequality. We show that for sufficiently large $k$,

$$\log i(x, y) \leq \sum_Z [d_Z(x, y)]_k + \sum_A \log [d_A(x, y)]_k$$

where $Z$ ranges over all subsurfaces in $S$ which are not annuli, and $A$ ranges over all annuli in $S$.

1 Introduction

Let $S = S_{g,n}$ be a surface with $g$ genus and $n$ boundary components, $\xi(S_{g,n}) = 3g + n - 3$ be the complexity and $\chi(S_{g,n})$ be the Euler characteristic of $S_{g,n}$. We assume all curves are simple, closed, essential, and non-peripheral. Also we assume all arcs are simple, essential and non-peripheral and the isotopy of arcs is relative to the boundaries setwise unless we say relative to the boundaries pointwise.

Harvey associated the set of curves in a surface with the simplicial complex, the curve complex [Har81]. Suppose $\xi(S) \geq 1$. The vertices are isotopy classes of curves and the simplices are collections of curves that mutually intersect the minimal possible number. We also review the arc complex, $A(S)$ and the arc and curve complex, $AC(S)$. Suppose $\xi(S) \geq 0$, the vertices of $A(S)$ ($AC(S)$) are isotopy classes of arcs (arcs and curves) and the simplices are collections of arcs (arcs and curves) that can be mutually realized to be disjoint in $S$. In this paper, $a \in C(S)$, $a \in A(S)$ and $a \in AC(S)$ means that $a$ is an element of 0-skelton in each complex.

*The author was partially supported from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: Geometric Structures and Representation Varieties" (the GEAR Network) while he was visiting Brown University.
We observe that every complex above is a geodesic metric space with the simplicial metric. If \( x, y \in C(S) \) we let \( d_S(x, y) \) be the length of a geodesic between \( x \) and \( y \). If \( A, B \subset C(S) \), we define \( d_S(A, B) = \max_{a \in A, b \in B} d_S(a, b) \). Suppose \( x, y \in AC(S) \), the intersection number, \( i(x, y) \) is the minimal geometric intersection number under the isotopy classes of \( x \) and \( y \), we say \( x \) and \( y \) are in minimal position if they realize the intersection number.

Our main result in this paper is to understand the relation between the intersection number of two curves and their subsurface projection distance. We review the following classical inequality, which goes back to the work of Lickorish [Lic62] and explicitly stated by Hempel.

**Lemma 1.1** ([Hem01]). Let \( x, y \in C(S) \) such that \( i(x, y) > 0 \), then \( d_S(x, y) \leq 2 \log_2 i(x, y) + 2 \).

We observe that the above inequality is not sharp. Indeed, we improve on Lemma 1.1 so that we have an asymptotic equality with the consideration of all subsurfaces which two curves project and their subsurface projection distance is sufficiently large. For this, we prove Theorem 1.3. First, we define notations.

**Definition 1.2.** Suppose \( n, m \in \mathbb{R} \).

- We denote \( n \overset{K,C}{=} m \) if there exist \( K \geq 1 \) and \( C \geq 0 \) such that \( \frac{m}{K} - C \leq n \leq Km + C \). We call \( K \) a multiplicative constant and \( C \) an additive constant. We also use \( n \asymp m \) instead of \( n \overset{K,C}{=} m \) when the constants are not emphasized. Similarly, we use \( n \preceq m \) for \( n \leq Km + C \) when the constants are not emphasized.
- We define \([n]_m = 0 \) if \( n \leq m \) and \([n]_m = n \) if \( n > m \). We call \( m \) a cut-off number.
- Let \( M \) be from the Bounded Geodesic Image Theorem. We define the cut-off function \( c(n) = Mn + 2M \).

For the rest of this paper if the base of log function is not specified we always assume it is bigger than 1. We state our main theorem.

**Theorem 1.3.** Suppose \( \xi(S) \geq 1 \). For any \( x, y \in C(S) \), if \( k \geq c(\xi(S)) \) then

\[
\log i(x, y) \asymp \sum_Z [d_Z(x, y)]_k + \sum_A \log [d_A(x, y)]_k,
\]

where \( Z \) ranges over all subsurfaces in \( S \) which are not annuli, and \( A \) ranges over all annuli in \( S \).

We remark that for a fixed cut-off number the additive and multiplicative constants will depend only on the topology of the surface. For a simpler presentation of this paper, we often avoid keeping track of the additive and multiplicative constants. Instead, we use the notation \( \asymp \) and \( \preceq \).

In [CR07], Choi-Rafi showed the above formula when \( x \) and \( y \) are two markings, by understanding \( x \) and \( y \) as short markings of two points in the thick part of the Teichmüller space.
and using the combinatorial formula for the Teichmüller distance between two points in the thick part of the Teichmüller space, which was derived by Rafi [Raf07]. Choi-Rafi formula and our formula can each be derived from the other, yet the proofs are done by different approaches.

In the beginning of this project, the author was interested in understanding the followings.

**Lemma 1.4.** Suppose $\xi(S) \geq 1$ and $x, y \in C(S)$.

1. There exists a computable $L_S(d) > 0$ such that if $d_Z(x, y) \leq d$ for all proper subsurfaces $Z$ then $\log L_S(d) i(x, y) \leq d_S(x, y)$.

2. There exists a computable $N_S(d) > 0$ such that if $i(x, y) \geq N_S(d)$ then $d_Z(x, y) \geq d$ for some $Z \subseteq S$.

The idea which arose in the proof of Lemma 1.4 lead the proof of Theorem 1.3. Therefore, even though Lemma 1.4 follows from Theorem 1.3, we also prove the first statement of Lemma 1.4 as warm up. (The proof of the second statement would be the restatement of that of the first statement.)

### 1.1 Plan of the paper

This paper is organized as follows. In Section 2, we review subsurface projections and tight geodesics defined by Masur-Minsky [MM00]. Both proofs of Theorem 1.3 and Lemma 1.4 will be done by the inductive arguments on the complexity of a surface (Theorem 1.3 will be proved by the double induction on the complexity of a surface and the distance between two curves). Therefore, in section 3 we make an observation of the geometry of the curve graphs of $S_{1,1}$ and $S_{0,4}$ as well as its applications with the Bounded Geodesic Image Theorem, which will the key for the base cases of both Lemma 1.4 and Theorem 1.3. In section 4, we prove Lemma 1.4. In Section 5, we prove Theorem 1.3.

**Acknowledgements**

The author primarily thanks Kenneth Bromberg for suggesting this project and useful discussions as well as continuous feedbacks throughout this paper. The author also thanks Mladen Bestvina for suggesting the statement of Theorem 1.3 and Richard Webb for insightful discussions. Finally, the author thanks Tarik Aougab and Samuel Taylor for useful conversations.

Much of this paper was written while the author was visiting Brown University under the supervision of Jeffrey Brock, the author thanks the hospitality of his and the institute.

### 2 Background

In this section we review subsurface projections and tight geodesics from [MM00]. We remark that tight geodesics will be an important tool for our inductive proofs for Lemma 1.4 and Theorem 1.3.
2.1 Subsurface projection

Suppose $A$ is a subset of $S$, then we let $R(A)$ be the regular neighborhood of $A$ in $S$. Let $\mathcal{P}(C(S))$ and $\mathcal{P}(AC(S))$ be the set of finite subsets in each complex. We define the subsurface projection. Let $Z$ be a subsurface of $S$ such that $\xi(Z) \neq 0$.

Suppose $\xi(Z) > 0$. Let $x \in AC(S)$ and assume $x$ and $\partial Z$ are in minimal position then we define the map, $i_Z : AC(S) \to \mathcal{P}(AC(Z))$ such that if $x \in AC(S)$ then $i_Z(x)$ is the set of arcs obtained by $x \cap Z$. Also we define the map, $p_Z : AC(Z) \to \mathcal{P}(C(Z))$ such that if $x \in AC(Z)$ then $p_Z(x) = \partial R(x \cup z \cup z')$ where $z, z' \subset \partial Z$ which contain $\partial x$. ($z$ could be same as $z'$). Also if $x \in C(Z)$ then $p_Z(x) = \partial R(x)$, which is just $x$.) We observe $|\{p_Z(x)\}| \leq 2$. If $C \subset AC(S)$, we define $p_Z(C) = \cup_{c \in C} p_Z(c)$.

**Definition 2.1.** Suppose $Z$ is a subsurface of $S$ such that $\xi(Z) > 0$. Then the subsurface projection to $Z$ is the map, $\pi_Z = p_Z \circ i_Z : AC(S) \to \mathcal{P}(C(Z))$. If $C \subset AC(S)$, we define $\pi_Z(C) = \cup_{c \in C} \pi_Z(c)$.

Now, we observe the lemma which follows, with this we define the subsurface projection distance.

**Lemma 2.2.** Suppose $Z$ is a subsurface of $S$ such that $\xi(Z) > 0$ and $x \in AC(S)$ then $|\{i_Z(x)\}| \leq 3|\chi(S)|$ and $|\{\pi_Z(x)\}| \leq 6|\chi(S)|$.

**Proof.** The first statement follows from $\dim(A(Z)) < 3|\chi(Z)| \leq 3|\chi(S)|$ which could be proved by a simple Euler characteristic argument. For instance, see [KP10].

**Definition 2.3.** Suppose $Z$ is a subsurface of $S$ such that $\xi(Z) > 0$. If $A, B \subset AC(S)$, we define $d_Z(A, B) = \max_{a \in \pi_Z(A), b \in \pi_Z(B)} d_Z(a, b)$.

Suppose $\xi(Z) < 0$ then $Z$ is an essential annulus in $S$. Fix a hyperbolic metric on $S$, compactify the cover of $S$ which corresponds to $\pi_1(Z)$ with its Gromov boundary and denote the resulting surface $S^Z$. We define the vertices of $C(Z)$ to be the set of the isotopy classes of the arcs whose endpoints lie on two boundaries of $S^Z$, here the isotopy is relative to $\partial S^Z$ pointwise. Two vertices of $C(Z)$ are distance one apart if they are disjoint in the interior of $S^Z$.

The subsurface projection to $Z$ is the map, $\pi_Z : AC(S) \to \mathcal{P}(C(Z))$ such that if $b \in AC(S)$ then $\pi_Z(b)$ is the set of all arcs obtained by the lift of $b$ which connects two boundaries of $S^Z$. As in the previous case, if $C \subset AC(S)$, we let $\pi_Z(C) = \cup_{c \in C} \pi_Z(c)$. If $A, B \subset AC(S)$, we let $d_Z(A, B) = \max_{a \in \pi_Z(A), b \in \pi_Z(B)} d_Z(a, b)$. Now, we observe the following lemma for annular projections.

**Lemma 2.4.** [MM00] Suppose $Z$ is an essential annulus in $S$ and the core curve of $Z$ is $a \in C(S)$. Let $T_a$ be the dehn twist of $a$. If $b \in C(S)$ is such that $\pi_Z(b) \neq \emptyset$ then $d_Z(b, T^n_a(b)) = |n| + 2$ for $n \neq 0$. If $b$ intersects a exactly twice with opposite orientation, a half twist to $b$ is well defined to obtain a curve $H_a(b)$, which is taking $a \cup b$ and resolving the intersections in a way consistent with the orientation (see [Luo10] for a generalization). Then $H^n_a(b) = T_a(b)$, and $d_Z(b, H^n_a(b)) = \lfloor|n|/2\rfloor + 2$ for $n \neq 0$. 


If $Z \subset S$ and $x \in AC(S)$ such that $\pi_Z(x) = \emptyset$ then we say $x$ misses $Z$ or $Z$ misses $x$.

We recall an important geometric property of the the 1-skeleton of the curve complex, the curve graph. The curve graph is hyperbolic in the sense of Gromov, which was first proved by Masur and Minsky [MM99]. We also refer to [Bow06] [Ham07]. Recently, the hyperbolicity constant was shown to be uniform for all surfaces.

**Theorem 2.5.** The curve graph is uniformly hyperbolic.

*Proof.* We refer the proofs by Aougab [Aou], by Bowditch [Bow], by Clay, Rafi and Schleimer [MCS] and by Hensel, Przytycki and Webb [HPW].

Lastly, we observe the Bounded Geodesic Image Theorem which was first proved by Masur and Minsky [MM00] and recently by Webb [Web] by more direct approach.

**Theorem 2.6.** (Bounded Geodesic Image Theorem) Let $\delta$ be a hyperbolicity constant of the curve graph of $S$. There exists $M(\delta)$ such that if $\{x_i\}_{0}^{n}$ is a geodesic and $\pi_Z(x_i) \neq \emptyset$ for $Z \subset S$ for all $0 \leq i \leq n$ then $d_Z(x_0, x_n) \leq M$.

In the rest of the paper, we mean $M$ as $M$ in the statement of the Bounded Geodesic Image Theorem.

### 2.2 Tight geodesics

A multicurve is the set of curves that form a simplex in the curve complex.

**Definition 2.7.** Suppose $V$ and $W$ are multicurves. We say $V$ and $W$ fill $S$ if there is no curve in the components of the complement of $R(V \cup W)$ in $S$. Note that we can construct the smallest essential subsurface, $F(V,W)$ which contains $V$ and $W$ and they fill by taking $R(V \cup W)$ and filling in a disk for every component of the complement of the complement of $R(V \cup W)$ in $S$ which is a disk.

**Lemma 2.8.** If $\xi(S) > 1$, then $V$ and $W$ fill $S$ if and only if $d_S(V,W) > 2$.

*Proof.* $V$ and $W$ fill $S$ if and only if any curve which is not contained in $V$ or $W$ intersects some curve in $V$ or $W$. \qed

Now, we define tight geodesics.

**Definition 2.9.** A multigeodesic is a sequence of multicurves $\{V_i\}$ such that $d_S(a,b) = |p-q|$ for all $a \in V_p$ and $b \in V_q$ for all $p, q$ such that $p \neq q$. A tight multigeodesic is a multigeodesic $\{V_i\}$ such that $V_i = \partial F(V_{i-1},V_{i+1})$ for all $i$. Lastly, let $x, y \in C(S)$ then a tight geodesic between $x$ and $y$ is a geodesic $\{x_i\}$ with $x_i \in V_i$ for all $i$ where $\{V_i\}$ is a tight multigeodesic between $x$ and $y$.

**Theorem 2.10.** [MM00]. There exists a tight geodesic between any two points in $C(S)$.

We observe the following property of tight geodesics.
Lemma 2.11. Let \( x, y \in C(S) \), suppose \( \{V_j\} \) is a tight multigeodesic between \( x \) and \( y \), let \( Z \subset S \) such that \( \pi_Z(V_i) \neq \emptyset \), then \( d_Z(x, V_i) \leq M \) or \( d_Z(V_i, y) \leq M \).

In particular let \( \{x_j\} \) be a tight geodesic between \( x \) and \( y \), then if \( \pi_Z(x_i) \neq \emptyset \) then \( d_Z(x, x_i) \leq M \) or \( d_Z(x_i, y) \leq M \).

Proof. By Definition 2.9, \( \pi_Z(V_{i-1}) \neq \emptyset \) or \( \pi_Z(V_{i+1}) \neq \emptyset \). With Lemma 2.8 it follows by the Bounded Geodesic Image Theorem.

For the second statement, there exists a tight multigeodesic \( \{V_j\} \) between \( x \) and \( y \), such that \( x_j \in V_j \) for all \( j \). Suppose \( d_Z(x_i, y) > M \), we show \( d_Z(x, x_i) \leq M \). If \( \pi_Z(V_{i+1}) = \emptyset \), then we are done with the first statement. If not, there exists \( a \in V_{i+1} \) such that \( \pi_Z(a) \neq \emptyset \) and \( g = x_i, a, \ldots, y \) is a geodesic. But since \( d_Z(x_i, y) > M \) some vertex of \( g \) needs to miss \( Z \) but \( \pi_Z(x_i) \neq \emptyset \) and \( \pi_Z(a) \neq \emptyset \), again we are done with Lemma 2.8.

3 Geometry of a Farey graph

The curve graphs of \( S_{1,1} \) and \( S_{0,4} \) are both Farey graphs such that the vertices are identified with \( \mathbb{Q} \cup \{\frac{1}{0} = \infty\} \subset S^1 \). Indeed, they are isometric, we refer [FM12] for a detailed treatment. The following observation is elementary yet useful in this section.

Geometric property of a Farey graph: Suppose \( \xi(S) = 1 \), let \( x, y \in C(S) \) such that \( d_S(x, y) = 1 \). If \( I \) and \( I' \) are the segments obtained by removing \( \{x, y\} \) form \( S^1 \), then any geodesic between a curve in \( I \) and a curve in \( I' \) needs to contain \( x \) or \( y \).

Proof. Since \( d_S(x, y) = 1 \), there exists the edge between \( x \) and \( y \). The statement follows from the fact that the interiors of any two distinct edges of a Farey graph are disjoint.

We will use the above observation without referring for the rest of this section. Now, we observe Lemma 3.1. For the rest of this paper, we denote \( g_{x,y} \) as a geodesic between \( x \) and \( y \) where \( x, y \in C(S) \).

Lemma 3.1. Suppose \( \xi(S) = 1 \). Let \( x, y \in C(S) \) such that \( d_S(x, y) > 1 \) and \( g_{x,y} = \{x_i\} \) such that \( d_S(x, x_i) = i \) for all \( i \), then \( \frac{i(x_{i-1}, y)}{i(x, y)} > \frac{3}{2} \) for all \( 0 < i < d_S(x, y) \).

Proof. First, we recall that if \( \frac{a}{b}, \frac{c}{d} \in C(S) \), then \( \frac{a}{b} \cdot \frac{c}{d} = k \cdot |s't' - s't| \) where \( k = 1 \) when \( S = S_{1,1} \) and \( k = 2 \) when \( S = S_{0,4} \).

Now, we prove the statement. We may assume \( x_{i-1} = \frac{0}{1} \) and \( x_i = \frac{1}{0} \) then \( y = \frac{i(x_{i-1}, y)}{i(x, y)} \) or \( y = -i(x_{i-1}, y) \). Therefore it suffices to show \( |y| > \frac{3}{2} \). Without loss of generality, we assume \( y = \frac{3}{2} \) and derive a contradiction.

If \( y \leq 1 \): Since there exists the edge between \( x_{i-1} = 0 \) and \( 1 \), \( g_{x,y} \) needs to contain \( 1 \), but since \( d_S(x_{i-1}, 1) = 1 \) and \( d_S(x_i, 1) = 1 \), it implies \( d_S(x_{i-1}, y) \leq d_S(x_i, y) \).
If $1 < y \leq \frac{3}{2}$: Since $g_{x_i, y}$ does not contain 1 and there exists the edge between 1 and 2, $g_{x_i, y}$ needs to contain 2. Also since there exists an edge between $\frac{3}{2}$ and 1, $g_{x_i, y}$ needs to contain $\frac{3}{2}$, but since $d_S(x_{i-1}, \frac{3}{2}) = 2$ and $d_S(x_i, \frac{3}{2}) = 2$, it implies $d_S(x_{i-1}, y) \leq d_S(x_i, y)$.

Now, we have the following key theorem.

**Theorem 3.2.** Suppose $\xi(S) = 1$. Let $x, y \in C(S)$ such that $d_S(x, y) > 1$ and $g_{x, y} = \{x_i\}$ such that $d_S(x, x_i) = i$ for all $i$. If $d_{R(x_i)}(x, y) = L$, then

$$L - 2M - 3 \leq \frac{i(x_i-1, y)}{i(x_i, y)} \leq 2(L + 2M)$$

for all $0 < i < d_S(x, y)$.

**Proof.** Since $d_{R(x_i)}(x, x_{i-1}) \leq M$ by the Bounded Geodesic Image Theorem, we have $L - M \leq d_{R(x_i)}(x_{i-1}, y) \leq L + M$. We may assume $x_{i-1} = \frac{0}{1}$ and $x_i = \frac{1}{0}$ then $|y| = \frac{i(x_i-1, y)}{i(x_i, y)}$.

We show $d_{R(x_i)}(y, [y]) \leq M$ or $d_{R(x_i)}(y, [y]) \leq M$. We claim $x_i \notin g_{[y], y}$ or $x_i \notin g_{|y|, y}$. With this claim, the statement follows by the Bounded Geodesic Image Theorem. To prove the claim, we observe there exists the edge between $[y]$ and $[y]$. Let $I_y$ be the interval obtained by removing $\{[y], [y]\}$ from $S^1$ which contains $y$. Since $x_i$ is not contained in $I_y$, the claim follows by the geometry of a Farey graph.

Therefore, we have either $L - 2M \leq d_{R(x_i)}(x_{i-1}, [y]) \leq L + 2M$ or $L - 2M \leq d_{R(x_i)}(x_{i-1}, [y]) \leq L + 2M$.

By Lemma 3.1 $|y| \neq 0$ and $[y] \neq 0$ so we may apply Lemma 2.4 on $d_{R(x_i)}(x_{i-1}, [y])$ and $d_{R(x_i)}(x_{i-1}, [y])$ and we have

$$d_{R(x_i)}(x_{i-1}, [y]) = \begin{cases} \lfloor y \rfloor + 2 & \text{if } S = S_{1,1} \\ \lceil y \rceil + 2 & \text{if } S = S_{0,4} \end{cases} \quad \text{and} \quad d_{R(x_i)}(x_{i-1}, [y]) = \begin{cases} \lfloor y \rfloor + 2 & \text{if } S = S_{1,1} \\ \lceil y \rceil + 2 & \text{if } S = S_{0,4} \end{cases}$$

If $S = S_{1,1}$ we have

$$L - 2M - 3 \leq |y| = \frac{i(x_i-1, y)}{i(x_i, y)} \leq L + 2M - 1.$$

If $S = S_{0,4}$ we have

$$2(L - 2M - 2) - 1 \leq |y| = \frac{i(x_i-1, y)}{i(x_i, y)} \leq 2(L + 2M - 1) + 1.$$
4 Lemma 1.4

We first prove Lemma 1.4 for $\xi(S) = 1$ as the base case of our inductive proof.

Lemma 4.1. Suppose $\xi(S) = 1$ and $x, y \in C(S)$. There exists a computable $L_S(d) > 0$ such that if $d_S(x, y) \leq d$ for all proper subsurfaces $Z$, then $\log_{L_S(d)} i(x, y) \leq d_S(x, y)$.

Proof. Let $g_{x,y} = \{x_i\}$ such that $d_S(x, x_i) = i$ for all $i$, then since $d_R(x_i)(x, y) \leq d$ for all $i$, by Theorem 3.2 we have $\frac{i(x_{i-1}, y)}{i(x_i, y)} \leq 2(d + 2M) = D$ for all $i$. Therefore we have

$$\frac{i(x, y)}{i(x_{d_S(x,y)-1}, y)} \leq D^{d_S(x,y)-1}$$

where $i(x_{d_S(x,y)-1}, y) = 1$ or 2, so we are done. \hfill \Box

We observe the following topological lemma. Suppose $A$ is a subset of $S$, we let $S - A$ denote the set of components of the compliments of $A$ in $S$ and we treat them as embedded subsurfaces in $S$.

Lemma 4.2. Let $x \in C(S)$ and $y \in A(S)$ such that they are in minimal position. If $\partial y$ lie on two distinct boundaries of $S$ then $i(x, \pi_S(y)) = 2i(x, y)$, and if $\partial y$ lie on one boundary of $S$ then $\max(i(x, Y), i(x, Y')) = i(x, y)$ where $\{\pi_S(y)\} = \{Y, Y'\}$.

Proof. Recall that two curves are in minimal position if and only if they do not form a bigon, for instance see [EMT12]. For the first statement, it suffices to show that the set of arcs obtained by $\pi_S(y) \cap (S - x)$ are all essential in $S - x$. For the second statement, we show the set of arcs obtained by either $Y \cap (S - x)$ or $Y' \cap (S - x)$ are all essential in $S - x$. We let $\{a_i\}$ be the set of arcs obtained by $y \cap (S - x)$, then they are all essential arcs in $S - x$ by minimality.

For the first statement, we let $a_p, a_q \in \{a_i\}$ such that one of their boundaries lie on $\partial S$. Let $s \in \partial S$ such that one of $\partial a_p$ lies on, then $\partial R(a_p \cup s)$ is an essential arc in the component of $S - x$ which contains $a_p$. If not, $s$ and $x$ would be isotopic but $x \in C(S)$. The same argument applies for $a_q$, which implies $\{\pi_S(y) \cap (S - x)\}$ are all essential in $S - x$. Therefore, $i(x, \pi_S(y)) = 2i(x, y)$.

For the second statement, we let $a_p, a_q \in \{a_i\}$ such that one of their boundaries $b \in \partial a_p, b' \in \partial a_q$ lie on $s \in \partial S$. Now, we have two cases, i.e. their other boundaries lie on two distinct boundaries (Case 1) or the same boundary (Case 2) of $S - x$ which come from cutting $S$ along $x$. Let $s_1, s_2$ both be the closure of the subsegments of $s$ obtained by removing $\{b, b'\}$ from $s$.

Case 1 $a_p \cup s_1 \cup a_q$ and $a_p \cup s_2 \cup a_q$ are both essential arcs in $S - x$. Therefore, by letting $\{\pi_S(y)\} = \{Y, Y'\}$, we have $i(x, Y) = i(x, Y') = i(x, y)$.

Case 2 At least one of $a_p \cup s_1 \cup a_q$ and $a_p \cup s_2 \cup a_q$ is an essential arc in $S - x$ otherwise $s$ and $x$ would be isotopic. Therefore, $\max(i(x, Y), i(x, Y')) = i(x, y)$. \hfill \Box
Now, we complete Lemma 1.3 for $\xi(S) > 1$. Let $k_S(d) = \max_W L_W(d)$ for all surfaces $W$ such that $\xi(W) < \xi(S)$.

**Lemma 4.3.** Suppose $\xi(S) > 1$ and $x, y \in C(S)$. There exists a computable $L_S(d) > 0$ such that if $d_Z(x, y) \leq d$ for all proper subsurfaces $Z$, then $\log L_S(d) i(x, y) \leq d_S(x, y)$.

**Proof.** Let $g_{x, y} = \{x_i\}$ such that $d_S(x, x_i) = i$ for all $i$. Indeed, we assume $g_{x, y}$ is a tight geodesic so that we may apply Lemma 2.11. We claim

- $i(x_{d_S(x, y) - 2}, y) \leq k_S(d + M)^{d + M}$
- $\frac{i(x_{i-1}, y)}{i(x_i, y)} \leq k_S(d + M)^{d + M}$ for all $0 < i < d_S(x, y) - 1$.

With the claim, we obtain $i(x, y) \leq (k_S(d + M)^{d + M})^{d_S(x, y) - 1}$ and we are done.

For the rest of this proof we let $S'$ denote the component of the compliments of $x_i$ in $S$ which contains $x_{i-1}$. Now, we start the proof of the above two inequalities. The first inequality is derived when $i = d_S(x, y) - 1$ and the second inequality is derived when $i \neq d_S(x, y) - 1$.

**First inequality:** First, we observe $y$ lies in $S'$. Let $Z \subseteq S'$ such that $\pi_Z(x_{d_S(x, y) - 2}) \neq \emptyset$ then $d_Z(x, x_{d_S(x, y) - 2}) \leq M$ or $d_Z(x_{d_S(x, y) - 2}, y) \leq M$ by tightness, Lemma 2.11. For the first case, with the hypothesis of this lemma we have $d_Z(x_{d_S(x, y) - 2}, y) \leq d_Z(x, y) + d_Z(x, x_{d_S(x, y) - 2}) \leq d + M$. Therefore, we have $d_Z(x_{d_S(x, y) - 2}, y) \leq d + M$ for all $Z \subseteq S'$.

In particular, $d_Z(x_{d_S(x, y) - 2}, y) \leq d + M$ for all proper subsurfaces $Z \subseteq S'$. Now, since $x_{d_S(x, y) - 2}$ and $y$ both lie in $S'$ and $\xi(S') < \xi(S)$ we use our inductive hypothesis here, and we have $\log k_S(d + M) i(x_{d_S(x, y) - 2}, y) \leq d_S'(x_{d_S(x, y) - 2}, y)$.

However, by our first observation, we must have $d_{S'}(x_{d_S(x, y) - 2}, y) \leq d + M$. Therefore, we need to have

$$i(x_{d_S(x, y) - 2}, y) \leq k_S(d + M)^{d + M}.$$

**Second inequality:** The argument is slightly more complicated since $y$ does not lie in $S'$. However the main idea is same. Let $A$ be the set of arcs obtained by $y \cap S'$. We observe $|A| = i(x_i, y)$ when $x_i$ does not separate $S$ and $2 \cdot |A| = i(x_i, y)$ when $x_i$ separates $S$, in either case we have $|A| \leq i(x_i, y)$.

As in the previous case, we have $d_Z(x_{i-1}, y) \leq d + M$ for all $Z \subseteq S'$. Also we observe $\pi_{S'}(y) = \bigcup_{a \in A} \pi_{S'}(a)$. Let $c$ be a curve obtained by $\pi_{S'}(a)$ for $a \in A$ then we have $d_Z(x_{i-1}, c) \leq d + M$ for all $Z \subseteq S'$. Since $x_{i-1}$ and $c$ lie in $S'$, as in the previous case we need to have $i(x_{i-1}, c) \leq k_S(d + M)^{d + M}$, which implies $i(x_{i-1}, a) \leq k_S(d + M)^{d + M}$ by Lemma 1.2.

Since $i(x_{i-1}, y) = \sum_{a \in A} i(x_{i-1}, a)$ we have $i(x_{i-1}, y) \leq |A| \cdot k_S(d + M)^{d + M}$. Then we have

$$\frac{i(x_{i-1}, y)}{i(x_i, y)} \leq k_S(d + M)^{d + M}.$$
5 Main Formula

We start with Theorem 1.3 for $\xi(S) = 1$. Recall the cut-off function $c(n) = Mn + 2M$ from Definition 1.2 for $n \in \mathbb{N}$.

**Theorem 5.1.** Suppose $\xi(S) = 1$. For any $x, y \in C(S)$, if $k \geq c(1) = 3M$ then

$$\log i(x, y) \asymp \sum_{Z} \log |d_{Z}(x, y)|_{k} + \sum_{A} \log |d_{A}(x, y)|_{k},$$

where $Z$ ranges over all subsurfaces in $S$ which are not annuli, and $A$ ranges over all annuli in $S$.

**Proof.** We observe that every proper subsurface which appears in the formula is an annulus, also by the Bounded Geodesic Image Theorem, if $d_{A}(x, y) > M$ then $\partial A$ is contained in every geodesic between $x$ and $y$.

Let $g_{x, y} = \{x_{i}\}$ such that $d_{S}(x, x_{i}) = i$ for all $i$. Suppose $d_{R(x_{i})}(x, y) = L_{i}$ for all $i$ such that $0 < i < d_{S}(x, y)$. By Lemma 3.1 and Theorem 3.2 we have

$$\max \left\{ \frac{3}{2}, L_{i} - 2M - 3 \right\} \leq \frac{i(x_{i-1}, y)}{i(x_{i}, y)} \leq 2(L_{i} + 2M).$$

We define

$$L_{i}^{-} = \begin{cases} \frac{3}{2} & \text{if } L_{i} \leq k \\ \frac{3}{2}\sqrt{L_{i}} & \text{if } L_{i} > k \end{cases} \quad \text{and} \quad L_{i}^{+} = \begin{cases} k^2 & \text{if } L_{i} \leq k \\ L_{i}^2 & \text{if } L_{i} > k \end{cases}$$

then we may assume $L_{i}^{-} \leq \frac{i(x_{i-1}, y)}{i(x_{i}, y)} \leq L_{i}^{+}$ and we have

$$\log i(x_{d_{S}(x, y)-1}, y) + \sum_{i=1}^{d_{S}(x, y)-1} \log L_{i}^{-} \leq \log i(x, y) \leq \log i(x_{d_{S}(x, y)-1}, y) + \sum_{i=1}^{d_{S}(x, y)-1} \log L_{i}^{+},$$

where $i(x_{d_{S}(x, y)-1}, y) = 1$ or $2$. Therefore, we have

- $\log 2 + \sum_{i=1}^{d_{S}(x, y)-1} \log L_{i}^{+} \leq k^2 \cdot d_{S}(x, y) + 2 \cdot \left( \sum_{A} \log |d_{A}(x, y)|_{k} \right).$

- $\sum_{i=1}^{d_{S}(x, y)-1} \log L_{i}^{-} = \log \left( \frac{3}{2} \cdot (d_{S}(x, y) - 1) + \frac{1}{2} \cdot \left( \sum_{A} \log |d_{A}(x, y)|_{k} \right) \right).$

$\square$
The proof for higher complexity surfaces will be done by the double induction on the complexity and the distance. First, we show Lemma 5.2 which is the base case of the distance. The proof will be done by the inductive hypothesis on the complexity. Suppose \( \xi(S) > 1 \) we let \( P \) and \( Q \) be the maximum multiplicative and additive constants among all lower complexity surfaces so that Theorem 1.3 holds respectively. We note that \( P \) and \( Q \) also depends on a cut-off number. We show

**Lemma 5.2.** Suppose \( \xi(S) > 1 \). Let \( x, y \in C(S) \) and \( g_{x,y} = \{x_i\} \) such that \( d_S(x, x_i) = i \) for all \( i \). Suppose \( d_S(x, y) = 2 \) then if \( k \geq c(\xi(S) - 1) \) we have

\[
\log i(x, y)^{P_{d_g(x,y) - Q}} \sum_Z [d_Z(x, y)]_k + \sum_A \log[d_A(x, y)]_k.
\]

*Proof.* We observe that all proper subsurfaces which show up in the formula misses \( x_1 \) by the Bounded Geodesic Image Theorem, also we observe that \( x \) and \( y \) both lies in the same component of the compliment of \( x_1 \) in \( S \). We are done with our inductive hypothesis on the complexity. (We could have \( Q \) as an additive constant for this identity but the constants we have in the statement of this lemma is more convenient to use for the rest of this paper.) \( \square \)

In order to simply our notations for the rest of this section, we define followings.

**Definition 5.3.** Let \( x, y \in C(S) \) and \( g_{x,y} = \{x_i\} \) where \( d_S(x, x_i) = i \) for all \( i \). Let \( A \subset S \) denote annuli and \( Z \subset S \) otherwise. For \( n \geq 3M \), we let

\[
G_i(n) = \sum_{Z \subset S - x_i} [d_Z(x_{i-1}, y)]_n + \sum_{A \subset S - x_i} \log[d_A(x_{i-1}, y)]_n \text{ and } G(n) = \sum_i G_i(n).
\]

By replacing \( x_{i-1} \) above by \( x \), we let

\[
H_i(n) = \sum_{Z \subset S - x_i} [d_Z(x, y)]_n + \sum_{A \subset S - x_i} \log[d_A(x, y)]_n \text{ and } H(n) = \sum_i H_i(n).
\]

Furthermore, let \( W(n) = \{W \subset S \mid d_W(x, y) > n\} \) and \( W_i(n) = \{W \in W(n) \mid \pi_W(x_i) = \emptyset \text{ and } \pi_W(x_j) \neq \emptyset \text{ for all } j < i\} \). Lastly, we let

\[
F_i(n) = \sum_{Z \in W_i(n)} [d_Z(x, y)]_n + \sum_{A \in W_i(n)} \log[d_A(x, y)]_n \text{ and } F(n) = \sum_i F_i(n).
\]

We observe \( \{W_i(n)\} \) is a partition of \( W(n) \), therefore \( F(n) = \sum_{Z \in W(n)} [d_Z(x, y)]_n + \sum_{A \in W(n)} \log[d_A(x, y)]_n. \)

We relate each notation above by the following lemma.

**Lemma 5.4.** Suppose \( \xi(S) > 1 \). Let \( x, y \in C(S) \), then for \( n \geq c(\xi(S)) \) we have

\[
G(n + M) < F(n) < G(n - M).
\]
Proof. We show the following three boxed inequalities, which finish the proof. Let \( g_{x,y} = \{x_i\} \) where \( d_{S}(x, x_j) = i \) for all \( i \).

- **\( F(n) \leq 2G(n - M) \)**
  
  We show \( F_i(n) \leq 2G_i(n - M) \) for all \( 0 < i < d_S(x,y) \). If \( W \in W_i(n) \), by the Bounded Geodesic Image Theorem and the definition of \( W_i(n) \) we have \( d_{W}(x, x_{i-1}) \leq M \), so \( d_{W}(x_{i-1}, y) > n - M \) therefore we have \( W_i(n) \subseteq \{ W \subseteq S - x_i|[d_{W}(x_{i-1}, y)]_{n-M} > 0 \} \).
  
  Lastly, we observe \( n \leq 2(n - M) \) and \( \log n \leq 2 \log(n - M) \).

- **\( \frac{\mathcal{H}(n)}{3} \leq F(n) \)**
  
  We observe that if \( W \in W_i(n) \) then \( W \) could miss \( x_{i+1} \) and \( x_{i+2} \) at most by Lemma 2.8.

- **\( G(n + M) \leq 2\mathcal{H}(n) \)**
  
  We show \( G_i(n + M) \leq 2H_i(n) \) for all \( 0 < i < d_S(x,y) \). We use a main property of tight geodesics, Lemma 2.11 on \( g_{x,y} \) by assuming it is a tight geodesic. Suppose \( Y \subseteq S - x_i \) such that \( [d_{Y}(x_{i-1}, y)]_{n+M} > 0 \) then we have \( d_{Y}(x, x_{i-1}) \leq M \). Therefore, \( [d_{Y}(x,y)]_{n} > 0 \) and the rest of the proof follows as in the proof of the first inequality.

One of the key observation for the proof of Theorem 1.3 will be Lemma 5.7. For this, we state the following algebraic identity and one of the main results of Shackleton in [Sha12].

**Lemma 5.5.** Suppose \( m_i \in \mathbb{N} \) and \( m_i > 0 \) for all \( 1 \leq i \leq l \), then

\[
\log(\sum_{i=1}^{l} m_i) \leq \sum_{i=1}^{l} \log m_i.
\]

**Proof.** We observe \( (\prod_{i=1}^{l} m_i)^{\frac{1}{l}} \leq \sum_{i=1}^{l} m_i \leq l \cdot \prod_{i=1}^{l} m_i \) (The first inequality follows from AM-GM inequality.)

**Theorem 5.6 ([Sha12]).** Suppose \( \xi(S) > 1 \). Let \( x,y \in C(S) \) and \( g_{x,y} = \{x_i\} \) be a tight geodesic such that \( d_{S}(x, x_i) = i \) for all \( i \), then there exists a computable increasing function \( F_1 : \mathbb{N} \to \mathbb{N} \) such that \( i(x_1, y) \leq F_1(i(x,y)) \).

Indeed, we have \( \log i(x_1, y) \leq r \cdot \log i(x,y) \) where \( r \in \mathbb{R} \) only depends the topology of the surface.

**Proof.** The second statement follows from the definition of \( F_1 \). For \( n \in \mathbb{N} \),

\[
F_1(n) = n \cdot R^{[2\log_2 n]}
\]

where \( R \) only depends on \( \xi(S) \).
We shall interpret the above as Lemma 3.1 for $\xi(S) > 1$ in the sense that $i(x_1, y)$ is controlled by $i(x, y)$. Now, we have the following key fact.

**Lemma 5.7.** Suppose $\xi(S) > 1$. Let $x, y \in C(S)$, for $n \geq c(\xi(S) - 1)$ we have

$$\log i(x, y) \leq \mathcal{K} \mathcal{G}(n)$$

for some $K$ and $C$ which depend only on the topology of the surface and $n$.

**Proof.** We let $g_{x,y} = \{x_i\}$ be a tight geodesic such that $d_S(x, x_i) = i$ for all $i$ and $S'$ be the component of the compliment of $x_1$ in $S$ which contains $x$.

Recall Lemma 2.2, we have $|i_{S'}(y) = \{a_j\}| \leq 3|\chi(S)|$ and $|\pi_{S'}(y) = \{c_j\}| \leq 6|\chi(S)|$. Let $n_j$ be the number of arcs obtained by $y \cap S'$ which are isotopic to $a_j$. Furthermore, we let $n_j' = n_j$ if $i(x, a_j) > 0$ and $n_j' = 0$ otherwise. We have

- $i(x, y) = \sum_j n_j' \cdot i(x, a_j)$ by the definition of $a_j$ and $n_j'$.

- $i(x_1, y) = \sum_j n_j$ when $x_1$ does not separate $S$ and $i(x_1, y) = 2 \cdot \sum_j n_j$ when $x_1$ separates $S$. Therefore, we have $\sum_j n_j \leq i(x_1, y) \leq 2 \cdot \sum_j n_j$.

- $\sum_j i(x, a_j) \leq \sum_j i(x, c_j) \leq 2 \cdot \sum_j i(x, a_j)$ by Lemma 4.2.

With Lemma 5.5, for each bullet we have

- $\log i(x, y) \asymp \sum_j \log n_j' + \sum_j \log i(x, a_j)$.

- $\log i(x_1, y) \asymp \sum_j \log n_j$.

- $\sum_j \log i(x, a_j) \asymp \sum_j \log i(x, c_j)$.

Therefore we have

$$\log i(x, y) - \log i(x_1, y) \asymp \sum_j \log n_j' - \sum_j \log n_j + \sum_j \log i(x, c_j).$$

By Theorem 5.6, there exists $r \in \mathbb{R}$ which depends only on the topology of the surface such that $-r \cdot \log i(x, y) \leq \sum_j \log n_j' - \sum_j \log n_j \leq 0$ and we obtain

$$\log i(x, y) - \log i(x_1, y) \asymp \sum_j \log i(x, c_j).$$
Now, we apply the double induction. Recall the definition of $P$ and $Q$ from Lemma 5.2.

Inductive step by complexity: Since $x, c_j \in C(S')$ for all $j$ and $n \geq c(\xi(S'))$ we use the inductive hypothesis on the complexity. Since $|\{c_j\}| \leq 6|\chi(S)|$ we have

$$\sum_j \log i(x, c_j) P^{|\chi(S)|-Q} \sum_j \left( \sum_{Z \subseteq S'} [d_Z(x, c_j)]_n + \sum_{A \subseteq S'} \log [d_A(x, c_j)]_n \right)$$

also since $\pi_{S'}(y) = \{c_j\}$, by the definition of subsurface projection distance we have

$$G_1(n) \leq \sum_j \left( \sum_{Z \subseteq S'} [d_Z(x, c_j)]_n + \sum_{A \subseteq S'} \log [d_A(x, c_j)]_n \right) \leq 6|\chi(S)| \cdot G_1(n).$$

Therefore, we have

$$\sum_j \log i(x, c_j) 6|\chi(S)|-P^{|\chi(S)|-Q} G_1(n)$$

and we obtain

$$\log i(x, y) - \log i(x_1, y) P^{|\chi(S)|} \equiv G_1(n)$$

where $P' \geq 6|\chi(S)| \cdot P$ and $Q' \geq 6|\chi(S)| \cdot Q$.

Inductive step by distance: We may also use the inductive hypothesis on the distance for $\log i(x_1, y)$, then we have

$$\log i(x_1, y) \equiv \sum G_i(n).$$

Uniform constants, $K$ and $C$: We observe that we could replace $P$ and $Q$ in Lemma 5.2 by $P'$ and $Q'$ respectively. By the inductive proof of this theorem, it implies we may take $A = P'$ and $B = Q'$. Therefore, we have $K = P'$ and $C = Q'$ as uniform constants for any two curves.

We have the following corollary by Lemma 5.4 and Lemma 5.7.

**Corollary 5.8.** Suppose $\xi(S) > 1$. Let $x, y \in C(S)$, then for $k \geq c(\xi(S))$ we have

$$\log i(x, y)^{K', d_S(x, y) \cdot C'} \equiv \mathcal{F}(k)$$

for some $K'$ and $C'$ which depends only on the topology of the surface and $k$.

**Proof.** By Lemma 5.7 we have $\log i(x, y) \asymp \mathcal{G}(k - M)$ and $\log i(x, y) \asymp \mathcal{G}(k + M)$. We note this is the place which the cut-off function needs to grow linearly with $\xi(S)$ so that we can have the first identity. The proof is done by Lemma 5.4
Now, we complete Theorem 1.3.

**Theorem 5.9.** Suppose $\xi(S) > 1$. Let $x, y \in C(S)$, for $k \geq c(\xi(S))$ we have

$$\log i(x, y) \asymp \sum_Z [d_Z(x, y)]_k + \sum_A \log [d_A(x, y)]_k,$$

where $Z$ ranges over all subsurfaces in $S$ which are not annuli, and $A$ ranges over all annuli in $S$.

**Proof.** First, we observe the right hand side of the identity is equal to $[d_S(x, y)]_k + \mathcal{F}(k)$. We apply Corollary 5.8 here.

If $[d_S(x, y)]_k = 0$, we have

$$\log i(x, y) \stackrel{K', C'}{=} [d_S(x, y)]_k + \mathcal{F}(k).$$

If $[d_S(x, y)]_k > 0$, first we have

$$\log i(x, y) \leq \max\{K', C'\} \cdot ([d_S(x, y)]_k + \mathcal{F}(k)).$$

For the opposite inequality, we restate corollary 5.8 as $\frac{\mathcal{F}(k)}{K} - d_S(x, y) \cdot C' \leq \log i(x, y)$. The rest of the proof follows with Lemma 1.1, $d_S(x, y) \leq 2 \log_2 i(x, y) + 2$. \hfill $\square$

**References**

[Aou] Tarik Aougab. Uniform hyperbolicity of the graphs of curves. arXiv:1212.3160.

[Bow] Brian H. Bowditch. Uniform hyperbolicity of the curve graphs. http://homepages.warwick.ac.uk/ masgak/preprints.html.

[Bow06] Brian H. Bowditch. Intersection numbers and the hyperbolicity of the curve complex. J. Reine Angew. Math., 598:105–129, 2006.

[CR07] Young-Eun Choi and Kasra Rafi. Comparison between Teichmüller and Lipschitz metrics. J. Lond. Math. Soc. (2), 76(3):739–756, 2007.

[FM12] Benson Farb and Dan Margalit. A primer on mapping class groups, volume 49 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012.

[Ham07] Ursula Hamenstädt. Geometry of the complex of curves and of Teichmüller space. In Handbook of Teichmüller theory. Vol. I, volume 11 of IRMA Lect. Math. Theor. Phys., pages 447–467. Eur. Math. Soc., Zürich, 2007.

[Har81] W. J. Harvey. Boundary structure of the modular group. In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), volume 97 of Ann. of Math. Stud., pages 245–251. Princeton Univ. Press, Princeton, N.J., 1981.
[Hem01] John Hempel. 3-manifolds as viewed from the curve complex. *Topology*, 40(3):631–657, 2001.

[HPW] Sebastian Hensel, Piotr Przytycki, and Richard C. H. Webb. Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs. *arXiv:1301.5577*.

[KP10] Mustafa Korkmaz and Athanase Papadopoulos. On the arc and curve complex of a surface. *Math. Proc. Cambridge Philos. Soc.*, 148(3):473–483, 2010.

[Lic62] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962.

[Luo10] Feng Luo. Simple loops on surfaces and their intersection numbers. *J. Differential Geom.*, 85(1):73–115, 2010.

[MCS] Kasra Rafi Matt Clay and Saul Schlelmer. Uniform hyperbolicity of the curve graph via surgery sequences. *arXiv:1302.5519*.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.

[Raf07] Kasra Rafi. A combinatorial model for the Teichmüller metric. *Geom. Funct. Anal.*, 17(3):936–959, 2007.

[Sha12] Kenneth J. Shackleton. Tightness and computing distances in the curve complex. *Geom. Dedicata*, 160:243–259, 2012.

[Web] Richard C. H. Webb. A short proof of the bounded geodesic image theorem. *arXiv:1301.6187*.

E-mail adress: ywatanab@math.utah.edu