Renormalization Group Evolution of Neutrino Parameters in Presence of Seesaw Threshold Effects and Majorana Phases

Shivani Gupta, Sin Kyu Kang, C. S. Kim

Department of Physics and IPAP, Yonsei University, Seoul 120-749, Korea
School of Liberal Arts, Seoul-Tech, Seoul 139-743, Korea
E-mail: shivani@yonsei.ac.kr, skkang@seoultech.ac.kr, cskim@yonsei.ac.kr

Abstract: We examine the renormalization group evolution (RGE) for different mixing scenarios in the presence of seesaw threshold effects from high energy scale (GUT) to the low electroweak (EW) scale in the Standard Model (SM) and the Minimal Supersymmetric Standard Model (MSSM). We consider four mixing scenarios namely Tri-Bimaximal Mixing, Bimaximal Mixing, Hexagonal Mixing and Golden Ratio Mixing which come from different flavor symmetries at the GUT scale. All these mixing scenarios give vanishing reactor angle ($\theta_{13}$) and maximal atmospheric mixing angle. The solar mixing angle has different value for all four cases. In the light of non zero value of $\theta_{13}$ it becomes interesting to study the present status of these symmetries, i.e. whether they can generate the current neutrino oscillation data at low energy scale or not. We find that the Majorana phases play an important role in the RGE running of these mixing patterns along with the seesaw threshold corrections. We present a comparative study of the RGE of all these mixing scenarios both with and without Majorana CP phases when seesaw threshold corrections are taken into consideration. We find that in the absence of these Majorana phases both the RGE running and seesaw effects may lead to $\theta_{13} < 5^\circ$ at low energies both in the SM and the MSSM. However, if the Majorana phases are incorporated to the mixing matrix the running can be enhanced both in the SM and the MSSM. Even by incorporating non zero Majorana CP phases in the SM, we do not get $\theta_{13}$ in its present $3\sigma$ range. The current values of the two mass squared differences and mixing angles including $\theta_{13}$ can be produced in the MSSM case with $\tan\beta = 10$ and non zero Majorana CP phases at low energy. Thus, all these mixing scenarios considered at the high energy scale can produce the current low energy data if Majorana phases are taken into consideration. We also calculate the order of effective Majorana mass and Jarlskog Invariant for each scenario under consideration.
1 Introduction

Many flavor symmetries studied in literature [1, 2] can result in some particular form of mixing in leptonic sector. The mixing scenarios obtained by some symmetries lead to the vanishing reactor neutrino mixing angle, $\theta_{13}$. In this present scenario where $\theta_{13}$ is non zero [3], it is meaningful to turn to systematic study of the effects of perturbation on these flavor symmetries or to search for alternative symmetry which gives non zero $\theta_{13}$. In the flavor basis, the lepton mixing matrix comprises of three mixing angles, three masses and three CP violating phases if neutrinos are Majorana particles. The $3 \times 3$ leptonic mixing matrix is given as

$$
U = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13}
\end{pmatrix} \cdot P. \tag{1.1}
$$

Here $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$; $\theta_{ij}$ are the three mixing angles, $\delta_{CP}$ is the Dirac CP phase. The matrix $P$ = $\text{diag}(1, e^{-i\phi_{1}/2}, e^{-i\phi_{2}/2})$ has two Majorana CP phases $\phi_{1}$ and $\phi_{2}$ respectively. The relatively large value of $\theta_{13}$ has also provided an opportunity for the measurement of $\delta_{CP}$ in the lepton mixing matrix. The Jarlskog rephasing invariant quantity, $J_{CP}$ is given as $J_{CP} = c_{12}s_{12}c_{23}s_{23}s_{13}^2 \sin \delta_{CP}$ [4] which controls the magnitude of CP violation in neutrino oscillations generated by the Dirac phase in the PMNS matrix. Recent global fit analysis for the neutrino mixing angles and mass squared differences is given in [5–7]. We show the best fit values along with the 3$\sigma$ constraints in Table 1.

The mixing in neutrino sector is still not completely understood. The hierarchy of the three neutrino masses is unknown, whether it is normal ($m_1 < m_2 < m_3$) or inverted.
### Table 1

The best fit along with the 3σ experimental constraints on neutrino mass squared differences and mixing angles. The data is taken from [7].

| Parameter | Best fit | 3σ Range |
|-----------|----------|-----------|
| $\Delta m_{12}^2/10^{-5}$ eV$^2$ | 7.50 | 7.00–8.09 |
| $\Delta m_{13}^2/10^{-3}$ eV$^2$ | 2.473 | 2.276–2.695 |
| $\theta_{12}$ | 33.36 | 31.09–35.89 |
| $\theta_{13}$ | 8.66 | 7.19–9.96 |
| $\theta_{23}$ | 40.0, 50.4 | 35.8–54.8 |

$(m_3 < m_1 < m_2)$. The CP violating phases are totally unknown at present. The absolute mass scale of neutrinos is still not known. The possible measurement of effective Majorana mass in neutrinoless double beta decay experiments will provide an additional constraint on the neutrino mass scale and Majorana CP phases.

$$M_{ee} = |m_{12}e^{2i\theta_{12}} + m_{23}e^{2i\delta_{CP} + \phi_2}|. \quad (1.2)$$

The Planck Collaboration [8] has given the cosmological constraint on the sum of neutrino masses to be $\sum m_\nu < 0.23$ eV at 95% C.L. This sum of the neutrino masses depend on the values chosen for the priors and can be in the range (0.23 – 0.933) eV. The South Pole Telescope Collaboration also give the sum of light neutrino masses as $\sum m_\nu = 0.32 \pm 0.11$ eV [9]. The data gives 3σ detection of positive neutrino masses in the range (0.01 – 0.63) eV at 99.7% C.L.. KATRIN experiment on tritium $\beta$ decay in preparation aims to probe absolute neutrino mass scale down to 0.2 eV [10]. The bounds and limits are needed to be tested in the forthcoming observations.

The fact that $\theta_{13}$ is not only non zero but relatively large motivates us to study how good are the flavor symmetries predicting zero value of $\theta_{13}$. Some of the mixing scenarios from flavor symmetries are Tri-Bimaximal Mixing (TBM) [11], Bimaximal Mixing (BM) [12], Hexagonal Mixing (HM) [13] and Golden Ratio (GR) [2, 14]. All these mixing scenarios predict the vanishing $\theta_{13}$ and maximal atmospheric mixing angle i.e. $\theta_{23} = \pi/4$. The solar mixing angle $\theta_{12}$ is different in all four cases. Four different forms of mixing matrices and the corresponding light neutrino mass matrices considered here are shown in Table 2. The general mixing matrix for all the above scenarios can be written as

$$U = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\delta_{CP} + \phi_2) & \sin(2\delta_{CP} + \phi_2) \\ 0 & \sin(2\delta_{CP} + \phi_2) & \cos(2\delta_{CP} + \phi_2) \end{pmatrix}. \quad (1.3)$$

For TBM $\sin^2\theta_{12} = \frac{1}{3}$; BM: $\theta_{12} = \pi/4$, HM: $\theta_{12} = \pi/6$ and GR $\theta_{12} = \tan^{-1}(1/\varphi)$, where $\varphi = (1 + \sqrt{5})/2$. Mixing angles in these scenarios are determined independent of the
which occur by subsequently integrating out heavy right handed Majorana masses at the GUT tri-bimaximal mixing is considered at some high scale \([19]\). We study the radiative corrections it becomes pertinent to study the RGE effects on these masses and mixing angles.

The RGE’s can be studied in these patterns from a high energy scale \(\Lambda\) to \(10^{16}\) GeV. In addition to the seesaw threshold corrections \([22–24, 27]\) in type I seesaw mechanism \([18]\) three heavy right handed neutrinos are introduced at a very high scale. The seesaw induced mass operator emerges and neutrino mass splittings and thus, they should not be neglected while studying the mixing scenarios in order to accommodate non zero \(\theta_{13}\).

Different attractive possibility is that these flavor symmetries are present at very high scale say grand unified scale \((\Lambda_{GUT} \sim 10^{16}\) GeV\). It has been found earlier that renormalization group evolution (RGE) corrections can be significant for leptonic mixing angles and neutrino mass splittings and thus, they should not be neglected while studying the models at high energy scale. The Standard Model (SM) needs to be extended to incorporate neutrino masses. Like in type I seesaw mechanism \([18]\) three heavy right handed neutrinos are introduced at a very high scale. The seesaw induced mass operator emerges at that high scale. Since the neutrino parameters are measured in the low energy experiments it becomes pertinent to study the RGE effects on these masses and mixing angles. The RGE’s can be studied in these patterns from a high energy scale \((\Lambda_{GUT} \sim 10^{16}\text{ GeV})\) to the low energy electroweak scale \((\Lambda_{EW} \sim 10^2\text{ GeV})\) which produces some corrections to the mixing angles. There have been studies about the effects of RGE’s on the neutrino mixing when tri-bimaximal mixing is considered at some high scale \([19]\). We study the radiative corrections to the masses and mixing angles of neutrinos in the charged lepton basis by the RGE \([20, 21]\) from \(\Lambda_{GUT}\) to \(\Lambda_{EW}\), in addition to the seesaw threshold corrections \([22–24, 27]\) which occur by subsequently integrating out heavy right handed Majorana masses at the

| Mixing | U | \(M_\nu\) | A, B, C |
|--------|---|-------------|--------|
| TBM    | \(\frac{\sqrt{2}}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0\) | \(\begin{pmatrix} A & B & B \\ \frac{1}{2}(A+B+C) & \frac{1}{2}(A+B-C) & \frac{1}{2}(A+B+C) \end{pmatrix} \) | \(A = \frac{1}{4}(2m_1 + m_2)e^{-i\theta_1}, B = \frac{1}{4}(m_2e^{-i\theta_1} - m_1), C = m_3e^{-i\theta_2}\) |
| BM     | \(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0\) | \(\begin{pmatrix} A & B & B \\ \frac{1}{2}(A+B+C) & \frac{1}{2}(A+B-C) & \frac{1}{2}(A+B+C) \end{pmatrix} \) | \(A = \frac{1}{4}(m_1 + m_3)e^{-i\theta_2}, B = \frac{1}{4}(m_3e^{-i\theta_2} - m_1), C = m_1 + m_2e^{-i\theta_2} + 2m_3e^{-i\theta_2}\) |
| HM     | \(\frac{1}{\sqrt{1+x^2}} \quad \frac{1}{\sqrt{1+y^2}} \quad 0\) | \(\begin{pmatrix} A & B & B \\ \frac{1}{4}(A+\sqrt{2}B+C) & \frac{1}{4}(A+\sqrt{2}B-C) & \frac{1}{4}(A+\sqrt{2}B+C) \end{pmatrix} \) | \(A = \frac{1}{4}(3m_1 + m_2)e^{-i\theta_1}, B = \frac{1}{4}\sqrt{2}(m_2e^{-i\theta_1} - m_1), C = m_3e^{-i\theta_2}\) |
| GR     | \(\frac{1}{\sqrt{1+x^2}} \quad \frac{1}{\sqrt{1+y^2}} \quad 0\) | \(\begin{pmatrix} A & B & B \\ \frac{1}{4}(A+\sqrt{2}B+C) & \frac{1}{4}(A+\sqrt{2}B-C) & \frac{1}{4}(A+\sqrt{2}B+C) \end{pmatrix} \) | \(A = \frac{1}{4}(m_1\psi + m_1)e^{-i\theta_1}, B = \frac{1}{4}\sqrt{2}(m_2e^{-i\theta_1} - m_1\psi), C = \frac{1}{4}(m_1 + m_2e^{-i\theta_1}\psi) + \frac{1}{2}m_3e^{-i\theta_2}\) |

Table 2. Different forms of mixing matrices, U and their corresponding forms of light neutrino mass matrices, \(M_\nu\). For GR, \(\psi = (1 + \sqrt{5})/2\) as given in the text.
respective seesaw scales both in the SM and the Minimal Supersymmetric Standard Model (MSSM). We consider all the above mentioned mixings at GUT scale, so that the neutrino mass matrix, $M_\nu$ is given in terms of three masses as shown in Table 2. The heavy right handed mass matrix can be determined by inverting the seesaw formula. We first take general $Y_\nu$ which is proportional to $M_D$, and than pick its specific form that leads to the specific mixing by scanning the parameter space. Below the seesaw threshold scales the RGE behavior is described by the effective theory which is governed by the effective mass operator. However, above the seesaw threshold scales, we have to consider the full theory. The interplay of the heavy and the light sectors can modify the RGE effects, further on top of what were in the effective theory. We consider the viability of these mixing scenarios at the high scale and running down to EW scale to produce the neutrino mixing angles and mass squared differences in the currently allowed 3$\sigma$ range.

In [23] an exhaustive analysis is presented for the RGE evolution of some of the mixing scenarios at high scale studying the RGE running and seesaw threshold effects and conclude that two of the considered mixing scenarios may lead to $\theta_{13} \sim 5^\circ$ at low energy. They however, do not take into consideration the Majorana phases in the RGE, which can be significant. Comparatively studying the RGE in the above mentioned mixing scenarios in both cases of absence and presence of the Majorana phases, we find that presence of Majorana CP violating phases can be significant to the RGE running of neutrino flavor mixing angles in the presence of seesaw threshold corrections. In the absence of these CP phases our results are somewhat similar to [23]. The reactor mixing angle, $\theta_{13}$ cannot be achieved in it’s present 3$\sigma$ range if the Majorana phases are absent at the high scale in the MSSM. However, its value in the current limit is obtained as we switch on the Majorana phases and thus all three mixing angles and two mass squared differences are obtained in the 3$\sigma$ ranges in the MSSM, respectively. The paper is organized as follows.

In section 2 we give the form of neutrino mass and mixing matrices for different mixing scenarios. In the next section the RGE equations governing at various energy scales in addition to the seesaw threshold effects are presented. Section 4 gives the numerical results of our study both in the SM and the MSSM, respectively. We summarize our results in the last section.

2 Lepton mixing matrices at the GUT scale

At the GUT scale ($\Lambda_{GUT}$) we consider the charged lepton mass matrix to be diagonal. The Yukawa coupling matrix for charged leptons is given as $Y_l = \frac{1}{v} \text{Diag}(m_e, m_\mu, m_\tau)$. Here $v$ is the Higgs vacuum expectation value (VEV) taken 246 GeV in the SM ($v\cos\beta$ in the MSSM). The effective light neutrino mass matrix, $M_\nu$ is in general given as

$$M_\nu = U^* P^* M_\nu^{diag} P^\dagger U^\dagger. \tag{2.1}$$

Here $U$ has one of the forms given in Table 2. $P$ is the phase matrix having two Majorana phases given as $\text{Diag}(1, e^{-i\phi_1}/2, e^{-i\phi_2}/2)$ and $M_\nu^{diag} = \text{Diag}(m_1, m_2, m_3)$. The different form of the corresponding light neutrino mass matrices are given in Table 2.
The Yukawa coupling matrix, $Y_\nu$ which is proportional to $M_D$ is taken to be of the form \[ Y_\nu = y_\nu \cdot R \cdot \text{Diag}(r_1, r_2, 1) \] \[ (2.2) \]

The three real, positive and dimensionless parameters $y_\nu, r_1$ and $r_2$ characterize the hierarchy of $Y_\nu$ and $R$ is given as

\[ R = R_{23}(\theta_2) \cdot R_{13}(\theta_3 e^{-i\delta}) \cdot R_{12}(\theta_1) \] \[ (2.3) \]

where $R_{ij}$ are the rotation matrices in the $ij$th plane. These three angles $(\theta_1, \theta_2, \theta_3)$ and a phase $\delta$ are free parameters varied randomly. Thus, $Y_\nu$ comprises of 7 free parameters, three eigenvalues, three angles and $\delta$. The free parameters $y_\nu, r_1$ and $r_2$ of $Y_\nu$ that determine the hierarchy to be small $\leq O(1)$. Thus, the effective RGEs between GUT scale and seesaw scales will be dependent on the structure and running of $Y_\nu$, $Y_l$ and $M_R$. $M_R$ can be determined by inverting seesaw formula given as

\[ M_R = -Y_\nu M^{-1}_\nu Y^T_\nu, \] \[ (2.4) \]

Going to the basis where $M_R$ is diagonal,

\[ U^T_R M_R U_R = \text{Diag}(M_{R_1}, M_{R_2}, M_{R_3}), \] \[ (2.5) \]

$Y_\nu$ gets simultaneously transformed as $Y_\nu U_R^*$. Since $M_R$ in our analysis is hierarchical i.e. $M_{R_1} < M_{R_2} < M_{R_3}$ we have to consider the seesaw threshold effects which arise due to sequential decoupling of these fields at respective scales. We consider the normal hierarchical spectrum where the lowest neutrino mass, $m_1$ is a free parameter. The other two masses $m_2$ and $m_3$ are determined by the relation $m_2 = \sqrt{m_1^2 + \Delta m_{12}^2}$ and $m_3 = \sqrt{m_1^2 + \Delta m_{13}^2}$ where $\Delta m_{12}^2$ and $\Delta m_{13}^2$ are the solar and the atmospheric mass squared differences, respectively.

### 3 RGE equations in the presence of seesaw threshold effects

Extended by three right handed neutrinos, the leptonic Yukawa terms of the Lagrangian in the SM can be written as

\[ -\mathcal{L}_{(SM)} = \bar{l}_i H Y_{l_i l} + \bar{l}_i \tilde{H} Y_{\nu R} + \frac{1}{2} \bar{\nu}^c R M_R \nu_R + \text{h.c.} \] \[ (3.1) \]

For the MSSM it is

\[ -\mathcal{L}_{(MSSM)} = \bar{l}_i H_1 Y_{l_i l} + \bar{l}_i \tilde{H}_2 Y_{\nu R} + \frac{1}{2} \bar{\nu}^c R M_R \nu_R + \text{h.c.} \] \[ (3.2) \]

where $H (\tilde{H} = i\sigma^2 H^*)$ is the SM Higgs doublet ($H_1, H_2$ for MSSM), $l_i, e_R, \nu_R$ are the lepton $SU(2)_L$ doublet, right handed charged leptons and right handed neutrinos, respectively. The three right handed singlet neutrinos with heavy Majorana masses are introduced at high scale far away from the EW scale. The current neutrino mixing angles and mass squared differences are determined from the neutrino oscillation experiments at the low
energy scale. The seesaw induced neutrino mass operator is given at the seesaw scale. Therefore, in addition to the RGE, these neutrino parameters will be modified by the seesaw threshold corrections originating from decoupling of the heavy right handed neutrino masses at different seesaw scales. The seesaw threshold corrections can be quite significant at the seesaw scales as the heavy singlets can be nondegenerate i.e. \( M_{R_1} < M_{R_2} < M_{R_3} \). At energy above the seesaw, i.e., say at the GUT scale we have to consider full theory. Thus interplay of the heavy and the light sector can lead to different running behavior from that in the effective theory.

In the flavor basis the effective light neutrino mass matrix, \( M_\nu \), above the highest seesaw scale is given to be

\[
M_\nu(\mu) = -\frac{\kappa(\mu)v^2}{4}, \tag{3.3}
\]

here \( v = 246 \text{ GeV} \) in the SM and \((246 \text{ GeV})\sin\beta \) in the MSSM, \( \mu \) is the renormalization scale and \( \kappa \) is the effective coupling matrix given as

\[
\kappa(\mu) = 2Y_\nu^T(\mu)M_R^{-1}(\mu)Y_\nu(\mu). \tag{3.4}
\]

The RGEs of Yukawa couplings and \( M_R \) are given as

\[
16\pi^2 \frac{dY_i}{dt} = Y_i(\alpha_i + C_l^i Y_i^\dagger Y_i + C_R^i Y_i^\dagger Y_\nu), \tag{3.5}
\]

\[
16\pi^2 \frac{dY_\nu}{dt} = Y_\nu(\alpha_\nu + C_{lR}^\nu Y_i^\dagger Y_i + C_{R\nu}^\nu Y_i^\dagger Y_\nu), \tag{3.6}
\]

\[
16\pi^2 \frac{dM_R}{dt} = C_R[(Y_i Y_i^\dagger)M_R + M_R(Y_i Y_i^\dagger)^T]. \tag{3.7}
\]

The coefficients \( C_l^i, C_{lR}^\nu, C_R^i, C_{R\nu}^\nu \) and \( C_R \) are \( \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{3}{2} \) and \( 1 \) in the SM and \( 3, 1, 1, 3 \) and \( 2 \) in the MSSM, respectively. \( t = \ln \mu/\mu_0 \) with \( \mu (\mu_0) \) is the running (fixed) scale. The parameters \( \alpha_i (i = l, \nu) \) are explicitly given as

\[
\alpha_l(\text{SM}) = \text{Tr}(3Y_i^\dagger Y_u + 3Y_i^\dagger Y_d + Y_i^\dagger Y_l + Y_i^\dagger Y_\nu) - \left( \frac{9}{4}g_1^2 + \frac{9}{4}g_2^2 \right),
\]

\[
\alpha_\nu(\text{SM}) = \text{Tr}(3Y_i^\dagger Y_u + 3Y_i^\dagger Y_d + Y_i^\dagger Y_l + Y_i^\dagger Y_\nu) - \left( \frac{20}{9}g_1^2 + \frac{9}{4}g_2^2 \right),
\]

\[
\alpha_l(\text{MSSM}) = \text{Tr}(3Y_i^\dagger Y_d + Y_i^\dagger Y_l) - \left( \frac{9}{5}g_1^2 + 3g_2^2 \right),
\]

\[
\alpha_\nu(\text{MSSM}) = \text{Tr}(3Y_i^\dagger Y_u + Y_i^\dagger Y_\nu) - \left( \frac{3}{5}g_1^2 + 3g_2^2 \right), \tag{3.8}
\]

where \( g_{l,2} \) are the \( U(1)_Y \) and \( SU(2)_L \) gauge coupling constants. The effective operator at the heaviest scale, \( M_{R_3} \) is given by the matching condition

\[
\kappa^{(3)} = 2Y_\nu^T M_{R_3}^{-1} Y_\nu, \tag{3.9}
\]

in the basis where \( M_R \) is diagonal. The effective neutrino mass matrix at the scale below \( M_{R_3} \) now constitutes of two parts

\[
M_\nu = -\frac{v^2}{4}[\kappa^{(3)} + 2Y_\nu^T(3)M_R^{-1(3)}Y_\nu^{(3)}], \tag{3.10}
\]
where $\kappa^{(3)}$ is given in the above Eq. (3.9). $Y_\nu^{(3)}$ is now $2 \times 3$ and $M_R^{(3)}$ is the $2 \times 2$ mass matrix remained after decoupling $M_{R_1}$. RGE at this stage is governed by running of $Y_\nu^{(3)}$, $M_R^{(3)}$ given earlier in Eqs. (3.6, 3.7) and $\kappa^{(3)}$ given as

$$16\pi^2 \frac{d\kappa^{(3)}}{dt} = \left[ (C^d_\nu Y_\nu^\dag Y_\nu Y_\nu^\dag Y_\nu Y_\nu^\dag Y_\nu Y_\nu^\dag Y_\nu) + \kappa^{(3)}(C^d_\nu Y_\nu^\dag Y_\nu Y_\nu^\dag Y_\nu Y_\nu^\dag Y_\nu Y_\nu^\dag Y_\nu + \alpha^{(3)}\kappa^{(3)}) \right],$$

where $C_\nu = \frac{1}{2}$, $\alpha^{(3)}$ in the SM and the MSSM is given as

$$\alpha^{(3)(SM)} = 2 \text{Tr}(3Y_d^\dag Y_d + 3Y_u^\dag Y_u + Y_l^\dag Y_l + Y_\nu^\dag Y_\nu)^{(3)} - 3g_2^2 + \lambda,$$

$$\alpha^{(3)(MSSM)} = 2 \text{Tr}(3Y_d^\dag Y_d + Y_\nu^\dag Y_\nu)^{(3)} - \frac{6}{5}g_1^2 - 6g_2^2.$$ 

Here $\lambda$ is the Higgs self coupling in the SM. The effective operator $\kappa^{(2)}$ is given by the matching condition as

$$\kappa^{(2)} = \kappa^{(3)} + 2Y_\nu^{(2)}M_R^{-1(2)}Y_\nu^{(2)},$$

where all the variables are set to scale $M_{R_2}$. The running of $\kappa^{(2)}$ is governed by the Eq. (3.11) except $Y_\nu^{(3)}$ is replaced by $Y_\nu^{(2)}$ which is $1 \times 3$ matrix left after integrating out the heavy state $M_{R_2}$. The low energy effective theory operator $\kappa^{(1)}$ is obtained after decoupling all the heavy right handed fields. The one loop RGE for $\kappa^{(1)}$ from $M_{R_1}$ scale down to EW scale is given as

$$16\pi^2 \frac{d\kappa^{(1)}}{dt} = (C^d_\nu (Y_\nu^\dag Y_\nu)^{(1)} + \kappa^{(1)}(C^d_\nu (Y_\nu^\dag Y_\nu)) + \alpha^{(1)},$$

where parameter $\alpha$ is explicitly given by

$$\alpha(SM) = 2 \text{Tr}(3Y_d^\dag Y_d + 3Y_u^\dag Y_u + Y_l^\dag Y_l - 3g_2^2 + \lambda,$$

$$\alpha(MSSM) = 2 \text{Tr}(3Y_d^\dag Y_d) - \frac{6}{5}g_1^2 - 6g_2^2.$$

The effective neutrino mass matrix obtained from $\kappa^{(1)}$ at the EW scale is diagonalized so as to obtain the neutrino mixing angles, CP violating phases and mass squared differences.

The neutrino mass matrices at two different scales $\Lambda_{GUT}$ and $\Lambda_{EW}$ are homogeneously related as [25, 26]

$$M_{\nu}^{\Lambda_{EW}} = I_K^T M_{\nu}^{\Lambda_{GUT}} I_K,$$

here $I_K$ is the scale factor common to all elements of $M_{\nu}^{\Lambda_{EW}}$. The matrix $I_K$ is given as

$$I_K = \begin{pmatrix} \sqrt{T_e} & 0 & 0 \\ 0 & \sqrt{T_\mu} & 0 \\ 0 & 0 & \sqrt{T_\tau} \end{pmatrix}.$$

In the presence of seesaw threshold corrections we have

$$\sqrt{T_j} = \text{Exp} \left( -\frac{1}{16\pi^2} \int [3(Y_j^\dag Y_j) - (Y_\nu^\dag Y_\nu)] dt \right) = e^{-\Delta_j},$$

- 7 -
\( j = e, \mu \) and \( \tau \).

For \( Y_\tau \sim 0.01 \) and \( Y_{\nu_\tau} \sim 0.3 \), the magnitude of \( \Delta_\tau \) can be of the order of \( 10^{-3} \) in the SM \((10^{-3}(1 + \tan^2 \beta)) \) in the MSSM, whereas it is of the order of \( 10^{-5} \) if \( Y_{\nu_\tau} \) is absent and threshold corrections are turned off from \( 10^{12}\text{GeV} \) to \( 10^2\text{GeV} \). Due to the seesaw corrections and thus the contribution from \( Y_{\nu_\tau} \) term, the mixing angles and masses can be largely affected above the seesaw scales. Below the seesaw scales when all the heavy right handed fields are integrated out, the running in the SM is mostly governed by \( Y_\tau \sim \sqrt{2}m_\tau/v \approx O(10^{-2}) \) and \( Y_\tau \sim \sqrt{2}m_\tau/(v \cos \beta) \) in the MSSM and there is no significant running effect on mixing angles in the SM even for quasidegenerate spectrum of masses in this region. In the MSSM case, however, there can be significant running effects in the case of large \( \tan \beta \). The analytic expressions of the RGE of masses, mixing angles and CP phases below the seesaw scales are given in \([28]\) and the expressions for running above the seesaw scales are given in \([24, 29]\) in detail.

4 Numerical results

We begin at \( \Lambda_{\text{GUT}} \) by varying the three angles and phase of \( Y_\nu \) along with two Majorana phases in the range of \( 0 \)–\( 2\pi \). The three hierarchical parameters of \( Y_\tau \) and \( m_1 \) are also randomly varied. For simplicity we start with \( \delta_{\text{CP}} = 0^\circ \) at \( \Lambda_{\text{GUT}} \), for our main purpose is to study the effects of the Majorana phases on the RGE running. We illustrate the parameter space for each mixing scenarios which can lead to the current data at low scale. In all the Tables in the paper we give the set of input parameters in the considered parameter space for which we achieve the maximum value of \( \theta_{13} \) at low scale and other mixing angles and mass squared differences in their \( 3\sigma \) ranges. However, we insist that this parameter space obtained is not unique and we give it for illustration. Search for the complete parameter space with such large number of free parameters is quite tedious and independent work in itself. We, however, restrict our study to the given parameter space as the main motive is to study the possibility of these mixings to be present at some high energy scale.

4.1 RGE and seesaw threshold corrections in the SM

In this analysis we start with different neutrino mass matrices, \( M_\nu \) that are diagonalized by the mixing matrices, \( U \) given in Table 2 respectively, generating vanishing \( \theta_{13} \) and maximal \( \theta_{23} \). All these scenarios have different value of the solar mixing angle \( \theta_{12} \). We begin with the four mixing scenarios at \( \Lambda_{\text{GUT}} \) in the SM. The study of the radiative corrections in our analysis can be divided into three regions governed by different RGE equations in the respective regions as
i) RGE from \( \Lambda_{\text{GUT}} \) down to the highest seesaw scale \( M_{R_3} \),
ii) in between the three seesaw scales
iii) from lowest seesaw scale \( M_{R_1} \) down to \( \Lambda_{\text{EW}} \).

The threshold effects can affect the running behavior in between and above the seesaw scales. The analytical expressions for the RGE’s below and above seesaw scales are given in \([28]\) and \([29]\). At the leading order the expression for \( \theta_{12} \) is inversely proportional to
The SM Input $\phi_1, \phi_2 = 0$  $\phi_1, \phi_2 \neq 0$ The SM Output $\phi_1, \phi_2 = 0$  $\phi_1, \phi_2 \neq 0$

| Parameter | Value | Parameter | Value | Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|-----------|-------|-----------|-------|
| $r_1$     | 0.57$x10^{-3}$ | $r_2$ | 0.6 | $M_{R_1}$ (GeV) | 2.6$x10^4$ | $M_{R_1}$ (GeV) | 2.1$x10^9$ |
| $\delta$ | 14.7$^\circ$ | $\phi_1$ | 281.7$^\circ$ | $M_{R_2}$ (GeV) | 8.2$x10^8$ | $M_{R_3}$ (GeV) | 2.5$x10^9$ |
| $\theta_1$ | 216$^\circ$ | $\phi_2$ | 281.7$^\circ$ | $\theta_1$ | 35.1$^\circ$ | $\theta_2$ | 43.9$^\circ$ |
| $\phi_1$ | 0$^\circ$ | $\phi_2$ | 0$^\circ$ | $\theta_2$ | 44.1$^\circ$ | $\theta_3$ | 4.3$^\circ$ |
| $\phi_2$ | 0$^\circ$ | $\phi_2$ | 0$^\circ$ | $\theta_3$ | 1.6$^\circ$ | $\theta_3$ | 1.6$^\circ$ |
| $m_1$ (eV) | 3.6$x10^{-4}$ | $m_1$ (eV) | 2.94$x10^{-4}$ | $m_1$ (eV) | 0.021 |
| $\Delta m_{12}^2$ (eV$^2$) | 8.2$x10^{-5}$ | $\Delta m_{12}^2$ (eV$^2$) | 7.63$x10^{-5}$ | $\Delta m_{12}^2$ (eV$^2$) | 7.86$x10^{-5}$ |
| $\Delta m_{13}^2$ (eV$^2$) | 3.7$x10^{-3}$ | $\Delta m_{13}^2$ (eV$^2$) | 2.6$x10^{-3}$ | $\Delta m_{13}^2$ (eV$^2$) | 2.48$x10^{-3}$ |

Table 3. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects in the SM for TBM mixing when zero and nonzero values of $\phi_1$ and $\phi_2$ are taken at $\Lambda_{GUT} = 2x10^{16}$.

Solar mass squared difference ($\Delta m_{12}^2$), whereas $\theta_{23}$ and $\theta_{13}$ are both inversely related to atmospheric mass squared difference ($\Delta m_{13}^2$). Thus, $\theta_{12}$ is maximally affected by the RGE among all the mixing angles. In case of quasidegenerate neutrino spectrum there can be visible corrections to other mixing angles also. In the SM, the RGE corrections to neutrino mixing angles in the absence of threshold corrections are negligible. However, inclusion of the threshold effects can significantly modify neutrino masses and mixing angles. It is worthwhile to note that the RGE of $\theta_{12}$ depends strongly on the CP violating phases.

We study the RGE of the neutrino mixing angles from $\Lambda_{GUT}$ to $\Lambda_{EW}$ in the SM for zero and non zero Majorana phases in all four mixing scenarios. The RGE behavior of mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ for the TBM scenario in the SM is given in Fig 1. The upper panel is when both $\phi_1$ and $\phi_2$ are zero while the lower panel is when these phases are non zero. The RGE of mixing angles and masses is obtained for the set of input parameters taken at $\Lambda_{GUT}$ given in Table 3. The second and third columns in the give the set of input parameters and the last two columns give the output parameters obtained at $\Lambda_{EW}$ for zero and non zero phases. The set of input parameters is taken a benchmark for illustration as it is not unique parameter space. There is no significant deviations in angles and $\theta_{13}$ as large as 1.6$^\circ$ is produced when both $\phi_1$ and $\phi_2$ are zero. However, for non zero $\phi_1$ and $\phi_2$, $\theta_{13} \approx 4.3^\circ$ is produced at $\Lambda_{EW}$. Except $\theta_{13}$ all the other neutrino parameters are produced in their current $3\sigma$ ranges in both the cases. In Fig. 1 we also show the RGE of the neutrino masses from $\Lambda_{GUT}$ to $\Lambda_{EW}$ for both zero and non zero Majorana phases. The
Figure 1. The RGE of the mixing angles and masses between $\Lambda_{GUT}$ and $\Lambda_{EW}$ in the SM for zero and non zero $\phi_1$, $\phi_2$ for TBM mixing. The input parameters are given in the second and third column of Table 3. The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

running of the mass eigenvalues in this region below the seesaw scales can be significant in the SM due to the factor $\alpha (3.15)$ which can be larger than $Y_\tau^2$. As we can see from the right panel of Fig. 1 that there is running of masses even below $M_{R_i}$, irrespective of values of $\phi_1$ and $\phi_2$ which indicate that the running of masses is not directly dependent on the Majorana phases [30]. $M_{ee}$ (eV) and $J_{CP}$ of the order of $\sim 10^{-3}$ for zero phases and $10^{-2}$ for non zero phases is obtained at the EW scale. Fig. 2 shows the RGE of the mixing angles in the BM scenario for zero and non zero Majorana phases. Here we find that $\theta_{13} \approx 3.5^\circ$ is produced at low scale when Majorana phases are zero while $\theta_{13} \approx 5.07^\circ$ is produced from zero value of $\theta_{13}$ at high scale and non zero Majorana phases for the set of input parameters given in Table 4. The solar mixing angle, $\theta_{12}$ that is $\pi/4$ at $\Lambda_{GUT}$ can run down to the value of $34.7^\circ$ for zero $\phi_1$, $\phi_2$. For non zero $\phi_1$ and $\phi_2$ it takes the value near best fit ($\approx 33^\circ$) at $\Lambda_{EW}$. The Jarlskog invariant $J_{CP}$ of the order $-7\times 10^{-4}$ is produced when Majorana phases are not considered and increases to $-10^{-2}$ in the presence of these phases. For zero Majorana phases in BM scenario we get $M_{ee} \approx 10^{-3}$ eV and $10^{-2}$eV for
Table 4. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects in the SM for BM mixing when zero and non zero values of $\phi_1$ and $\phi_2$ are taken at $\Lambda_{GUT} = 2 \times 10^{16}$.

| The SM Input | The SM Output |
|--------------|---------------|
| $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ |
| $r_1$ | $0.64 \times 10^{-2}$ | $0.24 \times 10^{-2}$ |
| $r_2$ | 0.65 | 0.703 |
| $\delta$ | $243^\circ$ | $268^\circ$ |
| $y_\nu$ | 0.37 | 0.74 |
| $\theta_1$ | $243^\circ$ | $163^\circ$ |
| $\theta_2$ | $60.2^\circ$ | $329^\circ$ |
| $\theta_3$ | $306^\circ$ | $333.5^\circ$ |
| $m_1$ (eV) | $0.0264$ | $4.8 \times 10^{-3}$ |
| $\Delta m^2_{12}$ (eV$^2$) | $1.5 \times 10^{-4}$ | $4.8 \times 10^{-7}$ |
| $\Delta m^2_{13}$ (eV$^2$) | $3.07 \times 10^{-3}$ | $3.4 \times 10^{-3}$ |
| $\phi_1$ | $0^\circ$ | $7.85^\circ$ |
| $\phi_2$ | $0^\circ$ | $112.6^\circ$ |
| $\delta_m$ | $3.05^\circ$ | $90^\circ$ |
| $J_{CP}$ | $-7.3 \times 10^{-4}$ | $-0.01$ |
| $M_{ee}$ (eV) | $4.14 \times 10^{-3}$ | $0.024$ |

Figure 2. The RGE of the mixing angles and masses between $\Lambda_{GUT}$ and $\Lambda_{EW}$ in the SM for zero and non nonzero $\phi_1, \phi_2$ for BM mixing. The input parameters are given in the second and third column of Table (4). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

non zero values. For the HM, the RGE of the mixing angles obtained in the output of Table 5 is given in Fig. 3. The left panel shows that small $\theta_1 \approx 1.4^\circ$ is produced at the
low energy scale for zero Majorana phases and \( \theta_{13}=0 \) at the high scale. \( \theta_{12} \) and \( \theta_{23} \) shows little corrections to their values of \( \pi/6 \) and \( \pi/4 \) at \( \Lambda_{\text{GUT}} \) as seen in Fig. 3. The right

| The SM Input \( \phi_1, \phi_2 = 0 \) | \( \phi_1, \phi_2 \neq 0 \) | The SM Output \( \phi_1, \phi_2 = 0 \) | \( \phi_1, \phi_2 \neq 0 \) |
|--------------------------------|-----------------|-----------------|-----------------|
| \( r_1 \)  & \( 0.29 \times 10^{-2} \) & \( 0.63 \times 10^{-2} \) & \( M_{R_1} \) (GeV) & \( 6.4 \times 10^5 \) & \( 4.7 \times 10^5 \) |
| \( r_2 \)  & 0.57 & 0.68 & \( M_{R_2} \) (GeV) & \( 3.3 \times 10^9 \) & \( 3.1 \times 10^9 \) |
| \( \delta \) & 23.1° & 337.5° & \( M_{R_3} \) (GeV) & \( 3.7 \times 10^{10} \) & \( 7.6 \times 10^9 \) |
| \( y_\nu \)  & 0.661 & 0.59 & \( \theta_{12} \) & 33.7° & 34.4° |
| \( \theta_1 \) & 146° & 147.2° & \( \theta_{23} \) & 45.2° & 44.8° |
| \( \theta_2 \) & 261° & 271.6° & \( \theta_{13} \) & 1.4° & 6.9° |
| \( \theta_3 \) & 175.3° & 92.25° & \( m_1 \) (eV) & \( 4.14 \times 10^{-3} \) & 0.0294 |
| \( \Delta m_{12}^2(eV^2) \) & 9.6 \times 10^{-5} & 8.6 \times 10^{-5} & \( \Delta m_{12}^2(eV^2) \) & \( 7.4 \times 10^{-5} \) & \( 7.63 \times 10^{-5} \) |
| \( \Delta m_{13}^2(eV^2) \) & \( 3.65 \times 10^{-3} \) & \( 3.8 \times 10^{-3} \) & \( \Delta m_{13}^2(eV^2) \) & \( 2.4 \times 10^{-3} \) & \( 2.6 \times 10^{-3} \) |
| \( \phi_1 \)  & \( 0° \) & \( 340.3° \) & \( \phi_1 \) & 160.2° & 25.9° |
| \( \phi_2 \)  & \( 0° \) & \( 219.6° \) & \( \phi_2 \) & 151.4° & 242° |


Table 5. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects in the SM for HM mixing when zero and non zero \( \phi_1 \) and \( \phi_2 \) are taken at \( \Lambda_{\text{GUT}} = 2 \times 10^{16} \).

Figure 3. The RGE of the mixing angles and masses between \( \Lambda_{\text{GUT}} \) and \( \Lambda_{\text{EW}} \) in the SM for zero and non zero \( \phi_1, \phi_2 \) for HM mixing. The input parameters are given in the second and third column of Table (5). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.
panel shows the deviations for mixing angles for non zero Majorana phases. Quite large value of $\theta_{13} \approx 6.9^\circ$ is produced at the low scale. Also deviation can be seen in $\theta_{12} \sim 34^\circ$ at the electroweak scale. $J_{CP} \approx 5 \times 10^{-3}$ ($\approx 10^{-2}$) is produced for zero (non zero) phases. The effective Majorana mass $M_{ee} \approx 5 \times 10^{-3}$eV ($\approx 10^{-2}$eV) is produced at low scale for HM mixing. In the GR mixing we have $\theta_{13}=0$, $\theta_{23}=\pi/4$ and $\theta_{12} \approx 31.7^\circ$ at the GUT scale. From the right panel of Fig. 4 we conclude that there is very small deviation in $\theta_{13} \approx 1.7^\circ$ from $\Lambda_{GUT}$ to $\Lambda_{EW}$ in the absence of Majorana phases. $\theta_{12}$ shows the RGE towards the upper direction to value of $34.4^\circ$ while there is no large deviation for $\theta_{23}$ in this case. The input values for the parameters are given in second column of Table 6. For the non zero Majorana phases $\theta_{13} \approx 6^\circ$ is obtained at the low scale as shown in the right panel of Fig 4. The Jarlskog invariant $J_{CP}$ of the order $7 \times 10^{-3}$ is produced when Majorana phases are not considered and increases to $-10^{-2}$ in the presence of these phases. For zero Majorana phases in GR scenario we get effective Majorana mass $M_{ee} \approx 10^{-3}$eV and $10^{-2}$eV for non zero values.

| The SM Input | $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ | The SM Output | $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ |
|-------------|---------------------|---------------------|---------------|---------------------|---------------------|
| $r_1$       | 0.27 $\times 10^{-2}$ | 0.68 $\times 10^{-3}$ | $M_{R_1}$ (GeV) | 8.7 $\times 10^5$ | 4.9 $\times 10^5$ |
| $r_2$       | 0.6                  | 0.35                 | $M_{R_2}$ (GeV) | 4 $\times 10^9$ | 7.7 $\times 10^8$ |
| $\delta$    | 55.2$^\circ$         | 63.0$^\circ$         | $M_{R_3}$ (GeV) | 6.6 $\times 10^{10}$ | 1.8 $\times 10^9$ |
| $y_\nu$     | 0.7                  | 0.65                 | $\theta_1$     | 34.4$^\circ$ | 34.2$^\circ$ |
| $\theta_1$  | 46.8$^\circ$         | 192.5$^\circ$        | $\theta_1$     | 34.4$^\circ$ | 34.2$^\circ$ |
| $\theta_2$  | 225$^\circ$          | 185.6$^\circ$        | $\theta_2$     | 44.6$^\circ$ | 41.9$^\circ$ |
| $\theta_3$  | 123$^\circ$          | 249$^\circ$          | $\theta_3$     | 17.3$^\circ$ | 6$^\circ$ |
| $m_1$ (eV)  | 2.1 $\times 10^{-3}$ | 0.079                | $m_1$ (eV)     | 1.62 $\times 10^{-3}$ | 0.067 |
| $\Delta m_{12}^2$(eV$^2$) | 8.9 $\times 10^{-5}$ | 9.3 $\times 10^{-5}$ | $\Delta m_{12}^2$(eV$^2$) | 7.25 $\times 10^{-5}$ | 7.6 $\times 10^{-5}$ |
| $\Delta m_{13}^2$(eV$^2$) | 3.7 $\times 10^{-3}$ | 3.8 $\times 10^{-3}$ | $\Delta m_{13}^2$(eV$^2$) | 2.3 $\times 10^{-3}$ | 2.3 $\times 10^{-3}$ |
| $\phi_1$    | 0$^\circ$            | 240.6$^\circ$        | $\phi_1$       | 171$^\circ$ | 136$^\circ$ |
| $\phi_2$    | 0$^\circ$            | 353.5$^\circ$        | $\phi_2$       | 185$^\circ$ | 107.5$^\circ$ |
| $\theta_{CP}$ | -                   | -                   | $J_{CP}$       | 0.007      | -0.02      |
| $M_{ee}$    | 3.8 $\times 10^{-3}$ | -                   |               | 0.06       |

Table 6. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects both in the SM for GR mixing when zero and non zero $\phi_1$ and $\phi_2$ are taken at $\Lambda_{GUT} = 2 \times 10^{16}$.

In all four cases, there are considerable RGE effects for $\theta_{12}$ and it is possible to get it near the best fit value the value (33$^\circ$) at the low scale when it ranges from (30$^\circ$-45$^\circ$) at the high GUT scale. In the absence of the Majorana phases it is not possible to have large values of $\theta_{13}$ at the low scale in the SM. In [23] it is shown that in the SM, $\theta_{13}$ as large as $5^\circ$ can only be obtained when very large $\theta_{12}=67^\circ$ is considered at the GUT scale. However, taking into consideration the Majorana phases in the SM results in large $\theta_{13}$ ($\approx 5^\circ$-6.9$^\circ$) at
Figure 4. The RGE of the mixing angles and masses between $\Lambda_{GUT}$ and $\Lambda_{EW}$ in the SM for zero and non zero $\phi_1$ and $\phi_2$ in GR mixing. The input parameters are given in the second and third column of Table (6). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

As written above, the study of RGE can be divided in three regions governed by different RGE equations in these respective regions. In the region below the seesaw scales the RGE corrections to the neutrino mixing angles are negligible in the SM. This is due to the small corrections arising due to $Y_l$ that is quite small. As seen from the figs 1, 2, 3 and 4 in the RGE of neutrino mixing angles for zero and non zero Majorana phases $\phi_1$ and $\phi_2$, there is practically no significant deviations below the lowest seesaw scale $M_{R_1}$ down to $\Lambda_{EW}$. At the energy scale in between and above the seesaw scales there will be additional contributions of $Y_\nu$ in addition to $Y_l$. Thus, the RGE is dependent on $Y_\nu$ which is free and can be as large as $O(1)$. Heavy right handed fields are subsequently integrated out at the three seesaw scales shown by the three grey regions in the Figures, and thus $(n-1)\times 3$ submatrix of $Y_\nu$ remains after each step of integrating out $M_{R_i}$. As can be seen from Eq. (3.10) that the running in between the seesaw scales is dependent on the sum of two terms $\kappa^{(n)}$ and $2Y_\nu^{T(n)} M_{R_i}^{(n)} Y_\nu^{(n)}$. As discussed in [28], in the SM the RGE scaling in these two parts is different due to interaction with trivial flavor structure. This implies that there can be large corrections for the mixing angles in between these threshold scales in the SM. Again we find there is significant running of mixing angles especially $\theta_{12}$ in between and above the seesaw scale for non zero Majorana phases in the SM whereas small running for vanishing $\phi_1$ and $\phi_2$. The three grey regions in all the Figures illustrate the validate ranges of the effective theories that emerge due to decoupling of the heavy right handed states. In addition, the byproduct of the analysis is the three seesaw scales $M_{R_i}$. 

the low scale. $\theta_{12}$ at $\Lambda_{GUT}$ is in the range of $(30^\circ$-$45^\circ)$. Thus, the Majorana phases can significantly affect the RGE of neutrino mixing angles [31] as observed above. However, in the SM the value of $\theta_{13}$ is still below the $3\sigma$ allowed range at low scale for both zero and non zero phases.
given in output which are not free parameters but determined from Eqs. (2.4, 2.5). The order of the effective Majorana mass and the Jarlskog rephasing invariant for each scenario are calculated at $\Lambda_{EW}$ for each set of the input parameters in the corresponding tables, respectively.

4.2 RGE and seesaw threshold corrections in the MSSM

The study of radiative corrections in the MSSM in presence of seesaw threshold effects can again be divided into three regions as above in the SM. All these three regions will be governed by different RGE equations, respectively. The RGE corrections to the mixing angles in the MSSM for region below seesaw scale can be larger than the SM due to the presence of factor $Y^2_{\tau}(1+tan^2\beta)$. This term can be large when $tan\beta$ is large, thus, resulting in significant changes in the mixing angles where $Y_l$ is the only contributing term.

| The MSSM Input | $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ | The MSSM Output | $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ |
|----------------|----------------------|-----------------------|----------------|----------------------|-----------------------|
| $r_1$          | $0.36 \times 10^{-3}$ | $0.42 \times 10^{-3}$ | $M_{R_1}$ (GeV) | $9.9 \times 10^3$    | $9.13 \times 10^3$    |
| $r_2$          | 0.47                 | 0.68                  | $M_{R_2}$ (GeV) | $2.1 \times 10^9$    | $2.04 \times 10^9$    |
| $\delta$       | 238°                 | 196°                  | $M_{R_3}$ (GeV) | $4.0 \times 10^{10}$ | $1.36 \times 10^{10}$ |
| $y_\nu$        | 0.56                 | 0.46                  | $\theta_{12}$   | 35.2°                | 34.3°                 |
| $\theta_1$     | 176°                 | 300.2°                | $\theta_{23}$   | 49.5°                | 40.6°                 |
| $\theta_2$     | 256°                 | 13.06°                | $\theta_{13}$   | 3.46°                | 9.46°                 |
| $\theta_3$     | 66.5°                | 124.9°                | $\theta_{13}$   | 3.46°                | 9.46°                 |
| $m_1$ (eV)     | $3.4 \times 10^{-3}$ | $4.5 \times 10^{-3}$  | $m_1$ (eV)      | $2.2 \times 10^{-3}$ | $5.9 \times 10^{-3}$  |
| $\Delta m^2_{12}$ (eV$^2$) | $2.1 \times 10^{-5}$ | $7.33 \times 10^{-5}$ | $\Delta m^2_{12}$ (eV$^2$) | $8 \times 10^{-5}$ | $7.48 \times 10^{-5}$  |
| $\Delta m^2_{13}$ (eV$^2$) | $2.5 \times 10^{-3}$ | $3.9 \times 10^{-3}$  | $\Delta m^2_{13}$ (eV$^2$) | $2.56 \times 10^{-3}$ | $2.57 \times 10^{-3}$  |
| $\phi_1$       | 0°                   | 256.8°                | $\phi_1$       | 50.0°                | 112.7°                |
| $\phi_2$       | 0°                   | 210.8°                | $\phi_2$       | 30°                   | 10.5°                 |
| $J_{CP}$       | -                    | -                     |                | -3.6 $\times 10^{-3}$ | -0.0156               |
| $M_{ee}$ (eV)  | -                    | -                     |                | 3.3 $\times 10^{-3}$  | 3.7 $\times 10^{-3}$  |

Table 7. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects in the MSSM for TBM mixing when zero and nonzero values of $\phi_1$ and $\phi_2$ are taken at $\Lambda_{GUT} = 2 \times 10^{16}$ and $\tan\beta=10$.

We study the RGE of the mixing angles with zero and non zero Majorana phases in the MSSM with $\tan\beta=10$. The RGE of the three neutrino mixing angles for the TBM scenario is given in Fig 5. The set of input parameters taken as a benchmark for illustration are given in Table 7. The first two columns in the give input parameters for zero and non zero $\phi_1$ and $\phi_2$ at $\Lambda_{GUT}$ while fourth and fifth columns give the corresponding output parameters at $\Lambda_{EW}$. The upper panel of Fig 5. is for zero Majorana phases. There are small corrections to all the three mixing angles here. $\theta_{13}$ as large as 3.46° can be obtained at the low scale. $\theta_{23}$ is deviated in the upper direction by $\approx 4.5°$ from its initial value of...
Figure 5. The RGE of the mixing angles and masses between $Λ_{GUT}$ and $Λ_{EW}$ in the MSSM with $\tan\beta=10$ for zero and non zero $φ_1$, $φ_2$ for TBM mixing. The input parameters are given in the second and third column of Table (7). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

$\pi/4$ at GUT scale. For the non zero Majorana phases there can be large corrections to $θ_{13}$ as seen in the lower panel of Fig. 5. $θ_{23}$ and $θ_{12}$ are deviated in the lower direction to the value of $40.6°$ and $34.3°$. Large corrections are possible in $θ_{13}$ as it attains the value $\approx 9.46°$ at the low scale. Thus, it is possible to obtain all the neutrino oscillation parameters in their $3σ$ ranges at the low scale in this case. We also see the radiative corrections to the masses in the Fig. 5. The running of masses, however, as in the SM is independent of the mixing parameters since $α$ is usually much larger than $Y_τ^2 (1+\tan^2β)$ except in the MSSM with large $\tanβ$. RGE effects of neutrino masses are smallest if $\tanβ=10$. The negligible running of masses is seen below the seesaw scales irrespective of values of $φ_1$ and $φ_2$ which indicate that the running of masses is not directly dependent on the Majorana phases [30]. The Jarlskog invariant $J_{CP}$ of the order $-10^{-3}$ is produced when Majorana phases are not considered and increases to $-10^{-2}$ in the presence of these phases. For zero and non zero Majorana phases in TBM scenario we get $M_{ee} \approx 10^{-3}$ eV at low scale. Fig. 6 gives the radiative corrections to the mixing angles in the BM scenario when threshold effects are
Table 8. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects in the MSSM for BM mixing when zero and non zero values of $\phi_1$ and $\phi_2$ are taken at $\Lambda_{GUT} = 2 \times 10^{16}$ and $\tan \beta = 10$.

| The MSSM Input | The MSSM Output |
|----------------|-----------------|
| $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ |
| $\phi_1$ | $\phi_1$ |
| $\phi_2$ | $\phi_2$ |
| $\phi_1, \phi_2 \neq 0$ | $\phi_1, \phi_2 \neq 0$ |

| Parameter | Value $\phi_1, \phi_2 = 0$ | Value $\phi_1, \phi_2 \neq 0$ |
|-----------|-----------------|-----------------|
| $r_1$     | $0.274 \times 10^{-3}$ | $0.59 \times 10^{-3}$ |
| $r_2$     | $0.59$ | $0.54$ |
| $\delta$  | $261.3^\circ$ | $157.5^\circ$ |
| $y_\nu$   | $0.584$ | $0.66$ |
| $\theta_1$ | $44.3^\circ$ | $115^\circ$ |
| $\theta_2$ | $352^\circ$ | $356^\circ$ |
| $\theta_3$ | $68.2^\circ$ | $296^\circ$ |
| $m_1$ (eV) | $1.76 \times 10^{-3}$ | $5.57 \times 10^{-3}$ |
| $\Delta m_{12}^2$ (eV$^2$) | $5.2 \times 10^{-7}$ | $4.3 \times 10^{-7}$ |
| $\Delta m_{13}^2$ (eV$^2$) | $3.2 \times 10^{-3}$ | $3.45 \times 10^{-3}$ |
| $\theta_1$ | $0^\circ$ | $253.2^\circ$ |
| $\theta_2$ | $0^\circ$ | $295.3^\circ$ |
| $\theta_3$ | $- -$ | $- -$ |
| $J_{CP}$ | $-0.56 \times 10^{-2}$ | $0.0233$ |
| $M_{ee}$ (eV) | $2.6 \times 10^{-3}$ | $4.3 \times 10^{-3}$ |

Figure 6. The RGE of the mixing angles between $\Lambda_{GUT}$ and $\Lambda_{EW}$ in the MSSM with $\tan \beta = 10$ for zero and non zero $\phi_1$, $\phi_2$ for BM mixing. The input parameters are given in second and third column of Table (8). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

present. For zero Majorana phases $\theta_{13} \approx 3.5^\circ$ can be produced. $\theta_{23} \approx 38.7^\circ$ and $\theta_{12} \approx 31.5^\circ$ are obtained at the EW scale from the initial value of $\pi/4$ at $\Lambda_{GUT}$. For non zero phases $\phi_1$
and $\phi_2$ there can be large corrections to all the mixing angles. $\theta_{13} \approx 7.41^\circ$ can be obtained at low scale. Both angles $\theta_{23}$ and $\theta_{12}$ get the corrections towards the lower direction. $\theta_{12}$ is obtained near it’s current best fit 33.4$^\circ$ and $\theta_{23}$ is obtained below maximal. The set of input parameters for Fig 6. is given in Table 8. The fourth and fifth column also show the values of the mass squared differences obtained at the low scale. The Jarlskog invariant $J_{CP}$ of the order $(-5 \times 10^{-3})$ is produced when Majorana phases are not considered and increases to $-10^{-2}$ in the presence of these phases. For zero and non zero Majorana phases in BM scenario we get $M_{ee} \approx 10^{-3}$ eV at the low scale. In the HM scenario the input mixing angles at GUT scale are $\theta_{13}=0^\circ$, $\theta_{23}=\pi/4$ and $\theta_{12}=\pi/6$ at $\Lambda_{GUT}$. The set of the input parameters at the GUT scale are given in the second column of Table 9 where $\phi_1$, $\phi_2$ are zero.

| The MSSM Input | $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ | The MSSM Output | $\phi_1, \phi_2 = 0$ | $\phi_1, \phi_2 \neq 0$ |
|----------------|----------------------|------------------------|------------------|----------------------|------------------------|
| $r_1$          | $0.51 \times 10^{-3}$ | $0.68 \times 10^{-3}$  | $M_{R_1}$ (GeV)  | $4.36 \times 10^4$  | $5.2 \times 10^4$      |
| $r_2$          | 0.49                 | 0.46                   | $M_{R_2}$ (GeV)  | $2.2 \times 10^9$   | $2.6 \times 10^9$      |
| $\delta$      | 34.7$^\circ$         | 214.3$^\circ$          | $M_{R_3}$ (GeV)  | $9.5 \times 10^{10}$| $3.3 \times 10^{10}$   |
| $y_{\nu}$      | 0.57                 | 0.675                  | $\theta_{12}$   | $33.8^\circ$        | $35.5^\circ$           |
| $\theta_1$    | 126$^\circ$          | 55.6$^\circ$           | $\theta_{23}$   | $39.7^\circ$        | $39.9^\circ$           |
| $\theta_2$    | 276$^\circ$          | 145$^\circ$            | $\theta_{13}$   | $3.7^\circ$         | $7.3^\circ$            |
| $\theta_3$    | 319$^\circ$          | 278.4$^\circ$          | $m_1$ (eV)      | $5.84 \times 10^{-4}$| $3.95 \times 10^{-3}$ |
| $m_{11} (eV)$  | $2.03 \times 10^{-3}$| $5.5 \times 10^{-3}$   | $m_{12} (eV)$   | $5.74 \times 10^{-5}$| $7.6 \times 10^{-5}$   |
| $\Delta m^2_{12} (eV^2)$ | $4.8 \times 10^{-8}$ | $4.7 \times 10^{-8}$ | $\Delta m^2_{12} (eV^2)$ | $2.6 \times 10^{-3}$ | $2.5 \times 10^{-3}$ |
| $\Delta m^2_{13} (eV^2)$ | $3.0 \times 10^{-3}$ | $3.25 \times 10^{-3}$ | $\Delta m^2_{13} (eV^2)$ | $7.45 \times 10^{-5}$ | $7.6 \times 10^{-5}$ |
| $\phi_1$      | 0$^\circ$            | 304.6$^\circ$          | $\phi_1$        | $71^\circ$          | $321.4^\circ$          |
| $\phi_2$      | 0$^\circ$            | 308.3$^\circ$          | $\phi_2$        | $99^\circ$          | $322.5^\circ$          |
| $J_{CP}$       | -                    | -                      | $M_{ee} (eV)$   | $0.9 \times 10^{-2}$| -0.01                  |
| $M_{ee} (eV)$  | -                    | -                      | -               | $2.5 \times 10^{-3}$| $5.2 \times 10^{-3}$   |

Table 9. Numerical values of input and output parameters radiatively generated via RGE and seesaw threshold effects in the MSSM for HM mixing when zero and non zero $\phi_1$ and $\phi_2$ are taken at $\Lambda_{GUT} = 2 \times 10^{16}$ and $\tan \beta=10$.

The solar mixing angle $\theta_{12}$ is again most affected angle that deviates to 33.8$^\circ$ very close to its best fit value at the low scale for zero $\phi_1$ and $\phi_2$ as can be seen from left panel of Fig. 7. Corrections of the order of $\approx 5.3^\circ$ and $\approx 3.7^\circ$ are obtained for $\theta_{23}$ and $\theta_{13}$, respectively. However, when $\phi_1$ and $\phi_2$ are non zero we find all the parameters to be in their allowed 3$\sigma$ ranges at the low scale. Right panel of Fig. 7 predicts $\theta_{13} \approx 7.3^\circ$ at the low scale. $\theta_{23}$ gets the correction of $\approx 5^\circ$ in the down direction and $\theta_{12}$ gets the same correction in the upper direction at the low scale. The Jarlskog invariant $J_{CP}$ of the order $9 \times 10^{-3}$ is produced when Majorana phases are not considered and increases to $\approx -10^{-2}$ in the presence of these phases. For zero and non zero Majorana phases in HM scenario we get $M_{ee} \approx 10^{-3}$eV at
Figure 7. The RGE of the mixing angles between $\Lambda_{GUT}$ and $\Lambda_{EW}$ in the MSSM with $\tan\beta=10$ for zero and non zero $\phi_1$, $\phi_2$ for HM mixing. The input parameters are given in the second and third column of Table (9). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

Finally, in Fig. 8 we show the RGE of the mixing angles with zero and non zero...
Figure 8. The RGE of the mixing angles between $\Lambda_{GUT}$ and $\Lambda_{EW}$ in the MSSM with $\tan \beta = 10$ for zero and non zero $\phi_1, \phi_2$ for GR mixing. The initial values of the parameters are given in the third column of Table (10). The grey shaded areas illustrate the ranges of effective theories when heavy right handed singlets are integrated out.

Majorana phases in the GR scanario. The corresponding set of input parameters are given in second and third columns of Table 10. We find the correction to $\theta_{12}$ is $\sim 3.6^\circ$ in the upper direction for zero Majorana phases shown in left panel of Fig. 8. Both $\theta_{13}$ and $\theta_{23}$ in this case have the corrections $\sim 3.7^\circ$ and $\sim 6^\circ$ at the low scale. However, non zero values of $\phi_1$ and $\phi_2$ result in $\theta_{13}$ in its current $3\sigma$ range ($\sim 8^\circ$) at the low scale along with the other neutrino parameters. $\theta_{23}$ is below maximal ($41.3^\circ$) and $\theta_{12}$ deviates to 34.8$^\circ$ in the upper direction. The Jarlskog invariant $J_{CP}$ of the order $-5 \times 10^{-3}$ is produced when Majorana phases are not considered and increases to $\approx 10^{-2}$ in the presence of these phases. For zero and non zero Majorana phases in GR scenario we get $M_{ee} \approx 10^{-3}$ eV at low scale.

As seen in all the Figures, the RGE running effects are small below lowest seesaw scale $M_{R_1}$ down to the EW scale for $\tan \beta = 10$. In the region above seesaw scale, $M_{R_3}$ there is large running of all the mixing angles as can be seen from the Figures of all the mixing scenarios in the MSSM. This is due to the contribution of $Y_{\nu}$ which can be large regardless to value of $\tan \beta$ in addition of $Y_l$. We have the large running of all mixing angles including $\theta_{13}$ in this region when Majorana phases are nonzero. In between the seesaw scale there is very small running in comparison to the SM. This behavior is described in [28] in detail as the enhanced running between the threshold due to the term with trivial flavor structure as in the SM do not exist in the MSSM. The Majorana phases play an important role in the enhancement of RGE and $\theta_{13}$ can be produced in its $3\sigma$ allowed range at the EW scale in all mixing scenario along with the other neutrino oscillation parameters. Thus, large value of $\theta_{13}$ which is in its present $3\sigma$ range at EW scale can be produced in the MSSM for $\tan \beta =10$ when $\phi_1$ and $\phi_2$ are both non zero at high scale. The order of the magnitude of effective Majorana mass and the Jarlskog rephasing invariant for each scenario is also calculated at $\Lambda_{EW}$ for the given input parameter space under consideration.
5 Conclusions

We assume different lepton mixing matrices at the high energy (GUT) scale and study effects of the RGE and seesaw threshold corrections to these mixing scenarios both in the SM and the MSSM. In the absence of seesaw threshold there are very small corrections both in the SM and the MSSM. Due to the presence of seesaw threshold corrections it is possible to have significant corrections both in the SM and the MSSM as there is contribution of $Y_\nu$ in addition to $Y_l$ above the seesaw scale. Thus, above the seesaw scale there are more number of parameters due to $Y_\nu$ that can significantly affect the RGE of mixing angles. Below the lowest seesaw scale, contribution of $Y_\nu$ is absent and the RGE corrections are only due to $Y_l$ which is very small in the SM and the MSSM with small $\tan \beta$. For large $\tan \beta$ however, there can be significant contribution below the lowest seesaw scale in the MSSM. Some of these mixing scenarios are studied in [23] at high scale without considering the effects of Majorana CP phases. In that case our results are somewhat similar that $\theta_{13} < 5^\circ$ can be obtained at the low energy scale. The Majorana phases, however, play a significant role in the running of the parameters as can be seen from the expressions of the masses and the mixing angles given in [24, 28, 29]. In the absence of these phases it is not possible to get $\theta_{13}$ in the allowed $3\sigma$ range at the low scale in both the SM and the MSSM with $\tan \beta=10$. However, when we take the non zero value of these phases at the high scale, it is possible to enhance $\theta_{13}$ to its allowed $3\sigma$ range in the MSSM. Here, we presented a comprehensive study by considering four different mixing scenarios at the GUT scale and study their running behavior in the SM and the MSSM with $\tan \beta=10$. Non zero value of $\phi_1$ and $\phi_2$ can modify the RGE both in the SM and the MSSM with seesaw effects. We conclude that when TBM, BM, HM and GR mixings are considered at some high scale, say the GUT scale, the RGE and seesaw threshold corrections can result in significant corrections to the mixing angles both in the SM and the MSSM at the low energy scale. It is found that in the MSSM with $\tan \beta=10$ it is possible to get all the mixing angles and the mass squared differences in their present $3\sigma$ range at the EW scale in all the considered four mixing scenarios when we take into account the Majorana phases.

6 Acknowledgements

The work of C.S.K and S.G. is supported by the National Research Foundation of Korea (NRF) grant funded by Korea government of the Ministry of Education, Science and Technology (MEST) (Grant No. 2011-0017430 and Grant No. 2011-0020333). The work of S.K.K. is supported by the NRF grant funded by Korea government of the MEST (No. 2011-0029758).

References

[1] G. Altarelli, F. Feruglio, Discrete Flavor Symmetries and Models of Neutrino Mixing, Rev. Mod. Phys. 82, 2701 (2010) [arXiv:1002.0211 [hep-ph]]; H. Ishimori, T. Kobayashi, H. Okuki, H. Okada, Y. Shimizu, M. Tanimoto, Non-Abelian Discrete Symmetries in Particle Physics, Prog. Theor. Phys. Suppl. 183, 1 (2010) [arXiv:1003.3552 [hep-th]]; S. F. King and C. Luhn,
Neutrino Mass and Mixing with Discrete Symmetry, Rept. Prog. Phys. 76, 056201 (2013) [arXiv:1301.1340 [hep-ph]].

[2] L. L. Everett, A. J. Stuart, Icosahedral (A5) Family Symmetry and the Golden Ratio Prediction for Solar Neutrino Mixing, Phys. Rev. D 79 085005 (2009) [arXiv:0812.1057 [hep-ph]]; F. Feruglio, A. Paris, The Golden Ratio Prediction for the Solar Angle from a Natural Model with A5 Flavour Symmetry, JHEP 1103 101 (2011) [arXiv:1101.0393 [hep-ph]]; G. -J. Ding, L. L. Everett and A. J. Stuart, Golden Ratio Neutrino Mixing and A5 Flavor Symmetry, Nucl. Phys. B 857, 219 (2012) [arXiv:1110.1688 [hep-ph]]; I. K. Cooper, S. F. King and A. J. Stuart, A Golden A5 Model of Leptons with a Minimal NLO Correction, arXiv:1212.1066 [hep-ph].

[3] F. P. An et al. [DAYA-BAY Collaboration], Observation of electron-antineutrino disappearance at Daya Bay, Phys. Rev. Lett. 108, 171803 (2012) [arXiv:1203.1660 [hep-ex]]; J. K. Ahn et al. [RENO Collaboration], Observation of Reactor Electron Antineutrino Disappearance in the RENO Experiment, Phys. Rev. Lett. 108, 191802 (2012) [arXiv:1204.0626 [hep-ex]]; Y. Abe et al. [DOUBLE-CHOOZ Collaboration], Indication for the disappearance of reactor electron antineutrinos in the Double Chooz experiment, Phys. Rev. Lett. 108, 131801 (2012) [arXiv:1112.6353 [hep-ex]]; K. Abe et al. [T2K Collaboration], Indication of Electron Neutrino Appearance from an Accelerator-produced Off-axis Muon Neutrino Beam, Phys. Rev. Lett. 107, 041801 (2011) [arXiv:1106.2822 [hep-ex]]; P.Adamson et al. [MINOS Collaboration], Improved search for muon-neutrino to electron-neutrino oscillations in MINOS, Phys. Rev. Lett. 107, 181802 (2011) [arXiv:1108.0015 [hep-ex]].

[4] C. Jarlskog, Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation, Phys. Rev. Lett. 55, 1039 (1985).

[5] G. L. Fogli, E. Lisi, A. Marrone, D. Montanino, A. Palazzo and A. M. Rotunno, Global analysis of neutrino masses, mixings and phases: entering the era of leptonic CP violation searches, Phys. Rev. D 86(2012) 013012 [arXiv:hep-ph/1205.5254].

[6] D. V. Forero, M. Tortola and J. W. F. Valle, Global status of neutrino oscillation parameters after Neutrino-2012, Phys. Rev. D 86, 073012 (2012) [arXiv:1205.4018 [hep-ph]].

[7] M. C. Gonzalez-Garcia, M. Maltoni, J. Salvado and T. Schwetz, JHEP 1212, 123 (2012) [arXiv:1209.3023 [hep-ph]].

[8] P. A. R. Ade et al. [Planck Collaboration], Planck 2013 results. XVI. Cosmological parameters, [arXiv:astro-ph.CO/1303.5076].

[9] Z. Hou, C. L. Reichardt, K. T. Story, B. Follin, R. Keisler et al., Constraints on Cosmology from the Cosmic Microwave Background Power Spectrum of the 2500-square degree SPT-SZ Survey, arXiv:1212.6267 [astro-ph.CO].

[10] R. G. H. Robertson [KATRIN Collaboration], arXiv:1307.5486 [physics.ins-det].

[11] P. F. Harrison, D. H. Perkins, W. G. Scott, Tri-Bimaximal Mixing and the Neutrino Oscillation Data, Phys. Lett. B 530, 167 (2002) [arXiv:0202074 [hep-ph]]; P. F. Harrison, W. G. Scott, Symmetries and Generalisations of Tri-Bimaximal Neutrino Mixing, Phys. Lett. B 535, 163 (2002) [arXiv:0203209 [hep-ph]]; Zhi zhong Xing, Nearly Tri-Bimaximal Neutrino Mixing and CP Violation, Phys. Lett. B 533, 85 (2002) [arXiv:0204049 [hep-ph]].

[12] F. Vissani, A Study of the scenario with nearly degenerate Majorana neutrinos, hep-ph/9708483; V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Bimaximal mixing of three neutrinos, Phys. Lett. B 437, 107 (1998) [hep-ph/9806387]; A. J. Baltz,
A. S. Goldhaber and M. Goldhaber, *The Solar neutrino puzzle: An Oscillation solution with maximal neutrino mixing*, Phys. Rev. Lett. **81**, 5730 (1998) [hep-ph/9806540].

[13] C. H. Albright, A. Dueck, W. Rodejohann, *Possible Alternatives to Tri-bimaximal Mixing*, Eur. Phys. J **C70** 1099 (2010) [arXiv:1004.2798 [hep-ph]].

[14] A. Datta, F. -S. Ling and P. Ramond, *Correlated hierarchy, Dirac masses and large mixing angles*, Nucl. Phys. **B671**, 383 (2003) [hep-ph/0306002]; Q. Duret and B. Machet, *The Neighborhood of the standard model: Mixing angles and quark-lepton complementarity for three generations of non-degenerate coupled fermions*, arXiv:0705.1237 [hep-ph]; Y. Kajiyama, M. Raidal, A. Strumia, *The golden ratio prediction for the solar neutrino mixing*, Phys. Rev. **D 76** 117301 (2007) [arXiv:0705.4559 [hep-ph]].

[15] W. Rodejohann, *Unified Parametrization for Quark and Lepton Mixing Angles*, Phys. Lett. **B671** 267 (2009) [arXiv:0810.5239 [hep-ph]]; A. Adulpravitchai, A. Blum, W. Rodejohann, *Golden Ratio Prediction for Solar Neutrino Mixing*, New J. Phys. **11** 063026 (2009) [arXiv:0903.0531 [hep-ph]].

[16] C. I. Low, R. R. Volkas, *Tri-bimaximal mixing, discrete family symmetries, and a conjecture connecting the quark and lepton mixing matrices*, Phys. Rev. **D 68**, 033007 (2003) [arXiv:0305243 [hep-ph]]; C. S. Lam, *Mass Independent Textures and Symmetry*, Phys. Rev. **D 74**, 113004 (2006). [arXiv:0611017 [hep-ph]].

[17] Z. -z. Xing, *Nearly tri bimaximal neutrino mixing and CP violation*, Phys. Lett. **B533**, 85 (2002) [hep-ph/0204049]; R. N. Mohapatra and W. Rodejohann, *Broken mu-tau symmetry and leptonic CP violation*, Phys. Rev. **D 72**, 053001 (2005) [hep-ph/0507312]; S. Antusch and S. F. King, *Charged lepton corrections to neutrino mixing angles and CP phases revisited*, Phys. Lett. **B 631**, 42 (2005) [hep-ph/0508044]; S. Boudjemaa and S. F. King, *Deviations from Tri-bimaximal Mixing: Charged Lepton Corrections and Renormalization Group Running*, Phys. Rev. **D 79**, 033001 (2009) [arXiv:0808.2782 [hep-ph]]; Y. Shimizu and R. Takahashi, *Deviations from Tri-Bimaximality and Quark-Lepton Complementarity*, Europhys. Lett. **93**, 61001 (2011) [arXiv:1009.5504 [hep-ph]]; D. Meloni, F. Plentinger and W. Winter, *Perturbing exactly tri-bimaximal neutrino mixings with charged lepton mass matrices*, Phys. Lett. **B 699**, 354 (2011) [arXiv:1012.1618 [hep-ph]]; Y. -j. Zheng and B. -Q. Ma, *Re-Evaluation of Neutrino Mixing Pattern According to Latest T2K result*, Eur. Phys. J. Plus **127**, 7 (2012) [arXiv:1106.4040 [hep-ph]]; Z. -Z. Xing, *The T2K Indication of Relatively Large \(\theta_{13}\) and a Natural Perturbation to the Democratic Neutrino Mixing Pattern*, Chin. Phys. **C 36**, 101 (2012) [arXiv:1106.3244 [hep-ph]]; T. Araki, *Getting at large \(\theta_{13}\) with almost maximal \(\theta_{23}\) from tri-bimaximal mixing*, Phys. Rev. **D 84**, 037301 (2011) [arXiv:1106.5211 [hep-ph]]; S. Dev, S. Gupta and R. R. Gautam, *Parametrizing the Lepton Mixing Matrix in terms of Charged Lepton Corrections*, Phys. Lett. **B 704**, 527 (2011) [arXiv:1107.1125 [hep-ph]].

[18] P. Minkowski, Phys. Lett. **B 67**, 421 (1977); T. Yanagida, *Proceedings of the Workshop on the Unified Theory and the Baryon Number in the Universe* (O.Sawada and A. Sugamoto, eds.), KEK, Tsukuba, Japan, 1979, p. 95; M. Gell-Mann, P. Ramond, and R. Slansky, *Complex spinors and unified theories in supergravity* (P. Van Nieuwenhuizen and D. Z. Freedman, eds.), North Holland, Amsterdam, 1979, p.315; R. N. Mohapatra and G. Senjanovic Phys. Rev. Lett. **44**, 912 (1980).

[19] S. Luo and Z. -z. Xing, *On the quasi-fixed point in the running of CP-violating phases of Majorana neutrinos*, Phys. Lett. **B 637**, 279 (2006) [arXiv:0603091 [hep-ph]]; S. Luo and Z. -z. Xing, *Generalized tri-bimaximal neutrino mixing and its sensitivity to radiative...*
corrections, Phys. Lett. B 632, 341 (2006) [hep-ph/0509065]; A. Dighe, S. Goswami and W. Rodejohann, Corrections to Tri-bimaximal Neutrino Mixing: Renormalization and Planck Scale Effects, Phys. Rev. D 75, 073023 (2007) [arXiv:0612328 [hep-ph]]; A. Dighe, S. Goswami and P. Roy, Radiatively broken symmetries of nonhierarchical neutrinos, Phys. Rev. D 76, 096005 (2007) [arXiv:0704.3735 [hep-ph]]; S. Luo and Z.-z. Xing, Impacts of the observed $\theta_{13}$ on the running behaviors of Dirac and Majorana neutrino mixing angles and CP-violating phases, Phys. Rev. D 86, 073003 (2012) [arXiv:1203.3118 [hep-ph]].

[20] P. H. Chankowski and Z. Pluciennik, Renormalization group equations for seesaw neutrino masses, Phys. Lett. B 316, 312 (1993) [arXiv:9306333 [hep-ph]]; K. S. Babu, C. N. Leung and J. T. Pantaleone, Renormalization of the neutrino mass operator, Phys. Lett. B 319, 191 (1993) [arXiv:9309223 [hep-ph]]; J. A. Casas, J. R. Espinosa, A. Ibarra and I. Navarro, General RG equations for physical neutrino parameters and their phenomenological implications, Nucl. Phys. B 573, 652 (2000) [arXiv:9910420 [hep-ph]]; J. A. Casas, J. R. Espinosa, A. Ibarra and I. Navarro, Nearly degenerate neutrinos, supersymmetry and radiative corrections, Nucl. Phys. B 569, 82 (2000) [arXiv:9905381 [hep-ph]]; A. S. Joshipura, S. D. Rindani and N. N. Singh, Predictive framework with a pair of degenerate neutrinos at a high scale, Nucl. Phys. B 660, 362 (2003) [hep-ph/0211378].

[21] S. Antusch, M. Drees, J. Kersten, M. Lindner and M. Ratz, Neutrino mass operator renormalization revisited, Phys. Lett. B 519, 238 (2001) [hep-ph/0108005]; S. Antusch, M. Drees, J. Kersten, M. Lindner and M. Ratz, Neutrino mass operator renormalization in two Higgs doublet models and the MSSM, Phys. Lett. B 525, 130 (2002) [hep-ph/0110366]; S. Antusch, J. Kersten, M. Lindner and M. Ratz, Neutrino mass matrix running for nondegenerate seesaw scales, Phys. Lett. B 538, 87 (2002) [hep-ph/0203233].

[22] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M. A. Schmidt, Running neutrino mass parameters in see-saw scenarios, JHEP 0503, 024 (2005) [hep-ph/0501272].

[23] J.-w. Mei and Z.-z. Xing, Radiative generation of $\theta_{13}$ with the seesaw threshold effect, Phys. Rev. D 70, 053002 (2004) [hep-ph/0404081].

[24] J. Bergstrom, M. Malinsky, T. Ohlsson and H. Zhang, Renormalization group running of neutrino parameters in the inverse seesaw model, Phys. Rev. D 81, 116006 (2010) [arXiv:1004.4628 [hep-ph]]; J. Bergstrom, T. Ohlsson and H. Zhang, Threshold effects on renormalization group running of neutrino parameters in the low-scale seesaw model, Phys. Lett. B 698, 297 (2011) [arXiv:1009.2762 [hep-ph]].

[25] J. R. Ellis and S. Lola, Can neutrinos be degenerate in mass?, Phys. Lett. B 458, 310 (1999) [hep-ph/9904279];

[26] P. H. Chankowski and S. Pokorski, Quantum corrections to neutrino masses and mixing angles, Int. J. Mod. Phys. A 17, 575 (2002) [hep-ph/0110249].

[27] S. Goswami, S. Khan and S. Mishra, Threshold effects and renormalization group evolution of neutrino parameters in TeV scale seesaw models, arXiv:1310.1468 [hep-ph].

[28] S. Antusch, J. Kersten, M. Lindner and M. Ratz, Running neutrino masses, mixings and CP phases: Analytical results and phenomenological consequences, Nucl. Phys. B 674, 401 (2003) [hep-ph/0305273].

[29] J.-w. Mei, Running neutrino masses, leptonic mixing angles and CP-violating phases: From $M(Z)$ to Lambda(GUT), Phys. Rev. D 71, 073012 (2005) [hep-ph/0502015].

[30] J. A. Casas, J. R. Espinosa, A. Ibarra and I. Navarro, General RG equations for physical
neutrino parameters and their phenomenological implications, Nucl. Phys. B 573, 652 (2000) [hep-ph/9910420].

[31] N. Haba, Y. Matsui and N. Okamura, The Effects of Majorana phases in three generation neutrinos, Eur. Phys. J. C 17, 513 (2000) [hep-ph/0005075].