Global Attractor for the Navier–Stokes Equations with horizontal filtering

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Abstract

We consider a Large Eddy Simulation model for a homogeneous incompressible Newtonian fluid in a box space domain with periodic boundary conditions on the lateral boundaries and homogeneous Dirichlet conditions on the top and bottom boundaries, thus simulating a horizontal channel. The model is obtained through the application of an anisotropic horizontal filter, which is known to be less memory consuming from a numerical point of view, but provides less regularity with respect to the standard isotropic one defined as the inverse of the Helmholtz operator.

It is known that there exists a unique regular weak solution to this model that depends weakly continuously on the initial datum. We show the existence of the global attractor for the semiflow given by the time-shift in the space of paths. We prove the continuity of the horizontal components of the flow under periodicity in all directions and discuss the possibility to introduce a solution semiflow.

Keywords. Navier–Stokes equations, anisotropic filters, global attractor, time-shift semiflow, Large Eddy Simulation (LES), turbulent flows in domains with boundary, approximate deconvolution methods (ADM).

Subjects. 76D05; 35Q30, 76F65, 76D03.

1 Introduction

Incompressible fluids with constant density are described by the Navier–Stokes equations

\[ \begin{align*}
\frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) - \nu \Delta u + \nabla \pi &= f, \\
\nabla \cdot u &= 0,
\end{align*} \]

supplemented with initial and boundary conditions, where \( u(t, x) = (u_1, u_2, u_3) \) is the velocity field, \( \pi(t, x) \) denotes the pressure, \( f(t, x) = (f_1, f_2, f_3) \) is the external force, and \( \nu > 0 \) the kinematic viscosity.

In order to perform numerical simulations of the 3-dimensional fluid equations (1)–(2), many regularization methods have been proposed. Among these models, in the recent years, there has been a lot of interest, from both the point of view of the pure and applied mathematicians, around the so called “\( \alpha \)-models”. These models are based on a filtering obtained through the application
of the inverse of the Helmholtz operator

\[ A = I - \alpha^2 \Delta, \]

where \( \alpha > 0 \) is interpreted as a spatial filtering scale.

In this paper, we are concerned with a regularized model for the 3D Navier–Stokes equations derived by the introduction of a suitable horizontal (anisotropic) differential filter and we prove the existence of a global attractor for the corresponding time-shift dynamical system in path-space. Let us consider

\[ x = (x_1, x_2, x_3), \quad x_h = (x_1, x_2), \]
\[ \partial_j = \partial_{x_j}, \quad \Delta_h = \partial_1^2 + \partial_2^2, \quad \nabla_h = (\partial_1, \partial_2), \]

where “h” stays for “horizontal” and, instead of choosing the filter given by (3), we take into account the horizontal filter given by (see [5])

\[ A_h = I - \alpha^2 \Delta_h \quad \text{with} \quad \alpha > 0. \]

As discussed in [2, 15, 16], from the point of view of the numerical simulations, this filter is less memory consuming with respect to the standard one. Another significant advantage of this choice is that there is no need to introduce artificial boundary conditions for the Helmholtz operator.

The idea behind anisotropic differential filters can be traced back to the approach used by Germano [15]. Recently, the Large Eddy Simulation (LES) community has manifested interest in models involving such a kind of filtering (e.g., [2, 4, 12, 20]) and the connection with the family of \( \alpha \)-models has been highlighted and investigated by Berselli in [5]. Exploiting the smoothing provided by the horizontal filtering (4), Berselli has proved global existence and uniqueness of a proper class of weak solutions to the considered regularized model (see the system of equations (6)–(7) below). Again, motivated by [5], the authors of [6, 7] have given a considerable mathematical support to the well-posedness of initial-boundary value problems, in suitable anisotropic Sobolev spaces, to the 3D Boussinesq equations with horizontal filter for turbulent flows.

In the sequel, we mainly consider the domain

\[ D = \{ x \in \mathbb{R}^3 : -\pi L < x_1, x_2 < \pi L, -d < x_3 < d \} \]

with \( 2\pi L \) periodicity with respect to \( x_h \) (i.e., with respect to \( x_1, x_2 \)), and homogeneous Dirichlet boundary conditions on \( \Gamma := \{ x \in \mathbb{R}^3 : -\pi L < x_1, x_2 < \pi L, x_3 = \pm d \} \). Observe that the filter given by (4) is acting just on the horizontal variables, so it makes sense to require the periodicity only in \( x_h = (x_1, x_2) \).

Set \( w = \mathbf{w}^h = A_h^{-1} u \) and \( q = \mathbf{q}^h = A_h^{-1} \pi \), so that \( u = A_h w \). Then, applying the horizontal filter \( \sim \) component by component to the various fields and tensor fields in (1)–(2), and solving the interior closure problem by the approximation

\[ \mathbf{u} \otimes \mathbf{w}^h \approx \mathbf{w}^h \otimes \mathbf{w}^h = \mathbf{w} \otimes \mathbf{w}^h, \]

we finally get the regularized model

\[ \partial_t w + \nabla \cdot (w \otimes w)^h - \nu \Delta w + \nabla q = \mathbf{f}^h, \]
\[ \nabla \cdot w = 0. \]
Here, we assume homogeneous Dirichlet boundary conditions on $\Gamma$ for the filtered fields as well as for the unfiltered ones, in order to prevent the introduction of artificial boundary conditions, and impose the initial datum $w_{|t=0} = w_0$ for the filtered velocity field $w$.

Let us note that this model represents a special case of Approximate Deconvolution LES Model (ADM), see Adams–Stolz [1], when the order of deconvolution is zero. We refer to [5, 2, 9, 8] for some recent results in this context concerning general orders of deconvolution.

Our aim is to prove the existence of an attractor in the class of regular weak solutions (see below for details) to the horizontally filtered model (6)–(7). However, the present case does not seem to fit the classical theory of attractors (see, e.g., [3]) and a different scheme is needed to carry out our analysis. In fact, despite the smoothing created by the horizontal filter, the regularity of the considered weak solutions does not ensure the continuous dependence on their initial data, even in the fully periodic setting (the dependence on the initial data is only weakly continuous). Hence, the standard dynamical theory fails to apply to this situation since the strong continuity on the initial data is needed to get the continuity of the solution semiflow.

To overcome this problem, we follow the approach proposed by Sell [18]: in this case, the dynamics becomes the time-shift in the space of paths, and the attractor is a suitable compact set that attracts the regular weak solutions under the action of the time-shift $S(t)w(\cdot) = w(t + \cdot)$.

Let $W$ be the space of the regular weak solutions to (6)–(7), denote by $H$ and $V$ the usual function spaces of fluid dynamics (see Subsection 2.1), and set

$$
\begin{align*}
V_h &= \{ \phi \in H : \nabla_h \phi \in L^2(D) \}, \\
H_h^2 &= \{ \phi \in V_h : \nabla_h \nabla \phi \in L^2(D) \}.
\end{align*}
$$

We prove the following result (see Section 5).

1.1 Theorem Given $w_0 \in V_h$ and $f = f(x) \in (V \cap H_h^2)^*$, the time-shift $S(t)$ in $W$ has a unique global attractor.

Here, for the sake of simplicity, we assume that the forcing term $f$ is independent of time.

In the last part of the paper, we consider problem (6)–(7) in the fully space-periodic setting. Notice that, following analogous computations, Theorem 1.1 can be readily adapted to the simplified situation given by space-periodic boundary conditions in all directions.

We want to discuss the possibility to obtain an analogous result for the semigroup $(S(t))_{t \geq 0}$ on $V_h$, where $S(t) : V_h \to V_h$ is given by $S(t)w_0 = w(t, \cdot)$, $w_0 \in V_h$, and $w(t, \cdot)$ is the solution at time $t$ to (6)–(7) corresponding to the initial datum $w_0$. In such a case, the main issue is to prove that $S(t)$ effectively defines a semiflow (the solution semiflow) and, in particular, that $S(t)$ is a continuous operator. In fact, the regular weak solutions to (6)–(7) depend just in a weakly continuously way on their initial data, giving no guarantees on the continuity of $S(t)$.

In Section 6, we show that the horizontal components of the flow, i.e. $w_h$, depend continuously on the initial datum, proving the following.

1.2 Proposition If $w$ is a regular weak solution to (6)–(7) under the periodic setting, then $w_h \in \mathcal{C}([0, T]; V_h)$ for each $T > 0$.

Observe that the continuity of the solution operator $S(t)$ could be shown by proving that $\partial_t \nabla_h w \in L^2 V^*$; nevertheless, we only have that $\partial_t \nabla_h w_h \in L^2 V^*$. However, since the considered problem admits a unique regular weak solution, we may wonder whether it is possible to get more regularity for such a solution, also in the vertical component, by exploiting again the special features of the fully periodic space-domain. To this end, we give an improved regularity result.
(see Theorem 6.3 below) which actually shows that, although the regularization created by the filter is strong in the horizontal components (and in the derivatives with respect to the horizontal components), this smoothing is not so effectively transported to the vertical component, even in the space-periodic case. Though Theorem 6.3 is not directly useful to prove the continuity of $S(t)$ (in fact, it might be more appropriate in order to get compactness properties of $S(t)$), it seems interesting by itself and we report it, at the end of Section 6, for the reader’s convenience.

Because of all the above facts, different techniques seem necessary to get the continuity of the solution operator $S(t)$ associated to (6)–(7), and to prove the existence of the global attractor in such a case. We will address these issues in a future paper, in which we will study more thoroughly the dynamics associated to problem (6)–(7), possibly supplemented with homogeneous Dirichlet boundary conditions.

### 1.1 Plan of the paper

In Section 2 we describe the functional setting, the horizontal filter and the notion of regular weak solution, and we recall a known result concerning the existence and uniqueness of such solutions. In Section 3, we recall the notion of global attractor and describe the main results. Section 4 is devoted to the proof of some basic estimates that will be used subsequently. In Section 5, under horizontal periodicity and homogeneous Dirichlet boundary conditions on $\Gamma$, the existence of the global attractor defined through the shifting semiflow is given. Finally, in Section 6, when the domain is periodic in all directions, we prove the continuity of the horizontal components of the flow, and we also provide an improved regularity result for the solutions to (6)–(7), i.e. Theorem 6.3.

### 2 Functional setting and anisotropic filtering

#### 2.1 Function spaces

We introduce the following function spaces:

\[
L^2(D) := \{ \phi : D \to \mathbb{R} \text{ measurable, } 2\pi L \text{ periodic in } x_h, \int_D |\phi|^2 \, dx < +\infty \},
\]

\[
L^2_0(D) := \{ \phi \in L^2(D) \text{ with zero mean with respect to } x_h \},
\]

\[
H := \{ \phi \in (L^2_0(D))^3 : \nabla \cdot \phi = 0 \text{ in } D, \phi \cdot n = 0 \text{ on } \Gamma \},
\]

($n$ is the outward normal to $\Gamma$), all with $L^2$ norm denoted by $\| \cdot \|$, and scalar product $(\cdot, \cdot)$ in $L^2$. Here and in the following, we use the same notation for scalar and vector valued functions, when they make sense. Moreover, we set

\[
V_h := \{ \phi \in H : \nabla_h \phi \in L^2(D) \},
\]

\[
V := \{ \phi \in H : \nabla \phi \in L^2(D) \text{ and } \phi = 0 \text{ on } \Gamma \},
\]

\[
H_h^2 := \{ \phi \in H_h^1 : \nabla_h \nabla \phi \in L^2(D) \},
\]

where the definitions of $V_h$ and $H_h^2$ have been recalled for the reader’s convenience, and denote by $V^*$ the topological dual space to $V$. We denote by $L^p$ and $H^m$ classical Lebesgue and Sobolev spaces. Continuous and weakly continuous functions are denoted respectively by the symbols $\mathcal{C}$ and $\mathcal{C}_w$. 

4
Given a Banach space $X$ with norm $\| \cdot \|_X$ and $p \in [1, +\infty]$, we denote by $L^p_{\text{loc}}(0,\infty; X)$ the usual Bochner space formed by functions $\phi: [0, \infty] \to X$ such that, for all $0 < a \leq b < +\infty$, the $L^p(a,b)$ norm of $\|\phi(\cdot)\|_X$ is finite. We will also denote by $L^p_{\text{loc}}(0,\infty; X)$ the space of functions in $L^p_{\text{loc}}(0,\infty; X)$ such that the $L^p(0,b)$-norm of $\|\phi(\cdot)\|_X$ is finite for every $b \in [0, +\infty]$, and analogously for $H^m_{\text{loc}}(0,\infty; X)$ (see, e.g., [19]).

### 2.2 Basic results for the filtered model

Applying the horizontal filter given by (4) to the various terms in the Navier–Stokes equations (1)–(2), and modifying the nonlinear quadratic term by means of the approximation (5) (interior closure problem in LES), we have deduced the approximate model (6)–(7), i.e.

\[
\begin{align*}
\partial_t w + \nabla \cdot (w \otimes w) - \nu \Delta w + \nabla q &= f^h, \\
\nabla \cdot w &= 0.
\end{align*}
\]

Let us recall the following definition.

#### 2.1 Definition
We say that $w: [0, T] \times D \to \mathbb{R}^3$ is a regular weak solution (omitting the pressure term $q$) to (6)–(7), with $f \in (V \cap H^2)\ast$ independent of time (for simplicity), and $w(0) = w_0 \in V_h$ in weak sense, when the following properties are verified.

- **Regularity:**
  \[
  w \in L^\infty(0,T; V_h) \cap L^2(0,T; V \cap H^2_h) \cap C_w(0,T; V_h),
  \]
  \[
  w_3 \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2),
  \]
  \[
  \partial_t w \in L^2(0,T; V^*).
  \]

- **Weak formulation:**
  \[
  \int_0^{+\infty} \{(w, \partial_t \varphi) - \nu (\nabla w, \nabla \varphi) + (w \otimes w, \nabla \varphi^h)\}(s) \, ds = -\int_0^{+\infty} (f, \varphi)(s) \, ds - (w(0), \varphi(0))
  \]
  for each $\varphi \in (H^\infty_0(D \times [0,T])})^3$ such that $\nabla \cdot \varphi = 0$ or, equivalently (up to a modification of $w$ on a set of zero measure, see Galdi [14]),
  \[
  \int_{t_0}^{t_1} \{(w, \partial_t \varphi) - \nu (\nabla w, \nabla \varphi) + (w \otimes w, \nabla \varphi^h)\}(s) \, ds
  \]
  \[
  = -\int_{t_0}^{t_1} (f, \varphi)(s) \, ds + (w(t_1), \varphi(t_1)) - (w(t_0), \varphi(t_0))
  \]
  for each $0 \leq t_0 \leq t_1 < T$.

#### 2.2 Remark
Since $w \in L^2(0,T; V)$ and $\partial_t w \in L^2(0,T; V^*)$, we deduce by classical interpolation results that $w \in C([0,T]; H)$ (see [17, 19]).

We state the following existence theorem.
2.3 Theorem Let be given \( w_0 \in V_h, \ f = f(x) \in (V \cap H^2_h)^* \) and \( \nu > 0 \). Then there exists a unique regular weak solution to (6)–(7), with \( w(0) = w_0 \), depending in a weakly continuous way on the data. Moreover, the solution satisfies the energy (of the model) identity

\[
\frac{1}{2} \left( \|w(t)\|^2 + \alpha^2 \|\nabla_h w(t)\|^2 \right) + \nu \int_0^t \left( \|\nabla w(s)\|^2 + \alpha^2 \|\nabla_h \nabla w(s)\|^2 \right) ds \\
= \frac{1}{2} \left( \|w(0)\|^2 + \alpha^2 \|\nabla_h w(0)\|^2 \right) - \int_0^t \left( f, A_h w(s) \right) ds.
\]

This result can be obtained as in [5, 6], where a space-domain \([-\pi,\pi[ \times ]-d,d[ \) periodic in \((x_1,x_2)\) and with homogeneous Dirichlet boundary conditions on \( x_3 = \pm d \) is considered.

Also, notice that the regularity of \( f \) implies \( \mathcal{F}^0 \in L^2_0(D) \).

3 Attractors and main results

Let \((W,d)\) be a metric space. A semigroup on \((W,d)\) is a family of operators \((S(t))_{t \geq 0}\), \( S(t) : W \to W \), that satisfies \( S(0)w = w \) and \( S(s)S(t)w = S(t+s)w \) for each \( w \in W \) and for every \( s,t \geq 0 \).

A semiflow on \((W,d)\) is a mapping \( \sigma : [0,\infty[ \times W \to W \) defined by \( \sigma(t,w) = S(t)w \), where \((S(t))_{t \geq 0}\) is a semigroup, and such that the restriction \( \sigma : [0,\infty[ \times W \to W \) is continuous.

We say that \( A \subset W \) is a global attractor for the semiflow if \( A \) is nonempty and compact, \( S(t)A = A \) for all \( t \geq 0 \) (i.e. \( A \) is invariant), and for all bounded sets \( B \subset W \), we have \( \lim_{t \to +\infty} \delta(S(t)B,A) = 0 \), where \( \delta(X,Y) := \sup_{x \in X} \inf_{y \in Y} d(x,y) \) is the Hausdorff semidistance between the pair of sets \( X,Y \subset W \). A global attractor is necessarily unique (and it coincides with the omega-limit of an absorbing set, see Section 5).

Let us note that, for each \( T > 0 \), a regular weak solution to (6)–(7) in \([0,T[ \times D\) is defined, so that, by uniqueness, we get a unique solution \( w \) defined in \([0,\infty[ \times D\), and, in particular, \( w \in L^\infty_{\text{loc}}[0,\infty;V_h] \cap L^2_{\text{loc}}[0,\infty;V \cap H^2_h] \). Indeed, we can prove that \( w \in L^\infty(0,\infty;V_h) \cap L^2_{\text{loc}}(0,\infty;V \cap H^2_h) \), as shown in Theorem 4.1.

Thus we can consider as \( W \) the set \( \mathcal{W} \) of regular weak solutions to (6)–(7) in \( L^\infty(0,\infty;V_h) \cap L^2_{\text{loc}}(0,\infty;V \cap H^2_h) \), with the metric \( d \) induced by \( L^2_{\text{loc}}(0,\infty;V_h) \):

\[
d(w_1,w_2) = \sum_{n=1}^{\infty} 2^{-n} \min \left\{ 1, \|w_1 - w_2\|_{L^2(0,n;V_h)} \right\}.
\]

Notice that \( \mathcal{W} \) is closed in \( L^2_{\text{loc}}(0,\infty;V_h) \), which is complete with respect to \( d \), thus \((\mathcal{W},d)\) is a complete metric space; this is one of the consequences of the proof of Proposition 5.4 below.

We use the time-shift operator to define the semigroup and hence the semiflow: \( S(t)w = w_{t+t} := w(\cdot + t) \), for each \( w \in \mathcal{W} \). The existence of the global attractor for the time-shift \( S(t) \), i.e. Theorem 1.1, is proved in Section 5.

4 Preliminary Estimates

We set \( \lambda_1 > 0 \) equal to the first eigenvalue of the Stokes operator with horizontal periodic conditions and homogeneous Dirichlet boundary conditions on \( \Gamma \), projected on the space of divergence free
functions. We recall that $\lambda_1$ can be used as the constant in the Poincaré inequality. Moreover, we set $\Lambda_h = (-\Delta_h)^{1/2}$ with domain $H^2_h$ and the same mixed periodic-Dirichlet conditions as above ($\Lambda_h$ is a positive self-adjoint operator), and

$$k_0(t) = \|w(t)\|^2 + \alpha^2 \|\nabla_h w(t)\|^2,$$

$$K_1 = \min \left\{ \|\Lambda_h^{-1} f\|^2, \|\Lambda_h^{-1/2} f\|^2 \right\},$$

$$k_1(t) = k_0(t) + \frac{K_1}{\nu \lambda_1}$$

for each $t \geq 0$.

4.1 Theorem If $w$ is the regular weak solution to (6)–(7) in the time interval $[0, \infty]$, $t \geq 0$ and $r > 0$, then we have

$$\|w(t+r)\|^2 + \alpha^2 \|\nabla_h w(t+r)\|^2 \leq k_0(t) e^{-\nu \lambda_1 (t+r)} + \frac{K_1}{\nu \lambda_1} (1 - e^{-\nu \lambda_1 (t+r)}) \leq k_1(t),$$

(10)

$$\int_t^{t+r} \left( \nu \|\nabla w(s)\|^2 + \nu \alpha^2 \|\nabla_h \nabla w(s)\|^2 \right) ds \leq r K_1 + k_1(t).$$

(11)

In particular, we have $w \in L^\infty(0,\infty; V_h) \cap L^2_{\text{loc}}([0,\infty); V \cap H^2_h)$.

Proof. Let us consider the Galerkin approximate solutions

$$w_m(t,x) = \sum_{j=1}^m g_j^t(t) E_j(x),$$

where $E_j$ are smooth eigenfunctions of the Stokes operator on $D$, with periodicity in $x$. If $P_m$ denotes the projection on span $\{ E_1, \ldots, E_m \}$, then $w_m$ solves the Cauchy problem

$$\frac{d}{dt}(w_m, E_i) + \nu (\nabla w_m, \nabla E_i) - (w_m \otimes w_m, \nabla E_i) = (f, E_i),$$

$$w_m(0) = P_m(w(0)),$$

for $i = 1, \ldots, m$. As shown in [5], we have that, up to considering a subsequence,

$$w_m \to w \quad \text{in} \quad L^2(0,T;L^2(D))$$

as $m \to +\infty$, where $w_m$ is the regular weak solution to (6)–(7), and this is sufficient to pass to the limit in the nonlinear term.

We test the first equation against $A_h w_m$ and use (see [5])

$$-(w_m \otimes w_m, \nabla A_h w_m^h) = (\overline{\nabla \cdot (w_m \otimes w_m)}, A_h w_m) = 0$$

to get

$$\frac{1}{2} \frac{d}{dt} \left( \|w_m\|^2 + \alpha^2 \|\nabla_h w_m\|^2 \right) + \nu \left( \|\nabla w_m\|^2 + \alpha^2 \|\nabla_h \nabla w_m\|^2 \right) = (T^h, A_h w_m) = (f, w_m).$$
We estimate the righthand side by

\[
|\langle f, w_m \rangle| \leq \left\{ \begin{array}{l}
\|\Lambda_h^{-2} f\| \|\Delta_h w_m\| \\
\|\Lambda_h^{-1} f\| \|\Lambda_h w_m\|
\end{array} \right\} \leq \left\{ \begin{array}{l}
\|\Lambda_h^{-2} f\|^2 + \frac{\nu \alpha^2 \|\Delta_h w_m\|^2}{2} \\
\|\Lambda_h^{-1} f\|^2 + \frac{\nu \|\Lambda_h w_m\|^2}{2}
\end{array} \right\} \leq \frac{1}{2} \left( K_1 + \nu \|\nabla_h w_m\|^2 + \nu \alpha^2 \|\nabla_h \nabla w_m\|^2 \right)
\]

by Cauchy–Schwarz inequality and definition of $K_1$. We deduce

\[
\frac{d}{dt} \left( \|w_m\|^2 + \alpha^2 \|\nabla_h w_m\|^2 \right) + \nu \left( \|\nabla w_m\|^2 + \alpha^2 \|\nabla_h \nabla w_m\|^2 \right) \leq K_1
\]

and, thanks to the Poincaré inequality,

\[
\frac{d}{dt} \left( \|w_m\|^2 + \alpha^2 \|\nabla_h w_m\|^2 \right) + \nu \lambda_1 \left( \|w_m\|^2 + \alpha^2 \|\nabla_h w_m\|^2 \right) \leq K_1
\]

an application of Gronwall’s lemma in the interval $[t, t + r]$ yields

\[
\|w_m(t + r)\|^2 + \alpha^2 \|\nabla_h w_m(t + r)\|^2 \\
\leq \left( \|w_m(t)\|^2 + \alpha^2 \|\nabla_h w_m(t)\|^2 \right) e^{-\nu \lambda_1 (t + r)} + \frac{K_1}{\nu \lambda_1} (1 - e^{-\nu \lambda_1 (t + r)})
\]

and, taking the limit as $m \to +\infty$, we obtain (10) and in particular

\[
\|w(t + r)\|^2 + \alpha^2 \|\nabla_h w(t + r)\|^2 \leq k_1(t),
\]

which implies

\[
w \in L^\infty([0, \infty); V_h)
\]

by taking $t = 0$ and using the fact that $k_1(0)$ is finite (and independent of time), since $w_0 \in V_h$.

Integrating (12) over $s \in [t, t + r]$, taking $m \to +\infty$ and using (10) yields (11), and consequently

\[
w \in L^2_{loc}([0, \infty); V \cap H^2_h).
\]

5 Global attractor for the time-shift semiflow

This section is devoted to the proof of Theorem 1.1. First, we recall some notions concerning semiflows and a fundamental result that we will exploit in order to prove the existence of the global attractor.

A bounded subset $B \subset W$ is called an absorbing set if for every $w \in W$, there exists $t_1 = t_1(w)$ such that $S(t)w \in B$ for all $t \geq t_1$. A semiflow is said compact if, for every bounded set $B \subset W$ and for every $t > 0$, $S(t)B$ lies in compact subset of $W$.

We have the following result [18, 21].
5.1 Theorem Let $S(t)$ define a compact semiflow admitting an absorbing set $\mathcal{B}$ on a complete metric space $W$. Then $S(t)$ has a global attractor $\mathcal{A}$ in $W$ and it coincides with the omega-limit set of $\mathcal{B}$:

$$\mathcal{A} = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} S(t)\mathcal{B}.$$ 

where the closure is taken in $W$.

Hence, in order to prove Theorem 1.1, it is sufficient to verify the hypotheses of the previous result for the time-shift semiflow, which is the content of the following propositions.

5.2 Proposition The mapping $\sigma$ given by $S(t)w = w(\cdot + t)$ is a semiflow.

Proof. Clearly, $S(t)$ is a semigroup. We need to prove that the mapping $(\tau, w) \mapsto S(\tau)w = w_{+\tau}$ is continuous for $(\tau, w) \in [0, +\infty[ \times L^2_{\text{loc}}[0, \infty; V_h])$. It is sufficient to show that, if $\tau_n$ and $w^n$ are sequences such that $\tau_n \to \tau$ in $[0, +\infty[$ and $w^n \to w$ in $L^2_{\text{loc}}[0, \infty; V_h)$, then $d(w^n_{+\tau_n}, w_{+\tau}) \to 0$ as $n \to +\infty$, which holds provided

$$\int_a^b \|w^n_{+\tau_n} - w_{+\tau}\|^2_{V_h} \to 0 \quad \text{as } n \to +\infty$$

for given $0 \leq a < b < \infty$, where $\|w\|^2_{V_h} = \|w\|^2 + \alpha^2 \|\nabla_h w\|^2$. Here and in the following, we omit "ds" in several integrals to keep the notation more compact.

We can follow the steps of the proof of [18, Lemma 7], the main difference consisting in the use of the norm $\|\cdot\|_{V_h}$ instead of $\|\cdot\|$. Let us note that, since $\tau > 0$, we can assume $0 < \frac{\tau}{2} \leq \tau_n \leq 2\tau$.

It is sufficient to prove that

(13) $d(w^n_{+\tau_n}, w_{+\tau}) \to 0$,

(14) $d(w^n_{+\tau_n}, w_{+\tau}) \to 0$,

when $n \to +\infty$.

As for (13), we have

$$\int_a^b \|w^n_{+\tau_n} - w_{+\tau}\|^2_{V_h} = \int_{a+\tau_n}^{b+\tau_n} \|w^n - w\|^2_{V_h} \leq \int_{a+\frac{\tau}{2}}^{b+\tau} \|w^n - w\|^2_{V_h} \to 0,$$

as $n \to +\infty$, since $w^n \to w$ in $L^2_{\text{loc}}[0, \infty; V_h)$, which implies (13).

In order to show (14), let us fix $\varepsilon > 0$ and take $\psi \in L^2((a + \frac{\tau}{2}, b + 2\tau; V_h)) \cap C^1([a + \frac{\tau}{2}, b + 2\tau]; V_h)$ such that

(15) $\int_a^b \|w_{+\sigma} - \psi_{+\sigma}\|^2_{V_h} \leq \varepsilon$

for all $\sigma \in [\tau/2, 2\tau]$ (this is possible since $C^1$ is dense in $L^2$).

Moreover, if $K$ denotes an upper bound for $\|\partial_t \psi(s)\|_{V_h}$ for $a + \frac{\tau}{2} \leq s \leq b + 2\tau$, we have

$$\|\psi(\tau_n + t) - \psi(\tau + t)\|_{V_h} \leq \int_{\tau_n}^{\tau} \|\partial_t \psi(s + t)\|_{V_h} ds \leq K|\tau_n - \tau|,$$
therefore

\[ \int_a^b \| \psi_{+\tau} - \psi_{+\tau} \|^2_{V_h} \leq K^2(b-a)|\tau_n - \tau|^2 \leq \varepsilon \]

for \( n \geq N \) sufficiently large.

Using the triangular inequality, (15) and (16), we infer that

\[
\int_a^b \| w_{+\tau_n} - w_{+\tau} \|^2_{V_h} \leq \int_a^b (\| w_{+\tau_n} - \psi_{+\tau_n} \|^2_{V_h} + \| \psi_{+\tau_n} - \psi_{+\tau} \|^2_{V_h} + \| w_{+\tau} - \psi_{+\tau} \|^2_{V_h})^2 \\
\leq 3 \int_a^b (\| w_{+\tau_n} - \psi_{+\tau_n} \|^2_{V_h} + \| \psi_{+\tau_n} - \psi_{+\tau} \|^2_{V_h} + \| w_{+\tau} - \psi_{+\tau} \|^2_{V_h}) \leq 9\varepsilon
\]

for all \( n \geq N \), which proves (14) and ends the proof. \( \blacksquare \)

5.3 Proposition There exists an absorbing set \( \mathcal{B} \subset \mathcal{W} \) that is bounded in \( \mathcal{W} \).

**Proof.** We define \( \mathcal{B} \) as the subset of \( \mathcal{W} \) such that the inequality

\[ \| w(t) \|^2 + \alpha^2 \| \nabla_h w(t) \|^2 \leq \frac{2K_1}{\nu \lambda_1} \]

is satisfied for every \( t \geq 0 \). According to the definition of the metric \( d \), \( \mathcal{B} \) is bounded in \( \mathcal{W} \).

We need to prove that, if \( w \in \mathcal{W} \), then \( S(t)w \in \mathcal{B} \) for each \( t \) sufficiently large. Actually, from (10), there exists \( t_1 > 0 \) such that the inequality (17) holds for all \( t \geq t_1 \). Thus \( S(t)w \) belongs to \( \mathcal{B} \) for each \( t \geq t_1 \), and \( \mathcal{B} \) is an absorbing set. \( \blacksquare \)

5.4 Proposition The semiflow defined by \( S(t) \) on the metric space \( \mathcal{W} \) is compact, i.e. for each bounded set \( \mathcal{B} \) in \( \mathcal{W} \) and for each \( t > 0 \), then \( S(t)\mathcal{B} \) lies in a compact subset of \( \mathcal{W} \).

**Proof.** Let \( \mathcal{B} \) be a bounded subset of \( \mathcal{W} \). Thanks to the semigroup property of \( S(t) \), if \( S(t)\mathcal{B} \) is contained in a compact set of \( \mathcal{W} \) for some \( t > 0 \), then \( S(t+s)\mathcal{B} \) lies in a compact set of \( \mathcal{W} \) too. Then, to prove the claim, it suffices to prove that \( S(t)\mathcal{B} \) lies in a compact set of \( \mathcal{W} \) for \( 0 < t \leq 1 \).

Let \( \{w^n\} \) be a bounded sequence in \( \mathcal{W} \). Thus, there exists a positive constant \( M_0 \) such that

\[
\int_0^1 (\| w^n(s) \|^2 + \alpha^2 \| \nabla_h w^n(s) \|^2) \, ds \leq M_0^2.
\]

Recalling that \( S(t)w^n(\tau) = w^n_{\tau+t} = w^n(\tau+t) \), by the estimate in (10), for \( s_0 \in [0,t] \) and \( s \geq 0 \), it follows that

\[
\| w^n_{\tau+t}(s) \|^2 + \alpha^2 \| \nabla_h w^n_{\tau+t}(s) \|^2 = \| w^n(s+t) \|^2 + \alpha^2 \| \nabla_h w^n(s+t) \|^2 \\
\leq k_0(s_0) e^{-\nu \lambda_1 (t+s-s_0)} + \frac{K_1}{\nu \lambda_1} (1 - e^{-\nu \lambda_1 (t+s-s_0)}) \\
\leq k_0(s_0) + \frac{K_1}{\nu \lambda_1}
\]

and, integrating on \([0,1]\) in \( s_0 \), we reach

\[
\| w^n_{\tau+t}(s) \|^2 + \alpha^2 \| \nabla_h w^n_{\tau+t}(s) \|^2 \leq \int_0^1 (k_0(s_0) + \frac{K_1}{\nu \lambda_1}) \, ds_0 \\
\leq M_0^2 + \frac{K_1}{\nu \lambda_1}.
\]
Hence, we get
\[ \|w^n_t(\cdot)\|^2_{L^\infty} + \alpha^2 \|\nabla h w^n_t(\cdot)\|^2_{L^\infty} \leq M_0^2 + \frac{K_1}{\nu \lambda_1}. \]
for all \( n \). Further, by (8)–(11) we have that
\[
\nu \int_m^{m+1} (\|\nabla w^n_{t(s)}\|^2 + \alpha^2 \|\nabla h \nabla w^n_{t(s)}\|^2) \, ds = \nu \int_m^{m+1} (\|\nabla w^n(s+t)\|^2 + \alpha^2 \|\nabla h \nabla w^n(s+t)\|^2) \, ds \\
\leq (\|w^n(t)\|^2 + \alpha^2 \|\nabla h w^n(t)\|^2) + (1 + \frac{1}{\nu \lambda_1}) K_1 \\
\leq (\|w^n(s_0)\|^2 + \alpha^2 \|\nabla h w^n(s_0)\|^2) e^{-\nu \lambda_1 (t-s_0)} + (1 + \frac{2}{\nu \lambda_1}) K_1 \\
\leq k_0(s_0) + (1 + \frac{2}{\nu \lambda_1}) K_1
\]
for \( s_0 \in [0, t[ \). Therefore, integrating the above inequality on the interval \([0, 1]\) in \( s_0 \), we get
\[
\int_m^{m+1} \left( \|\nabla w^n_t(\cdot)\|^2 + \alpha^2 \|\nabla h \nabla w^n_{t(s)}\|^2 \right) \, ds \leq \frac{M_0^2}{\nu} + (1 + \frac{2}{\nu \lambda_1}) \frac{K_1}{\nu}
\]
for all \( n \) and for each \( m = 0, 1, 2, \ldots \). The above estimates imply that \( \{S(t)w^n\} \) is bounded in \( L^\infty(0, \infty; V_h) \cap L^2_{\text{loc}}[0, \infty; V \cap H^1_0] \). Following the same line of reasoning as in the proof of existence in [5], one can also prove that \( S(t)w^n \in H^1_{\text{loc}}[0, \infty; V^*] \). Therefore, using Aubin–Lions compactness theorem, up to a subsequence, \( S(t)w^n \to \gamma(t) \) strongly in \( L^2_{\text{loc}}[0, T; V_h] \) as \( n \to +\infty \), with \( \gamma(t) \in L^\infty(0, \infty; V_h) \cap L^2_{\text{loc}}[0, \infty; V \cap H^1_0] \).

The same computations also show that, up to a subsequence, \( w^n \to w \) strongly in \( L^2_{\text{loc}}[0, T; V_h] \) as \( n \to +\infty \), with \( w \in L^\infty(0, \infty; V_h) \cap L^2_{\text{loc}}[0, \infty; V \cap H^1_0] \). By the continuity of \((\tau, w) \mapsto S(t)w\) in \([0, +\infty[ \times L^2_{\text{loc}}[0, \infty; V_h] \) and the uniqueness of the limit, we actually obtain that \( S(t)w^n \to \gamma(t) = S(t)w \in W \). This ends the proof.

6 The fully space-periodic case

In this section, in order to improve the regularity of the solutions, we consider a torus as a space domain, i.e., a domain periodic in all directions: \( D = \{ \mathbf{x} \in \mathbb{R}^3 : -\pi L < x_1, x_2, x_3 < \pi L \} \), \( L > 0 \), with \( 2\pi L \) periodicity with respect to \( \mathbf{x} \). This setting enables us to perform some computations (more precisely, the usage of some test functions) that are not allowed in the presence of Dirichlet boundary conditions; in such a way we retrieve some additional information on the horizontal components of velocity field, \( w_h \), and on the pressure, \( q \).

Function spaces are defined accordingly to our periodic setting, in particular
\[
L^2(D) = \{ \phi : D \to \mathbb{R} \text{ measurable}, 2\pi L \text{ periodic in } x, \int_D |\phi|^2 \, d\mathbf{x} < +\infty \} ,
\]
\[
L^2_0(D) = \{ \phi \in L^2(D) \text{ with zero mean with respect to } x \} .
\]

We can still consider the horizontal filtering and thus problem (6)–(7), and prove the existence of a unique regular weak solution (defined like in Section 2) essentially as done in the presence of Dirichlet boundary conditions on the top and bottom boundary \( \Gamma \).
6.1 Continuity of the horizontal flow

To prove the claimed continuity of \( w_h \), we need the following two lemmas.

6.1 Lemma A pressure term \( q \) in (6) corresponding to a regular weak solution to (6)–(7) satisfies \( \nabla h q \in L^2(0,T;L^2(D)) \) for each \( T > 0 \).

Proof. Taking the divergence operator in (6) and using (7), we obtain

\[-\Delta q = \nabla \cdot \left( \nabla \cdot (w \otimes w^h) \right) - \nabla \cdot f^h,\]

and, by applying the operator \( A_h \), we deduce

\[-\Delta A_h q = \nabla \cdot (w \otimes w) - \nabla \cdot f,\]

so that

\[
\|A_h q\|_{H^{-1}(D)} \leq C \left( \|w \otimes w\|_{H^{-1}(D)} + \|f\|_{H^{-2}(D)} \right)
\]

(18)

by elliptic estimates used along with \( \|w \otimes w\|_{H^{-1}(D)} \leq C\|w\|_{L^2(D)}\|\nabla w\|_{L^2(D)} \). This last control follows from the Hölder and the Gagliardo–Nirenberg inequalities: taking \( \varphi \in H^1(1) \), we have that

\[
\|(w \otimes w, \varphi)\| \leq \|w\|_{L^2} \|\varphi\|_{L^2} \leq C\|w\|_{L^2}\|\nabla w\|_{L^2}\|\varphi\|_{H^1},
\]

which proves the bound. Thus, from relation (18) we get

\[
\|A_h q\|_{L^2 H^{-1}} \leq C(\|w\|_{L^\infty L^2}\|\nabla w\|_{L^2 L^2} + \|f\|_{L^2 H^{-2}}),
\]

and hence \( A_h q \in L^2 H^{-1} \), so that \( \nabla h q \in L^2 L^2 \), where \( L^2 L^2 \) denotes \( L^2(0,T;L^2(D)) \).

6.2 Lemma If \( w \) is a regular weak solution to (6)–(7), then \( \partial_t \nabla h w_h \in L^2(0,T;V^*) \) for every \( T > 0 \).

Proof. From (6), we have

\[
\partial_t w_h = \mathbf{j}_h - \nu \Delta w_h - \nabla h q - \left[ \nabla \cdot (w \otimes w^h) \right]_h,
\]

(19)

Considering the last term in the righthand side of the above equation, we have

\[
\left\| \nabla \cdot (w \otimes w^h) \right\|_{L^2(D)} \leq \|w \otimes w\|_{L^2 L^2} \leq \|w \otimes \nabla\| \leq C\|w\|_{L^2(D)}\|\nabla w\|_{L^2(D)},
\]

and therefore

\[
\left\| \left[ \nabla \cdot (w \otimes w^h) \right]_h \right\|_{L^2 L^2} \leq C\|w\|_{L^\infty L^2}\|\nabla w\|_{L^2 L^2},
\]

which yields \( \nabla \cdot (w \otimes w^h) \in L^2 L^2 \).

Since all the other terms on the righthand side of (19) have the same regularity \( L^2 L^2 \), we deduce that \( \partial_t w_h \in L^2 L^2 \), or \( \partial_t \nabla w_h \in L^2 V^* \); in particular, \( \partial_t \nabla h w_h \in L^2 V^* \).

Now, we are ready to prove Proposition 1.2 (continuity of \( w_h \)).

Proof (Proposition 1.2). Since \( w \in L^2(\mathcal{V} \cap H^0_0) \), thus \( \nabla h w \in L^2 V \), and \( \partial_t \nabla h w_h \in L^2 V^* \), we obtain by interpolation (see [17, 19]) that \( \nabla h w_h \in \mathcal{C}^0 H \), i.e. \( w_h \in \mathcal{C}^0 V_h \). This implies the continuity of the map \( w_0 \mapsto w_h(t) \), with \( w_0 \in V_h \).
6.2 A higher order estimate

We refer to the beginning of Section 4 for the definitions of \( \Lambda_h, k_1(t) \), and set \( \lambda_1 = L^{-2} \) (first eigenvalue of the Laplace operator \(-\Delta\) on \( D \) fully periodic, and Poincaré constant) and

\[
k_2(t) = \| \nabla \mathbf{w}(t) \|^2 + \alpha^2 \| \nabla \nabla_h \mathbf{w}(t) \|^2,
\]

\[
K_2 = \frac{3}{\nu} \min \left\{ \frac{\| \Lambda_h^{-1} \mathbf{f} \|^2}{\alpha^2}, \| \mathbf{f} \|^2 \right\},
\]

(20)

\[
k_3(t) = K_2 + \frac{Ck_1(t)^3}{\alpha^2 \nu^3} \left( \frac{1}{\alpha^4 \lambda_1^4} + \frac{k_1(t)^2}{\nu^4} \right),
\]

\[
k_4(t) = k_2(t) + \frac{k_3(t)}{\nu \lambda_1}.
\]

We state and prove the following result.

6.3 Theorem Let \( \mathbf{w} \) be a regular weak solution to (6)–(7) in the time interval \([0, \infty]\); then we have that

\[
\| \nabla \mathbf{w}(t+r) \|^2 + \alpha^2 \| \nabla \nabla_h \mathbf{w}(t+r) \|^2 \leq k_2(t) e^{-\nu \lambda_1 (t+r)} + \frac{k_3(t)}{\nu \lambda_1} (1 - e^{-\nu \lambda_1 (t+r)}) \leq k_4(t)
\]

for each \( t, r > 0 \).

**Proof.** In the space periodic setting we can use \(-\Delta A_h \mathbf{w}\) as test function for Equation (6) (now formally, but the procedure actually goes through the Galerkin approximation).

First, observe that

\[
| (f, \Delta \mathbf{w}) | \leq \left\{ \begin{array}{l}
\| \Lambda_h^{-1} \mathbf{f} \| \| \Lambda_h \Delta \mathbf{w} \| \\
\| \mathbf{f} \| \| \Delta \mathbf{w} \|
\end{array} \right\} \leq \left\{ \begin{array}{l}
\| \Lambda_h^{-1} \mathbf{f} \|^2 \frac{\nu \alpha^2}{4} + \nu \alpha^2 \| \Delta \nabla_h \mathbf{w} \|^2 \\
3 \| \mathbf{f} \|^2 + \nu \| \Delta \mathbf{w} \|^2 \frac{2 \nu^2}{6}
\end{array} \right\}
\]

Thus, testing the equation (6) against \(-\Delta A_h \mathbf{w}\), we get

\[
\frac{1}{2} \frac{d}{dt} (\| \nabla \mathbf{w} \|^2 + \alpha^2 \| \nabla \nabla_h \mathbf{w} \|^2) + \nu (\| \Delta \mathbf{w} \|^2 + \alpha^2 \| \Delta \nabla_h \mathbf{w} \|^2)
\]

\[
\leq | (f, \Lambda_h \Delta \mathbf{w}) | + \| \mathbf{v} \cdot (w \otimes w) , \Lambda_h \Delta \mathbf{w} \|
\]

\[
\leq \frac{1}{2} K_2 + \frac{\nu \alpha^2}{4} \| \Delta \nabla_h \mathbf{w} \|^2 + \nu \| \Delta \mathbf{w} \|^2 + | ((w \cdot \nabla) w, \Delta \mathbf{w}) |.
\]

Take into account the nonlinear term in the righthand side of the above estimate. We have that

\[
| ((w \cdot \nabla) \mathbf{w}, \Delta \mathbf{w}) | = | (\nabla \left( (w \cdot \nabla) \mathbf{w} \right), \nabla \mathbf{w}) | = \left| \int_D \nabla \left( (w \cdot \nabla) \mathbf{w} \right) : \nabla \mathbf{w} dx \right|
\]

\[
\leq \left| (\nabla_h \left( (w \cdot \nabla) \mathbf{w} \right), \nabla_h \mathbf{w}) \right| + | \partial_3 \left( (w \cdot \nabla) \mathbf{w} \right), \partial_3 \mathbf{w} |
\]

\[
= \left| ((\nabla_h \mathbf{w} \cdot \nabla) \mathbf{w}, \nabla_h \mathbf{w}) \right| + \left| ((\partial_3 \mathbf{w} \cdot \nabla) \mathbf{w}, \partial_3 \mathbf{w}) \right|
\]

\[
= A + B,
\]

(23)
where in the last step we used the following relations

\[
((\mathbf{w} \cdot \nabla) \nabla_h \mathbf{w}, \nabla_h \mathbf{w}) = 0 \quad \text{and} \quad ((\mathbf{w} \cdot \nabla) \partial_3 \mathbf{w}, \partial_3 \mathbf{w}) = 0.
\]

Therefore, exploiting [5, §4, (4.12)] we obtain

\[
A = \left| \left( (\nabla_h \mathbf{w} \cdot \nabla_h) \mathbf{w} + [\nabla_h \mathbf{w}] \partial_3 \mathbf{w}, \nabla_h \mathbf{w} \right) \right| \\
\leq \left| \left( (\nabla_h \mathbf{w} \cdot \nabla_h) \mathbf{w}, \nabla_h \mathbf{w} \right) \right| + \left| \left( \nabla_h \mathbf{w} \partial_3 \mathbf{w}, \nabla_h \mathbf{w} \right) \right| \\
=: A_1 + A_2,
\]

where \([\cdot]_h\) and \([\cdot]_3\) indicate, respectively, the horizontal and vertical components of the considered vector fields.

Next, we estimate the nonlinear terms \(A_1\), \(A_2\) and \(B\) defined above. First, we estimate \(A_1\) and we get

\[
A_1 \leq \int_D |\nabla_h \mathbf{w}_{h}| |\nabla_h \mathbf{w}|^2 \\
\leq \|\nabla_h \mathbf{w}\|_{L^4}^2 \|\nabla_h \mathbf{w}_h\| \\
\leq \|\nabla \nabla_h \mathbf{w}\|^{3/2} \|\nabla_h \mathbf{w}\|^{1/2} \|\nabla_h \mathbf{w}_h\| \\
\leq \frac{1}{4} \nu \alpha^2 \lambda_1 \|\nabla \nabla_h \mathbf{w}\|^2 + \frac{C}{\nu^3 \alpha^6 \lambda_1^2} \|\nabla_h \mathbf{w}\|^4 \\
\leq \frac{1}{4} \nu \alpha^2 \|\Delta \nabla_h \mathbf{w}\|^2 + \frac{C}{\nu^3 \alpha^6 \lambda_1^2} \|\nabla_h \mathbf{w}\|^6.
\]

For the term \(A_2\) we have the following control

\[
A_2 \leq \int_D |\nabla_h \mathbf{w}| |\partial_3 \mathbf{w}| |\nabla_h \mathbf{w}_3| \\
\leq \int_D |\nabla \mathbf{w}|^2 |\nabla_h \mathbf{w}_3| \\
\leq \|\nabla \mathbf{w}\|_{L^4}^2 \|\nabla_h \mathbf{w}_3\| \\
\leq C \|\nabla \mathbf{w}\|^{7/4} \|\mathbf{w}\|^{1/4} \|\nabla_h \mathbf{w}_3\| \\
\leq \frac{1}{6} \nu \|\Delta \mathbf{w}\|^2 + \frac{C}{\nu^2 \|\mathbf{w}\|^2} \|\nabla_h \mathbf{w}_3\|^8,
\]

where we used the Gagliardo–Nirenberg inequality \(\|D^j \mathbf{w}\|_{L^p} \leq C \|D^m \mathbf{w}\|^{\alpha} \|\mathbf{w}\|^{1-\alpha}_{L^q}\), with \(j = 1, p = 4, m = r = q = 2\) and \(\alpha = 7/8\), and the Young’s inequality.

Let us now consider the term \(B\) in (23). We have that

\[
B = \left| \left( [\partial_3 \mathbf{w}]_h \cdot \nabla_h \right) \mathbf{w} + [\partial_3 \mathbf{w}]_3 \partial_3 \mathbf{w}, \partial_3 \mathbf{w} \right| \\
\leq \left| \left( \partial_3 \mathbf{w}_h \cdot \nabla_h \right) \mathbf{w}, \partial_3 \mathbf{w} \right| + \left| \left( \partial_3 \mathbf{w}_3 \partial_3 \mathbf{w}, \partial_3 \mathbf{w} \right) \right|.
\]
Now, arguing as in the case of $A_2$, the term $B$ can be controlled as follows.

\[
B \leq \int_D |\partial_3 w_b||\partial_3 w||\nabla_h w| + \int_D |\partial_3 w_3||\partial_3 w|^2
\leq \|\partial_3 w\|^2_1 \|\nabla_h w\| + \|\partial_3 w_3\|_2 \|
\partial_3 w_3\|
\leq \|\nabla w\|^2_1 (\|\nabla_h w\| + \|\partial_3 w_3\|)
\leq C \|\Delta w\|^{7/4} \|w\|^{1/4} (\|\nabla_h w\| + \|\partial_3 w_3\|)
\leq \frac{1}{6} \|\Delta w\|^2 + \frac{C}{\nu^2} \|w\|^2 (\|\nabla_h w\| + \|\partial_3 w_3\|)^8.
\]

(26)

Collecting (22), (24), (25), and (26), we obtain

\[
\frac{1}{2} \frac{d}{dt} (\|\nabla w\|^2 + \alpha^2 \|\nabla_h \nabla w\|^2) + \nu (\|\Delta w\|^2 + \alpha^2 \|\nabla_h \Delta w\|^2)
\leq \frac{1}{2} \nu (\|\Delta w\|^2 + \alpha^2 \|\nabla_h w_3\|^2) + \frac{C}{\nu^2 \alpha^4 \lambda_1^6} \|\nabla_h w\|^6 + \frac{C}{\nu^2} \|w\|^2 \|\nabla_h w_3\|^8
\]

\[
+ \frac{C_k}{\nu^2} \|w\|^2 (\|\nabla_h w\| + \|\partial_3 w_3\|)^8 + \frac{K_2}{2}
\leq \frac{1}{2} \nu (\|\Delta w\|^2 + \alpha^2 \|\Delta \nabla_h w\|^2) + \frac{K_2}{2} + \frac{C_k(t)^3}{\alpha^8 \nu^4} \left( \frac{1}{\alpha^4 \lambda_1^3} + \frac{k_1(t)^2}{\nu^4} \right),
\]

where in the last step we have used (10) and $\partial_3 w_3 = -\nabla_h \cdot w_b$ (which imply $w \in L^\infty(0,\infty; V_h)$ as well as $w_3 \in L^\infty(0,\infty; H^1)$). Thanks to the Poincaré’s inequality, we easily reach

\[
y'(t) + \nu \lambda_1 y(t) \leq k_3(t),
\]

having set

\[
y(t) = \|\nabla w(t)\|^2 + \alpha^2 \|\nabla_h w(t)\|^2.
\]

(28)

Hence, by applying Gronwall’s inequality in the interval $[t, t + r]$, we infer that

\[
\|\nabla w(t + r)\|^2 + \alpha^2 \|\nabla_h w(t + r)\|^2 \leq k_2(t) e^{-\nu \lambda_1 (t + r)} + \frac{k_3(t)}{\nu \lambda_1} (1 - e^{-\nu \lambda_1 (t + r)}),
\]

and finally

\[
\|\nabla w(t + r)\|^2 + \alpha^2 \|\nabla_h w(t + r)\|^2 \leq k_4(t),
\]

by definition of $k_4(t)$. Thus, the conclusion follows. \hfill \blacksquare

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