ALMOST SURE INVARIANCE PRINCIPLE OF $\beta$–MIXING TIME SERIES IN HILBERT SPACE

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ABSTRACT. Inspired by Berkes et al. (2014) and Wu (2007), we prove an almost sure invariance principle for stationary $\beta$–mixing stochastic processes defined on Hilbert space. Our result can be applied to Markov chain satisfying Meyn-Tweedie type Lyapunov condition and thus generalises the contraction condition in Berkes et al. (2014, Example 2.2). We prove our main theorem by the big and small blocks technique and an embedding result in Götze and Zaitsev (2011). Our result is further applied to the ergodic Markov chain and functional autoregressive processes.

Key words: Almost sure invariance principle, Hilbert space, $\beta$–mixing time series.

MSC2020: 60F17, 60G10

1. INTRODUCTION

Let $\mathbb{H}$ be a separable Hilbert space with orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$, denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the associated inner product and norm respectively. For any $x \in \mathbb{H}$, there exists an unique representation $x = \sum_{k=1}^{\infty} x_k e_k$ with $x_k \in \mathbb{R}$ for $k \geq 1$ and $\sum_{k=1}^{\infty} x_k^2 < \infty$, and thus we can represent $x$ by the sequence $(x_1, \ldots, x_k, \ldots)^T$, where $T$ is the transpose operator.

Let $X$ and $Y$ be two $\mathbb{H}$-valued random variables, denote by $\mathbb{P}_{X \times Y}$ the joint probability of $(X, Y)$ and by $\mathbb{P}_X, \mathbb{P}_Y$ the probabilities of $X$ and $Y$ respectively, define

$$\beta(X, Y) := \| \mathbb{P}_{X \times Y} - \mathbb{P}_X \times \mathbb{P}_Y \|_{TV},$$

where $\| \cdot \|_{TV}$ is the total variation norm of probability measures, i.e.

$$\| \mathbb{P}_{X \times Y} - \mathbb{P}_X \times \mathbb{P}_Y \|_{TV} = \sup_{f \in C_b(\mathbb{H} \times \mathbb{H}) : \| f \|_{\infty} \leq 1} | \mathbb{P}_{X \times Y}(f) - \mathbb{P}_X \times \mathbb{P}_Y(f) |.$$

Let $(X_k)_{k \in \mathbb{N}_0}$ be an $\mathbb{H}$-valued time series with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, its $\beta$–mixing coefficient is defined as

$$\beta(n) = \sup_{k \geq 1} \beta((X_i)_{0 \leq i \leq k}, (X_i)_{i \geq n+k}).$$

In this paper, we assume that

(A1) $(X_k)_{k \in \mathbb{N}_0}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is stationary with a stationary measure $\pi$ and exponentially $\beta$–mixing, i.e., there exist some $C > 0$ and $\beta > 0$ such that

$$\beta(n) \leq Ce^{-\beta n}, \quad n \in \mathbb{N}.$$

(A2) There exists some positive linear operator $\Gamma : \mathbb{H} \to \mathbb{H}$ such that

$$\lim_{n \to \infty} \frac{\text{cov}(\sum_{k=0}^{n-1} X_k)}{n} = \Gamma,$$
where $\text{cov}(\cdot)$ is the covariance operator of the $\mathbb{H}$-valued random variable. Moreover, $\Gamma$ have positive eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ with $\lambda_1 \geq \lambda_2 \geq \ldots > 0$. There exist $\delta_1 \geq \delta_2 > 1$ and positive constants $C_1$ and $C_2$, such that

$$C_1k^{-\delta_1} \leq \lambda_k \leq C_2k^{-\delta_2}, \quad k \in \mathbb{N}.$$ 

**Remark 1.1.** It is easy to obtain

$$\Gamma = \text{cov}(X_0) + \sum_{k=1}^{\infty} \left( \text{cov}(X_0, X_k) + \text{cov}(X_0, X_k)^T \right).$$

**Remark 1.2.** (A2) implies that $\Gamma$ has effective rank and the eigenvalues of covariance operator polynomially decay. Similar conditions can be found in Reiß and Wahl (2020), Lopes et al. (2019) and citations therein.

1.1. **Motivations and literature review.** Let $(Y_k)_{k \in \mathbb{N}_0}$ be random variables with $E|Y_k|^p < \infty$, $(Y_k)_{k \in \mathbb{N}_0}$ satisfies the almost sure invariance principle (ASIP) with rate $r_n$ if there exists, after suitably enlarging the probability space, independent Gaussian random variables $(\eta_k)_{k \in \mathbb{N}_0}$ with covariance $\text{cov}(Y_k)$, that is $\eta_k \sim \mathcal{N}(0, \text{cov}(Y_k))$ such that

$$\max_{1 \leq i \leq n} \left| \sum_{k=1}^{i} Y_k - \sum_{k=1}^{i} \eta_k \right| = o(r_n), \quad \text{a.s.},$$

where $o(\cdot)$ defined as follows, let $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ be two nonnegative real number sequences, if $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$, we write $a_n = o(b_n)$. We further denote $a_n \asymp b_n$ if there exist positive constants $c$ and $C$ such that $ca_n \leq b_n \leq Ca_n$.

The ASIP was introduced by Strassen (1964, 1967) to prove the functional law of iterated logarithm for independent, identically distributed (i.i.d.) random variables. Besides that, the ASIP implies Donsker’s theorem. Komlós et al. (1975, 1976) considered this problem for i.i.d. random variables with $p$-th moment and got the optimal rate $n^{1/p}$ for $p > 2$.

For the ASIP of dependent random variables, Wu (2007) used martingale approximation and the Skorokhod embedding to prove the ASIP of rate $n^{1/p}(\log n)^{1/2}$ for a class of stationary processes with $2 < p < 4$ on 1-dimensional space. However, due to Monrad and Philipp (1991) which showed that one cannot embed a general $\mathbb{R}^d$-valued martingale in an $\mathbb{R}^d$-valued Gaussian process, it is difficult to extend the martingale embedding method to $d$-dimensional space. Liu and Lin (2009) used the block technique to get the rate $n^{1/p}$ for $2 < p < 4$ on $\mathbb{R}^d$ space. Within this framework, Berkes et al. (2014) followed the block technique and some skills to construct the independence of the block sum, they got the rate $n^{1/p}$ for $p > 2$ on $\mathbb{R}$. For more research about the ASIP on $\mathbb{R}$, we refer the reader to Shao and Lu (1987); Merlevède and Rio (2012); Cuny et al. (2020a, b).

For the multidimensional invariance principle, Zaitsev (1998) obtained optimal rate for the multivariate version of the KMT theorem (see, Komlós et al. (1975, 1976)), and then established multivariate versions of Sakhanenko’s theorem, see Zaitsev (2001, 2002a,b). We refer the reader to Götze and Zaitsev (2009, 2010) for the follow-up work. These Gaussian approximation results for independent multivariate random variables give a tool to prove the ASIP for multidimensional dependent random variables. Gouëzel (2010) used the block technique and constructed the independence for the block sums, then got the ASIP for multidimensional dynamical system with the rate greater than $n^{1/4 + 1/(4p-4)}$ for $p > 2$ on $\mathbb{R}^d$–space under the mixing condition which is represented by the characteristic function. Following similar conditions, Hafouta (2020) got the rate greater than $n^{1/4}$ for uniformly bounded $\phi$–mixing sequence defined on $d$-dimensional space.
The motivations of studying the ASIP of stationary $\beta$—mixing time series are two folds. One is that there have been many ASIP results for dependent $\mathbb{R}$ and $\mathbb{R}^d$ valued random variables, see the references above, whereas there are very few this type of ASIP results for dependent time series in Hilbert space, see Dedecker and Merlevède (2010); Cuny and Merlevède (2014) and the references therein. The other is that $\beta$—mixing property is closely related to the concept of ergodicity for a stationary Markov chain, which can be verified by the Lyapunov condition, see Davydov (1974); Meyn and Tweedie (2009). Our result provides a weaker condition comparing with the condition of Berkes et al. (2014, Example 2.2), which is satisfied by many examples.

1.2. Main result and methodology. For random variables $\xi$ and $\eta$, let $\xi \overset{d}{=} \eta$ denote they have the same distribution. Our main result is given as follows.

**Theorem 1.3.** Assume that (A1) and (A2) hold and that the stationary measure $\pi$ has $p$—th moment with $p > 2$. Then there exists a probability space on which we can define random variables $X_i^*$ and i.i.d. Gaussian random variables $\eta_i$ such that $(X_i)_{1 \leq i \leq n} \overset{d}{=} (X_i^*)_{1 \leq i \leq n}$, $\eta_i \sim \mathcal{N}(0, \Gamma)$ and

$$\max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} (X_j^* - \pi(X)) - \sum_{j=1}^{i} \eta_j \right| = o(n^{\overline{\theta}}), \quad \text{a.s.,}$$

where $\Gamma$ defined in (A2) and $\overline{\theta} > \max \{ (2p-2)\delta_1 + 2\delta_2 + 2p-4, \frac{(2p-2)\delta_1 + 2\delta_2 + p(4p+4)/2 - 4}{4p-4+4p}, \frac{(2p-2)\delta_1 + 2\delta_2 + p(4p+4)/2 - 4}{4p-4+4p} \}$.

**Corollary 1.4.** Under the conditions of Theorem 1.3, for any $\varepsilon > 0$, there exists a $\delta_{p,\varepsilon}$ depends on $p$ and $\varepsilon$ and defined in (3.17) such that as $\lambda_k \asymp k^{-\delta}$ and $\delta > \delta_{p,\varepsilon}$, there exists a probability space on which we can define random variables $X_i^*$ and i.i.d. Gaussian random variables $\eta_i$ such that $(X_i)_{1 \leq i \leq n} \overset{d}{=} (X_i^*)_{1 \leq i \leq n}$, $\eta_i \sim \mathcal{N}(0, \Gamma)$ and

$$\max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} (X_j^* - \pi(X)) - \sum_{j=1}^{i} \eta_j \right| = o(n^{\frac{1}{4} + \frac{1}{4p-4+4p} - \varepsilon}), \quad \text{a.s..}$$

**Remark 1.5.** For random variables defined on $\mathbb{R}^d$ with condition (A1) and further assume $\Gamma$ is a positive semidefinite matrix. As we shall see in step 1 of the proof of Lemma 3.3, by the block technique and the invariance principle of i.i.d. random vectors in Zaitsev (2007, Corollary 3), when $\lambda > \frac{1}{4} + \frac{1}{4p-4+4p}$, the following ASIP holds,

$$\max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} (X_j^* - \pi(X)) - \sum_{j=1}^{i} \eta_j \right| = o((d^\theta \log d)n^{\lambda}), \quad \text{a.s..}$$

This rate is same as Gouëzel (2010, Theorem 1.2) which is under the mixing condition tied to spectral properties. As $\lambda_k \asymp e^{-|k|^\gamma}$ with $\gamma > 0$, the rate on the right hand side of (1.2) can be improved to $o(n^{\frac{1}{4} + \frac{1}{2} \frac{2}{3(p-2)}})$ by our method. One may intuitively see this by letting $\delta_1$ and $\delta_2$ tend to $\infty$ in Theorem 1.3.

Let us briefly describe our approach as follows. We divide the positive integer number $\mathbb{N}$ into the intervals $[2^m+1, 2^{m+1}]$ for $m = 0, 1, \ldots$. Each interval is partitioned into a sequence of big blocks with length $2^{a_1m}$ for $a_1 \in (0, 1)$ and small blocks with length $m$. Following the properties of $\beta$—mixing, we can construct i.i.d. random variables distributed as the big block sums in $\mathbb{H}$. The projection of these random variables on $\mathbb{H}_{\leq d}$ are comparable with Gaussian random variables following Götze and Zaitsev (2011), while the residual on $\mathbb{H}_{> d}$ and small block sums are negligible. After carefully choosing the relation between the dimension $d$ and the sample size $n$, we can get the result.
1.3. Organization of the paper and some notations. The paper is organized as follows. Our main result is stated in Section 1. In Section 2, we provide some preliminary lemmas and the technique for the proof of our main result. The proof of the ASIP is given in Section 3. In Section 4, we give some applications. The proof of the crucial lemmas in Section 2 is deferred to Appendix A and B, and the proofs of examples in Section 4 are given in Appendix C.

We finish this section by introducing some notations which will be frequently used in sequel. For variables \((X_i)_{i \in \mathbb{N}_0} \in \mathbb{H}, X_{i,j}\) denotes the \(j\)-th element of \(X_i\). We denote \(\xi^*\) the random variable has the same distribution with \(\xi\). For an operator \(A : \mathbb{H} \to \mathbb{H}, \|A\|_F\) for operator on \(\mathbb{H}\) denote the Frobenius norm of \(A\), i.e., \(\|A\|_F = \sqrt{\text{Tr}(AA^\top)}\). For \(x \in \mathbb{R}, \lfloor x \rfloor\) and \(\lceil x \rceil\) smallest integer greater than \(x\) and the integer part of \(x\) respectively. The symbols \(C\) and \(c\) denote positive numbers, \(C_p\) and \(c_p\) denote positive numbers depend on the parameter \(p\). Their values may vary from line to line.

For the stationary process \((X_k)_{k \in \mathbb{N}_0}\) with ergodic measure \(\pi\), without loss of generality, from now on we assume

\[
\pi(X_0) = 0.
\]

2. Auxiliary Lemmas for Theorem 1.3

The strategy of proving Theorem 1.3 is to decompose \(\sum_{i=1}^n X_i\) into two parts by the block technique, showing that the big block sum is comparable with Gaussian random variables and the small block sum is negligible following the properties of \(\beta\)-mixing which are shown below. In this section, we provide some preliminary lemmas and the technique for the proof of our main result.

2.1. Lemmas of \(\beta\)-mixing time series. In this subsection, we recall the result of Berbee (1987) first for the convenience of the reader, which is useful to construct the independence for \(\beta\)-mixing time series. At the end of this subsection, we give the following two lemmas, the first lemma paving a way for proving the last which is an extension to infinite dimensional space valued random variables of Shao and Yu (1996, Theorem 4.1).

Lemma 2.1. (Merlevède et al., 1997, Lemma 2) Let \(X\) and \(Y\) be two random variables defined on \(\mathbb{H}\) with quantile functions \(Q_{\|X\|}(u)\) and \(Q_{\|Y\|}(u)\). Then

\[
|\mathbb{E}(X, Y) - (\mathbb{E}X, \mathbb{E}Y)| \leq 18 \int_0^\bar{\alpha} Q_{\|X\|}(u)Q_{\|Y\|}(u)du.
\]

Here \(Q_{\|X\|}(u) = \inf\{t : \mathbb{P}(\|X\| > t) \leq u\}\) and \(Q_{\|Y\|}(u)\) is defined similarly. \(\bar{\alpha}\) is defined by \(\bar{\alpha} = \sup_{A \subset (X), B \subset (Y)} \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|\}\). Similarly,

\[
\|\text{cov}(X, Y)\|_F \leq 18 \int_0^\bar{\alpha} Q_{\|X\|}(u)Q_{\|Y\|}(u)du.
\]

It is easy to see that \(Q_{\|X\|}\) is the inverse function of \(\mathbb{P}(\|X\| > t)\) and is a non-increasing function. Moreover, \(Q_{\|X\|}(U) \overset{d}{=} \|X\|\), where \(U\) is a random variable uniformly distributed on \([0, 1]\).

Lemma 2.2. Suppose Assumption (A1) holds and \(X_0 \sim \pi\) with \(p\)-th moment, there exists a constant \(\bar{C} > 0\), such that for any \(2 \leq p' < p\) and large enough \(n\),

\[
\mathbb{E}\left\|\sum_{i=1}^n X_i\right\|^{p'} \leq \bar{C}n^{\frac{p'}{p}} \left(\pi(\|X\|^p)\right)^{\frac{p'}{p}}.
\]

Proof. The proof is given in Appendix B. \(\square\)
2.2. Blocking. The blocking technique is a typical way for dependent time series and applied in proving almost sure invariance principle, see e.g., Liu and Lin (2009); Gouëzel (2010); Berkes et al. (2014).

We subdivide $\mathbb{N}$ into the intervals $[2^m + 1, 2^{m+1}]$. For any positive integer $m$, let $n_1 = 2^{\lfloor \alpha_1 m \rfloor}$ and $n_2 = [C^* \log 2^m]$ with $0 < \alpha_1 < 1$ and $C^* > 0$, $\alpha_1$ and $C^*$ will be chosen later. For any $n \in [2^m + 1, 2^{m+1}]$, we denote
\[
\kappa(n) = \left\lfloor \frac{n - 2^m}{m_1 + m_2} \right\rfloor.
\]
For $1 \leq j \leq \kappa(2^{m+1})$, put
\[
I_{m,j} = \{ i : 2^m + (m_1 + m_2)(j - 1) + 1 \leq i \leq 2^m + (m_1 + m_2)(j - 1) + m_1 \},
\]
\[
J_{m,j} = \{ i : 2^m + (m_1 + m_2)(j - 1) + m_1 + 1 \leq i \leq 2^m + (m_1 + m_2)j \},
\]
where $I_{m,j}$ (resp., $J_{m,j}$) are big (resp., small) blocks. The tail is defined by
\[
I_{m,\kappa(2^{m+1}) + 1} = \{ i : 2^m + (m_1 + m_2)\kappa(2^{m+1}) + 1 \leq i \leq 2^{m+1} \land ((m_1 + m_2)\kappa(2^{m+1}) + m_1) \},
\]
\[
J_{m,\kappa(2^{m+1}) + 1} = \{ i : 2^m + (m_1 + m_2)\kappa(2^{m+1}) + m_1 + 1 \leq i \leq 2^{m+1} \}.
\]
Thus we decompose the interval $[2^m + 1, 2^{m+1}]$ as a union of $\kappa(2^{m+1}) + 1$ big blocks and $\kappa(2^{m+1}) + 1$ small blocks. We further denote $i_{m,j}$ is smallest element of $I_{m,j}$, $\mathcal{I}(m) = \bigcup_j I_{m,j}$ and $\mathcal{J}(m) = \bigcup_j J_{m,j}$. Let the block sums defined by
\[
Y_{m,j} = \sum_{i \in I_{m,j}} X_i, \text{ and } Z_{m,j} = \sum_{i \in J_{m,j}} X_i.
\]
We introduce following lemmas to give the moment bounds for $Y_{m,j}$ and $Z_{m,j}$ which paving a way for showing that $Y_{m,j}$ is comparable with i.i.d. $\tilde{Y}_{m,j}$ distributed as $Y_{m,j}$ for $j = 1, ..., \kappa(2^{m+1}) + 1$, we have
\[
\mathbb{E}\|Y_{m,j}\|^{p'} \leq Cm_1^{\frac{p'}{p}}, \quad \mathbb{E}\|Z_{m,j}\|^{p'} \leq Cm_2^{\frac{p'}{p}}.
\]

Lemma 2.3. Under the condition of Theorem 1.3, for any $2 \leq p' < p$ and $j = 1, ..., \kappa(2^{m+1}) + 1$, we have
\[
\mathbb{E}\|Y_{m,j}\|^{p'} \leq Cm_1^{\frac{p'}{p}}, \quad \mathbb{E}\|Z_{m,j}\|^{p'} \leq Cm_2^{\frac{p'}{p}}.
\]

Lemma 2.4. Under the condition of Theorem 1.3, for any $2 < p' < p$, we can construct independent random variables $\tilde{Y}_{m,j}$ defined on a richer probability space $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ such that $\tilde{Y}_{m,j} \overset{\mathcal{D}}{=} Y_{m,j}$ for $j = 1, ..., \kappa(2^{m+1})$ satisfies
\[
(2.1) \quad \max_{1 \leq i \leq \kappa(2^{m+1})} \| \sum_{j=1}^i (Y_{m,j} - \tilde{Y}_{m,j}) \| = o(2^{((1-\alpha_1)p'+\alpha_1)/2)m \sqrt{\log 2^m}), \quad a.s.
\]
Moreover,
\[
(2.2) \quad \max_{1 \leq i \leq j} \| \sum_{\ell=1}^i (Y_{m,\ell} - \tilde{Y}_{m,\ell}) \| = o(2^{((1-\alpha_1)p'+\alpha_1)/2)m \sqrt{\log 2^m}), \quad a.s.
\]

Lemma 2.5. Under the condition of Theorem 1.3, for any $n \in [2^m + 1, 2^{m+1}]$, we have
\[
\max_{2^m + 1 \leq i \leq n} \| \sum_{\ell \in \mathcal{J}(m) \cap [0,i]} X_\ell \| = o(2^{2(1-\alpha_1)m \log 2^m}), \quad a.s.
\]
Moreover, for any i.i.d. centered Gaussian random vectors $\eta_\ell$,
\[
\max_{2^m + 1 \leq i \leq n} \| \sum_{\ell \in \mathcal{J}(m) \cap [0,i]} \eta_\ell \| = o(2^{2(1-\alpha_1)m \log 2^m}), \quad a.s.
\]
3. PROOF OF THEOREM 1.3

The strategy of proving Theorem 1.3 is to decompose $\sum_{i=1}^{n} X_i$ into two parts by the block technique where the small blocks are negligible following Lemma 2.5. For big blocks, the projection of these random variables on $H_{\leq d}$ are comparable with Gaussian random variables following Götze and Zaitsev (2011), while the residual on $H_{> d}$ can be directly estimated. The dimension $d$ will be chosen carefully with respect to $n$. In this section we first introduce following lemmas to show big blocks are comparable with Gaussian random variables and then finish the proof of Theorem 1.3.

Lemma 3.1. Under the conditions of Theorem 1.3, one has

\begin{equation}
\|\text{cov}(\sum_{i=1}^{n} X_i) - n\Gamma\|_F \leq C,
\end{equation}

where $C$ is a positive constant depends on $p$ and $\beta$. For the diagonal elements of the matrix $\text{cov}(\sum_{i=1}^{n} X_i)$, we further have

\begin{equation}
\mathbb{E}[\sum_{i=1}^{n} X_{i,\ell}]^2 - n\lambda_\ell = r_\ell,
\end{equation}

where $X_{i,\ell}$ is the $\ell$-th element of $X_i$ and $r_\ell$ is a square summable constant depends on $\ell$.

**Proof.** A straight calculation yields that

$$\text{cov}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{cov}(X_i) + \sum_{1 \leq i < j \leq n} (\text{cov}(X_i, X_j) + \text{cov}(X_i, X_j)^T).$$

Since $(X_i)_{0 \leq i \leq n}$ is stationary, there exist linear operators $\Gamma_i$ for $i = 0, 1, 2, \ldots$ such that $\text{cov}(X_i) = \Gamma_0$ $\text{cov}(X_i, X_j) = \mathbb{E}[X_i X_j^T] = \mathbb{E}[X_0 X_{j-i}] = \Gamma_{j-i}$.

We further denote $\Gamma = \Gamma_0 + \sum_{k=1}^{\infty} (\Gamma_k + \Gamma_k^T)$. Thus,

$$\|\text{cov}(\sum_{i=1}^{n} X_i) - n\Gamma\|_F$$

$$= \|\sum_{i=1}^{n} \text{cov}(X_i) + \sum_{1 \leq i < j \leq n} (\text{cov}(X_i, X_j) + \text{cov}(X_i, X_j)^T) - n\Gamma_0 - n \sum_{k=1}^{\infty} (\Gamma_k + \Gamma_k^T)\|_F$$

$$= \|\sum_{k=1}^{\infty} (n-k)(\Gamma_k + \Gamma_k^T) - n \sum_{k=1}^{\infty} (\Gamma_k + \Gamma_k^T)\|_F$$

$$\leq \|n \sum_{k=1}^{\infty} (\Gamma_k + \Gamma_k^T)\| + \|\sum_{k=1}^{n} k(\Gamma_k + \Gamma_k^T)\|_F.$$  

To finish the proof, we show that $\|\text{cov}(X_0, X_k)\|_F = \|\Gamma_k\|_F \leq Ce^{-\beta k}$ and this gives the bound of (3.1). Following Lemma 2.1 and Hölder’s inequality, one has

$$\|\text{cov}(X_0, X_k)\|_F \leq 18 \int_0^{\alpha(k)} Q_{||X_0||}^2(u)du \leq 18\left( \int_0^1 1_{u \leq \alpha(k)} du \right)^{1-2/p} \left( \int_0^1 Q_{||X_0||}^p(u)du \right)^{2/p}$$

$$\leq Ce^{-\beta(1-2/p)}(\mathbb{E}[||X||^p])^{2/p}.$$
Thus, we obtain
\[
\| \text{cov}(\sum_{i=1}^{n} X_i) - n\mathbf{\Gamma} \|_F \leq C(\mathbb{E}\|X\|^p)^{2/p}.
\]

Notice that $\mathbf{\Gamma}$ is diagonal with element $\lambda_1, \lambda_2, \ldots$ and with the form $\mathbf{\Gamma} = \mathbf{\Gamma}_0 + \sum_{k=1}^{\infty}(\mathbf{\Gamma}_k + \mathbf{\Gamma}_k^T)$, thus for each element of $\mathbf{\Gamma}$, we have for the $\ell$–th element in the diagonal
\[
(3.3) \quad \mathbb{E}[X_{0,\ell}^2] + 2\sum_{k=1}^{\infty}\mathbb{E}[X_{0,\ell}X_{k,\ell}] = \lambda_{\ell},
\]
and for the remaining elements with $i \neq j$ and $i, j = 1, 2, \ldots$,
\[
(3.4) \quad \mathbb{E}[X_{0,i}X_{0,j}] + \sum_{k=1}^{\infty}\left(\mathbb{E}[X_{0,i}X_{k,j}] + \mathbb{E}[X_{0,j}X_{k,i}]\right) = 0.
\]
Following (3.1) and the definition of Frobenius norm, it is easy to see that
\[
\sum_{\ell=1}^{\infty}\left(\mathbb{E}\left[\sum_{i=1}^{n} X_{i,\ell}\right]^2 - n\lambda_{\ell}\right)^2 \leq C,
\]
which implies there exists a square summable number $r_{\ell}$ such that
\[
\mathbb{E}\left[\sum_{i=1}^{n} X_{i,\ell}\right]^2 - n\lambda_{\ell} = r_{\ell}.
\]

For any vectors $\xi_0, \ldots, \xi_n$ defined on $\mathbb{H}$, let
\[
\xi^{(d)}_i = (\xi_{i,1}, \ldots, \xi_{i,d})^T \in \mathbb{R}^d, \quad \xi^{[d]}_i = (0, \ldots, 0, \xi_{i,d+1}, \xi_{i,d+2}, \ldots)^T \in \mathbb{H}.
\]

**Lemma 3.2.** Under the conditions of Theorem 1.3 and $m_1\lambda_k > Ck + c$ for $k \in \mathbb{N}$ and constants $C$ and $c$, let $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and smallest eigenvalue of $\text{cov}(\sum_{i=1}^{m_1} X_i^{(d)})$ respectively, then one has one has
\[
\lambda_{\max} \leq C m_1 \lambda_1 \quad \lambda_{\min} \geq c m_1 \lambda_d.
\]

**Proof.** Since $X_i = ((X_i^{(d)})^T, 0, \ldots)^T + X_i^{[d]}$, a straight calculation implies that $\text{cov}(\sum_{i=1}^{m_1} X_i^{(d)})$ is the upper left $d \times d$ block of of $\text{cov}(\sum_{i=1}^{m_1} X_i)$. Following Garren (1968, Corollary II.A.3), we have
\[
\lambda_{\max} \leq \max_{k \in \{1, \ldots, d\}} \sum_{j=1}^{d} \left| \mathbb{E}\left[\sum_{i=1}^{m_1} X_{i,k} \sum_{i=1}^{m_1} X_{i,j}\right] \right|
\]
\[
\lambda_{\min} \geq \min_{k \in \{1, \ldots, d\}} \left( \left| \mathbb{E}\left[\sum_{i=1}^{m_1} X_{i,k}\right]^2 \right| - \sum_{j=1, j \neq k}^{d} \left| \mathbb{E}\left[\sum_{i=1}^{m_1} X_{i,k} \sum_{i=1}^{m_1} X_{i,j}\right] \right| \right).
\]

Since
\[
\sum_{j=1, j \neq k}^{d} \left| \sum_{i=1}^{m_1} \mathbb{E}[X_{i,k}X_{i,j}] + \mathbb{E}\left[\sum_{1 \leq i < r \leq m_1} X_{i,k}X_{r,j} + X_{i,j}X_{r,k}\right] \right|
\]
\[
= \sum_{j=1, j \neq k}^{d} \left| m_1 \mathbb{E}[X_{0,k}X_{0,j}] + \sum_{i=1}^{m_1} (m_1 - i) \mathbb{E}[X_{0,k}X_{i,j} + X_{0,j}X_{i,k}] \right|
\]

(3.4) and Lemma 2.1 yield
\[
\sum_{j=1, j \neq k}^{d} \left| \mathbb{E} \left[ \sum_{i=1}^{m_1} X_{i,k} \sum_{i=1}^{m_1} X_{i,j} \right] \right|
\]
\[
= \sum_{j=1, j \neq k}^{d} \left| \sum_{i=m_1 + 1}^{\infty} m_1 \mathbb{E}[X_{0,k}X_{i,j} + X_{0,j}X_{i,k}] + \sum_{i=1}^{m_1} i \mathbb{E}[X_{0,k}X_{i,j} + X_{0,j}X_{i,k}] \right|
\]
\[
\leq C(d-1).
\]
Combining with (3.2) and the fact \((r_k)_{k \geq 0}\) is a sequence of summable constants, we obtain
\[
\lambda_{\max} \leq \max_{k \in \{1, \ldots, d\}} \sum_{j=1}^{d} \left| \mathbb{E} \left[ \sum_{i=1}^{m_1} X_{i,k} \sum_{i=1}^{m_1} X_{i,j} \right] \right|
\]
\[
\leq \max_{k \in \{1, \ldots, d\}} \left( C(d-1) + m_1 \lambda_k + r_k \right) \leq Cd + m_1 \lambda_l + c
\]
\[
\lambda_{\min} \geq \min_{k \in \{1, \ldots, d\}} \left( \left| \mathbb{E} \left[ \sum_{i=1}^{m_1} X_{i,k} \right]^2 \right| - \sum_{j=1, j \neq k}^{d} \left| \mathbb{E} \left[ \sum_{i=1}^{m_1} X_{i,k} \sum_{i=1}^{m_1} X_{i,j} \right] \right| \right)
\]
\[
\geq \min_{k \in \{1, \ldots, d\}} \left( m_1 \lambda_k + r_k - C(d-1) \right) \geq m_1 \lambda_d - Cd + c.
\]
Since \(m_1 \lambda_1 \geq m_1 \lambda_d > Cd + c\), thus
\[
\lambda_{\max} \leq C m_1 \lambda_1, \quad \lambda_{\min} \geq c m_1 \lambda_d.
\]

\[\square\]

**Lemma 3.3.** Under the conditions of Theorem 1.3 and \(m_1 \lambda_k > Ck + c\) for \(k \in \mathbb{N}\). For any \(2 \leq p' < p\) and \(\tilde{Y}_{m,j}\), one can construct on a probability space \((\Omega_2, A_2, \mathbb{P}_2)\) a sequence of independent random vectors \((\tilde{Y}_{m,j})^*\) and the corresponding sequence of independent Gaussian random vectors
\[
\eta_{m,j} = \left( (\eta_{m,j})^T, 0, \ldots \right) + \eta_{m,j}^{[d]},
\]
for \(1 \leq j \leq \kappa(2^{m+1})\) so that
\[
\tilde{Y}_{m,j} \overset{\mathcal{D}}{=} (\tilde{Y}_{m,j})^*, \quad \eta_{m,j}^{[d]} \sim \mathcal{N}(0, \text{cov}(\tilde{Y}_{m,j}^{[d]})), \quad \eta_{m,j}^{(d)} \sim \mathcal{N}(0, \text{cov}(\tilde{Y}_{m,j}^{(d)}))
\]
and
\[
\max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (\tilde{Y}_{m,j})^* - \eta_{m,j} \right\|
\]
\[
= o \left( A_d \sqrt{\lambda_d^{-\frac{1}{2}} 2^{((1-\alpha)/p'+\alpha_1/2)m} \sqrt{\log 2m} + 2^{m/2}d^{(1-\delta_2)/2} \log 2m} \right) \quad \text{a.s.}
\]

**Proof.** It is easy to see that
\[
\tilde{Y}_{m,j} = \left( (\tilde{Y}_{m,j})^T, 0, 0, \ldots \right)^T + \tilde{Y}_{m,j}^{[d]},
\]
To compare \((\tilde{Y}_{m,j})_{1 \leq j \leq \kappa(2^{m+1})}\) with Gaussian random vectors on Hilbert space \(\mathbb{H}\), we show that \(\tilde{Y}_{m,j}^{(d)}\) are comparable with Gaussian random variables \(\eta_{m,j}^{(d)}\) on \(\mathbb{R}^d\) space following Götze and Zaitsev (2011, Theorem 2) and \(\tilde{Y}_{m,j}^{[d]}\) are negligible.
Step 1. We show that for $\tilde{Y}_{m,j}^{(d)}$ one can construct on a probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ a sequence of independent random vectors $(\tilde{Y}_{m,j}^{(d)})^* \cong \tilde{Y}_{m,j}^{(d)}$ and the corresponding sequence of independent Gaussian random vectors $\eta_{m,j}^{(d)} \sim \mathcal{N}(0, \text{cov}(\tilde{Y}_{m,j}^{(d)}))$ for $1 \leq j \leq \kappa(2^{m+1})$ such that

\begin{equation}
\max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{(d)})^* - \eta_{m,j}^{(d)}) \right\| = o\left(A_d \lambda_d^{-\frac{1}{2}} 2^{(1-\alpha_1)/p' + \alpha_1/2} m \sqrt{\log 2^m}\right), \quad \text{a.s.,}
\end{equation}

where $A_d = C \max\{d^{1/p'}, d^{(p'/4)2} (\log d)^{(p'/2)}\}$ and $C$ is a constant depends on $p'$.

For brevity, instead of writing out the properties of $(\tilde{Y}_{m,j}^{(d)})^*$ and $\eta_{m,j}^{(d)}$ listed above we simply say that there is a construction of $\tilde{Y}_{m,j}^{(d)}$ to show that one can construct a coupling between $\tilde{Y}_{m,j}^{(d)}$ and Gaussian random variables on a probability space enjoying the mentioned additional properties accordingly for $1 \leq j \leq \kappa(2^{m+1})$.

Let $\lambda_{\text{max},Y}$ and $\lambda_{\text{min},Y}$ be the maximal and minimal strictly positive eigenvalues of the covariance matrix $\text{cov}(\tilde{Y}_{m,j}^{(d)})$ respectively. According to Götze and Zaitsev (2011, Theorem 2), there is a construction of $\tilde{Y}_{m,j}^{(d)}$ on probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ such that

\begin{equation}
\mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{(d)})^* - \eta_{m,j}^{(d)}) \right\|^{p'} \right] \leq A_d (\lambda_{\text{max},Y}/\lambda_{\text{min},Y})^{\frac{p'}{2}} \kappa(2^{m+1}) \mathbb{E}\|\tilde{Y}_{m,j}\|^{p'} \leq CA_d (\lambda_{\text{max},Y}/\lambda_{\text{min},Y})^{\frac{p'}{2}} \kappa(2^{m+1}) \frac{m^{\frac{p'}{2}}}{2^m}.
\end{equation}

Following Lemma 3.2, we obtain

\begin{equation}
\mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{(d)})^* - \eta_{m,j}^{(d)}) \right\|^{p'} \right] \leq CA_d \lambda_d^{\frac{p'}{2}} \kappa(2^{m+1}) m^{\frac{p'}{2}}.
\end{equation}

Thus, we have

\begin{equation}
\sum_{m=1}^{\infty} \mathbb{P}\left( \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{(d)})^* - \eta_{m,j}^{(d)}) \right\| \geq A_d \lambda_d^{-\frac{1}{2}} 2^{(1-\alpha_1)/p' + \alpha_1/2} m \sqrt{\log 2^m} \right)
\leq \sum_{m=1}^{\infty} \mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{(d)})^* - \eta_{m,j}^{(d)}) \right\|^{p'} \right] A_d^{-1} \lambda_d^{\frac{p'}{2}} 2^{-(1-\alpha_1)/p' - \alpha_1/2} m (\log 2^m)^{-\frac{p'}{2}} \leq \sum_{m=1}^{\infty} C (\log 2^m)^{-\frac{p'}{2}} < \infty.
\end{equation}

By the Borel-Cantelli lemma, we obtain

\begin{equation}
\max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{(d)})^* - \eta_{m,j}^{(d)}) \right\| = o\left(A_d \lambda_d^{-\frac{1}{2}} 2^{(1-\alpha_1)/p' + \alpha_1/2} m \sqrt{\log 2^m}\right), \quad \text{a.s.}
\end{equation}

Step 2. We show that there is a construction of $\tilde{Y}_{m,j}^{[d]}$ on probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ such that

\begin{equation}
\max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} ((\tilde{Y}_{m,j}^{[d]})^* - \eta_{m,j}^{[d]}) \right\| = o\left(2^{(1-\alpha_1)m/2}(1 + 2^{\alpha_1 m} d_1^{-\delta_2})^{1/2} \log 2^m\right) \quad \text{a.s.}
\end{equation}
According to Götze and Zaitsev (2011, (25,28,29)), one has
\[
\mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\hat{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \|^2 \right] 
\leq C \mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\hat{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \|^2 \right] + C \mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} \eta_{m,j}^{[d]} \|^2 \right] 
\leq C \sum_{j=1}^{\kappa(2m+1)} \mathbb{E} \| \tilde{Y}_{m,j}^{[d]} \|^2.
\]

By (3.2) and the condition \( m_1 \lambda_k > Ck + c \geq r_k \) for \( k \in \mathbb{N} \), we can get
\[
\mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \|^2 \right] 
\leq C \kappa(2m+1) \sum_{k=d+1}^{\infty} \mathbb{E} \left[ \sum_{i=1}^{m_1} X_{i,k} \right]^2 
= C \kappa(2m+1) \sum_{k=d+1}^{\infty} (m_1 \lambda_k + r_k) 
\leq C 2^m d^{1-\delta_2}.
\]

Thus, the Markov inequality yields
\[
\sum_{m=1}^{\infty} \mathbb{P} \left( \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \| \geq 2^{m/2} d^{(1-\delta_2)/2} \log 2^m \right) 
\leq \sum_{m=1}^{\infty} \mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \|^2 \right] 2^{-m} d^{-1+\delta_2} \log (-2) 2^m < \infty.
\]

By the Borel-Cantelli lemma, we obtain
\[
\max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \| = o \left( 2^{m/2} d^{(1-\delta_2)/2} \log 2^m \right), \text{ a.s.}
\]

Step 3. Combining the estimates in step 1 and step 2 above, i.e., (3.6) and (3.7), one has
\[
\max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \| 
\leq \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \| + \max_{1 \leq i \leq \kappa(2m+1)} \| \sum_{j=1}^{i} (\tilde{Y}_{m,j}^{[d]} - \eta_{m,j}^{[d]}) \| 
= o \left( A_d^{1/2} \lambda_d^{-1/2} \gamma(1-\alpha_1/p') \log 2^{m/2} + 2^{m/2} d^{(1-\delta_2)/2} \log 2^m \right) \text{ a.s.}
\]

Proof of Theorem 1.3. For any \( i \in [2^m + 1; 2^{m+1}] \), it is easy to see that
\[
\sum_{\ell=1}^{i} \chi_{\ell} = \sum_{j=1}^{\kappa(i)} Y_{m,j} + \sum_{\ell \in I_{m,\kappa(i)+1} \cap [2^m+1,i]} X_{\ell} + \sum_{\ell \in J(m) \cap [2^m+1,i]} X_{\ell}.
\]
\[
\sum_{\ell=1}^{i} \chi_{\ell} = \sum_{j=1}^{\kappa(i)} Y_{m,j} + \sum_{\ell \in I_{m,\kappa(i)+1} \cap [2^m+1,i]} X_{\ell} + \sum_{\ell \in J(m) \cap [2^m+1,i]} X_{\ell}.
\]
Recall that $i_{m,j}$ is the smallest element of $I_{m,j}$, following Lemma 2.5, we can get

$$
\max_{2^m+1 \leq l \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i_{m,j}} X_{\ell} - \sum_{j=1}^{\kappa(i)} Y_{m,j} \right\|
\leq \max_{2^m+1 \leq l \leq 2^{m+1}} \left( \sum_{\ell \in I_{m,i^+1}} \left\| X_{\ell} \right\| + \max_{2^m+1 \leq l \leq 2^{m+1}} \left\| \sum_{\ell \in J(m)} X_{\ell} \right\| \right) = \max_{1 \leq j \leq \kappa(2^{m+1})} \max_{1 \leq i < |I_{m,j}|} \left\| \sum_{\ell \in i_{m,j}} X_{\ell} \right\| + o\left(2^{\frac{1}{2}(1-\alpha_1)m \log 2^m}\right), \quad \text{a.s..}
$$

(3.8)

Let $p'$ be a positive constant such that $2 < p' < p$. For the first term, Wu (2007, Proposition 1) and Lemma 2.3 imply

$$
\left( \mathbb{E}\left[ \max_{1 \leq i \leq 2^r} \left\| \sum_{j=1}^{i_{m,j}} X_j \right\|^{p'} \right] \right)^{\frac{1}{p'}} \leq \sum_{i=0}^{r} 2^{(r-i)/p'} \left( \mathbb{E}\left[ \sum_{j=1}^{i_{m,j}} \left\| X_j \right\|^{p'} \right] \right)^{\frac{1}{p'}} \leq C \sum_{i=0}^{r} 2^{(r-i)/p'} (2^i)^{\frac{1}{2}} \leq C 2^{rac{r}{2}},
$$

which yields

$$
\mathbb{E}\left[ \max_{1 \leq i < |I_{m,j}|} \left\| \sum_{\ell \in i_{m,j}} X_{\ell} \right\|^{p'} \right] \leq |I_{m,j}|^{\frac{p'}{2}} \leq 2^\frac{p'}{2} \alpha_1 m.
$$

Thus, the Markov inequality implies

$$
\sum_{m=1}^{\infty} \mathbb{P}\left( \max_{1 \leq j \leq \kappa(2^{m+1})} \max_{1 \leq i < |I_{m,j}|} \left\| \sum_{\ell \in i_{m,j}} X_{\ell} \right\| \geq 2^{(1-\alpha_1)/p' + \alpha_1/2)m \left( \log 2^m \right)^{1/2}} \right)
\leq \sum_{m=1}^{\infty} \mathbb{E}\left[ \max_{1 \leq j \leq \kappa(2^{m+1})} \max_{1 \leq i < |I_{m,j}|} \left\| \sum_{\ell \in i_{m,j}} X_{\ell} \right\|^{p'} \right] 2^{-(1-\alpha_1 + \alpha_1 p'/2)m \left( \log 2^m \right)^{-p'/2}}
\leq \sum_{m=1}^{\infty} \mathbb{E}\left[ \max_{1 \leq j \leq \kappa(2^{m+1})} \max_{1 \leq i < |I_{m,j}|} \sum_{\ell \in i_{m,j}} X_{\ell} \right] 2^{-(1-\alpha_1 + \alpha_1 p'/2)m \left( \log 2^m \right)^{-p'/2}}
\leq \sum_{m=1}^{\infty} 2^{(1-\alpha_1+p'/2)m} 2^{-(1-\alpha_1 + \alpha_1 p'/2)m \left( \log 2^m \right)^{-p'/2}} < \infty.
$$

By the Borel-Cantelli lemma, we obtain

$$
\max_{1 \leq j \leq \kappa(2^{m+1})} \max_{1 \leq i < |I_{m,j}|} \left\| \sum_{\ell \in i_{m,j}} X_{\ell} \right\| = o\left(2^{(1-\alpha_1)/p' + \alpha_1/2)m \left( \log 2^m \right)^{1/2}} \right), \quad \text{a.s..}
$$

(3.9)

Similar estimate holds for any i.i.d. centered Gaussian random vectors $\eta_{\ell}$, i.e.,

$$
\max_{1 \leq j \leq \kappa(2^{m+1})} \max_{1 \leq i < |I_{m,j}|} \left\| \sum_{\ell \in i_{m,j}} \eta_{\ell} \right\| = o\left(2^{(1-\alpha_1)/p' + \alpha_1/2)m \left( \log 2^m \right)^{1/2}} \right), \quad \text{a.s.}
$$

(3.10)

Combining (3.8) and (3.9), we have

$$
\max_{2^m+1 \leq l \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i_{m,j}} X_{\ell} - \sum_{j=1}^{\kappa(i)} Y_{m,j} \right\| = o\left(2^{\frac{1}{2}(1-\alpha_1)m} + 2^{\left(1-\alpha_1(p'/2)+\alpha_1/2\right)} \log 2^m \right), \quad \text{a.s.}
$$
Following Lemma 2.4, on a richer probability space of \((\Omega, \mathcal{A}, \mathbb{P})\), one has
\[
\max_{2^m+1 \leq i \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i} X_{\ell} - \sum_{j=1}^{\kappa(i)} \tilde{Y}_{m,j} \right\|
\leq \max_{2^m+1 \leq i \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i} X_{\ell} - \sum_{j=1}^{\kappa(i)} Y_{m,j} + \sum_{j=1}^{\kappa(i)} Y_{m,j} - \sum_{j=1}^{\kappa(i)} \tilde{Y}_{m,j} \right\|
= o\left(2^{\frac{1}{2} (1-\gamma) m} + 2^\frac{p'}{p} + 2^{\frac{\alpha_1}{2}} \right) \log 2^m, \quad \text{a.s.}
\]

Lemma 3.3 implies that there is a construction of \(Y_{m,j}\) on probability space \((\Omega_2, \mathcal{A}_2, \mathbb{P}_2)\) and one can compare \(\tilde{Y}_{m,j}\) with \(\eta_{m,j}\). Now we regularize \(\eta_{m,j}\) using the linear form \(\text{cov}(\eta_{m,j})\) by the linear form \(|I_{m,j}| \Gamma\). We denote \(\tilde{\eta}_{m,j} := \mathcal{N}(0, |I_{m,j}| \Gamma)\).

Following Lemma 3.1 and for large enough \(m\), one has
\[
\text{cov}(\eta_{m,j}) + M_{m,j} = |I_{m,j}| \Gamma + N_{m,j},
\]
where \(M_{m,j}\) and \(N_{m,j}\) are positive definite linear operators with \(\|M_{m,j}\|_F \leq C\), \(\|N_{m,j}\|_F \leq C\). Therefore, \(\mathcal{N}(0, |I_{m,j}| \Gamma + N_{m,j})\) is the sum of \(\tilde{\eta}_{m,j}\) and an independent random variable \(\mathcal{N}(0, N_{m,j})\). On the other hand, \((3.12)\) implies \(\mathcal{N}(0, |I_{m,j}| \Gamma + N_{m,j})\) is also the sum of \(\eta_{m,j}\) and an independent random variable \(\mathcal{N}(0, M_{m,j})\). Using Berkes and Philipp (1979, Lemma A1), we obtain a coupling between \(\eta_{m,j}\) and \(\tilde{\eta}_{m,j}\) such that the difference \(D_{m,j} = \eta_{m,j} - \tilde{\eta}_{m,j}\) is centered and \(\mathbb{E}\|D_{m,j}\|^2 \leq C\).

Thus, following the Lévy inequality, see Lin and Bai (2011, 5.4.a), we have
\[
\mathbb{P}\left(\max_{2^m+1 \leq i \leq 2^{m+1}} \left\| \sum_{j=1}^{\kappa(i)} D_{m,j} \right\| > 2^{\frac{1}{2} (1-\gamma) m} \log 2^m\right)
\leq 2 \mathbb{P}\left(\left\| \sum_{j=1}^{\kappa(2^{m+1})} D_{m,j} \right\| > 2^{\frac{1}{2} (1-\gamma) m} \log 2^m\right)
\leq 2 \mathbb{E}\left\| \sum_{j=1}^{\kappa(2^{m+1})} D_{m,j} \right\|^2 2^{-(1-\gamma) m} (\log 2^m)^{-2}
\leq C \kappa(2^{m+1}) 2^{-(1-\gamma) m} (\log 2^m)^{-2},
\]
which is summable with respect to \(m\). Then there is a construction for \(\eta_{m,j}\) on probability space \((\Omega_3, \mathcal{A}_3, \mathbb{P}_3)\) such that
\[
\max_{2^m+1 \leq i \leq 2^{m+1}} \left\| \sum_{j=1}^{\kappa(i)} (\eta_{m,j}^* - \tilde{\eta}_{m,j}^*) \right\| = o\left(2^{\frac{\gamma}{2} (1-\gamma) m} \log 2^m\right), \quad \text{a.s.}
\]
Combining (3.13) with (3.5) and Berkes et al. (2014, Lemma 4.1), there is a construction for \(\tilde{Y}_{m,j}^*\) on probability space \((\Omega_4, \mathcal{A}_4, \mathbb{P}_4)\) such that
\[
\max_{2^m+1 \leq i \leq 2^{m+1}} \left\| \sum_{j=1}^{\kappa(i)} (\tilde{Y}_{m,j}^* - \tilde{\eta}_{m,j}^*) \right\|
=o\left(\left(\frac{a_d}{\kappa d} \right)^{\frac{1}{2}} 2^{\frac{1}{2} (1-\gamma) m} + 2^{m/2} d^{(1-\delta)/2} + 2^{(1-\gamma) m/2} \log 2^m\right) \quad \text{a.s.}
\]
Using Berkes et al. (2014, Lemma 4.1) again with (3.10), (3.11), (3.14) and Lemma 2.5, we can finally construct a probability space \((\Omega, A, \mathbb{P})\) on which we can define \(X^*_\ell\) distributed as \(X\) to ensure \(p\) close enough to \(\bar{p}\) and \(d\).

Hence we can construct probability space \((\Omega', A', \mathbb{P}')\) on which

\[
\max_{1 \leq i \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i} (X^*_\ell - \eta_\ell) \right\| = o(2^{\frac{m}{2} (1-\delta_2)\theta_p'} \log 2^m) \quad \text{a.s.}
\]

For the logarithmic term, there exists a \(2 < p'' < p'\) such that,

\[
\max_{1 \leq i \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i} (X^*_\ell - \eta_\ell) \right\| = o(2^{\frac{m}{2} (1-\delta_2)\theta_{p''}}) \quad \text{a.s.}
\]

Let \(p''\) close enough to \(p\), we obtain

\[
\max_{1 \leq i \leq 2^{m+1}} \left\| \sum_{\ell=1}^{i} (X^*_\ell - \eta_\ell) \right\| = o(2^{m\bar{\theta}}) \quad \text{a.s.},
\]

where

\[
(3.15) \quad \bar{\theta} > \max \left\{ \frac{(2p-2)\delta_1 + 2\delta_2 + 23p-4}{44p-4 + (4p-4)\delta_1 + 2p\delta_2}, \frac{(2p-2)\delta_1 + 2\delta_2 + p(p+4)/2 - 4}{p(p+2) - 4 + (4p-4)\delta_1 + 2p\delta_2} \right\},
\]

and consequently

\[
\max_{1 \leq i \leq n} \left\| \sum_{\ell=1}^{i} (X^*_\ell - \eta_\ell) \right\| = o(n^{\bar{\theta}}), \quad \text{a.s.}
\]

\[\Box\]

Proof of Corollary 1.4. When \(\delta_1 = \delta_2 = \delta\), it is easy to see that

\[
(3.16) \quad \bar{\theta} > \max \left\{ \frac{2p\delta + 23p - 4}{44p-4 + (6p-4)\delta}, \frac{2p\delta + p(p+4)/2 - 4}{p(p+2) - 4 + (6p-4)\delta} \right\},
\]
which converges to \( \frac{1}{p} + \frac{2}{d(3p - 2)} \) as \( \delta \to \infty \). That is, for any \( \varepsilon > 0 \), as \( \delta > \delta_{p, \varepsilon} \) where

\[
(3.17) \quad \delta_{p, \varepsilon} = \begin{cases} 
\frac{25p^2 - 54p + 8 - (4p - 4)(3p - 2)\varepsilon}{2(3p - 2)^2\varepsilon}, & p \leq 42, \\
\frac{p^2/2 + 3p^2 - 12p^2 + 8 - (p^2 + 2p - 4)(3p - 2)\varepsilon}{2(3p - 2)^2\varepsilon}, & p > 42.
\end{cases}
\]

we obtain

\[
\max_{1 \leq j \leq n} \left\| \sum_{j=1}^{i} (X_j^\varepsilon - \eta_j) \right\| = o\left(n^{\frac{1}{p} + \frac{2}{d(3p - 2)} + \varepsilon}\right), \quad \text{a.s.}
\]

\( \Box \)

4. Examples

In this section, we give two examples where the first compares the mixing condition with the geometric moment contraction (GMC) condition of Berkes et al. (2014, Example 2.2) on \( \mathbb{H} \) and the second considers the functional autoregressive processes.

4.1. Markov chain. Let \( (X_k)_{k \geq 0} \) be a \( \mathbb{H} \)-valued time homogeneous Markov chain with \( p \)-th moment for \( p > 2 \) satisfying the condition:

(A3) \( (X_k)_{k \geq 0} \) is irreducible, aperiodic and Feller. There exists a Lyapunov function \( V : \mathbb{H} \to [1, +\infty) \) such that

\[
\mathbb{E}[V(X_1)|X_0 = x] \leq \gamma V(x) + K1_C(x),
\]

where \( 0 < \gamma < 1 \), \( K > 0 \) and \( C \) is a compact set.

It is easy to prove that \( (X_k)_{k \geq 0} \) is exponential ergodic with invariant measure \( \pi \) which yields exponential \( \beta \)-mixing, see Tuominen and Tweedie (1994, Theorem 2.1) and Davydov (1974, Proposition 1). We further assume that \( X_0 \sim \pi \) and \( \Gamma \) is a positive definite operator whose eigenvalues polynomial decay, then \( (X_k)_{k \geq 0} \) satisfies ASIP with rate (1.3).

Comparing (A3) with the contraction condition in Berkes et al. (2014),

\[
\mathbb{E}\|X_t^\varepsilon - X_t^y\|^p \leq r^p\|x - y\|^p,
\]

where \( 0 < r < 1 \), \( X_t^x \) denotes the Markov chain with initial value \( X_0 = x \). The term \( K1_C(x) \) of (A3) makes it be a weaker condition than the contraction condition.

4.2. Functional autoregressive processes. We consider the functional autoregressive processes

\[
X_{k+1} = \mu + AX_k + B\varepsilon_{k+1},
\]

where \( A \) and \( B \) are linear operators from \( \mathbb{H} \) to \( \mathbb{H} \) with kernel \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) respectively, \( \varepsilon_{k+1} \) is white noise. Let \( (e_k)_{k \in \mathbb{N}} \) be an orthonormal basis of \( L^2([0, \pi]) \) with the form \( e_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx) \) and

\[
Ae_k(x) = \lambda_k e_k(x),
\]

where \( Af(x) = \int_0^\pi a(s, x)f(s)ds \). According to Karhunen-Loève decomposition, one has

\[
a(s, t) = \sum_{k=1}^{\infty} \lambda_k e_k(s)e_k(t).
\]

(4.3) can be written as

\[
X_{k+1}(\cdot) = \mu(\cdot) + \int_0^\pi a(\cdot, s)X_k(s)ds + \int_0^\pi b(\cdot, s)e_{k+1}(s)ds.
\]
We refer the reader to Bosq (2000); Wang et al. (2020) for more details of functional autoregressive processes. We assume $\mu = 0$, $B^2 = A$ and $A$ is symmetric for the simplification of calculation. Further assuming that $0 < \lambda_k \asymp k^{-\delta} < 1$, conditions (A1) and (A2) are satisfied and $(X_k)_{k \in \mathbb{N}_0}$ satisfies the ASIP.

**APPENDIX A. THE PROOF OF LEMMA 2.2**

The proof of Lemma 2.2 following the properties of $\alpha-$mixing sequence, see Bradley (2005) for more details. Denote the $\alpha-$mixing coefficients by

$$\alpha(n) = \sup_{k \geq 1} \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \sigma(X_i, 1 \leq i \leq k), B \in \sigma(X_i, i \geq k + n)\}.$$ 

Since $\alpha(n) \leq \beta(n)$, assumption (A1) implies

(A.1) \quad $\alpha(n) \leq Ce^{-\beta n}$.

We first give following preparing lemma.

**Lemma A.1.** Let $(\theta_i)_{1 \leq i \leq n}$ be a sequence of random variables on $\mathbb{H}$ with finite $p-$moment and let $\mathcal{F}_i = \sigma(\theta_j, j \leq i)$. Then for any $p \geq 2$, there exists constant $C_p$ such that

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} \theta_i\right\|^p\right] \leq C_p \left( \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i|^2]^\frac{p}{2} \right)^2 + \sum_{i=1}^{n} \mathbb{E}[|\theta_i|^p] + n^{p-1} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[\theta_i | \mathcal{F}_{i-1}]]^{\frac{p}{2}} \right) + n^{p-1} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[|\theta_i|^2 | \mathcal{F}_{i-1}] - \mathbb{E}[|\theta_i|^2]]^{\frac{p}{2}}.$$

**Proof.** The strategy is to construct martingale differences $\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}]$ and using the Burkholder inequality to get the result. That is,

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} \theta_i\right\|^p\right] = \mathbb{E}\left[\left\|\sum_{i=1}^{n} (\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}] + \mathbb{E}[\theta_i | \mathcal{F}_{i-1}])\right\|^p\right] \leq 2^p \left( \mathbb{E}\left[\left\|\sum_{i=1}^{n} (\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}])\right\|^p\right] + n^{p-1} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[\theta_i | \mathcal{F}_{i-1}]]^{\frac{p}{2}} \right).$$

For the first term, Pinelis (1994, Theorem 4.1) implies

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} (\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}])\right\|^p\right] \leq C_p \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}]|^p] + \mathbb{E}\left[\left\|\sum_{i=1}^{n} (\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}])\right\|^2 | \mathcal{F}_{i-1}\right]^{\frac{p}{2}} \right)$$

$$\leq C_p \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i - \mathbb{E}[\theta_i | \mathcal{F}_{i-1}]|^p] + \mathbb{E}\left[\left\|\theta_i\right\|^2 | \mathcal{F}_{i-1}\right]^{\frac{p}{2}} \right)$$

$$= C_p \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i|^p] + \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i|^2]^{\frac{p}{2}} \right)^2 + \mathbb{E}\left( \sum_{i=1}^{n} \left( \mathbb{E}[|\theta_i|^2 | \mathcal{F}_{i-1}] - \mathbb{E}[|\theta_i|^2] \right)^{\frac{p}{2}} \right) \right)$$

$$\leq C_p \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i|^p] + \left( \sum_{i=1}^{n} \mathbb{E}[|\theta_i|^2]^{\frac{p}{2}} \right)^2 + n^{p-1} \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[|\theta_i|^2 | \mathcal{F}_{i-1}] - \mathbb{E}[|\theta_i|^2]]^{\frac{p}{2}} \right).$$
Thus we can get the result.

**Proof of Lemma 2.2.** We first prove the case $p' = 2$ which is a extension of Rio (1993) on $\mathbb{H}$. The fact that $(X_i)_{1 \leq i \leq n}$ is zero mean implies

\[
\| \sum_{i=1}^{n} X_i \|^2 = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} E\langle X_i, X_j \rangle
\]

\[
= \sum_{1 \leq i \leq n} E\|X_i\|^2 + 2 \sum_{1 \leq i < j \leq n} (E\langle X_i, X_j \rangle - \langle E X_i, E X_j \rangle)
\]

\[
\leq \sum_{1 \leq i \leq n} E\|X_i\|^2 + 2 \sum_{1 \leq i < j \leq n} 18 \int_{0}^{\alpha(j-i)} Q_{\|X_i\|}(u) du,
\]

where the last line follows Lemma 2.1 and $\alpha(j - i) \geq \bar{\alpha}$ therein. For the second term, a straight calculation yields

\[
\sum_{1 \leq i < j \leq n} \int_{0}^{\alpha(j-i)} Q_{\|X_i\|}(u) du
\]

\[
= \sum_{1 \leq i < j \leq n} \int_{0}^{1} \sum_{i < j \leq n} 1_{(\alpha(j-i) > u)} Q_{\|X_i\|}(u) du
\]

\[
\leq \sum_{1 \leq i \leq n} \left( \int_{0}^{1} \left( \sum_{i < j \leq n} 1_{(\alpha(j-i) > u)} \right)^{\frac{p}{p-2}} du \right)^{\frac{p-2}{p}} \left( \int_{0}^{1} Q_{\|X_i\|}(u) du \right)^{\frac{2}{p}}.
\]

Since $\sum_{1 \leq i \leq n} 1_{(\alpha(j) > u)} = k$ if and only if $\alpha(k + 1) \leq u < \alpha(k)$, then we have

\[
\int_{0}^{1} \left( \sum_{i < j \leq n} 1_{(\alpha(j-i) > u)} \right)^{\frac{p}{p-2}} du \leq \int_{0}^{1} \left( \sum_{j=1}^{n} 1_{(\alpha(j) > u)} \right)^{\frac{p}{p-2}} du
\]

\[
= \sum_{k=1}^{\infty} \int_{0}^{\alpha(k+1)} \left( \sum_{j=1}^{n} 1_{(\alpha(j) > u)} \right)^{\frac{p}{p-2}} du.
\]

\[
\leq \sum_{k=1}^{\infty} k^{\frac{p}{p-2}} \alpha(k) < \infty.
\]

Notice that $\int_{0}^{1} Q_{\|X_i\|}(u) du = E\|X_i\|^p$, combining estimates above with (A.2), we obtain

\[
2 \sum_{1 \leq i < j \leq n} 18 \int_{0}^{\alpha(j-i)} Q_{\|X_i\|}(u) du \leq \sum_{1 \leq i \leq n} C(E\|X_i\|^p)^{\frac{2}{p}}
\]

Thus, the stationary of $(X_i)_{0 \leq i \leq n}$ implies

\[
E\left\| \sum_{i=1}^{n} X_i \right\|^2 \leq Cn(\pi(\|X\|^p))^\frac{2}{p},
\]

We shall prove the case $p > p' > 2$ by induction on $n$. Suppose that for $1 \leq k < n$,

\[
E\left\| \sum_{i=1}^{k} X_i \right\|^{p'} \leq Ck^{\frac{p'}{2}}(\pi(\|X\|^p))^\frac{p'}{2}.
\]
When \( k = n \), let \( m = \lfloor \sqrt{n} \rfloor \) and \( \bar{\kappa}(n) = \lfloor \frac{n}{2m} \rfloor \) here. The block sums are defined by

\[
Y_{i,1} = \sum_{j=1+2(i-1)m}^{n/2m} X_j; \quad Y_{i,2} = \sum_{j=1+(2i-1)m}^{n/2m} X_j \quad i \in \{1, ..., \bar{\kappa}(n)+1\}.
\]

Then we can get

\[
\mathbb{E} \left\| \sum_{i=1}^{n} X_i \right\|_{p'}^{p'} = \mathbb{E} \left\| \sum_{j=1}^{\bar{\kappa}(n)+1} Y_{j,1} + \sum_{j=1}^{\bar{\kappa}(n)+1} Y_{j,2} \right\|_{p'}^{p'} \leq 2^{p'-1} \left( \mathbb{E} \left\| \sum_{j=1}^{\bar{\kappa}(n)+1} Y_{j,1} \right\|_{p'}^{p'} + \mathbb{E} \left\| \sum_{j=1}^{\bar{\kappa}(n)+1} Y_{j,2} \right\|_{p'}^{p'} \right) \tag{A.6}
\]

For \( I_1 \), Lemma A.1 implies

\[
I_1 \leq C_{p'} \left( \sum_{i=1}^{\bar{\kappa}(n)+1} \mathbb{E} \left\| Y_{i,1} \right\|_{p'}^{p'} \right) + (\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \mathbb{E} \left\| [Y_{i,1} | \mathcal{F}_{i-1}] \right\|_{p'}^{p'}
\]

\[
= C_{p'} \left( I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} \right),
\]

where \( \mathcal{F}_i = \sigma(Y_{j,1}, j \leq i) \). For \( I_{1,1} \), (A.4) implies

\[
I_{1,1} \leq \sum_{i=1}^{\bar{\kappa}(n)+1} C_{m} (\mathbb{E} \left\| X \right\|_{p'}^{p'}) \leq \left( C_{m}(\bar{\kappa}(n) + 1) \right) \mathbb{E} \left\| X \right\|_{p'}^{p'}
\]

For \( I_{1,2} \), since \( \mathbb{E} \left[ Y_{i,1} | \mathcal{F}_{i-1} \right] \) is \( \mathcal{F}_{i-1} \) measurable, one has

\[
I_{1,2} = (\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \mathbb{E} \left( \mathbb{E} \left[ Y_{i,1} | \mathcal{F}_{i-1} \right], \mathbb{E} \left[ Y_{i,1} | \mathcal{F}_{i-1} \right] \right) \left\| \mathbb{E} \left[ Y_{i,1} | \mathcal{F}_{i-1} \right] \right\|_{p'-2}
\]

\[
= (\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \mathbb{E} \left( Y_{i,1}, \mathbb{E} \left[ Y_{i,1} | \mathcal{F}_{i-1} \right] \right) \left\| Y_{i,1} \right\|_{p'-2}
\]

\[
= (\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \sum_{j=1+2(i-1)m}^{n/2m} \mathbb{E} \left( X_j, \mathbb{E} \left[ Y_{i,1} | \mathcal{F}_{i-1} \right] \right) \left\| Y_{i,1} \right\|_{p'-2}
\]

Since \( (X_i)_{i \geq 0} \) are zero mean, Lemma 2.1 and Young’s inequality yield

\[
I_{1,2} \leq 18(\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \sum_{j=1+2(i-1)m}^{n/2m} \int_{0}^{\alpha(m)} Q_{\|X_j\|}(u) Q_{\|Y_{i,1} | \mathcal{F}_{i-1}\|} \|Y_{i,1} \|_{p'-1}(u) du
\]

\[
\leq 18 \frac{p'}{p'} (\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \sum_{j=1+2(i-1)m}^{n/2m} \int_{0}^{\alpha(m)} Q_{\|X_j\|}(u) du
\]

\[
+ \frac{18(p' - 1)}{p'} (\bar{\kappa}(n) + 1)^{p'-1} \sum_{i=1}^{\bar{\kappa}(n)+1} \int_{0}^{\alpha(m)} Q_{\|Y_{i,1} | \mathcal{F}_{i-1}\|}^{\frac{p'}{p'-1}}(u) du,
\]
For the first term, Hölder’s inequality implies
\[
\int_0^{\alpha(m)} Q_{\|X\|}(u)du \leq \left( \int_0^{1} 1_{\{\alpha(m) > u\}} du \right)^{1 - \frac{p'}{p}} \left( \int_0^{1} Q_{\|X\|}^p(u)^{\frac{p'}{p}} du \right)^{\frac{1}{p'}} = \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\|X\|^p]^{\frac{1}{p'}}.
\]
For the second term, similar calculation implies
\[
\int_0^{\alpha(m)} Q_{\|Y_{i,1}\|,\|F_{i-1}\|}^{p' - 1}(u)du \leq \left( \int_0^{1} 1_{\{\alpha(m) > u\}} du \right)^{1 - \frac{p'}{p}} \left( \int_0^{1} Q_{\|Y_{i,1}\|,\|F_{i-1}\|}^{p' - 1}(u)du \right)^{\frac{p'}{p}} = \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\|Y_{i,1}\|^p]^{\frac{p'}{p} - 1} \mathbb{E}[\|F_{i-1}\|]^{\frac{p'}{p}} \leq \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\|Y_{i,1}\|^p]^{\frac{p'}{p}} \mathbb{E}[\|F_{i-1}\|]^{\frac{p'}{p}}.
\]
Then we have,
\[
I_{1,2} \leq \frac{18}{p'}(\kappa(n) + 1)^{p' - 1} \sum_{i=1}^{n \wedge 2(i-1)m} \sum_{j=1+2(i-1)m}^{\kappa(n)+1} \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\|X\|^p]^{\frac{p'}{p}}
+ \frac{18(p' - 1)}{p'}(\kappa(n) + 1)^{p' - 1} m \sum_{i=1}^{\kappa(n)+1} \alpha(m)^{1 - \frac{p'}{p}} m^{p'} \mathbb{E}[\|X\|^p]^{\frac{p'}{p}}
\leq \left( \frac{18}{p'}(\kappa(n) + 1)^{p' - 1} m + \frac{18(p' - 1)}{p'}(\kappa(n) + 1)^{p' - 1} m^{p'+1} \right) \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\|X\|^p]^{\frac{p'}{p}}
\leq 36(\kappa(n) + 1)^{p' + 1} (C e^{-\beta m})^{1 - \frac{p'}{p}} \mathbb{E}[\|X\|^p]^{\frac{p'}{p}}
\]
where the last line follows \((A.1)\). For \(I_{1,3}\), we denote
\[
\overline{Y}_{i,1} = \mathbb{E}[\|Y_{i,1}\|^2 | \mathcal{F}_{i-1}] - \mathbb{E}[\|Y_{i,1}\|^2],
\]
\(\mathcal{F}_{i-1}\)-measurable. Following the definition of \(Y_{i,1}\) and Davydov (1968, Corollary), one has
\[
\mathbb{E}[\overline{Y}_{i,1}]^{\frac{p'}{p}} = \mathbb{E} \left[ \left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1}) \left( \mathbb{E}[\|Y_{i,1}\|^2 | \mathcal{F}_{i-1}] - \mathbb{E}[\|Y_{i,1}\|^2] \right) \right]
= \sum_{j,l=1+2(i-1)m}^{n \wedge 2(i-1)m} \mathbb{E} \left[ \left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1}) \left( \mathbb{E}[\langle X_j, X_l \rangle | \mathcal{F}_{i-1}] - \mathbb{E}[\langle X_j, X_l \rangle] \right) \right]
= \sum_{j,l=1+2(i-1)m}^{n \wedge 2(i-1)m} \left\{ \mathbb{E} \left[ \left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1}) \langle X_j, X_l \rangle \right] - \mathbb{E} \left[ \langle X_j, X_l \rangle \right] \mathbb{E} \left[ \left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1}) \right] \right\}
\leq 12 \sum_{j,l=1+2(i-1)m}^{n \wedge 2(i-1)m} \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\langle X_j, X_l \rangle]^{\frac{p'}{p}} \mathbb{E}[\left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1})]^{\frac{p'}{p} - 1} \mathbb{E}[\langle X_j, X_l \rangle]^{\frac{p'}{p} - 1}
\leq 12 m^2 \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\langle X_j, X_l \rangle]^{\frac{p'}{p}} \mathbb{E}[\left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1})]^{\frac{p'}{p} - 1}.
\]
Thus, we can get
\[
\mathbb{E}[\overline{Y}_{i,1}]^{\frac{p'}{p}} \leq 12 m^2 \alpha(m)^{1 - \frac{p'}{p}} \mathbb{E}[\langle X_j, X_l \rangle]^{\frac{p'}{p}} \mathbb{E}[\left( \overline{Y}_{i,1} \right)^{\frac{p'}{p} - 1} sgn(\overline{Y}_{i,1})]^{\frac{p'}{p} - 1}.
\]
Then we have
\[ I_{1,3} \leq (1 + \kappa(n))^{L} 12^{\frac{L}{p'}} (m)^{L} \alpha(m)^{1-\frac{L}{p}} \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}} \]
\[ \leq 12^{\frac{L}{p'}} (m)^{L} \left( C \alpha(m)^{1-\frac{L}{p}} \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}} \right) \]
For \( I_{1,4} \), the induction hypothesis (A.5) implies
\[ I_{1,4} \leq C(\kappa(n) + 1)^{L} \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}} \]
\[ \leq Cn^{L+\frac{L}{p'}} \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}}. \]
Combining the estimate of \( I_{1,1}, I_{1,2}, I_{1,3} \) and \( I_{4} \), one has
\[ I_1 \leq C_{p'} \left( (Cm(\kappa(n) + 1))^{L} + 36(\kappa(n) + 1)^{p'} m^{p+1} (C \alpha(m))^{1-\frac{L}{p}} \right) \]
\[ + 12^{\frac{L}{p'}} (m)^{L} \left( C \alpha(m)^{1-\frac{L}{p}} \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}} \right) \]
\[ \leq C_{p'} \left( C_{1,p'} n^{L} + C_{2,p'} n^{L} e^{-\beta \sqrt{1-\frac{L}{p'}}} + C_{n}^{L+\frac{L}{p'}} \right) \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}}. \]
Similarly,
\[ I_2 \leq C_{p'} \left( C_{1,p'} n^{L} + C_{2,p'} n^{L} e^{-\beta \sqrt{1-\frac{L}{p'}}} + C_{n}^{L+\frac{L}{p'}} \right) \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}}. \]
Combining the estimate of \( I_1, I_2 \) with (A.6), we have
\[ \mathbb{E} \left\| \sum_{i=1}^{n} X_i \right\|^{p'} \leq 2^{p'} C_{p'} \left( C_{1,p'} n^{L} + C_{2,p'} n^{L} e^{-\beta \sqrt{1-\frac{L}{p'}}} + C_{n}^{L+\frac{L}{p'}} \right) \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}} \]
\[ \leq C n^{L} \left( \mathbb{E} \|X\|^p \right)^{\frac{1}{p'}}, \]
here we take \( C/3 \geq 2^{p'} C_{p'} C_{1,p'}, n \) is large enough.

\[ \square \]

**APPENDIX B. THE PROOF OF LEMMAS IN SECTION 2.2**

**Proof of Lemma 2.3.** Following Lemma 2.2, we can get the result immediately. \( \square \)

**Proof of Lemma 2.4.** For \( m \) large enough, Berbee (1987, Lemma 2.1) implies that one can construct independent random variables \( \tilde{Y}_{m,j} \) distributed as \( Y_{m,j} \) for \( j = 1, \ldots, \kappa(2^{m+1}) \) on a richer probability space and
\[ \mathbb{P}(Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } 1 \leq j \leq \kappa(2^{m+1})) \leq \kappa(2^{m+1}) \beta(m_2) \leq C \kappa(2^{m+1}) e^{-\beta m_2}. \]
The Markov inequality implies
\[ \mathbb{P} \left( \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\| \geq x \right) \]
\[ \leq \mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\|^{p'} \right] x^{-p'} \]
\[ = x^{-p'} \mathbb{E} \left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\|^{p'} \mathbb{1}_{\{Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } j = 1, \ldots, i\}} \right]. \]
Notice that
\[ 1\{Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } j=1,...,i\} \leq 1\{Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } j=1,...,r\} \quad \text{for } i \leq r. \]

Then we have
\[
\begin{aligned}
&\mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\|^{p'} \right] \\
\quad &\leq \mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\|^{p'} \right]^{1/p} \left\{ \mathbb{E}[1\{Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } j=1,...,\kappa(2^{m+1})\}] \right\}^{1-1/p}.
\end{aligned}
\]

A straightforward calculation yields
\[
\begin{aligned}
&\mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\|^{2p'} \right] \\
\quad &\leq C_{p'}\kappa(2^{m+1}) \sum_{j=1}^{\kappa(2^{m+1})} \left( \mathbb{E}\left[ Y_{m,j} \right]^{2p'} + \mathbb{E}\left[ \tilde{Y}_{m,j} \right]^{2p'} \right) \\
\quad &\leq C_{p'}\kappa(2^{m+1}) \left| \tilde{Y}_{m,j} \right|^{2p'},
\end{aligned}
\]

where the last line follows Lemma 2.3 and the fact \( Y_{m,j} \not\sim \tilde{Y}_{m,j} \). For the indicator function part, we have
\[
\mathbb{E}[1\{Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } j=1,...,\kappa(2^{m+1})\}] = \mathbb{P}(Y_{m,j} \neq \tilde{Y}_{m,j} \text{ for some } j = 1, ..., \kappa(2^{m+1})) \\
\quad \leq C\kappa(2^{m+1}) e^{-\beta m_{2}}.
\]

Thus,
\[
\begin{aligned}
&\mathbb{E}\left[ \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\|^{p'} \right] \\
\quad &\leq C_{p'}\kappa(2^{m+1}) \left| \tilde{Y}_{m,j} \right|^{p'} e^{-\beta m_{2}}.
\end{aligned}
\]

Taking \( x = 2^{((1-\alpha)/(p' + \alpha/2))m} \sqrt{\log 2^{m}} \) and \( C^* \geq 2(1 - \alpha)(p' - \frac{1}{2})/\beta \), we can get
\[
\begin{aligned}
&\sum_{m=1}^{\infty} \mathbb{P} \left( \max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\| \geq x \right) \\
\quad &\leq \sum_{m=1}^{\infty} Cx^{-p'} 2^{\frac{\alpha}{2}m} m^{2(p' + \frac{1}{2})(1-\alpha)m} \exp\left\{ -\frac{\beta}{2} m_{2} \right\} \\
\quad &\leq \sum_{m=1}^{\infty} C(\log 2^{m})^{-\frac{\alpha}{2}} 2^{(p' - \frac{1}{2})(1-\alpha)m} \exp\left\{ -\frac{\beta}{2} C^* \log 2^{m} \right\} < \infty.
\end{aligned}
\]

The Borel-Cantelli lemma yields
\[
\max_{1 \leq i \leq \kappa(2^{m+1})} \left\| \sum_{j=1}^{i} (Y_{m,j} - \tilde{Y}_{m,j}) \right\| = o\left(2^{((1-\alpha)/(p' + \alpha/2))m} \sqrt{\log 2^{m}}\right), \quad \text{a.s.}
\]

For any \( j \leq \kappa(2^{m+1}) \), (2.1) immediately implies (2.2). \( \square \)
Proof of Lemma 2.5. Similar with the proof of Lemma 2.4, we can construct independent random variables \( \tilde{Z}_{m,j} \) distributed as \( Z_{m,j} \) for \( j = 1, ..., 1 + \kappa(2^{m+1}) \) on a richer probability space and

\[
\mathbb{P}(Z_{m,j} \neq \tilde{Z}_{m,j} \text{ for some } 1 \leq j \leq \kappa(2^{m+1}) + 1 \leq C\kappa(2^{m+1})\beta(m_1).
\]

Considering the following probability,

\[
\mathbb{P}\left( \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{\ell \in J(m) \cap [2^{m+1},i]} X_\ell \right\| \geq x \right)
\]

\[
\leq \mathbb{P}\left( \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} (Z_{m,j} - \tilde{Z}_{m,j}) + \sum_{j=\kappa(i)(m_1+m_2)+m_1+1}^{\kappa(i)} X_j + \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} \tilde{Z}_{m,j} \right\| \geq x \right) \right)
\]

\[
\leq \mathbb{P}\left( \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} \tilde{Z}_{m,j} \right\| \geq x/2 \right)
\]

\[
+ C_{p'} x^{-p'} \left( \mathbb{E}\left[ \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} (Z_{m,j} - \tilde{Z}_{m,j}) \right\|^{p'} \right] + \mathbb{E}\left[ \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=\kappa(i)(m_1+m_2)+m_1+1}^{\kappa(i)} X_j \right\|^{p'} \right] \right).
\]

For the first term, since \( \tilde{Z}_{m,j} \) are centered i.i.d. random vectors, Lévy’s inequality and Lemma 2.3 imply

\[
\mathbb{P}\left( \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} \tilde{Z}_{m,j} \right\| \geq x/2 \right)
\]

\[
\leq 2^{\kappa(n)} \sum_{j=1}^{\kappa(n)} \mathbb{P}\left( \left\| \sum_{j=1}^{\kappa(i)} \tilde{Z}_{m,j} \right\| \geq x/2 \right) \leq C x^{-p'} \kappa(2^{m+1})^{\frac{p'}{2}} m_2^{\frac{p'}{2}}.
\]

Taking \( x = 2^{m(1-\alpha_1)/2} \log 2^m \), one has

(B.1)

\[
\sum_{m=1}^{\infty} \mathbb{P}\left( \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} \tilde{Z}_{m,j} \right\| \geq x/2 \right) < \infty.
\]

For the second term, similar with the estimate of Lemma 2.4, a straight calculation implies

\[
\mathbb{E}\left[ \max_{2^{m+1}+1 \leq i \leq n} \left\| \sum_{j=1}^{\kappa(i)} (Z_{m,j} - \tilde{Z}_{m,j}) \right\|^{p'} \right] = \mathbb{E}\left[ \max_{1 \leq i \leq \kappa(n)} \left\| \sum_{j=1}^{i} (Z_{m,j} - \tilde{Z}_{m,j}) \right\|^{p'} 1_{\left\{ Z_{m,j} \neq \tilde{Z}_{m,j} \text{ for some } j = 1, ..., i \right\}} \right]
\]

\[
\leq C m_2^{p'} \kappa(2^{m+1})^{p'/2} e^{-\frac{x}{2} m_1}.
\]

For the last term, following Wu (2007, Proposition 1) and Lemma 2.3, one has

\[
\left( \mathbb{E}\left[ \max_{1 \leq i \leq 2^r} \left\| \sum_{j=1}^{i} X_j \right\|^{p'} \right] \right)^{\frac{1}{p'}} \leq \sum_{i=0}^{r} 2^{(r-i)/p'} \left( \mathbb{E}\left[ \sum_{j=1}^{2^i} X_j \right]^{p'} \right)^{\frac{1}{p'}} \leq C \sum_{i=0}^{r} 2^{(r-i)/p'} (2^i)^{\frac{1}{2}} \leq C 2^\frac{r}{2},
\]
which yields
\[ E \left[ \max_{1 \leq i \leq m_2} \left\| \sum_{j=1}^{i} X_j \right\|^{p'} \right] \leq m_2^{p'/2}. \]

Then we have
\[ E \left[ \max_{2^m+1 \leq i \leq n} \left\| \sum_{j=\kappa(i)(m_1+m_2)+m_1+1} \right\|^{p'} X_j \right] \leq \kappa(2^m+1) E \left[ \max_{1 \leq i \leq m_2} \left\| \sum_{j=1}^{i} X_j \right\|^{p'} \right] \leq \kappa(2^m+1) m_2^{p'/2} \]

Thus,
\[ \sum_{m=1}^{\infty} x^{-p'} C_{\gamma}\left( E \left[ \max_{2^m+1 \leq i \leq n} \left\| \sum_{j=\kappa(i)(m_1+m_2)+m_1+1} \right\|^{p'} X_j \right] + E \left[ \max_{1 \leq i \leq m_2} \left\| \sum_{j=1}^{i} X_j \right\|^{p'} \right] \right) \leq \sum_{m=1}^{\infty} x^{-p'} C_{\gamma}\left( m_2^{p'/2} \kappa(2^m+1) p' + \frac{2^m}{p} + C_{\gamma} \left( m_2^{p'/2} + 1 \right) \right) < \infty. \]

Combining (B.1) and (B.2), the Borel-Cantelli lemma implies
\[ \max_{2^{m+1} \leq i \leq n} \left\| \sum_{\ell \in J(m) \cap [2^{m+1}, i]} X_{\ell} \right\| = o(2^{n(1-\alpha/2)} \log 2^n), \text{ a.s.} \]

Similarly estimate holds for any i.i.d. centered Gaussian random vectors \( \eta_\ell \).

\[ \square \]

APPENDIX C. PROOF OF EXAMPLES IN SECTION 4

C.1. Proof of Example 4.1.

Lemma C.1. Under Assumption (A3), \( (X_k)_{k \geq 0} \) is exponential ergodic with invariant measure \( \pi \), that is,
\[ \sup_{|f| \leq 1+V} |E[f(X_n)|X_0 = x] - \pi(f)| \leq V(x)e^{-C_\gamma n}, \]
where \( C_\gamma > 0 \) depends on \( \gamma \).

Proof. We give the proof of the ergodicity of \( (X_n)_{n \geq 0} \) following Tuominen and Tweedie (1994, Theorem 2.1). To verify condition (7) of Tuominen and Tweedie (1994), let
\[ V^n(x) = e^{C_\gamma n} V(x) \quad r(n) = C_\gamma e^{C_\gamma n}, \]
the constant \( C_\gamma > 0 \) will be chosen later. Following equation (4.1), one has
\[ PV^{n+1}(x) + r(n)V(x) \leq e^{C_\gamma (n+1)} (\gamma V(x) + K) + C_\gamma e^{C_\gamma n} V(x) \]
\[ = e^{C_\gamma n} V(x) + C_\gamma e^{C_\gamma n} \left( \frac{C_\gamma - 1}{C_\gamma} + \frac{1}{C_\gamma} \gamma e^{C_\gamma} V(x) + \frac{1}{C_\gamma} K e^{C_\gamma} \right). \]
Choosing \( C_\gamma \) small enough such that \( \gamma e^{C_\gamma} - 1 + C_\gamma < 0 \), we can get
\[ PV^{n+1}(x) + r(n)V(x) \leq V^n(x) + br(n) \mathbf{1}_{\{x \in C\}}, \]
where \( b = Ke^{C_\gamma}/C_\gamma \) and \( C = \{x : V(x) \leq \frac{Ke^{C_\gamma}}{1-C_\gamma - \gamma e^{C_\gamma}}\} \). We deduce that \( (X_n)_{n \geq 0} \) is ergodic with invariant measure \( \pi \). \( \square \)
Recall the definition of $\beta$–mixing in Davydov (1974, Proposition 1) which is equivalent with \((1.1)\).

**Definition C.2.** The $\beta$–mixing coefficients are given by:

$$
\beta(n) = \int \sup_{0 \leq f \leq 1} |P_n f(x) - \int f dQ| dQ,
$$

where $Q$ is a stationary distribution. The process $X_n$ is $\beta$–mixing if $\lim_{n \to \infty} \beta(n) = 0$; is $\beta$–mixing with exponential decay rate if $\beta(n) \leq Ce^{-cn}$ for some $C > 0$ and $c > 0$.

**Lemma C.3.** Under Assumption (A3), one has

\begin{equation}
\beta(n) \leq \pi(V)e^{-C\gamma n}.
\end{equation}

**Proof.** According to Lemma C.1 and $\pi(V) < \infty$, one has

$$
\beta(n) = \int \sup_{0 \leq f \leq 1} |\mathbb{E}[f(X_n)|X_0 = x] - \pi(f)| \pi(dx)
\leq \int \sup_{0 \leq f \leq V} |\mathbb{E}[f(X_n)|X_0 = x] - \pi(f)| \pi(dx)
\leq \int V(x)e^{-C\gamma n} \pi(dx)
= \pi(V)e^{-C\gamma n}.
$$

Thus $(X_n)_{n \geq 0}$ is $\beta$–mixing with exponential decay rate. \(\square\)

Thus condition (A1) is satisfied and $(X_n)_{n \geq 0}$ satisfies ASIP with rate greater than $1/4 + 1/(4p - 4)$.

C.2. **Proof of Example 4.2.** It is easy to calculate that \((4.3)\) has a unique stationary solution given by

\begin{equation}
X_k = \sum_{j=0}^{\infty} A^j B \varepsilon_{k-j}.
\end{equation}

For this exponential ergodic process, (A1) is easy to verified. Now we verify the condition (A2).

$$
cov(\sum_{k=0}^{n-1} X_k) = \sum_{k=0}^{n-1} cov(X_k) + \sum_{0 \leq i < j \leq n-1} (cov(X_i, X_j) + cov(X_i, X_j)^T)
= n\text{cov}(X_0) + \sum_{0 \leq i < j \leq n-1} \text{cov}(X_0, X_{j-i}) + \text{cov}(X_0, X_{j-i})^T).
$$

Following (C.2), one has

$$
cov(X_0) = cov\left(\sum_{j=0}^{\infty} A^j B \varepsilon_{-j}\right) = (I - A^2)^{-1}B^2,
$$

and

$$
cov(X_0, X_{j-i}) = cov\left(\sum_{r=0}^{\infty} A^r \varepsilon_{-r}, \sum_{r=0}^{j-i-1} A^r \varepsilon_{-j-i-r} + \sum_{r=0}^{\infty} A^{j-i+r} \varepsilon_{-r}\right)
= cov(X_0)A^{j-i}.
$$
Thus, we have as $n \to \infty$,
\[
\frac{\text{cov}\left(\sum_{k=0}^{n-1} X_k\right)}{n} = \frac{1}{n} \sum_{k=1}^{n-1} k\left(\text{cov}(X_0, X_{n-k}) + \text{cov}(X_0, X_{n-k})^T\right)
\to \text{cov}(X_0) + 2\text{cov}(X_0)(A^{-1} - I)^{-1},
\]
i.e., $\Gamma = \text{cov}(X_0) + 2\text{cov}(X_0)(A^{-1} - I)^{-1}$. It is easy to see that,
\[
\Gamma e_i(x) = \left(\frac{\lambda_i}{1 - \lambda_i^2} + \frac{2\lambda_i^2}{(1 - \lambda_i^2)(1 - \lambda_i)}\right)e_i(x).
\]
Thus (A2) is satisfied by taking $\lambda_i \asymp i^{-\delta}$ and ASIP holds.

**Acknowledgements:** L. Xu is supported in part by NSFC Grant No.12071499, Macao S.A.R. grant FDCT 0090/2019/A2 and University of Macau grant MYRG2020-00039-FST.

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