A Novel Arc-length Numerical Method for Shock Interruption Problems

Hui Xu*

School of Mathematics and Physics, Mianyang Teachers’ College, Mianyang 621900, China

*Corresponding author

Abstract. Accurately tracking the propagation process of a wave front is important for studying the propagation law of shock waves. In this study, a novel arc-length numerical algorithm to effectively calculate the interruption of shock waves is proposed, and a systematic theoretical analysis and application of this research are conducted. First, the arc-length numerical method is proposed based on the hyperbolic conservation model equation, and the basic concept of the arc-length numerical method of hyperbolic problems is provided. Then, an introduction method for arc-length in multidimensional space is described from the perspective of tensor analysis, and the mathematical model of shock wave propagation problems in arc-length space is established. The discrete solution of the spatial mathematical model of arc length is given, and control and smoothing factors are added to ensure shock wave propagation without oscillation. Finally, the numerical calculation of the Lax and double Mach reflection problems shows that the arc-length numerical algorithm can be widely used as a new calculation method for shock interruption problems.

Keywords: Arc-length; Shock interruption; Numerical method.

1. Introduction
Research on numerical shock wave algorithms began in the 1950s with the development of computational fluid mechanics. As computer hardware improved, various discrete methods including the finite difference method, the finite element method, and the finite volume method were proposed. Courant et al. [1] presented a first-order explicit windward method to study the numerical calculation of the Euler equation. Then, Lax and Wendroff [2] improved the windward method and proposed the famous Lax-Wendroff scheme. However, these numerical methods failed to solve the shock interruption problem due to the nonphysical solutions near the flow gap and dissipation. Although researchers have suggested some methods to solve these problems such as the artificial viscosity method [3], they are still unable to fundamentally solve the numerical oscillation problem. Therefore, accurately constructing a high resolution non-oscillatory numerical scheme became important to computational fluid dynamics. Van Leer [4] proposed the second order MUSCL scheme that was based on the first order Godunov method. Subsequently, Steger and Warming [5] presented the Steger-Warming scheme to deal with oscillation problems near the discontinuity surface. Numerous interface treatment methods were performed for specific problems based on the Lax scheme. The TVD (Total Variation Diminishing) method, proposed by Harten [6], is one of the typical methods. In addition, Harten and Osher presented the Essentially Non-Oscillatory (ENO) method which is essentially non-oscillatory. In recent years, Shu et al. [7] promoted the ENO difference over the Weighted ENO (WENO) for better computational results. The ENO difference scheme makes full use of discrete grid
information to make up for the deficiency of the ENO method. This method had been widely applied [8] to shock interruption problems. Now a new method, the NND scheme proposed by Zhang [9], provides a more reasonable method for calculating shock wave oscillation.

However, the solution is often discontinuous due to the time dependence of the governing equation. Particularly with commonly used governing equations such as the Euler or N-S equations, even if initial smooth conditions are given, the numerical solutions will still show strong discontinuity with increasing time, which results in singularity and rigidity problems. To address these issues, numerous mathematical theories and numerical methods such as the singular perturbation theory [10], wavelet analysis method [11], spectral method [12], and particle class method have been developed. The numerical results of these methods for some specific problems agree well with experimental data. For example, high resolution methods can better capture and track a shock wave, but there is a large amount of calculation time and memory needed to solve the problem. Modern computer hardware limits the application of these calculation methods because of the huge discrete grids that are required. Therefore, it is imperative to develop an algorithm that can be applied to shock wave interruption problems with low computation costs.

In this study, an arc-length method is proposed to solve shock wave propagation based on the arc length parameter, and the application of the numerical algorithm to shock wave front propagation is studied. The arc-length parameter is introduced to establish a spatial mathematical model, which can eliminate or weaken the singularity of the governing equation of the existing numerical methods. Finally, the numerical calculation of the Lax and double Mach reflection problems demonstrates that shock wave propagation can be simulated with high efficiency and a high resolution.

2. Arc-length Method

2.1. Mathematical Model

The one-dimensional hyperbolic conservation system is discussed in this study to provide a general mathematical model suitable for the arc-length method of hyperbolic partial differential equations. The expression can be described as:

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0
\]  

(1)

\[U(x, 0) = U_0(x)\]  

(2)

where \(U(x, t) \in \mathbb{R}^n\) is an \(n\)-dimensional vector.

In order to expand the arc-length method to high-dimensional space, a more general arc-length parameter is defined. First, a position vector \(L\) in multidimensional space is given, which is represented by the geometric space \(I = (x_1, x_2, \cdots, x_n)\) and the object understanding space \(U = (U_1, U_2, \cdots, U_n)^T\). The expression can be described as:

\[L = I + U = x_i \cdot \hat{i}_i + U_i \cdot e_i, \quad i = 1, 2, \cdots, n \quad (n \leq N)\]  

(3)

The covariant tensor of the position vector is [13]

\[G_{ij} = \frac{\partial L}{\partial s_i} \frac{\partial L}{\partial s_j} = g_{ij} + U_{ki} U_{kj}\]  

(4)

where \(g_{ij}\) is the metric tensor of the physical space. According to \(U_\xi = U_i \cdot f_\xi\), we can obtain \(G_\xi = (1 + \|U_i \|^2) g_\xi\) and \(\sqrt{G} = \sqrt{1 + (U_i)^2} I_\xi\).
The above equation can be written as an expression of arc-length on the solution curve of the one-dimensional space, and a scalar function can be defined to achieve the same result. Specifically, the solution curve has the following relationship with the change in arc-length curve:

\[(d s)^2 = (d x)^2 + (d U)^T (d U)\]  

(5)

2.2. Governing Equation of Arc-length Space.

Taking a one-dimensional space as an example, the specific form of the arc-length parameter is:

\[(d s)^2 = (d x)^2 + \sum_{i=1}^{N} \eta_i (d u_i)^2\]  

(6)

where \(\lambda_i\) is a positive number that can be adjusted depending on different problems.

The governing equations of physical space can be described by the Navier-Stokes equation, but the physical quantities such as time and velocity of the strong impact process can neglect transport effects such as viscosity, heat conduction, and diffusion in the process. Therefore, the following unsteady compressible Euler equations can be used to address these:

\[\frac{\partial}{\partial t} u + \nabla \cdot (f(u)) = S\]  

(7)

where \(u(x,t)\) is a conservation-type physical quantity \((u(x,t) \in \Re^n)\), and \(f\) and \(S\) are conservation-type fluxes abstracted as \(f = f(u,x,t)\) and \(S = S(u,x,t)\), respectively.

According to Eq. (6), the arc-length monitoring function and the control factor can be obtained by:

\[s_x = \xi(x,t) = \sqrt{1 + \sum_{i=1}^{N} \eta_i (\frac{\partial u_i}{\partial x})^2}\]  

(8)

\[\beta(s,t) = \frac{1}{\sqrt{1 - \sum_{i=1}^{N} \eta_i (\frac{\partial u_i}{\partial s})^2}}\]  

(9)

Therefore, there is a one-to-one relationship between the physical space and the arc-length space. If the physical space and the arc-length curve are equally divided, different spatial point distributions are obtained.

The mathematical model of a one-dimensional arc-length space can be obtained by combining the differential chain relation of physical quantities and the control equation of physical space:

\[u_x + s_x (A(u) - x) u_x = S\]  

(10)

where \(A(u) = \nabla_x f\) is the Jacobian matrix of the equation, and \(s_x\) can be expressed by the parameters \(s\) and \(t\). The arc-length space gradient \(u_x\) becomes a smooth continuous function, and there is no discontinuity problem. Therefore, the singularity problem can be solved by general numerical methods. In order to derive the general form of multi-dimensional space, it is assumed that the physical space and arc-length space are expressed as \(x = (x_1, x_2, \cdots, x_n)^T\) and \(s = (s_1, s_2, \cdots, s_n)^T\), respectively. The Jacobian matrix of space transformation is \(J = \frac{\partial x}{\partial s}\). In order to introduce the arc-length parameter of multi-dimensional space, a position vector \(L\) is also defined, which can be represented by geometric physical space \(x = (x_1, x_2, \cdots, x_d)^T\) and physical space \(u = (u_1, u_2, \cdots, u_n)^T\) as follows:
\[ L = x + \eta u = x_i + \eta u_i e_i, \quad i = 1, 2, \ldots, n \quad (n \leq N) \quad (11) \]

The parameters \( \eta = (\eta_1, \eta_2, \cdots, \eta_n)^T \) are also given here and can be adjusted according to different physical problems. The covariant tensor of position vector \( L \) for arc-length space is:

\[
G_{ij} = \left( \frac{\partial L_i}{\partial S_j} \right) = (x_{ij} + \eta u_{ij}) (x_{ij} + \eta u_{ij}) = g_{ij} + \delta_{im} \eta_\imath \eta_m u_{ij} u_{ij} \quad (12)
\]

where \( g_{ij} \) is the covariant tensor between physical space and arc-length space and \( \delta_{im} \) is the Kronick symbol. Since \( u_{ij} = u_s \cdot x_j \), and assuming \( \eta = \eta_\imath \eta_\imath \), Eq. (12) can become:

\[
G_{ij} = g_{ij} + \eta (u_s \cdot x_j) (u_s \cdot x_j) = Q(x) g_{ij} \quad (13)
\]

and

\[
|G_{ij}| = \left( 1 + \eta |u_s|^2 \right) |g_{ij}| = \text{det}(Q(x)) |g_{ij}| \quad (14)
\]

where tensor \( Q(x) \) is the tensor monitoring function of the multi-dimensional space. The function and the scalar monitoring function \( \xi(x, t) \) satisfy the following relationship: \( \xi(x, t)^2 = \text{det}(Q(x)) \).

It is assumed that there is a regular grid unit \( K_s \) based on the tensor monitoring function \( Q(x) \) of the arc-length space \( \Omega_s \), and there is a grid \( K \) corresponding to \( K_s \) in the physical space \( \Omega \). Then, by reverse mapping \( F_k : K_s \rightarrow K \), the equation \( K = F_k(K_s) \) can be satisfied. The mapping belongs to the map between grid units. If the cell is regarded as the entire spatial region, the map is the Jacobian matrix \( J \) of the spatial coordinate transformation. Using the principle of equipartition, the grid of multi-dimensional physical space satisfies the following relationship:

\[
\int_K \xi(x, t) \, d\mathbf{x} = \frac{\sigma_h}{M}, \quad \forall K \in \Omega \quad (15)
\]

where \( M \) is all the grid cells in the physical area \( \Omega \), and \( \sigma_h = \int_{\Omega} \xi(x, t) \, d\mathbf{x} \). The edges of the grid cell are assumed to be \( \gamma_1, \gamma_2, \cdots, \gamma_{d(d+1)/2} \) and replace the previous tensor monitoring function \( Q(x) \) with the average value of the function of the grid cell \( K \), which is expressed as:

\[
Q(x)_K = \frac{1}{|K|} \int_K Q(x) \, d\mathbf{x} \quad (16)
\]

where \(|K|\) is the volume of grid element \( K \). According to the principle of equipartition, each edge of the grid cell should be equal based on the mean of the tensor monitoring function \( M(x) \), namely:

\[
|\eta_1|_{\Omega} = |\eta_2|_{\Omega} = \cdots = |\eta_{n(n+1)/2}|_{\Omega}, \quad \forall K \in \Omega \quad (17)
\]

where \(|\eta_\imath|_{\Omega} \) is the length of the i-th edge \( \eta_i \). Assuming that any edge can be parameterized and taken as \( x = \phi_i(c), \quad c \in [c^i_0, c^i_1] \), then the length of this curve can be expressed as:
When the grid element is d-dimensional, the parameterized curve of any edge of the grid can be regarded as a straight line. For a fixed time layer, \( \frac{d\varphi}{dc} \) is a constant vector. Therefore, the length of the line segment in the above equation can be written as follows:

\[
|\eta_l|_o = \sqrt{\left(\frac{d\varphi}{dc}\right)^T Q_x \frac{d\varphi}{dc}} (c_l^i - c_o^i) = \sqrt{\eta^T Q_x \eta}
\]  

For two-dimensional space, the direct introduction method is to introduce an arc-length parameter in the opposite direction of \( x \) and \( y \) of the form \( s_1 = s_1(x, y, t) \) and \( s_2 = s_2(x, y, t) \). According to the above equation, it should satisfy the following equations:

\[
(d s_1)^2 = (d x)^2 + \sum_{i=1}^n \eta_{u_1} (d u_1(x))^2
\]

\[
(d s_2)^2 = (d y)^2 + \sum_{i=1}^n \eta_{u_2} (d u_2(y))^2
\]

where \( \eta_{u_1}, \eta_{u_2} \) is the setting parameter, which can be adjusted according to different physical problems like the one-dimensional case. \( du_1(x) \) and \( du_2(y) \) respectively represent the variation of physical quantities in all directions of two-dimensional space. After the above parameter transformation, the two-dimensional arc-length space is normalized to \( \Omega_2 = \{(s_1, s_2)|0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\} \), and there is one-to-one mapping with the physical space. Each physical space variable also becomes a function of the arc-length variables: \( x = x(s_1, s_2, t) \) and \( y = y(s_1, s_2, t) \).

### 2.3. Numerical Method

According to the characteristics and properties of the arc-length method, considering the deformation and movement of the physical space grid in the spatial transformation process and the fact that the difference method is only applicable to the regular grid, the finite volume method is used to solve the problem in this study. In the process of applying the finite volume method, the model equation of arc-length space is first converted into an integral form:

\[
\frac{d}{dt} \int_{K_s} Ju dK_s + \int_{\gamma_s} f_e n d\gamma_s = \int_{K_s} JS dK_s
\]  

where \( K_s \) represents the grid element in arc-length space, \( \gamma_s \) is the boundary of the grid element \( K_s \), and \( n \) is the outer normal direction of the boundary \( \gamma_s \). The flux of arc-length space is assumed to be:

\[
f_e = J \cdot d^i \cdot f + J \cdot u \frac{\partial s_e}{\partial t}
\]  

In order to facilitate the derivation of the later discrete method, we take two-dimensional space as an example, the entire arc-length space is normalized to \( \Omega_2 = \{(s_1, s_2)|0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\} \), and Eq. (21) is simplified into a scalar equation:

\[
\frac{d}{dt} \int_{K_s} Ju d s_1 d s_2 + \int_{\gamma_s} \nabla_s \cdot f_e d s_1 d s_2 = \int_{K_s} JS d s_1 d s_2
\]
The grid is evenly distributed in the arc-length space so the corresponding physical space grid is a quadrilateral grid. The average physical quantity of the physical space grid \((i, j)\) can be expressed as:

\[
\bar{u}_{ij} = \frac{1}{|K|} \int_{s_{ij}} u(x_1, x_2, t_n) \, dx_1 \, dx_2 = \frac{1}{|K|} \int_{s_{ij}} J u(s_1, s_2, t_n) \, ds_1 \, ds_2
\]  

(24)

The gradient of flux \(f\) in arc-length space and physical space satisfies the following relationship:

\[
\nabla \cdot f = \frac{1}{J} \frac{\partial}{\partial s_1} \left( J \begin{bmatrix} s_{1x} \\ s_{1y} \end{bmatrix} \cdot f \right) + \frac{1}{J} \frac{\partial}{\partial s_2} \left( J \begin{bmatrix} s_{2x} \\ s_{2y} \end{bmatrix} \cdot f \right)
\]  

(25)

By combining the above equations, Eq. (25) can be converted back to the physical space form, which is expressed as:

\[
\frac{d}{dt} \int_{\gamma} u \, dK + \int_{\gamma} f \cdot n \, dc = \int_{\gamma} S \, dK
\]  

(26)

where \(\gamma\) is the boundary of the physical space grid unit \(K\) and \(f = (f_1, f_2)^T\) is the physical space flux.

When combined with spatial dispersion and time dispersion, the mathematical model of arc-length space can be numerically solved.

3. Numerical Experiments

3.1. Lax Problem

The initial conditions of the Lax problem are as follows:

\[
(\rho, u, p)(x, 0) = \begin{cases} 
    (0.445, 0.698, 3.528) & 0 < x \leq 0.5 \\
    (0.500, 0.000, 0.571) & 0.5 < x < 1 
\end{cases}
\]  

(27)

The reflection boundary condition is adopted at \(x = 0\) and \(x = 1\). The calculation termination time is \(T = 0.15\) and the number of regular grids is \(N = 100\) at initial time. The format of the arc length parameter is:

\[
(ds)^2 = (dx)^2 + (d\rho)^2
\]  

(28)

In Fig. 1, the numerical results predicted by the finite volume method and the pseudo arc length method, are shown, where the solid line represents the exact solution, and “□” and “+” represent the data predicted by the finite volume method and the arc-length method, respectively. According to the grid trajectory in Fig. 2, it can be seen that the arc-length method realizes adaptive mesh movement and also improves the maximum precision while still maintaining efficiency. Compared to the finite volume method used on the regular grid, the arc-length method ensures that the numerical oscillation does not appear in the singularity and reduces dissipation by encrypting the grid. Ultimately, it can improve the capability of shock capturing and tracking, and ensure the total number of grid applications can also achieve high resolution in discontinuous problems.
3.2. Double Mach Reflection Problem

The initial boundary value conditions are as follows: the region of the calculation is \([0,4] \times [0,1]\), the bottom is a reflection surface, a strong oblique shock wave with Mach number = 10 is placed at \(x = 1/6\), \(y = 0\) and the x-axis is 60°. Standard shock wave conditions are applied to the bottom of the wall before \(x = 1/6\), and the other wall has a reflection boundary condition. Further details can be found in Ref. [14]. The initial conditions are as follows:

\[
U = \begin{cases} 
U_L, & \text{for } y \geq h(x,0) \\
U_R, & \text{otherwise}
\end{cases}
\]  

(29)

where

\[
U_L = (8.0, 57.1597, -33.0012, 563.544)^T \]

\[
U_R = (1.4, 0.0, 0.0, 2.5)^T, h(x,t) = \sqrt{3}(x - 1/6) - 20t
\]

The calculated termination time is \(T = 0.2\). The initial space grid is \([200,50]\) and the arc parameter \(\eta = 0.125\). Fig. 3 shows the numerical results calculated by arc-length method. From Fig. 4, it can be found that the pseudo arc length method can achieve propagation and reflection with a high Mach number in a two-dimensional space, and can solve for Mach stem and double Mach wave structure. We found that the numerical results calculated by pseudo arc length when the grid number is \((200, 50)\) are comparable to that calculated by a five-order WENO with a grid of \((960,240)\) [15]. The CPU time was only a third of that elapsed by the WENO method with the same grid number, which is an improvement of the arc-length method. The movement of the grid can realize high resolution shock wave interruption capture, and the overall small number of grids can reduce the calculation time. Therefore, for calculating problems of a certain scale, the arc-length method can be used effectively to solve the calculation with efficiency.
4. Summary
In this study, an arc-length numerical method is proposed for solving the shock singularity of hyperbolic partial differential equations. The governing equations of arc-length space are given, and the mathematical model of the uniform arc space shock wave propagation problem is established. Then, the discrete solution method of the arc-length space mathematical model is described. Numerical results of the Lax and double Mach reflection problems demonstrate that the arc-length numerical method can independently determine the singular point position and perform automatic mesh encryption processing. The numerical results of the arc-length numerical method agree well with that of the finite volume method and exact solution, which show that the arc-length numerical algorithm can be widely used as a new calculation method for shock interruption problems.

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