Theory of Abelian Projection

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Analytic methods for Abelian projection are developed. A number of results are obtained related to string tension measurements. It is proven that even without gauge fixing, abelian projection yields string tensions of the underlying non-Abelian theory. Strong arguments are given for similar results in the case where gauge fixing is employed. The methods used emphasize that the projected theory is derived from the underlying non-Abelian theory rather than \textit{vice versa}. In general, the choice of subgroup used for projection is not very important, and need not be Abelian. While gauge fixing is shown to be in principle unnecessary for the success of Abelian projection, it is computationally advantageous for the same reasons that improved operators, \textit{e.g.}, the use of fat links, are advantageous in Wilson loop measurements. Two other issues, Casimir scaling and the conflict between projection and critical universality, are also discussed.

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I. INTRODUCTION

In this paper we study analytically Abelian projection, attempting to resolve fundamental issues of its utility and interpretation. Abelian projection is a method for investigating the properties of gauge theories, particularly those properties associated with confinement. A gauge is chosen which reduces the gauge symmetry of a non-Abelian group $G$ to its maximal Abelian subgroup $H$. In such a gauge, it is possible to identify Abelian gauge fields and magnetic monopoles. These monopoles are presumed to play an essential role in confinement. As a lattice gauge theory technique, Abelian projection is an algorithm, described below, for extracting an ensemble of Abelian gauge field configuration from an ensemble of non-Abelian configurations.

Abelian dominance is a central concept in work on Abelian projection, positing that the essential non-perturbative physics on the non-Abelian gauge field is carried in its Abelian projection. A strong form of Abelian dominance states for every observable $O$ in the non-Abelian theory, there is a corresponding observable in the Abelian theory $O'$ such that $\langle O \rangle = \langle O' \rangle$, where the averages are taken over corresponding ensemble of non-Abelian and Abelian gauge fields, respectively. Such equalities can only be approximate at best. However, Abelian projection has had notable success in relating the string tensions and monopole densities in the projected theories to related quantities in the underlying non-Abelian theories.

There are several widely recognized issues having to do with Abelian projection. The first is the problem of Casimir scaling. The simplest illustration comes from $SU(2)$. Using Abelian dominance, the gauge field in its $3 \times 3$ adjoint representation ($j = 1$) can be decomposed into Abelian gauge fields of charge $\pm 1$, with a field identically zero representing the charge 0 sector. The corresponding adjoint representation Wilson loop receives a constant contribution from the charge 0 sector. Such behavior is in contradiction to the known area-law behavior of the adjoint Wilson loop at intermediate distances. This is based on a seductive misapprehension: that the non-Abelian gauge fields are obtained from the
Abelian fields by ”dressing” them with small non-Abelian fluctuations. In fact, it is the
Abelian fields which are derived from the non-Abelian fields in much the same way that the
renormalization group derives a coarse-grained field from a fine-grained field. The second
problem is the possible triviality of projection, that it is guaranteed to work for reasons
having nothing to do with Abelian dominance. This work provides support for that posi-
tion. A third issue is the correct, or best subgroup to use for projection. As will be shown
below, the success of Abelian projection is largely independent of the subgroup used. For
example, in the case of $SU(3)$, one could use as a subgroup for projection $U(1) \times U(1)$, $Z(3)$,
or even $SU(2) \times U(1)$. The next section briefly recapitulates the algorithm for Abelian pro-
jection. Section 3 analyzes projection without gauge fixing, and section 4 projection with
gauge fixing. Section 5 discusses Casimir scaling. Section 6 deals with a conflict between
projection and critical universality which occurs when finite temperature gauge theories are
projected to subgroups other than the center of the gauge group. A final section discusses
the implications of the results obtained, particularly in relation to these three issues.

II. ABELIAN PROJECTION IN PRACTICE

The standard approach to Abelian projection is a three step process. The gauge fields
are associated with links of the lattice, and take on values in a compact Lie group $G$.
An ensemble of lattice gauge field configurations is generated using standard Monte Carlo
methods. This ensemble of $G$-field configurations is generated by a functional integral

$$Z_g = \int [dg] e^{S_g[g]}$$

(2.1)

where $[dg]$ will consistently be used to denote the integral over all fields of a given type;
$(dg)$ will be used for integrals over individual fields and includes Haar measure. $S_g$ is a
gauge-invariant action for the gauge fields, e.g., the Wilson action for $SU(N)$ gauge fields:

$$S_g = \frac{\beta}{2N} \sum_{\text{plaq}} Tr \left( g_{\text{plaq}} + g_{\text{plaq}}^+ \right)$$

(2.2)
where $g_{\text{plaq}}$ is a plaquette variable composed from link variables, and the sum is over all plaquettes of the lattice. The expectation value of any observable $O$ is given formally by

$$\langle O \rangle = \frac{1}{Z_g} \int [dg] \, e^{S_g[g]} \, O$$

but in simulations is evaluated by an average over an ensemble of field configurations:

$$\langle O \rangle = \frac{1}{n} \sum_{i=1}^{n} O_i$$

Each field configuration in the $G$-ensemble is placed in a particular gauge. This gauge is chosen to preserve gauge invariance for some subgroup $H$ of $G$. For example, when $SU(2)$ is projected to $U(1)$, the gauge often used is defined by maximizing the quantity

$$\sum_l \text{Tr} \left[ g_l \sigma_3 g_l^+ \sigma_3 \right]$$

for each configuration over the class of all gauge transformations. The sum is over all the links of the lattice. This global maximization is often implemented as a local iterative maximization. While this subgroup has generally been chosen to be Abelian, I will show later that this is not essential. From this ensemble of field configurations, another ensemble of gauge fields is generated from the gauge-fixed ensemble, with the fields taking on values in the subgroup $H$. This is obtained by maximizing

$$\sum_l \text{Tr} \left( g_l^+ h_l + h_l^+ g_l \right)$$

where $h_l \in H$. For example, when $G = SU(2)$ and $H = U(1)$, an element $h_l$ can be represented as

$$h_l = \begin{pmatrix} e^{i \frac{1}{2} \theta_l} & 0 \\ 0 & e^{-i \frac{1}{2} \theta_l} \end{pmatrix}$$

using the natural embedding of $U(1)$ in $SU(2)$. Note that the irreducible representations of $G$ are generally reducible representations of $H$. No distinction will be made here between an element of $H$ and its representation in $G$ for notational convenience, although elsewhere it will be. The projection procedure can be carried out very efficiently for each link. It is
true, but perhaps not obvious, that the derived ensemble is, in the limit of large numbers of configurations \( n \), invariant under gauge transformations in \( H \).

For analytical purposes, it is necessary to generalize this procedure, so that a given single configuration of \( G \)-fields will be associated with an ensemble of configurations of \( H \)-fields. We will generate this ensemble using

\[
S_{\text{proj}} [g, h] = \sum_l \left[ \frac{p}{2N} Tr \left( g_i^+ h_l + h_i^+ g_l \right) \right]
\]  

(2.8)
as a weight function to select an ensemble of \( h \) fields. The normal procedure is formally regained in the limit \( p \to \infty \). Computationally, this would be implemented as a Monte Carlo simulation inside a Monte Carlo simulation. Note that the \( h \) fields should be thought of as quenched variables, since they do not effect the \( g \)-ensemble. We will treat the gauge fixing process similarly later.

### III. PROJECTION WITHOUT GAUGE FIXING

We begin with a discussion of projection without gauge fixing, as this is the simplest and most analytically tractable case. As discussed above, it is necessary for analytical purposes to generalize the projection procedure, so that a given configuration of \( G \)-fields will be associated with an ensemble of configurations of \( H \)-fields. This can be done by extending the definition of the expectation value of an operator \( O \) as

\[
\langle O \rangle = \frac{1}{Z_g} \int [dg] \frac{1}{Z_{\text{proj}}[g]} \int [dh] e^{S_{\text{proj}}[g,h]} O
\]  

(3.1)

where the presence of \( Z_{\text{proj}}[g] \), defined as

\[
Z_{\text{proj}}[g] = \int [dh] e^{S_{\text{proj}}[g,h]}
\]  

(3.2)
is crucial. If \( O \) depends only on the \( g \) fields, this definition reduces to the previous case. \( Z_{\text{proj}}[g] \) ensures that the \( h \) fields behave as quenched variables, and have no effect on the distribution of \( g \) fields. It may be helpful to compare the \( h \) fields to the role of quark fields in the quenched approximation of lattice QCD. The role of \( Z_{\text{proj}}[g] \) is analogous to the
fermion determinant. It is easy to check that if the observable $O$ is invariant under gauge transforms in $H$, so is its expectation value $\langle O \rangle$. Note that $Z_{\text{proj}}[g]$ is invariant under gauge transformations in $H$ acting on $g$.

A crucial tool in our analysis will be the character expansion. Each irreducible representation of $G$ or $H$ will have a label $\alpha$, so that $D_{ij}^\alpha(g)$ is the $i, j$ entry of a matrix representative of $g$ in the irreducible representation $\alpha$. Irreducible representations of $H$ will be denoted by $\tilde{D}_{ij}^\alpha(h)$. The fundamental orthogonality relation for matrix elements is

$$\int_G (dg) \ D_{ij}^\alpha(g) D_{kl}^\beta(g^+) = \frac{1}{d_\alpha} \delta_{\alpha\beta} \delta_{il} \delta_{jk} \quad (3.3)$$

where the integral is over Haar measure, conventionally normalized to 1, and $d_\alpha$ is the dimensionality of the irreducible representation $\alpha$. The group character $\chi^\alpha(g)$ is the trace of a group element in a particular representation: $\chi^\alpha(g) = D_{ii}^\alpha(g)$, where the summation convention is employed. It follows immediately that

$$\int_G (dg) \ \chi^\alpha(g) \chi^\beta(g^+) = \delta_{\alpha\beta} \quad (3.4)$$

The weight function for projecting each link can be expanded in the characters of the group $G$

$$\exp \left[ \frac{p}{2N} Tr \left( g^+ h + h^+ g \right) \right] = \sum_{\alpha} d_\alpha c_\alpha(p) \chi_\alpha(h^+ g) \quad (3.5)$$

in a generalization of Fourier decomposition. The coefficients of the expansion are given by

$$c_\alpha(p) = \frac{1}{d_\alpha} \int_G (dg) \chi_\alpha(g^+) \exp \left[ \frac{p}{2N} Tr \left( g + g^+ \right) \right] \quad (3.6)$$

and are known for several common groups. For example, in the case of $SU(2)$, the characters are labeled by a non-negative integer or half-integer $j$, the dimensionality $d_j = 2j + 1$, and the coefficients $c_j$ are given by $c_j(p) = 2 I_{2j+1}(p)/p$.

It is easy to see that $Z_{\text{proj}}[g]$ is given by

$$Z_{\text{proj}}[g] = \prod_l \tilde{c}_0(p, g_l) \quad (3.7)$$
where $\tilde{c}_0$ is given by

$$
\tilde{c}_0(p, g_l) = \int_H (dh_l) \exp \left[ \frac{p}{2N} Tr \left( g_l h_l + h_l^+ g_l^+ \right) \right]
$$

If $H$ is abelian, the weight function can also be expanded in characters of the subgroup $H$:

$$
\exp \left[ \frac{p}{2N} Tr \left( g^+ h + h^+ g \right) \right] = \sum_\alpha \tilde{d}_\alpha \tilde{c}_\alpha(p, g) \tilde{\chi}_\alpha(h)
$$

where the coefficients $\tilde{c}_\alpha(p, g)$ depend on $g$. For example, in the case $G = SU(2)$ and $H = Z(2)$, we have

$$
\exp \left[ \frac{p}{2} Tr \left( zg \right) \right] = \frac{1}{2} \cosh(p Tr g) + \frac{1}{2} z \sinh(p Tr g)
$$

making clear that the projection weight is not invariant under gauge transformation in $G$.

We now examine the computation of the expectation value of a Wilson loop $W$ which has no self-intersections; in particular consider

$$
\tilde{\chi}^\beta(h_1..h_n) = \tilde{D}^\beta_j^j(h_1) \tilde{D}^\beta_j^j(h_2) .. \tilde{D}^\beta_j^j(h_n)
$$

Each term in the product will be paired with terms from the character expansion of the projection weight for that link. A typical term has the form

$$
\tilde{D}^\beta_j^j_{j_{m+1}}(h) \sum_\alpha d_\alpha c_\alpha \chi_\alpha(h) g_m
$$

where $m$ is a particular link and $\beta$ is an index associated with an irreducible representation of $H$. At this point, we invoke the gauge invariance of the underlying theory and the non-intersecting character of the curve $W$. For the moment, we also set $Z_{proj}[g]$ equal to its lowest-order expression in the expansion in characters of $G$:

$$
Z_{proj}[g] = \prod_l c_0(p)
$$

Consider two adjacent links on the curve $g_m$ and $g_{m+1}$. We are free to make a change of variable on all the links associated with their common site which has the form of a gauge transformation: $g_m$ is replaced by $g_m \phi_m$ and $g_{m+1}$ by $\phi_m^+ g_{m+1}$ and so forth in such a way that the action $S_g$ is left invariant. We are free to integrate over the variable $\phi_m$, giving
\[ \int_G (d\phi_m) D_{l_m}^{\alpha_m} (g_m \phi_m) D_{l_{m+1}}^{\alpha_{m+1}} (\phi_{m+1}^* g_{m+1}) = \frac{1}{d_{\alpha_m}} \delta_{\alpha_m \alpha_{m+1}} \delta_{k_{m+1} l_{m+1}} D_{l_m k_{m+1}}^{\alpha_m} (g_m g_{m+1}) \] (3.14)

Systematic application of this result at all sites along the curve \( W \) collapses the sum into the simple result

\[ \langle \tilde{\chi}^\beta (h_1 \ldots h_n) \rangle = \sum_\alpha \left( \frac{c_\alpha(p) m}{c_0(p)} \right)^n \int_H (dh) \tilde{\chi}^\beta (h) \chi^\alpha (h^+) \langle \chi^\alpha (g_1 \ldots g_n) \rangle \] (3.15)

The single integral over \( h \) occurs because the integral over all the fields \( h_1 \ldots h_n \), which has the form

\[ \int_H (dh_1) \ldots (dh_n) \tilde{\chi}^\beta (h_1 \ldots h_n) \chi^\alpha (h_{n+1}^+ h_1^+) \] (3.16)

can be simplified by the change of variable \( h = h_1 \ldots h_n \). This integral returns a non-negative integer which is the number of times the representation \( \beta \) of \( H \) is contained in the representation \( \alpha \) of \( G \). While this result for the Wilson loop in the \( \beta \) representation of \( H \) treats the numerator exactly, it is only the lowest-order approximation to the denominator. However, it has obvious physics content: the Wilson loop as measured in the \( \beta \) representation of \( H \) is given as a sum of Wilson loops in the irreducible representations of \( G \), each weighted by the number of times \( \beta \) appears in \( \alpha \) and by a \( p \)-dependent factor which contribute to the perimeter dependence and goes to one as \( p \to \infty \). This result will be the starting point for all subsequent cases considered.

We can turn this approximate result into rigorous upper and lower bounds, by noting that \( \tilde{c}_0(p, g_m) \) is bounded:

\[ e^{p M_1} \leq \tilde{c}_0(p, g_m) \leq e^{p M_2} \] (3.17)

where \( M_1 \) and \( M_2 \) are the lower and upper bounds of the projection function. Thus we have the upper bound

\[ \langle \tilde{\chi}^\beta (h_1 \ldots h_n) \rangle \leq \sum_\alpha \left( c_\alpha(p) e^{-p M_1} \right)^n \int_H (dh) \tilde{\chi}^\beta (h) \chi^\alpha (h^+) \langle \chi^\alpha (g_1 \ldots g_n) \rangle \] (3.18)

with a corresponding lower bound when \( M_1 \) is replaced by \( M_2 \). These bounds hold for \( 0 < p < \infty \). The lower bound requires two extra assumptions. First is the Griffiths-type
inequality \( \langle \chi^\alpha(g_1..g_n) \rangle \geq 0 \), which has not been proven for non-Abelian gauge theories, but should hold on physical grounds for rectangular Wilson loops. Second is the assumption that the coefficients \( c_a(p) \) are non-negative; this holds for the projection function considered here, but might fail for others. Assuming that all the representations contributing to the sum have area law behavior, we have the result, independent of \( p \),

\[
\tilde{\sigma}_\beta = \min_\alpha \sigma_\alpha \tag{3.19}
\]

where the minimum is taken over all representations \( \alpha \) that have a non-zero contribution. This is the key result for this section.

Consider, as an example, the case of \( SU(2) \) projected to \( U(1) \). The string tension is non-zero for the half-integer representations \( j = 1/2, 3/2, .. \) due to the \( Z_2 \) center symmetry, but not for the integer representations. Because \( Z_2 \subset U(1) \), the integral

\[
\int_{U(1)} (dh) \bar{\chi}^\alpha(h) \chi^j(h^+) \tag{3.20}
\]

will vanish in many cases. Typical equalities include

\[
\tilde{\sigma}_{1/2} = \min_{j=1/2,3/2,..} \sigma_j \tag{3.21}
\]

and

\[
\tilde{\sigma}_{3/2} = \min_{j=3/2,5/2,..} \sigma_j \tag{3.22}
\]

but \( \tilde{\sigma}_1 = 0 \) because of string-breaking in the adjoint representation: \( \sigma_1 = 0 \). Note that \( U(1) \) charges have been normalized such that the highest \( U(1) \) charge associated with an \( SU(2) \) representation \( j \) has charge given by \( m = j \) so that \( U(1) \) integrations must go over \( 4\pi \).

In the case of \( SU(N) \) projected to \( Z(N) \), a very direct alternative proof has been given recently by Ambjorn and Greensite. [7] They observe that the \( Z(N) \) projection of an \( SU(N) \) matrix is a class function, and thus may be directly expanded in characters. For example, the \( Z(2) \) projection of an \( SU(2) \) element is given simply as \( sign(TrU) \), which is readily expanded in a character expansion. This yields exact relations between \( Z(N) \) \( SU(N) \) Wilson loops.

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There are obvious similarities between projection and renormalization group transformations. In particular, the projection function is also used in real space renormalization group calculations, where it is used to project blocks of spins back onto the space of site variables. For example, in the case of an $SO(N)$ spin model where the spin variable $\sigma$ is an $N$-dimensional real unit vector, the weight function for a typical real space renormalization group transformation would be

$$\exp\left[\frac{p}{N} \sigma' \cdot \left( \sum_{\text{block}} \sigma \right) \right]$$

(3.23)

where the sum of all the spins $\sigma$ in a block are projected back to a spin variable $\sigma'$. In view of this connection, it is interesting to take $H = G$ so that the subgroup is in fact the group. In that case, we have

$$\tilde{c}_0(g) = \int_G (dh) \exp\left[\frac{p}{2N} Tr \left( (g^+ h + h^+ g) \right) \right] = \int_G (dh) \exp\left[\frac{p}{2N} Tr \left( h + h^+ \right) \right] = c_0(p)$$

(3.24)

giving the exact result

$$\langle \tilde{\chi}^\alpha(h_1..h_n) \rangle = \left( \frac{c_\alpha(p)}{c_0(p)} \right)^n \langle \chi^\alpha(g_1..g_n) \rangle$$

(3.25)

showing that the only effect of projection in this case is a finite renormalization of the perimeter contribution. This corresponds to field renormalization for renormalization group transformations. Note that the perimeter renormalization is always $\leq 1$, attaining 1 in the limit $p \to \infty$. This is easily understood as a consequence of the smearing properties of the transformation.

**IV. PROJECTION WITH GAUGE FIXING**

Projection combined with gauge fixing is much more difficult to analyze. We assume we are given some gauge fixing function $S_{gf}[g]$ to maximize which, while not invariant under local gauge transformations in $G$, is invariant under local gauge transformations in $H$. In the case of $SU(2)$ projected to $U(1)$, we take
\[ S_{gf} = \lambda \sum_i Tr \left[ g_i \sigma_3 g_i^+ \sigma_3 \right] \] (4.1)

where the parameter \( \lambda \) has been introduced in the same way \( p \) was introduced for projection.

It is very convenient to introduce an auxiliary gauge-fixing field \( \phi(x) \), which takes values in \( G \). This field is applied to an unfixed field configuration as \( g_\mu(x) \rightarrow \bar{g}_\mu(x) = \phi(x)g_\mu(x)\phi^+(x + \hat{\mu}) \) so that \( \bar{g}_\mu(x) \) is used wherever the gauge-fixed field is required. The gauge fixing function depends on \( \bar{g} \), which is to say both \( g \) and \( \phi \). The expectation value of an observable \( O \) is now given by

\[ \langle O \rangle = \frac{1}{Z_g} \int [dg] e^{S_g[g]} \frac{1}{Z_{gf}[\bar{g}]} \int [d\phi] e^{S_{gf}[\bar{g}]} \frac{1}{Z_{proj}[\tilde{g},h]} \int [dh] e^{S_{proj}[\tilde{g},h]} O \] (4.2)

where \( Z_{gf}[g] \), defined as

\[ Z_{gf}[g] = \int [d\phi] e^{S_{gf}[\bar{g}]} \] (4.3)

is needed for the same reason that \( Z_{proj} \) was before. In the limit \( \lambda \rightarrow \infty \), the procedural implementation of this formula is equivalent to the commonly used algorithm for lattice gauge fixing. As before, it is easy to check that invariance under gauge transformations in \( H \) holds.

Formally, the field \( \phi \) is just a quenched, adjoint representation scalar field. It has two independent local symmetry groups: \( H_L \otimes G_R \). The generating function \( Z_{gf}[\bar{g}] \) is a lattice analog of the Fadeev-Popov determinant (actually the inverse of the determinant). However, there are some important differences. Note immediately that \( Z_{gf}[\bar{g}] \) depends on the gauge-fixing parameter \( \lambda \). More fundamentally, with the continuum Fadeev-Popov determinant, there is the vexing question of Gribov copies: what should be done about field configurations on the same gauge orbit satisfying the same gauge condition? The lattice formalism avoids this question. By construction, gauge-invariant observables are evaluated by integrating over all configurations. Gauge-variant quantities receive weighted contributions from Gribov copies. Thus the connection between lattice gauge fixing and gauge fixing in the continuum is not simple.
We have not been able to derive general results similar to those obtained without gauge fixing. However, there are strong reasons for believing that similar results hold in the gauge-fixed case as well.

First, rigorous bounds will be proven for one-dimensional gauge-fixing, where the gauge-fixing condition only depends on the links in one direction, which we take to be the timelike direction. An example of such a gauge, in an obvious notation, is

\[ S_{gf} = \lambda \sum_{\vec{x},t} \text{Re} \: \text{Tr} \left[ g_0(\vec{x},t) \right] \tag{4.4} \]

which is maximized when all the links associated with a given spatial value \( \vec{x} \) are equal. This particular gauge does not have a gauge-invariant subgroup \( H \) associated with it, but other gauges which do can also be used, for example

\[ S_{gf} = \lambda \sum_{\vec{x},t} \text{Tr} \left[ g_0(\vec{x},t) M g_0^+(\vec{x},t) M \right] \tag{4.5} \]

where \( M \) is a Hermitian matrix. The matrix \( M \) can be taken to be traceless, since the trace would only contribute a constant. The set of group elements that commute with \( M \) determines the subgroup \( H \). In the case of one-dimensional gauge fixing, it is natural to use a lattice at finite temperature, taken to be \( T \), and to use Polyakov loops instead of Wilson loops as observables. Note that \( T \) can be taken arbitrarily large, so the restriction to finite temperature is not significant. Consider a correlation function of the form

\[ \langle \bar{\chi}^\beta(P(\vec{x_1})) \rangle = \langle \bar{\chi}^\beta(h_0(\vec{x_1},1) h_0(\vec{x_1},T)) \rangle \tag{4.6} \]

which is the expectation value of a Polyakov loop. The one dimensional character of the gauge-fixing greatly simplifies the integrals over \( h \) and \( \phi \). Let \( M_3 \) and \( M_4 \) be lower and upper bounds for each link’s contribution to the gauge-fixing functional \( S_{gf} \), with \( M_1 \) and \( M_2 \) remaining as similar bounds for the projection function. Then it is easy to see that

\[ \left| \langle \bar{\chi}^\beta(P(\vec{x_1})) \rangle \right| \leq \sum_{\alpha} c_\alpha(p) e^{-T p M_1 + T \lambda (M_4 - M_3)} \int_H (dh) \bar{\chi}^\beta(h) \chi^\alpha(h^+) \cdot \langle \chi^\alpha(P(\vec{x_1})) \rangle \tag{4.7} \]

which establishes that the representation \( \beta \) of \( H \) is confined if all the representations \( \alpha \) in \( G \) which contain it are confined. If \( \beta \) transforms non-trivially under the center of \( G \), then \( \beta \)
is confined when the center symmetry of $G$ is unbroken, a result which can also be obtained on the basis of center symmetry alone. There is no lower bound in this case, because the integrand in the numerator is non-positive and difficult to approximate. For Polyakov loop two point functions, defined by

$$\langle \tilde{\chi}^\beta(P(x_1))\tilde{\chi}^\beta(P^+(x_2)) \rangle = \langle \tilde{\chi}^\beta(h_0(x_1,1)h_0(x_1,T))\tilde{\chi}^\beta(h_0(x_2,1)h_0(x_2,T)) \rangle$$

(4.8)

a similar upper bound can be obtained:

$$\left| \langle \tilde{\chi}^\beta(P(x_1))\tilde{\chi}^\beta(P^+(x_2)) \rangle \right| \leq \sum_{\alpha,\gamma} c_\alpha(p)^T c_\gamma(p)^T e^{-2T p_{M_1} + 2T \lambda(M_1 - M_3)} \int_H (dh) \tilde{\chi}^\beta(h) \chi^\alpha(h^+) \cdot \int_H (dh) \tilde{\chi}^\beta(h^+) \chi^\gamma(h) \left| \langle \chi^\alpha(P(x_1))\chi^\gamma(P^+(x_2)) \rangle \right| .$$

If the original two point function has confining behavior, \textit{i.e.,}

$$\langle \tilde{\chi}^\beta(P(x_1))\tilde{\chi}^\beta(P^+(x_2)) \rangle \sim \exp \left[ -\tilde{\sigma}_\beta T \left| x_1 - x_2 \right| \right]$$

(4.9)

then, using the same arguments as in the previous section,

$$\tilde{\sigma}_\beta \geq \min_{\alpha,\gamma} \sigma_{a\gamma}$$

(4.10)

where, as before, the minimum is taken over all representations $\alpha$ that have a non-zero contribution and $\sigma_{a\gamma}$ denotes the string tension measured by the mixed correlation function. Such bounds are useful in the case of one-dimensional gauge fixing because bounds on the gauge fixing term grow only exponentially with $T$, renormalizing perimeter terms. Unfortunately, they do not seem to extend to higher dimension.

In fact, it is very likely that the equality proved in the non-gauge-fixed case also holds for the case where gauge-fixing is used. As will now be shown, strong-coupling expansions in $\lambda$ indicate very little difference with the cases already considered. The strong-coupling expansion is convergent for sufficiently small $\lambda$. As before, we consider the computation of the expectation value of a Wilson loop $W$ which has no self-intersections:

$$\tilde{\chi}^\beta(h_1..h_n) = \tilde{D}^\beta_{j_1 j_2}(h_1)\tilde{D}^\beta_{j_2 j_3}(h_2)\..\tilde{D}^\beta_{j_n j_1}(h_n)$$

(4.11)
Each term in the product will be paired with terms from the character expansion of the projection weight for that link. A typical term has the form

\[
\tilde{D}_{j_mj_{m+1}}^{\beta}(h_m) \sum_{\alpha} d_{\alpha}c_{\alpha}\chi_{\alpha} \left( h_m^+ \phi_m g_m \phi_{m+1}^+ \right)
\]  \hspace{1cm} (4.12)

where \( m \) is a particular link and \( \beta \) is an index associated with an irreducible representation of \( H \). At order \( \lambda^0 \), the same argument given for the ungauge-fixed case applies. Integration over the \( \phi \) fields yields the same result as before:

\[
\langle \tilde{\chi}^{\beta}(h_1..h_n) \rangle = \sum_{\alpha} \left( \frac{c_{\alpha}(p)}{c_0(p)} \right)^n \int_H (dh) \tilde{\chi}^{\beta}(h) \chi^{\alpha}(h^+) \langle \chi^{\alpha}(g_1..g_n) \rangle
\]  \hspace{1cm} (4.13)

where terms of higher order in \( p \) and \( \lambda \) have been neglected.

Higher order terms in \( \lambda \) lead to corrections of this basic result, which should not alter the basic behavior. For simplicity, consider a typical contribution in the fundamental representation of \( SU(N) \) with \( N > 2 \). The latter restriction eliminates some graphs which are special to \( SU(2) \). Let the gauge fixing function have the form

\[
S_{gf} = \lambda \sum_l Tr \left[ g_l M g_l^+ M \right]
\]  \hspace{1cm} (4.14)

where, as before, \( M \) is a traceless, Hermitian matrix that commutes with every element of the subgroup \( H \) and the sum is taken over all links.

Consider a straight segment which contributes to \( \langle \chi(h_1..h_n) \rangle \):

\[
Tr h_1^+ \phi_a g_1 \phi_b^+ Tr h_2^+ \phi_b g_2 \phi_c^+ Tr h_3^+ \phi_c g_3 \phi_d^+
\]  \hspace{1cm} (4.15)

Integration over \( \phi_b \) and \( \phi_c \) gives

\[
\left( \frac{1}{N} \right)^2 Tr \phi_a g_1 g_2 g_3 \phi_d h_1^+ h_2^+ h_3^+
\]  \hspace{1cm} (4.16)

which is a piece of the \( O(\lambda^0) \) result. We now consider a \( O(\lambda^3) \) correction of the form:

\[
Tr h_1^+ \phi_a g_1 \phi_b^+ Tr h_2^+ \phi_b g_2 \phi_c^+ Tr h_3^+ \phi_c g_3 \phi_d^+ \\
Tr \phi_b g_A \phi_e^+ M \phi_e g_A \phi_b^+ M \phi_f g_B \phi_e^+ M \phi_f g_B \phi_f^+ M \\
Tr \phi_f g_C \phi_e^+ M \phi_e g_C \phi_f^+ M
\]
where integrations must now be performed over $G$ for the fields $\phi_b$, $\phi_c$, $\phi_e$, and $\phi_f$. Figure 1 shows the labeling of the sites and links. It is quite worthwhile to use graphical techniques for evaluating these integrals. Using the techniques developed by Creutz [10] and a large number of colored pens, the result is

$$\frac{\lambda^3 (Tr M^2)^2}{(N^2 - 1)^4} Tr \left[ \phi_a g_1 g_A g_B g_C g_3 \phi_d h_3^+ h_2^+ h_1^+ M^2 \right] Tr \left[ g_2 g_C^+ g_B^+ g_A^+ \right] +$$

$$- \frac{\lambda^3 (Tr M^2)^2}{N (N^2 - 1)^4} Tr \left[ \phi_a g_1 g_2 g_3 \phi_d h_3^+ h_2^+ h_1^+ M^2 \right]$$

Up to multiplicative coefficients and commutation of $M$ with $h$, this form is uniquely required by the $H_L \otimes G_R$ gauge invariance. From this, we can see that the expectation value $\langle \chi (h_1...h_n) \rangle$ will be given by

$$\langle \chi (h_1...h_n) \rangle \approx p^n K_1 (p, \lambda, M, N) \langle \chi (g_1...g_n) \rangle +$$

$$+ p^n \lambda^3 K_2 (p, \lambda, M, N) \sum_{\text{all insertions}} \langle \chi (g_1 g_A g_B g_C...g_n) \chi (g_2 g_C^+ g_B^+ g_A^+) \rangle$$

where the summation sign indicates that the same sort of insertion performed at the one link is to be repeated through the entire loop for all directions orthogonal to the loop. The constants $K_1$ and $K_2$ are power series in $p$ and $\lambda$, beginning at order 1. This formula is not exactly correct, because there are $O(\lambda^2)$ terms associated with corners. However, it does capture the lowest-order corrections to the area- and perimeter-dependence. These corrections are shown graphically in Figure 2. From this expression, we can see that the string tension should still satisfy

$$\tilde{\sigma}_\beta = \min_{\alpha} \sigma_a$$  \hspace{1cm} (4.17)$$

as in the case of no gauge fixing. The parameters $p$ and $\lambda$ control the perimeter dependence of the Wilson loop expectation value, but not the area dependence.

It is interesting to compare gauge fixing with the use of fat links, which leads to a similar result. Let the projection function be

$$S_{proj} [g, h] = \sum_i \left[ \frac{p}{2N} Tr \left( \pi_i^+ h_i + h_i^+ \pi_i \right) \right]$$  \hspace{1cm} (4.18)
where $\mathbf{g}_\mu$ is a fat link defined by

$$
\mathbf{g}_\mu(x) = g_\mu(x) + \gamma \sum_{\nu \neq \mu} g_\nu(x) g_\mu(x + \hat{\nu}) g_{-\mu}(x + \hat{\nu} + \hat{\mu})
$$

(4.19)

where $\gamma$ is an arbitrary real number, generally taken to be positive. It is now necessary to calculate perturbatively in $\gamma$ the corrections to the lowest order result, which are shown in Figure 3.

In both cases we have considered, the corrections to the lowest order result have been slightly different, but all should give the same result for the string tension. The perimeter dependence will be different for each case, and depends on the projection parameter $p$, as well as the gauge fixing parameter $\lambda$ or any other parameters involved. In practice, some schemes will be numerically advantageous. It is likely that extraction of the string tension without gauge fixing would be highly inefficient, unless fat links or some equivalent were used. Fundamentally, it appears that gauge fixing is providing the same kind of advantage that fat links do, by providing an improved operator for measuring the string tension, constructed implicitly by the gauge fixing procedure.

V. CASIMIR SCALING

The intuitive picture of Abelian dominance, if not Abelian projection, is that the dominant contribution to the partition function comes from Abelian field configurations dressed by non-Abelian fluctuations. In this kind of picture, it makes sense to conjecture that for $SU(2)$ projected to $U(1)$

$$
\langle \chi_{j=1} \rangle \sim \langle \tilde{\chi}_{m=1} \rangle + \langle \tilde{\chi}_{m=-1} \rangle + \langle \tilde{\chi}_{m=0} \rangle
$$

(5.1)

where needed multiplicative coefficients are suppressed for notational simplicity. However, as has been shown above, this is not the sort of relation which naturally emerges. The natural relations give $U(1)$ Wilson loops in terms of $SU(2)$ Wilson loops:

$$
\langle \tilde{\chi}_{m=0} \rangle = \langle \chi_{j=0} \rangle
$$
\[ \langle \tilde{\chi}_{m=1} \rangle \sim \langle \chi_{j=1} \rangle + \langle \chi_{j=2} \rangle + .. \]
\[ \langle \tilde{\chi}_{m=2} \rangle \sim \langle \chi_{j=2} \rangle + \langle \chi_{j=3} \rangle + .. \]

where coefficients have again been suppressed for notational simplicity and only the integer charges, which transform trivially under \( Z(2) \) are shown. Also suppressed are the notational complexities arising from the appearance of more complicated loops. Notice that the \( n = 0 \) relation is exact. I have assumed that there is no \( j = 1 \) contribution to \( n = 4 \), for example, but that is inessential. More importantly, there should be no \( j = 0 \) contribution for \( n > 0 \), since that would give a constant contribution, rather than the expected area or perimeter behavior. If that is the case, we can imagine inverting these relations, obtaining, for example

\[ \langle \chi_{j=1} \rangle \sim \langle \tilde{\chi}_{m=1} \rangle + \langle \tilde{\chi}_{m=2} \rangle + .. \] (5.2)

where no \( m = 0 \) contribution appears. This would resolve the issue of Casimir scaling discussed in the introduction, by establishing that there is no \( n = 0 \) contribution to \( j = 1 \).

As an example, it is possible to invert equation 3.14 yielding.

\[ \langle \chi_{j} \rangle = \frac{1}{2} \left( \frac{c_0(p)}{c_j(p)} \right)^n \left[ \langle \tilde{\chi}_{m=j} \rangle - \langle \tilde{\chi}_{m=j+1} \rangle \right] \] (5.3)
valid for \( j > 0 \). While this equation should not be taken too seriously, it does show that the problem of Casimir scaling may be resolved in a simple way.

VI. CRITICAL BEHAVIOR AND UNIVERSALITY

There is a puzzling aspect of Abelian projection associated with universality, which is not immediately apparent when considering only \( SU(N) \) gauge theories, but appears immediately when considering the analogous procedure in spin models. In a generic spin model, the spin \( \sigma \) takes on values in some space \( M \). A group \( G \) acts on \( \sigma \) in such a way that the Hamiltonian of the spin system is invariant. Imagine projecting \( \sigma \) to a new spin variable \( \mu \) which lies in a subspace \( N \) of \( M \); the new variable \( \mu \) will have a symmetry group
$H$ which is a subgroup of $G$. Obviously, any critical behavior of the original system will be reflected in the behavior of the projected variables.

For the sake of concreteness, take $\sigma \in S^{N-1}$, the unit sphere in $N$ dimensions, and the symmetry group to be $O(N)$. Take $\mu \in Z(2)$. The projection function can be taken to be

$$\sum_s p \mu_s (e \cdot \sigma)$$

where $e$ is a fixed element of $S^{N-1}$ and $p$ is an adjustable parameter as before. When considering spontaneous symmetry breaking where the direction of the field is determined by an infinitesimal symmetry breaking field or by boundary conditions, it is natural to choose $e$ in the appropriate direction. It is easy to derive relations between the correlation functions, e.g.,

$$\langle \mu_i \rangle = \langle \tanh(pe \cdot \sigma_i) \rangle$$

and

$$\langle \mu_i \mu_j \rangle = \langle \tanh(pe \cdot \sigma_i)\tanh(pe \cdot \sigma_j) \rangle$$

valid for $i \neq j$. If we examine the behavior of the projected $Z(2)$ theory in the vicinity of a second-order phase transition of the underlying $O(N)$ model, the correlation function equalities imply that the critical index $\beta$ measured from the projected $Z(2)$ variable correlation functions will be identical to the critical indices of the underlying $O(N)$ model. On the other hand, if the $O(N)$ model has a first-order transition, a jump in the order parameter $\langle \sigma \rangle$ will cause a jump in $\langle \mu \rangle$ as well. This is troubling, for the following reason. Assume for the moment that the correlation functions of $\mu$ can be derived from some short-ranged effective $Z(2)$-invariant Hamiltonian, whose parameters are smooth functions of the parameters of the underlying Hamiltonian. Standard universality arguments tell us that the critical indices should be those of a $Z(2)$ model, rather than those of an $O(N)$ model. The success of projection means a clear failure for universality. For example, $O(N)$ models are asymptotically free in two dimensions, but the Ising model is not asymptotically free, and has a second-order transition in two dimensions.
The conflict between projection and universality can be taken over from spin systems to finite temperature gauge theories, using the well-known arguments of Svetitsky and Yaffe. The basic idea is that the critical behavior of a d-dimensional gauge theory at finite temperature lies in the universality class of (d-1)-dimensional spin systems which have a global symmetry group equivalent to the center of the gauge group. For example, the four-dimensional $SU(2)$ gauge theory has a finite temperature deconfining transition which is in the universality class of the three-dimensional Ising model. Projection to $U(1)$ must lead to either a failure of universality or a failure of projection. On the other hand, there is no difficulty whatsoever with projection to $Z(2)$. In this sense, projection to the center of the gauge group is more natural than projection to other subgroups, such as the maximal Abelian subgroup.

There are three mechanisms by which this failure may be avoided. The first two have been discussed in the context of the renormalization group in the opus by van Enter, Fernandez and Sokal. First, the parameters of the effective theory may not be well-behaved functions of the underlying theory. Second, the effective Hamiltonian may not exist. Although the results of [12] are not directly applicable here, it seems unlikely, based on their work, that the projection mapping is discontinuous. It is possible that the effective Hamiltonian may not exist. For an interesting example of what happens when a $Z(3)$ invariant system, finite temperature $SU(3)$ lattice gauge theory, is reduced to a $Z(2)$ spin system, the reader should consult the recent work of Svetitsky and Weiss. A third possibility is that the effective Hamiltonian may include novel terms which remove it from its naive universality class. The work by Yee on projection from $d = 4$ $SU(2)$ lattice gauge theory to $U(1)$ is relevant here. The $d = 4$ $SU(2)$ model has no phase transition (at zero temperature), but the standard $U(1)$ model does. Using the demon method, Yee determined an approximate form for the $U(1)$ effective action, showing that the projected theory develops a magnetic monopole mass term which allows the effective theory to avoid the phase transition of the standard $U(1)$ model. Further work is needed here.
VII. CONCLUSIONS

The success of Abelian projection appears to have its origin in very general considerations. The key principle is local gauge invariance. Ultimately, it is Elitzur’s theorem [15] that ensures that observables constructed from the projected field can always be rewritten in terms of gauge-invariant observables of the underlying gauge fields. Abelian dominance is not necessary. Note that at no point in the arguments given above has space-time dimensionality been a consideration, a further indication that Abelian projection does not depend on some particular set of important field configurations.

There is one possible weak point in the gauge-fixed case: the standard gauge fixing algorithm corresponds formally to the limit $\lambda \to \infty$, but the strong-coupling expansion in $\lambda$ has a finite radius of convergence. Thus it is possible that these arguments fail for large $\lambda$. Indeed, if we interpret the gauge-fixing field $\phi$ as a quenched adjoint scalar, it seems possible, based on studies of similar models, that there is a phase transition along some critical line $\lambda_c(\beta)$, a function of the gauge coupling $\beta$. The nature of the phase transition will depend on the dimensionality of the gauge-fixing functional; note that no phase transition will exist in the $d = 1$ case. If there is a phase transition in the gauge-variant sector, it may be that the strong-coupling region and weak-coupling region are in fact connected because the critical line has an end-point. Even if that is not the case, it remains conceptually difficult to claim that confinement should be understood differently for large $\lambda$ and small $\lambda$, because by construction, the underlying ensemble of non-Abelian gauge fields does not depend on $\lambda$. There is no grounds for saying one value of $\lambda$ is more physical than another. This is similar to continuum gauge fixing. Although one may prefer Landau gauge to Feynman gauge for calculation, one cannot say that Landau gauge is correct, but Feynman gauge is wrong.

Is there value in Abelian projection? At least some of its success appears to have nothing to do with the dynamics of confinement, but rather follows from general field-theoretic principles. On the other hand, Abelian projection may capture important aspects of the QCD vacuum, although perhaps not in a unique way. There is much we need to know: 1)
Which features of projection follow from general principles? We have shown here that the string tension is one such quantity. It would be very interesting to extend the work presented here on Wilson loops to operators sensitive to monopoles and other interesting topological objects. 2) Which features depend on the subgroup used for projection? Although both center projection and projection to the maximal Abelian subgroup can be used to obtain the string tension, they differ in their approach to confinement. Each has advantages over the other. For example, center projection is consistent with universality, but projection to the maximal Abelian subgroup is more easily translated to the continuum. 3) Are there crucial tests for the various conceptual approaches to projection and confinement which can be used to falsify them? The ultimate utility of Abelian projection can only be determined in conjunction with theories which make testable quantitative predictions beyond the equality of string tensions.

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FIGURES

FIG. 1. Labelling of sites and links for integration over $\phi$ fields for $O(\lambda^3)$ correction to a Wilson loop.

Figure 1
FIG. 2. Approximate relation of $H$ Wilson loops to $G$ Wilson loops with gauge fixing. The summation is over all possible handle insertions.
FIG. 3. Approximate relation of $H$ Wilson loops to $G$ Wilson loops with fat links. The summation is over all possible staple insertions.