On Conformal Powers of the Dirac Operator on Spin Manifolds

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Abstract

The well known conformal covariance of the Dirac operator acting on spinor fields over a semi Riemannian spin manifold does not extend to powers thereof in general. For odd powers one has to add lower order curvature correction terms in order to obtain conformal covariance. We derive an algorithmic construction in terms of associated tractor bundles to compute these correction terms. Depending on the signature of the semi Riemannian manifold in question, the obtained conformal powers of the Dirac operator turn out to be formally self-adjoint with respect to the $L^2$–scalar product, or formally anti-self-adjoint, respectively. Working out this algorithm we present explicit formulas for the conformal third and fifth power of the Dirac operator.

Furthermore, we present a new family of conformally covariant differential operators acting on the spin tractor bundle which are induced by conformally covariant differential operators acting on the spinor bundle. Finally, we will give polynomial structures for the first examples of conformal powers in terms of first order differential operators acting on the spinor bundle.

1 Introduction

Considering a semi Riemannian spin manifold $(M^n, g)$ the Dirac operator is conformally covariant, see [Hit74], whereas the Laplacian has to be modified by a multiple of scalar curvature, called the Yamabe operator, in order to become conformally covariant, see [Yam60], [Ors76] and [Bra82]. Having these two examples of conformally covariant operators, Paneitz [Pan08], actually in 1983, constructed a conformal second power of the Laplacian, i.e., he presented explicit curvature correction terms for the square of the Laplacian resulting in a conformally covariant operator of fourth order acting on functions. This conformal second power is called the Paneitz operator. Almost
ten years later Graham, Jenne, Mason and Sparling [GJMS92] constructed a
series of conformally covariant differential operators $P_{2N}(g)$ acting on func-
tions with leading part an $N$–th power of the Laplacian, for $N \in \mathbb{N}$ ($n$ odd) and
$N \in \mathbb{N}$ with $N < \frac{n}{2}$ ($n$ even). The first two cases $N = 1, 2$ are covered
by the Yamabe and the Paneitz operator. Beside that construction there
were two other points of view describing these so-called GJMS operators.
One point of view was the tractor machinery used by Gover and Peterson
[GP03] and the other one was given by Graham and Zworski [GZ03] using a
spectral theoretical point of view. Again, both constructions do not produce
any conformal $N$–th power of the Laplacian when $n$ is even and $N > \frac{n}{2}$.
Although all three constructions are algorithmic explicit formulas have very
rarely been produced, due to their complexity. In case of Einstein manifolds,
Gover [Gov06] proved a product structure of shifted Laplacains of the GJMS
operators. Recent results of Juhl [Juh10, Juh13] simplified the structure by
showing that the GJMS operators can be described as polynomials in second
order differential operators.

Let us now move to the spinor case: It follows from [Slo93, Theorem
8.13] that no conformal even powers of the Dirac operator can be expected.
Holland and Sparling [HS01] proved the existence of conformal odd powers
of the Dirac operator. In the even dimensional case, their construction failed
to give conformal odd powers when the order exceed the dimension. The
first explicit formula for a conformal third power is due to Branson [Bra05],
which he derived using tractor techniques. Later on, Gillarmou, Moroianu
and Park [GMP12] gave a construction for conformal odd powers of the Dirac
operator using a spectral theoretical point of view. However, in the even di-
mensional case, this does not yield conformal powers when the order exceed
the dimension. They also gave an explicit formula for the conformal third
power of the Dirac operator, in agreement with the result of Branson. In
[EST10], Eelbode and Souček derived a product structure of shifted Dirac op-
erators for conformal powers of the Dirac operator in case of the Riemannian
sphere. But in general, due to the complexity of the underlying algorithms,
further examples were not known in the literature.

The mentioned constructions of conformal powers of the Laplacian and
the Dirac operator based on the ambient metric construction, introduced by
Fefferman and Graham [FG85, FG11]. In general, the construction of the
ambient metric is obstructed in case of even dimensional manifolds. This is
the reason that in those dimensions the conformal powers of the Laplacian
and Dirac operator only exist up to the order mentioned above.

The paper is organized as follows. We always assume that $(M, g)$ is a
semi Riemannian spin manifold.
In Section 2 we recall basic notation from semi Riemannian geometry and spin geometry. Furthermore, we recall parabolic geometries with main focus conformal geometry. That means, we will present the standard tractor bundle with is normal conformal Cartan connection. This construction goes back to Cartan [Car23] and Thomas [Tho26] and was put into a modern language by Čap and Slovák [CS09]. Dealing with conformal spin structures naturally leads to the spin tractor bundle, which is also introduced.

In Section 3 we recall the construction of so-called splitting operators, using Casimir techniques [CS07]. They will be used for the construction of a series of conformally covariant differential operators $P_{2N}^{S(M)}(g)$ acting on the spin tractor bundle by translation of the strongly invariant Yamabe operator in the sense of [ER87].

In Section 4 we use the splitting operators to construct conformal odd powers of the Dirac operator, again using the curved translation principle of Eastwood and Rice. In case of even dimensional manifolds this construction does not give any conformal odd powers when the order exceeds the dimension. Furthermore, depending on the signature of metric, we prove that the constructed operators are formally self-adjoint, or anti-self-adjoint, with respect to the $L^2$–scalar product, respectively. In the special case of Einstein manifolds, we prove that the first examples of conformal powers of the Dirac operator possesses a product structure, consisting of shifted Dirac operators. We then return to the general setting, and show that the splitting operators can be used to construct a new family of conformally covariant differential operators $L_k(g)$, for $k \in 2\mathbb{N} + 1$, acting on the spin tractor bundle. These differ slightly from the $P_{2N}^{S(M)}(g)$, however they do have the same conformal bi-degree. Finally, we give a new polynomial structure for the first examples of the conformal powers of the Dirac operator, analogous to the work of Juhl in case of the GJMS operators. For computations which are omitted and further references we refer to [Eis13].

Acknowledgements: I like to thank the BMS, SFB 647 and Eduard Čech Institute for their financial support. Furthermore, I would like to take this opportunity to express my gratitude to Helga Baum and Andreas Juhl. Helga Baum introduced me to the realm of semi Riemannian geometry, especially conformal geometry, whereas Andreas Juhl inspired me to work on the subject of conformally covariant differential operators.

2 Preliminaries

Let $(M, g)$ be a semi Riemannian spin manifold of signature $(p, q)$. We begin by fixing some curvature conventions and introduce tensors fields which will
be used throughout the paper. Next, we recall the concept of spinor bundles associated to \((M, g)\). A detailed treatment of spinor bundles and tools used within the paper can be found in [LM89, Bau81]. We then go on to recall the concept of conformally covariant differential operators in the sense of [Kos75]. Finally, we present a conformal invariant calculus in the language of parabolic geometry, see [ˇCSS97a, ˇCSS97b] and [ˇCS09], and related tractor bundles, upon which our construction of conformal powers of the Dirac operator is based.

### 2.1 Tensor conventions

Let us denote by \(\nabla^{LC} : \Gamma(TM) \to \Gamma(T^*M \otimes TM)\) the Levi-Civita connection canonically associated to \((M, g)\). The curvature tensor of the Levi-Civita connection is defined by

\[
R(X, Y)Z := \nabla^{LC}_X \nabla^{LC}_Y Z - \nabla^{LC}_Y \nabla^{LC}_X Z - \nabla^{LC}[X,Y]Z,
\]

and the Riemannian curvature tensor is defined by

\[
R(X, Y, Z, W) := g(R(X, Y)Z, W),
\]

for \(X, Y, Z, W \in \mathfrak{X}(M)\). Further tensor fields which can be built from the Riemannian curvature tensor (using covariant derivatives and contractions) are:

- \(\text{Ric}(X, Y) := \text{tr}_g (\mathcal{R}(X, \cdot, \cdot, Y))\) (Ricci tensor),
- \(\tau := \text{tr}_g (\text{Ric} (\cdot, \cdot))\) (scalar curvature),
- \(J := \frac{1}{2(n-1)} \tau\) (normalized scalar curvature),
- \(P(X, Y) := \frac{1}{n-2} (\text{Ric}(X, Y) - Jg(X, Y))\) (Schouten tensor),
- \(W(X, Y, Z, W) := \mathcal{R}(X, Y, Z, W) + P \otimes g(X, Y, Z, W)\) (Weyl tensor),
- \(C(X, Y, Z) := \nabla^{LC}_X P(Y, Z) - \nabla^{LC}_Y P(X, Z)\) (Cotton tensor),
- \(B(X, Y) := \text{tr}_g (\nabla^{LC} C (\cdot, X, Y)) + g(P(\cdot, \cdot), W(\cdot, X, Y, \cdot))\) (Bach tensor),

where the Kulkarni-Nomizu product \(\otimes\) is defined by

\[
P \otimes g(X, Y, Z, W) := P(X, Z)g(Y, W) + P(Y, W)g(X, Z) - P(X, W)g(Y, Z) - P(Y, Z)g(X, W),
\]

for \(X, Y, Z, W \in \mathfrak{X}(M)\). Finally, the semi Riemannian metric yields the usual isomorphisms \(\flat : T^*M \to TM\) and \(\sharp : TM \to T^*M\).
2.2 Clifford algebras, spin groups and their representations

Consider the vector space $\mathbb{R}^n$ ($n = p + q$) together with the scalar product $\langle \cdot, \cdot \rangle_{p,q}$ of index $p$, i.e., $\langle e_i, e_j \rangle_{p,q} = \varepsilon_i \delta_{ij}$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$, $\varepsilon_i = -1$ for $1 \leq i \leq p$; $\varepsilon_i = 1$, for $p + 1 \leq i \leq n$, and $\delta_{ij}$ denotes the Kronecker delta. Consider the Clifford algebra of $\mathbb{R}^{p,q} := (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{p,q})$ realized by $C_{p,q} := \mathcal{T}(\mathbb{R}^n)/J$, where $\mathcal{T}(\mathbb{R}^n)$ denotes the tensor algebra of $\mathbb{R}^n$, and $J$ is the two-sided ideal in $\mathcal{T}(\mathbb{R}^n)$ generated by the relations $x \otimes x = -\langle x, x \rangle_{p,q}$, for $x \in \mathbb{R}^n$. The Clifford algebra carries a $\mathbb{Z}_2$-grading, given by even and odd elements, i.e., $C_{p,q} = C^0_{p,q} \oplus C^1_{p,q}$. We denote the group of units of $C_{p,q}$ by $C^*_{p,q}$ and call it the Clifford group. This leads to two important subgroups, the pin group $Pin(p,q)$, given by products of elements $x \in \mathbb{R}^n$ of length $\pm 1$, and the spin group $Spin(p,q) := Pin(p,q) \cap C^0_{p,q}$. There is an algebra isomorphism of the complexified Clifford algebra

$$\Phi_{p,q}^{even/odd} : C^\mathbb{C}_{p,q} \to \begin{cases} \text{Mat}(2^m, \mathbb{C}), & n = 2m \\ \text{Mat}(2^m, \mathbb{C}) \oplus \text{Mat}(2^m, \mathbb{C}), & n = 2m + 1 \end{cases}$$

It is defined as follows: Set

$$g_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\alpha(j) := \begin{cases} 1, & j \in 2\mathbb{N} - 1 \\ 2, & j \in 2\mathbb{N} \end{cases}, \quad \tau(j) := \begin{cases} i, & j \leq p \\ 1, & j > p \end{cases}.$$

In the case of $n = 2m$, we use an orthonormal basis $\{e_i\}$ of $\mathbb{R}^{p,q}$ to define the isomorphism

$$\Phi_{p,q}^{even}(e_j) := \tau(j)E \otimes \ldots \otimes E \otimes g_{\alpha(j)} \otimes T \ldots \otimes T.$$

Here, the right hand side is a product of $m$ matrices, $\left[\frac{m-1}{2}\right]$ of them are copies of $T$, and the tensor product used is the Kronecker tensor product for matrices. In the case of $n = 2m + 1$, we set

$$\Phi_{p,q}^{odd}(e_j) := \left(\Phi_{p,q-1}^{even}(e_j), \Phi_{p,q-1}^{even}(e_j) \right), \quad j = 1, \ldots, 2m,$$

$$\Phi_{p,q}(e_{2m+1}) := (iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T),$$

where $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^{p,q}$. Hence, in the case of $n = 2m$, the Clifford algebra $C^\mathbb{C}_{p,q}$ has (up to equivalence) an unique irreducible representation $\Phi_{p,q} := \Phi_{p,q}^{even}$, whereas, in the case of $n = 2m + 1$ it has (up to
equivalence) two unique irreducible representations denoted by $\Phi^0_{p,q}$ and $\Phi^1_{p,q}$. In all cases the representation space is $\Delta_{p,q} := \mathbb{C}^{2^m}$. Note that in the case of $n = 2m + 1$, both irreducible representations $\Phi^0_{p,q}$ and $\Phi^1_{p,q}$ become equivalent when they are restricted to the even part $C^0_{p,q}$. Restricting $\Phi_{p,q}$, in the even case, or $\Phi^0_{p,q}$, in the odd case, to the spin group yields a representation of the spin group, which will be denoted by $\kappa_{p,q}$. This is the spinor representation we will work with. Again, in the case of $n = 2m$ we have that $\kappa_{p,q}$ decomposes into two non-equivalent irreducible representations, whereas in the case of $n = 2m + 1$ the representation $\kappa_{p,q}$ is irreducible.

On the representation space $\Delta_{p,q}$ there exists a $Spin_0(p,q)$–invariant hermitian scalar product $(v, w)_\Delta := (b \cdot v, w)_{\mathbb{C}^{2^m}}$, where $Spin_0(p,q)$ denotes the connected component containing the identity; $(\cdot, \cdot)_{\mathbb{C}^{2^m}}$ is the standard hermitian scalar product on $\mathbb{C}^{2^m}$, and $b := i^{p(q-1)} e_1 \cdots e_p$. For Riemannian signature (that is $p = 0$) it reduces to the standard hermitian scalar product, which is $Spin_0(0,n) = Spin(0,n)$–invariant.

### 2.3 Spin structures and spinor bundles

Let $(Q^g, f^g)$ be a spin structure for $(M,g)$, i.e., a $\lambda$–reduction of the orthonormal frame bundle $(P^g, \pi, M, SO(p,q))$, where $\lambda : Spin(p,q) \to SO(p,q)$ denotes the usual twofold covering of $SO(p,q)$. The associated vector bundle $S(M,g) := Q^g \times_{(Spin_0(p,q),\kappa_{p,q})} \Delta_{p,q}$ over $M$ is called the spinor bundle of $(M,g)$. The hermitian scalar product $(\cdot, \cdot)_\Delta$ induces a scalar product on the spinor bundle by $<\psi, \phi> := (v, u)_\Delta$, for $\psi = [q, v], \phi = [q, u] \in S(M,g)$. Due to the reduction property of $(Q^g, f^g)$ we obtain an isomorphism $TM \simeq Q^g \times_{(Spin_0(p,q),\rho,\lambda)} \mathbb{R}^n$, where $\rho$ denotes the standard representation of $SO(p,q)$ on $\mathbb{R}^{p,q}$, and thus we may define the Clifford multiplication $\mu : TM \otimes S(M,g) \to S(M,g)$ by

$$\mu(X \otimes \psi) := \begin{cases} [q, \Phi_{p,q}(x)v], & n = 2m \\ [q, \Phi^0_{p,q}(x)v], & n = 2m + 1, \end{cases}$$

for $X = [q, x] \in TM$, and $\psi = [q, v] \in S(M,g)$. If there is no confusion we will use $X \cdot \psi$ instead of $\mu(X \otimes \psi)$. Clifford multiplication extends to the exterior algebra of $T^*M$ by

$$w \cdot \psi := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varepsilon_{i_1} \cdots \varepsilon_{i_k} w(s_{i_1}, \ldots, s_{i_k}) s_{i_1} \cdots s_{i_k} \cdot \psi,$$

where $w \in (\Lambda^k M)_x$, $\psi \in S(M,g)_x$ and $\{s_i\}$ is an orthonormal basis in $T_x M$, for $x$ the base point. Note that Clifford multiplication varies smoothly on
M, thus it descends to sections of corresponding vector bundles. In order to define a covariant derivative on the spinor bundle in a canonical way we choose the Levi-Civita connection form \( A^g \in \Omega^1 (\mathcal{P}^g, \mathfrak{so}(p, q)) \), induced by \( \nabla^{LC} \), and define, using the isomorphism \( \lambda_* : \text{spin}(p, q) \rightarrow \mathfrak{so}(p, q) \) (the differential of the covering map at the identity), a connection form \( \bar{A}^g := \lambda_*^{-1} \circ A^g \circ \lambda_* \in \Omega^1 (\mathcal{Q}^g, \text{spin}(p, q)) \) on \( \mathcal{Q}^g \). This induces a covariant derivative on the associated vector bundle \( S(M, g) \) in the usual way, i.e., locally we have

\[
\nabla^S_{X(M,g)} \psi \big|_U = \left[ q, dv_q(X) + (\kappa_{p,q})_* \left( (\bar{A}^g)_q(X) \right) v \right]
\]

\[
\sim = X(\psi) + \frac{1}{2} \sum_{i<j} \varepsilon_i \varepsilon_j g(\nabla^{LC}_X s_i, s_j) s_i \cdot s_j \cdot \psi,
\]

for local sections \( \psi = [q, v] : U \rightarrow S(M, g) \) and \( s = \{s_i\} : U \rightarrow \mathcal{P}^g \), and for \( X(\psi) := [q, dv_q(X)] \). Here, \( (\bar{A}^g)_q(X) := A^g_q(dq(X)) \) is the connection 1–form induced by the local section \( q : U \rightarrow \mathcal{Q}^g \). The covariant derivative \( \nabla^S_{X(M,g)} \) leads to the definition of the Dirac operator by

\[\mathcal{D} : \Gamma (S(M, g)) \rightarrow \Gamma (S(M, g))\]

\[\psi \mapsto \mathcal{D} \psi := \mu \left( (\nabla^S_{X(M,g)} \psi)^2 \right),\]

where \( \cdot \) indicates the identification \( T^* M \simeq TM \) induced by \( g \). Locally the Dirac operator reads \( \mathcal{D} \psi \overset{\text{loc.}}{=} = \sum \varepsilon_i s_i \cdot \nabla^S_{X(M,g)} \psi \). The following list collects useful formulas, some are well known, see \[\text{Bau81, LM89,}\] and while the remainder are straightforward to derive: For \( \psi, \phi \in \Gamma (S(M, g)) \) and \( X, Y \in \mathfrak{X}(M) \), one has

1. \( \nabla^S_{X(M,g)} (Y \cdot \psi) = \nabla^{LC}_X Y \cdot \psi + Y \cdot \nabla^S_{X(M,g)} \psi, \)
2. \( \nabla^S_{X(M,g)} \) is metric with respect to \( \cdot, \cdot \),
3. \( <X \cdot \psi, \phi> + (-1)^p <\psi, X \cdot \phi> = 0, \)
4. \( \mathcal{R}^S_{X(M,g)}(X, Y) \psi = \frac{i}{2} \mathcal{R}(X, Y) \cdot \psi, \) where the Riemannian curvature tensor is considered as endomorphism of 2–forms,
5. \( [\mathcal{D}, f] \psi = \mathcal{D} (f \psi) - f \mathcal{D} \psi = \text{grad}^g f \cdot \psi, \) for any \( f \in C^\infty (M), \)
6. \( [\mathcal{D}, \nabla^S_{X(M,g)}] \psi = \frac{i}{2} \text{Ric}(X)^g \cdot \psi \) and
7. \( \mathcal{D}^2 \psi = -\Delta^S_{X(M,g)} \psi + \frac{i}{2} \psi \) is the Bochner formula, where \( \Delta^S_{X(M,g)} := tr_g(\nabla^{T^*M \otimes S(M, g)} \circ \nabla^S_{X(M,g)}) \) is the Bochner Laplacian on spinor fields.
Concerning questions of self-adjointness of certain operators on spinor fields we introduce a bracket notation. Let $T$ be a symmetric $(0,2)$--tensor and $\psi$ a spinor field. We define first a $1$--form $T \cdot \psi$ with values in the spinor bundle by $T \cdot \psi(X) := T(X)^h \cdot \psi$. Then the following brackets are defined:

$$ (T, \nabla \psi) := \mu \left( \text{tr}_g(T(\cdot)^h \otimes \nabla \psi) \right) \bigg|_{\text{loc.}} = \sum_i \varepsilon_i T(s_i)^h \cdot \nabla s_i \psi, \quad (1) $$

$$ (\nabla, T \cdot \psi) := -\delta^{S(M,g)}(T \cdot \psi), \quad (2) $$

where, for $\eta \in \Omega^1(M,S(M,g))$, $\delta^{S(M,g)} \eta \big|_{\text{loc.}} = -\sum_i \varepsilon_i (\nabla s_i \eta)(s_i)$ is the co-differential of $d^{S(M,g)}$. Note that the last bracket can be rewritten as

$$ (\nabla^{S(M,g)}, T \cdot \psi) = (T, \nabla^{S(M,g)} \psi) - (\delta^{\nabla^{LC}} T^2) \cdot \psi, $$

where $\delta^{\nabla^{LC}}$ denotes the co-differential of $d^{\nabla^{LC}}$. Next, we define a $(0,2)$--tensor $T^2$ by $T^2(X,Y) := T(T(X)^h,Y)$, and a further bracket by

$$ (C, P \cdot \psi) := \sum_i \varepsilon_i C(s_i) \cdot P(s_i) \cdot \psi, \quad (3) $$

where the Cotton tensor is considered as $C(X) := C(\cdot, \cdot, X) \in \Omega^2(M)$. Analogously one defines $(P, C \cdot \psi)$. Using the same notation for those brackets will not lead to any confusion. Two more product types, needed later on, are

$$ W \cdot W \cdot \psi := \sum_{i,j} \varepsilon_i \varepsilon_j W(s_i, s_j) \cdot W(s_i, s_j) \cdot \psi, \quad (4) $$

$$ C \cdot W \cdot \psi := \sum_{i,j} \varepsilon_i \varepsilon_j C(s_i, s_j, \cdot)^h \cdot W(s_i, s_j) \cdot \psi, \quad (5) $$

where Clifford multiplication of $2$--forms $W(X,Y) := W(X,Y,\cdot,\cdot) \in \Omega^2(M)$ appears. Similarly we define $W \cdot C \cdot \psi$.

### 2.4 Conformal structures and conformally covariant differential operators

We say that another metric $\hat{g}$ on $M$ is conformally related to $g$ if there is a smooth function $\sigma \in \mathcal{C}^\infty(M)$ such that $\hat{g} = e^{2\sigma} g$. This clearly defines an equivalence relation among metrics on $M$. We call $(M, c := [g])$ a conformal semi Riemannian manifold. Note that signature and orientation are invariant under a conformal change of a metric. A conformal structure $[g]$ on $M$ induces a $CO(p,q) \simeq \mathbb{R}^+ \times SO(p,q)$--reduction $(\mathcal{P}^0, \pi, M, CO(p,q))$ of the frame
bundle \((GL(M), \pi, M, GL(n, \mathbb{R}))\), in analogy to semi Riemannian structures \(g\) on \(M\) where \(GL(M)\) reduces to the orthonormal frame bundle \(\mathcal{P}^g\). We should point out, that in contrast to the semi Riemannian case there is no distinguished connection form on the conformal frame bundle, but there is one on its first prolongation which will be discussed in the next subsection.

We will now define a conformal spin structure on a conformal manifold \((M, c)\). Consider the conformal spin group

\[ CSpin(p, q) := \mathbb{R}^+ \times Spin(p, q) \]

and the map \(\lambda^c : CSpin(p, q) \to CO(p, q)\), defined by \(\lambda^c(a, g) := a\lambda(g)\). A conformal spin structure \((Q^0, f^0)\) on \((M, c)\) is defined to be a \(\lambda^c\)-reduction of the conformal frame bundle. Conformal spin structures on \((M, c)\) are equivalent to spin structures on \((M, g)\) in the following way: Given a spin structure \((Q^0, f^0)\) on \((M, g)\), we define a conformal spin structure \((Q^0, f^0)\) on \((M, c)\) by taking the extension \(Q^0 := Q^g \times_{Spin(p, q)} CSpin(p, q)\), and setting \(f^0 := f^g \times \lambda^c\). Conversely, given a conformal spin structure \((Q^0, f^0)\) on \((M, c)\), choosing \(g \in c\), we define, using the obvious reduction map \(\iota : \mathcal{P}^g \to \mathcal{P}^0\), a spin structure \((Q^g, f^g)\) on \((M, g)\) by \(Q^g := \{q \in Q^0 \mid f^0(q) \in \iota(\mathcal{P}^g)\}\) and \(f^g := f^0|_{Q^g}\).

**Remark 2.1** Since we have no distinguished connection form on the conformal frame bundle we cannot build up a conformally invariant differential calculus on the tangent bundle. However, as we will see in the next two subsections, there is a first prolongation of the conformal frame bundles which possess a distinguished Cartan connection. This Cartan connection induces a covariant derivative on the so-called tractor bundles. Then, by fixing a representative \(g \in c\), it is possible to identify within that covariant derivative, its curvature, or in the divergence of its curvature tensors like Schouten, Weyl, Cotton and Bach associated to \(g\).

Let us finish this subsection with the notion of conformally covariant differential operators acting between sections of two vector bundles \(E \to M\) and \(F \to M\) over \((M, g)\). We say that a linear differential operator \(D(g) : \Gamma(E) \to \Gamma(F)\) is \(g\)-geometrical if it is a polynomial in \(g, g^{-1}, \nabla^{LC}\) and \(\mathcal{R}\). A \(g\)-geometrical differential operator \(D(g)\) is said to be conformally covariant of bi-degree \((a, b)\) if there exists \(a, b \in \mathbb{R}\) such that

\[ D(e^{2\sigma}g)(e^{a\sigma}\psi) = e^{b\sigma}D(g)\psi, \]

for any metric \(e^{2\sigma}g\), and \(\psi \in \Gamma(E)\). If the bundles \(E\) and \(F\) depend on the chosen metric, but can be related by a bundle map for conformally related
metrics, then this map can be used to define conformally covariant operators between \( E \) and \( F \). An example is given by the spinor bundle \( S(M, g) \); here, there exists a bundle isomorphism \( F_\sigma : S(M, g) \to S(M, e^{2\sigma}g) \) induced from the map \( \Lambda_\sigma : \mathcal{P}^g \to \mathcal{P}^{e^{2\sigma}g} \) (which is given by \( \Lambda_\sigma(s_1, \ldots, s_n) := (e^{-\sigma}s_1, \ldots, e^{-\sigma}s_n) \)), and the covering property of spin structures, see [Bau81].

Another example is given by the maps \( T \) and \( T^{S(h)}(g, \sigma) \), see Subsection 2.3. These identify the metric decomposition of certain tractor bundles with respect to two representatives from the conformal class. Examples of conformally covariant operators are the Yamabe operator acting on functions, the Dirac operator and the twistor operator acting on spinor fields. In Section 3 and 4 we will deal with more conformally covariant differential operators.

### 2.5 Parabolic geometries for conformal spin structures

Parabolic geometries are special classes of Cartan geometries, which themselves are curved versions of Klein geometries \((G, \pi, G/H, H; w_G)\), where \( G \) is a Lie group, \( H \subset G \) is a closed subgroup, and \( w_G \) is the Maurer-Cartan form.

For \( H \subset G \) as above and \( M \) a smooth manifold, a Cartan geometry \((G, \pi, M, H; w)\) of type \((G, H)\), consists of an \( H \)-principal bundle \( G \) over \( M \) with a Cartan connection \( w \in \Omega^1(G, \mathfrak{g}) \), such that (1) \( w(X) = X \) for every \( X \in \mathfrak{h} \) (where \( X \) denotes the fundamental vector field of \( X \)), (2) \( w : T_uG \to \mathfrak{g} \) is an isomorphism, for every \( u \in G \), and (3) \( (R_h)^*w = Ad(h^{-1}) \circ w \), for every \( h \in H \).

A Cartan geometry \((G, w)\) of type \((G, H)\), for which \( H \) is a parabolic subgroup inside a semisimple Lie group \( G \), is referred to as a parabolic geometry. For more details see [Sha97] and [CS09].

A conformal manifold \((M, c)\) of signature \((p, q)\) can be described as a parabolic geometry as follows: Let us denote \( G := O(p+1, q+1) / \{ \pm I_d \} \) the projective orthonormal group. In terms of the standard orthonormal basis \( \{ e_\alpha \}_{\alpha=0}^{n+1} \) with respect to the standard semi Riemannian metric \( \langle \cdot, \cdot \rangle_{p+1, q+1} \) on \( \mathbb{R}^{n+2} \), we define the following basis

\[
 f_0 := \frac{1}{\sqrt{2}}(e_{n+1} - e_0), \quad f_i := e_i, \quad f_{n+1} := \frac{1}{\sqrt{2}}(e_{n+1} + e_0)
\]

on \( \mathbb{R}^{n+2} \). The stabilizer \( B := \text{stab}_{\mathbb{R} f_0}(G) \) of the isotropic line \( \mathbb{R} f_0 \) defines a parabolic subgroup of \( G \), and it is isomorphic, under the projection \( O(p+1, q+1) \to G \), to the following subgroup of \( O(p+1, q+1) \):

\[
 B \simeq \left\{ Z(a, A, v) := \begin{pmatrix} a^{-1} & v^t & b \\ 0 & A & x \\ 0 & 0 & a \end{pmatrix} \mid a \in \mathbb{R}^+, v \in \mathbb{R}^{p,q}, A \in O(p, q), \right. \\
\left. x := -aA J_{p,q} v, \quad b := -\frac{1}{2} a \langle v, v \rangle_{p,q} \right\},
\]

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where $J^{p,q} := \text{diag}(-I_p, I_q)$ and $I_r$ denotes the identity matrix of size $r$. This group carries a semi direct product structure: $B \simeq B_0 \ltimes_\rho B_1$ for

$$B_0 := \{X(a, A) := Z(a, A, 0) \in B\} \simeq CO(p, q),$$
$$B_1 := \{Y(v) := Z(1, I_n, v) \in B\} \simeq \mathbb{R}^n,$$

where $\rho : B_0 \times B_1 \to B_1$ is the conjugation map $\rho(b_0)b_1 := b_0b_1b_0^{-1}$. Finally, let us denote $B_{-1} := \{Y(v)^t \mid v \in \mathbb{R}^n\}$. This will be needed for the grading of the Lie algebra of $G$, i.e., $\mathfrak{g} := LA(G) = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is a $|1|$-graded Lie algebra. In terms of matrices one has

$$\mathfrak{g} = \left\{ M(x, (A, a), z) := \begin{pmatrix} -a & z & 0 \\ x & A & -J^{p,q}x^t \\ 0 & -J^{p,q}x^t & a \end{pmatrix} \mid x \in \mathbb{R}^n, z \in (\mathbb{R}^n)^*, \begin{pmatrix} a \in \mathbb{R}, A \in \mathfrak{o}(p, q) \end{pmatrix} \right\}$$

and

$$\mathfrak{b}_{-1} = \left\{ M(x, (0, 0), 0) \in \mathfrak{g} \right\} \simeq \mathbb{R}^n,$$
$$\mathfrak{b}_0 = \left\{ M(0, (a, A), 0) \in \mathfrak{g} \right\} \simeq \mathfrak{co}(p, q),$$
$$\mathfrak{b}_1 = \left\{ M(0, (0, 0), z) \in \mathfrak{g} \right\} \simeq (\mathbb{R}^n)^*.$$

In this setting it is shown in [CS09, Section 1.6] that there exists a parabolic geometry $(\mathcal{P}^1, w^{nc})$ of type $(G, B)$ uniquely associated to the conformal structure. Roughly speaking, the $B$–principal bundle $\mathcal{P}^1$, called the first prolongation of the conformal frame bundle, is the collection of horizontal and torsion free subspaces in $T\mathcal{P}^0$, and the normal conformal Cartan connection $w^{nc}$ is an extension of the soldering form of $\mathcal{P}^1$. Additionally, one has that $(\mathcal{P}^1, \pi^1, \mathcal{P}^0, B_1)$ is a $B_1$–principal bundle over $\mathcal{P}^0$, whereas $(\mathcal{P}^1, \pi^0, M, B)$ is a $B$–principal bundle over $M$, with the obvious projection maps.

As we promised earlier, choosing a metric $g$ from the conformal class, we can pull back the normal conformal Cartan connection to the orthonormal frame bundle which will yield a formula in terms of the metric $g$, i.e., in terms of the Levi-Civita connection and Schouten tensor. More precisely, the metric $g$ induces a reduction $\iota : \mathcal{P}^g \to \mathcal{P}^0$, and the Levi-Civita connection form $A^g \in \Omega^1(\mathcal{P}^g, \mathfrak{so}(p, q))$ determines a $B_0$–equivariant section $\sigma^g : \mathcal{P}^0 \to \mathcal{P}^1$ by $\sigma^g(u) := \ker(\gamma^g)$, where $\gamma^g$ is the extension of $A^g$ to the conformal frame bundle. Then we have

$$(\sigma^g \circ \iota)^* w^{nc}_s(Y) = [s]^{-1}(d\pi^g_s(Y)) + A^g_s(Y) - \sum_{i=1}^n P^g_{\pi^g_s}(d\pi^g_s(Y), s_i) \cdot e^*_i,$$

where $\pi^g : \mathcal{P}^g \to M$ is the projection map, $s \in \mathcal{P}^g$, $[s] : \mathfrak{b}_{-1} \to T_{\pi^g_s}M$ the induced isomorphism from $TM \simeq \mathcal{P}^g \times_{(O(p,q),Ad)} \mathfrak{b}_{-1}$, $Y \in T_s\mathcal{P}^g$, $P^g$
denotes the Schouten tensor with respect to \( g \), and \( \{ e_i \} \) is an orthonormal basis in \( b_{-1} \cong \mathbb{R}^{p,q} \) with dual basis \( \{ e^*_i \} \) in \( b_1 \) such that \( \{ s_i := [s, e_i] \} \) is an orthonormal basis for \( T_{\pi \circ f^g} M \) with respect to \( g \).

Now we will define the first prolongation of a conformal spin structure. This requires the pull back, denoted by \( \tilde{\gamma} \), of the groups \( G, B, B_0 \) and \( B_1 \) by the covering map \( \lambda : Spin(p+1,q+1) \to SO(p+1,q+1) \). Consider a conformal spin structure \( (Q^0, f^0) \) on \( (M, c) \) and define the set

\[
Q^1 := \{ \tilde{H}_q \subset T_q Q^0 \mid q \in Q^0, df^0_q(\tilde{H}_q) \in \mathcal{P}^1 \},
\]

and a \( \tilde{B} \)-action on it by

\[
\tilde{H}_q \cdot \tilde{b} := (df^0_{q,0})^{-1} (df^0_q(\tilde{H}_q) \cdot \lambda(\tilde{b})),
\]

for \( \tilde{H}_q \in Q^1 \), and \( \tilde{b} = \tilde{b}_0 \cdot \tilde{b}_1 \in \tilde{B} \) (\( \tilde{B} \) inherits the semi direct product structure from \( B \)). With the obvious projection maps this gives us a \( \tilde{B}_1 \)-principal bundle \( (Q^1, \tilde{\pi}^1, Q^0, B_1) \), and a \( \tilde{B} \)-principal bundle \( (Q^1, \tilde{\pi}^0, M, \tilde{B}) \) equipped with an equivariant bundle map \( f^1 := df^0 : Q^1 \to \mathcal{P}^1 \). Hence, \( (Q^1, f^1) \) is referred to as the first prolongation of the conformal spin structure. We can lift the normal conformal Cartan connection \( w^{nc} \) to a Cartan connection \( \tilde{w}^{nc} := \lambda_* \circ w^{nc} \circ df^1 \in \Omega^1(Q^1, \mathfrak{spin}(p+1,q+1)) \) on \( Q^1 \). Again, a choice of a metric \( g \) from the conformal class leads to the spin connection form \( \tilde{A}^g \in \Omega^1(Q^g, \mathfrak{spin}(p,q)) \) which extends to a connection form \( \tilde{\gamma}^g \) on \( Q^0 \). This in turn induces a \( B_0 \)-equivariant section \( \tilde{\sigma}^g : Q^0 \to Q^1 \). Using the reduction map \( \tilde{\iota} : Q^g \to Q^0 \) the pull back of \( \tilde{w}^{nc} \) by \( \tilde{\sigma}^g \circ \tilde{\iota} \) gives us

\[
(\tilde{\sigma}^g \circ \tilde{\iota})^* \tilde{w}^{nc}(\tilde{Y}) = \lambda_*^{-1} \left( [f^g(q)]^{-1} d\pi^g_{f^g(q)}(Y) + A^g_{f^g(q)}(Y) \right) - \sum_i P^g_{\pi \circ f^g(q)}(d\pi^g_{f^g(q)}(Y), s_i) \cdot e^*_i,
\]

where \( \pi^g : \mathcal{P}^g \to M \) is the projection, \( \tilde{Y} \in T_q Q^g, Y := df^g_q(\tilde{Y}) \), \( P^g \) denotes the Schouten tensor with respect to \( g \), and \( \{ e_i \} \) and \( \{ e^*_i \} \) are as above such that \( \{ s_i := [f^g(q), e_i] \} \) is an orthonormal basis for \( T_{\pi \circ f^g(q)} M \) with respect to \( g \).

Summarizing, we have defined first prolongations for the conformal frame bundle and the conformal spin structure of \( (M, c) \), and equipped them with distinguished Cartan connections. These structures are the analogues of the orthonormal frame bundle equipped with the Levi-Civita connection form, and the spin connection form, for a chosen spin structure.
2.6 Tractor bundles for conformal spin structures

Let \((M, c)\) be a conformal spin manifold and \(\mathcal{P}^1\) and \(Q^1\) their associated \(B\)- and \(\tilde{B}\)-principal bundles. Considering the standard representation \(\rho :\ SO(p + 1, q + 1) \to GL(n + 2, \mathbb{R})\) and spin representation \(\tilde{\rho} := \kappa_{p+1,q+1} : Spin(p+1, q+1) \to GL(\Delta_{p+1,q+1})\), we may define the standard tractor bundle and spin tractor bundle by

\[
\mathcal{T}(M) := \mathcal{P}^1 \times_{(B, \rho)} \mathbb{R}^{n+2},
\]

\[
\mathcal{S}(M) := Q^1 \times_{(\tilde{B}, \tilde{\rho})} \Delta_{p+1,q+1},
\]

where the subscript \(\cdot_0\) denotes the connected component of \(\tilde{B}\) containing the identity. Both bundles can be equipped with a bundle metric, defined by \(g^\mathcal{T}(t_1, t_2) := \langle y_1, y_2 \rangle_{p+1,q+1}\), for \(t_i = [H, y_i] \in \mathcal{T}(M), i = 1, 2;\) and \(g^\mathcal{S}(s_1, s_2) := \langle v_1, v_2 \rangle_{\Delta}\), for \(s_i = [H, v_i] \in \mathcal{S}(M), i = 1, 2,\) since \(\langle \cdot, \cdot \rangle_{p+1,q+1}\) and \(\langle \cdot, \cdot \rangle_{\Delta}\) are invariant under \(B\) and \(\tilde{B}\). Since we have used representations of the groups \(SO(p + 1, q + 1)\) and \(Spin(p + 1, q + 1)\) to form the associated vector bundles, we may define covariant derivatives \(\nabla^\mathcal{T}\) and \(\nabla^\mathcal{S}\) induced by the Cartan connections \(w^{\text{nc}}\) and \(\tilde{w}^{\text{nc}}\). It turns out that \(g^\mathcal{T}\) and \(g^\mathcal{S}\) are parallel with respect to the corresponding covariant derivatives.

Choosing a metric \(g\) from the conformal class, the orthonormal frame bundle \(\mathcal{P}^g\) is a \(SO(p, q)\)–reduction of the conformal frame bundle \(\mathcal{P}^0\), and a \(SO(p, q)\) to \(B\)–reduction of the first prolongation \(\mathcal{P}^1\). Similarly, \(Q^g\) is a \(Spin(p, q)\) to \(CSpin(p, q)\)–reduction of the conformal spin structure \(Q^0\), and a \(Spin(p, q)\) to \(\tilde{B}\)–reduction of the first prolongation \(Q^1\). Thus the following isomorphisms arise:

\[
\mathcal{T}(M) \simeq \mathcal{P}^g \times_{(O(p,q), \rho)} \mathbb{R}^{n+2},
\]

\[
\mathcal{S}(M) \simeq Q^g \times_{(Spin_0(p,q), \tilde{\rho})} \Delta_{p+1,q+1},
\]

\[
TM \simeq \mathcal{P}^g \times_{(O(p,q), \text{Ad})} b_{-1} \simeq Q^g \times_{(Spin_0(p,q), \text{Ad} \circ \lambda)} b_{-1},
\]

\[
T^* M \simeq \mathcal{P}^g \times_{(O(p,q), \text{Ad})} b_1 \simeq Q^g \times_{(Spin_0(p,q), \text{Ad} \circ \lambda)} b_1,
\]

\[
\mathfrak{so}(TM, g) \simeq \mathcal{P}^g \times_{(O(p,q), \text{Ad})} \mathfrak{so}(p, q) \simeq Q^g \times_{(Spin_0(p,q), \text{Ad} \circ \lambda)} \mathfrak{so}(p, q).
\]

Therefore, for \(V\) being one of the bundles \(TM, T^* M\) or \(\mathfrak{so}(TM, g)\), we may define actions \(\rho^g : V \to \text{End}(\mathcal{T}(M))\) and \(\tilde{\rho}^g : V \to \text{End}(\mathcal{S}(M))\) by

\[
\rho^g(\Theta)t := [\Theta, \rho_\ast([\Theta]^{-1}t)],
\]

\[
\tilde{\rho}^g(\Theta)s := [\Theta, \tilde{\rho}_\ast \Theta^{-1}([\Theta]^{-1}s)],
\]

where \(t = [e, y] \in \mathcal{T}(M)\), \(s = [q, v] \in \mathcal{S}(M)\), \(\Theta \in V\), and \([e] : W \to V\) and \([q] : W \to V\) are the induced isomorphisms \([e]w := [e, w]\) and \([q]w := [q, w]\),

\[13\]
for \( w \in W = b_{-1}, b_1, so(p, q) \), respectively. In terms of these actions we have

\[
\nabla_X^t t = \nabla_X^t t + \rho^g(X)t - \rho^g(P^g(X))t, \\
\nabla_X^s s = \nabla_X^s s + \tilde{\rho}^g(X)s - \tilde{\rho}^g(P^g(X))s,
\]

for sections \( t = [e, y] \in \Gamma(T(M)), s = [q, v] \in \Gamma(S(M)) \), and a vector field \( X \in \mathfrak{X}(M) \). Note that \( \nabla_X^t t \) and \( \nabla_X^s s \) are abbreviations for \( [e, X(y) + \rho_\ast(A_p^0(de(X))y] \) and \( [q, X(v) + \tilde{\rho}_\ast(A_p^0(dq(X))v] \), and \( P^g(X) \) is considered as a \( 1 \)-form.

A crucial step in this subsection is to define a \( g \)-metric decomposition of standard tractors and spin tractors with respect to a metric \( g \) from the conformal class. Firstly, we have the bundle isomorphism

\[
Φ^g : T(M) \to \overline{M} \oplus TM \oplus \overline{M} =: T(M)_g,
\]

\[
t = [e, y] \mapsto (\alpha, X, \beta) =: t_g,
\]

where \( \overline{M} := M \times \mathbb{R} \) is the trivial bundle, \( y \in \mathbb{R}^{n+2} \) has coordinates \( (\alpha, x = (x_1, \ldots, x_n), \beta) \) with respect to the basis \( \{f_-, e_i, f_+\} \) of \( \mathbb{R}^{n+2} \), and \( X := [\epsilon]^{-1}x \in TM \). Secondly, we have the bundle isomorphism

\[
Ψ^g : S(M) \to S(M, g) \oplus S(M, g) =: S(M)_g
\]

\[
s = [q, v] \mapsto (\psi, \phi) =: s_g,
\]

where \( \psi = [q, w_1] \) and \( \phi = [q, w_2] \), with \( w_1, w_2 \in \Delta_{p, q} \) being determined as follows: Consider the two \( Spin(p, q) \)-invariant subspaces \( W^\pm := \{v \in \Delta_{p+1, q+1} \mid f_\pm \cdot v = 0\} \) of \( \Delta_{p+1, q+1} \). Note that we naturally identify \( W^+ \) with \( \Delta_{p+1, q+1} \). Hence, \( \tilde{\rho} \) restricted to \( Spin(p, q) \) decomposes into two representations \( \tilde{\rho}^\pm : Spin(p, q) \to GL(W^\pm) \), such that \( \tilde{\rho}_{Spin(p, q)} = \tilde{\rho}^+ \oplus \tilde{\rho}^- \). From the definition of \( W^\pm \) it follows that \( \tilde{\rho}^\pm \) are equivalent with respect to the isomorphism \( W^+ \ni w \mapsto f_- \cdot w \in W^- \). Therefore, our element in question \( v \in \Delta_{p+1, q+1} \) can be uniquely decomposed as \( v = w_1 + f_- \cdot w_2 \) with \( w_1, w_2 \in W^+ \), due to the isomorphism \( W^+ \times W^+ \ni (w_1, w_2) \mapsto w_1 + f_- \cdot w_2 \in \Delta_{p+1, q+1} \).

With the help of the two maps \( Φ^g \) and \( Ψ^g \) we will interpret tractor objects with data coming from the metric \( g \). For example, we have that

\[
Φ^g \circ \nabla_X^t \circ (Φ^g)^{-1} = \begin{pmatrix}
\nabla_X^{LC} & -P^g(X, \cdot) & 0 \\
X \cdot & \nabla_X^{LC} + P^g(X)^2 & \nabla_X^{LC} \\
0 & -g(X, \cdot) & \nabla_X^{LC}
\end{pmatrix}
\]

and

\[
Ψ^g \circ \nabla_X^S \circ (Ψ^g)^{-1} = \begin{pmatrix}
\nabla_X^{S(M, g)} & X \cdot & \nabla_X^{S(M, g)} \\
\frac{1}{2}P^g(X)^2 & \nabla_X^{S(M, g)}
\end{pmatrix},
\]

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which follows from the actions $\rho^g$ and $\tilde{\rho}^g$ defined above. A further example is given by the bundle metrics, here we have that

$$g^T(t_1, t_2) = \alpha_1\beta_2 + g(X_1, X_2) + \beta_2\alpha_1,$$

for $t_i = [e, y_i]$, $i = 1, 2$. Moreover, for $s_i = [q, v_i] \in S(M)$, $i = 1, 2$, we have that

$$g^g(s_1, s_2) = -2\sqrt{2}p^p (\langle \phi_1, \psi_2 \rangle + (-1)^p < \psi_1, \phi_2 >).$$

Note that these results are based on the isomorphisms (6) and (7). Let us end this subsection with the realization of standard and spin tractors with respect to two metrics $g$ and $\hat{g} = e^{2\sigma}g$ from the conformal class. Here it holds that

$$T(g, \sigma) := \Phi^g \circ (\Phi^g)^{-1} = \begin{pmatrix} e^{-\sigma} & -e^{-\sigma}d\sigma & -\frac{1}{2}e^{-\sigma}|\text{grad}^g(\sigma)|^2_g \\
0 & e^{-\sigma} & e^{-\sigma}\text{grad}^g(\sigma) \\
0 & 0 & e^\sigma \end{pmatrix},$$

and

$$T^{S(M)}(g, \sigma) := \Psi^{\hat{g}} \circ (\Psi^{\hat{g}})^{-1} = F_{\sigma} \oplus F_{\sigma} \begin{pmatrix} e^{4\sigma} & 0 \\
0 & e^{-2\sigma}\text{grad}^{\hat{g}}(\sigma) \end{pmatrix},$$

where $F_{\sigma} : S(M, g) \rightarrow S(M, \hat{g})$ is the bundle isomorphism relating spinor bundles for two conformally related metrics $g$ and $\hat{g} = e^{2\sigma}g$.

## 3 Relevant differential operators

In this section we present some operators necessarily for the construction of conformal powers of the Dirac operator. First we recall the construction of the splitting operator for the standard tractor bundle (in the spirit of [CS07, CGS10]), and compute its formal adjoint. The notation is borrowed from these two papers. Both the splitting operator and its adjoint can be extended to $S(M)$, as well as to $S^k(M) := \bigotimes^k(T(M)) \otimes S(M)$, for $k \geq 0$. Secondly, we consider the translations of the strongly invariant Yamabe operator with these splitting operators and their formal adjoints. We do this in order to obtain higher order differential operators acting on the spin tractor bundle.

Let us assume that $M$ is even dimensional, and so, that $p + 1 + q + 1 = 2(m + 1)$. The odd dimensional case is treated similarly. The weighted standard tractor bundle $T(M)[w - 1]$ splits under the conformal group $B_0$ as
\(\mathcal{T}(M)_g[w - 1] = M[w - 2] \oplus TM[w - 1] \oplus M[w].\) The lowest weights for these summands are \((w - 2)[0, \ldots, 0], (w - 1)[1, 0, \ldots, 0]\) and \((w)[0, \ldots, 0]\), each of length \((m + 1)\). Moreover, we denote by \(\rho = (m, m - 1, \ldots, 1, 0)\) the half sum of all positive roots.

The curved Casimir operator \(C : \Gamma(\mathcal{T}(M)_g) \rightarrow \Gamma(\mathcal{T}(M)_g)\) obeys the following formula, given in [CGS10, Section 2.2],

\[
C(t_g) = \beta(t_g) - 2 \sum_{i=1}^{n} \rho^g(\xi_i) \left( \nabla^g \xi_i t_g - \rho^g(P(\xi^i)) t_g \right),
\]

(10)

where \(\{\xi^i\}\) denotes a basis of \(TM\) and \(\{\xi_i\}\) is its dual, \(t_g \in \Gamma(\mathcal{T}(M)_g)\), \(P(\xi^i)\) is considered to be a 1–form, and the map \(\beta : \Gamma(\mathcal{T}(M)_g) \rightarrow \Gamma(\mathcal{T}(M)_g)\) acts on the direct sum by the Casimir scalars

\[
\beta_1 = w(w + n) - 2(n + 2w - 2), \quad \beta_2 = w(w + n) - 2w, \quad \beta_3 = w(w + n),
\]

which can be derived from [CS07, Theorem 1]. Thus, using

\[
C(t_g) = \begin{pmatrix}
[w(w + n) - 2(n + 2w - 2)]\alpha - 2\text{ div}(X) - 2J\beta \\
w(w + n) - 2w|X + 2\text{ grad}(\beta) \\
w(w + n)\beta
\end{pmatrix},
\]

where \(t_g = (\alpha, X, \beta) \in \Gamma(\mathcal{T}(M)_g)\), one computes that

\[
(C - \beta_1) \circ (C - \beta_2) t_g = 4 \begin{pmatrix}
-\Delta^L_C \beta - wJ\beta \\
(n + 2w - 2)(d\beta)^2 \\
w(n + 2w - 2)\beta
\end{pmatrix} =: 4D(g)\beta.
\]

Note our sign convention for the Laplacian is \(\Delta^g := Tr_g(\nabla^LC \circ \nabla^LC)\). This defines a mapping \(D(g) : \Gamma(M[w]) \rightarrow \Gamma(T(M)_g[w - 1])\). In the same manner one constructs an operator \(D^k : \Gamma(S^k(M)) \rightarrow \Gamma(S^k(M) \otimes T(M)_g),\) for \(k \geq 0.\) The splitting operator for the spinor bundle is constructed similarly: The spin tractor bundle splits under the conformal spin group \(\tilde{B}_0\) as \(S(M)_g \simeq S(M, g)[\tfrac{d}{2}] \oplus S(M, g)[\tfrac{d}{2}]\). Thus, \(S(M)_g[\eta - \tfrac{d}{2}]\) decomposes into a direct sum corresponding to lowest weights \((\eta)[\tfrac{1}{2}, \ldots, \tfrac{1}{2}]\) and \((\eta - 1)[\tfrac{1}{2}, \ldots, -\tfrac{1}{2}]\). Again, the Casimir scalars are given by

\[
\beta_1 = \eta(\eta + n) + \frac{1}{2} \sum_{i=0}^{m-1} \left( \frac{1}{2} + 2i \right), \quad \beta_2 = (\eta - 1)(\eta - 1 + n) + \frac{1}{2} \sum_{i=0}^{m-1} \left( \frac{1}{2} + 2i \right).
\]

Hence, equation (10) adapted to the spin tractor setting gives us

\[
C(s) = \begin{pmatrix}
\beta_1 \psi \\
\beta_2 \phi + \psi \psi
\end{pmatrix},
\]
for $s = (\psi, \phi) \in \Gamma(S(M)_g)$. This shows that

$$(C - \beta_2)s = 2 \left( \left( \eta + \frac{n-1}{2} \right) \psi \right) = 2D^{\text{spin}}(g)\psi$$

defines a map $D^{\text{spin}}(g) : \Gamma(S(M, g)[\eta]) \to \Gamma(S(M)_g[\eta - \frac{1}{2}])$. Note that the construction of $D^k(g)$ and $D^{\text{spin}}(g)$ only depends on the tractor data, hence they are well defined.

From now on we will work with unweighted bundles. The conformal weights are absorbed into the splitting operators as follows:

$$D^k(g, w) : \Gamma(S^k(M)) \to \Gamma(S^k(M) \otimes \mathcal{T}(M)_g)$$

$$s \mapsto \begin{pmatrix} -\Box^w_s \\ (n - 2 + 2w)(\nabla s)^k \\ w(n - 2 + 2w)s \end{pmatrix},$$

where $\Box^w_s := \Delta^w_s + wJs$, and

$$D^{\text{spin}}(g, \eta) : \Gamma(S(M, g)) \to \Gamma(S(M)_g)$$

$$\psi \mapsto \left( \left( \eta + \frac{n-1}{2} \right) \psi \right).$$

Since we are restricting our attention to unweighted bundles we have the following conformal transformation laws:

**Proposition 3.1** Let $\hat{g} = e^{2\sigma}g$, $s \in \Gamma(S^k(M))$ and $\psi \in \Gamma(S(M, g))$. Then one has

$$D^k(\hat{g}, w)(e^{w\sigma}s) = e^{(w - 1)\sigma}T(g, \sigma)D^k(g, w)s,$$

$$D^{\text{spin}}(\hat{g}, \eta)(e^{\sigma}F_\sigma \psi) = e^{(\eta - \frac{n+1}{2})\sigma}T^{S(M)}(g, \sigma)D^{\text{spin}}(g, \eta)\psi,$$

for all $w, \eta \in \mathbb{R}$. Here $F_\sigma : \Gamma(S(M, g)) \to \Gamma(S(M, \hat{g}))$ is the isomorphism for conformally related metrics.

For later purposes, let us define

$$C^{\text{spin}}(g, \eta) : \Gamma(S(M)_g) \to \Gamma(S(M, g))$$

$$s_g = (\psi, \phi) \mapsto \frac{1}{2}D\psi - (\eta + \frac{n}{2})\phi,$$

and

$$C^k(g, w) : \Gamma(S^k(M) \otimes \mathcal{T}(M)_g) \to \Gamma(S^k(M))$$

$$(s_1, \eta, s_2) \mapsto (n + nw_1 + w_1w)s_1 + (n + 2w)\text{div}(\eta)$$

$$- (\Delta^w_s + (1 - n - w)J)s_2,$$

(12)
where \( \text{div}(Y \otimes s) := \text{div}(Y)s + \nabla Y s \in \Gamma(S^k(M)) \), for \( Y \otimes s \in \Gamma(TM \otimes S^k(M)) \), and the divergence of a vector field is defined by

\[
\text{div}(Y) := \sum_i \varepsilon_i g(\nabla_s Y, s_i),
\]

in terms of a local section \((s_1, \ldots, s_n) : U \subset M \to \mathcal{P}^g\). By the proposition below, they are the formal adjoints of corresponding splitting operators.

**Proposition 3.2** As formal adjoints with respect to the corresponding \(L^2\)–scalar product we have that

\[
(D^k(g, w))^* = C^k(g, 1 - n - w),
\]

\[
(C^k(g, w))^* = D^k(g, 1 - n - w),
\]

\[
(D^{\text{spin}}(g, \eta))^* = -2\sqrt{2} p C^{\text{spin}}(g, \frac{1}{2} - n - \eta),
\]

\[
(C^{\text{spin}}(g, \eta))^* = -\frac{1}{2\sqrt{2}} p D^{\text{spin}}(g, \frac{1}{2} - n - \eta).
\]

**Proof.** Using the formulas (8) and (9) for the scalar products \(g^T\) and \(g^S\) we compute, for \( k = 0 \), that

\[
g^{T \otimes S} \left( D^0(g, w)s, \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix} \right)_{L^2} = -g^S(\Box_s w s_1 + \nabla w s_2) + g^{TM \otimes S}(w_1(\nabla s)^2, \eta)_{L^2} + g^S(w_1 w_1 s, s_1)_{L^2}
\]

\[
= -g^S(s, \Box_w s_1 + \nabla w s_2)_{L^2} - g^{TM \otimes S}(s, w_1 \text{div}(\eta))_{L^2} + g^S(s, w_1 w_1 s_1)_{L^2}
\]

\[
= g^S(s, C^0(g, 1 - n - w) \begin{pmatrix} s_1 \\ \eta \\ s_2 \end{pmatrix})_{L^2},
\]

where we have used the known adjoints of \(\Delta_g^\nabla\) and \(d^\nabla\). Note that the index \( \cdot_{L^2} \) indicates the induced \(L^2\)–scalar product. The case for \( k > 0 \) runs along the same lines. The second assertion follows immediately. Coming to the
third one, we have, for \( \eta_1 := (\eta + \frac{n-1}{2}) \), that

\[
g^S \left( D^{\text{spin}}(g, \eta) \psi, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)_{L^2}
= -2\sqrt{2}i^p \left( \frac{1}{2} \hat{D} \psi, \phi_1 \right)_{L^2} + (-1)^p < \eta_1 \psi, \phi_2 >_{L^2}
= -2\sqrt{2}i^p \left( \frac{1}{2} \hat{D} \phi_1 >_{L^2} + (-1)^p < \psi, \eta_1 \phi_2 >_{L^2}
= -2\sqrt{2}i^p (-1)^p \left( \psi, \frac{1}{2} \hat{D} \phi_1 - (\frac{1}{2} - \eta - n + \frac{n}{2}) \phi_2 >_{L^2}
= \psi, (-2)\sqrt{2}i^p g^{\text{spin}}(g, \frac{1}{2} - n - \eta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} >_{L^2},
\]

where we have used the (anti-) self-adjointness of \( \hat{D} \). Also note the hermiticity of \(< \cdot, \cdot >_{L^2} \). An analogous computation shows that

\[
< C^{\text{spin}}(g, \eta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \psi >_{L^2} = \frac{1}{2} \hat{D} \phi_1, \psi >_{L^2} - < (\eta + \frac{n}{2}) \phi_2, \psi >_{L^2}
= -\frac{1}{2\sqrt{2}i^p} g^S \left( \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} - n - \eta + \frac{n-1}{2} \psi \\ \frac{1}{2} \hat{D} \psi \end{pmatrix} \right)_{L^2}
= g^S \left( \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, -\frac{1}{2\sqrt{2}i^p} D^{\text{spin}}(g, \frac{1}{2} - n - \eta) \psi \right)_{L^2},
\]

which completes the proof. \( \square \)

It now follows from this proposition and from the invariance of the corresponding scalar products with respect to \( g \) and \( \hat{g} = e^{2\sigma} g \), that:

**Proposition 3.3** For \( \hat{g} = e^{2\sigma} g \), \( s_g = (\psi, \phi) \in \Gamma(S(M)_g) \) and \((s_1, \eta, s_2) \in \Gamma(S^k(M) \otimes T(M)_g)\), one has

\[
C^k(\hat{g}, w)(e^{w\sigma} T(g, \sigma)(s_1, \eta, s_2)) = e^{(w-1)\sigma} C^k(g, w)(s_1, \eta, s_2),
\]

\[
C^{\text{spin}}(\hat{g}, \eta)(e^{\eta\sigma} T^{S(M)}(g, \sigma)s_g) = e^{(\eta-\frac{n}{2})\sigma} F_\sigma(C^{\text{spin}}(g, \eta)s_g).
\]

Since Proposition 3.1 holds for any real numbers \( w, \eta \in \mathbb{R} \), the conformal covariance of the Box operator \( \Box^\Sigma_{2w} \) (the strongly invariant Yamabe operator), and the Dirac operator \( \hat{D} \), follow from that of \( D^k(g, w) \) and \( D^{\text{spin}}(g, \eta) \), for the values \( w = \frac{2-n}{2} \) and \( \eta = \frac{n-1}{2} \). 19
As mentioned above, the operator $\Box^{2n} \overline{\nabla}$ acts conformally on $\Gamma(S^k(M))$, for $k \in \mathbb{N}_0$. Hence we can use the curved translation principle, introduced in [ER87], to define $P_{2N}^{S(M)}(g): \Gamma(S^k(M)_g) \to \Gamma(S^k(M)_g)$ for $N \in \mathbb{N}$, by

$$
P_{2N}^{S(M)}(g) := C_0^0(g, -\frac{2N - n}{2}) \circ \cdots \circ C_{N-2}^{N-2}(g, -\frac{2 + n}{2}) \circ \Box^{2n} \overline{\nabla} \circ D^{N-2}(g, -\frac{4 - n}{2}) \circ \cdots \circ D_0^0(g, -\frac{2N - n}{2}), \quad N > 1. \quad (13)
$$

These operators satisfy the following:

**Proposition 3.4** The operator $P_{2N}^{S(M)}(g)$ is conformally covariant of bi-degree $(-\frac{2N - n}{2}, -\frac{2N + n}{2})$, i.e., for $\hat{g} = e^{2\sigma} g$ we have

$$
P_{2N}^{S(M)}(\hat{g}) \left( e^{2\sigma} s \right) = e^{-\frac{2N + n}{2}} P_{2N}^{S(M)}(g) s, \quad \text{for } s \in \Gamma(S(M)).
$$

Its leading term is given by $c(n,N)(\Delta^S \overline{\nabla})^N$, where the constant is

$$
c(n,N) := (-1)^{N-1} \prod_{k=1}^{N-1} [k(2 + 2k - n)]. \quad (14)
$$

**Proof.** The conformal covariance follows from the well-chosenness of $w$ in the composition. The given expression for $c(n,N)$ follows directly from [11], producing $(-1)^{N-1}$, and [12] producing the product. \qed

**Remark 3.5** In case of even $n$, the operator $P_{2N}^{S(M)}(g)$, for $N \geq \frac{n}{2}$, is not identically zero as stated in [Fis13, Proposition 5.26]. It is just of order less than $2N$, due to the fact that the constant $c(n,N)$ is zero in this case.

## 4 The construction of conformal powers of the Dirac operator and related structures

This section makes further use of the curved translation principle, [ER87], to define conformally covariant operators, acting on the spinor bundle, which are conformal powers of the Dirac operator. Furthermore, we present explicit
formulas for lower order examples in general, and subsequently simplify to the Einstein case. We then go on to prove some formal self-adjointness results. Using these explicit formulas we are able to show that the conformal powers of the Dirac operator are polynomials in first order operators.

Consider the differential operator

\[ D_{2N+1}(g) := C^{\text{spin}}(g, -\frac{2N + n}{2}) \circ P_{2N}^{S(M)}(g) \circ D^{\text{spin}}(g, \frac{2N + 1 - n}{2}) \] (15)

constructed from \( P_{2N}^{S(M)}(g) \) by translation, which acts on the spinor bundle.

**Theorem 4.1** Let \( N \in \mathbb{N} \). The operator \( D_{2N+1}(g) \) is conformally covariant of bi-degree \((2N + 1 - n, -2N + 1 + n)\), i.e., for \( \hat{g} = e^{2\sigma} g \) and \( \psi \in \Gamma(S(M, g)) \) we have

\[ D_{2N+1}(\hat{g})(e^{-\frac{2N + 1 + n}{2}F_{\sigma}} \psi) = e^{-\frac{2N + 1 + n}{2}F_{\sigma}} \circ D_{2N+1}(g) \psi. \]

Its leading term is given by a constant multiple of \( \mathcal{D}^{2N+1} \).

**Proof.** The conformal covariance follows directly from the construction of \( D_{2N+1}(g) \). The leading term is given by a scalar multiple of \( \mathcal{D}^{2N+1} \), due to the fact that \( P_{2N}^{S(M)}(g) \) has leading term \( c(n, N)(\Delta^\sigma)^N \) and the explicit formula

\[ \Delta^\sigma_g = \left( -\mathcal{D}^2 + \frac{n-2}{2}J, (P, \nabla^{S(M, g)}) + \frac{1}{2} \text{grad}(J) \right) - \mathcal{D}^2 + \frac{n-2}{2}J \).

The scalar multiple of \( \mathcal{D}^{2N+1} \) is a product of \( c(n, N) \) and a term independently of \( n \). \( \square \)

**Remark 4.2** In case of even \( n \), the operator \( D_{2N+1}(g) \), for \( N \geq \frac{n}{2} \), is not identically zero as stated in [Fis13, Theorem 5.27]. It is just of order less than \( 2N + 1 \), due to the fact that the constant infront of \( \mathcal{D}^{2N+1} \) is zero in this case. Thus, in that case, the last theorem does not yield conformal powers of the Dirac operator.

Explicit formulas for \( D_{2N+1}(g) \), for \( N = 1, 2 \), can be derived from explicit knowledge of \( P_2^{S(M)}(g) \) and \( P_4^{S(M)}(g) \) found in [Fis13, Proposition 5.28] and [Fis13, Proposition 5.36]:
Theorem 4.3  Let $(M, g)$ be a semi Riemannian spin manifold. The operators $D_3(g)$ and $D_5(g)$ are given by

$$D_3(g) = -\frac{1}{2} [\nabla^3 - (P, \nabla^{S(M,g)}) - (\nabla^{S(M,g)}, P^\cdot)],$$

$$D_5(g) = (n - 4) \left[ \nabla D_3(g) \nabla + 2(\nabla^2 D_3(g) + D_3(g) \nabla^2) - 4 \nabla^5 + 4(2P^2 + \frac{1}{n - 4} B, \nabla^{S(M,g)}) + 4(\nabla^{S(M,g)}, 2P^2 \cdot + \frac{1}{n - 4} B^\cdot) - 2(C, P^\cdot) - 2(P, C^\cdot) \right]$$

where the bracket and product notations were introduced in Subsection 2.3.

A detailed presentation of the proof can be found in \cite{Fis13, Theorem 5.29} and \cite{Fis13, Theorem 5.39, Remark 5.40}.

We have to remark that the operator $P_4^{S(M)}(g)$ decomposes into $P_4^{S(M)}(g) = P_4(g) + R(g)$, where both operators are conformally covariant of the same bi-degree as $P_4^{S(M)}(g)$. However, $P_4(g)$ has leading term a multiple of $(\Delta g^\cdot)^2$, whereas $R(g)$ is a zero order operator involving Weyl and Cotton curvatures, see \cite{Fis13, Proposition 5.37, Remark 5.38}. Hence, the operator $D_5(g)$ decomposes into $D_5(g) = D_5^{red}(g) + R_{spin}(g)$, where

$$R_{spin}(g) := C_{spin}(g, -\frac{4 + n}{2}) \circ R(g) \circ D_{spin}(g, \frac{5 - n}{2})$$

$$= \nabla(W \cdot W^\cdot) + W \cdot W \cdot \nabla + 4(C \cdot W \cdot + W \cdot C^\cdot).$$

Terms involving the Weyl curvature in the formula for $D_5(g)$ are relics from the tractor machinery we used for the construction.

Finally, let us denote the first three examples of conformal powers of the Dirac operator as follows:

$$D_1 := \nabla; \quad D_3 := -2D_3(g); \quad D_5 := \frac{1}{n - 4} D_5^{red}(g), \quad (n \neq 4).$$

These operators have an odd power of the Dirac operator as the leading term. Due to the explicit formulas we can prove the following theorem.

Theorem 4.4  Let $(M^n, g)$ be an Einstein spin manifold. Then one has

$$D_3 = \left( \nabla - \sqrt{\frac{2J}{n}} \right) \nabla \left( \nabla + \sqrt{\frac{2J}{n}} \right),$$

$$D_5 = \left( \nabla - \sqrt{\frac{8J}{n}} \right) \left( \nabla - \sqrt{\frac{2J}{n}} \right) \nabla \left( \nabla + \sqrt{\frac{2J}{n}} \right) \left( \nabla + \sqrt{\frac{8J}{n}} \right),$$

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where $J$ is the normalized (constant) scalar curvature.

Proof. Since $(M, g)$ is Einstein, we have by definition that $Ric = \lambda g$, for some constant $\lambda \in \mathbb{R}$. Thus, the scalar curvature satisfies $\tau = n\lambda$, hence it is constant, and so is $J = \frac{n\lambda}{2(n-1)}$. It follows that $P = \frac{\lambda}{2(n-1)}g$. This shows, that

$$D_3 = \psi^3 - 2(P, \nabla) = \psi^3 - \frac{\lambda}{n-1} \psi = \left(\psi - \sqrt{\frac{2J}{n}}\right) \psi \left(\psi + \sqrt{\frac{2J}{n}}\right).$$

Since the Bach tensor and the Cotton tensor vanish for Einstein metrics, we have

$$D_5 = \psi D_3 \psi + 2\left(\psi^2 D_3 + D_3 \psi^2\right) - 4\psi^5 + 16(P^2, \nabla)$$

$$= \psi^5 - 5\frac{\lambda}{n-1} \psi^3 + 4\frac{\lambda^2}{(n-1)^2} \psi$$

$$= \left(\psi - \sqrt{\frac{8J}{n}}\right) \psi \left(\psi - \sqrt{\frac{2J}{n}}\right) \psi \left(\psi + \sqrt{\frac{2J}{n}}\right) \psi + \sqrt{\frac{8J}{n}}.$$

which completes the proof. \qed

The result of the last theorem is analogous to the product structure for the conformal powers of the Laplacian for Einstein manifolds, compare [Gov06]. For example, Theorem 4.4 in case of the standard sphere, i.e., $J = \frac{n^2}{2}$, agrees with the result obtained in [ES10], where it was proven that all conformal odd powers of the Dirac operator have such a product structure.

In order to prove some formal (anti-) self-adjointness results, we present the following theorem. It generalizes the formal (anti-) self-adjointness of the Dirac operator, which is given in terms of the bracket notation (1) by

$$\psi = \frac{1}{2} \left( (g, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, g \cdot) \right),$$

to arbitrary symmetric $(0,2)-$tensor fields $T$ instead of $g$.

**Theorem 4.5** Let $(M, g)$ be a semi Riemannian spin manifold without boundary, and let $T$ be a symmetric $(0,2)-$tensor field. The operator

$$(T, \nabla^{S(M,g)}) + (\nabla^{S(M,g)}, T \cdot) : \Gamma (S(M,g)) \to \Gamma (S(M,g))$$

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is formally self-adjoint, or anti self-adjoint, with respect to the $L^2$–scalar product, depending on the signature $(p,q)$ of $(M,g)$.

**Proof.** Let $\psi, \phi \in \Gamma_c(S(M,g))$ be the compactly supported spinors, and define a 1–form with values in $\mathbb{C}$ by $w(X) := \langle T(X)^2 \cdot \psi, \phi \rangle$. Considering its dual $Y_w$, with respect to $g$, and taking its divergence we obtain

$$\text{div}(Y_w) = \sum_i \varepsilon_i \left[ \langle T(s_i)^2 \cdot \nabla_s^{S(M,g)} \psi, \phi \rangle - (-1)^p \langle \psi, T(s_i)^2 \cdot \nabla_s^{S(M,g)} \phi \rangle \right]$$

$$+ (-1)^p \langle \psi, (\delta^{\nabla LC} T)^2 \cdot \phi \rangle.$$

Using Stokes’ Theorem we get

$$\int_M \text{div}(Y_w) dM = 0,$$ hence

$$< (T, \nabla^{S(M,g)} \psi) + (\nabla^{S(M,g)}, T \cdot \psi), \phi >_{L^2}$$

$$= \int_M < (T, \nabla^{S(M,g)} \psi) + (\nabla^{S(M,g)}, T \cdot \psi), \phi > dM$$

$$= (-1)^p \int_M < \psi, 2(T, \nabla^{S(M,g)} \phi) - (\delta^{\nabla LC} T)^2 \cdot \phi > dM$$

$$= (-1)^p < \psi, (T, \nabla^{S(M,g)} \phi) + (\nabla^{S(M,g)}, T \cdot \phi) >_{L^2},$$

which completes the proof. □

This leads us to the following result:

**Theorem 4.6** Let $(M,g)$ be a semi Riemannian spin manifold without boundary. The operators $D_k$, $k = 1, 3, 5$, are formally self-adjoint (anti self-adjoint) with respect to the $L^2$–scalar product, i.e.,

$$< D_k \psi, \phi >_{L^2} = (-1)^p < \psi, D_k \phi >_{L^2}$$

for $\psi, \phi$ compactly supported sections of the spinor bundle.

**Proof.** This follows from Theorem 4.5, Theorem 4.3 and the fact that we have

$$< (C, P \cdot) \psi + (P, C \cdot) \psi, \phi >$$

$$= \sum_i \varepsilon_i < C(s_i) \cdot P(s_i) \cdot \psi + P(s_i) \cdot C(s_i) \cdot \psi, \phi >$$

$$= (-1)^p \sum_i \varepsilon_i < \psi, P(s_i) \cdot C(s_i) \cdot \phi + C(s_i) \cdot P(s_i) \cdot \phi >$$

$$= (-1)^p < \psi, (C, P \cdot) \phi + (P, C \cdot) \phi >,$$
for any ψ, φ ∈ Γ(S(M, g)), where {s_i} is a g–orthonormal basis.

This theorem is a special case of the following result:

**Theorem 4.7** Let (M, g) be a semi Riemannian spin manifold without boundary. For N ∈ N the operator D_{2N+1}(g) is formally self-adjoint (anti self-adjoint) with respect to the L^2–scalar product, i.e.,

\[ <D_{2N+1}(g)\psi, \phi >_{L^2} = (-1)^p <\psi, D_{2N+1}(g)\phi >_{L^2} \]

for ψ, φ compactly supported sections of the spinor bundle.

**Proof.** First of all note that from Proposition 3.2 the operator P_{2N}^S(g) is formally self-adjoint. Hence, by further use of Proposition 3.2 we get that

\[ <D_{2N+1}(g)\psi, \phi >_{L^2} = <C^{spin}(g, -\frac{2N+n}{2}) \circ P_{2N}^S(g) \circ D^{spin}(g, \frac{2N+1-n}{2})\psi, \phi >_{L^2} \]

\[ = <\psi, i^{p-p}C^{spin}(g, -\frac{2N+n}{2}) \circ P_{2N}^S(g) \circ D^{spin}(g, \frac{2N+1-n}{2})\phi >_{L^2} \]

\[ = <\psi, (-1)^p D_{2N+1}(g)\phi >_{L^2}, \]

which completes the proof. □

Now we are going to introduce a new family of conformally covariant differential operators acting on sections of the spin tractor bundle. Consider a series of conformally covariant differential operators D_k(g) : Γ(S(M, g)) → Γ(S(M, g)), of bi-degree \((k-n^2, -k+n^2)\), for odd k ∈ N, not necessarily conformal powers of the Dirac operator. Using these we may define an operator

\[ L_k(g) := \frac{4}{k+1} D^{spin}(g, -\frac{k+n}{2}) \circ D_k(g) \circ C^{spin}(g, \frac{k+1-n}{2}), \quad (16) \]

acting on Γ(S(M, g)). It satisfies the following:

**Theorem 4.8** For any odd k ∈ N, the operator L_k(g) is conformally covariant of bi-degree \((\frac{k+1-n}{2}, -\frac{k+1+n}{2})\), i.e., for any \(\hat{g} = e^{2\sigma} g\) we have that

\[ L_k(\hat{g}) \left( e^{-\frac{k+1+n}{2}} T^{S(M)}(g, \sigma) \right) = e^{-\frac{k+1+n}{2}} T^{S(M)}(g, \sigma) \circ L_k(g). \]

The case k = 1 and D_1(g) = D was found in a joint work with Andreas Juhl analyzing the conformal transformation law for the operator P_2^S(g) in detail.
Proof. This is a direct consequence of its definition \(16\). \(\Box\)

**Remark 4.9** Note that both operators \(P_{2N}(g)\) and \(L_{2N-1}(g)\) have the same conformal weights, see Theorems 3.4 and 4.8. Their construction, given in equations (13) and (16), can be illustrated, for the case \(N = 2\), as follows:

\[
\begin{align*}
\Gamma(S^1(M)) & \xrightarrow{D^0(g, \frac{4-n}{2})} \Gamma(S(M)_g) & \xrightarrow{C^{\text{spin}}(g, \frac{4-n}{2})} & \Gamma(S(M, g)) \\
\Gamma(S^1(M)) & \xrightarrow{C^0(g, -2+\frac{n}{2})} \Gamma(S(M)_g) & \xrightarrow{D^{\text{spin}}(g, -\frac{n}{2})} & \Gamma(S(M, g)).
\end{align*}
\]

Thus the operators \(P_{4}^{S(M)}(g)\) and \(L_3(g)\) (up to a constant) arise by the dashed arrow depending on the path taken through the diagram. Note, that in general a translation of \(L_{2N-1}(g)\) to the spinor bundle vanishes identically, due to \(C^{\text{spin}}(g, \frac{2N-n}{2}) \circ D^{\text{spin}}(g, \frac{2N+1-n}{2}) = 0\), whereas a translation of \(P_{2N}^{S(M)}(g)\) to the spinor bundle yields a conformal power of the Dirac operator.

Now consider the conformal powers of the Dirac operator \(D_k\), for \(k = 1, 3\), and denote by \(L_k(g)\) the induced conformally covariant operator acting on the spin tractor bundle, given by equation (16).

**Theorem 4.10** On the spin tractor bundle one has that

\[
L_1(g) - P_2^{S(M)}(g) = \begin{pmatrix} 0 & 0 \\ D_3 & 0 \end{pmatrix},
\]

\[
4L_3(g) - \frac{4}{4-n}P_4(g) = \begin{pmatrix} 0 & 0 \\ D_5 & 0 \end{pmatrix},
\]

where in the second difference we have chosen the main part of \(P_4^{S(M)}(g) = P_4(g) + R(g)\).

The proof based on explicit formulas of the involved operators and can be found in [Fis13, Theorem 5.48, Theorem 5.49].

**Remark 4.11** Theorem 4.10 gives also a construction of a conformal third and fifth power of the Dirac operator. It differs to the construction (15), since we are looking at certain differences of \(L_{2N-1}(g)\) and \(P_{2N}^{S(M)}(g)\), for \(N = 1, 2\), instead of translating \(P_{2N}^{S(M)}(g)\), for \(N = 1, 2\), to the spinor bundle. Of course, translating those differences to the spinor bundle will give us nothing new, since \(L_{2N-1}(g)\) is canceled by translations.
Now, we come to the polynomial structure of the first examples of conformal powers of the Dirac operator. Using the explicit formulas for $D_k$, for $k = 1, 3, 5$, we can define differential operators $M_k$, for $k = 1, 3, 5$, by

\[ M_1 := D_1 - 0 \]
\[ = \frac{1}{2}(g, \nabla \nabla^S(M,g)) + \frac{1}{2}(\nabla \nabla^S(M,g) , g'), \]
\[ M_3 := D_3 - D_1^3 \]
\[ = - (P, \nabla \nabla^S(M,g)) - (\nabla \nabla^S(M,g) , P). \]
\[ M_5 := D_5 - D_1 D_3 D_1 - 2(D_1^2 D_3 + D_3 D_1^2) + 4D_1^5 \]
\[ = 4(2P^2 + \frac{1}{n-4}B, \nabla \nabla^S(M,g)) + 4(\nabla \nabla^S(M,g) , 2P^2 + \frac{1}{n-4}B) \]
\[ - 2(C, P) - 2(P, C). \]

By definition they are first order operators. Just as for each $D_k$, the $M_k$, for $k = 1, 3, 5$, are formally (anti-) self-adjoint with respect to the $L^2$--scalar product. More interesting, however, is the following result:

**Theorem 4.12**  On a spin manifold $(M, g)$ of dimension $\neq 4$ we have

\[ D_1 = M_1, \]
\[ D_3 = M_1^3 + M_3, \]
\[ D_5 = M_1^5 + M_1 M_3 M_1 + 2(M_1^2 M_3 + M_3 M_1^2) + M_3. \]

This structure for the conformal powers of the Dirac operator is very similar to that for the conformal powers of the Laplacian discovered by A. Juhl. He presented a complete series of second order differential operators, such that the GJMS-operators can be written as a polynomial in these operators, see [Juh13, Theorem 1.1]. That series was rediscovered by Fefferman and Graham in [FG13].

We believe that there is a completely analogous picture for the conformal powers of the Dirac operator. Hence, it is natural to ask about the nature of $M_k$, for $k \in 2\mathbb{N} - 1$. For example, is there a generating function for the series of $M_k$, and how can one understand the coefficients arising in the polynomial description of $D_k$?

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