TOTAL CURVATURE AND SIMPLE PURSUIT ON DOMAINS OF CURVATURE BOUNDED ABOVE

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ABSTRACT. We show how circumradius and asymptotic behavior of curves in $\text{CAT}(0)$ and $\text{CAT}(K)$ spaces ($K > 0$) are controlled by growth rates of total curvature. We apply our results to pursuit and evasion games of capture type with simple pursuit motion, generalizing results that are known for convex Euclidean domains, and obtaining results that are new for convex Euclidean domains and hold on playing fields vastly more general than these.

1. INTRODUCTION

The goals of this paper are twofold:

(1) We study total curvature of a curve (the integral of its curvature) in spaces of curvature bounded above, and relate the total curvature, the curve’s circumradius function, the asymptotic behavior of the curve, and the domain’s curvature bound.

(2) We apply these results to a foundational problem in pursuit-evasion games, where an evader moves in a domain and is followed by a pursuer along a pursuit curve. We study the capture problem: whether the pursuer ever catches (comes sufficiently close to) the evader. Although our total curvature and circumradius results are new even for convex Euclidean playing fields, our playing fields are vastly more general than these.

We assume that the reader is familiar with the basic notions of $\text{CAT}(K)$ and Alexandrov geometry (see, e.g., [8, 9]) as well as the theme that results which are true in Riemannian spaces of sectional curvature bounded above are often true — and often have more transparent proofs — in the broader class of $\text{CAT}(K)$ spaces. Alternatively, the reader may consult the short appendix containing the definitions and basic tools that we use. We hope that readers based in comparison geometry will find both the theorems on the asymptotics of total curvature, and their applications to the capture problem, of interest; and readers interested in pursuit-evasion games will find the power of comparison geometry compelling.

1.1. Motivation. The application to pursuit-evasion games requires some motivation and background. There is a significant literature on pursuit and evasion games with natural motivations coming from robotics, control theory, and defense applications [13, 17, 31]. Such games involve one or more evaders in a fixed domain being hunted by one or more pursuers who win the game if the appropriate capture criteria are satisfied. Such criteria may be physical capture (the pursuers move to
where the evaders are located) [15, 16, 26] or visual capture (there is a line-of-sight between a pursuer and an evader) [13, 29]. The types of pursuit games are many and varied: continuous or discrete time, bounded or unbounded speed, and constrained or unconstrained acceleration, energy expenditure, strategy, and sensing. For a quick introduction to the literature on pursuit games, see, e.g., [19, 13].

The applications in this paper focus on one particular variable in pursuit games: the geometry and topology of the domain on which the game is played. We keep all other variables fixed and as simple as possible. Thus, there will be a single pursuer-evader pair, a single simple pursuit strategy (‘move toward the evader’), and no constraints on acceleration or related system features.

The vast majority of the known results on pursuit-evasion are dependent on having Euclidean domains which are two-dimensional or, if higher-dimensional, then convex. There has of late been a limited number of results for pursuit games on surfaces of revolution [14], cones [25], and round spheres [20]. Our results are complementary to these, in the sense that we work with domains of arbitrary dimension, with no constraints on being either smooth or locally Euclidean.

There are several reasons for wanting to extend the study of pursuit-evasion games to the most general class of playing fields possible. The most obvious such application is in the generalization from 2-d to 3-d, in which the pursuit game is a model for physical pursuit, as well as in the expansion to nonconvex domains. For example, a closed, simply connected domain \( D \) with smooth boundary in \( \mathbb{E}^3 \) is \( \text{CAT}(0) \) if the tangent plane at every boundary point \( p \) contains points arbitrarily close to \( p \) that are not in the interior of \( D \). (This is a special case of the characterization of upper curvature bounds of manifolds with boundary in [3].) More generally, a domain in \( \mathbb{E}^3 \) is \( \text{CAT}(1) \) if it is not too far from convex, that is, its boundary is not too outwardly curved; see Theorem 12 below.

However, higher dimensional playing fields can also correspond to physical problems, via configuration spaces of physical systems. Consider the following (fanciful) example. If one wants to mimic the action of a dancer with a complex robot, one could attempt a generalized pursuit game in which the playing field is the configuration space of the dancer’s (or robot’s) motions. The dancer’s configuration plays the role of the evader, and the robot’s configuration plays the role of the pursuer. If the robot’s goal is to mimic the dancer in real time with knowledge only of the dancer’s instantaneous body configuration, then this translates into a simple pursuit problem with one pursuer and one evader. The results of this paper show that no matter how high the dimension of the configuration space, the pursuit strategy will be successful if the configuration space is \( \text{CAT}(0) \). It has been demonstrated recently that there is a significant class of configuration spaces in robotics and related fields which do have an underlying \( \text{CAT}(0) \) geometry [1, 7, 11, 12], rendering the cartoon example above a little less unrealistic. In like vein, work on consensus, rendezvous, and flocking [30] is a form of coordinated pursuit in which the evader is the consensus or rendezvous state(s).

2. Total curvature

For a curve in a \( \text{CAT}(0) \) space, successively stronger constraints on the total curvature function control long-term behavior.

2.1. Definitions.
Definition 1. The total rotation $\tau_\sigma$ of a polygonal (i.e., piecewise-geodesic) curve $\sigma$ is $\sum_i (\pi - \beta_i)$, where the $\beta_i \geq 0$ are the angles at the interior vertices. The total curvature of any curve $\gamma$ is the lim sup of $\tau_\sigma$ as $\mu_\sigma \to 0$, over all polygonal $\sigma$ inscribed in $\gamma$, where $\mu_\sigma$ is the maximum segment length of $\sigma$.

In $\text{CAT}(0)$ spaces, monotonicity of total rotation follows from triangle comparisons: if $\sigma$ is inscribed in a polygonal curve $\gamma$, then $\tau_\sigma \leq \tau_\gamma$ [4]. Thus the total curvature of any curve $\gamma$ in a $\text{CAT}(0)$ space is the supremum of $\tau_\sigma$ over all polygonal $\sigma$ inscribed in $\gamma$. Monotonicity of total rotation fails in $\text{CAT}(K)$ spaces for $K > 0$, but a more subtle argument proves that the total curvature of an arbitrary curve is the limit of $\tau_{\sigma_n}$ for any sequence $\sigma_n$ of inscribed polygonal curves with $\mu_\sigma \to 0$ [23, see Theorem 18 below].

In particular, the total rotation $\tau_\sigma$ of a polygonal curve coincides with its total curvature. Accordingly: from now on, we denote the total curvature of an arbitrary curve $\gamma$ in a $\text{CAT}(K)$ space by $\tau_\gamma$.

Example 2. If $\gamma$ is a unitspeed curve in $\mathbb{E}^n$, then $\tau_\gamma$ equals the length in the unit sphere of the curve $\gamma'$ of righthand unit tangent vectors, with jump discontinuities replaced by great circular arcs [?]. In particular, if $\gamma$ is smooth in $\mathbb{E}^2$, so that $\gamma'(t) = (\cos \theta(t), \sin \theta(t))$, then $\tau_\gamma = \int \kappa$, where $\kappa = |\gamma''| = |\theta'|$.

Curves of finite total curvature in $\text{CAT}(K)$ spaces are well-behaved, in the sense that they have unit-speed parametrizations, which have left and right unit velocity vectors at every point [23].

We are interested in how the asymptotic behavior of the total curvature function,

$$\tau(t) = \tau_{\gamma|_{[0,t]}}$$

controls the function that measures the maximum distance from its initial point realized by the curve in a given time period:

Definition 3. The circumradius function of a curve $\gamma$ is the real-valued function $c$, where $c(t)$ is the smallest number such that the path $\gamma|_{[0,t]}$ lies in the ball of radius $c(t)$ about $\gamma(0)$.

2.2. Growth rate of total curvature and circumradius. The following theorem will be applied in Section 4 to pursuit-and-evasion games, to obtain a necessary condition for the evader to win, in terms of how far from home the evader wanders during given time periods. Theorem 4 generalizes a theorem of Dekster for Riemannian manifolds [10]. However, we use Reshetnyak majorization to obtain a simple argument that moreover holds for any $\text{CAT}(0)$ domain.

Theorem 4. For any curve $\gamma$, parametrized by arclength $t$, in a $\text{CAT}(0)$ space, let $\tau$ and $c$ be its total curvature and circumradius functions.

(a) If $\lim_{t \to \infty} \tau(t)/t = 0$, then $\gamma$ is unbounded.

(b) If $\tau \in O(t^a)$ for some $a \in (0,1)$, then $c \in \Omega(t^{1-a})$.

Example 5. Consider the spiral $\gamma(u) = (u \cos 2\pi u, u \sin 2\pi u)$ in $\mathbb{E}^2$. The total curvature function is linear in $u$, as is the circumradius, while the arclength $t$ grows quadratically. This is case (b) for $a = \frac{1}{2}$, with $t/c(t)^2$ bounded.

Proof of Theorem 4. For part (a), we may suppose by approximation that any fixed initial segment $\gamma|_{[0,t]}$ is polygonal. Subdivide $[0,t]$ into at most $\frac{\tau(t)}{\tau_{\gamma_{0}}^2} + 1$ subintervals so that the restriction $\gamma_i$ of $\gamma$ to each subinterval has total curvature at most
\[ \frac{\pi}{2} \]. (If any angles are less than \( \frac{\pi}{2} \), we first refine the polygon by cutting across each such angle with a short segment to obtain two angles of at least \( \frac{\pi}{2} \).) Let \( \rho_i \) be the closed polygon consisting of \( \gamma_i \) and its chord \( \sigma_i \). By Reshetnyak majorization, there is a closed convex curve \( \tilde{\rho}_i \) in \( \mathbb{E}^2 \) that majorizes \( \rho_i \). Since a majorizing map preserves geodesics and does not increase angles, \( \tilde{\rho}_i \) is a closed polygon with the same sidelengths as \( \rho_i \), consisting of a polygonal curve \( \tilde{\gamma}_i \) and its chord \( \tilde{\sigma}_i \), where the total curvature of \( \tilde{\gamma}_i \) is at most \( \frac{\pi}{2} \).

Since \( \tilde{\gamma}_i \) is a convex curve in \( \mathbb{E}^2 \) having total curvature at most \( \frac{\pi}{2} \), the ratio of its length to that of its chord is at most \( \sqrt{2} \) (the ratio of two sides of an isosceles right triangle to its hypotenuse). Therefore

\[
(2.1) \quad t \leq \left( \frac{\tau(t)}{\frac{\pi}{2} + 1} \right) \sqrt{2} \sup |\sigma_i|,
\]

so

\[
\frac{\tau(t)}{t} \geq \frac{\pi}{2} \left( \frac{1}{\sqrt{2} \sup |\sigma_i|} - \frac{1}{t} \right).
\]

But if \( \gamma \) is bounded, so that \( \sup |\sigma_i| < \infty \), it follows that \( \tau(t)/t \) is bounded away from 0 for \( t \) sufficiently large. This proves Part (a).

For Part (b), if one substitutes \( \tau(t) \leq \Lambda t^\alpha \) and \( \sup |\sigma_i| \leq 2c(t) \) in (2.1), it is immediate that \( t/c(t)^{1/(1-\alpha)} \) is bounded. \( \square \)

### 2.3. Finite total curvature and asymptotic rays.

Total curvature also controls how close an infinite curve of finite total curvature must be to a geodesic ray. In the Riemannian setting, the conclusions of the following theorem were obtained by Langevin and Sifre [21] under stronger hypotheses. That is, they assume the pointwise curvature \( \kappa(t) \) of a smooth curve \( \gamma \) satisfies \( \kappa(t) \in O(t^{−1−\epsilon}) \) in Part (a), and the same with \( t^{2+\epsilon} \) in Part (b). Here again, we give a simple argument using CAT(0) techniques.

A curve \( \gamma \) is said to be asymptotic to a geodesic ray \( \sigma \) if \( d(\gamma(t), \sigma) \) is bounded. Now we show that a curve of finite total curvature \( \tau_\gamma \) always has sublinear distance to some geodesic ray, to which it is asymptotic if the total curvature function approaches its limit \( \tau_\gamma \) sufficiently rapidly.

**Theorem 6.** Let \( \gamma \) be a curve, parametrized by arclength \( t \), in a CAT(0) space. Suppose \( \gamma \) has finite total curvature \( \tau_\gamma = \lim_{t \to \infty} \tau(t) \).

(a) Through any point \( p \), there is geodesic ray \( \sigma \) such that \( d(\gamma(t), \sigma) \in o(t) \).

(b) The circumradius function satisfies \( c \in \Omega(t) \).

(c) If \( \int_0^\infty (\tau_\gamma - \tau(t)) \, dt < \infty \), then \( \gamma \) and \( \sigma \) are asymptotic.

**Proof.** Again, we may assume \( \gamma \) is polygonal. Choose an increasing sequence \( t_i \to \infty, t_i \geq 0, t_0 = 0 \). Let \( \sigma_i \) be the geodesic joining \( \gamma(0) \) and \( \gamma(t_i) \). For \( i \geq 1 \), let \( \rho_i \) be the closed polygon made up of \( \sigma_{i-1}, \gamma_i = \gamma|_{t_{i-1}, t_i} \), and \( \sigma_i \). We denote by \( \tilde{\rho}_i \) a convex polygon in \( \mathbb{E}^2 \) that majorizes \( \rho_i \), and by \( \tilde{\gamma}_i \) its subarc corresponding to \( \gamma_i \). We suppose the \( \tilde{\rho}_i \) are arranged in a counterclockwise “fan”, so that \( \tilde{\rho}_i \) intersects \( \tilde{\gamma}_i \) along the straight line segment \( \tilde{\sigma}_i \) in each that corresponds to \( \sigma_i \), and the points corresponding to \( \gamma(0) \) coincide at the centerpoint \( \tilde{O} \).

Consider the polygonal curve \( \tilde{\gamma} : [0, \infty) \to \mathbb{E}^2 \), parametrized by arclength, whose image is the union of the convex curves \( \tilde{\gamma}_i \). The restriction of the majorizing map of \( \tilde{\rho}_i \) to \( \tilde{\gamma}_i \) maps onto \( \gamma_i \) and does not increase angles at vertices. Thus it does
not decrease total curvature. Moreover, the interior angles of $\tilde{\rho}_i$ at $\tilde{\gamma}(t_{i-1})$ and $\tilde{\gamma}(t_i)$ are at least those of $\rho_i$ at the corresponding points. Let $\tilde{\beta}_i$ be the sum of the angles of $\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}$ at $\tilde{\gamma}(t_i)$. By the triangle inequality for angles in $M$, $\tilde{\beta}_i$ is at least the angle between the left and right directions of $\gamma$ at $t_i$. Thus the angle between the initial and final tangents of $\tilde{\gamma}|_{[t_{m}, t_n]}$ in $E^2$, namely, the sum of the $(\pi - \tilde{\beta}_i)$, which could be negative, and the positive total curvatures of the convex polygonal curves $\tilde{\gamma}_i$ for $m + 1 \leq i \leq n$, is no more than the total curvature of $\tilde{\gamma}|_{[t_{m}, t_n]}$. Now take $t_m = 0$. Since $\tilde{\gamma}(0) = \tilde{\sigma}$, the total angle at $\tilde{\sigma}$ of the first $n$ sectors of the fan is no more than the angle between the initial and final tangents of $\tilde{\gamma}|_{[0, t_n]}$, and hence no more than $\tau_\gamma$. Therefore the vertex angles of the $\tilde{\rho}_i$ are summable, and the angle between $\tilde{\gamma}$ and the ray from $\tilde{\sigma}$ through $\tilde{\gamma}(t)$ converges to $0$.

Let $\tilde{\tau}(t) = d(\tilde{\sigma}, \tilde{\gamma}(t))$. By the First Variation Formula (A.1), the one-sided derivatives $d\tilde{\tau}/dt$ converge to $1$. Thus for any $A < 1$ we have $\tilde{\tau}(t)$ increasing and $\tilde{\tau}(t) > At$ for $t$ sufficiently large. Furthermore, for each choice of sequence $t_i \to \infty$ our construction produces a function $\tilde{\tau}$ satisfying $\tilde{\tau}(t_i) = \tau(t_i)$, where $\tau(t) = d(\tilde{\sigma}, \gamma(t))$. It follows that $\tilde{\tau}(t)$ also eventually increases and $\tilde{\tau}(t) > At$ for $t$ sufficiently large.

Since the directions of the line segments $\tilde{\sigma}_i$ at their basepoint $\tilde{\sigma}$ converge and $|\tilde{\sigma}_i| \to \infty$, the $\tilde{\sigma}_i$ converge to a Euclidean ray $\sigma$ from $\tilde{\sigma}$. Let $s$ be the arclength parameter on $\sigma$. Since the angle at which a ray strikes $\gamma$ converges to $0$, it follows that for $t$ sufficiently large, $\gamma$ is the graph of a height function of order $o(s)$ over $\tilde{\gamma}$. Hence $d(\gamma(t), \sigma) = o(s(t)) = o(t)$.

Now since each $\tilde{\rho}_i$ majorizes $\rho_i$, the intersections of the geodesics $\sigma_i$ with any ball in $M$ about $\gamma(0)$ converge to a geodesic. Therefore the $\sigma_i$ converge to a geodesic ray $\sigma$ in $M$. Furthermore, $d(\gamma(t), \sigma)$ is realized by a line segment through the fan, infinitely partitioned by its intersections with a truncated sequence of the $\tilde{\sigma}_i$. Since $\tilde{\rho}_i$ majorizes $\rho_i$, $\gamma(t)$ is joined to $\sigma$ by a path of no greater length. Therefore $d(\gamma(t), \sigma) \leq d(\gamma(t), \tilde{\sigma}) + o(t)$, as claimed in Part (a).

Part (b) follows from (a) trivially, given the linear circumradius of geodesic rays.

Since $d(\gamma(t), \sigma) \leq d(\gamma(t), \tilde{\sigma})$, it suffices for Part (c) to show that the latter is bounded. The length of the projection of $\tilde{\gamma}$ in $E^2$ to a line normal to $\tilde{\sigma}$ is obtained by integrating $\sin \beta(t)$, where $\beta$ the angle between the righthand tangent of $\tilde{\gamma}$ and the direction of $\tilde{\sigma}$. Since

$$\sin \beta(t) \leq \beta(t) \leq \tau_{\tilde{\gamma}|_{[0, \infty]}} \leq \tau_{\tilde{\gamma}|_{[0, \infty]}} - \tau(t),$$

Part (c) follows.

Remark 7. By alternately gluing Euclidean and hyperbolic bands bounded by pairs of asymptotic geodesics, one can construct a $\text{CAT}(0)$ space in which the curves of Part (c) need not have strict asymptotes (the distance to which approaches $0$ rather than merely being bounded). This construction is carried out in [21], although the $\text{CAT}(0)$ nature of the resulting glued space is not mentioned.

3. Simple pursuit on $\text{CAT}(0)$ domains

The following rules define a basic discrete-time equal-speed pursuit game. (Continuous-time pursuit will be discussed in Section 6.) Let $(X, d)$ denote a geodesic metric space (representing the domain on which the game is played). There is a single
pursuer $P$ and a single evader $E$ starting at locations $P_0$ and $E_0$ respectively. At the $i$-th step, the evader moves from $E_{i-1}$ to $E_i$, a point within distance $D$ chosen by the evader. The pursuer moves to $P_i$, the point along a geodesic from $P_{i-1}$ to $E_{i-1}$ at distance $D$ from $P_{i-1}$. The moves are illustrated in Figure 1. Four points, $P_i, E_i, E_{i+1}$ and $P_{i+1}$, form a degenerate geodesic quadrangle with side lengths $L_i$, $d(E_i, E_{i+1}) \leq D$, $L_{i+1}$, and $D$, where $L_i = d(P_i, E_i)$ for each nonnegative integer $i$.

![Figure 1. A degenerate quadrangle arising from a discrete-time capture problem.](image)

This type of motion, in which the pursuer moves in an unconstrained fashion in the direction of the evader, is called simple pursuit. Given $P_0$ and the sequence $\{E_i\}$, we say that $P$ wins if $d(P_i, E_i) \leq D$ for some $i$; otherwise, $E$ wins. We note that this instantaneous, memoryless strategy for pursuit is not necessarily the pursuer’s optimal strategy [28, 19] — merely the simplest.

By the triangle inequality, 

$$L_{i+1} \leq d(P_{i+1}, E_i) + D = L_i.$$ 

Thus $\lim_{i \to \infty} L_i = L$ exists, and the evader wins if and only if this limit is greater than $D$. Moreover, 

$$\pi - \beta_i \leq \alpha_i \leq \bar{\alpha}_i,$$ 

where $\alpha_i$ is the angle between the geodesics joining $P_{i+1}$ to $E_i$ and $E_{i+1}$, and $\bar{\alpha}_i$ is the angle corresponding to $\alpha_i$ in the Euclidean triangle with the same sidelengths as $\Delta E_i P_{i+1} E_{i+1}$ (compare Definition 1). The first inequality in (3.2) is by the triangle inequality for the angle distance between the directions of geodesic segments with a common origin. These observations highlight the naturality of the $\text{CAT}(0)$ definition in the context of pursuit problems.

A discrete-time pursuit curve $P(t)$ is obtained by joining the $P_i$ by geodesic segments, where $t$ has speed 1. Thus $P_i = P(iD)$, where $D$ is the step size. The discrete-time evader curve $E(t)$ is defined similarly; however, since the evader’s step sizes are assumed $\leq D$, on each geodesic segment the parameter $t$ has $\Delta t = D$ and constant speed $\leq 1$.

The following simple result is well known for convex Euclidean domains. Theorem 4 provides us with the immediate extension to $\text{CAT}(0)$ domains:

**Theorem 8.** For discrete-time simple pursuit on a complete $\text{CAT}(0)$ domain, the domain is compact if and only if the pursuer always wins.
Proof. If the evader wins, then by (3.1) we have \( L_{i+1} \to L \) and \( d(P_{i+1}, E_i) + D \to L \). Therefore the angle \( \tilde{\alpha}_i \) vanishes in the limit, because the same is true for \( \alpha_i \). By (3.2), the total curvature of the \( P \) curve is sublinear. This curve is unbounded via Theorem 4, so the domain is noncompact.

Conversely, a noncompact \( \text{CAT}(0) \) domain contains an infinite geodesic ray, along which the evader and pursuer may move with constant separation and hence without capture. □

We now relate the total curvature functions \( \tau^E(t) \) of the evader curve \( E(t) \) and \( \tau^P(t) \) of the pursuer curve \( P(t) \). On the one hand, the evader may accumulate large total curvature by zigzagging, without much affecting the pursuer’s total curvature. On the other hand, the pursuer’s total curvature may exceed the evader’s: if \( E \) runs along a geodesic ray and \( P \) does not start on the ray, then the evader’s total curvature is 0 and the pursuer’s is positive. The following result makes these observations precise. The proof uses only angle comparisons, and is added evidence that simple pursuit has an affinity for the \( \text{CAT}(0) \) setting.

Theorem 9. For discrete-time simple pursuit on a \( \text{CAT}(0) \) domain, the total curvature functions of evader and pursuer satisfy

\[
\tau^P(t) \leq \tau^E(t) + \pi.
\]

Proof. Set \( (\tau^P)^{n+1} = \tau^P((n+1)D) \), the total curvature of the pursuit curve to \( P_{n+1} \), and similarly for \( (\tau^E)^{n+1} \). Label the internal angles of \( \triangle E_i P_{i+1} E_{i+1} \) by \( \alpha_i, \varphi_i, \delta_i \) as indicated in Figure 1. Then

\[
(\tau^P)^{n+1} = \sum_{0}^{n-1} (\pi - \beta_i) \leq \sum_{0}^{n-1} \alpha_i \leq \sum_{0}^{n-1} (\pi - \varphi_i - \delta_i),
\]

since \( \alpha_i + \varphi_i + \delta_i \leq \pi \) by the \( \text{CAT}(0) \) condition.

On the other hand, letting \( \theta_i \) be the interior angle of the evader curve at \( E_i \), we have

\[
(\tau^E)^n = \sum_{1}^{n-1} (\pi - \theta_i) \geq \sum_{1}^{n-1} (\pi - \delta_{i-1} - \varphi_i),
\]

since \( \theta_i \leq \delta_{i-1} + \varphi_i \) by the triangle inequality for angle distance between directions at \( E_i \). Therefore

\[
(\tau^P)^{n+1} - (\tau^E)^n \leq \pi - \varphi_0 - \delta_{n-1}.
\]

□

4. Escape

On a noncompact domain, the relevant question is whether the evader can escape when the pursuer adopts the simple pursuit-curve strategy, and, if so, what conditions lead to escape. Here we show that the pursuer still always wins if the circumradius of the evader does not grow fast enough, or, equivalently via Theorem 4, if the pursuit path is forced to curve too much. The proof of this necessary condition for escape uses an estimate on total curvature of pursuit curves that will be proved in Theorem 13 of the next section.

Theorem 10. Suppose the evader wins a discrete-time simple pursuit on a \( \text{CAT}(0) \) domain \( D \). Then the total curvature \( \tau(t) \) of the pursuit curve from \( P_0 \) to \( P(t) \) is \( O(t^{1/2}) \).
Proof. assuming Theorem 13. We invoke the facts that a CAT(0) domain is also a CAT(1) domain, and a rescaling of a CAT(0) domain is again a CAT(0) domain.

Theorem 13 states that for simple pursuit on a CAT(1) domain, if the evader wins and the initial distance \( L_0 = d(P_0, E_0) \) is less than \( \pi \), then the total curvature \( \tau(t) \) of the pursuit curve from \( P_0 \) to \( P(t) \) is \( O(t^{1/2}) \). Therefore we rescale the metric of \( \mathcal{D} \) by \( \pi/L_0 \). Since angles are invariant under rescaling, then by Definition 1, the total curvature of a given segment of the pursuit curve is also invariant under rescaling. \( \square \)

Corollary 11. Let \( c \) denote the circumradius function of an evader’s path on a CAT(0) domain. Then \( c \in \Omega(t^{1/2}) \) is a necessary condition for the evader to win in a discrete-time simple pursuit game.

Proof. Combine Theorem 10 with Part (b) of Theorem 4, using \( a = \frac{1}{2} \). \( \square \)

5. Domains with Positive Curvature Bounds

In applications, spaces with positive curvature are not merely possible but prevalent. In this section, we demonstrate that controlled amounts of positive curvature are admissible, as long as we control initial distances between the pursuer and evader.

First we provide a large class of nonconvex examples of CAT(K) domains in \( \mathbb{R}^n \):

Theorem 12. A closed domain \( \mathcal{D} \) in \( \mathbb{R}^n \) with smooth boundary \( \partial \mathcal{D} \), where \( \mathcal{D} \) carries its intrinsic metric, is a CAT(K) space for \( K > 0 \) if it is supported at every \( p \in \partial \mathcal{D} \) by a sphere of radius \( 1/\sqrt{K} \), that is, every point at distance \( \leq 1/\sqrt{K} \) from \( \mathcal{D} \) is the center of a closed ball that meets \( \mathcal{D} \) in a single point.

Proof. By [2, Theorem 3], the hypothesis of supporting balls implies that geodesics of \( \mathcal{D} \) of length \( < \pi/\sqrt{K} \) are uniquely (and hence continuously) determined by their endpoints. By the Alexandrov patchwork construction (see [8, p.199]), it follows that \( \mathcal{D} \) is a CAT(K) space. \( \square \)

We now study the asymptotic behavior of total curvature of pursuit curves in CAT(K) spaces for \( K > 0 \). Rescaling the metric by the factor \( 1/\sqrt{K} \), we may assume \( K = 1 \).

Just as in the CAT(0) case, the triangle inequality for \( \triangle P_{i+1} E_{i+1} P_i \) in a CAT(1) space easily implies that the distances \( L_i = d(P_i, E_i) \) are monotonically non-increasing; see Figure 1. The condition for equality from one step to the next is that \( P_{i+1} \) and \( E_i \) are on a geodesic segment \( P_i E_{i+1} \), and hence the angles \( \angle P_i P_{i+1} E_{i+1} \) and \( \angle P_{i+1} E_i E_{i+1} \) are both \( \pi \).

Theorem 13. On a CAT(1) domain, suppose the evader wins a discrete-time simple pursuit with initial distance \( L_0 = d(P_0, E_0) < \pi \). Then the total curvature \( \tau(t) \) of the pursuit curve from \( P(0) \) to \( P(t) \) is \( O(t^{1/2}) \).

Proof. Since the distances \( L_i = d(P_i, E_i) \) are monotonically non-increasing, all \( L_i < \pi \) and the triangles \( \triangle P_{i+1} E_i E_{i+1} \) have perimeters < \( 2\pi \). Thus, they have model triangles in the unit sphere. Let the angle corresponding to \( \alpha_i \) be \( \tilde{\alpha}_i \), so that \( \alpha_i \leq \tilde{\alpha}_i \) by the CAT(1) condition. Hence

\[
\tau(t) \leq \sum_{i=0}^{n-1} \alpha_i \leq \sum_{i=0}^{n-1} \tilde{\alpha}_i.
\]
Since the evader wins, we have $L_{\infty} > D$. We set $\Delta_i = L_i - L_{i+1}$.

Apply the spherical law of cosines to the model triangles:
\[
\cos D \leq \cos d(E_i, E_{i+1}) = \cos L_{i+1} \cos(L_i - D) + \sin L_{i+1} \sin(L_i - D) \cos \tilde{\alpha}_i
\]
\[
= \cos L_{i+1} \cos(L_i - D) + \sin L_{i+1} \sin(L_i - D)
\]
\[
- \sin L_{i+1} \sin(L_i - D)(1 - \cos \tilde{\alpha}_i)
\]
\[
= \cos(D - \Delta_i) - \sin L_{i+1} \sin(L_i - D)(1 - \cos \tilde{\alpha}_i)
\]
(5.1)

where $B_\infty = \inf_i(\sin L_{i+1} \sin(L_i - D))$. The last inequality depends on two elementary inequalities. To verify that $\cos(D - \Delta_i) \leq \cos D + \Delta_i \sin D$, apply the Mean Value Theorem for $\cos x$ on the interval $D - \Delta_i \leq x \leq D$. To verify that $1 - \cos \alpha_i \geq \alpha_i^2/5$, use the identity $1 - \cos x = 2 \sin^2(x/2)$. Then since $\sin(x/2)$ is concave on the interval $0 \leq x \leq \pi$, its graph is above the chord: $\sin(x/2) \geq x/\pi$. Since $\pi^2 < 10$, we have $2/\pi^2 > 1/5$.

Inequality (5.1) yields an inequality for $\alpha_i^2$:
\[
\alpha_i^2 \leq 5\Delta_i \sin D/B_\infty.
\]

The following calculation completes the estimate for total curvature. It uses the Cauchy inequality
\[
\tau(t) = \tau(nD) = \sum_{i=0}^{n-2} \alpha_i \leq \sqrt{(n - 1) \sum_{i=0}^{n-2} \alpha_i^2},
\]
and the telescoping sum
\[
\sum_{i=0}^{n-2} \Delta_i = L_0 - L_{n-1} \leq L_0 - L_{\infty}.
\]

Hence
\[
\tau(t) \leq \sqrt{(n - 1)/5 \sin D(L_0 - L_{\infty})/B_\infty} \leq C \sqrt{nD} = C \sqrt{t},
\]
where
\[
C = \sqrt{5(L_0 - L_{\infty})/B_\infty}.
\]

Example 14. The hypothesis $L_0 = d(P_0, E_0) < \pi$ in Theorem 13 is necessary. For example, let $\mathcal{D}$ be the complement in $\mathbb{R}^n$ of one or more disjoint open balls of radius $> 1$. Then $\mathcal{D}$ is a $\text{CAT}(1)$ domain by Theorem 12. Start with $P$ and $E$ antipodal on the boundary of one of the balls. $P$ moves toward $E$ a distance $D < \pi$ around the boundary, while $E$ moves to the antipodal point; from then on, $P$ and $E$ can oscillate between the two antipodal pairs. The total curvature of $P$ increases by $\pi$ at each step, so grows linearly rather than $O(t^{3/2})$.

Example 15. In $\text{CAT}(K)$ domains for $K > 0$, there is no circumradius estimate analogous to that of Corollary 11. As a simple example of escape with bounded circumradius, consider the domain $\mathcal{D}$ of Remark 14. Let $P$ and $E$ travel at constant distance $< \pi$ apart around a local geodesic of $\mathcal{D}$ which is a great circle in one of the ball boundaries.

Bounded escape in the $\text{CAT}(1)$ setting always exhibits some aspects of Example 15:
Proposition 16. Let $\mathcal{D}$ be a compact $\text{CAT}(1)$ space. Suppose the evader wins a discrete-time simple pursuit with initial distance $L_0 = d(P_0, E_0) < \pi$. Then there is a bilaterally infinite local geodesic in $\mathcal{D}$, any finite segment of which is the limit of segments of the pursuit curve.

The proof is immediate from Theorem 13 and the following lemma:

Lemma 17. Let $\mathcal{D}$ be a compact $\text{CAT}(1)$ space. Suppose the total curvature function $\tau(t)$ of a curve $\gamma : [0, \infty) \to \mathcal{D}$ has sublinear growth. Then there is a bilaterally infinite local geodesic in $\mathcal{D}$, any finite segment of which is the limit of segments of $\gamma$.

We draw on the following work of Maneesawarng et al:

Theorem 18. In a $\text{CAT}(1)$ space, total curvature has the following properties.

1. Semi-continuity [24]: if a sequence of polygonal curves with total curvatures $\tau_m$ converges uniformly on the same parameter interval to a curve $\gamma$, then $\tau_\gamma \leq \lim \inf \tau_m$.

2. Continuity under inscription [22, 23]: if a sequence of polygonal curves with total curvatures $\tau_m$ is inscribed in a curve $\gamma$ so the maximum diameters $d_m$ of the broken segments of $\gamma$ approach 0, then $\tau_\gamma = \lim \tau_m$.

3. Length estimate [22, 23]: Let $\gamma$ be a curve from $p$ to $q$, with $\tau_\gamma + d[p, q] < \pi$. Then $|\gamma|$ is at most the length of an isosceles once-broken geodesic in the unit sphere having the same total curvature and endpoint separation.

Proof of Lemma 17. We claim that for any $\epsilon > 0$ and $T > 0$, there is a sequence $t_n \to \infty$ such that the total curvature $\tau(t)$ of $\gamma$ satisfies $\tau(t_n + T) - \tau(t_n - T) < \epsilon$. Otherwise, any increasing sequence of such $t_n$ would have a finite supremum, after which the growth of $\tau$ would be linear, contradicting the hypothesis. Choosing sequences $\epsilon_i \to 0$ and $T_i = 2i$, and selecting one $t_i$ for each $i$, yields a sequence $t_i$ satisfying

$$
\tau(t_i + i) - \tau(t_i - i) \to 0, \quad t_i \to \infty.
$$

Let $\gamma_{ik}$ be the the restriction of $\gamma$ to $[t_i - k, t_i + k]$, $i \geq k$, reparametrized by arclength on $[-k, k]$. Writing $\tau_{ik} = \tau_{\gamma_{ik}}$, we have $\lim_{i \to \infty} \tau_{ik} = 0$ by (5.2).

By compactness of $\mathcal{D}$, the $\gamma_{i1}$ have a subsequence $\gamma_{i1}$ that converges to a curve $\rho_1$. By Theorem 18 (1), $\tau(\rho_1) = 0$ and so $\rho_1$ is a local geodesic. Theorem 18 (3) implies $|\gamma_{i1}| = |\rho_1|$, and so $\rho_1$ has length 2.

By construction, the $\gamma_{i1}$ for $i \geq 2$ extend to subsegments of the pursuit curve of length 4 that form a subsequence of the sequence $\gamma_{i2}$. From this subsequence we may extract a further subsequence $\gamma_{i2}$ that converges to a curve $\rho_2$ of length 4. As before, $\rho_2$ is a local geodesic. Since the restrictions to $[-1, 1]$ of the $\gamma_{i2}$ converge to $\rho_1$, then $\rho_2$ extends $\rho_1$.

In this manner, we obtain local geodesics $\rho_k$ of length $2k$ for any $k$, each an extension of the preceding one; and hence obtain a local geodesic $\rho$ which by construction has the desired property.

Remark 19. There may be no periodic local geodesic $\rho$ with the property described in Proposition 16 or Lemma 17. To see this, consider the Thue-Morse infinite binary word, which we write as a sequence of the integers 1 and 2. Let $\mathcal{D}$ be the $\text{CAT}(1)$ domain given by the complement in $\mathbb{R}^2$ of two disjoint open disks of radius 1. Let $\gamma : [0, \infty) \to \mathcal{D}$ be a local geodesic that winds around the two boundary circles $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_2$, according to the pattern dictated by Thue-Morse word. That
is, the appearance of an integer \( i \in \{1, 2\} \) indicates that \( \gamma \) makes positively oriented contact with \( \partial D_i \), and a subword consisting of \( k > 1 \) repeats of the integer \( i \) indicates that \( \gamma \) also consecutively performs \( k - 1 \) complete, positively oriented circuits around \( \partial D_i \). Our claim is immediate from the fact that no subword of the Thue-Morse word repeats three times in a row [6, Theorem 1.8.1].

6. Continuous-time pursuit

In the continuous version of simple pursuit, \( E \) moves along a rectifiable curve \( E(t) \) parametrized with speed \( \leq 1 \). It is assumed that \( P(t) \) moves at constant unit speed, and for each \( t \), the right-handed velocity vector \( P'(t) \) exists and points along a geodesic from \( P(t) \) to \( E(t) \). Thus a continuous pursuit curve is a time-dependent gradient curve for the distance function from a moving point \( E(t) \).

In the \( \text{CAT}(K) \) setting, we assume that the initial separation satisfies \( L(0) < \pi/\sqrt{K} \). As follows from the First Variation Formula (A.1), \( L(t) \) is non-increasing, where \( L(t) = d(P(t), E(t)) \). Hence the geodesic from \( P(t) \) to \( E(t) \) is unique. The evader wins if and only if \( \lim_{t \to \infty} L(t) > 0 \).

Suppose we are given a rectifiable curve \( E(t) \), with \( t \geq 0 \) and speed \( \leq 1 \), and an initial pursuer position \( P(0) \) and positive step size \( D \). The discrete-time pursuit game \( P_{D,i} \) generated by the data \( \{E(t), P(0), D\} \) has evader sequence \( E_{D,i} \) and initial pursuit point \( P_{D,0} = P(0) \). As in section 3, there are corresponding broken geodesic pursuit and evader curves \( P_D(t) \) and \( E_D(t) \). These discrete-time curves do not form a continuous-time simple pursuit game unless the evader curve is a geodesic with \( P(0) \) on a left-end geodesic extension. We denote the separation at time \( t \) by \( L_D(t) = d(P_D(t), E_D(t)) \).

Jun has provided a foundation for the theory of continuous simple pursuit and its approximation by discrete simple pursuit, including existence, uniqueness and curvature properties of continuous pursuit curves. In particular we use the following theorem:

**Theorem 20** ([18]). In a \( \text{CAT}(K) \) space, let \( E(t), t \geq 0 \), be a rectifiable curve with speed \( \leq 1 \), and \( P(0) \) be an initial pursuer position with initial separation \( L(0) = d(P(0), E(0)) < \pi/\sqrt{K} \). Consider the corresponding discrete-time pursuit games \( P_{D(m),i} \) with step sizes \( D(m) = 2^{-m} \). Then the sequence of discrete-time pursuit curves \( P_{D(m),i}(t) \) for \( E_{D(m)}(t) \) converges to a continuous unit-speed pursuit curve \( P(t) \) for \( E(t) \), uniformly on any initial arc \( t \leq T \). Moreover, \( P(t) \) is the unique pursuit curve with initial position \( P(0) \).

An immediate consequence of Theorem 20 is that if the continuous evader wins, then eventually the generated discrete evaders also win. Indeed, if \( m \) is sufficiently large then

\[
L_{D(m)}(t) > L(t)/2 > D(m).
\]

Therefore the continuous versions of Theorem 8 on compact domains, and Corollary 11 on escape circumradius functions \( c \), follow directly from these discrete theorems and Theorem 20.

Moreover, Jun showed that our estimates from the proof of Theorem 13 can be used to prove the continuous version of that theorem. We do not have to assume that the evader wins in order to get a bound on total curvature of an initial arc \( t \leq T = nD \) of the pursuer, only that \( L_{n-2} > D \), i.e., the evader has not been
caught yet. Then, evidently, we can replace $C$ by the time-dependent multiplier

$$C_D(T) = \sqrt{\frac{5(L_0 - L_{n-1})}{B(T)}},$$

where $B_D(T) = \min\{\sin L_{i+1} \sin(L_i - D) : i < n - 1\}$.

**Theorem 21 ([18]).** On a CAT$(1)$ domain, suppose the evader wins a continuous simple pursuit with initial distance $L_0 = d(P_0, E_0) < \pi$. Then the total curvature $\tau(t)$ of the pursuit curve from $P(0)$ to $P(t)$ is $O(t^{\frac{1}{2}})$.

**Proof.** By Theorem 20 and the inequalities (6.1) we know that for $m$ sufficiently large, $E_{D(m)}(T)$ is not caught. Then the limit of the bound on $\tau_D(T)$ given by (6.2) is

$$C(T)\sqrt{T} = \sqrt{\frac{5(L(0) - L(T))}{B(T)}} \sqrt{T},$$

where $B(T) = \min\{\sin^2 L(0), \sin^2 L(T)\}$. By Theorem 18 (1),

$$\tau(T) \leq \lim \inf \tau_{D(m)}(T) \leq \lim \inf \tau_{D(m)}(T) \sqrt{T} = C(T) \sqrt{T}.$$ 

Since the evader wins, then $\tau(t) \leq C\sqrt{T}$ where $C = \lim_{T \to \infty} C(T) > 0$. □

The continuous case of Theorem 9 may also be reduced to the discrete case:

**Theorem 22.** For continuous simple pursuit on a CAT$(0)$ domain, the total curvature functions of evader and pursuer satisfy

$$\tau^P(t) \leq \tau^E(t) + \pi.$$

**Proof.** The result is immediate from the following chain of inequalities.

$$\tau^P(t) \leq \lim \inf \tau^P_m(t) \leq \lim \inf \tau^E_m(t) + \pi = \lim \sup \tau^E_m(t) + \pi = \tau^E(t) + \pi.$$

The first inequality in the chain is by Theorem 18 (1); the second is by Theorem 9; the next (equality) is by Theorem 18 (2), which applies because the generated discrete evaders $E_m$ are inscribed in $E$; finally the last is the definition of total curvature. □

Proposition 16 depended only on the asymptotic estimate for the total curvature of the pursuit curve, whose continuous version is Theorem 21, and on Lemma 17, which applies to any curve, so we immediately obtain the continuous case:

**Proposition 23.** Let $D$ be a compact CAT$(1)$ space. Suppose the evader wins in continuous simple pursuit with initial distance $L_0 = d(P_0, E_0) < \pi$. Then there is a bilaterally infinite local geodesic in $D$, any finite segment of which is the limit of segments of the pursuit curve.

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### APPENDIX A. BACKGROUND

Here we set out the definitions and results that are assumed in the paper. Further
discussion may be found in [8, 9].

We consider *length spaces*, which are metric spaces for which the distance be-
tween any two points is the infimum of pathlengths joining them.

A curve \( \sigma \) in a length space is a *geodesic* if \( d(\sigma(t), \sigma(t')) = |t - t'| \) for any two
parameter values \( t, t' \), and a *local geodesic* if it is a geodesic when restricted to some
neighborhood of each of its parameter values. A length space is a *geodesic space* if
any two points are joined by a geodesic, and a *C-geodesic space* if any two points
with distance < \( C \) are joined by a geodesic.

Spaces with curvature bounded above are spaces whose geodesic triangles are
no ‘fatter’ than triangles with the same sidelengths in a model space of constant
curvature, according to the following definition.

**Definition 24.** A geodesic metric space is \( \text{CAT}(0) \) if the distance between any two
points of any geodesic triangle \( \triangle pqr \) is no greater than the distance between the
Corresponding points of the model triangle \( \tilde{\triangle} \tilde{p} \tilde{q} \tilde{r} \) with the same sidelengths in the
Euclidean plane \( M_0 = \mathbb{E}^2 \).

A \((\pi/VK)\)-geodesic metric space is \( \text{CAT}(K) \) for \( K > 0 \) if the distance between any
two points of any geodesic triangle \( \triangle pqr \) of perimeter < \( 2\pi/\sqrt{K} \) is no greater than the
distance between the corresponding points of the model triangle \( \tilde{\triangle} \tilde{p} \tilde{q} \tilde{r} \) with the same sidelengths in the 2-dimensional Euclidean sphere \( M_K \) of radius \( 1/\sqrt{K} \).

These definitions may be unified by setting \( \pi/\sqrt{K} = \infty \) if \( K = 0 \), as we will do
from now on. For \( K < 0 \), \( \text{CAT}(K) \) spaces are defined similarly by taking \( \pi/\sqrt{K} = \infty \]
and \( M_K \) to be the hyperbolic plane of curvature \( K \). \( \text{CAT}(K) \) spaces for \( K < 0 \) are
automatically \( \text{CAT}(0) \) and of no further interest in this paper.

Note that rescaling a \( \text{CAT}(1) \) space by multiplying all distances by \( 1/\sqrt{K} \), \( K > 0 \),
yields a \( \text{CAT}(K) \) space, since rescaling \( M_1 \) yields \( M_K \).

Since triangles with given sidelengths in the model spaces \( M_K \) become fatter as
\( K \) increases, it is clear that a \( \text{CAT}(K_1) \) space is also a \( \text{CAT}(K_2) \) space for \( K_2 > K_1 \).
It is an easy consequence of the definition that a \( \text{CAT}(0) \) space \( \mathcal{D} \) (respectively, a
\( \text{CAT}(K) \) space \( \mathcal{D} \)) has unique geodesics between any two points (respectively, any
two points with distance < \( \pi/\sqrt{K} \)), and these geodesics vary continuously with
their endpoints. In particular, \( \mathcal{D} \) is simply connected (respectively, the open ball of
radius \( \pi/\sqrt{K} \) about any point in \( \mathcal{D} \) is simply connected).

For a simple example, take \( \mathcal{D} \) to be the Euclidean plane with one or more dis-
joint open circular disks of radius 1 removed, where \( \mathcal{D} \) is equipped with the length
metric. Then \( \mathcal{D} \) is not a \( \text{CAT}(0) \) space but is \( \text{CAT}(1) \), since the \( \text{CAT}(1) \) perimeter
condition excludes any triangle that encloses a removed disk, and the remaining
triangles are even thinner than their Euclidean models. \( \mathcal{D} \) is not simply connected,
while open balls of radius \( \pi \) are simply connected since they do not include any
boundary circle. (Since \( \mathcal{D} \) is locally \( \text{CAT}(0) \), its simply connected covering is a
CAT(0) space, whose geodesics are sent to the local geodesics of $\mathcal{D}$ by the covering map.)

An effective tool in CAT(K) geometry is Reshetnyak majorization, which extends the defining comparison property of CAT(K) spaces:

**Theorem 25** (Reshetnyak [27]). Let $\gamma$ be a closed curve of length $< 2\pi/\sqrt{|K|}$ in a CAT(K) space $X$. Then there is a closed curve $\tilde{\gamma}$ which is the boundary of a convex region $D$ in $M_K$ and a distance-nonincreasing map $\varphi : D \to X$ such that the restriction of $\varphi$ to $\tilde{\gamma}$ is an arclength-preserving map onto $\gamma$.

In a CAT(K) space, the **angle** $\alpha \in [0, \pi]$ between two geodesic segments starting from a common endpoint is well-defined: it is the greatest lower bound of the corresponding angles in model triangles for triangles formed by initial subsegments of the two geodesics together with the attached third side. The thinness condition implies that these model angles descend monotonically as the subsegments are shortened, with the greatest lower bound equal to the limit.

The angle between geodesic segments with a common origin gives the **angle distance** between the directions of geodesic segments emanating from a point. In any CAT(K) space, the angle distance satisfies the triangle inequality.

The First Variation Formula for CAT(K) spaces governs the rate of change of distance $r(t)$ between unit-speed geodesics $\gamma_1(t), \gamma_2(t)$, assuming $r < \pi/\sqrt{|K|}$:

$$\left. \frac{dr}{dt} \right|_0 = -(\cos \alpha_1 + \cos \alpha_2),$$

where $\alpha_i$ is the angle between the geodesic $\gamma_i$ and the geodesic joining $\gamma_1(0)$ and $\gamma_2(0)$. We can free the First Variation Formula from the restriction that $r$ is the distance between two unit speed geodesics, and assume only that the two curves have righthand directions, by using the chain rule to insert speed factors multiplying the terms.

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