Non-Abelian Monopoles in the Higgs Phase

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Abstract

We use the moduli matrix approach to study the moduli space of 1/4 BPS kinks supported by vortices in the Higgs phase of $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theories when non-zero masses for the matter hypermultiplets are introduced. We focus on the case of degenerate masses. In these special cases vortices acquire new orientational degrees of freedom, and become “non-Abelian”. Kinks acquire new degrees of freedom too, and we will refer to them as “non-Abelian”. As already noticed for the Abelian case, non-Abelian kinks must correspond to non-Abelian monopoles of the unbroken phase of $SU(N)$ Yang-Mills. We show, in some special cases, that the moduli spaces of the two objects are in one-to-one correspondence. We argue that the correspondence holds in the most general case.

The consequence of our result is two-fold. First, it gives an alternative way to construct non-Abelian monopoles, in addition to other well-known techniques (Nahm transform, spectral curves, rational maps). Second, it opens the way to the study of the quantum physics of non-Abelian monopoles, by considering the simpler non-Abelian kinks.
1 Introduction

The history of magnetic monopoles is quite long, and it tracks back to the successful attempt of Dirac to introduce magnetic charges into a consistent quantum mechanics of charged particles [1]. Existence of monopoles was motivated by the explanation of the quantization of the electric charge. Another step toward the legitimization of monopoles as an important subject of study was made by ’t Hooft [2] and Polyakov [3], who showed that magnetic monopoles where necessarily present in many spontaneously broken gauge theories, including all theories of grand unification [4]. While from the experimental point of view monopoles are still problematic, since they have never been observed in nature, from the theoretical point of view they have been inspiring successful ideas in various area of physics. In cosmology, for example, they motivated the introduction of the concept of inflation by Sato and Guth [5]. Monopoles were also considered as playing a crucial role in the strongly coupled dynamics of gauge theories. In particular, condensation of monopoles in the vacuum of QCD can explain confinement in terms of a dual superconductivity mechanism [6]. At the same time, the study of particular monopole solutions, called BPS [7,8], whose energy is proportional to the magnetic charge, was particularly fruitful. BPS monopoles naturally arise in theories with extended supersymmetry, where the proportionality of masses and the existence of degenerate solutions are explained in terms of central charges and unbroken supersymmetry. More recent results in the non-perturbative dynamics of $\mathcal{N} = 2$ supersymmetric theories have been used to prove the actual role of magnetic monopoles in the mechanism of confinement [9].

The study of monopoles in theories with residual non-Abelian gauge symmetries has also motivated the idea of dualities as crucial properties of gauge theories. The first proposed example was a direct generalization of the electro-magnetic duality to the non-Abelian case [10,11]. The presence of electro-magnetic dualities, which also involve a weak-strong coupling duality, is now well established in the context of supersymmetric gauge theories and string theory. A few examples are Seiberg duality in $\mathcal{N} = 1$ theories [12], Seiberg-Witten duality in $\mathcal{N} = 2$ [9], S-duality in $\mathcal{N} = 4$ [13]. In all of these cases, quantum monopoles play a crucial role. Moreover, $\mathcal{N} = 2$ theories can support vacua where non-Abelian gauge symmetries are unbroken in the infrared [13]. In this context, semiclassical configurations of non-Abelian monopoles [15,16] play a crucial role. However, despite many efforts, the quantum nature of these objects is still quite mysterious. This is due to a number of reasons. For example, the residual non-Abelian dynamics is usually strongly coupled in the infrared, and semiclassical configurations cannot be trusted a-priori. Even when one circumvents this problem by considering infrared free theories, the standard semiclassical quantization of non-Abelian monopoles is still problematic [17,19] despite various attempts [20,21,16].

Despite these difficulties, there is a large amount of literature devoted to the construction of the most general configuration of classical monopoles. In the BPS limit, the second order equations of motion are simplified into the first order Bogomol’nyi equations, which admit a large set of continuously connected solutions called moduli space. Unfortunately, the explicit construction of monopole solutions is, in general, very difficult. Despite the complicate nature of the problem, a surprisingly large number of different approaches and auxiliary methods have been developed during the years to pursue this task. The Nahm transform [22], which is a direct adaptation of the Atiya-Drinfeld-Hitchin-Manin construction for instantons [23], and the spectral curve approach [24] are two such examples. Other alternative methods are
related to twistors \[25\] and integrable systems \[26\]. We will not discuss these methods in detail, rather we will make use of the rational map approach \[27\]–\[30\].

With the present paper we propose a different approach to study classical configurations of BPS non-Abelian monopoles. Following the ideas developed in Refs. \[31\] for the Abelian case, we put non-Abelian monopoles in the Higgs phase \[1\].

The minimal set-up for the construction of non-Abelian monopoles is an \(SU(N)\) pure Yang-Mills with an additional adjoint field. By embedding this monopole theory into a larger model, with the inclusion of matter fields in the fundamental representation and the extension of the gauge group to \(U(N)\), it is possible to track the fate of monopoles when we enter the Higgs phase. As well-known, the magnetic flux in the Higgs phase will be confined by flux tubes. Monopoles, for the same reason, will necessarily become kinks connecting different vortices \[33\]–\[31\]. This setup was successfully considered in Refs. \[34\]–\[35\] to give a physical explanation of the correspondence between the BPS spectra of two and four-dimensional gauge theories \[36\]. A strictly related fact is the equivalence of the spectrum of classical excitations of Abelian monopoles and kinks \[37\]. Moreover, it was found that the moduli spaces of Abelian monopoles aligned on a line and of domain walls are isomorphic \[38\]–\[39\].

The main purpose of this paper is the extension of this correspondence of moduli spaces of monopoles and kinks in the most general case, including the non-Abelian.

As a concrete mode, we consider \(\mathcal{N} = 2\) supersymmetric \(U(N)\) SQCD with \(N\) matter hypermultiplets. The theory has a Fayet-Iliopoulos (FI) parameter \(\xi\) which can be turned on to put the theory into the Higgs phase. The symmetry breaking pattern is controlled by the values of the hypermultiplets bare masses. When two or more masses coincide, the vacuum of the theory has, at the classical level, unbroken, non-Abelian symmetries. For vanishing FI, the theory is in the unbroken phase, and these symmetries are local. When we turn on the FI, the surviving symmetries are global. As we enter the Higgs phase, non-Abelian monopoles become “non-Abelian” kinks \[39\] interpolating between different type of non-Abelian vortices. As a consequence, we can identify the classical moduli space of non-Abelian monopoles with that of non-Abelian kinks. The construction of kinks is made in terms of the moduli matrix \[40\]. This allows us to propose a “moduli matrix construction for monopoles”. We also notice a close similarity between moduli matrices and rational maps. This observation provides us with a “physical” interpretation of the monopole rational map.

In section 2, we review the concept of non-Abelian monopoles and the determination of their moduli space in terms of rational maps. In section 3, we consider a supersymmetric theory which supports monopoles in the unbroken phase and a system of vortices and kinks in the Higgs phase. In section 4, we review the moduli matrix construction for kinks, and apply it to study their moduli spaces. In section 5, we compare the moduli spaces of monopoles and kinks in two particular cases, explicitly showing their equivalence. Finally, in section 6, we present and motivate a conjecture about the complete correspondence between the moduli space of kinks and monopoles in the most general case.

\[3\]The idea of considering non-Abelian monopoles in the Higgs phase is not new \[32\]. The main advantage of our set-up is that monopoles in the Higgs phase are still 1/4 BPS, while in the approach of Ref. \[32\] monopoles are confined by vortices in a non-BPS solitonic configuration.
2 Non-Abelian Monopoles in the Unbroken Phase

In this section we briefly review the concept of (non-Abelian) monopole as topological soliton supported by non-Abelian gauge theories. We also review the concept of moduli space, which is a crucial aspect in studying the dynamics of monopoles in the BPS saturated case.

2.1 Monopole solutions in spontaneously broken gauge theory

It is well-known, since the pioneering works of ’t Hooft [2] and Polyakov [3], that non-Abelian gauge theories with spontaneous symmetry breaking admit non-singular magnetic monopole solutions. The simplest example of such theories is pure SU(N) Yang-Mills with an additional adjoint scalar field φ:

\[
S = \int d^4x \left\{ \frac{1}{4g^2} (F^a_{\mu\nu})^2 + \frac{1}{g^2} |D_{\mu} \phi^a|^2 - \frac{\lambda}{4} (v - |\phi|^2)^2 \right\}
\]

(2.1)

with \(|A|^2 \equiv A^\dagger A\) for a square matrix \(A\). In the second line, we have written fields as matrices in the following way:

\[
F_{\mu\nu} = F_{\mu\nu}^a \tau^a, \quad A_\mu = A_\mu^a \tau^a, \quad \Phi = \sqrt{2} \phi^a \tau^a, \quad D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi], \quad \text{Tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}.
\]

(2.2)

The action contains a potential term which fixes the expectation value of the scalar field \(\Phi\), triggering a spontaneous breaking of the gauge symmetry.

\[
|\Phi_0|^2 = v/N.
\]

(2.3)

We will be particularly interested in the Bogomol’nyi-Prasad-Sommerfeld (BPS) limit [7,8]: \(\lambda \to 0\). In this case, a square root completion is possible [7]. For time independent configurations, the action (2.1) reduces to an energy and can be rewritten as

\[
E = \int d^3x \text{Tr} \left\{ \frac{1}{4g^2} (\epsilon_{ijk} F_{jk} - D_i \Phi)^2 + \frac{1}{g^2} \partial_i (\epsilon_{ijk} \Phi F_{jk}) \right\}.
\]

(2.4)

The energy is minimized when the first positively defined term is set to zero. This gives the Bogomol’nyi equations for BPS monopoles:

\[
D_i \Phi = \epsilon_{ijk} F_{jk}.
\]

(2.5)

The monopole mass is then given by the second term

\[
M = \frac{1}{g^2} \int d^3x \text{Tr} \partial_i (\epsilon_{ijk} \Phi F_{jk}) = \frac{1}{g^2} \int d^3x \text{Tr} \partial_i (B_i \Phi) = \frac{1}{g^2} \int_{S^2} dS^i \text{Tr} (B_i \Phi),
\]

(2.6)
where we have introduced the non-Abelian magnetic field $B_i \equiv \epsilon_{ijk} F_{jk}$. The last term is proportional to the magnetic flux of the configuration, and, as we shall see later in more detail, is proportional to the topological invariants associated with the monopole.

By an appropriate gauge transformation, the value of $\Phi$ at infinity can always be put into a diagonal form:

$$
\Phi_\infty = \begin{pmatrix}
\mu_1 & \ldots & 0 \\
0 & \ddots & \vdots \\
0 & \ldots & \mu_N
\end{pmatrix}, \quad \sum_{i=1}^{N} \mu_i = 0, \quad \mu_i \leq \mu_{i+1}.
$$

(2.7)

For generic values of the eigenvalues $\mu_i$, the gauge symmetry is maximally broken:

$$
SU(N) \to U_1(1) \times \cdots \times U_{N-1}(1),
$$

(2.8)

and monopoles are supported by the non-trivial homotopy group:

$$
\pi_2 \left( \frac{SU(N)}{U(1)^{N-1}} \right) = \pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1}.
$$

(2.9)

The topological sector is associated to $N - 1$ integers $m_i$ which correspond to the $N - 1$ magnetic charges of the monopole. From a physical point of view, these integers arise as a direct consequence of the Dirac quantization condition for the magnetic charge [1]. In the maximally broken case monopoles are called Abelian.

In contrast, when two or more $\mu_i$’s are equal, there is a surviving non-Abelian gauge symmetry:

$$
SU(N) \to S(U(n_1) \times U(n_2) \cdots \times U(n_q)), \quad \sum_{i=1}^{q} n_i = N
$$

(2.10)

and monopoles are supported by the homotopy group:

$$
\pi_2 \left( \frac{SU(N)}{S(U(n_1) \times U(n_2) \cdots \times U(n_q))} \right) = \mathbb{Z}^{q-1}.
$$

(2.11)

The case with $q = 2$ with $SU(N) \to SU(N - 1) \times U(1)$ is called minimal breaking. Monopoles in the non-Abelian case were originally constructed as embeddings of the $SU(2)$ solutions [15, 16]. Non-trivial solutions which cannot be obtained as embeddings have been also constructed as explicit solutions of the BPS equations [11].

Let us now review a more formal approach to deal with monopoles in the most general case of symmetry breaking. First, we can write the expectation value of $\Phi$ in terms of a vector $\vec{h}$:

$$
\Phi_\infty \equiv \vec{h} \cdot \vec{H}
$$

(2.12)

with generators $\vec{H}$ in the Cartan subalgebra of $G$. The symmetry breaking pattern is determined by the alignment of $\vec{h}$ with respect to the roots $\vec{\beta}_i$ of $SU(N)$. As well-known
from group theory, maximal symmetry breaking (2.8) is obtained when $\vec{h}$ has a non-zero projection along all of the roots

$$\vec{h} \cdot \vec{\beta}_i \neq 0 \quad \text{for each} \quad i = 1, \ldots, N - 1,$$

(2.13)

while a residual non-Abelian gauge symmetry remains unbroken as in (2.10) whenever $\vec{h}$ is orthogonal to one or more roots $\vec{\beta}_i$. By analyzing the Bogomol’nyi equations (2.5) together with the boundary conditions (2.12) it is possible to show that there is a gauge choice such that the asymptotic behavior for the adjoint field is the following [10, 42]:

$$\Phi \sim \Phi_\infty - \frac{G_0}{4\pi r} + O\left(\frac{1}{r^2}\right).$$

(2.14)

As a consequence of the Bogomol’nyi equations, $\Phi_\infty$ and $G_0$ commute [12], and, up to a gauge transformation, both of them can be chosen to belong to the Cartan subalgebra. $G_0$ is then related to the asymptotic value of the magnetic field:

$$B_r = \frac{G_0}{4\pi r^2} + O\left(\frac{1}{r^3}\right).$$

(2.15)

The matrix $G_0$, being an element of the Cartan subalgebra,

$$G_0 = \vec{g} \cdot \vec{H},$$

(2.16)

can be written in terms of a vector $\vec{m}$ of magnetic charges as below. A crucial early result about non-Abelian monopoles is the existence of a generalized Dirac quantization condition [10, 13, 11]. It states that the magnetic charge vector must belong to the dual root lattice of $SU(N)$:

$$\vec{g} = 4\pi \sum_{i=1}^{N-1} m_i \cdot \vec{\beta}^*_i,$$

(2.17)

where the integers $m_i$ are magnetic charges and the dual roots $\vec{\beta}^*_i$ are defined as usual:

$$\vec{\beta}^*_i = \frac{\vec{\beta}_i}{\vec{\beta}_i \cdot \vec{\beta}_i}.$$

(2.18)

It is important to recognize that this quantization condition is valid regardless of the specific symmetry breaking pattern defined by $\vec{h}$. A consequence of this is that only a subset of the $N - 1$ integers $m_i$ is related to a topological quantity [12]. First, $m_i$ can be divided into two classes $\{m_i\} = \{m_i^{\text{top}}, m_i^{\text{hol}}\}$ defined as follows. The charge $m_i$ corresponds to a topological integer if the root $\vec{\beta}_i$ defines a broken $SU(2)$ subgroup. In other words:

$$m_i^{\text{top}}: \quad \vec{\beta}_i \cdot \vec{h} \neq 0, \quad t = 1, \ldots, q - 1.$$

(2.19)

The remaining integers are called “holomorphic” charges and correspond to roots defining unbroken $SU(2)$ groups:

$$m_i^{\text{hol}}: \quad \vec{\beta}_i \cdot \vec{h} = 0, \quad h = 1, \ldots, N - q.$$

(2.20)
Using the above formulas we can evaluate the expression for the monopole mass,

\[
M_{\text{mon}} = \frac{2\pi}{g^2} \sum_{i=1}^{N-1} m_i \vec{h} \cdot \vec{\beta}_i = \frac{2\pi}{g^2} \sum_{t=1}^{q-1} m_{\text{top}}^t \vec{h} \cdot \vec{\beta}_t^*,
\]

which depends, as expected, only on the topological charges.

As we will discuss in more detail in the next session, BPS monopoles come as a continuous family of degenerate solutions of the equations of motion. This moduli space is given by the disjoint union of sectors labeled by the topological charges. On the other hand, holomorphic charges have a more subtle mathematical nature \[30\]. They may change under gauge transformations, nonetheless, they describe important properties of the monopole. Holomorphic charges define a stratification of each topological sector in connected subspaces. Their value changes discontinuously from one stratum to another.

2.2 Framed moduli spaces of monopoles

It is customary in literature to consider two different definitions of moduli spaces. Let us consider the set of field configurations \((A_i, \Phi)\) which satisfy the BPS equations and the boundary conditions \((2.11)\) and \((2.15)\). The stabilizer for \(\Phi_{\infty}\) is defined as the set of “unframed” gauge transformations which leave the adjoint field invariant (along some arbitrary chosen \(x_3\) direction):

\[
S_0(x) \in G : \quad S_0 \Phi(0, 0, x_3) = \Phi(0, 0, x_3).
\]

The moduli space space of “unframed” monopoles is thus defined as the set of gauge inequivalent configurations:

\[
\mathcal{M}_{\text{unframed}} = \{(A_i, \Phi) | (2.12)\} / S_0.
\]

For a fundamental \(SU(2)\) monopole, for example, it is just given by moduli related to spatial translations:

\[
\mathcal{M}^{SU(2)}_{\text{unframed}} = \mathbb{R}^3.
\]

However, it was soon realized, for the ’t Hooft-Polyakov monopole, that the \(S^1\) phase generated by the \(U(1)\) stabilizer has an important physical effect. In fact, upon quantization, this phase gives rise to an infinite tower of electrically charged states (dyons) \[44\]. The moduli space of the most physical interest is thus that of “framed” monopoles:

\[
\mathcal{M}_{\text{framed}} = \{(A_i, \Phi) | (2.12)\} / G_0,
\]

where the quotient is only taken with framed gauge transformation \(G_0\)

\[
G_0(x) \in G : \quad G_0(0, 0, x_3) \to 1, \quad x_3 \to \infty.
\]

In the \(SU(2)\) case, this definition correctly includes the relevant \(S^1\) phase:

\[
\mathcal{M}^{SU(2)}_{\text{framed}} = \mathbb{R}^3 \times S^1.
\]
The framed moduli spaces of monopoles capture important new features which arise in the non-Abelian case. Unframed gauge transformations $S_0$ generically do not leave the quantity $G_0$ invariant, and generate what may be called a “magnetic orbit” \cite{21}. The existence of these orbits gives rise to various subtleties in the quantization of non-Abelian monopoles. As can be intuitively seen from Eq. (2.14) the modes generated by variations of $G_0$ are non-normalizable, behaving as $\sim 1/r$ \cite{17}. This is problematic when one tries to apply standard quantization methods to the non-Abelian modes. Another related problem is the non-existence of a stabilizer $S_0$ which is globally defined on the whole two-sphere at spatial infinity \cite{18}. Moreover, as noticed in \cite{21,30}, the physical interpretation of holomorphic charges is also not completely clear. A reason for this is that monopole configurations with certain holomorphic charges cannot be considered as composite state of fundamental objects.

It is widely believed that a correct physical understanding of the moduli space of non-Abelian monopoles and its quantization would shed more light into the issue of dualities in non-Abelian gauge theories. We are not concerned here in this important issue, but rather we limit ourselves to consider the structure of the classical moduli space.

### 2.3 Rational map construction for the monopole moduli space

The rational map construction is based on the existence of a one-to-one correspondence between the moduli space of framed monopoles and the space of based rational maps from $\mathbb{C}P^1$ to a special class of flag manifolds\cite{30,28}: \[ R(z) : \mathbb{C}P^1 \to \text{Flag}_{n_1,\ldots,n_q}. \] (2.28)

These flag manifolds are given by the following homogeneous spaces \[ \text{Flag}_{n_1,\ldots,n_q} = \frac{SU(N)}{SU(n_1) \times \cdots \times SU(n_q)}. \] (2.29)

The quotient above can also be expressed in a complexified form: \[ \text{Flag}_{n_1,\ldots,n_q} = \frac{SL(N,\mathbb{C})}{P_{n_1,\ldots,n_q}}, \] (2.30)

where $P_{n_1,\ldots,n_q}$ is the parabolic group given by the set of upper-block-triangular complex matrices

\[
P_{n_1,\ldots,n_q} = \begin{pmatrix} P_{n_1} & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & 0 & P_{n_q} \end{pmatrix}.
\]

The quotient above can be realized by right side multiplication of an invertible matrix $f \sim fP$, and we can completely fix $P$ by putting $f$ into a lower-block triangular matrix, which gives a set of coordinates for the flag manifold

\[
f^l = \begin{pmatrix} 1_{n_1} & 0 & 0 \\ F_1 & \ddots & 0 \\ F_2 & F_3 & 1_{n_q} \end{pmatrix}.
\]

With based we mean that a chosen point of $\mathbb{C}P^1$ is mapped into a chosen point of the flag, for every map.
A based holomorphic rational map $R(z)$ is obtained by promoting the elements of $f^l$ to be ratios of polynomials. The condition that the map is based can be realized by imposing, for example, that all the ratios go to zero at large values of $z$

$$R(z) : f^l(z) = \begin{pmatrix} 1_{n_1} & 0 & 0 \\ R_1(z) & \ddots & 0 \\ R_2(z) & R_3(z) & 1_{n_q} \end{pmatrix}, \quad R_i(z) \to 0, \quad z \to \infty. \quad (2.31)$$

A crucial property of these rational maps is that their moduli space is partitioned in terms of topological numbers and stratified in terms of holomorphic integers exactly as the monopole moduli space [30,45]. Let us now be more concrete by considering the $SU(2)$ case originally considered by Donaldson [27]. We will follow Ref. [46] to describe the more constructive approach of Hurtubise [47], which makes use of the scattering data of the Hitchin equation [24]. Let us first consider an oriented straight line. This line will be parametrized by the coordinate $x_3$, while the orthogonal plane is described by the complex coordinate $z$. For each point on the plane, then, we consider the following Hitchin equation:

$$\nabla_M \psi = (D_3 + \Phi) \psi = 0 \quad (2.32)$$

where $\psi$ is a 2-component complex spinor field. The asymptotic value of $\Phi$ can be chosen to be proportional to $\tau^3$:

$$\Phi_{\infty} = 2\mu \tau_3 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}. \quad (2.33)$$

The Hitchin equation has 2 independent solutions, whose exponential behavior is dictated by the eigenvalues of $\Phi$. We thus have solutions which decay and grow exponentially for large $|x_3|$. We can approximately solve the Hitchin equation at large values of $|x_3|$ using the asymptotic expression given by Eq. (2.14):

$$\Phi = \left( 2\mu - \frac{m}{g|x_3|} \right) \tau^3 + O \left( \frac{1}{|x_3|^2} \right), \quad (2.34)$$

where $m$ is the monopole number. The solution which decay at large positive $x_3$ is thus asymptotically given by:

$$\psi(x_3) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} |x_3|^{m/g} e^{-\mu x_3}. \quad (2.35)$$

The same solution can be generally expressed at large negative $|x_3|$ in terms of the scattering coefficients $a$ and $b$:

$$\psi(x_3) \sim b \begin{pmatrix} 1 \\ 0 \end{pmatrix} |x_3|^{m/g} e^{\mu x_3} + a \begin{pmatrix} 0 \\ 1 \end{pmatrix} |x_3|^{m/g} e^{-\mu x_3}. \quad (2.36)$$

We are interested in the scattering coefficients as functions of $z$. First, we observe that the Hitchin operator commutes with the complex covariant derivative $D_z$

$$D_z \equiv \frac{1}{2}(D_1 + i D_2). \quad (2.37)$$
This follows from the BPS equations \((2.5)\)
\[
[D_z, \nabla_H] = -i(F_{13} + iF_{23}) + (D_1 + iD_2)\Phi = 0.
\]
(2.38)

At large values of \(|x_3|\), in the gauge choice made in Eq. (2.14), the covariant derivatives reduce to ordinary ones, \(D_z \to \partial_z\), and from the equations above it follows that
\[
\partial_z \nabla_H \psi = \nabla_H \partial_z \psi = 0,
\]
\[
\partial_z \psi = 0,
\]
(2.39)
which implies that the scattering coefficients \(a\) and \(b\) are holomorphic functions of the variable \(z\):
\[
a \to a(z), \quad b \to b(z).
\]
(2.40)

At large distances from the monopole, \(|z| \to \infty\), the Hitchin equation is trivial. This implies the boundary values for the coefficients:
\[
b(z)/a(z) \to 0, \quad z \to \infty.
\]
(2.41)

This condition, together with the assumption of analyticity and continuity, implies that \(a(z)/b(z)\) is a ratio of polynomials\(^4\):
\[
R(z) \equiv \frac{b(z)}{a(z)} = \frac{Q(z)}{P(z)} = \frac{q_1 z^{m-1} + \cdots + q_m}{z^m + p_1 z^{m-1} + \cdots + p_m}.
\]
(2.42)

The degree of the polynomials is determined by the monopole number \(m\), which is also the total magnetic charge of the corresponding configuration. The ratio \(R(z)\) is independent of the normalization of the solutions. It contains \(2m\) complex variables, which define the monopole moduli space. The expression above realizes the rational map construction for \(SU(2)\) monopoles, which according to Eq. (2.28) in this case is a map from \(\mathbb{C}P^1\) to \(\mathbb{C}P^1\).

The Donaldson construction gives a simple prescription for the determination of the monopole moduli space. It is quite non-trivial, however, to reconstruct the explicit solution starting by the data of the rational map. To this end, it is still more convenient to use the Nahm transform. The choice of a particular direction also hides most of the symmetries that a monopole configuration may have. For example, rotational symmetry is not manifest. The action of general symmetry transformations on the Donaldson rational map is still not known explicitly. There is an exception for translations and symmetries not broken by the choice of a line: rotations along the \(x_3\)-axis and reflections. Rotations of an angle \(\theta\) act on the rational map in the following way
\[
\gamma : R(z) \to R(e^{i\theta} z),
\]
(2.43)
while translations are given by
\[
\delta : R(z) \to e^{-\delta x_3 m/g} R(z - \delta z).
\]
(2.44)

\(^4\)It is possible to show that even at arbitrary small values of \(|x_3|\) the ratio \(R(z) = b(z)/a(z)\) is an holomorphic function of \(z\)
Reflections $\sigma : (z, x_3) \rightarrow (z, -x_3)$ act in a more involved way:

$$\sigma : R = \frac{Q}{P} \quad \rightarrow \quad \tilde{R} = \frac{\tilde{Q}}{P},$$

where $\tilde{Q}$ is given by the following relation:

$$Q \tilde{Q} = 1.$$

The monopole configuration is invariant if and only if the action of one of the above symmetry transformations is equivalent to a change of the framing of the map $[46]$:

$$R_i(z) \quad \rightarrow \quad e^{i\alpha} R_i(z).$$

The generalization to the non-Abelian case with minimal symmetry breaking is straightforward. The spinor $\psi$ will now have $N$ components, as many as the independent solutions of the Hitchin equation. The rational map will be given by a collection of $N - 1$ ratios $R_i(z) \equiv b_i(z)/a(z) (i = 1, \cdots, N - 1)$. The action of the unframed gauge transformations $S_0$ on the scattering coefficients can also be easily determined:

$$b_i \rightarrow U_{ij}b_j, \quad U \in S_0 = SU(N - 1) \times U(1).$$

Let us consider, for example, the rational map for a single $SU(N)$ monopole:

$$R^1_i = \frac{b_i}{z - z_0}.$$  \hspace{1cm} (2.49)

By the action of the transformations described above, it is possible to reduce the above map to the following form:

$$R^1_1 = \frac{1}{z}, \quad R^1_i = 0, \quad i \neq 1.$$  \hspace{1cm} (2.50)

which describe an embedded, unframed, $SU(2)$ 't Hooft-Polyakov monopole sitting at the origin. By using this information, it is possible to determine the physical meaning of the scattering coefficients $b_i$. The position of the monopole is given by:

$$(z_0, x_{3,0}) = \left( z_0, \frac{1}{4gN} \ln(|b_1|^2 + \cdots + |b_{N-1}|^2) \right),$$

while the set of ratios $b_i/b_j$ ($N - 2$ of them are independent) parameterize an $S^1$ fibration of $\mathbb{C}P^{N-2}$. We have the following result for the moduli space of fundamental $SU(N)$ monopole in the minimal symmetry breaking case:

$$\mathcal{M}^{SU(N), k=1}_{\text{framed}} = \mathbb{R}^3 \times (S^1_{\text{el}} \times \mathbb{C}P^{N-2}_{\text{mag}}) = \mathbb{R}^3 \times S^{2N-3}. \hspace{1cm} (2.52)$$

In the expression above we have used the terminology of Ref. [21] to distinguish between an “electrical” orbit generated by the Abelian factor and a “magnetic” orbit generated by the non-Abelian residual gauge symmetry. It is important to notice that the electric orbit is non-trivially fibered on the magnetic one. This fact is related to the difficulties to associate to non-Abelian monopoles well-defined algebraic objects, as required by quantum mechanics [21, 48].
2.4 Composite monopoles in $SU(3)$ gauge theory:

In this section we review in detail the moduli space of two $SU(3)$ monopoles, by explicitly constructing the associated rational maps. We follow the approach of [45].

Maximal Breaking: $SU(3) \to U(1) \times U(1)$

The moduli space of monopoles in this case is related to based rational maps into the following flag manifold:

$$\mathbb{F}lag_{1,1} = \frac{SU(3)}{U(1) \times U(1)}.$$  \hspace{1cm} (2.53)

As well known from mathematical literature, this space can be obtained in terms of a complexified quotient

$$\mathbb{F}lag_{1,1} = SL(3, \mathbb{C})/B_{1,1},$$ \hspace{1cm} (2.54)

where $B_{1,1}$ is the Borel group of upper triangular matrices. The space $\mathbb{F}lag_{1,1}$ can thus be realized as the set of 3 by 3 invertible matrices $M$, quotiented by the right action of $B_{1,1}$:

$$\mathbb{F}lag_{1,1} = \{M, M \sim MB_{1,1}; M \text{ invertible}\}. \hspace{1cm} (2.55)$$

Then we can completely fix $B$ by putting the matrix $M$ into the following lower triangular form:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}. \hspace{1cm} (2.56)$$

A based rational map can be simply obtained by promoting the coefficients $a, b$ and $c$ to be holomorphic functions of $z$ which vanish at infinity. The study of a composite configuration of two monopoles with the same charges, $(m_1, m_2) = (2, 0)$ or $(m_1, m_2) = (0, 2)$, can be reduced to that of $SU(2)$ monopoles. Because of this, here we just consider a composite monopole configuration with the following magnetic charge $(m_1, m_2) = (1, 1)$.

Since the rational map construction requires the choice of a preferred spatial direction we have two possibilities: the two monopoles can be aligned along the $x_3$ or separated on the $z$ plane. Let us start with the case of non-aligned monopoles. The correct rational map is given, as shown in Ref. [45], by

$$R_{(1,1)}^\text{sep}(z) = \begin{pmatrix} 1 & 0 & 0 \\ \beta/(z - z_1) & 1 & 0 \\ 0 & \alpha/(z - z_2) & 1 \end{pmatrix}, \hspace{1cm} \alpha, \beta \in \mathbb{C}^\ast. \hspace{1cm} (2.57)$$

It represents a $(0, 1)$ monopole located at the point $z_1$ and a $(1, 0)$ monopole located at $z_2$. There is no $(1, 0)$ or $(0, 1)$ monopole if either $\alpha$ or $\beta$ is zero. The moduli space is then:

$$\mathcal{M}_{(1,1)\text{sep}}^\text{mon} = (\mathbb{C} \times \mathbb{C}^\ast)^2. \hspace{1cm} (2.58)$$
The rational map representing a configuration of two aligned \((1,1)\) monopoles is, instead:

\[
R_{(1,1)}^{\text{align}}(z) = \begin{pmatrix}
1 & 0 & 0 \\
\beta/(z - z_0) & 1 & 0 \\
\gamma/(z - z_0) & \alpha/(z - z_0) & 1
\end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{C}^*, \quad \alpha \beta = 0. \tag{2.59}
\]

The conditions on the parameters are needed to ensure that the map describes the right topological sector. For example, if \(\alpha \beta \neq 0\), the topological number will be \((2,1)\). On the other hand, setting \(\gamma\) to zero would reduce the topological numbers to \((0,1)\) or \((1,0)\), in the case \(\alpha\) or \(\beta\) vanishes respectively. The two cases with either \(\alpha\) or \(\beta\) equal to zero can be interpreted as representing monopoles aligned along \(x_3\) with a different order. Setting \(\alpha = \beta = 0\) gives in fact a configuration of coincident monopoles. The moduli space in this case has complex dimension \(3\) and it is given by:

\[
\mathcal{M}_{(1,1)\text{align}}^{\text{mon}} = \{ \mathbb{C}(z_0) \times \mathbb{C}^2(\alpha, \beta) \times \mathbb{C}^*(\gamma) \mid \alpha \beta = 0 \} \tag{2.60}
\]

**Minimal Breaking: \(SU(3) \rightarrow SU(2) \times U(1)\)**

When two eigenvalues of \(\Phi\) coincide, say \(\mu_1 = \mu_2\), the unbroken symmetry is enhanced, and become non-Abelian. Moreover, of the two magnetic charges \((m_1, m_2)\), only \(m_2\) remains topological, while \(m_1\) is holomorphic. As a consequence, all the examples of the previous section degenerate into a fundamental non-Abelian monopole \((0,1)\).

Let us describe in detail what happens to the monopoles moduli space as we start from the maximally broken case of the previous section and we tune two masses to be coincident. The relevant flag is now

\[
\text{Flag}_{2,1} = SL(3, \mathbb{C})/B_{2,1}, \tag{2.61}
\]

where \(B_{2,1}\) is the upper block-triangular:

\[
B_{2,1} = \begin{pmatrix}
i & j & k \\
0 & l & m \\
0 & n & o
\end{pmatrix}
\]

It is easy to see that, with this enhanced symmetry we can always put \(b = 0\) in 2.56. This implies \(\alpha = 0\) in Eqs. (2.57) and (2.59). We can obtain the most general configuration for a fundamental non-Abelian monopole from the aligned case Eq. (2.59) only:

\[
R_1(z) = \begin{pmatrix}
1 & 0 & 0 \\
\beta/(z - z_0) & 1 & 0 \\
\gamma/(z - z_0) & 0 & 1
\end{pmatrix}, \quad \beta, \gamma \in \mathbb{C}^2 \backslash \{0,0\}. \tag{2.62}
\]

Notice that is now allowed to have \(\gamma = 0\). Setting \(\gamma\) to zero changes the value of the holomorphic charge \(m_1\) from 1 to 0, but leaves the topological charge unmodified. In physical

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5The monopole corresponding to (2.57) will reduce to a special non-Abelian monopole with \(\gamma = 0\).
terms, when $\gamma = 0$ the $(1, 0)$ monopole in the maximally broken case is sent to spatial infinity on the $x_3$ line. The moduli space can then be written as:

$$\mathcal{M}^{\text{mon}}_{(1)} = \{ \mathbb{C}(z_0) \times \mathbb{C}^2(\beta, \gamma) \setminus \{0, 0\} \} \sim \mathbb{R}^3 \times S^3,$$

which is the same result written in Eq. (2.52). \[\text{Eq. 2.63}\]

## 3 Non-Abelian Monopoles in the Higgs Phase

The most natural and convenient way to embed the monopole theory (2.1), in the BPS saturated case, is to consider a $U(N)$ gauge theory with extended $\mathcal{N} = 2$ supersymmetry. To ensure the existence of a supersymmetric vacuum, we also include $N_F = N$ fundamental hypermultiplets. The bosonic part of this model is:

$$S = \int d^4x \left\{ \frac{1}{4g^2}(F_{\mu\nu}^0)^2 + \frac{1}{4g^2}(F_{\mu\nu}^a)^2 + \frac{1}{g^2}D_\mu \phi^0|^2 + \frac{1}{g^2}D_\mu \phi^a|^2 + |\nabla_\mu q^A|^2 + g^2 \left( \frac{1}{2} \epsilon^{abc} \phi^b \phi^c + \bar{q}^A \frac{\tau^a}{2} q^A \right)^2 + g^2 (|q|^2 - N\xi)^2 + \frac{1}{2} \left( \phi^0 \frac{2}{\sqrt{2N}} + \phi^a \frac{\tau^a}{2} + \sqrt{2m_A} q^A \right) \right\},$$

with:

$$A = 1, 2, \ldots, N_F \quad \nabla_\mu = \partial_\mu - \frac{i}{\sqrt{2N}} A_\mu^0 - i \frac{\tau^a}{2} A_\mu^a.$$  \[\text{Eq. 3.1}\]

The real parameter $\xi$ is the Fayet-Iliopolous term [49]. As we will see shortly, a non-zero $\xi$ puts the theory into the Higgs phase. If the masses $m_A$ are taken real, we can consistently consider the adjoint fields $\phi^0, \phi^a$ to be real as well on the solitonic solutions. The above expression then simplifies:

$$S = \int d^4x \left\{ \frac{1}{4g^2}(F_{\mu\nu}^0)^2 + \frac{1}{4g^2}(F_{\mu\nu}^a)^2 + \frac{1}{g^2}D_\mu \phi^0|^2 + \frac{1}{g^2}D_\mu \phi^a|^2 + |\nabla_\mu q^A|^2 + g^2 \left( \frac{1}{2} \epsilon^{abc} \phi^b \phi^c + \bar{q}^A \frac{\tau^a}{2} q^A \right)^2 + g^2 (|q|^2 - N\xi)^2 + \frac{1}{2} \left( \phi^0 \frac{2}{\sqrt{2N}} + \phi^a \frac{\tau^a}{2} + \sqrt{2m_A} q^A \right) \right\}. $$

We can fit all the fields into $N \times N$ matrices

$$F_{\mu\nu}^0 \equiv \frac{F_{\mu\nu}^0}{\sqrt{2N}}, \quad F_{\mu\nu}^a \equiv \frac{F_{\mu\nu}^a}{\sqrt{2N}}, \quad \Phi \equiv \sqrt{2} \left( \phi^0 \frac{1}{\sqrt{2N}} + \phi^a \frac{\tau^a}{2} \right), \quad Q \equiv q^A_i.$$  \[\text{Eq. 3.3}\]
in terms of which the action (3.3) can be written in the following compact form

$$S = \int d^4x \text{Tr} \left\{ \frac{1}{2g^2} F_{\mu \nu}^2 + \frac{1}{g^2} |D_\mu \Phi|^2 + |\nabla_\mu Q|^2 + \frac{g^2}{4} (Q\bar{Q} - \xi)^2 + |\Phi Q + Q M|^2 \right\},$$  \hspace{1cm} (3.5)

where we have defined the mass matrix \( M \) as:

$$M_{AB} = \delta_{AB} m_A = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & m_N \end{pmatrix}, \quad m_1 \leq m_2 \leq \cdots \leq m_N. \hspace{1cm} (3.6)$$

Non-zero masses generically break the \( SU(N)_F \) flavor symmetry down to \( U(1)^{N-1}_F \). Notice that we can always absorb an equal contribution to the masses into a shift of the adjoint field \( \Phi \). With no loss of generality thus, we can always set \( \sum_{A=1}^N m_A = 0 \).

This model has a unique vacuum (up to gauge-flavor symmetry transformations)

$$\Phi_0 = -M, \quad Q = \sqrt{\xi} 1_N \hspace{1cm} (3.7)$$

invariant under a “color-flavor locked” global symmetry \( H_{C+F} \), which plays an important role in the study of the moduli space of solitons:

$$H_{C+F}(\Phi) = H_C \Phi H_C^{-1} = \Phi \quad H_{C+F}(Q) \equiv H_C Q H_F^{-1}, \quad H_C = H_F. \hspace{1cm} (3.8)$$

This residual color-flavor symmetry is determined by the vacuum value of \( \Phi \). In the most general case, it is given by the stabilizer of the adjoint field, similarly to what happens in the unbroken phase (see Eq. (2.10)):

$$H_{C+F} = S(U(n_1) \times U(n_2) \cdots \times U(n_q)), \hspace{1cm} (3.9)$$

in the case where there are \( q \) sets of fields with degenerate masses. The theory has two parameters with non-trivial mass dimension \( m \sim m_i \) and \( \sqrt{\xi} \). Playing with the relative value of these two parameters we can put the theory in two different regimes. When \( m \gg \sqrt{\xi} \), the symmetry breaking in the vacuum is:

$$U(N)_C \times SU(N)_F \overset{m}{\longrightarrow} U(1)_C \times H_C \times H_F \overset{\sqrt{\xi}}{\longrightarrow} H_{C+F}. \hspace{1cm} (3.10)$$

It supports “almost free” monopoles, with a typical size of order \( \Delta m^{-1} \), confined by very wide flux tubes with a width of order \( \sqrt{\xi} \). On the other hand, when \( m \ll \sqrt{\xi} \) we have:

$$U(N)_C \times SU(N)_F \overset{\sqrt{\xi}}{\longrightarrow} SU(N)_{C+F} \overset{m}{\longrightarrow} H_{C+F}. \hspace{1cm} (3.11)$$

In this second regime, the theory supports \( N \), very narrow, flux tubes. Monopoles are now squeezed into the flux tubes, and correspond to kinks, interpolating between different various string-like solitons [34,35]. When two or more masses are degenerate, the regime (3.10) will

\footnote{For convenience we chose the masses \( m_i \) to be all of the same order \( m \).}
supports confined non-Abelian monopoles, while the regime (3.11) will supports 2 different kinds of 1/2 BPS non-Abelian vortices and 1/4 kinks interpolating among them.

It is quite remarkable that the solitons in the two regimes can be studied by analyzing the same set of Bogomol’nyi equations, which follow from the square root completion the action (3.5) [31, 34, 35, 51, 40]:

\[
S = \int d^3x \, \text{Tr} \left\{ \frac{1}{g^2} \left( F_{12} - D_3 \Phi + \frac{g^2}{2} (Q \bar{Q} - \xi) \right)^2 + \right.
\]

\[+ \left. |\nabla_1 Q + i \nabla_2 Q|^2 + \frac{1}{g^2} (D_1 \Phi - F_{23})^2 + \frac{1}{g^2} (D_2 \Phi + F_{13})^2 + \right.
\]

\[+ \left. |\nabla_3 Q + \Phi Q + QM|^2 + \right. \xi F_{12} + \frac{1}{g^2} \partial_i (\epsilon_{ijk} \Phi F_{jk}) \right\}.
\]

The Bogomol’nyi equations are given, as usual, by imposing vanishing of the positive definite contributions:

\[
\nabla_1 Q + i \nabla_2 Q = 0 \quad \nabla_3 Q + \Phi Q + QM = 0 \quad D_1 \Phi - F_{23} = 0, \quad D_2 \Phi + F_{13} = 0 \quad F_{12} + D_3 \Phi + \frac{g^2}{2} (Q \bar{Q} - \xi) = 0
\]

The total mass is the sum of the last two terms in the action (3.12):

\[
M = \int d^3x \, \text{Tr} \left\{ \xi F_{12} + \frac{1}{g^2} \partial_i (\epsilon_{ijk} \Phi F_{jk}) \right\}.
\]

The first term is infinite and related to the tension of the string. It is proportional to the total Abelian flux flowing through the space:

\[
M_{\text{vort}} = \xi \int d^3x \, \text{Tr} \, F_{12} = 2\pi \xi L n^{(v)},
\]

where \(n^{(v)}\) is the total number of vortices and we have regulated the integration with a finite length \(L\) of the vortex. The second term gives the mass of the kinks and must be evaluated as a difference between the non-Abelian fluxes flowing through the two planes at positive and negative infinity

\[
M_{\text{kink}} = \frac{1}{g^2} \int d^3x \, \text{Tr} \, \partial_i (\epsilon_{ijk} \Phi F_{jk}) = \frac{1}{g^2} \int d^2x \, \text{Tr} \, ((\Delta B_3) \Phi),
\]

\[
\Delta B_3 = B_3(x_1, x_2, x_3 = +\infty) - B_3(x_1, x_2, x_3 = -\infty).
\]

\[10\] The theory admits a whole zoo of solitons, including domain walls, junctions of vortices on walls... [50,40]
The flux of $B_3$ can be determined, for large $|x_3|$, in terms of the vortex number. In the general case of symmetry breaking given by Eqs. (3.9) and (3.10) we have $q$ distinct topological numbers

$$n^{(v)} \equiv \sum_{t=1}^{q} n_t^{(v)}, \quad n_t^{(v)} \equiv \frac{1}{2\pi} \int d^2x \text{Tr} (B_3 \tau_t^0),$$  

(3.17)

where $\tau_t^0$ is a $U(1)$ generator in the unbroken $t$-th sector:

$$\tau_t^0 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{n_t} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

(3.18)

The total contribution to the mass coming from kinks is given in terms of the vortex numbers

$$M_{\text{kink}} = \frac{1}{g^2} \int \frac{d^2x}{2\pi} \text{Tr} ((\Delta B_3)\Phi) = \frac{2\pi}{g^2} \sum_t \Delta n_t^{(v)} \phi_t^0, \quad \Phi \equiv \phi_t^0 x_t^0, \quad (3.19)$$

where the symbol $\Delta$ always indicates the difference between the values of a quantity evaluated at positive and negative infinities. By comparing Eq. (2.21) with Eq. (2.11) we find a relationship between the vortex number and the monopole/kink charges:

$$\sum_t \Delta n_t^{(v)} \phi_t^0 = \sum_t n_t^{(k)} \vec{h} \cdot \vec{\beta}_t^* = \sum_t m_t^{\text{top}} \vec{h} \cdot \vec{\beta}_t^*. \quad (3.20)$$

To write the relations above we have used the fact that the masses of monopole and kinks coincide \[31, 35\]. Furthermore, we identify the kink numbers $n_t^{(k)}$ with the magnetic charges of the monopoles $m_t^{\text{top}}$. The number of kinks can also be easily defined in a more direct way if we recall that each fundamental kink interpolate between two vortices built in two neighboring $U(n_t)$ blocks. We thus may write the following relations:

$$n_t^{(k)} - n_{t-1}^{(k)} = \Delta n_t^{(v)}, \quad t = 1, \ldots, q - 1, \quad n_0^{(k)} = n_q^{(k)} = 0, \quad (3.21)$$

from which we get:

$$n_t^{(k)} = \sum_{l=1}^{t} \Delta n_l^{(v)};$$

$$n_t^{(k)} = \sum_{l=1}^{q-1} n_t^{(k)} = \sum_{l=1}^{q-1} (q - l) \Delta n_l^{(v)} = \sum_{l=1}^{q-1} l \Delta n_l^{(v)}.$$

(3.22)
4 Moduli Matrix Formalism for Kinks

The BPS equations (3.13) were first considered in [31, 35]. We follow the approach of Refs. [51, 54, 55, 40], where the moduli matrix technology was thoroughly developed to construct BPS solitonic configurations in the Higgs phase. The moduli matrix was first applied to domain walls [56, 39] and then extended to non-Abelian vortices [52, 54, 57–64] and BPS composite solitons [51, 65, 66]. The moduli matrix is believed to exhaust all possible solutions of the BPS equations, provided that a likely generalization of the so-called Hitchin-Kobayashi correspondence [67] holds in the non-compact case.

Let us start from the equation in the first line of the system (3.13). It can be solved by the following ansatz [51]:

\[
A_{\bar{z}} = i S^{-1} \partial_{\bar{z}} S, \quad Q(z, \bar{z}, x_3) = S^{-1}(z, \bar{z}, x_3) H_0(z) P(x_3),
\]

with \(S\) and \(P\) being invertible matrices depending only on the specified variables. \(S\) and \(H_0\) are defined modulo an important holomorphic “\(V\)-equivalence”:

\[
S(z, \bar{z}, x_3) \rightarrow V(z) S(z, \bar{z}, x_3), \quad H_0(z) \rightarrow V(z) H_0(z),
\]

where \(V(z)\) is an holomorphic matrix with determinant equal to one. The equation in the second line is also identically satisfied, with \(P\) explicitly determined in terms of \(x_3\) and the remaining adjoint fields expressed again in terms of the matrix \(S\):

\[
\partial_3 P + PM = 0 \Rightarrow P = e^{-M x_3} A_3 + i \Phi = i S^{-1} \partial_3 S.
\]

Remarkably, the equations on the third line are now identically satisfied with no further conditions on \(S\). In fact, this equation is the integrability condition for the system given by the first two lines (see Eq. (2.38)):

\[
[\nabla_{\bar{z}} \cdot, (\nabla_3 + \Phi) \cdot + \cdot M] = -i (F_{13} + i F_{23}) \cdot + (D_1 + i D_2) \Phi \cdot = 0.
\]

The full set of equations (3.13) is now reduced to the following “master equation” [51, 40]:

\[
4 \partial_2 (\Omega^{-1} \partial_2 \Omega) + \partial_3 (\Omega^{-1} \partial_3 \Omega) + g^2 (\Omega^{-1} \Omega_0 - \xi) = 0,
\]

which is nothing but the the last line of (3.13) written in terms of the gauge invariant quantities:

\[
\Omega = SS^\dagger, \quad \Omega_0 = H_0 PP^\dagger H_0^\dagger = H_0 e^{-2 M x_3} H_0^\dagger.
\]

We will assume the existence and uniqueness of the solution of the master equation. This assumption enables us to give a precise definition of the moduli space of the solitons supported by the model. It is given by the set of holomorphic “moduli matrices” \(H_0\), defined up to \(V(z)\)-equivalence relations:

\[
\bigoplus_{n_i^{(v)}, n_i^{(k)}} \mathcal{M}_{n_i^{(v)}, n_i^{(k)}} = \{H_0 \mid H_0 \sim VH_0\}.
\]

The notation above means that the full moduli space is a sum of disconnected topological sectors labelled by the number of vortices and kinks.

\[\text{Notice that the master equation can be solved algebraically in the } g \rightarrow \infty \text{ limit [68, 51, 40, 60] (strictly speaking, to obtain regular solutions we must consider a slight generalized model, with additional flavors, } N_F > N\). Moreover, numerical searches can be done, which confirm the uniqueness of the solution [69].]
4.1 Moduli spaces for kinks

Let us first review how to explicitly construct the moduli matrix for the vortex-kink system. We will then explain how to extract from it the numbers $n^{(v)}_t$, $n^{(k)}_t$.

First of all, we recall that for a generic holomorphic matrix $H_0$, we can completely fix the $V$-equivalence by putting the matrix in the following canonical form [54]:

$$H_0 = \begin{pmatrix} P_1(z) & Q_{1,2}(z) & \ldots & Q_{1,N}(z) \\ 0 & P_2(z) & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_N(z) \end{pmatrix},$$

(4.8)

where the $Q_{j,i}(z)$ are polynomials of degree less than that of the polynomials $P_i(z)$. To better understand the kink configuration described by the matrix above we need, first of all, to count the number of vortices at both infinities. From this knowledge, as described in the previous section, we can determine the number of kinks. To do this is convenient, as noticed in [40] to interpret the combination $H_0(z)P(x_3)$ introduced in Eq. (4.1) as an $x_3$ dependent moduli matrix for non-Abelian vortices. Using a $V$-transformation, we can actually consider the following moduli matrix, which includes the same informations about the kink moduli space as the original matrix in Eq. (4.8):

$$H^u_0(z, x_3) = P(x_3)^{-1}H_0(z)P(x_3) = \begin{pmatrix} P^u_1(z) & Q^u_{1,2}(z)e^{-(m_1-m_2)x_3} & \ldots & \vdots \\ 0 & P^u_2(z) & \ldots & Q^u_{2,N}(z)e^{-(m_2-m_N)x_3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P^u_N(z) \end{pmatrix}. \quad (4.9)$$

Thanks to our choice for the ordering of the masses ($m_i \leq m_j$ for $i < j$), the non-diagonal elements go to zero, or to a constant at most, for $x_3 \to -\infty$. The number $n^{(v)}_{-i}$ of vortices at negative infinity is related to the degree of $P_i^u(z)$:

$$P_i^u(z) \sim z^{n^{(v)}_{-i}}, \quad \text{for large } z,$n^{(v)}_{-i} = \sum_{i=1}^{N} n^{(v)}_{-i}. \quad (4.10)$$

Notice that the $n^{(v)}_{-i}$ all correspond to a topological integer only in the case of maximal symmetry breaking. In the general case, the topological vortex numbers defined above Eq. (3.22) are given by:

$$n^{(v)}_{-t} = \sum_{i=n_1+\ldots+n_{t-1}+1}^{n_{t+1}+\ldots+n_t} n^{(v)}_{-i}. \quad (4.11)$$

In the same way we can look at $x_3 \to +\infty$. In this case, the non-diagonal elements in Eq. (4.9) diverge. We can overcome the problem by using the $V$-equivalence to put the
matrix (4.9) into a lower triangular form

$$H_0^I(z, x_3) = V(z) H_0^u(z, x_3) =
\begin{pmatrix}
P_1^I(z) & 0 & \ldots & 0 \\
Q_{2,1}(z)e^{-(m_2-m_1)x_3} & P_2^I(z) & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\ldots & \ldots & Q_{N,2}(z)e^{-(m_N-m_2)x_3} & P_N^I(z)
\end{pmatrix}.$$ (4.12)

Notice that generically $P_i^u(z) \neq P_i^l(z)$, a crucial condition to have kinks. The number of vortices $n_{+i}^{(v)}$ at positive infinity is now given by the degree of $P_i^l(z)$:

$$P_i^l(z) \sim z^{n_{+i}^{(v)}}, \text{ for large } z,$$

$$n_{+i}^{(v)} = \sum_i n_{+i}^{(v)}, \quad n_{+l}^{(v)} = \sum_{i=n_1+\ldots+n_{l-1}+1} n_{+i}^{(v)}.$$ (4.13)

Of course, the total number of vortices is conserved:

$$n_{-}^{(v)} = n_{+}^{(v)} \equiv n^{(v)}.$$ (4.14)

The number of kinks will then be determined by Eq. (3.22).

It is possible to rewrite the formulas above in terms of $N$-component vectors $\phi_I(z)$, called orientational vectors, defined by the following condition [54, 40]:

$$H_0(z, x_3) \phi_I(z, x_3) = 0 \text{ Mod } \prod_i P_i(z),$$ (4.15)

The vectors $\phi_I$ are holomorphic functions of $z$ of degree at most $n^{(v)} - 1$. There are precisely $n^{(v)}$ linearly independent vectors, thus $I = 1, \ldots, n^{(v)}$. As it is clear from their defining condition, the vectors $\phi_I$ are defined up to a complex scaling: $\phi_I \sim \lambda_I \phi_I$ with $\lambda \in \mathbb{C}$. We can define the quantities $n_{-i}^{(v)}$ as the number of orientational vectors which at negative infinity are equivalent to $\phi_{0,i}^T \equiv (0, \ldots, 0, 1, 0, \ldots)$, where the only non-zero element is in the $i$-th position:

$$n_{-i}^{(v)} = \text{number of } \phi_I^T(z, -\infty) \sim \phi_{0,i}^T = (0, \ldots, 0, 1, 0, \ldots).$$ (4.16)

Similarly we define the vortex numbers at positive infinity:

$$n_{+i}^{(v)} = \text{number of } \phi_I^T(z, +\infty) \sim \phi_{0,i}^T = (0, \ldots, 0, 1, 0, \ldots).$$ (4.17)

Each vector can be interpreted as describing the orientation of a non-Abelian vortex. Comparing the expression of each $\phi_I$ at the two infinities gives us the number of kinks supported.

---

There is also a $GL(C, n^{(v)})$ equivalence due to the freedom of taking linear combinations of the vectors $\phi_I$. 

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by each vortex. If we have:

$$\phi^T_i(z, -\infty) \sim \phi^T_{0,i} = (0, \ldots, 0, 1, 0, \ldots), \quad \phi^T_i(z, +\infty) \sim \phi^T_{0,j} = (0, \ldots, 0, 1, 0, \ldots), \quad i \leq j,$$

the number of kinks supported by the $I$-th vortex will be:

$$n^{(k)}_{tI} = 0, \quad n^{(k)}_{tI} = 0, \quad j < n_1 + \cdots + n_t + 1,$$

$$n^{(k)}_{tI} = 1, \quad \text{otherwise.}$$

(4.19)

Again, it is easy to see that this definition gives the same result with Eq. (3.22).

Neutral Vortices

The reformulation in terms of orientational vectors enables us to give a precise definition of what we call “neutral vortices”. Neutral vortices do not support any kink, and can be decoupled completely by our system without changing the number and the type of kinks. Being able to remove these vortices is particularly useful when we compare the moduli space of kinks to the moduli space of monopoles in the unbroken phase, where vortices simply disappear from the spectrum.

We can identify two conditions for the existence of neutral vortices, that are relevant for the purpose of this paper. The first one requires that the moduli matrix (4.8) has a column with a common factor:

$$H_0(z, x_3) = \begin{pmatrix} \vdots & \cdots & p_i(z)Q'_{1,i}(z)e^{-(m_1-m_i)x_3} & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & p_i(z)P'_{i}(z) & \vdots \\ \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$  

(4.20)

Then, there exists the following orientational vectors:

$$\phi_{1,m}(z)^T = (0, \ldots, z^m P'_{i}(z), \ldots, 0), \quad 0 \leq m < \deg(p_i).$$

(4.21)

The orientational vectors above do not depend on the coordinate $x_3$, and remain the same at both infinities. This implies that the corresponding vortex does not support any kink. In fact, this same number of vortices cancels in the differences in Eq. (3.22). As a general statement, in fact, a neutral vortex corresponds to an $x_3$-independent orientational vector.

In the maximally broken case, the condition in (4.20) is sufficient to identify all neutral vortices. In the degenerate case, we have an additional situation. Consider, for example, the following moduli matrix:

$$H_0^u(z, x_3) = \begin{pmatrix} H^u_1(z, x_3) & 0 & \vdots \\ 0 & H^u_2(z) & \vdots \\ 0 & 0 & H^u_3(z, x_3) \end{pmatrix},$$

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where $H_u^u$ is an upper triangular moduli matrix located in a block corresponding to fields with degenerate masses. The elements of $H_u^u$ will not depend on $x_3$, and the same will be for the corresponding orientational vector:

$$\phi(z)^T = (0, \ldots, 0, \phi_2(z)^T, 0, \ldots, 0), \quad J = \sum_{i}^{j-1} n_i, \quad (4.22)$$

where $\phi_2(z)$ is an orientational vector for the moduli matrix $H_u^u$. In the following, we will always simplify the moduli matrix by removing the neutral vortices described above.

### 4.2 Coincident Kinks in $U(3)$ gauge theory

Let us now study the case of two kinks which corresponds to the monopoles of section 2.4. To be able to do so, we have to consider both the cases where the two kinks are separated on the $z$ plane (confined by distinct vortices) and where two kinks are aligned along $x_3$ (confined by a single vortex).

**Maximal breaking: $U(3) \rightarrow U(1)^3$**

As already explained, the number of vortices appearing in the Higgs phase is in principle arbitrary and determined by boundary conditions independent from the presence of kinks. Generically, the minimum number of vortices should be at least equal to the total number of kinks. Nevertheless, several kinks can be confined by a single vortex. In the Higgs phase they corresponds to a bead of at most $N - 1$ kinks (see Refs. [33, 40]).

We start from a configuration of two kinks separated on the $z_0$ plane. According to the counting in Eq. (4.19), we have to take the following moduli matrix:

$$H_u^{(1,1)}_{\text{sep}}(z, x_3) = \begin{pmatrix}
1 & \alpha e^{-(m_1 - m_2)x_3} & 0 \\
0 & z - z_2 & \beta e^{-(m_2 - m_3)x_3} \\
0 & 0 & z - z_1
\end{pmatrix}, \quad (4.23)$$

which corresponds to the rational map (2.57), and gives the same result for the moduli space.

$$\mathcal{M}^{\text{mon}}_{(1,1)\text{sep}} \equiv \mathcal{M}^{\text{kink}}_{(1,1)\text{sep}} \quad (4.24)$$

Let us consider now the case of coincident kinks. The most general configuration with at most two coincident vortices is described by the following moduli matrices. The first one is:

$$H_u^{(2,2)}_{\text{align}}(z, x_3) = \begin{pmatrix}
1 & 0 & (az + b)e^{-(m_1 - m_3)x_3} \\
0 & 1 & (cz + d)e^{-(m_2 - m_3)x_3} \\
0 & 0 & (z - z_0)^2
\end{pmatrix}, \quad (4.25)$$

Each vortex can support at most two kinks, thus the matrix above will generically describe a configuration with charges $(n_1^{(k)}, n_2^{(k)}) = (2, 2)$. According to Eq. (3.22), to have a configuration with charges $(1, 1)$ we need some constraint on the moduli parameters such that, at positive infinity, the matrix above reduce to the following

$$H_0^d(z, x_3) = \begin{pmatrix}
z - z_0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & z - z_0
\end{pmatrix} \quad \text{at} \quad x_3 \rightarrow +\infty. \quad (4.26)$$
This implies the existence of a neutral vortex. The constraint on the moduli parameters is thus the existence of a common factor \( z - z_0 \) on the same column. This enables us to reduce the degree of the polynomials by one, once we eliminate the neutral vortex:

\[
H_u^{(1,1)\text{align}}(z, x_3) = \begin{pmatrix}
1 & 0 & \gamma e^{-(m_1-m_3)x_3} \\
0 & 1 & \alpha e^{-(m_2-m_3)x_3} \\
0 & 0 & z - z_0
\end{pmatrix}.
\]

The same condition \([3.22]\) implies \( \gamma \neq 0 \). We may start from the following moduli matrix as well:

\[
H_u^{(2,1)\text{align}}(z, x_3) = \begin{pmatrix}
1 & \alpha e^{-(m_1-m_2)x_3} & \gamma e^{-(m_1-m_3)x_3} \\
0 & z - z_0 & \beta e^{-(m_2-m_3)x_3} \\
0 & 0 & z - z_0
\end{pmatrix}.
\]  

(4.27)

It describes generic configurations of kinks with charges \((2, 1)\). The condition to have a \((1, 1)\) configuration is that the matrix above reduces to the following at positive infinity

\[
H_d^0(z, x_3) = \begin{pmatrix}
z - z_0 & 0 & 0 \\
0 & z - z_0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{at } x_3 \to \infty.
\]

Using the \(V\)-equivalence explicitly, we can find the following conditions on the moduli parameters: \( \alpha \beta = 0 \) and again \( \gamma \neq 0 \). The case \( \alpha = 0 \) reduces to the previous one (there is an additional neutral vortex in the second position). The case \( \beta = 0 \) is new instead. The two cases differ by the order of alignment of the kinks along the vortices\(^{13}\). The case above correspond to the rational map \([2.59]\). The observation above implies the following:

\[
\mathcal{M}_{(1,1)\text{align}}^{\text{mon}} \equiv \mathcal{M}_{(1,1)\text{align}}^{\text{kink}} \quad (4.28)
\]

### Minimal breaking: \(U(3) \to U(2) \times U(1)\)

It is very simple to see what happens when we tune two masses to be equal \( m_1 = m_2 \). Expression \([4.28]\) reduce to

\[
H_u^{(1)}(z, x_3) = \begin{pmatrix}
1 & 0 & \alpha e^{-(m_1-m_2)x_3} \\
0 & 1 & \beta e^{-(m_2-m_3)x_3} \\
0 & 0 & z - z_1
\end{pmatrix}, \quad (4.29)
\]

which is nothing but a kink located at \( z_1 \) plus a neutral non-Abelian vortex located at \( z_2 \). Analogously to the monopole case, to obtain the most general kink we have to look at the matrix \([4.27]\). It the degenerate case, it corresponds to a neutral non-Abelian vortex plus a kink which can be described by the simplified matrix:

\[
H_u^{(1)}(z, x_3) = \begin{pmatrix}
1 & 0 & \gamma e^{-(m_1-m_3)x_3} \\
0 & 1 & \beta e^{-(m_2-m_3)x_3} \\
0 & 0 & z - z_0
\end{pmatrix}.
\]  

(4.30)

\(^{13}\)The order of kinks on a single vortex is fixed. To be able to invert this order, we need more coincident vortices to support the kinks.
For the matrix above to correctly describe a kink, we are allowed to consider both the following asymptotic expression at positive infinity:

\[
H^d_0(z, x_3) = \begin{pmatrix}
  z - z_0 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
  1 & 0 & 0 \\
  0 & z - z_0 & 0 \\
  0 & 0 & 1
\end{pmatrix}, \quad \text{at } x_3 \to \infty.
\]

This implies the condition \((\beta, \gamma) \neq (0, 0)\), and a perfect matching with the rational map (2.62) corresponding to a non-Abelian monopole:

\[\mathcal{M}^{\text{mon}}_{(1)} \equiv \mathcal{M}^{\text{kink}}_{(1)} \quad (4.31)\]

5 Correspondence of Moduli Spaces

We have already discussed in detail the correspondence of fundamental and composite object in the \(SU(3)\) case. In this section we use the rational map and the moduli matrix constructions for monopoles and kinks respectively, to show, in two additional examples, the isomorphism of the classical moduli spaces obtained by the two methods, when we correctly identify the monopole and kink numbers.

5.1 Separated monopoles/kinks in \(U(N)\) gauge theories

**Monopoles**

If we limit ourselves to study the moduli space of separate monopoles, we can describe it as a direct product of the moduli spaces of single charge monopoles. This product should be then quotiented by permutations of identical monopoles:\[^{14}\]

\[\mathcal{M}^{\text{mon}}_{\text{sep}} = \prod_{i=1}^{q-1} (\mathcal{M}^{m}_{i_1} \times \cdots \times \mathcal{M}^{m}_{i_n}) / \mathcal{P}_{m_t}, \quad (5.1)\]

where \(\mathcal{M}^{m}_{i_t}\) is the moduli space of the \(i\)-th single monopole in the \(t\)-th charge sector. The formula above can be considered as a consequence of the property of “addition” of rational maps with non-coincident poles [45].

We can construct a fundamental monopole for a gauge theory with the generic symmetry breaking \((2.10)\) by embedding an Abelian monopole constructed from the breaking \(SU(2) \to U(1)\) [15, 16, 21] into the larger gauge group. To construct a fundamental vortex, the broken \(SU(2)\) must be chosen such that the \(U(1) \subset SU(2)\) commutes with all the unbroken gauge symmetry, but not with two neighboring factors \(S(U(n_t) \times U(n_{t+1}))\). It is possible to show that this embedding results in a monopole configuration with a single non-vanishing topological charge \(m_t = 1\) related to the root \(\vec{\beta}\) corresponding to the broken \(SU(2)\). The holomorphic charges are also vanishing [70]. The moduli space in this case is the product of the translational sector \(\mathbb{R}^3\) times a non-Abelian quotient which parametrize \[^{14}\]

With identical monopoles we mean two objects with the same topological charge. The permutations exchange both spatial and internal (orientational) degree of freedom.
all the possible way to embed the Cartan subalgebra of the $SU(2)$ group into the product $S(U(n_t) \times U(n_{t+1}))$ [21]:

$$
\mathcal{M}_{(1),t}^{\text{mon}} = \mathbb{R}^3 \times S^1 \ltimes \left( \frac{SU(n_t)}{SU(n_t-1) \times U(1)} \times \frac{SU(n_{t+1})}{SU(n_{t+1}-1) \times U(1)} \right)
$$

$$
= \mathbb{R}^3 \times S^1 \ltimes (\mathbb{C}P_{\text{mag}}^{n_t-1} \times \mathbb{C}P_{\text{mag}}^{n_{t+1}-1}).
$$

(5.2)

The denominators in the quotients appear because they act trivially on the monopole configuration. Notice the existence of the non-trivial fibration of the magnetic orbit with the electric phase.

Kinks

Let us now switch to kinks. We show that the same expressions (5.1) and (5.2) give the moduli spaces of separated kinks. If we start from a configuration of well separated monopoles in the unbroken phase, we can always choose the $x_3$-direction such that, once we enter the Higgs phase, there are no monopoles/kinks aligned on the same vortex. All the kinks will be separated in the transverse $z$ plane, and thus confined by non-coincident vortices. As was shown in Refs. [54, 40], the moduli space of separated non-Abelian vortices splits into a symmetric product of single vortices. The same result trivially holds for kinks too:

$$
\mathcal{M}^{\text{kink}}_{\text{sep}} = \prod_{t=1}^{q-1} (\mathcal{M}^k_{n_t} \times \cdots \times \mathcal{M}^k_{n_t(n)}) / P_{n_t(n)},
$$

(5.3)

where $\mathcal{M}^k_{n_t}$ is the moduli space of the $i$-th non-Abelian kink in the $t$-th topological sector. We now show that:

$$
\mathcal{M}^k_{n_t} \equiv \mathcal{M}^m_{n_t}
$$

(5.4)

To study a fundamental kink we just need a single vortex supporting it. From (4.19) we see that the presence of a single kink requires an orientational vector which has non-zero elements only in two neighboring group of equal masses. This means that the fundamental kink can be constructed by embedding the kink present in the system $15$:

$$
U(n_t + n_{t+1}) \rightarrow U(n_t) \times U(n_{t+1}).
$$

(5.5)

A moduli matrix with the correct orientational vector is the following upper triangular submatrix:

$$
H^u_0(z, x_3) = \begin{pmatrix}
1_{n_t} & 0 & \tilde{b}_{n_t} e^{-(m_{t+1}-m_t)x_3} \\
0 & 1_{n_{t+1}-1} & \tilde{c}_{n_{t+1}-1} \\
0 & 0 & z - z_0
\end{pmatrix},
$$

(5.6)

15The embedding will not generate further moduli, because the kink will be invariant under the additional larger symmetries.
where $\vec{b}_{n_t}$ and $\vec{c}_{n_t+1-1}$ are vectors of moduli of lengths $n_t$ and $n_{t+1} - 1$, respectively. At negative infinity we have

$$H_0^u(z, x_3) = \begin{pmatrix} 1_{n_t} & 0 & 0 \\ 0 & 1_{n_t+1-1} & \vec{c}_{n_t+1-1} \\ 0 & 0 & z - z_0 \end{pmatrix}, \quad x_3 \to -\infty,$$

which corresponds to a non-Abelian vortex with orientations $\vec{c}_{n_t+1-1}$ for the gauge group factor $U(n_{t+1})$. The same matrix can be put into a lower triangular form using the $V$-equivalence:

$$H_0^d(z, x_3) = \begin{pmatrix} z - z_0 & 0 & 0 \\ \vec{b}_{n_t-1}' & 1_{n_t-1} & 0 \\ \vec{c}_{n_t+1}' e^{-(m_t-m_{t+1})x_3} & 0 & 1_{n_{t+1}} \end{pmatrix},$$

where the new variables are related to the old one by:

$$b_{n_t-1,t}' = b_{n_t,t+1}/b_{n_t,1}, \quad c_{n_t+1,t}' = c_{n_t,t}/b_{m_t,1}, \quad c_{n_t+1,n_{t+1}}' = 1/b_{n_t,1}.$$

The lower triangular matrix reduces at positive infinity to

$$H_0^d(z, x_3) = \begin{pmatrix} z - z_0 & 0 & 0 \\ \vec{b}_{n_t-1}' & 1_{n_t-1} & 0 \\ 0 & 0 & 1_{n_{t+1}} \end{pmatrix}, \quad x_3 \to +\infty,$$

which is a non-Abelian vortex constructed in the $U(n_t)$ gauge group factor. The moduli space of the non-Abelian kink is thus given by:

$$\mathcal{M}^{\text{kink}} = \mathbb{R}^2 \times \mathbb{R} \times S^1_{\text{phase}} \times (\mathbb{C}P^n_{\text{vort}} \times \mathbb{C}P^{n_{t+1}-1}_{\text{vort}}).$$

In the expression above, a factor of $\mathbb{R}$ is given by the position $x_{3,0}$ of the kink along the $x_3$ line, which can be estimated by the following condition [10]:

$$|\vec{b}_{n_t}|^2 e^{-2(m_{t+1}-m_t)x_{3,0}} \equiv 1 + |\vec{c}_{n_t+1-1}|^2, \quad \text{or equivalently}$$

$$1 + |\vec{b}_{n_t-1}|^2 \equiv |\vec{c}_{n_t+1-1}'|^2 e^{-2(m_t-m_{t+1})x_{3,0}}.$$

Notice that, as in the monopole case, the moduli space is completely given (excluding the translational moduli) by symmetries. However there exists a difference: while in the monopole case the moduli are generated by gauge symmetries, in the kink case they are generated by the color-flavor mixed global symmetries defined in Eq. (3.9).

The two $\mathbb{C}P$ factors in Eq. (5.11) are not part of the kink moduli space in the ordinary terminology but rather of the moduli spaces of the non-Abelian vortices attached to the monopole. However it is interesting to observe that they cannot be separated from a “genuine” kink moduli $\mathbb{R} \times S^1_{\text{phase}}$ because of the non-trivial fibering. From the point of view of kinks, these $\mathbb{C}P$ factors are the boundary moduli which are non-normalizable. Therefore we
simply call $\mathcal{M}^{\text{kink}}$ in (5.11) as “kink moduli” in this sense. The same structure was actually found for instantons trapped inside a non-Abelian vortex \[65\] \[16\].

Comparing Eqs. (5.2) and (5.11), we notice a precise correspondence between the magnetic orbits of non-Abelian monopoles in unbroken phase and the orientational moduli spaces of non-Abelian vortices attached to the monopole from both sides of the $x_3$-direction in the Higgs phase. That is, in the case of $SU(n_1 + n_2) \rightarrow SU(n_1) \times SU(n_2) \times U(1)$ we have

$$M_{\text{mag}}^{\text{mon}} = M_{U(n_1) \text{ vortex}} \times M_{U(n_2) \text{ vortex}}.$$ (5.13)

The relationship above extends the observation of matching of non-Abelian fluxes made in \[32, 53, 59\] to a one-to-one correspondence between non-Abelian monopole solutions and the non-Abelian vortices that confine them.

5.2 Minimal symmetry breaking

Monopoles

Let us analyze the case of minimal symmetry breaking $SU(N) \rightarrow SU(N-1) \times U(1)$. The flag manifolds introduced in Eq. (2.29) reduce to ordinary projective spaces:

$$\mathbb{F}_{\text{lag}_{N-1,1}} = \frac{SU(N)}{SU(N-1) \times U(1)} = \mathbb{C}P^{N-1}.$$ (5.14)

There is only one topological magnetic charge $m$ characterizing the number of monopoles, and $N-2$ holomorphic charges. A generic, based, rational map for the space $\mathbb{C}P^{N-1}$ can be constructed as a set of polynomials:

$$R_m : \mathbb{C}^1 \rightarrow \mathbb{C}P^{N-1};$$

$$R_m = (P(z), Q_1(z), \ldots, Q_{N-1}(z)), \quad \deg P(z) = m, \quad \deg Q_i(z) < m.$$ (5.15)

The zeroes of the polynomial $P(z)$ can be regarded as the positions of the $m$ monopoles in the $z$ plane \[16\], while the polynomials $Q_i(z)$, which cannot be all identically vanishing, include orientational degrees of freedom and positions of the monopoles along the $x_3$ axis.

As already emphasized, a crucial point in the identification of monopoles with rational maps is the correspondence between a stratification of the monopole moduli space, in terms of quantized holomorphic magnetic charges, and a stratification in the rational maps \[30\]. To give an explicit example, let us consider the case with two monopoles in $SU(3)$, with its corresponding stratification, following Ref. \[21\]. The value of the topological charge is $m = 2$. In this case there are two possible values of the holomorphic charge $m_{\text{hol}} = 0, 1$ \[30\]. The explicit rational map is:

$$R_2(z) = (z^2 + \alpha z + \beta, az + b, cz + d).$$ (5.16)

\[16\] The instantons can be regarded as sigma model lumps in the $\mathbb{C}P^{N-1}$ model on the vortex \[65][71\]. The moduli space for a single instanton inside a single vortex in $U(2)$ gauge theory in the Higgs phase was found in \[65\] to be $\mathcal{M}^{\text{lump}} = \mathcal{C}_{\text{vort}} \times \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}P^3_{\text{vort}}$. Here the “genuine” lump moduli $\mathbb{C} \times \mathbb{C}^*$ is made of the position moduli $\mathbb{C}$ and the size and phase moduli $\mathbb{C}^*$ of the single lump. The latter part is fibered over the boundary moduli $\mathbb{C}P^3_{\text{vort}}$ which is in fact the vortex orientational moduli.
The total moduli space has complex dimension 6. A generic point in this moduli space belongs to the large 6-dimensional stratum $S_{m,m_{hol}} = S_{2,1}$. Points belonging to the smaller stratum $S_{2,0}$ are defined by the following condition

$$\text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{Det} \, D \equiv 0.$$ 

The small stratum has thus complex dimension 5. It corresponds to configurations which can be embedded into a smaller $\mathbb{C}P^1$. As a consequence, these configurations can be obtained as embedding of $SU(2)$ monopoles.

### Kinks

Let us now determine the moduli space of kinks. A generic configuration containing $n^{(k)}$ kinks confined by the same number of vortices in the minimally broken case is given by the following $x_3$ dependent moduli matrix:

$$H_0^n(z, x_3) = \begin{pmatrix} 1 & \cdots & Q_1(z)e^{-(m_1-m_2)x_3} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_{N-1}e^{-(m_1-m_2)x_3} \\ 0 & \cdots & P(z) \end{pmatrix}, \quad \text{deg} \, P(z) = n^{(k)}, \quad \text{deg} \, Q_i(z) < n^{(k)}. \quad (5.17)$$

It is simple to check that, as soon as the $Q_i(z)$'s are not all vanishing, it is possible to put the matrix above in a lower triangular form:

$$H_0^d(z, x_3) = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & P(z) & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & Q_{N-1}e^{-(m_2-m_1)x_3} & \cdots & 1 \end{pmatrix}, \quad (5.18)$$

where the polynomials can lie in one of the first $N - 1$ columns. In terms of the analysis of the previous section, this ensures the presence of $n^{(k)}$ kinks. The matrix (5.17) exactly corresponds to the rational map describing non-Abelian monopoles in the minimal breaking case.

Let us work out explicitly the $SU(3)$ case with two kinks. The moduli matrix is

$$H_0(z, x_3) = \begin{pmatrix} 1 & 0 & (az + b)e^{-(m_1-m_2)x_3} \\ 0 & 1 & (cz + d)e^{-(m_1-m_2)x_3} \\ 0 & 0 & z^2 + az + \beta \end{pmatrix}. \quad (5.19)$$

It correctly represents a double kink, confined by two vortices centered along the zeros of the polynomial $z^2 + az + \beta$. It is also important to identify a concept of stratification for

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17 The three polynomials do not have common factors, as a consequence at least one of the parameters $a, b, c, d$ must be non-zero.

18 The moduli space of two $SU(2)$ monopoles has complex dimension 4. The smaller stratum has an additional complex parameter which determines how we can embed $SU(2)$ into the full $SU(3)$. 

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28
the moduli space of kinks, as defined by the moduli matrix. In the case at hand, the larger
stratum is defined by the set of points in the moduli space for which the matrix (5.19) can
be put into the following lower triangular form at positive infinity:

\[
H_0(z, +\infty) = \begin{pmatrix}
    z - \phi & \eta & 0 \\
    \tilde{\eta} & z - \tilde{\phi} & 0 \\
    0 & 0 & 1
\end{pmatrix}.
\]

(5.20)

Kinks corresponding to the case above correspond to monopoles that have a non vanishing
holomorphic charge \( n^{(k)}_{\text{hol}} = 1 \). The condition on the moduli parameters which gives the
smaller stratum is then the same as equation (5.2). In fact, as was shown in Ref. [57], the
condition \( \text{Det } D = 0 \) on the moduli matrix parameters implies parallel vortices which can be
embedded, together with the kinks that they support, into an \( SU(2) \) subgroup.

We notice here that the stratification of the composite \( SU(3) \) monopole considered in
this section corresponds to the stratification of the moduli space of two composite \( U(2) \) non-
Abelian vortices\(^{20}\) originally considered in Ref. [57,59,58]. In those works, the moduli space of
non-Abelian vortices was decomposed into submanifold and it was proposed to associate each
stratum to different representations of the color-flavor symmetry group. In the case at hand,
for example, two composite vortices associated with a fundamental representation of the
\( SU(2) \) group form a composite state with a moduli space which has two strata, associated
respectively with the singlet and the triplet representations. This issue was clarified in the
general case in Ref. [63], where the moduli space of \( k \) composite \( U(N) \) non-Abelian vortices is
decomposed into strata associated to all the \( k \)-tensor representations of \( SU(N) \). The Kähler
potential in each of this stratum was also computed, and it turns out to be proportional to
integer quantities. The size of these strata can be compared with a similar property of the
monopole strata [30].

6 Equivalence of the Moduli Matrix and the Rational
Map Construction

In the previous sections we have shown, in some special cases, that the moduli spaces of
monopoles and kinks are isomorphic. It is plausible that these results hold in the most
general case, with arbitrary number of monopoles and generic symmetry breaking pattern.
The physical reason behind this expectation is that, as already emphasized, monopoles and
kinks are essentially the same objects in two different phases of the theory. Supported by
the non-trivial checks we considered in this paper, we are led to state the following:

There is a one-to-one correspondence between:

1. The moduli space of non-Abelian monopoles, or equivalently,
   The moduli space of based rational maps into flag manifolds;

\(^{19}\)One can add a small mass difference for the first two flavors to transform the holomorphic charge into
a topological one. If we calculate the value of this second topological charge in the case (5.20) using the
formula given in the previous section we find \( n^{(k)}_{\text{hol}} = 1 \).

\(^{20}\)These vortices are exactly the ones required to confine the composite monopole in the Higgs phase.
2. The moduli space of non-Abelian kinks, or equivalently, 
The set of moduli matrices modulo $V$-equivalence;

provided that we eliminate, from the kink-vortex system, neutral vortices.

6.1 Hitchin and Bogomol’nyi equations

In this section we give a sketch of a proof for the correspondence. The technical link between 
the moduli matrix for kinks and the rational map for monopoles is the Hitchin operator 
\[(2.32)\]. It appears naturally in the second of the Bogomol’nyi equations \[(3.13)\]. In fact, after 
the field redefinition \[(4.1)\], the following equation

$$\nabla_3 Q + \Phi Q + Q M = 0 \quad (6.1)$$

reduces exactly to the Hitchin equation for the field $\Phi$:

$$\(\nabla_3 + \Phi\)\Psi = 0, \quad Q = \Psi P, \quad (6.2)$$

which must be solved with the following boundary condition:

$$|Q| = \sqrt{\xi} \quad \Rightarrow \quad |\Psi| = \sqrt{\xi} P^{-1}. \quad (6.3)$$

If we recall that we constructed the rational maps for monopoles by studying the scattering 
coefficient of the auxiliary Hitchin equation as an operator on an auxiliary field $\psi$ (Eq. 
\[(2.32)\]), we see that the construction of kinks reduces “almost” exactly to the same problem. 
Moreover, putting monopoles in the Higgs phase gives a simple physical interpretation for 
the rational map construction:

- The Hitchin equation naturally arises as part of the Bogomol’nyi equations for kinks.
- The auxiliary field $\psi$ is provided by the matter fields $\Psi$ introduced in order to enter 
  the Higgs phase.
- The arbitrary direction $x_3$ is the direction of formation of vortices.

There is, however, a difference when we analyze the Hitchin equation in the unbroken 
phase and in the Higgs phase. In the unbroken phase the Higgs field $\Phi$ has a polynomial tail 
which is proportional to the magnetic charges (see equation \[(2.14)\]). In the Higgs phase it 
has an exponential decay, determined by the Fayet-Iliopoulos term. The information about 
the topological charges of kinks must be determined, in this case, by comparing the vortex 
numbers at both infinities.

Notice, however, that the construction for kinks holds for arbitrary small values of the 
Fayet-Iliopolous term $\sqrt{\xi}$. In the regime $\sqrt{\xi} \ll |m_i - m_j|$, monopoles are weakly confined 
(the width of vortices is much larger than the size of monopoles), and it looks like that they 
are still in the unbroken phase. For very small $\xi$, indeed, the values of the matter fields 
are very small, and the Bogomol’nyi equations for kinks reduce, at first order, to those for 
free monopoles. In this regime we can then ignore the backreaction of the matter fields.
on the monopole, and directly apply the rational map construction for monopoles. The BPS equation for the matter fields, then, can be considered as the Hitchin equation on a fixed background, exactly as needed to interpret $\Psi$ as the scattering fields of the rational map construction. We can then rely on the fact that the parameter $\xi$ cannot change the dimension of the moduli space to claim the validity of the analysis at large $\xi$.

7 Discussion

We have studied the precise correspondence between the moduli spaces of monopoles in the unbroken phase and in the Higgs phase, including non-Abelian monopoles. The former described in the rational map construction has been found to coincide with the latter described by the moduli matrix formalism. Nontrivial fiber structure of an electric orbit over magnetic orbits in the moduli space of non-Abelian monopoles in the unbroken phase becomes the kink moduli fibered over the moduli of a non-Abelian vortex. We thus have found that the moduli space of monopoles coincides with the moduli space of kinks if we include the boundary moduli which are in fact the moduli of vortices attached to kinks.

In this paper we have studied monopoles in $SU(N)$ or $U(N)$ gauge theories. Changing gauge groups is one interesting extension. Especially the $SO$ and $USp$ cases have been studied for monopoles and also recently for vortices in [72,52,73], and there will be a similar relation between monopoles in unbroken and Higgs phases for $SO$ and $USp$ gauge theories. The case of arbitrary gauge groups [61] will be possible in principle.

The correspondence of monopoles and kinks studied in this paper can be extended to the one of Yang-Mills instantons and lumps. This is because in the Higgs phase instantons can stably exist inside a non-Abelian vortex [65,71]. Such trapped instantons can be regarded as lumps in the $\mathbb{C}P^{N-1}$ model. For the case of a single vortex, a similar correspondence can be understood from the work of Atiyah [74]; The moduli space of $SO(2)$ invariant $SU(2)$ Yang-Mills instantons, placed on a plane, is isomorphic to a space of a rational map into $\mathbb{C}P^1$. Extension to the case of instantons is surely interesting for instance in application to the instanton counting [75]. It is well known that monopoles and instantons are related by the Nahm transformation [22] or T-duality in the corresponding D-brane configuration. The same relation should hold when we enter the Higgs phase. In fact the T-duality between domain walls and vortices on a cylinder [76] and on a torus [77] was found already, and such a relation was found to hold for kinks and lumps inside a non-Abelian vortex [65].

Acknowledgments

W.V. would like to thank M. Shifman and Kenichi Konishi for their valuable comments on the preliminary version of the paper. The work of M.N. is partially supported by a Grant-in-Aid for Scientific Research No. 20740141 from the Ministry of Education, Culture, Sports, Science and Technology, Japan. The work of W.V. is supported by the DOE grant DE-FG02-94ER40823.
A Scattering data

The analysis of section 6 suggests, indeed, that the moduli matrix can be considered as an explicit realization of the scattering data of the Hitchin equation in the background of non-Abelian monopoles.

To reconstruct the moduli matrix, we simply patch together $N$ independent solutions of the Hitchin operator,

$$
\Psi_{\text{scatt}} = (\psi_1, \ldots, \psi_1, \ldots, \psi_N), \quad (A.1)
$$

which converge at negative infinity. We can chose, for example, the following set:

$$
\psi_i \sim \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \end{pmatrix} e^{m_i x_3}, \quad x_3 \to -\infty. \quad (A.2)
$$

The single non-zero field is in the $i$-th position. As already explained, we can track these solutions toward large positive values of $x_3$, and rewrite them in term of scattering data. The condition of holomorphicity of these data holds here too at large $|x_3|$ values. At positive infinity, the $\Psi$ matrix reads:

$$
\Psi_{\text{scatt}} \sim e^{M x_3} \begin{pmatrix}
1_i & \ldots & R_{1,i}(z) & \ldots & R_{1,N}(z) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
R_{i,1}(z) & 1_i & R_{i,N}(z) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
R_{N,1}(z) & \ldots & R_{N,i}(z) & \ldots & 1_N
\end{pmatrix}, \quad x_3 \to +\infty, \quad (A.3)
$$

where the $R_{i,j}(z)$ are rational holomorphic functions which tend to zero at large $z$. Furthermore, we may assume that all scattering coefficients $R_{i,j}$ of the same solution $\psi_j$ are continuous up to a finite number $n_i$ of points, the zeroes of the polynomial $P_j(z)$:

$$
R_{i,j} \equiv Q_{i,j}(z)/P_j(z). \quad (A.4)
$$

If we now fix the normalization of $\Psi_{\text{scatt}}$ using the boundary equation (6.3), we can directly identify $\Psi_{\text{scatt}}$ with $\Psi$:

$$
\Psi = \Psi_{\text{scatt}} \cdot \text{diag}(P_1(z), \ldots, P_N(z)) \quad (A.5)
$$

Given the relation $|\Psi| = |S^{-1}||H_0|$ we obtain $H_0$ and $S$ from $\Psi_{\text{scatt}}$.

$$
H_0 = \begin{pmatrix}
P_1 & \ldots & Q_{1,i}(z) & \ldots & Q_{1,N}(z) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
Q_{i,1}(z) & \ldots & P_i & \ldots & Q_{i,N}(z) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
Q_{N,1}(z) & \ldots & Q_{N,i}(z) & \ldots & P_N
\end{pmatrix}, \quad S \to e^{-M x_3}, \quad |x_3| \to \infty, \quad (A.6)
$$
The moduli matrix above is already in a form where the $V$-equivalence is completely fixed, and all the coefficients are true moduli of the configuration. If one wishes, one can put the matrices in an upper triangular form, to make full contact with the discussions in the bulk of the paper.

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