Effects of the Fourth Generation on $\Delta M_{B_{d,s}}$

in $B^0 - \bar{B}^0$ Mixing

Wu-Jun Huo

CCAST (World Lab.), P.O. Box 8730, Beijing 100080

and

Institute of High Energy Physics, Academia Sinica, P.O. Box 918(4), Beijing 100039, P.R. China

Abstract

We investigate the mass differences $\Delta M_{B_{d,s}}$ in the mixing $B^0_{d,s} - \bar{B}^0_{d,s}$ with a new up-like quark $t'$ in a sequential fourth generation model. We give the basic formulae for $\Delta M_{B_{d,s}}$ in this model. (1), then, we analysis the final numerical results of $\Delta M_{B_{d}}$, which is the function of $m_{t'}$ through a fourth generation CKM factor $V^*_{t'b}V_{t'd}$ constrained by the rare decay $B \rightarrow X_s l^+ l^-$ and two new Wilson coefficients $S_0(x_{t'})$ and $S_0(x_t,x_{t'})$. We found that one kind of the new results are satisfied with the present experimental low-bound of $\Delta M_{B_s}$. (2), we get the constraint of the fourth generation CKM factor $V^*_{t'b}V_{t'd}$, (which is also the function of $t'$), from the experimental measurements of $\Delta M_{B_d}$ and give the figs. of $V^*_{t'b}V_{t'd}$ to $m_{t'}$ and the general CKM factor $V^*_{tb}V_{td}$. We also give the numerical results of the Wilson coefficients $S_0(x_{t'}), S_0(x_t,x_{t'})$ and $\eta_{t'}, \eta_{tt'}$ as the function of $m_{t'}$. We also talk about the hierarchy of the fourth generation CKM matrix. As one of the directions beyond the SM, $\Delta M_{B_{d,s}}$ could provide a possible test of the fourth generation or perhaps a signal of the new physics.
1 Introduction

The Standard Model (SM) is a very successful theory of the elementary particles known today. But it must be incomplete because it has too many unpredicted parameters (nineteen!) to be put by hand. Most of these parameters are in the fermion part of the theory. We don’t know the source of the quarks and leptons, as well as how to determine their mass and number theoretically. We have to get their information all from experiment. There is still no successful theory which can be described them with a unified point, even if the Grand Unified Theory[1] and Supersymmetry[2]. Perhaps elementary particles have substructure (like preon)[3] and we need to progress more elementary theories. But this is beyond our current experimental level.

From the point of phenomenology, for fermions, there is a realistic question is number of the fermions generation or weather there are other additional quarks or leptons. The present experiments can tell us there are only three generation fermions with light neutrinos which mass are less smaller than $M_Z/2$[4] but the experiments don’t exclude the existence of other additional generation, such as the fourth generation, with a heavy neutrino, i.e. $m_{\nu_4} \geq M_Z/2$[5]. Many refs. have studied models which extend the fermions part, such as vector-like quark models[6], sterile neutrino models[7] and the sequential four generation standard model (SM4)[8] which we talk in this note. We consider a sequential fourth generation non-SUSY model[8], which is added an up-like quark $t'$, a down-like quark $b'$, a lepton $\tau'$, and a heavy neutrino $\nu'$ in the SM. The properties of these new fermions are all the same as their corresponding counterparts of other three generations except their masses and CKM mixing, see tab.1,

| up-like quark | down-like quark | charged lepton | neutral lepton |
|---------------|-----------------|---------------|---------------|
| $u$           | $d$             | $e$           | $\nu_e$       |
| $c$           | $s$             | $\mu$        | $\nu_\mu$     |
| $t$           | $b$             | $\tau$       | $\nu_\tau$    |
| new fermions  |                 | $\tau'$      | $\nu_{\tau'}$ |

Table 1: The elementary particle spectrum of SM4

There are a lot of refs. about the fourth generation. Some refs. devoted to the mass spectrum of the fourth generation particles[8], as well as some discussed the mass bounds of the fourth leptons[5]. There are many papers talked about other problems about the fourth generation[10, 11, 12] and the experimental search of the fourth generation particles[13, 14]. In our previous papers[10, 11], we investigated the rare $B$ meson decays with the fourth generation[14] and $\epsilon'/\epsilon$ in $K^0$ systems in SM4[14]. We got some interesting results, such as the new effects of the 4th generation particle on the meson decays and CP violation. We also got the constraints of the fourth generation CKM matrix factors, like $V_{ts}^*V_{tb}$ from $B \rightarrow X_s\gamma$[10] and $V_{ts}^*V_{td}$ from $\epsilon'/\epsilon[14]$. In other words, these rare decays provided possible test of the fourth generation existence, as well as CP violation.
In this note, we talk about the mass difference $\Delta M_{B_{d,s}}$ in $B^0 - \bar{B}^0$ with the fourth generation. We will give the prediction of $\Delta M_{B_{d}}$ in SM4 and the constraints of a new fourth generation CKM matrix factor $V_{t'd}^* V_{t'b}$ from $\Delta M_{B_{d}}$ in SM4. Like refs. [21, 16, 20], this note can also provide a test of 4th generation.

Particle-antiparticle mixing is responsible for the small mass differences between the mass eigenstates of neutral mesons, such as $\Delta M_K$ in $K_L - K_S$ mixing and $\Delta M_{B_{d,s}}$ in $B^0 - \bar{B}^0$ mixing. Being an FCNC process it involves heavy quarks in loops and consequently it is a perfect testing ground for heavy flavor physics. For example, $B^0 - \bar{B}^0$ mixing[17] gave the first indication of a large top quark mass. $K_L - K_S$ mixing is also closely related to the violation of CP symmetry which is experimentally known since 1964[18]. They are sensitive measures of the top quark couplings $V_{t'i}(i = d, s, b)$ and of the top quark mass $m_t$. The experimental measurements of $\Delta B_d$ is used to determine the CKM matrix elements $V_{td}[15]$. It offer an improved determination of the unitarity triangle with the future accurate measurement of $\Delta M_{B_s}[13, 16]$. For physics beyond the SM, there are a number of studies of the new physics effects in $B_d$ decays[13, 15, 20]. But $B_s$ system has received somewhat less attention from new physics point of view[21, 16, 20]. Experimentally, $\Delta M_{B_d}$ has been accurately measured, $\Delta M_{B_d} = 0.473 \pm 0.016(\text{ps})^{-1}[10, 22]$. But $\Delta M_{B_s}$ has only lower bound, $\Delta M_{B_s} > 14.3(\text{ps})^{-1}[13, 23, 22]$.

In this note, We want to investigate $\Delta M_{B_{d,s}}$ in $B^0 - \bar{B}^0$ mixing in SM4. First, if we add a sequential fourth up-like quark $t'$, there produce new prediction of the the mass difference $\Delta M_{B_{d,s}}$ through the new Wilson coefficients and a fourth generation CKM matrix factor $V_{t's}^* V_{t'b}$, which constrained for the rare decay $B \rightarrow X_s \gamma$ in [10]. We found, as like the analysis of $B \rightarrow X_s l^+l^-$ in [10], our results of the prediction of $\Delta M_{B_{d,s}}$ in SM4 are quite different from that of SM and can satisfy the lower experimental bound in one case of the values $V_{t's}^* V_{t'b}$ taken. In another case, it is almost the same as the SM one. The new effects of the fourth generation show clearly in the first case. Second, we can get the constraint of a fourth generation CKM matrix factor, $V_{t'b}^* V_{t'd}$ from the experimental measurement of $\Delta M_{B_d}$. We get one kind of reasonable analytical solutions of $V_{t'b}^* V_{t'd}$. They are very small, $1.0 \times 10^{-4} \leq V_{t'b}^* V_{t'd} \leq 0.5 \times 10^{-4}$. These result don’t contradicted the unitarity constraints for quark $d, s, b$[24]. Moreover, This small absolute value, $\lambda^{-4} \sim \lambda^{-5}$ order, is in agreement with the hierarchy in the CKM matrix elements[23, 26]. It seems to give the possible test of this hierarchy and the existence of the fourth generation[26]. We give the analysis of the hierarchy in the four generation CKM matrix.

In sec. 2, we give the basic formulae for the mass difference $\Delta M_{B_{d,s}}$ in $B^0 - \bar{B}^0$ with the sequential fourth generation up-like quark $t'$ in SM4 model. In sec. 3, we give the prediction of mass difference $\Delta M_{B_s}$ in SM4 and the numerical analysis. Sec. 4 is devoted to the numerical analysis of one fourth generation CKM matrix factor $V_{t'b}^* V_{t'd}$ from the experimental measurements of the mass difference $\Delta M_{B_d}$ in SM4. We also analyze the hierarchy of the four generation CKM matrix in this section. Finally, in sec. 5, we give our conclusion.
2 Basic formulae for $\Delta M_{B_{d,s}}$ with $t'$

$B_{d,s}^0 - B_{d,s}^0$ mixing proceeds to an excellent approximation only through box diagrams with internal top quark exchanges in SM. In SM, the effective Hamiltonian $\mathcal{H}_{\text{eff}}(\Delta B = 2)$ for $B_{d,s}^0 - B_{d,s}^0$ mixing, relevant for scales $\mu_b = \mathcal{O}(m_b)$ is given by \[13\]

$$
\mathcal{H}_{\text{eff}}^{\Delta B=2} = \frac{G_F^2}{16\pi^2} M_W^2 (V_{tb}^* V_{tq})^2 S_0(x_t) Q(\Delta B = 2) + \text{h.c.}
$$

(1)

where $Q(\Delta B = 2) = (\bar{b}_q q_\alpha) v_A (\bar{b}_3 q_3) v_A$, with $q = d, s$ for $B_{d,s}^0 - B_{d,s}^0$ respectively and $S_0(x_t)$ is the Wilson coefficient which is taken the form \[13\]

$$
S_0(x_t) = \frac{4x_t - 11x_t^2 + x_t^3}{4(1-x_t)^2} - \frac{3}{2} \cdot \frac{x_t^3}{(1-x_t)^3} \cdot \ln x,
$$

(2)

where $x_t = m_t^2/M_W^2$. The mass differences $\Delta M_{d,s}$ can be expressed in terms of the off-diagonal element in the neutral $B$-meson mass matrix

$$
\Delta M_{d,s} = \begin{cases} 2|M_{12}^{t,s}| \\ 2m_{B_{d,s}} |M_{12}^{t,s}| = |\langle \bar{B}_{d,s}^{0} | \mathcal{H}_{\text{eff}}(\Delta B = 2) | B_{d,s}^{0} \rangle |. \\
\end{cases}
$$

(3)

Now, we turn to the case of SM4. If we add a fourth sequential fourth generation up-like quark $t'$, the above equations would have some modification. There exist other box diagrams contributed by $t'$ (see fig. 1), similar to the leading box diagrams in MSSM \[20\].

The effective Hamiltonian in Standard Model, eq.(1), change into the form \[27\],

$$
\mathcal{H}_{\text{eff}}^{\Delta B=2} = \frac{G_F^2}{6\pi^2} M_W^2 m_{B_d}(\hat{B}_{B_d} \hat{F}_{B_d}) [\eta_t (V_{tb}^* V_{tq})^2 S_0(x_t) + \\
\eta_t' (V_{t_b}^* V_{t_q})^2 S_0(x_t') + \eta_{tt'} (V_{t_b}^* V_{t_q}) (V_{tb} V_{tq}) S_0(x_t, x_t')] Q(\Delta B = 2) + \text{h.c.}
$$

(4)

The mass differences $\Delta M_{d,s}$ in SM4 can be expressed

$$
\Delta M_d = \frac{G_F^2}{6\pi^2} M_W^2 m_{B_d}(\hat{B}_{B_d} \hat{F}_{B_d}) [\eta_t (V_{tb}^* V_{td})^2 S_0(x_t) + \\
\eta_t' (V_{t_b}^* V_{t_d})^2 S_0(x_t') + \eta_{tt'} (V_{t_b}^* V_{t_d}) (V_{tb} V_{td}) S_0(x_t, x_t')] + \\
\Delta M_s = \frac{G_F^2}{6\pi^2} M_W^2 m_{B_s}(\hat{B}_{B_s} \hat{F}_{B_s}) [\eta_t (V_{tb}^* V_{ts})^2 S_0(x_t) + \\
\eta_t' (V_{t_b}^* V_{t_s})^2 S_0(x_t') + \eta_{tt'} (V_{t_b}^* V_{t_s}) (V_{tb} V_{ts}) S_0(x_t, x_t')]
$$

(5)

where $(\hat{B}_{B_d} \hat{F}_{B_d}) = \xi_{t}^2 \cdot (\hat{B}_{B_d} \hat{F}_{B_d})$. The new Wilson coefficients $S_0(x_t')$ present the contribution of $t'$, which like $S_0(x_t)$ SM in eq. (5) except exchanging $t'$ quark not $t$ quark. $S_0(x_t, x_t')$ present the contribution of a mixed $t - t'$, which is taken the form \[28\]

$$
S_0(x, y) = x \cdot y \left[ -\frac{1}{y-x} \left( \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{1-x} - \frac{3}{4} \cdot \frac{1}{(1-x)^2} \right) \ln x + \\
+ (y \leftrightarrow x) - \frac{3}{4} \cdot \frac{1}{(1-x)(1-y)} \right].
$$

(7)
where \( x = x_t = m_t^2/M_W^2 \), \( y = x_{t'} = m_{t'}^2/M_W^2 \). The numerical results of \( S_0(x_t') \) and \( S_0(x_t, x_{t'}) \) is shown on the tab. 2.

| \( m_t (GeV) \) | 50   | 100  | 150  | 200  | 250  | 300  | 350  | 400  | 450  | 500  |
|----------------|------|------|------|------|------|------|------|------|------|------|
| \( S_0(x_t') \) | 0.33 | 1.07 | 2.03 | 3.16 | 4.44 | 5.87 | 7.47 | 9.23 | 11.15| 13.25|
| \( S_0(x_t, x_{t'}) \) | 0.48 | -7.03| -9.49| -5.09| -5.39| -5.87| -5.99| -6.25| -6.49| -6.72|
| \( m_{t'} (GeV) \) | 550  | 600  | 650  | 700  | 750  | 800  | 850  | 900  | 950  | 1000 |
| \( S_0(x_t') \) | 15.52| 17.97| 20.60| 23.41| 26.40| 29.57| 32.93| 36.47| 40.96| 44.11|
| \( S_0(x_t, x_{t'}) \) | -6.92| -7.11| -7.28| -7.44| -7.60| -7.74| -7.99| -8.12| -8.23| -8.23|

Table 2: The Wilson coefficients \( S_0(x_t') \) and \( S_0(x_t, x_{t'}) \) to \( m_{t'} \)

The short-distance QCD correction factors \( \eta_{t'} \) and \( \eta_{tt'} \) can be calculated like \( \eta_c \) and \( \eta_{ct} \) in the mixing of \( K^0 - \bar{K}^0 \), which the NLO values are given in refs\[13, 27\], relevant for scale not \( O(\mu_c) \) but \( O(\mu_b) \). In leading-order, \( \eta_t \) is calculated by

\[
\eta_t^0 = [\alpha_s(\mu_t)]^{(6/23)} \alpha_s(\mu_t) = \alpha_s(M_Z)[1 + \sum_{n=1}^{\infty} (\beta_0 \alpha_s(M_Z)^{2}\ln \frac{M_Z}{\mu_t})^n],
\]

with its numerical value in tab. 3. The formulae of factor \( \eta_{t'} \) is similar to the above equation except for exchanging \( t \) by \( t' \). In ref.\[26\], to leading order, \( \eta_{tt'} \) was taken as

\[
\eta_{tt'}^0 = [\alpha_s(\mu_t)]^{(6/23)} \frac{\alpha_s(\mu_{t'})}{\alpha_s(\mu_t)} = [\alpha_s(\mu_{t'})]^{(6/19)},
\]

with value 0.58 for the same as for \( \eta_{tt'}^K \) in \( K^0 - \bar{K}^0 \). For simplicity, we take \( \eta_{tt'} = \eta_{t'} \). We give the numerical results in tab.4.

| \( m_{t'} (GeV) \) | 50   | 100  | 150  | 200  | 250  | 300  | 350  | 400  | 450  | 500  |
|----------------|------|------|------|------|------|------|------|------|------|------|
| \( \eta_{t'} \) | 0.968| 0.556| 0.499| 0.472| 0.455| 0.443| 0.433| 0.426| 0.420| 0.416|
| \( m_{t'} (GeV) \) | 550  | 600  | 650  | 700  | 750  | 800  | 850  | 900  | 950  | 1000 |
| \( \eta_{t'} \) | 0.412| 0.408| 0.405| 0.401| 0.399| 0.396| 0.395| 0.393| 0.391| 0.389|

Table 3: The short-distance QCD factors \( \eta_{t'} \), \( \eta_{tt'} (= \eta_{t'}) \) to \( m_{t'} \)

In the last of this section, we give other input parameters necessary in this note. (See the following tab.).

### 3 Prediction of \( \Delta M_{B_s} \) with \( t' \)

Experimentally, the mass difference \( \Delta M_{B_s} \) of the \( B^0 - \bar{B}^0 \) mixing is unclear. It has only low bound, \( \Delta M_{B_s}^{\text{exp}} > 14.3(\text{ps})^{-1} \[14, 23\]. We have given the calculation formula of
Table 4: Numerical values of the input parameters[16].

| $m_c(m_c(pole))$ | $1.25 \pm 0.05$ GeV |
|-------------------|----------------------|
| $m_t(m_t(pole))$ | 175 GeV |
| $\Delta M_{B_d}$ | $(0.473 \pm 0.016) (ps)^{-1}$ |
| $\Delta M_{B_s}$ | $> 14.3( ps)^{-1}$ |
| $M_W$ | 80.2 GeV |
| $\hat{F}_{B_d} \sqrt{B_{B_d}}$ | 215 \pm 40 MeV |
| $\xi_s$ | 1.14 \pm 0.06 |
| $G_F$ | $1.166 \times 10^{-5}$ GeV$^{-2}$ |

$\Delta M_{B_s}$ in eq. (6) and the numerical results of Wilson coefficients $S_0$ and QCD correction coefficients $\eta$. Now, if we constrain the fourth generation CKM factor $V_{l_b}^{*} V_{l_s}^{*}$, we can predict $\Delta M_{B_s}$ in our four generations model. Fortunately, from our previous paper[10], we have given the constraints of $V_{l_b}^{*} V_{l_s}^{*}$ from experimental measurements of $B \rightarrow X_s \gamma$. Here, we give only the basic scheme and the final numerical results.

The leading logarithmic calculations can be summarized in a compact form as follows[13]:

$$R_{\text{quark}} = \frac{Br(B \rightarrow X_s \gamma)}{Br(B \rightarrow X_s \gamma)} = \frac{|V_{t_s}^{*} V_{l_b}^{*}|^2}{|V_{t_s}^{*} V_{l_b}^{*}|^2} \left( \frac{|V_{c_b}^{*} V_{b}^{*}|^2}{\pi f(z)} \right) \left| C_{7,8}^{(\text{eff})}(\mu_b) \right|^2.$$

In the case of four generation there is an additional contribution to $B \rightarrow X_s \gamma$ from the virtual exchange of the fourth generation up quark $t'$. The Wilson coefficients of the dipole operators are given by

$$C_{7,8}^{(\text{eff})}(\mu_b) = C_{7,8}^{(\text{SM})}(\mu_b) + \frac{V_{t_s}^{*} V_{l_b}^{*}}{V_{t_s}^{*} V_{l_b}^{*}} C_{7,8}^{(4)\text{eff}}(\mu_b),$$

where $C_{7,8}^{(4)\text{eff}}(\mu_b)$ present the contributions of $t'$ to the Wilson coefficients, and $V_{t_s}^{*} V_{l_b}^{*}$ are the fourth generation CKM matrix factor which we need now. With these Wilson coefficients and the experiment results of the decays of $B \rightarrow X_s \gamma$ and $Br(B \rightarrow X_s e \bar{\nu}_e)$[29,24], we obtain the results of the fourth generation CKM factor $V_{t_s}^{*} V_{l_b}^{*}$. There exist two cases, a positive factor and a negative one:

$$V_{t_s}^{+} V_{l_b}^{(+)} = \left[ C_7^{(0)\text{eff}}(\mu_b) - C_7^{(\text{SM})\text{eff}}(\mu_b) \right] \frac{V_{t_s}^{*} V_{l_b}^{*}}{C_7^{(\text{eff})}(\mu_b)},$$

$$V_{t_s}^{+} V_{l_b}^{(-)} = \left[ \sqrt{R_{\text{quark}} \left| V_{c_b}^{*} V_{b}^{*} \right|^2 \pi f(z)} - C_7^{(\text{SM})\text{eff}}(\mu_b) \right] \frac{V_{t_s}^{*} V_{l_b}^{*}}{C_7^{(\text{eff})}(\mu_b)},$$

as in table 3. With these values, we can give the prediction of $\Delta M_{B_s}$ in SM4 by the figs. 2. It is very interesting that the final analytic result is same as that in decay of $B \rightarrow X_s l^+ l^-[10]$.

The mass difference $\Delta M_{B_s}$ in the two cases of $V_{t_s}^{*} V_{l_b}^{*}$ are shown in the figs. 2(a) and 2(b) respectively. In the first case, which the value of $V_{t_s}^{*} V_{l_b}^{*}$ takes positive, i.e. $(V_{t_s}^{*} V_{l_b}^{+})$, the
Table 5: The values of $V_{t's}^* \cdot V_{t'b}$ due to masses of $t'$ for $Br(B \rightarrow X_s \gamma) = 2.66 \times 10^{-4}$

| $m_t$ (Gev) | 50  | 100 | 150 | 200 | 250 | 300 | 350 |
|-------------|-----|-----|-----|-----|-----|-----|-----|
| $V_{t's}^* V_{t'b}^{(+)} / 10^{-2}$ | -11.591 | -9.259 | -8.126 | -7.501 | -7.116 | -6.861 | -6.580 |
| $V_{t's}^* V_{t'b}^{(-)} / 10^{-3}$ | 3.564 | 2.850 | 2.502 | 2.309 | 2.191 | 2.113 | 2.205 |
| $m_t$ (Gev) | 400 | 500 | 600 | 700 | 800 | 900 | 1000 |
| $V_{t's}^* V_{t'b}^{(+)} / 10^{-2}$ | -6.548 | -6.369 | -6.255 | -6.178 | -6.123 | -6.082 | -6.051 |
| $V_{t's}^* V_{t'b}^{(-)} / 10^{-3}$ | 2.016 | 1.961 | 1.926 | 1.902 | 1.885 | 1.872 | 1.863 |

curve of $\Delta M_{B_s}$ to $m_{t'}$ is almost overlap with that of SM. That is, the results in SM4 are the same as that in SM, except a peak in the curve when $m_{t'}$ takes value about 170GeV. The reason is not because there is new prediction deviation from the SM but only because there is a term of $(x - y)$ in denominator of the formulae of eq. (7). In this case, it does not show the new effects of $t'$. The mass difference $\Delta M_{B_s}$ is not possible to be obtained. Also, we cannot obtain the information of existence of the fourth generation from $\Delta M_{B_s}$, although we cannot exclude them either. This is because, from tab. 5, the values of $V_{t's}^* V_{t'b}^{(-)}$ are positive. But they are of order $10^{-3}$ and is very small. The values of $V_{t's}^* V_{t'b}$ are about ten times larger than them ($V_{t's}^* = 0.038, V_{t'b} = 0.9995$, see ref. [24]). Furthermore, the last two terms about $m_{t'}$ in eq. (6) are approximately same order. The contribution of them counteract each other.

But in the second case, when the values of $V_{t's}^* V_{t'b}$ are negative, i.e. $(V_{t's}^* V_{t'b}^{(-)})$. The curve of $\Delta M_{B_s}$ is quite different from that of the SM. This can be clearly seen from fig. 2(b). The enhancement of $\Delta M_{B_s}$ increases rapidly with increasing of $t'$ quark mass. In this case, the fourth generation effects are shown clearly. The reason is that $V_{t's}^* V_{t'b}^{(+)10}$ is 2-3 times larger than $V_{t's}^* V_{t'b}$ so that the last two terms about $m_{t'}$ in eq. (6) becomes important and it depends on the $t'$ mass strongly. Thus, the effect of the fourth generation is significant.

Meanwhile, The prediction of $\Delta M_{B_s}$ in SM4 can satisfy the experimental low bound of $\Delta M_{B_s} \geq 14.3 (ps^{-1})$. So, the sequential fourth generation model could be one of the ways of searching new physics about $\Delta M_{B_s}$. If $V_{t's}^* V_{t'b}$ choose this case, the mass difference $\Delta M_{B_s}$ in $B^0 - \bar{B}^0$ mixing could be a good probe to the existence of the fourth generation.

## 4 Constrains of the fourth generation CKM factor $V_{t's}^* V_{t'd}$ from experimental measurements of $\Delta M_{B_d}$

Unlike $\Delta M_{B_s}$, the mass difference $\Delta M_{B_d}$ of $B^0_d - \bar{B}^0_d$ mixing is experimental clear, $\Delta M_{B_d}^{exp} = 0.473 \pm 0.016 (ps^{-1})$ [10]. We can get the constraints of the fourth generation CKM factor $V_{t's}^* V_{t'd}$ from the present experimental value of $\Delta M_{B_d}$.
We change the form of eq. (5) as a quadratic equation about \( V^*_{t'd} V_{t'd} \). By solving it, we can get two analytical solutions \( V^*_{t'd} V_{t'd}^{(1)} \) (absolute value is the large one) and \( V^*_{t'd} V_{t'd}^{(2)} \) (absolute value is the small one), just like the other 4th generation CKM matrix factor \( V_{t's} V_{t'b}^{(\pm)} \) in last section.

However, experimentally, it is not accurate for the measurement of CKM matrix element \( V_{td}[15, 24] \). So, we have to search other ways to solve this difficulty. Fortunately, the CKM unitarity triangle[30], i.e. the graphic representation of the unitarity relation for \( d, b \) quarks, which come from the orthogonality condition on the first and third row of \( V_{CKM} \),

\[
V_{ud} V^*_{ub} + V_{cd} V^*_{cb} + V_{td} V^*_{tb} = 0, \tag{14}
\]

can be conveniently depicted as a triangle relation in the complex plane, as shown in the fig. 3(a).

From the above equation, we can give the constraints of \( V_{td} V^*_{tb} \),

\[
0.005 \leq |V_{td} V^*_{tb}| \leq 0.013 \tag{15}
\]

Then, we give the final results as shown in the figs. 4(a) and 4(b).

We must announce that figs. 4 only show the curves with \( V^*_{t'd} V_{t'd}^{(2)} \) (absolute value is the small one) firstly. Because the absolute value of \( V^*_{t'd} V_{t'd}^{(1)} \) is generally larger than 1. This is contradict to the unitarity of CKM matrix. So, we don’t think about this solution. From the figs. 4, we found all curves are in the range from \(-1 \times 10^{-4}\) to \(0.5 \times 10^{-4}\) when we considering the constraint of \( V_{td} V^*_{tb} \). That is to say, the absolute value of \( V^*_{t'd} V_{t'd} \) is about \(\sim 10^{-4}\) order. This is a very interesting result.

First, these CKM matrix elements obey unitarity constraints. With the fourth generation quark \( t' \), eq. (12) change to,

\[
V^*_{ud} V_{ub} + V^*_{cd} V_{cb} + V_{td} V^*_{tb} + V^*_{t'd} V_{t'b} = 0. \tag{16}
\]

This a quadrilateral, (see fig. 3(b)). We take the average values of the SM CKM matrix elements from Ref. [24]. The sum of the first three terms in eq. (12) is about \(\sim 10^{-2}\) order. If we take the value of \( V^*_{t's} V_{t'b} \) the result of the left of (14) is better and more close to 0 than that in SM, when \( V^*_{t'd} V_{t'b} \) takes negative values. Even if \( V^*_{t's} V_{t'b} \) takes positive values, the sum of (16) would change very little because the values of \( V^*_{t'd} V_{t'b} \) are about \(\sim 10^{-4}\) order, two orders smaller than the sum of the first three ones in the left of (14). Considering that the data of CKM matrix is not very accurate, we can get the error range of the sum of these first three terms. It is much larger than \( V^*_{t'd} V_{t'b} \). Thus, in the case the values of \( V^*_{t'd} V_{t'b} \) satisfy the CKM matrix unitarity constraints.

Second, this small order of \( V^*_{t'd} V_{t'b} \) doesn’t contradict to the hierarchy of the CKM matrix elements or the quarks mixing angles[24, 32]. Moreover, it seem to prove the hierarchy. The hierarchy in the quarks mixing angles is clearly presented in the Wolfenstein
parameterization of the CKM matrix. Let’s see CKM matrix firstly,

\[
V_{\text{CKM}} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} & \cdots \\
V_{cd} & V_{cs} & V_{cb} & \cdots \\
V_{td} & V_{ts} & V_{tb} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \sim \begin{pmatrix}
1 & \lambda & \lambda^2 & \cdots \\
-\lambda & 1 & \lambda^2 & \cdots \\
\lambda^3 & -\lambda^2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(17)

with \(\lambda = \sin^2 \theta = 0.23\). Now, the hierarchy can be expressed in powers of \(\lambda\). We found, the magnitudes of the mixing angles are about 1 among the same generations, \(V_{ud}, V_{cs}\) and \(V_{tb}\). For different generations, the magnitudes are about \(\lambda\) order between 1st and 2nd generation, \(V_{us}\) and \(V_{cd}\), as well as about \(\lambda^2\) order between 2nd and 3rd generation, \(V_{cb}\) and \(V_{ts}\). The magnitudes are about \(\lambda^3\) order between the 1st and third generation, \(V_{ub}\) and \(V_{td}\). Then, there should be an interesting problem: If the fourth generation quarks exist, how to choose the order do the magnitude of the mixing angles concern the fourth generation quarks? Because there is not direct experimental measurement of the fourth generation quark mixing angles, one have to look for other indirect methods to solve the problem. Many ref.s have already talked about these additional CKM mixing angles, like the vector-like quark models, the four neutrinos models and the sequential four generations models. For simple, we give a guess for the magnitude of the fourth generation mixing angles. Similar to the general CKM matrix elements magnitude order, the fourth generation ones are about \(\lambda^4 \sim \lambda^5\) order between the 1st and 4th generation, such as \(V_{t'd'}\), as well as \(\lambda^2 \sim \lambda^3\) between the 2nd and 4th generation, such as \(V_{t's'}\). For the mixing between the 3rd and 4th generation quarks, such as \(V_{t'b}\), we take the magnitude as 1 because the mass of the fourth generation quark \(t'\) is the same order, \(10^2\), as the top quark \(t\). So \(V_{t'b}\) should take the order of \(V_{tb}\). Then, the magnitude order of the fourth generation CKM factor \(V_{t'd'}V_{t'b}^*\) is about \(\lambda^4 \sim \lambda^5\), i.e. \(< \lambda^4\). From figs. 4, we found that the numerical results, \(V_{t'd'}V_{t'b}^{(2)}\), satisfy this guess.

At last, the factor \(V_{t'd'}V_{t'b}\) constrained from \(\Delta M_{B_d}\) does not contradict to the CKM matrix texture. Moreover, it seem to support the existence of the fourth generation.

5 Conclusion

In this note, we have investigated the mass differences \(\Delta M_{B_{d,s}}\) in the mixing \(B_{d,s}^0 - \bar{B}_{d,s}^0\) with a new up-like quark \(t'\) in a sequential fourth generation model. We give the basic formulae for \(\Delta M_{B_{d,s}}\) in this model and calculated the new Wilson coefficients \(S_0(m_{t'}/m_W)\) and \(S_0(m_{t'}/m_W)\) in the effective \(\Delta S = 2\) Hamiltonian. We also calculated the short distance QCD factors \(\eta_{t'}\) and \(\eta_{t't'}\). With these values, first, we analysis the final numerical results of \(\Delta M_{B_s}\), which is the function of \(m_{t'}\) through a fourth generation CKM factor \(V_{t's'}^*V_{t'b}\). This factor was constrained from the rare decay \(B \to X_s\gamma\) and had two kinds of numerical results. We found the new results in SM4 were almost same as in SM with one case of \(V_{t's'}^*V_{t'b}\) numerical values and satisfied with the present experimental low-bound of \(\Delta M_{B_s}\) in other case. We gave the figs of \(\Delta M_{B_s}\) to \(m_{t'}\) and the numerical analysis. Second, we investigated the mass difference \(\Delta M_{B_d}\) to get the constraint of the other fourth
generation CKM factor $V_{tb}'^{\ast} V_{td}'$, (which is the function of $t'$), from the experimental measurements of $\Delta M_{B_d}$. We first got the constraints of the general CKM factor $V_{tb}^{\ast} V_{td}$ from the CKM unitarity triangle. Then we give the figs. of $V_{tb}'^{\ast} V_{td}'$ to the mass of $t'$ and to the general factor $V_{td}^{\ast} V_{tb}$ and the numerical analysis too. We also talked about the texture of the fourth generation CKM matrix.

The fourth generation quark $t'$ will give obviously new effects on the mass difference $\Delta M_{B_{d,s}}$ if it really exists. At least, the present experimental statue of $\Delta M_{B_{d,s}}$ could not exclude the space of the fourth generation. Furthmore, the progress of theoretical calculation and experimental measurement $\Delta M_{B_{d,s}}$ could provide the strong test of the existence of the fourth generation. In other words, as one of the directions beyond the SM, $\Delta M_{B_{d,s}}$ could provide a possible test of the fourth generation or perhaps a signal of the new physics.

Acknowledgments

This research was supported in part by the National Nature Science Foundation of China. I am grateful to prof. C.S. Huang and Prof. Y.L. Wu for useful discussions and valuable modification and comments on the manuscript.

References

[1] P. Langacker, Phys. Rep. 72, No. 4, (1981) 185.

[2] M.F. Sohnius, Phys. Rep. 128, No. 2&3 (1985) 39.

[3] L. Lyons, Prog. Part. Nucl. Phys. 10 (1983) 227; C. Heuson, hep-ph/9904493; and references therein.

[4] G.S. Abrams et al., Mark II Collab., Phys. Rev. Lett. 63 (1989) 2173; B. Advera et al., L3 Collab., Phys. Lett. B 231 (1989) 509; I. Decamp et al., OPAL Collab., ibid., 231 (1989) 519; M.Z. Akrawy et al., DELPHI Collab., ibid., 231 (1989) 539; C. Caso et al., (Particle Data Group), Eur. Phys. J.C 3 (1998) 1.

[5] Z. Berezhiani and E. Nardi, Phys. Rev. D 52 (1995) 3087; C.T. Hill, E.A. Paschos, Phys. Lett. B 241 (1990) 96.

[6] Y. Nir and D. Silverman, Phys. Rev. D 42 (1990) 1477; W-S, Choong and D. Silverman, Phys. Rev. D 49 (1994) 2322; L.T. Handoko, hep-ph/9708447.

[7] V. Barger, Y.B. Dai, K. Whisnant and B.L. Young, hep-ph/9901380; R.N. Mohapatra, hep-ph/9702229; S. Mohanty, D.P. Roy and U. Sarkar, hep-ph/9810309; S.C. Gibbons, et al., Phys. lett. B 430 (1998) 296; V. Barger, K. Whisnant and T.J. Weiler, 10
Phys. lett. B427, (1998) 97; V. Barger, S. Pakvasa, T.J. Weiler and K. Whisnant, Phys. Rev. D58 (1998) 093016.

[8] J.F. Gunion, Douglas W. McKay, H. Pois, Phys. Lett. B 334 (1994) 339; Phys. Rev. D 51 (1995) 201.

[9] J. Swain, L. Taylor, hep-ph/9712383; R.N. Mohapatra, X. Zhang, hep-ph/9301286; V. Novikov, hep-ph/9606318.

[10] C.S. Huang, W. J. Huo and Y.L. Wu, Mod. Phys. Lett. A14 (1999)2453.

[11] C.S. Huang, W. J. Huo and Y.L. Wu, hep-ph/0005227.

[12] L. L. Smith, D. Jain, hep-ph/9501291; K.C. Chou, Y.L. wu, and Y.B. Xie, Chinese Phys.Lett. 1 (1984) 2. A. Datta, hep-ph/9411433; T. Yoshikawa, Prog. Theor. Phys. 96 (1996) 269; D. Grosser,Phys. Lett. B86 (1979) 301; C.D. Froggatt, H.B. Nielsen, D.J. Smith, Z. Phys. C 73 (1997) 333; S. Dimopoulos, Phys. Lett. B 129 (1983) 417.

[13] LEP1.5 Collab., J. Nachtman, hep-ex/960615.

[14] DØ Collab., S. Abachi et al., Phys. Rev. Lett. 78 (1997) 3818.

[15] A.J. Buras, hep-ph/9806471.

[16] A. Ali and D. London, hep-ph/0002167.

[17] H. Alberecht, et. al., Phys. Lett. B192 (1987) 245; M. Artuso, et. al, Phys. Rev. Lett. 62 (1989) 2233.

[18] J.H. Christenson, J.W. Cronin, V.L. Fitch and R. Turlay, Phys. Rev. Lett. 13 (1964) 128.

[19] Y. Grossman and M. Worah, Phys. Lett. B395, (1997) 241; M.Worah, hep-ph/9711266; N.G. Deshpande, B. Dutta and S. Oh, Phys. Rev. Lett. 77, (1996) 4499; M. Ciuchini, et. al., Phys. Rev. Lett. 79, (1997) 978; D. London and A. Soni, Phys. Lett. B407, (1997) 61; A. Abd El-Hady and G. Valencia, Phys. Lett. B414, (1997) 173; J. P. Silva and L. Wolfenstein, Phys. Rev. D55, (1997) 5331; A.I. Sanda and Z.Z. Xing, Phys. Rev. D56 (1997) 6866; S.A. Abel, W.N. Cottingham and I.B. Whittingham, Phys. Rev. D58, (1998) 073006.

[20] I. Hinchliffe and N. Kersting, hep-ph/0003090.

[21] G. Barenboim, J. Bernabeu, J. Matias and M. Raidal, hep-ph/9901263; Y. Grossman, Phys. Lett. B380 (1996) 99; Z.Z. Xing, Eur. Phys. J. C4 (1998) 283; Y. Grossman, Y. Nir and R. Rattazzi, hep-ph/9701234.

[22] G. Blaylock, http://www.cern.ch/LEPBOSC/.

[23] S. Willocq, hep-ex/0002053.

[24] C.Caso et al., (Particle Data Group), Eur. Phys. J. C3 (1998) 1.
[25] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.

[26] T. Hattori, T. Hasuike and S. Wakaizumi, Phys. Rev. D60 (1999) 113008.

[27] S. Herrich and U. Nierste, hep-ph/9604330, hep-ph/9310311; S. Herrich, hep-ph/9609378.

[28] J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model (Cambridge University Press, New York, 1992).

[29] M.S. Alam, Phys. Rev. Lett. 74, (1995) 2885.

[30] A. Ali, hep-ph/9606324; hep-ph/9612262.

[31] G. Barenboim, G. Eyal and Y. Nir, hep-ph/9905397.

[32] M. Leurer, Y. Nir, N. Seiberg, Nucl. Phys. B420 (1994) 468; P. Kielanowski, et al., hep-ph/0002062.
Figure 1: The Additional Box Diagrams to $B_{d,s}^0 - \bar{B}_{d,s}^0$ with the fourth up-like quark $t'$. 
Figure 2: Prediction of $\Delta M_{B_s}$ to $m_{t'}$ in SM4 when $V_{t's}^*V_{t'b}$ takes (a) positive and (b) negative values.

![Prediction of $\Delta M_{B_s}$ to $m_{t'}$ in SM4](image)

Figure 3: (a) Unitarity triangle for $d, b$ quarks in Standard Model and (b) Unitarity quadrilateral in Sequential 4th Generation Model.

![Unitarity triangle and quadrilateral](image)
Figure 4: Constraint of the 4th generation CKM factor $V_{td}^*V_{tb}$ to (a) $|V_{td}^*V_{tb}|$ with $m_{t'}$ range from 50GeV to 800GeV, (b) to $m_{t'}$ with $|V_{td}^*V_{tb}|$ range from 0.005 to 0.013.