ON THE SPACE OF THETA FUNCTIONS FOR A PRIME LEVEL

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Abstract. c

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1. Introduction

In [6], Hecke expected that an explicit set of theta series obtained from maximal orders of the definite quaternion algebra over \( \mathbb{Q} \) which is ramified at a prime \( N \) will be a basis of space \( M_2(\Gamma_0(N)) \). However, later Eichler noticed that Hecke’s conjecture does not hold in general ([4]). It is natural to ask for the dimension of the subspace of \( M_2(\Gamma_0(N)) \) spanned by the theta series. This question is called Hecke’s basis problem ([7] p.143). In [2], Böcherer and Schulze-Pillot have given an answer using the theory of theta liftings. In this paper we will give another proof of their results using arithmetic and geometric properties of the modular curve.

Let \( N \) be a prime and \( \{E_1, \cdots, E_n\} \) the set of isomorphism classes of supersingular elliptic curves defined over the algebraic closure \( \overline{\mathbb{F}} \) of \( \mathbb{F}_N \). Set \( E_i = [i] \) and we let \( X \) be the free abelian group generated by \( \{[1], \cdots, [n]\} \),

\[
X = \oplus_{i=1}^{n} \mathbb{Z}[i]
\]

and define the monodromy pairing on \( X \) to be

\[
([i], [j]) = w_i \delta_{ij},
\]

where \( w_i \) is half of the order of the automorphism group of \( E_i \) and \( \delta_{ij} \) is Kronecker’s delta. Clearly this is symmetric and its extension to \( X \otimes \mathbb{R} \) is positive definite.
For a positive integer \( m \), we define the Hecke operator \( T_m \) by
\[
T_m(E_j) = \sum_{C \subseteq E_j} E_j/C,
\]
where \( C \) runs through subgroup schemes of \( E_j \) of order \( m \). The representing matrix \( B(m) = (B(m)_{ij}) \) of \( T_m \) with respect to the basis \( \{[1], \ldots, [n]\} \) is called the \( m \)-th Brandt matrix,
\[
T_m([j]) = \sum_{i=1}^n B(m)_{ij}[i].
\]
For every positive integer \( m \), \( B(m) \) is self-adjoint for the monodromy paring (see (2.3)). Note that our definition of the Brandt matrix is the transposition of Gross’ one ([7] Proposition 4.4). Set
\[
B(0) = \frac{1}{2} \begin{pmatrix}
1/w_1 & \cdots & 1/w_1 \\
\vdots & \ddots & \vdots \\
1/w_n & \cdots & 1/w_n
\end{pmatrix}
\]
and define the theta function \( \theta_{ij} \) to be
\[
\theta_{ij} = \sum_{m=0}^{\infty} B_{ij}(m)q^m, \quad q = e^{2\pi iz}.
\]
It is an element of \( M_2(\Gamma_0(N)) \), and \( \{\theta_{ij}\}_{ij} \) generates \( M_2(\Gamma_0(N)) \) (see also Theorem 3.1). For \( 1 \leq i \leq n \), we let \( \Theta_i \) be the \( \mathbb{C} \)-linear subspace of \( M_2(\Gamma_0(N)) \) spanned by \( \{\theta_{i1}, \ldots, \theta_{in}\} \):
\[
\Theta_i := \langle \theta_{i1}, \ldots, \theta_{in} \rangle \subset M_2(\Gamma_0(N)).
\]
Let \( R_i \) be the endomorphism ring of \( E_i \). It is a maximal order of the definite quaternion algebra \( \mathbb{B} \) ramified at \( N \), and each conjugacy classe of maximal orders in \( \mathbb{B} \) appears once or twice in \( \{R_1, \ldots, R_n\} \). The space \( \Theta_i \) will be called as the space of theta functions of \( R_i \). As we have mentioned before, Hecke expected that \( \Theta_i \) will coincide with \( M_2(\Gamma_0(N)) \) for all \( i \). However Eichler noticed that this conjecture does not hold in general. In fact if \( N = 37 \), there is a maximal order \( R_i \) such that \( \Theta_i \) is strictly smaller than \( M_2(\Gamma_0(37)) \) (it is known that \( N = 37 \) is the smallest prime level that Hecke’s conjecture fails [12]. See also Example 4.2 and Theorem 3.5 below). We will determine the dimension and a basis of \( \Theta_i \). In order to state our results we recall basic facts on the Hecke algebra.

Let \( \mathbb{T} \) be the commutative subalgebra of \( \text{End}_\mathbb{Z}(X) \) generated by the Hecke operators, called the Hecke algebra. Then \( \mathbb{T} \) is commutative, and since the action of \( T \in \mathbb{T} \) on \( X \) is symmetric for the monodromy paring, there is an orthonormal basis \( \{f_1, \ldots, f_n\} \) of \( X \otimes \mathbb{R} \) for the monodromy paring such that
\[
T(f_i) = \alpha_i(T)f_i, \quad \forall T \in \mathbb{T}
\]
where $\alpha_i$ is an algebraic homomorphism from $T$ to $R$. Hereafter an algebraic homomorphism from $T$ to $C$ is called a character, and if it is real valued we say it real. Let $T_0(N)$ be the Hecke algebra for Hecke’s congruence subgroup $\Gamma_0(N)$. It is a commutative subalgebra of the endomorphism ring of $M_2(\Gamma_0(N))$. In §2, we will show that $T \otimes Q$ is naturally isomorphic to $T_0(N) \otimes Q$, and we will identify them and denote them by $T \otimes Q$. There is an isomorphism of $T \otimes Q$-modules

$$X \otimes C \simeq M_2(\Gamma_0(N)),$$

which maps $f_i$ to a normalized Hecke eigenform $f_i$ (cf. Proposition 2.1). This fact is well-known (for example [5] Theorem 3.1 and Corollary 3.2) but we give a proof for the sake of convenience. The multiplicity one theorem implies that the characters $\{\alpha_i\}_i$ are mutually distinct and $f_i$ is determined up to sign. Let us fix $1 \leq i \leq n$. Writing

$$f_k = \sum_{i=1}^n f_{ik}[i], \quad f_{ik} \in R$$

we set

$$\Sigma(i) = \{k : ([i], f_k) \neq 0\} = \{k : f_{ik} \neq 0\}.$$

Note that $\Sigma(i)$ depends on the ordering and is independent of the choice of $\{f_1, \cdots, f_n\}$. Here is our main theorem.

**Theorem 1.1.** $\{f_k\}_{k \in \Sigma(i)}$ is a basis of $\Theta_i$. In particular

$$\dim \Theta_i = |\Sigma(i)|,$$

where $|\cdot|$ denotes the cardinality.

This yields results (see Theorem 3.5 and Theorem 3.4) which explain Pizer’s result ([12] Theorem 3.2) and an observation ([11] §1) due to Ohta. As we have mentioned before, Theorem 1.1 has been obtained by Böcherer and Schulze-Pillot ([2] Proposition 10.1) by the theory of theta liftings. In this paper, we will adopt a different approach using arithmetic geometry.

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2. **Brandt matrices and modular forms**

2.1. **The Brandt matrix.** In this subsection we will recall the theory of Brandt matrices following [7]. Let \( N \) be a prime and let \( \mathcal{B} \) be the quaternion algebra over \( \mathbb{Q} \) ramified at two places \( N \) and \( \infty \). Let \( R \) be a fixed maximal order in \( \mathcal{B} \) and \( \{I_1, \ldots, I_n\} \) the set of left \( R \)-ideals representing the distinct ideal classes. We call \( n \) the class number of \( \mathcal{B} \). We choose \( I_1 = R \). For \( 1 \leq i \leq n \), let \( R_i \) denote the right order of \( I_i \):

\[
R_i = \{b \in \mathcal{B} \mid I_i b \subset I_i\},
\]

and \( w_i \) the order of \( R_i^\times/\{\pm 1\} \). The product

\[
W = \prod_{i=1}^{n} w_i
\]

is independent of the choice of \( R \) and is equal to the exact denominator of \( \frac{N-1}{12} \) ([7] p.117). Eichler’s mass formula states that

\[
\sum_{i=1}^{n} \frac{1}{w_i} = \frac{N - 1}{12}.
\]

The set

\[
I_j^{-1} = \{b \in \mathcal{B} \mid I_j b I_j \subset I_j\}
\]

is a right \( R \)-ideal whose left order is \( R_j \). Then the product \( M_{ij} = I_j^{-1} I_i \) is a left \( R_j \)-ideal with the right order \( R_i \). For \( x \in M_{ij} \), let \( \mathbb{N}(x) \) be its reduced norm and let \( \mathbb{N}(M_{ij}) \) denote the unique positive rational number such that the quotients \( \mathbb{N}(x)/\mathbb{N}(M_{ij}) \) are all integers with no common factor. We define the theta function \( \theta_{ij} \) by

\[
\theta_{ij} = \frac{1}{2w_i} \sum_{x \in M_{ij}} q^{\mathbb{N}(x)/\mathbb{N}(M_{ij})} = \frac{1}{2w_i} + \sum_{m=1}^{\infty} B_{ij}(m) q^m, \quad q = e^{2\pi i z}
\]

and the \( m \)-th Brandt matrix \( B(m) \) is defined to be

\[
B(m) = (B(m)_{ij})_{1 \leq i, j \leq n}.
\]

For \( m \geq 1 \), \( B(m) \) has the following geometric description. Let \( \mathcal{F} \) be an algebraic closure of \( \mathbb{F}_N \). There are \( n \) distinct isomorphism classes \( \{E_1, \ldots, E_n\} \) of supersingular elliptic curves over \( \mathcal{F} \) such that End\( (E_i) \) is \( R_i \). Then one has an isomorphism

\[
M_{ij} \cong \text{Hom}(E_j, E_i), \quad x \mapsto \phi_x
\]

satisfying

\[
\deg \phi_x = \mathbb{N}(x)/\mathbb{N}(M_{ij}), \quad x \in M_{ij}.
\]
For a positive integer \(m\) let \(\text{Hom}(E_j, E_i)(m)\) denote the set of homomorphisms from \(E_j\) to \(E_i\) of degree \(m\). Then

\[
B(m)_{ij} = \frac{1}{2w_i} |\text{Hom}(E_j, E_i)(m)|. 
\]

Since \(\text{Hom}(E_j, E_i)(m)\) has a faithful action of \(R_i^\times\) from the right, \(B(m)_{ij}\) is a nonnegative integer and is equal to the number of subgroup schemes \(C\) of order \(m\) in \(E_j\) satisfying \(E_j/C \cong E_i\) (Proposition 2.3). Thus (2.2) coincides with (1.3). In particular, \(T_N(E_i)\) is the image of the \(N\)-th power Frobenius \(F\) of \(E_i\):

\[
T_N(E_i) = E/\text{Ker}F = E_i^F. 
\]

Since each of \(\{E_i\}_{1 \leq i \leq n}\) is defined over \(\mathbb{F}_{N^2}\), \(B(N)\) is a permutation matrix of order dividing 2. More precisely, \(E_i\) and \(E_j\) are conjugate by an automorphism of \(F\) if and only if \(i = j\) or \(B(N)_{ij}\) is 1 (Proposition 2.4).

Taking the dual isogeny we have a bijective correspondence

\[
I : \text{Hom}(E_i, E_j)(m) \rightarrow \text{Hom}(E_j, E_i)(m), \quad I(\phi) = \tilde{\phi}, 
\]

which implies

\[
w_i B(m)_{ij} = w_j B(m)_{ji}, \quad \forall m \geq 1 
\]

and \(T_m\) is symmetric for the monodromy pairing. Let \(T\) be the subalgebra of \(\text{End}_{\mathbb{Z}}(X)\) generated by \(\{T_p\}_p\) (\(p\) runs through all primes), which is known to be commutative (Proposition 2.7).

**Remark 2.1.** Our definition of a Brandt matrix is the transposition of Gross’ one.

**2.2. Brandt matrices and modular forms.** Let \(M_2(\Gamma_0(N))\) and \(S_2(\Gamma_0(N))\) denote the space of modular and cusp forms of weight 2 for the Hecke congruence subgroup

\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \quad c \equiv 0 \pmod{N} \right\}, 
\]

respectively. It is known that \(\dim M_2(\Gamma_0(N)) = n\) and that

\[
M_2(\Gamma_0(N)) = S_2(\Gamma_0(N)) \oplus \mathbb{C}F, 
\]

where \(F\) is the Eisenstein series defined by

\[
F = \frac{N-1}{24} + \sum_{m=1}^{\infty} \sigma(m) N q^m, \quad \sigma(m) = \sum_{d|m, (d,N)=1} d. 
\]

Both the spaces \(M_2(\Gamma_0(N))\) and \(S_2(\Gamma_0(N))\) have an action by Hecke operators, which we will recall (see [15] for details).

Let \(Y_0(N)\) be the generic fiber of the coarse moduli scheme over \(\mathbb{Z}\) which parametrizes isomorphism classes of pairs \(E = (E, \Gamma_N)\) of an elliptic curve \(E\)
together with a cyclic subgroup scheme $\Gamma_N$ of order $N$. It is a smooth affine curve defined over $\mathbb{Q}$, and its set of $\mathbb{C}$-valued points is the quotient of the upper half plane by $\Gamma_0(N)$. The compactification $X_0(N)$ of $Y_0(N)$ is a smooth projective curve defined over $\mathbb{Q}$ which has a finite number of cusps as points at infinity. For a prime $p$ different from $N$, $X_0(N)$ furnishes the $p$-th Hecke operator defined by

$$T_p(E, \Gamma_N) := \sum_C (E/C, (\Gamma_N + C)/C),$$

where $C$ runs through all subgroup schemes of $E$ of order $p$. On the other hand the operator $T_N$ (denoted by $U_N$ in the literatures) is defined by

$$T_N(E, \Gamma_N) := \sum_{D \neq \Gamma_N} (E/D, (\Gamma_N + D)/D),$$

where $D$ runs through subgroup schemes of $E$ of order $N$ different from $\Gamma_N$. These correspondences define endomorphisms of $M_2(\Gamma_0(N))$ and $S_2(\Gamma_0(N))$ which are denoted by the same symbols. The effects of the Hecke operator on a modular form $f = \sum_{m=0}^{\infty} a_m(f)q^m$ are

$$f|T_p = \sum_{m=0}^{\infty} (a_{pm}(f) + pa_{m/p}(f))q^m, \quad p \neq N$$

and

$$f|T_N = \sum_{m=0}^{\infty} a_{mN}q^m.$$

Here $a_{m/p}$ is understood to be 0 if $m/p$ is not an integer. We define the Hecke algebra as $T_0(N) = \mathbb{Z}[\{T_p\}_p] \subset \text{End}(M_2(\Gamma_0(N)))$. Then $T_0(N)$ preserves the decomposition (2.4) and we denote its restriction to $S_2(\Gamma_0(N))$ by $T_0(N)$. The Eisenstein series $F$ satisfies

$$(2.5) \quad F|T_m = \sigma(m)_N F, \quad m \geq 1$$

and is a Hecke eigenform of character $\sigma$ which is defined by

$$\sigma(T_m) = \sigma(m)_N.$$

We have an embedding

$$(2.6) \quad T_0(N) \otimes \mathbb{Q} \hookrightarrow (T_0(N) \otimes \mathbb{Q}) \times \mathbb{Q}, \quad T = (T|S_2(\Gamma_0(N)), \sigma(T)).$$

We claim that this is an isomorphism. In fact, it is known that $S_2(\Gamma_0(N))$ has a spectral decomposition

$$S_2(\Gamma_0(N)) = \bigoplus_{i=1}^{n-1} \mathbb{C} f_i.$$

Here $\{f_1, \ldots, f_{n-1}\}$ are normalized Hecke eigenforms such that

$$T(f_i) = \alpha_i(T)f_i, \quad \forall T \in T_0,$$

where $\alpha_i$ is a character of $T_0(N) \otimes \mathbb{Q}$ (see also the arguments following Remark 2.2). By the multiplicity one theorem ([11], [10]) $\{\alpha_1, \ldots, \alpha_{n-1}\}$ are mutually
different and the Eichler-Shimura congruence relation and the Weil conjecture imply
\[ |\alpha_i(T_p)| \leq 2\sqrt{p}, \quad 1 \leq i \leq n - 1 \]
for any prime \( p \) different from \( N \). On the other hand the character \( \sigma \) satisfies
\[ |\sigma(T_p)| = 1 + p > 2\sqrt{p}, \quad \forall p \neq N. \]
Thus \( T_0(N) \otimes \mathbb{Q} \) has distinct \( n \) characters \( \{\alpha_1, \ldots, \alpha_{n-1}, \sigma\} \) and \( \dim_{\mathbb{Q}} T_0(N) \otimes \mathbb{Q} = n \). Hence (2.6) is an isomorphism and we have a decomposition
\[ T_0(N) \otimes \mathbb{Q} = (T_0(N) \otimes \mathbb{Q}) \times \mathbb{Q}, \quad T = (T|_{S_2(\Gamma_0(N))}, \sigma(T)). \]
Using this we will relate \( T_0(N) \otimes \mathbb{Q} \) with \( T \otimes \mathbb{Q} \).

The canonical model of \( X_0(N) \) over \( \mathbb{Z} \) is studied in detail in [3] and [9]. Applying these results to our case we see that the reduction \( X_0(N)_{\overline{\mathbb{F}}_N} \) of the model at the prime \( N \) has two irreducible components \( C_F \) and \( C_V \), which are isomorphic to the projective line \( \mathbb{P}^1 = X_0(1) \). Over \( C_F \) (resp. \( C_V \)) \( \Gamma_N \) is the kernel of the Frobenius \( F \) (resp. the Verschiebung \( V \)) and \( C_F \) and \( C_V \) transversally intersect at supersingular points \( \Sigma_N = \{E_1, \ldots, E_n\} \). Thus the group \( X \) in the introduction is the free abelian group generated by \( \Sigma_N \). Now consider the homomorphism
\[ \partial : X \to \mathbb{Z}C_F \oplus \mathbb{Z}C_V, \quad \partial(E_i) = C_F - C_V, \]
which is compatible with the action of \( T \). This is the simplicial complex of the dual graph of \( X_0(N)_{\overline{\mathbb{F}}_N} \). Since \( X_0 \) is the kernel of \( \partial \), we have an exact sequence of \( T \)-modules
\[ 0 \to X_0 \to X \xrightarrow{\partial} \mathbb{Z} \epsilon \to 0, \quad \epsilon = C_F - C_V. \]
As in the introduction, let \([i]\) denote \( E_i \). Then
\[ \partial([i]) = \epsilon, \quad 1 \leq i \leq n \]
and
\[ X_0 = \{ \sum_{i=1}^{n} a_i [i] \mid a_i \in \mathbb{Z}, \sum_{i=1}^{n} a_i = 0 \}. \]
Let \( T_0 \) be the restriction of \( T \) to \( X_0 \). As we will explain below it is closely related to \( T_0^e(\mathbb{N}) \).

The space of cusp forms \( S_2(\Gamma_0(N)) \) can be naturally identified with the space of holomorphic 1-forms \( H^0(X_0(N), \Omega) \) and in particular with the holomorphic cotangent space \( \text{Cot} J_0(N) \) at the origin of the Jacobian variety \( J_0(N) \) of \( X_0(N) \). By functoriality, Hecke operators act on \( J_0(N) \) and they generate a commutative subalgebra of \( \text{End}(J_0(N)) \), which is temporarily denoted by \( T' \). The action of Hecke operators induces one on \( \text{Cot} J_0(N) \) which coincides with action of \( T_0^e(\mathbb{N}) \) on \( S_2(\Gamma_0(N)) \). Since the action is faithful, \( T' \) and \( T_0^e(\mathbb{N}) \) are isomorphic and they will be identified. Let \( J_0(N) \) be the Néron model of \( J_0(N) \) over \( \mathbb{Z} \). It is known that
the identity component of the reduction of $J_0(N)$ at $N$ is a torus, which is denoted
by $\mathcal{T}$. By the Néron property, it admits an action of $\mathbb{T}'$. By $[14]$ Proposition
3.1, $X_0$ is the character group of $\mathcal{T}$ and the induced action of $\mathbb{T}'$ on $X_0$ coincides
with one of $\mathbb{T}_0$. Therefore $\mathbb{T}_0$ is the image of $\mathbb{T}'$ in $\mathrm{End}_\mathbb{Z}(X_0)$. Moreover by $[14]$ Theorem 3.10 the action of $\mathbb{T}'$ on $X_0$ is faithful and $\mathbb{T}'(=\mathbb{T}_0(N))$ is isomorphic
to $\mathbb{T}_0$. Hence hereafter we will identify $\mathbb{T}_0$ and $\mathbb{T}_0(N)$.

Let us investigate the action of Hecke operators on $\epsilon$. Let $p$ be a prime. Then
a simple computation shows that

$$T_p(C_F) = (p + 1)C_F, \quad T_p(C_V) = (p + 1)C_V, \quad T_p(\epsilon) = (p + 1)\epsilon,$$

for $p \neq N$, and

$$T_N(C_F) = C_F, \quad T_N(C_V) = C_V, \quad T_N(\epsilon) = \epsilon.$$

Thus we have

$$T_m(\epsilon) = \sigma(m)_N\epsilon, \quad \sigma(m)_N = \sum_{d|m, (d,N)=1} d$$

and $\epsilon$ is a Hecke eigenvector for the character $\sigma$. We extend the monodromy pairing
to a positive definite symmetric bilinear form on $X \otimes \mathbb{R}$. Remember that $T \in \mathbb{T}$ is
self adjoint for the monodromy pairing:

$$(Tx, y) = (x, Ty), \quad \forall x, y \in X.$$

Since $X_0 \otimes \mathbb{Q}$ is stable under the action of $\mathbb{T}$, so is the orthogonal complement
$(X_0 \otimes \mathbb{Q})^\perp$. It has dimension one and we choose a base vector $b$. Then (2.8) and
(2.9) imply

$$T_m(b) = \sigma(T_m)b.$$

Thus we have an orthogonal decomposition

$$X \otimes \mathbb{Q} = (X_0 \otimes \mathbb{Q}) \oplus \mathbb{Q}b$$

stable under $\mathbb{T}$ ($\oplus$ means an orthogonal direct sum) and an injective homomorphism

$$\mathbb{T} \otimes \mathbb{Q} \hookrightarrow (\mathbb{T}_0 \otimes \mathbb{Q}) \times \mathbb{Q}, \quad T = (T|_{X_0 \otimes \mathbb{Q}}, \sigma(T)).$$

The proof of (2.7) shows that (2.11) is an isomorphism and therefore $\mathbb{T} \otimes \mathbb{Q}$ and
$\mathbb{T}_0(N) \otimes \mathbb{Q}$ are isomorphic. We set

$$f = \frac{b}{\|b\|} \in (X_0 \otimes \mathbb{R})^\perp.$$

**Remark 2.2.** Suppose $w_i = 1$ for all $1 \leq i \leq n$. Then the Brandt matrix is
symmetric. One easily check that $\delta := \sum_{i=1}^n [i]$ is contained in $(X_0 \otimes \mathbb{R})^\perp$ and
$T_m(\delta) = \sigma(m)_N\delta$. 

Therefore
\[ f = \frac{\delta}{||\delta||}. \]

Since \( T_0 \) is commutative and since all of its elements are symmetric for the monodromy pairing, we have a spectral decomposition,
\[(2.12) \quad X_0 \otimes \mathbb{R} = \oplus_{i=1}^{n-1} \mathbb{R}f_i, \quad ||f_i|| = 1,\]
where \( f_i \) is a simultaneous eigenvector. i.e. there is a real character \( \alpha_i : T_0 \to \mathbb{R} \) such that
\[ T(f_i) = \alpha_i(T)f_i, \quad \forall T \in T_0. \]

Using the multiplicity one theorem ([1] [10]), we have proved the following result.

**Fact 2.1. ([16], Proposition 3.2)** The characters \( \{\alpha_1, \cdots, \alpha_{n-1}\} \) are mutually distinct, and \( X_0 \otimes \mathbb{C} \) and \( S_2(\Gamma_0(N)) \) are isomorphic as \( T_0 \otimes \mathbb{C} \)-modules.

Thus \( \{f_1, \cdots, f_{n-1}\} \) is an orthonormal basis of \( X_0 \otimes \mathbb{R} \) and there are normalized Hecke eigenforms \( \{f_1, \cdots, f_{n-1}\} \) such that
\[ S_2(\Gamma_0(N)) = \oplus_{i=1}^{n-1} \mathbb{C} f_i \]
and
\[ T(f_i) = \alpha_i(T) f_i, \quad \forall T \in T_0. \]

Set \( \alpha_n = \sigma \) and
\[ f_n = f, \quad f_n = F. \]

Then \( f_n \) (resp \( f_n \)) is a Hecke eigenvector (resp. eigenform) of character \( \alpha_n \), and we have real characters \( \{\alpha_1, \cdots, \alpha_n\} \) of \( T \) which are also the characters of \( T_0(N) \) via the isomorphism \( T \otimes \mathbb{Q} \cong T_0(N) \otimes \mathbb{Q} \). As we have seen before \( \{\alpha_1, \cdots, \alpha_n\} \) are mutually different, hence the corresponding set of eigenvectors \( \{f_1, \cdots, f_n\} \) form an orthonormal basis of \( X \otimes \mathbb{R} \). We summarize these results.

**Proposition 2.1. (cf. [5] Theorem 3.1, Corollary 3.2)** There is an isomorphism of \( T \otimes \mathbb{C} \)-modules
\[ X \otimes \mathbb{C} = \oplus_{i=1}^{n} \mathbb{C} f_i \cong M_2(\Gamma_0(N)) = \oplus_{i=1}^{n} \mathbb{C} f_i, \]
defined by
\[ f_i \mapsto f_i. \]

Here \( \{f_1, \cdots, f_n\} \) is an orthonormal basis of \( X \otimes \mathbb{R} \) satisfying
\[ T(f_i) = \alpha_i(T) f_i, \]
and \( f_i \) is the normalized Hecke eigenform of the character \( \alpha_i \). Moreover, \( \{\alpha_1, \cdots, \alpha_n\} \) are mutually different real characters.
We have a decomposition
\[ \mathbb{T} \otimes \mathbb{C} \cong \mathbb{C}^n \]
such that
\[ \alpha_i = \pi_i \circ \rho, \]
where \( \pi_i \) is the \( i \)-th projection. We adopt \( \{\alpha_1, \ldots, \alpha_n\} \) as a basis of \( \text{Hom}_{\mathbb{C}}(\mathbb{T} \otimes \mathbb{C}, \mathbb{C}) \) and define a linear isomorphism
\[
(2.13) \quad \mu : \text{Hom}_{\mathbb{C}}(\mathbb{T} \otimes \mathbb{C}, \mathbb{C}) \cong M_2(\Gamma_0(N))
\]
by
\[ \mu(\alpha_i) = f_i. \]
Note that since \( M_2(\Gamma_0(N)) \cap \mathbb{C} = 0 \), an element of \( M_2(\Gamma_0(N)) \) is determined by the Fourier expansion without a constant term. Thus we may write
\[
(2.14) \quad f_n = F = \sum_{m=1}^{\infty} \sigma(m)Nq^m, \quad \theta_{ij} = \sum_{m=1}^{\infty} B(m)_{ij}q^m.
\]
Using this convention, (2.12) is described as
\[ \mu(\alpha) = \sum_{m=1}^{\infty} \alpha(T_m)q^m, \quad \alpha = \sum_{i=1}^{n} a_i \alpha_i. \]
In fact, since \( \mu(\alpha) = \mu(\sum_{i=1}^{n} a_i \alpha_i) = \sum_{i=1}^{n} a_i f_i, \) we have to verify
\[ \sum_{m=1}^{\infty} \alpha(T_m)q^m = \sum_{i=1}^{n} a_i f_i, \]
which is easily checked
\[ \sum_{m=1}^{\infty} \alpha(T_m)q^m = \sum_{i=1}^{n} a_i \sum_{m=1}^{\infty} \alpha_i(T_m)q^m = \sum_{i=1}^{n} a_i \sum_{m=1}^{\infty} \alpha_i(T_m)q^m = \sum_{i=1}^{n} a_i f_i. \]
Define an action of \( \mathbb{T} \) on \( \text{Hom}_{\mathbb{C}}(\mathbb{T} \otimes \mathbb{C}, \mathbb{C}) \) by
\[ (Tf)(t) = f(Tt), \quad f \in \text{Hom}_{\mathbb{C}}(\mathbb{T} \otimes \mathbb{C}, \mathbb{C}), \quad T \in \mathbb{T}, \quad t \in \mathbb{T} \otimes \mathbb{C}, \]
and one sees that \( \mu \) commutes with the action of a Hecke operator. Therefore we have shown the following result.

**Proposition 2.2.** There is an isomorphism as \( \mathbb{T} \otimes \mathbb{C} \)-modules
\[
\mu : \text{Hom}_{\mathbb{C}}(\mathbb{T} \otimes \mathbb{C}, \mathbb{C}) \cong M_2(\Gamma_0(N))
\]
declared by
\[ \mu(\alpha) = \sum_{m=1}^{\infty} \alpha(T_m)q^m. \]
3. A CORRESPONDENCE BETWEEN THE CHARACTER GROUP AND THE SPACE OF MODULAR FORMS

We extend the monodromy pairing to $X \otimes \mathbb{C}$ as a non-degenerate symmetric $\mathbb{C}$-bilinear pairing and denote the extension by the same symbol.

**Definition 3.1.** Fix $a \in X \otimes \mathbb{Q}$. Then we define the $\mathbb{Q}$-linear map

$$\phi_a : X \otimes \mathbb{Q} \to \text{Hom}_\mathbb{C}(T \otimes \mathbb{C}, \mathbb{C})$$

by

$$\phi_a(x)(T) = (a, Tx), \quad x \in X \otimes \mathbb{Q}, \quad T \in T \otimes \mathbb{C}.$$ 

It is clear that this map is also linear for $a$, and after a scalar extension to $\mathbb{C}$ we have a $\mathbb{C}$-linear map

$$\phi : (X \otimes \mathbb{C}) \otimes_\mathbb{C} (X \otimes \mathbb{C}) \to \text{Hom}_\mathbb{C}(T \otimes \mathbb{C}, \mathbb{C}), \quad a \otimes x \mapsto \phi_a(x).$$

**Lemma 3.1.** $\phi$ is surjective.

**Proof.** Identify $X \otimes \mathbb{C}$ with the dual $(X \otimes \mathbb{C})^*$ by the extension of the monodromy pairing. Writing $\text{End}_\mathbb{C}(X \otimes \mathbb{C}) = (X \otimes \mathbb{C}) \otimes_\mathbb{C} (X \otimes \mathbb{C})^*$, the dual $\text{End}_\mathbb{C}(X \otimes \mathbb{C})^*$ is isomorphic to $(X \otimes \mathbb{C}) \otimes_\mathbb{C} (X \otimes \mathbb{C})$. Now observe that $\phi$ is the dual of the natural embedding $T \otimes \mathbb{C} \hookrightarrow \text{End}_\mathbb{C}(X \otimes \mathbb{C})$ and the claim is proved. \hfill \Box

**Lemma 3.2.**

$$(\mu \phi)([i] \otimes [j]) = \mu(\phi([j])([i])) = w_i \theta_{ij}.$$ 

**Proof.** The claim follows from a simple computation. Using the convention to write a modular form omitting a constant term,

$$\mu(\phi([j])([i])) = \sum_{m=1}^{\infty} \phi([j])(T_m)q^m = \sum_{m=1}^{\infty} ([i], T_m[j])q^m$$

$$= \sum_{m=1}^{\infty} \sum_{k=1}^{n} B(m)_{kj}k[q^m] = \sum_{m=1}^{\infty} B(m)_{ij}q^m = w_i \theta_{ij}.$$ 

Using **Lemma 3.1** and **Lemma 3.2**, **Proposition 2.2** yields the following well-known fact.

**Theorem 3.1.** The set $\{\theta_{ij}\}_{1 \leq i, j \leq n}$ spans $M_2(\Gamma_0(N))$.

**Definition 3.2.** We define a linear subspace $\Theta_i$ of $M_2(\Gamma_0(N))$ by

$$\Theta_i = \langle \theta_{i1}, \ldots, \theta_{in} \rangle = \{\sum_{j=1}^{n} c_j \theta_{ij} \mid c_j \in \mathbb{C}\}.$$
The symmetry of the monodromy paring implies (cf. (2.3))

$$\Theta_i = (\theta_{1i}, \ldots, \theta_{ni}) = \{ \sum_{j=1}^{n} c_j \theta_{ji} \mid c_j \in \mathbb{C} \}.$$ 

The following proposition is an immediate consequence of Lemma 3.2.

**Proposition 3.1.** For any $1 \leq i \leq n$,

$$\Theta_i = \mu(\text{Im} \phi_{[i]} \otimes \mathbb{C}).$$

For brevity the extension of $\phi_{[i]}$ to an $\mathbb{R}$-linear map is denoted by the same symbol.

**Lemma 3.3.**

$$\mu \phi_{[i]}(f_j) = ([i], f_j)f_j.$$

**Proof.**

$$\mu \phi_{[i]}(f_j) = \sum_{m=1}^{\infty} \phi_{[i]}(f_j)(T_m)q^m = \sum_{m=1}^{\infty} ([i], T_m f_j)q^m$$

$$= \sum_{m=1}^{\infty} ([i], \alpha_j(T_m f_j))q^m = ([i], f_j) \sum_{m=1}^{\infty} \alpha_j(T_m)q^m$$

$$= ([i], f_j)f_j.$$

□

**Lemma 3.2** and **Lemma 3.3** imply the following theorem.

**Theorem 3.2.** (1) Let us write the eigenvector $f_j$ by

$$f_j = \sum_{k=1}^{n} c_{jk}[k].$$

Then

$$([i], f_j)f_j = w_i \sum_{k=1}^{n} c_{jk} \theta_{ik}.$$

(2)

$$w_j \theta_{ji} = w_i \theta_{ij} = \sum_{k \in \Sigma(i) \cap \Sigma(j)} ([j], f_k)([i], f_k)f_k.$$
\textbf{Proof.} A simple computation shows the claims. In fact
\begin{equation*}
([i],f_j)f_j = \mu \phi_{[i]}(f_j)
\end{equation*}
\begin{equation*}
= \mu \phi_{[i]}(\sum_{k=1}^{n} c_{jk}[k]) = \sum_{k=1}^{n} c_{jk} \cdot \mu \phi_{[j]}([k])
\end{equation*}
\begin{equation*}
= w_i \sum_{k=1}^{n} c_{jk} \theta_{ik},
\end{equation*}
which implies (1). We will show (2). Since \(\{f_1, \cdots, f_n\}\) is an orthonormal basis of \(X \otimes \mathbb{R}\) for the monodromy paring,
\begin{equation*}
[j] = \sum_{k=1}^{n} ([j], f_k)f_k = \sum_{k \in \Sigma(j)} ([j], f_k)f_k
\end{equation*}
and a computation using Lemma 3.2 and Lemma 3.3 yields
\begin{equation*}
w_i \theta_{ij} = \mu \phi_{[i]}([j])
= \sum_{k \in \Sigma(j)} ([j], f_k)f_k
= \sum_{k \in \Sigma(j)} ([j], f_k)([i], f_k)f_k = \sum_{k \in \Sigma(i) \cap \Sigma(j)} ([j], f_k)([i], f_k)f_k.
\end{equation*}
\(\square\)

\textbf{Lemma 3.4.} Let \(x\) be an element of \(X \otimes \mathbb{C}\). Then \(\partial(x) = 0\) if and only if \((x, f_n) = 0\).

\textbf{Proof.} By Proposition 2.1 and (2.12) there is an orthogonal decomposition
\begin{equation*}
X \otimes \mathbb{C} = (X_0 \otimes \mathbb{C}) \oplus \mathbb{C} f_n.
\end{equation*}
We obtain the claim because \(X_0 = \ker \partial\). \(\square\)

\textbf{Theorem 3.3.} For an arbitrary \(1 \leq i \leq n\), \(\ker \phi_{[i]}\) is a linear subspace of \(X \otimes \mathbb{Q}\) which is stable by the action of \(T\). After scalar extension to \(\mathbb{R}\), it has a spectral decomposition
\begin{equation*}
\ker \phi_{[i]} \otimes \mathbb{R} = \oplus_{\tau \in \Sigma'(i)} \mathbb{R} f_\tau,
\end{equation*}
where \(\Sigma'(i)\) is the complement of \(\Sigma(i)\); \(\Sigma'(i) = \{\tau \mid ([i], f_\tau) = 0\}\). Moreover, \(\Sigma'(i)\) is contained in \(\{1, \cdots, n-1\}\).

\textbf{Proof.} Remember that the action of \(T \in T\) on \(X\) is symmetric for the monodromy pairing. Then by definition
\begin{equation*}
\phi_{[i]}(x)(T) = ([i], Tx) = (T[i], x), \quad T \in T, \quad x \in X \otimes \mathbb{Q}
\end{equation*}
and \(\ker \phi_{[i]}\) is equal to the orthogonal complement of \(T[i] \otimes \mathbb{Q}\):
\begin{equation*}
(T[i] \otimes \mathbb{Q})^\perp = \{x \in X \otimes \mathbb{Q} \mid (x, y) = 0 \ \forall y \in T[i] \otimes \mathbb{Q}\}.
\end{equation*}
Since $T[i] \otimes \mathbb{Q}$ is $T$-stable so is $\operatorname{Ker} \phi[i] = (T[i] \otimes \mathbb{Q})^\perp$. Hence after scalar extension to $\mathbb{R}$, it admits a spectral decomposition
\[ \operatorname{Ker} \phi[i] \otimes \mathbb{R} = \oplus_{\tau \in \Sigma} \mathbb{R} f_\tau, \quad \Sigma \subset \{1, \cdots, n\}. \]
We determine the index set $\Sigma$. The computation
\[ \phi[i](f_\tau)(T) = ([i], T f_\tau) = \alpha_\tau(T)([i], f_\tau), \quad \forall T \in T, \]
shows that $\phi[i](f_\tau) = 0$ is equivalent to $([i], f_\tau) = 0$. Thus we see $\Sigma = \Sigma'(i)$.

Finally let us show that $n$ is not contained in $\Sigma'(i)$. By Lemma 3.4 it is sufficient to show that $\partial([i]) \neq 0$ but this is clear since $\partial([i]) = \epsilon \neq 0$.

\[ \square \]

**Remark 3.1.** There is another way to show that $\operatorname{Ker} \phi[i]$ is stable under the action of $T$. Remember that the $T$-module structure on $\text{Hom}_\mathbb{C}(T \otimes \mathbb{C}, \mathbb{C})$ is defined by
\[ (T f)(t) := f(T t), \quad f \in \text{Hom}_\mathbb{C}(T \otimes \mathbb{C}, \mathbb{C}), \quad T \in T, \quad t \in T \otimes \mathbb{C}. \]
Then it is easy to check that $\phi[i] : X \otimes \mathbb{Q} \to \text{Hom}_\mathbb{C}(T \otimes \mathbb{C}, \mathbb{C}), \quad \phi[i](x)(t) = ([i], tx)$ commutes with the action of $T$. In fact, the computation
\[ [\phi[i](T x)](t) = ([i], t(T x)) = ([i], (T t)x) = \phi[i](x)(T t) = ([T \phi[i]](x))(t), \]
shows that $\phi[i]$ commutes with $\forall T \in T$ and $\operatorname{Ker} \phi[i]$ is stable by $T$.

Now we finish the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 2.1,
\[ X \otimes \mathbb{C} = \oplus_{i=1}^n \mathbb{C} f_i. \]
We extend $\phi[i]$ to a $\mathbb{C}$-linear map. Then Proposition 3.1, Theorem 3.3 and Lemma 3.3 imply
\[ \Theta_i = \mu \phi[i](X \otimes \mathbb{C}) = \mu \phi[i](\oplus_{\kappa \in \Sigma(i)} \mathbb{C} f_\kappa) = \oplus_{\kappa \in \Sigma(i)} \mathbb{C} f_\kappa. \]
Remember that $[i]$ denotes the supersingular elliptic curve $E_i$ and
\[ T_N(E_i) = E_i^F \]
where $F$ is the $N$-th power Frobenius. Since every supersingular elliptic curve is defined over $\mathbb{F}_{N^2}$, $B(N)$ is a permutation matrix of order dividing 2 and the eigenvalues are $\pm 1$. In particular $B(N)_{i i} = 1$ if and only if $E_i$ is defined over the prime field $\mathbb{F}_{N}$ (cf. \cite[Proposition 2.4]{[1]}). Suppose that $T_N(f_r) = -f_r$ and let $E_i$
be defined over $\mathbb{F}_N$. Then writing $f_\tau = \sum_{i=1}^n f_{ir}[i]$ we see that $f_{ir} = 0$. Since the Atkin-Lehner involution $w_N$ is related to $T_N$ by $w_N = -T_N$

(14) Proposition 3.7), we see

$$\{\tau \mid w_N f_\tau = f_\tau\} = \{\tau \mid T_N f_\tau = -f_\tau\} \subset \Sigma'(i).$$

These arguments yield the following result.

**Theorem 3.4.** Let $\rho$ be the number of normalized Hecke eigenforms of which the sign of the Atkin-Lehner involution is $+1$. Suppose that $E_i$ is defined over the prime field $\mathbb{F}_N$. Then

$$n - \text{dim}\Theta_i \geq \rho.$$

**Remark 3.2.** Theorem 3.4 has been obtained by Ohta (see [11] §1) and Pizer ([12] Proposition 3.1).

**Theorem 3.5.** Suppose that there is a totally real number field $F$ of degree $n - 1$ over $\mathbb{Q}$ satisfying $\mathbb{T}_0 \otimes \mathbb{Q} \simeq F$. Then

$$\Theta_i = M_2(\Gamma_0(N)).$$

**Proof.** As we have seen (cf.(2.8)) $X_0 \otimes \mathbb{Q}$ is a $\mathbb{T}_0 \otimes \mathbb{Q}$-module and the proof of Theorem 3.3 shows that $\ker \phi_i$ is a $\mathbb{T}_0 \otimes \mathbb{Q}$-submodule of $X_0 \otimes \mathbb{Q}$ (see also Remark 3.1). On the other hand, since we have assumed that $\mathbb{T}_0 \otimes \mathbb{Q}$ is isomorphic to a totally real field $F$ with $[F : \mathbb{Q}] = n - 1$, $\ker \phi_i$ is a $F$-vector space satisfying $\dim_F \ker \phi_i \leq 1$. Therefore $\Sigma(i) = \{1, \cdots, n\}$ or $\Sigma(i) = \{n\}$ according to $\dim_F \ker \phi_i = 0$ or 1. Now Theorem 1.1 implies that one of the following occurs.

(1) $\Theta_i = M_2(\Gamma_0(N)).$

(2) $\Theta_i = \mathbb{C}f_n, \quad f_n = \frac{N - 1}{24} + \sum_{m=1}^{\infty} \sigma(m)Nq^m.$

(The following proof is suggested by the referee.) We remark that (2) automatically implies (1). In fact, if (2) holds, comparing the constant terms

$$\theta_{ij} = \frac{12}{(N - 1)w_i} f_n, \quad \forall j.$$

Let us look at the coefficients of $q^N$. Since $\sigma(N) = 1$

$$\frac{12}{(N - 1)w_i} = B(N)_{ij} \in \mathbb{Z}$$

and $N - 1$ divides 12. Thus $N$ is one of

$$2, 3, 5, 7, 13$$
and the genus of $X_0(N)$ is known to be zero in these cases (see also the remark below). Thus

$$M_2(\Gamma_0(N)) = \mathbb{C}f_n = \Theta_i.$$  

\[\square\]

**Remark 3.3.** One finds that a prime $N$ which satisfies the assumption of Theorem 3.5 is contained in

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\},$$

which are listed up in [12] Theorem 3.2 (we learned the following argument from Ohta). Due to Ribet it is known that $\text{End}(J_0(N)) \otimes \mathbb{Q} = T_0 \otimes \mathbb{Q}$ and the assumption implies that $\text{End}(J_0(N)) \otimes \mathbb{Q} = F.$ (13 Corollary 3.3). Therefore $J_0(N)$ is absolutely simple. On the other hand, let $\Gamma_0(N)^+$ be the subgroup of $\text{GL}_2^+(\mathbb{Q}) := \left\{ \gamma \in \text{GL}_2(\mathbb{Q}) \mid \det \gamma > 0 \right\}$ generated by $\Gamma_0(N)$ and the involution $w_N$, and let $X_0(N)^+$ be the compactification of the quotient of the upper half plane by $\Gamma_0(N)^+$. Since the Jacobian of $X_0(N)^+$ is a proper subvariety of $J_0(N)$, the genus of $X_0(N)^+$ is zero. This will be happen if $N < 37$ or $N = 41, 47, 59, 71$, which proves the claim. Moreover a numerical experiment shows that each of

$$N = 11, 17, 19, 23, 29, 31, 41, 47, 59$$

satisfies the assumption of Theorem 3.5 with $n \geq 2$. Thus the theorem explains Pizer’s result except the case $N = 71$ (if $N = 71$, $T_0 \otimes \mathbb{Q} = F_1 \times F_2$, where $F_i$ is a totally real field of degree 3 for $i = 1, 2$).

4. Examples

Here are examples which illustrate our theory.

**Example 4.1.** Let $N = 11$. By Eichler’s mass formula we see that $n = 2$ and $(w_1, w_2) = (2, 3)$. Therefore there are two isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_{11}$, which are denoted by $\{(1), [2]\}$. From [7], §6, we find

$$B(0) = \frac{1}{2} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 1/3 \end{pmatrix}, \quad B(3) = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$  

(Remember that our Brandt matrix is the transposition of Gross’s one, and the index of the theta function $\theta_{ij}$ is interchanged from his notation). The eigenvectors of $T_3$ in $X$ are

$$f_1 = \frac{[1] - [2]}{\sqrt{5}}, \quad f_2 = \frac{3[1] + 2[2]}{\sqrt{30}}.$$  

(4.1)

which satisfies

$$T_3(f_1) = -f_1, \quad T_3(f_2) = 4f_2.$$
Comparing the eigenvalues with the coefficient of $q^3$ of the Fourier expansion, we find that the eigenvector $f_i$ correspond to Hecke eigenforms $f_i$ by the isomorphism of Hecke modules $X_C \simeq M_2(\Gamma_0(11))$, where

$$f_1 = \theta_{11} - \theta_{12} = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \cdots,$$

and

$$f_2 = \theta_{11} + \theta_{21} = \frac{5}{12} + \sum_{m=1}^{\infty} \sigma(m)_{11} q^m.$$

(See [7] (6.4) and (6.6)). Theorem 1.1 implies that

$$\Theta_1 = \Theta_2 = \mathbb{C} f_1 \oplus \mathbb{C} f_2 (= M_2(\Gamma_0(11)))$$

and

$$\dim \Theta_1 = \dim \Theta_2 = 2.$$

Let us investigate (4.2) and (4.3) from our point of view. Application of Theorem 3.2 (1) to (4.1) gives

$$f_1 = \theta_{11} - \theta_{12}, \quad f_2 = \theta_{11} + \frac{2}{3} \theta_{12},$$

which implies

$$\Theta_1 = \langle f_1, f_2 \rangle.$$

Moreover, since $2\theta_{12} = 3\theta_{21}$, the second equation is

$$f_2 = \theta_{11} + \theta_{21}$$

and (4.4) recovers (4.2) and (4.3). On the other hand Theorem 3.2 (2) yields

$$\theta_{11} = \frac{2}{5} f_1 + \frac{3}{5} f_2, \quad \theta_{12} = -\frac{3}{5} f_1 + \frac{3}{5} f_2.$$

This equation is also derived from (4.4).

Example 4.2. Suppose $N = 37$. By (2.1) and Eichler’s mass formula, we find that $n = 3$ and $w_i = 1$ for $i = 1, 2, 3$. According to Pizer ([12], Theorem 3.2), this is the smallest prime level for which the Hecke conjecture fails. That is, there is a certain maximal order $\mathcal{O}$ of the definite quaternion algebra $\mathbb{B}$ ramified at 37 such that the dimension of the space of the theta functions is less than 3. We investigate this example from our viewpoint. There are three isomorphism classes of supersingular elliptic curves over $\mathbb{F}_{37}$, which are denoted by $\{[1], [2], [3]\}$. The action of $T_3$ on $X$ is

$$T_3([1]) = [2] + [2] + [3], \quad T_3([2]) = [1] + 3[3], \quad T_3([3]) = [1] + 3[2],$$

and the corresponding Brandt matrix is

$$B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}.$$
The eigenvalues of $B(3)$ are $\{1, -3, 4\}$ and the corresponding eigenvectors are

$$f_1 = \frac{-2[1] + [2] + [3]}{\sqrt{6}}, \quad f_2 = \frac{-[2] + [3]}{\sqrt{2}}, \quad f_3 = \frac{[1] + [2] + [3]}{\sqrt{3}},$$

respectively. Comparing the eigenvalues with the coefficient of $q^3$ of the Fourier expansion, we find that the eigenvectors $\{f_1, f_2, f_3\}$ correspond to the Hecke eigenforms $\{f_1, f_2, f_3\}$ by the isomorphism of Hecke modules $X_C \simeq M_2(\Gamma_0(37))$, where $f_1 = q + q^3 - 2q^4 - q^7 - 2q^9 + \cdots$, $f_2 = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + \cdots$ and

$$f_3 = \frac{3}{2} + \sum_{m=1}^{\infty} \sigma(m)_{37} q^m.$$

Now Theorem 1.1 shows that

\begin{equation}
\Theta_1 = \mathbb{C} f_1 \oplus \mathbb{C} f_3, \quad \Theta_2 = \Theta_3 = \mathbb{C} f_1 \oplus \mathbb{C} f_2 \oplus \mathbb{C} f_3 (= M_2(\Gamma_0(37)))
\end{equation}

and

$$\dim \Theta_1 = 2, \quad \dim \Theta_2 = \dim \Theta_3 = 3.$$

Therefore we see that the Hecke conjecture fails for $\Theta_1$, which does not contain $f_2$. Let us investigate the relation between the theta functions and Hecke eigenforms for $\Theta_1$. We find that Theorem 3.2 (1) and Theorem 3.2 (2) imply

$$f_1 = \frac{1}{2}(2\theta_{11} - \theta_{12} - \theta_{13}), \quad f_3 = \theta_{11} + \theta_{12} + \theta_{13},$$

and

$$\theta_{11} = \frac{2}{3} f_1 + \frac{1}{3} f_3, \quad \theta_{12} = \theta_{13} = -\frac{1}{3} f_1 + \frac{1}{3} f_3,$$

respectively.

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