Strong cosmic censorship and topology change in four dimensional gravity

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Abstract

A physical interpretation of the recently discovered vast class of vacuum space-times, which stably violate the strong cosmic censor conjecture (in its usual broad formulation) in four dimensions, is exhibited. Namely, by elementary Morse theory we demonstrate that these geometries describe transitions between topologically different spacelike hypersurfaces. More precisely, in these four dimensional smooth vacuum space-times there exist three dimensional spacelike hypersurfaces which display an increasingly violent unbounded oscillation between topologically different closed, orientable three-manifolds as one moves towards the asymptotic region in their ambient space-times. Moreover this spatial oscillation appears as a cosmological redshift for late time observers. Therefore these new vacuum solutions shed some light onto the deep dynamic regime of general relativity and the structure of the four dimensional continuum itself.

This picture, beyond offering a physical clarification how global hyperbolicity breaks down in these solutions, is also consistent with the fact that the Riemannian counterparts of these Lorentzian vacuum geometries are not only Ricci-flat but even self-dual four-manifolds hence give rise to gravitational instantons. Consequently the role of these Riemannian solutions is similar to that of Yang–Mills instantons in semi-classical non-Abelian gauge theories over Minkowski space-time.

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1 Introduction

Recently there has been an activity on the strong cosmic censor conjecture in its usual broad formulation (SCCC for short); far from being complete, cf. e.g. [4, 5, 7, 8, 9, 11, 12, 14, 17, 21, 24] regarding fresh results on the four dimensional case (for more complete historical accounts see [6, 19, 26]). The current situation can perhaps be best summarized as a puzzling dichotomy: Although there are some signs or hints for its (in)validity in physically relevant situations (like various black holes in asymptotically flat or de Sitter space-times filled with vacuum or various matter fields, etc.), these
are still not sharp enough to decide the status of the SCCC in these important cases. On the other hand there exists an abundance of “exotic” smooth solutions in which the SCCC clearly fails [11, 12] however the physical meaning of these quite purely mathematical solutions is not clear yet. The reason for this latter issue is that, although being smooth solutions of the vacuum Einstein equation hence are physically relevant as well, the SCCC violating properties of these “exotic” solutions rest neither on some physical phenomenon nor on standard analytico-geometric properties of Lorentzian metrics but rather based on subtle novel differen-rio-topological features (often called exotica) of four dimensional manifolds have gradually been recognized in the underlying mathematical model of physical space-times from the 1980’s. Despite that no a priori principle has been introduced so far to exclude these curious and apparently fundamental mathematical discoveries from the game, they have not found their right places in theoretical physics yet [1].

The aim of this short note is an effort to fill in this gap by offering a plausible and simple physical interpretation of the new SCCC violating solutions [11, 12]. As an interesting observation it will turn out that, meanwhile the aforementioned classical situations in which SCCC breakdown has been examined belong to the well known static or stationary regime of general relativity, the new SCCC violating solutions are related with the yet unexplored deep dynamical regime of general relativity; more precisely related with so far hypothetical topology changing phenomena (again far from being complete, cf. e.g. [10, 15, 18]), as explained in Section 2. We also find in Section 2 that this topology changing dynamics appears as a cosmological redshift for late time internal observers in these space-times (for a derivation of the cosmological constant via exotica cf. [2]). Therefore, quite unsurprisingly, one is tempted to say that as one moves from the static towards the dynamical regime of general relativity, SCCC violating phenomena get more and more relevant.

In the rest of this Introduction as a preparation for Section 2 we evoke a recent result concerning the global solvability of the four dimensional Riemannian vacuum Einstein equation [13] (for earlier, yet not fully satisfactory versions cf. [11, 12]):

**Theorem 1.1.** Let \( M \) be a connected, simply connected, oriented, closed (i.e., compact without boundary), smooth four-manifold which is spin (or equivalently having even intersection form) and put \( M^\times := M \setminus \{\text{point}\} \). If \( X^\times \) is a smooth four-manifold homeomorphic but not necessarily diffeomorphic to \( M^\times \) such that it carries a smooth structure à la Gompf then \( X^\times \) can be equipped with a smooth complete Ricci-flat Riemannian metric \( g \).

A typical example for the space \( X^\times \) in Theorem 1.1 arises as follows. Referring to [16, Chapter 9] let \( \mathbb{R}^4 \) denote the largest member of the so-called Gompf–Taubes radial family of large exotic \( \mathbb{R}^4 \)'s. By “exotic” one means that this is a differentiable 4-manifold which is homeomorphic but not diffeomorphic to the standard \( \mathbb{R}^4 \) (regarded as a differentiable manifold) and by “large” that \( \mathbb{R}^4 \) even does not admit any smooth embedding into \( \mathbb{R}^4 \). The fact that \( \mathbb{R}^4 \) is not diffeomorphic to \( \mathbb{R}^4 \) implies the counterintuitive phenomenon that \( \mathbb{R}^4 \neq W \times \mathbb{R} \) i.e., \( \mathbb{R}^4 \) does not admit any smooth splitting into a 3-manifold \( W \) and \( \mathbb{R} \) (with their unique smooth structures) in spite of the fact that such continuous splittings obviously exist. Indeed, from the contractibility of \( \mathbb{R}^4 \) we know that \( W \) must be a contractible open 3-manifold (a so-called Whitehead continuum [28]) however, by an early result of McMillen [22] spaces of this kind always satisfy \( W \times \mathbb{R} \cong \mathbb{R}^4 \) i.e., their product with a line is always diffeomorphic to the standard \( \mathbb{R}^4 \). We will call this property of (any) exotic \( \mathbb{R}^4 \) occasionally below as “creased”. A more precise description of \( \mathbb{R}^4 \) is however a difficult task because these large exotic \( \mathbb{R}^4 \)'s—although being honest differentiable 4-manifolds—are very transcendental objects [16, p. 366]: For instance they require infinitely many 3-handles in any handle decomposition (like any other known large exotic \( \mathbb{R}^4 \)) and there is presently\(^1\)

\(^{1}\)More precisely in the year 1999, cf. [16].
no clue as how one might draw explicit handle diagrams of them (even after removing their 3-handles). This fact, among other things, forces any Morse function on $R^4$ to suffer from infinitely many critical points in spite of the fact that $R^4$ is topologically trivial. We note that the structure of small exotic $R^4$'s i.e., which admit smooth embeddings into $R^4$, is better understood, cf. [16, Chapter 9].

Take now any connected, simply connected, closed spin 4-manifold $M$ and form the connected sum $X^\times := M \# R^4$ (see Figure 1). This space is easily seen to be homeomorphic to the punctured space $M^\times = M \setminus \{\text{point}\}$ however cannot be diffeomorphic to it (with its usual inherited smooth structure from the smooth embedding $M^\times \subset M$) since $M^\times$ is diffeomorphic to $M \# R^4$ meanwhile $X^\times$ by construction is diffeomorphic to $M \# R^4$ hence the ends of the two open spaces, although homeomorphic, are not diffeomorphic. Actually, from a general viewpoint, the appearance of non-compact 4-manifolds carrying smooth structures like $X^\times$ seems to be much more typical. Theorem 1.1 says that $X^\times$ always carries a complete Ricci-flat Riemannian metric $g$.

Let us make a digression here on SCCC and observe that the Ricci-flat Riemannian metric $g$ always can be converted into a smooth Ricci-flat Lorentzian metric $g_L$ on $X^\times$ (see Theorem 2.1 below). Moreover $X^\times$ having a creased end it cannot be written as a smooth product $\Sigma \times R$ where $\Sigma$ is a 3-manifold and $R$ is the real line (with their unique smooth structures); however the existence of such a smooth splitting is a necessary condition of global hyperbolicity [3]. Consequently we arrive at a sort of breakdown of the strong cosmic censorship conjecture (in its usual broad formulation, cf. e.g. [11, 12]), namely

**SCCC.** The smooth Ricci-flat Lorentzian 4-manifold $(X^\times, g_L)$ is not globally hyperbolic and no (sufficiently large in an appropriate topological sense) perturbation of it can be globally hyperbolic.

Returning to Riemannian signature again, as a by-product of the construction behind Theorem 1.1, it turns out that with fixing an appropriate orientation on $X^\times$ the metric $g$ is self-dual, too. However a simply-connected, complete Riemannian 4-manifold which is both Ricci-flat and self-dual is in fact hyper-Kähler hence a straightforward reformulation of Theorem 1.1 is [13]

**Theorem 1.2.** The complete Ricci-flat metric $g$ of Theorem 1.1 on $X^\times$ with its fixed orientation is self-dual as well consequently $X^\times$ carries a hyper-Kähler structure, too.

Physically speaking the Riemannian 4-manifolds $(X^\times, g)$ exhibited in Theorem 1.1 or equivalently, in Theorem 1.2 are therefore examples of gravitational instantons. Consequently, even if these Riemannian (or Euclidean) vacuum spaces might not play any role in classical general relativity, they are not negligible in any quantum theory if lurking behind classical general relativity.

## 2 Topology change and cosmological redshift

Let us begin by recalling [13, Theorem 6.1], for completeness and latter reference together with its proof, how to convert all the Ricci-flat Riemannian metrics of Theorem 1.1 here into Ricci-flat Lorentzian ones.

**Theorem 2.1.** Consider the complete Ricci-flat Riemannian 4-manifold $(X^\times, g)$ of Theorem 1.1 moreover pick a nowhere vanishing smooth vector field along $X^\times$ (such vector field exists because $X^\times$ is open). Then out of these data one can construct a smooth Lorentzian metric $g_L$ on $X^\times$ such that $(X^\times, g_L)$ is a Ricci-flat Lorentzian 4-manifold, perhaps incomplete.\(^2\)

\(^2\)Note that $g_L$ depends not only on the original Riemannian metric $g$ but on an arbitrarily chosen vector field $v$ on $X^\times$ as well hence this construction is not canonical with respect to $g$. 

Proof. By virtue of its global triviality (cf. [13, Lemma 3.4]), the tangent bundle \( TX^\times \) admits lot of non-vanishing smooth sections; picking one
\[
v : X^\times \longrightarrow TX^\times \setminus \{0\}
\]
gives rise to a splitting \( TX^\times = L \oplus L^\perp \) into a real line bundle \( L \subset TX^\times \) spanned the vector field \( v \) and its \( g \)-orthogonal complement \( L^\perp \subset TX^\times \). Take the complexification \( T^C X^\times := TX^\times \otimes_{\mathbb{R}} \mathbb{C} \) of the real tangent bundle as well as the complex bilinear extension of the Riemannian Ricci-flat metric \( g \) found on \( TX^\times \) to a Ricci-flat metric \( g^C \) on \( T^C X^\times \). This means that if \( w^C \) is a complexified tangent vector then both \( w^C \mapsto g^C(w^C, \cdot) := g(w^C, \cdot) \) and \( w^C \mapsto g^C(\cdot, w^C) := g(\cdot, w^C) \) are declared to be \( \mathbb{C} \)-linear and Ric\(_g^C \) = Ric\(_g \) = 0. There is an induced splitting
\[
T^C X^\times = L \oplus L^\perp \oplus \sqrt{-1} L \oplus \sqrt{-1} L^\perp
\]
over \( \mathbb{R} \) of the complexification i.e., if \( T^C X^\times \) is considered as a real rank-8 bundle over \( X^\times \). Define a metric on the real rank-4 sub-bundle \( L^\perp \oplus \sqrt{-1} L \subset T^C X^\times \) by taking the restriction \( g^C|_{L^\perp \oplus \sqrt{-1} L} \).
It readily follows from the orthogonality and reality of the splitting that this is a non-degenerate real-valued \( \mathbb{R} \)-bilinear form of Lorentzian type on this real sub-bundle. To see this, we simply have to observe that taking real vector fields \( v_1, v_2 : X^\times \rightarrow L \) and \( w_1, w_2 : X^\times \rightarrow L^\perp \) we can exploit the \( \mathbb{C} \)-bilinearity of \( g^C \) to write
\[
g^C|_{L^\perp \oplus \sqrt{-1} L}(\sqrt{-1} v_1, \sqrt{-1} v_1) = g^C(\sqrt{-1} v_1, \sqrt{-1} v_1) = -g^C(v_1, v_1) = -g(v_1, v_1)
\]
and
\[
g^C|_{L^\perp \oplus \sqrt{-1} L}(\sqrt{-1} v_1, w_1) = g^C(\sqrt{-1} v_1, w_1) = \sqrt{-1} g^C(v_1, w_1) = \sqrt{-1} g(v_1, w_1) = 0
\]
and finally
\[
g^C|_{L^\perp \oplus \sqrt{-1} L}(w_1, w_2) = g^C(w_1, w_2) = g(w_1, w_2) .
\]
Consider the \( \mathbb{R} \)-linear bundle isomorphism \( W_L : T^C X^\times \rightarrow T^C X^\times \) of the complexified tangent bundle defined by, with respect to the splitting (2), as
\[
W_L(v_1, w_1, \sqrt{-1} v_2, \sqrt{-1} w_2) := (v_2, w_1, \sqrt{-1} v_1, \sqrt{-1} w_2) .
\]
Obviously \( W_L^2 = \text{Id}_{T^C X^\times} \) or more precisely \( W_L \) is a real reflection with respect to \( g^C \) making the diagram
\[
\begin{array}{c}
T^C X^\times \xrightarrow{W_L} T^C X^\times \\
\downarrow \\
X^\times \xrightarrow{\text{Id}_{X^\times}} X^\times
\end{array}
\]
commutative. In particular it maps the real tangent bundle \( TX^\times = L \oplus L^\perp \subset T^C X^\times \) onto the real bundle \( L^\perp \oplus \sqrt{-1} L \subset T^C X^\times \) and vice versa. Consequently with arbitrary two tangent vectors \( v, w : X^\times \rightarrow TX^\times \)
\[
\text{g}_L(v, w) := g^C(W_L v, W_L w)
\]
satisfies \( \text{g}_L(v, w) = g^C|_{L^\perp \oplus \sqrt{-1} L}(W_L v, W_L w) \) i.e., we obtain a non-degenerate real-valued \( \mathbb{R} \)-bilinear form of Lorentzian type hence a smooth Lorentzian metric \( \text{g}_L \) on the original real tangent bundle \( TX^\times \).
Concerning the Ricci tensor of $g_L$, the Levi–Civita connections $\nabla^L$ of $g_L$ and $\nabla^C$ of $g_C$ satisfy

$$g_L(\nabla^L_u v, w) + g_L(v, \nabla^L_w u) = dg_L(v, w)u = dg^C(\nabla^C_L v, \nabla^C_L w)u$$

$$= g^C(\nabla^C_W v, \nabla^C_W w) + g^C(\nabla^C_L v, \nabla^C_L w)$$

$$= g^C(\nabla^C_L v, W_L w) + g^C(\nabla^C_W v, W_L w)$$

$$= g_L((W_L \nabla^C_W) v, w) + g_L(v, (W_L \nabla^C_W) w)$$

yielding $\nabla^L = W_L \nabla^C W_L$ (this is an $\mathbb{R}$-linear operator). Consequently the curvature $\operatorname{Riem}_{gL}$ of $g_L$ looks like

$$\operatorname{Riem}_{gL}(v, w) = [\nabla^L_v, \nabla^L_w]u - \nabla^L_{[v, w]}u = W_L(\operatorname{Riem}_{g_C}(v, w) W_L u) .$$

Let $\{e_0, e_1, e_2, e_3\}$ be a real orthonormal frame for $g_L$ at $T_pX^\times$ satisfying $g_L(e_0, e_0) = -1$ and $+1$ for the rest; then $W_L e_0 = \sqrt{-1} e_0$ and $W_L e_j = e_j$ for $j = 1, 2, 3$ together with the definition of $g_L$ imply that

$$g_L(\operatorname{Riem}_{gL}(e_0, e_0) w, e_0) = g^C(\nabla^C_L (\operatorname{Riem}_{gL}(e_0, v) w, W_L e_0) = g^C(\operatorname{Riem}_{g_C}(e_0, v) W_L w, \sqrt{-1} e_0)$$

and likewise

$$g_L(\operatorname{Riem}_{gL}(e_j, v) w, e_j) = g^C(\nabla^C_L (\operatorname{Riem}_{gL}(e_j, v) w, W_L e_j) = g^C(\operatorname{Riem}_{g_C}(e_j, v) W_L w, e_j) .$$

Using an orthonormal frame $\{f_1, \ldots, f_m\}$ for a metric $h$ of any signature, its Ricci tensor looks like

$$\operatorname{Ric}_h(v, w) = \sum_{k=1}^m h(f_k, f_k) h(\operatorname{Riem}_h(f_k, v) w, f_k) ;$$

hence

$$\operatorname{Ric}_gL(v, w) = g_L(e_0, e_0) g_L(\operatorname{Riem}_{gL}(e_0, v) w, e_0) + \sum_{j=1}^3 g_L(e_j, e_j) g_L(\operatorname{Riem}_{gL}(e_j, v) w, e_j)$$

$$= g^C(\sqrt{-1} e_0, e_0) g^C(\operatorname{Riem}_{g_C}(e_0, v) W_L w, \sqrt{-1} e_0) + \sum_{j=1}^3 g^C(e_j, e_j) g^C(\operatorname{Riem}_{g_C}(e_j, v) W_L w, e_j)$$

$$= (-\sqrt{-1} - 1) g^C(e_0, e_0) g^C(\operatorname{Riem}_{g_C}(e_0, v) W_L w, e_0) + \operatorname{Ric}_g(v, W_L w)$$

$$= (-1 + \sqrt{-1}) g_L(\operatorname{Riem}_{gL}(e_0, v) w, e_0)$$

and we also used $\{e_0, e_1, e_2, e_3\}$ as a complex orthonormal basis for $g_C$ on $T^*_pX^\times$ to write

$$\sum_{j=0}^3 g^C(e_j, e_j) g^C(\operatorname{Riem}_{g_C}(e_j, v) W_L w, e_j) = \operatorname{Ric}_g(v, W_L w) = 0 .$$

Being the left hand side in $\operatorname{Ric}_gL(v, w) = (-1 + \sqrt{-1}) g_L(\operatorname{Riem}_{gL}(e_0, v) w, e_0)$ real, the right hand side must be real as well for all $v, w \in T_pX^\times$ which is possible if and only if both sides vanish. This demonstrates that $gL$ is indeed Ricci-flat. ◇

The previous construction is based on a nowhere-vanishing vector field (1) along $X^\times$ whose choice is otherwise arbitrary. Therefore, taking into account the global triviality of the tangent bundle $TX^\times$, we have a great freedom in specifying it what we now exploit as follows. Consider the original simply connected and closed $M$ used in Theorem 1.1. Simply connectedness implies the vanishing of the first de Rham cohomology of $M$ therefore if we put any Riemannian metric onto $M$ and consider the corresponding Laplacian on 1-forms, its kernel is trivial. The Hodge decomposition theorem then says
that any 1-form $\xi$ on $M$ uniquely splits as $\xi = df + d^* \eta$ where $f$ is a function and $\eta$ a 2-form on $M$. The corresponding dual decomposition of a smooth vector field $v$ on $M$ therefore looks like $v = \nabla f + \text{div} T$ where $T$ is a $(2,0)$-type tensor field.

Motivated by this, consider now the space $X^\times$ of Theorem 2.1 and recall that it is homeomorphic to $M^\times$ consequently has vanishing first de Rham cohomology, too. Therefore, as a first and naive choice, we set the nowhere vanishing vector field (1) used to construct the Ricci-flat Lorentzian metric $g_L$ on $X^\times$ out of the Ricci-flat Riemannian one $g$ to be of the form

$$v := \nabla f$$

where $f : X^\times \to (-\infty, 0]$ is a Morse function (to be defined shortly) on $X^\times$ such that $f^{-1}(-\infty)$ corresponds to the creased end of $X^\times$ while $f^{-1}(t) \subset X^\times$ are compact level sets for all $-\infty < t \leq 0$ and in particular the point $f^{-1}(0)$ is the “top” of $X^\times$ (see Figure 1).

![Figure 1. The manifold $X^\times$ with a zig-zag representing its creased end and a Morse function $f : X^\times \to (-\infty, 0]$ on it.](image)

Moreover $\nabla f$ in (3) is defined by $df = g(\nabla f, \cdot)$ to be the dual vector field of the 1-form $df$ with respect to the original Riemannian metric $g$ on $X^\times$. If the choice in (3) is possible then we gain a very nice picture on the vacuum space-time $(X^\times, g_L)$. Namely, $\nabla f : X^\times \to L \subset L \oplus L^= = TX^\times$ is a vector field such that for a generic $t \in (-\infty, 0]$ it does not vanish and the level set $f^{-1}(t) \subset X^\times$ is a 3 dimensional closed (i.e., compact without boundary) submanifold with $T f^{-1}(t) = L^= \subset L \oplus L^= = TX^\times$. Hence with respect to $g_L$ we find that $\nabla f$ is a timelike and by definition future-directed vector field $g_L$-orthogonal for the level sets which are spacelike. In other words: The vector field $v$ in (1) is an infinitesimal observer in the space-time $(X^\times, g_L)$. If $v$ has the form (3) then $v$ can be identified with a global classical observer in the sense that the level value $t \in (-\infty, 0]$ corresponds to its global classical proper time as moves along its future directed own timelike curves (i.e., the integral curves of $v = \nabla f$) and the level sets $f^{-1}(t) \subset X^\times$ correspond to its global classical spacelike submanifolds. However this picture is too naive because $f$ may attain critical points i.e., $p \in X^\times$ where $\nabla f(p) = 0$ as we know from Morse theory. Hence the nowhere-vanishing vector field (1) cannot globally look like (3).

**A rapid course on Morse theory.** The following things are well known [16, 23] but we summarize them here for completeness and convenience. Let $N$ be a smooth $n$-manifold. The point $p \in N$ is a critical point of a smooth function $f : N \to \mathbb{R}$ iff in a local coordinate system $(U, x_1, \ldots, x_n)$ centered at $p$ all the partial derivatives vanish there i.e., $\partial_i f(p) = 0$ for all $i = 1, \ldots, n$ and it is non-degenerate.
iff the matrix $(\partial^2_{ij} f(p))_{i,j=1,\ldots,n}$ is not singular. Moreover $c \in \mathbb{R}$ is a critical value iff the level set $f^{-1}(c) \subset N$ contains a critical point. The smooth function $f : N \to \mathbb{R}$ is a Morse function along $N$ iff it admits only non-degenerate critical points such that each critical value level set contains at most one critical point. (Being non-degenerate already implies that the critical points are isolated [23, Corollary 2.3].) For simplicity we shall also assume below that the level set $f^{-1}(c) \subset N$ is compact as well, for all $c \in \mathbb{R}$.

We know the following things. If $c \in \mathbb{R}$ is non-critical then $f^{-1}(c) \subset N$ is a smooth $n-1$ dimensional submanifold. If $c \in \mathbb{R}$ critical with a single critical point $p \in f^{-1}(c) \subset N$ then (cf. [23, Lemma 2.2]) there exists a local coordinate system $(U, y_1, \ldots, y_n)$ about $p$ i.e., $y_1(p) = \cdots = y_n(p) = 0$, in which

$$f|_U(y_1, \ldots, y_n) = f(0, \ldots, 0) - \sum_{i=1}^k y_i^2 + \sum_{i=k+1}^n y_i^2$$

and the number $0 \leq k \leq n$ is called the index of the critical point. Therefore a critical point of index $k = 0$ is a local minimum while with index $k = n$ is a local maximum of $f$. Take $c \in \mathbb{R}$, $\varepsilon > 0$ and suppose that $[c - \varepsilon, c + \varepsilon] \subset \mathbb{R}$ consists of non-critical values only. Then (cf. [23, Theorem 3.1]) $f^{-1}(c - \varepsilon)$ and $f^{-1}(c + \varepsilon)$ are diffeomorphic. If the only critical value in $[c - \varepsilon, c + \varepsilon]$ is $c$ and its unique critical point $p \in f^{-1}(c)$ is of index $k$ then (cf. [23, Theorem 3.2]) $f^{-1}(c + \varepsilon)$ is obtained from $f^{-1}(c - \varepsilon)$ by taking the boundary of $f^{-1}((\infty, c - \varepsilon])$ with a closed $n$-ball $B^n$ in the form of a $k$-handle $B^k \times B^{n-k}$ attached to it. More precisely take an embedding $\varphi_k : S^{k-1} \times B^{n-k} \to f^{-1}(c - \varepsilon)$ and glue $B^n$ to $f^{-1}((\infty, c - \varepsilon])$ by identifying $S^{k-1} \times B^{n-k} \sqcup \partial(B^k \times B^{n-k}) = (S^{k-1} \times B^{n-k}) \cup (B^k \times S^{n-k-1})$ with the image $\varphi_k(S^{k-1} \times B^{n-k}) \subset \partial f^{-1}((\infty, c - \varepsilon]) = f^{-1}(c - \varepsilon)$. Then after “smoothing off the corners” we obtain an $n$ dimensional manifold-with-boundary $f^{-1}(((\infty, c - \varepsilon]) \cup \varphi_k B^n$ and $f^{-1}(c + \varepsilon)$ is diffeomorphic to $\partial(f^{-1}((\infty, c - \varepsilon]) \cup \varphi_k B^n)$. For instance if $k = 0$ then $B^n$ is glued along $S^{n-1} \times B^0$ where $S^{n-1} = \emptyset$ i.e., it is not glued hence this critical point is a local minimum; while if $k = n$ then $B^n$ is attached along $S^{n-1} \times B^0$ where $B^0$ is a point i.e., it is attached along its full boundary $S^{n-1}$ hence this is a local maximum of $f$. Note that replacing the bottom-up function $f$ with the top-down function $-f$ critical points with index $k$ and $n-k$ interchange.

Critical points necessarily occur. The fundamental result of Morse theory (cf. [23, Theorem 5.2]) states that if $m_k(N)$ denotes the number of critical points of index $k$ and $b_k(N)$ the $k$th Betti number of a compact manifold $N$ then $b_k(N) \leq m_k(N) < +\infty$. However if $N$ is not compact then some $m_k(N)$’s can be even infinite. For further details cf. [16, Chapter 4] or [23].

Returning to our problem, we therefore correct (3) as follows. Although critical points of $f$ are unavoidable, they are at least isolated i.e., for all $p, q \in X^\times$ pairs of critical points there exist small surrounding open neighbourhoods $U_p, U_q \subset X^\times$ such that $U_p \cap U_q = \emptyset$. Then taking the union (which is therefore disjoint)

$$C_f := \bigcup_{p \text{ is a critical point of } f} U_p$$

and supposing that this set is sharply concentrated around the critical points of $f$ in $X^\times$, let us correct (3) to $\nabla f + w$ where $w$ is a smooth vector field on $X^\times$ (of the form $w = \text{div} T$) such that $w(p) \neq 0$ in the critical point $p$ but $\text{supp } w \subset C_f$ i.e., vanishes outside of $C_f \subset X^\times$. In other words we define the vector field (1) to construct the Lorentzian metric $g_L$ out of the Riemannian one $g$ on $X^\times$ not as in (3) but rather to be of the form

$$v := \nabla f + w$$

in Theorem 2.1. Fortunately this changes our physical picture on $(X^\times, g_L)$ only locally (i.e. close to a critical point only). More precisely, the classical observer picture of $v$ breaks down only in the vicinity
of critical points of its Morse function part. Therefore from now on: If \( v = \text{grad} f + w \) is a non-vanishing vector field on \( X^\times \) with associated smooth vacuum space-time \((X^\times, g_L)\) then the infinitesimal observer provided by \( v \) gives rise to a global classical observer at least on the open subset

\[
\left( X^\times \setminus \overline{C_f}, g_L|_{X^\times \setminus \overline{C_f}} \right) \subsetneq \left( X^\times, g_L \right)
\]

because \( v = \text{grad} f \) along this restriction.

Let us ask ourselves now about the “experiences” of this partial global classical observer, constructed from a Morse function, as it moves in \((X^\times, g_L)\). That is, consider a Morse function \( f \) on \( X^\times \) as above (see Figure 1) with an associated global classical observer on the restricted domain \( X^\times \setminus \overline{C_f} \). This observer has a global proper time \( t \in (-\infty, 0] \) measured by \( f \) with the infinite past \( t = -\infty \) being the creased end of \( X^\times \) and also has corresponding global spacelike \( \Sigma_f := f^{-1}(t) \subset X^\times \setminus \overline{C_f} \) for appropriate \( t \)'s which are closed 3-manifolds. First, fix \( -\infty < K < 0 \) such that \( \Sigma_K \) is a submanifold and consider the compact part \( f^{-1}([K, 0]) \subseteq X^\times \). As the observer moves forward in time i.e., from \( t = K \) to \( t = 0 \) along the integral curves of \( \text{grad} f \) then only finitely many critical points occur. As we have seen, around these points the spacelike \( \Sigma_f \)'s change topology by picking up a \( k \)-handle according to the index of the critical point.

Now consider the much more interesting non-compact \( f^{-1}((-\infty, K]) \subset X^\times \) regime, the downward “neck” part in Figure 1. If \( K < 0 \) is sufficiently small (we mean |\( K \)| > 0 is sufficiently large) we can suppose that \( f^{-1}((-\infty, K]) \) is fully contained in the exotic but topologically trivial summand \( R^4 \) of \( X^\times \) in its decomposition \( X^\times = M \# R^4 \). Therefore if \( -\infty < t \leq K \) then \( \Sigma_t \) is fully contained in the \( R^4 \) summand. We can without loss of generality suppose that \( \Sigma_K \) surrounds the attaching region of \( M \) and \( R^4 \) hence \( \Sigma_K \) is diffeomorphic to \( S^3 \). Now take an observer in \((X^\times, g_L)\) moving backwards in time along the integral curves of \( \text{grad} f \) i.e. from \( t = K \) downwards \( t = -\infty \). A generic value of \( t \) is not critical for \( f \) consequently the corresponding spacelike submanifold \( \Sigma_t \) exists. Consider a fixed time \( -\infty < t_0 < K \) which is a critical value of \( f \). How the corresponding transition between the \( \Sigma_t \)'s then looks like? As we have seen, in this moment always a single 4-ball \( B^4 \), attached through its boundary \( S^3 \) in various ways to \( \Sigma_t \) depending on the index \( k \) of the critical point, is going to be removed from the latter space-time portion \( f^{-1}([t_0, K]) \). Therefore, as we move backwards in time provided by \( f \) (or move forward in time provided by \( -f \)) and pass through the moment \( t_0 \) the space \( \Sigma_{t_0 + \varepsilon} \) undergoes one of the following transitions:

(i) If \( k = 1 \) then at \( t_0 \) an \( S^3 \), attached along two disjoint \( B^3 \)'s to \( \Sigma_{t_0 + \varepsilon} \), is annihilated (or equivalently, attached along a thickened \( S^2 \), is created);

(ii) If \( k = 2 \) then at \( t_0 \) an \( S^3 \), attached along a thickened knot to \( \Sigma_{t_0 + \varepsilon} \), is annihilated (or equivalently, attached along a thickened knot, is created);

(iii) If \( k = 3 \) then at \( t_0 \) an \( S^3 \), attached along a thickened \( S^2 \) to \( \Sigma_{t_0 + \varepsilon} \), is annihilated (or equivalently, attached along two disjoint \( B^3 \)'s, is created)

and in this way the latter space \( \Sigma_{t_0 + \varepsilon} \) evolves into to the earlier \( \Sigma_{t_0 - \varepsilon} \) as we move backwards in time. Strictly mathematically speaking this \( k \)-handle attachment is to be performed “instantaneously” somewhere along the singular level surface \( \Sigma_{t_0} \) carrying a unique critical point \( p \) at the moment \( t_0 \); however from a physical viewpoint we can rather suppose that it occurs within the “non-classical” (with respect to the observer provided by \( \text{grad} f \)) region \( \Sigma_t \cap U_p \subset C_f \) at some unspecified time \( t \in (t_0 - \varepsilon, t_0 + \varepsilon) \) such that \( \Sigma_{t_0 + \varepsilon} \cap U_p \) are still not empty (see Figure 2). Beside the \( f \) Morse function picture, we have formulated all processes in the dual picture of the reversed Morse function \( -f \) as well in order to gain
full symmetry in the formulation. Moreover we note that applying diffeomorphisms on $X^\times$ (or equivalently, modifying $f$) we can assume that along $f^{-1}((\infty,K])$ with $K < 0$ the $k = 0,4$ handle attachment steps corresponding to local minima and maxima do not occur.

![Figure 2. Topology change about the critical point $p \in \Sigma_0 \cap U_p \subset X^\times$.](image)

Taking $-\infty \leftarrow t$ i.e., as moving backwards in time till the creased end of $X^\times$ in Figure 1, in this process the collection $\{\Sigma_t\}_{-\infty < t \leq K}$ of spacelike submanifolds looks like an evolution (in reversed time) from $\Sigma_K = S^3$ into a three dimensional “boiling foam” limit $\Sigma_{-\infty}$ or something like that. That is, these spacelike submanifolds unboundedly continue to switch their topology or in other words the spatial oscillation between these states never stops and it is reasonable to expect that all closed orientable 3-manifolds occur as $-\infty \leftarrow t$. Indeed, as we noted in the Introduction, large exotic $\mathbb{R}^4$‘s always require countably infinitely many handles in their handle decomposition therefore moving backwards in time the $\Sigma_t$‘s permanently continue changing their topological type. Moreover later or soon $\Sigma_t$ very likely can be arbitrary since the $k = 2$ processes above are nothing but surgeries along knots and all connected, closed, orientable 3-manifolds arise this way from $S^3 = \Sigma_K$ by the Lickorish–Wallace theorem [20, 27]. This “boiling foam” picture therefore seems to be very weird and dynamical and the sole “driving force” behind this dynamics is the non-standard smooth structure along the end of $X^\times$. (Exactly the same thing is responsible for the role of these spaces in $\text{SCCC}$, too.) The existence of topologically different Cauchy surfaces in $\mathbb{R}^4$ is already known to physicists, too [25].

All the things have described up to this point might seem as mere mathematical nonsense. However things get even physically interesting if we add that this vivid spatial topology oscillation in $(X^\times, g_L)$ appears as a cosmological redshift phenomenon to our observer moving in (4), as it looks back to the early creased end of $X^\times$ at late times. Let $E \in X^\times \setminus C_f$ be a space-time event with a normalized future-directed timelike vector $n_E$ where a photon is emitted; in the geometrical optics approximation this photon travels along a future-directed null geodesic $\gamma$ in $(X^\times, g_L)$ till it is received in a later $R \in X^\times \setminus C_f$ with corresponding receiver $n_R$. The emitted frequency measured by $n_E$ is $\omega_E = -g_L(\gamma', n_E)$ while $\omega_R = -g_L(\gamma', n_R)$ is the frequency measured by the receiver. Then we define the redshift factor $z$ in the standard way by the frequency ratio

$$1 + z = \frac{\omega_E}{\omega_R} = \frac{g_L(\gamma', n_E)}{g_L(\gamma', n_R)}$$

and say that the photon is redshifted along $\gamma$ if $z > 0$. It is worth reminding here that this definition
depends on the parameterization of the null geodesic and in general it is tacitly assumed that \( \gamma \) is in any affine parameterization (i.e., \( \nabla^L \gamma' = 0 \)). We adapt this general framework at least qualitatively to our setup as follows. Assume that the observer in the above process is given by \( n = \frac{\text{grad} f}{|\text{grad} f|_{gL}} \) and for simplicity \( \gamma \) is parameterized in a compatible way with \( f \) i.e., such that \( f(\gamma(t)) = t \). To emphasize this, will write \( \gamma' \) as \( \dot{\gamma} \) from now on. Then, using the notation in the proof of Theorem 2.1 we know that \( \text{grad} f \) is a section of \( L \subset TX \times \) hence \( W_L \text{grad} f = \sqrt{-1} \text{grad} f \). Moreover if \( \dot{\gamma} = \dot{\gamma}_L + \gamma_L \perp \) is the unique decomposition according to \( TX \times = L \oplus L \perp \) then \( W_L \dot{\gamma} = W_L \dot{\gamma}_L + W_L \gamma_L \perp = \sqrt{-1} \dot{\gamma}_L + \gamma_L \perp \in \sqrt{-1}L \oplus L \perp \). Consequently

\[
g_L(\dot{\gamma}, n) = \frac{g_L(\dot{\gamma}, \text{grad} f)}{|\text{grad} f|_{gL}} = \frac{g^C(W_L \dot{\gamma}, W_L \text{grad} f)}{|W_L \text{grad} f|_{g^C}} = -\frac{g(\dot{\gamma}_L, \text{grad} f)}{|\text{grad} f|_g} = \frac{g(\dot{\gamma}, \text{grad} f)}{|\text{grad} f|_g}
\]

moreover differentiating \( f(\gamma(t)) = t \) gives \( g(\dot{\gamma}, \text{grad} f) = 1 \). Therefore we eventually come up with

\[
1 + z = \frac{|\text{grad} f(R)|_g}{|\text{grad} f(E)|_g}.
\]

As we emphasized throughout this note, the level surfaces \( f^{-1}(t) \subset X \times \) attain critical points more and more frequently as \(-\infty \leftarrow t \). Consequently, the earlier space-time event \( E \in f^{-1}(t_E) \) is “more likely” to be in the vicinity of a critical point \( p_E \in f^{-1}(t_E) \) satisfying \( \text{grad} f(p_E) = 0 \) than the later event \( R \in f^{-1}(t_R) \) with \( t_R > t_E \). Therefore, acknowledging that a more careful statistical analysis is surely required, it is reasonable that “typically” \(|\text{grad} f(E)|_g \approx 0 \) meanwhile \(|\text{grad} f(R)|_g \approx 1 \) implying that the gradient ratio on the right hand side of \( 1 + z \), when calculated for the “typical” early photon emitting event \( E \in X \times \setminus \overline{C}_f \) and late photon receiving event \( R \in X \times \setminus \overline{C}_f \), is large resulting in \( z > 0 \). By the same reasoning this ratio even seems to be capable to be unbounded hence “typically” even \( z > 2 \) seems reasonable which is exclusively characteristical for \textit{cosmological} (i.e., not gravitational caused by a compact body, etc.) redshift.

Finally, one may raise the question about the place or role or relevance of this topology changing phenomenon within the full theory of (classical or even quantum) general relativity. Regarding this it is worth calling attention that the Riemannian solutions \((X \times, g)\) underlying our smooth vacuum space-times \((X \times, g_L)\) are not only Ricci-flat but even self-dual (see Theorem 1.2 here). Consequently they are gravitational instantons and their appearance at the semi-classical level as a leading term of quantum corrections looks therefore reasonable. The whole picture presented here strongly resembles the structure of the vacuum sector of a non-Abelian gauge theory in temporal gauge over Minkowski space: Instantons of the corresponding Euclidean theory give rise to semi-classical tunnelings between topologically (hence classically) separated vacua along a space-like submanifold in the original theory over Minkowski space.

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