Abstract: In this research paper, we deal with the problem of determining the function \( \chi : G \rightarrow \mathbb{R} \), which is the solution to the maximum functional equation (MFE) \( \max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y) \), when the domain is a discretely normed abelian group or any arbitrary group \( G \). We also analyse the stability of the maximum functional equation \( \max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y) \) and its solutions for the function \( \chi : G \rightarrow \mathbb{R} \), where \( G \) be any group and also investigate the connection of the stability with commutators and free abelian group \( K \) that can be embedded into a group \( G \).

Keywords: maximum functional equations; discretely normed abelian group; stability of functional equation

MSC: 20D60; 54E35; 11T71

1. Literature Review

In [1], Volkmann proved that every function \( \chi \) defined on an abelian group \( G \) can be described in the form \( \chi(x) = |\alpha(x)| \), which is the solution of the maximum FE

\[
\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y),
\]

where \( \alpha : G \rightarrow \mathbb{R} \) is an additive function. Consequently, in [2], Toborg showed that Equation (1) also characterises the absolute value of additive functions when \( G \) be any group. Their’s main theorem is stated, as follows:

**Theorem 1** (Toborg [2]). Let \( G \) be any group (\( G \) is an abelian group (Volkmann [1])), then a function \( \chi : G \rightarrow \mathbb{R} \) fulfills (1) if and only if there exists an additive function \( \alpha : G \rightarrow \mathbb{R} \), such that \( \chi(x) = |\alpha(x)| \) for any \( x \in G \).

We recommend the readers to see [3,4], and the references cited therein in order to obtain comprehensive results related to functional Equation (1), which characterises the additive function’s absolute value.

According to Simon and Volkmann [5], solutions of the MFE

\[
\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x)\chi(y)
\]

with the additional assumption that \( G \) is an additive abelian group was exhibited in the theorem, which is stated as:
Theorem 2 ([5], Theorem 2). Assume that $G$ is an additive abelian group, whose each element is divisible by 6 (divisible by 2 and 3), and then a function $\chi : G \rightarrow \mathbb{R}$ fulfills

$$\max\{\chi(x + y), \chi(x - y)\} = \chi(x)\chi(y)$$

if and only if there exists an additive function $\alpha : G \rightarrow \mathbb{R}$ such that $\chi(x) = 0$ or $\chi(x) = e^{\alpha(x)}$ for any $x \in G$.

Jarczyk and Volkmann [6] demonstrated the stability results of MFE (1) on additive abelian groups. Consequently, Badara et al. [7] also generalized their results in order to prove the stability for a certain class of groupoids. In [3], Gilanyi et al. determined the stability results of maximum functional equation $\max\{\chi((x \circ y) \circ y), \chi(x)\} = \chi(x \circ y) + \chi(y)$ on a square-symmetric groupoid. Consequently, they also examined the stability of maximum equation $\max\{\chi(x + y), \chi(x - y)\} = \chi(x) + \chi(y)$ on additive abelian groups. Various appropriate and useful results regarding stability can be found in papers by Przebieracz [8,9].

This paper is arranged, as follows: in Section 2, we prove the functional Equation (1) for any arbitrary group $G$ without any characterisation of an additive function’s absolute value. Besides, we also demonstrate some consistent and useful results concerning the normal subgroup of $G$.

In Section 3, we analyse the functional Equation (2) in order to obtain its solution. For this purpose, we drop the additional assumption that $G$ is an additive abelian group and is divisible by 6. We present the generalization of Theorem 2 by proposing a discretely normed abelian group $G$. Moreover, we investigate the functional Equation (2) for any arbitrary group $G$, which satisfies the Kannappan condition [10].

Section 4 deals with the stability results of the MFE (1) for a function $\chi : G \rightarrow \mathbb{R}$, where $G$ be any group and find some useful connection of stability with commutators and embedding of a group.

2. Solutions of the Functional Equation (1)

Throughout this article, let $G$ be any group, and 1 is considered to be the identity element of a group $G$.

Definition 1 ([10]). Let $G$ be an arbitrary group, we say a function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition if

$$\chi(uvx) = \chi(uxv)$$

for all $u, v, x \in G$.

Theorem 3. Let $G$ be a group and function $\chi : G \rightarrow \mathbb{R}$ satisfies the Kannappan condition then

$$\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y)$$

for all $x, y \in G$ (3)

if and only if

$$\min\{\chi(xy), \chi(xy^{-1})\} = |\chi(x) - \chi(y)|$$

for all $x, y \in G$ (4)

and also satisfies

$$\chi(x^2) = 2\chi(x)$$

for any $x \in G$. (5)

Proof. Assume that $\chi$ satisfies Equation (3), and then by setting $x = y = 1$ in (3), we can conclude that $\chi(1) = 0$. It is also clear that $\chi(x) \geq 0$ and $\chi(x^2) = 2\chi(x)$ for any $x \in G$. The proof of Equation (4) consists of the following simple computation:
\[
\max \{ \chi(xy), \chi(xy^{-1}) \} + \min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(xy) + \chi(xy^{-1})
\]
\[
\chi(x) + \chi(y) + \min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(xy) + \chi(xy^{-1})
\]
\[
\chi(x) + \chi(y) + \min \{ \chi(xy), \chi(xy^{-1}) \} = \max \{ \chi(xyxy^{-1}), \chi(xyyx^{-1}) \}
\]
\[
\chi(x) + \chi(y) + \min \{ \chi(xy), \chi(xy^{-1}) \} = \max \{ \chi(x^2), \chi(y^2) \}
\]
\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \max \{ 2\chi(x), 2\chi(y) \} - \chi(x) - \chi(y)
\]
\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = |\chi(x) - \chi(y)|.
\]

Conversely, assume that Equations (4) and (5) hold. Subsequently, we can obtain
\[
\max \{ \chi(xy), \chi(xy^{-1}) \} - \min \{ \chi(xy), \chi(xy^{-1}) \} = |\chi(xy) - \chi(xy^{-1})|.
\]

Using Equation (4) in the following computation, we get that
\[
\max \{ \chi(xy), \chi(xy^{-1}) \} - |\chi(x) - \chi(y)| = |\chi(xy) - \chi(xy^{-1})|
\]
\[
= \min \{ \chi(xyxy^{-1}), \chi(xyyx^{-1}) \}
\]
\[
= \min \{ \chi(x^2), \chi(y^2) \}
\]
\[
\max \{ \chi(xy), \chi(xy^{-1}) \} = 2\min \{ \chi(x), \chi(y) \} + |\chi(x) - \chi(y)|
\]
\[
\max \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x) + \chi(y).
\]

\[\square\]

**Corollary 1.** Let \( G \) be a group and function \( \chi: G \to \mathbb{R} \) satisfies the Kannappan condition, and then \( \chi \) satisfies the Equation (3) if and only if
\[
\chi(x) + \chi(y) = \chi(xy) + \chi(xy^{-1}) - |\chi(x) - \chi(y)| \quad \text{for any } x, y \in G
\]

and also \( \chi(1) = 0 \).

**Proof.** Suppose that \( \chi(1) = 0 \) and Equation (6) holds. Subsequently, (6) can be written as
\[
2\max \{ \chi(x), \chi(y) \} = \chi(xy) + \chi(xy^{-1}) \quad \text{for any } x, y \in G.
\]

Because \( \chi(1) = 0 \), then, for \( x = y \), we can obtain that \( \chi(x^2) = 2\chi(x) \). Afterwards, writing \( xy \) instead of \( x \) and \( xy^{-1} \) instead of \( y \) in (7), we can conclude that
\[
2\max \{ \chi(xy), \chi(xy^{-1}) \} = \chi(xyxy^{-1}) + \chi(xyyx^{-1})
\]
\[
= \frac{1}{2} \{ \chi(x^2) + \chi(y^2) \}
\]
\[
= \frac{1}{2} \{ 2\chi(x) + 2\chi(y) \}
\]
\[
\max \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x) + \chi(y).
\]

Conversely, assume that maximum Equation (3) holds, then, by Theorem 3, we can determine that \( \chi(1) = 0 \) and Equation (6) holds. \[\square\]

**Corollary 2.** Let \( G \) be a group and function \( \chi: G \to \mathbb{R} \) satisfies the Kannappan condition, and then \( \chi \) is a solution of equation
\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = |\chi(x) - \chi(y)| \quad \text{for any } x, y \in G
and also satisfies
\[ \chi(x^2) = 2\chi(x) \quad \text{for any } x \in G \]

if and only if there exists an additive function \( \alpha : G \to \mathbb{R} \) such that \( \chi(x) = |\alpha(x)| \) for any \( x \in G \).

**Theorem 4.** Let \( G \) be a group and function \( \chi : G \to \mathbb{R} \) satisfies the Kannappan condition. If the Equations (4) and (5) are satisfied, then there exists a normal subgroup \( N_\chi = \{ v \in G \mid \chi(v) = 0 \} \) of \( G \).

**Proof.** Because the function \( \chi : G \to \mathbb{R} \) satisfies the Equations (4) and (5), then by Theorem 3
\[ \max\{ \chi(xy), \chi(xy^{-1}) \} = \chi(x) + \chi(y) \quad \text{for any } x, y \in G. \]

When we investigate Equations (3) and (4), then we can see that either \( \chi(xy) = |\chi(x) - \chi(y)| \) then \( \chi(xy^{-1}) = \chi(x) + \chi(y) \) or \( \chi(xy^{-1}) = |\chi(x) - \chi(y)| \) then \( \chi(xy) = \chi(x) + \chi(y) \). First, writing \( v \) instead of \( y \) and assume that \( \chi(x) \neq 0 \) for some \( x \in G \) and also \( \chi(vx) = \chi(xv^{-1}) \). Subsequently, we can compute that \( \chi(x) + \chi(v) = |\chi(x) - \chi(v)| \). From Equation (3), we can easily deduce that \( \chi(v), \chi(x) \geq 0 \). Our assumption that \( \chi(x) \neq 0 \) forces \( \chi(v) \) to be 0.

From Equation (5), we can compute that \( \chi(1) = 0 \), therefore \( 1 \in N_\chi \). Let \( u, v \in N_\chi \). We can also deduce from Equation (3) that \( \chi \) is even, so we have \( \chi(v^{-1}) = \chi(v) = 0 \), therefore \( v^{-1} \in N_\chi \). Additionally, \( 0 = \chi(u) + \chi(v) = \max\{ \chi(uv), \chi(uv^{-1}) \} = \chi(uv) \), but \( \chi(x) \geq 0 \) for all \( x \in G \); therefore, we can obtain that \( \chi(uv) = 0 \), so \( uv \in N_\chi \). Hence \( N_\chi \) is a subgroup of \( G \). Let \( v \in N_\chi \) and \( x \in G \), then Equation (3) yields that
\[
\chi(x) + \chi(x^{-1}vx) = \max\{ \chi(xx^{-1}vx), \chi(xx^{-1}v^{-1}x) \}
= \max\{ \chi(vx), \chi(v^{-1}x) \}
= \max\{ \chi(x^{-1}v^{-1}), \chi(x^{-1}v) \}
= \chi(v^{-1}) + \chi(v)
\]
\[\chi(x) + \chi(x^{-1}vx) = \chi(x) + \chi(v)\]
\[\chi(x^{-1}vx) = \chi(v),\]

which implies that \( \chi(x^{-1}vx) = \chi(v) \) for all \( x \in G \) and \( v \in N_\chi \). Hence, \( N_\chi \) is a normal subgroup of \( G \). \( \square \)

**Corollary 3.** For any group \( G \), let a function \( \chi \) on group \( G \) satisfying Equation (3), then \( \chi : G/N_\chi \to \mathbb{R} \) also satisfies
\[ \chi(xv) = \chi(x) \quad \text{for any } x \in G, v \in N_\chi. \]  

Additionally, \( G/N_\chi \) is an abelian quotient group.

**Proof.** Let \( x \in G \) be an arbitrary and assume that \( v \in N_\chi \). We can deduce from (3)
\[ \chi(xv) \leq \max\{ \chi(xv), \chi(xv^{-1}) \} = \chi(x) + \chi(v) = \chi(x) \]
\[ \leq \max\{ \chi(xv^2), \chi(x) \} = \chi(xv) + \chi(v) = \chi(xv) \]

that \( \chi(xv) = \chi(x) + \chi(v) \). Hence, \( \chi(xv) = \chi(x) \) for any \( x \in G \) and \( v \in N_\chi \). Moreover, we already proved that \( \chi(x^{-1}vx) = \chi(v) \) for any \( x \in G \) and \( v \in N_\chi \), then, by replacing \( v \) with \( xv \), we can evaluate that \( \chi(xv) = \chi(xv) \) for any \( x \in G \) and \( v \in N_\chi \). Additionally, \( N_\chi \) is a normal subgroup of \( G \); therefore, \( G/N_\chi \) is an abelian quotient group. \( \square \)
Corollary 4. Let \( G \) be any group and function \( \chi : G \rightarrow \mathbb{R} \) satisfy the Kannappan condition. If the maximum functional Equation (3) holds, then \( \min \{ \chi(xy), \chi(xy^{-1}) \} \in N_x \) for every \( x, y \in G \).

Proof. Assume that \( \chi \) satisfies the Equation (3) and Kannappan condition. Subsequently, from Theorem 3, we can compute that

\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x) - \chi(y), \quad \text{when} \ (\chi(x) \geq \chi(y))
\]
or

\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(y) - \chi(x), \quad \text{when} \ (\chi(x) < \chi(y))
\]

combining both cases, we can see that \( 2 \min \{ \chi(xy), \chi(xy^{-1}) \} = 0 \), therefore \( \min \{ \chi(xy), \chi(xy^{-1}) \} = 0 \). In either case, it follows that \( \min \{ \chi(xy), \chi(xy^{-1}) \} \in N_x \) for every \( x, y \in G \). 

\[\blacksquare\]

3. Solutions of the Functional Equation (2)

Theorem 5. Let \( G \) be any arbitrary group, then a function \( \chi : G \rightarrow \mathbb{R} \) satisfies

\[
\max \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x)\chi(y) \quad \text{for any} \ x, y \in G
\] (9)

if and only if simultaneously \( \chi(x) = 0 \) or

\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x)\chi(y)^{-1} \vee [\chi(x)\chi(y)^{-1}]^{-1} \quad \text{for any} \ x, y \in G
\] (10)

holds and satisfies

\[
\chi(x^2) = \chi(x)^2 \quad \text{for all} \ x \in G.
\] (11)

Proof. Suppose that \( \chi \) satisfies Equation (9). Assume 1 as neutral element of group \( G \), then by putting \( x = y = 1 \) in (9), we can obtain that \( \chi(1)\chi(1) = \chi(1) \), then \( \chi(1) = 0 \) or \( \chi(1) = 1 \). Let \( \chi(1) = 0 \), then by Equation (9) we can compute \( \chi(x) = 0 \) for any \( x \in G \). Assume that \( \chi(1) = 1 \), then from (9), It is easy to see that \( \chi(x^{-1}) = \chi(x) \geq 0, \chi(x^2) \geq 1 \) and \( \chi(x^2) = \chi(x)x \) for any \( x \in G \). The proof of Equation (10) consists of the following simple computation:

\[
\max \{ \chi(xy), \chi(xy^{-1}) \} \cdot \min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(xy)\chi(xy^{-1})
\]

\[
\chi(x)\chi(y) \cdot \min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(xy)\chi(xy^{-1})
\]

\[
\chi(x)\chi(y) \cdot \min \{ \chi(xy), \chi(xy^{-1}) \} = \max \{ \chi(xy^xy^{-1}), \chi(xy^yx^{-1}) \}
\]

\[
\chi(x)\chi(y) \cdot \min \{ \chi(xy), \chi(xy^{-1}) \} = \max \{ \chi(x^2), \chi(y^2) \}
\]

\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \max \{ \chi(x)^2, \chi(y)^2 \} \chi(x)^{-1}\chi(y)^{-1}
\]

\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x)\chi(y)^{-1} \vee [\chi(x)\chi(y)^{-1}]^{-1} \quad \text{when} \ (\chi(x) \geq \chi(y))
\]

\[
\min \{ \chi(xy), \chi(xy^{-1}) \} = \chi(x)\chi(y)^{-1} \vee [\chi(x)\chi(y)^{-1}]^{-1} \quad \text{when} \ (\chi(x) < \chi(y)).
\]

Conversely, suppose that the Equations (10) and (11) are satisfied and \( \chi(x) \neq 0 \). Subsequently, it can be determined that

\[
\max \{ \chi(xy), \chi(xy^{-1}) \} \cdot \min \{ \chi(xy), \chi(xy^{-1}) \}^{-1} = \chi(xy)\chi(xy^{-1})^{-1}
\]

\[
\vee \chi(xy)^{-1}\chi(xy^{-1}).
\]

Here, we have two cases, in the first case, using Equation (10) and (11), we derive the required result, as follows:
Definition 2. If and only if there exist an additive function \( \chi \) the main theorem of Toborg \cite{2}, a function \( G \) notion of discretely normed abelian group

\[ G \rightarrow \mathbb{R} \] is called the discretely normed abelian group \cite{11}. Let \( G \) be an abelian group. Subsequently, a function \( \chi : G \rightarrow \mathbb{R} \) is said to be a discrete norm on \( G \) if there exists some \( \alpha > 0 \) such that \( \chi(x) > \alpha \) whenever \( x \) is not identity element of \( G \). Afterwards, \((G, \chi)\) is called the discretely normed abelian group \cite{11}.

Simon and Volkmann \cite{5} proved Theorem 2 with additional assumption that \( G \) is an additive abelian group and is divisible by 6, but we present the generalization of Theorem 2 by introducing the notion of discretely normed abelian group \( G \), as follows:

Theorem 6. Assume that \((G, \chi)\) be a discretely normed abelian group, then a function \( \chi : G \rightarrow \mathbb{R} \) fulfills (9) if and only if there exists an additive function \( \alpha : G \rightarrow \mathbb{R} \) such that \( \chi(x) = e^{\alpha(x)} \) for any \( x \in G \).

Proof. Because \( G \) is a discretely normed abelian group, therefore there exists a discrete norm function \( \chi : G \rightarrow \mathbb{R} \) such that \( \chi(x) > 0 \) whenever \( x \in G \setminus \{1\} \). Assume that \( \eta(x) = \log \chi(x) \), then applying the main theorem of Toborg \cite{2}, a function \( \chi : G \rightarrow \mathbb{R} \) satisfies

\[ \max\{ \eta(xy), \eta(xy^{-1}) \} = \eta(x) + \eta(y) \]

if and only if there exist an additive function \( \alpha : G \rightarrow \mathbb{R} \) such that \( \eta(x) = |\alpha(x)| \). Subsequently, we have

\[ \max\{ \log \chi(xy), \log \chi(xy^{-1}) \} = \log \chi(x) + \log \chi(y) \]

if and only if \( \log \chi(x) = |\alpha(x)| \), which implies that

\[ \log \max\{ \chi(xy), \chi(xy^{-1}) \} = \log \chi(x) \chi(y), \]

thus, we conclude that

\[ \max\{ \chi(xy), \chi(xy^{-1}) \} = \chi(x) \chi(y) \]

if and only if there exists an additive function \( \alpha : G \rightarrow \mathbb{R} \), such that \( \chi(x) = e^{|\alpha(x)|} \) for any \( x \in G \setminus \{1\} \). □

Corollary 5. Let \( G \) be a group and function \( \chi : G \rightarrow \mathbb{R} \) satisfies the maximum Equation (9) if and only if

\[ \chi(xy)\chi(xy^{-1}) = \chi(x)\chi(y) \min\{ \chi(xy), \chi(xy^{-1}) \} \quad \text{for any} \ x, y \in G \]  

(12)

and also \( \chi(1) = 1 \).

Corollary 6. Let \( G \) be a group and function \( \chi : G \rightarrow \mathbb{R} \) is a solution of Equation (10) satisfies (11) if and only if there exists an additive function \( \alpha : G \rightarrow \mathbb{R} \), such that \( \chi(x) = e^{\alpha(x)} \) for any \( x \in G \).

Definition 2 (\cite{11}). Let \( G \) be an abelian group. Subsequently, a function \( \chi : G \rightarrow \mathbb{R} \) is called the discrete norm on \( G \) if there exists some \( \alpha > 0 \) such that \( \chi(x) > \alpha \) whenever \( x \) is not identity element of \( G \). Afterwards, \((G, \chi)\) is called the discretely normed abelian group \cite{11}.
Corollary 7. Let \((G, \chi)\) be a discretely normed abelian group, then a function \(\chi\) is a solution of Equation (10) satisfying (11) if and only if there exists an additive function \(\alpha : G \to \mathbb{R}\), such that \(\chi(x) = e^{\alpha(x)}\) for any non-identity element \(x \in G\).

**Proof.** From Theorem 6, we concluded that maximum functional Equation (9) holds; therefore, using Theorem 5, we can also obtain required proof. \(\square\)

We have well-known theorem presented by Steprans Juris in [11] about a group \(G\), which is a discretely normed abelian group. Therefore, we have following corollaries.

Corollary 8. For free abelian group \(G\), a function \(\chi\) is a solution of Equation (9) if and only if there exists an additive function \(\alpha : G \to \mathbb{R}\), such that \(\chi(x) = e^{\alpha(x)}\) for any \(x \in G \setminus \{1\}\).

Corollary 9. Suppose that \(G\) is a free abelian group, then \(\chi\) is a solution of functional Equation (10) satisfying (11) if and only if there exists an additive function \(\alpha : G \to \mathbb{R}\), such that \(\chi(x) = e^{\alpha(x)}\) for any \(x \in G \setminus \{1\}\).

Theorem 7. Let \(G\) be any group and let a function \(\chi : G \to \mathbb{R}\) is a solution of Equation (10) and (11), which is not identically zero, then there exists a normal subgroup \(H = \{v \in G \mid \chi(v^2) = 1\}\) of \(G\).

**Proof.** Because the function \(\chi : G \to \mathbb{R}\) satisfying the Equations (10) and (11), then by Theorem 5, we can obtain that

\[
\max\{ \chi(xy), \chi(xy^{-1}) \} = \chi(x)\chi(y) \quad \text{for all } x, y \in G. \tag{13}
\]

For neutral element 1, we can write \(\chi(1^2) = \chi(1) = 1\), then \(1 \in H\). Let \(v \in H\), then \(\chi(v^2) = 1\). From Equation (13), follows immediately that \(\chi\) is even; therefore, \(\chi(v^{-2}) = \chi(v^2) = 1\), hence \(v^{-1} \in H\). Let \(u, v \in H\), therefore, by property of \(H\), \(\chi(u^2) = 1\) and \(\chi(v^2) = 1\). We can deduce from maximum Equation (13) that

\[
\chi(u^2v^2) = \chi(u^2v^2)\chi(v^2) = \max\{ \chi(u^2v^4), \chi(u^2) \} \\
\geq \chi(u^2) \\
\chi(u^2v^2) \geq \chi(u^2)\chi(v^2). \tag{14}
\]

\[
\chi(u^2)\chi(v^2) = \max\{ \chi(u^2v^2), \chi(u^2v^{-2}) \} \\
\geq \chi(u^2v^2) \\
\chi(u^2)\chi(v^2) \geq \chi(u^2v^2). \tag{15}
\]

From inequalities (14) and (15), we can calculate \(\chi(u^2v^2) = \chi(u^2)\chi(v^2) = 1\), which implies that \(uv \in H\). Therefore, \(H\) is a subgroup of \(G\). Additionally, Equation (13) yields that \(\chi\) is central, therefore \(\chi(xv) = \chi(vx)\), for every \(x \in G\) and \(v \in H\). Hence, \(H\) is a normal subgroup of \(G\). \(\square\)

Corollary 10. For any group \(G\), let a function \(\chi\) on group \(G\) satisfying (13), which is not identically zero, then \(\chi : G/H \to \mathbb{R}\) also satisfies

\[
\chi(xv) = \chi(x)\chi(v) \quad \text{for any } x \in G, \ v \in H. 
\]
Moreover,
\[ \chi(xv^2) = \chi(v^2x) = \chi(x) \quad \text{for any } x \in G, \ v \in H_x. \] (16)

**Proof.** Let \( x \in G \) be an arbitrary and assume that \( v \in H_x \), then \( \chi(v^2) = 1 \). We can deduce from (13) that
\[ \chi(xv)\chi(v) = \max\{ \chi(xv^2), \chi(x) \} \geq \chi(x)\chi(v^2), \]
which provides that \( \chi(xv)\chi(v) \geq \chi(x)\chi(v^2) \). Applying condition (11), we compute \( \chi(xv) \geq \chi(x)\chi(v) \). Additionally, from (13), we have \( \chi(x)\chi(v) \geq \chi(xv) \), so we can evaluate \( \chi(xv) = \chi(x)\chi(v) \) for any \( x \in G \) and \( v \in H_x \).

Because \( H_x \) is a subgroup of \( G \), then \( v^2 \in H_x \), so we can see that \( \chi(xv^2) = \chi(x)\chi(v^2) \) for any \( x \in G \) and \( v^2 \in H_x \). Therefore \( \chi(xv^2) = \chi(x) \). Writing \( x \) instead of \( y \) and \( v \) instead of \( x \) in (13), we can conclude \( \chi(x^{-1}vx) = \chi(v) \) for any \( x \in G \), then writing \( xv^2 \) instead of \( v \), we have \( \chi(vx^2) = \chi(xv^2) = \chi(x) \). \( \square \)

**Corollary 11.** Let \( G \) be a group and function \( \chi : G \to \mathbb{R} \) satisfies the Kannappan condition. If the maximum functional Equation (13) holds, then \( \min\{ \chi(xy), \chi(xy^{-1}) \} \in H_x \) for any \( x, y \in G \).

**Proof.** Assume that \( \chi \) satisfies the Equation (9) and Kannappan condition. Subsequently, from Theorem 5, we can conclude that
\[ \min\{ \chi(xy), \chi(xy^{-1}) \} = \chi(x)\chi(y)^{-1}, \quad \text{(when } \chi(x) \geq \chi(y)) \]
or
\[ \min\{ \chi(xy), \chi(xy^{-1}) \} = [\chi(x)\chi(y)^{-1}]^{-1}, \quad \text{(when } \chi(x) < \chi(y)) \]
combining both cases, we can see that \( \min\{ \chi(xy), \chi(xy^{-1}) \} = [\min\{ \chi(xy), \chi(xy^{-1}) \}]^{-1} \), therefore \( \min\{ \chi(xy), \chi(xy^{-1}) \}^2 = 1 \). In either case, it follows that \( \min\{ \chi(xy), \chi(xy^{-1}) \} \in H_x \) for every \( x, y \in G \). \( \square \)

4. Stability of Maximum Functional Equation (1)

In order to check the stability of MFE (1) in two-variables \( x \) and \( y \), at the first stage, we put \( y = x \) in (1) and derive the stability of MFE (1) in a single variable, as follows:

**Theorem 8.** Let \( G \) be any group and a function \( \eta : G \to \mathbb{R} \) satisfies
\[ |\max\{ \eta(x^2), \eta(1) \} - 2\eta(x) | \leq \lambda \quad \text{for any } x \in G, \] (17)
for some \( \lambda \geq 0 \). Subsequently, we can obtain a solution \( \chi : G \to \mathbb{R} \) of
\[ \max\{ \chi(x^2), \chi(1) \} = 2\chi(x) \quad \text{for any } x \in G \] (18)
such that
\[ -3\lambda \leq \chi(x) - \eta(x) \leq \lambda \quad \text{for any } x \in G. \] (19)

Additionally, \( \chi \) is given by
\[ \chi(x) = \lim_{n \to \infty} \frac{1}{2^n} \eta(x^{2^n}) \quad x \in G. \] (20)

Additionally, by (18), \( \chi \) is uniquely determined, and by (19), the requirement of \( \chi - \eta \) to be bounded is also satisfied.
Proof. First, put $x = 1$ in (17), then we have $|\max\{\eta(1), \eta(1)\} - 2\eta(1)| \leq \lambda$, which implies that $|\eta(1)| \leq \lambda$. Furthermore, (17) also implies that
\begin{align*}
-\lambda + 2\eta(x) &\leq \max\{\eta(x^2), \eta(1)\} \leq \lambda + 2\eta(x) \\
-\lambda &\leq \eta(1) \leq \lambda + 2\eta(x) \\
-\lambda &\leq \eta(x) \quad \text{for any } x \in G.
\end{align*}
(21)

Writing $x^2$ instead of $x$, then also $-\lambda \leq \eta(x^2)$, so we can get that
\begin{align*}
\eta(1) &\leq \lambda = 2\lambda - \lambda \leq 2\lambda + \eta(x^2) \\
\eta(1) &\leq 2\lambda + \eta(x^2) \quad \text{for any } x \in G.
\end{align*}
(22)

From (17), we have
\begin{align*}
2\eta(x) &\leq \lambda + \max\{\eta(x^2), \eta(1)\} \\
2\eta(x) &\leq \lambda + \eta(1) \leq 3\lambda + \eta(x^2) \quad \text{using (22)} \\
\text{or} &\quad 2\eta(x) \leq \lambda + \eta(x^2) \leq 3\lambda + \eta(x^2).
\end{align*}
Combining both cases, we can conclude
\begin{equation}
-3\lambda \leq \eta(x^2) - 2\eta(x). \tag{23}
\end{equation}

From (17), we have $\max\{\eta(x^2), \eta(1)\} \leq \lambda + 2\eta(x)$, which is only possible when
\begin{equation}
\eta(x^2) - 2\eta(x) \leq \lambda. \tag{24}
\end{equation}

From inequalities (23) and (24), we have
\begin{equation}
-3\lambda \leq \eta(x^2) - 2\eta(x) \leq \lambda \quad \text{for all } x \in G. \tag{25}
\end{equation}

When we observe (25), it can be seen that the function $\chi: G \to \mathbb{R}$ presented in (20) exists and this $\chi$ satisfies
\begin{equation}
\chi(x^2) = 2\chi(x) \quad \text{for all } x \in G, \tag{26}
\end{equation}
also $\chi$ fulfills (19). Furthermore, writing $x^{2^n}$ instead of $x$ in (21), dividing by $2^n$ and taking a limit $n \to \infty$ and, using (20), we can obtain that $\chi(x) \geq 0$ for any $x \in G$. Hence we can obtain (18) from (26). Furthermore, when we consider (18), then we can easily see the uniqueness of $\chi$, due to the result that $\chi(x^2) = 2\chi(x) \geq 0$ for any $x \in G$.

By utilizing Theorem (8), we are going to derive the stability of Equation (1) in two-variables $x$ and $y$. □

**Theorem 9.** Let $G$ be any group and function $\eta: G \to \mathbb{R}$ satisfies
\begin{equation}
|\max\{\eta(xy), \eta(xy^{-1})\} - \eta(x) - \eta(y)| \leq \lambda \quad \text{for any } x, y \in G, \tag{27}
\end{equation}
for some $\lambda \geq 0$. Subsequently, we can evaluate a unique solution $\chi: G \to \mathbb{R}$ of
\begin{equation}
\max\{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y) \quad \text{for any } x, y \in G \tag{28}
\end{equation}
Theorem 10. Let $G$ be a group and function $\eta : G \rightarrow \mathbb{R}$ fulfills (27), then, for any $x, y \in G$, it holds the condition
\[
\lim_{n \rightarrow \infty} \frac{1}{2^n} \max \{\eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n})\} = \max \{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\} = 0
\] (31)
if and only if the Equation (28) is satisfied.

Proof. Assume that maximum Equation (28) is satisfied. For every $x, y \in G$, we have
\[
|\max \{\eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n})\} - \max \{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\}| \\
\leq |\max \{\eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n})\} - \eta(x^{2^n}) - \eta(y^{2^n})| \\
+ |\max \{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\} - \eta(x^{2^n}) - \eta(y^{2^n})| \\
\leq \lambda + |\max \{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\} - \eta(x^{2^n}) - \eta(y^{2^n})|.
\]
To obtain the required result, first, we divide both sides by $2^n$, apply limit as $n \rightarrow \infty$ and utilize the given condition (28), we obtain that, for any $x, y \in G$, we can get
\[
\lim_{n \rightarrow \infty} \frac{1}{2^n} \max \{\max \{\eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n})\} - \max \{\eta([xy]^{2^n}), \eta([xy^{-1}]^{2^n})\}\} = 0.
\]
Conversely, assume that condition (31) holds. Putting $x^{2^n}$ and $y^{2^n}$ in (27) instead of $x$ and $y$, respectively, then dividing by $2^n$ and taking limit $n \rightarrow \infty$, we can compute
\[
\lim_{n \rightarrow \infty} \frac{1}{2^n} \max \{\eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n})\} = \chi(x) + \chi(y),
\]
\[
\lim_{n \rightarrow \infty} \frac{1}{2^n} \max \{\eta(x^{2^n}y^{2^n}), \eta([xy]^{2^n})\} = \max \{\chi(xy), \chi(xy^{-1})\}.
\]
Given condition (31), yielding that $\max \{\chi(xy), \chi(xy^{-1})\} = \chi(x) + \chi(y)$.

Furthermore, the given condition (31) for function $\eta$ is not directly associated to the properties of a group $G$, but we can see some useful results regarding group $G$. The resulting condition presented below is equivalent to the above condition (31), as there exists a subsequence of $\mathbb{N}$, such that
\[
\lim_{n \rightarrow \infty} \frac{1}{2m(n)} \max \{\max \{\eta(x^{2m(n)}y^{2m(n)}), \eta(x^{2m(n)}y^{-2m(n)})\} - \max \{\eta([xy]^{2m(n)}), \eta([xy^{-1}]^{2m(n)})\}\} = 0.
\]
This refers to the function $\eta$ and completely satisfactory, due to the fact that both limits
\[
\lim_{n \rightarrow \infty} \frac{1}{2m(n)} \max \{\eta(x^{2m(n)}y^{2m(n)}), \eta(x^{2m(n)}y^{-2m(n)})\}
\]
and
\[
\lim_{n \to \infty} \frac{1}{2^m(n)} \max \{ \eta([xy]^{2m(n)}), \eta([xy^{-1}]^{2m(n)}) \}
\]
are finite and exists. □

**Corollary 12.** A function \( \eta : G \to \mathbb{R} \) fulfills (18), then it satisfies the condition

\[
\lim_{n \to \infty} \frac{1}{2^n} \max \{ \eta(x^{2^n}y^{2^n}), \eta(x^{2^n}y^{-2^n}) \} - \eta([xy]^{2^n}) = 0 \quad \text{for all } x, y \in G \tag{32}
\]

if and only if \( \chi(xy) = \chi(x) + \chi(y) \) for any \( x, y \in G \).

**Remark 1.** (i) The given condition (31) holds when group \( G \) is an \( n \)—Abelian group (for any integer \( n \), a group \( G \) is called an \( n \)—Abelian group if \( (ab)^n = a^nb^n \), for any \( a, b \in G \), see [12,13] ).

(ii) The condition (31) also satisfies when group \( G \) belongs to the class \( C_n \) for any natural number \( n \in \mathbb{N} \) (for all \( n \in \mathbb{N} \), \( C_n \) is denoted as the class of groups, which satisfies the relation \( b^n a^n = a^n b^n \)).

(iii) When \( \chi \) is central, then condition (31) also holds.

**Theorem 11.** If maximum functional Equation (1) is stable on group \( G \), then every free abelian group \( K \) can be embedded into a group \( G \).

**Proof.** Assume that \( \eta : G \to \mathbb{R} \) and let \( \lambda \geq 0 \) be such that

\[
| \max \{ \eta(xy), \eta(xy^{-1}) \} - \eta(x) - \eta(y) | \leq \lambda \quad \text{for any } x, y \in G. \tag{33}
\]

Because every free abelian group is a torsion-free, \( K \) is torsion-free group. Subsequently, from HNN-extensions, any torsion-free group \( K \) can be embedded into a group \( G \), if for all \( k \in K \), there exists \( g \in G \) such that \( gk^{-1} = k^2 \) [14,15]. Assume that \( K \) is embedded into a group \( G \). Subsequently, inequality (33) can be rewritten as

\[
| \max \{ \eta(kg), \eta(kg^{-1}) \} - \eta(k) - \eta(g) | \leq \lambda \quad \text{for any } k, g \in G. \tag{34}
\]

We need to show that \( \eta \) is bounded. The result is obvious for \( \lambda = 0 \), so let \( \lambda \) be positive. More specifically, we prove that \( |\eta(k)| \leq 2\lambda \) for all \( k \in G \). Afterwards, from (34), we have two possibilities, either \( |\eta(kg^{-1}) - \eta(k) - \eta(g)| \leq \lambda \) or \( |\eta(kg) - \eta(k) - \eta(g)| \leq \lambda \). Let us consider first possibility and put \( k = g = 1 \), then we can conclude that \( |\eta(1)| \leq \lambda \). Put \( g = k \), then we have

\[
|\eta(1) - 2\eta(k)| \leq \lambda
\]

\[
|2\eta(k)| - |\eta(1)| \leq \lambda
\]

\[
2|\eta(k)| \leq \lambda + |\eta(1)|
\]

|\eta(k)| \leq \lambda.

For second possibility, we have

\[
|\eta(kg) - \eta(k) - \eta(g)| \leq \lambda. \tag{35}
\]
On the contrary, let $|\eta(k)| \geq 2\lambda$ for some arbitrary $k \in G$. Put $g = k$ in (35), then we have

$$
|\eta(k^2) - 2\eta(k)| \leq \lambda \\
|2\eta(k) - \eta(k^2)| \leq \lambda \\
2|\eta(k)| - \lambda \leq |\eta(k^2)| \\
3\lambda \leq |\eta(k^2)|.
$$

Again, put $g = k^2$ in (35), then

$$
|\eta(k^3) - \eta(k) - \eta(k^2)| \leq \lambda \\
|\eta(k) + \eta(k^2) - \eta(k^3)| \leq \lambda \\
|\eta(k)| + |\eta(k^2)| \leq \lambda + |\eta(k^3)| \\
5\lambda - \lambda \leq |\eta(k^3)| \\
4\lambda \leq |\eta(k^3)|.
$$

Repeating again, for $g = k^3$, we can determine $|\eta(k^4)| \geq 4\lambda$. Continuing this process, we can evaluate that

$$(m + 1)\lambda \leq |\eta(k^m)|, \quad \text{for } m=1,\ldots,$$

which determines that when $m$ varies, then $\eta(k^m)$ is unbounded.

Furthermore, pick an arbitrary element $g \in G$, such that $k^2 = gkg^{-1}$. Subsequently, for every integer $m > 0$, we have $k^{2m} = g^{m}g^{-1}$. Therefore, for every $m$, put $g = k^m$ and $k = k^m$ in (35), then

$$
|\eta(k^{2m}) - 2\eta(k^m)| \leq \lambda \\
|\eta(g^{m}g^{-1}) - 2\eta(k^m)| \leq \lambda. \quad (36)
$$

Additionally, inequality (35) follows that

$$
|\eta(g^{m}g^{-1}) - \eta(g) - \eta(k^m)| \leq \lambda \quad \text{and} \quad |\eta(g^{m}g^{-1}) - \eta(k^m) - \eta(g^{-1})| \leq \lambda,
$$

thus, we can conclude that

$$
|\eta(g^{m}g^{-1}) - \eta(g) - \eta(k^m) - \eta(g^{-1})| \leq |\eta(g^{m}g^{-1}) - \eta(g) - \eta(k^m)| + |\eta(g^{m}g^{-1}) - \eta(k^m) - \eta(g^{-1})| \\
|\eta(g^{m}g^{-1}) - \eta(g) - \eta(k^m) - \eta(g^{-1})| \leq 2\lambda. \quad (37)
$$

From (36) and (37), we have

$$
|\eta(g^{m}g^{-1}) - 2\eta(k^m) + \eta(k^m)| \leq 2\lambda + |\eta(g)| + |\eta(g^{-1})| \\
|\eta(k^m)| \leq 2\lambda + |\eta(g)| + |\eta(g^{-1})| + |\eta(g^{m}g^{-1}) - 2\eta(k^m)| \\
|\eta(k^m)| \leq 5\lambda \quad \text{for } m=1,2,\ldots,
$$

which gives a contradiction. Hence, $\eta$ is bounded, which completes the required proof. □

**Corollary 13.** If maximum functional Equation (1) is stable on group $G$, then any discretely normed abelian group $K$ can be embedded into a group $G$.

**Proof.** Because $K$ is a discretely normed abelian group, then, by applying theorem from [11], $K$ is a free group; hence it can be embedded into a group $G$. □
Theorem 12. A function \( \eta : G \to \mathbb{R} \) satisfying (27) is bounded on the commutator group \( G_1 \) of the subgroup \( G' \) of \( G \) if the condition (31) is satisfied.

Proof. Assume that condition (31) is satisfied. Subsequently, by Theorem 10, we can get that \( \chi(x) + \chi(y) = \max\{\chi(xy), \chi(xy^{-1})\} \) holds for any \( x, y \in G \). Because this maximum equation holds; therefore, by Theorem 1, there exists an additive function \( a : G \to \mathbb{R} \), such that \( \chi(x) = |a(x)| \) for any \( x \in G \). For \( a, b \in G \), take \( a^{-1}b^{-1}ab \in G \), then

\[
\chi(a^{-1}b^{-1}ab) = |a(a^{-1}b^{-1}ab)| = |a(a^{-1}) + a(b^{-1}) + a(a) + a(b)| \\
= |a(a^{-1})a + a(b^{-1})b| = |a(1) + a(1)| = |0 + 0| \\
\chi(a^{-1}b^{-1}ab) = 0.
\]

Because \( \chi \) is zero on the commutator group \( G_1 \) of the subgroup \( G' \) of \( G \), \( \eta \) is bounded on commutator group \( G_1 \).

Corollary 14. If a function \( \eta : G \to \mathbb{R} \) satisfies the condition

\[
\lim_{n \to \infty} \frac{1}{2^n} \max\{\eta(x^{2^n}y^m), \eta(x^{2^n}y^{-2^n})\} - \eta([xy]^{2^n}) = 0 \quad \text{for any } x, y \in G,
\]

then \( \eta \) is bounded on the commutator group \( G_1 \).

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