GRAPH INVERSE SEMIGROUPS: THEIR CHARACTERIZATION AND COMPLETION

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ABSTRACT. Graph inverse semigroups generalize the polycyclic inverse monoids and play an important role in the theory of $C^*$-algebras. This paper has two main goals: first, to provide an abstract characterization of graph inverse semigroups; and second, to show how they may be completed, under suitable conditions, to form what we call the Cuntz-Krieger semigroup of the graph. This semigroup is the ample semigroup of a topological groupoid associated with the graph, and the semigroup analogue of the Leavitt path algebra of the graph.

1. INTRODUCTION

1.1. What graph inverse semigroup are. We begin by recalling the definition of the polycyclic inverse monoids first introduced by Nivat and Perrut [40]. The polycyclic monoid $P_n$, where $n \geq 2$, is defined as a monoid with zero by the following presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} : a_i^{-1}a_i = 1 \text{ and } a_i^{-1}a_j = 0, i \neq j \rangle.$$ 

Intuitively, we may think of $a_1, \ldots, a_n$ as partial bijections of a set $X$ and $a_1^{-1}, \ldots, a_n^{-1}$ as their respective partial inverses. The first relation says that each partial bijection $a_i$ has domain the whole of $X$ and the second says that the ranges of distinct $a_i$ are orthogonal. Every non-zero element of $P_n$ is of the form $yx^{-1}$ where $x, y \in A_n^*$, the free monoid on the set $\{a_1, \ldots, a_n\}$ where the identity is $\varepsilon \varepsilon^{-1}$; here $\varepsilon$ is the empty string. The product of two elements $yx^{-1}$ and $vu^{-1}$ is zero unless $x$ and $v$ are prefix-comparable, meaning that one is a prefix of the other. If they are prefix-comparable then

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzv^{-1} & \text{if } v = xz \text{ for some string } z \\
y(uz)^{-1} & \text{if } x = vz \text{ for some string } z \end{cases}$$

A few years after the appearance of [40], the polycyclic inverse monoids were then re-introduced, independently, by Cuntz [6]. For this reason, they are often referred to as Cuntz (inverse) semigroups in the $C^*$-algebra literature, such as page 2 of [41]. However, we shall not use this terminology in this paper because for us the Cuntz inverse semigroup will be a different semigroup. The Cuntz $C^*$-algebra $O_n$ is generated by $n$ partial isometries $s_1, \ldots, s_n$ on a Hilbert space satisfying $s_i^*s_j = \delta_{ij}$ and $\sum_{i=1}^n s_is_i^* = 1$. Polycyclic inverse monoids are also implicit in the work of

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Leavitt \cite{33, 34, 35} on rings without invariant basis number. Given a field \( K \), define the unital algebra \( L(1, n) \) generated by elements \( x_1, \ldots, x_n, y_1, \ldots, y_n \) satisfying the relations \( x_i y_j = \delta_{ij} \) and \( \sum_{i=1}^n y_j x_i = 1 \). There are clear parallels between the Cuntz algebras \( O_n \) and the Leavitt algebras \( L(1, n) \) observed in \cite{1} and \cite{49}.

The polycyclic inverse monoid \( P_n \) is constructed from the free monoid on an \( n \)-letter alphabet. Such a monoid can be viewed as the free category on the directed graph consisting of one vertex and \( n \) loops. This suggests that polycyclic monoids might be generalized by replacing free monoids by free categories and this is how graph inverse semigroups arise. This was first carried out by Ash and Hall \cite{2} in 1975 at which point history repeated itself. In \cite{7}, Cuntz and Krieger introduced a class of \( \mathbb{C}^* \)-algebras, constructed from suitable directed graphs, now known as Cuntz-Krieger algebras. In 2005, Abrams and Pini \cite{1} introduced what they called Leavitt path algebras as the algebra analogues of the Cuntz-Krieger algebras. These are also the subject of \cite{49}. The connection between graph inverse semigroups and the Cuntz-Krieger algebras is spelled out by Paterson \cite{42} and, significantly for this paper, in the work by Lenz \cite{37}.

In this paper, we approach graph inverse semigroups from a slightly more general perspective. In \cite{21, 24}, building on work by Leech, the second author described the general procedure by which an inverse semigroup could be constructed from a suitable left cancellative category. Free categories on arbitrary directed graphs are suitable such categories and their inverse semigroups are the graph inverse semigroups.

1.2. Background. For the remainder of this section we shall summarize the key definitions needed to read this paper. For more details on inverse semigroups, the reader is directed to \cite{20, 44}. Throughout this paper all inverse semigroups will be assumed to have a zero and \( \leq \) will denote their natural partial order. We shall also need basic definitions from general semigroup such as Green’s relations \( L, R, H, D \) and \( J \). For each of Green’s relations \( K \) there is a Green’s preorder \( \leq_K \). A semigroup is 0-simple if it has no non-trivial ideals. A semigroup is 0-bisimple if any two non-zero elements are \( D \)-related. A semigroup is combinatorial if the \( H \)-relation is equality. An inverse semigroup \( S \) is said to be completely semisimple if \( s D t \) and \( s \leq t \) implies \( s = t \) for all \( s, t \in S \). A semigroup homomorphism is 0-restricted if only the zero of the domain semigroup maps to the zero of the codomain semigroup. A semigroup homomorphism is idempotent pure if only idempotents map to idempotents. A good reference for general semigroup theory is \cite{11}.

Partially ordered sets, posets, play an important role in this paper. Let \( (E, \leq) \) be a poset. For \( x \in E \) define

\[ x^\downarrow = \{ y \in E : y \leq x \}, \]

the principal order ideal generated by \( x \), and

\[ x^\uparrow = \{ y \in E : y \geq x \}, \]

the principal filter generated by \( x \). We extend this notation to subsets \( A \subseteq E \) and define \( A^\downarrow \) and \( A^\uparrow \). A subset \( A \) such that \( A = A^\downarrow \) is called an order ideal. If \( A \) is a finite set then \( A^\downarrow \) is said to be a finitely generated order ideal. The posets we consider will always have a smallest element 0. Such a poset \( X \) is said to be unambiguous\footnote{Strictly speaking ‘unambiguous except at zero’ but that is too much of a mouthful.} if for all \( x, y \in X \) if there exists \( 0 \neq z \leq x, y \) then either \( x \leq y \) or...
Given $e, f \in E$ we say that $e$ covers $f$ if $e > f$ and there is no $g \in E$ such that $e > g > f$. For each $e \in E$ define $\hat{e}$ to be the set of elements of $E$ that are covered by $e$. A poset is said to be pseudofinite if whenever $e > f$ there exists $g \in \hat{e}$ such that $e > g \geq f$, and for which the sets $\hat{e}$ are always finite; this simply means that the order ideal $e^+ \setminus \{e\}$ is finitely generated.

If $S$ is an inverse semigroup then $S^* = S \setminus \{0\}$. If $X$ is any subset of $S$ then $X^0$ denotes $X \cup \{0\}$, and $E(X)$ denotes the set of idempotents in $X$. If $s \in S$ we write $d(s) = s^{-1}s$ and $r(s) = ss^{-1}$. The partial binary operation defined on an inverse semigroup by defining $st$ only when $d(s) = r(t)$ is called the restricted product. If $st$ is an arbitrary product then $st = (se)(et)$, where $e = s^{-1}stt^{-1}$, is a restricted product. These results imply that in order to check that a function $\theta: S \to T$ between inverse semigroups is a homomorphism it is enough to prove three things: it preserves the restricted product, it preserves the natural partial order, and it preserves the meet operation on the semilattice of idempotents. If $T$ is an inverse subsemigroup of $S$ we say that it is wide if $E(T) = E(S)$. Wide inverse subsemigroups are automatically order ideals.

If $e$ is an idempotent in the inverse semigroup $S$ then $eS$ is called a local submonoid. Let $S$ be an inverse semigroup and $e \in E(S)$. We say that $S$ is an enlargement of $eS$ if $S = eSe$. In this situation, $S$ is Morita equivalent to $eSe$.

We say that elements $s$ and $t$ in an inverse semigroup are compatible, denoted $s \sim t$, if both $s^{-1}t$ and $st^{-1}$ are idempotents. A subset of $S$ is compatible if each pair of elements in the subset are compatible. An inverse semigroup is said to be distributive if the following holds. Let $\{a_1, \ldots, a_m\}$ be a finite subset of $S$ and let $a \in S$ be any element. If $\bigvee_{i=1}^m a_i$ exists then both $\bigvee_{i=1}^m aa_i$ and $\bigvee_{i=1}^m a_i a$ exist and we have the following two equalities

$$a \left( \bigvee_{i=1}^m a_i \right) = \bigvee_{i=1}^m aa_i \quad \text{and} \quad \left( \bigvee_{i=1}^m a_i \right) a = \bigvee_{i=1}^m a_i a.$$

An inverse semigroup is said to be complete if every finite (N.B.) compatible subset has a join and the semigroup is distributive. A homomorphism $\phi: S \to T$ is said to be join-preserving if for every finite subset $A \subseteq S$ the existence of $\bigvee A$ implies the existence of $\bigvee \phi(A)$ and $\bigvee \phi(A) = \phi(\bigvee A)$.

A pair of elements $s, t \in S$ is said to be orthogonal if $s^{-1}t = 0 = st^{-1}$. Observe that $s$ and $t$ are orthogonal iff $d(s)d(t) = 0$ and $r(s)r(t) = 0$. A subset of $S$ is said to be orthogonal iff each pair of distinct elements in it is orthogonal. If the join of a finite set of orthogonal elements exists we talk about orthogonal joins. An inverse semigroup with zero $S$ will be said to be orthogonally complete if it has joins of all finite orthogonal subsets and multiplication distributes over finite orthogonal joins. Homomorphisms between inverse semigroups with zero map finite orthogonal subsets to finite orthogonal subsets. If orthogonal joins are preserved then we say that the homomorphism is orthogonal join-preserving. The symmetric inverse monoids are (orthogonally) complete.

Throughout this paper categories are small and objects are replaced by identities. The elements of a category $C$ are called arrows and the set of identities of $C$ is denoted by $C_0$. Each arrow $a$ has a domain, denoted by $d(a)$, and a codomain denoted by $r(a)$, both of these are identities and $a = ad(a) = r(a)a$. Given identities $e$ and $f$ the set of arrows $eCf$ is called a hom-set and $eCe$ is a monoid called the
local monoid at $e$. We shall say that a pair of identities $e$ and $f$ in a category $C$ are strongly connected if and only if $eCf \neq \emptyset$ and $fCe \neq \emptyset$. A category is said to be strongly connected if each pair of identities is strongly connected. An arrow $a$ is invertible or an isomorphism if there is an arrow $a^{-1}$ such that $a^{-1}a = d(a)$ and $aa^{-1} = r(a)$. A category in which every arrow is invertible is called a groupoid.

We denote the subset of invertible elements of $C$ by $G(C)$. This forms a groupoid. If $G(C) = C_0$ then we shall say that the groupoid of invertible elements is trivial. We say that a category $C$ has trivial subgroups if the only invertible elements in the local monoids are the identities. An identity $e$ in a category $C$ is said to be a root if for every identity $f$ the set $eCf \neq \emptyset$. A principal right ideal in a category $C$ is a subset of the form $aC$ where $a \in C$. A category $C$ is said to be right rigid if $aC \cap bC \neq \emptyset$ implies that $aC \subseteq bC$ or $bC \subseteq aC$; this terminology is derived from Cohn [5]. A left Rees category is a left cancellative, right rigid category in which each principal right ideal is properly contained in only finitely many distinct principal right ideals.

A representation of an inverse semigroup $S$ is a homomorphism $\theta: S \to I(X)$, where $I(X)$ is the symmetric inverse semigroup on the set $X$. Equivalently, this can be defined by means of a partially defined action where $s \cdot x$ is defined if and only if $x \in \text{dom}(\theta(s))$ in which case it is equal to $\theta(s)(x)$. Abstractly, an action of this type is a partial function $S \times X \to X$ mapping $(s, x)$ to $s \cdot x$ when $\exists s \cdot x$ satisfying the following two axioms:

(A1): If $\exists e \cdot x$ where $e$ is an idempotent then $e \cdot x = x$.

(A2): $\exists(zt) \cdot x$ if and only if $\exists s \cdot (t \cdot x)$ in which case they are equal.

An action is effective if for each $x \in X$ there exists $s \in S$ such that $\exists s \cdot x$. We usually require our actions to be effective and, if they are not, they can easily be rendered so by restricting the action to a subset of $X$.

The proofs of the following may be found in [20].

**Lemma 1.1.** Let $S$ be an inverse semigroup.

(1) For each element $s$ in an inverse semigroup $S$ the subset $s^\perp$ is compatible.

(2) If $s$ and $t$ are compatible then $s \land t$ exists and $d(s \land t) = d(s)d(t)$, and dually.

(3) If $s$ and $t$ are compatible and $d(s) \leq d(t)$ then $s \leq t$, and dually.

(4) If $s \land t$ exists then $a(s \land t) = a(s \land t)$, and dually.

The following is the finitary version of Proposition 1.4.20 [20].

**Lemma 1.2.** Let $S$ be a finitely complete inverse semigroup. Then the following are equivalent:

(1) $S$ is distributive.

(2) $E(S)$ is a distributive lattice (with possibly no top element).

(3) For all finite subsets $A, B \subseteq S$ if $\lor A$ and $\lor B$ both exist then $\lor AB$ exists and $(\lor A)(\lor B) = \lor AB$.

We use the term boolean algebra to mean what is often referred to as a generalized boolean algebra, and a unital boolean algebra is what is usually termed a boolean algebra. The proof of the following can be obtained by generalizing the proofs of Lemmas 2.2 and 2.3 of [30].

**Lemma 1.3.** Let $S$ be an orthogonally complete inverse semigroup whose semilattice of idempotents is a boolean algebra. Then $S$ is actually complete.
A directed graph $G$ is a collection of vertices $G_0$ and a collection of edges $G_1$ together with two functions $d, r : G_1 \to G_0$ called the domain or source and the range or target, respectively. The in-degree of a vertex $v$ is the number of edges $x$ such that $r(x) = v$ and the out-degree of a vertex $v$ is the number of edges $x$ such that $d(x) = v$. A sink is a vertex whose out-degree is zero and a source is a vertex whose in-degree is zero. Two edges $x$ and $y$ match if $d(x) = r(y)$. A path is any sequence of edges $x_1 \ldots x_n$ such that $x_i$ and $x_{i+1}$ match for all $i = 1, \ldots, n_1$. Paths should look like this

\[ \ldots \]

The length $|x|$ of a path $x$ is the total number of edges in it. The empty path, or path of length zero, at the vertex $v$ is denoted by $1_v$. The free category $G^*$ generated by the directed graph $G$ is the set of all paths equipped with concatenation as the partial binary operation. If $x, y \in G^*$ are such that either $x = yz$ or $y = zx$ for some path $z$ then we say that $x$ and $y$ are prefix-comparable. Given a directed graph $G$, we define $G^\omega$ to be the set of all right-infinite paths in the graph $G$. Such paths have the form $w = w_1w_2w_3\ldots$ where the $w_i$ are edges in the graph and $d(w_i) = r(w_{i+1})$. If $x \in G^*$, and so is a finite path in $G$, we write $xG^\omega$ to mean the set of all right-infinite paths in $G^\omega$ that begin with $x$ as a finite prefix.

**Lemma 1.4.** If $xG^\omega \cap yG^\omega \neq \emptyset$ then $x$ and $y$ are prefix comparable and so either $xG^\omega \subseteq yG^\omega$ or $yG^\omega \subseteq xG^\omega$.

If $G$ has any vertices of in-degree 0, that is, sources, then a finite path may get stuck and we may not be able to continue it to an infinite path. For this reason, we shall require that our directed graphs have the property that the in-degree of each vertex is at least 1. There is a map $G^* \to G^\omega$ given by $x \mapsto xG^\omega$. It need not be injective but it will be useful to us to have a sufficient condition when it is.

**Lemma 1.5.** Let $G$ be a directed graph in which the in-degree of each vertex is at least 2. Then if $x$ and $y$ are finite paths in the free category on $G$ such that $xG^\omega = yG^\omega$ then $x = y$.

**Proof.** The finite paths $x$ and $y$ must be prefix comparable and have the same target vertex $v$. Therefore to show that they are equal, it is enough to prove that they have the same length. Without loss of generality assume that $|x| < |y|$. Then $y = xz$ for some finite path $z$. Denote the source vertex of $x$ by $u$. Suppose that $z = az$ where $a$ is one edge with target $u$. By assumption, there is at least one other edge with target $u$; call this edge $b$. We may extend $b$ by means of an infinite path $\omega$. Thus by assumption $xb\omega = yb\omega$ for some infinite string $\omega$. But this implies that $b = a$ which is a contradiction. It follows that $x$ and $y$ have the same length and so must be equal. 

We shall now define a topology on $G^\omega$. The subsets of the form $xG^\omega$ where $x$ is a finite path in $G$ form a basis for a topology on $G^\omega$ by Lemma 1.4. The topology that arises is hausdorff and the sets $xG^\omega$ are compact. More generally, the compact-open subsets of this topology are precisely the sets $AG^\omega$ where $A$ is a finite subset of $G^*$. See [17] for details. It follows that with this topology $G^\omega$ is a boolean space, and the compact-open subsets of such a space form a boolean algebra under the usual set-theoretic operations.

An inverse semigroup $S$ is said to be *unambiguous* if for all non-zero idempotents $e$ and $f$ if $ef \neq 0$ then $e \leq f$ or vice-versa. An inverse semigroup $S$ is said
to satisfy the Dedekind height condition if for all non-zero idempotents \( e \) the set \( [e^\top \cap E(S)] \) has finite cardinality. We define a Perrot semigroup to be an inverse semigroup that is unambiguous and has the Dedekind height property.

An inverse semigroup \( S \) is said to have maximal idempotents if for each non-zero idempotent \( e \) there is an idempotent \( e^\circ \) such that \( e \leq e^\circ \) where \( e^\circ \) is a maximal idempotent such that if \( e \leq i^\circ, j^\circ \) then \( i^\circ = j^\circ \). Observe that this is a special case of what might ordinarily be regarded as a semigroup having maximal idempotents.

An inverse semigroup will be called a *Leech semigroup* if it has maximal idempotents and each \( D \)-class contains a maximal idempotent. Such a semigroup is said to be a strict if each \( D \)-class contains a unique maximal idempotent.

A semilattice \( E \) is said to be 0-disjunctive if for each \( 0 \neq f \in E \) and \( e \) such that \( 0 \neq e < f \), there exist \( 0 \neq e' < f \) such that \( ee' = 0 \). It can be proved that an inverse semigroup \( S \) is congruence-free if it is 0-simple, fundamental and its semilattice of idempotents is 0-disjunctive [47]. Combinatorial inverse semigroups are fundamental and we shall not need the more general notion in this paper.

An inverse semigroup is \( E^\ast \)-unitary if \( 0 \neq e \leq s \), where \( e \) is an idempotent, implies that \( s \) is an idempotent. The following is Remark 2.3 of [37] which is worth repeating since it was a surprise to many.

**Lemma 1.6.** If \( S \) is an \( E^\ast \)-unitary inverse monoid then \((S, \leq)\) is a meet semilattice.

**Proof.** Let \( s, t \in S \). Suppose first that \( a \leq s, t \) implies \( a = 0 \). Then in fact \( s \land t = 0 \). We shall therefore suppose that \( s \) and \( t \) have non-zero lower bounds. Let \( 0 \neq s, t \). Then \( s^{-1}t, st^{-1} \) are both idempotents since \( S \) is \( E^\ast \)-unitary. It follows that \( s \) and \( t \) are compatible. By Lemma 1.1, this implies that \( s \land t \) exists, as required.

There are many naturally occurring examples of \( E^\ast \)-unitary inverse monoids and it is a condition that is easy to verify. In particular, the graph inverse semigroups are \( E^\ast \)-unitary. More generally, an inverse semigroup is called an inverse \( \land \)-semigroup if each pair of elements has a meet.

**Lemma 1.7.** Let \( S \) be an unambiguous inverse semigroup. Then the partially ordered set \((S, \leq)\) is unambiguous if and only if \( S \) is \( E^\ast \)-unitary.

**Proof.** Let \( S \) be an \( E^\ast \)-unitary inverse semigroup. Let \( 0 \neq a \land b \leq a, b \). Then \( 0 \neq d(a \land b) \leq d(a), d(b) \). By unambiguity, it follows that either \( d(a) \leq d(b) \) or vice-versa. We assume the former without loss of generality. Thus \( d \leq d(b) \).

However \( a^{-1}b \) and \( ab^{-1} \) are both above non-zero idempotents. Thus from the fact that the semigroup is \( E^\ast \)-unitary we have that \( a \) is compatible with \( b \). By Lemma 1.1, we have that \( a \leq b \), as required.

Let \((S, \leq)\) be an unambiguous poset. We prove that \( S \) is \( E^\ast \)-unitary. Let \( 0 \neq e \leq s \) where \( e \) is an idempotent. We prove that \( s \) is an idempotent. Clearly \( e \leq s^{-1} \). Thus \( s \) and \( s^{-1} \) are comparable. If \( s \leq s^{-1} \) then by taking inverses we also have that \( s^{-1} \leq s \) and vice-versa. It follows that \( s = s^{-1} \). Thus \( s^2 = ss^{-1} \) is an idempotent. Now \( s \) and \( s^2 \) are also comparable. If \( s \leq s^2 \) then \( s \) is an idempotent and we are done. If \( s^2 \leq s \) then \( s = ss^{-1}s = s^3 \leq s^2 \) and so \( s \leq s^2 \) and \( s \) is again an idempotent.

Graph inverse semigroups satisfy the conditions of the above lemma.
2. Characterization

The main goal of this section is to prove Theorem 2.20, which combined with Proposition 2.10(1), provides an abstract characterization of graph inverse semigroups: they are precisely the combinatorial Perrot semigroups which are also strict Leech semigroups.

2.1. A general construction. Graph inverse semigroups are constructed as a special case of a general procedure for constructing inverse semigroups from left cancellative categories \[21, 22, 24, 23\] which has its origins in the work of Leech \[36\]. The left cancellative categories to which this procedure can be applied are required to satisfy the additional condition that any pair of arrows with a common range that can be completed to a commutative square have a pullback. There is no standard term for such categories so in this paper we shall call them Leech categories.

With each Leech category \(C\), we may associate an inverse semigroup \(\mathcal{S}(C)\) as follows; all proofs may be found in \[24\]. Put \(U = \{(a, b) \in C \times C : d(a) = d(b)\}\).

Define a relation \(\sim\) on \(U\) as follows
\[(a, b) \sim (a', b') \iff (a, b) = (a', b')u\]
for some isomorphism \(u \in C\). This is an equivalence relation on \(U\) and we denote the equivalence class containing \((a, b)\) by \([a, b]\). The product \([a, b][c, d]\) is defined as follows: if there are no elements \(x\) and \(y\) such that \(bx = cy\) then the product is defined to be zero; if such elements exist choose such a pair that is a pullback. The product is then defined to be \([ax, dy]\).

Define \(\mathcal{S}(C)\) to be the set of equivalence classes together with an additional element that plays the role of zero. Then the following can be deduced from \[24\].

**Theorem 2.1.** Let \(C\) be a Leech category. Then \(\mathcal{S}(C)\) is an inverse semigroup with zero.

The inverse semigroup \(\mathcal{S}(C)\) has the following important features: \([a, b]^{-1} = [b, a]\); the non-zero idempotents are the elements of the form \([a, a]\); the natural partial order is given by \([a, b] \leq [c, d]\) if and only if \((a, b) = (c, d)p\) for some arrow \(p\).

**Lemma 2.2.** Let \(C\) be a Leech category. Then the semilattice of idempotents of the inverse semigroup \(\mathcal{S}(C)\) is order-isomorphic to the set of principal right ideals of \(C\) together with the emptyset under subset inclusion.

**Proof.** The non-zero idempotents of \(\mathcal{S}(C)\) are the elements of the form \([a, a]\). We have that \([a, a] \leq [b, b]\) if and only if \(a = bp\) for some \(p \in C\). Define a map from idempotents of \(\mathcal{S}(C)\) to principal right ideals of \(C\) by \([a, a] \mapsto aC\) and maps the zero to the emptyset. This is well-defined because if \([a, a] = [a', a']\) then \(a = a'u\) for some isomorphism \(u\) and we have that \(aC = a'uC = a'C\). Next observe that \(aC = bC\) if and only if \(a = bu\) for some isomorphism \(u\) using the fact that \(C\) is left cancellative. Also \([a, a] \leq [b, b]\) if and only if \(a = bp\) if and only if \(aC \subseteq bC\). \(\square\)

**Lemma 2.3.** Let \(C\) be a Leech category. Then in the inverse semigroup \(\mathcal{S}(C)\), we have the following:

1. \([a, b] \leq [c, d]\) if and only if \(b = du\) for some isomorphism \(u \in C\).
Proof. The proofs of (1) and (2) are straightforward.

(3) Suppose that \( d(b) \) and \( d(d) \) are isomorphic where \( u : d(d) \to d(b) \). Then \( \{a, b\} R \{u, d\} L \{c, d\} \) and so \( \{a, b\} D \{c, d\} \). Conversely, suppose that \( \{a, b\} D \{c, d\} \). Then for some \( [x, y] \) we have that \( a = xu \) and \( d = yv \) for isomorphisms \( u \) and \( v \). Then \( v^{-1}u \) is an isomorphism from \( d(b) \) to \( d(d) \).

(4) From \([20]\), this is equivalent to \( \{b, b\} D \{x, x\} \leq \{d, d\} \) for some \( x \in C \). Thus there is an isomorphism \( u \) from \( d(x) \) to \( d(b) \) and \( x = dp \) for some \( p \in C \). Thus \( pu^{-1} \) is a path from \( d(b) \) to \( d(d) \). Conversely, let \( p \) be a path from \( d(b) \) to \( d(d) \). Put \( x = dp \), a well-defined element of \( C \). Then \( [x, x] \leq \{d, d\} \). But \( d(x) = d(b) \).

The proof of (5) follows immediately from the proof of (4). \( \square \)

Define \( [a, a]^\circ = [r(a), r(a)] \) and observe that \( [a, a] \leq [a, a]^\circ \).

**Lemma 2.4.** Let \( C \) be a Leech category. Then the inverse semigroup \( S(C) \) has maximal idempotents and each non-zero \( D \)-class contains a maximal idempotent. Thus these semigroups are Leech semigroups.

**Proof.** Let \( e \) be an identity of the category \( C \). Then \( [e, e] \) is an idempotent. Let \( [e, e] \leq [x, x] \). Then \( e = xp \) for some arrow \( p \) in \( C \). Let \( f = d(x) \). Then \( x = xpx \) and \( p = pxp \). By left cancellation, \( f = px \) and \( e = xp \). Thus both \( x \) and \( p \) are isomorphisms and so \( [e, e] = [x, x] \). Suppose that \( [a, a] \leq [e, e], [f, f] \) where \( e \) and \( f \) are both identities in \( C \). Then it follows immediately that \( e = f = r(a) \). We have shown that \( S(C) \) is an inverse semigroup with maximal idempotents. Finally, we show that each \( D \)-class contains a maximal idempotent. Let \( [a, a] \) be a non-zero idempotent. Observe that \( [a, a] D [d(a), d(a)] \). \( \square \)

The ideal structure of Leech semigroups can be described in terms of certain subsets of the set of maximal idempotents. This is described in the first author’s thesis \([13]\).

**Lemma 2.5.** The poset \( S(C)/J \) has a maximum element if and only if \( C \) contains a root.

**Proof.** Put \( S = S(C) \). Let \( S[e, e] S \) be a principal ideal where we have used the fact that \( S \) is a Leech semigroup and so every principal ideal has a maximum idempotent as a generator. This ideal is a maximum iff for every maximum idempotent \( [f, f] \) we have that \( S[f, f] S \subseteq S[e, e] S \). That is if and only if there is an arrow from \( f \) to \( e \) which means precisely that \( e \) is a root. \( \square \)

Let \( C \) be a Leech category with a root \( e \). We may construct an inverse monoid \( S(C, e) \) as the local submonoid

\[ [e, e] S(C)[e, e]. \]

The nonzero elements of this inverse monoid are the equivalence classes \([a, b]\) where in addition we have that \( r(a) = e = r(b) \). This is in fact Leech’s original construction of an inverse monoid from what we might call a rooted Leech category.
Observe that in any inverse semigroup $S$ we have that $SeS$ is a maximum principal ideal if and only if $S = SeS$. It is known that $S$ is then an enlargement of the local submonoid $eSe$. This implies in particular that $S$ and $eSe$ are Morita equivalent. We have therefore proved the following result.

**Proposition 2.6.** Let $C$ be a Leech category and let $e$ be any identity. Then $S(C)$ is Morita equivalent to the local submonoid $[e, e]S(C)[e, e]$ if and only if $e$ is a root of the category $C$.

We now turn to structural properties of the inverse semigroups $S(C)$.

**Lemma 2.7.** Let $C$ be a Leech category. Then in the inverse semigroup $S = S(C)$, we have the following:

1. The semigroup $S(C)$ is $E^*$-unitary if and only if the Leech category $C$ is right cancellative.
2. The semigroup $S(C)$ is combinatorial if and only if the invertible elements in each local monoid of $C$ are identities.
3. Each $D$-class of $S(C)$ contains a unique maximal idempotent if and only if the only invertible elements are in the local monoids of $C$.
4. The groupoid of invertible elements in $C$ is trivial if and only if $S(C)$ is combinatorial and each $D$-class contains exactly one maximal idempotent.
5. The semigroup $S(C)$ is unambiguous if and only if the category $C$ is right rigid.
6. The inverse semigroup $S(C)$ is completely semisimple if and only if for all identities $e$ and $f$ whenever $eCf$ contains an isomorphism then every element of $eCf$ is an isomorphism.
7. The inverse semigroup $S(C)$ is $0$-bisimple if and only if $C$ is equivalent to a monoid.

**Proof.**

1. Suppose that $C$ is right cancellative. Let $[a, a] \leq [x, y]$. Then $a = xp$ and $a = yp$ for some arrow $p$. But then $xp = yp$. By right cancellation we have that $x = y$ and so $[x, y]$ is an idempotent as required. To prove the converse, suppose that $S$ is $E^*$-unitary. Let $xp = yp$ in the category $C$. Put $a = xp = yp$. Then $[a, a] \leq [x, y]$. But $[a, a]$ is a non-zero idempotent. It follows by assumption that $[x, y]$ is an idempotent and so $x = y$, as required.

2. Suppose that the only invertible elements in the local submonoids are identities. Let $[a, b] \mathcal{H} [e, c]$. Then there are isomorphisms $u$ and $v$ such that $au = c = bv$. It follows that $u$ and $v$ are isomorphisms that begin and end at the same identities. By assumption $u^{-1}v$ is an invertible element in a local monoid and so must be an identity. It follows that $u = v$. Thus $au = bu$ and $u$ is an isomorphism and so $a = b$. It follows that each subgroup of $S$ is trivial and so $S$ is combinatorial. To prove the converse, suppose that $S$ is combinatorial. Let $u$ be an isomorphism from $e$ to itself. Observe that $[e, u] \mathcal{H} [e, e]$. Thus since $S$ is combinatorial, we have that $[e, u] = [e, e]$. Thus there is an isomorphism $v$ such that $e = ev$ and $u = ev$. It follows $u = v$ and $v = e$. Thus $u$ is an identity.

3. Suppose first that the only isomorphisms in $C$ are in the local submonoids. Let $[e, e] \mathcal{D} [f, f]$ where $e$ and $f$ are identities in $C$. By Lemma 2.3(3), $e$ and $f$ are isomorphic and so by our assumption are equal. Conversely, suppose that each $D$-class contains a unique maximal idempotent. Let $e$ and $f$ be isomorphic identities. Then $[e, e] \mathcal{D} [f, f]$ by Lemma 2.3(3). By assumption $[e, e] = [f, f]$. Thus there is an isomorphism $u$ such that $f = eu$. Thus $f = u$ and so $e = f$, as required.
(4) This is immediate by (2) and (3) above.
(5) This is immediate by Lemma 2.2.
(6) Suppose that each hom-set either doesn’t contain any isomorphisms or every element is an isomorphism. Let \([a, a] \subseteq [b, b]\) and \([a, a] \leq [b, b]\). Then there is an isomorphism \(u\) from \(d(a)\) to \(d(d)\) and \(a = bp\) for some arrow \(p\). Thus \(p\) is an arrow from \(d(a)\) to \(d(d)\). By assumption, \(p\) must be an isomorphism and so \([a, a] = [b, b]\), as required. We now prove the converse. Suppose that \(S\) is completely semisimple. Let \(a\) and \(u\) be arrows from \(f\) to \(e\) where \(u\) is an isomorphism. Then \([a, a] \subseteq [e, e]\) and \([a, a] \leq [e, e]\). Thus \([a, a] = [e, e]\) and so there is an isomorphism \(v\) such that \(a = ev\). It follows that \(a = v\) is an isomorphism, as required.

(7) The inverse semigroup with zero \(S\) is 0-bisimple if and only if there is an isomorphism between any two identities of \(C\), by Lemma 2.3(3), if and only if \(C\) is equivalent to a monoid.

When a Leech category has a trivial groupoid of invertible elements the equivalence class \([a, b]\) is just the singleton set \([\{a, b\}\). It is convenient in this case to denote \([a, b]\) by \(ab^{-1}\). The virtues of this notation will become apparent.

2.2. Graph inverse semigroups. The results obtained in Section 2.1 enable us to give a quick presentation of the salient properties of graph inverse semigroups, as we shall now show. Let \(G\) be a directed graph and \(G^*\) the free category it generates. The proof of the following is straightforward.

Lemma 2.8. Free categories are left Rees categories with trivial groupoids of invertible elements which are also right cancellative.

Given a directed graph \(G\), we define the graph inverse semigroup \(P_G\) to be the inverse semigroup \(S(G^*)\). The free category has no non-trivial invertible elements and so each equivalence class is denoted by \(xy^{-1}\). Thus the non-zero elements of \(P_G\) are of the form \(uv^{-1}\) where \(u, v\) are paths in \(G\) with common domain. With elements in this form, multiplication assumes the following shape:

\[
xy^{-1} \cdot uv^{-1} = \begin{cases} 
    xzv^{-1} & \text{if } u = yz \text{ for some path } z \\
    (uvz)^{-1} & \text{if } y = uz \text{ for some path } z \\
    0 & \text{otherwise.}
\end{cases}
\]

Let \(xy^{-1}\) and \(uv^{-1}\) be non-zero elements of \(P_G\). Then

\[
xy^{-1} \leq uv^{-1} \iff \exists p \in G^* \text{ such that } x = up \text{ and } y = vp.
\]

If \(xy^{-1} \leq uv^{-1}\) or \(uv^{-1} \leq xy^{-1}\) then we say \(xy^{-1}\) and \(uv^{-1}\) are comparable.

Lemma 2.9. Let \(P_G\) be a graph inverse semigroup.

(1) The inverse semigroup \(P_G\) has no 0-minimal idempotents if and only if the in-degree of each vertex is at least one.

(2) The inverse semigroup \(P_G\) has a 0-disjunctive semilattice of idempotents if and only if the in-degree of each vertex is either 0 or at least 2.

(3) The semilattice of idempotents of \(P_G\) is pseudofinite if and only if the in-degree of each vertex is finite.

Proof. (1) Let \(e\) be a vertex with in-degree at least 1, and let \(b\) be an edge with target \(e\). Let \(x\) be a path with source \(e\), where we include the possibility that \(x\) is the empty path at \(e\). Then \(xb(xy)^{-1} \leq xx^{-1}\). It follows that if the in-degree of
each vertex is at least 1 then there can be no 0-minimal idempotents. Now let $e$ be a vertex with in-degree 0. Then $1_11_1^{-1}$ is a 0-minimal idempotent.

(2) Suppose that $E$ is 0-disjunctive. Let $v$ be any vertex. Let $x$ be any path that starts at $v$ including the empty path $1_v$. Suppose that the in-degree of $v$ is not zero. Then there is at least one edge $w$ into $v$. It follows that $xw(xw)^{-1} \leq xx^{-1}$. By assumption, there exists $zz^{-1} \leq xx^{-1}$ such that $zz^{-1}$ and $xw(xw)^{-1}$ are orthogonal. Now $z = xp$ for some non-empty path $p$. It follows that $w$ is not a prefix of $p$ and so there is at least one other edge coming into the vertex $v$.

Suppose now that the in-degree of each vertex is either zero or at least two. Let $yy^{-1} < xx^{-1}$ where $y = xp$ where the target of $x$ is the vertex $v$. Since $p$ is a non-empty path that starts at $v$ it follows that there is at least one other edge $w$ with target $v$ that differs from the first edge of $p$. Thus $xw(xw)^{-1} \leq xx^{-1}$ and $xw(xw)^{-1}$ and $yy^{-1}$ are orthogonal.

(3) Straightforward.

The proof of (1) below is now immediate from Theorem 2.1, Lemmas 2.2, 2.8, 2.7 parts (4),(5). The proof of (2) below follows from Lemma 1.6, Lemma 2.7(1) and Lemma 2.8.

**Proposition 2.10.**

1. A graph inverse semigroup is a combinatorial Perrot semigroup with maximal idempotents such that each $D$-class contains a unique maximal idempotent. It is therefore a combinatorial Perrot semigroup which is also strict Leech semigroup.

2. Graph inverse semigroups are $E^*$-unitary and so are inverse $\wedge$-semigroups.

An important class of rooted graphs are the Bratteli diagrams as defined on page 20 of [15]. Such a diagram gives rise to a rooted graph and so to an associated inverse monoid. These inverse monoids arise naturally in the construction of AF-algebras; see Proposition 2.12 of [15]. See Section 2.4.

### 2.3. A characterization of graph inverse semigroups.

We shall begin by obtaining an abstract characterization of free categories in Theorem 2.16. First we recall some results that were proved in a much more general frame in [29].

**Lemma 2.11.** Let $C$ be a left cancellative category.

1. If $e = xy$ is an identity then $x$ is invertible with inverse $y$.

2. We have that $aC = bC$ iff $a = bg$ where $g$ is an invertible element.

3. $aC = eC$ for some identity $e$ iff $a$ is invertible.

**Proof.** (1) We have that $r(x) = e$ and $d(y) = e$. Now

$$xyx = ex = x$$

and

$$xyy = ey = y.$$ 

Thus by left cancellation, $x$ is invertible with inverse $y$.

(2) Suppose that $aC = bC$. Then $a = bx$ and $b = ay$. Thus $a = ayx$ and so by left cancellation $d(a) = yx$. Thus by (1) above, $x$ is invertible. Conversely, suppose that $a = bg$ where $g$ is invertible with inverse $g^{-1}$. Then $d(b) = gg^{-1}$ and $d(a) = g^{-1}g$. But $ag^{-1} = bgg^{-1} = b$, and so $aC = bC$.

(3) Suppose that $aC = eC$. Then by (2), we have that $a = eg$ for some invertible element $g$. Thus $a = g$ is invertible. Conversely, if $a$ is invertible then $aC = aa^{-1}aC \subseteq aa^{-1}C \subseteq aC$. Thus $aC = aa^{-1}C$, as required.
Lemma 2.12. Let $S$ be a left cancellative category. Then the maximal principal right ideals are those generated by identities.

Proof. Observe that for any element $a$ we have that $aC \subseteq r(a)C$. It follows that if $aC$ is maximal then $aC = r(a)C$. By Lemma 2.11(3), this implies that $a$ is invertible. Conversely, let $e$ be an identity. Suppose that $eC \subseteq aC$. Then $e = ab$ for some $b$ and so $r(a) = e$. Thus $eC \subseteq aC \subseteq eC$. Hence $eC = aC$, and so $eC$ is maximal. □

The proof of the following is immediate by the above result.

Lemma 2.13. Let $S$ be a left cancellative right rigid category. Then two maximal principal right ideals either have an empty intersection or are equal.

An element $a \in C$ is said to be indecomposable iff $a = bc$ implies that either $a$ or $b$ is invertible. A principal right ideal $aC$ is said to be submaximal if $aC \neq r(a)C$ and there are no proper principal right ideals between $aC$ and $r(a)S$.

Lemma 2.14. Let $C$ be a left cancellative category. The non-invertible element $a$ is indecomposable iff $aC$ is submaximal.

Proof. Suppose that $a$ is indecomposable, and that $aC \subseteq bC$. Then $a = bc$. By assumption either $b$ or $c$ is invertible. If $c$ is invertible then $aC = bcC = bC$ by Lemma 2.11. If $b$ is invertible then $bC$ is a maximal principal right ideal by Lemmas 2.11 and 2.12. Thus $aC$ is submaximal.

Conversely, suppose that $aC$ is submaximal. Let $a = bc$. Then $aC = bcC \subseteq bC$. By assumption either $aC = bC$ or $b$ is invertible. If the latter we are done; suppose the former. Then $a = bg$ where $g$ is invertible by Lemma 2.11. By left cancellation $c = g$ and so $c$ is invertible. It follows that $a$ is indecomposable. □

Lemma 2.15. Let $C$ be a left cancellative category. The set of invertible elements is trivial iff for all identities $e$ we have that $e = xy$ implies that either $x$ or $y$ is an identity.

Proof. Suppose that the set of invertible elements is trivial. Let $e$ be an identity such that $e = xy$. Then by Lemma 2.11(1), $x$ and $y$ are both invertible. By assumption, they must be identities.

Conversely, suppose that for all identities $e$ we have that $e = xy$ implies that either $x$ or $y$ is an identity. If $a$ is invertible then it has an inverse $a^{-1}$. Thus $e = a^{-1}a$ is an identity. By assumption, either $a^{-1}$ or $a$ is an identity. But the set of invertible elements is a groupoid and so $a^{-1}$ is an identity iff $a$ is an identity. It follows that $a$ is an identity. Thus the set of invertible elements is trivial. □

The following theorem was inspired by similar characterizations of free monoids [19].

Theorem 2.16 (Characterization of free categories). A category is free if and only if it is a left Rees category having a trivial groupoid of invertible elements.

Proof. By Lemma 2.8, free categories are left Rees categories with trivial groupoids of invertible elements.

Let $C$ be a left Rees category having a trivial groupoid of invertible elements. We prove that it is isomorphic to a free category generated by a directed graph. Let $X$ be a transversal of generators of the submaximal principal right ideals.
We may regard $X$ as a directed graph: the set of vertices is $C_x$ and if $a \in X$ then $r(a) \xrightarrow{a} d(a)$. We shall prove that the free category $X^*$ generated by $X$ is isomorphic to $C$. Let $a \in C$. If $aC$ is submaximal then $a$ is indecomposable and since we are assuming that the invertible elements are trivial it follows that $a \in X$. Suppose that $aC$ is not submaximal. Then $aC \subseteq a_1C$ where $a_1 \in X$. Thus $a = a_1b_1$. We now repeat this argument with $b_1$. We have that $b_1C \subseteq a_2C$ where $a_2 \in X$. Thus $b_1 = a_2b_2$. Observe that $aC \subseteq a_1C \subseteq a_1a_2C$. Continuing in this way and using the fact that each principal right ideal is contained in only a finite set of principal right ideals we have shown that $a = a_1 \ldots a_n$ where $a_i \in X$. It remains to show that each element of $C$ can be written uniquely as an element of $X^*$. Suppose that

$$a = a_1 \ldots a_m = b_1 \ldots b_n$$

where $a_i, b_j \in X$. Then $a_1C \cap b_1C \neq \emptyset$. But both principal right ideals are submaximal and so $a_1C = b_1C$. Hence $a_1 = b_1$. By left cancellation we get that

$$a_2 \ldots a_m = b_2 \ldots b_n.$$ 

If $m = n$ then $a_i = b_i$ for all $i$ and we are done. If $m \neq n$ then we deduce that a product of indecomposables is equal to an identity. Suppose that $e = c_1 \ldots c_r$ where $e$ is an identity and the $c_i$ are indecomposables. Then $c_1 \ldots c_rC$ is a maximal principal right ideal. But $c_1 \ldots c_rC \subseteq c_1C$. Thus $c_1C$ is maximal and so $c_1$ is invertible which is a contradiction.

We now return to the main goal of this section that of characterizing graph inverse semigroups. Let $S$ be an inverse semigroup. Put

$$C(S) = \{(e, s) \in E(S^*) \times S^*: r(s) \leq e\}$$

and define $d(e, s) = (d(s), d(s))$ and $r(e, s) = (e, e)$. Define a partial product $(e, s)(f, t) = (e, st)$ iff $d(e, s) = r(f, t)$. Then $C(S)$ is a Leech category called the Leech category associated with $S$ [24].

**Lemma 2.17.** An element $(e, s) \in C(S)$ is an isomorphism if and only if $e = ss^{-1}$.

**Proof.** Suppose that $(e, s)$ is an isomorphism. Then there is an element $(f, t) \in C(S)$ such that $(e, s)(f, t) = (e, e)$ and $(f, t)(e, s) = (s^{-1}s, s^{-1}s)$. Thus $st = e$ and $ts = s^{-1}s$. But then $sts = s$ and $tst = t$. It follows that $t = s^{-1}$. Thus $ss^{-1} = e$. Conversely, suppose that $e = ss^{-1}$. Then $(e, s)$ is invertible with inverse $(s^{-1}s, s^{-1})$.

The proof of the following is immediate by the lemma above.

**Lemma 2.18.** Let $S$ be a combinatorial inverse semigroup. Then the invertible elements of $C(S)$ are those elements $(e, s)$ where $e = ss^{-1}$ and $s^{-1}s \neq e$.

**Lemma 2.19.** If $S$ is a perrot semigroup then $C(S)$ is a left Rees category. If, in addition, $S$ is combinatorial then $C(S)$ has trivial subgroups.

**Proof.** Suppose that $(e, s)C(S) \cap (e, t)C(S) \neq \emptyset$. Then $(e, s)(i, a) = (e, t)(j, b)$ for some $(i, a), (j, b) \in C(S)$. Thus $sa = tb$. Observe that $ss^{-1}sa = sa$. It follows that $ss^{-1}tt^{-1} \neq 0$. But $S$ is unambiguous and so either $ss^{-1} \leq tt^{-1}$ or $tt^{-1} \leq ss^{-1}$. Without loss of generality we assume that $ss^{-1} \leq tt^{-1}$. Thus $ss^{-1} = tt^{-1}ss^{-1}$ and so $s = t(t^{-1}s)$. Observe that $r(t^{-1}s) \leq d(t)$. Thus $(d(t), t^{-1}s) \in C(S)$. But $(e, s) = (e, t)(d(t), t^{-1}s)$ and so $(e, s)C(S) \subseteq (e, t)C(S)$, as required.
Suppose now that \((e, s)C(S) \subseteq (e, t)C(S)\). Then \((e, s) = (e, t)(d(t), a)\) for some \((d(t), a) \in C(S)\). It follows that \(s = ta\) and so \(r(s) \leq r(t)\). Suppose now that \(r(s) = r(t)\). Then \(t = s(s^{-1}t)\). Observe that \(r(s^{-1}t) \leq d(s)\). Thus \((s^{-1}s, s^{-1}t) \in C(S)\) and \((e, t) = (e, s)(s^{-1}s, s^{-1}t)\). Thus \((e, s)C(S) = (e, t)C(S)\). The result now follows.

When \(S\) is combinatorial the claim follows from Lemma 2.18.

Our characterization theorem can now be stated.

**Theorem 2.20.** Let \(S\) be a combinatorial Perrot semigroup with maximal idempotents such that each \(D\)-class contains a unique maximal idempotent. Then there is a free category \(C\) such that \(S\) is isomorphic to the inverse semigroup \(S(C)\).

**Proof.** Let \(S\) be an inverse semigroup satisfying the conditions of the theorem. Let \(s \in S\) be a non-zero element. By assumption \(sDe\) for a unique maximal idempotent \(e\). Thus there is an element \(a\) such that \(sRaLe\). Put \(b = a^{-1}s\). Thus \(s = ab\). Observe that \(a^{-1}a = e = bb^{-1}\) and that \(r(s) = r(a)\) and \(d(s) = d(b)\). Suppose that \(s = a'b'\) where \(d(a') = e = r(b')\). Then because \(S\) is combinatorial we have that \(a = a'\) and \(b = b'\). We shall say that each element of \(S\) can be uniquely factored through the maximal idempotents.

We have seen that the category \(C(S)\) is a left Rees category with trivial subgroups. However, there may be isomorphisms between distinct identities. For this reason, we shall define a full subcategory, denoted by \(C'(S)\), whose elements are those pairs \((e, s) \in C(S)\) such that \(d(s)\) and \(e\) are maximal idempotents. In other words, we take the full subcategory of \(C(S)\) determined by those identities \((e, e)\) where \(e\) is a maximal idempotent of \(S\). It follows that \(C'(S)\) is a left Rees category with only trivial isomorphisms. Thus by Theorem 2.16, this category is free.

Put \(S' = S C'(S)\). We shall prove that \(S\) and \(S'\) are isomorphic. A typical element of \(S'\) is an ordered pair \(((e, s), (f, t))\) such that \(s^{-1}s = t^{-1}t\) and where \(e, f, s^{-1}s\) and \(t^{-1}t\) are all maximal identities. We shall map this element to \(st^{-1} \in S\). On the other hand the non-zero element \(s \in S\) which has the factorization through \(e\) of \(s = ab\) will be mapped to the element

\[ ((r(a)^\circ, a), (d(b)^\circ b^{-1})) \]

We denote this map by \(\theta\). The zero elements in both cases are paired off. We have therefore shown that there is a bijection between \(S\) and \(S'\). It remains to show that this is a homomorphism and we shall have proved the theorem.

Let \(s = ab\) be the factorization through \(e\) and let \(t = cd\) be the factorization through \(f\). The semigroup \(S\) is unambiguous and so there are three cases to consider: (1) \(d(b)r(c) = 0\), (2) \(d(b) < r(c)\) and (3) \(r(c) < d(b)\). In case (1), \(st = 0\).

In case (2), \(st = a(bcd)\) is a factorization through \(e\). In case (3), \(st = (abc)d\) is a factorization through \(f\).

Now \(s \mapsto ((r(a)^\circ, a), (d(b)^\circ b^{-1})) = \theta(s)\) and \(t \mapsto ((r(c)^\circ, c), (d(d)^\circ d^{-1})) = \theta(t)\).

We now calculate \(\theta(s)\theta(t)\) in each of the three cases. In case (1), the product is zero. In case (2), we have that \(d(b)^\circ = r(c)^\circ\). Observe that

\[ (d(b)^\circ, b^{-1})(r(b)^\circ, r(b)^\circ) = (r(a)^\circ, c)(r(c)^\circ, c^{-1}b^{-1}) \]

Their product is therefore

\[ ((r(a)^\circ, a), (d(d)^\circ, d^{-1}c^{-1}b^{-1})). \]

Case (3) is similar to case (2). In all three cases, we have that \(\theta(st) = \theta(s)\theta(t)\). □
2.4. Bratteli inverse monoids. Let $C$ be a rooted Leech category with root $e$. Then we may construct an inverse monoid $S(C, e)$ as indicated after Lemma 2.5. The elements of this monoid are those equivalence classes $[a, b]$ where $a$ and $b$ have the same domain, and the same range $e$. This is Leech’s original construction [36].

Let $G$ be a rooted graph with root $e$ that satisfies the following conditions:

- (B1): $e$ is a sink, and the only sink in graph.
- (B2): There are no sources.
- (B3): The set of vertices of $G$ is partitioned into sets $V_i$ where $i \in \mathbb{N}$ such that $V_0 = \{e\}$.
- (B4): For each edge $x$ there is an $n$ such that $x$ starts in $V_n$ and ends in $V_{n-1}$.

We call such a graph a Bratteli graph. Observe that we shall regard it as a pair $(G, e)$. Our definition is essentially the same as that of Bratteli diagram given in [45]. Given a Bratteli graph we may form the rooted free category $G$. Then we may construct an inverse monoid $S(G^*, e)$. We shall characterize the inverse monoids that arise.

We extend some ideas due to Dooley [5] in the process.

A function $\mu : S^* \to \mathbb{N}$ is called a weight function if it satisfies the following axioms:

- (W1): $s < t$ implies that $\mu(s) > \mu(t)$.
- (W2): $s \Delta t$ implies that $\mu(s) = \mu(t)$.

Lemma 2.21. In an inverse semigroup equipped with a weight function $\mathcal{D} = \mathcal{J}$.

Proof. Suppose that $s \mathcal{J} t$. Then we may find elements $a$ and $b$ such that $s \mathcal{D} a \leq t$ and $t \mathcal{D} b \leq s$. Thus $\mu(s) = \mu(a) \geq \mu(t)$ and $\mu(t) = \mu(b) \geq \mu(s)$. We therefore have that $\mu(s) = \mu(t) = \mu(a) = \mu(b)$. It follows that $a = t$ and so $s \mathcal{D} t$, as required. □

Lemma 2.22. In an unambiguous inverse semigroup equipped with a weight function we have that if $st \neq 0$ then $\mu(st) = \max\{\mu(s), \mu(t)\}$.

Proof. Suppose that $st \neq 0$. Put $e = s^{-1}stt^{-1} \neq 0$. Then $st = (se)(te)$. But $st \Delta se$ and so $\mu(st) = \mu(se)$. But $(se)^{-1}se = e$ and so $\mu(st) = \mu(e)$. Now $e = s^{-1}s \wedge tt^{-1}$. Thus by unambiguity, we have that $s^{-1}s \leq t^{-1}t$ or $t^{-1}t \leq s^{-1}s$. Suppose, without loss of generality, $s^{-1}s \leq t^{-1}t$. Then $\mu(st) = \mu(s^{-1}s) = \mu(s)$. But $\mu(s^{-1}s) \geq \mu(t^{-1}t)$ and so $\mu(s) \geq \mu(t)$. It follows that in this case $\mu(st) = \max\{\mu(s), \mu(t)\}$. □

An inverse monoid $S$ is said to be a Bratteli monoid if it satisfies the following conditions:

- (BM1): It is a combinatorial.
- (BM2): It is unambiguous.
- (BM3): It has no 0-minimal idempotents.
- (BM4): It has a weight function $\mu : S \to \mathbb{N}$ such that $\mu^{-1}(0) = 1$ and for each $n \in \mathbb{N}$ the set $\mu^{-1}(n)$ is finite and non-empty.
- (BM5): If $s < t$ then there exists $s \leq t' < t$ such that $\mu(t') = \mu(t) + 1$.

Proposition 2.23. The inverse monoids $S(G^*, e)$ are Bratteli monoids.

Proof. Observe that $S(G^*, e)$ is a local submonoid of the graph inverse semigroup $F_G$. Thus the fact that it is a combinatorial and unambiguous is inherited from $F_G$. Thus (BM1) and (BM2) hold. The fact that (BM3) holds follows from the fact that the graph has no sources and Lemma 2.9(1).

If $xy^{-1}$ is a nonzero element of $S(G^*, e)$ then $x$ and $y$ are paths that must begin at the same vertex of $G$ and end at $e$. However, because of (B4), it follows that $x$ and $y$
must also be the same length. We define $\mu(xy^{-1}) = |x| = |y|$. It is clear that (W1) holds. The fact that (W2) holds follows from the fact that $xy^{-1}Duv^{-1}$ if and only if $d(y) = v$. Thus (BM4) holds. The proof that (BM5) holds is straightforward. □

Lemma 2.24. Bratelli monoids are in particular Perrot monoids.

Proof. This is a consequence of (BM4) and the fact that $\mu^{-1}(n)$ is always finite. Thus above any non-zero idempotent there can only be a finite number of idempotents.

The main theorem of this section is the following.

Theorem 2.25. Let $S$ be a Bratelli monoid. Then there is a Bratelli graph $G$ with root $e$ such that $S$ is isomorphic to $S(G^*, e)$.

Proof. We form the category $C(S)$ as before. By Lemma 2.19, (BM1) and Lemma 2.24, this category is a Rees category with trivial subgroups. To get rid of any remaining non-trivial isomorphisms we use a similar idea to that in the proof of Theorem 2.20.

Let $F$ be an idempotent transversal of the non-zero $D$-classes. In particular, $1 \in F$. Define $D(S)$ to be the full subcategory of $C(S)$ determined by the identities $(e, e)$ where $e \in F$. From the general theory $S$ is isomorphic to $S(D(S), (1, 1))$. We also know that $D(S)$ is a free category by Theorem 2.16. Thus it only remains to prove that the graph $G$ determined by the indecomposable elements of $D(S)$ is a Bratelli graph.

We claim that the indecomposable elements of $D(S)$ are those elements of the form $(e, s)$ where $e, s^{-1}s \in F$ and $\mu(s) = \mu(e) + 1$. We shall prove this claim below but first we show how this proves the theorem. Put $G$ equal to the set of all indecomposable elements of $D(S)$. The theory tells us that $G^* = D(S)$. Observe that the vertices of $G$ are of the form $(e, e)$ where $e \in F$. We now verify that $G$ is a Bratelli graph.

Observe that $(1, 1)$ is a sink and the only sink. Suppose that there were an indecomposable element of the form $(e, s)$ where $s^{-1}s = 1$. If $ss^{-1} < 1$ then $\mu(ss^{-1}) > \mu(1)$ but this contradicts the fact that $\mu(ss^{-1}) = \mu(1)$. Thus $ss^{-1} = 1$ and so $\mu(s) = 0$ giving $s = 1$. It follows that $(1, 1)$ is a sink. Suppose that $e$ is any idempotent in $F$ such that $\mu(e) \neq 0$. Since $e < 1$ there exists an idempotent $e'$ such that $e < e' \leq 1$ and $\mu(e') = \mu(e) - 1$. There exists $f \in F$ such that $e'Df$. Let $s$ be the element such that $s^{-1}s = e'$ and $ss^{-1} = f$. Now consider the pair $(f, se)$. Observe that $(se)^{-1}se = e$ and $se(se)^{-1} \leq f$. Thus the vertex $(e, e)$ is not a sink. We have proved that (B1) holds.

Let $(e, e)$ be any vertex. By assumption there are no 0-minimal idempotents and so there is a non-zero idempotent $e' < e$. We may therefore assume that $\mu(e') = \mu(e) + 1$. By assumption $e'Df$ where $f \in F$. Let $s$ be such that $s^{-1}s = f$ and $ss^{-1} = e'$. Then $(e, s)$ is an indecomposable element. It follows that $(e, e)$ is not a source and so (B2) holds.

Define $V_i = \{(e, e) : e \in F, \mu(e) = i\}$. Then (B3) holds by (BM4) and the fact that $F$ is a transversal of all the non-zero $D$-classes.

Finally, (B4) holds by the structure of the indecomposables.

It only remains to prove our claim above. Let $(e, s)$ be an element where $e, s^{-1}s \in F$ and $\mu(s) = \mu(e) + 1$. We prove that it is indecomposable. Suppose that we have $(e, s) = (e, t)(f, u)$
where $\mu(u) \geq \mu(f) + 1$ and $\mu(t) \geq \mu(e) + 1$. Then $s = tu = (tu^{-1})u$, a restricted product. It follows that $\mu(s) = \mu(u)$. But then

$$\mu(s) = \mu(u) \geq \mu(f) + 1 = \mu(t) + 1 \geq \mu(e) + 2$$

which is a contradiction. Thus $(e, s)$ is indecomposable.

Now let $(e, s)$ be an element where $e, s^{-1}s \in F$ and $\mu(s) = \mu(e) + r$, where $r \geq 2$. Since $ss^{-1} < e$ there exists an idempotent $e'$ such that $ss^{-1} \leq e' < e$ where $\mu(e') = \mu(e) + 1$. By assumption there exists an element $a$ such that $e' = aa^{-1}$ and $a^{-1}a = f \in F$. We may therefore form the element $(e, a)$ where $\mu(a) = \mu(e) + 1$ by construction. Consider the pair $(f, a^{-1}s)$. Clearly $a^{-1}s(a^{-1}s)^{-1} \leq f$ and $(a^{-1}s)^{-1}a^{-1}s = s^{-1}s$. Thus $(f, a^{-1}s)$ belongs to the category $D(S)$. Furthermore, $\mu(s) = \mu(f) + (r - 1)$. \hfill $\Box$

3. Completion: the Cuntz-Krieger semigroups

Let $G$ be a directed graph satisfying the condition that the in-degree of each vertex is at least 2 and finite. We define the Cuntz-Krieger inverse semigroup $CK_G$ in the following way:

(CK1): It is complete.
(CK2): It contains a copy of $P_G$ and every element of $CK_G$ is a join of a finite subset of $P_G$.
(CK3): $e = \sqcup_{f' \in \hat{e}} f'$ for each maximal idempotent $e$ of $P_G$, where $\hat{e}$ is the set of idempotents covered by $e$.
(CK4): It is the freest inverse semigroup satisfying the above conditions.

We shall prove that this inverse semigroup exists and show how to construct it. In addition, we shall explain how it is related to the representation theory of the graph inverse semigroup $P_G$, and its relation to the Cuntz-Krieger $C^*$-algebra via the associated topological groupoid.

In the case where $G$ has one vertex and $n$ loops, the graph inverse semigroup is just the polycyclic monoid $P_n$ and we denote its completion by $C_n$ and call it the Cuntz semigroup of degree $n$. This semigroup was constructed in [26]. Our goal is to generalize the techniques described there to the more general case.

We proceed in the following steps:

- We start with the graph inverse semigroup $P_G$.
- We construct its orthogonal completion $D(P_G)$ in Section 3.2.
- We define a congruence $\equiv$ on $D(P_G)$. The definition of this congruence uses the Lenz arrow relation described in Section 3.1, and the congruence itself is defined in Section 3.2 and related to work carried out on the polycyclic inverse monoids in [26]. The Cuntz-Krieger semigroup is defined to be $D(P_G)/\equiv$ and a homomorphism $\delta: P_G \rightarrow CK_G$ is defined.
- In Proposition 3.10, it is proved that $CK_G$ is orthogonally complete and that properties (CK2) and (CK3) hold.
- In Section 3.5, a concrete representation of $CK_G$ is obtained as an inverse subsemigroup of $I(G^*)$. This enables us to describe the semilattice of idempotents of $CK_G$ and so complete the proof that property (CK1) holds.
- That the homomorphism $\delta: P_G \rightarrow CK_G$ satisfies the required universal property, property (CK4), is the goal of Section 3.6.
3.1. **The Lenz arrow relation.** The following definitions assume that the inverse semigroup is an inverse $\wedge$-semigroup. Since the inverse semigroups to which these definitions will be used are $E^*$-unitary this will not be a problem by Lemma 1.6. The key concept we shall need in this is the **Lenz arrow relation** introduced in [37]. Let $a, b \in S$. We define $a \to b$ iff for each non-zero element $x \leq a$, we have that $x \wedge b \neq 0$. Observe that $a \leq b \Rightarrow a \to b$. We write $a \leftrightarrow b$ iff $a \to b$ and $b \to a$. More generally, if $a, a_1, \ldots, a_m \in S$ then we define $a \to (a_1, \ldots, a_m)$ iff for each non-zero element $x \leq a$ we have that $x \wedge a_i \neq 0$ for some $i$. Finally, we write $(a_1, \ldots, a_m) \to (b_1, \ldots, b_n)$ iff both $(a_1, \ldots, a_m) \to (b_1, \ldots, b_n)$ and $(b_1, \ldots, b_n) \to (a_1, \ldots, a_m)$. A subset $Z \subseteq A$ is said to be a **cover** of $A$ if for each $a \in A$ there exists $z \in Z$ such that $a \wedge z \neq 0$. A special case of this definition is the following. A finite subset $A \subseteq a^\downarrow$ is said to be a **cover** of $a$ if $a \to A$. A homomorphism $\theta : S \to T$ is said to be a **cover-to-join map** if for each element $s \in S$ and each finite cover $A$ of $s$ we have that $\lor \theta(A)$ exists and $\theta(s) = \lor \theta(A)$.

A much more detailed discussion of cover-to-join maps and how they originated from the work of Exel [9] and Lenz [37] can be found in [32] which can be viewed as a substantial generalization of this paper.

An inverse $\wedge$-semigroup $S$ is said to be **separative** if and only if the Lenz arrow relation is equality.

**Lemma 3.1.** Let $S$ be an unambiguous $E^*$-unitary inverse semigroup. Then $S$ is separative if and only if the semilattice of idempotents $E(S)$ is $0$-disjunctive.

**Proof.** Suppose first that $S$ is separative. We prove that $E(S)$ is $0$-disjunctive. Let $0 \neq e < f$. Then $e \to f$. By assumption, we cannot have that $f \to e$. Thus for some $e' \leq f$ we must have that $e' \wedge e = 0$. It follows that $E(S)$ is $0$-disjunctive.

We shall now prove the converse. We shall prove that if $s \not\leq t$ where $s$ and $t$ are non-zero then there exists $0 \neq s' \leq s$ such that $s' \wedge t = 0$. Before we do this, we show that this property implies that $S$ is separative. Suppose that $s \leftrightarrow t$ and that $s \neq t$. Then we cannot have both $s \leq t$ and $t \leq s$. Suppose that $s \not\leq t$. Then we can find $0 \neq s' \leq s$ such that $s' \wedge t = 0$ which contradicts our assumption.

We now prove the claim. We shall use Lemma 1.7 that tells us that the inverse semigroup itself is an unambiguous poset. Suppose that $s \wedge t = 0$. But then $0 \neq s \leq s$ and $s \wedge t = 0$. We may therefore assume that $s \wedge t \neq 0$. But then $s \leq t$ or $t < s$. The former cannot occur by assumption and so $t < s$. It follows that $d(t) < d(s)$. The semilattice of idempotents is $0$-disjunctive and so there exists an idempotent $e < d(s)$ such that $d(t)e = 0$. Put $s' = se$. Then $0 \neq s' \leq s$. We have to calculate $s' \wedge t$. Suppose that $a \leq s', t$. Then $d(a) \leq d(s')d(t) = ed(t) = 0$, as required. \hfill $\Box$

3.2. **Orthogonal completions.** We begin by recalling some results from [25]. Observe that all orthogonal sets will be assumed finite. The following is Lemma 2.1 of [25].

**Lemma 3.2.** Let $A$ and $B$ be orthogonal subsets containing zero of an inverse semigroup with zero.
(i): \( AB \) is an orthogonal subset containing zero.
(ii): \( AA^{-1} = \{ aa^{-1} : \ a \in A \} \) and \( A^{-1}A = \{ a^{-1}a : \ a \in A \} \).
(iii): \( A = AA^{-1}A \) and \( A^{-1} = A^{-1}AA^{-1} \).

Let \( D(S) \) denote the set of finite orthogonal subsets of the inverse semigroup \( S \) that contain zero. The following is Lemma 2.2 and Lemma 2.3 of [25].

**Lemma 3.3.** With the above definition, \( D(S) \) is an inverse semigroup with zero under multiplication of subsets. In addition, the following hold:

1. If \( A, B \in D(S) \) then \( A \leq B \) iff for each \( a \in A \) there exists \( b \in B \) such that \( a \leq b \).
2. If \( A, B \in D(S) \) then \( A \) and \( B \) are orthogonal iff \( A \cup B \) is an orthogonal subset of \( S \).
3. If \( A, B \in D(S) \) and \( A \) and \( B \) are orthogonal then \( A \lor B = A \cup B \).
4. Multiplication distributes over finite orthogonal joins in \( D(S) \).

Define the function \( \iota : S \to D(S) \) by \( s \mapsto \{0,s\} \). This is an injective homomorphism. The following is Theorem 2.5 of [25] and describes the universal property enjoyed by this map.

**Theorem 3.4.** Let \( S \) be an inverse semigroup with zero. Then \( D(S) \) is orthogonally complete. Let \( \theta : S \to T \) be a homomorphism to an orthogonally complete inverse semigroup \( T \). Then there is a unique orthogonal join preserving homomorphism \( \phi : D(S) \to T \) such that \( \phi \iota = \theta \).

Finally, the following is Lemma 3.4 of [25].

**Lemma 3.5.** Let \( S \) be an orthogonally complete inverse semigroup. Let \( \rho \) be a 0-restricted congruence on \( S \) such that if \( \rho(a) = \rho(a') \) and \( \rho(b) = \rho(b') \) and \( a \) and \( b \) are orthogonal, and \( a' \) and \( b' \) are orthogonal then \( (a \lor b) \rho (a' \lor b') \). Then \( S/\rho \) is also an orthogonally complete inverse semigroup and the natural homomorphism from \( S \) to \( S/\rho \) preserves finite orthogonal joins.

### 3.3. Definition of an equivalence relation

In this section, we shall show how the definition of the congruence given in [26] can be phrased in terms of the Lenz arrow relation. This will show that the construction described in the next section really is a generalization of the one to be found in [26].

The idempotents of \( D(P_n) \) correspond to prefix codes in the free monoid on \( n \) letters by Corollary 3.4 of [25]. By Lemma 4.1 of [25], the maximal prefix codes correspond to the essential idempotents of \( D(P_n) \). However, it is immediate from this that these correspond to those sets of orthogonal idempotents of \( P_n \) that cover, in the sense this term was defined above, the identity of \( P_n \).

Let \( G \) be a directed graph. We may construct the inverse semigroup \( P_G \) and therefore the inverse semigroup \( D = D(P_G) \). The elements of \( D \) will be written \( A^0 \) where \( A \) is a finite set of non-zero orthogonal elements of \( P_G \). For \( A^0, B^0 \in D \) define

\[
A^0 \preceq B^0
\]

if and only if \( A^0 \leq B^0 \) and \( B \to A \).

**Lemma 3.6.** In a graph inverse semigroup, we have the following.

1. Let \( (a_1, \ldots, a_m) \) and \( (b_1, \ldots, b_n) \) be orthogonal sets. Let
\[
(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)
\]
and let $a_1, \ldots, a_n$ be all the elements that lie beneath $b_i$. Then\{ $e_1 = d(a_1), \ldots, e_q = d(a_n)$\} covers $d(b_i)$.

(2) Let $a_1, \ldots, a_m$ be a set of orthogonal elements below the element $a$. If\{ $d(a_1), \ldots, d(a_m)$\} covers $d(a)$ then $\{a_1, \ldots, a_m\}$ covers $a$.

Proof. (1) Let $0 \neq e \leq d(b_i)$. Then $b_i e \leq b_i$. It follows that there exists $k$ such that $0 \neq b_i e \wedge a_k$. By our assumption that the $b_i$ are orthogonal, we must have that $a_k \leq b_i$. Thus $a_k = a_{ik}$, say. By Lemma 1.1, we have that $e \leq e_{ik}$ as required.

(2) Let $0 \neq b \leq a$. Then $0 \neq d(b) \leq d(a)$. By assumption, there exists $i$ such that $d(a_i) \wedge d(b) \neq 0$. But $a_i, b \leq a$ implies that $a_i$ and $b$ are compatible. Thus by Lemma 1.1, we have that $d(a_i \wedge b) = d(a_i) \wedge d(b) \neq 0$. Thus $a_i \wedge b \neq 0$, as required.

Lemma 3.7. Let $\{e_1, \ldots, e_m\}$ cover the idempotent $e$. Suppose that $e = a^{-1} a$ and $f = a a^{-1}$. Then $\{a_1 a^{-1}, \ldots, a_m a^{-1}\}$ covers $f$.

Proof. Let $0 \neq p \leq f$. Then $a^{-1} pa \leq e$ and it is easy to check that $a^{-1} pa \neq 0$. By assumption, there exists an $i$ such that $a^{-1} pa \wedge e_i \neq 0$. But $a^{-1} pa \wedge e_i \leq a^{-1} a$ and so $a(a^{-1} pa \wedge e_i)a^{-1} \neq 0$. But $a(a^{-1} pa \wedge e_i)a^{-1} = p \wedge ae_i a^{-1} \neq 0$, as required, by Lemma 1.1.

We now use the above two lemmas to study the relation $\preceq$ on the semigroup $D(P_n)$. Let $A = \{x_i y_i^j : 1 \leq i \leq p\}$ and $B = \{u_j v_j^i : 1 \leq j \leq q\}$ be two elements of $D(P_n)$. We may partition the elements of $A$ according to which elements of $B$ they lie beneath. Thus if we choose $b = u_j v_j^{-1}$ we may consider all the $x_i y_i^{-1}$ that lie below it. Denote these elements by $x_1 y_1^{-1}, \ldots, x_l y_l^{-1}$ for some $l$. Then by Lemma 3.5(1), the set of idempotents $d(a_1), \ldots, d(a_l)$ covers $d(b)$. That is, $\{y_1 y_1^{-1}, \ldots, y_l y_l^{-1}\}$ covers $v_j v_j^{-1}$. But every non-zero element in a polycyclic monoid is $D$-related to the identity. It follows that the set of idempotents $\{y_1 y_1^{-1}, \ldots, y_l y_l^{-1}\}$ may be obtained from a set of idempotents associated with a maximal prefix code by conjugation. Thus there is a maximal prefix code $\{z_1, \ldots, z_l\}$ where for each $k$ we have that $y_k = v_j z_k$. It follows that $x_k y_k^{-1} = u_j z_k z_k^{-1} v_j^{-1}$. We have proved that if $A \preceq B$ in the sense of this paper then $A \preceq B$ in the sense of the definition given in Section 3 of \cite{23}. On the other hand, the converse is true by Lemma 3.5(2).

If $A^0, B^0 \neq 0$ define $A^0 \equiv B^0$ if and only if there exists $C^0 \neq 0$ such that $C^0 \preceq A^0$ and $C^0 \preceq B^0$. In addition, define $\{0\} \equiv \{0\}$.

Lemma 3.8. In a graph inverse semigroup, we have the following. We have $A^0 \equiv B^0$ if and only if $A^0 \leftrightarrow B^0$.

Proof. Observe that if $(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$ then

\[(a_1, \ldots, a_m) \rightarrow (a_i \wedge b_j : 1 \leq i \leq m, 1 \leq j \leq n) \]

and

\[(b_1, \ldots, b_n) \rightarrow (a_i \wedge b_j : 1 \leq i \leq m, 1 \leq j \leq n) \]

Thus if $(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$ then there is $(c_1, \ldots, c_p)$ such that $(c_1, \ldots, c_p) \preceq (a_1, \ldots, a_m)$ and $(c_1, \ldots, c_p) \preceq (b_1, \ldots, b_n)$.

On the other hand, if $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$ then in fact $(a_1, \ldots, a_m) \leftrightarrow (b_1, \ldots, b_n)$.

It follows that the complicated equivalence relation defined in Section 3 of \cite{23} is nothing other than the relation $\leftrightarrow$. It is therefore this relation we shall use in
our main construction in the next section, confident that we are generalizing exactly.

3.4. The construction. In this section, we shall construct the Cuntz-Krieger semigroup $CK_G$. We shall use the characterization of relation $\equiv$ described in Lemma 3.8.

Lemma 3.9. In a graph inverse semigroup $P_G$ we have the following. The relation $\equiv$ is a 0-restricted, idempotent pure congruence on $D(P_G)$. Furthermore, if $A^0 \equiv B^0$ and $C^0 \equiv D^0$ and $A^0$ and $B^0$ are orthogonal and $B^0$ and $D^0$ are orthogonal then $A^0 \lor C^0 \equiv B^0 \lor D^0$.

Proof. Observe first that if $a \to 0$ then $a = 0$. It follows that the relation $\equiv$ will be 0-restricted. From Lemma 3.8 or by direct calculation, the relation $\to$ is reflexive and transitive. It follows readily from this that $\equiv$ is an equivalence relation. From Lemma 3.8 or by direct calculation, the relation $\to$ is right and left compatible with the multiplication. It follows that $\equiv$ is a 0-restricted congruence. This congruence is idempotent pure. To see why, let $A^0 \equiv B^0$ where $A^0$ is a non-zero idempotent. Let $xy^{-1} \in B$. Then there exists $uu^{-1} \in A$ such that $xy^{-1} \land uu^{-1} \neq 0$. But this implies that $xy^{-1}$ lies above a non-zero idempotent and $P_G$ is $E^*$-unitary. It follows that $xy^{-1}$ is an idempotent. Since $xy^{-1}$ was arbitrary, $B^0$ is an idempotent as claimed. If $A^0$ and $B^0$ are orthogonal then $A^0 \lor B^0 = A^0 \cup B^0$. It readily follows that the last stated property holds.

Define $CK_G$ to be $D(P_G)/\equiv$ and define $\delta: P_G \to CK_G$ by $\delta(s) = [\{0, s\}]$, where $[x]$ denotes the $\equiv$-class containing $x$.

Proposition 3.10. For any directed graph $G$, there is an orthogonally complete inverse semigroup $CK_G$ together with a homomorphism $\delta: P_G \to CK_G$ such that every element of $CK_G$ is a finite join of a finite orthogonal subset of the image of $\delta$. For each maximal idempotent $e$ in $P_G$, we have that

$$\delta(e) = \bigvee_{f \in \hat{e}} \delta(f).$$

If $G$ has the additional property that the in-degree of each vertex is either 0 or at least 2, then the homomorphism $\delta$ is injective.

Proof. By Lemma 3.4, the semigroup $CK_G$ is orthogonally complete. The homomorphism $\delta$ is injective if and only if the Lenz arrow relation is equality. Since $P_G$ is unambiguous and $E^*$-unitary, it follows by Lemma 3.1 that the semilattice of idempotents of $P_G$ must be 0-disjunctive. By Lemma 2.15(2) this means that the in-degree of each vertex of $G$ is either 0 or at least 2.

3.5. A concrete description. Our goal now is to obtain a more concrete description of the inverse semigroup $CK_G$ as well as a description of it semilattice of idempotents. We shall do this by first defining an action of $P_G$ on the set $G^\omega$ of right-infinite paths in the graph $G$ and thereby define a homomorphism $\theta: P_G \to I(G^\omega)$. Let $xy^{-1} \in P_G$ and $w \in G^\omega$. We define $xy^{-1} \cdot w$ if and only if we may factorize $w = yw'$ where $w' \in G^\omega$; in which case, $xy^{-1} \cdot w = xw'$. This is well-defined since $d(x) = d(y)$. It is easy to check that the two axioms (A1) and (A2) for an action hold. We call this action the natural action of the graph inverse semigroup on the space of infinite paths.
Lemma 3.11. Let $\theta : P_G \to I(G^\omega)$ be the homomorphism associated with the above natural action.

1. The action leads to a 0-restricted homomorphism $\theta$ if and only if there is no vertex of in-degree $0$.

2. If the in-degree of each vertex is at least 2 then the homomorphism $\theta$ is injective.

Proof. (1) Suppose that $\theta(xx^{-1}) = 0$ for some $x$. This means that there are no right-infinite strings with prefix $x$. This implies that there is some vertex of the graph which has in-degree zero. On the other hand, if each vertex of the graph has in-degree at least one then the action is 0-restricted: given any finite path $w$ we may extend it to an infinite path $w = xw'$. Then $xx^{-1} \cdot w$ is defined.

(2) Suppose that $\theta(xy^{-1}) = \theta(uv^{-1})$. Then $yG^\omega = vG^\omega$ and so $y = v$ by Lemma 1.5. Similarly $xG^\omega = uG^\omega$ and so $x = u$ again by Lemma 1.5. \qed

From now on, we shall assume that the in-degree of each vertex of the graph is finite and at least 2. The representation $\theta : P_G \to I(G^\omega)$ is injective and so $P_G$ is isomorphic to its image $P'$. Define $O_G$ to be the inverse subsemigroup of $I(G^\omega)$ consisting of all non-empty finite unions of pairwise orthogonal elements of $P'$.

Let $X = \{x_1y_1^{-1}, \ldots, x_my_m^{-1}\}$ be an orthogonal sets in $P_G$. Define a function $f_X \in I(G^\omega)$ as follows:

$$f_X : \bigcup_{i=1}^m y_iG^\omega \to \bigcup_{i=1}^m x_iG^\omega$$

is given by $f_X(w) = x_iw'$ if $w = y_iw'$

Lemma 3.12. Let $X = \{x_1y_1^{-1}, \ldots, x_my_m^{-1}\}$ and $Y = \{u_1v_1^{-1}, \ldots, u_nv_n^{-1}\}$ be two orthogonal sets in $P_G$. Then $f_X = f_Y$ if and only if $X \leftrightarrow Y$.

Proof. By definition

$$f_X : \bigcup_{i=1}^m y_iG^\omega \to \bigcup_{i=1}^m x_iG^\omega$$

and

$$f_Y : \bigcup_{j=1}^n v_jG^\omega \to \bigcup_{j=1}^n u_jG^\omega$$

We suppose first that $f_X = f_Y$. Thus

$$\{y_1, \ldots, y_m\}G^\omega = \{v_1, \ldots, v_n\}G^\omega$$

Let $0 \neq wz^{-1} \leq x_iy_i^{-1}$. Then for some finite string $p$ we have that $w = x_ip$ and $z = y_p$. By definition, $f_X$ restricts to define a map from $x_ipG^\omega$ to $y_iG^\omega$ such that for any infinite string $\omega$ for which the product is defined we have that $f_X(y_ip\omega) = x_ip\omega$. By assumption, $f_Y(y_ip\omega) = x_ip\omega$. It follows that there are two possibilities. Either $zG^\omega$ has a non-empty intersection with $v_jG^\omega$ with exactly one of the $j$, in which case $zG^\omega \subseteq v_jG^\omega$ or it intersects a number of them in which case $v_jG^\omega \subseteq zG^\omega$ for a number of the $j$.

Suppose the first possibility occurs. Then $z = v_jq$ for some finite path $q$. The map from $zG^\omega$ to $wG^\omega$ must be a restriction of the map from $v_jG^\omega$ to $u_jG^\omega$. It follows that $wG^\omega = u_jqG^\omega$ and so by the above lemma we have that $w = u_jq$. It follows that $wz^{-1} = u_jv_j^{-1}$. \qed
We now suppose that the second possibility occurs. Then for at least one \( j \) we have that \( u_j = sq \) for some finite path \( q \). In this case, we have that \( u_j G^\omega = wqG^\omega \) and so by the above lemma we have that \( u_j = wq \). It follows that \( u_j v_j^{-1} \leq wz^{-1} \).

We have therefore shown that \( X \to Y \). The result follows by symmetry.

We now prove the converse. Suppose that \( X \leftrightarrow Y \). We prove that \( f_X = f_Y \).

Let \( w \) be an infinite string in \( \text{dom}(f_X) \). Then we may write it as \( w = y_i \bar{w} \) for some infinite string \( \bar{w} \). By definition \( f_X(y_i \bar{w}) = x_i \bar{w} \). For every prefix \( p \) of \( \bar{w} \), the element \( x_i p(y_i p)^{-1} \leq x_i y_i^{-1} \) and so there is an element \( \leq u_j v_j^{-1} \in Y \), depending on \( p \), such that \( x_i p(y_i p)^{-1} \wedge u_j v_j^{-1} \neq 0 \). By Lemma 1.7, this meet is equal to \( x_i p(y_i p)^{-1} \) or \( u_j v_j^{-1} \), whichever is smaller. Therefore if we choose \( p \) sufficiently long, we can ensure that the element \( x_i p(y_i p)^{-1} \) cannot be greater than or equal to any element in \( Y \). It follows that there is a \( j \) such that \( x_i p(y_i p)^{-1} \leq u_j v_j^{-1} \) using the fact that \( X \to Y \). Thus \( x_i p = u_j q \) and \( y_i p = v_j q \) for some finite path \( q \). Put \( \bar{w} = pw' \). Then \( w = y_i pw' \). Thus

\[
f_Y(w) = f_Y(y_i pw') = f_Y(v_j qw') = u_j qw' = x_i pw' = x_i \bar{w} = f_X(w).
\]

The result now follows by symmetry. \( \square \)

It follows by the above lemma that the function \( F: CK_G \to O_G \) given by \( F([A^0]) = f_A \) is well-defined and a bijection.

**Theorem 3.13.** Let \( G \) be a directed graph in which the in-degree of each vertex is at least 2 and is finite. Then the inverse semigroup \( CK_G \) is isomorphic to the inverse semigroup \( O_G \) defined as an inverse semigroup of partial bijections of the set \( G^\omega \). In particular, the semilattice of idempotents of \( CK_G \) is a (non-unital, in general) boolean algebra. It follows that \( CK_G \) is a complete inverse semigroup.

**Proof.** We have defined a bijection from \( F: CK_G \to O_G \). It remains to show that this is a homomorphism. From [20], we can simplify this proof by splitting it up into three cases.

(Case 1) Check that if \( d(X) = r(Y) \) then \( F(XY) = F(X)F(Y) \).

Let \( X = \{x_1 y_1^{-1}, \ldots, x_m y_m^{-1} \} \) and \( Y = \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1} \} \). By assumption, \( d(X) = r(Y) \) and so \( m = n \) and \( y_i y_i^{-1} = u_i u_i^{-1} \). It follows that

\[
XY = \{x_1 v_1^{-1}, \ldots, x_n v_n^{-1} \},
\]

remembering orthogonality. It is now easy to check that \( F(XY) = F(X)F(Y) \).

(Case 2) Check that if \( X \leq Y \) then \( F(X) \leq F(Y) \).

Suppose that \( X \leq Y \). Let \( xy^{-1} \in X \). Then \( xy^{-1} \leq uv^{-1} \) for some \( w^{-1} \in Y \). Thus \( x = up \) and \( y = vq \) for some finite path \( p \). It follows that \( xG^\omega \subseteq uG^\omega \) and \( yG^\omega \subseteq vG^\omega \). It is simple to check that \( F(xy^{-1}) \leq F(uv^{-1}) \). Thus by glueing the separate functions together we have that \( F(X) \leq F(Y) \).

(Case 3) Check that if \( X \) and \( Y \) are idempotents then \( F(X \wedge Y) = F(X)F(Y) \).

Observe first that \( xx^{-1} \leq yy^{-1} \) if and only if \( xG^\omega \subseteq yG^\omega \) and that \( xx^{-1} \wedge yy^{-1} = zz^{-1} \) if and only if \( xG^\omega \cap yG^\omega = zG^\omega \). Let \( X = \{x_1 x_1^{-1}, \ldots, x_m x_m^{-1} \} \) and \( Y = \{u_1 u_1^{-1}, \ldots, u_n u_n^{-1} \} \). Then \( X \wedge Y \) is constructed by forming all possible meets \( x_i x_i^{-1} \wedge u_j u_j^{-1} \). But this translates into forming all possible intersections
\[ x_1G^\omega \cap u_jG^\omega. \] The result is now clear.

The semilattice of idempotents of \( O_G \) is in bijective correspondence with the subsets of \( G^\omega \) of the form \( XG^\omega \) where \( X \) is a finite set of finite paths in \( G \). However, these are precisely the compact-open subsets of the topological space \( G^\omega \) which has a basis of compact-open subsets and is hausdorff. It follows that the semilattice of idempotents of \( O_G \) and so of \( CK_G \) is a boolean algebra. Thus by Lemma 1.3, \( CK_G \) is complete. \( \square \)

3.6. Universal characterization. It remains to show that the map \( \delta: P_G \to CK_G \) has the right universal property. If \( xx^{-1} \) is a non-zero idempotent in \( P_G \) then we define its weight to be the number \( |x| \).

**Lemma 3.14.** Let \( P_G \) be a graph inverse semigroup in which the in-degree of each vertex is finite. Let \( F = \{e_1, \ldots, e_m\} \) be an orthogonal cover of the maximal idempotent \( e \). Suppose that \( e_1 = xx^{-1} \) is an idempotent in \( F \) of maximum weight at least 1. Put \( x = \bar{xa} \) where \( a_1 \) is an edge with target \( f \). Let \( a_1, \ldots, a_n \) be all the edges with target \( f \).

1. Then \( f_j = \bar{xa}_ja_j^{-1}x^{-1} \in F \) for \( 1 \leq j \leq n \).
2. Put \( F' = F \setminus \{f_1, \ldots, f_n\} \cup \{\bar{x}\} \). Then \( F' \) is a cover of \( e \) and \( |F'| < |F| \).
3. \( F = F' \cup \{\bar{x}\} \). The result is now clear.

**Proof.** (1) The string \( \bar{xa}_j \) has target the vertex corresponding to \( e \). Thus \( f_j = \bar{xa}_ja_j^{-1}x^{-1} \leq e \). By assumption \( f_j \land e_i \neq 0 \) for some \( i \). Let \( e_i = yy^{-1} \). Then \( y \) and \( \bar{xa}_j \) are prefix-comparable. By assumption, \( e_i \) has maximum weight amongst all the idempotents in \( F \) and so \( \bar{xa}_j = yz \) for some path \( z \). If \( z \) were not empty, \( y \) would be a prefix of \( \bar{x} \) and so we would have that \( e_i < e_j \) which is a contradiction. It follows that \( z \) is empty and so \( f_j = e_i \).

(2) Let \( 0 \neq f \leq e \) and suppose that \( 0 = f \land e_i = 0 \) for all \( i \geq 1 \). We must have that \( 0 \neq f \land e_1 \). We shall show that \( 0 \neq \bar{x}x^{-1} \land f \). Let \( f = yy^{-1} \). Then \( x \) and \( y \) are prefix-comparable. If \( |y| < |x| \) then \( y \) is a prefix of \( \bar{x} \) and \( \bar{x} \) and \( y \) are prefix-comparable. If \( |y| \geq |x| \) then \( \bar{x} \) is a prefix of \( y \) and again \( \bar{x} \) and \( y \) are prefix-comparable.

(3) This is immediate. \( \square \)

**Lemma 3.15.** Let \( \theta: P_G \to T \) be a homomorphism to a complete inverse semigroup where for each idempotent \( e \) and cover \( F \) of \( e \) we have that \( \theta(e) = \bigvee_{f \in F} \theta(f) \). Then \( \theta \) is a cover-to-join map.

**Proof.** Let \( \{a_1, \ldots, a_m\} \) be a cover of \( a \). Then \( \{d(a_1), \ldots, d(a_m)\} \) is a cover of \( d(a) \). By assumption \( \theta(d(a)) = \bigvee_{i=1}^m \theta(d(a_i)) \). Now multiplying on the left by \( \theta(a) \) and using distributivity we get that \( \theta(a) = \bigvee_{i=1}^m \theta(a_i) \). \( \square \)

**Lemma 3.16.** Let \( \theta: P_G \to T \) be a homomorphism to a complete inverse semigroup where \( \theta(e) = \bigvee_{f \in F} \theta(f) \) for each maximal idempotent \( e \) in \( P_G \). Then \( \theta \) is a cover-to-join map.

**Proof.** Suppose first that we can prove the following claim: for every maximal idempotent \( e \) and every cover \( F \) of \( e \) we have that \( \theta(e) = \bigvee_{f \in F} \theta(f) \). Then we can prove that \( \theta \) is a cover-to-join map. Let \( \{e_1, \ldots, e_m\} \) be the cover of the idempotent \( f \). In a graph inverse semigroup, there is a maximal idempotent \( e \) such that \( f \not\Delta e \). Thus we may find an element \( a \) such that \( a^{-1}a = f \) and \( aa^{-1} = e \). By an argument
similar to Lemma 3.7, we have that \( \{ae_1a^{-1}, \ldots, ae_ma^{-1}\} \) is a cover of \( e \). By assumption,

\[
\theta(e) = \bigvee_{i=1}^{m} \theta(ae_i a^{-1}).
\]

Multiplying on the left by \( \theta(a^{-1}) \) and on the right by \( \theta(a) \) and using distributivity we get that

\[
\theta(f) = \bigvee_{i=1}^{m} \theta(e_i).
\]

To prove the lemma, it therefore remains to prove the claim. This can now be achieved using induction and Lemma 3.14 and, since the maximal idempotents are pairwise orthogonal, we can fix attention on all covers of a fixed maximal idempotent \( e \). Suppose that our claim holds for all orthogonal covers of \( e \) with at most \( p \) elements. Let \( F \) be an orthogonal cover of \( e \) with \( p+1 \) elements. By Lemma 3.14, we may write

\[
F = F' \setminus \{\bar{x}\bar{x}^{-1}\} \cup \bar{x}\bar{f}\bar{x}^{-1}
\]

where \( F' \) is cover of \( e \) and \( |F'| < |F| \). By our induction hypothesis, we may write

\[
\theta(e) = \bigvee_{f' \in F'} \theta(f').
\]

By assumption, we may write

\[
\theta(f) = \bigvee_{g \in f} \theta(g).
\]

Thus using distributivity, we have that

\[
\theta(\bar{x}\bar{x}^{-1}) = \bigvee_{g \in f} \theta(\bar{x}g\bar{x}^{-1}).
\]

It follows that

\[
\theta(e) = \bigvee_{f \in F} \theta(f),
\]

as required. \( \square \)

The following theorem can be deduced from the general theory described in [32], but we give a direct proof.

**Theorem 3.17.** Let \( \theta : P_G \to T \) be a homomorphism to a complete inverse semigroup where \( \theta(e) = \bigvee_{f \in e} \theta(f) \) for each maximal idempotent \( e \) in \( P_G \). Then there is a unique join-preserving homomorphism \( \bar{\theta} : CK_G \to T \) such that \( \bar{\theta} \delta = \theta \).

**Proof.** Put \( S = P_G \). By Lemma 3.15, the map \( \theta \) is a cover-to-join map. The theorem will be proved if we can show that for any two orthogonal sets \( X \) and \( Y \) in \( P_G \) we have that \( X \leftrightarrow Y \) implies that \( \bigvee_{x \in X} \theta(x) = \bigvee_{y \in Y} \theta(y) \). By Lemma 3.8, it is enough to show this in the special case where \( X \preceq Y \) and we may further assume that \( Y = \{y\} \). But then the result is immediate by the definition of a cover-to-join map. \( \square \)

The above theorem justifies our claims made about the semigroup \( CK_G \) at the beginning of this section.
3.7. **The topological connection.** This topic is taken up in more depth in [32]. Here we shall just sketch out the key result. We shall use the following notation. Let \( w = uw' \) where \( w, w' \) are infinite strings and \( u \) is a finite string. Define \( u^{-1}w = w' \). Given a directed graph \( G \) a groupoid \( \mathcal{G} \) is defined as follows. Its elements consist of triples \( (w, k, w') \in G^\omega \times \mathbb{Z} \times G^\omega \) where there are finite strings \( u \) and \( v \) such that \( u^{-1}w = v^{-1}w' \) and \( |v| - |u| = k \). The groupoid product is given by \((w, k, w')(u, l, w'') = (w, k + l, w'') \) and \((w, k, w')^{-1} = (w', -k, w) \). A basis for a topology is given as follows. For each pair \( x, y \in G^* \) define \( Z(x, y) \) to consist of all groupoid elements \( (w, k, w') \) where \( x^{-1}w = y^{-1}w' \) and \( k = |y| - |x| \). Observe that under our assumptions on \( G \), the sets \( Z(x, y) \) are always non-empty. It can be shown that this is a basis, and that with respect to the topology that results the groupoid \( \mathcal{G} \) is an étale, Hausdorff topological groupoid in which the sets \( Z(x, y) \) are compact-open bisections. The space of identities of this groupoid is homeomorphic to the usual topology defined on \( G^\omega \) [17]. With our usual assumptions on the directed graph \( G \) this makes \( \mathcal{G} \) what we have called a boolean groupoid in [32]. The compact-open bisections of the groupoid \( \mathcal{G} \) form an inverse semigroup called the *ample semigroup* of \( \mathcal{G} \). We shall prove that this semigroup is the Cuntz-Krieger semigroup \( CK_G \). The following two lemmas are the key to our main theorem.

**Lemma 3.18.**

\[
Z(x, y)Z(u, v) = \begin{cases} 
Z(x, vz) & \text{if } y = uz \\
Z(yz, v) & \text{if } u = yz \\
\emptyset & \text{else}
\end{cases}
\]

*Proof.* It is easy to check that the product is empty if \( y \) and \( u \) are not prefix-comparable. We shall therefore assume, without loss of generality, that \( y = uz \) for some finite path \( z \). It is straightforward to check that \( Z(x, y)Z(u, v) \subseteq Z(x, vz) \). We prove the reverse inclusion. Let \((w_1, m, w_2) \in Z(x, vz)\) where \( m = |vz| - |x| \). Let \( w_1 = x\bar{w} \) and \( w_2 = vz\bar{w} \). Then a routine calculation shows that \((w_1, m, w_2)\) is equal to the product

\[
(x\bar{w}, |y| - |x|, y\bar{w})(uz\bar{w}, |vz| - |uz|, vz\bar{w})
\]

where the first element is from \( Z(x, y) \) and the second is from \( Z(u, v) \). \( \square \)

The following is Lemma 2.5 of [17].

**Lemma 3.19.**

\[
Z(x, y) \cap Z(u, v) = \begin{cases} 
Z(x, y) & \text{if } xy^{-1} \leq uv^{-1} \\
Z(u, v) & \text{if } uv^{-1} \leq xy^{-1} \\
\emptyset & \text{else}
\end{cases}
\]

Denote by \( B(\mathcal{G}) \) the inverse semigroup of compact-open bisections of the topological groupoid \( \mathcal{G} \).

**Theorem 3.20.** *The Cuntz-Krieger semigroup \( CK_G \) is the ample semigroup of the topological groupoid \( \mathcal{G} \) constructed from the directed graph \( G \).*

*Proof.* Define \( \theta: P_G \rightarrow B(\mathcal{G}) \) by \( \theta(xy^{-1}) = Z(x, y) \) and map the zero to the empty set. Then by Lemma 3.18, this map is a homomorphism. We claim that it is injective. Suppose that \( Z(x, y) = Z(u, v) \). Since these sets are non-empty, we have
that $xy^{-1}$ and $uv^{-1}$ are comparable since by Lemma 1.7 the poset $P_G$ is unambiguous. It is now immediate by Lemma 3.19 that $xy^{-1} = uv^{-1}$. Let $e$ be a vertex of $G$ and let $a_1, \ldots, a_m$ be the edges of $G$ with source $e$. Then

$$Z(1_e, 1_e) = \bigcup_{i=1}^{m} Z(a_i, a_i).$$

The conditions of Theorem 3.17 hold and so $\theta$ may be extended to a homomorphism $\tilde{\theta}: CK_G \to B(G)$. Lemma 3.19 implies that each element of $B(G)$ is a finite disjoint union of basis elements and so $\theta$ is surjective.

It remains to show that $\tilde{\theta}$ is injective. We shall prove that if

$$\bigcup_{i=1}^{m} Z(x_i, y_i) = \bigcup_{j=1}^{n} Z(u_j, v_j),$$

then

$$(x_1y_1^{-1}, \ldots, x_my_m^{-1}) \leftrightarrow (u_1v_1^{-1}, \ldots, u_nv_n^{-1}).$$

By symmetry it is enough to prove that if

$$Z(x, y) \subseteq \bigcup_{j=1}^{n} Z(u_j, v_j),$$

then

$$xy^{-1} \rightarrow (u_1v_1^{-1}, \ldots, u_nv_n^{-1}).$$

Let $wz^{-1} \leq xy^{-1}$. Then $w = xp$ and $z = yp$ for some finite path $p$. Let $w'$ be any infinite path so that $xpw'$ and $ypw'$ are defined. Then

$$(xpw', |y| - |x|, ypw') \in Z(x, y)$$

and so belongs to $Z(u_j, v_j)$ for some $j$. It is now easy to show that $wz^{-1}$ and $u_jv_j^{-1}$ are comparable where we make essential use of the central number in the triple. □

3.8. **Strong representations.** A 0-restricted homomorphism $\theta: P_G \to I(X)$ is called a *strong representation* of $P_G$ if for each maximal idempotent $e$ of $P_G$ we have that $\theta(e) = \bigvee_{f \in \hat{e}} \theta(f')$ for each maximal idempotent $e$ of $P_G$. This terminology was first applied in the case where the inverse semigroup was a polycyclic inverse monoid. This case was introduced in the book [20]. It was independently formulated by Kawamura [15, 16] in terms of branching function systems, who then classified what we would call the transitive strong representations. In [27], the second author reproved Kawamura’s results using the theory of transitive inverse semigroup actions. In [14], the authors showed that the monograph [4] could be understood purely in terms of strong representations of polycyclic monoids.

The notion of a branching function system can be generalized to graph inverse semigroups. It coincides with what are called $E$-algebraic branching systems in [10]. In our terms, they take the following form. For each vertex $v$ of $G$ let $X_v$ be a non-empty set. We put $X$ equal to the disjoint union of the $X_v$. For each edge $v \xleftarrow{e} u$ there is an injective function $X_u \xrightarrow{\theta(e)} X_v$. The strong representation condition translates into the following. Let $v \xleftarrow{e_i} u_i$ be all the $n$ edges with target $v$. Then we require that

$$X_v = \bigcup_{i=1}^{n} \text{im} \theta(e_i).$$
This strongly suggests developing the theory of strong representations of graph inverse semigroups along the lines of [14] and [4].

By Lemma 3.16 and Theorem 3.17 and the fact that the symmetric inverse monoids are complete, every strong representation \( \theta: P_G \rightarrow I(\mathbb{X}) \) gives rise to a join-preserving homomorphism \( \bar{\theta}: CK_G \rightarrow I(\mathbb{X}) \). The relationship between \( CK_G \) and \( P_G \) is therefore analogous to the relationship between the group-ring of a group and the group itself. The partial join operation on complete inverse semigroups gives them a ring-like character.

The semigroup \( CK_G \) may also explain why the role of inverse semigroups in the theory of graph \( C^* \)-algebras is not as apparent as maybe it should be. Although Paterson makes an explicit connection in [41, 42], they are not mentioned in [45] or [49]. We believe that one explanation is that the Cuntz-Krieger relations \( e = \sum_{f' \in \mathbb{E}} f' \) cannot be expressed in pure inverse semigroup theory and these would seem to be the essence of Cuntz-Krieger algebras. However, we have shown that by working with a suitable completion of the graph inverse semigroup, these relations can be naturally expressed.

There are therefore three algebraic structures that arise in studying algebras, in the most general sense, arising from directed graphs:

- Cuntz-Krieger semigroups,
- Leavitt path algebras,
- graph \( C^* \)-algebras.

From the theory developed in the previous section, we know that the Cuntz-Krieger semigroups can be used to construct, and may be constructed from, the topological groupoids usually associated with the graph \( C^* \)-algebras. As a result, we believe that they are basic structures to study.

### 3.9. Unambiguity and self-similarity

The notion of unambiguity was first formalized within semigroup theory in [3] and briefly considered in the context of inverse semigroup theory in [22]. However, it was the paper [28] by the second author that made us realize just how important unambiguous inverse semigroups were. This paper shows that the self-similar groups of [39] are in bijective correspondence with what we would now call 0-bisimple Perrot monoids. This latter terminology was introduced to record the fact that this result is implicit in Perrot’s thesis [43], even though this predates the formal introduction of self-similar groups. In its turn, Perrot’s work can be seen as a wide-ranging generalization of Rees’s [45] pioneering paper, which led to the terminology for the categories we have used in this paper. Further evidence for the importance of the class of unambiguous semigroups comes from the work of [12] who studies topological groupoids associated with certain kinds of ultra-metric spaces. The poset of open balls in such a space is unambiguous. If the 0-bisimple Perrot monoids are constructed from self-similar groups, the question arises of what we can say about more general kinds of Perrot semigroup. This is the theme of [29].

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