Inverse scattering transform for the Toda lattice with steplike initial data

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Abstract. We study the solution of the Toda lattice Cauchy problem with steplike initial data. The initial data are supposed to tend to zero as $n \to +\infty$. By the inverse scattering transform method formulas allowing us to find solution of the Toda lattice is obtained.

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1. Introduction

The Toda lattice has some very important applications in the theory of physics of nonlinear processes (see [1]). It is known the inverse scattering method allows one to investigate in detail the Cauchy problem for the Toda lattice in the different classes of initial data (see [1]-[15] and references therein). The last problem for the doubly-infinite Toda lattice

\[
\begin{align*}
& \dot{a}_n = \frac{a_n}{2} (b_{n+1} - b_n), \quad \dot{b}_n = a_n^2 - a_{n-1}^2, \quad a_n = a_n(t) > 0, \\
& b_n = b_n(t), \quad n = 0, \pm 1, \pm 2, \ldots
\end{align*}
\]

(1.1)

with fast stabilized or steplike fast stabilized initial data is investigated in [1]-[9] (see also references therein) by the method of inverse scattering transform. However, this problem is not studied in the case of steplike initial data, where \(a_n\) tend to zero as \(n \to +\infty\) (or \(n \to -\infty\)).

In this paper we study the Cauchy problem for the system (1.1) with initial data

\[
a_n(0) \to 0, \quad b_n(0) \to 0 \quad \text{as} \quad n \to +\infty,
\]

\[
\sum_{n<0} |n| \left( |a_n(0) - 1| + |b_n(0)| \right) < \infty.
\]

(1.2)

The solution is considered in the class

\[
\|a_n(t)\|_{C[0,T]} \to 0, \quad \|b_n(t)\|_{C[0,T]} \to 0, \quad \text{as} \quad n \to +\infty,
\]

\[
\|Q(t)\|_{C[0,T]} < \infty, \quad \text{for arbitrary} \quad T > 0, \quad \text{where}
\]

\[
Q(t) = \sum_{n<0} |n| \left( |a_n(t) - 1| + |b_n(t)| \right).
\]

(1.3)

Note, we cannot apply directly method given in [1]-[9] for the case \(\inf a_n > 0\), because the Jost solution with the asymptotic behaviour on an \(+\infty\) does not exist in our case. On the other hand, method of inverse problem is used (see [10]) in the case when Jacobi operator associated with (1.1) has the continuous spectrum \([a, b]\) of multiplicity two. But this method cannot be used when the spectrum of the Jacobi operator has a continuous spectrum of multiplicity one and a discrete spectrum.

The paper is organized as follows. In section 2 we formulate some auxiliary facts to the inverse scattering problem for the Jacobi operator associated with (1.1)-(1.2). In section 3 we describe the evolution of the scattering data of problem (1.1)-(1.2).

In the last section we prove existence of the solution of the problem (1.1)-(1.2) in class (1.3).
2. The scattering problem

Consider Jacobi operator $L$ generated in $\ell^2(-\infty, \infty)$ by the finite-difference operations

$$(Ly)_n = a_{n-1}y_{n-1} + b_ny_n + b_{n+1}y_{n+1},$$

in which the real coefficients $a_n > 0$, $b_n$ satisfy the conditions

$$a_n \to 0, \ b_n \to 0 \quad as \quad n \to +\infty,$$

$$\sum_{n<0} |n| \{(|a_n - 1| + |b_n|) < \infty .$$

The interval $[-2, 2]$ is the continuous spectrum of multiplicity one of operator $L$ (see [16],[17]). Beyond the continuous spectrum, $L$ can have a finite number of simple eigenvalues $\mu_k(t)$, $k = 1, \ldots, p$.

Let us formulate some auxiliary facts related to the inverse scattering problem for the equation

$$(Ly)_n = \lambda y_n, \quad n = 0, \pm 1, \ldots, \lambda \in C$$  \hspace{1cm} (2.1)

Many of these facts can be found in [16],[17].

Let $P_n(\lambda)$ and $Q_n(\lambda)$ be solutions of Eq. (2.1) with initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1,$$

$$Q_0(\lambda) = 0, \quad Q_1(\lambda) = \frac{1}{a_0}.$$  \hspace{1cm} (2.2)

We denote by $L_0$ semi-infinite Jacobi operator generated $\ell^2[0, \infty)$ by Eq. (2.1) as $n \geq 0$ and the boundary condition $y_{-1} = 0$. This operator is completely continuous. Moreover, the spectral function $\rho(\lambda)$ of $L_0$ represented [18] in the form

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \beta_n^{-2},$$

where $\lambda_n$ is the eigenvalue of $L_0$ and $\beta_n$ is the norm of the eigenfunction corresponding to the $\lambda_n$.

As is known from [18]-[19], the right Weyl function of the problem (2.1) has the form

$$m(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(\tau)}{\tau - \lambda},$$  \hspace{1cm} (2.2)

or

$$m(\lambda) = \sum_{n=1}^{\infty} \frac{\beta_n^{-2}}{\lambda_n - \lambda}.$$  \hspace{1cm} (2.3)

where $\lambda_n \to 0$ as $n \to \infty$. It follows from [12]-[13] that for $\lambda \neq \lambda_k$, $k = 1, 2, \ldots$, Eq.(2.1) has Weyl solution

$$\psi_n(\lambda) = Q_n(\lambda) + m(\lambda)P_n(\lambda),$$  \hspace{1cm} (2.3)
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“on the right semiaxis” (such that $\sum_{n=0}^{\infty} |\psi_n(\lambda)|^2 < \infty$).

Suppose that $\Gamma$ is the complex $\lambda$-plane with cut along the interval $[-2, 2]$. In the plane $\Gamma$, consider the function

$$z(\lambda) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - 1}$$

choosing the regular branch of the radical so that $\sqrt{\frac{\lambda^2}{4} - 1} < 0$ for $\lambda > 2$. We often omit the dependence of $z(\lambda)$ on $\lambda$ in what follows. Thus, in the formulas involving $z$ and $\lambda$, we always assume that $z$ is as in the above equation.

It is well known (see, for example, [20]) that Eq.(2.1) has a Jost solution represented in the form

$$f_n(\lambda) = a_n z^{-n} \left(1 + \sum_{m<0} A_{nm} z^{-m}\right). \tag{2.4}$$

The coefficients are given by

$$a_n = \frac{\alpha_n}{\alpha_{n+1}}, \quad b_n = A_{n,-1} - A_{n+1,-1}. \tag{2.5}$$

Without restriction of generality we can suppose that $\lambda_m \in (-2, 2)$ for any $m = 1, 2, \ldots$. As known [16],[17], for $\lambda \in \partial \Gamma$, $\lambda^2 \neq 4, \lambda \neq \lambda_m$ identity

$$\psi_n(\lambda) = a(\lambda)f_n(\lambda) + \overline{a(\lambda)f_n(\lambda)} \tag{2.6}$$

holds, where the function $a(\lambda)$ can be regularly continued to $\Gamma$. Note also, $a(\lambda)$ can have a finite number of coinciding simple zeros outside the interval $[-2, 2]$, because, these zeros constitute the discrete spectrum $\mu_k, \ k = 1, \ldots, p$, of the operator $L$.

Introduce reflection $R(\lambda)$ coefficient by the formula

$$R(\lambda) = \frac{a(\lambda)}{\overline{a(\lambda)}}.$$

The function $R(\lambda)$ is continuous for $\lambda \in \partial \Gamma$. Setting $n = -1$ and $n = 0$ in the identity (2.6) yields the expression

$$m(\lambda) = -\frac{1}{a_{-1}} \frac{f_0(\lambda) + R(\lambda)f_0(\lambda)}{f_{-1}(\lambda) + R(\lambda)f_{-1}(\lambda)} \tag{2.7}$$

The norming constants $M_k(t)$ corresponding to the $\mu_k(t)$ are given as

$$M_k^{-2} = \sum_{n=-\infty}^{\infty} f_n^2(\mu_k), \ k = 1, \ldots, p.$$

The set of quantities $\{R(\lambda); \mu_k; \ M_k, \ k = 1, \ldots, p\}$ is called the scattering data for the Jacobi operator $L$. The inverse scattering problem for $L$ is to recover the coefficients $a_n, \ b_n$ from the scattering data.
In solving the inverse problem, an important role is played by the Marchenko-type basic equation. Define

\[ F_n = \sum_{k=1}^{p} M_k^{-2} z_k^{-n} + \frac{1}{2\pi i} \int_{\partial \Gamma} \frac{R(\lambda)}{\zeta - 1} \zeta^{-n} d\lambda, \]  

(2.8)

where \( z_k = z(\mu_k), \ k = 1, \ldots, p. \)

Then \( A_{nm} \) and \( \alpha_n \) involved in (2.4) satisfy the relations

\[ F_{2n+m} + A_{nm} + \sum_{k<0} A_{nk} F_{2n+m+k} = 0, \quad m < n \leq 0, \]  

(2.9)

\[ \alpha_n^{-2} = 1 + F_{2n} + \sum_{k<0} A_{nk} F_{2n+k}, \quad n \leq 0. \]  

(2.10)

To reconstruct the operator \( L \), we consider Eq. (2.8) which is constructed by the scattering data. We find \( A_{nm} \) and \( \alpha_n \) from Eqs. (2.9) and (2.10), respectively, the first one having a unique solution with respect to \( A_{nm} \). The coefficients \( a_n \) and \( b_n \) are defined for \( n < 0 \) by (2.5). \( f_n(\lambda) \) for \( n \leq 0 \) are defined by (2.4). From the formula (2.7) we obtain Weyl function \( m(\lambda) \). The spectral measure \( d\rho(\lambda) \) can be found by the formula

\[ d\rho(\lambda_n) = \lim_{\lambda \to \lambda_n} (\lambda_n - \lambda)m(\lambda), \quad n = 1, 2, \ldots. \]

Using the approach in [12],[13],[19], we can reconstruct semi-infinite Jacobi operator \( L_0 \) by its spectral measure \( d\rho(\lambda) \). Therefore, we find \( a_n, b_n \) for \( n \geq 0 \).

3. Evolution of the scattering data

In this section we use the inverse scattering transform method to solve the problem (1.1)-(1.2). Let \( a_n(t), b_n(t) \) be a solution of the problem (1.1)-(1.2) satisfying (1.3). Consider the Jacobi operator \( L = L(t) \) associated with \( a_n = a_n(t), b_n = b_n(t) \). Jost and Weyl solutions, reflection coefficient, spectral measure now depend on the additional parameter \( t \in [0, \infty) \).

**Theorem 1.** If the coefficients \( a_n = a_n(t), b_n = b_n(t) \) of Eq. (2.1) are solutions to problem (1.1)-(1.2) in the class (1.3), then the evolution of the scattering data is described by the formulas

\[ R(\lambda, t) = R(\lambda, 0) e^{(z^{-1}-z)t}, \]  

(3.1)

\[ \mu_k(t) = \mu_k(0), \ k = 1, \ldots, p \]  

(3.2)

\[ M_k^{-2}(t) = M_k^{-2}(0) e^{(z^{-1}-z_k)t}, \quad z_k = z(\mu_k), \ k = 1, \ldots, p. \]  

(3.3)

**Proof.** System (1.1) is represented (see, for example [8],[13]) in the Lax form

\[ \dot{L} = [L, A] = AL - LA, \]  

(3.4)

where \( A = A(t) \) are Jacobi operator in \( \ell^2(-\infty, \infty) \):

\[ (Ay)_n = \frac{1}{2} a_n y_{n+1} - \frac{1}{2} a_{n-1} y_{n-1}. \]
Since (3.4) implies that the family of operators \( L = L(t) \) are unitarily equivalent (see [5],[8]), the spectrum of \( L = L(t) \) does not depend on \( n \) and (3.2) is valid.

Let \( f_n(\lambda, \ t) \) and \( \psi_n(\lambda, \ t) \) respectively be the Jost and Weyl solutions of the Eq.(2.1) with the parameter \( t \). Consider the identity (2.6) with the parameter \( t \). As follows from [8],[12] the function \( \frac{d}{dt} \psi_n - (A\psi)_n \) is also a solution of the Eq.(2.1) with the parameter \( t \). Appling the operator \( \frac{d}{dt} - A \) to (2.6), taking into account that the Jost solution \( f_n(\lambda, \ t) \) does not depend (see [8], on \( t \) asymptotically, we obtain

\[
\frac{d}{dt} \psi_n - (A\psi)_n = \left( a(\lambda, \ t) + \frac{1}{2} (z^{-1} - z) a(\lambda, \ t) \right) f_n(\lambda, \ t) + \left( \dot{a}(\lambda, \ t) - \frac{1}{2} (z^{-1} - z) a(\lambda, \ t) \right) f_n(\lambda, \ t). \tag{3.5}
\]

On the other hand, we find

\[
\frac{d}{dt} P_0 - (AP)_0 = \frac{b_0 - \lambda}{2}, \quad \frac{d}{dt} P_{-1} - (AP)_{-1} = -a_{-1}.
\]

Since \( P_n(\lambda, \ t) \) and \( Q_n(\lambda, \ t) \) are linearly independent, the function \( \frac{d}{dt} P_n - (AP)_n \) can be represented as

\[
\frac{d}{dt} P_n - (AP)_n = A(\lambda, \ t) P_n + D(\lambda, \ t) Q_n.
\]

Setting \( n = -1 \) and \( n = 0 \) in the last relation, we find that

\[
A(\lambda, \ t) = \frac{b_0 - \lambda}{2}, \quad D(\lambda, \ t) = a_{-1}^2.
\]

Therefore,

\[
\frac{d}{dt} P_n - (AP)_n = \frac{b_0 - \lambda}{2} P_n + a_{-1}^2 Q_n.
\]

The same arguments are valid for solution \( Q_n(\lambda, \ t) \). Thus, we have the formula

\[
\frac{d}{dt} Q_n - (AQ)_n = -P_n + \frac{\lambda - b_0}{2a_0} Q_n.
\]

Now by the formula (2.3) with the parameter \( t \) we find that

\[
\frac{d}{dt} \psi_n - (A\psi)_n = \left( a_{-1}^2 m(\lambda, \ t) + \frac{\lambda - b_0}{2a_0} \right) Q_n + \left( \dot{m}(\lambda, \ t) + \frac{b_0 - \lambda}{2} m(\lambda, \ t) - 1 \right) P_n. \tag{3.6}
\]

Since \( L = L(t) \) is selfadjoint and bounded, \( \frac{d}{dt} \psi_n - (A\psi)_n \) must satisfy the relation

\[
\frac{d}{dt} \psi_n - (A\psi)_n = \theta(\lambda, \ t) \psi_n. \tag{3.7}
\]

Hence, we can represent the function \( \frac{d}{dt} \psi_n - (A\psi)_n \) as

\[
\frac{d}{dt} \psi_n - (A\psi)_n = \theta(\lambda, \ t) Q_n + \theta(\lambda, \ t) m(\lambda, \ t) P_n. \tag{3.8}
\]
Comparing this identity with (3.6), we have

\[ \theta(\lambda, t) = a_{-1}^2 m(\lambda, t) + \frac{\lambda - b_0}{2a_0}, \]  

(3.9)

Further, according to (2.6), (3.5), (3.7),

\[ \theta(\lambda, t) a(\lambda, t) f_n + \theta(\lambda, t) a(\lambda, t) f_n = \left( \dot{a}(\lambda, t) + \frac{1}{2} (z^{-1} - z) a(\lambda, t) \right) f_n + \]

\[ \left( \frac{a(\lambda, t) - \frac{1}{2} (z^{-1} - z) a(\lambda, t)}{f_n} \right). \]

Since \( f_n \) and \( \overline{f_n} \) are linearly independent, so substituting (3.9) into the last identity, we obtain

\[ \dot{a}(\lambda, t) + \frac{1}{2} (z^{-1} - z) a(\lambda, t) = \left( a_{-1}^2 m(\lambda, t) + \frac{\lambda - b_0}{2a_0} \right) a(\lambda, t), \]

\[ \dot{a}(\lambda, t) - \frac{1}{2} (z^{-1} - z) a(\lambda, t) = \left( a_{-1}^2 m(\lambda, t) + \frac{\lambda - b_0}{2a_0} \right) a(\lambda, t). \]

From this relations, we get

\[ \dot{R}(\lambda, t) = (z^{-1} - z) R(\lambda, t), \]

which imply (3.1).

Now, let \( g_n(\mu_k, t) \) be a normalized eigenfunction of \( L \). Since the eigenvalues \( \mu_k, k = 1, ..., p \), of this operator are simple, we have

\[ \frac{d}{dt} g_n - (Ag)_n = c g_n. \]

Taking the scalar products of \( g_n \) with both sides of this equality in \( \ell^2(-\infty, \infty) \) and using \( \| \psi_n \|_{\ell^2(-\infty, \infty)} = 1 \) and \( A^* = -A \), we obtain \( c = 0 \). Therefore,

\[ \frac{d}{dt} g_n - (Ag)_n = 0 \]

(3.10)

On the other hand, if a normalized eigenfunction \( g_n(\mu_k, t) \) corresponds to the eigenvalue \( \mu_k \), then

\[ g_n(\mu_k, t) = c_k(t) f_n(\mu_k, t). \]

This implies that \( M_k^2(t) = c_k^2(t) \). By virtue of (2.4), we find that

\[ \frac{d}{dt} g_n - (Ag)_n \sim \left( \dot{c}_k(t) + \frac{z_k - z_k^{-1}}{2} c_k(t) \right) z_k^{-n} \]

as \( n \to -\infty \). Taking into account (3.10), we have

\[ \dot{c}_k(t) + \frac{z_k - z_k^{-1}}{2} c_k(t) = 0 \]

This equation implies the relation (3.3).

The theorem is proved.

Using Theorem 1, we obtain the following procedure for solving problem (1.1),(1.2) based on the inverse scattering transform method: Initial data (1.2) is given. Construct \( R(\lambda, 0), \mu_k(0), M_k(0), \) \( k = 1, ..., p \). Calculate \( R(\lambda, t), \mu_k(t), M_k(t) \) using formulas (3.1)-(3.3). Construct a solution by solving the inverse problem by applying approach of the section 2 with \( R(\lambda, 0), \mu_k(0), M_k(0), \) \( k = 1, ..., p \) replaced by (3.1)-(3.3).
4. Solvability of the Cauchy problem for the Toda lattice

In section 3, while constructing a solution to problem (1.1)-(1.2), we assumed that this solution exists in the class (1.3). Let us now investigate its existence.

**Theorem 2.** The problem (1.1)-(1.2) has a unique solution in the class (1.3).

**Proof.** Denote by $B$ the Banach space of pairs of sequences $y = (y_{1,n}, y_{2,n})_{n=0}^{\infty}$ for which the norm $\|y\|_B = \sup_{n \geq 0} (|y_{1,n}| + |y_{2,n}|) + \sum_{n < 0} |n| (|y_{1,n}| + |y_{2,n}|)$ is finite. Then (see [21]) the set $C([0, T]; B)$ of the continuous on an interval $[0, T]$ with respect to the norm $\|\cdot\|_B$ functions is the Banach space.

Let us assume that

$$x_{1,n} = \begin{cases} a_n(t) & \text{for } n \geq 0, \\ a_n(t) - 1 & \text{for } n < 0, \end{cases} \quad (4.1)$$

$$x_{2,n} = b_n(t).$$

Then system (1.1) is equivalent to the system

$$\begin{cases}
\dot{x}_{1,n} = \frac{1}{2} x_{1,n} (x_{2,n+1} - x_{2,n}) + \frac{1}{2} \left(1 - \delta_{n,|n|}\right) (x_{2,n+1} - x_{2,n}), \\
\dot{x}_{2,n} = x_{1,n}^2 - x_{1,n-1}^2 + 2 \left(1 - \delta_{n,|n|}\right) (x_{1,n} - x_{2,n-1}),
\end{cases} \quad (4.2)$$

where $\delta_{n,m}$ is the Kronecker symbol.

Denote by $F$ the operator generated the right-hand sides of system (4.2). Note, operator $F$ is strongly continuously differentable in the space $C([0, T]; B)$.

Now passing to the integral equation in the standard manner, we find problem (4.2) with initial conditions

$$x_{1,n}(0) = \begin{cases} a_n(0) & \text{for } n \geq 0, \\ a_n(0) - 1 & \text{for } n < 0, \end{cases} \quad (4.3)$$

$$x_{2,n}(0) = b_n(0).$$

is equivalent to the equation

$$x(t) = x(0) + \int_0^t F(x(\tau))d\tau \quad (4.4)$$

Applying the principle of compressed maps, we find that problem (4.4) on some interval $[0, \delta]$ has a unique solution $x(t)$ with finite norm $\|x(t)\|_{C([0, \delta]; B)} < \infty$. Let us show that this solution can be extended to the entire positive semi-axis. Assume the opposite. Then there exists a point $t^* \in (0, \infty)$ such that problem (4.2)-(4.3) has a solution $x(t) = (x_{1,n}(t), x_{2,n}(t))$ on the interval $[0, t^*)$ but $\lim_{t \to t^* - 0} \|x(t)\|_B = \infty$. It follows from [8],[13] problem (1.1)-(1.2) has a unique solution $(a_n(t), b_n(t))$ in $C^\infty([0, \infty); M)$,
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where \( M = \ell^\infty (\infty, \infty) \oplus \ell^\infty (\infty, \infty) \). Hence, according to the (4.1) problem (4.2)-(4.3) has a unique solution \( x(t) = (x_{1,n}(t), x_{2,n}(t)) \) satisfying

\[
|x_{1,n}(t)| + |x_{2,n}(t)| < C
\]

for any \( t \in [0, \infty) \), where \( C \) does not depend on \( t \). We integrate the system (4.2) over a interval \([0, t]\). Then, using the last inequality, after some simple transformations, we get

\[
\|x(t)\|_B \leq 2 \|x(0)\|_B + (4C + 4) \int_0^t \|x(\tau)\|_B d\tau, \quad 0 < t < t^*,
\]

which, according to the Gronwall’s inequality implies

\[
\|x(t)\|_B \leq 2 \|x(0)\|_B e^{(4C+4)t},
\]

Therefore, our assumption that \( \lim_{t \to t^* - 0} \|x(t)\|_B = \infty \) is not correct and problem (4.2)-(4.3) has a unique solution \( x(t) = (x_{1,n}(t), x_{2,n}(t)) \in C([0, T]; B) \) for any \( T > 0 \). Integrating the system (1.1) over a interval \([0, t]\) and using (4.1), we obtain that problem (1.1)-(1.2) be uniquely solvable in the class (1.3).

Thus, the theorem is proved.

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