On inclusive distance vertex irregularity strength of small identical copies of star graphs

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Abstract. For a simple graph $G$, an inclusive distance vertex irregular $k$-labeling of $G$ is a mapping $\lambda : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that all the vertex-weights are pairwise distinct, where the weight of a vertex $v$, denoted by $wt(v)$, is the sum of labels of vertices in the close neighborhood of the vertex $v$. The minimum $k$ for which the graph $G$ has an inclusive distance vertex irregular $k$-labeling is called the inclusive distance vertex irregularity strength of $G$, $\hat{\text{dis}}(G)$. Here we introduce a new lower bound for $\hat{\text{dis}}(G)$ and determine the exact value of the inclusive distance vertex irregularity strength for identical copies of star graphs, especially $2S_n$ and $3S_n$.

1. Introduction

Let $G$ be a finite, undirected graph with neither loops nor multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and edges of $G$, respectively. By a labeling we mean any mapping that carries some elements (vertices, edges, or both) of a graph to the set of integers (usually positive integers) called labels. Such a labeling is called a vertex (resp. an edge) labeling if the domain is the vertex-set (resp. the edge-set). Moreover, if the domain is $V(G) \cup E(G)$ then it is called a total labeling. There are many types of graph labeling techniques that have been established. The most recent complete surveys on labelings is given by Gallian [6].

In [10], Slamin introduced distance vertex irregular labelings as a combination of distance based-labelings [1, 7] and irregularity strengths of graphs [5]. A vertex $k$-labeling $\lambda : V(G) \rightarrow \{1, 2, \ldots, k\}$ is said to be a distance vertex irregular $k$-labeling of $G$ if the weights of all vertices are all distinct where the weight of a vertex $v$ of $G$ is $wt(v) = \sum_{u \in N(v)} \lambda(u)$, where $N(v)$ denotes the open neighborhood of $v$, i.e., the set of vertices adjacent to $v$ in $G$. The minimum $k$ for which $G$ has a distance vertex irregular $k$-labeling is called the distance vertex irregularity strength of $G$, $\text{dis}(G)$. There are some results on distance vertex irregularity strengths that have been found for both connected graphs [4, 8, 9, 10] and disconnected graphs [11].

Later on, Baća et al. [2] developed a variation of distance vertex irregular labelings namely the inclusive distance vertex irregular labelings of graphs. Such a labeling $\lambda$ is called an inclusive distance vertex irregular $k$-labeling of $G$ if there are no two vertices having the same weight, where the weight of a vertex $v$ is now calculated by summing all the vertices in the close neighborhood of $v$, i.e., $wt(v) = \sum_{u \in N[v]} \lambda(u)$. The inclusive distance vertex irregularity strength of $G$ is
the smallest $k$ such that $G$ has an inclusive distance vertex irregular $k$-labeling and is denoted by $\text{dis}(G)$. There are some results on the inclusive distance vertex irregularity strengths that have been found, including, for example, paths, cycles, complete and complete bipartite graphs, special type of complete tripartite graphs, fans and wheels, caterpillars, join products $G \oplus K_1$ [2, 3], and triangular ladders [12].

In [3], there is also investigated a generalization of the non-inclusive and inclusive distance vertex irregular labelings. The most recent result on this parameter was due to Utami et al. [13].

The following theorem showing a lower bound for the inclusive distance vertex irregularity strengths was developed by Bong et al. [3].

**Theorem 1.** [3] Let $G$ be a graph with order $n$, maximum degree $\Delta$, and minimum degree $\delta$. Then

$$\widehat{\text{dis}}(G) \geq \left\lceil \frac{\delta + n}{\Delta + 1} \right\rceil.$$  

In this paper we introduce a new lower bound for the inclusive distance irregularity strengths and prove the exact values on this parameter for small identical copies of star graphs.

2. Main Results

We introduce a new lower bound for the inclusive distance vertex irregularity strength of graphs that generalizes the lower bound in Theorem 1.

**Theorem 2.** Let $G$ be a graph with maximum degree $\Delta$ and minimum degree $\delta$. Let $n_i$ be the number of vertices of degree $i$ in $G$ for every $\delta \leq i \leq \Delta$. Then

$$\widehat{\text{dis}}(G) \geq \max_{\delta \leq i \leq \Delta} \left\lceil \frac{\delta + \sum_{j=\delta}^{\delta} n_j}{i + 1} \right\rceil.$$  

**Proof.** Suppose that $k$ is the largest vertex label under an inclusive distance vertex irregular labeling of a graph $G$. The value of $k$ will be minimum if the vertex weights are induced on vertices such that the vertex with smaller degree receive smaller weight. For each $i$, the largest among the weights of $n_i$ vertices having degree $i$ must be at least $\delta + \sum_{j=\delta}^{\delta} n_j$, and this weight is obtained from the sum of $i + 1$ integers. Therefore

$$k \geq \widehat{\text{dis}} \geq \max_{\delta \leq i \leq \Delta} \left\lceil \frac{\delta + \sum_{j=\delta}^{\delta} n_j}{i + 1} \right\rceil.$$  

3. Identical Copies of Stars

In this part, we present our main results dealing with the inclusive distance vertex irregularity strength for the identical copies of stars. For positive integers $m \geq 2$ and $n \geq 3$, let $mS_n$ be the identical copies of stars with

$$V(mS_n) = \{u_j : 1 \leq j \leq m\} \cup \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

such that

$$E(mS_n) = \{u_jv_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}.$$  

Here we consider the graph $mS_n$ for $m = 2, 3$ only.

**Theorem 3.** Let $n$ be a positive integer, $n \geq 3$. Then $\hat{\text{dis}}(2S_n) = n + 1$.  

Proof. We have $\hat{\text{dis}}(2S_n) \geq \max\{\left\lceil \frac{(2n+1)}{2} \right\rceil, \left\lceil \frac{(2n+3)}{(n+1)} \right\rceil\} = n + 1$ by Theorem 2. To prove that $\hat{\text{dis}}(2S_n) \leq n + 1$ it suffices to show that there exists an optimal inclusive distance vertex irregular $(n + 1)$-labeling of $2S_n$. For $n = 3$, a corresponding inclusive distance vertex irregular 4-labeling of $2S_3$ is described in Figure 1.

![Figure 1](image)

**Figure 1.** An inclusive distance vertex irregular 4-labeling of $2S_3$.

Let $n \geq 4$. Let $\lambda$ be a labeling on the vertices of $2S_n$ defined as follows.

$$
\lambda(u_j) = \begin{cases} 
1 & \text{for } j = 1, \\
2 & \text{for } j = 2,
\end{cases}
$$

$$
\lambda(v_{i,j}) = i + j - 1 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq 2.
$$

Clearly, there is no label greater than $n + 1$. For the the vertex-weights we have the following.

$$
\text{wt}(u_j) = \begin{cases} 
1 + \frac{n(n+1)}{2} & \text{for } j = 1, \\
2n + \frac{n(n+1)}{2} & \text{for } j = 2,
\end{cases}
$$

$$
\text{wt}(v_{i,j}) = (j - 1)n + i + 1 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq 2.
$$

It is not difficult to see that the vertex weights are all distinct and therefore the labeling $\lambda$ is the desired inclusive distance vertex irregular $(n + 1)$-labeling of $2S_n$. The proof is complete.

**Theorem 4.** Let $n$ be a positive integer, $n \geq 3$. Then

$$
\hat{\text{dis}}(3S_n) = \begin{cases} 
6 & \text{for } n = 3, \\
\left\lceil \frac{3n+1}{2} \right\rceil & \text{for } n \geq 4.
\end{cases}
$$

Proof. We distinguish our proof into two cases.

**Case 1:** $n = 3$. By Theorem 2, we obtain $\hat{\text{dis}}(3S_3) \geq \max\{[10/2], [13/4]\} = 5$. However, we will show that $\hat{\text{dis}}(3S_3) \geq 6$. Suppose to the contrary that $\hat{\text{dis}}(3S_3) = 5$ and let $\lambda$ be an inclusive distance vertex irregular 5-labeling of $3S_3$. Then the integers 2, 3, ..., 10 must be realizable as the weights of all the 9 pendant vertices of $3S_3$. We suppose without loss of generality that $\text{wt}(v_{1,1}) = 2$ and $\text{wt}(v_{3,3}) = 10$ which mean that $\lambda(u_1) = \lambda(v_{1,1}) = 1$ and $\lambda(v_3) = \lambda(v_{3,3}) = 5$. Thus we have $\{\lambda(v_{1,2}), \lambda(v_{1,3})\} = \{4, 5\}$ (and hence $\{\text{wt}(v_{1,2}), \text{wt}(v_{1,3})\} = \{5, 6\}$), otherwise there will be two vertices having the same weight. Furthermore, the weights 3 and 4 must be lied on the two of the three vertices $v_{2,1}, v_{2,2}$ and $v_{2,3}$. Let say, $\text{wt}(v_{2,1}) = 3$ and $\text{wt}(v_{2,2}) = 4$. Then we have two subcases.

- $\lambda(u_2) = 1$. Then $\lambda(v_{2,1}) = 2$ and $\lambda(v_{2,2}) = 3$. Regardless the label of $v_{2,3}$, there will always be two vertices having the same weight, a contradiction.
• $\lambda(u_2) = 2$. Then $\lambda(v_{2,1}) = 1$ and $\lambda(v_{2,2}) = 2$. Similarly, regardless the label of $v_{2,3}$, there will always be two vertices having the same weight, a contradiction.

From the aforementioned arguments, we can conclude that $\hat{\text{dis}}(3S_3) \geq 6$. A corresponding inclusive distance vertex irregular 6-labeling of $3S_3$ given in Figure 2 proves the equality.

**Case 2:** $n \geq 4$. According to Theorem 2, $\hat{\text{dis}}(3S_n) \geq \max\{\lceil (3n+1)/2 \rceil, \lceil (3n+4)/(n+1) \rceil \} = \lceil (3n + 1)/2 \rceil$. To prove that $\hat{\text{dis}}(3S_n) \leq \lceil (3n + 1)/2 \rceil$, it is enough for us to construct a corresponding inclusive distance vertex irregular $\lceil (3n + 1)/2 \rceil$-labeling of $3S_n$. For $n = 4$ and $n = 5$, their corresponding inclusive distance vertex irregular labelings are shown in Figure 3 and 4, respectively.

Now let $n \geq 6$ and let $\lambda : V(3S_n) \to \{1, 2, \ldots, \lceil (3n + 1)/2 \rceil \}$ be a labeling on the vertices of the graph $3S_n$ defined such that

$$
\lambda(u_j) = \begin{cases} 
1 & \text{for } j = 1, \\
n + 1 & \text{for } j = 2, \\
\left\lfloor \frac{3n+1}{2} \right\rfloor & \text{for } j = 3,
\end{cases}
$$

![Figure 2](image1.png) **Figure 2.** An inclusive distance vertex irregular 6-labeling of $3S_3$.

![Figure 3](image2.png) **Figure 3.** An inclusive distance vertex irregular 7-labeling of $3S_4$.

![Figure 4](image3.png) **Figure 4.** An inclusive distance vertex irregular 8-labeling of $3S_5$. 

Now let $n \geq 6$ and let $\lambda : V(3S_n) \to \{1, 2, \ldots, \lceil (3n + 1)/2 \rceil \}$ be a labeling on the vertices of the graph $3S_n$ defined such that

$$
\lambda(u_j) = \begin{cases} 
1 & \text{for } j = 1, \\
n + 1 & \text{for } j = 2, \\
\left\lfloor \frac{3n+1}{2} \right\rfloor & \text{for } j = 3,
\end{cases}
$$
\[ \lambda(v_{i,j}) = \begin{cases} 
1 & \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq 2, \\
\lceil n+1 \rceil + i & \text{for } 1 \leq i \leq n \text{ and } j = 3. 
\end{cases} \]

Then we have the following vertex-weights.

\[ wt(u_j) = \begin{cases} 
1 + \frac{n(n+1)}{2} & \text{for } j = 1, \\
n + 1 + \frac{n(n+1)}{4} & \text{for } j = 2, \\
2n + 1 + \frac{n(n+1)}{2} + (n-1)\lceil \frac{n+1}{2} \rceil & \text{for } j = 3, 
\end{cases} \]

\[ wt(v_{i,j}) = \begin{cases} 
(j-1)n + i + 1 & \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq 2, \\
2n + i + 1 & \text{for } 1 \leq i \leq n \text{ and } j = 3. 
\end{cases} \]

Thus the set of weights of all the pendant vertices is \{2, 3, \ldots, 3n + 1\}. Furthermore, it is easy to see that the weights of all the three center vertices are distinct and also as \( n \geq 6 \), we have

\[ 1 + \frac{n(n+1)}{2} \geq 1 + \frac{7n}{2} > 3n + 1, \]

meaning that there are no two vertices having the same weight. This completes the proof. \( \square \)

4. Conclusion

In this paper we introduced a new lower bound for the inclusive distance vertex irregularity strength of arbitrary graphs and determined the exact value for identical copies stars \( mS_n \), particularly for \( m = 2, 3 \). We have also tried to find this parameter for \( m \geq 4 \) but unsuccessful. Therefore we propose the problem below.

**Problem 1.** Determine the inclusive distance vertex irregularity strength of \( mS_n \) for \( m \geq 4 \) and \( n \geq 3 \).

More general, the following problem is also considerable to be investigated.

**Problem 2.** Determine the inclusive distance vertex irregularity strength of \( \bigcup_{i=1}^{m} S_{n_i} \) for \( m \geq 2 \) and \( n_i \geq 3 \).

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References

[1] Arumugam S and Kamatchi N 2012 On \((a,d)\)-distance antimagic graphs *Australas. J. Combin.* 54 279-288.
[2] Baća M, Semaničová-Feňovčíková A, Slamin and Sugeng K A 2018 On inclusive distance vertex irregular labelings *Electron. J. Graph Theory Appl.* (EJGTA) 6 61-83.
[3] Bong N H, Lin Y and Slamin On inclusive and non-inclusive vertex irregular \( d \)-distance vertex labelings *J. Combin. Math. Combin. Comput.* (to appear)
[4] Bong N H, Lin Y and Slamin 2017 On distance-irregular labelings of cycles and wheels *Australas. J. Combin.* 69 315-322.
[5] Chartrand G, Jacobson M S, Lehel J, Oellermann O R, Ruiz S and Saba F 1998 Irregular networks *Congr. Numer.* 64 197-210.
[6] Gallian J A 2019 A dynamic survey of graph labeling *Electron. J. Combin.* #DS6.
[7] Miller M, Rodger C and Simanjuntak R 2003 Distance magic labelings of graphs *Australas. J. Combin.* 28 305-315.
[8] Novindasari S, Marjono and Abusini S 2016 On distance irregular labeling of ladder graph and triangular ladder graph *Pure Math. Sci.* 5 75-81.
[9] Slamin, Windartini T and Purnomo K D On distance irregular labeling of wheel related graphs *submitted*. 
[10] Slamin 2017 On distance irregular labelling of graphs *Far East J. Math. Sci.* **102** 919-932.

[11] Susanto F, Wijaya K, Purnama P M and Slamin 2022 On distance irregular labeling of disconnected graphs *Kragujevac J. Math.* **46** 507-523.

[12] Utami B, Sugeng K A and Utama S 2018 Inclusive vertex irregular 1-distance labelings on triangular ladder graphs *AIP Conf. Proc.* **2021** 060006.

[13] Utami B, Sugeng K A and Utama S 2020 On inclusive $d$-distance irregularity strength on triangular ladder graph and path *AKCE Int. J. Graphs Comb.* doi: 10.1016/j.akcej.2019.10.003