Relative Combinatorial Asphericity

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Abstract

We introduce relative versions of diagrammatic reducibility (DR) and vertex asphericity (VA). The definition of diagrammatic reducibility of a 2-complex goes back to Sieradski 1983 who developed it as a tool for detecting asphericity of a 2-complex. We characterize relative DR in terms of finite subcomplexes of the universal covering analogous to such a characterization of Corson and Trace in the non-relative case and compare relative DR and VA with existing notions of relative asphericity that are in the literature. We present tests for relative DR and DA and apply them to LOT-complexes. We give a simpler proof for the asphericity of injective LOTs, a result obtained by the authors in 2017, using a relative weight test.

1 Introduction

A 2-complex $K$ is diagrammatically reducible, DR for short, if every combinatorial map from a 2-sphere into $K$ contains a pair of adjacent faces with an edge in common so that the faces are mapped mirror-wise across this edge. Diagrammatic reducibility of $K$ implies topological asphericity. The concept was introduced by Sieradski [11] in 1983. See also Gersten [5]. In 2002 Huck-Rosebrock [8] considered the weaker notion of vertex asphericity VA. In this paper we introduce relative versions of DR and VA. Relative vertex asphericity already appeared in the previous article [6] by the authors, where it was used to establish asphericity of injective LOT-complexes. If a 2-complex $K$ is DR or VA relative to a sub-complex $K_0$, then $\pi_2(K)$ is generated, as $\pi_1(K)$-module, by the image of $\pi_2(K_0)$ under the map induced by inclusion. In particular, if $K_0$ is aspherical, then so is $K$.

Other and related notions of relative combinatorial asphericity are in the literature. Diagrammatic reducibility for relative presentations was considered by Bogley-Pride [1] in 1992 and has found many applications over the years. See Bogley-Edjvet-Williams [2] for a good overview. Very recently the idea of directed diagrammatic reducibility was introduced and studied by the authors in [7]. The relative versions of DR and VA given here are weaker than directed DR and DR for relative presentations in the sense of
Bogley and Pride. They do not imply (at least not without further assumptions) that the inclusion induced map \( \pi_1(K_0) \to \pi_1(K) \) is injective. But we feel they are natural and immediate generalization of absolute DR and VA.

Here is an outline of the paper. We begin by defining relative DR and relative VA in the next section. In Section 3 we provide a comparison between the various notions of combinatorial relative asphericity. In Section 4 we characterize relative DR in terms of finite subcomplexes of the universal covering. In the absolute setting this is due to Corson-Trace \([4]\) and reads: A complex \( K \) is DR if and only if every finite subcomplex of the universal covering collapses into the 1-skeleton. In Section 5 we present tests for relative DR and VA. Theorem 5.3 states that if the positive or negative part of the vertex link in a standard 2-complex is a relative tree, then \( K \) is relative VA. Definition 5.5 gives a weight test for relative DR. The last two sections are devoted to applications concerning labelled oriented trees. In Section 6 we give examples that illustrate applications of our relative DR and VA tests. In Section 7 we reprove the main result from \([3]\), which states that injective labeled oriented trees are aspherical. The new proof uses the weight test mentioned above and is much simpler than the original proof. In fact, it aligns with the proof of the main result of Huck-Rosebrock \([9]\) from 2001 which says that prime injective labeled oriented trees are aspherical. This adds transparency.

2 Relative Asphericity

A map \( f: X \to Y \) between complexes is combinatorial if \( f \) maps open cells of \( X \) homeomorphically to open cells of \( Y \). Given a presentation \( P = \langle X \mid R \rangle \) we denote by \( K(P) \) the 2-complex defined by \( P \). It has a single 0-cell, 1-cells in correspondence with \( X \) and 2-cells in correspondence with \( R \).

**Definition 2.1** A spherical diagram over a 2-complex \( K(P) \) defined by a presentation \( P = \langle X \mid R \rangle \) is a combinatorial map \( f:C \to K(P) \), where \( C \) is a 2-sphere with a cell structure.

Note that if we orient the cells in \( C \) and label each cell \( c \) of \( C \) by \( f(c) \), the labeling on \( C \) carries all information of \( f \). We refer to such a labeled 2-sphere also as a spherical diagram over \( K(P) \).

If a 2-complex \( K \) is non-aspherical, then there exists a spherical diagram which realizes a nontrivial element of \( \pi_2(K) \). In fact, \( \pi_2(K) \) is generated by spherical diagrams. So in order to check whether a 2-complex is aspherical or not it is enough to check spherical diagrams.

If \( v \) is a vertex of a 2-complex \( K \) then the link \( Lk(K, v) \) is the boundary of a regular neighborhood of \( v \) in \( K \) equipped with the induced cell decompo-
Definition 2.2 Let $\Gamma$ be a graph and $\Gamma_0$ be a subgraph (which could be empty).

1. An edge cycle $c = e_1 \ldots e_q$ in $\Gamma$ is called homology reduced if it does not contain a pair of edges $e_i$ and $e_j$ so that $e_j = \widetilde{e}_i$, where $\widetilde{e}_i$ is the edge $e_i$ with opposite orientation.

2. An edge cycle $c = e_1 \ldots e_q$ is said to be homology reduced relative to $\Gamma_0$ if it does not contain a pair of edges $e_i$ and $e_j$ of $\Gamma - \Gamma_0$ so that $e_j = \widetilde{e}_i$.

Let $K_0$ be a full subcomplex of the 2-complex $K$. By full we mean that if $D \in K$ is a 2-cell where all boundary cells are elements of $K_0$ then $D \in K_0$. $\text{Lk}(K_0)$ is a subgraph of $\text{Lk}(K)$. Let $f: C \to K$ be a spherical diagram and $v$ be a vertex of the 2-sphere $C$. The map $f$ induces a combinatorial map $f_v: \text{Lk}(C, v) \to \text{Lk}(K)$. Note that $\text{Lk}(C, v)$ is a circle and the image of that circle, oriented clockwise, is a cycle of corners $\alpha(v) = \alpha_1 \ldots \alpha_q$, that is a closed edge path, in $\text{Lk}(K)$. We say that the diagram $f: C \to K$ is vertex reduced at $v$ relative to $K_0$ if the cycle $\alpha(v)$ in $\text{Lk}(K)$ is homology reduced relative to $\text{Lk}(K_0)$. We say that the diagram is vertex reduced relative to $K_0$ if it is vertex reduced relative to $K_0$ at all its vertices. The 2-complex $K$ is called vertex aspherical relative to $K_0$, VA relative to $K_0$ for short, if there does not exist a vertex reduced spherical diagram $f: C \to K$ such that $f(C) \not\subseteq K_0$.

If we omit “relative to” we implicitly imply relative to the empty set $\emptyset$, even if a subcomplex is present. For example if we say a spherical diagram is vertex reduced we mean vertex reduced relative to $\emptyset$. Consequently a 2-complex $K$ is called vertex aspherical (VA) if there is no vertex reduced spherical diagram over $K$. 
Definition 2.3 Let $\Gamma$ be a graph and $\Gamma_0$ be a subgraph (which could be empty).

1. An edge cycle $c = e_1 \ldots e_q$ in $\Gamma$ is called reduced if it does not contain a consecutive pair of edges $e_i$ and $e_{i+1}$ ($i \mod q$) so that $e_{i+1} = \bar{e}_i$, where $\bar{e}_i$ is the edge $e_i$ with opposite orientation.

2. An edge cycle $c = e_1 \ldots e_q$ is said to be reduced relative to $\Gamma_0$ if it does not contain a consecutive pair of edges $e_i$ and $e_{i+1}$ ($i \mod q$) of $\Gamma - \Gamma_0$ so that $e_{i+1} = \bar{e}_i$.

Let $K_0$ be a full subcomplex of the 2-complex $K$. Let $f: C \to K$ be a spherical diagram and $v$ be a vertex of the 2-sphere $C$. As before there is a cycle of corners $\alpha(v) = \alpha_1 \ldots \alpha_q$, that is a closed edge path, in $\text{Lk}(K)$. We say that the diagram $f: C \to K$ is reduced at $v$ relative to $K_0$ if the cycle $\alpha(v)$ in $\text{Lk}(K)$ is reduced relative to $\text{Lk}(K_0)$. So if $f$ is not reduced at $v$ relative to $K_0$ there is a folding edge $e \in C$ with $v$ in its boundary, such that the adjacent 2-cells $d, d' \in C$ that share $e$ in their boundary map to the same 2-cell in $K(P) - K(P_S)$ by folding over $e$.

We say that the diagram is reduced relative to $K_0$ if it is reduced relative to $K_0$ at all its vertices. The 2-complex $K$ is called diagrammatically reducible relative to $K_0$, DR relative to $K_0$ for short, if there does not exist a reduced spherical diagram $f: C \to K$ such that $f(C) \not\subseteq K_0$.

If we omit “relative to” we implicitly imply relative to the empty set $\emptyset$, even if a subcomplex is present. For example if we say a spherical diagram is reduced we mean reduced relative to $\emptyset$. Consequently a spherical diagram $f: C \to K$ is reducible, if there exists a pair of 2-cells in $C$ with a common edge $e$, such that both 2-cells are mapped to $K$ by folding over $e$ and a 2-complex $K$ is called diagrammatically reducible (DR) if there is no reduced spherical diagram over $K$.

For a 2-complex we have DR $\Rightarrow$ VA $\Rightarrow$ aspherical. Vertex asphericity in case $K_0 = \emptyset$ was considered in Huck, Rosebrock [8]. We also have: If $K$ is a 2-complex with full subcomplex $K_0$ and if $K$ is DR relative to $K_0$ then $K$ is VA relative to $K_0$.

The following theorem is proved in [6]:

**Theorem 2.4** If $K$ is VA relative to $K_0$, then $\pi_2(K)$ is generated, as $\pi_1(K)$-module, by the image of $\pi_2(K_0)$ under the map induced by inclusion. In particular, if $K_0$ is aspherical, then so is $K$. 
3 Comparison with directed diagrammatic reducibility

In [7] we defined the notion of directed diagrammatic reducibility. We recall the definition here: For a set $X$ call a subset $S$ proper if $S \neq X$ ($S$ may be empty). If $S$ is a proper subset of the set $X$ of generators of a presentation $P$ let $P_S$ be the sub-presentation of $P$ carried by $S$. Observe that $K(P_S)$ is a full subcomplex of $K(P)$. We say $P$ is DR directed away from $S$ if every spherical diagram $f: C \to K(P)$ that contains an edge labelled by an element of $X - S$ also contains a folding edge labelled by an element of $X - S$.

**Proposition 3.1** Let $P = \langle X \mid R \rangle$ be a presentation and $S$ a proper subset of $X$. If $P$ is DR directed away from $S$, then $K(P)$ is DR relative to $K(P_S)$.

**Proof:** Assume $K(P)$ is not DR relative to $K(P_S)$. Then there exists a spherical diagram $f: C \to K(P)$ such that $f(C) \not\subseteq K(P_S)$ where all pairs of 2-cells which may be reduced lie in $K(P_S)$. Since $f(C) \not\subseteq K(P_S)$ and $K(P_S)$ is a full subcomplex of $K(P)$ we have that $C$ contains an edge labelled by an element of $X - S$. Since all pairs of 2-cells which may be reduced lie in $K(P_S)$ we have that all folding edges of $C$ are labelled by elements of $S$. So $P$ is not DR directed away from $S$. \qed

If $K(P)$ is DR relative to $K(P_S)$ then $P$ does not have to be DR directed away from $S$. This is because a pair of cancelling 2-cells in a spherical diagram might map to $K(P) - K(P_S)$ but the common edge of these 2-cells maps to $S$.

**Example 3.2** Let $P = \langle a, b, c \mid b a c^{-1}, c b^{-1} a^{-1} \rangle$ be a presentation. $K(P)$ is the torus. There is a disk diagram $D$ with boundary reading $a b a^{-1} b^{-1}$ achieved by gluing the two relator disks along $c$ together. Glue $D$ to $-D$ to obtain a spherical diagram which is reducible at $a$ and $b$ only. This shows that $P$ is not DR away from $S = \{a, b\}$. Since $P_S = \langle a, b \mid \rangle$ has no relators we have that $K(P)$ is DR relative $K(P_S)$ if and only if $K(P)$ is DR. But this presentation of the torus is certainly DR, so $K(P)$ is DR relative $K(P_S)$.

There is one more notion of relative combinatorial asphericity, and that is diagrammatic reducibility for relative presentations as defined by Bogley-Pride [1]. A relative presentation $\hat{P} = \langle H, \hat{x} \mid \hat{r} \rangle$ consists of a group $H$, a generating set $\hat{x}$ and relator set $\hat{r} \subseteq H * F(\hat{x})$. Given an ordinary presentation $P = \langle X \mid R \rangle$ and a proper subset $S \subseteq X$ we can associate to it a relative presentation. Let $H = G(P_S)$, $\hat{x} = X - S$ and $\hat{r}$ be the set obtained from
R in the following way: We have a homomorphism \( \phi : F(X) \to H \ast F(x) \) by sending \( x \in S \) to the group element in \( H \) that it presents, and \( x \) to \( x \) if \( x \in X - S \). Let \( \hat{r} = \phi(r) \), \( r \in R \). The following result was shown in Harlander-Rosebrock [7].

**Theorem 3.3** The presentation \( P = \langle X \mid R \rangle \) is DR directed away from \( S \) if and only if the relative presentation \( \hat{P} = \langle H, x \mid \hat{r} \rangle \) is DR.

### 4 Corson-Trace for Relative Diagrammatic Reducibility

Let \( P \) be a presentation and \( K(P) \) the corresponding 2-complex. Let \( \tilde{K}(P) \) be the universal covering of \( K(P) \) and \( p \) the corresponding covering projection. For a complex \( K \) let \( K^{(1)} \) be its 1-skeleton.

Corson and Trace have shown in [4] the following result:

**Theorem 4.1** \( K(P) \) is DR if and only if every finite subcomplex of \( \tilde{K}(P) \) collapses into \( \tilde{K}(P)^{(1)} \).

The proof of this result can be modified to give the following version in the relative setting.

**Theorem 4.2** Let \( T \) be a full subpresentation of a presentation \( P \). Then \( K(P) \) is DR relative to \( K(T) \) if and only if every finite subcomplex of \( \tilde{K}(P) \) collapses into \( p^{-1}(K(T)) \cup \tilde{K}(P)^{(1)} \).

Recall that an edge in a 2-complex is called free if it occurs exactly once in the boundary of exactly one 2-cell. A 2-complex is called closed if it does not have a free edge.

**Lemma 4.3** Let \( K \) be a finite 2-complex and \( d \) be a 2-cell in \( K \). If \( K \) is closed then there exists a reduced closed surface diagram \( f : F \to K \) so that \( d \) is contained in \( f(F) \).

This lemma is in Corson-Trace [3], Theorem 2.1, stated without the fixed 2-cell \( d \). The fact that \( f \) hits a specified 2-cell will be important in the relative case. A detailed proof of Lemma 4.3 can be found in Harlander-Rosebrock [7], Lemma 3.2.

**Proof:** (of Theorem 4.2) Suppose the statement is false. Among all finite subcomplexes that do not collapse into \( p^{-1}(K(T)) \cup \tilde{K}(P)^{(1)} \) choose one with the minimal number of 2-cells. Call it \( X \). Note that \( X \) does not have a free edge, because a collapse could be performed at that free edge to produce a complex with fewer 2-cells, contradicting minimality.
Let $\tilde{d}$ be a 2-cell in $X$ not contained in $p^{-1}(K(T)) \cup \tilde{K}(P)^{(1)}$. If follows from Lemma 4.3 that there exists a reduced surface diagram $\tilde{f}: F \to X \subseteq \tilde{K}(P)$, where $F$ is a closed orientable surface and $\tilde{d}$ is contained in $\tilde{f}(F)$. Let $f = p \circ \tilde{f}: F \to K(P)$. Note that $d = p(\tilde{d})$ is a 2-cell in $f(F)$ not contained in $K(T)$. We now proceed as in the proof of Lemma 2.1 in [4]: Attach Van Kampen diagrams along cutting curves of $f$ to produce a simply connected 2-complex $L_0$ and combinatorial maps $F \xrightarrow{\alpha_0} L_0 \xrightarrow{\beta_0} K(P)$ such that $\beta_0 \circ \alpha_0 = f$. The Van Kampen diagrams exist because $f$ lifts to $\tilde{f}$, so every closed curve in $F$ maps to a closed curve in $K(P)$ that is homotopically trivial. Note furthermore that $L_0$ is the 2-skeleton of a cell decomposition of the 3-sphere $S^3$. Let $L$ be a 2-complex with the minimal number of 2-cells satisfying the following conditions:

1. $L$ is a simply connected 2-skeleton of a cell-decomposition of the 3-sphere $S^3$;

2. There exist combinatorial maps $F \xrightarrow{\alpha} L \xrightarrow{\beta} K(P)$ such that $\beta \circ \alpha = f$.

Corson-Trace show that the attaching maps of the 3-cells of $S^3$ utilize all 2-cells of $L$ and each attaching map $g: C \to L$ results in a reduced spherical diagram $\beta \circ g: C \to K(P)$. Choose a 3-cell so that $\beta \circ g(C)$ contains $d$. Then $\beta \circ g: C \to K(P)$ is a reduced spherical diagram that is not a diagram over $K(T)$, contradicting the assumption that $K(P)$ is DR relative to $K(T)$.

For the other direction assume that $f: C \to K(P)$ is a spherical diagram which is not already a diagram over $K(T)$. We can lift it to a spherical diagram $\tilde{f}: C \to \tilde{K}(P)$. Now $\tilde{f}(C) = X$ is a finite subcomplex of $\tilde{K}(P)$ not already contained in $p^{-1}(K(T)) \cup \tilde{K}(P)^{(1)}$. Since $X$ collapses into $p^{-1}(K(T)) \cup \tilde{K}(P)^{(1)}$, it has a free edge $\tilde{e}$. Any edge $e$ in $C$ so that $\tilde{f}(e) = \tilde{e}$ is a folding edge. Thus $f: C \to K(P)$ is not reduced. \hfill $\square$

The following corollary can be used as a tool to show that a given presentation defines an infinite group.

**Corollary 4.4** Let $P$ be a presentation of a finite group and $T$ a subpresentation. Then $K(P)$ is DR relative to $K(T)$ if and only if $K(P)$ collapses into $K(T)$.

For completeness we finish this section with the directed DR version of Corson-Trace which again shows the subtle differences between the different notions. For a proof see [7].

**Theorem 4.5** Let $P = \langle X \mid R \rangle$ be a presentation and $S$ a proper subset of the generators $X$. Then $P$ is DR directed away from $S$ if and only if every finite subcomplex of $\tilde{K}(P)$ collapses into $p^{-1}(K(P_S)) \cup \tilde{K}(P)^{(1)}$, where only edges of the form $(g, x), x \in X - S$ are used as collapsing edges.
5 Methods for showing relative DR and VA

**Definition 5.1** Let $\Gamma$ be a graph and $\Gamma_0$ a subgraph of $\Gamma$. $\Gamma$ is called a forest relative to $\Gamma_0$ if every homology reduced cycle is contained in $\Gamma_0$. $\Gamma$ is called a tree relative to $\Gamma_0$ if $\Gamma$ is connected and every homology reduced cycle is contained in $\Gamma_0$.

Let $C$ be an oriented cell decomposition of the 2-sphere. A *source* in $C$ is a vertex so that all adjacent edges point away from it. A *sink* is a vertex so that all adjacent edges point towards it. A 2-cell $d \in C$ is said to have *exponent sum 0* if, when traveling along the boundary of $d$ in clockwise direction, one encounters the same number of positive and negative edges.

The following theorem is due to Gersten (see [5]):

**Theorem 5.2** Let $C$ be a cell decomposition of the 2-sphere with oriented edges, such that all 2-cells have exponent sum 0. Then $C$ contains a sink and a source.

**Proof:** We can define a map $h: C \to S$, where $S$ is the unit circle with a single vertex $u$ and a single edge $e$ oriented clockwise, by mapping each vertex of $C$ to $u$, and an oriented edge of $C$ in an orientation preserving manner to $e$. This map extends over the 2-skeleton of $C$ because 2-cells have exponent sum 0. We can now lift $h$ to $\tilde{h}: C \to \tilde{S}$, where $\tilde{S}$ is the universal covering of $S$. Note that $\tilde{S}$ is the real number line $\mathbb{R}$ with its vertices located at the integers. The map $\tilde{h}$ takes on a maximum at some vertex $v$ and a minimum at some vertex $v'$ which are a sink and a source in $C$, respectively. \qed

If $P_1 = \langle X_1 \mid R_1 \rangle$ and $P_2 = \langle X_2 \mid R_2 \rangle$ are presentations, then $P_1 \cup P_2 = \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle$. Let $P = \langle X \mid R \rangle$ be a finite presentation with cyclically reduced relators and let $\{T_1, \ldots, T_n\}$ be a set of disjoint *full* sub-presentations of $P$. Full means that if $r$ is a relator in $P$ that only involves generators from $T_i$, then $r$ is already a relator in $T_i$. Disjoint means that the generating sets of $T_i$ and $T_j$ are disjoint subsets of $X$ in case $i \neq j$.

Let $T = T_1 \cup \ldots \cup T_n$. The complex $K(T) = K(T_1) \lor \ldots \lor K(T_n)$ (the $n$-fold wedge product defined by identifying the vertices of the $K(T_i)$) is a sub-complex of $K(P)$.

Let $T_i = \langle t_{1i}, \ldots, t_{mi} \mid S_i \rangle$ and let $U_i$ be the set of all words with letters in $t_{1i}^{\pm 1}, \ldots, t_{mi}^{\pm 1}$ of exponent sum zero, including words that are cyclically reducible. Let $T_i = \langle t_{1i}, \ldots, t_{mi} \mid S_i \cup U_i \rangle$. Let $T = T_1 \cup \ldots \cup T_n$ and note that $P \cup T = \langle X \mid R \cup U_1 \cup \ldots \cup U_n \rangle$ since $t_{ij} \subseteq X$ and $S_i \subseteq R$ for every $i, j$. The presentation $P \cup T$ is infinite and in the corresponding
group \( G(P \cup T) \) the generators of each \( T_i \) are identified, since we have the relator \( t^{-1}t' \) in \( P \cup T \) for every pair \( t, t' \) of generators in \( T_i \). Note that \( T_i \) is a sub-presentation of \( P \cup T \) and the subgraph \( W(T_i) \) of the Whitehead graph \( W(P \cup T) \), which is spanned by the vertices \( t_{1i}^{\pm}, \ldots, t_{mi}^{\pm} \), contains the complete graph on these vertices. In fact, every pair of vertices in \( W(T_i) \) is connected by infinitely many edges.

**Theorem 5.3** Let \( P \) be a finite presentation with cyclically reduced relators of exponent sum 0, and let \( \{T_1, \ldots, T_n\} \) be a set of disjoint full sub-presentations. Let \( T = T_1 \cup \ldots \cup T_n \). If \( W^+(P \cup T) \) is a forest relative to \( W^+(T) \) or \( W^-(P \cup T) \) is a forest relative to \( W^-(T) \) then \( K(P) \) is VA relative to \( K(T) \). Furthermore, the inclusion induced homomorphism \( \pi_1(K(T_i)) \to \pi_1(K(P)) \) is injective for every \( i = 1, \ldots, n \).

The proof of this theorem uses the following lemma (a very similar result is Lemma 4 in [6]):

**Lemma 5.4** Let \( P \) be a finite presentation with cyclically reduced relators. Let \( \{T_1, \ldots, T_n\} \) be a set of disjoint full sub-presentations, and \( T = T_1 \cup \ldots \cup T_n \). If \( K(P) \) is not VA relative to \( K(T) \), then there is a spherical diagram \( f: C \to K(P \cup T) \) such that

1. \( f: C \to K(P \cup T) \) is vertex reduced relative to \( K(T) \),
2. \( f(C) \) is not contained in \( K(T) \), and
3. if \( v \in C \) is a vertex then the corner cycle \( \alpha(v) \) has length at least two and \( f(\alpha(v)) \) is not contained in \( K(T) \).

Lemma 4 in [6] states a stronger result. The proof of the lemma stated here is technically easier and we give it below:

**Proof:** Since \( K(P) \) is not VA relative to \( K(T) \) there exists a spherical diagram \( C^* \to K(P) \) that is vertex reduced relative to \( K(T) \) but does not map entirely into \( K(T) \). Since \( K(P) \) is a subcomplex of \( K(P \cup T) \) this diagram can be viewed as a diagram \( C^* \to K(P \cup T) \). Under all these diagrams take one with the minimal number of 2-cells, and among those one with the minimal number of edges.

This spherical diagram \( f: C \to K(P \cup T) \) does not contain a vertex of valency one. If \( v \) were a vertex of valency one in \( C \), then it would be a vertex in the boundary of a cell \( E \) that maps to some \( K(T_i) \) since we assumed the relators in \( P \) outside \( T \) to be cyclically reduced. Let \( e \) be the edge in \( C \) that contains \( v \). See Figure[1] We can remove \( v \) and the interior of \( e \) and transform \( E \) into \( E' \). Note that the boundary words of \( E \) and \( E' \) are the same up to free or cyclic reduction, hence removing \( v \) and the interior of \( e \)
produces a spherical diagram $f': C' \to K(P \cup T)$ with fewer edges but the same number of 2-cells, contradicting the choice of $f$.

The spherical diagram $C$ has the first and the second property by choice. Let us look at the third property. Assume there is a vertex $v \in C$ where all 2-cells with $v$ in their boundary map to $K(T)$. Let $G$ be the maximal region of $C$ with $v$ in its interior which maps entirely to $K(T)$. If $G$ is homeomorphic to a disk we can erase interior arcs and vertices of $G$ and we get a spherical diagram with less 2-cells contradicting the choice of $f$. If $G$ has $m > 1$ boundary components, erase the interior of $G$ and insert $m - 1$ arcs $a_1, \ldots, a_m$, such that $G - \{a_1, \ldots, a_m\}$ is a disk, again contradicting the choice of $f$. □

**Proof of Theorem 5.3**: Suppose $K(P)$ is not VA relative to $K(T)$. Then there exists a spherical diagram $f: C \to K(P \cup T)$ that satisfies the conditions 1, 2, and 3 stated in Lemma 5.4. Since $P$ and $T$ contain only relators of exponent sum 0, we know that $C$ contains only cells of exponent sum 0, hence $C$ contains a sink and a source by Theorem 5.2. Let us assume without loss of generality that $W^+(P \cup T)$ is a forest relative to $W^+(T)$. Assume that the vertex $v \in C$ is a source. The cycle $\alpha(v) = \alpha_1 \ldots \alpha_l$ satisfies $l \geq 2$, is contained in $W^+(P \cup T)$, and is homology reduced relative to $W^+(T)$ because $f: C \to K(P \cup T)$ is vertex reduced relative to $K(T)$. Since $W^+(P \cup T)$ is a forest relative to $W^+(T)$ we know that $\alpha(v)$ is entirely contained in a connected component of $W^+(T)$, and hence in some $W^+(T_i)$, because $W^+(T)$ is a disjoint union of the $W^+(T_i)$, $i = 1, \ldots, n$. Thus, if $\alpha(v) = \alpha_1 \ldots \alpha_l$, then all corners $\alpha_j$, $j = 1, \ldots, l$ are in $W^+(T_i)$. This contradicts condition 3 stated in Lemma 5.4. Thus $K(P)$ is VA relative to $K(T)$.

Suppose the map $\pi_1(K(T_i)) \to \pi_1(K(P))$ is not injective for some $i = 1, \ldots, n$. Then there exists a cyclically reduced word $w$ in the generators of $T_i$ that represents the trivial element of $\pi_1(K(P))$ but is not trivial in $\pi_1(K(T))$. Hence $w$ is the boundary word of a vertex reduced disc diagram $g: D \to K(P)$. Note that $D$ has to contain 2-cells that are not mapped to $K(T)$ because the map $\pi_1(K(T_i)) \to \pi_1(K(T)) = \pi_1(K(T_1)) \ast \ldots \ast \pi_1(K(T_n))$ is injective. The word $w$ has exponent sum zero because relators in $P$ have exponent sum 0 and hence $w$ is a relator of $T_i$. We can attach a disc $D'$ to $D$.
and obtain a spherical diagram $f : C \to K(P \cup T)$. Note that this spherical diagram is vertex reduced. If it were not, then there would have to be a vertex on the boundary of $D$ where the spherical diagram $f : C \to K(P \cup T)$ is not vertex reduced. But that would mean that $D$ contains a 2-cell with boundary word $w$. This would imply that $w$ is a relator in $P$. Since we assumed $T_i$ to be a full sub-presentation, the word $w$ would have to be a relator in $T_i$, which is not the case because $w$ does not represent the trivial element of $\pi_1(K(T_i))$. Thus $f : C \to K(P \cup T)$ is indeed vertex reduced. Since $K(P)$ is VA relative to $K(T)$ we have that $f(C) \subseteq K(T)$, which implies that $g(D) \subseteq K(P) \cap K(T) = K(T)$. We have reached a contradiction. □

As a second result in this section we have a relative weight test. If $E$ is the set of edges of a graph $\Gamma$ then a weight function is a function $g : E \to \mathbb{R}$. If $w = e_1 \ldots e_n$ is a path in $\Gamma$ with $e_i \in E$ then let $g(w) = \sum_{i=1}^{n} g(e_i)$.

Let $P$ be a finite presentation with cyclically reduced relators. Let $\{T_1, \ldots, T_n\}$ be a set of disjoint full sub-presentations. Let $T = T_1 \cup \ldots \cup T_n$. We denote by $P - T$ the presentation with relators in $P$ not in $T$ and with generator set the generators occurring in the relators of $P - T$. Define the Whitehead graph of $P$ relative to $T$, abbreviated $W(P, T)$, to be the following graph:

- The vertex set of $W(P, T)$ is equal to the vertex set of $W(P)$;
- edges of $W(P)$ which come from relators of $P - T$ are also in $W(P, T)$;
- let $Y_i = \{t_{i1}^+, \ldots, t_{im_i}^+\}$ be the set of vertices of $W(T_i)$. For all $1 \leq i \leq n$ and for all pairs $a, b \in Y_i$ (not necessarily different) there is exactly one edge in $W(P, T)$ connecting $a$ and $b$.

The graphs $W(P \cup T)$ and $W(P, T)$ are not the same graphs. The parts of the graphs coming from relators of $P - T$ are the same. In the $T$-part of the Whitehead graphs between any two vertices of $Y_i$ all but one edges are deleted by going from $W(P \cup T)$ to $W(P, T)$.

**Definition 5.5** Let $P$ be a finite presentation with cyclically reduced relators. Let $\{T_1, \ldots, T_n\}$ be a set of disjoint full sub-presentations where each relator has exponent sum 0. Let $T = T_1 \cup \ldots \cup T_n$. Let $W(P, T)$ be the Whitehead-graph of $P$ relative to $T$ with edge set $E$. Then $P$ is said to satisfy the weight-test relative to $T$ if there is a weight function $g : E \to \mathbb{R}$ satisfying:

(a) If $e \in E$ connects a vertex $t_{ik}^+$ with a vertex $t_{ip}^-$ then $g(e) = 1$,

(b) If $e \in E$ connects a vertex $t_{ik}^+$ with a vertex $t_{ip}^+$ or $e$ connects a vertex $t_{ik}^-$ with a vertex $t_{ip}^-$ then $g(e) = 0$. 
(c) Let \( e_1, \ldots, e_q \in E \) be the edges coming from a relator of \( P - T \). Then
\[
\sum_{i=1}^{q} g(e_i) \leq q - 2
\]

(d) If \( c \) is a reduced cycle in \( W(P, T) \) containing at least one edge coming from a relator of \( P - T \), then \( g(c) \geq 2 \).

**Theorem 5.6** Let \( P \) be a finite presentation which satisfies the weight-test relative to \( T \). Then \( K(P) \) is diagrammatically reducible relative to \( K(T) \). If in addition relators of \( P \) have exponent sum 0, then the inclusion induced homomorphism \( \pi_1(K(T_i)) \to \pi_1(K(P)) \) is injective for every \( i = 1, \ldots, n \).

We need a lemma analogously to Lemma 5.4:

**Lemma 5.7** Let \( P \) be a finite presentation with cyclically reduced relators. Let \( \{T_1, \ldots, T_n\} \) be a set of disjoint full sub-presentations with relators of exponent sum 0, and \( T = T_1 \cup \ldots \cup T_n \). If \( K(P) \) is not DR relative to \( K(T) \), then there is a spherical diagram \( f:C \to K(P \cup T) \) such that

1. \( f: C \to K(P \cup T) \) is reduced relative to \( K(T) \),
2. \( f(C) \) is not contained in \( K(T) \), and
3. if \( v \in C \) is a vertex then the corner cycle \( \alpha(v) \) has length at least two and \( f(\alpha(v)) \) is not contained in \( K(T) \).

The proof is the same as the proof of Lemma 5.4. Instead of a spherical diagram which is vertex reduced relative to \( K(T) \) we choose one which is reduced relative to \( K(T) \). The same arguments apply.

**Proof:** (of Theorem 5.6) Suppose \( K(P) \) is not DR relative to \( K(T) \). Then there exists a spherical diagram \( f:C \to K(P \cup T) \) that satisfies the conditions 1, 2, and 3 stated in Lemma 5.7. Pull back the weights of \( W(P, T) \) to all corners of all vertices in \( C \).

Condition 3 stated in Lemma 5.7 implies that each cycle of corners \( c \in C \) contains at least one corner the image of which under \( f \) is not contained in \( K(T) \). Then condition (d) of Definition 5.5 implies \( g(c) \geq 2 \) for all cycles of corners in \( C \).

We claim that if \( d \in C \) is a 2-cell with corners \( e_1, \ldots, e_q \), then \( \sum_{i=1}^{q} g(e_i) \leq q - 2 \). If \( d \) is not mapped to \( K(T) \) under \( f \) then this is true by condition (c) of Definition 5.5. If \( d \) is mapped to \( K(T_i) \), then \( d \) comes from a relator of exponent sum 0. So there will be at least one corner in \( d \), which is mapped to an edge which connects a vertex \( t^+_{ik} \) with a vertex \( t^-_{ip} \) and there will be another corner of \( d \) which is mapped to an edge which connects a vertex \( t^-_{ik} \).
with a vertex $t_{ip}$. Those two corners each have weight 0 by condition (b) of Definition 5.5. All other corners of $d$ will have weight 1 at most by condition (a) of Definition 5.5. So in total we have $\sum_{i=1}^q g(e_i) \leq q - 2$.

We think of weights at the corners of $C$ as angles. The curvature $\kappa(v)$ at a vertex $v \in C$ is defined as $2 - g(\beta)$, where $\beta$ is the cycle of corners at $v$. The curvature $\kappa(d)$ of a 2-cell $d \in C$ is defined as $\sum g(e_i) - (|\partial d| - 2)$, where the sum is taken over all the corners $e_i$ of $d$ and $|\partial d|$ denotes the number of edges in the boundary of $d$. The combinatorial Gauss-Bonnet Theorem (see Proposition 4.4 in [5]) asserts that summing up the curvature at all the vertices and 2-cells yields twice the Euler characteristic of the 2-sphere $C$. But since $g(v) \geq 2$ for all cycles of corners in $C$, we have $\kappa(v) \leq 0$ at all vertices in $C$. Also $\sum_{i=1}^q g(e_i) \leq q - 2$ implies $\kappa(d) \leq 0$ for all 2-cells of $C$. This is a contradiction.

Suppose the map $\pi_1(K(T_i)) \to \pi_1(K(P))$ is not injective for some $i = 1, \ldots, n$. Then there exists a cyclically reduced word $w$ in the generators of $T_i$ that represents the trivial element of $\pi_1(K(P))$ but is not trivial in $\pi_1(K(T))$. Hence $w$ is the boundary word of a reduced disc diagram $g: D \to K(P)$. Note that $D$ has to contain 2-cells that are not mapped to $K(T)$ because the map $\pi_1(K(T_i)) \to \pi_1(K(T)) = \pi_1(K(T_1)) \ast \ldots \ast \pi_1(K(T_n))$ is injective. Relators of $P$ have exponent sum 0, so the word $w$ has exponent sum zero and hence is a relator of $T_i$. We can attach a disc $D'$ to $D$ and obtain a spherical diagram $f: C \to K(P \cup T)$. Note that this spherical diagram is reduced. If it were not, then there would have to be a folding edge on the boundary of $D$. But that would mean that $D$ contains a 2-cell with boundary word $w$. This would imply that $w$ is a relator in $P$. Since we assumed $T_i$ to be a full sub-presentation, the word $w$ would have to be a relator in $T_i$, which is not the case because $w$ does not represent the trivial element of $\pi_1(K(T_i))$. Thus $f: C \to K(P \cup T)$ is indeed reduced. Since $K(P)$ is DR relative to $K(T)$ we have that $f(C) \subseteq K(T)$, which implies that $g(D) \subseteq K(P) \cap K(T) = K(T)$. We have reached a contradiction. \hfill $\square$

## 6 Applications to Labelled Oriented Trees

A standard reference for labeled oriented graphs, LOGs for short, is [10]. Here are the basic definitions. A LOG is an oriented graph $P$ on vertices $x$ and edges $e$, where each oriented edge is labeled by a vertex. Associated with a LOG is the LOG-presentation $P = \langle x \mid \{ r_e \}_{e \in E} \rangle$. If $e$ is an edge that starts at $x$, ends at $y$ and is labeled by $z$, then $r_e = xz(zy)^{-1}$. A labelled oriented graph is called compressed if no edge is labelled with one of its vertices. It is boundary reduced if every boundary vertex occurs as an edge label. It is called interior reduced if there is no vertex with two adjacent
edges with the same label that either point away or towards that vertex. It is reduced if it is compressed, boundary reduced, and interior reduced. It can be shown that a LOG can be transformed into a reduced LOG without altering the homotopy-type of the LOG-complex, the standard 2-complex for the LOG-presentation. A LOG is injective if a vertex occurs as an edge label at most once. Finally, a labeled oriented tree, LOT, is a labeled oriented graph where the underlying graph is a tree.

The following lemma is Lemma 2 in [6]. It will be convenient to have for the examples constructed in this section.

**Lemma 6.1** Let $\Gamma$ be a graph, $\Gamma_0$ a subgraph with connected components $\Gamma_1, \ldots, \Gamma_n$. Let $\Gamma'$ be the graph obtained by collapsing each component $\Gamma_i$ to a vertex $g_i \in \Gamma_i$. Then $\Gamma$ is a forest relative to $\Gamma_0$ if and only if $\Gamma'$ is a forest.

The next result is Theorem 5.3 for labeled oriented trees.

**Theorem 6.2** Let $\mathcal{P}$ be a compressed LOT with corresponding presentation $\mathcal{P}$ and $\mathcal{T} = \{T_1, \ldots, T_n\}$ a set of proper sub-LOTs with presentations $T = \{T_1, \ldots, T_n\}$. Assume the $T_i$ are pairwise disjoint. If $W^+(\mathcal{P} \cup \mathcal{T})$ is a forest relative to $W^+(\mathcal{T})$ or $W^-(\mathcal{P} \cup \mathcal{T})$ is a forest relative to $W^-(\mathcal{T})$ then $K(\mathcal{P})$ is VA relative to $K(\mathcal{T})$. In particular, if $K(\mathcal{T})$ is aspherical then so is $K(\mathcal{P})$. Furthermore, the inclusion induced homomorphism $\pi_1(K(\mathcal{T})) \to \pi_1(K(\mathcal{P}))$ is injective.

**Proof:** We claim that each $T_i$ is a full sub-presentation of $\mathcal{P}$. This is true because a LOT is connected and $\mathcal{P} - \mathcal{T}$ is a forest. So no edge of $\mathcal{P} - \mathcal{T}$ can have all three of its vertex labels from one $T_i$.

Since the LOT $\mathcal{P}$ is compressed, the relators are cyclically reduced. LOT relators have exponent sum 0, so all conditions of Theorem 5.3 are satisfied. So $K(\mathcal{P})$ is VA relative to $K(\mathcal{T})$. The statement that $K(\mathcal{P})$ is aspherical if $K(\mathcal{T})$ is follows from Theorem 2.4. □

This theorem is in general not true for LOGs which contain cycles. Assume you have a LOG $\mathcal{P}$ which consists of exactly one cycle. Let $\mathcal{T} = \mathcal{P} - e$, where $e$ is an arbitrary edge of $\mathcal{P}$. Then the presentation defined by $\mathcal{T}$ is not a full sub-presentation.

It is very easy to find examples to Theorem 6.2. Take your favorite compressed aspherical LOT presentation $T$ (for instance one coming from an injective LOT or a LOT which is $C(4), T(4)$). Add edges to the corresponding LOT and obtain a LOT presentation $\mathcal{P}$. This has to happen in such a way that $W^+(\mathcal{P} \cup \mathcal{T})$ is a forest relative to $W^+(\mathcal{T})$ or $W^-(\mathcal{P} \cup \mathcal{T})$ is a forest relative to $W^-(\mathcal{T})$. Lemma 6.1 tells you how to do this. For instance in order to prevent cycles in $W^+(\mathcal{P} \cup \mathcal{T})$ relative to $W^+(\mathcal{T})$, identify
the subgraph coming from relators of $T$ to a vertex in the positive graph and check that adding edges doesn’t build cycles. Those new edges may contain labels which are generators from $T$. Also $P$ might contain more sub-LOT presentations than just $T$. Then Theorem 6.2 implies that $K(P)$ is aspherical.

**Example 6.3** Let $P$ be the presentation defined by the LOT depicted in Figure 2. Let $T$ be the presentation of the sub-LOT consisting of the first two edges between $x_1$ and $x_3$ (colored red). $K(T)$ is aspherical because $T$ is injective. Since $W^-(P \cup T)$ is a forest relative to $W^-(T)$ Theorem 6.2 implies that $K(P)$ is an aspherical LOT.

![Figure 2: An aspherical LOT with sub-LOT.](image)

Building an example out of an aspherical sub-LOT can even be done in a 2-step (or more steps) process such that the edges you add lead to cycles in the negative and the positive graph:

**Example 6.4** Let $T = \langle X \mid R \rangle$ be an arbitrary compressed aspherical LOT presentation. Assume $x_i, x_j, x_p, x_q \in X$ (not necessarily pairwise distinct). Let $P_1 = \langle X, a, b \mid R, ab = b x_i, ba = a x_j \rangle$. Since $W^-(P_1 \cup T)$ is a forest relative to $W^-(T)$ Theorem 6.2 implies that $K(P_1)$ is an aspherical LOT-complex. Now let

$$P_2 = \langle X, a, b, c, d \mid R, ab = b x_i, ba = a x_j, x_p c = cd, x_q d = dc \rangle$$

![Figure 3: An aspherical LOT build in two steps.](image)

(see Figure 3). Since $W^+(P_2 \cup P_1)$ is a forest relative to $W^+(P_1)$, Theorem 6.2 implies that $K(P_2)$ is an aspherical LOT-complex.
In this example edges where added to $T$ which have cycles in both, negative and positive graph but still asphericity may be shown by using Theorem 6.2 twice.

**Example 6.5** Let $T = \langle X \mid R \rangle$ be an arbitrary compressed aspherical LOT presentation. Assume $x_i, x_j \in X$ (not necessarily distinct). Let

$$P = \langle X, u, v, w, y \mid R, wy = yu, x_iw = wu, vw = wy, vu = ux_j \rangle$$

(see Figure 4). Since $W^+(P \cup T)$ is a forest relative to $W^+(T)$ Theorem 6.2 implies that $K(P)$ is an aspherical LOT-complex.

![Figure 4: An aspherical LOT.](image)

Let $H$ be the group defined by $T$ and let

$$\hat{P} = \langle H, u, v, w, y \mid wy = yu, x_iw = wu, vw = wy, vu = ux_j \rangle$$

be a relative presentation in the sense of Bogley and Pride (see [1]). In their paper they define a weight test. $\hat{P}$ does not satisfy this weight test if $x_i^3 = x_j^3$ (which is of course satisfied for $x_i = x_j$). This can be seen by drawing the Whitehead graph in the sense of Bogley and Pride and using the simplex method to show that the weight test does not apply.

**Example 6.6** Let $T = \langle X \mid R \rangle$ be an arbitrary compressed aspherical LOT presentation. Assume $x_i, x_j \in X$ (not necessarily distinct). Let

$$P = \langle X, u, v, w, y, \mid R, uv = vw, x_iy = yw, vw = wy, vu = ux_j \rangle$$

(see Figure 5).

![Figure 5: An aspherical LOT.](image)

$P$ does not satisfy the conditions of Theorem 6.2 since cycles occur in $W^+(P \cup T)$ relative to $W^+(T)$ and $W^-(P \cup T)$ relative to $W^-(T)$. There
is also no way to build up $P$ in two steps as in Example 6.4. On the other hand it can easily be seen that $P$ satisfies the weight test of Bogley and Pride (see [1]). Give all edges which occur twice in the Whitehead-graph weight 1 and all other edges weight 0. $P$ satisfies the relative weight test of Theorem 5.6 since it is injective relative to $T = \langle X \mid R \rangle$. See page 11, after Lemma 5 in [6] for the definition of “injective relative to”.

7 The Asphericity of Injective LOTs

In [6] the authors showed the following result.

Theorem 7.1 Injective LOTs are aspherical.

Here we reprove this theorem using the relative weight test. The new proof is simpler and now perfectly aligns with the proof of a weaker version of Theorem 7.1 given by Huck and Rosebrock [9] already in 2001:

Theorem 7.2 A compressed and injective prime LOT is DR.

*Prime* means that the LOT does not contain proper sub-LOTs. Actually Huck and Rosebrock proved a stronger result which allowed boundary reducible sub-LOTs in the LOT under consideration.

Below $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{T}$ will be LOTs with LOT presentations $P, Q$ and $T$, respectively.

**Proof of Theorem 7.1** We proceed by induction on the number of vertices of an injective LOT $\mathcal{P}$. If $\mathcal{P}$ consists of a single vertex, then $K(\mathcal{P})$ is aspherical. We may assume that $\mathcal{P}$ is reduced, because the process of reducing a LOT does not change the homotopy type of the corresponding LOT-complex. So we assume from now on that $\mathcal{P}$ is a reduced injective LOT that is not prime (otherwise we are done by Theorem 7.2). Let $\mathcal{T} = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$ be the set of maximal proper sub-LOTs of $\mathcal{P}$. Each $\mathcal{T}_i$ has fewer vertices than $\mathcal{P}$ and hence is aspherical by induction hypothesis.

**Case 1.** Suppose that for some $i \neq j$ we have $\mathcal{T}_i \cap \mathcal{T}_j \neq \emptyset$.

This is the easy case. One can show that $\pi_1(K(\mathcal{P}))$ is an amalgamated product of two maximal sub-LOT groups and asphericity follows from a result of Whitehead. See [6] for details.

**Case 2.** The $\mathcal{T}_i$, $i = 1, \ldots, n$, are pairwise disjoint.

This is the interesting case where we will make use of the relative weight test. Recall that a labelled oriented graph $\mathcal{Q}$ is a *reorientation* of a labelled
oriented graph $\mathcal{P}$ if $\mathcal{Q}$ is obtained from $\mathcal{P}$ by changing the orientation of some edges. We will follow the proof of Theorem 7.2 given in [9]. This proof was based on two observations.

1. If $\mathcal{P}$ is a reduced injective prime LOT, then there is a reorientation $\mathcal{Q}$ of $\mathcal{P}$ such that $W^+(\mathcal{Q})$ and $W^-(\mathcal{Q})$ are trees. In particular $\mathcal{Q}$, the LOT-presentation for $\mathcal{Q}$, satisfies the weight test by assigning weight 0 to all corners in $W^+(\mathcal{Q})$ and $W^-(\mathcal{Q})$, and weight 1 to all other corners. A relative version of this observation is Corollary 1 in [6].

1. If $\mathcal{P}$ be a reduced injective labeled oriented tree then there is a reorientation $\mathcal{Q}$ of $\mathcal{P}$, where only certain edges of $\mathcal{P} - T$ are reoriented, so that $W^+(\mathcal{Q} \cup T)$ and $W^-(\mathcal{Q} \cup T)$ are trees relative to $W^+(T)$ and $W^-(T)$, respectively.

In particular $\mathcal{Q}$ satisfies the weight test relative to $T$ by assigning weight 0 to corners in $W^+(Q - T)$ and $W^-(Q - T)$, and weight 1 to all other corners in $W(Q - T)$.

The second main observation in the proof of Theorem 7.2 is

2. If $\mathcal{Q}$ is a reorientation of a reduced injective LOT $\mathcal{P}$ and $\mathcal{Q}$ satisfies the weight test, then $\mathcal{P}$ satisfies the weight test.

This is not difficult to see. Suppose we can return to $\mathcal{P}$ from $\mathcal{Q}$ by reversing an edge in $\mathcal{Q}$ with label $x$. Let $\mathcal{Q}^*$ be the presentation obtained from $\mathcal{Q}$ by replacing $x$ with $x^{-1}$ in every relator of $\mathcal{Q}$. Since $\mathcal{Q}$ satisfies the weight test, $\mathcal{Q}^*$ does so as well because we have an isomorphism of Whitehead graphs $W(\mathcal{Q}) \rightarrow W(\mathcal{Q}^*)$ that sends $x^\epsilon$ to $x^{-\epsilon}$, $\epsilon \in \{+,-\}$. Note that $\mathcal{Q}^*$ is not $\mathcal{P}$, because in relations of $\mathcal{Q}$ where $x$ occurs only once, replacing $x$ by $x^{-1}$ does not give a LOT relation. For example if we have the relator $xz(zy)^{-1}$, coming from an edge in $\mathcal{Q}$ from $x$ to $y$ and labeled with $z$, we get the relator $x^{-1}z(zy)^{-1}$. See Figure 6.

![Figure 6: $W(P) = W(Q^*)$](image-url)

In order to obtain a relator in $\mathcal{P}$ we need to change the orientation of $x$ in $x^{-1}z(zy)^{-1}$. Note that doing so has no effect on the Whitehead graph, so
$W(Q^*) = W(P)$. Thus $P$ satisfies the Weight test. This argument works without change in the relative setting and we obtain:

2. If $Q$ is a reorientation of a reduced injective LOT $P$ and $Q$ satisfies the weight test relative to $T$, then $P$ satisfies the weight test relative to $T$.

This completes the proof. Starting with $P$, we reorient to $Q$ so that $Q$ satisfies the weight test relative to $T$. But then $P$ satisfies the weight test relative to $T$. It follows that $K(P)$ is DR relative to $K(T)$ by Theorem 5.6. Since $K(T) = K(T_1) \lor \ldots \lor K(T_n)$, and each $K(T_i)$ is aspherical by induction, $K(P)$ itself is aspherical. □

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