THE COLORED HOMFLY POLYNOMIAL IS $q$-HOLONOMIC

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Abstract. We prove that the colored HOMFLY polynomial of a link, colored by symmetric or exterior powers of the fundamental representation, is $q$-holonomic with respect to the color parameters. As a result, we obtain the existence of an $(a, q)$ super-polynomial of all knots in 3-space. Our result has implications on the quantization of the $SL(2, \mathbb{C})$ character variety of knots using ideal triangulations or the topological recursion, and motivates questions on the web approach to representation theory.

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1. Introduction

1.1. The colored Jones polynomial. In [GL05] it was shown that the colored Jones function $J_{L,\lambda}(q)$ of an oriented link $L$ in 3-space is $q$-holonomic with respect to the color parameters $\lambda$ of a fixed simple Lie algebra $\mathfrak{g}$. In particular, if we fix a knot $K$ and a dominant weight $\lambda$ of the simple Lie algebra $\mathfrak{sl}_N$, then the sequence $J^{\mathfrak{sl}_N}_{K,n\lambda}(q) \in \mathbb{Z}[q^{\pm 1/2}]$ is $q$-holonomic with respect to $n$. In other words, the sequence $(J^{\mathfrak{sl}_N}_{K,n\lambda}(q))_{n \in \mathbb{N}}$ satisfies a linear recursion with coefficients polynomials in $q$ and $q^n$. When $N = 2$, the minimal order content-free recursion relation is the non-commutative $A$-polynomial of $K$ (see [GL05, Gar04]) which plays a key

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role to several conjectures in Quantum Topology. For a detailed discussion, see [Gar] and references therein.

The minimal order content-free recursion for the sequence \((J_{K,n\lambda}^{sl_N}(q))_{n \in \mathbb{N}}\) exists for every \(N \geq 2\). There are two natural questions.

- How do the coefficients of these recursions depend on \(N\)?
- What is the specialization of these recursions to \(q = 1\)?

The goal of this paper is to answer the first question and pose a natural conjecture regarding the second question. More precisely, we prove that the colored HOMFLY polynomial of a link, colored by symmetric powers of the fundamental representation of \(sl_N\), is \(q\)-holonomic with respect to the color parameters, see Theorem 1.1 below. In particular, for a knot there is a single recursion for the colored HOMFLY polynomial colored by the \(n\)-th symmetric power, such that the coefficients of the recursion are polynomials in \(q\) and \(a\). Moreover, specializing \(a = q^N\) for fixed \(N\) gives a recursion for \((J_{K,n\lambda}^{sl_N}(q))\) and further specializing the coefficients of the recursion to \(q = 1\) gives a 2-variable polynomial which is independent of \(N\). As a result, we obtain a rigorously-defined \((a, q)\)-variable deformation of a two-variable polynomial of a knot, and conjecture that the latter is the \(A\)-polynomial of a knot, see Conjecture 1.5 below.

Our result has implications on the quantization of the \(\text{SL}(2, \mathbb{C})\) character variety of knots using \(\text{ideal triangulations}\) (see [Dim, DG]) or the \(\text{topological recursion}\) (see [BE]), and motivates questions on the \(\text{web}\) i.e., skein-theory approach to representation theory (see [CKM, GM]). We plan to discuss these implications in subsequent publications.

1.2. Super-polynomials in mathematical physics. Mathematical physics has formulated several interesting questions and conjectures regarding the structure of the colored HOMFLY polynomial of a knot. For instance, the LMOV Conjecture of Labastida-Marino-Ooguri-Vafa is an integrality statement for the coefficients of the colored HOMFLY polynomial, [LMV00]. To formulate enumerative integrality conjectures concerning counting of BPS states, physicists often use the term \emph{super-polynomial} in various contexts.

A super-polynomial (an element of \(\mathbb{Z}[a^{\pm 1}, q^{\pm 1}, t^{\pm 1}]\)) that specializes to the HOMFLY polynomial of a knot when \(t = -1\) and its \(sl_N\)-Khovanov-Rozansky homology when \(a = q^N\) was conjectured to exist in [DGR06, Conj.1.2], motivated by the prior work of [GSV05].

Another super-polynomial (an element of \(\mathbb{Z}[q^{\pm 1}, a^{\pm 1}, t^{\pm 1}][M\langle L \rangle]\), where \(LM = qML\) that specializes to a recursion for the sequence \((J_{K,n\lambda}^{sl_N}(q))_{n \in \mathbb{N}}\) when \(t = -1\) and \(a = q^N\) was conjectured to exist in [FGSa, FGSb, GSb]. A similar conjecture (without the \(t\)-variable) was also studied in some cases by [IMMMb, IMMMa, NRS].

Physics reveals that a super-polynomial is an exciting structure which ties together perturbative and non-perturbative aspects of quantum knot theory. For an up-to-date review to this wonderful subject, see the survey [GSa].

The existence of a super-polynomial has been speculated, but so far not proven. Theorem 1.1 settles the existence of the super-polynomial of a link colored by the symmetric powers of the fundamental representation of \(sl_N\).

In a separate publication we will discuss the \(q\)-holonomicity of the colored HOMFLY polynomial with respect to an arbitrary representation of \(sl_N\).
1.3. The colored HOMFLY polynomial. The HOMFLY polynomial $X_L \in \mathbb{Q}(a, q)$ of a framed oriented link in $S^3$ is uniquely characterized by the skein axioms

\[
X_{\bigcirc} - X_{\bigotimes} = (q - q^{-1}) X_{\bigotimes} \quad X_{\bigstar} = aX_{\bigotimes} \quad X_{\text{unknot}} = \frac{a - a^{-1}}{q - q^{-1}}.
\]

The HOMFLY polynomial has an extension to the colored HOMFLY polynomial $X_{L, \lambda}(a, q) \in \mathbb{Q}(a, q)$, defined for framed oriented links $L$ in $S^3$ with $r$ ordered components colored by an $r$-tuple of partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$. Roughly, the colored HOMFLY polynomial is a sum of the HOMFLY of parallels of the link, dictated in a universal way by the color. For a precise and detailed definition, see [ML03, MM08].

1.4. Recursions and the $q$-Weyl algebra. Recall the notion of a $q$-holonomic sequence introduced by Zeilberger [Zei90]. We say that a sequence $f_n(q) \in E(q)$ for $n \in \mathbb{N}$ is $q$-holonomic (where $E$ is a field of characteristic zero) if there exist $d \in \mathbb{N}$ and $a_j(u, v) \in E[u, v]$ such that for all $n \in \mathbb{N}$ we have:

\[
\sum_{j=0}^{d} a_j(q, q^n)f_{n+j}(q) = 0
\]

We can write the above recursion in operator form

\[
P f = 0, \quad P = \sum_{j=0}^{d} a_j(q, M)L^j
\]

where the operators $M$ and $L$ act on a sequence $(f_n(q))_{n \in \mathbb{N}}$ by

\[
(Mf_n)(q) = q^n f_n(q), \quad (Lf_n)(a, q) = f_{n+1}(q)
\]

The operators $M$ and $L$ generate the $q$-Weyl algebra $\mathbb{W} = E(q)[M]/\langle L \rangle$ where $LM = qML$. The set $\{P \in \mathbb{W} | Pf = 0\}$ is a left ideal, non-zero if and only if $f$ is $q$-holonomic. Although $\mathbb{W}$ is not a principal ideal domain, it was observed in [Gar04] that its localization $E(q, M)/\langle L \rangle$ is a principal ideal domain [Gar04]. If we choose a generator of $\{P \in \mathbb{W} | Pf = 0\}$, we can lift it to a unique content-free element $P_f$ of $\mathbb{W}$. Below, we will consider two versions

\[
\mathbb{W} = \mathbb{Z}[a, q, M]/\langle L \rangle, \quad \mathbb{W} = \mathbb{Z}[q, M]/\langle L \rangle
\]

of the $q$-Weyl algebra. There is an obvious commutative diagram

\[
\begin{array}{ccc}
\mathbb{W} & \overset{a=q^N}{\longrightarrow} & \mathbb{W} \\
\downarrow_{a=1, q=1} & & \downarrow_{q=1} \\
\mathbb{Z}[M, L] & & \\
\end{array}
\]

Finally, we point out the existence of a multivariable generalization of $q$-holonomic sequences $f : \mathbb{N} \longrightarrow E(q)$, see [Sab93] and also [GL05].
1.5. Our results.

**Theorem 1.1.** Fix a framed oriented link $L$ with $r$ ordered components. Then,

$$(n_1, \ldots, n_r) \mapsto X_{L,(n_1),\ldots,(n_r)}(a, q) \quad \text{and} \quad (n_1, \ldots, n_r) \mapsto X_{L,(1^{n_1}),\ldots,(1^{n_r})}(a, q)$$

and are $q$-holonomic functions.

**Remark 1.2.** There is an involution $\lambda \mapsto \lambda^T$ where $\lambda^T$ is the transpose of $\lambda$, obtained by interchanging columns and rows. For instance, if $\lambda = (4, 2, 1)$ then $\lambda^T = (3, 2, 1, 1)$. It is a consequence of rank-level duality (see [LMV00, Eqn.4.41]) that $X_{K,\lambda}(a, q) = (-1)^{|\lambda|}X_{K,\lambda^T}(a, q^{-1})$, where $|\lambda|$ is the number of boxes of $\lambda$. Since $(n)^T = (1^n)$, it suffices to show that $X_{L,(1^{n_1}),\ldots,(1^{n_r})}$ is $q$-holonomic with respect to $(n_1, \ldots, n_r)$.

**Remark 1.3.** Theorem 1.1 was previously known for torus links in [BMS11], and for finitely many twist knots (that include the $3_1, 4_1, 5_2$ and $6_1$ knots) in [NRS].

**Theorem 1.4.** (a) For every knot $K$, there exists a unique content-free minimal order recursion relation $A_K(a, q, M, L) \in \mathbb{W}$ of $X_{K,(n)}(a, q)$.
(b) For every fixed $N \in \mathbb{N}$, $A_K(q^N, q, M, L) \in \mathbb{W}$ is a recursion of the sequence $(J_{K,n,\lambda_1}^{sl}(q))$ with respect to $n$.
(c) For every fixed $N \in \mathbb{N}$, the specialization

$$A_K(q^N, q, M, L)|_{q=1} = A_K(1, 1, M, L)$$

is independent of $N$.

For a fixed natural number $N$, it might be the case that the recursion $A_K(q^N, q, M, L) \in \mathbb{W}$ of the sequence $(J_{K,n,\lambda_1}^{sl}(q))$ is not of minimal order. If for $N = 2$ the above recursion is of minimal order, then $A_K(q^2, q, M, L) \in \mathbb{W}$ coincides with the non-commutative $A$-polynomial of $K$. In that case, the AJ Conjecture (see [Gar04] and also [Gel02]) combined with Theorem 1.4 imply the following conjecture: relating the super-polynomial $A_K(a, q, M, L)$ to the $A$-polynomial $A_K(M, L)$ of $K$, introduced and studied in [CGG+94].

**Conjecture 1.5.** For every knot $K$, we have:

$$A_K(1, 1, M, L) = A_K(M, L)b_K(M) \in \mathbb{Z}[M, L]$$

where $A_K(M, L)$ is the $A$-polynomial of $K$ and $b_K(M) \in \mathbb{Z}[M]$.

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## 2. MOY graphs, spiders and webs

2.1. MOY graphs. In [MOY98], Murakami-Ohhtsuki-Yamada introduced the notion of a MOY graph which is an enhancement of the HOMFLY skein theory of links colored by exterior powers of the fundamental representation. MOY graphs are similar to the spiders of Kuperberg [Kup96] and the webs of Morrison et al [Mor, CKM].

A **MOY graph** is a trivalent planar graph $G$ with oriented edges, possibly multiple edges and loops with no sinks nor sources. Locally, a MOY graph is made out of forks and fuses,
shown in Figure 1, using the terminology of [Mor, CKM]. A coloring $\gamma$ of a MOY graph $G$ is an assignment of a natural number to each edge of the graph such that at each fork the flow condition is satisfied as shown in Figure 1. Here, the color $i \in \mathbb{N}$ of an edge corresponds to the $i$-th exterior power of the fundamental representation.

![Figure 1](image.png)

**Figure 1.** Forks on the left and fuses on the right with a coloring.

**Remark 2.1.** MOY graphs visually resemble oriented train tracks. The latter were introduced by Thurston [Thu, Chpt.8] and further studied by Penner-Harer [PH92]. We will not make further use of this observation here.

For $N \in \mathbb{N}$, [MOY98, p.328] define the evaluation $\langle G, \gamma \rangle_N \in \mathbb{N}[q^{\pm 1}]$ of a colored MOY graph. In the formulas below, our $q$ equals to $q^2$ in [MOY98].

**Lemma 2.2.** (a) There is a 1-1 correspondence between the set of colorings of a MOY graph $G$ by integers and $\mathbb{Z}^r$, where $r$ is the number of bounded regions of $G$.
(b) Given $(G, \gamma)$ there exists $\langle G, \gamma \rangle \in \mathbb{Z}(a, q)$ such that for every $N \in \mathbb{N}$ we have

$\langle G, \gamma \rangle|_{a=q^N} = \langle G, \gamma \rangle_N$

**Proof.** (a) follows from Kirchhoff’s theorem. Fix a connected graph $G$ with oriented edges, not necessarily planar or trivalent, possibly with loops and multiple edges. Then, we have a chain complex

$$C : \quad 0 \to C_1 \xrightarrow{\partial} C_0 \to 0$$

where $C_1$ and $C_0$ is the free abelian group on the set $E$ of edges and $V$ vertices of $G$, and $\partial(e) = v_1 - v_0$ if $e$ is an oriented edge with head $v_1$ and tail $v_0$. An assignment $\gamma : E \to \mathbb{Z}$ gives rise to $[\gamma] = \sum_{e \in E} \gamma(e)e$ and $\partial([\gamma]) = 0$ if and only if $\gamma$ is a coloring. It follows that the set of colorings by integers is $H_1(C, \mathbb{Z})$. Moreover, $H_0(C, \mathbb{Z}) = \mathbb{Z}$ since $G$ is connected. Taking Euler characteristic, it follows that the rank $r$ of $H_1(C, \mathbb{Z})$ is given by $-|E| + |V| - 1$, where $|X|$ is the number of elements of $X$. If $G$ is planar, $-|E| + |V| - 1 = |F| - 1$ where $F$ is the set of regions of $G$. Thus, $r$ is the set of bounded regions.

(b) follows from the definition of the evaluation of a MOY graph, and the identity

$$\sum_{i=-N+1}^{N-1} q^{2ji+k} = q^k q^{Nj} - q^{-Nj} = q^k a^j - a^{-j} |_{a=q^N}$$
In [MOY98, p.341], the authors give the following replacement rule of a crossing by a \(\mathbb{Z}[q^{\pm 1}]\) linear combination of local MOY graphs. For a positive crossing we have

\[
\left\langle \begin{array}{c} i \\ j \end{array} \right\rangle = \sum_{k=0}^{j} (-1)^{k+(j+1)i} q^{i-k} \left\langle \begin{array}{c} j+k-i \\ j-k \end{array} \right\rangle \quad \text{for} \quad i \leq j
\]

and for a negative crossing, replace \(q\) by \(q^{-1}\). It follows that if \(\beta\) is a braid word in the braid group of a fixed number of strings and \(L\) is the corresponding oriented link obtained by the closure of \(\beta\), with components colored by \(n_i\), then the colored HOMFLY polynomial \(X_{L,(1^{n_1}),(1^{n_2})...}\) equals to a linear combination the evaluations of the corresponding MOY graphs.

Observe that

- the coefficients of the replacement rule are \(q\)-proper hypergeometric functions in all color variables. In fact, they are \(q\)-holonomic monomials in all color variables,
- the replacement rule constructs a finite set of MOY graphs that depend on \(\beta\) and not on the colorings of \(L\). The colorings of these graphs are linear forms of the colorings of \(L\).

Since the class of \(q\)-holonomic functions is closed under summation and multiplication, Theorem 1.1 follows from the following theorem and Remark 1.2.

**Theorem 2.3.** For every MOY graph \(G\), \(\gamma \mapsto \langle G, \gamma \rangle\) is \(q\)-holonomic.

Theorem 2.3 follows from the existence of any \(q\)-holonomic evaluation algorithm. For a detailed discussion of algorithms in quantum topology and planar algebras see [BPMS12, Sec.4]. Not all of those algorithms decrease the complexity. Some do, some keep the complexity fixed for a while, and some (like the jelly-fish algorithm of [BPMS12]) initially increase the complexity.

An algorithm for evaluating MOY graphs is described in [JK12]. A detailed discussion of the Jeong-Kim algorithm, along with the fixing of some intermediate steps will be described in a forthcoming publication [GM]. Our description of the Jeong-Kim algorithm is implicit in the recent work of [CKM] and uses only the relations (2.3)-(2.8) of [CKM].

For the next lemma, we will say that an evaluation algorithm of MOY graphs is \(q\)-holonomic if

(a) the algorithm uses linear relations whose coefficients are \(q\)-holonomic functions of \(a = q^{N}\) and \(q\), and
(b) replaces \((G, \gamma)\) by graphs \((G', \gamma')\) where \(G'\) is in a finite set (determined by \(G\)) and \(\gamma'\) are linear forms on \(\gamma\).
Like the proof of Theorem 1.1, Theorem 2.3 is automatically implied by the following.

**Lemma 2.4.** The evaluation algorithm of [JK12] is $q$-holonomic.

**Proof.** $q$-holonomicity of the evaluation algorithm follows from the fact that the coefficients in relations (2.3)-(2.8) of [CKM] are $q$-proper hypergeometric (in fact, products of $q$-binomials) in linear forms of the color variables. □

### 3. Computations

There are several papers in the physics literature that discuss the colored HOMFLY polynomial of a knot, see for example [IMMMb, IMMMa, NRS, FGSa, FGSb]. Although the colored HOMFLY polynomial is a well-defined object, the formulas for the colored HOMFLY polynomial presented in the above papers are often void of rigor, although their specializations match rigorous computations of the colored Jones polynomial, and Khovanov Homology. If a formula is written as a multi-dimensional sum of a $q$-proper hypergeometric summand, then the corresponding function is $q$-holonomic, and a rigorous computation of a recursion is possible (see [WZ92]) using computer-implemented methods [PWZ96]. This was precisely done in [NRS], using some formulas that ought to give the colored HOMFLY polynomial of twist knots.

Rigorous formulas for the colored HOMFLY polynomial of a knot are at least as hard as formulas for the $\mathfrak{sl}_2$-colored Jones polynomial and the latter are already difficult. For the 1-parameter family of twist knots, an iterated 2-dimensional sum for the $\mathfrak{sl}_2$-colored Jones polynomial was obtained by Habiro [Hab02, Mas03]. A HOMFLY extension of Habiro’s formulas appears in [Kaw]. Using Kawagoe’s formulas for the twist knots $3_1$, $4_1$, $5_2$ and $6_1$ and applying the creative telescoping method of Zeilberger, one can obtain a rigorous computation of the super-polynomial $A_K(a, q, M, L) \in \overline{W}$ for the twist knots $3_1$, $4_1$, $5_2$ and $6_1$. The results appear in [NRS].

In addition, a matrix model formulation of the colored HOMFLY polynomial of torus knots, combined with a topological recursion, provides a rigorous proof of Theorem 1.1 for all torus links, see [BMS11].

### References

[BDE] Gaétan Borot and Bertrand Eynard, All-order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials, arXiv:1205.2261, Preprint 2012.

[BMS11] Andrea Brini, Marcos Mariño, and Sébastien Stevan, The uses of the refined matrix model recursion, J. Math. Phys. 52 (2011), no. 5, 052305, 24.

[BPMS12] Stephen Bigelow, Emily Peters, Scott Morrison, and Noah Snyder, Constructing the extended Haagerup planar algebra, Acta Math. 209 (2012), no. 1, 29–82.

[CCG+94] Daryl Cooper, Marc Culler, Henry Gillet, Darren D. Long, and Peter B. Shalen, Plane curves associated to character varieties of 3-manifolds, Invent. Math. 118 (1994), no. 1, 47–84.

[CKM] Sabin Cautis, Joel Kamnitzer, and Scott Morrison, Webs and quantum skew Howe duality, arXiv: 1210.6437, Preprint 2012.

[DG] Tudor Dimofte and Stavros Garoufalidis, The quantum content of the gluing equations, arXiv: 1202.6268, Preprint 2012.

[DGR06] Nathan M. Dunfield, Sergei Gukov, and Jacob Rasmussen, The superpolynomial for knot homologies, Experiment. Math. 15 (2006), no. 2, 129–159.
[Dim] Tudor Dimofte, *Quantum Riemann surfaces in Chern-Simons theory*, arXiv:1102.4847, Preprint 2011.

[FGSa] Hiroyuki Fuji, Sergei Gukov, and Piotr Sułkowski, *Super-A-polynomial for knots and BPS states*, arXiv:1205.1515, Preprint 2012.

[FGSb] ———, *Volume conjecture: Refined and categorified*, arXiv:1203.2182, Preprint 2012.

[Gar] Stavros Garoufalidis, *Quantum knot invariants*, arXiv:1201.3314, Mathematische Arbeitstagung 2012.

[Gar04] Stavros Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 291–309 (electronic).

[Gel02] Răzvan Gelca, *On the relation between the A-polynomial and the Jones polynomial*, Proc. Amer. Math. Soc. 130 (2002), no. 4, 1235–1241 (electronic).

[GL05] Stavros Garoufalidis and Thang T. Q. Lê, *The colored Jones function is q-holonomic*, Geom. Topol. 9 (2005), 1253–1293 (electronic).

[GM] Stavros Garoufalidis and Scott Morrison, *In preparation*.

[GSa] Sergei Gukov and Ingmar Saberi, *Lectures on knot homology and quantum curves*, arXiv:1211.6075, Preprint 2012.

[GSb] Sergei Gukov and Marko Stosic, *Homological algebra of knots and BPS states*, arXiv:1112.0030, Preprint 2011.

[GSV05] Sergei Gukov, Albert Schwarz, and Cumrun Vafa, *Khovanov-Rozansky homology and topological strings*, Lett. Math. Phys. 74 (2005), no. 1, 53–74.

[Hab02] Kazuo Habiro, *On the quantum sl$_2$ invariants of knots and integral homology spheres*, Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr., vol. 4, Geom. Topol. Publ., Coventry, 2002, pp. 55–68 (electronic).

[IMMMa] H. Itoyama, A. Mironov, A. Morozov, and An. Morozov, *Character expansion for HOMFLY polynomials. III. all 3-strand braids in the first symmetric representation*, arXiv:1204.4785, Preprint 2012.

[IMMMb] ———, *HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations*, arXiv:1203.5978, Preprint 2012.

[JK12] Myeong-Ju Jeong and Dongseok Kim, *The quantum sl$(n,\mathbb{C})$ representation theory and its applications*, 2012, arXiv:math/0506403, Preprint 2012, pp. 993–1015.

[Kaw] Kenichi Kawagoe, *On the formulae for the colored homfly polynomials*, arXiv:1210.7574, Preprint 2012.

[Kup96] Greg Kuperberg, *Spiders for rank 2 Lie algebras*, Comm. Math. Phys. 180 (1996), no. 1, 109–151.

[LMV00] José M. F. Labastida, Marcos Mariño, and Cumrun Vafa, *Knots, links and branes at large N*, J. High Energy Phys. (2000), no. 11, Paper 7, 42.

[Mas03] Gregor Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebr. Geom. Topol. 3 (2003), 537–556 (electronic).

[ML03] Hugh R. Morton and Sascha G. Lukac, *The Homfly polynomial of the decorated Hopf link*, J. Knot Theory Ramifications 12 (2003), no. 3, 395–416.

[MM08] H. R. Morton and P. M. G. Manchón, *Geometrical relations and plethysms in the Homfly skein of the annulus*, J. Lond. Math. Soc. (2) 78 (2008), no. 2, 305–328.

[Mo] Scott Morrison, *A diagrammatic category for the representation theory of $U_q(sl_n)$*, arXiv:0704.1503, Thesis 2007.

[MOY98] Hitoshi Murakami, Tomotada Ohtsuki, and Shuji Yamada, *Homfly polynomial via an invariant of colored plane graphs*, Enseign. Math. (2) 44 (1998), no. 3-4, 325–360.

[NRS] Satoshi Nawata, P. Zodinmawia Ramadevi, and Xinyu Sun, *Super-A-polynomials for twist knots*, arXiv:1209.1409, Preprint 2012.

[PH92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992.
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[PWZ96] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, \( A = B \), A K Peters Ltd., Wellesley, MA, 1996, With a foreword by Donald E. Knuth, With a separately available computer disk.

[Sab93] Claude Sabbah, *Systèmes holonomes d’équations aux \( q \)-différences*, D-modules and microlocal geometry (Lisbon, 1990), de Gruyter, Berlin, 1993, pp. 125–147.

[Thu] William P. Thurston, *The geometry and topology of three-manifolds*, Lecture notes from Princeton University 1978-1980, available from http://www.msri.org/publications/books/gt3m.

[WZ92] Herbert S. Wilf and Doron Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and “\( q \)” multiset/integral identities*, Invent. Math. **108** (1992), no. 3, 575–633.

[Zei90] Doron Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), no. 3, 321–368.

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