Quantum Energies of Interfaces

N. Graham,*a R.L. Jaffe,b M. Quandt,b and H. Weigel†c

aDepartment of Physics and Astronomy University of California at Los Angeles, Los Angeles, CA 90095
bCenter for Theoretical Physics Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology Cambridge, Massachusetts 02139
cInstitute for Theoretical Physics Tübingen University Auf der Morgenstelle 14, D–72076 Tübingen, Germany

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Abstract

We present a method for computing the one-loop, renormalized quantum energies of symmetrical interfaces of arbitrary dimension and codimension using elementary scattering data. Internal consistency requires finite-energy sum rules relating phase shifts to bound state energies.

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*e-mail: graham@physics.ucla.edu, jaffe@mit.edu, maqua@mitlns.mit.edu, herbert.weigel@uni-tuebingen.de

†Heisenberg Fellow
Recent work in particle theory has highlighted the importance of a class of time independent extended objects that are symmetric in $m$ “non-trivial” spatial dimensions and independent of the coordinates in the $n$ remaining, “trivial” spatial dimensions. Examples include domain walls in lattice field theories [1,2], branes in string theory and extradimensional gravity [3], and vortices and other phenomena in statistical mechanics. Generically, we refer to these objects as “interfaces.” In a series of earlier works, we developed methods to evaluate the renormalized, one-loop quantum contribution to the energy of solitons ($n = 0$) [4–6]. In this Letter, we extend our method to interfaces. We identify the one-loop quantum energy or equivalently the functional determinant or partition function of such a background with elementary quantities in scattering theory. Our methods yield simple, unambiguous results renormalized in conventional schemes, well suited to numerical computation. Eq. (8), for example, gives the one loop quantum energy for $n = 1$ and $m < 4$ in terms of the partial wave phase shifts, the first and second Born approximations, the energies of bound states and the Feynman two-point function. Extensions to arbitrary $n$ and $m$ do not become more complicated in fundamental ways.

In the course of our analysis we find that the renormalizability of the underlying field theory requires certain identities to hold within the scattering data in $m$ dimensions. These take the form of “finite energy sum rules” that generalize Levinson’s theorem: They relate integrals over the phase shifts, regulated at high momentum by subtracting one or more Born approximations, to the energies of bound states. The required sum rules were first obtained by Puff within scattering theory some time ago [7]. In Ref. [8] we analyze the sum rules in detail. We generalize them to cases where more Born approximations are subtracted than are demanded to regulate the high momentum piece, we treat the symmetric channel in one spatial dimension, which is anomalous, and we discuss how the sum rules can be understood as generalizations of Levinson’s theorem. A related family of subtracted sum rules has been obtained by Buslaev and Faddeev [9]. However, these sum rules mix various orders of the Born approximations and are thus not of particular use for computing quantum energies.

For a static, pointlike object in $m$ dimensions described by a classical background $\phi$, we would compute the “effective energy,” $E_{m}[\phi]$, which is the effective action per unit time. In the present case, the relevant quantity is $\mathcal{E}_{n,m}[\phi] = E_{n,m}[\phi]/L^{n}$, the effective energy per unit volume of the trivial dimensions. $\mathcal{E}_{n,m}$ looks like an interface tension when viewed from the outside and like an induced cosmological constant intrinsically. It can be expressed as an infinite sum over (one particle irreducible) Feynman diagrams where the fluctuating field runs in a loop with all possible insertions of $\phi$, or equivalently as a sum/integral over the shifts in the zero-point energies of the fluctuating field in the background $\phi$. Both of these expressions are formally infinite for cases of interest. We regulate these divergences using dimensional regularization separately in $n$ and $m$, use the tools of quantum mechanics to connect these two pictures, and then renormalize $\mathcal{E}_{n,m}$ unambiguously. We will take the dynamical field $\psi$ to be either a complex boson or Dirac fermion of mass $\mu$, coupled to the classical background $\phi$ by $g\psi^{*}\phi\psi$ or $g\bar{\psi}\phi\psi$. In the case of a self-coupled scalar we employ a source to stabilize the classical background if it is not a solution to the classical equations of motion [3].

Other approaches to this problem exist for the case $m = 1$. Ref. [10] uses properties of one-dimensional functional determinants to integrate over the non-trivial dimensions first.
Ref. [11] uses zeta function regularization and analytic properties of the scattering amplitudes to rotate to the imaginary $k$ axis. Both renormalize by subtracting a local term to cancel divergences. In contrast, we subtract the entire Feynman graph, and then add it back in and renormalize using standard techniques. For the leading subtraction, the tadpole graph, there is no difference, since it is entirely local. But for further subtractions, which become necessary in higher dimensions, local subtractions involve an arbitrary scale, and thus are difficult to relate to definite renormalization schemes (such as on-shell, or $\overline{\text{MS}}$). Refs. [10] and [11] have not been generalized beyond $m=1$, whereas the generalization is straightforward in our case. In the cases where a direct comparison can be made ($m=1$ and renormalization of the tadpole graph only) all three approaches yield superficially distinct expressions, but appear to us actually to be equivalent.

The zero-point energies and phase shifts, which are central to our method, are determined by solving the time independent Klein-Gordon or Dirac equation in the background $V(x) = g\phi(x)$. This background is restricted such that the scattering data and in particular the associated Jost functions can be uniquely determined [8]. The Born approximation is an expansion in $g$. So too is the expansion of the effective energy in terms of the Feynman diagrams, where each insertion carries a factor of the potential $V(\vec{q}) = g \int d^m x e^{-i\vec{q} \cdot \vec{x}} \phi(\vec{x})$.

By identifying orders in $g$ we rewrite the ultra–violet divergent contributions to the quantum energy as Feynman diagrams whose divergences are unambiguously canceled by counterterms [4–6,12].

In the continuum the sum over zero-point energies becomes a sum over bound state energies and an integral over $\omega(k, p) = \sqrt{k^2 + p^2 + \mu^2}$, weighted by the density of states. Here $k$ and $p$ refer to the magnitudes of the momenta in the non-trivial and trivial dimensions, respectively. The density of states factorizes into $(L/2\pi)^n$ times the density of states $\rho_m(k)$ in the $m$ non-trivial dimensions. Note that $\rho_m(k)$ is independent of $p$. We assume enough symmetry in the background $\phi$ that the scattering problem in the non-trivial directions decomposes into a sum over partial waves. Then it is well known that the change in the density of states due to $\phi$ can be written as $\pi \delta \rho_m(k) = \sum_{\ell} D^\ell_m d\delta^\ell_m(k)/dk$, where $D^\ell_m$ is the degeneracy of the $\ell^{th}$ partial wave in $m$ dimensions. By convention, we take $\delta^\ell_m(k)$ to be the sum over both signs of the energy.

As in the pointlike case, $n=0$, it is necessary to soften the infrared ($k=0$ and $p=0$) behavior [4]. The first step is to use Levinson’s theorem in each partial wave,

$$\int_0^{\infty} \frac{dk}{\pi} \frac{d}{dk} \delta^\ell_m(k) + \sum_j 1 = 0.$$  \hspace{1cm} (1)

We multiply it by $\frac{1}{2} \mu(p) = \frac{1}{2} \sqrt{p^2 + \mu^2}$ and subtract it from the formal expression for the Casimir energy [3]. Then we can write the fundamental expression for the effective energy per

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1For $m=1$, the right-hand side of eq. (1) is modified to 1, which will cancel the contributions from the half-bound states at $\omega = \pm \mu$ that exist in the free background in this case [3,13].
unit volume,
\[ \mathcal{E}_{n,m}[\phi] = \pm \int \frac{d^n p}{(2\pi)^n} \sum_{\ell} D_{m}^{\ell} \left[ \int_{0}^{\infty} \frac{dk}{2\pi} (\omega(k, p) - \mu(p)) \frac{d}{dk} \delta_{m}^{\ell}(k) \right. \\
+ \frac{1}{2} \sum_{j} (|\omega_{j,m}^{\ell}(p)| - \mu(p)) \bigg] + \mathcal{C}_{n,m}[\phi] \quad (2) \]

where \( |\omega_{j,m}^{\ell}(p)| = \sqrt{-(\kappa_{j,m}^{\ell})^2 + p^2 + \mu^2} \) are the absolute values of the bound state energies and the \( \kappa_{j,m}^{\ell} \) are the absolute values of their (imaginary) momenta. The overall sign in eq. (2) is for bosons and fermions respectively. \( \mathcal{C}_{n,m} \) represents the contributions of Lagrangian counterterms necessary to cancel infinities and enforce a particular renormalization scheme.

The integrals in eq. (2) diverge in the cases of interest and should be regularized throughout our analysis. We employ dimensional regularization, so we assume both \( n \) and \( m \) are chosen in regimes where the integrals converge. This is the case for \( 0 < m + n < 1 \). Subsequently we analytically continue to physically interesting cases \((n, m \text{ integers})\).

Our procedure is to identify potentially divergent diagrams in the effective action expansion with terms in the Born approximation to the phase shift. We subtract these terms under the \( k \)-integral in eq. (2) and then add back in exactly what we subtracted, this time as Feynman diagrams \( \mathcal{F}_{n,m}[\phi] \), which we combine with the counterterms in the standard way. The renormalized Feynman diagram contributions, \( \mathcal{F}_{n,m}[\phi] = \mathcal{F}_{n,m}[\phi] + \mathcal{C}_{n,m}[\phi] \), are a straightforward piece of our result. The number of subtractions required will depend on how large we want to allow the final space dimension \( m + n \) to get. For \( m + n < 3 \), subtraction of the tadpole graph is sufficient. For \( 3 < m + n < 5 \) the two-point function must be subtracted, and so on. (For fermions, we must include also contributions from higher-order graphs that eventually simplify because the symmetries of the interaction relate them to lower-order graphs.)

We begin this procedure by subtracting the first Born approximation \( \delta_{m}^{(1)\ell}(k) \) from the phase shift \( \delta_{m}(k) \) and adding back in the contribution of the tadpole graph, \( \mathcal{F}_{n,m}^{(1)}[\phi] \). In Ref. [3] we proved the two are equal using dimensional regularization for \( n = 0 \). This proof extends in a straightforward way to the case of \( n > 0 \). Thus the effective energy per unit volume with one subtraction becomes

\[ \mathcal{E}_{n,m}[\phi] = \pm \int \frac{d^n p}{(2\pi)^n} \sum_{\ell} D_{m}^{\ell} \left[ \int_{0}^{\infty} \frac{dk}{2\pi} (\omega(k, p) - \mu(p)) \frac{d}{dk} (\delta_{m}^{\ell}(k) - \delta_{m}^{(1)\ell}(k)) \right. \\
+ \frac{1}{2} \sum_{j} (|\omega_{j,m}^{\ell}(p)| - \mu(p)) \bigg] + \mathcal{F}_{n,m}^{(1)}[\phi]. \quad (3) \]

The two terms in \( \mathcal{E}_{n,m} \) given by eq. (3) should now be separately finite for \( m + n < 3 \). Integrating over \( p \), we obtain

\[ \mathcal{E}_{n,m}[\phi] = \mp \frac{\Gamma\left(\frac{1+n}{2}\right)}{2(4\pi)^{\frac{1+n}{2}}} \sum_{\ell} D_{m}^{\ell} \left[ \int_{0}^{\infty} \frac{dk}{\pi} (\omega^{n+1}(k) - \mu^{n+1}) \frac{d}{dk} (\delta_{m}^{\ell}(k) - \delta_{m}^{(1)\ell}(k)) \right. \\
+ \sum_{j} (|\omega_{j,m}^{\ell}(n+1)| - \mu^{n+1}) \bigg] + \mathcal{F}_{n,m}^{(1)}[\phi] \quad (4) \]
which presents a puzzle: if we take $n \to 1$ (say with $m = 1$), $\mathcal{E}_{1,1}$ appears to diverge because of the pole in the gamma function. The divergence is spurious, so the quantity in brackets must vanish for $n = 1$. Furthermore, since each partial wave is independent, each must vanish separately. **This is guaranteed by the sum rule** [7]

$$
\int_0^\infty \frac{dk}{\pi} k^2 \frac{d}{dk} \left( \delta_m^\ell(k) - \delta_m^{(1)}(k) \right) - \sum_j (\kappa_j^\ell m)^2 = 0.
$$

(5)

With the aid of eq. (5) we can take the $n \to 1$ limit, 

$$
\mathcal{E}_{1,m} = \pm \frac{1}{4\pi} \sum_\ell D^\ell_m \left[ \int_0^\infty \frac{dk}{\pi} k \log \frac{\omega(k)^2}{\mu^2} \left( \delta_m^\ell(k) - \delta_m^{(1)}(k) \right) \right.
$$

$$
- \frac{1}{2} \sum_j (\omega_j^\ell m)^2 \log \left( \frac{\omega_j^\ell}{\mu^2} \right) + (\kappa_j^\ell m)^2 \left] \right] .
$$

(6)

Here we have adopted the renormalization condition that the counterterm exactly cancels the tadpole graph [6]. Finally note that the arbitrary scale of the logarithms cancels because of eqs. (1) and (5).

To extend to higher dimensions, we need to make a second Born subtraction and add back the Feynman two-point function, which will suffice for dimensions with $m + n < 5$. We can continue this procedure indefinitely – subtracting higher Born approximations and adding back the appropriate Feynman diagrams, which are renormalized by local counterterms. Questions of renormalizability enter only if we include ab initio the vertices associated with these new counterterms.

To avoid new infrared problems for $m = 1$, we perform an additional subtraction, by subtracting eq. (5) divided by $2\mu(p)$ from eq. (3), so that $\omega(k,p) - \mu(p)$ is replaced by $\omega(k,p) - k^2/2\mu(p)$ under the $k$-integral. Next we subtract the second Born approximation, add back the Feynman two-point function, and perform the $p$ integration:

$$
\mathcal{E}_{1,m}[\phi] = \pm \frac{\Gamma(-\frac{1+n}{2})}{2(4\pi)^{\frac{n+1}{2}}} \sum_\ell D^\ell_m \left[ \sum_j \left( (\omega_j^\ell)^{n+1} - \mu^{n+1} + \frac{n+1}{2}(\kappa_j^\ell m)^2 \mu^{n-1} \right) \right.
$$

$$
+ \int_0^\infty \frac{dk}{\pi} \left( (\omega)^{n+1} - \mu^{n+1} - \frac{n+1}{2} k^2 \mu^{n-1} \right) \frac{d}{dk} \left( \delta_m^\ell(k) - \delta_m^{(1)}(k) - \delta_m^{(2)}(k) \right) \right]
$$

$$
+ \mathcal{F}_{1,m}[\phi].
$$

(7)

The coefficient of the gamma function now vanishes as $n \to 1$ by construction. The $n \to 1$ limit then gives

$$
\mathcal{E}_{1,m}[\phi] = \pm \frac{1}{4\pi} \sum_\ell D^\ell_m \left[ \int_0^\infty \frac{dk}{\pi} k \log \frac{\omega(k)^2}{\mu^2} \left( \delta_m^\ell(k) - \delta_m^{(1)}(k) - \delta_m^{(2)}(k) \right) \right.
$$

$$
- \frac{1}{2} \sum_j (\omega_j^\ell)^2 \log \left( \frac{\omega_j^\ell}{\mu^2} \right) + (\kappa_j^\ell m)^2 \left] \right] + \mathcal{F}_{1,m}[\phi].
$$

(8)



\^See ref. [8] for a thorough discussion of the infrared anomalies that occur for $m = 1$. 


Eqs. (6) and (8) are identical for values of $m$ where only one Born subtraction is necessary. The contribution of the second Born approximation has been replaced by the second Feynman diagram. However, eqs. (7) and (8) can be continued to values of $n + m$ where two subtractions are necessary.

The finiteness of eq. (7) as $n \rightarrow 3$ implies another scattering theory identity,

$$\int_0^\infty \frac{dk}{\pi} k^4 \frac{d}{dk} \left( \delta_m^\ell(k) - \delta_m^{(1)\ell}(k) - \delta_m^{(2)\ell}(k) \right) + \sum_j (\kappa_{j,m}^\ell)^4 = 0$$

which again is true channel by channel and can be derived directly from scattering theory [7]. In the limit $n \rightarrow 3$ we then obtain

$$\mathcal{E}_{3,m} = \pm \frac{1}{32\pi^2} \sum_{\ell} D_m^\ell \left[ - \int_0^\infty \frac{dk}{2\pi} 4k\omega(k)^2 \log \frac{\omega(k)^2}{\mu^2} \left( \delta_m^\ell(k) - \delta_m^{(1)\ell}(k) - \delta_m^{(2)\ell}(k) \right) \right. \\
+ \left. \frac{1}{2} \sum_j \left( (\omega_{j,m}^\ell)^4 \log \frac{\omega_{j,m}^\ell}{\mu^2} + \mu^2 (\kappa_{j,m}^\ell)^2 - \frac{1}{2}(\kappa_{j,m}^\ell)^4 \right) \right] + \mathcal{F}^{(2)}_{3,m}[\phi].$$

To illustrate our approach, we consider a dynamical scalar in a background potential $V(x) = -\frac{\ell + 1}{4}\frac{\mu^2}{\ell^2} \text{sech}^2 \frac{\mu x}{\ell}$. For integer $\ell$, this potential is reflectionless and the phase shifts
can be obtained analytically, but we will take general $\ell$ and work numerically. We perform two Born subtractions, and adopt the renormalization conditions that the tadpole graph vanishes, and the two-point function vanishes when the external four-momentum satisfies $q^2 = M^2$. For a background field $\phi$ of mass $M$, this choice represents an on-shell renormalization scheme, which keeps fixed the location of the pole in the two-point function. (We do not perform any wavefunction renormalization.) Our approach allows us to hold this scheme fixed as we vary either the background field or the number of transverse dimensions $n$. Figure 1 shows the quantum energy per unit volume in units of $\mu$ of the configuration for $M = \mu$ and $\ell = 1.5$, as a function of $n$. We have also used these calculations to numerically verify the equivalence of the expressions (6) and (8).

We have described a simple, practical approach to computing quantum energies of interfaces. In the process, we have applied tools from ordinary scattering theory that enabled us to precisely and efficiently apply these techniques to a general class of physical problems.

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