A Perturbative Gluon Condensate?

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Abstract
I propose that the properties of QCD perturbation theory should be investigated when the boundary state (‘perturbative vacuum’) at \( t = \pm \infty \) includes gluons. Any boundary state that has an overlap with the true QCD ground state generates a perturbative series that (when summed to all orders) is formally exact. Through an analogy with the boundary condition corresponding to a fermion condensate, I propose an explicit form for a ‘perturbative gluon condensate’ that suppresses low momentum gluon production, thus generating an effective mass gap. Standard perturbative calculations are modified only through a change in the \( i\varepsilon \) prescription of low momentum (\( |\vec{p}| \lesssim \Lambda_{QCD} \)) gluon propagators. Gauge invariance is expected to be preserved since this modification is equivalent to adding on-shell external particles. Renormalizability is unaffected since only low-momentum propagators are modified. Due to the asymptotic low momentum gluons boost invariance is not explicit. Lorentz invariance should be restored in the sum to all orders in analogy to standard bound state calculations.

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1. Introduction

Hadron wave functions appear phenomenologically to be frame dependent. In the infinite-momentum (or light-cone) frame the proton is observed to have, in addition to its $uud$ valence quarks, important gluon and sea quark components [1]. In particular, gluons carry about half of the proton momentum. This measurement of the proton is rather rigorously justified by the QCD factorization theorem [2]. The success of perturbative QCD predictions for a large number of hard scattering processes has established QCD as the correct theory for the strong interactions.

The non-relativistic quark model (NRQM) [1, 3, 4] provides a less rigorous but phenomenologically very successful and simple rest frame picture of hadrons as non-relativistic bound states of ‘constituent’ quarks. The masses of the $u,d$ constituent quarks are $\mathcal{O}(300 \text{ MeV})$, which is considerably larger than the ‘current’ quark masses which are relevant for short-distance processes. The ‘missing’ gluon and sea quark degrees of freedom appear to be frozen in the structure of the constituent quarks.

The simple regularities of the hadron spectrum, coupled with the success of QCD as applied to hard processes, obviously invites efforts to find a QCD justification of the NRQM (see, eg, [3, 5, 6, 7, 8]). The challenge is to find a formulation which in a first approximation retains the simplicity of the quark model, yet allows corrections specified by QCD to be evaluated to arbitrary order.

The similarities of the hadron spectrum with QED bound states, together with the success of perturbative calculations in that theory, suggests the use of a perturbation expansion also in QCD. Such an expansion is determined by the lagrangian and by the boundary conditions at asymptotic times. Central
properties of QCD like the finite range of the color force and confinement are commonly associated with a non-trivial ground state of the theory, the ‘gluon condensate’ [3,10]. In this paper I shall propose a specific boundary condition on QCD perturbation theory which is motivated by the gluon condensate and by a suppression of soft gluon production. The purpose is not to model the condensate in detail. Just as in QED it may suffice to achieve so much overlap with the true ground state that low orders of the perturbative series already incorporates the main physical features of the theory. Corrections are then given systematically by the higher orders of the expansion.

This work points to a perturbative expansion of QCD which appears not substantially more difficult to evaluate than the standard one, but which has a number of novel features. The usefulness of this approach can only be judged after a further study of the properties of that expansion.

2. A fermion condensate

The asymptotic states that we impose on perturbative expansions at initial and final times \((t = \pm \infty)\) should have an overlap with the true ground state of the theory. This guarantees that the full perturbative expansion formally gives exact results. In euclidean formulations this fact is particularly clear since the time development of energy eigenstates is given by \(\exp(-E\tau)\), implying a dominance of the true ground state (of lowest energy \(E\)) in the limit \(\tau \to \infty\). In minkowski space the same result is obtained using an \(i\varepsilon\) prescription\(^2\).

This freedom in the choice of boundary states allows for a whole set of formally equivalent perturbative expansions. Since all expansions are expected to diverge, their equivalence is of more formal than practical significance.

\(^2\)For a discussion of boundary states in field theory see, eg, Ref. [1].
From a practical point of view what matters is that the lowest orders already incorporate the main physical characteristics of the theory.

We have little understanding of the structure of the QCD gluon condensate in terms of Fock state wave functions. It nevertheless seems plausible that the vacuum wave function components involving gluons and quarks of 3-momenta smaller than the characteristic QCD scale $\Lambda_{QCD}$ are strongly modified. In this respect, the gluon condensate may resemble a fermi condensate with fermi momentum of $O(\Lambda_{QCD})$. In a fermi condensate the exclusion principle prevents pair production below the fermi momentum. It seems desirable to have a similar property for gluons, to suppress soft gluon production which can give rise to long-range color correlations.

In this section I recall how perturbation theory is modified in the presence of a fermion condensate. Only the $i\varepsilon$ prescription is affected – which is enough to have significant consequences. I shall then in the next section use this as a guide for constructing a ‘perturbative gluon condensate’, namely one that results in an analogous modification of the $i\varepsilon$ prescription for the gluon propagator. Having shown that there exists a boundary condition which implies such an $i\varepsilon$ modification for gluons it can for many practical purposes be forgotten, and the usual feynman diagrams be evaluated with modified (low momentum) propagators.

The standard free fermion propagator

$$iS_F(x - y) = \langle 0 | T [ \psi(x) \bar{\psi}(y) ] | 0 \rangle$$

(1)

is in momentum space

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} = \frac{\not{p} + m}{(p^0 - E_p + i\varepsilon)(p^0 + E_p - i\varepsilon)} .$$

(2)
If we add an antifermion to the initial and final states,

\[
\langle 0 | d_\lambda^\dagger (\vec{k}) T [\psi(x) \bar{\psi}(y)] d_\lambda (\vec{k}) | 0 \rangle = i S_F (x - y) 2E_k (2\pi)^3 \delta^3 (\vec{k} - \vec{k}') \delta_{\lambda \lambda'} + v(\lambda', \vec{k}) \bar{v}(\lambda, \vec{k}) e^{ik' \cdot x - ik \cdot y},
\]

the Feynman propagator is multiplied by the annihilation amplitude for the inserted antifermions, and there is a new term corresponding to a mixing of the antifermion propagating from \(x\) to \(y\) with the antifermion in the in- and out-states.

For a condensate we would fill both helicity states at a given momentum \(\vec{k}\). The free propagator

\[
i S(x - y) \equiv \langle 0 | d_{1/2} (\vec{k}) d_{-1/2} (\vec{k}) T [\psi(x) \bar{\psi}(y)] d_{1/2}^\dagger (\vec{k}) d_{-1/2}^\dagger (\vec{k}) | 0 \rangle
\]

is then in momentum space

\[
S(p) = \left[ (2\pi)^3 2E_k \delta^3 (\vec{0}) \right]^2 \begin{cases} S_F(p) & (\vec{p} \neq -\vec{k}) \\ S_E(p) & (\vec{p} = -\vec{k}) \end{cases}
\]

where

\[
S_E(p) = \frac{p + m}{(p^0 - E_k + i\epsilon)(p^0 + E_k + i\epsilon)}
\]

differs from the Feynman propagator only in the \(i\epsilon\) prescription at \(p^0 = -E_k\). Since the antifermions inserted in the definition are on-shell, it is clear that a mixing between them and the propagating fermion only can occur at the antifermion pole of \(S(p)\). Adding antifermions for all momenta \(|\vec{p}| \leq \Lambda\), the corresponding propagator will equal \(S_E(p)\) for all \(|\vec{p}| \leq \Lambda\).

The addition of (anti)fermions at \(t = \pm \infty\) influences the fermion propagators in Feynman diagrams at any order of perturbation theory exactly as it does the lowest order propagator above. This can be easily seen using the...
generating functional of green functions (in a theory like QED or QCD),

\[ Z[\zeta, \bar{\zeta}; J] = \exp \left[ i S_{\text{int}} \left( \frac{\delta}{\delta \zeta}, \frac{\delta}{\delta \bar{\zeta}} \right) \right] Z_B[J] Z_F[\zeta, \bar{\zeta}] \]  

(7)

where \( S_{\text{int}} \) is the interaction part of the action and \( Z_B, Z_F \) are free functionals of the boson (\( J \)) and fermion (\( \zeta, \bar{\zeta} \)) sources, respectively. The free fermion functional is

\[ Z_F[\zeta, \bar{\zeta}] = \exp \left[ i \int \frac{d^4p}{(2\pi)^4} \bar{\zeta}(-p) S_F(p) \zeta(p) \right] \]  

(8)

with the feynman propagator given by Eq. (2).

It is instructive first to rederive the result (5) for the free propagator with the boundary states (4) using the generating functional. The propagator \( S_F(p) \) is diagonalized by the sources \( z, \bar{z} \) of definite helicity (\( \lambda \)) and energy signature (\( \pm \)),

\[ \zeta(p) = \frac{\gamma^0}{\sqrt{2E_p}} \sum_{\lambda} \left[ u(\lambda, \vec{p}) z^\lambda_+(p) + v(\lambda, -\vec{p}) z^\lambda_-(p) \right] \]

\[ \bar{\zeta}(-p) = \frac{1}{\sqrt{2E_p}} \sum_{\lambda} \left[ \bar{z}^\lambda_+(-p) u^\dagger(\lambda, \vec{p}) + \bar{z}^\lambda_-(-p) v^\dagger(\lambda, -\vec{p}) \right] \]  

(9)

In the new basis we have

\[ Z_F[z, \bar{z}] = \exp \left\{ i \int \frac{d^4p}{(2\pi)^4} \sum_{\lambda} \left[ \frac{\bar{z}^\lambda_+(-p) z^\lambda_+(p)}{p^0 - E_p + i\epsilon} + \frac{\bar{z}^\lambda_-(-p) z^\lambda_-(p)}{p^0 + E_p - i\epsilon} \right] \right\} \]  

(10)

The generating functional \( Z_E \) of the modified propagator (3) (for some given \( \vec{k} \)) differs by the sign of \( i\epsilon \) in the second term of Eq. (10). Hence (I suppress the 3-momentum \( \vec{p} \) and factors \( (2\pi)^3 2E \delta^3(0) \) in the following),

\[ Z_E[z, \bar{z}] = \exp \left[ i \int \frac{dp^0}{2\pi} \bar{\zeta}(-p) S_E(p) \zeta(p) \right] \]

\[ = \prod_{\lambda} \left[ 1 + \bar{z}^\lambda_-(E) z^\lambda_-(E) \right] Z_F[z, \bar{z}] \]  

(11)
where I used \((p^0 + E + i\varepsilon)^{-1} - (p^0 + E - i\varepsilon)^{-1} = -2\pi i\delta(p^0 + E)\), and \(\exp(\bar{z}z) = 1 + \bar{z}z\) for grassmann sources \(\bar{z}, z\).

As a function of time,

\[
\begin{align*}
    z(p^0) &= \int dt' z(t')e^{i\nu t^0} \\
    \bar{z}(-p^0) &= \int dt'' \bar{z}(t'')e^{-i\nu t^0}
\end{align*}
\]

(12)

the free generating functionals are

\[
\begin{align*}
    Z_F &= \exp\left\{ dt' dt'' \sum_\lambda \left[ \bar{z}_+(t'') \theta(t'' - t') e^{-iE(t'' - t')} z_+(t') - \bar{z}_-(t'') \theta(t' - t'') e^{-iE(t' - t'')} z_-(t') \right] \right\} \\
    Z_E &= \prod_\lambda \left[ 1 + \int dt' dt'' \bar{z}_+(t'') e^{-iE(t' - t'')} z_+(t') \right] Z_F
\end{align*}
\]

(13)

(14)

This expression for \(Z_E\) can now be compared with the one obtained by explicitly differentiating \(Z_F\) wrt. its sources at \(t = \pm\infty\), corresponding to the boundary states of Eq. (4). One readily finds

\[
Z_E = \lim_{t_i \to -\infty} \prod_\lambda \left[ e^{iE(t_f - t_i)} \frac{\delta^2}{\delta z_+(t_i) \delta \bar{z}_+(t_f)} \right] Z_F
\]

(15)

This result extends immediately to the full interactive functional (13), since the derivatives in Eq. (13) commute through the derivatives in \(\exp(iS_{int})\). This means that using the modified fermion propagator \(S_E\) of Eq. (3) everywhere in a perturbative calculation of an arbitrary green function (for some given 3-momentum \(\vec{k}\) of the propagators) is exactly equivalent to calculating the same green function using ordinary feynman propagators but with additional incoming and outgoing antifermions as in Eq. (14).

The change of \(i\varepsilon\) prescription suppresses fermion pair production at the corresponding value(s) of \(\vec{k}\), as required by the exclusion principle. This
can be seen directly for, *eg*, a fermion loop correction to a gauge boson propagator. The loop gives no contribution at those values of the fermion momenta \( \vec{k} \) at which external fermions have been introduced, since the poles in the loop momentum \( p^0 \) then are all below the real axis and the \( p^0 \) integral may be closed in the upper half plane.

Since I have shown that the propagator modification is equivalent to adding particles at \( t = \pm \infty \) gauge invariance is likely to be preserved. Formally, the ward identities involve inverse propagators, for which the sign of \( i\varepsilon \) is irrelevant.

### 3. A boson ‘condensate’

Boundary conditions like that of Eq. (4) with (anti)fermions added to the in- and out-states are relevant in situations involving fermion condensates, but not for typical applications of QCD. The QCD vacuum has zero baryon number, and thus no overlap with states having extra (anti)quarks. The propagator modification nevertheless seems phenomenologically interesting for gluons, since it suggests a ‘freezing’ of the low momentum gluon d.o.f.’s. Effective gluon and constituent quark masses can be generated through loop corrections due to the propagator modification, presumably without loss of gauge invariance (but with loss of lorentz invariance order by order, see section 4).

I shall show that there is a boundary condition which implies an analogous modification of the \( i\varepsilon \) prescription for boson propagators as the one for fermions discusses above. Not surprisingly, this ‘perturbative boson condensate’ involves an indefinite number of external bosons. For simplicity, I shall consider scalar bosons only. The generalization to real (transverse) gluons should be straightforward.
The free boson functional appearing in Eq. (7) is (for scalars)
\[ Z_B[J] = \exp \left[ \frac{i}{2} \sum_{\pm \vec{p}} \int \frac{dp^0}{2\pi} J(-p^0, -\vec{p}) D_F(p) J(p^0, \vec{p}) \right] \tag{16} \]
Since we shall be dealing with the free functional (the generalization to the interacting one will again be straightforward), it is sufficient to consider a single 3-momentum \( \vec{p} \), and keep only the bose symmetrization over \( \pm \vec{p} \) as indicated in Eq. (16). The feynman propagator is
\[ D_F(p) = \frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2E} \left( \frac{1}{p^0 - E + i\epsilon} - \frac{1}{p^0 + E - i\epsilon} \right) \tag{17} \]
where \( E = \sqrt{p^2 + m^2} \). A modification of the \( i\epsilon \) prescription at the \( p^0 = -E \) pole gives
\[ D_E(p) \equiv \frac{1}{(p^0 - E + i\epsilon)(p^0 + E + i\epsilon)} = D_F(p) + \frac{2\pi i}{2E} \delta(p^0 + E) \tag{18} \]
Note that the same generating functional is obtained if the \( i\epsilon \) prescription is changed instead at the \( p^0 = +E \) pole. Thus
\[ \tilde{D}_E(p) \equiv \frac{1}{(p^0 - E - i\epsilon)(p^0 + E - i\epsilon)} = D_E(-p) \tag{19} \]
so that
\[ \sum_{\pm \vec{p}} \int \frac{dp^0}{2\pi} J(-p) \tilde{D}_E(p) J(p) = \sum_{\pm \vec{p}} \int \frac{dp^0}{2\pi} J(-p) D_E(p) J(p) \tag{20} \]
In \( (t, \vec{p}) \)-space,
\[ J(p^0, \vec{p}) = \int dt J(t, \vec{p}) e^{ipt_0} \tag{21} \]
we have
\[ Z_B[J] = \exp \left\{ \sum_{\pm \vec{p}} \frac{1}{4E} \int dt' dt'' J(t'', -\vec{p}) \times \left[ \theta(t'' - t') e^{-iE(t'' - t')} + \theta(t' - t'') e^{iE(t'' - t')} \right] J(t', \vec{p}) \right\} \tag{22} \]
The generating functional for the modified scalar propagator (18) is then

\[ Z_E[J] \equiv \exp \left[ \frac{i}{2} \sum_{\pm p} \int \frac{dp^0}{2\pi} J(-p^0, -\vec{p}) D_E(p) J(p^0, \vec{p}) \right] \]

(23)

\[ = \exp \left[ -\sum_{\pm \vec{p}} \frac{1}{4E} J(E, -\vec{p}) J(-E, \vec{p}) \right] Z_B[J] \]

\[ = \exp \left[ -\sum_{\pm \vec{p}} \frac{1}{4E} \int dt' dt'' J(t'', -\vec{p}) e^{iE(t'' - t')} J(t', \vec{p}) \right] Z_B[J] \]

(24)

Eq. (24) may be compared with Eq. (14) in the fermion case. Due to the Grassmann algebra, the exponential factor multiplying \( Z_F \) contains only a single power of the fermion sources \( \bar{z}, z \). In the boson case the factor multiplying \( Z_B \) in Eq. (24) contains arbitrary powers of the sources \( J \). It can be reproduced only by differentiating \( Z_B[J] \) an arbitrary number of times, corresponding to an indefinite number of incoming and outgoing bosons.

A single boson of momentum \( \vec{p} \) in the in-state is obtained as

\[ \lim_{t_i \to -\infty} \delta Z_B[J] \delta J(t_i, -\vec{p}) = \left[ \frac{1}{2E} \int dt e^{-iE(t-t_i)} J(t, \vec{p}) \right] Z_B[J] \]

(25)

where we used \( \lim_{t_i \to -\infty} \theta(t_i - t) = 0 \). Similarly an outgoing boson corresponds to

\[ \lim_{t_f \to +\infty} \delta Z_B[J] \delta J(t_f, \vec{p}) = \left[ \frac{1}{2E} \int dt e^{-iE(t_f-t)} J(t, -\vec{p}) \right] Z_B[J] \]

(26)

Having a boson both incoming and outgoing is then given by

\[ \lim_{t_i \to -\infty, t_f \to +\infty} e^{iE(t_f-t_i)} \frac{\delta^2 Z_B[J]}{\delta J(t_f, \vec{p}) \delta J(t_i, -\vec{p})} = \left[ 1 + \frac{1}{2E} \int dt' dt'' J(t'', -\vec{p}) e^{iE(t'' - t')} J(t', \vec{p}) \right] Z_B[J] \]

(27)

Further differentiation wrt. \( J(t_i, -\vec{p}) \) and \( J(t_f, \vec{p}) \) now operates also on the first factor in Eq. (27). However, this gives back the factors in Eqs. (24).
and (26), respectively. Hence applying the double derivative of Eq. (27) any number of times on $Z_B$ generates a polynomial factor in $xy$, where

$$x \equiv \frac{1}{\sqrt{2E}} \int dt'' J(t'', -\vec{p}) e^{iEt''}$$

$$y \equiv \frac{1}{\sqrt{2E}} \int dt' e^{-iEt'} J(t', \vec{p}) .$$ (28)

According to Eq. (24),

$$Z_E[J] = \exp \left( -\frac{1}{2} \sum_{x \neq \vec{p}} xy \right) Z_B[J] .$$ (29)

We need to consider only how to generate the $+\vec{p}$ term in Eq. (24) through repeated differentiation of $Z_B$ as in Eq. (27). The $-\vec{p}$ term will then be obtained similarly through repeated $\delta^2/\delta J(t_f, -\vec{p})$ differentiation.

The function

$$f(xy) \equiv \exp \left( \lambda \frac{\partial^2}{\partial x \partial y} \right) \exp(xy)$$ (30)

provides an adequate model for the present problem. As seen from Eq. (22), $Z_B$ is not of the form $\exp(xy)$ due to the $\theta$-functions, but as in Eqs. (25 – 27), $Z_B$ acts precisely like $\exp(xy)$ when differentiated in the limits $t_i \to -\infty$, $t_f \to +\infty$. Hence the polynomial in $xy$ generated by the derivatives in Eq. (30) will be the same as that generated from $Z_B$.

It is straightforward to evaluate $f(xy)$ in Eq. (30) by using the identity

$$\exp \left( \frac{1}{2} x^2 \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \exp \left( -\frac{1}{2} u^2 + ux \right)$$ (31)

to express

$$\exp(xy) = \exp \left[ \frac{1}{2} (x + y)^2 \right] \exp \left( -\frac{1}{2} x^2 \right) \exp \left( -\frac{1}{2} y^2 \right)$$ (32)
as a three-fold integral. The integral resulting from applying the derivatives in Eq. (30) is gaussian and gives

\[
\exp \left( \lambda \frac{\partial^2}{\partial x \partial y} \right) \exp(xy) = \frac{1}{1 - \lambda} \exp \left( \frac{xy}{1 - \lambda} \right)
\]

\[
= \frac{1}{1 - \lambda} \exp \left( \frac{\lambda}{1 - \lambda} xy \right) \exp(xy) \quad (33)
\]

Requiring that the first exponent in Eq. (33) be \(-xy/2\) according to Eq. (29) gives the result \(\lambda = -1\).

We have thus shown that the generating functional \(Z_E[J]\) (23) of the scalar propagator \(D_E(p)\) (18), which differs from the Feynman propagator \(D_F(p)\) (17) by the sign of \(i\varepsilon\) at the \(p^0 = -E\) pole, is equivalent to the standard generating functional \(Z_B[J]\) (16) of Feynman propagators differentiated wrt. sources at \(t = \pm \infty\),

\[
Z_E[J] = 4 \exp \left[ -\sum_{\pm} \lim_{t_i \to -\infty} \lim_{t_f \to +\infty} e^{iE(t_f - t_i)} \delta^2 \frac{\delta E}{\delta J(t_f, p) \delta J(t_i, -p)} \right] Z_B[J] \quad (34)
\]

As noted in Eq. (20), the same result obtains if the \(i\varepsilon\) prescription is modified at the \(p^0 = +E\) pole instead.

The source derivatives in (34) commute through the interaction term in the definition (3) of the full generating functional of Green functions in the interacting theory. Hence the above result establishes that a perturbative calculation (to arbitrary order) which uses the propagator \(D_E(p)\) (18), with its non-standard \(i\varepsilon\) prescription, is equivalent to a standard perturbative calculation using Feynman propagators in the presence of a ‘perturbative condensate’ of incoming and outgoing particles as specified by the source derivatives in Eq. (34).
4. Discussion

I have argued that it may be useful to consider perturbative expansions of QCD using non-trivial boundary conditions at \( t = \pm \infty \), given that the ground state of the theory is a gluon condensate. All expansions in which the boundary states overlap the true ground state are formally equivalent and \textit{a priori} equally good.

I investigated a particular case which is the bosonic equivalent of a fermion condensate, and in which the \( i\varepsilon \) prescription of low momentum boson propagators is modified. Such a propagator modification corresponds to a superposition of standard perturbative calculations where 0, 1, 2, etc. bosons are added both to the initial and final state, as expressed by Eq. (34). I have not shown that these boundary states have an overlap with the true QCD vacuum (but then, neither do we know that the standard perturbative vacuum has such an overlap).

The relevance of this expansion depends on its theoretical and phenomenological viability, which remains to be demonstrated. Gauge invariance is among the important properties that should be explicitly verified.

Since only low-momentum (\(|\vec{p}| \lesssim \Lambda_{QCD} \)) propagators are modified, the successful results of ‘hard’ QCD processes remain unaltered. In particular, the renormalization procedure will not be affected in any way by the modifications suggested here.

The most striking difference compared to standard perturbation theory is the lack of boost invariance order by order. Contrary to what might first appear, this need not signal a breakdown of lorentz symmetry for the full series. The true asymptotic degrees of freedom are the hadron bound states, which do not occur at any finite order of perturbation theory. Physical symmetry
requirements should be imposed only on resummations of the series.

The subtleties of lorentz invariance in bound state calculations is known from QED. As an example \cite{12}, consider the lippman-schwinger equation

\[ G_T(E) = K(E) + K(E)S(E)G_T(E) \]  \hspace{1cm} (35)

for the (truncated) green function \( G_T \) of a \( 2 \to 2 \) process with c.m. energy \( E \). Iterating this equation generates an expansion of \( G_T \) in powers of the propagator \( S \) and the kernel \( K \). While the standard perturbative expansion in \( \alpha \) is unique (up to renormalization conventions) for the green function \( G_T \), this is not so for \( S \) and \( K \) separately. Rather, we can choose the form of the propagator \( S \) freely, be it of relativistic (dirac) or non-relativistic (schrödinger) form. Eq. (35) then determines the corresponding perturbative expansion of the kernel \( K \). At a pole of the (full) green function of the form

\[ G(E) = \frac{\psi_n \bar{\psi}_n}{E - E_n} + \text{regular terms} \]  \hspace{1cm} (36)

the lippman-schwinger equation implies a bound state equation of the form

\[ S^{-1}(E_n) = K(E_n)\psi_n \]  \hspace{1cm} (37)

For a non-relativistic propagator \( S \) this will have the form of a schrödinger equation, but it will give exact results provided the full perturbative series for the interaction kernel \( K \) is used.

It should furthermore be realized that the transformation properties of equal-time bound state wave functions under lorentz boosts is quite non-trivial. The requirement that the constituents should be evaluated at equal time in all frames is inconsistent with explicit space-time covariance, even for non-relativistic QED bound states.
A simple example serves to illustrate the novel aspects of the frame
dependence of equal-time wave functions. There is a bound state equation in QED for
which it is possible to relate explicitly the solutions in different
lorentz frames, and thus verify that they have the correct transformation
properties [13]. The wave function of a two fermion bound state is written

$$\psi(t, x_1, x_2) = \exp(-iEt) \exp\left(i k \frac{x_1 + x_2}{2} \right) \chi(x_1 - x_2),$$

(38)

where \(x_1, x_2\) are the positions of the constituents and \(t\) their common
time. Both the bound state energy \(E\) and the 2 × 2 dirac wave function \(\chi\) depend
on the bound state c.m. momentum parameter \(k\). The bound state equation
for \(\chi\) is

$$-i\partial_x [\alpha, \chi(x)] + \frac{1}{2} k \{\alpha, \chi(x)\} + m_1 \gamma^0 \chi(x) - m_2 \chi(x) \gamma^0 = (E - V(x)) \chi(x)$$

(39)

where \(m_1, m_2\) are the constituent masses and \(V(x) = \frac{1}{2} e^2 |x|\) is the instantan-
eous Coulomb potential. In 1+1 dimensions we may represent the dirac
matrices using pauli matrices, \(\gamma^0 = \sigma_3\) and \(\alpha = \gamma^0 \gamma^1 = \sigma_1\). Despite the fact
that Eq. (39) has no explicit lorentz covariance (space and time coordinates
are treated differently in Eqs. (38,39)) the bound state energies for different
c.m. momenta \(k\) are correctly related: \(E = \sqrt{k^2 + M^2}\), with \(M\) indepen-
dent of \(k\). In the limit of non-relativistic internal motion \(e/m_{1,2} \ll 1\) the
wave function \(\chi(x)\) lorentz contracts in the standard way as a function of \(k\).
Related examples may be found in Ref. [14].

The ground state wave function of QCD is invariant under boosts. This
is obviously not the case for our asymptotic states which according to Eq.

\footnote{This is in fact the only case where the solutions are normalizable and thus meaningful, due to the Klein paradox. The bound state momentum \(k\) can be arbitrarily large, however.}
contain bosons of definite 3-momenta ($\lesssim \Lambda_{QCD}$). In the present formulation, the same boundary states must be used in all frames, since they model the same (invariant) ground state. Hence the perturbative expansion of a given QCD process will depend on the lorentz frame. If the method works, measurable quantities such as hadronic cross sections will be lorentz invariant. This does not mean that the hadron wave functions themselves will be invariant – the starting point of this paper was in fact that they appear phenomenologically to be strongly frame dependent.

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