Kähler geometry for \( su(1, N|M) \)-superconformal mechanics

Erik Khastyan,† Sergey Krivonos,‡ and Armen Nersessian\(^1,\,2,\,3\)^∗

\(^1\)Yerevan Physics Institute, 2 Alikhanian Brothers St., Yerevan 0036 Armenia
\(^2\)Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia
\(^3\)Institute of Radiophysics and Electronics, Astsharak-2, 0203, Armenia

We suggest the \( su(1, N|M) \)-superconformal mechanics formulated in terms of phase superspace given by the non-compact analogue of complex projective superspace. We parameterized this phase space by the specific coordinates allowing to interpret it as a higher-dimensional super-analogue of the Lobachevsky plane parameterized by lower half-plane (Klein model). Then we introduced the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables we demonstrated that proposed scheme allows to construct the \( su(1, N|M) \) superconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb-like systems with a \( su(1, N|M) \) dynamical superalgebra, and found that oscillator-like systems admit deformed \( N = 2M \) Poincaré supersymmetry, in contrast with Coulomb-like ones.

I. INTRODUCTION

Kähler manifolds are the Hermitian manifolds which possesses the symplectic structure obeying the specific compatibility condition with the Riemann (and/or complex) structure. Being highly common objects in almost all areas of theoretical physics, these manifolds usually appear as configuration spaces of the particles and fields. Only in a limited number of physical problems they appear as phase spaces, mostly for the description of various generalizations of tops, Hall effect (including its higher-dimensional generalizations, see, e.g. refs therein), etc. Respectively, the number of the known nontrivial (super)integrable systems with Kähler phase spaces is very restricted, and their study does not attract much attention. The widely known integrable model with Kähler phase space extensively studying nowadays is compactified Ruijsenaars-Schneider model with excluded center of mass, whose phase space is complex projective space.\(^4\)

On the other hand, there are some indications that Kähler phase spaces can be useful for the study of conventional Hamiltonian systems, i.e. for the systems formulated on cotangent bundle of Riemann manifolds. A very simple example of such system is one-dimensional conformal mechanics formulated in terms of Lobachevsky plane \( \text{("noncompact complex projective plane")} \) treated as a phase space.\(^5\) Such description, being quite elegant, allows immediate construction of \( N = 2M \) superconformal extension associated with \( su(1, 1|M) \) superalgebra. Recently the similar formulation of some higher-dimensional systems was given in terms of \( su(1, N) \)-symmetric Kähler phase space treated as the non-compact version of complex projective space.\(^6\) In such approach all symmetries of the generic superintegrable conformal-mechanical systems acquire interpretation in terms of the powers of the \( su(1, N) \) isometry generators. The maximally superintegrable generalizations of the Euclidean oscillator/Coulomb systems has also been considered, all the symmetries of these superintegrable systems were expressed via \( su(1, N) \) isometry generators as well. However, the supersymmetrization aspects of that system was not considered there at all. In the present paper we construct the \( N \)-extended superconformal extensions of the systems considered in \( \text{[6]} \), as it was done in \( \text{[4]} \) for one-dimensional case. Namely, we consider the systems with \( su(1, N|M) \)-symmetric \( (N|M)_{\mathbb{C}} \)-dimensional Kähler phase superspace (in what follow we denote it by \( \mathbb{CP}^{N|M} \)) and relate their symmetries with the isometry generators of the super-Kähler structure. We construct this superspace reducing the \( (N + 1|M)_{\mathbb{C}} \)-dimensional complex pseudo-Euclidean superspace by the \( U(1) \)-group action and then identify the reduced phase superspace with noncompact analogue of complex projective superspace constructed in \( \text{[6]} \). We parameterize this superspace by the complex bosonic variable \( w \). Im \( w < 0 \), by the \( N - 1 \) complex bosonic variables \( z^\alpha \in [0, \infty) \), \( \arg z \in [0; 2\pi] \), and by \( M \) complex fermionic coordinates \( \eta^A \). Thus, it can be considered as the \( N \)-dimensional extension of the Klein model of Lobachevsky plane.\(^7\) This allows us to connect the complex coordinate \( w \) with the radial coordinate and

\*Electronic address: khastyanerik@gmail.com
†Electronic address: krivonos@theor.jinr.ru
‡Electronic address: arnerses@yerphi.am
momentum of the conformal-mechanical system spanned by \( su(1,1) \) subalgebra, and separate the \( su(1, 1) \) generators interpreting them as Hamiltonian, conformal boosts and dilatation operators. The rest bosonic generators \( z^\alpha \) parameterize the angular part of integrable conformal mechanics with Euclidean configuration spaces\(^1\). Relating the angular coordinates and momenta with the action-angle variables, we describe all symmetries of the generic superintegrable conformal-mechanical systems in terms of the powers of the \( su(1,N) \) isometry generators. An important aspect of proposed approach is the choice of canonical coordinates where all fermionic degrees of freedom appear only in the angular part of the Hamiltonian.

Furthermore, we construct the super-analogues of the maximally superintegrable generalizations of the Euclidean oscillator/Coulomb systems considered in \( \mathbb{R}^3 \) as follows: we preserve the form of Hamiltonian expressed via generators of \( su(1,1) \) subalgebra but extend the phase space \( \mathbb{C}P^N \) to phase superspace \( \mathbb{C}P^{N|M} \). As a result, we find that these superextensions preserve all symmetries of the initial bosonic Hamiltonians and possess maximal set of functionally-independent fermionic integrals, i.e. they remains superintegrable in the sense of super-Liouville theorem. We also find that the constructed oscillator-like systems (in contrast with Coulomb-like ones) possess deformed \( N = 2M, d = 1 \) Poincaré supersymmetry (see \( \mathbb{R}^3 \)), and express all the symmetries of these superintegrable systems via \( su(1,N) \) isometry generators as well.

The paper organized as follows.

In Section 2 we present the basic facts on Kähler supermanifolds and construct, by the Hamiltonian reduction, the non-compact complex projective superspace \( \mathbb{C}P^{N|M} \) in the parametrization similar to those of Klein model. In Section 3 we analyze the symmetry algebra of \( \mathbb{C}P^{N|M} \) and extract from it the \( su(1,N|M) \)-superconformal systems. In Section 4 we introduce the canonical coordinates which naturally split radial and angular parts of the Hamiltonian and relate the angular part with the systems formulating in terms of action-angle variables. In the Section 5 we construct superintegrable supergeneralizations of oscillator- and Coulomb-like systems. In Section 6 we represent the Kähler structure of phase superspace in the Fubini-Study-like form. We conclude the paper by the outlook and final remarks in Section 7.

\section{Noncompact Complex Projective Superspace}

The (even) \( (N|M) \)-dimensional Kähler supermanifold can be defined as a complex supermanifold with symplectic structure given by the expression

\[
\Omega = i(-1)^{p_{IJ}(p_{J}+1)}g_{IJ}dZ^I \wedge d\bar{Z}^J, \quad d\Omega = 0, \tag{1}
\]

with \( Z^I \) denoting \( N \) complex bosonic coordinates and \( M \) complex fermionic ones. The \( p_I := p(Z^I) \) is Grassmanian parity of coordinate: it is equal to zero for bosonic coordinate and to one for the fermionic one. Through the paper we will use the following conjugation rule:

\[
\bar{Z}^I Z^J = Z^I \bar{Z}^J, \quad \bar{Z}^I Z^J = Z^I \bar{Z}^J, \quad \bar{Z}^I \bar{Z}^J = Z^I \bar{Z}^J,
\]

for both bosonic and fermionic variables.

The “metrics components” \( g_{IJ} \) can then be locally represented in the form

\[
g_{IJ} = \frac{\partial L}{\partial Z^I} \frac{\partial R}{\partial \bar{Z}^J} K(Z, \bar{Z}), \tag{2}
\]

where \( \partial^{L(R)}/\partial Z^I \) denotes left(right) derivatives.

The Poisson brackets associated with this Kähler structure looks as follows

\[
\{ f, g \} = i \left( \frac{\partial R f}{\partial Z^I} \frac{\partial L g}{\partial \bar{Z}^J} - (-1)^{p_{IJ} p_{J}} \frac{\partial R f}{\partial Z^I} \frac{\partial L g}{\partial \bar{Z}^J} \right), \quad \text{where} \quad g_{IJ} g_{JK} = \delta^I_K, \quad g^{IJ} = (-1)^{p_{I} p_{J}} g^{IJ}, \tag{3}
\]

As in the pure bosonic case, the isometries of Kähler manifolds are given by the holomorphic Hamiltonian vector fields,

\[
V_{\mu} := \{ h_{\mu}(Z, \bar{Z}), \} = V^I(Z) \frac{\partial L}{\partial Z^I} + \bar{V}^I(\bar{Z}) \frac{\partial L}{\partial \bar{Z}^I}. \tag{4}
\]

\(^1\) The convenience of the separation of the radial coordinates from the angular one in the study of conformal mechanics and in their supersymmetrization was demonstrated, e.g., in \( \mathbb{R}^3 \).
where \( h_\mu(Z, \bar{Z}) \) are real functions called Killing potentials (see, e.g. [10] for the details).

Our goal is to study the systems on the Kähler phase space with \( su(1, N|M) \) isometry superalgebra. For the construction of such phase space it is convenient, at first, to present the linear realization of \( u(1, N|M) \) superconformal algebra on the complex pseudo-Euclidean superspace \( \mathbb{C}^{1, N|M} \), equipped with the canonical Kähler structure (and thus, by the canonical supersymplectic structure) and then reduced it by the action of \( U(1) \) generator.

It is instructive to present this reduction in details. Let us equip, at first, the \((N+1|M)\)-dimensional complex superspace with the canonical symplectic structure

\[
\Omega_0 = i \sum_{a,b=0}^N \gamma_{a\bar{b}} dv^a \wedge d\bar{v}^b + \sum_{A=1}^M dh^A \wedge d\bar{h}^A,
\]

with \( v^a, \bar{v}^a \) being bosonic variables, and \( h^A, \bar{h}^A \) being fermionic ones, and with the matrix \( \gamma_{a\bar{b}} \) chosen in the form

\[
\gamma_{a\bar{b}} = \begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix}, \quad a, b = N, 0, 1, \ldots, N - 1.
\]

With this supersymplectic structure we can associate the Poisson brackets given by the relations

\[
\{\bar{v}^a, v^b\} = -i \delta^{ab}, \quad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{AB}, \quad \gamma^{\bar{a}b} \gamma_{bc} = \delta_{\bar{c}}^a.
\]

Equivalently,

\[
\{v^0, \bar{v}^N\} = 1, \quad \{v^N, \bar{v}^0\} = -1, \quad \{v^a, \bar{v}^\beta\} = i \delta^{a\bar{\beta}}, \quad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{AB}.
\]

Here we introduced the indices \( \alpha, \beta = 1, \ldots, N - 1 \).

On this superspace we can define the linear Hamiltonian action of \( u(1, N|M) = u(1) \times su(1, N|M) \) superalgebra

\[
\{h_{a\bar{b}}, h_{c\bar{d}}\} = -i \left(h_{a\bar{d}} \gamma^{\bar{c}b} - h_{c\bar{b}} \gamma^{\bar{a}d}\right), \quad \{\Theta_{A\bar{a}}, \Theta_{B\bar{b}}\} = h_{\bar{a}b} \delta^{BA} - R_{AB} \gamma^{\bar{b}a}, \quad \{\Theta_{A\bar{a}}, h_{\bar{b}c}\} = -i \Theta_{AC} \gamma^{\bar{b}a},
\]

\[
\{R_{AB}, R_{CD}\} = i \left(R_{A\bar{D}} \delta^{B\bar{C}} - R_{C\bar{D}} \delta^{B\bar{A}}\right), \quad \{\Theta_{A\bar{a}}, R_{\bar{C}D}\} = -i \Theta_{C\bar{a}} \delta^{B\bar{A}},
\]

where

\[
h_{a\bar{b}} = \bar{v}^a v^b, \quad \Theta_{A\bar{a}} = \bar{\eta}^A v^a, \quad R_{AB} = i \eta^A \eta^B.
\]

The \( u(1) \) generator defining the center of \( u(N,1|M) \) is given by the expression

\[
J = \gamma_{a\bar{b}} v^a \bar{v}^b + i \eta^A \bar{\eta}^A : \{J, h_{a\bar{b}}\} = \{J, \Theta_{A\bar{a}}\} = \{J, R_{AB}\} = 0.
\]

Hence, reducing the system by the action of this generator we will get the ”non-compact” projective super-space \( \mathbb{C}P^N|M \) (i.e. the supergeneralization of noncompact projective space \( \mathbb{C}P^N \)), which is \((2N|2M)\)-(real)dimensional space.

For performing the reduction by the action of generator \( \{12\} \) we have to choose, at first, the \( 2N \) real \((N \) complex\) bosonic and \(2M\) real \((M \) complex\) fermionic functions commuting with \( J \). Then, we have to calculate their Poisson brackets and restrict the latter to the level surface

\[
J = g.
\]

As a result we will get the Poisson brackets on the reduced \((2N|2M)\)-(real) dimensional space, with that \( U(1) \)-invariant functions playing the role of the latter’s coordinates.

The required functions could be easily found as

\[
w = \frac{\bar{v}^N}{v^0}, \quad z^a = \frac{v^a}{v^0}, \quad \theta^A = \frac{\eta^A}{v^0} : \{w, J\} = \{z^a, J\} = \{\theta^A, J\} = 0, \quad \text{and c.c.}
\]

(14)
Calculating their Poisson brackets and having in mind the expression following from \(13\),
\[
A := \frac{1}{\nu^0\bar{v}^0} \bigg|_{J=g} = \frac{1}{g} \left( i(w - \bar{w}) - \sum_{\gamma=1}^{N-1} z^\gamma \bar{z}^\gamma + i \sum_{C=1}^{M} \theta^C \bar{\theta}^C \right),
\]
we get the reduced Poisson brackets defined by the following non-zero relations (and their complex conjugates)
\[
\{w, \bar{w}\} = -A(w - \bar{w}), \quad \{z^\alpha, \bar{z}^\beta\} = iA\delta^\alpha_\beta, \quad \{\theta^A, \bar{\theta}^B\} = A\delta^{AB}, \quad \{w, \bar{z}^\alpha\} = Az^\alpha, \quad \{w, \bar{\theta}^A\} = A\bar{\theta}^A.
\]
These Poisson brackets are associated with the supersymplectic structure
\[
\Omega = \frac{i}{\nu^0\bar{v}^0} \left[ \frac{1}{A^2} dw \wedge d\bar{w} - \frac{i z^\alpha}{A^2} dw \wedge dz^\alpha - \frac{\theta^A}{A^2} dw \wedge d\bar{\theta}^A \right.
\]
\[
\left. + \frac{i \bar{z}^\alpha}{A^2} dz^\alpha \wedge d\bar{w} + \left( \frac{\theta^\alpha_\beta A}{A^2} + \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2} \right) dz^\alpha \wedge d\bar{z}^\beta - \left( \frac{\delta^\alpha_\beta A}{A^2} \right) d\theta^A \wedge d\bar{\theta}^B \right].
\]
It is defined by the Kähler potential
\[
K = -g \log(i(w - \bar{w}) - z^\alpha \bar{z}^\alpha + \nu \theta^A \bar{\theta}^A).
\]
In what follows we will call this space “noncompact projective superspace \(\tilde{\mathbb{CP}}^N[M]\)”. The isometry algebra of this space is \(su(1, N|M)\), which can be easily obtained by the restriction of the generators \(9, 10\) to the level surface \(13\). It is defined by the following Killing potentials
\[
H := v^N \bar{v}^N \big|_{J=g} = \frac{w \bar{w}}{A}, \quad K := v^0 \bar{v}^0 |_{J=g} = \frac{1}{A}, \quad D := (v^N \bar{v}^0 + v^0 \bar{v}^N) |_{J=g} = \frac{w + \bar{w}}{A},
\]
\[
H_\alpha := \bar{v}^\alpha v^N |_{J=g} = \frac{\bar{z}^\alpha w}{A}, \quad K_\alpha := \bar{v}^\alpha v^0 |_{J=g} = \frac{\bar{z}^\alpha}{A}, \quad h_\alpha_\beta := \bar{v}^\alpha \bar{v}^\beta |_{J=g} = \frac{\bar{z}^\alpha \bar{z}^\beta}{A},
\]
\[
Q_A := \bar{v}^A v^N |_{J=g} = \frac{\bar{\theta}^A w}{A}, \quad S_A := \bar{v}^A v^0 |_{J=g} = \frac{\bar{\theta}^A}{A}, \quad \Theta_{A\bar{A}} := \bar{v}^A v^\alpha |_{J=g} = \frac{\bar{\theta}^A \bar{z}^\alpha}{A},
\]
\[
R_{A\bar{B}} := \bar{v}^A \eta^\alpha |_{J=g} = \frac{\bar{\theta}^A \bar{\theta}^B}{A}.
\]
Constructed super-Kähler structure can be viewed as a higher dimensional analogue of the Klein model of Lobachevsky space, where the latter is parameterized by the lower half-plane. One can choose, instead of non-diagonal matrix \(6\), the diagonal one, \(\gamma_{\bar{a}b} = diag(1, -1, \ldots, -1)\). In that case the reduced Kähler structure will have the Fubini-Study-like form (see Section VI). In the next Section we will analyze the isometry algebra defined by these generators in details. Presented choice \(6\) is motivated by its convenience for the analyzing superconformal mechanics. Indeed, in that case the generators \(19\) define conformal subalgebra \(su(1, 1)\) and are separated from the rest \(su(N, 1)\) generators. Thus they can be interpreted as the Hamiltonian of conformal mechanics, the generator of conformal boosts and the generator of dilatation.

In the next section we will analyze in details these superconformal mechanics and their dynamical defined by the generators \(19, 20, 21, 22\).

### III. \(su(1, N|M)\) SUPERCONFORMAL ALGEBRA

The generators (Killing potentials) \(19, 20, 21, 22\) form \(su(1, N|M)\) superalgebra given by \(9, 10\) with \(\gamma_{\bar{a}b}\) defined in \(6\). Its explicit expression with separated \(su(1, 1)\) subalgebra is represented below. For the convenience it is divided into three sectors: "bosonic", "fermionic" and "mixed" ones.
The bosonic sector is the direct product of the \( su(1,N) \) algebra defined by the generators \([19],[20]\), and the \( u(M) \) algebra defined by the R-symmetry generators \([22]\). Explicitly, the \( su(1,N) \) algebra is given by the relations

\[
\{ H, K \} = -D, \quad \{ H, D \} = -2H, \quad \{ K, D \} = 2K, \\
\{ H, K_\alpha \} = -H_\alpha, \quad \{ H, H_\alpha \} = \{ H, h_{\bar{\alpha}\beta} \} = 0, \\
\{ K, H_\alpha \} = K_\alpha, \quad \{ K, K_\alpha \} = \{ K, h_{\bar{\alpha}\beta} \} = 0, \\
\{ D, K_\alpha \} = -K_\alpha, \quad \{ D, H_\alpha \} = H_\alpha, \quad \{ D, h_{\bar{\alpha}\beta} \} = 0, \\
\{ K_\alpha, K_\beta \} = \{ H_\alpha, H_\beta \} = \{ K_\alpha, H_\beta \} = 0,
\]

where

\[
I := g + \sum_{\gamma=1}^{N-1} h_{\gamma\bar{\gamma}} + \sum_{C=1}^{M} R_{C\bar{C}}
\]

The R-symmetry generators form \( u(M) \) algebra and commutes with all generators of \( su(1,N) \):

\[
\{ R_{\bar{A}B}, R_{C\bar{D}} \} = i(R_{\bar{A}D}\delta_{C\bar{B}} - R_{C\bar{B}}\delta_{\bar{A}D}), \quad \{ R_{\bar{A}B}, (H_\alpha; D; K_\alpha; H_\alpha; h_{\bar{\alpha}\beta}) \} = 0.
\]

It is clear that the generators \( H, D, K \) form conformal algebra \( su(1,1) \), the generators \( h_{\bar{\alpha}\beta} \) form the algebra \( u(N-1) \), and all together - the \( su(1,1) \times u(N-1) \) algebra. Notice, that the generator \( I \) in \([40]\) defines the Casimir of conformal algebra \( su(1,1) \):

\[
I := \frac{1}{2}I^2 = \frac{1}{2}D^2 - 2HK.
\]

Hence, choosing \( H \) as a Hamiltonian, we get that \( H_\alpha, h_{\bar{\alpha}\beta}, R_{\bar{A}B} \) define its constant of motion. Similarly, choosing the generator \( K \) as a Hamiltonian, we get that it has constants of motion \( K_\alpha, h_{\bar{\alpha}\beta}, R_{\bar{A}B} \).

"Fermionic" sector

The Poisson brackets between fermionic generators \([21]\) have the form

\[
\{ S_A, \bar{S}_B \} = K\delta_{AB}, \quad \{ Q_A, \bar{Q}_B \} = H\delta_{AB}, \quad \{ S_A, \bar{Q}_B \} = -iR_{\bar{A}B} + \frac{i}{2} (I - iD) \delta_{AB}, \\
\{ \Theta_{\bar{A}B}, \bar{S}_B \} = R_{\bar{A}B}\delta_{\bar{A}\bar{B}} + h_{\bar{B}\bar{A}}\delta_{AB}, \quad \{ S_A, \bar{S}_B \} = K_\alpha\delta_{AB}, \quad \{ Q_A, \bar{S}_B \} = H_\alpha\delta_{AB}, \\
\{ S_A, S_B \} = \{ S_A, Q_B \} = \{ \Theta_{\bar{A}B}, Q_B \} = \{ S_A, \Theta_{\bar{B}\bar{A}} \} = \{ Q_A, \Theta_{\bar{B}\bar{A}} \} = 0.
\]

Hence, the functions \( Q_A \) play the role of supercharges for the Hamiltonian \( H \), and the functions \( S_A \) define the supercharges of the Hamiltonian given by the generator of conformal boosts \( K \).

"Mixed" sector

The mixed sector is given by the relations

\[
\{ H, Q_A \} = \{ H, \Theta_{\bar{A}\bar{A}} \} = 0, \quad \{ H, S_A \} = -Q_A, \quad \{ K, S_A \} = \{ K, \Theta_{\bar{A}\bar{A}} \} = 0, \quad \{ K, Q_A \} = S_A, \\
\{ D, S_A \} = -S_A, \quad \{ D, Q_A \} = Q_A, \quad \{ D, \Theta_{\bar{A}\bar{A}} \} = 0, \\
\{ Q_A, \bar{K}_{\alpha} \} = -\Theta_{\bar{A}\alpha}, \quad \{ Q_A, H_\alpha \} = \{ Q_A, \bar{H}_{\alpha} \} = \{ Q_A, K_\alpha \} = \{ Q_A, h_{\alpha\beta} \} = 0, \\
\{ S_A, \bar{Q}_B \} = \Theta_{\alpha\bar{B}}, \quad \{ S_A, K_\alpha \} = \{ S_A, \bar{K}_{\alpha} \} = \{ S_A, H_{\alpha} \} = \{ S_A, h_{\alpha\beta} \} = 0, \\
\{ \Theta_{\alpha\bar{A}}, K_\beta \} = iS_A\delta_{\alpha\beta}, \quad \{ \Theta_{\alpha\bar{A}}, H_\beta \} = iQ_A\delta_{\alpha\beta}, \quad \{ \Theta_{\alpha\bar{A}}, \bar{H}_{\beta} \} = \{ \Theta_{\alpha\bar{A}}, \bar{K}_{\beta} \} = \{ \Theta_{\alpha\bar{A}}, h_{\beta\gamma} \} = i\Theta_{\alpha\bar{A}}\delta_{\beta\gamma}, \\
\{ S_A, R_{\bar{B}\bar{C}} \} = -iS_B\delta_{AC}, \quad \{ Q_A, R_{\bar{B}\bar{C}} \} = -iQ_B\delta_{AC}, \quad \{ \Theta_{\alpha\bar{A}}, R_{\bar{B}\bar{C}} \} = -i\Theta_{\alpha\bar{A}}\delta_{AC}.
\]
Looking to the all Poisson bracket relations together we conclude that

- The bosonic functions $H_\alpha$, $h_{\alpha\bar{\beta}}$, and the fermionic functions $Q_\alpha$, $\Theta_{\bar{A}\bar{\alpha}}$ commute with the Hamiltonian $H$ and thus, provide it by the superintegrability property $^2$.

- The bosonic functions $K_\alpha$, $h_{\alpha\bar{\beta}}$ and the fermionic functions $S_\alpha\Theta_{\bar{A}\bar{\alpha}}$ commute with the generator $K$. Hence, the Hamiltonian $K$ defines the superintegrable system as well.

- The triples $(H, H_\alpha, Q_\alpha)$ and $(K, K_\alpha, S_\alpha)$ transform into each other under the discrete transformation

$$ (w, z^\alpha, \theta^A) \to (-\frac{1}{w}, \frac{z^\alpha}{w}, \frac{\theta^A}{w}) \Rightarrow D \to -D, \quad \begin{cases} (H, H_\alpha, Q_\alpha) \to (K, -K_\alpha, -S_\alpha), \\ (K, K_\alpha, S_\alpha) \to (H, H_\alpha, Q_\alpha). \end{cases} \quad (43) $$

- The functions $h_{\alpha\bar{\beta}}$, $\Theta_{\bar{A}\bar{\alpha}}$ are invariant under discrete transformation $[43]$. Moreover, they appear to be constants of motion both for $H$ and $K$. Hence, they remain to be constants of motion for any Hamiltonian being the functions of $H$, $K$. In particular, adding to the Hamiltonian $H$ the appropriate function of $K$, we get the superintegrable oscillator- and Coulomb-like systems with dynamical superconformal symmetry (see Section V).

- The superalgebra $su(1, N|M)$ admits $5$-graded decomposition $[12, 13]$

$$ su(1, N|M) = \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+1} \oplus \mathfrak{f}_{+2} \quad \text{with} \quad [f_i, f_j] \subseteq f_{i+j} \quad \text{for} \ i, j \in \{-2, -1, 0, 1, 2\}, \quad (44) $$

where $f_i = 0$ for $|i| > 2$ is understood. The subset $f_0$ includes the generators $D, h_{\alpha\bar{\beta}}, \Theta_{\bar{A}\bar{\alpha}}, \bar{\Theta}A\bar{\alpha}, R_{\bar{A}\bar{B}}$, the subsets $f_{-2}$ and $f_2$ contain only generators $H$ and $K$, respectively, while the subsets $f_{-1}$ and $f_1$ contain the generators $H_\alpha, H_{\bar{\alpha}}, Q_\alpha, Q_{\bar{\alpha}}$ and $K_\alpha, K_{\bar{\alpha}}, S_\alpha, S_{\bar{\alpha}}$.

Let us conclude this section by the following remark. It is easy to see, that the generator $\Theta_{\bar{A}\bar{\alpha}}$ commutes the generators $H, D, K, S_\alpha, Q_\alpha, R_{\bar{A}\bar{B}}$. Hence, these generators form superconformal algebra $su(1,1|M)$ with central charge $\sqrt{2N}$ $[22]$ (being the Casimir of $su(1,1|M)$ as well)

$$ \{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K, \quad \{S_\alpha, S_{\bar{\alpha}}\} = K\delta_{\alpha\bar{\beta}}, \quad \{Q_\alpha, Q_{\bar{\alpha}}\} = H\delta_{\alpha\bar{\beta}}, $$

$$ \{S_\alpha, Q_{\bar{\alpha}}\} = -iR_{\bar{A}\bar{B}} + \frac{i}{2} \left(\sqrt{2N} - iD\right)\delta_{\alpha\bar{\beta}}, $$

$$ \{H, S_\alpha\} = -Q_{\alpha}, \quad \{K, Q_\alpha\} = S_\alpha, \quad \{H, Q_\alpha\} = \{K, S_\alpha\} = 0, \quad \{D, S_\alpha\} = -S_\alpha, \quad \{D, Q_\alpha\} = Q_\alpha, $$

$$ \{R_{\bar{A}\bar{B}}, R_{\bar{C}\bar{D}}\} = i(R_{\bar{A}\bar{D}}\delta_{\bar{C}\bar{D}} - R_{\bar{C}\bar{D}}\delta_{\bar{A}\bar{D}}), \quad \{S_\alpha, R_{\bar{B}\bar{C}}\} = -iS_\beta\delta_{\alpha\bar{B}}, \quad \{Q_\alpha, R_{\bar{B}\bar{C}}\} = -iQ_\beta\delta_{\alpha\bar{B}}. \quad (45) $$

In the next section we will express presented $su(1, N|M)$ generators in appropriate canonical coordinates and in this way we will relate presented formulae with the superextensions of conventional conformal mechanics.

**IV. CANONICAL COORDINATES AND ACTION-ANGLE VARIABLES**

To define the canonical coordinates we pass from the complex bosonic coordinates $w, z^\alpha$

$$ w = x + iy, \quad z^\alpha = q_\alpha e^{i\varphi_\alpha}, \quad \text{where} \quad y < 0, \quad q_\alpha \geq 0, \quad \varphi_\alpha \in [0, 2\pi), \quad q^2 := \sum_{\alpha=1}^{N-1} q_\alpha^2 < -2y. \quad (46) $$

Then we re-define fermionic ones such that the new variables will have canonical Poisson brackets.

For this purpose we write down the symplectic/Kähler one-form and identify it with the canonical one

$$ A = -\frac{g}{2} \frac{du + d\bar{w} - i(z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{i(w - \bar{w}) - z^\alpha \bar{z}^\mu + i\theta^\beta \bar{\theta}^\mu} := p_x dx + \pi_\alpha d\varphi_\alpha + \frac{1}{2} \chi^A d\chi^A + \frac{1}{2} \bar{\chi}^A d\bar{\chi}^A \quad (47) $$

$^2$ In accord with super-analogue of Liouville theorem $[13]$ the system on $(2N,M)$ phase superspace is integrable iff it possess $N$ commuting bosonic integrals (with nonvanishing and functionally independent bosonic parts) and $M$ fermionic ones.
After some calculations and canonical transformation \((p_x, x) \to (-\frac{z}{r}, \frac{x}{r})\), one can obtain

\[ w = \frac{p_r}{r} - \frac{I}{r^2}, \quad z^\alpha = \frac{\sqrt{2} \pi \alpha}{r} e^{i \varphi_a}, \quad \theta^A = \frac{\sqrt{2}}{r} \chi^A, \]

(48)

where \(r, p_r, \pi_\alpha, \varphi, \chi^A, \bar{\chi}^A\) are canonical coordinates.

\[
\{r, p_r\} = 1, \quad \{\varphi, \pi_\alpha\} = \delta_{\alpha \beta}, \quad \{\chi^A, \bar{\chi}^B\} = \delta^{AB}, \quad \pi_\alpha \geq 0, \quad \varphi^a \in [0, 2\pi), \quad r > 0.
\]

(49)

They express via initial ones as follows

\[
p_r = \frac{w + \bar{w}}{2} \sqrt{\frac{2}{A}}, \quad r = \sqrt{\frac{2}{A}}, \quad \pi_\alpha = \frac{z^\alpha \bar{z}^\alpha}{A}, \quad \varphi = \arg(z^\alpha), \quad \chi^A = \frac{\theta^A}{\sqrt{A}}, \quad c.c.,
\]

(50)

where

\[
I = g + \sum_{\alpha=1}^{N-1} \pi_\alpha + \sum_{A=1}^{M} i \bar{\chi}^A \chi^A, \quad A := \frac{i(w - \bar{w}) - z^\gamma \bar{z}^\gamma + i \theta^C \bar{\theta}^C}{g} = \frac{2}{r^2}.
\]

(51)

In these canonical coordinates the isometry generators read

\[
H = \frac{p_r^2}{2} + \frac{l^2}{2r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r,
\]

\[
H_\alpha = \sqrt{\frac{\pi_\alpha}{2}} e^{-i \varphi_a} \left( p_r - \frac{I}{r} \right), \quad K_\alpha = \sqrt{\frac{\pi_\alpha}{2}} e^{-i \varphi_a}, \quad h_{\alpha \bar{\beta}} = \sqrt{\frac{\pi_\alpha \pi_{\bar{\beta}}}{2}} e^{-i (\varphi_a - \varphi_{\bar{\beta}})},
\]

\[
Q_A = \frac{\bar{\chi}^A}{\sqrt{2}} \left( p_r - \frac{i 2 I}{r} \right), \quad S_A = \frac{\bar{\chi}^A}{\sqrt{2}}, \quad \Theta_{A \bar{\alpha}} = \bar{\chi}^A \sqrt{\frac{\pi_\alpha}{2}} e^{i \varphi_a}, \quad R_{A \bar{B}} = i \bar{\chi}^A \chi^B.
\]

(53)

(54)

Interpreting \(r\) as a radial coordinate, and \(p_r\) as radial momentum, we get the superconformal mechanics with angular Hamiltonian given by

\[
\mathcal{I} = \frac{l^2}{2} := \frac{1}{2} \left( I_0 + (\bar{\chi} \chi) \right)^2, \quad \text{with} \quad I_0 := g + \sum_{\alpha=1}^{N-1} \pi_\alpha, \quad (\bar{\chi} \chi) := \sum_{A=1}^{M} i \bar{\chi}^A \chi^A.
\]

(55)

So, the fermionic part of superconformal Hamiltonian is encoded in its angular part.

The explicit dependence of the Hamiltonian \(H\) and the supercharges \(Q_A\) on the fermions is as follows

\[
H = H_0 + \frac{I_0 (\bar{\chi} \chi)}{r^2} + \frac{(\bar{\chi} \chi)^2}{2r^2}, \quad Q_A = -\frac{\bar{\chi}^A}{\sqrt{2}} \left( p_r - \frac{I_0}{r} - i \frac{(\bar{\chi} \chi)}{r} \right),
\]

(56)

while the dependence of bosonic integrals \(H_\alpha\) on fermions is given by the expression

\[
H_\alpha = H_\alpha^0 - \frac{K_\alpha (\bar{\chi} \chi)}{2K},
\]

(57)

where

\[
H_0 := \frac{p_r^2}{2} + \frac{I_0^2}{2r^2}, \quad H_\alpha^0 = \sqrt{\frac{\pi_\alpha}{2}} e^{-i \varphi_a} \left( p_r - \frac{I_0}{r} \right) : \{H_\alpha^0, H^0\} = 0.
\]

(58)

So, proposed superconformal Hamiltonian \(H\) inherits all symmetries of initial Hamiltonian \(H_0\) (given by \(H_\alpha^0, h_{\alpha \bar{\beta}}\)).

Looking at the functional dependence of the angular Hamiltonian \(\mathcal{I}\) from the angular variables \(\varphi^a, \pi_\alpha\) one can expect that the set of conformal mechanics admitting proposed \(su(1, N|\mathcal{M})\) superconformal extensions seems to be very restricted. However, it is not the case, since we it is not necessary to interpret \(\varphi^a\) as a coordinate of the configuration space, and \(\pi_\alpha\) as its canonically conjugated momentum. Instead, since \(\pi_\alpha\) define a constant of motion
of the bosonic Hamiltonian $H_0$ (and of the respective angular Hamiltonian $I_0 = H_0K/2 - D^2$), we can interpret it as the action variable $I_\alpha$, and consider $\varphi_\alpha$ as a respective angle variable $\Phi_\alpha$.

Furthermore, suppose that $\pi_\alpha, \varphi_\alpha$ are related with the action-angle variables $(I_\alpha, \Phi_\alpha)$ of some $(N-1)$-dimensional angular mechanics by the relations
\[
\pi_\alpha = n_\alpha I_\alpha, \quad \varphi_\alpha = \frac{\Phi_\alpha}{n_\alpha}, \quad \text{where} \quad n_\alpha \in \mathbb{N}, \quad \{\Phi_\alpha, I_\beta\} = \delta_\alpha\beta, \quad \Phi_\alpha \in [0, 2\pi).
\] (59)

Upon this identification the bosonic part of the angular Hamiltonian $I_0$ takes a form
\[
\tilde{I}_0 = \frac{1}{2} \left( g + \sum_{\alpha=1}^{N-1} n_\alpha I_\alpha \right)^2, \quad \text{with} \quad n_\alpha \in \mathbb{N},
\] (60)

but the bosonic generators $H_\alpha, S_\alpha, h_{\alpha\beta}$, become locally defined, $\varphi_\alpha \in [0, 2\pi/n_\alpha)$, and fail to be constants of motion. To get the globally defined bosonic generators we have to take their relevant powers,
\[
\tilde{H}_\alpha := (H_\alpha)^n_\alpha, \quad \tilde{K}_\alpha := (K_\alpha)^n_\alpha, \quad \tilde{h}_{\alpha\beta} := (h_{\alpha\beta})^{n_\alpha n_\beta}.
\] (61)

as well as replace the fermionic generator $\Theta_{A\tilde{a}}$ by the following one
\[
\tilde{\Theta}_{A\tilde{a}} = (H_\alpha)^{n_\alpha-1}\Theta_{A\tilde{a}}.
\] (62)

As a result, the dynamical (super)symmetry algebra becomes nonlinear deformation of $su(1, N|M)$.

The angular Hamiltonian (60) defines the class the superintegrable generalizations of the conformal mechanics, and of the oscillator- and Coulomb-like systems on the $N$-dimensional Euclidean spaces \cite{14}. As a particular case, this class of systems includes the "charge-monopole" system \cite{13}, Smorodinsky-Winternitz system \cite{16} (for the explicit expressions of the action-angle variables of these systems see, respectively, \cite{17} and \cite{18}), as well as the rational Calogero models \cite{3}. Thus, proposed systems can be considered as their $2M$ superconformal extensions.

Since the generators $Q_A, S_A, R_{AB}$ remain unchanged upon above identification (as well as the expression of the angular Hamiltonian (62) via generators $H, K, D$), we conclude that listed generators form superconformal algebra $su(1, 1|N)$ with central charge \cite{15}.

Finally, notice that in (60) the nonzero constant $g \neq 0$ appears, and the range of validity of the action variables is fixed to be $I_\alpha \in [0, \infty)$. As a result, standard free particle and conformal mechanics cannot be included in the proposed description, since for these systems we should choose $g = 0, I_\alpha \in [0, \infty)$. To exclude this constant we should replace the initial generators by the following ones
\[
\tilde{\mathcal{H}} := H - \frac{g(g - 2I)}{4K}, \quad \mathcal{H}_\alpha := \tilde{H}_\alpha + ig \frac{K_\alpha}{2K}, \quad \mathcal{Q}_A := Q_A - ig \frac{S_A}{2K}.
\] (63)

This deformation will further “non-linearize” the dynamical supersymmetry algebra $su(1, N|M)$.

V. OSCILLATOR- AND COULOMB-LIKE SYSTEMS

In the previous section we mentioned that the angular Hamiltonian \cite{60} defines the superintegrable deformations of $N$-dimensional oscillator and Coulomb system \cite{14}, while in \cite{6} the examples of such systems on noncompact projective space $\mathbb{CP}^N$ playing the role of phase space were constructed. So, one can expect that on the phase superspace $\mathbb{CP}^N$ one can construct the super-counterparts of that systems, which presumably, possess (deformed) $\mathcal{N} = 2M, d = 1$ Poincaré supersymmetry. Below we examine this question and show that our claim is corrects in some particular cases.

---

\cite{3} To our best knowledge, action-angle variables for the angular part of the rational Calogero models are not yet constructed explicitly. However, we have at hand the spectrum of the angular part of rational Calogero model \cite{19}. Taking its (semi)classical limit we can conclude that it has the form \cite{69}, see, e.g. \cite{14}.
A. Oscillator-like systems

We define the supersymmetric oscillator-like system by the the phase space $\tilde{\mathbb{C}P}^{N|M}$ (equipped with the Poisson brackets [16]) by the Hamiltonian

$$H_{osc} = H + \omega^2 K,$$

(64)

where the generators $H, K$ are given by [19]. In canonical coordinates [50] it reads

$$H_{osc} = \frac{p_r^2}{2} + \left( g + \sum_{\alpha=1}^{N-1} \pi_\alpha + \sum_{\alpha=1}^{M} i\chi^\alpha \chi^{\alpha}\right)^2 + \frac{\omega^2 r^2}{2}.$$ 

(65)

This system possesses the $u(N)$ symmetry given by the generators $h_{\alpha\beta}$ defined in [20] (among them $N-1$ constants of motion $\pi_\alpha$ are functionally independent), the $U(M)$ $R$-symmetry given by the generators $R_{AB}$ [22] as well as $N-1$ hidden symmetries given by the generators

$$M_{\alpha\beta} = (H_\alpha + i\omega K_\alpha)(H_\beta - i\omega K_\beta) = \frac{\varepsilon^\alpha \varepsilon^\beta}{A^2}(w^2 + \omega^2) : \{H_{osc}, M_{\alpha\beta}\} = 0,$$

(66)

The generators [66] and the $su(N)$ generators $h_{\alpha\beta}$ form the following symmetry algebra

$$\{h_{\alpha\beta}, M_{\gamma\delta}\} = i \left(M_{\alpha\gamma}M_{\beta\delta} + M_{\alpha\delta}M_{\beta\gamma}\right), \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0,$$

(67)

$$\{M_{\alpha\beta}, M_{\gamma\delta}\} = i \left(4\omega^2 h_{\alpha\delta}h_{\beta\gamma} - M_{\alpha\delta}h_{\beta\gamma} - M_{\alpha\gamma}h_{\beta\delta} - M_{\alpha\delta}h_{\beta\gamma} + M_{\alpha\beta}h_{\gamma\delta} - M_{\alpha\gamma}h_{\beta\delta}\right),$$

(68)

with $I$ given by [60] and summation over repeated indices is not assumed.

Besides, this system has a fermionic constants of motion $\Theta_{AC}$ defined in [21]. Hence, it is superintegrable system in the sense of super-Liouville theorem, i.e. it has $2N-1$ bosonic and $2M$ fermionic, functionally independent, constants of motion [11]. Further generalization to the systems with angular Hamiltonian (60) is straightforward.

Let us show, that for the even $M = 2k$ this system possess the deformed $\mathcal{N} = 2k$ Poincaré supersymmetry, in the sense of papers [9]. For this purpose we choose the following Ansatz for supercharges

$$Q_A = Q_A + \omega C_{AB} \bar{S}_B,$$

(69)

with the constant matrix $C_{AB}$ obeying the conditions

$$C_{AB} + C_{BA} = 0, \quad C_{AB}C_{BD} = -\delta_{AD}$$

(70)

For sure, the condition [71] assumes that $M$ is an even number, $M = 2k$.

Calculating Poisson brackets of the functions [69] we get

$$\{Q_A, Q_B\} = H_{osc} \delta_{AB}, \quad \{Q_A, Q_B\} = -i\omega \tilde{G}_{AB}, \quad \{\bar{Q}_A, \bar{Q}_B\} = i\omega \tilde{G}_{AB},$$

(71)

where

$$\tilde{G}_{AB} := C_{AC} R_{BC} + C_{BC} R_{AC}, \quad \tilde{G}_{AB} := \tilde{G}_{AB} = \tilde{C}_{AC} R_{CB} + \tilde{C}_{BC} R_{CA}, \quad \tilde{G}_{AB} = \tilde{C}_{AC} \tilde{C}_{DB} \tilde{G}_{DC}.$$ 

(72)

Then we get that the algebra of generators $Q_A, H_{osc}, R^B_A$ is closed indeed:

$$\{Q_A, H_{osc}\} = \omega C_{AB} Q_B, \quad \{\tilde{G}_{AB}, H_{osc}\} = 0,$$

(73)

$$\{Q_A, \tilde{G}_{BC}\} = i(C_{AB} Q_C + C_{AC} Q_B), \quad \{Q_A, \tilde{G}_{BC}\} = -i(C_{BD} Q_D \delta_{AC} + C_{CD} Q_D \delta_{AB}).$$

(74)

Hence, for the $M = 2k$ the above oscillator-like system [62] possesses deformed $\mathcal{N} = 4k$ supersymmetry. In the particular case $M = 2$ the choice of the matrix $C_{AB}$ is unique (up to unessential phase factor): $C_{AB} := e^{\varepsilon_{A\beta} \varepsilon_{AB}}$. In that case the above relations define the superalgebra $su(1|2)$-deformation of $\mathcal{N} = 4$ Poincaré supersymmetric mechanics studied in details in [9]. For the $k \geq 2$ the choice of matrices $C_{AB}$ is not unique, and we get the family of
deformed $\mathcal{N} = 4k$ Poincaré supersymmetric mechanics.

Let us present other deformed $\mathcal{N} = 2M$ Poincaré supersymmetric systems whose bosonic part is different from those of \([61]\) but nevertheless, has the oscillator potential.

For this purpose we choose another Ansatz for supercharges (in contrast with previous case $M$ is not restricted to be even number)

$$\tilde{Q}_A = Q_A + \omega S_A. \quad (75)$$

These supercharges generates the $su(1|\mathcal{M})$ superalgebra, and thus generalizes the systems considered in \([61]\) to arbitrary $\mathcal{M}$,

$$\{\tilde{Q}_A, \tilde{Q}_B\} = \mathcal{H}_{osc} \delta_{AB} - \omega R^A_{\bar{C}}, \quad \{\tilde{Q}_A, \bar{Q}_B\} = 0, \quad \{R^A_{\bar{C}}, R^D_{\bar{C}}\} = i(\bar{R}^D_{\bar{C}} \delta^B_C - R^B_{\bar{C}} \delta^D_A) \quad (76)$$

$$\{\tilde{Q}_A, R^C_{\bar{B}}\} = i \left( \frac{1}{M} \tilde{Q}_A \delta_{BC} + \tilde{Q}_B \delta_{AC} \right), \quad \{\tilde{Q}_A, \mathcal{H}_{osc}\} = i \omega \frac{2M-1}{M} \tilde{Q}_A, \quad (77)$$

where

$$\mathcal{H}_{osc} := H_{osc} - \omega(I + \frac{1}{M} \sum_C R_{\bar{B}C}), \quad R^A_{\bar{C}} := R_{\bar{A}B} - \frac{1}{M} \delta^B_A \sum_C R_{\bar{C}C} \quad (78)$$

with $I$ defined by \([50]\). Hence, the Hamiltonian get the additional bosonic term proportional to the Casimir of conformal group. In canonical coordinates \([50]\) it reads

$$\mathcal{H}_{osc} = \frac{p^2}{2} + \frac{I}{r^2} + \frac{\omega^2 p^2}{2} - \omega \left( \sqrt{2I} + \frac{1}{M} \bar{\chi} \right). \quad (79)$$

This Hamiltonian, seemingly, describes the oscillator-like systems specified by the presence of external magnetic field.

So, choosing $\mathbb{CP}^{N|\mathcal{M}}$ as a phase superspace, we can easily construct superintegrable oscillator-like systems which possess deformed $\mathcal{N} = 2M, d = 1$ Poincaré supersymmetry.

### B. Coulomb-like systems

Now, let us construct on the phase space $\mathbb{CP}^{N|\mathcal{M}}$ with the Poisson bracket relations \([16]\), the Coulomb-like system given by the Hamiltonian

$$H_{Coul} = H + \frac{\gamma}{\sqrt{2I}}, \quad (80)$$

where the generators $H, K$ are defined by \([19]\).

The bosonic constants of motion of this system are given by the $u(N - 1)$ symmetry generators $h_{\alpha\beta}$, and by the $N - 1$ additional constants of motion

$$R_\alpha = H_\alpha + r_\gamma \frac{K_\alpha}{\sqrt{2I}} : \quad \{H_{Coul}, R_\alpha\} = \{H_{Coul}, h_{\alpha\beta}\} = 0, \quad (81)$$

where $H_\alpha, K_\alpha, \eta_{\alpha\beta}$ are defined by \([20]\). These generators form the algebra

$$\{R_\alpha, R_\beta\} = -i\delta_{\alpha\beta} \left( H_{Coul} - \frac{r_\gamma^2}{2I^2} \right) + \frac{r_\gamma^2 h_{\alpha\beta}}{2I^3}, \quad \{h_{\alpha\beta}, R_\gamma\} = i\delta_{\alpha\beta} R_\gamma, \quad \{R_\alpha, R_\beta\} = 0. \quad (82)$$

Besides, proposed system has $2M$ fermionic constants of motion given by $\Theta_A\bar{A}$, and $u(M)$ R-symmetry given by $R_{\bar{A}\bar{B}}$. Hence, it is superintegrable in the sense of super-Liouville theorem \([11]\). So, we constructed the maximally superintegrable Coulomb problem with dynamical $SU(1, N|\mathcal{M})$ superconformal symmetry which inherits all symmetries of initial bosonic system.
One can expect, that in analogy with oscillator-like system, our Coulomb-like system would possess (deformed) $\mathcal{N} = 2M$-super-Poincaré symmetry for $M = 2k$ and $\gamma > 1$. However, it is not a case. Indeed, let us choose the following Ansatz for supercharges

$$Q_A = Q_A + \sqrt{2} \gamma C_{AB} \frac{S_B}{(2K)^{3/4}} ,$$

with the constant matrix $C_{AB}$ obeying the conditions (10), $M = 2k$ and $\gamma > 0$.

Calculating their Poisson brackets we find

$$\{ Q_A, \tilde{Q}_B \} = H_{Coul} \delta_{AB} + \frac{3}{2} \frac{\sqrt{2} \gamma}{(2K)^{3/4}} (S_A \tilde{C}_{BD} S_D + \tilde{S}_B C_{AD} \tilde{S}_D) ,$$

$$\{ Q_A, Q_B \} = - \frac{i \sqrt{2} \gamma}{2(2K)^{3/4}} (C_{BD} \mathcal{R}^D_A + C_{AC} \mathcal{R}^C_B) , \quad \{ Q_A, \mathcal{R}^C_B \} = - i Q_B \delta_{AC} ,$$

where $\mathcal{R}^A_B$ is defined in (78).

Further calculating the Poisson brackets of $Q_A$ with the generators appearing in the r.h.s. of the above expressions we get that the superalgebra is not closed. For example,

$$\{ Q_A, H_{Coul} \} = \frac{3 \gamma}{(2K)^{3/2}} S_A + \frac{\sqrt{2} \gamma}{(2K)^{3/4}} C_{AB} \left( \tilde{Q}_B - \frac{3}{4K} \tilde{S}_B D \right) .$$

Hence, proposed supercharges do not yield closed deformation of $\mathcal{N} = 2M$-super-Poincaré algebra.

Let us choose another Ansatz for supercharges (as above we assume that $\gamma > 0$)

$$\tilde{Q}_A = Q_A + i \sqrt{2} \gamma e^{i \chi} \frac{S_A}{(2K)^{3/4}} ,$$

which yields

$$\{ \tilde{Q}_A, \tilde{Q}_B \} = H_{Coul} \delta_{AB} + \frac{\sqrt{2} \gamma}{2(2K)^{3/4}} \mathcal{R}^B_A , \quad \{ \tilde{Q}_A, \tilde{Q}_B \} = 0 , \quad \{ \tilde{Q}_A, \mathcal{R}^C_B \} = i \left( \frac{1}{M} \tilde{Q}_A \delta_{BC} - \tilde{Q}_B \delta_{AC} \right) ,$$

where

$$H_{Coul} = H_{Coul} - \frac{\sqrt{2} \gamma}{(2K)^{3/4}} \left( I - \frac{1}{2M} \sum_C R_{CC} \right) ,$$

with $I$ and $\mathcal{R}^A_B$ are defined, respectively, in (55) and (78). In canonical coordinates (50) this Hamiltonian reads

$$H_{Coul} = \frac{p_r}{2} + \frac{I}{r^2} + \frac{\gamma}{r} \frac{\sqrt{2} \gamma}{r^{3/2}} \left( g + \sum_{\alpha} \pi_{\alpha} + \frac{2M - 1}{2M} (\chi \chi) \right) .$$

However, one can easily check that proposed supercharges do not yield closed deformation of Poincaré superalgebra as well, e.g.

$$\{ \tilde{Q}_A, \frac{\mathcal{R}^C_B}{(2K)^{3/4}} \} = \frac{i}{(2K)^{3/4}} \left( \frac{1}{M} \tilde{Q}_A \delta_{BC} - \tilde{Q}_B \delta_{AC} \right) + \frac{3}{2} \frac{S_A}{(2K)^{3/4}} \mathcal{R}^C_B$$

So, proposed superextensions of Coulomb-like systems, being well-defined from the viewpoint of superintegrability, do not possess neither $\mathcal{N} = 2M$ supersymmetry, no its deformation. The $su(1, N|M)$ superalgebra plays the role of dynamical algebra of that systems.
VI. FUBINI-STUDY-LIKE KÄHLER STRUCTURE

The above considered super-Kähler structure is obviously the higher-dimensional super-analogue of the Klein model of Lobachevsky space. On the other hand, Lobachevsky space has other common parametrization as well, which is known as Poincaré disc [7]. The higher-dimensional generalization of Poincaré disc parameterizing the noncompact complex projective space is quite similar to the Fubini-Study structure for \( CP^N \). It is defined by the Kähler potential

\[
\mathcal{K} = -g \log(1 - \sum_{a=1}^{N} z^a \bar{z}^a)
\]  

(92)

For the obtaining of the super-analogue of this potential from \( CP^{1,N|M} \), one should pass from the matrix \( [\gamma_{ab}] \) to the diagonal matrix \( \gamma_{ab} = \text{diag}(1,-1,\ldots,-1) \). This can be done by the transformation

\[
\begin{align*}
 v^0 &\rightarrow \frac{v^0 + v^N}{\sqrt{2}}, \\
v^N &\rightarrow \frac{v^0 - v^N}{i \sqrt{2}}.
\end{align*}
\]  

(93)

On the reduced phase space \( (93) \) corresponds to the transformation

\[
\begin{align*}
 w &\rightarrow \frac{z^N - 1}{z^N + 1}, \\
z^a &\rightarrow \sqrt{\frac{2}{z^N + 1}} z^a, \\
\theta^A &\rightarrow \sqrt{\frac{2}{z^N + 1}} \bar{\theta}^A.
\end{align*}
\]  

(94)

Thus we will get the Fubini-Study-like Kähler potential

\[
\mathcal{K} = -g \log(1 - z^c \bar{z}^c + i \theta^C \bar{\theta}^C),
\]  

(95)

which defines the following Kähler structure

\[
\Omega = \frac{i}{g} \left[ \left( \frac{g \delta_{ab}}{A} + \frac{z^a \bar{z}^b}{A^2} \right) dz^a \wedge d\bar{z}^b + \frac{\bar{\theta}^A z^a}{A^2} d\theta^A \wedge dz^a - \frac{i z^a \theta^A}{A^2} d\bar{z}^a \wedge d\bar{\theta}^A - \left( \frac{g \delta_{AB}}{A} + \frac{\bar{\theta}^A \theta^B}{A^2} \right) d\theta^A \wedge d\bar{\theta}^B \right],
\]  

(96)

where we have used a similar notation as in \( (15) \)

\[
\tilde{A} := \frac{1 - z^c \bar{z}^c + i \theta^C \bar{\theta}^C}{g}.
\]  

(97)

The respective Poisson brackets read:

\[
\{ z^a, z^b \} = i \tilde{A} (\delta^{ab} - z^a \bar{z}^b),
\]

\[
\{ z^a, \theta^A \} = i \tilde{A} z^a \bar{\theta}^A,
\]

\[
\{ \theta^A, \bar{\theta}^B \} = \tilde{A} (\delta^{AB} + \theta^A \bar{\theta}^B).
\]  

(98)

Now let us introduce the canonical coordinates, but now taking the symplectic/Kähler one form associated with the Kähler potential \( (95) \), i.e. the one that define "Fubini-Study"-like metric. Then, as before, one needs to identify it with the canonical one, and this canonical coordinates will play the role of "Cartesian" coordinates instead of the "spherical" ones discussed above.

\[
\tilde{A} = -\frac{g}{2} \frac{i (z^a dz^a - z^a d\bar{z}^a) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{1 - z^c \bar{z}^c + i \theta^C \bar{\theta}^C} := p_a d\varphi_a + \frac{1}{2} \chi^A d\bar{\chi}^A + \frac{1}{2} \bar{\chi}^A d\chi^A.
\]  

(99)

It leads to the relations

\[
z^a = \sqrt{\frac{p_a}{g + p - i \chi^A \bar{\chi}^A e^{i \varphi_a}}},
\]

\[
\theta^A = \sqrt{\frac{2}{r}} \chi^A, \\
p = \sum_a p_a,
\]  

(100)

or

\[
p_a = \frac{z^a z^a}{\tilde{A}},
\]

\[
\varphi_a = \arg(z^a), \\
\chi^A = \frac{\theta^A}{\sqrt{\tilde{A}}},
\]  

(101)

where \( \tilde{A} \) is defined by \( (97) \).
These coordinates are related with $\bar{\mathbb{CP}}^N$ as follows:

$$p_\alpha = \pi_\alpha, \quad p_N = \frac{1}{4} \left( p_r^2 + \left( \frac{r - \sqrt{2T}}{r} \right)^2 \right), \quad \varphi_N = \arctan \left( \frac{2xy}{(x-y)(x+y)} \right).$$  \hspace{1cm} (102)

where

$$x = 1 - \frac{p_r^2}{r^2} - \frac{2T}{r^4}, \quad y = \frac{p_r}{r},$$  \hspace{1cm} (103)

while $\chi^A$ and $\varphi_\alpha$ remains unchanged after transition from one parameterization to the other.

Finally, let us draw readers attention to the complete similarity of the bosonic part of $\bar{\mathbb{CP}}^N$ with the equations mapping parameterizing compactified Ruijsenaars-Schneider model with excluded centre of mass to the complex projective (phase) space $\mathbb{CP}^N$. This prompt us, at first, to construct the conformal-invariant analogue of that model by replacing the complex projective space by its noncompact analogue $\bar{\mathbb{CP}}^N$. Then one can try to construct its $su(1,N|M)$-superconformal extension by further replacement of $\bar{\mathbb{CP}}^N$ by $\bar{\mathbb{CP}}^{N|M}$.

\section{VII. CONCLUDING REMARKS}

In this paper we suggested to construct the $su(1,N|M)$-superconformal mechanics formulating them on phase superspace given by the non-compact analogue of complex projective superspace $\bar{\mathbb{CP}}^{N|M}$. The $su(1,N|M)$ symmetry generators were defined there as a Killing potentials of $\bar{\mathbb{CP}}^{N|M}$. We parameterized this phase space by the specific coordinates allowing to interpret it as a higher-dimensional super-analogue of the Lobachevsky plane parameterized by lower half-plane (Klein model). Then we transitied to the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables we demonstrated that proposed scheme allows to construct the $su(1,N|M)$ superconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb-like systems with a $su(1,N|M)$ dynamical superalgebra, and found that oscillator-like systems admit deformed $N = 2M$ Poincaré supersymmetry, in contrast with Coulomb-like ones.

In fact, proposed scheme demonstrated the effectiveness of the supersymmetrization via formulation of the initial systems in terms of Kähler phase space and further superextension of the latters. In order to relate considered systems with the conventional ones (with Euclidean configuration spaces), we restricted ourself by the non-compact complex projective superspace. So, we are sure that applying the same approach to the conventional (compact) complex projective spaces we can find many new integrable systems as well and construct their unexpected extended supersymmetric extensions.

Proposed scheme could obviously be extended to the systems on complex Grassmanians (and on their noncompact analogues). In particular, we expect to find, in this way, the $N$-supersymmetric extensions of compactified spin-Ruijsenaars-Schneider models. Moreover, it seems to be straightforward task to apply proposed approach to the systems with generic $U(N)$-invariant Kähler phase spaces locally defined by the Kähler potential $\mathcal{K}(z^a \bar{z}^a)$. We expect that it can be done in terms of Kähler phase superspace locally defined by the potential

$$\bar{\mathcal{K}} = \mathcal{K} \left( z^a \bar{z}^a + i\eta^a \bar{\eta}^A \right).$$  \hspace{1cm} (104)

In this way we expect to construct the $\bar{\mathcal{N}} = 2M$ supersymmetric extensions of the systems with curved (Riemann) configuration space as well, in particular, of the so-called $\kappa$-deformations (i.e. spherical/hyperbolic generalizations) of conformal mechanics, oscillator and Coulomb systems$^{[20, 21]}$.

Finally, notice that considered phase superspace is not associated with external algebra of initial bosonic manifold, and thus, it is not related with the superfield approach. Thus, it is interesting to consider the systems with $(N|kM)$-dimensional Kähler phase superspaces defined by the potentials

$$\bar{\mathcal{K}} = \mathcal{K} (z^a \bar{z}^a) + F \left( kg_{a\bar{b}} \eta^a \bar{\eta}^b \right), \quad F'(0) = \text{const},$$  \hspace{1cm} (105)

and construct, in this way, the $\bar{\mathcal{N}} = kN$ supersymmetric mechanics. Very preliminary attempt in this direction was done in$^{[22]}$ where the $\bar{\mathcal{N}} = 2$ supersymmetric extensions of the systems with generic Kähler phase space was considered. However, this promising direction was not further developed since that time. We plan to consider listed problems elsewhere.
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[1] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Vol. 2* (Wiley Classics Library)[1969].
[2] D. Karabali and V. P. Nair, *Quantum Hall effect in higher dimensions*, Nucl. Phys. B 641 (2002) 533 [hep-th/0203264].
[3] B. P. Dolan and A. Hunter-McCabe, *Ground state wave functions for the quantum Hall effect on a sphere and the Atiyah-Singer index theorem*, J. A.: Math. Theor. 53 (2020)215306 [arXiv:2001.02208 [hep-th]].
[4] S. N. M. Ruijsenaars and H. Schneider, *A New Class of Integrable Systems and Its Relation to Solitons*, Annals Phys. 170 (1986) 370.
[5] S. N. M. Ruijsenaars, *Complete Integrability of Relativistic Calogero-moser Systems and Elliptic Function Identities*, Commun. Math. Phys. 110 (1987) 191; *Action-angle maps and scattering theory for some finitedimensional integrable systems. III. Sutherland type systems and their duals*, Publ.Res.Inst.Math.Sci.Kyoto.31 (1995), 247-353.
[6] J. F. van Diejen, L. Vinet, “ *The Quantum Dynamics of the Compactified Trigonometric Ruijsenaars-Schneider Model*, Commun. Math. Phys. 197(1998), 33-74 [arXiv:math/9709221 [math-ph]].
[7] E. Ivanov and S. Sidorov, *Deformed Supersymmetric Mechanics*, Phys. Rev. D 101 (2020) no.2, 025003 [arXiv:1911.06290 [hep-th]].
[8] E. Ivanov, A. Nersessian, S. Sidorov and H. Shmavonyan, *Noncompact $CP^N$ as a phase space of superintegrable systems*, Int. J. Mod. Phys. A 36 (2021) 2150055 [arXiv:2010.00002 [math-ph]].
[9] O. M. Khudaverdian and A. P. Nersessian, “Even and odd symplectic and Kahlerian structures on projective superspaces,” J. Math. Phys. 34 (1993) 5533 [hep-th/9210091].
[10] E. Khastyan, A. Nersessian and H. Shmavonyan, *Cuboctahedric Higgs oscillator from the Calogero model*, Phys. Lett. B. 748 (2015) no.6, 065052 [arXiv:1503.05622 [hep-th]].
[11] T. Hakobyan, A. Nersessian and V. Yeghikyan, *Super Kähler oscillator from\(SU(2|1)\) superspace*, Phys. Rev. D 91 (2015) no.8, 085032 [arXiv:1501.05622 [hep-th]].
[12] T. Hakobyan, O. Lechtenfeld and A. Nersessian, *SU(2|2) supersymmetric mechanics*, JHEP 11 (2016), 031 [arXiv:1609.00490 [hep-th]].
[13] T. Hakobyan, O. Lechtenfeld and A. Nersessian, *Symmetries of deformed supersymmetric mechanics on Kähler manifolds*, Phys. Rev. D 101 (2020) no.2, 025003 [arXiv:1911.06290 [hep-th]].
[14] E. Ivanov and S. Sidorov, *Deformed Supersymmetric Mechanics*, Class. Quant. Grav. 31 (2014) 075013 [arXiv:1307.7690 [hep-th]]; *Super Kähler oscillator from SU(2|1) superspace*, J. Phys. A 47 (2014), 292002 [arXiv:1312.6821 [hep-th]]; *Long multiplets in supersymmetric mechanics*, Phys. Rev. D 93 (2016) no.6, 065052 [arXiv:1509.05561 [hep-th]].
[15] E. Ivanov, S. Sidorov and F. Toppan, *Superconformal mechanics in SU(2|1) superspace*, Phys. Rev. D 91 (2015) no.8, 085032 [arXiv:1501.05622 [hep-th]].
[16] E. Ivanov, O. Lechtenfeld and S. Sidorov, *Super Kähler oscillator from\(SU(2|1)\) superspace*, JHEP 11 (2016), 031 [arXiv:1609.00490 [hep-th]].
[17] E. Ivanov, O. Lechtenfeld and S. Sidorov, *Symmetries of deformed supersymmetric mechanics on Kähler manifolds*, Phys. Rev. D 101 (2020) no.2, 025003 [arXiv:1911.06290 [hep-th]].
[18] A. Nersessian *Elements of (super-)Hamiltonian Formalism* Lect. Notes Phys. 698 (2006) 139-188
[19] V. N. Shander, *Complete integrability of ordinary differential equations on supermanifolds*, Functional Analysis and Its Applications 17 (1983), 74 Darboux and Liouville theorems on supermanifolds DAN Bulgaria, 36 (1983), 309; O.M. Khudaverdian, A. P. Nersessian, *Formulation of Hamiltonian Mechanics With Even and Odd Poisson Brackets*, Preprint EFI-1031-81-87-YEREVAN, 1987
[20] B. Bina, M. Günaydin,*Real forms of nonlinear superconformal and quasi-superconformal algebras and their unified realization*, Nucl. Phys. B 502 (1997) 713, [hep-th/9703185].
[21] J. Palmkvist, *A realization of the Lie algebra associated to a Kantor triple system*, J. Math. Phys. 47 (2006) 023505, [math/0504544 [math.RA]].
[22] T. Hakobyan, O. Lechtenfeld and A. Nersessian, *Superintegrability of generalized Calogero models with oscillator or Coulomb potential*, Phys. Rev. D 90 (2014) no.10, 101701 [arXiv:1405.8288 [hep-th]].
[23] J. Schwinger, *A Magnetic model of matter Science* 165 (1969) 757
D. Zwanziger, *Exactly Soluble Nonrelativistic Model of Particles with Both Electric and Magnetic Charges*, Phys. Rev. 176 (1968) 1480
[24] I. Fris, V. Mandrosov, Ya. A. Smorodinsky, M. Uhlir and P. Winternitz, *On higher symmetries in quantum mechanics*, Phys. Lett. 16 (1965) 354; P. Winternitz, Ya. A. Smorodinsky, M. Uhlir and I. Fris, *Symmetry groups in classical and quantum mechanics*, Soviet J.
A. A. Makarov, Ya. A. Smorodinsky, Kh. Valiev, and P. Winternitz, *A systematic search for non-relativistic system with dynamical symmetries*, Nuovo Cim. A 52 (1967) 1061.

[17] A. Saghatelian, *Near-horizon dynamics of particle in extreme Reissner-Nordström and Clement-Gal’tssov black hole backgrounds: action-angle variables*, Class. Quant. Grav. 29 (2012) 245018 [arXiv:1205.6270 [hep-th]].

[18] A. Galajinsky, A. Nersessian and A. Saghatelian, *Superintegrable models related to near horizon extremal Myers-Perry black hole in arbitrary dimension*, JHEP 1306 (2013) 002 [arXiv:1303.4901 [hep-th]]; *Action-angle variables for spherical mechanics related to near horizon extremal Myers–Perry black hole*, J. Phys. Conf. Ser. 474 (2013) 012019.

[19] M. Feigin, O. Lechtenfeld and A. P. Polychronakos, *The quantum angular Calogero-Moser model*, JHEP 1307 (2013) 162 [arXiv:1305.5841 [math-ph]].

[20] P. Dombrowski and J. Zitterbarth, *On the planetary motion in the 3-Dim standard spaces of constant curvatur*, Demonstratio Mathematica 24 (1991) 375;
A. Ballesteros, F.J. Herranz, M.A. del Olmo, and M. Santander, *Quantum structure of the motion groups of the two-dimensional Cayley-Klein geometries*, J. Phys. A 26 (1993) 5801;
M.F. Rañada and M. Santander, *Superintegrable systems on the two-dimensional sphere S2 and the hyperbolic plane H2*, J. Math. Phys. 40 (1999) 5026;
M. F. Ranada, *The Tremblay-Turbiner-Winternitz system on spherical and hyperbolic spaces: Superintegrability, curvaturedependent formalism and complex factorization*, J. Phys. A 47 (2014) 165203, [arXiv:1403.6266 [math-ph]].

[21] T. Hakobyan, A. Nersessian and H. Shmavonyan, *Symmetries in superintegrable deformations of oscillator and Coulomb systems: Holomorphic factorization*, Phys. Rev. D 95 (2017) no.2, 025014 [arXiv:1612.00791 [hep-th]].

[22] S. Bellucci and A. Nersessian, *Kahler geometry and SUSY mechanics*, Nucl. Phys. Proc. Suppl. 102 (2001) 227 [hep-th/0103005].