Behavioural investors in conic market models*

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Abstract
We treat a fairly broad class of financial models which includes markets with proportional transaction costs. We consider an investor with cumulative prospect theory preferences and a non-negativity constraint on portfolio wealth. The existence of an optimal strategy is shown in this context in a class of generalized strategies.

1 Introduction
In this paper we continue the investigations of [CR17] where behavioural investors were studied in a model with price impact. In the current work we treat the case of conic models, see [KS09], which subsume foreign exchange markets as well as multi-asset markets with proportional transaction costs.

The mathematical difficulty stems from the fact that behavioural preferences lack concavity and involve probability distortions, see [KT79], [Qui82], [TK92]. Hence, instead of almost sure techniques, we need to employ weak convergence in the arguments. In Theorem 3.2 below we establish the existence of optimizers in a suitable class of generalized strategies. We rely on results of [Jak97], see Theorem 4.1 below.

In Section 2 we present our model. In Section 3 we construct optimal strategies for investment problems with behavioural preferences. Section 4 collects auxiliary material.

2 Conic market model
We will assume throughout the paper that trading takes place continuously in the time interval [0, 1]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, P)$ be a filtered probability space, where the filtration is complete and right-continuous, $\mathcal{F}_0$ is trivial. The notation $EX$ will refer to the expectation of the random variable $X$. If there is ambiguity about the probability measure then $E_Q X$ will denote the expectation of $X$ under the probability $Q$. Similarly, Law$(X)$ denotes the law of $X$ and Law$_Q$(X) refers to its law under $Q$. When $x, y$ are vectors in the same Euclidean space then the concatenation $xy$ denotes their scalar product, $|x|$ is the Euclidean norm.

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In the sequel we will need that the filtration is of a specific type and that the probability space is large enough.

**Assumption 2.1.** There exists a càdlàg $\mathbb{R}^m$-valued process $Y$ with independent increments such that $\mathcal{H}_t$ is the $P$-completion of $\sigma(Y_u, 0 \leq u \leq t)$, for $t \in [0, 1]$.

For $m \in \mathbb{N}$, we denote by $\mathcal{D}^m$ the space of $\mathbb{R}^m$-valued RCLL functions on $[0,1]$ equipped with Skorohod’s topology, see Chapter 3 of [Bil99].

**Remark 2.2.** The Borel-field of $\mathcal{D}^m$ is generated by the coordinate mappings $x \in \mathcal{D}^m \to x(t) \in \mathbb{R}^m$, $t \in [0,1]$, see Theorem 12.5 of [Bil99]. It follows that the function $\omega \in \Omega \to Y(\omega) \in \mathcal{D}^m$ is a random variable and so is $\omega \in \Omega \to Y(\omega) \in \mathcal{D}^m$, for all $t \in [0,1]$, where $Y$ is the process defined as $(Y)_u = Y_1\mathbb{1}_{[0, t)} + Y_1\mathbb{1}_{[t, 1]}$, $u \in [0,1]$. Furthermore, $\mathcal{H}_t = \sigma(Y)$, for all $t \in [0,1]$.

**Assumption 2.3.** There exists a random variable $U$ that is uniformly distributed on $[0,1]$ and independent of $\mathcal{H}_1$.

Let us define the augmented filtration $\mathcal{F}_t := \mathcal{H}_t \vee \sigma(U)$, $t \in [0,1]$. Standard arguments show that $\mathcal{F}_t$, $t \in [0,1]$ also satisfies the usual hypotheses of completeness and right-continuity.

We now recall the market model presented in Subsection 3.6.3 of [KS09]. Let $\xi^k_t$, $t \in [0,1]$ be $\mathcal{H}$-adapted $\mathbb{R}^d$-valued processes for each $k \in \mathbb{N}$ such that, for a.e. $\omega$ and for all $t$, only finitely many terms of the sequence $\xi^k_t(\omega)$, $k \in \mathbb{N}$ differ from 0. Let $G_t(\omega), t \in [0,1], \omega \in \Omega$ denote the polyhedral cone generated by $\xi^k_t(\omega)$, $k \in \mathbb{N}$. We assume that $\mathbb{R}^d_+ \subset G_t$ a.s. for each $t \in [0,1]$. Let the dual cones be defined by $G^*_t(\omega) := \{x \in \mathbb{R}^d : xy \geq 0 \text{ for all } y \in G_t(\omega)\}$. We imagine that $G_t(\omega)$ represents the set of solvent positions in $d$ financial assets at time $t$ in the state of the world $\omega \in \Omega$.

**Assumption 2.4.** There is a family of $\mathcal{H}$-adapted continuous processes $\zeta^k_t$, $t \in [0,1]$, $k \in \mathbb{N}$ such that $G^*_t(\omega)$ is generated by $\zeta^k_t(\omega)$, $k \in \mathbb{N}$ and only finitely many terms of this sequence differ from 0, for a.e. $\omega$ and for every $t$.

Although the dual generators $\zeta^k$, $k \in \mathbb{N}$ are assumed to be continuous processes, the above assumption allows them to depend on a driving process $Y$ with possibly discontinuous paths (consider e.g. a stochastic volatility model with jumps in the volatility).

The following assumption requires that there is efficient friction in the market, see page 158 of [KS09].

**Assumption 2.5.** For each $t \in [0,1]$ and for a.e. $\omega \in \Omega$, $\text{int} G^*_t(\omega) \neq \emptyset$.

Let $\mathcal{D}$ denote the set of $\mathcal{H}$-adapted martingales $Z_t$, $t \in [0,1]$ such that $Z_t \in \text{int} G^*_t$ and $Z_{t-} \in \text{int} G^*_t$ a.s. for each $t \in [0,1]$. The next assumption is essentially condition $\mathcal{B}$ on page 160 of [KS09]. It stipulates that there is a rich enough class of objects in $\mathcal{D}$.

**Assumption 2.6.** Assume that $\mathcal{D}$ is nonempty. For each $s \in [0,1]$, and for each $\mathcal{H}_s$-measurable random variable $\xi$ if $\xi Z_s \geq 0$ for all $Z \in \mathcal{D}$ then $\xi \in G_s$ a.s.

For an $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted càdlàg process $X$ with bounded variation we denote by $\|X\|$ its total variation process (scalar-valued) and let $\dot{X}$ denote the
pathwise Radon-Nykodim derivative of $X$ with respect to $||X||$, this can be chosen to be an $\mathbb{R}^d$-valued process. Let $\mathcal{X}^0$ denote the family of $\mathcal{F}$-adapted processes with bounded variation $X$ such that $X_0 = 0$ and $\dot{X}_t \in -G_t$ a.s. for all $t \in [0, 1]$. These processes represent the evolution of portfolio positions in a self-financing way, starting from initial position 0.

For each integer $k \geq 1$, consider $\mathcal{C}^k$, the space of $\mathbb{R}^k$-valued continuous functions on the unit interval. This is a separable Banach space with the supremum norm. Let $\mathfrak{M}^{2d}$ denote the Banach space of $2d$-tuples of finite signed measures on $\mathcal{B}([0, 1])$. This is the dual space of $\mathcal{C}^d$ with the total variation norm, henceforth denoted by $|| \cdot ||_1$. However, in the sequel we equip $\mathfrak{M}^{2d}$ with the weak-$*$ topology in the natural dual pairing between $\mathcal{C}^d$ and $\mathfrak{M}^{2d}$.

**Remark 2.7.** Let us notice that if $X \in \mathcal{X}^0$ then, for each $\omega \in \Omega$, $X(\omega)$ can be naturally identified with an element of $\mathfrak{M}^{2d}$. Indeed, we may consider

$$\mathfrak{X}^{2j-1}(\omega)(A) := \int_A (\dot{X}_t^{j})^+ d||X||_t(\omega), \ A \in \mathcal{B}([0, 1]), \ j = 1, \ldots, d,$$

and

$$\mathfrak{X}^{2j}(\omega)(A) := \int_A (\dot{X}_t^{j})^- d||X||_t(\omega), \ A \in \mathcal{B}([0, 1]), \ j = 1, \ldots, d.$$

Furthermore, we claim that the mapping $\mathfrak{X} \ : \ \Omega \to \mathfrak{M}^{2d}$ is $\mathcal{F}_t$-measurable. Indeed, it suffices to show that for each continuous $\phi : [0, 1] \to \mathbb{R}^d$, the mapping $\omega \to \int_0^1 \phi(\dot{X}_t^j)^+ d||X||_t(\omega)$ is $\mathcal{F}_t$-measurable for each $j = 1, \ldots, d$ (similarly for $(\dot{X}_t^j)^-$), which is clear since $X$ is càdlàg and adapted. By similar arguments, $\omega \to \mathfrak{X}(\omega)$ is $\mathcal{F}_t$-measurable, for every $t \in [0, 1]$, where $\mathfrak{X}(\omega)(A) := \mathfrak{X}(\omega)(A \cap [0, t])$. We will identify $X$ with $\mathfrak{X}$ in the sequel: when we write $X$ it may refer to either the stochastic process or to the $\mathfrak{M}^{2d}$-valued random variable. A similar identification of $\mathfrak{X}$ with $\mathfrak{T}^\times \mathfrak{X}$ will also be used.

For each initial position $x \in G_0$, we furthermore define $A(x) := \{ X \in \mathcal{X}^0 : x + X_t \in G_t \ \text{a.s. for all} \ t \in [0, 1] \}$, the portfolio value processes which never become insolvent.

**Remark 2.8.** Investment decisions will be based on the augmented filtration $\mathcal{F}$. It is pointed out in [CR15] that by using a uniform $U$ (independent of $\mathcal{F}_1$) for randomizing the strategies an investor can increase her satisfaction, however, further randomizations are pointless. See Remarks 22 and 23 of [CR17] and Section 5 of [CR15] for detailed explanations. Unlike other studies, we assume that the “dual process” $Z$ is $\mathcal{F}$-adapted, since information from $U$ does not weaken market viability.

We fix a function $\ell : \mathcal{D}^m \times \mathbb{R}^d \to \mathbb{R}$ (interpreted as a *liquidation function*) which transfers the terminal portfolio position into cash. We assume that it is continuous. The liquidation value of a position $x \in \mathbb{R}^d$ is $\ell(Y, x)$ (so it depends on the market situation via $Y$).

### 3 Optimal investments

For $z \in \mathbb{R}$ we denote $z^+ := \max\{z, 0\}, z^- := \max\{-z, 0\}$. Let $u_+ : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, increasing functions such that $u_+(0) = 0$. Let $w_+, w_- : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, increasing functions such that $w_-(0) = 0$. Let $u_+, w_-, w_+$ be continuous, increasing functions such that $u_+(0) = 0$, $w_-(0) = 0$, and $w_+(0) = 0$. Let $\omega \in \Omega$ be such that $X_0 = 0$. The investor starts from initial position $x = X_0$.
[0,1] → [0,1] be continuous with \( w_+(0) = 0, w_+(1) = 1 \). Functions \( u_\pm \) express the agent’s attitude towards gains and losses while \( w_\pm \) are functions distorting the probabilities of events, see [TK92, CR15].

We define, for any random variable \( X \geq 0 \),
\[
V_+(X) := \int_0^\infty w_+(P(u_+(X) \geq y)) \, dy,
\]
and
\[
V_-(X) := \int_0^\infty w_-(P(u_-(X) \geq y)) \, dy.
\]

For each real-valued random variable \( X \) with \( V_+(X^+) < \infty \) we set
\[
V(X) := V_+(X^+) - V_-(X^-).
\]

**Assumption 3.1.** The function \( u_+ \) is bounded from above.

Assumption 3.1 could be substantially relaxed at the price of requiring stronger assumptions about \( \mathcal{D} \) but this would significantly complicate the arguments. Let \( W \) be an \( \mathcal{F}_t \)-measurable \( d \)-dimensional random variable representing a reference point for the investor in consideration. Notice that under Assumption 3.1 the functional \( V(\ell(Y, X_1 - W)) \) is well-defined for every \( X \in \mathcal{A}(x) \).

The quantity \( V(\ell(Y, X_1 - W)) \) expresses the satisfaction of an agent with CPT preferences when (s)he has a portfolio process \( X \), see [JZ08, CR15] for more detailed discussions. Positive \( \ell(Y, X_1 - W) \) means outperforming the benchmark \( W \), negative \( \ell(Y, X_1 - W) \) means falling short of it. Doob’s theorem implies that there is a measurable \( h : \mathcal{D}^m \to \mathbb{R}^d \) such that \( W = h(Y) \).

We aim to find an optimal investment strategy, i.e. \( X^\dagger \in \mathcal{A}(x) \) with
\[
V(\ell(Y, X^\dagger_1 - W)) = \sup_{X \in \mathcal{A}(x)} V(\ell(Y, X_1 - W)).
\]

The next theorem is our main result on the existence of optimizers for behavioural investors in conic models.

**Theorem 3.2.** Let Assumptions 2.1, 2.3, 2.4, 2.5, 2.6 and 3.1 be valid. Fix \( x \in G_0 \). There exists \( X^\dagger \in \mathcal{A}(x) \) such that
\[
V(\ell(Y, X^\dagger_1 - W)) = \sup_{X \in \mathcal{A}(x)} V(\ell(Y, X_1 - W)).
\]

**Remark 3.3.** Let \( u : \mathbb{R}^d \to \mathbb{R} \) be continuous and bounded from above. The arguments in the proof below can also establish that there is \( X^\dagger \in \mathcal{A}(x) \) such that
\[
E[u(X^\dagger_1)] = \sup_{X \in \mathcal{A}(x)} E[u(X_1)].
\]

**Proof of Theorem 3.2.** Let \( X(n) \in \mathcal{A}(x), n \in \mathbb{N} \) be such that
\[
V(\ell(Y, X_1(n) - W)) \to \sup_{X \in \mathcal{A}(x)} V(\ell(Y, X_1 - W)), \, n \to \infty.
\]

Applying Lemma 3.6.4 of [KS09] to the set \( \{X(n), n \in \mathbb{N}\} \) with the choice \( \kappa := \|x\|_1 \), there exists a probability measure \( Q \sim P \) such that \( \sup_{n \in \mathbb{N}} E_Q[|X(n)|]^1 < \)
Let \( c_n, n \in \mathbb{N} \) be an arbitrary sequence of positive real numbers converging to 0. Letting \( \varepsilon > 0 \), the Markov inequality yields

\[
\lim_{n \to \infty} Q(c_n ||X(n)||_1) \geq \varepsilon \leq \lim_{n \to \infty} c_n E_Q( ||X(n)||_1 ) / \varepsilon = 0.
\]

In other words, \( c_n ||X(n)||_1 \) converges to 0 in \( Q \)-probability and hence in \( P \)-probability as well by the equivalence of \( Q \) and \( P \). Lemma 3.9 of [Kall02] shows that the sequence of \( \mathbb{R} \)-valued random variables \( ||X(n)||_1 \), \( n \in \mathbb{N} \) is tight.

For any \( r > 0 \), the set \( \{ m \in \mathbb{M}^{2d} : ||m||_1 \leq r \} \) is weak*-compact by the Banach-Alaoglu theorem hence the \( \mathbb{M}^{2d} \)-valued sequence \( X(n) \) is tight. So is the sequence \( (X(n), Y) \). Applying Theorem 4.1, there exist a probability space \( (O, \mathcal{O}, R) \) and \( \mathbb{M}^{2d} \times \mathbb{D}^m \)-valued random variables \( (\tilde{X}(n), Y(n)) \) that converge \( R \)-a.s. to \((X^*, Y^*) \) along a subsequence (for which we keep the same notation) and \( \text{Law}_R(\tilde{X}(n), Y(n)) = \text{Law}(X(n), Y), n \in \mathbb{N} \). By subtracting a further subsequence we may and will also assume that

\[
\tilde{X}_1(n) \to X^*_1 \text{ in law as } n \to \infty.
\]

For each \( k \in \mathbb{N} \), let \( f_k : \mathbb{D}^m \to \mathbb{C}^d \) be such that \( \zeta^k = f_k(Y) \). Such functions exist by Doob’s lemma. Passing to a further subsequence through a diagonal argument, we may and will assume that, for each \( k \in \mathbb{N} \), \( \zeta^k(n) := f_k(Y(n)) \to \zeta^k \) \( R \)-a.s. in \( \mathbb{C}^d \) when \( n \to \infty \) by Lemma 4.4 and by the fact that each \( Y(n) \) has the same law (on \( \mathbb{D}^m \)). Analogously, we may and will assume \( W(n) := h(Y(n)) \to W^* := h(Y^*) \) \( R \)-a.s. in \( \mathbb{R}^d \).

Let us define the analogue of the functionals \( V^*, V \) for non-negative random variables \( X \) on \((O, \mathcal{O}, R)\).

\[
V^+_R(X) := \int_0^\infty w_+(R(u_+(X) \geq y)) \, dy,
\]

and

\[
V^-_R(X) := \int_0^\infty w_-(R(u_-(X) \geq y)) \, dy.
\]

For each real-valued random variable \( X \) on \((O, \mathcal{O}, R)\) with \( V^+_R(X^+) \leq \infty \) we set

\[
V_R(X) := V^+_R(X^+) - V^-_R(X^-).
\]

Assumption 3.1 and the reverse Fatou lemma imply that

\[
V_R(\ell(Y^*, X^*_1 - W^*)) \geq \limsup_n V_R(\ell(Y(n), X_1(n) - W(n))), \tag{2}
\]

so \( V_R(\ell(Y^*, X^*_1 - W^*)) \geq \sup_{X \in A(\varepsilon)} V(\ell(Y, X_1 - W)) \).

Let us invoke Lemma 4.5 with the choice \( \phi := X^* \), \( \tilde{H} := Y^* \) and \( H := Y \). We get a \( \mathcal{F}_t \)-measurable random element \( X^t := \phi \in \mathbb{M}^{2d} \) satisfying \( \text{Law}(X^t, Y) = \text{Law}_R(X^*, Y^*) \). Let us fix \( 0 \leq t < u \leq 1 \). We recall that \( X(n) \) is independent from \( Y_u - Y_t \), or equivalently,

\[
\text{Law}(\ell(X(n), Y_u - Y_t)) = \text{Law}(\ell(X(n)) \otimes \text{Law}(Y_u - Y_t)).
\]

\[^1\text{In } [KS09], Z \text{ and } X \text{ are adapted to the same filtration } \mathcal{F}. \text{ Here, we allow } X \text{ to be a } \mathcal{F} \text{-adapted process but this causes no problem.}\]
By construction, \( \text{Law}(t^X(n), Y_u - Y_t) = \text{Law}_R(t^\tilde{X}(n), Y_u(n) - Y_t(n)) \). This implies also

\[
\text{Law}_R(t^\tilde{X}(n), Y_u(n) - Y_t(n)) = \text{Law}_R(t^\tilde{X}(n)) \otimes \text{Law}_R(Y_u(n) - Y_t(n)).
\]

Passing to the limit as \( n \to \infty \),

\[
\text{Law}_R(t^X^*, Y_u^* - Y_t^*) = \text{Law}_R(t^X^*) \otimes \text{Law}_R(Y_u^* - Y_t^*),
\]

which implies independence of \( t^X \in \mathbb{R}^{2d} \) from \( tY \in \mathcal{D}^m \) as well where \( (tY)_s := 0 \) if \( 0 \leq s \leq t \) and \( (tY)_s := Y_s - Y_t, t < s \leq 1 \).

Since \( Y \) is clearly a measurable function of \( \{Y^t, Y\} \in \mathcal{D}^m \times \mathcal{D}^m \), applying Lemma 4.1 with the choice \( b := t, Y \) and \( a := (U, \bar{t}) \) we get that \( t^X \) is \( \mathcal{F}_t \)-measurable, for all \( t \).

The set \( \mathcal{L} := \{Z_1 : Z \in \mathcal{D}\} \) is a subset of the separable metric space \( L^1(P) \) hence it is also separable. Let \( \{Z_1^k, k \in \mathbb{N}\} \) be a countable dense subset of \( \mathcal{L} \). For each \( k \in \mathbb{N} \), there exist measurable functions \( g_{k,s} : \mathcal{D} \to \mathbb{R}^d \) such that \( E[Z_1^k] = g_{k,s}(Y) \). Let \( \xi \) be an \( \mathcal{F}_t \)-measurable random variable. By the density of the family \( \{Z_1^k, k \in \mathbb{N}\} \) and Assumption 2.6, if \( g_{k,s}(Y) \geq 0 \) a.s. for each \( k \) then \( \xi \in \mathcal{G}_s \) a.s. Indeed, let \( Z \) be an arbitrary element of \( \mathcal{D} \) and \( Z_1^k, n \in \mathbb{N} \) be in the dense subset such that \( Z_{1}^{k_n} \to Z_1 \) in \( L^1(P) \), and hence, \( E[Z_1^{k_n}] \to E[Z_1] \) in \( L^1(P) \) as well. One can extract a subsequence \( k_{n_l}, l \in \mathbb{N} \) along which almost sure convergence holds, i.e. \( g_{k_{n_l},s}(Y) \to Z_s, \) P-a.s. Therefore, the fact \( \xi g_{k_{n_l},s}(Y) \geq 0 \) a.s. for each \( l \) implies \( \xi Z_s \geq 0 \) a.s. and then \( \xi \in \mathcal{G}_s \) a.s. by Assumption 2.6.

Fix \( k \in \mathbb{N} \) for a moment. Since \( X_s(n) \in \mathcal{G}_s \), obviously \( X_s(n)g_{k,s}(Y) \geq 0 \) P-a.s. for each \( n \in \mathbb{N} \). Hence, we obtain \( \tilde{X}_s(n)g_{k,s}(Y(n)) \geq 0 \), R-a.s. for all \( n \).

By construction, \( \tilde{X}(n) \) tends to \( X^* \) R-a.s. in \( \mathbb{R}^{2d} \) (equipped with the weak* topology). Moreover, from the properties of weak convergence of probabilities on \( \mathbb{R} \) we know that, for \( R \)-a.e. \( \omega \), \( \lim_{n \to \infty} \tilde{X}_s(n)(\omega) = X^*_s(\omega) \) for every \( s \in [0, 1] \setminus I(\omega) \) where \( I(\omega) \) is a countable set. Fubini’s theorem then implies that there is a fixed set \( T \) of Lebesgue measure 0 such that for \( s \notin T \), \( \lim_{n \to \infty} \tilde{X}_s(n) = X^*_s \) R-a.e. By \( 1 \) we may assume that \( 1 \notin T \).

An application of Lemma 4.1 gives \( X^*_s g_{k,s}(Y^*) \geq 0 \), R-a.s. for every \( s \in [0, 1] \setminus T \). Notice that \( X^*_s = j(U, Y) \) for some \( j : [0, 1] \times \mathcal{D}^m \to \mathbb{R} = \mathcal{B}[0, 1] \otimes \mathcal{G}_s \)-measurable where \( \mathcal{G}_s \) is generated by the coordinate mappings of \( \mathcal{D}^m \) up to \( s \).

This means that for \( B := \cap_{k \in \mathbb{N}} \{((u, y) : j(u, y)g_{k,s}(y) \geq 0) \} \) we have \( \text{Leb} \times \text{Law}(Y))(B) = 1 \). But then, for \( \text{Leb}(Y) \)-a.e. \( u \), for \( \text{Law}(Y) \)-a.e. \( y \),

\[
j(u, y)g_{k,s}(y) \geq 0, \quad k \in \mathbb{N},
\]

which implies \( j(u, Y)Z_s^k \geq 0 \) a.s. for Leb-a.e. \( u \) and for each \( k \in \mathbb{N} \). Noting that \( j(u, Y) \) is \( \mathcal{F}_t \)-measurable, Assumption 2.6 gives \( j(u, Y) \in \mathcal{G}_s \), for Leb-a.e. \( u \). This means \( X^*_s \in \mathcal{G}_s \) a.s.

Fix now some \( t \in T \) and let \( s_n, n \in \mathbb{N} \) be a sequence in \( [0, 1] \setminus T \) such that \( s_n \downarrow t \). Right-continuity implies that \( X^*_s \xi^k \to \lim_{n \to \infty} X^*_s \xi^k \geq 0 \). We thus conclude that \( X^*_s \in \mathcal{G}_s \) a.s. for all \( s \in [0, 1] \).

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To prove $\dot{X}_t^\dagger \in -G_t$, it suffices to show that the integrals $\int_0^t \zeta^k_u dX_u(n) \leq 0$, $P$-a.s. for any $0 \leq s < t \leq 1$. Lemma 4.7 gives us

$$\int_s^t \zeta^k_u dX_u(n) \leq 0, \quad \text{R-a.s.}$$

Again, the facts that $\dot{X}(n)$ tends to $X^*$ $R$-a.s. in $\mathbb{M}^{2d}$ and $\zeta^k(n) := f_k(Y(n))$ tends to $\zeta^k := f_k(Y^*)$ $R$-a.s. in $\mathbb{C}^{2d}$ imply

$$\int_s^t \zeta^k_u^\dagger dX^*_u \leq 0, \quad \text{R-a.s.}$$

Thus,

$$\int_s^t \zeta^k_u dX^*_u \leq 0, \quad \text{R-a.s.}$$

that is, $\int_0^t \zeta^k_u dX^*_u$ is non-increasing.

The previous arguments show $X^\dagger \in A(x)$. As $\text{Law}(X^\dagger, Y) = \text{Law}(X^*, Y^*)$, $\text{Law}_R(X^\dagger - W^*) = \text{Law}(X^*_1 - W)$, and [2] shows that $X^\dagger$ is the maximizer we have been looking for. \qed

4 Auxiliary results

We denote by $\mathcal{B}(Z)$ the Borel-field of a topological space $Z$. A sequence of probabilities $\mu_k$, $k \in \mathbb{N}$ on $\mathcal{B}(Z)$ is said to be tight if, for all $\varepsilon > 0$, there is a compact set $K(\varepsilon) \subset Z$ such that, for all $k$, $\mu_k(Z \setminus K(\varepsilon)) < \varepsilon$. Take $Z := \mathbb{M}^{2d} \times \mathbb{D}^m$.

**Theorem 4.1.** Let $\mu_k$, $k \in \mathbb{N}$ be a tight sequence of measures on $\mathcal{B}(Z)$. Then there is a subsequence $k_j$, $j \in \mathbb{N}$ and a probability space on which there exist $Z$-valued random variables $\xi_j$, with $\text{Law}(\xi_j) = \mu_{k_j}$, $j \in \mathbb{N}$ and $\xi_j \to \xi$ a.s., $j \to \infty$.

**Proof.** This follows as in Corollary 3 and Example 5 of [CR17], using results of [Jak97]. \qed

**Remark 4.2.** Note that the space $Z$ is not metrizable so the well-known versions of Skorohod’s representation theorem (see e.g. Lemma 4.30 in [Kal02]) are not applicable.

**Lemma 4.3.** Let $(A, \mathcal{A})$, $(B, \mathcal{B})$ be measurable spaces and $j : A \times B \to \mathbb{R}$ a measurable mapping. Let $(a, b)$ be an $A \times B$-valued random variable. If $\sigma(j(a, b), a)$ is independent of $b$ then $j(a, b)$ is $\sigma(a)$-measurable.

**Proof.** See Lemma 29 of [CR17]. \qed
We also recall Théorème 1 of [BÉK+98].

**Lemma 4.4.** Let $A, B$ be separable metric spaces and $\xi_n \in A$, $n \in \mathbb{N}$ a sequence of random variables converging to $\xi \in A$ in probability such that $\text{Law}(\xi_n)$ is the same for all $n$. Then for each measurable $h : A \to B$ the random variables $h(\xi_n)$ converge to $h(\xi)$ in probability (hence also a.s. along a subsequence). \hfill $\square$

**Lemma 4.5.** Let $B$ be a measurable space. Let $H, \tilde{H}$ be random elements in $B$ with identical laws, defined on the probability spaces $(\Xi, \mathcal{E}, \mathbb{R})$, $(\tilde{\Xi}, \tilde{\mathcal{E}}, \tilde{\mathbb{R}})$, respectively. Let $\tilde{\phi}$ be a random element in $Z$, defined on $(\tilde{\Xi}, \tilde{\mathcal{E}}, \tilde{\mathbb{R}})$. Let $U$ be independent of $H$ with uniform law on $[0, 1]$. There exists a measurable function $f : B \times [0, 1] \to Z$ such that $\phi = f(H, U)$ satisfies $\text{Law}_R(H, \phi) = \text{Law}_R(H, \tilde{\phi})$.

**Proof.** Notice that the topological space $Z$ is the union of its closed, increasing subspaces $A_n$, $n \in \mathbb{N}$ which are Polish spaces (with appropriate metrics). Now use Lemma 31 of [CR17]. \hfill $\square$

We give a criterion of admissibility for $\dot{X}$.

**Lemma 4.6.** A $\mathcal{F}$-adapted process $X$ of bounded variation satisfying $\dot{X}_t \in -G_t$ for all $t \in [0, 1]$ if and only if the integrals $\int_0^t \xi^k dX_t$ are non-increasing, for all $k \in \mathbb{N}$.

**Proof.** Identical to the proof of Lemma 3.6.1 of [KS09]. \hfill $\square$

**Lemma 4.7.** Let $Y, \tilde{Y}$ be càdlàg processes, $X, \tilde{X}$ bounded variation processes defined on two probability spaces $(\Xi, \mathcal{E}, \mathbb{R})$, $(\tilde{\Xi}, \tilde{\mathcal{E}}, \tilde{\mathbb{R}})$, respectively. Assume that $(\tilde{Y}, \tilde{X})$ has the same law as $(Y, X)$. Let $f : \mathbb{D}^m \to \mathbb{C}^d$ be measurable. Then for all $0 \leq s < t \leq 1$, it holds that

$$\text{Law}_R \left( \int_s^t f(\tilde{Y})_u \, d\tilde{X}_u \right) = \text{Law}_R \left( \int_s^t f(Y)_u \, dX_u \right).$$  \hfill (3)

**Proof.** We approximate $f$ by step functions and then pass to the limit. \hfill $\square$

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