On m-quasi-ideals in m-regular ordered semigroups

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Abstract

In this paper we characterize left(right) ideals, bi-ideals and quasi-ideals of an ordered semigroup by an index $m$ and give some important interplays between these ideals. The concept of $m$-regularity of an ordered semigroups has been introduced. Moreover $m$-regular ordered semigroups are characterized by their $m$-quasi-ideals and the fact that for any $m$-regular ordered semigroups $A$, the set $Q_A$ of all $m$-quasi-ideals of $A$, with multiplication defined by: $Q_1 \circ Q_2 = (Q_1Q_2)$, for all $Q_1, Q_2 \in Q_A$, is a $m$-regular semigroup is obtained here.

Keywords and Phrases: $m$-left ideal, $m$-right ideal, $m$-bi-ideal, $m$-quasi-ideal, $m$-left simple, $m$-right simple, $m$-regular.

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1 Introduction and preliminaries

An ordered semigroup is a partially ordered set $(A, \leq)$, and at the same time a semigroup $(A, \cdot)$ such that for all $a, b, x \in A$; $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. It is denoted by $(A, \cdot, \leq)$. Throughout this paper we consider an ordered semigroup $A$ with $\{1\}$. For an ordered semigroup $A$ and $H \subseteq S$, denote $(H)_A := \{t \in A : t \leq h, \text{ for some } h \in H\}$. Also it is denoted by $(H)$ if there is no scope of confusion. Let $I$ be a nonempty subset of an ordered semigroup $A$. $I$ is a left (right) ideal [2] of $S$, if $AI \subseteq I$ ($IA \subseteq I$) and $(I) = I$ and $I$ is an ideal of $A$ if it is both a left and a right ideal of $A$. $A$ is left (right) simple [3] if it has no non-trivial proper left (right) ideal. For an ordered semigroup $A$ and a subsemigroup $S$ of $A$, $S^m = SSSS \cdots S$($m$-times) where $m$ is a positive integer. Its easy to check that for any subsemigroup $S$ of an ordered semigroup $A$, $S^n \subseteq S$ for all positive integer $n$. Similarly $S^r \subseteq S^t$, for all positive integers $r$ and $t$, such that $r \geq t$ but the converse is not true.
Lemma 1.1. [4] Let $T$ be an ordered semigroup and $A$ and $B$ be subsets of $T$. Then the following statements hold:
1. $A \subseteq (A]$.
2. $([(A)] = (A]$.
3. If $A \subseteq B$ then $(A] \subseteq (B]$.
4. $(A \cap B] \subseteq (A] \cap (B]$.
5. $(A \cup B] = (A] \cup (B]$.
6. $(A]((B] \subseteq (AB]$.
7. $(((A]((B]] = (AB]$.

2 $m$-left/right ideal

Definition 2.1. Let $A$ be an ordered semigroup. A subsemigroup $L$ of $A$ is called $m$-left ideal of $A$ if $A^m L \subseteq L$ and $(L] = L$, where $m$ is a positive integer not necessarily 1.

Similarly, a subsemigroup $R$ of $A$ is called $m$-right ideal of $A$ if $RA^m \subseteq R$ and $(R] = R$, where $m$ is a positive integer. A subsemigroup $I$ of $A$ is called an $m$-two sided ideal or simply an $m$-ideal of $A$ if it is both $m$-left ideal and $m$-right ideal of $A$.

Proposition 2.2. Let $A$ be an ordered semigroup. Then following assertions hold:
1. Every left(right) ideal is an $m$-left(right) ideal of $A$.
2. Every $m$-left(right) ideal is an $n$-left(right) ideal of $A$, for all $n \geq m$.
3. Intersection of $m$-left ideals of $A$ (If non-empty) is an $m$-left ideal of $A$.
4. Intersection of $m$-right ideals of $A$ (If non-empty) is an $m$-right ideal of $A$.

Proof. (1): Let $L$ be a left ideal of $A$ then $A^m L \subseteq AL \subseteq L$ and $(L] = L$. Hence $L$ is a $m$-left ideal of $A$.

Similarly every right ideal of $A$ is an $m$-right ideal of $A$.

(2): First consider $L$ be an $m$-left ideal of $A$, then $A^m L \subseteq L$ and $(L] = L$. Now for $n \geq m; A^n L \subseteq A^m L \subseteq L$ and $(L] = L$. Hence $L$ is an $n$-left ideal of $A$.

Likewise, every $m$-right ideal of $A$ is an $n$-right ideal of $A$.

(3): Let $\{L_\lambda : \lambda \in \Lambda\}$ be a family of $m$-left ideals of an ordered semigroup $A$. Let $L = \cap_{\lambda \in \Lambda} L_\lambda$ is a subsemigroup of $A$ being intersection of a family of subsemigroups of $A$. Now we have, $A^{m_\lambda} L_\lambda \subseteq L_\lambda$, for all $\lambda \in \Lambda$ and $(L_\lambda] = L_\lambda$. Now $L \subseteq L_\lambda$, for all $\lambda \in \Lambda$. Let $m = \max\{m_\lambda : \lambda \in \Lambda\}$. Hence $A^m L \subseteq A^{m_\lambda} L_\lambda \subseteq L_\lambda$, for all $\lambda \in \Lambda$. Therefore $A^m L \subseteq \cap_{\lambda \in \Lambda} L_\lambda = L$ implies $A^m L \subseteq L$. Now $(L] = (\cap_{\lambda \in \Lambda} L_\lambda] \subseteq \cap_{\lambda \in \Lambda} (L_\lambda] = \cap_{\lambda \in \Lambda} L_\lambda = L$ implies $(L] = L$. Hence $L$ is a $m$-left ideal of $A$.

(4):
Analogously.
The converse of the above statement is incorrect. This is verified by the examples.

**Example 2.3.** Let \( A = \{x, y, z, w\} \) be an ordered semigroup with the multiplication \( \cdot \) and the order relation defined by

\[
\begin{array}{c|cccc}
\cdot & x & y & z & w \\
\hline
x & w & z & w & w \\
y & z & w & w & w \\
z & w & w & w & w \\
w & w & w & w & w \\
\end{array}
\]

\( \leq \) = \{(x, x), (y, y), (z, z), (w, x), (x, y), (x, z), (w, w)\}.

Let \( L = \{x, w\} \). For integer \( m > 1 \), we obtain that \( L \) is an \( m \)-left ideal, \( m \)-right ideal of \( A \) but not a left(right) ideal of \( A \).

Let \( K \) be a subsemigroup of an ordered semigroup \( A \) and \( \mathcal{L} = \{L : L \text{ is an } m \text{-left ideal of } A \text{ containing } K\} \). Therefore \( \mathcal{L} \) is a non-empty because \( A \in \mathcal{L} \). Let \( (K)_{m-l} = \cap_{L \in \mathcal{L}} L \). Hence by Theorem 2.2 \( (K)_{m-l} \) is an \( m \)-left ideal of an ordered semigroup \( A \). Moreover we can easily check that \( (K)_{m-l} \) is the smallest \( m \)-left ideal of \( A \) containing \( K \). The \( m \)-left ideal \( (K)_{m-l} \) is called principal \( m \)-left ideal of \( A \) generated by \( K \).

**Theorem 2.4.** Let \( K \) be a subsemigroup of an ordered semigroup \( A \). Then

1. \( m \)-left ideal generated by \( K \) is defined by \( (K)_{m-l} = (K \cup A^m K) \).
2. \( m \)-right ideal generated by \( K \) is defined by \( (K)_{m-r} = (K \cup KA^m) \).

**Proof.** (1): Suppose \( (K)_{m-l} = (K \cup A^m K) \). We must explain that \( (K)_{m-l} \) is the minimal \( m \)-left ideal of \( A \) which contains \( K \). Now \( (K \cup A^m K)(K \cup A^m K) \subseteq (KK \cup KA^m K \cup A^m KK \cup A^m KA^m K) = (KK) \cup (KA^m K) \cup (A^m KK) \cup (A^m KA^m K) \subseteq (K) \cup (AA^m K) \cup (A^m AK) \cup (A^m AA^m K) \subseteq (K) \cup (A^m+K) \cup (A^m+1K) \subseteq (K) \cup (A^m K) = (K \cup A^m K) \), using Lemma \[.\] Hence \( (K)_{m-l} \) is a subsemigroup of \( A \). Next we have to show that \( A^m(K)_{m-l} \subseteq (K)_{m-l} \). Suppose \( A^m(K)_{m-l} = A^m(K \cup A^m K) \subseteq A^m K \cup A^{2m} K \subseteq (A^m K) \subseteq (K \cup A^m K) \). Hence \( K \cup A^m K \) is an \( m \)-left ideal of \( A \). Now we need to show that \( (K)_{m-l} \) is the minimal \( m \)-left ideal of \( A \) containing \( K \). Consider \( K' \) be any \( m \)-left ideal of \( A \) containing \( K \). Now \( (K \cup A^m K) \subseteq (K' \cup A^m K') \subseteq (K' \cup K) = (K') = K' \). Evidently \( (K)_{m-l} \subseteq K' \). Hence \( (K)_{m-l} \) is the minimal \( m \)-left ideal containing \( K \).

(2): Similar As previous.

\[\square\]

**Corollary 2.5.** Let \( A \) be an ordered semigroup. If \( x \in A \), the \( m \)-left ideal generated by \( x \) denoted by \( (x)_{m-l} \) and defined by \( (x)_{m-l} = (x \cup A^m x) \).
Definition 2.6. Let $A$ be an ordered semigroup. An $m$-left ideal of $A$ is called principal $m$-left ideal of $A$ if it is generated by a single element of $A$.

Theorem 2.7. In an ordered semigroup $A$, the following hold:

1. For a subsemigroup $K$ of $A$, $(K)_{m-l} \subseteq (K)_l$.
2. For any element $a \in A$, $(a)_{m-l} \subseteq (a)_l$.

Proof. (1): Since for any positive integer $m$, $(K \cup A^mK) \subseteq (K \cup AK)$. Hence $(K)_{m-l} \subseteq (K)_l$.

(2): Analogously.

Definition 2.8. Let $m$ be non-negative integer. An ordered semigroup $(A, \cdot, \leq)$ is said to be $m$-left-simple ($m$-right-simple) if it does not contain any proper non trivial $m$-left($m$-right) ideal.

Lemma 2.9. Let $A$ be an ordered semigroup and $m$ be any non-negative integer. The following statements hold:

1. $A$ is $m$-left-simple if and only if $A = (A^m x]$ for all $x \in A$.
2. $A$ is $m$-right-simple if and only if $A = (xA^m]$ for all $x \in A$.

Proof. (1): Assume that $A$ is $m$-left simple and let $x \in A$. Now $A^m(A^m x] \subseteq (A^{2m} x] \subseteq (A^m x]$. Hence $(A^m x]$ is an $m$-left ideal of $A$. Hence $A = (A^m x]$, by assumption.

Conversely, assume that $A = (A^m x]$ for all $x \in A$. Let $K$ be any $m$-left ideal of $A$ then $A^m K \subseteq K$ and $(K] = K$. Let $x \in K \subseteq A$ then $A = (A^m x] \subseteq (A^m K] \subseteq (K] = K$. Hence $A$ is $m$-left simple.

(2): This can be proved similarly.

Corollary 2.10. Let $A$ be an ordered semigroup. The following statements hold for an ordered semigroup:

1. $A$ is left-simple if and only if $A = (As]$ for all $s \in A$.
2. $A$ is right-simple if and only if $A = (sA]$ for all $s \in A$.

3 m-bi-ideal

Definition 3.1. Let $A$ be an ordered semigroup and $B$ be subsemigroup of $A$, then $B$ is called $m$-bi-ideal of $A$ if $BA^m B \subseteq B$ and $(B] = B$, where $m \geq 1$ is a positive integer, called bipotency of bi-ideal $B$.

Theorem 3.2. Every bi-ideal in an ordered semigroup $A$ is $m$-bi-ideal for any $m \geq 1$. 
Proof. Let $B$ be a bi-ideal of $A$, then by definition $BAB \subseteq B$ that is $BA^1B \subseteq B$ and $(B) = B$ which implies $B$ is an m-bi-ideal of $A$. 

But the converse is not true.

**Example 3.3.** Consider an ordered semigroup $A = \{x, y, z, w\}$ with the multiplication ‘.’ and the order relation defined by

$$
\begin{array}{c|cccc}
. & x & y & z & w \\
\hline 
x & x & x & x & x \\
y & x & x & x & z \\
z & x & x & w & y \\
w & x & z & y & x \\
\end{array}
$$

\[ \leq = \{(x, x), (x, y), (y, y), (z, z), (w, w)\} \]

Then $\{x, w\}$ is an m-bi-ideal of $A$ for $m > 1$ but not a bi-ideal of $A$.

**Proposition 3.4.** Let $B_1, B_2$ be two m-bi-ideals with bipotencies $m_1, m_2$ respectively of an ordered semigroup $A$. The product of any $m_1$-bi-ideal and $m_2$-bi-ideal of $A$ is a $\max(m_1, m_2)$-bi-ideal of $S$. The product defined by $B_1 \ast B_2 = (B_1B_2)$.

Proof. Let $B_1$ and $B_2$ be two m-bi-ideals of an ordered semigroup $A$ with bipotencies $m_1, m_2$ respectively then $B_1A^mB_1 \subseteq B_1$, $(B_1) = B_1$ and $B_2A^{m2}B_2 \subseteq B_2$, $(B_2) = B_2$. $(B_1B_2)$ is obviously a non-empty subset of $A$. Now $(B_1B_2)^2 \subseteq (B_1B_2)(B_1B_2) \subseteq (B_1B_2B_1B_2) \subseteq (B_1AB_1B_2) \subseteq (B_1A \cdot 1 \cdot 1 \cdot 1 \cdot 1B_1B_2) \subseteq (B_1A^mB_1B_2) \subseteq (B_1B_2)$ thus $(B_1B_2)$ is a subsemigroup of $A$. Now we have $(B_1B_2)A^{\max(m_1, m_2)}(B_1B_2) \subseteq (B_1B_2A^{\max(m_1, m_2)}(B_1B_2) \subseteq (B_1A^{1+m}\max(m_1, m_2)(B_1B_2) \subseteq (B_1A^{m_1}B_1B_2) \subseteq (B_1B_2)$, using $\prod$. Again $((B_1B_2)) = (B_1B_2)$. Hence $B_1 \ast B_2$ is an m-bi-ideal with bipotency $\max\{m_1, m_2\}$.

**Proposition 3.5.** Let $A$ be an ordered semigroup and $X$ be an arbitrary subset of $A$ and $B$ be an m-bi-ideal of $A$, $m$ is not necessarily 1. Then the $(BX)$ is also an m-bi-ideal of $A$.

Proof. Clearly $(BX)$ is a non-empty subset of $A$. Now $(BX)^2 \subseteq (BX)(BX) \subseteq (BXBX) \subseteq (BABX) \subseteq (BA \cdot 1 \cdot 1 \cdot 1BX) \subseteq (BA^mBX) \subseteq (BX)$. Now $(BX)A^{m}(BX) \subseteq (BXA^{m}BX) \subseteq (BA^mBX) \subseteq (BA^{1+m}BX) \subseteq (BA^mBX) \subseteq (BX)$. Hence $(BX)$ is an m-bi-ideal of $A$ with bipotency $m$.

Similarly we can show that $(XB)$ is an m-bi-ideal of $A$. 

5
Theorem 3.6. The intersection of a family of m-bi-ideals of an ordered semigroup \( A \) with bipotencies \( t_1, t_2, \ldots \) is also an m-bi-ideal with bipotency \( \max\{t_1, t_2, \ldots\} \). (Provided the intersection is non-empty.)

Proof. Let \( \{B_{\alpha} : \alpha \in \Delta\} \) be a family of m-bi-ideals of an ordered semigroup \( A \) with bipotencies \( \{t_{\alpha} : \alpha \in \Delta\} \). Now we have to show that \( B = \cap_{\alpha \in \Delta} B_{\alpha} \) is again an m-bi-ideal of \( A \) with bipotency \( \max\{t_{\alpha} : \alpha \in \Delta\} \). Now \( B = \cap_{\alpha \in \Delta} B_{\alpha} \) is also a subsemigroup of \( A \) being intersection of subsemigroups of \( A \). Since each \( B_{\alpha} \) is an m-bi-ideal with bipotencies \( t_{\alpha} \) for all \( \alpha \in \Delta \) then \( B_{\alpha} \subseteq B_{\alpha} \) and \( \cap_{\alpha \in \Delta} B_{\alpha} \subseteq \cap_{\alpha \in \Delta} B_{\alpha} \subseteq B_{\alpha} \). Thus \( B = B \). Hence \( \cap_{\alpha \in \Delta} B_{\alpha} \) is an m-bi-ideal of \( A \) with bipotency \( \max\{t_{\alpha} : \alpha \in \Delta\} \). \( \Box \)

4 m-quasi-ideal

Definition 4.1. A subsemigroup \( Q \) of an ordered semigroup \( A \) is called m-quasi-ideal of \( A \) if \( (A^mQ) \cap (QA^m) \subseteq Q \) and \( (Q) = Q \), where \( m \geq 1 \), called quasipotency of quasi-ideal \( Q \).

Theorem 4.2. For any \( m \geq 1 \), a quasi-ideal is an m-quasi-ideal.

Proof. Let \( Q \) be a quasi-ideal of \( A \), then \( (QA^m) \cap (A^mQ) \subseteq (QA) \cap (AQ) \subseteq Q \) and \( (Q) = Q \). Hence \( Q \) is an m-quasi-ideal of \( A \). \( \Box \)

Corollary 4.3. Every m-quasi-ideal of an ordered semigroup is an n-quasi-ideal for all \( n \geq m \).

Proof. Let \( Q \) be an m-quasi-ideal of an ordered semigroup \( A \) then \( (QA^n) \cap (A^nQ) \subseteq (QA^m) \cap (A^mQ) \subseteq Q \) that is \( (QA^n) \cap (A^nQ) \subseteq Q \) and \( (Q) = Q \). \( Q \) is an n-quasi-ideal of \( A \). \( \Box \)

Theorem 4.4. Every m-left/m-right ideal and hence every m-ideal of an ordered semigroup \( A \) is an m-quasi-ideal of \( A \).

Proof. Let \( L \) be an m-left ideal of an ordered semigroup \( A \). Now \( (A^mL) \cap (LA^m) \subseteq (A^mL) \subseteq (L) = L \) infers that \( (A^mL) \cap (LA^m) \subseteq L \) and \( (L) = L \). Hence \( L \) is an m-quasi-ideal of \( A \). \( \Box \)

Hence From the previous example 2.3 we can see that \( \{x, w\} \) is an m-quasi-ideal but not an quasi-ideal of \( A \).

Theorem 4.5. The intersection of a family of m-quasi-ideals of an ordered semigroup \( A \) with quasipotencies \( t_1, t_2, \ldots \) is also an m-quasi-ideal with quasipotency \( \max\{t_1, t_2, \ldots\} \). (Provided the intersection is non-empty.)
Proof. Let \( \{Q_\lambda : \lambda \in I\} \) be a family of \( m \)-quasi-ideals of an ordered semigroup \( A \) with quasipotencies \( \{t_\lambda : \lambda \in I\} \) then \((A^m Q_\lambda) \cap (Q_\lambda A^m) \subseteq Q_\lambda \) and \((Q_\lambda) = Q_\lambda \) for all \( \lambda \in I \). Let \( t = \text{max}\{t_\lambda : \lambda \in I\} \) and \( Q = \cap_{\lambda \in I} Q_\lambda \) and \( Q \) is a subsemigroup of \( A \) being intersection of subsemigroups of \( A \). Now \((A^t Q) \cap (Q A^t) \subseteq (A^t Q_\lambda) \cap (Q_\lambda A^t) \subseteq Q_\lambda \) for all \( \lambda \in I \). Thus \((A^t Q_\lambda) \cap (Q_\lambda A^t) \subseteq Q \) and \((Q) = (\cap_{\lambda \in I} Q_\lambda) \subseteq \cap_{\lambda \in I} (Q_\lambda) = \cap_{\lambda \in I} Q_\lambda = Q \). Hence \( \cap_{\lambda \in I} Q_\lambda \) is an \( m \)-quasi-ideal with quasipotency \( \text{max}\{t_\lambda : \lambda \in I\} \). \( \square \)

**Lemma 4.6.** Let \( A \) be an ordered semigroup and \( L \) be an \( m \)-left ideal and \( R \) be an \( m \)-right ideal of \( A \). Then \( R \cap L \) is an \( m \)-quasi-ideal of \( A \).

**Proof.** Since \( RL \subseteq RS \subseteq R \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot S \subseteq RS^m \subseteq R \) and similarly \( RL \subseteq L \) infers that \( RL \subseteq R \cap L \) thus \( R \cap L \neq \emptyset \). Now \((R \cap L) \subseteq (R) \cap (L) \subseteq R \cap L \) hence \((R \cap L) = R \cap L \). Then by Theorem 4.4 follows that \( R \cap L \) is an \( m \)-quasi-ideal of \( A \). \( \square \)

**Definition 4.7.** A subsemigroup \( P \) of an ordered semigroup \( A \) has the \( m \)-intersection property if \( P \) is intersection of an \( m \)-left ideal and an \( m \)-right ideal of \( A \).

**Theorem 4.8.** An \( m \)-quasi-ideal \( Q \) of an ordered semigroup \( A \) has the \( m \)-intersection property if and only if \((Q \cup A^m Q) \cap (Q \cup QA^m) = Q \).

**Proof.** First consider \( Q \) has the \( m \)-intersection property. Its evident that \( Q \subseteq (Q \cup A^m Q) \) and \( Q \subseteq (Q \cup QA^m) \) infers that \( Q \subseteq (Q \cup A^m Q) \cap (Q \cup QA^m) \). Since \( Q \) has the \( m \)-intersection property then there exists an \( m \)-left ideal \( L \) and \( m \)-right ideal \( R \) such that \( Q = L \cap R \). Thus \( Q \subseteq L \) and \( Q \subseteq R \) so that \((A^m Q) \subseteq (A^m L) \subseteq (L) = L \) and \((QA^m) \subseteq (RA^m) \subseteq (R) = R \). Now \((Q \cup A^m Q) \subseteq Q \cup (A^m Q) \subseteq L \) and \((Q \cup QA^m) \subseteq Q \cup (QA^m) \subseteq R \). Hence \((Q \cup A^m Q) \cap (Q \cup QA^m) \subseteq L \cap R \). Hence \( Q = (Q \cup A^m Q) \cap (Q \cup QA^m) \).

Conversely, Suppose \( Q = (Q \cup A^m Q) \cap (Q \cup QA^m) \). Since it is clear that from Theorem 2.4 \((Q \cup A^m Q) \) and \((Q \cup QA^m) \) are \( m \)-left and \( m \)-right ideal respectively. Hence \( Q \) has the \( m \)-intersection property. \( \square \)

**Lemma 4.9.** Let \( Q \) be an \( m \)-quasi-ideal of an ordered semigroup \( A \) then \( Q = L(Q) \cap R(Q) = (Q \cup A^m Q) \cap (Q \cup QA^m) \).

**Proof.** The inclusion \( Q \subseteq (Q \cup A^m Q) \cap (Q \cup QA^m) \) is always true.

Conversely, let \( a \in (A^m Q) \cap (Q \cup QA^m) \) then \( a \leq w \) or \( a \leq xb \) and \( a \leq cy \) for some \( x, y \in A^m \), \( w, b, c \in Q \). Since \( Q \) is \( m \)-quasi-ideal then \( a \leq w \) infers that \( a \in (Q) = Q \). Again \( a \leq xb \) and \( a \leq cy \) implies that \( a \in (A^m Q) \cap (QA^m) \subseteq Q \). Thus \((Q \cup A^m Q) \cap (Q \cup QA^m) = Q \). \( \square \)
Let $B$ be a non-empty subset of an ordered semigroup $A$. We denote the least m-quasi-ideal of containing $B$ by $Q(B)$. If $B = b$, we denote $Q(\{b\})$ by $Q(b)$.

**Corollary 4.10.** Let $A$ be an ordered semigroup. Then

1. For every $x \in A$, $Q(x) = L(x) \cap R(x) = (x \cup A^m x) \cap (x \cup x A^m)$
2. For every $\emptyset \neq X \subseteq A$, $Q(X) = L(X) \cap R(X) = (X \cup A^m X) \cap (X \cup X A^m)$.

**Proof.** (1): Let $x \in A$, Since $x \in L(x) \cap R(x)$, hence non-empty then by 4.6. $L(x) \cap R(x)$ is an m-quasi-ideal infers that $Q(x) \subseteq L(x) \cap R(x)$. Again by Theorem 2.4, $L(x) \cap R(x) = (x \cup A^m x) \cap (x \cup x A^m) \subseteq (Q(x) \cup A^m Q(x)) \cap (Q(x) \cup Q(x) A^m) = Q(x)$. Thus $Q(x) = L(x) \cap R(x)$.

(2): It can be proved easily. \qed

It is simple to verify this every m-left and m-right ideal of an ordered semigroup $A$ have the intersection property.

Let $L$ be an m-left ideal of an ordered semigroup $A$. By Theorem 4.4, it is obvious that $L$ is an m-quasi ideal of $A$. Now $(L \cup A^m L) \cap (L \cup LA^m) = ([L] \cup (A^m L)) \cap ([L] \cup (LA^m)) = \{L \cup (A^m L)\} \cap \{L \cup (LA^m)\} = L \cup ((A^m L) \cap (LA^m)) \subseteq L \cup L = L$. Again $L \subseteq L \cup ((A^m L) \cap (LA^m)) = \{L \cup (A^m L)\} \cap \{L \cup (LA^m)\} = ([L] \cup (A^m L)) \cap ([L] \cup (LA^m)) = (L \cup A^m L) \cap (L \cup LA^m)$. Hence by previous result $[8]$, $L$ has the m-intersection property. Likewise we can show that every m-right ideal has the m-intersection property.

**Theorem 4.11.** For m-quasi-ideal $Q$ of an ordered semigroup $A$ if $A^m Q \subseteq QA^m$ or $QA^m \subseteq A^m Q$ then $Q$ has the m-intersection property.

**Proof.** Without loss of generality, consider that $A^m Q \subseteq QA^m$ then $A^m Q = A^m Q \cap QA^m \subseteq (A^m Q) \cap (QA^m) \subseteq Q$ infers that $A^m Q \subseteq Q$ that is $Q$ is an m-left ideal of $A$. Thus $Q$ has the m-intersection property. \qed

**Theorem 4.12.** Every m-quasi-ideal of an ordered semigroup $A$ is its m-bi-ideal.

**Proof.** Suppose that $Q$ be an m-quasi-ideal of an ordered semigroup $A$ then $QA^m Q \subseteq QA^m A \cap AA^m Q \subseteq QA^{m+1} \cap A^{m+1} Q \subseteq QA^m \cap A^m Q \subseteq (QA^m) \cap (A^m Q) \subseteq Q$ and $[Q] = Q$. $Q$ is an m-bi-ideal of $A$. \qed

**Definition 4.13.** An element $a$ of an ordered semigroup $A$ is called m-regular if $a \leq axa$, for some $x \in A^m$. An ordered semigroup $A$ is m-regular if every element of $A$ is m-regular that is m-regular if $a \in \{aA^m a\}$, for all $a \in A$.

Every regular ordered semigroup is an m-regular ordered semigroup but the converse is not true.
Lemma 4.14. Let $A$ be an $m$-regular ordered semigroup and $T$ be any non-empty subset of $A$. Then

1. $X \subseteq (T^m)$, for any positive integer $m$.
2. $X \subseteq (A^mT)$, for any positive integer $m$.

Proof. (1): Let $a \in T$ then $a \leq axa$ for some $x \in A^m$, Since $A$ is m-regular. Hence $a \in (aA^mA) \subseteq (T^mT) \subseteq (T^mT) \subseteq (T^m+1) \subseteq (T^m)$ infers that $T \subseteq (T^m)$.

(2): Similarly we can prove that $T \subseteq (A^mT)$.

Theorem 4.15. The following conditions for an ordered semigroup $A$ are equivalent:

1. $A$ is m-regular.
2. For every $m$-right ideal $R$ and $m$-left ideal $L$ of $A$, $(RL) = R \cap L$.
3. For every $m$-right ideal $R$ and $m$-left ideal $L$ of $A$,
   
   (a) $(R^2) = R$.
   (b) $(L^2) = L$
   (c) $(RL)$ is a m-quasi-ideal of $A$.
4. The set of $m$-quasi ideals $Q_A$ of $A$ with the multiplication $\circ$ defined on it by $Q_1 \circ Q_2 = (Q_1Q_2)$, for all $Q_1, Q_2 \in Q_A$, is a m-regular semigroup.
5. Every $m$-quasi-ideal $Q$ has the form $(QA^mQ) = Q$.

Proof. (1) $\Rightarrow$ (2): First consider that $A$ is an m-regular ordered semigroup. Let $R$ and $L$ be $m$-right ideal and $m$-left ideal of $A$ respectively. Now by Lemma 4.14 $(RL) \subseteq (A^mRL) \subseteq (A^mAL) \subseteq (A^{m+1}L) \subseteq (A^mL) \subseteq (L) = L$. Similarly $(RL) \subseteq R$. Hence $(RL) \subseteq R \cap L$. Again let $x \in R \cap L$. Since $A$ is m-regular then $x \leq xyx$ for some $y \in A^m$ which implies that $x \in (xA^mx) \subseteq (RA^mL) \subseteq (RL)$. Therefore $R \cap L = (RL)$.

(2) $\Rightarrow$ (3): First assume that $(RL) = R \cap L$. Now by Lemma 4.6 $(RL)$ is an m-quasi-ideal of $A$. Now for ordered semigroup $A$, then according to Theorem 2.4 m-left ideal generated by $R$ is $(R \cup A^mR)$. Now $R = R \cap (R \cup A^mR) = (R(R \cup A^mR))$ which infers that $(R^2) \subseteq (R(R \cup A^mR)) = R$. Conversely, let $p \in (R(R \cup A^mR))$. Then $p \leq r_1x$, for some $r_1 \in R$ and $x \in (R \cup A^mR)$. Now $x \in (R \cup A^mR) = (R) \cup (A^mR) = R \cup (A^mR)$ which implies that $x \leq r_2$ for some $r_2 \in R$ or $x \leq sr_3$ where $s \in A^m$, $r_3 \in R$. Hence $p \leq r_1sr_3 \in (RA^mR) \subseteq (R^2)$. So $p \in (R^2)$. Thus $R \subseteq (R^2)$ that is $(R^2) = R$.

Similarly we can show that $(L^2) = L$.

(3) $\Rightarrow$ (4): Suppose $Q_A$ be the set of all $m$-quasi-ideals of $A$. Now for any $Q \in Q_A$, $(Q \cup A^mQ)$ is the $m$-left ideal of $A$ generated by $Q$. Now by condition 3(b), we have $Q \subseteq (Q \cup A^mQ) = ((Q \cup A^mQ)^2) \subseteq ((Q \cup A^mQ)(Q \cup A^mQ)) \subseteq (Q^2 \cup A^mQ^2 \cup QA^mQ \cup (A^mQ)^2) = (Q^2) \cup (A^mQQ) \cup (QA^mQ) \cup ((A^mQ)(A^mQ)) \subseteq (Q^2) \cup (A^mQQ) \cup (QA^mQ) \cup ((A^mQ)(A^mQ)) \subseteq (Q^2) \cup (A^mQ) \subseteq (A.1.1.1...$
·1·Q∪(A^mQ) ⊆ (A^mQ). Similarly we can prove that Q ⊆ (QA^m). Hence Q ⊆ (A^mQ∩(QA^m). Since Q is an m-quasi-ideal of A then (A^mQ∩(QA^m) ⊆ Q which follows that Q = (A^mQ∩(QA^m).

(i)

Now, from condition (3)(c), it follows that (RL) = ((RL)AQ) ∩ (A^m(RL)].

(ii) for every m-left ideal L and m-right ideal R of A. Now we have to prove that the product Q_1 ◦ Q_2 = (Q_1Q_2) of two m-quasi-ideals Q_1 and Q_2 is also an m-quasi ideal of S. (Q_1Q_2)^2 = (Q_1Q_2)(Q_1Q_2) ⊆ (Q_1AQ_2)Q_2 ⊆ (Q_1A · 1 · 1 · · · · · Q_1Q_2) ⊆ (Q_1 · A · A · A · · · · QAQ_1Q_2) ⊆ ((Q_1A^mQ_1)Q_2) ⊆ (Q_1Q_2), by Theorem 4.12. Hence (Q_1Q_2)^2 = (Q_1Q_2) that is (Q_1Q_2) is a subsemigroup of A. Now (A^mQ_1Q_2) is an m-left ideal. Using result (3)(a) and (3)(b),

(A^mQ_1Q_2) = ((A^mQ_1Q_2)[A^mQ_1Q_2]) ⊆ ((A^mQ_1Q_2)[A^mQ_1Q_2]^2) ⊆ (A^mQ_1Q_2A^mA^mQ_1Q_2) ⊆ (A^mQ_1Q_2A^mA^mQ_1Q_2 ⊆ (A^mQ_1Q_2A^mA^mQ_1Q_2) ⊇ (A^mQ_1Q_2A^mA^mQ_1Q_2)

Q_1Q_2 ⊆ (A^m((Q_1Q_2)^m)[(A^mQ_1Q_2)]).

Similarly, (Q_1Q_2A^m) ⊆ (((Q_1Q_2A^m)[Q_1Q_2A^m])A^m].

Now (A^m(Q_1Q_2)]∪((Q_1Q_2)^m[A^m] ⊆ (A^mQ_1Q_2)∩(Q_1Q_2A^m] = (Q_1Q_2A^m]∩(A^mQ_1Q_2] ⊆ (((Q_1Q_2A^m][Q_1Q_2A^m]

(A^m((Q_1Q_2A^m][A^mQ_1Q_2)]) ⊆ ((Q_1Q_2A^m][A^mQ_1Q_2]) ⊆ (Q_1Q_2A^mA^mA^mQ_1Q_2] ⊆ (Q_1AA^mA^mA^mQ_1Q_2] ⊆ (Q_1A^mQ_1Q_2] ⊆ (Q_1A^mQ_1Q_2] ⊆ (Q_1Q_2). Hence (Q_1Q_2] is an m-quasi-ideal of A. Since the multiplication defined in Q_A is associative, so (Q_A, ◦) is a semigroup. Now we have to show that (Q_A, ◦) is regular. Using the conditions (3)(a) and (3)(b), we have A^2 = A and

(A^mQ] = (A^mQ)^2 = ((A^mQ][A^mQ] ⊆ (A^mQ][A^mQ] ⊆ (A^mQA^mQ] ⊆ (A^mQ[A^mQ]. Similarly (QA^m] ⊆ ((QA^m][A^mQ]A^m]. therefore from relations (i), (ii) it follows that Q = (A^mQ] ∩ (QA^m] ⊆ (A^m(QA^m]

(A^mQ] ∩ ((QA^m][A^mQ]A^m] ⊆ ((QA^m][A^mQ]) ⊆ (QA^m] ⊆ (QA^m] Q = Q. Hence Q = (QA^m] = Q ◦ A^m ◦ Q. This means that (Q_A, ◦) is a regular semigroup.

(4) ⇒ (5): Let Q be an m-quasi-ideal of A. By the assumption (4), there exists an m-quasi-ideal P of A such that Q = Q ◦ P^m ◦ Q = (QP^mQ] ⊆ (QA^mQ] ⊆ (QA^m] ∩ (A^mQ] ⊆ Q which infers that Q = (QA^mQ].

(5) ⇒ (1): Let x ∈ A and L(x) and R(x) be the principal m-left and principal m-right ideal of A generated by x. Now Lemma 4.16 infers that R(x) ∩ L(x) is an m-quasi-ideal of A. Then by condition (5), we have x ∈ R(x) ∩ L(x) = ((R(x) ∩ L(x))A^m(R(x) ∩ L(x))] ⊆ (R(x)A^mL(x)] ⊆ ((x ∪ xA^m]A^m(x ∪ A^mx]) ⊆ (xA^m]. Thus A is m-regular.

Lemma 4.16. Every two sided m-ideal J of an m-regular ordered semigroup A is an m-regular subsemigroup of A.
Proof. By Theorem 4.4, \( J \) is an \( m \)-quasi-ideal of \( A \). Let \( a \in J \). Since \( A \) is \( m \)-regular, then there exists \( x \in A^m \) such that \( a \leq axa \leq a(xax)a \). Since \( xax \in A^mJA^m \subseteq J \). Hence we have \( a \in (aJa) \). Thus \( a \) is regular in \( J \).

**Theorem 4.17.** Let \( A \) be an \( m \)-regular ordered semigroup, then the following assertions hold:

1. Every \( m \)-quasi ideal \( Q \) of \( A \) can be written in the form \( Q = R \cap L = (RL) \), where \( R \) is \( m \)-right ideal and \( L \) is \( m \)-left ideal of \( A \).
2. If \( Q \) is \( m \)-quasi-ideal of \( A \), then \( (Q^2) = (Q^3) \).
3. Every \( m \)-bi-ideal of \( A \) is a \( m \)-quasi-ideal of \( A \).
4. Every \( m \)-bi-ideal of any two-sided ideal of \( A \) is an \( m \)-quasi-ideal of \( A \).

Proof. (1): Since \( Q \) is an \( m \)-quasi ideal of an \( m \)-regular ordered semigroup \( A \). Now by Lemma 4.9 and Theorem 4.15, it follows that \( Q = R \cap L = (RL) \).

(2): \((Q^3) \subseteq (Q^2)\) always hold, we have to show that \((Q^2) \subseteq (Q^3)\). By previous theorem 4.15 \((Q^2)\) is an \( m \)-quasi-ideal of \( A \) and in-addition, \((Q^2) = (Q^2A^mQ^2) = (QQA^mQQ) = (Q(QA^m)Q) \subseteq (QQQ) = (Q^3)\) which implies that \((Q^2) \subseteq (Q^3)\). Hence \((Q^2) = (Q^3)\).

(3): Let \( B \) be an \( m \)-bi-ideal of \( A \) then \((A^mB)\) is an \( m \)-left ideal and \((BA^m)\) is an \( m \)-right ideal of \( A \). By Theorem 4.15 \((BA^m) \cap (A^mB) = (BA^mA^mB) \subseteq (BA^2mB) \subseteq (BA^mB) \subseteq (B) = B \). Hence \( B \) is an \( m \)-quasi-ideal of \( A \).

(4): Let \( I \) be two sided \( m \)-ideal of \( A \), and \( B \) be an \( m \)-bi-ideal of \( I \). According to Lemma 4.16 \( I \) is an \( m \)-regular subsemigroup of \( A \). By previous property (3) \( B \) is \( m \)-quasi-ideal of \( I \). Now \( BA^mB \subseteq BA^mI \) and \( BA^mB \subseteq IA^mB \), so \( BA^mB \subseteq BA^mI \cap IA^mB \subseteq BI \cap IB \subseteq (BI) \cap (IB) \subseteq B \) that is \( BA^mB \subseteq B \). Hence \( B \) is an \( m \)-bi-ideal of \( A \). Therefore by previous condition (3), \( B \) is an \( m \)-quasi-ideal of \( A \).

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