A NOTE ON THE SUPERSINGULAR K3 SURFACE OF ARTIN INVARIANT 1

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Abstract. We prove that the supersingular K3 surface of Artin invariant 1 in characteristic $p$ (where $p$ denotes an arbitrary prime) admits a model over $\mathbb{F}_p$ with Picard number 21.

1. Introduction

This note concerns an explicit problem about the supersingular K3 surface $X$ of Artin invariant 1 in characteristic $p$ where $p$ denotes an arbitrary prime. Over $\overline{\mathbb{F}}_p$ this surface is unique by [7] (see [9] for characteristic 2). Here we prove the existence of a model over $\mathbb{F}_p$ with ideal properties:

**Theorem 1.1.** The K3 surface $X$ admits a model over $\mathbb{F}_p$ with Picard number 21.

The Picard number is the rank of the Néron-Severi group $\text{NS}(X)$ consisting of divisors up to algebraic equivalence (or, in the context of K3 surfaces, linear or numerical equivalence). The above model attains the geometric Picard number 22 over the quadratic extension $\mathbb{F}_p^2$. We remark that this maximum cannot be attained over $\mathbb{F}_p$, or in fact over any finite field $\mathbb{F}_{p^e}$ with $e$ odd, by [1, (6.8)] (see also [10, Thm. 4.4]). In this sense, Theorem 1.1 represents the ideal situation over $\mathbb{F}_p$.

This note is organised as follows. The next section gives a motivation for the problem – in fact a solution for most characteristics. Then Section 3 gives a general proof that builds on Shioda–Inose structures and a sandwich picture developed by Shioda in [14], [15] and extended in [4].

2. Motivation: singular K3 surfaces

The problem of Theorem 1.1 arose from discussions with H. Ohashi who considered a very specific elliptic fibration on the supersingular K3 surface $X$ of Artin invariant 1 in characteristic 11 in [8]. The author’s initial idea was that the fact that $X$ has a model with NS fully defined over $\mathbb{F}_{11^2}$ might simplify some arguments in [8]. Subsequently this led to the general statement of Theorem 1.1.
A basic approach to see the claim in characteristic 11 is reduction from characteristic zero (see Example 2.2). Indeed this provides a convenient way to produce supersingular K3 surfaces. Here we use as input singular K3 surfaces, i.e. complex K3 surfaces whose Picard number attains Lefschetz’ bound of $h^{1,1} = 20$. These can be classified completely in terms of their transcendental lattices by [16]. In particular, each one is defined over some number field. However, over all number fields of degree not exceeding some given bound, there are only finitely many singular K3 surfaces up to $\mathbb{Q}$-isomorphism (see [11]). Note the following subtlety which is in contrast to the result of Theorem 1.1: there are singular K3 surfaces over arbitrarily large number fields whose moduli point is defined over $\mathbb{Q}$. By this rough statement we mean that the surface does not admit a model over any smaller field, yet it is $\mathbb{Q}$-isomorphic to all its Galois-conjugates. In comparison, Theorem 1.1 states that the $\mathbb{F}_p$-moduli point formed by the supersingular K3 surface $X$ corresponds indeed to a K3 surface over the prime field $\mathbb{F}_p$.

Specifically consider singular K3 surfaces over $\mathbb{Q}$ with all of the Néron-Severi group defined over $\mathbb{Q}$ as well. By [1] there are 13 such surfaces up to $\mathbb{Q}$-isomorphism. These are in 1-to-1 correspondence with elliptic curves over $\mathbb{Q}$ with complex multiplication. Such a singular K3 surface $X$ gives rise to the discriminant $d < 0$ of the Néron-Severi group, and thus to an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$.

Lemma 2.1. The reduction of $X$ modulo some good prime $p \nmid d$ is supersingular if and only if $p$ is inert in $K/\mathbb{Q}$.

Example 2.2. The supersingular K3 surface with Artin invariant 1 in characteristic 11 (as studied in [8]) arises from the singular K3 surfaces with discriminant $-3$ or $-4$ by reduction.

Lemma 2.1 can be seen geometrically by way of Shioda-Inose structures as we will exploit in Section 3. Alternatively one can argue with modularity after Livné [6]. Since the corresponding Hecke eigenform has weight 3 and nebentypus character $\chi$ associated to $K$, one finds the characteristic polynomial of Frobenius on $H^2_{\text{ét}}(X \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)$ as

$$P(X, T) = (T - p)^{20}(T^2 - a_p T + \chi(p)p^2) = (T - p)^{21}(T + p).$$

Here first equality holds generally with $a_p$ denoting the eigenvalues of the eigenform and the 20-fold factor $(T - p)$ coming from NS in characteristic zero. Meanwhile the second equality depends on the choice of an inert prime. Since the Tate conjecture is known for elliptic K3 surfaces with section by [2], and every singular K3 surface has such a fibration (as exploited in Section 3) we infer that the reduction of $X$ is supersingular; more precisely we find that $X$ has Picard number 21 over $\mathbb{F}_p$ and 22 over $\mathbb{F}_p^2$. It remains to compute the Artin invariant of the supersingular reduction $X$. For this we refer to a result of Shimada [13, Proposition 1.0.1]:

**Proposition 2.3.** The reduction of a singular K3 surface modulo a supersingular prime has Artin invariant 1.
In conclusion, the supersingular K3 surfaces $X$ derived as above do exactly fit with Theorem 1.1. It is only the primes which split in each imaginary quadratic field of class number one where the above construction fails to produce the required models. In order to cover these primes ($2^{-9}$th of all primes, the first one being 15073) we will employ a geometric argument along the lines of [4] in the next section.

3. Geometric approach

Another prototype of (supersingular) K3 surfaces are Kummer surfaces. Here we start with an abelian variety $A$, quotient by the involution and desingularise to obtain a K3 surface $Km$. If our initial abelian variety is a product of two elliptic curves $E, E'$, then $Km$ is supersingular if and only if both $E$ and $E'$ are. However, these Kummer surfaces are not sufficient for our purpose of proving Theorem 1.1. Namely they inherit too big a Galois action on NS from the abelian variety. Possibly this can also stem from the 2-torsion points of $E$ and $E'$, but always from the cohomology of $A$ since the characteristic polynomial of Frobenius on $\wedge^2 H^{2}_{\text{ét}}(A \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell) \subset H^{2}_{\text{ét}}(A \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)$ implies $\rho(E \times E'/\mathbb{F}_p) = 4$ so that the natural model of the Kummer surface has $\rho(Km(E \times E')/\mathbb{F}_p) \leq 20$.

It is instructive to note the parallel that complex Kummer surfaces do also not suffice to treat all singular K3 surfaces. Continuing the analogy, we will invoke the concept of Shioda–Inose structures in order to give a complete proof of Theorem 1.1.

3.1. Shioda–Inose and sandwich structure. Over $\mathbb{C}$, the notion of Shioda–Inose structure refers to a pair of an abelian surface $A$ and a K3 surface $X$ (not necessarily singular) with the same transcendental lattice such that $X$ admits a rational map of degree two to $Km(A)$:

$$A \underset{\text{Km}(A)}{\longrightarrow} X$$

In [10], Shioda–Inose prove that any singular K3 surface $X$ fits into such a structure and can thus be described in terms of products of isogenous CM-elliptic curves. In [14] Shioda extends this construction to show that $X$ is in fact sandwiched by $Km(A)$. In particular this implies that $X$ and $Km(A)$ have the same geometric Picard number regardless of the characteristic (a fact that would follow over $\mathbb{C}$ from a notion of isogeny due to Inose [5]).

All these constructions are exhibited in terms of explicit algebraic equations (which we give below), so they directly apply to characteristic $p$. Here we only have to take extra care of characteristics 2 and 3 where the elliptic fibrations involved degenerate. However, for those characteristics, explicit models guaranteeing Theorem 1.1 have been exhibited, for instance, in [10] and [12], so we shall omit them in the sequel.
The next diagram gives a brief schematic sketch of some of the elliptic fibrations in the product case:

\[ \begin{array}{ccc}
A & \xrightarrow{\pi_1} & \text{Km}(E \times E') \\
\downarrow & & \downarrow \\
E & \text{Km}(E \times E') & \xrightarrow{\pi_X} \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\pi_0 & & \mathbb{P}^1
\end{array} \]

We continue by explaining how these fibrations arise over some field \( k \) of characteristic different from 2. Throughout we work with Weierstrass models

\[ E : y^2 = f(x), \quad E' : y^2 = g(x) \]

with cubic polynomials \( f, g \in k[x] \). Then \( \text{Km}(E \times E') \) admits a birational model

\[ \text{Km}(E \times E') : f(t)y^2 = g(x). \]

In terms of this model the elliptic fibrations are as follows:

1. The elliptic fibration \( \pi_0 \) is given by projection onto the projective \( t \)-line. This fibration is always isotrivial with four singular fibres of type \( I^*_0 \) in Kodaira’s notation.

2. The elliptic fibration \( \pi_1 \) is given in terms of (2) by projection onto the \( y \)-line. If \( E \not\sim E' \), then this fibration has only two reducible fibres, both of Kodaira type \( IV^* \).

3. Finally \( X \) arises from \( \text{Km}(E \times E') \) as the quotient by the involution which composes \( y \mapsto -y \) with the hyperelliptic involution of fibration \( \pi_1 \) (once a zero section is chosen). Clearly this induces an elliptic fibration \( \pi_X \) on \( X \) which has two type \( II^* \) fibres if \( E \not\sim E' \). (This is often referred to as Inose’s fibration.)

The next step consists in specialising to the situation where \( E \cong E' \). Unless \( j(E) = 0, 12^3 \), the fibration \( \pi_X \) attains exactly one additional reducible fibre which has type \( I_2 \). This fibre is duplicated on the fibration \( \pi_1 \).

Since the two special \( j \)-invariants \( j(E) = 0, 12^3 \) can be understood completely by reducing the corresponding singular K3 surfaces over \( \mathbb{Q} \) (of discriminant \(-3, -4\)), we shall exclude these cases in the sequel without further mention.

3.2. Proof of Theorem 1.1 From now on, we fix the prime \( p \). The overall idea is to pick some supersingular elliptic curve \( E \) over \( \mathbb{F}_p \) and consider the K3 surface \( X \) arising from the Shioda–Inose structure for \( E \times E \). Such a curve is characterised by its trace being zero, or equivalently \( \#E(\mathbb{F}_p) = p + 1 \). Thus its existence over \( \mathbb{F}_p \) follows, for instance, from Honda’s theorem which states that there exists an elliptic curve over \( \mathbb{F}_p \) with any trace \( a \in \mathbb{Z}, |a| \leq 2\sqrt{p} \).

The elliptic fibration \( \pi_X \) thus obtained has Mordell-Weil rank 3 over \( \overline{\mathbb{F}}_p \). It seems feasible to apply abstract lattice theoretic arguments along the lines of...
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§3], combined with lifting to a singular K3 surface, to analyse the possible Galois action on MWL($X, \pi_X$) and produce a quadratic twist, if necessary, with MW-rank 2 over $\mathbb{F}_p$. However, we decided to pursue a direct geometric approach.

A detailed analysis of the Mordell-Weil groups of the elliptic fibrations in question has been carried out over algebraically closed fields by Shioda in [15]. As in [4, §4], we throw in just a little bit of extra thought to make the argument work over a non-algebraically closed field $k$. Later on we will specialise to the case $k = \mathbb{F}_p$.

Consider the lattice $\text{Hom}(E, E)$ endowed with a norm given by the degree. By [15, Prop. 3.1], there is an isomorphism of lattices

$$\text{Hom}(E, E) \cong \text{MWL}(\text{Km}(E \times E), \pi_0).$$

By construction this is clearly Galois-equivariant. Hence, for $E/\mathbb{F}_p$ supersingular, we deduce from the argument given at the beginning of this section, that $\pi$ has MW-rank 2 over $\mathbb{F}_p$ and 4 over $\mathbb{F}_p^2$.

Our aim is to compare these lattices to a certain sublattice of $\text{NS}(\text{Km}(E \times E))$ arising from fibration $\pi_1$. For this purpose consider $\pi_X$ for the moment. The singular fibres of type $II^*$ together with the zero section generate the unimodular sublattice $U + E_8(-1)^2 \subset \text{NS}(X)$ which is completely defined over $k$. We denote its orthogonal complemen by $L$. Usually the lattice $L$ is exactly the Mordell-Weil lattice of $\pi_X$ up to sign, but if $E \cong E'$, as is presently the case, $L(-1)$ is comprised of the root lattice $A_1$ corresponding to an additional reducible fibre, and the Mordell-Weil lattice of rank 3 over $\bar{k}$ (assuming that $E$ is supersingular).

We can pull-back the lattice $L$ via the quotient map $\text{Km}(E \times E) \longrightarrow X$ to obtain a natural sublattice $L(2) \subset \text{NS}(\text{Km}(E \times E))$. The crucial point in Shioda’s argument in [15, §4 & 7] is a geometric transition $\bar{k}$ between the fibrations $\pi_0$ and $\pi_1$ on $\text{Km}(E \times E)$ which implies that

$$L(2) \cong \text{Hom}(E, E)(4).$$

In order to guarantee the Galois-equivariance of this lattice isomorphism, it suffices that the fibration $\pi_1$ given by (2) has a base point over $k(y)$, i.e. $E$ has a 2-torsion point giving a zero of $f = g$. Recall that presently we are concerned with a supersingular elliptic curve $E$ over $\mathbb{F}_p$. Any such curve has an $\mathbb{F}_p$-rational 2-torsion point (outside characteristic 2) simply because, the trace being zero, $\#E(\mathbb{F}_p) = p + 1$ is even.

In conclusion we find for the given model of $X$ over $\mathbb{F}_p$ that $L$ has rank 2 over $\mathbb{F}_p$ and rank 4 over $\mathbb{F}_p^2$. Since both components of the $I_2$ fiber are clearly defined over $\mathbb{F}_p$, this implies Mordell-Weil rank 1 over $\mathbb{F}_p$ and rank 3 over $\mathbb{F}_p^2$. But then the quadratic twist $X'$ of $X$ over $\mathbb{F}_p^2$ automatically has reversed MW-ranks: 2 over $\mathbb{F}_p$ and 3 over $\mathbb{F}_p^2$. Summing up, $X'$ gives an alternative model of $X$ defined over $\mathbb{F}_p$ with $\rho(X'/\mathbb{F}_p) = 21$. This verifies Theorem 1.1.\[\square\]
Remark 3.1. Implicitly the above argument uses the fact that quadratic twists do not affect fibers of type $II^*$ and $I_2$ (such as on fibration $\pi_X$ on $X$). This does not hold for fibers of type $IV^*$, for instance, so quadratic twisting has a fundamentally different effect on fibration $\pi_1$ on $\text{Km}(E \times E)$.

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