Linear steering inequalities for high-dimensional systems

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In the present work, the averaged fidelity is introduced as the steering parameter. According to the definitions of steering from Alice to Bob, a general scheme for designing linear steering criteria is developed for high-dimensional systems. For a given set of measurements on Bob’s side, two quantities, the so-called non-steering-thresholds, can be defined to quantify its ability for detecting steering. If the measured averaged fidelity exceeds these thresholds, the state shared by Alice and Bob is steerable from Alice to Bob, and the measurements performed by Alice are also verified to be incompatible. Within the general scheme, we also construct a linear steering inequality when the set of measurements performed by Bob has a continuous setting. Some applications are also provided.

I. INTRODUCTION

The concept of steering introduced by Schrödinger [1] can date back to 1930s, as a generalization of the Einstein-Podolsky-Rosen (EPR) paradox [2]. For a bipartite state shared by Alice and Bob, steering infers the fact that an observer on one side can affect the state of a far remote system by local measurements. Specifically, if the shared state is entangled, Alice can remotely steer Bob’s state by performing measurements only in her part of the system. Actually, it was realized recently that steering can be regarded as a form of quantum correlation between entanglement and Bell nonlocality [3–5]. In 2007, Wiseman, Jones and Doherty [3] formally defined quantum steering as a type of quantum nonlocality that is logically distinct from inseparability [6, 7] and Bell nonlocality [8]. In the modern view, quantum steering can be understood as the impossibility to describe the conditional states at one party by a local hidden state (LHS) model.

Unlike quantum nonlocality and entanglement, steering is inherently asymmetric with respect to the observers [9, 10], and there are entangled states which are one-way steerable [10, 11]. Besides its foundational significance in quantum information theory, steering has a vast range of information-theoretic applications in one-sided device-independent scenarios where the player being steered has trust on his/her own quantum device while the other’s device is untrusted, such as one-sided device-independent quantum key distribution [12], advantage in subchannel discrimination [13], secure quantum teleportation [14, 15], quantum communication [14], detecting bound-entanglement [16], one-sided device-independent randomness generation [17], and one-sided device-independent self-testing of pure maximally as well as nonmaximally entangled state [18].

Meanwhile, the detection and characterization of steering, especially the steering inequalities, have been widely discussed. In 1989, the variance inequalities that are violated with EPR correlations for continuous variable system were derived by Reid [19], and this was generalized to discrete variable systems [20]. In quantum information processing, EPR steering can be defined as the task for a referee to determine whether two parties share entanglement [3]. Based on it, EPR-steering inequalities were defined [21], where the violation of any such inequality implies steering. Following this work, further schemes have been proposed to signalize steering from experimental correlations, for instance, the linear and nonlinear steering criteria [5, 22–25], steering criteria from uncertainty relations [26–31], steering with Clauser-Horne-Shimony-Holt (CHSH)-like inequalities [32–35], moment matrix approach [36–38], and steering criteria based on local uncertainty relations [39, 40]. The discussed criteria or small variation thereof have been used in several experiments [5, 22, 41–43].

Our work is originated from one of the open questions summarized in Ref. [44]: Though a complete characterization of quantum steerability has been obtained for two-qubit systems and projective measurements, it is still desirable to extend such a characterization to higher-dimensional systems. Although there is indication that such an extension is possible, much remains to be works out. Here, we shall investigate this question by focusing on the case where the linearly steering inequalities (LSIs) are applied for detecting steering [5, 21, 45]. The LSIs have an advantage that they can work even when the bipartite state is unknown. They also have a deep relation with the joint measurement problem: If a LSI is violated, the state is steerable from Alice to Bob and the measurements performed by Alice are also verified to be incompatible [46–50].

In the present work, according to the definitions of steering [51], we develop a general scheme for designing linear steering criteria for high-dimensional systems by introducing the averaged fidelity as the steering parameter. The content of this work is organized as follows. In Sec. II, we give a brief review of the definitions of steering from Alice to Bob, the Werner states and the isotropic states. In Sec. III, a detail introduction of the non-steering threshold is given there. In Sec. IV, we address the problem of constructing linear criteria for high-dimensional system, and an explicit LSI is constructed. Some of its applications are discussed in Sec. V. Finally, we end our work with a short conclusion.
II. PRELIMINARY

A. Steering from Alice to Bob

A bipartite state $W$ shared by Alice and Bob can be expressed by a pure (entangled) state and a one-sided quantum channel [52, 53],

$$ W = I_d \otimes \varepsilon (|\Psi\rangle\langle\Psi|), $$

where $I_d$ is an identity map, and $|\Psi\rangle$ is the purification of $\sqrt{\rho_A}$ with $\rho_A$ the reduced density matrix on Alice’s side and $\rho_A^T$ its transpose. Usually, $\rho_A$ has an eigen decomposition $\rho_A = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|$, with $d$ the dimension of the Hilbert space $\mathcal{H}$, and the state $|\Psi\rangle$ is expressed as $|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$.

In the field of quantum information and computation, entanglement is one of the most important quantum resources, and it is important to verify whether a bipartite state $W$ is entangled or not. Based on this decomposition, the state $W$ should be a mixture of products states if and only if the channel $\varepsilon$ is entanglement-breaking (EB) [52, 53].

Before one can show how to demonstrate a state is steerable from Alice to Bob, some necessary denotations are required. First, Alice can perform $N$ projective measurements on her side labelled by $\mu = 1, 2, ..., N$, each having $d$ outcomes $a = 0, 1, ..., d - 1$, and the measurements are denoted by $\Pi^\mu$, $\sum_{\mu=1}^{d} \Pi^\mu = I_d$, with $I_d$ the identity operator on a $d$-dimensional Hilbert space. Then, from the diagonal density matrix $\rho_A$ and the measurement $\Pi^\mu_a$, one can introduce

$$ \hat{\psi}^a = \frac{\sqrt{\rho_A} (\Pi^\mu_a)^* \sqrt{\rho_A}}{p(a|\mu)}, $$

with the probability $p(a|\mu) = \text{Tr}(\Pi^\mu_a \rho_A)$, $\sum_{\mu=1}^{d} p(a|\mu) = 1$, and “*” represents the complex conjugate. For the $\mu$-th measurement with the outcome $a$, the corresponding normalized conditional state on Bob’s side is denoted by $\rho^\mu_a$, which comes with the probability $p(a|\mu)$. From the decomposition in Eq. (1), we have

$$ \rho^\mu_a = \varepsilon(\hat{\psi}^a), $$

where $\rho^\mu_a$ can be viewed as the output of the channel $\varepsilon$ with $\hat{\psi}^a$ as the input. Moreover, the set of unnormalized quantum states \{\rho^\mu_a\}_{a,\mu}, with $\rho^\mu_a = p(a|\mu)\rho^\mu_a$, is usually called an assemblage.

In 2007, Wiseman, Jones and Doherty [3] formally defined as quantum steering as the possibility of remotely generating ensembles that could not be produced by a LHS model. An LHS model refers to the case where a source sends a classical message $\xi$ to one of the parties, say, Alice, and a corresponding quantum state $\rho_\xi$ to another party, say Bob, and when Alice decides to apply the $\mu$-th measurement, the variable $\xi$ instructs the output $a$ of Alice’s apparatus with the probability $p(a|\mu, \xi)$. Usually, $\xi$ can be chosen according to a distribution $\Omega(\xi)$, $\int \Omega(\xi) d\xi = 1$. Bob does not have access to the classical variable $\xi$, and his final assemblage is composed by

$$ \hat{r}^\mu_a = \int d\xi \Omega(\xi) p(a|\mu, \xi) \rho_\xi, $$

with the probability $p(a|\mu) = \int d\xi \Omega(\xi) p(a|\mu, \xi)$ that the outcome is $a$ when the $\mu$-th measurement is performed by Alice. The definition of steering in present work is directly cited from the review article [51]: An assemblage is said to demonstrate steering if it does not admit a decomposition of the form in Eq. (4). Furthermore, a quantum state $W$ is said to be steerable from A to B if the experiments in Alice’s part produce an assemblage that demonstrate steering. On the contrary, an assemblage is said to be LHS if it can be written as in Eq. (4), and a quantum state is said to be unsteerable if an LHS assemblage is generated for all local measurements.

B. Werner states

Two classes of states, Werner std the isotropic states, have important applications in the studying of quantum steering [4]. The Werner states can be defined as [4, 54]

$$ W^w_d = \frac{d - 1 + w I_d \otimes I_d}{d - 1} - \frac{w}{d - 1} \mathbf{V}, $$

with $0 \leq w \leq 1$, and $\mathbf{V}$ is the “flip” operator defined by $\mathbf{V}|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$. Werner states are nonseparable iff $w > 1/(d + 1)$. If Alice performs a projective measurement $\Pi^A_\omega = |a\rangle\langle a|$, $\forall a \in \{0, 1, ..., d - 1\}$ on her side, the unnormalized conditional state on Bob’s side is

$$ \tilde{\rho}^a_w = \frac{d - 1 + w I_d}{d(d - 1)} - \frac{w}{d(d - 1)} |a\rangle\langle a|. $$

It was showed that the original derivation by Werner in Ref. [54] can be equivalently expressed in terms of steering [3]. Denote the $d$-dimensional unitary group by $U(d)$, and with a unitary operator $U_\omega \in U(d)$, an state $|\psi_\omega\rangle$ can be expressed as $|\psi_\omega\rangle = U_\omega |0\rangle$, where $|0\rangle$ is a fixed state in the $d$-dimensional Hilbert space and $\omega$ represents the group parameters. The complete set of pure states in the $d$-dimensional system is denoted by $F^* \equiv \{\psi_\omega\langle\psi_\omega|d\mu_{\text{Haar}}(\omega)\}$, with $d\mu_{\text{Haar}}(\omega)$ the Harr measure on the group $U(d)$. If Alice is trying to simulate the conditional state above, the optimal state of LHS should be $F^*$ [3]. Formally, the simulation can be described as

$$ \tilde{\rho}^a_w = \int d\omega \Omega(\omega) p(a|A, \psi_\omega) |\psi_\omega\rangle\langle\psi_\omega| $$

with the constraint $\sum_{a,\omega} p(a|A, \psi_\omega) = 1$. For the explicit conditional state in Eq. (6), the optimal choices of the $\{p(a|A, \psi_\omega)\}$ are

$$ p^*(a|A, \psi_\omega) = \begin{cases} 1 & \text{if } |\langle\psi_\omega|\hat{\Pi}^A_\omega|\psi_\omega\rangle| < |\langle\psi_\omega|\tilde{\Pi}^A_\omega|\psi_\omega\rangle|, \quad a \neq a' \\ 0 & \text{otherwise}. \end{cases} $$
As it was shown by Werner [54], for any positive normalized distribution \( \{p(a|A, \psi_{\omega})\} \), there should be
\[
\langle a | \int d\mu_{\text{Haar}}(\omega) |\psi_{\omega}\rangle \langle \psi_{\omega} | p(a|A, \psi_{\omega}) |a\rangle \geq \frac{1}{d^3}.
\] (9)

The equality is attained for the optimal \( p^*(a|A, \psi_{\omega}) \) specified in Eq. (8). From it, it can be found that Alice cannot simulate the conditional state in Eq. (6) iff \((1 - w)/d^2 < 1/d^3\).

### C. Isotropic states

The isotropic states, which were introduced in Ref. [55], can be parameterized similarly to the Werner states with a mixing parameter \( \eta \).

\[
W^\eta_A = (1 - \eta) I_d \otimes I_d + \eta P_+.
\] (10)

Here \( P_+ = |\psi_+\rangle \langle \psi_+| \), where \(|\psi_+\rangle = \sum_{i=1}^d |i\rangle i/\sqrt{d} \) is a maximally entangled state. For \( d = 2 \), it is straightforward to verify that the isotropic states are identical to Werner states up to local unitaries. These states are entangled if \( \eta > 1/(d+1) \).

If Alice makes a projective measurement, the conditional state for Bob is

\[
\rho_{\psi}^A = \frac{1 - \eta}{d} I_d + \frac{\eta}{d} |a\rangle \langle a|.
\] (11)

When Alice tries to simulate this state, the ensemble \( F^\ast := \{|\psi_{\omega}\rangle \langle \psi_{\omega}| |d\mu_{\text{Haar}}(\omega)\} \) has been proved to be the most powerful LHS [3]. Especially, the choices of the \( p(a|A, \psi_{\omega}) \)

\[
p^*(a|A, \psi_{\omega}) = \begin{cases} 
1 & \text{if } \langle \psi_{\omega}| \hat{\Pi}_A^a |\psi_{\omega}\rangle > \langle \psi_{\omega}| \hat{\Pi}_A^{a'} |\psi_{\omega}\rangle, \ a \neq a' \\
0 & \text{otherwise}
\end{cases}
\] (12)

are optimal for Alice to simulate the conditional states in Eq. (11). It has been found that for any positive normalized distribution \( \{p(a|A, \psi_{\omega})\} \),

\[
\langle a | \int d\mu_{\text{Haar}}(\omega) |\psi_{\omega}\rangle \langle \psi_{\omega} | p(a|A, \psi_{\omega}) |a\rangle \leq \frac{H_d}{d^2},
\] (13)

where \( H_d = \sum_{n=1}^d (1/n) \) is the Harmonic series and the equality attained for the optimal \( p^*(a|A, \psi_{\omega}) \) specified in Eq. (12). Therefore, Alice cannot simulate the conditional states iff \( \eta/d + (1 - \eta)/d^2 > H_d/d^2 \) [3].

### III. NON-STEERING THRESHOLD

#### A. Sufficient criteria for steering

For a given output \( \bar{\rho}_{\mu}^n \), we suppose that Bob will measure the fidelity with a set of rank-one projective operators \( \{\hat{M}_{\mu}^a\} \),

\[
\hat{M}_{\mu}^a := \hat{\Phi}_{\mu}^a = |\phi_{\mu}^a\rangle \langle \phi_{\mu}^a|, \langle \phi_{\mu}^a | \phi_{\mu}^{a'}\rangle = \delta_{ab}, \sum_a \hat{M}_{\mu}^a = I_d,
\] (14)

which are usually called target states in previous works. The fidelity \( F_{\mu} \) is defined as the overlap between the targets and the unnormalized conditional states, \( F_{\mu} = \sum_{a=1}^{d-1} \text{Tr}(\hat{\Phi}_{\mu}^a \rho_{\mu}(a|\mu) \psi_{\mu}(\hat{\psi}_{\mu}^a)| \). Let \( \langle A \otimes B \rangle = \text{Tr}(A \otimes BW) \) be the expectation value of the operator \( A \otimes B \), and in experiment, \( F_{\mu} \) can be measured as

\[
F_{\mu} = \sum_{a=0}^{d-1} (\hat{\Pi}_{\mu}^a \otimes \hat{\Phi}_{\mu}^a). \] (15)

Denote \( q_{\mu} \) the probability of the case where the \( \mu \)-th setting has been measured, \( \sum_{\mu=1}^N q_{\mu} = 1 \), and the averaged fidelity \( \bar{F} \) can be defined as

\[
\bar{F} = \sum_{\mu=1}^N q_{\mu} F_{\mu}.
\] (16)

The averaged fidelity is a traditional quantity which has already been applied to detect entanglement. Let \( \bar{F}_{\text{average}} \) be an entanglement-breaking channel, with \( F_{\text{EB}}(a|\mu) = \text{Tr}(\hat{\Phi}_{\mu}^a \rho_{\mu}(a|\mu) \psi_{\mu}(\hat{\psi}_{\mu}^a)| \) and \( F_{\text{average}} = \sum_{\mu} \sum_{a} q_{\mu} F_{\mu}(a|\mu) \), the classic fidelity threshold (CFT) can be defined as \( \bar{F}_\text{CFT} = \max_{\text{EB channel}} F_{\text{average}} \) with \( \{\text{EB}\} \) the set of all EB channels. As shown in previous works [55–63], the CFT depends on the actual choices of the input and target states. If the experiment data \( \bar{F} \) exceeds this threshold, \( \bar{F} > \bar{F}_\text{CFT} \), one may conclude that the channel does not belong to the \( \{\varepsilon_{\text{EB}}\} \) and the state \( W \) is an entangled state.

In the task of detecting the steering, similar idea can be employed, and by taking the averaged fidelity as the steering parameter, a steering inequality can be constructed by just considering the measurement performed by Bob [5, 21, 45]. Assume that the assemblage \( \{p_{\mu}(a|\mu) \psi_{\mu}(\hat{\psi}_{\mu}^a)| \} \) has a LHS decomposition, and one can define

\[
F_{\text{average}} = \int d\xi \Omega(\xi) \text{Tr}(\rho_{\xi} \bar{\rho}),
\] (17)

where

\[
\bar{\rho} = \sum_{\mu=1}^N d_{\mu} q_{\mu} p_{\mu}(a|\mu, \xi) \hat{\Phi}_{\mu}^a,
\] (18)

with the probability \( p_{\mu}(a|\mu, \xi) \) the value of \( \hat{\Phi}_{\mu}^a \) in the LHV model, \( \bar{\rho} \) is a density matrix, and formally, can be expanded as \( \bar{\rho} = \sum_{\nu} \lambda_{\nu}|\lambda_{\nu}\rangle \langle \lambda_{\nu}| \) with \( \lambda_{\nu} \) the eigenvalue and \( |\lambda_{\nu}\rangle \) the corresponding eigenvector. In this work, the largest eigenvalue of \( \bar{\rho} \) is defined as the nonsteering threshold (NST) and denoted by the symbol \( \bar{F}^+_{\text{NST}} \) hereafter,

\[
\bar{F}^+_{\text{NST}}(\{q_{\mu}, \hat{M}_{\mu}^a\}) = \max_{\{\phi\}} \max_{p(a|\mu, \xi)} \langle \phi | \bar{\rho} | \phi \rangle.
\] (19)

Together with the facts \( \text{Tr}(\rho_{\xi} \bar{\rho}) \leq \bar{F}^+_{\text{NST}} \) and \( \int \Omega(\xi)d\xi = 1 \), one can conclude that \( \bar{F}^+_{\text{NST}} \) is an upper bound of \( F_{\text{average}} \), say, \( \bar{F}^+_{\text{NST}} \geq F_{\text{average}} \). In a similar way, the minimum eigenvalue of \( \bar{\rho} \) is another NST, and denoted by \( \bar{F}^-_{\text{NST}} \) hereafter,

\[
\bar{F}^-_{\text{NST}}(\{q_{\mu}, \hat{M}_{\mu}^a\}) = \min_{\{\phi\}} \min_{p(a|\mu, \xi)} \langle \phi | \bar{\rho} | \phi \rangle.
\] (20)
With $\text{Tr}[\hat{\rho}\hat{\rho}^\dagger] \geq \tilde{\mathcal{F}}_{\text{NST}}^+$ and $\int \Omega(\xi)d\xi = 1$, one can conclude that $\tilde{\mathcal{F}}_{\text{NST}}^+$ is a lower bound of $F_{\text{avg}}^{\text{LHS}}$, say, $\tilde{\mathcal{F}}_{\text{NST}}^+ \leq F_{\text{avg}}^{\text{LHS}}$. Therefore, an LSI can be defined

$$\tilde{\mathcal{F}}_{\text{NST}}^+(\{q_\mu, M_\mu^a\}) \leq F \leq \tilde{\mathcal{F}}_{\text{NST}}^+(\{q_\mu, M_\mu^a\}). \quad (21)$$

For the set of measurements $\{q_\mu, M_\mu^a\}$, its ability to detect steering is quantified by $\tilde{\mathcal{F}}_{\text{NST}}^+(\{q_\mu, M_\mu^a\})$. Following the two conditions—(a) The state is steerable from Alice to Bob and (b) the set of measurements $\{\Pi_\mu^a\}$ performed by Alice is incompatible—are necessary for that the assemblage $\{\hat{\rho}_a\}$ does not admit a LHS model, one may conclude that the violation of the steering inequality, is a sufficient condition for Bob to make the statements (a) and (b).

To show a state $W$ is steerable from Alice to Bob, the extremal values of the averaged fidelity should be considered. For a fixed measurement $\{\hat{\Phi}_\mu^a\}_{\mu=0}^{d-1}$ performed by Bob, let $F_{\mu}^+$ ($F_{\mu}^-$) be the maximum (minimum) value of $F_{\mu}$ with corresponding measurements $\hat{\Pi}_\mu^a$ performed by Alice. The extremal values of the fidelity are

$$F^\pm(\{q_\mu, M_\mu^a\}) = \sum_{\mu=1}^{N} q_\mu F_{\mu}^\pm, \quad (22)$$

and obviously, $F^- \leq F \leq F^+$. Now, two types of steering criteria can be introduced. For the reason which can be clarified in the following, one can define the Wiseman-Jones-Doherty (WJD) type criterion

$$F^+(\{q_\mu, M_\mu^a\}) > \tilde{\mathcal{F}}_{\text{NST}}^+(\{q_\mu, M_\mu^a\}), \quad (23)$$

and the Werner-type one

$$F^-(\{q_\mu, M_\mu^a\}) < \tilde{\mathcal{F}}_{\text{NST}}^-(\{q_\mu, M_\mu^a\}). \quad (24)$$

The two criteria above are independent, which means that if either is verified, the state $W$ is demonstrated to be steerable from Alice to Bob. It will be shown that both types of the criteria should be considered for the high-dimensional system ($d > 2$).

### B. The Optimal Eigenvectors

First, let us consider the probabilistic LHV model. For the $\mu$-th measurement $\{\Pi_\mu^a\}$, $\sum_{\mu=0}^{d-1} \Pi_\mu^a = I_d$, and

$$0 \leq p(a|\mu, \xi) \leq 1, \quad \sum_{\mu=0}^{d-1} p(a|\mu, \xi) = 1. \quad (25)$$

From the Eqs. (18) and (19), a quantity $f_{\mu}(\phi)$ can be introduced

$$f_{\mu}(\phi) = \langle \phi | \sum_{a=0}^{d-1} p(a|\mu, \xi) \hat{\Phi}_\mu^a | \phi \rangle, \quad (26)$$

for a fixed $|\phi\rangle$, and its maximum value

$$f_{\mu}^{\text{max}}(\phi) = \max_a \{ \langle \phi | \hat{\Phi}_\mu^a | \phi \rangle \}, \quad (27)$$

can be obtained with the optimal choice of the probabilities $\{p(a|\mu, \xi)\}$

$$p^*(a|\mu, \xi) = \begin{cases} 1 & \text{if } \langle \phi | \hat{\Phi}_\mu^a | \phi \rangle > \langle \phi | \hat{\Phi}_\mu^{a'} | \phi \rangle, a \neq a' \ , \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

where $a, a' \in \{0, 1, \ldots, d-1\}$. $\tilde{\mathcal{F}}_{\text{NST}}^+$ can be rewritten as

$$\tilde{\mathcal{F}}_{\text{NST}}^+ = \max_{|\phi\rangle} \sum_{\mu=1}^{d-1} q_\mu f_{\mu}^{\text{max}}(\phi). \quad (29)$$

Next, one may seek the optimal state $|\phi_+\rangle$ corresponding to the largest eigenvalue of $\tilde{\rho}$, and then, the result can be formally expressed as

$$\tilde{\mathcal{F}}_{\text{NST}}^+ = \sum_{\mu=1}^{d-1} \sum_{a=0}^{d-1} q_\mu \langle \phi_+ | p^*(a|\mu, \xi) \hat{\Phi}_\mu^a | \phi_+ \rangle. \quad (30)$$

On the other hand, $\tilde{\mathcal{F}}_{\text{NST}}^-$ can be derived similarly. With the optimal probabilities

$$p^*(a|\mu, \xi) = \begin{cases} 1 & \text{if } \langle \phi | \hat{\Phi}_\mu^a | \phi \rangle < \langle \phi | \hat{\Phi}_\mu^{a'} | \phi \rangle, a \neq a' \ , \\ 0 & \text{otherwise} \end{cases}, \quad (31)$$

the same quantity $f_{\mu}(\phi)$ in Eq. (26) achieves its minimum value

$$f_{\mu}^{\text{min}}(\phi) = \min_a \{ \langle \phi | \hat{\Phi}_\mu^a | \phi \rangle \}. \quad (32)$$

Therefore, we have

$$\tilde{\mathcal{F}}_{\text{NST}}^- = \min_{|\phi\rangle} \sum_{\mu=1}^{d-1} q_\mu f_{\mu}^{\text{min}}(\phi), \quad (33)$$

and by choosing the optimal vector $|\phi_-\rangle$ corresponding to the minimum eigenvalue of $\tilde{\rho}$, one can come to the final result

$$\tilde{\mathcal{F}}_{\text{NST}}^- = \sum_{\mu=1}^{d-1} \sum_{a=0}^{d-1} q_\mu \langle \phi_- | p^*(a|\mu, \xi) \hat{\Phi}_\mu^a | \phi_- \rangle. \quad (34)$$

Until now, we have considered the case where the number of the experiment settings are finite, and the above conclusions can be easily generalized to the case where the experiment settings are continuous.

### C. Deterministic LHV Model

In the above discussion, a general protocol to calculate NSTs through finding the optimal eigenvalues has been constructed. From the optimal choices of $\{p(a|\mu, \xi)\}$, it is shown that the NSTs are unchanged if a deterministic LHV is applied. For the $\mu$-th measurement $\{\Pi_\mu^a\}$, $\sum_{\mu=0}^{d-1} \Pi_\mu^a = I_d$, and

$$p(a|\mu, \xi) \in \{0, 1\}; \sum_{\mu=0}^{d-1} p(a|\mu, \xi) = 1. \quad (35)$$
So, one may have another way to derive the NSTs. For the measurements \( \{ q_\mu, \hat{\phi}_\mu \} \), a density matrix can be introduced

\[
\hat{\rho}_{k_1, k_2, \ldots, k_N} = \sum_{\mu=1}^{N} q_\mu \hat{\phi}_\mu^{k_\mu},
\]

(36)

where \( k_\mu \in \{0, 1, \ldots, d-1\} \) for all \( \mu = 1, 2, \ldots, N \). There are totally \( d^N \) matrices in such kind. For \( \hat{\rho}_{k_1, k_2, \ldots, k_N} \), its largest eigenvalue and the minimum eigenvalue is

\[
\lambda_{\text{max}}^{k_1, k_2, \ldots, k_N} = \max_{|\phi\rangle} \langle \phi | \hat{\rho}_{k_1, k_2, \ldots, k_N} |\phi\rangle,
\]

(37)

\[
\lambda_{\text{min}}^{k_1, k_2, \ldots, k_N} = \min_{|\phi\rangle} \langle \phi | \hat{\rho}_{k_1, k_2, \ldots, k_N} |\phi\rangle,
\]

(38)

respectively. The NSTs can be expressed as

\[
\tilde{\sigma}_{\text{NST}}^+ = \max_{a, b} \left\{ \lambda_{\text{max}}^{a, b} \right\}, \quad \tilde{\sigma}_{\text{NST}}^- = \min_{a, b} \left\{ \lambda_{\text{min}}^{a, b} \right\}.
\]

(39)

(40)

Next, let us consider a two-settings case as a specific example. Two sets of orthogonal basis \( \{|\phi_1^a\rangle\} \) and \( \{|\phi_2^b\rangle\} \) with \( a, b \in \{1, 2, \ldots, d\} \) can be chosen, which are supposed to be related by a unitary matrix \( U \). \( U_{ab} \) are matrix elements and \( \langle \phi_2^b | = \sum_{a=1}^{d} U_{ab} \langle \phi_1^a | \). Fixing the probability for each setting as \( q_1 = q_2 = 1/2 \) and with the deterministic LHV model, a series of states \( \tilde{\rho}_{ab} = \frac{1}{2}(\hat{\phi}_1^a + \hat{\phi}_2^b) \) can be introduced and NST can be obtained

\[
\tilde{\sigma}_{\text{NST}}^+ = \max_{a, b} \left\{ \lambda_{\text{max}}^{a, b} \right\}, \quad \tilde{\sigma}_{\text{NST}}^- = \min_{a, b} \left\{ \lambda_{\text{min}}^{a, b} \right\}.
\]

(41)

For a mixed state \( \rho = (|e_1\rangle \langle e_1| + |\phi\rangle \langle \phi|) / 2 \), where the state \( |\phi\rangle = s|e_1\rangle + \sqrt{1-s^2}|e_2\rangle \), with two orthogonal bases \( |e_1\rangle \) and \( |e_2\rangle \), one can have its maximum eigenvalue \( \lambda_{\text{max}}(\rho) = \frac{1}{2}(1 + |s|) \). Based on this fact, \( \lambda_{\text{max}}^{a, b} = \frac{1}{2}(1 + |U_{ab}|) \) and

\[
\tilde{\sigma}_{\text{NST}}^+ = \frac{1}{2}(1 + \max_{a, b, c \in \{1, 2, \ldots, d\}} |U_{ab}|).
\]

(42)

One can select out the optimal unitary element \( U_{ab}^{\text{opt}} \), whose mode \( |U_{ab}^{\text{opt}}| \) has the largest value, from all the unitary matrix elements. Then, \( \tilde{\sigma}_{\text{NST}}^+ = (1 + |U_{ab}^{\text{opt}}|) / 2 \). Note that each \( \tilde{\rho}_{ab} \) is a density matrix in \( d \)-dimensional system, and another NST can be obtained

\[
\tilde{\sigma}_{\text{NST}}^- = \left\{ \begin{array}{ll}
\frac{1}{2}(1 - |U_{ab}^{\text{opt}}|) & \text{if } d = 2 \\
0 & \text{if } d > 2
\end{array} \right.
\]

(43)

It is known that a set of mutually unbiased bases (MUBs) consists of two or more orthonormal basis \( \{|\phi_+^a\rangle\} \) in \( d \)-dimensional Hilbert space satisfying

\[
|\langle \phi_+^a | \phi_+^b \rangle|^2 = \frac{1}{d}, \quad \forall a, b \in \{0, 1, \ldots, d-1\}, \quad x \neq y,
\]

(44)

for all \( x \) and \( y \) [65]. Collecting above results together, one can have the WJD-type steering criterion as

\[
\sum_{a=1}^{d} (\Pi_\mu^{a, b} \otimes \hat{\phi}_\mu^{a, b}) > 1 + \frac{1}{\sqrt{d}},
\]

(45)

with \( \hat{\phi}_\mu^{a, b} \) one of MUBs. This result has appeared in previous works with different approaches [66, 67].

### D. Geometric steering inequality

Here, the geometric averaged fidelity, which is related to the averaged fidelity \( \tilde{F} \) in a simple way, can be defined as

\[
\tilde{f} = \frac{d\tilde{F} - 1}{d - 1}.
\]

(46)

Correspondingly, one can define the so-called geometric NSTs,

\[
\mathcal{g}_{\text{NST}}^\pm(\{q_\mu, M_\mu^a\}) = \frac{d\mathcal{g}_{\text{NST}}^\pm(\{q_\mu, M_\mu^a\}) - 1}{d - 1},
\]

(47)

and the criteria about Eq. (21) can be equivalently expressed as: If the geometric inequality

\[
\mathcal{g}_{\text{NST}}^\pm \leq \tilde{f} \leq \mathcal{g}_{\text{NST}}^\pm
\]

(48)

is violated, the state \( W \) is steerable from Alice to Bob. This type of inequality is convenient for the qubit case. With \( \sigma_x, \sigma_y \) and \( \sigma_z \), the Pauli matrices and a three-dimensional Bloch vector \( r = r_x \hat{x} + r_y \hat{y} + r_z \hat{z} \) (\( \hat{x}, \hat{y}, \) and \( \hat{z} \) are unit vectors along coordinate axes), a density matrix can be expressed as

\[
\rho = (I_2 + r \cdot \sigma) / 2,
\]

with \( r \cdot \sigma = r_x \sigma_x + r_y \sigma_y + r_z \sigma_z \). The geometric length of \( r \) is \( |r| = \sqrt{r_x^2 + r_y^2 + r_z^2} \). Furthermore, the measurement results of Alice are usually denoted by \( a = +, - \). Then, the measurement performed by Alice can be expressed as \( \Pi_\mu = (I_2 \pm \sigma \cdot \hat{n}_\mu) / 2 \), and the target states can be written as

\[
\hat{\phi}_\mu^\pm = (I_2 \pm \sigma \cdot \hat{n}_\mu) / 2.
\]

Based on this assumption, one can define a quantity \( \mathcal{A}(\mu, \xi) = p(\pm |\mu, \xi\rangle - p(\pm |\mu, \xi\rangle) \), and by the constraints

\[
p(\pm |\mu, \xi\rangle, p(\pm |\mu, \xi\rangle) + p(- |\mu, \xi\rangle) = 1,
\]

it can be obtained that \( -1 \leq \mathcal{A}(\mu, \xi) \leq 1 \). In fact, \( \mathcal{A}(\mu, \xi) \) may be viewed as the predetermined value of the operator \( \hat{\tau}_\mu \cdot \sigma \) in a LHV model. With the vector \( \hat{r} = \sum_{\mu=1}^{N} q_\mu \mathcal{A}(\mu, \xi) \hat{n}_\mu \), the state \( \tilde{\rho} \) in Eq. (18) can be expressed as

\[
|\hat{r}| = \sum_{\mu=1}^{N} q_\mu \mathcal{A}(\mu, \xi) \hat{n}_\mu \cdot \sigma
\]

the geometric NSTs can be obtained as follows

\[
\mathcal{g}_{\text{NST}}^\pm = \pm |\hat{r}|.
\]

(49)

With the averaged geometric fidelity \( \tilde{f} = \sum_{\mu=1}^{N} q_\mu (\hat{r}_\mu \cdot \sigma \otimes \hat{n}_\mu \cdot \sigma) \), the geometric steering inequality for the qubit case becomes

\[
-|\hat{r}| \leq \sum_{\mu=1}^{N} q_\mu (\hat{r}_\mu \cdot \sigma \otimes \hat{n}_\mu \cdot \sigma) \leq |\hat{r}|
\]

(50)

For the deterministic LHV model, \( \mathcal{A}(\mu, \xi) \in \{-1, +1\} \), and introducing \( \sum_{\mu=1}^{N} q_\mu \hat{n}_\mu \), the optimal length of \( \hat{r} \) can be expressed as

\[
|\hat{r}| = \max_{\pm \ldots \pm} |\hat{r}|_{\pm \ldots \pm}.
\]

(52)

The known result in Ref. [5] is recovered here.
As a simple example, let us consider the case where Bob’s measurements are MUBs: \( \hat{n} \cdot \sigma \) and \( \hat{n}_1 \cdot \sigma \), where \( \hat{n} \) and \( \hat{n}_1 \) are two orthogonal unit vectors. With \( q(\hat{n}) \) and \( q(\hat{n}_1) \) the probabilities for each measurement, respectively, all the possible four vectors are \( \hat{\mathbf{r}}_{\pm \pm} = \pm q(\hat{n}) \hat{n} \pm q(\hat{n}_1) \hat{n}_1 \), and it is easy to calculate the geometric length for each vector \( |\hat{\mathbf{r}}_{\pm \pm}| = \sqrt{q^2(\hat{n}) + q^2(\hat{n}_1)} \). Thus, the optimal length is \( |\hat{\mathbf{r}}|_{\text{opt}} = \sqrt{q^2(\hat{n}) + q^2(\hat{n}_1)} \) and the geometric steering inequality above has a more explicit form

\[
-1 \leq \frac{q(\hat{n}) \langle \hat{\mathbf{r}} \otimes \hat{n} \rangle + q(\hat{n}_1) \langle \hat{\mathbf{r}} \otimes \hat{n}_1 \rangle}{\sqrt{q^2(\hat{n}) + q^2(\hat{n}_1)}} \leq 1, \tag{53}
\]

where \( \langle \hat{\mathbf{a}} \otimes \hat{n} \rangle = \langle \hat{\mathbf{a}} \cdot \sigma \otimes \hat{n} \cdot \sigma \rangle \).

If the CHSH inequality [68],

\[
-2 \leq \langle \hat{\mathbf{a}} \otimes (\hat{n}_1 - \hat{n}_2) \rangle + \langle \hat{\mathbf{b}} \otimes (\hat{n}_1 + \hat{n}_2) \rangle \leq 2 \tag{54}
\]

is violated, the state is Bell-nonlocal. As in Refs. [69, 70], a pair of orthogonal unit vectors \( \hat{n}' \) and \( \hat{n}'_1 \) can be introduced, and the vectors \( \hat{\mathbf{n}} \) can be expanded as \( \hat{n}_1 = \cos \theta \hat{n}' + \sin \theta \hat{n}'_1 \), \( \hat{n}_2 = -\cos \theta \hat{n}' + \sin \theta \hat{n}'_1 \). Equivalently, \( \hat{n}_1 - \hat{n}_2 = 2 \cos \theta \hat{n}'_1 \), and \( \hat{n}_1 + \hat{n}_2 = 2 \sin \theta \hat{n}'_1 \). Instead of \( \hat{n}', \hat{n}'_1 \), one can introduce another pair of orthogonal unit vectors

\[
\hat{n}_1 = \frac{\cos \theta}{|\cos \theta|} \hat{n}', \hat{n}'_1 = \frac{\sin \theta}{|\sin \theta|} \hat{n}',
\]

and then, \( \hat{n}_1 - \hat{n}_2 = 2 |\cos \theta| \hat{n}_1 \), \( \hat{n}_1 + \hat{n}_2 = 2 |\sin \theta| \hat{n}_1 \). Putting them back into Eq. (54), the CHSH inequality takes an equivalent form

\[
-1 \leq |\cos \theta| \langle \hat{\mathbf{a}} \otimes \hat{n} \rangle + |\sin \theta| \langle \hat{\mathbf{b}} \otimes \hat{n}_1 \rangle \leq 1. \tag{55}
\]

Comparing the steering criterion in Eq. (53) with it, it could be found that the two criteria are very similar. Introducing two operators

\[
\hat{T}_{\text{steer}} = \frac{g(\hat{n}) \hat{\mathbf{a}} \cdot \sigma \otimes \hat{n} \cdot \sigma + q(\hat{n}_1) \hat{\mathbf{b}} \cdot \sigma \otimes \hat{n}_1 \cdot \sigma}{\sqrt{q^2(\hat{n}) + q^2(\hat{n}_1)}}, \quad \hat{T}_{\text{CHSH}} = |\cos \theta| \langle \hat{\mathbf{a}} \cdot \sigma \otimes \hat{n} \cdot \sigma \rangle + |\sin \theta| \langle \hat{\mathbf{b}} \cdot \sigma \otimes \hat{n}_1 \cdot \sigma \rangle,
\]

one may find that the two operators are equal with each other: \( \hat{T}_{\text{steer}} = \hat{T}_{\text{CHSH}} \), under the one-to-one mapping,

\[
|\cos \theta| = \frac{q(\hat{n})}{\sqrt{q^2(\hat{n}) + q^2(\hat{n}_1)}}, \quad |\sin \theta| = \frac{q(\hat{n}_1)}{\sqrt{q^2(\hat{n}) + q^2(\hat{n}_1)}}.
\]

Based on the results above, one may conclude that if the geometric inequality in Eq. (53) is violated, the state must be Bell-nonlocal. A similar result has also been found in [32, 33].

IV. CONTINUOUS SETTINGS

A. Qubit Case

In the above sections, we have developed a general scheme for constructing LSTs for the discrete case. In this section, two explicit LSTs will be constructed for the case where the measurement performed by Bob has a continuous form. Before the LSTs for an arbitrary dimensional system can be derived, a detail discussion about the qubit case is required first, and this is useful to show what are necessary to construct the LSTs.

Now, instead of the symbol \( \mu \), a three dimensional unit vector \( \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) with \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \), is employed to label Bob’s measurement as \( \hat{\Phi}_a^\phi \) with \( a = \pm \) the outcomes. One can introduce the measure \( 1/\pi \int d^2 \hat{n} \equiv 1/\pi \int_{0}^{\pi} \int_{0}^{\pi/2} \sin \theta d\theta d\phi \), and certainly, \( \int_{0}^{\pi} \int_{0}^{\pi/2} \sin \theta d\theta d\phi = 1 \). In general, the measurement \( \hat{\Phi}_a^a \) has a probability distribution \( q(\hat{n}) \), and in this work, we just consider the case that the experimental settings are equal-weighted, say, \( q(\hat{n}) = 1 \). Now, the density matrix in Eq. (18) becomes

\[
\hat{\rho} = \frac{1}{4\pi} \int d^2 \hat{n} \sum_a p(a|\hat{n}, \xi) \hat{\Phi}_a^\phi_{\hat{n}}. \tag{56}
\]

Correspondingly, the expression for its maximum and minimum eigenvalue can be obtained from the Eq. (19) and Eq. (20), respectively.

The set of measurements performed by Bob can be denoted by \( \{ \hat{\Phi}_a^\phi, d^2 \hat{n} \} \). This set of measurements has a special property: The optimal vector \( |\phi_+\rangle \) should be the eigenvector of one of the measurement which belongs to the set \( \{ \hat{\Phi}_a^\phi, d^2 \hat{n} \} \). Without lose of generality, one may fix it as the eigenvector of \( \sigma_+ \), \( |\phi_+\rangle \equiv |+\rangle \), where \( \sigma_\pm = \pm |\pm\rangle \langle \pm| \). As a comparison, one may recall the case that Bob’s measurements are MUBS: \( \sigma \cdot \hat{n} \) and \( \sigma \cdot \hat{n}_1 \), where \( |\phi_\pm\rangle \) should be the eigenvector of \( \sigma \cdot \hat{\mathbf{r}}_{\pm \pm} \). However, this property does not hold anymore. With \( \hat{U}_n \) a unitary matrix transforming \( |+\rangle \) to a state represented by a unit Bloch vector \( \hat{n} \), \( |\phi_n\rangle = \hat{U}_n |+\rangle \), one may redefine \( \hat{\Phi}_a^\phi_{\hat{n}} = \hat{U}_n^\dagger |a\rangle \langle a| \hat{U}_n \), and obtain a complete set of pure states \( \{ |\phi_n\rangle \} \). By some simple algebra, \( \langle +| \hat{\Phi}_a^\phi_{\hat{n}} |+\rangle = \langle a| \hat{\Phi}_a^\phi_{\hat{n}} |a\rangle \), the Eq. (30) becomes

\[
\hat{\Phi}_a^\phi_{\hat{n}} + = \frac{1}{4\pi} \left\{ \langle +| \left( \int d^2 \hat{n} \hat{\Phi}_a^\phi_{\hat{n}}^\dagger (+| \hat{n}, \xi) \langle \phi_n| \hat{\Phi}_a^\phi_{\hat{n}} |+\rangle \right) \right\} + \left\{ \langle -| \left( \int d^2 \hat{n} \hat{\Phi}_a^\phi_{\hat{n}}^\dagger (-| \hat{n}, \xi) \langle \phi_n| \hat{\Phi}_a^\phi_{\hat{n}} |-\rangle \right) \right\}, \tag{57}
\]

with the optimal probabilities

\[
p^*(a, |\hat{n}, \xi\rangle) = \begin{cases} 1 & \text{if } \langle a| \hat{\Phi}_a^\phi_{\hat{n}} |a\rangle > \langle a'| \hat{\Phi}_a^\phi_{\hat{n}} |a'\rangle, \quad a \neq a', \\ 0 & \text{otherwise} \end{cases}
\]

where \( a, a' \in \{ +, - \} \), and \( \hat{\Phi}_a^\phi_{\hat{n}} = |\phi_n\rangle \langle \phi_n| \). Now, only the pure states on the northern hemisphere of the Bloch sphere \( 0 < \theta \leq \pi/2 \) contribute to the first term in Eq. (57),

\[
\frac{1}{4\pi} \left\{ \langle +| \left( \int d^2 \hat{n} \hat{\Phi}_a^\phi_{\hat{n}}^\dagger (+| \hat{n}, \xi) \langle \phi_n| \hat{\Phi}_a^\phi_{\hat{n}} |+\rangle \right) \right\} = \frac{1}{2} \int_{0}^{\pi/2} \sin \theta d\theta \int_{0}^{\pi/2} \frac{1}{2} (1 + \cos \theta) = \frac{3}{8}, \tag{59}
\]

while, only the pure states on the southern hemisphere of the
Bloch sphere contribute to the second term in Eq. (57).

\[
\frac{1}{4\pi} \left\{ - \left( \int \int d^2 \hat{n} p^* (-|\hat{n}, \xi|\phi_\alpha|\phi_\alpha|) \right) \right\} = \frac{1}{2} \int_{\pi/2}^{\pi} \sin \theta d\theta \frac{1}{2} (1 - \cos \theta) = \frac{3}{8}. \tag{60}
\]

Collecting the results above together, one can obtain the NST

\[ \hat{\mathcal{F}}_{\text{NST}} = \frac{3}{4}. \]

With suitable basis, the optimal vector \(|\phi_-\rangle\) can be fixed as the eigenvector of \(\hat{\sigma}_z \), \(|\phi_-\rangle \equiv |+\rangle, \hat{\sigma}_z |\pm\rangle = |\pm\rangle\). In a similar way to the one for deriving \(\hat{\mathcal{F}}_{\text{NST}}\), and with the optimal probabilities

\[
p^*(a, |\hat{n}, \xi\rangle) = \begin{cases} 1 & \text{if } \langle a|\hat{\Phi}_\alpha|a\rangle < \langle a'|\hat{\Phi}_\alpha|a'\rangle, \ a \neq a' \\ 0 & \text{otherwise} \end{cases} \tag{61}
\]

where \(a, a' \in \{+, -\}\), and one can have \(\hat{\mathcal{F}}_{\text{NST}} = \frac{1}{4}\). From the derivation above, one can see that the optimal probabilities in Eq. (58) and Eq. (61) play an important role in deducing the NSTs. Finally, one can come to a state-independent LSI for qubit case,

\[
\frac{1}{4} \leq \hat{F} \left( \frac{1}{4\pi} d^2 \hat{n}, \hat{\Phi}_\alpha \right) \leq \frac{3}{4}, \tag{62}
\]

where the measurements by Bob are fixed as \(\{\frac{1}{4\pi} d^2 \hat{n}, \hat{\Phi}_\alpha\}\).

### B. High-dimensional system case

With a set of basis vectors \(\{|a\rangle, a = 0, ..., d - 1\}\), the parameter \(\omega\) can be used to label the experiment setting by Bob’s measurement, \(\hat{\Phi}_\alpha = U_t^\dagger |a\rangle\langle a|U_\omega\rangle\) where \(U_\omega\) can take all the unitary operators in the \(d\)-dimensional unitary group \(U(d)\), and \(a\) represents the outcomes. It is assumed that the probability for each measurement is equal-weighted, and a Haar measure \(d\mu_{\text{Haar}}(\omega)\) on \(U(d)\) can be introduced,

\[
\int d\mu_{\text{Haar}}(\omega) \sum_{a=0}^{d-1} \hat{\Phi}_\alpha = I_d. \tag{63}
\]

Formally, the measurements by Bob is denoted by \(\{d\mu_{\text{Haar}}(\omega), \hat{\Phi}_\alpha\}\). Meanwhile, \(|\psi_\omega\rangle = U_\omega |0\rangle\) is a pure state in the \(d\)-dimensional Hilbert space. Analogously, without lose of generality, the optimal eigenvector is chosen as \(|\phi_+\rangle\) as \(|\phi_-\rangle \equiv |0\rangle\). Now, Eq. (30) may be rewritten into a form more appropriate for the continuous setting

\[ \hat{\mathcal{F}}_{\text{NST}} = \sum_{a=0}^{d-1} |a\rangle\langle a|d \mu_{\text{Haar}}(\omega)p^*(a|\omega, \xi\rangle|\phi_\alpha\rangle\langle \phi_\alpha|)|a\rangle, \tag{64} \]

where \(|0\rangle\hat{\Phi}_\alpha|0\rangle = \langle a|\phi_\omega\rangle\langle \phi_\omega|a\rangle|a\rangle\) has been applied. Correspondingly, as a generalization of Eq. (28), the optimal probabilities are

\[
p^*(a|\omega) = \begin{cases} 1 & \text{if } \langle \phi_\omega|a\rangle\langle a|\phi_\omega\rangle > \langle \phi_\alpha|a'\rangle\langle a'|\phi_\alpha\rangle, \ a \neq a' \\ 0 & \text{otherwise} \end{cases} \tag{65}
\]

where \(a, a' \in \{0, 1, ..., d - 1\}\).

Now, let us come back to the general results about the isotropic states in Sec. II. One may easily verify that the Eq. (64) is similar to the one in Eq. (12). As a direct application of the inequality in Eq. (13), the NST can be derived

\[ \hat{\mathcal{F}}_{\text{NST}} = \frac{1}{d} \hat{H}_d \tag{66} \]

from Eq. (63).

Similarly, one can fix the optimal eigenvector \(|\phi_-\rangle \equiv |0\rangle\) and rewrite Eq. (34) as

\[ \hat{\mathcal{F}}_{\text{NST}} = \sum_{a=0}^{d-1} |a\rangle d \mu_{\text{Haar}}(\omega)p^*(a|\omega, \xi\rangle|\phi_\alpha\rangle\langle \phi_\alpha|)|a\rangle, \tag{67} \]

with the optimal probabilities

\[
p^*(a|\omega) = \begin{cases} 1 & \text{if } \langle \phi_\omega|a\rangle\langle a|\phi_\omega\rangle > \langle \phi_\alpha|a'\rangle\langle a'|\phi_\alpha\rangle, \ a \neq a' \\ 0 & \text{otherwise} \end{cases} \tag{68}
\]

where \(a, a' \in \{0, 1, ..., d - 1\}\). With the inequality in Eq. (9), another NST can be obtained

\[ \hat{\mathcal{F}}_{\text{NST}} = \frac{1}{d^2} \hat{H}_d. \tag{69} \]

Collecting the above results together, a LSI for the continuous settings \(\{d\mu_{\text{Haar}}, \hat{\Phi}_\alpha\}\) takes the form

\[ \frac{1}{d^2} \leq \hat{F} \left( \frac{1}{4\pi} d^2 \hat{n}, \hat{\Phi}_\alpha \right) \leq \frac{H_d}{d}. \]

For any state \(W\), if the LSI is violated, the state is verified to be steerable from Alice to Bob, and the measurement performed by Alice is also incompatible. As a special case, the LSI in Eq. (62) can be recovered from the general one above with \(d = 2\).

### V. APPLICATIONS

#### A. T-state problem

An arbitrary two-qubit state can be expressed in the standard form

\[ W = \frac{1}{4} (I_2 \otimes I_2 + a \cdot \sigma \otimes I_2 + I_2 \otimes b \cdot \sigma + \sum_{j k} T_{jk} \sigma_i \otimes \sigma_j), \tag{70} \]

where \(a \) and \(b \) are the Bloch vectors for Alice and Bob’s reduced states, respectively, and \(T\) is the correlation matrix. The T-state is a special class of two-qubit states,

\[ W = \frac{1}{4} (I_2 \otimes I_2 + \sum_j t_j \sigma_j \otimes \sigma_j), \tag{71} \]

where \(a = b = 0\) and \(T\) is a diagonal matrix with the \(t_j\) the diagonal elements. In 2015, Jevtic et. al. gave a necessary condition of EPR steerability for T-states

\[ \frac{1}{2\pi} \int \int d^2 \hat{n} \sqrt{\hat{n}^T T^2 \hat{n}} = 1. \tag{72} \]
The authors also conjectured that the derived condition was precisely the border between steerable and non-steerable states, and this was later shown analytically [72]. Here, we shall revisit this problem from the view of LSI.

When Bob’s measurement is fixed as \( \hat{a} \cdot \sigma \), the expectation \( \langle \hat{a} \otimes \hat{n} \rangle_+ = \max_{\hat{a}} \langle \hat{a} \otimes \hat{n} \rangle \) is the maximum one. Further assume that Bob’s measurement is the continuous set \( \{ \frac{1}{2\pi} d^2 \hat{a} \otimes \hat{n} \}_{\hat{a}} \), and from the definition of the geometric fidelity in Eq. (46), one can have the maximum value of the geometric fidelity \( F^+ = \frac{1}{2\pi} \int d^2 \hat{a} \langle \hat{a} \otimes \hat{n} \rangle_+ \). With the geometric NST \( \tilde{\nu}_{\text{NST}}^+ = 1/2 \), which can be directly calculated from Eq. (62), a WJD-type criterion now is constructed

\[
\frac{1}{2\pi} \int d^2 \hat{n} \langle \hat{a} \otimes \hat{n} \rangle_+ > 1.
\]

This criterion is general in the sense that it is suitable for any two-qubit state. For the T-state, the correlation \( \langle \hat{a} \otimes \hat{n} \rangle = \hat{a} \cdot \hat{n} \) is the inner product between the two vectors \( \hat{a} = (a_x, a_y, a_z) \) and \( \hat{n} = (t_x n_x, t_y n_y, t_z n_z) \). Via the Cauchy-Swarz inequality, the optimal choice of \( \hat{a} \) could be \( a_i = \sqrt{\frac{\eta}{1 + \eta}} t_i n_i \) with \( i = x, y, z \). Thus, \( \langle \hat{a} \otimes \hat{n} \rangle_+ = \sqrt{\frac{1}{1 + \eta}} \), and obviously, \( \langle \hat{a} \otimes \hat{n} \rangle_+ = \sqrt{\frac{1}{1 + \eta}} \). Therefore, a sufficient condition for the T-state to be steerable from Alice to Bob is

\[
\frac{1}{2\pi} \int d^2 \hat{n} \sqrt{\frac{1}{1 + \eta}} > 1,
\]

with the equality in Eq. (72) the border of it.

### B. Bounds of the general NSTs

When the state is the isotropic state and a set of measurements \( \{q_\mu, \hat{\Phi}_\mu^a\} \) is used by Bob to detect steering, there is a sufficient criterion that the isotropic state is steerable from Alice to Bob

\[
\tilde{N}^+(\{q_\mu, \hat{\Phi}_\mu^a\}) > \tilde{\nu}_{\text{NST}}^+(\{q_\mu, \hat{\Phi}_\mu^a\}), \tag{75}
\]

where the subscript \( \eta \) indicates that the isotropic states are considered. For the \( \mu \)-th setting of measurements by Bob \( \{\hat{\Phi}_\mu^a\}_{\mu=0}^{d-1} \), the conditional states defined in Eq. (11) can be expressed as

\[
\rho_\mu^a = \frac{1 - \eta}{d} I_d + \frac{\eta}{d} \hat{\Phi}_\mu^a, \quad a \in \{0, 1, ..., d-1\}. \tag{76}
\]

The extreme values of \( f_\eta = \sum_{\alpha=0}^{d-1} \text{Tr}[\hat{\Phi}_\mu^a \rho_\mu^a] \) will be derived in the following. Obviously, the maximum value \( f_\eta^{\text{max}} \) can be attained if \( \hat{\Phi}_\mu^a = \hat{\Phi}_\mu^a \), and \( f_\eta^{\text{max}} = \frac{1 + (d-1)\eta}{d} \). The minimum value \( f_\eta^{\text{min}} \) can be attained by setting \( \text{Tr}[\hat{\Phi}_\mu^a \hat{\Phi}_\mu^a] = 0 \), and \( f_\eta^{\text{min}} = (1 - \eta)/d \). Moreover, these extremal values do not depend on the actual form of the measurements \( \hat{\Phi}_\mu^a \), and therefore,

\[
\begin{align*}
\tilde{F}^+_{\eta}(\{q_\mu, \hat{\Phi}_\mu^a\}) &= \frac{1 + (d-1)\eta}{d}, \\
\tilde{F}^0_{\eta}(\{q_\mu, \hat{\Phi}_\mu^a\}) &= \frac{1 - \eta}{d}, \tag{77}
\end{align*}
\]

However, for the continuous-settings case, the criterion

\[
\tilde{F}^+_{\eta}(\{d_{\text{Haar}}^a, \hat{\Phi}_\mu^a\}) > \frac{H_d}{d} \tag{78}
\]

is different from the one in Eq. (75). The WJD threshold \( H_d/d \) has been proven to be a tight bound: If it is achieved, the conditional states should admit a LHS model [3]. In other words, the equivalent form of Eq. (78)

\[
\frac{1 + (d-1)\eta}{d} > \frac{H_d}{d}, \tag{79}
\]

is a necessary and sufficient condition for the isotropic state to be steerable, while \( \frac{1 + (d-1)\eta}{d} > \tilde{\nu}_{\text{NST}}^+(\{q_\mu, \hat{\Phi}_\mu^a\}) \) is just a sufficient one. Therefore, a state-independent relation does exist

\[
\tilde{\nu}_{\text{NST}}^+(\{q_\mu, \hat{\Phi}_\mu^a\}) \geq \frac{H_d}{d}, \tag{80}
\]

where the WJD threshold is a lower bound of the general \( \tilde{\nu}_{\text{NST}}^+(\{q_\mu, \hat{\Phi}_\mu^a\}) \). This is the reason why we call the criterion in Eq. (23) as the WJD-type one.

When the state is the Werner state and the same measurement \( \{q_\mu, \hat{\Phi}_\mu^a\} \) is performed by Bob, there is a criterion which is sufficient for the Werner state to be steerable from Alice to Bob

\[
\tilde{F}^-_{\eta}(\{q_\mu, \hat{\Phi}_\mu^a\}) < \tilde{\nu}_{\text{NST}}^-(\{q_\mu, \hat{\Phi}_\mu^a\}), \tag{81}
\]

where the subscript \( w \) is used to indicate that only the Werner state is considered. For the \( \mu \)-th run of experiment, the conditional states, as they are defined Eq. (6), can be expressed as

\[
\rho_\mu^a = \frac{d - 1 + w}{d(d - 1)} I_d - \frac{w}{d(d - 1)} \hat{\Phi}_\mu^a. \tag{82}
\]

The extreme values of \( f_w \) \( \equiv \sum_{\alpha=0}^{d-1} \text{Tr}[\hat{\Phi}_\mu^a \rho_\mu^a] \) can be derived as follows. The minimum value \( f_w^{\text{min}} \) can be attained if \( \hat{\Phi}_\mu^a = \hat{\Phi}_\mu^a \), and \( f_w^{\text{min}} = (1 - w)/d \). The maximum value \( f_w^{\text{max}} \) is obtained by setting \( \text{Tr}[\hat{\Phi}_\mu^a \hat{\Phi}_\mu^a] = 0 \), and \( f_w^{\text{max}} = (d - 1 + w)/(d(d - 1)) \). Furthermore, these extremal values do not depend on the actual form of \( \hat{\Phi}_\mu^a \), and thus, there should be

\[
\begin{align*}
\tilde{F}^+_{\text{w}}(\{q_\mu, \hat{\Phi}_\mu^a\}) &= \frac{d - 1 + w}{d(d - 1)}, \\
\tilde{F}^-_{\text{w}}(\{q_\mu, \hat{\Phi}_\mu^a\}) &= \frac{1 - w}{d}, \tag{83}
\end{align*}
\]

The criterion for the continuous settings takes the form

\[
\tilde{F}^-_{\text{w}}(\{d_{\text{Haar}}^a, \hat{\Phi}_\mu^a\}) < \frac{1}{d^2}, \tag{84}
\]

where the Werner threshold \( 1/d^2 \) has been proven to be a tight bound. If it is achieved, the conditional states should admit a LHS model [3]. In other words,

\[
\frac{1 - w}{d} < \frac{1}{d^2} \tag{85}
\]
is a necessary and sufficient condition for the Werner state to be steerable, while \( \frac{1}{d^2} < \bar{\tilde{\gamma}}_{\text{NST}}(\{q_\mu, \hat{\phi}^a_\mu\}) \) is just a sufficient one. Therefore, one can have a state-independent relation

\[
\bar{\tilde{\gamma}}_{\text{NST}}(\{q_\mu, \hat{\phi}^a_\mu\}) \leq \frac{1}{d^2}, \tag{86}
\]

where the Werner threshold \( 1/d^2 \) is the upper bound of an arbitrary \( \bar{\tilde{\gamma}}_{\text{NST}}(\{q_\mu, \hat{\phi}^a_\mu\}) \). So, it is reasonable that the criterion in Eq. (24) is referred as the Werner-type one.

C. Detecting the steerability of Werner state

For the qubit case, one can easily verify that \( \bar{\gamma}_{\text{NST}} = -\bar{\gamma}_{\text{NST}}^d \) and \( \bar{f}^- = -\bar{f}^+ \) for an arbitrary measurement \( \{q_\mu, \hat{\phi}^a_\mu\} \). The WJD-type geometric criterion, \( \bar{f}^+ > \bar{\gamma}_{\text{NST}} \), and the Werner-type one, \( \bar{f}^- < \bar{\gamma}_{\text{NST}} \), are equivalent. Therefore, only one of the above two criteria, usually the WJD-type one, is required to detect the steerability of the two-qubits states including the Werner state for \( d = 2 \). This equivalence can also be easily explained since \( \bar{\gamma}_{\text{NST}}^d + \bar{\gamma}_{\text{NST}}^d = 1 \) holds for \( d = 2 \). However, for the high-dimensional system, this equality does not hold any more. For the Werner state, the maximal value of the averaged fidelity is shown in Eq. (83), \( \bar{f}_{\text{WJD}}^+(\{q_\mu, \hat{\phi}^a_\mu\}) = (d - 1 + w)/(d(d - 1)) \). Certainly, \( \bar{f}_{\text{WJD}}^+(\{q_\mu, \hat{\phi}^a_\mu\}) \leq 1/d \). With WJD bound \( H_d/d = (1 + 1/2 + \ldots + 1/d)/d \), one can easily check that \( \bar{f}_{\text{WJD}}^+(\{q_\mu, \hat{\phi}^a_\mu\}) < H_d/d \) if \( d > 2 \). Using Eq. (80), one has

\[
\bar{f}_{\text{WJD}}^+(\{q_\mu, \hat{\phi}^a_\mu\}) < \bar{\tilde{\gamma}}_{\text{NST}}^d(\{q_\mu, \hat{\phi}^a_\mu\}), \text{ if } d > 2. \tag{87}
\]

So, if \( d > 2 \), the steerability of Werner state cannot be detected by the WJD-type criterion \( \bar{f}^+ > \bar{\tilde{\gamma}}_{\text{NST}}(\{q_\mu, \hat{\phi}^a_\mu\}) \).

This is the reason why both types of the criteria in Eq. (23) and Eq. (24) are required for high-dimensional systems.

VI. CONCLUSIONS

According to the fundamental idea that a steering inequality can be constructed by just considering the measurement performed by Bob, proposed in Refs. [5, 21, 45], and from the definitions of steering from Alice to Bob [51], we have developed a general scheme for designing linear steering criteria for high-dimensional systems. For a given set of measurements (on Bob’s side), we have defined two quantities, the so-called non-steering-thresholds, to quantify its ability for detecting steering. If the measured averaged fidelity exceeds these thresholds, the state shared by Alice and Bob is steerable from Alice to Bob, and the measurements performed by Alice are also verified to be incompatible. Within the general scheme, we also constructed a LSI when the set of measurements performed by Bob has a continuous setting. In the derivation of this LSI, the main results in Refs [3, 54] have been applied. Two kinds of steering criteria, the WJD-type and Werner-type, have been defined in the present work. For the qubit case, it has been shown that the two types of steering criteria are equivalent to each other. However, when \( d > 2 \), these criteria have different properties.

The LSI in this work is limited for the case where the set of measurements by Bob has a continuous and equal-weighted form. From the view of experiment, the LSIs with a finite number of experimental settings are necessary. Such kinds of LSIs, especially adapted to the Werner state, are left as our future works. We expect that the results in this work could lead to further theoretical or experimental consequences.

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