Synchronizing quantum clocks with classical one-way communication: Bounds on the generated entropy

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Abstract

We describe separable joint states on bipartite quantum systems that cannot be prepared by any thermodynamically reversible classical one-way communication protocol. We argue that the joint state of two synchronized microscopic clocks is always of this type when it is considered from the point of view of an “ignorant” observer who is not synchronized with the other two parties.

We show that the entropy generation of a classical one-way synchronization protocol is at least $\Delta S = \hbar^2/(4\Delta E \Delta t)^2$ if $\Delta t$ is the time accuracy of the synchronism and $\Delta E$ is the energy bandwidth of the clocks. This dissipation can only be avoided if the common time of the microscopic clocks is stored by an additional classical clock.

Furthermore, we give a similar bound on the entropy cost for resetting synchronized clocks by a classical one-way protocol. The proof relies on observations of Zurek on the thermodynamic relevance of quantum discord. We leave it as an open question whether classical multi-step protocols may perform better.

We discuss to what extent our results imply problems for classical concepts of reversible computation when the energy of timing signals is close to the Heisenberg limit.

1 The thermodynamic advantage of quantum information transfer

Among the most important properties of quantum channels is their ability to create entangled states in a bipartite or multipartite system. It is well-known that every non-entangled (i.e. separable) joint state can be prepared using local quantum operations on each subsystem and classical communication among them. However, although it is possible to prepare all separable joint states with classical communication it may

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nevertheless be advantageous to use quantum communication for thermodynamic reasons. Bennett et al. [1] considered the following problem: Alice and Bob receive each a classical message $i$. Alice is instructed to prepare the quantum state $|\alpha_i\rangle$ when she receives the message $i$ and Bob should prepare the state $|\beta_i\rangle$ whenever he receives the message $i$. They found a set of tensor product states $|\psi_i\rangle := |\alpha_i\rangle \otimes |\beta_i\rangle$ with the property that all $|\psi_i\rangle$ are mutually orthogonal but neither all states $|\alpha_i\rangle$ nor all states $|\beta_i\rangle$ are orthogonal. The message $i$ received by Alice and Bob is certainly represented by any physical system and can hence be modeled without loss of generality by mutually orthogonal quantum states $|i\rangle$. Hence Alice and Bob share the state $|i\rangle \otimes |i\rangle$ after they have received the instruction. Clearly there is a unitary transformation $U$ acting on the composed Hilbert space transforming $|i\rangle \otimes |i\rangle$ into $|\alpha_i\rangle \otimes |\beta_i\rangle$. But there are no local unitary transformations $U_A$ and $U_B$ (independent of $i$) for Alice and Bob, converting $|i\rangle$ into $|\alpha_i\rangle$ and $|\beta_i\rangle$. The authors of [1] conjecture that Alice and Bob necessarily have to use thermodynamical irreversible operations in order to achieve their tasks. Note that, in their setting, the task is not to prepare a density matrix of the form

$$\sum_{i} p_i |\alpha_i\rangle \langle \alpha_i| \otimes |\beta_i\rangle \langle \beta_i|,$$

it is rather to prepare the state $|\alpha_i\rangle \otimes |\beta_i\rangle$ whenever the message was $i$. This problem does only make sense when a third party keeps the message $i$ in mind, i.e., if one has prepared the tripartite density matrix

$$\sum_{i} p_i |i\rangle \langle i| \otimes |\alpha_i\rangle \langle \alpha_i| \otimes |\beta_i\rangle \langle \beta_i|,$$

where the left component is a memory for the message. From this point of view, they consider the preparation of tripartite density matrices.

In this paper we consider, in contrast to the setting above, the preparation of bipartite density matrices. The type of density matrices considered here appears naturally when two synchronized “microscopic clocks” (quantum dynamical systems) are considered from the point of view of an observer without clock or without knowledge of the common time of the other two parties. These bipartite states have the property that they have no decomposition into locally distinguishable product states. States of this type have already been considered by Ollivier and Zurek [2]. The authors observed that 3 classically equivalent ways to define mutual information differ in the quantum case. They called the difference quantum discord. Zurek observed [3] the thermodynamic relevance of this quantity: He considered how much entropy has to be transferred into the environment when both parties want to obtain a pure state provided they are restricted to classical one-way communication from Alice to Bob. He showed that the thermodynamic cost is lower when they are allowed to use quantum communication and the difference of the entropy generation is exactly the quantum discord. In Sections 3 and 4 we consider in some sense the inverse problem to prepare correlations of this kind with classical one-way communication and given a bound on the entropy generation. In the following we will define the class of states that we consider and explain why they should be considered as synchronized clocks. The formal setting for defining synchronization is mostly taken from our paper [4]. In Section 6
we consider the problem to resolve the correlations of the joint states (“to reset” the synchronized clocks) using classical one-way communication and prove a lower bound on the quantum discord of the two-clocks state. In Section 6 we explain why our model assumptions seem to appear naturally in real applications. In Section 7 we discuss to what extent our results imply serious constraints or problems for classical concepts of extremely low power computation as soon as the signal energy of clock signals is close to the Heisenberg limit.

2 How to define synchronism of microscopic clocks

In the first place we note that synchronization involves clocks in their broadest sense, namely physical systems that are evolving in time. Consider for instance a classical system which has the unit circle $\Gamma$ in $\mathbb{R}^2 = \mathbb{C}$ as its “phase space” and the dynamics is just the rotation

$$z \mapsto z \exp(i\omega t).$$

This would be a very primitive clock since it shows the time only up to multiples of the period $T = 2\pi/\omega$. However, at least this is done perfectly.

Now consider two parties, $A$ and $B$ (Alice and Bob) each having a clock of this kind. Then we may say that both are synchronized with the external time.

We want to define synchronization between Alice and Bob in a way that does not refer to an absolute (external) time but formalizes only the fact that both clocks agree perfectly. An observer who does not know the time $t$ will not realize that both clocks are in the position $z = \exp(i\omega t)$ at time $t$, he will only observe that the positions $z_A$ and $z_B$ of both “pointers” agree. In the language of probability theory, he may describe this fact by a measure $\delta$ on $\Gamma \times \Gamma$ which is defined by

$$\delta(M \times K) = \lambda(M \cap K)$$

where $\lambda$ is the normalized Lebesgue measure on the unit circle. If $z_A$ and $z_B$ agree always up to a well-defined phase difference $\phi$ the clocks are also synchronized. Formally, we define:

**Definition 1** Two classical clocks (with equal frequency $\omega$) which are described by rotating pointer on the unit circle are called perfectly synchronized if the external observer describes the expected pointer positions $z_A$ and $z_B$ of their clocks by a joint probability distribution of the form

$$\delta_{\phi}$$

with

$$\delta_{\phi}(M \times K) := \lambda(M \cap (K \exp(i\phi)))$$

and $\phi$ is the phase difference which is known to the external observer.
An imperfect synchronism may, for instance, be a joint measure described by a probability density
\[ p(z_A, z_B) = f(z_A z_B^{-1}) \]
where \( f \) is an appropriate continuous function with support essentially at the point 1 or any other point \( \exp(i\phi) \) where \( \phi \) is the expected phase difference between the clocks. These remarks should only show why synchronized classical clocks may be described by a time invariant joint measure with non-trivial correlations.

An interesting kind of synchronism is given if both parties have “quantum clocks”, i.e., quantum systems with non-trivial time evolution. Consider for instance the case that each one has a two level system where \( |0\rangle \) and \( |1\rangle \) denote the lower and upper level, respectively. Each is evolving in time according to
\[ |\psi_t\rangle := \frac{1}{\sqrt{2}}(|0\rangle + \exp(-i\omega t)|1\rangle). \]
From the point of view of the “ignorant” observer the bipartite system is in the joint state
\[ \sigma := \frac{1}{T} \int_0^T \rho_t \otimes \rho_t \, dt \quad \text{with} \quad \rho_t := |\psi_t\rangle\langle\psi_t|. \]
In some sense, this is a perfectly synchronized system since the phases of both two-level systems agree exactly. On the other hand, none of both can read out the phase of its system and so they cannot use this synchronism as resource to obtain perfectly synchronized classical clocks. But they could obtain imperfectly synchronized classical clocks by performing measurements on their systems and adjust their clocks according to the measurement results. The most natural way to do this is to apply a covariant positive operator valued measure \((M_t)_{t \in [0, 2\pi/\omega]}\) satisfying the covariance condition
\[ \exp(iH t) M_s \exp(-iH t) = M_{s+t}, \]
where \( H := \text{diag}(1, 0) \) is the Hamiltonian of the two-level system. The probability density for obtaining the time \( t' \) if the true time is \( t \) is now given by
\[ p(t'|t) := \text{tr}(\rho_t M_{t'}). \]
The observer who only notices these two classical clocks which have been adjusted according to the estimated times will only note that the probability that \( A \) has the time \( t_A \) and \( B \) has the time \( t_B \) is given by
\[ q(t_A, t_B) = \int_0^T \text{tr}((\rho_t \otimes \rho_t) M_{t_A} \otimes M_{t_B}) \, dt. \]
If the POVM \((M_t)\) is not completely useless, i.e., if it is chosen such that \( \text{tr}(\rho_t M_{t'}) \) is not constant for all \( t, t' \) the probability distribution \( q \) contains some synchronism in the following sense: There are some correlations between the time estimations of \( A \) and \( B \). A large number of copies of such pairs of “weakly synchronized” clocks are as worthy as one well synchronized pair. Considerations of this kind are made precise in [4] where we have defined as so-called “quasi-order of clocks” and a “quasi-order of synchronism”. To rephrase the relevant part of this concept we have to define a clock:
Definition 2 A (quantum or classical) clock is a physical system with non-trivial dynamics. We denote it as a pair \((\rho, \alpha)\) where \(\rho\) is the state at the time 0 and \(\alpha := (\alpha_t)_{t \in \mathbb{R}}\) denotes the time evolution, i.e., \(\alpha_t(\rho)\) is the state at the time \(t\). In the classical case \(\rho\) is a probability distribution on the phase space \(\Omega\) of the system and \(\alpha_t : \Omega \to \Omega\) is a flow in the phase space shifting this measure.

In the quantum case \(\rho\) is a density matrix and \(\alpha_t(\rho) := \exp(-iHt)\rho\exp(iHt)\) is the time evolution according to the Hamiltonian \(H\).

The ability of the clock \((\rho, \alpha)\) to show the time is given by the distinguishability of the states \(\rho_t := \alpha_t(\rho)\).

In [4] we have developed a formal setting and theory to classify clocks with respect to their quality. A unifying framework describing quantum and classical physical systems is given by the \(C^*\)-algebraic approach [4].

To define synchronism of clocks formally we state that two physical systems with non-trivial separate time evolutions \(\alpha\) and \(\beta\) are to some extent synchronized if and only if their joint state is correlated in such a way that the correlations carry some information about a common time. The following setting includes also the trivial case that there is no common timing information:

Definition 3 A synchronism is a triple \((\sigma, \alpha, \beta)\) where \(\sigma\) is the joint state of a bipartite physical system, \(\alpha\) and \(\beta\) are the time evolutions corresponding to the first and second system, respectively and \(\sigma\) is invariant under \(\alpha_t \otimes \beta_t\).

Then we formalize whether a synchronized pair of clocks can be used as resource to synchronize another pair of clocks sufficiently:

Definition 4 A bipartite system \((\sigma, \alpha, \beta)\) is at least as good synchronized as \((\tilde{\sigma}, \tilde{\alpha}, \tilde{\beta})\) if there is a completely positive trace preserving map \(G\) satisfying the covariance condition

\[
G \circ (\alpha_t \otimes \beta_{-t}) = (\tilde{\alpha}_t \otimes \tilde{\beta}_{-t}) \circ G
\]

such that

\[
G(\sigma) = \tilde{\sigma}
\]

We say \((\sigma, \alpha, \beta)\) is sufficient as resource to prepare \((\tilde{\sigma}, \tilde{\alpha}, \tilde{\beta})\)

Note that the covariance condition ensures that the conversion process does not refer to additional synchronized clocks [4]. The least elements in this quasi-order are given by those synchronisms \((\sigma, \alpha, \beta)\) which satisfy

\[
(\alpha_t \otimes \beta_{-t})(\sigma) = \sigma.
\]

This is intuitively plausible since it does not require any synchronization procedure or synchronized clocks to prepare a system that is invariant with respect to relative time translations among both parties. Formally it is also easy to see since one can define \(G\) such that it maps every state of the considered resource system onto the state \(\sigma\). This satisfies clearly the covariance condition independent of the time evolution of the resource system.

Consequently, we define:
Definition 5 A bipartite system $(\sigma, \alpha, \beta)$ is not synchronized if $(\alpha_t \otimes \beta_{-t})(\sigma) = \sigma$. Otherwise we call it “to some extent synchronized”.

Now we briefly consider the question how the quality of a synchronization can be measured. Assume Alice and Bob perform measurements on their clocks in order to obtain information about the time. It is natural to restrict the attention to time-covariant measurements. Alice’s and Bob’s measurements are described by positive operator valued measurements $(M_t)_{t \in [0,T)}$ and $(N_s)_{s \in [0,T)}$, respectively. They satisfy $M_t = \alpha_t(M_0)$ and $N_s = \beta_s(N_0)$. We define the mean quadratic deviation between Alice’s and Bob’s measurement result by

$$D := \int_0^T \int_0^T (s - t)^2 \text{tr}(\sigma(M_t \otimes N_s)) \, dsdt.$$ 

Since $s, t$ are only defined modulo $T$ the expression $(s - t)^2$ is to be understood as

$$\min_{l \in \mathbb{Z}} \{(s - t + lT)^2\}.$$ 

In abuse of notation we denote

$$\Delta t := \sqrt{D},$$

and call $\Delta t$ the standard time deviation.

Now we restrict our attention to systems which are purely quantum, i.e., Alice’s and Bob’s clocks are moving according to their Hamiltonians $H_A$ and $H_B$ acting on the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

After the synchronization Alice and Bob share a joint density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. The following quantity will be a useful measure for the degree of synchronization since it quantifies the non-invariance of the joint state with respect to the relative time translation $\alpha_{t/2} \otimes \beta_{-t/2}$:

We consider the trace-norm of the derivative of the joint state with respect to relative time translation, i.e.,

$$\| \frac{d}{dt}(\alpha_{t/2} \otimes \beta_{-t/2})(\rho) \|_1 = \frac{1}{2} \|[H_A \otimes 1 - 1 \otimes H_B, \sigma]\|_1$$

(1)

These equations follow easily from the invariance of $\sigma$ with respect to the dynamical evolution which is generated by the Hamiltonian $H := H_A \otimes 1 + 1 \otimes H_B$. The quantity in eq. (1) has a less intuitive meaning than the standard time deviation but it will help to prove our main theorem in Section 4. The following Lemma draws a connection to the standard time deviation.

Lemma 1 Let the standard deviation $\Delta t$ of a synchronism $(\sigma, \alpha, \beta)$ be much smaller than the period $T$ of the clocks (Here we assume $\Delta t \leq T/12$ for technical reasons to get a simple proof). Then we have the following inequality:

$$\frac{1}{4\Delta t} \leq \| \frac{d}{dt}(\alpha_{t/2} \otimes \beta_{-t/2})(\sigma) \|_1.$$
Proof: Define the observable
\[ A := \int_{|s-t| \leq 2\Delta t} M_t \otimes N_s \, dsdt - \int_{|s-t| \geq 2\Delta t} M_t \otimes N_s \, dsdt. \]
Its operator norm is not greater than 1. Consider \( s - t \) as a random variable with values in \([-T/2, T/2]\). The generalized Tschebyscheff inequality states that for every random variable \( X \) the event \( |X| \geq \epsilon \) occurs at most with probability \( E(X^2)/\epsilon^2 \) when \( E(X^2) \) denotes the expectation value of \( X^2 \). We conclude that \( |s - t| \) exceeds \( 2\Delta t \) at most with probability \( 1/4 \) (note that here \( |s - t| \) is to be understood as the minimum \( |s - t - lT| \) for \( l \in \mathbb{Z} \)). This implies
\[ \text{tr}(\sigma A) \geq 3/4 - 1/4 = 1/2. \]
Now consider the state that is obtained from \( \rho \) by relative time translation of the amount \( 4\Delta t \). Set \( r := \Delta t \) and
\[ \tilde{\sigma} := (\alpha_{2r} \otimes \beta_{-2r})(\sigma). \]
Due to the covariance of the operators \( M_t \) and \( N_s \) and the condition \( \Delta t \leq T/12 \) we know that with probability at least \( 3/4 \) the values \( s \) and \( t \) satisfy
\[ -6r \leq s - t \leq -2r. \]
This implies obviously
\[ \text{tr}(\tilde{\sigma} A) \leq -3/4 + 1/4 = -1/2. \]
Since the expectation value
\[ \text{tr}((\alpha_{t/2} \otimes \beta_{-t/2})(\sigma) A) \]
decreases from \( 1/2 \) to \(-1/2\) within an interval of length \( 4r \) the average derivative of
\[ \frac{d}{dt} \text{tr}(\alpha_{t/2} \otimes \beta_{-t/2}(\sigma) A) \]
is less or equal to \(-1/(4r)\) on this interval. Note that the modulus of expression (2) is bounded by
\[ \| \frac{d}{dt} \alpha_{t/2} \otimes \beta_{-t/2}(\sigma) \|_1, \]
which is constant for all \( t \). Hence expression (3) has to be at least \( 1/(4r) \). \( \square \)

In the following section we will prove a lower bound on the entropy increase based on Lemma 5. We do not claim that the bound is tight since it uses inequalities connecting trace-norm distances between quantum states with relative entropies. However, we were not able to find tighter bounds in this general setting.
3  A simple classical synchronization protocol

The following scheme shows a straightforward method to achieve synchronization for a
simple type of quantum clocks on Hilbert spaces of arbitrary finite dimension. Let the
clocks of Alice and Bob each be described by the Hilbert space $C^n$ and the Hamiltonian
be $H := diag(0, 1, \ldots, n - 1)$. Let $|\psi\rangle$ be a uniform superposition of basis states $|j\rangle$ in
$C^n$:

$$|\psi\rangle := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |j\rangle. \quad (4)$$

Let $\rho_t$ be the density matrix obtained from $|\psi\rangle\langle\psi|$ after the time $t$, i.e.,

$$\rho_t := e^{-iHt}|\psi\rangle\langle\psi|e^{iHt}.$$

If these clocks are optimally synchronized Alice and Bob share the joint state

$$\sigma := \int_0^{2\pi} \rho_t \otimes \rho_t \, dt.$$

Due to a theorem of Carathéodory [5] this state can also be obtained by a finite convex
combination of product states. Elementary Fourier analysis arguments show that $\sigma$
can also be obtained by

$$\sigma = \frac{1}{2n-1} \sum_{j=1}^{2n-2} \rho_{t_j} \otimes \rho_{t_j}, \quad (5)$$

with $t_j := 2\pi j/(2n - 1)$.

We are looking for a protocol with the following properties:

1. Alice and Bob start with a product state. Both are allowed to perform any arbitrary
   local operations on their physical systems. Their physical systems may be
   of arbitrary dimension and they have unrestricted access to ancilla systems. The
   only physical systems with non-trivial dynamics are given by one Hamiltonian
   quantum dynamical system on Alice's side (Alice's quantum clock), one Hamiltonian
   quantum dynamical system on Bob's side (Bob's quantum clock), and a
   classical clock on Alice's side. All the other systems have trivial time evolution.

2. Alice sends Bob a package consisting of the classical clock and a memory with
   some additional information.

3. Bob receives the clock and the memory and keeps both. Then he is allowed to
   implement any transformation on the extended system consisting of his quantum
   clock and the received package.

4. At the end of the protocol the joint state of Alice's and Bob's quantum clocks
   should be uncorrelated with the classical clock. Otherwise there would be trivial
   thermodynamically reversible way to achieve synchronization using an ideal
   “circle clock” as introduced at the beginning of Section 2. Alice performs the
   transformation $\exp(-iHt)$ if the classical clock has the time $t$. She sends the
   classical clock to Bob and he implements $\exp(-iHt')$ according to the actual
time \( t' \). For the external observer the result is a tripartite joint state of the three clocks. In Section 5 we will explain in detail why we exclude protocols where both quantum clocks are afterwards committed to a classical clock. The idea is that we have applications in mind where the classical clock is a signal which is \textit{absorbed} by the receiver (Bob) and hence does not exist any longer.

The following protocol satisfies all these requirements and prepares the state (5):

We assume that Alice has a classical memory with \( 2^n - 1 \) possible states. We assume that it is not initialized, i.e., it is in the mixed state

\[
\gamma := \frac{1}{2n-1} \sum_{j=0}^{2n-2} |j\rangle\langle j|
\]

Her clock is assumed to be in the state \( |\psi\rangle\langle \psi| \) (defined as in eq. (4)). Then she performs a unitary transformation conditional on the state of the memory. If the memory state is \( |j\rangle\langle j| \) she implements the unitary operation

\[
\exp(-iH \frac{2\pi j}{2n-1})
\]

(6)
on the quantum clock. Furthermore we assume that she has a classical clock. If it shows the time \( t \) she implements

\[
\exp(-iHt)
\]

(7)
on her quantum clock. Afterwards she sends the memory and the classical clock to Bob. When he receives both he implements

\[
\exp(-iHt')
\]

(8)
when the classical clock shows the time \( t' \). The operations (7) and (8) ensure that the joint state obtained at the end of the protocol does not depend on the time the message needs to reach Bob. Then he implements also the conditional transformation (6) whenever the message is \( j \). Now the reduced state of the system consisting of Alice’s and Bob’s clocks is already the desired state \( \sigma \) but the two clocks are still correlated with the memory. So far, the protocol is thermodynamically reversible and the joint state of Alice’s and Bob’s quantum clock and the memory is uncorrelated with the classical clock as desired.

However, the joint state of memory and both clocks evolves in time. Hence it is only pure from the point of view of somebody who knows the time which has passed by since the synchronization has taken place. From the point of view of an ignorant observer the protocol is hence only reversible when the joint state is still correlated with an additional classical clock showing the time that has been passed by. Here we do not allow this and have hence entropy increase by “forgetting the time”. The reduced state \( \gamma \) of the memory is obviously stationary since the memory is a stationary system by assumption. The reduced state of the composed system consisting of both clocks is also stationary since it is the desired state \( \sigma \). Forgetting the time destroys the correlations between memory and both clocks and leads to the state

\[
\sigma \otimes \gamma.
\]
Hence the entropy increase is exactly the mutual information between memory and both clocks. The entropy of the memory is $S(\gamma)$, the entropy of the joint state of Alice’s and Bob’s clock is $S(\sigma)$. The entropy of the joint state (before the time has been forgotten) is $S(\gamma)$. Hence the mutual information is $S(\sigma)$. It can be calculated as follows. Let $H_j$ be the eigenspace of $H_A \otimes 1 + 1 \otimes H_B$ corresponding to the eigenvalue $j$. They have dimension $j + 1$ for $j \leq n$ and dimension $2n - j + 1$ for $j \geq n$. Since $\sigma$ is stationary it is block diagonal with respect to this decomposition into subspaces. By elementary Fourier analysis one can see that all entries of each block matrix are $1/n^2$. The eigenvalues of a matrix of dimension $\sigma$ which has only 1 as entries are given by 0, 0, …, 0, d. Hence the eigenvalues of $\sigma$ are

\[ \frac{j}{n^2} \]

for $j = 0, 1, \ldots, n$. For each $1 \leq j \leq n - 1$ the corresponding eigenvalue occurs twice, the eigenvalue $1/n$ occurs only once and the eigenvalue 0 occurs $(n - 1)^2$ times. The entropy generated by the protocol can easily be calculated from these eigenvalues.

## 4 Entropy increase in a classical one-way synchronization protocol

Alice sends Bob a classical signal (“clock”) that is correlated with her clock. It is a classical physical system with non-trivial dynamics, i.e., a flow $\gamma_t : \Omega \rightarrow \Omega$ on its phase space $\Omega$. Instead of sending such a clock Alice could also send Bob a composed system consisting of the following two systems which are easier to deal with:

1. A system which is described by the same phase space $\Omega$ as the original system but with trivial time evolution, i.e, the measure $\mu$ is stationary in time and

2. a perfect classical clock which tells Bob exactly the time that has been passed by since Alice has sent the message on the phase space $\Omega$.

Then Bob can implement the dynamical evolution $\gamma_t$ corresponding to the original system. For simplicity we assume the message space $\Omega$ to consist of finitely many points $\omega_1, \ldots, \omega_l$. Since we shall derive a bound that is independent of the message size we expect that it holds also in the limit of infinite messages.

After Alice has sent the message we have a joint state on $\Omega$ and $H_A$ such that Alice’s clock is in the state $\rho_j$ if the message contains the symbol $\omega_j$. This case occurs with probability $p_j$. We denote the joint state by

\[ \sum_{j \leq t} p_j \rho_j \otimes \omega_j, \]

keeping in mind that the left component of the tensor product is a quantum density matrix and the right component a point in a classical space.

When Bob receives the message he may certainly perform an operation on his quantum clock conditional on the message and throw the message away. However,
this is not the most general operation. We allow also that he could, for instance, implement a swap operation exchanging the state of the signal and his clock. To describe operations like this we have to change the point of view. From now on we consider the medium which carries the message as a quantum system with Hilbert space $\mathcal{H}_\Omega$ such that $\omega_j$ are mutually orthogonal density matrices. The orthogonality expresses the fact that only classical messages are allowed. At the time instant where the message arrives the joint state of Alice and Bob is

$$\nu := \sum_j p_j \rho_j \otimes (\omega_j \otimes \eta),$$

where $\eta$ is an arbitrary density matrix of Bob’s quantum clock. Bob owns the two rightmost components and Alice the leftmost component of the three-fold tensor product. Regardless of the unitary operation that Bob performs on $\mathcal{H}'_B := \mathcal{H}_\Omega \otimes \mathcal{H}_B$ the states $\omega_j \otimes \eta$ are always transformed into mutually orthogonal states $\sigma_j$. He may for instance apply some transformation $U$ according to the state of the classical clock. So far, we assumed that Alice and Bob implement only reversible transformations. Since we did not specify the physical systems which they use this restriction does not imply any loss of generality. After the protocol is finished the joint state $\nu$ evolves according to its autonomous dynamical evolution:

$$(\alpha_t \otimes 1 \otimes \beta_t)(\nu).$$

We have emphasized that Alice and Bob are not allowed to keep the classical clock, they have to forget the time in order to obtain a joint state of their quantum clocks that is no longer correlated with the classical clock. A priori, it is not clear that the state $\nu$ cannot be stationary and “forgetting” the time $t$ produces entropy. However, in the following we show that $\nu$ cannot be stationary and prove a lower bound on the entropy difference between $\nu$ and the time average

$$\overline{\nu} := \int_0^T (\alpha_t \otimes 1 \otimes \beta_t)(\nu) \, dt.$$ 

The idea is that every joint state that is prepared in a reversible one-way protocol has a decomposition into product states which are mutually orthogonal when they are restricted to Bob’s system. On the other hand, there is no time-invariant joint state with non-trivial synchronization with this property.

In analogy to our results on the minimal entropy generation when timing information is read out from a microscopic clock [6] our bound on the generated entropy relies on the energy bandwidth of the joint state. Explicitly the bandwidth is defined as follows.

**Definition 6** Let $(Q^A_r)_{r \in \mathbb{R}}$ and $(Q^B_r)_{r \in \mathbb{R}}$ the families of spectral projections of $H_A$ and $H_B$, respectively, i.e., $Q^A_r$ projects onto the subspace of $H_A$ that corresponds to eigenvalues not greater than $r$.

Let $\sigma$ be a joint state of Alice and Bob. The bandwidth of Alice’s clock is the least number $\Delta E_A$ such that there exists $E \in \mathbb{R}$ such that $(Q^A_{E+\Delta E_A} \otimes 1 - Q^A_{E} \otimes 1)\sigma = \sigma$. Define $\Delta E_B$ similarly.
Note that for finite spectral widths $\Delta E_A$ and $\Delta E_B$ the time evolution can equivalently be described by norm bounded Hamiltonians:

$$H_A := Q_{E+\Delta E_A} Q_E H_A - 1(E + \Delta E_A/2).$$

Therefore we assume without loss of generality $\|H_A\| \leq \Delta E_A/2$ and $\|H_B\| \leq \Delta E_B/2$.

We will need the following Lemma:

**Lemma 2** Let $W$ be a selfadjoint operator on a (not necessarily finite dimensional) Hilbert space $W$ with discrete (not necessarily finite) spectrum. Let $\rho$ be an arbitrary density matrix on $W$. Then the entropy difference between $\rho$ and the average

$$\overline{\rho} := \lim_{T \to \infty} \frac{1}{T} \int_0^T \exp(-iWt)\rho \exp(iWt)dt$$

is given by the Kullback-Leibler relative entropy

$$K(\rho||\overline{\rho}) = tr(\rho \ln \rho) - tr(\rho \ln \overline{\rho}).$$

The proof is an immediate conclusion from Lemma 1 in [6] using the observation that the average state $\overline{\rho}$ coincides with the post-measurement state after measuring the observable $W$. Explicitly one has

$$\overline{\rho} = \sum_j p_j \rho_j,$$

where $Q_j$ are the spectral projections of $W$. Note that discreteness of the spectrum ensures that the time average exists [7]. Now we can state our main theorem:

**Theorem 1 (Entropy generated by synchronization)**

Let $\Delta t$ be the standard time deviation of the synchronism $(\sigma, \alpha, \beta)$. Let $\Delta E$ be the total energy bandwidth of $\sigma$, i.e., $\Delta E := \Delta E_A + \Delta E_B$. Then every classical one-way protocol to prepare the state $\sigma$ generates at least the entropy

$$\Delta S = \frac{1}{16(\Delta E \Delta t)^2}.$$

**Proof:** Let us modify our notation for simplicity. In contrast to the definition above, we denote by $\beta_t$ the joint time evolution of Bob’s clock and his memory (and not the dynamics of Bob’s clock alone). It acts on $\mathcal{H}'_B$. Given the time $t$ after the synchronization has been taken place the joint state on $\mathcal{H}_A \otimes \mathcal{H}'_B$ is given by

$$\nu = \sum_j p_j \rho_j \otimes \mu_j$$

where $\mu_j$ are mutual orthogonal density matrices on $\mathcal{H}_B$ due to the arguments above.

The resulting joint state $\sigma$ (which is the desired synchronism) is obtained from $\nu$ by forgetting the time $t$. It is hence the time average $\sigma := \overline{\nu}$. First we consider the trace norm distance between $\nu$ and $\overline{\nu}$:

$$\|\sum_j p_j \rho_j \otimes \mu_j - \overline{\nu}\|_1.$$ (9)
Let $A$ be an arbitrary observable with norm 1. With $H := H_A \otimes 1 + 1 \otimes H_B$ we have $\|[H, A]\| \leq 2\|H\| = \Delta E$.

Note that for any two matrices $C$, $D$ one has $\text{tr}(CD) \leq \|C\|_1 \|D\|$ where $\|D\|$ denotes the usual operator norm of $D$. Hence expression 9 is at least

$$\frac{\text{tr}(i[H, A](\sum_j p_j \rho_j \otimes \mu_j - \nu))}{\Delta E} = \frac{\text{tr}(i[H, A] \sum_j p_j \rho_j \otimes \mu_j)}{\Delta E}.$$  

This equality is due to the time invariance of $\nu$.

Now we choose observables $A_j$ with $\|A_j\| = 1$ such that

$$\text{tr}(-iA_j[H_A, \rho_j]) = \text{tr}(i[H_A, A_j] \rho_j) = \|[H_A, \rho_j]\|_1.$$  

Let $P_j$ be a complete set of mutually orthogonal projections separating the states $\mu_j$, i.e., $P_j \mu_j = \mu_j$ and $P_i \mu_j = 0$ for $i \neq j$. With the definition $A := \sum_j A_j \otimes P_j$ we obtain

$$\text{tr}(i[H, A] \sum_j p_j \rho_j \otimes \mu_j) = \text{tr}(i[H, \sum_l A_l \otimes P_l] \sum_j p_j \rho_j \otimes \mu_j) = \sum_{l,j} \text{tr}((i[H_A \otimes 1, A_l] \otimes P_l)(\rho_j \otimes \mu_j)) + \sum_{l,j} \text{tr}(iA_l \otimes [H_B, P_l](\rho_j \otimes \mu_j)).$$  

By simple calculations we find that the only remaining term is

$$\sum_j p_j \text{tr}(i[H_A, A_j] \rho_j) = \sum_j p_j \|[H_A, \rho_j]\|_1.$$  

Using the bound $K(\nu|\nu) \geq \frac{\|\nu - \nu\|^2}{2}$ (see [8]) and Lemma 2 we conclude for the entropy generation

$$\Delta S \geq \frac{(\sum_j p_j \|[H_A, \rho_j]\|_1)^2}{2\Delta E}. \quad (10)$$  

On the other hand we know that the quality of synchronization can be defined by

$$\frac{\|d}{dt} \alpha_{t/2} \otimes \beta_{-t/2}(\sigma)\|_1 = \|[\frac{1}{2}(H_A \otimes 1 - 1 \otimes H_B), \nu]\|_1.$$  

(11)

In the following we will use the abbreviation $\overline{C}$ for the “time mean” of an operator $C$ obtained by averaging over the evolution $\alpha_t \otimes \beta_t$. The latter term in eq. (11) can be estimated as follows:

$$\|[\frac{1}{2}(H_A \otimes 1 - 1 \otimes H_B), \nu]\|_1 = \|[H_A \otimes 1, \nu]\|_1 = \|[\overline{H_A \otimes 1}, \nu]\|_1 \leq \|[H_A \otimes 1, \nu]\|_1 \leq \|[H_A \otimes 1, \nu]\|_1.$$  

(12)
The last equality is due to the observation that the time evolution $\alpha_t \otimes \beta_t$ commutes with the superoperator $[H_A \otimes 1, \cdot]$. The inequality is due to the fact that averaging over unitary dynamical evolution is a contractive map on the set of matrices. We have

$$\| [H_A \otimes 1, \sum_j p_j \rho_j \otimes \mu_j] \|_1 \leq \| [H_A \otimes 1, \sum_j p_j \rho_j \otimes \mu_j] \|_1 \leq \sum_j p_j \| [H_A, \rho_j] \|_1.$$  

With inequality (10) and (12) we conclude

$$\Delta S \geq \frac{1}{16(\Delta E \Delta t)^2}.$$  

So far, we used natural units, i.e., Planck’s constant was assumed to be 1. Using SI-units we obtain

$$\Delta S \geq \frac{\hbar^2}{16(\Delta E \Delta t)^2}$$

as lower bound on the entropy generation for a classical one-way protocol.

5 Physical models for sending timing information by classical communication

Consider two microscopic clocks (like the systems in $C^n$ considered in Section 3) that should be synchronized. Both systems are controlled by electronic devices which are connected by an optical fiber. The fiber allows them to communicate (see Fig. 1). We send a signal to one of both devices (or to both) which triggers the synchronization procedure. Essential in our setting is that this signal is much less localized in time than $\Delta t$, the time accuracy of the synchronization. Otherwise we deal with a synchronization procedure that is run with absolute time. Note that each signal with energy much less than $\hbar/\Delta t$ satisfies this criterion by Heisenberg’s uncertainty relation. Say, for instance, the signal arrives at Alice’s clock and starts the protocol. Then the device connected to Alice’s clock sends a light pulse to Bob carrying some information about the actual time of clock A.

The optical fiber itself is a quantum channel. Its quantum state may be described by a density matrix $\rho$ in an appropriate Fock space with time evolution

$$\rho_t := \exp(-iH_L t) \rho \exp(iH_L t)$$

according to the corresponding Hamiltonian $H_L$ of the light field. The states $\rho_t$ are necessarily non-commuting density matrices since this holds for every non-trivial Hamiltonian evolution [9]. The timing information is hence necessarily to some extent quantum
information. However, now we include an assumption to the setting which makes the protocol to be a classical communication protocol: Assume that Bob’s device starts his clock as soon as the light signal arrives by measuring at every moment whether the light pulse has arrived or not.

At first sight it seems that the entropy generation by this protocol could simply be derived from results in [6]. There we have shown that every measurement that extracts timing information from a clock with energy bandwidth $\Delta E$ produces at least the entropy

$$\Delta S \geq \frac{\hbar^2}{2(\Delta E \Delta t)^2},$$

whenever the measurement allows to determine the time up to an error of $\Delta t$. Hence it seems that the measurement of the time of arrival measurement must necessarily generate the entropy $\Delta S$ of inequality (13). However, this argument is not correct since we want to calculate the entropy increase from the point of view of somebody who does not know the absolute time. From his point of view the quantum state of the optical fiber is not the state $\rho_t$ of the light field at a specific time instant $t$. It is rather a mixture over all possible time instants.

The following example shows that the entropy production in a classical synchronization protocol is not necessarily due to a measurement but rather, as argued in Section 4 by the fact that no absolute time is available. Let Alice and Bob have “n-level” clocks as in Section 3. Let Alice’s clock be in a maximally mixed state. Let Alice perform a measurement with respect to the Fourier basis $|\psi_{t_j}\rangle$ with $t_j := 2\pi/n$ and $|\psi_t\rangle := \exp(-iHt)|\psi\rangle$ with $|\psi\rangle$ as in eq. (4). This measurement does not generate any entropy since the clock was already in its maximally mixed state before the measurement. If Alice writes the measurement outcome into a memory that is sent to Bob we can continue with the protocol as in Section 3. Then entropy is generated not earlier than the moment where the correlation with the memory is lost in order to get a stationary joint state.

It may seem a little bit artificial to assume that a classical clock is available during the transfer of timing information but has to be cleared after the protocol has been
finished. In the situation of Fig. 1 the light signal may have large energy bandwidth compared to the clocks of Alice and Bob. Hence it may be considered approximatively as a classical clock. However, this signal is absorbed by Bob’s device. This does not necessarily mean that the signal energy itself is lost. The device may be designed in an energy-saving way such that the signal energy is used to reload a capacitor (by a solar cell, for instance). The fact that the signal is absorbed by Bob’s device means indeed that the classical clock used for the time transfer is no longer available. Of course it is not necessary that the light signal is absorbed. It could also be captured by a cavity such that the cavity contains an oscillating light field. But in this case Bob’s clock should be considered as the composed system consisting of the cavity and Bob’s original clock. Anyway, the classical clock which is approximatively given by a classical light field in the optical fiber, is no longer existent after the light pulse has arrived. The authors think that this shows that the setting presented here is rather natural.

6 Cost for resetting synchronized clocks: quantum discord

As already mentioned in Section quantum communication may also be advantageous when the synchronized clocks should be reset. This is shown in the following example. Let Alice and Bob each have a “two-level clock”, i.e., two-level systems in the state

\[ |\psi_t\rangle := \frac{1}{\sqrt{2}}(|0\rangle + \exp(-i\omega t)|1\rangle).\]

Such a clock is reset, for instance, when it is set into the state |0\rangle (one may think of a stop-watch which was running for a while and should be stopped and reset afterwards). The problem is that neither Alice nor Bob is able to perform a thermodynamically reversible operation which converts \(|\psi_t\rangle\) into \(|0\rangle\) when no additional clock is available since they do not know \(t\). However, Alice and Bob could each have an ancilla qubit which is a degenerated two-state system with the zero operator as Hamiltonian. Let these qubits be initialized to |0\rangle. Then both of them may exchange the state of the two-level system with the state of the degenerated system. Obviously, both clocks are set to |0\rangle by this procedure. However, in case Alice and Bob have not agreed upon a common time instant where they perform the operation this whole process is irreversible since the correlation between Alice’s and Bob’s clock is lost. From the point of view of the ignorant observer the mixture of \(|\psi_1\rangle \otimes |\psi_t\rangle\) over all \(t\) is transformed into an uncorrelated state where Alice’s and Bob’s ancilla qubits are both in the maximally mixed state. But, in analogy to the requirements listed in Section we allow Alice and Bob to transfer a classical clock. However, we assume that the synchronized quantum clocks are not correlated with the classical clock when the resetting procedure is started. Using classical clock transfer they can synchronize their “clock resetting” process and assure that the joint state of their ancilla qubits is a mixture of all states \(|\psi_1\rangle \otimes |\psi_t\rangle\). As long as they keep this correlated joint state they have indeed reset their clocks in a reversible way. However, this is an unsatisfactory end of the process: Maybe Alice would like to
synchronize her clock afterwards with a third party, say Carol and would like to reset this synchronization after a while in order to synchronize with Dave. Then she would need an additional ancilla qubit for each party in order to keep the correlations. To avoid those unrestricted resource requirements we would like to resolve all correlations between Alice’s and Bob’s clocks or ancillas. When quantum communication is allowed Bob may for instance have two initialized ancilla qubits and the mixture of all states $|\psi_t\rangle \otimes |\psi_t\rangle$ may be converted into the same mixture in Bob’s two ancilla qubits. But here we will only allow classical one-way communication and show that resolving the correlations between ancillas or clocks will unavoidably lead to dissipation. We will show this by proving that the joint state of synchronized microscopic clocks has always quantum discord (“quantum correlations without entanglement” [2]). Let us explain briefly this concept introduced by Ollivier and Zurek. For two classical systems $A$ and $B$ (formally described by random variables $A$ and $B$) one may define mutual information in two equivalent ways [10]:

1. The symmetric expression

$$I(A : B) := H(A) + H(B) - H(A, B),$$

where $H(A)$, $H(B)$, $H(A, B)$ are the Shannon entropies of $A$, $B$, or joint entropy of $A$ and $B$, respectively.

2. The asymmetric expression

$$I(A : B) := H(B) - H(B|A),$$

where $H(B|A)$ is the entropy of $B$ given $A$.

The quantum analogue of 1. is given by

$$S(\sigma^A) + S(\sigma^B) - S(\sigma)$$

(14)

where $S(.)$ denotes the von-Neumann entropy [8], $\sigma$ is the joint density matrix of the system $A$ and $B$ and $\sigma^A$ and $\sigma^B$ are the restrictions of the state $\sigma$ to the subsystem $A$ and $B$, respectively.

The analogue of 2. refers to measurements on the system $A$. As in [2] we restrict our attention to von-Neumann measurements described by a family $(P_j)$ of orthogonal projections acting on Alice’s Hilbert space $\mathcal{H}$. Define the probabilities

$$p_j := \text{tr}((P_j \otimes 1)\sigma)$$

and the selected post-measurement states

$$\sigma_j := (P_j \otimes 1)\sigma(P_j \otimes 1)/p_j.$$

Then the entropy of $B$ given the measurement outcome is given by

$$\sum_j p_j S(\sigma_j^B)$$

\[\text{1}\]In our setting this is no loss of generality since one may count the quantum clock together with arbitrarily many ancillas as a new clock.
and the difference
\[ S(\sigma^B) - \sum_j p_j S(\sigma_j^B) \]
may be considered as the quantum analogue of 2.

The discord \( \partial(B|A) \) as introduced by Ollivier and Zurek \[2\] is the minimum of all values \( \partial(P_j)(B|A) \) over all measurements \((P_j)\), with
\[
\partial(P_j)(B|A) := S(\sigma^A) - S(\sigma) + \sum_j p_j S(\sigma_j^B).
\]

This quantity is the difference between both possible translations of mutual entropy.

A rather artificial combination of 1. and 2. leads to
\[
I(A : B) = H(A) + H(B) - (H(A) + H(B|A)).
\]

(15)

Note \( H(A) \) has two possible translations into the quantum setting. It may either be the entropy of \( \rho^A \) or of the unselected post-measurement \( \sum_j p_j \sigma_j^A \). As Zurek noted \[3\] one may also choose the first possibility for the first term \( H(A) \) and the second possibility for the second term \( H(A) \). This leads to
\[
S(\sigma^A) + S(\sigma^B) - S\left( \sum_j p_j \sigma_j^A \right) - \sum_j p_j S(\sigma_j^B).
\]

(16)

In \[3\] Zurek considered the difference between expression (14) and expression (16):
\[
\delta(P_j)(B|A) := S\left( \sum_j p_j \sigma_j^A \right) + \sum_j p_j S(\sigma_j^B) - S(\sigma).
\]

(17)

He called the minimum of all values \( \delta(P_j)(B|A) \) over all measurements also discord and denoted it by the symbol \( \delta(B|A) \). He showed this expression of discord to be thermodynamically relevant \[3\]. It is the difference between the entropy cost for erasing the joint state of a bipartite quantum memory when only classical one-way communication from Alice to Bob is allowed to the erasure cost in optimal quantum protocols. In our setting it is the difference between the entropy that has to be transferred to the environment when Alice and Bob reset their synchronized clocks using classical communication to the amount they would have to transfer to the environment if quantum communication was allowed. We will prove a lower bound on the discord of two synchronized quantum clocks. The following Lemma will be useful in our proof:

Lemma 3 The expression \( \delta(P_j)(B|A) \) as in eq. (17) associated with a measurement \((P_j)\) can be written as a sum of Kullback-Leibler distances:
\[
\delta(P_j)(B|A) = K(\sigma|| \sum_j p_j \sigma_j) + \sum_j p_j K(\sigma_j|| \sigma_j^A \otimes \sigma_j^B)
\]
\[
= K(\sigma|| \sum_j p_j \sigma_j) + K(\sum_j p_j \sigma_j|| \sum_l p_l \sigma_l^A \otimes \sigma_l^B).
\]
This expression has a rather intuitive meaning: The first summand is the distance between the pre-measurement state and the unselected post-measurement state (as mentioned in the remarks after Lemma 2 this coincides with the entropy generated by the measurement). The second term is the average distance between the selected joint state and the tensor product of the reduced (selected) post-measurement states. If a joint state has discord this means that each measurement either generates entropy or it does not resolve the correlations, i.e., the selected post-measurement state is still correlated. This suggests already that correlations with discord cannot be resolved in a thermodynamically reversible way by measurements of one party.

Proof: (of Lemma 3): Obviously one has

\[
\delta(P_j)(B|A) = S(\sum_j p_j \sigma_j^A) + \sum_j p_j S(\sigma_j^B) - S(\sum_j p_j \sigma_j) + S(\sum_j p_j \sigma_j) - S(\sigma) = S(\sum_j p_j \sigma_j^A) + \sum_j p_j S(\sigma_j^B) - S(\sum_j p_j \sigma_j) + K(\sigma||\sum_j p_j \sigma_j).
\]

Since the states \(\sigma_j\) are orthogonal and also their restrictions to \(A\) are orthogonal we have

\[
S(\sum_j p_j \sigma_j) = \sum_j p_j S(\sigma_j) + H(p)
\]

and

\[
S(\sum_j p_j \sigma_j^A) = \sum_j p_j S(\sigma_j^A) + H(p),
\]

where \(H(p)\) is the amount of information of the measurement result. We conclude

\[
\delta(P_j)(B|A) = \sum_j p_j S(\sigma_j^A) + \sum_j p_j S(\sigma_j^B) - \sum_j p_j S(\sigma_j) + K(\sigma||\sum_j p_j \sigma_j).
\]

Using the identity

\[
tr(\sigma_j \ln(\sigma_j^A \otimes \sigma_j^B)) = tr((\sigma_j^A \otimes \sigma_j^B) \ln(\sigma_j^A \otimes \sigma_j^B))
\]

we obtain

\[
\delta(P_j)(B|A) = \sum_j p_j K(\sigma_j||\sigma_j^A \otimes \sigma_j^B) + K(\sigma||\sum_j p_j \sigma_j)
\]

by elementary calculation.

The last equality of the statement, namely

\[
\sum_j p_j K(\sigma_j||\sigma_j^A \otimes \sigma_j^B) = K(\sum_j p_j \sigma_j||\sum_j p_j \sigma_j^A \otimes \sigma_j^B),
\]

is easy to check since the states \(\sigma_j\) and \(\sigma_j^A \otimes \sigma_j^B\) act on the images of the projections \(P_j \otimes 1\), i.e., they act on mutually orthogonal subspaces for different \(j\). \(\Box\)

Now we shall prove a lower bound on the quantum discord of the joint state of two synchronized clocks.
Theorem 2 (Discord of synchronized clocks)

Let \((ρ, α, β)\) be a synchronism of quantum clocks with equal period. Then the quantum discord between \(A\) and \(B\) is non-vanishing and we have

\[
\delta(A|B) \geq \frac{1}{256(ΔtΔE)^2} \quad \text{and} \quad \delta(B|A) \geq \frac{1}{256(ΔtΔE)^2},
\]

where \(Δt\) is the accuracy of the synchronization, i.e., the standard time deviation and \(ΔE\) is the energy bandwidth of the clocks.

Proof: Due to the symmetry with respect of \(A\) and \(B\) it is sufficient to prove the second inequality. We prove the bound by showing that it holds for every von-Neumann measurement \((P_j)\). We define

\[
d_1 := \|\sigma - \sum_j p_j σ_j\|_1
\]

and

\[
d_2 := \|\sum_j p_j σ_j - \sum_j σ_j^A ⊗ σ_j^B\|_1.
\]

The idea of the proof is that \(d_1\) and \(d_2\) cannot be simultaneously small. Otherwise the joint state \(σ\) of the synchronized clocks would be close to the state

\[
\sum p_j σ_j^A ⊗ σ_j^B.
\]

This state consists of product states that are locally distinguishable on Alice's subsystem. The fact that no time invariant state with non-trivial synchronization can have this property was already the key idea in the proof of Theorem 1.

Now we have

\[
\|[1 \otimes H_B, σ]\|_1 \leq \|[1 \otimes H_B, \sum_j p_j σ_j^A ⊗ σ_j^B]\|_1 + (d_1 + d_2)ΔE
\]

\[
= \sum_j p_j\|[1 \otimes H_B, σ_j^A ⊗ σ_j^B]\|_1 + (d_1 + d_2)ΔE.
\]

The first inequality follows from the fact that \(H_B\) is bounded with \(\|H_B\| ≤ ΔE/2\). Therefore the commutator with \(σ\) cannot differ from the commutator with

\[
\sum_j p_j σ_j^A ⊗ σ_j^B
\]

by more than the amount \((d_1 + d_2)ΔE\). The last equality is due to the orthogonality of the states \(σ_j^A\) since \(1 \otimes H_B\) acts only on the second tensor component.

In analogy to the proof of Theorem 1 we choose observables \(A_j\) such that

\[
tr(i[H_B, A_j σ_j^B]) = \|[H_B, σ_j^B]\|_1,
\]

and define \(A := \sum_j P_j ⊗ A_j\), where \(P_j\) are the measurement operators.
Some calculations show

\[ \| [1 \otimes H_B, \sum_j p_j \sigma_j^A \otimes \sigma_j^B] \|_1 = \text{tr}(i[1 \otimes H_B, A] \sum_j p_j \sigma_j^A \otimes \sigma_j^B) = \text{tr}(i[H, A] \sum_j p_j \sigma_j^A \otimes \sigma_j^B). \]

Since the difference between

\[ \text{tr}(i[H, A] \sum_j p_j \sigma_j^A \otimes \sigma_j^B] \]

and

\[ \text{tr}(i[H, A] \sigma) \]

cannot be greater than \((d_1 + d_2) \Delta E\) according to the same arguments as above we have

\[ \| [1 \otimes H_B, \sigma] \|_1 \leq \text{tr}(i[H, A] \sigma) + (d_1 + d_2)2\Delta E = (d_1 + d_2)2\Delta E. \]

The last equality is due to the stationarity of the state \(\sigma\). Due to the inequality

\[ K(\gamma || \tilde{\gamma}) \geq \frac{\| \gamma - \tilde{\gamma} \|^2}{2} \]

we have

\[ \delta_{(P_j)}(B|A) \geq \frac{d_1^2 + d_2^2}{2} \]

With

\[ d_1 + d_2 \geq \frac{[1 \otimes H_B, \sigma]_1}{2\Delta E} \]

we have

\[ d_1^2 + d_2^2 \geq \frac{[1 \otimes H_B, \sigma]_1^2}{8(\Delta E)^2}. \]

We conclude

\[ \delta_{(P_j)}(B|A) \geq \frac{[1 \otimes H_B, \sigma]_1^2}{16(\Delta E)^2}. \]

Using Lemma \(\Pi\) and eq. \(\Pi\) we conclude

\[ \delta(B|A) \geq \frac{1}{256(\Delta E \Delta t)^2}. \]

\(\Box\)
7 Implications for low power computation

When we discuss hypothetical low power computers here we mean devices with energy consumption much below the consumption of any present technology or prototypes for the middle future. Nevertheless we find it worth to discuss under which circumstances fundamental lower bounds on the power consumption of computers can be proved. Currently, the only fundamental bound that is known is Landauer's principle [11] stating that every logical irreversible computation leads unavoidably to power consumption. The converse statement that power consumption could in principle be avoided at all by using logically reversible circuits, is questionable. It is well-known that all classical computations can be implemented using Toffoli-gates [12] which can be considered as unitary transformations in a Hilbert space of the computer due to their logical reversibility. However, the signals controlling the implementation time is always excluded in the thermodynamical considerations [9]. This is correct if the signal energy is sufficiently high such that the signal can be considered classical. If its quantum nature is taken into account severe problems with thermodynamical reversibility may appear (some thoughts on this problem can be found in [9]). To our knowledge, the only theoretical models for a closed physical system can be found in [13, 14, 15, 16]. The model in [13] uses a Hamiltonian which is unbounded below. Such Hamiltonians only exists in the limit of high system energy since the energy is then much above the ground state. The concept of [15, 16] avoids a global clocking mechanism at all. It leads to states which are superpositions of different results. The register is highly entangled, i.e., a lot of quantum information is transferred among different parts of the register.

Now we show in which way our results may give lower bounds for the energy consumption of all computers that rely on too conventional concepts, for instance, in the sense that they do not transfer quantum coherent signals. Since our results may apply to different levels of a computer (communication between transistors, devices, gates, processors, computers) we will not specify the components at all.

Consider two components $A$ and $B$ each producing an output $a_j$ and $b_j$ in time step $j$. A third logical device $C$ receives $a_j$ and $b_j$ as inputs (see Fig. 2).

Hence the components $A$ and $B$ have to be synchronized up to an accuracy $\Delta t$, the length of the time steps. Assume $A$ and $B$ to be quantum systems that evolve approximatively according to their Hamiltonians $H_A$ and $H_B$. Of course, this can only be a rough approximation since both systems receive signals from other components and send signals to $C$. Nevertheless we tend to believe that our results above suggests that the required synchronization requires either quantum communication between $A$ and $B$ or leads unavoidably to power consumption. Although we were only able to prove our bounds for one-way protocols it seems likely that also classical multi-step protocols generate some entropy.

We admit that arguments like this should be analyzed thoroughly. This should be subject of further research.

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Figure 2: Two components $A$ and $B$ that have to be synchronized. When their evolution can approximatively be described by separate Hamiltonians our lower bounds on the required synchronization entropy are valid.

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