Abstract
We completely describe a new domain for abstract interpretation of numerical programs. Fixpoint iteration in this domain is proved to converge to finite precise invariants for (at least) the class of stable linear recursive filters of any order. Good evidence shows it behaves well also for some non-linear schemes. The result, and the structure of the domain, rely on an interesting interplay between order and topology.

Categories and Subject Descriptors D.2.4 [Software/Program Verification]: [Validation]; F.3.1 [Specifying and Verifying and Reasoning about Programs]; [Mechanical verification]; F.3.2 [Semantics of Programming Languages]; [Program analysis]

General Terms Theory, Verification

Keywords Abstract interpretation, numerical programs

1. Introduction
An everlasting challenge of the verification of programs involving numerical computations is to efficiently find accurate invariants for values of variables. Even though machine computations use finite precision arithmetic, it is important to rely on the properties of real numbers and estimate the real number values of the program variables first, before even trying to characterize the floating-point number invariants. We refer the reader to [9], which describes a way to go from this to floating-point analysis, or to the static linearization techniques of [15].

In [9] as well, some first ideas about an abstract interpretation domain which would be expressive enough for deriving these invariants, were sketched. It relied on a more accurate alternative to interval arithmetic: affine arithmetic, the concretization of which is a center-symmetric polytope. But, contrarily to existing numerical relational abstract domains with polyhedral concretization (polyhedra [4] of course, but also zones, octagons [16] etc.), dependencies in affine arithmetic are implicit, making the semantics very economical. Also, affine arithmetic is close to Taylor models, which can be exploited to give precise abstractions of non-linear computations.

But these advantages are at a theoretical cost: the partial order and the correctness of the abstract computations are intricate to find and prove. In this article, we construct a “quasi” lattice abstract domain, and study the convergence of fixpoint computations. We show how the result of the join operators we define can be considered as a perturbation of the affine forms, and thus how the fixpoint iteration can be seen as a perturbation of the numerical schemes we analyze. A crucial point is that our abstract domain is both almost a bounded complete lattice, and an ordered Banach space, where approximation theorems and convergence properties of numerical schemes naturally fit. As an application of the framework, we prove that our approach allows us to accurately bound the values of variables for stable linear recursive filters of any order.

Contributions This article fully describes a general “completeness” result of the abstract domain, for a class of numerical programs (linear recursive filters of any order), meaning that we prove that the abstract analysis results will end up with finite numerical bounds whenever the numerical scheme analyzed has this property. We also show good evidence that, on this class of programs, we can get as close an over-approximation of the real result as we want.

The abstract domain on which we prove this result is a generalization of the one of [9], better join and meet operators are described, and the full order-theoretic structure is described (sketches of proofs are given).

A new feature of this domain, with respect to the other numerical abstract domains, is that it does not only have an order-theoretic structure, but also a topological one, the interaction of which plays an important role in our results. The domain is an ordered Banach space, “almost” a Riesz space, which are structures of interest in functional analysis and optimization theory. This is not just a coincidence: correctness of the abstraction relies on the correctness of functional evaluations in the future, i.e. continuations. This opens up promises for useful generalizations and new techniques for solving the corresponding semantic equations.

Contents Section 2 introduces the general problematic of finding precise invariants for numerical programs, and defines an interesting sub-class of programs, that is linear recursive filters of any order. We also introduce the classical affine forms [15] introduced in numerical mathematics, on which our work elaborates.

Section 3 extends these affine forms to deal with static analysis invariants. We show that the set of such generalized forms has the structure of an ordered Banach space, which almost has least upper bounds and greatest lower bounds: it actually only has maximal lower bounds and minimal upper bounds, in general. An equivalent of bounded completeness is proved using the interplay between the partial order and the topology (from the underlying Banach space).

We develop particular Kleene iteration techniques in Section 4. With these, we prove that we can find finite bounds for the invariants of stable linear recursive filters. We also show evidence...
2. Problematic

We are interested in numerical schemes in the large. This includes signal processing programs, control programs such as the ones used in aeronautics, automotive and space industry, libraries for computing transcendental functions, and as a long-term goal, simulation programs (including the solutions of ordinary or partial differential equations). The context of our work is the determination of the accuracy reached by the finite precision (generally IEEE 754) implementation of these numerical schemes, see for instance [3, 6, 10, 12]. But it is already a difficult problem for these numerical programs, to determine run-time errors (RTEs) statically, just because the bounds of the results of numerical computation are hard to find. These bounds are not only hard to find for floating-point arithmetic, but also for real arithmetic, which is the first critical step towards solving the complete problem.

In the sequel, we are expressing a precise abstract domain of affine forms for bounding real number calculations, in the sense of abstract interpretation [2].

We give in Section 2.1 a class of simple programs that are pervasive in the field of numerical computing: linear recursive filters of any order. They are encountered generally in signal processing and control programs, but encompass also linear recurrence schemes that can be found in simulation programs. We will study extensively the behavior of our abstract domain on such programs.

Of course, we are also interested in non-linear schemes, and already studied some coming for example from the solution of a conjugate gradient algorithm, or algorithms for estimating tran-
sformations, and as a long-term goal, simulation programs. We will study

2.1 A class of numerical schemes of interest

Let us consider the following class of program, that we will study in depth with our abstract domain in Section 2.3.

```plaintext
filter(float x[n+1]) {
    real e[n+1];
    e[*] = input(m, M); [1]
    while (true) {
        e[n+1] = input(m, M);
        x[n+1] = a[1]*x[1] + a[2]*x[2] + ... + a[n]*x[n]
        + b[1]*e[1] + b[2]*e[2] + ... + b[n+1]*e[n+1]; [3]
        x[n] = x[n+1]; ... x[1] = x[2]; [4] }
}
```

In the program above, a[] is an array of n constants ai, i = 1,..., n (indices of arrays start at 1), b[] is an array of n + 1 constants bi, i = 1,..., n + 1. M and m are parameters, giving the bounds M and m of the successive inputs over time. For purposes of simplicity, as was discussed in the introduction, types of variables are real number types. We use the notation e[*] = input(m, M); to denote the sequence of n + 1 input assignments between m and M. At iterate k of the filter, variable x[1]

represents the value xk+1 of the output. Our main interest here is in

The program filter describes the infinite iteration of a filter of order n with coefficients a1, ..., an, b1, ..., bn+1 and a new input e between m and M at each iteration:

\[
x_{k+n+1} = \sum_{i=1}^{n} a_i x_{k+i} + \sum_{j=1}^{n+1} b_j e_{k+j},
\]

starting with initial conditions x1, ..., x_n, x_n.

We rewrite (1) as:

\[
X_{k+1} = AX_k + BE_{k+1},
\]

with

\[
X_k = \begin{pmatrix} x_k & x_{k+1} & \ldots & x_{k+n} \\ \end{pmatrix}, \quad E_{k+1} = \begin{pmatrix} e_k & e_{k+1} & \ldots & e_{k+n+1} \end{pmatrix}
\]

\[
A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & \vdots \\ b_1 & b_2 & \ldots & b_{n+1} \end{pmatrix}
\]

Now, of course, (2) has solution

\[
X_k = A^{k-1} X_1 + \sum_{i=1}^{k} A^{k-i} B E_i,
\]

where X_1 is the vector of initial conditions of this linear dynamical system. If A has eigenvalues (roots of \( x^n - \sum_{i=1}^{n} a_i x^{n-i} \)) of module strictly less than 1, then the term \( A^{k-1} X_1 \) will tend to zero when k tends to infinity, whereas the partial sums \( \sum_{i=1}^{k} A^{k-i} B E_i \) will tend towards a finite value (obtained as a convergent infinite series).

**EXAMPLE 1.** Consider the following filter of order 2 (see [11]):

\[
x_k = 0.7e_i - 1.3e_{i-1} + 1.1e_{i-2} + 1.4x_{i-1} - 0.7x_{i-2},
\]

where e_i are independent inputs between 0 and 1. A typical run of this algorithm with e_i = 1/2 (for all i) and x_0 = 0, x_1 = 0 converges towards 0.8333..., stays positive, and reaches at most 1.1649 (its dynamics is shown in Figure 7).

2.2 Classical affine arithmetic

An affine form is a formal series over a set of noise symbols \( \varepsilon_i \)

\[
\hat{x} = \alpha_0 + \sum_{i=1}^{\infty} \alpha_i \varepsilon_i,
\]

with \( \alpha_i \in \mathbb{R} \).
Let $A^\mathbb{R}$ denote the set of such affine forms. Each noise symbol $\varepsilon_i$ stands for an independent component of the total uncertainty on the quantity $\hat{x}$, its value is unknown but bounded in $[-1,1]$; the corresponding coefficient $\alpha_i^e$ is a known real value, which gives the magnitude of that component. The idea is that the same noise symbol can be shared by several quantities, indicating correlations among them. These noise symbols can be used not only for modeling uncertainty in data or parameters, but also uncertainty coming from computation.

When the cardinal of the set $\{\alpha_i^e \neq 0\}$ is finite, such affine forms correspond to the affine forms introduced first in [18] and defined for static analysis in [9] by the authors.

The concretization of a set of affine forms sharing noise symbols is a center-symmetric polytope, which center is given by the $e_0$ vector of the affine forms. For example, the concretization of

$$
\begin{align*}
\hat{x} &= 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4 \\
\hat{y} &= 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4
\end{align*}
$$

is given in Figure 3. $A^\mathbb{R}$ is a $\mathbb{R}$-vector space with the operations $+$ and $\times$:

$$
(\alpha_0^e + \sum_{i=1}^{\infty} \alpha_i^e \varepsilon_i) + (\alpha_0^e + \sum_{i=1}^{\infty} \alpha_i^e \varepsilon_i) = \left(\alpha_0^e + \sum_{i=1}^{\infty} (\alpha_i^e + \alpha_i^{e'}) \varepsilon_i \right)
$$

$$
\lambda \times (\alpha_0^e + \sum_{i=1}^{\infty} \alpha_i^e \varepsilon_i) = \lambda \alpha_0^e + \sum_{i=1}^{\infty} \lambda \alpha_i^e \varepsilon_i
$$

A sub-vector space $A^\mathbb{R}_{1}$ of $A^\mathbb{R}$ can be classically endowed with a Banach space structure, using the $\ell_1$ norm

$$
||\hat{x}||_1 = \sum_{i=0}^{\infty} |\alpha_i^e|,
$$

for the elements $\hat{x}$ such that the above sum is finite.

We define the projection

$$
\pi_L(\hat{x}) = \sum_{i=1}^{\infty} \alpha_i^e \varepsilon_i,
$$

and the associated semi-norm

$$
||\hat{x}||_L = ||\pi_L(\hat{x})||_1.
$$

We also define $||\hat{x}||_A = ||\alpha_0^e|| + ||\hat{x}||_L$.

Let us come back now to the linear recursive filters of Section 2.1. For simplicity’s sake, suppose $\varepsilon_{k+n+1} = \varepsilon_{k+n+1}$, so the $\varepsilon_{k+n+1}$ are independent inputs between 0 and 1. As we will see later on, but as should already be obvious because of the definitions of sum and product of affine forms by a scalar, the semantics, using affine forms, of the completely unfolded filter program exactly gives, at unfolding $k$:

$$
\hat{X}_k = A^{k-1} \hat{X}_1 + \sum_{i=1}^{k} A^{k-i} B \hat{E}_i
$$

where

$$
\hat{E}_i = \left( \begin{array}{c}
\varepsilon_{i-n} \\
\varepsilon_{i-n+1} \\
\vdots \\
\varepsilon_i
\end{array} \right)
$$

and $\hat{X}_i$ are the obvious affine forms vector counterparts of $X_i$.

This means that in the case $A$ has complex roots of module strictly less than 1, the affine forms (with a finite number of $\varepsilon_i$) giving the semantics of values at unfolding $k$, converge in the $\ell_1$ sense to an affine form with infinitely many noise symbols.

**Example 2.** Consider again the filter of order 2 of Example 1. We supposed that the successive inputs $\varepsilon_i$ are independent inputs between 0 and 1, so that we can write $\hat{e}_i = \frac{1}{2} + \frac{1}{2}\varepsilon_{i+1}$ (with different noise symbols at each iterate), and $x_0 = x_1 = 0$. For instance, if we compute the affine form after 99 unfolds, we find:

$$
\hat{x}_{99} = 0.83 + 7.81 e^{-9} \varepsilon_1 - 2.1 e^{-8} \varepsilon_2 - 1.58 e^{-8} \varepsilon_3 + \ldots -0.16 \varepsilon_{99} + 0.35 \varepsilon_{100}
$$

whose concretization gives an exact (under the assumption that the coefficients of the affine form are computed with arbitrary precision) enclosure of $x_{99}$:

$$
x_{99} \subseteq [-1.0907188500, 2.7573854753].
$$

The limit affine form has a concretization converging towards (see Figure 2):

$$
x_{\infty} = [-1.0907188498\ldots, 2.7573855165\ldots].
$$

Unfortunately, if asymptotically (i.e. when $k$ is large enough), the concretization of the affine forms $\hat{X}_k$ converges to a good estimate of the values that program variable $x$ can take (meaning, after a large number of iteration $k$), this form is in no way an invariant of the loop, and does not account for all values that this variable can take along the loops.

**Example 3.** This can be seen for the particular filter of order 2 of Example 1. In Figure 2, the reader with good eyes can spot that around iterations 8-10, the concretization of $x$ can go above $2.7573855165\ldots$, which is the asymptotic supremum. Actually, the sup value is $2.8243184\ldots$, reached at iteration 8, whereas the infimum is $-1.1212409\ldots$, reached at iteration 13.

The aim of this paper is to describe a suitable extension of these affine forms that can account for such invariants.

3. **An ordered Banach space of generalized affine forms**

We now extend our Banach space of affine forms in order to represent unions of affine forms, as a perturbed affine form. We consider $A_1 = A^\mathbb{R}_{1} \oplus \mathbb{R}$ and write these new affine forms as:

$$
\hat{x} = \alpha_0^e + \sum_{i=1}^{\infty} \alpha_i^e \varepsilon_i + \beta^e \varepsilon_U
$$

Norms $\ell_1$ are extended over this new domain in an obvious manner. We now have $||\hat{x}||_1 = ||\alpha_0^e|| + ||\hat{x}||_L + ||\beta^e||$.

**Remark:** In the rest of this section, unless otherwise stated, we restrict the study to elements in the cone $A_1$. $A_1$ whose elements $\hat{x}$ have a positive $\beta^e$. We will sketch some ways to extend the results obtained, and their meaning, for all of $A_1$, in Sections 3.3.2 and 5.

We first give concrete semantics to these generalized affine forms in Section 3.3, then we give in Section 3.2 the abstract transfer functions for arithmetic expressions. The counterpart of the inclusion ordering, the continuation ordering, is defined in Section 3.3. The main technical ingredient that will allow us to find...
effective join and meet operations in Section 3.5 is the equivalence between this seemingly intractable ordering, and an ordering with a much simpler definition, the perturbation ordering, see Theorem 15. Finally we prove in Section 3.4 that these generalized affine forms also have the structure of an ordered Banach space. This will be useful for proving convergence results with our iteration schemes in Section 4.3.

3.1 Concrete semantics of expressions and concretization function

**Definition 4.** We define the concretization function $\gamma : A_+ \rightarrow \mathbb{R}$ in intervals as follows, for $\hat{x} = a_0 + \sum_{i=1}^{\infty} a_i^x e_i + \beta^x \varepsilon_U$:

$$
\gamma(\hat{x}) = [a_0^x - \|\hat{x}\|_\ell - \beta^x, a_0^x + \|\hat{x}\|_\ell + \beta^x]
$$

whose lower (respectively upper) bound corresponds to the infimum (respectively supremum) of the affine form $\hat{x}$ seen as a function from $e_i \in [-1, 1]$ to $\mathbb{R}$.

Let Var be the set of program variables. An abstract environment $\sigma$ associates a concrete constant, and operations $+, -$ and $\times$ on abstract forms, with values in the set of subsets of $\mathbb{R}^k$, as follows. We note $\hat{x}(t_1, \ldots, t_n, \ldots, u_x)$, for $t_1, \ldots, t_n, \ldots, u_x \in [-1, 1]$, the application $\hat{x}$, seen as an affine function of $e_1, e_2, \ldots, e_n$, to $t_1, t_2, \ldots, t_n, u_x$.

Let $\hat{x}$, $\hat{y}$ be two abstract forms in $\mathbb{R}$. Their joint concretization is the inner polyhedron of $X$.

3.2 Abstract interpretation of simple arithmetic expressions

Let $\text{Expr}$ be the set of polynomial expressions, i.e. expressions built inductively from the set of program variables Var, real number constants, and operations $+, -$ and $\times$. We now define the respective operators $+, -$ and $\times$ (extending the ones of Section 2.2).

**Definition 6.**

$$
\begin{align*}
\bar{x} + \bar{y} &= a_0^x + a_0^y + \sum_{i=1}^{\infty} (a_i^x + a_i^y) e_i + (\beta^x + \beta^y) \varepsilon_U \\
\bar{x} - \bar{y} &= -a_0^x - \sum_{i=1}^{\infty} a_i^x e_i + \beta^x \varepsilon_U \\
\bar{x} \times \bar{y} &= \\text{Left for the full version of this article.}
\end{align*}
$$

1. Nothing prevents us from defining abstract transfer functions for other operations, such as $\sqrt{\cdot}$, $\sin$, $\acos$ etc. as affine forms are naturally Taylor forms. This is not described in this article, for lack of space.

(note that the sign $\in + \beta^y \varepsilon_U$ is certainly not a typo). And we define $\bar{x}$ for affine forms $\bar{x}$ and $\bar{y}$ having a finite number of non-zero $\alpha_i$ coefficients (we call them affine forms with finite support)

$$
\hat{x} \times \hat{y} = a_0^x a_0^y + \sum_{i=1}^{\infty} (a_i^x a_i^y + a_i^x a_i^y) e_i + \sum_{i,k=1}^{\infty} | a_i^x a_k^y | e_i + \sum_{j=0}^{\infty} (| a_j^x | | \beta^y + \beta^x | | a_j^y | + | \beta^y | \beta^x) \varepsilon_U,
$$

where $\varepsilon_f$ is a symbol which is unused in $\hat{x}$ nor in $\hat{y}$ ("fresh noise symbol").

**Lemma 7.** We have the following correctness result on the abstract semantics of expressions:

$$
\begin{align*}
\text{Im} \hat{x} + \hat{y} &\subseteq \gamma(\hat{x} + \hat{y}) &\subseteq \gamma(\hat{x}) + \gamma(\hat{y}) \\
\text{Im} - \hat{x} &\subseteq \gamma(-\hat{x}) &\subseteq -\gamma(\hat{y}) \\
\text{Im} \hat{x} \times \hat{y} &\subseteq \gamma(\hat{x} \times \hat{y})
\end{align*}
$$

where $+$ and $-$ on the right hand side of inequalities above are the corresponding operations on intervals, and the last inclusion holds only for affine forms with finite support.

**Sketch of proof.** This is mostly the similar classical result in affine arithmetic [13], and easily extended to $\varepsilon_U$ symbols and infinite series (convergent in the $\ell_1$ sense).

3.3 The continuation and the perturbation ordering

The correctness of the semantics of arithmetic expressions defined in Section 3.2 and more generally of the semantics of a real language (Section 4.2) relies on an information ordering, which we call the continuation ordering, Definition 8. Unfortunately, its definition makes it difficult to use, and we define an a priori weaker ordering, that we call perturbation ordering, Definition and Lemma 9 that will be easily decidable, and shown equivalent to the continuation ordering (Proposition 13). The perturbation ordering has minimal upper bounds, but not least upper bounds. A simple construction will allow us to define in Section 3.6 a lattice with a slightly stronger computational ordering, based on the perturbation ordering.

**Definition 8** (continuation order). Let $\sigma_1$ and $\sigma_2$ be two abstract environments. We say that $\sigma_1 \preceq \sigma_2$ if and only if for all $e \in \text{Expr}$

$$
\gamma(e) |\sigma_1 \subseteq \gamma(e) |\sigma_2
$$

We naturally say that $\hat{x} \preceq \hat{y}$ if and only if

$$
\gamma(e)[u \leftarrow \hat{x}] \subseteq \gamma(e)[u \leftarrow \hat{y}]
$$

for all $e \in \text{Expr}$ and all $\sigma \in \text{Var}$, and for some $u \in \text{Var}$.

**Definition & Lemma 9** (perturbation ordering). We define the following binary relation $\lesssim$ on elements of $A_1$

$$
\begin{align*}
x \lesssim y \Leftrightarrow & \lVert x - y \rVert_\ell \leq \beta^y - \beta^x \\
\text{Then } \lesssim & \text{ is a partial order on } A_1.
\end{align*}
$$

We extend this partial order componentwise to abstract environments as follows: for all $\sigma_1, \sigma_2 : \text{Var} \rightarrow A_1$,

$$
\sigma_1 \preceq \sigma_2 \Leftrightarrow \forall x \in \text{Var}, \sigma_1(x) \leq \sigma_2(x)
$$

**Sketch of proof.** Reflexivity and transitivity of $\lesssim$ are trivial. For antisymmetry, suppose $\hat{x} \lesssim \hat{y}$ and $\hat{y} \lesssim \hat{x}$, then we have

$$
\begin{align*}
\lVert \hat{x} - \hat{y} \rVert_\ell &\leq | \beta^y - \beta^x | \\
\lVert \hat{x} - \hat{y} \rVert_\ell &\leq | \beta^x - \beta^y |
\end{align*}
$$

2 Better abstractions are available, but make the presentation more complex, this is left for the full version of this article.
This implies that both $\beta^y - \beta^x$ and $\beta^x - \beta^y$ are positive, hence necessarily zero. Hence also $\|\hat{x} - \hat{y}\|_A = 0$ meaning $\pi_A \hat{x} = \pi_A \hat{y}$. Overall: $\hat{x} = \hat{y}$.

Now, we prove intermediary results in order to prove equivalence between the two orders above. Half of this equivalence is easy, see Lemma $[10]$ The other half is a consequence of Lemma $[11]$ and of Lemma $[12]$ Theorem $[15]$ is the same as Proposition $[13]$ not just for individual affine forms, but for all abstract environments.

**Lemma 10.** $\hat{x} \preceq \hat{y} \Rightarrow \hat{x} \preceq \hat{y}$

**Proof.** Given $\hat{x}$ and $\hat{y}$ in $A_1$, consider the expression $e = u - v$ and the environment $\sigma$ such that $\sigma(v) = \pi_{L, \hat{y}}(\hat{y}) + \alpha^y_v$. We have $\gamma[e][u \leftarrow \hat{x}] \subseteq \gamma[e][u \leftarrow \hat{y}]$, which means:

$$\begin{align*}
\alpha^x_0 - \alpha^y_0 - \|x - y\|_L - \beta^y - \beta^x, \\
\alpha^0_0 - \alpha^0_0 + \|x - y\|_L + \beta^y - \beta^x
\end{align*} \subseteq [-\beta^y, \beta^y].$$

Thus we have

$$\begin{align*}
\alpha^x_0 - \alpha^y_0 & \leq -\|x - y\|_L + \beta^y - \beta^x, \quad (3) \\
\alpha^0_0 - \alpha^0_0 & \geq \|x - y\|_L + \beta^y - \beta^x. \quad (4)
\end{align*}$$

Inequality (4) is equivalent to

$$\alpha^y_0 - \alpha^x_0 \leq -\|x - y\|_L + \beta^y - \beta^x,$$

hence together with inequality (3)

$$\|\alpha^x_0 - \alpha^y_0\| \leq -\|x - y\|_L + \beta^y - \beta^x,$$

this exactly translates into $\hat{x} \preceq \hat{y}$. □

**Lemma 11.** For all $\hat{x}, \hat{y} \in A_1$, $\hat{x} \preceq \hat{y}$ implies $\gamma(\hat{x}) \subseteq \gamma(\hat{y})$.

**Proof.** We compute:

$$\begin{align*}
\sqrt{\hat{y}}(\hat{y}) - \sqrt{\hat{x}}(\hat{x}) &= \alpha^x_0 - \alpha^y_0 + \|\|\hat{y}\|_L - \|\hat{x}\|_L + \beta^y - \beta^x.
\end{align*}$$

Using the triangular inequality $\|\hat{x}\|_L \leq \|\hat{x} - \hat{y}\|_L + \|\hat{y}\|_L$, and $\|\hat{x} - \hat{y}\|_L \leq \beta^y - \beta^x$, we write:

$$\begin{align*}
\sqrt{\hat{y}}(\hat{y}) - \sqrt{\hat{x}}(\hat{x}) &\geq \alpha^x_0 - \alpha^y_0 - \|\hat{x} - \hat{y}\|_L + \beta^y - \beta^x \\
&\geq \alpha^x_0 - \alpha^y_0 + \|\hat{x} - \hat{y}\|_L + \beta^y - \beta^x
\end{align*}$$

and similarly for the inf bound of the concretization. □

Notice that the converse of Lemma $[11]$ is certainly not true: just take $\hat{x} = 1 + \varepsilon_1$ and $\hat{x}' = 1 + \varepsilon_2$. It is easy to see that $\hat{x}$ and $\hat{x}'$ are incomparable, but have same concretizations.

**Lemma 12.** $\hat{+}, \hat{-}$ and $\hat{\cdot}$ are increasing functions on $A_1, \leq$.

**Proof.** We have easily, for $\hat{x} \preceq \hat{y}$ and $\hat{z} \in A_1$:

$$\begin{align*}
\|\hat{x} + \hat{z} - \hat{y} + \hat{z}\|_A &= \|\hat{x} - \hat{y}\|_A \leq \beta^y - \beta^x = \beta^{y + z} - \beta^{z + x}, \\
\|\hat{x} - \hat{z} - \hat{y} + \hat{z}\|_A &= \|\hat{x} - \hat{y}\|_A \leq \beta^y - \beta^x = \beta^{y - z} - \beta^{z - x}.
\end{align*}$$

Now:

$$\|\hat{x} \hat{y} \hat{z} - \hat{y} \hat{z} \hat{x}\|_A = \|\hat{z}\|_A \|\hat{x} - \hat{y}\|_A \leq \|\hat{z}\|_A (\beta^y - \beta^x).$$

$$\beta^{y + x} - \beta^{x + y} = \|\hat{x}\|_A (\beta^y - \beta^x) + \beta^y (\|\hat{y}\|_A - \|\hat{x}\|_A + \beta^y - \beta^x)$$

But $\hat{x} \preceq \hat{y}$ so $\beta^y - \beta^x \geq \|\hat{y} - \hat{x}\|_A \geq \|\hat{x}\|_A - \|\hat{y}\|_A$, the last inequality being entailed by the triangular inequality. Thus,

$$\beta^{y + x} - \beta^{x + y} \geq \|\hat{z}\|_A (\beta^y - \beta^x)$$

which, by combining with inequality (5), completes the proof. □

**Proposition 13.** $\hat{x} \preceq \hat{y}$ if and only if $\hat{x} \preceq \hat{y}$

**Sketch of proof.** We know from Lemma $[10]$ that $\hat{x} \preceq \hat{y}$ implies $\hat{x} \preceq \hat{y}$. Now, let $e \in Expr$, and suppose $\hat{x} \preceq \hat{y}$. We reason by induction on $e$: the base case is constants and variables (trivial). A consequence of Lemma $[12]$ is that for all $\hat{z} \in A_1$, $\hat{x} + \hat{z} \preceq \hat{z}$ and $\hat{x} \hat{z} \preceq \hat{z}$, hence $\hat{x} \preceq \hat{y}$. □

Finally, we can prove the following more general equivalence, which is nothing but obvious at first. The example below shows the subtlety of this result.

**Example 14.** To illustrate one of the aspects of next theorem, that is, $\hat{x} \preceq \hat{x}'$ implies that any joint concretization of $\hat{x}'$ with other affine forms, (say just one, $\hat{y}$, here), contains the joint concretization of $\hat{x}$ with $\hat{y}$, take again $\hat{x}$ as in Example $[5]$ and

$$\hat{x}' = \frac{3}{2} + \frac{1}{2} \varepsilon_1 + \varepsilon_2 + 2 \varepsilon_3.$$  

**Of course, $\hat{x} \preceq \hat{x}'$. Figure 4 shows the inclusion of the joint concretization of $(\hat{x}, \hat{y})$ in the joint concretization of $(\hat{x}', \hat{y})$. Note that several of the faces produced are fairly different.**

**Theorem 15.** Let $\sigma_1, \sigma_2$ be two abstract environments, then $\sigma_1 \preceq \sigma_2$ if and only if $\sigma_1 \leq \sigma_2$.

**Sketch of proof.** It can be shown first (classical result in affine arithmetic $[13]$), that $\gamma(\sigma_2)$ is a polyhedron (a particular kind, called a zonotope). It means that it can be equivalently described by a system of affine constraints ($i = 1, \ldots, k$):

$$\sum_{x \in Var} a_i^x x \leq b_i.$$  

Consider the expressions (in Expr): $e^i = \sum_{x \in Var} a_i^x x - b_i$. We know that for all $x \in Var$, $\sigma_1(x) \leq \sigma_2(x)$, hence by Proposition $[13]$, $\sigma_1(x) \leq \sigma_2(x)$. This entails, by induction on $x$, that $\gamma(e^i) \sigma_1 \subseteq \gamma(e^i) \sigma_2$. Thus the constraint $\sum_{x \in Var} a_i^x x \leq b_i$ is satisfied by elements of $\gamma(\sigma_1)$, by Lemma $[7]$ So $\gamma(\sigma_1) \subseteq \gamma(\sigma_2)$. Let $e$ be any expression in $Expr$. The result follows from Proposition $[13]$ and the result above, by induction on $e$.

**3.4 Ordered Banach structure**

The aim of this section is to prove Proposition $[16]$. This will be central to the proofs in Sections $[3.6]$ and $[4]$.

**Proposition 16.** $(A_1, \ately)$ is an ordered Banach space.

**Sketch of proof.** First we show that the partial order $\leq$ of Definition and Lemma $[9]$ makes $A_1$ into an ordered vector space. For showing this, we have to show compatibility of $\leq$ with the linear structure, i.e., for $\lambda \geq 0$ and $\hat{x} \preceq \hat{y}$, and for all $\hat{z}$: $\hat{x} + \hat{z} \leq \hat{y} + \hat{z}$, $\lambda \hat{x} \leq \lambda \hat{y}$, and $-\hat{y} \preceq -\hat{x}$, which is immediate verification.
The only remaining property to prove is that $\leq$ is closed in $A_1 \times A_1$, in the product topology, $A_1$ being given the Banach topology of the $\ell_1$ norm. Suppose $x_n$ converges towards $x$ as $n$ goes towards $\infty$, in the sense of the $\ell_1$ norm, and suppose for all $n$, $x_n \leq y$. Then

$$\|y - x\|_A \leq \|y - x_n\|_A + \|x_n - x\|_A$$

for all $\epsilon > 0$ and $n \geq N(\epsilon)$. By continuity of $\pi_U$, we thus know that there exists $K(\epsilon)$ such that for all $n \geq \text{sup}(N(\epsilon), K(\epsilon))$:

$$\|y - x\|_A \leq \|y - x_n\|_A + \|x_n - x\|_A$$

This concludes the proof. □

A different way of stating that $(A_1, \leq)$ is an ordered vector space is to introduce the subset $C$ of $A_1$ such that

$$x \leq y \iff y - x \in C,$$

and show it is indeed the cone of $\leq$. We see that

$$C = \{ x \mid \|x\|_A \leq \beta^* \}.$$

This is the analogue of the Lorentz cone in special relativity theory, but with the $\ell_1$ norm instead of the $\ell_2$ norm.

To use the vocabulary from relativity theory, identifying the $A_1^\mathbb{R}$ part with the space coordinates and the $\beta$ coefficient with the time coordinate, $\leq$ is the causal order and $\hat{x} \leq y$ if the space-time interval $[\hat{x}, \hat{y}]$ is time-like or light-like, whereas $\hat{x} \geq y$ if $[x, y]$ is space-like or light-like. Other considerations, using domain-theoretic methods, on the causal order in the case of the $\ell_2$ Lorentz cone can be found for instance in [14].

### 3.5 The quasi lattice structure

We will show in this section that $(A_1, \leq)$ is almost a bounded complete partial order (bcpo). It is not a bcpo because there is not in general any least upper bound. This is a consequence of [13]: as the cone $C$ of our partial order has $2^n$ generators (the generators of the polyhedron which is the unit $l_1$ ball), it cannot be simplicial, hence $(A_1, \leq)$ is not a lattice. Instead, there are in general infinitely many minimal upper bounds, which will suffice for our semantics purposes. We prove furthermore that many bounded subset of $A_1$ ("enough") again admit minimal upper bounds.

We first recall the definition of a minimal upper bound or mub (maximal lower bounds, or mlb, are defined similarly): 

**Definition 17.** Let $\subseteq$ be a partial order on a set $X$. We say that $z$ is a mub of two elements $x, y$ of $X$ if and only if

- $z$ is an upper bound of $x$ and $y$, i.e. $x \subseteq z$ and $y \subseteq z$,
- for all $z'$ upper bound of $x$ and $y$, $z' \subseteq z$ implies $z' = z'$.

We note that for the order $\leq$, we have a very simple characterization of mubs, if they exist (proving existence, and deriving some formulas, when available, are the aims of the section to come).

**Lemma 18.** Let $\hat{x}$ and $\hat{y}$ be two elements of $A_1$. Then $\hat{z}$ is a mub of $\hat{x}$ and $\hat{y}$ if and only if

- $\hat{x} \leq \hat{z}$ and $\hat{y} \leq \hat{z}$,
- $\beta^*$ is minimal among the $\beta^*$, for all $i$ upper bounds of $\hat{x}$ and $\hat{y}$.

**Proof.** Suppose we have $\hat{z}$ such as defined above. Take any upper bound $t$ of $\hat{x}$ and $\hat{y}$ and suppose $t \leq \hat{z}$. Then $\|\hat{z} - t\|_A \leq \beta^* - \beta^*$. Hence, $\beta^* \geq \beta^*$. But by hypothesis, $\beta^*$ is minimal among all upper bounds, so $\beta^* = \beta^*$. Then this implies $\|\hat{z} - t\|_A = 0$ so $\pi_A(\hat{z}) = \pi_A(t)$ as well, hence $\hat{z} = t$. □

In what follows, we will need an extra definition:

**Definition 19.** Let $x$ and $y$ be two intervals. We say that $x$ and $y$ are in generic positions if, whenever $x \subseteq y$, inf $x = \inf y$ or sup $x = \sup y$.

By extension, we say that two affine forms $\hat{x}$ and $\hat{y}$ are in generic position when $\gamma(\hat{x})$ and $\gamma(\hat{y})$ are intervals in generic positions.

#### 3.5.1 The join operation

For any interval $i$, we note mid($i$) its center. Let $\alpha^*_i \land \alpha^*_j$ denote the minimum of the two real numbers, and $\alpha^*_i \lor \alpha^*_j$ their maximum. We define

$$\begin{align*}
\arg\min & \quad \{ \alpha \mid \alpha \in [\alpha^*_i \land \alpha^*_j, \alpha^*_i \lor \alpha^*_j] \}, \\
\arg\max & \quad \{ \alpha \mid \alpha \in [\alpha^*_i \land \alpha^*_j, \alpha^*_i \lor \alpha^*_j] \}.
\end{align*}$$

**Proposition 20.** Let $\hat{x}, \hat{y} \in A_1$. There exist minimal upper bounds $\hat{z}$ of $\hat{x}$ and $\hat{y}$ if and only if

$$\|\hat{x} - \hat{y}\|_A \geq \|\beta^* - \beta^*\|^2.$$  \hspace{1cm} (7)

Moreover, the minimal upper bounds, when they exist, all satisfy

$$\begin{align*}
\beta^* & = \frac{1}{2}(\|\hat{x} - \hat{y}\|_A + \beta^* + \beta^*), \\
\alpha^*_i \land \alpha^*_j & \leq \alpha^*_i \lor \alpha^*_j, \forall i, j \geq 0, \\
\|\hat{x} - \hat{z}\|_A & = \beta^* - \beta^* \text{ and } \|\hat{y} - \hat{z}\|_A = \beta^* - \beta^*.
\end{align*}$$

**Proof.** We first characterize $\beta^*$ by expressing $\hat{z} = \hat{x} \land \hat{y}$ and $\hat{y} \leq \hat{z}$:

$$\|\hat{x} - \hat{y}\|_A \leq \|\hat{x} - \hat{z}\|_A + \|\hat{y} - \hat{z}\|_A \leq \beta^* - \beta^* + \beta^* - \beta^*.$$ 

The smallest possible $\beta^*$ thus is $\beta^* = \frac{1}{2}(\|\hat{x} - \hat{y}\|_A + \beta^* + \beta^*).$

Let us now characterize solutions with such a $\beta^*$, they satisfy:

$$\|\hat{x} - \hat{z}\|_A + \|\hat{y} - \hat{z}\|_A \leq 2\beta^* - \beta^* - \beta^* = \|\hat{x} - \hat{y}\|_A,$$

thus implying that $\|\hat{x} - \hat{z}\|_A + \|\hat{y} - \hat{z}\|_A = \|\hat{x} - \hat{y}\|_A$, which is equivalent to (7). Also, these solutions are such that $\|\hat{x} - \hat{z}\|_A = \beta^* - \beta^*$ and $\|\hat{y} - \hat{z}\|_A = \beta^* - \beta^*$. Thus, there exist solutions only is $\beta^* - \beta^* \geq 0$ and $\beta^* - \beta^* \geq 0$, and the combination of these two equalities, with $\beta^*$ defined by (8), is equivalent to (7).

Let us now check that there exist minimal upper bounds under this assumption : we must prove that if (7) holds, there exists $\hat{z}$ satisfying (8) and (9) such that

$$\|\hat{x} - \hat{z}\|_A - \frac{1}{2}\|\hat{x} - \hat{y}\|_A = \frac{1}{2}\|\hat{x} - \hat{z}\|_A - \beta^* = \frac{1}{2}(\beta^* - \beta^*).$$

First part of this equality is always satisfied when (7) holds. Second part is about the existence of solutions to $f(\hat{z}) = 2\|\hat{y} - \hat{z}\|_A - \|\hat{x} - \hat{z}\|_A + \beta^* - \beta^* = 0$. Using (7), we have $f(\hat{y}) \leq 0$ and $f(\hat{x}) \geq 0$, so there exists indeed such minimal upper bounds $\hat{z}$ when (7) is satisfied. □

**Example 21.** Take $\hat{x} = \varepsilon_1$ and $\hat{y} = 2\varepsilon_U$, condition (7) is not satisfied, so there exists no minimal upper bounds. Indeed, minimal upper bounds would be $\hat{z} = a + b\varepsilon_1 + 1.5$, with $0 \leq a \leq 0$ and $0 \leq b \leq 1$. And expressing $\|\hat{x} - \hat{z}\|_A = \beta^* = 1.5$ gives $b = -0.5$, which is not admissible (not in $[0,1]$).

We note that when $\hat{x}$ and $\hat{y}$ do not have $\varepsilon_U$ symbols, there always exist minimal upper bounds. In the case when they do not exist, we will use a widening introduced in Definition 22.

**Example 22.** Take $\hat{x} = 1 + \varepsilon_1$ and $\hat{y} = 2\varepsilon_U$. We have $\gamma(\hat{x}) = [0,2]$ and $\gamma(\hat{y}) = [-2,2]$, so $\hat{x}$ and $\hat{y}$ are in generic positions. Minimal upper bounds $\hat{z}$ of $\hat{x}$ and $\hat{z}$ are

$$\hat{z} = a + b\varepsilon_1 + \varepsilon_U,$$
where \( \| \hat{z} - \hat{z} \|_A = \| \hat{y} - \hat{z} \|_A = \beta^2 \). This implies \( a - b = -1 \), with \( 0 \leq a \leq 1 \), and \( 1 \leq b \leq 2 \). Among these solutions, we find a unique one that minimizes the width of the concretization, by taking \( b = 1 \) and thus \( a = 0 \). This solution satisfies \( \gamma(\hat{z}) = \gamma(\hat{z}) \cup \gamma(\hat{y}) \), \( a^{\beta}_0(a = \text{mid}(\gamma(\hat{z}) \cup \gamma(\hat{y})) \) and \( a_0(b = 1) = \text{argmin}_{a \in [-1, 1]} \| a \|_A \).

Indeed, the solution with minimal concretization is particularly interesting when computing fixpoint in loops, by preserving the stability of the concretizations of variables values in iteres, as we will see in Theorem.\( \square \)

**Sketch of Proof.** We want to find \( \alpha^*_\text{p} \) such that the concretization is the smallest possible, with the above conditions still holding. For that, we have to minimize \( |\alpha^*_\text{p}| \) with constraints \( \| \gamma(\hat{z}) \|_A \)

\[
\alpha^*_\text{p} = \text{argmin}_{\alpha \in [-1, 1]} |\alpha|, \text{ for all } i \geq 1.
\]

For this choice of the \( \alpha^*_\text{p} \), then for all \( i \geq 1 \), we can prove the two following properties:

\[
|\alpha_i^* - \alpha_i^*| - |\alpha_i^* - \alpha_i| = |\alpha_i^*| - |\alpha_i^*| = \alpha_i^* - |\alpha_i^*,| \quad \text{(11)}
\]

\[
|\alpha_i^*| = \frac{1}{2} (|\alpha_i^*| + |\alpha_i^*| - |\alpha_i^*| - |\alpha_i^*|) \quad \text{(12)}
\]

We now define \( \alpha_0^* \) and \( \gamma(\hat{z}) \) and \( \gamma(\hat{y}) \) are not in generic positions. Minimal upper bounds \( z \) are:

\[
\hat{z} = a + b \epsilon_1 + 2 \epsilon_2,
\]

where \( a - b = -2, 0 \leq a \leq 1, \) and \( 1 \leq b \leq 4 \). Now, let us minimize the width of the concretization in the previous example. The problem is that we cannot choose in this case \( b = \text{argmin}_{a \in [-1, 1]} |a| = 1 \) because then the value of \( a (1) \) deduced from \( a - b = -2 \) is not admissible (it is not between 0 and 1).

The solution minimizing the width of the concretization is in fact \( z = 2 + 2 \epsilon_2 + 2 \epsilon_3 \), and it is such that \( a_0^*(a = \text{mid}(\gamma(\hat{z}) \cup \gamma(\hat{y})) \) and \( \gamma(\hat{z}) = \gamma(\hat{z}) \cup \gamma(\hat{y}) \).

We now give an intuition on the general case by taking examples with several noise symbols.

**Example 24.** Take \( \hat{x} = \hat{x} = 3 + \epsilon_1 + 2 \epsilon_2 \) and \( \hat{y} = 1 - 2 \epsilon_1 + \epsilon_2 + 2 \epsilon_3 \). We have \( \gamma(\hat{x}) = [0, 6] \) and \( \gamma(\hat{y}) = [-2, 4] \). \( \gamma(\hat{x}) \) and \( \gamma(\hat{y}) \) are in generic positions. Minimal upper bounds are \( \hat{z} = a + b \epsilon_1 + c \epsilon_2 + d \epsilon_3 + 3 \epsilon_4 \), where \( a + b + c + d = -3, -2 \leq a \leq 1, -6 \leq b \leq 1, 1 \leq c \leq 2, \) and \( 1 \leq d \leq 4 \). Among these solutions, we can still find a unique one that minimizes the width of the concretization, taking \( a = 2, b = 0 \) and \( c = 1; \hat{z} = 2 + 2 \epsilon_2 + 2 \epsilon_3 \).

**Example 25.** Take \( \hat{x} = 1 + \epsilon_1 + 2 \epsilon_2 + \epsilon_3 \) and \( \hat{y} = -2 - 6 \epsilon_1 + \epsilon_2 + 2 \epsilon_3 + 6 \epsilon_4 \). We have \( \gamma(\hat{x}) = [-3, 5] \) and \( \gamma(\hat{y}) = [-11, 7] \). Here \( \gamma(\hat{x}) \) and \( \gamma(\hat{y}) \) are not in generic positions. Minimal upper bounds are \( \hat{z} = a + b \epsilon_1 + c \epsilon_2 + d \epsilon_3 + 6 \epsilon_4 \), where \( a + b + c + d = -3, \) \( -2 \leq a \leq 1, -6 \leq b \leq 1, 1 \leq c \leq 2, \) and \( 1 \leq d \leq 4 \). Again, as in Example 23 minimizing the concretization of \( \gamma \) by minimizing the absolute value of \( b, c, \) and \( d \) does not give an admissible solution (when \( b = 0, c = 1, \) and \( d = 1 \), relation \( a + b + c + d = -3 \) gives \( a = -3 \) which is not admissible). But, as the minimal concretization for \( \gamma \) in any case \( \gamma(\hat{z}) \cup \gamma(\hat{y}) \), we can try to impose it. This gives \( a = \text{mid}(\gamma(\hat{z}) \cup \gamma(\hat{y})) = -2, \) and an additional relation \( -2 + b + c + d = 7 \).

All solutions \( \hat{z} = -2 + b \epsilon_1 + c \epsilon_2 + d \epsilon_3 + 6 \epsilon_4 \), with \( -6 \leq b \leq 1, 1 \leq c \leq 2, 1 \leq d \leq 4, |b| + c + d = 3 \) and \( b + c + d = -3 \) are minimal upper bounds with minimum concretization, and there are an infinite number of them. We can choose for example \( \hat{z} = -2 + \epsilon_1 + \epsilon_2 + 3 \epsilon_3 + 6 \epsilon_4, \) or \( \hat{z} = -2 - \epsilon_1 + 2 \epsilon_3 + 6 \epsilon_4, \) etc.

**Proposition 26.** Let \( \hat{x}, \hat{y} \in K_3 \), such that \( \| \hat{y} \| \) holds. If \( \gamma(\hat{x}) \) and \( \gamma(\hat{y}) \) are in generic positions, then \( \hat{z} \) defined by \( \hat{z} \)

\[
\alpha_0^* = \text{mid}(\gamma(\hat{z}) \cup \gamma(\hat{y}))
\]

\[
\alpha^*_\text{p} = \text{argmin}_{a \in [-1, 1]} |a|, \text{ for all } i \geq 1
\]

is the unique minimal upper bound of \( \hat{z} \) and \( \hat{y} \) whose concretization is the union of the concretization of \( \hat{z} \) and \( \hat{y} \).

If \( \gamma(\hat{x}) \) and \( \gamma(\hat{y}) \) are not in generic positions and \( \gamma(\hat{x}) \subset \gamma(\hat{y}) \) (we get symmetric properties when \( \gamma(\hat{y}) \subset \gamma(\hat{x}) \)), then all \( \hat{z} \) satisfying \( \| \hat{y} \| \leq \| \hat{y} \| \) and \( \alpha_0^* = \text{mid}(\gamma(\hat{z}) \cup \gamma(\hat{y})) = a_0^* \)

\[
\alpha^*_\text{p} \leq \alpha_i \leq 0 \text{ or } 0 \leq \alpha_i \leq \alpha^*_\text{p}, \text{ for all } i \geq 1
\]

are minimal upper bounds with concretization the union of the concretization of \( \hat{z} \) and \( \hat{y} \).

---

1 This is a slight abuse of notation here: we do not have in general the finite chain property, but a similar one, in our framework (convergence in a finite time, in finite arithmetic).
3.5.2 The meet operation

If \((A_1, \leq)\) admitted binary least upper bounds, then we would have a Riesz space, for which \(x \wedge y\) would be defined as \(-((-x) \cup (-y)))\). Here, we have a different formula, linking \(\cap\) with \(\cup\) in some interesting cases. Intersections will produce negative \(\beta\) coefficients, where unions were producing positive \(\beta\) coefficients.

**Lemma 27.** For all \(\hat{x}, \hat{y} \in A_1\), there exist maximal lower bounds (or mlb) \(\hat{z}\) of \(\hat{x}\) and \(\hat{y}\) if and only if

\[
\|\hat{y} - \hat{x}\|_A \geq |\beta^y - \beta^x|.
\]

They then all satisfy (for all \(i \geq 1\)):

\[
\beta^i = \frac{1}{2}(\beta^y + \beta^x - \|y - x\|_A)
\]

\[
\alpha^i \land \alpha^j \leq \alpha^i \lor \alpha^j
\]

**Sketch of proof.** Being a lower bound of \(x\) and \(y\) means:

\[
\|z - x\|_A \leq \beta^y - \beta^x
\]

\[
\|z - y\|_A \leq \beta^y - \beta^x
\]

Summing the two inequalities, and using the triangular inequality:

\[
\|y - x\|_A \leq \beta^y - \beta^x
\]

\[
\leq \beta^y + \beta^x - 2\beta^x.
\]

Hence \(\beta^x \leq \frac{1}{2} (\beta^x + \beta^y - \|y - x\|_A)\), giving an upper bound. As we want a maximal \(z\), the natural question is whether we can reach this bound. This is the case when the triangular inequality for \(\ell_1\) norm is an equality, which is the case when \(\alpha^i \land \alpha^j \leq \alpha^i \lor \alpha^j\). A solution exists to these constraints only if (14) is satisfied, as for the proof of Proposition 20.

Contrarily to the join operators, we cannot in general impose (even in generic position) for a mlb \(\hat{z}\) to have a given concretization, such as \(\gamma(\hat{z}) \cap \gamma(\hat{y})\), or even a smaller value, that contains all values that \(\hat{x}(t_1, \ldots, u_x)\) and \(\hat{x}(t_1, \ldots, u_y)\) share for some \(t_1, \ldots, u_x, u_y \in [-1, 1]\).

**Example 28.** Consider \(\hat{x} = 1 + e_1 - 2e_2 \leq [-2, 4]\) and \(\hat{y} = 2 + 2e_1 + e_2 \leq [-1, 5]\). They are in generic position. Then \(\hat{z}\) is a mlb with concretization \(\gamma(\hat{z}) = \gamma(\hat{x}) \cap \gamma(\hat{y})\). Hence \(\alpha^i \land \alpha^j \geq 0\) for all \(i \geq 1\), there exists a maximal lower bound \(\hat{z}\) with \(\gamma(\hat{z}) = \gamma(\hat{x}) \cap \gamma(\hat{y})\), given by the formulas:

\[
\alpha^i = \text{mid}(\gamma(\hat{x}) \cap \gamma(\hat{y}))
\]

\[
\alpha^i = \text{argmax}_{\alpha^i \land \alpha^j \leq \alpha^i \lor \alpha^j} |\alpha|
\]

\[
\beta^i = \frac{1}{2}(\beta^x + \beta^y - \|y - x\|_A)
\]

In this case, we have:

\[
x \cap y + y \cup y = x + y
\]

**Proof.** The formula for \(\beta^i\) is given by Lemma 27. The fact that the concretization of \(\hat{z}\) is \(\gamma(\hat{z}) \cap \gamma(\hat{y})\) implies the formula for \(\alpha^i\).

The formulas for \(\alpha^i\), \(i \geq 1\) can be checked easily as follows: as \(\gamma(\hat{x})\) and \(\gamma(\hat{y})\) are in generic positions, \(\gamma(\hat{x}) \cap \gamma(\hat{y})\) and \(\gamma(\hat{z})\) (similarly with \(\gamma(\hat{y})\)) are in generic positions. Thus we can use the formula of Proposition 26 for the join operator, to compute \(z \cup y\) and \(z \cup y\). It is easily seen now that for \(w, v \geq 0\), and \(w, v\),

\[
\text{argmin}_{u \in A \land v \geq u} |\alpha| = \text{argmax}_{u \in A \land v \geq u} |\alpha|
\]

\[
\text{argmin}_{u \in A \land v \geq u} |\alpha| = \text{argmax}_{u \in A \land v \geq u} |\alpha|
\]

Hence \(\alpha^{i=1} = \alpha^i\) and \(\alpha^{i=2} = \alpha^i\) for all \(i \geq 1\).

Furthermore, \(\gamma(x \cup y) = \gamma(y)\) and \(\gamma(z \cup z) = \gamma(\hat{x})\) hence \(\alpha^{i=1} = \alpha^i\) and \(\alpha^{i=2} = \alpha^i\). Finally, again because of this equality and concretizations, and that all coefficients but \(\beta^{i=1}\) (respectively \(\beta^{i=2}\)) have been shown equal to the ones of \(\hat{x}\) (respectively \(y\)), we have necessarily that \(\beta^{i=1} = \beta^x\) (respectively \(\beta^{i=2} = \beta^y\)).

Therefore \(x \cap x \geq z \cup y\), and \(y \cup y \geq z \cup y\) so \(z\) is a lower bound of \(x\) and \(y\). Because of the value of \(\beta^x\), by Lemma 27 and 18 (adapted to mlbs), \(z\) is an mlb (with the right concretization). \(\square\)

3.6 Quasi bounded completeness

We prove that we have almost bounded completeness of \(A_1\). Unfortunately, as shown in Example 31 it is barely usable in practice, and we resort to a useful sub-structure of \(A_1\) in Section 3.7 (in particular, with a view to Section 4).

**Proposition 30.** \((A_1, \leq)\) is a quasi bounded-complete partial order (or is quasi-"Dedekind-complete"), meaning that any bounded subset \(A\) of \(A_1\) such that for all \(\hat{x}, \hat{y} \in A\)

\[
\|\hat{x} - \hat{y}\|_A \geq |\beta^x - \beta^y|
\]

has a minimal upper bound in \(A_1\).

**Sketch of proof.** Let \(z\) be a lower bound of all \(a_i \in A\). This means again, for all pairs \(i, j\) that

\[
\beta^i \geq \frac{1}{2} (|a_i - a_j|_A + \beta^{a_i} + \beta^{a_j})
\]

We can always suppose that \(b = \beta^{a_i} + \beta^{a_j}\). As \(b\) dominates all \(a_i \in A\), we have, using the triangular inequality:

\[
\|a_i - a_j\|_A \leq \|a_i - b\|_A + \|b - a_j\|_A \leq 2\|a_i - b\|_A - \beta^{a_i} - \beta^{a_j}
\]

so \(\beta^i \geq \beta^b\). This means that we can write:

\[
\beta^i = \inf_{a_j} \left\{ \frac{1}{2} \left( \|a_i - a_j\|_A + \beta^{a_i} + \beta^{a_j} \right) \right\}
\]

which exists in \(\mathbb{R}\). Similarly to the proof of Proposition 20 condition 3 allows to prove existence of a solution to the lub equations. \(\square\)

Unfortunately, even if we pick one of the possible join operators, they are not in general associative operators, which means that even for countable subsets \(A\) of \(A_1\), according to the iteration strategy we choose, we might end up with a non-minimal upper bound.

**Example 31. Take**

\[
\begin{align*}
\hat{x} &= 1 + 2e_1 - e_2 + 2e_3 \\
\hat{y} &= e_1 + e_2 + e_3 \\
\hat{z} &= 5 + e_1 - 2e_2
\end{align*}
\]

When computing with one of the possible join operators previously defines, we obtain

\[
\begin{align*}
(\hat{x} \cup \hat{y}) \cup \hat{z} &= 2 + e_1 + 2e_2 + 0.5(7.5 + a_1 + b_1 + c_1)eU \\
(y \cup \hat{z}) \cup \hat{z} &= 2 + e_1 + 5eU \\
(\hat{x} \cup \hat{z}) \cup \hat{y} &= 2 + e_1 + 2e_2 + (0.5 + \beta_2)e_3 + 4.5eU,
\end{align*}
\]
with $a_1 - b_1 + c_1 = 2.5$, $1 \leq a_1 \leq 2$, $-1 \leq b_1 \leq 0$, $1 \leq c_1 \leq 2$, and $-1 \leq b_2 \leq 0$.

We see on this example that, in the case where all concretizations are not in generic positions, the join operator is not associative. Moreover, the result of two successive join operations may not be a minimal upper bound of three affine forms because we do not always get the same $\beta$ coefficient (5 when computing $(\hat{y} \cup \hat{\alpha}) \cup \hat{\beta}$, and 4.5 when computing $(\hat{\alpha} \cup \hat{\beta}) \cup \hat{\alpha}$). Indeed, we have here $(\hat{\alpha} \cup \hat{\beta}) \cup \hat{\alpha} \leq (\hat{y} \cup \hat{\alpha}) \cup \hat{\beta}$ when $-0.5 \leq b_2 \leq 0$.

We fix this difficulty in next section.

3.7 A bounded lattice sub-structure

In practice, we obtain a stronger sub-structure by using a widening instead of the minimal upper bound:

**Definition & Lemma 32.** We define the widening operation $\hat{\gamma} = \hat{x} \nabla \hat{y}$ by

- $\alpha_0^\gamma = \text{mid}(\gamma(\hat{x}) \cup \gamma(\hat{y}))$
- $\alpha_i^\gamma = \arg\min_{\alpha \in \alpha_i^\gamma} |\alpha|$ for all $i \geq 1$,
- $\beta^\gamma = \sup_{\alpha, \alpha \in \alpha_i^\gamma} |\alpha - \|z\|_L|

In the case when $\hat{\alpha}$ and $\hat{y}$ are in generic positions, $\nabla$ is the union defined in Section 3.5.7. Otherwise, $\nabla \hat{y}$ has as concretization the union of the concretizations of $\hat{\alpha}$ and $\hat{y}$, it is an upper bound of $\hat{\alpha}$ and $\hat{y}$ (but it is not a minimal upper bound with respect to $\leq$, because $\beta^\gamma$ is not minimal).

This operator has the advantages of presenting a simple an explicit formulation, a stable concretization with respect to the operands, and of being associative.

**Definition & Lemma 33 (computational order).** Let $\ll$ be the binary relation defined by:

$$\hat{x} \ll \hat{y} \Leftrightarrow \hat{x} \nabla \hat{y} = \hat{y}.$$ 

Then, $\ll$ is a partial order.

**Sketch of Proof.** Reflexivity comes from $\hat{x} \nabla \hat{x} = \hat{x}$. Antisymmetry is trivial. Transitivity comes from the associativity of $\nabla$. $\square$

**Lemma 34.** $\hat{x} \ll \hat{y}$ if and only if:

- $\gamma(\hat{x}) \subseteq \gamma(\hat{y})$ and
- for all $i \geq 1$, $0 \leq \alpha_i^y \leq \alpha_i^x$ or $\alpha_i^x \leq \alpha_i^y \leq 0$

**Sketch of Proof.** The first condition ensures that $\alpha_i^{x \cup y} = \alpha_i^x$ and $\beta_{i \cup y} = \beta_i^y$. Take $i \geq 1$. If $\alpha_i^x \alpha_i^y \leq 0$ then because $\alpha_i^{x \cup y}$ has to be zero. Otherwise, this translates precisely to the second condition.

**Definition 35.** We define operation $\hat{z} = \hat{x} \Delta \hat{y}$ by

- $\alpha_0^\Delta = \text{mid}(\gamma(\hat{x}) \cap \gamma(\hat{y}))$
- $\alpha_i^\Delta = \arg\max_{\alpha \in \alpha_i^\Delta} |\alpha|$ for all $i \geq 1$, if $\alpha_i^x \alpha_i^y \geq 0$, otherwise $\alpha_i^\Delta = 0$
- $\beta^\Delta = \sup_{\alpha \in \alpha_i^\Delta} \gamma(\hat{x} \cap \hat{y}) - \alpha_i^\Delta - \|z\|_L$

**Proposition 36.** $(\mathcal{A}_1, \ll)$ is a bounded complete lattice, with:

- $\nabla$ being the union,
- $\Delta$ being the intersection.

**Sketch of Proof.** Easy verification for the binary unions and intersections. Take now $A \subseteq \mathcal{A}_1$ and $b$ such that $b \geq A$. Then $\gamma(a) \subseteq \gamma(b)$ for all $a \in A$, so $I = \bigcup_{a \in A}(a)$ has finite bounds. This gives us $\alpha_0 = \text{mid}(I)$.

Consider now any countable filtration of $A$ by an increasing sequence of finite subsets $A_k$, $k \in K$, and consider $\alpha_k = \bigcup A_k$ any minimal upper bound of $A_k$. We know that $|\alpha_k^\beta|$ is a decreasing sequence of positive real numbers when $k$ increases, so it converges. We also know that the sign of $\alpha_k^\beta$ remains constant, so $\alpha_k^\beta$ converges, say to $\alpha_\beta$. Last but not least, let $\beta = \arg\min_{\alpha \in A} |\alpha_\beta|$, we argue that $\alpha_0 + \sum_{i=1}^\infty \alpha_i \epsilon_i + \beta \epsilon_U$ is a minimal upper bound of $A$ in $A_\beta$. $\square$

This allows to use $\nabla$ (and $\Delta$) as effective widenings during the iteration sequence for solving the least fixed point problem.

4. Iteration schemes and convergence properties

This is where all properties we studied fit together, to reach the important Theorem [4] stating good behavior of the Kleene-like iteration schemes defined in Section 4.2. First, we show that we must improve the computation of the abstract semantic functional, between two union points, this is explained in Section 4.1. We also improve things a little bit, on the practical side, by defining new widening operators, in Section 4.3.

4.1 The shift operator and the iteration scheme

One problem we encounter if we are doing the blind Kleene iteration in the lattice of Proposition [5] is that we introduce $\epsilon_U$ coefficients, for which the semantics of arithmetic expressions is far less well behaved than for “ordinary” noise symbols $\epsilon_i$.

**Example 37.** Let us give a first simple example of what can go wrong. Consider the following program:

```plaintext
F(real a) {
    real x;
    x = input(-1,1); [1]
    while (true)
        x = x-a*x; [2]
}
```

**Suppose that a can only be given values between (strictly) 0 and 1, then it is easy to see that this scheme will converge towards zero, no matter what the initial value of x is. As the scheme is essentially equivalent, in real numbers to $x_{n+1} = (1-a)x_n$, with [1 - a] < 1, a simple Kleene iteration scheme should converge. Let us look at the successive iterates $\hat{x}_i$ at control point [2], of this scheme. First, note that $\hat{x}_{n+1} = \epsilon_1 \nabla (\hat{x}_n - a \hat{x}_n)$ (or equivalently $\hat{x}_{n+1} = \epsilon_1 \nabla (\hat{x}_n - a \hat{x}_n)$ starting with $\hat{x}_0 = \epsilon_1$, where $\epsilon_1$ stands for the noise symbol introduced by assignment at control point [1]). Then $\hat{x}_{n+1} = \epsilon_1 \nabla (\hat{x}_n - a \hat{x}_n)$ starting with $\hat{x}_0 = \epsilon_1$, where $\epsilon_1$ stands for the noise symbol introduced by assignment at control point [1]).$

$$\hat{x}_1 = \epsilon_1$$
$$\hat{x}_2 = (1-a)\epsilon_1 + a \epsilon_U$$
$$\hat{x}_3 = \epsilon_1 \nabla ((1-a)^2 \epsilon_1 + a \epsilon_U - a^2 \epsilon_U)$$
$$= \epsilon_1 \nabla ((1-a)^2 \epsilon_1 + a \epsilon_U)$$

because the semantics of $\hat{}$ on $\epsilon_U$ symbols cannot cancel out its coefficients a priori. We will see a bit later that under some conditions, we can improve the semantics locally.

To carry on with this example, let us particularize the above scheme to the case $a = \frac{1}{2}$:

$$\hat{x}_3 = \epsilon_1 \nabla \left(\frac{1}{2} \epsilon_1 + \frac{1}{2} \epsilon_U\right)$$
$$= \frac{1}{4} \epsilon_1 + \frac{3}{4} \epsilon_U$$

We already see that the concretization of $\hat{x}_3$ is bigger than $[-1,1]$ showing loss of precision, even to simple interval computations. The next iterations make this interval grow to infinity. If we could have written

$$\hat{x}_{i+1} = \epsilon_1 \nabla (\hat{x}_i - a \hat{x}_i)$$

instead of

$$\hat{x}_{i+1} = \epsilon_1 \nabla (\hat{x}_i - a \hat{x}_i)$$

we would get a much more accurate description of the noise symbol.
the iteration sequence would have been convergent. We are going to explain that we can make sense of this.

We first introduce a shift operator, that decreases the current abstract value.

**Definition & Lemma 38.** Let $\hat{x} = a_0^\varepsilon + \sum_{i=1}^{\infty} a_i^\varepsilon e_i + \beta^\varepsilon e_U$. Define, for $\hat{x}$ with finitely many non-zero coefficients:

$$\hat{x} = a_0^\varepsilon + \sum_{i=1}^{\infty} a_i^\varepsilon e_i + \beta^\varepsilon e_f$$

where $e_f$ is a “fresh” symbol. Then for all $\hat{x} \in A_+, \hat{x} \leq \hat{x}$.

The idea is that, after some unions, during a Kleene iteration sequence, such as right after the union in semantic equation [18] of Example [37] we would like to apply the ! operator, allowing us to get an equation equivalent to (17).

The full formalization of this refined iteration scheme is outside the scope of this paper. It basically relies on the following observation of our abstract semantics:

All concrete executions of a program correspond to a unique choice of values between -1 and 1, of $e_1, \ldots, e_n, \ldots, e_U$ (for all join control points).

Hence, locally, between two join control points, $e_f$ coefficients act as normal $e_i$ coefficients. Hence one can use ! to carry on the evaluation of the abstract functionals after each control point where a union was computed, corresponding to a branching between several concrete executions.

**Example 39.** We carry on with the example [1]. We use now the new semantic equation: $\hat{x}_{i+1} = x_1^\varepsilon \nabla (\hat{x}_i - a_0^\varepsilon \hat{x}_i)$. Therefore, the successive iterates, for $\alpha = \frac{3}{4}$ are:

$$\hat{x}_1 = e_1$$
$$\hat{x}_2 = \frac{1}{2} e_1 + \frac{3}{4} e_U$$
$$\hat{x}_3 = e_1 \nabla (\frac{1}{2} e_1 + \frac{3}{4} e_2)$$

where $e_2 = e_U$ in the last iterate. Therefore, $\hat{x}_3 = \frac{1}{2} e_1 + \frac{15}{16} e_U$.

The successive iterates will converge very quickly to $\hat{x}_\infty = e_U$ with concretion being $[-1, 1]$ and no surviving relation.

### 4.2 Iteration schemes

As we have almost bounded completeness, and not unconditional completeness, our iteration schemes will be parametrized by a large interval $I$: as soon as the current iterate leaves $I$, we end iteration by $\top$ (that we can choose to represent by $\infty \in E$ by an abuse of notation). The starting abstract value of the Kleene like iteration $I$ is as usual notation). The starting abstract value of the Kleene-like iteration $I$ is as usual.

**Definition 40.** The (\top, c, \cup)-iteration scheme of some functional $F : \text{Var} \to \text{Var}$ is as follows:

- First unroll $i$ times the Kleene iteration sequence, starting from $\bot$, i.e. compute $x_i = F^i(\bot)$.
- Then iterate: $x_{n+1} = x_n \cup F^i(\{x_n\})$ starting with $n = 1$.
- End when a fixpoint is reached or with $\top$ if $\gamma(x_{n+1}) \not\subseteq I$.

Note that initial unfolding are important for better precision but will not be used in the sequel.

### 4.3 Convergence for linear recursive filters

We prove that our approach allows us to find good estimates for the real bounds of general affine recurrences (i.e. linear recursive filters of any order), see Section 2.2. The only abstract domains known to be able to give accurate results are the one of [3], which only deals with filters of order 2, and the one of [17], which is specialized for digital filters (which is not the case of our abstraction).

We consider again the class of programs of Section 2.2.

**Theorem 41.** Suppose scheme $\{ \hat{x} \}$ has bounded outputs, i.e. the (complex) roots of $x^n - \sum_{i=0}^{n-1} a_i x^i$ have module strictly less than 1. Then there exists $q$ such that the $(0, q, \nabla)$-iteration scheme (see Section 4.2) converges towards a finite over-approximation of the output.

In other words, the perturbed numerical scheme solving the fixpoint problem is also bounded.

**Sketch of proof.** Being a fixed point of abstract functional $F$ (giving the abstract semantics of the one iteration of the loop) means

$$x_{k+n+1} = x_{n+1} \nabla \left( \sum_{i=0}^{n-1} a_i x_{k+i} + \sum_{j=0}^{n} b_j e_{k+j} \right)$$

Define

$$y_1 = x_n$$
$$y_2 = x_{n+1} \nabla x_n$$
$$\vdots$$
$$y_n = x_{n} \nabla x_2 \nabla \ldots x_n$$
$$y_{n+1} = x_n$$

Then the fixpoints of $F$ are determined by the fixed points $z$:

$$z = x_{n+1} \nabla \left( \sum_{i=1}^{n} a_i (y_{n+1-i} \nabla z) + \sum_{j=1}^{n+1} b_j e_{k+j} \right).$$

Suppose first that $\sum_{i=1}^{n} |a_i| < 1$. Consider now the interval fixpoint equation resulting from (19). As $\gamma$ commutes with $\nabla$, by definition, and because of Lemma 40, it transforms into

$$\gamma(z) \subseteq \gamma(x_{n+1} \cup \sum_{i=1}^{n} (a \gamma(y_{n+1-i}) \cup \gamma(z)) + \sum_{j=1}^{n+1} b_j e_{k+j}).$$

This equation shows that $\gamma(z)$ is a pre-fixpoint of the interval abstraction of our linear scheme. It is well known that in the case $\sum_{i=1}^{n} |a_i| < 1$, this interval abstraction admits a bounded least fixpoint $z'$. Hence, $z$ in this case is bounded by $z'$ (for order $\leq$, when $z'$ is written as $\mid d(\mid z') + \operatorname{dev}(\mid z')\epsilon_U$, with $\operatorname{dev}([a, b]) = \frac{\ln(2)}{\ln(\epsilon)}$, hence has finite concretion. In fact, not only $z$ but all the ascending sequence of the $(0, 1, \nabla)$-iteration scheme from $\bot$ is bounded by $z'$. Note that any ascending sequence for any $(p, q, \nabla)$-iteration scheme is also ascending for the partial order $\ll$. By Proposition 36 it has a least upper bound, which is the least fixed point of $F$ for partial order $\ll$, because of the obvious continuity (in the $\ell_1$ sense) of $F$. Hence again, this fixed point is bounded by $z'$ so has finite concretion.

Secondly, when the roots of $x^n - \sum_{i=0}^{n-1} a_i x^i$ have module strictly less than 1, then there exists $q$ such that $F^q$ is a filter of order $nq$ in the inputs $e$, and $n$ in the outputs with coefficients $c_j, j = 1, \ldots, n$ such that $\sum_{i=nq+1}^{nq+1} c_i$ is strictly less than 1. One can check that the semantics on affine forms is exact on affine computations (because of the use of the shift operator). We can then apply the result above to reach the conclusion. □
More generally, and this is beyond the scope of this article, we can show that there exist \((i, q, \nabla)\)-iteration schemes that will come as close as we want to the exact range of values that \(x\) can take.

**Example 42.** We carry on with Example 7. We see that matrix \(A\) in our case is
\[
A = \begin{pmatrix} 0 & 1 \\ -0.7 & 1.4 \end{pmatrix}
\]
and of course, the \(\ell_1\) norm of the rows of \(A\) is bigger than 1. Iterating \(A\), we see that:
\[
A^2 = \begin{pmatrix} -0.5488 & 0.2156 \\ -0.15092 & -0.24696 \end{pmatrix}
\]
is the first iterate of \(A\) with \(\ell_1\) norm of rows less than 1. We know by Theorem 7 that a \((0, 5, \nabla)\)-iteration scheme will eventually converge to an upper approximation of the invariant (which we can estimate, see Example 3 to \([-1.12124069..., 2.82431841...])\).
Here is what \((0, i, \nabla)\)-iteration schemes give as last invariant and concretization, when \(i\) is greater than 5 (rounded, for purposes of readability):

| \(i\) | invariant | concretization |
|------|-----------|----------------|
| 5    | \(0.8334+2.4661e_{\mathbb{E}}\) | \([-1.6328, 3.2995]\) |
| 16   | \(0.7621+2.0622e_{\mathbb{E}}\) | \([-1.3, 2.8244]\) |

Note that although the convergence to this invariant is asymptotic (meaning that we would need in theory an infinite Kleene iteration to reach the invariant), in finite precision, the invariant is reached in a finite number of steps. In the case of the \((0, 16, \nabla)\)-iteration scheme, the fixpoint is reached after 18 iterations. In some ways, we have replaced the numerical scheme (a filter of order 2 here), by an abstract numerical scheme which has similar convergence properties, and can be simulated in a finite time and in a guaranteed manner, accurately. We can also use extrapolation or widening techniques, for which we will show some results in Example 44.

Note also that none of the noise symbols survived in the final invariant: there is no dependency left with the successive inputs, when looking at the overall invariant. This is very easily shown on the first few Kleene iterates already. We denote by \(\hat{z}\), the affine form at control point \([2]\), \(\hat{y}\), the affine form at control point \([3]\), at iteration \(i\), for the \((0, 16, \nabla)\)-iteration scheme. We have (as produced by our prototype implementation):
\[
\begin{align*}
\hat{x}_1 &= 0.8808 + 1.8593e_{\mathbb{E}} \\
\hat{y}_1 &= 0.8808 + 0.01038e_{\mathbb{E}} + 0.0429e_2 + 0.0369e_4 + \ldots + 0.1052e_{21} + 0.2041e_{23} + 0.2589e_{25} + 0.2254e_{27} + 0.0811e_{29} - 0.16e_{31} + 0.35e_{33} \\
\hat{x}_2 &= 0.8422 + 1.9407e_{\mathbb{E}} \\
\hat{x}_3 &= 0.8323 + 1.9688e_{\mathbb{E}} \\
\end{align*}
\]

Finally, you should note that Theorem 7 is not limited by any means to finding invariants of such filter programs with independent inputs, or independent initial conditions. For instance, if all the inputs over time are equal, but unknown numbers between 0 and 1, the final invariant has concretization \([-0.1008, 2.3298]\).

### 4.4 Simple widening operators

We can define numerous widening operators, among which the following:

**Definition & Lemma 43.** The operator \(W\) defined by \(\hat{z} = zW\hat{y}\) such that:
- \(\alpha_0^z = \text{mid}(\gamma(\hat{z}) \cup \gamma(\hat{y}))\),
- \(\alpha_i^z = \alpha_i^y\) for all \(i \geq 1\) such that \(\alpha_i^z = \alpha_i^y\),
- \(\alpha_i^z = 0\) for all \(i \geq 1\) such that \(\alpha_i^y \neq \alpha_i^z\).

\(\beta^z = \text{sup } \gamma(\hat{z} \cup \hat{y}) - \alpha_0^z - \|z\|_L\)
gives an upper bound of \(\hat{z}\) and \(\hat{y}\) that can be used as an efficient widening.

**Example 44.** Now we are carrying on with Example 7, but this time we apply the widening defined above after 1 normal iteration step. For \(i\) equal to 5, fixpoint is reached at iteration 9, and for \(i\) equal to 16, it is reached at iteration 4, with precision equivalent to the case without widening. This time, convergence is reached in finite time, by construction (and not because of "topological" convergence).

### 5. Directions currently investigated

We discuss in this section very promising improvements of the above schemes, which we feel are important to mention here. But, as they are not fully formalized yet, we mostly demonstrate them on examples.

#### 5.1 Iteration strategies for a refined join operation

As we introduced (Definition 38), the possibility to shift the union symbols to "classical" noise symbols in the iteration scheme, it becomes important to create as few union symbols as possible, in order not to lose relations. This can be partly solved by an adapted refined iteration strategy : when there is a cycle of explicit dependency between variables, make the union only on one variable and apply immediately the shift operator of Definition 38 before this union is propagated to the other dependent variables.

**Example 45.** Consider the following program, implementing a second-order filter (where \(\text{xpnl}\) stands for \(x_{n+1}\), \(x\) stands for \(x_n\) and \(\text{xmm1}\) stands for \(x_{n-1}\)):

```c
real xpnl,xn, xmm1;

xn = [0,1];
while (true) {
    xpnl = 1.2*xn - 0.8*xmm1;
    xmm1 = xn; xn = xpnl;
}
```

In this program, we have \(x_n^k = x_{n-1}^{k+1}\), where \(k\) is the current iteration of the loop, so it is clearly a bad idea to make unions independently on \(x_n\) and \(x_{n-1}\).

Before the loop, \(\hat{x}_0 = 0.5 + 0.5\varepsilon_1\), and after first iteration of the loop, \(\hat{x}_1 = 0.5 + 0.5\varepsilon_1 + 0.5\varepsilon_2\), \(\hat{x}_2 = 0.6 + 0.6\varepsilon_1\).

Applying the classical join operations to \(\hat{x}_n\) and \(\hat{x}_{n-1}\) at the beginning of the loop after first iteration gives \(\hat{x}_n^1 = \hat{x}_n \cup \hat{x}_n^0 = 0.6 + 0.5\varepsilon_1 + 0.1\varepsilon_2\), \(\hat{x}_{n-1} = \hat{x}_{n-1} \cup 0 = 0.5 + 0.5\varepsilon_2\).

Then, after applying the shift operator (with a new symbol \(e_2\) for \(\hat{x}_n\), and a new symbol \(e_3\) for \(\hat{x}_{n-1}\)), we get \(\hat{x}_n^2 = \hat{x}_n \cup 0.6 + 0.6\varepsilon_1 + 0.12\varepsilon_2 - 0.4\varepsilon_3\), and \(\gamma(\hat{x}_n^2) = [0.8, 1.44]\).

\(\hat{x}_n^2 = \hat{x}_n^0 \cup (\hat{x}_n^1) = 0.32 + 0.5\varepsilon_1 + 0.1\varepsilon_2 + 0.5\varepsilon_2\), \(\hat{x}_{n-1} = \hat{x}_{n-1} \cup 0 = 0.6 + 0.6\varepsilon_1\), and after applying a shift that creates new symbols \(e_4\) and \(e_5\), we get \(\hat{x}_n^3 = x_{n+1} = 0.096 + 0.6\varepsilon_1 + 0.12\varepsilon_2 + 0.624\varepsilon_2 - 0.48\varepsilon_3\) and \(\gamma(\hat{x}_n^3) = [-1.92, 1.728]\).

Of course, in practice, cyclically unrolling the loop allows to care with the bad behavior of the scheme, but it is better to refine as well the iteration as follows.

**Applying the join and shift operations on \(x_n\) only, we write**:

\(\hat{x}_n^1 = (\hat{x}_n^0 \cup \hat{x}_n^1) = 0.6 + 0.5\varepsilon_1 + 0.1\varepsilon_2\), \(\hat{x}_n^2 = 0.32 + 0.2\varepsilon_1 + 0.12\varepsilon_2\), and \(\gamma(\hat{x}_n^2) = [0.64]\).

Then \(\hat{x}_n^3 = (\hat{x}_n^2 \cup \hat{x}_n^3) = 0.6 + 0.2\varepsilon_1 + 0.1\varepsilon_2 + 0.3\varepsilon_3\) and \(\hat{x}_n^3 = 0.24 - 0.16\varepsilon_1 + 0.04\varepsilon_2 + 0.36\varepsilon_3\), \(\gamma(\hat{x}_n^3) = [-0.32, 0.8]\).

We dealt here with an example where the dependencies between variables were explicit, but we can also generalize this and intro-
duce symbols to explicit and preserve implicit dependency, as in Example 46.

**Example 46.** Take the following program:

```c
real x,y;
x = 1; y = 0;
for (x=1; x<=10000 ; x++)
    y = y + 2;
```

If we apply standard union, we get for example after first iteration,
```c
real x,y;
y
```
are always such that
```c
use union and shift on
deduce
y
```

The result for
```c
x
```
```c
if (y > 0)
else { x = 0; y = 1; }
```
```c
if (z < 0.5) {
real x,y,z;
```
```c
5.2 Refining again the join operator for disjunctive analysis

**Example 47.** Take the following program:

```c
real x,y,z;
z = [0,1];
if (z < 0.5) {
    x = 1; y = -1;
} else { x = 0; y = 1; }
if (y > 0)
x = x + y;
```

The result for \( x \) of the execution of this toy example, is always \( x = 1 \), whatever \( x \in [0,1] \).

Expliciting the dependency between \( x \) and \( y \) (we can always do so with constants) : in the two branches taken after the first test, we have \( y = -2x + 1 \), then we write \( \hat{x} = 0.5 + 0.5\varepsilon_{x} \) and \( \hat{y} = -\varepsilon_{z} \) after joining the results from the two branches. Then, interpreting test \( (y > 0) \) leads to add constraint \( \varepsilon_{z} < 0 \), but this is not enough to deduce \( x = 1 \).

Then, we can note that for example \( \hat{x} = 0.5 + 0.5\varepsilon_{x} \) should denote a disjoint union of two values 1 and 0, thus in this case symbol \( \varepsilon_{z} \) no longer takes all values in \([-1,1]\), but only the two values \(-1\) and 1. Then, when test \( (y > 0) \) is true, then \( \varepsilon_{z} = -1, \hat{x} = 0.5 - 0.5\varepsilon_{z} = 1 \), and when it is false, then \( \hat{x} \) is naturally equal to 1.

6. Conclusion, Related and Future Work

We have proved that our abstract domain behaves well for an interesting class of numerical programs. More work has yet to be done on the formalisation of the shift operator (Section 4.4) and on more general schemes, such as some non-linear schemes of interest. Many questions arise also from this work. For instance, can we replace affine forms (but \( \ell_{p} \)) by higher-order Taylor models?

Also, most of our proofs only rely on the general properties of norms, and not specifically on \( \ell_{1} \). What do we get with the \( \ell_{p} \) norms, and in particular with the standard Lorentz cone, when considering \( \ell_{2} \)? Many techniques are available here that could help, in particular the techniques of Second-Order Cone Programming.

Our main convergence result (Theorem 14) can be recast as a fixpoint property of some general \( (\min, \max, +) \) functions. Can we use policy iteration techniques [8,10] to help solve these? Last but not least, Property 15 brings a bell and looks like phenomena appearing with spectral measures (measures with value in a Banach space). Is this also generalizable to affine forms where \( \varepsilon_{i} \) are random variables of some sort?

Acknowledgments are due to Stéphane Gaubert for interesting remarks during an earlier presentation of these results.

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