NEW LOWER BOUNDS FOR THE MAXIMAL DETERMINANT PROBLEM

WILLIAM P. ORRICK, BRUCE SOLOMON, ROLAND DOWDESWELL, AND WARREN D. SMITH

Abstract. We report new world records for the maximal determinant of an $n \times n$ matrix with entries $\pm 1$. Using various techniques, we beat existing records for $n = 22, 23, 27, 29, 31, 33, 34, 35, 39, 45, 53, 63, 69, 73, 77, 79, 93,$ and $95$, and we present the record-breaking matrices here. We conjecture that our $n = 22$ value attains the globally maximizing determinant in its dimension. We also tabulate new records for $n = 67, 75, 83, 87, 91$ and $99$, dimensions for which no previous claims have been made. The relevant matrices in all these dimensions, along with other pertinent information, are posted at www.indiana.edu/~maxdet.

1. Introduction

Let $md(n)$ denote the global maximum of the determinant function on the set of all $n \times n$ matrices populated entirely by $\pm 1$s. It is easy to show that $md(n) \leq n^{n/2}$, and the well-known Hadamard conjecture predicts that when $n > 2$, equality occurs if $n \equiv 0 \pmod{4}$. This conjecture has been confirmed for all $n \leq 424$ and many larger $n$, including a number of infinite sequences [SY].

When $n > 2$ is not divisible by 4, however, the inequality is necessarily strict, and relatively few exact values for $md(n)$ are known.

Using a combination of techniques, we find a number of matrices whose determinants exceed previous records, thereby establishing new lower bounds for $md(n)$. We sketch our methods Sec. 2 below, and will expand on them further in a forthcoming survey paper [OOS]. Here, our purpose is simply to record and establish priority for our new lower bounds.

Other authors have derived upper bounds for $md(n)$. When $n \not\equiv 0 \pmod{4}$ these are more stringent than the simple Hadamard estimate $md(n) \leq n^{n/2}$. We compare our lower bounds to these upper bounds below, using formulae which come from Barba [Ba] when $n \equiv 1 \pmod{4}$, from Ehlich [Eh1] and Wojtas [Wo] (independently) when $n \equiv 2 \pmod{4}$, and from Ehlich [Eh2] when $n \equiv 3 \pmod{4}$.
These bounds cannot be sharp in the cases we treat here, except possibly when $n = 91$. Indeed, we conjecture that $\text{md}(22) = 2^{23} \times 5^{11}$, i.e. that the value we present for $n = 22$ below is in fact best possible, though it achieves only 95% of the theoretical bound.

The upper bound formulae we use, as well as the matrices associated with all determinants listed in Sec. 3 below, are available at \url{http://www.indiana.edu/~maxdet}, a website we have created as an archive for data pertaining to the determinant maximization problem. We intend to update it with increasingly better values, matrices, and references as they become available.

After a brief summary of our methods briefly in Sec. 2 below, we present some of our findings in Sec. 3. Our records in dimensions $n = 22, 23, 27, 29, 31, 33, 34, 35, 39, 45, 47, 53, 63, 69, 73, 77, 79, 93$, and 95 either eclipse a previous record, or manifest visible structure, or both. Except for the $n = 93$ case (which barely misses) each of these examples also exceeds 75% of the theoretical upper bound in its dimension. Many do much better. In these cases we therefore present our record-breaking matrix of $\pm 1$s, along with its determinant, a comparison to the theoretical bound, and a brief note, including a reference to the previous record.

Our work has also produced matrices of large determinant in dimensions not listed above. While they may not beat any previously published records, we believe several of them do provide best-known lower bounds for $\text{md}(n)$. We list such values for $n = 67, 75, 83, 87, 91$ and 99 in a table at the end of Sec. 3.

One further note: In dimensions $n = 29, 33, 45, 53, 69, 73, 77$ and 93 (each congruent to 1 mod 4), our determinants improve on values published by Farmakis & Kounias, who constructed their matrices by adding a suitable row and column to a Hadamard matrix of largest known excess [FK]. The excess of a Hadamard matrix is simply the sum of its entries, and has been studied in connection with the classification problem for Hadamard matrices. By deleting the appropriate row and column from matrices we have found in dimensions 73 and 77, one gets Hadamard matrices with larger excess than any previously known in those dimensions. For instance, we get a $72 \times 72$ Hadamard with excess 580, improving the 576 given in [FK], by deleting the first row and column from the $n = 73$ example in Sec. 3 below. Proceeding similarly with an $n = 77$ example posted on our website, one gets a $76 \times 76$ Hadamard with excess 628, beating the value of 620 given in [FK]. We present a different $n = 77$ example in Sec. 3 because its determinant is larger.

2. Methods.

Where not otherwise noted, the matrices in Sec. 3 were discovered numerically, using a hill-climbing computer program which combines a discrete version of gradient-ascent with simulated annealing. The first program of this type was used by Smith, and described in his thesis [Sm].
Stated roughly, the algorithm runs as follows. First one chooses a starting matrix; a random array of ±1s, or better, a well-chosen guess as to what the maximizer might look like. One then passes repeatedly through this candidate, modifying it row by row to raise its determinant. Since changes in a row don’t affect the cofactors of that row, one can exploit the cofactor expansion formula for determinants to do this very efficiently.

This simple strategy encounters a fundamental limitation, however: It almost always stalls in a local maximum of the determinant function, far below \( \text{md}(n) \). We overcome this problem to some extent by randomly perturbing the ascent when it gets stuck in this way, and attempting to continue from there.

By carefully examining our best result found using this method in the case \( n = 31 \), we discovered an explicit construction which produces new records for \( n = 47, 63, 79 \) and 95. More generally, the construction applies when \( n \equiv 15 \pmod{16} \), and uses conference matrices [MS]. We describe it briefly as follows.

Suppose \( n \equiv 15 \pmod{16} \), and write \( n = 4k + 3 \). Then \( k+1 \equiv 0 \pmod{4} \), and (at least in the dimensions of interest here) there exists an antisymmetric \( (k+1) \times (k+1) \) conference matrix \( C = [c_{ij}] \). Normalize \( C \), by multiplying rows (and corresponding columns, to preserve antisymmetry) by \(-1\) so that \( c_{1j} = 1 \) for all \( j > 1 \), and \( c_{i1} = -1 \) for all \( i > 1 \). Now delete the first row and column of \( C \) to produce a \( k \times k \) matrix \( B \), with the property that \( BB^T = (k-1)I_{k-1} - J_{k-1} \), \( J \) here denoting a matrix comprised entirely of +1s.

Given \( B \), we now obtain a ±1 matrix in four steps: First, tensor \( B \) with a 4 \( \times \) 4 Hadamard matrix \( H_4 \). Second, replace each 0 in the tensor product by a –1. Third, pad this 4\( k \) \( \times \) 4\( k \) matrix with 3 initial rows, obtained by normalizing \( H_4 \), deleting a row, and then tensoring with a row of \( k \) 1s. Finally, we similarly pad with three initial columns. The examples in Sec. 3 should clarify this.

Note that if \( k \) is a prime power, the Jacobsthal matrix of the finite field \( \text{GF}(k) \) provides a matrix \( B \) with exactly the properties we need.

In any case, this construction produces a matrix of very high determinant when \( n \) is not too big. As \( n \to \infty \), however, the resulting determinant tends to 0 relative to Ehlich’s theoretical upper bound.
3. Record-holding matrices.

\[ n = 22: \]
\[
\begin{array}{c}
++-----++++++++++-----
\end{array}
\]
\[
\begin{array}{c}
++-----+++++-----+++++
\end{array}
\]
\[
\begin{array}{c}
--+----+----++-----+-+
\end{array}
\]
\[
\begin{array}{c}
---+----+----++--+--
\end{array}
\]
\[
\begin{array}{c}
----+----+--+---+-+-+
\end{array}
\]
\[
\begin{array}{c}
-----+----+---++--+--
\end{array}
\]
\[
\begin{array}{c}
+----+----+---+++-+
\end{array}
\]
\[
\begin{array}{c}
+++----+------+-+++---
\end{array}
\]
\[
\begin{array}{c}
++-+----+---+--+--++--
\end{array}
\]
\[
\begin{array}{c}
++--+----+---+-+-+---+
\end{array}
\]
\[
\begin{array}{c}
++---+----+--+--+--++-
\end{array}
\]
\[
\begin{array}{c}
++----+----++-+-----++
\end{array}
\]
\[
\begin{array}{c}
+---+++++-+-+----+----+
\end{array}
\]
\[
\begin{array}{c}
+-+--++-++-+-+----+---+
\end{array}
\]
\[
\begin{array}{c}
+--+++-+-+-+--+----+--
\end{array}
\]
\[
\begin{array}{c}
+-++--++-++----+----+-
\end{array}
\]
\[
\begin{array}{c}
+-+++---+-++----+----+
\end{array}
\]
\[
\begin{array}{c}
-+++-+---++++----+----
\end{array}
\]
\[
\begin{array}{c}
-+-++-++--++-+----+---
\end{array}
\]
\[
\begin{array}{c}
-++-+-+-+++---+----+--
\end{array}
\]
\[
\begin{array}{c}
-++-++-++--+---+----+-
\end{array}
\]
\[
\begin{array}{c}
+-+-+-+++++------+----+
\end{array}
\]

Determinant: \( 2^{23} \times 5^{11} \)

Theoretical bound: \( 2^{21} \times 3 \times 5^{10} \times 7 \)

Fraction of bound: 0.95

We conjecture that the determinant above is \( \text{md}(22) \). Smith [Sm], and more recently, Cohn [Co1, Co2] published earlier records. Cohn’s value was \( 2^{21} \times 3^2 \times 23^2 \times 197^2 \) (90.1% of bound).

\[ n = 23: \]
\[
\begin{array}{c}
++-----++++++++++-----
\end{array}
\]
\[
\begin{array}{c}
++-----+++++-----+++++
\end{array}
\]
\[
\begin{array}{c}
+++----+------+-+++---
\end{array}
\]
\[
\begin{array}{c}
++-+----+---+--+--++--
\end{array}
\]
\[
\begin{array}{c}
++--+----+---+-+-+---+
\end{array}
\]
\[
\begin{array}{c}
++---+----+--+--+--++-
\end{array}
\]
\[
\begin{array}{c}
++----+----++-+-----++
\end{array}
\]
\[
\begin{array}{c}
+---+++++-+-+----+----+
\end{array}
\]
\[
\begin{array}{c}
+-+--++-++-+-+----+---+
\end{array}
\]
\[
\begin{array}{c}
+--+++-+-+-+--+----+--
\end{array}
\]
\[
\begin{array}{c}
+-++--++-++----+----+-
\end{array}
\]
\[
\begin{array}{c}
+-+++---+-++----+----+
\end{array}
\]
\[
\begin{array}{c}
-+++-+---++++----+----
\end{array}
\]
\[
\begin{array}{c}
-+-++-++--++-+----+---
\end{array}
\]
\[
\begin{array}{c}
-++-+-+-+++---+----+--
\end{array}
\]
\[
\begin{array}{c}
-++-++-++--+---+----+-
\end{array}
\]
\[
\begin{array}{c}
+-+-+-+++++------+----+
\end{array}
\]

Determinant: \( 2^{22} \times 3 \times 5^6 \times 67 \times 211 \)

Theoretical bound: \( 2^{22} \times 3 \times 5^6 \times 675 \sqrt{505} \)

Fraction of bound: 0.931983

Smith [Sm] published an earlier record of \( 2^{22} \times 19 \times 5741^2 \) (88% of bound).
n = 27:

Determinant: \(2^{26} \times 6^{11} \times 518\)
Theoretical bound: \(2^{26} \times 5 \times 6^{10} \times 44\sqrt{237}\)
Fraction of bound: 0.917665

Smith \[Sm\] published an earlier record of \(2^{27} \times 11 \times 90481^2\) (88% of bound).

n = 29:

Determinant: \(2^{28} \times 7^{12} \times 320\)
Theoretical bound: \(2^{28} \times 7^{12} \times 49\sqrt{57}\)
Fraction of bound: 0.865001

Smith \[Sm\], Farmakis & Kounias \[FK\], and most recently, Koukouvinos \[Ko\], have published earlier records. Koukouvinos' value is \(2^{28} \times 7^{13} \times 43\) (81.4% of bound).
n = 31:

Determinant: $2^{30} \times 7^{12} \times 5324$
Theoretical bound: $2^{30} \times 7^{12} \times 144\sqrt{1589}$
Fraction of bound: 0.927499

Though discovered numerically, our conference-matrix method also produces this example (see Sec. 2). Smith [Sm] published an earlier record of $2^{30} \times 5^{4} \times 7^{2} \times 11^{2} \times 29^{2} \times 149^{2}$ (87% of bound).

n = 33:

Determinant: $2^{32} \times 8^{14} \times 441$
Theoretical bound: $2^{32} \times 8^{14} \times 64\sqrt{65}$
Fraction of bound: 0.854677

Farmakis & Kounias [FK] published an earlier record of $2^{32} \times 8^{15} \times 51$ (79% of bound).
We find no prior records in the literature, though one produces a ±1 matrix with determinant $2^{33} \times 8^{16} \times 25$ (75.76% of bound) by tensoring an $n = 17$ maximizer with a $2 \times 2$ maximizer.
\[ n = 39: \]

Determinant: \( 2^{38} \times 9^{17} \times 1241 \)
Theoretical bound: \( 2^{38} \times 9^{17} \times 80\sqrt{357} \)
Fraction of bound: 0.821009

Welsh \cite{Welsh} earlier reported a record of \( 2^{28} \times 9^{17} \times 1197 \) (79.2\% of bound).
\( n = 45: \)

```
-+-+-+-+-+-+-+---+-+-+-+-----+---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
-++-++-++-++-+-+-+-++---++---+++---++++
```

Determinant: \( 2^{44} \times 11^{21} \times 83 \)

Theoretical bound: \( 2^{44} \times 11^{22} \times \sqrt{89} \)

Fraction of bound: 0.799817

Farmakis & Kounias [FK] published an earlier record of \( 2^{44} \times 11^{21} \times 81 \) (78% of bound).
$n = 47$ :

Determination: $2^{46} \times 11^{18} \times 15^4 \times 60$

Theoretical bound: $2^{46} \times 11^{20} \times 19^2 \sqrt{5665}$

Fraction of bound: 0.923897

Our conference-matrix construction produces this example (see Sec. 2). Welsh [We] earlier reported a record of $2^{46} \times 11^{21} \times 1896$ (76.8% of bound).
n = 53:

Determinant: $2^{52} \times 13^{25} \times 105$
Theoretical bound: $2^{52} \times 13^{26} \times \sqrt{105}$
Fraction of bound: 0.788227

Farmakis & Kounias [FK] published an earlier record of $2^{52} \times 13^{25} \times 104$ (78.1% of bound).
Determinant: $2^{64} \times 15^{24} \times 19^{7}$

Theoretical bound: $2^{64} \times 15^{28} \times 12^{3} \times 2\sqrt{33}$

Fraction of bound: 0.889364

Our conference-matrix construction produces this example (see Sec. 2). We find no prior records in the literature.
\[ n = 69 : \]

\[
\begin{array}{c}
\text{Determinant: } 2^{68} \times 17^{33} \times 155 \\
\text{Theoretical bound: } 2^{68} \times 17^{34} \times \sqrt{137} \\
\text{Fraction of bound: } 0.778973
\end{array}
\]

Farmakis & Kounias [EK] published an earlier record of \( 2^{68} \times 17^{33} \times 153 \) (76.9% of bound).
Determinant: $2^{72} \times 18^{35} \times 163$
Theoretical bound: $2^{72} \times 18^{36} \times \sqrt{145}$
Fraction of bound: 0.752023

Farmakis & Kounias [FK], published an earlier record of $2^{72} \times 18^{35} \times 162$ (74.7% of bound). The lower right $72 \times 72$ block above gives a Hadamard matrix with excess 580. The largest excess previously known for Hadamards of this size was 576 [FK].
\( n = 77 \):

Determinant: \( 2^{76} \times 19^{37} \times 177 \)

Theoretical bound: \( 2^{76} \times 19^{38} \times \sqrt{153} \)

Fraction of bound: 0.753137

Farmakis & Kounias \([FK]\) published an earlier record of \( 2^{76} \times 19^{37} \times 174 \) (74.04\% of bound).
\( n = 79 \):

Determinant: \( 2^{78} \times 4 \times 19^{30} \times 23^9 \)

Theoretical bound: \( 5^{78} \times 4 \times 19^{36} \times 225 \sqrt{40145} \)

Fraction of bound: 0.84924

Our conference-matrix construction produces this example (see Sec. 2). We find no prior records in the literature.
\[ n = 93 : \]

\begin{align*}
\text{Determinant:} & \quad 2^{92} \times 23^{45} \times 231 \\
\text{Theoretical bound:} & \quad 2^{92} \times 23^{46} \times \sqrt{185} \\
\text{Fraction of bound:} & \quad 0.738411
\end{align*}

Farmakis & Kounias [FK] published an earlier record of \( 2^{92} \times 23^{45} \times 230 \) (73.52\% of bound).
Theoretical bound: $2^{96} \times 2^{36} \times 27^{11}$

Fraction of bound: 0.810642

Our conference-matrix construction produces this example (see Sec. 2). We find no prior records in the literature.
Other cases of interest: The table below gives the largest determinants we have found among ±1 matrices in several additional dimensions, with $ub(n)$ denoting the theoretical upper bound for $md(n)$ mentioned in Sec. 1. They do not quite achieve 75% of the theoretical bound, and to the best of our knowledge, no previous bounds have been published. Further details, including the matrices themselves, can be found at our website, www.indiana.edu/~maxdet.

| Size  | Lower bound for $md(n)$          | Fraction of $ub(n)$ |
|-------|----------------------------------|---------------------|
| $n = 67$ | $2^{66} \times 16^{31} \times 4765$ | 0.7677              |
| $n = 75$ | $2^{74} \times 18^{35} \times 6064$ | 0.7303              |
| $n = 83$ | $2^{82} \times 20^{38} \times 15788$ | 0.7322              |
| $n = 87$ | $2^{86} \times 21^{41} \times 8777$ | 0.7220              |
| $n = 91$ | $2^{90} \times 22^{43} \times 9826$ | 0.7203              |
| $n = 99$ | $2^{98} \times 24^{47} \times 12118$ | 0.7160              |

ACKNOWLEDGEMENTS

The websites of N.J.A. Sloane, J. Seberry, C. Koukouvinos, and E. Spence provided us with Hadamard and other large-determinant matrices that we used to construct starting points for our own searches. We gratefully acknowledge them for this contribution. We made extensive use of Mathematica during this project, and we also thank Indiana University for the use of their Sun E10000 computing platform. Finally, we thank Michael Neubauer for helpful remarks and interest in our work.

REFERENCES

[Ba] G. Barba, Intorno al teorema di Hadamard sui determinanti a valore massimo, Giorn. Mat. Battaglini 71 (1933) 70–86.
[Co1] J. H. E. Cohn, On determinants with elements ±1, Bull. London Math. Soc. 21 (1989) 36–42.
[Co2] J. H. E. Cohn, Almost D-optimal designs, Utilitas Math. 57 (2000) 121–128.
[Eh1] H. Ehlich, Determinantenabschätzungen für binäre Matrizen, Math. Z. 83 (1964) 123–132.
[Eh2] H. Ehlich, Determinantenabschätzungen für binäre Matrizen mit $N \equiv 3 \bmod 4$, Math. Z. 84 (1964) 438–447.
[EZ] H. Ehlich & K. Zeller, Binäre Matrizen, Z. Angew. Math. Mech. 42 (1962) T20–T21.
[FK] N. Farmakis & S. Kounias, The excess of Hadamard matrices and optimal designs, Discrete Math. 67 (1987) 165–176.
[Ko] C. Koukouvinos, On almost $D$-optimal first order saturated designs and their efficiency, Utilitas Math. 52 (1997), 113–121.
[MS] F. J. MacWilliams & N. J. A. Sloane, The Theory of Error-Correcting Codes, Elsevier-North Holland, 1978, pp. 55–56.
[OOS] W. P. Orrick, J. Osborn & B. Solomon, The maximal determinant problem, to appear.

[Sm] Warren D. Smith, *Studies in Computational Geometry Motivated by Mesh Generation*, Ph. D. dissertation, Princeton University, 1988.

[SY] J. Seberry & M. Yamada, Hadamard matrices, sequences, and block designs, *Contemporary design theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, New York, 1992, pp. 431–560.

[We] Trevor Welsh, personal communication, 2002.

[Wo] W. Wojtas, On Hadamard’s inequality for the determinants of order non-divisible by 4, *Colloq. Math.*, **12** (1964) 73–83.

Department of Mathematics, Indiana University, Bloomington, IN 47405

*E-mail address: worrick@indiana.edu*

Department of Mathematics, Indiana University, Bloomington, IN 47405

*E-mail address: solomon@indiana.edu*

11 West 17th St., Floor 2, New York, NY 10011

*E-mail address: elric@imrryr.org*

Department of Mathematics, Temple University, Philadelphia, PA 19122

*E-mail address: wds@math.temple.edu*