APPLICATION OF UNIFORM ASYMPTOTICS TO THE SECOND PAINLEVÉ TRANSCENDENT

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\textbf{Abstract.} In this work we propose a new method for investigating connection problems for the class of nonlinear second-order differential equations known as the Painlevé equations. Such problems can be characterized by the question as to how the asymptotic behaviours of solutions are related as the independent variable is allowed to pass towards infinity along different directions in the complex plane. Connection problems have been previously tackled by a variety of methods. Frequently these are based on the ideas of isomonodromic deformation and the matching of WKB solutions. However, the implementation of these methods often tends to be heuristic in nature and so the task of rigorising the process is complicated. The method we propose here develops uniform approximations to solutions. This removes the need to match solutions, is rigorous, and can lead to the solution of connection problems with minimal computational effort.

Our method is reliant on finding uniform approximations of differential equations of the generic form

$$\frac{d^2 \phi}{d\eta^2} = -\xi^2 F(\eta, \xi) \phi$$

as the complex-valued parameter $\xi \to \infty$. The details of the treatment rely heavily on the locations of the zeros of the function $F$ in this limit. If they are isolated then a uniform approximation to solutions can be derived in terms of Airy functions of suitable argument. On the other hand, if two of the zeros of $F$ coalesce as $|\xi| \to \infty$ then an approximation can be derived in terms of parabolic cylinder functions. In this paper we discuss both cases, but illustrate our technique in action by applying the parabolic cylinder case to the “classical” connection problem associated with the second Painlevé transcendent. Future papers will show how the technique can be applied with very little change to the other Painlevé equations, and to the wider problem of the asymptotic behaviour of the general solution to any of these equations.

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1. Introduction.

Asymptotic behaviour of solutions of the second Painlevé transcendent (PII),

\[ q'' = 2q^3 + xq + \beta, \]  \hspace{1cm} (1.1)

where \( q' \equiv \frac{d}{dx}q \) and \( \beta \) is a complex constant, have been much studied, for example in [3, 4, 6–17, 19–21, 23]. In particular connection problems have been investigated in which one attempts to relate the asymptotic behaviour in one \( x \)-direction to that in another. Some of the results are heuristic, and some rigorous. The heuristic arguments tend to use the method of isomonodromic deformations, linked with asymptotic arguments that use the WKB method and matching, and although Suleimanov [23] has given a rigorous version of this for one problem associated with the second Painlevé transcendent (1.1), the task of extending these techniques rigorously to more complicated problems, and in particular to problems associated with the higher equations, seems formidable.

In this paper we develop a new technique for investigating such problems. The technique uses the method of isomonodromy, but thereafter develops a uniform approximation which dispenses with matching, is rigorous and even from a computational point of view is simpler than previous methods. We will use it in this paper to study the asymptotic behaviour of solutions of PII (1.1) when \( \beta = 0 \), giving the algorithm which enables one to compare asymptotic behaviour in different directions, but we emphasise that the method is certainly not restricted to PII, and we will return in later papers to its application to the other transcendents.

In particular, of course, we can solve once again the “classic” problem for PII (1.1), which for convenience and completeness we state here. Its statement depends upon the following theorem, a proof of which can be found in [8].

**Theorem A.** There exists a unique solution of (1.1) with \( \beta = 0 \) which is asymptotic to a \( \text{Ai}(x) \) as \( x \to +\infty \), \( a \) being any positive number. If \( a < 1 \), this solution exists for all real \( x \) as \( x \) decreases to \( -\infty \), and, as \( x \to -\infty \),

\[ q(x) \sim d|x|^{-1/4} \sin \left\{ \frac{2}{3} |x|^{3/2} - \frac{3}{4} d^2 \log |x| + \gamma \right\} \]

for some constants \( d, \gamma \) which depend on \( a \).

From this result one can easily compute more detailed asymptotics which hold as \( x \to +\infty \):

\[ q(x) = \frac{1}{2} a \pi^{-1/2} x^{-1/4} \exp \left( -\frac{2}{3} x^{3/2} \right) \left[ 1 - \frac{5}{48} x^{-3/2} + O(x^{-3}) \right], \]  \hspace{1cm} (1.2a)

\[ r(x) = \frac{dq}{dx} = -\frac{1}{2} a \pi^{-1/2} x^{1/4} \exp \left( -\frac{2}{3} x^{3/2} \right) \left[ 1 + \frac{7}{48} x^{-3/2} + O(x^{-3}) \right]. \]  \hspace{1cm} (1.2b)

The usual connection problem is the question of the specific dependence of \( d \) and \( \gamma \) on \( a \), and this is given as follows:

**Theorem B.**

\[ d^2 = -\pi^{-1} \log \left( 1 - a^2 \right), \]  \hspace{1cm} (1.3a)

\[ \gamma = \frac{3}{4} \pi - \frac{3}{2} d^2 \log 2 - \arg \Gamma(-\frac{1}{2} id^2). \]  \hspace{1cm} (1.3b)
We have already mentioned that our technique involves the concept of isomonodromy, and we now quickly review the relevant facts [6]. (Again we give the details for PII (1.1) but emphasise that comparable results are known [11] for all the other Painlevé transcendents, and indeed that there is a hierarchy of equations [1] of higher order which fit into the same general framework.) Suppose that \( x \) and \( \lambda \) are independent complex variables and there exists a \( 2 \times 2 \) matrix function \( \Psi(x, \lambda) \) which satisfies both

\[
\frac{\partial \Psi}{\partial x} = (-i \lambda \sigma_3 + q \sigma_1) \Psi, \quad \text{i.e.} \quad D_x \Psi = 0, \tag{1.4}
\]

and

\[
\frac{\partial \Psi}{\partial \lambda} = \left\{ -i(4\lambda^2 + x + 2q^2)\sigma_3 + 4\lambda q \sigma_1 - 2r \sigma_2 - \frac{\beta}{\lambda} \sigma_1 \right\} \Psi, \quad \text{i.e.} \quad D_\lambda \Psi = 0. \tag{1.5}
\]

Here

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

are the standard Pauli spin matrices which, in particular, satisfy

\[
\sigma_1 \sigma_2 = i\sigma_3, \quad \sigma_2 \sigma_3 = i\sigma_1, \quad \sigma_3 \sigma_1 = i\sigma_2.
\]

Then there is a compatibility condition

\[
[D_x, D_\lambda] \Psi = (D_x D_\lambda - D_\lambda D_x) \Psi = 0, \tag{1.6}
\]

and an easy calculation shows that (1.6) reduces to (1.1). Conversely, if \( q(x) \) evolves according to (1.1), then (1.4) and (1.5) are compatible. Thus (1.1) is equivalent to compatibility and compatibility is easily seen to imply isomonodromy.

For suppose that we have two fundamental solutions \( \Psi^{(1)}(x, \lambda) \) and \( \Psi^{(2)}(x, \lambda) \) of (1.5) in two different but overlapping sectors in the \( \lambda \)-plane. (The equation has an irregular singularity at \( \lambda = \infty \) and a regular singularity at \( \lambda = 0 \) but, as far as monodromy is concerned, we need only deal with the irregular singularity.) Since \( \Psi^{(1)} \) and \( \Psi^{(2)} \) are both fundamental solutions there must be a matrix \( S \) independent of \( \lambda \) but in general dependent on \( x \), such that

\[
\Psi^{(2)}(x, \lambda) = \Psi^{(1)}(x, \lambda) S(x), \tag{1.7}
\]

and \( S \) is referred to as the monodromy matrix. (Of course, \( S \) depends on the particular fundamental solutions which are compared, and we return to this point later.) There is a monodromy matrix for each pair of sectors, and the assemblage of all the monodromy matrices forms the monodromy data. If we now differentiate (1.7) with respect to \( x \) and use the fact that \( \Psi^{(1)} \) and \( \Psi^{(2)} \) satisfy (1.4), we obtain immediately that \( S \) is independent of \( x \), which is to say that the problem is isomonodromic in \( x \). It should be noted that this involves care in choosing \( \Psi^{(1)} \) and \( \Psi^{(2)} \), for if we multiply \( \Psi^{(1)}(x, \lambda) \) by a function of \( x \), it still satisfies (1.5), but no longer (1.4).

We make the remark also that we shall be able to arrange that the monodromy matrix takes the form of a triangular matrix with 1 as the principal diagonal. Thus
the monodromy data reduces to the one remaining entry in the matrix, the so-called Stokes multiplier.

Given isomonodromy, we can now prove Theorem B as follows. We work out the monodromy data for (1.5) as \( x \to +\infty \), using the known asymptotic dependence of \( q \) on \( x \), and then the monodromy data as \( x \to -\infty \), and equate them to give the required result. The way in which this has so far been carried out is to compute the fundamental solutions in different sectors and use (1.7) to obtain \( S \). This means that we have to compute the solutions (or at least their asymptotic behaviours) as \(|\lambda| \to \infty\) and also as \(|x| \to \infty\). This uses WKB asymptotics, and also matching, since the form of the asymptotics depends on the relative values of \( \lambda \) and \( x \), and we have to match different forms in different regions. The procedure can be complicated and rigorising it difficult.

The procedure would be much simplified if one could find approximations to solutions which are uniformly valid for all relevant large \(|\lambda|,|x|\). This we can in fact do, and in a general form which is certainly applicable to more than just PII (1.1). Once it is done, there is no further rigorous analysis required; it is merely a matter of computing the monodromy data by relating it to the (known) data for the approximations.

In Section 2 we describe, in the context of PII (1.1), the heuristic reasoning which leads to the uniform approximation. Then in Sections 3,4 we state and prove two theorems on uniform approximations, which we believe to be the only such theorems necessary for the discussion of any of the Painlevé equations. In the final sections of the paper we use these theorems to compute monodromy data both in a general setting and in the particular case of PII, and finally as an application prove Theorem B.

It should be remarked that for the purposes of Theorem B only the first of the two approximation theorems (that relating to double turning-points) is required. For more general solutions of PII, and for a general discussion of the other Painlevé equations, the second theorem is also required. We intend to return to such developments in later papers.

### 2. Deriving a Uniform Approximation.

To see the nature of the uniform approximation, we turn (1.5) into a single second-order equation. We first make the scaling

\[
\xi = x^{3/2}, \quad \eta = x^{-1/2} \lambda,
\]

so that (1.5) becomes

\[
\frac{d\Psi}{d\eta} = \xi \left\{ -i \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \sigma_3 + \left( \frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta \xi} \right) \sigma_1 - \frac{2r}{x} \sigma_2 \right\} \Psi,
\]

which, with

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]

is equivalent to

\[
\begin{align*}
\frac{d\psi_1}{d\eta} &= \xi \left\{ -i \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \psi_1 + \left( \frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta \xi} + \frac{2ir}{x} \right) \psi_2 \right\}, \\
\frac{d\psi_2}{d\eta} &= \xi \left\{ i \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \psi_2 + \left( \frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta \xi} - \frac{2ir}{x} \right) \psi_1 \right\}.
\end{align*}
\]

(2.1a)

(2.1b)
Eliminating $\psi_1$, we obtain
\[
\frac{\mathrm{d}^2 \psi_2}{\mathrm{d}\eta^2} = \xi \left\{ i \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \frac{\mathrm{d}\psi_2}{\mathrm{d}\eta} + 8i\eta \psi_2 + \left( \frac{4q}{\sqrt{x}} + \frac{\beta}{\eta^2 \xi} \right) \psi_1 \\
- i\xi \left( \frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta \xi} - \frac{2ir}{x} \right) \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \psi_1 \\
+ \xi \left( \frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta \xi} + \frac{2ir}{x} \right) \left( 4\eta q - \frac{\beta}{\eta \xi} - \frac{2ir}{x} \right) \psi_2 \right\}
\]
\[
= \xi \left\{ -\xi \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right)^2 \psi_2 + 8i\eta \psi_2 + \xi \left[ \left( \frac{4\eta q}{\sqrt{x}} - \frac{\beta}{\eta \xi} \right)^2 + \frac{4r^2}{x^2} \right] \psi_2 \\
+ \frac{1}{\xi^2} \left( 1 + \frac{\beta}{4\eta^2 q x} \right) \left( \eta - \frac{ir}{2q \sqrt{x}} - \frac{\beta}{4\eta qx} \right)^{-1} \\
\times \left[ \frac{\mathrm{d}\psi_2}{\mathrm{d}\eta} - i\xi \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \psi_2 \right] \right\}.
\]

The term $\frac{\mathrm{d}\psi_2}{\mathrm{d}\eta}$ can be removed by setting
\[
\phi = \left( \eta - \frac{ir}{2q \sqrt{x}} - \frac{\beta}{4\eta qx} \right)^{-1/2} \psi_2,
\]
whence
\[
\frac{\mathrm{d}^2 \phi}{\mathrm{d}\eta^2} = \xi^2 \phi \left\{ -(4\eta^2 + 1)^2 + 8i\eta + \left( \frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} - \frac{8q\beta}{x^2} + \frac{\beta^2}{\eta^2 x^3} \right) \right\}
\]
\[
- \frac{i}{\xi} \left( 1 + \frac{\beta}{4\eta^2 q x} \right) \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \left( \eta - \frac{ir}{2q \sqrt{x}} - \frac{\beta}{4\eta qx} \right)^{-1} \\
+ \frac{1}{\xi^2} \frac{\beta}{4\eta^3 q x} \left( \eta - \frac{ir}{2q \sqrt{x}} - \frac{\beta}{4\eta qx} \right)^{-1} \\
+ \frac{3}{4\xi^2} \left( 1 + \frac{\beta}{4\eta^2 q x} \right)^2 \left( \eta - \frac{ir}{2q \sqrt{x}} - \frac{\beta}{4\eta qx} \right)^{-2} \right\}. \tag{2.2}
\]

In (2.2), attention should be drawn to the terms
\[
\frac{4r^2}{x^2} - \frac{4q^2}{x^2} - \frac{4q^4}{x^2} - \frac{8q\beta}{x^2} = M(\xi), \tag{2.3}
\]
say, which depend only on $x$ or $\xi$, and not on $\eta$. How $M(\xi)$ behaves for large $\xi$ (which is always our interest) will depend on the asymptotics of the functions $q(x), r(x)$ as $|x| \to \infty$, and therefore on the particular solution of PII. For the remainder of this heuristic discussion we will consider the case where $M(\xi) \to 0$, since in the case of Theorem B this is certainly true from the given asymptotics both as $x \to +\infty$ and as $x \to -\infty$. But it is not true for a general solution that $M(\xi) \to 0$, and our methods do not need it, and we will point out where the essential difference lies.

Assuming then that $M(\xi) \to 0$ as $|\xi| \to \infty$, it is to be expected from the form of (2.2) that, as $|\eta| \to \infty$ with $|\xi|$ large, the dominant term on the right-hand side will be
\[
-\xi^2 (4\eta^2 + 1)^2 \phi,
\]
so that, from the usual WKB approximation, the solution should be asymptotically of the form

\[
\eta^{-1} \exp \left\{ \pm i \xi \int_{\eta}^{\eta} (4\sigma^2 + 1) \, d\sigma \right\} = \eta^{-1} \exp \left\{ \pm i \xi \left( \frac{4}{3} \eta^3 + \eta \right) \right\}.
\]

The two exponentials are thus equipollent in directions

\[
\arg (\xi \eta^3) = 0, \pm \pi, \pm 2\pi, \ldots,
\]
i.e.

\[
\arg \eta = -\frac{1}{3} \arg \xi \pm \frac{1}{3} k\pi, \quad k = 0, 1, 2, \ldots
\]
and these are the so-called Stokes directions. We can determine the Stokes multipliers by relating the asymptotic behaviour of a solution in one Stokes direction to its asymptotic behaviour in the next, since it is in Stokes directions (and only in Stokes directions) that the full asymptotics appear and solutions can be defined by their asymptotics.

However, in order to connect the behaviours as \(|\eta| \to \infty\) on, say, \(\arg \eta = -\frac{1}{3} \arg \xi\) and \(\arg \eta = -\frac{1}{3} \arg \xi + \frac{1}{3} \pi\), we need to follow the solution along a curve for which \(\Re \{ i \xi \int_{\eta}^{\eta} (4\sigma^2 + 1) \, d\sigma \} = 0\), for if we depart significantly from such curves (so-called Stokes curves), we shall lose equipollence, and so the effect of exponentially small solutions and therefore the Stokes multiplier. Now there is some choice of Stokes curve depending on the initial point of integration, but to obtain a uniform approximation we shall consider Stokes curves which pass through turning-points of equation (2.2); by a turning-point we mean a value of \(\eta\) which is a zero of the right-hand side of (2.2) although we will slightly adapt this definition later.

The idea of uniform asymptotics through turning-points was first proposed by Langer [18] and Titchmarsh [24] in work on the distribution of eigenvalues for the Schrödinger equation (see also [22]). They dealt with the equation

\[
\frac{d^2 y}{dz^2} + [\mu - q(z)] y = 0, \quad -\infty < z < \infty, \tag{2.4}
\]
where, for example, we may think of \(\mu\) as a large positive parameter and \(q(z) \to \infty\) as \(z \to \infty\). If \(q\) is strictly monotonic, then there is a simple turning-point at \(q(z) = \mu\). Langer pointed out that the prototypical case of this is \(q(z) = z\), so that the equation becomes

\[
\frac{d^2 y}{dz^2} + (\mu - z) y = 0
\]
whose general solution is a linear combination of \(\text{Ai}(z - \mu)\) and \(\text{Bi}(z - \mu)\), where \(\text{Ai}, \text{Bi}\) are the usual Airy functions. He then went on to show that one could obtain a uniform approximation to solutions of (2.4), valid for large \(\mu\) and \(z \to \pm \infty\), by introducing Airy functions of a suitable argument.

We need to modify the idea further, because Langer’s approximation relates to situations where the turning-point is simple, whereas in our case (2.2) there are two turning-points which, for large \(\xi\), are close to \(\eta = \frac{1}{2} i\) (and two others close to \(\eta = -\frac{1}{2} i\)). (This of course is a consequence of our assumption that \(M(\xi) \to 0\). If \(M(\xi) \not\to 0\), then the turning-points are simple, and it is then a matter of adapting
Langer’s approximation using Airy functions. In our present situation, therefore, it seems that the parabolic cylinder equation

\[
\frac{d^2y}{dz^2} = \left[ \frac{1}{4}z^2 - \left( \nu + \frac{1}{2} \right) \right] y,
\]  

(2.5)

with linearly independent solutions \( D_\nu(z) \) and \( D_{-\nu-1}(iz) \), is an appropriate one for coping with coalescing turning-points, and in fact this possibility has already been explored by Olver [22] and Dunster [5], primarily for real values of \( z \). With our particular applications in mind, it will be better to consider (2.5) in the form

\[
\frac{d^2y}{dz^2} = -\xi^2 \left( z^2 - \frac{2\nu+1}{i\xi} \right) y 
\]

(2.6)

and

\[
\frac{d^2y}{dz^2} = -\xi^2 (z^2 - \alpha^2) y,
\]

(2.7)

where \( i\xi \alpha^2 = 2\nu + 1 \), which has solutions \( D_\nu(e^{\pi i/4}/\sqrt{2\xi z}) \) and \( D_{-\nu-1}(e^{-\pi i/4}/\sqrt{2\xi z}) \).

To see how this applies to (2.2), we will restrict ourselves to the particular case when \( \beta = 0 \). We try as a uniform approximation to a solution of (2.2) the expression

\[
\phi(\eta) = \rho(\eta) D_\nu \left( e^{\pi i/4}/\sqrt{2\xi \zeta(\eta)} \right) = \rho(\eta) F_\nu(\zeta(\eta)), \quad \text{say,}
\]

(2.8)

where the functions \( \rho \) and \( \zeta \) are to be determined. Substituting (2.8) in (2.2) with \( \beta = 0 \) we have

\[
\rho'' F_\nu + 2 \rho' \zeta' F'_\nu + \rho((\zeta')^2 F''_\nu + \zeta'' F'_\nu) 
\]

\[
\quad = \xi^2 \rho F_\nu \left\{ -(4\eta^2 + 1)^2 + \frac{8i\eta}{\xi} + \left( \frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} \right) \right. 
\]

\[
\quad \left. - \frac{i}{\xi} \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-1} + \frac{3}{4\xi^2} \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-2} \right\}. \quad (2.9)
\]

Recalling that \( F_\nu \) satisfies (2.6) we can compare coefficients of \( F'_\nu \) and \( F_\nu \) in (2.9). The vanishing of the coefficient of \( F'_\nu \) gives

\[
2\rho' \zeta' + \rho \zeta'' = 0,
\]

so that we can take

\[
\rho = (\zeta')^{-1/2}, \quad (2.10)
\]

for the choice of integration constant at this point is inconsequential. The vanishing of the coefficient of \( F_\nu \) gives

\[
\xi^2 (\zeta^2 - \alpha^2)(\zeta')^2 
\]

\[
\quad = \xi^2 \left\{ (4\eta^2 + 1)^2 - \frac{8i\eta}{\xi} + \left( \frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} \right) \right. 
\]

\[
\quad \left. + \frac{i}{\xi} \left( 4\eta^2 + 1 + \frac{2q^2}{x} \right) \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-1} - \frac{3}{4\xi^2} \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-2} + \rho'' \right\}, \quad (2.11)
\]
If we ignore the last two terms in \{\ldots\} as being of smaller order (for large $\xi$) than the others, then we are left with

$$(\zeta^2 - \alpha^2)(\zeta')^2 = (4\eta^2 + 1)^2 - \frac{8i\eta}{\xi} - \left(\frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2}\right)$$

$$+ \frac{i}{\xi}\left(4\eta^2 + 1 + \frac{2q^2}{x}\right)\left(\eta - \frac{ir}{2q\sqrt{x}}\right)^{-1}$$

$$= G(\eta, \xi), \quad \text{say},$$

which, apart from a constant of integration, defines $\zeta$ as a function of $\eta$ once we have specified $\alpha$. (We recall always that $r, q, x, \xi$ are constants as far as $\eta, \zeta$ are concerned.) We note however from (2.10) that we certainly want to avoid zeros of $\zeta'$ and, from (2.12), $\zeta'$ has a zero wherever $G(\eta, \xi) = 0$; i.e. essentially at a turning-point of the equation, unless we can choose $\alpha$ so that the zeros of $\zeta^2 - \alpha^2$ coincide with those of $G$. For large $\xi$, there are two turning-points, say $\eta_1$ and $\eta_2$, close to $\frac{1}{2}i$, and two close to $-\frac{1}{2}i$. If we are interested in a Stokes curve which passes through (or close to) $\frac{1}{2}i$, then we must choose $\alpha$ so that $\zeta = -\alpha$ corresponds to $\eta = \eta_1$ and $\zeta = +\alpha$ corresponds to $\eta = \eta_2$. We can ensure one of these holds by use of the constant of integration implicit in the evaluation of $\zeta$ from (2.12). The second can be achieved by defining $\alpha$ so that

$$\int_{-\alpha}^{\alpha} (\zeta^2 - \alpha^2)^{1/2} \, d\zeta = \int_{\eta_1}^{\eta_2} G^{1/2}(\eta, \xi) \, d\eta. \quad (2.13)$$

Since the left-hand side integrates easily to $\frac{1}{2}\pi\alpha^2$, we have $\alpha$ given by

$$\frac{1}{2}\pi\alpha^2 = \int_{\eta_1}^{\eta_2} G^{1/2}(\eta, \xi) \, d\eta.$$

With $\alpha$ so defined, and $\zeta$ chosen according to

$$\int_{\alpha}^{\zeta} (\tau^2 - \alpha^2)^{1/2} \, d\tau = \int_{\eta_1}^{\eta_2} G^{1/2}(\sigma, \xi) \, d\sigma,$$

we can hope that solutions of (2.2) are approximated, uniformly on $\eta$ for large $\xi$, by some linear combination of

$$(\zeta')^{-1/2}D_\nu(e^{i\pi/4}\sqrt{2\xi}\zeta) \quad \text{and} \quad (\zeta')^{-1/2}D_{-\nu-1}(e^{-i\pi/4}\sqrt{2\xi}\zeta).$$

A precise statement and proof of this conjecture is given in the next section.

We remark finally that it is a consequence of this uniform approximation that the monodromy data for (2.2) as $|\xi| \to \infty$ will be the same as that for the parabolic cylinder functions, which can be found in any text on special functions, modified only by some allowance for the various changes of variable involved. We work this out more precisely in Sections 5 & 6.

3. The Uniform Approximation Theorem for a Double Turning-Point.

We are interested in differential equations of the form

$$\frac{d^2\phi}{d\eta^2} = -\xi^2 F(\eta, \xi)\phi \quad (3.1)$$

and, guided by the heuristic discussion in Section 2, we shall make the following assumptions about $F$. Suppose that our concern is with the limit $|\xi| \to \infty$ with $\arg\xi \to \theta$; we then hypothesise that
H1. There is a sequence of values $\xi_n, |\xi_n| \to \infty$, $\arg \xi_n \to \theta$, such that

$$F(\eta, \xi_n) = F_0(\eta) (\eta - \eta_0)^2 - \frac{\tilde{F}(\eta, \xi_n)}{\xi_n},$$

where

(i) $F_0(\eta)$ is a polynomial in $\eta$, with $F_0(\eta_0) \neq 0$ and

$$F_0(\eta) \sim A\eta^m \quad \text{as} \quad \eta \to \infty, \quad (3.2)$$

(ii) $\tilde{F}(\eta, \xi_n)$ is a rational function of $\eta$, with the location of its poles possibly dependent on $\xi_n$.

(Further assumptions on $\tilde{F}$ are given in due course.)

Remarks.
1. The assumption that $F_0$ is polynomial is not essential. Polynomial-like behaviour of some sort would certainly be sufficient, but in applications to the Painlevé transcendents it is always the case that $F_0$ is a polynomial, and since no new ideas would be involved in generalization, we do not consider this here. Similar comments hold with regard to the rational behaviour of $\tilde{F}$.
2. The assumption $F_0(\eta_0) \neq 0$ is crucial. It implies that (3.1) is to have a double turning-point at $\eta = \eta_0$ (or, more precisely, for large $|\xi|$, two turning-points close to $\eta_0$), but no other turning-points close to $\eta_0$.
3. Our assumption is only about a sequence of values $\xi_n$ since it will turn out in our applications to be a consequence of isomonodromy that behaviour as $|\xi| \to \infty$ through a sequence is sufficient to determine behaviour as $|\xi| \to \infty$ generally. However, in the present approximation theorem, which is in itself quite independent of the concept of isomonodromy, we will not be involved in comparing different sequences, and so we can without confusion drop the subscript in $\xi_n$, and this will be done henceforth.
4. The usual WKB approximation for (3.1) would suggest, in view of (3.2), that, for large $\eta$, the solutions of (3.1) are asymptotic to linear combinations of

$$\eta^{-(m+2)/4} \exp \left( \pm i \xi \int_{\eta_0}^{\eta} F_0^{1/2}(s) (s - \eta_0) \, ds \right),$$

and so for Stokes directions we must have

$$\arg \left( \xi A^{1/2} \eta^{(4+m)/2} \right) = 0, \pm \pi, \pm 2\pi, \ldots$$

or

$$\left( \frac{1}{2} m + 2 \right) \arg \eta = - \arg \xi - \frac{1}{2} \arg A + k \pi \quad (k = 0, \pm 1, \ldots). \quad (3.3)$$

To compute monodromy data for (3.1), we need the behaviour of solutions in two successive Stokes directions, and this leads to the next hypothesis.

H2. There exists a Stokes curve $C_{k,k+1}$, defined by

$$\text{Re} \left( i \xi \int_{\eta_0}^{\eta} F^{1/2}(\sigma, \xi) \, d\sigma \right) = 0,$$

which connects $\infty$ in two successive Stokes directions (given by $k \pi$ and $(k+1) \pi$ in (3.3) above), passes through $\eta_0$ and is (for large $\xi$) bounded from any zero of $F_0$. 


H3. On and in a neighbourhood of $C_{k,k+1}^+$, $\tilde{F}$ has no poles, at least for large $\xi$, and, for all $\eta$ and uniformly for large $\xi$,

$$\frac{\tilde{F}(\eta, \xi)}{F_0(\eta)} = O(|\eta| + 1),$$

while, for large $\eta$ and uniformly for large $\xi$,

$$F'/F = O(\eta^{-1}), \quad F''/F = O(\eta^{-2}), \quad F' = dF/d\eta.$$

Remarks.
1. It is now clear from Rouché’s theorem that, for $\xi$ sufficiently large, $F(\eta, \xi)$ has, as a function of $\eta$, precisely two zeros $\eta_1, \eta_2$ close to $\eta_0$. In fact, if $\tilde{F}(\eta, \xi) \to L$ as $\eta \to \eta_0$, $|\xi| \to \infty$, we have

$$\eta_j = \eta_0 \pm \left(\frac{L}{F_0(\eta_0)}\right)^{1/2} \xi^{-1/2}\{1 + o(1)\}, \quad j = 1, 2. \quad (3.4)$$

(We can take $\eta_2$ to correspond to the upper sign.)
2. In line with the heuristic discussion in Section 2, we define a number $\alpha$ by

$$\frac{1}{2}\pi i\alpha^2 = \int_{-\alpha}^{\alpha} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{\eta_1}^{\eta_2} F^{1/2}(\eta, \xi) d\eta, \quad (3.5)$$

and a new variable $\zeta$ by

$$\int_{\alpha}^{\zeta} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{\eta_2}^{\eta} F^{1/2}(s, \xi) ds. \quad (3.6)$$

There is a choice of signs for the various square roots, but any consistent choice will do. Other choices merely lead to a permutation amongst the solutions $D_\nu(z), D_\nu(-z), D_{-\nu-1}(iz)$ and $D_{-\nu-1}(-iz)$ of (2.5) (or, of course, (2.6)) and do not therefore affect the space of approximating functions in our theorem below. We note also that $F$ does not vanish on or near $C_{k,k+1}$ if $\xi$ is large, except at $\eta_1, \eta_2$, and so there is no ambiguity in the sign of $F^{1/2}$ once some initial value has been chosen.

3. There is a certain arbitrariness in the precise choice of a Stokes curve. All that is required is that on it both WKB approximations are equipollent, so that both appear in an asymptotic expansion of a solution. With this in mind, it would be equally good to choose a curve connecting two Stokes directions on which $\text{Re}(i\xi \int_{\eta_0}^{\eta} F^{1/2}(\sigma, \xi) d\sigma)$ is bounded independently of $\eta$ and $\xi$, and we shall make use of this possibility.

Given these three assumptions concerning the problem (3.1) we can now show that solutions of this equation can be approximated uniformly by parabolic cylinder functions so long as $\eta$ remains on $C_{k,k+1}$. This result can be summarised thus:
Theorem 1. Under hypotheses H1-H3, and given any solution \( \phi \) of (3.1), there exist constants \( c_1, c_2 \) such that, uniformly for \( \eta \) on \( C_{k,k+1} \), as \( |\xi| \to \infty \),

\[
\left( \frac{\zeta^2 - \alpha^2}{F(\eta, \xi)} \right)^{-1/4} \phi(\eta, \xi) = \left\{ [c_1 + o(1)] D_{\nu} \left( e^{\pi i/4} \sqrt{2\zeta} \right) + [c_2 + o(1)] D_{-\nu-1} \left( e^{-\pi i/4} \sqrt{2\zeta} \right) \right\}.
\]

Proof. We have to compare the equations

\[
\frac{d^2 \phi}{d\eta^2} = -\xi^2 F(\eta, \xi) \phi \tag{3.7}
\]

and

\[
\frac{d^2 \psi}{d\zeta^2} = -\xi^2 (\zeta^2 - \alpha^2) \psi. \tag{3.8}
\]

Set

\[
p = \frac{d\eta}{d\zeta} = \left( \frac{\zeta^2 - \alpha^2}{F} \right)^{1/2}, \tag{3.9}
\]

which we note is bounded both above and below on any bounded part of \( C_{k,k+1} \). The only problem can occur near \( \eta_0 \), and there we notice that \( \zeta^2 - \alpha^2 \) and \( F \) have the same zeros, so that \( p \) and \( p^{-1} \) are analytic in a neighbourhood of \( \eta_0 \). Since trivially \( p \) and \( p^{-1} \) are bounded on some fixed small circle \( |\eta - \eta_0| = k \), say, it follows from the maximum principle that \( p \) and \( p^{-1} \) are bounded inside \( |\eta - \eta_0| = k \). Also as \( \eta, \xi \to \infty \), it is immediate from (3.6) that

\[
\frac{1}{2} \zeta^2 \sim \frac{2A^{1/2} \eta^{2+m/2}}{4 + m},
\]

so that \( p \) is asymptotically some power of \( \eta \) (or \( \zeta \)), and so, considering \( p = p(\zeta) \) and \( p' = dp/d\zeta \), we have, as \( |\zeta| \to \infty \), from H3,

\[
\frac{p'}{p} = O \left( \frac{1}{\zeta} \right), \quad \frac{p''}{p} = O \left( \frac{1}{\zeta^2} \right),
\]

and the bounds implicit in the \( O \)-terms are independent of \( \xi \). Now

\[
\frac{d\phi}{d\eta} = \frac{1}{p} \frac{d\phi}{d\zeta}, \quad \frac{d^2 \phi}{d\eta^2} = \frac{1}{p^2} \frac{d^2 \phi}{d\zeta^2} - \frac{p'}{p^3} \frac{d\phi}{d\zeta},
\]

so that (3.7) becomes

\[
\frac{d^2 \phi}{d\zeta^2} = -\xi^2 (\zeta^2 - \alpha^2) \phi + \frac{p'}{p} \frac{d\phi}{d\zeta}.
\]

Setting

\[
\phi = p^{1/2} \Phi, \tag{3.10}
\]

we have

\[
\frac{d^2 \Phi}{d\zeta^2} = -\xi^2 (\zeta^2 - \alpha^2) \Phi - \frac{1}{2} \left[ \frac{p''}{p} - \frac{3 (p')^2}{2 p^2} \right] \Phi. \tag{3.11}
\]
Now we have already seen that linearly independent solutions of (3.8) are

\[ D_\nu \left( \frac{\pi^i}{\sqrt{2\xi\zeta}} \right), \quad D_{-\nu-1} \left( \frac{-\pi^i}{\sqrt{2\xi\zeta}} \right), \quad (3.12) \]

where, by (2.7),

\[ \nu = -\frac{1}{2} + \frac{1}{2}i\xi\alpha^2, \quad (3.13) \]

and the asymptotics of the functions in (3.12), as \( |\sqrt{2\xi\zeta}| \to \infty \), are always linear combinations of

\[ \exp \left( -\frac{1}{2}i\xi\zeta^2 \right) \left( \sqrt{2\xi\zeta} \right)^\nu \quad \text{and} \quad \exp \left( \frac{1}{2}i\xi\zeta^2 \right) \left( \sqrt{2\xi\zeta} \right)^{-\nu-1}. \]

(For the asymptotics of parabolic cylinder functions, one can consult, for example, [25].) We want to assert that these are bounded on \( C_{k,k+1} \), which is so if

\[ \Re \left( \frac{1}{2}i\xi\zeta^2 - \nu \log \left( \sqrt{2\xi\zeta} \right) \right) \quad \text{is bounded.} \quad (3.14) \]

But by definition

\[ i\xi \int_\alpha^\zeta (\tau^2 - \alpha^2)^{1/2} d\tau = i\xi \int_{\eta_2}^\eta F^{1/2}(s, \xi) ds = i\xi \int_{\eta_2}^\eta F^{1/2}(s, \xi) ds + i\xi \int_{\eta_0}^{\eta_2} F^{1/2}(s, \xi) ds, \]

and the last term is bounded independent of \( \xi \). (Merely set \( s - \eta_0 = t\xi^{-1/2} \) in the integrand, and use the fact that \( (\eta_2 - \eta_0)\xi^{1/2} \) is bounded.) Thus, on \( C_{k,k+1} \),

\[ \Re \left( i\xi \int_\alpha^\zeta (\tau^2 - \alpha^2)^{1/2} d\tau \right) \quad \text{is bounded,} \quad (3.15) \]

and it is an elementary integration that

\[ \int_\alpha^\zeta (\tau^2 - \alpha^2)^{1/2} d\tau = \frac{1}{2} \left\{ \zeta(\zeta^2 - \alpha^2)^{1/2} - \alpha^2 \log \left( \zeta + (\zeta^2 - \alpha^2)^{1/2} \right) + \alpha^2 \log \alpha \right\}. \quad (3.16) \]

Substituting for \( \alpha \) from (3.13), we see easily that (3.15) implies (3.14).

We can now turn (3.11) into an integral equation in the usual way. In fact, any solution of (3.11) satisfies, for some constants \( c_1, c_2 \), the integral equation

\[ \Phi(\zeta) = c_1 D_\nu \left( \frac{\pi^i}{\sqrt{2\xi\zeta}} \right) + c_2 D_{-\nu-1} \left( \frac{-\pi^i}{\sqrt{2\xi\zeta}} \right) \]

\[ - \frac{i}{2\sqrt{2\xi}} \int_\alpha^\zeta \left\{ D_\nu \left( \frac{\pi^i}{\sqrt{2\xi\zeta}} \right) D_{-\nu-1} \left( \frac{-\pi^i}{\sqrt{2\xi t}} \right) + D_{-\nu-1} \left( \frac{\pi^i}{\sqrt{2\xi t}} \right) D_\nu \left( \frac{-\pi^i}{\sqrt{2\xi \zeta}} \right) \right\} \left[ \frac{p'}{p} - \frac{3}{2} \left( \frac{p'}{p} \right)^2 \right] \Phi(t) dt. \quad (3.17) \]
In deriving (3.17) we have made use of the standard result that the Wronskian
\[ W \left( D_\nu \left( e^{\pi i/4} \sqrt{2\xi} \right), D_{-\nu-1} \left( e^{-\pi i/4} \sqrt{2\xi} \right) \right) = i\sqrt{2\xi} \]
and the integral is to be taken along \( C_{k,k+1} \). Since \( D_\nu, D_{-\nu-1} \) are bounded on this curve, and
\[ \frac{p''}{p} - \frac{3}{2} \left( \frac{p'}{p} \right)^2 = O \left( \frac{1}{\zeta^2} \right) \]
and so is integrable to infinity on \( C_{k,k+1} \), we can solve (3.17) by iteration (see, for example [24], to conclude that \( \Phi \) is bounded on \( C_{k,k+1} \). Furthermore, we deduce that
\[ \Phi(\zeta) = c_1 D_\nu \left( e^{\pi i/4} \sqrt{2\xi} \right) + c_2 D_{-\nu-1} \left( e^{-\pi i/4} \sqrt{2\xi} \right) + O \left( \frac{|c_1| + |c_2|}{\sqrt{\xi}} \right) \]
and, returning to \( \phi \) via the transformation (3.10), we see that the theorem is proved.

4. The Uniform Approximation Theorem for a Simple Turning-Point.

Consider differential equations of the form
\[ \frac{d^2\phi}{d\eta^2} = -\xi^2 F(\eta, \xi) \phi, \quad (4.1) \]
where we make the following assumptions about \( F \) in the limit as \( |\xi| \to \infty, \arg \xi = \theta \).

H1. There is a sequence of values \( \xi_n, |\xi_n| \to \infty, \arg \xi_n \to \theta, \) such that
\[ F(\eta, \xi_n) = F_0(\eta, \xi_n) (\eta - \eta_0(\xi_n)) - \frac{\tilde{F}(\eta, \xi_n)}{\xi_n}, \quad (4.2) \]
where
(i) \( \eta_0(\xi_n) \to \eta_\infty \) as \( \xi_n \to \infty, \eta_\infty \) finite,
(ii) \( F_0(\eta, \xi_n) \) is a polynomial in \( \eta \) whose zeros tend to finite limits as \( \xi_n \to \infty, \) all distinct from \( \eta_\infty \), and
\[ F_0(\eta, \xi_n) \sim A \eta^m \text{ as } \eta \to \infty, \quad (4.3) \]
(iii) \( \tilde{F}(\eta, \xi_n) \) is a rational function of \( \eta \).

Remarks.
1. We are allowing the possibility that the turning-point \( \eta_0 \) may depend on \( \xi \). (We drop the subscript \( n \) as in Section 3.) We could do this also in Theorem 1, but this does not seem relevant in the applications of Theorem 1, whereas it certainly is in applications of the present case.
2. The usual WKB approximation for (4.1) would suggest, from (4.3), that for large \( \eta \) solutions of (4.1) are asymptotic to linear combinations of
\[ \eta^{-(m+1)/4} \exp \left( \pm i \xi \int_{\eta_0}^{\eta} \frac{1}{2} F_0^{1/2}(s, \xi)(s - \eta_0)^{1/2} ds \right) \]
and so for Stokes directions we must have
\[ \arg \left( \xi A^{1/4} \eta^{(3+m)/2} \right) = 0, \pm \pi, \pm 2\pi, \ldots \]
or
\[ \frac{1}{2} (m + 3) \arg \eta = - \arg \xi - \frac{1}{2} \arg A + k \pi \quad (k = 0, \pm 1, \ldots). \quad (4.4) \]

Monodromy data for (4.1) can be computed once the behaviours of solutions in two successive Stokes directions are known. We therefore assume the following:
H2. There exists a Stokes curve $C_{k,k+1}$, defined by

$$\text{Re} \left( i\xi \int_{\eta_0}^{\eta} F^{1/2}(\sigma, \xi) d\sigma \right) = 0,$$

which connects $\infty$ in two successive Stokes directions (given by $k\pi$ and $(k+1)\pi$ in (4.4) above) and which passes through $\eta_0$ and is (for large $\xi$) bounded from any zero of $F_0$. (We drop the explicit dependence of $\eta_0$ on $\xi$.)

H3. On and in a neighbourhood of $C_{k,k+1}$, $\tilde{F}$ has no poles, at least for large $\xi$, and, for all $\eta$ and uniformly for large $\xi$,

$$\frac{\tilde{F}(\eta, \xi)}{F_0(\eta, \xi)} = O(1),$$

whilst, for large $\eta$ and uniformly for large $\xi$,

$$F'/F = O(\eta^{-1}), \quad F''/F = O(\eta^{-2}).$$

Based on the above, it follows from Rouché’s theorem that, for sufficiently large $\xi$, $F(\eta, \xi)$ has, as a function of $\eta$, precisely one zero $\eta^*$ close to $\eta_0$ and so close to $\eta_\infty$. In fact,

$$\eta^* = \eta_0 + O(\xi^{-1}). \quad \text{(4.5)}$$

Now if we define a new variable $\zeta$ by

$$\frac{2}{3} \zeta^{3/2} = \int_0^{\zeta} \tau^{1/2} d\tau = \int_{\eta^*}^{\eta} F^{1/2}(s, \xi) ds,$$ \text{(4.6)}

then we can obtain uniform approximations to the solution of (4.1) according to

Theorem 2. Under hypotheses H1-H3, and given any solution $\phi$ of (4.1), there exist constants $c_1, c_2$ such that, uniformly for $\eta$ on $C_{k,k+1}$, as $|\xi| \to \infty$,

$$\left( \frac{\zeta}{F(\eta, \xi)} \right)^{-1/4} \phi(\eta, \xi) = \left\{ [c_1 + o(1)] \text{Ai} \left( e^{\pi i/3} \zeta^{2/3} \right) + [c_2 + o(1)] \text{Bi} \left( e^{\pi i/3} \zeta^{2/3} \right) \right\}$$

where $\text{Ai}$ and $\text{Bi}$ are the usual Airy functions.

Proof. We need to compare the equations

$$\frac{d^2 \phi}{d\eta^2} = -\xi^2 F(\eta, \xi) \phi \quad \text{and} \quad \frac{d^2 \psi}{d\zeta^2} = -\xi^2 \zeta \psi, \quad \text{(4.7a,b)}$$

and do so by setting

$$p = \frac{d\eta}{d\zeta} = \left( \frac{\zeta}{F} \right)^{1/2}.$$ 

Now $p$ is bounded both above and below on any bounded part of $C_{k,k+1}$. The only difficulty might arise near $\eta_0$ and there we note that $\zeta$ and $F$ have the same simple zero (from definition (4.6)) so that $p$ and $p^{-1}$ are analytic near $\eta_0$. Since trivially $p$ and $p^{-1}$ are bounded on some fixed small circle $|\eta - \eta_0| = \varepsilon$, say, it is a consequence
of the maximum principle that both $p$ and $p^{-1}$ are bounded inside $|\eta - \eta_0| = \varepsilon$. As $\eta, \zeta \to \infty$, it is obvious from (4.6) that
\[
\frac{2}{3} \zeta^{3/2} \sim \frac{2A^{1/2} \eta^{(m+3)/2}}{m + 3}, \tag{4.8}
\]
so that $p$ is asymptotically some power of $\eta$ (or $\zeta$). Therefore, considering $p = p(\zeta)$ and $p' \equiv dp/d\zeta$ we have as $|\zeta| \to \infty$, from H3,
\[
\frac{p'}{p} = O \left( \frac{1}{\zeta} \right), \quad \frac{p''}{p} = O \left( \frac{1}{\zeta^2} \right),
\]
and the bounds implicit in the O-terms are independent of $\xi$. Since
\[
\frac{d\phi}{d\eta} = \frac{1}{p} \frac{d\phi}{d\zeta}, \quad \frac{d^2\phi}{d\eta^2} = \frac{1}{p^2} \frac{d^2\phi}{d\zeta^2} - \frac{p'}{p^2} \frac{d\phi}{d\zeta},
\]
equation (4.7a) becomes
\[
\frac{d^2\phi}{d\zeta^2} = -\xi^2 \zeta \phi + \frac{p'}{p} \frac{d\phi}{d\zeta},
\]
and on setting
\[
\phi = p^{1/2} \Phi \tag{4.9}
\]
we obtain
\[
\frac{d^2\Phi}{d\zeta^2} = -\xi^2 \zeta \Phi - \frac{1}{2} \left[ \frac{p''}{p} - \frac{3}{2} \left( \frac{p'}{p} \right)^2 \right] \Phi. \tag{4.10}
\]
It is a standard result that linearly independent solutions of (4.7b) are
\[
\text{Ai} \left( e^{i\pi/3} \zeta^{2/3} \right), \quad \text{Bi} \left( e^{i\pi/3} \zeta^{2/3} \right) \tag{4.11}
\]
and the asymptotics of these functions as $|\zeta^{2/3}| \to \infty$ are always linear combinations of
\[
\xi^{-1/6} \zeta^{-1/4} \exp \left\{ \pm \frac{2}{3} i\xi^{3/2} \right\}.
\]
We would like to assert that these are bounded on $C_{k,k+1}$, which is the case if Re $i\xi^{3/2}$ is bounded. However, we have from (4.6) that
\[
\frac{2}{3} i\xi^{3/2} = i \int_{\eta^*}^{\eta} F^\frac{1}{2}(s, \xi) \, ds = i\xi \int_{\eta^*}^{\eta_0} F^{1/2}(s, \xi) \, ds + i\xi \int_{\eta_0}^{\eta} F^{1/2}(s, \xi) \, ds
\]
so that, on $C_{k,k+1}$, Re $i\xi^{3/2}$ is bounded if and only if Re $i\xi \int_{\eta^*}^{\eta_0} F^{1/2}(s, \xi) \, ds$ is bounded. This latter expression is $O \left( |\xi|^{-1/2} \right)$ for large $|\xi|$ (using (4.2) and (4.5)) so that Re $i\xi^{3/2}$ is indeed bounded on the Stokes curve.

To complete the proof of Theorem 2 we turn (4.10) into an integral equation in the usual way. It follows that any solution of (4.10) satisfies, for some constants $c_1$ and $c_2$, the equation
\[
\Phi(\zeta) = c_1 \text{Ai} \left( e^{\pi i/3} \zeta^{2/3} \right) + c_2 \text{Bi} \left( e^{\pi i/3} \zeta^{2/3} \right)
\]
\[
- \frac{i}{4\xi^{5/6}} \int_0^\zeta \left\{ \text{Ai} \left( e^{\pi i/3} \zeta^{2/3} \right) \text{Bi} \left( e^{\pi i/3} \zeta^{2/3} t \right) - \text{Bi} \left( e^{\pi i/3} \zeta^{2/3} \right) \text{Ai} \left( e^{\pi i/3} \zeta^{2/3} t \right) \right\} \left[ \frac{p''}{p} - \frac{3}{2} \left( \frac{p'}{p} \right)^2 \right] \Phi(t) \, dt. \tag{4.12}
\]
In deriving (4.12) we have used the standard result for Wronskians that
\[ W \left( \text{Ai} \left( e^{\pi i/3} \xi^{2/3} \zeta \right), \text{Bi} \left( e^{\pi i/3} \xi^{2/3} \zeta \right) \right) = 2i \xi^{5/6}. \]

The integral within (4.12) is taken along the Stokes curve \( C_{k,k+1} \) and since \( \text{Ai} \) and \( \text{Bi} \) are bounded there and
\[ \frac{p''}{p} - \frac{3}{2} \left( \frac{p'}{p} \right)^2 = O \left( \frac{1}{\zeta^2} \right) \]
and so is integrable to infinity on \( C_{k,k+1} \), we can solve (4.12) by iteration to conclude that \( \Phi \) is bounded on \( C_{k,k+1} \). Furthermore, we have that
\[ \Phi(\zeta) = c_1 \text{Ai} \left( e^{\pi i/3} \xi^{2/3} \zeta \right) + c_2 \text{Bi} \left( e^{\pi i/3} \xi^{2/3} \zeta \right) + O \left( \frac{|c_1| + |c_2|}{\xi^{5/6}} \right) \]
and, returning to the variable \( \phi \) via the transformation (4.9), we conclude that the theorem is proved.

5. Monodromy Data for Parabolic Cylinder Functions.

This section sets out the well-known results that we shall need concerning Stokes multipliers for the parabolic cylinder function. We shall be interested in computing the multipliers for the curve \( C_{k,k+1} \); i.e. we wish to compare the asymptotic behaviours on
\[ \left( \frac{1}{2} m + 2 \right) \arg \eta + \arg \xi + \frac{1}{2} \arg A = k \pi \quad \text{and} \quad (k + 1) \pi \]
and, since for large \( \eta, \xi, 2 \arg \zeta \sim \frac{1}{2} \arg A + \left( \frac{1}{2} m + 2 \right) \arg \eta \), this is equivalent to comparing behaviours on
\[ \arg \left( \sqrt{2 \xi \zeta} \right) = \frac{1}{2} k \pi \quad \text{and} \quad \frac{1}{2} (k + 1) \pi. \]

Let us set \( z \equiv e^{\pi i/4} \sqrt{2 \xi \zeta} \); the complete asymptotic behaviours of \( D_\nu(z) \) as \( |z| \to \infty \) are well known (see for example [2]) and are given by
\[
D_\nu(z) \sim \begin{cases}
    z^\nu \exp\left(-\frac{1}{4} z^2\right), & \text{if } |\arg z| < \frac{3}{4} \pi, \\
    z^\nu \exp\left(-\frac{1}{4} z^2\right) - \frac{\sqrt{2 \pi}}{\Gamma(-\nu)} e^{i \pi \nu} z^{-\nu-1} \exp\left(\frac{1}{4} z^2\right), & \text{on } \arg z = \frac{3}{4} \pi, \\
    e^{-2i \pi \nu} z^\nu \exp\left(-\frac{1}{4} z^2\right) - \frac{\sqrt{2 \pi}}{\Gamma(-\nu)} e^{i \pi \nu} z^{-\nu-1} \exp\left(\frac{1}{4} z^2\right), & \text{on } \arg z = \frac{5}{4} \pi, \\
    e^{-2i \pi \nu} z^\nu \exp\left(-\frac{1}{4} z^2\right), & \text{if } \frac{5}{4} \pi < \arg z < \frac{11}{4} \pi.
\end{cases}
\]

Then, on \( \arg z = \pm \frac{1}{4} \pi + 2 \ell \pi \), with \( \ell \) integral,
\[ D_\nu(z e^{-2i \ell \pi}) \sim (z e^{-2i \ell \pi})^\nu \exp\left(-\frac{1}{4} z^2\right) \]
and so, since \( D_\nu(z) \) is single-valued,
\[ D_\nu(z) \sim \exp\left(-\frac{1}{4} z^2\right) z^\nu e^{-2i \pi \nu}. \]
Similarly, on arg \( z = \frac{3}{4} \pi + 2\ell \pi \), we have

\[
D_{\nu}(z) \sim \exp(-\frac{1}{4}z^2) - \nu e^{-2\ell \pi i\nu} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu \pi i} \exp(\frac{1}{4}z^2) z^{-\nu - 1} e^{2\ell \pi i(\nu + 1)}.
\] (5.3)

To evaluate the Stokes multiplier we proceed as follows. In any sector between two adjacent Stokes directions there is (modulo multiplication by a constant) a unique solution \( f_1 \) which is asymptotic to the small exponential. All other solutions are necessarily asymptotic to some multiple of the large exponential, but if we take such a solution on the first Stokes line, then we will find that on the second Stokes line its asymptotics will have added a multiple of \( f_1 \). That multiple is the Stokes multiplier. Thus, relative to the asymptotic forms \( \exp(-\frac{1}{4}z^2) z^\nu \) and \( \exp(\frac{1}{4}z^2) z^{-\nu - 1} \), the Stokes multiplier for the sector \( \frac{1}{4} \pi + 2\ell \pi \) to \( \frac{3}{4} \pi + 2\ell \pi \), in which \( \exp(-\frac{1}{4}z^2) z^\nu \) is dominant, can be immediately deduced from (5.2) and (5.3). Consequently,

\[
\text{SM} \left( \frac{1}{4} \pi + 2\ell \pi, \frac{3}{4} \pi + 2\ell \pi \right) = -\frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu \pi i} e^{4\ell \pi i\nu}.
\] (5.4)

Similar calculations for each of the other pairs of sectors yields the complete monodromy data in the form

\[
\begin{align*}
\text{SM} \left( \frac{3}{4} \pi + 2\ell \pi, \frac{5}{4} \pi + 2\ell \pi \right) &= \frac{\Gamma(-\nu)}{\sqrt{2\pi}} e^{-\nu \pi i} e^{-4\ell \pi i\nu} (1 - e^{-2\pi i\nu}) \\
&= i\sqrt{\frac{2}{\pi}} \Gamma(-\nu) e^{-(4\ell + 2)\pi i\nu} \sin \pi \nu, \\
\text{SM} \left( -\frac{3}{4} \pi + 2\ell \pi, -\frac{1}{4} \pi + 2\ell \pi \right) &= \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-\nu \pi i} e^{4\ell \pi i\nu}, \\
\text{SM} \left( -\frac{1}{4} \pi + 2\ell \pi, \frac{1}{4} \pi + 2\ell \pi \right) &= -i\sqrt{\frac{2}{\pi}} \Gamma(-\nu) e^{-4\ell \pi i\nu} \sin \pi \nu.
\end{align*}
\] (5.5)

(5.6)

(5.7)

To obtain (5.7) we need the asymptotics of \( D_{-\nu - 1} \), which are that

\[
D_{-\nu - 1}(iz) \sim \begin{cases} e^{-\pi i(\nu + 1)/2} z^{-\nu - 1} \exp(\frac{1}{4}z^2), & \text{on arg } z = -\frac{1}{4} \pi, \\ -\frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{-\pi i(\nu + 2)/2} z^{\nu} \exp(-\frac{1}{4}z^2), & \text{on arg } z = +\frac{1}{4} \pi. \end{cases}
\]

6. Monodromy Data for (3.1).

Although one might expect the double turning-point case to be more complicated than the simple case (and in some sense it is), yet in the double turning-point case one can work out the monodromy data quite explicitly, in terms of the coefficients of the monodromy equations, even for a general form of equation. It is this that leads to the wealth of explicit connection formulae given, for example, in [11]. They are explicit because they are connecting directions where the behaviour of the solution of the Painlevé equation leads to a double turning-point in the isomonodromy equations.

In the present section, we show how this monodromy data can be calculated. To do this, we add to hypotheses H1-H3 in Theorem 1 the following additional hypothesis.
H4. Suppose that in H1 we can express $\tilde{F}$ in the form

$$\tilde{F}(\eta, \xi) = F_1(\eta, \xi) + \xi^{-\gamma}F_2(\eta, \xi)$$

for some $\gamma > 0$, where $F_1$ and $F_2$ are rational in $\eta$. Suppose also that $F_0$ is a perfect square, so that $F_1(\eta, \xi)/\{F_0^{1/2}(\eta)(\eta - \eta_0)\}$ is rational in $\eta$ with partial fraction decomposition

$$\frac{F_1(\eta, \xi)}{F_0^{1/2}(\eta)(\eta - \eta_0)} = \sum_{i=1}^{N} \frac{A_i}{\eta - s_i}, \quad \text{with} \quad s_1 = \eta_0. \quad (6.1)$$

Finally, suppose that, on and in a neighbourhood of $C_{k,k+1}$, $F_2/F_0^{1/2}$ is bounded uniformly in $\xi$.

Remarks.
1. The quantities $A_i, s_i$ will in general depend on $\xi$, but we suppress that dependence. We shall, however, assume that they tend to finite limits as $|\xi| \to \infty$.

2. It is obvious that

$$A_1 = \frac{F_1(\eta_0, \xi)}{F_0^{1/2}(\eta_0)}, \quad (6.2a)$$

and we will set

$$B = \sum_{i=1}^{N} A_i. \quad (6.2b)$$

3. To compute the monodromy data, we need the relation between $\zeta$ and $\eta$ for large $\xi$. This is the content of the next theorem.

Theorem 3. Under the hypothesis H4, and hypotheses H1-H3 of Theorem 1, we have, for large $\xi$ and $\eta$,

$$\zeta^2 - \alpha^2 \log \zeta + \frac{1}{4} \alpha^2 \log F_0(\eta_0) + o(\xi^{-1})$$

$$= 2 \int_{\eta_0}^{\eta} F_0^{1/2}(s)(s - \eta_0) \, ds - \frac{B}{\xi} \log \eta + \frac{1}{\xi} \sum_{i=2}^{N} A_i \log(\eta_0 - s_i). \quad (6.3)$$

Proof. From the definition of $\zeta$ and (3.16), we have

$$\frac{1}{4} \left\{ 2\zeta^2 - 2\alpha^2 \log(2\zeta) + 2\alpha^2 \log \alpha - \alpha^2 + O(\alpha^4\zeta^{-2}) \right\} = \int_{\eta_2}^{\eta} F^{1/2}(s, \xi) \, ds. \quad (6.4)$$

In calculating the right-hand side, we will replace $F(\eta, \xi)$ by

$$\tilde{F}(\eta, \xi) = F_0(\eta)(\eta - \eta_0)^2 - \frac{F_1(\eta, \xi)}{\xi}.$$
i.e. we will ignore $F_2$. This is justifiable because we will find that even the $F_1$ term contributes only a term of size $O(\xi^{-1})$ which is all that we are interested in. The term $F_2$ if we included it, would similarly contribute a term of only $O(\xi^{-1-\gamma})$. Thus

$$
\int_{n_2}^{\eta} \hat{F}^{1/2}(s, \xi) \, ds = \left( \int_{n_2}^{\eta^*} + \int_{\eta^*}^{\eta} \right) \hat{F}^{1/2}(s, \xi) \, ds = I_1 + I_2,
$$

(6.5)
say, where

$$
\eta^* = \eta_0 + T\xi^{-1/2}
$$

and $T$ is a large positive number to be specified more precisely later. In $I_1$ we make the change

$$
s - \eta_0 = t\xi^{-1/2},
$$

and then

$$
I_1 = \frac{1}{\xi} \int_{\{F_1(\eta_0)/F_0(\eta_0)\}^{1/2}}^{T} \{F_0(s)t^2 - F_1(s)\}^{1/2} \, dt.
$$

Since $s - \eta_0 = O(\xi^{-1/2})$ and we are only concerned with evaluating (6.5) correct to $O(\xi^{-1})$, we can safely replace $s$ by $\eta_0$ in $I_1$ which, on integration using (3.16), gives

$$
I_1 = \frac{F_0^{1/2}(\eta_0)}{4\xi} \left\{ 2T^2 - \frac{2F_1(\eta_0)}{F_0(\eta_0)} \log(2T) + \frac{F_1(\eta_0)}{F_0(\eta_0)} \log \left( \frac{F_1(\eta_0)}{F_0(\eta_0)} \right) - \frac{F_1(\eta_0)}{F_0(\eta_0)} \right\} + O(\xi^{-1-2}T^{-2}) + o(\xi^{-1}).
$$

(6.6)

Taking $T = -(F_1(\eta_0)/F_0(\eta_0))^{1/2}$, we can compute $I_1$ explicitly (with $s = \eta_0$) and so conclude from (3.5) that

$$
\frac{1}{2}i\pi \alpha^2 = \frac{F_0^{1/2}(\eta_0)}{4\xi} \left\{ 2i\pi \frac{F_1(\eta_0)}{F_0(\eta_0)} \right\} + o(\xi^{-1}),
$$

or

$$
\alpha^2 = \frac{F_1(\eta_0)}{\xi F_0^{1/2}(\eta_0)} + o(\xi^{-1}).
$$

(6.7)

Also, by the binomial expansion, the integral $I_2$ in (6.5) is given by

$$
I_2 = \int_{\eta^*}^{\eta_0} F_0^{1/2}(s)(s - \eta_0) \left[ 1 - \frac{F_1(s)}{2\xi F_0(s)(s - \eta_0)^2} \right] \, ds
$$

$$
+ O \left( \int_{\eta^*}^{\eta} \frac{F_1^2(s)ds}{\xi^2 F_0^{3/2}(s)(s - \eta_0)^3} \right).
$$

(6.8)

According to hypothesis H4, $F_1$ is bounded by $F_0^{1/2}$ and so the final term in this expression is of size

$$
O \left( \frac{1}{\xi^2} \int_{\eta^*}^{\eta} \frac{|ds|}{|F_0^{1/2}(s)||s - \eta_0|^3} \right) = O \left( \frac{1}{\xi^2 |\eta^* - \eta_0|^2} \right) = O(\xi^{-1}T^{-2}).
$$
Using the proposed form for $F_1(\eta, \xi)$ as given by (6.1), the second term in $I_2$ is equal to

$$
- \frac{1}{2\xi} \sum_{i=1}^{N} \int_{\eta^*}^{\eta} \frac{A_i}{s-s_i} \, ds = -\frac{B}{2\xi} \log \eta + O(\eta^{-1}\xi^{-1}) + o(\xi^{-1})
$$

$$
+ \frac{1}{2\xi} \log \prod_{j=2}^{N} (\eta_0 - s_j)^{A_j} + \frac{F_1(\eta_0)}{4\xi |F_0(\eta_0)|^{1/2}} \log \left( \frac{T^2}{\xi} \right), \tag{6.9a}
$$

whilst the first term may be written as

$$
\int_{\eta^*}^{\eta} F_0^{1/2}(s)(s-\eta_0) \, ds = \int_{\eta_0}^{\eta} F_0^{1/2}(s)(s-\eta_0) \, ds - \frac{1}{2} F_0^{1/2}(\eta_0) T^2 \xi^{-1} + o(\xi^{-1}). \tag{6.9b}
$$

Combining (6.5), (6.6), (6.9a) and (6.9b) yields

$$
\int_{\eta^*}^{\eta} F^{1/2}(s) \, ds = \frac{F_1(\eta_0)}{4\xi F_0^{1/2}(\eta_0)} \left\{ -2 \log 2 - \log \xi + \log \left( \frac{F_1(\eta_0)}{F_0(\eta_0)} - 1 \right) \right\} - \frac{B}{2\xi} \log \eta
$$

$$
+ \int_{\eta_0}^{\eta} F_0^{1/2}(s)(s-\eta_0) \, ds + \frac{1}{2\xi} \log \prod_{j=2}^{N} (\eta_0 - s_j)^{A_j} + O(\xi^{-1}T^{-2}) + o(\xi^{-1}),
$$

and so, using (6.4) and (6.7) and making a choice of $T$ large, we see that Theorem 3 is verified.

We have shown that linearly independent solutions of (3.8) are $D_{\nu}(e^{\pi i/4} \sqrt{2\xi} \zeta)$ and $D_{-\nu-1}(e^{-\pi i/4} \sqrt{2\xi} \zeta)$ and the asymptotic behaviours of these functions in various sectors have been noted in (5.1). Using the result of Theorem 3 and recalling that $i\xi \alpha^2 = 2\nu + 1$, it follows that as $\xi, \zeta \to \infty$,

$$
\zeta^{1/2} e^{-i\xi \zeta^{2}/2} \left( e^{\pi i/4} \sqrt{2\xi} \zeta \right)^{\nu} \sim e^{-i\xi F(\eta) \eta^{-1/2}}
$$

$$
x \left\{ e^{i\xi F(\eta_0) \xi^{\nu/2}} \prod_{j=2}^{N} (\eta_0 - s_j)^{-1A_j/2} F_{00}^{(2\nu+1)/82\nu/2} e^{i\pi \nu/4} \right\}, \tag{6.10a}
$$

$$
\zeta^{1/2} e^{i\xi \zeta^{2}/2} \left( e^{\pi i/4} \sqrt{2\xi} \zeta \right)^{-\nu-1} \sim e^{i\xi F(\eta) \eta^{-1/2}}
$$

$$
x \left\{ e^{-i\xi F(\eta_0) \xi^{-(1+\nu)/2}} \prod_{j=2}^{N} (\eta_0 - s_j)^{iA_j/2} F_{00}^{-(2\nu+1)/82^{-(1+\nu)/2}} e^{-i\pi (\nu+1)/4} \right\}, \tag{6.10b}
$$

where $F(\eta) \equiv \int_{0}^{\eta} F_0^{1/2}(s)(s-\eta_0) \, ds$ and $F_{00}$ denotes the value $F_0(\eta_0)$. These relations, together with the Stokes multipliers (5.4)-(5.7) for the parabolic cylinder function, enable us to write down the complete monodromy data for (3.1). With $z \equiv e^{\pi i/4} \sqrt{2\xi} \zeta$, the Stokes multipliers relative to the solutions

$$
e^{-i\xi F(\eta) \eta^{-1/2}} \quad \text{and} \quad e^{i\xi F(\eta) \eta^{-1/2}}$$
are

i) from arg $z = \frac{1}{4}\pi + 2\ell\pi$ to $\frac{3}{4}\pi + 2\ell\pi$ (where $e^{-i\xi F(\eta)}\eta^{iB/2}$ is the dominant solution)

$$
-\frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi (4\ell + 1)\nu - 2i\xi F(\eta_0)} \xi^{-\nu - 1/2} \\
\times \prod_{j=2}^{N} (\eta_0 - s_j)^{iA_j} F_{00}^{-(2\nu + 1)/4} 2^{-\nu - 1/2} e^{-i\pi (2\nu + 1)/4},
$$

(6.11a)

ii) from arg $z = \frac{3}{4}\pi + 2\ell\pi$ to $\frac{5}{4}\pi + 2\ell\pi$ ($e^{i\xi F(\eta)}\eta^{-iB/2}$ dominant)

$$
i\sqrt{\frac{2\pi}{\Gamma(-\nu)}} e^{-i\pi (4\ell + 2)\nu + 2i\xi F(\eta_0)} \xi^{\nu + 1/2} \\
\times \prod_{j=2}^{N} (\eta_0 - s_j)^{-iA_j} F_{00}^{(2\nu + 1)/4} 2^{\nu + 1/2} e^{i\pi (2\nu + 1)/4} \sin \pi \nu;
$$

(6.11b)

iii) from arg $z = -\frac{3}{4}\pi + 2\ell\pi$ to $-\frac{1}{4}\pi + 2\ell\pi$ ($e^{-i\xi F(\eta)}\eta^{iB/2}$ dominant)

$$
\sqrt{\frac{2\pi}{\Gamma(-\nu)}} e^{i\pi (4\ell - 1)\nu - 2i\xi F(\eta_0)} \xi^{-\nu - 1/2} \\
\times \prod_{j=2}^{N} (\eta_0 - s_j)^{iA_j} F_{00}^{-(2\nu + 1)/4} 2^{-\nu - 1/2} e^{-i\pi (2\nu + 1)/4},
$$

(6.11c)

iv) from arg $z = -\frac{1}{4}\pi + 2\ell\pi$ to $\frac{1}{4}\pi + 2\ell\pi$ ($e^{i\xi F(\eta)}\eta^{-iB/2}$ dominant)

$$
-i\sqrt{\frac{2\pi}{\Gamma(-\nu)}} e^{-4i\pi \ell \nu + 2i\xi F(\eta_0)} \xi^{\nu + 1/2} \\
\times \prod_{j=2}^{N} (\eta_0 - s_j)^{-iA_j} F_{00}^{(2\nu + 1)/4} 2^{\nu + 1/2} e^{i\pi (2\nu + 1)/4} \sin \pi \nu.
$$

(6.11d)

We remark finally that, from Theorem 1 and (6.10), the asymptotic forms of $\phi$ are

$$F^{-1/4} e^{-i\xi F(\eta)}\eta^{iB/2}$$

and

$$F^{-1/4} e^{i\xi F(\eta)}\eta^{-iB/2}.
$$

(6.12)

7. Application to the Painlevé Equations.

Suppose that equation (3.1) arises after scaling from the monodromy equation of some Painlevé equation. (It is our contention that all such monodromy equations reduce to the form (3.1) with simple or double turning-points.) In the preceding sections we have evaluated the monodromy data with respect to the usual WKB solutions

$$F^{-1/4} \exp \left\{ \pm i\xi \int_{\eta_0}^{\eta} F^{1/2}(t) dt \right\}.
$$

(These solutions are given in terms of the variable $\eta$ but when expressed in terms of the original variable $\lambda$ they are the usual WKB forms.) The theory of the Painlevé
equations tells us that the monodromy data is independent of $\xi$ provided that the $\lambda$–sector in which the Stokes multipliers are being calculated remains fixed. Since $\xi, \lambda, \eta$ are inter-related (in the case of (2.2) $\eta = \xi^{-1/3}\lambda$), the condition that the $\lambda$–sector remains fixed means that the $\eta$–sector changes with $\xi$, or at least with $\arg \xi$, and also the turning-point $\eta_0$ depends in general on $\xi$. Indeed, it may change from simple to double as $\arg \xi$ changes. (In the case of (2.2), this is a question of the behaviour of $M(\xi)$ as $|\xi| \to \infty$ in a specific direction; $M(\xi) \to 0$ gives double turning-points.)

Thus, the monodromy data depends on $\xi$ in various ways, but so long as the $\lambda$–sector remains fixed, these various dependences must cancel. This leads, for example, to relations between $M(\xi)$ and $\xi$ for large $|\xi|$, and so to statements about the possible asymptotic behaviours of the Painlevé functions, which we will pursue in later papers.

We now turn to examine the application of our method to the specific case of PII (1.1) with $\beta = 0$ with the aim of using it to establish Theorem B. The relevant version of the generic equation (3.1) is

$$\frac{d^2 \phi}{d\eta^2} = \xi^2 \phi \left\{ -(4\eta^2 + 1)^2 + \frac{8i\eta}{\xi} - \frac{i}{\xi} (4\eta^2 + 1) \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-1} + \frac{4r^2}{x^2} - \frac{4q^2}{x} - \frac{4q^4}{x^2} \right. \right.$$

$$\left. - \frac{2iq^2\sqrt{x}}{\xi^2} \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-1} + \frac{3}{4\xi^2} \left( \eta - \frac{ir}{2q\sqrt{x}} \right)^{-2} \right\}, \quad (7.1)$$

and this form follows directly from (2.2). If either $x \to +\infty$ or $x \to -\infty$, we will have a double turning-point $\eta_0 = \pm \frac{1}{2}i$, and although we can choose, say, $\eta_0 = \frac{1}{2}i$ when we are considering $x \to +\infty$, the turning-point that we will have to use when $x \to -\infty$ is then fixed. Thus we have

$$F_0(\eta) = 16 \left( \eta + \frac{1}{2}i \right)^2 \quad \text{if} \quad \eta_0 = \frac{1}{2}i, \quad F_0(\eta) = 16 \left( \eta - \frac{1}{2}i \right)^2 \quad \text{if} \quad \eta_0 = -\frac{1}{2}i. \quad (7.2)$$

In view of the behaviours of the solution of PII as $x \to \pm \infty$ (given by Theorem A and (1.2)), we will write

$$q = x^{-1/4}Q(\xi) = \xi^{-1/6}Q(\xi), \quad (7.3a)$$

so that

$$r = \frac{dq}{dx} = \frac{3}{2} \xi^{1/6} \left( Q'(\xi) - \frac{1}{6\xi} Q \right). \quad (7.3b)$$

With these definitions we will assume that there exists a sequence $\xi_n \to \infty e^{i\theta}$ with

$$\frac{ir}{2q\sqrt{x}} = \frac{3}{4}i \left( \frac{Q'}{Q} - \frac{1}{6\xi} \right) \to \ell_1, \quad (7.4a)$$

and

$$\xi \left( \frac{4r^2}{x^2} - \frac{4q^4}{x^2} - \frac{4q^2}{x} \right) \equiv 9 \left[ (Q')^2 - \frac{1}{3\xi} Q Q' + \frac{Q^2}{36\xi^2} \right] - \frac{4Q^4}{\xi} - 4Q^2 \to \ell_2. \quad (7.4b)$$
(This assumption is certainly justified if \( \theta = 0, \frac{3}{2} \pi \), and \( q \) is the solution given by Theorem A. Of course, \( \ell_1 \) and \( \ell_2 \) will depend on \( \theta \).) Then, with the notation of Section 6, as \( |\zeta_n| \to \infty \),

\[
F_1(\eta, \zeta_n) \to 8 i \eta - \frac{i(4 \eta^2 + 1)}{\eta - \ell_1} + \ell_2,
\]

and, from (3.13) and (6.7), in the limit as \( |\zeta_n| \to \infty \),

\[
2 \nu + 1 - i \frac{F_1(\eta_0, \zeta_n)}{F_0^{1/2}(\eta_0)} \to 0,
\]

so that

\[
\nu + 1 \to \frac{i \ell_2}{16 \eta_0}.
\]

(Recall that we may have \( \eta_0 = \frac{1}{2} i \) or \( \eta_0 = -\frac{1}{2} i \).) Furthermore, \( F_1/F^{1/2} \) has poles at \( \eta = \pm \eta_0 \) and \( \eta = \ell_1 \) and

\[
\frac{F_1}{F^{1/2}} = -\frac{i(2 \nu + 1)}{\eta - \eta_0} + \frac{i(2 \nu + 3)}{\eta + \eta_0} - \frac{i}{\eta - \ell_1},
\]

so that \( s_2 = -\eta_0, \ s_3 = \ell_1, \ A_2 = i(2 \nu + 3), \ A_3 = -i \) and

\[
B = \sum_{j=1}^{3} A_j = i.
\]

We are now in a position to write down the respective monodromy data by appealing to formulae (6.11). However, rather than expressing the data relative to the solutions (see (6.12))

\[
\phi \sim F^{-1/4} e^{-i \xi F(\eta)} \eta^{1/2} \quad \text{and} \quad \phi \sim F^{-1/4} e^{i \xi F(\eta)} \eta^{-1/2},
\]

it is more convenient to use modified reference solutions. We must use \( \psi \) rather than \( \phi \) (see (2.1)), since it is in terms of \( \psi \) and \( \lambda \) that the monodromy data is independent of \( \xi \). As \( \psi_2 = (\eta - l_1)^{1/2} \phi \) and since \( B = i \), linearly independent asymptotic solutions for \( \psi_2 \) are

\[
\psi_2^{(1)} \sim \eta^{-1} e^{-i \xi F(\eta)} \quad \text{and} \quad \psi_2^{(2)} \sim e^{i \xi F(\eta)}.
\]

In order that \( \Psi \) should satisfy (1.4), where the matrix has zero trace, we need the component \( \psi_2^{(1)} \sim \exp(-i \xi F) \). It is then immediate from (2.1) that \( \psi_2^{(1)} \sim \frac{1}{2} q i \xi^{-1/3} \eta^{-1} \exp(-i \xi F) \), and so we choose to establish monodromy data with respect to

\[
\frac{1}{2} q i \xi^{-1/3} \eta^{-1} e^{-i \xi F(\eta)} \quad \text{and} \quad e^{i \xi F(\eta)},
\]

whence, from (6.11), for \( \arg z \equiv \arg \left( e^{i \pi/4} \sqrt{2 \zeta} \right) = \frac{\pi}{4} + 2 \ell \pi \) to \( \frac{3}{4} \pi + 2 \ell \pi \), the Stokes multiplier is

\[
- \frac{i \sqrt{2 \pi}}{\Gamma(-\nu)} e^{(4 \xi + 1) i \nu - 2 i \xi F(\eta_0)} \xi^{-\nu - 1/2} \left[ \prod_{j=2}^{3} \frac{1}{\eta_0 - s_j} A_j \right]
\times F_{00}^{-(\nu + 1/2)/2} \nu - 3/2 e^{-i \pi (2 \nu + 1)/4} q \xi^{-1/3},
\]

(7.9a)
whilst for the sector from \( \arg z = \frac{\pi}{4} + 2\ell \pi \) to \( \frac{5\pi}{4} + 2\ell \pi \) it is

\[
\sqrt{\frac{2}{\pi}} \Gamma(-\nu) e^{-(4\ell+2)i\pi\nu+2i\xi F(\eta_0)} \xi^{\nu+1/2} \left[ \prod_{j=2}^{3} (\eta_0 - s_j)^{-iA_j} \right] \\
\times F_{00}^{(\nu+1/2)/2} 2^{\nu+3/2} e^{i\pi(2\nu+1)/4} q^{-1} \xi^{1/3} \sin \pi \nu.
\]

\[(7.9b)\]

8. Monodromy Data as \( x \to +\infty \).

Here we take \( \eta_0 = \frac{1}{2}i \), and the Stokes curves through \( \eta_0 = \frac{1}{2}i \) are given by

\[
\text{Re} \left\{ i\xi \int_{i/2}^{\eta} (4\sigma^2 + 1) \, d\sigma \right\} = \text{Re} \left[ i\xi \left( \frac{4}{3} \eta^3 + \eta - \frac{1}{3}i \right) \right] = 0,
\]

which are asymptotic to the directions (with \( \arg \xi = 0 \)) \( \arg \eta = \frac{1}{3}j\pi \), for integral \( j \).

We shall choose the sector bounded by \( \arg \eta = 0 \) and \( \arg \eta = \frac{1}{3}\pi \), which corresponds to the \( \lambda \)–sector \( 0 \leq \arg \lambda \leq \frac{\pi}{3} \). This \( \lambda \)–sector must then be the same when we consider \( x \to -\infty \).

The asymptotics in Theorem A tell us that, as \( x \to +\infty \),

\[
\frac{ir}{2q\sqrt{x}} \to -\frac{1}{2}i,
\]

so that, in the notation of Section 7,

\[
\ell_1 = -\frac{1}{2}i, \quad \ell_2 = 0, \quad \nu = -1, \quad s_2 = -\frac{1}{2}i, \quad s_3 = -\frac{1}{2}i, \quad A_2 = i, \quad A_3 = -i, \quad B = i.
\]

Also, from (6.10),

\[
F(\eta_0) = \frac{1}{3}i, \quad F_{00} = -16.
\]

Thus from (7.9) the Stokes multiplier for the relevant \( z \)–sector \( \left( \frac{1}{4}\pi \leq \arg z \leq \frac{3}{4}\pi \right) \) is given by

\[
\text{SM}_{\infty} = -a.
\]

(8.1)

9. Monodromy Data as \( x \to -\infty \).

Since \( \eta = x^{-1/2} \lambda \) and we now have \( \arg x = \pi \), the requirement that the \( \lambda \)–sector be fixed now demands

\[
-\frac{1}{2}\pi \leq \arg \eta \leq -\frac{1}{6}\pi.
\]

(9.1)

We assert that the relevant turning-point must now be \(-\frac{1}{2}i\). For if we suppose for contradiction that it is still \( +\frac{1}{2}i \), then we note that the Stokes curve from \( \frac{1}{2}i \) to \( \infty e^{-i\pi/2} \) passes through \(-\frac{1}{2}i\), since

\[
\text{Re} \left\{ i\xi \int_{i/2}^{-i/2} (4\sigma^2 + 1) \, d\sigma \right\} = 0.
\]

(Recall that \( \arg \xi = \frac{3}{2}\pi \).) Thus also the Stokes curve associated with \(-\frac{1}{2}i\) and the sector (9.1) must pass through \( +\frac{1}{2}i \), and this is impossible since the real direction from \(-\frac{1}{2}i\) is also a direction for which

\[
\text{Re} \left\{ i\xi \int_{-i/2}^{-i/2} (4\sigma^2 + 1) \, d\sigma \right\} = 0.
\]
(Set $\sigma + \frac{1}{2}i = \tau$ for small real $\tau$.)

The asymptotics in Theorem A tell us that, as $x \to -\infty$,

$$\frac{ir}{2q\sqrt{x}} \sim -\frac{1}{2} \cot \left( \frac{2}{3} |x|^{3/2} - \frac{3}{4} d^2 \log |x| + \gamma \right),$$

so that

$$\ell_1 = -\frac{1}{2} \lim_{n \to \infty} \left\{ \cot \left( \frac{2}{3} |x_n|^{3/2} - \frac{3}{4} d^2 \log |x_n| + \gamma \right) \right\},$$

where the sequence $\{x_n\}$ (or $\{\xi_n\}$) has to be chosen so that the limit exits. Also

$$\ell_2 = 4e^{3\pi i/2}d^2, \quad \nu = -1 + \frac{1}{2}id^2, \quad s_2 = \frac{1}{2}i,$$

$$s_3 = \ell_1, \quad A_2 = i(2\nu + 3), \quad A_3 = -i,$$

$$F(\eta_0) = -\frac{1}{3}i, \quad F_{00} = -16.$$ 

Note also that since, from Theorem 3, arg $\zeta = \frac{3}{2}$ arg $\eta$ for large $|\eta|$, and since

$$\arg z = \frac{1}{4}\pi + \frac{1}{2} \arg \xi + \arg \zeta$$

$$= \frac{1}{4}\pi + \frac{3}{4} \arg x + \frac{3}{2} \arg \eta$$

$$= \frac{1}{4}\pi + \frac{3}{2} \arg \lambda,$$

we see that keeping the $\lambda$–sector fixed also fixes the $z$–sector and so we have that the relevant $z$–sector is again $\frac{1}{4}\pi \leq \arg z \leq \frac{3}{4}\pi$. Thus from (7.9) the Stokes multiplier is

$$ \frac{i\sqrt{2\pi}}{\Gamma(1-\frac{1}{2}id^2)} e^{\pi d^2/2} e^{-2\xi/3 \xi(1-id^2)/2} (-i)^{-(2\nu+3)} (\frac{1}{2}i - \ell_1)$$

$$\times (-16)^{(1-id^2)/4} 2^{-(1+id^2)/2} e^{i\pi/4 + \pi d^2/4} q\xi^{-1/3}. \quad (9.2)$$

Since

$$\frac{1}{2}i + \ell_1 = -\frac{1}{2} \lim_{n \to \infty} \left\{ \frac{\exp\{-i(\frac{2}{3} |\xi_n| - \frac{1}{2}d^2 \log |\xi_n| + \gamma)\}}{\sin(\frac{2}{3} |\xi_n| - \frac{1}{2}d^2 \log |\xi_n| + \gamma)} \right\}$$

we see that (9.2) reduces to

$$\text{SM}_{-\infty} = \frac{2\sqrt{\pi}}{d\Gamma(-\frac{1}{2}id^2)} e^{-\pi i/4} e^{-i\gamma} 2^{-3id^2/2} e^{-\pi d^2/4}. \quad (9.3)$$

Since the Stokes multiplier must be independent of the $x$–direction, comparison of (8.1) and (9.3) gives (1.3) and proves Theorem B.
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