Supercurrent non-reciprocity and vortex formation in superconductor heterostructures

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Recent excitement in observation of non-reciprocal supercurrent (NRS) is motivated by a suggestion that “superconducting diode effect” may be an intrinsic property of non-centrosymmetric superconductors with strong spin-orbit interactions[1]. Theoretically it has been understood that linear-in-momentum energy terms, such as Rashba spin-orbit interaction or, more generally, any symmetry-allowed Lifshitz invariants[2], do not contribute to the supercurrent, yet the role of higher-order terms remains unclear[3, 4]. In this work we study non-reciprocity of critical current in nanowires fabricated from InAs/Al heterostructures. We show experimentally that the sign of NRS does not depend on the crystallographic axis, thus ruling out intrinsic NRS due to cubic-in-momentum terms. The overall shape of NRS and its multiple sign changes as a function of magnetic field are similar to NRS in superconducting loops, and point to the formation of circular currents. We present a model which shows that NRS is a generic property of two coupled dissimilar superconductors, and NRS sign change can be attributed to the formation of vortices. This extrinsic mechanism qualitatively explains ours and provides a compelling explanation to some previously published results.

Diodes are the most basic elements of semiconductor electronics and development of a superconducting counterpart can extend functionality of superconducting circuitry. Non-reciprocal supercurrent (NRS) in multiply-connected superconductors is a well known effect and can be readily observed in, e.g., asymmetric superconducting rings[5]. An implicit suggestion that NRS may be an intrinsic property of non-centrosymmetric superconductors[1] generated renewed theoretical and experimental interest motivated by an analogy with non-reciprocal resistivity due to the magnetochiral effect, which can appear in uniform materials with broken spatial and time-reversal symmetry[6]. However, a direct analogy between corrections to resistivity and superconducting current is misleading because the anisotropy of scattering is caused by the spin-orbit effects, while the proposed origin of non-reciprocity in singlet-pairing superconductors is a spin-independent Lifshitz invariant[7, 8]. It has been already demonstrated in the literature[2] that in uniform superconductors Lifshitz invariants can be eliminated by a gauge transformation from both the Ginzburg-Landau (GL) equation and the expression for the supercurrent. Thus, they cannot lead to non-reciprocal contributions to the critical current or a kinetic inductance.
Phenomenological treatment of cubic in momentum terms indicates an appearance of NRS corrections\[3\]. Microscopic calculations and symmetry analysis of GL functional and equations in the presence of terms due to the cubic Dresselhaus spin-orbit interactions and Zeemann effect show that the sign of NRS should depend on the crystallographic orientation of the supercurrent flow\[9\]. Another suggested mechanism of NRS is the formation of non-uniform currents in superconducting multilayers \[10; 12\].

In this paper we restrict our discussion to NRS in nanowire heterostructures, where the critical current is limited by the de-pairing velocity of Cooper pairs\[13\]. The term “superconducting diode effect” has been also used to describe NRS in systems with fundamentally different origins of critical current. For example, critical current in thin films in out-of-plane magnetic fields\[14; 16\] or in superconductors in proximity to a ferromagnet\[17; 19\] is limited by flux pinning and NRS can be attributed to the ratchet effect from an asymmetric pinning potential\[20, 21\]. In Josephson junctions interplay between magnetic field and spin-orbit interactions results in a non-reciprocal Doppler shift of Cooper pair momentum \[22\] and NRS due to phase rigidity in contacts\[23–25\], see also Refs. \[26, 27\].

We studied switching currents $I_{sw}$ (a transition from superconducting to normal state) in nanowires fabricated from Al/InGaAs/InAs/InGaAs heterostructures\[28\], where Al induces superconductivity in high mobility InAs quantum well via proximity effect. The nanowires are formed by patterning a top Al layer, an AFM micrograph of a typical device is shown in the inset in Fig. 1. A typical current-voltage characteristic exhibits a sharp switching transition limited by the current resolution ($< 5 \text{ nA}$ for the fastest sweep rates used in our experiments). A histogram of switching currents $I_{sw}^\pm$ for positive (+) and negative (-) current sweeps is shown in Fig. 1(a) for 10,000 sweeps. Field dependence of average values $\langle I_{sw}^+ \rangle$ and $\langle I_{sw}^- \rangle$ is plotted in Fig. 1(b) for the in-plane field $B_{\perp}$ perpendicular to the wire. The $\langle I_{sw}^+ \rangle$ and $\langle I_{sw}^- \rangle$ can be separated into a symmetric $\langle I_{sw} \rangle = \langle (I_{sw}^+ + I_{sw}^-) \rangle / 2$ and asymmetric $\Delta I = \langle I_{sw}^+ \rangle - \langle I_{sw}^- \rangle$ parts, the latter being the non-reciprocal component of the supercurrent NRS. Both $\langle I_{sw} \rangle$ and $\Delta I$ are non-monotonic functions of magnetic field, however our data suggests that these features have different physical origins. As shown in the Supplement, a minima of $\langle I_{sw} \rangle$ at low fields vanishes above 350 mK ($0.3 \ T_C$), while there is no change in $\Delta I$ at least up to 750 mK $> 0.6 T_C$. Additionally, $\langle I_{sw} \rangle$ and $\Delta I$ have different dependencies on magnetic field direction: $\Delta I \propto \sin(\theta)$ while $\langle I_{sw} \rangle \propto \cos(2\theta)$. Some devices were fabricated with a top gate, which allows electrostatic control of electron density in the InAs not covered
FIG. 1. Non-reciprocal critical current in Al/InAs nanowires. (a) Histograms of switching currents for 10,000 positive $I_{sw}^+$ and negative $I_{sw}^-$ current sweeps performed at $T = 30$ mK and $B_\perp = 100$ mT. Inset shows a typical current-voltage characteristic. (b) Average switching current for positive $\langle I_{sw}^+ \rangle$ and negative $\langle I_{sw}^- \rangle$ sweeps, non-reciprocal difference $\Delta I = \langle I_{sw}^+ \rangle + \langle I_{sw}^- \rangle$ and an average of all sweeps $\langle I_{sw} \rangle$ is plotted as a function of in-plane magnetic field $B_\perp$. In (c) enlarged $\Delta I$ data is colored to signify non-monotonic field dependence and multiple sign changes. (d) Dependence of $\Delta I$ on in-plane field orientation is measured at a constant $B = 100$ mT. Blue line is a fit with a sine function. Insert shows an AFM image of a 3 $\mu$m-long wire connected to wide contacts, yellow areas are Al, in darker areas Al is removed and InAs is exposed.

by Al; we found that depletion of the 2D electron gas in the exposed InAs results in a slight increase of $\langle I_{sw} \rangle$ but does not affect $\Delta I$. Field-effect has been observed previously in superconductors nanodevices\cite{29} and was attributed to the presence of quasiparticles\cite{30}, a conclusion consistent with the observed gate dependence of the $\langle I_{sw} \rangle$.

Field dependence of non-reciprocal critical current $\Delta I$ is plotted in Fig. 1(c,d). At low
fields $\Delta I$ can be approximated as $\propto c[I \times B]$, where $c$ is a vector normal to the surface, deceptively suggesting an analogy to the magnetochiral correction to resistivity in materials with strong spin-orbit interactions. As was mentioned in the introduction, presence of linear Rashba spin-orbit coupling $\alpha_R \hat{c} [p \times \sigma]$, or any other linear spin-orbit energy terms allowed by symmetry (Lifshitz invariants), as well as linear spin-independent terms does not lead to non-reciprocal corrections to the critical current$[2]$. Non-monotonic field dependence and multiple sign changes of $\Delta I$ at high fields experimentally further exclude these mechanisms, since otherwise an intrinsic property of the material (vector $c$) would require to change sign as a function of $B$.

FIG. 2. Dependence of NRS on the nanowire length and crystallographic orientation. (a) NTS is plotted for two 2 $\mu$m-long wires oriented along [110] and [110] crystallographic axes. Insets define wires and fields orientation. (b) NRS for 2, 3, and 5 $\mu$m wires. The top and bottom curves are shifted vertically by 0.2 $\mu$A. Brackets with arrows indicate a maximum $\Delta B$ needed to insert a flux $\phi_0 = h/2e$ in the area defined by the corresponding wire lengths, as indicated by a dashed loop in the inset. An effective length for the period marked by a magenta bracket is $l = 0.5 \mu$m for the same loop.

Unlike linear terms, some higher order terms cannot be removed by gauge transformation and it was shown that presence of cubic $\alpha_3 Q^3 \Delta^2$ and linear $\beta_1 Q \Delta^4$ terms in a generic Taylor expansion of Ginsburg-Landau coefficients $\alpha(Q)$ and $\beta(Q)$ can lead to non-zero $\Delta I$ which is a non-monotonic function of $B$ and can even change sign$[3]$ (here $Q = -i\hbar \nabla - 2eA$
is a generalized momentum, $A$ is electromagnetic vector-potential, and the supercurrent $I$ includes terms linear and higher order in $Q$. Investigation of realistic cubic terms showed\cite{9} that the dominant contribution is due to Dresselhaus terms which are highly anisotropic. For comparison with experiments, it is instructive to express GL functional in coordinates 90 degrees rotated in respect to the primary crystallographic axes of InAs, where $\hat{x}|[1\bar{1}0]$ and $\hat{y}|[110]$, see insert in Fig. 2(a). In these coordinates Dresselhaus contribution to the kinetic term is given by

$$f_k = |\kappa(B_yQ_x^3 + B_xQ_y^3 - Q_xQ_yB_xQ_y + B_yQ_x)|\Delta^2,$$

(1)

where coefficient $\kappa$ contains the Dresselhaus constant $\beta_D$ and other material parameters. After spatial quantization terms $Q_iQ_j^2$ become linear in $Q$ and have no effect on supercurrent in much the same way as the Rashba term, and the resulting NRS correction to the supercurrent is

$$\Delta I \propto (B_yI_x^2 + B_xI_y^2).$$

(2)

This correction does not depend on the sign of $I$ and will be either added or subtracted to the $B = 0$ current depending on the direction of the current flow. In the above expression $B_x$ and $B_y$ enter symmetrically for wires oriented along $x$ and $y$. However, for the current $I||\hat{x}$ compared to $I||\hat{y}$ fields $B_x$ and $B_y$ have opposite signs for the same mutual orientation of $I$ and $B$, see inset in Fig. 2(a), and the Dresselhaus contribution is expected to result in NRS with opposite sign for wires oriented along [1\bar{1}0] and [110] crystallographic axis. Such crystallographic dependence is not observed in experiment, Fig. 2(a). Thus, we conclude that the observed NRS is not an intrinsic property of our non-centrosymmetric heterostructure.

Another possible mechanism for NRS is a magnetic field-dependent Doppler shift of Cooper pair momentum as they travel between two reservoirs\cite{22}. Such mechanism was proposed to explain Josephson current non-reciprocity, in this case $\Delta I \propto \sin(2ql)$, where $q$ is a field-dependent momentum shift and $l$ is the length of the junction\cite{24}. We studied length-dependence of $\Delta I$ for wires with $l = 2$, 3 and 5 $\mu$m, Fig. 2(b). The three curves are almost identical, ruling out Doppler shift as an origin of NRS in our devices.

While recent interest in NRS is motivated by a possibility of the intrinsic origin of the effect, NRS naturally arises in multiply-connected superconductors. In superconducting loops critical current is modulated by an external flux $\phi = BS_{\text{loop}}$ piercing the loop. In a loop with asymmetric arms current maximum is shifted from $B = 0$, the sign of the shift
FIG. 3. **NRS in an asymmetric superconducting loop.** (a) An average switching current for positive $\langle I_{sw}^+ \rangle$ and negative $\langle I_{sw}^- \rangle$ sweeps, non-reciprocal difference $\Delta I = \langle I_{sw}^- \rangle + \langle I_{sw}^+ \rangle$ and an average of all sweeps $\langle I_{sw} \rangle$ is plotted as a function of out-of-plane magnetic field $B_o$ for a loop shown in the insert in (b). Note that $\langle I_{sw} \rangle$ is maximum while $\Delta I = 0$ when the flux $\phi = n\phi_0$. In (b) $\Delta I$ for the nanowire and the loop are plotted together as a function of a reduced flux $\phi/\phi_0$, where we used $S_{wire} = 0.0052 \ \mu m^2$ for the effective area in the wire and $S_{loop} = 2.59 \ \mu m^2$ in the loop.

depends on the direction of the current as shown in Fig. 3(a). A non-reciprocal component of the switching current $\Delta I$ is linear in the vicinity of $B = 0$, reaches extrema at $\phi \approx \phi_0/4$, changes sign and oscillates with a period $\Delta \phi = \phi_0$. As such, asymmetric loop is the simplest “superconducting diode”. There is a clear similarity between $\Delta I$ measured in an asymmetric superconducting loop and in an Al/InAs nanowire as emphasized in Fig. 3(b), suggesting that NRS in our nanowires may be due to formation of circular currents.

InAs/InGaAs/Al heterostructure can be modeled as two one-dimensional (1D) superconducting wires separated by a tunneling barrier of width $d$, see Fig. 4(a). The total energy of the system can be written as a sum of kinetic and Josephson energies

$$E_{tot} = \int dx \left[ \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - E_J \cos(\phi_1 - \phi_2) \right],$$

where $L_i$ are kinetic inductances per unit length, supercurrents $I_i = (2eL_i)^{-1}(h\partial_x \phi_i - 2eA_x)$, $E_J$ is the Josephson coupling per unit length, and $I_1(x) + I_2(x) = I_{ext}$ sets current conservation and the external current. Actual current distribution between wires 1 and 2 can be

\[ \text{FIG. 3. NRS in an asymmetric superconducting loop. (a) An average switching current for positive } \langle I_{sw}^+ \rangle \text{ and negative } \langle I_{sw}^- \rangle \text{ sweeps, non-reciprocal difference } \Delta I = \langle I_{sw}^- \rangle + \langle I_{sw}^+ \rangle \text{ and an average of all sweeps } \langle I_{sw} \rangle \text{ is plotted as a function of out-of-plane magnetic field } B_o \text{ for a loop shown in the insert in (b). Note that } \langle I_{sw} \rangle \text{ is maximum while } \Delta I = 0 \text{ when the flux } \phi = n\phi_0. \text{ In (b) } \Delta I \text{ for the nanowire and the loop are plotted together as a function of a reduced flux } \phi/\phi_0, \text{ where we used } S_{wire} = 0.0052 \ \mu m^2 \text{ for the effective area in the wire and } S_{loop} = 2.59 \ \mu m^2 \text{ in the loop.} \]

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FIG. 4. Two-wire model and vortex formation. (a) Two one-dimensional superconducting wires separated by a tunneling barrier are coupled with a Josephson coupling $E_J$. (b) Depending on the relative strength of kinetic $E_K$ and Josephson energies phases in two wires can be locked ($\phi_1 = \phi_2$), vary independently or undergo a phase slip. (c) Circular currents due to the vortex formation (in units of $2\pi E_J/\phi_0$). Dashed rectangle outlines an effective vortex area $S_v$. (d) NRS $\Delta I$ versus the flux $\Phi = S_v B_\perp$ is plotted for several $B_\parallel$, Eq. (S12).

found by minimizing $E_{\text{tot}}$ with respect to the phases $\phi_{1,2}$. The details of the theory are presented in the Supplementary Materials, here we highlight the main results. In the case of two independent non-coupled wires ($E_J = 0$) $\phi_1, \phi_2 \propto x$, as shown schematically by a blue line in Fig. 4(b). If the Josephson term dominates, $E_{\text{tot}}$ is minimized by locking phases in the two layers, $\Delta \phi = \phi_1 - \phi_2 = 0$ (red line in Fig. 4(b)). Magnetic field $B_\perp \parallel \hat{y}$ generates screening currents $\Delta I = B_\perp d/L_2$ which result in non-reciprocal critical current (here we assume that $L_2 \gg L_1$). Thus, a superconducting diode effect is a generic property of asymmetric multi-layer superconductors.

In the intermediate case of comparable kinetic and Josephson energies it becomes energetically favorable to create a Josephson vortex and keep phases locked in the rest of the wire (green line in Fig. 4(b)). Experimentally, kinetic energy can be increased by applying $B_\perp$. The critical field for the vortex formation $B_c = (3/\pi^2)\Phi_0/(l_J d)$ depends on the
Josephson length $l_J \approx \Phi_0/(2\pi \sqrt{2}E_J L_2)$. Distribution of circular currents in the vicinity of a vortex is plotted in Fig. 4(c); formation of a vortex makes $\Delta I$ a non-monotonic function of $B_\perp$ Fig. 4(d). A gradual change of $\Delta I$ near $\Phi_0/2$ is due to quantum fluctuations of the winding number due to strong coupling of the vortex to current-carrying wires. This smearing is similar to the gradual change of critical current in a ring connected to superconducting leads (Fig. 3), as compared to an abrupt reversal of persistent currents at $\Phi_0/2$ in isolated rings [31]. The period of $\Delta I$ oscillations corresponds to the flux threading an effective vortex area $S_v = (\pi^2/3)l_Jd = l_vd$. Experimentally, the period $\Delta B_\perp = 400 \text{ mT}$ translates into the length $l_v \approx 500 \text{ nm}$, where $\Delta \phi$ substantially deviates from zero. Since $l_v < \xi_{\text{InAs}} = \sqrt{\xi_{\text{InAs}}^0 l} \approx 750 \text{ nm}$, where $\xi_{\text{InAs}}^0 = \hbar v_F/\pi \Delta \approx 1.8 \mu\text{m}$ is the superconducting coherence length and $l \approx 300 \text{ nm}$ is the mean free path in InAs, we expect proximity-induced superconductivity in InAs to be preserved in the presence of a vortex.

Finally, we use the above two-wire model to estimate temperature and in-plane field $B_\parallel \parallel \hat{x}$ dependence of NRS assuming that both parameters affect superfluid density in InAs $n_2$. In the vicinity of $B_\perp = 0$ the amplitude of $\Delta I \propto L_z^{-1} \propto n_2$ and is expected to decrease with an increase of $T$ or $B_\parallel$. The critical field $B_c \propto \sqrt{E_J L_2}$ depends on $E_J \propto n_2$, and oscillation period is expected to be $T$ and $B_\parallel$ independent [Fig. 4(d)]. These qualitative estimates are consistent with experimental observations, see Figs. S2 and S3 in the Supplemental Material.

**METHODS**

**Materials.** The wafer was grown using Molecular Beam Epitaxy (MBE) on an InP substrate. The growth started with a 1 $\mu$m graded In$_x$Al$_{1-x}$As insulating buffer. Then a stack of In$_{0.75}$Ga$_{0.25}$As(4nm)/ InAs(7nm)/ In$_{0.75}$Ga$_{0.25}$As(4nm) was grown which defines the quantum well. A 7nm Al layer was deposited *in-situ* on top without breaking the chamber vacuum. The two-dimensional electron gas has a peak mobility of 28000 cm$^2$/Vs at a density $8 \times 10^{11}$ cm$^{-2}$.

**Sample Fabrication.** The nanowires were fabricated using standard electron lithography techniques. The mesas were defined by first removing the top Al layer with Al etchant Transene D and then a deep wet etching using a III-V chemical wet etchant $\text{H}_3\text{PO}_4:\text{H}_2\text{O}_2:\text{H}_2\text{O}:\text{C}_6\text{H}_8\text{O}_7$ (1ml:8ml:85ml:2g). Nanowires are defined in the second lithog-
raphy step by patterning the Al layer. Some devices have a top electrostatic gate, in this devices a 20 nm HfO$_2$ is grown by atomic layer deposition followed by a deposition of a Ti/Au (10/100 nm) gate.

**Measurements.** Current-voltage sweeps were performed using a homemade high speed high resolution DAC/ADC (digital-to-analog and analog-to-digital converter) setup, we used 100 kΩ resistor as a current source. The sweeps were automatically interrupted at the superconductor-normal transition ($I_{sw}$) in order to minimize device heating. Current sweep rate and delay between sweeps have been optimized to obtain < 5 nA current resolution and to keep device temperature < 50 mK at the base temperature of the fridge.

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**AUTHOR CONTRIBUTIONS**

L.P.R conceived, A.S. performed experiments, J.I.V. and Y.L.G developed the theory. All authors participated in writing the manuscript.

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Supercurrent non-reciprocity and vortex formation in superconductor heterostructures

Supplementary Materials

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COMPARISON OF SYMMETRIC AND ASYMMETRIC CONTRIBUTIONS TO THE CRITICAL CURRENT.

Switching current can be decomposed into symmetric $\langle I_{sw}^+ \rangle + \langle I_{sw}^- \rangle$ and asymmetric $\Delta I = \langle I_{sw}^- \rangle$ parts, their dependence on magnetic field is plotted in Fig. S1. Both $\langle I_{sw} \rangle$ and $\Delta I$ are non-monotonic functions of $B_\perp$, however they have very different $T$ and field angle $\theta$ dependencies. $\Delta I$ is almost unaffected by temperature up to $T \sim 0.8T_c$, while a dip around $B_\perp = 0$ in $\langle I_{sw} \rangle$ is developed at $T < 0.2T_c$. At constant $B = 100 \text{ mT}$ angle dependence $\Delta I \propto \sin(\theta)$ while $\langle I_{sw} \rangle/\langle I_{sw} \rangle_0 \propto \cos(2\theta)$. These differences in energy scales ($T$-dependence) and angular dependencies indicate that suppression of $\langle I_{sw} \rangle$ near $B = 0$ and asymmetric $\Delta I$ have different physical origins. Indeed, suppression of critical current near $B = 0$ has been attributed to the presence of quasiparticles and/or magnetic impurities [30, 32], different from geometrical effects responsible for $\Delta I(B)$ dependence.

FIG. S1. Field dependence of symmetric and asymmetric parts of the switching current. (a,b) Temperature dependence of $\Delta I$ and $\langle I_{sw} \rangle$ measured as a function of $B_\perp (\theta = 90$ deg). The curves are offset by -0.1 $\mu$A ($\Delta I$) and 0.01 ($\langle I_{sw} \rangle$). (c,d) Angle dependence is measured by rotating magnetic field of constant magnitude $B = 100$ mT at the base temperature.
FIG. S2. Effect of temperature on NRS. (a) Amplitude of NRS $[\Delta I(-100\text{mT}) - \Delta I(100\text{mT})]$, (b) standard deviation of the switching currents at $B_\perp = 0$, and (c) the average switching current at $B_\perp = 0$ are plotted as a function of reduced temperature. NRS amplitude follows T-dependence of superfluid density $n_s(T)$, consistent with Eq. (S13). $\langle I_{sw}\rangle(T)$ follows the Bardeen relation.$^\text{[13]}$. 

DEPENDENCE OF NRS ON TEMPERATURE.
DEPENDENCE OF NRS ON $B\parallel I$. 

Non-reciprocity of the switching current is linearly suppressed by in-plane magnetic field $B\parallel I$ and vanishes at $\approx 750$ mT. Within the same range of $B\parallel$ magnitude of the switching current remains almost constant (decreases < 2.5% at $B\parallel = 750$ mT).

FIG. S3. Effect of in-plane current $B\parallel I$ on non-reciprocal supercurrent. (a) Evolution of $\Delta I$ in the presence of $B\parallel$. The plots are vertically shifted for clarity (b) The NRS amplitude falls approximately linearly with $|B\parallel|$. (c) Dependence of average switching current $\langle I_{sw} \rangle$ at $B_{\perp} = 0$ on $B\parallel$. All data is taken at the base temperature.
NRS IN NANOWIRES OF VARIOUS WIDTH

We studied NRS in multiple nanowires of different width and length. Since all devices were fabricated from similar wafers Josephson coupling and, thus, $l_J$ are similar for all devices and we expect the amplitude of $\Delta I$ and period $\Delta B$ to be similar. Indeed, that is the case for most devices, see Fig. S4a. One nanowire showed $\approx 2 \times$ enhancement of $\Delta I$ and $\approx 2 \times$ reduction of $\Delta B$, which would be consistent with a local enhancement of $l_J$ by a factor of 2.

![Graphs showing NRS in nanowires of different width and multiple sign reversal in another nanowire.](image)

**FIG. S4. NRS in other devices.** (a) NRS in nanowires of different width. The plots are shifted vertically for clarity. (b) Multiple sign reversal of $\Delta I$ in another nanowire. (c) One out of $\sim 20$ nanowires fabricated from similar wafers showed enhanced magnitude of $\Delta I$ and reduced $\Delta B$. 

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GATE DEPENDENCE OF NRS AND CRITICAL CURRENT.

On one of the samples measured, we fabricated HfO\textsubscript{x} dielectric and deposited Ti/Au acting as a gate electrode. We see no observable effect on the NRC on varying the gate voltage. The InAs is expected to be fully depleted with a gate voltage of -1.5V. We measured NRC at different gate voltages varying from 0 to -4.5V and observe no variation of $\alpha$ or $\Delta B$. Slight (up to 0.26%) increase of $\langle I_{sw} \rangle$ at large negative gate voltages is consistent with the reduction of the number of quasiparticles when InAs is depleted.

![Diagram](image.png)

**FIG. S5. Effect of gate on NRS.** (a) NRS shows no observable difference on varying the gate voltage. (b) $\alpha$ shows very little variation on gate voltage. (c) When a negative gate voltage is applied the $\langle I_{sw} \rangle$ increases.
FIG. S6. **No NRS in a control device.** A control 150 nm wide and 3 µm long nanowire is fabricated from a 20 nm thick Al film deposited on a semi-insulating Si wafer. This device shows no NRS.
THEORY: GEOMETRIC EFFECTS AND $I_c$ NON-RECIPROCITY IN COUPLED SUPERCONDUCTING WIRES

In this section we derive critical current in two Josephson-coupled superconducting wires ("two-wire model") in the presence of external magnetic field. We show that at high enough magnetic field, it becomes energetically favorable to form a Josephson vortex (Fig. S7a), which in turn can lead to an oscillatory non-reciprocity of the critical current (Fig. 4 of the main text). Furthermore, the oscillations will be damped due to one of the wires turning normal upon increasing the magnetic field.

Let us consider a pair of parallel superconducting wires 1 (Al wire) and 2 (proximitized InAs) along the $x$-direction with a magnetic field $B_\perp$ in the $y$-direction, normal to the

![Diagram](image)

FIG. S7. (a) Schematic picture of the model to explain non-reciprocity. The dark grey regions depict two superconducting wires labeled 1 and 2 (corresponding to Al and proximitized InAs, respectively) with order parameter phases $\phi_1$, $\phi_2$. The region between the wires denotes the insulating barrier of thickness $d$. In most positions $x$, the phases are locked to $\phi_1 = \phi_2 (\text{mod } 2\pi)$ due to a strong Josephson coupling. In the region of length $l_v$ spanned by the Josephson vortex the phases are not equal and as a result the phase difference winds an additional $2\pi n$ over the vortex. The vertical arrows denote the resulting Josephson currents flowing between the two wires in the vortex. (b) Total energy vs magnetic field in the two-wire model. The dashed curves show the spectrum obtained from Eq. [S7]. The three parabolas correspond to Josephson vortices/antivortices with $n = -1, 0, 1$. The solid curves show the energies when coherent vortex tunneling (strength $E_t = 0.4$ in units of $\frac{\eta}{1+\eta} \frac{\mu_0^2}{e L_1 l_J}$) is included, leading to avoided crossings of states with different $n$. 

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plane of the wires. The corresponding vector potential is \( A_x(z) = B_\perp z \) with the two wires separated by distance \( d \) being at positions \( z = z_{1,2} = -(-1)^{1,2}d/2 \), see Fig. S7a. We ignore here the screening of the magnetic field by the Josephson vortex; this effect would merely modify the Josephson length \( l_J \) (introduced below). We also approximate the wires as one-dimensional, given that their typical thickness is smaller than the penetration depth and the width is smaller that the size of the Peal vortex. This makes the supercurrent distribution approximately uniform within the wire. Denoting \( \phi_{1,2} \) the phases of the superconducting order parameters, we have a supercurrent in wire \( i \) given by

\[
I_i = \frac{1}{2eL_i} \left( \hbar \partial_x \phi_i - 2eA_x(x, z_i) \right),
\]

in terms of the kinetic inductances (per length) \( L_i = m_i/(e^2S_in_i) \) for wires \( i = 1, 2 \). Here \( S_i \), \( m_i \), and \( n_i \) denote the cross sectional area, effective mass, and the superfluid densities. For Al wire \( (i = 1) \) we will account for disorder by multiplying the superfluid density by \( \sqrt{l/\xi} \), where \( l \approx 2\text{nm} \) is the mean free path and \( \xi \approx 1\mu\text{m} \) is the coherence length [13]. Thus, we use \( L_1 \to L_1\sqrt{\xi/l} \) in our final estimates.

The phases \( \phi_{1,2}(x) \) can be found by minimizing the total energy

\[
E_{\text{tot}} = \int dx \left[ \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 - \mathcal{E}_J \cos(\phi_1 - \phi_2) \right],
\]

consisting of kinetic energies of each wire and a Josephson energy density \( \mathcal{E}_J \) coupling the two wires. With an applied external supercurrent \( I_{\text{ext}} \), we also have the constraint \( I_1(x) + I_2(x) = I_{\text{ext}} \) at every point \( x \). The constrained energy minimization leads to a Sine-Gordon equation for \( \varphi = \phi_1 - \phi_2 \),

\[
\frac{\partial^2 \varphi}{\partial x^2} = l_J^{-2} \sin \varphi,
\]

where \( l_J = 1/\sqrt{8e^2\mathcal{E}_J(L_1 + L_2)/\hbar^2} \) is the Josephson length that determines the characteristic size of a Josephson vortex. Next, we solve Eq. (S3) with the appropriate boundary conditions. We assume that the Josephson coupling in Eq. (S2) is strong, such that \( \phi_1 = \phi_2 (\text{mod} \ 2\pi) \) for most \( x \). If the two phases were locked for all \( x \), i.e. \( \varphi(x) = 0 (\text{mod} \ 2\pi) \), we would find a non-reciprocal critical current \( I_c(B_\perp) \) with the non-reciprocity \( \Delta I = I_{c,+} - I_{c,-} \) that increases monotonically with \( B_\perp \). Experimentally, a non-monotonic dependence is observed, see Fig. 1b.

The non-monotonic \( \Delta I \) can be explained by a formation of a Josephson vortex, see Fig. 4. In the Josephson vortex, the phase difference \( \varphi \) increases by \( 2\pi \) over approximately the distance \( 2\pi l_J \); explicitly, \( \varphi(x) = 4\arctan e^{x/l_J} \) for a vortex at \( x = 0 \).
The Josephson vortex solution yields a current distribution

\[ I_1(x) = \frac{1}{1 + \eta I_{\text{ext}}} + \delta I_n(x), \quad (S4) \]

\[ I_2(x) = \frac{\eta}{1 + \eta} I_{\text{ext}} - \delta I_n(x), \quad (S5) \]

\[ \delta I_n(x) = \frac{2\eta}{1 + \eta} \frac{1}{2} \frac{\hbar}{eL_1 l_J} \left( n \tanh \frac{x}{l_J} - \frac{3}{\pi} \frac{\Phi}{\Phi_0} \right), \quad (S6) \]

where we introduced the integer \( n = \pm 1 \) for Josephson vortex/antivortex and \( n = 0 \) in the absence of a vortex. The vortex is centered at \( x = 0 \) which is also the position of the maximal circulating currents in wires 1 and 2. We also denote \( \eta = L_1/L_2 = S_2 \frac{m_2}{m_1} / S_1 \frac{m_1}{m_1} \) and introduced the flux \( \Phi/\Phi_0 = S_v B_\perp / (\pi \hbar/e) \) through the effective vortex area \( S_v = (\pi^2/3) l_J d \).

The formation of a Josephson vortex becomes energetically favorable at a large enough magnetic field \( B_\perp \). The energy cost is determined from Eq. (S2) by the balance of the Josephson energy \( E_J \) lost and the kinetic energy gained in the creation of a vortex. Ignoring \( n \)-independent terms, we find (see Fig. S7b),

\[ E_{\text{Vortex}}(n) = \frac{\eta}{1 + \eta} \frac{\hbar^2}{2e^2 L_1 l_J} \left[ \left( n - \frac{3}{2} \frac{\Phi}{\Phi_0} \right)^2 + \frac{1}{2} |n| \right], \quad (S7) \]

where \( n = 0, \pm 1 \). This energy is analogous to the (inductive) energy of a superconducting ring with a phase winding \( 2\pi n \) [33] apart from the last term in Eq. (S7) which is the cost in Josephson energy. In the absence of quantum fluctuations and at zero temperature, one finds from Eq. (S7) that the thermal average \( \langle n \rangle = [\Phi/\Phi_0] \) is given by the nearest integer to \( \Phi/\Phi_0 \), and one would find a sawtooth-like dependence for \( \Delta I \) versus \( B_\perp \) (see below). Fluctuations will smear out the sawtooth. In analogy to a superconducting ring [33], in the case of strong quantum or thermal fluctuations we expect to find a harmonic dependence on the flux on a linear background,

\[ \langle n \rangle = \frac{\Phi}{\Phi_0} - \delta n \sin \frac{2\pi \Phi}{\Phi_0}, \quad (S8) \]

where \( \delta n \ll 1 \) due to strong fluctuations. Importantly, in the case of quantum fluctuations, \( \delta n \) is independent of the temperature, whereas for thermal fluctuations one has exponential dependence on \( 1/T \). As we discuss below, the harmonic dependence on the flux translates to a similar dependence in the non-reciprocal part \( \Delta I \) of the critical current, in agreement with experimental data. The observed weak \( T \)-dependence in Fig. S1a indicates that quantum fluctuations exceed thermal fluctuations in the experiment.
The critical current through the ring is determined by the condition that at large enough $I_{ext}$, one of the wires (or arms of the ring) turns normal. (Experiment indicates that the switching happens in Al, i.e., wire 1, see below.) Assuming that $I_{ext} > 0$ and wire 1 turns normal, the condition for this is that $I_{ext} = I_{c,+}$, where

$$I_{c,±} = (1 + \eta)(±I_{1,c} - \delta I), \quad (S9)$$

where $I_{1,c}$ is the critical current of wire 1 and $\delta I = \langle \delta I_n(0) \rangle$ is the circulating current at its peak value at $x = 0$. Likewise, for $I_{ext} < 0$ we find $I_{ext} = I_{c,-}$. This yields

$$\Delta I = I_{c,+} + I_{c,-} \quad (S10)$$

$$= -2\eta \frac{1}{L_e} \frac{\hbar}{L_{1,1}} \left( \langle n \rangle - \frac{3}{\pi} \frac{\Phi}{\Phi_0} \right), \quad (S11)$$

which determines the slope $\alpha = d\Delta I/dB_\perp$. We note that if wire 2 is normal, $n_2 = 0$, we have $\eta = 0$ and $\Delta I$ vanishes. The $B_\perp$-dependence of the critical current $I_{c,±}$ in that case would merely show a monotonic decrease (without non-reciprocity) as the superconducting gap is suppressed in Al. Experiments show a few distinct oscillations in the non-reciprocity $\Delta I$ of the critical current, see Figs. 1c and S3. We attribute the experimentally observed vanishing amplitude of $\Delta I$ (loss of non-reciprocity) at fields higher than $B_\perp \approx 750 \text{mT}$ to the destruction of proximity effect. We can model this by taking $\eta$ in Eq. (S11) to be magnetic field-dependent, detailed below.

**FIG. S8.** Left: The critical current non-reciprocity $\Delta I$, Eq. (S12), versus the flux $\Phi = B_\perp dl_\psi$ through the Josephson vortex. The crosses denote the approximation, Eq. (S13). The applied field $B_\parallel$ suppresses proximity effect and thus $\Delta I$. In the figure $E_t = 0.4$ in units of $\frac{\eta}{1+\eta} \frac{\hbar^2}{e^2 L_1 l_J}$. Right: $\Delta I$ (at $B_\parallel = 0$) for different strengths $E_t$ (in units of $\frac{\eta}{1+\eta} \frac{\hbar^2}{e^2 L_1 l_J}$) of coherent vortex tunneling that controls vortex number fluctuations; weak tunneling leads to a sawtooth-like $\Delta I$. 

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Proximity effect is also destroyed by an in-plane field $B_{\parallel}$ along the wire (along $x$) at roughly the same 750mT scale, see Fig. S2a. Since in our model wire 2 corresponds to the proximitized system, we will include a field-suppression of the superfluid density $n_2$ at lower fields than wire 1 (Al) gap suppression. We model this by taking a linear suppression for $n_2$ (or $1/L_2$) which translates to $\eta = \eta_0 (1 - |B|/B_{\text{InAs,c}})$ in Eq. S11. We take $B_{\text{InAs,c}} \approx 750$ mT and denote here $|B| = \sqrt{B_{\perp}^2 + B_{\parallel}^2}$ assuming that the suppression of proximity is isotropic (at least in-plane). The linear suppression is taken to match with experimental observations. In particular, linear field-dependence is seen in Fig. S3b where the slope $\alpha$ is plotted as a function of $B_{\parallel}$. The measurement shows also that the switching current does not differ much from its $B_{\perp} = 0$ value (see Fig. S2c), which indicates that the critical current is determined by wire 1, as we assumed in Eq. (S9).

We thus obtain the following expression for the non-reciprocity contribution to the critical current, plotted in Fig. 4 and Fig. S8,

$$\Delta I(B_{\perp}, B_{\parallel}) = -2\eta_0 \frac{1}{L_1 e l_J} \left( \langle n \rangle - \frac{3}{\pi} \frac{\Phi}{\Phi_0} \right) \left( 1 - \frac{|B|}{B_{\text{InAs,c}}} \right),$$

(S12)

$$\approx -2\eta_0 \frac{1}{L_1 e l_J} \left( c \frac{\Phi}{\Phi_0} - \delta n \sin \frac{2\pi \Phi}{\Phi_0} \right) \left( 1 - \frac{|B|}{B_{\text{InAs,c}}} \right),$$

(S13)

where in the last line $c = (1 - \frac{3}{\pi}) \approx 0.05$ and we assumed strong quantum fluctuations of $n$, see discussion below Eq. (S8). The approximate period is $B_{\perp} = \Phi_0 (3/\pi^2)/(l_J d)$, experimentally observed to be approximately 400 mT. This period indicates 500 nm for the effective size of the vortex, given that $d = 10$ nm.

From Eq. (S13) we obtain a zero-field slope $d\Delta I/dB_{\perp} \approx -c_0 \eta_0 d/L_1$ where $c_0 = (2\pi/3) (c - \delta n 2\pi)$ is in principle an unknown numerical coefficient (since $\delta n$ is unknown). However, the dimensionless number $\delta n_q \ll 1$ characterizes the amplitude of the persistent current in the loop (Fig. 4) and is suppressed due to quantum phase slips [33]. We can therefore take $c_0 \approx 2\pi c/3 \approx 0.1$. Using values $S_1 = 150$ nm $\times$ 10 nm, $n_1 = 18 \cdot 10^{28}$ m$^{-3}$ and $m_1 = 9.1 \cdot 10^{-31}$ kg (Al electron density and effective mass), we obtain $d/L_1 = \sqrt{1/|S_1| e^2 m_1 d / \pi} \approx 3.4$ mA/T. By comparing to the zero-field slope $d\Delta I/dB_{\perp} \approx 1.6$ $\mu$A/T in Fig. S2b, we obtain $\eta_0 \approx 10^{-2}$. This is consistent with an estimate $\eta_0 \approx 10^{-2}$ based on the ratio of Al and InAs kinetic inductances. Finally, we note that non-reciprocity component is proportional to superfluid density, $\Delta I \propto L_2^{-1} \propto n_2$, which is consistent with the temperature-dependence of both quantities plotted in Fig. S2a.
Finally, we note that asymmetry between wires 1 and 2 is essential to get non-reciprocity in our model. If the wires were identical, the wire that switches to normal state first [in Eq. (S9)] would change upon reversing the current direction. We also note that a non-reciprocity is obtained even if there is no clear loop ($I_J \rightarrow 0$), in which case there is no phase winding, $n = 0$ from Eq. (S7), but there is nevertheless a circulating current $\delta I_0$, Eq. (S6), and non-reciprocity, Eq. (S12), due to the assumed Josephson coupling induced phase locking $\phi_1 = \phi_2$ between the wires.