A description of a space-and-time neighborhood of generic singularities formed by mean curvature flow

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Abstract

We consider one of the generic regimes of formation of singularities, and we obtain a detailed description of a possibly small, but fixed, neighborhood of the blowup point, up to (and including) the blowup time, and find that it is mean convex in the considered set. This confirms a conjecture by Ilmanen.

Estimates up to any order of derivatives are provided. They are small up to a proper rescaling, if sufficiently close to the blowup time and the blowup point. And we find that the singularity is isolated from the other ones.

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1 Introduction

Here we study mean curvature flow (MCF) for a \(n\)-dimensional hypersurface embedded in \(\mathbb{R}^{n+1}\):

\[
\partial_t X_t = -h(X_t),
\]

where \(X_t\) is the immersion at time \(t\), \(h(X_t)\) is the mean curvature vector at the point \(X_t\).

We are mainly interested in the generic blowups. The objective is to find a detailed description of a (possibly small but fixed) neighborhood of the singularity in both space and time variables, in preparation for studying the flow through singularity, either by level set approach or surgery. These problems will be addressed in subsequent papers.

It is proved by Colding and Minicozzi in [7] that if the singularity is generic, and suppose that the singularity is at time \(T\) and at 0, then under the scaling \(X_t \rightarrow \frac{1}{\sqrt{T-t}} X_t\), the manifold will converge to a unique cylinder \(\mathbb{R}^{n-k} \times S^k \sqrt{2k}\), \(k = 1, 2, \ldots, n-1\), or \(S^{n-1} \sqrt{2(n-1)}\), here \(S^k \sqrt{2k}\) is \(k\)-dimensional torus with radius \(\sqrt{2k}\).

Here we choose to study the regime where the limit cylinder is \(\mathbb{R}^3 \times S^1 \sqrt{2}\), even though we expect that the techniques work equally well for all the generic blowups. One motivation is that such singularities are not understood as well as \(\mathbb{R} \times S^{n-1} \sqrt{2(n-1)}\). See the results in [20, 11, 17, 18]. For the other related works, see [16, 11, 22].

Next we present some technical details.

Suppose that the blowup point is the origin and the time is \(T\), and the limit cylinder is \(\mathbb{R}^3 \times S^1 \sqrt{2}\), then for \(t\) sufficiently close to the blowup time \(T\), a neighborhood of the blowup point can be parametrized by some positive function \(u\) as

\[
\Psi(z, t) = \begin{bmatrix} z \\ u(z, \theta, t, \cos \theta) \\ u(z, \theta, t, \sin \theta) \end{bmatrix}, \quad z \in \mathbb{R}^3, \quad |z| \leq c(t) \text{ for some } c(t) > 0,
\]

(1.2)

where \(u\) is periodic in \(\theta\).

We study the problem in two steps. In the first step we consider the rescaled MCF, namely \(X_t \rightarrow \frac{1}{\sqrt{T-t}} X_t\), and define a new function \(v\) by

\[
u(z, \theta, t) = \sqrt{T-t} \nu(y, \theta, \tau)
\]

(1.3)
here $y$ and $\tau$ are the rescaled space and time variables defined as
\begin{equation}
y := \frac{z}{\sqrt{T - t}}, \quad \tau := -\ln(T - t).
\end{equation}
Then the part parametrized in (1.2) becomes
\begin{equation}
\begin{bmatrix}
y \\
v(y, \theta, \tau) \cos \theta \\
v(y, \theta, \tau) \sin \theta
\end{bmatrix}.
\end{equation}

In the present paper we manage to prove that for $|y| \leq 3\tau_1^{1/5} + \frac{1}{20}$, the rescaled flow is of the form (1.5), and moreover the dominant part of $v$ is $\sqrt{2 + \tau^{-1}y^T \tilde{B} y}$ in the sense that
\begin{equation}
|v - \sqrt{2 + \tau^{-1}y^T \tilde{B} y}| \leq \tau^{-\frac{1}{10}},
\end{equation}
where $\tilde{B}$ is a $3 \times 3$ diagonal matrix
\begin{equation}
\tilde{B} = \text{diag}[b_1, b_2, b_3], \quad \text{with } b_k = 0 \text{ or } 1.
\end{equation}
And we have estimates on the derivatives if $v$ in this region, see Theorem 2.1 below.

In the second step we consider the regime where
\begin{equation}
b_1 = b_2 = b_3 = 1,
\end{equation}
and then consider MCF from some fixed time $t_1 = t(\tau_1)$, with $\tau_1$ sufficiently large and hence $T - t_1$ sufficiently small, with initial condition provided by $v(\cdot, \tau_1)$ which is a rescaled version of $u(\cdot, t_1)$. Our estimates of $v$ in time $\tau \leq \tau_1$ provide the “past history” of the flow. It is crucially important since the regularity estimates, including local smooth extension, see [9], become applicable.

More importantly, we observe that
\begin{equation}
\sqrt{2 + \tau_1^{-1}y^T \tilde{B} y}|_{|y|=2\tau_1^{1/5} + \frac{1}{20}} \gtrsim \sqrt{2 + \tau_1^{-1}y^T \tilde{B} y}|_{|y|=0}.
\end{equation}
This, together with the other estimates on the function $v$, and the regularity estimates, enable us find that when the singularity forms at $z = 0$ and $t = T$, the function $u$ will stay smooth and positive in a neighborhood of $z$ corresponding to $y$ with $|y| = 2\tau_1^{1/5} + \frac{1}{20}$. Moreover we argue that, effected by rescaling, every sufficiently small $z \neq 0$ will have its moment $\tau$ of becoming $y$ with $|y| = 2\tau^{1/5} + \frac{1}{20}$ and then the arguments above become applicable. For the detailed description we refer to Theorem 2.2 below.
For the regimes where at least one of $b_k$ in (1.8) is zero, we will address them in subsequent papers.

On the technical level, the techniques used in the present paper is different from that used in [20, 17, 18]. For example, the so-called PDE-ODE techniques do not play any role here. In the present paper and the preceding ones, we try to demonstrate that the weighted $L^\infty$—norms, defined as,

$$\langle y \rangle^k L^\infty := \{ f \|\langle y \rangle^{-k} f \|_{\infty} < \infty \}, \quad k \geq 1.$$  \hspace{1cm} (1.9)

perhaps are the “right norms” to consider the blowup of MCF.

The present paper depends on [12], where, among other results, we proved that the $3 \times 3$ symmetric matrix $B(\tau)$, in (2.5) below, is close to being semi-positive definite. This allows us to consider a large region of the rescaled flow. Some of the key techniques were devised in [14, 13, 15], see also [4]. In the present paper and the earlier one [12] we simplify the arguments significantly, with the help of regularity estimates in [9] and some techniques learned from [7].

The paper is organized as the following: In Section 2 we state two Main Theorems 2.1 and 2.2. The Theorem 2.1 is reformulated in Section 3. And the results there will be proved in Sections 4 and 5. The two Main Theorems will be proved in Sections 6 and 7 respectively. In Sections 8, 9, 10 we prove some technical results.

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2 Main Theorem

As defined in [6], $\lambda(\Sigma)$ is the supremum of the Gaussian surface areas of hypersurface $\Sigma$ over all centers and scales. Under the condition that there exists a constant $\lambda_0 > 0$ such that for any $\tau > 0$,

$$\lambda(\Sigma_\tau) \leq \lambda_0,$$ \hspace{1cm} (2.1)

it was proved in [6, 5] the only possible singularities are cylinders. Then it was proved in [7] the limit cylinder is unique. Here $\Sigma_\tau$ is the rescaled hypersurface at time $\tau$. 
We suppose that the blowup point is the origin and the blowup time is \( T > 0 \), and suppose that the limit cylinder is \( \mathbb{R}^3 \times S^1_{\sqrt{T}} \), parameterized by,
\[
\begin{bmatrix}
y \\
\sqrt{2} \cos \theta \\
\sqrt{2} \sin \theta
\end{bmatrix}, \quad y := (y_1, \ y_2, \ y_3)^T \in \mathbb{R}^3.
\]
(2.2)

In the present paper we consider the region, for \( \tau \geq \xi_0 \), with \( \xi_0 \) being sufficiently large,
\[
|y| \leq \Omega(\tau)
\]
with \( \Omega \) defined as
\[
\Omega(\tau) := \sqrt{100 \ln \tau + 9(\tau - \xi_0)^{\frac{11}{10}}}
\]
(2.3)

The result is the following:

**Theorem 2.1.** Suppose the condition (2.1) holds, the blowup point is the origin and the blowup time is \( T \), and the limit cylinder is the one parametrized by (2.2).

Then there exists a large time \( \xi_0 \) such that for \( \tau \geq \xi_0 \) and for \( |y| \leq \Omega(\tau) \), the rescaled MCF is parametrized by
\[
\sqrt{T-t} \begin{bmatrix}
y \\
v(y, \theta, \tau) \cos \theta \\
v(y, \theta, \tau) \sin \theta
\end{bmatrix},
\]
(2.4)

with \( v, \ y, \ \tau \) defined in (1.5), and, up to a rotation \( y \rightarrow Ry \), one has that, for \( |y| \leq \Omega(\tau) \),
\[
v(y, \theta, \tau) = \sqrt{\frac{2 + y^T B(\tau) y}{2a(\tau)}} + \eta(y, \theta, \tau),
\]
(2.5)

where, for some \( C > 0 \), the parameter \( a \) satisfies the estimate
\[
|a(\tau) - \frac{1}{2}| \leq C\tau^{-1},
\]
(2.6)

and the symmetric \( 3 \times 3 \) matrix \( B \) satisfies the estimates
\[
B(\tau) = \tau^{-1} \begin{bmatrix}
b_1 & 0 & 0 \\
0 & b_2 & 0 \\
0 & 0 & b_3
\end{bmatrix} + \mathcal{O}(\tau^{-2}), \text{ with } b_k = 0 \text{ or } 1, \ k = 1, 2, 3,
\]
(2.7)
and $\eta$ satisfies the estimates
\[
\left\| e^{-\frac{1}{2}|y|^2} 1_\Omega \eta(\cdot, \tau) \right\|_2 \leq C\tau^{-2},
\]
and
\[
\begin{align*}
\left\| \langle y \rangle^{-3} 1_\Omega \partial^l_\theta \eta(\cdot, \tau) \right\|_\infty & \leq C(\tau^{-2} + \Omega^{-4}), \quad l = 0, 1, 2, \\
\left\| \langle y \rangle^{-2} 1_\Omega \partial^l_y \nabla \eta(\cdot, \tau) \right\|_\infty & \leq C\Omega^{-3}, \quad l = 0, 1, \\
\left\| \langle y \rangle^{-1} 1_\Omega \nabla^k \eta(\cdot, \tau) \right\|_\infty & \leq C\Omega^{-2}, \quad |k| = 2.
\end{align*}
\] (2.8)
\]

The theorem will be reformulated in Section 3 and proved in Section 6.

Here $1_{\leq \Omega}$ is the Heaviside function defined as
\[
1_{\leq \Omega}(y) = \begin{cases} 
1 & \text{if } |y| \leq \Omega, \\
0 & \text{otherwise.}
\end{cases}
\] (2.10)

Next we discuss the original MCF.

**Theorem 2.2.** Suppose that in (3.8) of Theorem 3.1 we have
\[
b_1 = b_2 = b_3 = 1.
\] (2.11)

Then there exist constants $\epsilon_1, \epsilon_2 > 0$, such that when $0 \leq T - t \leq \epsilon_1$ and $|z| \leq \epsilon_2$, the manifold is parameterized as in (1.2), and $u$ is continuous in all the variables.

And in the same space and time intervals, except at $(z, t) = (0, T)$ where $u(0, \theta, T) = 0$, the following two statements hold:

(A) the function $u$ is positive, smooth in all variables, and is strictly decreasing in $t$,

(B) for any fixed $N$, there exists some positive constant $\delta(|z|, t)$ satisfying
\[
\lim_{|z| \to 0, \quad t \to T} \delta(|z|, t) = 0,
\]

such that for $|m| + n = 1, 2, \ldots, N$,
\[
u^{|m|-1} |\partial^m_\theta \nabla^m u(z, \theta, t)| \leq \delta(|z|, t),
\] (2.12)

and the normal direction $n(z, \theta, t)$ satisfies
\[
|n(z, \theta, t) - (0, 0, 0, -\sin \theta, \cos \theta)^T| \leq \delta(|z|, t).
\] (2.13)

And in the considered set the manifold is mean convex.
The theorem will be proved in Section 7. The fact that the manifold is mean convex is directly implied by the estimates in (2.12) and (2.13), as shown below. For the importance of mean convexity (or having positive mean curvature), namely non-fattening, we refer to the results in [19].

We expect the estimates in (2.12) and (2.15) will help to study flow singularities: observe that if we rescale the flow such that $u(z, \theta, t)$ is rescaled into

$$u(z, \theta, t) = \lambda u_1(\lambda^{-1}z, \theta, \lambda^{-2}t),$$

then we have that, since these functions in (2.12) are “rescaling invariant”,

$$u_1^{m-1} |\partial_\theta \nabla_x u_1(z, \theta, t)| = u^{m-1} |\partial_\theta \nabla_x u(x, \theta, s)|_{x=\lambda^{-1}z, s=\lambda^{-2}t} \leq \delta(\lambda^{-1}|z|, \lambda^{-2}t).$$

Moreover if we choose $\lambda$ to make $u_1(\theta_0, t_0)$ = 1 for some $z_0$, $\theta_0$ and $t_0$, then by the estimates

$$|\nabla z u_1|, u_1^{-1} |\partial_\theta u_1| \leq \delta(\lambda^{-1}|z|, \lambda^{-2}t_0),$$

provided in (2.15), we find that an increasingly large neighborhood of $(z_0, \theta_0)$ will become increasingly resemble to a cylinder with radius 1. This together with the other estimates in (2.12) establishes uniform smoothness. Moreover the manifold is mean convex, i.e. has positive mean curvature.

The flow through singularities will be addressed in subsequent papers.

3 Reformulation of Theorem 2.1

We will prove Theorem 2.1 by bootstrap arguments. Specifically, under assumption of some regularity estimates, specifically (3.4) below, in a time interval, we prove Theorem 3.1 below. The estimates in Theorem 3.1, in turn, will make Lemma 3.2 applicable, and will make (3.4), and hence Theorem 3.1, hold in a larger interval.

To initiate the bootstrap arguments, we need the estimates from the previous paper [12]. For the details we refer to the proof of Theorem 2.1 in Section 4.

In what follows we derive equations for the function $u$ in (1.2).

Recall that we suppose the blowup point is the origin, and the blowup time is $T$, and the limit cylinder is the one parametrized by (2.2).

Then for $t < T$, there exists some $\epsilon(t) > 0$ such that in the region $|z| \leq \epsilon(t)$, the manifold can be parameterized as in (1.2). And the function $u$ satisfies the parabolic differential
equation, by the mean curvature equation (1.1) and the results in [11],

\[
\frac{\partial u}{\partial t} = \frac{1}{1 + |\nabla_x u|^2 + (\frac{\partial u}{u})^2} \sum_{k=1}^{3} \left[ 1 + |\nabla_x u|^2 - (\partial_{x_k} u)^2 + \frac{2\partial u}{u} \right] \partial_{x_k}^2 u
\]

\[
+ u^{-2} \frac{1}{1 + |\nabla_x u|^2 + (\frac{\partial u}{u})^2} \partial_{y}^2 u + u^{-2} \frac{2\partial u}{1 + |\nabla_x u|^2 + (\frac{\partial u}{u})^2} \sum_{l=1}^{3} \partial_{x_l} u \partial_{x_l} \partial_{y} u \quad (3.1)
\]

Now we rescale solution as in (1.3), and derive an equation for the function \(v\),

\[
\partial_{x} v = \Delta_y v + v^2 \partial_{\theta}^2 v - \frac{1}{2} y \cdot \nabla_y v + \frac{1}{2} v - \frac{1}{v} + N_1(v) \quad (3.2)
\]

with \(N_1(v)\) defined as

\[
N_1(v) := - \sum_{k=1}^{3} (\frac{\partial_{y_k} v}{1 + |\nabla_y v|^2 + (\frac{\partial_{y_k} v}{v})^2}) \partial_{y_k}^2 v - \sum_{i \neq j} \frac{\partial_{y_i} v \partial_{y_j} v}{1 + |\nabla_y v|^2 + (\frac{\partial_{y_i} v}{v})^2} \partial_{y_i} \partial_{y_j} v
\]

\[
+ v^{-2} \frac{2\partial_{y} v}{1 + |\nabla_y v|^2 + (\frac{\partial_{y} v}{v})^2} \sum_{l=1}^{3} \partial_{y_l} v \partial_{y_l} \partial_{y} v - v^{-2} \frac{(v^{-1} \partial_{\theta} v)^2 \partial_{\theta}^2 v}{1 + |\nabla_y v|^2 + (\frac{\partial_{y} v}{v})^2}
\]

\[
+ \frac{1}{1 + |\nabla_y v|^2 + (\frac{\partial_{y} v}{v})^2} \frac{\partial_{\theta} v}{v^3} \quad (3.3)
\]

The result is the following:

**Theorem 3.1.** Suppose that \(\xi_0\) in (2.3) is a sufficiently large constant.

There exists a small constant \(\delta\), such that if in the region \(|y| \leq (1 + \epsilon)\Omega(\tau)\) and in the


time interval \(\tau \in [\xi_0, \tau_1]\) the following estimates hold, for \(m, |k| + l = 1, \cdots, 5,\) and \(|k| \geq 1,\)

\[
\left| \frac{v(\cdot, \tau)}{\sqrt{2 + \tau^{-1} y^T \hat{B} y}} - 1 \right|, \ v^{-1}(\cdot, \tau) |\partial^m_{\theta} v(\cdot, \tau)|, \ |\nabla^k_{y} \partial_{\theta} v(\cdot, \tau)| \leq \delta, \quad (3.4)
\]

where \(\hat{B}\) is a 3 \(\times\) 3 diagonal matrix

\[
\hat{B} = \begin{bmatrix}
b_1 & 0 & 0 \\
0 & b_2 & 0 \\
0 & 0 & b_3
\end{bmatrix}, \ b_k = 1 \text{ or } 0, \ k = 1, 2, 3, \quad (3.5)
\]
then the following statements hold in the time interval $[\xi_0, \tau_1]$.

There exist unique parameters $a, \alpha_1, \alpha_2$, a $3 \times 3$ symmetric matrix $B$ and 3-dimensional vectors $\vec{\beta}_k$, $k = 1, 2, 3$, such that

$$v(y, \theta, \tau) = V_a(\tau) B(\tau) + \vec{\beta}_1(\tau) \cdot y + \vec{\beta}_2(\tau) \cdot y \cos \theta + \vec{\beta}_3(\tau) \cdot y \sin \theta + \alpha_1(\tau) \cos \theta + \alpha_2(\tau) \sin \theta + w(y, \theta, \tau), \quad (3.6)$$

where the function $e^{-\frac{1}{8} \rho^2} \chi_\Omega w$ is orthogonal to the following 18 functions,

$$e^{-\frac{1}{8} \rho^2} w, e^{-\frac{1}{8} \rho^2} \cos \theta, e^{-\frac{1}{8} \rho^2} \sin \theta, e^{-\frac{1}{8} \rho^2} y_k, e^{-\frac{1}{8} \rho^2} (\frac{1}{2} y_k^2 - 1), e^{-\frac{1}{8} \rho^2} y_k \cos \theta, e^{-\frac{1}{8} \rho^2} y_k \sin \theta, k = 1, 2, 3, \quad (3.7)$$

Here $V_a, B, \chi_\Omega$ are two functions to be defined in (3.16) and (3.17) below.

Up to a unitary rotation, $y \rightarrow U y$, the $3 \times 3$ real symmetric matrix $B$ is “almost” semi-positive definite,

$$B(\tau) = \tau^{-1} \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} + \mathcal{O}(\tau^{-2}), \text{ and } b_k = 0 \text{ or } 1, k = 1, 2, 3, \quad (3.8)$$

The other parameters and vectors satisfy the estimates, for some $C > 0$,

$$|a(\tau) + \frac{1}{2}| \leq C \tau^{-1}, \quad (3.9)$$

and

$$|\vec{\beta}_1(\tau)| \leq C \tau^{-2}, \ |\vec{\beta}_2(\tau)|, \ |\vec{\beta}_3(\tau)|, \ |\alpha_1(\tau)|, \ |\alpha_2(\tau)| \leq C \tau^{-3}. \quad (3.10)$$

And they satisfy the equations,

$$\left| \frac{d}{d\tau} B + B^T B \right| \leq C \tau^{-3}$$

$$\left| \left( \frac{d}{d\tau} - \frac{1}{2} \right) \left( a - \frac{1}{2} (b_{11} + b_{22} + b_{33}) \right) \right| \leq C \tau^{-2}, \quad (3.11)$$

$$\left| \frac{d}{d\tau} \vec{\beta}_1 - a (1 + \mathcal{O}(|B|)) \vec{\beta}_1 \right| \leq C \tau^{-3}, \quad \left| \frac{d}{d\tau} \vec{\beta}_2 \right|, \ \left| \frac{d}{d\tau} \vec{\beta}_3 \right|, \ \left| \frac{d}{d\tau} \alpha_1 - \frac{1}{2} \alpha_1 \right|, \ \left| \frac{d}{d\tau} \alpha_2 - \frac{1}{2} \alpha_2 \right| \leq C \tau^{-3}.$$
The remainder $w$ satisfies the estimates,

$$
\sum_{|k| + l = 0, 1, 2} \| e^{-\frac{1}{2}|y|^2} \nabla_y^k \phi^l (\chi \omega w(\cdot, \tau)) \|_2 \leq C \tau^{-2},
$$

(3.12)

and in the weighted $L^\infty$-norms, for some constant $\kappa(\epsilon)$ to be defined in (3.20) below,

$$
\| \langle y \rangle^{-3} \partial_\theta^m \chi \omega w(\cdot, \tau) \|_{L^\infty} \leq C (\tau^{-2} + \kappa(\epsilon) \Omega^{-4}), \text{ with } m = 0, 1, 2,
$$

(3.13)

$$
\| \langle y \rangle^{-2} \nabla_y \partial_\theta^m \chi \omega w(\cdot, \tau) \|_{L^\infty} \leq C \kappa(\epsilon) \Omega^{-3}, \text{ with } m = 0, 1.
$$

(3.14)

$$
\| \langle y \rangle^{-1} \nabla_y \chi \omega w(\cdot, \tau) \|_{L^\infty} \leq C \kappa(\epsilon) \Omega^{-2}, \text{ with } |l| = 2.
$$

(3.15)

Here $V_{a,B}$ is a function defined as, for $a \in \mathbb{R}^+$ and $3 \times 3$ symmetric matrix $B$,

$$
V_{a,B}(y) := \sqrt{\frac{2 + y^T By}{2a}}.
$$

(3.16)

Before defining the cutoff function $\chi_\Omega$, we define a smooth, spherically symmetric cutoff function $\chi$,

$$
\chi(z) = \chi(|z|) = \begin{cases} 
1, & \text{if } |z| \leq 1, \\
0, & \text{if } |z| \geq 1 + \epsilon.
\end{cases}
$$

(3.17)

We require it is decreasing in $|z|$, and there exist constants $M_k = M_k(\epsilon)$, $k = 0, 1, \cdots, 5$, such that for any $z$ satisfying $0 \leq 1 + \epsilon - |z| \ll 1$, $\chi$ satisfies the estimates

$$
\frac{d^k}{dz^k} \chi(|z|) = M_k (|z| - 1 - \epsilon)^{20-k} + O((|z| - 1 - \epsilon)^{21-k}), \quad k = 0, 1, 2, 3, 4.
$$

(3.18)

Such a function is easy to construct, we skip the details here.

Now we define cutoff functions $\chi_\Omega$, for any $\Omega > 0$, as

$$
\chi_\Omega(y) := \chi(\frac{y}{\Omega}).
$$

(3.19)

The constant $\kappa(\epsilon)$ is defined to control terms produced by the cutoff function $\chi$,

$$
\kappa(\epsilon) := \sum_{k=1}^5 \sup_{|z|} \left| \frac{d^k}{dz^k} \chi(|z|) \right| + \sup_{z, |l| = 1, 2, 3} \left| \chi^{-\frac{3}{4}}(z) \nabla_z^l \chi \right| < \infty.
$$

(3.20)

To justify that $\kappa(\epsilon)$ is finite, it is easy to see that the first term is finite; for the second, the properties in (3.18) imply that $|\nabla_z^l \chi(|z|)|$, $|l| \leq 3$, approach to zero faster than $\chi^{\frac{3}{4}}(|z|)$ as $|z| \to 1 + \epsilon$, and recall that $\chi(|z|) > 0$ when $|z| < 1 + \epsilon$.

To make Theorem 3.1 applicable one needs to verify its conditions (3.4). For that purpose, we need the following results, recall that the definition of $\Omega$ in (2.3) depends on $\xi_0$, for any $\Omega$
Lemma 3.2. Suppose that the graph function $v$ of the rescaled MCF satisfies the estimates,

$$\left| v(\cdot, \tau) - \sqrt{2 + \tau^{-1}y^T \hat{B}y} \right| \leq \Omega^{-\frac{2}{3}}(\tau), \quad \text{and} \quad \left| \nabla^k y^\beta \partial_\theta^\alpha v(\cdot, \tau) \right| \leq \Omega^{-\frac{9}{10}}(\tau), \quad |k| + l = 1, 2,$$

(3.21)

for $|y| \leq \Omega(\tau)$ and for $\tau \in [\xi_0, \tau_1]$, with $\tau_1 \geq \xi_0 + 20$, and $\hat{B}$ takes the form

$$\hat{B} = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}, \quad b_k = 1 \text{ or } 0, \quad k = 1, 2, 3.$$

(3.22)

Then for any $\delta > 0$, there exists a sufficiently large $\xi_0$, independent of $\tau_1$, to make the following estimates hold.

There exist some constant $C$, independent of $\delta$, and some small constant $\kappa = \kappa(\delta) > 0$ such that at time $\tau = \tau_1 + 10\kappa$ and in the region

$$|y| \leq (1 + 5\kappa)(\Omega(\tau) - C \sup_{|y| \leq \Omega(\tau)} \sqrt{2 + \tau^{-1}y^T \hat{B}y}),$$

(3.23)

$v$ satisfies the estimates, for $m = 1, 2, \cdots, 5$ and $|k| + l = 1, 2, \cdots, 5$ and $|k| \geq 1$,

$$\left| \frac{v(\cdot, \tau)}{\sqrt{2 + \tau^{-1}y^T \hat{B}y}} - 1 \right|, \quad v^{-1} |\partial_\beta^m v|, \quad \left| \nabla^k y^\beta \partial_\theta v(\cdot, \tau) \right| \leq \delta.$$  

(3.24)

The lemma will be proved in Section 5.

4 Proof of Theorem 3.1

We start with deriving an effective equation for $\chi_\Omega w$, almost identically to those in [12].

Plug the decomposition of $v$ in (3.6) into (3.2) to find

$$\partial_\tau w = -Lw + F(B, a) + G(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \alpha_1, \alpha_2) + N_1(v) + N_2(\eta),$$

(4.1)

where the linear operator $L$ is defined as

$$L := -\Delta_y + \frac{1}{2} y \cdot \nabla_y - V_{a,B}^{-2} \partial_\theta^2 - \frac{1}{2} - V_{a,B}^{-2},$$

(4.2)

the nonlinearity $N_2(\eta)$ is defined as

$$N_2(\eta) := -v^{-1} + V_{a,B}^{-1} - V_{a,B}^{-2} \eta + (v^{-2} - V_{a,B}^{-2}) \partial_\theta^2 \eta$$

$$= -V_{a,B}^{-2} v^{-1} \eta^2 - v^{-2} V_{a,B}^{-2} (v + V_{a,B}) \eta \partial_\theta^2 \eta,$$

(4.3)
the function \( \eta \) is defined as
\[
\eta(y, \theta, \tau) := \tilde{\beta}_1(\tau) \cdot y + \tilde{\beta}_2(\tau) \cdot y \cos \theta + \tilde{\beta}_3(\tau) \cdot y \sin \theta + \alpha_1(\tau) \cos \theta + \alpha_2(\tau) \sin \theta + w(y, \theta, \tau),
\]
and the function \( F(B, \alpha) \) is defined as
\[
F(B, \alpha) := -\frac{y^T(\partial_\tau B + B^T B)y}{2\sqrt{2a + y^T By}} + \frac{1}{2\sqrt{2a + y^T By}} \left[ \frac{a_\tau}{a} + 1 - 2a + b_{11} + b_{22} \right]
+ \frac{(y^T B^T B y)(y^T B y)}{2\sqrt{2a + y^T By}} + \frac{a_\tau}{(2a)^{\frac{3}{2}} \sqrt{2 + y^T By}}.
\]
and \( G(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \alpha_1, \alpha_2) \) is defined as
\[
G(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \alpha_1, \alpha_2) := \left[ \frac{2a}{2 + y^T By} \tilde{\beta}_1 - \frac{d}{d\tau} \tilde{\beta}_1 \right] \cdot y - \frac{d}{d\tau} \tilde{\beta}_2 \cdot y \cos \theta - \frac{d}{d\tau} \tilde{\beta}_3 \cdot y \sin \theta
+ \left[ \frac{1}{2} \alpha_1 - \frac{d}{d\tau} \alpha_1 \right] \cos \theta + \left[ \frac{1}{2} \alpha_2 - \frac{d}{d\tau} \alpha_2 \right] \sin \theta.
\]
Impose the cutoff function \( \chi_\Omega \) onto (4.1) and find
\[
\partial_\tau (\chi_\Omega w) = -L(\chi_\Omega w) + \chi_\Omega \left[ F(B, a) + G(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \alpha_1, \alpha_2) + N_1(v) + N_2(\eta) \right] + \mu(w),
\]
here the term \( \mu(w) \) is defined as
\[
\mu(w) := \frac{1}{2} (y \cdot \nabla_y \chi_\Omega) w + \left( \partial_\tau \chi_\Omega \right) w - (\Delta_y \chi_\Omega) w - 2\nabla_y \chi_\Omega \cdot \nabla_y w.
\]
In what follows we prove (3.7)-(3.10) and (3.12) of Theorem 3.1.
To prove the existence and uniqueness of the decomposition in (3.6), we need to find \( a, B, \tilde{\beta}_k, k = 1, 2, 3, \) and \( \alpha_l, l = 1, 2, \) to make the function
\[
e^{-\frac{1}{2}|w|^2} \chi_\Omega \left[ v - \left( V_{a, B} + \tilde{\beta}_1 \cdot y + \tilde{\beta}_2 \cdot y \cos \theta + \tilde{\beta}_3 \cdot y \sin \theta + \alpha_1 \cos \theta + \alpha_2 \sin \theta \right) \right]
\]
orthogonal to the 18 functions listed in (3.7). In the previous paper [12] we proved this, by the fixed point argument. The only difference is that in [12], the cutoff function is \( \chi_R \), where the scalar function \( R : [\tau_0, \infty) \rightarrow \mathbb{R}^+ \) is defined as, for some \( \tau_0 \gg 1, \)
\[
R(\tau) := \sqrt{\frac{26}{5} \ln \tau + 100 \ln(1 + \tau - \tau_0)}.
\]
And we proved that these parameters, matrix and vectors satisfied the estimates in (3.7)-(3.10) and (3.12).

Under the condition (3.4), the very rapid decay of $e^{-\frac{1}{4}|y|^2}$ and that the difference between $\chi_\Omega$ and $\chi\bar{R}$ is on $|y| \geq 10\sqrt{\ln \tau}$ makes the estimates in (3.8)-(3.10) and (3.12) hold, because, in applying the fixed point theorem, one find that the difference between those proved in [12] and the present is of order $\tau^{-20}$, which is negligibly small for the present purpose.

Similarly every equation in (3.11) is derived by taking inner product $\langle e^{-\frac{1}{8}|y|^2}(4.6), g \rangle$, where $g$ is some function listed in (3.7). Together with the analysis above, we see that the difference between those proved in [12] and the present is of order $\tau^{-20}$, hence is negligibly small.

Next we prove (3.13)-(3.15) of Theorem 3.1.

Decompose $w$ into four parts, according to the frequencies in $\theta$, when $|y| \leq (1 + \epsilon)\Omega(\tau)$,

$$w(y, \theta, \tau) = w_0(y, \tau) + e^{i\theta}w_1(y, \tau) + e^{-i\theta}w_{-1}(y, \tau) + P_{\theta, \geq 2}w(y, \theta, \tau) \quad (4.9)$$

where the functions $w_m$, $m = -1, 0, 1$ are defined as,

$$w_m(y, \tau) := \frac{1}{2\pi} \langle w(y, \cdot, \tau), e^{im\theta} \rangle_\theta. \quad (4.10)$$

Here $P_{\theta, \geq 2}$ is the orthogonal projection onto the subspace orthogonal to $1$, $e^{\pm i\theta}$ : for any smooth function $f(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} f_n$,

$$P_{\theta, \geq 2}f(\theta) = \sum_{|n| \geq 2} e^{in\theta} f_n.$$

The reason in decomposing $w$ is that we will apply different techniques on these components. Specifically, we will estimate $\chi_\Omega w_0$ and $\chi_\Omega w_{\pm 1}$ by propagator estimates, generated by certain linear operators; and apply the maximum principle to estimate $\chi_\Omega P_{\theta, \geq 2}w(y, \theta, \tau)$.

Then, recall that we start the consideration at $\tau \geq \xi_0$, we define controlling functions $\mathcal{M}_k$, $k = 1, 2, 3, 4$, as,

$$\mathcal{M}_1(\tau) := \max_{\xi_0 \leq s \leq \tau} \frac{1}{\kappa(\epsilon)\Omega^{-4}(s) + s^{-2}} \left( \sum_{m=-1,0,1} \| \langle y \rangle^{-3} \chi_{\Omega(s)} w_m(\cdot, s) \|_\infty \right. \left. + \| (100 + |y|^2)^{-\frac{3}{2}} \|_{L^\infty} \right) \quad (4.11)$$
Recall that there exists some constant $\xi$ such that, if $\tau$ is sufficiently large, then

$$
\|\langle y \rangle^{-3} R w(\cdot, \tau)\|_{C_{\theta}} \leq C\kappa(\epsilon) R^{-4}(\tau),
\|\langle y \rangle^{-2} \nabla_y^m \partial_\theta^n R w(\cdot, \tau)\|_{C_{\theta}} \leq C\kappa(\epsilon) R^{-3}(\tau), \quad |m| + n = 1,
\|\langle y \rangle^{-1} \nabla_y^m \partial_\theta^n R w(\cdot, \tau)\|_{C_{\theta}} \leq C\kappa(\epsilon) R^{-2}(\tau), \quad |m| + n = 2.
$$

(4.15)

Recall that $R(\tau)$ is defined in (4.8).

Besides these we need the following estimates,

**Proposition 4.1.** There exists some constant $C$ such that, if $\tau$ is sufficiently large, then

$$
\|\langle y \rangle^{-3} \partial_\theta^3 P_{\theta; \geq 2} R w(\cdot, \tau)\|_{L^2_\theta} \leq C\kappa(\epsilon) R^{-4}(\tau),
\|\langle y \rangle^{-2} \partial_\theta^2 \nabla_y P_{\theta; \geq 2} R w(\cdot, \tau)\|_{L^2_\theta} \leq C\kappa(\epsilon) R^{-3}(\tau),
\sum_{|k|=2} \|\langle y \rangle^{-1} \partial_\theta \nabla_y^k P_{\theta; \geq 2} R w(\cdot, \tau)\|_{L^2_\theta} \leq C\kappa(\epsilon) R^{-2}(\tau).
$$

(4.16), (4.17), (4.18)

This proposition will be proved in Appendix A, based on the results proved in [12].

\[15\]
Note here we slightly abuse the notations: in [12], we require $e^{-\frac{1}{8} |y|} \chi_{Rw}$ to be orthogonal to the functions listed in (3.7), while in the present paper we require that $e^{-\frac{1}{8} |y|} \chi_{\Omega w}$ to satisfy the same orthogonal conditions. As argued in the estimates of parameters above, the difference between $\chi_{Rw}$ and $\chi_{\Omega w}$ in the overlapping part is of order $\tau^{-20}$, which is negligibly small for the present purpose.

The estimates in (4.15) and Proposition 4.1 directly imply that, for some $C > 0$, if $\xi_0$ is sufficiently large,

$$\mathcal{M}_k(\xi_0) \leq C. \quad (4.19)$$

The functions $\mathcal{M}_l$, $k = 1, 2, 3, 4$, satisfy the following estimates.

**Proposition 4.2.** Under the conditions in Theorem 3.1, there exists a constant $C > 0$, independent of $\delta$, such that for any $\tau \in [\xi_0, \tau_1]$, $l = 1, 2, 3, 4$,

$$\mathcal{M}_l(\tau) \leq C + C\delta \sum_{l=1}^{4} \sum_{k=1}^{4} \mathcal{M}_l^k. \quad (4.20)$$

This proposition will be proved in subsequent sections.

Now we are ready to prove (3.13)-(3.15), part of Theorem 3.1, assuming Proposition 4.2.

**Proof.** (4.19) and (4.20) imply that, if $\delta$ is sufficiently small, then for all $\tau \in [\xi_0, \tau_1]$

$$\mathcal{M}_k(\tau) \lesssim 1, k = 1, 2, 3, 4. \quad (4.21)$$

This together with the definitions of $\mathcal{M}_k$ and Lemma 4.3 below implies the desired estimates.

The proof is complete.

In the proof above the following lemma was used.

**Lemma 4.3.** There exists a constant $C$ such that for any smooth function $f : \mathbb{T} \rightarrow \mathbb{C}$, we have that, for any $l \in \mathbb{N}$,

$$\|P_{\theta \geq 2}^l f\|_{\infty} \leq C \|\partial_{\theta}^l P_{\theta \geq 2} f\|_{L^2}, \text{ and } \|P_{\theta \geq 2} f\|_{L^2} \leq \|\partial_{\theta}^l P f_{\theta \geq 2} f\|_{L^2}. \quad (4.22)$$

**Proof.** It is easy to prove the second estimate, by Fourier expanding $f$ and analyzing each frequency.

We only prove the first one, with $l = 1$. The others follow from the second estimate.
Apply $P_{\theta, \geq 2}$ to $f$, whose Fourier expansion takes the form $f(\theta) = \sum_{n=\infty}^{\infty} e^{in\theta} f_n$, to have

$$P_{\theta, \geq 2} f(\theta) = \sum_{|n| \geq 2} e^{in\theta} f_n. \quad (4.23)$$

Compute directly to obtain, for any $\zeta > 0$,

$$|P_{\theta, \geq 2} f(\theta)| \leq \sum_{|n| \geq 2} |f_n| = \sum_{|n| \geq 2} \frac{1}{|n|} |n f_n| \leq \zeta \sum_{|n| \geq 2} \frac{1}{|n|^2} + \frac{1}{\zeta} \sum_{|n| \geq 2} |n f_n|^2 \quad (4.24)$$

Now we choose $\zeta := \|\partial_\theta P_{\theta, \geq 2} f\|_{L^2} = \sqrt{\sum_{|n| \geq 2} |n f_n|^2}$, and use that $\sum_{|n| \geq 2} \frac{1}{|n|^2} < \infty$ to find the desired result.

5 Local Smooth Extension: Proof of Lemma 3.2

The main tools are the local smooth extension, see [9], and comparing the rescaled MCF to MCF, which were used in [7]. See also [2, 23, 21, 5].

Similar results were proved in our previous paper [12], where the considered region is $|y| \leq R(\tau) = O(\sqrt{\ln\tau})$, that part of the hypersurface is very close to a cylinder with radius $V_{a, B} \approx \sqrt{2}$. In the present situation, we control the region $|y| \leq \Omega(\tau) = O(\tau^{\frac{1}{2} + \frac{\lambda}{\tau}})$, the radius are different on the different scales if at least one of $b_k$’s in (3.8) is 1. Hence the difficulty here is to show that the techniques work uniformly well for different scales.

To facilitate later discussions we define a new MCF by rescaling the old one and shifting time. Let $\tau_1$ be the chosen time in Lemma 3.2. We define

$$t_1 := t(\tau_1) \quad (5.1)$$

recall that $\tau$ and $t$ are related (one to one) by $\tau := -\ln(T - t)$. We rescale the MCF such that the part of the manifold being a graph is parametrized as

$$\begin{bmatrix} z \\ q(z, \theta, s) \cos \theta \\ q(z, \theta, s) \sin \theta \end{bmatrix}, \quad (5.2)$$

and it is related to the original evolution by

$$q(z, \theta, s) = \frac{1}{\lambda} u(\lambda z, \theta, \lambda^2 s + t_1). \quad (5.3)$$
Here \( s \) and \( \lambda \) are defined as
\[
s := \frac{t - t_1}{\lambda^2}, \quad \lambda := \sqrt{T - t_1}.
\] (5.4)
This is a well defined MCF since it is scaling invariant by \( u(z, \theta, t) \rightarrow \lambda^{-1} u(\lambda z, \theta, \lambda^2 t) \) for any \( \lambda > 0 \).

The evolution of \( q \) is related to that of \( v \) by the identity,
\[
q(z, \theta, s) = \sqrt{1 - s} \ v\left(\frac{z}{\sqrt{1 - s}}, \theta, -\ln(1 - s) + \tau_1\right).
\] (5.5)
resulted by the rescaling identity in (1.3) and elementary calculations, specifically,
\[
\frac{1}{\lambda} u(\lambda z, \theta, \lambda^2 s + t_1) = \sqrt{T - t} \ v\left(\frac{\lambda z}{\sqrt{T - t}}, \theta, -\ln (T - \lambda^2 s - t_1)\right)
= \sqrt{1 - s} \ v\left(\frac{z}{\sqrt{1 - s}}, \theta, -\ln(1 - s) + \tau_1\right).
\] (5.6)
(5.5) implies that the new MCF will blow up at time \( t = 1 \) and at point \( z = 0 \).

Another fact we will use is that, for \(-1 \leq s \leq 0\), if \(|z| \leq \Omega(\tau_1)\), then
\[
\frac{|z|}{\sqrt{1 - s}} \leq \Omega(-\ln(1 - s) + \tau_1),
\] (5.7)
which is equivalent to that, \( \Omega(\tau_1) \leq \sqrt{1 - s} \ \Omega(-\ln(1 - s) + \tau_1) \), \( s \in [-1, 0] \). This, in turn, is justified by the definition of \( \Omega \) in (2.3), and the different growth rates of \( \Omega(-\ln(1 - s) + \tau_1) \) and \( \sqrt{1 - s} \) for \( s \leq 0 \). And we observe that, by setting \( s = 0 \) in (5.5),
\[
q(z, \theta, 0) = v(z, \theta, \tau_1).
\] (5.8)

**Remark 1.** The important consequence is that, in what follows we need information for \( q(z, \theta, s) \) and its derivatives, in \( s \in [-1, 0] \) and \( |z| \leq \Omega(\tau_1) \) to make regularity estimate. This is provided to us by the estimates on \( v(y, \theta, \tau) \), \( |y| \leq \Omega(\tau) \), \( \tau \in [\xi_0, \tau_1] \), through the identities in (5.5) and (5.8), and the estimate in (5.7).

Recall that we assume \( \tau_1 - \xi_0 \geq 20 \).

Now we start proving Lemma 3.2.

Observe for \( |z| \leq \Omega(\tau_1) \), \( q \) can take some values much greater than \( \sqrt{2} \). To compare the results, we consider the following regions, for \( n = 1, 2, \ldots, N \), with \( N \) being the smallest integer such that \( 2^N \sqrt{2} \geq \sup_{|y| \leq \Omega(\tau)} \sqrt{2 + \tau_1^{-1} y^T \tilde{B} y} \).
\[
\mathbb{R}^3 \supset \Lambda_n := \{ y \mid |y| \leq \Omega(\tau_1), \ \text{and} \ \sqrt{2 + \tau_1^{-1} y^T \tilde{B} y} \in [2^{n-1} \sqrt{2}, 2^{n+1} \sqrt{2}] \}. \] (5.9)
It is easy to prove the local extension for the region $\Lambda_1$ as in [12]. By Remark [11] and the estimates for $v$ and its derivatives in (3.21), we have that when
\[
|z| \leq \Omega(\tau_1) \text{ and } s \in [-1, 0],
\]
the function $q$ satisfies the estimate
\[
1 \leq q \leq 10, \quad |\nabla^k z^l q| \leq \Omega^{-1}(\tau_1), \quad |k| + l = 1, 2.
\]
This together with the techniques of regularity estimate and of interpolation between the estimates of derivatives, implies that there exists some $C_1 > 0$, independent of $\Omega(\tau_1)$, such that when
\[
s = 0, \quad z \in \{ z \mid z \in \Lambda_1, \text{dist}(z, \partial \Lambda_1) \geq C_1 \},
\]
we have, there exists some $\tilde{\Omega}(\tau_1)$ satisfying $\tilde{\Omega}(\tau_1) \to \infty$ as $\Omega(\tau_1) \to \infty$, such that
\[
|q(\cdot, 0) - \sqrt{2 + \tau_1^{-1} z^T \tilde{B} z}|, \quad |\nabla^k z^l q(\cdot, 0)| \leq \frac{1}{\tilde{\Omega}(\tau_1)}, \quad \text{if } |k| + l = 1, 2, 3, 4, 5.
\]
Here $\partial \Lambda_1$ signifies the boundary of $\Lambda_1$.

For the positive time, $s \geq 0$, we choose $\xi_0$ to be large enough to make $\frac{1}{\Omega(\tau_1)} \leq \frac{1}{10}\delta$, with $\delta$ being the desired small constant in Lemma [3.2]. By the local smooth extension, there exists a constant $C_2 \geq C_1$ and a small constant $\kappa = \kappa(\delta)$ such that
\[
|z| \leq \Omega(\tau_1) \text{ and } s \in [-1, 0],
\]
the following estimates hold, if $|k| + l = 1, 2, 3, 4, 5$, then
\[
\left| \frac{1}{\sqrt{1-s}} q(\cdot, s) - \sqrt{2 + (\tau_1 - \ln(1-s))^{-1}} (1-s)^{-1} z^T \tilde{B} z \right|, \quad \text{and } |\nabla^k z^l q(\cdot, s)| \leq \frac{1}{5}\delta.
\]
Now we convert the estimates to that on $v$ by the identity (5.5), for $\tau = \tau_1 - \ln(1 - s)$ and $y = \frac{z}{\sqrt{1-s}}$, with $0 \leq s \leq 10\kappa$, and,
\[
(1-s)y \in \tilde{\Lambda}_1 := \{ z \mid z \in \Lambda_1, \text{dist}(z, \partial \Lambda_1) \geq C_2 \}
\]
we have
\[
|v(y, \theta, \tau) - \sqrt{2 + \tau^{-1} y^T \tilde{B} y}|, \quad |\nabla^k \partial_\theta^l v(y, \theta, \tau)| \leq \frac{1}{2}\delta.
\]
Now we consider the region $\Lambda_n$, $n \geq 2$. 19
To make the arguments comparable to that for $\Lambda_1$, we rescale $q$ in (5.3) by defining $\lambda_n := 2^n$ and,

$$q_n(z, \theta, s_1) := \lambda_n^{-1} q(\lambda_n z, \theta, \lambda_n^2 s_1),$$

$$= \sqrt{1 - s_1} \lambda_n \left( \frac{\lambda_n}{\sqrt{1 - s_1}} z, \theta, -\ln(1 - \lambda_n^2 s_1) + \tau_0 \right).$$

(5.17)

We observe that:

(1) When $n \geq 2$, it becomes easier to consider the extension into positive time to fulfill the smallness estimates similar to those in (5.14), than choosing $10\kappa$ above for $n = 1$. Since later we need to rescale back, here we only need to consider the time scale $s_1 \leq 10\lambda_n^{-2}\kappa$. The larger value $\lambda_n$ takes, the easier it becomes to consider the extension.

(2) For $s_1 \in [-1, 0]$, the informations for $q_n$ are provided by $v(\cdot, \tau), \tau \in [\xi_0, \tau_1]$. Since if the set $\Lambda_n, n \geq 2$, is not empty, then the definitions of $\Omega(\tau)$ and $\Lambda_n$ force that $\tau_1 - \xi_0 = O(1)$. This allows $s = \lambda_n^2 s_1$ in (5.5) to take any value in $[-e^\sqrt{\tau_1}, 0]$. On this other hand $\lambda_n$ is at most (a modest) $3\tau_1^{1/2}$.

(3) The estimates in (3.21) imply that, at $s_1 = 0$, for any $z \in \frac{1}{\lambda_n} \Lambda_n$,

$$\left| q_n(z, \theta, 0) - \lambda_n^{-1} \sqrt{2 + \tau_1^{-1} \lambda_n^2 z^T \tilde{B} z} \right| \leq \Omega^{-\frac{3}{4}}(\tau_1), \text{ and } |\nabla_z^k \partial_\theta^l q_n(z, \theta, 0)| \leq \Omega^{-\frac{3}{4}}(\tau_1).$$

(5.18)

And for $s_1 \in [-1, 0]$, and for any $z \in \lambda_n^{-1} \Lambda_n$, $|k| + l = 1, 2,$

$$1 \leq q_n(z, \theta, s_1) \leq 10, \ |\nabla_z^k \partial_\theta^l q_n(z, \theta, s_1)| \leq \Omega^{-\frac{3}{4}}(\tau_1), \ |k| + l = 1, 2.$$ (5.19)

Now we can run the arguments as in (5.12)-(5.14) for $n = 1$, but for $s_1 \leq 10\lambda_n^{-2}\kappa$ (see the discussion in Item [1] above). We find that,

$$1 \leq q_n(z, \theta, s_1) \leq 10, \ |\nabla_z^k \partial_\theta^l q_n(z, \theta, s_1)| \leq \Omega^{-\frac{3}{4}}(\tau_1), \ |k| + l = 1, 2.$$ (5.19)

Now we can run the arguments as in (5.12)-(5.14) for $n = 1$, but for $s_1 \leq 10\lambda_n^{-2}\kappa$ (see the discussion in Item [1] above). We find that,

$$1 \leq q_n(z, \theta, s_1) \leq 10, \ |\nabla_z^k \partial_\theta^l q_n(z, \theta, s_1)| \leq \Omega^{-\frac{3}{4}}(\tau_1), \ |k| + l = 1, 2.$$ (5.19)

Now we can run the arguments as in (5.12)-(5.14) for $n = 1$, but for $s_1 \leq 10\lambda_n^{-2}\kappa$ (see the discussion in Item [1] above). We find that,

$$1 \leq q_n(z, \theta, s_1) \leq 10, \ |\nabla_z^k \partial_\theta^l q_n(z, \theta, s_1)| \leq \Omega^{-\frac{3}{4}}(\tau_1), \ |k| + l = 1, 2.$$ (5.19)
Now we rescale back to $q$ and then find estimates for $v$ by the identity (5.17). One adverse effect is that, after rescaling back, some functions will become adversely large, for example $|\partial_\theta q|$, $k = 1, \cdots, 5$, will be of the order $\lambda_n \delta$. To offset this, we apply a factor $q^{-1}$ or $\frac{1}{\sqrt{2 + \tau^{-1}y^TB \tilde{y}}}$ to some of them since $\frac{\sqrt{2 + \tau^{-1}y^TB \tilde{y}}}{\lambda_n} = \frac{q}{\lambda_n} = \mathcal{O}(1)$ in the considered region. And to some of the functions, for example $\nabla_z q(z, \theta, t)$, we do not need the help, since it is “scaling invariant” in the sense that

$$\nabla_z q(z, \theta, t) = \nabla_y \lambda_n q(\lambda_n^{-1}y, \theta, t)|_{z = \lambda_n^{-1}y},$$

and for $\nabla_z^k \partial_\theta^m q$, $|k| \geq 2$, the scaling works favorably since $\lambda_n \geq 1$ makes

$$|\nabla_z^k q(z, \theta, t)| \leq |\nabla_y \lambda_n q(\lambda_n^{-1}y, \theta, t)|_{z = \lambda_n^{-1}y}|.$$

Consequently there exists some constant $\tilde{C}_2$, independent of $n$, such that for $\tau = \tau_1 - \ln(1 - s)$ with $0 \leq s \leq 10\kappa$, and

$$(1 - s)y \in \tilde{\Lambda}_n := \{z | z \in \Lambda_n, \text{dist}(z, \partial \Lambda_n) \geq \lambda_n \tilde{C}_2\}$$

(5.23)

$v$ satisfies the desired estimates (3.23), for $m = 1, \cdots, 5$, and $|k| + l = 1, \cdots, 5$, and $|k| \geq 1,$

$$|\nabla^k_z v(\cdot, \tau)| \leq |\nabla^k_y \lambda_n q(\lambda_n^{-1}y, \theta, t)|_{z = \lambda_n^{-1}y}|$$

and for $\nabla_z^k \partial_\theta^m q$, $|k| \geq 2$, the scaling works favorably since $\lambda_n \geq 1$ makes

$$|\nabla_z^k q(z, \theta, t)| \leq |\nabla_y \lambda_n q(\lambda_n^{-1}y, \theta, t)|_{z = \lambda_n^{-1}y}|.$$

$$\frac{v(\cdot, \tau)}{\sqrt{2 + \tau^{-1}y^TB \tilde{y}}} - 1, |v^{-1}| |\partial_\theta^m v|, |\nabla^k_y \partial_\theta^l v(\cdot, \tau)| \leq \delta.$$

(5.24)

Now we patch the different sets $\tilde{\Lambda}_n$, defined in (5.23), together to get the desired set (3.23). Since each $\tilde{\Lambda}_n$ is just a subset of $\Lambda_n$, we need to make sure there will not be a “hole” left inside. This is not a problem, since, in defining $\Lambda_n$ we make the overlapping of different ones large enough. Recall that $\lambda_n$ is at most $3\tau_1^{\frac{1}{20}}$, and the overlapping of different $\Lambda_n$ is of order $\tau_1$.

The proof is complete.

6 Proof of Main Theorem 2.1

We observe that the definitions of $\Omega(\tau)$ in (2.3) and of $R(\tau)$ in (4.8) imply that,

$$R(\tau) \geq \Omega(\tau) \text{ for } \tau \in [\xi_0, \xi_0 + 40].$$

This makes the results proved in [12], especially (4.15), applicable in this interval. Thus the results in Theorem 3.1 hold in $[\xi_0, \xi_0 + 40]$. 
This, in turn, makes Lemma 3.2 applicable, provided that $\xi_0$ is sufficiently large. By choosing $\epsilon = \kappa$, Lemma 3.2 makes the condition (3.4) hold in a larger interval $[\xi_0, \xi_0 + 40 + 10\kappa]$ with $\kappa$ being determined by $\delta$. Hence the results in Theorem 3.1 hold in this larger interval, and then Lemma 3.2 becomes applicable again.

By boot-strapping these arguments, we find that the results in Theorem 3.1 hold in the interval $[\xi_0, \infty)$. We derive the estimates in Theorem 2.1 from those in Theorem 3.1.

7 Proof of Main Theorem 2.2

The key argument is that, the estimates in Theorem 3.1 imply a good control in the region $|y| \leq 3\tau^{\frac{1}{2} + \frac{1}{20}}$ if $\tau$ is large enough. Specifically (3.13) implies that, if $|y| \leq 3\tau^{\frac{1}{2}} + \frac{1}{20}$, then

$$|w(y, \sigma, \tau)| \lesssim \langle y \rangle^3 \|\langle y \rangle - w(\cdot, \tau)\|_\infty \lesssim \tau^{-\frac{7}{20}},$$

(7.1)

and similarly the other ones imply

$$\sum_{k+|l|=1} |\partial^k_y \nabla^l_y w(\cdot, \tau)| \lesssim \tau^{-\frac{7}{20}}, \quad \sum_{k+|l|=2} |\partial^k_y \nabla^l_y w(\cdot, \tau)| \lesssim \tau^{-\frac{3}{5}}. \quad (7.2)$$

These, together with the decomposition of $v$ and the estimates in (3.8)-(3.10), make,

$$|v(y, \theta, \tau) - \sqrt{2 + \tau^{-1}} |y|^2| \lesssim \tau^{-\frac{7}{20}}, \quad \sum_{k+|l|=1,2} |\partial^k_y \nabla^l_y v(\cdot, \tau)| \lesssim \tau^{-\frac{1}{10}}, \quad \text{for } |y| \leq 3\tau^{\frac{1}{2} + \frac{1}{20}}. \quad (7.3)$$

Now we prove Theorem 2.2.

To facilitate later discussions we consider two different MCF.

Let $\tau_1$ be a sufficiently large time, we define a new MCF by scaling and shifting time, as in (5.3) in the previous section, so that the part being a graph is parametrized by

$$
\begin{bmatrix}
q(z, \theta, s) \cos \theta \\
q(z, \theta, s) \sin \theta
\end{bmatrix},
$$

(7.4)

where $q$ is related to $u$, and to $v$ by the identities

$$q(z, \theta, s) = \frac{1}{\lambda} u(\lambda z, \theta, \lambda^2 s + t_1)$$

(7.5)

$$= \sqrt{1 - s} \ v(\frac{z}{\sqrt{1 - s}}, \theta, - \ln(1 - s) + \tau_1), \quad (7.6)$$
Here $s$, $t_1$, $\lambda$ are defined as
\begin{equation}
    s := \frac{t - t_1}{\lambda^2}, \quad t_1 := t(\tau_1), \quad \lambda := \sqrt{T - t_1}.
\end{equation}
(7.7)

Recall that $\tau$ is defined by a bijection $\tau = - \ln(T - t)$, $t_1$ is the time $t$ when $\tau(t) = \tau_1$. Obviously the blowup time here is 1. For a detail reasoning, see the discussion after (5.3).

Now we define a second new flow, by rescaling the one in (7.5), specifically,
\begin{equation}
    \begin{bmatrix}
        z \\
        p(z, \theta, s) \cos \theta \\
        p(z, \theta, s) \sin \theta
    \end{bmatrix},
\end{equation}
(7.8)
with $p(z, \theta, s)$ defined as
\begin{align*}
    p(z, \theta, s) &= \tau_1^{-\frac{1}{10}} q(\tau_1^{-\frac{1}{10}} z, \theta, \tau_1^{-\frac{1}{10}} s) \\
    &= \frac{1}{\tau_1^{-\frac{1}{10}} u(\tau_1^{-\frac{1}{10}} \lambda z, \theta, \tau_1^{-\frac{1}{10}} \lambda^2 s + t_1)} \\
    &= \sqrt{1 - \tau_1^{-\frac{1}{10}} s} \left( \frac{\tau_1^{-\frac{1}{10}} z}{\sqrt{1 - \tau_1^{-\frac{1}{10}} s}}, \theta, -\ln(1 - \tau_1^{-\frac{1}{10}} s) + \tau_1 \right),
\end{align*}
(7.9)
where in the second and last steps we use (7.5) and (7.6) respectively.

In what follows we use implicitly that $\tau_1$ is a sufficiently large constant.

Obviously, the new flow will blowup at time $s = \tau_1^{-\frac{1}{10}}$ since the one for $q$ will blowup at time 1. At $s = 0$ and $|z| \in \left[ \sqrt{2} \tau_1^{\frac{1}{10}} - \tau_1^{\frac{1}{2}}, \sqrt{2} \tau_1^{\frac{1}{10}} + \tau_1^{\frac{1}{2}} \right]$, the estimates on $v$ in (7.3) imply that
\begin{equation}
    |p(z, \theta, 0) - \sqrt{2}| \leq \tau_1^{-\frac{1}{4}},
\end{equation}
(7.10)
and
\begin{equation}
    |\nabla_z^k \partial_\theta p(z, \theta, 0)| \leq \tau_1^{-\frac{3}{10}}, |k| + l = 1, 2.
\end{equation}
(7.11)

Now we consider the negative time interval $s \in [-1, 0]$. This, by the identity in (7.9), corresponds to the time interval $\tau \in [- \ln(1 + |s| \tau_1^{\frac{1}{10}}) + \tau_1, \tau_1]$ for the rescaled MCF. The estimates on $v$ imply that, for $s \in [-1, 0]$ and $|z| \in \left[ \sqrt{2} \tau_1^{\frac{1}{10}} - \tau_1^{\frac{1}{2}}, \sqrt{2} \tau_1^{\frac{1}{10}} + \tau_1^{\frac{1}{2}} \right]$, 
\begin{equation}
    1 \leq p \leq 10, \quad |\partial_z^k \partial_\theta^l p| \leq \tau_1^{-\frac{3}{10}}, |k| + l = 1, 2.
\end{equation}
(7.12)
Now we apply the regularity estimate, interpolation between the estimates of the derivatives, as in the proof of Lemma 3.2. We will apply them up to the blowup time \( s = \tau_1^{-\frac{1}{10}} \) to find that, for some constant \( C \), independent of \( \tau_1 \), such that in the set

\[
s \in [0, \tau_0^{-\frac{1}{10}}], \quad \text{and } |z| \in [\sqrt{2\tau_1^{-\frac{1}{2}} - \frac{1}{2}\tau_1^{-\frac{1}{4}}}, \sqrt{2\tau_1^{-\frac{1}{2}} + \frac{1}{2}\tau_1^{-\frac{1}{4}}}],
\]

the following estimates hold

\[
|p(z, \theta, s) - \sqrt{2}| \leq C\tau_1^{-\frac{1}{10}};
\]

and for any fixed \( N \in \mathbb{N} \), there exists a constant \( \epsilon(\tau_1) \) satisfying \( \lim_{\tau \to \infty} \epsilon(\tau) = 0 \), such that

\[
|\nabla^m \partial^n_x p(z, \theta, s)| \leq \epsilon(\tau_1), \quad 1 \leq |m| + n \leq N,
\]

and hence, there exists a constant \( C_N \) such that

\[
p^{m-1}|\partial^n_x \nabla^m_x p(z, \theta, s)| \leq C_N \epsilon(\tau_1).
\]

We claim that (7.15) implies the desired (2.12) since the functions in (7.15) are “scaling invariant”, to be shown in (7.16) below. Use the second identity in (7.9) to find that, for any \( z \) satisfying

\[
\tau_1^{-\frac{1}{20}} \sqrt{T - t(\tau_1)} \sqrt{2\tau_1^{-\frac{1}{2}} - \frac{1}{2}\tau_1^{-\frac{1}{4}}} \leq |z| \leq \tau_1^{-\frac{1}{20}} \sqrt{T - t(\tau_1)} \sqrt{2\tau_1^{-\frac{1}{2}} + \frac{1}{2}\tau_1^{-\frac{1}{4}}},
\]

and for the time interval \( t \in [t(\tau_1), T] \), we have, for example, if \( |k| = 3 \), then

\[
u^2 |\nabla^k_x u(z, \theta, t)| = p^2 \left| \nabla^k_x p(x, \theta, s) \right|_{z = t(\tau)} \leq C_N \epsilon(\tau_1),
\]

and the other estimates in (2.12) will be derived similarly. The estimate in (2.13) will be derived easily, hence will be skipped.

Next we would like to argue that the estimates in Theorem 2.2 holds in a larger set, specifically they hold for time \( t \in [t_1, T] \) and \( |z| \leq \sqrt{2\tau_1^{-\frac{1}{2}}} \sqrt{T - t(\tau_1)} \tau_1^{-\frac{1}{4}} \). The reason is that for each fixed \( z \neq 0 \) in the set, by the rescaling \( y = \sqrt{\frac{1}{T - t(\tau)}} z = e^{\frac{1}{2} \tau} z \) and that \( e^{\frac{1}{2} \tau} \) grows much faster than \( \tau \), it will have its moment of becoming a \( y \) with \( |y| = 2\tau_1^{-\frac{1}{4}} \tau_1^{-\frac{1}{4}} \) at some time \( t(\tau) \geq t(\tau_1) \). After that moment we apply the arguments above, before that moment, we derive the desired estimates for \( u \) from that of \( v \), see (2.5) and (2.9), and use that “scaling invariant”, for example, for \( |k| = 2 \),

\[
u(z, \theta, t)|\nabla^k_x u(z, \theta, t)| = v(y, \theta, \tau)|\partial^n_y v(y, \tau, \tau(t))|_{z = e^{-\frac{1}{2} \tau} y}.
\]
All the other estimates in (2.12) will be derived similarly. It is easy to see that the estimates improve as $|z|$ becomes smaller, and $\tau$ becomes larger.

To see that $u(\cdot, t)$ is decreasing in $t$, we derive, from the equation for $u$ in (3.1) and the estimates in (2.12), and find,

$$\partial_t u = -\frac{1}{u} \left( 1 + O(\delta(|z|, t)) \right).$$

(7.18)

8 Estimates for $\mathcal{M}_1$ and $\mathcal{M}_4$, Proof of part of (4.20)

To simplify the notations, in the rest of paper we use $P(M)$ to denote

$$P(M) := 1 + \sum_{k=1}^4 \mathcal{M}_k + \sum_{k=1}^4 \mathcal{M}_k^2. \quad (8.1)$$

And we choose $\xi_0$, which is the time we start considering the rescaled flow, to be sufficiently large such that for $\tau \geq \xi_0$,

$$\left(\kappa(\epsilon) + 1\right)\Omega^{-\frac{1}{16}}(\tau) \leq \delta. \quad (8.2)$$

Recall that $\Omega$ and $\kappa(\epsilon)$ are defined in (2.3) and (3.20) respectively.

The different components in the definitions of $\mathcal{M}_1$ and $\mathcal{M}_4$ satisfy the estimates,

**Proposition 8.1.**

$$\|\langle y \rangle^{-3} w_0(\cdot, \tau)\|_{\infty} \lesssim \left(\kappa(\epsilon)\Omega^{-4}(\tau) + \tau^{-2}\right)\left(1 + \delta P(M)\right), \quad (8.3)$$

$$\|\langle y \rangle^{-3} w_{\pm}(\cdot, \tau)\|_{\infty} \lesssim \left(\kappa(\epsilon)\Omega^{-4} + \tau^{-2}\right)\left(1 + \delta P(M)\right), \quad (8.4)$$

and

$$\left\| (100 + |y|^2)^{-\frac{3}{2}} \| P_{\theta, \geq 2\Omega} \chi_\Omega w(\cdot, \tau) \|_{L^2_\theta} \right\|_{\infty} \lesssim \delta \left(\kappa(\epsilon)\Omega^{-4} + \tau^{-2}\right)P(M),$$

$$\left\| (100 + |y|^2)^{-1} \| P_{\theta, \geq 2\Omega} \chi_\Omega w(\cdot, \tau) \|_{L^2_\theta} \right\|_{\infty} \lesssim \delta \kappa(\epsilon)\Omega^{-3}P(M). \quad (8.5)$$

The proposition will be proved in subsequent subsections.

In the rest of the paper we need the following results:
Lemma 8.2. For \( l_1 = 0, 1, 2 \), and \( l_2 = 0, 1, 2, 3 \),

\[
\| \langle y \rangle^{-3} \partial_{\theta}^{l_1} \chi \Omega w(\cdot, \tau) \|_{L^\infty} \leq \tau^{-2} + \kappa(\epsilon) \Omega^{-4} \mathcal{M}_1(\tau), \quad (8.6)
\]

for \( n_1 = 1, 2 \), and \( n_2 = 1, 2, 3 \),

\[
\| \langle y \rangle^{-2} \partial_{\theta}^{n_1} \chi \Omega w(\cdot, \tau) \|_{L^\infty} \leq \kappa(\epsilon) \Omega^{-3} \mathcal{M}_4(\tau), \quad (8.7)
\]

for \( m_1 = 0, 1 \), and \( m_2 = 0, 1, 2 \),

\[
\| \langle y \rangle^{-2} \partial_{\theta}^{m_1} \nabla_y \chi \Omega w(\cdot, \tau) \|_{L^\infty} \leq \kappa(\epsilon) \Omega^{-3} \mathcal{M}_2(\tau), \quad (8.8)
\]

and for \( |k| = 2 \), and \( d = 0, 1 \),

\[
\| \langle y \rangle^{-1} \nabla^k_y \chi \Omega w(\cdot, \tau) \|_{L^\infty} \leq \kappa(\epsilon) \Omega^{-2} \mathcal{M}_3(\tau). \quad (8.9)
\]

Proof. To prove the first estimate in (8.6) we decompose \( w \) as in (4.9), and apply Lemma 4.3 to find, when \( l = 0, 1, 2 \),

\[
\| \langle y \rangle^{-3} \partial_{\theta} \chi \Omega w \|_{L^\infty} \leq \sum_{m = -1, 0, 1} \| \langle y \rangle^{-3} \chi \Omega w_m \|_{L^\infty} + \bigg\| (100 + |y|^2)^{-\frac{3}{2}} \| P_{\theta \geq 2} \partial_{\theta}^3 \chi \Omega w \|_{L^\infty} \bigg\|, \quad (8.10)
\]

which together with the definitions of \( \mathcal{M}_1 \) implies the desired estimate.

Similarly, for the second in (8.6)

\[
\| \chi \Omega \partial_{\theta}^l w \|_{L^\infty} \leq \sum_{m = -1, 0, 1} |w_m| + \| P_{\theta \geq 2} \partial_{\theta}^3 \chi \Omega w \|_{L^\infty}, \quad l = 0, 1, 2, 3. \quad (8.11)
\]

then we consider the weighted \( L^\infty \)-norm to obtain the desired result.

The proof of the others are similar, except that we need to use, for example, in the proof of (8.7), that \( \partial_{\theta} w_0 = 0 \) since \( w_0 \) is \( \theta \)-independent. Hence we skip the details here. \( \Box \)

8.1 Proof of (8.3)

From (4.6) we derive an equation for \( \chi \Omega w_0 \) as

\[
\partial_{\tau} \chi \Omega w_0 = -H \chi \Omega w_0 + \chi \Omega \Sigma + \frac{1}{2\pi} \chi \Omega \langle N_1(v) + N_2(\eta), 1 \rangle_{\theta} + \mu(w_0), \quad (8.12)
\]

where the function \( \Sigma \) is defined as

\[
\Sigma := \frac{1}{2\pi} \langle F(B, a) + G(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2), 1 \rangle_{\theta} = F(B, a) + \frac{2a}{2 + y^T B y} \beta_1 - \frac{d}{d\tau} \beta_1 \cdot y,
\]

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where we use that $F(B,a)$ is independent of $\theta$, and the $\theta$–dependent part of $G$ does not contribute, and the linear operator $H$ is defined as

$$H := -\Delta_y + \frac{1}{2} y \cdot \nabla_y - \frac{1}{2} - a - \tau^{-\frac{1}{2}} + V_1,$$

(8.13)

resulted from the observations

$$\langle L\chi_\Omega w, 1 \rangle_\theta = H\chi_\Omega w_0, \text{ and } \chi_\Omega \chi_\Omega = \chi_\Omega.$$  

Here we use $\langle V^{-2}_\theta \phi, 1 \rangle_\theta = 0$, $V^{-2}_\theta$ is independent of $\theta$ and $\chi_D(y) := \chi(y/D)$ for any $D > 0$.

The reason we need $H$ to be this form is to facilitate the estimate in Lemma 8.3 below. The matrix $B$ is sufficiently close to being semi-positive definite by estimates in (3.8) when we consider the region $|y| \leq 2\Omega$. However for general $y \in \mathbb{R}^3$, $V^{-2}_\theta$ might not be well defined.

As discussed in the previous paper [12], we divide $\mu(w_0)$ into two parts:

$$\mu(w_0) = \frac{1}{2}(y \cdot \nabla_y \chi_\Omega)w_0 + \Gamma(w_0)$$

(8.14)

with $\Gamma(w_0)$ defined as

$$\Gamma(w_0) := (\partial_{\tau}\chi_\Omega)w_0 - (\Delta_y \chi_\Omega)w_0 - 2\nabla_y \chi_\Omega \cdot \nabla_y w_0.$$  

The term $\Gamma(w_0)$ is small, by the definition $\chi_\Omega(y) = \chi(y/D)$ and the estimates in (3.4).

The difficulty in controlling $\frac{1}{2}(y \cdot \nabla_y \chi_\Omega)w_0$ is caused by two observations, as pointed out in [12]: (1) the function $\frac{1}{2}y \cdot \nabla_y \chi_\Omega$ is not small, in $L^\infty$ norm, since the definition $\chi_\Omega(y) = \chi(y/D)$ makes

$$\sup_y |\frac{1}{2}y \cdot \nabla_y \chi_\Omega(y)| \leq \sup_x |\frac{1}{2}x \cdot \nabla_x \chi(x)|,$$  

(8.15)

(2) the mapping $\chi_\Omega w \rightarrow \frac{1}{2}(y \cdot \nabla_y \chi_\Omega)w = \frac{1}{2}y \cdot \nabla_{\chi_\Omega} \chi_\Omega w$ is unbounded since $|y \cdot \nabla_{\chi_\Omega} \chi_\Omega| \rightarrow \infty$ as $|y| \rightarrow (1 + \epsilon)\Omega$.

The key observation is that $\frac{1}{2}y \cdot \nabla_y \chi_\Omega$ has a favorable non-positive sign, by the requirement $\chi(z) = \chi(|z|)$ being decreasing in $|z|$ (see (3.17)).

To absorb “most” of it into the linear operator, we define a new non-negative smooth cutoff function $\tilde{\chi}_\Omega(y)$ such that

$$\tilde{\chi}_\Omega(y) = \begin{cases} 1, & \text{if } |y| \leq \Omega(1 + \epsilon - \Omega^{-\frac{1}{4}}), \\ 0, & \text{if } |y| \geq \Omega(1 + \epsilon - 2\Omega^{-\frac{1}{4}}) \end{cases}$$  

(8.16)
and it satisfies the estimate
\[ |\nabla^k_y \tilde{\chi}_\Omega(y)| \lesssim \Omega^{-\frac{3}{4}|k|}, \quad |k| = 1, 2. \] (8.17)

Such a function is easy to construct, hence we skip the details.

Then we decompose \( \frac{1}{2}(y \cdot \nabla_y \chi_\Omega)w_0 \) into two parts
\[
\frac{1}{2}(y \cdot \nabla_y \chi_\Omega)w_0 = \frac{1}{2} \left( \frac{y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} \right) \tilde{\chi}_\Omega \chi_\Omega w_0 + \frac{1}{2}(y \cdot \nabla_y \chi_\Omega)(1 - \tilde{\chi}_\Omega)w_0. \] (8.18)

The following three observations will be used in later development:

(A) The first part in (8.18) is a bounded (but not uniformly bounded) multiplication operator since, for some \( c(\epsilon) > 0 \),
\[
\left| \frac{y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} \tilde{\chi}_\Omega \right| \leq c(\epsilon) \Omega^{\frac{1}{4}}. \] (8.19)

(B) If \( \Omega \) is sufficiently large, then by that \( \left| \frac{d}{dz} \chi(|z|) \right| \to 0 \) rapidly as \( |z| \to 1 + \epsilon \), see (3.18), we have
\[
\left| \nabla^k_y (y \cdot \nabla_y \chi_\Omega)(1 - \tilde{\chi}_\Omega) \right| \leq \Omega^{-5}. \] (8.20)

(C) If \( \Omega \) is sufficiently large, then by the properties of \( \chi \) in (3.18), we have that,
\[
\left| \nabla^k_y \left[ \frac{y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} \tilde{\chi}_\Omega \right] \right| \leq \Omega^{-\frac{1}{4}}, \quad |k| = 1, 2. \] (8.21)

Returning to the equation for \( \chi_\Omega w_0 \) in (8.12), we absorb the first part in (8.18) into the linear operator and leave the second to the remainder, thus
\[
\partial_\tau (\chi_\Omega w_0) = -H_2(\chi_\Omega w_0) + \chi_\Omega \Sigma + \frac{1}{2\pi} \chi_\Omega (N_1(v) + N_2(\eta), 1)_\theta + \Lambda(w_0), \quad (8.22)
\]
with the linear operator \( H_2 \) defined as
\[
H_2 := H - \frac{1}{2} \tilde{\chi}_\Omega \frac{y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} = H + \frac{1}{2} \left| \frac{\tilde{\chi}_\Omega y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} \right|,
\]
and \( \Lambda \) is a linear operator defined as
\[
\Lambda(w_0) := \left( \frac{1}{2} y \cdot \nabla_y \chi_\Omega \right) (1 - \tilde{\chi}_\Omega) w_0 + (\partial_\tau \chi_\Omega) w_0 - (\Delta_y \chi_\Omega) w_0 - 2\nabla_y \chi_\Omega \cdot \nabla_y w_0. \] (8.23)
Observing the operator $e^{-\frac{1}{8}|y|^2} H_2 e^{\frac{1}{8}|y|^2}$, mapping $L^2$ space into itself, is self-adjoint, we transform the equation accordingly

$$
\partial_t (e^{-\frac{1}{8}|y|^2} \chi w_0) = -\mathcal{L} (e^{-\frac{1}{8}|y|^2} \chi w_0) + e^{-\frac{1}{8}|y|^2} \left[ \chi \left( \Sigma + \frac{1}{2\pi} \langle N_1 (v) + N_2 (\eta), 1 \rangle_\theta \right) + \Lambda (w_0) \right],
$$

with the linear operator $\mathcal{L}$ defined as

$$
\mathcal{L} := e^{-\frac{1}{8}|y|^2} H_2 e^{\frac{1}{8}|y|^2} = -\Delta_y + \frac{1}{16} |y|^2 - \frac{3}{4} a - \frac{1}{2} \tau - \frac{3}{4} + V_1 + \frac{1}{2} \chi \phi \cdot \nabla_y \chi.
$$

The orthogonality conditions imposed on $e^{-\frac{1}{8}|y|^2} \chi \omega$ in (3.7) imply that

$$
e^{-\frac{1}{8}|y|^2} \chi \omega_0 \perp e^{-\frac{1}{8}|y|^2} y_k, e^{-\frac{1}{8}|y|^2} \left( \frac{1}{2} y_k^2 - 1 \right), k = 1, 2, 3,
$$

$$
e^{-\frac{1}{8}|y|^2} y_m y_n, m \neq n, m, n = 1, 2, 3.
$$

Define the orthogonal projection onto the $L^2$ subspace orthogonal to these 13 functions by $P_{13}$, which makes

$$
P_{13} e^{-\frac{1}{8}|y|^2} \chi \omega_0 = e^{-\frac{1}{8}|y|^2} \chi \omega_0.
$$

Apply the operator $P_{13}$ on (8.24), and then apply Duhamel’s principle to have

$$
e^{-\frac{1}{8}|y|^2} \chi \omega_0 = U_1 (\tau, \xi_0) e^{-\frac{1}{8}|y|^2} \chi \omega_0 (\xi_0) + \int_{\xi_0}^\tau U_1 (\tau, s) P_{13} e^{-\frac{1}{8}|y|^2} \Lambda (w_0) (s) \, ds
$$

$$
+ \int_{\xi_0}^\tau U_1 (\tau, s) P_{13} e^{-\frac{1}{8}|y|^2} \chi \left( \Sigma (s) + \frac{1}{2\pi} \langle N_1 + N_2, 1 \rangle_\theta (s) \right) \, ds,
$$

where $U_1 (\sigma_1, \sigma_2)$ is the propagator generated by the linear operator $-P_{13} \mathcal{L} P_{13}$ from $\sigma_2$ to $\sigma_1$, with $\sigma_1 \geq \sigma_2$.

The propagator satisfies the following estimate:

**Lemma 8.3.** There exists a constant $C$, such that for any function $g$ and for any times $\sigma_1 \geq \sigma_2 \geq \xi_0$, we have that

$$
\| \langle y \rangle^{-3} e^{\frac{1}{8}|y|^2} U_1 (\sigma_1, \sigma_2) P_{13} g \|_\infty \leq C e^{-\frac{1}{8}(\sigma_1 - \sigma_2)} \| \langle y \rangle^{-3} e^{\frac{1}{8}|y|^2} g \|_\infty.
$$

**Proof.** Recall that

$$
\mathcal{L} = -\Delta_y + \frac{1}{16} |y|^2 - \frac{3}{4} a - \frac{1}{2} \tau - \frac{3}{4} + V_1 + V_2
$$
with the multipliers $V_1$ and $V_2$ are nonnegative and defined as

$$V_1 := \left[ \frac{ay^TBy}{2 + y^TBy} + \tau^{-\frac{1}{2}} \right] \chi_\Omega, \quad V_2 := \frac{1}{2} \left| \frac{\chi_\Omega \cdot \nabla y \chi_\Omega}{\chi_\Omega} \right| \geq 0,$$

here we need the estimates on $B$ and $a$ in (3.8) and (3.9) to prove that $V_1 \geq 0$.

In the previous papers [4, 8, 15, 14, 13, 10], similar results were proved for the propagator generated by the linear operator $L_1$, which is defined as

$$L_1 := -\Delta_y + \frac{1}{16} |y|^2 - \frac{3}{4} - a - \frac{1}{2} + \frac{ay^T B_1 y}{2 + y^T B_1 y},$$

with $B_1$ is positive definite: $B_1 = \tau^{-1}I + O(\tau^{-2})$ and moreover in [12], we considered

$$L_2 := -\Delta_y + \frac{1}{16} |y|^2 - \frac{3}{4} - 1 + V_2.$$

As shown in [12], the techniques in estimating the propagators are almost identical, resulted by that both $V_1$ and $V_2$ are favorably nonnegative, and $\|(1+|y|)^{-5}V_k(\tau)\|_\infty$, $|\nabla_y V_k| \to 0$ as $\tau \to \infty$, $k = 1, 2$.

Here, the techniques in proving Lemma 8.3 is identical to these used in [12]. Hence here we choose to skip the proof.

Apply the propagator estimate to (8.26) to find

$$\| \langle y \rangle^{-3} \chi_R w_0(\cdot, \tau) \|_\infty \leq e^{-\frac{2}{3}(\tau - \tau_0)} \| \langle y \rangle^{-3} \chi_R w_0(\cdot, \xi_0) \|_\infty + \int_{\xi_0}^{\tau} e^{-\frac{2}{3}(\tau - s)} \| \langle y \rangle^{-3} \Lambda(w_0)(s) \|_\infty ds$$

$$+ \int_{\xi_0}^{\tau} e^{-\frac{2}{3}(\tau - s)} \| \langle y \rangle^{-3}(\Sigma + \frac{1}{2\pi} (N_1 + N_2, 1_\theta)(s) \|_\infty ds.$$

(8.29)

The terms on the right hand side satisfy the following estimates:

**Proposition 8.4.**

$$\| \langle y \rangle^{-3} \Lambda(w_0) \|_\infty \leq \delta \kappa(\epsilon) \Omega^{-4},$$

$$\| \langle y \rangle^{-3} \chi_R \Sigma \|_\infty \leq \tau^{-2},$$

$$\| \langle y \rangle^{-3} \chi_R N_1 \|_\infty \leq \delta (\tau^{-2} \kappa(\epsilon) \Omega^{-4}) P(M),$$

$$\| \langle y \rangle^{-3} \chi_R (N_2, 1_\theta)(s) \|_\infty \leq \delta (\tau^{-2} + \kappa(\epsilon) \Omega^{-4}) M_1.$$
The proposition will be proved in Subsection 8.1.1. Here $\kappa(\epsilon)$ and $P(M)$ are defined in (3.20) and (2.3) respectively.

We continue to study (8.29). Apply the estimates in Proposition 8.4 and the estimate
\[
\|\langle y \rangle^{-3} \chi w(\cdot, \xi_0)\|_\infty \lesssim \kappa(\epsilon) \Omega^{-4}(\xi_0)
\]
from (4.19) to obtain the desired result,
\[
\|\langle y \rangle^{-3} \chi w_0(\cdot, \tau)\|_\infty \lesssim e^{-\frac{2}{5}(\tau - \xi_0)} \Omega^{-4}(\xi_0) + (\tau^{-2} + \kappa(\epsilon) \Omega^{-4})(1 + \delta P(M))
\]
\[
\lesssim (\tau^{-2} + \kappa(\epsilon) \Omega^{-4})(1 + \delta P(M)),
\]
here we use the following facts: (1) $M_k$, $k = 1, 2, 3, 4$, are increasing functions, (2) for any $k > 0$, there exists a constant $C_k$ such that
\[
\int_{\xi_0}^\tau e^{-\frac{2}{5}(\tau - s)} \Omega^{-k}(s) \, ds \leq C_k \Omega^{-k}(\tau), \quad \text{and} \quad \int_{\xi_0}^\tau e^{-\frac{2}{5}(\tau - s)} s^{-2} \, ds \lesssim \tau^{-2}.
\]

To prove the first estimates in (8.35), we find a function equivalent to the function $\Omega^{-k}$, namely there exist constants $C_k$ such that, for $\tau \geq \xi_0,$
\[
\frac{1}{C_k} \leq \frac{\Omega^{-k}(\tau)}{\min\{\Omega^{-k}(\xi_0), (1 + \tau - \xi_0)^{-\frac{11}{20}k}\}} \leq C_k.
\]

Compute directly to obtain
\[
\int_{\xi_0}^\tau e^{-\frac{2}{5}(\tau - s)} \Omega^{-k}(\xi_0) ds \lesssim \Omega^{-k}(\xi_0),
\]
and apply L’Hospital’s rule to obtain, for some $a_k > 0$,
\[
\int_{\xi_0}^\tau e^{-\frac{2}{5}(\tau - s)} \left(2 + s - \xi_0\right)^{-\frac{11}{20}k} ds \leq a_k \left(2 + \tau - \xi_0\right)^{-\frac{11}{20}k}.
\]

We take the minimum of these two estimates, together with (8.36), to obtain the desired first estimate in (8.35).

The second estimate in (8.35) will be proved similarly, hence we skip the details here. 

\[\square\]
8.1.1 Proof of Proposition 8.4

Proof. To prove (8.30), we need (8.20) and the definition of \( \chi_\Omega(y) = \chi\left(\frac{y}{\Omega}\right) \) and the definitions of \( \Omega \) and \( \kappa(\epsilon) \) in (2.3) and (3.20). These, together with that \( V_{a,B}^{-1}w, |\partial_y^k w| \leq \delta, |k| = 1, 2, \) implied by (3.4) and (8.2), and \( V_{a,B}(y) \leq \sqrt{1 + \tau^{-1}\Omega^2} \), imply the desired result,

\[
\|y^{-3}\Lambda(w_0)\|_\infty \lesssim \delta \sqrt{1 + \tau^{-1}\Omega^2} \left[ \|y^{-3}(\frac{1}{2}y \cdot \nabla_y \chi_\Omega (1 - \tilde{\chi}_\Omega))\|_\infty + \|y^{-3}\partial_\tau \chi_\Omega\|_\infty + \|y^{-3}\Delta_y \chi_\Omega\|_\infty \right] + \delta\|y^{-3}\nabla_y \chi_\Omega\|_\infty \lesssim \delta \kappa(\epsilon) \sqrt{1 + \tau^{-1}\Omega^2} \Omega^{-5} + \delta \kappa(\epsilon) \Omega^{-4} \leq \delta \kappa(\epsilon) \Omega^{-4} .
\]

It is easy to prove (8.31), by the estimates on the scalar functions in (3.8)-(3.10) and the estimates on the equations in (3.11).

To prove (8.32), we observe that

\[
|N_1(v)| \lesssim \delta |\nabla_y v|^2 + v^{-1} |\partial_\theta v|^2.
\]

Decompose \( v \), and use the estimates in (3.8)-(3.10) to find

\[
|\nabla_y v|^2 \lesssim \tau^{-2} |y|^2 V_{a,B}^{-2} + (1 + |y|^2) \tau^{-4} + |\nabla_y w|^2,
\]

\[
v^{-1} |\partial_\theta v|^2 \lesssim (1 + |y|^2) \tau^{-4} + v^{-1} |\partial_\theta w|^2.
\]

We claim that

\[
\|y^{-3}\chi_\Omega |\nabla_y w|^2\|_\infty, \|y^{-3}\chi_\Omega v^{-1} |\partial_\theta w|^2\|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-4} P(M).
\]

Suppose these hold, then these together with (8.39) imply the desired estimates (8.32).

It is easier to prove the second estimate in (8.41), since \( \partial_\theta \) and \( \chi_\Omega \) commute with each other. Hence we choose to skip the details here.

To prove the first estimate in (8.41), it is easy to see that

\[
\|y^{-3}\chi_\Omega |\nabla_y w|^2\|_\infty \lesssim \|y^{-2}\chi_\Omega \nabla_y w\|_\infty \|y^{-1} \mathbb{1}_{(1+\epsilon)\Omega} \nabla_y w\|_\infty ,
\]

where \( \mathbb{1}_{(1+\epsilon)\Omega} \) is the Heaviside function taking value 1 for \( |y| \leq (1 + \epsilon)\Omega \), and 0 otherwise.

We bound the first factor by

\[
\|y^{-2}\chi_\Omega \nabla_y w\|_\infty \lesssim \kappa(\epsilon) \Omega^{-3} \mathcal{M}_2 + \Omega^{-3 \frac{1}{2}} \mathcal{M}_1^{\frac{3}{2}}.
\]
To see this, we change the order of $\nabla_y$ and $\chi_\Omega$ to find
\[
\chi_\Omega \nabla_y w = \nabla_y (\chi_\Omega w) - w \nabla_y \chi_\Omega.
\]
Apply Lemma 8.2 to control the first term
\[
\|\langle y \rangle^{-2} \nabla_y (\chi_\Omega w)\|_\infty \lesssim \kappa(\epsilon) \Omega^{-3} M_2.
\] (8.43)
For the second term, we have
\[
\langle y \rangle^{-2} |w \nabla_y \chi_\Omega| \leq \Omega^{-3} |\chi_\Omega w|^{\frac{3}{2}} |w|^{\frac{1}{2}} \sup_z \left\{ \left| \frac{\nabla_z \chi(z)}{\chi^\frac{3}{2}(z)} \right| \right\} \lesssim \Omega^{-\frac{2}{3}} (\kappa(\epsilon) \Omega^{-4} + \tau^{-2}) \frac{2}{3} M_1^2,
\] (8.44)
based on the following four facts, besides using the condition in (8.2) and the definition of $\Omega$ in (2.3), (1) that $\nabla_y \chi_\Omega = \frac{1}{\Omega} (\nabla_z \chi)|_{z=\frac{y}{\Omega}}$ and is supported by the set $|y| \in \Omega, (1 + \epsilon) \Omega$, and (2) that $|\chi^{-\frac{4}{3}} \nabla_z \chi| \leq \kappa(\epsilon)$ in (3.20), and (3) $|w| \lesssim \delta V_{a,B} \lesssim \delta \sqrt{1 + \tau^{-1} \Omega^2}$ by (3.4), and lastly (4)
\[
|\chi_\Omega w| \lesssim \Omega^2 \langle y \rangle^{-3} \chi_\Omega w \|_\infty \lesssim \Omega^2 (\kappa(\epsilon) \Omega^{-4} + \tau^{-2}) M_1.
\]
To estimate $\|\langle y \rangle^{-1} 1_{\leq (1+\epsilon) \Omega} \nabla_y w\|_\infty$, we insert $1 = \chi_\Omega + 1 - \chi_\Omega$ before $\nabla_y w$ to find
\[
\|\langle y \rangle^{-1} 1_{\leq (1+\epsilon) \Omega} \nabla_y w\|_\infty \lesssim \|\langle y \rangle^{-1} \chi_\Omega \nabla_y w\|_\infty + \|\langle y \rangle^{-1} 1_{\leq (1+\epsilon) \Omega} (1 - \chi_\Omega) \nabla_y w\|_\infty
\lesssim \Omega \|\langle y \rangle^{-2} \chi_\Omega \nabla_y w\|_\infty + \delta \Omega^{-1}
\lesssim \kappa(\epsilon) \Omega^{-2} (M_2 + \Omega^{-3-\frac{4}{3}} M_1^2) + \delta \Omega^{-1},
\] (8.45)
where, in the second step we use that the cutoff function $\chi_\Omega$ is supported on the set $|y| \leq (1 + \epsilon) \delta$, and use $|\nabla_y w| \lesssim \delta$ by (3.4), and in the last step use (8.42).

This and (8.42) imply the desired (8.41).

Next we prove (8.33). From the definition of $N_2(\eta)$ in (4.3) we derive
\[
\langle N_2(\eta), 1 \rangle_\theta = -\langle V_{a,B}^{-2} v^{-1} \eta^2, 1 \rangle_\theta + \langle (v^{-2} - V_{a,B}^{-2}) \partial_\theta \eta, 1 \rangle_\theta
= -\langle V_{a,B}^{-2} v^{-1} \eta^2, 1 \rangle_\theta + 2 \langle v^{-3} (\partial_\theta \eta)^2, 1 \rangle_\theta,
\] (8.46)
where we simplify the second term by observing that $V_{a,B}^{-2}$ is independent of $\theta$, integrating by parts in $\theta$ and using $\partial_\theta v = \partial_\theta \eta$.

Apply the estimates in (3.4), decompose $\eta$ as in (4.4), and apply (8.6) again to have the desired result,
\[
\|\langle y \rangle^{-3} \chi_\Omega \langle N_2(\eta), 1 \rangle_\theta\|_\infty \lesssim \delta \sum_{l=0,1} \|\langle y \rangle^{-3} \chi_\Omega \partial_\theta^l \eta\|_\infty \lesssim \delta \left[ \tau^{-2} + \kappa(\epsilon) \Omega^{-4} \right](1 + M_1).
\] (8.47)
8.2 Proof of (8.4)

Since \( w_{-1} = \overline{w} \), we only need to estimate \( w_{1} \).

We derive an equation for \( w_{1} \) from (4.6),
\[
\partial_{\tau} \chi_{\Omega} w_{1} = - \left( - \Delta_{y} + \frac{1}{2} y \cdot \nabla_{y} - \frac{1}{2} \right) \chi_{\Omega} w_{1} + \frac{1}{2 \pi} \chi_{\Omega} (G + N_{1}(v) + N_{2}(\eta), e^{i\theta}) + \mu(w_{1}),
\]
where the linear operator is derived from the operator \( L \) in (4.6) by
\[
\frac{1}{2 \pi} \langle Lw, e^{i\theta} \rangle_{\theta} = \left( - \Delta_{y} + \frac{1}{2} y \cdot \nabla_{y} - \frac{1}{2} \right) w_{1},
\]
by the cancellation \( \langle -V_{a,B}^{2} \partial_{2} \theta \chi \Omega u_{1}, e^{i\theta} \rangle_{\theta} = 0, \) and we use that \( \langle F(B,a), e^{i\theta} \rangle_{\theta} = 0 \) since \( F(B,a) \) is independent of \( \theta \).

As in (8.18), to control the term \( \frac{1}{2} (y \cdot \nabla_{y} \chi_{\Omega} w_{1}) \) in \( \mu(w_{1}) \), we decompose it into two parts
\[
\frac{1}{2} (y \cdot \nabla_{y} \chi_{\Omega} w_{1}) = \frac{1}{2} (y \cdot \nabla_{y} \chi_{\Omega}) \chi_{\Omega} w_{1} + \frac{1}{2} (y \cdot \nabla_{y} \chi_{\Omega}) (1 - \chi_{\Omega}) w_{1}
\]
and move the first part into the linear operator and the second to the remainder. This makes
\[
\partial_{\tau} \chi_{\Omega} w_{1} = - H_{1} w_{1} + \frac{1}{2 \pi} \chi_{\Omega} (G + N_{1}(v) + N_{2}(\eta), e^{i\theta}) + \Lambda(w_{1}),
\]
where the linear operator \( H_{1} \) is defined as
\[
H_{1} := - \Delta_{y} + \frac{1}{2} y \cdot \nabla_{y} - \frac{1}{2} + \frac{1}{2} \left| \chi_{\Omega} y \cdot \nabla_{y} \chi_{\Omega} \right|
\]
and \( \Lambda(w_{1}) \) is defined in the same fashion to that \( \Lambda(w_{0}) \) in (8.23).

The orthogonality conditions imposed on \( \chi_{\Omega} w \) imply that
\[
e^{-\frac{1}{8} |y|^{2}} \chi_{\Omega} w_{1} \perp e^{-\frac{1}{8} |y|^{2}}, \ y_{k} e^{-\frac{1}{8} |y|^{2}}, \ k = 1, 2, 3.
\]
We denote, by \( P_{4} \), the orthogonal projection onto the subspace orthogonal to these four functions, which makes
\[
P_{4} e^{-\frac{1}{8} |y|^{2}} \chi_{\Omega} w_{1} = e^{-\frac{1}{8} |y|^{2}} \chi_{\Omega} w_{1}.
\]

On (8.51) we apply \( e^{-\frac{1}{8} |y|^{2}} \), and then \( P_{4} \), and then Duhamel’s principle to have
\[
e^{-\frac{1}{8} |y|^{2}} \chi_{\Omega} w_{1}(\tau) = U_{2}(\tau, \xi_{0}) e^{-\frac{1}{8} |y|^{2}} \chi_{\Omega} w_{1}(\xi_{0})
+ \int_{\xi_{0}}^{\tau} U_{2}(\tau, \sigma) P_{4} \left[ \frac{1}{2 \pi} \chi_{\Omega} (G + N_{1}(v) + N_{2}(\eta), e^{i\theta}) + \Lambda(w_{1}) \right](\sigma) \, d\sigma.
\]
where $U_2(\tau, \sigma)$ is the propagator generated by $-P_4 e^{-\frac{i}{\hbar} |y|^2} H_1 e^{\frac{i}{\hbar} |y|^2} P_4$ from $\sigma$ to $\tau$.

The propagator satisfies the following estimates, as discussed in the proof of Lemma 8.3, its proof is very similar to the proved cases, thus we choose to skip it.

**Lemma 8.5.** For $l = 2, 3$, and any function $g$,

\[
\|\langle y \rangle^{-l} e^{\frac{i}{\hbar} |y|^2} U_2(\tau, \sigma) P_4 g\|_\infty \lesssim e^{-\frac{3}{\hbar} (\tau-\sigma)} \|\langle y \rangle^{-l} e^{\frac{i}{\hbar} |y|^2} g\|_\infty.
\] (8.54)

For the terms on the right hand side we have, recall that $P(M)$ is defined in (8.1),

**Proposition 8.6.** For $l = 2, 3$,

\[
\|\langle y \rangle^{-l} \langle N_1(v), e^{i\theta} \rangle\|_\infty \lesssim \delta \tau^{-2} + \delta \kappa(\epsilon) \Omega^{-l-1} P(M),
\] (8.55)

\[
\|\langle y \rangle^{-l} \langle N_2(\eta), e^{i\theta} \rangle\|_\infty \lesssim \delta \tau^{-4} + \delta \kappa(\epsilon) \Omega^{-l-1} P(M),
\] (8.56)

\[
\|\langle y \rangle^{-l} \Lambda(\omega_1)\|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-l-1},
\] (8.57)

\[
\|\langle y \rangle^{-l} \langle G, e^{i\theta} \rangle\|_\infty \lesssim \delta \tau^{-2}.
\] (8.58)

The proposition will be proved in subsubsection 8.2.1

Suppose the proposition holds, then we prove the desired results for $\mathcal{M}_1$ and $\mathcal{M}_4$ in (8.1) as in (8.31). Here we choose to skip the details.

**8.2.1 Proof of Proposition 8.6**

**Proof.** In what follows we only consider the case $l = 3$. The proof for $l = 2$ is considerably easier since the wanted decay estimate is slower. Hence we skip that part.

For (8.55), since $\|\langle y \rangle^{-3} \langle N_1, e^{i\theta} \rangle\|_\infty \lesssim \|\langle y \rangle^{-3} N_1\|_\infty$, we use (8.32) as the desired results.

The proof of (8.57) is identical to that of (8.30), hence we skip the details here.

For (8.56), some cancellations simplify the expression

\[
\langle N_2(\eta), e^{i\theta} \rangle = \langle -v^{-1} + V_{a,B}^{-1} - V_{a,B}^{-2} \eta + (v^{-2} - V_{a,B}^{-2}) \partial_\eta^2 \eta, e^{i\theta} \rangle = \langle -v^{-1} + v^{-2} \partial_\eta^2 \eta, e^{i\theta} \rangle = -2 \langle v^{-3} (\partial_\eta^2 \eta)^2, e^{i\theta} \rangle,
\] (8.59)

where we use the following identities, $\langle -V_{a,B}^{-2} \eta - V_{a,B}^{-2} \partial_\eta^2 \eta, e^{i\theta} \rangle = 0$ and $\langle V_{a,B}^{-1}, e^{i\theta} \rangle = 0$ by that $V_{a,B}$ is independent of $\theta$, $(v^{-1}, e^{i\theta}) = -\langle \partial_\eta^2 v^{-1}, e^{i\theta} \rangle$, and $\partial_\theta v = \partial_\theta \eta$.

Decompose $\eta$ as in (4.4) and apply Young’s inequality to obtain

\[
|\langle v^{-3} (\partial_\theta \eta)^2, 1 \rangle_\theta \chi_\Omega| \lesssim (1 + |y|^2) \tau^{-4} + \chi_\Omega \langle v^{-3} (\partial_\theta w)^2, 1 \rangle_\theta.
\] (8.60)
Then apply the results in (8.41) on the last term to have the desired results.

For (8.58), the desired estimate follows from the estimates on the scalar functions in (3.8)-(3.10) and (3.11).

\[\square\]

### 8.3 Proof of (8.5)

In what follows we prove the estimate for
\[
\| (100 + |y|^2)^{\frac{3}{2}} \| \partial_3^3 \theta P \|_{L_\theta^2} \geq 2 \chi_{\Omega} w \| L_\infty^2 \theta \|.
\]
It is easier to prove the desired estimate for
\[
\| (100 + |y|^2)^{-1} \| \partial_3^3 \theta P \|_{L_\theta^2} \geq 2 \chi_{\Omega} w \| L_\infty^2 \theta \|.
\]
since the wanted decay estimate is considerably slower. Hence we skip this part.

We start with deriving an equation for \( \partial_3^3 \theta P \|_{\Omega} \geq 2 \chi_{\Omega} w \| \) instead from the equation for \( v \) in (3.2), instead from the equation for \( w \), since it makes it easier to see some positivity and cancellation. The decomposition of \( v \) in (3.6) implies that
\[
P \geq 2 \omega = P \geq 2 v.
\]
(8.61)

To simplify the notation we define a new function \( \Phi_3 \) as
\[
\Phi_3 := (100 + |y|^2)^{-3} \langle P \geq 2 \partial_3^3 \theta \| \chi_{\Omega} v, \partial_3^3 \theta \| \rangle.
\]
(8.62)

It satisfies the equation
\[
\partial_t \Phi_3 = -(L_3 + V_3) \Phi_3 - 2(100 + |y|^2)^{-3} \| P \geq 2 \partial_3^3 \theta \| \chi_{\Omega} v \|_{L_\theta^2}^2 + 2 \sum_{k=1}^3 \Psi_{3k},
\]
(8.63)

where the linear operator \( L_3 + V_3 \) is related to \(-\Delta + \frac{1}{2} y \cdot \nabla y - 1 \) by
\[
L_3 + V_3 := (100 + |y|^2)^{-3}(-\Delta + \frac{1}{2} y \cdot \nabla y - 1)(100 + |y|^2)^3,
\]
and \( L_3 \) is a differential operator, and \( V_3 \) is a multiplier, defined as
\[
L_3 := -\Delta + \frac{1}{2} y \cdot \nabla y - 2(100 + |y|^2)^{-3} \chi_{\Omega}(100 + |y|^2)^3 \cdot \nabla y,
\]
\[
V_3 := -1 + \frac{3|y|^2}{100 + |y|^2} - \frac{18}{100 + |y|^2} - \frac{24|y|^2}{(100 + |y|^2)^2}.
\]
(8.64)

The functions \( \Psi_{3k}, k = 1, 2, 3 \), in (8.63) are defined as
\[
\Psi_{31} := (100 + |y|^2)^{-3} \langle P \geq 2 \partial_3^3 \theta \| \chi_{\Omega} v, \partial_3^3 \theta \| \rangle, \chi_{\Omega}(v^{-2} \partial_3^3 v - v^{-1}) \rangle,
\]
\[
\Psi_{32} := (100 + |y|^2)^{-3} \langle P \geq 2 \partial_3^3 \theta \| \chi_{\Omega} v, \partial_3^3 \theta \| \rangle, \chi_{\Omega} N_1(v) \rangle,
\]
\[
\Psi_{33} := (100 + |y|^2)^{-3} \langle P \geq 2 \partial_3^3 \theta \| \chi_{\Omega} v, \mu \rangle \rho \rangle.
\]
(8.65)
Here $\mu$-term is defined in the same fashion as that in (4.7). These terms satisfy the following estimates, recall that $P(M)$ is defined in (8.1),

**Proposition 8.7.** There exists some constant $C > 0$ such that

$$
\Psi_{31} \leq -\left(\frac{18}{25} - C\delta\right)(100 + |y|^2)^{-3}V_{a,B}^{-2}\|P_{\theta \geq 2}\partial_\theta^4 \chi v\|_{L_2^a}^2 \\
+ C\delta^2 \left[\tau^{-2} + \kappa(\epsilon)\Omega^{-4}\right]^2 P^2(M),
$$

(8.66)

$$
\Psi_{32} \leq \frac{1}{100}\Phi_3 + C\delta^2 \kappa^2(\epsilon)\Omega^{-8} P^2(M),
$$

(8.67)

$$
\Psi_{33} \leq \frac{1}{100}(100 + |y|^2)^{-3} \left[V_{a,B}^{-2}\|P_{\theta \geq 2}\partial_\theta^3 \chi v\|_{L_2^a}^2 + \|P_{\theta \geq 2}\partial_\theta^3 \nabla_y \chi v\|_{L_2^a}^2\right] \\
+ C\delta^2 \left[\tau^{-2} + \kappa(\epsilon)\Omega^{-4}\right]^2 P^2(M).
$$

(8.68)

The proposition will be proved in subsubsection 8.3.1.

By the proposition, the fact $\|P_{\theta \geq 2}\partial_\theta f\|_{L_2^a}^2 \geq 4\|P_{\theta \geq 2}\tilde{f}\|_{L_2^a}^2$ for any smooth function $f$, and $\delta$ is sufficiently small, we have that, for some $C > 0$,

$$
2 \sum_{k=1,2,3} \Psi_{3k} \leq -\frac{28}{5}(100 + |y|^2)^{-3}V_{a,B}^{-2}\|P_{\theta \geq 2}\partial_\theta^3 \chi v\|_{L_2^a}^2 + \frac{1}{50}\Phi_3 + C\delta^2 \left[\tau^{-2} + \kappa(\epsilon)\Omega^{-4}\right]^2 P^2(M).
$$

Moreover we observe that, since for $|y| \leq \tau^4$, $V_{a,B} \approx \frac{1}{\sqrt{2}}$, and in the other region, $V_3 \approx 2$,

$$
\frac{28}{5}V_{a,B}^{-2}(y) - \frac{1}{50} + V_3(y) \geq 1.
$$

(8.69)

Consequently, for some constant $C > 0$,

$$
\partial_\tau \Phi_3 \leq -(L_3 + 1)\Phi_3 + C\delta^2 \left[\tau^{-2} + \kappa(\epsilon)\Omega^{-4}\right]^2 P^2(M).
$$

(8.70)

The presence of cutoff function $\chi_\Omega$ makes $\Phi_3(y, \tau) = 0$ if $|y| \geq (1 + \epsilon)\Omega$. A standard application of the maximum principle yields,

$$
\Phi_3(\tau) \lesssim e^{-\left(\tau - \xi_0\right)}\Phi_3(\xi_0) + \delta^2 \left[\tau^{-2} + \kappa(\epsilon)\Omega^{-4}\right]^2 P^2(M).
$$

(8.71)

This yields the desired estimate after taking a square root on both sides, recall that $\Phi_3(\xi_0) \lesssim \kappa^2(\epsilon)\Omega^{-8}(\xi_0)$ implied by (4.16). Recall the definitions of $R(\tau)$ and $\Omega(\tau)$ in (4.8) and (2.3) respectively, and that $\xi_0 \gg \tau_0$. 37
8.3.1 Proof of Proposition 8.7

Proof. We start proving (8.66) by separating the negative part of \( \Psi_{31} \) from the rest,

\[
\Psi_{31}(v) = (100 + |y|^2)^{-3} \left[ - \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, \partial_{\theta}^2 \chi \Omega v^{-2} \partial_{\theta}^2 v \rangle_{\theta} - \langle P_{\theta,\geq 2} \partial_{\theta}^3 \chi \Omega v, \partial_{\theta}^3 \chi \Omega v^{-1} \rangle_{\theta} \right] \\
= -(100 + |y|^2)^{-3} \left[ D_1 + D_2 + D_3 \right],
\]

where in the first term we integrate by parts in \( \theta \), and \( D_l, \ l = 1, 2, 3, \) are defined as

\[
D_1 := \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, v^{-2} P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v \rangle_{\theta} - \langle P_{\theta,\geq 2} \partial_{\theta}^3 \chi \Omega v, v^{-2} P_{\theta,\geq 2} \partial_{\theta}^3 \chi \Omega v \rangle_{\theta},
\]

\[
D_2 := \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, v^{-2}(1 - P_{\theta,\geq 2}) \partial_{\theta}^4 \chi \Omega v \rangle_{\theta} + \langle P_{\theta,\geq 2} \partial_{\theta}^3 \chi \Omega v, v^{-2}(1 - P_{\theta,\geq 2}) \partial_{\theta}^3 \chi \Omega v \rangle_{\theta},
\]

\[
D_3 := \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, \chi \Omega \partial_{\theta}^2 (v^{-2} \partial_{\theta}^2 v) - \partial_{\theta}^4 v \rangle_{\theta} - \langle P_{\theta,\geq 2} \partial_{\theta}^3 \chi \Omega v, \chi \Omega (\partial_{\theta}^3 v^{-1} + v^{-2} \partial_{\theta}^3 v) \rangle_{\theta}.
\]

Apply \(|V_{a,B} - 1| \lesssim \delta\) in (3.4), and use that for any smooth function \( f \), \( \|P_{\theta,\geq 2} \partial_{\theta}^4 f\|_{L^2_\theta} \geq 4\|P_{\theta,\geq 2} \partial_{\theta}^4 f\|_{L^2_\theta}^2 \) to find, for some \( C > 0 \),

\[
D_1(v) \geq \left( \frac{3}{4} - C\delta \right) V_{a,B}^2 \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v \rangle_{\theta}.
\]

For \( D_2 \) we observe that

\[
(1 - P_{\theta,\geq 2}) \partial_{\theta}^4 \chi \Omega v = \frac{1}{2\pi} \left[ e^{i\theta} \langle v, e^{i\theta} \rangle_{\theta} + e^{-i\theta} \langle v, e^{-i\theta} \rangle_{\theta} \right]
\]

thus

\[
\langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, v^{-2}(1 - P_{\theta,\geq 2}) \partial_{\theta}^4 \chi \Omega v \rangle_{\theta} = \frac{1}{2\pi} \left[ K \chi \Omega \langle v, e^{i\theta} \rangle_{\theta} + \chi \Omega \overline{K(v, e^{-i\theta})} \right].
\]

Here the term \( K \) is defined as, using that \( P_{\theta,\geq 2} P_{\theta,\geq 2} = P_{\theta,\geq 2} \),

\[
K : = \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, v^{-2} e^{i\theta} \rangle_{\theta} = \langle P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v, P_{\theta,\geq 2} v^{-2} e^{i\theta} \rangle_{\theta}.
\]

Since \( v = V_{a,B} + \eta, V_{a,B} \) is independent of \( \theta \), and \(|\eta|_{a,B} \lesssim \delta\) in (3.4), we have

\[
|P_{\theta,\geq 2} v^{-2} e^{i\theta}| \lesssim \delta V_{a,B}^{-2}, \quad \text{and hence} \quad |K| \lesssim \delta V_{a,B}^{-2} \|P_{\theta,\geq 2} \partial_{\theta}^4 \chi \Omega v\|_{L^2_\theta}.
\]

Returning to (8.74), we decompose \( v \) and apply (8.6) to find that

\[
(100 + |y|^2)^{-\frac{3}{2}} \langle v, e^{i\theta} \rangle \lesssim \tau^{-2} + (100 + |y|^2)^{-\frac{3}{2}} \|w\|_{L^2_\theta} \lesssim (\tau^{-2} + \kappa(\epsilon) \Omega^{-4})(1 + M_2).
\]

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These together with Young’s inequality make, for some $C > 0$,
\[
(100 + |y|^2)^{-3}|\langle P_{\theta, \geq 2} \partial^4_\theta \chi \Omega v, v^{-2}(1 - P_{\theta, \geq 2})\partial^4_\theta v \rangle| \leq \frac{1}{100} V_{a, B}(100 + |y|^2)^{-3}\|P_{\theta, \geq 2} \partial^4_\theta \chi \Omega v\|_{L^2_\theta}^2 + C \delta^2 (\tau^{-2} + \kappa(\epsilon) \Omega^{-4})^2 (1 + \mathcal{M}_1)^2.
\]
(8.76)

The second term in $D_2$ will be processed similarly. Consequently,
\[
|(100 + |y|^2)^{-3}D_2| \leq \frac{1}{50} (100 + |y|^2)^{-3} V_{a, B}^2\|P_{\theta, \geq 2} \partial^4_\theta \chi \Omega v\|_{L^2_\theta}^2 + C \delta^2 (1 + \mathcal{M}_1^2)(\tau^{-2} + \kappa(\epsilon) \Omega^{-4})^2.
\]
(8.77)

To estimate $D_3$ we compute directly to have
\[
v|\partial^2_\theta (v^{-2} \partial^4_\theta v) - v^{-2} \partial^4_\theta v|, \ v|\partial^2_\theta v^{-1} + v^{-2} \partial^4_\theta v| \lesssim \delta \sum_{k=1,2} |\partial^2_\theta v| \lesssim \delta \left[(1 + |y|)\tau^{-2} + \sum_{k=1,2} |\partial^2_\theta w|\right].
\]
This, after applying Lemma 4.3 and Young’s inequality, implies, for some $C > 0$,
\[
|(100 + |y|^2)^{-3}D_3| \leq \frac{1}{100} (100 + |y|^2)^{-3} V_{a, B}^2\|P_{\theta, \geq 2} \partial^4_\theta \chi \Omega v\|_{L^2_\theta}^2 + C \delta^2 \left[(\tau^{-2} + \kappa(\epsilon) \Omega^{-4})^2 \mathcal{M}_1^2\right].
\]
(8.78)

Collect the estimates above to find that $\Psi_{31}$ satisfies the desired estimates (8.66).

Now we prove (8.68). We divide $\Psi_{32}$ into two parts
\[
\Psi_{32} = (100 + |y|^2)^{-3}\left[\langle P_{\theta, \geq 2} \partial^3_\theta \chi \Omega v, \partial^3_\theta \chi \Omega N_{11}\rangle + \langle P_{\theta, \geq 2} \partial^3_\theta \chi \Omega v, \partial^3_\theta \chi \Omega N_{12}\rangle\right]
:= (100 + |y|^2)^{-3}[W_1 + W_2],
\]
(8.79)
where $W_l$, $l = 1, 2$, are naturally defined, and $N_{1l}$, $l = 1, 2$, are two parts of $N_1$:
\[
N_1 = N_{11} + N_{12},
\]
(8.80)
where $N_{11}$ is defined as
\[
N_{11} := -\sum_{k=1}^3 \frac{(\partial_{y_k} v)^2 \partial^2_{y_k} v}{1 + |\nabla y_k|^2 + (\frac{\partial_{y_k} y}{v})^2} - \sum_{i \neq j} \frac{\partial_{y_i} v \partial_{y_j} v}{1 + |\nabla y_j|^2 + (\frac{\partial_{y_j} y}{v})^2} \partial_{y_i} \partial_{y_j} v,
\]
and $N_{12}$ is different from $N_{11}$ by that each term has a factor $v^{-l}$, $l = 2, 3, 4$,
\[
N_{12} := -v^{-4} \frac{(\partial_{\theta} v)^2 \partial^2_{\theta} v}{1 + |\nabla y|^2 + (\frac{\partial_{\theta} y}{v})^2} + v^{-2} \frac{2\partial_{\theta} v}{1 + |\nabla y|^2 + (\frac{\partial_{\theta} y}{v})^2} \sum_{l=1}^3 \partial_{y_l} v \partial_{y_l} \partial_{\theta} v
+ v^{-3} \frac{(\partial_{\theta} v)^2}{1 + |\nabla y|^2 + (\frac{\partial_{\theta} y}{v})^2}.
\]

We start with estimating $W_1$.

Since all the terms in $N_{11}$ are of the form $\sum_{1+|\partial_v|+v, |\partial_{\theta}v|}^{c_{k,l}} \partial_{\theta} v \partial_{y_i} v \partial_{y_j} \partial_{y_k} v$ for some constants $c_{k,l}$, $k, l = 1, \ldots, 3$, we apply (3.3) and compute directly to obtain

$$\chi_{\Omega} |\partial_{\theta}^3 N_{11}| \lesssim \chi_{\Omega} \sum_{m=0}^3 \sum_{k,l=1,2,3} |\partial_{\theta}^m (\partial_{y_k} v \partial_{y_l} v \partial_{y_i} \partial_{y_j} v)| \lesssim \delta \chi_{\Omega} \sum_{m+n=1,2,3} |\partial_{\theta}^m \nabla y v||\partial_{\theta}^n \nabla y v|. \quad (8.81)$$

We start with processing $\chi_{\Omega} |\nabla y v||\partial_{\theta}^3 \nabla y v|$, the only term containing a fourth order derivative. We use $|\nabla y v| \lesssim \delta$ in (3.4), and decompose $v$, and change the order $\chi_{\Omega}$ and $\nabla y$ to find,

$$(100 + |y|^2)^{-\frac{3}{2}} \chi_{\Omega} \|\nabla y v \partial_{\theta}^3 \nabla y v\|_{L^2_\theta} \lesssim \delta (100 + |y|^2)^{-\frac{3}{2}} \|P_{\theta,2} \partial_{\theta}^3 \nabla y \chi_{\Omega} w\|_{L^2_\theta} + \|\langle y \rangle^{-1} \chi_{\Omega} \nabla y v\|_{L^\infty} \sum_{m=\pm 1} \|\langle y \rangle^{-2} \nabla y \chi_{\Omega} w_m\|_{L^\infty}$$

$$+ \delta (100 + |y|^2)^{-\frac{3}{2}} \|\nabla y \chi_{\Omega} \|\partial_{\theta}^3 w\|_{L^2_\theta} + \delta \tau^{-2} \lesssim \delta (100 + |y|^2)^{-\frac{3}{2}} \|P_{\theta,2} \partial_{\theta}^3 \nabla y \chi_{\Omega} v\|_{L^2_\theta} + (\tau^{-2} + \kappa(\epsilon) \Omega^{-4}) P(M),$$

where, we control $\|\langle y \rangle^{-1} \chi_{\Omega} \nabla y v\|_{L^\infty}$ as in (8.45) after decomposing $v$, and we observe that, for $m = \pm 1$, by the definition of $\mathcal{M}_4$,

$$\|\langle y \rangle^{-2} \nabla y \chi_{\Omega} w_m\|_{L^\infty} \lesssim \|\langle y \rangle^{-2} \nabla y \chi_{\Omega} \partial_{\theta} w\|_{L^\infty} \lesssim \kappa(\epsilon) \Omega^{-3} \mathcal{M}_4,$$

and moreover in the last step we use $P_{\theta,2} w = P_{\theta,2} v$, and we argue as in (8.44) to find

$$(100 + |y|^2)^{-\frac{3}{2}} \|\nabla y \chi_{\Omega} \|\partial_{\theta}^3 w\|_{L^2_\theta} \lesssim \kappa(\epsilon) \Omega^{-\frac{9}{2}} \mathcal{M}_4^{\frac{3}{4}}, \quad (8.82)$$

Similarly, for the other terms in (8.81), we find, for any $m + n = 0, 1, 2, 3$, and $m, n \leq 2$,

$$(100 + |y|^2)^{-\frac{3}{2}} \chi_{\Omega} \|\partial_{\theta}^m \nabla y v \partial_{\theta}^n \nabla y v\|_{L^2_\theta} \lesssim (\tau^{-2} + \kappa(\epsilon) \Omega^{-4}) P(M). \quad (8.83)$$

Collect the estimates above to obtain

$$(100 + |y|^2)^{-\frac{3}{2}} \chi_{\Omega} \|\partial_{\theta}^3 N_{11}\|_{L^2_\theta} \lesssim \delta (100 + |y|^2)^{-\frac{3}{2}} \|\partial_{\theta}^3 \nabla y P_{\theta,2} \chi_{\Omega} w\|_{L^2_\theta} + \delta (\tau^{-2} + \kappa(\epsilon) \Omega^{-4}) P(M). \quad (8.84)$$

Returning to the definition of $W_1$ and applying Young’s inequality, we find

$$(100 + |y|^2)^{-3} |W_1| \leq \frac{1}{100} (100 + |y|^2)^{-3} \|\partial_{\theta}^3 \nabla y P_{\theta,2} \chi_{\Omega} w\|_{L^2_\theta}^2 + C \delta^2 (\tau^{-2} + \kappa(\epsilon) \Omega^{-4})^2 P(M). \quad (8.85)$$
Now we estimate $W_2$, which, after integrating by parts and adding a factor $v^{-1}$ to the first component inside the inner product, becomes

$$W_2 = -\langle v^{-1}P_{\theta,2}\partial_\theta^4\chi\Omega v, v\partial_\theta^2\chi\Omega N_{12}\rangle_{\theta}$$

Compute directly, and apply the estimates in (3.4), to find

$$|v\partial_\theta^2 N_{12}| \lesssim \sum_{m=0,1,2} \left[ v^{-2}|\partial_\theta^m (\nabla_y v\partial_\theta v\nabla_y \partial_\theta v)| + v^{-4}|\partial_\theta^m ((\partial_\theta v)^2\partial_\theta^2 v)| + v^{-3}|\partial_\theta^m (\partial_\theta v)^2| \right]$$

$$\lesssim \delta \sum_{m=1,2,3} |\partial_\theta^m v|.$$  

(8.86)

Decompose $v$, and consider in the space $|| \cdot ||_{L_\theta^2}$ and apply Lemma 8.2 to find

$$\chi_{\Omega}(y)^{-3}\|v\partial_\theta^2 N_{12}\|_{L_\theta^2} \lesssim \delta(\tau^{-2} + \kappa(\epsilon)\Omega^{-4})(1 + M_1).$$

Consequently, for some $C > 0$,

$$(100 + |y|^2)^{-3}|W_2| \leq \frac{1}{100}(100 + |y|^2)^{-3}V_{a,B}^2\|P_{\theta,2}\partial_\theta^4 \chi\Omega v\|_{L_\theta^2}^2 + C\delta^2(\tau^{-2} + \kappa(\epsilon)\Omega^{-4})^2P^2(M).$$

(8.87)

This, together with (8.79) and (8.85), implies the desired estimate (8.68).

Now we prove (8.67). The key observation is that the $O(1)$—term has a favorable sign, specifically, since $y \cdot \nabla_y \chi\Omega \leq 0$,

$$\langle P_{\theta,2}\partial_\theta^3 \chi\Omega v, (y \cdot \nabla_y \chi\Omega)\partial_\theta^2 v \rangle \leq 0.$$

This, after estimating the derivatives of $\chi\Omega$ and applying Young’s inequality, yields,

$$\Psi_{33} \leq (100 + |y|^2)^{-3}\langle P_{\theta,2}\partial_\theta^3 \chi\Omega v, P_{\theta,2}\partial_\theta^3 \left((\partial_\tau \chi\Omega)v - (\Delta_y \chi\Omega)v - 2\nabla_y \chi\Omega \cdot \nabla_y v\right) \rangle_{\theta}$$

$$\leq \frac{1}{100}(100 + |y|^2)^{-3}\|P_{\theta,2}\partial_\theta^3 \chi\Omega v\|_{L_\theta^2}^2 + \delta^2 \kappa^2(\epsilon)\Omega^{-8},$$

(8.88)

where we use $|P_{\theta,2}w| = |P_{\theta,2}\eta| \lesssim \max_{\theta} |\eta(\theta)| \lesssim \delta \sqrt{1 + \tau^{-1}\Omega^2}$ and $|\nabla_y v| \lesssim \delta$, see (3.4).

\section{Estimate for $M_2$, Proof of part of (4.20)}

The following results obviously imply the desired estimate for $M_2$. 

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Proposition 9.1.
\[ \| \langle y \rangle^{-2} \nabla y \chi_\Omega w_m(\cdot, \tau) \|_{\infty}, \quad \| \langle y \rangle^{-2} \nabla y \partial^2 \chi_\Omega w(\cdot, \tau) \|_{L^2} \lesssim \delta \kappa(\epsilon) \Omega^{-3} P(M), \quad m = -1, 0, 1. \] (9.1)

The proposition will be proved in subsequent subsections.

9.1 Proof of the estimate for \( \| \langle y \rangle^{-2} \nabla y \chi_\Omega w_0(\cdot, \tau) \|_{\infty} \) in (9.1)

We derive an equation for \( \nabla y \chi_\Omega w_0 \) by taking \( \nabla y \) on the both sides of (8.22),
\[ \partial_\tau (\nabla y \chi_\Omega w_0) = -(H_2 + \frac{1}{2}) (\nabla y \chi_\Omega w_0) + \nabla y \chi_\Omega \left( \Sigma + \frac{1}{2\pi} \langle N_1(v) + N_2(\eta), 1 \rangle_\theta \right) + \Lambda_1(w_0), \] (9.2)
where, in changing \( H_2 \) to \( H_2 + \frac{1}{2} \) here, we observe that, for any \( l = 1, 2, 3 \), and any function \( g \),
\[ \partial_y \frac{1}{2} y \cdot \nabla_y g = \left( \frac{1}{2} y \cdot \nabla_y + \frac{1}{2} \right) \partial_y g, \] (9.3)
and the term \( \Lambda_1(w_0) \) is defined as
\[ \Lambda_1(w_0) := \nabla_y \Lambda(w_0) + \frac{1}{2} \left( \nabla_y \frac{\bar{x}_{\Omega} y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} \right) \chi_\Omega w_0 + (\nabla_y V_{a,B}^{-2}) \chi_\Omega w_0 \]
\[ = \nabla_y \left( \partial_\tau \chi_\Omega \right) w_0 - (\Delta_y \chi_\Omega) w_0 - 2 \nabla_y \chi_\Omega \cdot \nabla_y w_0 \]
\[ + \frac{1}{2} \left( \nabla_y (y \cdot \nabla_y \chi_\Omega) \right) w_0 - \frac{1}{2} \left( y \cdot \nabla_y \chi_\Omega \right) \frac{\bar{x}_{\Omega} \nabla_y \chi_\Omega}{\chi_\Omega} w_0 + (\nabla_y V_{a,B}^{-2}) \chi_\Omega w_0. \] (9.4)

where we observe some cancellation in \( \nabla_y (y \cdot \nabla_y \chi_\Omega (1 - \bar{x}_{\Omega}) \chi_\Omega w_0) + (\nabla_y y \nabla_y \chi_\Omega \frac{\bar{x}_{\Omega}}{\chi_\Omega}) \chi_\Omega w_0. \) (9.5)

By the orthogonality conditions imposed on \( \chi_\Omega w \) we have that
\[ e^{-\frac{1}{8} |y|^2} \nabla_y \chi_\Omega w_0 \perp e^{-\frac{1}{8} |y|^2}, \quad y_k e^{-\frac{1}{8} |y|^2}, \quad k = 1, 2, 3. \] (9.6)

Denote the orthogonal projection onto the subspace orthogonal to these 4 functions by \( P_4 \), which makes
\[ P_4 e^{-\frac{1}{8} |y|^2} \nabla_y \chi_\Omega w_0 = e^{-\frac{1}{8} |y|^2} \nabla_y \chi_\Omega w_0. \] (9.7)
Returning to (9.2), we apply $e^{-\frac{1}{2}|y|^2}$, and then $P_4$, and then Duhamel’s principle to obtain
\begin{align}
e^{-\frac{1}{2}|y|^2} \nabla_y \chi \omega w_0 = & \ U_3(\tau, \xi_0) e^{-\frac{1}{2}(\tau-\xi_0)} e^{-\frac{1}{2}|y|^2} \nabla_y \chi \omega w_0(\xi_0) \\
+ & \int_{\xi_0}^{\tau} U_3(\tau, s) e^{-\frac{1}{2}(\tau-s)} P_4 e^{-\frac{1}{2}|y|^2} \nabla_y \left( \chi \omega \left( \frac{1}{2\pi} (N_1 + N_2, 1) + \Lambda_1(w_0) \right) \right)(s) \, ds,
\end{align}
(9.8)

where $U_3(\tau, \sigma)$ is the propagator generated by $-P_4 e^{-\frac{1}{2}|y|^2} (H^2 + \frac{1}{2}) e^{\frac{1}{2}|y|^2} P_4$ from $\sigma$ to $\tau$.

We have the following estimate for the propagator. As discussed in the proof of Lemma 8.3, its proof is very similar to the proved cases, thus we choose to skip the proof.

**Lemma 9.2.** For any function $g$ and $\sigma_1 \geq \sigma_2 \geq \xi_0$,
\begin{align*}
\| \langle y \rangle^{-2} e^{\frac{1}{2}|y|^2} U_3(\sigma_1, \sigma_2) P_4 g \|_{\infty} & \lesssim e^{\frac{3}{2}(\sigma_1-\sigma_2)} \| \langle y \rangle^{-2} e^{\frac{1}{2}|y|^2} g \|_{\infty}.
\end{align*}
(9.9)

Next we estimate the terms on the right hand side. Recall that $P(M)$ is defined in (8.1).

**Proposition 9.3.**
\begin{align*}
\| \langle y \rangle^{-2} \nabla_y \chi \omega \|_{\infty} & \lesssim \tau^{-2},
\end{align*}
(9.10)
\begin{align*}
\| \langle y \rangle^{-2} \nabla_y \chi \omega N_1 \|_{\infty} & \lesssim \delta \kappa(\epsilon) \Omega^{-3} P(M),
\end{align*}
(9.11)
\begin{align*}
\| \langle y \rangle^{-2} \nabla_y \chi \omega \langle N_2, 1 \rangle_{\theta} \|_{\infty} & \lesssim \delta \kappa(\epsilon) \Omega^{-3} (1 + M_4),
\end{align*}
(9.12)
\begin{align*}
\| \langle y \rangle^{-2} \Lambda_1(w_0) \|_{\infty} & \lesssim \delta \kappa(\epsilon) \Omega^{-3} (1 + M_1).
\end{align*}
(9.13)

The proposition will be proved in subsection 9.1.1.

Suppose the proposition holds, then we prove the desired result for $M_2$ in (9.1) as in (8.34). Here we choose to skip the details.

### 9.1.1 Proof of Proposition 9.3

**Proof.** It is easy to prove (9.10) by the estimates on the scalar functions and their functions in (3.8)-(3.10) and (3.11).

Now we prove (9.11). Reason as in (8.80), (8.81) and (8.86), to find that
\begin{align}
|\nabla_y N_1| \lesssim \delta \left[ \sum_{|k|=1,2} |\nabla_y^k v|^2 + \sum_{|k|=0,1} |\nabla_y^k \partial_y v| \right].
\end{align}
(9.14)

Compute directly and apply the same techniques used in (8.40) and (8.41), and find that,
for $|k| = 1, 2$,
\begin{align}
\| \langle y \rangle^{-2} \chi \omega |\nabla_y^k v|^2 \|_{\infty} \lesssim \tau^{-2} + \delta \kappa(\epsilon) \Omega^{-3} P(M).
\end{align}
(9.15)
For the second term, we discuss separately the case $|k| = 0$ and $|k| = 1$. When $|k| = 0$, we decompose $\eta$ as in (8.44), and apply Lemma 8.2 to find that
\[ \| \langle y \rangle^{-2} \chi_{\Omega} \partial_{\theta} \eta \|_{\infty} \lesssim \tau^{-2} + \| \langle y \rangle^{-2} \partial_{\theta} \chi_{\Omega} w \|_{\infty} \lesssim \tau^{-2} + \kappa(\epsilon) \Omega^{-3} \mathcal{M}_4. \] (9.16)

For the case $|k| = 1$ we decompose $\partial_{\theta} v = \partial_{\theta} \eta$, change the order of $\nabla_y$ and $\chi_{\Omega}$, to find that
\[ \langle y \rangle^{-2} \chi_{\Omega} |\partial_{\theta} \nabla_y v| \lesssim \tau^{-2} + \| \langle y \rangle^{-2} |\partial_{\theta} \nabla_y \chi_{\Omega} w| + \| \langle y \rangle^{-2} |\nabla_y \chi_{\Omega}| |\partial_{\theta} w| \|. \]

We control the second term in terms of $\mathcal{M}_2$ by Lemma 8.2. For the third term, we apply the same techniques as in proving (8.44) to obtain,
\[ \langle y \rangle^{-2} |\partial_{\theta} w \nabla_y \chi_{\Omega}| \leq \Omega^{-3} |\chi_{\Omega} \partial_{\theta} w|^{\frac{3}{2}} \| \partial_{\theta} w \|_{L^2} \sup_z \{ |\nabla_z \chi(z)| \chi^{-\frac{3}{2}}(z) \} \leq \Omega^{-\frac{3}{2}} \mathcal{M}_4^{\frac{3}{2}}. \] (9.17)

Consequently
\[ \| \langle y \rangle^{-2} \chi_{\Omega} \nabla_y \partial_{\theta} v \|_{\infty} \lesssim \tau^{-2} + \kappa(\epsilon) \Omega^{-3} (\delta + \mathcal{M}_2 + \Omega^{-\frac{1}{2}} \mathcal{M}_4^{\frac{3}{2}}). \] (9.18)

Collect the estimates above to have the desired (9.11).

Now we prove (9.12). Rewrite the expression as in (8.46), compute directly, apply the estimates in (3.4) and decompose $\eta$ as in (4.4) to have
\[ |\nabla_y \langle N_2, 1 \rangle_\theta| \lesssim \delta \| v^{-1} |\eta| (|\nabla_y V_{a,B}| + |\nabla_y \eta|) + \delta |\partial_{\theta} \eta| \|_{L^2} \lesssim \delta \| \tau^{-\frac{1}{2}} |\partial_{\theta} w| + |\nabla_y \partial_{\theta} w| \|_{L^2} \delta + \delta \tau^{-2} (1 + |y|). \]

Apply the techniques in proving (8.42) to find
\[ \langle y \rangle^{-2} \chi_{\Omega} |\nabla_y \langle N_2, 1 \rangle_\theta| \lesssim \delta \kappa(\epsilon) \Omega^{-3} P(M). \] (9.19)

Now we prove (9.13). The terms in (9.4) will be processed as those in (8.30), hence we skip the details.

The terms in (9.5) are new. Use that $|\nabla_y V_{a,B}| \lesssim \tau^{-\frac{1}{2}}$ and the condition in (8.2) to obtain,
\[ \| \langle y \rangle^{-2} (\nabla_y V_{a,B}) \chi_{\Omega} w_0 \|_{\infty} \lesssim \tau^{-\frac{3}{2}} \Omega \| \langle y \rangle^{-3} \chi_{\Omega} w_0 \|_{\infty} \lesssim \delta \kappa(\epsilon) \Omega^{-3} \mathcal{M}_1, \] (9.20)

and for the other two terms, by the same strategies as those in proving (8.44),
\[ \| (\nabla_y (y \cdot \nabla_y \chi_{\Omega})) w_0 \| \lesssim \Omega^{-3} |\chi_{\Omega} w_0|^{\frac{3}{2}} |w_0|^{\frac{1}{2}} \sup_z \left| \frac{\nabla_z (z \cdot \nabla_z \chi(z))}{\chi^{\frac{3}{4}}(z)} \right| \lesssim \Omega^{-\frac{3}{2}} \mathcal{M}_4^{\frac{3}{2}}, \]
\[ \| (\nabla_y y \cdot \nabla_y \chi_{\Omega}) \chi_{\Omega} \nabla_y \chi_{\Omega} w_0 \| \lesssim \Omega^{-3} |\chi_{\Omega} w_0|^{\frac{3}{2}} |w_0|^{\frac{1}{2}} \sup_z \left| \frac{z \cdot \nabla_z \chi(z) \nabla_z \chi}{\chi^{\frac{3}{4}}(z)} \right| \lesssim \Omega^{-\frac{3}{2}} \mathcal{M}_4^{\frac{3}{2}}. \] (9.21)
9.2 Proof of the estimate for \( \|\langle y \rangle^{-2} \nabla_y \chi_\Omega w_{-1}(\cdot, \tau)\|_\infty \) in (9.1)

Since \( w_{-1} = \overline{w}_1 \), we only need to estimate \( w_1 \).

We derive an equation for \( \nabla_y \chi_\Omega w_1 \), by taking a derivative \( \nabla_y \) on (8.51) and using commutation relation in (9.3),

\[
\partial_\tau \nabla_y \chi_\Omega w_1 = -\left( H_1 + \frac{1}{2} \right) \nabla_y \chi_\Omega w_1 + \frac{1}{2\pi} \nabla_y \chi_\Omega \langle G + N_1(v) + N_2(\eta), e^{i\theta} \rangle + \Lambda_2(w_1),
\]

where \( \Lambda_2(w_1) \) is defined similarly to \( \Lambda_1(w_0) \) in (9.4) and (9.5),

\[
\Lambda_2(w_1) := \nabla_y \left( (\partial_\tau \chi_\Omega) w_1 - (\Delta_y \chi_\Omega) w_1 - 2 \nabla_y \chi_\Omega \cdot \nabla_y w_1 \right) + \frac{1}{2} (y \cdot \nabla_y \chi_\Omega)(1 - \tilde{\chi}_\Omega) \nabla_y w_1
\]

\[
+ \frac{1}{2} (\nabla_y (y \cdot \nabla_y \chi_\Omega)) w_1 - \frac{1}{2} \left( \frac{y \cdot \nabla_y \chi_\Omega}{\chi_\Omega} \right) \nabla_y w_1.
\]

The terms satisfy the following estimates:

**Proposition 9.4.**

\[
\|\langle y \rangle^{-2} \Lambda_2(w_1)\|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-3} (1 + M_4),
\]

\[
\|\langle y \rangle^{-2} \nabla_y \chi_\Omega \langle G, e^{i\theta} \rangle\|_\infty \lesssim \tau^{-2},
\]

\[
\|\langle y \rangle^{-2} \nabla_y \chi_\Omega N_1\|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-3} P(M),
\]

\[
\|\langle y \rangle^{-2} \nabla_y \chi_\Omega \langle N_2, e^{i\theta} \rangle\|_\infty \lesssim \delta \tau^{-2} + \delta \kappa(\epsilon) \Omega^{-3} (1 + M_4).
\]

**Proof.** Since the proof of (9.24) is almost identical to proving (9.13), and since it is easy to prove (9.25) by the estimates of the scalar functions and their equations in (3.8)-(3.10) and (3.11), we skip the details here.

(9.26) is the same as the previous (9.11).

Now we prove (9.27). We write \( \langle N_2, e^{i\theta} \rangle \) as in (8.59), and compute directly to find that

\[
|\nabla_y \langle N_2, e^{i\theta} \rangle| \lesssim \delta |\partial_\theta \eta| \lesssim \delta \tau^{-2} (1 + |y|) + \delta \|\partial_\eta w\|_{L^2_\theta},
\]

This, together with the estimates in (8.7) and the definition of \( \Omega \) in (2.3) and estimating the terms produced in changing the order of \( \nabla_y \) and \( \chi_\Omega \), implies (9.27).

\( \square \)
Return to the equation (9.22). The orthogonality conditions of $\chi_\Omega w_1$ imply,
\[
e^{-\frac{1}{8}|y|^2} \nabla_y \chi_\Omega w_1 \perp e^{-\frac{1}{8}|y|^2}.
\] (9.29)

Denote by $P_1$ the orthogonal projection onto the subspace orthogonal to $e^{-\frac{1}{8}|y|^2}$. This makes
\[
P_1 e^{-\frac{1}{8}|y|^2} \nabla_y \chi_\Omega w_1 = e^{-\frac{1}{8}|y|^2} \nabla_y \chi_\Omega w_1.
\]

Then we have that
\[
e^{-\frac{1}{8}|y|^2} \nabla_y \chi_\Omega w_1(\tau) = U_4(\tau, \xi_0) e^{-\frac{1}{8}|y|^2} \nabla_y \chi_\Omega w_1(\xi_0)
\]
\[
+ \int_{\xi_0}^{\tau} U_4(\tau, \sigma) P_1 \left( \frac{1}{2\pi} \nabla_y \chi_\Omega (G + N_1(v) + N_2(\eta), e^\theta \varphi + \Lambda_2(w_1)) \right)(\sigma) \, d\sigma.
\] (9.30)

Here $U_4(\tau, \sigma)$ is the propagator generated by the linear operator $-P_1 e^{-\frac{1}{8}|y|^2}(H_1 + \frac{1}{2}) e^{\frac{1}{8}|y|^2} P_1$.

We have the following estimate for the propagator. As discussed in the proof of Lemma 8.3, its proof is very similar to the previously proved ones, thus we choose to skip the proof.

**Lemma 9.5.** For any function $g$, and $\tau \geq \sigma$,
\[
\|\langle y \rangle^{-2} e^{\frac{1}{8}|y|^2} U_4(\tau, \sigma) P_1 g \|_{L^\infty} \lesssim e^{-\frac{3}{8}(\tau-\sigma)} \|\langle y \rangle^{-2} e^{\frac{1}{8}|y|^2} g \|_{L^\infty}.
\] (9.31)

What is left is to obtain the desired estimate for $\|\langle y \rangle^{-2} \nabla_y \chi_\Omega w_{\pm 1}(\cdot, \tau) \|_{L^2}$ in (9.1). The procedure is standard, as in (8.34), we skip the details here.

### 9.3 Proof of the estimate for $\|\langle y \rangle^{-2} \nabla_y \partial^2_\theta \chi_\Omega w(\cdot, \tau) \|_{L^2}$ in (9.1)

By the identity in (8.61), it is sufficient to estimate $\|\langle y \rangle^{-2} \nabla_y \partial^2_\theta \chi_\Omega v(\cdot, \tau) \|_{L^2}$ in (9.1). By the equation for $v$ in (8.2) and the commutation relation in (9.3), we find that the function
\[
\Phi_2 := (100 + |y|^2)^{-2} \|P_{\theta \geq 2} \nabla \partial^2_\theta \chi_\Omega v\|_{L^2}^2
\]
satisfy the equation
\[
\partial_t \Phi_2 = -(L_2 + V_2) \Phi_2 - 2(100 + |y|^2)^{-2} \sum_{k=1,2,3} \|P_{\theta \geq 2} \nabla \partial_{yk} \partial^2_\theta \chi_\Omega v\|_{L^2}^2 + 2 \sum_{k=1}^3 \Psi_{2k},
\] (9.32)

where the linear operator $L_2 + V_2$ is related to $-\Delta + \frac{1}{2} y \cdot \nabla_\theta$ by the identity
\[
L_2 + V_2 := (100 + |y|^2)^{-2} (-\Delta + \frac{1}{2} y \cdot \nabla_\theta)(100 + |y|^2)^2,
\] (9.33)
and the linear operators $L_2$ and $V_2$ are defined as,

$$L_2 := -\Delta + \frac{1}{2}y \cdot \nabla_y - 2(100 + y^2)^{-2}(\nabla_y(100 + |y|^2)^2) \cdot \nabla_y,$$

$$V_2 := \frac{2|y|^2}{100 + |y|^2} - \frac{12}{100 + |y|^2} - \frac{8|y|^2}{(100 + |y|^2)^2},$$

and the functions $\Psi_{2k}$, $k = 1, 2, 3$, are defined as

$$\Psi_{21} := (100 + |y|^2)^{-2} \langle P_{\theta \geq 2} \nabla_y \partial_\theta^3 \chi_\Omega v, \nabla_y \partial_\theta \chi_\Omega (v^{-2} \partial_\theta^2 v - v^{-1}) \rangle_\theta,$$

$$\Psi_{22} := (100 + |y|^2)^{-2} \langle P_{\theta \geq 2} \nabla_y \partial_\theta^3 \chi_\Omega v, \nabla_y \partial_\theta \chi_\Omega N_1(v) \rangle_\theta,$$

$$\Psi_{23} := (100 + |y|^2)^{-2} \langle P_{\theta \geq 2} \nabla_y \partial_\theta^3 \chi_\Omega v, \nabla_y \mu (P_{\theta \geq 2} \partial_\theta^2 v) \rangle_\theta.$$ 

Here $\mu$ is defined in the same fashion as that in (4.7).

For the terms on the right hand side we have

**Proposition 9.6.** There exists a constant $C > 0$ such that

$$\Psi_{21} \leq - \left(\frac{18}{25} - C\delta\right)V_{a,B}^{-2}(100 + |y|^2)^{-2} \| P_{\theta \geq 2} \nabla_y \partial_\theta^3 \chi_\Omega v \|_{L_2^{1/3}}^2 + C\delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M), \quad (9.34)$$

$$\Psi_{23} \leq \frac{1}{100} \left[ \Phi_1 + V_{a,B}^{-2}(100 + |y|^2)^{-2} \| P_{\theta \geq 2} \nabla_y \partial_\theta^3 \chi_\Omega v \|_{L_2^{1/3}}^2 \right] + C\delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M), \quad (9.35)$$

$$\Psi_{22} \leq \frac{1}{50} V_{a,B}^{-2}(100 + |y|^2)^{-2} \| P_{\theta \geq 2} \nabla_y \partial_\theta^3 \chi_\Omega v \|_{L_2^{1/3}}^2 + C\delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M). \quad (9.36)$$

The proposition will be proved in subsubsection 9.3.1.

We continue to study (9.32). As deriving (8.70), we derive, for some $C_1 > 0$,

$$\partial_\tau \Phi_2 \leq -(L_2 + 1) \Phi_2 + C_1 \delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M). \quad (9.37)$$

Apply the maximum principle to find, for some $C_2 > 0$,

$$\Phi_2(\tau) \leq e^{-\frac{1}{2}(\tau - \xi_0)} \Phi_2(\xi_0) + C_2 \delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M), \quad (9.38)$$

then obtain the desired result after using that $\Phi_2(\xi_0) \leq \kappa^2(\epsilon) \Omega^{-6} (\xi_0)$ implied by (4.17), and taking square roots.

**9.3.1 Proof of Proposition 9.6**

*Proof.* To prove (9.36), we decompose $N_1$ as $N_1 = N_{11} + N_{12}$ as in (8.80), and this makes

$$\Psi_{22} := (100 + |y|^2)^{-2} \langle P_{\theta \geq 2} \nabla_y \partial_\theta^2 \chi_\Omega v, \nabla_y \partial_\theta \chi_\Omega \left(N_{11}(v) + N_{12}(v)\right) \rangle_\theta = D_1 + D_2 \quad (9.39)$$

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with $D_1$ and $D_2$ naturally defined as the $N_{11}$- and $N_{12}$-terms respectively.

To estimate $D_1$, we argue as in (8.41) to obtain
\[
|\nabla_y \partial^2_\theta N_{11}| \lesssim \delta \left[ |\nabla_y v|^2 + |\nabla_y v| \sum_{|k|=2} (|\nabla_y^k v| + |\nabla_y^k \partial_\theta v|) + \sum_{l=1,2} |\nabla_y \partial^l_\theta v| \right]
\]
\[
\lesssim \delta \left[ \sum_{|k|=2} |\nabla_y^k v|^2 + |\nabla_y v| \sum_{|k|=2} |\nabla_y^k \partial_\theta v| + \sum_{l=1,2} |\nabla_y \partial^l_\theta v| \right].
\] (9.40)

We start with processing the terms containing only the first and second order derivatives of $v$. As in (8.40)-(8.41) and (9.17), we find
\[
\chi_\Omega \langle y \rangle^{-2} \left[ \sum_{|k|=1,2} |\nabla_y^k v|^2 + |\nabla_y v| \right] \lesssim \tau^{-2} + \kappa(\epsilon) \Omega^{-3} P(M).
\] (9.41)

For the other terms, specifically $|\nabla_y v|,|\nabla_y^k \partial_\theta v|$, $|k|=2$, and $|\nabla_y \partial^2_\theta v|$, we use that $|\nabla_y v| \lesssim \delta$, $|\nabla_y^k w| \lesssim \sup_{\theta} |\nabla_y^k w(\cdot, \theta, \tau)| \lesssim \delta$ by (3.4), and change the order of $\nabla_y$ and $\chi_\Omega$, and use the techniques in (8.42) and (9.17), and use the definition of $\Omega$ in (2.3), to have
\[
(100 + |y|^2)^{-1} \chi_\Omega \| \nabla_y v \nabla_y^k \partial_\theta v \|_{L_\theta^2}
\]
\[
\lesssim \delta \tau^{-2} + \delta (100 + |y|^2)^{-1} \chi_\Omega \| \nabla_y^k \partial_\theta P_{\theta \geq 2} w \|_{L_\theta^2}
\]
\[
+ \tau^{-1} \sum_{m=1,2} \| \langle y \rangle^{-1} \chi_\Omega \nabla_y^k w_m \|_\infty + \delta \| \langle y \rangle^{-2} \chi_\Omega \nabla_y w \|_\infty
\] (9.42)
\[
\lesssim \delta \kappa(\epsilon) \Omega^{-3} P(M),
\]
similarly,
\[
(100 + |y|^2)^{-1} \chi_\Omega \| \nabla_y \partial^2_\theta v \|_{L_\theta^2} \lesssim \tau^{-2} + \kappa(\epsilon) \Omega^{-3} P(M).
\] (9.43)

These, after applying Young’s inequality, make
\[
|D_1| \leq \frac{1}{100} (100 + |y|^2)^{-2} \| P_{\theta \geq 2} \nabla_y \partial^2_\theta \chi_\Omega v \|_{L_\theta^2} + C\delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M).
\] (9.44)

To estimate $D_2$, we integrate by parts in $\theta$ to have
\[
D_2 = - \langle 100 + |y|^2 \rangle^{-2} \langle v^{-1} P_{\theta \geq 2} \nabla_y \partial^2_\theta \chi_\Omega v, v \nabla_y \partial_\theta \chi_\Omega N_{12}(v) \rangle_{\theta}.
\]
Reason as in (8.86) to find that
\[
v |\nabla_y \partial_\theta N_{12}| \lesssim \delta v^{-1} \left[ \sum_{l=1,2} |\partial^l_\theta v| + |\nabla_y \partial_\theta v| \right].
\]
Change the order of $\nabla_y$ and $\chi_\Omega$, decompose $v$, and apply (3.31) and Lemma 8.2 to find

$$\|(100 + |y|^2)^{-1}v\nabla_y \partial_\theta \chi_\Omega \nu_{12}\|_\infty \lesssim \delta \kappa(\epsilon)\Omega^{-3}[1 + M_2 + M_4].$$  \hspace{1cm} (9.45)

This makes, for some $C > 0$,

$$D_2 \leq \frac{1}{100}(100 + |y|^2)^{-2}\|v^{-1}P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v\|_{L^2_\theta} + C\delta^2 \kappa^2(\epsilon)\Omega^{6}P^2(M).$$  \hspace{1cm} (9.46)

Collect the estimates above to have the desired estimate (9.36).

Next we prove (9.34). Here we adopt the same strategy as that in proving (8.66).

We rewrite the first term by integrating by parts and then decompose,

$$\Psi_{21} = -(100 + |y|^2)^{-2}\sum_{k=1}^{3}E_k,$$  \hspace{1cm} (9.47)

where $E_k$, $k = 1, 2, 3$, are defined as

\[
E_1 := \langle P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v, v^{-2}P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v \rangle_\theta - \langle P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v, v^{-2}P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v \rangle_\theta;
\]

\[
E_2 := \langle P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v, v^{-2}(1 - P_{\theta;\geq 2})\nabla_y \partial_\theta^3 \chi_\Omega v \rangle_\theta - \langle P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v, v^{-2}(1 - P_{\theta;\geq 2})\nabla_y \partial_\theta^2 \chi_\Omega v \rangle_\theta;
\]

\[
E_3 := \langle P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v, [\nabla_y \partial_\theta \chi_\Omega v^{-2}\partial_\theta^2 v - v^{-2}\nabla_y \partial_\theta \chi_\Omega v] \rangle_\theta + \langle P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v, [\nabla_y \partial_\theta^2 \chi_\Omega v^{-1} + v^{-2}\nabla_y \partial_\theta \chi_\Omega v] \rangle_\theta.
\]

By arguing as in (8.73), we have that, for some $C > 0$,

$$E_1 \geq \left(\frac{3}{4} - C\delta\right)V_{a,B}^{-2}\|P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v\|_{L^2_\theta}^2.$$  \hspace{1cm} (9.48)

To estimate $E_2$, we use

$$(1 - P_{\theta;\geq 2})\nabla_y \partial_\theta^2 \chi_\Omega v = -\frac{1}{2\pi}\sum_{m=\pm 1} e^{im\theta} \nabla_y \chi_\Omega (v, e^{im\theta})_\theta$$

and then follow the steps in (8.77), and apply the estimate (8.8) to obtain

$$(100 + |y|^2)^{-2}|E_2| \leq \frac{1}{100}V_{a,B}^{-2}(100 + |y|^2)^{-2}\|P_{\theta;\geq 2}\nabla_y \partial_\theta^3 \chi_\Omega v\|_{L^2_\theta}^2 + C\delta^2 \kappa^2(\epsilon)\Omega^{-6}P^2(M).$$  \hspace{1cm} (9.49)
For $E_3$, compute directly and use the definition of $\Omega$ in (8.2) to find
\[
(100 + |y|^2)^{-1}|E_3| \lesssim \delta V_{a,B}^{-1} \| P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v \|_{L^2_\theta} \sum_{l=2,3} \| \chi \partial_\theta^l v \|_{L^2_\theta} \tag{9.50}
\]
Apply Young’s inequality to find
\[
(100 + |y|^2)^{-2}|E_3| \leq \frac{1}{100} V_{a,B}^{-2} (100 + |y|^2)^{-2} \| P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v \|_{L^2_\theta}^2 + C \delta^2 \kappa^2(\epsilon) \Omega^{-6} P^2(M). \tag{9.51}
\]
Collect the estimate above to have the desired (9.34).

Now we prove (9.35). The present problem is slightly more involved than proving (8.67) resulted by that $\chi \Omega$ and $\nabla_y$ do not commute with each other. We decompose $\Psi_{23}$ into two parts,
\[
\Psi_{23} = (100 + |y|^2)^{-2}(U_1 + U_2), \tag{9.52}
\]
with $U_1$ and $U_2$ defined as
\[
U_1 := \frac{1}{2} \langle P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v, \nabla_y ((y \cdot \nabla_y \chi \Omega) P_{a,B} \partial_\theta^3 \chi \Omega v) \rangle_\theta,
\]
\[
U_2 := \langle P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v, \nabla_y \left[ (\partial_\tau \chi \Omega) v - (\Delta_y \chi \Omega) v - 2 \nabla_y \chi \Omega \cdot \nabla_y v \right] \rangle.
\]
It is easy to estimate $U_2$. Use the definitions of $\chi \Omega$ and $\Omega$, the estimates in (3.4), and then apply Young’s inequality to find, for some $C > 0$,
\[
(100 + |y|^2)^{-2}|U_2| \leq \frac{1}{100} (100 + |y|^2)^{-2} \| P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v \|_{L^2_\theta}^2 + C \delta^2 \kappa^2(\epsilon) \Omega^{-6}. \tag{9.53}
\]
For $U_1$, we observe that the $O(1)$ term has a favorable nonpositive sign, specifically
\[
\langle \chi \Omega P_{a,B} \nabla_y \partial_\theta^3 \nabla_y v, (y \cdot \nabla_y \chi \Omega) P_{a,B} \partial_\theta^3 \nabla_y v \rangle_\theta \leq 0.
\]
Hence, for some $C > 0$
\[
U_1 \leq \frac{1}{2} \langle P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v, (\nabla_y (y \cdot \nabla_y \chi \Omega)) P_{a,B} \partial_\theta^3 \chi \Omega v \rangle_\theta
\]
\[
+ \frac{1}{2} \langle (\nabla_y \chi \Omega) P_{a,B} \partial_\theta^3 \chi \Omega v, (y \cdot \nabla_y \chi \Omega) P_{a,B} \partial_\theta^3 \nabla_y v \rangle_\theta
\]
\[
\leq \frac{1}{100} \| P_{a,B} \nabla_y \partial_\theta^3 \chi \Omega v \|_{L^2_\theta}^2
\]
\[
+ C \left[ |\nabla_y (y \cdot \nabla_y \chi \Omega)|^2 \| \partial_\theta^3 \nabla_y v \|_{L^2_\theta}^2 + |\nabla_y \chi \Omega| \| (y \cdot \nabla_y \chi \Omega) \| \| \partial_\theta^3 v \|_{L^2_\theta} \| \partial_\theta^3 \nabla_y v \|_{L^2_\theta} \right]. \tag{9.54}
\]
We estimate the terms on the right hand side by methods similar to the proof of (8.44). For the first term in (9.54), we decompose $u$ and then use the same strategy as in (8.44),

\[(100 + |y|^2)^{-2} |\nabla_y(y \cdot \nabla_y \chi)|^2 \|\partial_y^2 v\|_{L^2_y}^2 \leq \Omega^{-6} \|\chi \partial_y^2 w\|_{L^2_y}^2 \|\partial_y^2 w\|_{L^2_y} \sup_z \left|\frac{\nabla_z(z \cdot \nabla_z \chi)}{\chi^{\frac{7}{2}}}\right|^2 + \kappa^2(\epsilon)\Omega^{-6} \tau^{-4} \]

\[
\lesssim \kappa^2(\epsilon)\Omega^{-\frac{15}{4}} (1 + \tau^{-1} \Omega^2) \frac{1}{4} (M_4^3 + 1) + \kappa^2(\epsilon)\Omega^{-6} \tau^{-4},
\]

and similarly for the second term,

\[(100 + |y|^2)^{-2} |\nabla_y \chi\Omega| \left| (y \cdot \nabla_y \chi\Omega) \right| \|\partial_y^2 v\|_{L^2_y} \|\partial_y^2 \nabla_y v\|_{L^2_y} \leq \left( \kappa^2(\epsilon)\Omega^{-\frac{15}{4}} (1 + \tau^{-1} \Omega^2) + \kappa^2(\epsilon)\Omega^{-6} \tau^{-4} \right) P(M), \]

(9.55)

where we change the order of $\nabla_y$ and $\chi\Omega$, and apply the estimates in (3.4) and (3.20).

Collect the estimates above and use the condition in (8.2) to find that

\[(100 + |y|^2)^{-2} U_1 \leq \frac{1}{100} \left\|P_{\theta \geq \epsilon} \nabla_y \partial_y^2 \chi\Omega v\right\|_{L^2_y}^2 + C\delta^2 \kappa^2(\epsilon)\Omega^{-6} P^2(M). \]

(9.56)

This together with (9.53) and (9.52) implies the desired estimate (9.35). \hfill \Box

10 Estimate for $M_3$, Proof of part of (4.20)

We reformulate the estimate into the following results,

**Proposition 10.1.** For any $|k| = 2$ we have

\[
\|\langle y \rangle^{-1} \nabla_y^k \chi\Omega w_0\|_{L^\infty}, \quad \|\langle y \rangle^{-1} \|\nabla_y^k \partial_y \chi\Omega w\| \|_{L^2_y} \lesssim \delta \kappa(\epsilon)\Omega^{-2} P(M). \]

(10.1)

The proposition will be proved in subsequent subsections.

10.1 Proof of the first estimate in (10.1)

Similar to deriving the equation for $\nabla_y \chi\Omega w_0$ in (9.2),

\[
\partial_y \nabla_y^k \chi\Omega w_0 = - (H_2 + 1) \nabla_y^k \chi\Omega w_0 + \nabla_y^k \chi\Omega \left( \Sigma + \frac{1}{2\pi} \langle N_1 + N_2, 1 \rangle_\theta \right) + \Lambda_3(w_0)
\]

(10.2)
where the term $\Lambda_3(w_0)$ is defined as, suppose that $k = k_1 + k_2$ with $|k_1| = |k_2| = 1$,$\Lambda_3(w_0) := \nabla_\chi y^k \left( (\partial_\tau \chi) w_0 - (\Delta_y \chi \omega) w_0 - 2\nabla_y \chi \omega \cdot \nabla y w_0 \right) + \frac{1}{2} \nabla_\chi y^{k_2} \left( (y \cdot \nabla y \chi \omega)(1 - \bar{\chi}) \nabla_\chi y^{k_1} w_0 \right) + \frac{1}{2} \nabla_\chi y^{k_2} \left( \frac{\nabla y^{k_1}(y \cdot \nabla y \chi \omega) w_0}{\chi \omega} - \frac{1}{2} \nabla y^{k_1} \chi \omega \nabla_\chi y^{k_1} \chi \omega w_0 + (\nabla y^{k_1} V_{a,B}) \chi \omega w_0 \right) + \frac{1}{2} \nabla_\chi y^{k_2} \left( \bar{\chi} \omega \cdot \nabla y \chi \omega \right) \nabla_\chi y^{k_1} \chi \omega w_0 + (\nabla_\chi y^{k_2} V_{a,B}) \nabla_\chi y^{k_1} \chi \omega w_0, \tag{10.3}

and we use the commutation relation in (9.3) again to make the linear operator change from $H_2 + \frac{1}{2}$ in (9.2) to that in (10.2).

The orthogonality conditions imposed on $e^{-\frac{1}{8}|y|^2} \chi \omega w$ imply that, recall that $|k| = 2$,$e^{-\frac{1}{8}|y|^2} \nabla_\chi y \omega w_0 \perp e^{-\frac{1}{8}|y|^2} \tag{10.4}

We denote by $P_1$ the orthogonal projection onto the subspace orthogonal to $e^{-\frac{1}{8}|y|^2}$.

Apply $e^{-\frac{1}{8}|y|^2}$ and then $P_1$ on (10.2), and then apply Duhamel’s principle to find

$$e^{-\frac{1}{8}|y|^2} \nabla_\chi y \omega w_0(\cdot, \tau) = U_5(\tau, \tau_0)e^{-\frac{1}{8}|y|^2} \nabla_\chi y \omega w_0(\cdot, \tau_0) + \int_{\tau_0}^\tau U_5(\tau, \sigma) P_1 e^{-\frac{1}{8}|y|^2} \left[ \nabla_\chi y \omega \left( \Sigma + \frac{1}{2\pi} \langle N_1 + N_2, 1 \rangle \right) + \Lambda_3(w_0) \right](\sigma) \, d\sigma, \tag{10.5}

where $U_5(\tau, \sigma)$ is the propagator generated by $-P_1 e^{-\frac{1}{8}|y|^2} (H_2 + 1) e^{\frac{1}{8}|y|^2} P_1$ from $\sigma$ to $\tau$.

The propagator satisfies the following estimate, recall that $|k| = 2$,

**Lemma 10.2.** For any function $g$, and $\tau \geq \sigma \geq \xi_0$,$\| \langle y \rangle^{-1} e^{\frac{1}{8}|y|^2} U_5(\tau, \sigma) P_1 g \|_\infty \lesssim e^{-\frac{3}{8}(\tau - \sigma)} \| \langle y \rangle^{-1} e^{\frac{1}{8}|y|^2} g \|_\infty. \tag{10.6}

As discussed in the proof of Lemma 8.3, its proof is very similar to the ones considered in the known results. Hence we skip the details.

Recall the definition of $P(M)$ in (8.1). The terms in (10.5) satisfy the following estimates:

**Proposition 10.3.**

\[ \| \langle y \rangle^{-1} \nabla_\chi y \omega \Sigma \|_\infty \lesssim \tau^{-2}, \tag{10.7} \]
\[ \| \langle y \rangle^{-1} \nabla_\chi y \omega \langle N_1, 1 \rangle \|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-2} P(M), \tag{10.8} \]
\[ \| \langle y \rangle^{-1} \nabla_\chi y \omega \langle N_2, 1 \rangle \|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-2} P(M), \tag{10.9} \]
\[ \| \langle y \rangle^{-1} \Lambda_3(w_0) \|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-2} (1 + \mathcal{M}_4 + \mathcal{M}_2). \tag{10.10} \]

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The proposition will be proved in subsubsection 10.1.1.

By the estimates above, and going through a procedure similar to that in (8.34), we obtain the desired result (10.1). We skip the details here.

10.1.1 Proof of Proposition 10.3

The proof of (10.7) is easy, by the estimates in (3.8)-(3.10) and (3.11).

Now we prove (10.8). By arguing as in (8.81) and (8.86) and using (3.4), we have, for any $|k| = 2$,
\[
|\nabla_y^k N_1| \lesssim \delta \left[ \sum_{|l|=1,2} |\nabla_y^l v|^2 + \sum_{|l|=0,1} |\nabla_y^l \partial \theta v| \right] 
\lesssim \delta \left[ \sum_{|l|=1,2} |\nabla_y^l V_{a,B}|^2 + \tau^{-2}(1+|y|) + \sum_{|l|=1,2} |\nabla_y^l w| + \sum_{|l|=0,1} |\nabla_y^l \partial \theta w| \right].
\] (10.11)

Apply the same techniques as in proving (8.42) to obtain the desired estimate,
\[
\| \langle y \rangle^{-1} \nabla_y^k \chi \Omega N_1 \|_{\infty} \lesssim \delta \tau^{-\frac{3}{2}} + \delta \kappa(\epsilon) \Omega^{-2} P(M).
\] (10.12)

Now we prove (10.9). We rewrite the expression as in (8.46) to find
\[
\nabla_y^k \chi \Omega \langle N_2(\eta), 1 \rangle_{\theta} = -\nabla_y^k \chi \Omega \langle V_{a,B}^{-2} v^{-1} \eta^2, 1 \rangle_{\theta} + 2\nabla_y^k \chi \Omega \langle v^{-3} (\partial \theta \eta)^2, 1 \rangle_{\theta}
= W_1 + W_2
\]
with the terms $W_1$ and $W_2$ naturally defined.

For $W_2$, we decompose $\eta$ as in (4.4) and use (3.4) to find that, for $|k| = 2$,
\[
\| \langle y \rangle^{-1} W_2 \|_{\infty} \lesssim \delta \Omega \left[ \| \langle y \rangle^{-2} \partial \theta \nabla_y \chi \Omega w \|_{\infty} + \| \langle y \rangle^{-2} \partial \theta \chi \Omega w \|_{\infty} \right] + \delta \kappa(\epsilon) \Omega^{-2}
\lesssim \delta \kappa(\epsilon) \Omega^{-2}(1 + M_2 + M_4).
\] (10.13)

For $W_1$ we use that $\nabla_y v = \nabla_y V_{a,B} + \nabla_y \eta$ and that $\frac{\eta}{V_{a,B}}$, $\frac{\eta}{\nabla_y v} \lesssim \delta$ in (3.4),
\[
|\nabla_y^k V_{a,B}^{-2} \eta^2| \lesssim V_{a,B}^{-2} |\nabla_y^k \eta| \left[ |\nabla_y^k V_{a,B}| + |\nabla_y^k \eta| \right] + V_{a,B}^{-2} \left[ |\nabla_y \eta|^2 + |\nabla_y V_{a,B}|^2 \right]
\lesssim \tau^{-1} |w| + \delta |\nabla_y^k w| + \delta |\nabla_y w| + \tau^{-2}(1+|y|^2),
\] (10.14)
where we used that $|\nabla_y^k V_{a,B}| \lesssim \tau^{-1}$ for $|k| = 2$, and $|\nabla_y V_{a,B}| \lesssim \tau^{-1} |y|$.
For the first term on the right hand side,
\[
\tau^{-1} \| \langle y \rangle^{-1} \chi \omega \|_\infty \lesssim \tau^{-1} \Omega^2 \| \langle y \rangle^{-3} \chi \omega \|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-2} M_1, \tag{10.15}
\]
where we use that \(\tau^{-3} \Omega^2 \leq \Omega^{-\frac{3}{2}}\) by the definition of \(\Omega\) in (2.3). For the second and third terms we use the techniques in proving (8.12), and control the last term by direct computation. This, together with changing order of \(\chi \omega\) and \(\nabla^k_y\) and applying (3.4), implies
\[
\| \langle y \rangle^{-1} W_1 \|_\infty \lesssim \delta \kappa(\epsilon) \Omega^{-2} P(M). \tag{10.16}
\]
This together with (10.13) implies the desired (10.9).

Now we prove (10.10). Here the proof is considerably easier than that of (9.13), because the wanted decay estimate is significantly slower than that in (9.13), and because here two \(y\)-derivatives, instead of only one in (9.13), generate better smallness estimates, by (3.4) and that \(|\nabla_y \chi \omega|\) becomes small as \(|l|\) increases.

Hence here we choose to skip the proof.

### 10.2 Proof of the second estimate in (10.1)

As in deriving (9.32), we find \(\Phi_1\), defined as, for \(|k| = 2\),
\[
\Phi_1 := (100 + |y|^2)^{-1} \| \nabla^k_y \partial \theta \chi \omega v \|_{L^2_\theta}^2, \tag{10.17}
\]
satisfies the equation
\[
\partial_t \Phi_1 = -(L_1 + V_1) \Phi_1 - 2(100 + |y|^2)^{-1} \| \partial \theta \nabla^k_y \chi \omega v \|_{L^2_\theta}^2 + 2 \sum_{k=1}^{3} \Psi_{1k}, \tag{10.18}
\]
where the linear operator \(L_1 + V_1\) is defined as
\[
L_1 + V_1 := (100 + |y|^2)^{-1}(-\Delta + \frac{1}{2} y \cdot \nabla_y - \frac{2y}{100 + |y|^2} \cdot \nabla_y) (100 + |y|^2),
\]
where \(L_1\) is a differential operator and \(V_1\) is a multiplier, defined as,
\[
L_1 := -\Delta + \frac{1}{2} y \cdot \nabla_y - \frac{2y}{100 + |y|^2} \cdot \nabla_y,
\]
\[
V_1 := 1 + \frac{|y|^2}{100 + |y|^2} - \frac{6}{100 + |y|^2}, \tag{10.19}
\]

And the terms $\Psi_{ik}$, $k = 1, 2, 3$, are defined as

\[
\begin{align*}
\Psi_{11} & := (100 + |y|^2)^{-1}\langle \nabla_y^k \partial_\theta \chi \Omega v, \nabla_y^k \partial_\theta \chi (v^{-2} \partial_\theta^2 v - v^{-1}) \rangle, \\
\Psi_{12} & := (100 + |y|^2)^{-1}\langle \nabla_y^k \partial_\theta \chi \Omega v, \nabla_y^k \partial_\theta \chi N_1(v) \rangle, \\
\Psi_{13} & := (100 + |y|^2)^{-1}\langle \nabla_y^k \partial_\theta \chi \Omega v, \nabla_y^k \partial_\theta \mu(v) \rangle,
\end{align*}
\]

where $\mu$ is defined as in (4.7). They satisfy the following estimates, recall the definition of $P(M)$ in (8.1):

**Proposition 10.4.** There exists some constant $C$ such that

\[
\begin{align*}
\Psi_{11}(v) & \leq - \left( \frac{18}{25} - C\delta \right) (100 + |y|^2) V_{a,B}^2 \nabla_y^k \partial_\theta \chi \Omega v \|_{L_0^2}^2 + C\delta \Phi_1 + C\delta^2 \kappa^2(\epsilon) \Omega^{-4} P^2(M), \\
|\Psi_{12}| & \leq \frac{1}{100} \left[ (100 + |y|^2)^{-1} V_{a,B}^2 \nabla_y^k \partial_\theta \chi \Omega v \|_{L_0^2}^2 + \Phi_1 \right] + C\delta^2 \kappa^2(\epsilon) \Omega^{-4} P^2(M), \\
\Psi_{13} & \leq \frac{1}{100} (100 + |y|^2)^{-1} \nabla_y^k \partial_\theta \chi \Omega v \|_{L_0^2}^2 + C\delta^2 \kappa^2(\epsilon) \Omega^{-4} P^2(M).
\end{align*}
\]

The proposition will be proved in subsubsection 10.2.1.

We continue to study (10.18). We observe that

\[
V_1 \geq \frac{9}{10},
\]

This together with the results in Proposition 10.4 implies, for some $C_1 > 0$,

\[
\partial_\tau \Phi_1 \leq -(L_1 + \frac{1}{2}) \Phi_1 + C_1\delta^2 \kappa^2(\epsilon) \Omega^{-4} P^2(M).
\]

Apply the maximum principle, using that $\Phi_1(y) = 0$ if $|y| \geq (1 + \epsilon) \Omega(\tau)$, to find that, for some $C_2 > 0$,

\[
\Phi_1(\tau) \leq e^{-\frac{1}{2}(\tau - \xi_0)} \Phi_1(\xi_0) + C_2\delta^2 \kappa^2(\epsilon) \Omega^{-4} P^2(M).
\]

Now we derive the desired estimate for $\nabla_y^k \partial_\theta \chi \Omega w$. The decomposition of $v$ implies that

\[
\nabla_y^k \partial_\theta \chi \Omega w = \nabla_y^k \partial_\theta \chi \Omega \left( v - \tilde{\beta}_2(\tau) \cdot y \cos \theta + \tilde{\beta}_3(\tau) \cdot y \sin \theta + \alpha_1(\tau) \cos \theta + \alpha_2(\tau) \sin \theta \right).
\]

(10.26)
This together with the estimates in (3.10) implies that, for some $C_3 > 0$,
\[ \Phi_1 \geq \frac{1}{2}(100 + |y|^2)^{-1} \| P_{\theta, \geq 2} \nabla^k_y \bar{\partial}_\theta \chi \Omega^i w \|_{L^2_\theta}^2 - C_3 \tau^{-4}. \] (10.27)

This, together with (10.25) and that $\Phi_1(\xi_0) \lesssim \kappa^2(\epsilon)\Omega^{-4}(\xi_0)$ implied by (4.18), directly implies the desired second estimate in (10.1).

10.2.1 Proof of Proposition 10.4

To prove (10.21) we decompose $N_1$ into $N_{11} + N_{12}$ as in (8.80). This makes
\[ \Psi_{12} = D_1 + D_2, \] (10.28)
where $D_1$ and $D_2$ are defined in terms of $N_{11}$ and $N_{12}$ respectively.

By arguing as in (8.81), we have that, for any $|k| = 2$,
\[ |\nabla^k_y \bar{\partial}_\theta N_{11}| \leq \delta \sum_{|l|=1,2} \left[ |\nabla^l_y v|^2 + |\nabla^l_y \bar{\partial}_\theta v| \right]. \] (10.29)

Then we decompose $v$ and apply the same techniques as in proving (8.41)-(8.42), to find
\[ |D_1| \lesssim \delta \kappa(\epsilon)\Omega^{-2} P(M) \sqrt{\Phi_1}. \] (10.30)

For $D_2$, we integrate by parts in $\theta$ to find that
\[ D_2 = -(100 + |y|^2)^{-1} \langle v^{-1} \nabla^k_y \bar{\partial}_\theta^2 \chi \Omega^i v, \ v \nabla^k_y \chi \Omega N_{12} \rangle_\theta. \]

By arguing as in (8.86) we find
\[ v|\nabla^k_y N_{12}| \lesssim \delta ||\nabla^l_y \theta v| + \sum_{i=1,2} |\partial^l_y v|, \] (10.31)
and then decompose $v$ and apply the techniques as in proving (8.42) to have
\[ |D_2| \lesssim \delta \kappa(\epsilon)\Omega^{-2} P(M) (100 + |y|^2)^{-\frac{3}{2}} \| \nabla^k_y \bar{\partial}_\theta^2 \chi \Omega^i v \|_{L^2_\theta}. \] (10.32)

Collect the estimates above and apply Young’s inequality to obtain the desired (10.21).

Now we prove (10.20). The general strategy is the same to that in proving (8.66) and (9.34). The difference is that $P_{\theta, \geq 2}$ is not in the definition of $\Phi_1$, but it appears in the first term on the right hand side of (10.20). This is resulted by a cancellation, see (10.34) below.

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To see this, we compute directly to find
\[ \Psi_{11} = \langle \partial_\theta^2 \nabla_y^k \chi \Omega v, v^{-2} \partial_\theta^2 \nabla_y^k \chi \Omega v \rangle_\theta - \langle \partial_\theta \nabla_y^k \chi \Omega v, v^{-2} \partial_\theta \nabla_y^k \chi \Omega v \rangle_\theta. \]  
(10.33)

We decompose \( \partial_\theta \chi \Omega v \) as
\[ \partial_\theta \chi \Omega v = \partial_\theta \left[ \frac{1}{2\pi} \sum_{m=\pm 1} e^{im\theta} \chi \Omega v_m + P_{\theta \geq 2} \chi \Omega v \right], \]
with \( v_m := \frac{1}{2\pi} \langle v, e^{i \theta} \rangle_\theta \) and observe for \( m = \pm 1 \),
\[ \langle \partial_\theta^2 \nabla_y^k \Omega \Theta e^{i m \theta} v_m, v^{-2} \partial_\theta^2 \nabla_y^k \Omega \Theta e^{i m \theta} v_m \rangle_\theta - \langle \partial_\theta \nabla_y^k \Omega \Theta e^{i m \theta} v_m, v^{-2} \partial_\theta \nabla_y^k \Omega \Theta e^{i m \theta} v_m \rangle_\theta = 0, \]  
(10.34)

and hence find
\[ \Psi_{11} = W_1 + W_2, \]  
(10.35)

with
\[ W_1 := \langle P_{\theta \geq 2} \partial_\theta^2 \nabla_y^k \chi \Omega v, v^{-2} P_{\theta \geq 2} \partial_\theta^2 \nabla_y^k \chi \Omega v \rangle_\theta - \langle P_{\theta \geq 2} \partial_\theta \nabla_y^k \chi \Omega v, v^{-2} P_{\theta \geq 2} \partial_\theta \nabla_y^k \chi \Omega v \rangle_\theta \]
and \( W_2 \) collecting the other terms and satisfying the estimate,
\[ |W_2| \lesssim \langle P_{\theta \geq 2} \partial_\theta^2 \nabla_y^k \chi \Omega v, v^{-2} e^{i \theta} \nabla_y^k \chi \Omega v_1 \rangle_\theta + \langle P_{\theta \geq 2} \partial_\theta \nabla_y^k \chi \Omega v, v^{-2} e^{i \theta} \nabla_y^k \chi \Omega v_1 \rangle_\theta \]
\[ + |\langle e^{2i \theta} \nabla_y^k \chi \Omega v_1, v^{-2} \nabla_y^k \chi \Omega v_{-1} \rangle_\theta|, \]
where we use that \( v_{-1} = \overline{v_1} \) resulted by that \( v \) is a real function.

Similar to proving (8.66) and (9.31), \( W_1 \) is bounded from below, for some \( C > 0 \),
\[ W_1 \geq \left( \frac{3}{4} - C\delta \right) V_{a, \Omega}^{-2} \langle P_{\theta \geq 2} \partial_\theta^2 \nabla_y^k \chi \Omega v, P_{\theta \geq 2} \partial_\theta^2 \nabla_y^k \chi \Omega v \rangle_\theta. \]  
(10.36)

Now we estimate \( W_2 \), we control the first two terms as in the proof of (8.75), namely the presence of \( P_{\theta \geq 2} \) and \( e^{i k \theta}, k = 1, 2 \), force the \( \theta \)-dependent part of \( v^{-2} \) to contribute. For the last term, besides using the this technique, we use that
\[ |\nabla_y^k \chi \Omega v_{\pm 1}| \leq \| \partial_\theta^2 \nabla_y^k \chi \Omega v \|_{L_\theta^2} \]  
(10.37)

Consequently, we have,
\[ (100 + |y|^2)^{-1} |W_2| \lesssim \delta \kappa(\epsilon) \Omega^{-2} (1 + M_3)(100 + |y|^2)^{-\frac{1}{2}} \| v^{-1} P_{\theta \geq 2} \partial_\theta^2 \nabla_y^k \chi \Omega v \|_{L_\theta^2} + \delta \Phi_1. \]  
(10.38)

Collect the estimates above and apply Young’s inequality to prove (10.20).

The proof of (10.22) is significantly easier than those of (8.67) and (9.35), because the wanted decay estimate is considerably slower, and presently one has more \( y \)-derivatives. The latter is important since \( \nabla_y^l \chi \Omega \) becomes smaller as \( |l| \) increases, and, as shown in (3.4), the \( y \)-derivatives of \( v \) is smaller than the ones without a \( y \)-derivative. Hence we skip the details.
A Proof of Proposition 4.1

The proof of Proposition 4.1 is significantly easier than estimating the functions \( (100 + |y|^2)^{-3+|k|} \| P_{\theta, \geq 2} \theta^3 |k| \nabla_y^k \chi \nabla v \|_{L^2_\infty} \), \(|k| = 0, 1, 2\), by the following two reasons: (1) the wanted decay estimates are significantly slower and hence it is relatively easy to obtain, and (2) in the presently considered region \(|y| \leq (1 + \epsilon) R(\tau) = O(\sqrt{\ln \tau})\), we have \( V_{a,B} \approx \sqrt{2} \) implied by its definition (3.16), while in the region \(|y| \leq (1 + \epsilon) \Omega(\tau) = O(\tau^{1+\frac{1}{2}})\), \( V_{a,B} \) might be (adversely large) for large \(|y|\).

In the proof we need some results proved in [12].

Recall that we proved that, if \( \tau_0 \) is sufficiently large, then for \( \tau \geq \tau_0 \), we can decompose \( v \) as

\[
v(y, \theta, \tau) = V_{a,B}(\tau)(y) + \tilde{\beta}_1(\tau) \cdot y + \tilde{\beta}_2(\tau) \cdot y \cos \theta + \tilde{\beta}_3(\tau) \cdot y \sin \theta + \alpha_1(\tau) \cos \theta + \alpha_2(\tau) \sin \theta + w(y, \theta, \tau),
\]

(A.1)

and if

\[
\tau_1 \geq \tau_0
\]

(A.2)

is sufficiently large, then \( a, B, \tilde{\beta}_k, \alpha_l, k = 1, 2, 3, l = 1, 2 \) satisfy the estimates in (3.8)-(3.10), and there exists a constant \( C \) such that,

\[
\| \langle y \rangle^{-3} \chi_R w(\cdot, \tau) \|_\infty \leq C \kappa(\epsilon) R^{-4}(\tau), \\
\| \langle y \rangle^{-2} \nabla^m y \partial^\nu_0 \chi_R w(\cdot, \tau) \|_\infty \leq C \kappa(\epsilon) R^{-3}(\tau), \ |m| + n = 1, \\
\| \langle y \rangle^{-1} \nabla^m y \partial^\nu_0 \chi_R w(\cdot, \tau) \|_\infty \leq C \kappa(\epsilon) R^{-2}(\tau), \ |m| + n = 2.
\]

(A.3)

The remainder \( \chi_R w \) satisfies the equation

\[
\partial_\tau \chi_R w = - L \chi_R w + \chi_R \left[ F + G + N_1(v) \chi_R + N_2(\eta) \right] + \mu_R(w),
\]

(A.4)

where the operator \( L \) is defined as

\[
L := -\Delta_y + \frac{1}{2} y \cdot \nabla_y - \frac{1}{2} - V_{a,B}^{-2},
\]

(A.5)

the functions \( F, G, N_1 \) and \( N_2 \) are defined in (4.1), and \( \mu_R(w) \) is defined as

\[
\mu_R(w) := \frac{1}{2} (y \cdot \nabla_y \chi_R) w + (\partial_\tau \chi_R) w - (\Delta_y \chi_R) w - 2 \nabla_y \chi_R \cdot \nabla_y w.
\]
And we proved that, for some sufficiently small constant $\delta$,
\[
|v - V_{a,B}|, |\partial_y^l\nabla_y^k v| \lesssim \delta \text{ for } |y| \leq (1 + \epsilon)R \text{ and } |k| + l = 1, 2, 3, 4.
\] (A.6)

As in (4.9) we defined three functions $w_m$, $m = -1, 0, 1$, by decomposing $w$,
\[
w(y, \theta, \tau) = w_0(y, \tau) + e^{i\theta}w_1(y, \tau) + e^{-i\theta}w_{-1}(y, \tau) + P_{\theta \geq 2}w(y, \theta, \tau).
\] (A.7)

Impose $P_{\theta \geq 2}$ on both sides and use that $P_{\theta \geq 2}(F + G) = 0$ to find
\[
\partial_\tau(P_{\theta \geq 2}\chi_R w) = -L(P_{\theta \geq 2}\chi_R w) + P_{\theta \geq 2}\left[N_1(\eta)\chi_R + N_2(v)\chi_R + \mu_R(w)\right].
\] (A.8)

In the rest of the section we prove Proposition 4.1.

A.1 Proof of (4.16)

For notational purpose we define
\[
\tilde{\Phi}_3(y, \tau) := (100 + |y|^2)^{-3}\|P_{\theta \geq 2}\partial^3_\theta \chi_R w(y, \cdot, \tau)\|_{L^2_{\theta}}^2,
\]
and derive an equation for it from (A.8),
\[
\partial_\tau \tilde{\Phi}_3 = -(L_3 + W_3)\tilde{\Phi}_3 - 2(100 + |y|^2)^{-3}\left[V_{a,B}^{-2}\|P_{\theta \geq 2}\partial^4_\theta \chi_R w\|_{L^2_{\theta}}^2 + \|P_{\theta \geq 2}\partial^3_\theta \nabla_y \chi_R w\|_{L^2_{\theta}}^2\right]^{\frac{2}{3}} + 2(100 + |y|^2)^{-3}D.
\] (A.9)

Here $L_3$ is a differential operator, and $W_3$ is a multiplier, defined as
\[
L_3 := -\Delta + \frac{1}{2}y \cdot \nabla_y - 2(100 + |y|^2)^{-3}\left(\nabla_y (100 + |y|^2)^{3}\right) \cdot \nabla_y,
\]
\[
W_3 := -1 + \frac{3|y|^2}{100 + |y|^2} - \frac{18}{100 + |y|^2} - \frac{24|y|^2}{(100 + |y|^2)^2} - 2V_{a,B}^{-2},
\] (A.10)

and the term $D$ is defined as
\[
D := \langle P_{\theta \geq 2}\partial^3_\theta \chi_R w, \partial^3_\theta \left(\chi_R (N_1(v) + N_2(\eta)) + \mu_R(w)\right)\rangle_\theta.
\]

We claim that $D$ satisfies the estimate, for $\tau \geq \tau_1$, with $\tau_1$ defined in (A.2),
\[
(100 + |y|^2)^{-3}D \leq \frac{1}{50}(100 + |y|^2)^{-3}\left[\|P_{\theta \geq 2}\partial^4_\theta \chi_R w\|_{L^2_{\theta}}^2 + \|P_{\theta \geq 2}\partial^3_\theta \nabla_y \chi_R w\|_{L^2_{\theta}}^2\right] + C\delta^2 \kappa^2(\epsilon)R^{-8}.
\] (A.11)
Suppose the claim holds, then we use that \( \| P_{\theta \geq 2} \partial_\theta^4 \chi_{Rw} \|_{L^2_\theta}^2 \geq 4 \| P_{\theta \geq 2} \partial_\theta^3 \chi_{Rw} \|_{L^2_\theta}^2 \) and that \( V_{a,B}^{-2} = \frac{1}{2} + \mathcal{O}(\tau^{-\frac{1}{2}}) \) if \( |y| \leq (1 + \epsilon) R = \mathcal{O}(\sqrt{n \tau}) \) to have that, for some \( C > 0 \),
\[
\partial_\tau \Phi_3 \leq -(L_3 + \frac{1}{2}) \Phi_3 + c\delta^2 \kappa^2 (\epsilon) R^{-8}. \tag{A.12}
\]
Apply the maximum principle, and that \( \Phi_3(y, \tau) = 0 \) if \( |y| \geq (1 + \epsilon) R(\tau) \), to have that, for some \( C_1 > 0 \),
\[
\Phi_3(\tau) \leq e^{-\frac{1}{2}(\tau - \tau_1)} \Phi_3(\tau_1) + C_1 \delta^2 \kappa^2 (\epsilon) R^{-8}. \tag{A.13}
\]
This, together with \( \Phi_3(\tau_1) \leq \delta \) if \( \tau \geq \tau_0 \) implied by \( \text{(A.6)} \) and choosing \( \tau \) to be sufficiently large, implies the desired estimate.

A.1.1 Proof of (A.11)

We decompose \( D \) into two terms
\[
D := D_1 + D_2, \tag{A.14}
\]
where \( D_1 \) is defined as
\[
D_1 := \langle P_{\theta \geq 2} \partial_\theta^3 \chi_{Rw}, \partial_\theta^3 \chi_{R(N_1 + N_2)} \rangle_\theta = -\langle P_{\theta \geq 2} \partial_\theta^4 \chi_{Rw}, \chi_{R} \partial_\theta^2 (N_1 + N_2) \rangle_\theta,
\]
here we integrate by parts in \( \theta \) in the second step, and \( D_2 \) is defined as
\[
D_2 := \langle P_{\theta \geq 2} \partial_\theta^3 \chi_{Rw}, \mu_R (\partial_\theta^2 w) \rangle_\theta.
\]

For \( D_1 \) we observe that
\[
\chi_{R} |\partial_\theta^2 N_1| \lesssim \delta \chi_{R} \sum_{0 \leq l \leq 2} \left[ |\partial_\theta^l \nabla_y v|^2 + \sum_{0 \leq l \leq 2} |\partial_\theta^{l+1} v| \right], \tag{A.15}
\]
\[
\chi_{R} \| \partial_\theta^2 N_2 \|_{L^2_\theta} \lesssim \delta \chi_{R} \sum_{l=1,2,3,4} \| \partial_\theta^l \eta \|_{L_\theta^2} \tag{A.16}
\]
Here in deriving (A.15) we use the definition of \( N_1 \) and (A.6) to find that
\[
|\partial_\theta^2 N_1| \lesssim \sum_{m=0}^2 \left[ \sum_{k,l=1,2,3} |\partial_\theta^m (\partial_y v \partial_y v \partial_y v \partial_y v)| + \sum_{k=1,2,3} |\partial_\theta^m (\partial_y v \partial_y v \partial_y v \partial_y v)| \right] \tag{A.17}
\]
\[
+ |\partial_\theta^m ((\partial_\theta v)^2 \partial_\theta^2 v) | + |\partial_\theta^m (\partial_\theta v)^2 | \right],
\]

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then by considering the distribution of the derivatives $\partial_\theta^m$ among the terms. And \((A.16)\) is derived similarly, besides that $\partial_\theta \eta = \partial_\theta \eta$.

\((A.15)\), together with decomposition of $v$, and that $\|\partial_\theta f\|_{L_\theta^2} \leq \|\partial_{\theta}^2 f\|_{L_\theta^2}$ for any smooth function $f$, and \((A.6)\), makes

\[
(100 + |y|^2)^{-\frac{3}{2}} \chi_R \|\partial_\theta^2 \chi_R N_1(v)\|_{L_\theta^2}^2 \\
\lesssim \delta (100 + |y|^2)^{-\frac{3}{2}} \left[ \|\nabla_y \partial_\theta^2 P_{\theta \geq 2} \chi_R w\|_{L_\theta^2} + \|\partial_\theta^4 \chi_R P_{\theta \geq 2} w\|_{L_\theta^2} \right] + \delta \kappa(\epsilon) R^{-4},
\]

\((A.18)\)

where, besides using that

\[
\tau^{-\frac{1}{2}} \leq \kappa(\epsilon) R^{-10}(\tau) = \mathcal{O}(\ln \tau)^{-5},
\]

\((A.19)\)

we use $\kappa(\epsilon) R^{-3}$ to control $\nabla_y \chi_R$ produced in changing the order of $\nabla_y$ and $\chi_{\Omega}$ and to control $\|100 + |y|^2\|_{\chi R}^{-\frac{3}{2}} \chi_R w_m \|_{\infty}$ and $\|100 + |y|^2\|_{\chi R}^{-\frac{3}{2}} \chi_R |\nabla_y w_m|^2 \|_{\infty}$, $m = -1, 0, 1$. Specifically,

\[
(100 + |y|^2)^{-\frac{3}{2}} \chi_R |\nabla_y w_m|^2 \\
\lesssim (100 + |y|^2)^{-\frac{3}{2}} \chi_R \|\nabla_y w_m\|_{\infty} \lesssim 1 \lesssim \|y\|^{-2} \chi_R \|\nabla_y w_m\|_{\infty} + \delta \kappa(\epsilon) R^{-3} \left[ \|y\|^{-1} \chi_R \|\nabla_y w_m\|_{\infty} + \|y\|^{-1} \chi_R \|\nabla_y (1 - \chi_R) w_m\|_{\infty} \right] \\
\lesssim \kappa(\epsilon) R^{-4} \left( \kappa(\epsilon) R^{-1} + \delta \right),
\]

\((A.20)\)

where in the second step, we change the order of $\chi_R$ and $\nabla_y$ in the first factor, and insert $1 = \chi_R + (1 - \chi_R)$ before $w$ in the second factor, and then apply the estimates in \((A.3)\) in the last step, and similarly

\[
\|100 + |y|^2\|_{\chi R}^{-\frac{3}{2}} \chi_R w_m \|_{\infty} \lesssim \|100 + |y|^2\|_{\chi R}^{-\frac{3}{2}} \chi_R w \|_{\infty} \leq C \kappa(\epsilon) R^{-4}.
\]

\((A.21)\)

Similarly, from \((A.16)\) we derive

\[
(100 + |y|^2)^{-\frac{3}{2}} \chi_R \|\partial_\theta^2 \chi_R N_2\|_{L_\theta^2} \lesssim \delta (100 + |y|^2)^{-\frac{3}{2}} \|\partial_\theta^4 \chi_R P_{\theta \geq 2} w\|_{L_\theta^2} + \delta \kappa(\epsilon) R^{-4}.
\]

\((A.22)\)

Collect the estimates above and apply Young’s inequality to have, for some $C > 0$,

\[
(100 + |y|^2)^{-3} |D_1| \\
\leq \frac{1}{100} (100 + |y|^2)^{-3} \left[ \|P_{\theta \geq 2} \partial_\theta^4 \chi_R w\|_{L_\theta^2}^2 + \|P_{\theta \geq 2} \partial_\theta^3 \chi_R N_2(\eta)\|_{L_\theta^2}^2 \right] + C \delta^2 \kappa^2(\epsilon) R^{-8}.
\]

\((A.23)\)
For $D_2$, as pointed out in [12], the term $\frac{1}{2}(y\nabla_y \chi_R)$ in the definition of $\mu_R(\partial^3_\theta P_{\theta, \geq 2}w)$ is of order $\mathcal{O}(1)$, but it has a favorable non-positive sign. This makes

$$D_2 \leq \langle P_{\theta, \geq 2} \partial^3_\theta \chi_R w, P_{\theta, \geq 2} \partial^3_\theta \left( (\partial_\tau \chi_R) w - (\Delta_y \chi_R) w - 2\nabla_y \chi_R \cdot \nabla_y w \right) \rangle_\theta. \quad (A.24)$$

Compute directly, and use (A.6), to have that, for some $C > 0$,

$$(100 + |y|^2)^{-3}D_2 \leq \frac{1}{100} \tilde{\Phi}_3 + C\delta^2 \kappa^2 (\epsilon) R^{-8}. \quad (A.25)$$

Collect the estimates in (A.23), (A.25) and (A.14) to obtain the desired estimate (A.11).

### A.2 Proof of (4.17)

Compute directly to find that the function $\tilde{\Phi}_2$, defined as

$$\tilde{\Phi}_2 := (100 + |y|^2)^{-2} \| \nabla_y \partial^2_\theta \mu_R \|_{L^2_\theta}^2 \quad (A.26)$$

satisfies the equation

$$\partial_\tau \tilde{\Phi}_2 = - (L_2 + W_2) \tilde{\Phi}_2$$

$$- 2(100 + |y|^2)^{-2} \left[ V_{a,B}^2 \| \nabla_y \partial^2_\theta \mu_R \|_{L^2_\theta}^2 + \sum_{l=1}^3 \| P_{\theta, \geq 2} \partial^2_\theta \nabla_y \partial^l_\theta \mu_R \|_{L^2_\theta}^2 \right]$$

$$+ 2(100 + |y|^2)^{-2}(U_1 + U_2 + U_3), \quad (A.27)$$

where $L_2$ is a differential operator and $W_2$ is a multiplier, defined as,

$$L_2 := - \Delta + \frac{1}{2} y \cdot \nabla_y - 2(100 + |y|^2)^{-2} \left( \nabla_y (100 + |y|^2)^2 \right) \cdot \nabla_y,$$

$$W_2 := \frac{2|y|^2}{100 + |y|^2} - \frac{12}{100 + |y|^2} - \frac{8|y|^2}{(100 + |y|^2)^2} - 2V_{a,B}^{-2},$$

and we use that $\partial_y y \cdot \nabla_y g = (y \cdot \nabla_y + 1) \partial_y g$. Moreover $U_l$, $l = 1, 2, 3$, are defined as

$$U_1 := \langle \nabla_y \partial^2_\theta P_{\theta, \geq 2} \chi_R w, (\nabla_y V_{a,B}) \partial^2_\theta \chi_R w \rangle_\theta,$$

$$U_2 := \langle \nabla_y \partial^2_\theta P_{\theta, \geq 2} \chi_R w, \nabla_y \partial^2_\theta \chi_R (N_1(v) + N_2(\eta)) \rangle_\theta,$$

$$U_3 := \langle \nabla_y \partial^2_\theta P_{\theta, \geq 2} \chi_R w, \nabla_y \partial^2_\theta \mu_R (w) \rangle_\theta.$$
We claim that, for some $C > 0$, and $\tau \geq \tau_1$ (see (A.2))

$$|U_1| \leq C \delta \tau^{-\frac{1}{2}} \|\nabla_y \partial_\theta^2 P_{\theta,\geq 2} \chi R w\|_{L^2_\theta};$$

$$(100 + |y|^2)^{-1}|U_2| \leq \frac{1}{100} (100 + |y|^2)^{-1} \|P_{\theta,\geq 2} \nabla_y \partial_\theta^3 \chi R w\|_{L^2_\theta} + C \delta^2 \kappa^2 (\epsilon) R^{-6},$$

$$(100 + |y|^2)^{-2} U_3 \leq C \delta^2 \kappa^2 (\epsilon) R^{-6}. \tag{A.28}$$

The claims will be proved in subsubsection A.2.1.

Suppose the claims hold, then as proving (A.12), we find that, for some $C > 0$,

$$\partial_\tau \tilde{\Phi}_2 \leq -(L_2 + \frac{1}{2}) \tilde{\Phi}_2 + C \delta^2 \kappa^2 (\epsilon) R^{-6}. \tag{A.29}$$

Apply the maximum principle to have that, for some $C_1 > 0$,

$$\tilde{\Phi}_2(\tau) \leq e^{-\frac{1}{2} (\tau - \tau_1)} \tilde{\Phi}_2(\tau_1) + C_1 \delta^2 \kappa^2 (\epsilon) R^{-6}. \tag{A.30}$$

We observe that $\tilde{\Phi}_2(\tau_1) \leq \delta$ for $\tau_1 \geq \tau_0$ implied by (A.6). Hence if $\tau$ is sufficiently large, we have the desired estimate.

### A.2.1 Proof of (A.28)

It is easy to prove the estimate for $U_1$ since $|\nabla_y V_{a,B}| \lesssim \tau^{-\frac{1}{2}}$.

To prove the second estimate, we use the idea in (A.17) and compute directly to have

$$|\partial_\theta \nabla_y N_1| \lesssim \delta \sum_{l=0,1,2,3} |\partial_\theta^l \nabla_y v| + \delta \sum_{l=1,2} |\partial_\theta^l v|,$$

$$|\partial_\theta \nabla_y N_2| \lesssim \delta \sum_{|k|,l=1,0} \|\nabla_y \partial_\theta^k \eta\|. \tag{A.31}$$

By decomposing $v$ and changing the order of $\chi_R$ and $\nabla_y$ we find that,

$$\langle y \rangle^{-2} \|\chi_R \partial_\theta \nabla_y N_1\|_{L^2_\theta} \lesssim \delta \langle y \rangle^{-2} \|\partial_\theta^3 \chi R w\|_{L^2_\theta} + \delta \kappa (\epsilon) R^{-3} + \delta \tau^{-1}, \tag{A.32}$$

where we use the following bounds, by the estimates in (A.3), and for $m = -1, 0, 1$,

$$\|\langle y \rangle^{-2} \nabla_y \chi R w_m\|_\infty \lesssim \|\langle y \rangle^{-2} \nabla_y \chi R w\|_\infty \lesssim \kappa (\epsilon) R^{-3},$$

$$\langle y \rangle^{-2} \chi R |w_m| \lesssim R \|\langle y \rangle^{-3} \chi R w\|_\infty \lesssim \kappa (\epsilon) R^{-3}, \tag{A.33}$$

and by (4.16),

$$\langle y \rangle^{-2} \|P_{\theta,\geq 2} \partial_\theta^3 \chi R w\|_{L^2_\theta} \lesssim \langle y \rangle^{-2} \|P_{\theta,\geq 2} \partial_\theta^3 \chi R w\|_{L^2_\theta} \lesssim \kappa (\epsilon) R^{-3} \tag{A.34}$$
Similarly,
\[ \langle y \rangle^{-2} \chi_{\mathcal{R}} \| \partial_{\theta} \nabla_{y} N_{2} \|_{L_{\theta}^{2}} \lesssim \| \langle y \rangle^{-2} \nabla_{y} \tilde{\theta}^{3} P_{\tilde{\theta} \geq 2} \chi_{\mathcal{R}} w \|_{L_{\tilde{\theta}}^{2}} + \delta \kappa(\epsilon) R^{-3} + \delta \tau^{-1}. \] (A.35)

Collect the estimates above to find the desired estimate for \( U_{2} \) in (A.28).

For \( U_{3} \), the method is the same as that in (A.24), and we choose to skip the details here.

### A.3 Proof of (4.18)

Compute directly to find that the function \( \tilde{\Phi}_{l} \), \( l \in (\mathbb{N} \cup \{0\})^{3} \) and \(|l| = 2\), defined as
\[ \tilde{\Phi}_{l}(y, \tau) := (100 + |y|^{2})^{-1} \langle P_{\tilde{\theta} \geq 2} \nabla_{y} \chi_{\mathcal{R}} w, \nabla_{y} \tilde{\theta} \rangle, \] (A.36)
satisfies the equation
\[ \partial_{\tau} \tilde{\Phi}_{l} = -(L_{1} + W_{1}) \tilde{\Phi}_{l} - 2(100 + |y|^{2})^{-1} \left( V_{a,B}^{-2} \| P_{\tilde{\theta} \geq 2} \tilde{\theta} \nabla_{y} \chi_{\mathcal{R}} w \|_{L_{\tilde{\theta}}^{2}} + \| P_{\tilde{\theta} \geq 2} \nabla_{y} \nabla_{y} \chi_{\mathcal{R}} w \|_{L_{\tilde{\theta}}^{2}} \right) \]
\[ + 2(100 + |y|^{2})^{-1} \sum_{k=1,2} \Upsilon_{k}, \] (A.37)

where the linear operators \( L_{1} \) and \( W_{1} \) are defined as
\[ L_{1} := -\Delta + \frac{1}{2} y \cdot \nabla_{y} - \frac{2y}{100 + |y|^{2}} \cdot \nabla_{y} \]
\[ W_{1} := 1 + \frac{|y|^{2}}{100 + |y|^{2}} - \frac{6}{100 + |y|^{2}} - 2V_{a,B}^{-2}, \] (A.38)

and we use the relation that, for any function \( h \), \( \partial_{y_{k}} \left( \frac{1}{2} y \cdot \nabla_{y} h \right) = \left( \frac{1}{2} y \cdot \nabla_{y} + \frac{1}{2} \right) \partial_{y_{k}} h \), and the terms \( \Upsilon_{k}, k = 1, 2 \), are defined as
\[ \Upsilon_{1} := \langle P_{\tilde{\theta} \geq 2} \nabla_{y}^{l} \tilde{\theta} \chi_{\mathcal{R}} w, \nabla_{y}^{l} \tilde{\theta} \nabla_{y}^{2} \chi_{\mathcal{R}} w - V_{a,B}^{-2} \nabla_{y}^{l} \tilde{\theta} \chi_{\mathcal{R}} w \rangle, \]
\[ \Upsilon_{2} := \langle P_{\tilde{\theta} \geq 2} \nabla_{y}^{l} \tilde{\theta} \chi_{\mathcal{R}} w, \nabla_{y}^{l} \tilde{\theta} \left( \chi_{\mathcal{R}} \left( N_{1}(v) + N_{2}(\eta) \right) + \mu_{R}(w) \right) \rangle. \]

We claim that they satisfy the estimates, for some \( C > 0 \) and \( \tau \geq \tau_{1} \) (see (A.2)),
\[ |\Upsilon_{1}| \leq \delta \tau^{-\frac{3}{2}} \| P_{\tilde{\theta} \geq 2} \nabla_{y}^{l} \tilde{\theta} \chi_{\mathcal{R}} w \|_{L_{\tilde{\theta}}^{2}}, \] (A.39)
\[ (100 + |y|^{2})^{-1} \Upsilon_{2} \leq \frac{1}{100} (100 + |y|^{2})^{-1} \sum_{|n| = 2} \| P_{\tilde{\theta} \geq 2} \nabla_{y}^{n} \tilde{\theta} \chi_{\mathcal{R}} w \|_{L_{\tilde{\theta}}^{2}}^{2} + C \delta^{2} \kappa^{2}(\epsilon) R^{-4}. \] (A.40)
The claim will be proved in the subsubsection A.3.1.

Suppose the claims hold, then as proving (A.12), we have that, for some $C > 0$,
\[
\partial_\tau \sum_{|l|=2} \tilde{\Phi}_l \leq -(L_1 + \frac{1}{2}) \sum_{|l|=2} \tilde{\Phi}_l + C \delta^2 \kappa^2(\epsilon) R^{-4}. \tag{A.41}
\]
Apply the maximum principle to have that, for some $C_1 > 0$,
\[
\sum_{|l|=2} \tilde{\Phi}_l(\tau) \leq e^{-\frac{1}{2}(\tau-\tau_1)} \sum_{|l|=2} \tilde{\Phi}_l(\tau_1) + C_1 \delta \kappa^2(\epsilon) R^{-4}. \tag{A.42}
\]
This, together with $\tilde{\Phi}_l(\tau_1) \leq \delta$ if $\tau_1 \geq \tau_0$ implied by (A.6) and choosing $\tau$ to be sufficiently large, implies the desired estimate.

A.3.1 Proofs of (A.39) and (A.40)

It is easy to prove (A.39), by the fact that $|\nabla^l_y V_{a,B}| \lesssim \tau^{-\frac{1}{2}}$, $|l| = 1, 2$.
Next we prove (A.40). For $N_1$ we use the idea in (A.17) to find that, for any $|k| = 2$,
\[
|\nabla^k_y N_1| \lesssim \delta \left[ \sum_{|n|=1,2, l=0,1,2} |\partial_\theta^n \nabla^l_y \eta| + \sum_{l=1,2} |\partial_\theta^l v| \right]. \tag{A.43}
\]
We decompose $v$, and consider in $L^2_\theta$ norm to have
\[
\langle y \rangle^{-1} \chi_R \|\nabla^k_y N_1\|_{L^2_\theta} \lesssim \delta \langle y \rangle^{-1} \sum_{|n|=2} \|\nabla^n_y \partial_\theta P_{\theta \geq 2} \chi_R w\|_{L^2_\theta} + \delta \kappa(\epsilon) R^{-2} + \delta \tau^{-1}, \tag{A.44}
\]
where $\delta \kappa(\epsilon) R^{-2}$ is used to the ones produced in changing order of $\nabla_y$ and $\chi_\Omega$, and we use the estimates in (A.3), and (4.16) and (4.17). The method is very similar to those used in (A.33) and (A.34), see also (A.20) and (A.21), and we skip the details here.
For the $N_2$-term, we have, for $|k| = 2$,
\[
|\nabla^k_y N_2| \lesssim \delta \sum_{|l|=0,1,2} |\nabla^l_y \eta|. \tag{A.45}
\]
And hence similarly,
\[
\langle y \rangle^{-1} \chi_R \|\nabla^k_y N_2\|_{L^2_\theta} \lesssim \delta \kappa(\epsilon) R^{-2} + \delta \tau^{-1}. \tag{A.46}
\]
For the $\mu_R$-term, we use the same methods as that in proving (A.24), and skip the details here.
Collecting the estimates above, we find the desired estimate.
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