Nuclear-structure corrections to the double-beta-decay operator

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Abstract. In order to extract an effective neutrino mass from a measurement of neutrinoless double-beta decay, one must calculate nuclear matrix elements that help determine the decay rate. QRPA and shell-model calculations of the matrix elements are contrasted, and the possibility is raised of improving the shell-model calculations by incorporating missing configurations into an effective decay operator. Preliminary results from a perturbative treatment of the missing configurations are presented.

1. Introduction
Neutrinoless double beta decay is a very slow lepton-number-violating nuclear transition that occurs if neutrinos have mass (which they do) and are their own antiparticles. An initial nucleus \((Z, A)\), with proton number \(Z\) and total nucleon number \(A\), decays to \((Z + 2, A)\), emitting two electrons in the process [1].

The neutrino masses and mixing matrix figure prominently in the decay. The rate, assuming that the process is mediated by the exchange of a light virtual neutrino, is

\[
[T_{1/2}^{0\nu}]^{-1} = G_{0\nu}(Q, Z) |M_{0\nu}|^2 \langle m_{\beta\beta} \rangle^2 ,
\]

where \(Q\) is the energy difference between the initial and final nuclei, \(Z\) is the charge of the initial nucleus, \(G_{0\nu}(Q, Z)\) is a tabulated phase-space measure, \(M_{0\nu}\) is the all-important nuclear matrix element to which we return shortly, and \(m_{\beta\beta}\) is a linear combination of neutrino masses:

\[
m_{\beta\beta} \equiv | \sum_{k=1}^{3} m_k U_{ek}^2 | .
\]

Here, \(m_k\) is the mass of the \(k^{th}\) neutrino (these mass eigenstates are linear combinations of the electron-, mu- and tau-neutrinos) and \(U_{ek}\) is the element of the unitary mixing matrix that connects that neutrino to the electron neutrino. The quantity \(m_{\beta\beta}\) is what we want to extract from the decay rate. We cannot do so, however, without knowing the matrix element \(M_{0\nu}\).

A precise expression for the matrix element is complicated, but with a few approximations it can be written to within an accuracy of about 30% as

\[
M_{0\nu} \approx M_{0\nu}^{GT} - \frac{g^2}{g_A} M_{0\nu}^F
\]

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with $g_V$ and $g_A$ the vector and axial-vector coupling constants, and

$$M_{0\nu}^E = \langle f | \sum_{a,b} H(r_{a,b},E) \tau_a^+ \tau_b^+ | i \rangle, \quad (4)$$

$$M_{0\nu}^{GT} = \langle f | \sum_{a,b} H(r_{a,b},E) \vec{\sigma}_a \cdot \vec{\sigma}_b \tau_a^+ \tau_b^+ | i \rangle. \quad (5)$$

Here $|i\rangle$ and $|f\rangle$ are the initial and final nuclear ground states, $a, b$ label particles, $E$ is an average excitation energy, and $H$ is a “neutrino potential”, given by

$$H(r, E) \approx \frac{2R}{\pi r} \int_0^\infty dq \frac{\sin qr}{q + E - (E_i + E_f)/2}. \quad (6)$$

$R$ is the nuclear radius, inserted to make the matrix element dimensionless. Equation (3) is the form we work with in what follows.

### 2. Two ways to calculate $M_{0\nu}$

Most calculations of $M_{0\nu}$ are in the neutron-proton Quasiparticle Random Phase Approximation (QRPA); the rest are in the shell model. The two methods have complementary virtues. The QRPA includes many single-particle levels outside a relatively small “inert” core, but limits itself to a particular kind of correlation. The shell model includes arbitrary complicated correlations, but only in a few single-particle levels outside a much larger inert core. The current predictions of the two models, after a recent shaking out period, are related as shown in figure 1 below.

![Figure 1](image.png)

**Figure 1.** Shell-model (triangles, from reference [2]) and QRPA (circles with error bars, from reference [3]) $0\nu$ matrix elements.

The error bars in the figure come from the use of two or more prescriptions for treating short-range correlations and for the effective value of the axial-vector coupling constant $g_A$ in nuclei. The shell-model calculations simply use one of the prescriptions for each (short-range correlations treated via the method of reference [4] and an effective $g_A$ of 1.25), and include an estimate for the relatively small corrections from forbidden terms in the weak current. Another
difference between the two calculations is in the Hamiltonian. The QRPA Hamiltonian is not as realistic as possible, but adjusted to reproduce two-neutrino double-beta decay. The shell-model Hamiltonian is fixed by a combination of theory and fitting to spectra. When all is said and done, the shell-model results are pretty uniformly smaller than those of the QRPA, sometimes by a factor of 2 or more.

Which set of calculations is closer to the truth? To find out, one has to correct one or the other for the physics it omits. Although it is possible to add missing correlations to the QRPA, it is not easy to do so because one would have to undo the simplified treatment of correlations (e.g., BCS pairing) first. By contrast, there is a systematic procedure for adding the effects of missing single-particle levels to a shell-model space [5, 6]. While the procedure is perturbative and not always reliable for that reason, it often works and we try it here.

3. Effective-Operator Formalism in the Shell Model

We give the briefest of introductions to the theory of shell-model effective operators. The formalism is based on the division of the many-body Hilbert space into the shell-model space (always called $P$) and the rest ($Q$). One begins by defining operators with the same names that project onto these spaces:

$$ P = \sum_i |i\rangle\langle i|, \quad Q = \sum_{\text{other} i} |i\rangle\langle i|. $$

(7)

Here the sums are over orthogonal sets, so that $P$ and $Q$ are true projectors. Next we define an effective Hamiltonian that when acting on the $P$-space projection of an eigenstate $\Psi_a$ gives back that projection with the correct eigenvalue:

$$ H_{\text{eff}}(E_a)P|\Psi_a\rangle = E_a P|\Psi_a\rangle. $$

(8)

Similarly, for any “bare” operator $M$ (where “bare” means “acting in the full model space”), we can define an effective operator $M_{\text{eff}}$ that acts in the $P$ space with matrix elements related to those of the bare operator by:

$$ \frac{\langle \Psi_a | P M_{\text{eff}} P | \Psi_b \rangle}{\sqrt{\langle \Psi_a | P | \Psi_a \rangle \langle \Psi_b | P | \Psi_b \rangle}} = \langle \Psi_a | M | \Psi_b \rangle. $$

(9)

Next, through a series of steps, one can develop a perturbative expansion for the matrix elements of $M_{\text{eff}}$. The starting point is the Bloch-Horowitz equation (or, more precisely, its extension to the wave function) relating an arbitrary eigenstate $\Psi(E)$ to its $P$-space projection:

$$ \Psi(E) = \mathcal{N} \left[ P |\Psi\rangle + \frac{1}{E - QHP} QHP |\Psi\rangle \right]. $$

(10)

Here $E$ is the energy of the eigenstate and $\mathcal{N}$ a normalization coefficient.

The perturbative expansion based on equation (10) is similar to the Bethe-Breuckner-Goldstone expansion for nuclear matter. Here, things are a little more complicated; we are computing a model-space interaction rather than the energy, so the diagrams look a bit different, and the isolation of a model space that contains empty levels as well as occupied levels leads to “folded diagrams”, the complications of which we need not go into here. The first steps for obtaining an effective Hamiltonian $H_{\text{eff}}$, however, are almost exactly the same as in nuclear matter. One defines a $G$ matrix as the sum over all “ladder diagrams”, as in figure 2 below. The railed lines in the figure indicate states well above the shell-model single-particle space (e.g., more than six or seven shells above).
Although here are an infinite number of diagrams, the sum can be carried out indirectly, e.g., through the solution of the Beth-Goldstone equation. The idea is that the hard short-range core that makes the nucleon-nucleon interaction intractable is treated exactly through the sum over all two-particle states at high energy (short-range). The effective interaction $G[V]$ between low-energy states that results has a soft core because the effects of short-range physics have already been accounted for in the ladder sum (the argument $V$ is meant to reinforce the fact that $G$ depends on the “bare interaction” $V$).

After this nonperturbative construction of the $G$ matrix, one can use perturbation theory in $G$ to add the effects of states that are at low energy but still outside the model space. The set of diagrams in figure 3 includes all such effects up to second order in $G$, and some third-order effects. In this figure, the upward-going lines are low-lying “particle” states, that is, single-particle states in the six- or seven-shell gap between the shell-model levels and the high-energy levels (including the shell-model levels themselves). Downward going lines are “hole” states that correspond to the vacating of levels below the shell-model space.

Typically, more complicated graphs, including the folded ones, are added to the procedure. Even then, problems with convergence, three- and higher-body operators that are generally too complicated to include, etc., mean that the resulting interaction often must be tuned phenomenologically to reproduce spectra, especially in the monopole-monopole channel [7]. Sometimes, however, the perturbation theory by itself is enough to produce a pretty good interaction [6].

A similar construction can be carried out for any two-body operator $\mathcal{M}$. The expansion of equation (10) leads to a set of diagrams in which a horizontal line representing the bare operator appears once alongside an arbitrary number of interaction lines. The first step, in analogy with the construction of the $G$-matrix, is to construct an operator $\mathcal{M}_{\text{high}}$ by summing all diagrams with two particles excited to high energies in which one line in each diagram is the operator $\mathcal{M}$ rather than the interaction. The diagrams with different numbers of interaction lines before or after the operator line can be summed and replaced by $G$-matrices (or, more, precisely, the equivalent operator that acts on the entire space). If $\mathcal{M}$ is one of the double-beta operators appearing in equations (3), then two neutron lines become proton lines whenever it acts, and the ladder sum becomes that depicted in figure 4. Here all particle lines preceding the double-beta operator (red online) are neutrons and lines following the operator (blue online) are protons.
This sum of diagrams can be calculated simply from a $G$-matrix code through the trick

$$\mathcal{M} = \frac{d}{d\lambda} G[V + \lambda \mathcal{M}]|_{\lambda=0}, \tag{11}$$

that is by computing the $G$ matrix corresponding to the interaction $V + \lambda \mathcal{M}$. The derivative filters out graphs that have one interaction replaced by a double-beta decay.

Having calculated this renormalized double-beta operator that includes the effects of short-range physics, one can use it, in parallel with the $G$-matrix, to calculate the additional renormalization from low-lying excitations. Here, we include the diagrams in figure 3, but with one $G$-matrix line replaced by $\mathcal{M}_{\text{high}}$. These diagrams, all first order in $G$, except for second-order ladders appear in figure 5. Short-range correlations are included at every vertex through the use of $G$ and $\mathcal{M}_{\text{high}}$ in place of $V$ and $\mathcal{M}$. Most of these diagrams can be calculated through the trick in equation (11). The core-polarization graphs (the last two in the figure) must be treated explicitly because the combination of neutron and proton lines that enters in those graphs for the effective operator in figure 5 are not the same as in the graphs for the effective interaction in figure 3.

4. Results

The shell-model calculation of reference [2] used an interaction that was constructed largely through the effective-interaction theory discussed above (though it included higher-order and folded diagrams). The Bonn-C nucleon-nucleon interaction [8] was used as a starting point for the ladder-diagrams that enter the $G$-matrix. After cranking the theoretical machinery, the authors tweaked the interaction by fitting certain components to spectra.
In our calculation of the effective double-beta operator, to be as consistent as possible, we began from the same Bonn-C nucleon-nucleon interaction. After calculating the $G$-matrix and its extension to the $Q$ space, we evaluated $M_{\text{high}}$ in figure 4 and the final effective-operator diagrams in figure 5. We then folded the matrix elements of this final effective operator with the two-body ground-state-to-ground-state transition densities from the shell-model calculation to obtain a transition matrix element for the decay $^{82}\text{Se} \rightarrow ^{82}\text{Kr}$. The preliminary results for the Gamow-Teller part of the matrix element (equation (5)) appear below in table 1.

| $\omega$ | low-\(E\) border | bare | high-\(E\) ladders | all ladders | + 4p-2h | + 3p-1h |
|---------|------------------|------|---------------------|-------------|---------|---------|
| $5\hbar\omega$ | 3.33 | 3.06 | 4.17 | 5.38 | — |
| $6\hbar\omega$ | 3.33 | 3.05 | 4.16 | 5.39 | — |
| $7\hbar\omega$ | 3.33 | 3.06 | 4.17 | 5.39 | — |
| $8\hbar\omega$ | 3.33 | 3.06 | 4.17 | 5.39 | — |

The table shows the contributions of various diagrams to the decay matrix element (except for the contributions of the core-polarization diagrams, which have not yet been correctly evaluated). Each row corresponds to a boundary, measured in $\hbar\omega$ from the valence $fp$ shell, between the levels that are considered “high-energy” (the ruled lines in figures 2 and 4) and those that are not. The ladder and 4-particle-2-hole contributions increase the matrix element by some 60%. The core polarization (3-particle-1-hole) diagrams turn out to make it smaller again, though by how much we are not yet sure.

One thing that can be firmly deduced from these calculations is the size of the damping caused by short-range correlations (the high-\(E\) ladders in the third column), which are included in a self-consistent way for the first time here. As is apparent from the table, the short-range correlations reduce the matrix element by about 8%. This rigorous result is more in line with that of references [9, 10], which predicted a relatively small effect, than with that obtained from the phenomenological Jastrow function of reference [4].

Without the 3p-1h contributions, it is hard to draw a firm conclusion about the size of the overall correction to the matrix element\(^1\). Whatever the eventual result, however, it is clear that the individual corrections are significant. Since they are not all that much smaller than the matrix element itself, we shouldn’t place too much trust in the perturbative treatment undertaken here. A more reliable nonperturbative evaluation of the corrections is very important, and we are working in that direction.

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\(^1\) Complete results with the 3p-1h contributions, obtained after the writing of this paper, appear in reference [11]
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