ON COINCIDENCE OF CLASSES OF FUNCTIONS DEFINED BY
A GENERALISED MODULUS OF SMOOTHNESS AND THE
APPROPRIATE INVERSE THEOREM

FATON M. BERISHA

Abstract. We give the theorem of coincidence of a class of functions defined
by a generalised modulus of smoothness with a class of functions defined by
the order of the best approximation by algebraic polynomials. We also prove
the appropriate inverse theorem in approximation theory.

0. In [4], an asymmetric operator of generalised translation was introduced, by
means of it the generalised modulus of continuity was defined, and the theorem of
coincidence of a class of functions defined by that modulus with a class of functions
with given order of the best approximation by algebraic polynomials was proved.

In our paper the analogous results are obtained for a generalised modulus of
smoothness of order $r$. In addition, in the present paper we prove a theorem inverse
to the Jackson’s theorem related to that modulus of smoothness.

1. By $L_p$ we denote the set of functions $f$ measurable on the segment $[-1, 1]$ such
that for $1 \leq p < \infty$
\[ \|f\|_p = \left( \int_{-1}^{1} |f(x)|^p \, dx \right)^{1/p} < \infty, \]
and for $p = \infty$
\[ \|f\|_\infty = \text{ess sup}_{-1 \leq x \leq 1} |f(x)| < \infty. \]

Denote by $L_{p,\alpha}$ the set of functions $f$ such that $f(x)(1 - x^2)\alpha \in L_p$, and put
\[ \|f\|_{p,\alpha} = \|f(x)(1 - x^2)^\alpha\|_p. \]

By $E_n(f)_{p,\alpha}$ we denote the best approximation of the function $f \in L_{p,\alpha}$ by
algebraic polynomials of degree not greater than $n - 1$, in $L_{p,\alpha}$ metrics, i.e.
\[ E_n(f)_{p,\alpha} = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_{p,\alpha}, \]
where $\mathcal{P}_n$ is the set of algebraic polynomials of degree not greater than $n - 1$.

By $E'(p, \alpha, \lambda)$ we denote the class of functions $f \in L_{p,\alpha}$ satisfying the condition
\[ E_n(f)_{p,\alpha} \leq Cn^{-\lambda}, \]
where $\lambda > 0$ and $C$ is a constant not depending on $n$.

1991 Mathematics Subject Classification. Primary 41A35, Secondary 41A50, 42A16.
Key words and phrases. Generalised modulus of smoothness, asymmetric operator of gen-
eralised translation, coincidence of classes, best approximations by algebraic polynomials.
For a function $f$ we define the operator of generalised translation $\hat{T}_t (f, x)$ by

$$\hat{T}_t (f, x) = \frac{1}{\pi (1 - x^2)} \int_0^\pi \left( 1 - \left( x \cos t - \sqrt{1 - x^2} \sin t \cos \varphi \right)^2 \right. $$

$$\left. - 2 \sin^2 t \sin^2 \varphi + 4 \left( 1 - x^2 \right) \sin^2 t \sin^2 \varphi \right)$$

$$\times f \left( x \cos t - \sqrt{1 - x^2} \sin t \cos \varphi \right) d\varphi.$$ 

By means of that operator of generalised translation we define the generalised difference of order $r$ by

$$\Delta_t^1 (f, x) = \Delta_t (f, x) = \hat{T}_t (f, x) - f(x),$$

$$\Delta_t^{r+1} (f, x) = \Delta_t \left( \Delta_t^{r-1} (f, x), x \right) \quad (r = 2, 3, \ldots),$$

and the generalised modulus of smoothness of order $r$ by

$$\dot{\omega}_t (f, \delta)_{p, \alpha} = \sup_{|t| \leq \delta} \| \Delta_t^{r+1} (f, x) \|_{p, \alpha} \quad (r = 1, 2, \ldots).$$

Denote by $H(p, \alpha, r, \lambda)$ the class of functions $f \in L_{p, \alpha}$ satisfying the condition

$$\dot{\omega}_t (f, \delta)_{p, \alpha} \leq C \delta^\lambda,$$

where $\lambda > 0$ and $C$ is a constant not depending on $\delta$.

2. Put $y = \cos t, z = \cos \varphi$ in the operator $\hat{T}_t (f, x)$, we denote it by $T_y (f, x)$ and rewrite it in the form

$$T_y (f, x) = \frac{1}{\pi (1 - x^2)} \int_{-1}^1 \left( 1 - R^2 \right) \left( 1 - y^2 \right) \left( 1 - z^2 \right)$$

$$+ 4 \left( 1 - x^2 \right) \left( 1 - y^2 \right) \left( 1 - z^2 \right) \ f(R) \frac{dz}{\sqrt{1 - z^2}},$$

where $R = xy - z \sqrt{1 - x^2} \sqrt{1 - y^2}$.

We define the operator of generalised translation of order $r$ by

$$T_y^r (f, x) = T_y (f, x),$$

$$T_{y_1, \ldots, y_r} (f, x) = T_{y_r} \left( T_{y_1, \ldots, y_{r-1}} (f, x), x \right) \quad (r = 2, 3, \ldots).$$

By $P_{\nu}^{(\alpha, \beta)} (x) \ (\nu = 0, 1, \ldots)$ we denote the Jacobi’s polynomials, i.e. algebraic polynomials of degree $\nu$ orthogonal with the weight function $(1 - x)^\alpha (1 + x)^\beta$ on the segment $[-1, 1]$ and normed by the condition $P_0^{(\alpha, \beta)} (1) = 1 \ (\nu = 0, 1, \ldots)$.

Denote by $a_n (f)$ the Fourier–Jacobi coefficients of a function $f$, integrable with the weight function $(1 - x^2)^2$ on the segment $[-1, 1]$, with respect to the system of Jacobi polynomials $\left\{ P_n^{(\alpha, \beta)} (x) \right\}_{n=0}^\infty$: i.e.,

$$a_n (f) = \int_{-1}^1 f(x) P_n^{(\alpha, \beta)} (x) (1 - x^2)^2 \ dx \quad (n = 0, 1, \ldots).$$

We define the following operators, having an auxiliary role later on

$$T_{1, y} (f, x) = \frac{1}{\pi (1 - x^2)} \int_{-1}^1 \left( 1 - R^2 \right) \left( 1 - y^2 \right) \left( 1 - z^2 \right) \ f(R) \frac{dz}{\sqrt{1 - z^2}},$$

$$T_{2, y} (f, x) = \frac{8}{3\pi} \int_{-1}^1 (1 - z^2)^2 \ f(R) \frac{dz}{\sqrt{1 - z^2}}.$$
where \( R = xy - z\sqrt{1 - x^2} \sqrt{1 - y^2} \), and the corresponding operators of order \( r \)
\[ T_{k,y}^r (f, x) = T_{k,y} (f, x), \]
\[ T_{k,y_1,\ldots,y_r}^r (f, x) = T_{k,y_r} \left( T_{k,y_1,\ldots,y_{r-1}}^{r-1} (f, x) \right) \quad (r = 2, 3, \ldots) \]
for \( k, 1, 2 \).

3.

Lemma 3.1. Let \( P_n(x) \) be an algebraic polynomial of degree not greater than \( n - 1 \), \( 1 \leq p \leq \infty \), \( \alpha > -\frac{1}{p} \) and \( \rho \geq 0 \). Then the following inequalities hold true
\[ \|P_n(x)\|_{p,\alpha} \leq C_1 n \|P_n\|_{p,\alpha}, \]
\[ \|P_n\|_{p,\alpha} \leq C_2 n^{2\rho} \|P_n\|_{p,\alpha+\rho}, \]
where the constants \( C_1 \) and \( C_2 \) do not depend on \( n \).

Lemma is proved in [2].

Lemma 3.2. The operators \( T_{1,y} \) and \( T_{2,y} \) have the following properties
\[ T_{1,y} \left( P_{\nu}^{(2,2),x} \right) = P_{\nu}^{(2,2)}(x) P_{\nu+2}^{(0,0)}(y), \]
\[ T_{2,y} \left( P_{\nu}^{(2,2),x} \right) = P_{\nu}^{(2,2)}(x) P_{\nu}^{(2,2)}(y) \]
for \( \nu = 0, 1, \ldots \).

Lemma 3.2 is proved in [3].

Lemma 3.3. Let \( g(x)T_{k,y}^0 (f, x) \in L_{1,2} \) for every \( y \). Then for \( k = 1, 2 \) the following equality holds true
\[ \int_{-1}^1 f(x)T_{k,y}^0 (g, x) (1 - x^2)^2 \, dx = \int_{-1}^1 g(x)T_{k,y}^0 (f, x) (1 - x^2)^2 \, dx. \]

Proof. Let \( k = 1 \) and
\[ I_1 = \int_{-1}^1 f(x)T_{1,y}^0 (g, x) (1 - x^2)^2 \, dx \]
\[ = \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 f(x)g(R) (1 - R^2 - 2 (1 - y^2) (1 - z^2)) (1 - x^2) \frac{dz \, dx}{\sqrt{1 - z^2}}, \]
where \( R = xy - z\sqrt{1 - x^2} \sqrt{1 - y^2} \). Performing change of variables in the double integral by the formulas
\[ x = Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2}, \]
\[ z = -\frac{R \sqrt{1 - y^2} - V y \sqrt{1 - R^2}}{\sqrt{1 - \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right)^2}}, \]
we get
\[ I_1 = \frac{1}{\pi} \int_{-1}^1 \int_{1}^1 (1 - R^2) f \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right) g(R) \]
\[ \times \left( 1 - \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right)^2 - 2 (1 - y^2) (1 - V^2) \right) \frac{dV \, dR}{\sqrt{1 - V^2}} \]
\[ = \int_{-1}^1 g(R)T_{1,y}^0 (f, R) (1 - R^2)^2 \, dR, \]
which proves the equality of the lemma for \( k = 1 \).
Let \( k = 2 \) and
\[
I_2 = \int_{-1}^{1} f(x) T_{2,y} (g, x) \left( 1 - x^2 \right)^2 \, dx \\
= \frac{8}{3\pi} \int_{-1}^{1} \int_{-1}^{1} f(x) g(R) \left( 1 - x^2 \right)^2 \left( 1 - z^2 \right)^2 \, dz \, dx \, \sqrt{1 - z^2}.
\]
Performing change of variables in that double integral by the formulas \((3.1)\) we get
\[
I_2 = \frac{8}{3\pi} \int_{-1}^{1} \int_{-1}^{1} f(x) \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right) g(R) \left( 1 - R^2 \right)^2 \times \left( 1 - V^2 \right)^2 \, dV \, dR = \int_{-1}^{1} g(R) T_{2,y} (f, R) \left( 1 - R^2 \right)^2 \, dR.
\]
Lemma \((3.3)\) is proved. \(\square\)

**Corollary 3.1.** If \( f \in L_{1,2} \), then for every natural number \( r \) we have \( T_{k,y} (f, x) \in L_{1,2} \) \((k = 1, 2)\).

**Proof.** Put \( g(x) \equiv 1 \) on \([-1, 1]\), considering that by Lemma \((3.2)\) (see \([1, \text{vol. II, p. 180}]\))
\[
T_{1,y} (1, x) = T_{1,y} \left( P_{0}^{(2,2)}, x \right) = P_{0}^{(2,2)}(x) P_{2}^{(0,0)}(y) = \frac{3}{2} y^2 - \frac{1}{2},
\]
we have \( f(x) T_{k,y} (1, x) \in L_{1,2} \) \((k = 1, 2)\). Hence, applying Lemma \((3.3)\) we derive
\[
\int_{-1}^{1} T_{k,y} (f, x) \left( 1 - x^2 \right)^2 \, dx = \int_{-1}^{1} f(x) T_{k,y} (1, x) \left( 1 - x^2 \right)^2 \, dx \quad \text{for } k = 1, 2.
\]
Therefrom it follows that \( T_{k,y} (f, x) \in L_{1,2} \). Now the corollary is proved by induction. \(\square\)

**Lemma 3.4.** Let \( f \in L_{1,2} \). For every natural number \( n \) the following equality holds true
\[
\int_{-1}^{1} T_{1,y} (f, x) P_{n}^{(1,1)} (y) \, dy = \sum_{m=0}^{n-2} a_m (f) \gamma_m (x),
\]
where \( \gamma_m (x) \) is an algebraic polynomial of degree not greater than \( n-2 \), and \( \gamma_m (x) \equiv 0 \) for \( n = 0 \) or \( n = 1 \).

Lemma \((3.4)\) is proved in \([1]\).

**Lemma 3.5.** Let \( q \) and \( m \) given natural numbers and let \( f \in L_{1,2} \). For every natural numbers \( l \) and \( r \) \((l \leq r)\) the function
\[
Q^{(l)}_1 (x) = \int_{0}^{\pi} \cdots \int_{0}^{\pi} T_{1,\cos t_1, \ldots, \cos t_l} (f, x) \prod_{s=1}^{r} \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s \, dt_1 \cdots dt_r
\]
is an algebraic polynomial of degree not greater than \((q+2)(m-1)\).

**Proof.** Since
\[
A_s = \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} = \sum_{k=0}^{(q+2)(m-1)} a_k \cos kt_s = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k,
\]
it follows that
\[ A_0 \sin^2 t_s = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k (1 - \cos^2 t_s) = \sum_{k=0}^{(q+2)(m-1)+2} c_k (\cos t_s)^k \]
\[ = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \beta_k^{(1)}(\cos t_s) \quad (s = 1, 2, \ldots, r). \]

Hence we have
\[ Q^{(l)}_1(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \int_0^\pi \ldots \int_0^\pi \prod_{s=1 \atop s \neq l}^r \left( \frac{\sin t_s}{\sin t_{s/2}} \right)^{2q+4} \]
\[ \times \sin^3 t_s \, dt_1 \ldots dt_{l-1} \, dt_{l+1} \ldots dt_r \int_0^\pi T^{l-1}_{1, \cos t_1, \ldots, \cos t_l} (f, x) P^{(1)}_k(\cos t_l) \sin t_l \, dt_l. \]

Let
\[ \varphi_{l, k}(x) = \int_0^\pi T^{l-1}_{1, \cos t_1, \ldots, \cos t_l} (f, x) P^{(1)}_k(\cos t_l) \sin t_l \, dt_l. \]

Substituting \( y = \cos t_l \) we obtain
\[ \varphi_{l, k}(x) = \int_{-1}^1 T_{1, y} \left( T^{l-1}_{1, \cos t_1, \ldots, \cos t_{l-1}} (f, x), y \right) P^{(1)}_k(y) \, dy. \]

Using Lemma 3.3 we get
\[ \varphi_{l, k}(x) = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^1 T^{l-2}_{1, \cos t_1, \ldots, \cos t_{l-2}} (f, R) P^{(2)}_m(R) (1 - R^2)^2 \, dR. \]

Considering Corollary 3.1 we have that \( T^{l-1}_{1, \cos t_1, \ldots, \cos t_{l-1}} (f, R) \in L_{1, 2} \). Applying \( l - 1 \) times Lemma 3.3 and Lemma 3.2 we obtain
\[ \varphi_{l, k}(x) = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^1 T^{l-2}_{1, \cos t_1, \ldots, \cos t_{l-2}} (f, R) T^{(2)}_m(R) \left( P^{(2)}_m(R) \right) \]
\[ \times \left( 1 - R^2 \right)^2 dR = \sum_{m=0}^{k-2} \gamma_m(x) P^{(0)}_{m+2}(\cos t_{l-1}) \]
\[ \times \int_{-1}^1 T^{l-2}_{1, \cos t_1, \ldots, \cos t_{l-2}} (f, R) P^{(2)}_m(R) (1 - R^2)^2 \, dR \]
\[ = \sum_{m=0}^{k-2} \gamma_m(x) P^{(0)}_{m+2}(\cos t_1) \ldots P^{(0)}_{m+2}(\cos t_{l-1}) \]
\[ \times \int_{-1}^1 f(R) P^{(2)}_m(R) (1 - R^2)^2 \, dR = \sum_{m=0}^{k-2} \gamma_m(x) a_m(f) \prod_{s=1}^{l-1} P^{(0)}_{m+2}(\cos t_s), \]

where \( a_m(f) \) is the Fourier–Jacobi coefficient of the function \( f \) with respect to the system \( \left\{ P^{(2)}_{m+2}(x) \right\}_{m=0}^\infty \). Substituting \( \varphi_{l, k}(x) \) in the expression for \( Q^{(l)}_1(x) \) we get
\[ Q^{(l)}_1(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \sum_{m=0}^{k-2} \beta_m \gamma_m(x). \]

Since \( \gamma_m(x) \) is an algebraic polynomial of degree not greater than \( k - 2 \) for \( k \geq 2 \) and \( \gamma_m(x) \equiv 0 \) for \( k = 0 \) and \( k = 1 \), then the last equality yields that \( Q^{(l)}_1(x) \) is an algebraic polynomial of degree not greater than \( (q + 2)(m - 1) \).

Lemma 3.3 is proved. \( \square \)
Lemma 3.6. Let \( q \) and \( m \) given natural numbers. Let \( f \in L_{1,2} \). For every natural numbers \( l \) and \( r \) \((l \leq r)\) the function
\[
Q_2^{(l)}(x) = \int_0^\pi \cdots \int_0^\pi T_{2,\cos t_1, \ldots, \cos t_l}(f, x) \prod_{s=1}^r \left( \frac{\sin mt_s}{\sin \frac{\pi}{2}} \right)^{2q+4} \sin^5 t_s \, dt_1 \ldots dt_r
\]
is an algebraic polynomial of degree not greater than \((q+2)(m-1)\).

Proof. As shown in Lemma 3.5
\[
A_l = \left( \frac{\sin \frac{mt_s}{s+1}}{\sin \frac{\pi}{2}} \right)^{2q+4} = \sum_{k=0}^{(q+2)(m-1)} b_k(\cos t_s)^k
\]
Hence
\[
Q_2^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)} \beta_k \int_0^\pi \cdots \int_0^\pi \prod_{s \neq l} \left( \frac{\sin \frac{mt_s}{s+1}}{\sin \frac{\pi}{2}} \right)^{2q+4} \times \sin^5 t_s \, dt_1 \ldots dt_{l-1} \, dt_{l+1} \ldots dt_r \int_0^\pi T_{2,\cos t_1, \ldots, \cos t_l}(f, x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l \, dt_l.
\]
Let
\[
\psi_{l,k}(x) = \int_0^\pi T_{2,\cos t_1, \ldots, \cos t_l}(f, x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l \, dt_l
\]
Substituting \( y = \cos t_l \) we obtain
\[
\psi_{l,k}(x) = \int_{-1}^1 T_{2,y} \left( T_{2,\cos t_1, \ldots, \cos t_{l-1}}(f, x), x \right) P_k^{(2,2)}(y) (1 - y^2)^2 \, dy.
\]
Since operator \( T_{2,y}(f, x) \) is symmetrical on \( x \) and \( y \), i.e. for every function \( g \) holds \( T_{2,y}(g, x) = T_{2,x}(g, y) \), we have
\[
\psi_{l,k}(x) = \int_{-1}^1 T_{2,x} \left( T_{2,\cos t_1, \ldots, \cos t_{l-1}}(f, y), y \right) P_k^{(2,2)}(y) (1 - y^2)^2 \, dy.
\]
Since Corollary 3.1 yields \( T_{2,\cos t_1, \ldots, \cos t_{l-1}}(f, y) \in L_{1,2} \), applying Lemma 3.3 we obtain
\[
\psi_{l,k}(x) = \int_{-1}^1 T_{2,\cos t_1, \ldots, \cos t_{l-1}}(f, y) T_{2,x} \left( P_k^{(2,2)}, y \right) (1 - y^2)^2 \, dy.
\]
Considering the property of the operator \( T_{2,x} \) from Lemma 3.2 we get
\[
\psi_{l,k}(x) = P_k^{(2,2)}(x) \int_{-1}^1 T_{2,\cos t_1, \ldots, \cos t_{l-1}}(f, y) P_k^{(2,2)}(y) (1 - y^2)^2 \, dy.
\]
Applying \( l-1 \) times Lemma 3.3 and Lemma 3.2 we obtain
\[
\psi_{l,k}(x) = P_k^{(2,2)}(x) P_k^{(2,2)}(\cos t_1) \ldots P_k^{(2,2)}(\cos t_{l-1}) \times \int_{-1}^1 f(y) P_k^{(2,2)}(y) (1 - y^2)^2 \, dy = P_k^{(2,2)}(x) a_k(f) \prod_{s=1}^{l-1} P_k^{(2,2)}(\cos t_s).
\]
where \( a_k(f) \) is the Fourier–Jacobi coefficient of the function \( f \) with respect to the system \( \{ P^{(2,2)}_k(x) \}_{k=0}^{\infty} \). Substituting \( \psi_{l,k}(x) \) in the expression for \( Q^{(l)}_2(x) \) we get

\[
Q^{(l)}_2(x) = \sum_{k=0}^{(q+2)(m-1)} \delta_k P^{(2,2)}_k(x).
\]

Since \( P^{(2,2)}_k(x) \) is an algebraic polynomial of degree not greater than \( k \), the last equality implies that \( Q^{(l)}_2(x) \) is an algebraic polynomial of degree not greater than \( (q+2)(m-1) \).

Lemma is proved.

**Lemma 3.7.** Operator \( T_y \) has the following properties

1. The operator \( T_y(f,x) \) is linear on \( f \);
2. \( T_1(f,x) = f(x) \);
3. \( T_y \left( P^{(2,2)}_n(x) \right) = P^{(2,2)}_n(x) R_n(y) \) \((n = 0, 1, \ldots)\),
   where \( R_n(y) = P^{(0,0)}_{n+2}(y) + \frac{3}{2} (1 - y^2) P^{(2,2)}_n(y) \);
4. \( T_y(1,x) = 1 \);
5. \( a_k \left( T_y(f,x) \right) = R_k(y) a_k(f) \) \((k = 0, 1, \ldots)\).

Lemma 3.7 is proved in [4].

**Corollary 3.2.** If \( P_n(x) \) is an algebraic polynomial of degree not greater than \( n-1 \), then for every natural number \( r \), for fixed \( y_1, y_2, \ldots, y_r \), functions \( T^{y_1,\ldots,y_r}_n(P_n,x) \) and \( \Delta^{y_1,\ldots,y_r}_n(P_n,x) \) are algebraic polynomials on \( x \) of degree not greater than \( n-1 \).

**Lemma 3.8.** If \(-1 \leq x \leq 1, -1 \leq z \leq 1, 0 \leq t \leq \pi \) and \( R = xy + z\sqrt{1-x^2} \times \sqrt{1-y^2} \), then \(-1 \leq R \leq 1 \) and

\[
\left( x\sqrt{1-y^2} + yz\sqrt{1-x^2} \right)^2 \leq (1 - R^2),
\]

\[
\left( \sqrt{1-x^2y} + xz\sqrt{1-y^2} \right)^2 \leq (1 - R^2),
\]

\[
(1 - x^2)(1 - z^2) \leq (1 - R^2),
\]

\[
(1 - y^2)(1 - z^2) \leq (1 - R^2).
\]

Lemma 3.8 is proved in [4] and [3].

**Lemma 3.9.** Let given numbers \( p \) and \( \alpha \) be such that \( 1 \leq p \leq \infty \);

\[
\frac{1}{2} < \alpha \leq 1 \quad \text{for } p = 1,
\]

\[
1 - \frac{1}{2p} < \alpha < 3 - \frac{1}{2p} \quad \text{for } 1 < p < \infty,
\]

\[
1 \leq \alpha \leq \frac{3}{2} \quad \text{for } p = \infty.
\]

Let \( f \in L_{p,\alpha} \). The following inequality holds true

\[
\|T_y(f,x)\|_{p,\alpha} \leq C \|f\|_{p,\alpha},
\]

where the constant \( C \) does not depend on \( f \) and \( y \).

Lemma 3.9 is also proved in [4].
Corollary 3.3. Let given numbers \( p \) and \( \alpha \) be such that \( 1 \leq p \leq \infty \);
\[
\begin{align*}
\frac{1}{2} &< \alpha \leq 1 \quad \text{for } p = 1, \\
1 - \frac{1}{2p} &< \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 < p < \infty, \\
1 &\leq \alpha < \frac{3}{2} \quad \text{for } p = \infty.
\end{align*}
\]
Let \( f \in L_{p,\alpha} \). The following inequality holds true
\[
\|T_{y_1,\ldots,y_r}^r (f,x)\|_{p,\alpha} \leq C \|f\|_{p,\alpha},
\]
where the constant \( C \) does not depend on \( f \) and \( y_j (j = 1, 2, \ldots, r) \).

The corollary is proved by applying \( r \) times Lemma 3.9 taking into consideration Corollary 3.1 (see [4]).

4. Theorem 4.1. Let \( q, m \) and \( r \) given natural numbers and let \( f \in L_{1,2} \). The function
\[
Q(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi (\Delta_{t_1,\ldots,t_r}^r (f,x) - (-1)^r f(x))
\times \prod_{s=1}^r \left( \frac{\sin \frac{m}{2} t_s}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \cdots dt_r,
\]
is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

Proof. To prove the theorem it is sufficient to show that for every \( l = 1, 2, \ldots, r \) the function
\[
Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi T_{\cos t_1,\ldots,\cos t_l}^l (f,x) \prod_{s=1}^r \left( \frac{\sin \frac{m}{2} t_s}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \cdots dt_r
\]
is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

It is obvious that the function \( Q^{(l)}(x) \) can be written in the form
\[
Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \left( Q_1^{(l)}(x) + \frac{3}{2} Q_2^{(l)}(x) \right),
\]
where \( Q_1^{(l)}(x) \) and \( Q_2^{(l)}(x) \) are the functions from Lemmas 3.5 and 3.6 respectively. But, then Lemmas 3.5 and 3.6 yield that \( Q^{(l)}(x) \) is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

Theorem is proved. \( \square \)

Theorem 4.2. Let given numbers \( p, \alpha, r \) and \( \lambda \) be such that \( 1 \leq p \leq \infty \), \( \lambda > 0 \), \( r \in \mathbb{N} \);
\[
\begin{align*}
\alpha &\leq 2 \quad \text{for } p = 1, \\
\alpha &< 3 - \frac{1}{p} \quad \text{for } 1 < p \leq \infty.
\end{align*}
\]
Let \( f \in L_{p,\alpha} \) and
\[
\hat{\omega}_r(f,\delta)_{p,\alpha} \leq M \delta^{\lambda}.
\]
Then
\[ E_n(f)_{p,\alpha} \leq CMn^{-\lambda}, \]
where the constant \( C \) does not depend on \( f, M \) and \( n \).

Proof. It can easily be proved that under the conditions of the theorem, if \( f \in L_{p,\alpha} \), then \( f \in L_{1,2} \).

We choose a natural number \( q \) such that \( 2q > \lambda \), and for each natural number \( n \) we choose the natural number \( m \) satisfying the condition
\[
(4.1) \quad \frac{n - 1}{q + 2} < m \leq \frac{n - 1}{q + 2} + 1.
\]

For those \( q \) and \( m \) polynomial \( Q(x) \) defined in Theorem 4.1 is an algebraic polynomial of degree not greater than \( n - 1 \). Hence
\[
E_n(f)_{p,\alpha} \leq \| f(x) - (-1)^{r+1}Q(x) \|_{p,\alpha}
\]
\[
= \left\| \frac{1}{(\gamma_m)^{\gamma}} \int_0^\pi \cdots \int_0^\pi \Delta_{t_1,\ldots,t_r}^r (f, x) \times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \ldots dt_r \right\|_{p,\alpha}.
\]

Applying the generalised inequality of Minkowski we obtain
\[
E_n(f)_{p,\alpha} \leq \frac{1}{(\gamma_m)^{\gamma}} \int_0^\pi \cdots \int_0^\pi \| \Delta_{t_1,\ldots,t_r}^r (f, x) \|_{p,\alpha}
\]
\[
\times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \ldots dt_r
\]
\[
\leq \frac{1}{(\gamma_m)^{\gamma}} \int_0^\pi \cdots \int_0^\pi \mathcal{\hat{\omega}}_r \left( f, \sum_{j=1}^r t_j \right) \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \ldots dt_r.
\]

Hence, considering the conditions of the theorem we have (see [1, p. 31])
\[
E_n(f)_{p,\alpha} \leq \frac{M}{(\gamma_m)^{\gamma}} \int_0^\pi \cdots \int_0^\pi \left( \sum_{j=1}^r t_j \right)^\lambda
\]
\[
\times \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \ldots dt_r
\]
\[
\leq C_1M \sum_{j=1}^r \frac{1}{(\gamma_m)^{\gamma}} \int_0^\pi \cdots \int_0^\pi t_j^\lambda \prod_{s=1}^r \left( \frac{\sin \frac{mt_s}{2}}{\sin \frac{t_s}{2}} \right)^{2q+4} \sin^3 t_s dt_1 \ldots dt_r.
\]

Applying the standard evaluation of the Jackson’s kernel, considering inequality (4.1), we obtain
\[
E_n(f)_{p,\alpha} \leq C_2Mm^{-\lambda} \leq C_3Mn^{-\lambda}.
\]

Theorem 4.2 is proved. \( \square \)
Theorem 4.3. Let given numbers $p$, $\alpha$, $r$ and $\lambda$ be such that $1 \leq p \leq \infty$, $r \in \mathbb{N}$, $0 < \lambda < 2r$;

\[
\frac{1}{2} < \alpha \leq 1 \quad \text{for } p = 1,
\]

\[
1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 < p < \infty,
\]

\[
1 \leq \alpha < \frac{3}{2} \quad \text{for } p = \infty.
\]

If $f \in L_{p,\alpha}$ and

\[
E_n(f)_{p,\alpha} \leq \frac{M}{n^\alpha},
\]

then

\[
\hat{\omega}_r(f,\delta)_{p,\alpha} \leq CM\delta^\lambda,
\]

where the constant $C$ does not depend on $f$, $M$ and $\delta$.

Proof. Let $P_n(x)$ be the polynomial of degree not greater than $n - 1$ such that

\[
\|f - P_n\|_{p,\alpha} = E_n(f)_{p,\alpha} \quad (n = 1, 2, \ldots).
\]

We construct the polynomials $Q_k(x)$ by

\[
Q_k(x) = P_{2^k}(x) - P_{2^k-1}(x) \quad (k = 1, 2, \ldots)
\]

and $Q_0(x) = P_1(x)$. Since for $k \geq 1$ we have

\[
\|Q_k\|_{p,\alpha} = \|P_{2^k} - P_{2^k-1}\|_{p,\alpha} \leq \|P_{2^k} - f\|_{p,\alpha} + \|f - P_{2^k-1}\|_{p,\alpha}
\]

\[
= E_{2^k}(f)_{p,\alpha} + E_{2^{k-1}}(f)_{p,\alpha},
\]

then under the conditions of the theorem it follows that

\[
\|Q_k\|_{p,\alpha} \leq C_1M2^{-k\lambda}.
\]

It is obvious that without lost in generality we may assume that $t_s \neq 0$ $(s = 1, 2, \ldots, r)$. For $0 < |t_s| < \delta$ $(s = 1, 2, \ldots, r)$ we estimate

\[
I = \|\Delta^r_{t_1,\ldots,t_r} (f, x)\|_{p,\alpha}.
\]

For every natural number $N$, considering that linearity of the operator $\hat{T}_{t_1} (f, x)$ implies the linearity of the operator $\hat{T}^r_{t_1,\ldots,t_r} (f, x)$, i.e. the linearity of the difference $\Delta^r_{t_1,\ldots,t_r} (f, x)$, we have

\[
I \leq \|\Delta^r_{t_1,\ldots,t_r} (f - P_{2^N}, x)\|_{p,\alpha} + \|\Delta^r_{t_1,\ldots,t_r} (P_{2^N}, x)\|_{p,\alpha}.
\]

Since $P_{2^N}(x) = \sum_{k=0}^N Q_k(x)$, we get

\[
I \leq \|\Delta^r_{t_1,\ldots,t_r} (f - P_{2^N}, x)\|_{p,\alpha} + \sum_{k=1}^N \|\Delta^r_{t_1,\ldots,t_r} (Q_k, x)\|_{p,\alpha}.
\]

Applying Corollary 3.3 we have

\[
I \leq C_2E_{2^N}(f)_{p,\alpha} + \sum_{k=1}^N I_k.
\]

Let $N$ be chosen so that

\[
\frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}.
\]

We prove that the following inequality holds true

\[
I_k \leq C_3M\delta^\lambda 2^{k(2r-\lambda)}.
\]
Let 
\[ \psi_k(x) = \Delta^r_{1,\ldots,t_r}(Q_k, x). \]

It can be proved that
\begin{equation}
\psi_k(x) = \frac{1}{2\pi(1-x^2)} \int_{t_0}^{t_r} \int_{-u}^{u} \int_{0}^{\pi} \left( A(v)(R_v')^2 \frac{d^2}{dR_v^2} \Delta_{1,\ldots,t_r-1}^{r-1}(Q_k, R_v) 
- (A(v)R_v - 2A'(v)R_v') \frac{d}{dR_v} \Delta_{1,\ldots,t_r-1}^{r-1}(Q_k, R_v) 
+ A''(v)\Delta_{1,\ldots,t_r-1}^{r-1}(Q_k, R_v) \right) \, d\varphi \, dv \, du,
\end{equation}

where \( R_v = x \cos \varphi - \sqrt{1-x^2} \cos \varphi \sin \varphi, \)
\[ A(v) = 1 - R_v^2 - 2 \sin^2 v \sin^2 \varphi + 4 \left(1 - x^2 \right)^2 \sin^2 \varphi. \]

Applying estimates from Lemma 3.8 and performing change of variables \( z = \cos \varphi \) we obtain
\[ |\psi_k(x)| \leq C_4 \frac{1}{1-x^2} \int_{t_0}^{t_r} \int_{-u}^{u} \int_{0}^{1} B(R_v) \frac{dz}{\sqrt{1-z^2}} \, dv \, du, \]

where
\[ B(R_v) = (1 - R_v^2)^2 \left| \frac{d^2}{dR_v^2} \Delta_{1,\ldots,t_r-1}^{r-1}(Q_k, R_v) \right| 
+ (1 - R_v^2) \left| \frac{d}{dR_v} \Delta_{1,\ldots,t_r-1}^{r-1}(Q_k, R_v) \right| 
+ \left| \Delta_{1,\ldots,t_r-1}^{r-1}(Q_k, R_v) \right| = B_1(R_v) + B_2(R_v) + B_3(R_v). \]

Therefore using the generalised Minkowski’s inequality we get
\begin{equation}
I_k = \|\psi_k(x)\|_{p, \alpha} \leq C_4 \int_{t_0}^{t_r} \int_{-u}^{u} \int_{0}^{1} \left\| B(R_v) \right\|_{p, \alpha} \frac{dz}{\sqrt{1-z^2}} \, dv \, du.
\end{equation}

Let \( p = 1. \) Considering that \( \alpha \leq 1 \) we obtain
\[ I_k \leq C_4 \int_{t_0}^{t_r} \int_{-u}^{u} \int_{0}^{1} \left\{ \int_{0}^{1} \left| B(R_v) \right|^p (1 - x^2)^{p(\alpha-1)} (1 - z^2)^{\alpha-1} \frac{dx \, dz}{\sqrt{1-z^2}} \right\}^{\frac{1}{p}} \, dv \, du. \]

Let \( 1 < p < \infty. \) Applying the Hölder’s inequality in the inside integral in equation (4.5), considering that \( \alpha < \frac{3}{2} - \frac{1}{4p} \) we obtain
\begin{align*}
I_k &\leq C_4 \int_{t_0}^{t_r} \int_{-u}^{u} \int_{0}^{1} \left\{ \int_{0}^{1} \frac{\left| B(R_v) \right|^p (1 - x^2)^{p(\alpha-1)} (1 - z^2)^{\alpha-1} \, dx \, dz}{\sqrt{1-z^2}} \right\}^{\frac{1}{p}} \, dv \, du. \\
&\leq C_5 \int_{t_0}^{t_r} \int_{-u}^{u} \left\{ \int_{0}^{1} \frac{\left| B(R_v) \right|^p (1 - x^2)^{p(\alpha-1)} \, dx \, dz}{\sqrt{1-z^2}} \right\}^{\frac{1}{p}} \, dv \, du.
\end{align*}

Thus, under the conditions of the theorem, for \( 1 \leq p < \infty \) we have
\begin{align*}
I_k &\leq C_6 \int_{t_0}^{t_r} \int_{-u}^{u} \left\{ \int_{0}^{1} \int_{0}^{1} \frac{\left| B(R_v) \right|^p (1 - x^2)^{p(\alpha-1)} \, dx \, dz}{\sqrt{1-z^2}} \right\}^{\frac{1}{p}} \, dv \, du.
\end{align*}
Performing the change of variables in double integral by the formulas
\[
R = x \cos v - z \sqrt{1 - x^2} \sin v,
\]
\[
V = \frac{x \sin v + z \sqrt{1 - x^2} \cos v}{\sqrt{1 - (x \cos v - z \sqrt{1 - x^2} \sin v)^2}},
\]
we obtain
\[
I_k \leq C_6 \int_0^{t_r} \int_{-u}^{u} \left\{ \int_{-1}^{1} \int_{-1}^{1} |B(R)|^p (1 - R^2)^{p(\alpha - 1)} \right. \\
\left. \times (1 - V^2)^{p(\alpha - 1) - \frac{d}{2}} dR dV \right\}^{\frac{1}{p}} du.
\]
Since, under the conditions of theorem \( \alpha > 1 - \frac{3}{2p} \), it follows that
\[
I_k \leq C_7 \int_0^{t_r} \int_{-u}^{u} \left\{ \int_{-1}^{1} \int_{-1}^{1} |B(R)|^p (1 - R^2)^{p(\alpha - 1)} dR \right\}^{\frac{1}{p}} du \\
\leq C_8 t_r^2 \|B(R)\|_{p, \alpha - 1}.
\]
Let now \( p = \infty \). Considering the estimates from Lemma 3.8 and that \( \alpha \geq 1 \), inequality (4.3) yields
\[
I_k \leq C_4 \int_0^{t_r} \int_{-u}^{u} \int_{-1}^{1} \text{ess sup}_{1-1 \leq x \leq 1} |B(R_u)| (1 - x^2)^{\alpha - 1} \frac{dz}{\sqrt{1 - z^2}} du \\
\leq C_4 \|B(x)\|_{\infty, \alpha - 1} \int_0^{t_r} \int_{-u}^{u} \int_{-1}^{1} (1 - z^2)^{-\alpha + \frac{3}{2}} dz du.
\]
Hence, considering that \( \alpha < \frac{3}{2} \) we get
\[
I_k \leq C_9 t_r^2 \|B(x)\|_{\infty, \alpha - 1}.
\]
Thus for all \( 1 \leq p \leq \infty \) we proved that
\[
I_k \leq C_{10} t_r^2 \|B(x)\|_{p, \alpha - 1}.
\]
Applying Lemma 3.1 and Corollaries 3.2 and 3.3 under the conditions of the theorem we obtain
\[
I_k = \|\Delta_{t_1, \ldots, t_r} (Q_k, x)\|_{p, \alpha} \leq C_{10} t_r^2 \|B(x)\|_{p, \alpha - 1} \\
\leq C_{10} t_r^2 \left( \|B_1(x)\|_{p, \alpha - 1} + \|B_2(x)\|_{p, \alpha - 1} + \|B_3(x)\|_{p, \alpha - 1} \right) \\
\leq C_{10} t_r^2 \left( \left\| \frac{d^2}{dx^2} \Delta_{t_1, \ldots, t_r} (Q_k, x) \right\|_{p, \alpha + 1} \\
+ \left\| \frac{d}{dx} \Delta_{t_1, \ldots, t_r} (Q_k, x) \right\|_{p, \alpha} + \left\| \Delta_{t_1, \ldots, t_r} (Q_k, x) \right\|_{p, \alpha - 1} \right) \\
\leq C_{11} t_r^2 \|\Delta_{t_1, \ldots, t_r} (Q_k, x)\|_{p, \alpha}.
\]
Applying \( r \) times this inequality it follows that
\[
I_k \leq C_{12} t_1^2 \cdots t_r^2 \|Q_k\|_{p, \alpha}.
\]
Therefore we have
\[
I_k \leq C_{13} M 2^r \|Q_k\|_{p, \alpha}.
\]
Inequality (4.3) is proved.
Inequalities (4.3) and (4.2) yield
\[ I \leq C_{14}M \left( \delta^\lambda + \delta^{2r} \sum_{k=1}^{N} \delta^{k(2r-\lambda)} \right) \leq C_{15}M \left( \delta^\lambda + \delta^{2r}2^{N(2r-\lambda)} \right) \leq C_{16}M \delta^\lambda. \]

Theorem 4.3 is completed. □

Now we formulate the theorem of coincidence of the class \( H(p, \alpha, r, \lambda) \) with the class \( E(p, \alpha, \lambda) \), and the inverse theorem.

**Theorem 4.4.** Let given numbers \( p, \alpha, r \) and \( \lambda \) be such that \( 1 \leq p \leq \infty \), \( 0 < \lambda < 2r \), \( r \in \mathbb{N} \);
\[
\frac{1}{2} < \alpha \leq 1 \quad \text{for } p = 1,
\]
\[
1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 < p < \infty,
\]
\[
1 \leq \alpha < \frac{3}{2} \quad \text{for } p = \infty.
\]
The class \( H(p, \alpha, r, \lambda) \) coincides with the class \( E(p, \alpha, \lambda) \).

Theorem 4.4 is implied by Theorems 4.2 and 4.3 proved above.

**Theorem 4.5.** Let given numbers \( p, \alpha, r \) and \( \lambda \) be such that \( 1 \leq p \leq \infty \), \( 0 < \lambda < 2r \), \( r \in \mathbb{N} \);
\[
\frac{1}{2} < \alpha \leq 1 \quad \text{for } p = 1,
\]
\[
1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 < p < \infty,
\]
\[
1 \leq \alpha < \frac{3}{2} \quad \text{for } p = \infty.
\]
If \( f \in L_{p, \alpha} \), then the following inequality holds
\[
\hat{\omega}_r \left( f, \frac{1}{n}, \frac{1}{n^r} \right)_{p, \alpha} \leq \frac{C_1}{n^{2r}} \sum_{\nu=1}^{n} \nu^{2r-1} E_{\nu} \{ f \}_{p, \alpha},
\]
where the constant \( C \) does not depend on \( f \) and \( n \).

**Proof.** Let \( P_n(x) \) be the polynomial of degree not greater than \( n - 1 \) such that
\[
\| f - P_n \|_{p, \alpha} = E_n(f)_{p, \alpha} \quad (n = 1, 2, \ldots),
\]
and
\[
Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \ldots),
\]
\( Q_0(x) = P_1(x). \)
For given \( n \) we chose the natural number \( N \) such that
\[
\frac{n}{2} < 2^N \leq n + 1.
\]
By the proof of Theorem 4.3 it follows that

\[ \hat{\omega}_r \left( f, \frac{1}{n} \right)_{p,\alpha} \leq C_2 \left( E_{2N} (f)_{p,\alpha} + \frac{1}{n^{2r}} \sum_{\mu=1}^{N} 2^{2\mu r} \|Q_k\|_{p,\alpha} \right) \]

\[ \leq 2C_2 \left( E_{2N} (f)_{p,\alpha} + \frac{1}{n^{2r}} \sum_{\mu=1}^{N} 2^{2\mu r} \left( E_{2^\mu} (f)_{p,\alpha} + E_{2^{\mu-1}} (f)_{p,\alpha} \right) \right) \]

\[ \leq 4C_2 \left( E_{2N} (f)_{p,\alpha} + \frac{1}{n^{2r}} \sum_{\mu=0}^{N-1} 2^{2(\mu+1)r} E_{2^\mu} (f)_{p,\alpha} \right) \]

\[ \leq \frac{C_3}{n^{2r}} \sum_{\mu=0}^{N} 2^{2(\mu+1)r} E_{2^\mu} (f)_{p,\alpha} . \]

Considering that for \( \mu \geq 1 \) we have

\[ \sum_{\nu=2^{r-1}}^{2^\mu-1} \nu^{2r-1} E_{\nu} (f)_{p,\alpha} \geq E_{2^\mu} (f)_{p,\alpha} 2^\mu - 1 \geq C_4 2^{2(\mu+1)r} E_{2^\mu} (f)_{p,\alpha} , \]

it follows that

\[ \hat{\omega}_r \left( f, \frac{1}{n} \right)_{p,\alpha} \leq \frac{C_5}{n^{2r}} \left( 2^{2r} E_1 (f)_{p,\alpha} + \sum_{\mu=1}^{N} \sum_{\nu=2^{r-1}}^{2^\mu-1} \nu^{2r-1} E_{\nu} (f)_{p,\alpha} \right) \]

\[ \leq \frac{C_6}{n^{2r}} \sum_{\nu=1}^{N} \nu^{2r-1} E_{\nu} (f)_{p,\alpha} . \]

Theorem 4.5 is proved. □

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, Three volumes, Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981, (Russian translation, Gosudarstv. Izdat. Inostrannoi Literatury, Moscow, 1969). MR 84h:33001

2. B. A. Halilova, *O nekotorykh otsenkakh dlya polinomov*, Izv. Akad. Nauk Azerbaizhana SSR Ser. Fiz.-Tekhn. Mat. Nauk (1974), no. 2, 46–55. MR 50 #4863

3. M. K. Potapov, *Ob uslovnykh sovpadenii nekotorykh klassov funktsii*, Trudy Sem. Petrovsk. (1981), no. 6, 223–238. MR 82f:46053

4. , *O sovpadenii klassov funktsii opredelyaemykh operatorom obobshchennogo sdviga ili poryadkom nailuchshego priblizheniya algebraicheskimi mnogochlenami*, Mat. Zametki 66 (1999), no. 2, 242–257. MR 2000k:41008

F. M. Berisha, Faculty of Mathematics and Sciences, University of Prishtina, Nëna Terezë 5, 10000 Prishtinë, Kosovë

E-mail address: faton.berisha@uni-pr.edu