A new closed formula for the Hermite interpolating polynomial with applications on the spectral decomposition of a matrix

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Abstract

We present a new closed form for the interpolating polynomial of the general univariate Hermite interpolation that requires only calculation of polynomial derivatives, instead of derivatives of rational functions. This result is used to obtain a new simultaneous polynomial division by a common divisor over a perfect field. The above findings are utilized to obtain a closed formula for the semi–simple part of the Jordan decomposition of a matrix. Finally, a number of new identities involving polynomial derivatives are obtained, based on the proposed simultaneous polynomial division. The proposed explicit formula for the semi–simple part has been implemented using the Matlab programming environment.

1 Introduction

The Hermite interpolation of total degree is described in the following Theorem [1]:

Theorem 1.1. Given n distinct elements $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ in a perfect field $k$, positive integers $m_i$, $i = 0, \ldots, n - 1$, and $a_{ij} \in k$ for $0 \leq i \leq n - 1$, $0 \leq j \leq m_i - 1$, then there exists one and only one polynomial $r \in k[x]$ of degree less than $\sum_{i=0}^{n-1} m_i$, such that

$$r^{(j)}(\lambda_i) = a_{ij}, \ 0 \leq j \leq m_i - 1, \ 0 \leq i \leq n - 1. \quad (1.1)$$

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This polynomial \( r \) is explicitly given by,

\[
r(x) = \sum_{i=0}^{n-1} \sum_{j=0}^{m_i} \sum_{k=0}^{j-1} a_{ij} \frac{1}{j! k!} \left[ \frac{(x - \lambda_i)^{m_i}}{\Omega(x)} \right]^{(k)}_{x=x_i} \times \frac{\Omega(x)}{(x - \lambda_i)^{m_i-j-k}}.
\]

where

\[
\Omega(x) = \prod_{i=0}^{n-1} (x - \lambda_i)^{m_i},
\]

and \( r^{(k)}(a) \) is the \( k \)-th derivative of \( r \) at \( a \).

The applications of hermite interpolation to numerical analysis is well known. In this paper we show the use of a computationally more efficient formula for the Hermite interpolation to a number of algebraic applications.

A number of forms of the interpolating polynomial \( r(x) \) have been reported in the literature, however they all require calculation of derivatives of rational polynomial functions (e.g. [1]), or recursive calculation of the coefficients \( a_{ij} \) of Theorem 1.1 (e.g. [8]). We firstly propose a new closed form of the interpolating polynomial \( r \) of the univariate Hermite interpolation that requires only calculation of polynomial derivatives instead of derivatives of rational functions, as required in Theorem 1.1, thus achieving a substantial computational acceleration compared to the Theorem 1.1, without imposing any restrictions in the generality of the interpolation. We then exploit this result to show a Theorem for the calculation of the remainders of the simultaneous division of a number of polynomials by a separable polynomial divisor, as derivatives of a unique polynomial. Namely, let us note that the remainder \( r \in \mathbb{k}[x] \) of the Euclidean division of any polynomial \( P \in \mathbb{k}[x] \) of degree \( n \) by a separable polynomial \( Q \in \mathbb{k}[x] \) of degree \( m \), where \( n \geq m \), can be calculated in closed form using the Langrange interpolation formula as following:

\[
r(x) = \sum_{i=1}^{m} P(\lambda_i) \prod_{j=1}^{m} \frac{(x - \lambda_j)}{(\lambda_i - \lambda_j)},
\]

where \( \lambda_1, \ldots, \lambda_m \) are the roots of \( Q \) in the algebraic closure \( \mathbb{K} \) of \( \mathbb{k} \). In this work we extent this simplified idea of polynomial remainder calculation by Langrange interpolation, to achieve simultaneous polynomial division by a common separable divisor, using the closed form of the interpolating polynomial of the univariate Hermite interpolation \( r \) that we propose.

Furthermore, the above results will be used to obtain a new closed formula for the semi–simple part of the Jordan decomposition of an algebraic element in an arbitrary algebra. If \( \mathbb{k} \) is a perfect field and \( \mathbf{A} \) an algebra over \( \mathbb{k} \) with unit 1, then the well–known result on the Jordan decomposition, (e.g. [2], [4], [7]) is presented in the following theorem.
Theorem 1.2. For each algebraic $A \in A$ there exist unique $S_A, N_A \in k[A]$ such that $A = S_A + N_A$, $S_A$ is semi–simple and $N_A$ is nilpotent.

It should be noted that in the case of $k$ being algebraically closed and $A$ is a subalgebra of the matrix-algebra $k^{n \times n}$, then $S_A$ is semi–simple if and only if $S_A$ is diagonalizable.

Theorem 1.2 has various interesting applications, such as efficient computation of high powers of $A$, therefore, efficient algorithms for the calculation of $S_A$ in terms of $A$ are of great importance. A proof of the existence of $S_A$ that is presented in the book of Hoffman and Kunze [4], is based on the Newton’s method and yields direct methods for computations. An algorithm, which is essentially based on these ideas, is given by Levelt [6]. The algorithm of Bourgoyne and Cushman [3] is faster, because higher derivatives are used. In [9] D. Schmidt has used the Newton’s method to construct the semi–simple part of the Jordan decomposition of an algebraic element in an arbitrary algebra, showing quadratic convergence of the algorithm. Another approach uses the partial fractions decomposition of the reciprocal of the minimal polynomial [5,10,11].

These works however, do not provide a closed form formula for $S_A$ as a polynomial of $A$. An explicit construction of the spectral decomposition of a matrix using Hermite interpolation appears in [11]. However, this approach requires the use of Taylor coefficients of the reciprocal of the matrix minimal polynomial.

In this work, we exploit our proposed simultaneous polynomial division by a common separable divisor, to obtain a new closed formula for the semi–simple part $S_A$ of the Jordan decomposition of an algebraic element $A$ in an arbitrary algebra. The proposed closed formula requires only evaluation of the derivatives of the basic Hermite–like interpolation polynomials that are associated by the eigenvalues of $A$, up to the maximum algebraic multiplicity of the roots of the minimal polynomial of $A$, as well as matrix multiplication operations. The derived closed formula for the semi–simple part of the Jordan decomposition has been implemented in the Matlab programming environment. The source code is provided and results are shown from it’s executions on examples of matrices taken from the literature.

Furthermore, a number of new identities involving polynomial derivatives are also shown, based on the the simultaneous polynomial division by a common separable divisor.

The rest of the paper is organized as follows. In section 2 we present a new closed form for Hermite interpolation. In section 3 we show a Theorem for the calculation of the remainders of the simultaneous division of a number of polynomials by a separable polynomial divisor, as derivatives of a unique polynomial. In section 4 we present an explicit Formula for $S_A$, the relative source code in MATLAB environment, as well as some numerical examples from the literature. Finally, in section 5 the main result of section 3 is generalized and applied to the derivation of some interesting polynomial formulas.
2 A new closed form for Hermite interpolation

Let \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) be distinct elements in a perfect field \( k \) and \( m_0, m_1, \ldots, m_{n-1} \) be positive integers. Let us denote by \( L_k \) the polynomials given by

\[
L_k(x) = \prod_{\substack{i=0 \atop i \neq k}}^{n-1} \frac{(x - \lambda_i)^{m_i}}{(\lambda_k - \lambda_i)^{m_i}} \in k[x].
\]  

Furthermore, we denote by \( \Lambda_k \), \( 0 \leq k \leq n - 1 \) the \( m_k \times m_k \) lower triangular matrices \([l_{ij}] \in k^{m_k \times m_k} \), \( 0 \leq i, j \leq m_k - 1 \), given by

\[
l_{ij} := \begin{cases} \binom{i}{j} (L_k)^{(i-j)}(\lambda_k) & \text{if } 0 \leq j \leq i \leq m_k - 1 \\ 0 & \text{if } 0 \leq i < j \leq m_k - 1 \end{cases},
\]

where \((L_k)^{(i-j)}(\lambda_k)\) is the derivative of order \((i-j)\) of the polynomial \(L_k(x)\) at \( \lambda_k \). Thus \( \Lambda_k \) has the following representation

\[
\begin{pmatrix}
\binom{0}{0} L_k(\lambda_k) & 0 & \ldots & 0 \\
\binom{1}{0} (L_k)^{(1)}(\lambda_k) & \binom{1}{1} L_k(\lambda_k) & \ldots & 0 \\
& \ddots & \ddots & \ddots \\
\binom{m_k-1}{0} (L_k)^{(m_k-1)}(\lambda_k) & \binom{m_k-1}{1} (L_k)^{(m_k-2)}(\lambda_k) & \ldots & \binom{m_k-1}{m_k-1} L_k(\lambda_k)
\end{pmatrix}.
\]  

(2.2)

For our purpose the following technical Lemmas are required:

**Lemma 2.1.** The matrices \( \Lambda_k \), \( 0 \leq k \leq n - 1 \) are invertible with \( \Lambda_k^{-1} = \sum_{i=0}^{m_k-1} (I_{m_k} - \Lambda_k)^i \), where \( I_{m_k} \) is the \( m_k \times m_k \) unit matrix.

**Proof.** Clearly, for \( 0 \leq k \leq n - 1 \) holds \( L_k(\lambda_k) = 1 \). Therefore, all matrices \( \Lambda_k \), \( 0 \leq k \leq n - 1 \) are invertible and unit diagonal lower triangular. Therefore, it follows \((I_{m_k} - \Lambda_k)^{m_k} = 0\) and consequently \( \Lambda_k \sum_{i=0}^{m_k-1} (I_{m_k} - \Lambda_k)^i = I_{m_k} \).  

Using the Leibnitz’s rule for derivatives, we easily get the following Lemma:

**Lemma 2.2.** For \( 0 \leq i, s \leq m_k - 1 \) and \( 0 \leq j, t \leq n-1 \), the following holds:

\[
\frac{(x - \lambda_i)^s}{s!} L_t(x)^{(i)} \bigg|_{x=\lambda_j} = \begin{cases} 0 & \text{if } t \neq j \\
0 & \text{if } t = j \text{ and } i < s \\
\binom{i}{s} (L_j)^{(i-s)}(\lambda_j) & \text{if } t = j \text{ and } s \leq i
\end{cases}.
\]  

(2.3)

Our proposed form of Hermite interpolation can now be presented in the following Theorem.

**Theorem 2.3.** Given \( n \) distinct elements \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) in a perfect field \( k \), positive integers \( m_i \), \( i = 0, \ldots, n - 1 \), and \( a_{ij} \in k \) for \( 0 \leq j \leq n - 1 \), \( 0 \leq i \leq n)
m_j - 1, then there exists one and only one polynomial \( r \in k[x] \) of degree less than \( \sum_{i=0}^{n-1} m_i \), such that

\[
r^{(i)}(\lambda_j) = a_{ij}, \quad 0 \leq j \leq n - 1, \quad 0 \leq i \leq m_j - 1.
\]  
(2.4)

This polynomial \( r \) is explicitly given by,

\[
r = \sum_{j=0}^{n-1} X_j \Lambda_j^{-1} A_j
\]

(2.5)

where the matrices \( X_j, A_j \) are given by

\[
X_j = \left[ L_j(x) \ \frac{(x-\lambda_j)}{1!} L_j(x) \ \ldots \ \frac{(x-\lambda_j)^{m_j-1}}{(m_j-1)!} L_j(x) \right],
\]

and

\[
A_j = \begin{bmatrix} a_{0j} & a_{ij} & \ldots & a_{m_j-1j} \end{bmatrix}^T.
\]

Proof. It can be observed that (2.5) can be equivalently written in the following form:

\[
r(x) = \sum_{j=0}^{n-1} \sum_{i=0}^{m_j-1} c_{ij} \frac{(x-\lambda_j)^i}{i!} L_j(x) \in k[x],
\]

where

\[
\begin{bmatrix} c_{0j} \\ c_{1j} \\ \vdots \\ c_{m_j-1j} \end{bmatrix} = \Lambda_j^{-1} \begin{bmatrix} a_{0j} \\ a_{1j} \\ \vdots \\ a_{m_j-1j} \end{bmatrix}, \quad 0 \leq j \leq n - 1.
\]  
(2.6)

Now, by calculating the derivative of order \( i \) of the polynomial \( r \) at \( \lambda_j \) and using (2.3) in Lemma 2.2 we obtain

\[
r^{(i)}(\lambda_j) = \sum_{k=0}^{i} \binom{i}{k} (L_j)^{(i-k)}(\lambda_j),
\]  
(2.7)

for all \( 0 \leq j \leq n - 1, \quad 0 \leq i \leq m_j - 1 \). Taking into consideration the definition of \( \Lambda_j \) in (2.2), the system of equations (2.7) can be rewritten in the following matrix form

\[
\begin{bmatrix} r(\lambda_j) \\ r'(\lambda_j) \\ \vdots \\ r^{(m_j-1)}(\lambda_j) \end{bmatrix} = \Lambda_j \begin{bmatrix} c_{0j} \\ c_{1j} \\ \vdots \\ c_{m_j-1j} \end{bmatrix}, \quad 0 \leq j \leq n - 1
\]  
(2.8)
By substituting (2.6) in (2.8), we get

\[
\begin{bmatrix}
  r(\lambda_j) \\
  r'(\lambda_j) \\
  \quad \vdots \\
  r^{(m_j-1)}(\lambda_j)
\end{bmatrix}
= \begin{bmatrix}
  a_{0j} \\
  a_{1j} \\
  \quad \vdots \\
  a_{m_j-1}
\end{bmatrix}.
\]

Hence, we derive that

\[
r(i)(\lambda_j) = a_{ij}, \quad 0 \leq j \leq n-1, 0 \leq i \leq m_j - 1.\]

The second equality of (2.5) is obtained by Lemma 2.1.

Moreover, it can be easily confirmed that any polynomial \((x - \lambda_j)L_j(x)\), \(0 \leq j \leq n-1, 0 \leq i \leq m_j - 1\) has degree less than \(\sum_{i=0}^{n-1} m_i\), and since \(r\) is a \(k\)-linear combination of these polynomials, we conclude that the degree of \(r\) is less than \(\sum_{i=0}^{n-1} m_i\).

\section{Simultaneous division of polynomials by separable polynomial}

At this point we are ready to present the following generalization of Euclidean polynomial division by a separable divisor, based on Theorem 2.3.

**Theorem 3.1.** Let \(g \in k[x]\) be a separable polynomial and \(\lambda_0, \ldots, \lambda_{n-1}\) be the roots of \(g\) in the algebraic closure \(\overline{k}\) of \(k\). Then for any polynomials \(f_0, f_1, \ldots, f_{m-1} \in k[x]\), there exists unique \(r \in k[x]\) of degree less than \(mn\) and unique polynomials \(q_0, q_1, \ldots, q_{m-1} \in k[x]\) such that

\[
f_i = r(i) + q_i g, \quad 0 \leq i \leq m-1.
\]

This result is optimal, in the sense that if \(m \geq 1\) and \(g\) is inseparable, then this result is not true. The polynomial \(r\) is given by

\[
r(x) = \sum_{j=0}^{n-1} X_j A_j^{-1} A_j \in k[x],
\]

where

\[
X_j = \begin{bmatrix} L_j(x) & \frac{(x-\lambda_j)}{1!} L_j(x) & \ldots & \frac{(x-\lambda_j)^{m-1}}{(m-1)!} L_j(x) \end{bmatrix},
\]

\[
A_j = \begin{bmatrix} f_0(\lambda_j) & f_1(\lambda_j) & \ldots & f_{m-1}(\lambda_j) \end{bmatrix}^T
\]

and \(L_j, A_j\) are respectively as in (2.1), (2.2) by \(m_0 = m_1 = \ldots = m_{n-1} = m\).

**Proof.** Let \(k\) be the dimension of the field \(k(\lambda_0, \ldots, \lambda_{n-1})\) as vector space over \(k\), that is

\[
k = [k(\lambda_0, \ldots, \lambda_{n-1}), k]\]
then there exist \( \tau_1, ..., \tau_{k-1} \) in \( k(\lambda_0, \ldots, \lambda_{n-1}) \) such that \( \{1, \tau_1, ..., \tau_{k-1}\} \) is a basis of \( k(\lambda_0, \ldots, \lambda_{n-1}) \) as a vector space over \( k \).

Now, using Theorem 2.3 by \( m_0 = m_1 = \ldots = m_{n-1} = m \), we have that there is unique polynomial \( \hat{r} \in k(\lambda_0, \ldots, \lambda_{n-1})[x] \) given by (2.3) for \( m_0 = m_1 = \ldots = m_{n-1} = m \) having degree less than \( mn \) such that

\[
(\hat{r})^{(i)}(\lambda_j) = f_i(\lambda_j) \tag{3.2}
\]

for all \( 0 \leq i \leq m-1, 0 \leq j \leq n-1 \). Therefore there exist polynomials \( \hat{q}_i \in k(\lambda_0, \ldots, \lambda_{n-1})[x], 0 \leq i \leq m-1 \) such that

\[
\hat{r}^{(i)} = f_i + g\hat{q}_i. \tag{3.3}
\]

Moreover, by (3.1) we have that the dimension of \( k(\lambda_0, \ldots, \lambda_{n-1})[x] \) as free module over \( k[x] \) is \( k \) and \( \{1, \tau_1, ..., \tau_{k-1}\} \) is a basis of \( k(\lambda_0, \ldots, \lambda_{n-1})[x] \) over \( k[x] \). Therefore the polynomials \( \hat{r}, \hat{q}_i \in k(\lambda_0, \ldots, \lambda_{n-1})[x] \) can be uniquely written in the following form:

\[
\hat{r} = r + \sum_{s=1}^{n-1} \tau_s r_s, \tag{3.4}
\]

and

\[
\hat{q}_i = q_i + \sum_{s=1}^{n-1} \tau_s q_{si}, \ 0 \leq i \leq m-1, \tag{3.5}
\]

where \( r, r_s, q_i, q_{si} \in k[x] \) with \( \deg r, \deg r_i < mn \).

Setting (3.4) and (3.5) in (3.3) and using that \( \{1, \tau_1, ..., \tau_{k-1}\} \) is linearly independent over \( k[x] \), we get

\[
r^{(i)} = f_i + gq_i \tag{3.6}
\]

for all \( 0 \leq i \leq m-1 \).

Now from (3.6) we clearly get that \( r \) satisfies the identities (3.2) and since the polynomial of degree less than \( mn \) satisfying the identities (3.2) is unique, we conclude that \( r = \hat{r} \).

Finally, suppose that Theorem 3.1 is true for one polynomial \( g \in k[x] \) having a root \( \lambda \in k \) of multiplicity > 1. Then there exist a polynomial \( g_0 \) in \( k[x] \) such that

\[
g(x) = (x - \lambda)^2 g_0(x).
\]

Applying Theorem 3.1 for \( g \) and \( f_0 = f_1 = 1 \) we obtain

\[
r(x) = 1 + (x - \lambda)^2 g_0(x)q_0(x), \tag{3.7}
\]

\[
r^{(1)}(x) = 1 + (x - \lambda)^2 g_0(x)q_1(x) \tag{3.8}
\]

for some \( r, g_0, q_1 \in k[x] \).

Differentiating (3.7), and setting \( x = \lambda \) in the resulting identity, as well as, in (3.8) we respectively get

\[
r^{(1)}(\lambda) = 0 \text{ and } r^{(1)}(\lambda) = 1.
\]

That is not true. \( \square \)
Corollary 3.2. Let $g$ be as in Theorem 3.1. Then for any $f \in k[x]$, $c \in k - \{0\}$, $m \in \mathbb{N}$ with $m \geq 2$ there exists unique $r \in k[x]$ of degree less than $mn$ such that $r = f \mod g$ and $r' = cr^{m-1}$.

Proof. According to Theorem 3.1 there exists unique $r \in k[x]$ of degree less than $mn$, such that

$$f = r, cf = r' \mod g, \ldots, c^{m-1}f = r^{(m-1)} \mod g.$$  \tag{3.9}

Now multiplying the $i$–th relation of (3.9) by $c$ and afterwards substracting from the result the $(i+1)$–th relation we get

$$(r' - cr)^{(i)} = 0 \mod g,$$

for all $0 \leq i \leq m - 2$ and since $g$ is separable we direct get the conclusion. \qed

4 An explicit formula for $S_A$

We will use Theorem 3.1 to give a closed formula for the semi–simple part of the Jordan decomposition of an algebraic number of an algebra $A$ over a perfect field $k$.

Lemma 4.1. Let $g \in k[x]$ be a separable polynomial and $\lambda_0, \ldots, \lambda_{n-1}$ are the roots of $g$ in $k$. Let $f \in k[x,y]$ be a polynomial of two variables and $m$ be a positive integer. Then there exists unique $r \in k[x]$ of degree less than $mn$ such that

$$f(x, y) = r(x + y) \mod I,$$

where $I \subset k[x,y]$ is the ideal generated from the polynomials $g(x), y^m$. Further the polynomial $r$ is given by

$$r(x) = \sum_{j=0}^{n-1} X_j A_j^{-1} A_j \in k[x],$$  \tag{4.1}

where the matrices $X_j$, $A_j$ are given by

$$X_j = \left[ L_j(x) \left( \frac{x-\lambda_j}{1!} L_j(x) \right) \ldots \left( \frac{(x-\lambda_j)^{m-1}}{(m-1)!} L_j(x) \right) \right],$$

$$A_j = \left[ f(\lambda_j, 0) \frac{\partial}{\partial y} f(\lambda_j, 0) \ldots \frac{\partial^{m-1}}{\partial y^{m-1}} f(\lambda_j, 0) \right]^T,$$

and $L_j, A_j$ are as in (2.1), (2.2) by $m_0 = m_1 = \ldots = m_{n-1} = m$.

Proof. The polynomial $f$ can be rewritten in the following form

$$f(x, y) = \sum_{i=0}^{m-1} \frac{1}{i!} \frac{\partial^i}{\partial y^i} f(x, 0)y^i + y^m h(x, y),$$  \tag{4.2}

for some $h \in k[x,y]$.  

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Furthermore, according to Theorem 3.1 there exists unique polynomial \( r \in k[x] \) of degree less than \( mn \), given by (4.1), such that:

\[
\frac{\partial^i}{\partial y^i} f(x, 0) = r^{(i)}(x) \mod g(x), \ 0 \leq i \leq m - 1,
\]

(4.3)

Combining (4.2) with (4.3) we get

\[
f(x, y) = \sum_{i=0}^{m-1} \frac{r^{(i)}(x)}{i!} y^i \mod I.
\]

(4.4)

Furthermore, by using the Taylor formula we have:

\[
\sum_{i=0}^{m-1} \frac{r^{(i)}(x)}{i!} y^i = r(x + y) \mod y^m.
\]

(4.5)

Substituting (4.5) in (4.4), completes the required proof.

Let \( p \in k[x] \) be the minimal polynomial of an algebraic element \( A \) of an algebra \( A \) over \( k \) with unit 1. Let \( \lambda_i, i = 0, 1, \ldots, n - 1 \) be the distinct roots of \( p \) in \( k \), and let \( k_i, i = 1, 2, \ldots, n - 1 \) be their respective multiplicities. We denote \( \hat{p}(x) := \prod_{i=0}^{n-1} (x - \lambda_i) \) and \( m(p) := \max\{k_0, \ldots, k_{n-1}\} \).

**Theorem 4.2.** The semi–simple part \( S_A \) of the Jordan decomposition of \( A \) is given by

\[
S_A = r(A),
\]

where

\[
r(x) = \sum_{j=0}^{n-1} X_j A_j^{-1} A_j \in k[x],
\]

(4.6)

where

\[
X_j = \begin{bmatrix} L_j(x) & (x-\lambda_1) L_j(x) & \cdots & (x-\lambda_{m-1}) L_j(x) \end{bmatrix},
\]

\[
A_j = \begin{bmatrix} \lambda_j & 0 & \cdots & 0 \end{bmatrix}^T,
\]

and \( L_j, A_j \) are as in (2.1), (2.2) by \( m_0 = m_1 = \cdots = m_{n-1} = m(p) \).

**Proof.** Since \( S_A \) is the semi–simple part of \( A \) and \( N_A \) is the nilpotent part of \( A \), the minimal polynomials of \( S_A \) and \( N_A \) are respectively \( \hat{p}(x) \) and \( x^{m(p)} \). So we have \( \hat{p}(S_A) = 0 \) and \( N_A^{m(p)} = 0 \). Now if we apply Lemma 3.1 by choosing \( f(x, y) = x \), \( g(x) = \hat{p}(x) \) and \( m = m(p) \), and taking into account that \( f(x, 0) = x \), and \( \frac{\partial^i}{\partial y^i} f(x, 0) = 0 \) for \( 1 \leq i \leq m - 1 \) we have that for the polynomial \( r \) given by (4.6) holds:

\[
x = r(x + y) \mod I,
\]

(4.7)

where \( I \) is the ideal generated from the polynomials \( \hat{p}(x) \) and \( y^{m(p)} \). Finally setting \( x = S_A \) and \( y = N_A \) in (4.7) we get \( S_A = r(S_A + N_A) = r(A) \).
4.1 An implementation of the explicit formula for $S_A$ in the MATLAB programming environment and some arithmetic examples

In this subsection we present a computational implementation in the MATLAB programming environment for the calculation of the semi-simple part $S_A$ of the Jordan decomposition of $A$. The MATLAB source code of function `semisimple` is provided, with appropriate explanatory comments and pointers to the relevant Equations. Instead of using the interpolating polynomial given in Theorem 4.2, we implemented the equivalent formulation for the interpolating polynomial of Eq. (2.6), by setting

$$
[c_{0j}, c_{1j}, \ldots, c_{m(p)-1j}]^T = \Lambda_j^{-1}[\lambda_j, 0, \ldots, 0]^T.
$$

The function’s input arguments are the matrix $A$ whose semisimple part is required, as well as the number of decimal digits that will be taken into consideration for calculating the multiplicity of the eigenvalues of $A$. Integer $m(p)$ in Theorem 4.2 which is equal to the maximum multiplicity of the roots of the minimal polynomial of $A$, is calculated as the maximum number of non zero elements of the rows and columns of the Jordan decomposition of $A$. The main function (`semisimple`) uses two more functions that are also provided, one for the calculation of the Lagrange polynomial coefficients and one for the calculation of the coefficients $n^{th}$ power of a given polynomial.

```matlab
function R=semisimple(A,tol)
    % by A. Kechriniotis, K. Delibasis, 2009
    % A: matrix whose semisimple part is required
    % tol: number of digits to use for finding multiple eigenvalues of A

    p=poly(A);
    a=roots(p);

    % determine multiplicity of roots according to tolerance tol
    tol=10^-tol;
    r1=round(a*tol)/tol;
    [r_uniq,i_uniq]=unique(r1);
    for i=1:length(r_uniq)
        multi_uniq(i)=sum(r1==r_uniq(i));
    end
    a=r_uniq;
    n=length(a);

    % determine m as the max root multiplicity of the minimal polynomial
    J=jordan(A);
    m=max(max(sum(J~=0)), max(sum(J~=0,2)));
```

10
% create a table with n rows to hold Langrange polynomial coefficients as in (2.1)
for k=1:n
    myLangr_coefs(k,:)=myLangr(a,k,n);
end

% Calculate cnm array in (2.6)
for j0=1:n
    generate Lamda mxm matrix, as in (2.2)
    for i=1:m
        for j=1:m
            if j>i
                Lamda(i,j)=0;
            else
                l1=myLangr_coefs(j0,:);
                l1=polypower(l1,m);
                f1=nchoosek(i-1,j-1);
                if i>1 calc (i-1) order derivative of l1
                    lk=l1;
                    for k=1:i-j
                        lk=polyder(lk);
                    end
                    Lamda(i,j)=f1*polyval(lk,a(j0));
                else no derivative of l1
                    Lamda(i,j)=f1*polyval(l1,a(j0));
                end
            end
        end
    end
    I=diag(ones(size(A,1),1));
    col1=[a(j0),zeros(1,m-1)]';
    c(:,j0)=(Lamda^-1)*col1;
    col1;
end
R=zeros(size(A));
for i=1:m
    for j=1:n
        Ln=myLangr_coefs(j,:);
        LnM=polypower(Ln,m);
        R=R+c(i,j)*polyvalm(LnM,A)*((A-a(j)*I)^(i-1))/factorial(i-1);
    end
end
function Ln=myLangr(a,n,N)
% a: is the vector of roots
% N: length of a
% n: integer between 1 and N inclusive
% Ln is an array of coefficients of the Lagrange polynomial
Ldenom=1;
Lnom=1;
for i=1:N
    if i~=n
        Ldenom=Ldenom*(a(n)-a(i));
        Lnom=conv(Lnom,[1,-a(i)]);
    end
end
Ln=Lnom/Ldenom;

function p1=polypower(p,n);
% p: an array of coefficients of the Lagrange polynomial
% n: integer power
% p1: array of coefficients of polynomial p raised to the nth power
p1=p;
if n>1
    for i=1:n-1
        p1=conv(p1,p);
    end
end

In order to verify the accuracy of the above implementation, we present the following arithmetic examples of the application of the above source code for the calculation of the semisimple part $S_A$, taken from [5].

Let $A = \begin{bmatrix} 3 & 4 & 3 \\ 2 & 7 & 4 \\ -4 & 8 & 3 \end{bmatrix}$. It is easy to verify that $A$ has 2 unique eigenvalues: $\lambda_0 = 1$, $\lambda_1 = 11$ with multiplicity 2 and 1 respectively. The matrices $\Lambda_0, \Lambda_1$ according to Eq. (2.2) are calculated as $\Lambda_0 = \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix}$ and $\Lambda_1 = \begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix}$, whereas the matrix $c_{ij}$ obtains the following values according, to Eq. (2.6):

$$c = \begin{bmatrix} 1 & 11 \\ 0.2 & -2.2 \end{bmatrix}.$$  

The semisimple part is calculated according to Eq. (2.6):

$$S_A = \begin{bmatrix} 1 & 5.6 & 2.8 \\ 0 & 8.6 & 3.8 \\ 0 & 4.8 & 3.4 \end{bmatrix}.$$  

It is easy to demonstrate that $S_A$ is semisimple (e.g. by calculating the Jordan decomposition of $S_A$ which is a diagonal $3 \times 3$ matrix), as well as that $(A - S_A)$ is nilpotent, since $(A - S_A)^2$ is the $3 \times 3$ zero matrix. The result obtained for $S_A$ by the proposed method is identical to the result.
produced in [5].

The second arithmetic example is also taken from [5]. Let

\[ A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.25 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

It is easy to show that the roots of the characteristic polynomial of \( A \) are the following: \( a_0 = 5, a_1 = 4, a_2 = -1, a_3 = 3 \) (multiplicity equal to 2), \( a_4 = 2 \) (multiplicity equal to 2), \( a_5 = 1 \) (multiplicity equal to 2), \( a_6 = 0 \). The matrices \( \Lambda_i \) for \( i = 0, \ldots, 6 \) according to Eq. (2.2) are calculated as:

\[ \Lambda_0 = \begin{bmatrix} 1 \\ -4.9 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 1 \\ -2.5667 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 1 \\ -1.667 \end{bmatrix}, \quad \Lambda_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Lambda_4 = \begin{bmatrix} 1 \\ 1.667 \end{bmatrix}, \quad \Lambda_5 = \begin{bmatrix} 1 \\ 2.5667 \end{bmatrix}, \quad \Lambda_6 = \begin{bmatrix} 1 \\ 4.9 \end{bmatrix}. \]

The matrix \( c_{ij} \) obtains the following values according to Eq. (2.6):

\[ c = \begin{bmatrix} -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ -0.25 & 0 & -1 & 1 & 0 & 0 & 0 \\ -0.0156 & 0.0156 & 0 & 0 & 2 & -1 & 0 \\ -0.0039 & -0.0156 & 0.0156 & 0 & 0 & 3 & -1 \\ 0.0052 & -0.0039 & -0.0156 & 0.0156 & 0.25 & 0 & -1 \\ -0.0004 & 0.0002 & 0.0039 & -0.0026 & 0.0156 & 0.0156 & 0 \\ -0.0007 & 0.0004 & 0.0052 & 0.0039 & 0.0156 & 0.0156 & 0.25 \end{bmatrix}. \]

The semisimple part \( S_A \) is calculated according to Eq. (2.5):

\[ S_A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.0156 & 0.0156 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ -0.0039 & -0.0156 & 0.0156 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0.0052 & -0.0039 & -0.0156 & 0.0156 & 0.25 & 0 & -1 & 3 & 0 \\ -0.0004 & 0.0002 & 0.0039 & -0.0026 & 0.0156 & 0.0156 & 0 & 4 & -1 \\ -0.0007 & 0.0004 & 0.0052 & 0.0039 & 0.0156 & 0.0156 & 0.25 & 0 & -1 \end{bmatrix}. \]

Again this result is identical with Example 3 from [5]. It is also easy to verify that the nilpotency holds, since \((A - S_A)^2 = 0\).

## 5 Further generalizations of Theorem 3.1 and derived identities

The main result of this section is the generalization of Theorem 3.1, which is expressed in the following Theorem.
Theorem 5.1. Let \( g \in k[x] \) be separable of degree \( n \). Let \( \Pi \in (k[x])^{m \times m} \) such that \( \det(\Pi) \neq 0 \). Then for any polynomials \( f_0, f_1, \ldots, f_{m-1} \in k[x] \) there exist unique polynomial \( r \in k[x] \) of degree less than \( mn \), such that

\[
\Pi \begin{bmatrix}
    r \\
r'
    .
    .
    .
    r^{(m-1)}
\end{bmatrix} = E \begin{bmatrix}
    f_0 \\
f_1
    .
    .
    f_{m-1}
\end{bmatrix} \mod g,
\]

where \( E := \gcd(g, \det(\Pi)) \).

Proof. We start from

\[
\Pi \tilde{\Pi} = \tilde{\Pi} \Pi = \det(\Pi)I_M, \tag{5.1}
\]

where \( \tilde{\Pi} \) is the adjugate of \( \Pi \) and \( I_m \) is the \( m \times m \) unit matrix.

Moreover, since \( E = \gcd(g, \det(\Pi)) \) one has that there exist \( H, G \in k[x] \) such that

\[
H \det(\Pi) + Gg = E. \tag{5.2}
\]

Combining (5.1) with (5.2) we get,

\[
H \Pi \tilde{\Pi} = (E - gG)I_m. \tag{5.3}
\]

Now, according to Theorem 5.1 there exist unique \( r \in k[x] \) of degree less than \( mn \) such that

\[
\begin{bmatrix}
    r \\
r'
    .
    .
    .
    r^{(m-1)}
\end{bmatrix} = H \tilde{\Pi} \begin{bmatrix}
    f_0 \\
f_1
    .
    .
    f_{m-1}
\end{bmatrix} \mod g. \tag{5.4}
\]

Multiplying (5.4) from the left by \( \Pi \) and setting (5.3) in the resulting identity we get the conclusion. \( \square \)

Remark 5.2. Choosing \( \Pi = I_m \) in Theorem 5.1 we get Theorem 3.1. Therefore Theorem 3.1 can be regarded as a generalization of Theorem 5.1.

Now we will apply Theorem 5.1 to produce some formulas for polynomials involving derivatives.

Corollary 5.3. Let \( g, g_0, g_1, \ldots, g_{m-1} \in k[x] \) be polynomials. Assume that \( g \) is separable and that

\[(g_i, g) = 1, \ 0 \leq i \leq m - 1.\]

Then for any \( f_0, f_1, \ldots, f_{m-1} \in k[x] \) there exists unique \( r \in k[x] \) of degree less than \( mn \), such that

\[g_i r^{(i)} = f_i \mod g, \ 0 \leq i \leq m - 1.\]
Proof. Applying Theorem 5.1 by 

$$\Pi := \begin{bmatrix}
g_0 & 0 & \ldots & 0 \\
0 & g_1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & g_{m-1}
\end{bmatrix},$$

and afterwards using that $E = \gcd(g, \det(\Pi) = \prod_{i=0}^{m-1} g_i) = 1$ we directly get the conclusion.

Corollary 5.4. Let $g, g_0, g_1, \ldots, g_{m-1} \in \mathbb{k}[x]$ be as in Corollary 5.3. Then for any $f_0, f_1, \ldots, f_{m-1} \in \mathbb{k}[x]$ there exists unique $r$ of degree less than $mn$ such that 

$$(g_i r)^{(i)} = f_i \mod g, \quad 0 \leq i \leq m - 1.$$  

Proof. Let $\Pi \in (\mathbb{k}[x])^{m \times m}$ be the matrix defined by 

$$\Pi = \begin{bmatrix}
g_0 & 0 & \ldots & 0 \\
(1_0)g_1 & (1_1)g_1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
(1_{m-1})g_{m-1} & (1_{m-1})g_{m-1} & \ldots & (1_{m-1})g_{m-1}
\end{bmatrix}. \quad (5.5)$$

From the assumptions 

$$(g_i, g) = 1, \quad 0 \leq i \leq m - 1$$

we have 

$$(\det(\Pi), g) = 1. \quad (5.6)$$

Applying Theorem 5.1 for $\Pi$, as given in (5.5), using (5.6) and the Leibnitz’s rule, we get the conclusion.

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