Second-order Tail Asymptotics of Deflated Risks

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Abstract: Random deflated risk models have been considered in recent literatures. In this paper, we investigate second-order tail behavior of the deflated risk \( X = RS \) under the assumptions of second-order regular variation on the survival functions of the risk \( R \) and the deflator \( S \). Our findings are applied to approximation of Value at Risk, estimation of small tail probability under random deflation and tail asymptotics of aggregated deflated risk.

Key words and phrases: Random deflation; Value-at-Risk; Risk aggregation; Second-order regular variation; Estimation of tail probability.

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1 Introduction

Let \( R \) be a non-negative random variable (rv) with distribution function (df) \( F \) being independent of the rv \( S \in (0, 1) \) with df \( G \). If \( R \) models the loss amount of a financial risk, and \( S \) models a random deflator for a particular time-period, then the product \( X = RS \) represents the deflated value of \( R \) at the end of the time-period under consideration.

Random deflation is a natural phenomena in most of actuarial applications attributed to the time-value of money. When large values or extremes are of interest, for instance for reinsurance pricing and risk management purposes, it is important to link the behaviors of the risk \( R \) and the random deflator \( S \). Intuitively, we expect that large values observed for \( R \) are not significantly influenced by the random deflation. However, this is not always the case; a precise analysis driven by some extreme value theory models is given in Tang and Tsitsiashvili (2004), Tang (2006, 2008), Hashorva et al. (2010), Hashorva (2013), Yang and Hashorva (2013), Yang and Wang (2013), Tang and Yang (2012), Zhu and Li (2012) and the references therein. The results of the aforementioned papers are obtained mainly under a first-order asymptotic condition for the survival function or the quantile function in extreme value theory, i.e., the df \( F \) under consideration belongs to the max-domain of attraction (MDA) of a univariate extreme value distribution \( Q_\gamma, \gamma \in \mathbb{R} \), abbreviated as \( F \in D(Q_\gamma) \), which means

\[
F^n(a_n x + b_n) \to Q_\gamma(x) := \exp \left( -(1 + \gamma x)^{-1/\gamma} \right), \quad 1 + \gamma x > 0, \quad n \to \infty
\]  

(1.1)

holds for some constants \( a_n > 0 \) and \( b_n \in \mathbb{R}, n \geq 1 \), see Resnick (1987). The parameter \( \gamma \) is called the extreme value index; according to \( \gamma > 0, \gamma = 0 \) and \( \gamma < 0 \), the df \( F \) belongs to the MDA of the Fréchet distribution, the Gumbel distribution and the Weibull distribution, respectively.

In order to derive some more informative asymptotic results, second-order regular variation (2RV) conditions are widely used in extreme value theory. Here we only mention de Haan and Resnick (1996) for the uniform convergence rate of \( F^n \) to its ultimate extreme value distribution \( Q_\gamma \) under 2RV, and both Beirlant et al. (2009, 2011), Ling et al. (2012) and the references therein for the asymptotic distributions of the extreme value index estimators under consideration.

Indeed, almost all the common loss distributions including log-gamma, absolute \( t \), log-normal, Weibull, Benktander II, Beta, (cf. Table 2 in the Appendix) possess 2RV properties; actuarial applications based on those properties are developed in the recent contributions Hua and Joe (2011), Mao and Hu (2012a, 2012b) and Yang (2012).

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The main contributions of this paper concern the second-order expansions of the tail probability of the deflated risk $X = RS$ which are then illustrated by several examples. Our main findings are utilized for the formulations of three applications, namely approximation of Value-at-Risk, estimation of small tail probability of the deflated risk, and the derivation of the tail asymptotics of aggregated risk under deflation.

The rest of this paper is organized as follows. Section 2 gives our main results under second-order regular variation conditions. Section 3 shows the efficiency of our second-order asymptotics through some illustrating examples. Section 4 is dedicated to three applications. The proofs of all results are relegated to Section 5. We conclude the paper with a short Appendix.

## 2 Main results

We start with the definitions and some properties of regular variation followed by our principal findings. A measurable function $f : [0, \infty) \to \mathbb{R}$ with constant sign near infinity is said to be of second-order regular variation with parameters $\alpha \in \mathbb{R}$ and $\rho \leq 0$, denoted by $f \in 2RV_{\alpha, \rho}$, if there exists some function $A$ with constant sign near infinity satisfying $\lim_{t \to \infty} A(t) = 0$ such that for all $x > 0$ (cf. Bingham et al. (1987) and Resnick (2007))

$$
\lim_{t \to \infty} \frac{f(tx)/f(t) - x^\alpha}{A(t)} = x^\alpha \int_1^\infty u^{\rho-1} du =: H_{\alpha, \rho}(x). \tag{2.1}
$$

Here, $A$ is referred to as the auxiliary function of $f$. Noting that (2.1) implies $\lim_{t \to \infty} f(tx)/f(t) = x^\alpha$, i.e., $f$ is regularly varying at infinity with index $\alpha \in \mathbb{R}$, denoted by $f \in RV_\alpha$; $RV_0$ is the class of slowly varying functions. When $f$ is eventually positive, it is of second-order $\Pi$-variation with the second-order parameter $\rho \leq 0$, denoted by $f \in 2ERV_{0, \rho}$, if there exist some functions $a$ and $A$ with constant sign near infinity and $\lim_{t \to \infty} A(t) = 0$ such that for all $x$ positive

$$
\lim_{t \to \infty} \frac{f(tx) - f(t)}{a(t)} - \log x = \psi(x) := \begin{cases} 
\frac{x^\rho}{\log^2 x}, & \rho < 0, \\
0, & \rho = 0
\end{cases} \tag{2.2}
$$

(cf. Resnick (2007)), where the functions $a$ and $A$ are referred to as the first-order and the second-order auxiliary functions of $f$, respectively. From Theorem 3.1 in de Haan and Ferreira (2006), $a \in 2ERV_{0, \rho}$ with auxiliary function $A$, and the second-order auxiliary function $A$ satisfies $|A| \in RV_{\rho}$. In fact, (2.2) implies $\lim_{t \to \infty} (f(tx) - f(t))/a(t) = \log x$ for all $x > 0$, which means $f$ is $\Pi$-varying with auxiliary function $a$, denoted by $f \in \Pi(a)$.

We shall keep the notation of the introduction for $R$ and $S \in (0, 1)$, denoting their df’s by $F$ and $G$, respectively, whereas the df of $X = RS$ will be denoted by $H$. Throughout this paper, let $F_0 = 1 - F_0$ denote the survival function of a given distribution $F_0$.

Next, we present our main results in three theorems below. Theorem 2.1 gives a second-order counterpart of Breiman’s Lemma (see Breiman (1965)) while Theorem 2.3 and Theorem 2.6 include refinements of the tail asymptotics of products derived in Hashorva et al. (2010).

**Theorem 2.1.** If $F \in D(Q_{1/\alpha_1})$ satisfies $F \in 2RV_{-\alpha_1, \tau_1}$ with auxiliary function $\tilde{A}$ for some $\alpha_1 > 0$ and $\tau_1 \leq 0$, then

$$
\frac{H(x)}{F(x)} = \mathbb{E} \{S^{\alpha_1}\} \left[ 1 + \mathcal{E}(x) \right], \tag{2.3}
$$

where $\mathcal{E}(x) = \frac{1}{\tau_1} \left( \frac{\mathbb{E}(S^{\alpha_1-\tau_1})}{\mathbb{E}(S^{\alpha_1})} - 1 \right) \tilde{A}(x)(1 + o(1))$ as $x \to \infty$, and thus $H \in 2RV_{-\alpha_1, \tau_1}$ with auxiliary function

$$
A^*(x) = \frac{\mathbb{E} \{S^{\alpha_1-\tau_1}\}}{\mathbb{E} \{S^{\alpha_1}\}} \tilde{A}(x).
$$

**Remark 2.2.** a) The expression for $\tau_1 = 0$ is understood throughout this paper as its limit as $\tau_1 \to 0$.

b) Under the assumptions of Theorem 2.1, Breiman’s Lemma only implies

$$
\frac{H(x)}{F(x)} = \mathbb{E} \{S^{\alpha_1}\} \left[ 1 + \mathcal{E}^*(x) \right]
$$

with $\lim_{x \to \infty} \mathcal{E}^*(x) = 0$, while the error term $\mathcal{E}(x)$ in (2.3) not only converges to 0 as $x \to \infty$, but it shows also the speed of convergence being determined by $\tilde{A}(x)$.

Next, we shall consider the cases that $F$ belongs to the MDA of the Gumbel distribution and the Weibull distribution, respectively.
We write $Y \sim Q$ for some rv $Y$ with df $Q$, whereas $Q^{-}$ denotes the generalized left-continuous inverse of $Q$ (also for $Q$ which is not df). Since $H$ has the same upper endpoint $x_F := \sup \{ y : F(y) < 1 \}$ as that of the df $F$, then all the limit relations below are for $x \to x_F$ unless otherwise specified. Further, for some $\alpha_2 > 0$ we set

$$L(x) = x^{\alpha_2} \overline{G}(1 - 1/x), \quad K(\alpha_2, \rho) = \begin{cases} (1 - \rho)^{\alpha_2 - 1} \Gamma(\alpha_2 + 1), & \rho < 0, \\ \alpha_2^{\alpha_2(\rho-2)+1} \rho, & \rho = 0, \end{cases}$$

(2.4)

where $\Gamma(\cdot)$ is the Euler Gamma function, and define

$$w(x) = 1/E \{ R - x | R > x \}, \quad \eta(x) = xw(x).$$

(2.5)

Hereafter the generalized left-continuous inverse of $F$ and $H$ are denoted by

$$U = U_R = (1/F)^{-}, \quad U_X = (1/H)^{-}.$$

**Theorem 2.3.** Let $F$ be strictly increasing and continuous in the left neighborhood of $x_F$ and let $U \in 2ERV_{0,\rho}, \rho \leq 0$ with auxiliary functions $1/w(U)$ and $A$. If $L \in 2RV_{\tau_2}, \tau_2 < 0$ with auxiliary function $A$, then

$$\overline{H}(x) / F(x) \overline{G}(1 - 1/\eta(x)) = \Gamma(\alpha_2 + 1) + \mathcal{E}(x),$$

(2.6)

where $K(\alpha_2, \rho), \eta(x)$ are defined in (2.4), (2.5) and

$$\mathcal{E}(x) = \left[ \frac{\Gamma(\alpha_2 - \tau_2 + 1) - \Gamma(\alpha_2 + 1)}{\tau_2} A(\eta(x)) - \frac{\alpha_2 \Gamma(\alpha_2 + 2)}{\eta(x)} + K(\alpha_2, \rho) \bar{A} \left( \frac{1}{F(x)} \right) \right] (1 + o(1)).$$

In view of our second result, the error term $\mathcal{E}(x)$ converges to 0 as $x \to x_F$ with a speed which is determined by $A(\eta(x)), 1/\eta(x)$ and $\bar{A}(1/F(x))$. In general, it is not clear which of these terms is asymptotically relevant for the definition of the error term $\mathcal{E}(x)$. For instance in Example 3.3 below $\bar{A}(1/F(x))$ determines $\mathcal{E}(x)$. However, Example 3.4 shows the opposite, namely $A(1/F(x))$ does not appear in our second-order approximation.

**Corollary 2.4.** Under the conditions of Theorem 2.3 with $\psi$ and $w$ given by (2.2) and (2.5), then for $z \in \mathbb{R}$

$$\overline{H}(x + z/w(x)) / \exp(-z) \overline{H}(x) = 1 + \mathcal{E}(x), \quad \mathcal{E}(x) = \left[ \left( \psi(e^{-z}) + \alpha_2 \frac{e^{\rho z} - 1}{\rho} \right) \bar{A} \left( \frac{1}{F(x)} \right) - \frac{\alpha_2 z}{\eta(x)} \right] (1 + o(1)),$$

(2.7)

where $(e^{\rho z} - 1)/\rho$ is interpreted as $z$ for $\rho = 0$. Thus $U_X \in 2ERV_{0,0}$ with auxiliary functions $\bar{a}$ and $\bar{A}$ given by

$$\bar{a}(x) = \bar{a}(x) \left( 1 - \frac{\alpha_2 \bar{a}(x)}{U_X(x)} + \alpha_2 \bar{A} \left( \frac{1}{F(U_X(x))} \right) \right), \quad \bar{A}(x) = \frac{\alpha_2^2 \bar{a}^2(x)}{U_X^2(x)} + \bar{A} \left( \frac{1}{F(U_X(x))} \right).$$

(2.8)

where $\bar{a} = 1/w(U_X)$.

Numerous df’s in the MDA of the Gumbel distribution have Weibull distribution (see Embrechts et al. (1997) and Table 1 in the Appendix); specifically such a distribution function $F$ has the representation

$$\overline{F}(x) = \exp(-V(x)),$$

(2.9)

where $V^\ast(x) = x^\theta \ell(x)$, with $\ell$ denoting a positive slowly varying function at infinity.

**Corollary 2.5.** Under the conditions of Theorem 2.3 if $F$ is given by (2.9) and $\ell \in 2RV_{0,\rho'}, \rho' \leq 0$ with auxiliary function $b$, then

$$\overline{H}(x) = \exp(-V(x)) \overline{G} \left( 1 - \frac{1}{V(x)} \right) \Gamma(\alpha_2 + 1) \theta^{\alpha_2} \left[ 1 + \mathcal{E}(x) \right],$$

(2.10)

with

$$\mathcal{E}(x) = \left( \frac{\alpha_2}{\theta} b(V(x)) + \frac{\Gamma(\alpha_2 - \tau_2 + 1)}{\theta^{\tau_2} \Gamma(\alpha_2 + 1)} A(V(x)) - \frac{\alpha_2 (\alpha_2 + 1)(\theta + 1)}{2V(x)} \right) (1 + o(1)),$$

and thus

$$\overline{H}(x) = \exp(-V^\ast(x)), \quad (V^\ast)^\ast(x) = x^\theta \ell^\ast(x).$$
where \( \ell^* \in 2RV_{0,\max(\rho',-1)} \) with auxiliary function \( b^*(x) = b(x) + \theta a_2(\log x)/x \).

Theorem 2.1 and Corollary 2.4 illustrate that the tail asymptotics of the product \( X = RS \) mainly depends on the heavier factor \( R \). Corollary 2.5 shows that for the Weibull tail distributions, the Weibull tail properties of \( X \) are inherited from the factor \( R \) in the presence of random deflation. The result of Corollary 2.5 is of particular interest for the estimation of tail probabilities, see Section 4.2.

Our last theorem shows for both \( R \) and \( S \) belonging to the MDA of the Weibull distribution, the tail of the product \( X = RS \) is heavier than those of the factors \( R \) and \( S \).

**Theorem 2.6.** Let \( F \) be strictly increasing and continuous in the left neighborhood of \( x_F = 1 \). Assume that for some \( \alpha_1 > 0, \tau_1 \leq 0, 1 - U \in 2RV_{-1/\alpha_1,\tau_1/\alpha_1} \) with auxiliary function \( A \). If further \( L \in 2RV_{0,\tau_2,\tau_2} \leq 0 \) with auxiliary function \( A \), then

\[
\frac{\overline{H}(x)}{F(x)G(x)} = a_1 B(\alpha_1, \alpha_2 + 1) + \mathcal{E}(x),
\]

where

\[
\mathcal{E}(x) = \left[ -\frac{\alpha_1^2 \alpha_2}{\tau_1}(B(\alpha_2, \alpha_1 - \tau_1 + 1) - B(\alpha_2, \alpha_1 + 1))\bar{A}\left(\frac{1}{F(x)}\right) + \frac{\alpha_1}{\tau_2}(B(\alpha_1, \alpha_2 - \tau_2 + 1) - B(\alpha_1, \alpha_2 + 1))\bar{A}\left(\frac{1}{1 - x}\right) \right. \\
+ \left. \alpha_1 \alpha_2 B(\alpha_1 + 1, \alpha_2 + 1)(1 - x) \right] (1 + o(1)),
\]

with \( B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \), \( a, b > 0 \).

**Remark 2.7.** Recall that for a df \( F \) with a finite endpoint \( x_F \) belonging to MDA of the Weibull distribution, then for some \( \alpha_1 > 0,\tau_1 \leq 0, x_F - U \in 2RV_{-1/\alpha_1,\tau_1/\alpha_1} \) with auxiliary function \( A \) is equivalent to \( \overline{F}(x_F - 1/x) \in 2RV_{-\alpha_1,\tau_1} \) with auxiliary function \( \bar{A}^*(x) = -\frac{\alpha_1^2}{\tau_1}A\left(\frac{1}{F(x_F - 1/x)}\right) \) and \( |\bar{A}^*| \in RV_{\tau_1} \) (cf. Theorem 2.3.8 in de Haan and Ferreira (2006)). Thus \( (2.11) \) holds with

\[
\mathcal{E}(x) = \left[ \frac{\alpha_2}{\tau_1}(B(\alpha_2, \alpha_1 - \tau_1 + 1) - B(\alpha_2, \alpha_1 + 1))\bar{A}\left(\frac{1}{1 - x}\right) + \frac{\alpha_3}{\tau_2}(B(\alpha_1, \alpha_2 - \tau_2 + 1) - B(\alpha_1, \alpha_2 + 1))\bar{A}\left(\frac{1}{1 - x}\right) \right. \\
+ \left. \alpha_1 \alpha_2 B(\alpha_1 + 1, \alpha_2 + 1)(1 - x) \right] (1 + o(1)).
\]

**Remark 2.8.** Under the assumptions of Theorem 2.7, \( \overline{H}(1 - 1/x) \in 2RV_{-\alpha,\tau} \) with \( \alpha = \alpha_1 + \alpha_2 \) and \( \tau = \max(-1,\tau_1,\tau_2) \).

### 3 Examples

In this section, six examples are presented to illustrate estimation errors of the second-order expansions given by Section 2 and the first-order asymptotics by Breiman (1965) and Hashorva et al. (2010). We use the software R to calculate the exact value of \( \overline{H}(x) \).

**Example 3.1.** (Fréchet case with Pareto Distribution) Let \( R \) be a random variable with a Pareto df \( F \) given by

\[
\overline{F}(x) = \left(\frac{\theta}{x + \theta}\right)^\alpha, \quad x > 0, \alpha, \theta > 0
\]

denoted in the sequel as \( F \sim \text{Pareto}(\alpha, \theta) \). Suppose that \( S \sim \text{beta}(a,b) \) where \( \text{beta}(a,b) \) stands for the Beta distribution with positive parameters \( a \) and \( b \) and probability density function (pdf)

\[
g(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1, a, b > 0.
\]

We have that \( \overline{F} \in 2RV_{-\alpha,-1} \) with auxiliary function \( \bar{A}(x) = \alpha \theta/x \) and \( \mathbb{E}\{S^\alpha\} = B(a + \kappa, b)/B(a, b) \) for all \( \kappa > 0 \). By Theorem 2.1 with \( \alpha_1 = \alpha \) and \( \tau_1 = -1 \)

\[
\overline{H}(x) = \overline{F}(x)\mathbb{E}\{S^\alpha\}[1 + \mathcal{E}(x)] = \left(\frac{\theta}{x + \theta}\right)^\alpha \frac{B(a + \alpha, b)}{B(a, b)}[1 + \mathcal{E}(x)],
\]
with
\[ \mathcal{E}(x) = \left(1 - \frac{\mathbb{E} \{S^{\alpha+1}\}}{\mathbb{E} \{S^{\alpha}\}}\right) \tilde{A}(x)(1 + o(1)) = \frac{\alpha \theta b}{(\alpha + a + b)x}(1 + o(1)). \]

**Example 3.2.** (Fréchet case with Beta distribution of second kind) Let \( R \) be a random variable with Beta distribution of second kind with positive parameters \( a, b \), i.e., \( R \overset{d}{=} R_0^{-1}, R_0 \sim \text{beta}(b, a) \), denoted by \( R \sim \text{beta}_2(a, b) \) (here \( \overset{d}{=} \) stands for equal in distribution). It follows from (3.1) that
\[
\mathbb{P}(R_0 < x) = \frac{x^b}{bB(b, a)} \left[ 1 - \frac{(a - 1)b}{(1 + b)x}(1 + o(1)) \right], \quad x \downarrow 0,
\]
and thus
\[
\widetilde{F}(x) = \mathbb{P}(R > x) = \mathbb{P} \left( R_0 < \frac{1}{1 + x} \right) = \frac{x^{-b}}{bB(b, a)} \left[ 1 - \frac{(a + b)b}{(1 + b)x}(1 + o(1)) \right], \quad x \to \infty, \tag{3.2}
\]
i.e., \( \widetilde{F} \in 2RV_{-b,-1} \) with auxiliary function \( \tilde{A}(x) = (a + b)b/(1 + b)x \). If \( S \sim \text{beta}(c, d) \), then \( \mathbb{E} \{S^\alpha\} = B(c + \kappa, d)/B(c, d) \) for all \( \kappa > 0 \). In view of Theorem 2.1 with \( \alpha_1 = b \) and \( \tau_1 = -1 \)
\[
\overline{H}(x) = \widetilde{F}(x)\mathbb{E} \{S^b\}[1 + \mathcal{E}(x)] = \frac{x^{-b}}{bB(b, a)} \left[ 1 - \frac{(a + b)b}{(1 + b)x}(1 + o(1)) \right] \frac{B(c + b, d)}{B(c, d)}[1 + \mathcal{E}(x)],
\]
with
\[
\mathcal{E}(x) = \left(1 - \frac{\mathbb{E} \{S^{b+1}\}}{\mathbb{E} \{S^b\}}\right) \tilde{A}(x)(1 + o(1)) = \frac{d}{b + c + d}(1 + o(1)).
\]
In particular, for \( a = c + d \),
\[
\overline{H}(x) = \frac{x^{-b}}{bB(b, c)} \left[ 1 - \frac{(c + b)b}{(1 + b)x}(1 + o(1)) \right],
\]
which is the second-order expansion of survival function of \( \text{beta}_2(c, b) \) (cf. (3.2)), and consistent with \( X \sim \text{beta}_2(c, b) \) (see Lemma 5 in Balakrishnan and Hashorva (2011)).

**Example 3.3.** (Gumbel case with \( \rho = 0 \)) Let \( R \sim F \) with
\[
\overline{F}(x) = \exp \left( -\frac{cx}{1 - x} \right), \quad 0 < x < 1, c > 0. \tag{3.3}
\]
We write below (3.3) as \( F \sim E(1, c) \). If follows that \( F \in D(Q_0) \) with \( w(x) = c/(1 - x)^2 \), and \( U \in 2ERV_{0,0} \) with auxiliary functions
\[
a(x) = 1/w(U(x)), \quad \tilde{A}(x) = -\frac{2}{c + \log x}.
\]
If \( S \sim \text{beta}(a, b) \), then we have that the df \( G \) of \( S \) satisfies
\[
\overline{G} \left( 1 - \frac{1}{x} \right) = \frac{x^{-b}}{bB(a, b)} \left( 1 - \frac{b(a - 1)}{(b + 1)x}(1 + o(1)) \right), \quad x \uparrow \infty, \tag{3.4}
\]
i.e., \( \overline{G}(1 - 1/x) = x^bL(x), L \in 2RV_{0,-1} \) with auxiliary function
\[
A(x) = \frac{b(a - 1)}{(b + 1)x}.
\]
Consequently
\[
\frac{1}{\eta(x)} = \frac{(1 - x)^2}{cx}, \quad \tilde{A} \left( \frac{1}{\overline{F}(x)} \right) = -\frac{2(1 - x)}{c}, \quad A(\eta(x)) = \frac{b(a - 1)(1 - x)^2}{(b + 1)cx}.
\]
By Theorem 2.3 with \( \alpha_2 = b, \tau_2 = -1 \) and \( \rho = 0 \),
\[
\overline{H}(x) = \overline{F}(x)\overline{G} \left( 1 - \frac{(1 - x)^2}{cx} \right) \Gamma(b + 1)[1 + \mathcal{E}(x)]
\]
with
\[
\mathcal{E}(x) = K(b, 0)\tilde{A} \left( \frac{1}{\overline{F}(x)} \right)(1 + o(1)) = \frac{b(b + 1)}{c}(1 - x)(1 + o(1)).
\]
Example 3.4. (Gumbel case with \( \rho < 0 \)) Let \( R \sim F \) with
\[
\mathcal{F}(x) = \frac{1 - \exp(-\exp(-x))}{p}, \quad x > 0, \ p = 1 - e^{-1}.
\]

It follows that \( F \in D(Q_0) \) with constant scaling function \( w(x) = 1 \) and its tail quantile function is
\[
U(x) = \log \frac{x}{p} - \frac{p}{2x}(1 + o(1)).
\]

Furthermore, \( U \in 2ERV_{0,-1} \) with auxiliary functions
\[
a(x) = 1, \quad A(x) = \frac{p}{2x}.
\]

Next, suppose that \( S \sim \text{beta}(a,b) \) (cf. (3.4)). Thus,
\[
\frac{1}{\eta(x)} = \frac{1}{x} \quad \bar{A} \left( \frac{1}{\mathcal{F}(x)} \right) = \frac{1}{2} e^{-x}, \quad A(\eta(x)) = \frac{b(a-1)}{(b+1)x},
\]

By Theorem 2.3 with \( \alpha_2 = b, \tau_2 = -1 \) and \( \rho = -1 \)
\[
\overline{H}(x) = \mathcal{F}(x) \overline{G} \left( 1 - \frac{1}{x} \right) \Gamma(b+1)[1 + \mathcal{E}(x)],
\]

with
\[
\mathcal{E}(x) = - \left[ \frac{b^2(a-1)}{(b+1)x} + \frac{b(b+1)}{x} \right] (1 + o(1)).
\]

Example 3.5. (Gumbel case with Weibull tail) Let \( R \sim \Gamma(\alpha, \lambda) \) with pdf
\[
f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0, \lambda, \alpha > 0.
\]

The tail quantile function of \( F \) is
\[
U(x) = \frac{1}{\lambda} \left( \log x - \log \Gamma(\alpha) \right) \left[ 1 + \frac{(\alpha-1) \log \log x}{\log x - \log \Gamma(\alpha)} (1 + o(1)) \right].
\]

Thus \( F \in D(Q_0) \) with \( w(x) = \lambda \) and \( U \in 2RV_{0,0} \) with second-order auxiliary function
\[
\bar{A}(x) = \frac{1 - \alpha}{\log^2 x}
\]

(cf. Table 4 in the Appendix). Next, suppose that \( S \sim \text{beta}(a,b) \), where the survival function satisfies (3.4). Consequently,
\[
\frac{1}{\eta(x)} = \frac{1}{\lambda x}, \quad \bar{A} \left( \frac{1}{\mathcal{F}(x)} \right) = \frac{1 - \alpha}{(\lambda x)^2}, \quad A(\eta(x)) = \frac{b(a-1)}{(b+1)\lambda x}.
\]

By Theorem 2.3 with \( \alpha_2 = b, \tau_2 = -1 \) and \( \rho = 0 \)
\[
\overline{H}(x) = \mathcal{F}(x) \overline{G} \left( 1 - \frac{1}{\lambda x} \right) \Gamma(b+1)[1 + \mathcal{E}(x)],
\]

with
\[
\mathcal{E}(x) = - \frac{b}{\lambda x} \left[ \frac{b(a-1)}{b+1} + (b+1) \right] (1 + o(1)).
\]

Thus
\[
\overline{H}(x) = \frac{(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \left[ 1 + \frac{\alpha-1}{\lambda x} (1 + o(1)) \right] \frac{\lambda x^{-b} \Gamma(b+1)}{bB(a,b)} \left( 1 - \frac{b(a-1)}{(b+1)\lambda x} (1 + o(1)) \right) \
\times \left[ 1 - \frac{b}{\lambda x} \left( \frac{b(a-1)}{b+1} + (b+1) \right) (1 + o(1)) \right]
\]
\[
= \frac{(\lambda x)^{\alpha-b-1} e^{-\lambda x}}{\Gamma(\alpha) \Gamma(b+a)} \left[ 1 + \frac{\alpha - b(a+b) - 1}{\lambda x} (1 + o(1)) \right].
\]

(3.5)
On the other hand, in view of Corollary 2.5 both \( R \) and \( X \) are Weibull tail distributions with (cf. Table 1 in the Appendix)

\[
\theta = 1, \rho' = -1, b(x) = \frac{(1 - \alpha) \log x}{x} \quad \text{and} \quad \rho^* = -1, b^*(x) = b(x) + \frac{\theta_0 b \log x}{x} = \frac{(1 - \alpha + b) \log x}{x},
\]

which is consistent with (3.3). In particular, if \( \alpha = a + b \), then (3.3) and (3.4) are consistent with the well-known result \( X \sim \Gamma(a, \lambda) \) (cf. Hashorva (2013)).

Example 3.6. (Weibull case) Let \( R \sim \text{beta}(a_1, b_1) \) and \( S \sim \text{beta}(a_2, b_2) \). By (3.4), \( 1 - U \in 2RV_{-1/b_1,-1/b_1} \) with auxiliary function

\[
\tilde{A}(x) = -\frac{a_1 - 1}{b_1(b_1 + 1)} \left( \frac{x}{b_1 B(a_1, b_1)} \right)^{-1/b_1},
\]

and \( \tilde{C}(1 - 1/x) = x^{b_2} L(x), L \in 2RV_{0,-1} \) with auxiliary function

\[
A(x) = \frac{b_2(a_2 - 1)}{(b_2 + 1)x}.
\]

Hence

\[
\bar{A} \left( \frac{1}{\tilde{F}(x)} \right) = -\frac{a_1 - 1}{b_1(b_1 + 1)} (1 - x), \quad A \left( \frac{1}{1 - x} \right) = \frac{b_2(a_2 - 1)}{b_2 + 1} (1 - x).
\]

By Theorem 2.6 with \( \alpha_1 = b_1, \alpha_2 = b_2, \tau_1 = \tau_2 = -1 \) and

\[
\bar{H}(x) = \tilde{F}(x) \tilde{G}(x) \left[ b_1 B(b_1, b_1 + 1) + \mathcal{E}(x) \right],
\]

with

\[
\mathcal{E}(x) = b_1 b_2 B(b_1 + 1, b_2 + 1) \left( 1 + \frac{a_1 - 1}{b_1 + 1} + \frac{a_2 - 1}{b_2 + 1} \right) (1 - x)(1 + o(1)).
\]

In particular, for \( a_2 + b_2 = a_1 \),

\[
\bar{H}(x) = \frac{(1 - x)^{b_1+b_2} B(b_1, b_2 + 1)}{b_2 B(a_1, b_1) B(a_2, b_2)} \left[ 1 + \left( \frac{b_1 + b_2}{b_1 + b_2 + 1} \left( 1 + \frac{a_1 - 1}{b_1 + 1} + \frac{a_2 - 1}{b_2 + 1} \right) \right) (1 - x)(1 + o(1)) \right]
\]

\[
= \frac{(1 - x)^{b_1+b_2}}{(b_1 + b_2) B(a_2, b_1 + 1)} \left[ 1 - \frac{(b_1 + b_2)(a_2 - 1)}{b_1 + b_2 + 1} (1 - x)(1 + o(1)) \right],
\]

which is the second-order expansion of survival function of \( \text{beta}(a_2, b_1 + b_2) \) (cf. (3.4)), and consistent with \( X \sim \text{beta}(a_2, b_1 + b_2) \) (cf. Hashorva (2010)).
4 Applications

4.1 Asymptotics of Value-at-Risk

In insurance and risk management applications, Value-at-Risk (denoted by VaR) is an important risk measure, see e.g., Denuit et al. (2006). We shall analyse first the asymptotics of \( \text{VaR}_p(X) \) in case that \( R \) has a heavy tail and a Weibull tail, respectively. Recall that VaR at probability level \( p \) for some rv \( R \) is defined by

\[
\text{VaR}_p(R) = \inf \{ y : F(y) \geq p \} = U(1/(1-p)). \tag{4.1}
\]

With the same notation as before, if \( \overline{F} \in RV_{\alpha}, \alpha > 0 \), then by Breiman’s Lemma

\[
\overline{H}(x) \sim \mathbb{E}\{S^\alpha\} \overline{F}(x) \sim \overline{F}(\mathbb{E}\{S^\alpha\})^{-1/\alpha} x, \quad x \to \infty
\]

implying the following first-order asymptotics

\[
\text{VaR}_p(X) \sim (\mathbb{E}\{S^\alpha\})^{1/\alpha} \text{VaR}_p(R), \quad p \uparrow 1. \tag{4.2}
\]

Refining the above, we derive the following second-order asymptotics

\[
\text{VaR}_p(X) = (\mathbb{E}\{S^\alpha\})^{1/\alpha} \text{VaR}_p(R)[1 + \mathcal{E}(p)], \quad \mathcal{E}(p) = \left( \frac{\mathbb{E}\{S^\alpha - \tau\}}{(\mathbb{E}\{S^\alpha\})^{1-\tau/\alpha}} - 1 \right) \frac{\tilde{A}(\text{VaR}_p(R))}{\alpha \tau} (1 + o(1)), \quad p \uparrow 1, \tag{4.3}
\]

provided that \( \overline{F} \in 2RV_{\alpha,\tau}, \alpha > 0, \tau < 0 \) with auxiliary function \( \tilde{A} \).

Indeed, there exists some positive constant \( c \) such that (cf. Hua and Joe (2011))

\[
\overline{F}(x) = cx^{-\alpha} \left[ 1 + \frac{\tilde{A}(x)}{\tau}(1 + o(1)) \right]
\]

for sufficiently large \( x \). Thus, by Theorem 2.1

\[
\overline{H}(x) = cx^{-\alpha} \mathbb{E}\{S^\alpha\} \left[ 1 + \frac{\mathbb{E}\{S^\alpha - \tau\}}{\mathbb{E}\{S^\alpha\}} \frac{\tilde{A}(x)}{\tau}(1 + o(1)) \right].
\]

Therefore, by Theorem 1.5.12 in Bingham et al. (1987)

\[
\text{VaR}_p(R) = \left( \frac{c}{1-p} \right)^{1/\alpha} \left[ 1 + \frac{\tilde{A}(\text{VaR}_p(R))}{\alpha \tau} (1 + o(1)) \right], \quad p \uparrow 1
\]

and

\[
\text{VaR}_p(X) = \left( \frac{\mathbb{E}\{S^\alpha\}}{1-p} \right)^{1/\alpha} \left[ 1 + \frac{\mathbb{E}\{S^\alpha - \tau\}}{\mathbb{E}\{S^\alpha\}} \frac{\tilde{A}(\text{VaR}_p(X))}{\alpha \tau} (1 + o(1)) \right], \quad p \uparrow 1.
\]

Consequently, by \( |\tilde{A}| \in RV_\tau \) and (4.2) we obtain the second-order asymptotics (4.3) follows.

In what follows we will consider the case that \( \overline{F} \) is in the MDA of the Gumbel distribution. Since most of such distributions are Weibull tail distributions (cf. Table 1 and Table 2 in the Appendix), we focus on the derivation of the asymptotics of \( \text{VaR}_p(X) \) by \( \text{VaR}_p(R) \) (see 1.4 below) under the conditions of Corollary 2.5. Note that \( \overline{F} \) has a Weibull tail satisfying the second-order condition (cf. 4.4)

\[
\overline{F}(x) = \exp(-V(x)) \quad \text{with} \quad V^+(x) = x^\theta \ell(x)
\]

and \( \ell \in 2RV_{\alpha,\rho}, \rho \leq 0 \) with auxiliary function \( b \). By (4.1)

\[
\text{VaR}_p(R) = V^+(-\log(1-p)) = (-\log(1-p))^{\theta \ell(-\log(1-p))}, \quad p \uparrow 1.
\]

In view of Corollary 2.5 (see 2.10),

\[
\overline{H}(x) = \exp(-V(x) - \alpha_2 \log V(x) + \log L^*(V(x))),
\]
where \( L^* \) denotes a slowly varying function. Recall that \( \log L^*(V(x)) = o(\log V(x)) \) (see Bingham et al. (1987)), we have

\[
\text{VaR}_p(X) = V_{\theta}^{-} \left( -\log(1-p) \left[ 1 - \alpha_2 \frac{\log(-\log(1-p))}{-\log(1-p)} (1 + o(1)) \right] \right) \\
= \left( \log \frac{1}{1-p} \right)^\theta \left[ 1 - \theta \alpha_2 \frac{\log \log \frac{1}{1-p}}{\log \frac{1}{1-p}} (1 + o(1)) \right] \ell \left( \log \frac{1}{1-p} \right) \left[ 1 + \left( 1 - \alpha_2 \frac{\log \log \frac{1}{1-p}}{\rho(1)} \right)^{\rho} - 1 \right] b \left( \log \frac{1}{1-p} \right) (1 + o(1)) \\
= \text{VaR}_p(R) \left[ 1 - \theta \alpha_2 \frac{\log \frac{1}{1-p}}{\log \frac{1}{1-p}} (1 + o(1)) \right], \quad p \uparrow 1. \tag{4.4}
\]

### 4.2 Estimations of tail probability

In many insurance applications it is important to estimate the tail probability of the extreme risks. In what follows, we investigate this problem under the random scaling framework. Let \( \{(R_i, S_i), i = 1, \cdots, n\} \) be a random sample from \( (R, S) \), our goal is to estimate \( p = \mathbb{P}(X > x) = \mathbb{P}(RS > x) \) with sufficiently large \( x \). One possible estimation is via the empirical df if \( x \) is in the region of the sample \( X_i, i \leq n \) with \( X_i = R_i S_i, i = 1, \cdots, n \). In general, we consider how to estimate \( p_n := \mathbb{P}(X > x_n) \) as \( x_n \to \infty \). Let \( R_{n-k+1,n}, S_{n-k+1,n} \) and \( X_{n-k+1,n}, k = 1, \cdots, n \) be the associated increasing order statistics and \( R \sim F \) and \( S \in (0, 1) \) are independent.

First we consider the case that \( \overline{F} \in 2RV_{\alpha, \tau} \) with \( \alpha > 0, \tau < 0 \) and the second-order auxiliary function \( \tilde{A} \), thus by Hua and Joe (2011), there exists a positive constant \( c \) such that

\[
\overline{F}(x) = cx^{-\alpha} (1 + \tilde{A}(x)/\tau (1 + o(1))) =: cx^{-\alpha} (1 + o(\delta(x))),
\]

i.e., \( F \in \mathcal{F}_{1/\alpha, \tau} \) with \( \delta(x) = \tilde{A}(x)/\alpha \tau \) (cf. Beirlant et al. (2009)). By Theorem \[4.2\]

\[
\overline{F}(x) = \overline{F}(x) \left( \mathbb{E} \{ S^\alpha \} + \mathbb{E} \{ S^\alpha (S^{-\tau} - 1) \} \right) \alpha \delta(x)(1 + o(1))). \tag{4.5}
\]

In order to estimate \( \overline{H}(x) \) with \( x = x_n \), we use the estimators of \( 1/\alpha, \delta, \tau \) and \( \overline{F} \) proposed by Beirlant et al. (2009). Let \( y_{k,n} = x_n/R_{n-k,n}, \tilde{r}_{k,n} = \tilde{\rho}_n/H_{k,n} \) with \( \tilde{\rho}_n \) some weakly consistent estimator of \( \rho = \tau/\alpha \) based on samples from the parent \( R \), denote

\[
H_{k,n} = \frac{1}{k} \sum_{i=1}^{k} \log R_{n-i+1,n} - \log R_{n-k,n}, \quad E_{k,n}(s) = \frac{1}{k} \sum_{i=1}^{k} (R_{n-i+1,n}/R_{n-k,n})^s, s \leq 0
\]

and

\[
\hat{\alpha}_{k,n} = \left( H_{k,n} - \hat{\delta}_{k,n} \frac{\hat{\rho}_n}{1 - \hat{\rho}_n} \right)^{-1}, \quad \hat{\delta}_{k,n} = H_{k,n} (1 - 2 \hat{\rho}_n) (1 - \hat{\rho}_n)^{-2} \hat{\rho}_{n}^{-4} \left( E_{k,n}(\hat{\rho}_n/H_{k,n}) - \frac{1}{1 - \hat{\rho}_n} \right).
\]

Thus, by \[4.5\], the tail probability \( p_n \) can be estimated as (denoted by \( \hat{p}_{k,n}(R, S) \))

\[
\hat{p}_{k,n}(R, S) = \overline{F}(x_n) \left( \mathbb{E} \{ S^\alpha \} + (\mathbb{E} \{ S^\alpha - \tau \} - \mathbb{E} \{ S^\alpha \} \right) \hat{\delta}_{k,n}, \frac{\hat{\alpha}_{k,n}}{H_{k,n}}), \tag{4.6}
\]

with

\[
\hat{F}(x_n) = \frac{k}{n} \left( y_{k,n} \left( 1 + \hat{\delta}_{k,n} (1 - \tilde{y}_{k,n}) \right) \right)^{-\hat{\alpha}_{k,n}}, \quad \mathbb{E} \{ S^\alpha \} = \frac{1}{n} \sum_{i=1}^{n} S_i \hat{\alpha}_{k,n} - \tilde{y}_{k,n} \mathbb{E} \{ S^\alpha - \tau \} = \frac{1}{n} \sum_{i=1}^{n} S_i \hat{\alpha}_{k,n} - \tilde{y}_{k,n}. \tag{4.7}
\]

On the other hand, by Theorem \[4.2\] \( X \) has the same second-order tail behavior as \( R \). Consequently, \( p_n \) can be directly estimated by using samples from \( X \). We denote that estimator (cf. \[4.7\]) by \( \hat{p}_{k,n}(X) \), given as

\[
\hat{p}_{k,n}(X) = \frac{k}{n} \left( y'_{k,n} \left( 1 + \hat{\delta}_{k,n} (1 - (y'_{k,n}) \tilde{y}_{k,n}^*)) \right) \right)^{-\hat{\alpha}_{k,n}}, \tag{4.8}
\]

with \( y_{k,n}^* = x_n/X_{n-k,n} \) and \( \hat{\delta}_{k,n}, \tilde{r}_{k,n}, \hat{\alpha}_{k,n} \) are \( \hat{\delta}_{k,n}, \tilde{r}_{k,n}, \hat{\alpha}_{k,n} \) with the order statistics replaced by \( \{X_{n-k+1,n}, k = 1, \cdots, n-1\} \).
Relying on (4.6) and (4.8), we shall perform some simulations to compare \( \hat{\alpha}_{k,n}, \hat{\beta}_{k,n}(R,S) \) and \( \hat{\alpha}_{k,n}', \hat{\beta}_{k,n}(X) \). Since \( \tau = -1 \) holds in most applications, we take \( \hat{\tau}_{k,n} = -1 \) and \( \hat{\rho}_n = -H_{k,n} \).

Next, we investigate the case of \( F \sim D(Q_0) \). For convenience, we consider only the estimation comparisons for \( F \) being Weibull tail distributions. By Corollary 2.5, both \( R \) and \( X \) are Weibull tail distributions with the same Weibull tail coefficient \( \theta \) and further the second-order parameter \( \rho^* \) is greater than \(-1\), we consider the bias-reduced Weibull tail coefficient estimators \( \hat{\theta} \) due to Diebolt et al. (2008):

\[
\hat{\theta} = \hat{\theta}(k, R) = Z_k - \hat{b}(\log(n/k))\bar{\tau}_k, \tag{4.9}
\]

with

\[
\hat{b}(\log(n/k)) = \frac{\sum_{i=1}^{k} (x_i - \tau_k)Z_i}{\sum_{i=1}^{k} (x_i - \tau_k)^2}
\]

and

\[
x_j = \frac{\log(n/k)}{\log(n/j)}, \quad Z_j = j \log(n/j)(\log R_{n-j+1,n} - \log R_{n-j,n}), \quad \bar{\tau}_k = \frac{\sum_{j=1}^{k} x_j}{k}, \quad Z_k = \frac{\sum_{j=1}^{k} Z_j}{k}.
\]

Based on the bias-reduced tail quantile estimators provided by Diebolt et al. (2008), given by

\[
\hat{x}_{p_n} = R_{n-k,n} \left( \log(1/p_n) / \log(n/k) \right) \exp \left( \hat{b}(\log(n/k)) \left( \log(1/p_n) / \log(n/k) \right)^{\hat{\rho}^*} - 1 \right)
\]

with \( p_n \) known, we can solve the dual problem and estimate the tail probability \( \hat{F}(x) \) for given \( x \) as follows

\[
\hat{F}(x) = \exp \left( -\log(n/k) \left( \frac{x}{R_{n-k,n}} \right)^{1/\hat{\theta}} \exp \left( -\hat{b}(\log(n/k)) \left( x/R_{n-k,n} \right)^{\hat{\rho}^*/\hat{\theta}^*} - 1 \right) \right), \tag{4.10}
\]

with \( \hat{\rho}^* \) a consistent estimator of \( \rho^* \). Note that \( \hat{F}(x) = \exp(-V(x)) \) and \( S \sim G \) with \( \hat{G}(1 - 1/x) \in 2RV_{\alpha_2,\tau_2} \) is equivalent that \( S^* = 1/(1 - S) \sim G^* \) with \( \hat{G}^* \in 2RV_{\alpha_2,\tau_2} \). Hence by (4.10) and Beirlant et al. (2009), we have

\[
\check{V}(x) = -\log \hat{F}(x), \quad \check{b}(V(x)) = \hat{b}(\log(n/k)) \left( \frac{\check{V}(x)}{\log(n/k)} \right)^{\hat{\rho}^*} \tag{4.11}
\]

and

\[
\hat{G}(1 - 1/V(x)) = k_n \left( y_{k,n} + 1 + \hat{\tau}_{k,n}(1 - y_{k,n})^{1/\hat{\tau}_2(k)} \right) \hat{\alpha}_2(k), \quad \hat{A}(V(x)) = \hat{\alpha}_2(k)\hat{\tau}_2(k)\hat{\delta}_{k,n}(y_{k,n})\hat{\tau}_2(k), \tag{4.12}
\]

where \( y_{k,n} = \check{V}(x)/S^*_{n-k,n} \) and \( \hat{\delta}_{k,n}, \hat{\tau}_2(k), \hat{\alpha}_2(k) \) are estimated with the order statistics replaced by \( S^*_{n-k,n} := 1/(1 - S_{n-k,n}) \) in (4.7). Therefore, combining (4.9), (4.10), (4.11) and (4.12), the estimator of \( p = \check{H}(x) \), denoted by \( \hat{p}_k(R,S) \), is given as (cf. Corollary 2.5)

\[
\hat{p}_k(R,S) = \check{F}(x) \hat{G}(1 - 1/V(x)) \hat{\Gamma}(\hat{\alpha}_2(k) + 1)\hat{\theta}^{\hat{\alpha}_2(k)} \times \left[ 1 + \frac{\hat{\alpha}_2(k)\hat{\tau}_2(k) + 1}{\hat{\theta}^{\hat{\tau}_2(k)}} \frac{\hat{b}(V(x))}{\hat{A}(V(x))} + \frac{\hat{\alpha}_2(k)\hat{\tau}_2(k) + 1}{\hat{\theta}^{\hat{\alpha}_2(k)}} \frac{1}{2\check{V}(x)} \right]. \tag{4.13}
\]

On the other hand, by Corollary 2.5 we can estimate \( p = \check{H}(x) \) directly based on samples from \( X \) as

\[
\hat{p}_k(X) = \exp \left( -\log(n/k) \left( \frac{x}{X_{n-k,n}} \right)^{1/\hat{\theta}^*} \exp \left( -\hat{b}^*(\log(n/k)) \left( x/X_{n-k,n} \right)^{\hat{\rho}^*/\hat{\theta}^*} - 1 \right) \right), \tag{4.14}
\]

where \( \hat{\rho}^* \) is a consistent estimator of \( \rho^* \) and \( \hat{\theta}^*, \hat{b}^* \) are computed by (4.9) with samples replaced by \( X_i = R_iS_i, i = 1, 2, \cdots, n \).

Now, we perform the simulations of the estimators of \( \theta \) and \( p = P(X > x) \) given by (4.13) and (4.14) with one sample of size \( n = 5000 \) from Table 1 and Table 2 in the Appendix. In the simulation we take \( \hat{\tau}_2(k) = -1, \hat{\rho}^* = -1 \) and plot sample paths of \( \hat{\theta} \) and \( \log(\hat{p}_k/p) \), \( k = 100, \cdots, 4500 \), with \( \hat{p}_k = \hat{p}_k(R, S), \hat{p}_k(X) \), respectively (cf. (4.13) and (4.14)).
4.3 Linear combinations of random contractions

Motivated by the dependence structure of elliptical random vectors Hashorva et al. (2010) discussed the first-order tail asymptotics of the aggregated risks of certain bivariate random vectors which we shall introduce next. Let therefore \((V_1, V_2)\) be a bivariate scale mixture random vector with stochastic representation

\[
(V_1, V_2) \overset{d}{=} R(I_1 S, I_2 \sqrt{1 - S^2}),
\]

(4.15)

where \(R \sim F\), is almost surely positive, \(S \sim G\) is a scaling random variable taking values in \((0, 1)\), while \(I_1, I_2\) assume values in \([1, -1]\). Hashorva et al. (2010) studied the tail asymptotics of the aggregated risk

\[
V(\lambda) = \lambda V_1 + \sqrt{1 - \lambda^2} V_2 = R(I_1 \lambda S + I_2 \sqrt{1 - \lambda^2} \sqrt{1 - S^2}) =: RS(\lambda)
\]

(4.16)

for \(\lambda \in (0, 1)\). In what follows, we derive the second-order tail asymptotics of \(V(\lambda)\) in (4.16) if the following condition holds for small \(x > 0\)

\[
\mathbb{P}(|S - \lambda| \leq x) = c_\lambda x^{\alpha_\lambda}(1 + L_\lambda(x)x^{\tau_\lambda}), \quad \alpha_\lambda, \tau_\lambda \in (0, \infty) \quad \text{and} \quad \lambda \in [0, 1],
\]

(4.17)

where \(c_\lambda\) is a positive constant and \(|L_\lambda|\) is slowly varying at 0. Set

\[
q_\lambda = \mathbb{P}(I_1 = I_2 = 1)\mathbb{I}\{\lambda \in (0, 1)\} + \mathbb{P}(I_2 = 1)\mathbb{I}\{\lambda = 0\} + \mathbb{P}(I_1 = 1)\mathbb{I}\{\lambda = 1\}.
\]

(4.18)

with \(\mathbb{I}\{\cdot\}\) the indicator function.

**Lemma 4.1.** Let \(I_1, I_2\) be two random variables taking values \(-1, 1\) with probability \(q_\lambda \in [0, 1]\) defined by (4.18) and independent of the scaling random variable \(S \sim G\). For given \(\lambda \in [0, 1]\), if further \(dG\) satisfies (4.17) for small \(x > 0\), then the \(S(\lambda)\) defined in (4.16) satisfies

a) For \(\lambda \in (0, 1)\),

\[
\mathbb{P}(S(\lambda) > 1 - x) = q_\lambda c_\lambda (2x(1 - \lambda^2))^{\alpha_\lambda/2}[1 + A_\lambda(x)],
\]

with

\[
A_\lambda(x) = \left(L_\lambda(\sqrt{x})(2x(1 - \lambda^2))^{\tau_\lambda/2} - \frac{\alpha_\lambda \lambda}{\sqrt{2(1 - \lambda^2)}} x^{1/2}\right)(1 + o(1)).
\]

b) For \(\lambda = 0\),

\[
\mathbb{P}(S(\lambda) > 1 - x) = q_\lambda c_\lambda (2x)^{\alpha_\lambda/2}[1 + A_\lambda(x)], \quad A_\lambda(x) = \left(L_\lambda(\sqrt{x})(2x)^{\tau_\lambda/2} - \frac{\alpha_\lambda x}{4}\right)(1 + o(1)).
\]

c) For \(\lambda = 1\),

\[
\mathbb{P}(S(\lambda) > 1 - x) = q_\lambda c_\lambda x^{\alpha_\lambda}[1 + A_\lambda(x)], \quad A_\lambda(x) = L_\lambda(x)x^{\tau_\lambda}.
\]

In view of Lemma 4.1, we have \(\mathbb{P}(S(\lambda) > 1 - 1/x) \in 2RV_{-\alpha, \tau}\) with \(\alpha, \tau\) and auxiliary function \(A\) as follows

\[
\alpha = \begin{cases} 
\frac{\alpha_\lambda}{2}, & \lambda \in [0, 1), \\
\alpha_\lambda, & \lambda = 1;
\end{cases} \quad \tau = \begin{cases} 
-\min(\tau_\lambda, 1)/2, & \lambda \in (0, 1), \\
-\min(\tau_\lambda, 2)/2, & \lambda = 0, \\
-\tau_\lambda, & \lambda = 1;
\end{cases} \quad A(x) = \tau A_\lambda(1/x).
\]

(4.19)

Now, utilizing Theorem 2.39, Theorem 2.40 and Lemma 4.1, we derive the following second-order tail asymptotics of \(V(\lambda)\).

**Theorem 4.2.** Let \(V(\lambda)\) be defined in (4.16) for \(\lambda \in [0, 1]\) and satisfying the conditions of Lemma 4.1

a) If \(F \in D(Q_0)\) and its tail quantile function \(U \in 2ERV_{0, \rho}\), \(\rho \leq 0\) with auxiliary functions \(1/w(U)\) and \(\tilde{A}\), then for \(x \uparrow x_F\) (set \(\eta(x) = xw(x)\))

\[
\mathbb{P}(V(\lambda) > x) = \bar{F}(x)\mathbb{P}
\left(S(\lambda) > 1 - \frac{1}{\eta(x)}\right)
\times \left[\Gamma(\alpha + 1) + \frac{\Gamma(\alpha + 1)}{\tau}\tilde{A}(\eta(x)) + K(\alpha, \rho)\tilde{A}\left(\frac{1}{\bar{F}(x)}\right)(1 + o(1))\right].
\]
b) If $F \in D(Q_{-1/\alpha_1}, \alpha_1 > 0$ and $x_F = 1$. Furthermore, we assume its tail quantile function $U$ satisfies $1 - U \in 2RV_{-1/\alpha_1, \tau_1/\alpha_1}$ with auxiliary function $\tilde{A}$, then for $x \downarrow 0$

$$P(V(\lambda) > 1 - x) = \frac{1}{\tilde{A}(1 - x)} P(S(\lambda) > 1 - x)$$

$$\times \left[ \alpha_1 B(\alpha_1, \alpha_1 + 1) + \left( \frac{\alpha_1^2}{\tau_1} B(\alpha, \alpha_1 + 1) - B(\alpha, \alpha_1 - \tau_1 + 1) \right) \tilde{A} \left( \frac{1}{F(1 - x)} \right) \right. \right.$$ 

$$+ \left. \frac{\alpha_1}{\tau} B(\alpha_1, \alpha - \tau + 1) - B(\alpha_1, \alpha + 1) \right) A \left( \frac{1}{x} \right) (1 + o(1)) \right].$$

Here $\alpha, \tau$ and $A$ are those defined in (4.19), and $P(S(\lambda) > 1 - x)$ is given by Lemma 4.4.

**Remark 4.3.**

a) If $S$ has Beta distribution with positive parameters $a$ and $b$, then (4.17) holds for $\lambda = 0, 1$ and $\alpha_0 = a, \alpha_1 = b, \tau_0 = \tau_1 = 1,$

$$c_0 = \frac{1}{aB(a, b)}, \quad L_0(x) = -\frac{(b - 1)a}{a + 1} (1 + o(1)), \quad c_1 = \frac{1}{bB(a, b)}, \quad L_1(x) = -\frac{a - b}{b + 1} (1 + o(1)).$$

b) If $G$ has a continuous 3rd differentiable pdf $g$, then condition (4.17) holds with all $\lambda \in (0, 1)$ and

$$\alpha_\lambda = 1, \quad c_\lambda = 2g(\lambda), \quad L_\lambda(x) = \frac{g'''(\lambda)}{6g'(\lambda)} (1 + o(1)), \quad \tau_\lambda = 2.$$

c) If $S$ has Beta distribution with parameters $1/2, 1/2$ and $I_1, I_2$ are independent with mean $0$ being further independent of $S$, then $(V_1, V_2)$ is spherically distributed. And $V(\lambda) \overset{d}{=} \int_1 R^S \overset{d}{=} I_2 R \sqrt{1 - S^2}$ for all $\lambda \in [0, 1]$. Thus the tail asymptotics of $V(\lambda)$ can be directly obtained by Theorem 2.1 and Theorem 2.2 in Section 4.

## 5 Proofs

**Proof of Theorem 2.1** It follows from Breiman’s Lemma that

$$\lim_{x \to \infty} \frac{\tilde{P}(x)}{F(x)} = E \{ S^{\alpha_1} \}.$$

We consider two cases $\tau_1 < 0$ and $\tau_1 = 0$ separately. For $\tau_1 < 0$, by Lemma 5.2 of Draisma et al. (1999), for every $\epsilon > 0$, there exists $x_0 = x_0(\epsilon) > 0$ such that for all $x > x_0$ and all $s \in (0, 1)$

$$\left| \frac{\tilde{P}(x/s)}{F(x/s)} s^{\alpha_1} - s^{\alpha_1} - 1 \right| \leq \epsilon (C_1 + C_2 s^{\alpha_1} + C_3 s^{\alpha_1 - \tau_1 - \epsilon}),$$

with some positive constants $C_1, C_2$ and $C_3$ independent of $x$ and $s$. Therefore, by the dominated convergence theorem

$$\lim_{x \to \infty} \frac{1}{A(x)} \left( \frac{\tilde{P}(x)}{F(x)} - E \{ S^{\alpha_1} \} \right) = \int_0^1 \lim_{x \to \infty} \frac{\tilde{P}(x/s)}{F(x/s)} - s^{\alpha_1} - 1 \frac{dG(s)}{A(x)} = E \{ S^{\alpha_1} - \frac{1}{\tau_1} \}.$$

For $\tau_1 = 0$, note that for all $\alpha_1 > 0$, the function $f(s) = s^{\alpha_1} \log s^{-1}$ is continuous in $(0, 1]$ and $\lim_{s \to 0^+} f(s) = 0$ implies that $f(s)$ is bounded in $[0, 1]$ and $E \{ f(S) \}$ exists. Similarly to the proof of the case $\tau_1 < 0$

$$\lim_{x \to \infty} \frac{1}{A(x)} \left( \frac{\tilde{P}(x)}{F(x)} - E \{ S^{\alpha_1} \} \right) = E \{ S^{\alpha_1} \log S^{-1} \}$$

holds for the case $\tau_1 = 0$, hence the proof is complete. \hfill \Box

**Proof of Theorem 2.2** Denote $t = 1/F(x)$, noting that

$$\mathcal{H}(x) = \int_x^\infty \mathcal{G} \left( \frac{x}{y} \right) dF(y) = \int_0^\infty \mathcal{G} \left( \frac{U(t)}{U'(t)} \right) d(1 - 1/s) = t^{-1} \int_0^1 \mathcal{G} \left( 1 - \frac{U(t/s) - U(t)}{U'(t/s)} \right) ds$$

and rewrite the left-hand side of (2.20) as

$$\frac{\mathcal{H}(x)}{\mathcal{F}(x)\mathcal{G}(1 - \frac{1}{\pi(x)})} = \int_0^1 \mathcal{G} \left( 1 - \frac{U(t/s) - U(t)}{U'(t/s)} \right) ds$$
\[
\begin{align*}
&= \int_0^1 \left( \frac{U(t/s) - U(t)}{a(t)} \right) \alpha_2 L \left( \frac{U(t)}{a(t)} / \left( \frac{U(t/s) - U(t)}{a(t)} \right) \right) ds \\
&= \int_0^1 (\Theta_t(s))^{\alpha_2} \frac{L(\Xi_t(s))}{L(\varphi_t)} ds,
\end{align*}
\]

where

\[
\Theta_t(s) = q_t(s)\phi_t(s), \quad \Xi_t(s) = \frac{\varphi_t}{\Theta_t(s)}, \quad \varphi_t = \frac{U(t)}{a(t)}
\]

and

\[
q_t(s) = \frac{U(t/s) - U(t)}{a(t)}, \quad a = 1/w(U), \quad \phi_t(s) = \frac{U(t)}{U(t/s)}.
\]

Further we decompose (5.1) as

\[
\frac{H(x)}{F(x)G(1 - \frac{1}{a(x)})} - \Gamma(\alpha_2 + 1) = \int_0^1 ((q_t(s))^{\alpha_2} - \log^{\alpha_2}(1/s)) ds - \int_0^1 (q_t(s))^{\alpha_2}(1 - (\phi_t(s))^{\alpha_2}) ds
\]

\[
+ \int_0^1 (\Theta_t(s))^{\alpha_2} \left( \frac{L(\Xi_t(s))}{L(\varphi_t)} - 1 \right) ds =: I - II + III.
\]

Since (5.1) tends to \(\Gamma(\alpha_2 + 1)\) by Theorem 3.1 in Hashorva et al.(2010). The rest is to derive the convergence rates of the three terms on the right-hand side of (5.2). By Lemma 5.2 in Draisma et al.(1999), for every \(\epsilon > 0\), there exists \(t_0 = t_0(\epsilon) > 0\) such that for all \(t > t_0\) and all \(s \in (0, 1)\)

\[
\left| \frac{q_t(s) - \log(1/s)}{A(t)} - \psi(1/s) \right| \leq \epsilon(C_1 + C_3 s^{-\rho - \epsilon}),
\]

with some positive constants \(C_1\) and \(C_3\), independent of \(t\) and \(s\). Therefore, by Taylor’s expansion and the dominated convergence theorem, we have

\[
\lim_{t \to \infty} \frac{I}{A(t)} = \int_0^1 \alpha_2 \log^{\alpha_2-1}(1/s)\psi(1/s) ds = K(\alpha_2, \rho),
\]

(5.3)

with \(\psi\) and \(K(\alpha_2, \rho)\) defined in (5.2) and (5.4), respectively.

For the second term \(II\), recall that \(U \in \Pi(a)\) implies that \(U \in RV_0\) and \(\varphi_1 \to \infty\) as \(t \to \infty\). By Corollary B.2.10 of de Haan and Ferreira (2006), for all \(s \in (0, 1)\) and sufficiently large \(t\)

\[
0 \leq q_t(s) \leq cs^{-\epsilon}, \quad 0 \leq \phi_t(s) = \left( 1 + \frac{q_t(s)}{\varphi_t} \right)^{-1} \leq 1
\]

(5.4)

for some \(c > 1\) and any \(\epsilon > 0\). Hence,

\[
\frac{1 - \phi_t(s)}{1/\varphi_t} \leq q_t(s) \leq cs^{-\epsilon}.
\]

Therefore by Taylor’s expansion and the dominated convergence theorem

\[
\lim_{t \to \infty} \frac{II}{1/\varphi_t} = \alpha_2 \int_0^1 \log^{\alpha_2+1}(1/s) ds = \alpha_2 \Gamma(\alpha_2 + 2),
\]

(5.5)

Finally, we shall show below (5.6) holds for the third term \(III\)

\[
\lim_{t \to \infty} \frac{III}{A(\varphi_t)} = \frac{\Gamma(\alpha_2 - \tau_2 + 1) - \Gamma(\alpha_2 + 1)}{\tau_2}
\]

\[
= \lim_{t \to \infty} \int_0^1 (\Theta_t(s))^{\alpha_2} \left( \frac{L(\Xi_t(s))}{A(\varphi_t)} - 1 \right) ds = 0.
\]

(5.6)

Recall that \(L \in 2RV_{0, \tau_2}\) with auxiliary function \(A\), by Lemma 5.2 in Draisma et al.(1999), for every \(\epsilon > 0\), there exists \(t_0 = t_0(\epsilon) > 0\) such that for all \(\varphi_1 > t_0\), the integral of the right-hand side of (5.6) is dominated by

\[
\int_{\{s: 0 < 1, \Xi_t(s) > t_0\}} e(\Theta_t(s))^{\alpha_2}(C_1 + C_3(\Theta_t(s))^{-\tau_2} \exp(\epsilon |\log(\Theta_t(s))|)) ds
\]
\[
+ \int_{\{s, x \in (0, 1), \Xi(s) < t_0\}} \left( \Theta_t(s) \right)^{\alpha_2} \left| \frac{L(\Xi_t(s))/L(\varphi_t) - 1}{A(\varphi_t)} \right| ds \\
+ \int_{\{s, x \in (0, 1), \Xi(s) < t_0\}} \left( \Theta_t(s) \right)^{\alpha_2} \left| \frac{(\Theta_t(s))^{-\tau_2} - 1}{\tau_2} \right| ds =: J_1 + J_2 + J_3. \tag{5.7}
\]
Recall that \(5.4\) implies that \(f_t = (\Theta_t(s))^{\alpha}, s \in (0, 1)\) is integrable for all \(\alpha > 0\) and sufficiently large \(t\). Thus, \(J_1\) tends to 0 since \(\epsilon\) is arbitrarily small, whereas \(J_3\) tends to 0 due to \(\varphi_t/t_0 \to \infty\).

To deal with \(J_2\), we need two inequalities of \(L\) and \(A\) stated below in \(5.8\) and \(5.9\). Indeed, note that \(L \in 2RV_{0, \tau_2}, \tau_2 < 0\) implies that \(L\) is ultimately bounded away from 0 and

\[
L(t) = t^{\alpha_2} C(1 - 1/t) \leq t^{\alpha_2}, \quad L(t) > 1/M
\]

hold for some given \(M > 0\) and sufficiently large \(t\). By Potter bounds (cf. Proposition B.1.9 in de Haan and Ferreira (2006)), for any \(\epsilon > 0\), there exists \(t_0 = t_0(\epsilon) > 0\) such that \(\min(\varphi_t, \Xi_t(s)) > t_0\)

\[
\frac{L(\Xi_t(s))}{L(\varphi_t)} \leq c \max((\Theta_t(s))^{\alpha}, (\Theta_t(s))^{-\epsilon}),
\]

otherwise for \(\varphi_t > t_0, \Xi(s) \leq t_0\) such that

\[
\frac{L(\Xi_t(s))}{L(\varphi_t)} \leq \frac{(\Xi_t(s))^{\alpha_2}}{M t_0^{\alpha_2}}. \tag{5.8}
\]

For \(A\), note that \(|A| \in RV_{\tau_2}\) and it is ultimately decreasing. By the Karamata Representation (cf. Resnick (1987), p.17), for any given \(\delta > 0\) and \(t_0 < \varphi_t < \Theta_t(s) t_0\)

\[
|A(\varphi_t)| \geq |A(\Theta_t(s) t_0)| \geq K_2(\Theta_t(s))^{-\tau_2 - \delta}|A(t_0)|,
\]

with some \(K_2 \in (0, 1)\) a constant. Therefore, the integrand of \(J_2\) is dominated by

\[
\frac{Mt_0^{\alpha_2} + 1}{K_2|A(t_0)|}(\Theta_t(s))^{\alpha_2 - \tau_2 + \delta} \leq \frac{Mt_0^{\alpha_2} + 1}{K_2|A(t_0)|}(cs^{-\epsilon})^{\alpha_2 - \tau_2 + \delta}.
\]

So, by the dominated convergence theorem, \(J_2\) tends to 0 as \(t \to \infty\). Thus this together with the proved results for \(J_1\) and \(J_3\) concludes the proof of \(5.7\), and thus \(5.8\) holds. Theorem \(2.3\) follows from \(5.4\), \(5.6\) and \(5.8\). \(\square\)

**Proof of Corollary 2.4.** For \(a = 1/w(U)\) the first-order auxiliary function of \(U\), it follows from Theorem B.3.1 in de Haan and Ferreira (2006) that \(a \in 2RV_{0, \rho}, \rho \leq 0\) with auxiliary function \(\tilde{A}\). Thus for sufficiently large \(x\)

\[
w \left( \frac{x + \frac{\delta}{w(x)}}{w(x)} \right) = 1 - \frac{e^{\rho z} - 1}{\rho} \tilde{A} \left( \frac{1}{F(x)} \right) (1 + o(1)) \tag{5.10}
\]

holds for all \(z \in \mathbb{R}\) (here \((e^{\rho z} - 1)/\rho\) is interpreted as \(z\) for \(\rho = 0\)). Note that \(\tilde{G}(1 - 1/x) \in 2RV_{-\alpha_2, \tau_2}\) and \(|A| \in RV_{\tau_2}\) yield that

\[
\tilde{G} \left( 1 - \frac{\eta(x + z/w(x))}{\eta(x)} \right) \leq \left( \frac{\eta(x + z/w(x))}{\eta(x)} \right)^{-\alpha_2} \left( 1 + \frac{\eta(x + z/w(x))}{\eta(x)} \right)^{\tau_2} - 1 \tilde{A}(\eta(x)(1 + o(1)))
\]

\[
= \left( \frac{x + z/w(x)}{x} \frac{w(x + z/w(x))}{w(x)} \right)^{-\alpha_2} \left[ 1 + o(1/\eta(x)) + o(\tilde{A}(1/F(x))) \right] = 1 - \frac{\alpha_2 z}{\eta(x)} - \frac{\rho z}{\eta(x)} - \frac{\rho z}{\eta(x)} \tilde{A} \left( \frac{1}{F(x)} \right) \tag{5.11}
\]

Recall that \(U \in 2ERV_{0, \rho}\) with auxiliary function \(\tilde{A}\),

\[
\tilde{F} \left( \frac{x + \frac{\delta}{w(x)}}{F(x)} \right) = e^{-x} \left( 1 + \psi(e^{-z})\tilde{A} \left( \frac{1}{F(x)} \right) \right). \tag{5.12}
\]

The claim \(2.7\) follows from \(2.6\), \(5.10\), \(5.11\), \(5.12\) and the fact that

\[
\lim_{z \to \infty} \eta(x) \tilde{A} \left( \frac{1}{F(x)} \right) = \lim_{z \to \infty} \frac{\tilde{A}(x)}{a(x)/U(x)} = 0 \tag{5.13}
\]
for $\rho < 0$ (cf. Lemma B.3.16 in de Haan and Ferreira (2006)).

Using (5.13) and \( h(h^{-1}(t)) \sim t \) for \( h = \frac{1}{\Pi} \) for (2.7), one can verify that \( U_X \in 2ERV_{0,0} \) with auxiliary functions stated by (2.8).

\[ \square \]

**Proof of Corollary 2.5** First, note that 

\[ U(t) = V^\rho (\log t) = (\log t)^\rho (\log t) \]

and thus

\[ U(tx) = V^\rho (\log tx) = (\log t)^\rho (\log t) \left( 1 + \frac{\log x}{\log t} \right) \theta (\log t (1 + \log x/\log t)) \]

\[ = U(t) \left( 1 + \theta \frac{\log x}{\log t} + \frac{\theta(\theta - 1) \log^2 x}{2 \log^2 t} (1 + o(1)) \right) \left( 1 + b(\log t) \frac{(1 + \log x/\log t)^{\rho^*} - 1}{\rho^*} (1 + o(1)) \right). \]

Therefore, \( U \in 2ERV_{0,0} \) with auxiliary functions \( a \) and \( \tilde{A} \) as

\[ a(t) = \frac{\theta + b(\log t)}{\log t} U(t), \quad \tilde{A}(t) = \frac{\theta - 1}{\log t}. \]

This implies that

\[ \eta(x) = \frac{x}{a(1/F(x))} = \frac{V(x)}{\theta + b(V(x))}, \quad \tilde{A} \left( \frac{1}{F(x)} \right) = \frac{\theta - 1}{V(x)}. \]

By Theorem 2.8

\[ \Pi(x) = \frac{\exp(-V(x))}{(1 - \frac{1}{V(x)}) \Gamma(\alpha_2 + 1)} \left( 1 + \theta \frac{\log x}{\log t} \right) \theta^{\alpha_2} \]

\[ \times \left[ 1 + \left( \frac{\Gamma(\alpha_2 - \tau_2 + 1)}{\alpha_2 + 1} \right) \left( \frac{\eta(x)}{V(x)} \right)^{\tau_2} A(V(x)) - \frac{(\theta + b(V(x)) \alpha_2 (\alpha_2 + 1)}{2V(x)} + \frac{(\theta - 1) \alpha_2 (\alpha_2 + 1)}{2V(x)} \right) (1 + o(1)) \right] \]

\[ = \exp(-V(x)) \left( 1 - \frac{1}{V(x)} \right) \Gamma(\alpha_2 + 1) \theta^{\alpha_2} \]

\[ \times \left[ 1 + \left( \frac{\alpha_2}{\theta} b(V(x)) + \frac{\Gamma(\alpha_2 - \tau_2 + 1)}{\theta^2 \Gamma(\alpha_2 + 1)} \right) \left( \frac{\eta(x)}{V(x)} \right)^{\tau_2} A(V(x)) - \frac{(\theta + 1) \alpha_2 (\alpha_2 + 1)}{2V(x)} \right) (1 + o(1)) \right] \]

\[ = \exp(-V(x)) (V(x))^{-\alpha_2} L^\rho(V(x)), \]

where (5.15) is due to (5.14) and \( G(1 - 1/x) \in 2RV_{\alpha_2,\tau_2} \) with auxiliary function \( A \). Clearly, \( L^\rho \) is a slowly varying function. Therefore, let the right-hand side of (5.16) equal to 1/s, and solve the equation of \( x \), then \( V(x) \sim \log s \) and

\[ U_X(s) = V^\rho \left( \log \frac{s^{L^\rho(V(x))}}{(V(x))^{\alpha_2}} \right) = \left( \log s - \alpha_2 \log V(x) \left( 1 - \frac{\log L^\rho(V(x))}{\alpha_2 \log V(x)} \right) \right)^\theta \ell\left( \log s - \alpha_2 \log V(x) \left( 1 - \frac{\log L^\rho(V(x))}{\alpha_2 \log V(x)} \right) \right) \]

\[ = (\log s - \alpha_2 \log s(1 + o(1)))^\theta t(\log s)(1 + o(\log \log s/\log s)). \]

The last step is due to \( \ell \in 2RV_{0,\rho^*} \) and the property of slowly varying function: \( \log L^\rho(V(x))/\log V(x) \to 0 \) (see Bingham et al. (1987)). Hence

\[ \Pi(x) = \exp(-V^\rho(x)), \quad (V^\rho)^{\rightarrow}(x) = x^\theta \left( 1 - \alpha_2 \frac{\log x}{x} \right)^\theta \ell^\alpha(x). \]

Thus the claim in Corollary 2.5 follows from \( \ell^\alpha \in 2RV_{0,\rho^*} \) with \( \rho^* = \max(\rho^* - 1) \) and auxiliary function

\[ b^\alpha(x) = b(x) + \frac{\theta \alpha_2 \log x}{x}. \]

\[ \square \]
Proof of Theorem 2.6 First, by arguments similar to the case $F \in D(Q_0)$ (cf. (5.14)), we have

$$\frac{\mathcal{P}(x)}{F(x)} = \int_0^1 (\Theta_t(s))^{\alpha_2} \frac{L\left(\frac{\varphi_t(s)}{\Theta_t(s)}\right)}{L(\varphi_t)} \, ds,$$

with $t = 1/F(x), x = U(t)$ and

$$\Theta_t(s) = q_t(s)\varphi_t(s), \quad \varphi_t = \frac{1}{1 - U(t)} \quad \text{with} \quad \varphi_t(s) = \frac{1}{U(t/s)}, \quad q_t(s) = \frac{U(t/s) - U(t)}{1 - U(t)}.$$

Next

$$\frac{\mathcal{P}(x)}{F(x)} - \alpha_1 B(\alpha_1, \alpha_2 + 1) = \int_0^1 (q_t(s))^{\alpha_2} - (1 - s^{1/\alpha_1})^{\alpha_2} \, ds$$

$$+ \int_0^1 (q_t(s))^{\alpha_2}((\varphi_t(s))^{\alpha_2} - 1) \, ds + \int_0^1 (\Theta_t(s))^{\alpha_2} \left(\frac{L\left(\frac{\varphi_t(s)}{\Theta_t(s)}\right)}{L(\varphi_t)} - 1\right) \, ds$$

$$=: I + II + III. \tag{5.17}$$

It remains thus to derive the convergence rate of each term in (5.17). By Lemma 5.2 in Draisma et al. (1999), for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that for all $t > t_0$ and all $s \in (0, 1)$

$$\left|q_t(s) - (1 - s^{1/\alpha_1}) + s^{1/\alpha_1}s^{-\tau_1/\alpha_1} - 1\right| \leq \epsilon(C_1 + C_2 s^{1/\alpha_1} + C_3 s^{(1-\tau_1)/\alpha_1-\epsilon}),$$

with some positive constants $C_1, C_2$ and $C_3$, independent of $t$ and $s$. Therefore, by Taylor’s expansion and the dominated convergence theorem

$$\lim_{t \to \infty} \frac{I}{A(t)} = -\alpha_2 \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2-1} s^{1/\alpha_1}s^{-\tau_1/\alpha_1} - 1 \, ds = -\frac{\alpha_2 \alpha_1^2}{\tau_1}(B(\alpha_2, \alpha_1 - \tau_1 + 1) - B(\alpha_2, \alpha_1 + 1)). \tag{5.18}$$

Here, (5.18) for $\tau_1 = 0$ is understood as

$$-\alpha_2 \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2-1} s^{1/\alpha_1} \lim_{\tau_1 \to 0} s^{-\tau_1/\alpha_1} - 1 \, ds = \lim_{\tau_1 \to 0} -\frac{\alpha_2 \alpha_1^2}{\tau_1}(B(\alpha_2, \alpha_1 - \tau_1 + 1) - B(\alpha_2, \alpha_1 + 1))$$

(cf. Corollary 4.4 in Mao and Hu (2012(a))). For $II$, note that $q_t(s) \in (0, 1), \varphi_t \to \infty$ and thus for all $s \in (0, 1)$

$$0 \leq \frac{\varphi_t(s) - 1}{1/\varphi_t} = (1 - (1 - q_t(s))/\varphi_t)^{-1} - 1 = \frac{1 - q_t(s)}{1 - (1 - q_t(s))/\varphi_t} \leq \frac{1}{1 - 1/\varphi_t} \to 1$$

as $t \to \infty$. So, by Taylor’s expansion and the dominated convergence theorem

$$\lim_{t \to \infty} \frac{II}{1/\varphi_t} = \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2} s^{1/\alpha_1} \, ds = \alpha_2 B(\alpha_1 + 1, \alpha_2 + 1). \tag{5.19}$$

Now we consider the third term $III$. By Lemma 5.2 in Draisma et al. (1999), for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that for all $\varphi_t > t_0$ and all $s \in (0, 1)$

$$\left|\Theta_t(s)^{\alpha_2} \left(\frac{L\left(\frac{\varphi_t(s)}{\Theta_t(s)}\right)/L(\varphi_t) - 1}{A(\varphi_t)} \right) \right| \leq \epsilon(C_1 + C_2 \Theta_t(s)^{\alpha_2} + C_3 (\Theta_t(s))^{\alpha_2 - \tau_2 - \epsilon}) \leq \epsilon(C_1 + C_2 + C_3).$$

The last step is due to $\Theta_t(s) \leq 1$ for all $s \in (0, 1)$ and $t > 0$. Hence, by the dominated convergence theorem

$$\lim_{t \to \infty} \frac{III}{A(t)} = \int_0^1 \lim_{t \to \infty} \left(\Theta_t(s)^{\alpha_2} \left(\frac{\Theta_t(s)^{-\tau_2} - 1}{\tau_2}\right) \right) \, ds$$

$$= \int_0^1 (1 - s^{1/\alpha_1})^{\alpha_2} (1 - s^{1/\alpha_1})^{-\tau_2} - 1 \, ds = \frac{\alpha_1}{\tau_2}(B(\alpha_1, \alpha_2 - \tau_2 + 1) - B(\alpha_1, \alpha_2 + 1)). \tag{5.20}$$
The claim follows from (5.18), (5.19) and (5.20). □

**Proof of Lemma 4.1** We only give the proofs of the case \( \lambda \in (0, 1) \). The other cases are left to the readers and one can verify it by the similar arguments. Clearly, for \( \lambda \in (0, 1) \), \( S(\lambda) \leq 1 \) and it is bounded away from unity unless \( I_1 = I_2 = 1 \), and when the event \( \{I_1 = I_2 = 1\} \) occurs, \( S(\lambda) \uparrow 1 \) if and only if \( |S - \lambda| \downarrow 0 \). For small \( x > 0 \), the event

\[
\{S(\lambda) > 1 - x\} = \{(S - \lambda)^2 + 2\lambda x S < 2x - x^2\}
\]

is equivalent to

\[
(S - \lambda)^2 < 2x((1 - \lambda^2) - \lambda \sqrt{2x(1 - \lambda^2)}(1 + o_p(1)))
\]

Consequently, the claim follows from (4.17). □
6 Appendix

This appendix includes two tables. Table 1 contains Weibull tail distributions satisfying the second-order regular varying conditions and Table 2 shows several distributions in maximum domain attraction of the Fréchet distribution, the Gumbel distribution and the Weibull distribution in the second-order framework.

Table 1: Weibull tail distributions

| Weibull tail distributions | Tail $F(x)$ or pdf $f(x)$ | $\theta$ | $\rho$ | $b(x)$ |
|---------------------------|---------------------------|----------|--------|--------|
| Gamma ($\Gamma(\alpha, \lambda)$) | $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x}, \lambda, \alpha > 0, \alpha \neq 1$ | 1        | -1     | $(1 - \alpha) \log x / x$ |
| Absolute Normal ($|N(0, 1)|$) | $f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$ | 1/2      | -1     | $\log x / (4x)$ |
| Weibull ($W(\beta, c)$) | $F(x) = \exp(-cx^\beta), c, \beta > 0$ | 1/\beta  | -\infty | 0 |
| Perturbed Weibull ($PW(\beta, \alpha)$) | $F(x) = e^{-x^\beta(C + Dx^{-\alpha})}, \alpha, \beta, C > 0, D \in \mathbb{R}$ | 1/\beta  | -\alpha/\beta | $\frac{\alpha D}{\beta^2} C^\alpha/\beta^{-1} x^{-\alpha/\beta}$ |
| Modified Weibull ($MW(\beta)$) | $Y \log Y \sim F, Y \sim W(\beta)$ | 1/\beta  | 0      | $1 / \log x$ |
| Benktander II ($BII(\beta, \lambda)$) | $F(x) = x^{-(1 - \beta)} \exp(-\frac{\lambda}{\beta}(x^\beta - 1)), \lambda > 0, 0 < \beta < 1$ | 1/\beta  | -1     | $(1 - \beta) \log x / (\beta^2 x)$ |
| Extended Weibull ($EW(\beta, \alpha)$) | $F(x) = r(x) \exp(-x^\beta), \beta \in (0, 1), r \in RV_{-\alpha}, \alpha \in \mathbb{R}$ | 1/\beta  | -1     | $\alpha \log x / (\beta^2 x)$ |
| Logistic | $F(x) = \frac{2}{1+e^{-x}}$ | 1        | -1     | $-(\log 2) / x$ |
| Gumbel ($G(\mu)$) | $F(x) = 1 - \exp(-\exp(\mu - x)), \mu \neq 0$ | 1        | -1     | $-\mu / x$ |

Weibull tail distribution: $F(x) = \exp(-V(x)), V^{-1}(x) = x^\theta \ell(x)$ and $\ell \in 2RV_{\theta, \rho}$ with auxiliary function $b$. 
Table 2: Risks satisfying the second-order regular variation conditions

| Fréchet attraction | Tail $\overline{F}(x)$ or pdf $f(x)$ | $\alpha$ | $\tau$ | $A(x)$ |
|--------------------|--------------------------------------|--------|------|--------|
| Pareto | $F(x) = \left( \frac{\theta}{\tau+x} \right)^\alpha$, $\theta, \alpha > 0$ | $\alpha$ | $-1$ | $\alpha \theta / x$ |
| Fréchet | $F(x) = 1 - \exp(-x^{-\alpha})$ | $\alpha$ | $-\alpha$ | $\alpha x^{-\alpha}/2$ |
| Burr | $F(x) = (1 + x^b)^{-\alpha}$ | $ab$ | $-b$ | $ab x^{-b}$ |
| Hall-Weiss | $F(x) = \frac{1}{2} x^{-\alpha}(1 + x^\tau)$, $\alpha > 0, \tau < 0$ | $\alpha$ | $\tau$ | $\tau x^\tau$ |
| $F(m, n)$ | $f(x) = \frac{1}{n!} \left[ \frac{m}{n} \right]^{m/2} x^{m/2-1} (1 + \frac{m x}{n})^{-(m+n)/2}$ | $n/2$ | $-1$ | $\frac{(m+n) n^2}{2m(n+2)x}$ |
| Log-gamma | $f(x) = \frac{1}{\Gamma(\beta/2)} \{ \log x \}^{\beta-1} x^{-\alpha-1}$, $\alpha, \beta > 0$ | $\alpha$ | $0$ | $(\beta - 1)/\log x$ |
| Inv-gamma | $f(x) = \frac{1}{\Gamma(\beta)} x^{-\alpha-1} e^{-\beta/x}$, $\alpha, \beta > 0$ | $\alpha$ | $-1$ | $\frac{(\alpha+1) x}{\beta}$ |
| Absolute t | $f(x) = \frac{2^{v/2}}{\sqrt{\pi} \Gamma((v+1)/2)} (1 + x^{2/v})^{-(v+1)/2}$, $v \in \mathbb{N}$ | $v$ | $-2$ | $\frac{v^2(v+1)}{(v+2)x^2}$ |

| Weibull attraction | Tail $\overline{F}(xF - 1/x)$ or pdf $f(x)$ | $\alpha$ | $\tau$ | $A(x)$ |
|-------------------|-------------------------------------------|--------|------|--------|
| Beta | $f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$, $a, b > 0$ | $b$ | $-1$ | $(a \neq 1)$ |
| Reverse-Burr | $\overline{F}(xF - 1/x) = (1 + x^b)^{-a}$ | $ab$ | $-b$ | $ab x^{-b}$ |
| Extreme value Weibull | $\overline{F}(xF - 1/x) = 1 - \exp(-x^{-a})$ | $\alpha$ | $-\alpha$ | $\alpha x^{-\alpha}/2$ |

| Gumbel attraction | Tail $\overline{F}(x)$ or pdf $f(x)$ | $\rho$ | $a(x)$ | $A(x)$ |
|-------------------|--------------------------------------|--------|-------|--------|
| Gamma | $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $\lambda, \alpha > 0$ | $0$ | $\left( 1 + \frac{\alpha-1}{\log x} \right)/\lambda$ | $(1 - \alpha)/\log^2 x$ |
| Absolute Normal | $f(x) = \sqrt{\frac{\pi}{2}} e^{-x^2/2}$ | $0$ | $2 \log(2x) - 1/2 \log x$ | $-1/2 \log^2 x$ |
| Log-normal | $f(x) = \frac{1}{\sqrt{2\pi x}} \exp\left( -\frac{x^2}{2} \right)$ | $0$ | $\log(\exp(U_1(x)))$ | $1/\sqrt{2 \log x}$ |
| Logistic | $F(x) = \frac{1}{1 + e^{-x}}$ | $0$ | $1$ | $1/(2x)$ |
| Truncated Gumbel | $f(x) = \frac{1}{1-e x^{\alpha}}$ | $-1$ | $1$ | $(1 - e^{-1})/(2x)$ |
| Exponential with finite $x_F$ | $\overline{F}(x) = \exp(-\frac{c}{x_F x} + \frac{c}{x_F})$, $c > 0, x_F > 0$ | $0$ | $\frac{\log x + c / x_F}{x_F^{1+c}}$ | $-2/\log x$ |
| Weibull | $\overline{F}(x) = \exp(-c x^\beta)$, $c > 0, \beta \in (0, 1)$ | $0$ | $\frac{U_2(x)}{2 \beta \log x}$ | $1/(\beta + 1)/\log x$ |
| Benktander I | $\overline{F}(x) = \left( 1 + \frac{2\beta}{\alpha} \log x \right) \exp(-c x^\beta) - (1 + \beta) \log U(x)$ | $0$ | $U_1(x)$ | $1/(2 \sqrt{\beta \log x})$ |
| Benktander II | $\overline{F}(x) = x^{-(1-\beta)} \exp(-c x^{\beta - 1})$, $a > 0, 0 < \beta < 1$ | $0$ | $a^\beta(x)$ | $(1 - \beta)/\log x$ |

For Fréchet attraction, $\overline{F} \in 2RV_{-\alpha, \tau}$ with auxiliary function $A$. For Weibull attraction, $\overline{F}(xF - 1/x) \in 2RV_{-\alpha, \tau}$ with auxiliary function $A$ and a finite upper endpoint $x_F$. For Gumbel attraction, the tail quantile function $U \in 2ERV_{0, \rho}$ with the first-order auxiliary function $a$ and the second-order auxiliary function $A$. 

| $a^\times(x) = \frac{1-(1-\beta)(\alpha/\beta + \log x)}{\beta(\alpha/\beta + \log x)} U(x)$, $U(x) = \left( \frac{\alpha}{\beta} \log x + (\beta/\alpha + \log x) \right)^{1/\beta}$ |
| $U_1(x) = \sqrt{2 \log x} - \frac{\log(4x^2 \log x)}{2 \sqrt{2 \log x}}$, $U_2(x) = \exp(-\frac{\alpha}{2 \beta} + \frac{\log x + \log(4x^2 \log x)}{2 \sqrt{2 \log x}} + (\alpha+1)/\beta)$ |
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