EXTREMAL PROBLEMS FOR SURFACES OF PRESCRIBED TOPOLOGICAL TYPE (1)

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1. INTRODUCTION

The study of extremal properties of surfaces with bounded smooth curvature shows that geometrical properties in the large are solidly connected to their topological structure. We take up some of these questions here.

Notation: $F_R$ – all compact $C^2$ $n$-dimensional surfaces contained in $E^{n+1}$, $n \geq 2$, with principal radii of curvature all $\geq R$. \cite{L1, L2, L3, L4}; let

$$\kappa(F_R) = \inf \{ \rho : \text{there is a ball of radius } \rho \text{ interior to a surface in } F_R \}.$$ 

It was shown that $\kappa(F_R) = \kappa_0 R$, where $\kappa_0 = 2/\sqrt{3} - 1 \approx 0.155$.

The sharpness of this bound was shown by constructing examples of surfaces $F(\epsilon) \in F_R$, containing spheres of radii $\kappa_0 R + \epsilon$, for $\epsilon > 0$ arbitrarily small. The surfaces $F(\epsilon)$ have nonzero Betti numbers and bound a body of complicated topological structure; precisely, $F(\epsilon)$ is homeomorphic to a boundary $S_n^k$ of a ball with $k$ handles $h_n^{k+1}$, but bounds a solid not homeomorphic to $h_n^{k+1}$ (precisely, see below, p. 188). There remains the question of whether the above bound can be improved if instead of $F_R$ some subset of $F_R$ is considered, consisting of surfaces of sufficiently simple topological structure of sufficiently simple imbedding in $E^{n+1}$. Some results in this direction were already presented by us in the Second All-Union topological conference in Tbilisi in 1959 \cite{LF}.

We introduce notation: For $F \in F_R$ let $\kappa(F)$ be the radius of a maximal ball interior to $F$. $M \subset F_R$, $\kappa(M) = \inf_{F \in M} \kappa(F)$.

Let $S$ be the subset of surfaces in $F_R$ homeomorphic to $S^n$; $H_k$ those homeomorphic to a sphere with $k$ handles, $S_n^k$; $H_0^k$ those which bound a solid ball with $k$ handles, $h_k^{n+1}$; $T^{0}$ those which bound a solid toroidal ring, cf. §9, part 1.

$$\kappa_1 = \sqrt{3/2} - 1 \approx 0.2246.$$ 

Then

**Theorem 1.** If the first Betti number of $F \text{ mod } 2$ is zero, then $\kappa(F) \geq \kappa_1 R$.

In case $H_1(F, Z_2) \neq 0$ we turn to the universal covering solid $T$ of the boundary $F$; in connection with this a condition is included on the homotopy type.

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\footnotesize
Russian version in Siberian Math. J., Vol 4, 1963, pp. 145-176. Translation and remarks enclosed in brackets [ ] are by Richard L. Bishop, University of Illinois at Urbana-Champaign. Many parts have been abbreviated and in some case alternative proofs(?) were devised emphasizing intuition. Horizontal lines indicate original pages.
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Theorem 2. Let $F \in F_R$. If the homomorphism $h : \pi_1(F) \to \pi_1(T)$ induced by the inclusion of $F$ in $T$ is an isomorphism, and $\pi_2(T) = 0$, then $\kappa(F) \geq \kappa_1 R$.

For $n = 2$ the condition of Theorem 2 can be weakened.

Theorem 3. Let $n = 2$, $F \in F_R$, and $h : \pi_1(F) \to \pi_1(T)$ be onto. Then $\kappa(F) \geq \kappa_1 R$.

From Theorems 1, 2, in combination with corresponding examples:

Theorem 4. $\kappa(S) \geq \kappa_1 R$; $\kappa(H^0_k) \geq \kappa_1 R$; $\kappa(T^0) \geq \kappa_1 R$.

For $n = 2$ these inequalities reduce to equality.

Examples proving the second assertion of Theorem 4 will be constructed in the second part of this work; Theorems 1 – 3 and the first assertion of Theorem 4 are proved in this first part.

Sharp bounds in Theorem 4 for $n > 2$ are unknown. We note that $\kappa(H_k) = \kappa_0 R$, $k = 1, 2, \ldots$, cf. [L3], Introduction. This shows that a surface homeomorphic to a sphere with $k$ handles and bounding a solid “sufficiently correctly” in a topological sense contains a ball of radius $\kappa_1 R$; but surfaces can be constructed for which the topological type of the body bounded is “incorrect”, which contain only balls of radius differing from $\kappa_0 R$ by an arbitrarily small amount. The “critical numbers” $\kappa_0$ and $\kappa_1$ have a simple geometrical meaning: $\kappa_0 R$ is the radius of the greatest circle in the plane included between three tangent circular arcs of radius $R$; $\kappa_1 R$ is the radius of the greatest ball in $E^3$ included between four tangent spheres of radius $R$.

In [LF] the equation $\kappa(S) = \kappa_1 R$ was published for the case $n = 2$ of Theorem 3; there also was indicated the possibility of generalizing these results for any $n$.

In this work we depend on geometric methods developed in [L3] and in a series of results of the work [L3] assumed to be known.

§2 is carried out with a purely geometric character. In it is established Lemma 6, from which the proofs of our theorems upon establishing that the multiplicity of the central set $Z$ is greater than three (cf. [L3, p. 225 (3:5)]). As proved in [L3], the multiplicity of $Z$ must be greater than 2; consequently, it remains to obtain conditions on $F$ and $T$ excluding multiplicity 3 and that it then follows that $\kappa(F) \geq \kappa_1 R$. In §3 the local structure of $Z$ is studied under the assumption that the multiplicity is 3. In §4 it is proved that $Z$ (in the case of multiplicity 3) has a topological structure defined there and called a 3-complex.

In §§5 - 7 the topological properties of a 3-complex are studied, abstracting from the fact that a 3-complex is a central set of $T$; in these paragraphs only the basic topological properties of the configuration $\{F, T, Z\}$ are used which are recounted at the beginning of §4. In the results it is clarified what the topological conditions are needed on $F, T$ in order that a 3-complex $Z$ will fail to exist (Lemma 14). The proof of our theorems are completed in §8 by combining the results of §4 (cf. above) with Lemma 14.

2. Geometrical lemmas

1. Let $g_1, \ldots, g_k$ be unit vectors in $E^{n+1}$, $\beta$ = the minimum angle between pairs of them, and $a^{n+1}(k)$ the supremum of such $\beta$ (cf. [L3, p. 226]). We need
Lemma 5. \(\alpha^{n+1}(4) = 2 \csc^{-1} \sqrt{3/2}.\) (Equality with \(\beta\) occurs for the vectors which go from the center of a regular tetrahedron to its vertices.)

[A proof due to Reshetnyak is given.]

2. Let \(F^n \in F_R\) bound a body \(T^{n+1}\); we designate by \(Z\) the central set [cut locus] of \(F^n\) [LF, p. 224]. In the following it is assumed everywhere that \(F^n\) is a flattened surface, and consequently, \(Z\) has multiplicity \(> 2\) (cf. [L3, pp. 206, 225]). Such assumptions do not limit the generality of considerations, since for nonflattened \(F^n\) the results of this work are evident, but for flattened ones the multiplicity of \(Z\) is \(> 2\). ([L3, pp. 231-232]).

Lemma 6. If the multiplicity of \(Z\) is \(> 3\), then \(F^n\) contains a sphere of radius \(\kappa_1 R\).

[In the proof Lemma 5 is applied to 4 unit vectors going from a point of multiplicity \(\geq 4\) along lines normal to \(F^n\).]

3. Local structure of the central set

[Standard properties of the cut locus of multiplicity 3 are developed. Cf. Ozols paper on that subject. They are the properties abstracted as a normally imbedded 3-complex in §4.]

4. Triangulations and 3-complexes

p. 148.

1. Continuing we will need some properties of \(Z\) shared with \(\tilde{Z}\), the covering of \(Z\) in the universal covering \(\tilde{T}\) of \(T\). To avoid repetition and provide convenient reference we formulate a 3-complex \(Z^n\) in \(T^{n+1}\) as satisfying:

1): \(Z^n\) is an \(n\)-dimensional locally finite polyhedron, triangulated by \(\tau\).
2): \(Z^n\) contains a subcomplex \(Z^{n-1}\), decomposed into a finite or countable nonoverlapping union of \((n-1)\)-manifolds \(Z^{n-1}_i\).
3): \(Z^n \setminus Z^{n-1}\) is a finite or countable union of \(n\)-dimensional manifolds.
4): Each \((n-1)\)-simplex of \(Z^{n-1}\) is a face of exactly 3 \(n\)-simplices of \(Z^n\).

We say that the 3-complex \(Z^n\) is normally imbedded in \(T^{n+1}\) if 5)–12) as follows hold.

5): \(T^{n+1}\) is an \((n + 1)\)-manifold with boundary \(F^n\). \((T^{n+1}\) is not generally compact, \(F^n\) not generally connected.)
6): \(Z^n\) is a closed subset of \(T^{n+1} \setminus F^n\).
7): The triangulation \(\tau\) is extended to one of \(T^{n+1}\), \(\tau^0\), for which \(F^n\) is a subcomplex.
8): \(Z^n\) is a deformation retract of \(T^{n+1}\) by \(\varphi_t : T^{n+1} \rightarrow T^{n+1}\) with \(\varphi_1 : T^{n+1} \rightarrow Z^n\) and \(\varphi = \varphi_1 | F^n\) is simplicial.
9): \(F^n\) is a deformation retract of \(T^{n+1} \setminus Z^n\) by \(\psi_t : T^{n+1} \setminus Z^n \rightarrow T^{n+1} \setminus Z^n\), deforming the identity \(\psi_0\) to \(\psi_1 = \psi : T^{n+1} \setminus Z^n \rightarrow F^n\).
10): \(\varphi\) is a 2-fold covering on \(\varphi^{-1}(Z^n \setminus Z^{n-1})\).
11): \(\varphi\) is a 3-fold covering on \(\varphi^{-1}(Z^{n-1})\). Moreover, each \(Q \in Z^{n-1}\) has a neighborhood \(W\) such that \([Z^n \cap W\) is a triad bundle over \(Z^{n-1}\).]
12): The $Z^n$-star (closed) of each vertex $Q$ in $Z^n$ belongs to a neighborhood $W(Q)$ evenly covered by $\varphi$ (cf. (10)) if $Q \in Z^n \setminus Z^{n-1}$, or decomposed as in 11) if $Q \in Z^{n-1}$. In each component of $Z^n \setminus Z^{n-1}$ there is at least one vertex $Q$ for which the closed star doesn’t meet $Z^{n-1}$.

p. 153.

Remark. The properties are not independent; for example 4) follows from 11).

2.

Lemma 7. If the central set $Z$ of a body $T^{n+1}$ of $(n+1)$-dimensional Euclidean space, bounded by a surface $F^n \in F^n_R$, has multiplicity 3, then $Z^n$ is a 3-complex normally imbedded in $T^{n+1}$.

For the triangulation use the methods of Whitney [W, pp. 175-191].

p. 154.

The rest of the proof has been set up by the preceding material.

In the continuation the triangulation of $T^{n+1}$ is assumed to extend triangulations of $Z^n, F^n$ so that $\varphi : F^n \to Z^n$ is simplicial.

3. We construct for the polyhedron $T^{n+1}$ of part 2 the universal covering $\kappa : \tilde{T}^{n+1} \to T^{n+1}$. $\tilde{F}^n = \kappa^{-1}(F^n)$ is the boundary of the manifold $\tilde{T}^{n+1}$, $\tilde{Z}^n = \kappa^{-1}(Z^n)$ is the universal covering of $Z^n$, and the deformation retracts $\varphi, \psi$ can be lifted.

p. 155.

The triangulation can be lifted too, so

Lemma 8. $\tilde{Z}^n$ is a 3-complex normally imbedded in $\tilde{T}^{n+1}$.

5. Coverings in 3-complexes

1. Consider a normally imbedded 3-complex $Z^n \subset T^{n+1}$. Denote connected components by subscripts: $Z^n_j, Z^n_k, F^n_j, F^n_k$. The closures of the $n$-dimensional ones are subcomplexes. $F^{n-1} = \varphi^{-1}(Z^{n-1})$, all have triangulated closures.

Lemma 9. 1): If $P \in F^n \setminus F^{n-1}$ then $\varphi(P)$ is a double point of $Z^n$.

2): If $P \in F^n \setminus F^{n-1}$, then $\varphi(P)$ is a triple point.

3): For each $F^{n-1}_j$, $\varphi(F^{n-1}_j)$ coincides with some $Z^{n-1}_k$ and $\varphi : F^{n-1} \to Z^{n-1}_k$ is a covering.

4): For each $Z^{n-1}_k$ there is at least one $F^{n-1}_j$ such that $k = j$.

5): If $F^{n-1}_j$ is oriented for all $j$ such that $k = j$, then so is $Z^{n-1}_k$.

Of these only 5) seems to need explaining. Since $\varphi : \varphi^{-1}(Z^{n-1}_k) \to Z^{n-1}_k$ is a 3-fold covering, the restrictions to components of $\varphi^{-1}(Z^{n-1}_k)$ must be coverings whose multiplicities add up to 3. One of the multiplicities must be odd (1 or 3), so that $Z^{n-1}_k$ is oriented.

2. [In this part some combinatorics of simplices are developed. It is a clumsy but precise way of getting the essential properties of tubular neighborhoods of the $Z^{n-1}_j$. I believe a better alternative is to use cells rather than simplices, and adapt the cells to the local product structure of the triad bundle.]

p. 157.

3. [A similar development is given for tubular neighborhoods of $Z^n_t$ in $T^{n+1}$.]
4. [More combinatorics.]

5. [The idea of the holonomy of the triad bundle is pursued using the combinatorics of the previous parts. A component $Z_j^{n-1}$ is said to be a manifold of the first class if the holonomy is trivial. It is said to be of the second class if the holonomy consists of a group of order 2, so that two arms of the triad can be transposed and neither is connected to the third arm. If the holonomy group is transitive on the three arms, it is said to be of the third class.]

The finer classification of the third class into those with holonomy the alternating subgroup of the three arms and those with holonomy all permutations of the three arms is not discussed. Probably the latter is ruled out later by orientability considerations, along with those of the second class.]

6. Basic topological lemmas

1. We consider homology groups $H_q(M, G)$ using $G = J$, the integers, and $G = J_2$, the integers mod 2. For infinite but locally finite complexes there are further homology theories: $H_q^{tinf}(M, G)$, the homology of finite chains, and $H_q^{inf}(M, G)$, the homology of infinite chains. The basic reference is [E, §9]. The symbol $\sim$ is used to denote “homologous”.

**Lemma 10.** If $H_n^{inf}(T^{n+1}, J) = 0$ and all $Z_j^{n-1}$ are orientable, then manifolds of the third class don’t exist.

**Proof.** Since $Z^n$ is a deformation retract of $T^{n+1}$, we also have that $H_n^{tinf}(Z^n, J) = 0$. Since $Z_j^{n-1}$ is orientable, it is a cycle for a chosen orientation. But then it must be a boundary in $Z^n$, $Z_j^{n-1} = \partial c^n$ for some $n$-chain of $Z^n$.

If $Z_j^{n-1}$ is of the third class, then any $[n$-cell] adjacent to $Z_j^{n-1}$ [has a coefficient in $c^n$ which must propagate to adjacent cells in a tubular neighborhood of $Z_j^{n-1}$, continuing to all of the cells adjacent to $Z_j^{n-1}$. If the $n$-manifold formed by these cells is orientable, then the boundary of $c^n$ must have every $(n - 1)$-cell of $Z_j^{n-1}$ with coefficient which is a multiple of 3. If the $n$-manifold is nonorientable, then the holonomy contains a transposition and the boundary of $c^n$ could not have all $(n - 1)$-cells of the interior of the $n$-manifold cancel, so the boundary could not be the fundamental class of $Z_j^{n-1}$.]

**□**

2. **Lemma 11.** If $T^{n+1}$ and $Z^{n-1}$ are orientable, then there are no manifolds of the second class.

[If the holonomy has a transposition, then the normal bundle of $Z^{n-1}$ is nonorientable, so just one of $T^{n+1}$ and $Z^{n-1}$ is orientable along the loop giving that transposition.]
3. We turn to the study of manifolds of the first class. If $Z_j^{n-1}$ is of the first class, then a [tubular neighborhood of $Z_j^{n-1}$ with $Z_j^{n-1}$ removed] has three connected components, the 3 sheets adjacent to $Z_j^{n-1}$. [Another Lemma, omitted, formulates this in terms of the combinatorics of simplices.]

p. 163.

4. Designate the 3 sheets in a tubular neighborhood $U$ of $Z_j^{n-1}$ by $M_{j\alpha}$, $\alpha = 1, 2, 3$. The closures of these $U$ are assumed to be disjoint.

**Lemma 12.** Let $Z_j^{n-1}$ be of the first class. Then

\[ \partial M_{j\alpha} = Z_j^{n-1} + z_j^{n-1} \]

where $Z_j^{n-1}$ is the fundamental $(n-1)$-cycle mod 2 of $Z_j^{n-1}$ and $z_j^{n-1}$ is a cycle mod 2, not 0, and having no common points with $Z_j^{n-1}$.

[Again this is given and proved in terms of the simplicial triangulation. In terms of the structure of bundles over the $Z_j^{n-1}$ with triad bundle, the assumption that $Z_j^{n-1}$ is of the first class tells us that the bundle is trivial, so that $M_{j\alpha} = Z_j^{n-1} \times [0,1]$, and in these terms 4) is geometrically transparent: $\partial M_{j\alpha}^{n-1} = Z_j^{n-1} \times \{0\} + Z_j^{n-1} \times \{1\}$.

p. 165.

5.

**Lemma 13.** Let $F^n$ be orientable and $H^j_{\text{fin}}(F^n, J_2) = 0$; then all $Z_j^{n-1}$ are orientable.

[The proof given invokes Poincaré duality ([E, §33]) in the form $H^j_{\text{fin}}(F^n, J_2) \approx H^{n-j}_{\text{fin}}(F^n, J_2)$, and goes on to argue that $(n-1)$-submanifolds $F_j^{n-1}$ of $F^n$ are orientable. Then since $\varphi; F_j^{n-1} \to Z_j^{n-1}$ is a 3-fold cover, $Z_j^{n-1}$ must be orientable too.

We can avoid the use of Poincaré duality by a more direct argument to show that $F_j^{n-1}$. Suppose we have a loop $\gamma$ in $F_j^{n-1}$. What $H^j_{\text{fin}}(F^n, J_2) = 0$ means is that $\gamma = \partial c^2$, where $c^2$ is a finite 2-chain mod 2. Hence $c^2$ is carried by a compact immersed 2-manifold $S$ with boundary. We can put $S$ in general position relative to $F_j^{n-1}$, which means that the intersection is a graph including $\gamma$ in such a way that vertices on $\gamma$ are all triple points and there are no other branch points. Using this graph we decompose $\gamma$ into a sum of simple cycles along which $S$ provides a normal field to $F_j^{n-1}$. Since $F^n$ is orientable, these simple cycles preserve orientation on $F_j^{n-1}$, and hence so does $\gamma$.].
7. Representing graph

1. The complex $Z^n$ is built from subcomplexes $Z^n_i$, attached to one another by subcomplexes $Z^{n-1}_j$; for a more detailed study of this situation we construct the representing graph $\Gamma$ of the 3-complex $Z^n$. This has two kinds of vertices:
   - $e_i$ – principal vertices, one for each $Z^n_i$;
   - $\epsilon_j$ – auxiliary vertices, one for each $Z^{n-1}_j$;
and edges $k_{j\alpha}$ corresponding to the sheets $M_{j\alpha}$ and joining a principal vertex to an auxiliary vertex if and only if the sheet of the auxiliary vertex $\epsilon_j$ is contained in the $Z^n_i$ corresponding to the principal vertex $e_i$.

   There are just 3 edges ending in each auxiliary vertex; even if some of the 3 sheets coincide we still take 3 edges [but see the next paragraph].

   Somewhat retreating from the customary definition of a graph, we call the set of all vertices and edges of $\Gamma$ the representing graph of the 3-complex $Z^n$. We note that manifolds $Z^{n-1}_j$ of the second and third class do not play a rôle in the preceding definition, which will be used only under conditions guaranteeing the nonexistence of such manifolds.

2. We say that a subgraph $\Gamma' \subset \Gamma$ is a proper tree if $\Gamma'$ has no cycles and each auxiliary vertex of $\Gamma'$ is incident with exactly two edges. [This definition seems incomplete: I think they intend to include connectedness and/or maximality with respect to the specified properties.]

Lemma 14. $\Gamma$ has either a cycle or a proper tree.

Proof. Build $\Gamma'$ recursively as an increasing union of connected subgraphs. Start with $\Gamma_1$ consisting of a single principal vertex, all of the edges from it, and the auxiliary vertices at the other end of those edges.

   Stop whenever a cycle is obtained. Otherwise get $\Gamma_{\mu+1}$ from $\Gamma_\mu$ by choosing a second edge for each auxiliary vertex of $\Gamma_\mu$ which has no second edge, add in the other ends of those new second edges, and add in all the edges (and their ends) incident to the new principal vertices.

In this process, if we are forced to take into $\Gamma_{\mu+1}$ the third edge of some auxiliary vertex already in $\Gamma_\mu$, then within $\Gamma_{\mu+1}$ there are two distinct paths from the starting vertex to the auxiliary vertex in question: one in $\gamma_\mu$ and one in $\Gamma_{\mu+1}$ using the third edge. Hence $\Gamma_{\mu+1}$ must contain a cycle.

Taking $\Gamma' = \bigcup_\mu \Gamma_\mu$, either $\Gamma'$ has a cycle or it is a proper tree such that

1): $\Gamma'$ is connected and

2): whenever a principal vertex belongs to $\Gamma'$, then so do all the edges incident to it.

$\square$

3. Corresponding to each cycle $\Gamma_0 \subset \Gamma$ we construct a 1-cycle with compact support in the polyhedron $Z^n$. Let $\Gamma_0$ consist of edges $k_{j,s\alpha}, s = 1, \ldots, t$; $t$ is even (equal to twice the number of principal vertices incident to the edges of $\Gamma_0$). Let the numbering of the edges of $\Gamma_0$ be carried out so that the ends of $k_{j,s\alpha}$ are the
principal vertex $e_i$, and the auxiliary vertex $e_j$, $e_j = e_{j+1}$ for $1 \leq s < t, s$ odd, $e_i = e_{i+1}$ for $1 < s < t, s$ even, $e_i = e_{i+1}$. For convenience in writing out we will understand by $k_{j+1, \alpha+1}$, $e_{j+1}$, respectively, $k_{j, \alpha}, e_j$.

**Lemma 15.** For each vertex of $\Gamma_0$ choose a point in the interior of the corresponding submanifold $Z_i^n$ or $Z_j^{n-1}$. For each edge of $\Gamma_0$ with ends $e_i$ and $e_j$ choose a path in $Z_i^n$ from $e_i$ to $e_j$ which contains points of only the corresponding sheet of $Z_j^{n-1}$ besides points in $Z_i^n$ — no other sheets of $Z_j^{n-1}$ nor any other $Z_k^{n-1}, j \neq k$. Then these paths will form a loop in $Z^n$, and if the loop in $\Gamma_0$ is simple, the loop in $Z^n$ can be chosen to be simple as well. Moreover, at points of $Z_j^{n-1}$ on this loop in $Z^n$ the loop passes from one sheet of $Z_j^{n-1}$ to another, and (if simple) can never hit $Z_j^{n-1}$ again because $e_j$ can only occur once in the loop of $\Gamma_0$.

This means that each $Z_j^{n-1}$ crossed by one of these loops does not separate $Z_j^{n-1}$. [What if $n = 2$ and $Z_j^{n-1}$ is not closed?]

4. For a proper tree $\Gamma' \subset \Gamma$ we construct an $n$-dimensional submanifold $M' \subset Z^n$. $M'$ consists of the union of $Z_i^n$ for which $e_i$ is a vertex of $\Gamma'$. By the requirement that $\Gamma'$ has all of the edges attached to such an $e_i$, all of the boundary of $Z_i^n$ is contained in $M'$. By the requirement that each $e_j$ in $\Gamma'$ is incident to exactly two edges in $\Gamma'$, $M'$ is a manifold in a neighborhood of each point of $Z_j^{n-1}$, since exactly two of the 3 sheets along $Z_j^{n-1}$ are contained in $M'$.

**Lemma 16.** Clearly $M'$ forms an $n$-cycle mod 2 in $T^{n+1}$. If $F^n$ is connected, then $M'$ does not separate $T^{n+1}$. Indeed, starting at a point $A$ of $M'$ in $Z^n$ we can run out on either side to the points $\varphi^{-1}(A) = \{A', A''\} \subset F^n$. Then $A', A''$ can be connected by a path in $F^n$, closing a loop which crosses $M'$ simply.

8. Proofs of Theorems 1, 2, 3

1. In this paragraph theorems 1, 2, 3 are proved, giving sufficient topological conditions for the validity of the bound $\kappa(F) \geq \kappa_1 R$ in the class $F_R$. First we prove two lemmas.

**Lemma 17.** For a 3-complex $Z^n$ normally imbedded in $T^{n+1}$ the following conditions cannot hold simultaneously:

1): the boundary $F^n$ of $T^{n+1}$ is connected;
2): $F^n$ is orientable;
3): $T^{n+1}$ is orientable;
4): $H_{jn}^j(F^n, J_2) = 0$;
5): $H_{jn}^j(T^{n+1}, J_2) = 0$; [i.e., $H_{jn}^j(Z^n, J_2) = 0$]
6): $H_{jn}^j(T^{n+1}, J) = 0$. [i.e., $H_{jn}^j(Z^n, J) = 0$]

**Proof.** We suppose 1) - 6) hold.

a) From conditions 2) and 4) and Lemma 15 it follows that all $Z_j^{n-1}$ are orientable.

Due to condition 6) and Lemma 10 there do not exist manifolds $Z_j^{n-1}$ of the third class. From condition 3) and Lemma 11 it follows that also there do not exist manifolds $Z_j^{n-1}$ of the second class.
b) We consider the first possibility specified in Lemma 16: let the representing graph $\Gamma$ of the 3-complex $Z^n$ contain a cycle $\Gamma_0$. According to Lemma 17, $\Gamma_0$ corresponds to a 1-cycle $\zeta$ of the complex $Z^n$. From condition 5) it follows that there is a finite 2-chain $c^2$ in $Z^n$ such that $\partial c^2 = \zeta \mod 2$. (21) [Some of the numbering of equations in the original is retained.]

p. 171.

This is my proof. We may assume $\zeta$ is a simple loop. We can realize $c^2$ as a union of immersed compact surfaces, one of which has boundary $\zeta$, the others without boundary. By taking them in general position we can assume that the intersection with any $Z_j^{n-1}$ is a union of regular curves. Since $Z_j^{n-1}$ is a closed $(n-1)$ manifold (not necessarily compact), the intersections with these surfaces are circles except for the one with boundary $\zeta$. Because $\zeta$ crosses $Z_j^{n-1}$ just once, there is only one endpoint for the intersection of that part of $c^2$ with $Z_j^{n-1}$ which is impossible.]

[The proof given.] We take an arbitrary manifold $Z^n$ intersecting $\zeta$; designate by $c^2_j$ the 2-chain mod 2 consisting of all simplices of $\bar{Z}_i^n$ belonging to $c^2$. [From part 3, §7, we had $\zeta_i$, the part of $\zeta$ in $\bar{Z}_i^n$.] Then from (21) we see that
\begin{equation}
(2)
\partial c^2_i = \zeta_i + c_i
\end{equation}
where $c_i \subset Z^{n-1}$. In defining "sheets" $M^n_{j\alpha}$ we had a cycle "parallel" to $Z_j^{n-1}$ forming the boundary of that sheet
\begin{equation}
\partial M^{n-1}_{j\alpha} = Z_j^{n-1} + z_{j\alpha}^{n-1}.
\end{equation}
Let $Z_j^{n-1}$ be one of the components of $Z^{n-1}$ containing an end of $\zeta_i$ so that the sheet $M^n_{j\alpha} \subset \bar{Z}_i^n$. Then $\zeta$ has intersection number 1 with $z_{j\alpha}^{n-1}$, just as it does with $Z_j^{n-1}$. We use notation $\times$ for intersection numbers: $z_{j\alpha}^{n-1} \times \zeta = 1$. Since $z_{j\alpha}^{n-1} \subset Zn_i \subset Z^n \setminus Z^{n-1}$, from (2) we obtain
\begin{equation}
(3)
\partial c^2_i \times z_{j\alpha}^{n-1} = 1 \text{ in } Z^n_i.
\end{equation}
Let $C$ be the union of all the closed stars of the complex $\bar{Z}_i^n$ intersecting the support of $c^2_i$. Since $c^2_i$ is finite and $Z^n$ is a locally finite polyhedron, so also $C$ consists of a finite number of simplices.

It is clear that
\begin{equation}
(4)
\partial C \cap c^2_i \cap Z^n_i = 0.
\end{equation}
$[C$ is a tubular neighborhood of $c^2_i$, so its boundary only intersects $c^2_i$ at the ends of the tube, which lie in $Z^{n-1}$, excluded from the open manifold $Z^n_i$.]

Let $c^{n-1}$ consist of all the simplices of $Z_{j\alpha}^{n-1}$ belonging to $C$; since $z_{j\alpha}^{n-1}$ is a cycle (generally speaking, infinite), $\partial c^{n-1}$ is contained in the support of $\partial C$, and from (4) we arrive at
\begin{equation}
\partial c^2_i \times z_{j\alpha}^{n-1} = \partial c^2_i \times c^{n-1} = c^2_i \times \partial c^{n-1} = 0,
\end{equation}
which contradicts (3).

c) We consider the second possibility specified in Lemma 14: let $\Gamma$ contain a proper tree $\Gamma'$. The result of part a) allows the use of Lemma 16. According to Lemma 16, $\Gamma'$ corresponds to an $n$-dimensional (generally speaking infinite) cycle $\zeta^n \mod 2$ of the polyhedron $Z^n$. $[\zeta^n$ is an $n$-manifold. From a point on it, $A_0$, we can move on paths on either side (locally) in $T^{n+1}$ out to points $B_1, B_2 \in$
Let $F^n$, connecting $B_1, B_2$ by an arc in $F^n$ we get a loop $\delta$ in $T^{n+1}$ having a simple intersection with $\zeta^n$. $\delta$ can be represented simplicially and we have $\delta \times \zeta^n = 1$ (27).

By condition 5), $\delta = \partial b^2$ mod 2 for some finite mod 2 2-chain $b^2$. By the same argument as in b) we reach a contradiction. [$b^2$ is essentially a compact immersed surface with boundary $\delta$. It can be taken in general position relative to $Z^n$, so the intersection is a regular curve. But that curve only has one end by (27).] $\square$

**Lemma 18.** Let $T^{n+1}$ be a manifold with boundary $F^n$, lying in Euclidean space $E^{n+1}$, and with $F^n$ connected. If the homomorphism $h : \pi_1(F^n) \to \pi_1(T^{n+1})$, induced by the inclusion of $F^n$ in $T^{n+1}$, is an isomorphism [1-1 and onto], then in the universal covering $\tilde{T}^{n+1}$ of the polyhedron $T^{n+1}$ the polyhedron $\tilde{F}^n$ covering $F^n$ is connected and simply connected. If in addition $\pi_2(T^{n+1}) = 0$, then $H^2_{\tilde{F}^n}(\tilde{T}^{n+1}, J) = 0$.

**Proof.** Let $\kappa : \tilde{T}^{n+1} \to T^{n+1}$ be the covering map; then $\kappa^{-1}(F^n) = \tilde{F}^n$ is, evidently, the union of a finite or countable number of (connected) manifolds. We show that $\tilde{F}^n$ is connected. [Just lift a path between the images of two points. This reduces it to the case of connecting two points $\tilde{a}_1, \tilde{a}_2 \in \kappa^{-1}(A), A \in F^n$. Then there is a loop in $T^{n+1}$ at $A$ such that its lift to $\tilde{A}_1$ is a path to $\tilde{A}_2$. Since $\pi_1(F^n) \to \pi_1(T^{n+1})$ is onto, the loop in $T^{n+1}$ is homotopic to a loop in $F^n$ which lifts to a path in $\tilde{F}^n$ connecting $\tilde{A}_1, \tilde{A}_2$.]

Now let $\tilde{\lambda}$ be a closed path in $\tilde{F}^n$ based at $\tilde{A}$, $\kappa(\tilde{\lambda}) = \lambda, \kappa(\tilde{A}) = A$; then $\lambda$ is homotopic to the trivial loop in $T^{n+1}$. Since $h$ is 1-1, $\lambda$ is nonhomotopic to the trivial loop in $F^n$; but then $\lambda$ is homotopic to the trivial loop in $\tilde{F}^n$.

Finally, $\pi_2(T^{n+1}) = \pi_2(T^{n+1}) = 0$, $\pi_1(T^{n+1}) = 0$, and by Hurewicz's theorem (cf., for example [H, p. 57]), $H^2_{\tilde{F}^n}(\tilde{T}^{n+1}, J) = 0$. $\square$

**Theorem 19 (\textit{= Theorem 1}).** Let $F^n$ be a surface of class $F_R$ in Euclidean space $E^{n+1}$ and suppose $H_1(F^n, J_2) = 0$. Then $\kappa(F) \geq \kappa_1 R$.

**Proof.** According to Poincaré duality $H_{n-1}(F^n, J_2) = H_1(F^n, J_2) = 0$ ([A1, p. 484 3.332]). Applying Alexander duality to the polyhedron $F^n \subset E^{n+1}$ ([A1, p. 490, 4:13]), we are led to

$$H_1(E^{n+1} \setminus F^n, J_2) = H_{n-1}(F^n, J_2) = 0.$$ 

But by the Jordan-Brouwer theorem ([A1, p. 519, 3:44]), $T^{n+1}$ is a connected component of $E^{n+1} \setminus F^n$, from whence $H_1(T^{n+1}, J_2) = 0$.

[There is a more direct argument that $H_1(F^n, J_2) = 0 \Rightarrow H_1(T^{n+1}, J_2) = 0$. Suppose we have a 1-cycle mod 2 in $T^{n+1}$; that is, a formal sum of loops $z^1$. We can fill a loop in $E^{n+1}$ with a surface $S$ which can be assumed to have general position relative to $F^n$. The intersection of that surface with $F^n$ then consists of several loops which form the boundary of the inside $S \cap T^{n+1}$ except for the given loop. Each of those loops in $S \cap F^n$ is the boundary of a surface in $F^n$ since $H_1(F^n, J_2) = 0$, and if we replace the outside $S \setminus T^{n+1}$ by these surfaces in $F^n$ we get a surface in $T^{n+1}$ whose boundary is the original loop.]
Thus, conditions 4), 5) of Lemma 17 are satisfied. Moreover, \( H_1(F^n, J_2) = 0 \Rightarrow H_1(F^n, J) \) has no \( J \)-summand. ([A1, p.358, theorem 4:41].) By Alexander duality then \( H_{n-1}(T^{n+1}, J) \) has no \( J \)-summand ([A1, p. 490, 4:1]); the torsion group \( \Theta_{n-1}(T^{n+1}) \) is always trivial (cf. the corollary of 4:1 immediately after the formulation of 4:1, [A1, p. 490]). Hence \( H_{n-1}(T^{n+1}, J) = 0 \), and condition 6) of Lemma 17 holds.

Conditions 1), 3) hold by an obvious means. Finally, condition 2) follows from the theorem of Jordan-Brouwer.

By Lemma 17 \( T^{n+1} \) cannot contain a normally imbedded 3-complex, so that either the central set of \( F^n \) has points of multiplicity \( >3 \) and hence \( \kappa(F^n) \geq \kappa_1 R \), or the cutlocus has focal points and \( \kappa(F^n) \geq R \). This completes the proof of Theorem 1.

\[ \square \]

**Theorem 20** (= Theorem 2). Let \( F \in F_R \). If the homomorphism \( h : \pi_1(F) \to \pi_1(T) \) induced by the inclusion of \( F \) in \( T \) is an isomorphism, and \( \pi_2(T) = 0 \), then \( \kappa(F) \geq \kappa_1 R \).

**Proof.** Let the multiplicity of \( Z^n \) equal 3. According to Lemma 8, in the universal covering \( \tilde{T}^{n+1} \) of the polyhedron \( T^{n+1} \) there is contained a cutlocus \( \tilde{Z}^n \), normally imbedded in \( \tilde{T}^{n+1} \) as a 3-complex.

Therefore, as in the proof of Theorem 1, it suffices to verify for \( \tilde{T}^{n+1} \) that properties 1) – 6) of Lemma 17 hold.

From Lemma 18 it follows that \( \tilde{F}^n \) is connected and simply-connected; therefore conditions 1), 2), 4) hold. Conditions 3) and 5) hold in view of the simple-connectedness of \( \tilde{T}^{n+1} \). It remains to verify condition 6). According to Lemma 18, \( H_2^{\text{fin}}(\tilde{T}^{n+1}, J) = 0 \) We apply Poincaré duality for infinite manifolds to \( \tilde{T}^{n+1} \) (cf, e.g., [E, §§9, 33]), accounting for condition 5) of Lemma 17; we obtain \( H_{n-1}(\tilde{T}^{n+1}, J) = 0 \), that is, condition 6) also holds, which concludes the proof of the theorem.

\[ \square \]

In the case \( n = 2 \) (of a surface \( F^2 \) in 3-dimensional space) the condition of Theorem 2 can be significantly weakened.

**Theorem 21** (= Theorem 3). Let \( n = 2 \), \( F \in F_R \), and \( h : \pi_1(F) \to \pi_1(T) \) be onto. Then \( \kappa(F) \geq \kappa_1 R \).

p. 174.

**Proof.** We verify the conditions for applying Lemma 17 to \( \tilde{T}^3 \). For \( n = 2 \) condition 6) of Lemma 17 is found to be unnecessary; concerning this, this condition was needed to prove the nonexistence of manifolds \( \tilde{Z}_j^{n-1} \) of the third class (Lemma 10). But a manifold \( \tilde{Z}_j^1 \) of the third class would need to be a simply closed curve, since otherwise [the holonomy would be trivial]. Consequently, it may be assumed that \( \tilde{Z}_j^1 \) is a \textit{finite} cycle of \( Z^2 \). The orientability of \( \tilde{Z}_j^1 \) is evident, and the exclusion of manifolds of the third follows from the triviality of \( H_1^{\text{fin}}(\tilde{T}^3, J) \) for a simply-connected polyhedron \( \tilde{T}^3 \).

Moreover, conditions 2) and 4) are also found to be unnecessary. Concerning this, in the proof of Lemma 17 conditions 2) and 4) were used only in point a), in order to claim the orientability of \( \tilde{Z}_j^{n-1} \), which for \( n = 2 \) holds automatically.
Conditions 3, 5) hold for simply-connected polyhedron \( \tilde{T} \), and it is only needed to verify condition 1. But for the proof of connectedness of \( \tilde{F}^n \) in Lemma 18 only the onto-ness of the homomorphism \( h \) was used, which is assumed for Theorem 3.

9. Applications to some simple types of surfaces

1. We consider now several particular cases, presenting interest from a geometrical point of view. We give a definition of a solid homeomorphic to a ball with \( k \) handles.

Let \( K^{n+1} \) be a regular closed ball in \( E^{n+1} \) and \( f_j, j = 1, \ldots, k \) be homeomorphisms from \( K^n \times [0,1] \) into \( E^{n+1} \) such that the sets \( Q_j = f_j(K^n \times (0,1)) \subset E^{n+1} \setminus K^{n+1}, Q_j \) are pairwise nonintersecting, and \([f_j(K^n \times 0) \cap f_j(K^n \times 1)] \subset K^{n+1}, j = 1, \ldots, k \). The polyhedron \( h_k^{n+1} = K^{n+1} \cap \bigcup_{j=1}^k Q_j \) is called an \((n+1)\)-dimensional ball with \( k \) handles, and the boundary of \( h_k^{n+1} \) in \( E^{n+1} \) is called an \( n \)-dimensional sphere with \( k \) handles and is designated by \( S^n_k \).

A regular \((n+1)\)-dimensional toroidal ring \( T \) is the direct product of the disk \( K^2 \) by \( E^{n-1} \) [sic. Should this be \( S^{n-1} \)? or \( (S^1)^{n-1} \)?]

Finally, we designate the \( n \)-dimensional sphere by \( S^n \).

We now introduce the following classes of surfaces (cf. §1): \( S \) consists of all surfaces of class \( F_R \) homeomorphic to \( S^n \); \( H_k \) consists of all surfaces of class \( F_R \) homeomorphic to \( S^n_k \); \( H_0^k \) consists of all surfaces of class \( F_R \) bounding a solid homeomorphic to \( h_k^{n+1} \); \( T^0 \) consists of all surfaces of class \( F_R \) bounding a solid homeomorphic to \( T^{n+1} \).

We recall that \( \kappa(M) = \inf_{F \in M} \kappa(F), M \subset F_R \).

2. Theorem 22 (= Theorem 4). \( \kappa(S) \geq \kappa_1 R; \kappa(H_k^0) \geq \kappa_1 R; \kappa(T^0) \geq \kappa_1 R \).

For \( n = 2 \) these inequalities reduce to equality.

**Proof.** In this part of the work we limit ourselves to the proofs of the inequalities \( \kappa(S) \geq \kappa_1 R, \kappa(H_k^0) \geq \kappa_1 R, \kappa(T^0) \geq \kappa_1 R \); the sharp bound for classes \( S, H_k^0 \) for \( n = 2 \) and any \( k = 1, 2, \ldots \) will be established in the second part of the work by the construction of corresponding examples (we note that for \( n = 2 \) we have \( T^0 = H_0^0 \)).

p. 175

a): If \( F^n \in S \), then the assertion of the theorem follows from Theorem 1.

b): If \( F^n \in H_k^0 (k > 0) \), then, as is easily seen, \( F^n \) contains a subset \( T^1 \), homeomorphic to the union of \( k \) circles with one common point, such that there exists a deformation

\[ \omega_t : T^{n+1} \times [0,1] \to T^{n+1}, \]

\[ \omega_t(P) = P (P \in T^1, 0 \leq t \leq 1), \] of the identity map \( \omega_0 \) into \( \omega_1, \omega_1(T^{n+1}) \subset T^1 \). For \( n = 2 \) thus it follows that \( h \) is onto and Theorem 3 can be applied.

Hence it follows that the homomorphism \( h : \pi_1(F^n) \to \pi_1(T^{n+1}) \), induced by \( F^n \subset T^{n+1} \), is an isomorphism; \( \pi_2(T^{n+1}) = \pi_2(T^1) \). Considering the universal covering polyhedron \( \tilde{T}^1 \), it is easy to convince oneself that \( \pi_2(\tilde{T}^1) = 0 \), whence \( \pi_2(T^1) = 0 \); hence Theorem 2 can be applied to a surface \( F^n \in H_k^0 \), which leads to the required bound.
[h is not an isomorphism for \( n = 2 \), only onto, but that case has been covered.]

**c):** Let \( F^n \in T_{01} \). We construct the universal covering polyhedron \( \tilde{T}^{n+1} \) for the solid \( T^{n+1} \), bonded by \( F^n \) in \( E^{n+1} \). Evidently, \( \tilde{T}^{n+1} \) is homeomorphic to \( K^2 \times E^{n-1} \), \( \tilde{F}^n \) is homeomorphic to \( S^1 \times E^{n-1} \) [the text was written \( E^{n+1} \)]. We verify the conditions for the applicability of Lemma 17 to \( \tilde{T}^{n+1} \).

Evidently, all conditions besides 4) hold. But 4) was used only in point a) of the proof of Lemma 17 for establishing the orientability of manifolds \( \tilde{Z}_j^{n-1} \). By Lemma 6, 5), for this it suffices to prove the orientability of the manifolds \( \tilde{F}_j^{n-1} \subset \tilde{F}^n \).

By a small isotopic deformation \( \tilde{F}_j^{n-1} \) can be moved to general position relative to the cycle \( \tilde{z}^{n-1} = P \times E^{n-1}, P \in S^1 \) (the construction of such a deformation is simplified thanks to the special form of \( z^{n-1} \)). We also designate the cycle obtained as a result of the deformation by \( \tilde{F}_j^{n-1} \) and we note that in the deformation the characteristic of orientability of \( \tilde{F}_j^{n-1} \) is not changed. We construct, furthermore, a simplicial subdivision of \( \tilde{F}^n \), subcomplexes of which are \( \tilde{F}_j^{n-1} \) and \( z^{n-1} \), since \( z^{n-1} \) is the unique basis of homology cycles mod 2 of the polyhedron \( \tilde{F}^n \), there exists a chain \( c^n \) (infinite) mod 2 constructed in the above subdivision, such that

\[
\partial c^n = \tilde{F}_j^{n-1} + z^{n-1}.
\]

We take an integral chain \( z_{\alpha*}^{n-1} \), which in mod 2 reduces to \( z^{n-1} \). For each component of string connectedness \( c^n_\alpha \) of the chain \( c^n \) we choose an orientation of the simplices of \( C^n_\alpha \) such that for the integral chain \( c^n_\alpha* \) obtained

\[
\partial c^n_\alpha = \tilde{F}_j^{n-1} + c^n_{\alpha*}^{-1},
\]

where \( \tilde{F}_j^{n-1} \) consists of oriented simplices of \( \tilde{F}_j^{n-1} \), and \( z_{\alpha*}^{n-1} \) consists of oriented simplices of \( z^{n-1} \). It can be achieved in this that the orientation of the simplices of \( z_{\alpha*}^{n-1} \) should be in accord with the orientation fixed above \( z^{n-1} \) of the cycle \( z^{n-1} \); concerning this, \( c^n_\alpha \) lies in one of the two domains into which \( z^{n-1} \) separates \( \tilde{F}^n \), and therefore for all oriented simplices \( \zeta^n \) of the chain \( c^n_\alpha* \) and the simplices \( \zeta^n_{\alpha*} \) of the chain \( z_{\alpha*}^{n-1} \) adjacent with the coefficients of incidence \([\zeta^n : \zeta^n_{\alpha*}]\) are 1 throughout.

Since for each simplex of \( \tilde{F}_j^{n-1} \) and \( z^{n-1} \) the incidence equals 1 with simplices of \( c^n \), the chains \( \tilde{F}_j^{n-1*} : \tilde{z}_{\alpha*}^{n-1} \), \( \tilde{F}_j^{n-1*} : \tilde{z}_{\beta*}^{n-1} \), \( \tilde{z}_{\beta*}^{n-1} \) for \( \alpha \neq \beta \) do not have simplices in common. We put \( c^n_\alpha = \sum_\alpha c^n_\alpha \), then

\[
\partial c^n_\alpha = \sum_\alpha \tilde{F}_j^{n-1*} + \sum_\alpha z_{\alpha*}^{n-1}.
\]

Inasmuch as \( c^n_\alpha \) contains all simplices of \( c^n \), from (5) and (6) it follows that

\[
\tilde{F}_j^{n-1*} = \sum_\alpha \tilde{F}_j^{n-1*} : \tilde{z}_{\alpha*}^{n-1} = \sum_\alpha z_{\alpha*}^{n-1}
\]

in reduction mod 2 is transformed, correspondingly, to \( \tilde{F}_j^{n-1} \) and \( z^{n-1} \). Due to the choice of orientation of the chains \( c^n_\alpha \), \( z_{\alpha*}^{n-1} \) coincides with \( z^{n-1} \), and is, consequently, an integral cycle. But then from (6) it follows that also \( \tilde{F}_j^{n-1} \) is an integral cycle, which proves the orientability of \( \tilde{F}_j^{n-1} \).

\[\square\]

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