Analysis of Inpainting via Clustered Sparsity and Microlocal Analysis

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Abstract

Recently, compressed sensing techniques in combination with both wavelet and directional representation systems have been very effectively applied to the problem of image inpainting. However, a mathematical analysis of these techniques which reveals the underlying geometrical content is completely missing. In this paper, we provide the first comprehensive analysis in the continuum domain utilizing the novel concept of clustered sparsity, which besides leading to asymptotic error bounds also makes the superior behavior of directional representation systems over wavelets precise. First, we propose an abstract model for problems of data recovery and derive error bounds for two different recovery schemes, namely $\ell_1$ minimization and thresholding. Second, we set up a particular microlocal model for an image governed by edges inspired by seismic data as well as a particular mask to model the missing data, namely a linear singularity masked by a horizontal strip. Applying the abstract estimate in the case of wavelets and of shearlets we prove that – provided the size of the missing part is asymptotically to the size of the analyzing functions – asymptotically precise inpainting can be obtained for this model. Finally, we show that shearlets can fill strictly larger gaps than wavelets in this model.

Key Words. $\ell_1$ Minimization, Cluster Coherence, Inpainting, Parseval Frames, Sparse Representation, Data Recovery, Shearlets, and Meyer Wavelets

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1 Introduction

A common problem in many fields of scientific research is that of missing data. The human visual system has an amazing ability to fill in the missing parts of images, but automating this process is not trivial. Also, depending on the type of data, the human senses may be unable to fill in the gaps. Conservators working to repair damaged paintings use the term inpainting to describe the process. This word now also means digitally recovering missing data in videos and images. The
removal of overlaid text in images, the repair of scratched photos and audio recordings, and the recovery of missing blocks in a streamed video are all examples of inpainting. Seismic data are also commonly incomplete due to land development and bodies of water preventing optimal sensor placement \cite{HH08, HH10}. In seismic processing flow, data recovery plays an important role.

One very common approach to inpainting is using variational methods \cite{BBS01, BBC+01, BSCB00, CS02}. However, recently the novel methodology of compressed sensing, namely exact recovery of sparse or sparsified data from highly incomplete linear non-adaptive measurements by $l_1$ minimization or thresholding, has been very effectively applied to this problem. The pioneering paper is \cite{ESQD05}, which uses curvelets as sparsifying system for inpainting. Various intriguing successive empirical results have since then been obtained using applied harmonic analysis in combination with convex optimization \cite{CCS10, DJL+12, ESQD05}. These three papers do contain theoretical analyses of the convergence of their algorithms to the minimizers of specific optimization problems but not theoretical analyses of how well those optimizers actually inpaint. Other theoretical analysis of those types of methods (imposing sparsity with a discrete dictionary) typically use a discrete model of the original image which does not allow the geometry of the problem to be taken into account. However, variational methods are built on continuous methods and may be analyzed using a continuous model, for example, \cite{CKS02}. Also, some work has been done to compare variational approaches with those built on $l_1$ minimization \cite{CDOS12, Mey01}. Finally, in works such as \cite{HH08, HH10}, intuition behind why directional representation systems such as curvelets and shearlets outperform wavelets when inpainting images strongly governed by curvilinear structures such as seismic images is given. So, although there are many theoretical results concerning inpainting, they mainly concern algorithmic convergence or variational methods.

The preliminary results presented in the SPIE Proceedings paper \cite{KKZ11} combined with the theory in this paper provide the first comprehensive analysis of discrete dictionaries inpainting the continuum domain utilizing the novel concept of clustered sparsity, which besides leading to asymptotic error bounds also makes the superior behavior of directional representation systems over wavelets precise. Along the way, our abstract model and analysis lay a common theoretical foundation for data recovery problems when utilizing either analysis-side $l_1$ minimization or thresholding as recovery schemes (Section 2). These theoretical results are then used as tools to analyze a specific inpainting model (Sections 3–6).

### 1.1 A Continuum Model

One of the first practitioners of curvelet inpainting for applications was the seismologist Felix Herrmann, who achieved superior recovery results for images which consisted of curvilinear singularities in which vertical strips are missing due to missing sensors. These techniques were soon also exploited for astronomical imaging, etc., the common bracket being the governing by curvilinear singularities. It is evident, that no discrete model can appropriately capture such a geometrical content.

Thus a continuum domain model seems appropriate. In fact, in this paper we choose a distributional model which is a distribution $wL$ acting on Schwartz functions $g \in S'(\mathbb{R}^2)$ by

$$\langle wL, g \rangle = \int_{-\rho}^{\rho} w(x_1)g(x_1, 0)dx_1,$$

the weight $w$ and length $2\rho$ being specified in the main body of the paper. Essentially, the weight $w$ sets up the linear singularity that is smooth in the vertical direction, while the value of $\rho$ corresponds
to the length of the singularity. Mimicking the seismic imaging situation, we might then choose
the shape of the missing part to be
\[ \mathcal{M}_h = \mathbb{1}_{\{|x_1| \leq h\}}, \]
i.e., a vertical strip of width 2h. Clearly, h cannot be too large relative to \( \rho \) or else we are erasing
too much of \( w\mathcal{L} \). Further, we let \( P_{\mathcal{M}_h} \) and \( P_{\mathbb{R}^2 \setminus \mathcal{M}_h} \) denote the orthogonal projection of \( L^2(\mathbb{R}^2) \)
on the missing part and the known part, respectively. One task can now be formulated mathematically
precise in the following way. Given
\[ f = P_{\mathbb{R}^2 \setminus \mathcal{M}_h} w\mathcal{L}, \]
recover \( w\mathcal{L} \).

It should be mentioned that such a microlocal viewpoint was first introduced and studied in
the situation of image separation [DK12].

1.2 Sparsifying Systems

It was recently made precise that the optimal sparsifying systems for such images governed by
anisotropic structures are curvelets [CD04] and shearlets [KL12, KL11]. Of these two systems
shearlets have the advantage that they provide a unified concept of the continuum and digital do-
main, which curvelets do not achieve. However, many inpainting algorithms even still use wavelets,
and one might ask whether shearlets provably outperform wavelets. In fact, we will make the
superior behavior of shearlets within our model situation precise.

For our analysis, we will use systems of wavelets and shearlets which are defined below. Both
systems are smooth Parseval frames. Parseval frames generalize orthonormal bases in a manner
which will be useful in the sequel.

**Definition 1.1.** A collection of vectors \( \Phi = \{\varphi_i\}_{i \in I} \) in a separable Hilbert space \( \mathcal{H} \)
forms a Parseval frame for \( \mathcal{H} \) if for all \( x \in \mathcal{H} \),
\[ \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 = \|x\|^2. \]

With a slight abuse of notation, given a Parseval frame \( \Phi \), we also use \( \Phi \) to denote the synthesis
operator
\[ \Phi : \ell_2(\mathbb{N}) \to \mathcal{H}, \quad \Phi(\{c_i\}_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} c_i \varphi_i. \]

With this notation, \( \Phi^* \) is called the analysis operator.

1.2.1 Wavelets

Meyer wavelets are some of the earliest known examples of orthonormal wavelets; they also happen
to have high regularity [Dau92, Mey87]. We modify the classic system to get a decomposition of the
Fourier domain that is comparable to the shearlet system that we will use. For the construction,
let \( \nu \in C^\infty(\mathbb{R}) \) satisfy \( \nu(\cdot) + \nu(1 - \cdot) = 1_{\mathbb{R}}(\cdot) \), where the indicator function \( 1_A \) is defined to take
the value 1 on \( A \) and 0 on \( A^c \), and
\[ \nu(x) = \begin{cases} 
0 & : \ x \leq 0, \\
1 & : \ x \geq 1.
\end{cases} \]
Then the Fourier transform of the 1D Meyer generator is defined by

\[
W(\xi) = \begin{cases} 
  e^{-16\pi i \xi/3} \sin \left( \frac{2}{3} \nu (16 |\xi| - 1) \right) &: \frac{1}{16} \leq |\xi| \leq \frac{1}{8}, \\
  e^{-8\pi i \xi/3} \cos \left( \frac{2}{3} \nu (8 |\xi| - 1) \right) &: \frac{1}{8} \leq |\xi| \leq \frac{1}{4}, \\
  0 &: \text{else,}
\end{cases}
\]

and the Fourier transform of the scaled 1D Meyer scaling function is

\[
\hat{\phi}(\xi) = \begin{cases} 
  1 &: |\xi| \leq \frac{1}{16}, \\
  \cos \left( \frac{2}{3} \nu (16 |\xi| - 1) \right) &: \frac{1}{16} \leq |\xi| \leq \frac{1}{8}, \\
  0 &: \text{else,}
\end{cases}
\]

where we use the following Fourier transform definition for \( f \in L^1(\mathbb{R}^n) \)

\[
\mathcal{F} f := \hat{f} = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \cdot \rangle} dx,
\]

(\( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product) which can be naturally extended to functions in \( L^2(\mathbb{R}^n) \). The inverse Fourier transform is given by

\[
\mathcal{F}^{-1} f := \tilde{f} = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \langle \cdot, \xi \rangle} d\xi.
\]

We will not detail the interpretation of a scaling function but refer the interested reader to \cite{Dau92,Mey87}. Then we define the \( C^\infty \cap L^2(\mathbb{R}^2) \)-functions \( W^v, W^h, \) and \( W^d \) by

\[
W^v(\xi) = \hat{\phi}(\xi_1)W(\xi_2), \quad W^h(\xi) = W(\xi_1)\hat{\phi}(\xi_2), \quad \text{and} \quad W^d(\xi) = W(\xi_1)W(\xi_2).
\]

We denote

\[
\hat{\psi}_\lambda(\xi) = 2^{-j}W^\epsilon(\xi/2^j) e^{-2\pi i (k, \xi/2^j)}, \quad \lambda = (\epsilon, j, k).
\]

Then the **Parseval Meyer wavelet system** is given by

\[
\{ \psi_\lambda : \lambda = (\epsilon, j, k), \epsilon \in \{ h, v, d \}, j \in \mathbb{Z}, k \in \mathbb{Z}^2 \}
\]

We have not yet shown that this system forms a Parseval frame. It is known (in various forms, for example \cite{Chr03,CS93,Dau92,Jin99,Kin09}) that if for \( \{ \psi^\epsilon \in L^2(\mathbb{R}^d) \}_\epsilon \)

\[
\sum_\epsilon \sum_{k \neq 0} \sum_j |\hat{\psi}_\epsilon(2^j \xi)\hat{\psi}_\epsilon(2^j \xi - k)| = 0 \text{ a.e. } \xi
\]

and

\[
\sum_j |\hat{\psi}_\epsilon(2^j \xi)|^2 = 1 \text{ a.e. } \xi,
\]

then

\[
\{ 2^{j/2}\hat{\psi}_{\epsilon}(2^j \cdot - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \epsilon \}
\]

is a Parseval frame for \( L^2(\mathbb{R}^d) \). The Meyer wavelet system defined above easily satisfies this.
1.2.2 Shearlets

We will use the construction of Guo and Labate of smooth Parseval frames of shearlets [GL12] which is a modification of cone-adapted shearlets (see, for example [KL12]). Let the parabolic scaling matrices $A^h_a$ and $A^v_a$ and shearing matrices $S^h_s$ and $S^v_s$ be defined as

$$
A^h_a = \begin{bmatrix} a^2 & 0 \\ 0 & a \end{bmatrix}, \quad A^v_a = \begin{bmatrix} a & 0 \\ 0 & a^2 \end{bmatrix},
$$

$$
S^h_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad S^v_s = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}.
$$

We use these dilation matrices as these are used in [GL12] and given particulars of their construction, it is not straightforward to adopt their methods to the dilation matrix $\begin{bmatrix} a & 0 \\ 0 & \sqrt{a} \end{bmatrix}$. In addition, given the fact that the matrices defined above always have integer values when $a$ is an integer, they are reasonable from the point of view of implementation. Let $V \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ satisfy $\text{supp} V \subseteq [-1, 1]$, and

$$
\sum_{k=-1}^{1} |V(\xi + k)|^2 = 1, \quad \xi \in [-1, 1].
$$

Further set $V^h(\xi) = V(\xi_2/\xi_1)$ and $V^v(\xi) = V(\xi_1/\xi_2)$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, define

$$
\hat{\phi}(\xi) = \hat{\phi}(\xi_1)\hat{\phi}(\xi_2)
$$

and

$$
\mathcal{W}(\xi) = \sqrt{|\hat{\phi}(2^{-2}\xi)|^2 - |\hat{\phi}(\xi)|^2}.
$$

We define the following shearlet system for $L^2(\mathbb{R}^2)$

$$
\{\phi_k : k \in \mathbb{Z}\} \cup \{\sigma^j_{\ell,k}, : j \geq 0, |\ell| < 2^j, k \in \mathbb{Z}^2, \ell \in \{h,v\}\} \cup \{\sigma^j_{\ell,k} : j \geq 0, \ell = \pm 2^j, k \in \mathbb{Z}^2\}, \quad (1)
$$

where

$$
\phi_k = \phi(\cdot - k);
$$

for $\ell \in \{h, v\}$,

$$
\hat{\sigma}^\ell_{j,\ell,k}(\xi) = 2^{-3j/2}W(2^{-2j}\xi)V^\ell(\xi A^\ell_{2^{-j}}S^\ell_{-\ell}k)e^{2\pi i(\xi A^\ell_{2^{-j}}S^\ell_{-\ell}k)};
$$
for \( j = 0 \) and \( \ell = \pm 1 \),
\[
\hat{\sigma}_{0,\ell,k}(\xi) = \begin{cases} 
W(\xi) V(\xi_2 - \ell) e^{2\pi i (\xi_1,k)} & : |\xi_2| \leq 1 \\
W(\xi) V(\xi_2 - \ell) e^{2\pi i (\xi_1,k)} & : |\xi_2| > 1 
\end{cases}
\]
and for \( j \geq 1, \ell = \pm 2^j \), \( \hat{\sigma}_{j,\ell,k}(\xi) = \)
\[
\begin{cases} 
2^{-\frac{3}{2}j - \frac{1}{2}} W(2^{-2j} \xi) V(2^j \xi_2 - \ell) e^{\pi i (\xi_{A_{2^{-j}} S_{\ell,k}})} & : |\xi_2| \leq 1 \\
2^{-\frac{3}{2}j - \frac{1}{2}} W(2^{-2j} \xi) V(2^j \xi_2 - \ell) e^{\pi i (\xi_{A_{2^{-j}} S_{\ell,k}})} & : |\xi_2| > 1 
\end{cases}
\]
The \( \sigma_{j,k} \) are the “seam” elements that piece together the \( \sigma_{j,\ell,k}^\iota \) and \( \phi_k \). We now have the following result from [GL12, Theorem 5p].

**Theorem 1.2.** The system defined in \([\mathcal{S}]\) is a Parseval frame for \( L^2(\mathbb{R}^2) \). Furthermore, the elements of this system are \( C^\infty \) and band-limited.

We will sometimes employ the notation
\[
\hat{\sigma}_\eta = \hat{\sigma}_{j,\ell,k}^\iota, \quad \eta = (\iota, j, \ell, k),
\]
where \( \iota \in \{h, v, \emptyset\}, j \in \mathbb{Z}, \ k \in \mathbb{Z}^2, \) and \( \ell \in \mathbb{Z}. \)

Fix a \( j \geq 0 \). Then the support of each \( \hat{\sigma}_{j,\ell,k}^\iota \) and \( \hat{\sigma}_{j,\ell,k} \) lies in the Cartesian corona
\[
C_j = [-2^{j-1}, 2^{j-1}]^2 \setminus [-2^{j-4}, 2^{j-4}]^2.
\]
The position of the support inside the corona is determined by the values of \( \ell \) and \( \iota \), with the “seam” elements \( \hat{\sigma}_{j,\ell,k} \) having support in the corners. Thus, the shearlet system induces the frequency tiling in Figure 2 (cf. Figure 1 for the frequency tiling of Meyer wavelets).

**Figure 2:** Frequency tiling of the shearlet system.

### 1.3 Recovery Algorithms

We next decide upon a recovery strategy. Compressed sensing offers a variety of such, the most common ones being \( \ell_1 \) minimization and thresholding. We will also use these. However, for preparation purposes to derive an asymptotic scale dependent analysis – the fact that the energy of our model is arbitrary high frequencies requires this approach –, we first perform a band-pass
filtering on \( wL \) (see Eqn. (7)). The band-pass filter will be roughly speaking chosen according to
the band given by the wavelets and shearlets, see Figures 1 and 2, leading to the sequence
\[
(f_j)_j = (P_{\mathcal{R}^2 \setminus \mathcal{M}_0} wL)_j.
\]

The \( \ell_1 \) minimization problem we choose has the form
\[
L_j = \arg \min_{L} \| \Phi^* L \|_1 \text{ subject to } f_j = P_{\mathcal{R}^2 \setminus \mathcal{M}_0} L,
\]
where \( \Phi \) is a Parseval frame. We emphasize that this approach to inpainting minimizes the analysis coefficients and is hence related to the newly introduced cosparsity model \textit{NDEG11, NDEG12}. The choice will be explained further in Subsection 2.2.

The thresholding strategy we choose is brutally simple. We only perform one step of hard thresholding, namely, setting
\[
T_j = \{ i : | \langle f_j, \phi_i \rangle | \geq \beta_j \}
\]
for some threshold \( \beta_j \), the reconstructed image is
\[
L_j = \Phi_1 T_j \Phi^* wL_j.
\]

For the asymptotic analysis, the \( \beta_j \) are explicitly computed in the proofs of Lemmas 4.4 and 5.5. In practice, as is usual with parameters in algorithms, one must be careful when selecting the \( \beta \).

It will be surprising that the geometry of wavelets and shearlets is strong enough to achieve the same asymptotic recovery results as for \( \ell_1 \) minimization for the respective systems. However, thresholding techniques can be viewed as approximations of \( \ell_1 \) minimization and many parallel results have been found for \( \ell_1 \) minimization and thresholding. For example, \( \ell_1 \) minimization \textit{DK12} and thresholding \textit{Kut12} applied to the geometric separation problem both achieve asymptotic separation. In fact, thresholding can be used to separate wavefront sets \textit{Kut12}. Iterative thresholding algorithms have successfully approximated solutions to such diverse sparsity problems as multidimensional NMR spectroscopy \textit{Dro07} and finding row-sparse solutions to underdetermined linear systems \textit{Foul11}.

### 1.4 Microlocal Analysis

One might ask where the geometry we mentioned before will come into play. This can best be explained and illustrated using microlocal analysis in phase space. For a more detailed explanation of the fundamentals of microlocal analysis, see \textit{Hör03}, and for an application of microlocal analysis to derive a fundamental understanding of sparsity-based algorithms using shearlets and curvelets,
see [CD05, Gro11, KL09]. Phase space in this context is indexed by position-orientation pairs \((b, \theta)\) which describe the singular behavior of a distribution. The orientation component \(\theta\) is an element of real projective space, which for simplicity’s sake we shall identify in what follows with \([0, \pi]\). The wavefront set \(WF(f)\) of a distribution \(f\) is roughly the set of elements in the phase space at which \(f\) is nonsmooth. First consider a curvilinear singularity \(C\) along a closed curve \(\tau : [0, 1] \rightarrow \mathbb{R}^2\):

\[
C = \int \delta_{\tau(t)}(\cdot)dt,
\]

where \(\delta_x\) is the usual Dirac delta distribution located at \(x\). As illustrated in Figure 1.4 the wavefront set of \(C\) is

\[
WF(C) = \{(\tau(t), \theta(t)) : t \in [0, 1]\},
\]

where \(\theta(t)\) is the normal direction of \(C\) at \(\tau(t)\). Now consider the model from Section 1.1

\[
f = P_{\mathbb{R}^2 \setminus \mathcal{M}_h} w\mathcal{L}.
\]

As can be seen in Figure 1.4 the wavefront set of \(f\) almost looks like \(f\) itself except that the wavefront set fills all possible angles (i.e., forms a spike) at the end points of the missing mask. This is because at the end points, the distribution is singular in all but the parallel direction. Note that the wavefront set of the linear singularity does not have spikes at the end due to the smooth weight. The difference between the approximate phase space portrait of shearlets and wavelets is demonstrated in Figure 4. The intuition behind the image comes from the fact that shearlets resolve the wavefront set [Gro11, KL09]. Even though our shearlets and wavelets are smooth and thus do not have a wavefront set, by doing a continuous shearlet transform \((f \mapsto \langle f, u^{3/2}\sigma(S_{\ell A_a} \cdot -k)\rangle)\), one can get an approximation of phase space information which takes into account orientation, this is shown in Figure 4. This is similar in spirit to a wavelet spectrogram.

Furthermore, in Figure 5 (Left) the small overlap of the wavefront set of a cluster of shearlets with a spike in the phase space, which represents an end point of the mask of missing information \(\mathcal{M}_h\), can be clearly seen. Thus shearlet clusters are incoherent with the end points, meaning that the clusters do not overlap the spikes strongly in the phase space. However, there is a lot of phase space overlap with the wavefront set away from the endpoints. So it is easy to see how easily a cluster of shearlets can span a gap of missing data (Figure 5 (Right)). Herrmann and Hennenfent call this property the “principle of alignment” which explains why curvelets “attain high compression on synthetic data as well as on real seismic data” [HH08]. The phase space information of curvelets and shearlets are essentially the same [GK12].

### 1.5 Asymptotical Analysis

The width of the area to be inpainted plays a key role, even when using other inpainting techniques. In [CK06], variational inpainting methods are analyzed theoretically, showing that the local thickness of the area to be inpainted affects the success of the inpainting more than the overall size of the area to be inpainted.

Thus our analysis shall also take this into account. We accomplish this by also making the gap size \(h\) dependent on the scale \(j\). This leads to the problem of recovering \(w\mathcal{L}_j\) from knowledge of

\[
f_j = P_{\mathbb{R}^2 \setminus \mathcal{M}_h} w\mathcal{L}_j,
\]
for each scale $j$. Letting $L_j$ denote the recovered image by either one of the proposed algorithms, we will show that asymptotically precise inpainting, i.e.,

$$\frac{\|L_j - wL_j\|_2}{\|wL_j\|_2} \to 0, \quad j \to \infty,$$

is achieved for wavelets provided that $h_j = o(2^{-2j})$ (Theorems 4.3 and 4.7) as $j \to \infty$ and for shearlets provided that $h_j = o(2^{-j})$ (Theorems 5.4 and 5.8) as $j \to \infty$. In fact, this is exactly what one would imagine. Inpainting succeeds provided that the gap size is comparable to the size of the analyzing elements. The scale-dependent gap size allows us to analyze dependency on the size of the shearlets and wavelets in a clear way, providing a theoretical understanding of how inpainting algorithms work even though in practice the gap size is fixed.

### 1.6 Wavelets versus Shearlets

This observation seems to indicate that shearlets indeed perform better than wavelets. However, the previously mentioned theorems just state positive results. In order to show that shearlets outperform wavelets in the model situation which we consider, we require a negative result of the following type: If $h_j > O(2^{-j})$ as $j \to \infty$ and $L_j$ is recovered by wavelets, then

$$\frac{\|L_j - wL_j\|_2}{\|wL_j\|_2} \not\to 0, \quad j \to \infty.$$
And in fact, this is what we will prove in Theorem 6.2. In this sense, we now have a mathematically precise statement showing that shearlets are strictly better for inpainting in our model.

The only slight disappointment is the fact that this statement will only be proven for thresholding as the recovery scheme. We strongly suspect that this result also holds for $\ell_1$ minimization. However, we are not aware of any analysis tools strong enough to derive these results also in this situation.

1.7 Our Approach

Our analysis has focused primarily on revealing the fundamental mathematical concepts which lead to successful image inpainting using wavelets or shearlets. The viewpoint we take, however, is that this is just the “tip of the iceberg,” and the main results are susceptible of very extensive generalizations and extensions. For example, our asymptotic analysis is based on a vertical mask of missing data from a horizontal wavefront. Other masks applied to curved wavefronts could be considered. The microlocal bending techniques employed in [DK12] seem to suggest that this approach will yield desirable results.

1.8 Contents

We begin in Section 2 with an abstract analysis of data recovery via $\ell_1$ minimization introducing clustered sparsity and concentration in a Hilbert space as tools. We then apply the results in Section 2 to a particular class of inpainting problems which are described in Section 3. In Sections 4 and 5 we prove that both wavelets and shearlets, respectively, are able to inpaint a missing band but that shearlets can handle wider gaps. It is shown in Section 6 that the inpainting result for wavelets in Section 4 is tight; i.e., shearlets strictly outperform wavelets in the considered model situation. We discuss future directions of research and limitations of the current model in Section 7. Finally, Section 8 is an appendix that contains auxiliary results concerning shearlets needed for Section 5.

2 Abstract Analysis of Data Recovery

We start by analyzing missing data recovery via $\ell_1$ minimization and thresholding in an abstract model situation. The error estimates we will derive can be applied in a variety of situations. In this paper, we will focus on utilizing them to analyze inpainting via wavelets and shearlets following a continuum domain model. In fact, these error estimates will later on be applied to each scale while deriving an asymptotic analysis.

2.1 Abstract Model

Let $x^0 \in \mathcal{H}$ be a signal in a Hilbert space $\mathcal{H}$. To model the data recovery problem correctly, we assume that $\mathcal{H}$ can be decomposed into a direct sum

$$\mathcal{H} = \mathcal{H}_M \oplus \mathcal{H}_K$$

of a subspace $\mathcal{H}_M$ which is associated with the missing part of $x^0$ and a subspace $\mathcal{H}_K$ which relates to the known part of the signal. Further, let $P_M$ and $P_K$ denote the orthogonal projections onto those subspaces, respectively. The problem of data recovery can then be formulated as follows: Assuming that $P_K x^0$ is known to us, recover $x^0$. 
Following the philosophy of compressed sensing, suppose that there exists a Parseval frame $\Phi$ which – in a way yet to be made precise – sparsifies the original signal $x^0$. Either $\Phi$ can be selected non-adaptively such as choosing a wavelet or shearlet system which will be our avenue in the sequel, or $\Phi$ can be chosen adaptively using dictionary learning algorithms such as [AE06, FAH99, OF97].

To already draw the connection to the special situation of inpainting at this point, assume that $H = L^2(\mathbb{R}^2)$. If the measurable subset $B \subseteq \mathbb{R}^2$ is the missing area of the image, we set $H_K = L^2(\mathbb{R}^2 \setminus B)$ and $H_M = L^2(B)$.

2.2 Inpainting via $\ell_1$ Minimization

A first methodology from compressed sensing to achieve recovery is $\ell_1$ minimization, which recovers the original signal by solving

\[
(\text{Inp}) \quad x^* = \text{argmin}_x \|\Phi^* x\|_1 \text{ subject to } P_K x = P_K x^0.
\]

We wish to remark that in this problem, the norm is placed on the analysis coefficients rather than on the synthesis coefficients as in [DE03, EB02] and other papers on basis pursuit. Since we intend to also apply this optimization problem in the situation when $\Phi$ does not form a basis but merely a frame, the analysis and synthesis approaches are different. One reason to do this is to avoid numerical instabilities which are expected to occur since, for each $x \in H$, the linear system of equations $x = \Phi c$ has infinitely many solutions, only the specific solution $\Phi^* x$ is analyzed. Also, since we are only interested in correctly inpainting and not in computing the sparsest expansion, we can circumvent possible problems by solving the inpainting problem by selecting a particular coefficient sequence which expands out to the $x$, namely the analysis sequence. A similar strategy was pursued in [KKZ11] and [Kut12]. Various inpainting algorithms which are based on the core idea of (Inp) combined with geometric separation are heuristically shown to be successful in [CCS10, DFL+12, ESQ05].

Interestingly, this minimization problem can be also regarded as a mixed $\ell_1$-$\ell_2$ problem [KT09], since the analysis coefficient sequence $\Phi^* x$ is exactly the minimizer of

\[
\min \{ \|c\|_2 : c \in \ell_2, x = \Phi c \},
\]

that is, the coefficient sequence which is minimal in the $\ell_2$ norm. The optimization problem in (Inp) may also be thought of a relaxation of the cosparsity problem

\[
x^* = \text{argmin}_x \|\Phi^* x\|_0 \text{ subject to } P_K x = P_K x^0.
\]

Theoretical results concerning cosparsity may be found in [NDEG11, NDEG12].

We also consider the noisy case. Assume now that we know $\tilde{x} = P_K x^0 + n$, where $x^0$ and $n$ are unknown, but $n$ is assumed to be small in the sense of $\|\Phi^* n\|_1 \leq \epsilon$ for small $\epsilon$. Also, clearly $n = P_K n$. Then we solve

\[
(\text{InpNOISE}) \quad \tilde{x}^* = \text{argmin}_x \|\Phi^* x\|_1 \text{ subject to } P_K x = \tilde{x}.
\]

To analyze this optimization problem, we require the following notion, which intuitively measures the maximal fraction of the total $\ell_1$ norm which can be concentrated to the index set $\Lambda$ restricted to functions in $H_M$. In this sense, the geometric relation between the missing part $H_M$ and expansions in $\Phi$ is encoded.
Definition 2.1. Let \( \Phi \) be a Parseval frame, and let \( \Lambda \) be a index set of coefficients. We then define the concentration on \( \mathcal{H}_M \) by

\[
\kappa = \kappa(\Lambda, \mathcal{H}_M) = \sup_{f \in \mathcal{H}_M} \frac{\|1_\Lambda \Phi^* f\|_1}{\|\Phi^* f\|_1}.
\]

This notion allows us to formulate our first estimate concerning the \( \ell_2 \) error of the reconstruction via (INP). The reader should notice that the considered error \( \|x^* - x^0\|_2 \) is solely measured on \( \mathcal{H}_M \), the masked space, since \( P_K x^* = P_K x^0 \) due to the constraint in (INP). Another important notion is that of clustered sparsity.

Definition 2.2. Fix \( \delta > 0 \). Given a Hilbert space \( \mathcal{H} \) with a Parseval frame \( \Phi \), \( x \in \mathcal{H} \) is \( \delta \)-clustered sparse in \( \Phi \) (with respect to \( \Lambda \)) if

\[
\|1_\Lambda \Phi^* x\|_1 \leq \delta,
\]

where given a space \( X \) and a subset \( A \subseteq X \), \( A^c \) denotes \( X \setminus A \).

We now present a pair of lemmas which were first published in [KKZ11] without proof.

Lemma 2.3. Fix \( \delta > 0 \) and suppose that \( x^0 \) is \( \delta \)-clustered sparse in \( \Phi \). Let \( x^* \) solve (INP). Then

\[
\|x^* - x^0\|_2 \leq \frac{2\delta}{1 - 2\kappa}.
\]

The noiseless case Lemma 2.3 holds as a corollary to the case with noise, which follows.

Lemma 2.4. Fix \( \delta > 0 \) and suppose that \( x^0 \) is \( \delta \)-clustered sparse in \( \Phi \). Let \( \tilde{x}^* \) solve (INPNoise). Also assume that the noise satisfies \( \|\Phi^* n\|_1 \leq \epsilon \). Then

\[
\|\tilde{x}^* - x^0\|_2 \leq \frac{2\delta + (3 + \kappa^2)\epsilon}{1 - 2\kappa}.
\]

Proof. Since \( \Phi \) is Parseval,

\[
\|\tilde{x}^* - x^0\|_2 \leq \|\Phi^* (\tilde{x}^* - x^0)\|_1. \tag{5}
\]

We invoke the relation \( P_K \tilde{x}^* = P_K x^0 + n \), which implies that \( P_K (\tilde{x}^* - x^0) = n \). Using the definition of \( \kappa \), we obtain

\[
\|1_\Lambda \Phi^* (\tilde{x}^* - x^0)\|_1 \leq \|1_\Lambda \Phi^* P_M (\tilde{x}^* - x^0)\|_1 + \|1_\Lambda \Phi^* n\|_1 \leq \kappa \|\Phi^* P_M (\tilde{x}^* - x^0)\|_1 + \|\Phi^* n\|_1 \leq \kappa \|\Phi^* (\tilde{x}^* - x^0)\|_1 + (1 + \kappa)\epsilon. \tag{6}
\]

It follows that

\[
\|\Phi^* (\tilde{x}^* - x^0)\|_1 = \|1_\Lambda \Phi^* (\tilde{x}^* - x^0)\|_1 + \|1_\Lambda \Phi^* (\tilde{x}^* - x^0)\|_1 \leq \kappa \|\Phi^* (\tilde{x}^* - x^0)\|_1 + (1 + \kappa)\epsilon.
\]

The clustered sparsity of \( x^0 \) now implies

\[
\|\Phi^* (\tilde{x}^* - x^0)\|_1 \leq \frac{1}{1 - \kappa} \left( \|1_\Lambda \Phi^* (\tilde{x}^* - x^0)\|_1 + (1 + \kappa)\epsilon \right) \leq \frac{1}{1 - \kappa} \left( \|1_\Lambda \Phi^* \tilde{x}^*\|_1 + \delta + (1 + \kappa)\epsilon \right).
\]
Applying the sparsity of $x^0$ again and the minimality of $\tilde{x}^*$, we have

\[
\|1_{\Lambda^c} \Phi^* \tilde{x}^*\|_1 = \|\Phi^* \tilde{x}^*\|_1 - \|1_{\Lambda^c} \Phi^* \tilde{x}^*\|_1 \leq \|\Phi^*(x^0 + n)\|_1 - \|1_{\Lambda^c} \Phi^* \tilde{x}^*\|_1 \\
\leq \|\Phi^* x^0\|_1 - \|1_{\Lambda^c} \Phi^* \tilde{x}^*\|_1 + 1 \leq \|\Phi^* x^0\|_1 + \|1_{\Lambda^c} \Phi^*(\tilde{x}^* - x^0)\|_1 - \|1_{\Lambda^c} \Phi^* x^0\|_1 + 1 \\
\leq \|\tilde{x}^* - x^0\|_1 + \delta + \epsilon.
\]

Using (6) and (2.2), this leads to

\[
\|\Phi^* (\tilde{x}^* - x^0)\|_1 \leq \frac{1}{1 - \kappa} (\|1_{\Lambda^c} \Phi^* \tilde{x}^*\|_1 + \delta + (1 + \kappa)\epsilon) \\
\leq \frac{1}{1 - \kappa} (\|1_{\Lambda^c} \Phi^* (\tilde{x}^* - x^0)\|_1 + 2\delta + \frac{(2 + \kappa)\epsilon}{1 - \kappa}) \\
\leq \frac{1}{1 - \kappa} (\|\tilde{x}^* - x^0\|_1 + 2\delta + \frac{(3 + 2\kappa)\epsilon}{1 - \kappa}).
\]

Combining this with (5), we finally obtain

\[
\|\tilde{x}^* - x^0\|_2 \leq \left(1 - \frac{\kappa}{1 - \kappa}\right)^{-1} \frac{2\delta + (3 + 2\kappa)\epsilon}{1 - \kappa} = \frac{2\delta + (3 + 2\kappa)\epsilon}{1 - 2\kappa}
\]

We now establish a relation between the concentration $\kappa(\Lambda, H_M)$ on $H_M$ and the notion of cluster coherence $\mu_c$ first introduced in [DK12]. For this, by abusing notation, we will write $P_M \Phi = \{P_M \varphi_i\}_i$ and $P_K \Phi = \{P_K \varphi_i\}_i$ for the projected frame elements.

To first introduce the notion of cluster coherence, recall that in many studies of $\ell_1$ optimization, one utilizes the mutual coherence $\mu(\Phi_1, \Phi_2) = \max_{j} \max_{i} |\langle \varphi_{1i}, \varphi_{2j} \rangle|$, whose importance was shown by [DH01]. This may be called the singleton coherence. We modify the definition to take into account clustering of the coefficients arising from the geometry of the situation.

**Definition 2.5.** Let $\Phi_1 = \{\varphi_{1i}\}_{i \in I}$ and $\Phi_2 = \{\varphi_{2j}\}_{j \in J}$ lie in a Hilbert space $\mathcal{H}$ and let $\Lambda \subseteq I$. Then the cluster coherence $\mu_c(\Lambda, \Phi_1; \Phi_2)$ of $\Phi_1$ and $\Phi_2$ with respect to $\Lambda$ is defined by

\[
\mu_c(\Lambda, \Phi_1; \Phi_2) = \max_{j \in J} \sum_{i \in \Lambda} |\langle \varphi_{1i}, \varphi_{2j} \rangle|.
\]

The following relation is a specific case of Proposition 3.1 in [KKZ11]. We include a proof for completeness.

**Lemma 2.6.** We have

\[
\kappa(\Lambda, H_M) \leq \mu_c(\Lambda, P_M \Phi; P_M \Phi) = \mu_c(\Lambda, P_M \Phi; \Phi).
\]
Proof. For each \( f \in \mathcal{H}_M \), we choose a coefficient sequence \( \alpha \) such that \( f = \Phi \alpha \) and \( \| \alpha \|_1 \leq \| \beta \|_1 \) for all \( \beta \) satisfying \( f = \Phi \beta \). Invoking the fact that \( \Phi \) is a tight frame, hence \( f = \Phi \Phi^* \Phi \alpha \), and the fact that \( f = (P_M \Phi) \alpha \), we obtain

\[
\| \mathbf{1}_\Lambda \Phi^* f \|_1 = \| \mathbf{1}_\Lambda (P_M \Phi)^* f \|_1 = \| \mathbf{1}_\Lambda (P_M \Phi)^* (P_M \Phi) \alpha \|_1 \\
\leq \sum_{i \in \Lambda} \left( \sum_j |\langle P_M \phi_i, P_M \phi_j \rangle| |\alpha_j| \right) = \sum_j \left( \sum_{i \in \Lambda} |\langle P_M \phi_i, P_M \phi_j \rangle| \right) |\alpha_j| \\
\leq \mu_c(\Lambda, P_M \Phi; P_M \Phi) \| \alpha \|_1 \leq \mu_c(\Lambda, P_M \Phi; P_M \Phi) \| \Phi^* \Phi \alpha \|_1 \\
= \mu_c(\Lambda, P_M \Phi; P_M \Phi) \| \Phi^* f \|_1.
\]

Combining Lemmata 2.3 and 2.6 proves the final noiseless estimate and combining Lemmata 2.4 and 2.6 proves the final estimate with noise:

**Proposition 2.7.** Fix \( \delta > 0 \) and suppose that \( x^0 \) is \( \delta \)-clustered sparse in \( \Phi \). Let \( x^* \) solve \((\text{INP})\). Then

\[
\| x^* - x^0 \|_2 \leq \frac{2\delta}{1 - 2\mu_c(\Lambda, P_M \Phi; \Phi)}.
\]

**Proposition 2.8.** Fix \( \delta > 0 \) and suppose that \( x^0 \) is \( \delta \)-clustered sparse in \( \Phi \). Let \( \tilde{x}^* \) solve \((\text{INPNoise})\). Also assume that the noise satisfies \( \| \Phi^* n \|_1 \leq \epsilon \). Then

\[
\| \tilde{x}^* - x^0 \|_2 \leq \frac{2\delta + (3 + 2\kappa)\epsilon}{1 - 2\mu_c(\Lambda, P_M \Phi; \Phi)}.
\]

Let us briefly interpret this estimate, first focusing on the noiseless case. As expected the error decreases linearly with the clustered sparsity. It should also be emphasized that both clustered sparsity and cluster coherence depend on the chosen “geometric set of indices” \( \Lambda \). Thus this set is crucial for determining whether \( \Phi \) is a good dictionary for inpainting. This will be illustrated in the sequel when considering a particular situation; however, \( \Lambda \) is merely an analysis tool and explicit knowledge of it is not necessary to recover data. Note that in general, the larger the set \( \Lambda \) is, the smaller \( \| \mathbf{1}_\Lambda \Phi^* x^0 \|_1 \) is (i.e., \( x^0 \) is \( \delta \)-relatively sparse for a smaller \( \delta \)) and the larger the cluster coherence is. This seems to be a contradiction, but if \( \Phi \) sparsifies \( x^0 \) well, then a small set \( \Lambda \) can be chosen which keeps \( \| \mathbf{1}_\Lambda \Phi^* x^0 \|_1 \) small. Finally, considering the noisy case, as also expected the error estimate depends linearly on the \( \ell_2 \) bound for the noise.

### 2.3 Inpainting via Thresholding

Another fundamental methodology from compressed sensing for sparse recovery is thresholding. The beauty of this approach lies in its simplicity and its associated fast algorithms. Typically, it is also possible to prove success of recovery in similar situations as in which \( \ell_1 \) minimization succeeds.

Various thresholding strategies are available such as iterative thresholding, etc. It is thus surprising that the most simple imaginable strategy, which is to perform just one step of hard thresholding, already allows for error estimates as strong of for \( \ell_1 \) minimization. We start by presenting this thresholding strategy. For technical reasons, – note also that this is no restriction
One-Step-Thresholding

Parameters:

- Incomplete signal $\tilde{x} = P_K x^0$ (noiseless) or $P_K x^0 + n$ (with noise).
- Thresholding parameter $\beta$.

Algorithm:

1) Threshold Coefficients with Respect to Frame $\Phi$:
   a) Compute $\langle \tilde{x}, \phi_i \rangle$ for all $i$.
   b) Apply threshold and set $T = \{ i : |\langle \tilde{x}, \phi_i \rangle| \geq \beta \}$.

2) Reconstruct Original Signal:
   a) Compute $x^* = \Phi^* P_M x^0$.

Output:

- Significant thresholding coefficients: $T$.
- Approximation to $x^0$: $x^*$.

Figure 6: One-Step-Thresholding Algorithm to reconstruct $x^0$ from noiseless $P_K x^0$ or noisy $P_K x^0 + n$.

at all – we now assume that the Parseval frame $\Phi = (\phi_i)_i$ consists of frame vectors with equal norm, i.e., $\|\phi_i\| = c$ for all $i$.

The following result provides us with an estimate for the $\ell_2$ error of the synthesized signal $x^*$ computed via One-Step-Thresholding.

Proposition 2.9. Let $T$ and $x^*$ be computed via the algorithm One-Step-Thresholding (Figure 6) for noiseless data, and for $\delta > 0$ assume that $x^0$ is relatively sparse in $\Phi$ with respect to $T$. Then

$$\|x^* - x^0\|_2 \leq c \left[ \delta + \|1_T \Phi^* P_M x^0\|_1 \right].$$

As before, Proposition 2.9 follows as a corollary to the case with noise:

Proposition 2.10. Let $T$ and $x^*$ be computed via the algorithm One-Step-Thresholding for data with noise, and for $\delta > 0$ assume that $x^0$ is relatively sparse in $\Phi$ with respect to $T$. Also assume that the noise satisfies $\|\Phi^* n\|_1 \leq \epsilon$. Then

$$\|x^* - x^0\|_2 \leq c \left( \|1_T \Phi^* P_M x^0\|_1 + \delta + \epsilon \right).$$
Proof. Invoking the decomposition of $H$ and the fact that $\Phi$ is Parseval,

$$\|x^* - x^0\|_2 = \|\Phi 1^\tau \Phi^* (P_K x^0 + n) - \Phi \Phi^* P_K x^0 - P_M x^0\|_2 = \|\Phi 1^\tau \Phi^* P_K x^0 + \Phi 1\tau \Phi^* n - P_M x^0\|_2.$$ 

Since 

$$P_M x^0 = \Phi 1^\tau \Phi^* P_M x^0 + \Phi 1\tau \Phi^* P_M x^0$$

and $P_K x^0 + P_M x^0 = x^0$, it follows that 

$$\|x^* - x^0\|_2 \leq \|\Phi 1^\tau \Phi^* x^0\|_2 + \|\Phi 1\tau \Phi^* P_M x^0\|_2 + \|\Phi 1\tau \Phi^* n\|_2.$$ 

It follows from the equal-norm condition on the frame $\Phi$ that for any $\ell_1$ sequence $x$,

$$\|\Phi x\|_2 \leq c\|x\|_1.$$ 

Applying the clustered sparsity of $x^0$ we obtain 

$$\|x^* - x^0\|_2 \leq c \left(\|1^\tau \Phi^* P_M x^0\|_1 + \delta + \epsilon\right),$$

which is what we intended to prove. 

As before, let us also interpret this estimate. Now the situation is slightly different from the estimate for the $\ell_1$ approach. Again, the estimate depends linearly on the clustered sparsity and the noise. The difference now is the appearance of the term $\|1^\tau \Phi^* P_M x^0\|_1$ in the numerator instead of the cluster coherence in the denominator. Note, however, that 

$$\|1^\tau \Phi^* P_M x^0\|_1 \leq \kappa \|\Phi^* P_M x^0\|_1 \leq \mu_c(T, P_M \Phi; \Phi) \|\Phi^* P_M x^0\|_1.$$ 

Thus both in the $\ell_1$ minimization case Proposition 2.7 and in the thresholding case Proposition 2.9, the bound on the error is lower when the cluster coherence is lower. Furthermore, $\|\Phi^* P_M x^0\|_1$ is a quantification of how much of the signal is missing, which clearly can not be too high.

3 Mathematical Model

We next provide a specific mathematical model which is motivated by the fact that images are typically governed by edges, which can most prominently be seen in, for example, seismic imaging (Figure 7). Following this line of thought, our model is based on line singularities – which can as explained later be extended to curvilinear singularities – with missing data of the forms as gaps or holes. In this section, such a model for the original image and the mask will be introduced. Since the analysis we derive later is based on the behavior in Fourier domain, the Fourier content of the models is another focus.

3.1 Image Model

Inspired by seismic data with missing traces, an example of which is found in Figure 7, we define our mathematical model. The data can be viewed as a collection of curvilinear singularities which are missing nearly vertical strips of information. We first simplify the model by considering linear singularities. As shearlets are directional systems, we then simplify the model so that the linear singularity is horizontal. The specific mathematical model that we shall analyze is as follows. Let
Let $w : \mathbb{R} \mapsto [0, 1]$ be a smooth function that is supported in $[-\rho, \rho]$, where we always assume that $\rho$ is sufficiently large, in particular, much larger than $h$ (a measure of the missing data which will be defined in the next subsection). For now, we consider as a prototype of a line singularity the weighted distribution $w \mathcal{L}$ acting on tempered distributions $\mathcal{S}'(\mathbb{R}^2)$ by

$$\langle w \mathcal{L}, f \rangle = \int_{-\rho}^{\rho} w(x_1) f(x_1, 0) dx_1.$$  

Notice that this distribution is supported on the segment $[-\rho, \rho] \times \{0\}$ of the $x$-axis, hence can be employed as a model for a horizontal linear singularity. The weighting was chosen to ensure that we are dealing with an $L_2$-function after filtering. The Fourier transform of $w \mathcal{L}$ can be computed to be

$$\langle \hat{w} \mathcal{L}, f \rangle = \langle w \mathcal{L}, \hat{f} \rangle = \int_{\mathbb{R}} \hat{w}(\xi_1) \int_{\mathbb{R}} f(\xi_1, \xi_2) d\xi_2 d\xi_1.$$  

Let now $\hat{F}_j$ be a filter corresponding to the frequency corona $C_j$ at level $j$ (see Equation (2)), defined by its Fourier transform $F_j$,

$$F_j = \sum_{\iota \in \{h, v, d\}} (W^{\iota}(2^{-2j} \xi) + W^{\iota}(2^{-2j-1} \xi)).$$

To simplify the proofs for wavelets, we also define

$$\hat{F}_j = \sum_{\iota \in \{h, v, d\}} W^{\iota}(2^{-j} \xi),$$

so that $F_j = \hat{F}_{2j} + \hat{F}_{2j+1}$. We use two bands for the wavelets so that the wavelet and shearlet systems will be compared on the same frequency corona. This makes sense as the base ($j = 1$) dilation for the 2D wavelets has determinant 4, while the base dilation for the shearlets has determinant 8. We consider the filtered version of $w \mathcal{L}$ which we denote by $w \mathcal{L}_j$, i.e.,

$$w \mathcal{L}_j = w \mathcal{L} \ast \hat{F}_j = \int_{\mathbb{R}^2} w \mathcal{L}(\cdot - t) \hat{F}_j(t) dt. \quad (7)$$

The next result provides us with an estimate of the norm of $w \mathcal{L}_j$. 

---

**Figure 7:** Synthetic seismic data with randomly distributed missing traces – Hennenfent and Hermann [H106]
Lemma 3.1. For some $c > 0$,
$$\|w L_j\|_2 \geq c 2^j, \quad j \to \infty.$$ 

Proof. We have
$$\|w L_j\|_2 \geq \left( \int_{\xi_1 \in \mathbb{R}} |\hat{w}(\xi_1)|^2 d\xi_1 \int_{|\xi_2| \in [2^{2j-4} c_0, 2^{2j-1} c_0]} d\xi_2 \right)^{1/2} \approx c 2^j.$$

3.2 Masks

Inspired by the missing sensor scenario in seismic data we will define the mask of a missing piece of the image as follows. The mask $M_h$ is a vertical strip of diameter $2h$ and is given by
$$M_h = \mathbb{1}_{\{|x_1| \leq h\}}.$$

For an illustration, we refer to Figure 8.

Figure 8: Mask $M_h$ (gray shaded region), together with the linear singularity $wL$ (horizontal line with dashed center indicating part masked out).

For the convenience of the reader, we compute the associated Fourier transforms, where as usual we set $\text{sinc}(y) = \sin(\pi y)/(\pi y)$ for $y \in \mathbb{R}$.

Lemma 3.2. We have
$$\hat{M}_h = 2h \text{sinc}(2h \xi_1) \hat{L}_y,$$
where $L_y$ is the distribution acting as
$$\langle L_y, f \rangle = \int f(0, y) dy$$
and $\langle \hat{L}_y, f \rangle = \int f(x, 0) dx$.

Proof. Define the planar Heaviside by $H(x) = \mathbb{1}_{\{x_1 \geq 0\}}$. Since $L_y = \frac{\partial}{\partial x_1} H$, we have $\hat{H}(\xi) = (2\pi i \xi_1)^{-1} \hat{L}_y$. We now express $M_h$ in terms of $H$ by
$$M_h = H(x + (h, 0)) - H(x - (h, 0)).$$
This leads to
$$\hat{M}_h = (e^{2\pi i h \xi_1} - e^{-2\pi i h \xi_1})(2\pi i \xi_1)^{-1} \hat{L}_y = 2 \sin(2\pi h \xi_1)/(2\pi \xi_1) \hat{L}_y = 2h \text{sinc}(2h \xi_1) \hat{L}_y.$$
The proof is finished.
3.3 Transfer of Abstract Setting

All of the main proofs in Sections 4 and 5 will follow a particular pattern. Either Proposition 2.7 (in the case of ℓ₁ minimization) or Proposition 2.9 (in the case of thresholding) is applied to the situation in which \( x^0 \) is chosen to be the filtered linear singularity \( wL_j \), the Hilbert space \( \mathcal{H}_M \) is defined by \( \{ fM_k : f \in L^2(\mathbb{R}^2) \} \), and \( \Phi \) is either the Parseval system of Meyer wavelets or of shearlets at scale \( j \).

In the analysis that follows, \( \delta_j \) will denote the optimal \( \delta \)-clustered sparsity for filtered coefficients. That is, for \( \ell_1 \) minimization with a fixed filter level \( j \), we will fix a set \( \Lambda_j \) of significant coefficients of \( \Phi = \{ \psi_\lambda \} \) and set

\[
\delta_j = \sum_{\lambda \in \Lambda_j} |\langle wL_j, \psi_\lambda \rangle|.
\]

Similarly, we will analyze thresholding schemes by setting

\[
\delta_j = \sum_{\lambda \in T_j} |\langle wL_j, \psi_\lambda \rangle|,
\]

where the \( T_j \) are the significant coefficients in One-Step-Thresholding Algorithm. The inpainting accomplished (i.e., the solution in Proposition 2.7 or Proposition 2.9) on the filtered levels \( j \) will be denoted by \( L_j \). \( wL_j \) will denote the filtered real image; that is, \( wL \ast \hat{F}_j \), where \( wL \) is the original, complete image. Thus, the main theorems in Sections 4 and 5 will show that

\[
\frac{\|L_j - wL_j\|_2}{\|wL_j\|_2} \to 0, \quad j \to \infty.
\]

The results will specifically depend on the asymptotic behavior of the gap \( h_j \). For the proofs involving the Meyer system, the following notation will also be useful

\[
\tilde{wL}_j = wL \ast \hat{F}_j^\vee.
\]

4 Positive Results for Wavelet Inpainting

We begin by proving theoretically for the first time what has been known heuristically; namely, that wavelets can successfully inpaint an edge as long as not too much is missing. In Section 4.1, we investigate the inpainting results of \( \ell_1 \) minimization by estimating the \( \delta \)-clustered sparsity \( \delta_j \) and cluster coherence \( \mu_c \) with respect to \( \Phi = \{ \psi_\lambda : \lambda = (\iota,j,k), \iota = h, v, d; k \in \mathbb{Z}^2 \} \) and a proper chosen index set \( \Lambda_j \). In Subsection 4.2, we similarly give the estimation of \( \delta_j \) and \( \mu_c \) for inpainting using thresholding.

4.1 ℓ₁ Minimization

In what follows, we use the compact notation \( \langle a \rangle := (1 + |a|^2)^{1/2} \). We first need to choose the set of significant coefficients appropriately. We do this by setting

\[
\Lambda_j = \{(\iota; j, k) : |k_1| \leq \rho n_j 2^j, |k_2| \leq n_j, \iota = h, v, d\},
\]

where \( n_j = 2^{\epsilon j} \). This choice of \( \Lambda_j = \tilde{\Lambda}_{2j} \cup \tilde{\Lambda}_{2j+1} \) forces the clustered sparsity to grow slower than the growth rate of \( \|wL_j\|_2 \):
Lemma 4.1. $\delta_j = o(1) = o(\|wL_j\|_2)$, $j \to \infty$.

Proof. By definition, we have

$$\delta_j = \sum_{\lambda \in \Lambda_j^c} |\langle wL_j, \psi_\lambda \rangle| \leq \sum_{\lambda \in \Lambda_j^c} (|\langle \tilde{w}L_{2j}, \psi_\lambda \rangle| + |\langle \tilde{w}L_{2j+1}, \psi_\lambda \rangle|) \leq \sum_{\lambda \in \Lambda_j^c} (|\langle \tilde{w}L_{2j}, \psi_\lambda \rangle| + \sum_{\lambda \in \Lambda_{2j+1}} |\langle \tilde{w}L_{2j+1}, \psi_\lambda \rangle|) =: \tilde{\delta}_{2j} + \tilde{\delta}_{2j+1}.$$  

We now compute

$$\tilde{\delta}_j = \sum_{\lambda \in \Lambda_j^c} |\langle \tilde{w}L_j, \psi_\lambda \rangle| = \sum_{\lambda \in \Lambda_j^c} |\langle \tilde{w}L_j, \psi_\lambda \rangle|;$$  

that is,

$$\tilde{\delta}_j = \sum_{\lambda \in \Lambda_j^c} \left| \int_{\mathbb{R}^2} 2^{-j} \tilde{w}(\xi) \tilde{F}_j(\xi) W^\varepsilon(\xi/2^j) e^{-2\pi i (k, \xi/2^j)} d\xi \right|,$$

where $\tilde{G}_j(\xi) = 2^{-j} \tilde{w}(\xi) \tilde{F}_j(\xi) W^\varepsilon(\xi/2^j)$ is a smooth and compactly supported function that is essentially supported on $[-1/\rho, 1/\rho] \times [-2^j c_0, 2^j c_0]$.

Applying the change of variable $(\xi_1, \xi_2) \rightarrow (\rho^{-1} \xi_1, 2^j \xi_2)$ ensures that $\tilde{G}_j(\rho^{-1} \xi_1, 2^j \xi_2)$ is smooth and compactly supported independent of $j$. Then

$$\int_{\mathbb{R}^2} \tilde{G}_j(\xi) e^{-2\pi i (k, \xi/2^j)} d\xi \leq c_N \|\tilde{G}_j\|_\infty (\rho^{-1} 2^j)^{\|\rho^{-1} k_1/2^j, k_2\|} - N \leq c_N (\rho^{-1} 2^j)^{\|\rho^{-1} k_1/2^j, k_2\|} - N.$$

Consequently, $\tilde{\delta}_j/c_N$ is bounded above by

$$\rho^{-1} 2^j \sum_{\lambda \in \Lambda_j^c} (\|\rho^{-1} k_1/2^j, k_2\|) - N,$$

$$\leq \rho^{-1} 2^j \left( \sum_{|k_1| \geq \rho n_j 2^j} (\|\rho^{-1} k_1/2^j, k_2\|) - N + \sum_{|k_1| \leq \rho n_j 2^j, |k_2| \geq n_j} (\|\rho^{-1} k_1/2^j, k_2\|) - N \right) \leq \rho^{-1} 2^j \left( \int_{\rho n_j 2^j}^{\infty} \int_{\mathbb{R}} (\|\rho^{-1} k_1/2^j, x_2\|) - N dx_2 dx_1 + \int_{\rho n_j 2^j}^{\infty} \int_{n_j}^{\infty} (\|\rho^{-1} k_1/2^j, x_2\|) - N dx_2 dx_1 \right) \leq \int_{n_j}^{\infty} \int_{\mathbb{R}} (\|x_1, x_2\| - N dx_2 dx_1 + \int_{n_j}^{\infty} \int_{\mathbb{R}} (\|x_1, x_2\| - N dx_2 dx_1 \leq 2^{j(1-N)}.$$

Thus,

$$\delta_j \leq c 2^{2^j(1-N)} + 2^{(2^j+1)(1-N)}$$

and for $N$ large enough, $\delta_j \to 0$ as $j \to \infty$.  

20
On the other hand, the choice of $\Lambda_j$ offers low cluster coherence as well:

**Lemma 4.2.** For $h_j = o(2^{-2j})$ as $j \to \infty$, we have

$$\mu_c(\Lambda_j, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}) \to 0, \quad j \to \infty.$$  

*Proof.* We again first consider $\tilde{\Lambda}_j$. By definition, we have

$$\mu_c(\tilde{\Lambda}_j, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}) = \max_{\lambda'} \sum_{\lambda \in \tilde{\Lambda}_j} |\langle M_{h_j}\psi_\lambda, \psi_{\lambda'} \rangle| = \max_{\lambda'} \sum_{\lambda \in \tilde{\Lambda}_j} |\langle \tilde{M}_{h_j} \psi_\lambda, \tilde{\psi}_{\lambda'} \rangle|.$$  

Note that for $\lambda = (\tau, j, k)$, we can choose $\lambda' = (\tau', j, 0)$.

$$\langle \tilde{M}_{h_j} \ast \tilde{\psi}_\lambda, \tilde{\psi}_{\lambda'} \rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{M}_{h_j}(\xi) \tilde{\psi}_\lambda(\tau - \xi) d\xi d\tau'$$

$$\quad = \int_{\mathbb{R}^2} \int_{\mathbb{R}} 2h_j \text{sinc}(2h_j \xi_1) \tilde{\psi}_\lambda(\tau - (\xi_1, 0)) d\xi_1 d\tau'$$

$$\quad = \int_{\mathbb{R}} 2h_j \text{sinc}(2h_j \xi_1) \left[ \int_{\mathbb{R}^2} 2^{-2j} W'(\tau - (\xi_1, 0)) e^{-2\pi i k_1 \xi_1} d\xi_1 d\tau' \right]$$

$$\quad = 2^{j} 2h_j \int_{\mathbb{R}^2} \text{sinc}(2^{j} 2h_j \xi_1) W'((\tau - (\xi_1, 0))) e^{2\pi i k_1 \xi_1} d\xi_1 W'(\tau) e^{-2\pi i (k, \tau)} d\tau'$$

$$\quad =: 2^{j} 2h_j \int_{\mathbb{R}^2} \hat{g}_j(\tau) e^{-2\pi i (k, \tau)} d\tau,$$

where

$$\hat{g}_j(\tau) := W'(\tau) \int_{\mathbb{R}} \text{sinc}(2^{j} 2h_j \xi_1) \text{sinc}(2^{j} 2h_j \xi_1) e^{2\pi i (k_1, \xi_1)} d\xi_1$$

(8)

is a smooth function supported on a box independent of $j$. Hence, $|\int \hat{g}_j(\tau) e^{-2\pi i k \tau} d\tau| \leq c_N \|\hat{g}_j\|_\infty \langle |k| \rangle^{-N}$, and

$$||\hat{g}_j||_\infty \leq c \sup_{\tau} \int |\text{sinc}(2^{j} 2h_j \xi_1)| \langle W(\tau - (\xi_1, 0)) \rangle d\xi_1 \leq c \| \text{sinc}(2^{j} 2h_j \xi_1) \|_2 \leq c(2^{j} h_j)^{-1/2}.$$  

Consequently, we have

$$\mu_c(\tilde{\Lambda}_j, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}) \leq c_N 2^{j} 2h_j (2^{j} h_j)^{-1/2} \sum_{k \in \mathbb{Z}^2} \langle |k| \rangle^{-N} \leq c_N (2^{j} h_j)^{1/2},$$

where

$$\mu_c(\Lambda_j, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}) = \mu_c(\tilde{\Lambda}_{2j}, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}) + \mu_c(\tilde{\Lambda}_{2j+1}, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}).$$

which goes to 0 as $j \to \infty$ by assumption. \hfill \qed

We would like to remark at this point that we do not need the strong condition that $h_j = o(2^{-2j})$ as $j \to \infty$. In fact, carefully handling the constants in the proof of Lemma 4.2 will lead us to the condition

$$\mu_c(\Lambda_j, \{M_{h_j}\psi_\lambda\}; \{\psi_\lambda\}) \leq c_N (2^{j} h_j)$$

21
with precise knowledge of the value of $c_N$. Since ultimately, we “only” need the cluster coherence to boundedly stay away from $1/2$, we only require the weaker condition of

$$2^j h_j \leq \frac{1}{2c_N} - \epsilon$$

for some $\epsilon > 0$ and for all $j \geq j_0$.

This condition would then be also sufficient for deriving the following theorem.

We now apply Proposition 2.7 to Lemmata 3.1, 4.1, and 4.2 to obtain the desired convergence for the normalized $\ell_2$ error of the reconstruction $L_j$ derived from (3), where in this case $L = wL_j$ and $\Phi$ are wavelets $\psi_\lambda$ at scale $j$.

**Theorem 4.3.** For $h_j = o(2^{-2j})$ and $L_j$ the solution to (3) with $\Phi$ the 2D Meyer Parseval system,

$$\frac{\| L_j - wL_j \|_2}{\| wL_j \|_2} \to 0, \quad j \to \infty.$$  

This result shows that if the size of the gap shrinks faster than $2^{-2j}$ – i.e., the size of the gap is asymptotically smaller than $2^{-2j}$ – or if the gap shrinks at the same rate than $2^{-2j}$ with an exactly prescribed factor, we have asymptotically perfect inpainting.

### 4.2 Thresholding

We will now study thresholding as an inpainting method, which is from a computational point of view much easier to apply than $\ell_1$ minimization. Our analysis will show that we can derive the same asymptotic performance as for $\ell_1$ minimization.

Our first claim concerns the set of the thresholding coefficients $T_j$.

**Lemma 4.4.** For $h_j = o(2^{-2j})$ as $j \to \infty$, there exist thresholds $\{ \beta_j \}_j$ such that, for all $j \geq j_0$,

$$\{ k : |k_1| \leq \rho 2^{2j(1+n_1)}, |k_2| \leq 2^{2jn_1} \} \subseteq T_j$$

for positive $j_0$ and $n_1$.

**Proof.** We again first analyze $\hat{wL}_j$. By Plancherel, we can rewrite the coefficients which we have to threshold as follows:

$$\langle (1 - M_{h_j}) \hat{wL}_j, \hat{\psi}_\lambda \rangle = \langle \hat{\delta_0} \ast \hat{wL}_j, \hat{\psi}_\lambda \rangle - \langle \hat{M}_{h_j} \ast \hat{wL}_j, \hat{\psi}_\lambda \rangle.$$

Choose a function $F$ such that $F(\cdot/2^j) = \hat{F}_j$. Then,

$$\hat{wL}_j(\xi) = \hat{wL}(\xi) \hat{F}_j(\xi) = \hat{wL}(\xi) F(\xi/2^j).$$

As we are analyzing a horizontal line singularity, we only need to consider

$$\hat{\psi}_\lambda = 2^{-j}W^v(\xi/2^j)e^{-2\pi i (k, \xi/2^j)}$$

for large wavelet coefficients. Then, the first term equals

$$\langle \hat{\delta_0} \ast \hat{wL}_j, \hat{\psi}_\lambda \rangle = 2^{-j} \int \hat{w}(\xi_1) \int (FW^v)(\xi/2^j)e^{-2\pi i (k/2^j, \xi)} d\xi$$

$$= \int \left[ \int \hat{w}(\xi_1)(FW^v)((\xi_1/2^j, \xi_2))e^{-2\pi i (k_1/2^j, \xi_1)} d\xi_1 \right] e^{-2\pi i (k_2, \xi_2)} d\xi_2.$$
By using Lemma 3.2, we derive for the second term:

\[
\langle \hat{\mathcal{M}}_{h_j} \ast \hat{w} \mathcal{L}_j, \hat{\psi}_\lambda \rangle = 2h_j \int \text{sinc}(2h_j \tau_1) \int \hat{w}(\xi_1)(\hat{\psi}_\lambda F_j)((\tau_1, 0) + (\xi_1, \xi_2))d\xi d\tau_1
\]

\[= 2h_j 2^{-j} \int \text{sinc}(2h_j \tau_1) \int \hat{w}(\xi_1)F(\xi_1/2^j, \xi_2/2^j)W^v((\tau_1 + \xi_1)/2^j, \xi_2/2^j)e^{-2\pi i(k/2^j, \tau_1 + \xi_1 \xi_2)}d\xi d\tau_1
\]

\[= 2h_j \int \left[ \int \hat{w}(\xi_1) \int \text{sinc}((h_j/\pi)\tau_1)F(\xi_1/2^j, \xi_2)W^v(((\tau_1 + \xi_1)/2^j, \xi_2))e^{-2\pi i(k(\tau_1 + \xi_1)/2^j)}d\tau_1 d\xi_1 \right] e^{-2\pi i(k, \xi_2)}d\xi_2.
\]

Let \( \hat{G} \) now be the function

\[
\hat{G}(\xi_2) = \int \hat{w}(\xi_1) \left( (FW^v)((\xi_1/2^j, \xi_2)) + |\xi_1|^{-2}H_{\xi_2}(\xi_1) e^{-2\pi i(k_1/2^j)\xi_1}d\xi_1 \right)
\]

with

\[
H_{\xi_2}(\xi_1) = (FW^v)((\xi_1/2^j, \xi_2)) - 2h_j \int \text{sinc}((h_j/\pi)\tau_1)F(\xi_1/2^j, \xi_2)W^v(((\tau_1 + \xi_1)/2^j, \xi_2))e^{-2\pi i(k(\tau_1 + \xi_1)/2^j)}d\tau_1.
\]

The function \( \hat{G} \) is supported on the set \([1/2, 2] \), which is independent of \( j \). By standard arguments, we can deduce that

\[
|\langle (1 - \hat{\mathcal{M}}_{h_j})\hat{w} \mathcal{L}_j, \hat{\psi}_\lambda \rangle| \leq c_{N_1} \| \hat{G} \|_\infty |k_2|^{-N_1}.
\]

Let us now investigate the term \( \| \hat{G} \|_\infty \) further. Using Plancherel and the support properties of \( w \),

\[
\left| \int \hat{w}(\xi_1) \hat{H}_{\xi_2}(\xi_1) e^{-2\pi i(k_1/2^j)\xi_1}d\xi_1 \right| = \left| (\hat{w} \hat{H}_{\xi_2})^\vee (-k_1/2^j) \right| = \left| (w \ast H_{\xi_2})(-k_1/2^j) \right|
\]

\[= \left| \int w(-k_1/2^j - x)H_{\xi_2}(x)dx \right| \approx c \int_{-k_1/2^j - \rho}^{k_1/2^j + \rho} H_{\xi_2}(x)dx.
\]

For the analysis of the function \( H_{\xi_2} \), we use well-known properties of the Fourier transform to derive

\[
H_{\xi_2}(x) = \left( (FW^v)(\cdot/2^j, \xi_2) \right)^\vee (x) - \left( 2h_j \text{sinc}(2h_j \cdot) e^{-2\pi i(k_1/2^j)\cdot} \right) \ast \left( (FW^v)(\cdot/2^j, \xi_2) \right)^\vee (x)
\]

\[= \left( (FW^v)(\cdot/2^j, \xi_2) \right)^\vee (x) - \left( 2h_j \text{sinc}(2h_j \cdot) e^{-2\pi i(k_1/2^j)\cdot} \right) \ast \left( (FW^v)(\cdot/2^j, \xi_2) \right)^\vee (x)
\]

\[= \left( (FW^v)(\cdot/2^j, \xi_2) \right)^\vee (x) - \mathbf{1}_{[-h_j, h_j]}(x - k_1/2^j) \left( (FW^v)(\cdot/2^j, \xi_2) \right)^\vee (x).
\]

Hence, since \( h_j < \rho \),

\[
c \int_{-k_1/2^j - \rho}^{k_1/2^j + \rho} H_{\xi_2}(x)dx = c \int_{k_1/2^j - \rho}^{k_1/2^j + \rho} ((FW^v)(\cdot/2^j, \xi_2))^\vee (x) - \int_{k_1/2^j - h_j}^{k_1/2^j + h_j} ((FW^v)(\cdot/2^j, \xi_2))^\vee (x)dx
\]

\[= c \int_{k_1/2^j - h_j}^{k_1/2^j + h_j} ((FW^v)(\cdot/2^j, \xi_2))^\vee (x)dx.
\]
Notice that the bounds of integration indeed make sense, since the values of $k_1$ which lie “in between $h_j$ and $\rho$” should play an essential role. Due to the regularity of $W$, there exist some $N_2$ and $c$ (possibly differing from the one before, but we do not need to distinguish constants here) such that

$$|(FW^v)(\langle \cdot, \xi_2 \rangle \rangle^\vee (x)| \leq c(|x|)^{-N_2},$$

and hence by (10),

$$\|\hat{G}\|_\infty \leq c\min\{|k_1 - 2^j \rho|, |k_1 + 2^j \rho|\}^{-N_2}.$$

Finally, we have to study how the function $\hat{H}$ relates to $h$, which will show the behavior of the coefficients as they approach the center of the mask. For this, setting

$$\hat{J}_{\xi_2}(\tau_1) = (FW^v)((\tau_1 + \xi_1)/2^j, \xi_2)e^{-2\pi i(k_1/2^j, \tau_1)},$$

we obtain

$$|(FW^v)(\xi_1/2^j, \xi_2) - 2h_j \int \text{sinc}((h_j/\pi)\tau_1)(FW^v)((\tau_1 + \xi_1)/2^j, \xi_2)e^{-2\pi i(k_1/2^j, \tau_1)}d\tau_1|$$

$$= |\hat{J}_{\xi_2}(0) - 2h_j \int \text{sinc}((h_j/\pi)\tau_1)\hat{J}_{\xi_2}(\tau_1)d\tau_1|$$

$$= |\hat{J}_{\xi_2}(0) - \int_{-h_j}^{h_j} (\hat{J}_{\xi_2}(\tau_1)d\tau_1|$$

$$= |\hat{J}_{\xi_2}(0) - \int_{-h_j}^{h_j} J_{\xi_2}(x)dx| = |\int_{|x|>h_j} J_{\xi_2}(x)dx|.$$

Hence another way to estimate $\|\hat{G}\|_\infty$ is by

$$\frac{\|\hat{G}\|_\infty}{c} \leq \|\hat{H}_{\xi_2}\|_\infty$$

$$\leq \max_{\xi_1, \xi_2} \int_{|x|>h_j} ((FW^v)((\cdot + \xi_1)/2^j, \xi_2))^\vee (x - k_1/2^j)dx$$

$$\leq \max_{\xi_2} \int_{|x|>2^j h_j} ((FW^v)((\cdot, \xi_2))^\vee (x - k_1)dx.$$

Certainly, the minimum is attained in the center of the mask, i.e., with $k = 0$. So combining this with (9) and (11),

$$|(1-M_{h_j})\tilde{\mathcal{W}}_{\mathcal{L}_j, \psi_\lambda})| \leq c \max_{\xi_2} \int_{|x|>2^j h_j} ((FW^v)((\cdot, \xi_2))^\vee (x)dx \|k_2\|^{-N_1} (\min\{|k_1 - 2^j \rho|, |k_1 + 2^j \rho|\})^{-N_2}.$$

which is what we intend to use as a “model.” Observe that this indeed is also intuitively the right estimate, since the $k_2$ component has to decay rapidly away from zero, thereby sensing the singularity in zero in this direction. In contrast, the $k_1$ component stays greater or equal to $\langle 2^j \rho \rangle^{-N_2}$ up to the point $2^j \rho$ and then decays rapidly in accordance with the fact that up to the point $k_1 = \rho 2^j$ we are “on” the line singularity which decays smoothly with $\tilde{w}$. Moreover, the first term models the behavior in the mask, which is also nicely supported by the fact that the crucial product $2^j h_j$ is appearing therein.

24
We now apply the triangle inequality
\[ |(1 - \mathcal{M}_{h_j})w_L, \psi_\lambda| \leq |(1 - \mathcal{M}_{h_j})\tilde{w}_L_{2j}, \psi_\lambda| + |(1 - \mathcal{M}_{h_j})\tilde{w}_L_{2j+1}, \psi_\lambda|. \]
Since \(2^{2j}h_j \rightarrow 0\) and \(2^{2j+1}h_j \rightarrow 0\) as \(j \rightarrow \infty\), we have as \(j \rightarrow \infty\)
\[ \max_{\xi_2} \left| \int_{|x| > 2^{2j}h_j} ((FW^v)((\cdot, \xi_2)))^v(x) dx \right| \rightarrow C. \]
We now set the thresholds \(\beta_j\) to be
\[ c(C - \epsilon) \frac{(2^{2j})N_1(\min\{|(2^{2j} - 1)2^{2j}\rho|, |(2^{2j} + 1)2^{2j}\rho|\})^{N_2}}. \]
This choice immediately proves the claim of the lemma.

Note that \(A_j \subseteq \{k : |k_1| \leq 2^{2j(1+\nu_1)}, |k_2| \leq 2^{2j\nu_1}\} \subseteq T_j\) for some \(n_1 > 0\). For such a choice of \(T_j\), we have the following lemma.

**Lemma 4.5.**
\[ \delta_j = \sum_{k \in T_j} |\langle w_L, \psi_\lambda \rangle| = o(\|w_L\|_2), j \rightarrow \infty. \]

**Proof.** We observe from the proof of Lemma 4.1 that the desired property is automatically satisfied provided that, for all \(j \geq j_0\), the set \(T_j\) satisfies
\[ T_j \supseteq \{k : |k_1| \leq 2^{2j(1+\nu_1)}, |k_2| \leq 2^{2j\nu_1}\} \supseteq A_j, \]
for some \(\nu_1 > 0\), which is implied by Lemma 4.4.

We next analyze the second term in the estimate from Proposition 2.9.

**Lemma 4.6.** For \(h_j = o(2^{-2j})\) as \(j \rightarrow \infty\),
\[ \sum_{k \in T_j} |\langle \mathcal{M}_{h_j}w_L, \psi_\lambda \rangle| = o(2^{j/2}), \quad j \rightarrow \infty. \]

**Proof.** We first need to derive some estimates dependent on \(k\) for the term \(|\langle \mathcal{M}_{h_j}w_L, \psi_\lambda \rangle|\). By using the definitions of \(M_{h_j}\) and \(w_L\) and a change of variables, we first obtain \(\langle \mathcal{M}_{h_j}w_L, \psi_\lambda \rangle = \)
\[ 2h_j \left[ \int \hat{w}(\xi_1) \int \text{sinc}((2h_j)\tau_1) F(\xi_1/2^j, \xi_2)W^v((\tau_1 + \xi_1)/2^j, \xi_2) e^{-2\pi i (k_1(\tau_1 + \xi_1)/2^j)} d\tau_1 d\xi_1 \right] e^{-2\pi i (k_2 \xi_2)} d\xi_2. \]
Here \(F(\cdot/2^j) = \hat{F}\). Let \(\hat{G}\) now be the function
\[ \hat{G}(\xi_2) = \int \hat{w}(\xi_1)2h_j \int \text{sinc}((h_j/\pi)\tau_1) F(\xi_1/2^j, \xi_2)W^v((\tau_1 + \xi_1)/2^j, \xi_2) e^{-2\pi i (k_1/2^j, \tau_1 + \xi_1)} d\tau_1 d\xi_1 \]
\[ = \int \hat{w}(\xi_1)\hat{H}_\xi_2(\xi_1)e^{-2\pi i (k_1/2^j, \xi_1)} d\xi_1. \]
with
\[ 
\hat{H}_{\xi_2}(\xi_1) = 2h_j \int \text{sinc}\left((h_j/\pi)\tau_1\right) F(\xi_1/2^j, \xi_2) W^v(\tau_1 + \xi_1/2^j, \xi_2) e^{-2\pi i(k_1/2^j, \tau_1)} d\tau_1. 
\]

The function \( \hat{G} \) is supported on the set \([-1/4, -1/16] \cup [1/16, 1/4] \), which is independent of \( j \). Hence, we have
\[ |\langle \mathcal{M}_{h_j} w \mathcal{L}_j, \psi_\lambda \rangle| \leq c_{N_1} \| \hat{G} \|_\infty \langle |k_2| \rangle^{-N_1}. \]  

(12)

By Plancherel’s theorem and the support properties of \( w \),
\[ |\int \hat{w}(\xi_1) \hat{H}_{\xi_2}(\xi_1) e^{-2\pi i(k_1/2^j, \xi_1)} d\xi_1| = |\hat{w} \hat{H}_{\xi_2}(\xi_1)| = |w \ast H_{\xi_2}(-k_1/2^j)| 
\]
\[ = \left| \int w(-k_1/2^j - x) H_{\xi_2}(x) dx \right| \approx c \left| \int_{-k_1/2^j - \rho}^{-k_1/2^j + \rho} H_{\xi_2}(x) dx \right|. \]

Next, using well-known properties of the Fourier transform, we can manipulate \( H_{\xi_2} \):
\[ H_{\xi_2}(x) = \left( 2h_j \text{sinc}(2h_j \cdot) e^{-2\pi i k_1/2^j} \right) \ast (\mathcal{F}W^v(\cdot/2^j, \xi_2)) (x) 
\]
\[ = \left( 2h_j \text{sinc}(2h_j \cdot) e^{-2\pi i k_1/2^j} \right) (x) \left( (\mathcal{F}W^v(\cdot/2^j, \xi_2)) (x) \right)^\vee (x) 
\]
\[ = \mathbb{1}_{[-h_j, h_j]}(-x - k_1/2^j) \left( (\mathcal{F}W^v(\cdot/2^j, \xi_2)) (x) \right)^\vee (x). \]

Hence, since \( h_j < \rho \),
\[ c \left| \int_{-k_1/2^j - \rho}^{-k_1/2^j + \rho} H_{\xi_2}(x) dx \right| \approx c \left| \int_{k_1/2^j - h_j}^{k_1/2^j + h_j} (\mathcal{F}W^v(\cdot/2^j, \xi_2))^\vee (x) dx \right| 
\]
\[ = c \left| \int_{k_1/2^j - h_j}^{k_1/2^j} (\mathcal{F}W^v(\cdot, \xi_2))^\vee (x) dx \right|. \]

Notice that this indeed makes sense, since due to the masking the length of the line singularity isn’t allowed to play a role here. Due to the regularity of \( W \), there exists some constants \( N_2 \) and \( c \) such that
\[ ||(\mathcal{F}W^v(\cdot, \cdot))^\vee (x)|| \leq c (|x|)^{-N_2}. \]

Hence,
\[ ||\hat{G}||_\infty \leq c \langle \min\{ |k_1 - 2^j h_j|, |k_1 + 2^j h_j| \} \rangle^{-N_2}. \]

Combining this estimate with \([12]\), we obtain
\[ |\langle \mathcal{M}_{h_j} w \mathcal{L}_j, \psi_\lambda \rangle| \leq c\langle k_2 \rangle^{-N_1} \langle \min\{ |k_1 - 2^j h_j|, |k_1 + 2^j h_j| \} \rangle^{-N_2}, \]

which is what we intend to use.

Finally,
\[ \sum_{k \in T_j} |\langle \mathcal{M}_{h_j} w \mathcal{L}_j, \psi_\lambda \rangle| \leq c \left( \sum_{k \in T_j} \langle k_2 \rangle^{-N_1} \langle \min\{ |k_1 - 2^j h_j|, |k_1 + 2^j h_j| \} \rangle^{-N_2} + \right. 
\]
\[ + \sum_{k \in T_j} \langle k_2 \rangle^{-N_1} \langle \min\{ |k_1 - 2^j + 1 h_j|, |k_1 + 2^j + 1 h_j| \} \rangle^{-N_2} \right) \]
\[ \leq c. \]

\[ \square \]
Notice that this result holds for any $T_j$, which again is intuitively clear since if it holds for the claimed on, then extending the set $T_j$ does not change the estimate due to the fact that $M_{h_j} w L_j$ is zero “outside.”

We now apply Proposition 2.9 to Lemmata 3.1, 4.5, and 4.6 to obtain the desired convergence for the normalized $\ell_2$ error of the reconstruction $L_j$ from One-Step-Thresholding in Figure 6. Again, in this case $x = w L_j$ and $\Phi$ are wavelets $\psi_\lambda$ at scale $j$.

**Theorem 4.7.** For $h_j = o(2^{-2j})$ and $L_j$ the solution to (4) with $\Phi$ the 2D Meyer Parseval system,

$$\frac{\|L_j - w L_j\|_2}{\|w L_j\|_2} \to 0, \quad j \to \infty.$$  

This result shows that One-Step-Thresholding fills in gaps of the same size as $\ell_1$ minimization (INP) in an asymptotic sense when considering the $\ell_2$ error.

### 5 Shearlet Inpainting Positive Results

In this section, $\Phi$ is the shearlet frame as in (1) in Subsection 1.2.2. The general approach in this section is the same as in the preceding section. We show that use of the analysis coefficients of the shearlet system through either $\ell_1$ minimization or thresholding will successfully inpaint a line across a missing strip. Namely, in Subsection 5.1 we investigate the inpainting results of $\ell_1$ minimization by estimating the $\delta$-clustered sparsity $\delta_j$ and cluster coherence $\mu_c$ with respect to $\{\sigma_\eta : \eta = (i, j, \ell, k), i \in \{h, v, \emptyset\}; |\ell| \leq 2^j; k \in \mathbb{Z}^2\}$ and a properly chosen index set $\Lambda_j$. In Subsection 5.2, we similarly give the estimation of $\delta_j$ and $\mu_c$ for inpainting using thresholding.

Some of the proofs in this section are very similar in spirit to the corresponding ones in Section 4 but decidedly more technical due to the structural difference between wavelets and shearlets. The auxiliary functions (8) and (13) in the proofs of Lemma 4.2 and Theorem 5.3 demonstrate this relationship quite well.

#### 5.1 $\ell_1$ Minimization

For our analysis we choose the set of significant shearlet coefficients to be

$$\Lambda_j = \{(i; j, k, \ell) : |k_1| \leq \rho n_j 2^j, |k_2| \leq n_j, \ell = 0; i = v\}$$

where we revive the notion $n_j = 2^{2j}$ from the previous subsection.

Now we can show the clustered sparsity of the shearlet coefficients with the choice of $\Lambda_j$.

**Lemma 5.1.** For $\epsilon < 1/4$,

$$\delta_j = o(2^j), \quad j \to \infty.$$  

**Proof.** By the definition, we have

$$\delta_j = \sum_{|k_1| \geq \rho n_j 2^j, |k_2| \leq n_j, \ell = 0} |\langle w L_j, \sigma_{j,\ell,k}^v \rangle| + \sum_{|k_2| \geq n_j, \ell = 0} |\langle w L_j, \sigma_{j,\ell,k}^v \rangle| + \sum_{k \in \mathbb{Z}^2, \ell \neq 0} |\langle w L_j, \sigma_{j,\ell,k}^c \rangle| + \sum_{k \in \mathbb{Z}^2, \ell \neq 0} |\langle w L_j, \sigma_{j,\ell,k}^c \rangle| + \sum_{k \in \mathbb{Z}^2} |\langle w L_j, \sigma_{j,\ell,k}^h \rangle| + \sum_{k \in \mathbb{Z}^2} |\langle w L_j, \sigma_{j,\ell,k}^h \rangle| + \sum_{k \in \mathbb{Z}^2} |\langle w L_j, \sigma_{j,\ell,k}^\pm 2^j \rangle|$$

$$=: T_1 + T_2 + T_3 + T_4 + T_5.$$  

27
To estimate $T_1$, we first estimate $\langle w \mathcal{L}, \sigma_0 \rangle$ for the case $\ell = 0$ and $\nu = v$. By Lemma 8.3 in Section 8, we have

\[
\langle w \mathcal{L}, \sigma_0 \rangle \leq c_N a_j^{-1/2} \langle |k_2|^{-1} \langle |k_2|^2 + a_j^{-4} \min(\pm a_j k_1 \pm \rho)^2 \rangle^{1/2} \rangle^{2-N}
\]

Therefore, we have

\[
T_1 \leq c_N a_j^{-1/2} a_j^{-\epsilon} \sum_{|k_1| \geq \rho a_j^{-1-2\epsilon}} \langle a_j^{-2} \min(\pm a_j k_1 \pm \rho) \rangle^{2-N}
\]

Note that $a_j^{-2\epsilon} = n_j = 2^{2j\epsilon}$. Since

\[
\int_{|x| > \rho a_j^{-1-2\epsilon}} \langle a_j^{-1} x - a_j^{-2} \rangle \rangle^{2-N} dx = a_j \int_{|y| > \rho a_j^{-2-2\epsilon}} \langle y - a_j^{-2} \rangle \rangle^{2-N} dy
\]

we obtain

\[
T_1 \leq c_N a_j^{1/2 - 2\epsilon + 2(N-3)}.
\]

For $T_2$, we have

\[
\frac{T_2}{c_N a_j^{-1/2}} \leq \sum_{k_1 \in \mathbb{Z}, |k_2| \geq a_j^{-2\epsilon}} \langle |k_2|^2 + \min(\pm a_j^{-1} k_1 \pm a_j^{-2} \rho)^2 \rangle^{1/2} \rangle^{2-N}
\]

For $T_{2,1}$, we have

\[
T_{2,1} \leq c \int_{|x_1| < \rho a_j^{-1-2\epsilon}} \int_{|x_2| > a_j^{-2\epsilon}} \langle |x_2| \rangle^{2-N} dx_2 dx_1 \leq c a_j^{-1+2(N-4)\epsilon}.
\]

For $T_{2,2}$, we have

\[
T_{2,2} \leq c a_j \int_{x_1 > \rho a_j^{-2-2\epsilon}} \int_{x_2 > a_j^{-2\epsilon}} \langle |(x_1, x_2)| \rangle^{2-N} dx_2 dx_1 \leq c a_j^{2(N-3)(1+2\epsilon)}.
\]
Therefore,
\[ T_2 \leq c_N a_j^{-3/2 + 2(N-1)\epsilon}. \]

For \( T_3 \), we convert the result in Lemma 8.4 in Section 8 to the discrete case.

**Lemma 5.2.** Let \( t_1 = a_j^2(k_1 - \ell k_2) \) and \( t_2 = a_j k_2 \) with \( a_j = 2^{-j} \).

(i) For \( t_1 \neq 0 \) and \( t_2 \neq 0 \), we have
\[
|\langle w\mathcal{L}_j, \sigma_{j, t_1, k}^h \rangle| \leq c_N e^{-ca_j^{-1}} a_j^{-1/2} |a_j^2(k_1 - \ell k_2)|^{-N} |a_j k_2|^{-N} a_j^2.
\]
and
\[
|\langle w\mathcal{L}_j, \sigma_{j, t_1, k}^v \rangle| \leq c_N e^{-ca_j^{-2}} a_j^{-1/2} |a_j(k_1 - \ell k_2)|^{-N} |a_j k_2|^{-N} a_j^2.
\]

(ii) When exactly one of \( t_1 \) or \( t_2 \) is 0 and \( \ell \in \{h, v\} \), we have
\[
|\langle w\mathcal{L}, \sigma_{j, 0, k}^{\ell} \rangle| \leq c_L \left[ \max\{a_j^2 |k_1 - \ell k_2|, a_j |k_2|\} \right]^{-L} a_j^{-1/2} e^{-ca_j^{-1} \ell}.
\]

(iii) For \( t_1 = t_2 = 0 \) and \( \ell \in \{h, v\} \), we have
\[
|\langle w\mathcal{L}, \sigma_{j, 0, k}^{\ell} \rangle| \leq ca_j^{-1/2} e^{-ca_j^{-1}}.
\]

For \( t_1 := a_j^2(k_1 - \ell k_2) \neq 0 \) and \( t_2 := a_j k_2 \neq 0 \), we have
\[
a_j^3 \sum_{k \in \mathbb{Z}^2, t_1 \neq 0, t_2 \neq 0} |a_j^2(k_1 - \ell k_2)|^{-N} |a_j k_2|^{-N} \leq a_j^3 \int_{\{x : x_1 \neq 0, x_2 \neq 0\}} |a_j^2(x_1 - \ell x_2)|^{-N} |a_j x_2|^{-N} dx_1 dx_2
\]
\[
< c \cdot \int_{|x_1| \geq 1, |x_2| \geq 1} |x_1|^{-N} |x_2|^{-N} dx_1 dx_2
\]
\[
< \infty.
\]
Hence
\[
\sum_{k \in \mathbb{Z}^2, t_1 \neq 0, t_2 \neq 0} |a_j^2(k_1 - \ell k_2)|^{-N} |a_j k_2|^{-N} < ca_j^{-3}.
\]

Similarly, for \( t_1 = 0 \) or \( t_2 = 0 \), we have
\[
\sum_{k \in \mathbb{Z}^2, t_1 \neq 0} \left[ \max\{a_j^2 |k_1 - \ell k_2|, a_j |k_2|\} \right]^{-N} < ca_j^{-3}.
\]

The estimate for (iii) follows by direct computation. Therefore, by the above estimates (i), (ii), and (iii), and that
\[
T_3 = \sum_{\ell=1}^{a_j^{-1}} \sum_{k \in \mathbb{Z}^2,(t_1,t_2)\neq0} |\langle w\mathcal{L}_j, \sigma_{j, t_1, k}^{\ell} \rangle| + \sum_{\ell=1}^{a_j^{-1}} |\langle w\mathcal{L}_j, \sigma_{j, 0, k}^{\ell} \rangle|,
\]
we obtain
\[
T_3 \leq \sum_{\ell=1}^{a_j^{-1}} c_N a_j^{-1/2} e^{-ca_j^{-1}} (a_j^{-3} + 1) \leq c_N a_j^N \quad \forall N \geq 0.
\]

29
Similarly, for \( T_4 \),
\[
T_4 \leq \sum_{\ell=1}^{a_j^{-2}} c_N a_j^{-1} e^{-c a^{-1}} (a_j^{-3} + 1) \leq c_N a_j^N \quad \forall N \geq 0.
\]

Finally, since the “seam” elements \( \sigma_{j,\ell,k} \) are only slight modifications of the \( \sigma_{j,\ell,k}^* \), \( T_5 \leq c_N a_j^N \) for all \( N \geq 0 \).

Combining the estimates for \( T_1, \ldots, T_5 \), we are done.

Next we estimate the cluster coherence
\[
\mu_c(\Lambda_j, \{\mathcal{M}_{h_j} \sigma_\eta\}; \{\sigma_\eta\})
\]
and show that it converges to zero as \( j \to \infty \) when \( h_j \) is related \( j \) by \( h_j = o(2^{-j}) \) as \( j \to \infty \). We wish to remark that the size of the gaps which can be filled with asymptotically high precision is dramatically larger than the corresponding size for wavelet inpainting.

**Theorem 5.3.** For \( h_j = o(2^{-j}) \)
\[
\mu_c(\Lambda_j, \{\mathcal{M}_{h_j} \sigma_\eta\}; \{\sigma_\eta\}) \to 0, \quad j \to \infty
\]
with \( \eta = (t, j, \ell, k) \) and \( t \in \{h, v, 0\} \).

**Proof.** We have
\[
\mu_c(\Lambda_j, \{\mathcal{M}_{h_j} \sigma_\eta\}; \{\sigma_\eta\}) = \max_{\eta_2 \in \Lambda_j} \sum_{\eta_1 \in \Lambda_j} |\langle \mathcal{M}_{h_j} \sigma_{\eta_1}, \sigma_{\eta_2} \rangle|
\]
\[
\leq \max_{\eta_2 \in \Lambda_j} \sum_{\eta_1 \in \Lambda_j} |\langle \mathcal{M}_{h_j} \sigma_{\eta_1}, \sigma_{\eta_2} \rangle| + \max_{\eta_1 \in \Lambda_j} \sum_{\eta_2 \in \Lambda_j} |\langle \mathcal{M}_{h_j} \sigma_{\eta_1}, \sigma_{\eta_2} \rangle| + \max_{\eta_1 \in \Lambda_j} \sum_{\eta_2 \in \Lambda_j} |\langle \mathcal{M}_{h_j} \sigma_{\eta_1}, \sigma_{\eta_2} \rangle|
\]
\[
= : T_1 + T_2 + T_3.
\]

We bound \( T_1 \) using simple substitutions:
\[
T_1 \leq \sum_{(i,j,\ell,k) \in \Lambda_j} |\langle \mathcal{M}_{h_j} \sigma_{j,\ell,k}^v, \sigma_{j,0,0}^v \rangle|
\]
\[
\leq \sum_{(i,j,\ell,k) \in \Lambda_j} \left| \int_{\mathbb{R}} 2h_j \text{sinc}(2h_j \xi_1) \int_{\mathbb{R}^2} 2^{-3j} \mathcal{W}(\frac{\tau}{2^{2j}}) \mathcal{W}\left(\frac{\tau_1 - \xi_1}{2^{2j}}, \frac{\tau_2}{2^{2j}}\right) V(\ell + 2j \frac{\tau_1 - \xi_1}{2^{2j}}) V(2j \frac{\tau_1}{\tau_2}) \times \right.
\]
\[
ex^{-2\pi i (\xi_1, 0) \Lambda_{2^{-j} S_\ell}^*} d\tau \left| d\xi_1 \right|
\]
\[
\leq 2(2^ih_j) \sum_{(i,j,\ell,k) \in \Lambda_j} \left| \int_{\mathbb{R}} \text{sinc}(2^i 2h_j \xi_1) \int_{\mathbb{R}^2} \mathcal{W}(\frac{\tau_1}{2^{2j}}, \frac{\tau_2}{2^{2j}}) \mathcal{W}\left(\frac{\tau_1 - \xi_1}{2^{2j}}, \frac{\tau_2}{2^{2j}}\right) V(\ell + \frac{\tau_1 - \xi_1}{\tau_2}) V(\frac{\tau_1}{\tau_2}) \times \right.
\]
\[
ex^{2\pi i \ell \xi_1} e^{-2\pi i (t, A_{i/a_j}^\tau)} d\tau \left| d\xi_1 \right|
\]
\[
\leq 2(2^ih_j) \sum_{(i,j,\ell,k) \in \Lambda_j} \left| \int_{\mathbb{R}} \hat{g}_j(\tau) e^{-2\pi i (t, A_{i/a_j}^\tau)} d\tau \right|,
\]
\]
\]
where \( t = A_{i/a_j}^v S_{i/k}^v \) with \( a_j = 2^{-j} \) and

\[
\hat{g}_j(\tau) := \int_{\mathbb{R}} \text{sinc}(2^j 2h_j \xi_1) V \left( \ell + \frac{\tau - \xi_1 \tau_2}{\tau_2} \right) e^{2\pi i t_1 \xi_1} d\xi_1 \mathcal{W} \left( \frac{\tau_1 - \xi_1}{\tau_2}, \tau_2 \right) \mathcal{W} \left( \frac{\tau_1 - \xi_1}{\tau_2}, \tau_2 \right) V \left( \frac{\tau_1 - \xi_1}{\tau_2} \right). \quad (13)
\]

Note that the support of \( \mathcal{W}(\tau_1/2^j, \cdot) \) and of \( \mathcal{W}(\tau_1/2^j, \cdot) \) of variable \( \tau_2 \) is independent of \( j \) and the support of \( V(\cdot/\tau_2) \) of variable \( \tau_1 \) is depending only on \( \tau_2 \). Hence, \( \hat{g}_j(\tau) \) is smooth and compactly supported on a box \( \Xi \) of volume independent of \( j \),

\[
| \int \hat{g}_j(\tau) e^{2\pi i t \tau} d\tau | \leq c_N \| \hat{g}_j \|_\infty \langle |t| \rangle^{-N}.
\]

Note that

\[
\| \hat{g}_j \|_\infty \leq c(2^j h_j)^{-1/2};
\]

therefore,

\[
T_1 \leq c(2^j h_j)^{1/2} \sum_{k \in \mathbb{Z}^2} \langle |k| \rangle^{-N} \rightarrow 0, j \rightarrow \infty.
\]

We now bound \( T_2 \):

\[
T_2 \leq \sum_{(i,j,\ell,k) \in \Lambda_j} | \langle \hat{M}_{h_j} \hat{\sigma}_{j,\ell}^v, \hat{\sigma}_{j,\ell,0}^h \rangle | \leq \sum_{(i,j,\ell,k) \in \Lambda_j} \int_{\mathbb{R}^2} 2h_j \text{sinc}(2h_j \xi_1) \left[ \int_{\mathbb{R}} \hat{\sigma}_{a_j,0}^v(\tau - (\xi_1, 0)) \hat{\sigma}_{a_j,0}^h(\tau) e^{-2\pi i (t, \tau - (\xi_1, 0))} d\tau \right] d\xi_1 \leq \sum_{(i,j,\ell,k) \in \Lambda_j} \int_{\mathbb{R}^2} \int_{\mathbb{R}} 2h_j \text{sinc}(2h_j \xi_1) \hat{\sigma}_{a_j,0}^v(\tau - (\xi_1, 0)) \hat{\sigma}_{a_j,0}^h(\tau) d\xi_1 e^{-2\pi i (t, \tau - (\xi_1, 0))} d\tau =: \sum_{(i,j,\ell,k) \in \Lambda_j} \int_{\mathbb{R}^2} \hat{g}_j(\tau) e^{-2\pi i (t, \tau)} d\tau,
\]

where

\[
\hat{g}_j(\tau) := \int_{\mathbb{R}} 2h_j \text{sinc}(2h_j \xi_1) \hat{\sigma}_{a_j,0}^v(\tau - (\xi_1, 0)) \hat{\sigma}_{a_j,0}^h(\tau) e^{2\pi i t_1 \xi_1} d\xi_1.
\]

Using integration by parts, we obtain

\[
| \int_{\mathbb{R}^2} \hat{g}_j(\tau) e^{-2\pi i (t, \tau)} d\tau | \leq c_{L,M} \langle |t_1| \rangle^{-L} \langle |t_2| \rangle^{-M} \| D_L^{M} \hat{g}_j \|_\infty \text{supp} \ (\hat{g}_j) \leq c_{L,M} \langle |t_1| \rangle^{-L} \langle |t_2| \rangle^{-M} \| D_L^{M} \hat{g}_j \|_\infty a_j^{-4},
\]

where

\[
| D_L^{M} \hat{g}_j | \leq 2h_j \int_{\mathbb{R}} | \text{sinc}(2h_j \xi_1) | | D_L^{M} (\hat{\sigma}_{a_j,0}^v(\tau - (\xi_1, 0)) \hat{\sigma}_{a_j,0}^h(\tau)) | d\xi_1 \leq 2h_j \| \text{sinc}(2h_j \cdot) \|_2 | D_L^{M} (\hat{\sigma}_{a_j,0}^v(\tau - (\xi_1, 0)) \hat{\sigma}_{a_j,0}^h(\tau)) \|_2 \leq c_{L,M} 2h_j^{1/2} | D_L^{M} (\hat{\sigma}_{a_j,0}^v(\tau - (\xi_1, 0)) \hat{\sigma}_{a_j,0}^h(\tau)) \|_\infty a_j^{-1}.
\]
Since
\[
\frac{\partial^N}{\partial t_1^N}(\hat{\sigma}_{a,s,0}^v \hat{\sigma}_{a,s',0}^h) = O(a_j^{3/2} a_j^N) \quad \text{and} \quad \frac{\partial^N}{\partial t_2^N}(\hat{\sigma}_{a,s,0}^v \hat{\sigma}_{a,s',0}^h) = O(a_j^{3/2} a_j^N).
\]
Consequently, as \( j \to \infty \),
\[
T_2 \leq h_j^{1/2} \sum_{(i; j, \ell, k) \in \Lambda_j} c_N(|t_1|^{-N} |\langle t_2|^{-N} a_j^{-4} a_j^{-3/2} a_j^{2N} \sum_{(i; j, \ell, k) \in \Lambda_j} \langle \hat{\sigma}_{a,s,0}^v \hat{\sigma}_{a,s',0}^h \rangle \rangle \to 0.
\]
By construction, \( T_3 \leq 2^{-1/2}(T_1 + T_2) \).

Notice that – in contrast to the wavelet result – here we require the stronger condition \( 2^j h_j \to 0 \) as \( j \to \infty \) to handle the additional angular component.

We now apply Proposition 2.7 to Lemmata 3.1, 5.1, and 5.3 to obtain the desired convergence for the normalized \( \ell_2 \) error of the reconstruction \( L_j \) from (3). In this case \( L = wL_j \) and \( \Phi \) are shearlets \( \sigma_{j,\ell,k}^v \) at scale \( j \).

**Theorem 5.4.** For \( h_j = o(2^{-j}) \) and \( L_j \) the solution to (3) with \( \Phi \) the shearlet system defined using the Meyer wavelet
\[
\|L_j - wL_j\|_2 \to 0, \quad j \to \infty.
\]

This result shows that we have asymptotically perfect inpainting as long as the size of the gap shrinks faster than \( 2^{-j} \). The similar result for wavelet inpainting, Theorem 4.3, only guarantees such successful inpainting when the gap is asymptotically smaller than \( 2^{-2j} \).

### 5.2 Thresholding

Our first claim concerns the set of the thresholding coefficients \( T_j := \{ \eta = (i; j, \ell, k) : |\langle wL_j, \sigma_\eta \rangle| \geq \beta_j \} \) for some \( \beta_j > 0 \).

**Lemma 5.5.** For \( h_j = o(2^{-j}) \) as \( j \to \infty \), there exist thresholds \( \{ \beta_j \}_j \) such that, for all \( j \geq j_0 \),
\[
\{(i; j, \ell, k) : |k_1| \leq \rho 2^{2j(1+\nu_1)}, |k_2| \leq 2^{2j\nu_1}, \ell = 0; \ell = v\} \subseteq T_j
\]
for some \( j_0, \nu_1, \) and \( \nu_2 < 1/4 \).

**Proof.** We first observe that
\[
|\langle (1 - \mathcal{M}_{h_j})wL_j, \sigma_{j,\ell,k}^v \rangle| = |\langle \delta_0 \ast \tilde{wL}_j, \hat{\sigma}_{j,\ell,k}^v \rangle - \langle \mathcal{M}_{h_j} \ast \tilde{wL}_j, \hat{\sigma}_{j,\ell,k}^v \rangle|.
\]
The first term equals
\[
\langle \delta_0 \ast \tilde{wL}_j, \hat{\sigma}_{j,\ell,k}^v \rangle = 2^{j/2} \int \left[ \int \tilde{w}(\xi_1)F(\xi_1/2^{2j}, \xi_2)W(\xi_1/2^{2j}, \xi_2)W(\xi_1/2^{2j}, \xi_2) e^{-2\pi i (b_1, \xi_1)} d\xi_1 \right] e^{-2\pi i (b_2, \xi_2)} d\xi_2;
\]
(14)
whereas, by using Lemma 3.2, we derive for the second term

\[
\langle M_{h_j} w L_j, \sigma_{\ell,k}^\nu \rangle = 2h_j \int \text{sinc}(2h_j \tau_1) \int \hat{w}(\xi_1) F_j(\xi_1, \xi_2) \hat{\sigma}_{\ell,k}^\nu(\xi_1 + \tau_1, \xi_2) d\xi d\tau_1
\]

\[
= 2^{j/2} \int \left[ \int \hat{w}(\xi_1) 2h_j \int \text{sinc}(2h_j \tau_1) F(\xi_1/2^j, \xi_2) \times \right.
\]
\[
\times W(\xi_1/2^j, \xi_2) V(\ell + 2^{-j} \frac{\tau_1 + \xi_1}{\xi_2}) e^{-2\pi i (b_1 \cdot \tau_1 + \xi_1)} d\tau_1 d\xi_1
\]
\[
\left. \left. e^{-2\pi i (b_2 \cdot \xi_2)} d\xi_2 \right] \right] e^{-2\pi i (b_2 \cdot \xi_2)} d\xi_2.
\]

By standard arguments, we can deduce that

\[
|\langle (1 - M_{h_j}) w L_j, \sigma_{\ell,k}^\nu \rangle | \leq c_{N_1} 2^{j/2} \| \hat{G} \|_\infty \langle |2^j b_2| \rangle^{-N_1}.
\]

By \( b_2 = k_2/2^j \) due to \( b = (A^\nu_{-j}, S^\nu_{-j})^T k \), we have

\[
|\langle (1 - M_{h_j}) w L_j, \sigma_{\ell,k}^\nu \rangle | \leq c_{N_1} 2^{j/2} \| \hat{G} \|_\infty \langle |k_2| \rangle^{-N_1}
\]

Let us now investigate the term \( \| \hat{G} \|_\infty \) further. We define

\[
\hat{H}_{\ell_2}(\xi_1) = F(\xi_1/2^j, \xi_2) W(\xi_1/2^j, \xi_2) V(\ell + 2^{-j} \xi_1/\xi_2) - 2h_j \int \text{sinc}(2h_j \tau_1) F(\xi_2/2^j, \xi_2) W(\xi_1/2^j, \xi_2) \times \]
\[
\times V(\ell + 2^{-j} \frac{\xi_1 + \tau_1}{\xi_2}) e^{-2\pi i (b_1 \cdot \tau_1)} d\tau_1
\]
and hence need to analyze

\[
\| \hat{G} \|_\infty = \left| \int \hat{w}(\xi_1) \hat{H}_{\ell_2}(\xi_1) e^{-2\pi i (b_1 \cdot \xi_1)} d\xi_1 \right|.
\]

By Plancherel’s theorem and the support properties of \( w \),

\[
\left| \int \hat{w}(\xi_1) \hat{H}_{\ell_2}(\xi_1) e^{-2\pi i (b_1 \cdot \xi_1)} d\xi_1 \right| = \| (\hat{w} \hat{H}_{\ell_2})^\vee (-b_1) \| \]
\[
\approx c \left| \int _{-b_1 - \rho}^{b_1 + \rho} H_{\ell_2}(x) dx \right|.
\]

We now need to compute \( H \). Using well-known properties of the Fourier transform, we manipulate \( H_{\ell_2} \) to obtain

\[
H_{\ell_2}(x) = (F(\cdot/2^j, \xi_2) W(\cdot/2^j, \xi_2) V(\ell + 2^{-j} \cdot \xi_2))^\vee (x) +
\]
\[
- \left( 2h_j \text{sinc}(2h_j \cdot e^{-2\pi ib_1}) \ast (F(\cdot/2^j, \xi_2) W(\cdot/2^j, \xi_2) V(\ell + 2^{-j} \cdot \xi_2)) \right)^\vee (-x)
\]
\[
= (F(\cdot/2^j, \xi_2) W(\cdot/2^j, \xi_2) V(\ell + 2^{-j} \cdot \xi_2))^\vee (x) +
\]
\[
- \left( 2h_j \text{sinc}(2h_j \cdot e^{-2\pi ib_1}) \right)^\vee (-x) \left( F(\cdot/2^j, \xi_2) W(\cdot/2^j, \xi_2) V(\ell + 2^{-j} \cdot \xi_2) \right)^\vee (-x)
\]
\[
= (F(\cdot/2^j, \xi_2) W(\cdot/2^j, \xi_2) V(\ell + 2^{-j} \cdot \xi_2))^\vee (x) +
\]
\[
- \mathbb{1}_{[-h_j, h_j]}(x - b_1) \left( F(\cdot/2^j, \xi_2) W(\cdot/2^j, \xi_2) V(\ell + 2^{-j} \cdot \xi_2) \right)^\vee (-x).
\]

33
Hence, since \( h_j < \rho \),
\[
\left| \int_{-b_1 + \rho}^{b_1 + \rho} H_{\xi_2}(x) dx \right| = \left| \int_{b_1 - \rho}^{b_1 + \rho} (F(\cdot/2^{2j}, \xi_2)W(\cdot/2^{2j}, \xi_2)V(\ell + 2^{-j}(\cdot/\xi_2))^\vee(x)
- \int_{b_1 - h_j}^{b_1 + h_j} (F(\cdot/2^{2j}, \xi_2)W(\cdot/2^{2j}, \xi_2)V(\ell + 2^{-j}(\cdot/\xi_2))^\vee(x) dx \right|
= \left| \int_{b_1 - \rho}^{b_1 + \rho} \left(F(\cdot/2^{2j}, \xi_2)W(\cdot/2^{2j}, \xi_2)\times
V(\ell + 2^{-j}(\cdot/\xi_2))^\vee(x) dx \right|
\]
\]

Notice that this fact also implies that the function
\[
(F(\cdot/2^{2j}, \xi_2)W(\cdot/2^{2j}, \xi_2)V(\ell + 2^{-j}(\cdot/\xi_2))^\vee(x)
\]

is independent of \( j \). Due to the regularity of \( W \), there exist some \( N_2 \) and \( c \) such that
\[
| (F(\cdot/2^{2j}, \xi_2)W(\cdot/2^{2j}, \xi_2)V(\ell + 2^{-j}(\cdot/\xi_2))^\vee(x) | \leq c(|x|^{-N_2},
\]

and hence by (16) and the previous computation,
\[
\| \tilde{G} \|_{\infty} \leq c(\min\{|k_1 - 2^j \rho|, |k_1 + 2^j \rho|\})^{-N_2}. \tag{17}
\]

Finally, we study how the term \( \tilde{H} \) relates to \( h_j \). For this, we set
\[
\hat{J}_{\xi_2}(\tau_1) = F(\xi_1/2^{2j}, \xi_2)W(\xi_1/2^{2j}, \xi_2)V(\ell + 2^{-j}\xi_1/\xi_2)e^{-2\pi i (b_1, \tau_1)}
\]

Now,
\[
|\hat{H}_{\xi_2}(\xi_1)| = \left| F(\xi_1/2^{2j}, \xi_2)W(\xi_1/2^{2j}, \xi_2)V(\ell + 2^{-j}\xi_1/\xi_2) - 2h_j \int \text{sinc}(2h_j \tau_1)F(\xi_1/2^{2j}, \xi_2)W(\xi_1/2^{2j}, \xi_2) \times
V(\ell + 2^{-j}\xi_1/\xi_2)e^{-2\pi i (b_1, \tau_1)}d\tau_1 \right|
= |\hat{J}_{\xi_2}(0)| - 2h_j \int \text{sinc}(2h_j \tau_1)\hat{J}_{\xi_2}(\tau_1) d\tau_1
= |\hat{J}_{\xi_2}(0)| - \int_{h_j}^{h_j} \hat{J}_{\xi_2}(\tau_1) d\tau_1
= |\hat{J}_{\xi_2}(0)| - \int_{h_j}^{h_j} J_{\xi_2}(x) dx
= |\int_{|x| > h_j} J_{\xi_2}(x) dx|.
\]
Hence another way to estimate (16) is by
\[
\|\hat{G}\|_{\infty} \leq c\|\hat{H}\|_{\infty} \\
\leq c\max_{\xi_1, \xi_2} \left| \int_{|x| > h_j} (F(\xi_1/2^j, \xi_2)W(\xi_1/2^j, \xi_2)V(\ell + 2^{-j} + \frac{\xi_1}{\xi_2}))^\nu(x - b_1)dx \right| \\
\leq c\max_{\xi_1, \xi_2} \left| \int_{|x| > 2^j h_j} (F(\xi_1/2^j, \xi_2)W(\xi_1/2^j, \xi_2)V(\ell + \frac{\cdot + 2^{-j}\xi_1}{\xi_2}))^\nu(x - 2^j b_1)dx \right|.
\]
Certainly, the minimum is attained in the center of the mask, i.e., with \(b = 0\). So by combining this with (15) and (17),
\[
|\langle (1 - M_{h_j})wL_j, \sigma_{j, \ell, k}^x \rangle| \leq c2^j\left| \int_{|x| > 2^j h_j} \max_{\xi_1, \xi_2} \left| \int_{|x| > 2^j h_j} (F(\xi_1/2^j, \xi_2) \times \\
W(\xi_1/2^j, \xi_2)V(\ell + \frac{\cdot + 2^{-j}\xi_1}{\xi_2}))^\nu(x - 2^j b_1)dx \right| \times \\
\times \langle \min\{|k_1 - 2^j \rho|, |k_1 + 2^j \rho|\} \rangle^{-N_2} \langle |k_2| \rangle^{-N_1},
\right|
\]
which is what we intend to use as a “model.” Observe that this indeed is the right intuitive estimate, since the \(k_2\) component has to decay rapidly away from zero thereby sensing the singularity in zero in this direction. In contrast, the \(k_1\) component stays greater or equal to \(\langle 2^j \rho \rangle^{-N_2}\) up to the point \(2\rho 2^j\) and then decays rapidly in accordance with the fact that until the point \(k_1 = \rho 2^j\) we are “on” the line singularity which decays smoothly up with \(\hat{w}\). Also, the required angle sensitivity is represented. Finally, the first term models the behavior in the mask, which is also nicely supported by the fact that the crucial product \(2^j h_j\) is appearing therein. Set
\[
J(\cdot) = F(\cdot/2^j, \xi_2)W(\cdot/2^j, \xi_2)V(\ell + 2^{-j}(\cdot/\xi_2)).
\]
Since \(2^j h_j \to 0\) as \(j \to \infty\), letting \(j \to \infty\) we have
\[
\left| \int_{|x| > 2^j h_j} J(x)dx \right| \leq C.
\]
We now use
\[
\beta = c2^{j/2}(C - \epsilon)\langle |2^j| \rangle^{-N_1} \langle \min\{|(2^j - 1)2^j \rho|, |(2^j + 1)2^j \rho|\} \rangle^{-N_2}
\]
as a threshold. It follows immediately that, for all \(j \geq j_0\),
\[
\{(\nu; j, \ell, k) : |k_1| \leq \rho 2^{j(1+\nu_1)}, |k_2| \leq 2^{j\nu_1}, \ell = 0; \nu = \nu_1 \} \subseteq \mathcal{F}_j
\]
for some \(j_0\) and \(\nu_1\).

\[\text{Lemma 5.6.}\]
\[
\sum_{\eta \in \mathcal{F}_j} |\langle wL_j, \sigma_{j, \eta} \rangle| = o(2^j), \quad j \to \infty.
\]

35
Proof. We observe from the proof of Lemma 5.1 that the desired property is automatically satisfied provided that, for all \( j \geq j_0 \), the set \( T_j \) contains
\[
\{(i, j, \ell, k) : |k_1| \leq \rho 2^{2(j/2+\nu_1)}, |k_2| \leq 2^{2j\nu_1}, \ell = 0, i = v\},
\]
for some \( \nu_1 > 0 \), which is the content of Lemma 4.1.

We next analyze the second term in the estimate from Proposition 2.9.

Lemma 5.7. For \( h_j = o(2^{-i}) \) as \( j \to \infty \),
\[
\sum_{\eta \in T_j} |\langle M_{h_j} wL_j, \sigma_{\eta} \rangle| = o(2^j), \quad j \to \infty.
\]

Proof. First, we need to derive some estimates dependent on \((k, \ell)\) for the term \( |\langle M_{h_j} wL_j, \sigma_{j, \ell, k} \rangle|\).
By using the definitions of \( M_{h_j} \) and \( wL_j \) and a change of variables, we obtain
\[
\langle M_{h_j} wL_j, \sigma_{j, \ell, k} \rangle = 2^{j/2} \int \left[ \int \tilde{w}(\xi_1) 2h_j \int \frac{\sin(2h_j \tau_1)}{2} F(\xi_1/2^{2j}, \xi_2) \times \right.
\]
\[
\times W(\xi_1/2^{2j}, \xi_2) V(\ell + 2^{-j} \tau_1 + \xi_1) e^{-2\pi i b_1(\tau_1 + \xi_1) / \xi_2} d\tau_1 d\xi_1 \left] e^{-2\pi i (2^{2j} b_2 \xi_2)} d\xi_2. \right.
\]

Let \( \hat{G} \) now be the function
\[
\hat{G}(\xi_2) = \int \tilde{w}(\xi_1) 2h_j \int \frac{\sin(2h_j \tau_1)}{2} F(\xi_1/2^{2j}, \xi_2) W(\xi_1/2^{2j}, \xi_2) V(\ell + 2^{-j} \tau_1 + \xi_1) e^{-2\pi i (b_1 \tau_1 + \xi_1) / \xi_2} d\tau_1 d\xi_1.
\]
This function is supported on the set \([1/16, 1/2] \), which is independent of \( j \). By standard arguments, we can deduce that
\[
|\langle M_{h_j} wL_j, \sigma_{j, \ell, k} \rangle| \leq c_{N_1} 2^{j/2} \| \hat{G} \|_\infty (|k_2|)^{-N_1}. \tag{18}
\]
Let us now investigate the term \( \| \hat{G} \|_\infty \) further. We define
\[
\hat{H}_{\xi_2}(\xi_1) = 2h_j \int \frac{\sin(2h_j \tau_1)}{2} F(\xi_1/2^{2j}, \xi_2) W(\xi_1/2^{2j}, \xi_2) V(\ell + 2^{-j} \tau_1 + \xi_1) e^{-2\pi i (b_1 \tau_1 + \xi_1) / \xi_2} d\tau_1,
\]
and hence need to analyze
\[
\| \hat{G} \|_\infty = \left| \int \tilde{w}(\xi_1) \hat{H}_{\xi_2}(\xi_1) e^{-2\pi i (b_1 \xi_1) / \xi_2} d\xi_1 \right|. \tag{19}
\]

By Plancherel's theorem and the support properties of \( w \),
\[
\left| \int \tilde{w}(\xi_1) \hat{H}_{\xi_2}(\xi_1) e^{-2\pi i (b_1 \xi_1) / \xi_2} d\xi_1 \right| = |(\hat{w} \hat{H}_{\xi_2})^\vee(-b_1)| \approx c \int_{-b_1 - \rho}^{-b_1 + \rho} H_{\xi_2}(x) dx.
\]

Next,
\[
H_{\xi_2}(x) = \left( 2h_j \sin(2h_j \cdot) e^{-2\pi i b_1 \cdot} * \left( F(\cdot/2^{2j}, \xi_2) W(\cdot/2^{2j}, \xi_2) V(\ell + 2^{-j} \cdot / \xi_2) \right) \right)^\vee(-x)
\]
\[
= \left( 2h_j \sin(2h_j \cdot) e^{-2\pi i b_1 \cdot} \right)^\vee(-x) \left( F(\cdot/2^{2j}, \xi_2) W(\cdot/2^{2j}, \xi_2) V(\ell + 2^{-j} \cdot / \xi_2) \right)^\vee(-x)
\]
\[
= 1_{[-h_j, h_j]}(-x - b_1) \left( F(\cdot/2^{2j}, \xi_2) W(\cdot/2^{2j}, \xi_2) V(\ell + 2^{-j} \cdot / \xi_2) \right)^\vee(-x).
\]
Hence, since \( h_j < \rho \),
\[
\left| \int_{b_1 - \rho}^{-b_1 + \rho} H\xi_2(x) \, dx \right| = \left| \int_{b_1 - \rho}^{b_1 + h_j} (F(. / 2^{2j}, \xi_2) W(. / 2^{2j}, \xi_2) V(\ell + 2^{-j}(. / \xi_2)) \right|^\vee (-x) \, dx \right|
\]
\[
= \left| \int_{2^{j}(b_1 - \rho)}^{2^{j}(b_1 + h_j)} (F(. / 2^{2j}, \xi_2) W(. / 2^{2j}, \xi_2) V(\ell + (. / \xi_2)) \right|^\vee (-x) \, dx \right|.
\]

Notice that this indeed makes sense, since due to the masking, the length of the line singularity is not allowed to play a role here. Since \((k, \ell) \in T_j\), we have
\[
\left| \int_{b_1 - \rho}^{-b_1 + \rho} H(x) \, dx \right| = \left| \int_{k_1 - 2^{-j}h_j}^{k_1 + 2^{-j}h_j} (F(. / 2^{2j}, \xi_2) W(. / 2^{2j}, \xi_2) V(\ell + (. / \xi_2)) \right|^\vee (-x) \, dx \right|.
\]

Due to the regularity of \( W \), there exists some \( N_2 \) and \( c \) (possibly differing from the one before, but we do not need to distinguish those) such that
\[
|\left( F(. / 2^{2j}, \xi_2) W(. / 2^{2j}, \xi_2) V(\ell + (. / \xi_2)) \right|^\vee (-x) | \leq c|\langle x \rangle|^{-N_2},
\]
and hence by (19) and the previous computation,
\[
\| \hat{G} \|_\infty \leq c \min\{ |k_1 - 2^{-j}h_j|, |k_1 + 2^{-j}h_j| \}^{-N_2}.
\]

Combining this estimate with (18), we obtain
\[
|\langle \mathcal{M}_{h_j} wL_j, \sigma_{j, \ell, k} \rangle| \leq c 2^{j/2} (|k_2|)^{-N_2} \min\{ |k_1 - 2^{-j}h_j|, |k_1 + 2^{-j}h_j| \}^{-N_2},
\]
which is what we intend to use.

Hence,
\[
\frac{1}{c} \sum_{\eta \in T_j} |\langle \mathcal{M}_{h_j} wL_j, \sigma_{\eta} \rangle| \leq 2^{j/2} \sum_{\eta \in T_j} (|k_2|)^{-N_2} \min\{ |k_1 - 2^{-j}h_j|, |k_1 + 2^{-j}h_j| \}^{-N_2}
\]
\[
\leq 2^{2j(1/4 + \nu_2)}.
\]

Since \( \nu_2 < 1/4 \), the lemma is proven. \( \square \)

We now apply Proposition 2.9 to Lemmata 3.1, 5.6 and 5.7 to obtain the desired convergence for the normalized \ell_2 error of the reconstruction \( L_j \) from One-Step-Thresholding in Figure 6.

In this case \( x = wL_j \) and \( \Phi \) are shearlets \( \sigma_{j, \ell, k} \) at scale \( j \).

**Theorem 5.8.** For \( h_j = o(2^{-j}) \) and \( L_j \) the solution to (4) with \( \Phi \) the shearlet system defined using the Meyer wavelet
\[
\frac{\| L_j - wL_j \|_2}{\| wL_j \|_2} \rightarrow 0, \quad j \rightarrow \infty.
\]

This result shows that if the size of the gap shrinks faster than \( 2^{-j} \), the gap can be asymptotically perfect inpainted.
6 A Comparison of Shearlet vs. Wavelets

From the results of previous sections, we see that the size of the gaps which can be filled by shearlets ($h_j = o(2^{-j})$) with asymptotically high precision is larger than the corresponding size for wavelets ($h_j = o(2^{-2j})$); however, certainly we still need to prove that we cannot do better than the presented rates for wavelet in order to show that shearlets perform better than wavelets. In fact, we show that the rates presented for wavelets are indeed the “critical scales” for the thresholding case.

Theorem 6.1. Let $\psi_\lambda$ be the Meyer Parseval wavelets. Let $T$ be a index set such that

$$T \supseteq \{(i, j, 0, (k_1, 0)) : |k_1| \leq 2^j h_j - K_0\}$$

for some $K_0 > 0$ and $h_j > 0$. Then, we have

$$\sum_{\lambda \in T} |\langle M_{h_j} wL_j, \psi_\lambda \rangle| = O(2^{2j} h_j).$$

Proof. Recall that at level $j$, the signal $wL$ is filtered with the three corresponding frequency strips:

$$\tilde{F}_j = \sum_{i \in \{h,v,d\}} (W^i(2^{-2j} \xi) + W^i(2^{-2j-1} \xi))$$

with

$$\tilde{F}_j = \sum_{i \in \{h,v,d\}} W^i(2^{-j} \xi)$$

so that

$$F_j = \tilde{F}_j + \tilde{F}_{j+1}.$$ 

We can consider each of the filtered signals; i.e., consider $wL_j^i := wL \ast F_j^i$ with $i = v, h, d$. Since the signal is a horizontal line segment, we only need to consider $wL_j^h$. For simplicity, we denote $wL_j := wL_j^h$, $F_j := F_j^h$, and $\psi_\lambda = \psi_{j,k}^h := \psi_{j,k}$. Note that $\tilde{F}_j^v(x, y) = 2^{2j} \phi(2^j x) \widetilde{W}(2^j y)$. We want to estimate the coefficients $|\langle M_{h_j} wL_j, \psi_\lambda \rangle|$. As with other proofs for wavelets, we first consider $\tilde{wL_j}$. By definition, we have

$$\langle M_{h_j} \tilde{wL}_j, \psi_\lambda \rangle = \int_{|x| < h_j} \int_{y \in \mathbb{R}} \tilde{wL}_j(x, y) \psi_\lambda(x, y) dydx$$

$$= \int_{|x| < h_j} \int_{y \in \mathbb{R}} (wL \ast \tilde{F}_j^v(x, y)) \psi_\lambda(x, y) dydx$$

$$= \int_{|x| < h_j} \int_{y \in \mathbb{R}} \int_{z \in \mathbb{R}^2} wL(z_1, z_2) \tilde{F}_j^v((x, y) - (z_1, z_2)) dz \psi_\lambda(x, y) dydx.$$

Now, by the definition of $wL$, we have

$$\langle M_{h_j} \tilde{wL}_j, \psi_\lambda \rangle = \int_{|x| < h_j} \int_{y \in \mathbb{R}} \int_{-\rho}^\rho w(z) \tilde{F}_j^v(x - z, y) dz \psi_\lambda(x, y) dydx$$

$$\approx c \int_{|x| < h_j} \int_{y \in \mathbb{R}} \int_{-\rho}^\rho \tilde{F}_j^v(x - z, y) dz \psi_\lambda(x, y) dydx$$

38
be of order such that $T \supseteq \{\text{the wavelet threshold}, \text{to guarantee that the first term}

$T$ in the wavelet threshold case, the first term corresponds to

For each $x$ for all $k$ as long as the index set $c\{2^j(x - z)\}dz\phi(2^j x - k_1)dx$

$= c\int_{|x| < h_j} \int_{y \in \mathbb{R}} \int_{-\rho}^\rho 2^j \phi(2^j (x - z)) \hat{W}(2^j y) dy dz \phi(2^j x - k_1) \hat{W}(2^j y - k_2) dy dx$

$= c2^j \int_{y \in \mathbb{R}} \hat{W}(2^j y) \hat{W}(2^j y - k_2) dy 2^j \int_{|x| < h_j} \int_{-\rho}^\rho \phi(2^j (x - z)) dz \phi(2^j x - k_1) dx$

$= c2^j \int_{|x| < h_j} \int_{-\rho}^\rho \phi(2^j (x - z)) dz \phi(2^j x - k_1) dx$

$= c2^j \int_{|x| < h_j} \int_{-2^j \rho + 2^j x}^0 \phi(z) dz \phi(2^j x - k_1) dx$

$= c \int_{-k_1 - 2^j h_j}^{-k_1 + 2^j h_j} \int_{-2^j \rho + x + k_1}^{2^j \rho + x + k_1} \phi(z) dz \phi(x) dx.$

For each $x \in [-2^j h_j, -k_1 + 2^j h_j]$, we have $x + k_1 \in [-2^j h_j, 2^j h_j]$. Consequently, we have

$[-2^j \rho + x + k_1, 2^j \rho + x + k_1] \supseteq [-2^j (\rho - h_j), 2^j (\rho - h_j)]$

for all $x \in [-2^j h_j, -k_1 + 2^j h_j]$. Note that $\rho > h_j$. Hence, when $j$ is large enough, we have

$\int_{-2^j \rho + x + k_1}^{2^j \rho + x + k_1} \phi(z) dz \approx c \neq 0$ due to $\int \phi(x) dx \neq 0$. Therefore, we have

$|\langle M_{h_j} w L_j, \psi_\lambda \rangle| \approx c \left( \int_{-k_1 - 2^j h_j}^{-k_1 + 2^j h_j} + \int_{-k_1 - 2^j h_j}^{-k_1 + 2^j h_j} \right) \phi(x) dx.$

As $\int \phi(x) dx \neq 0$, there exists $K_0 > 0$ such that

$\int_{|x| < K} \phi(x) dx \geq c_0$

for some $c_0 > 0$ as long as $K > K_0$. Hence, when $j$ is large enough so that $2^j h_j > K_0$ and $k_1 \in [-2^j h_j - K_0, 2^j h_j - K_0]$, we have about $2^j h_j - K_0$ many coefficients that are larger than $c_0$. Consequently, when $j$ is large enough, we have

$\sum_{k \in T} |\langle M_{h_j} w L_j, \psi_\lambda \rangle| = O(2^j h_j)$

as long as the index set $T \supseteq \{(i,j,0,(k_1,0)) : |k_1| \leq 2^j h_j - K_0\}$.

For the other orientations $w L_{j_i}^0$ and $w L_{j_i}^0$, the coefficients are negligible following calculations similar to above.

In the proof of Proposition 2.10, we have

$\|x^* - x^0\|_2 = \|\Phi 1_T \Phi^* P_K x^0 + \Phi 1_T \Phi^* P_M x^0\|_2 =: \|T_1 + T_2\|_2 \geq \|T_2\|_2 - \|T_1\|_2.$

In the wavelet threshold case, the first term corresponds to $T_1 = \sum_{k \in T} |\langle w L_j, \psi_\lambda \rangle|$ while the second term corresponds to $T_2 = \sum_{k \in T} |\langle M_{h_j} w L_j, \psi_\lambda \rangle|$ for some index set $T$. As shown in the wavelet threshold, to guarantee that the first term $\|T_1\|_2$ is small, the index set $T$ is chosen such that $T \supseteq \{(k_1, k_2) : |k_1| \leq \rho 2^{2j(1+\alpha)}, |k_1| \leq 2^{2j\rho e}\}$. But then the second term $\|T_2\|_2$ will be of order $O(2^j h_j)$ as shown above. If $h_j$ decays slower than order of $O(2^{-j})$, then we have $\|L_j - w L_j\| = O(2^j)$. Thus, we have the following theorem:
Theorem 6.2. For $h_j = \omega(2^{-j})$ and $L_j$ the solution to (4) where $\Phi$ is the 2D Meyer Parseval system,
$$\frac{\|L_j - wL_j\|_2}{\|wL_j\|_2} \to 0, \quad j \to \infty.$$  

That is, the wavelet threshold method does not fill the gap. Heuristically, one can think about the situation when the gap size $h_j$ is fixed as 1. Consider the wavelets $2^j\phi(2^jx - k_1)\tilde{W}(2^jy)$. Then as $j \to \infty$, the number of such wavelets that fall in the gap is about $O(2^{2j})$. The norm $\langle M_{h_j}wL, \psi \rangle$ for any such wavelets in the gap is about the same. Consequently, the total energy concentrated in the gap will be about $O(2^{2j})$.

When $2^{2j}h_j \to 0$ and since $|\phi(x)| \leq c_N \langle |x| \rangle^{-N}$ for any $N$, we have
$$|\langle M_{h_j}wL_j, \psi \rangle| \leq c2^{2j}h_j\langle \min\{|k_1 - 2^{2j}h_j|, |k_1 + 2^{2j}h_j|\} \rangle^{-N}.$$  

For the Meyer mother wavelets $W^w = \tilde{W}(x)\phi(y)$ and $W^d = \tilde{W}(x)\psi(y)$, the above inequality still holds. In this case, the threshold method fills the gap.

Contrasting Theorem 5.8 and Theorem 6.2, we see that when the gap size $h_j$ decays like $2^j$, the using the One-Step-Thresholding algorithm produces a good approximation of the original image if shearlets are used but does not if wavelets are used.

Figure 9 shows a comparison of wavelet- and shearlet-based inpainting results. In the left column, a seismic image containing mainly curvilinear features is masked by 3 vertical bars. Using 2D Meyer tensor wavelets or shearlets – we refer to the ShearLab package in www.shearlab.org for codes of shearlet transforms –, the coefficients of the masked image are computed. After applying the threshold and applying the backward transform we derive a first approximation of an inpainted image by leaving the known part unchanged. These steps are then iterated with the threshold becoming smaller at each iteration. The outcome is illustrated in the middle column of Figure 9. The last column is the zoom-in comparison. From this, we can also visually confirm that the shearlet system is superior to the chosen wavelet system when inpainting images governed by curvilinear structures such as the exemplary seismic image.

Figure 9: Left column: original image and missing data. Middle column: wavelet inpainting and shearlet inpainting. Right column: wavelet zoom in and shearlet zoom in.
7 Extensions and Future Directions

As mentioned previously, we believe that this work and [KKZ11] make important steps in a new direction of theoretical analysis of inpainting problems. When taking into account the similar results concerning geometric separation in [DK12] and [Kut12], clustered sparsity could provide a new paradigm to prove theoretical results in a variety of problems involving sparsity. With this in mind, we mention possible extensions of this work as well as current limitations.

- **More General Singularity Models.** We anticipate that our results can be generalized to a much broader setting. In [DK12, Kut12], curvilinear singularities were segmented and flattened out using the Tubular Neighborhood Theorem. This was done in such a way as to be able to apply results concerning the clustering of curvelet coefficients along linear singularities to curvilinear singularities. Using this technique, the results in this paper concerning line singularities \( w \mathcal{L} \) should be able to be extended to curvilinear singularities.

- **Different Masks.** In this paper, we focus on a vertical strip as mask. However, after rotation other typical masks are locally vertical strips, and the analysis in our proofs occurred locally around the missing singularity. It is possible to think of a ball with radius \( h \) as mask, in which case similar results should be obtained. Other imaginable shapes could be horizontal strips, flat ellipsoids, and other polygonal objects.

- **Different Recovery Techniques.** Both hard and soft iterative thresholding techniques are quite common and usually produce convincing results. The results in this paper concern one-step-(hard)-thresholding rather than iterative thresholding. As iterative thresholding is stronger than one-pass thresholding, we strongly believe that a similar abstract analysis can be derived leading to asymptotically inpainting results in this case.

- **Other Dictionaries.** It should also be pointed out that the results in Section 2 hold for all Parseval frames. Furthermore, the asymptotic analysis in Sections 4 and 5 hold not only for the Meyer Parseval wavelets and shearlets, but also, for instance, for radial wavelets – or any types of wavelets with isotropic feature at each scale similar to the radial wavelets – and other directional multiscale representation systems such as curvelets. The necessary changes in the proofs are foreseeable. Also, the novel framework of parabolic molecules advocated in [GK12] could be applied. Furthermore given the construction of 3-dimensional shearlets in [GL11, KLL10, KLL12, KLLar], it seems likely that the proofs in Sections 5 and 8 will generalize in a straightforward but technical manner to the 3-dimensional case.

- **Noise.** Data is typically affected by noise, a situation we considered in the abstract setting. This analysis can be directly applied also for the wavelet and shearlet inpainting results, leading to the same asymptotical behavior, provided that the noise \( n \) is small comparing to the signal; i.e., the \( \ell_1 \) norm of \( \Phi^*n \) is of order smaller than the \( \ell_2 \) norm of filtered signal. However, in the literature, noise is typically measured by the \( \ell_2 \) not the \( \ell_1 \) norm.
8 Appendix: Decay of Shearlet Coefficients Related to Line Singularity

We present the idea of a continuous shearlet system in order to prove various auxiliary results. For \( t \in \{ h, w \}, a > 0, s \in \mathbb{R}, \) and \( t \in \mathbb{R}^2, \) define

\[
\hat{\sigma}^t_{a,s,t}(\cdot) = a^{3/2}W(a^2 \cdot)V^t(\cdot\cdotA^t_aS^t_{a-s})e^{2\pi i \langle \cdot, t \rangle}.
\]

It is easy to show that \( \sigma^t_{a,s,t} = a^{-3/2}\sigma^{t,a,s}(S^t_{a-1}(-t)) \) for some smooth function \( \sigma^{t,a,s}. \) For \( s = \pm a, \) we similarly define the continuous version of the “seam” elements \( \sigma_{a,\pm a,t}. \) The discrete shearlet system \( \{ \sigma^t_{j,t,k} \} \) is then obtained by sampling \( \sigma^t_{a,s,t} \) on the discrete set of points

\[
\{ t = h, v \} \times \{ a = 2^j : j \in \mathbb{N} \} \times \{ s = \ell : \ell \in \mathbb{Z}, |\ell| < 2^j \} \times \{ t \in A^t_2, S^t_{-\ell}Z^2 \}
\]

To prove that the choice of \( \Lambda_j \) offers clustered sparsity for the shearlet frame, we need some auxiliary results. The following lemma gives the decay estimate of the shearlet elements.

**Lemma 8.1.** Let the line segment with respect to \((a, s, t; v)\) be \( \text{Seg}(a, s, t; v) := \{ S^v_{a-1}(x - t_1, t_2) : |x| \leq \rho \}. \) Then

1. Given the line

\[ \text{Line}(a, s, t; v) := \{ S^v_{a-1}(x - t_1, -t_2) : x \in \mathbb{R} \}, \]

the closest point \( P_L \) to the origin on this line satisfies

\[ d_1^2 := \|P_L\|^2 = \frac{a^{-4}}{1 + s^2} t^2. \]

2. Set \( x_0 = \frac{a^{-1} t_2}{1 + s^2} t_2 + t_1. \) If \( P_S \) is the closest point on the segment \( \text{Seg}(a, s, t; v) \) to the origin, then

\[ d_2^2 := \|P_S - P_L\|^2 = \min_{+} a^{-2}(1 + s^2)(\pm \rho - x_0)^2 \quad x_0 \in [-\rho, \rho] \]

\[ = 0 \quad x_0 \notin [-\rho, \rho]. \]

**Proof.** Let \( L(x) := S^v_{a-1}(x - t_1, -t_2). \) Then

\[
\|L(x)\|^2 = \|(a^{-1}(x - t_1), a^{-1}s(x - t_1) - a^{-2}t_2)\|^2 = a^{-2}(x - t_1)^2 + a^{-2}s^2(x - t_1)^2 + a^{-4}t_2^2 - 2a^{-3}s(x - t_1)t_2 = a^{-2}(1 + s^2)(x - t_1)^2 + a^{-4}t_2^2 - 2a^{-3}s(x - t_1)t_2.
\]
Solving \( \frac{d}{dx} \|L(x)\|_2 = 2(x - t_1)a^{-2}(1 + s^2) - 2a^{-3}s t_2 = 0 \), we have \( x_0 = \frac{a^{-1}s}{1 + s^2} t_2 + t_1 \). It follows that
\[
\|P_L\|_2^2 = \|L(x_0)\|_2^2 = \|L(\frac{a^{-1}s}{1 + s^2} t_2 + t_1)\|_2^2 = \frac{a^{-4}}{1 + s^2} t_2^2 =: d_1^2.
\]

Note that \( P_L \in \text{Seg}(a, s, t; \nu) \) if and only if \( x \in [-\rho, \rho] \), in which case \( d_2 = 0 \). Otherwise,
\[
d_2^2 = \min_{\pm} \|L(\pm \rho) - P_L\|_2^2
\]
\[
= \min_{\pm} \|L(\pm \rho) - P_L\|_2^2
\]
\[
= \min_{\pm} \|(a^{-1}(\pm \rho - x_0), -a^{-1}s(\pm \rho - x_0))\|_2^2
\]
\[
= \pm a^{-2}(1 + s^2)(\pm \rho - x_0)^2,
\]
which completes the proof.

We need another auxiliary lemma. Note that
\[
\langle wL, \sigma^t_{a, s, t} \rangle = \langle wL_j, \sigma^t_{a, s, t} \rangle.
\]

**Lemma 8.2.** Define \( R_N(x_0, y_0) := \int_{y_0}^\infty \langle |(x_0, \alpha)| \rangle^{-N} \, d\alpha \) (which may be thought of as a ray integral). Then for \( y_0 \geq 0 \),
\[
R_N(x_0, y_0) \leq \pi \langle |x_0| \rangle^{-1} \langle |(x_0, y_0)| \rangle^{2-N}.
\]

**Proof.** Choose \( \beta \in (0, 1) \). Then
\[
\int_0^\infty |f(\alpha)| \, d\alpha \leq (\sup_{t \in (0, \infty)} |f(\alpha)|^\beta) \int_0^\infty |f(\alpha)|^{1-\beta} \, d\alpha.
\]
If we set \( (1 - \beta)N = 2 \) and \( f(t) = \langle |(x_0, y_0 + \alpha)| \rangle^{-N} \), then we obtain
\[
R_N(x_0, y_0) \leq (\sup_{v \in R(x_0, y_0)} \langle |v| \rangle^{2-N}) \int_0^\infty \langle |(x_0, y_0 + \alpha)| \rangle^{-2} \, d\alpha.
\]
Since
\[
\int_{-\infty}^\infty \langle |(x_0, y)| \rangle^{-M} \, dy = \langle |x_0| \rangle^{-M} \int_{-\infty}^\infty \langle \frac{y}{|x_0|} \rangle^{-M} \, dy
\]
\[
= \langle |x_0| \rangle^{-M+1} \int_{-\infty}^\infty \langle \alpha \rangle^{-M} \, d\alpha,
\]
fixing \( M = 2 \) and recalling the classic identity \( \pi = \int_{-\infty}^\infty (1 + \alpha^2)^{-1} \, d\alpha \) yield the bound
\[
\int_0^\infty \langle |(x_0, y_0 + \alpha)| \rangle^{-2} \, d\alpha \leq \pi \langle |x_0| \rangle^{-1}.
\]
Furthermore, since \( y_0 \geq 0 \),
\[
\sup_{v \in R(x_0, y_0)} \langle |v| \rangle^{2-N} = \langle |(x_0, y_0)| \rangle^{2-N}.
\]
This completes the proof. \( \square \)
Now we can estimate the decay of the shearlet coefficients aligned with the line singularity \( wL \) as follows.

**Lemma 8.3.** Retaining the notation as above, we have

\[
\langle wL, \sigma_{a,s,t}^v \rangle \leq c_N \frac{a^{-1/2}}{\sqrt{1 + s^2}} R_N(d_1, a^{-1} \sqrt{1 + s^2} d_2) \\
\leq c_N \frac{a^{-1/2}}{\sqrt{1 + s^2}} \langle |d_1| \rangle^{-1} \langle (d_1, a^{-1} \sqrt{1 + s^2} d_2) \rangle^{2-N}.
\]

**Proof.** We have

\[
|\langle wL, \sigma_{a,s,t}^v \rangle| = | \int_{-\rho}^{\rho} w_1(x) \sigma_{a,s,t}^v(x,0) dx |
\leq \int_{-\rho}^{\rho} |\sigma_{a,s,t}^v(x,0)| dx
\leq c_N a^{-3/2} \int_{Seg(a,s,t;v)} \langle |w| \rangle^{-N} dw,
\]

where we use an affine transformation of variables to turn the anisotropic norm \( |(x,0)|_{a,s,t;v} \) into the Euclidean norm \( |w| \). Application of the same transformation to \( [-\rho, \rho] \times \{0\} \) yields \( Seg(a,s,t;v) \). The integral in (20) is along a curve traversing \( Seg(a,s,t;v) \) at speed \( \nu_1 = a^{-1} \sqrt{1 + s^2} \). If we let \( Ray(a,s,t;v) \) denote the ray starting from \( P_S \) and initially traversing \( Seg(a,s,t;v) \), then

\[
a^{-3/2} \int_{Seg(a,s,t;v)} \langle |w| \rangle^{-N} dw \leq a^{-3/2} \int_{Ray(a,s,t;v)} \langle |w| \rangle^{-N} dw
\leq a^{-3/2} \nu_1^{-1} \int_{Ray(a,s,t;v)} \langle |w| \rangle^{-N} dw
\leq \frac{a^{-1/2}}{\sqrt{1 + s^2}} \int_{\nu_1 d_2}^{\infty} \langle (d_1, t) \rangle^{-N} dw
\leq \frac{a^{-1/2}}{\sqrt{1 + s^2}} R_N(d_1, \nu_1 d_2).
\]

\( \square \)

Next, we estimate the decay of the shearlet coefficients associated with those shearlets not aligned with the line singularity.

**Lemma 8.4.** Let \( t = (t_1, t_2) \). We consider the following three cases:

(i) \( t_1 \neq 0 \) and \( t_2 \neq 0 \). Then we have

\[
|\langle wL, \sigma_{a,s,t}^v \rangle| \leq c_{L,M} |t_1|^{-L} |t_2|^{-M} a^{-1/2} e^{-ca^{-1}s} a^{2M},
\]

when \( 1 \leq |s| < a^{-1} \)

\[
|\langle wL, \sigma_{a,s,t}^h \rangle| \leq c_{L,M} |t_1|^{-L} |t_2|^{-M} a^{-1/2} e^{-ca^{-2}} a^{M}
\]

and for \( s = \pm a^{-1} \)

\[
|\langle wL, \sigma_{a,s,t} \rangle| \leq c_{L,M} |t_1|^{-L} |t_2|^{-M} a^{-1/2} e^{-ca^{-1}} a^M.
\]

44
(ii) If exactly one of $t_1$ or $t_2$ is 0, then we have
\[ |\langle w L, \sigma_{a,s,t}^t \rangle| \leq c L |t_1|^2 + t_2^2 - L/2 a^{-1/2} e^{-ca^{-1} s}, t = h, v. \]

(iii) $t_1 = t_2 = 0$. Then we have
\[ |\langle w L, \sigma_{a,s,t}^t \rangle| \leq c a^{-1/2} e^{-ca^{-1} s}, t = h, v. \]

Proof. First, it is easy to show that
\[ \frac{\partial L}{\partial r_1} \frac{\partial M}{\partial r_2} |\hat{\sigma}_{a,s,0}^v| \leq c L, M a^{3/2} a^2, \]

By definition of the line singularity $w L$, we have
\[ \langle w L, \sigma_{a,s,t}^v \rangle = \int \int \hat{w}(\xi_1) \hat{\sigma}_{a,s,t}^v(\xi_1, \xi_2) d\xi_1 d\xi_2 \]
\[ = \int e^{-2\pi i t_2 \xi_2} \left[ \int \hat{w}(\xi_1) \hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2) e^{-2\pi i t_1 \xi_1} d\xi_1 \right] d\xi_2. \]

For $t_1 \neq 0$ and $t_2 \neq 0$, when we repeatedly apply integration by parts, we have
\[ |\langle w L, \sigma_{a,s,t}^v \rangle| \leq C |t_2|^{-M}|t_1|^{-L} \|h_{L,M}\|_{L^1(\mathbb{R})}, \]

where
\[ h_{L,M}(\xi_2) = \int D^{L,M}(\hat{w}(\xi_1) \hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2)) d\xi_1, \]

and for some function $f$ which is sufficiently differentiable we define the multi index,
\[ D^{L,M} f(\eta_1, \eta_2) = \left( \frac{\partial}{\partial \eta_1} \right)^L \left( \frac{\partial}{\partial \eta_2} \right)^M f(\eta_1, \eta_2). \]

The next step is to estimate the term $|h_{L,M}(\xi_2)|$.

Let $\Xi_{a,s}(\xi_2)$ be the support of the function
\[ \xi_1 \mapsto D^{L,M}(\hat{w}(\xi_1) \hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2)). \]

Note that for fixed $a, s$, the function $\xi_1 \mapsto \hat{w}(\xi_1) \hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2)$ is supported inside $[ca^{-1}|s|, \frac{1}{2}a^{-1}s]$ for a constant $c < \frac{1}{2}$. $h_{L,M}$ can then be written as
\[ h_{L,M}(\xi_2) = \int_{\Xi_{a,s}(\xi_2)} D^{L,M}(\hat{w}(\xi_1) \hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2)) d\xi_1. \]

We then rewrite the integrand as
\[ D^{L,M}(\hat{w}(\xi_1) \hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2)) = \sum_{\ell=0}^L \binom{L}{\ell} \hat{w}^{(\ell)}(\xi_1) D^{L-\ell,M}(\hat{\sigma}_{a,s,0}^v(\xi_1, \xi_2)) \]

45
Thus $|h_{L,M}(\xi_2)|$ is bounded by

$$
|h_{L,M}(\xi_2)| \leq \sum_{\ell=0}^{L} \binom{L}{\ell} \left| \int_{\Xi(a,s)(\xi_2)} \hat{w}^{(\ell)}(\xi_1) D^{L-\ell,M}(\tilde{\sigma}_{a,s,0}(\xi_1,\xi_2)) d\xi_1 \right|
$$

$$
\leq \sum_{\ell=0}^{L} \binom{L}{\ell} \|\hat{w}^{(\ell)}()\|_{L^1(|a^{-1}|s|,a^{-1}|s|)} N^{L-\ell,M}(a,s)
$$

$$
\leq c_{L,M} e^{-ca^{-1}s} \sum_{\ell=0}^{L} \binom{L}{\ell} N^{L-\ell,M}(a,s)
$$

$$
\leq c_{L,M} e^{-ca^{-1}s} a^{3/2} a^{2M}
$$

where

$$
N^{L-\ell,M}(a,s) = \|D^{L-\ell,M} \tilde{\sigma}_{a,s,0}(\xi_1,\xi_2)\|_{L^\infty(\Xi_{a,s}(\xi_2))}
$$

Consequently, we have

$$
\|h_{L,M}\|_{L^1(\mathbb{R})} \leq c_{L,M} a^{-2} e^{-ca^{-1}s} a^{3/2} a^{2M}
$$

$$
\leq c_{L,M} a^{-1/2} e^{-ca^{-1}s} a^{2M}.
$$

Therefore,

$$
|\langle w\mathcal{L}, \sigma_{a,s,t}^v \rangle| \leq c_{L,M} |t_1|^{-L} |t_2|^{-M} a^{-1/2} e^{-ca^{-1}s} a^{2M}.
$$

Using the same approach, it is not difficult to show that for $|s| < a^{-1},$

$$
|\langle w\mathcal{L}, \sigma_{a,s,t}^h \rangle| \leq c_{L,M} |t_1|^{-L} |t_2|^{-M} a^{-1/2} e^{-ca^{-2}} a^{M},
$$

and for $s = \pm a^{-1}$

$$
|\langle w\mathcal{L}, \sigma_{a,s,t}^v \rangle| \leq c_{L,M} |t_1|^{-L} |t_2|^{-M} a^{-1/2} e^{-ca^{-1}} a^{M}.
$$

The proofs for other cases are similar with simple modifications of the above procedure.

The proofs for other cases are similar with simple modifications of the above procedure.

References

[AEB06] M. Aharon, M. Elad, and A. Bruckstein, K-SVD: An algorithm for designing over-complete dictionaries for sparse representation, IEEE Trans. Signal Process. 54 (2006), 4311–4322.

[BBC+01] C. Ballester, M. Bertalmio, V. Caselles, G. Sapiro, and J. Verdera, Filling-in by joint interpolation of vector fields and gray levels, IEEE Trans. Image Process. 10 (2001), no. 8, 1200–1211.

[BBS01] M. Bertalmio, A.L. Bertozzi, and G. Sapiro, Navier-stokes, fluid dynamics, and image and video inpainting, Proceedings of the 2001 IEEE Computer Society Conference on Computer Vision and Pattern Recognition, 2001 (CVPR 2001), IEEE, 2001, pp. I–355–I362.

[BSCB00] M. Bertalmío, G. Sapiro, V. Caselles, and C. Ballester, Image inpainting, Proceedings of SIGGRAPH 2000, New Orleans, July 2000, pp. 417–424.
[CCS10] Jian-Feng Cai, Raymond H. Cha, and Zuowei Shen, *Simultaneous cartoon and texture inpainting*, Inverse Probl. Imag. 4 (2010), no. 3, 379 – 395.

[CD04] Emmanuel J. Candès and David L. Donoho, *New tight frames of curvelets and optimal representations of objects with piecewise $C^2$ singularities*, Comm. Pure Appl. Math. 57 (2004), no. 2, 219–266. MR 2012649 (2004k:42052)

[CD05] , *Continuous curvelet transform. I. Resolution of the wavefront set*, Appl. Comput. Harmon. Anal. 19 (2005), no. 2, 162–197. MR 2163077 (2006d:42058a)

[CDOS12] Jian-Feng Cai, Bin Dong, Stanley Osher, and Zuowei Shen, *Image restoration: Total variation, wavelet frames, and beyond*, J. Amer. Math. Soc. (2012), To appear.

[Chr03] Ole Christensen, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc., Boston, MA, 2003. MR 1946982 (2003k:42001)

[CK06] Tony F. Chan and Sung Ha Kang, *Error analysis for image inpainting*, J. Math. Imaging Vision 26 (2006), no. 1-2, 85–103. MR 2283872 (2007k:68131)

[CKS02] Tony F. Chan, Sung Ha Kang, and Jianhong Shen, *Euler’s elastica and curvature based inpainting*, SIAM J. Appl. Math. 63 (2002), no. 2, 564–592.

[CS02] Tony F. Chan and Jianhong Shen, *Mathematical models for local nontexture inpaintings*, SIAM J. Appl. Math. 62 (2001/02), no. 3, 1019–1043 (electronic). MR 1897733 (2003f:65110)

[CS93] Charles K. Chui and Xian Liang Shi, *Inequalities of Littlewood-Paley type for frames and wavelets*, SIAM J. Math. Anal. 24 (1993), no. 1, 263–277. MR MR1199539 (94d:42039)

[Dau92] Ingrid Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR MR1162107 (93e:42045)

[DE03] David L. Donoho and Michael Elad, *Optimally sparse representation in general (nonorthogonal) dictionaries via $l^1$ minimization*, Proc. Natl. Acad. Sci. USA 100 (2003), no. 5, 2197–2202 (electronic). MR 1963681 (2004c:94068)

[DH01] David L. Donoho and Xiaoming Huo, *Uncertainty principles and ideal atomic decomposition*, IEEE Trans. Inform. Theory 47 (2001), no. 7, 2845–2862. MR 1872845 (2002k:94012)

[DJL+12] Bin Dong, Hui Ji, Jia Li, Zuowei Shen, and Yuhong Xu, *Wavelet frame based blind image inpainting*, Appl. Comput. Harmon. Anal. 32 (2012), no. 2, 268–279.

[DK12] D. L. Donoho and G. Kutyniok, *Microlocal analysis of the geometric separation problem*, Comm. Pure Appl. Math. (2012), To appear.

[Dro07] Iddo Drori, *Fast $\ell_1$ minimization by iterative thresholding for multidimensional NMR spectroscopy*, EURASIP Journal on Advances in Signal Processing 2007 (2007).
[EAHH99] K. Engan, S.O. Aase, and J. Hakon Husoy, *Method of optimal directions for frame design*, IEEE International Conference on Acoustics, Speech, and Signal Processing, 1999. (ICASSP ’99), vol. 5, 1999, pp. 2443–2446.

[EB02] Michael Elad and Alfred M. Bruckstein, *A generalized uncertainty principle and sparse representation in pairs of bases*, IEEE Trans. Inform. Theory 48 (2002), no. 9, 2558–2567. MR 1929464 (2003h:15002)

[ESQD05] M. Elad, J.-L. Starck, P. Querre, and D. L. Donoho, *Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA)*, Appl. Comput. Harmon. Anal. 19 (2005), no. 3, 340–358. MR 2186449 (2007h:94007)

[Fou11] Simon Foucart, *Recovering jointly sparse vectors via hard thresholding pursuit*, Proceedings of SampTA 2011 (Singapore), 2011.

[GK12] P. Grohs and G. Kutyniok, *Parabolic molecules*, preprint.

[GL11] Kanghui Guo and Demetrio Labate, *Analysis and detection of surface discontinuities using the 3D continuous shearlet transform*, Appl. Comput. Harmon. Anal. 30 (2011), no. 2, 231–242. MR 2754778

[GL12] ———, *The Construction of Smooth Parseval Frames of Shearlets*, Math. Model. Nat. Phenom. (2012), To appear.

[Gro11] Philipp Grohs, *Continuous shearlet frames and resolution of the wavefront set*, Monatsh. Math. 164 (2011), no. 4, 393–426. MR 2861594

[HFH10] Gilles Hennenfent, Lloyd Fenelon, and Felix J. Herrmann, *Nonequispaced curvelet transform for seismic data reconstruction: A sparsity-promoting approach*, Geophysics 75 (2010), no. 6, WB203–WB210.

[HH06] Gilles Hennenfent and Felix J. Herrmann, *Application of stable signal recovery to seismic interpolation*, SEG International Exposition and 76th Annual Meeting, SEG, 2006.

[HH08] F. J. Herrmann and G. Hennenfent, *Non-parametric seismic data recovery with curvelet frames*, Geophys. J. Int. 173 (2008), 233–248.

[Hör03] Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003, Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. MR 1996773

[Jin99] Zhang Jing, *On the stability of wavelet and Gabor frames (Riesz bases)*, J. Fourier Anal. Appl. 5 (1999), no. 1, 105–125. MR MR1682246 (2000a:42055)

[Kin09] Emily J. King, *Wavelet and frame theory: frame bound gaps, generalized shearlets, Grassmannian fusion frames, and p-adic wavelets*, Ph.D. thesis, University of Maryland, College Park, 2009.

[KKZ11] Emily J. King, Gitta Kutyniok, and Xiaosheng Zhuang, *Analysis of data separation and recovery problems using clustered sparsity*, SPIE Proceedings: Wavelets and Sparsity XIV 8138 (2011).
[KL09] Gitta Kutyniok and Demetrio Labate, *Resolution of the wavefront set using continuous shearlets*, Trans. Amer. Math. Soc. 361 (2009), no. 5, 2719–2754. MR 2471937 (2010b:42043)

[KL11] Gitta Kutyniok and Wang-Q. Lim, *Compactly supported shearlets are optimally sparse*, J. Approx. Theory 163 (2011), 1564–1589.

[KL12] Gitta Kutyniok and Demetrio Labate (eds.), *Shearlets: Multiscale analysis for multivariate data*, Applied and Numerical Harmonic Analysis, Birkhäuser, 2012.

[KLL10] Gitta Kutyniok, Jakob Lemvig, and Wang-Q Lim, *Compactly supported shearlets*, Approximation Theory XIII (San Antonio, TX, 2010), Springer, 2010.

[KLL12] _____, *Shearlets and optimally sparse approximations*, Shearlets: Multiscale Analysis for Multivariate Data, Springer, 2012.

[KLLar] G. Kutyniok, J. Lemvig, and W.-Q Lim, *Compactly supported shearlet frames and optimally sparse approximations of functions in $l^2(r^3)$ with piecewise $c^\alpha$ singularities*, SIAM J. Appl. Math. (to appear).

[KT09] Matthieu Kowalski and Bruno Torrésani, *Sparsity and persistence: mixed norms provide simple signal models with dependent coefficients*, Signal, Image and Video Processing 3 (2009), 251–264.

[Kut12] G. Kutyniok, *Geometric separation by single pass alternating thresholding*, Preprint, 2012.

[Mey87] Yves Meyer, *Principe d’incertitude, bases hilbertiennes et algèbres d’opérateurs*, Astérisque (1987), no. 145-146, 4, 209–223, Séminaire Bourbaki, Vol. 1985/86. MR 880034 (88g:42012)

[Mey01] _____, *Oscillating patterns in image processing and nonlinear evolution equations*, University Lecture Series, vol. 22, American Mathematical Society, Providence, RI, 2001, The fifteenth Dean Jacqueline B. Lewis memorial lectures. MR 1852741 (2002j:43001)

[NDEG11] Sangnam Nam, Michael Davies, Michael Elad, and Rémi Gribonval, *Cosparse analysis modeling - uniqueness and algorithms*, International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2011), IEEE, 2011.

[NDEG12] Sangnam Nam, Mike E. Davies, Michael Elad, and Rémi Gribonval, *The Cosparse Analysis Model and Algorithms*, Appl. Comput. Harmon. Anal. (2012), In Press.

[OF97] Bruno A. Olshausen and David J. Field, *Sparse coding with an overcomplete basis set: A strategy employed by V1?*, Vision Res. 37 (1997), no. 23, 3311–3325.