Vacuum Polarization for a Massless Scalar Field in the Global Monopole Spacetime at Finite Temperature

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Abstract

In this paper we calculate the effects produced by the temperature in the renormalized vacuum expectation value of the square of the massless scalar field in the pointlike global monopole spacetime. In order to develop this calculation, we had to construct the Euclidean thermal Green function associated with this field in this background. We also calculate the high-temperature limit for the thermal average the zero-zero component of the energy-momentum tensor.
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1. Introduction

It is well known that different types of topological defects may have been created in the early Universe after the Planck time by the vacuum phase transition\[1, 2\]. These include domain walls, cosmic strings and monopoles. Among them cosmic string and monopole seem to be the best candidates to be detected.

A global monopole is a heavy object formed in the phase transition of a system composed by a self-coupling scalar field triplet \( \varphi^a \), whose original global \( O(3) \) symmetry is spontaneously broken to \( U(1) \). The simplest model which gives rise to a global monopole, was presented by Barriola and Vilenkin \[3\]. The gravitational effects produced by this object may be approximated by a solid angle deficit in the \((3+1)\)-dimensional spacetime whose line element is given by

\[
    ds^2 = -\alpha^2 dt^2 + \frac{1}{\alpha^2} dr^2 + r^2(\theta + \sin^2 \theta d\varphi^2)
\]

Here the parameter \( \alpha^2 = 1 - 8\pi G \eta_0^2 \) is smaller than unity and depends on the energy scale \( \eta_0 \) where the global symmetry is spontaneously broken. The energy-momentum tensor of this monopole has a diagonal form and reads: \( T^0_0 = T^1_1 = (\alpha^2 - 1)/r^2 \) and \( T^2_2 = T^3_3 = 0 \).

The non-trivial topology of this spacetime implies that the renormalized vacuum expectation value (VEV) of the energy-momentum tensor, \( \langle T_{\mu\nu}(x) \rangle_{\text{ren}} \), associated with an arbitrary collection of conformal massless quantum fields should not vanish\[4\]. The explicit calculations for the massless scalar\[5\] and fermionic\[6\] fields have been already obtained.

In the framework of the quantum field theory at finite temperature, the fundamental quantity is the thermal Green function, \( G_\beta(x, x') \). For the scalar field it should be periodic in the imaginary time with period \( \beta \), which is proportional to the inverse of the temperature. Because we are interested to obtain the thermal Green function, it is convenient to work in the Euclidean analytic continuation of the Green function performing a Wick rotation. So, we shall work on the Euclidean version of the monopole metric,

\[
    ds^2 = d\tau^2 + \frac{1}{\alpha^2} dr^2 + r^2(\theta + \sin^2 \theta d\varphi^2)
\]

where we have absorbed the \( \alpha \)-parameter redefining the Euclidean temporal coordinate.
The main objectives of this paper are to study the effects produced by the temperature on the renormalized VEV of the square of the massless scalar field operator and the respective energy-momentum tensor in the spacetime defined by (2). In order to do that, we calculate first the respective Euclidean thermal Green function in this manifold, which is the most relevant quantity for the obtaining of these expressions.

This paper is organized as follows. In section 2, we obtain the Euclidean thermal Green function $G_\beta(x, x')$ adopting the imaginary-time approach\cite{7}, using the Schwinger-De Witt formalism for a massless scalar field. In section 3, we calculate the renormalized thermal average value $\langle \phi^2(x) \rangle_\beta$. Because this term cannot be written in a closed form, its dependence on the temperature is not evident. So, in order to obtain some quantitative information about its behavior, we have to proceed a numerical evaluation. In section 4, we present a formal expression for the thermal average of the zero-zero component of the energy-momentum tensor $\langle T_{00}(x) \rangle_\beta$. Again, because of its complicated dependence on the temperature, only numerical calculations enable us to present its appropriate behavior. In section 5, we present our conclusions and remarks about this paper. Finally, we left for the Appendix some details of our numerical calculations.

2. The Euclidean Thermal Green Function

The Euclidean Green function associated with a massless scalar field in the global monopole spacetime has been obtained by Mazzitelli and Lousto a few years ago\cite{5}. There this Green function was obtained assuming a non-minimal coupling between the field and the geometry through the scalar curvature, $R = 2(1 - \alpha^2)/2r^2$, of this spacetime.

In this section we extend this result to obtain the Euclidean Green function at finite temperature in the monopole spacetime defined by (2). The thermal Green function, $G_\beta(x, x')$, must obey the non-homogeneous Klein-Gordon differential equation, and be periodic in the "Euclidean" time $\tau$ with a period $\beta = 1/\kappa_B T$, $\kappa_B$ being the Boltzmann constant and $T$ the absolute temperature. Considering the non-minimal coupling, the scalar Green function must obey the differential equation:

$$(\Box - \xi R)G_\beta(x, x') = -\delta^{(4)}(x, x'), \quad (3)$$
where $\Box$ denotes the covariant d’Alambertian operator in the metric defined by (2), $\xi$ is an arbitrary coupling constant and $\delta^{(4)}(x, x')$ is the bidensity Dirac distribution.

Because our spacetime is an ultrastatic one\footnote{An ultrastatic spacetime admits a globally defined coordinate system in which the components of the metric tensor are time independent and the conditions $g_{00} = 1$ and $g_{0i} = 0$ hold.}, the Euclidean thermal Green function can be obtained by the Schwinger-De Witt formalism as follows:

$$G_\beta(x, x') = \int_0^\infty ds K_\beta(x, x'; s) ,$$

where the thermal heat kernel, $K_\beta(x, x'; s)$, obeys the equation

$$\left( \frac{\partial}{\partial s} - \Box + \xi R \right) K_\beta(x, x', s) = 0 \quad (s > 0) ,$$

is subject to the boundary condition

$$\lim_{s \to 0} K_\beta(x, x'; s) = \delta^{(4)}(x, x') ,$$

and periodic in the Euclidean time with period $\beta$.

Following the prescription given in the papers by Braden\textsuperscript{8} and Page\textsuperscript{9}, the thermal heat kernel can be expressed in terms of the sum

$$K_\beta(x, x'; s) = \sum_{n=-\infty}^{\infty} K_\infty(x, x' - n\lambda_\beta; s) ,$$

where $\lambda$ is the "Euclidean" time unit vector. The zero temperature heat kernel, $K_\infty(x, x'; s)$, in this spacetime, can be factorized as

$$K_\infty(x, x'; s) = K_{(1)}(\tau, \tau'; s)K_{(3)}(\vec{x}, \vec{x}'; s) ,$$

where $K_{(1)}(\tau, \tau'; s)$ and $K_{(3)}(\vec{x}, \vec{x}'; s)$ obey, respectively, the differential equations:

$$\left( \frac{\partial}{\partial s} - \frac{\partial^2}{\partial \tau^2} \right) K_{(1)}(\tau, \tau'; s) = 0$$

and

$$\left( \frac{\partial}{\partial s} - \nabla^2 + \xi R \right) K_{(3)}(\vec{x}, \vec{x}'; s) = 0 .$$
Here the covariant spatial Laplace operator has the form
\[ \nabla^2 = \frac{\alpha^2}{r^2} \partial_r (r^2 \partial_r) - \frac{\bar{L}^2}{r^2}, \]
\( \bar{L}^2 \) is the square of the flat-space angular momentum operator.

Also, as was pointed out in [10], the heat kernel can be expressed in terms of the eigenfunctions of the operators \(-\Box + \xi R\) as
\[ K(x, x'; s) = \sum_{\sigma} \phi_\sigma(x) \phi_\sigma^*(x') e^{-s\sigma^2}, \quad (11) \]
\( \sigma^2 \) is the corresponding positively defined eigenvalues. Writing
\[ (-\Box + \xi R) \phi_\sigma(x) = \sigma^2 \phi_\sigma(x), \quad (12) \]
we get in our metrical spacetime the following results:
\[ \phi_\sigma(x) = \sqrt{\frac{\alpha p}{2\pi r}} Y_{l,m}(\theta, \phi) e^{-i\omega r} J_{\nu_l}(pr), \quad (13) \]
and
\[ \sigma^2 = \alpha^2 p^2 + \omega^2, \quad (14) \]
where \( Y_{l,m}(\theta, \phi) \) are the spherical harmonics, \( J_{\nu_l} \) the Bessel functions and \( \nu_l = \alpha^{-1} \sqrt{(l + 1/2)^2 + 2(1 - \alpha^2)(\xi - 1/8)} \). So, according to (11) our zero temperature heat kernel is given by
\[ K_\infty(x, x'; s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} dp \sum_{l,m} \phi_\sigma(x) \phi_\sigma^*(x') e^{-s\sigma^2} \]
\[ = \frac{e^{-(\Delta \tau)^2/4s}}{2\sqrt{\pi s}} \frac{e^{-\frac{r^2 + r'^2}{4\alpha^2}}}{8\pi s\alpha (rr')^{1/2}} \sum_{l=0}^{\infty} (2l + 1) I_{\nu_l} \left( \frac{rr'}{2s\alpha^2} \right) P_l(\cos \gamma), \quad (15) \]
where we have used the addition theorem for the spherical harmonics, with \( \gamma \) satisfying the well known relation with the original angles \((\theta, \phi)\) and \((\theta', \phi')\).
In (15) \( I_{\nu_l} \) is the modified Bessel function.

Comparing (15) with (8) we can identify
\[ K_{(1)}(\tau, \tau'; s) = \frac{e^{-(\tau - \tau')^2/4s}}{2\sqrt{\pi s}} \quad (16) \]
and
\[
K_{(3)}(\vec{x}, \vec{x}'; s) = \frac{e^{-\frac{r^2+r'^2}{4s\alpha^2}}}{8\pi s\alpha^{(rr')^{1/2}}} \sum_{l=0}^{\infty} (2l+1)I_{\nu_l} \left( \frac{rr'}{2s\alpha^2} \right) P_l(\cos \gamma).
\] (17)

Now we want to call attention for the fact that for \(\alpha = 1\), \(\nu_l\) become equal to \(l + 1/2\) and it is possible to get a closed expression for the sum in (17) \[11\], getting the ordinary heat kernel function for the flat-space case
\[
K_{(\alpha=1)}^{(\infty)}(x, x'; s) = \frac{1}{16\pi^2} e^{-\frac{(x-x')^2}{4s}}.
\] (18)

Now we are in position to calculate the heat kernel function at non-zero temperature in the global monopole spacetime. According to (7)-(9) only \(K_{(1)}(\tau, \tau'; s)\) will be affected by temperature, so we get
\[
K_{(1)}^{(\beta)}(\tau, \tau'; s) = \frac{1}{2\sqrt{\pi s}} \sum_{n=-\infty}^{\infty} e^{-\frac{(\tau-\tau'+n\beta)^2}{4s}}.
\] (19)

Combining Eqs. (19) and (17), we have \(K_{(1)}(x, x'; s)\) and consequently our thermal Green function, \(G_{\beta}(x, x')\), using (4). Our result is:
\[
G_{\beta}(x, x') = G_{\infty}(x, x') + \frac{1}{8\pi^2 r r'} \sum_{n \neq 0} \sum_{l=0}^{\infty} (2l+1)Q_{\nu_l-1/2}(u_\beta) P_l(\cos \gamma),
\] (20)
where
\[
u_l = \frac{\alpha^2(\tau - \tau' + n\beta)^2 + r^2 + r'^2}{2rr'},
\] (21)
and
\[
G_{\infty}(x, x') = \frac{1}{8\pi^2 r r'} \sum_{l=0}^{\infty} (2l+1)Q_{\nu_l-1/2}(u_\infty) P_l(\cos \gamma).
\] (22)

(Unfortunately it is not possible to write this thermal Green function in terms of a single special function.)

Our temperature independent Green function \(G_{\infty}(x, x')\) coincides, up to a redefinition of the "Euclidean" time \(\tau\) by \(\alpha\tau\), with the Green function for a massless scalar field at zero temperature given in Ref. \[3\].

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For the case $\alpha = 1$, Eq. (20) provides
\[
G_{\beta}^{(\alpha=1)}(x, x') = \frac{1}{4\pi\beta|\vec{r} - \vec{r}'|} \frac{\sinh \left(\frac{2\pi}{\beta|\vec{r} - \vec{r}'|}\right)}{\cosh \left(\frac{2\pi}{\beta|\vec{r} - \vec{r}'|}\right) - \cos \left(\frac{2\pi}{\beta}(\tau - \tau')\right)}.
\]  
(23)

From the expression above it is possible to obtain its zero-temperature limit, taking $\beta \to \infty$. In this limit we get the ordinary Euclidean Green function given in the literature:
\[
G(x, x') = \frac{1}{4\pi^2 (x - x')^2}.
\]  
(24)

3. The Computation of $\langle \phi^2(x) \rangle_\beta$ at Nonzero Temperature

The thermal average of $\langle \phi^2(x) \rangle_\beta$ can be obtained computing the coincidence limit of the Euclidean thermal Green function as shown below,
\[
\langle \phi^2(x) \rangle_\beta = \lim_{x' \to x} G_\beta(x, x') = \lim_{x' \to x} [G_\infty(x, x') + \mathcal{G}_\beta(x, x')],
\]  
(25)

where we have separated the purely thermal part of (20), defined as $\mathcal{G}_\beta(x, x')$.

The above procedure gives us a divergent result which comes exclusively from the zero-temperature contribution of the thermal Green function, $G_\infty(x, x)$, which is proportional to the behavior of the Legendre function evaluated at the unity. So, in order to obtain a well defined value for $\langle \phi^2(x) \rangle_\beta$, we extended to our thermal problem the renormalization procedure given in Ref. [12]: we subtract from the complete thermal Green function the Hadamard function, $G_\infty^{(1)}(x, x')$, which is given in terms of the square of the biscalar geodesic interval, $\sigma(x, x')$. So, the renormalized thermal average value for $\langle \phi^2(x) \rangle_\beta$ is given by
\[
\langle \phi^2(x) \rangle_{\beta,\text{Ren.}} = \lim_{x' \to x} [G_\beta(x, x') - G_\infty^{(1)}(x, x')],
\]  
(26)

which is explicitly equal to
\[
\langle \phi^2(x) \rangle_{\beta,\text{Ren.}} = \langle \phi^2(x) \rangle_{\infty,\text{Ren.}} + \frac{1}{4\pi^2 r^2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1) Q_{n-1/2} \left(1 + \alpha^2 \frac{n^2 \beta^2}{2r^2}\right).
\]  
(27)
In their paper, Mazzitelli and Lousto\cite{5} have computed only the zero temperature contribution, \(\langle \phi^2(x) \rangle_{\infty, \text{Ren.}}\), up to first order in the parameter \(\eta^2 = 1 - \alpha^2\), considered smaller than unity. So, we will not repeat their calculation. In this paper we compute the thermal contribution for the square of the scalar field operator up to the first order in the parameter \(\eta^2\) like in\cite{5}:

\[
\langle \phi^2(x) \rangle_{\beta} = \frac{1}{4\pi^2r^2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1) Q_{\nu_l-1/2} \left( 1 + \alpha^2 \frac{n^2 \beta^2}{2r^2} \right).
\]

(28)

In order to do that let us take the integral representation for the Legendre function\cite{13}

\[
Q_{\nu-1/2}(\cosh \rho) = \frac{1}{\sqrt{2}} \int_{\rho}^{\infty} dt \frac{e^{-\nu t}}{\sqrt{\cosh t - \cosh \rho}}.
\]

(29)

Substituting this expression in (28), it is possible to proceed the summation on the angular quantum number, by using the expansion for \(\nu_l\),

\[
\nu_l \approx \left( l + \frac{1}{2} \right) \left( 1 + \frac{\eta}{2} \right) + \frac{2\xi - 1/4}{2l + 1} \eta + \cdots
\]

After some other intermediate steps we obtain\footnote{In the Appendix A we present in more details the steps adopted by us in this calculation.}

\[
\langle \phi^2(x) \rangle_{\beta} = \frac{1}{12\beta^2} (1 + \eta^2) - \eta^2 \xi \frac{\sqrt{2}}{8\pi^2 r^2} S_1 + \eta^2 \frac{1}{32\pi^2 r^2} S_2,
\]

(30)

where

\[
S_1 = \sum_{n=1}^{\infty} \int_{\rho_n}^{\infty} dt \frac{1}{\sqrt{\cosh t - \cosh \rho_n \sinh(t/2)}}
\]

(31)

and

\[
S_2 = \sum_{n=1}^{\infty} \int_{\rho_n}^{\infty} dt \frac{1}{\sqrt{\cosh t - \cosh \rho_n \sinh^2(t/2)}}
\]

(32)

with the notation \(\rho_n = \text{arccosh}(1 + n^2 \beta^2 / 2r^2)\).

Unfortunately it was not possible to obtain an explicit result for both integrals above in terms of elementary or special functions. The best that we could do was to proceed a numerical evaluation of both integrals for specific
values of the ratio $\zeta := \beta/r$ and different values of $n$. In our numerical analyses the results were developed in the high temperature, or large distance limit, $\zeta \ll 1$. They are shown in Figs.(1a) and (1b) for $S_1(\zeta)$ and $S_2(\zeta)$ respectively. From the graphs displayed in Fig. (2), which exhibit the logarithmic behavior for the previous figures, it is possible to infer the following dependences for $S_1(\zeta)$ and $S_2(\zeta)$ with $\zeta$: \[
S_1(\zeta) = \frac{c_1}{\zeta^{q_1}} \tag{33}\]
and
\[
S_2(\zeta) = \frac{c_2}{\zeta^{q_2}}, \tag{34}\]
where $c_1 = 14.26$ with $q_1 = 0.92 \pm 0.07$ and $c_2 = 17.82$ with $q_2 = 2.02 \pm 0.02$. From these results we can see that the most relevant contribution to $\langle \phi^2 \rangle_{\beta}$ is given by the term independent on parameter $\xi$. It is, approximately, proportional to $1/\beta^2$ and consequently independent of the distance from the point to the global monopole.

4. The Thermal Average of $\langle T_{00}(x) \rangle_{\beta}$

The energy-momentum tensor, $T_{\mu\nu}(x)$, is a bilinear function of the fields, so we can evaluate its vacuum expectation value, $\langle T_{\mu\nu}(x)\rangle_{\beta}$, by the standard method using the Green’s function.[14] The thermal vacuum average, consequently, can also be obtained using the thermal Green function.

In Ref. [5] the general structure of the renormalized VEV of the energy-momentum tensor associated with a massless scalar field in the global monopole spacetime at zero temperature is presented. (There, it was obtained considering the VEV of square of the field operator on the base of dimensional arguments, symmetries and trace anomaly.) The objective of this section is to calculate the nonzero temperature correction to this vacuum polarization effect. So we concentrate on the thermal correction. The purely thermal average of energy-momentum tensor is given by:
\[
\langle T_{\mu\nu}(x) \rangle_\beta = \lim_{x' \to x} \left[ (1 - 2\xi)\nabla_\mu \nabla_\nu \overline{G}_\beta(x, x') - 2\xi \nabla_\rho \nabla_\nu \overline{G}_\beta(x, x') \right.
\]

\[
+ \left( 2\xi - \frac{1}{2} \right) g_{\mu\nu}(x)g^{\rho\sigma}(x, x') \nabla_\rho \nabla_\sigma \overline{G}_\beta(x, x')
\]

\[
- \xi G_{\mu\nu}(x) \overline{G}_\beta(x, x') - 2\xi^2 g_{\mu\nu}R(x) \overline{G}_\beta(x, x') \right],
\]

(35)

where \( G_{\mu\nu} \) and \( R \) are the Einstein tensor and the scalar curvature, respectively.

As we have already mentioned, we shall consider, for the sake of simplicity only, the zero-zero component of (35); so, taking \( \mu = \nu = 0 \) and using the fact that \( \partial_t \overline{G}(x, x') = -\partial_t \overline{G}(x, x') \), we obtain

\[
\langle T_{00}(x) \rangle_\beta = \lim_{x' \to x} \left[ -\partial_x^2 \overline{G}_\beta + \left( 2\xi - \frac{1}{2} \right) g_{00}(x)g^{\rho\sigma}(x, x') \nabla_\rho \nabla_\sigma \overline{G}_\beta(x, x') \right.
\]

\[
+ \frac{1}{2}\xi(1 - 4\xi)R(x) \overline{G}_\beta(x, x') \right].
\]

(36)

On the other hand we know from (20) that

\[
\overline{G}_\beta(x, x') = \frac{1}{8\pi^2rr'} \sum_{n \neq 0} \sum_{l=0}^{\infty} (2l+1)Q_{l-1/2}(\alpha, \zeta) P_l(\cos \gamma). \]

(37)

So, after some intermediate steps, we arrive at

\[
\langle T_{00}(x) \rangle_\beta = \frac{\alpha^2}{4\pi^2r^4} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l+1)[\Xi_n(\alpha, \zeta)Q^{(2)}_{l-1/2}(z_n) + \Phi_n(\alpha, \xi, \zeta)Q^{(1)}_{l-1/2}(z_n) + \Psi_n(\alpha, \xi)Q_{l-1/2}(z_n)],
\]

(38)

where

\[
\Xi_n(\alpha, \zeta) = \frac{1}{(\alpha^2n^2\zeta^2 + 4)} \left[ 2\xi(n^2\zeta^2 - 4) - \frac{1}{2}(n^2\zeta^2 + 4) \right],
\]

(39)

\[
\Phi_n(\alpha, \xi, \zeta) = \frac{1}{n\alpha^2\sqrt{(n^2\zeta^2 + 4)}} \left\{ 2\xi \left[ (3n^2\zeta^2 - 2)\alpha^2 - 2 \right]
\]

\[
- \frac{1}{2} \left[ (3n^2\zeta^2 - 2)\alpha^2 - 2 \right] \right\}
\]

(40)
and
\[ \Psi_n(\alpha, \xi) = \left(2\xi - \frac{1}{2}\right) + \frac{l(l + 1)}{\alpha^2} + \frac{1}{2\xi(1 - 4\xi)} \frac{\alpha^2 - 1}{\alpha^2}. \]  
(41)

Here \( z_n = 1 + \frac{n^2\alpha^2}{2} \xi^2 \) and \( \zeta = \beta/r \). In Eq. (38) \( Q^{(m)}_\nu(z) \) are the associated Legendre functions given by
\[ Q^{(m)}_\nu(z) = \frac{(z^2 - 1)^{m/2}}{d^m dz^m Q_\nu(z)} \]
for integer positive \( m \).

The equations (38)-(41) give a formal expression for the purely thermal correction to vacuum expectation value of \( \langle T_{00}(x) \rangle_\beta \). However from these expressions it is not possible to obtain a concrete conclusion about its behavior with temperature. So, in order to provide this quantitative information, let us expand \( \langle T_{00}(x) \rangle_\beta \) in powers of the parameter \( \eta^2 \) up to the first order. However, the dependence of \( \langle T_{00}(x) \rangle_\beta \) on \( \eta^2 = 1 - \alpha^2 \) appears in two different ways: (i) in the coefficients (39)-(41) and (ii) in the argument of the Legendre functions. After long calculations\(^3\) which also include the summation in the angular quantum number \( l \), we obtain an expression containing a large number of simple algebraic functions and integrals similar to the previous ones given in (31) and (32). Our final result is:
\[ \langle T_{00}(x) \rangle_\beta = -\frac{\pi^2}{30\beta^4} + \frac{1}{24} - \frac{\xi}{6} \frac{1}{r^2\beta^2} + \left[ \pi^2 \left( -\frac{11}{180} + \frac{\xi}{9} \right) \right] \frac{1}{\beta^4} + \Omega(\xi) \frac{1}{r^2\beta^2} + F(\xi, \zeta) \eta^2, \]  
(42)
where
\[ \Omega(\xi) = \frac{1}{24}(1 - 27\xi - \xi^2). \]  
(43)
The function \( F(\xi, \zeta) \) is expressed in terms of integrals as we show below
\[ F(\xi, \zeta) = \frac{1}{4\sqrt{2\pi^2}\eta^4} \left\{ \left[ \frac{1}{4}f_{0,3}(\zeta) + \xi f_{0,1}(\zeta) - \xi f_{2,3}(\zeta) - \xi^2 f_{2,1}(\zeta) \right] \right. \]
\[ - \left. \left[ \frac{1}{2}g_3(\zeta) + \xi g_1(\zeta) \right] + \left( 2\xi - \frac{1}{2} \right) \left[ \frac{1}{4}h_{4,1,5}(\zeta) - \xi \frac{1}{2}h_{1,0,3}(\zeta) \right] \right\}, \]
(44)
\(^3\)In the Appendix B we present these calculations in more details.
where
\[ f_{i,j}(\zeta) = \sum_{n=1}^{\infty} (3n^2\zeta^2 - i^2) \left[ \frac{d}{dy} \int_{\arccosh(y)}^{\infty} \frac{udu}{\sqrt{\cosh u - y \sinh^2(u/2)}} \right]_{y=1+n^2\zeta^2/2}, \]  
(45)

\[ g_j(\zeta) = \sum_{n=1}^{\infty} \bar{g}_n \tilde{X}_n^2 \left[ \frac{d^2}{dy^2} \int_{\arccosh(y)}^{\infty} \frac{udu}{\sqrt{\cosh u - y \sinh^2(u/2)}} \right]_{y=1+n^2\zeta^2/2}, \]  
(46)

and

\[ h_{l,p,q}(\zeta) = \sum_{n=1}^{\infty} \left[ \int_{\arccosh(y)}^{\infty} \frac{udu}{\sqrt{\cosh u - y \sinh^2(u/2)}} \right]_{y=1+n^2\zeta^2/2}, \]  
(47)

with \( i = 0,2; j = 1,3; l = 1,2; p = 0,1 \) and \( q = 3,5 \). We call attention for the fact that \( \bar{g}_n = 2\xi(n^2\zeta^2 - 4) - \frac{1}{2}(n^2\zeta^2 + 4) \) and \( \tilde{X}_n^2 = n^2\zeta^2/2 \).

Unfortunately, we could not express the integrals above in terms of elementary functions. Once more, in order to provide a quantitative information about the behavior of \( \langle T_{00}(x) \rangle_\beta \), we have to proceed a numerical analysis of the integrals as a function of \( \zeta = \beta/r \). However, the numerical calculations become more complicated here because of the derivatives which come from the development of \( Q_\nu^{(1)} \) and \( Q_\nu^{(2)} \). So making an appropriate transformation of variable, \( u := \arccosh \left( \frac{\zeta}{q} + 1 \right) \), the new integrals in the variable \( q \) present fixed limits. This procedure allows us to take their derivative in a simple way. In the high temperature regime, \( \zeta \ll 1 \), we have obtained the following numerical results:

\[ \langle T_{00}(x) \rangle_\beta = -\frac{\pi^2}{30\beta^4} + \left( \frac{1}{24} - \frac{1}{6} \right) \frac{1}{r^2\beta^2} + \left[ \Omega_1(\xi) \frac{1}{\beta^4} + \Omega_2(\xi) \frac{1}{r^2\beta^2} \right] + F(\xi, \zeta) \eta^2, \]  
(48)

where

\[ \Omega_1(\xi) = -\frac{\pi}{180}(11 + 52\xi), \]  
(49)

\[ \Omega_2(\xi) = -\frac{1}{768}(64\xi^2 + 1725\xi - 832) \]  
(50)

and

\[ F(\xi, \zeta) = \frac{1}{4\sqrt{2}\pi^2 r^4} \left\{ \left[ \frac{1}{4} \bar{f}_{0,3}(\zeta) + \xi \bar{f}_{0,1}(\zeta) - \xi^2 \bar{f}_{2,3}(\zeta) - \xi^2 \bar{f}_{2,1}(\zeta) \right] \right\} \]
\[- \frac{1}{2} \tilde{g}_3(\zeta) + \xi \tilde{g}_1(\zeta) \right) + \left( 2\xi - \frac{1}{2} \right) \left\{ \frac{1}{4} \tilde{h}_{2,1,5}(\zeta) - \frac{\xi}{2} \tilde{h}_{1,0,3}(\zeta) \right\} \tag{51} \]

Here the behaviors of all the above expressions with \( \zeta = \beta/r \) are given below:

(i) For the functions \( \tilde{g}_j(\zeta) \) we have:

\[ \tilde{g}_1 = -\xi \left[ \frac{132.71}{\zeta^{a_1}} + \frac{10.19}{\zeta^{b_1}} + \frac{8.11}{\zeta^{c_1}} \right] + \frac{28.20}{\zeta^{a_2}} + \frac{2.26}{\zeta^{b_2}} + \frac{8.11}{\zeta^{c_2}} \tag{52} \]

and

\[ \tilde{g}_3 = -\xi \left[ \frac{129.84}{\zeta^{a_3}} + \frac{10.19}{\zeta^{b_3}} \right], \tag{53} \]

with \( a_1 = 2.26 \pm 0.04, b_1 = 2.03 \pm 0.03, c_1 = 3.00 \pm 0.00 \) and \( a_2 = 2.22 \pm 0.02, b_2 = 1.96 \pm 0.23, c_2 = 2.99 \pm 0.01 \). Also \( a_3 = 4.01 \pm 0.01 \) and \( b_3 = 4.0 \pm 0.00 \).

(ii) For the functions \( \tilde{f}_{ij} \) we have:

\[ \tilde{f}_{03} = -\frac{55.21}{\zeta^{a_4}} \tag{54} \]

with \( a_4 = 1.96 \pm 0.01 \),

\[ \tilde{f}_{21} = -\frac{118.85}{\zeta^{a_5}} + \frac{21.27}{\zeta^{b_5}} \tag{55} \]

with \( a_5 = 2.27 \pm 0.07 \) and \( b_5 = 3.04 \pm 0.03 \),

\[ \tilde{f}_{23} = -\frac{48.87}{\zeta^{a_6}} - \frac{915.95}{\zeta^{b_6}} \tag{56} \]

with \( a_6 = 3.99 \pm 0.04 \) and \( b_6 = 2.13 \pm 0.07 \).

(iii) Finally, for the functions \( \tilde{h}_{l,p,q}(\zeta) \), we have:

\[ \tilde{h}_{2,1,5} = \frac{77.63}{\zeta^{a_7}} \tag{57} \]

and

\[ \tilde{h}_{1,0,3} = \xi \frac{16.36}{\zeta^{b_7}}, \tag{58} \]

\footnote{We exhibit only the contributions which depend on \( \zeta \) with power higher than or the same order as \( O(1/\zeta^2) \).}
where $a_\tau = 2.00 \pm 0.02$ and $b_\tau = 3.97 \pm 0.03$.

From our numerical results, which were found for $\zeta = \beta/r$ belonging to the interval $[0.01, 0.1]$, we can conclude that most relevant terms are of order $1/\zeta^4$. According to (51) these terms represent contributions to Eq. (52) which are proportional to $1/\beta^4$ and are of the same order of the leading term, obtained analytically and independent of the distance. So the thermal average $\langle T_{00}(x) \rangle_\beta$ can be written in the form below,

$$\langle T_{00}(x) \rangle^T_R = \langle T_{00}(x) \rangle^0_R + T^4 g(\gamma),$$

(59)

where $g$ is a function of the dimensionless variable $\gamma = rT$, whose expression has been found by us approximately up to first order in the parameter $\eta^2$.

5. Concluding Remarks

In the present paper we have obtained the Euclidean thermal Green function $G_\beta(x, x')$ associated with a massless scalar field in a global monopole spacetime. Our result was expressed as a sum of a zero-temperature Green function plus a purely thermal contribution. Using this Green function we were able to obtain the corrections due to temperature in $\langle \phi^2(x) \rangle_\beta$ and $\langle T_{00}(x) \rangle_\beta$. Because in both calculations, the thermal contributions could not be expressed in a simple way, but some of them in terms of not solvable integrals, their dependence on temperature could not be evaluated explicitly. So, in order to get some quantitative information, we had to develop a numerical analysis. We decided to consider these analyses for a high-temperature regime. Our final results for the thermal vacuum average $\langle \phi^2(x) \rangle_\beta$ and $\langle T_{00} \rangle_\beta$, present the following behavior:

$$\langle \phi^2(x) \rangle^T_R = \langle \phi^2(x) \rangle^0_R + T^2 f(\gamma)$$

(60)

and

$$\langle T_{00}(x) \rangle^T_R = \langle T_{00}(x) \rangle^0_R + T^4 g(\gamma).$$

(61)

For both case the functions $f$ and $g$, which depend on the specific geometry, were found by us up to the first order in the parameter $\eta^2 = 1 - \alpha^2$, considered smaller than unity. In fact for a typical grand unified theory the parameter $\eta_0$ is of order $10^{16} GeV$, so $\eta^2 = 8\pi \eta_0^2 \approx 10^{-5}$.  

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A few years ago, Smith and Linet have obtained the Euclidean thermal Green function and also the thermal average $\langle \phi^2(x) \rangle_\beta$ and $\langle T_{00}(x) \rangle_\beta$, for a massless scalar field in a cosmic string spacetime. For this case, Linet could present his results in a closed form. These two beautiful papers give us the motivation to analyse similar phenomena in a global monopole spacetime.

We want to emphasize that in all the above numerical evaluations, we considered sufficiently large number of terms in the summation in order to obtain good approximate results. (Independent numerical simulations provided us the minimal values for $n$ in each series.)

The obtained results may have some applications in the early cosmology, where the temperature of the Universe was really high. We can see that the thermal contributions to $\langle \phi^2(x) \rangle_\beta$ and $\langle T_{00}(x) \rangle_\beta$ modify significantly the zero temperature quantities in that epoch. Because of this, they should be taken into account, e.g., in the theory of structure formation.
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A High-temperature limit of the thermal average of $\langle \phi^2(x) \rangle_\beta$

In this appendix we give a brief explanation of our procedure used to obtain the high-temperature limit of the thermal average of $\langle \phi^2(x) \rangle_\beta$. According to (27), the purely thermal contribution to this quantity is:

$$\langle \phi^2(x) \rangle_\beta = \frac{1}{4\pi^2 r^2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1)Q_{\nu_l-1/2}(z_n - \eta^2 \lambda_n^2), \quad (A.62)$$

with

$$z_n = 1 + \frac{n^2 \zeta}{2} \quad \text{and} \quad \lambda_n^2 = \frac{n^2 \zeta^2}{2}, \quad (A.63)$$

where $\zeta = \beta/r$.

Expanding first the argument of the Legendre function in (A.62) up to the first order in $\eta^2$, we get:

$$\langle \phi^2(x) \rangle_\beta = \frac{1}{4\pi^2 r^2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1) \left[ Q_{\nu_l}(z_n) - \eta^2 \lambda_n^2 \frac{d}{dy} Q_{\nu_l-1/2}(y) \right]_{y=z_n} \quad (A.64)$$

We can substitute $\nu_l = l + 1/2$ into the second term of (A.64). This allows us to make the summation in the angular quantum number $l$ [13], and consequently make the second summation in $n$, resulting

$$\langle \phi^2(x) \rangle_\beta = \frac{1}{12\beta^2} \eta^2 + \frac{1}{4\pi^2 r^2} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1)Q_{\nu_l-1/2}(z_n). \quad (A.65)$$

Using the integral representation (29) for the Legendre function and the expansion

$$\nu_l \approx \left(l + \frac{1}{2}\right) \left(1 + \frac{\eta}{2}\right) + \frac{2\xi - 1/4}{2l + 1} \eta^2 + \cdots,$$

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we can obtain a simpler expression for (A.65) by the summation in $l$ as shown below:

$$
\sum_{l=0}^{\infty} (2l + 1)e^{-\nu l} = q^{1/2} \left\{ 1 + \frac{\eta^2 \ln q}{(1-q^2)} \left[ \xi (1-q)^2 + q \right] \right\}.
$$

(A.66)

where $q = e^{-t}$. (This summation was also present in Ref.[5]. There is, however a misprint, the factor $\eta^2$ appears dividing by 2, for this reason we reproduce the correct result. We want to call attention that this misprint, has been corrected in further calculations and no mistake was found by us).

Now going back to (A.64) we get the following expression

$$
\langle \phi^2(x) \rangle^\beta = \frac{1}{12 \beta^2 (1 + \eta^2)} - \eta^2 \frac{\xi \sqrt{2}}{8 \pi^2 r^2} S_1(\xi) + \eta^2 \frac{1}{32 \pi^2 r^2} S_2(\xi),
$$

(A.67)

where $S_1(\xi)$ and $S_2(\xi)$ are integrals given by (31) and (32) respectively.

**B  The expansion of (38) in powers of $\eta^2$**

The expansion of (38) in powers of $\eta^2$ is very long. In order to make our explanation clear to the reader, let us separate this calculations into three parts:

(i) The first one refers to the term in $Q_{\nu-1/2}^{(2)}$,

$$
I_1 = \frac{1}{4 \pi^2 r^4 (1 - \eta^2)} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1) \Xi_n (1 - \eta^2; \xi, \zeta) Q_{\nu-1/2}^{(2)} (z_n - \eta^2 \lambda_n^2).
$$

(B.68)

Making the expansion in the argument of the Legendre function in powers of $\eta^2$, we get:

$$
I_1 = \frac{1}{4 \pi^2 r^4} \left( 1 - \eta^2 \right) \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1) \Xi_n (1 - \eta^2; \xi, \zeta) Q_{\nu-1/2}^{(2)} (z_n) \right\}

- \left[ \left( \frac{4}{n^2 \zeta^2 + 4} \right) Q_l^{(2)} (z_n) + \lambda_n^2 \frac{d}{dy} Q_l^{(2)} (y) \right|_{y=z_n} \right\} \eta^2,
$$

(B.69)

where

$$
\Xi_n (x, \zeta) = 2\xi (n^2 \zeta^2 - 4) - \frac{1}{2} (n^2 \zeta^2 + 4).
$$

(B.70)
Using the definition for $Q_l^{(2)}$ and the integral representation for the Legendre function, we obtain, after the expansion in powers of $\eta^2$ and the summation on $l$, the following expression:

\[ I_1 = -\frac{\pi^2}{45}(1 + 4\xi)\frac{1}{\beta^4} - \frac{1}{12}(1 + 4\xi)\frac{1}{r^2\beta^2} + \left[ -\frac{\pi^2}{2}(\frac{1}{30} + \frac{\xi}{45})\frac{1}{\beta^4}ight.
\]
\[ -\frac{1}{12}(1 - 4\xi)\frac{1}{r^2\beta^2} \right] \eta^2 + \frac{1}{4\sqrt{2\pi^2r^4}}\eta^2 \left[ \frac{1}{2}g_3(\zeta) - \xi g_1(\zeta) \right], \quad (B.71) \]

where the functions $g_1(\zeta)$ and $g_2(\zeta)(z_n)$ are given in (16).

The other two terms of (38) are developed in similar way. After some intermediate steps we obtain the following results:

(ii) For the term proportional to $Q_{l-1/2}^{(1)}$ we have found

\[ I_2 = \frac{\pi}{2}\frac{2\pi^2}{45}(-1 + 4\xi)\frac{1}{\beta^4} + \left[ \frac{\pi^2}{24}(-1 + 20\xi)\frac{1}{\beta^4} + \frac{1}{8}(1 - 12\xi)\frac{1}{r^2\beta^2} \right] \eta^2
\]
\[ + \frac{1}{4\sqrt{2\pi^2r^4}}\eta^2 \left[ \frac{1}{4}f_{0,3}(\zeta) + \xi f_{0,1}(\zeta) - \xi f_{2,3}(\zeta) - \xi^2 f_{2,1}(\zeta) \right], \quad (B.72) \]

where the functions $f_{0,3}(\zeta)$, $f_{0,1}(\zeta)$, $f_{2,3}(\zeta)$ and $f_{1,3}(\zeta)$ are given in Eq. (46).

(iii) For the term proportional to $Q_{l-1/2}(z_n)$ we have:

\[ I_3 = \frac{\pi^2}{90(-1 + 4\xi)\frac{1}{\beta^4} + \frac{1}{24}(1 - 4\xi)\xi\frac{1}{r^2\beta^2} \eta^2 + \frac{\pi^2}{45}(-1 + \xi)\frac{1}{\beta^4} \eta^2
\]
\[ + \frac{1}{4\sqrt{2\pi^2r^4}}\eta^2 \left[ \frac{1}{4}h_{4,1,5}(\zeta) - \xi \frac{1}{2} h_{1,0,3}(\zeta) \right], \quad (B.73) \]

where the functions $h_{2,1,5}(\zeta)$ and $h_{1,0,3}(\zeta)$ are given in the Eq. (47).
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Figure 1: These figures exhibit, respectively, the behavior of $S_1$ and $S_2$ with $\zeta$ in the region $[0.01, 0.1]$. 
Figure 2: This figure exhibit the logarithmic behavior for both functions, $S_1$ and $S_2$ with $\zeta$, in the region $[0.01, 0.1]$. From it, it is possible to estimate the leading dependence for both quantities with $\zeta$. 