On the strong instability of the multi-layer Hele-Shaw flows.
Gelu I. Pașa, Simion Stoilow Institute of Mathematics of Romanian Academy
e-mail: gelu.pasa@imar.ro

We study the effects of some injection policies used in oil recovery process. The Saffman-Taylor instability occurs when a less viscous fluid is displacing a more viscous one, in a rectangular Hele-Shaw cell. The injection of \( N \) successive intermediate phases with constant viscosities (the multi-layer Hele-Shaw model) was studied in some recent papers, where a minimization of the Saffman-Taylor instability was obtained for large enough \( N \). However, in this paper we get a particular eigenfunction of the linear stability system which leads to eigenvalues which become infinite for large wave numbers. We obtain a strong instability of the multi-layer Hele-Shaw displacement, even if \( N \) is very large.

AMS Subject Classification: 34B09; 34D20; 35C09; 35J20; 76S05.
Key Words: Hele-Shaw displacements; Linear Hydrodynamic instability.

1. Introduction.

We consider a Stokes flow in a Hele-Shaw cell (see [14]) parallel with the plane \( xOy \). The thickness of the gap between the cell plates is denoted by \( b \). The gravity effects are neglected. The viscosity, velocity and pressure are denoted by \( \nu, u = (u, v, w), p \). As \( b \) is very small, we neglect \( w \). The flow equations are

\[
px = -\frac{12\nu}{b^2} < u >, \quad py = -\frac{12\nu}{b^2} < v >, \quad pz = 0,
\]

\[
<u>_x + <v>_y = 0,
\]

where the lower indices \( x, y, z \) are denoting the partial derivatives and \( < F > = (1/b) \int_0^b F dz \).

The above equations are similar to the Darcy’s law for the flow in a porous medium with the permeability \((b^2/12)\) - see [1], [13].

A sharp interface exists between two immiscible displacing fluids in a Hele-Shaw cell. This flow-model can be used to study the secondary oil-recovery process: the oil (with low pressure) contained in a porous medium is obtained by pushing it with a second displacing fluid. Saffman and Taylor [22] have proven the well know result: the interface is unstable when the displacing fluid is less viscous. Moreover, the fingering phenomenon appears in this case - see [15], [21].

The optimization of displacements in porous media were studied in [3], [9], [10], [17], [25]. Some effects due to the very small surface tensions are studied in [27], [29].
An intermediate fluid with a variable viscosity in a middle layer between the displacing fluids can minimize the Saffman-Taylor instability - see the experimental and numerical results given in [11], [12], [13], [20], [23], [24], [30].

A theoretical optimal variable intermediate viscosity was obtained in Carasso and Pasa [2]. An important improvement of stability of the was proved, compared with the Saffman-Taylor case.

A linear stability analysis of the three-layer Hele-Shaw flow with a constant intermediate viscosity was performed in [4], by using an algebraic method.

The Hele-Shaw displacement with \( N \) intermediate layers was studied in [5], [6], [7], [8]. We use the notation MLHS for this Multi-Layer Hele-Shaw model. In the case of constant intermediate viscosities with positive jumps in the flow direction, an improvement of the stability was obtained in [5], [6], [7], [8], if the number of the intermediate layers is very large and the surface tensions verify some conditions.

In the present paper we obtain a new eigenfunction of the linear system which governs the linear stability of MLHS. The corresponding eigenvalues become infinite for large wave numbers. Our result is mainly builds upon the boundary conditions on the interfaces where the viscosity jumps exist. Then we get a strong instability of MLHS flow, even if \( N \) is very large. Moreover, we prove that large values of the surface tensions on the interfaces are amplifying the instability, in contradiction with the experimental results and with the Saffman-Taylor criterion.

In the last part of this paper, we show that a possible strategy to minimize the Saffman-Taylor instability is the use of an intermediate fluid with a suitable variable viscosity.

The paper is laid out as follows. In section 2 we recall the three-layer Hele-Shaw model introduced in [12]. In section 3 we use this result for a model with \( N \) intermediate layers with constant viscosities and we prove the flow instability. In section 4 we study the effect of an intermediary linear viscosity profile. We conclude in section 5.
2. The three-layer Hele-Shaw model.

The three-layer Hele-Shaw flow with variable intermediate viscosity was first described in [12] and studied also in [13]. We recall here the basic elements.

A polymer solute with a variable concentration $c$ and variable viscosity $\nu$ is injected with the positive velocity $U$ in a rectangular Hele-Shaw cell which is saturated with oil of viscosity $\nu_O$, during a time interval $TI$. As in [12], adsorption, dispersion and diffusion of the solute in the equivalent porous medium are neglected. The expression of the intermediate viscosity $\nu$ as a function of $c$ is

$$\nu(c) = a_0 + a_1 c + a_2 c^2 + .... \quad (2)$$

where $a_i$ are constant coefficients - see [10], [11]. In the case of a dilute solute, which is studied here, we have $\nu = a_0 + a_1 c$, then $\nu$ is invertible with respect to $c$. The continuity equation for the solute is $Dc/Dt = 0$, then we have $D\nu/Dt = 0$. That means

$$\nu_t + u\nu_x + v\nu_y = 0. \quad (3)$$

After $TI$, a displacing fluid with viscosity $\mu_W$ is injected in the porous medium, with the same velocity $U$.

We consider incompressible fluids, then the amount of the polymer solute between the two interfaces cannot change, according to the principle of mass conservation. Therefore an arbitrary (small) movement of the first interface must induce a movement with the same velocity of the second interface.

However, it is well known - see Saffman and Taylor [22] - that interfaces change over time and turn into fingers of fluid (or polymer solute). We study the evolution of perturbations only in a small time interval after $TI$ and believe that the initial shape of interfaces has not changed so much. On this way we obtain an intermediate fluid layer, moving with the velocity $U$, where the viscosity is variable.

Consider $u = U, v = 0$, then from (3) we get $\nu = \nu(x - Ut)$. In some experiments (see [20]) an exponentially - decreasing (from the front interface) viscosity $\nu(x - Ut)$ was used and the instability was almost suppressed. The displacements with variable viscosity in Hele-Shaw cells and porous media are studied in [2], [19], [26].

The displacing and the displaced fluids are denoted with the lower indices $W, O$.

Suppose the intermediate region is the interval

$$Ut < x < Ut + L,$$
moving with the constant velocity $U$ far upstream. We have three incompressible fluids with the viscosities $\nu_W$ (displacing fluid), $\nu$ (intermediate layer) and $\nu_O$ (displaced fluid). The flow is governed by the Darcy’s equations, where $(u, v)$ are the averaged velocities - see (1):

$$
p_x = -\mu_d u; \quad p_y = -\mu_d v; \quad p_z = 0; \quad u_x + v_y = 0; \quad (4)
$$

$$
\mu_d = \mu_W, \quad x < Ut; \quad \mu_d = \mu, \quad x \in (Ut, Ut + L); \quad \mu_d = \mu_O, \quad x > Ut + L;
$$

$$
\mu_W = 12\nu_W/b^2; \quad \mu = 12\nu/b^2; \quad \mu_O = 12\nu_O/b^2. \quad (5)
$$

We consider the following basic state. The basic velocity and interfaces are

$$
u = U, \quad v = 0; \quad x = Ut, \quad x = Ut + L.
$$

On the interfaces we consider the Laplace’s law: the pressure jump is given by the surface tension multiplied with the interfaces curvature and the component $u$ of the velocity is continuous. Moreover, the interface is a material one. The basic interfaces are straight lines, then the basic pressure $P$ is continuous (but his gradient is not) and

$$
P_x = -\mu_d U, \quad P_y = 0. \quad (6)
$$

We use the equation (3), then the basic (unknown) viscosity $\mu$ in the middle layer verifies the equation

$$
\mu_t + U\mu_x = 0. \quad (7)
$$

We introduce the moving reference frame

$$
\bar{x} = x - Ut, \quad \tau = t. \quad (8)
$$

The equation (7) leads to $\mu_\tau = 0$, then $\mu = \mu(\bar{x})$. The intermediate region in the moving reference frame is the segment $0 < \bar{x} < L$. We still use the notation $x, t$ instead of $\bar{x}, \tau$.

The small perturbations of the basic velocity, pressure and viscosity are denoted by $u', v', p', \mu'$. We insert the perturbations in the equations (1), (7). As in (12), we obtain the linear stability system:

$$
p'_x = -\mu u' - \mu' U, \quad p'_y = -\mu v', \quad u'_x + v'_y = 0, \quad (9)
$$

$$
\mu'_t + U\mu'_x = 0. \quad (10)
$$

The above system is linear in disturbance quantities. Consider a perturbation velocity $u'$ of the form:

$$
u(x, y, t) = f(x)[\cos(ky) + \sin(ky)]e^{\sigma t}, \quad k \geq 0, \quad (11)$$


where \( f(x) \) is the amplitude, \( \sigma \) is the growth constant and \( k \) are the wave numbers.

The velocity along the axis \( Ox \) is continuous, then the amplitude \( f(x) \) is continuous. From (9)3, (9)2, (10), (11) we get

\[
v' = (1/k) f_x [\sin(ky) + \cos(ky)] e^{\sigma t},
\]
\[
p' = (\mu/k^2) f_x [\cos(ky) - \sin(ky)] e^{\sigma t},
\]
\[
\mu' = (-1/\sigma) \mu_x f [\cos(ky) + \sin(ky)] e^{\sigma t}.
\]

We perform the cross derivation of the relations (9)1, (9)2 and obtain

\[
\mu u_y' + \mu' U = \mu_x v' + \mu v'_x.
\]

Then from (11), (12)1, (12)3 we get the equation which governs the amplitude \( f \):

\[
- (\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \notin \{0, L\}.
\]

The viscosity is constant outside the intermediate region, then from (13) we get

\[
- f_{xx} + k^2 f = 0, \quad x \notin (0, L).
\]

The perturbations must decay to zero in the far field. Then we have

\[
f(x) = f(0)e^{kx}, \quad x \leq 0; \quad f(x) = f(L)e^{-k(x-L)}, \quad x \geq L.
\]

We describe the Laplace law in a point \( x = a \) where a a viscosity jump exists. The amplitude \( f \) is continuous in \( a \) but \( f_x \) could be discontinuous. The perturbed interface near \( a \) is denoted by \( \eta'(a, y, t) \). In the first approximation we have \( \eta'_t = u' \). As in [12], we consider

\[
\eta'(a, y, t) = (1/\sigma_a) f(a) [\cos(ky) + \sin(ky)] e^{\sigma t},
\]

where \( \sigma_a \) is the growth rate associated to the point \( x = a \).

We search for the right and left limit values of the pressure in the point \( a \), denoted by \( p^+(a) \), \( p^-(a) \). For this we use the basic pressure \( P \) in the point \( a \), the Taylor first order expansion of \( P \) near \( a \) and the expression (12)2 of \( p' \) in \( x = a \). From (6) it follows

\[
P_{x}^{+,-}(a) = -\mu_{+,-}(a)U
\]

then we get

\[
p^+(a) = P^+(a) + P_x^+(a) \eta + p_{+}^t(a), \quad p^-(a) = P^-(a) + P_x^-(a) \eta + p_{-}^t(a),
\]
\[ p^+(a) = P^+(a) - \mu^+(a)\left\{ \frac{Uf(a)}{\sigma_a} + \frac{f_x^+(a)}{k^2} \right\}[\cos(ky) + \sin(ky)]e^{\sigma t}, \quad (16) \]
\[ p^-(a) = P^-(a) - \mu^-(a)\left\{ \frac{Uf(a)}{\sigma_a} + \frac{f_x^-(a)}{k^2} \right\}[\cos(ky) + \sin(ky)]e^{\sigma t}. \quad (17) \]

The Laplace’s law is
\[ p^+(a) - p^-(a) = T(a)\eta_{yy}, \quad (18) \]
where \( T(a) \) is the surface tension acting in the point \( a \) and \( \eta_{yy} \) is the approximate value of the curvature of the perturbed interface. As \( P^-(a) = P^+(a) \), from the equations (16) - (18) we get the relationship between \( f_x^-(a), f_x^+(a) \) and \( \sigma \):
\[ -\mu^+(a)\left\{ \frac{Uf(a)}{\sigma_a} + \frac{f_x^+(a)}{k^2} \right\} + \mu^-(a)\left\{ \frac{Uf(a)}{\sigma_a} + \frac{f_x^-(a)}{k^2} \right\} = -\frac{T(a)}{\sigma_a}f(a)k^2. \quad (19) \]

When \( L = 0 \), from (14) it follows
\[ f_x^-(0) = kf(0), \quad f_x^+(0) = -kf(0), \quad \mu^-(0) = \mu_W, \quad \mu^+(0) = \mu_O. \]

Then from (19) we recover the Saffman - Taylor formula
\[ \sigma_{ST} = \frac{kU(\mu_O - \mu_W) - T(a)k^3}{\mu_O + \mu_W}. \quad (20) \]
If \( \mu_O > \mu_W \), then \( \sigma_{ST} > 0 \) in the range \( k^2 < U(\mu_O - \mu_W)/T(a) \) and the flow is unstable.

3. The \( N \)-layers Hele-Shaw displacements with constant viscosities.

It is possible to inject several polymer-solutes with constant-concentrations \( c_1, c_2, ..., c_N \) during the time intervals \( TI_1, TI_2, ..., TI_N, \ N > 1 \). We get a steady flow of \( N \) layers of immiscible fluids with constant viscosities \( \mu_i \) such that (see the equation (13))
\[ \mu_W = \mu_0 < \mu_1 < \mu_2 < ... < \mu_N < \mu_{N+1} = \mu_O; \quad \mu(x) = \mu_{i+1}, \ x \in (x_i, x_{i+1}); \quad (21) \]
\[ 0 = x_0 < x_1 < x_2 < ... < x_i < x_{i+1} < ... < x_N = L; \quad (22) \]
\[ -\mu_{i+1}f_x + \mu_{i+1}k^2f = 0, \ x \in (x_i, x_{i+1}), \quad i = 0, 1, 2, ..., N - 1. \quad (23) \]
The viscosities jumps are positive in the flow direction. On each interface \( x_i \) we have the boundary conditions (19), which can be written in the form
\[ \mu^-(x_i)f_x^+(x_i) - \mu^+(x_i)f_x^+(x_i) = \frac{k^2U[\mu^+(x_i) - \mu^-(x_i)]}{\sigma_i} - k^4T(x_i)f(x_i), \quad (24) \]
where \( \sigma_i \) is the eigenvalue corresponding to the interface \( x_i \).
This \( N \)-layers model is studied in [5], [6], [7], [8]. The boundary conditions \((24)\) of our paper are equivalent with the equations \((20)\) of [5].

In this section we show that the \( N \)-layer Hele-Shaw flow with constant intermediate viscosities is much more unstable, compared with the Saffman-Taylor displacement. We get a new particular eigenfunction of the linear stability system which gives us positive eigenvalues which become infinite for large wave numbers, even if \( N \) and the surface tensions are very large.

Propositin 1. There exists at least a growth rate \( \sigma_i \) corresponding to \((23)-(24)\) s.t.

\[
\sigma_i \to \infty \text{ for large } k.
\]

Proof. We consider the following eigenfunctions which verify the relations \((14)\) and \((23)\):

\[
f(x) = \begin{cases} 
  f(0)e^{kx}, & x \leq 0; \\
  f(0)e^{kx}, & x \in [0, L]; \\
  f(0)e^{kL - k(x - L)}, & x \geq L.
\end{cases}
\]  

\((25)\)

We use the notation \( f_i = f(x_i), \ T_i = T(x_i) \). From \((24)\) and \((26)\) we obtain

\[
(\mu_i - \mu_{i+1})f_i = \frac{kU(\mu_{i+1} - \mu_i) - k^3T_i}{\sigma_i}f_i, \quad 0 \leq i \leq N - 1, \tag{26}
\]

\[
(\mu_N + \mu_O)f_N = \frac{kU(\mu_O - \mu_N) - k^3T_N}{\sigma_N}f_N. \tag{27}
\]

From \((21)\) we have \((\mu_i - \mu_{i+1}) < 0\) for \(0 \leq i \leq N\), then \((26)\) gives us

\[
\sigma_i \to \infty \text{ for } k \to \infty, \quad i = 0, ..., N - 1. \tag{28}
\]

Only \( \sigma_N \) has a maximum positive value with respect to the wave number \( k \).

If all jumps \((\mu_i - \mu_{i+1})\) and all surface tensions \( T_i \) are equal, then \( \sigma_0 = \sigma_1 = ... = \sigma_{N-1} \) and we have only two different growth rates.

The formula \((27)\) is quite similar with the Saffman-Taylor value \((20)\), only \( \mu_W \) is replaced with \( \mu_N \).

Remark 1. Consider increasing surface tensions \( T_i \) in \((26)\). Then for a fixed (large enough) \( k \), the eigenvalues will increase as well. This means that the surface tensions amplify the
instability. This is in contradiction with the experimental results and also with the Saffman-Taylor formula.

\[ \square \]

**Remark 2.** The possible solutions of the equation (13) are

\[ f(x) = A \exp(kx) + B \exp(-kx). \]

It is possible to have \( A = 0, B = 1 \) and in this case we get

\[ f(x) = \begin{cases} 
  f(0)e^{kx}, & x \leq 0; \\
  f(0)e^{-kx}, & x \in [0, L]; \\
  f(0)e^{-kL}e^{-k(x-L)}, & x \geq L.
\end{cases} \tag{29} \]

Some elementary calculations show that the above eigenfunction give us bounded eigenvalues with respect to \( k \). Moreover, the maximum (positive) values of the eigenvalues can be arbitrary small if the viscosity jumps are small enough. It seems that in [6] it was omitted the existence of the eigenfunction (25), which is used in our paper. \[ \square \]

4. The 3-layers Hele-Shaw flow with variable intermediate viscosity.

We consider \( \mu_W < \mu_O \) and the continuous linear intermediate viscosity

\[ \mu(x) = (\mu_O - \mu_W)x/L + \mu_W, \quad \forall x \in [0, L]. \tag{30} \]

We multiply with the unknown amplitude \( f \) in (13), we integrate on \([0, L]\) and get

\[ (\mu f_x f)^+(0) - (\mu f_x f)^-(L) + \int_0^L \mu f_x^2 + k^2 \int_0^L \mu f^2 = k^2/\sigma \int_0^L \mu_x f^2, \tag{31} \]

where we use the notation \((FGH)(x) := F(x)G(x)H(x)\).

It is important to see that we have not jumps of the viscosities in \( x = 0, x = L \). Then we consider, like in [12], that the surface tensions are equal to zero in these points and from the conditions (14), (19) we get

\[ (\mu^+ f_x^+(0)) = k\mu_W f(0), \quad (\mu^- f_x^-(L)) = -k\mu_O f(L). \]

Therefore the relation (31) lead us to the following growth rate, denoted by \( \sigma_L \):

\[ \sigma_L = \frac{k^2 \int_0^L \mu_x f^2}{\mu_W k f^2(0) + \mu_O k f^2(L) + \int_0^L (\mu f_x^2 + k^2 \mu f^2)}. \tag{32} \]
We neglect some positive terms in the denominator and obtain the estimate

$$\sigma_L \leq \frac{\int_0^L \mu_x f^2}{\int_0^L \mu f^2} \leq \frac{\mu_O - \mu_W}{L \mu_W}. \tag{33}$$

In the case of a continuous (and not linear) viscosity $\mu$, we get

$$\sigma_C \leq \frac{\max_x (\mu_x)}{\min_x (\mu)}. \tag{34}$$

Both above estimates are not depending on $k$. Moreover, from (33) it follows

$$L \to \infty \Rightarrow \sigma_L \to 0. \tag{35}$$

On the page 3 of [29] is considered a linear viscosity profile in a porous medium.

5. Conclusions

The interface between two Newtonian immiscible fluids in a rectangular Hele-Shaw cell is unstable when the displacing fluid is less viscous.

An intermediate fluid with a variable viscosity between the displacing fluids can minimize the Saffman-Taylor instability - see [2], [11], [12], [13], [23], [24], [30].

The multi-layer Hele-Shaw model, consisting of $N$ intermediate fluids with constant viscosities was studied in [5], [6], [7], [8]. The main result of these papers was following: If all surface tensions verify some conditions and $N$ is large enough, then an arbitrary small (positive) upper bound of the growth rates can be obtained.

In this paper we point out a significant difference between the displacement with constant intermediate viscosities and the displacement with a suitable variable intermediate viscosity. In the first case, if the viscosity-jumps are positive in the flow direction, then some eigenvalues become infinite for large wavenumbers - see Proposition 1. In the second case we can almost suppress the Saffman-Taylor instability - see the formula (35).

The growth constants given by the formulas (26) tend to $\infty$ for large values of $k$. Moreover, if the surface tensions $T_i$ appearing in these formulas are very large, then the instability is amplified. This is in contradiction with the Saffman-Taylor stability criterion.

Our main conclusion is following. A possible strategy to minimize the Saffman-Taylor instability is to use the three-layer Hele-Shaw model with a suitable variable intermediate viscosity. On this way, the instability of the two-layer flow can be almost suppressed.
References

[1] J. Bear, Dynamics of Fluids in Porous Media, Elsevier, New York, 1972.

[2] C. Carasso and G. Paşa, An optimal viscosity profile in the secondary oil recovery, *RAIRO M2AN - Mod. Math. et Analyse Numérique*, 32(1998), 2, 211-221.

[3] C.-Yao Chen, C.-W. Huang, L.-C. Wang, and Jos A. Miranda, Controlling radial fingering patterns in miscible confined flows, *Phys. Rev. E*, 82 (2010), pp. 056308.

[4] P. Daripa, Studies on stability in three-layer Hele-Shaw flows, *Phys. Fluids.*, 20 (2008), 112101.

[5] P. Daripa, Hydrodynamic stability of multi-layer Hele-Shaw flows, *J. Stat. Mech.*, Art. No. P12005 (2008).

[6] P. Daripa and X. Ding, Universal stability properties for Multi-layer Hele-Shaw flows and Applications to Instability Control, *SIAM J. Appl. Math.*, 72 (2012), 1667-1685.

[7] P. Daripa and X. Ding, A Numerical Study of Instability Control for the Design of an Optimal Policy of Enhanced Oil Recovery by Tertiary Displacement Processes, *Transport in Porous Media* 93(2012), 675-703.

[8] P. Daripa, Some Useful Upper Bounds for the Selection of Optimal Profiles, *Physica A: Statistical Mechanics and its Applications* 391 (2012). 4065-4069.

[9] E. O. Diaz, A. Alvaeez-Lacalle, M.S. Carvalho, Jose A. Miranda, Minimization of viscous fluid fingering: a variational scheme for optimal flow rates, *Physical Review Letters*, PRL 109 (2012), pp. 144502.

[10] P. J. Flory, Principles of Polymer Chemistry, Ithaca, New York, Cornell University Press, 1953.

[11] E. Gilje, Simulations of viscous instabilities in miscible and immiscible displacement, *Master Thesis in Petroleum Technology*, University of Bergen, 2008.

[12] S. B. Gorell and G. M. Homsy, A theory of the optimal policy of oil recovery by secondary displacement process, *SIAM J. Appl. Math.* 43 (1983), 79-98.

[13] S. B. Gorell and G. M. Homsy, A theory for the most stable variable viscosity profile in graded mobility displacement process, *AIChE Journal*, 31 (1985), 1598-1503.
[14] H. S. Hele-Shaw, Investigations of the nature of surface resistance of water and of streamline motion under certain experimental conditions, Inst. Naval Architects Transactions 40(1898), 21-46.

[15] G. M. Homsy, Viscous fingering in porous media, Ann. Rev. Fluid Mech., 19 (1987), 271-311.

[16] T. T. Al-Housseiny, P. A. Tsai, H. A. Stone, Control of interfacial instabilities using flow geometry, Nature Physics Letters, 8 (2012), 747750.

[17] T. T. Al-Housseiny and H. A. Stone, Controlling viscous fingering in Hele-Shaw cells, Physics of Fluids, 25 (2013), pp. 092102.

[18] H. Lamb, Hydrodynamics, Dower Publications, New York, 1933.

[19] D. Loggia, N. Rakotomalala, D. Salin and Y. C. Yortsos, The effect of mobility gradients on viscous instabilities in miscible flows in porous media, Physics of Fluids, 11 (1999), 740-742.

[20] N. Mungan, Improved waterflooding through mobility control, Canad J. Chem. Engr., 49 (1971), 32-37.

[21] P. G. Saffman, Viscous fingering in Hele-Shaw cells, J. Fl. Mech., 173 (1986), 73-94.

[22] P. G. Saffman and G. I. Taylor, The penetration of a fluid in a porous medium or Helle-Shaw cell containing a more viscous fluid, Proc. Roy. Soc. A, 245 (1958), 312-329.

[23] G. Shah and R. Schecter, eds., Improved Oil Recovery by Surfactants and Polymer Flooding, Academic Press, New York, 1977.

[24] R. L. Slobod and S. J. Lestz, Use of a graded viscosity zone to reduce fingering in miscible phase displacements, Producers Monthly, 24 (1960), 12-19.

[25] B. Sudaryanto and Y. C. Yortsos, Optimization of Displacements in Porous Media Using Rate Control, Society of Petroleum Engineers, Annual Technical Conference and Exhibition, 30 September-3 October, New Orleans, Louisiana (2001).

[26] L. Talon, N. Goyal and E. Meiburg, Variable density and viscosity, miscible displacements in horizontal Hele-Shaw cells. Part 1. Linear stability analysis, J. Fluid Mech, 721 (2013), 268-294.
[27] S. Tanveer, Evolution of Hele-Shaw interface for small surface tension, *Philosophical Trans. Roy. Soc. A*, Published 15 May 1993.DOI: 10.1098/rsta.1993.0049.

[28] S. Tanveer, Surprises in viscous fingering, *J. Fluid Mech.*, **409** (2000), 273-368.

[29] U. S. Geological Survey, Applications of SWEAT to select Variable-Density and Viscosity Problems, U. S Department of the Interior, *Specific Investigations Report 5028* (2009).

[30] A. C. Uzoigwe, F. C. Scanlon, R. L. Jewett, Improvement in polymer flooding: The programmed slug and the polymer-conserving agent, *J. Petrol. Tech.*, **26** (1974), 33-41.

Gelu I. Paşa
Simion Stoilow Institute of Mathematics of Romanian Academy
Calea Grivitei 21, Bucuresti Sector 1, Romania
e-mail: gelu.pasa@imar.ro