ROOTS OF CROSSCAP SLIDES AND CROSSCAP TRANSPOSITIONS

ANNA PARLAK    MICHAL STUKOW

ABSTRACT. Let \( N_g \) denote a closed nonorientable surface of genus \( g \). For \( g \geq 2 \) the mapping class group \( \mathcal{M}(N_g) \) is generated by Dehn twists and one crosscap slide (\( Y \– \)homeomorphism) or by Dehn twists and a crosscap transposition. Margalit and Schleimer observed that Dehn twists have nontrivial roots. We give necessary and sufficient conditions for the existence of roots of crosscap slides and crosscap transpositions.

1. Introduction

Let \( N_{g,s}^n \) be a connected, nonorientable surface of genus \( g \) with \( s \) boundary components and \( n \) punctures, that is a surface obtained from a connected sum of \( g \) projective planes \( N_g \) by removing \( s \) open disks and specifying the set \( \Sigma = \{ p_1, \ldots, p_n \} \) of \( n \) distinguished points in the interior of \( N_g \). If \( s \) or \( n \) equals zero, we omit it from notation. The mapping class group \( \mathcal{M}(N_{g,s}^n) \) consists of isotopy classes of self–homeomorphisms \( h : N_{g,s}^n \to N_{g,s}^n \) fixing boundary components pointwise and such that \( h(\Sigma) = \Sigma \). The mapping class group \( \mathcal{M}(S_{g,s}^n) \) of an orientable surface is defined analogously, but we consider only orientation–preserving maps. If we allow orientation–reversing maps, we obtain the extended mapping class group \( \mathcal{M}^\pm(S_{g,s}^n) \). By abuse of notation, we identify a homeomorphism with its isotopy class.

In the orientable case, the mapping class group \( \mathcal{M}(S_g) \) is generated by Dehn twists [3]. As for nonorientable surfaces, Lickorish proved that Dehn twists alone do not generate \( \mathcal{M}(N_g) \), \( g \geq 2 \). This group is generated by Dehn twists and one crosscap slide (\( Y \– \)homeomorphism) [4]. A presentation for \( \mathcal{M}(N_g) \) using these generators was obtained by Stukow [11]. This presentation was derived from the presentation given by Paris and Szeptycki [7], which used as generators Dehn twists and yet another homeomorphisms of nonorientable surfaces, so–called crosscap transpositions.

Margalit and Schleimer discovered a surprising property of Dehn twists: in the mapping class group of a closed, connected, orientable surface \( S_g \) of genus \( g \geq 2 \), every Dehn twist has a nontrivial root [5]. It is natural to ask

2000 Mathematics Subject Classification. Primary 57N05; Secondary 20F38, 57M99.

Key words and phrases. Mapping class group, nonorientable surface, punctured sphere, elementary braid, \( Y \– \)homeomorphism, crosscap slide, crosscap transposition, roots.

The second author is supported by grant 2015/17/B/ST1/03235 of National Science Centre, Poland.

1
if crosscap slides and crosscap transpositions also have a similar property. The main goal of this paper is to prove the following:

**Main Theorem.** In $\mathcal{M}(N_g)$ a nontrivial root of a crosscap transposition [or a crosscap slide] exists if and only if $g \geq 5$ or $g = 4$ and the complement of the support of this crosscap transposition [or crosscap slide] is orientable.

2. Preliminaries

**Crosscap transpositions and crosscap slides.** Let $N = N_g$ be a nonorientable surface of genus $g \geq 2$. Let $\alpha$ and $\mu$ be two simple closed curves in $N$ intersecting in one point, such that $\alpha$ is two–sided and $\mu$ is one–sided. A regular neighborhood of $\mu \cup \alpha$ is homeomorphic to the Klein bottle with a hole denoted by $K$. A convenient model of $K$ is a disk with 2 crosscaps, see Figure 1. In this figure shaded disks represent crosscaps, thus the boundary points of these disks are identified by the antipodal map.

![Figure 1. Crosscap transposition and crosscap slide.](image)

A crosscap transposition $U_{\mu,\alpha}$ specified by $\mu$ and $\alpha$ is a homeomorphism of $K$ which interchanges two crosscaps keeping the boundary of $K$ fixed $^7$. It may be extended by the identity to a homeomorphism of $N$. If $t_\alpha$ is the Dehn twist about $\alpha$ (with the direction of the twist indicated by small arrows in Figure 1), then $Y_{\mu,\alpha} = t_\alpha U_{\mu,\alpha}$ is a crosscap slide of $\mu$ along $\alpha$, that is the effect of pushing $\mu$ once along $\alpha$ keeping the boundary of $K$ fixed. Note that $U_{\mu,\alpha}^2 = Y_{\mu,\alpha}^2 = t_{\partial K}$.

If $g$ is even, then the complement of $K$ in $N_g$ can be either a nonorientable surface $N_{g-2,1}$ or an orientable surface $S_{g-2,1}$, therefore on surfaces of even genus two conjugacy classes of crosscap slides and crosscap transpositions exist.

**Notation.** Represent $N_g$ as a connected sum of $g$ projective planes and let $\mu_1, \ldots, \mu_g$ be one–sided circles that correspond to crosscaps as in indicated in Figure 2. By abuse of notation, we identify $\mu_i$ with the corresponding crosscap.

If $\alpha_1, \ldots, \alpha_{g-1}$ are two–sided circles indicated in the same figure, then for each $i = 1, \ldots, g-1$ by $t_{\alpha_i}, u_i, y_i$ we denote the Dehn twist about $\alpha_i$, the crosscap transposition $U_{\mu_{i+1},\alpha_i}$, and the crosscap slide $Y_{\mu_{i+1},\alpha_i}$, respectively.
Relations in the mapping class group of a nonorientable surface. A full presentation for $\mathcal{M}(N_g)$ is given in [7, 11]. Among others, the following relations hold in $\mathcal{M}(N_g)$:

(R1) $u_iu_j = u_ju_i$ for $i, j = 1, \ldots, g - 1, \ |i - j| > 1$,

(R2) $u_1u_{i+1}u_i = u_{i+1}u_iu_{i+1}$ for $i = 1, \ldots, g - 2$,

(R3) $(u_1 \ldots u_{g-1})^g = 1$,

(R4) $t_\alpha u_j = u_j t_\alpha$, and hence $y_i u_j = u_j y_i$ for $i, j = 1, \ldots, g - 1, \ |i - j| > 1$

It is straightforward to check that relations (R1)–(R3) imply

(R5) $(u_1^2u_2 \ldots u_{g-1})^{g-1} = 1$

Geometrically $u_1 \ldots u_{g-1}$ is a cyclic rotation of $\mu_1, \ldots, \mu_g$ and $u_1^2u_2 \ldots u_{g-1}$ is a cyclic rotation of $\mu_2, \ldots, \mu_g$ around $\mu_1$. In particular,

(R6) $(u_1 \ldots u_{g-1})^g = (u_1^2u_2 \ldots u_{g-1})^{g-1} = t_{\partial N_g, 1}$ in $\mathcal{M}(N_g, 1)$.

We also have the following chain relation between Dehn twists (Proposition 4.12 of [1]): if $k \geq 2$ is even and $c_1, \ldots, c_k$ is a chain of simple closed curves in a surface $S$, such that the boundary of a closed regular neighborhood of their union is isotopic to $d$, then

(R7) $(t_{c_1} \ldots t_{c_k})^{2k+2} = td$.

3. Proof of the Main Theorem

Remark 1. Automorphisms of $H_1(N_g; \mathbb{R})$ induced by crosscap transpositions and crosscap slides have determinants equal to $-1$, so if a root of a crosscap slide or a crosscap transposition exists, it must be of odd degree.

Let $K$ be a subsurface of $N_g$ that is a Klein bottle with one boundary component $\delta$ and which contains $\mu_1$ and $\mu_2$ (Figure 2). In particular $u_1^2 = y_1^2 = t_\delta$.

The case of $g \geq 5$ odd. Let $p, q \in \mathbb{Z}$ be such that $2p + q(g - 2) = 1$. By relations (R6) and (R1),

$$u_1^2 = t_\delta = (u_3 \ldots u_{g-1})^{g-2}$$

$$u_1^{2p} = (u_3 \ldots u_{g-1})^{p(g-2)}$$

$$u_1 = ((u_3 \ldots u_{g-1})^{p}u_1^{q})^{g-2}$$
Analogously, by relations (R6), (R1) and (R4), \( y_1 = (u_3 \ldots u_{g-1})^p y_1^g \).

The case of \( g \geq 6 \) even and \( N_g \setminus K \) nonorientable. Let \( p, q \in \mathbb{Z} \) be such that \( 2p + q(g-3) = 1 \). By relations (R6) and (R1),

\[
\begin{align*}
    u_1^2 &= t \delta = (u_3^2 u_4 \ldots u_{g-1})^{g-3} \\
    u_1^{2p} &= (u_3^2 u_4 \ldots u_{g-1})^p (g-3) \\
    u_1 &= ((u_3^2 u_4 \ldots u_{g-1})^p y_1^q)^{g-3}.
\end{align*}
\]

Analogously, by relations (R6), (R1) and (R4), \( y_1 = ((u_3^2 u_4 \ldots u_{g-1})^p y_1^q)^{g-3} \).

The case of \( g \geq 4 \) even and \( N_g \setminus K \) orientable. Suppose now that crosscap transposition \( u \) and crosscap slide \( y \) are supported in a Klein bottle with a hole \( K \) such that \( N_g \setminus K \) is orientable. If \( c_1, \ldots, c_{g-2} \) is a chain of two-sided circles in \( N_g \setminus K \), then by relation (R7),

\[
\begin{align*}
    u_1^2 &= t \partial K = (t_{c_1} \ldots t_{c_{g-2}})^{2g-2} \\
    (u_1^q)^2 &= ((t_{c_1} \ldots t_{c_{g-2}})^{2g-2})^q \\
    u_1 &= ((t_{c_1} \ldots t_{c_{g-2}})^q y_1^{-1})^{g-1}.
\end{align*}
\]

Analogously, \( y_1 = ((t_{c_1} \ldots t_{c_{g-2}})^g y_1^{-1})^{g-1} \).

The case of \( g = 2 \). Crosscap slides and a crosscap transpositions are primitive in \( \mathcal{M}(N_2) \) because \( [4] \)

\[
\mathcal{M}(N_2) \cong \langle t_{a_1}, y_1 \mid t_{a_1}^2 = y_1^2 = (t_{a_1} y_1)^2 = 1 \rangle
\]

\[
\cong \langle t_{a_1}, u_1 \mid t_{a_1}^2 = u_1^2 = (t_{a_1} u_1)^2 = 1 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]

The case of \( g = 3 \).

Remark 2. It is known that the mapping class group \( \mathcal{M}(N_3) \) is hyperelliptic \( [12] \) and has the central element \( g \) such that \( \mathcal{M}(N_3)/\langle g \rangle \) is the extended mapping class group \( \mathcal{M}^\pm(S^3, K_\partial) \) of a sphere with 4 punctures. Two upper subscripts mean that we have four punctures on the sphere, but one of them must be fixed. This implies \( [2] \) that the maximal finite order of an element in \( \mathcal{M}^\pm(S^3, K_\partial) \) is 3, and hence the maximal finite order of an element in \( \mathcal{M}(N_3) \) is 6. Moreover, each two rotations of order 3 in \( \mathcal{M}^\pm(S^3, K_\partial) \) are conjugate, which easily implies that each two elements of order 6 in \( \mathcal{M}(N_3) \) are conjugate. The details of the proof of the last statement are completely analogous to that used in \( [9] \), hence we skip them.

The same conclusion can be obtained also purely algebraically: it is known \( [8] \) that \( \mathcal{M}(N_3) \cong \text{GL}(2, \mathbb{Z}) \) and the maximal finite order of an element in \( \text{GL}(2, \mathbb{Z}) \) is 6. Moreover, there is only one conjugacy class of such elements in \( \text{GL}(2, \mathbb{Z}) \) — for details see for example Theorem 2 of \( [6] \).
We will show that crosscap transpositions do not have nontrivial roots in $\mathcal{M}(N_3)$. Suppose that $x \in \mathcal{M}(N_3)$ exists such that $x^{2k+1} = u_1$, where $k \geq 1$ (see Remark 1). Then

$$x^{4k+2} = u_1^2 = t_\delta = 1.$$ 

By Remark 2 $k = 1$. Moreover, by relation (R7),

$$(t_{\alpha_1}t_{\alpha_2})^6 = t_\delta = 1,$$

hence $x$ is conjugate to $t_{\alpha_1}t_{\alpha_2}$. This contradicts Remark 1 because Dehn twists induce automorphisms of $H_1(N_3; \mathbb{R})$ with determinant equal to 1 and $x^3 = u_1$.

In the case of a crosscap slide the argument is completely analogous, hence we skip the details.

**The case of $g = 4$ and $N_4 \setminus K$ nonorientable.** If $N_4 \setminus K$ is nonorientable, then $\delta$ cuts $N_4$ into two Klein bottles with one boundary component: $K$ and $K_1$. Moreover, as was shown in Appendix A of [10],

$$\mathcal{M}(K) = \langle t_{\alpha_1}, u_1 \mid u_1t_{\alpha_1} = t_{\alpha_1}^{-1}u_1 \rangle$$

$$\mathcal{M}(K_1) = \langle t_{\alpha_3}, u_3 \mid u_3t_{\alpha_3} = t_{\alpha_3}^{-1}u_3 \rangle.$$ 

If $x \in \mathcal{M}(N_4)$ exists such that $x^{2k+1} = u_1$ and $k \geq 1$ (see Remark 1), then

$$x^{4k+2} = u_1^2 = t_\delta.$$ 

In particular, $x$ commutes with $t_\delta$ and

$$t_\delta = x t_\delta x^{-1} = t_\delta^{x(\delta)}.$$

By Proposition 4.6 of [10], up to isotopy of $N_4$, $x(\delta) = \delta$. Because $u_1$ does not interchange two sides of $\delta$ and does not reverse the orientation of $\delta$, $x$ has exactly the same properties. Therefore, we can assume that $x$ is composed of maps of $K$ and $K_1$. Moreover $u_1 = x^{2k+1}$ interchanges $\mu_1$ and $\mu_2$ and does not interchange $\mu_3$ and $\mu_4$, hence

$$x = t_{\alpha_1}^{k_1}t_{\alpha_3}^{2m_1+1}t_{\alpha_3}^{k_2}t_{\alpha_3}^{2m_2} = t_{\alpha_1}^{k_1}t_{\alpha_3}^{k_2}t_{\alpha_3}^{m_1+m_2}$$

$$x^2 = t_{\alpha_3}^{2k_2}t_{\alpha_3}^{2m_1+2m_2+1}$$

But then

$$t_\delta = (x^2)^{2k+1} = t_{\alpha_3}^{2k_2(2k+1)}t_{\alpha_3}^{2m_1+2m_2+1)}(2k+1)$$

which is a contradiction, because Dehn twists about disjoint circles generate a free abelian group (Proposition 4.4 of [10]).

In the case of a crosscap slide the argument is completely analogous, hence we skip the details.
4. Roots of elementary braids in the mapping class group of \( n \)-punctured sphere.

Margalit and Schleimer observed in [5] that if \( g \geq 5 \), then roots of elementary braids in \( \mathcal{M}(S^0_g) \) exist. The Main Theorem implies slightly stronger version of that result.

**Corollary 3.** An elementary braid in the mapping class group \( \mathcal{M}(S^0_n) \) or in the extended mapping class group \( \mathcal{M}^\pm(S^0_n) \) of \( n \)-punctured sphere has a nontrivial root if and only if \( n \geq 5 \).

**Proof.** By Proposition 2.4 of [7], there is a monomorphism

\[
\varphi: \mathcal{M}^\pm(S^0_n) \to \mathcal{M}(N_g)
\]

which is induced by blowing up each puncture to a crosscap. In particular, this monomorphism sends elementary braids to crosscap transpositions. Moreover, all roots of crosscap transpositions constructed in the proof of the Main Theorem are elements of \( \varphi(\mathcal{M}(S^0_n)) \). \( \square \)

**References**

[1] B. Farb and D. Margalit. A Primer on Mapping Class Groups, volume 49 of Princeton Mathematical Series. Princeton Univ. Press, 2011.

[2] R. Gillette and J. Van Buskirk. The word problem and consequences for the braid groups and mapping-class groups of the 2-sphere. Trans. Amer. Math. Soc., 131:277–296, 1968.

[3] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. Ann. of Math., 76:531–540, 1962.

[4] W. B. R. Lickorish. Homeomorphisms of non–orientable two–manifolds. Math. Proc. Cambridge Philos. Soc., 59:307–317, 1963.

[5] D. Margalit and S. Schleimer. Dehn twists have roots. Geom. Topol., 13(3):1495–1497, 2009.

[6] S. Meskin. Periodic automorphisms of the two-generator free group, volume 372 of Lecture Notes in Math., pages 494–498. Springer–Verlag, 1974.

[7] L. Paris and B. Szepietowski. A presentation for the mapping class group of a nonorientable surface. Bull. Soc. Math. France., 143:503–566, 2015.

[8] M. Scharlemann. The complex of curves on non-orientable surfaces. J. London Math. Soc., 25(2):171–184, 1982.

[9] M. Stukow. Conjugacy classes of finite subgroups of certain mapping class groups. Turkish J. Math., 28(2):101–110, 2004.

[10] M. Stukow. Dehn twists on nonorientable surfaces. Fund. Math., 189:117–147, 2006.

[11] M. Stukow. A finite presentation for the mapping class group of a nonorientable surface with Dehn twists and one crosscap slide as generators. J. Pure Appl. Algebra, 218(12):2226–2239, 2014.

[12] M. Stukow. A finite presentation for the hyperelliptic mapping class group of a nonorientable surface. Osaka J. Math., 52(2):495–515, 2015.

Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland

E-mail address: anna.parlak@gmail.com, trojkat@mat.ug.edu.pl