In this talk I discuss neutrino oscillations due to possible nonuniversality of the gravitational coupling constants. It has been pointed out [1] [2] that a breakdown of the universality of the gravitational couplings to different neutrino flavors could lead to neutrino oscillations. In particular the authors of [2] studied the possibility in which the solar neutrinos can be used to test this kind of breakdown of the universality. Since the flux of the solar neutrinos is relatively small and the energy spectrum is beyond our control, the utility of the solar neutrino for this purpose is limited. Here we would like to propose a possible long-baseline experiments of neutrino oscillations to test the breakdown of the universality of the gravitational couplings to neutrinos [3] [4] [5]. As we will see, the breakdown of the universality of the gravitational couplings to neutrinos of different flavors leads to a violation of Einstein’s equivalence principle (see, e.g., [6]) which states that all the laws of physics must take on their familiar special-relativistic forms in any and every local Lorentz frame, anywhere and any time in the universe. No consistent theory is known to predict such nonuniversality of the gravitational coupling constants, so our motivation is to give an upper bound on such nonuniversality. In the present case, it turns out that we can probe the magnitude of the breakdown of Einstein’s equivalence principle to the order of $10^{-15}$, assuming that there are neutrino mixings. Among various experiments to test the equivalence principle (see e.g., Ref. [7] for a review), there have been few tests of Einstein’s equivalence principle for neutrinos [8]. The universality of the gravitational couplings that I discuss in this talk is of different type from these experiments in the past, so our discussions here are complementary to them.

In this talk we assume that there are two neutrino flavors which have different couplings to gravity and that the eigenstates of these different gravitational couplings do not coincide with those of the electroweak flavors. Throughout the present discussions we consider neutrino oscillations between two flavors for simplicity. Let us suppose that two kinds of neutrinos are in a certain background field with the different gravitational couplings, and that they are described by the following Lagrangian

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\begin{align}
\mathcal{L} &= e(G_1)\mathcal{V}_1 \left[ ie^{\alpha \mu} (G_1) \gamma_\alpha D_\mu (G_1) - m_1 \right] \nu_1 \\
&+ e(G_2)\mathcal{V}_2 \left[ ie^{\alpha \mu} (G_2) \gamma_\alpha D_\mu (G_2) - m_2 \right] \nu_2 \\
&+ \text{(interaction terms with electroweak gauge bosons)},
\end{align}

where we have included mass terms to keep generality, $e^{\alpha \mu}(G_i)$ ($i = 1, 2$) are the vierbein fields of some background metric with different Newton constants $G_i$ ($i = 1, 2$), and $e(G_i) \equiv \det e^{\alpha}_\mu(G_i)$. For simplicity we assume that the eigenstates of the gravitational couplings coincide with those of the masses. Notice that even if these neutrinos are massless, we cannot rotate these two fields so that these are the eigenstates of the electroweak theory, since the gravitational coupling terms are not invariant under the rotation in the flavor space. Since the gravitational couplings for these two kinds of neutrinos are different, even if we choose a coordinate system in which the Dirac equation for $\nu_1$ in (1) becomes the one in a flat space-time, the Dirac equation for $\nu_2$ in the same coordinate system does not necessarily do so. Thus Einstein’s equivalence principle is violated in (1).

The configuration of the long-baseline experiment we will discuss is depicted in Fig. 1, and the neutrino beams go underneath the ground. The density of the Earth is not constant, so in principle we have to regard the density as a function of the radius \[1\]. When we consider the case like the longbase-line experiment in the DUMAND project \[10\], where the neutrino beam goes deep in the Earth ($L \equiv$ (length of the trajectory of the neutrino beam) $\sim$ 6,000 km), this is actually the case. However, in the case like the one in the SOUDAN2 project \[10\], where the neutrino beam goes just slightly under the surface of the Earth ($L \sim$ 800 km), we can assume that the density is approximately constant. In this talk we will take this assumption for simplicity, and the analysis without this approximation is given in \[4\].

First let us consider the Dirac equation of left-handed neutrinos without any flavor in the interior Schwarzschild background \[11\]:

\[(ie^{\alpha \mu} \gamma_\alpha D_\mu - m) \psi = 0,\]  

where $e_{\alpha \mu}$ is the vierbein of the interior Schwarzschild metric

\[ds^2 = (e^{\theta}_t)^2 dt^2 - (e^{\phi}_r)^2 dr^2 - (e^{\phi}_\theta)^2 d\theta^2 - (e^{\phi}_\phi)^2 d\phi^2\]
and is given by

\[ e_0^t = \frac{3}{2} \sqrt{1 - \frac{\alpha}{R}} - \frac{1}{2} \sqrt{1 - \frac{\alpha r^2}{R^2}}, \quad e_1^r = \frac{1}{\sqrt{1 - \frac{\alpha r^2}{R^2}}}, \quad e_2^\theta = r, \quad e_3^\phi = r \sin \theta. \] (4)

\[ D_\mu \psi \equiv (\partial_\mu - \frac{1}{2} \omega_{\mu ab} \sigma^{ab}) \psi \] is the covariant derivative acting on a spinor \( \psi \), \( \omega_{\mu ab} \) is the spin connection given by \( e^b_{[\mu} \omega^a_{\nu]} = \partial_{[\mu} e^a_{\nu]} \), and \( \alpha \) in (4) is the Schwarzschild radius.

One characteristic dimensionless parameter in our case is \( ER \), where \( E \) is the energy of the neutrino, and \( R \) is the radius of the Earth. For \( E=10 \text{ GeV} \) and \( R=6,400 \text{ Km} \), \( ER \sim 3 \times 10^{23} \), and derivative terms in the spin connections are all of the order of \( 1/ER \), so we will neglect them throughout this talk. \( \alpha \) in eq. (4) is the Schwarzschild radius of the Earth which is about 9 mm, so we also expand (2) to the first order in \( \alpha/r \). In this approximation the positive energy part of the Dirac equation finally becomes

\[ \frac{dv}{du} = i \left[ 1 - \frac{m^2}{2E^2} + \frac{3\alpha}{4R} - \frac{\alpha \ell^2}{4R^3} \left( 1 - \frac{u^2}{B^2} \right) \right] \nu, \]

where \( u \equiv Ex \) is a dimensionless coordinate along the x-axis, \( \ell \equiv \sqrt{R^2 - (L/2)^2} \) is the distance of the trajectory of the neutrino beam from the center of the Earth, and \( B \equiv E\ell \) is a very large number.

In case of two kinds of neutrinos described by the Lagrangian (1), it is straightforward to see that the Dirac equation for (1) is given by

\[ \frac{d}{du} \left( \begin{array}{c} \nu_1 \\ \nu_2 \end{array} \right) = i \left[ 1 - \frac{m_1^2 + m_2^2}{4E^2} + \left( -3 + \frac{\ell^2}{R^2} \left( 1 - \frac{u^2}{B^2} \right) \right) \frac{f_1 + f_2}{4} \Phi + \Delta(u)\sigma_3 \right] \left( \begin{array}{c} \nu_1 \\ \nu_2 \end{array} \right), \] (5)

where

\[ \Delta(u) \equiv \frac{\Delta m^2}{4E^2} + \left( 3 - \frac{\ell^2}{R^2} \left( 1 - \frac{u^2}{B^2} \right) \right) \frac{\Delta f}{4} \Phi. \] (6)

Here \( \Delta m^2 \equiv m_2^2 - m_1^2 \) is the difference of the masses, we have defined the Newton potential \( \Phi \equiv -GM/R \) on the surface of the Earth, and we have also defined the difference \( \Delta f = f_2 - f_1 \) of the dimensionless gravitational couplings of the two neutrino species

\[ \begin{pmatrix} f_1 \Phi \\ f_2 \Phi \end{pmatrix} = - \begin{pmatrix} \alpha_1/2R \\ \alpha_2/2R \end{pmatrix} = - \begin{pmatrix} G_1 M/R \\ G_2 M/R \end{pmatrix}. \]

The equation (5) can be easily integrated from \( u = -EL/2 \) to \( u = EL/2 \).

Now let us introduce the flavor eigenstates \( \nu_a, \nu_b \) of the weak interaction by

\[ \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \]

Then the probability of detecting a different flavor \( \nu_b \) at a distance \( L \) after producing one neutrino flavor \( \nu_a \) is given by

\[ P(\nu_a \rightarrow \nu_b) = \sin^2 2\theta \sin^2 \left[ \left( \frac{\Delta m^2}{4E^2} + \left( 1 + \frac{L^2}{6R^2} \right) \frac{\Delta f \Phi}{2} \right) EL \right]. \] (7)
This formula applies to the transition between $\nu_\mu$ and $\nu_\tau$, where no MSW effect is expected to occur.

In case of the transition between $\nu_e$ and $\nu_\mu$, we have to take the MSW effect into consideration, and the Dirac equation is modified as

$$\frac{d}{du} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = i \begin{pmatrix} \Delta(u) \cos 2\theta - \frac{G_F N_e}{\sqrt{2}E} & \Delta(u) \sin 2\theta \\ \Delta(u) \sin 2\theta & -\Delta(u) \cos 2\theta + \frac{G_F N_e}{\sqrt{2}E} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \quad (8)$$

where $G_F$ is the Fermi coupling constant, $N_e$ is the density of electrons in the Earth. (8) can be solved in the same way as before by introducing the variables

$$\Delta_N(u) \cos 2\theta_N = \Delta(u) \cos 2\theta - \frac{G_F N_e}{\sqrt{2}E},$$

$$\Delta_N(u) \sin 2\theta_N = \Delta(u) \sin 2\theta. \quad (9)$$

Note that $\theta_N$ does depend on the variable $u$ in this case. It is easy to integrate (9), and we have the transition probability of detecting $\nu_e$ at a distance $L$ from the source of $\nu_\mu$ beams

$$P(\nu_\mu \rightarrow \nu_e) = \sin^2 2\theta_N \left( u = \frac{EL}{2} \right) \sin^2 \left( \int_{-EL/2}^{EL/2} du \Delta_N(u) \right), \quad (10)$$

where we have used the fact $\theta_N(u = \frac{EL}{2}) = \theta_N(u = -\frac{EL}{2})$, and $\Delta_N, \theta_N$ are defined through (9). The integration in the exponent in (10) can be performed numerically (3).

We have studied the quantity $P(\nu_\alpha \rightarrow \nu_\beta)$ for various cases (3) (4). As in the case of neutrino oscillations due to masses, we can exclude certain regions in the $(\sin^2 2\theta, \Delta f)$ plot. A typical figure is given in Fig. 2, which shows the region $(\sin^2 2\theta, \Delta f)$ for $\nu_\mu$-$\nu_\tau$ oscillations which can be excluded by the proposed longbase-line experiment in the SOUDAN2 project. From Fig. 2 we see that we can give an upper limit as small as $10^{-14}$ on the nonuniversality of the gravitational coupling constants of neutrinos. In case of the DUMAND project this upper limit could reach $10^{-15}$ (4).
One important feature of this experiment is the dependence of the probability on the detection threshold energy of muons, which is, roughly speaking, proportional to the energy of the incident neutrinos. To emphasize this aspect, let us make comparison of the energy dependence of the probability for various cases. So far we have considered the case where the particle which mediates the force between neutrinos is a spin-2 particle, i.e., graviton. In this case the energy defined in the Dirac equation

\[ i \frac{\partial \psi}{\partial t} = E \psi \]  

is given by

\[ E_{J=2} = \left(1 + \frac{1}{2} \Phi_{J=2}\right) \sqrt{\vec{p}^2 + m^2} \]  

(12)

On the other hand, if the force is mediated by a scalar or a vector particle, then the energy would be given by

\[
\begin{align*}
E_{J=0} &= \sqrt{\vec{p}^2 + (m + \Phi_{J=0})^2} \\
E_{J=1} &= \sqrt{\vec{p}^2 + m^2 + \Phi_{J=1}},
\end{align*}
\]  

(13)

respectively. Here \( \Phi_J \) \((J = 0, 1, 2)\) denotes certain potentials for spin-J forces. Examples for such forces are a scalar field in [13] which couples only to tau neutrinos and a torsion tensor field (i.e., a dual of an axial vector filed) in [14] whose eigenstates are different from those of the electroweak interaction. It is well-known that the nonrelativistic behavior of scalar forces is similar to that of tensor forces, i.e., gravity. In the ultrarelativistic limit, however, the situation changes drastically. In this limit we have from (12) and (13)

\[
\begin{align*}
E_{J=0} - |\vec{p}| - \frac{m^2}{2|\vec{p}|} &\approx \frac{1}{2|\vec{p}|} \Phi_{J=0}^2 + \frac{m}{|\vec{p}|} \Phi_{J=0} \\
E_{J=1} - |\vec{p}| - \frac{m^2}{2|\vec{p}|} &\approx \Phi_{J=1} \\
E_{J=2} - |\vec{p}| - \frac{m^2}{2|\vec{p}|} &\approx \frac{|\vec{p}|}{2} \Phi_{J=2},
\end{align*}
\]  

(14)

where only terms proportional to \( m \) and \( \Phi_J \) could be relevant to neutrino oscillations. So if all neutrinos are messless and if neutrino oscillations occur solely due to the presence of \( \Phi_J \), then the probability of neutrino oscillations would be given by

\[
\begin{align*}
\text{Prob}(J = 0) &\sim \sin^2 2\theta \sin^2 \left(\frac{\Phi_{J=0}^2 L}{|\vec{p}|}\right) \\
\text{Prob}(J = 1) &\sim \sin^2 2\theta \sin^2 (\Phi_{J=1} L) \\
\text{Prob}(J = 2) &\sim \sin^2 2\theta \sin^2 (|\vec{p}| \Phi_{J=2} L).
\end{align*}
\]  

(15)
In Fig. 3 we give the energy dependence of the probability in case of gravity. In case of a scalar force and a vector force the energy dependence would look like Figs. 4 and 5, respectively. From these figures we conclude that the shape of the energy spectrum depends on the spin of the particle which mediates the force, and hence it should be possible to identify the spin of the particle which mediates a possible new force by looking at the energy dependence. (15) also explains why we have obtained such a severe constraint on the universality of the gravitational coupling constants of neutrinos, as the argument of sine in Prob($J=2$) in (15) is proportional to the energy of the neutrino. We note in passing that ordinary neutrino oscillations due to masses is analogous to the case of a scalar force, as far as the energy dependence is concerned. In accelerator experiments like those we have proposed in this talk, the length $L$ is much smaller and the energy of neutrinos $|\vec{p}|$ is much larger than typical quantities in astrophysical observations discussed in [7]. In longbase-line experiments, therefore, while it is easy to test gravity, it would be more difficult to detect a new scalar force or a new vector force such as those in [13] and [14].
In this talk we have proposed long-baseline experiments to test the universality of the gravitational couplings of neutrinos, and we found that we could probe the dimensionless parameter $\Delta f$ as small as $10^{-14}$ or $10^{-15}$ which is smaller by a few orders of magnitudes than the upper limit on a breakdown of the equivalence principle from different types of experiments. Although we have not evaluated systematic errors in detail, we hope our analysis will stimulate and motivate long-baseline experiments in the near future.

References

[1] M. Gasperini, Phys. Rev. D38 (1988) 2635; ibid D39 (1989) 3606.

[2] A. Halprin and C.N. Leung, Phys. Rev. Lett. 67 (1991) 1833.

[3] K. Iida, H. Minakata and O. Yasuda, Mod. Phys. Lett. A8 (1993) 1037.

[4] K. Iida, H. Minakata and O. Yasuda, in preparation.

[5] J. Pantaleone, A. Halprin and C.N. Leung, Phys. Rev. D47 (1993) 4199.

[6] C.W. Misner, K.S. Thorne and J.A. Wheeler, “Gravitation”, W.H. Freeman and Company (1973).

[7] C.M. Will, Phys. Rep. 113 (1984) 345; Int. J. Mod. Phys. D1 (1992) 13.

[8] M.J. Longo, Phys. Rev. Lett. 60 (1987) 173; L.M. Krauss and S. Tremaine, ibid. 176.

[9] F.D. Stacey, “Physics of the Earth”, 2nd ed., John Wiley & Sons, Inc. (1977).

[10] R. Bernstein et al., Conceptual Design Report: Main Injector Neutrino Program, Fermilab, June 1991.

[11] C. Møller, “The Theory of Relativity”, 2nd ed., Oxford University Press (1972).

[12] S.P. Mikheyev, A. Yu. Smirnov, Sov. J. Nucl. Phys. 42 (1985) 913; L. Wolfenstein, Phys. Rev. D17 (1987) 2369.

[13] M. Kawasaki, H. Murayama and T. Yanagida, Mod. Phys. Lett. A7 (1992) 563.

[14] V. De Sabbata and M. Gasperini, Nuovo Cim. 65 A (1981) 479.