Dark energy survivals in massive gravity after GW170817: SO(3) invariant

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The recent detection of the gravitational wave signal GW170817 together with an electromagnetic counterpart GRB 170817A from the merger of two neutron stars puts a stringent bound on the tensor propagation speed. This constraint can be automatically satisfied in the framework of massive gravity. In this work we consider a general SO(3)-invariant massive gravity with five propagating degrees of freedom and derive the conditions for the absence of ghosts and Laplacian instabilities in the presence of a matter perfect fluid on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological background. The graviton potential containing the dependence of three-dimensional metrics and a fiducial metric coupled to a temporal scalar field gives rise to a scenario of the late-time cosmic acceleration in which the dark energy equation of state $w_{\text{DE}}$ is equivalent to $-1$ or varies in time. We find that the deviation from the value $w_{\text{DE}} = -1$ provides important contributions to the quantities associated with the stability conditions of tensor, vector, and scalar perturbations. In concrete models, we study the dynamics of dark energy arising from the graviton potential and show that there exist viable parameter spaces in which neither ghosts nor Laplacian instabilities are present for both $w_{\text{DE}} > -1$ and $w_{\text{DE}} < -1$. We also generally obtain the effective gravitational coupling $G_{\text{eff}}$ with non-relativistic matter as well as the gravitational slip parameter $\eta_s$ associated with the observations of large-scale structures and weak lensing. We show that, apart from a specific case, the two quantities $G_{\text{eff}}$ and $\eta_s$ are similar to those in general relativity for scalar perturbations deep inside the sound horizon.

I. INTRODUCTION

The large-distance modification of gravity has been under active study over the past two decades \cite{1–8}. This is mostly attributed to the constantly accumulating observational evidence of the late-time cosmic acceleration \cite{9–14}. In General Relativity (GR), we need to introduce an unknown matter component dubbed dark energy to account for this phenomenon. In modified gravitational theories, new degrees of freedom (DOFs) arising from the breaking of gauge symmetry in GR can be the source for the acceleration.

The new DOFs appearing in modified gravitational theories usually possess the properties of scalar or vector fields besides two tensor polarizations. The most general second-order scalar-tensor theories with one scalar and two tensor DOFs are known as Horndeski theories \cite{15}. The application of theories within the framework of Horndeski theories—such as $f(R)$ gravity \cite{16,17} and Galileons \cite{18,19}—to the late-time cosmic acceleration has been extensively performed in the literature \cite{20,21}. In Refs. \cite{26,29} the authors constructed second-order massive vector-tensor theories with a vector field $A^\mu$ coupled to gravity, which propagate five DOFs (one scalar, two vectors, two tensors). Such theories can also lead to the cosmic acceleration \cite{30,31}, while satisfying local gravity constraints in the solar system \cite{32,33}.

Massive gravity is also the way of modifying gravity at large distances by giving a tiny mass to the graviton \cite{34}. In general there are six propagating DOFs in massive gravity, one of which behaves as a ghost. Fierz and Pauli (FP) \cite{35} constructed a Lorentz-invariant linear theory of massive gravity without the ghost on the Minkowski background. However, the FP theory is plagued by a problem of the so-called van Dam-Veltman-Zakharov (vDVZ) discontinuity with which the massless limit does not recover GR \cite{36,37}. Vainshtein \cite{38} argued that the vDVZ discontinuity originates from the breakdown of linear EP theory in the massless limit and that nonlinearities can cure the problem. However, the nonlinear extension of FP theory generally suffers from the appearance of a so-called Bouliware-Deser (BD) ghost \cite{39}.

In 2010, de Rham, Gabadadze, and Tolley (dRGT) \cite{40} constructed a unique nonlinear theory of Lorentz-invariant massive gravity without the BD ghost (see also Refs. \cite{41,42}). Applying the dRGT theory to cosmology, it was first recognized that there are no viable expanding solutions on the flat FLRW background \cite{43} (see also \cite{44,50}). If we replace the Minkowski fiducial metric in the original dRGT theory with a de Sitter or FLRW fiducial metric, this allows for the existence of self-accelerating or non self-accelerating solutions \cite{53,52}. The resulting self-accelerating solution can be relevant to the late-time cosmic acceleration, but it suffers from the strong coupling problem where two kinetic terms among five propagating DOFs exactly vanish \cite{53}. This is associated with the nonlinear ghost instability, which manifests itself on the anisotropic cosmological background \cite{55}. Apart from the case in which matter species have a specific coupling to the metric \cite{57,61}, the cosmological solutions in dRGT theory are generally plagued by the problem of ghost instabilities (including the appearance of a Higuchi ghost \cite{62} for non self-accelerating solutions).
If we break the Lorentz-invariance, it is possible to avoid the aforementioned problems of Lorentz-invariant massive gravity theories. Indeed, the extension of the FP action to a Lorentz-violating form does not give to the vDVZ discontinuity \[63\]. The expansion around the Minkowski background shows that, for the graviton mass \( m \), the low-energy effective theory is valid up to the strong coupling scale \( \sqrt{mM_{\text{pl}}} \), where \( M_{\text{pl}} = 2.435 \times 10^{18} \text{ GeV} \) is the reduced Planck mass. The construction of \( SO(3) \)-invariant theories of massive gravity was performed in Ref. \[64\] by imposing the residual reparametrization symmetry \( x^i \rightarrow x^i + \xi^i(t) \), where \( x^i \) are spatial coordinates and \( \xi^i \) is a time-dependent shift. The application of such rotational invariant massive gravity to cosmology was carried out in Refs. \[65, 66\] (see also Refs. \[70–72\]). The resulting solutions can lead to the late-time cosmic acceleration \[66\] without theoretical pathologies.

In \( SO(3) \)-invariant massive gravity theories, Comelli et al. \[73, 74\] derived a most general form of the graviton potential with five propagating DOFs. From the Hamiltonian analysis the extra ghost DOF among six DOFs is eliminated at fully nonperturbative level. Applying such general massive gravity theories to cosmology, the background dynamics is parametrized by a time-dependent function \( \mathcal{U} \). This can lead to the cosmic acceleration with a dynamically varying dark energy equation of state \( w_{\text{DE}} \). If we consider a fiducial metric coupled to a temporal scalar field \( \phi \) in the graviton potential, there exists a self-accelerating solution \( (w_{\text{DE}} = -1) \) free from the strong coupling problem associated with vanishing kinetic terms \[76\].

So far, the stability analysis against perturbations on the FLRW background in general \( SO(3) \)-invariant massive gravity theories has been restricted to the case without matter. In this paper we take into account a matter perfect fluid and scalar perturbations and apply them to a concrete model of Lorentz-violating massive gravity with the late-time cosmic acceleration. As in the model discussed in Ref. \[76\] we allow for the existence of a non-trivial fiducial metric multiplied by a temporal scalar, but we will study more general cases in which the dark energy equation of state varies in the region \( w_{\text{DE}} > -1 \) or \( w_{\text{DE}} < -1 \). For the purpose of constraining \( SO(3) \)-invariant massive gravity from observations of large-scale structures and weak lensing, we will also derive the effective gravitational coupling with nonrelativistic matter and the gravitational slip parameter.

We note that the recent detection of gravitational waves from a binary neutron star merger placed the bound \(-3 \times 10^{-15} \leq c_T/c - 1 \leq 7 \times 10^{-16} \), where \( c_T \) is the tensor propagation speed and \( c \) is the speed of light \[79\]. In our \( SO(3) \)-invariant massive gravity the tensor speed \( c_T \) is equivalent to \( c \), so it evades the tight bound constrained from GW170817. For the graviton mass the observations of GW170104 put the upper limit \( m \leq 7.7 \times 10^{-23} \text{ eV} \) \[80\]. The massive gravity relevant to the late-time cosmic acceleration has the graviton mass of order \( m \lesssim 10^{-33} \text{ eV} \), which is well consistent with the GW170104 bound. The Lorentz-violating dark energy scenario with such a tiny graviton mass does not contradict with observations associated with the Lorentz violation either. The same is true for the doubly coupled massive gravity scenario \[61, 62\].

This paper is organized as follows. In Sec. \[II\] we briefly review general Lorentz-violating massive gravity with the \( SO(3) \) invariance. In Sec. \[III\] we discuss the cosmology on the flat FLRW background by paying particular attention to the dynamics of dark energy arising from the graviton potential. In Sec. \[IV\] we revisit how the Lorentz-violating mass terms arise after expanding the action up to second order in perturbations. In Sec. \[V\] we derive conditions for avoiding ghosts and Laplacian instabilities of tensor, vector, and scalar perturbations and apply them to a concrete model of Lorentz-violating massive gravity with cosmology. In Sec. \[VI\] we obtain stability conditions of scalar perturbations and identify parameter spaces consistent with all the stability conditions for a concrete model. In Sec. \[VII\] we study the growth of non-relativistic matter perturbations and derive the effective gravitational coupling and the gravitational slip parameter under a quasi-static approximation deep inside the sound horizon. Sec. \[VIII\] is devoted to conclusions.

## II. GENERAL \( SO(3) \)-IN Variant Massive Gravity

The Lorentz-violating massive gravity with three-dimensional Euclidean symmetry are characterized by the four-dimensional metric tensor \( g_{\mu\nu} \) and scalar fields \( (\phi, \phi^i) \), where \( i = 1, 2, 3 \). The theory respects the three-dimensional internal symmetry

\[
\phi^i \rightarrow \Lambda^i_j \phi^j, \quad \phi^i \rightarrow \phi^i + C^i,
\]

where \( \Lambda^i_j \) is a \( SO(3) \) rotational operator, and \( C^i \) are constants. Due to this symmetry, the fields \( \phi^i \) appear in the action only as a form of the fiducial metric \( \epsilon_{\mu\nu} = \delta_{ij} \partial_\mu \phi^i \partial_\nu \phi^j \). Then, the massive graviton potential \( V \) generally depends on \( g_{\mu\nu}, \epsilon_{\mu\nu}, \phi, \partial_\mu \phi \). The higher-order derivative terms are not taken into account to avoid the Ostrogradski instability.
We introduce the ten scalar quantities
\[ N \equiv \frac{1}{\sqrt{-g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi}}, \quad N^i \equiv N^{\mu\nu}\partial_{\mu}\phi^i, \quad \Gamma^{ij} \equiv (g^{\mu\nu} + n^\mu n^\nu) \partial_{\mu}\phi^i \partial_{\nu}\phi^j, \] (2.2)
where \( n^\mu = N g^{\mu\nu} \partial_{\nu}\phi \) is a unit vector, and \( \Gamma^{ij} = \Gamma^{ji} \). In the following, we choose the unitary gauge in which the scalar components \( \phi \) and \( \phi^i \) are identical to the time \( t \) and the spatial vector \( x^i \), respectively,
\[ \phi = t, \quad \phi^i = x^i. \] (2.3)
For this gauge choice the quantities in Eq. (2.2) reduce to \( N = 1/\sqrt{-g^{\mu\nu}}, N^i = N n^i \), and \( \Gamma^{ij} = g^{ij} + n^i n^j \), where \( n^i = N g^{i0} \). In this case, the quantities \( N, N^i \), and \( \Gamma^{ij} \) correspond to the lapse \( N \), the shift vector \( N^i \), and the three-dimensional metric \( \gamma^{ij} \) in the ADM formalism \([51]\), respectively, with the line element
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma^{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right). \] (2.4)

The most general massive graviton potential \( V \) with five propagating DOFs is parametrized by two \( SO(3) \)-invariant functions \( U \) and \( E \) \([73–75]\). The function \( U \) depends on the special combination
\[ K^{ij} = \gamma^{ij} - \xi^i \xi^j, \] (2.5)
where the new shift variables \( \xi^i \) are related to \( N^i \) according to
\[ N^i - N \xi^i = -U^{-1} \xi_j. \] (2.6)
Here, \( U^{-1} \) is the inverse of the matrix \( U_{ij} \equiv \partial^2 U / \partial \xi^i \partial \xi^j, E \) is a function containing the dependence of \( \gamma^{ij} \) and \( \xi^i \), and \( \xi_j = \partial E / \partial \xi^j \). If \( E_j = 0 \), then the vector \( \xi^i \) is directly related to the shift \( N^i \) as \( \xi^i = N^i / N \). In this case, the quantity \( K^{ij} \) reduces to the three-dimensional metric \( g^{ij} = \gamma^{ij} - N^i N^j / N^2 \).

The massive graviton potential that propagates five DOFs is restricted to the form \([73–75]\)
\[ V = U + \frac{1}{N} \left( E - U \xi_j \right), \] (2.7)
where \( U \equiv \partial U / \partial \xi^i \). In the unitary gauge the functions \( U \) and \( E \) also depend on the fiducial metric \( e_{ij} = \delta_{ij} \) and the temporal scalar \( \phi = t \), such that
\[ U = U(K^{ij}, \delta_{ij}, \phi), \quad E = E(\gamma^{ij}, \xi^i, \delta_{ij}, \phi). \] (2.8)
We will consider the combination of \( \delta_{ij} \) and \( \phi \) given by \([77]\)
\[ f_{ij} = b(\phi) \delta_{ij}, \] (2.9)
where \( b(\phi) \) is an arbitrary function of \( \phi \). In this case, the \( SO(3) \)-invariant functions \( U \) and \( E \) have the dependence
\[ U = U(K^{ij}, f_{ij}), \quad E = E(\gamma^{ij}, \xi^i, f_{ij}). \] (2.10)

The massive gravity theory invariant under the shift symmetry \( \phi \to \phi + c \) corresponds to \( b(\phi) = 1 \). For the choice \( b(\phi) = e^{M \phi} \), the above prescription accommodates the theory invariant under the diatonic global symmetry \( \phi \to \phi + c \) and \( \phi^i \to e^{-M \phi^i} \) \([77]\).

Besides the massive graviton potential, we also take into account the Ricci scalar \( R \) in the action and the matter field \( \Psi_M \) minimally coupled to gravity. Namely, we focus on the massive gravity theories with the action
\[ S = S_{EH} + S_{mg} + S_M, \] (2.11)
where the Einstein-Hilbert term \( S_{EH} \) and the massive graviton term \( S_{mg} \) are given, respectively, by
\[ S_{EH} = M_{pl}^2 \int d^4x \sqrt{-g} \frac{R}{2}, \] (2.12)
\[ S_{mg} = -M_{pl}^2 \int d^4x \sqrt{-g} m^2 V. \] (2.13)

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1 This formulation does not accommodate a Lorentz-violating minimal theory of massive gravity proposed in Refs. \([56,57]\), as the latter contains the time derivative of \( \gamma_{ij} \). The non-local theory of massive gravity \([54,55]\) is also beyond our framework.
Here, $g$ is the determinant of $g_{\mu\nu}$, $m$ is the graviton mass, and $V$ is the graviton potential of the form (2.7).

To describe a perfect fluid for the matter field $\Psi_M$, we consider the Schutz-Sorkin action \cite{77, 78}

$$S_M = - \int d^4x \left[ \sqrt{-g} \rho_M(n) + J^\mu (\partial_\mu \ell + A_1 \partial_\mu B_1 + A_2 \partial_\mu B_2) \right],$$

(2.14)

where $\rho_M$ is the fluid density that depends on its number density defined by

$$n = \frac{\sqrt{g} J^\alpha J^\beta g_{\alpha\beta}}{g},$$

(2.15)

with $J^\mu$ being a vector field of weight one. The quantity $\ell$ corresponds to the scalar mode, whereas $A_1, A_2, B_1, B_2$ are scalar quantities whose perturbations are associated with the vector modes. In massive gravity theories the vector modes generally propagate, so it is convenient to use the Schutz-Sorkin action to describe the propagation of vector modes in the matter sector (as in the case of vector-tensor theories \cite{30, 31}).

### III. FLRW BACKGROUND AND DARK ENERGY DYNAMICS

For the massive gravity theory given by the action (2.11) the equations of motion on the flat FLRW background were derived in Ref. \cite{76}. In Ref. \cite{76} the authors proposed a concrete SO(3)-invariant massive gravity model with a nontrivial fiducial metric to discuss the property of de Sitter solutions. Here we will revisit them in the presence of matter and study the background dynamics relevant to the late-time cosmic acceleration. Compared to Ref. \cite{76}, our analysis is more general in that the expansion rate arising from $b(t)$ in the metric $f_{ij}$ is not necessarily equivalent to the Hubble expansion rate.

#### A. Background equations

Let us consider the flat FLRW spacetime described by the line element

$$-ds^2 = -\bar{N}^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j,$$

(3.1)

where $\bar{N}(t)$ is the background value of the lapse $N$, and $a(t)$ is the time-dependent scale factor. Here and in the following, we use an overbar to represent background quantities. Since $\bar{N}^i = \bar{E}^i = \bar{U}^{-1} \bar{E}_j = 0$ on the FLRW background, the massive graviton potential (2.7) reduces to $\bar{V} = \bar{U} + \bar{E} / \bar{N}$. Up to boundary terms, the sum of the two actions (2.12) and (2.13) reads

$$\bar{S}_{EH} + \bar{S}_{mg} = -M_p^3 \int d^4x \left[ \frac{3\dot{a}^2}{\bar{N}} + a^3 m^2 \left( \bar{N}\dot{U} + \dot{\bar{E}} \right) \right],$$

(3.2)

where an overdot represents a derivative with respect to $t$. The two functions in Eq. (2.10) have the dependence $\bar{U} = \bar{U}(\bar{\gamma}^{ij}, f_{ij})$ and $\dot{\bar{E}} = \dot{\bar{E}}(\bar{\gamma}^{ij}, f_{ij})$, where $\bar{\gamma}^{ij} = a^{-2}(t)\delta^{ij}$ and $f_{ij} = b^2(t)\delta_{ij}$.

On the FLRW background (3.1) the matter action (2.14) reduces to $\bar{S}_M = - \int d^4x [\sqrt{-\bar{g}} \rho_M + J^\mu \partial_\mu \ell]$. The temporal component $\dot{J}^0$ corresponds to the total fluid number $N_0$, which is constant. From Eq. (2.15) the number density $n_0$ is given by

$$n_0 = \frac{N_0}{a^3},$$

(3.3)

and hence $\dot{J}^0 = n_0 a^3$. Then, the matter action (2.14) is expressed as

$$\bar{S}_M = - \int d^4x \left( \bar{N} a^3 \bar{\rho}_M + n_0 a^3 \bar{\ell} \right).$$

(3.4)

For the variation of the total action $\bar{S} = \bar{S}_{EH} + \bar{S}_{mg} + \bar{S}_M$, we exploit the fact that $\partial \bar{U} / \partial a = -6 \partial \bar{U} / \partial a$ and $\partial \dot{\bar{E}} / \partial a = -6 \partial \dot{\bar{E}} / \partial a$, where the derivatives $\partial \bar{U}$ and $\partial \dot{\bar{E}}$ are defined by

$$\frac{\partial \bar{U}}{\partial \bar{\gamma}^{ij}} = \partial \bar{U} \bar{\gamma}^{ij}, \quad \frac{\partial \dot{\bar{E}}}{\partial \bar{\gamma}^{ij}} = \partial \dot{\bar{E}} \bar{\gamma}^{ij}. $$

(3.5)
Varying the action $\bar{S}$ with respect to $\bar{N}$ and $a$, respectively, we obtain the background equations of motion

\begin{align}
3M_{pl}^2 H^2 &= \rho_{mg} + \bar{\rho}_M, \\ M_{pl}^2 \left( \frac{\dot{H}}{\bar{N}} + 3H^2 \right) &= -P_{mg} - \bar{P}_M,
\end{align}

where

\[ H \equiv \frac{\dot{a}}{Na} \]

is the Hubble expansion rate, and $\bar{P}_M$ is the matter pressure defined by

\[ \bar{P}_M \equiv -\bar{n}_0 \frac{\dot{\bar{U}}}{\bar{U}} - \bar{\rho}_M. \]

The quantities $\rho_{mg}$ and $P_{mg}$, which correspond to the energy density and the pressure arising from the graviton potential respectively, are given by

\[ \rho_{mg} = M_{pl}^2 m^2 \bar{U}, \quad P_{mg} = M_{pl}^2 m^2 \left[ 2d\bar{U} - \bar{U} + \frac{1}{\bar{N}} (2d\bar{E} - \bar{E}) \right]. \]

The matter sector obeys the continuity equation

\[ \dot{\bar{\rho}}_M + 3\bar{N} H (\bar{\rho}_M + \bar{P}_M) = 0. \]

On using Eqs. (3.6) and (3.7) with Eq. (3.11), the massive gravity sector also satisfies $\dot{\rho}_{mg} + 3\bar{N} H (\rho_{mg} + P_{mg}) = 0$. This translates to

\[ \dot{\bar{U}} + 6\bar{N} H d\bar{U} + 3H (2d\bar{E} - \bar{E}) = 0. \]

The quantity $\bar{U}$ depends on $t$ through the metrics $\gamma^{ij} = a^{-2}(t) \delta^{ij}$ and $f_{ij} = b^2(t) \delta_{ij}$. Computing the time derivative $\dot{\bar{U}} = (\partial \bar{U} / \partial \gamma^{ij}) \dot{\gamma}^{ij} + (\partial \bar{U} / \partial f_{ij}) \dot{f}_{ij}$, it follows that

\[ \dot{\bar{U}} + 6\bar{N} H d\bar{U} = 6\bar{N} H_b d\bar{U}, \]

where

\[ H_b \equiv \frac{\dot{b}}{Nb} \]

is the expansion rate of $b(t)$ in the metric $f_{ij}$. From Eqs. (3.12) and (3.13) we obtain

\[ H (2d\bar{E} - \bar{E}) = -2\bar{N} H_b d\bar{U}. \]

Since $P_{mg} = M_{pl}^2 m^2 [2(1 - H_b/H)d\bar{U} - \bar{U}]$ in the expanding Universe ($H > 0$), using Eqs. (3.6) and (3.7) leads to the following relation

\[ \frac{2M_{pl}^2 \dot{H}}{N} = -2M_{pl}^2 m^2 (1 - r_b) d\bar{U} - \bar{P}_M - \bar{P}_M, \]

where

\[ r_b \equiv \frac{H_b}{H}. \]

If the massive gravity sector is responsible for the late-time cosmic acceleration, the equation of state in the dark sector is given by

\[ w_{DE} \equiv \frac{P_{mg}}{\rho_{mg}} = -1 + 2 (1 - r_b) \frac{d\bar{U}}{\bar{U}}. \]
so that the accelerated expansion occurs for \((1 - r_b)\dd U/\dd t < 1/3\).

For the theories in which \(b\) is constant, we have \(H_b = 0\) and hence \(r_b = 0\) for \(H \neq 0\). In this case we have \(2d\mathcal{E} - \dd \mathcal{E} = 0\) from Eq. (3.15), so the energy density \(\rho_{mg}\) and the pressure \(P_{mg}\) depend on \(\dd U\) and \(\dd t\) alone with the equation of state \(w_{DE} = -1 + 2d\mathcal{E}/\dd U\). Then the de Sitter solution \((w_{DE} = -1)\) exists only for \(\dd U = 0\), but in this case the theory is plagued by the strong coupling problem in which two of kinetic terms in the second-order action of perturbations vanish (as we will see later in Sec IV). The dynamical dark energy scenario with \(w_{DE}\) different from \(-1\) is realized for the time-varying function \(d\dd U/\dd t\) different from \(0\), in which case the problem of vanishing kinetic terms can be avoided.

If \(b\) is not a constant, which is the case for massive gravity with dilaton-like symmetry, the de Sitter solution can be realized for \(r_b = 1\). In this case the expansion rate \(H_b\) is identical to \(H\), so that \(2d\mathcal{E} - \dd \mathcal{E} = -2Nd\dd U\) and \(\dd P_{mg} = -M^2_{pl}m^2\dd U = -\rho_{mg}\). Then the quantity \(\dd U\) does not need to vanish on this de Sitter solution, so it is not plagued by the strong coupling problem of vanishing kinetic terms.

We will consider a more general situation in which \(r_b\) is not necessarily equivalent to \(1\). We require the condition that the quantity \(\dd \mathcal{E}\) does not vanish from the past (below the strong coupling scale \(\sim \sqrt{mM_{pl}}\)) to today. As we will see in Sec. III B for concrete models, it is possible to realize the dynamical dark energy scenario with \(w_{DE} > -1\) for \(r_b < 1\) and \(w_{DE} < -1\) for \(r_b > 1\). The stability of such cosmological solutions against ghost and Laplacian instabilities will be discussed in Sec. VI C.

### B. Concrete models

We consider concrete models of the SO(3)-invariant massive gravity given by the graviton potential (2.7) with the two functions \(\mathcal{U}\),

\[
\mathcal{U} = u_0 + u_1K^{ij}f_{ij} + \frac{1}{2}u_{2a}\left(K^{ij}f_{ij}\right)^2 + \frac{1}{2}u_{2b}K^{ij}f_{jk}K^{kl}f_{li},
\]

(3.19)

\[
\mathcal{E} = \mathcal{F}(t) \left[ v_0 + v_1\gamma^{ij}f_{ij} + \frac{1}{2}v_{2a}\gamma^{ij}f_{ij}\right]^2 + \frac{1}{2}v_{2b}\gamma^{ij}f_{jk}\gamma^{kl}f_{li} + \frac{1}{2}w_2\xi^i\xi^jf_{ij},
\]

(3.20)

where \(u_0, u_1, u_{2a}, u_{2b}\) and \(v_0, v_1, v_{2a}, v_{2b}, w_2\) are constants, and \(\mathcal{F}(t)\) is a function of \(t\). From Eq. (2.8) the quantity \(\dd \mathcal{E}\) can contain the time-dependence through \(\phi\) in the unitary gauge, so the time-dependent function \(\mathcal{F}(t)\) has been taken into account in Eq. (3.20). On using the property

\[
\gamma^{ij}f_{jk} = Y^2\delta^i_k, \quad Y \equiv \frac{b}{a},
\]

(3.21)

and the definition (3.6), the background values of \(\mathcal{U}, \mathcal{E}\) and the derivatives \(\dd \mathcal{U}, \dd \mathcal{E}\) are given, respectively, by

\[
\dd \mathcal{U} = u_1Y^2 + u_2Y^4, \quad \dd \mathcal{E} = \mathcal{F}(t) \left( v_0 + 3v_1Y^2 + \frac{3}{2}v_2Y^4 \right),
\]

(3.22)

where

\[
u_2 \equiv 3u_{2a} + u_{2b}, \quad v_2 \equiv 3v_{2a} + v_{2b}.
\]

(3.23)

We study the background cosmological dynamics in the presence of nonrelativistic matter with energy density \(\bar{\rho}_m\) and pressure \(\bar{P}_m = 0\) as well as radiation with energy density \(\bar{\rho}_r\) and pressure \(\bar{P}_r = \bar{\rho}_r/3\). Then, the background Eqs. (3.6) and (3.10) yield

\[
3M^2_{pl}H^2 = M^2_{pl}m^2 \left( u_0 + 3u_1Y^2 + \frac{3}{2}u_2Y^4 \right) + \bar{\rho}_m + \bar{\rho}_r,
\]

(3.24)

\[
\frac{2M^2_{pl}\dot{H}}{N} = -2M^2_{pl}m^2(1 - r_b) \left( u_1Y^2 + u_2Y^4 \right) - \bar{\rho}_m - \frac{4}{3}\bar{\rho}_r.
\]

(3.25)

From Eq. (3.15) there is the relation

\[
v_0 + v_1Y^2 - \frac{1}{2}v_2Y^4 = 2\frac{\dot{N}(t)}{\mathcal{F}(t)}r_b \left( u_1Y^2 + u_2Y^4 \right).
\]

(3.26)
We choose the function \( \mathcal{F}(t) \) to have the same time dependence as \( \bar{N}(t) \), i.e.,

\[
\mathcal{F}(t) = \bar{N}(t).
\]

If the time \( t \) in Eq. (3.21) plays the role of standard cosmic time, the choice (3.27) simply corresponds to \( \mathcal{F}(t) = \bar{N}(t) = 1 \). The constraint (3.26) reduces to

\[
v_0 + (v_1 - 2r_b u_1) Y^2 - \left( \frac{1}{2} v_2 + 2r_b u_2 \right) Y^4 = 0.
\]

(3.28)

Introducing the density parameters

\[
\Omega_{DE0} = \frac{m^2 u_0}{3H^2}, \quad \Omega_{DE1} = \frac{m^2 u_1 Y^2}{H^2}, \quad \Omega_{DE2} = \frac{m^2 u_2 Y^4}{2H^2}, \quad \Omega_m = \frac{\bar{\rho}_m}{3M^2_{pl}H^2}, \quad \Omega_r = \frac{\bar{\rho}_r}{3M^2_{pl}H^2},
\]

(3.29)

we can express Eqs. (3.24) and (3.25) in the forms

\[
\Omega_m = 1 - \Omega_{DE0} - \Omega_{DE1} - \Omega_{DE2} - \Omega_r, \quad \mu = \frac{H}{NH^2} = -(1 - r_b) (\Omega_{DE1} + 2\Omega_{DE2}) - \frac{3}{2} \Omega_m - 2\Omega_r.
\]

(3.30)

(3.31)

To avoid the negative dark energy density, we will focus on the case in which \( \Omega_{DE0}, \Omega_{DE1}, \Omega_{DE2} \) are all positive, i.e.,

\[
u_0 \geq 0, \quad u_1 \geq 0, \quad u_2 \geq 0.
\]

(3.32)

The dark energy equation of state (3.18) reduces to

\[
w_{DE} = -1 + \frac{2}{3} (1 - r_b) \frac{\Omega_{DE1} + 2\Omega_{DE2}}{\Omega_{DE0} + \Omega_{DE1} + \Omega_{DE2}}.
\]

(3.33)

Defining the e-folding number \( x \equiv \ln a \), the density parameters obey the differential equations

\[
\frac{d\Omega_{DE0}}{dx} = -2\mu \Omega_{DE0}, \quad \frac{d\Omega_{DE1}}{dx} = 2 (r_b - 1 - \mu) \Omega_{DE1}, \quad \frac{d\Omega_{DE2}}{dx} = 2 (2r_b - 2 - \mu) \Omega_{DE2}, \quad \frac{d\Omega_r}{dx} = -(4 + 2\mu) \Omega_r.
\]

(3.34)

(3.35)

(3.36)

(3.37)

In the following, we will consider theories in which \( r_b \) is constant. Since the two scale factors \( a \) and \( b \) are related to each other as \( b \propto a^\gamma \), it follows that

\[
Y = Y_0 a^{\gamma_b - 1},
\]

(3.38)

where \( Y_0 \) is constant. Since the evolution of \( Y \) is different depending on the values of \( r_b \), we will discuss the three cases (1) \( r_b = 1 \), (2) \( r_b < 1 \), and (3) \( r_b > 1 \), separately. Apart from the case (1) we choose the constant \( Y_0 \) in Eq. (3.38) to be 1 without loss of generality, so the value of \( Y \) today (\( a = 1 \)) is equivalent to 1.

I. \( r_b = 1 \)

For \( r_b = 1 \) the expansion rates \( H_b \) and \( H \) are equivalent to each other, so the ratio \( Y = b/a \) remains to be a constant value \( Y_0 \). Then, Eq. (3.23) gives a constraint among the coefficients \( v_0, v_1, v_2, u_1, u_2 \), i.e.,

\[
v_0 + (v_1 - 2u_1) Y_0^2 - \left( \frac{1}{2} v_2 + 2u_2 \right) Y_0^4 = 0.
\]

(3.39)

Since \( Y \) is constant, the terms \( u_1 Y^2 \) and \( u_2 Y^4 \) in Eq. (3.24) behave as a cosmological constant with their vanishing contributions to the r.h.s. of Eq. (3.24). At the background level, the model with \( r_b = 1 \) is equivalent to the ΛCDM model (\( w_{DE} = -1 \)), see case (a) in Fig. 1.
this asymptotic de Sitter solution is given by Eq. (3.40). In case (c) of Fig. 1 we plot the evolution of $w_{DE}$ versus $z+1 = 1/a$ for the models (a) $r_b = 1$, (b) $r_b = 0.7$, $v_0 = 1.86 H_0^2/m^2$, $u_1 = 5.97 \times 10^{-2} H_0^2/m^2$, $u_2 = 0$, (c) $r_b = 0$, $v_0 = 2.01 H_0^2/m^2$, $u_1 = 1.11 \times 10^{-2} H_0^2/m^2$, $u_2 = 0$, (d) $r_b = 1.2$, $v_0 = 1.89 \times 10^{-1} H_0^2/m^2$, $u_1 = 6.17 \times 10^{-1} H_0^2/m^2$, $u_2 = 0$, (e) $r_b = 1.2$, $v_0 = 7.13 \times 10^{-2} H_0^2/m^2$, $u_1 = 9.72 \times 10^{-2} H_0^2/m^2$, $u_2 = 1.12 H_0^2/m^2$. The present epoch ($a = 1$ and $H = H_0$) is identified according to the condition $\Omega_m = 0.32$ with $\Omega_c \simeq 10^{-4}$.

In cases (b) and (c) the dark energy equation of state is in the region $w_{DE} > -1$, whereas in cases (d) and (e) the phantom equation of state is realized.

2. $r_b < 1$

For $r_b < 1$, the quantity $Y$ varies in proportion to $a^r < 1$. Since the relation $\Omega_{DE0} \leq 1$, $\Omega_{DE1}$ and $\Omega_{DE2}$ in Eq. (3.41) decrease with the growth of $a$, so the system approaches the de Sitter solution driven by the constant $u_0$. The fixed point of the dynamical system (3.41) corresponding to this asymptotic de Sitter solution is given by

$$\begin{align*}
(A) \quad (\Omega_{DE0}, \Omega_{DE1}, \Omega_{DE2}, \Omega_r, \Omega_m) &= (1, 0, 0, 0, 0), \quad \text{with} \quad \mu = 0, \quad w_{DE} = -1.
\end{align*}$$

The density parameters $\Omega_{DE1}$ and $\Omega_{DE2}$ dominate over $\Omega_{DE0}$ in the asymptotic past ($a \to 0$). In this regime, $w_{DE}$ is given by

$$w_{DE}(a \to 0) = \begin{cases} 
- (1 + 2r_b)/3 & \text{for } \Omega_{DE1} \gg \Omega_{DE2}, \\
+ (1 - 4r_b)/3 & \text{for } \Omega_{DE1} \ll \Omega_{DE2},
\end{cases}$$

both of which are larger than $-1$ for $r_b < 1$. Since the quantity $Y$ evolves in time, the existence of terms $\Omega_{DE1}$ and $\Omega_{DE2}$ in Eq. (3.41) leads to a dynamical dark energy scenario in which $w_{DE}$ starts to evolve from the value $w_{DE} = -1$ and then it finally approaches the asymptotic value $-1$. In this case, the dark energy equation of state is always in the region $w_{DE} > -1$. In case (b) of Fig. 1 we show the evolution of $w_{DE}$ for $r_b = 0.7$ and $u_2 = 0$ (i.e., $\Omega_{DE2} = 0$), in which case the initial value of $w_{DE}$ is $-1 + 2r_b)/3 = -0.8$ and the solution finally approaches the de Sitter fixed point (A).

For the theories with $r_b = 0$ (i.e., $b = \text{constant}$) the asymptotic values (3.42) reduce to $w_{DE} = -1/3$ for $\Omega_{DE1} \gg \Omega_{DE2}$ and $w_{DE} = 1/3$ for $\Omega_{DE1} \ll \Omega_{DE2}$. Note that, for $r_b = 0$, the coefficients $v_0, v_1, v_2$ are constrained to be 0 from Eq. (3.40). In case (c) of Fig. 1 we plot the evolution of $w_{DE}$ for $r_b = 0$ and $u_2 = 0$, which shows that $w_{DE}$ evolves from the value close to $-1/3$ and asymptotically approaches $-1$. 
The quantity $\delta \mu_m^2 = (u_1 Y^2 + u_2 Y^4) m^2$, which appears in the second-order perturbed action discussed later in Sec. [V] is associated with the strong coupling scale $\Lambda_{SC} \sim \sqrt{m M_{pl}(\delta \mu)^{1/2}}$, see Sec. [VI]. For $r_b < 1$, $\Lambda_{SC}$ decreases with the growth of $a$. For $u_1$ and $u_2$ not much less than unity the today’s value of $\delta \mu$ is not significantly smaller than 1, so the strong coupling scale today is as high as $\sqrt{m M_{pl}}$. The strong coupling problem arises only in the asymptotic future at which $\delta \mu$ sufficiently approaches 0. This property is different from the self-accelerating solution in dRGT theory in which two coefficients of kinetic terms exactly vanish [53].

3. $r_b > 1$

For $r_b > 1$ the quantities $u_1 Y^2$ and $u_2 Y^4$ increase with the growth of $a$, so the Universe finally enters the regime in which these terms dominate over the constant $u_0$ in Eq. (3.24). Note that there is the relation (3.40) among the coefficients $v_0, v_1, v_2, u_1, u_2$. Let us first discuss the theories satisfying

$$u_2 = 0,$$  \hspace{1cm} (3.43)

in which case $v_2 = 0$. Then, the solutions finally approach the fixed point

$$(B) \quad (\Omega_{DE0}, \Omega_{DE1}, \Omega_{DE2}, \Omega_r, \Omega_m) = (0, 1, 0, 0, 0), \quad \text{with} \quad \mu = r_b - 1, \quad w_{DE} = -\frac{1}{3}(1 + 2r_b).$$  \hspace{1cm} (3.44)

Since $\mu > 0$ and $w_{DE} < -1$, the Hubble parameter grows according to the relation $H \propto Y / a^{r_b - 1}$. Then, the expanding solution associated with the fixed point (B) is given by

$$a \propto (t_s - t)\frac{1}{t_s - t}, \quad H = \frac{1}{(r_b - 1)(t_s - t)},$$  \hspace{1cm} (3.45)

where $t_s$ is a constant. The Hubble parameter exhibits the divergence at $t = t_s$, which correspond to a big-rip singularity. If the condition $\Omega_{DE0} \gg \Omega_{DE1}$ is satisfied in the early Universe, $w_{DE}$ starts to evolve from the value close to $-1$ and then it finally approaches the phantom equation of state $-1 + (1 + 2r_b)/3$. The case (d) plotted in Fig. 1 corresponds to $r_b = 1.2$, in which case the asymptotic value of $w_{DE}$ is $-1.13$. If $\Omega_{DE0} \ll \Omega_{DE1}$ initially, $w_{DE}$ is always close to $-1 + 2r_b)/3$.

If we consider the theories with

$$u_2 \neq 0,$$  \hspace{1cm} (3.46)

the term $u_2 Y^4$ in Eq. (3.24) finally dominates over the other terms. The associated fixed point is given by

$$(C) \quad (\Omega_{DE0}, \Omega_{DE1}, \Omega_{DE2}, \Omega_r, \Omega_m) = (0, 0, 1, 0, 0), \quad \text{with} \quad \mu = 2r_b - 2, \quad w_{DE} = \frac{1}{3}(1 - 4r_b),$$  \hspace{1cm} (3.47)

so that $\mu > 0$ and $w_{DE} < -1$. We note that $w_{DE}$ at the point (C) is smaller than that at the point (B). The Hubble parameter obeys $H \propto Y^2 / a^{2(r_b - 1)}$ on the fixed point (C), so the integrated solutions read

$$a \propto (t_s - t)^\frac{1}{2(r_b - 1)}, \quad H = \frac{1}{2(r_b - 1)(t_s - t)},$$  \hspace{1cm} (3.48)

which exhibit the big-rip singularity at $t = t_s$. The early evolution of $w_{DE}$ is different depending on which density parameters dominate over the others in Eq. (3.33). If $\Omega_{DE0}$ is initially much larger than $\Omega_{DE1}$ and $\Omega_{DE2}$, then $w_{DE}$ starts to evolve from the value close to $-1$ and finally approaches $-1 - (4r_b)/3$. If there is a period in which $\Omega_{DE1}$ gets larger than the other density parameters, $w_{DE}$ temporarily approaches the value $-1 + 2r_b)/3$. In case (e) plotted in Fig. 1 which corresponds to $r_b = 1.2$, $w_{DE}$ starts to evolve from the value close to $-1$ and approaches the asymptotic value $-1.27$.

For $r_b > 1$ the quantity $\delta \mu_m^2 = (u_1 Y^2 + u_2 Y^4) m^2$ increases with the growth of $a$, so it goes to 0 in the asymptotic past. As we will see in Sec. [VII C] the decrease of $\delta \mu_m^2$ toward the past is not significant for $r_b$ close to 1. Then, the strong coupling scale $\Lambda_{SC} \sim \sqrt{m M_{pl}(\delta \mu)^{1/2}}$ in the past is not very different from its today’s value. In the asymptotic future the quantity $\delta \mu_m^2$ goes to infinity with the approach to the big-rip singularity, around which the perturbative analysis breaks down.

The above discussion shows that the Lorentz-violating model given by the functions (3.33) and (3.20) allows for a variety of the dark equation of state: (1) $w_{DE} = -1$ for $r_b = 1$, (2) $w_{DE} > -1$ for $r_b < 1$, and (3) $w_{DE} < -1$ for $r_b > 1$, during the past cosmic expansion history. In Sec. [VII C] we will study whether this model is free from the problems of ghosts and Laplacian instabilities.
IV. SECOND-ORDER LINEARIZED ACTION OF THE GRAVITON POTENTIAL

In this section, we expand the action (2.13) with the graviton potential given by Eq. (2.7) up to second order in perturbations on the flat FLRW background (3.1). The ADM line element (2.4) can accommodate the perturbation $\delta N$ in the lapse function, as $N = \bar{N}(t) + \delta N$, with the shift perturbation $\dot{N}$. We introduce the perturbation $\delta g_{ij}$ in the three-dimensional metric $\gamma_{ij}$, as

$$\gamma_{ij} = a^2(t) (\delta_{ij} + \delta g_{ij}) . \quad (4.1)$$

Since the unitary gauge is chosen, the perturbation $\delta \phi$ in the scalar field $\phi$ vanishes. This means that the dependence of $f_{ij} = b^2(\phi) \delta_{ij}$ in $\mathcal{U}$ and $\mathcal{E}$ does not generate any perturbed quantity after varying the action $S_{mg}$. The action (2.13) can be expressed as

$$S_{mg} = -M_p^2 m^2 \int d^4x \sqrt{\gamma} \left( NU + \mathcal{E} - \mathcal{U}_i \mathcal{U}_j^{-1} \mathcal{E}_j \right) , \quad (4.2)$$

where $\gamma$ is the determinant of the three-dimensional ADM metric $\gamma_{ij}$. In the following, we expand the action (4.2) up to second order in perturbations for the functions $\mathcal{U} = \mathcal{U}(K^{ij})$ and $\mathcal{E} = \mathcal{E}(\gamma^{ij}, \xi^i)$. In doing so, we use the following properties

$$\mathcal{U}_i = -2d\mathcal{U} \bar{\gamma}_{ik} \xi^k + O(\epsilon^2) , \quad \mathcal{U}_{ij} = -2d\mathcal{U} \bar{\gamma}_{ij} + O(\epsilon) , \quad \mathcal{E}_j = d^2 \mathcal{E} \bar{\gamma}_{ij} \xi^i + O(\epsilon^2) , \quad (4.3)$$

where $\epsilon$ describes the order of perturbations, and

$$\frac{\partial^2 \mathcal{E}}{\partial \xi^i \partial \xi^j} = d^2 \bar{\gamma} \bar{\gamma}_{ij} . \quad (4.4)$$

The last test term in Eq. (4.2) is of second order, i.e., $\mathcal{U}_i \mathcal{U}_j^{-1} \mathcal{E}_j = d^2 \mathcal{E} \bar{\gamma}_{ij} \xi^i \xi^j$. On using Eqs. (2.6) and (4.3), the vector $\xi^i$ is related to $N^i$, as

$$\xi^i = \frac{2d\mathcal{U}}{2N \mathcal{U} + d^2 \mathcal{E}} N^i + O(\epsilon^2) . \quad (4.5)$$

If $d^2 \mathcal{E} = 0$, then $\xi^i = N^i / \bar{N} + O(\epsilon^2)$.

The square root of the determinant $\gamma$ contains the contributions of zero-th order, first order, and second order, as

$$(\sqrt{\gamma})^{(0)} = a^3 , \quad (\sqrt{\gamma})^{(1)} = \frac{1}{2} a^3 \delta g_i^i , \quad (\sqrt{\gamma})^{(2)} = \frac{1}{8} a^3 \left( \delta g_i^j \delta g_j^i - 2 \delta g_{ij} \delta g^{ij} \right) . \quad (4.6)$$

where $\delta g_i^j \equiv \delta^i_j \delta g_{ij}$. In the action $S_{mg}$, there is the second-order contribution

$$S_{\mathcal{U}} = -M_p^2 m^2 \int d^4x (N \sqrt{\gamma})^{(2)} \bar{\mathcal{U}} = - \int d^4x (N \sqrt{\gamma})^{(2)} \rho_{mg} , \quad (4.7)$$

which we will separate from other contributions. Up to second order in perturbations, the quantities $\mathcal{U}$ and $\mathcal{E}$ are expanded as

$$\mathcal{U} = \bar{\mathcal{U}} + d\mathcal{U} \bar{\gamma}_{ij} \delta \gamma^{ij} - d\mathcal{U} \bar{\gamma}_{ij} \xi^i \xi^j + \frac{1}{2} \frac{\partial^2 \mathcal{U}}{\partial K^{ij} \partial K^{kl}} \delta \gamma^{ij} \delta \gamma^{kl} + O(\epsilon^3) , \quad (4.8)$$

$$\mathcal{E} = \bar{\mathcal{E}} + d\mathcal{E} \bar{\gamma}_{ij} \delta \gamma^{ij} + \frac{1}{2} \frac{d^2 \mathcal{E}}{\partial \gamma^{ij} \partial \gamma^{kl}} \delta \gamma^{ij} \delta \gamma^{kl} + O(\epsilon^3) \quad (4.9)$$

The second-order term containing the contribution $\delta N^2$ arises only from the action $S_{\mathcal{U}}$. From the action $-M_p^2 m^2 \int d^4x (N \sqrt{\gamma})^{(1)} \bar{\mathcal{U}}$ in $S_{mg}$, there is the second-order product $-M_p^2 m^2 \int d^4x a^3 d\mathcal{U} \bar{\gamma}_{ij} \delta N \delta \gamma^{ij}$ containing the perturbation $\delta N$. We also employ the fact that the perturbation of the three-dimensional ADM metric $\gamma^{ij}$ is written in the form

$$\delta \gamma^{ij} = a^{-2}(t) (-\delta g^{ij} + \delta g^{ik} \delta g_{kj}) + O(\epsilon^3) . \quad (4.10)$$
On using the notations

\[ \frac{\partial^2 U}{\partial K^{ij} \partial K_{kl}} = d^2 U \tilde{\gamma}_{ij} \tilde{\gamma}_{kl} + \frac{1}{2} d^2 U \tilde{\gamma}_{ikj} \tilde{\gamma}_{jl} + \frac{1}{2} d^2 U (\tilde{\gamma}_{ikj} \tilde{\gamma}_{jl} + \tilde{\gamma}_{ilk} \tilde{\gamma}_{jkl}) , \]  

(4.11)

\[ \frac{\partial^2 \xi}{\partial \gamma_{ij} \partial \gamma_{kl}} = d^2 \xi \tilde{\gamma}_{ij} \tilde{\gamma}_{kl} + \frac{1}{2} d^2 \xi (\tilde{\gamma}_{ikj} \tilde{\gamma}_{jl} + \tilde{\gamma}_{ilk} \tilde{\gamma}_{jkl}) , \]  

(4.12)

the resulting second-order action of \( S_{mg} \) is given by

\[
S_{mg}^{(2)} = M_{pl}^2 n^2 \int d^4x \bar{N} a^3 \left[ \frac{dU}{N} \delta N \delta g_{ij} + \frac{2dU^2}{2N} \alpha^2 N^i N^j + \frac{1}{2} \left( -\frac{\bar{\xi}}{4N} + dU + \frac{d\xi}{N} - d^2 U - \frac{d^2 \xi}{N} \right) \delta g_{ij} \right] + \left( \frac{\bar{\xi}}{4N} - dU - \frac{d\xi}{N} - \frac{1}{2} d^2 U - \frac{d^2 \xi}{2N} \right) \delta g_{ij} \delta g_{ij} \]  

\[ - \int d^4x (N \sqrt{\gamma}) \rho_{mg} . \]  

(4.13)

If we choose

\[
\bar{N} = a, \]  

(4.14)

the time \( t \) plays the role of conformal time \( \eta \) with the background line element

\[
\frac{ds^2}{a^2(\eta)} = -d\eta^2 + \delta_{ij} dx^i dx^j . \]  

(4.15)

We also write the four-dimensional perturbed metric \( g_{\mu\nu} \) in the form

\[
g_{\mu\nu} = a^2(\eta) (\eta_{\mu\nu} + \delta g_{\mu\nu}) , \]  

(4.16)

so that the perturbations \( \delta g_{00} \) and \( \delta g_{0i} \) are related to \( \delta N \) and \( N^i \) according to the relations \( \delta g_{00} = -2 \delta N / \bar{N} \) and \( \delta g_{0i} = \delta_{ij} N^j \). Then, the second-order action in terms of the conformal time can be expressed as

\[
S_{mg}^{(2)} = M_{pl}^2 \int d^4x \frac{d^4}{4} (m_0^2 \delta g_{00} + 2m_1^2 \delta g_{00} \delta g_{0i} - 2m_2^2 \delta g_{00} \delta g_{ii} + m_3^2 \delta g_{ii} \delta g_{jj} - m_4^2 \delta g_{ij} \delta g_{ij}) - \int d^4x (N \sqrt{\gamma}) \rho_{mg} , \]  

(4.17)

where the terms with same subscripts are summed over, and

\[
m_0^2 = 0 , \quad m_1^2 = \frac{4\bar{N}dU^2}{2N} m^2 , \quad m_2^2 = dU m^2 , \quad m_3^2 = 2 \left( -\frac{\bar{\xi}}{4N} + dU + \frac{d\xi}{N} - d^2 U - \frac{d^2 \xi}{N} \right) m^2 , \quad m_4^2 = -4 \left( \frac{\bar{\xi}}{4N} - dU + \frac{d\xi}{N} - \frac{1}{2} d^2 U - \frac{d^2 \xi}{2N} \right) m^2 . \]  

(4.18)

with \( \bar{N} = a \). The fact that \( m_3^2 = 0 \) is attributed to the absence of the sixth ghost DOF, which is guaranteed by the construction of the graviton potential of the form \[ 73, 74 \]. The four mass terms \( m_1^2, m_2^2, m_3^2, m_4^2 \), besides \( m^2 \) appearing in \( \rho_{mg} \), affect the dynamics of linear perturbations. In Secs. V and VI we will derive the equations of motion of tensor (two DOFs), vector (two DOFs), and scalar (1 DOF) perturbations in the presence of a matter perfect fluid by using the second-order action \[ 11, 18 \].

The Lorentz-invariant FP theory \[ 35 \] is given by the Lagrangian \( L_{FP} = (M_{pl}^2 m^2 / 4)(-h_{\mu\nu} h^{\mu\nu} + h^\nu h^\nu) \), where \( h_{\mu\nu} \) is the perturbed part of the four-dimensional metric \( g_{\mu\nu} \). The linear expansion on the Minkowski background leads to the particular relations \( m_0^2 = 0 \) and \( m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2 \) \[ 35 \], which is also the case for the dRGT massive gravity \[ 40 \]. For the self-accelerating branch of dRGT theory the three masses \( m_0^2, m_1^2, m_2^2 \) vanish with the relation \( m_3^2 = m_4^2 \) \[ 52 \], so there is a strong coupling problem. Indeed, the ghost instability arises at nonlinear level on the anisotropic cosmological background \[ 56 \].

In the following, we will focus on the Lorentz-violating massive gravity theories in which \( m_1^2, m_2^2, m_3^2, m_4^2 \) are not generally equivalent to each other. As shown in Ref. \[ 63 \], the difference between these mass terms allows for the absence of the vDVZ discontinuity. In some works of Lorentz-violating massive gravity \[ 61, 67 \], the dependence of \( X = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) was taken into account in the graviton potential for realizing the kinetically driven cosmic acceleration. Since this generally gives rise to a ghost state associated with nonvanishing \( m_0^2 \), we will not include such dependence in \( V \).

---

2 The quantities \( c_i \) used in the action (56) of Ref. \[ 76 \] are related to our mass terms, as \( m_0^2 = c_0 m^2, m_1^2 = 2c_2 m^2, m_2^2 = c_1 m^2, m_3^2 = 4c_3 m^2, \) and \( m_4^2 = -4c_4 m^2 \).
V. TENSOR AND VECTOR PERTURBATIONS

To study the propagation of five DOFs present in massive gravity theories given by the action (2.11), we consider the perturbations $\delta g_{\mu\nu}$ in the form (4.16) on the flat FLRW background. The linear perturbation equations of motion follow by expanding the action (2.11) up to quadratic perturbed order. The corresponding second-order action can be expressed in the form

$$S^{(2)} = S^{(2)}_{\text{EH}} + S^{(2)}_{\text{mg}} + S^{(2)}_M,$$

where $S^{(2)}_{\text{EH}}$ and $S^{(2)}_M$ are the second-order contributions to $S_{\text{EH}}$ and $S_M$, respectively. The quadratic action arising from the graviton potential is given by Eq. (4.17).

We decompose the perturbations in terms of irreducible representations of the $SO(3)$ group and study the perturbations in tensor, vector, and scalar sectors separately. The general perturbed line element on the flat FLRW background (4.15) is given by [87]

$$ds^2 = a^2(\eta) \left\{ - (1 + 2A) d\eta^2 + 2 \left( B_{ij} + S_i \right) d\eta dx^i + \left[ (1 + 2B) \delta_{ij} + 2E_{ij} + 2F_{ij} + h_{ij} \right] dx^i dx^j \right\},$$

where the subscript "|" is the covariant derivative with respect to the three-dimensional spatial metric $g_{ij}$. The scalar perturbations are characterized by the four quantities $A, B, \psi, E$. The vector perturbations are given by $S_i$ and $F_i$, both of which satisfy the transverse conditions

$$S_{ij} = 0, \quad F_{i}^{ij} = 0.$$

The tensor perturbation $h_{ij}$ obeys the transverse and traceless conditions

$$h_{ij}^{\ j} = 0, \quad h_{i}^{\ i} = 0.$$

For the computation of the second-order matter action $S^{(2)}_M$, we decompose the temporal and spatial components of $J^\mu$ in Eq. (2.14) of the form

$$J^0 = N_0 + \delta J , \quad J^i = \frac{1}{a^2(\eta)} \delta^{ik} (\partial_k \delta j + W_k),$$

where $\delta J$ and $\delta j$ correspond to scalar perturbations, and $\partial_k \delta j \equiv \partial \delta j / \partial x^k$. The vector perturbation $W_k$ obeys the transverse condition $\partial^k W_k = 0$. This is satisfied for the choice

$$W_k = (W_1(\eta, z), W_2(\eta, z), 0),$$

where $z$ is the third component of the spatial vector $x^i$. We decompose the scalar quantity $\ell$ in the form

$$\ell = - \int^0 a \hat{\rho}_{M,n}(\eta) d\hat{\eta} - \hat{\rho}_{M,n} v,$$

where $\hat{\rho}_{M,n} \equiv \partial \hat{\rho}_{M} / \partial n$ and $v$ is the velocity potential. At the background level we have $\partial \ell / \partial \eta = -a \hat{\rho}_{M,n}$, so the matter pressure (3.9) reduces to

$$\hat{P}_M = n_0 \hat{\rho}_{M,n} - \hat{\rho}_M,$$

where we used $\hat{N} = a$.

For the vector mode, we can choose $A_1, A_2, B_1, B_2$ in the following forms [30]

$$A_1 = \delta A_1(\eta, z), \quad A_2 = \delta A_2(\eta, z), \quad B_1 = x + \delta B_1(\eta, z), \quad B_2 = y + \delta B_2(\eta, z),$$

where $\delta A_{1,2}$ and $\delta B_{1,2}$ are perturbed quantities that depend on $\eta$ and $z$. Varying the matter action (2.14) with respect to $J^\mu$, it follows that

$$u_\mu = \frac{J_\mu}{n \sqrt{-g}} = \frac{1}{\hat{\rho}_{M,n}} (\partial_\mu \ell + A_1 \partial_\mu B_1 + A_2 \partial_\mu B_2),$$

where $u_\mu$ corresponds to the four velocity. Substituting Eq. (5.7) into (5.10), the spatial component of $u_\mu$ can be expressed as

$$u_i = - \partial_i v + v_i,$$
where the vector components $v_i$ (with $i = 1, 2$) are related to $\delta A_i$, as

$$\delta A_i = \bar{\rho}_{M,n} v_i. \quad (5.12)$$

The transverse condition $\partial^iv_i = 0$ is satisfied for the choice of $A_i$ given in Eq. (5.14).

In this section, we compute the second-order actions of tensor and vector perturbations and derive conditions for the absence of ghosts and Laplacian instabilities.

### A. Tensor Perturbations

Let us first compute the second-order action of tensor perturbations. The simplest choice of $h_{ij}$ obeying the transverse and traceless conditions \(5.3\) is given by

$$h_{11} = a^2(\eta)h_1(\eta, z), \quad h_{22} = -a^2(\eta)h_1(\eta, z), \quad h_{12} = h_{21} = a^2(\eta)h_2(\eta, z), \quad (5.13)$$

where the functions $h_1$ and $h_2$ characterize the two propagating DOFs in the tensor sector.

The second-order tensor action arising from the Einstein-Hilbert term and the graviton potential are given, respectively, by

$$S_{EH}^{(2)} = \int d\eta d^3x M^2_{\text{pl}} a^2 \sum_{i=1}^{2} \left[ \frac{1}{4} h_i^2 + \frac{1}{2} h_i \nabla^2 h_i - \frac{1}{2} (2\mathcal{H}' + \mathcal{H}^2) h_i^2 \right], \quad (5.14)$$

$$S_{\text{mg}}^{(2)} = \int d\eta d^3x a^4 \sum_{i=1}^{2} \left( -\frac{1}{2} M^2_{\text{pl}} m_{2i}^2 h_i^2 + \frac{1}{2} \rho_{\text{mg}} h_i^2 \right), \quad (5.15)$$

where $\nabla^2 = \delta^{ij}\partial_i \partial_j$, $H \equiv \alpha'/a$, and a prime represents the derivative with respect to $\eta$.

For the matter sector we need to expand the action $S_M = -\int d^4x \sqrt{-g} \rho_M(n)$ up to second order in $h_{ij}$.

The corresponding quadratic action can be split into the form $S_M^{(2)} = -\int d^4x [(\sqrt{-g})^{(2)} \bar{\rho}_M + \sqrt{-g} \bar{\rho}_{M,n} \delta n]$, where $\sqrt{-g}^{(2)} = -a^4(h_1^2 + h_2^2)/2$ and $\delta n = n_0(h_1^2 + h_2^2)/2$ with $\sqrt{-g} = a^4$. Then, it follows that

$$S_M^{(2)} = -\int d\eta d^3x \sum_{i=1}^{2} \frac{1}{2} a^4 \bar{P}_M h_i^2, \quad (5.16)$$

where $\bar{P}_M$ is the matter pressure given by Eq. (5.8).

The background equations follow from Eqs. (3.6) and (3.7) by replacing $H$ with $H/a$ and setting $\bar{N} = a$, i.e.,

$$3M^2_{\text{pl}} \mathcal{H}' = a^2(\rho_{\text{mg}} + \bar{\rho}_M), \quad (5.17)$$

$$M^2_{\text{pl}} (2\mathcal{H}' + \mathcal{H}^2) = -a^2 (P_{\text{mg}} + \bar{P}_M). \quad (5.18)$$

On using Eqs. (5.17)-(5.18) and the relation

$$\rho_{\text{mg}} + P_{\text{mg}} = 2M^2_{\text{pl}} m^2(1 - r_b) d\bar{U}, \quad (5.19)$$

the total second-order action (5.11) in the tensor sector reads

$$S_T^{(2)} = \int d\eta d^3x \frac{M^2_{\text{pl}}}{2} \sum_{i=1}^{2} \left( \frac{1}{2} a^2 h_i^2 + \frac{1}{2} a^2 h_i \nabla^2 h_i - \frac{1}{2} a^4 m_T^2 h_i^2 \right), \quad (5.20)$$

where

$$m_T^2 \equiv 2m^2 - 4m^2(1 - r_b) d\bar{U}. \quad (5.21)$$

The first two terms in Eq. (5.20) are exactly the same as those in GR, so there is neither ghost nor Laplacian instability in the tensor sector. The effect of massive gravity arises only through the effective graviton mass squared $m_T^2$. If $w_{DE} = -1$, i.e., $r_b = 1$ or $d\bar{U} = 0$, then the effective tensor mass squared (5.21) is equivalent to $2m^2_T$. The deviation from $w_{DE} = -1$ leads to the modification to the value $2m^2_T$. 

The fact that the kinetic and gradient terms remain unchanged relative to those in GR is an important property of the $SO(3)$-invariant massive gravity. The tensor propagation speed $c_T$ is equivalent to the speed of light $c$. This is consistent with the recent tight bound on the propagation speed of gravitational waves constrained from GW170817 79. The mass $m_T$ relevant to the late-time cosmic acceleration is not much different from the today’s Hubble expansion rate $H_0 \sim 10^{-33}$ eV, so it also satisfies the upper limit $m_T \leq 7.7 \times 10^{-23}$ eV constrained from GW170104 80.

Varying the action $S^{(2)}$ with respect to $h_i$, the resulting equation of motion in Fourier space with the comoving wavenumber $k$ is given by

$$h''_i + 2\mathcal{H}h'_i + \left(k^2 + a^2 m_T^2\right) h_i = 0. \quad (5.22)$$

If $m_T^2 < 0$, then the tachyonic instability is present for the modes $k^2/a^2 < |m_T^2|$. Provided that $|m_T|$ is of the same order as $H_0$, the tachyonic instability is harmless for tensor perturbations inside the today’s Hubble radius.

### B. Vector Perturbations

The perturbed line element $S$ contains the vector perturbations $S_i$ and $F_i$. We choose them in the forms $S_i = (S_i(\eta, z), S_2(\eta, z), 0)$ and $F_i = (F_i(\eta, z), F_2(\eta, z), 0)$ to satisfy the transverse conditions. Then, the quadratic actions $S^{(2)}_{EH}$ and $S^{(2)}_{mg}$ of the vector sector are given, respectively, by

$$S^{(2)}_{EH} = \int d\eta d^3x M^{2}_{pl} a^2 \sum_{i=1}^{2} \left[ -\frac{1}{4} (S_i - F'_i) \nabla^2 (S'_i - F'_i) + \frac{3}{2} \mathcal{H}^2 S_i^2 + \frac{1}{2} (2\mathcal{H}' + \mathcal{H}^2) F_i \nabla^2 F_i \right], \quad \quad (5.23)$$

$$S^{(2)}_{mg} = \int d\eta d^3x a^2 \sum_{i=1}^{2} \left[ -\frac{1}{2} M^{2}_{pl} (m_i^2 S_i^2 + m_2^2 F_i \nabla^2 F_i) - \frac{1}{2} \rho_{mg} \left( S_i^2 + F_i \nabla^2 F_i \right) \right]. \quad \quad (5.24)$$

The vector perturbations $W_i, A_i, B_i$ in the Schutz-Sorkin action are chosen to be of the forms $S^{(2)}$ and $S^{(2)}_{mg}$. Then, the second-order matter action reads

$$S^{(2)}_M = \int d\eta d^3x \sum_{i=1}^{2} \left[ \frac{1}{2a^4 N_0} \left\{ \bar{\rho}_{M,n} (W_i + N_0 a^2 S_i)^2 - N_0 a^7 \bar{\rho}_M S_i^2 \right\} - N_0 \delta A_i \delta B_i - \frac{1}{a^2} W_i \delta A_i + \frac{1}{2} a^4 \bar{P}_M F_i \nabla^2 F_i \right]. \quad (5.25)$$

The perturbations $W_i, \delta A_i, \delta B_i$ appear only in the action $S^{(2)}_M$. Varying Eq. (5.25) with respect to $W_i$ and using Eq. (5.12), it follows that

$$W_i = a N_0 (v_i - a S_i). \quad \quad (5.26)$$

Variations of the action (5.25) with respect to $\delta A_i$ and $\delta B_i$ lead, respectively, to

$$v_i = a (S_i - \delta B'_i), \quad \quad (5.27)$$

$$\delta A_i = \frac{a^3 (\bar{\rho}_M + \bar{P}_M)}{N_0} v_i = C_i, \quad \quad (5.28)$$

where $C_i$ are constants. Substituting Eqs. (5.26)-(5.28) into Eq. (5.25), the second-order matter action reduces to

$$S^{(2)}_M = \int d\eta d^3x \sum_{i=1}^{2} \left[ \frac{1}{2} a^2 (\bar{\rho}_M + \bar{P}_M) v_i^2 - \frac{1}{2} a^4 \bar{\rho}_M S_i^2 + \frac{1}{2} a^4 \bar{P}_M F_i \nabla^2 F_i \right]. \quad (5.29)$$

On using the background Eqs. (5.17)-(5.18), the total second-order action of vector perturbations reduces to

$$S^{(2)}_V = \int d\eta d^3x \sum_{i=1}^{2} \left[ -\frac{1}{4} M^2_{pl} a^2 (S_i - F'_i) \nabla^2 (S'_i - F'_i) + \frac{1}{2} a^2 (\bar{\rho}_M + \bar{P}_M) v_i^2 + \frac{1}{2} M^2_{pl} a^4 m_1^2 S_i^2 + \frac{1}{4} M^2_{pl} a^4 m_2^2 F_i \nabla^2 F_i \right], \quad (5.30)$$

where $m_2^2$ is defined by Eq. (5.21).

Taking note that $v_i$ is related to $S_i$ through Eq. (5.27) and varying the action (5.30) with respect to $S_i$, we obtain

$$M^2_{pl} \nabla^2 (S_i - F'_i) - 2M^2_{pl} a^2 m_1^2 S_i - \frac{2N_0}{a^2} C_i = 0. \quad (5.31)$$
The Fourier components of $S_i$ corresponding to the comoving wavenumber $k$ obey

$$S_i = \frac{M_{pl}^2 a^2 k^2 F_i' - 2 N_0 C_i}{M_{pl}^2 a^2 (k^2 + 2a^2 m_i^2)}.$$  \hspace{1cm} (5.32)

Substituting this relation into Eq. (5.30), the resulting second-order action is given by

$$S_{(2)}^{(2)} = \int d^3x \sum_{i=1}^2 \left[ \frac{M_{pl}^2}{2} a^4 q_V (F_i'^2 - c_V^2 F_i^2) + \frac{N_0^2 C_i^2 \{ M_{pl}^2 (k^2 + 2a^2 m_i^2) + 2a^2 (\bar{\rho}_M + \bar{P}_M) \}^2}{2M_{pl}^2 a^4 (\bar{\rho}_M + \bar{P}_M) (k^2 + 2a^2 m_i^2)} \right],$$  \hspace{1cm} (5.33)

where

$$q_V = \frac{k^2 m_i^2}{k^2 + 2a^2 m_i^2}, \quad c_V^2 = \frac{m_i^2}{2m_i^2} \left( 1 + \frac{2a^2 m_i^2}{k^2} \right).$$  \hspace{1cm} (5.34)

For the theories satisfying $m_i^2 \neq 0$, there are two dynamical fields $F_1$ and $F_2$ with the propagation speed squared $c_V^2$. In the small-scale limit characterized by $k^2/a^2 \gg m_i^2$, the quantities (5.34) reduce to $q_V \simeq m_i^2$ and $c_V^2 \simeq m_i^2/(2m_i^2)$. As long as the two conditions

$$m_i^2 > 0, \quad m_i^2 > 0$$  \hspace{1cm} (5.35)

are satisfied, there are neither ghosts nor Laplacian instabilities for the modes $k^2/a^2 \gg m_i^2$. Under Eq. (5.35), the conditions $q_V > 0$ and $c_V^2 \geq 0$ hold for any value of $k$. The two quantities $q_V$ and $c_V$ coincide with those derived in Ref. [76] in the absence of matter, so the presence of matter does not substantially modify the stability conditions of vector perturbations. The naive expectation is that adding the Lorentz-invariant Schutz-Sorkin action to the Lorentz-violating graviton action might give rise to some nontrivial modification to the kinetic and gradient terms of vector perturbations. We have explicitly shown that this is not the case.

It is worth mentioning that there is an associated strong coupling scale of the propagating vector fields $F_i$. This becomes apparent after normalizing the kinetic terms $M_{pl}^2 q_V F_i'^2/2$ to canonical forms. Since $q_V \simeq m_i^2$ for the modes $k^2/a^2 \gg m_i^2$, the strong coupling scale $\Lambda_{SC}$ will be of the order $\sqrt{m_i M_{pl}}$. In other words, our low-energy effective theory and the resulting cosmological solutions are valid up to the scale $\sqrt{m_i M_{pl}}$. If we consider the theory with $d^2 E = 0$, which is realized for $w_2 = 0$ in the function (5.20), then we have $m_i^2 = 2d^2 m^2$ and $\Lambda_{SC} \sim \sqrt{m M_{pl} (d^2)}/2$. For the dynamical dark energy scenario in which $d^2 \psi$ varies in time, the scale $\Lambda_{SC}$ is time-dependent. In Sec. VI C we will discuss the variation of $\Lambda_{SC}$ in concrete models of the late-time cosmic acceleration.

**VI. SCALAR PERTURBATIONS**

We proceed to the derivation of the second-order action of scalar perturbations and obtain the no-ghost and stability conditions in the presence of a matter perfect fluid.

**A. Second-order action**

With the notation of Eq. (4.1), the scalar metric perturbations $A, B, \psi, E$ in the line element (5.2) can be expressed as

$$\delta g_{00} = -2A, \quad \delta g_{0i} = \partial_i B, \quad \delta g_{ij} = 2\psi \delta_{ij} + 2\partial_i \partial_j E.$$  \hspace{1cm} (6.1)

The covariant derivative $E_{ij}$ in Eq. (5.2) has been replaced with the partial derivative $\partial_i \partial_j E$ by reflecting the fact that the term arising from the Christoffel symbols $\Gamma^j_{ik}$ are at most second-order in perturbations. The quadratic action arising from the Einstein-Hilbert term (2.12) is given by

$$S_{EH}^{(2)} = \int d^4x \left[ \frac{M_{pl}^2}{2} a^2 \left\{ 2(\psi' - \mathcal{H}A) \nabla^2 (B - E') - 3\psi'^2 - 2A \nabla^2 \psi - \psi \nabla^2 \psi + 6\mathcal{H} A \psi' - \frac{9}{2} \mathcal{H}^2 A^2 + 9\mathcal{H}^2 A \psi \right. \right. \hspace{1cm} (6.2)

+ \frac{3}{2} \mathcal{H}^2 \left\{ 2A \nabla^2 E + (\partial_i B)^2 \right\} + \frac{1}{2} \left\{ 3\psi^2 + 2\psi \nabla^2 E - (\nabla^2 E)^2 \right\} \left( 2\mathcal{H}' + \mathcal{H}^2 \right) \right].$$
The second-order action arising from the graviton potential \( \text{[2.13]} \) yields

\[
S_{(2)}^{(2)} = \int d^4x M_{pl}^2 a^4 \left[ m_2^2 A^2 + \frac{1}{2} m_1^2 (\partial_i B)^2 - m_2^2 \left\{ 3 (\psi')^2 + 2 \psi \nabla^2 E + (\nabla^2 E)^2 \right\} + m_3^2 (3 \psi + \nabla^2 E)^2 + 2 m_2^2 A (3 \psi + \nabla^2 E) - \frac{\rho_{mg}}{2 M_{pl}^2} \left\{ 2 A \nabla^2 E - B \nabla^2 B - A^2 + 6 A \psi + 3 \psi^2 + 2 \psi \nabla^2 E - (\nabla^2 E)^2 \right\} \right].
\] (6.3)

We have not omitted the term \( m_2^2 \) to discuss the ghost instability for the theories with \( m_2^2 \neq 0 \) later. For the matter sector, we define the matter density perturbation \( \delta \rho_M \), as

\[
\delta \rho_M \equiv \frac{\rho_{M,n}}{a^3} \left[ \delta J - N_0 (3 \psi + \nabla^2 E) \right].
\] (6.4)

On using Eq. \( \text{[5.5]} \), the perturbation of the number density \( \text{[2.15]} \) expanded up to quadratic order is given by

\[
\delta n = \frac{\delta \rho_M}{\rho_{M,n}} = \frac{2 a^7 N_0 \delta \rho_M (3 \psi + \nabla^2 E) + \{ a^4 N_0^2 [(\partial_i B)^2 + (3 \psi - \nabla^2 E) (\psi + \nabla^2 E)] + 2 N_0 a^2 \partial_i B \partial_j \delta j + (\partial_i \delta j)^2 \} \rho_{M,n}}{2 a^7 N_0 \rho_{M,n}},
\] (6.5)

which is equivalent to \( \delta \rho_M / \rho_{M,n} \) at linear order. The second-order action following from the expansion of the Schutz-Sorkin action \( \text{[2.14]} \) in scalar perturbations reads

\[
S^{(2)}_M = \int d^4x \left[ \frac{a^4 \rho_{M}}{2} A^2 + \frac{a^4 \tilde{P}_M}{2} (3 \psi^2 + (\partial_i B)^2) - \frac{a^4 \rho_{M,nn}}{2 \rho_{M,n}} \delta \rho_M^2 + \frac{\tilde{P}_M}{2 a^2 N_0} (\partial_i \delta j)^2 + \frac{\rho_{M,n}}{a^2} \partial_i \delta j \partial_i v + \frac{\rho_{M,n}}{a} \partial_i \delta j \partial_i B + a^3 \psi \delta \rho_M - a^4 A \delta \rho_M - 3 a^3 \psi \left(3 n_0^2 \rho_{M,nn,v} + a \rho_{M,A} - n_0 \rho_{M,n,v}' \right) \right] - \frac{3 a^4 n_0^2 \rho_{M,n} H \delta \rho_M}{\rho_{M,n}}
\] \[
\left[ \frac{1}{2} \nabla^2 E \left\{ n_0 \rho_{M,n} (\nabla^2 E - 2 \psi) + \tilde{P}_M (2 A + 2 \psi - \nabla^2 E) \right\} + a^3 \nabla^2 E \left\{ n_0 \rho_{M,n,v}' - 3 n_0^2 \rho_{M,n} H v \right\} \right],
\] (6.6)

where \( \rho_{M,n} \equiv \partial^2 \rho_M / \partial n^2 \big|_{n=n_0} \). Varying this action with respect to \( \delta j \), it follows that

\[
\partial_i \delta j = - a^4 n_0 (\partial_i v + a \partial_i B).
\] (6.7)

Substituting Eq. \( \text{[6.7]} \) into Eq. \( \text{[6.6]} \), the second-order matter action reduces to

\[
S^{(2)}_M = \int d^4x \left[ a^2 (v' - 3 \tilde{c}_M^2 v - a A) \delta \rho_M - \frac{a^4 c_M^2}{2 n_0 \rho_{M,n}} \delta \rho_M^2 - \frac{a^2}{2 n_0 \rho_{M,n}} \left\{ (\partial_i v)^2 + 2 a \partial_i v \partial_i B \right\} + \frac{a^4 \rho_{M}}{2} \left\{ A^2 - 6 A \psi \right\} + n_0 a^3 (v' - 3 \tilde{c}_M^2 H v) \rho_{M,n} (3 \psi + \nabla^2 E) - \frac{a^4 \tilde{P}_M}{2} \left\{ 2 A \nabla^2 E + (\partial_i B)^2 \right\} + \frac{a^4}{2} \tilde{P}_M \left\{ 3 \psi^2 + 2 \psi \nabla^2 E - (\nabla^2 E)^2 \right\} \right].
\] (6.8)

where \( c_M \) is the matter sound speed defined by

\[
c_M^2 = \frac{n_0 \rho_{M,n}}{\rho_{M,n}} = \frac{\tilde{P}_M}{\rho_{M,n}}.
\] (6.9)

Taking the sum of Eqs. \( \text{[6.2]}, \text{[6.3]}, \text{[6.8]} \) and using the background Eqs. \( \text{[5.17]} - \text{[5.18]} \), the total second-order action of scalar perturbations is given by

\[
S^{(2)}_S = \int d^4x \left[ M_{pl}^2 a^2 \left\{ 2 (\psi' - H A) \nabla^2 (B - E') - 3 H A^2 + 6 HA \psi' - 2 A \nabla^2 E - 3 \psi^2 - \psi \nabla^2 \psi \right\} - a^4 A \delta \rho_M + a^3 (v' - 3 \tilde{c}_M^2 v) \{ \delta \rho_M + n_0 \rho_{M,n} (3 \psi + \nabla^2 E) \} - \frac{a^4 c_M^2}{2 n_0 \rho_{M,n}} \delta \rho_M^2 + \frac{a^2}{2 n_0 \rho_{M,n}} (v \nabla^2 v + 2 a v \nabla^2 B) + M_{pl}^2 a^4 \left\{ m_0^2 A^2 + \frac{1}{2} m_1^2 B \nabla^2 B - 3 m_2^2 \psi^2 - 2 m_2^2 \psi \nabla^2 E - m_3^2 (\nabla^2 E)^2 + m_4^2 (3 \psi + \nabla^2 E)^2 \right\} \right],
\] (6.10)
where
\[
m_{2+}^2 \equiv m_2^2 + m^2 (1 - r_b) dU = m_2^2 + \frac{m^2}{2} \bar{U} (1 + w_{DE}),
\]
\[
m_{2-}^2 \equiv m_2^2 - m^2 (1 - r_b) dU = m_2^2 - \frac{m^2}{2} \bar{U} (1 + w_{DE}).
\]
If \(w_{DE}\) is equivalent to \(-1\), then \(m_{2+}^2 = m_{2-}^2 = m_2^2\). The deviation from \(w_{DE} = -1\) leads to the modification to the mass term \(m_2^2\). Note that there is the relation \(3m_{2-}^2 - m_{2+}^2 = m_2^2\).

### B. Stability conditions of scalar perturbations

Varying the action \((6.10)\) with respect to \(v\) and \(\delta \rho_M\) and using the relations \(n_0 \bar{\rho}_M, n = \bar{\rho}_M + \bar{P}_M\) and \(P'_M = c_M^2 \bar{\rho}'_M\), it follows that
\[
\delta \bar{\rho}'_M + 3 \mathcal{H} (1 + c_M^2) \delta \rho_M + (\bar{\rho}_M + \bar{P}_M) \left( 3v' + \nabla^2 \sigma - \frac{1}{a} \nabla^2 \psi \right) = 0,
\]
\[
v' - 3 \mathcal{H} c_M^2 v - a A - \frac{a c_M^2}{\bar{\rho}_M + \bar{P}_M} \delta \rho_M = 0,
\]
where
\[
\sigma \equiv E' - B.
\]
Variations of the action \((6.10)\) with respect to \(A, B, \psi, E\) lead to
\[
3 \mathcal{H} (\psi' - \mathcal{H} A) - \nabla^2 \psi + \nabla^2 \sigma + a^2 \left[ m_0^2 A + m_4^2 (3\psi + \nabla^2 E) \right] = \frac{a^2}{2 M_{pl}} \delta \rho_M,
\]
\[
\psi' - \mathcal{H} A - \frac{1}{2} a^2 m_1^2 B = -\frac{a}{2 M_{pl}} (\bar{\rho}_M + \bar{P}_M) \psi,
\]
\[
\psi'' + 2 \mathcal{H} \psi' - \mathcal{H} A' - (2 \mathcal{H}' + \mathcal{H}^2) A + a^2 \left[ \{ m_0^2 - m^2 (1 - r_b) dU \} A + (3m_2^2 - m_{2+}^2) \psi + (m_3^2 - m_{2-}^2) \nabla^2 E \right]
\]
\[
= -\frac{a^2 c_M^2}{2 M_{pl}} \delta \rho_M,
\]
\[
\sigma' + 2 \mathcal{H} \sigma - A - \psi + a^2 m_T^2 E = 0,
\]
where we used Eq. \((6.13)\).

In the following, we switch to the Fourier space with the comoving wavenumber \(k\). To discuss the evolution of matter density perturbations, we introduce the gauge-invariant density contrast
\[
\delta \equiv \frac{\delta \rho_M}{\bar{\rho}_M} + 3 \left( 1 + \frac{\bar{P}_M}{\bar{\rho}_M} \right) \psi.
\]

We express Eq. \((6.13)\) in terms of \(\delta\) and \(\delta'\) and then solve it for \(v\). The Hamiltonian and momentum constraints \((6.16)\) and \((6.17)\) are also used to eliminate the perturbations \(A\) and \(B\) from the action \((6.10)\). The kinetic term appearing in the second-order scalar action can be written in the form \(S_{2K}^{(2)} = \int d^3x \tilde{x} \cdot \tilde{K} \tilde{\chi}^2\), where \(\tilde{x} = (\delta/k, kE, \psi)\) and \(\tilde{K}\) is a \(3 \times 3\) matrix. The three matrix components \(K_{11}, K_{22}, K_{33}\) of \(\tilde{K}\) are given, respectively, by
\[
K_{11} = \frac{a^4 M_{pl}^2 [a^2 m_1^2 (3H^2 - a^2 m_0^2) + 2H^2 k^2]}{2a^2 (\bar{\rho}_M + \bar{P}_M + M_{pl}^2 m_1^2) (3H^2 - a^2 m_0^2) + 4M_{pl}^2 H^2 k^2 \bar{\rho}_M + \bar{P}_M},
\]
\[
K_{22} = \frac{a^4 M_{pl}^2 [a^2 (\bar{\rho}_M + \bar{P}_M) (3H^2 - a^2 m_0^2) + 2M_{pl}^2 H^2 k^2]}{2a^2 (\bar{\rho}_M + \bar{P}_M + M_{pl}^2 m_1^2) (3H^2 - a^2 m_0^2) + 4M_{pl}^2 H^2 k^2 m_1^2},
\]
\[
K_{33} = \frac{a^4 M_{pl}^2 [3a^2 (\bar{\rho}_M + \bar{P}_M + M_{pl}^2 m_1^2) + 2M_{pl}^2 k^2]}{a^2 (3H^2 - a^2 m_0^2) (\bar{\rho}_M + \bar{P}_M + M_{pl}^2 m_1^2) + 2M_{pl}^2 H^2 k^2 m_0^2}.
\]
Provided that $m_1^2 \neq 0$ and $m_0^2 \neq 0$, there are three dynamical propagating fields $\delta, E, \psi$. To see the ghost instability arising from the nonvanishing $m_0^2$, let us consider the situation without matter. In this case, the two fields $E$ and $\psi$ propagate and the nonvanishing matrix components are

$$K_{22} = \frac{M_{pl}^2 m_0^2 a^4 \cal{H}^2 k^2}{a^2 m_1^2 (3 \cal{H}^2 - a^2 m_0^2) + 2 \cal{H}^2 k^2}, \quad K_{23} = K_{32} = -\frac{a^2 m_0^2}{k \cal{H}^2} K_{22}, \quad K_{33} = \frac{M_{pl}^2 m_0^2 a^4 (3 a^2 m_1^2 - 2 k^2)}{a^4 m_1^2 (3 \cal{H}^2 - a^2 m_0^2) + 2 \cal{H}^2 k^2}.$$  

The absence of ghosts requires that the matrix $K$ is positive definite. In the Minkowski limit ($\cal{H} \rightarrow 0$), the no-ghost conditions translate to $M_{pl}^2 a^2 < 0$ and $M_{pl}^2 (3 |a|^2 + 2 k^2)/m_1^2 < 0$. Since the first condition is not satisfied, the theories with $m_0^2 \neq 0$ are plagued by the ghost problem on the Minkowski background.

The $SO(3)$-invariant massive gravity given by the action (2.11) satisfies the condition $m_0^2 = 0$ and hence $K_{33} = 0$ in Eq. (6.23), under which the field $\psi$ does not corresponds to a dynamical field. In this case, the field $\psi$ can be eliminated from the second-order scalar action. In the presence of matter, we are left with two dynamical fields $\delta$ and $E$ with a quadratic action in the form

$$S^{(2)}_s = \int d\nu^4 x \left( \delta' \delta + m_1^2 \delta M \delta - \bar{\chi} \overleftrightarrow{K} \chi - k^2 \bar{\chi} B \chi \right),$$  

where $K, L, M, B$ are $2 \times 2$ matrices, and

$$\bar{\chi} = (\delta/k, kE).$$  

The matrices $M$ and $B$ contain the contributions up to the order of $k^0$. In the small-scale limit ($k \rightarrow \infty$), the components of $K$ are given, respectively, by

$$K_{11} = \frac{a^4 \dot{\rho}_M^2}{2(\rho_M + \bar{P}_M)} + O \left( \frac{1}{k^2} \right),$$  

$$K_{12} = K_{21} = \frac{3a^6 \rho_M [2M_{pl}^2 m_1^4 (\cal{H}' - \cal{H}) + a^2 (m_1^2 (\dot{\rho}_M + \ddot{P}_M) + 4M_{pl}^2 m_0^2 (m_1^2 - m_0^2))]}{4 [2M_{pl}^2 (\cal{H}^2 - \cal{H}') - a^2 (M_{pl}^2 m_1^2 + \dot{\rho}_M + \ddot{P}_M)]} + \frac{1}{k^2} + O \left( \frac{1}{k^4} \right),$$  

$$K_{22} = \frac{a^4 M_{pl}^2 [2M_{pl}^2 (\cal{H}^2 - \cal{H}') m_1^2 + 2a^2 m_1^2 (m_1^2 - m_0^2) - a^2 m_0^2 (\dot{\rho}_M + \ddot{P}_M)]}{4M_{pl}^2 (\cal{H}^2 - \cal{H}') - 2a^2 (M_{pl}^2 m_1^2 + \dot{\rho}_M + \ddot{P}_M)} + \frac{1}{k^2} + O \left( \frac{1}{k^4} \right).$$  

If the matrix $K$ is positive definite, the ghost is absent. In the small-scale limit, this amounts to the conditions that the leading-order terms of both $K_{11}$ and $K_{22}$ are positive. The first condition corresponds to $\dot{\rho}_M + \ddot{P}_M > 0$, whereas the second translates to

$$qs = \frac{2m_1^2 (m_1^2 - m_0^2) + m_1^2 m_0^2 (1 - r_b) d\nu}{m_1^2 - 2m_0^2 (1 - r_b) d\nu} > 0,$$  

where we used the background Eqs. (5.17)-(5.18). If $w_{DE} = -1$, i.e., $(1 - r_b) d\nu = 0$, the condition (6.30) translates to $m_1^2 (m_1^2 - m_0^2)/m_1^2 > 0$. Since we require that $m_1^2 > 0$ to avoid the vector ghost, the scalar ghost is absent under the condition $0 < m_0^2 < m_1^2$. This condition agrees with that derived in Ref. [76] on the de Sitter background. From Eq. (6.30) we observe that the deviation from $w_{DE} = -1$ affects the no-ghost condition of scalar perturbations.

The components of the matrix $L$ in the $k \rightarrow \infty$ limit read

$$L_{11} = -c_3^2 \frac{a^4 \dot{\rho}_M^2}{2(\rho_M + \bar{P}_M)} + O \left( \frac{1}{k^2} \right),$$  

$$L_{12} = L_{21} = O \left( \frac{1}{k^2} \right),$$  

$$L_{22} = a^4 M_{pl}^2 (m_1^2 - m_0^2) + O \left( \frac{1}{k^2} \right).$$  

The leading-order contribution to the product $\delta E$ is at most of the order $k^0$ in the action (6.23), so it appears only through the matrix components $L_{12}$ and $M_{21}$ in $M$. The scalar propagation speed squared $c_s^2$ can derived from the dispersion relation

$$\det (c_s^2 K + L) = 0.$$  

(6.34)
In the small-scale limit, we obtain the two solutions
\[ c_{S1}^2 = c_M^2, \]  
\[ c_{S2}^2 = \frac{2(m_2^2 - m_3^2)[2M^2_{pl}(H^2 - \dot{H}) - a^2(M^2_{pl}m_4^2 + \bar{\rho}_M + \bar{P}_M)]}{2M^2_{pl}(H^2 - \dot{H})m_4^2 + 2a^2m_2^4(m_4^2 - m_1^2)} - a^2m_1^4(\bar{\rho}_M + \bar{P}_M). \]  
(6.35)
(6.36)

The Laplacian instability can be avoided for \( c_M^2 \geq 0 \) and \( c_{S2}^2 \geq 0 \). On using the background equations of motion, the second condition translates to
\[ c_{S2}^2 = \frac{(m_2^2 - m_3^2)m_4^2 - 2m^2(1 - r_b)d\mathcal{U}}{2m_2^2(m_4^2 - m_1^2) - m^2m_1^2(1 - r_b)d\mathcal{U}} \geq 0. \]  
(6.37)

For \( w_{DE} = -1 \) this condition reduces to \( c_{S2}^2 = (m_2^2 - m_3^2)m_4^2/[2m_2^2(m_4^2 - m_1^2)] \geq 0 \), which agrees with that derived in Ref. \[76\].

C. Stability of concrete models

We have shown that, in the small-scale limit, there are neither ghosts nor Laplacian instabilities for tensor, vector, and scalar perturbations under the conditions
\[ q_V \simeq m_1^2 > 0, \]  
\[ c_V^2 \simeq m_2^2 = \frac{m_2^2 - 2m^2(1 - r_b)d\mathcal{U}}{m_4^2} \geq 0, \]  
\[ q_S = \frac{2m_2^2(m_2^2 - m_4^2) - m_2^2m^2(1 - r_b)d\mathcal{U}}{m_4^2 - 2m^2(1 - r_b)d\mathcal{U}} > 0, \]  
\[ c_{S2}^2 = \frac{[m_2^2 - m_3^2 - m_2^2(1 - r_b)d\mathcal{U}]m_4^2 - 2m^2(1 - r_b)d\mathcal{U}}{2m_2^2(m_4^2 - m_1^2) - m^2m_1^2(1 - r_b)d\mathcal{U}} \geq 0. \]  
(6.38)
(6.39)
(6.40)
(6.41)

For \( w_{DE} = -1 \), the above conditions coincide with those derived in Ref. \[76\]. We have obtained the above stability conditions in the presence of matter without restricting to de Sitter solutions. The matter contribution \( \bar{\rho}_M + \bar{P}_M \) appearing in Eqs. (6.35) and (6.36) has been eliminated by using the background Eq. (6.16). The resulting no-ghost and stability conditions (6.38)–(6.41) are expressed in terms of \( m_1^2, m_2^2, m_4^2, \) and \( m^2(1 - r_b)d\mathcal{U} \).

For concreteness, we consider the model with the functions \( \mathcal{U}, \mathcal{E} \) given by Eqs. (3.19)–(3.21). The quantities defined in Eqs. (4.14), (4.11), and (4.12) reduce to
\[ d^2\mathcal{E} = \mathcal{F}(t)w_2Y^2, \quad d^2\mathcal{U}_e = w_{2a}Y^4, \quad d^2\mathcal{U}_s = \mathcal{F}(t)v_{2b}Y^4, \quad d^2\mathcal{E}_t = \mathcal{F}(t)v_{2b}Y^4, \]  
(6.42)

with \( d\mathcal{U} = u_1Y^2 + u_2Y^4 \) and \( d\mathcal{E} = \mathcal{F}(t)(v_1Y^2 + v_2Y^4) \). By choosing the function \( \mathcal{F}(t) \) in the form (3.27), the mass terms in Eq. (4.18) yield
\[ m_0^2 = 0, \quad m_1^2 = \frac{4Y^2(u_1 + u_2Y^2)^2}{2(u_1 + u_2Y^2) + w_2}m_2^2, \quad m_2^2 = Y^2(u_1 + u_2Y^2)m_2^2, \]  
\[ m_3^2 = \frac{1}{4}[(8u_2 - 8u_{2a} + 5v_2 - 8v_{2b})Y^4 + 2(4u_1 + v_1)Y^2 - 2v_0]m^2, \]  
\[ m_4^2 = \frac{1}{4}[(8u_2 + 4u_{2a} + 5v_2 + 4v_{2b})Y^4 + 2(4u_1 + v_1)Y^2 - 2v_0]m^2. \]  
(6.43)

The stability of de Sitter solutions satisfying \( r_b = 1 \) was studied in Ref. \[76\], which showed the existence of the viable parameter space. Here, we study the stability conditions for
\[ r_b \neq 1. \]  
(6.44)

In this case there exist three constraints (4.40) among the coefficients \( v_0, v_1, v_2, u_1, u_2 \). In what follows, we will also focus on the case \( w_2 = 0 \). Recalling the relation (3.23), there are five free parameters \( u_0, u_1, u_2, u_{2a}, v_{2a} \) in the model.
under consideration. The conditions (6.38) - (6.41) translate to

\[ q_V = 2Y^2 \left( u_1 + u_2 Y^2 \right) m^2 > 0, \]
\[ c_V^2 = \frac{(1 + 2r_b)u_1 + [2(1 - 4r_b)u_2 - 3(u_{2a} + v_{2a})]Y^2}{u_1 + u_2 Y^2} \geq 0, \]
\[ q_S = Y^2 \left( u_1 + u_2 Y^2 \right) m^2 > 0, \]
\[ c_S^2 = \frac{(1 + 2r_b)u_1 + [3(1 - 4r_b)u_2 - 4(u_{2a} + v_{2a})]Y^2}{u_1 + u_2 Y^2} \geq 0, \]

which depend on the two parameters \( u_1, u_2 \) and the combination \( u_{2a} + v_{2a} \). Provided that

\[ u_1 > 0, \quad \text{and} \quad u_2 > 0, \]

the no-ghost conditions (6.45) and (6.47) are satisfied. We also note that the density parameters \( \Omega_{DE1} \) and \( \Omega_{DE2} \) defined in Eq. (3.29) are positive under these conditions.

Let us first discuss the case \( r_b < 1 \), under which the quantity \( Y = a^{r_b - 1} \) decreases with the growth of \( a \). In the asymptotic past \( (a \to 0 \text{ and } Y \to \infty) \), the conditions (6.45) and (6.48) reduce, respectively, to

\[ c_V^2 = 2(1 - 4r_b) - \frac{3(u_{2a} + v_{2a})}{u_2} \geq 0, \]
\[ c_S^2 = 3(1 - 4r_b) - \frac{4(u_{2a} + v_{2a})}{u_2} \geq 0. \]

If \( r_b < 1/4 \) these conditions are satisfied for \( (u_{2a} + v_{2a})/u_2 \leq (2/3)(1 - 4r_b) \), whereas, if \( r_b > 1/4 \), they are satisfied for \( (u_{2a} + v_{2a})/u_2 \leq (3/4)(1 - 4r_b) \). In the asymptotic future \( (a \to \infty \text{ and } Y \to 0) \), the conditions (6.45) and (6.48) translate to

\[ c_V^2 = c_S^2 = 1 + 2r_b \geq 0, \]

which are satisfied for \( r_b \geq -1/2 \).
In the left panel of Fig. 2 we plot the evolution of $c_V^2$, $c_S^2$ and $w_{DE}$ for $r_b = 0$ and $u_2a + v_2a = 0.6u_2$. In this case we have $c_V^2 = 0.2$ and $c_S^2 = 0.6$ from Eqs. (6.50) and (6.51), whereas $c_V^2 = c_S^2 = 1$ from Eq. (5.52). As we see in Fig. 2, both $c_V^2$ and $c_S^2$ continuously increase from their past asymptotic values to the final same value 1, so the ghosts and Laplacian instabilities associated with vector and scalar perturbations are absent. In this numerical simulation the condition $\Omega_{DE1} \ll \Omega_{DE2}$ is initially satisfied, so $w_{DE}$ starts to evolve from the value close to $(1 - 4r_b)/3 = 1/3$ [see Eq. (3.42)]. After $\Omega_{DE2}$ gets smaller than $\Omega_{DE1}$, $w_{DE}$ temporally stays around the value $-(1 + 2r_b)/3 = -1/3$. After the dominance of $\Omega_{DE2}$ over $\Omega_{DE1}$ and $\Omega_{DE2}$, $w_{DE}$ approaches the asymptotic value $-1$.

For $r_b < 1$, both $q_V$ and $q_S$ asymptotically approach $+0$ with the decrease of $Y = a^{r_b - 1}$. Then, the strong coupling scales ($M_{pl}(\sqrt{q_V})^{1/2}$ and ($M_{pl}(\sqrt{q_S})^{1/2}$ also decrease in time. Provided that the today’s values $\sqrt{q_V} = \sqrt{2q_S} = \sqrt{2(u_1 + u_2)m}$ are not much different from the order $H_0 \sim 10^{-42}$ GeV, the strong coupling scales now can be estimated as $\Lambda_{SC} \sim \sqrt{M_{pl}H_0} \sim 10^{-12}$ GeV. This corresponds to the energy scale $E_{SC} \sim \sqrt{\Lambda_{SC}M_{pl}} \sim 10^3$ GeV, so our perturbative analysis is valid from the redshift $z \sim 10^{15}$ to today. In the future the strong coupling scale gradually decreases from the today’s value $\Lambda_{SC} \sim 10^{-12}$ GeV, so the region for the validity of our perturbative analysis shifts to lower energy scales.

For $r_b > 1$ the quantity $Y$ increases with the growth of $a$, so the past asymptotic values of $c_V^2$ and $c_S^2$ are given by Eq. (6.52), i.e., $c_V^2 = c_S^2 = 1 + 2r_b > 3$. In the asymptotic future, $c_V^2$ and $c_S^2$ approach the values (6.50) and (6.51), respectively. Provided that

$$\frac{u_2a + v_2a}{u_2} \leq \frac{3}{4} (1 - 4r_b),$$

(6.53)

the stability conditions of vector and scalar perturbations are satisfied. In the right panel of Fig. 2 we show the evolution of $c_V^2$, $c_S^2$ and $w_{DE}$ for $r_b = 1.2$ and $u_2a + v_2a = -3u_2$. In this case, $c_V^2$ and $c_S^2$ start to evolve from the same value 3.4 and then they approach the asymptotic values $c_V^2 = 1.4$ and $c_S^2 = 0.6$, respectively [as estimated from Eqs. (6.50) and (6.51)]. Initially, $\Omega_{DE0}$ dominates over $\Omega_{DE1}$ and $\Omega_{DE2}$, so $w_{DE}$ is close to $-1$. Finally, the solution approaches the fixed point (C) given by Eq. (3.47) with $w_{DE} = (1 - 4r_b)/3 = -1.267$.

For $r_b > 1$ the quantities $q_V$ and $q_S$ decrease toward the asymptotic past. Since $q_V$ and $q_S$ are of the order $u_1m^2Y^2$ in the regime $u_2Y^2 \ll u_1$, the strong coupling scales can be estimated as $\Lambda_{SC} \sim \sqrt{(u_1)^{1/2}mM_{pl}a^{(r_b - 1)/2}}$. Provided that $r_b$ is close to 1, the term $a^{(r_b - 1)/2}$ does not give rise to a significant change to $\Lambda_{SC}$ relative to the value $\sqrt{(u_1)^{1/2}mM_{pl}}$. For $r_b = 1.2$ we have $a^{(r_b - 1)/2} = 10^{-1}$ at $a = 10^{-10}$, so the strong coupling scales at the redshift $z \sim 10^{10}$ are as high as $\Lambda_{SC} \sim 10^{-13}$ GeV for $(u_1)^{1/2}m \sim H_0$. The quantities $q_V$ and $q_S$, which increase toward the asymptotic future, exhibit divergences at the big-rip singularity. The perturbative analysis breaks down around the big-rip singularity.

**VII. GROWTH OF NON-RELATIVISTIC MATTER PERTURBATIONS**

We study the evolution of nonrelativistic matter perturbations and derive the effective gravitational coupling as well as the gravitational slip parameter associated with the observations of large-scale structures and weak lensing. Let us consider non-relativistic matter satisfying

$$P_M = 0, \quad c_M^2 = 0.$$  

(7.1)

Introducing the Bardeen’s gauge-invariant gravitational potentials [87]

$$\Psi \equiv A - \sigma' - \mathcal{H}\sigma, \quad \Phi \equiv \psi - \mathcal{H}\sigma,$$

(7.2)

we can express Eq. (6.19) in the form

$$\Psi + \Phi = a^2m^2E.$$  

(7.3)

In Fourier space, Eqs. (6.13) and (6.14) reduce, respectively, to

$$\delta' - k^2\sigma + k^2\frac{\Psi}{a} = 0,$$

(7.4)

$$\nu' = aA,$$

(7.5)

where $\delta = \delta\rho_M/\rho_M + 3\psi$. Taking the time derivative of Eq. (7.4) and using Eq. (7.3), we obtain

$$\delta'' + \mathcal{H}\delta' + k^2\Psi = 0.$$  

(7.6)
We define the effective gravitational coupling $G_{\text{eff}}$ and the gravitational slip parameter $\eta_s$, as
\[
k^2\Psi = -4\pi G_{\text{eff}}a^2\beta_M\delta, \tag{7.7}
\]
\[
\eta_s = -\frac{\Phi}{\Psi}. \tag{7.8}
\]
Provided that $m_0^2 \neq 0$, $\eta_s$ is different from 1. The effective gravitational potential associated with the deviation of light rays is defined by \[88\,89\]
\[
\Phi_{\text{eff}} \equiv \Phi - \Psi = - (\eta_s + 1)\,\Psi. \tag{7.9}
\]
From Eqs. (7.7) and (7.9) it follows that
\[
k^2\Phi_{\text{eff}} = 8\pi G a^2 \Sigma \beta_M \delta, \tag{7.10}
\]
where $G = 1/(8\pi M_{\text{pl}}^2)$ is the Newton’s gravitational constant, and
\[
\Sigma \equiv \frac{G_{\text{eff}}}{G} \eta_s + \frac{1}{2}. \tag{7.11}
\]
Since we are interested in the growth of perturbations relevant to the observations of large-scale structures and weak lensing, we focus on the modes deep inside the Hubble radius ($k^2 \gg \mathcal{H}^2$). For the theories satisfying the condition $m_0^2 = 0$ there are two dynamical scalar quantities $\delta$ and $E$, whose equations of motion follow from the second-order action (6.25). Varying the action (6.25) with respect to $\delta$ and $E$, the resulting equations of motion for these perturbations can be written in the forms
\[
\delta'' + \alpha_1 \delta' + \alpha_2 \delta + \alpha_3 E' + \alpha_4 E = 0, \tag{7.12}
\]
\[
E'' + \beta_1 E' + \beta_2 E + \beta_3 \delta' + \beta_4 \delta = 0, \tag{7.13}
\]
where $\alpha_{1,2,3,4}$ and $\beta_{1,2,3,4}$ are time-dependent coefficients. The coefficients $\alpha_{1,2,3,4}$, expanded around large $k$, are given by
\[
\alpha_1 = \mathcal{H} + 3Ha^2\bar{\rho}_M \frac{1}{2M_{\text{pl}}^2} \frac{\kappa}{k^2} + \mathcal{O}\left(\frac{1}{k^4}\right), \quad \alpha_2 = -a^2\beta_M + \left(\frac{3a^4\bar{\rho}_M^2}{4M_{\text{pl}}^4} + \frac{3a^4m_0^2\bar{\rho}_M}{2M_{\text{pl}}^2}\right) \frac{1}{k^2} + \mathcal{O}\left(\frac{1}{k^4}\right),
\]
\[
\alpha_3 = -\frac{3a^2\mathcal{H}[a^2m_0^2\bar{\rho}_M - 2M_{\text{pl}}^2(m_0^2(\mathcal{H}^2 - \mathcal{H}') + 2a^2m_0^2(m_0^2 - m_1^2))]}{4M_{\text{pl}}^2(\mathcal{H}^2 - \mathcal{H}') - 2a^2(M_{\text{pl}}^2m_1^2 + \rho_M)} + \mathcal{O}\left(\frac{1}{k^2}\right),
\]
\[
\alpha_4 = -a^2 \left(m_2^2 + 3m_2^2 + m_1^2\right) + \mathcal{O}\left(k^0\right). \tag{7.14}
\]
For the modes $k \gg \mathcal{H}$, the second term in $\alpha_1$ is much smaller than $\mathcal{H}$. The term $3a^4\bar{\rho}_M/(4M_{\text{pl}}^4k^2)$ in $\alpha_2$ is also negligible relative to the first contribution to $\alpha_2$. Provided that all the mass terms $m_0^2$ as well as the combination $m_2^2 + 3m_2^2 + m_1^2$ are of the similar orders to $m^2_0$, the term $\alpha_3 E'$ is much smaller than $\alpha_4 E$ for $k \gg \mathcal{H}$. Then, Eq. (7.12) approximately obeys
\[
\delta'' + \mathcal{H}\delta' - 4\pi G a^2 \beta_M \delta \simeq a^2 \left(m_2^2 + 3m_2^2 + m_1^2\right) \frac{\kappa}{k^2} E - \frac{3a^4m_0^2\bar{\rho}_M}{2M_{\text{pl}}^2} \frac{\delta}{k^2}, \tag{7.15}
\]
where we have not ignored the term $3a^4m_0^2\bar{\rho}_M/(2M_{\text{pl}}^2k^2)$ in $\alpha_2$. The growth of matter perturbations is modified from that in GR by the graviton mass terms appearing on the r.h.s. of Eq. (7.15).

For large $k$ the coefficients $\beta_{1,2,3,4}$ in Eq. (7.13) are given by
\[
\beta_1 = \mathcal{O}\left(k^0\right), \quad \beta_2 = c_{\text{s2}}^2 k^2 + \mathcal{O}\left(k^0\right),
\]
\[
\beta_3 = \mathcal{O}\left(k^0\right), \quad \beta_4 = -\frac{(m_2^2 + 3m_2^2 + m_1^2)[2M_{\text{pl}}^2(\mathcal{H}^2 - \mathcal{H}') - a^2(M_{\text{pl}}^2m_1^2 + \bar{\rho}_M)]}{\left[2(\mathcal{H}^2 - \mathcal{H}')m_1^2 + 4a^2m_0^2(m_0^2 - m_1^2)\right]M_{\text{pl}}^2 - a^2m_0^2\bar{\rho}_M}]M_{\text{pl}}^2} \frac{\beta_M}{k^2} + \mathcal{O}\left(\frac{1}{k^4}\right), \tag{7.16}
\]
where $c_{\text{s2}}^2$ is the sound speed squared defined by Eq. (6.36) with $\bar{P}_M = 0$. The leading-order contribution to $\beta_1$, which does not contain the $k$-dependence, is at most of the order $\mathcal{H}$. Since the leading-order contribution to $\beta_3$ has the $k^{-4}$ dependence, the term $\beta_3 \delta'$ is suppressed relative to $\beta_4 \delta$ in the small-scale limit. Provided that the variation of
$E$ is small such that the conditions $|E'| \lesssim |HE|$ and $|E''| \lesssim |H^2E|$ are satisfied, the first two terms on the l.h.s. of Eq. (7.13) can be neglected relative to $\beta_2E$ for the modes deep inside the sound horizon,

$$c^2_{S0}k^2 \gg H^2. \tag{7.17}$$

In this case the perturbation $E$ is related to $\delta$ according to $\beta_2E + \beta_4\delta \simeq 0$, such that

$$E \simeq -\frac{\beta_4}{\beta_2}\delta \simeq \frac{(m^2_{2+} - 3m^2_{2} + m^2_{4})a^2\bar{\rho}_M}{2(m^2_{2} - m^2_{3})\overline{M}_p^2k^4}\delta. \tag{7.18}$$

This approximation, which is so-called the quasi-static approximation \cite{90–92}, is valid for the modes satisfying the condition (7.17). Substituting (7.18) into Eq. (7.13), we can express the matter perturbation equation in the form (7.6) with the effective gravitational coupling given by

$$G_{\text{eff}} = G \left[1 + \frac{m^4_1 - (m^2_{2+} - 3m^2_{2} + 2m^2_{4})a^2}{m^2_{2} - m^2_{3}} \right]. \tag{7.19}$$

On using Eqs. (7.3), (7.7), and (7.19), the two gauge-invariant gravitational potentials are given by

$$\Psi = -\frac{4\pi Ga^2\bar{\rho}_M}{k^2}\left[1 + \frac{m^4_1 - (m^2_{2+} - 3m^2_{2} + 2m^2_{4})a^2}{m^2_{2} - m^2_{3}} \right], \tag{7.20}$$

$$\Phi = \frac{4\pi Ga^2\bar{\rho}_M}{k^2}\left[1 + \frac{m^4_1 - (m^2_{2+} - 3m^2_{2})a^2}{m^2_{2} - m^2_{3}} \right]. \tag{7.21}$$

In the large $k$ limit the terms containing the graviton masses in $G_{\text{eff}}$, $\Psi$, $\Phi$ vanish, so the general relativistic behavior is recovered. If all the mass terms $m_i^2$ are of the similar order to $m^2$, the massive gravity corrections in the square brackets of Eqs. (7.19)-(7.21) are at most of the order $m^2a^2/(c^2_{S0}k^2)$. On using the condition (7.17), they are much smaller than $m^2/H^2$, where $H = H/a$ is the Hubble expansion rate in terms of the cosmic time $t = \int a\,dn$. For $m$ of the order of the today’s Hubble expansion rate $H_0$, it follows that the graviton mass terms in the square brackets of Eqs. (7.19)-(7.21) are much smaller than $1$ from the matter era to today.

Taking the small-scale limit, the gravitational slip parameter $\eta_s$ and the parameter $\Sigma$ reduce to

$$\eta_s \simeq 1 + \frac{(m^2_{2+} - 3m^2_{2} + m^2_{4})a^2}{m^2_{2} - m^2_{3}} \approx 0, \tag{7.22}$$

$$\Sigma \simeq 1 - \frac{(m^2_{2+} - 3m^2_{2} + 3m^2_{4})m^2_{4} - 2m^4_{4}}{2(m^2_{2} - m^2_{3})} \approx 0, \tag{7.23}$$

so that the deviations of $\eta_s$ and $\Sigma$ from $1$ are much smaller than $1$ for the modes deep inside the sound horizon.

The above discussion is valid for the modes satisfying the condition (7.17). There is a specific situation in which $c^2_{S0}$ is much smaller than $1$. This can be realized for $m^2_2 = m^2_3$, which is the case for the self-accelerating solution in dRGT theory. Since $m^2_{2} - m^2_{3} = -m^2(1 - r_k)d\Omega$ in this case, the small deviation from $w_{DE} = -1$ leads to the non-vanishing value of $c^2_{S0}$ close to $0$. The wavenumber $k$ associated with the linear regime of the observations of large-scale structures and weak lensing corresponds to $k \lesssim 300H_0$, where $H_0$ is the today’s value of $H$ (at the redshift $z = 0$). If the today’s sound speed satisfies the condition

$$c^2_{S0}(z = 0) \ll 10^{-5}, \tag{7.24}$$

then the analytic solution derived above loses its validity for the perturbations relevant to the growth of large-scale structures and weak lensing. In such cases the condition $c^2_{S0}k^2 \ll H^2$ is satisfied for the modes $k \lesssim 300H_0$ in the past, so the term $c^2_{S0}k^2$ is not the dominant contribution to $\beta_2$ in Eq. (7.10). If the masses $m_2$ are of the order $H_0$, the term $\beta_2E$ as well as $E''$ and $\beta_4E'$ would give rise to contributions at most of the order $H^2E$. Writing the sum of the first three terms on the l.h.s. of Eq. (7.13) as $E'' + \beta_1E' + \beta_2E = sH^2E$, where $s$ is a time-dependent dimensionless factor, it follows that

$$sH^2E + \beta_4\delta \simeq 0, \tag{7.25}$$

for the modes $k^2 \gg H^2$. Taking the leading-order term in $\beta_4$ and ignoring the last term on the r.h.s. of Eq. (7.15), we can write Eq. (7.15) in the form (7.6) with the effective gravitational coupling

$$G_{\text{eff}} = G \left[1 + \frac{2\{(2H^2 - 2H' - a^2m^2_{4})\overline{M}_p^2 - a^2\bar{\rho}_M\}(m^2_{2+} - 3m^2_{2} + m^2_{4})(m^2_{3} - m^2_{4}) - a^2m^2_{4}\bar{\rho}_M}{2\overline{M}_p^4\{(H^2 - H') - 2a^2m^2_{4}(m^2_{3} - m^2_{4}) - a^2m^2_{4}\bar{\rho}_M\}} \right]. \tag{7.26}$$
The second term in the squared bracket of Eq. (7.20) does not vanish in the small-scale limit. Provided that \( m_{\text{eff}}^2 = \mathcal{O}(m^2) \), the second term in the square bracket of Eq. (7.20) is of the order \( m^2/(sH^2) \). This correction can be important in the late Universe for \( m \sim H_0 \). To compute \( G_{\text{eff}} \) precisely we need to know how the parameter \( s \) varies in time, whose property depends on the models under consideration. It will be of interest to study the evolution of perturbations for concrete models satisfying the condition (7.23).

VIII. CONCLUSIONS

In this paper, we studied the cosmology in general theories of Lorentz-violating massive gravity by taking into account a perfect fluid in form of the Schutz-Sorkin action (2.14). This is for the purposes of studying the dynamics of late-time cosmic acceleration arising from the graviton potential in the presence of matter and discussing observational signatures relevant to the growth of matter density perturbations. The general \( SO(3) \)-invariant massive gravity theories that propagate five DOFs have the graviton potential of the form (2.7), which are parametrized by the two functions \( U \) and \( E \). By choosing the unitary gauge, we have taken into account the dependence of the fiducial metric \( \delta_{ij} \) and the temporal scalar \( \phi \) of the form \( f_{ij} = b^2(\phi)\delta_{ij} \) in the functions \( U \) and \( E \).

The equations of motion on the flat FLRW background are expressed as Eqs. (3.19) and (3.20), where the energy density \( \rho_{\text{mg}} \) and the pressure \( P_{\text{mg}} \) associated with the massive graviton are given by Eq. (3.10). Since there is the relation \( \rho_{\text{mg}} \propto \rho_{\text{DE}} \) between the functions \( E \) and \( U \), the dark energy equation of state arising from the graviton potential is of the form \( \rho_{\text{mg}} \propto \rho_{\text{DE}} \). The deviation of \( \rho_{\text{DE}} \) from \( -1 \) occurs for the theories in which the two conditions \( r_b \neq 0 \) and \( \delta t \neq 0 \) are satisfied. While the previous studies mostly focused on the cases \( r_b = 0 \) and \( \delta t = 0 \), we extended the analysis to the theories with more general values of \( r_b \). This amounts to considering different expansion rates between the scale factor \( a(t) \) and the other time-dependent factor \( b(t) \) in \( f_{ij} \).

For the concrete model in which the functions \( U \) and \( E \) are given by Eqs. (3.19) and (3.20), respectively, we studied the background cosmology relevant to the late-time cosmic acceleration in the presence of nonrelativistic matter and radiation. For \( r_b = 1 \) the ratio \( Y = b/a \) remains constant, in which case the background cosmological dynamics is equivalent to that in the \( \Lambda \)CDM model. If \( r_b \) is a constant different from 1 the ratio \( Y \) varies as \( Y \propto a^{r_b-1} \), so the relation (3.28) places the constraints (3.40) among the coefficients in \( U \) and \( E \). For \( r_b < 1 \), we have the dynamical dark energy scenario in which the equation of state changes in the region \( w_{\text{DE}} > -1 \). On the other hand, if \( r_b > 1 \), the phantom equation of state \( (w_{\text{DE}} < -1) \) can be realized.

To discuss the stability of solutions against linear cosmological perturbations, we derived general conditions for the absence of ghosts and Laplacian instabilities of tensor and vector perturbations in Sec. V. We showed that the presence of a perfect fluid does not substantially modify the stabilities of tensor and vector modes. In our \( SO(3) \)-invariant massive gravity theories the tensor propagation speed \( c_T \) is equivalent to the speed of light \( c \) with the tiny graviton mass of order \( H_0 \sim 10^{-33} \) eV, so they safely evade the bounds recently constrained by LIGO [79, 80]. Under the conditions (3.35) there are neither ghosts nor Laplacian instabilities for vector perturbations.

In Sec. VI we derived the second-order action of scalar perturbations in the presence of matter perturbations with the density contrast \( \delta \). By construction (3)-invariant massive gravity with the graviton potential (2.7) satisfies the condition \( m_{\text{eff}}^2 = 0 \), under which there are two dynamical fields \( E \) and \( \delta \). The conditions for the absence of ghosts and Laplacian instabilities associated with the field \( E \) are given, respectively, by Eqs. (6.30) and (6.37). The deviation from \( \rho_{\text{DE}} = -1 \), which is weighed by the factor \( 2(1-r_b)\delta t \), affects the stability of scalar perturbations. We applied general stability conditions to the concrete model given by Eqs. (3.19), (3.20) and showed that there are theoretically consistent parameter spaces in which all the stability conditions of vector and scalar perturbations are satisfied.

In Sec. VII we also studied the evolution of the gauge-invariant nonrelativistic matter density contrast \( \delta \) and gravitational potentials \( \Psi \) and \( \Phi \) to confront our massive gravity theories with the observations of large-scale structures and weak lensing. For the perturbations satisfying the condition (7.14), we derived the effective gravitational coupling \( G_{\text{eff}} \) and the gravitational slip parameter \( \eta_s = -\Phi/\Psi \) by using the quasi-static approximation. As we see in Eqs. (7.19) and (7.22), the corrections induced by the massive graviton potential to their GR values \( (G_{\text{eff}} = G \text{ and } \eta_s = 1) \) are suppressed by factors of the order \( m^2/a^2/(c_{S2}^2 k^2) \) for the modes deep inside the sound horizon. There is a specific case \( m_2^2 = m_3^2 \), under which the sound speed squared \( c_{S2}^2 \) can be much smaller than 1 for \( w_{\text{DE}} \) close to \(-1 \). In this case the massive graviton gives rise to a correction to the gravitational constant without the suppression in the small-scale limit, see Eq. (7.20).

We thus showed that the \( SO(3) \)-invariant theory of massive gravity offers an interesting possibility for giving rise to cosmological solutions with a variety of the dark energy equation of state, while satisfying the stability conditions of tensor, vector, and scalar perturbations. It will be of interest to place observational constraints on concrete models like Eqs. (3.19), (3.20) by using observational data associated with the cosmic expansion and growth histories.
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