DEGENERATIONS OF TRIPLE COVERINGS AND THOMAE’S FORMULA

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Dedicated to Professor Toshiyuki Katsura on his 60th birthday

Abstract. In this paper, we prove Thomae’s formula for triple coverings of the complex projective line $P^1$ and give the absolute constant in this formula for a specific choice of symplectic bases. This formula gives a relation between theta constants, the products of the determinant of a period matrix and difference products of branch points. To specify symplectic bases of them, we use the combinatorics of binary trees on $P^1$. These symplectic bases behave so well for degenerations that we reduce the formula to a special case treated in [BR, N].

1. Introduction

Let $E : y^2 = x(1-x)(1-\lambda x)$ be an elliptic curve and $\tau$ be the normalized period matrix of $E$. Then we have Jacobi’s formula for an elliptic curve $E$:

$$\vartheta(\tau)[0,0]^2 = \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{x(1-x)(1-\lambda x)}} \, dx,$$

where $\vartheta(\tau)[\alpha, \beta]$ is defined in (2.1).

In 1870, Thomae ([T]) generalized this formula to those for hyperelliptic curves of arbitrary genus. He showed that the squares of theta constants at the normalized period matrix of a hyperelliptic curve are equal to the products of the determinant of a period matrix and certain difference products of branch points up to an easy constant multiple. Bershadsky-Radul and Nakayashiki [BR, N] independently proved an analogous formula for cyclic coverings of the projective line $P^1$ with special branching indices, which is called Thomae’s formula for cyclic coverings. They prove a power of a theta constant with characteristic $\Lambda$ is the product of the determinant of a period matrix, a certain difference product and a constant $\kappa_\Lambda$, and show that the constant $\kappa_\Lambda$ depends only on the genus, a choice of symplectic basis of the covering and the theta characteristic $\Lambda$. This result is generalized to arbitrary branching indices and covering degrees by [K].

For hyperelliptic curves, Fey computed the absolute constants $\kappa_\Lambda$ in Thomae’s formula using degeneration arguments in his book [F]. In this paper, we give a closed formula for the absolute constants $\kappa_\Lambda$ for triple coverings of arbitrary $P^1$. 

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branching indices. To formulate the exact statement of Thomae’s formula, we construct symplectic bases of a family of triple coverings using the combinatorics of binary trees on $\mathbb{P}^1$. This family is extended to stable curves with trivial monodromy action and each of its special fibers is the union of two triple coverings of $\mathbb{P}^1$’s. By this degeneration, a binary tree decomposes to two trees and according to this decomposition, the symplectic bases are extended to the union of symplectic bases of two irreducible components. We use this property for the study of the absolute constant $\kappa_\Lambda$.

The contents of this paper are as follows. In Section 2, we recall results of Bershadsky-Radul and Nakayashiki. In Section 3, we define a specific choice of symplectic basis $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$ associated to a planar binary tree. We study the combinatorial process of degenerations. In Section 4, we study stable degenerations of algebraic curves associated to the decomposition of binary trees. In Section 5, we prove Thomae’s formula (Theorem 6.2) for triple coverings of $\mathbb{P}^1$ and compute the absolute constants for arbitrary branching indices using degeneration argument. The proof is based on results of Bershadsky-Radul and Nakayashiki and the formula (5.2), which is a variant of the Chowla-Selberg formula. Our method is different from that of [K].

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2. Result of Bershadsky-Radul-Nakayashiki

In this section, we recall results of Bershadsky-Radul and Nakayashiki. Let $n \geq 2$ be an integer, $\mathbb{P}^1$ be the projective line with a coordinate $x$, and $\Sigma = \{\lambda_1, \ldots, \lambda_{3n}\}$ be a set of distinct $3n$ points in $\mathbb{P}^1$ different from $x = \infty$. The values of $x$ at $\lambda_1, \ldots, \lambda_{3n}$ are also denoted by the same letter. Let $C$ be the cyclic triple covering of $\mathbb{P}^1$ defined by

$$y^3 = (x - \lambda_1) \cdots (x - \lambda_{3n}).$$

By Hurwitz’s formula, the genus $g = g(C)$ of $C$ is equal to $3n - 2$ and a basis of the space of holomorphic differential forms on $C$ is given by $\{\omega_1, \ldots, \omega_{3n - 2}\}$, where

$$\omega_i = \frac{x^{n-1-i} dx}{y} \quad \text{for} \ i = 1, \ldots, n - 1,$$

$$\omega_i + n - 1 = \frac{x^{2n-1-i} dx}{y^2} \quad \text{for} \ i = 1, \ldots, 2n - 1.$$

We fix a symplectic basis $\{A_i, B_i\}_{i=1, \ldots, 3n-2}$ of $H_1(C, \mathbb{Z})$. The period matrices $P_A$ and $P_B$ are defined as

$$P_A = (\int_{A_i} \omega_j)_{i,j=1,\ldots,3n-2}, \quad P_B = (\int_{B_i} \omega_j)_{i,j=1,\ldots,3n-2}.$$
The normalized period matrix $\tau$ is defined by $\tau = P_A P_B^{-1}$. For vectors $\alpha, \beta \in \mathbb{Q}^g$, we define the theta function as follows:

$$(2.1) \quad \vartheta(\tau)[\alpha, \beta](z) = \sum_{m \in \mathbb{Z}^g} e^{\frac{1}{2}(m + \alpha)\tau^t(m + \alpha) + (m + \alpha)^t(z + \beta)}.$$}

Here, we use the notation $e(x) = \exp(2\pi \sqrt{-1}x)$. The value of the theta function at $z = 0$ is called the theta constant and denoted by $\vartheta(\tau)[\alpha + a, \beta + b]$. Note that if $\alpha, \beta \in \frac{1}{6}\mathbb{Z}^g$, then the sixth power of the theta constant is periodic with respect to $\alpha$ and $\beta$, that is

$$\vartheta(\tau)[\alpha, \beta]^6 = \vartheta(\tau)[\alpha + a, \beta + b]^6$$

for $a, b \in \mathbb{Z}^g$. By the isomorphism

$$(2.2) \quad \mathbb{Q}^g \oplus \mathbb{Q}^g \to H_1(C, \mathbb{Q}) : (a_1, \ldots, a_g, b_1, \ldots, b_g) \mapsto \sum_i a_i A_i + \sum_i b_i B_i,$$

the theta constant $\vartheta(\tau)[\alpha, \beta]^6$ is a function on the set of 6-torsion points $H_1(C, \mathbb{Q}/\mathbb{Z})_6$ of $H_1(C, \mathbb{Q}/\mathbb{Z})$. The value at $\lambda \in H_1(C, \mathbb{Q}/\mathbb{Z})_6$ is denoted by $\vartheta(\tau)[\lambda]$.

Let $\rho : C \to C$ be the automorphism of the curve $C$ defined by $(x, y) \mapsto (x, \omega y)$, where $\omega = \frac{-1 + \sqrt{-3}}{2}$. The induced homomorphism on $H_1(C, \mathbb{Z})$ is also denoted by $\rho$. We give a description of the group of $(1 - \rho)$ torsion part $H_1(C, \mathbb{Q}/\mathbb{Z})(1-\rho)$ of $H_1(C, \mathbb{Q}/\mathbb{Z})$ as a $\mathbb{F}_3$-vector space.

Let $b$ be a base point in $\mathbb{P}^1 - \Sigma$. A path connecting $b$ and small anti-clockwise circle around $\lambda_i$ defines a path $\overline{\gamma_i}$ in $\mathbb{P}^1 - \Sigma$. We choose a base point $\overline{b}$ in $C$ over the base point $b$. Then the path $\overline{\gamma_i}$ is lifted uniquely beginning from $\overline{b}$, and its end point is $\rho(\overline{b})$. If any paths $\gamma_i - \{\overline{b}, \rho(\overline{b})\}$ are disjoint to one another, then we may assume that paths $\gamma_1, \ldots, \gamma_{3n}$ are arranged in the anti-clockwise order by renumbering the paths $\gamma_1, \ldots, \gamma_{3n}$. Then the cycle

$$\gamma_1 + \rho \gamma_2 + \rho^2 \gamma_3 + \cdots + \rho^{3n-1} \gamma_{3n}$$

is homologous to zero. For $i = 1, \ldots, 3n - 1$, the path $\gamma_i - \gamma_{3n}$ defined an element of $H_1(C, \mathbb{Z})$. Therefore $\delta_i = \frac{1}{3}(1 - \rho)(\gamma_i - \gamma_{3n})$ defines an element of $H_1(C, \mathbb{Q}/\mathbb{Z})(1-\rho)$. By the above relation, the cycle $\delta_1 + \rho \delta_2 + \cdots + \rho^{3n-2} \delta_{3n-1}$ is homologous to zero. Let $V$ be the $\mathbb{F}_3$-vector space $\bigoplus_{i=1}^{3n} \mathbb{F}_3 e_i$ generated by $e_1, \ldots, e_{3n}$. We define the map $\Pi$ and $\Delta$ by

$$\Pi : V \ni \sum_i a_i e_i \mapsto \sum_i a_i \in \mathbb{F}_3,$$

$$\text{Diag} : \mathbb{F}_3 \ni a \mapsto a \sum_{i=1}^{3n} e_i \in V.$$

Then we have an isomorphism

$$H_1(C, \mathbb{Q}/\mathbb{Z})(1-\rho) \simeq \text{Ker}(\Pi)/\text{Im}(\text{Diag})$$

by assigning $\delta_i$ to $e_i - e_{3n}$. 
We are ready to state Thomae’s formula for the curve $C$ ([N],[BR]). Let $	ilde{\Lambda} = \sum_i a_i e_i$ be a representative in $V$ of an element $\Lambda$ of $H_1(C,\mathbb{Q}/\mathbb{Z})_{(1-\rho)}$. We define a subset $\tilde{\Lambda}_i$ of $\{1, \ldots, 3n\}$ for $i = 0, 1, 2$ by

$$\tilde{\Lambda}_i = \{ p \in \{1, \ldots, 3n\} \mid a_p \equiv i \pmod{3} \}.$$  

The difference product $(\tilde{\Lambda}_i\tilde{\Lambda}_j)$ is defined by

$$(\tilde{\Lambda}_i\tilde{\Lambda}_j) = \begin{cases} \prod_{a \in \tilde{\Lambda}_i, b \in \tilde{\Lambda}_j} (\lambda_a - \lambda_b) & \text{if } i \neq j, \\ \prod_{a, b \in \tilde{\Lambda}_i, a < b} (\lambda_a - \lambda_b) & \text{if } i = j. \end{cases}$$

Then it is defined up to sign. We define the difference product $\Delta(\Lambda)$ attached to $\Lambda \in H_1(C,\mathbb{Q}/\mathbb{Z})_{(1-\rho)}$ by

$$\Delta(\Lambda) = \prod_{i=0}^2 (\tilde{\Lambda}_i\tilde{\Lambda}_i)^3 \prod_{0 \leq i < j \leq 2} (\tilde{\Lambda}_i\tilde{\Lambda}_j)$$

for a representative $\tilde{\Lambda}$ of $\Lambda$. It does not depend on the choice of the representative up to sign.

**Theorem 2.1.** Let $\Lambda$ be an element $H_1(C,\mathbb{Q}/\mathbb{Z})_{(1-\rho)}$ and $\tilde{\Lambda}$ be a representative of $\Lambda$ in $V$. Suppose that

$$(2.3) \quad \#\tilde{\Lambda}_0 = \#\tilde{\Lambda}_1 = \#\tilde{\Lambda}_2 = n.$$  

Then we have

$$\vartheta(\tau)[\Lambda + \varrho]^6 = \kappa_\Lambda \det(P_B)^3 \cdot \Delta(\Lambda).$$

Here $\varrho$ is the Riemann constant and $\kappa_\Lambda$ is a constant independent of $\lambda_1, \ldots, \lambda_{3n}$. Moreover $\kappa_\Lambda^6$ is independent of $\Lambda$.

**Definition 2.2.** The absolute constant $\kappa_\Lambda^6$ is called Thomae’s constant for the symplectic basis $\{A_i, B_i\}_i$.

3. **Binary trees and symplectic bases**

3.1. **Symplectic basis associated to a marked binary tree.** In this section, we define a symplectic basis associated to a binary tree and study its properties. For general terminology on trees, refer to [S] and [D].

**Definition 3.1.**

1. A vertex of a tree $\Gamma$ is called an inner vertex if it is adjacent to more than one edges. The set of inner vertices of $\Gamma$ is denoted by $V(\Gamma)$. A vertex of $\Gamma$ is a terminal if it is not a inner vertex.

2. A tree is called a trivalent tree if the valency of any inner vertex is three.

3. A planar trivalent tree $\Gamma$ with two colors, white and black, on its vertices is called a binary tree if two ends of any edge are differently colored.
Let \((\Gamma, C)\) be a binary tree. A marking of \((\Gamma, C)\) is to specify choices of one edge adjacent to each inner vertex. A binary tree \((\Gamma, C)\) with a marking \(M\) is called a marked binary tree and denoted by \((\Gamma, C, M)\). We connect the rest of edges by a small arc to illustrate the marking. (See Figure 1 for an example of marked binary tree.)

**Figure 1.** Marked binary tree

Let \(\lambda_1, \ldots, \lambda_m\) be points in \(\mathbb{P}^1\) and set \(\Sigma = \{\lambda_1, \ldots, \lambda_m\}\). We choose a base point \(b\) in \(\mathbb{P}^1 - \Sigma\) and a coordinate \(t\) so that the base point \(b\) corresponds to the infinity. Let \((\Gamma, C)\) be a binary tree on \(\mathbb{P}^1\), whose set of terminals is equal to \(\{\lambda_1, \ldots, \lambda_m\}\). We construct a cyclic triple covering \(C^0\) of \(\mathbb{P}^1 - \Sigma\) associated to the binary tree \((\Gamma, C)\). We prepare three copies \(C^{(0)}, C^{(1)}, C^{(2)}\) of \(\mathbb{P}^1 - \Gamma\). We patch them according to the following rule. (See Figure 2.)

1. If a point moves across an edge of \(\Gamma\) looking white vertex left, we change the sheet as \(C^{(0)} \rightarrow C^{(1)} \rightarrow C^{(2)} \rightarrow C^{(0)}\).

2. If a point moves across an edge of \(\Gamma\) looking white vertex right, we change the sheet as \(C^{(0)} \rightarrow C^{(2)} \rightarrow C^{(1)} \rightarrow C^{(0)}\).

**Figure 2.** Patching of sheets

In this way, we get a compact Riemann surface \(C\). Since the tree \(\Gamma\) is trivalent, the covering \(C \rightarrow \mathbb{P}^1\) is not branching at any point \(v\) in \(V(\Gamma)\), and the branching index \(a_i/3\) of \(\lambda_i\) in \(\Sigma\) is equal to 1/3 or 2/3 if the color of the vertex \(\lambda_i\) is white or black, respectively. Then the Riemann surface \(C\) is isomorphic to the compactification of the Kummer covering of \(\mathbb{P}^1\):

\[
y^3 = \prod_{i=1}^{m} (x - \lambda_i)^{a_i}.
\]

The automorphism \(\rho\) defined by \((x, y) \mapsto (x, \omega y)\) corresponds to the change of sheets \(C^{(0)} \rightarrow C^{(1)} \rightarrow C^{(2)} \rightarrow C^{(0)}\).

**Lemma 3.2.** The genus \(g = g(C)\) is equal to the number \(#V(\Gamma)\) of \(V(\Gamma)\).
Proof. Since the tree is trivalent, the number of edges is equal to \((3\#V(\Gamma) + \#\Sigma)/2\). Therefore the number of the vertices is

\[
(3\#V(\Gamma) + \#\Sigma)/2 + 1 = \#V + \#\Sigma.
\]
Therefore we have \(\#V(\Gamma) + 2 = \#\Sigma\) and \(g = \#\Sigma - 2 = \#V(\Gamma)\). \(\square\)

For a marking \(M\) of the binary tree \((\Gamma, C)\), we define a symplectic basis \(\{A_v, B_v\}_{v \in V(\Gamma)}\) as follows. We define cycles \(A_v\) and \(B_v\) as follows. By deleting a small neighborhood of \(v\) from the tree \(\Gamma\), we decompose it into three blocks \(B_1, B_2\) and \(B_3\) as in Figure 3. We define a cycle \(A_v\) by the path in Figure 3.

![Figure 3. Cycles](image)

If the vertex \(v\) is white (resp. black), we define \(B_v\) by \(B_v = \rho^2(A_v)\) (resp. \(B_v = \rho(A_v)\)). It is easy to see the following proposition.

**Proposition 3.3.** The set of cycles \(\{A_v, B_v\}_{v \in V(\Gamma)}\) forms a symplectic basis. That is

\[
(A_v, A_w) = (B_v, B_w) = 0, \quad (A_v, B_w) = -\delta(v, w)
\]
for \(v, w \in V(\Gamma)\).

3.2. Riemann constants. In this subsection, we describe the Riemann constant of the curve \(C\) with the symplectic basis \(\{A_v, B_v\}_{v \in V(\Gamma)}\) constructed in the last subsection. Let \(\tau\) be the normalized period matrix with respect to \(\{A_v, B_w\}_{v \in V(\Gamma)}\) and \(p_0\) be one of the branched points of the covering \(C \to \mathbf{P}^1\). Let \(S^{g-1}(C)\) be the \((g - 1)\)-th symmetric product of \(C\), \(Q = \sum_{i=1}^{g-1} q_i\) be an element in \(S^{g-1}\) and \(\text{sym}_Q^{g-1} : S^{g-1}(C) \to \text{Jac}(X)\) be the map defined by

\[
S^{g-1}(C) \ni \sum_{i=1}^{g-1} p_i \mapsto \sum_{i=1}^{g-1} (p_i) - (q_i) \in \text{Pic}(C).
\]
We assume that \( q_i \) are branch points and \( 2Q = K_C \). We define the theta zero divisor by
\[
(\vartheta(\tau)[0,0](\tau, z))_0 = \{ [z] \in \text{Jac}(X) \mid \vartheta(\tau)[0,0](\tau, z) = 0 \}.
\]
Then by Riemann’s theorem, there is a point \( \varrho(p_0, \{A_i, B_i\}) \) in \( \text{Jac}(X) \) such that
\[
(\vartheta(\tau)[0,0](\tau, z))_0 = \varrho(Q, \{A_i, B_i\}) - \text{sym}^g_{Q}^{-1}(S^{g-1}(C)).
\]
The point \( \varrho \) is called the Riemann constant. By [Mu], I, Cor. 3.1.1, p.166, we have \( 2\varrho(Q, \{A_i, B_i\}) = 0 \) in \( \text{Jac}(C) \).

Let \( \tau' \) be the normalized period matrix with respect to the symplectic basis \( \{\varrho(A_i), \varrho(B_i)\} \). By the theta transformation formula, we have
\[
(\vartheta(\tau)[0,0](\tau, z))_0 - (\vartheta(\tau)[0,0](\tau', z))_0 = m
\]
where
\[
m = \sum_{v \in \text{black vertex}} \frac{1}{2} B_i + \sum_{v \in \text{white vertex}} \frac{1}{2} A_i.
\]
Thus, we have
\[
m = \varrho(Q, \{A_i, B_i\}) - \varrho(Q, \{\varrho(A_i), \varrho(B_i)\}) = (1 - \rho)\varrho(Q, \{A_i, B_i\}).
\]
Therefore in the group of two torsion points in \( \text{Jac}(X) \), we have
\[
\varrho(Q, \{A_i, B_i\}) = (1 - \rho^2)m = \sum_{v} \frac{1}{2}(A_i + B_i).
\]

3.3. Description of \((1 - \rho)\) torsion cycles. We use the same notation as in the last subsection. We fix a numbering \( \lambda_1, \ldots, \lambda_m \) compatible with the cyclic ordering on the set of the terminals of the binary tree \( \Gamma \). Let \( \gamma_i \) be a loop in \( \mathbb{P}^1 - \Sigma \) starting the base point \( b \in \mathbb{P}^1 \) which turns around \( \lambda_i \) once anticlockwise. The point in \( C^{(i)} \) over the point \( b \) is denoted by \( b(i) \) for \( i = 0, 1, 2 \). The lift of \( \gamma_i \) starting from \( b(1) \) (resp. \( b(2) \)) is denoted by \( \gamma_i \) if the color of the vertex \( \lambda_i \) is white (resp. black). Then the path \( \gamma_i \) end at the point \( b(2) \) (resp. \( b(1) \)) if the color of the vertex \( \lambda_i \) is white (resp. black).

![Figure 4. Path \( \gamma_i \)](image)

Then by choosing suitable integers \( k_1, \ldots, k_m \), we have a relation
\[
(3.1) \quad \rho^{k_1}\gamma_1 + \rho^{k_2}\gamma_2 + \cdots + \rho^{k_m}\gamma_m = 0
\]
in the group \( H_1(C, \mathbb{Z}) \).
The sign $\epsilon_i$ of a terminal $\lambda_i$ is assigned as 1 or $-1$ if its color is white or black, respectively. Then the boundary of the chain $\epsilon_i \gamma_i$ is equal to $b(2) - b(1)$. We define cycles $\delta_1, \ldots, \delta_{m-1}$ by

$$\delta_i = \epsilon_i \gamma_i - \epsilon_m \gamma_m.$$ 

By the equation (3.1), we have

$$\rho^{k_1} \epsilon_1 \delta_1 + \rho^{k_2} \epsilon_2 \delta_2 + \cdots + \rho^{k_{m-1}} \epsilon_{m-1} \delta_{m-1} = 0.$$ 

The element in $H_1(C, Q/Z)_{(1-\rho)}$ represented by $\frac{1}{3} (1 - \rho) \delta_i$ is denoted by $\overline{\delta_i}$. Here we give a combinatorial description of the group $H_1(C, Q/Z)_{(1-\rho)}$. Let $V$ be the $F_3$-vector space $\oplus_{i=1}^m e_i F_3$ generated by $e_1, \ldots, e_m$. We define homomorphisms $\Pi$ and $\text{Diag}$ by

$$\Pi : V \ni \sum_i k_i e_i \mapsto \sum_i \epsilon_i k_i \in F_3,$$

$$\text{Diag} : F_3 \ni a \mapsto a \sum_i e_i \in V.$$ 

Then the set $\{ \epsilon_i e_i - \epsilon_m e_m \}_{i=1, \ldots, m-1}$ is a basis of $\text{Ker} (\Pi)$. By assigning the class of $\overline{\epsilon_i e_i - \epsilon_m e_m}$ to $\epsilon_i e_i - \epsilon_m e_m$, we have a map $\text{Ker} (\Pi) \to H^1(C, Q/Z)_{(1-\rho)}$ and as a consequence, we have an isomorphism

$$(3.4) \quad \text{Ker} (\Pi) / \text{Im} (\text{Diag}) \cong H^1(C, Q/Z)_{(1-\rho)}.$$ 

They are $(m-2)$-dimensional vector spaces.

**Definition 3.4.** The quotient space $\text{Ker} (\Pi) / \text{Im} (\text{Diag})$ is obtained from the information of the binary tree $(\Gamma, C)$. It is denoted by $H(\Gamma, C)$.

**3.4. Combinatorial computation of $\frac{1}{3} - \rho A_v$.** In this subsection, we give a combinatorial rule to compute the class $\overline{A_v}$ of $\frac{1}{3} - \rho A_v$ in $H_1(C, Q/Z)_{(1-\rho)}$. We illustrate an element $\Lambda = \sum_i k_i e_i$ by writing $k_i$ to each terminal $\lambda_i$ in the tree $\Gamma$. (For example, see Figure 5)

First, we consider the simplest case: $m = 3$ and the color of $\lambda_i$ is white for $i = 1, 2, 3$. There is only one inner black vertex. We assume that $\lambda_3$ is marked. Then we have $A_v = \gamma_2 - \gamma_1$ as in Figure 5. Via the isomorphism

$$\text{Figure 5. Simplest case}$$

(3.4), we have $\overline{A_v} = e_2 - e_1$. Similarly we can compute the image of $\overline{A_v}$.
in the case: $m = 3$ and the color of $\lambda_i$ is black. In this case, we also have $A_v = e_2 - e_1$. Note that the class of $\rho\left(\frac{1-\rho}{3}A_v\right)$ is equal to $A_v$ in each case.

Next, we consider a general situation. Let $v$ be a black vertex connected to three blocks $B_1, B_2, B_3$. (A block may be one point.) Recall that the cycle $A_v$ is given in Figure 6. We reduce the computation $A_v$ to the following

$\gamma$ be a path starting from $b(2)$ ending at $b(1)$ turning around $C_1$ and $C_2$ as in Figure 7. Note that the path $\gamma$ is homotopic to $\rho(\gamma_{C_1}) + \rho^2(\gamma_{C_2})$, where $\gamma_{C_i}$ is a path turning around the block $C_i$ anti-clockwise. Similarly, if $p$ is a white vertex, $\gamma$ is homotopic to $\rho^2(\gamma_{C_1}) + \rho(\gamma_{C_2})$. By this local deformation rule,

the coefficient of $e_i$ in $A_v$ is equal to $-1, 1$ and $0$ if the terminal $\lambda_i$ belongs to $B_1, B_2$ and $B_3$, respectively.

Finally, we have the following proposition.

**Proposition 3.5.** (1) The class $A_v \in H_1(C, \mathbb{Q}/\mathbb{Z})_{(1-\rho)}$ of $\frac{1}{3}(1-\rho)A_v$ is equal to the element

$$- \sum_{\lambda_i \in B_1 \cap \Sigma} e_i + \sum_{\lambda_i \in B_2 \cap \Sigma} e_i = - \sum_{\lambda_i \in B_2 \cap \Sigma} e_i + \sum_{\lambda_i \in B_1 \cap \Sigma} e_i = - \sum_{\lambda_i \in B_3 \cap \Sigma} e_i + \sum_{\lambda_i \in B_1 \cap \Sigma} e_i$$

via the isomorphism $(3,4)$. 

\[ \text{Figure 6. General case} \]

\[ \text{Figure 7. Local deformation} \]
(2) If $v$ is a black vertex (resp. a white vertex), then $\overline{A_v} \equiv \frac{1}{3}(-A_v + B_v)$ \mod $H_1(C, \mathbb{Z})$ (resp. $\overline{A_v} \equiv \frac{1}{3}(A_v - B_v)$ \mod $H_1(C, \mathbb{Z})$).

(3) The set $\{\overline{A_v}\}_{v \in V(\Gamma)}$ forms a basis of $H_1(C, \mathbb{Q}/\mathbb{Z})(1-\rho)$.

3.5. A decomposition of a marked binary tree. In this subsection, we consider a decomposition of a marked tree. Let $(\Gamma, C, M)$ be a marked binary tree and $E$ be its edge. Let $p$ and $q$ be white and black vertices adjacent to $E$, respectively. Let $C_p$ and $C_q$ be two blocks containing $p$ and $q$, respectively. We define subspaces $H_p$ and $H_q$ of $H(\Gamma, C)$ by

$$H_p = \{ \sum_{v \in \Sigma} a_v e_v \in H(\Gamma, C) \mid a_v = a_{v'} \text{ for } v, v' \in C_q \cap \Sigma \},$$

$$H_q = \{ \sum_{v \in \Sigma} a_v e_v \in H(\Gamma, C) \mid a_v = a_{v'} \text{ for } v, v' \in C_p \cap \Sigma \}.$$

We remark that the condition in the definition of $H_p$ and $H_q$ does not depend on the representative modulo $Im(\text{Diag})$. By cutting the edge $E$ of tree and inserting black and white vertices to $C_p$ and $C_q$, we get marked binary trees $(\Gamma_p, C_p, M_p)$ and $(\Gamma_q, C_q, M_q)$ as in the following picture.

![Figure 8. Decomposition of a marked binary tree](image)

It is easy to see the following lemma.

**Lemma 3.6.**

1. The element $\overline{A_v}$ belongs to the space $H_p$ and $H_q$ if $v$ belongs to $C_p$ and $C_q$, respectively.

2. The space $H_1(C, \mathbb{Q}/\mathbb{Z})(1-\rho)$ is the direct sum of $H_p$ and $H_q$. The sets $\{\overline{A_v}\}_{v \in V(\Gamma_p)}$ and $\{\overline{A_v}\}_{v \in V(\Gamma_q)}$ form bases of $H_p$ and $H_q$, respectively.

3. The vector spaces $H_p$ and $H_q$ are naturally isomorphic to $H(\Gamma_p, C_p)$ and $H(\Gamma_q, C_q)$, respectively.

4. Stable degeneration of triple coverings of $\mathbb{P}^1$

In this section, we compare a decomposition of a marked binary tree and a stable degeneration of triple coverings of $\mathbb{P}^1$.

Let $\lambda_1, \ldots, \lambda_m$ be points in $\mathbb{C} - \{0\}$. Let $(\Gamma, C, M)$ be a marked binary tree whose set of terminals is $\{\lambda_1, \ldots, \lambda_m\}$. The branching indices of $\lambda_1, \ldots, \lambda_m$ are written as $a_1/3, \ldots, a_m/3$. We choose an edge $E$ and obtain two blocks $C_p, C_q$ by cutting $E$ as in the last section. By renumbering the set of terminals, we can assume that $C_p \cap \Sigma = \{\lambda_1, \ldots, \lambda_l\}$ and $C_q \cap \Sigma = \{\lambda_{l+1}, \ldots, \lambda_m\}$ and that the cyclic order of the terminals is equal to $\lambda_1, \ldots, \lambda_m$. We consider a deformation of the set of terminals parametrized by $t$ as

$$\lambda_i(t) = t\lambda_i \quad (i = 1, \ldots, l), \quad \lambda_i(t) = \lambda_i \quad (i = l + 1, \ldots, m).$$
Suppose that \( \lambda_1, \ldots, \lambda_l \) are sufficiently close to 0. We consider a family of triple coverings \( C(t) \) of \( \mathbb{P}^1 \) parametrized by \( t \in \Delta^*(t) = \{ t \in \mathbb{C} \mid 0 < |t| < 1 \} \) as

\[
y^3 = \prod_{i=1}^{m} (x - \lambda_i(t))^{a_i}.
\]

This family extends to a stable model by the base change \( \varepsilon^3 = t \). We write down an affine birational model as follows. Let \( X \) be a submanifold of \( \mathbb{C}^1 \times C^1 \times \Delta(x(\varepsilon)) \) defined by

\[
\mathcal{X} = \{ (x, \xi, \varepsilon) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \Delta \mid x\xi = \varepsilon^3 \}.
\]

Let \( a, b \) be elements in \( \{1, 2\} \) satisfying \( \sum_{i=1}^{l} a_i \equiv a \pmod{3} \) and \( \sum_{i=l+1}^{m} a_i \equiv b \pmod{3} \). Let \( Y \) be the triple covering of \( \mathcal{X} \) defined by new coordinates \( y_1, y_2 \) satisfying

\[
y_1^3 = x^a \prod_{i=1}^{l} (1 - \lambda_i \xi)^{a_i} \prod_{i=l+1}^{m} (x - \lambda_i)^{a_i},
\]

\[
y_2^3 = \xi^b \prod_{i=1}^{l} (1 - \lambda_i \xi)^{a_i} \prod_{i=l+1}^{m} (x - \lambda_i)^{a_i},
\]

with \( \xi y_1 = \varepsilon^a y_2, xy_2 = \varepsilon^b y_1 \). Then \( \varphi : Y \to \Delta(x(\varepsilon)) \) is a family of affine curves over \( \Delta(x(\varepsilon)) \). The fiber of \( Y \) at the origin \( \tau = 0 \) is the union of two triple coverings of \( \mathbb{P}^1 \) connected at one point. One component is the restriction of (4.2) to \( \xi = 0 \) and the other is that of (4.3) to \( x = 0 \).

We define a family of marked binary trees \( (\Gamma_p, C_p, M_p)(t) \prod_{q} (\Gamma_q, C_q, M_q)(t) \) with two components parametrized by \( t \). We set \( (\Gamma_q, C_q, M_q)(t) \) as a constant family, since \( C_q \cap \Sigma = \{ \lambda_{l+1}, \ldots, \lambda_m \} \). We set the tree \( \Gamma_p(t) \) as the multiplication of the original tree by \( t \) for \( t \in \Delta^* \) since \( C_p \cap \Sigma = \{ \lambda_1, \ldots, \lambda_l \} \). Its coloring and marking are naturally defined by the multiplication by \( t \). By using the coordinate \( \xi \), we see that this tree is continued to \( \varepsilon = 0 \).

**Proposition 4.1.** The monodromy around \( \varepsilon = 0 \) acts trivially on the family of the relative homology \( R_1(\varphi_p)\mathbb{Z} \).

**Proof.** For a point \( t = 1 \), the symplectic basis \( \{ A_v, B_v \}_{v \in V(\Gamma)} \) associated to \( (\Gamma, C, M) \) is equal to the union of those associated to \( (\Gamma_p, C_p, M_p) \) and \( (\Gamma_q, C_q, M_q) \). They are extended continuously to \( \varepsilon = 0 \). \( \Box \)

**Remark 4.2.** Repeating this degeneration process, any triple covering of \( \mathbb{P}^1 \) degenerates to the union of elliptic curves \( E_v \) (\( v \in V(\Gamma) \)) with complex multiplication with \( \mathbb{Z}[\omega] \), which is called a totally degenerated curve. The binary tree is equal to the dual graph of the totally degenerated curve. The color of an inner vertex \( v \) corresponds to the eigen value of the action of \( \rho \) on \( H^0(E_v, \Omega^1) \).
5. DEGENERATIONS OF PERIOD INTEGRALS AND THEIR DETERMINANTS

5.1. Deformation of period integrals. In this section, we study degenerations of period integrals for the simplest case. We use the same notations $\lambda_1, \ldots, \lambda_m$ for branch points, $a_1/3, \ldots, a_m/3$ for branching indices and a marked binary tree $(\Gamma, C, M)$. Therefore the color of a vertex $\lambda_i$ is white if and only if $a_i = 1$. In this section, we assume that $\lambda_i$ and $\lambda_{i+1}$ is adjacent to a common inner vertex $p$. Thus the colors of $\lambda_i$ and $\lambda_{i+1}$ are same. The third vertex adjacent to $p$ is denoted by $q$. Moreover we assume that the edge to $q$ is marked for the vertex $p$. If the vertices $\lambda_i$ and $\lambda_{i+1}$ are white, the situation is illustrated in Figure 9.

![Simple degeneration](image)

**Figure 9. Simple degeneration**

Let $\{A_v, B_v\}$ be the symplectic basis of $H_1(C, \mathbb{Z})$. Let $\tilde{\lambda}$ be a point different from $\lambda_j$ for $j \neq i, i + 1$ and consider the family of points $\lambda_j(t)$ defined by $\lambda_j(t) = \lambda_j$ for $j \neq i, i + 1$ and $\lambda_k(t) = \tilde{\lambda} + t(\lambda_k - \tilde{\lambda})$ for $k = i, i + 1$. We consider the family of curves branched at $\Sigma(t) = \{\lambda_1(t), \ldots, \lambda_m(t)\}$. By changing parameter $t$ to $\varepsilon$ defined as $t = \varepsilon^3$, we get a stable degeneration associated to the above decomposition of the binary tree as in the last section. The tree containing the vertex $q$ obtained by this decomposition is denoted by $(\Gamma', C', M')$.

Let $y_1$ and $y_2$ be rational functions on $C$ defined by

$$y_1^2 = \prod_{j=1}^{m} (x - \lambda_j(t))^{a_j},$$

$$y_2^2 = \prod_{j=1}^{m} (x - \lambda_j(t))^{b_j},$$

where $b_j = 3 - a_j$. Let $d_1 = \sum_j a_j - 1$ and $d_2 = \sum_j b_j - 1$. We define a family of differential forms $\eta_j(t)$ for $j = 1, \ldots, d_1 + d_2$ by

$$\eta_j = \frac{x^{j-1}dx}{y_1} \quad \text{for } j = 1, \ldots, d_1,$$

$$\eta_{j+d_1} = \frac{x^{j-1}dx}{y_2} \quad \text{for } j = 1, \ldots, d_2.$$

Then $\{\eta_1, \ldots, \eta_{d_1+d_2}\}$ is a basis of $H^0(X, \Omega^1)$.
Proposition 5.1. Assume that the colors of $\lambda_i$ and $\lambda_{i+1}$ are white. We set

$$\overline{\eta}_{j+d_1} = \frac{(x - \tilde{\lambda})^{j-1}dx}{y_1} \quad \text{for } j = 1, \ldots, d_2.$$ 

(1) We have

$$\lim_{t \to 0} \int_{B_p} \eta_j(t) = 0 \quad \text{for } j = 1, \ldots, d_1,$$

$$\lim_{t \to 0} \int_{B_p} \overline{\eta}_j(t) = 0 \quad \text{for } j = d_1 + 2, \ldots, d_1 + d_2,$$

and

$$\lim_{t \to 0} t \cdot (\lambda_i - \lambda_{i+1}) \left( \int_{B_p} \overline{\eta}_{d_1+1}(t) \right)^3 = B^*(\frac{1}{3}, \frac{1}{3})^3 \prod_{k \neq i, i+1} (\tilde{\lambda} - \lambda_k)^{-b_k}.$$ 

Here we set $B^*(\frac{1}{3}, \frac{1}{3}) = (\omega - 1)B(\frac{1}{3}, \frac{1}{3})$.

(2) The limit

$$\lim_{t \to 0} \int_{B_v} \eta_{d_1+1}(t)$$

is finite for $v \neq p$.

(3) We set

$$\overline{\eta}_j = \frac{x^{j-1}dx}{y_1} \quad \text{for } j = 1, \ldots, d_1,$$

$$\overline{\eta}_{j+d_1} = \frac{(x - \tilde{\lambda})^{j-1}dx}{y_2} \quad \text{for } j = 1, \ldots, d_2 - 1,$$

where

$$\overline{y}_1^3 = (x - \tilde{\lambda})^2 \prod_{k \neq i, i+1} (x - \lambda_k)^{a_k},$$

$$\overline{y}_2^3 = (x - \tilde{\lambda}) \prod_{k \neq i, i+1} (x - \lambda_k)^{b_k}.$$ 

Then we have

$$\lim_{t \to 0} \int_{B_v} \eta_j(t) = \int_{B_v} \overline{\eta}_j \quad \text{for } j = 1, \ldots, d_1,$$

$$\lim_{t \to 0} \int_{B_v} \overline{\eta}_j(t) = \int_{B_v} \overline{\eta}_{j-1} \quad \text{for } j = d_1 + 2, \ldots, d_1 + d_2,$$

for $v \neq p$.

Proof. We prove (1). We compute the integral

$$\int_{B_p} \frac{dx}{y_2(t)} = \omega^i (1 - \omega) \int_{\lambda_i(t)}^{\lambda_{i+1}(t)} \frac{dx}{(x - \lambda_i(t))^{2/3} (x - \lambda_{i+1}(t))^{2/3} \prod_{k \neq i, i+1} (x - \lambda_k)^{b_k/3}}$$
by the variable change $x = \tilde{\lambda} + t(\xi - \tilde{\lambda})$. Since $x - \lambda_i(t) = t(\xi - \lambda_i), \xi$ varies
$
\lambda_i < \xi < \lambda_{i+1}$. Using $x - \lambda_k = \tilde{\lambda} - \lambda_k + t(\xi - \tilde{\lambda})$, we have

$$
\lim_{t \to 0} t^{1/3} \int_{B_p} \frac{dx}{y_2(t)} = \frac{\omega^i (1 - \omega)}{\prod_{k \neq i, i+1} (\lambda - \lambda_k)^{b_k/3}} \int_{\lambda_i}^{\lambda_{i+1}} \frac{d\xi}{(\xi - \lambda_i)^{2/3}(\xi - \lambda_{i+1})^{2/3}}
$$

$$
= \frac{-\omega'' (1 - \omega)}{(\lambda_i - \lambda_{i+1})^{1/3} \prod_{k \neq i, i+1} (\lambda - \lambda_k)^{b_k/3}} B(\frac{1}{3}, \frac{1}{3}).
$$

We can similarly prove the rests. \square

5.2. Deformation of period matrices. We define the period matrices $P_A = P_A(\Sigma, \Gamma, C, M)$ and $P_B = P_B(\Sigma, \Gamma, C, M)$ associated to the configuration of points $\Sigma = \{\lambda_1, \ldots, \lambda_m\}$ and a marked binary tree $(\Gamma, C, M)$ by

$$
P_A(\Sigma, \Gamma, C, M) = (\int_{A_v} \eta_1 \cdots \int_{A_v} \eta_{d_1 + d_2 - 2})_{v \in V(\Gamma)},
$$

$$
P_B(\Sigma, \Gamma, C, M) = (\int_{B_v} \eta_1 \cdots \int_{B_v} \eta_{d_1 + d_2 - 2})_{v \in V(\Gamma)}.
$$

We consider a family of configuration $\Sigma(t)$ defined in the last subsection. Then we have a family of period matrices $P_B(\Sigma(t), \Gamma, C, M)$. We set

$$
\Sigma' = \{\lambda_1, \ldots, \lambda_{i-1}, \tilde{\lambda}, \lambda_{i+2}, \ldots, \lambda_m\}.
$$

Then we have the following proposition.

Proposition 5.2. We have

$$
\lim_{t \to 0} t(\lambda_i - \lambda_{i+1}) \det(P_B(\Sigma(t), \Gamma, C, M))^3
$$

$$
= B^* \left(\frac{1}{3}, \frac{1}{3}\right)^3 \prod_{k \neq i, i+1} (\lambda - \lambda_k)^{-b_k} \cdot \det(P_B(\Sigma', \Gamma', C', M'))^3,
$$

where the marked binary tree $(\Gamma', C', M')$ is obtained by the decomposition of the marked binary tree of $(\Gamma, C, M)$ illustrated in Figure 2.

5.3. Equi-distributed characteristics and theta constants. Let $\lambda_1, \ldots, \lambda_m$ be a configuration of points in $\mathbb{P}^1, a_1/3, \ldots, a_m/3$ be the branching index of $\lambda_1, \ldots, \lambda_m$ and $(\Gamma, C, M)$ be a marked binary tree as in the beginning of this section. Let $\Sigma$ and $\overline{\Sigma}$ be sets given by

$$
\Sigma = \{\lambda_i \mid a_i = 1\}, \quad \overline{\Sigma} = \{\lambda_i \mid a_i = 2\}.
$$

Let $\Lambda = \sum_i k_ie_i$ be an element in $\oplus_{i=1}^m \mathbb{F}_3 e_i$. We set $\Lambda_i, \overline{\Lambda}_i$ for $i = 0, 1, 2$ by

$$
\Lambda_i = \{\lambda_j \in \Sigma \mid k_j = i\}, \quad \overline{\Lambda}_i = \{\lambda_j \in \overline{\Sigma} \mid k_j = i\}.
$$

The element $\Lambda$ is in the kernel of $\Pi : \oplus_{i=1}^m \mathbb{F}_3 e_i \to \mathbb{F}_3$ defined in (3.3) if and only if

$$
\#\Lambda_1 - \#\overline{\Lambda}_1 + 2(\#\Lambda_2 - \#\overline{\Lambda}_2) \equiv 0 \pmod{3}.
$$

An element $\Lambda$ is said to be equi-distributed if and only if

$$
\#\Lambda_0 - \#\overline{\Lambda}_0 = \#\Lambda_1 - \#\overline{\Lambda}_1 = \#\Lambda_2 - \#\overline{\Lambda}_2.
$$
By (5.1), an equi-distributed element $\Lambda$ gives an element in $H_1(C, \mathbb{Q}/\mathbb{Z})_{(1-\rho)}$. If the colors of all terminals are white, an element $\Lambda \in V$ is equi-distributed if and only if it satisfies the condition (2.3).

We consider the decomposition of the marked binary tree as in Figure [9]. For an element $\Lambda = \sum k_j e_j \in H(\Gamma, C)$, we define an element $\Lambda' \in H(\Gamma', C')$ by $\Lambda' = -(k_i + k_{i+1}) e_\lambda + \sum_{j \neq i, i+1} k_j e_j$. The following lemma is easy to see.

**Lemma 5.3.** Let $\Lambda$ be an equi-distributed element in $\bigoplus_i \mathbb{F}_3 e_i$. Suppose that $k_i \neq k_{i+1}$. Then $\Lambda'$ is also equi-distributed.

We consider the limit of the normalized period matrix. The fiber of the stable curve at $\varepsilon = 0$ becomes the union of $C_1$ and $C_2$, where

$C_1 : y^3 = (x - \lambda_i)(x - \lambda_{i+1})$,

$C_2 : \eta^3 = (\xi - \tilde{\lambda})^2 \prod_{j \neq i, i+1} (\xi - \lambda_j)^{a_i}$,

as in the last section. Since the variation of Hodge structure is smooth, we can compute the limit by considering logarithmic differentials. We consider a continuous extension of the symplectic basis $\{A_v, B_v\}_{v \in V(\Gamma)}$. Then $\{A_p, B_p\}$ is a symplectic basis of $C_1$ and $\{A_v, B_v\}_{v \in V(\Gamma')}$ is that of $C_2$ in Figure [9]. Let $\tau_i$ be the normalized period matrix of $C_i$ for $i = 1, 2$. Then we have

$$\lim_{\varepsilon \to 0} \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.$$ 

Under the above notations, we have the following proposition, by the computation of Riemann constants in §3.2.

**Proposition 5.4.** Let $\varrho, \varrho_1$ and $\varrho_2$ be the Riemann constants for symplectic bases $\{A_v, B_v\}_{v \in V(\Gamma)}$, $\{A_p, B_p\}$ and $\{A_v, B_v\}_{v \in V(\Gamma')}$. Then we have

$$\lim_{\varepsilon \to 0} \vartheta(\tau(\varepsilon)) [\Lambda + \varrho]^{6} = \vartheta(\tau_1) \left[\frac{1-\varrho}{3} A_p + \varrho_1\right]^{6} \cdot \vartheta(\tau_2) [\Lambda' + \varrho_2]^{6}.$$

Here, we use the identification (2.2) and the remark just after it. Moreover, we have

$$\vartheta(\tau_1) \left[\frac{1-\varrho}{3} A_p + \varrho_1\right]^{6} = \frac{3^{9/4}}{(2\pi)^6} \Gamma(1/3)^9 \exp\left(-\frac{5\pi i}{12}\right).$$

**Proof.** As for the last statement, we use the formula (c.f. [MTY]):

$$\vartheta(\omega) \left[\frac{1}{6}, \frac{-1}{6}\right]^{6} = \frac{3^{9/4}}{(2\pi)^6} \Gamma(1/3)^9 \exp\left(-\frac{5\pi i}{12}\right).$$

\[\square\]

6. **Thomae’s formula and Thomae’s constant for arbitrary branching indices**

Now we are ready to give the statement of Thomae’s theorem for arbitrary indices. By the inductive structure, we also evaluate the constants appeared in Thomae’s theorem.
Let $C \to \mathbb{P}^1$ be the cyclic triple covering branching at a set $\Sigma = \{\lambda_1, \ldots, \lambda_m\}$ of $\mathbb{P}^1$ and $a_1/3, \ldots, a_m/3$ be branching indices at $\lambda_1, \ldots, \lambda_m$, respectively. Let $(\Gamma, C, M)$ be a marked binary tree whose set of terminals is $\Sigma$, $\{A_v, B_v\}$ be the associated symplectic basis, and $\Lambda$ be an equi-distributed element in $H(\Gamma, C)$.

We set $\Lambda_1, \overline{\Lambda}_1$ as in the last section.

**Definition 6.1.**

1. Let $A_1, A_2$ be $\Lambda_i$ or $\overline{\Lambda}_i$ ($i = 0, 1, 2$) and $A_1 \neq A_2$.
   We define $(A_1, A_2)$ by
   
   $$(A_1, A_2) = \prod_{\lambda \in A_1, \mu \in A_2} (\lambda - \mu),$$

   and $(A_1, A_1)$ by

   $$(A_1, A_1) = \prod_{i < j, \lambda_i, \lambda_j \in A_1} (\lambda_i - \lambda_j).$$

2. The difference product $\Delta(\Sigma, \Lambda)$ of an equi-distributed characteristic $\Lambda$ is defined by

   $$\Delta(\Sigma, \Lambda) = \prod_{i=0}^{2} (\Lambda_i \Lambda_{i+1})^3 \cdot \prod_{0 \leq i < j \leq 2} (\Lambda_i \Lambda_j) \cdot \prod_{0 \leq i \neq j \leq 2} (\Lambda_i \overline{\Lambda}_j)^2.$$  

**Theorem 6.2.**

1. Under the above notations, we have

   $$\vartheta(\tau) \left[ \Lambda + \varrho \right]^6 = \pm \kappa_{\Lambda} \cdot \Delta(\Sigma, \Lambda) \cdot \det(P_B(\Sigma, \Gamma, C, M))^3,$$

   where $\kappa_{\Lambda}$ is an absolute constant depending only on $\Lambda$. Moreover $\kappa_{\Lambda}^6$ does not depend on the choice of $\Lambda$.

2. The constant $\kappa_{\Lambda}^6$ is equal to $\kappa^6(v_1 + v_2)$, where

   $$(6.1) \quad \kappa = ((2\pi)^{3}3^{3/4} \exp(\frac{11\pi i}{12}))^{-1}$$

   and $v_1$ and $v_2$ are the numbers of white and black inner vertices, respectively.

Let us prove the above theorem. We consider the degeneration studied in Section 5. We use the notations $\Gamma', C', \Sigma'$ and $\Lambda'$ as in the last section. Let $\Lambda = \sum_i k_i e_i$ be an equi-distributed element in $H(\Gamma, C)$. We prove the case: $k_i = 2, k_{i+1} = 1$ and $a_i = a_{i+1} = 1$. The other cases can be similarly proved.

We prepare Lemma 6.3 and Proposition 6.4.

**Lemma 6.3.** We have

$$\lim_{\varepsilon \to 0} \left[ \frac{\Delta(\Sigma(\varepsilon), \Lambda)}{t(\lambda_i - \lambda_{i+1})} \right] = \Delta(\Sigma', \Lambda').$$

**Proof.** We consider a family of the difference product $(\Lambda_i(t), \Lambda_j(t))$ and so on.

Then we have

$$\Lambda'_2 = \Lambda_2 - \{\lambda_i\}, \quad \Lambda'_1 = \Lambda_1 - \{\lambda_{i+1}\}, \quad \overline{\Lambda}'_0 = \overline{\Lambda}_0 \cup \{\overline{\lambda}\}.$$
Therefore, we have
\[
\lim_{t \to 0} (\Lambda_i(t)\Lambda_i(t))^3 = (\Lambda'_i\Lambda'_i)^3(\Lambda'_i, \lambda)^3 \quad i = 1, 2,
\]
\[
\lim_{t \to 0} (\Lambda_0(t)\Lambda_i(t)) = (\Lambda_0\Lambda'_i)(\Lambda_0, \lambda) \quad i = 1, 2,
\]
\[
\lim_{t \to 0} \frac{1}{t} (\Lambda_1(t)\Lambda_2(t)) = (\Lambda'_1\Lambda'_2)(\Lambda'_1, \lambda)(\Lambda'_2, \lambda)(\lambda_i - \lambda_{i+1}),
\]
\[
\lim_{t \to 0} (\Lambda_i(t)\overline{\Lambda}_0(t))^2 = (\Lambda'_i\overline{\Lambda}_0)^2(\Lambda'_i, \lambda)^{-2}(\lambda\overline{\Lambda}_0)^2 \quad i = 1, 2,
\]
\[
\lim_{t \to 0} (\Lambda_i(t)\overline{\Lambda}_j(t))^2 = (\Lambda'_i\overline{\Lambda}_j)^2(\lambda\overline{\Lambda}_j)^2 \quad (i, j) = (1, 2) \text{ or } (2, 1),
\]
\[
\lim_{t \to 0} (\overline{\Lambda}_0(t)\overline{\Lambda}_0(t))^3 = (\overline{\Lambda}_0\overline{\Lambda}_0)^3(\lambda\overline{\Lambda}_0)^{-3},
\]
\[
\lim_{t \to 0} (\overline{\Lambda}_0(t)\overline{\Lambda}_i(t)) = (\overline{\Lambda}_0\overline{\Lambda}_i)(\lambda\overline{\Lambda}_i)^{-1} \quad i = 1, 2.
\]
Since
\[
\prod_{k \neq i, i+1} (\lambda - \lambda_i)^{b_i} = (\Lambda'_1, \lambda)^2(\Lambda'_2, \lambda)^2(\lambda_0, \lambda)^2(\lambda_1, \lambda)(\lambda_2, \lambda),
\]
we have the lemma. \qed

**Proposition 6.4.** (1) The statement (1) of Theorem 6.2 for \((\Gamma, C, M)\) implies the same statement for \((\Gamma', C', M')\).

(2) Suppose that the statement (1) of Theorem 6.2 holds for \(\Gamma\) and \(\Gamma'\). Then we have the recursive relation
\[
\kappa_A = \pm \kappa_A' \kappa,
\]
where \(\kappa\) is defined in (6.1)

**Proof.** (1) We assume that the statement holds for the set of terminals \(\Sigma(\epsilon)\) of \((\Gamma, C, M)\) with \(\epsilon \neq 0\). We consider the following limit:
\[
(6.2) \quad \vartheta(\tau_1)[\Lambda_1 + \varrho_1]^6 \vartheta(\tau')[\Lambda'_1 + \varrho'_1]^6
\]
\[
= \lim_{\epsilon \to 0} \vartheta(\tau(\epsilon))[\Lambda + \varrho]^6
\]
\[
= \pm \kappa_A \lim_{\epsilon \to 0} \left[ \Delta(\Sigma(\epsilon), \Lambda) \cdot \det(P_B(\Sigma(\epsilon), \Gamma, C, M))^3 \right]
\]
\[
= \pm \kappa_A \lim_{\epsilon \to 0} \left[ t(\lambda_i - \lambda_{i+1}) \det(P_B(\Sigma(\epsilon), \Gamma, C, M))^3 \right] \cdot \lim_{\epsilon \to 0} \left[ \frac{\Delta(\Sigma(\epsilon), \Lambda)}{t(\lambda_i - \lambda_{i+1})} \right]
\]
\[
= \pm B^s(\frac{1}{3}, \frac{1}{3})^3 \kappa_A \cdot \lim_{\epsilon \to 0} \left[ \frac{\Delta(\Sigma(\epsilon), \Lambda)}{t(\lambda_i - \lambda_{i+1})} \right] \cdot \det(P_B(\Sigma', \Gamma', C', M'))^3.
\]
By Lemma 6.3 and the equality (6.2), we have
\[
\vartheta(\tau')[\Lambda']^6 = \pm \kappa_A' \Delta(\Sigma', \Lambda') \cdot \det(P_B(\Sigma', \Gamma', C', M'))^3,
\]
where
\[
\kappa_{\Lambda'} = \pm \vartheta(\tau_1)[\Lambda_1]^{-6}B'\left(\frac{1}{3}, \frac{1}{3}\right)^3\kappa_{\Lambda}
\]
\[
= \pm \frac{(2\pi)^6}{3^{9/4}} \exp\left(\frac{5\pi i}{12}(\omega - 1)^3\right) \frac{3^{3/2}}{(2\pi)^3}\kappa_{\Lambda}
\]
\[
= \pm (2\pi)^3 3^{3/4} \exp\left(\frac{11\pi i}{12}\right)\kappa_{\Lambda}.
\]

\[\square\]

Proof of Theorem 6.2. We consider the case there exists a procedure of degenerations
\[
(6.3) \quad \Lambda \to \Lambda_1 + \Lambda^{(1)} \to \Lambda_1 + \Lambda_2 + \Lambda^{(2)} \to \cdots \to \Lambda_1 + \cdots + \Lambda_g,
\]
with \(\kappa_{\Lambda_i} \neq 0\). In this case, by applying Proposition 6.4, we have the theorem.

If all terminals are white, since we know that \(\kappa_{\Lambda_i}\) is independent of the choice of \(\Lambda\), we have the theorem once we know the existence of \(\Lambda\) and the procedure of degenerations \((6.3)\). Actually we can choose such \(\Lambda\) and procedure \((6.3)\).

Now we prove the general case. For any marked binary graph and any equi-distributed element \(\Lambda\), we can choose procedure
\[
\overline{\Lambda} \to \Lambda_1 + \Lambda^{(1)} \to \Lambda_1 + \Lambda_2 + \Lambda^{(2)} \to \cdots \Lambda_1 + \cdots + \Lambda_k + \Lambda,
\]
where \(\kappa_{\Lambda_i} \neq 0\) and \(\overline{\Lambda}\) is an equi-distributed element for a marked binary graph whose terminals are white. Since we know the theorem on \(\overline{\Lambda}\), we have the theorem for \(\Lambda\). \[\square\]

7. An example

We consider the following marked binary tree in Figure 10 and an equi-distributed element \(-e_1 + e_2 + e_3 - e_4 \in H(\Gamma, \mathcal{C})\). We assume that \(\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \in \mathbb{R}\) to fix a branch of the third roots.

\[
\begin{array}{cccc}
-1 & 1 & 1 & -1 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\end{array}
\]

![Figure 10. Example](image)

Then the branches of \(y_1, y_2\) are given in Figure 11. Thus we have \(A = \)
\[(a_{ij}), B = (b_{ij})\] with

\[
a_{11} = \int_{A_1} \frac{dx}{y_1} = (\omega^2 - 1) \int_{\lambda_1}^{\lambda_2} \frac{dx}{3\sqrt{(x - \lambda_1)^2(x - \lambda_2)^2(x - \lambda_3)(x - \lambda_4)}},
\]

\[
a_{21} = \int_{A_2} \frac{dx}{y_1} = (\omega^2 - 1) \int_{\lambda_3}^{\lambda_4} \frac{dx}{3\sqrt{(x - \lambda_1)^2(x - \lambda_2)^2(x - \lambda_3)(x - \lambda_4)}},
\]

\[
a_{12} = \int_{A_2} \frac{dx}{y_2} = (1 - \omega) \int_{\lambda_3}^{\lambda_4} \frac{dx}{3\sqrt{(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^2(x - \lambda_4)^2}},
\]

\[
a_{22} = \int_{A_1} \frac{dx}{y_2} = (1 - \omega) \int_{\lambda_1}^{\lambda_2} \frac{dx}{3\sqrt{(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^2(x - \lambda_4)^2}}.
\]

\[
b_{11} = \int_{B_1} \frac{dx}{y_1} = \omega^2 \int_{A_1} \frac{dx}{y_1}, \quad b_{21} = \int_{B_2} \frac{dx}{y_1} = \omega \int_{A_2} \frac{dx}{y_1},
\]

\[
b_{12} = \int_{B_1} \frac{dx}{y_2} = \omega \int_{A_1} \frac{dx}{y_2}, \quad b_{22} = \int_{B_2} \frac{dx}{y_2} = \omega^2 \int_{A_2} \frac{dx}{y_2}.
\]

The above equi-distributed element is \(A_1 - A_2 = \frac{1}{3}(A_1 + A_2 - B_1 - B_2)\) and we have

\[
\vartheta(\tau)[5/6, 5/6, 1/6, 1/6]^6 = \frac{1}{3\sqrt{3}(2\pi)^6} \det(B)^3 \exp\left(\frac{\pi i}{6}\right) (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)(\lambda_3 - \lambda_1)^2(\lambda_4 - \lambda_2)^2.
\]

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