Lessons from $T^\mu_\mu$ on inflation models I: two-scalar theory and Yukawa theory

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Abstract

We demonstrate two properties of correlation functions of the trace of the energy-momentum tensor $T^\mu_\mu$ in the flat spacetime. One is the decoupling of heavy degrees of freedom; i.e., heavy degrees of freedom leaves no effect for low-energy correlation functions. This is intuitively apparent from the effective field theory point of view, but one has to take into account the so-called trace anomaly to explicitly demonstrate the decoupling. As a result, for example, in the $R^2$ inflation model, scalaron decay is insensitive to heavy degrees of freedom when a matter sector minimally couples to gravity (up to a non-minimal coupling of a matter scalar field other than scalaron). The other property is a quantum contribution to a non-minimal coupling of a scalar field. The non-minimal coupling disappears from the action in the flat spacetime, but leaves the so-called improvement term in $T^\mu_\mu$. We study the renormalization group equation of the non-minimal coupling to discuss its quantum-induced value and implications for inflation dynamics. We work it out in the two-scalar theory and Yukawa theory.
1 Introduction

The energy-momentum tensor $T_{\mu\nu}$ is an important object in quantum field theory [1,2]. It provides generators of spacetime symmetry (Poincaré symmetry in the flat spacetime). It is conserved, $\nabla_\mu T^{\mu\nu} = 0$ ($\nabla_\mu$: diffeomorphism covariant derivative), and finite up to an improvement term. Its trace $T^\mu_\mu$ is divergence of the dilatation current and vanishes when matter respects scale invariance. Moreover, it determines the coupling of matter to gravity.

The aim of this article is to highlight mainly two properties of $T^\mu_\mu$ in the flat spacetime. One is the decoupling of heavy degrees of freedom. Low-energy $T^\mu_\mu$-inserted correlation functions are robustly calculated in terms of effective field theory. The other is the renormalization of an improvement term of $T^\mu_\mu$. In this article, we define the energy-momentum tensor by a functional derivative of the matter action with respect to a metric $g_{\mu\nu}$. The matter action is assumed to be “minimally” coupled to gravity up to a non-minimal coupling of the scalar field to gravity, $\xi R\phi^2$ ($R$: Ricci scalar). The improvement term, $2\eta\partial^2\phi^2$ ($\phi$: scalar field), originates from a non-minimal coupling. Here $\xi = \xi_c + \eta/(d-1)$ with the conformal coupling $\xi_c = (d-2)/(4(d-1))$ in $d$ dimension. We work it out in the two-scalar theory and Yukawa theory.

These properties are relevant when one considers inflation models. The decoupling of heavy degrees of freedom is important when one considers the reheating of $R^2$ inflation (or generically $f(R)$ inflation) [3–8]. In this model, a Weyl degrees of freedom of the metric in the Jordan frame, called scalaron, is identified as inflaton. Through a Weyl transformation one can move to the scalaron frame, where the gravity has the Einstein-Hilbert action (up to a non-minimal coupling of a matter scalar field to $R$) and the scalaron has a canonical kinetic term and couples to matter through $T^\mu_\mu$ [5,16,17]. Scalaron (inflaton) decay is determined by $T^\mu_\mu$-inserted diagrams.

On the other hand, the decoupling of heavy degrees of freedom from low-energy $T^\mu_\mu$-inserted correlation functions is not obvious at first sight. This is because $T^\mu_\mu$ consists of mass terms (classical breaking of scale invariance). Some loop diagrams with a mass term being inserted do not vanish in the heavy mass limit, leaving non-decoupling contributions at low energy. A key is to take into account quantum breaking of scale invariance, known as trace anomaly [1,2,18–27]. The cancellation between classical breaking and quantum breaking is explicitly demonstrated for gauge trace anomaly [28].

The importance of the renormalization of the improvement term is evident in the following viewpoint. Measurements of cosmic microwave background (CMB) anisotropies [30,31] disfavor the chaotic inflation with a simple power-law potential [32] due to the predicted large tensor-to-scalar ratio. The situation gets improved simply with a small non-minimal coupling of $\xi \sim -10^{-3}$ [33,35,4] While one can take any value of $\xi$ at the tree

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1 Our discussion is also applicable to $f(\sigma)R$ inflation ($\sigma$: inflaton) [9,15], when $\sigma$ does not couple to matter in the Jordan frame for some reason.

2 The trace of energy-momentum tensor and trace anomaly are often not distinguished. In this paper we use the former to refer to the whole (classical + quantum) contribution, while we use the latter to refer to only a quantum contribution.

3 Ref. [28] works in $R^2$ inflation to be concrete, while it is applicable to a broad class of scalar-tensor gravity as discussed above. See Ref. [29] for $f(\sigma)R$ inflation.

4 The minus sign originates from our convention following Refs. [26,36]: the metric signature is
level, it is quite intriguing if such a small \( \xi \) appears at a quantum level. The quantum-induced value of \( \eta \) is studied in \( \lambda \phi^4 \) theory \([20,25,26]\): \( \Delta \eta = -\lambda^3/(864(4\pi)^6) \) at the leading (three-loop) order. It appears from the renormalization of trace-anomaly terms (i.e., composite operators \([37,40]\)) and related with a inhomogeneous term of the \( \beta \) function of \( \eta \). Unless the leading value is at the one-loop order, it is hard to imagine that \( \xi \sim -10^{-3} \) originates from a quantum contribution. We find that \( \eta \) appears at the one-loop order in the two-scalar theory and Yukawa theory, as a threshold correction when additional degrees of freedom are heavy and decouple from the low-energy dynamics. Nevertheless, its sign is positive and thus opposite from the one required for the chaotic inflation. When additional degrees of freedom are light and do not decouple, \( \eta \) does not appear at the one-loop order.

This article is organized as follows. In the next section, we evaluate \( T_{\mu \nu} \)-inserted diagrams. We demonstrate how the contribution from trace anomaly (quantum breaking of scale invariance) terms cancels with that from (classical breaking of scale invariance) mass terms for heavy degrees of freedom. We find that heavy degrees of freedom leave a one-loop threshold correction to \( \eta \), which is regarded as the quantum-induced value of \( \eta \). In Section 3 we study the renormalization group equation (RGE) of \( \eta \) to discuss the quantum-induced value of \( \eta \) when the additional degrees of freedom do not decouple. Section 4 is devoted to a summary and further remarks. Throughout this article, we adopt the modified minimal subtraction (\( \overline{\text{MS}} \)) scheme \([41,43]\) with \( d = 4 - \epsilon \). We summarize related one-loop calculations in Appendix A.

2 Decoupling of heavy degrees of freedom

We assume that the matter sector is minimally coupled to gravity, while maintaining renormalizability up to graviton loops that are suppressed by \( 1/M_{\text{pl}}^2 \):

\[
S_{\text{mat}} \left[ \{ \phi_0 \}; g_{\mu \nu}; \{ \lambda_0 \} \right],
\]

where \( \{ \phi_0 \} \) and \( \{ \lambda_0 \} \) collectively denote bare matter fields and parameters, respectively. In particular we require renormalizability of energy-momentum tensor that is defined as a linear response of the matter action to the metric:

\[
T^{\mu \nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{mat}} \left[ \{ \phi_0 \}; g_{\mu \nu}; \{ \lambda_0 \} \right]}{\delta g_{\mu \nu}}.
\]

In the following subsections, we take into account gravity only to derive \( T^{\mu \nu} \). We consider a non-minimal coupling \( \xi \) as a part of \( S_{\text{mat}} \). We evaluate the trace of energy-momentum tensor \( T^\mu_\mu \) in the flat spacetime. We remark that in the flat spacetime, \( \xi \) (+, −, −, −); the Einstein-Hilbert action with a free singlet scalar is

\[
S_{\text{E-H}} = -\frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} \xi R \phi^2 \right),
\]

with the reduced Planck mass \( M_{\text{pl}} \); and the four-dimensional conformal coupling is \( \xi_c = +1/6 \).

5This does not mean the matter sector consists solely of a finite number of renormalizable terms. Non-renormalizable terms are allowed when an infinite number of non-renormalizable terms are introduced for renormalization in the usual sense of effective field theory.
appears only in $T^\mu_\mu$ as an improvement term. $\xi$ does not change the usual multiplicative renormalization of all the fields and parameters. Thus $T^\mu_\mu$ in the flat spacetime, which consist of the renormalized fields and parameters, is almost pre-determined. The single exception is $\xi$, which is determined by the renormalization of $T^\mu_\mu$ itself.

In the $\overline{\text{MS}}$ scheme, we first calculate $d$-dimension $T^{\mu\nu}$, take the trace $T^\mu_\mu$, and then take the limit of $\epsilon \to 0$. As stressed in Ref. [28], a key point is that $T^\mu_\mu$ contains terms proportional to $\epsilon$. These terms vanish in the limit of $\epsilon \to 0$ at the classical level, but not at the quantum level due to the renormalization of composite operators. This is the origin of trace anomaly. As we will see in explicit examples below, trace anomaly plays an important role in decoupling of heavy degrees of freedom.

In the following, for diagrammatic convenience, we introduce “scalarmon” $\sigma$ that couples to $T^\mu_\mu$ as

$$L_{\text{scalarmon}} = \sigma T^\mu_\mu. \tag{4}$$

With this coupling, we calculate the scalaron decay amplitude into light scalars $\phi$: $\mathcal{M}(\sigma \to \phi\phi)$. From the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [44], $\mathcal{M}$ is given by the amputated amplitude times the product of the square-root residues of the mass pole of light degrees of freedom, $(Z_{\text{pole}})^{1/2}$s. In $R^2$ inflation, scalaron $\sigma$ couples to $T^{\mu\nu}$ as

$$L_{\text{scalarmon}} = \frac{\sigma}{\sqrt{6}M_{\text{pl}}}T^\mu_\mu. \tag{5}$$

Thus, the corresponding invariant amplitude of scalaron decay is given by

$$\mathcal{M}_{\text{dec}} = \frac{1}{\sqrt{6}M_{\text{pl}}} \mathcal{M}. \tag{6}$$

We remark that although we calculate the “scalarmon” decay amplitude for diagrammatic convenience, our results are not limited within the inflation model with scalaron. Through this decay amplitude, we study the properties of $T^\mu_\mu$ such as decoupling of heavy degrees of freedom and the renormalization of a non-minimal coupling of $\phi$. One can regard $\phi$ as inflaton as we will do in the next section.

### 2.1 Two-scalar theory

Let us consider the following action with two real scalar fields $\phi$ and $\psi$:

$$S_{\text{mat}} = \int d^d x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_0 \nabla_\nu \phi_0 + \frac{1}{2} \xi_{\phi_0} R \phi_0^2 + \frac{1}{2} g^{\mu\nu} \nabla_\mu \psi_0 \nabla_\nu \psi_0 + \frac{1}{2} \xi_{\psi_0} R \psi_0^2 - \frac{1}{2} M_0^2 \phi_0^2 - \frac{1}{2} m_0^2 \psi_0^2 - \frac{1}{4!} \lambda_{\phi_0} \phi_0^4 - \frac{1}{4!} \lambda_{\psi_0} \psi_0^4 - \frac{1}{4} \chi_{\phi_0} \phi_0^2 \psi_0^2 \right), \tag{7}$$

Parameters are scalar masses, $M$ and $m$, and quartic couplings, $\lambda_s$ ($s = \phi, \psi$) and $\chi$. We provide details of multiplicative renormalization in Appendix A.1. The $d$-dimension
flat-spacetime energy-momentum tensor is given by

\[ T_{\mu\nu} = \partial_\mu \phi_0 \partial_\nu \phi_0 + \partial_\mu \psi_0 \partial_\nu \psi_0 - \left( \frac{d-2}{4(d-1)} + \frac{\eta_{\phi0}}{d-1} \right) (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) \phi_0^2 \]

\[ - \left( \frac{d-2}{4(d-1)} + \frac{\eta_{\phi0}}{d-1} \right) (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) \psi_0^2 - g_{\mu\nu} \mathcal{L}, \]

where the flat-spacetime Lagrangian density \( \mathcal{L} \) is given by Eq. (50). Taking the trace one finds

\[ T^\mu = \eta_{\phi0} \partial^2 \phi_0^2 + \eta_{\psi0} \partial^2 \psi_0^2 + M_0^2 \phi_0^2 + m_0^2 \psi_0^2 \]

\[ + \epsilon \left( \frac{1}{4!} \lambda_{\phi0} \phi_0^4 + \frac{1}{4!} \lambda_{\psi0} \psi_0^4 + \frac{1}{4} \chi_{\phi0} \phi_0^2 \psi_0^2 \right) + (\text{e.o.m.}), \]

where the last term is proportional to the equation of motion:

\[ (\text{e.o.m.}) = \left( 1 - \frac{\epsilon}{2} \right) \phi_0 \left[ \partial^2 \phi_0 + M_0^2 \phi_0 + \frac{4}{4!} \lambda_{\phi0} \phi_0^3 + \frac{2}{4} \chi_{\phi0} \phi_0 \psi_0^2 \right] \]

\[ + \left( 1 - \frac{\epsilon}{2} \right) \psi_0 \left[ \partial^2 \psi_0 + m_0^2 \psi_0 + \frac{4}{4!} \lambda_{\psi0} \psi_0^3 + \frac{2}{4} \chi_{\psi0} \phi_0 \psi_0^2 \right]. \]  

We consider scaleron \( \sigma \) decay into two light scalars \( \phi \). We assume that \( \phi \) is much lighter than \( \psi \), \( M_{\text{phys}} \ll m_{\text{phys}} \) (pole mass), and the self-couplings, \( \lambda_{\phi} \) and \( \lambda_{\psi} \), are negligible. The leading contributions originate from

\[ T^\mu \supset \eta_{\phi} \partial^2 \phi^2 + \eta_{\psi} \partial^2 \psi^2 + M^2 \phi^2 + m^2 \psi^2 + \frac{1}{4} \epsilon \mu \phi \partial^2 \phi \psi^2 \]

\[ + (Z_{\eta\phi} - 1) \eta_{\phi} \partial^2 \phi^2 + (Z_{M^2} - 1) M^2 \phi^2. \]

Here we use the renormalized fields and parameters [see Appendix A.1]. The one-loop decay amplitude is given by

\[ i \mathcal{M}(\sigma(p) \to \phi(q)\phi(k)) = i \mathcal{M}^{\text{tree}} + i \mathcal{M}^{\text{loop}} + i \mathcal{M}^{\text{c.t.}}. \]  

Here the counter-term contribution is given by

\[ i \mathcal{M}^{\text{c.t.}} = 2i(Z_{M^2} - 1)M^2 - 2i(Z_{\eta\phi} - 1)\eta_{\phi} p^2 = 2i \left( \frac{\chi}{16\pi^2} \right) m^2 \frac{1}{\epsilon} - 2i(Z_{\eta\phi} - 1)\eta_{\phi} p^2. \]  

In the second equality, we use \( Z_{M^2} \) which is determined to absorb the \( 1/\epsilon \) pole in the one-loop self-energy of \( \phi \) and is given by Eq. (57). While \( Z_{M^2} \) is pre-determined, \( Z_{\eta\phi} \) is the counter term to cancel the \( 1/\epsilon \) pole appeared in \( \mathcal{M}^{\text{loop}} \).

The tree-level contribution is given by

\[ i \mathcal{M}^{\text{tree}} = 2i(M^2 - \eta_{\phi} p^2) \]

\[ = 2i(M_{\text{phys}}^2 - \eta_{\phi} p^2) - i \left( \frac{\chi}{16\pi^2} \right) m^2 \left[ \ln \left( \frac{m^2}{\mu^2} \right) - 1 \right]. \]
Figure 1: One-loop diagrams for scalaron decay $\sigma \to \phi \phi$. Crossed dots denote insertion of the energy-momentum tensor.

Note that no $Z_{\phi}^{\text{pole}}$ appears at the one-loop order since the wave function is not renormalized at this order [see Eq. (57)]. In the second line, we use the relation between the renormalized and pole masses of the scalar field in Eq. (60). The second term in the second line diverges as we take the heavy limit of $\psi$. This originates from the fact that the scalar mass squared is sensitive to ultraviolet physics and one needs fine-tuning to realize $M_{\text{phys}} \ll m_{\text{phys}}$. As we will see shortly below, the second term in the second line is canceled by other contributions and the amplitude is insensitive to ultraviolet physics.

Fig. 1 shows the one-loop diagrams contributing to $M_{\text{loop}}^1$. Fig. 1 (a) gives

$$iM_{1}^{\text{loop}} = -i \frac{\chi}{16 \pi^2} (m^2 - \eta \psi p^2) \left[ \frac{2}{\epsilon} - \ln \left( \frac{m^2}{\mu^2} \right) - 2J_s \left( \frac{p^2}{m^2} \right) \right], \quad (15)$$

where

$$J_s(r) = \frac{1}{2} \int_0^1 dx \ln [x(x - 1)r + 1 - i \epsilon_{\text{ad}}]$$

$$= \begin{cases} 
-1 + \sqrt{\frac{4 - r}{r}} \arcsin \frac{\sqrt{r}}{2} & (r < 4) \\
-1 + \sqrt{\frac{r - 4}{r}} \left[ \arccosh \frac{\sqrt{r}}{2} + i \frac{\pi}{2} \right] & (r > 4)
\end{cases} \quad (16)$$

For $r > 4$, one needs to take into account an adiabatic parameter $\epsilon_{\text{ad}} > 0$ properly. This arises from the fact that the loop scalar can be real. We remark that $J_s(r) \to -r/12$ ($r \to 0$) in the heavy $\psi$ limit ($p^2 \ll m_\psi^2$).

Fig. 1 (b) is the contribution from the term proportional to $\epsilon$.

$$iM_{2}^{\text{loop}} = -i \frac{\chi}{16 \pi^2} m^2. \quad (17)$$

We remark that this contribution does not vanish in the limit of $\epsilon \to 0$, is related with trace anomaly of $\beta_{M^2}$, and plays an important role in decoupling of heavy degrees of freedom.

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6 Note that an adiabatic parameter $\epsilon_{\text{ad}}$ associated with a Wick rotation is different from $\epsilon = 4 - d$ for dimensional regularization.
Summing up the decay amplitude, we obtain

\[ \mathcal{M}(\sigma \rightarrow \phi \phi) = 2(M_{\text{phys}}^2 - \eta \phi p^2) - \frac{\chi}{16\pi^2} \eta \phi p^2 \left[ \ln \left( \frac{m^2}{\mu^2} \right) + 2J_s \left( \frac{p^2}{m^2} \right) \right] + \frac{\chi}{16\pi^2} m^2 2J_s \left( \frac{p^2}{m^2} \right). \]

(18)

Here, we determine the counter term for the non-minimal coupling as

\[ Z_{\eta \phi} - 1 = \frac{\chi}{16\pi^2} \eta \phi \epsilon, \]

(19)

to make \( T^\mu_\mu \) finite. From Eqs. (55) and (57), one obtains

\[ \frac{d\eta \phi}{d\ln \mu} = \frac{\chi}{16\pi^2} \eta \phi. \]

(20)

When \( p^2 \ll m^2 \), the decay amplitude is approximated by

\[ \mathcal{M}(\sigma \rightarrow \phi \phi) = 2M_{\text{phys}}^2 - 2 \left( \eta \phi + \frac{1}{2} \frac{\chi}{16\pi^2} \eta \phi \ln \left( \frac{m^2}{\mu^2} \right) + \frac{1}{12} \frac{\chi}{16\pi^2} \right) p^2. \]

(21)

Now the dangerous term proportional to \( m^2 \) is absent and thus heavy degrees of freedom decouples from the low-energy dynamics. Meanwhile, in the low-energy effective theory with almost free (\( \lambda \phi \approx 0 \)) light \( \phi \), the leading contribution comes from

\[ T^\mu_\mu_{\text{low}} = \eta_{\text{low}} \phi^2 \phi^2 + M_{\text{phys}}^2 \phi^2. \]

(22)

We note that the pole mass squared \( M_{\text{phys}}^2 \) in the low-energy effective theory is identical to the one in the high-energy theory in Eq. (60). The decay amplitude is

\[ \mathcal{M}(\sigma(p) \rightarrow \phi \phi) = 2M_{\text{phys}}^2 - 2\eta_{\text{low}} p^2. \]

(23)

By matching Eqs. (21) and (23), we find

\[ \eta_{\text{low}} = \eta \phi + \frac{1}{2} \frac{\chi}{16\pi^2} \eta \phi \ln \left( \frac{m^2}{\mu^2} \right) + \frac{1}{12} \frac{\chi}{16\pi^2}. \]

(24)

2.2 Yukawa theory

Let us consider the following action with a scalar field \( \phi \) and a singlet fermion \( \psi \):

\[ S_{\text{mat}} = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_0 \nabla_\nu \phi_0 + \frac{1}{2} \xi_0 R \phi_0^2 - \frac{1}{2} M_0^2 \phi_0^2 - \frac{1}{4!} \lambda_0 \phi_0^4 \right. \\
+ \frac{1}{2} \bar{\psi}_0 \gamma^\mu iD_\mu \psi_0 - \frac{1}{2} iD_\mu \bar{\psi}_0 \gamma_\mu \psi_0 - m_0 \bar{\psi}_0 \psi_0 - ig_0 \phi_0 \bar{\psi}_0 \gamma_5 \psi_0 \right), \]

(25)
with $D_\mu$ being the Local Lorentz and diffeomorphism covariant derivative. Parameters are a scalar mass $M$, a fermion mass $m$, a quartic coupling $\lambda$, and a Yukawa coupling $y$.

We provide details of multiplicative renormalization in Appendix A.2. The $d$-dimension flat-spacetime energy-momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu \phi_0 \partial_\nu \phi_0 - \frac{i}{4} \left( \partial_\mu \overline{\psi}_0 \gamma_\nu \psi_0 - \overline{\psi}_0 \partial_\mu \gamma_\nu \psi_0 \right)$$

$$- \left( \frac{d - 2}{2(d - 1)} + \frac{\eta_0}{d - 1} \right) \left( \partial_\mu \partial_\nu - g_{\mu\nu} \partial^2 \right) \phi_0^2 - g_{\mu\nu} L,$$

where the flat-spacetime Lagrangian density $L$ is given by Eq. (76). Here, braces denote symmetrization of Lorentz indices. The trace of the energy momentum tensor is

$$T^\mu_{\mu} = \eta_0 \partial^2 \phi_0^2 + M_0^2 \phi_0^2 + m_0 \overline{\psi}_0 \psi_0 + \frac{1}{4!} \lambda_0 \phi_0^4 + \frac{\epsilon}{2} i y_0 \phi_0 \overline{\psi}_0 \gamma_5 \psi_0 + \text{(e.o.m.)},$$

where the last term is proportional to the equation of motion as

$$(\text{e.o.m.}) = \left( 1 - \frac{\epsilon}{2} \right) \phi_0 \left[ \partial^2 \phi_0 + M^2 \phi_0 + \frac{4}{4!} \lambda_0 \phi_0^4 + i y_0 \phi_0 \overline{\psi}_0 \gamma_5 \psi_0 \right]$$

$$+ \left( \frac{3}{2} - \frac{\epsilon}{2} \right) \left[ i \phi \overline{\psi}_0 + m_0 \overline{\psi}_0 + i y_0 \phi_0 \overline{\psi}_0 \gamma_5 \psi_0 \right] \psi_0$$

$$+ \left( \frac{3}{2} - \frac{\epsilon}{2} \right) \overline{\psi}_0 \left[ -i \phi \psi_0 + m_0 \psi_0 + i y_0 \phi_0 \gamma_5 \psi_0 \right].$$

We consider scalaron $\sigma$ decay into two light scalars $\phi$. We assume that $\phi$ is much lighter than $\psi$, $M_{\text{phys}} \ll m_{\text{phys}}$ (pole mass), and the self-coupling $\lambda$ is negligible. The leading contributions originate from

$$T^\mu_{\mu} \supset \eta \partial^2 \phi^2 + M^2 \phi^2 + m \overline{\psi} \psi + \frac{\epsilon}{2} i y \phi \overline{\psi} \gamma_5 \psi + (Z_{M^2} - 1) M^2 \phi^2.$$  

Here we use the renormalized fields and parameters [see Appendix A.2]. The one-loop decay amplitude is given by

$$i \mathcal{M}(\sigma(p) \to \phi(q) \phi(k)) = i \mathcal{M}^{\text{tree}} + i \mathcal{M}^{\text{loop}} + i \mathcal{M}^{\text{c.t.}}.$$

Here, the counter-term contribution is given by

$$i \mathcal{M}^{\text{c.t.}} = 2 i (Z_{M^2} - 1) M^2 = -16 i \frac{y^2}{16 \pi^2} m^2 \frac{1}{\epsilon}.$$  

In the second equality, we use $Z_{M^2}$ which is determined to absorb the $1/\epsilon$ pole in the one-loop self-energy of $\phi$ and is given by Eq. (84). In contrast to the two-scalar theory, we do not include the counter term for the non-minimal coupling since no $p^2/\epsilon$ term appears at this order.

The tree-level contribution is given by

$$i \mathcal{M}^{\text{tree}} = 2 i (M^2 - \eta p^2) \times Z^\text{pole}_\phi$$

$$= 2 i (M^2_{\text{phys}} - \eta p^2) + 8 i \frac{y^2}{16 \pi^2} m^2 \left[ \ln \left( \frac{m^2}{\mu^2} \right) - 1 \right] - 4 i \eta p^2 \frac{y^2}{16 \pi^2} \ln \left( \frac{m^2}{\mu^2} \right).$$

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In the second line, we use the relation between the renormalized and pole masses in Eq. (86). $Z_{\phi}^{\text{pole}}$ in the Yukawa theory is computed in Eq. (87). The second term in the second line diverges as one takes the heavy limit of $\psi$. This originates from the fact that the scalar mass squared is sensitive to ultraviolet physics and one needs fine-tuning to realize $M_{\text{phys}} \ll m_{\text{phys}}$. As we will see shortly below, the second term in the second line is canceled by other contributions and the amplitude is insensitive to ultraviolet physics.

As shown in Fig. 2, several loop diagrams contribute to $M_{\text{loop}}$. We obtain the one-loop contribution to the scalaron decay given by Fig. 2 (a).

$$iM_{\text{loop}}^1 = -8im^2 \frac{y^2}{16\pi^2} \left[ -\frac{2}{\epsilon} + J_f \left( \frac{p^2}{m^2} \right) + \ln \left( \frac{m^2}{\mu^2} \right) + 2J_s \left( \frac{p^2}{m^2} \right) \right], \quad (33)$$

where $J_s(r)$ is given by Eq. (16) and

$$J_f(r) = \frac{r}{2} \int_0^1 dx \int_0^{1-x} dy \frac{1}{-rxy + 1 - i\epsilon_{\text{ad}}}$$

$$= \begin{cases} 
\arcsin \frac{\sqrt{r}}{2} & \text{(for } r < 4) \\
\left( \arccosh \frac{\sqrt{r}}{2} - i\frac{\pi}{2} \right)^2 & \text{(for } r > 4) 
\end{cases} \quad (34)$$

For $r > 4$, one needs to take into account an adiabatic parameter $\epsilon_{\text{ad}} > 0$ properly. This arises from the fact that the loop scalar can be real. For $m^2 \gg p^2$ ($r \rightarrow 0$), $J_f$ is approximated by $J_f(r) \rightarrow r/4$.

Fig. 2 (b, c) arises from a term proportional to $\epsilon$. The divergent part of the diagrams gives finite corrections to the decay amplitude.

$$iM_{\text{loop}}^2 = 8i \frac{y^2}{16\pi^2} m^2. \quad (35)$$

We remark that this contribution does not vanish in the limit of $\epsilon \rightarrow 0$, is related with trace anomaly of $\beta_{M^2}$, and plays an important role in decoupling of heavy degrees of freedom.

For the Yukawa model with a light scalar, the total amplitude is given by

$$M(\sigma \rightarrow \phi\phi) = 2(M_{\text{phys}}^2 - \eta p^2) - 8m^2 \frac{y^2}{16\pi^2} \left[ J_f \left( \frac{p^2}{m^2} \right) + 2J_s \left( \frac{p^2}{m^2} \right) \right]. \quad (36)$$
Here, $Z_\eta = 1$ at this order. From Eqs. (81) and (84), one obtains

$$\frac{d\eta}{d\ln\mu} = 4\frac{y^2}{16\pi^2}\eta. \quad (37)$$

When $p^2 \ll m^2$, the total amplitude is approximated by

$$\mathcal{M}(\sigma \rightarrow \phi\phi) = 2M_{\text{phys}}^2 - 2\left(\eta + 2\frac{y^2}{16\pi^2}\eta\ln\left(\frac{m^2}{\mu^2}\right) + \frac{1}{3}\frac{y^2}{16\pi^2}\right)p^2. \quad (38)$$

Now the dangerous term proportional to $m^2$ is absent and thus heavy degrees of freedom decouples from the low-energy dynamics. Meanwhile, in the low-energy effective theory with almost free ($\lambda \approx 0$) light $\phi$, the leading contribution comes from

$$T_{\text{low}}^\mu = \eta_{\text{low}}\partial^2\phi^2 + M_{\text{phys}}^2\phi^2. \quad (39)$$

We note that the pole mass squared $M_{\text{phys}}^2$ in the low-energy effective theory is identical to the one in the high-energy theory in Eq. (86). The decay amplitude is

$$\mathcal{M}(\sigma(p) \rightarrow \phi\phi) = 2M_{\text{phys}}^2 - 2\eta_{\text{low}}p^2. \quad (40)$$

By matching Eqs. (38) and (40), we find

$$\eta_{\text{low}} = \eta + 2\frac{y^2}{16\pi^2}\eta\ln\left(\frac{m^2}{\mu^2}\right) + \frac{1}{3}\frac{y^2}{16\pi^2}. \quad (41)$$

### 3 Quantum-induced value of $\eta$

We again note that seemingly we have studied the scalaron decay amplitude in the previous section, but this is just for diagrammatic convenience. In the following, we regard $\phi$ as inflaton and the matching condition in Eqs. (24) and (41) as the one for the non-minimal coupling of inflaton. Interestingly a threshold correction that is independent of $\eta$ appears at the one-loop order [see the last terms Eqs. (24) and (41)]. This threshold correction is a quantum-induced value of $\eta$ in the low-energy theory because it is induced irrespectively of our choice of $\eta$ in the high-energy theory.

A caveat is that this low-energy theory itself may not be “natural” in the sense that one has to fine-tune the scalar mass squared to keep $\phi$ light. This originates from the fact that the quantum correction to the scalar mass squared is quadratic divergent as intensively discussed in the context of the Higgs mass squared in the standard model [45–47] (see Ref. [48] for a review). In the two scalar theory and Yukawa theory, we need fine-tuning between the renormalized mass of $\phi$ and the mass of heavy degrees of freedom [see Eqs. (60) and (86)]. This is known to be finite naturalness (see Refs. [49,50] for the standard model Higgs). Thus one may regard this threshold correction as a quantum-induced value in a “unnatural” theory.

What is a quantum-induced value of $\eta$ in a “natural” theory, i.e., when degrees of freedom that couples to $\phi$ are as light as $\phi$? It is the inhomogeneous solution of the RGE of $\eta$:

$$\frac{d\eta}{d\ln\mu} = \gamma_\phi^T \phi^2 \eta + \tilde{\beta}_\eta, \quad (42)$$
where \( \eta \) and \( \gamma_{\phi^2} \) should be understood as a vector and matrix, respectively, for multiple scalar fields. Here

\[
\begin{align*}
\phi_0^2 &= Z_{\phi^2}[\phi^2], \\
\frac{d \ln Z_{\phi^2}}{d \ln \mu} &= \gamma_{\phi^2}.
\end{align*}
\] (43)

The homogeneous term is proportional to the anomalous dimension of \( \phi^2 \) in the RGE. This is because the renormalization of the scalar field squared is multiplicative, \( Z_{\phi^2}^{-1} \partial^2 \phi_0^2 = \partial^2[\phi^2] \). It means that all the counter terms to renormalize \( \phi^2 \) is included in \( Z_{\phi^2}^{-1} Z_{\phi} \).

\( \tilde{\beta}_{\eta} \) denotes the inhomogeneous term of the RGE and induces \( \Delta \eta \) through the running irrespectively of our initial choice of \( \eta \). This \( \Delta \eta \) is nothing but a quantum-induced value of \( \eta \).

Before going to the inhomogeneous solution, let us study the homogeneous solution. In the two-scalar theory, from Eqs. (55) and (73), one obtains

\[
\gamma_{\phi^2} = \begin{pmatrix}
0 \\
\frac{\chi}{16\pi^2} \\
\frac{1}{16\pi^2} \\
0
\end{pmatrix},
\] (44)

for \( \eta = (\eta_{\phi}, \eta_{\psi})^T \). Now one sees that Eq. (42) reproduces Eq. (20). By using the \( \beta \) function of \( \chi \) [see Eq. (64)], we find the homogeneous solution:

\[
\begin{align*}
\eta_{\phi} &= \eta_{\phi i} \left( \frac{\chi}{\chi_i} \right)^{1/2} + \eta_{\psi i} \left( \frac{\chi}{\chi_i} \right)^{-1/2}, \\
\eta_{\psi} &= \eta_{\psi i} \left( \frac{\chi}{\chi_i} \right)^{1/2} - \eta_{\psi i} \left( \frac{\chi}{\chi_i} \right)^{-1/2},
\end{align*}
\] (45)

where the subscript \( i \) denotes the boundary condition of the RGE, i.e., \( \eta_{\phi} = \eta_{\phi i} \) and \( \eta_{\psi} = \eta_{\psi i} \) at \( \chi = \chi_i \).

In the Yukawa theory, from Eqs. (81) and (84) with the notion that \( Z_{\phi^2} = Z_{\phi} \) at the one-loop level, one obtains

\[
\gamma_{\phi^2} = \frac{4y^2}{16\pi^2}.
\] (46)

Now one sees that Eq. (42) reproduces Eq. (37). By using the \( \beta \) function of \( y \) [see Eq. (96)], we get the homogeneous solution:

\[
\eta = \eta_i \left( \frac{y}{y_i} \right)^{4/5},
\] (47)

where the subscript \( i \) denotes the boundary condition of the RGE, i.e., \( \eta = \eta_i \) at \( y = y_i \). \( \eta \) diminishes toward low energy as the theory becomes weakly coupled. It sounds very reasonable that \( \eta \) vanishes at the Gaussian fixed point, leaving \( T_{\mu} = 0 \).

Now let us discuss the inhomogeneous solution. To find the inhomogeneous solution of \( \eta \) at the \( n \)-loop order (leading contribution does not need to be at the one-loop order), one
has to determine the inhomogeneous term at the \((n+1)\)-loop level. This is because the right-hand side of Eq. (42) is of the order of \( (\beta \lambda/\lambda) \eta \) (\( \lambda \) correctly denotes couplings). The inhomogeneous term arises from a \( p^2/\epsilon \) pole in diagrams with trace anomaly being inserted (\( p \): incoming momentum) and two scalars outgoing. In \( \lambda \phi^4 \) theory, trace anomaly is \( \epsilon \lambda_0 \phi^4 \). \( \beta_\eta \) arises at the four-loop level (\( \lambda^4 \)) and leads to the inhomogeneous solution at the three-loop order \[20,25,26\]:

\[
\eta = \eta_i \left( \frac{\lambda}{\lambda_i} \right)^{1/3} - \frac{\lambda^3}{864(4\pi)^6},
\]

(48)

where the subscript \( i \) denotes the boundary condition of the RGE, i.e., \( \eta = \eta_i \) at \( \lambda = \lambda_i \), again. The first term is the homogeneous solution, while the second term is the inhomogeneous one. Trace anomaly is \( \epsilon \chi_0 \phi^2 \psi^2 \) in the two-scalar theory and \( \epsilon i y_0 \phi \bar{\psi} \gamma_5 \psi \) in the Yukawa theory. We find no \( p^2/\epsilon \) pole in two-loop diagrams neither in the two-scalar theory nor Yukawa theory. Thus \( \beta_\eta \) may arises only at the three-loop level in both theories, which may provide the inhomogeneous solution of \( \eta \) at the two-loop order.

4 Conclusion and remarks

The energy-momentum tensor \( T_{\mu\nu} \) determines the coupling of matter to gravity and provides a valuable site where we can study the properties of a non-minimal coupling of a scalar field to gravity even in the flat spacetime. We have studied the properties of its trace \( T^\mu_\mu \), which is tightly related to the conformal symmetry, particularly discussing its implications for inflation models. To be concrete, we have worked it out in the two-scalar theory and Yukawa theory.

The first property that we have stressed is decoupling of heavy degrees of freedom. This should be held in the effective field theory point of view, but is not apparent at first sight: heavy degrees of freedom appears in \( T^\mu_\mu \) in proportion to its mass, which leaves non-decoupling effects at a loop level. We have demonstrated that trace-anomaly terms, which are in proportion to \( \epsilon \) in dimensional regularization (and thus vanishes at the tree level), cancel the non-decoupling effects from mass terms. The similar conclusion is derived in gauge anomaly \[28\]. This conclusion is relevant, e.g., when one considers \( R^2 \) inflation, where inflaton (scalaron) couples to \( T^\mu_\mu \). One can safely evaluate, e.g., the decay rate of scalaron, in the effective field theory.

As a byproduct in demonstrating the decoupling, we have found that a non-minimal coupling, i.e., \( \xi = 1/6 + \eta/3 \) in four dimension, receives a one-loop-order threshold correction independent of \( \eta \). This threshold correction is a quantum-induced value of \( \eta \) in the low-energy theory in the sense that this value is irrespective of our choice of \( \eta \) in the high-energy theory. On the other hand, one may doubt that this low-energy theory itself is “unnatural”, since one has to fine-tune the scalar mass squared to keep a light scalar while taking other degrees of freedom heavy. The situation can change during inflation. The field value of inflaton can make some degrees of freedom, which is light in vacuum, heavy. It leaves a threshold correction to inflaton. The threshold correction can even change as the field value of inflaton changes during inflation, as seen in Eqs. \[24\].
and (41), supposing that $m^2$ is a function of the field value of inflaton. It deserves a further investigation, but we do not go in detail in this article.

To discuss a quantum-induced value of $\eta$ in a “natural” theory, we have taken notice of the inhomogeneous solution of the RGE of $\eta$. The inhomogeneous term of the RGE induces the inhomogeneous solution of $\Delta \eta$ irrespectively of our choice of $\eta$ and thus $\Delta \eta$ is a quantum-induced value of $\eta$. Since the inhomogeneous term arises from the (composite-operator) renormalization of trace-anomaly terms, $\Delta \eta$ originates from trace-anomaly.

Finally, let us discuss applications to inflation models. To resurrect inflation with a single power-law potential, one can introduce $\xi \sim -10^{-3}$ [33–35]. It is very intriguing if this small negative value originates from a quantum-induced value of $\eta$. To this end, a quantum-induced value should appear at the one-loop order. The threshold corrections found in the two-scalar theory and Yukawa theory are at the one-loop order, but their signs are positive. Furthermore, we have found no inhomogeneous solution at the one-loop order, i.e., the inhomogeneous term at the two-loop level, neither in the two-scalar theory nor Yukawa theory. We expect that this is not a general conclusion since it depends on the structure of available Feynman diagrams. We will study the inhomogeneous solution of the RGE of $\eta$ in scalar quantum electrodynamics or chromodynamics in the companion article [51].

One has to be careful also in the homogeneous term of $\eta$ when considering inflation models. $\xi \sim -10^{-3}$ means that one sets $\eta \simeq -1/2$ at the tree level, and thus $\eta_i \simeq -1/2$ in the homogeneous solutions of Eqs. (45) and (47). It can run with the coupling, i.e., $\chi$ in the two-scalar theory and $y$ in the Yukawa theory, during inflation. The renormalization scale $\mu$ would be set to be the Hubble expansion rate, where the flat-spacetime approximation breaks down, or the mass of loop particles, which is induced by the field value of inflaton, as for the Coleman-Weinberg potential [52]. Again it deserves a further investigation, but we do not go in detail in this article.

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A One-loop calculations

In the following calculations, we use the \( \overline{\text{MS}} \) scheme with a spacetime dimension of \( d = 4 - \epsilon \) and a renormalization scale of \( \mu \), while compensating a mass dimension by a modified renormalization scale \( \tilde{\mu} \) defined by

\[
\tilde{\mu}^2 = \mu^2 \frac{\epsilon \gamma_E}{4\pi} \tag{49}
\]

with \( \gamma_E \approx 0.577 \) being Euler’s constant. One-loop functions are summarized in Appendix \( \text{[A.3]} \).

A.1 Two-scalar theory

The Lagrangian density is

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_0)^2 - \frac{1}{2} M_0^2 \phi_0^2 + \frac{1}{2} (\partial_\mu \psi_0)^2 - \frac{1}{2} m_0^2 \psi_0^2 - \frac{1}{4!} \lambda \phi_0^4 - \frac{1}{4!} \lambda \psi_0^4 - \frac{1}{4} \chi_0 \phi_0^2 \psi_0^2. \tag{50}
\]

Multiplicative renormalization is set for fields as

\[
\psi_0 = Z_\psi^{1/2} \psi, \quad \phi_0 = Z_\phi^{1/2} \phi, \tag{51}
\]

and for parameters as

\[
Z_\phi M_0^2 = Z_M^2 M^2, \quad Z_\psi m_0^2 = Z_m^2 m^2, \quad Z_\lambda \lambda_s = Z_\lambda \tilde{\mu}^s \lambda_s \quad (s = \phi, \psi), \quad Z_\phi Z_\psi \chi_0 = Z_\chi \tilde{\mu}^s \chi, \tag{52}
\]

and

\[
Z_s \eta_s = Z_\eta \eta_s \quad (s = \phi, \psi). \tag{53}
\]

The Lagrangian density in terms of the renormalized quantities is

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} M^2 \phi^2 + \frac{1}{2} (\partial_\mu \psi)^2 - \frac{1}{2} m^2 \psi^2 - \frac{1}{4!} \tilde{\mu}^s \lambda \phi^4 - \frac{1}{4!} \tilde{\mu}^s \lambda \psi^4 - \frac{1}{4} \tilde{\mu}^s \chi \phi^2 \psi^2 \\
+ \frac{1}{2} (Z_\phi - 1)(\partial_\mu \phi)^2 - \frac{1}{2} (Z_M^2 - 1) M^2 \phi^2 + \frac{1}{2} (Z_\psi - 1)(\partial_\mu \psi)^2 - \frac{1}{2} (Z_m^2 - 1) m^2 \psi^2 \\
- \frac{1}{4!} (Z_\lambda \phi - 1) \tilde{\mu}^s \lambda \phi^4 - \frac{1}{4!} (Z_\lambda \psi - 1) \tilde{\mu}^s \lambda \psi^4 - \frac{1}{4} (Z_\chi - 1) \tilde{\mu}^s \chi \phi^2 \psi^2. \tag{54}
\]

It follows that

\[
\beta_\lambda = \lambda \left( -\epsilon + 2 \frac{d \ln Z_\lambda}{d \ln \mu} - \frac{d \ln Z_{\lambda s}}{d \ln \mu} \right) \quad (s = \phi, \psi), \tag{55}
\]

\[
\beta_\chi = \chi \left( -\epsilon + \frac{d \ln Z_\phi}{d \ln \mu} + \frac{d \ln Z_\psi}{d \ln \mu} - \frac{d \ln Z_\chi}{d \ln \mu} \right), \tag{55}
\]

\[
\beta_M = M^2 \left( \frac{d \ln Z_\phi}{d \ln \mu} - \frac{d \ln Z_{M^2}}{d \ln \mu} \right), \tag{55}
\]

\[
\beta_m = m^2 \left( \frac{d \ln Z_\psi}{d \ln \mu} - \frac{d \ln Z_m^2}{d \ln \mu} \right), \tag{55}
\]

\[
\beta_\eta = \eta \left( \frac{d \ln Z_s}{d \ln \mu} - \frac{d \ln Z_{\eta s}}{d \ln \mu} \right) \quad (s = \phi, \psi). \tag{55}
\]
The one-loop self-energy of $\phi$ is $i\Pi^{\text{loop}} + i\Pi^{\text{c.t.}}$ with

$$i\Pi^{\text{loop}} = \frac{1}{2}(-i\chi)\tilde{\mu}^\epsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} = \frac{i\chi}{16\pi^2} \frac{1}{2} A,$$

and

$$i\Pi^{\text{c.t.}} = i(Z_\phi - 1)p^2 - i(Z_{M^2} - 1)M^2. \quad (56)$$

Here, $A$ is one-point integral defined in Appendix A.3. $Z_\phi$ and $Z_{M^2}$ are determined to cancel $1/\epsilon$ poles as

$$Z_\phi - 1 = 0, \quad Z_{M^2} - 1 = \frac{\chi}{16\pi^2} \frac{m^2}{M^2} \frac{1}{\epsilon}. \quad (57)$$

Similarly, for the one-loop self-energy of $\psi$,

$$Z_\psi - 1 = 0, \quad Z_{m^2} - 1 = \frac{\chi}{16\pi^2} \frac{M^2}{m^2} \frac{1}{\epsilon}. \quad (58)$$

The resultant self-energy of $\phi$ is

$$\Gamma_2(p^2) = p^2 - M^2 - \frac{\chi}{16\pi^2} \frac{1}{2} m^2 \left[ \ln \left( \frac{m^2}{\mu^2} \right) - 1 \right]. \quad (59)$$

The pole mass squared of $\phi$ satisfies $\Gamma_2(M^2_{\text{phys}}) = 0$. The difference between the pole and MS masses squared is

$$M^2_{\text{phys}} = M^2(\mu) + \frac{\chi}{16\pi^2} \frac{1}{2} m^2 \left[ \ln \left( \frac{m^2}{\mu^2} \right) - 1 \right]. \quad (60)$$

We compute $\beta$ function of $\chi$. Fig. 3 gives

$$i\mathcal{M}^{\text{loop}} = (-i\chi\tilde{\mu}^\epsilon)^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} \frac{i}{(\ell + p)^2 - M^2},$$

$$= i\frac{\chi^2}{16\pi^2} \tilde{\mu}^\epsilon B_0(p^2, m^2, M^2),$$

and its divergent part is cancelled by the counter term,

$$i\mathcal{M}^{\text{c.t.}} = -i(Z_\chi - i)\tilde{\mu}^\epsilon \chi, \quad (62)$$
Figure 4: One loop diagrams for the renormalization of scalar field squared. Black dots denote insertion of composite operators, $\psi^2$ (a) and $\phi^2$ (b).

as

$$Z_X - 1 = 2 \frac{X}{16\pi^2} \frac{1}{\epsilon}. \quad (63)$$

From Eq. (55), we obtain the $\beta$ function of $X$ at one-loop order.

$$\beta_X^\epsilon = -\epsilon X + 2 \frac{X}{16\pi^2} = -\epsilon X + \beta_X. \quad (64)$$

We consider the (composite-operator) renormalization of scalar field squared, which is useful in understanding the RGE of the non-minimal couplings:

$$\begin{pmatrix} \phi_0^2 \\ \psi_0^2 \end{pmatrix} = Z_{\phi^2} \begin{pmatrix} [\phi^2] \\ [\psi^2] \end{pmatrix}, \quad (65)$$

where $Z_{\phi^2}$ is understood as a matrix. In practice, it is convenient to rewrite it as

$$\begin{pmatrix} [\phi^2] \\ [\psi^2] \end{pmatrix} = Z_{\phi^2}^{-1} \begin{pmatrix} Z_{\phi} & 0 \\ 0 & Z_{\psi} \end{pmatrix} \begin{pmatrix} \phi^2 \\ \psi^2 \end{pmatrix} \equiv \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} \phi^2 \\ \psi^2 \end{pmatrix}. \quad (66)$$

Note that the diagonal components are unity, $z_{11} = z_{22} = 1$, at the tree level, while the off-diagonal components are zero, $z_{12} = z_{21} = 0$, at the tree level. Again we assume that self-couplings, $\lambda_s (s = \phi, \psi)$, are negligible and focus on the quartic coupling $X$. No divergence appears in $z_{11}$ and $z_{22}$ at the one-loop level, and thus $z_{11} = z_{22} = 1$.

The one-loop amplitude with $[\psi^2]$ being inserted ($p$: incoming momentum) and two $\phi$'s outgoing is $iM = iM^{\text{loop}} + iM^{\text{c.t.}}$ with

$$iM^{\text{c.t.}}(\phi^2) = 2iz_{21}. \quad (67)$$

Fig. 4 (a) gives

$$iM^{\text{loop}}([\psi^2] \rightarrow \phi\phi) = (-iX)\mu^\epsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} \frac{i}{(\ell + p)^2 - m^2} \quad (68)$$

$$= -i \frac{X}{16\pi^2} B_0.$$
Here, $B_0$ is two-point integral defined in Appendix [A.3] and we use a short-hand notation $B_0 \equiv B_0(p^2; m^2, m^2)$. Thus, $z_{21}$ is determined to cancel the divergence:

$$ z_{21} = \frac{\chi}{16\pi^2} \frac{1}{\epsilon}. \quad (69) $$

Similarly, for $[\phi]^2$ insertion, one obtains

$$ i\mathcal{M}^{c.t.}(\psi^2) = 2iz_{12}. \quad (70) $$

and

$$ i\mathcal{M}^{\text{loop}}([\phi]^2 \rightarrow \psi\psi) = (-i\chi)\bar{\mu}' \int_{\mathcal{D}^d} \frac{d^d \ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} \frac{i}{(\ell + p)^2 - M^2} \frac{i}{\ell^2 - M^2} = -i\frac{\chi}{16\pi^2} B_0(p^2; M^2, M^2), \quad (71) $$

from Fig. 4(b). Thus, $z_{21}$ is determined to cancel the divergence:

$$ z_{12} = \frac{\chi}{16\pi^2} \frac{1}{\epsilon}. \quad (72) $$

As in Eqs. (57) and (58), $Z_{\phi} = Z_{\psi} = 1$ at the one-loop level. Hence,

$$ Z_{\phi^2}^{-1} = \left( \begin{array}{cc} 1 & \frac{\chi}{16\pi^2} \frac{1}{\epsilon} \\ \frac{\chi}{16\pi^2} \frac{1}{\epsilon} & 1 \end{array} \right), \quad Z_{\phi^2} = \left( \begin{array}{cc} 1 & -\frac{\chi}{16\pi^2} \frac{1}{\epsilon} \\ -\frac{\chi}{16\pi^2} \frac{1}{\epsilon} & 1 \end{array} \right). \quad (73) $$

We compute one-loop corrections to $T_\mu^\mu$-inserted correlation functions which appear in the text. Fig. 1(a) gives

$$ i\mathcal{M}_1^{\text{loop}} = (2m^2 - 2\eta_\psi p^2)(-i\chi)\bar{\mu}' \frac{1}{2} \int_{\mathcal{D}^d} \frac{d^d \ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} \frac{i}{(\ell + p)^2 - m^2} \frac{i}{\ell^2 - M^2} \frac{i}{\ell^2 - M^2} B_0 \frac{1}{2} \frac{1}{2} \frac{1}{2} \left( \frac{2}{\epsilon - \ln \left( \frac{m^2}{\mu^2} \right)} - 2J_s \left( \frac{p^2}{m^2} \right) \right), \quad (74) $$

while Fig. 1(b) gives

$$ i\mathcal{M}_2^{\text{loop}} = (-i\epsilon\chi)\bar{\mu}' \frac{1}{2} \int_{\mathcal{D}^d} \frac{d^d \ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} \frac{i}{\ell^2 - m^2} = -i\frac{\chi}{16\pi^2} \delta m^2. \quad (75) $$

**A.2 Yukawa theory**

The Lagrangian density is

$$ \mathcal{L} = \frac{1}{2} \bar{\psi}_0 \gamma^\mu \partial_\mu \psi_0 - \frac{1}{2} i \bar{\psi}_0 \gamma^\mu \gamma^5 \psi_0 - m_0 \bar{\psi}_0 \psi_0 + \frac{1}{2} (\partial_\mu \phi_0)^2 - \frac{1}{2} M_0^2 \phi_0^2 - \frac{1}{4!} \lambda_0 \phi_0^4 - i y_0 \phi_0 \bar{\psi}_0 \gamma^5 \psi_0. \quad (76) $$

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Multiplicative renormalization is set for fields as
\[ \psi_0 = Z^{1/2}_\psi \psi, \quad \phi_0 = Z^{1/2}_\phi \phi. \] (77)
and for parameters as
\[ Z_\phi M_0^2 = Z_M^2 M^2, \quad Z_\psi m_0 = Z_m m, \quad Z_\phi^2 \lambda_0 = Z_\lambda \tilde{\mu}^\epsilon \lambda, \quad Z_\phi^{1/2} Z_\psi y_0 = Z_y \tilde{\mu}^{\epsilon/2} y, \] (78)
and
\[ Z_\phi \eta_0 = Z_\eta \eta. \] (79)
The Lagrangian density in terms of the renormalized quantities is
\[
\mathcal{L} = \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{2} i \bar{\psi} \tilde{\gamma}^\mu \partial_\mu \psi - m \bar{\psi} \psi + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} M^2 \phi^2 - \frac{1}{4!} \tilde{\mu}^\epsilon \lambda \phi^4 - i \bar{\psi} \gamma^5 \phi \gamma_5 \psi
\] 
\[ + \frac{1}{2} (Z_\phi - 1) \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{2} (Z_\phi - 1) i \bar{\psi} \tilde{\gamma}^\mu \partial_\mu \psi - (Z_m - 1) m \bar{\psi} \psi + \frac{1}{2} (Z_\phi - 1) (\partial_\mu \phi)^2 - \frac{1}{2} (Z_M^2 - 1) M^2 \phi^2
\] 
\[ - \frac{1}{4!} (Z_\lambda - 1) \tilde{\mu}^\epsilon \lambda \phi^4 - i (Z_y - 1) \tilde{\mu}^{\epsilon/2} y \phi \gamma_5 \psi, \] (80)
It follows that
\[
\beta_y^\epsilon = y \left( -\frac{1}{2} \epsilon + \frac{1}{2} \frac{d \ln Z_\phi}{d \ln \mu} + \frac{d \ln Z_\psi}{d \ln \mu} - \frac{d \ln Z_y}{d \ln \mu} \right),
\]
\[
\beta_\lambda^\epsilon = \lambda \left( -\epsilon + 2 \frac{d \ln Z_\phi}{d \ln \mu} - \frac{d \ln Z_\lambda}{d \ln \mu} \right),
\]
\[
\beta_M^2 = M^2 \left( \frac{d \ln Z_\phi}{d \ln \mu} - \frac{d \ln Z_M^2}{d \ln \mu} \right),
\]
\[
\beta_m^2 = m^2 \left( \frac{d \ln Z_\psi}{d \ln \mu} - \frac{d \ln Z_m^2}{d \ln \mu} \right),
\]
\[
\beta_\eta = \eta \left( \frac{d \ln Z_\phi}{d \ln \mu} - \frac{d \ln Z_\eta}{d \ln \mu} \right). \] (81)
The one-loop self-energy of \( \phi \) is given by \( i \Pi_{\text{loop}} + i \Pi_{\text{c.t.}} \) with
\[
i \Pi_{\text{c.t.}} = i(Z_\phi - 1)p^2 - i(Z_M^2 - 1)M^2,
\]
\[
i \Pi_{\text{loop}} = -y^2 \tilde{\mu}^\epsilon \int \frac{d^d \ell}{(2\pi)^d} \left( \frac{1}{\ell^2 - m^2} + \frac{1}{(\ell + p)^2 - m^2} \right) \text{tr}[(\ell + m)\gamma_5(\ell + p + m)\gamma_5]
\] 
\[
= 4i \frac{y^2}{16\pi^2} \left( m^2 B_0 - p^2 B_1 - p^2 B_{21} - dB_{22} \right)
\]
\[ = 4i \frac{y^2}{16\pi^2} \left( \frac{1}{2} p^2 B_0 - A \right). \] (82)
The divergent part is
\[
(P_{\text{loop}})^{\text{pole}}_{\epsilon = 0} = 8 \frac{y^2}{16\pi^2} \left( \frac{1}{2} p^2 - m^2 \right) \frac{1}{\epsilon}. \] (83)
\( Z_\phi \) and \( Z_{M^2} \) are determined to absorb the divergent part as
\[
Z_\phi - 1 = -4 \frac{y^2}{16\pi^2} \frac{1}{\epsilon}, \quad Z_{M^2} - 1 = -8 \frac{y^2}{16\pi^2} \frac{m^2}{M^2} \frac{1}{\epsilon}.
\] (84)

The resultant self-energy of \( \phi \) is
\[
\Gamma_2(p^2) = p^2 - M^2 + 4 \frac{y^2}{16\pi^2} \left[ \frac{p^2}{2} \left[ -\ln \left( \frac{m^2}{\mu^2} \right) - 2J_s \left( \frac{p^2}{m^2} \right) \right] + m^2 \ln \frac{m^2}{\mu^2} - m^2 \right],
\] (85)

where \( J_s(r) \) is given by Eq. (16). For the heavy mass limit \( (m^2 \gg p^2) \), \( J_s(r) \rightarrow -r/12 \) \((r \rightarrow 0)\). The pole mass squared of the scalar field \( \phi \) satisfies \( \Gamma_2(M^2_{\text{phys}}) = 0 \). The relation between the pole and MS masses is given by
\[
M^2_{\text{phys}} = M^2(\mu) - 4 \frac{y^2}{16\pi^2} m^2 \left[ \ln \left( \frac{m^2}{\mu^2} \right) - 1 \right] + 2 \frac{y^2}{16\pi^2} M^2 \ln \left( \frac{m^2}{\mu^2} \right).
\] (86)

The pole wave function is given by
\[
Z_{\phi}^\text{pole} = \left( \frac{\partial}{\partial p^2} \Gamma_2 \right)_{p^2=M^2_{\text{phys}}}^{-1} = 1 + 2 \frac{y^2}{16\pi^2} \ln \left( \frac{m^2}{\mu^2} \right).
\] (87)

The one-loop self-energy of \( \psi \) is given by \( i\Pi^{\text{loop}} + i\Pi^{\text{c.t.}} \) with
\[
i\Pi^{\text{loop}}(p) = m_\psi \mu \int \frac{d^4\ell}{(2\pi)^4} \frac{i(\ell + m)}{\ell^2 - m^2} \frac{i}{\ell - p} \frac{i}{\ell - m} \left[ \frac{i}{m} \gamma_5 \right],
\]
\[
= -i \frac{y^2}{16\pi^2} \left[ \frac{1}{p} B_1(p^2; M^2) + \frac{1}{m} B_0(p^2; m^2, M^2) \right].
\] (88)

The divergent part,
\[
(i\Pi^{\text{loop}})^\text{pole} = \frac{y^2}{16\pi^2} \left( \phi - 2m \right) \frac{1}{\epsilon},
\] (89)

is canceled by the counter term,
\[
i\Pi^{\text{c.t.}}(p) = i(Z_{\psi} - 1)\phi - i(Z_m - 1)m.
\] (90)

\( Z_\psi \) and \( Z_m \) are given by
\[
Z_\psi - 1 = -\frac{y^2}{16\pi^2} \frac{1}{\epsilon}, \quad Z_m - 1 = -2 \frac{y^2}{16\pi^2} \frac{1}{\epsilon}.
\] (91)

We compute \( \beta \) function of Yukawa coupling. Fig. 5 gives
\[
iM^{\text{loop}} = (\mu^\epsilon/2)^3 \int \frac{d^4\ell}{(2\pi)^4} \gamma_5 \frac{e^{i(\ell + \gamma^5 + m)}}{(\ell + p + q)^2 - m^2 \gamma_5 \ell^2 - m^2 \gamma_5} \frac{i}{(\ell + q)^2 - M^2}
\]
\[
= (\mu^\epsilon/2)^3 \frac{y^2}{16\pi^2} \gamma_5 \left[ -B_0(p^2; M^2, m^2) - (\phi + \bar{q})\gamma^\mu C_\mu + (\phi + \bar{q})mC_0 \right].
\] (92)
Figure 5: One-loop correction to the Yukawa interaction.

The divergent part,

\[(i\mathcal{M}_\text{loop})_{\text{pole}}^{\text{pole}} = -2(y\bar{\mu}/2) \frac{y^2}{16\pi^2} \frac{1}{\epsilon} \gamma_5 , \] (93)

is canceled by the counter term,

\[i\mathcal{M}^\text{ct.} = (Z_y - 1)\bar{\mu}/2 y\gamma_5. \] (94)

\[Z_y \] is given by

\[Z_y - 1 = 2 \frac{y^2}{16\pi^2} \frac{1}{\epsilon}. \] (95)

From Eqs. (81), (84) and (91), we get

\[\beta_\epsilon = -\frac{\epsilon}{2} y + \frac{5y^2}{16\pi^2} = -\frac{\epsilon}{2} y + \beta_y. \] (96)

We compute one-loop corrections to \(T^\mu_\mu\)-inserted correlation functions which appear in the text. Fig. 2 (a) gives

\[i\mathcal{M}_1^{\text{loop}} = (im) y^2 \bar{\mu}(-1) \int \frac{d^d\ell}{(2\pi)^d} \text{tr} \left[ \gamma_5 \frac{i}{\ell + \bar{\mu} - m} \frac{i}{\ell - m} \gamma_5 \frac{i}{\ell + \bar{\mu} - m} \right] + (q \leftrightarrow k) \]

\[= -i4m^2 \frac{y^2}{16\pi^2} \left[ \left( m^2 - \frac{p^2}{2} \right) C_0 - p^2 C_{12} - p^2 C_{23} + dC_{24} \right] + (q \leftrightarrow k). \] (97)

Noting

\[p^2 C_{12} + p^2 C_{23} = -m^2 C_0 - \frac{1}{2}, \] (98)

one obtains

\[\mathcal{M}_1^{\text{loop}} = -8m^2 \frac{y^2}{16\pi^2} \left[ \left( m^2 - \frac{p^2}{2} \right) C_0 + m^2 C_0 + \frac{1}{2} - \frac{d}{4} \left( B_0(p^2) + 2m^2 C_0 + 1 \right) \right] \]

\[= -8m^2 \frac{y^2}{16\pi^2} \left[ -\frac{p^2}{2} C_0 - \frac{2}{\epsilon} + \ln \left( \frac{m^2}{\mu^2} \right) + 2J_s \left( \frac{p^2}{m^2} \right) \right] \]

\[= -8m^2 \frac{y^2}{16\pi^2} \left[ -\frac{2}{\epsilon} + J_f \left( \frac{p^2}{m^2} \right) + \ln \left( \frac{m^2}{\mu^2} \right) + 2J_s \left( \frac{p^2}{m^2} \right) \right] . \] (99)
The $B_0$ and $C_0$ functions lead to $J_s(r)$ and $J_f(r)$ given by Eq. (16) and Eq. (34), respectively.

Fig. 2 (b, c) gives

$$iM_2^{\text{loop}} = -\frac{1}{2}\epsilon y^2 \mu^\epsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - m^2][(\ell + q)^2 - m^2]} + (q \leftrightarrow k) = -2i\frac{y^2}{16\pi^2} \epsilon \left[ m^2 B_0 - q^\mu B_\mu + q^{\mu\nu} B_{\mu\nu} \right] + (q \leftrightarrow k)$$

$$= -4i\frac{y^2}{16\pi^2} \left[ - \left( 1 + \frac{\epsilon}{6} \right) A \right]$$

$$= 8i\frac{y^2}{16\pi^2} m^2. \quad (100)$$

### A.3 Summary of one-loop functions

One-loop functions are based on Refs. \[53, 54\] (see also Appendix F of Ref. \[55\]). One point integral is defined as

$$\tilde{\mu}^\epsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2 - m^2} = \frac{i}{16\pi^2} A(m^2). \quad (101)$$

The explicit form is

$$A = m^2 \left( \frac{2}{\epsilon} - \ln \left( \frac{m^2}{\mu^2} \right) + 1 \right). \quad (102)$$

Two point integrals are defined as

$$\tilde{\mu}^\epsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - m_1^2][(\ell + p)^2 - m_2^2]} = \frac{i}{16\pi^2} B_{0;\mu;\mu'}(p^2; m_1^2, m_2^2), \quad (103)$$

where

$$B_\mu = p_\mu B_1,$$

$$B_{\mu\nu} = g_{\mu\nu} B_{22} + p_\mu p_\nu B_{21}. \quad (104)$$

For our purpose, we can take $m_1 = m_2 = m$:

$$B_1 = -\frac{1}{2} B_0,$$

$$B_{22} = \frac{1}{6} \left[ A + 2m^2 B_0 - \frac{p^2}{2} B_0 + 2m^2 - \frac{p^2}{3} \right], \quad (105)$$

$$B_{21} = \frac{1}{3k^2} \left[ A - m^2 B_0 + p^2 B_0 - m^2 + \frac{p^2}{6} \right].$$

The explicit form with a Feynman parameter integral is

$$B_0 = \frac{2}{\epsilon} - \int_0^1 dx \ln \left( \frac{m^2 - x(1-x)p^2}{\mu^2} - i\epsilon \right). \quad (106)$$
Three point integrals are defined as

\[ \tilde{\mu} \int \frac{d^4 \ell}{(2\pi)^d} \frac{1; \ell; \ell; \ell}{([\ell^2 - m_1^2][((\ell + q)^2 - m_2^2)]([\ell + q + k]^2 - m_3^2])} = \frac{i}{16\pi^2} C_{0; \mu; \rho \sigma}(q^2, k^2, p^2; m_1^2, m_2^2, m_3^2), \]

where \( p + q + k = 0 \) and

\[
C_\mu = q_\mu C_{11} + k_\mu C_{12},
C_{\mu \nu} = g_{\mu \nu} C_{24} + q_\mu q_\nu C_{21} + k_\mu k_\nu C_{22} + (q_\mu k_\nu + k_\mu q_\nu) C_{23}. \]

For our purpose, again we can take \( m_1 = m_2 = m_3 = m \):

\[
C_{11} = \frac{1}{p^2} \left[ B_0(q^2) - B_0(p^2) - p^2 C_0 \right],
C_{12} = \frac{1}{p^2} \left[ B_0(p^2) - B_0(k^2) \right],
C_{24} = \frac{1}{4} \left[ B_0(p^2) + 2m^2 C_0 + 1 \right],
C_{21} = -\frac{1}{2p^2} \left[ 3B_0(p^2) - 3B_0(p^2) - 2p^2 C_0 \right],
C_{23} = -\frac{1}{2p^2} \left[ 2B_0(p^2) - 2B_0(k^2) + 2m^2 C_0 + 1 \right],
C_{22} = -\frac{1}{2p^2} \left[ B_0(p^2) - B_0(k^2) \right].
\]

Here, we use a short-hand notation \( B_0(p^2) \equiv B_0(p^2; m^2, m^2) \). The explicit form with Feynman parameter integrals is

\[
C_0 = -\int_0^1 dx \int_0^{1-x} dy \frac{1}{-p^2 xy + m^2 - i\epsilon_{ad}}.
\]