Mackey Theory for $p$-adic Lie groups

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Abstract
This paper gives a $p$-adic analogue of the Mackey theory, which relates representations of a group of type $G = H \times_t A$ to systems of imprimitivity.

Keywords
cosmooth projection valued measure, system of imprimitivity, cosmooth system of imprimitivity, smooth representation

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0 Introduction

There are many methods for representations of the groups which are semi-direct products. For Heisenberg groups, there are Stone-Von Neumann Theorem and Weil’s acta paper. For general cases, one can use Jacquet functor. In this paper, we consider smooth representations of a group $G = H \times_t A$ with $H$ a locally compact and totally disconnected group and $A$ an Abelian topological group such that $A$ and its dual $\hat{A}$ are both locally compact and totally disconnected Abelian groups. Our method is different from Weil’s paper and doesn’t need Jacquet functor, but is along Mackey’s idea. For Jacquet functor, [3] is a good reference.

Mackey has considered representations of Lie groups of type $G = H \times_t \mathbb{R}^n$ with $H$ a Lie group. He relates the representations of $G$ to systems of imprimitivity Of $(H, \mathbb{R}^n)$ (cf. Lemma 1 and Lemma 2 in our case). There is a one to one correspondence between them. A system of imprimitivity of $(H, \hat{A})$ means a representation $\pi$ of $H$ and a projection valued measure $P$ based on $\hat{A}$ such that

$$\pi(h)P_E\pi(h)^{-1} = P_{h[E]}$$

where $h \in H$ and $E$ is a Borel subset of $\hat{A}$. In [7], Varadarajan relates systems of imprimitivity of $(H, A)$ to “cocycles”. The calculate of “cocycles” is not an easy work. In our case, “cocycles” aren’t needed, because the topology is better (totally disconnected) and the representations are smooth, i.e., “locally
constant". We make the sheaf theory in the sense of Bernstein and Zelevinsky instead.

Section 1 gives the spectral decomposition of a smooth representation of $A$, which corresponds to Fourier analysis of $\mathbb{R}^n$ in real case. Section 2 states Mackey Theory of $p$-adic groups.

The sheaf theory of B-Z makes a projection valued measure $(P; \hat{A}, V)$ i.e. a $C_c^\infty(A)$-module $V$ into a sheaf on $\hat{A}$. By it, we change the representation space into the space of sections of the corresponding sheaf. We then change the sections into certain “compact” supported functions on $H$. Then we find that all of our representations are induced representations.

At last, we shall point out that all of our representations are complex and smooth, and that all functions are complex valued.

1 Spectral decomposition

1.1 A fact

Let $B$ be a compact and totally non-connected Abelian group. Let $C_{loc}(B)$ denote the set of all locally constant functions on $B$, then $\hat{B}$, the dual of $B$, is contained in $C_{loc}(B)$. We show that $\hat{B}$ generates $C_{loc}(B)$, or more precisely that every function in $C_{loc}(B)$ is a finite linear sum of elements of $\hat{B}$. To see this, we need only to show that $\chi_E$ for every open compact subset $E$ is so. Let $B_1$ be an open subgroup such that $\chi_E$ is constant on every $xB_1$ for $x \in B$. We can regard $\chi_E$ as a function on $B/B_1$, which is a finite group. There are $n = [B : B_1]$ elements $\hat{b}_1, ..., \hat{b}_n$ in $\hat{B}$ such that $\hat{b}_1, ..., \hat{b}_n$ are all characters of $B/B_1$. Then $\chi_E$ is a linear sum of $\hat{b}_1, ..., \hat{b}_n$. Therefore, we see that $\hat{B}$ generates $C_{loc}(B)$.

1.2 Cosmooth projection valued measure and spectral decomposition

Let $A$ and $\hat{A}$, the dual of $A$, be locally compact and totally non-connected Abelian groups. We will make this assumption in the following. $A = \mathbb{Q}_p$ is such an example, but $\mathbb{Q}_p^\times$ is not.

Let $U$ be a compact open subgroup of $A$, then the dual of $A/U$ is $U^\perp = \{ \hat{a} \in \hat{A} | \forall u \in U, < \hat{a}, u > = 1 \}$. Since $A/U$ is discrete, $U^\perp$ is a compact group. Since the topology of $\hat{A}$ is the open-compact topology, $U^\perp$ is an open subgroup of $\hat{A}$. Therefore $U^\perp$ is compact open subgroup of $\hat{A}$.

Let $(\pi, V)$ be a smooth representation of $A$. Fix a vector $v \in V$. There is a compact open subgroup $U$ of $A$ which fixes $v$. Let $V^U$ be the subspace whose vectors are fixed by $U$. Then $V^U$ is stable under $A$. We regard $A/U$ as functions on $U^\perp$. By the result of section 1.1, $A/U$ generates $C_{loc}(U^\perp)$, so we can extend $\pi|V^U$ to a representation $\pi^U$ of the algebra $C_{loc}(U^\perp)$. Write $P^U_E$ for $\pi^U(\chi_E)$,
where $E$ is an open subset of $U^\perp$. We see that

\begin{align}
P_{U^\perp}^U & = I, P_{\phi}^U = 0 \quad (1.1.1) \\
P_E^U P_F^U & = P_{E \cap F}^U \quad (1.1.2) \\
P_{U_i}^U & = \sum P_{E_i}^U \quad (1.1.3)
\end{align}

where $E, F, E_i$ are open subsets of $U^\perp$ and $E_i \cap E_j = \phi$ unless $i = j$. We call such a system $(P; U^\perp, V^U)$ a projection valued measure based on $U^\perp$.

We have

\[ \pi(a)|_{V^U} = \int_{U^\perp} x(a) dP^U(x) \quad (1.2). \]

It is easy to see that there exists an unique $(P; U^\perp, V^U)$ such that (1.2) is satisfied, by applying the basic fact of section 1.1.

We extend $P^U$ to a projection valued measure based on $\hat{A}$ by setting $P_{E}^U = P_{E \cap U^\perp}^U$. Then it is easy to see that

\begin{align}
P_{\hat{A}}^U & = I, P_{\phi}^U = 0 \quad (1.3.1) \\
P_E^U P_F^U & = P_{E \cap F}^U \quad (1.3.2) \\
P_{U_i}^U & = \sum P_{E_i}^U \quad (1.3.3)
\end{align}

where $E, F, E_i$ are open subsets of $\hat{A}$ and $E_i \cap E_j = \phi$ unless $i = j$.

We can define $P^U$ for other compact open subgroup $U'$ of $A$ in the same way.

If $v \in V^U \cap V^{U'}$, then

\[ \pi(a)v = \int x(a)(dP^U(x))v, \]

\[ \pi(a)v = \int x(a)(dP^{U'}(x))v. \]

Let $\hat{v} \in \hat{V}$, the dual of $V$. Then

\[ < \pi(a)v, \hat{v}> = \int x(a) < dP^U(x)v, \hat{v}> = \int x(a) < dP^{U'}(x)v, \hat{v}> \]

Applying the fact in section 1.1 with $U^\perp \cdot U'^\perp$ instead of $B$ and the above formula, we see that for each $E$,

\[ < P^U(E)v, \hat{v}> = < P^{U'}(E \cap U^\perp \cdot U'^\perp)v, \hat{v}> = < P^{U'}(E \cap U^\perp \cdot U'^\perp)v, \hat{v}> = < P^{U'}(E)v, \hat{v}> \]
Since \( \hat{v} \) is arbitrary,
\[
P^{\hat{v}}(E)v = P^{\hat{v}'}(E)v
\]
(1.4).

By (1.4), we can patch all \( P^{\hat{v}} \) to a \( P \) such that
\[
P_{\hat{A}} = I, \ P_{\phi} = 0
\]
(1.5.1)
\[
P_E P_F = P_{E \cap F}
\]
(1.5.2)
\[
P_{\bigcup E_i} = \sum P_{E_i}
\]
(1.5.3)
where \( E, F, E_i \) are as in (1.3). Such a \( P \), i.e. a \( P \) satisfying (1.5.1), (1.5.2) and (1.5.3) is called a projection valued measure.

Furthermore, for any given \( v \in V \), there exists a compact open subset \( E(v) \) such that
\[
P_{E} v = P_{E \cap E(v)} v
\]
(1.5.4)

We call a projection valued measure \( (P; \hat{A}, V) \) satisfying (1.5.4) a **cosmooth projection valued measure**.

Now, let \( (P; \hat{A}, V) \) be a cosmooth projection valued measure. For any \( v \in V \), define
\[
\pi(a)v = \int_{E(v)} x(a) dP(x) v
\]
(1.6).

We can show that \( \pi(a)v \) does not depend on the choice of \( E(v) \). In fact, for another choice \( E'(v) \),
\[
P_{E} v = P_{E \cap E(v) \cap E'(v)} v
\]
and therefore (1.6) become
\[
\pi(a)v = \int_{E(v) \cap E'(v)} x(a) dP(x) v
\]
(1.6.1)
So it is not depend on \( E(v) \).

Write (1.6) simply as
\[
\pi(a) = \int_{\hat{A}} x(a) dP(x)
\]
(1.6').

For every compact open subgroup \( E \) of \( \hat{A} \), by formula (1.5.1)-(1.5.3), we can define
\[
\pi(f)v = \int_{E} f(x) dP(x) v, \quad v \in P_{E} V
\]
which defines a representation of the algebra \( C_{loc}(E) \) on \( P_{E} V \) and therefore \( \pi \) is a representation of \( A \) on \( P_{E} V \). So (1.6) defines a representation of \( A \) on \( V \).

We obtain the main result of this section:

**Theorem 1.** For a smooth representation \( (\pi, V) \) of \( A \), there exists a unique cosmooth projection valued measure \( (P; \hat{A}, V) \) such that
\[
\pi(a)v = \int_{\hat{A}} x(a) dP(x) v \quad a \in A, \ v \in V.
\]
Conversely, given a cosmooth projection valued measure \((P; \hat{A}, V)\), the above formula defines a smooth representation \(\pi\) of \(A\).

Furthermore, we see that an operator on \(V\) commutes with \(\pi\) if and only if it commutes with \(P\).

In the next part, \((1.6')\) is always in the sense of \((1.6)\).

2 Representation and system of imprimitivity

2.1 Semidirect product

Let \(H\) be a locally compact and totally disconnected group and \(A\) be as in section 1. Assume that there is a continuous homomorphism \(t\) of \(H\) into the automorphism group of \(A\). We write \(h[a]\) simply for \(t_h(a)\). We now define a group \(G = H \ltimes_t A\) by

\[
(h, a)(h', a') = (hh', at_h(a'))
\]

(2.1).

It is easy to verify that \(G\) is really a group with the identity \(e = (e_H, e_A)\).

Furthermore

\[
(h, a)^{-1} = (h^{-1}, h^{-1}[a])
\]

(2.2).

\(G\) is called the semidirect product of \(H\) and \(A\) relative to \(t\). Since \(t\) is continuous, \(G\) becomes a topological group with the product topology.

A quick calculation shows that

\[
(h, a)(h', a')(h, a)^{-1} = (hh'h^{-1}, ah[a']t_{hh'h^{-1}[a^{-1}]})
\]

(2.3).

It follows that \(\hat{A} = \{(e_H, a) : a \in A\}\) is a closed normal subgroup of \(G\), and that

\[
(h, a)(e_H, a')(h, a)^{-1} = (e_H, ah[a']a^{-1})
\]

(2.4).

We put

\[
\hat{H} = \{(h, e_A) : h \in H\},
\]

then \(\hat{H}\) is a closed subgroup of \(G\). We identity \(H\) with \(\hat{H}\) and \(A\) with \(\hat{A}\), then we have

\[
G = AH
\]

(2.5.1),

\[
\{e\} = A \cap H
\]

(2.5.2),

\[
h[a] = hah^{-1}
\]

(2.5.3).

2.2 Representation of \(G\) and system of imprimitivity

In this section, we relate a smooth representation of \(G\) to a cosmooth system of imprimitivity.

Definition. Let \(X\) be a continuous \(H\)-space. A system of imprimitivity for \((H, X)\) acting on \(V\) is a pair \((\pi, P; V)\), where \(\pi\) is a smooth representation
of $H$ on $V$ and $P(E \to P_E)$ is a projection valued measure based on $X$, such that they satisfy a relation:
\[
\pi_h P_E \pi_h^{-1} = P_{h,E}
\] (2.6),
where, $h \in H$ and $E$ is an open subset of $X$. Furthermore, if $P$ is cosmooth, then $(\pi, P; V)$ is called **cosmooth system of imprimitivity**.

Two systems $(\pi, P; V)$ and $(\pi', P'; V')$ based on the same $H$-space $X$ are said to be **equivalent** if and only if there exists an isomorphism $T$ from $V$ to $V'$ such that
\[
\pi'(h) = T \pi(h) T^{-1}
\] (2.7.1),

\[
P'_E = TP_E T^{-1}
\] (2.7.2),
where, $h \in H$ and $E$ is an open subset of $X$. We say that a cosmooth system of imprimitivity $(\pi, P; V)$ is **irreducible** if and only if there is no subspace other than $0$ and $V$ which is invariant under all $P_E$ and $\pi_h$.

We define a homomorphism $t'$ of $H$ to the the automorphism group of $\hat{A}$ by
\[
t'_h(\hat{a}) (a) = \hat{a} (t_{h^{-1}} (a))
\] (2.8),
and we write simply $h[a]$ for $t'_h(\hat{a})$. Then $\hat{A}$ becomes a continuous $H$-space.

If $\pi$ is a smooth representation of $G$, then $\pi$ restrictions to $A$ and $H$ are also smooth.

**Lemma 1.** Let $\pi_1$ and $\pi_2$ be smooth representations of $A$ and $H$ respectively in a vector space $V$, and let $P$ be the corresponding cosmooth projection valued measure on $A$ for $\pi_1$. Then a necessary and sufficient condition such that there exists a smooth representation $\pi$ of $G$ in $V$ whose restrictions to $A$ and $H$ are $\pi_1$ and $\pi_2$ respectively, is that $(\pi_2, P; V)$ is a cosmooth system of imprimitivity for $H$ based on $\hat{A}$. In this case, $\pi$ is unique.

**Proof.** Let $\pi$ be a smooth representation of $G$ in $V$, and let $\pi_1, \pi_2$ be the restrictions to $A, H$ respectively. Now
\[
hah^{-1} = h[a]
\] (2.9)
so that
\[
\pi_2(h) \pi_1(a) \pi_2(h)^{-1} = \pi_1(h[a])
\] (2.10)
for all $(h, a) \in H \times A$. Let $P$ be the corresponding cosmooth projection valued measure on $\hat{A}$ for $\pi_1$. Now an easy calculation show that the projection valued measure for the representation $\{a \to \pi_2(h) \pi_1(a) \pi_2(h)^{-1}\}$ of $A$ is $\{E \to \pi_2(h) P_E \pi_2(h)^{-1}\}$, and that for the representation $\{a \to \pi_1(h[a])\}$ of $A$ is $\{E \to P_{h[E]}\}$. In view of the uniqueness of the cosmooth projection valued measure which corresponds to a representation of $A$, we infer that
\[
\pi_2(h) P_E \pi_2(h)^{-1} = P_{h[E]}
\] (2.11),
so $(\pi_2, P; V)$ is a cosmooth system of imprimitivity of $H$ based on $\hat{A}$.
Now let us state with \( \pi_1, \pi_2 \) and \( P \) such that

\[
\pi_2(h)P_E\pi_2(h)^{-1} = P_{h[E]},
\]

then we gain (2.10). Define \( \pi \) on \( G \)

\[
\pi(ah) = \pi_1(a)\pi_2(h).
\]

Then (2.10) is enough to secure the fact that \( \pi \) is a representation. Since the restriction of \( \pi \) to \( A \) and \( H \) are smooth and \( G \) is equipped with the product topology, \( \pi \) is smooth, too. \( \square \)

The lemma just stated enables us to relate a smooth representation of \( G \) to a cosmooth system of imprimitivity of \( H \) based on \( \hat{A} \). The following lemma tells us that the relation is one to one in the sense of equivalence.

**Lemma 2.** A smooth representation \( \pi \) of \( G \) on \( V \) is irreducible if and only if the corresponding cossmooth system of imprimitivity for \( H \) based on \( \hat{A} \) is irreducible. Two smooth representations of \( G \) are equivalent if and only if the corresponding cossmooth systems of imprimitivity are equivalent.

**Proof.** For the first assertion we need only to prove that any subspace \( V_1 \) of \( V \) is invariant under \( \pi_1 = \pi|A \) if and only if it is invariant under the corresponding cossmooth projection valued measure \( P \) based on \( \hat{A} \).

If \( V_1 \) is invariant under \( P \), then by the definition (1.6), \( V_1 \) is invariant under \( \pi_1 \). Let \( V_1^\perp \) be the subspace of \( \hat{V} \) (the space of linear functions on \( V \)) which is zero on \( V_1 \). For any \( v \in V_1, \hat{v} \in V_1^\perp, \)

\[
< \pi_1(a)v, \hat{v} >= 0
\]

Now

\[
< \pi_1(a)v, \hat{v} > = \int_{E(v)} x(a) < dP(x)v, \hat{v} > \tag{2.12}
\]

and the basic fact in section 1.1 tell us that if \( v \in V_1 \), then

\[
< P(E)v, \hat{v} >=< P(E \cap E(v))v, \hat{v} >= 0
\]

for any open subset of \( \hat{A} \), and any \( \hat{v} \) in \( V_1^\perp \). Therefore \( P(E)v \in V_1 \). In other word, \( V_1 \) is invariant under \( P \).

For the second assertion, let \( \pi^i \) be smooth representations of \( G \) in \( V^i \), and let \((\pi^i_2, P^i; V^i)\) be the corresponding cossmooth systems of imprimitivity, \((i=1,2)\). Let \( T \) be a isomorphism from \( V^1 \) to \( V^2 \). As in the proof of the lemma 1, the cossmooth projection valued measure corresponding to \( T(\pi^1|A)T^{-1} \) is \( TP^1T^{-1} \). Therefore by the uniqueness stated in Theorem 1 shows that, \( \pi^2|A = T(\pi^1|A)T^{-1} \) if and only if \( P^2 = TP^1T^{-1} \). From this the second assertion follows. \( \square \)

Lemma 2 tells us that, to study smooth representations of \( G \) is equivalent to study the cossmooth systems of imprimitivity of \( (H, \hat{A}) \).
2.3 Sheaf

In this section, we use the concepts of presheaf and sheaf in the sense of Bernstein and Zelevinsky.

We assume that $X$ is a totally disconnected locally compact space and that $\mathcal{I}_c$ is the set of all compact open subsets of $X$.

Let $C_c^\infty(X)$ be the sheaf of smooth complex valued functions on $X$ with compact support, let $\mathcal{M}$ be a sheaf of vector spaces over $X$ with base $\mathcal{I}_c$. Then $\mathcal{M}$ is naturally a sheaf of module for $C_c^\infty(X)$.

We call a $C_c^\infty(X)$-module $M$ cosmooth if for every $m \in M$, there exists a compact open subset $U$ of $X$ such that $1_U m = m$.

We have the following important proposition. For a proof, see [3].

**Proposition 1.** Let $M$ be a cosmooth $C_c^\infty(X)$-module. We associate a presheaf $\mathcal{M}$ in the follow way. If $U \in \mathcal{I}_c$, let $\mathcal{M}(U) = 1_U \cdot M$. If $U \supseteq V$, with $U, V \in \mathcal{I}_c$, we define a restriction map $\rho_{U,V} : \mathcal{M}(U) \to \mathcal{M}(V)$ by $\rho_{U,V}(m) = 1_V m$. Then $\mathcal{M}$ is a sheaf.

2.4 Irreducible smooth representation

Let $\pi$ be an irreducible admissible representation of $G$ and let $(\pi_2, P; V)$ be the corresponding cosmooth system of imprimitivity of $H$ based on $\hat{A}$ (see lemma 1). Due to (1.5.1)-(1.5.4), $V$ becomes a cosmooth $C_c^\infty(\hat{A})$-module by setting the action of $\chi_E$ on $V$ to be $P_E$. We can associate a sheaf $\mathcal{V}$ to $V$ via proposition 1. $H$ has an action $\Pi_2$ on $\mathcal{V}$ in the natural way, under which, $\mathcal{V}(E)$ is mapped to $\mathcal{V}(h(E))$ and $\mathcal{V}_x$ is mapped to $\mathcal{V}_{h(x)}$ by $\Pi_2(h)$.

**Definition.** Let $X$ be a $T_1$ $H$-space. $X$ is said to be a smooth $H$-space or in other word, $H$ acts smoothly on $X$, if for any two points $x_1, x_2$ in $X$, either $x_1$ and $x_2$ lie in the same orbit of $H$ in $X$, or there is $H$-invariant open subset of $X$ such that exact one of $x_1, x_2$ lies in it.

Now, we add a condition that $H$ acts smoothly on $\hat{A}$. Note that if $H$ is a compact group, then it acts always smoothly on $\hat{A}$.

**Lemma 3.** Let $(\pi_2, P; V)$, $(\pi_1^2, P^1; V^1)$ and $(\pi_2^2, P^2; V_2)$ be three irreducible cosmooth systems of imprimitivity, and let $\mathcal{V}$, $\mathcal{V}^1$ and $\mathcal{V}^2$ be the sheaves associated to them constructed by proposition 1. Then $\text{supp}(\mathcal{V})$ lies on exact one orbit of $H$. If $\text{supp}(\mathcal{V}^1) \neq \text{supp}(\mathcal{V}^2)$, then $(\pi_1^2, P^1; V^1)$ and $(\pi_2^2, P^2; V_2)$ are two inequivalent cosmooth systems of imprimitivity.

**Proof.** We assert that $\text{supp}(\mathcal{V})$ lies on exact one $H$-orbit. Otherwise, there are two orbits $H\hat{a}_1, H\hat{a}_2 \subset \text{supp}(\mathcal{V})$, then there is an $H$-invariant open subset $\hat{E}$ such that exact one of $H\hat{a}_1, H\hat{a}_2$, say $H\hat{a}_1$, lies in $\hat{E}$. Now, let $V = P_{\hat{E}} V$, then $V$ is a nontrivial subspace of $V$, which is invariant under $(\pi_2, P)$. This contradicts the irreducibility of $(\pi_2, P; V)$.

Let $(\pi_1^1, P^1; V^1), (\pi_2^2, P^2; V^2)$ be two irreducible cosmooth systems of imprimitivity. If $\text{supp}(\mathcal{V}^1), \text{supp}(\mathcal{V}^2)$ lie in two different $H$-orbits $Hx_1, Hx_2$. Suppose $\hat{E}$ is a $H$-invariant open subset such that exact one of these two orbits say
$Hx_1$ lies in it, then $P_1^1 \neq 0$, but $P_1^2 = 0$. Therefore $(\pi_1^1, P_1^1; V^1), (\pi_2^2, P_2^2; V^2)$ are two inequivalent cosmooth systems of imprimitivity.

Now let $(\pi_2, P; V)$ be an irreducible cosmooth system of imprimitivity, with $\text{supp}(V)$ lying in an orbit $Hx_0$. Let $H_0$ be the stable subgroup of $x_0$ in $H$. It is easy to see that $V_{x_0}$, denoted by $V_0$, is invariant under $H_0$. Let $\pi_0$ denote the action of $H_0$ on $V_0$. By proposition 1, we can identify the sections of $V$ with the vectors in $V$. For every section $s$ of $V$, define a function on $H$ with value in $V$, by

$$F_s(h) = \Pi_2(h)s(h^{-1}[x_0])$$

(2.13).

Let $C_c^\infty(H/H_0, \pi_0, V_0)$ denote the space of locally constant functions $f$ with values in $V_0$ whose support is compact mod $H_0$, and satisfies

$$f(h_0h) = \pi_0(h_0)f(h) \forall h_0 \in H_0 \forall h \in H$$

(2.14).

**Lemma 4.** $F_s$ belongs to $C_c^\infty(H/H_0, \pi_0, V_0)$.

**Proof.** It is easy to see that $F_s$ satisfies (2.14) and its support is compact mod $H_0$. Let $v_s$ be the vector in $V$ corresponding to $s$. Then there is a compact open subgroup $H_s$ which fixes $v_s$. Note that

$$(\pi_2(h)s)(x) = \Pi_2(h)s(h^{-1}[x]).$$

(2.15)

Therefore we have $\forall \ h \in H_s$,

$$\Pi_2(h)s(h^{-1}[x]) = s(x) \forall x \in \hat{A}.$$ 

Especially, $\forall \ h \in H_s$

$$F_s(hh) = \Pi_2(hh)s(h^{-1}g^{-1}[x_0]) = \Pi_2(g)s(g^{-1}[x_0]) = F_s(g)$$

Thus $F_s \in C_c^\infty(H/H_0, \pi_0, V_0)$. \qed

Conversely, for a function $f \in C_c^\infty(H/H_0, \pi_0, V_0)$, we can define

$$F_f(h[x_0]) = \Pi_2(h)f(h^{-1})$$

(2.16).

By (2.14), it is well defined.

**Lemma 5.** $F_f$ is a section of $V$.

**Proof.** It is easy to see that it has compact support.

There is a compact open subgroup $H_f$ such that for each $h_f \in H_f$, $f(hh_f) = f(h)$. Then (2.16) tells us that

$$\Pi_2(h_f)F_f(x) = F_f(h_f[x]) \forall h_f \in H_f, x \in Hx_0$$

(2.17).
We are now to prove \( \mathcal{F}_f \) is a section. Fix an \( x \in Hx_0 \). We can select a section \( s \) such that \( s(x) = \mathcal{F}_f(x) \). Let \( H_s \) be a compact open subgroup on \( H \) such that \( s \) is fixed by \( H_s \). By (2.15), we gain

\[
\Pi_2(h_s)s(x) = s(h_s[x]) \quad \forall h_s \in H_s, \ x \in Hx_0
\]

Comparing (2.17) and (2.18), we see that

\[
s(h[x]) = \mathcal{F}_f(h[x]) \quad \forall h \in H_s \cap H_f.
\]

Thus \( \mathcal{F}_f \) is really a section. \( \Box \)

It is obviously that \( \mathcal{F} \mathcal{F} \) and \( \mathcal{F} \mathcal{F} \) are both identity, or equivalently, \( \mathcal{F} = \mathcal{F}^{-1} \).

Now \( (\mathcal{F} \pi_2 \mathcal{F}, \mathcal{F} P \mathcal{F}; C^\infty\left(\Pi/\Pi_0, \pi_0, V_0\right)) \) is a cosmooth system of imprimitivity that is equivalent to \((\pi_2, \mathcal{P}; \mathcal{V})\). Write \((\pi_2, \mathcal{P}; \mathcal{V})\) for \((\mathcal{F} \pi_2 \mathcal{F}, \mathcal{F} P \mathcal{F}; C^\infty\left(\Pi/\Pi_0, \pi_0, V_0\right))\).

A direct calculation implies that: \( \forall f \in \mathcal{V}, \)

\[
\begin{align*}
(\pi_2(h_1)f)(h) &= f(hh_1) \quad (2.19.1), \\
\mathcal{P}f &= \chi \mathcal{E} f \quad (2.19.2),
\end{align*}
\]

where, \( h_1 \in \mathcal{H}, \mathcal{E} = \{h \in H : h[x_0] \in E\} \) and \( \mathcal{E}^{-1} = \{h \in H : h^{-1} \in \mathcal{E}\} \).

We see that \( \pi_2 \) is just the compact induced representation of \( \pi_0 \). Denote by \( \pi \), the representation of \( G \) corresponding to \((\pi_2, \mathcal{P}; \mathcal{V})\). A direct calculation shows:

\[
\begin{align*}
(\pi(a)f)(h_1) &= < a, h_1^{-1}x_0 > \cdot f(h_1) \quad (2.20.1), \\
(\pi(h)f)(h_1) &= f(h_1h) \quad (2.20.2),
\end{align*}
\]

where \( a \in A, h \in H \) and \( f \in \mathcal{V} \).

**Lemma 6.** The representation \( \pi \) is equivalent to the representation \( \pi \) mentioned at the beginning of this subsection.

This is just a consequence of Lemma 2.

Let \((\pi_2^1, \mathcal{P}^1; \mathcal{V}^1)\) and \((\pi_2^2, \mathcal{P}^2; \mathcal{V}^2)\) be two irreducible cosmooth systems of imprimitivity, supported both on \( Hx_0 \). Then it is easy to see that:

\[
\text{Hom}(\pi_2^1, \mathcal{P}^1; \mathcal{V}^1), (\pi_2^2, \mathcal{P}^2; \mathcal{V}^2)) \cong \text{Hom}(\Pi^1, \mathcal{V}^1), (\Pi^2, \mathcal{V}^2))
\]

\[
\cong \text{Hom}(\pi_0^1, \pi_0^2) \cong \text{Hom}(\pi^1, \pi^2) \quad (2.21)
\]

Therefore \( \pi^1 \) or \((\pi^1, \mathcal{P}^1; \mathcal{V}^1)\) is irreducible if and only if \( \pi_0^1 \) is irreducible. Moreover, \((\pi_2^1, \mathcal{P}^1; \mathcal{V}^1)\) and \((\pi_2^2, \mathcal{P}^2; \mathcal{V}^2)\) are two equivalent irreducible cosmooth systems of imprimitivity if and only if \( \pi_0^1 \) and \( \pi_0^2 \) are two equivalent irreducible smooth representations.

Write \( \chi \) for \( x_0 \) now. For a representation \( \pi_0 \) of \( H_x \), let \( \pi_0 \cdot \chi \) be the representation of \( H_x \times_t A \):

\[
(\pi_0 \cdot \chi)(h \times a) = \chi(a)\pi_0(h) \quad \forall h \in H_x, a \in A.
\]

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It is easy to check that $\pi_0 \cdot \chi$ is a representation. Due to formulas (2.20.1) and (2.20.2), a simple calculate shows that $\pi$ is equivalent to the compact induced representation $\text{Ind}_{H \times \chi}^G (\pi_0 \cdot \chi)$ of $G$.

Now, let $\mathcal{A}$ be a locally compact and totally disconnected Abelian group whose dual $\hat{\mathcal{A}}$ having the same property. Let $H$ be a locally compact and totally disconnected group with a continuous action $t$ on $\mathcal{A}$, and a dual action $t'$ on $\hat{\mathcal{A}}$. Let $G$ be $H \times t \mathcal{A}$.

We obtain the main result:

**Theorem 2.** For each orbit of $t'$, select out a point $\chi$ on it. Every irreducible smooth representation $\pi_0$ of $H \chi$ gives an irreducible smooth representation $\text{Ind}_{H \times \chi}^G (\pi_0 \cdot \chi)$ of $G$. Every irreducible smooth representation of $G$ is equivalent to one obtained in such a way. If furthermore $t'$ is smooth, the representations obtained in such a way are not equivalent with each other.

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