Supersymmetrical separation of variables for Scarf II model: Partial solvability

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received 20 February 2012; accepted 12 March 2012
published online 12 April 2012

PACS 03.65.-w – Quantum mechanics
PACS 03.65.Fd – Algebraic methods
PACS 11.30.Pb – Supersymmetry

Abstract – Recently, a new quantum model — two-dimensional generalization of the Scarf II — was completely solved analytically by the SUSY method for the integer values of a parameter. Now, the same integrable model, but with arbitrary values of a parameter, will be studied by means of supersymmetrical intertwining relations. The Hamiltonian does not allow the conventional separation of variables, but the supercharge operator does, leading to the partial solvability of the model. This approach, which can be called the first variant of SUSY separation, together with the shape invariance of the model, provides the analytical calculation of part of the spectrum and corresponding wave functions (quasi-exact solvability). The model is shown to obey two different variants of shape invariance which can be combined effectively in the construction of energy levels and wave functions.

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Introduction. – Supersymmetrical quantum mechanics gave a new impetus to the development of new analytical methods in the study of different quantum models [1–3]. The main ingredients and results of the SUSY approach in quantum mechanics are isospectral (or, almost isospectral) quantum systems and supersymmetrical intertwining relations [1], shape invariance [4–8], higher-order supercharges [9], supersymmetry of multidimensional [10–14], multiparticle [15] and matrix [16] quantum systems.

One of the directions of the research within the framework of this approach is the investigation of two-dimensional quantum models which are not amenable to the standard [17] separation of variables. Recently, two different variants of the supersymmetrical (SUSY) separation of variables were proposed [8,18–20] for the analysis of such two-dimensional quantum systems. Both procedures are based on supersymmetrical intertwining relations [1,10] with second-order supercharges [13,14,18] and shape invariance [4–8,18] property. The first variant is applicable if the intertwining operator (supercharge) allows the standard separation of variables as it happens for supercharges with Lorentz metric. It was used in practice for two-dimensional generalizations of Morse [8,18], Pöschl-Teller [21], and periodic Lame [19] models. The second variant works if one of the intertwined Hamiltonians allows the standard separation due to the specific choice of parameters, but its superpartner still does not. Such situation gives a chance to solve the problem completely — to find energies and wave functions of all bound states. This procedure was applied recently to Morse [20], Pöschl-Teller [22] models, and quite recently [23], to a new model — the two-dimensional generalization of the Scarf II model. In the latter case, one of intertwined Hamiltonians is separable if one of the parameters takes only negative integer values $a = -k$.

In the present paper, the same two-dimensional Scarf II model but with arbitrary values of parameter $a$ will be studied by means of the first procedure of the SUSY separation of variables. This method provides a part of the energy spectrum and corresponding wave functions in analytical form. Thus, the model belongs to the class of partially (i.e., quasi-exactly-solvable) models — the intermediate class between completely (exactly solvable) and analytically unsolvable models. This class was considered starting from 1980s [24–28]. In particular, the elegant algebraic method of construction of one-dimensional quasi-exactly-solvable (and sometimes, of exactly solvable) quantum models was elaborated in [25–27]. This approach is applicable to two-dimensional problems as well, but only in curved spaces with nontrivial metrics [26]. The
approach of the present paper allows to study the different class of quantum models which do not allow standard separation of variables. It is necessary to notice that by construction these models are integrable with the symmetry operators of fourth order in momenta. In the second section, the zero modes of supercharge will be found, and their linear combinations, which are the eigenfunctions of the Hamiltonian, will be built. In the third section, the shape invariance of the model will be used to enlarge the variety of the known wave functions. It will be shown that the second shape invariance exists for this model leading to additional relations between different wave functions.

**Wave functions in subspace of zero modes of $Q^+$.**

*Formulation of the model.* The main tools of the supersymmetrical approach in two-dimensional quantum mechanics are the supersymmetrical intertwining relations:

$$H^{(1)}Q^+ = Q^+H^{(2)}, \quad Q^-H^{(1)} = H^{(2)}Q^-,$$  \hspace{2.0cm} (1)

for two partner two-dimensional Hamiltonians of the Schrödinger form

$$H^{(i)} = -\Delta^{(i)} + V^{(i)}(\vec{x}), \quad i = 1, 2, \quad \vec{x} = (x_1, x_2),$$  \hspace{2.0cm} (2)

with mutually conjugate supercharges $Q^\pm$ of second order in derivatives. First of all, one has to find solutions of eq. (1), i.e. to find such potentials $V^{(1,2)}(\vec{x})$ and such coefficient functions of second-order supercharges $Q^\pm$ so that (1) are fulfilled. In terms of these unknown functions, we deal with a complicate system of nonlinear differential equations of second order. Due to a suitable choice of Ansätze, a list of particular solutions of (1) was found [13,18]. A part of these solutions was shown to allow the analytical construction of spectra and wave functions: depending on the chosen values of the parameters, the partial [8,18,19,21], and/or complete [20,22] solutions of corresponding models were obtained.

Recently, new results [6] obtained in one-dimensional shape invariance allowed to find also new solutions [7] of (1) in the two-dimensional framework. In particular, the two-dimensional generalization of the Scarf II model was among these new solutions, and just this system will be considered below. The potentials\(^1\)

$$V^{(1),(2)} = -2\lambda^2a(a+1)\left(\frac{1}{\cosh^2(\lambda x_+)} - \frac{1}{\sinh^2(\lambda x_-)}\right) + 2B(A + \lambda)\sinh(2\lambda x_1) + (B^2 - A^2 - 2A\lambda)\cosh^2(2\lambda x_1) + 2B(A + \lambda)\sinh(2\lambda x_2) + (B^2 - A^2 - 2A\lambda)\cosh^2(2\lambda x_2),$$  \hspace{2.0cm} (3)

and the second-order supercharges,

$$Q^+ = (Q^-)^\dagger = 4\partial_+ \partial_- + 4aA\tanh(\lambda x_+)\partial_- + 4aA\coth(\lambda x_-)\partial_+ + 4a^2\lambda^2\tanh(\lambda x_+)\coth(\lambda x_-)$$

$$- 2B(A + \lambda)\sinh(2\lambda x_1) + (B^2 - A^2 - 2A\lambda)\cosh^2(2\lambda x_1)$$

$$+ 2B(A + \lambda)\sinh(2\lambda x_2) + (B^2 - A^2 - 2A\lambda)\cosh^2(2\lambda x_2),$$  \hspace{2.0cm} (4)

solve the intertwining relations (1) \((x_\pm \equiv x_1 \pm x_2, \partial_\pm = \partial/\partial x_\pm; \lambda, a, B, \lambda \) are real parameters, and $A, B > 0$). As is typical for the approach, both potentials (3) are not amenable to the standard separation of variables, but they correspond to the integrable Hamiltonians (2) with symmetry operators of fourth order in derivatives: $R^{(1)} = Q^+Q^-, R^{(2)} = Q^-Q^+$.

Quite recently, it was proven that this model is completely (exactly) solvable for the values of parameter $a = -k$. The whole spectrum of the bound states and the wave functions were found analytically by means of intertwining relations (1) and the shape invariance of (3) under the change $a \rightarrow a - 1$ (see the third section below).

**Zero modes of supercharge $Q^+$.** The same model will be studied below for arbitrary values of $a$ in the framework of the SUSY separation of variables of the first kind [8,18,19,21]. The basic idea is that the supercharge $Q^+$ for the Lorentz (hyperbolic) form of the metric, like in (4), is amenable to the conventional separation of variables [8,18]. Indeed, after a suitable similarity transformation,

$$Q^+ = \exp(\chi(x))q^+\exp(-\chi),$$

$$\chi(\vec{x}) = -a \ln|\cosh(\lambda x_+)|\sinh(\lambda x_-)|,$$

we obtain the operator with separated variables:

$$q^+ = \partial_1^2 - \partial_2^2 - f(x_1) + f(x_2),$$

$$f(x) = \frac{2B(A + \lambda)\sinh(2\lambda x_2) + (B^2 - A^2 - 2A\lambda)}{\cosh^2(2\lambda x_2)},$$  \hspace{2.0cm} (5)

Therefore, depending on the explicit form of $f(x)$, we get a chance to find analytically the zero modes $\Omega_n(\vec{x})$ of supercharge $Q^+$:

$$\Omega_n(\vec{x}) = \exp(\chi(\vec{x}))\omega_n(\vec{x}) = |\cosh(\lambda x_+)|\sinh(\lambda x_-)|^{-a}\omega_n(\vec{x}); \ q^+\omega_n(\vec{x}) = 0.$$  \hspace{2.0cm} (6)

The zero modes $\omega_n(\vec{x}) = \eta_n(x_1)\eta_n(x_2)$ of operator $q^+$ are expressed in terms of solutions of one-dimensional Schrödinger equations:

$$(-\partial_1^2 + f(x_1))\eta_n(x_1) = \epsilon_n\eta_n(x_1).$$  \hspace{2.0cm} (7)

\(^1\)Slightly different notations for coupling constants are chosen here as compared with [23].
as Scarf II (hyperbolic Scarf) system [31], and its spectrum and wave functions were built analytically in terms of Jacobi polynomials [32]:

\[ \eta_n(x) = i^n (\cosh(2\alpha x))^{-A/2} \exp[-(B/2\lambda) \arcsinh(2\lambda x)] P_n^{(\gamma,\beta)}(i \sinh(2\lambda x)), \]

where \( \gamma \equiv -(A/2\lambda + iB/2\lambda + 1/2) \), \( \beta \equiv \gamma^* \), \( \epsilon_n = -(A - 2n\lambda)^2 \).

Therefore, all normalizable zero modes \( \Omega_n(\vec{x}) \) (and their components \( \omega_n \)) are known and can be used to find the wave functions of the Hamiltonian \( H^2(\omega) \).

It follows [8,18] directly from the intertwining relations (1) that the variety of zero modes is closed under the action of operator \( H^2(\omega) \). In other words, the action of \( H^2(\omega) \) onto \( \Omega_n \) gives the linear combination of \( \Omega \):

\[ H^2(\Omega_n(\vec{x})) = \sum_{k=0}^{N} c_{nk} \Omega_k(\vec{x}), \]

where the coefficients \( c_{nk} \) form the \((N+1) \times (N+1)\) constant matrix \( \tilde{C} \). The diagonalization of this matrix, if possible, will provide both wave functions and energy eigenvalues of \( H^2(\omega) \), though not all, in general.

**Calculation of matrix \( \tilde{C} \).** From this point and below, we shall simplify our presentation by choosing the specific value \( \lambda = 1/2 \) in all formulas above. It is more convenient to look for the explicit form of matrix \( \tilde{C} \) in terms of \( \omega_n \), replacing \( H^2(\omega) \) by its similarity transform:

\[ h^{(2)}(\omega_n(\vec{x})) = \sum_{k=0}^{N} c_{nk} \omega_k(\vec{x}), \]

\[ h^{(2)}(\omega_n(\vec{x})) = \left[ 2\epsilon_n - a^2 \right] \sinh x_1 - \sinh x_2 (\cosh x_1 \partial_1 - \cosh x_2 \partial_2) \omega_n. \]

Introducing new variables, \( z_1 = i \sinh x_1 \), \( z_2 = i \sinh x_2 \), we consider separately a part of (11):

\[ (\cosh x_1 \partial_1 - \cosh x_2 \partial_2)\omega_n = i(-1)^n \omega_0 \]

\[ i[(z_1 - z_2) \Delta P_n^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) + \Pi(z_1, z_2)]. \]

The function II appeared in (12) can be written as

\[ \Pi(z_1, z_2) = (1 - z_1^2) \partial_1 P_n^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) - (1 - z_2^2) P_n^{(\gamma,\beta)}(z_1) \partial_2 P_n^{(\gamma,\beta)}(z_2), \]

and by means of relations 22.8.1 of [33] between Jacobi polynomials \( P_n(z) \) and their derivatives over argument, it can be transformed as follows:

\[ \Pi(z_1, z_2) = -n(z_1 - z_2) P_n^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) \]

\[ + 2(n + \gamma)(n + \beta) \left( P_{n-1}^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) \right) \]

\[ - P_n^{(\gamma,\beta)}(z_1) P_{n-1}^{(\gamma,\beta)}(z_2) \].

The expression in brackets is transformed by Christoffel-Darboux formula 22.12.1 [33]:

\[ P_{n-1}^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) - P_n^{(\gamma,\beta)}(z_1) P_{n+1}^{(\gamma,\beta)}(z_2) = \]

\[ (z_2 - z_1) \left( \frac{k_n h_{n-1} - k_{n-1} h_n}{k_{n-1}} \right) \sum_{m=0}^{n-1} \frac{1}{h_m} P_m^{(\gamma,\beta)}(z_1) P_m^{(\gamma,\beta)}(z_2), \]

where the constants \( h_n, k_n \) for Jacobi polynomials are defined in [33] and [32]:

\[ h_n = 2^{\gamma+\beta+1} \frac{\Gamma(n+\gamma+1)\Gamma(n+\beta+1)}{(2n+\gamma+\beta+1)\Gamma(n+\gamma+\beta+1)}, \]

\[ k_n = 2^{-n} \frac{(2n+\gamma+\beta)!}{n!(n+\gamma+\beta)!}. \]

Inserting (13) and (14) into (12) we obtain

\[ (\cosh x_1 \partial_1 - \cosh x_2 \partial_2)\omega_n = \]

\[ i(-1)^n \omega_0 \left[ (A-n) P_n^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) \right] \]

\[ - 2(n+\gamma)(n+\beta) \frac{k_n h_{n-1}}{k_{n-1}} \sum_{m=0}^{n-1} \frac{1}{h_m} P_m^{(\gamma,\beta)}(z_1) P_m^{(\gamma,\beta)}(z_2). \]

Therefore, the relation (11) takes the form

\[ h^{(2)}(\omega_n) = (2\epsilon_n - a^2)\omega_n - 2a(1-\epsilon)^n \omega_0 \]

\[ \left[ (A-n) P_n^{(\gamma,\beta)}(z_1) P_n^{(\gamma,\beta)}(z_2) \right] \]

\[ - 2(n+\gamma)(n+\beta) \frac{k_n h_{n-1}}{k_{n-1}} \sum_{m=0}^{n-1} \frac{1}{h_m} P_m^{(\gamma,\beta)}(z_1) P_m^{(\gamma,\beta)}(z_2), \]

and taking into account that \( \omega_0 P_j^{(\gamma,\beta)}(z_1) P_j^{(\gamma,\beta)}(z_2) = (-1)^j \omega_j \), it can be rewritten as

\[ h^{(2)}(\omega_n) = (2\epsilon_n - a^2 - 2a(A-n))\omega_n \]

\[ - 4a(\gamma+n)(n+\beta) \frac{k_n h_{n-1}}{k_{n-1}} \sum_{m=0}^{n-1} \frac{(-1)^{n+m}}{h_m} \omega_m. \]

Thus, according to definition (9) the matrix elements of \( \tilde{C} \) are

\[ c_{n,m} = 0, \quad \text{for } m > n, \quad c_{n,m} = 2\epsilon_n - a^2 - 2a(A-n), \]

\[ c_{n,m} = - \frac{2a k_n h_{n-1}}{k_{n-1}} (1-\epsilon)^m, \quad \text{for } m < n. \]

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Similarly to the models with Morse [8,18] and Pöschl-Teller [21] potentials, matrix $\hat{C}$ is triangular, and its diagonal elements give immediately the energy eigenvalues of $H^{(2)}$:

$$E_n^{(2)} = 2\epsilon_n - a^2 - 2a(A - n) = -(A - n)^2 - (A - n + a)^2. \quad \text{(17)}$$

**Diagonalization of $\hat{C}$**. In order to find the corresponding wave functions, one has to diagonalize the matrix $\hat{C}$, i.e. to find matrices $\hat{B}$ and $\hat{\Lambda} = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N)$, such that

$$\hat{B}\hat{C} = \hat{C}\hat{\Lambda} \iff \sum_{k=0}^{N} b_{k,l} c_{kl} = \lambda_l b_{l}. \quad \text{(18)}$$

The procedure of diagonalization is specific due to triangularity of $\hat{C}$, and the corresponding algorithm was proposed for the Morse model in [8]. If this task is solved, and matrix $\hat{B}$ is found, one can construct a variety of eigenfunctions of $H^{(2)}$ as linear combinations of zero modes $\Omega_n$:

$$\Psi_{N-n}^{(2)}(\vec{x}) = \sum_{l=0}^{N} b_{n,l} \Omega_{l}(\vec{x}). \quad \text{(19)}$$

It is convenient to start an algorithm for the solution of the system of the linear equations (18) by solving the first line — with $i = 0$. Indeed, taking successively $l = N, N - 1, \ldots, l = 0$ in (18), we obtain that $b_{0,N}$ is an arbitrary normalization factor, and other $b_{0,l}$ are found iteratively:

$$b_{0(N-1)} = b_{0N} = \frac{c_{N(N-1)}}{c_{NN} - c_{N-1}(N-1)},$$

$$b_{0(N-2)} = b_{0N} \left[ \frac{c_{N(N-2)}}{c_{NN} - c_{N-2}(N-2)} + \frac{c_{N(N-1)}}{c_{NN} - c_{N-1}(N-1)} \right].$$

Analogously, for the second line with $i = 1$, it follows from (18) that $b_{1N} = 0$, the element $b_{1(N-1)}$ plays the role of an arbitrary normalization factor, and further

$$b_{1(N-2)} = b_{1(N-1)} = \frac{c_{N(N-1)}}{c_{NN} - c_{N-1}(N-1) - c_{N-2}(N-2)},$$

$$b_{1(N-3)} = \ldots .$$

This procedure can be continued for $l = (N-3), (N-4), \ldots, l = 0$, and after that, for the following lines $i = 2, 3, \ldots, N$. Finally, the obtained matrix $\hat{B}$ is also triangular, but with vanishing matrix elements lying under the cross-diagonal. Elements on the cross-diagonal are arbitrary normalizing factors ($b_{0N}, b_{1(N-1)}$ above). All other elements can be compactly written as follows:\footnote{Let us mention the misprint in the upper limit of summation in the analogous formula (46) in [8].}

$$b_{m,p} = b_{m,N-m} \left[ \sum_{i=1}^{N-p-m} (\tau^{(m)})_l \right]_{N-m,p} \quad \text{(20)}$$

where the $(N+1)$ triangular matrices labelled $\tau^{(m)}$, $m = 0, 1, \ldots, N$ are defined via the matrix elements of $\hat{C}$:

$$\tau^{(m)}_{n,k} = \frac{c_{n,k}}{c_{N-m,N-m} - c_{k,k}}.$$ We stress that in (20) the expression $(\tau^{(m)})_l$ means the $l$-th power of the matrix $\tau^{(m)}$. The repeated index $(N - m)$ is frozen in (20) and not summed over. This expression always allows to write all elements of the $m$-th line $b_{m,p}$ in terms of the matrix $\tau^{(m)}$ and the arbitrary value of the element $b_{m,N-m}$ on the crossed diagonal. These arbitrary values can be fixed by the normalization condition for the wave functions $\Psi_{N-n}^{(2)}(\vec{x})$ in (19).

**Shape invariance of the model.** — The idea of shape invariance is one of the most essential new contributions to the modern quantum mechanics made by the SUSY quantum mechanics approach\footnote{To be honest, it should be noted that this property was already known in the framework of the well-known factorization method of Schrödinger [34, 29], but in a slightly different (and not so transparent) form.}. It was formulated originally [41] in a one-dimensional context, and it provided the connection between SUSY and the exact solvability of the model. Namely, all exactly solvable models were shown to obey [2] shape invariance, which allows to solve the models in a pure algebraic way, without solution of any differential equations. This elegant method was generalized [8,18,19] to the two-dimensional case where it leads only to partial (quasi-exact) solvability due to many zero modes of second-order supercharges.

**First shape invariance.** It is easy to check that the shift of parameter $a \to a = a - 1$ transforms $H^{(2)}$ into $H^{(1)}$:

$$H^{(2)}(\vec{x}; a) = H^{(1)}(\vec{x}; a), \quad \text{(21)}$$

where both Hamiltonians (2) with potentials (3) are intertwined according to (1). As usual [8,18], the shape invariance property allows to build the whole tower of new wave functions starting from any known wave function $\Psi^{(2)}_{n,0}(\vec{x}; a)$:

$$\Psi^{(2)}_{n,m}(\vec{x}; a) = Q^{-}(a)Q^{-}(a-1)\ldots \Psi^{(2)}_{n,0}(\vec{x}; a-m); \quad m = 1, 2, 3, \ldots. \quad \text{(22)}$$

Herewith, the energy eigenvalues of $\Psi^{(2)}_{n,m}(\vec{x}; a)$ are

$$E^{(2)}_{n,m}(a) = E^{(2)}_{n,0}(a-m). \quad \text{(23)}$$

As a principle wave function $\Psi^{(2)}_{n,0}$, in (23), one can take an arbitrary wave function $\Psi^{(2)}(\vec{x}; a)$ obtained in the previous section (see (19) with index $n$). Its energy $E^{(2)}_{n,0}$ is given by (17), and therefore, the energies of states (22) are

$$E^{(2)}_{n,m}(a) = -(A - n)^2 - (A - n - m + a)^2. \quad \text{(24)}$$

One may verify that for $a = -k$ these energy levels coincide with a part of the full spectrum which was found in [23].
**Second shape invariance.** It was mentioned in a recent paper [7] that among different solutions of two-dimensional SUSY intertwining relations obeying the shape invariance property, some pairs are equivalent to each other up to a linear transformation of coordinates. In particular, the model considered in the present paper (the model A5 of [7]) is equivalent to the model A8 in [7].

One can check that the partner Hamiltonians \( \tilde H^{(1),(2)} \) on the plane \( \vec y = (y_1, y_2) \) with potentials

\[
\tilde V^{(1),(2)}(\vec y) = g \left( \frac{1}{\sinh^2(\lambda y_2)} - \frac{1}{\cosh^2(\lambda y_1)} \right) + \frac{\alpha \lambda (2b \mp 1) \sinh(\lambda y_+)}{\cosh^2(\lambda y_+)} \left( \frac{1}{2} \mp 1 \right) + \frac{\alpha \lambda (2b \mp 1) \sinh(\lambda y_-)}{\cosh^2(\lambda y_-)} \left( \frac{1}{2} \mp 1 \right),
\]

\[ y_\pm \equiv y_1 \pm y_2, \]

are solutions of intertwining relations of the form (1):

\[
\tilde H^{(1)}(\vec y) \tilde Q^+(\vec y) = \tilde Q^+(\vec y) \tilde H^{(2)}(\vec y)
\]

with the supercharges \( \tilde Q^\pm(\vec y) = 4 \partial_{y_+} \partial_{y_-} \mp 4 \tilde C_+(y_+) \partial_{y_+} + 4 \tilde C_-(y_-) \partial_{y_-} + \tilde B(\vec y) \), \( \tilde B(\vec y) = 4 \tilde C_+(y_+ \tilde C_-(y_-) + \tilde f_1(y_1) + \tilde f_2(y_2) \), \( \tilde C_+(y_+) = b \lambda \tanh(\lambda y_-) + \frac{\alpha}{2 \cosh(\lambda y_+)} \), \( \tilde C_-(y_-) = b \lambda \tanh(\lambda y_+) + \frac{\alpha}{2 \cosh(\lambda y_-)} \), \( \tilde f_1(y_1) = \frac{g}{\cosh^2(\lambda y_1)} \), \( \tilde f_2(y_2) = \frac{g}{\sinh^2(\lambda y_2)} \).

According to [7], this model is shape-invariant, \( \tilde H^{(2)}(\vec y, b \mp 1) = \tilde H^{(1)}(\vec y, b) \), and therefore, if one knows any (principal) wave function \( \tilde \Psi^{(2)}_{\alpha,0}(\vec y) \) of \( \tilde H^{(2)}(\vec y) \), then the whole tower of wave functions can be built due to shape invariance:

\[
\tilde \Psi^{(2)}_{n,m}(\vec y, b) = Q^- (\vec y, -b) \ldots Q^- (\vec y, -b - m + 1) \tilde \Psi^{(2)}_{n,0}(\vec y, b - m),
\]

\[ m = 1, 2, \ldots, \tilde E_{n,m}(b) = \tilde E_{n,0}(b - m). \]  

At first sight, the model (25) has no relation to the Scarf II model (3). But as was noticed in [7], they are related by substitution of coordinates: \( x_+ \equiv y_1, x_- \equiv y_2 \), since (25) can be rewritten as

\[
\tilde V^{(1),(2)}(\vec x) = g \left( \frac{1}{\sinh^2(\lambda x_-)} - \frac{1}{\cosh^2(\lambda x_+)} \right) + \frac{\alpha \lambda (2b \mp 1) \sinh(2\lambda x_+)}{2 \cosh^2(2\lambda x_+)} \left( \frac{1}{2} \mp 1 \right) + \frac{\alpha \lambda (2b \mp 1) \sinh(2\lambda x_-)}{2 \cosh^2(2\lambda x_-)} \left( \frac{1}{2} \mp 1 \right),
\]

\[ \alpha^2 - 4 \lambda^2 b(b \mp 1) = (B^2 - A^2 - 2 \Lambda \lambda), \]

it becomes evident, that the second superpartners are proportional: \( H^{(2)}(\vec x) = 2 H^{(3)}(\vec x) \), and thus, the Hamiltonian \( H^{(2)}(\vec x) \) satisfies simultaneously two intertwining relations — with \( H^{(1)} \) and \( \tilde H^{(1)} \):

\[
H^{(1)}(\vec x, a, b) \oplus H^{(2)}(\vec x, a, b) \equiv 2 \tilde H^{(2)}(\vec x, a, b) \oplus 2 \tilde H^{(1)}(\vec x, a, b),
\]

where the sign \( \oplus \) means intertwining, and both shape invariance parameters \( a, b \) of potentials are written explicitly.

For the particular case of \( 2 \lambda = 1 \), the solution of system (29) is \( b = A, \alpha = B \), and therefore, the energies in (27) coincide with the levels in (24). By means of laborious but elementary calculations, one can check straightforwardly, that

\[
\tilde Q^- (\vec x, a, b) \tilde Q^- (\vec x, a, b - 1) = Q^- (\vec x, a, b) \tilde Q^- (\vec x, a - 1, b).
\]

Similarly to the situation [35] for the Morse model, this identity means that starting from the arbitrary principle state \( \tilde \Psi^{(2)}_{\alpha,0}(\vec x, a, b) \), one may combine both shape invariances (in parameter \( a \), and parameter \( b \)) to obtain wave functions \( \tilde \Psi^{(2)}_{n,m}(\vec x, a, b) \). In other words, one can move by different paths in the plane \( (a, b) \) of parameters.

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EJV is indebted to the nonprofit foundation “Dynasty” for financial support. MVI acknowledges Saint-Petersburg State University for a research grant.

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