Entropy and the discrete central limit theorem

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Abstract

A strengthened version of the central limit theorem for discrete random variables is established, relying only on information-theoretic tools and elementary arguments. It is shown that the relative entropy between the standardised sum of $n$ independent and identically distributed lattice random variables and an appropriately discretised Gaussian, vanishes as $n \to \infty$.

Keywords — Central limit theorem, entropy, Fisher information, relative entropy, Bernoulli part decomposition, lattice distribution, convolution inequality

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1 Introduction

Suppose $X_1, X_2, \ldots$ are zero-mean, independent and identically distributed (i.i.d.), continuous random variables, with finite variance $\sigma^2$. The study of the entropy $h(\bar{S}_n)$ of the standardised sums $\bar{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ has a long history, starting with the 1959 work of Linnik [21]. Recall that the entropy of a continuous random variable $Y$ with density $f$ is $h(Y) = - \int f \log f$, where ‘log’ denotes the natural logarithm.

Barron [3] showed that, as $n \to \infty$,

$$h(\bar{S}_n) \to h(Z) = \frac{1}{2} \log(2\pi e\sigma^2),$$

where $Z \sim N(0, \sigma^2)$ is a zero-mean Gaussian random variable with variance $\sigma^2$. Barron’s proof combined earlier results by Brown [6] together with an integral form of de Bruijn’s identity for the entropy and a convolution inequality for the Fisher information [29, 4].

The fact that the Gaussian has maximal entropy among all random variables with variance no greater than $\sigma^2$ invites an appealing analogy between (1) and the second law of thermodynamics. Indeed, this analogy was carried further when it was shown that the entropy $h(\bar{S}_n)$ in fact increases to the maximum entropy $h(Z) = \frac{1}{2} \log(2\pi e\sigma^2)$. This was first established using analytical tools by Artstein et al. [1], and later using information-theoretic techniques in [32] and [22].

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Let $D(f\|g) = \int f \log(f/g) \, dx$ denote the relative entropy between two probability densities $f, g$ on $\mathbb{R}$. For a continuous random variable $Y$ with density $f$ and variance $\sigma^2 < \infty$, we write $D(Y)$ for $D(f\|\phi)$, the relative entropy between $f$ and the Gaussian density $\phi$ with the same mean and variance as $Y$. Then we always have,

$$D(Y) = \frac{1}{2} \log(2\pi e\sigma^2) - h(Y),$$

which implies that the convergence of $h(\hat{S}_n)$ to $h(Z)$ is equivalent to,

$$D(\hat{S}_n) \to 0, \quad \text{as } n \to \infty. \quad (3)$$

In view of Pinsker’s inequality, $2\|f - g\|_{TV}^2 \leq D(f\|g)$ [9, 20], the relative entropy convergence in (3) is enough, e.g., to recover the central limit theorem (CLT) in the sense of total variation convergence.

Note that, not only do the results in (1) and (3) not rely on the CLT, but they imply a strong form of the CLT, established without using any of the usual probabilistic techniques.

In the case of discrete random variables $\{X_n\}$, there is no immediately obvious starting point for identifying a corresponding connection between the CLT and the entropy of the standardised sums $\hat{S}_n$; for example, the distribution of $\hat{S}_n$ is orthogonal to the Gaussian and the relative entropy between them is always infinite. The main contribution of this work is the development of natural discrete analogs of the “entropic” CLTs in (1) and (3).

For i.i.d. random variables $\{X_n\}$, write $S_n$ for the partial sums $X_1 + X_2 + \cdots + X_n$, so that $\hat{S}_n = \frac{1}{\sqrt{n}} S_n$. If the $\{X_n\}$ are continuous with finite variance, then by the elementary scaling property of the entropy [8], (1) can equivalently be written,

$$\lim_{n \to \infty} \left[ h(S_n) - \log \sqrt{n} \right] = \frac{1}{2} \log(2\pi e\sigma^2). \quad (4)$$

The entropy of a discrete random variable $Y$ with probability mass function $p$ on a countable set $A$ is $H(Y) = -\sum_{y \in A} p(y) \log p(y)$. Our first result is the analog of (4) for lattice random variables. We say that $Y$ has a lattice distribution with span $h > 0$ if its support is a subset of $\{a + kh : k \in \mathbb{Z}\}$ for some $a \in \mathbb{R}$; the span $h$ is maximal if it is the largest such $h$.

**Theorem 1.1 (Entropy convergence)** If $S_n, n \geq 1$, are the partial sums of a sequence $\{X_n\}$ of i.i.d. lattice random variables with finite variance $\sigma^2$ and maximal span $h$, then:

$$\lim_{n \to \infty} \left[ H(S_n) - \log \frac{\sqrt{n}}{h} \right] = \frac{1}{2} \log(2\pi e\sigma^2). \quad (5)$$

Since the discrete entropy does not scale in the same way as the continuous entropy (e.g., $H(S_n) = H(\hat{S}_n)$), the equivalence between the convergence in (5) and a discrete version of the entropic CLT $D(\hat{S}_n) \to 0$ is no longer immediate. Nevertheless, it is possible to establish a result analogous to that in (3) in the discrete case, as shown in Theorem 1.2 below.

For discrete random variables $X, Y$ with probability mass functions $p, q$, respectively, on the same countable set $A$, the relative entropy $D(p\|q)$ between $p$ and $q$ is defined as $D(p\|q) = \sum_{x \in A} p(x) \log(p(x)/q(x))$, where the sum is interpreted as the Lebesgue integral of $\log(p/q)$ with respect to the probability measure induced by $p$ on $\mathbb{R}$. 

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Suppose $Y$ is lattice random variable maximal span $h$, values in $A = \{a + kh : k \in \mathbb{Z}\}$, mean $\mu$, and finite variance $\sigma^2$. We write $D(Y)$ for the relative entropy $D(p\|q)$ between the probability mass function $p$ of $Y$ and the probability mass function $q$ of a Gaussian random variable $Z \sim N(\mu, \sigma^2)$ quantised on $A$ as,

$$q(a + kh) = \int_{a + kh}^{a + (k + 1)h} \phi(x)dx, \quad k \in \mathbb{Z},$$

where $\phi$ is the $N(\mu, \sigma^2)$ density. Observe that, by definition, $D(Y+c) = D(Y)$ for any constant $c$.

**Theorem 1.2 (Discrete entropic CLT)** If $\hat{S}_n$, $n \geq 1$, are the standardised sums of a sequence $\{X_n\}$ of i.i.d. lattice random variables with finite variance, then:

$$D\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)\right) = D(\hat{S}_n) \to 0, \quad \text{as } n \to \infty. \tag{7}$$

As in the continuous case, Pinsker’s inequality combined with the triangle inequality for the total variation norm imply a strong version of the CLT: Taking $\mu = 0$ without loss of generality, let $Z \sim N(0, \sigma^2)$ and let $Z_n$ be the quantised Gaussian as in the definition of $D(\hat{S}_n)$. Then,

$$\|\hat{S}_n - Z\|_{TV} \leq \sqrt{\frac{1}{2} D(\hat{S}_n) + \|Z_n - Z\|_{TV}} \to 0, \quad \text{as } n \to \infty,$$

since the first term vanishes by Theorem 1.2 and the second term vanishes by the definition of $Z_n$. Alternatively, the fact that $\|\hat{S}_n - Z_n\|_{TV} \to 0$ implied by Theorem 1.2 readily translates to local-CLT-like results.

**Paper outline and proof ideas.** In the end of this Introduction we discuss the intriguing connection between the CLT and Shannon’s entropy power inequality. In Section 2 we prove that the entropic CLT statements (5) and (7) in Theorems 1.1 and 1.2, respectively, are equivalent. There we also establish a relation between $D(\hat{S}_n)$ and $D(\hat{S}_n + U)$, when $\hat{S}_n$ are the standardised sums of lattice random variables $X_i$ and and $U$ is an appropriate (continuous) uniform random variable, independent of the $X_i$ (Lemma 2.3). In Section 3 we establish two special cases of Theorem 1.1: first, when the $X_i$ in $S_n = \sum_{i=1}^{n} X_i$ are symmetric Bernoulli random variables, $X_i \sim Bern(1/2)$, and then when each $X_i$ can be written as the sum $X_i = V_i + B_i$ of a lattice random variable $V_i$ with maximal span $h = 1$ and a $B_i \sim Bern(1/2)$ independent of $V_i$.

At first sight it might be tempting to hope that the result of Theorem 1.2 could be derived from its continuous counterpart (3) via a simple quantisation argument using the “data processing” property of relative entropy [8], but this does not appear to be the case. Instead, Barron’s continuous result (3) is employed in a more indirect way in the proof of the special case of Theorem 1.1 given in Section 3. This is then used in the proof of our main result, the general case of Theorem 1.1, in Section 4. In addition to Barron’s result (3), and to simple information-theoretic properties and some well-known bounds and identities for the Fisher information, the other main ingredient in the proof of Theorem 1.1 is an elementary technique known as “Bernoulli part decomposition,” described in Section 4. Theorem 1.2 is an immediate consequence of Theorem 1.1 combined with Theorem 2.1.

The CLT for discrete random variables has been investigated from an information-theoretic point of view by, among others, Shimizu [28] and Brown [6], who obtained the convergence of $\hat{S}_n$
to a Gaussian in distribution (but not for entropy or relative entropy) by proving convergence for the Fisher information of smoothed versions of $S_n$. The Bernoulli part decomposition technique was first used (implicitly) by Mineka [25] in a different context, and by McDonald [24] and Davis and McDonald [10], who derived conditions under which the standardised sums of independent discrete random variables satisfy the local CLT [11, 13]. In the reverse direction, Takano [30] used the local CLT to derive entropy expansions for the standardised sums $S_n$ as in our Theorem 1.1.

There is a significant line of work re-examining core probabilistic results through the lens of information theory. In terms of ideas as well as techniques, perhaps the works closest in spirit to the present development are those providing information-theoretic treatments of Poisson approximation [15, 19] and compound Poisson approximation [2, 18].

The CLT and the entropy power inequality. The earliest indication of a nontrivial connection between the CLT and information-theoretic ideas comes from Shannon’s entropy power inequality (EPI) [27, 29, 4]. For i.i.d. continuous random variables $X_1, X_2$, the EPI states that,

$$h(X_1 + X_2) \geq h(X_1) + \frac{1}{2} \log 2,$$

with equality if and only if $X_1, X_2$ are Gaussian. Using the scaling property of the entropy, this implies that $h(S_n) \geq h(S_n)$ for all $n$, which likely provided some of the initial motivation for the works [21, 28, 6, 3] mentioned earlier. Further, a generalisation of the EPI was used to prove the monotonic increase of the entire sequence $\{h(n)\}$ to $\frac{1}{2} \log(2\pi e \sigma^2)$ in [22].

For i.i.d. discrete random variables $X_1, X_2$, it is easy to see (by considering random variables with entropy close to zero) that the obvious discrete analog, $H(X_1 + X_2) \geq H(X_1) + \frac{1}{2} \log 2$, fails to hold in general. On the other hand, Tao [31] showed that, for any $\epsilon > 0$,

$$H(X_1 + X_2) \geq H(X_1) + \frac{1}{2} \log 2 - \epsilon,$$

for all i.i.d. pairs $X_1, X_2$ such that $H(X_1)$ is large enough depending on $\epsilon$. Tao’s proof relies on the inverse sumset theory for entropy developed in [31]. A careful examination of the proof shows that not only is the lower bound on $H(X_1)$ at least,

$$\Omega\left(\frac{\epsilon^2}{\epsilon}\right),$$

but the implied absolute constants are also very large. Although the answer to the natural question of how much this bound can be improved remains unclear (see, e.g., [14] for some related bounds), in view of the results in this paper, particularly the nonasymptotic versions of Theorem 2.1 and Lemma 2.3, we expect that perhaps if one restricts attention to lattice random variables with finite variance, it may be possible to significantly improve on (8).

Interestingly, Tao [31] further conjectured that, for any $n \geq 2$,

$$H(X_1 + \ldots + X_n) \geq H(X_1 + \ldots + X_{n-1}) + \frac{1}{2} \log \left(\frac{n}{n-1}\right) - \epsilon,$$

as long as $H(X_1)$ is sufficiently large depending on $n$ and $\epsilon$. The present results again suggest that this conjecture might be easier to prove if attention is restricted to lattice random variables with finite variance. Specifically, in this setting (9) can be interpreted as an “approximate monotonicity” refinement of our Theorems 1.1 and 1.2: By the nonasymptotic form of Theorem 2.1 and the fact that $H(X_1) \to \infty$ implies $\text{Var}(X_1) \to \infty$, for lattice $X_1$ with finite variance the conjecture (9) is equivalent to:

$$D(S_n) \leq D(S_{n-1}) + \epsilon.$$
2 Entropy, relative entropy, and Fisher information

Let $X_1, X_2, \ldots$, be i.i.d. lattice random variables with values in $\{a + kh : k \in \mathbb{Z}\}$, mean $\mu$, and finite variance $\sigma^2$. As before, write $S_n$ for the partial sums $\sum_{i=1}^n X_i$ and $\hat{S}_n$ for the standardised sums $\frac{1}{\sqrt{n}}S_n$, and recall the definition of the relative entropy $D(Y)$ between a lattice random variable $Y$ and an appropriately quantised Gaussian as in (6).

Our first observation is that the “entropy deficit,”

$$\frac{1}{2} \log (2\pi e \sigma^2) - \left[ H(S_n) - \log \frac{\sqrt{n}}{h} \right],$$

can be viewed as a measure of the “Gaussianity” of the lattice sum $S_n$. Theorem 2.1 shows that the entropic CLTs stated in Theorems 1.1 and 1.2 are equivalent.

**Theorem 2.1 (Entropy and relative entropy solidarity)** Suppose $\{X_n\}$ are i.i.d. lattice random variables with finite variance $\sigma^2$ and maximal span $h > 0$. Then the partial sums $S_n$ and the standardised sums $\hat{S}_n$ of the $X_i$ satisfy, as $n \to \infty$,

$$D(\hat{S}_n) = \frac{1}{2} \log (2\pi e \sigma^2) - \left[ H(S_n) - \log \frac{\sqrt{n}}{h} \right] + O\left( \frac{1}{\sqrt{n}} \right).$$

In fact, for all $n \geq 1$, the $O(1/\sqrt{n})$ error term is absolutely bounded by:

$$\frac{h}{2 \sigma \sqrt{n}} \left[ 1 + \frac{h}{2 \sigma \sqrt{n}} \right].$$

**Proof.** Because $H(Y)$ is translation invariant and, as noted in the introduction, so is $D(Y)$, we may assume that $\mu = 0$ without loss of generality.

Since the $X_i$ take values in $\{a + kh : k \in \mathbb{Z}\}$, $S_n$ takes values in $\{na + kh : k \in \mathbb{Z}\}$ and $\hat{S}_n$ in $\{\sqrt{na} + kh/\sqrt{n} : k \in \mathbb{Z}\}$. Let $p, q$ denote the probability mass functions of $S_n$ and of the quantised Gaussian in the definition of $D(\hat{S}_n)$, respectively. Writing $\phi$ for the standard normal density, for each $k \in \mathbb{Z}$ we have,

$$q(\sqrt{na} + k \frac{h}{\sqrt{n}}) = \int_{\sqrt{na} + kh/\sqrt{n}}^{\sqrt{na} + (k+1)h/\sqrt{n}} \frac{1}{\sigma \sqrt{2\pi}} \phi\left( \frac{x}{\sigma} \right) dx = \frac{h}{\sigma \sqrt{n}} \phi\left( \frac{na + \xi_k h}{\sigma \sqrt{n}} \right),$$

for some $\xi_k \in [k, k+1]$. Using this, we can bound the absolute difference,

$$\Delta_n := \left| D(\hat{S}_n) - \frac{1}{2} \log (2\pi e \sigma^2) - H(S_n) + \log \frac{\sqrt{n}}{h} \right|$$

$$= \left| \sum_{k \in \mathbb{Z}} p(na + kh) \log \left( \frac{p(na + kh)}{\frac{h}{\sigma \sqrt{n}} \phi\left( \frac{na + \xi_k h}{\sigma \sqrt{n}} \right)} \right) - \sum_{k \in \mathbb{Z}} p(na + kh) \log \left( \frac{p(na + kh)}{\frac{h}{\sigma \sqrt{n}} \phi\left( \frac{na + kh}{\sigma \sqrt{n}} \right)} \right) \right|,$$

where the second sum contains the last three terms in $\Delta_n$. Simplifying we obtain,

$$\Delta_n \leq \frac{1}{2n\sigma^2} \sum_{k \in \mathbb{Z}} p(na + kh) \left( (na + kh)^2 - (na + \xi_k h)^2 \right)$$

$$\leq \frac{1}{2n\sigma^2} \sum_{k \in \mathbb{Z}} p(na + kh) (2h|na + kh| + h^2) \leq \frac{h}{\sigma \sqrt{n}} + \frac{h^2}{2\sigma^2 n},$$

as required, where the last step follows from the Cauchy-Schwarz inequality and the fact that the variance of $S_n$ is $n\sigma^2$. $\Box$
By the nonnegativity of relative entropy we obtain the following standard upper bound, which can be viewed as a discrete analog of the maximum entropy property of the Gaussian:

$$H(S_n) - \log \frac{\sqrt{n}}{h} \leq \frac{1}{2} \log (2\pi e\sigma^2) + \frac{h}{\sigma \sqrt{n}} \left[ 1 + \frac{h}{2\sigma \sqrt{n}} \right].$$

(10)

In fact, we can easily obtain a stronger bound. Let $U$ be an independent uniform random variable on $(-1/2, 1/2)$. Then, by the definitions of the continuous and discrete entropies,

$$H(S_n) - \log \frac{\sqrt{n}}{h} = h \left( \hat{S}_n + \frac{h}{\sqrt{n}} U \right).$$

(11)

And using the maximum maximum entropy property of the Gaussian yields:

**Proposition 2.2** If $S_n$ is the sum of $n$ i.i.d. lattice random variables with maximal span $h > 0$ and finite variance $\sigma^2$, then:

$$H(S_n) - \log \frac{\sqrt{n}}{h} \leq \frac{1}{2} \log \left[ 2\pi e \left( \sigma^2 + \frac{h^2}{12n} \right) \right].$$

(12)

In the special case $h = 1$, $n = 1$, the bound (12) appeared in [23]. It was recently exploited further in [26], where an improved inequality was also established for large $\sigma^2$ via the Poisson summation formula. For any $n$ and $h = 1$, (12) also appeared in [5], as a special case of an inequality for Rényi entropies.

The following lemma will be used in the proof of Theorem 3.2. It highlights the asymptotic equivalence between the discrete and continuous versions of the relative entropy $D(\hat{S}_n)$.

**Lemma 2.3** Under the assumptions of Theorem 2.1, let $U$ be an independent uniform random variable on $(-1/2, 1/2)$. Then, as $n \to \infty$,

$$D(\hat{S}_n) = D\left( \hat{S}_n + \frac{h}{\sqrt{n}} U \right) + O\left( \frac{1}{\sqrt{n}} \right).$$

In fact, for all $n \geq 1$, the $O(1/\sqrt{n})$ error term is absolutely bounded by:

$$\frac{h}{\sigma \sqrt{n}} \left[ 1 + \frac{13h}{24\sigma \sqrt{n}} \right].$$

**Proof.** As in the proof of Theorem 2.1, we may assume without loss of generality that the $X_i$ have zero mean. Note that $\hat{S}_n + (h/\sqrt{n})U$ has variance $\sigma^2 + h^2/(12n)$. Using the finite-$n$ bound in Theorem 2.1 and the general property (2) of the relative entropy,

$$\left| D(\hat{S}_n) - D\left( \hat{S}_n + \frac{h}{\sqrt{n}} U \right) \right| \leq D\left( \hat{S}_n + \frac{h}{\sqrt{n}} U \right) - \frac{1}{2} \log (2\pi e\sigma^2) + H(S_n) - \log \frac{\sqrt{n}}{h} + \frac{h}{\sigma \sqrt{n}} + \frac{h^2}{2\sigma^2 n},$$

$$= \frac{1}{2} \log \left( 1 + \frac{h^2}{12n\sigma^2} \right) - h(\hat{S}_n + \frac{h}{\sqrt{n}} U) + H(S_n) - \log \frac{\sqrt{n}}{h} + \frac{h}{\sigma \sqrt{n}} + \frac{h^2}{2\sigma^2 n},$$

and using (11),

$$\left| D(\hat{S}_n) - D\left( \hat{S}_n + \frac{h}{\sqrt{n}} U \right) \right| \leq \frac{1}{2} \log \left( 1 + \frac{h^2}{12n\sigma^2} \right) + \frac{h}{\sigma \sqrt{n}} + \frac{h^2}{2\sigma^2 n} \leq \frac{h}{\sigma \sqrt{n}} + \frac{13h^2}{24\sigma^2 n},$$

where the last inequality follows from the elementary bound $\log (1 + x) \leq x$, $x > 0$. \qed
We close this section by recalling some simple convolution inequalities that will be used in the following sections. If $X, Y$ are independent discrete random variables, then [8]:

$$H(X + Y) \geq H(X).$$

(13)

Similarly, if $X$ is a continuous random variable and $Y$ an arbitrary independent random variable, then [8]:

$$h(X + Y) \geq h(X).$$

(14)

Finally, for a continuous random variable $X$ with a continuously differentiable density $f$, we define the Fisher information of $X$ as $I(X) = \int (f')^2 / f$. If the independent random variables $X, Y$ have continuously differentiable densities with bounded derivatives, then [6, Lemma 5.5]:

$$I(V + W) \leq I(V).$$

(15)

3 Binomial sums and Bernoulli smoothing

We first establish a nonasymptotic version of Theorem 1.1 in the special case when $S_n$ is the sum of independent Bern(1/2) random variables, so that $S_n \sim \text{Bin}(n, 1/2)$ has a binomial distribution with parameters $n$ and 1/2. Although this elementary result is largely known [7, 16, 17], we state and prove it explicitly as it is the first step towards the proof of our main result, Theorem 1.1. Also, as earlier proofs of (16) actually use the CLT, relying on such arguments would defeat our main claim, namely, that of obtaining a complete proof of the entropic CLT without using any of the standard probabilistic normal approximation techniques or, of course, the CLT itself.

**Proposition 3.1 (Binomial entropy)** If $S_n \sim \text{Bin}(n, 1/2)$, then for all $n \geq 2$:

$$\left| H(S_n) - \log \sqrt{n} \right| - \frac{1}{2} \log \left( \frac{1}{2} \pi e \right) \leq \frac{4}{\sqrt{n}}.$$

(16)

Note that (16) combined with the finite-$n$ version of Theorem 2.1 also yields:

$$D(\hat{S}_n) \leq \frac{8}{\sqrt{n}}, \quad n \geq 2.$$

**Proof.** The general upper bound in (12) in this particular case gives,

$$H(S_n) - \log \sqrt{n} \leq \frac{1}{2} \log \left( \frac{1}{2} \pi e \right) + \frac{1}{2} \log \left( 1 + \frac{1}{12n} \right) \leq \frac{1}{2} \log \left( \frac{1}{2} \pi e \right) + \frac{1}{24n}.$$  

(17)

For the proof of the corresponding lower bound we only consider even $n$; the case of odd $n$ is similar. Let $b_n(k) = \binom{n}{k}2^{-n}$ denote the Bin($n, 1/2$) probabilities, and for fixed $n \geq 2$ write $a_k = b_n(\frac{n}{2} + k)$, for $-n/2 \leq k \leq n/2$. Following a simple argument by Feller [12, VII, 2], we first observe that for $k \geq 1$,

$$a_k = a_0 \times \frac{\binom{n}{k+1}}{(\frac{n}{2} + 1)(\frac{n}{2} + 2) \cdots (\frac{n}{2} + k)} = a_0 \times \frac{(1 - \frac{2}{n})(1 - \frac{4}{n}) \cdots (1 - \frac{2(k-1)}{n})}{(1 + \frac{2}{n}) \cdots (1 + \frac{2k}{n})},$$

and then use the elementary bounds, $1 - x \leq e^{-x}$ and $1 + x \geq e^{x-x^2}$, for $x \in [0,1)$, to obtain that,

$$a_k \leq a_0 \exp \left\{ -\frac{4}{n} \left[ 1 + \cdots + (k-1) \right] - \frac{2k}{n} + \frac{3k^3}{n^2} \right\} = a_0 e^{-\frac{2k^2}{n^2} + \frac{3k^3}{n^2}}.$$
By Robbins’ finite-\( n \) version of Stirling’s formula, e.g. [12, II, (9.15)], we can easily bound,

\[
a_0 \leq \left( \frac{\pi n}{2} \right)^{-1/2} e^{\frac{3}{4n}},
\]

so that, for \( k \geq 0 \),

\[
a_k \leq \left( \frac{\pi n}{2} \right)^{-1/2} e^{-\frac{2k^2}{n}} e^{\frac{3k^3}{2n^2} + \frac{1}{12n}}.
\] (18)

Since \( a_k = a_{-k} \), the same bound holds for all \(-n/2 \leq k \leq n/2\), with \(|k|\) in place of \( k \). And substituting (18) into the logarithmic term in the definition of \( H(S_n) \) gives,

\[
H(S_n) = - \sum_{k=-n/2}^{n/2} a_k \log a_k
\]

\[
\geq \log \sqrt{n} + \frac{1}{2} \log \left( \frac{1}{2\pi} \right) + \frac{2}{n} \sum_{k=n/2}^{n/2} a_k k^2 - \frac{3}{n^2} \sum_{k=-n/2}^{n/2} a_k |k|^3 - \frac{1}{12n}
\]

\[
\geq \log \sqrt{n} + \frac{1}{2} \log \left( \frac{1}{2\pi} \right) + \frac{1}{2} - \frac{4}{\sqrt{n}},
\] (19)

where in the last step we used the fact that the variance of \( S_n \) is \( n/4 \) and its third absolute central moment is bounded above by \( n^{3/2} \).

The result follows from (17) and (19). \( \square \)

Next, we extend the result of Proposition 3.1 to the case when each \( X_i \) in \( S_n \) can be written as the independent sum \( X_i = V_i + B_i \) of a lattice random \( V_i \) and a \( B_i \sim \text{Bern}(1/2) \). The proof of Theorem 3.2 is a key step towards the proof of the general case of Theorem 1.1 in the next section. We refer to the addition of an independent Bernoulli to a lattice random variable as “Bernoulli smoothing,” in analogy to the Gaussian smoothing step used in [28, 6, 3] along the corresponding development in the continuous case. There, one considers \( X_i + \sqrt{t}Z_i \), where the \( Z_i \) are standard normals, so that the resulting random variables have differentiable densities that smoothly interpolate between the distribution of \( X_i \) and the Gaussian, as \( t \) varies. In our case, the addition of a binomial random variable to the partial sums \( S_n \) facilitates the use of Proposition 3.1, and also allows us to establish a uniform integrability property which can be used to exploit the fact that Fisher information decreases on convolution.

**Theorem 3.2 (Bernoulli smoothing)** Suppose \( \{V_n\} \) are i.i.d. lattice random variables with finite variance \( \sigma_V^2 \) and maximal span \( h = 1 \), and let \( \{B_n\} \) be i.i.d. Bern(1/2), independent of \( \{V_n\} \). Then:

\[
\lim_{n \to \infty} \left[ H \left( \sum_{i=1}^{n} [V_i + B_i] \right) - \log \sqrt{n} \right] = \frac{1}{2} \log \left( 2\pi e \left( \sigma_V^2 + \frac{1}{4} \right) \right).
\]

For a continuous random variable with continuously differentiable density \( f \), the score function \( \rho \) of \( Y \) is \( \rho = f'/f \), so that the Fisher information \( I(Y) \) can be expressed \( I(Y) = \int f \rho^2 \). In particular, if \( \phi \) is the \( N(\mu, \sigma^2) \) density, then its score function \( \rho_\phi \) is linear, \( \rho_\phi(z) = -(z - \mu)/\sigma^2 \), \( z \in \mathbb{R} \). For the proof of Theorem 3.2, we will find it convenient to use the standardised Fisher information \( J(Y) \), which, when \( Y \) has mean \( \mu \) and variance \( \sigma^2 \), is defined as \( J(Y) = \sigma^2 \int f(\rho-\rho_\phi)^2 \), or, equivalently,

\[
J(Y) = \sigma^2 I(Y) - 1.
\] (20)
Proof. Let $U$ be an independent random variable, uniformly distributed on $(-1/2, 1/2)$, and write $S_n$ for the binomial sum $S_n = \sum_{i=1}^{n} B_i$. In view of Theorem 2.1 and Lemma 2.3, it suffices to show that, as $n \to \infty$,

$$D\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i + \frac{1}{\sqrt{n}} S_n + \frac{1}{\sqrt{n}} U\right) \to 0.$$ 

Using Barron’s integral form of de Bruijn’s identity [3, Eq. (4.1)], this can be expressed as,

$$D\left(\frac{1}{\sqrt{n}} \left[\sum_{i=1}^{n} V_i + S_n + U\right]\right) = D\left(\frac{1}{\sqrt{2n}} \left[\sum_{i=1}^{n} V_i + S_n + U + \frac{1}{\sqrt{2}} Z\right]\right)$$

$$+ \int_{0}^{1/2} J\left(\sqrt{\frac{1-t}{n}} \left[\sum_{i=1}^{n} V_i + S_n + U\right] + \sqrt{t}Z\right) \frac{dt}{2(1-t)},$$

where $Z$ is an independent normal random variable with the same mean and variance as, $\frac{1}{\sqrt{n}} \left[\sum_{i=1}^{n} V_i + S_n + U\right]$. Writing $\mu_V, \sigma^2_V$ for the mean and variance of the $V_i$, respectively, $Z$ can be expressed,

$$Z = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i + \frac{1}{\sqrt{n}} S_n + \frac{1}{\sqrt{n}} U.$$ 

Writing $Y_i$ for the continuous i.i.d. random variables $Y_i = (V_i + B_i + Z_i)/\sqrt{2}$ and let $\sigma^2_Y$ denote the variance of $T_n$ and of $Y_i$, respectively. By (2) and the convolution inequality (14) we have, as $n \to \infty$,

$$D(T_n) = \frac{1}{2} \log(2\pi e \sigma^2_Y) - h(T_n)$$

$$\leq \frac{1}{2} \log \left(2\pi e \left(\frac{1}{12n}\sigma^2_Y + \frac{1}{12n}\right)\right) - h\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i\right)$$

$$= D\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i\right) + o(1),$$

where the last relative entropy is also $o(1)$ by the continuous entropic CLT (3). Therefore, the relative entropy in (21) vanishes as $n \to \infty$, and now it suffices to show that so does the integral in (22).

An analogous argument to the one used above for the relative entropy can be used to show that, for each $t$, the standardised Fisher information in the integrand in (22) vanishes with $n$. For fixed $t \in (0, 1)$, let $R_n = R_n(t)$ denote the argument of the standardised Fisher information in (22), so that $R_n$ can be written,

$$R_n = \sqrt{\frac{1-t}{n}} \sum_{i=1}^{n} [V_i + B_i] + \sqrt{t} \hat{Z}_n + \sqrt{\frac{1-t}{n}} U + \sqrt{t} W_n.$$
where now \( \hat{Z}_n \sim N(\sqrt{n}(\mu V + 1/2), \sigma^2_V + 1/4) \). Write \( Y'_i \) for the i.i.d. random variables \( Y'_i := V_i + B_i \) and let \( \sigma^2_R \) and \( \sigma^2_Y \), denote the variances of \( R_n \) and \( Y'_i \), respectively. By the representation (20) and the convolution inequality (15), we have that,

\[
0 \leq J(R_n) = \sigma^2_R I(R_n) - 1 \leq \left( \sigma^2_Y + \frac{1}{12n} \right) I\left( \sqrt{\frac{1 - t}{n}} \sum_{i=1}^n Y'_i + \sqrt{t} \hat{Z}_n \right) - 1,
\]

which vanishes as \( n \to \infty \) by the Fisher information convergence in [3, Lemma 2], since \( J(\cdot) \) is translation invariant.

Finally, we show that the nonnegative sequence \( \{J(R_n(t)) : n \geq 1\} \) is uniformly integrable with respect to the probability measure \( \nu(dt) \propto \frac{dt}{2(1-t)} \) on \((0, 1/2)\). In fact, we will show that it is bounded above by the uniformly integrable sequence \( \{J(R'_n(t))\} \) defined next.

Let \( Z' \sim N\left(\frac{\sqrt{n}}{2}, \frac{1}{12n}\right) \) and \( Z'' \sim N\left(\sqrt{n}\mu V, \sigma^2_V\right) \) be independent random variables such that \( Z = Z' + Z'' \). Then we can write,

\[
R_n = R'_n + \sqrt{\frac{1 - t}{n}} \sum_{i=1}^n V_i + \sqrt{t} Z'',
\]

where,

\[
R'_n = R'_n(t) = \sqrt{\frac{1 - t}{n}} [S_n + U] + \sqrt{t} Z',
\]

so that, by the convolution inequality (15) and using the the representation (20) twice,

\[
J(R_n) = \sigma^2_R I(R_n) - 1 \leq (\sigma^2_{R'} + \sigma^2_V) I(R'_n) - 1 = \left( 1 + \frac{\sigma^2_V}{\sigma^2_{R'}} \right) J(R'_n) + \frac{\sigma^2_V}{\sigma^2_{R'}},
\]

where \( \sigma^2_{R'} = \frac{1}{4} + \frac{1}{12n} \) is the variance of \( R'_n \).

But by Proposition 3.1, Lemma 2.3 and de Bruijn’s integral identity,

\[
\int_0^{1/2} J(R'_n(t)) \frac{dt}{2(1-t)},
\]

vanishes as \( n \to \infty \). Therefore, \( \{J(R'_n(t))\} \) is uniformly integrable with respect to the probability measure \( \nu(dt) \propto \frac{dt}{2(1-t)} \) on \((0, 1/2)\), and hence so is \( \{J(R_n(t))\} \).

The result follows. \(\square\)

4 Bernoulli part decomposition

At the end of this section we give the proof of Theorem 1.1. In view of (10), our goal is to obtain an appropriate lower bound on the entropy \( H(S_n) \). The main idea is to show that \( S_n \) can be asymptotically approximately decomposed as a sum involving a Bin\((n, 1/2)\) random variable and then apply Theorem 3.2. The required decomposition will be based on the following elementary technique.

Let \( X \) be an integer-valued random variable with probability mass function \( p \) on \( Z \) and maximal span \( h = 1 \). The Bernoulli part decomposition of \( X \) is the representation,

\[
X \overset{\text{D}}{=} V + WB,
\]
where $V$ takes values in $\mathbb{Z}$, $W \sim \text{Bern}(q)$, and $B \sim \text{Bern}(1/2)$ is independent of $(V,W)$. The joint probability mass function of $V$ and $W$ is given by,

$$p_{V,W}(k, 1) = \min\{p(k), p(k + 1)\},$$

$$p_{V,W}(k, 0) = p(k) - \frac{1}{2}[p_{V,W}(k - 1, 1) + p_{V,W}(k, 1)], \quad k \in \mathbb{Z},$$

and the parameter $q$ is,

$$q := \sum_{k \in \mathbb{Z}} \min\{p(k), p(k + 1)\} > 0,$$

where the positivity of $q$ follows from the fact that the maximal span is 1.

For the proof we need the following elementary lemma. It says that, if we wait long enough, there will be an (approximately) symmetric Bernoulli step hidden in $S_n$.

**Lemma 4.1** Under the assumptions of Theorem 1.1, suppose the $X_i$ have zero mean and take values in \{a + k : k \in \mathbb{Z}\}, for some $a \in \mathbb{R}$, with maximal span $h = 1$. Then, for each $n \geq 1$, there is a random variable $V^{(n)}$ with values in \{na + k : k \in \mathbb{Z}\} and a $W^{(n)} \sim \text{Bern}(q^{(n)})$, such that,

$$S_n \overset{D} = V^{(n)} + W^{(n)}B,$$

where $B \sim \text{Bern}(1/2)$ is independent of $(V^{(n)}, W^{(n)})$ and $q^{(n)} \to 1$ as $n \to \infty$. Furthermore,

$$\text{Var}(S_n|W^{(n)} = 1) = n\sigma^2(1 + o(1)) \quad \text{as} \quad n \to \infty.$$

**Proof.** Using the Bernoulli part decomposition $X_i = V_i + W_iB_i$ for each $X_i$, we can write,

$$S_n \overset{D} = \sum_{i=1}^n V_i + \sum_{i=1}^n B_i,$$

where $N_n = \sum_{i=1}^n W_i \sim \text{Bin}(n, q)$. But also,

$$\sum_{i=1}^n V_i + \sum_{i=1}^{N_n} B_i \overset{D} = \sum_{i=1}^n V_i + \left(\sum_{i=1}^{N_n-1} B_i\right)\mathbb{I}_{\{N_n \geq 1\}} + \mathbb{I}_{\{N_n \geq 1\}}B,$$

where $B \sim \text{Bern}(1/2)$ is independent of everything else. This is exactly of the required form (24), with $V^{(n)} = \sum_{i=1}^n V_i + (\sum_{i=1}^{N_n-1} B_i)\mathbb{I}_{\{N_n \geq 1\}}$ and $W^{(n)} = \mathbb{I}_{\{N_n \geq 1\}}$, where $q^{(n)} = 1 - (1 - q)^n \to 1$ as $n \to \infty$ by (23).

For (25) we only have to consider the case $q < 1$, since otherwise the result holds trivially. For the mean we have,

$$0 = \mathbb{E}S_n = \mathbb{E}\left[\sum_{i=1}^n V_i \left| W^{(n)} = 0 \right.\right](1-q)^n + \mathbb{E}\left(S_n|W^{(n)} = 1\right)[1 - (1 - q)^n].$$

On the event $\{W^{(n)} = 0\} = \{W_1 = \cdots = W_n = 0\}$ the $V_i$ are i.i.d., so, $\mathbb{E}\sum_{i=1}^n V_i|W^{(n)} = 0] = O(n)$, and since $[1 - (1 - q)^n] \to 1$, we must have,

$$\mathbb{E}(S_n|W^{(n)} = 1) = o(1), \quad \text{as} \quad n \to \infty.$$  

(26)
For the second moment we similarly have,

\[ n\sigma^2 = \mathbb{E}S_n^2 = \mathbb{E} \left[ \left( \sum_{i=1}^{n} V_i \right)^2 \bigg| W^{(n)} = 0 \right] (1 - q)^n + \mathbb{E} \left( S_n^2 \bigg| W^{(n)} = 1 \right) [1 - (1 - q)^n], \]

and since the \( V_i \) are i.i.d. on \( \{W^{(n)} = 0\} \), we have, \( \mathbb{E}[(\sum_{i=1}^{n} V_i)^2 | W^{(n)} = 0] = O(n^2) \). Therefore,

\[ \mathbb{E} \left( S_n^2 \bigg| W^{(n)} = 1 \right) = n\sigma^2 (1 + o(1)), \quad \text{as} \ n \to \infty. \]  

(27)

The result follows from (26) and (27). \( \square \)

We can finally give the proof of the general case of our main result.

**Proof of Theorem 1.1.** In view of (10), we only need to show that, as \( n \to \infty \),

\[ H(S_n) \geq \log \frac{\sqrt{n}}{h} + \frac{1}{2} \log(2\pi e\sigma^2) + o(1). \]  

(28)

Without loss of generality, we assume that the \( X_i \) have mean zero and maximal span \( h = 1 \). Let \( \epsilon > 0 \) be arbitrary and \( M \) a large integer to be chosen later. For \( 1 \leq i \leq n/M \), let \( S_i^{(M)} = \sum_{j=(i-1)M+1}^{iM} X_j \), so that \( S_n = \sum_{i=1}^{n/M} S_i^{(M)} \). In the notation of Lemma 4.1, for \( n \geq M \),

\[ H(S_n) = H \left( \sum_{i=1}^{n/M} S_i^{(M)} \right) \]

\[ = H \left( \sum_{i=1}^{n/M} \left( V_i^{(M)} + W_i^{(M)} B_i \right) \right) \]

\[ \geq H \left( \sum_{i=1}^{n/M} \left( V_i^{(M)} + W_i^{(M)} B_i \right) \bigg| W^{(M)} = w_1, \ldots, W^{(M)} = w_{n/M} \right). \]

Let \( W^{(M)} \) denote the vector \( (W_1^{(M)}, \ldots, W_{n/M}^{(M)}) \) and write \( A_M \) the collection of vectors \( w = (w_1, \ldots, w_{n/M}) \in \{0, 1\}^{M/n} \) with \( w_i = 1 \) for at least \( n(q^{(M)} - \epsilon/2)/M \) indices \( i \), where \( q^{(M)} = q_i^{(M)} \) is the parameter in the Bernoulli decomposition of Lemma 4.1. Then we can bound,

\[ H(S_n) \geq \sum_{w \in A_M} \mathbb{P}(W^{(M)} = w) H \left( \sum_{i=1}^{n/M} \left( V_i^{(M)} + W_i^{(M)} B_i \bigg| W^{(M)} = w \right) \right. \]

(29)

Now observe that, on the event \( \{W^{(M)} = w\} \), the \( n/M \) random variables \( \{V_i^{(M)} + W_i^{(M)} B_i\} \) are independent, though not necessarily identically distributed. But by (13), we can leave out of the sum inside the entropy in (29) the summands that correspond indices \( i \) for which \( w_i = 0 \). Thus, writing \( \bar{W}^{(M)} \) for the vector consisting of \( W_i \) with \( 1 \leq i \leq n(q^{(M)} - \epsilon/2)/M \), and 1 for the vector of all 1s,

\[ H(S_n) \geq \sum_{w \in A_M} \mathbb{P}(W^{(M)} = w) H \left( \sum_{1 \leq i \leq n/M, w_i = 1} \left( V_i^{(M)} + W_i^{(M)} B_i \bigg| W^{(M)} = w \right) \right. \]

\[ \geq \mathbb{P} \left( \sum_{i=1}^{n/M} W_i^{(M)} \geq \frac{n}{M} \left( q^{(M)} - \frac{\epsilon}{2} \right) \right) H \left( \sum_{i=1}^{n/(q^{(M)} - \epsilon/2)/M} \left( V_i^{(M)} + B_i \bigg| \bar{W}^{(M)} = 1 \right) \right). \]

(30)
where the second inequality follows form another application of (13), and the fact that, for different \( i \), the distribution of \( V_i^{(M)} + W_i^{(M)} B_i \) only depends on \( W_i^{(M)} \).

Since each \( W_i^{(M)} \sim \text{Bern}(q^{(M)}) \), the probability in (30) converges to 1 exponentially fast. And since the summands inside the entropy in (30) are i.i.d. with variance \( O(1) \), from the upper bound in (10) it follows that, as \( n \to \infty \),

\[
H(S_n) \geq H \left( \sum_{i=1}^{n(q^{(M)} - \epsilon/2)/M} (V_i^{(M)} + B_i) \mid \bar{W}^{(M)} = 1 \right) - o(1).
\]

To complete the proof, we apply Theorem 3.2 to the sequence of i.i.d. random variables \( \{V_i^{(M)}\} \) conditional on \( \{W_i^{(M)} = 1\} \), and the independent sequence \( \{B_i\} \), to obtain that, as \( n \to \infty \),

\[
H(S_n) \geq \frac{1}{2} \log \left( \frac{n}{M} \left( q^{(M)} - \frac{\epsilon}{2} \right) \right) + \frac{1}{2} \log \left( 2\pi e \text{Var}(V_1^{(M)} + B_1 \mid W_1^{(M)} = 1) \right) - o(1),
\]

and using the variance bound in Lemma 4.1,

\[
H(S_n) \geq \frac{1}{2} \log \left( \frac{n}{M} \right) + \frac{1}{2} \log \left( 2\pi e \sigma^2 \right) + \frac{1}{2} \log(1 - \epsilon) + \frac{1}{2} \log \left( q^{(M)} - \frac{\epsilon}{2} \right) - o(1),
\]

where \( M \) is taken large enough for the \( o(1) \) term in Lemma 4.1 to be smaller than \( \epsilon \). And taking \( M \) large enough so that \( q^{(M)} > 1 - \epsilon/2 \),

\[
H(S_n) \geq \frac{1}{2} \log n + \frac{1}{2} \log(2\pi e \sigma^2) + \log(1 - \epsilon) - o(1).
\]

Since \( \epsilon > 0 \) was arbitrary, this gives (28) and completes the proof.

Finally we remark that, in order to avoid non-essential technicalities, throughout the proof we have implicitly assumed that both \( n/M \) and \( n(q^{(M)} - \epsilon/2)/M \) are integers. This does not harm generality as we could have replaced these quantities with their integer parts and “=” with “\( \geq \)” where necessary to obtain exactly the same result. \( \square \)

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