Trace formula for linear Hamiltonian systems with its applications to elliptic Lagrangian solutions

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Abstract

In the present paper, we build up trace formulas for both the linear Hamiltonian systems and Sturm-Liouville systems. The formula connects the monodromy matrix of a symmetric periodic orbit with the infinite sum of eigenvalues of the Hessian of the action functional. A natural application is to study the non-degeneracy of linear Hamiltonian systems. Precisely, by the trace formula, we can give an estimation for the upper bound such that the non-degeneracy preserves. Moreover, we could estimate the relative Morse index by the trace formula. Consequently, a series of new stability criteria for the symmetric periodic orbits is given. As a concrete application, the trace formula is used to study the linear stability of elliptic Lagrangian solutions of the classical planar three-body problem. It is well known that the linear stability of elliptic Lagrangian solutions depends on the mass parameter $\beta = 27(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2 \in [0, 9]$ and the eccentricity $e \in [0, 1)$. Based on the trace formula, we estimate the stable region and hyperbolic region of the elliptic Lagrangian solutions.

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Key Words. trace formula, Hamiltonian systems, Sturm-Liouville systems, planar three-body problem, linear stability

1 Introduction

In the study of symmetric periodic solutions or quasi-periodic solutions in $n$-body problem, it is natural to consider the $S$-periodic solution in Hamiltonian system

$$\dot{z}(t) = JH'(t, z(t)), \quad \gamma(t) \equiv \gamma_z(t)$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $S$ is a symplectic orthogonal matrix on $\mathbb{R}^{2n}$, and $H(t, x) \in C^2(\mathbb{R}^{2n+1}; \mathbb{R})$. Please refer \cite{4}, \cite{5}, \cite{9} and references therein for the background of $S$-periodic orbits in $n$-body problems. For the solution $z$ of \cite{1}, \cite{1}, \cite{1}, let $\gamma \equiv \gamma_z(t)$ be the corresponding fundamental solution, that is $\dot{\gamma}(t) = JB(t)\gamma(t), \gamma(0) = I_{2n}$, where $B(t) = B(t)^T = H''(t, z(t))$. $\gamma(T)$ is called the monodromy matrix.

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The linear stability of $S$-periodic solution $z(t)$ depends on the location of eigenvalues of $S\gamma(T)$ (see e.g. [14]). But due to the non-commutativity, in general, the fundamental solution could not be obtained directly. In the present paper, we obtain a kind of trace formula for linear Hamiltonian system. Using the trace formula, we can estimate the relative Morse index, and hence, based on the theory of Maslov-type index [18], we give some new stability criteria for Hamiltonian system. Finally, the trace formula will be used to study the stable region and hyperbolic region of Lagrangian solutions in planar three-body problem.

For $k \in \mathbb{N}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $M(k, \mathbb{F})$ be the set of $k \times k$ matrices on $\mathbb{F}^k$. We denote by Sp$(2k) = \{ \mathcal{P} \in M(2k, \mathbb{R}) \mid J\mathcal{P}J = \mathcal{P} \}$ the symplectic group, $S(k)$ the set of $k \times k$ real symmetric matrices and $\mathcal{B}(k) = C([0, T]; S(k))$, the space of continuous paths on $[0, T]$ of matrices in $S(k)$. For $B(t), D(t) \in \mathcal{B}(2n)$, consider the eigenvalue problem of the following linear Hamiltonian systems,

$$
\dot{z}(t) = J(B(t) + \lambda D(t))z(t),
$$

(1.3)

$$
z(0) = Sz(T).
$$

(1.4)

Denote by $A = -J\frac{d}{dt}$, which is densely defined in the Hilbert space $E = L^2([0, T]; \mathbb{C}^{2n})$ with the domain

$$
D_S = \{ z(t) \in W^{1,2}([0, T]; \mathbb{C}^{2n}) \mid z(0) = Sz(T) \}.
$$

$B$ is a bounded linear operator defined by $(Bz)(t) = B(t)z(t)$ on $E$. Then $A$ is a self-adjoint operator with compact resolvent; moreover for $\lambda \in \rho(A)$, the resolvent set of $A$, $(\lambda - A)^{-1}$ is Hilbert-Schimidt.

As above, let $\gamma(t)$ be the fundamental solution of (1.3). To state the trace formula for Hamiltonian system, we need some notations. Write $M = S\gamma(T)$ and $\hat{D}(t) = \gamma(t)D(t)\gamma(t)$. For $k \in \mathbb{N}$, let

$$
M_k = \int_0^T J\hat{D}(t_1) \int_0^{t_1} J\hat{D}(t_2) \cdots \int_0^{t_{k-1}} J\hat{D}(t_k) dt_k \cdots dt_2 dt_1,
$$

and

$$
G_k = M_k M \left( M - e^{\nu T} I_{2n} \right)^{-1}.
$$

**Theorem 1.1.** For $\nu \in \mathbb{C}$ such that $A - B - \nu J$ is invertible, we have for any positive integer $m$,

$$
\text{Tr} \left[ \left( D(A - B - \nu J)^{-1} \right)^m \right] = m \sum_{k=1}^{m} \frac{(-1)^k}{k} \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}).
$$

(1.5)

There are two reasons why we consider the parameter $\nu$ in Theorem 1.1. Firstly, for a given $B \in \mathcal{B}(2n)$, we can not expect that $A - B$ is invertible. However, for every $\nu \in \mathbb{C}$ except countable points, $A - B - \nu J$ is invertible. Secondly, the operator $D(A - B - \nu J)^{-1}$ comes from the following boundary value problem naturally

$$
\dot{z}(t) = J(B(t) + \lambda D(t))z(t),
$$

(1.6)

$$
z(0) = \omega S z(T),
$$

(1.7)

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $\omega = e^{\nu T}$. In fact, if we set $A_\omega = -J\frac{d}{dt}$ with the domain $D_S = \{ z(t) = \omega S z(T) \mid z(t) \in W^{1,2}([0, T]; \mathbb{C}^{2n}) \}$, then $e^{-\nu t} A_\omega e^{\nu t} = A - \nu J$. Thus $z \in \ker(A_\omega - B - \lambda D)$ if and only if $e^{-\nu t} z(t) \in \ker(A - \nu J - B - \lambda D)$, which is equivalent to that $\frac{1}{\lambda}$ is an eigenvalue $D(A - \nu J - B)^{-1}$ provided that $A - \nu J - B$ is invertible.
Remark 1.2. (1). For \( m = 1 \), \( D(A - \nu J - B)^{-1} \) is not a trace class operator but a Hilbert-Schmidt operator. And hence \( Tr(D(A - \nu J - B)^{-1}) \) is not the usual trace but a kind of conditional trace\(^\text{[14]}\).

(2). For \( m \geq 2 \), \( (D(A - \nu J - B)^{-1})^m \) are trace class operators. By the preceding argument, \( \lambda \) is a nonzero eigenvalue of system (1.6)-(1.7) if and only if \( \frac{1}{\lambda} \) is an eigenvalue of \( D(A - \nu J - B)^{-1} \). And hence, if we let \( \{ \lambda_i \} \) be the set of nonzero eigenvalues of the system (1.6)-(1.7),

\[
Tr\left[(D(A - B - \nu J)^{-1})^m\right] = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^m} = m \sum_{k=1}^{m} \frac{(-1)^k}{k} \sum_{j_1 + \cdots + j_k = m} Tr(G_{j_1} \cdots G_{j_k}), \quad m \geq 2. \tag{1.8}
\]

For large \( m \), the right hand side of (1.5) is a little complicated. However, for \( m = 1, 2 \), we can write it down more precisely.

Corollary 1.3. For \( \nu \in \mathbb{C} \) such that \( A - B - \nu J \) is invertible,

\[
Tr\left[D(A - B - \nu J)^{-1}\right] = -Tr\left[J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1}\right], \tag{1.9}
\]

and

\[
Tr((D(A - B - \nu J)^{-1})^2) = -2Tr\left[J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)J \int_0^\infty \gamma_0^T(s)D(s)\gamma_0(s)ds dt \cdot M(M - e^{\nu T}I_{2n})^{-1}\right] + Tr\left[J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt M(M - e^{\nu T}I_{2n})^{-1}\right]^2. \tag{1.10}
\]

Especially, in the case that \( M = \pm I_{2n} \),

\[
Tr\left[(D(A - \nu J - B)^{-1})^2\right] = \frac{\pm e^{\nu T}}{1 + e^{2\nu T}} Tr\left[J \int_0^T \gamma_0^T(s)D(s)\gamma_0(s)ds\right]^2. \tag{1.11}
\]

In some concrete problem, such as the estimation of hyperbolic region of elliptic Lagrangian solution, the trace formula for Lagrangian system is more convenient to be used. In order to introduce the trace formula for Lagrangian system, it is natural to consider the following eigenvalue problem of Sturm-Liouville system with \( \tilde{S} \)-periodic boundary condition

\[
-(Py + Qy) + Q^T \dot{y} + (R + AR_1)y = 0, \quad y(0) = \tilde{S}y(T), \quad \dot{y}(0) = \tilde{S} \dot{y}(T), \tag{1.12}
\]

where \( \tilde{S} \) is an orthogonal matrix on \( \mathbb{R}^n \), \( P, R, R_1 \in \mathcal{B}(n), \quad Q \in C([0, T]; M(n, \mathbb{R})) \). Instead of Legendre convexity condition, we assume for any \( t \in [0, T] \), \( P(t) \) is invertible. Moreover we assume

\[
\tilde{S} P(T) = P(0) \tilde{S} \quad \text{and} \quad \tilde{S} Q(T) = Q(0) \tilde{S}. \tag{1.13}
\]

Such a boundary value problem with condition (1.13) comes naturally from the study of symmetric periodic orbits in \( n \)-body problem.

By the standard Legendre transformation, the linear system (1.12) corresponds to the linear Hamiltonian system,

\[
\dot{z} = JB_z(t)z, \quad z(t) = \tilde{S}d z(T), \tag{1.14}
\]
with
\[
\tilde{S}_d = \begin{pmatrix} \tilde{S} & 0_n \\ 0_n & \tilde{S} \end{pmatrix}, \quad \text{and} \quad B_d(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t) P(t) - R(t) - \lambda R_1(t) \end{pmatrix}.
\] (1.15)

Obviously, \( \tilde{S}_d \) is a symplectic orthogonal matrix on \( \mathbb{R}^{2n} \), and the eigenvalue problem (1.14) is a special case of the eigenvalue problem (1.3-1.4). Without confusion, for Lagrangian system, denote by \( \gamma_j(t) \) the fundamental solution of (1.14).

Using the notations in Theorem 1.1, take \( D = \begin{pmatrix} 0_n & 0_n \\ 0_n & -R_1 \end{pmatrix} \). Temporarily, we assume the unperturbed systems is non-degenerate, that is, 0 is not the eigenvalue of (1.12), which is equivalent to that 1 is not the eigenvalue of \( M = \tilde{S}_d \gamma_0(T) \).

**Theorem 1.4.** Let \( \{\lambda_j\} \) be the eigenvalues for the boundary value problem (1.12), then
\[
\sum_j \frac{1}{\lambda_j^m} = m \sum_{k=1}^m (-1)^k \left[ \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right], \quad \forall m \in \mathbb{N},
\] (1.16)
especially for \( m = 1 \),
\[
\sum_j \frac{1}{\lambda_j} = -\text{Tr} \left[ J \int_0^T \gamma_0^T(t) D(t) \gamma_0(t) dt \cdot M(M - I_{2n})^{-1} \right].
\] (1.17)

It should be pointed out that from Proposition 3.5 for \( m \geq 2 \), the trace formula (1.16) is a special case of the formula (1.8). However, for \( m = 1 \), the meanings of the formula (1.9) and (1.17) are totally different. In fact, \( \text{Tr}(D(A - B - \nu J)^{-1}) \) is a kind of conditional trace. Details could be found in Remark 3.6. The formula (1.17) is proved for Sturm-Liouville system, and we do not know for general Hamiltonian system whether it holds true or not. Fortunately, (1.17) is easy to be calculated.

During the study of the above trace formula, thanking for Chongchun Zeng’s suggestion, we can find the original work by Krein\([20, 21]\) in 1950s. In fact, Krein considered the following system
\[
\begin{align*}
\frac{d}{dt}z(t) &= \lambda JD(t)z(t), \\
z(0) &= -z(T),
\end{align*}
\] (1.18)
where \( D \geq 0 \) and \( \int_0^T D(t)dt > 0 \). The system (1.18) is a special case of our system (1.3-1.4). For the system (1.18), Krein proved that \( \lim_{r \to \infty} \sum \frac{1}{\lambda_j^2} = 0 \), and
\[
\sum \frac{1}{\lambda_j^2} = \frac{T^2}{2} \text{Tr}(A_{11}A_{22} - A_{12}^2),
\] (1.20)
where \( \lambda_j \) are the eigenvalues for the system (1.18), and \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{T} \int_0^T D(t)dt \). Moreover, under the condition \( D \geq 0 \), \( \int_0^T D(t)dt > 0 \), Krein gave an interesting stability criteria:
\[
\frac{T^2}{2} \text{Tr}(A_{11}A_{22} - A_{12}^2) < 1.
\]
Obviously, by taking \( \nu = 0 \) and \( M = -I_{2n} \) in the formula (1.11), it is easy to see that Theorem 1.4 generalizes Krein’s formula (1.20).
Remark 1.5. Krein considered the simplest Hamiltonian system with some special conditions such as $D \geq 0$ and $\int_0^T D(t) dt > 0$. For the system coming from $n$-body problem, the conditions are not satisfied. Hence, Krein’s trace formula can not be used to study the $n$-body problem. However, Krein’s trace formula is a powerful tool to study the stability. It is surprising that, to the best of our knowledge, there is no further study along this line.

Next, we will introduce some applications of the trace formula. As one application, we will give some estimations on the non-degeneracy of the linear system. It is well-known that the system preserves the non-degeneracy under small perturbations. A natural question will be arisen: can we give an upper bound for the perturbation, such that, under the smaller perturbation, the systems preserve the non-degeneracy? By the trace formula, we can answer this question partly. Details could be found in Section 4. As another application, the trace formula could be used to estimate the relative Morse index for Hamiltonian systems and Morse index for Lagrangian systems. It is well-known that the relative Morse index (or Morse index) is equal to the Maslov-type index for path of symplectic matrices and the Maslov-type index is a successful tool in judging the linear stability [18], [12]. In Section 4 by using the trace formula, we can give some new stability criteria.

Before giving the further application of the trace formula on $n$-body problem, we want to interpret the proof of the trace formula intuitively. For a matrix $F$, to calculate the trace $Tr F^m$ for $m > 0$, the most effective method is to consider the determinant $\det(I + \alpha F)$, where $I$ is the identity matrix and $\alpha$ is a parameter. In the case of trace formula of differential equation, the idea does work too. From this viewpoint, Hill-type formula is the cornerstone to get the trace formula. The study of such a formula begins with the original work of Hill [10] in 1877. In his study of the motion of lunar perigee, Hill considered the following equation:

$$\ddot{x}(t) + \theta(t)x(t) = 0,$$

(1.21)

where $\theta(t) = \sum_{j \in \mathbb{Z}} \theta_j e^{2j \sqrt{-1} t}$ with $\theta_0 \neq 0$ is a real $\pi$-periodic function. Let $\gamma(t)$ be the fundamental solution of the associated first order system of (1.21), that is,

$$\dot{\gamma}(t) = \begin{pmatrix} 0 & -\theta(t) \\ 1 & 0 \end{pmatrix} \gamma(t),$$

$$\gamma(0) = I_2.$$

Suppose $\rho = e^{c \sqrt{-1} \pi}$, $\rho^{-1} = e^{-c \sqrt{-1} \pi}$ are the eigenvalues of the monodromy matrix $\gamma(\pi)$. In order to compute $c$, Hill obtained the following formula which connects the infinite determinant, corresponding to the differential operator, and the characteristic polynomial:

$$\frac{\sin^2\left(\frac{c\pi}{2}\right)}{\sin^2\left(\frac{\theta_0\pi}{2}\right)} = \det \left[ \left( -\frac{d^2}{dt^2} - \theta_0 \right)^{-1} \left( -\frac{d^2}{dt^2} - \theta_0 \right) \right],$$

(1.22)

where the right hand side of (1.22) is the Fredholm determinant. We should point out that the right hand side of the original formula of Hill [10] is a determinant of an infinite matrix. In [10], Hill did not prove the convergence of the infinite determinant, and the convergence was proved by Poincaré [26]. The Hill formula for a periodic solution of Lagrangian system on manifold was given by Bolotin [2]. In [3], Bolotin and Treschev studied the Hill-type formula for both continuous and discrete Lagrangian systems with Legendre convexity condition. For the periodic solution of ODE, the Hill-type formula was given by Denk [7].
For \( S \)-periodic orbit of Hamiltonian system, the Hill-type formula was given by the first and the third authors \([14]\), for \( B, D \in \mathcal{B}(2n) \)
\[
\det \left[ (A - (B + \lambda D) - \nu J)(A + P_0)^{-1} \right] = C(S) e^{-\nu T} \det(S \gamma_\lambda(T) - e^{\nu T} I_{2n}).
\]
(1.23)
where \( C(S) > 0 \) is a constant depending only on \( S \), and \( \gamma_\lambda(t) \) satisfies \( \dot{\gamma}_\lambda(t) = J(B(t) + \lambda D(t))\gamma_\lambda(t) \), and \( \gamma_\lambda(0) = I_{2n} \). The equality (1.23) is our starting point to get the trace formula of Hamiltonian system. In fact, both sides of (1.23) are analytic functions on \( \lambda \). Then, by taking Taylor expansion and comparing the coefficients on both sides of (1.23), we get the trace formula in Theorem \([17]\). Based on this idea, in order to obtain the trace formula for Lagrangian system, in the present paper we will get the following Hill-type formula.

**Theorem 1.6.** Let \( \{\lambda_j\} \) be the nonzero eigenvalues for the boundary value problem (1.12), then
\[
\prod_j \left( 1 - \frac{1}{\lambda_j} \right) = \det(\tilde{S}_d \gamma_1(T) - I_{2n}) \cdot \det(\tilde{S}_d \gamma_0(T) - I_{2n})^{-1},
\]
(1.24)
where \( \gamma_\lambda \) is the fundamental solution of the system (1.14).

**Remark 1.7.** The Hill-type formula for periodic orbits of Lagrangian system with the Legendre convex condition was given by Bolotin \([2]\) in 1988, and Theorem 1.6 can be considered as a generalization of Bolotin’s work to indefinite Lagrangian systems.

At the end of this paper, we will study the stability of Lagrangian orbits in planar three body problems. In 1772, Lagrange \([15]\) discovered some celebrated periodic solutions, now named after him, to the planar three-body problem, namely the three bodies form an equilateral triangle at any instant of the motion and at the same time each body travels along a specific Keplerian elliptic orbit about the center of masses of the system. All these orbits are homographic solutions. When \( 0 \leq e < 1 \), the Keplerian orbit is elliptic, following Meyer and Schmidt \([24]\), we call such elliptic Lagrangian solutions *elliptic relative equilibria*. Specially when \( e = 0 \), the Keplerian elliptic motion becomes circular motion and then all the three bodies move around the center of masses along circular orbits with the same frequency, which are called *relative equilibria* traditionally. Moreover, Meyer and Schmidt (cf. \([24]\)) used heavily the central configuration nature of the elliptic Lagrangian orbits and decomposed the fundamental solution of the elliptic Lagrangian orbit into two parts symplectically, one of which is the same as that of the Keplerian solution and the other is the essential part for the stability.

For the planar three-body problem with masses \( m_1, m_2, m_3 > 0 \), it turns out that the stability of elliptic Lagrangian solutions depends on two parameters, namely the mass parameter \( \beta \in [0, 9] \) defined below and the eccentricity \( e \in [0, 1] \),
\[
\beta = \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2}.
\]
In the current paper, the fundamental solution of the linearized Hamiltonian system of the essential part of the elliptic Lagrangian orbit is denoted by \( \gamma_{\beta,e}(t) \) for \( t \in [0, 2\pi] \), which is a path of \( 4 \times 4 \) symplectic matrices starting from the identity. The Lagrangian orbits is called spectrally stable (or elliptic) if all the eigenvalues of \( \gamma_{\beta,e}(2\pi) \) belong to the unite circle \( \mathbb{U} \), is called linear stable if moreover \( \gamma_{\beta,e}(2\pi) \) is semi-simple. In contrast, Lagrangian orbits are called hyperbolic if no eigenvalue of \( \gamma_{\beta,e}(2\pi) \) locates on \( \mathbb{U} \).

The linear stability of relative equilibria \((e = 0)\) were known more than a century ago and it is due to Gascheau \([8]\), 1843 and Routh \([29]\), 1875 independently. For the elliptic relative equilibria \((e > 0)\), the
linear stability problem is difficult, many interesting results could be found in [24], [22], [23], [28]. For the historical literature on linear stability of Lagrangian orbits, readers are referred to [11]. Recently, Y.Long, S.Sun and the first author introduced Maslov-type index and operator theory in studying the stability in n-body problem [11], [13]. In [11], the authors gave an analytic proof for the the stability bifurcation diagram of Lagrangian equilateral triangular homographic orbits in the \((\beta; e)\) rectangle \([0, 9] \times [0, 1]\) and proved that bifurcation curve is real analytic. But it is difficult to estimate the bifurcation curve.

To the best of our knowledge, we don’t know any result before to estimate the stability region. For the hyperbolic region, till now, we only know two results. Firstly, it was proved in [11] that the Lagrangian orbits is hyperbolic for \(\beta = 9\) (equal mass case) with any eccentricity \(e \in [0, 1]\). Secondly, based on the result in [11], it was proved by the second author [25] that Lagrangian orbits are hyperbolic for \(\beta > 8\). However, for \(\beta\) near 1, we know nothing about the estimation of the hyperbolic region before. In the present paper, based on works in [11], [13] and via trace formula, we estimate the stability region and hyperbolic region for the elliptic Lagrangian orbits.

**Theorem 1.8.** The elliptic Lagrangian orbits is linear stable if

\[
e < \frac{1}{1 + f(\beta, -1)^{\frac{1}{2}}}, \, \beta \in [0, 3/4),
\]

or

\[
e < \min \left\{ \frac{1}{\sqrt{f(\beta, -1)}}, \frac{1}{1 + \sqrt{f(\beta, e^{i\sqrt{2}\pi})}} \right\}, \, \beta \in (3/4, 1),
\]

where \(f(\beta, \omega)\) is a function on \([0, 9] \times \mathbb{U}\) given by (5.13). Let \(\hat{f}(\beta) = \sup\{f(\beta, \omega), \omega \in \mathbb{U}\}\), then for \(\beta \in (1, 9]\), \(\gamma_{\beta, e}\) is hyperbolic if

\[
e < \hat{f}(\beta)^{-1/2}.
\]  

(1.25)

It will be seen that \(f(\beta, \omega)\) is a elementary function determined by the trace formula. By Theorem 1.8, we can draw a picture as follows.

![Figure 1: The stable region S and hyperbolic region H given by Theorem 1.8](image)
In Figure 1, the points $O_1 \approx (0, 0.3333), O_2 \approx (0.8730, 0.0504), O_3 \approx (9, 0.4907)$. The curves
\[ \Gamma_1 = \left\{ (\beta, e) \mid e = 1 \left/ \left(1 + \sqrt{f(\beta, -1)} \right) \right., 0 \leq \beta \leq 3/4 \right\}, \quad \Gamma_2 = \left\{ (\beta, e) \mid e = 1 \left/ \sqrt{f(\beta, -1)} \right., 3/4 \leq \beta \leq 1 \right\}, \]

and
\[ \Gamma_3 = \left\{ (\beta, e) \mid e = 1 \left/ \left(1 + \sqrt{f(\beta, e^i \sqrt{2\pi n})} \right) \right., 3/4 \leq \beta < 1 \right\}, \quad \Gamma_4 = \left\{ (\beta, e) \mid e = 1 \left/ \sqrt{f(\beta)} \right., 1 \leq \beta \leq 9 \right\}. \]

This paper is organized as follows. In Section 2, we give the proof of the trace formula for linear Hamiltonian systems. Moreover, some application of the trace formula on the identity which related to the Zeta function is given. In Section 3, we prove the Hill-type formula and trace formula for Sturm-Liouville systems. The applications of the trace formula on the study of stability for Hamiltonian systems are given in Section 4, where we estimate the relative Morse index (Morse index for Sturm-Liouville systems) and some new stability criteria will be given. The study of stability of elliptic Lagrangian solutions will be given in Section 5.

2 Trace formula for linear Hamiltonian system

In this section, we will give the proof of the trace formula for linear Hamiltonian system. As been pointed out in the introduction, we will consider the Taylor expansion for the conditional Fredholm determinant of Hamiltonian system and the Monodromy matrices separately in §2.1 and §2.2. Based on it, we prove Theorem 1.1 in §2.3, some example on infinite identity and relation with the Zeta function is discussed.

2.1 Taylor expansion for conditional Fredholm determinant of the linear perturbation of Hamiltonian system

In this subsection, we will mainly consider the Taylor expansion of the conditional Fredholm determinant for linearly parameterized Hamiltonian system. Let $B(\alpha) : \Omega \rightarrow C([0, T], M(2n, \mathbb{C}))$ be an analytic function. For that $(A - B - \nu J)$ is invertible, denote by
\[ p(\alpha) = \det \left( id - (B_0 - B_\alpha)(A - B_\alpha - \nu J)^{-1} \right). \]

Notice that $(B_\alpha - B_0)(A - B_0 - \nu J)^{-1}$ is not trace class but Hilbert-Schmidt. Hence $p(\alpha)$ is not the usual Fredholm determinant, but a kind of conditional Fredholm determinant. The theory of conditional Fredholm determinant was studied in [14]. For readers convenience, we recall it briefly. For integer $N > 0$, let $P_N$ be the projection onto the subspace
\[ W_N = \bigoplus_{\nu \in \sigma(A), |\nu| \leq N} \ker(A - \nu). \]

We need the following definition, which comes from [14].

**Definition 2.1.** For a Hilbert-Schmidt operator $F$, it is said to have the trace finite condition, if the limit $\lim_{N \to \infty} Tr(P_N FP_N)$ exists, which is called the conditional trace and denoted by $Tr(F)$ without confusion.
Let $f(\alpha) = \det \left[ (A - B - \alpha D - \nu J)(A + P_0)^{-1} \right]$, suppose $A - B - \nu J$ is invertible, then the Taylor expansion of $f$ at 0 is

$$f(\alpha) = \sum_{m=0}^{\infty} \hat{b}_m \alpha^m,$$

where

$$\hat{b}_m = \frac{a_m}{m!} \det((A - B - \nu J)(A + P_0)^{-1}),$$

and

$$a_m = (-1)^m \det \begin{pmatrix} Tr(F) & m-1 & \cdots & 0 \\ Tr(F^2) & Tr(F) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Tr(F^m) & Tr(F^{m-1}) & \cdots & Tr(F) \end{pmatrix}, \quad (2.1)$$

with $F = D(A - B - \nu J)^{-1}$.

We first prove the following simple lemma.

**Lemma 2.3.** Let $B(\alpha) : \Omega \to C([0, T], \mathcal{M}(2n, \mathbb{C}))$ be an analytic mapping. Write

$$p_N(\alpha) = \det \left( id - P_N(B_0 - B_0)(A - B_0 - \nu J)^{-1} P_N \right),$$

then $p_N(\alpha)$ is analytic on $\Omega$.

**Proof.** Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis, defined by the eigenvectors of $A$. Set

$$F(\alpha) = (B_0 - B_0)(A - B_0 - \nu J)^{-1},$$

then $F(\alpha)$ can be considered as an infinite matrix $(\langle F(\alpha)e_j, e_i \rangle)_{i,j}$. Notice that $\langle F(\alpha)e_j, e_i \rangle$ is an analytic function on $\alpha$, which implies that $P_N F(\alpha) P_N$ is an analytic function on $\Omega$. By the definition of $p_N(\alpha)$, we know that $p_N(\alpha)$ is analytic. \qed

To prove Theorem 2.2 write

$$f_N(\alpha) = \det \left( id - \alpha P_N D(A - B - \nu J)^{-1} P_N \right).$$

Firstly, please note that $f_N(\alpha)$ is analytic. Secondly, we will show that there is a subsequence of $\{f_N(\alpha)\}$, which is convergent uniformly on any compact subset of $\Omega$. Obviously, $f_N(\alpha) \to f(\alpha)$ point-wisely on $\Omega$. Thirdly, by the theory in [30], we will give the expansion of $f_N(\alpha)$. Finally, by the convergence of $f_N(\alpha)$, we get the Taylor expansion of $f(\alpha)$.
To prove that there is a subsequence of \( \{f_N(\alpha)\} \), which is convergent uniformly to \( f(\alpha) \) on any compact subset, we will recall some properties of conditional Fredholm determinant and conditional trace.

Recall that, if \( \hat{F} \) is a Hilbert-Schmidt operator, then \((id + \hat{F})e^{-\hat{F}} - id\) is a trace class operator, thus, we can define
\[
\det_2(id + \hat{F}) = \det((id + \hat{F})e^{-\hat{F}}).
\]
In the classical settings, if \( \hat{F} \) is trace class, then \( \det_2(id + \hat{F}) = \det(id + \hat{F})e^{-\text{Tr}(\hat{F})} \). Inspired from this, in the case that \( \hat{F} \) has the trace finite condition, we proved in [14],
\[
\det_2(id + \hat{F}) = \det((id + \hat{F})e^{-\text{Tr}(\hat{F})})
\] (2.2)
still holds, however, where \( \text{Tr}(\hat{F}) = \lim_{N \to \infty} \text{Tr}(P_N \hat{F} P_N) \) is the conditional trace.

The conditional Fredholm determinant preserves almost all the properties that the determinant of matrix has. Such as, the multiplicity of the determinant. Let \( \hat{D} \) and \( \hat{F} \) be two Hilbert-Schmidt operators which have trace finite condition. Then
\[
\det(id + \hat{D}) \det(id + \hat{F}) = \det(id + \hat{D} + \hat{F} + \hat{D}\hat{F}),
\] (2.3)
where “det” represents conditional Fredholm determinant. Similar to [14, Proposition 3.2], we have the following lemma. The proof of the lemma is almost the same as that was given for [14, Proposition 3.2], and we will omit the proof.

**Lemma 2.4.** Under the assumption of Lemma 2.3, \( \{p_N\} \) is a normal family, that is, for any sequence in \( \{p_N\} \) there is a subsequence which is uniformly convergent on any compact subset of \( \mathbb{C} \).

For \( \nu \in \mathbb{C} \) such that \( A - B - \nu J \) is invertible,
\[
(A - B - \alpha D - \nu J)(A + P_0)^{-1} = (id - \alpha D(A - B - \nu J)^{-1})(A - B - \nu J)(A + P_0)^{-1}.
\]
Now set
\[
g(\alpha) = \det(id - \alpha D(A - B - \nu J)^{-1}),
\]
where the “det” is the conditional Fredholm determinant, and
\[
g_N(\alpha) = \det(id - \alpha P_N D(A - B - \nu J)^{-1} P_N).
\]
By Lemma 2.4, \( f_N \) and \( f \) are entire functions, and there is a subsequence \( \{f_{N_k}\} \) which is convergent to \( f \) uniformly on any compact subset in \( \Omega \). Set
\[
F_N = P_N D(A - B - \nu J)^{-1} P_N,
\]
then all of \( F_N \) are finite-rank operators, hence they are trace class operators, by [30, Theorem 5.4], we have the following lemma.

**Lemma 2.5.** Let \( g_N(\alpha) = \det(id + \alpha(-F_N)) \). Then the Taylor expansion near \( 0 \) for \( g_N(\alpha) \) is
\[
g_N(\alpha) = \sum_{m=0}^{\infty} \alpha^m a_{N,m}/m!,
\]
where
\[
a_{N,m} = \sum_{n=0}^{\infty} \alpha^n a_{N,m}/n!,
\]
and
\[
F_N = P_N D(A - B - \nu J)^{-1} P_N.
\]
where

\[
\begin{pmatrix}
\text{Tr}(F) & m-1 & 0 & \cdots & 0 \\
\text{Tr}(F^2) & \text{Tr}(F) & m-2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\text{Tr}(F^{m-1}) & \text{Tr}(F) & \cdots & \text{Tr}(F) & 1 \\
\text{Tr}(F_N) & \text{Tr}(F_N^{m-1}) & \cdots & \text{Tr}(F_N^2) & \text{Tr}(F_N) \\
\end{pmatrix}
\]

Let \( h_n \) be a sequence of analytic functions, which is convergent to \( h \) uniformly on any compact subset. Write the power series expansions as

\[ h_n(\alpha) = \sum_{m=0}^{\infty} c_{n,m} \alpha^m, \quad \text{and} \quad g(\alpha) = \sum_{m=0}^{\infty} c_m \alpha^m, \]

then, it is easy to see that \( c_{n,m} \) converges to \( c_m \) as \( n \to \infty \).

**Proof of Theorem** Now, notice that \( F = D(A - B - \nu J)^{-1} \) is a Hilbert-Schmidt operator with trace finite condition, hence the conditional trace

\[ \text{Tr}(F) = \lim_{N \to \infty} \text{Tr}(F_N). \]

Set

\[ a_m = (-1)^m \det \begin{pmatrix} \text{Tr}(F) & m-1 & \cdots & 0 \\ \text{Tr}(F^2) & \text{Tr}(F) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \text{Tr}(F^{m-1}) & \text{Tr}(F) & \cdots & \text{Tr}(F) \\ \end{pmatrix}, \]

Then \( a_{N,m} \) tends to \( a_m \) as \( N \to \infty \). By Lemma 2.4, there is a subsequence \( g_{N,j}(\alpha) \) of \( g_N(\alpha) \), which is convergent to \( g(\alpha) \) on any compact subset. Then

\[ g(\alpha) = \sum_{m=0}^{\infty} \frac{a_m}{m!} \alpha^m. \]

Since

\[ f(\alpha) = g(\alpha) \det((A - B - \nu J)(A + P_0)^{-1}), \]

we have

\[ f(\alpha) = \sum_{m=0}^{\infty} \alpha^m \left[ \frac{a_m}{m!} \det((A - B - \nu J)(A + P_0)^{-1}) \right] \]

The proof is finished. \( \square \)

Note that for \( \alpha \) small, by [30, p.47, (5.12)], for a matrix \( D \),

\[ \det(I + \alpha D) = \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \alpha^m \text{Tr}(D^m) \right), \tag{2.4} \]

Thus for \( \alpha \) small enough, write \( g_N(\alpha) = \exp(\alpha \alpha N(\alpha)), \) then

\[ h_N(\alpha) = \sum_{m=1}^{\infty} (-1)^{m+1} d_m(N) / m, \]

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with \( d_m(N) = Tr((-F_N)^m) \). On the other hand, since \((A - B - \nu J)\) is invertible, hence, \( id - \alpha D(A - B - \nu J)^{-1} \) is invertible in a neighborhood of 0. It follows that \( g(\alpha) \) vanishes nowhere in a neighborhood of 0. Write \( g(\alpha) = e^{h(\alpha)} \) near 0 with
\[
h(\alpha) = \sum_{m=1}^{\infty} (-1)^{m+1} d_m \alpha^m / m
\]
be the Taylor expansion for \( h(\alpha) \). Since \( g_N \) converge to \( g \) and is normal family, we have that \( d_m = (-1)^m Tr(F^m) \). We get the following theorem, which is the main result in this section.

**Theorem 2.6.** Under the above assumption, we have
\[
f(\alpha) = \det((A - B - \nu J)(A + P_0)^{-1}) \exp \left\{ \sum_{m=1}^{\infty} b_m \alpha^m \right\},
\]
where \( b_m = -\frac{1}{m} Tr(F^m) \).

Notice that \( F \) is a Hilbert-Schmidt operator with trace finite condition. Hence, \( Tr(F) \) is not the usual trace of \( F \), but the conditional trace. However, in [30, Theorem 5.4] \( F \) is a trace class operator, and then \( Tr(F) \) is the usual trace.

### 2.2 Taylor expansion for linearly parameterized Monodromy matrices

Set \( B(\alpha) = B + \alpha D \), for \( \alpha \in \mathbb{C} \), let \( \gamma(\alpha) \) be the corresponding fundamental solutions, that is
\[
\dot{\gamma}(\alpha)(t) = JB(\alpha)(t)\gamma(\alpha)(t).
\]
Fixed \( \alpha_0 \in \mathbb{C} \), direct computation shows that
\[
\frac{d}{dt}(\gamma^{-1}_{\alpha_0}(t)\gamma(\alpha)(t)) = \gamma^{-1}_{\alpha_0}(t)J(\gamma_{\alpha_0}(t) - B_{\alpha_0}(t))\gamma(\alpha)(t)
\]
\[
= J(\gamma^T_{\alpha_0}(t)(\gamma_{\alpha_0}(t) - B_{\alpha_0}(t))\gamma_{\alpha_0}(t))\gamma^{-1}_{\alpha_0}(t)\gamma(\alpha)(t)
\]
\[
= (\alpha - \alpha_0)J(\gamma^T_{\alpha_0}(t)D(t)\gamma_{\alpha_0}(t))\gamma^{-1}_{\alpha_0}(t)\gamma(\alpha)(t).
\]
Without loss of generality, assume \( \alpha_0 = 0 \). In what follows, write
\[
\dot{\gamma}_{\alpha}(t) = \gamma^{-1}_{0}(t)\gamma_{\alpha}(t),
\]
and
\[
\dot{D}(t) = \gamma^T_{0}(t)D(t)\gamma_{0}(t),
\]
thus
\[
\frac{d}{dt}\dot{\gamma}_{\alpha}(t) = \alpha J\dot{D}(t)\dot{\gamma}_{\alpha}(t). \tag{2.5}
\]
To simplify the notation, we use \( \alpha^{(k)} \) to denote the \( k \)-th derivative on \( \alpha \). Taking derivative on \( \alpha \) for both sides of (2.5), we get
\[
\frac{d}{dt}\dot{\gamma}^{(1)}_{\alpha}(t) = J\dot{D}(t)\dot{\gamma}_{\alpha}(t) + \alpha J\dot{D}_{\alpha}(t)\dot{\gamma}^{(1)}_{\alpha}(t). \tag{2.6}
\]
By taking $\alpha = 0$, $\tilde{\gamma}_0(t) \equiv I_{2n}$, we have
\[\tilde{\gamma}_0^{(1)}(t) = J \int_0^t \hat{D}(s) ds.\]

Now, taking derivative on $\alpha$ for both sides of (2.6), we get
\[\frac{d}{dt}\tilde{\gamma}_0^{(2)}(t) = 2J\hat{D}(t)\tilde{\gamma}_0^{(1)}(t) + J\alpha \hat{D}(t)\tilde{\gamma}_0^{(2)}(t).\]

Take $\alpha = 0$, and we get
\[\tilde{\gamma}_0^{(2)}(t) = 2J \int_0^t \hat{D}(s) \tilde{\gamma}_0^{(1)}(s) ds.\]

By induction,
\[\frac{d}{dt}\tilde{\gamma}_0^{(k)}(t) = kJ\hat{D}(t)\tilde{\gamma}_0^{(k-1)}(t),\]
and
\[\tilde{\gamma}_0^{(k)}(t) = kJ \int_0^t \hat{D}(s) \tilde{\gamma}_0^{(k-1)}(s) ds.\]

For $t = T$, by Taylor's formula,
\[\tilde{\gamma}_0(T) = I_{2n} + \alpha\tilde{\gamma}_0^{(1)}(T) + \cdots + \alpha^k\tilde{\gamma}_0^{(k)}(T)/k! + \cdots,\]
where
\[\tilde{\gamma}_0^{(1)}(T) = \int_0^T J\hat{D}(t) dt\]
and
\[\tilde{\gamma}_0^{(k)}(T)/k! = \int_0^T J\hat{D}(t)\tilde{\gamma}_0^{(k-1)}(t)/(k-1)! dt, k \in \mathbb{N}.\]

By induction, we have
\[\tilde{\gamma}_0^{(k)}(T)/k! = \int_0^T J\hat{D}(t_1) \int_0^{t_1} J\hat{D}(t_2) \cdots \int_0^{t_{k-1}} J\hat{D}(t_k) dt_k \cdots dt_2 dt_1, k \in \mathbb{N}.\]

Obviously $\tilde{\gamma}_0(T)$ is an entire function on the variable $\alpha$. We summarize the above reasoning as the following proposition.

**Proposition 2.7.** Let $B_\alpha = B + \alpha D$, $\gamma_\alpha(T)$ be the corresponding fundamental solutions. Write $\tilde{\gamma}_\alpha = \gamma^{-1}_0 \gamma_\alpha$. Then, the Taylor expansion for $\tilde{\gamma}_\alpha(T)$ at 0 is
\[\tilde{\gamma}_\alpha(T) = I_{2n} + \alpha\tilde{\gamma}_0^{(1)}(T) + \cdots + \alpha^k\tilde{\gamma}_0^{(k)}(T)/k! + \cdots,\]
where
\[\tilde{\gamma}_0^{(k)}(T)/k! = \int_0^T J\hat{D}(t_1) \int_0^{t_1} J\hat{D}(t_2) \cdots \int_0^{t_{k-1}} J\hat{D}(t_k) dt_k \cdots dt_2 dt_1, k \in \mathbb{N}.\]
In what follows, to simplify the notation, set
\[ M(\alpha) = \hat{\gamma}_\alpha(T), \quad M_0 = I_{2n} \quad \text{and} \quad M_j = \hat{\gamma}_0^{(j)}(T)/j!, \quad j \in \mathbb{N}, \]
then
\[ M(\alpha) = \sum_{j=0}^{\infty} \alpha^j M_j. \]

Direct computation shows that
\[ M(\alpha)^T J M(\alpha) = J + \alpha C_1 + \alpha^2 C_2 + \cdots + \alpha^k C_k + \cdots \]
where \( C_1 = M_1^T J + JM_1, \) \( C_2 = M_2^T J + JM_2 + M_1^T JM_1, \) and in general
\[ C_k = \sum_{j=0}^{k} M_j^T JM_{k-j}, \quad k \in \mathbb{N}. \]

By the fact that \( M(\alpha) \in \text{Sp}(2n), \) \( M(\alpha)^T J M(\alpha) = J, \) thus \( C_k = 0 \) for \( k \in \mathbb{N}. \) We have the following proposition.

**Proposition 2.8.** Under the above assumptions
\[ \sum_{j=0}^{k} M_j^T JM_{k-j} = 0, \quad \forall k \in \mathbb{N}. \tag{2.7} \]

Please note that, by taking \( k = 1 \) in (2.7), we have
\[ JM_1 + M_1^T J = 0, \tag{2.8} \]
which coincides with the fact that \( JM_1 \) is a symmetric matrix. Now, multiplying \(-J\) on both sides of (2.7) and taking trace, we have

**Corollary 2.9.** Under the above assumptions
\[ \sum_{j=0}^{m} \text{Tr}(-JM_j^T JM_{m-j}) = 0, \quad \forall m \in \mathbb{N}. \]

Especially, for \( m = 2, \) we get
\[ 2\text{Tr}(M_2) = \text{Tr}(JM_1^T JM_1) = \text{Tr}(M_1^2). \]

Set \( M = S\gamma_0(T), \) then \( S\gamma_0(T) = MM(\alpha). \) For \( \lambda \in \mathbb{C}, \) which is not an eigenvalue of \( M, \) by some easy computations, we have that
\[
\det(S\gamma_0(T) - \lambda I_{2n}) = \det(MM(\alpha) - \lambda I_{2n}) \\
= \det(M - \lambda I_{2n} + \alpha MM_1 + \cdots + \alpha^k MM_k + \cdots) \\
= \det(M - \lambda I_{2n}) \det(I + \cdots + \alpha^k (M - \lambda I_{2n})^{-1} MM_k + \cdots).
\]

Let \( G_k = (M - \lambda I_{2n})^{-1} MM_k, \)
\[ f(\alpha) = \det(I + \cdots + \alpha^k G_k + \cdots), \]
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which is an analytic function on \( \mathbb{C} \). Next, we will compute the Taylor expansion for \( f(\alpha) \). Let \( G(\alpha) = \sum_{k=1}^{\infty} \alpha^{k-1} G_k \), then for \( \alpha \) small enough, by (2.4), we have
\[
\begin{align*}
    f(\alpha) &= \det(I + \alpha G(\alpha)) \\
             &= \exp \left( \sum_{m=1}^{\infty} \left( \frac{(-1)^{m+1}}{m} \alpha^m \right) \right) \\
             &= \exp \left( \sum_{m=1}^{\infty} \left( \frac{(-1)^{m+1}}{m} \alpha^m \right) \left( \sum_{k=1}^{\infty} \alpha^{k-1} G_k \right) \right) \\
             &= \exp \left( \sum_{m=1}^{\infty} \left( \frac{(-1)^{m+1}}{m} \right) \left( \sum_{k_1, \ldots, k_m = 1}^{\infty} \alpha^{k_1 + \cdots + k_m} \text{Tr}(G_{k_1} \cdots G_{k_m}) \right) \right). \\
\end{align*}
\]  
(2.9)

Since \( f(\alpha) \) vanishes nowhere near 0, we can write \( f(\alpha) = e^{g(\alpha)} \), then by (2.9), some direct computation shows that
\[
\begin{align*}
    g^{(m)}(0)/m! &= \sum_{k=1}^{m} \left( \frac{(-1)^{k+1}}{k} \right) \left( \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right). \\
\end{align*}
\]  
(2.10)

For \( \alpha \) small enough, let \( g(\alpha) \) be the function satisfying
\[
\lambda^{-n} \det(S \gamma_\alpha(T) - \lambda I_{2n}) = \lambda^{-n} \det(M - \lambda I_{2n}) \cdot \exp(g(\alpha)),
\]  
(2.11)
then the coefficients of \( g^{(k)}(0)/k! \) could be determined by (2.10). And we have the following theorem, which is the main result in this subsection.

**Theorem 2.10.** Under the above assumption, let \( g(\alpha) \) be the function in (2.11). Let \( g(\alpha) = \sum_{m=1}^{\infty} c_m \alpha^m \) be its Taylor expansion. Then
\[
\begin{align*}
    c_m &= \sum_{k=1}^{m} \left( \frac{(-1)^{k+1}}{k} \right) \left( \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right).
\end{align*}
\]

We only list the first 4 terms
\[
\begin{align*}
    g^{(1)}(0) &= \text{Tr}(G_1), \\
    g^{(2)}(0)/2 &= \text{Tr}(G_2) - \frac{1}{2} \text{Tr}(G_1^2), \\
    g^{(3)}(0)/3! &= \text{Tr}(G_3) - \text{Tr}(G_1 G_2) + \frac{1}{3} \text{Tr}(G_1^3), \\
    g^{(4)}(0)/4! &= \text{Tr}(G_4) - \frac{1}{2} \text{Tr}(G_2^2) - \text{Tr}(G_1 G_3) + \text{Tr}(G_1^2 G_2) - \frac{1}{4} \text{Tr}(G_4).
\end{align*}
\]

By the definition of \( G_k \),
\[
\text{Tr}(G_1) = \text{Tr}(M_1 M(M - \lambda I_{2n})^{-1}) = \text{Tr}(H \int_0^T \hat{D}(s) ds \cdot M(M - \lambda I_{2n})^{-1}),
\]
\[ \text{Tr}(G_2) = \text{Tr}(M_2M(M - \lambda I_{2n})^{-1}) = \text{Tr}\left(J \int_0^T \dot{D}(s) J \int_0^s \dot{D}(\sigma) d\sigma d\sigma \cdot M(M - \lambda I_{2n})^{-1}\right). \]

Generally,
\[ \text{Tr}(G_k^n) = \text{Tr}\left(\left[ \int_0^T \dot{J} \dot{D}(t_1) \int_0^{t_1} J \dot{D}(t_2) \cdots \int_0^{t_{k-1}} J \dot{D}(t_k) dt_k \cdots dt_2 dt_1 \cdot M(M - \lambda I_{2n})^{-1}\right]^m\right), \]
and \( \text{Tr}(G_{j_1} \cdots G_{j_k}) \) could be given similarly.

### 2.3 The proof of the Trace formula for Hamiltonian system

In this subsection, we will give proof of Theorem 1.1.

**Proof of Theorem 1.1** We begin with the formula
\[ \det \left( (A - B - \alpha D - \nu J)(A + P_0)^{-1} \right) = C(S)e^{-\nu T} \det(S \gamma(T) - e^{\nu T} I_{2n}). \]

On the one hand, by Theorem 2.6
\[ \det((A - B - \alpha D - \nu J)(A + P_0)^{-1}) = \det((A - B - \nu J)(A + P_0)^{-1}) \exp \left\{ \sum_{m=1}^{\infty} b_m \alpha^m \right\}, \]

where \( b_m = -\frac{1}{m} \text{Tr}((D(A - B - \nu J)^{-1})^m) \). On the other hand, by Theorem 2.10
\[ C(S)e^{-\nu T} \det(S \gamma(T) - e^{\nu T} I_{2n}) = C(S)e^{-\nu T} \det(S \gamma(T) - e^{\nu T} I_{2n}) \exp \left( \sum_{n=1}^{\infty} c_n \alpha^m \right), \]

where
\[ c_m = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \left( \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right). \]

Since
\[ \det((A - B - \nu J)(A + P_0)^{-1}) = C(S)e^{-\nu T} \det(S \gamma(T) - e^{\nu T} I_{2n}), \]
we have that \( b_m = c_m \), that is
\[ -\frac{1}{m} \text{Tr}((D(A - B - \nu J)^{-1})^m) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \left( \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right). \]

It follows that,
\[ \text{Tr}\left[ (D(A - B - \nu J)^{-1})^m \right] = m \sum_{k=1}^{m} \frac{(-1)^k}{k} \left( \sum_{j_1 + \cdots + j_k = m} \text{Tr}(G_{j_1} \cdots G_{j_k}) \right). \tag{2.12} \]

The proof is completed. \( \square \)

By the equation (2.12), theoretically, we can calculate the trace of \((D(A - B - \nu J)^{-1})^m\), at least, numerically by computer. Notice that the right hand side of (2.12) is a kind of multiple integral, and it is a little complicated. Next, we will write down the first four terms.
Proposition 2.11.

\[ \text{Tr}(D(A - B - \nu J)^{-1}) = -\text{Tr}(G_1). \]

\[ \text{Tr}(\left[D(A - B - \nu J)^{-1}\right]^2) = \text{Tr}(G_1^2) - 2\text{Tr}(G_2). \]

\[ \text{Tr}(\left[D(A - B - \nu J)^{-1}\right]^3) = -3\text{Tr}(G_3) + 3\text{Tr}(G_1G_2) - \text{Tr}(G_1^3). \]

\[ \text{Tr}(\left[D(A - B - \nu J)^{-1}\right]^4) = -4\text{Tr}(G_4) + 2\text{Tr}(G_2^2) + 4\text{Tr}(G_1G_3) - 4\text{Tr}(G_1^2G_2) + \text{Tr}(G_1^4). \]

Moreover, for the first two terms, we can write it more precisely.

\[ \text{Tr}[D(A - B - \nu J)^{-1}] = -\text{Tr}\left( J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1} \right), \quad (2.13) \]

and

\[ \text{Tr}(\left[D(A - B - \nu J)^{-1}\right]^2) = -2\text{Tr}\left( J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)J \int_0^T \gamma_0^T(s) D(s) \gamma_0(s) ds dt \cdot M(M - e^{\nu T}I_{2n})^{-1} \right) \]

\[ + \text{Tr}\left( J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt M(M - e^{\nu T}I_{2n})^{-1} \right)^2, \quad (2.14) \]

which are (1.9) and (1.10) in Corollary 1.3.

It is worth to be pointed out that, on the left hand side of (2.13), the trace is the conditional trace, and on the right hand side of it, it is the trace of matrix on \( \mathbb{C}^{2n} \). Next, we will consider some special cases.

Proposition 2.12. Assume that \( B(t) \equiv B_0 \) is a constant matrix and \( S = \pm I_{2n} \), then,

\[ \text{Tr}(D(A - B - \nu J)^{-1}) = -\text{Tr}\left( J \int_0^T D(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1} \right). \]

Proof. Since \( B(t) \equiv B_0 \), obviously \( \gamma_0(t) = e^{JBo_t} \), thus \( \gamma_0(t) \) commutes with \( \gamma_0(T) \) and also commutes with \( M \) since \( S = \pm I_{2n} \). Easy computation shows that

\[ \text{Tr}\left( J \int_0^T D(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1} \right) = \text{Tr}\left( J \int_0^T e^{-JBo_t} JD(t)e^{JBo_t} dt \cdot M(M - e^{\nu T}I_{2n})^{-1} \right) \]

\[ = \text{Tr}\left( J \int_0^T e^{-JBo_t} JD(t)M(M - e^{\nu T}I_{2n})^{-1}e^{JBo_t} dt \right) \]

\[ = \text{Tr}\left( J \int_0^T D(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1} \right). \]

By (2.13), the proposition is proved.\[ \square\]

The following proposition considers the case that \( MJ = JM \), \( M^T = M \).
Proposition 2.13. If \(MJ = JM, M^T = M\), then

\[
\begin{align*}
\text{Tr}\left((D(A - vJ - B)^{-1})^2\right) &= \text{Tr}\left[\left(J \int_0^T \hat{D}(s) ds \cdot M(M - e^{vT}I_{2n})^{-1}\right)^2\right] \\
&= -\text{Tr}\left[\left(J \int_0^T \hat{D}(s) ds \right)^2 M(M - e^{vT}I_{2n})^{-1}\right].
\end{align*}
\]

Proof. Suppose \(MJ = JM, M = M^T\) then

\[
\begin{align*}
\text{Tr}(M_2 M(M - \omega I_{2n})^{-1}) &= \text{Tr}(-JM_2 JM(M - e^{vT}I_{2n})^{-1}) \\
&= \text{Tr}(-M(M - e^{vT}I_{2n})^{-1}JM^T_2 J) \\
&= \text{Tr}(-JM^T_2 JM(M - e^{vT}I_{2n})^{-1}).
\end{align*}
\]

By Proposition 2.8

\[
M^T_1 J + JM_1 = 0,
\]

and

\[
-JM^T_2 J + M_2 = JM^T_1 JM.
\]

Thus

\[
2\text{Tr}(G_2) = \text{Tr}\left(JM^T_1 JM_1 M(M - e^{vT}I_{2n})^{-1}\right) \\
= \text{Tr}\left(M^T_2 M(M - e^{vT}I_{2n})^{-1}\right) \\
= \text{Tr}\left[\left(J \int_0^T \hat{D}(s) ds \right)^2 M(M - e^{vT}I_{2n})^{-1}\right].
\]

By the formula (2.14), the proposition is proved. \(\blacksquare\)

Some easy computation shows that, if moreover \(M\) commutes with \(\int_0^T \hat{D}(s) ds\), then

\[
\begin{align*}
\text{Tr}\left((D(A - vJ - B)^{-1})^2\right) &= \text{Tr}\left[\left(J \int_0^T \hat{D}(s) ds \right)^2 M(M - e^{vT}I_{2n})^{-1}(M(M - e^{vT}I_{2n})^{-1} - I_{2n})\right] \\
&= e^{vT} \text{Tr}\left[\left(J \int_0^T \hat{D}(s) ds \right)^2 M(M - e^{vT}I_{2n})^{-2}\right].
\end{align*}
\]

(2.15)

More specially, we have the following corollary.

Corollary 2.14. If \(M = \pm I_{2n}\), then

\[
\text{Tr}(D(A - vJ - B)^{-1}) = \frac{\pm e^{vT}}{(1 \mp e^{vT})^2} \text{Tr}\left[\left(\int_0^T \gamma_0^T(s) D(s) \gamma_0(s) ds\right)^2\right].
\]

(2.16)

Especially in the case \(B = 0, \hat{D} = D\) and \(S = \pm I_{2n}\),

\[
\text{Tr}(D(A - vJ)^{-1}) = \frac{\pm e^{vT}}{(1 \mp e^{vT})^2} \text{Tr}\left[\left(\int_0^T D(s) ds\right)^2\right].
\]

(2.17)

Notice that (2.16) is just the formula (1.11) in Corollary 1.3.
Example 2.15. In the case $D(t) = I_{2n}$, then $\hat{D}(t) = \gamma_0^T(t)\gamma_0(t)$, so we have

$$Tr((A - B - \nu J)^{-1}) = Tr(J \int_0^T \gamma_0^T(s)\gamma_0(s)ds \cdot M(M - e^{T}I_{2n})^{-1}),$$

and for $k \geq 2$,

$$Tr\left[ (A - B - \nu J)^{-1} \right] = \sum_{j=-\infty}^{\infty} \frac{1}{\lambda_j},$$

where $\lambda_j$ are eigenvalues of $A - B - \nu J$. From the trace formula, we have

$$\sum_{j=-\infty}^{\infty} \frac{1}{\lambda_j^m} = m \sum_{k=1}^{m} (-1)^k \left[ \sum_{j_1+\cdots+j_k=m} Tr(G_{j_1} \cdots G_{j_k}) \right], \ \forall m \geq 2. \tag{2.18}$$

The equation (2.18) has its own interests. In fact, we can deduce some interesting equalities from this.

Example 2.16. Let $B = 0$, $D = I_2$, $S = I_2$ and $T = 1$. Then, for each fixed $\alpha \in \mathbb{C}$, it is easy to check that the eigenvalues for $A - \nu J - \alpha$ are $\{2k\pi \pm \sqrt{-1}\nu - \alpha | k \in \mathbb{Z}\}$. For $\nu \notin 2\pi \sqrt{-1}\mathbb{Z} - \alpha$, $A - \nu J - \alpha$ is invertible, and the left hand side of (2.18) is

$$Tr((A - \nu J - \alpha)^{-m}) = \sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi + \sqrt{-1}\nu - \alpha)^m} + \sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi - \sqrt{-1}\nu - \alpha)^m}, \ \forall m \in \mathbb{N},$$

where for $m = 1$, the infinite sum in the right side is understand by $\lim_{\beta \to \infty} \sum_{|k| \leq \beta}$. For the right hand side, the traces $Tr(G_{j_1} \cdots G_{j_k})$ can be calculated directly. We only list the first 3 equalities. For $m = 1$, direct computation shows that $Tr(G_1) = \frac{2e^\nu \sin \alpha}{(\cos \alpha - e^\nu)^2 + \sin^2 \alpha}$, thus we have

$$\sum_{k \in \mathbb{Z}} \frac{1}{2k\pi + \sqrt{-1}\nu - \alpha} + \sum_{k \in \mathbb{Z}} \frac{1}{2k\pi - \sqrt{-1}\nu - \alpha} = \frac{-2e^\nu \sin \alpha}{(\cos \alpha - e^\nu)^2 + \sin^2 \alpha}. \tag{2.19}$$

For $m = 2$, by (2.15), direct computation shows that

$$Tr((A - \nu J - \alpha)^{-2}) = \frac{-2e^\nu (\cos \alpha (1 + e^{2\nu}) - 2e^\nu)}{e^\nu \cos \alpha (4e^\nu \cos \alpha - 4e^{2\nu} - 4) + (1 + e^{2\nu})^2} = \frac{1 - \cosh \nu \cos \alpha}{(\cos \alpha - \cosh \nu)^2},$$

thus we have the identity

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi + \sqrt{-1}\nu - \alpha)^2} + \sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi - \sqrt{-1}\nu - \alpha)^2} = \frac{1 - \cosh \nu \cos \alpha}{(\cos \alpha - \cosh \nu)^2}. \tag{2.19}$$

Especially in the case $\alpha = 0$,

$$Tr((A - \nu J)^{-2}) = 2 \sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi + \sqrt{-1}\nu - \alpha)^2},$$

and the right hand side of (2.19) is reduced to $\frac{-2e^\nu}{(1-e^{2\nu})}$, thus we have the identity

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi + \sqrt{-1}\nu)^2} = \frac{1 + \cos \sqrt{-1}\nu}{2 \sin^2 \sqrt{-1}\nu}. \tag{2.19}$$
Similarly, for $m = 3$, we get
\[
\sum_{k \in \mathbb{Z}} \frac{1}{(2k \pi + \sqrt{-1} \nu - \alpha)^3} + \sum_{k \in \mathbb{Z}} \frac{1}{(2k \pi - \sqrt{-1} \nu - \alpha)^3} = \frac{1/2 \sin \alpha (\cosh^2 \nu + \cosh \nu \cos \alpha - 2)}{\cosh^3 \nu - 3 \cosh^2 \nu \cos \alpha + 3 \cosh \nu \cos^2 \alpha - \cos^3 \alpha}.
\]

The equality in the above example can be deduced by using techniques in complex analysis. However, the above example is only a kind of easiest case. If we take a non-constant path $B$, then the formula will be far from trivial.

**Remark 2.17.** Recall that, in [7], Atiyah, Patodi and Singer defined a kind of zeta function for self-adjoint elliptic differential operator $\mathcal{A}$ (the operator may not be positive). Let $\{\lambda\}$ be the eigenvalues for $\mathcal{A}$, then
\[
\eta_{\mathcal{A}}(s) = \sum_{\lambda \neq 0} (\text{sign} \lambda) |\lambda|^{-s},
\]
for $\text{Re}(s)$ large, and it can be extended meromorphically to the whole $s$-plane. Now, for the differential operator $A$, if we can take some proper $B$, $D$ and $S$ in our framework, such that $\lambda$ are the eigenvalues of $\mathcal{A} = D^{-1}(A - B - \nu J)$ is real, then by the trace formula, we can obtain the values for $\eta_{\mathcal{A}}(s)$ at odd integers.

# 3 Hill-type formula and Trace formula for Sturm-Liouville systems

In the study of $S$-periodic orbits in Lagrangian systems, it is natural to consider the standard Sturm systems:
\[
- (P \dot{y} + Q y) + Q^T \dot{y} + Ry = 0, \quad y(0) = \bar{S} y(T), \quad \dot{y}(0) = \bar{S} \dot{y}(T), \tag{3.1}
\]
where $\bar{S}$ is an orthogonal matrix on $\mathbb{R}^n$. We assume $P(t)$ is invertible for any $t$, which is a more general condition than the usual Legendre convexity assumptions. Denote $\hat{Q} = P^{-1}(Q^T - Q)$, $\hat{R} = P^{-1}(R - Q)$. Obviously, the system (3.1) is equivalent to
\[
- \ddot{z}(t) + \hat{Q}(t) \dot{z}(t) + \hat{R}(t) z(t) = 0, \quad z(0) = \bar{S} z(T), \quad \dot{z}(0) = \bar{S} \dot{z}(T), \tag{3.2}
\]
Please note that if $\ddot{z}(t)$ satisfies the equation (3.2) with $\ddot{z}(0) = e^{-\nu T} \bar{S} \ddot{z}(T)$, $\dot{z}(0) = e^{-\nu T} \bar{S} \dot{z}(T)$, then $z(t) = e^{\nu t} \ddot{z}(t)$ satisfies the following second order ODE
\[
- \left( \frac{d}{dt} + \nu \right)^2 z(t) + \hat{Q}(t) \left( \frac{d}{dt} + \nu \right) \dot{z}(t) + \hat{R}(t) z(t) = 0, \quad z(0) = \bar{S} z(T), \quad \dot{z}(0) = \bar{S} \dot{z}(T). \tag{3.5}
\]
Let $y(t) = \dot{z}(t) + \nu z(t)$, then we can write (3.4) as the following first order ODE
\[
\begin{pmatrix}
\dot{y}(t) \\
\dot{z}(t)
\end{pmatrix} = \begin{pmatrix}
\hat{Q}(t) - \nu & \hat{R}(t) \\
\hat{S} & -\nu
\end{pmatrix} \begin{pmatrix}
y(t) \\
z(t)
\end{pmatrix},
\tag{3.6}
\]
\[
\begin{pmatrix}
y(0) \\
z(0)
\end{pmatrix} = \begin{pmatrix}
\bar{S} & 0_n \\
0_n & \bar{S}
\end{pmatrix} \begin{pmatrix}
y(T) \\
z(T)
\end{pmatrix}. \tag{3.7}
\]
For simplicity, we denote
\[
\hat{B}(t) = \begin{pmatrix} I_n & 0_n \\ -\hat{Q}(t) & -\hat{R}(t) \end{pmatrix}, \quad \hat{S}_d = \begin{pmatrix} \bar{S} & 0_n \\ 0_n & \bar{S} \end{pmatrix}, \quad \text{and} \quad x(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}.
\]

The system \((3.6, 3.7)\) can be written as the following Hamiltonian system,
\[
\begin{align*}
\dot{x}(t) &= J(\hat{B}(t) + \nu J)x(t), \\
x(0) &= \hat{S}_dx(0).
\end{align*}
\]  
(3.8)  
(3.9)

It follows that \(z(t)\) is solution of \((3.4, 3.5)\) if and only if \(x(t) = \begin{pmatrix} \dot{z}(t) + \nu z(t) \\ z(t) \end{pmatrix}\) is solution of \((3.8, 3.9)\).

Therefore, we have
\[
\dim \ker \left(-\frac{d}{dt} + \nu I_n\right)^2 + \hat{Q}(t)\left(\frac{d}{dt} + \nu I_n\right) + \hat{R}(t) = \dim \ker \left(-\frac{d}{dt} - \hat{B} - \nu J\right).
\]  
(3.10)

Now, we will give the Hill-type formula for indefinite Lagrangian system. For \(N \in \mathbb{N}\), let
\[
\hat{W}_N = \bigoplus_{\nu \in \sigma(\frac{d}{dt} + \nu I_n)|\nu| \leq N} \ker \left(\frac{d}{dt} - \nu I_n\right),
\]
and denote by \(\hat{P}_N\) the orthogonal projection onto \(\hat{W}_N\). Then \(Q(t)\left(\frac{d}{dt} + \nu I_n\right)^{-1}\) is a Hilbert-Schmidt operator with the trace finite condition with respect to \(\{\hat{P}_N\}\). We define the conditional Fredholm determinant with respect to \(\hat{P}_N\),
\[
\det\left[-\left(\frac{d}{dt} + \nu I_n\right)^2 + Q(t)\left(\frac{d}{dt} + \nu I_n\right) + R(t)\right] = \det \left[-\left(\frac{d}{dt} + \nu I_n\right)^2 \right].
\]

At first, we recall Hill-type formula for linear Hamiltonian systems \([14]\). For \(B \in C([0, T]; M(2n, \mathbb{C}))\), which is not have to be real symmetric, we have that
\[
\det\left( -J\frac{d}{dt} - B - \nu J \right) = C(S)e^{-\frac{\gamma}{2} \int_0^T \text{Tr}(JB(t))dt}e^{-\nu \gamma T} \det \left(S\gamma(T) - e^T I_{2n} \right).
\]  
(3.11)

where \(\gamma\) is the fundamental solution corresponding to \(B\). We firstly prove the following proposition.

**Proposition 3.1.** For \(\nu \in \mathbb{C}\) such that \(\frac{d}{dt} + \nu I_n\) is invertible, we have
\[
\det\left[-J\frac{d}{dt} - \hat{B} - \nu J\right] = \det\left[-\left(\frac{d}{dt} + \nu I_n\right)^2 + Q(t)\left(\frac{d}{dt} + \nu I_n\right) + R(t)\right] = \det \left[-\left(\frac{d}{dt} + \nu I_n\right)^2 \right].
\]  
(3.12)

**Proof.** Let \(K_n = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}\), note that
\[
-J\frac{d}{dt} - \nu J - K_n = \begin{pmatrix} -I_n & \frac{d}{dt} + \nu I_n \\ -\left(\frac{d}{dt} + \nu I_n\right)^{-1} & 0_n \end{pmatrix}.
\]  
(3.13)

It follows that \(\frac{d}{dt} + \nu I_n\) is invertible if and only if \(-J\frac{d}{dt} - \nu J - K_n\) is invertible; moreover
\[
\left(-J\frac{d}{dt} - \nu J - K_n\right)^{-1} = \begin{pmatrix} 0_n & -\left(\frac{d}{dt} + \nu I_n\right)^{-1} \\ \left(\frac{d}{dt} + \nu I_n\right)^{-1} & -\left(\frac{d}{dt} + \nu I_n\right)^{-2} \end{pmatrix}.
\]
Combining (3.14) and (3.15), we have the desired result.

\[ (K_n - \hat{B})(- J \frac{d}{dt} - \nu J - K_n)^{-1} = \begin{pmatrix} 0_n \\ \hat{R}(t)(\frac{d}{dt} + \nu I_n)^{-1} - \hat{Q}(t)(\frac{d}{dt} + \nu I_n)^{-1} - \hat{R}(t)(\frac{d}{dt} + \nu I_n)^{-2} \end{pmatrix}. \]

Thus we have

\[
\det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} - K_n - \nu J \right)^{-1} \right]
= \det\left[ id - (\hat{B} - K_n\hat{B}) \left( - J \frac{d}{dt} - K_n - \nu J \right)^{-1} \right]
= \det\left[ id - \hat{Q}(t) \left( \frac{d}{dt} + \nu I_n \right)^{-1} - \hat{R}(t) \left( \frac{d}{dt} + \nu I_n \right)^{-2} \right]
= \det\left[ \left( - \left( \frac{d}{dt} + \nu I_n \right)^2 + \hat{Q}(t) \left( \frac{d}{dt} + \nu I_n \right) + \hat{R}(t) \right) \left( - \left( \frac{d}{dt} + \nu I_n \right)^2 \right)^{-1} \right]. \tag{3.14}
\]

Now, direct computation shows that,

\[
\det\left[ \left( - J \frac{d}{dt} - K_n - \nu J \right) \left( - J \frac{d}{dt} - \nu J \right)^{-1} \right] = 1.
\]

Therefore,

\[
\det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} - \nu J \right)^{-1} \right]
= \det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} - K_n - \nu J \right)^{-1} \right] \cdot \det\left[ \left( - J \frac{d}{dt} - K_n - \nu J \right) \left( - J \frac{d}{dt} - \nu J \right)^{-1} \right]
= \det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} - K_n - \nu J \right)^{-1} \right]. \tag{3.15}
\]

Combining (3.14) and (3.15), we have the desired result. \[\square\]

For \( R_1 \in \mathcal{B}(n) \), let \( \hat{B}_I(t) = \begin{pmatrix} I_n & 0_n \\ -\hat{Q}(t) & -\hat{R}(t) - \lambda P^{-1} R_1 \end{pmatrix} \), let \( \hat{\gamma}_I \) be the corresponding fundamental solutions. With the above preparation, we have the following theorem.

**Theorem 3.2.** For \( \nu \in \mathbb{C} \) such that \( \frac{d}{dt} + \nu I_n \) is invertible, we have

\[
\det\left[ \left( - \left( \frac{d}{dt} + \nu I_n \right)^2 + \hat{Q}(t) \left( \frac{d}{dt} + \nu I_n \right) + \hat{R}(t) \right) \left( - \left( \frac{d}{dt} + \nu I_n \right)^2 \right)^{-1} \right]
= e^{-\frac{\nu^2}{2} \text{Tr}(\hat{Q}) dt} \det(\hat{S}_d\hat{\gamma}_0(T) - e^{\nu T} I_{2n}) \det(\hat{S}_d - e^{-\nu T} I_{2n})^{-1}. \tag{3.16}
\]

**Proof.** By the multiplicative property of conditional Fredholm determinant

\[
\det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} - \nu J \right)^{-1} \right] = \det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} + P_0 \right)^{-1} \right] \cdot \det\left[ \left( - J \frac{d}{dt} + P_0 \right) \left( - J \frac{d}{dt} - \nu J \right)^{-1} \right]. \tag{3.17}
\]

By the Hill-type formula for Hamiltonian system (3.11), we have that

\[
\det\left[ \left( - J \frac{d}{dt} - \hat{B} - \nu J \right) \left( - J \frac{d}{dt} + P_0 \right)^{-1} \right] = C(\hat{S}_d) e^{-\frac{\nu^2}{2} \text{Tr}(\hat{Q}^2) dt} e^{-\nu T} \det(\hat{S}_d\hat{\gamma}_0(T) - e^{\nu T} I_{2n}) \tag{3.18}
\]
and
\[
\det\left[\left(-J\frac{d}{dt} + P_0\right)\left(-J\frac{d}{dt} - \nu J\right)^{-1}\right] = \det\left[\left(-J\frac{d}{dt} - \nu J\right)\left(-J\frac{d}{dt} + P_0\right)^{-1}\right]^{-1} = C(\bar{S}_d) e^{\nu T} \det(\bar{S}_d - e^{\nu T} I_{2n})^{-1}.
\] (3.19)

Substituting (3.19) and (3.18) into (3.17), by Proposition 3.1 we have the result.

We come back to the Lagrangian systems. To simplify the notation, let
\[
\mathcal{A}(\nu) = -\left(\frac{d}{dt} + \nu\right)\left(P\left(\frac{d}{dt} + \nu\right) + Q\right) + Q^T\left(\frac{d}{dt} + \nu\right) + R(t).
\]

**Theorem 3.3.** Under the condition (1.13), for any $\nu \in \mathbb{C}$ such that $\mathcal{A}(\nu)$ is invertible,
\[
\det\left[\left(\mathcal{A}(\nu) + R_1\right)\mathcal{A}(\nu)^{-1}\right] = \det(\hat{S}_d \gamma_1(T) - e^{\nu T} I_{2n}) \cdot \det(\hat{S}_d \gamma_0(T) - e^{\nu T} I_{2n})^{-1},
\] (3.20)

where $\gamma_A(t)$ is the fundamental solution of (1.14).

**Proof.** Please note that $F = R_1\mathcal{A}(\nu)^{-1}$ is a trace class operator, thus $\det(id + F)$ is the usual Fredholm determinant. Therefore
\[
\det(id + F) = \det(id + P^{-1}FP),
\]
hence
\[
\det\left[\left(\mathcal{A}(\nu) + R_1\right)\mathcal{A}(\nu)^{-1}\right] = \det\left[P^{-1}(\mathcal{A}(\nu) + R_1)\mathcal{A}(\nu)^{-1}\right] = \det\left[(P^{-1}(\mathcal{A}(\nu) + R_1))\left(P^{-1}\mathcal{A}(\nu)\right)^{-1}\right].
\]

Easy computation shows that
\[
P^{-1}\mathcal{A}(\nu) = -\left(\frac{d}{dt} + \nu\right)^2 + \hat{Q}\left(\frac{d}{dt} + \nu\right) + \hat{R},
\]
where $\hat{Q} = P^{-1}(Q^T - Q - \dot{P})$, $\hat{R} = P^{-1}(R - \dot{Q})$. By the multiplicative property (2.3) of Fredholm determinant,
\[
\det\left[P^{-1}(\mathcal{A}(\nu) + R_1)\left(P^{-1}\mathcal{A}(\nu)\right)^{-1}\right] = \det\left[P^{-1}(\mathcal{A}(\nu) + R_1)\left(-\left(\frac{d}{dt} + \nu\right)^2\right)^{-1}\right]
\cdot \det\left[(P^{-1}\mathcal{A}(\nu)) \cdot \left(-\left(\frac{d}{dt} + \nu\right)^2\right)^{-1}\right].
\] (3.21)

Substituting (3.16) into (3.21), we have
\[
\det\left[\left(\mathcal{A}(\nu) + R_1\right)\mathcal{A}(\nu)^{-1}\right] = \det(\hat{S}_d \hat{\gamma}_1(T) - e^{\nu T} I_{2n}) \cdot \det(\hat{S}_d \hat{\gamma}_0(T) - e^{\nu T} I_{2n})^{-1}.
\] (3.22)

To prove the theorem, we will make clear the relationship between $\hat{\gamma}_A(T)$ with $\gamma_A(T)$. Let $\eta(t) = \begin{pmatrix} P(t) & Q(t) \\ 0_n & I_n \end{pmatrix}$, then direct computation shows that
\[
\frac{d}{dt}(\eta(t)\hat{\gamma}_A(t)\eta(0)^{-1}) = JB_A(t)\eta(t)\hat{\gamma}_A(t)\eta(0)^{-1},
\]
which implies $\gamma_A(t) = \eta(t)\hat{\gamma}_A(t)\eta(0)^{-1}$. Moreover, from (1.13), $\hat{S}_d\eta(T) = \eta(0)\hat{S}_d$, easy computation shows that
\[
\hat{S}_d\gamma_A(T) = \eta(0)\hat{S}_d\hat{\gamma}_A(T)\eta(0)^{-1}.
\] (23.23)

It follows that
\[
\det(\hat{S}_d\gamma_A(T) - e^{\nu T} I_{2n}) = \det(\hat{S}_d\hat{\gamma}_A(T) - e^{\nu T} I_{2n}).
\]

Combining with (3.22), we have the desired result.
Obviously, by taking \( v = 0 \) in Theorem 3.3 we have Theorem 1.6.

To get the trace formula, let \( \lambda R_1 \) take place of \( R_1 \) in the Hill-type formula (3.20), and we have

\[
\det(id + \lambda R_1 \mathcal{A}(v)^{-1}) = \det \left( \hat{S}_d \gamma_0(T) - e^{iT} I_{2n} \right) \cdot \det \left( \hat{S}_d \gamma_0(T) - e^{iT} I_{2n} \right)^{-1}.
\]  (3.24)

Almost the same as the proof of Theorem 1.1, the trace formula for Lagrangian system could be obtained by taking Taylor expansion on the variable \( \lambda \) and comparing the coefficients of \( \lambda^m \) on both sides of (3.24), and the proof will be omitted. We have the trace formula for Lagrangian system, for \( m \in \mathbb{N} \),

\[
Tr\left( [R_1 \mathcal{A}(v)^{-1}]^m \right) = m \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k} \left[ \sum_{j_1 + \cdots + j_k = m} Tr(G_{j_1} \cdots G_{j_k}) \right],
\]  (3.25)

where for Lagrangian system, we always denote \( D = \begin{pmatrix} 0_n & 0_n \\ 0_n & -R_1 \end{pmatrix} \), \( G_k \) is defined in Theorem 1.4.

Since \( \mathcal{A}(v)^{-1} \) is a trace class operator, let \( \{ \lambda_i \} \) be the nonzero eigenvalues of \( \mathcal{A}(v)y + \lambda R_1 y = 0 \), then for positive integers \( m \),

\[
\sum_j \frac{1}{\lambda_j^m} = (-1)^m \cdot Tr\left( [R_1 \mathcal{A}(v)^{-1}]^m \right).
\]  (3.26)

Combining (3.25) and (3.26) we prove Theorem 1.4.

Especially,

\[
Tr[R_1 \mathcal{A}(v)^{-1}] = Tr\left( J \int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt \cdot M(M - e^{iT} I_{2n})^{-1} \right).
\]  (3.27)

Comparing with the Trace formula in Hamiltonian systems, we have

**Corollary 3.4.** For positive integers \( m \),

\[
(-1)^m \cdot Tr\left( [R_1 \mathcal{A}(v)^{-1}]^m \right) = Tr\left( [D(A - B_0 - vJ)^{-1}]^m \right).
\]  (3.28)

where \( D = \begin{pmatrix} 0_n & 0_n \\ 0_n & -R_1 \end{pmatrix} \), \( B_0 \) is defined in (1.15) and \( A = -J \frac{d}{dt} \) with \( \hat{S}_d \)-boundary condition.

Obviously \( \mathcal{A}(v) + \lambda R_1 \) is degenerate if only if \(-J \frac{d}{dt} - vJ - B_0 - \lambda D\) is degenerate, moreover, we have

**Proposition 3.5.** Let \( v \in \mathbb{C} \), such that \( \mathcal{A}(v) \) is invertible. Then \(-\frac{1}{\lambda_0}\) is an eigenvalue of \( R_1 \mathcal{A}(v)^{-1} \) of algebraic multiplicity \( k \) if and only if \(-\frac{1}{\lambda_0}\) is an eigenvalue of \( D(-J \frac{d}{dt} - vJ - B_0)^{-1} \) of algebraic multiplicity \( k \).

**Remark 3.6.**

1. For \( m \geq 2 \), notice that both \( [R_1 \mathcal{A}(v)^{-1}]^m \) and \( [D(A - B - vJ)^{-1}]^m \) are trace class, and hence by Proposition 3.5 we can get the trace formula for Lagrangian system from that of Hamiltonian system directly.

2. For \( m = 1 \), since the operator \( D(A - B - vJ)^{-1} \) is not trace class operator, but a Hilbert-Schmidt operator with trace finite condition. Therefore,

\[
Tr(D(A - B - vJ)^{-1}) = \lim_{N \to \infty} Tr \left[ P_N D(A - B - vJ)^{-1} P_N \right].
\]  (3.29)

For a general Hamiltonian system, we don’t know whether \( Tr(D(A - B - vJ)^{-1}) = \sum_j \frac{1}{\lambda_j} \) true or not. It follows that, the trace formula (3.27) can not be obtained by the trace formula from Hamiltonian system.
Example 3.8. We will compute the simplest case, that is \( \lambda_0 \) is a zero point of \( \det(id - \lambda F) \) of degree \( k \) if and only if \( \frac{1}{\lambda_0} \) is an eigenvalue of \( F \) of algebraic multiplicity \( k \).

Proof. Since \( F \) is a Hilbert-Schmidt operator, so \( \sigma_1 = \{ \frac{1}{\lambda_0} \} \) and \( \sigma_2 = \sigma(F) \setminus \sigma_1 \) are two disjoint closed subsets of the spectral of \( F \). By Riesz Decomposition Theorem for operators, let

\[
P_1 = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - F)^{-1} d\lambda,
\]

where \( \Gamma \) is a contour in the resolvent set of \( F \) such that \( \sigma_1 \) in its interior and \( \sigma_2 \) in its exterior. Then \( P_1 \) is its Riesz projection, and let \( P_2 = id - P_1 \). Since \( \frac{1}{\lambda_0} \) is a nonzero eigenvalue, then \( P_1 \) is a finite projection, and \( P_1 F = FP_1 \). Now, let \( F_1 = FP_1 \) and \( F_2 = FP_2 \), then \( F_1 F_2 = 0 \). By the multiplicative property of conditional Fredholm determinant,

\[
det(id - \lambda F) = det(id - \lambda F_1 - \lambda F_2 - \lambda^2 F_1 F_2) = det(id - \lambda F_1) \det(id - \lambda F_2).
\]

Since \( \frac{1}{\lambda_0} \) is not in the spectrum of \( F \), hence \( \frac{1}{\lambda_0} \) is not zero point of \( \det(id - \lambda F_2) \); moreover, it is not hard to see that \( \det(id - \lambda F_1) = (1 - \frac{1}{\lambda_0})^k \) where \( k \) is the algebraic multiplicity of the eigenvalue \( \frac{1}{\lambda_0} \) of \( F \). The proof is complete.

Proof of Proposition 3.5. By (3.24) and Lemma 3.7, \( \frac{1}{\lambda_0} \) is an eigenvalue of \( R_1 \mathcal{A}(t)^{-1} \) of algebraic multiplicity \( k \) if and only if it is a zero point of \( \det(\bar{S}_d \gamma_\lambda(T) - e^{\nu T} I_{2n}) \) of degree \( k \). On the other hand, by (1.23) and the multiplicative property, for \( B_1 \) defined in (1.15) we have

\[
det \left( id - \lambda D \left(- \frac{d}{dt} - \nu J - B_0 \right)^{-1} \right) = det \left[ \left(- \frac{d}{dt} - \nu J - B_0 \right)^{-1} \right] \cdot det \left[ \left(- \frac{d}{dt} + P_0 \right)^{-1} \right] = \left( \bar{S}_d \gamma_\lambda(T) - e^{\nu T} I_{2n} \right) \left( \bar{S}_d \gamma_\lambda(0) - e^{\nu T} I_{2n} \right).
\]

Again, by Lemma 3.7, \( \frac{1}{\lambda_0} \) is an eigenvalue of \( D(-\frac{d}{dt} - \nu J - B_0)^{-1} \) of algebraic multiplicity \( k \) if and only if it is also a zero point \( \det(\bar{S}_d \gamma_\lambda(T) - e^{\nu T} I_{2n}) \) of degree \( k \). The desired result is proved.

Example 3.8. We will compute the simplest case, that is \( \mathcal{A}(t) = -(\frac{d}{dt} + \nu)^2 \), \( R_1 = -R \). Recall that \( K_n = \left( \begin{array}{cc} I_n & 0_n \\ 0_n & 0_n \end{array} \right) \) and \( D_n = \left( \begin{array}{cc} 0_n & 0_n \\ 0_n & \nu \end{array} \right) \). Recall that \( \gamma_0(t) \) satisfied \( \dot{\gamma}_0(t) = JK_n \gamma_0(t) \) with \( \gamma_0(0) = I_{2n} \). Direct computation shows that \( \gamma_0(t) = \left( \begin{array}{cc} I_n & 0_n \\ tI_n & I_n \end{array} \right) \), and obviously \( \gamma_0(t)^{-1} = \left( \begin{array}{cc} I_n & 0_n \\ -tI_n & I_n \end{array} \right) \). Therefore,

\[
J \dot{D} = \gamma_0(t)^{-1} JD\gamma_0(t) = \left[ \begin{array}{cc} -tR & -R \\ t^2 R & tR \end{array} \right],
\]

thus

\[
J \int_0^T \dot{D} dt = \left[ \begin{array}{cc} -\int_0^T tR dt & -\int_0^T R dt \\ \int_0^T t^2 R dt & \int_0^T tR dt \end{array} \right].
\]
Let $\tilde{S}^T$ be the transposition of $\tilde{S}$, then $\tilde{S}^T = \tilde{S}^{-1}$. For $\omega = e^{\nu T}$

$$
M(M - \omega)^{-1} = \begin{pmatrix}
(I_n - \omega \tilde{S}^T)^{-1} & 0_n \\
0_n & (I_n - \omega \tilde{S}^T)^{-1}
\end{pmatrix}
\begin{pmatrix}
I_n & 0_n \\
-\omega T \tilde{S}^T (I_n - \omega \tilde{S}^T)^{-1} & I_n
\end{pmatrix},
$$

and

$$
G_1 = \begin{pmatrix}
-\int_0^T t R dt & -\int_0^T R dt \\
\int_0^T t^2 R dt & \int_0^T R dt
\end{pmatrix}
\begin{pmatrix}
(I_n - \omega \tilde{S}^T)^{-1} & 0_n \\
0_n & (I_n - \omega \tilde{S}^T)^{-1}
\end{pmatrix}
\begin{pmatrix}
I_n & 0_n \\
-\omega T \tilde{S}^T (I_n - \omega \tilde{S}^T)^{-1} & I_n
\end{pmatrix}. \quad (3.30)
$$

Thus

$$
Tr(G_1) = \omega Tr \left( T \int_0^T R dt \cdot \tilde{S}^T (I_n - \omega \tilde{S}^T)^{-2} \right).
$$

To simplify the notation, we denote by

$$
R_{ave} = \frac{1}{T} \int_0^T R(t) dt,
$$

which is a constant matrix. Then

$$
Tr(RA(\nu)^{-1}) = -\omega T^2 \cdot Tr(R_{ave} \cdot \tilde{S} (\tilde{S} - \omega)^{-2}). \quad (3.31)
$$

Please note that by take derivative with respect to $\nu$ on both sides of (3.31) we get

$$
Tr \left( R A(\nu)^{-2} \right) = \frac{\omega T^4}{6} Tr \left( R_{ave} \tilde{S}^2 + 4 \omega \tilde{S} + \omega^2 (\tilde{S} - \omega)^{-4} \right). \quad (3.32)
$$

**Remark 3.9.** In [20], Krein also consider the boundary value problem

$$
y'' + \lambda R(t)y = 0, \quad y(0) + y(T) = y'(0) + y'(T) = 0, \quad (3.33)
$$

where $R(t) \in B(n)$. Let $\lambda_j, \ j \in \mathbb{Z}$ or $\mathbb{N}$ (assume $\lambda_j \leq \lambda_{j+1}$), be the eigenvalues of boundary value problem (3.33), that means the system

$$
y'' + \lambda_j R(t)y = 0, \quad y(0) + y(T) = y'(0) + y'(T) = 0, \quad (3.34)
$$

has a nontrivial solution. Each $\lambda_j$ appears as many times as its multiplicity. To state Krein’s work, set

$$
X(t) = \int_0^T (R(s) - R_{ave}) ds + C, \quad (3.35)
$$

where $C$ is a constant matrix which is chosen such that $X_{ave} = 0$. Krein proved [20]

$$
\sum \frac{1}{\lambda_j} = \frac{T}{4} \int_0^T Tr(R(t)) dt, \quad (3.36)
$$

and

$$
\sum \frac{1}{\lambda_j^2} = \frac{T}{2} \int_0^T Tr(X^2(t)) dt + \frac{T^2}{48} Tr \left( \int_0^T R(t) dt \right)^2. \quad (3.37)
$$

Please note that (3.37) is a generalization of (3.36). Please note that, in the formula (1.16), the expression of $\frac{1}{\lambda_j}$ is different from (3.37). The precise generalization with the same form as Krein’s formula will be given in the forthcoming paper.
4 Application

The Maslov-type index is a very useful tool in studying the multiplicity and stability of periodic solution in Hamiltonian systems [17],[18]. It is well-known that the relative Morse index for linear Hamiltonian system equals to the Maslov-type index for the corresponding fundamental solutions. It will be seen that, by the trace formula, we could estimate the relative Morse index, and therefore the trace formula could be used to judge the linear stability via the Maslov-type index. For reader’s convenience, we review the relative Morse index and stability criteria via Maslov-type index in §4.1, details could be found in [12],[18]. The estimation of relative Morse index by the trace of operator is given in §4.2, some new criteria for the stability is given in §4.3, at §4.4, we give some estimation of Morse index for Sturm-Liouville systems.

In the whole of this section, \(\nu\) will be assumed to be an imaginary number.

4.1 Brief review of the relative Morse index, spectral flow and stability criteria via Maslov-type index

As we have reviewed in [14], let \(\tilde{A}, \tilde{B}\) be bounded self-adjoint operators on Hilbert space \(E\), \(\tilde{A}\) is a Fredholm operator and \(\tilde{B}\) is compact, then the relative Morse index \(I(\tilde{A}, \tilde{A} - \tilde{B})\) is defined by

\[
I(\tilde{A}, \tilde{A} - \tilde{B}) = \dim (E_-(\tilde{A} - \tilde{B}), E_-(\tilde{A})).
\]

where \(E_-(\tilde{A})\) and \(E_-(\tilde{A} - \tilde{B})\) are respectively the subspaces on which \(\tilde{A}\) and \(\tilde{A} - \tilde{B}\) is negative definite, and

\[
\dim (E_1, E_2) = \dim (E_1 \cap E_2^\perp) - \dim (E_2 \cap E_1^\perp).
\]

For the Hamiltonian system, let \(A = -J \frac{d}{dt}\). Denote by \(M\) the linear space generated by the eigenvectors of \(A\). The 1/2 inner product on \(M\) is defined by,

\[
\langle x, y \rangle_{1/2} = \langle (|A| + id)x, y \rangle, \quad \text{for} \ x, y \in M,
\]

where \(\langle \cdot, \cdot \rangle\) is the inner product in \(E\). Denote by \(\tilde{E}\) the Hilbert space completed by \(M\) under the 1/2 norm. Let \(\tilde{A} = (id + |A|)^{-1}A, \tilde{B}_j = (id + |A|)^{-1}B_j \ (j=0,1)\), then both of them are self-adjoint operators on \(\tilde{E}\). Define

\[
I(A - B_0, A - B_1) = I(\tilde{A} - \tilde{B}_0, \tilde{A} - \tilde{B}_1).
\]

The relationship between the conditional Fredholm determinant and the relative Morse index had been given in [14].

On the other hand, the relative Morse index could be defined by spectral flow [12]. As is well known, spectral flow was introduced by Atiyah, Patodi and Singer [1] in their study of index theory on manifolds with boundary. It is a very useful tool to understand the relative Morse index. Let \(\{A(\theta), \theta \in [0,1]\}\) be a continuous path of self-adjoint Fredholm operators on a Hilbert space \(\mathcal{H}\). Roughly speaking, the spectral flow of path \(\{A(\theta), \theta \in [0,1]\}\) counts the net change in the number of negative eigenvalues of \(A(\theta)\) as \(\theta\) goes from 0 to 1, where the enumeration follows from the rule that each negative eigenvalue crossing to the positive axis contributes +1 and each positive eigenvalue crossing to the negative axis contributes −1, and for each crossing, the multiplicity of eigenvalue is counted. More precisely, as shown in [11], let

\[
\varphi = \bigcup_{\theta \in [0,1]} \sigma(A(\theta)),
\]

27
Thus we have

\[ S f(A(\theta)) = \{ \varphi : \varphi(\theta) = \varphi(\theta_0) + \lambda \}, \quad \lambda \in [0, 1) \]

The spectral flow \( S f((A(\theta))) \) is defined to be the intersection number of \( \varphi \) with the line \( \lambda = -e \) with respect to the usual orientation for some small positive \( e \). Obviously, \( S f((A(\theta))) = S f((A(\theta) + \epsilon i)) \) if \( i \) is the identity operator on \( \mathcal{H} \), and \( 0 \leq e \leq e_0 \) for some sufficiently small positive number \( e_0 \).

We come back to the Hamiltonian systems, suppose \( B(s, t) \in C([0, 1] \times [0, T], S(2n)) \). For \( s \in [0, 1] \), let \( B_0 \in B(2n) \). For such two operators \( A - B_0 \) and \( A - B_1 \), we can define the relative Morse index via spectral flow. In fact, by \([12]\), we have,

\[
I(A - B_0, A - B_1) = -S f((A - B(s), s \in [0, 1])).
\]

For \( B_0, B_1, B_2 \), then

\[
I(A - B_0, A - B_1) + I(A - B_1, A - B_2) = I(A - B_0, A - B_2).
\]

Let \( D = B_1 - B_0 \), and we can simply let \( B(s) = B_0 + sD \). The next proposition is obvious from the definition of spectral flow.

**Proposition 4.1.** Let \( k = \{ s_0 \in [0, 1], \ker(A - B(s_0)) \neq 0 \} \),

\[
I(A - B_0, A - B_1) \leq \sum_{s_0 \in k} \dim \ker(A - B(s_0)).
\]

It is not hard to see that, if \( D > 0 \), then \( I(A - B, A - B - D) \geq 0 \). By careful analysis\([12]\), the crossing form

\[
I(A - B, A - B - D) = \sum_{s_0 \in k \cap (0, 1)} \dim \ker(A - B(s_0)). \tag{4.1}
\]

Similarly

\[
I(A - B, A - B + D) = -\sum_{s_0 \in k \cap (0, 1)} \dim \ker(A - B(s_0)). \tag{4.2}
\]

Thus we have

**Corollary 4.2.** Suppose \( D_1 \leq D \leq D_2 \), then

\[
I(A - B, A - B - D_1) \leq I(A - B, A - B - D) \leq I(A - B, A - B - D_2). \tag{4.3}
\]

To get the stability criteria, we consider the following Hamiltonian system,

\[
\dot{z}(t) = JB(t)z(t),
\]

\[
z(0) = \omega Sz(T).
\]

Denote \( A_\omega, B_\omega \) be the operators corresponding to \( A, B \) respectively under the \( \omega \)-boundary condition, then \( A_\omega \) is a self-adjoint operator with the domain \( D_{\omega S} \). Since \( \nu \) is an imaginary number, \( e^{\nu t} \) is a unitary operator on \( E \) and \( e^{\nu t}D_S = D_{\omega S} \). Simple calculations show that \( e^{-\nu t}A_\omega e^{\nu t} = A - \nu J \). Thus we have

\[
I(A_\omega, A_\omega - B_\omega) = I(A - \nu J, A - \nu J - B). \tag{4.4}
\]

To judge the stability, we use the Maslov-type index \( i_\omega(\gamma) \), which is essentially same as the relative Morse index \([18]\). Roughly speaking, for a continuous path \( \gamma(t) \in \text{Sp}(2n) \), \( \omega \in \mathbb{U} \), the Maslov-type index \( i_\omega(\gamma) \) is defined by the intersection number of \( \gamma \) and \( \text{Sp}_{\omega}^0(2n) = \{ M \in \text{Sp}(2n) | \det(M - \omega I_{2n}) = 0 \} \). Details could be found in \([16],[18] \), some brief review could be found in \([13]\). For simplicity, we assume \( S = I_{2n} \).
Proposition 4.3. Suppose $S = I_{2n}$, then, for imaginary number $\nu$ such that $\omega = e^{i\nu} \in \mathbb{U} \setminus \{1\}$, we have

$$I(A, A - B) = i_1(\gamma) + n.$$ 

and

$$I(A - \nu J, A - \nu J - B) = i_\omega(\gamma).$$ (4.5)

Proof. From [12, Theorem 2.5 and Lemma 4.5], we have

$$I(A, A - B) = i_1(\gamma) + n,$$

and

$$I(A_\omega, A_\omega - B_\omega) = i_\omega(\gamma), \omega \in \mathbb{U} \setminus \{1\}.$$ 

By (4.4), we have (4.5). \qed

We will continue to review the stability criteria by the Maslov-type index. Details for the stability criteria and the Maslov-type index are given in [18]. For $\omega \in \mathbb{U}$, the unit circle, $\omega = e^{i\theta}$ with $\theta \in [-\pi, \pi]$, let $U_\omega = \{e^{i\theta}, \theta \in [-|\theta_0|, |\theta_0|]\}$, denote by $e_\omega(M)$ the total algebraic multiplicities of all eigenvalues of $M$ in $U_\omega$. We also simply denote by $e(M)$ the total algebraic multiplicities of all eigenvalues of $M$ on $\mathbb{U}$. Obviously, for $M = \gamma(T)$ if $e(M) = 2n$ then $M$ is spectral stable.

For a bounded variation function $g(w)$ defined on some closed interval $[a, b]$, we define its variation by

$$\text{var}(g(w), [a, b]) = \sup \left\{ \sum_{j=0}^{k-1} |g(w_{j+1}) - g(w_j)| \left| a = w_0 < \ldots < w_k = b \text{ is any partition} \right. \right\}.$$ 

Notice that $i_{\nu \sqrt{-1}}$ is a bounded variation function on $[0, \theta_0]$. And the next proposition can be proved easily by the property of Maslov-type index (readers are referred to [18] or [12]).

Proposition 4.4. Let $\gamma$ be an arbitrary path in Sp$(2n)$ connecting $I_{2n}$ to $M$,

$$e_\omega(M)/2 \geq \text{var}(i_{\nu \sqrt{-1}}(\gamma), \theta \in [0, \theta_0]).$$ (4.6)

Corollary 4.5. With the notations as above,

$$e(M)/2 \geq \text{var}(i_{\nu \sqrt{-1}}(\gamma), \theta \in [0, \pi]).$$

Obviously, for $\omega \neq \{1, -1\}$

$$e(M)/2 \geq |i_{-1}(\gamma) - i_\omega(\gamma)| + |i_1(\gamma) - i_\omega(\gamma)|.$$ (4.7)

Especially,

$$e_\omega(M)/2 \geq |i_\omega(\gamma) - i_1(\gamma)|.$$ (4.8)

$$e(M)/2 \geq |i_{-1}(\gamma) - i_1(\gamma)|.$$ (4.9)

Remark 4.6. All the above results, for that the relative Morse index equals to Maslov-type index and for the stability criteria, could be proved for any $S$ boundary condition with $S \in \text{Sp}(2n) \cap O(2n)$, and details could be found in [12].
4.2 Estimate relative Morse index by trace formula

In this subsection, we will give the application of the trace formula on the estimation of the non-degeneracy. Moreover, we will estimate Maslov-type index by using the trace formula. Suppose $A - \nu J - B$ is non-degenerate, we will estimate the relative Morse index $I(A - \nu J - B, A - \nu J - B - D)$. Firstly assume $D > 0$, thus $I(A - \nu J - B, A - \nu J - B - D) \geq 0$.

**Lemma 4.7.** Suppose $D > 0$, $\nu$ is an imaginary number, then all the eigenvalues of $D(A - \nu J - B)^{-1}$ are real.

*Proof.* Let $D^{1/2}$ be the unique positive operator such that $D^{1/2}D^{1/2} = D$, then $D(A - \nu J - B)^{-1}$ is similar to $D^{1/2}(A - \nu J - B)^{-1}D^{1/2}$, which is a self-adjoint compact operator. Hence

$$
\sigma(D(A - \nu J - B)^{-1}) = \sigma(D^{1/2}(A - \nu J - B)^{-1}D^{1/2}) \subset \mathbb{R}.
$$

\[ \square \]

Let $\frac{1}{\lambda_j}$ be the eigenvalues of $D(A - \nu J - B)^{-1}$. By Lemma 4.7, $\lambda_j \in \mathbb{R}$, we can make the order such that

$$
\cdots \leq \lambda_2^- \leq \lambda_1^- < 0 < \lambda_1^+ \leq \lambda_2^+ \leq \cdots.
$$

Moreover, we have

**Lemma 4.8.** Suppose $D > 0$, then $\lim_{j \to \infty} \lambda_j^+ = +\infty$ and $\lim_{j \to \infty} \lambda_j^- = -\infty$.

*Proof.* We will use the contradiction argument. Suppose there is $\lambda_0^+$ such that, for each $j \in \mathbb{N}$, $\lambda_j^+ < \lambda_0^+$. We claim that

$$
\sigma(A - \nu J - B - \lambda_0^- D) \subset (-\infty, 0].
$$

In fact, notice that

$$
\sigma(A - \nu J - B - \lambda_0^- D) \subset (-\infty, 0]
$$

if and only if

$$
\sigma(D^{-\frac{1}{2}}(A - \nu J - B)D^{-\frac{1}{2}} - \lambda_0^+) \subset (-\infty, 0].
$$

Moreover, it is easy to see that

$$
\sigma(D^{-\frac{1}{2}}(A - \nu J - B)D^{-\frac{1}{2}}) = \{\lambda_j\},
$$

and hence

$$
\sigma(D^{-\frac{1}{2}}(A - \nu J - B)D^{-\frac{1}{2}} - \lambda_0^+) \subset (-\infty, 0].
$$

Now, notice that $A$ is an unbounded operator $\pm \infty$ is the limitation of its eigenvalues, and $\nu J - B - \lambda_0^+$ is a bounded operator. By the spectral theory for unbounded operator with perturbation by bounded operator, we have that

$$
\sigma(A) \subset \{\lambda \mid |\lambda - \lambda_0| \leq \|\nu J - B - \lambda_0^+\|, \text{ for some } \lambda_0 \in \sigma(A - \nu J - B - \lambda_0^- D)\}.
$$

This is a contradiction. The other part of the lemma can be proved similarly. \[ \square \]

Recall the formula (1.3) that,

$$
Tr \left[ (D(A - \nu J - B)^{-1})^m \right] = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^m}, \quad m \geq 2.
$$

(4.10)
**Proposition 4.9.** Suppose $D > 0$, we have that, for $\forall k \in \mathbb{N}$

$$I(A - vJ - B, A - vJ - B - D) + \dim \ker(A - vJ - B - D) < Tr((D(A - vJ - B)^{-1})^{2k}).$$

**Proof.** From Lemma [4.7] $\lambda_j$ are real numbers, and hence $\lambda_j^{2k} > 0$. By Lemma [4.8] and [4.10], we have

$$Tr((D(A - vJ - B)^{-1})^{2k}) > \sum_{|\lambda_j| \leq 1} \frac{1}{\lambda_j^{2k}}, \ \forall k \in \mathbb{N}.$$ 

Obviously, $\sum_{|\lambda_j| \leq 1} \frac{1}{\lambda_j^{2k}}$ is no less than the total multiplicity of eigenvalues with $|\lambda_j| \leq 1$. Please note that $\lambda_j \in D(A - vJ - B)^{-1}$ if and only if $\ker(A - vJ - B - \lambda_j D)$ is degenerate. Moreover, the multiplicity of the eigenvalue for $D(A - vJ - B)^{-1}$ at $\lambda_j$ is equal to $\dim \ker(A - vJ - B - \lambda_j D)$. By Proposition [4.1] and [4.1], the proposition is proved.

Similar to Proposition [4.9], we have the following proposition.

**Proposition 4.10.** Suppose $D > 0$, then

$$-Tr((D(A - vJ - B)^{-1})^{2k}) < I(A - vJ - B, A - vJ - B + D) \leq 0, \ \forall k \in \mathbb{N}.$$ 

We have the following corollary.

**Corollary 4.11.** Suppose $D > 0$, if for some $k \in \mathbb{N}$, $Tr((D(A - vJ - B)^{-1})^{2k}) \leq 1$, then

$$I(A - vJ - B, A - vJ - B + D) = I(A - vJ - B, A - vJ - B - D) + \dim \ker(A - vJ - B - D) = 0.$$ 

Now we can give the estimation on the upper bound that preserves the non-degeneracy.

**Theorem 4.12.** Suppose $A - B - vJ$ is non-degenerate. Suppose that there are $D_1, D_2 \in \mathcal{B}(2n)$ such that $D_1 < D < D_2$, with $D_1 < 0$, $D_2 > 0$, if there exists $k \in 2\mathbb{N}$, such that $Tr((D_j(A - B - vJ)^{-1})^{k}) \leq 1$ for $j = 1, 2$, then $A - B - D - vJ$ is non-degenerate.

**Proof.** By the condition $Tr((D_j(A - B - vJ)^{-1})^{k}) \leq 1$, for $j = 1, 2$, applying Corollary [4.11] we have that, for any $s \in [0, 1],

$$I(A - vJ - B, A - vJ - B - sD_1) = I(A - vJ - B, A - vJ - B - sD_2) + \dim \ker(A - vJ - B - sD) = 0. \ \ \ (4.11)$$

Next, we will prove the result by contradiction argument. Assume that $A - vJ - B - D$ is degenerate. Now, let

$$s_0 = \inf\{s \in [0, 1], \dim \ker(A - vJ - B - sD) \neq 0\}.$$ 

Notice that $A - vJ - B$ is non-degenerate, thus $s_0 > 0$. From the spectral theory of self-adjoint operators [19], the eigenvalues of $A - vJ - B - sD$ can be considered as a smooth function on $s$. Denote the eigenvalue functions by $\lambda_j(s)$. Since $A - B - vJ - s_0D$ is degenerate, there is some $\lambda_j(s_0) = 0$. We may assume that $\lambda_j(s_0) = 0$ for $j = 1, ..., m$. By the definition of $s_0$, $\lambda_j(s) \neq 0$ on $[0, s_0)$ for $j = 1, ..., m$. Without loss of generality, assume $\lambda_j(s) > 0$ on $[0, s_0)$ for $j = 1, ..., m$ and $\lambda_j(s) < 0$ on $[0, s_0)$ for $j = m_1 + 1, ..., m$, where $m_1$ can take value 0 or $m$. 

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Firstly, if \( m_1 > 0 \), by the property of relative Morse index, we have

\[
I(A - vJ - B, A - vJ - B - s_0D_2) = I(A - vJ - B, A - vJ - B - s_0D) + I(A - vJ - B - s_0D, A - vJ - B - s_0D_2),
\]

and \( I(A - vJ - B, A - vJ - B - s_0D) = m_1 - m \) by the definition of \( s_0 \). On the other hand, since \( D_2 > 0 \) form (4.11),

\[
I(A - vJ - B, A - vJ - B - s_0D) = m_1 - m + \dim \ker(A - vJ - B - s_0D) = m_1 > 0,
\]

which contradicts to (4.11).

Next, if \( m_1 = 0 \), noting that \( D_1 < D \), by the property of spectral flow, \( I(A - vJ - B, A - vJ - B - s_0D) = -m \).

By some similar discussion as above, we get

\[
I(A - vJ - B, A - vJ - B - s_0D_1)
\]

which also contradicts to (4.11). The proof is complete. \( \square \)

Next, we are going to give the estimation of the relative Morse index by the trace formula.

**Theorem 4.13.** Suppose \( A - B - vJ \) is non-degenerate and \( D_1 \leq D \leq D_2 \), where \( D_1 < 0 \), \( D_2 > 0 \). Let

\[
m^- = \inf\{[\operatorname{Tr}(D_1(A - B - vJ)^{-k})], k \in 2\mathbb{N}\} \quad \text{and} \quad m^+ = \inf\{[\operatorname{Tr}(D_2(A - B - vJ)^{-k})], k \in 2\mathbb{N}\},
\]

then

\[
-m^- \leq I(A - B - vJ, A - B - D - vJ) \leq m^+.
\]

**Proof.** Firstly, we will prove that

\[
I(A - B - vJ, A - B - D_2 - vJ) \leq m^+.
\]

Infact, by Proposition 4.9 we have that, for any \( k \in 2\mathbb{N} \),

\[
I(A - B - vJ, A - B - D_2 - vJ) < \operatorname{Tr}((A - vJ - B)^{-2k}).
\]

It follows that

\[
I(A - B - vJ, A - B - D_2 - vJ) \leq m^+.
\]

By Proposition 4.10 and some similar reasoning, we have

\[
I(A - B - vJ, A - B - D - vJ) \geq -m^-.
\]

Since \( D_1 \leq D \leq D_2 \), we get the result by (4.11). \( \square \)

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Motivated by Krein’s work [20], we consider the symmetric case, that is, \( D(t) = D(T - t) \). Suppose first that \( D \) is real and invertible. Then

\[
\sigma(D(A + \nu J)^{-1}) = \{\tilde{\lambda} | \lambda \in \sigma(D(A - \nu J)^{-1})\}.
\]

(4.12)

In fact, a nonzero \( \lambda \in \sigma(D(A - \nu J)^{-1}) = \sigma((A - \nu J)^{-1} D) \) if and only if there is \( x \neq 0 \), such that

\[
(A - \nu J)^{-1} Dx = \lambda x,
\]

if and only if

\[
(A + \nu J) \bar{x} = \bar{\lambda}^{-1} D \bar{x},
\]

if and only if \( \bar{\lambda} \in \sigma(D(A + \nu J)^{-1}) \). Therefore, (4.12) holds true. Now, suppose \( D > 0 \). Then, \( \sigma(D(A - \nu J)^{-1}) \subset \mathbb{R} \), and hence \( \sigma(D(A - \nu J)^{-1}) = \sigma(D(A + \nu J)^{-1}) \). If moreover \( D(t) = D(T - t) \), then, by some direct computation, \( x(t) \in \ker(A - \nu J - \lambda D) \) if and only if \( x(T - t) \in \ker(A + \nu J + \lambda D) \). We summarize the above reasoning as the following lemma.

**Lemma 4.14.** Suppose \( D > 0 \) and \( D(t) = D(T - t) \), then \( \lambda \in \sigma(D(A - \nu J)^{-1}) \) if and only if \(-\lambda \in \sigma(D(A - \nu J)^{-1}) \), and with the same multiplicity.

As an application, we have

**Proposition 4.15.** Suppose \( S = I_{2n}, B = 0, D > 0, \) and \( \omega \neq 1 \), if one of the following conditions holds

1) \[ \frac{\omega}{(1 - \omega)^2} Tr\left[ \left( J \int_0^T D(s) ds \right)^2 \right] \leq 1 \]

2) \[ D(t) = D(T - t), \frac{\omega}{2(1 - \omega)^2} Tr\left[ \left( J \int_0^T D(s) ds \right)^2 \right] \leq 1, \]

then \( i_\omega(\gamma) = 0 \), where \( \gamma \) is the fundamental solution with respect to \( D \).

**Proof.** Since \( M = S = I_{2n} \), by (2.17),

\[
Tr((D(A - \nu J)^{-1})^2) = \frac{\omega}{(1 - \omega)^2} Tr\left[ (J \int_0^T D(s) ds)^2 \right].
\]

The proofs of both cases are similar, we only list the proof under the second condition. By Lemma 4.14 and the (4.10),

\[
Tr\left[ (D(A - \nu J)^{-1})^2 \right] = 2 \sum_{j \in \mathbb{N}} \frac{1}{\lambda_j^2}.
\]

Thus we have

\[
\frac{\omega}{2(1 - \omega)^2} Tr\left[ (J \int_0^T D(s) ds)^2 \right] = \sum_{j \in \mathbb{N}} \frac{1}{\lambda_j^2}.
\]

Notice that \( \frac{\omega}{2(1 - \omega)^2} Tr\left[ (J \int_0^T D(s) ds)^2 \right] \leq 1 \). By the same discussion as in the proof of Proposition 4.9 we have

\[
I(A - \nu J, A - \nu J - D) = 0.
\]

By Proposition 4.3, \( i_\omega(\gamma) = I(A - \nu J, A - \nu J - D) = 0 \). The proof is complete. \( \square \)
4.3 Stability criteria

In this section, we only consider the case $S = I_{2n}$, and the general case is similar. Recall that $\gamma$ is the fundamental solution with respect $B$ and $M = \gamma(T)$, we denote $\tilde{\gamma}$ be the fundamental solution with respect to $B + D$, and write $\tilde{M} = \tilde{\gamma}(T)$.

**Proposition 4.16.** Suppose $D_1 \leq D \leq D_2$, where $D_1 < 0$, $D_2 > 0$. If for $j = 1, 2,$

$$Tr((D_j(A - B)^{-1})^2) \leq 1 \quad \text{and} \quad Tr((D_j(A - vJ - B)^{-1})^2) \leq 1$$

then $i_\omega(\gamma) = i_\omega(\tilde{\gamma})$, and

$$e_\omega(\tilde{M})/2 \geq |i_1(\gamma) - i_\omega(\gamma)|,$$

where $\omega = e^{iT}$.

Especially, if for $j = 1, 2,$

$$Tr\left((D_j(A - \sqrt{-1}\pi J - B)^{-1})^2\right) \leq 1,$$

then

$$e(\tilde{M})/2 \geq |i_1(\gamma) - i_{-1}(\gamma)|.$$  

**Proof.** Since $Tr((D_j(A - B)^{-1})^2) \leq 1$ for $j = 1, 2$, by Corollary 4.11

$$I(A - B, A - B - D_j) = 0.$$  

Hence, for $j = 1, 2,$

$$I(A, A - B - D_j) = I(A, A - B) + I(A - B, A - B - D_j) = I(A, A - B),$$

thus by (4.3)

$$I(A, A - B - D) = I(A, A - B),$$

and from Proposition 4.3 we have

$$i_1(\gamma) = i_1(\tilde{\gamma}).$$

Similar, $Tr((D_j(A - vJ - B)^{-1})^2) \leq 1$ for $j = 1, 2$ implies

$$i_\omega(\gamma) = i_\omega(\tilde{\gamma}).$$

From (4.8),

$$e_\omega(\tilde{M})/2 \geq |i_1(\tilde{\gamma}) - i_\omega(\tilde{\gamma})| = |i_1(\gamma) - i_\omega(\gamma)|.$$  

The desired result is proved. \hfill $$\Box$$

$Tr((D_j(A - B)^{-1})^2)$ could be estimated by using the trace formula. If moreover $MJ = JM$ and $M^T = M$, we could have a more simple estimation.
Corollary 4.17. Under the condition of Proposition 4.16 if moreover $MJ = JM$, $M^T = M$, for $j = 1, 2$,

$$Tr\left((J \int_{0}^{T} \hat{D}_j(s)ds \cdot M(M - \omega I_{2n})^{-1}\right)^2) - Tr\left((J \int_{0}^{T} \hat{D}_j(s)ds \cdot M(M - \omega I_{2n})^{-1}\right)^2) \leq 1, \quad (4.13)$$

and

$$Tr\left((J \int_{0}^{T} \hat{D}_j(s)ds \cdot M(M - I_{2n})^{-1}\right)^2) - Tr\left((J \int_{0}^{T} \hat{D}_j(s)ds \cdot M(M - I_{2n})^{-1}\right)^2) \leq 1, \quad (4.14)$$

where $\hat{D}_j(t) = \gamma_j^T(t)D_j(t)\gamma_0(t)$, then

$$e_\omega(M)/2 \geq |i_1(\gamma) - i_\omega(\gamma)|.$$

Proof. From Proposition 2.13 in case $MJ = JM$, $M^T = M$, the equality (4.13) implies

$$Tr((D_j(A - \nu J - B)^{-1})^2) \leq 1.$$

By Proposition 4.16, $i_\omega(\gamma) = i_\omega(\tilde{\gamma})$. Similarly, by (4.14), $i_1(\gamma) = i_1(\tilde{\gamma})$. The result is from (4.8). \qed

Theorem 4.18. If $M = I_{2n}$, $D > 0$ (or $D < 0$), $\frac{\omega}{(1 - \omega)}Tr\left((J \int_{0}^{T} \hat{D}(s)ds\right)^2) \leq 1$ then

$$e_\omega(M)/2 = n \quad (4.15)$$

Proof. Firstly, we will prove the result in the case of $D > 0$. Since $M = I_{2n}$, by [18, Chapter 9], we have

$$i_\omega(\gamma) = i_1(\gamma) + n.$$

On the other hand, since $D > 0$, by (4.1)

$$I(A, A - B - D) \geq I(A, A - B) + \dim \ker(A - B) = I(A, A - B) + 2n.$$

Thus

$$i_1(\tilde{\gamma}) \geq i_1(\gamma) + 2n.$$

By the condition $\frac{\omega}{(1 - \omega)}Tr((J \int_{0}^{T} \hat{D}(s)ds)^2) \leq 1$, we have

$$i_\omega(\tilde{\gamma}) = i_\omega(\gamma). \quad (4.16)$$

The result follows from (4.8).

In the case $D < 0$, we have

$$I(A, A - B - D) \leq I(A, A - B),$$

this is equivalent to $i_1(\tilde{\gamma}) \leq i_1(\gamma)$. On the other hand, we have $i_\omega(\tilde{\gamma}) = i_\omega(\gamma)$. The result follows from (4.8). The proof is complete. \qed

By taking $\omega = -1$, we have

Corollary 4.19. If $M = I_{2n}$, $D > 0$ (or $D < 0$), $-\frac{1}{4}Tr\left((J \int_{0}^{T} \hat{D}(s)ds)^2\right) \leq 1$, then $e(M)/2 = n$, that is $\tilde{M}$ is elliptic.
In the special case $B(t) \equiv 0$, then $\gamma(t) \equiv I_{2n}$ is a constant path, it is well known $i_1(\gamma) = -n$, and $i_\omega(\gamma) = 0$ for $\omega \in \mathbb{U} \setminus \{1\}$ (see [18]).

**Corollary 4.20.** Suppose $B = 0$ and $D > 0$ (or $D < 0$) if one of the following conditions satisfies:

(i) $\frac{\omega}{(1 - \omega^2)} Tr\left[\left(J \int_0^T D(s)ds\right)^2\right] \leq 1$,

(ii) $D(t) = D(T - t)$ and $\frac{\omega}{2(1 - \omega^2)} Tr\left[\left(J \int_0^T D(s)ds\right)^2\right] \leq 1$,

then

$$e_\omega(M)/2 = n.$$  

**Proof.** The result under condition (i) comes directly from Theorem 4.18 since $\dot{D} = D$ for $B = 0$. For condition (ii), by Proposition 4.15, $i_\omega(\tilde{\gamma}) = i_\omega(\gamma) = 0$. In this case $\gamma \equiv I_{2n}$ is a constant solution. By some similar argument to the proof of Theorem 4.18 we prove the result. □

We will give some hyperbolic criteria

**Proposition 4.21.** Suppose $M$ is hyperbolic, $Tr\left[(D - vJ - B)^{-1}\right] \leq 1$ for $v \in \left[0, \frac{\sqrt{-2}}{2}\right]$, then $\tilde{M}$ is hyperbolic.

**Proof.** Please note that $Tr((D - vJ - B)^{-1}) \leq 1$, thus $A - vJ - B - sD$ is non-degenerate for $s \in [0, 1]$. This is equivalent to $A_\omega - B_\omega - sD_\omega$ is non-degenerate. Therefore $\tilde{M} - \omega I_{2n}$ is nonsingular for $\omega \in \mathbb{U}$, thus $\tilde{M}$ is hyperbolic. □

When $B$ is constant path, our stability criteria can be easily used. Next example will give a new stability criteria.

**Example 4.22.** Suppose $B(t) \equiv B$ is constant path of matrices, $JB = BJ$ and $\exp(JBT) = I_{2n}$. This happens when $B = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_n, \alpha_1, \alpha_2, ..., \alpha_n)$, and $\alpha_j T/2\pi \in \mathbb{Z}$ for $j = 1, ..., n$. Consider the linear Hamiltonian systems

$$\dot{z}(t) = J(B + D(t))z(t),$$  

(4.17)

with $D(t) = D(t + T) \geq 0$ and $\int_0^T D(t)dt > 0$. Let $\lambda(t) = \lambda_{\text{max}}(D(t))$ which is the largest eigenvalue of $B(t)$, then (4.17) is spectrally stable if

$$\int_0^T \lambda(t)dt < 2.$$  

(4.18)

In fact, noting that $D(t) \leq \lambda(t) I_{2n}$, let $\tilde{\gamma}(t)$ and $\tilde{\gamma}_1(t)$ be the fundamental solutions corresponding to $B + D(t)$ and $B + \lambda(t) I_{2n}$ respectively, then

$$i_\omega(\tilde{\gamma}) \leq i_\omega(\tilde{\gamma}_1), \quad \forall \omega \in \mathbb{U}.$$  

By some easy computation, the condition (4.18) implies $i_{-1}(\tilde{\gamma}_1) = i_{-1}(\gamma)$. On the other hand, by the proof of Theorem 4.18 we have $i_1(\tilde{\gamma}) \geq i_1(\gamma) + 2n$ and $i_{-1}(\gamma) = i_1(\gamma) + n$, which yields the result by (4.9). Please note that, in the case $B = 0$, if we instead (4.18) by the condition (i) of Corollary 4.20 we also get $e_\omega(M)/2 = n$, which is a generalization of Krein’s stability criteria.

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4.4 Estimate the Morse index for $\tilde{S}$-periodic orbits in Lagrangian system

In this section, we will estimate the Morse index of $\tilde{S}$-periodic orbits in Lagrangian systems by using the trace formula. For $T > 0$, suppose $x(t)$ is a critical point of the functional

$$F(x) = \int_0^T L(t, x, \dot{x}), \forall x \in E = \{x \mid x \in W^{1,2}(\mathbb{R}, \mathbb{R}^n), x(t) = \tilde{S} x(t + T)\}$$

where $L \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and satisfies circle type symmetry \[12\]

$$L(t, x, \xi) = L(t + T, \tilde{S}^T x, \tilde{S}^T \xi).$$  \hfill (4.19)

It is well known that $x(t)$ is a solution of the corresponding Euler-Lagrangian equation:

$$d \frac{d}{dt} L_p(t, x, \dot{x}) - L_x(t, x, \dot{x}) = 0, x(0) = \tilde{S} x(T), \dot{x}(0) = \tilde{S} \dot{x}(T).$$ \hfill (4.20)

For such an extremal loop, define

$$P(t) = L_{p,p}(t, x(t), \dot{x}(t)), Q(t) = L_{x,p}(t, x(t), \dot{x}(t)), R(t) = L_{x,x}(t, x(t), \dot{x}(t)).$$

Note that

$$F''(x) = -\frac{d}{dt} \left( P \frac{d}{dt} + Q \right) + Q^T \frac{d}{dt} + R.$$  

For $\omega \in \mathbb{U}$, set $D_{\omega S} = \{y \in W^{1,2}([0, T]; \mathbb{C}^n) \mid y(0) = \omega \tilde{S} y(T)\}$. We define the $\omega$-Morse index $\phi_\omega(x)$ of $x$ to be the dimension of the negative definite subspace of

$$\langle F''(x)y_1, y_2 \rangle, y_1, y_2 \in D_{\omega S}.$$

For $\omega = e^{i\nu T}$ with imaginary number $\nu$, recall that

$$\mathcal{A}(\nu) = -\left( \frac{d}{dt} + \nu \right) P(t) \left( \frac{d}{dt} + \nu \right) + Q^T(t) \left( \frac{d}{dt} + \nu \right) + R(t),$$

with domain $D_\tilde{S}$. We also denote by $\phi_\omega(\mathcal{A})$ the $\omega$-Morse index of $\mathcal{A}$, which is defined to be the dimension of the negative definite space of

$$\langle \mathcal{A} y_1, y_2 \rangle, y_1, y_2 \in D_{\omega S}.$$

Obviously,

$$\phi_\omega(\mathcal{A}(0)) = \phi_1(\mathcal{A}(\nu)).$$ \hfill (4.21)

The next lemma is obvious.

**Lemma 4.23.** Suppose $R_1 \geq 0$, then

$$\phi_1(\mathcal{A}(\nu) + R_1) \leq \phi_1(\mathcal{A}(\nu)).$$  \hfill (4.22)
When we transform the Sturm-Liouville system to linear Hamiltonian system, it is obvious
\[
\dim \ker(A - \nu J - B) = \dim \ker(\mathcal{A}(\nu)).
\] (4.23)
Moreover, the Morse index is essentially same as the relative Morse index (Maslov-type index) (see \[18\] or \[12\]). Recall that 
\[
B_4(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}Q(t) \\ -Q(t)^TP^{-1}(t) & Q(t)^TP^{-1}(t)Q(t) - R(t) - \lambda R_1(t) \end{pmatrix}
\]. We have the following proposition.

**Proposition 4.24.**
\[
I(A - \nu J - B, A - \nu J - B_1) = \phi_1(\mathcal{A}(\nu) + R_1) - \phi_1(\mathcal{A}(\nu)) = i_\omega(\gamma_1) - i_\omega(\gamma_0).
\] (4.24)

**Proof.** Let \(\gamma_i\) be the fundamental solution corresponding to \(B_4\), then from \[18\] P172, we have
\[
\phi_1(\mathcal{A}(\nu) + \lambda R_1) = i_\omega(\gamma_i).
\] (4.25)
Thus
\[
\phi_1(\mathcal{A}(\nu) + R_1) - \phi_1(\mathcal{A}(\nu)) = i_\omega(\gamma_1) - i_\omega(\gamma_0),
\] (4.26)
the result is from Proposition 4.3.

By (4.23) and (4.24), all the result in §4.2 can be used to estimate the Morse index and non-degenerate of linear Lagrangian systems, however, there are some new estimation for the Lagrangian system.

**Theorem 4.25.** Let \(\nu \in \mathbb{C}\), assume \(\mathcal{A}(\nu) > 0\), if \(R_1 \geq -K\), where \(K \in \mathcal{B}(n)\) and \(K > 0\). Then
\[
\phi_1(\mathcal{A}(\nu) + R_1) \leq \inf[Tr((K(\mathcal{A}(\nu))^{-1})^k), k \in \mathbb{N}].
\] (4.27)

**Proof.** Please note that in this case, all the eigenvalues \(\{1/\lambda, 1\}\) of \(D(\mathcal{A}(\nu))^{-1}\) are positive, and \(K(\mathcal{A}(\nu))^{-1}\) is a trace class operator. Hence for any positive integers \(l\),
\[
Tr[(K(\mathcal{A}(\nu))^{-1})^l] > \sum_{|\lambda| \leq 1} \frac{1}{\lambda^l}.
\]
Similar argument to the proof of Proposition 4.9 implies the result.

**Corollary 4.26.** Under the conditions of Theorem 4.25 if \(Tr((D(\mathcal{A}(\nu))^{-1}) < 1\), then
\[
\phi_1(\mathcal{A}(\nu) + R_1) = \phi_1(\mathcal{A}(\nu)) = 0
\]
and \(\mathcal{A}(\nu) + R_1\) is non-degenerate.

Next, we will consider some special case that \(\mathcal{A}(\nu) = -(\frac{d}{dt} + \nu)^2 - R(t)\). Let \(R^+(t) = \frac{1}{2}(R(t) + |R(t)|)\), then \(R^+(t) \geq 0\), and \(R(t) \leq R^+(t)\), we have

**Theorem 4.27.** For imaginary number \(\nu\), such that \(-(\frac{d}{dt} + \nu)^2\) is invertible,
\[
\phi_1(-((\frac{d}{dt} + \nu)^2 - R(t)) \leq -\omega T \cdot Tr\left[\int_0^T R^+(t)dt \cdot S(S - \omega)^{-2}\right].
\] (4.28)

where \(\omega = e^{\nu T}\).

**Proof.** For any \(\varepsilon > 0\), \(R^+(t) + \varepsilon I_n > 0\), and \(\phi_1(-(\frac{d}{dt} + \nu)^2 - R(t)) \leq \phi_1(-(\frac{d}{dt} + \nu)^2 - (R^+(t) + \varepsilon I_n))\). The result follows from (3.31) and Theorem 4.25.

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5 Stability of Lagrangian orbits

In this section, we will give the application of the trace formula on the stability for elliptic Lagrangian orbits. To do this, in §5.1 we will recall some elementary results on Maslov-type index and Morse index of Lagrangian orbits. In §5.2 we will prove Theorem 1.8. Details on the function $f(\beta, \omega)$ in Theorem 1.8 via the trace formula (1.10) will be listed in §5.3. At last, in §5.4, by the first order trace formula (1.17) we will give another estimation for the hyperbolic region which is not too sharper but with more simple estimation.

5.1 A brief review on Lagrangian orbits

Following Meyer and Schmidt [24], the linear variational equation of the elliptic equilibria is decoupled into three subsystems, the first and second subsystems are from the first integral and the third is the essential part. The essential part $\gamma = \gamma_{\beta,e}(t)$ of the fundamental solution of the Lagrangian orbit [24, P.275] satisfies

\[
\begin{align*}
\dot{y}(t) &= J_B_{\beta,e}(t)y(t), \\
y(0) &= I_4,
\end{align*}
\]  

(5.1)

with

\[
B_{\beta,e}(t) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 2e \cos(t) - \frac{\sqrt{9 - \beta}}{2(1 + e \cos t)} & 0 \\
1 & 0 & 0 & \frac{2e \cos(t) + \sqrt{9 - \beta}}{2(1 + e \cos t)}
\end{pmatrix},
\]

where $e$ is the eccentricity, and $t$ is the truly anomaly.

Let

\[
J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{K}_{\beta,e}(t) = \begin{pmatrix} \frac{3 + \sqrt{9 - \beta}}{2(1 + e \cos t)} & 0 \\ 0 & \frac{3 - \sqrt{9 - \beta}}{2(1 + e \cos t)} \end{pmatrix},
\]

and the corresponding Sturm-Liouville system is

\[-\ddot{y} - 2J_2\dot{y} + \hat{K}_{\beta,e}y = 0.\]

For $(\beta, e) \in [0, 9) \times [0, 1)$, $\omega \in \mathbb{U}$, we set

\[
\overline{D}(\omega, 2\pi) = \{y \in W^{2,2}([0, 2\pi]; \mathbb{C}^n) | y(0) = \omega y(2\pi), \dot{y}(0) = \dot{y}(2\pi)\}.
\]

and

\[
\mathcal{A}(\beta, e, \nu) = -\left(\frac{d}{dt} + \nu\right)^2 - 2J_2\left(\frac{d}{dt} + \nu\right) + \hat{K}_{\beta,e}(t).
\]

Then for pure imaginary number $\nu$, $\mathcal{A}(\beta, e, \nu)$ are self-adjoint operators on $L^2([0, 2\pi], \mathbb{C}^n)$ with domain $\overline{D}(\omega, 2\pi)$ and depend on the parameters $\beta$ and $e$. We simply denote the operator by $\mathcal{A}_\omega(\beta, e, \nu)$ and omit $\omega$ when $\omega = 1$. Let $\phi(\mathcal{A}_\omega) = \phi_1(\mathcal{A}_\omega)$ be the Morse index of $\mathcal{A}_\omega$. It is obvious that $\mathcal{A}_\omega > 0$ if and only if $\phi(\mathcal{A}_\omega) = v(\mathcal{A}_\omega) = 0$, where

\[
v(\mathcal{A}_\omega) = \dim \ker(\mathcal{A}_\omega).
\]

For any $x(t) \in \overline{D}(1, 2\pi)$, direct computations show that

\[
e^{-\lambda t} \mathcal{A}(\beta, e, 0)e^{\lambda t}x(t) = \mathcal{A}(\beta, e, \nu)x(t), \quad (5.3)
\]
thus for \( \omega = e^{2\pi t} \), we have
\[
\phi(\mathcal{A}_\omega(\beta, e, 0)) = \phi(\mathcal{A}(\beta, e, 0)) \quad \text{and} \quad \nu(\mathcal{A}_\omega(\beta, e, 0)) = \nu(\mathcal{A}(\beta, e, 0)). \tag{5.4}
\]

Obviously
\[
\phi(\mathcal{A}_\omega(\beta, e, 0)) = I \left( -\frac{d^2}{dt^2}, \mathcal{A}_\omega(\beta, e, 0) \right).
\]

By the relationship between Morse index with Maslov-type index \([18, \text{p.172}]\), we have that for any \( \beta \) and \( e \) the Morse index \( \phi(\mathcal{A}_\omega(\beta, e, 0)) \) and nullity \( \nu(\mathcal{A}_\omega(\beta, e, 0)) \) satisfy
\[
\phi(\mathcal{A}_\omega(\beta, e, 0)) = i_\omega(\gamma_{\beta,e}), \quad \text{and} \quad \nu(\mathcal{A}_\omega(\beta, e, 0)) = \nu_\omega(\gamma_{\beta,e}), \quad \forall \omega \in \mathbb{U}.
\]

In particular, by (55) and (58) in \([13, \text{Lemma 4.1}]\), we obtain
\[
i_1(\gamma_{\beta,e}) = \phi(\mathcal{A}(\beta, e, 0)) = i_1(\gamma_{\beta,e}) = 0, \quad \forall (\beta, e) \in [0, 9] \times [0, 1). \tag{5.5}
\]

In the case \( e = 0 \), \( B_{\beta,0}(t) \) is a constant matrix and \( i_\omega(\gamma_{\beta,0}), \nu_\omega(\gamma_{\beta,0}) \) could be computed directly. We list the result for \( \omega = -1 \) and \( \omega = e^{i\sqrt{2}\pi} \) below.

**Theorem 5.1.** \([11]\) For any \( \omega = e^{2\pi \nu} \in \mathbb{U}, \beta \in (1, 9), \mathcal{A}(\beta, 0, \nu) > 0 \) or equivalently
\[
i_\omega(\gamma_{\beta,0}) = \phi(\mathcal{A}(\beta, 0, \nu)) = \nu(\mathcal{A}(\beta, 0, \nu)) = 0. \tag{5.6}
\]

For \( \omega = e^{i\sqrt{2}\pi/2}, \nu(\mathcal{A}(1, 0, i\sqrt{2}\pi/2)) = 1 \), and
\[
i_{e^{i\sqrt{2}\pi/2}}(\gamma_{\beta,0}) = \phi(\mathcal{A}(\beta, 0, i\sqrt{2}\pi/2)) \geq 1, \quad \text{for} \ \beta \in [0, 1). \tag{5.7}
\]

For \( \omega = -1, \nu(\mathcal{A}(3/4, 0, i/2)) = 2 \) and \( \nu(\mathcal{A}(\beta, 0, i/2)) = 0 \) if \( \beta \neq 3/4, \)
\[
i_{-1}(\gamma_{\beta,0}) = \phi(\mathcal{A}(\beta, 0, i/2)) = \begin{cases} 2 & \text{if} \ \beta \in [0, 3/4), \\ 0 & \text{if} \ \beta \in [3/4, 9]. \end{cases} \tag{5.8}
\]

### 5.2 Stability analysis via trace formula

Set
\[
D_{\beta,e}(t) = B_{\beta,e}(t) - B_{\beta,0}(t) = \frac{e \cos(t)}{1 + e \cos(t)} K_\beta,
\]
where
\[
K_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 + \sqrt{9 - \beta} & 0 \\ 0 & 0 & 0 & 3 - \sqrt{9 - \beta} \end{pmatrix},
\]
then
\[
-J \frac{d}{dt} B_{\beta,e} = -J \frac{d}{dt} B_{\beta,0} - D_{\beta,e}.
\]

Let \( \cos^\pm(t) = (\cos(t) \pm |\cos(t)|)/2 \), and denote
\[
K^\pm_\beta = \cos^\pm(t) K_\beta,
\]

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which can be considered as two bounded self-adjoint operators on \( L^2([0, 2\pi], \mathbb{C}^{2n}) \); moreover \( K_\beta^+ \geq 0 \) and \( K_\beta^- \leq 0 \). It is obvious that

\[
-J \frac{d}{dt} - \nu J - B_\beta \geq -J \frac{d}{dt} - \nu J - B_{\beta,0} - \frac{e \cos^+(t)}{1 + e \cos^+(t)} K_\beta
\]

\[
\geq -J \frac{d}{dt} - \nu J - B_{\beta,0} - e K_\beta^+,
\]

which is equivalent to

\[
\mathcal{A}(\beta, e, \nu) \geq \mathcal{A}(\beta, 0, \nu) - \frac{e}{1 + e \cos^+(t)} \hat{K}_{\beta,0}
\]

\[
\geq \mathcal{A}(\beta, 0, \nu) - e \cos^+(t) \hat{K}_{\beta,0},
\]

(5.9)

equivalently,

\[
\mathcal{A}(\beta, e, \nu) \geq \mathcal{A}(\beta, 0, \nu) - \frac{e}{1 + e \cos^+(t)} \hat{K}_{\beta,0}
\]

\[
\geq \mathcal{A}(\beta, 0, \nu) - e \cos^+(t) \hat{K}_{\beta,0}.
\]

(5.10)

Similarly,

\[
-J \frac{d}{dt} - \nu J - B_\beta \leq -J \frac{d}{dt} - \nu J - B_{\beta,0} - \frac{e}{1 + e \cos^-(t)} K_{\beta}^-
\]

\[
\leq -J \frac{d}{dt} - \nu J - B_{\beta,0} - e K_{\beta}^-,
\]

(5.11)

which is equivalent to

\[
\mathcal{A}(\beta, e, \nu) \leq \mathcal{A}(\beta, 0, \nu) - \frac{e}{1 + e \cos^-(t)} \hat{K}_{\beta,0}
\]

\[
\leq \mathcal{A}(\beta, 0, \nu) - e \cos^-(t) \hat{K}_{\beta,0}.
\]

(5.12)

**Lemma 5.2.** For an imaginary number \( \nu \), such that \( -J \frac{d}{dt} - \nu J - B_{\beta,0} \) is invertible, we have

\[
Tr\left[\left(K_{\beta}^+\left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}\right)^2\right] = Tr\left[\left(K_{\beta}^-\left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}\right)^2\right]
\]

Proof. Define an operator \( G : x(t) \rightarrow x(t + \pi) \) on the domain \( \overline{D}(1, 2\pi) \), then \( G^2 = \text{id} \). Direct calculation shows that

\[
\left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1} G = G \left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}.
\]

Moreover, \( K_{\beta,0} G = G K_{\beta,0} \) because \( K_{\beta,0} \) is a constant matrix. Therefore,

\[
Tr\left[\left(G \cos^+(t) K_{\beta} \left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1} G\right)^2\right] = Tr\left[\left(G \cos^+(t) K_{\beta} \left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}\right)^2\right]
\]

\[
= Tr\left[\left(\cos^-(t) K_{\beta} \left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}\right)^2\right].
\]

\[\square\]

Under the assumption of Lemma 5.2 we denote

\[
f(\beta, \omega) = Tr\left[\left(K_{\beta}^+ \left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}\right)^2\right] = Tr\left(\left(K_{\beta}^+ \left( -J \frac{d}{dt} - \nu J - B_{\beta,0} \right)^{-1}\right)^2\right),
\]

(5.13)

which is a positive function. The following theorem holds true.
Theorem 5.3. For $\beta \in [0, 3/4)$, $\gamma_{\beta,e}$ is spectrally stable if

$$0 \leq e < \frac{1}{1 + f(\beta, -1)}$$  \hspace{1cm} (5.14)

Proof. Obviously,

$$\text{Tr}\left(\left(\frac{e}{1 - e} K_{\beta}^- - \frac{\sqrt{-1}}{2} J - B_{\beta,0}\right)^{-1}\right)^2 = \frac{e^2}{(1 - e)^2} \text{Tr}\left(\left(\frac{e}{1 - e} K_{\beta}^- - \frac{\sqrt{-1}}{2} J - B_{\beta,0}\right)^{-1}\right)^2$$

$$= \frac{e^2}{(1 - e)^2} \text{f}(\beta, -1).$$

Thus, (5.14) is equivalent to $\frac{e^2}{(1 - e)^2} f(\beta, -1) < 1$ which implies $\text{Tr}\left(\left(\frac{e}{1 - e} K_{\beta}^- - \frac{\sqrt{-1}}{2} J - B_{\beta,0}\right)^{-1}\right)^2 < 1$. By the continuity of the trace, for $e > 0$ small enough, $\text{Tr}\left(\left(\frac{e}{1 - e} K_{\beta}^- - \epsilon I_{2n}\right)\left(\frac{e}{1 - e} K_{\beta}^- - \frac{\sqrt{-1}}{2} J - B_{\beta,0}\right)^{-1}\right)^2 < 1$. Obviously, $\frac{e}{1 - e} K_{\beta}^- - \epsilon I_{2n} < 0$. By Theorem 4.12 and Theorem 4.13, $\left(\frac{e}{1 - e} K_{\beta}^- - \frac{\sqrt{-1}}{2} J - B_{\beta,0} - \frac{e}{1 - e} K_{\beta}^-\right)$ is non-degenerate and

$$I\left(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0} - \frac{e}{1 - e} K_{\beta}^-\right) = 0.$$

From (5.11), $I\left(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0} - \frac{e}{1 - e} K_{\beta}^-\right) \geq 0$, consequently,

$$I\left(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0} - \frac{e}{1 - e} K_{\beta}^-\right) \geq 0.$$

By (5.8)

$$i_{-1}(\gamma_{\beta,e}) \geq i_{-1}(\gamma_{\beta,0}) = 2.$$

By (5.5) and (4.9), $e(\gamma_{\beta,e})/2 = 2$. The desired result is proved. \hfill \Box

Theorem 5.4. For $\beta \in (3/4, 1)$, $\gamma_{\beta,e}$ is spectrally stable if

$$0 \leq e < f(\beta, -1)^{-\frac{1}{2}}.$$  \hspace{1cm} (5.15)

and

$$0 \leq e < \frac{1}{1 + f(\beta, e^{\sqrt{2n}})^{\frac{1}{2}}}.$$  \hspace{1cm} (5.16)

Proof. Firstly, we’ll show that (5.15) implies

$$i_{-1}(\gamma_{\beta,e}) = 0,$$  \hspace{1cm} (5.17)

and the proof is similar to the proof of Theorem 5.3. In fact, please note

$$\text{Tr}\left(\left(e K_{\beta}^+ - \frac{\sqrt{-1}}{2} J - B_{\beta,0}\right)^{-1}\right)^2 = e^2 \text{Tr}\left(\left(K_{\beta}^+ - \frac{\sqrt{-1}}{2} J - B_{\beta,0}\right)^{-1}\right)^2$$

$$= e^2 f(\beta, -1).$$
Thus, (5.14) implies \( Tr((eK^+_\beta - J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0})^{-1})^2 < 1 \), then for \( \epsilon > 0 \) small enough,

\[
Tr(((eK^+_\beta + \epsilon l_2n)(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0})^{-1})^2) < 1.
\]

Obviously, \( eK^+_\beta + \epsilon l_2n > 0 \). Again, by Theorem 4.12 and Theorem 4.13 \(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0} - eK^+_\beta \) is non-degenerate and \( I(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0}, -J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0} - eK^+_\beta) = 0 \). By (5.9),

\[
I(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0}, -J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,e}) \leq 0.
\]

Therefore

\[
I(-J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,0}, -J \frac{d}{dt} - \frac{\sqrt{-1}}{2} J - B_{\beta,e}) \leq 0.
\]

By (5.8), we have (5.17).

On the other hand, almost the same proof as that of Theorem 5.3 shows that (5.16), (5.7) implies

\[
i_\omega(\gamma_{\beta,e}) \geq i_\omega(\gamma_{\beta,0}) \geq 1.
\]

The result comes from (5.17), (5.18), (5.5) and (4.7).

\[\square\]

**Remark 5.5.** It has been proved in [11,13] that \( \gamma_{\beta,e}(2\pi) \) is linear stable when \( (\beta, \epsilon) \) is in the stable region and not on the bifurcation curves. This implies that under the condition in Theorem 5.3 and Theorem 5.4 \( \gamma_{\beta,e} \) is linear stable. Moreover, the normal form of \( \gamma_{\beta,e}(2\pi) \) was given in [11,13]. Precisely, for \( (\beta, \epsilon) \) in the stable region given in Theorem 5.3 \( \gamma_{\beta,e}(2\pi) \approx R(\theta_1) \circ R(\theta_2) \) for some \( \theta_1, \theta_2 \in (\pi, 2\pi) \); for \( (\beta, \epsilon) \) in the stable region given in Theorem 5.4 \( \gamma_{\beta,e}(2\pi) \approx R(\theta_1) \circ R(\theta_2) \) for some \( \theta_1 \in ((2 - \sqrt{2})\pi, \pi), \theta_2 \in (\sqrt{2}\pi, 2\pi) \).

To estimate the hyperbolic region, denote

\[
\hat{f}(\beta) = \sup\{ f(\beta, \omega), \omega \in \mathbb{U} \},
\]

and we have

**Theorem 5.6.** For \( \beta \in (1, 9) \), \( \gamma_{\beta,e} \) is hyperbolic if

\[
e < \hat{f}(\beta)^{-1/2}.
\]

**Proof.** Similar to the proof of Theorem 5.4, the condition (5.20) implies that for any \( \omega \in \mathbb{U} \)

\[
i_\omega(\gamma_{\beta,e}) \leq i_\omega(\gamma_{\beta,0}) = 0,
\]

and

\[\nu(A_\omega(\beta, e, 0)) = \nu(A_\omega(\beta, 0, 0)) = 0,\]

which implies that \( \gamma_{\beta,e} \) is hyperbolic. \[\square\]

Combining Theorem 5.3, Theorem 5.4, with Theorem 5.6 and Remark 5.5, we have Theorem 1.8. The function \( f(\beta, \omega) \) will be dealt with in the next subsection, and based on this, with the help of Mathlab, we can draw a picture of the stable region and hyperbolic region in Figure 1.
5.3 The precise form of \( f(\beta, \omega) \)

In this subsection, we compute \( f(\beta, \omega) \) by trace formula (1.10). In order to make the calculation easier, we need to use some transformation first. By the definition, for \( e = 0 \),

\[
B_{\beta,0}(t) = B_{\beta} = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & -1 & \frac{-1 - \sqrt{9 - \beta}}{2} & 0 \\
1 & 0 & 0 & \frac{-1 + \sqrt{9 - \beta}}{2}
\end{pmatrix}.
\]

For \( \beta \in (0, 9) \setminus \{1\} \), let \( P_{\beta} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \) be the \( 4 \times 4 \) transformation matrices, where

\[
P_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{(2 + 2 \sqrt{1 - \beta})^{1/4}(\sqrt{9 - \beta} - \sqrt{1 - \beta})}{2(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}} \\ \frac{(2 + 2 \sqrt{1 - \beta})^{1/4}(\sqrt{9 - \beta} - \sqrt{1 - \beta})}{2(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}} & 0
\end{pmatrix},
\]

\[
P_{12} = \begin{pmatrix} P_{12}(1, 1) & 0 \\ 0 & P_{12}(2, 2) \end{pmatrix} = \begin{pmatrix} \frac{-2(2 \sqrt{1 - \beta})^{1/4}(\sqrt{9 - \beta} + \sqrt{1 - \beta})}{2(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}} & 0 \\ 0 & \frac{(2 + 2 \sqrt{1 - \beta})^{1/4}(2 + \sqrt{9 - \beta} + \sqrt{1 - \beta})}{2(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}}
\end{pmatrix},
\]

\[
P_{21} = \begin{pmatrix} P_{21}(1, 1) & 0 \\ 0 & P_{21}(2, 2) \end{pmatrix} = \begin{pmatrix} \frac{(2 - 2 \sqrt{1 - \beta})^{1/4}(4 + \sqrt{9 - \beta} + \sqrt{1 - \beta})}{2(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}} & 0 \\ 0 & \frac{-2(2 + 2 \sqrt{1 - \beta})^{1/4}}{(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}}
\end{pmatrix},
\]

and

\[
P_{22} = \begin{pmatrix} 0 & P_{22}(1, 2) \\ P_{22}(2, 1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{-2(2 + 2 \sqrt{1 - \beta})^{1/4}(\sqrt{9 - \beta} + \sqrt{1 - \beta})}{2(1 - \beta)^{1/4} \sqrt{4 + \sqrt{1 - \beta} - \sqrt{9 - \beta}}} & 0
\end{pmatrix}.
\]

In fact, \( P_{\beta} \) is obtained with the help of matlab. Direct computation shows that

\[
P_{\beta}^T J P_{\beta} = J.
\]

For \( \beta \in (0, 1) \), \( P_{\beta} \) is real, thus it is a symplectic matrix, and for \( \beta \in (1, 9) \), \( P_{\beta} \) is complex matrix. To continue, we need the notation of symplectic sum, which was introduced by Long [16] and [18]. Given any two \( 2m_k \times 2m_k \) matrices of square block form \( M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \) with \( k = 1, 2 \), the symplectic sum of \( M_1 \) and \( M_2 \) is defined by

\[
M_1 \odot M_2 = \begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.
\]
Set \( \theta_1(\beta) = -\sqrt{\frac{1}{2}(1 - \sqrt{1 - \beta})} \), and \( \theta_2(\beta) = \sqrt{\frac{1}{2}(1 + \sqrt{1 - \beta})} \). Let

\[
B_j(\beta) = \begin{pmatrix}
0 & -\theta_j(\beta) \\
\theta_j(\beta) & 0
\end{pmatrix}, \text{ for } j = 1, 2,
\]

and set

\[
S_{\beta} = B_1(\beta) \circ B_2(\beta).
\]

Direct computation shows that

\[
P^{-1}_\beta JBP_\beta = JP^T_\beta B_\beta P_\beta = S_{\beta}, \quad \beta \in (0, 1) \cup (1, 9].
\] (5.21)

Obviously

\[
\exp(J_2B_k(\beta)t) = R(\theta_k t) = \begin{pmatrix}
\cos(\theta_k t) & -\sin(\theta_k t) \\
\sin(\theta_k t) & \cos(\theta_k t)
\end{pmatrix}, \quad k = 1, 2,
\]

and hence

\[
P^{-1}_\beta^{-1} \gamma_{\beta,0}(t)P_\beta = R(\theta_1 t) \circ R(\theta_2 t).
\] (5.22)

In order to get the diagonal matrix, we introduce a unitary matrix \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & \sqrt{-1}I_2 \\ I_2 & -\sqrt{-1}I_2 \end{pmatrix} \), then we have

\[
UP^{-1}_\beta^{-1} \gamma_{\beta,0}(t)P_\beta U^{-1} = e^{i\Theta t}.
\] (5.23)

where \( \Theta = \text{diag}(\theta_1, \theta_2, \theta_3, \theta_4) \), \( \theta_3 = -\theta_1, \theta_4 = -\theta_2 \).

Especially

\[
UP^{-1}_\beta^{-1} \gamma_{\beta,0}(2\pi)P_\beta U^{-1} = e^{2\pi i\Theta}.
\] (5.24)

Change the basis by \( P_\beta U^{-1} \), then \( f(\beta, \omega) \) could be computed by (1.10), we have

\[
f(\beta, \omega) = \text{Tr}(K_{\beta}^T(-J \frac{d}{dt} - \nu J - \beta B_{\beta,0})^{-1})^2
\]

\[
= -2\text{Tr}(J \int_0^{2\pi} \gamma_{\beta,0}(t)K_{\beta}^T(t)\gamma_{\beta,0}(t)J \int_0^{2\pi} \gamma_{\beta,0}(s)K_{\beta}^T(s)\gamma_{\beta,0}(s)dsdt \cdot \gamma_{\beta,0}(2\pi)\gamma_{\beta,0}(2\pi) - \omega I_4)^{-1})
\]

\[+ \text{Tr}((J \int_0^{2\pi} \gamma_{\beta,0}(t)K_{\beta}^T(t)\gamma_{\beta,0}(t)dt \cdot \gamma_{\beta,0}(2\pi)\gamma_{\beta,0}(2\pi) - \omega I_4)^{-1})^2)
\]

\[= 2\text{Tr}(J \int_0^{2\pi} \gamma_{\beta,0}(t)\widetilde{D}_{\beta,0}(t)\gamma_{\beta,0}(t)J \int_0^{2\pi} \gamma_{\beta,0}(s)\widetilde{D}_{\beta,0}(s)\gamma_{\beta,0}(s)dsdt \cdot \gamma_{\beta,0}(2\pi)\gamma_{\beta,0}(2\pi) - \omega I_4)^{-1})
\]

\[- \text{Tr}(J \int_0^{2\pi} \gamma_{\beta,0}(t)\widetilde{D}_{\beta,0}(t)\gamma_{\beta,0}(t)dt \cdot \gamma_{\beta,0}(2\pi)\gamma_{\beta,0}(2\pi) - \omega I_4)^{-1})^2)
\]

\[= 2 \int_0^{2\pi} \int_0^{2\pi} \text{Tr}(\gamma_{\beta,0}(s)\gamma_{\beta,0}^T(t) \cdot J\widetilde{D}_{\beta,0}(t) \cdot \gamma_{\beta,0}(t)\gamma_{\beta,0}(s)dsdt \cdot M_0(\omega))
\]

\[- 2 \int_0^{2\pi} \int_0^{2\pi} \text{Tr}(\gamma_{\beta,0}(s)\gamma_{\beta,0}^T(t) \cdot J\widetilde{D}_{\beta,0}(t) \cdot M_0(\omega) \cdot \gamma_{\beta,0}(t)\gamma_{\beta,0}(s)dsdt \cdot M_0(\omega)), (5.25)
\]
where $\tilde{D}_\beta(s) = U^{-T}P_\beta^T K_\beta(s)P_\beta U^{-1}$, $\tilde{\gamma}_\beta(0)(t) = U P_\beta^{-1} \gamma_\beta(0)(t) P_\beta U^{-1} = e^{i0t}$, and $M_\beta(\omega) = \overline{\tilde{\gamma}}_{\beta,0}(2\pi)(\overline{\tilde{\gamma}}_{\beta,0}(2\pi) - \omega I_4)^{-1}$. The last equation from the facts that $J^T \overline{\tilde{\gamma}}_{\beta,0}(t) = \overline{\tilde{\gamma}}_{\beta,0}(-t)J$ and $\overline{\tilde{\gamma}}_{\beta,0}(s)$ commutes with $\overline{\tilde{\gamma}}_{\beta,0}(2\pi)$. In order to get the trace, we need to calculate $J\tilde{D}_\beta(t)$ and $M_\beta(\omega)$. Let $\omega = e^{2\pi iu}$, $u \in \mathbb{R}$, direct calculation shows that

$$M_\beta(\omega) = \begin{cases} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{cases}, \quad k_j = \frac{e^{2\pi i u_j}}{e^{2\pi i u_j} - e^{2\pi i u}},$$

(5.26)

and $P_\beta^T K_\beta P_\beta = \begin{pmatrix} a & 0 & 0 & b \\ 0 & h & f & 0 \\ 0 & f & g & 0 \\ b & 0 & 0 & c \end{pmatrix}$, where $a, b, c, h, f, g$ have explicit expression and depend on parameter $\beta$,

$$\begin{aligned}
\begin{cases}
a = P_{21}(1,1) P_{21}(1,1)d_1 \\
b = P_{21}(1,1) P_{22}(1,2)d_1 \\
c = P_{22}(1,2) P_{22}(1,2)d_1 \\
h = P_{21}(2,2) P_{21}(2,2)d_2 \\
f = P_{21}(2,2) P_{22}(2,1)d_2 \\
g = P_{22}(2,1) P_{22}(2,1)d_2 \\
d_1 = \frac{3 + \sqrt{\beta - 1}}{2} \\
d_2 = \frac{3 - \sqrt{\beta - 1}}{2}
\end{cases}
\end{aligned}$$

(5.27)

Let $\tilde{D}_\beta = U^{-T}P_\beta^T K_\beta P_\beta U^{-1}$, by direct computation, we have

$$J\tilde{D}_\beta = \frac{1}{2} \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix},$$

(5.28)

where

$$\begin{aligned}
D_{11} &= -(a + g), & D_{22} &= -(h + c), & D_{33} &= a + g, & D_{44} &= h + c, \\
D_{12} &= -D_{21} = -i(f - b), & D_{23} &= D_{32} = -i(f + b), & D_{24} &= -D_{42} = c - h, \\
D_{13} &= -D_{31} = g - a, & D_{14} &= D_{41} = -i(f + b), & D_{34} &= -D_{43} = i(b - f).
\end{aligned}$$

(5.29)

Obviously,

$$J\tilde{D}_\beta(s) = \cos^-(s) \cdot J\tilde{D}_\beta.$$

(5.30)

In order to make the computation clearer, we introduce

$$f_1(\beta, \omega) = \int_0^{2\pi} \int_0^\infty Tr[\overline{\tilde{\gamma}}_{\beta,0}(s)\overline{\tilde{\gamma}}_{\beta,0}^T(-t)J\tilde{D}_\beta(t) \cdot \overline{\tilde{\gamma}}_{\beta,0}(t)\overline{\tilde{\gamma}}_{\beta,0}^T(-s)J\tilde{D}_\beta(s) \cdot M_\beta(\omega)]dsdt,$$

(5.31)

and

$$f_2(\beta, \omega) = \int_0^{2\pi} \int_0^\infty Tr[\overline{\tilde{\gamma}}_{\beta,0}(s)\overline{\tilde{\gamma}}_{\beta,0}^T(-t)J\tilde{D}_\beta(t) \cdot M_\beta(\omega) \cdot \overline{\tilde{\gamma}}_{\beta,0}(t)\overline{\tilde{\gamma}}_{\beta,0}^T(-s)J\tilde{D}_\beta(s) \cdot M_\beta(\omega)]dsdt.$$

(5.32)

Therefore

$$f(\beta, \omega) = 2f_1(\beta, \omega) - f_2(\beta, \omega).$$

(5.33)
Direct computation shows that

\[
f_1(\beta, \omega) = \frac{1}{4} \sum_{n=1}^{4} D_{nm} D_{mn} k_n k_m \frac{2e^{\pi(\theta_m - \theta_n)i}}{2[\theta_m - \theta_n]^2 - 1]^{\frac{1}{2}}},
\]

and

\[
f_2(\beta, \omega) = \frac{1}{4} \sum_{n=1}^{4} D_{nm} D_{mn} k_n k_m \frac{2 + e^{\pi(\theta_m - \theta_n)i} + e^{-\pi(\theta_m - \theta_n)i}}{[\theta_m - \theta_n]^2 - 1]^{\frac{1}{2}}}.
\]

where the blocks \( D_{nm} \) are defined by (5.29). Thus \( f_1(\beta, \omega), f_2(\beta, \omega) \) and \( f(\beta, \omega) \) are elementary functions. Based on the precise form of the above functions, we can draw the the curves \( \Gamma_i, i = 1, \ldots, 4 \) in Figure 1 with the help of Matlab.

### 5.4 Hyperbolicity analysis via the first order trace formula

Recall that in (5.19), \( \hat{f}(\beta) \) is defined by taking maximum, and maybe it is not an elementary function. Another way to estimate the hyperbolic region is to use the trace formula for Lagrangian system (1.17). It will be seen that the estimation of the hyperbolic region given by the trace formula (1.10) for Hamiltonian system is sharper than that given by the trace formula (1.17) for Lagrangian system. However, the later is more computable.

From (5.6), for \( \beta \in (1, 9) \), \( \nu \) is imaginary number, \( A(\beta, 0, \nu) > 0 \). Recall that \( \hat{K}_{\beta,0}(t) = \begin{pmatrix} \frac{3+\sqrt{9-\beta}}{2} & 0 \\ 0 & \frac{3-\sqrt{9-\beta}}{2} \end{pmatrix} \),

for \( \omega = e^{2\pi \nu} \in \mathbb{U} \), we define

\[
g(\beta, \nu) = -Tr\left(JK_{\beta,0}(2\pi)(\gamma_{\beta,0}(2\pi) - e^{2\pi \nu I_4})^{-1}\right).
\]

From (1.17) or (3.27),

\[
Tr\left(\frac{e^{\cos(t)}}{1 + e^{\cos(t)}} \hat{K}_{\beta,0}A(\beta, 0, \nu)^{-1}\right) = -Tr\left(J \int_0^{2\pi} \gamma_{\beta,0}(t) \frac{e^{\cos(t)}}{1 + e^{\cos(t)}} K_{\beta,0}(t) \gamma_{\beta,0}(t) dt \cdot \gamma_{\beta,0}(2\pi)(\gamma_{\beta,0}(2\pi) - \omega I_4)^{-1}\right)
\]

\[
= - \int_0^{2\pi} \frac{e^{\cos(t)}}{1 + e^{\cos(t)}} dt \cdot Tr\left(JK_{\beta,0}(2\pi)(\gamma_{\beta,0}(2\pi) - \omega I_4)^{-1}\right)
\]

\[
= \left(\pi - \frac{4}{\sqrt{1 - e^2}} \tan^{-1} \sqrt{\frac{1 - e}{1 + e}}\right) g(\beta, \nu),
\]

where the second equality is from the fact that \( \gamma_{\beta,0}(t) = \exp(JB_{\beta,0}t) \) commutes with \( \gamma_{\beta,0}(2\pi) \), and the third equality is from

\[
\int_{-\pi/2}^{\pi/2} \frac{e^{\cos(s)}}{1 + e^{\cos(s)}} ds = \pi - \frac{4}{\sqrt{1 - e^2}} \tan^{-1} \sqrt{\frac{1 - e}{1 + e}}.
\]

Noting that \( \pi - \frac{4}{\sqrt{1 - e^2}} \tan^{-1} \sqrt{\frac{1 - e}{1 + e}} \geq 0 \) for \( e \in [0, 1) \), and setting

\[
\hat{g}(\beta) = \sup\{g(\beta, \nu), \nu \in \sqrt{-1}\mathbb{R}\}.
\]

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In order to calculate $g(\beta, \nu)$, we change the basis by $P_{\beta}U^{-1}$, then

$$g(\beta, \nu) = Tr(iJD_\beta M_\beta(\omega)).$$

(5.39)

From (5.28), direct computation shows that

Lemma 5.7. For $\beta \in (1, 9]$ and $\nu = \sqrt{-1} u \in \sqrt{-1}\mathbb{R}$,

$$g(\beta, \nu) = 2\operatorname{Re} \left(\frac{\sqrt{2}(-3 - \beta + 3\sqrt{1 - \beta})}{4\sqrt{1 - \beta}\sqrt{-1 + \sqrt{1 - \beta}2\cos(2\pi u) - e^{-\sqrt{2}\pi\sqrt{-1 + \sqrt{1 - \beta}}}e^{\sqrt{2}\pi\sqrt{1 + \sqrt{1 - \beta}}}}\right).$$

(5.40)

Similar to the proof of Theorem 5.6, we have the following theorem.

Theorem 5.8. For $\beta \in (1, 9]$, $\gamma_{\beta, e}$ is hyperbolic if

$$\pi - \frac{4}{\sqrt{1 - e^2}} \tan^{-1} \sqrt{\frac{1 - e}{1 + e}} < 1/\hat{g}(\beta).$$

(5.41)

We can use Theorem 5.8 or Theorem 5.6 to estimate the hyperbolic region. Next, we draw the following figure to compare the hyperbolic regions given by the two theorems respectively.

![Figure 2: The hyperbolic region given by Theorem 5.8 and Theorem 5.6](image)

In Figure 2, the points $O_3 \approx (9, 0.4907)$, $O_4 \approx (9, 0.2800)$. The curves

$$\Gamma_4 = \{(\beta, e) \mid e = \hat{\gamma}(\beta)^{-1/2}, 1 \leq \beta \leq 9\}, \quad \Gamma_5 = \{(\beta, e) \mid \pi - \frac{4}{\sqrt{1 - e^2}} \tan^{-1} \sqrt{\frac{1 - e}{1 + e}} = \hat{g}(\beta)^{-1}, 1 \leq \beta \leq 9\},$$

where $\Gamma_4$ is given by theorem 5.6, which is obtained by (1.11), and $\Gamma_5$ is given by theorem 5.8 which is obtained by using (1.17).

Since $\hat{g}(\beta)$ is not easy to be computed, we will control $\hat{g}(\beta)$ by some elementary function.
Lemma 5.9. For $\beta \in (1, 9]$,

$$\hat{g}(\beta) \leq \frac{\beta^{\frac{1}{2}}}{2(\beta - 1)} \frac{\sqrt{\beta + 15}}{2} \frac{e^{-2 \sqrt{2\pi} \hat{\gamma}} + e^{2 \sqrt{2\pi} \hat{\gamma}} - 2 \cos(2 \sqrt{2\pi} \hat{d})}{|\langle e^{-\sqrt{2\pi} \hat{\gamma}} - e^{\sqrt{2\pi} \hat{\gamma}} \rangle \sin(\sqrt{2\pi} \hat{d})|},$$

where $\hat{c} = \text{Re}(\sqrt{-1 + 1 - \beta})$, $\hat{d} = \text{Im}(\sqrt{-1 + 1 - \beta})$.

Proof. Let $\hat{c} = \text{Re}(\sqrt{-1 + 1 - \beta})$, $\hat{d} = \text{Im}(\sqrt{-1 + 1 - \beta})$, direct computation shows that $|\sqrt{-1 + 1 - \beta}| = \beta^{\frac{1}{2}}$ for $\beta \in (1, 9]$, applying Lemma 5.7, we have

$$g(\beta, \nu) = 2 \text{Re} \left( \frac{\sqrt{2}(3 - \beta + 3 \sqrt{1 - \beta})}{4 \sqrt{1 - \beta} \sqrt{1 + 1 - \beta}} - \frac{2 \cos(2\pi u)}{2 \sqrt{1 + 1 - \beta}} \right)$$

$$\leq \left| \left( \frac{\sqrt{2}(3 - \beta + 3 \sqrt{1 - \beta})}{4 \sqrt{1 - \beta} \sqrt{1 + 1 - \beta}} - \frac{2 \cos(2\pi u)}{2 \sqrt{1 + 1 - \beta}} \right) \right|$$

$$= \frac{\beta^{\frac{1}{2}}}{2(\beta - 1)} \frac{\sqrt{\beta + 15}}{2} \frac{e^{-2 \sqrt{2\pi} \hat{\gamma}} + e^{2 \sqrt{2\pi} \hat{\gamma}} - 2 \cos(2 \sqrt{2\pi} \hat{d})}{|\langle e^{-\sqrt{2\pi} \hat{\gamma}} - e^{\sqrt{2\pi} \hat{\gamma}} \rangle \sin(\sqrt{2\pi} \hat{d})|}.$$
**Corollary 5.11.** For $\beta \in [\beta_0, 9]$, $\gamma_{\beta,e}$ is hyperbolic if

$$\pi - \frac{4}{\sqrt{1-e^2}} \tan^{-1}\sqrt{1-e^2} < 0.3154.$$  \hfill (5.44)

That is, $\gamma_{\beta,e}$ is hyperbolic if $(\beta, e) \in [3.0334, 9] \times [0, 0.1797]$.

By using Corollary 5.10 and 5.11, we can draw a picture of the hyperbolic region as follows.

Figure 3: The hyperbolic region given by Cor. 5.10

Figure 4: The hyperbolic region given by Cor. 5.11

**Remark 5.12.** From the proof of Theorems 5.3 and Theorem 5.4, $\gamma_{\beta,e}$ is $-1$-nondegenerate if $(\beta, e)$ belongs to the set $\{(\beta, e) \mid 0 \leq e < 1/(1 + \sqrt{f(\beta, -1)}), 0 \leq \beta < 3/4 \}$ or $\{(\beta, e) \mid 0 \leq e < 1/\sqrt{f(\beta, -1)}, 3/4 < \beta \leq 9 \}$. However, using (1.17), we get that $\gamma_{\beta,e}$ is $-1$-nondegenerate if $(\beta, e)$ belongs to the set $\{(\beta, e) \mid \pi - \frac{4}{\sqrt{1-e^2}} \tan^{-1}\sqrt{1-e^2} < 1/g(\beta, \sqrt{-1}), 3/4 < \beta \leq 9 \}$.

Figure 5: The $-1$-nondegenerate region given by Remark 5.12.

In Figure 5, the points $O_5 \approx (9, 0.5309)$, $O_6 \approx (9, 0.2961)$. The curves

$$\Gamma_6 = \{(\beta, e) \mid e = f(\beta, -1)^{-1/2}, 1 \leq \beta \leq 9 \},$$
\[ \Gamma_7 = \left\{ (\beta, e) \left| \pi - \frac{4}{\sqrt{1 - e^2}} \tan^{-1} \frac{1 - e}{1 + e} = g\left(\beta, \frac{\sqrt{1 - e^2}}{2}\right)^{-1}, 1 \leq \beta \leq 9 \right. \right\}. \]

The same reasoning as above implies that we can estimate the non-degenerate region by the trace formulas in Theorem 1.1 for \( k \). As \( k \) is larger, the estimation of the non-degenerate region is sharper, however, the trace formula is more complex and less computable.

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