This paper develops a discontinuous Galerkin (DG) finite element differential calculus theory for approximating weak derivatives of Sobolev functions and piecewise Sobolev functions. By introducing numerical one-sided derivatives as building blocks, various first and second order numerical operators such as the gradient, divergence, Hessian, and Laplacian operator are defined, and their corresponding calculus rules are established. Among the calculus rules are product and chain rules, integration by parts formulas and the divergence theorem. Approximation properties and the relationship between the proposed DG finite element numerical derivatives and some well-known finite difference numerical derivative formulas on Cartesian grids are also established. Efficient implementation of the DG finite element numerical differential operators is also proposed. Besides independent interest in numerical differentiation, the primary motivation and goal of developing the DG finite element differential calculus is to solve partial differential equations. It is shown that several existing finite element, finite difference and DG methods can be rewritten compactly using the proposed DG finite element differential calculus framework. Moreover, new DG methods for linear and nonlinear PDEs are also obtained from the framework.

1. Introduction

Numerical differentiation is an old but basic topic in numerical mathematics. Compared to the large amount of literature on numerical integration, numerical differentiation is a much less studied topic. Given a differentiable function, the available numerical methods for computing its derivatives are indeed very limited. There are essentially only two such methods (cf. [31]). One method is to approximate derivatives by difference quotients. The other is to first approximate the given function (or its values at a set of points) by a more simple function (e.g., polynomial, rational function and piecewise polynomial) and then to use the derivative of the approximate function as an approximation to the sought-after derivative. The two types of classical methods work well if the given function is sufficiently smooth. However, the two classical methods produce large errors or divergent approximations if the given function is rough, which is often the case when the function is a solution of a linear or nonlinear partial differential equation (PDE).

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For boundary value and initial-boundary value problems, classical solutions often do not exist. Consequently, one has to deal with generalized or weak solutions, which are defined using a variational setting for linear and quasilinear PDEs. Although numerical methods for PDEs implicitly give rise to methods for approximating weak derivatives (in fact, combinations of weak derivatives) of the solution functions (cf. [4, 6, 15, 32]), to the best of our knowledge, there is no systematic study and theory in the literature on how to approximate weak derivatives of a given (not-so-smooth) function. Moreover, for linear second order PDEs of non-divergence form and fully nonlinear PDEs, it is not possible to derive variational weak formulations using integration by parts. As a result, weak solution concepts for those types of PDEs are different. The best known and most successful one is the \textit{viscosity solution} concept (cf. [13, 16] and the references therein). To directly approximate viscosity solutions, which in general are only continuous functions, one must approximate their derivatives in some appropriately defined sense offline (cf. [17, 24]), and then substitute the numerical derivatives for the (formal) derivatives appearing in the PDEs. Clearly, to make such an intuitive approach work, the key is to construct “correct” numerical derivatives and to use them judiciously to build numerical schemes.

This paper addresses the above two fundamental issues. The specific goals of this paper are twofold. \textit{First}, we systematically develop a computational framework for approximating weak derivatives and a new discontinuous Galerkin (DG) finite element differential calculus theory. Keeping in mind the approximation of fully nonlinear PDEs, we introduce locally defined, one-sided numerical derivatives for piecewise weakly differentiable functions. Using the newly defined one-sided numerical derivatives as building blocks, we then define a host of first and second order sided numerical differential operators including the gradient, divergence, curl, Hessian, and Laplace operators. To ensure the usefulness and consistency of these numerical operators, we establish basic calculus rules for them. Among the rules are the product and the chain rule, integration by parts formulas and the divergence theorem. We establish some approximation properties of the proposed DG finite element numerical derivatives and show that they coincide with well-known finite difference derivative formulas on Cartesian grids. Consequently, our DG finite element numerical derivatives are natural generalizations of well-known finite difference numerical derivatives on general meshes. These results are of independent interest in numerical differentiation. \textit{Second}, we present some applications of the proposed DG finite element differential calculus to build numerical methods for linear and nonlinear partial differential equations. This is done based on a very simple idea; that is, we replace the (formal) differential operators in the given PDE by their corresponding DG finite element numerical operators and project (in the $L^2$ sense) the resulting equation onto the DG finite element space $V_h$. We show that the resulting numerical methods not only recover several existing finite difference, finite element and DG methods, but also give rise to some new numerical schemes for both linear and nonlinear PDE problems.

The remainder of this paper is organized as follows. In Section 2 we introduce the mesh and space notation used throughout the paper. In Section 3 we give the definitions of our DG finite element numerical derivatives and various first and second order numerical differential operators. In Section 4 we establish an approximation property and various calculus rules for the DG finite element numerical
derivatives and operators. In Section 5 we discuss the implementation aspects of the numerical derivatives and operators. Finally, in Section 6 we present several applications of the proposed DG finite element differential calculus to numerical solutions of prototypical linear and nonlinear PDEs including the Poisson equation, the biharmonic equation, the $p$-Laplace equation, second order linear elliptic PDEs in non-divergence form, first order fully nonlinear Hamilton-Jacobi equations, and second order fully nonlinear Monge-Ampère equations.

2. Preliminaries

Let $d$ be a positive integer, $\Omega \subset \mathbb{R}^d$ be a bounded open domain, and $\mathcal{T}_h$ denote a locally quasi-uniform and shape-regular partition of $\Omega$. Let $\mathcal{E}_h^I$ denote the set of all interior faces/edges of $\mathcal{T}_h$, $\mathcal{E}_h^B$ denote the set of all boundary faces/edges of $\mathcal{T}_h$, and $\mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^B$.

Let $p \in [1, \infty]$ and $m \geq 0$ be an integer. Define the following piecewise $W^{m,p}$ and piecewise $C^m$ spaces with respect to the mesh $\mathcal{T}_h$:

$$W^{m,p}(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} W^{m,p}(K), \quad C^m(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} C^m(K).$$

When $p = 2$, we set $H^m(\mathcal{T}_h) := W^{m,2}(\mathcal{T}_h)$. We also define the analogous piecewise vector-valued spaces as $H^m(\mathcal{T}_h) := [H^m(\mathcal{T}_h)]^d$, $W^{m,p}(\mathcal{T}_h) = [W^{m,p}(\mathcal{T}_h)]^d$, $C^m(\mathcal{T}_h) = [C^m(\mathcal{T}_h)]^d$, and the matrix-valued spaces $\tilde{H}^m(\mathcal{T}_h) := [H^m(\mathcal{T}_h)]^{d \times d}$, $\tilde{W}^{m,p}(\mathcal{T}_h) := [W^{m,p}(\mathcal{T}_h)]^{d \times d}$, and $\tilde{C}^m(\mathcal{T}_h) = [C^m(\mathcal{T}_h)]^{d \times d}$. The piecewise $L^2$-inner product over the mesh $\mathcal{T}_h$ is given by

$$(v, w)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \int_K vw \, dx,$$

and for a set $\mathcal{S}_h \subset \mathcal{E}_h$, the piecewise $L^2$-inner product over $\mathcal{S}_h$ is given by

$$(v, w)_{\mathcal{S}_h} := \sum_{e \in \mathcal{S}_h} \int_e vw \, ds.$$

Angled brackets without subscripts $\langle \cdot, \cdot \rangle$ represent the dual pairing between some Banach space and its dual.

For a fixed integer $r \geq 0$, we define the standard discontinuous Galerkin (DG) finite element space $V^h_r \subset W^{m,p}(\mathcal{T}_h) \subset L^2(\Omega)$ by

$$V^h_r := \prod_{K \in \mathcal{T}_h} \mathbb{P}_r(K),$$

where $\mathbb{P}_r(K)$ denotes the set of all polynomials on $K$ with degree not exceeding $r$. The analogous vector-valued and matrix valued DG spaces are given by $V^h_r := [V^h_r]^d$ and $\tilde{V}^h_r := [V^h_r]^{d \times d}$. In addition, we define

$$V^h := W^{1,1}(\mathcal{T}_h) \cap C^0(\mathcal{T}_h), \quad V : L^2(\Omega) \to V^h$$

and $\tilde{V}^h := [V^h]^d$, $\tilde{V} : L^2(\Omega) \to \tilde{V}^h$. We note that $V^h_r \subset V^h$. We denote by $\mathcal{P}^h_r : L^2(\Omega) \to V^h_r$ the $L^2$ projection operator onto $V^h_r$, $\mathcal{P}^h_r : [L^2(\Omega)]^d \to V^h_r$ the $L^2$ projection operator onto $V^h_r$ and $\mathcal{F}^h_r : [L^2(\Omega)]^{d \times d} \to \tilde{V}^h$ the $L^2$ projection operator onto $\tilde{V}^h_r$. 


Let $K, K' \in \mathcal{T}_h$ and $e = \partial K \cap \partial K'$. Without loss of generality, we assume that the global labeling number of $K$ is smaller than that of $K'$. We then introduce the following standard jump and average notations across the face/edge $e$:

\[
[v] := v|_K - v|_{K'}, \quad \{v\} := v \quad \text{on } e \in \mathcal{E}^I_h, \quad [v] := v \quad \text{on } e \in \mathcal{E}^B_h,
\]

\[
\{v\} := \frac{1}{2}(v|_K + v|_{K'}) \quad \text{on } e \in \mathcal{E}^I_h, \quad \{v\} := v \quad \text{on } e \in \mathcal{E}^B_h
\]

for $v \in \mathcal{V}_h$. We also define $n_e := n_K|_e = -n_{K'}|_e$ as the unit normal on $e$.

### 3. Definitions of discrete differential operators

Let $v \in \mathcal{V}_h$. For $e \in \mathcal{E}^I_h$, that is, $e = \partial K \cap \partial K' \in \mathcal{E}^I_h$ for some $K, K' \in \mathcal{T}_h$, we write $n_e = (n_e^{(1)}, n_e^{(2)}, \ldots, n_e^{(d)})^t$ to be the unit normal of $e$. We then define the following three trace operators on $e$ in the direction $x_i$:

\[
Q_i^-(v)(x) := \begin{cases} 
\lim_{y \to x} v(y) & \text{if } n_e^{(i)} < 0, \\
\lim_{y \to x} v(y) & \text{if } n_e^{(i)} \geq 0,
\end{cases}
\]

\[
Q_i^+(v)(x) := \begin{cases} 
\lim_{y \to x} v(y) & \text{if } n_e^{(i)} < 0, \\
\lim_{y \to x} v(y) & \text{if } n_e^{(i)} \geq 0,
\end{cases}
\]

\[
Q_i(v)(x) := \frac{1}{2}(Q_i^-(v)(x) + Q_i^+(v)(x))
\]

for any $x \in e$ and $i = 1, 2, \ldots, d$. We note that $Q_i^-$ and $Q_i^+$ can be regarded respectively as the “left” and “right” limit of $v$ at $x \in e$ in the direction of $x_i$. If $e \in \mathcal{E}^B_h$, we simply let

\[
Q_i^-(v)(x) = Q_i^+(v)(x) = Q_i(v)(x) := \lim_{y \to x} v(y) \quad \forall x \in e.
\]

**Remark 3.1.** On an interior edge $e \in \mathcal{E}^I_h$, we may alternatively write

\[
Q_i^\pm(v) = \{v\} \pm \frac{1}{2} \text{sgn}(n_e^{(i)})[v], \quad \text{where} \quad \text{sgn}(n_e^{(i)}) = \begin{cases} 
1 & \text{if } n_e^{(i)} \geq 0, \\
-1 & \text{if } n_e^{(i)} < 0.
\end{cases}
\]

With the help of the trace operators $Q_i^-, Q_i^+$ and $Q_i$, we are ready to introduce our discrete partial derivative operators $\partial_{h,x_i}, \partial_{h,x_i}^+, \partial_{h,x_i}^- : \mathcal{V}_h \to \mathcal{V}_h^r$.

**Definition 3.1.** For any $v \in \mathcal{V}_h$, we define the discrete partial derivatives $\partial_{h,x_i} v, \partial_{h,x_i}^+ v, \partial_{h,x_i}^- v \in \mathcal{V}_h^r$ by

\[
(\partial_{h,x_i}^\pm v, \varphi_h)_{\mathcal{T}_h} := \langle Q_{i}^\pm(v)n^{(i)}, [\varphi_h]\rangle_{\mathcal{E}_h} - \langle v, \partial_{x_i}\varphi_h\rangle_{\mathcal{T}_h} + \langle \gamma_i^\pm[v], [\varphi_h]\rangle_{\mathcal{E}_h^I} \quad \forall \varphi_h \in \mathcal{V}_h^r,
\]

\[
\partial_{h,x_i} v := \frac{1}{2}(\partial_{h,x_i}^+ v + \partial_{h,x_i}^- v)
\]

for $i = 1, 2, \ldots, d$. Here, $\partial_{x_i}$ denotes the usual (weak) partial derivative operator in the direction $x_i$, $n^{(i)}$ is the piecewise constant function satisfying $n^{(i)}|_e = n_e^{(i)}$ and $\gamma_i^-$ and $\gamma_i^+$ are piecewise constants with respect to the set of interior edges.
In addition, we define the discrete partial derivatives when boundary data is provided.

**Definition 3.2.** Let \( g \in L^1(\partial \Omega) \) be given. Then for any \( v \in \mathcal{V}_h \), we define the discrete partial derivatives \( \partial_{h,x_i}^- v, \partial_{h,x_i}^+ v, \partial_{h,x_i}^0 v \in V^h \) by

\[
(\partial_{h,x_i}^\pm v, \varphi_h)_{\mathcal{T}_h} := (\partial_{h,x_i}^\pm v, \varphi_h)_{\Omega} + \langle (g - v)n^i, \varphi_h \rangle_{\partial \Omega} \quad \forall \varphi_h \in V^h,
\]

\[
\partial_{h,x_i}^0 v := \frac{1}{2} \left( \partial_{h,x_i}^- v + \partial_{h,x_i}^+ v \right).
\]

**Remarks 3.1.**

(a) Since every function \( v \in \mathcal{V}_h \) has a well-defined trace in \( L^1(\partial K) \) and every function \( \varphi_h \in V^h \) has a well-defined trace in \( L^\infty(\partial K) \) for all \( K \in \mathcal{T}_h \), the last term on the right-hand side of (3.6) is well defined.

(b) Since \( V^h \) is a totally discontinuous piecewise polynomial space, the discrete derivatives \( \partial_{h,x_i}^\pm v \) can also be written in their equivalent local versions:

\[
(\partial_{h,x_i}^\pm v, \varphi_h)_K = (\mathcal{Q}_i^\pm(v)n^i_K, \varphi_h)_{\partial K} - (v, \partial_{x_i} \varphi_h)_K + \sum_{i \in E_r(K)} \gamma_{i,e}^{\pm} [-[v], [\varphi_h]]_e \quad \forall \varphi_h \in \mathcal{P}_r(K)
\]

for \( i = 1, 2, \ldots, d \) and \( K \in \mathcal{T}_h \). Here, \( \gamma_{i,e}^{\pm} = \gamma_{i,e}^{\pm} \).

(c) The discrete derivatives \( \partial_{h,x_i}^- v, \partial_{h,x_i}^+ v \) and \( \partial_{h,x_i}^0 v \) can be regarded, respectively, as “left”, “right”, and “central” discrete partial derivatives of \( v \) with respect to \( x_i \). The definitions are analogous to the weak derivative definition.

(d) We note that the discrete one-sided partial derivatives are defined for functions in the space \( \mathcal{V}_h \), in particular, for functions in the DG finite element space \( V^h \subset \mathcal{V}_h \).

(e) By the identity (3.5), we have

\[ (\partial_{h,x_i}^\pm v, \varphi_h)_{\mathcal{T}_h} = \langle [v]n^i, [\varphi_h] \rangle_{\mathcal{E}_h} + \langle (\gamma_{i,e}^{\pm} \pm \frac{1}{2} |n^i|) [v], [\varphi_h] \rangle_{\partial \mathcal{E}_h} - (v, \partial_{x_i} \varphi_h)_{\mathcal{T}_h}. \]

Therefore, \( \partial_{h,x_i}^+ v = \partial_{h,x_i}^- v \) provided \( \gamma_{i,e}^{+} - \gamma_{i,e}^{-} = -|n^i| \) for all \( e \in \mathcal{E}_h \).

**Definition 3.3.** We define the following first order discrete operators:

\[
\nabla_h^+ v := (\partial_{h,x_1}^+ v, \partial_{h,x_2}^+ v, \ldots, \partial_{h,x_d}^+ v)^t,
\]

\[
\nabla_h^- v := \frac{1}{2} \left( \nabla_h^- v + \nabla_h^+ v \right),
\]

\[
\text{div}_h^+ v := \partial_{h,x_1}^+ v_1 + \partial_{h,x_2}^+ v_2 + \cdots + \partial_{h,x_d}^+ v_d,
\]

\[
\text{div}_h^- v := \frac{1}{2} (\text{div}_h^- v + \text{div}_h^+ v)
\]

for any \( v \in \mathcal{V}_h \) and \( v = (v_1, v_2, \ldots, v_d)^t \in \mathcal{V}_h \). In addition, we define the discrete curl operators for \( v \in \mathcal{V}_h \) and \( v \in \mathcal{V}_h \):

\[
\text{curl}_h^+ v := (\partial_{h,x_2}^+ v_3 - \partial_{h,x_3}^+ v_2, \partial_{h,x_3}^+ v_1 - \partial_{h,x_1}^+ v_3, \partial_{h,x_1}^+ v_2 - \partial_{h,x_2}^+ v_1)^t
\]

when \( d = 3 \), and

\[
\text{curl}_h^- v := \partial_{h,x_1}^- v_2 - \partial_{h,x_2}^- v_1, \quad \text{curl}_h^0 v := \left( \partial_{h,x_2}^- v_1, -\partial_{h,x_1}^- v_2 \right)^t
\]
when \(d = 2\). We also set
\[
\text{curl}_h v := \frac{1}{2} \left( \text{curl}_h v + \text{curl}_h^t v \right), \quad \text{curl}_h v = \frac{1}{2} \left( \text{curl}_h v + \text{curl}_h^t v \right).
\]
When boundary data \(g \in L^1(\partial \Omega)\) is provided, the analogous operators are given by
\[
\begin{align*}
\nabla_{h,g}^\pm v := & \left( \partial_{h,x_1}^{\pm,g} v_1, \partial_{h,x_2}^{\pm,g} v_2, \cdots, \partial_{h,x_d}^{\pm,g} v_d \right)^t, \\
\nabla_{h,g} v := & \frac{1}{2} \left( \nabla_{h}^- g v + \nabla_{h}^+ g v \right).
\end{align*}
\]
In addition, for given \(g = (g_1, g_2, \ldots, g_d)^t \in L^1(\partial \Omega)\), we set
\[
\begin{align*}
\text{div}_{h,g}^\pm v := & \partial_{h,x_1}^\pm g v_1 + \partial_{h,x_2}^\pm g v_2 + \cdots + \partial_{h,x_d}^\pm g v_d, \\
\text{div}_{h,g} v := & \frac{1}{2} \left( \text{div}_{h}^- g v + \text{div}_{h}^+ g v \right).
\end{align*}
\]

**Definition 3.4.** Denote by \(D_h^\pm\) and \(D_h\) the transposes of the operators \(\nabla_h^\pm\) and \(\nabla_h\), respectively; that is, \(D_h^\pm v = (\nabla_h^\pm v)^t\) and \(D_h v = (\nabla_h v)^t\). We then define the following second order discrete operators:
\[
\begin{align*}
D_{h}^\pm v := & \nabla_h^\pm v, \\
\Delta_{h}^\pm v := & \text{tr} \left( D_{h}^\pm v \right), \\
D_{h}^2 v := & \nabla_h v, \\
\Delta_{h} v := & \text{tr} \left( D_{h}^2 v \right)
\end{align*}
\]
for any \(v \in \mathcal{V}_h\). Here, \(\text{tr}(\cdot)\) denotes the matrix trace operator. When boundary data \(g \in L^1(\partial \Omega)\) is given, we define
\[
\begin{align*}
D_{h,g}^\pm v := & \nabla_{h,g}^\pm v, \\
\Delta_{h,g}^\pm v := & \text{tr} \left( D_{h,g}^\pm v \right), \\
D_{h,g}^2 v := & \nabla_{h,g} v, \\
\Delta_{h,g} v := & \text{tr} \left( D_{h,g}^2 v \right).
\end{align*}
\]

**Remarks 3.2.**
\begin{itemize}
\item[(a)] The matrix-valued functions \(D_{h}^- v, D_{h}^+ v, D_{h}^- v\) and \(D_{h}^+ v\) define four copies of discrete \(d \times d\) Hessian matrices of the function \(v\). Similarly, \(\Delta_{h}^- v, \Delta_{h}^+ v, \Delta_{h}^- v\) and \(\Delta_{h}^+ v\) are copies of discrete Laplacians of \(v\).
\item[(b)] Since \(\nabla_h v, \nabla_h^t v \in V_h^p \subset V_h\), all the above second order discrete operators are well-defined.
\item[(c)] It is simple to see that the definitions of the discrete Laplacians given in Definition 3.3 are equivalent to \(\Delta_{h}^\pm v = \text{div}_{h}^\pm \nabla_h^\pm v, \Delta_{h} v = \text{div}_{h} \nabla_h v\) and \(\Delta_{h} v = \text{div}_{h} \nabla_h v\).
\end{itemize}

### 4. Properties of discrete differential operators

In this section we shall establish a number of properties for the DG finite element (FE) discrete derivatives defined in Definitions 3.1 and 3.3. We start with some characterization results for the DG FE discrete derivatives. We then establish an approximation property followed by several basic calculus rules such as product and chain rules, the integration by parts formula, the divergence theorem and the relationship of the DG FE discrete derivatives with standard finite difference derivative formulas.
4.1. Characterization of DG finite element derivatives. For any \( v \in H^1(\Omega) \), we have \( Q^r(v)|_e = v|_e \) and \( [v] = 0 \) on \( e \in E_h^i \). Hence,

\[
(\partial^\pm_{h,x_i} v, \varphi_h)_{T_h} = \langle v n^{(i)}, [\varphi_h] \rangle_{E_h} - (v, \partial_i \varphi_h)_{T_h} \quad \forall \varphi_h \in V^h.
\]

Integration by parts element-wise immediately yields

\[
(\partial^\pm_{h,x_i} v, \varphi_h)_{T_h} = (\partial_{x_i} v, \varphi_h)_{T_h} \quad \forall \varphi_h \in V^h,
\]

and therefore

\[
(\partial^\pm_{h,x_i} v, \varphi_h)_K = (\partial_{x_i} v, \varphi_h)_K \quad \forall \varphi_h \in P_r(K), \forall K \in T_h.
\]

Hence, we have the following proposition.

**Proposition 4.1.** For any \( v \in \mathcal{V}_h \cap H^1(\Omega) \), \( \partial^\pm_{h,x_i} v \) and \( \partial_{h,x_i} v \) coincide with the \( L^2 \)-projection of \( \partial_{x_i} v \) onto \( V^h \). We write \( \partial^\pm_{h,x_i} v = \partial_{h,x_i} v = \mathcal{P}_r^h \partial_{x_i} v \), where \( \mathcal{P}_r^h \) denotes the \( L^2 \) projection onto \( V^h_r \).

From the above proposition, we easily derive the following corollary.

**Corollary 4.1.** Let \( v_h \in V^h \cap H^1(\Omega) \) with \( 0 \leq \ell \leq r+1 \). Then \( \partial^-_{h,x_i} v_h = \partial^+_{h,x_i} v_h = \partial_{h,x_i} v_h = \partial_{x_i} v_h \in V^h_{\ell-1}, \) where \( V^h_{\ell+1} := \{0\} \), the set with only the zero function.

For an arbitrary piecewise function \( v \in \mathcal{V}_h \), the above characterization results are not expected to hold in general because \( \partial_{x_i} v \) may not exist. Below we shall derive a similar characterization to (4.1) for \( \partial^\pm_{h,x_i} v \) and \( \partial_{h,x_i} v \) when \( v \) is an arbitrary function in \( \mathcal{V}_h \).

For any \( v \in \mathcal{V}_h \), let \( \mathcal{D}_{x_i} v \) denote the distributional derivative of \( v \) with respect to \( x_i \) (which does exist). Let \( \Xi \subset \Omega \) be a \((d-1)\)-dimensional continuous and bounded surface. We define the **delta function** \( \delta(\Xi, g, x) \) of variable strength supported on \( \Xi \) by (cf. [14])

\[
\langle \delta(\Xi, g, \cdot), \varphi \rangle := \int_\Xi g(s) \varphi(x(s)) \, ds \quad \forall \varphi \in C^0(\Omega),
\]

where \( x(s) \in \Xi \). We also extend the above definition to test functions from \( V^h_{\ell} \) as follows

\[
\langle \delta(\Xi, g, \cdot), \varphi_h \rangle := \int_\Xi g(s) \{ \varphi_h(x(s)) \} \, ds \quad \forall \varphi_h \in V^h_r.
\]

Using \( \delta(\Xi, g, x) \) we give the following characterization for \( \mathcal{D}_{x_i} v \).

**Lemma 4.1.** For every \( v \in \mathcal{V}_h \) there holds the following representation formula:

\[
\mathcal{D}_{x_i} v = \sum_{K \in T_h} \partial_{x_i} v \chi_K - \sum_{e \in E_h^i} n^{(i)}_e \delta(e, [v], x) \quad \text{for a.e. } x \in \Omega,
\]

where \( \chi_K \) denotes the characteristic function supported on \( K \).
Proof. By the definition of $\mathcal{D}_x v$ and integration by parts on each $K \in \mathcal{T}_h$, we have for any $\varphi \in C^1(\Omega) \cap H^1_0(\Omega)$

$$\langle \mathcal{D}_x v, \varphi \rangle := -\langle v, \partial_x \varphi \rangle = -\langle v, \partial_x \varphi \rangle_{\mathcal{T}_h}$$

$$= -\sum_{K \in \mathcal{T}_h} \langle v, \varphi n^{(i)}_K \rangle_{\partial K} + \langle \partial_x v, \varphi \rangle_{\mathcal{T}_h}$$

$$= -\sum_{e \in E_h^i} \langle [v], \varphi n^{(i)}_e \rangle_e + \langle \partial_x v, \varphi \rangle_{\mathcal{T}_h}$$

$$= -\sum_{e \in E_h^i} \langle n^{(i)}_e \delta(e, [v], \cdot), \varphi \rangle + \sum_{K \in \mathcal{T}_h} \langle \partial_x v \chi_K, \varphi \rangle.$$

Here we have used the fact that $v \in W^{1,1}(\Omega)$ for every $K \in \mathcal{T}_h$. Clearly the above identity infer (4.5). The proof is complete. $\square$

**Proposition 4.2.** Set $\gamma^\pm = 0$ in (3.6). For any $v \in \mathcal{V}_h$, $\partial_{h,x} v$ coincides with the $L^2$-projection of $\mathcal{D}_x v$ onto $V^h_r$ in the sense that

$$\langle \partial_{h,x} v, \varphi_h \rangle_{\mathcal{T}_h} = \langle \mathcal{D}_x v, \varphi_h \rangle \quad \forall \varphi_h \in V^h_r,$$

where the right hand-side is understood according to (4.4). We write $\partial_{h,x} v = \mathcal{P}^h_r \mathcal{D}_x v$.

Proof. By the definition of $\partial_{h,x} v$ and using the integration by parts formula for piecewise functions, we get, for $i = 1, 2, \ldots, d$,

$$\langle \partial_{h,x} v, \varphi_h \rangle_{\mathcal{T}_h} = \langle \{ v \} n^{(i)}, \{ \varphi_h \} \rangle_{\mathcal{E}_h} - \langle v, \partial_x \varphi_h \rangle_{\mathcal{T}_h}$$

$$= -\langle \{ v \} n^{(i)}, \{ \varphi_h \} \rangle_{\mathcal{E}^i_h} + \langle \partial_x v, \varphi_h \rangle_{\mathcal{T}_h}$$

$$= -\sum_{e \in E_h^i} \langle n^{(i)}_e \delta(e, [v], \cdot), \varphi_h \rangle + \sum_{K \in \mathcal{T}_h} \langle \partial_x v \chi_K, \varphi_h \rangle$$

$$= \langle \mathcal{D}_x v, \varphi_h \rangle \quad \forall \varphi_h \in V^h_r.$$

Here we have used (4.4) and (4.5) to get the final two equalities. The proof is complete. $\square$

**Remark 4.1.** Define the “lifting operator” $\mathcal{L}_{h,i} : L^1(\mathcal{E}_h) \rightarrow V^h_r$ by

$$\langle \mathcal{L}_{h,i} v, \varphi_h \rangle_{\mathcal{T}_h} := \langle \sum_{e \in E^i_h} n^{(i)}_e \delta(e, [v], \cdot), \varphi_h \rangle = \langle \{ v \} n^{(i)}, \{ \varphi_h \} \rangle_{\mathcal{E}^i_h}.$$

Then we have

$$\partial_{h,x} v = \sum_{K \in \mathcal{T}_h} \partial_x v \chi_K - \mathcal{L}_{h,i} v.$$

Finally, combining Propositions 4.1 and 4.2 and the well-known limiting characterization theorem of distributional derivatives (cf. [28, Theorem 6.32]), we obtain another characterization for our DG FE derivatives.

**Proposition 4.3.** Set $\gamma^\pm = 0$ in (3.6). Then for any $v \in \mathcal{V}_h$, there exists a sequence of functions $\{ v_j \}_{j \geq 1} \subset C_0^\infty(\Omega)$ such that, for $i = 1, 2, \cdots, d$, (i) $v_j \rightarrow v$ as $j \rightarrow \infty$ in $L^1(\Omega)$.
(ii) \( \partial_{h,x} v_j \to \mathcal{P}^h_r \mathcal{D}_x v \) as \( j \to \infty \) weakly in \( V^h_r \) in the sense that
\[
\lim_{j \to \infty} \left( \partial_{h,x} v_j, \varphi_h \right)_{\mathcal{T}_h} = \left( \mathcal{P}^h_r \mathcal{D}_x v, \varphi_h \right)_{\mathcal{T}_h} \quad \forall \varphi_h \in V^h_r.
\]

Proof. Let \( \{ \rho_j \}_{j \geq 1} \) denote a symmetric mollifier (or approximate identity) with compact support. For any \( v \in V_h \subset L^1(\Omega) \), set \( v_j := v * \rho_j \in C^\infty_0(\Omega) \), the convolution of \( v \) and \( \rho_j \). Then it is easy to check that the sequence \( \{ v_j \} \) fulfills the properties (i) and (ii). The proof is complete. \( \square \)

4.2. Approximation properties. By Proposition 4.1 we readily have the following approximation properties for the DG FE derivatives.

**Theorem 4.1.** For any \( v \in V_h \cap H^1(\Omega) \) there holds the following inequalities:

\[
\| \partial_{x,i} v - \partial_{h,x,i} v \|_{L^2(\mathcal{T}_h)} \leq \| \partial_{x,i} v - \partial_{h,x,i} \psi_h \|_{L^2(\mathcal{T}_h)} \quad \forall \psi_h \in V^h_{r+1},
\]
\[
\| \partial_{x,i} v - \partial_{h,x,i} v \|_{L^2(\mathcal{K})} \leq \| \partial_{x,i} v - \partial_{x,i} \psi_h \|_{L^2(\mathcal{K})} \quad \forall \psi_h \in \mathbb{P}_{r+1},
\]

where \(*\) can take \(+, −, \) and empty value.

Proof. Notice that (4.1) can be rewritten as
\[
\left( \partial_{x,i} v - \partial_{h,x,i} v, \varphi_h \right)_{\mathcal{T}_h} = 0 \quad \forall \varphi_h \in V^h_r.
\]
Hence, for any \( \psi_h \in V^h_{r+1} \), there holds
\[
\| \partial_{x,i} v - \partial_{h,x,i} v \|_{L^2(\mathcal{T}_h)}^2 = \left( \partial_{x,i} v - \partial_{h,x,i} v, \partial_{x,i} v - \partial_{h,x,i} v \right)_{\mathcal{T}_h}
= \left( \partial_{x,i} v - \partial_{h,x,i} v, \partial_{x,i} v \right)_{\mathcal{T}_h}
= \left( \partial_{x,i} v - \partial_{h,x,i} \psi_h, \partial_{x,i} v - \partial_{h,x} \psi_h \right)_{\mathcal{T}_h}
\leq \| \partial_{x,i} v - \partial_{h,x,i} \psi_h \|_{L^2(\mathcal{T}_h)} \| \partial_{x,i} v - \partial_{x,i} \psi_h \|_{L^2(\mathcal{T}_h)}.
\]

The identity (4.10) follows from the above inequality because \( \partial_{x,i} V^h_{r+1} \subset V^h_r \). The second inequality (4.11) is obtained by similar arguments. \( \square \)

It is also possible to derive some error estimates for \( \mathcal{D}_x v - \partial_{h,x}^* v \) with general \( v \in V_h \) in some weaker (than \( L^2 \)) norm. However, since the derivation is lengthy and technical, we leave it to the interested reader to exploit.

4.3. Product rule and chain rule. The product rule and the chain rule are two very basic properties of classical derivatives and weak derivatives (cf. 21 Chapter 7). The goal of this subsection is to establish both of the rules for our DG FE derivatives. As expected, these discrete rules take different forms from their continuous counterparts.

First, we consider the case when functions are from the space \( V_h \cap C^0(\Omega) \). In this case, we have the following product rule and chain rule.

**Theorem 4.2.** Let \( F \in C^1(\mathbb{R}), F' \in L^\infty(\mathbb{R}) \). For \( u, v \in V_h \cap C^0(\Omega) \), there holds, for \( i = 1, 2, \cdots, d \),
\[
\partial_{h,x,i} (uv) = \mathcal{P}^h_r (u \partial_{x,i} v + v \partial_{x,i} u),
\]
\[
\partial_{h,x,i} F(u) = \mathcal{P}^h_r (F'(u) \partial_{x,i} u).
\]
Proof. Notice that $uv \in V_h \cap C^0(\Omega)$ for any $u, v \in V_h \cap C^0(\Omega)$. By Proposition 4.1 and the product rule for weak derivatives (cf. [21, Chapter 7]) we get

$$(\partial_{h,x_i}(uv), \varphi_h)_{\mathcal{T}_h} = (\partial_{x_i}(uv), \varphi_h)_{\mathcal{T}_h} = (u\partial_{x_i}v + v\partial_{x_i}u, \varphi_h)_{\mathcal{T}_h} \quad \forall \varphi_h \in V^h_r.$$ 

Hence, (4.12) holds by the definition of $\mathcal{P}_r^h$.

Similarly, since $F(u) \in V_h \cap C^0(\Omega)$ for $u \in V_h \cap C^0(\Omega)$ we have

$$(\partial_{h,x_i}F(u), \varphi_h)_{\mathcal{T}_h} = (\partial_{x_i}F(u), \varphi_h)_{\mathcal{T}_h} = (F'(u)\partial_{x_i}u, \varphi_h)_{\mathcal{T}_h} \quad \forall \varphi_h \in V^h_r,$$

which then infers (4.12). The proof is complete. \qed

An immediate consequence of Theorem 4.2 and Corollary 4.1 is the following corollary.

**Corollary 4.2.** Let $F \in C^1(\mathbb{R})$, $F' \in L^\infty(\mathbb{R})$. For $u, v \in V^h_\ell \cap C^0(\Omega)$ with $0 \leq \ell - 1 \leq r$, there holds, for $i = 1, 2, \ldots, d$,

\begin{align}
\partial_{h,x_i}F(u) &= \mathcal{P}_r^h(F'(u)\partial_{h,x_i}u), \\
\partial_{h,x_i}(uv) &= \mathcal{P}_r^h(u\partial_{h,x_i}v + v\partial_{h,x_i}u),
\end{align}

Next, we consider the general case when functions are from the space $V_h$. As expected, the discrete product and chain rules appear in more complicated forms.

**Theorem 4.3.** Let $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. For $u, v \in V_h$, there holds for $i = 1, 2, \ldots, d,$

\begin{align}
\partial_{h,x_i}F(u) &= \mathcal{P}_r^h(F'(u)\mathcal{D}_xu), \\
\partial_{h,x_i}(uv) &= \mathcal{P}_r^h(u\partial_{h,x_i}v + v\partial_{h,x_i}u),
\end{align}

where $\mathcal{D}_xu$ is given by (4.15) and $\mathcal{P}$ represents a modification of $v$ which assumes the same value in each element $K \in \mathcal{T}_h$ and takes the average value of $v$ on each interior edge $e \in E^I_h$. Thus,

$$\mathcal{D}_xu = \sum_{K \in \mathcal{T}_h} u \partial_{x_i}v \chi_K - \sum_{e \in E^I_h} n^{(i)}_e \delta(e, \{v\}[v], x) \quad \text{for a.e. } x \in \Omega.$$ 

Proof. Without loss of the generality, we assume $F \in C^\infty(\mathbb{R})$. For $u, v \in V_h$, let $\{v_j\}, \{u_j\} \subset C^\infty_0(\Omega)$ be defined in Proposition 4.3. Then, by Theorem 4.2 we have

\begin{align}
\partial_{h,x_i}(u_jv_j) &= \mathcal{P}_r^h(u_j\partial_{x_i}v_j + v_j\partial_{x_i}u_j), \\
\partial_{h,x_i}F(u_j) &= \mathcal{P}_r^h(F'(u_j)\partial_{x_i}u_j).
\end{align}

It follows from (4.9) and (4.6) that for any $\varphi_h \in V^h_r$,

\begin{align}
\lim_{j \to \infty} (\partial_{h,x_i}(u_jv_j), \varphi_h)_{\mathcal{T}_h} &= (\mathcal{P}_r^h\mathcal{D}_x(uv), \varphi_h)_{\mathcal{T}_h} \\
&= \langle \mathcal{D}_x(uv), \varphi_h \rangle = \langle \partial_{h,x_i}(uv), \varphi_h \rangle_{\mathcal{T}_h}, \\
\lim_{j \to \infty} (\partial_{h,x_i}F(u_j), \varphi_h)_{\mathcal{T}_h} &= (\mathcal{P}_r^h\mathcal{D}_xF(u), \varphi_h)_{\mathcal{T}_h} \\
&= \langle \mathcal{D}_xF(u), \varphi_h \rangle = \langle \partial_{h,x_i}F(u), \varphi_h \rangle_{\mathcal{T}_h}. 
\end{align}
On the other hand, we have

\[
\lim_{j \to \infty} \left( \mathcal{P}_r^h (u_j \partial_x v_j + v_j \partial_x u_j), \varphi_h \right)_{\mathcal{T}_h} = \lim_{j \to \infty} \left( u_j \partial_x v_j + v_j \partial_x u_j, \varphi_h \right)_{\mathcal{T}_h} \\
= \langle \mathcal{D}_x, v, u \varphi_h \rangle + \langle \mathcal{D}_x, u, v \varphi_h \rangle \\
= \langle \overline{\nu} \mathcal{D}_x v + \overline{\nu} \mathcal{D}_x u, \varphi_h \rangle,
\]

(4.24)

\[
\lim_{j \to \infty} \left( \mathcal{P}_r^h (F'(u_j) \partial_x u_j), \varphi_h \right)_{\mathcal{T}_h} = \lim_{j \to \infty} \left( \mathcal{F}'(u_j) \partial_x u_j, \varphi_h \right)_{\mathcal{T}_h} \\
= \langle \mathcal{D}_x, u, F'(u) \varphi_h \rangle \\
= \langle \mathcal{F}'(u) \mathcal{D}_x, u, \varphi_h \rangle.
\]

Then, (4.16) follows from combining (4.19), (4.21) and (4.24), and (4.17) follows from combining (4.20), (4.22) and (4.24). The proof is complete. \(\Box\)

**Remark 4.2.** The \(C^1\) smoothness assumption on \(F\) in Theorem 4.3 may be weakened to \(F \in C^{0,1}(\mathbb{R})\) by following the techniques used in [1]. We leave the generalization to the interested reader to exploit.

### 4.4. Integration by parts formula and discrete divergence theorems

In this subsection, we derive integration by parts formulas for the discrete differential operators that resemble the standard integration by parts formula in the continuous setting. These results will play a crucial role in developing DG methods for a variety of PDE problems as well as in relating these methods to other existing DG methods in the literature.

**Theorem 4.4.** Suppose that \(\gamma^+_i = -\gamma^-_i\), that is, \(\gamma^+_{i,e} = -\gamma^-_{i,e}\) for all \(e \in \mathcal{E}_h^l\). Then the integration by parts formulas

\[
(\partial^+_{h,x}, v, \varphi_h)_{\mathcal{T}_h} = -\langle v_h, \partial^+_h v, \varphi_h \rangle_{\mathcal{T}_h} + \langle v_h, \varphi_h n^{(i)} \rangle_{\mathcal{E}_h^l}
\]

(4.25)

and

\[
(\partial^-_{h,x}, v, \varphi_h)_{\mathcal{T}_h} = -\langle v_h, \partial^-_h v, \varphi_h \rangle_{\mathcal{T}_h} + \langle v_h, \varphi_h n^{(i)} \rangle_{\mathcal{E}_h^l}
\]

(4.26)

hold for all \(v_h, \varphi_h \in V_{r,h}\).

**Proof.** It suffices to show (4.25), since (4.26) trivially follows from (4.25).

By (3.6) and integration by parts, we obtain

\[
(\partial^+_{h,x}, v, \varphi_h)_{\mathcal{T}_h} = (\partial_x, v, \varphi_h)_{\mathcal{T}_h} + \langle \mathcal{Q}_h^+(v) - \{v\}, [\varphi_h]n^{(i)} \rangle_{\mathcal{E}_h^l} \\
- \langle [v], \{\varphi_h\}n^{(i)} + [\gamma^+_i \varphi_h] \rangle_{\mathcal{E}_h^l}.
\]

Using the identity (3.5), we have

\[
(\partial^+_{h,x}, v, \varphi_h)_{\mathcal{T}_h} = (\partial_x, v, \varphi_h)_{\mathcal{T}_h} + \frac{1}{2} \langle [v], [\varphi_h] \text{sgn}(n^{(i)})n^{(i)} \rangle_{\mathcal{E}_h^l} \\
- \langle [v], \{\varphi_h\}n^{(i)} + [\gamma^+_i \varphi_h] \rangle_{\mathcal{E}_h^l} \\
= (\partial_x, v, \varphi_h)_{\mathcal{T}_h} + \langle [v], (\gamma^+_i + \frac{1}{2} n^{(i)})[\varphi_h] - \{\varphi_h\}n^{(i)} \rangle_{\mathcal{E}_h}.
\]

(4.27)
Now let \( v = v_h \in V^h_r \). Then by (3.6), (3.5) and rearranging terms, we have
\[
(\partial_x, v_h, \varphi_h)_{\mathcal{T}_h} = - (\partial^+_x, v_h, \varphi_h)_{\mathcal{T}_h} + \langle \mathcal{Q}_h^x (\varphi_h), [v_h] n^{(i)} \rangle_{\mathcal{E}_h^i} + \langle \gamma^+_i [\varphi_h], [v_h] \rangle_{\mathcal{E}_h^i} \\
= - (\partial^+_x, v_h, \varphi_h)_{\mathcal{T}_h} + \langle \{ \varphi_h \}, [v_h] n^{(i)} \rangle_{\mathcal{E}_h^i} \\
+ \langle (\gamma^+_i + \frac{1}{2}[n^{(i)}]) [\varphi_h], [v_h] \rangle_{\mathcal{E}_h^i} + \langle \varphi_h, v_h n^{(i)} \rangle_{\mathcal{E}_h^i}.
\]
Using the identity (4.28) in equation (4.27), we obtain
\[
(\partial^+_x, v_h, \varphi_h)_{\mathcal{T}_h} = - (v_h, \partial^+_x, \varphi_h)_{\mathcal{T}_h} + \langle v_h, \varphi_h n^{(i)} \rangle_{\mathcal{E}_h^i} \\
+ \langle (\gamma^+_i + \gamma^+_i) [\varphi_h], [v_h] \rangle_{\mathcal{E}_h^i}.
\]
The identity (4.25) then easily follows.

**Theorem 4.5.** Suppose that \( \gamma^+_i = -\gamma^-_i \) for all \( i = 1, 2, \ldots d \). Then the integration by parts formulas
\begin{align}
(\text{div}_h^{\pm} v_h, \varphi_h)_{\mathcal{T}_h} &= - (v_h, \nabla^+_h \varphi_h)_{\mathcal{T}_h} + \langle v_h \cdot n, \varphi_h \rangle_{\mathcal{E}_h^i} \\
(\text{div}_h v_h, \varphi_h)_{\mathcal{T}_h} &= - (v_h, \nabla_h \varphi_h)_{\mathcal{T}_h} + \langle v_h \cdot n, \varphi_h \rangle_{\mathcal{E}_h^i}
\end{align}
hold for all \( v_h \in V^h_r \) and \( \varphi_h \in V^h_r \).

**Proof.** If \( \gamma^+_i = -\gamma^-_i \), then by (3.11c), (4.25) and (3.11a), we have
\[
(\text{div}_h^{\pm} v_h, \varphi_h)_{\mathcal{T}_h} = \sum_{i=1}^d (\partial^+_x, v_{h,i}, \varphi_h)_{\mathcal{T}_h} \\
= \sum_{i=1}^d \left( - (\partial^+_x, \varphi_h, v_{h,i})_{\mathcal{T}_h} + \langle v_{h,i} n^{(i)}, \varphi_h \rangle_{\mathcal{E}_h^i} \right) \\
= - (v_h, \nabla^+_h \varphi_h)_{\mathcal{T}_h} + \langle v_h \cdot n, \varphi_h \rangle_{\mathcal{E}_h^i}.
\]
Formula (4.31) is obtained similarly.

**Theorem 4.6.** The formal adjoint of the operator \( \text{div}_h^{\pm} \) (resp., \( \text{div}_h \)) is \( -\nabla^+_h,0 \) (resp., \(-\nabla_h,0\)) with respect to the inner product \( (\cdot, \cdot)_{\mathcal{T}_h} \) provided \( \gamma^+_i = -\gamma^-_i \) for all \( i = 1, 2, \ldots d \); that is,
\begin{align}
(\text{div}_h^{\pm} v_h, \varphi_h)_{\mathcal{T}_h} &= - (v_h, \nabla^+_h,0 \varphi_h)_{\mathcal{T}_h} \\
(\text{div}_h v_h, \varphi_h)_{\mathcal{T}_h} &= - (v_h, \nabla_h,0 \varphi_h)_{\mathcal{T}_h}
\end{align}
for all \( v_h \in V^h_r \), \( \varphi_h \in V^h_r \). In addition, if \( \gamma^+_i = -\gamma^-_i \), then the formal adjoint of the operator \( \text{div}_h^{\pm} \) (resp., \( \text{div}_h \)) is \( -\nabla^+_h \) (resp., \(-\nabla_h\)).

**Proof.** This result immediately follows from Theorem 4.5 and the identities
\[
(v, \nabla^+_h,0 \varphi_h)_{\mathcal{T}_h} = (v, \nabla^+_h \varphi_h)_{\mathcal{T}_h} - \langle v \cdot n, \varphi_h \rangle_{\mathcal{E}_h^i}, \\
(\text{div}_h^{\pm} v, \varphi_h)_{\mathcal{T}_h} = (\text{div}_h v, \varphi_h)_{\mathcal{T}_h} - \langle v \cdot n, \varphi_h \rangle_{\mathcal{E}_h^i}
\]
for all \( v \in V_h \) and \( \varphi_h \in V^h_r \).
Theorem 4.7. Suppose that $\gamma^+ = -\gamma^-$. We then have
\begin{align}
-(\Delta_{h}^+ v, \varphi_h)_{T_h} &= \langle \nabla_{h}^+ v, \nabla_{h}^+ \varphi_h \rangle_{T_h} - \langle \nabla_{h}^+ v \cdot n, \varphi_h \rangle_{E_{h}^+}, \\
-(\Delta_{h}^- v, \varphi_h)_{T_h} &= \langle \nabla_{h}^- v, \nabla_{h}^- \varphi_h \rangle_{T_h} - \langle \nabla_{h}^- v \cdot n, \varphi_h \rangle_{E_{h}^-}
\end{align}
and
\begin{align}
-(\Delta_{h}^\pm v, \varphi_h)_{T_h} &= \langle \nabla_{h} v, \nabla_{h} \varphi_h \rangle_{T_h} - \langle \nabla_{h} v \cdot n, \varphi_h \rangle_{E_{h}^\pm}
\end{align}
for all $v \in V_h$ and $\varphi_h \in \mathcal{V}_h$. In addition, under the same hypotheses on $\gamma^\pm$, we have
\begin{align}
-(\Delta_{h,g}^+ v, \varphi_h)_{T_h} &= \langle \nabla_{h,g}^+ v, \nabla_{h,g}^+ \varphi_h \rangle_{T_h}, \\
-(\Delta_{h,g}^- v, \varphi_h)_{T_h} &= \langle \nabla_{h,g}^- v, \nabla_{h,g}^- \varphi_h \rangle_{T_h}
\end{align}
and
\begin{align}
-(\Delta_{h,g}^\pm v, \varphi_h)_{T_h} &= \langle \nabla_{h,g} v, \nabla_{h,g} \varphi_h \rangle_{T_h}.
\end{align}

Proof. These formulas follow from Theorems 4.5–4.6 with $\varphi = 1$ in (4.10) with $v = \nabla_{h}^\pm v \in \mathcal{V}_h$ and $\varphi = 1$ in (4.36) we get
\begin{align}
\langle \nabla_{h}^\pm v, 1 \rangle_{K} &= \langle \mathcal{Q}_{\delta}^\pm (v) n_K, 1 \rangle_{\partial K}.
\end{align}

4.5. Relationships with finite difference operators on Cartesian grids. We now show that when $\gamma^\pm = 0$ the operators $\partial_{h,x_i}^\pm$ and $\partial_{h,x_i}$ are natural extensions (on general grids) of the backward/forward and central difference operators defined on Cartesian grids.

Suppose $T_h$ is a rectangular mesh over $\Omega$ which is aligned with the underlying Cartesian coordinate system. Let $h_i$ denote the mesh size of $T_h$ in the direction of $x_i$. Notice that when $r = 0$, the function $\partial_{h,x_i} v$ is a piecewise constant function over the mesh $T_h$. Setting $\varphi = 1$ in (4.10) we get
\begin{align}
\langle \partial_{h,x_i}^\pm v, 1 \rangle_{K} &= \langle \mathcal{Q}_{\delta}^\pm (v) n_K, 1 \rangle_{\partial K}.
\end{align}

We only consider the two dimensional case. Then $K$ has four edges with $n_K = (-1, 0)^t$, $(1, 0)^t$, $(0, 1)^t$, $(0, -1)^t$. Let $v$ be a grid function over $T_h$ and $v_{ij}$ denote the value of $v$ on the $(i, j)$ cell/element. By (4.36) we obtain
\begin{align}
\partial_{h,x_1}^- v_{ij} &= \frac{v_{i+1,j} - v_{ij}}{h_1}, & \partial_{h,x_1}^+ v_{ij} &= \frac{v_{ij} - v_{i-1,j}}{h_1}, \\
\partial_{h,x_2}^- v_{ij} &= \frac{v_{ij+1} - v_{ij}}{h_2}, & \partial_{h,x_2}^+ v_{ij} &= \frac{v_{ij} - v_{ij-1}}{h_2}.
\end{align}

Consequently, we have
\begin{align}
\partial_{h,x_1}^- v_{ij} &= \frac{v_{i+1,j} - v_{i-1,j}}{2h_1}, & \partial_{h,x_1}^+ v_{ij} &= \frac{v_{i+1,j} - v_{ij}}{2h_1}, \\
\partial_{h,x_2}^- v_{ij} &= \frac{v_{ij+1} - v_{ij-1}}{2h_2}, & \partial_{h,x_2}^+ v_{ij} &= \frac{v_{ij} - v_{ij-1}}{2h_2}.
\end{align}

Hence, $\partial_{h,x_i}^-$ and $\partial_{h,x_i}^+$ coincide respectively with backward and forward difference operators in the direction $x_i$, while $\partial_{h,x_i}$ results in the central difference operator in the direction $x_i$.

Using the operators $\partial_{h}^\pm$, as building blocks, we can also build DG FE operators that are natural extensions of the standard finite difference approximations for
second order partial derivatives. From (4.37) and (4.38), we have

\begin{align}
\frac{\partial^+_{h,x_1} \partial^-_{h,x_1} v_{ij}}{2} &= \frac{\partial^-_{h,x_1} \partial^+_{h,x_1} v_{ij}}{2} = \frac{v_{i-1,j} - 2v_{ij} + v_{i+1,j}}{h_1^2}, \\
\partial_{h,x_1} \partial_{h,x_1} v_{ij} &= \frac{v_{i-2,j} - 2v_{ij} + v_{i+2,j}}{4h_1^2}, \\
\frac{\partial^+_{h,x_1} \partial^-_{h,x_2} v_{ij} + \partial^-_{h,x_1} \partial^+_{h,x_2} v_{ij}}{2} &= \frac{v_{i+1,j} - v_{i-1,j+1} - v_{i-1,j-1} + v_{i+1,j+1} + v_{i+1,j} - 2v_{ij}}{2h_1h_2}. \\
\partial_{h,x_2} \partial_{h,x_2} v_{ij} &= \frac{\partial^+_{h,x_2} \partial^-_{h,x_2} v_{ij} + \partial^-_{h,x_2} \partial^+_{h,x_2} v_{ij}}{2} = \frac{v_{i+1,j+1} - v_{i+1,j-1} - v_{i+1,j} - v_{i,j+1} + v_{i+1,j} + v_{i,j-1}}{4h_1h_2}.
\end{align}

Thus, for \( k \neq \ell \), the discrete differential operator \( \frac{\partial^+_{h,x_1} \partial^-_{h,x_2} \partial^+_{h,x_2} \partial^-_{h,x_1}}{2} \) coincides with the standard second order 3-point central difference operator for non-mixed second order derivatives, \( \partial_{h,x_1} \partial_{h,x_2} \) coincides with the second order 3-point central difference operator with mesh size \( 2h \), \( \frac{\partial^+_{h,x_2} \partial^-_{h,x_2} \partial^+_{h,x_1} \partial^-_{h,x_1}}{2} \) coincides with the second order 7-point central difference operator for mixed second order derivatives, and \( \partial_{h,x_1} \partial_{h,x_2} \) coincides with the standard second order central difference operator for mixed second order derivatives.

5. Implementation Aspects

In this section we explain how the discrete partial derivatives are computed. Denote by \( \tilde{K} := \{ x \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^d x_i < 1 \} \) the reference simplex, and let \( \{ \varphi_h^{(j)} \}_{j=1}^{n_r} \) denote a basis of \( \mathbb{P}_r(\tilde{K}) \). Here \( n_r = \dim \mathbb{P}_r = \binom{d+r}{r} \). For example, we could take the basis to be the space of monomials of degree less than or equal to \( r \): \( \{ x^\alpha : |\alpha| \leq r \} \). For \( K \in \mathcal{T}_h \), let \( F_K : \tilde{K} \to K \) be the affine mapping from \( \tilde{K} \) onto \( K \), and and let \( \varphi_h^{(j,K)} : K \to \mathbb{R} \) be defined by \( \varphi_h^{(j,K)}(x) = \varphi_h^{(j)}(\tilde{x}) \), where \( x = F_K(\tilde{x}) \in K \). It is then easy to see that \( \{ \varphi_h^{(j,K)} \}_{j=1}^{n_r} \) is a basis of \( \mathbb{P}_r(K) \). We then define the mass matrix \( \tilde{M}_K \in \mathbb{R}^{n_r \times n_r} \) associated with \( K \) as

\[
(\tilde{M}_K)_{\ell,m} = (\varphi_h^{(\ell,K)}, \varphi_h^{(m,K)})_K, \quad \ell, m = 1, 2, \ldots, n_r.
\]

By a change of variables we easily find \( \tilde{M}_K = |\det(DF_K)|\tilde{M} = d!|K|\tilde{M} \), where \( \tilde{M} \) is the mass matrix associated with \( \tilde{K} \), \( DF_K \) is the Jacobian of the mapping \( F_K \) and \( |K| \) is the \( d \)-dimensional volume of the simplex \( K \).

Next, given \( v \in \mathcal{V}_h \), write \( \partial_{h,x_i} v |_K = \sum_{j=1}^{n_r} \alpha_{i,j} \varphi_h^{(j,K)} \in \mathbb{P}_r(K) \) with \( \alpha_{i,j} = \alpha_{i,j,K} \in \mathbb{R} \) \( (i = 1, 2, \ldots, d, \ j = 1, 2, \ldots, n_r) \). We then define the vector \( \tilde{b}_i^{(j)} = b_{i,h}(v) \in \mathbb{R}^{n_r} \) by

\[
\tilde{b}_i^{(j)} = (Q_i^{(j)}(v) n_i^{(j,k)}, \varphi_h^{(j,K)})_{\partial K} - (v, \partial_{x_i} \varphi_h^{(j,K)})_K + \sum_{e \subset \partial K \setminus \partial i} \gamma_{i,e}^{(j)} (v, \varphi_h^{(j,K)})_e, \quad j = 1, 2, \ldots, n_r.
\]
Then by (3.10), the coefficients $\{\alpha_i^{(j)}\}_{j=1}^{n_r}$ are uniquely determined by the linear equation $M_K \alpha_i^{\pm} = b_i^{\pm}$, where $\alpha_i^{\pm} = (\alpha_i^{(1)}, \alpha_i^{(2)}, \ldots, \alpha_i^{(n_r)})^t$. Equivalently, we have $\alpha_i^{\pm} = \frac{1}{|\partial K|} M^{-1} b_i^{\pm}$, where $M^{-1}$ is the inverse matrix of the reference mass matrix which can be computed offline.

Now if $v = v_h \in V_r^h$, then we may write $v_h|_K = \sum_{j=1}^{n_r} \beta_j^{(j)} (\varphi_h)^{(j,K)}$ for some constants $\beta_j^{(j)} = \beta_j^{(j,K)} \in \mathbb{R}$. We then define the matrix $Q_i^{\pm} = Q_i^{\pm,K} \in \mathbb{R}^{n_r \times n_r}$ by

$$
(\hat{Q}_i^{\pm})_{\ell,m} = (Q_i^{\pm} (\varphi_h^{(m,K)})^{(i)}, \varphi_h^{(\ell,K)})_{\partial K} - (\varphi_h^{(m,K)}, \partial x, \varphi_h^{(\ell,K)})_{\partial K},
$$

for $\ell, m = 1, 2, \ldots, n_r$. Again, writing $\partial_h^{\pm} v|_K = \sum_{j=1}^{n_r} \alpha_j^{(j)} (\varphi_h)^{(j,K)}$, we find that the coefficients satisfy $\alpha_i^{\pm} = \frac{1}{|\partial K|} M^{-1} \hat{Q}_i^{\pm} \beta$, where $\beta = (\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(n_r)})^t$.

6. Applications

In this section, we apply our DG FE differential calculus framework to constructing numerical methods for several different types of PDEs. Essentially, we replace the (continuous) differential operator by its discrete counterpart. In addition, we add a stability term which ensures that the discrete problem is well-posed and also enforces the given boundary conditions weakly within the variational formulation. Throughout the section, we assume the penalty parameters $\gamma_i$ (which appear in the definition of the discrete differential operators) are zero in the discussion below.

6.1. Second order linear elliptic PDEs.

6.1.1. The Poisson equation. As our first example, we consider the simplest linear second order PDE, the Poisson equation with Dirichlet boundary conditions:

$$
\begin{align}
(6.1a) & \quad -\Delta u = f \quad \text{in } \Omega, \\
(6.1b) & \quad u = g \quad \text{on } \partial \Omega,
\end{align}
$$

where $f \in L^2(\Omega)$ and $g \in L^2(\partial \Omega)$ are two given functions. We then consider the following discrete version of (6.1): Find $u_h \in V_r^h$ such that

$$
(6.2) \quad -\Delta_h g u_h + j_{h,g}(u_h) = P_h f.
$$

Here, $j_{h,g} : \mathcal{V}_h \to V_r^h$ is the unique operator satisfying

$$
(6.3) \quad (j_{h,g}(v), \varphi_h)_{\mathcal{E}_h} = \langle \eta_1 [v], [\varphi_h] \rangle_{\mathcal{E}_h} + \langle \eta_1 (v - g), \varphi_h \rangle_{\mathcal{E}_h},
$$

and $\eta_1$ is a penalty parameter that is piecewise constant with respect to the set of edges.

Problem (6.2) has several interpretations. On the one hand, by the definition of the discrete Laplace operator, the problem is the twofold saddle point problem

$$
\begin{align}
-\text{tr}(\hat{r}_h) + j_{h}(u_h) & = P_h f, \\
\hat{r}_h & = D_h q_h, \quad q_h = \nabla_h g u_h,
\end{align}
$$
with \( q_h \in V^h_r \) and \( \tau_h \in \tilde{V}^h_r \). Namely, problem (6.2) is equivalent to finding \( u_h \in V_h \), \( q_h \in V^h_r \) and \( \tau_h \in \tilde{V}^h_r \) such that

\[
(6.4a) \quad (q_h, \tau_h)_h = \left\langle \{q_h\}, \{\tau_h\} \cdot n \right\rangle_{E^h} - (u_h, \text{div} \tau_h)_h + \langle g, \tau_h \cdot n \rangle_{E^h} \quad \forall \tau_h \in V^h_r,
\]

\[
(6.4b) \quad (\tau_h, \mu_h)_h = \left\langle \{q_h\}, [\mu_h] \right\rangle_{E^h} - (q_h, \text{div} \mu_h)_h \quad \forall \mu_h \in \tilde{V}^h_r,
\]

\[
(6.4c) \quad - \left( \text{tr}(\tau_h), \varphi_h \right)_h + (j_{h,g}(u_h), \varphi_h)_h = (f, \varphi_h)_h \quad \forall \varphi_h \in V^h_r.
\]

On the other hand, by Theorem 4.5, we can write problem (6.2) in its primal form:

\[
(6.5) \quad (\nabla h,g u_h, \nabla h \varphi_h)_h - \langle \nabla h,g u_h \cdot n, \varphi_h \rangle_{E^h} + (j_{h,g}(u_h), \varphi_h)_h = (f, \varphi_h)_h \quad \forall \varphi_h \in V^h_r.
\]

Finally, in the case \( g = 0 \), problem (6.5) can be viewed as finding a minimizer of the functional

\[
(6.6) \quad v_h \rightarrow \frac{1}{2} \int_{\Omega} |\nabla h,0 v_h|^2 \, dx + \sum_{e \in E_h} \frac{1}{2} \int_{e} \eta_1 \|v_h\|^2 \, ds - \int_{\Omega} f v_h \, dx
\]

over all \( v_h \in V^h_r \).

We now discuss the well-posedness of problem (6.2) as well as relate the discretization to other DG schemes. Let \( q_h = \nabla h,g u_h \), that is, \( q_h \in V^h_r \) satisfies (6.4a). Then by (6.5), we have

\[
(6.7) \quad (q_h, \nabla h \varphi_h)_h - \langle \{q_h\} \cdot n, \varphi_h \rangle_{E^h} + (j_{h,g}(u_h), \varphi_h)_h = (f, \varphi_h)_h \quad \forall \varphi_h \in V^h_r.
\]

In summary, problem (6.2) is equivalent to the mixed formulation (6.4a), (6.8). This formulation is nothing more than the local discontinuous Galerkin (LDG) method \([10, 2]\). We then have (see, e.g., \([8]\))

**Theorem 6.1.** Let \( r \geq 1 \), \( \eta_1 > 0 \) and \( \gamma_i = 0 \) (\( i = 1, 2, \ldots, d \)). Then there exists a unique \( u_h \in V^h_r \) satisfying (6.2). Moreover, if \( \eta_1 = O(h^{-1}) \) and if \( u \in H^{r+2}(\Omega) \), there holds

\[
\|u - u_h\|_{L^2(\Omega)} + h\|\nabla u - \nabla h,g u_h\|_{L^2(\Omega)} \leq Ch^{r+1}\|u\|_{H^{r+2}(\Omega)}.
\]

**Remark 6.1.** A similar methodology can be used to construct DG schemes for the Neumann problem

\[
(6.9) \quad -\Delta u = f \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial n} = q \quad \text{on} \ \partial \Omega.
\]

In this case, the DG method is to find \( u_h \in V^h_r \) satisfying

\[
-\text{div} h,q \nabla h u_h + j_{h}(u_h) = \mathcal{P}^h f,
\]
where \( \tilde{\gamma}_h(u_h) : V^h \to V^c_h \) is the operator satisfying \( (\tilde{j}_h(v), \varphi_h)_{\mathcal{T}_h} = \langle \eta_1[v], [\varphi_h] \rangle_{\mathcal{E}_h^i} \) for all \( \varphi_h \in V^c_h \), and \( q = q_n \). We again recover the LDG method for the Neumann problem \( (6.9) \) (cf. \[8\]).

6.1.2. The DWDG method for the Poisson problem. In this subsection we formulate a new DG method for the Poisson problem \( (6.1) \) that inherits better stability than the LDG method described above. The new scheme, called the symmetric dual-wind discontinuous Galerkin (DWDG) method, is simply given by

\[
(6.10) \quad -\frac{\Delta_{h,g} u_h + \Delta_{h,g}^{-} u_h}{2} + j_{h,g}(u_h) = f^h, \tag{6.10}
\]

where \( j_{h,g}(u_h) \) is defined by \( (6.3) \). By Theorem \[4.7\], problem \( (6.10) \) is equivalent to finding \( u_h \in V^c_h \) such that

\[
(6.11) \quad \frac{1}{2} \left( (\nabla_{h,g}^+ u_h, \nabla_{h,0}^+ v_h)_{\mathcal{T}_h} + (\nabla_{h,g}^- u_h, \nabla_{h,0}^- v_h)_{\mathcal{T}_h} \right) + j_{h,g}(u_h) = (f, v_h)_{\mathcal{T}_h}
\]

for all \( v_h \in V^c_h \). Equivalently, in the case \( g = 0 \), problem \( (6.10) \) asks to find the unique minimizer of the functional

\[
v_h \to \frac{1}{4} \int_{\Omega} \left( |\nabla_{h,0}^+ v_h|^2 + |\nabla_{h,0}^- v_h|^2 \right) dx + \sum_{e \in \mathcal{E}_h} \frac{1}{2} \int_{e} |\eta_1[v_h]|^2 ds - \int_{\Omega} f v_h dx
\]

over all \( v_h \in V^c_h \) (compare to \( (6.6) \)). A complete convergence analysis of the symmetric DWDG method for the Poisson problem is presented in \[26\]. Here, we summarize the main results, namely, well-posedness and optimal rates of convergence.

Theorem 6.2 (\[26\]). Set \( \gamma_{\min} := \min_{e \in \mathcal{E}_h} h^{-1}_e \eta_1(e) \). Suppose that there exists at least one simplex \( K \in \mathcal{T}_h \) with exactly one boundary edge/face. Then there exists a unique solution to \( (6.10) \) provided \( \gamma_{\min} \geq 0 \). Furthermore, if the triangulation is quasi-uniform, and if each simplex in the triangulation has at most one boundary face/edge, then there exists a constant \( C_* > 0 \) independent of \( h \) and \( \eta_1 \) such that problem \( (6.10) \) has a unique solution provided \( \gamma_{\min} > -C_* \).

Remark 6.2. We emphasize that problem \( (6.10) \) is well-posed without added penalty terms. As far as we are aware, this is the first symmetric DG method that has this property in any dimension (cf. \[27\] \[25\] \[5\]).

Theorem 6.3 (\[26\]). Let \( u_h \) be the solution to \( (6.10) \), \( u \in H^{s+1}(\Omega) \) be the solution to \( (6.1) \) and \( \gamma_{\max} = \max_{e \in \mathcal{E}_h} h^{-1}_e \eta_1(e) \). Then \( u_h \) satisfies the following estimate provided \( \gamma_{\min} > 0 \):

\[
(6.12) \quad \|u - u_h\|_{L^2(\Omega)} \leq Ch^{s+1} \left( \sqrt{\gamma_{\max} + \frac{1}{\sqrt{\gamma_{\min}}}} \right)^2 |u|_{H^{s+1}(\Omega)} \quad (1 \leq s \leq r),
\]

and if the triangulation is quasi-uniform and \( \gamma_{\min} > -C_* \), then there holds

\[
(6.13) \quad \|u - u_h\|_{L^2(\Omega)} \leq Ch^{s+1} \left( \sqrt{\gamma_{\min}} + \frac{1}{C_* + \gamma_{\min}} \right)^2 |u|_{H^{s+1}(\Omega)},
\]

where \( C \) denotes a generic positive constant independent of \( h \), and \( C_* \) is the positive constant from Theorem 6.2.
Remark 6.3. In light of (4.40) and (4.41), we can see that when approximating with piecewise constant basis functions on Cartesian grids, the DWDG method coincides with the standard finite difference method for Poisson’s equation while the LDG method coincides with a staggered finite difference method for Poisson’s equation that uses coarser second derivative approximations.

6.2. Fourth order linear PDEs. In this subsection we show how to use the discrete differential calculus to develop DG methods for fourth order linear PDEs. For simplicity, we focus our derivation to the model biharmonic problem with clamped boundary conditions:

\[ \Delta^2 u = f \quad \text{in } \Omega, \]
\[ u = g \quad \text{on } \partial \Omega, \]
\[ \frac{\partial u}{\partial n} = q \quad \text{on } \partial \Omega, \]

where we assume that \( f \in L^2(\Omega) \) and \( g, q \in L^2(\partial \Omega) \). The biharmonic problem with other boundary conditions (e.g., simply supported or Cahn-Hilliard type) are briefly discussed at the end of the section.

To develop DG methods using the discrete differential machinery, we first need to define some additional discrete operators corresponding to the biharmonic operator. Given the two functions \( g \) and \( q \) defined on the boundary, we set

\[ \Delta_{h,g,q} : = \text{div}_{h,q} \nabla_h \]
\[ \Delta^2_{h,g,q} : = \Delta_h(\Delta_{h,g,q}) = \text{div}_h \nabla_h \text{div}_{h,q} \nabla_h, \]

In addition, we define the operator \( r_{h,q}(\cdot) : W^{2,1}(T_h) \cap C^1(T_h) \to V_r^h \) by

\[ (r_{h,q}(v), \varphi_h)_{T_h} = \langle \eta_2[\partial v/\partial n], [\partial \varphi_h/\partial n]\rangle_{E_h} + \langle \eta_2(\partial v/\partial n - q), \partial \varphi_h/\partial n \rangle_{E_B}, \]

which we will use to enforce the Neumann boundary condition weakly in the DG formulation.

The DG method for (6.14) is then defined as seeking \( u_h^r \in V_r^h \) such that

\[ (\Delta_{h,g,q} u_h^r, \varphi_h)_{T_h} + (j_{h,q}(u_h))_{T_h} + (r_{h,q}(u_h), \varphi_h)_{T_h} = (f, \varphi_h)_{T_h} \quad \forall \varphi_h \in V_r^h. \]

Similar to the discussion in Section 6.1, we may write the DG method in various mixed forms (with up to six unknown variables). Instead, we focus mainly on the primal formulation.

By (6.15) and Theorem 4.6, we have

\[ (\Delta_{h,g,q} u_h, \varphi_h)_{T_h} = -\langle \nabla_h \Delta_{h,g,q} u_h, \nabla_h \varphi_h \rangle_{T_h} = (\Delta_{h,g,q} u_h, \text{div}_h \nabla_h \varphi_h)_{T_h} = (\Delta_{h,g,q} u_h, \Delta_{h,0,0} \varphi_h)_{T_h}. \]

Thus, we may write (6.17) in its primal formulation as follows: Find \( u_h \in V_r^h \) such that

\[ (\Delta_{h,g,q} u_h, \Delta_{h,0,0} \varphi_h)_{T_h} + (j_{h,g}(u_h), \varphi_h)_{T_h} + (r_{h,q}(u_h), \varphi_h)_{T_h} = (f, \varphi_h)_{T_h} \quad \forall \varphi_h \in V_r^h. \]
Remark 6.4. The DG method \((6.17)\) closely resembles the local continuous discontinuous Galerkin (LCDG) method proposed in \([22]\) (also see \([33]\)). Here, the authors consider a mixed formulation of the biharmonic problem with the Hessian of \(u\) as an additional unknown. The derivation of the LCDG method closely resembles the derivation of the LDG method for the Poisson problem; the main difference is that, as the name suggests, the LCDG method uses continuous finite element spaces.

Theorem 6.4. Suppose that \(\eta_1 > 0\) and \(\eta_2\) is non-negative. Then there exists a unique \(u_h \in V_r^h\) satisfying \((6.17)\).

Proof. Since the problem is finite dimensional and linear, it suffices to show that if \(f = 0, g = 0\) and \(q = 0\), then the solution is identically zero.

To this end, we set \(q_h = \nabla_{h,0} u_h\), \(v_h = \div_{h,0} q_h\), and \(z_h = \nabla_h v_h\) so that \(\div_h z_h + j_h,0(u_h) + r_h,0(u_h) = 0\). To ease notation, we define the bilinear forms

\[
\begin{align*}
    b(\mu_h, \varphi_h) &:= - (\div \mu_h, \varphi_h)_{T_h} + \{\mu_h\} \cdot n, \{\varphi_h\}_{E_h}, \\
    b_I(\mu_h, \varphi_h) &:= - (\div \mu_h, \varphi_h)_{T_h} + \{\mu_h\} \cdot n, \{\varphi_h\}_{E_h}, \\
    c(\psi_h, \varphi_h) &:= (j_h,0(\psi_h), \varphi_h)_{T_h} + (r_h,0(\psi_h), \varphi_h)_{T_h}.
\end{align*}
\]

It is then easy to verify that \((\div v_h, \mu_h, \varphi_h)_{T_h} = -b_I(\mu_h, \varphi_h)\) and \((\div v_h, \mu_h, \varphi_h)_{T_h} = -b(\mu_h, \varphi_h)\) for all \(\mu_h \in V_r^h\) and \(\psi_h \in V_r^h\). Furthermore, by Theorem 4.6, we have \((\mu_h, \nabla_h \psi_h)_{T_h} = b(\mu_h, \psi_h)\) and \((\mu_h, \nabla_h \psi_h)_{T_h} = b_I(\mu_h, \psi_h)\). It then follows that we may write \((6.17)\) in the following mixed-form:

\[
\begin{align*}
    (q_h, \mu_h)_{T_h} - b_I(\mu_h, u_h) &= 0 \quad \forall \mu_h \in V_r^h, \\
    (v_h, \psi_h)_{T_h} + b(q_h, \psi_h) &= 0 \quad \forall \psi_h \in V_r^h, \\
    (z_h, \tau_h)_{T_h} - b(\tau_h, v_h) &= 0 \quad \forall \tau_h \in V_r^h, \\
    -b_I(z_h, \varphi_h) + c(u_h, \varphi_h) &= 0 \quad \forall \varphi_h \in V_r^h.
\end{align*}
\]

Setting \(\mu_h = z_h\) in \((6.19a)\) and \(\varphi_h = u_h\) in \((6.19d)\), we have

\[
(q_h, z_h)_{T_h} - b_I(z_h, u_h) = 0, \quad \text{and} \quad -b_I(z_h, u_h) + c(u_h, u_h) = 0.
\]

Therefore, subtracting the two equations we get \((q_h, z_h)_{T_h} - c(u_h, u_h) = 0\). Next, we set \(\tau_h = q_h\) in \((6.19c)\) and \(\psi_h = v_h\) in \((6.19b)\) to obtain

\[
(z_h, q_h)_{T_h} - b(q_h, v_h) = 0, \quad \text{and} \quad (v_h, v_h)_{T_h} + b(q_h, v_h) = 0.
\]

Adding the two equations yields \(\|v_h\|_{L^2(\Omega)}^2 = -(z_h, q_h)_{T_h} - c(u_h, u_h) \leq 0\). Therefore, \(v_h \equiv 0\) and \(c(u_h, u_h) \equiv 0\). In particular, \(u_h\) vanishes on all of the boundary edges. Since \(v_h \equiv 0\), we also have \(z \equiv 0\) by \((6.19c)\). Setting \(\mu_h = q_h\) in \((6.19a)\) and \(\psi_h = u_h\) in \((6.19d)\), we have \(\|q_h\|_{L^2(\Omega)}^2 - b_I(q_h, u_h) = 0\), \(b(q_h, u_h) = 0\).

Since \(u_h\) vanishes on the boundary edges, \(b(q_h, u_h) = b_I(q_h, u_h)\). It then easily follows that \(q_h \equiv 0\). Finally, we have \(b_I(\mu_h, u_h) = 0\) for all \(\mu_h \in V_r^h\). This in turn implies that

\[
0 = (\mu_h, \nabla u_h)_{T_h} - \{\mu_h\} \cdot n, [u_h]_{E_h} = (\mu_h, \nabla u_h)_{T_h} \quad \forall \mu_h \in V_r^h.
\]

Therefore, \(\nabla u_h|_K = 0\) on all \(K \in T_h\). Since \([u_h]_e = 0\) across all edges, we conclude that \(u_h \equiv 0\). \(\square\)
Remarks 6.1.
(a) To obtain optimal order error estimates, we expect that the penalty parameters must scale like \( \eta_1 = O(h^{-3}) \) and \( \eta_2 = O(h^{-1}) \).
(b) The construction of DG schemes with other types of boundary conditions can easily be constructed by specifying the boundary data to different discrete differential operators. For example, if simply supported plate boundary conditions \( u = g \) and \( \Delta u = q \) are provided, then the corresponding discrete biharmonic operator is \( \Delta_{h,q} \Delta_{h,g} = \text{div}_h \nabla_{h,q} \text{div}_h \nabla_{h,g} u_h \). On the other hand, if Cahn-Hilliard-type boundary conditions \( \partial u/\partial n = g \) and \( \partial \Delta u/\partial n = q \) are given, then the discrete biharmonic operator is \( \text{div}_{h,q} \nabla_h \text{div}_{h,g} \nabla_h u_h \), where \( q = q_n \) and \( g = g_n \).

6.3. Quasi-linear second order PDEs.

6.3.1. The \( p \)-Laplace equation. We now extend the discrete differential framework to some non-linear elliptic problems. Although we can formulate the method for a very general class of quasi-linear PDEs, we shall focus our attention to a prototypical example, the \( p \)-Laplace equation \((2 \leq p < \infty):\)

\[
\begin{align*}
(6.20a) & \quad -\text{div}(|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega, \\
(6.20b) & \quad u = g \quad \text{on } \partial \Omega.
\end{align*}
\]

Similar to the Poisson problem, the DG method for (6.20) is obtained by simply replacing the \textit{grad} and \textit{div} operators by their respective discrete versions and adding a stability term to ensure that the resulting bilinear form is coercive over \( V_r^h \). In this case, the discrete problem reads: find \( u_h \in V_r^h \) satisfying

\[
-\text{div}_h(|\nabla_{h,q} u_h|^{p-2} \nabla_{h,q} u_h) + j^{(p)}_{h,q}(u_h) = p^h f, \tag{6.21}
\]

where \( j^{(p)}_{h,q}(-) \) is defined by

\[
(6.22) \quad (j^{(p)}_{h,q}(v), \varphi_h)_{T_h} = \langle \eta_1 |v|^{p-2} |v|, [\varphi_h] \rangle_{E^I_h} + \langle \eta_2 |v|^{p-2} (v - g), \varphi_h \rangle_{E^R_h}
\]

for all \( \varphi_h \in V_r^h \). Here, \( \eta_1 > 0 \) is a penalization parameter. Similar to the discrete Poisson problem, the discretization has several interpretations. By the definition of the discrete divergence and gradient operators, we can write (6.21) in the mixed formulation

\[
(6.23) \quad (|q_h|^{p-2} q_h, \nabla \varphi_h)_{T_h} - \left\langle \{ |q_h|^{p-2} q_h \} \cdot n, [\varphi_h] \right\rangle_{E^I_h} + (j^{(p)}_{h,q}(u_h), \varphi_h)_{T_h} = (f, \varphi_h)_{T_h} \quad \forall \varphi_h \in V_r^h,
\]

where \( q \in V_r^h \) satisfies (6.4a). We emphasize that the gradient appearing in the left-hand side of equation (6.23) is the piecewise gradient.

Remark 6.5. In [6], Burman and Ern proposed and analyzed the following LDG method for the \( p \)-Laplace equation (with \( r = 1 \) and \( g = 0 \):)

\[
(6.24) \quad (|\nabla_{h,0} u_h|^{p-2} \nabla_{h,0} u_h, \nabla_{h,0} \varphi_h)_{T_h} + \langle \eta_1 |u_h|^{p-2} |u_h|, [\varphi_h] \rangle_{E^I_h} = (f, \varphi_h)_{T_h}
\]

for all \( \varphi_h \in V_r^h \). Here, the authors showed the existence and uniqueness of the DG method (6.24) provided \( \eta_1 = O(h^{1-r}) \). In addition, Burman and Ern showed that the approximate solutions converge to \( u \) strongly in \( L^p(\Omega) \), and \( \nabla_{h,u} u_h \) converges to \( \nabla u \) strongly in \( L^p(\Omega) \). Furthermore, in the one dimensional setting, numerical
experiments indicate a convergence rate of at least $h^{3/4}$ for $p \in \{3, 4, 5\}$ and smooth exact solution.

Clearly, method (6.24) has a similar structure to (6.23), but they are different methods when $p \neq 2$. Indeed, since $|\nabla_h u_h|^{p-2}\nabla_h u_h \notin V^h$, we cannot use Theorems 4.5–4.6 and simply write

$$-(\text{div}_h(|\nabla_h u_h|^{p-2}\nabla_h u_h), \varphi_h)_h = (|\nabla_h u_h|^{p-2}\nabla_h u_h, \nabla_h \varphi_h)_h.$$  

In the following section, we show by way of numerical experiments that the DG method (6.23) converges with optimal order provided the exact solution is sufficiently smooth.

6.3.2. Numerical experiments of the $p$-Laplace equation. In this subsection we perform some numerical experiments to gauge the effectiveness of the DG method (6.21). We take the domain to be the unit square $\Omega = (0, 1)^2$ and choose the data $f$ such that the exact solution is $u = \sin(\pi x_1) \sin(\pi x_2)$ and $p = 5$. In all numerical experiments, we take the penalty parameter to be $\eta_1 = 20/h^{p-1} = 20/h^4$.

The resulting errors in the cases $r = 1$ and $r = 2$ are recorded in Table 1. The table clearly suggests that the following rates of convergence hold for smooth test problems:

$$\|u - u_h\|_{L^2(\Omega)} = O(h^{r+1}), \quad \|\nabla u - \nabla_h u_h\|_{L^2(\Omega)} = O(h^r).$$

Table 1. The errors of the computed solution and rates of convergence of the DG method (6.21) with solution $u = \sin(\pi x_1) \sin(\pi x_2)$ on the unit square with $p = 5$ and $\eta = 20/h^4$.

| $r$ | $h$     | $\|u - u_h\|_{L^2}$ | Order | $\|\nabla u - \nabla_h u_h\|_{L^2}$ | Order |
|-----|---------|----------------------|-------|-----------------------------------|-------|
| 1   | 1.00E-01| 6.48E-03             |       | 2.33E-01                          |       |
|     | 5.00E-02| 1.08E-03             | 2.59  | 1.16E-01                          | 1.01  |
|     | 2.50E-02| 1.95E-04             | 2.47  | 5.86E-02                          | 0.99  |
|     | 1.25E-02| 4.99E-05             | 1.97  | 2.94E-02                          | 0.99  |
| 2   | 1.00E-01| 2.25E-03             |       | 1.36E-02                          |       |
|     | 5.00E-02| 2.83E-04             | 2.99  | 3.01E-03                          | 2.18  |
|     | 2.50E-02| 3.60E-05             | 2.97  | 7.30E-04                          | 2.04  |
|     | 1.25E-02| 4.51E-06             | 3.00  | 1.80E-04                          | 2.02  |

6.4. Second order linear elliptic PDEs in non-divergence form. As a fourth example, we consider second order elliptic PDEs written in non-divergence form. Namely, we consider the problem of finding a strong solution satisfying

$$-\tilde{A} : D^2 u = f \quad \text{in } \Omega,
(6.25a)$$

$$u = g \quad \text{on } \partial \Omega.
(6.25b)$$

Here, $\tilde{A} \in [C^{0,\alpha}(\Omega)]^{d \times d}$ ($\alpha \in (0, 1)$) is a given positive definite matrix, and $\tilde{A} : D^2 u$ denotes the Frobenius inner product, i.e., $\tilde{A} : D^2 u = \sum_{i,j=1}^d A_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}$.

We note that if $\tilde{A}$ is sufficient smooth, e.g. if $\text{div} \tilde{A} \in L^\infty(\Omega)$, then we may write the PDE (6.25a) as $-\text{div}(\tilde{A} \nabla u) + (\text{div} \tilde{A}) \cdot \nabla u = f$. We can then apply any of the standard numerical methods for convection-diffusion equations to problem (6.25a). On the other hand, if $\tilde{A}$ only has the regularity $\tilde{A} \in [C^{0,\alpha}]^{d \times d}$, then this argument fails and the construction of numerical methods is less obvious. As far as
we are aware, only two finite element methods have appeared in the literature that addressed the numerical approximation of problems such as (6.25). In [23], Jensen and Smears propose a $P_1$ finite element method for the Hamilton-Jacobi-Bellman equation. To handle the lack of regularity of the coefficient matrix, the authors “freeze the coefficients” element-wise, and then perform the usual integration-by-parts technique. By modifying the framework of Barles-Sougandidis [3], Jensen and Smears show that the numerical solutions converge to the exact solution strongly in $H^1$. Another finite element method for problem (6.25), which is closely related to ours, is the one proposed by Lakkis and Pryer in [24]. Here, the authors used the notation of a discrete Hessian and rewrite problem (6.25) in a mixed form. The advantage of their approach is that the finite element spaces are simply the globally continuous Lagrange elements, which are simple to implement. A possible disadvantage of their approach is that the notion of their discrete Hessian is not local, and therefore writing the problem in its primal form results in a dense stiffness matrix.

To formulate the DG method for the PDE using the discrete differential calculus framework, we again replace the continuous differential operators by the discrete ones. In addition, we have to project both sides of the equation onto the finite element space. This then leads to the following problem: find a function $u_h \in V_r^h$ satisfying

$$-P_r^h(\hat{A} : D_{h,g}^2 u_h) + j_{h,g}(u_h) = P_r^h f$$

with $j_{h,g}(\cdot)$ defined by (6.3). Equivalently, the DG method is to find $u_h \in V_r^h$ such that

$$-P_r^h(\hat{A} : \hat{r}_h) + j_{h,g}(u_h) = P_r^h f,$$

where $r_h \in \tilde{V}_h$ satisfies (6.4a)–(6.4b).

**Remark 6.6.** If the coefficient matrix $\hat{A}$ is constant, then the DG method (6.26) reduces to the LDG method for the PDE $-\text{div}(\hat{A} \nabla u) = f$ with appropriate boundary conditions.

Below we present some numerical test results on the DG method (6.26) with the following parameters: $\Omega = (-0.5, 0.5)^2$, $f = 0$ and

$$g = \begin{cases} 
-\frac{4}{3} & \text{if } x_2 = 0, \\
-x_2^{4/3} & \text{if } x_1 = 0, \\
x_1^{4/3} - 1 & \text{if } x_2 = 1, \\
1 - x_2^{4/3} & \text{if } x_1 = 1 
\end{cases}, \quad \hat{A} = \frac{16}{9} \begin{pmatrix} 
-x_1^{-4/3} \frac{2}{3} & -x_1^{1/3} x_2^{1/3} \\
-x_1^{-1/3} \frac{1}{3} x_2 & x_2^{2/3}
\end{pmatrix}. 
$$

It can readily be checked that the exact solution is given by $u = x_1^{4/3} - x_2^{4/3} \in C^{1,1}(\Omega)$. We note that this is a particularly challenging example since $\hat{A}$ is not uniformly elliptic. The resulting errors for decreasing values of $h$ are listed in Table 2 and a computed solution and error is depicted in Figure 1. The table clearly indicates the convergence of the method, although the exact rates of convergence are not obvious.

**6.5. Fully nonlinear time dependent first order PDEs.** As a fifth example, we consider Hamilton-Jacobi equations. Namely, we consider the problem of finding
Table 2. The errors of the computed solution and rates of convergence with \( r = 1 \).

| \( h \)   | \( \| u - u_h \|_{L^2} \) | order | \( \| \nabla u - \nabla u_h \|_{L^2} \) | order |
|----------|-------------------------|--------|---------------------------------|--------|
| 1.00E-01 | 5.17E-03                |        | 1.45E-01                        |        |
| 5.00E-02 | 3.49E-03                | 0.56   | 9.52E-02                        | 0.60   |
| 2.50E-02 | 2.59E-03                | 0.43   | 6.50E-02                        | 0.55   |
| 1.25E-02 | 2.08E-03                | 0.32   | 4.81E-02                        | 0.43   |

Figure 1. Computed solution (height) and error (surface) of the DG method (6.26) with data (6.27) and parameters \( \eta = 20 \), \( r = 1 \) and \( h = 0.0125 \).

the viscosity solution \( u \in A \subset C^0(\Omega \times (0, T]) \) for the PDE problem

\[
\begin{align*}
(6.28a) & \quad u_t + H(\nabla u) = 0 \quad \text{in } \Omega \times (0, T], \\
(6.28b) & \quad u = g \quad \text{on } \Gamma \subset \partial \Omega \times (0, T], \\
(6.28c) & \quad u = u_0 \quad \text{on } \Omega \times \{0\},
\end{align*}
\]

where the operator \( H \) is a continuous and possibly nonlinear function and \( A \) is a function class in which the viscosity solution \( u \) is unique. We note that the following scheme can also be adapted for \( H \) a function of \( u \), \( x \), and \( t \).

Let \( B(\Omega) \), \( BUC(\Omega) \), \( USC(\Omega) \) and \( LSC(\Omega) \) denote, respectively, the spaces of bounded, bounded uniformly continuous, upper semi-continuous and lower semi-continuous functions on \( \Omega \). We now recall the well-known existence and uniqueness theorem for the corresponding Cauchy problem which was first proved in [11].

**Theorem 6.5.** Let \( H \in C(\mathbb{R}^d) \), \( u_0 \in BUC(\mathbb{R}^d) \). Then there is exactly one function \( u \in BUC(\mathbb{R}^d \times [0, T]) \) for all \( T > 0 \) such that \( u(x, 0) = u_0(x) \), and for every \( \phi \in C^1(\mathbb{R}^d \times (0, \infty)) \) and \( T > 0 \), if \( (x_0, t_0) \) is a local maximum (resp. local
minimum) point of \( u - \phi \) on \( \mathbb{R}^d \times (0, T] \), then
\[
\phi_t(x_0, t_0) + H(\nabla \phi(x_0, t_0)) \leq 0
\]
(resp. \( \phi_t(x_0, t_0) + H(\nabla \phi(x_0, t_0)) \geq 0 \)).

**Definition 6.1.** The function \( u \) whose existence and uniqueness is guaranteed by Theorem 6.5 is called the viscosity solution of the Cauchy version of (6.28).

Recently, a nonstandard LDG method was proposed by Yan and Osher in [34] for approximating the viscosity solution of the Hamilton-Jacobi problem (6.28). The main idea of [34] is to approximate the “left” and “right” side derivatives of the viscosity solution and to judiciously combine them through a monotone and consistent numerical Hamiltonian (cf. [29]) such as the Lax-Friedrichs numerical Hamiltonian
\[
(6.29) \quad \hat{H}(q^-, q^+) := H \left( \frac{q^- + q^+}{2} \right) - \frac{1}{2} \beta \cdot (q^+ - q^-),
\]
where \( \beta \in \mathbb{R}^d \) is an undetermined nonnegative vector chosen to enforce the monotonicity property of \( \hat{H} \), or the Godunov numerical Hamiltonian
\[
(6.30) \quad \hat{H}(q^- \cdot q^+) := \text{ext}_{q_1 \in I(q^-_1, q^+_1)} \cdots \text{ext}_{q_d \in I(q^-_d, q^+_d)} H(q),
\]
where
\[
\text{ext}_{v \in I(a, b)} := \begin{cases} \min_{a \leq v \leq b}, & \text{if } a \leq b, \\ \max_{b \leq v \leq a}, & \text{if } a > b, \end{cases}
\]
for \( I(a, b) := [\min(a, b), \max(a, b)] \).

**Remark 6.7.** For \( r = 0 \) and \( \beta = 1 \) on a uniform rectangular grid, the second term in (6.29) is equivalent to a second order finite difference approximation for the negative Laplacian operator scaled by \( h \). Thus, the second term in (6.29) is called a numerical viscosity, and the method is a direct realization of the vanishing viscosity method from PDE theory. However, for high order elements and variable coefficient vector \( \beta \), while we do not exactly recover a scaled Laplacian operator, we do recover some of the stabilizing properties from adding a second-order-like perturbation.

With the correct choice of discrete derivatives, we can rewrite the nonstandard LDG method of Yan and Osher [34] as follows: For each time step \( n = 1, 2, 3, \ldots \) with \( u^0_h := \mathcal{P}^h u_0 \), find \( u^n_h \in V^h_r \) using the recursive relation
\[
u^n_h = u^{n-1}_h + \Delta t \mathcal{P}^h \hat{H}(\nabla^-_{h,g} u^{n-1}_h, \nabla^+_{h,g} u^{n-1}_h).
\]
In addition to the above Euler time-stepping method with \( \hat{H} \) given by the Lax-Friedrichs numerical Hamiltonian defined in (6.29), in [34] Yan and Osher also implemented the explicit third-order TVD Runge-Kutta time-stepping method given in [30]. Tests include one and two dimensional problems for \( r \geq 0 \).

**Remark 6.8.** For \( r = 0 \) on a Cartesian grid, the method reduces to the convergent FD method of Crandall and Lions proposed and analyzed in [12].
6.6. Fully nonlinear second order PDEs. As the last example, we consider fully nonlinear second order elliptic PDEs. Namely, we consider the problem of finding the viscosity solution \( u \in \mathcal{A} \subset C^0(\Omega) \) for the PDE problem

\[
\begin{align*}
F[u] := F(D^2u, x) &= 0 \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega,
\end{align*}
\]

where the operator \( F \) can be nonlinear in all arguments and \( \mathcal{A} \) is a function class in which the viscosity solution is unique. Throughout this subsection, we assume that \( F[v] \) is elliptic for all \( v \in \mathcal{A} \), \( F \) satisfies a comparison principle and problem (6.31) has a unique viscosity solution \( u \in \mathcal{A} \). The definitions for the above terms are given below. We again use the same function and space notations from Section 6.5. We also note that the following scheme can also be adapted for \( F \) a function of \( u \) and \( \nabla u \).

For ease of presentation, we write (6.31) as

\[
F(D^2u, x) = 0 \quad \text{in } \Omega,
\]

where we have used the convention of writing the boundary condition as a discontinuity of the PDE (cf. [3, p.274]). Also for ease of presentation, we assume that \( F \) is a continuous function and refer the reader to [3, 17, 18] for the case when \( F \) is only a bounded function.

The following three definitions are standard and can be found in [21, 7, 3].

**Definition 6.2.** Equation (6.32) is said to be elliptic if for all \( x \in \Omega \) there holds

\[
F(\tilde{A}, x) \leq F(\tilde{B}, x) \quad \forall \tilde{A}, \tilde{B} \in S_{d \times d}, \tilde{A} \geq \tilde{B},
\]

where \( \tilde{A} \geq \tilde{B} \) means that \( \tilde{A} - \tilde{B} \) is a nonnegative definite matrix, and \( S_{d \times d} \) denotes the set of real symmetric \( d \times d \) matrices.

We note that when \( F \) is differentiable, ellipticity can also be defined by requiring that the matrix \( \frac{\partial F}{\partial D^2u} \) is negative semi-definite (cf. [21, p. 441]).

**Definition 6.3.** A function \( u \in C^0(\Omega) \) is called a viscosity subsolution (resp. supersolution) of (6.32) if, for all \( \varphi \in C^2(\Omega) \), if \( u - \varphi \) (resp. \( u - \varphi \)) has a local maximum (resp. minimum) at \( x_0 \in \Omega \), then we have

\[
F(D^2\varphi(x_0), x_0) \leq 0 \quad \text{(resp. } F(D^2\varphi(x_0), x_0) \geq 0)\text{).}
\]

The function \( u \) is said to be a viscosity solution of (6.32) if it is simultaneously a viscosity subsolution and a viscosity supersolution of (6.31).

**Definition 6.4.** Problem (6.32) is said to satisfy a comparison principle if the following statement holds. For any upper semi-continuous function \( u \) and lower semi-continuous function \( v \) on \( \Omega \), if \( u \) is a viscosity subsolution and \( v \) is a viscosity supersolution of (6.32), then \( u \leq v \) on \( \Omega \).

Inspired by the work of Yan and Osher [34], the first and second authors of this paper recently proposed in [17] a nonstandard LDG method for approximating the viscosity solution of the fully nonlinear second order problem (6.31) in one-dimension. The main idea of [17] is to use all four of the various “sided” approximations for the second order derivative, (3.13a), of the viscosity solution and to judiciously combine them through a g-monotone (generalized monotone) and
consistent numerical operator such as the following Lax-Friedrichs-like numerical operator that has been adopted from [17] for an arbitrary dimensional problem:

\[
\hat{F}(\tilde{P}^{--}, \tilde{P}^{--}, \tilde{P}^{-+}, \tilde{P}^{++}, \xi) := F\left(\frac{\tilde{P}^{--} + \tilde{P}^{-+}}{2}, \xi\right) + \hat{A} : (\tilde{P}^{--} - \tilde{P}^{-+} - \tilde{P}^{--} + \tilde{P}^{++}),
\]

where \( \hat{A} \in \mathbb{R}^{d \times d} \) is a nonnegative constant matrix that is chosen to enforce the g-monotonicity property of \( \hat{F} \). The consistency of \( \hat{F} \) is defined by fulfilling the following property:

\[
\hat{F}(\tilde{P}, \tilde{P}, \tilde{P}, \tilde{P}, x) = F(\tilde{P}, x) \quad \forall \tilde{P} \in \mathbb{R}^{d \times d}, x \in \Omega;
\]

and the g-monotonicity requires that \( \hat{F} \) is monotone increasing in its first and fourth arguments (i.e., \( \tilde{P}^{--}, \tilde{P}^{++} \)) and monotone decreasing in its second and third arguments (i.e., \( \tilde{P}^{-+}, \tilde{P}^{++} \)).

Below we give a reformulation of the nonstandard LDG method of [17] using our DG finite element differential calculus machinery. To this end, we simply replace the continuous differential operators by multiple copies of discrete ones. Due to the lack of integration by parts caused by the nonlinearity, we have to project the equation onto the DG finite element space. This then leads to the following scheme of finding \( u_h \in V_r^h \) such that

\[
\mathcal{P}_r^h \hat{F} \left( D^{-+}_{h,g} u_h, D^{++}_{h,g} u_h, D_{h,g}^{-+} u_h, D_{h,g}^{++} u_h, x \right) = 0,
\]

where \( D^{-+}_{h,g}, D^{++}_{h,g}, D_{h,g}^{-+}, D_{h,g}^{++} \) are the four sided numerical Hessians (with the prescribed boundary data \( g \)) defined in Section 3. It can be shown that (6.35) is indeed equivalent to the LDG scheme of [17] in one-dimension.

Remarks 6.2.

(a) When \( r = 0 \) and \( d = 1 \), scheme (6.35) reduces to the FD method given in [18], which was proved to be convergent.

(b) For \( r = 0 \) and \( \hat{A} = \hat{I}_{d \times d} \) on a rectangular grid, the second term on the right hand side in (6.34) is equivalent to a second order finite difference approximation for the biharmonic operator scaled by \( h^2 \). Thus, the second term in (6.34) is called a numerical moment, and the method is a direct realization of the vanishing moment method proposed in [19, 20]. However, for high order elements and variable coefficient matrices \( \hat{A} \), while we do not exactly recover a scaled biharmonic operator, we do recover some of the stabilizing properties from adding a fourth-order-like perturbation.

(c) Numerical tests of [17] show the above discretization can eliminate many, and in some cases all, of the numerical artifacts that plague standard discretizations for fully nonlinear second order PDEs (cf. [16] and the references therein).

(d) Fully discrete schemes of [17] for parabolic fully nonlinear second order PDEs can also be recast using the DG finite element differential calculus machinery.

To solve the algebraic system (6.35), a nonlinear solver must be used. Numerical tests of [17] show that when the initial guess for \( u_h \) is not too close to a nonviscosity solution of the PDE problem, a Newton-based solver performs well as long as the solution is not on the boundary of the admissible set \( \mathcal{A} \). However, in
the degenerate case, a split solver based on the DWDG discretization for the Poisson equation from Section 6.1.2 and the Lax-Friedrichs-like operators in (6.34) appear to be better suited. This new solver for (6.35) is given below in Algorithm 6.1.

We note that this solver appears to work well even for some cases when the initial guess for \( u_h \) is not in \( A \). Thus, the solver uses key tools from the discretization to address the issue of conditional uniqueness of viscosity solutions.

**Algorithm 6.1.** Pick \( u_h^{(0)} \in V_r^h \). Let \( \Lambda_{h,g}^{++}v \) and \( \Lambda_{h,g}^{+-}v \) denote the diagonal matrices formed by the diagonals of \( D_{h,g}^{++}v \) and \( D_{h,g}^{+-}v \), respectively, and \( \Lambda_{h,g}^{--}v \) and \( \Lambda_{h,g}^{-+}v \) denote the diagonal matrices formed by the diagonals of \( D_{h,g}^{--}v \) and \( D_{h,g}^{-+}v \), respectively, for all \( v \in V_r^h \). Let \( \tilde{\Lambda}I \) denote the diagonal matrix formed by the vector \( \lambda \in \mathbb{R}^d \).

For \( n = 1, 2, 3, \ldots \),

**Step 1:** For \( i = 1, 2, \ldots, d \), set

\[
\left[ G_i \right]^{(n)} := F \left( \left( (D_{h,g}^{++} - \Lambda_{h,g}^{+-})/2 + (D_{h,g}^{--} - \Lambda_{h,g}^{++})/2 \right) u_h^{(n-1)} + \lambda^{(n)}I, x \right) \\
+ A \left[ \Lambda_{h,g}^{+-}u_h^{(n-1)} + \Lambda_{h,g}^{++}u_h^{(n-1)} - 2 \lambda^{(n)}I \right]_{ii}
\]

for a fixed constant \( A > 0 \) such that \( G_i^{(n)} \) is monotone decreasing with respect to \( \lambda \).

**Step 2:** Solve for \( \lambda^{(n)} \in V_r^h \) such that

\[
\left( \left[ G_i \right]^{(n)}, \phi_i \right)_{T_h} = 0
\]

for all \( \phi_i \in V_r^h \), \( i = 1, 2, \ldots, d \).

**Step 3:** Solve for \( u_h^{(n)} \in V_r^h \) such that

\[
\frac{1}{2} \left( \nabla_{h,g}^{++}u_h^{(n)}, \nabla_{h,0}^{++}v_h \right)_{T_h} + \frac{1}{2} \left( \nabla_{h,g}^{--}u_h^{(n)}, \nabla_{h,0}^{--}v_h \right)_{T_h} = \sum_{i=1}^d \left( \lambda_i^{(n)}, v_h \right)_{T_h}
\]

for all \( v_h \in V_r^h \).

Note that we are solving the discretization that results from the choice \( \tilde{\Lambda} = A \tilde{I}_{d \times d} \) in (6.34).

The above solver is a fixed point method for the diagonal of the Hessian approximation formed by \( (D_{h,g}^{++} + D_{h,g}^{--})/2 \). We can see that the nonlinear equation in Step 2 is entirely monotone and local in its nonlinear components when \( A \) is sufficiently large. Step 3 is well-defined due to the well-posedness of the DWDG method for Poisson’s equation. Lastly, as written, we can see that the numerical moment inspired by the vanishing moment methodology serves as the motivation for using the diagonal of the Hessian approximation as the fixed-point parameter. Numerical tests indicate that the solver destabilizes numerical artifacts even when they exist (cf. [17]). Thus, the solver directly addresses the issue of conditional uniqueness by enforcing the preservation of monotonicity in the Hessian approximation at each iteration.

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