Abstract

ABSTRACT In previous studies of the theory of Coulomb excitation, the term in the electromagnetic interaction which is quadratic in the vector potential has been ignored. In this paper, we use qualitative arguments and detailed numerical calculations to show that this quadratic term must be included when the bombarding energy is in excess of about 20 GeV per nucleon.

1 Introduction

The subject of relativistic Coulomb excitation has received extensive study in the two decades since its theoretical foundations were established in the classic work of Alder and Winther [1]. In recent years, this subject has been shown to be relevant to the practical question of the stability of beams of relativistic heavy ions, since some of the processes which lead to loss of beam ions are initiated by Coulomb excitation [2, 3, 4].

In the semi-quantal approach to relativistic nuclear Coulomb excitation, the relative motion of the projectile and target is treated classically. Indeed, it is usually assumed to be straight-line motion at constant speed \( v \). The evolution of the internal degrees of freedom of each nucleus, under the influence of the classical electromagnetic field produced by the other nucleus, is then followed using quantum mechanics. The effect of this classical electromagnetic field on the nuclear charge and current densities has two components: one that is linear in the electromagnetic potentials, and one that is quadratic. All previous work on the theory of Coulomb excitation has been based on the linear term. However, it is known that there are some electromagnetic processes in which one cannot neglect the term quadratic in the potentials. Examples are Thomson scattering of photons by electrons [5], and the Zeeman effect in hydrogen atom states of high principal quantum number [6].
Our object in this paper is to determine whether the interaction term quadratic in the potentials must be included when we study the nuclear effects of the highly retarded electromagnetic field due to a rapidly passing projectile.

2 The electromagnetic interaction

We follow the usual approach of the semi-classical theory of Coulomb excitation, in which we solve the quantum-mechanical problem of the response of the target nucleus to the perturbation provided by the electromagnetic field of the passing projectile (ref.[1]). In the absence of this perturbation, the target Hamiltonian has the form

\[ H_0 = \sum_j \frac{1}{2m} p_j \cdot p_j + W(\zeta) \]  

(1)

Here \( \zeta \) represents the internal degrees of freedom of the target. When the effect of the projectile electromagnetic field is included, the target Hamiltonian becomes

\[ H = \sum_j \left( \frac{1}{2m} (p_j - e_j c A_{ret}^{\zeta}(r'_j, t))^2 + e_j \varphi^{\zeta}_{ret}(r'_j, t) + W(\zeta) \right) \]

\[ = H_0 + V_1 + V_2 \]  

(2)

with

\[ V_1 = -\sum_j \frac{e_j}{2mc} \left[ p_j \cdot A_{ret}^{\zeta}(r'_j, t) + A_{ret}^{\zeta}(r'_j, t) \cdot p_j \right] \]

\[ + e_j \varphi^{\zeta}_{ret}(r'_j, t), \]  

(3)

\[ V_2 = \sum_j \frac{e_j^2}{2mc^2} A_{ret}^{\zeta}(r'_j, t) \cdot A_{ret}^{\zeta}(r'_j, t). \]  

(4)

\( e_j \) is the proton charge or 0 depending upon whether the jth nucleon is a proton or neutron. Previous investigations (ref.[7, 8]) of relativistic Coulomb excitation have focussed only on the effect of the perturbation \( V_1 \), the term linear in \( e_j \). This term is evaluated using either the Lienard-Wiechert potential for \( \varphi^{\zeta}_{ret}(r'_j, t) \) and \( A_{ret}^{\zeta}(r'_j, t) \), or the Fermi-Weissaker-Williams method of equivalent photons [9, 10]. In both approaches, it is assumed that the perturbation produced by \( V_2 \), is of secondary importance, and can be neglected. We note that the classical non-relativistic equations of motion, with the full Lorentz force,

\[ m \frac{dv}{dt} = eE(r, t) + \frac{e}{c} v \times B(r, t) . \]

is obtained from the Hamiltonian (2.2) only if \( V_2 \) is included.

Our object in this study is to determine whether the \( V_2 \) term in the interaction makes a numerically significant contribution to the Coulomb excitation amplitude. We will show in the following that for lead projectiles with kinetic energy below about 20 GeV per nucleon, the \( V_2 \) term can safely be neglected. However, for kinetic energies at or above 80 GeV per nucleon, the contributions from \( V_2 \) are comparable to the those from \( V_1 \).
We assume that the perturbing electromagnetic field is produced by a spherically-symmetric projectile of charge \( Z_P e \) moving along a trajectory given by

\[
r = b + vt \hat{z}.
\]

where \( b \) is the impact parameter vector perpendicular to \( \hat{z} \), and \( v \) is the constant projectile speed. The scalar and vector potentials due to this projectile at the point \( r' \) of the target at time \( t \) can be taken to be the Lienard-Wiechert expressions (ref.[9])

\[
\varphi_{\text{ret}}(r', t) = \frac{Z_P e \gamma}{\sqrt{(x-x')^2 + (y-y')^2 + \gamma^2 (vt-z')^2}}
\]

\[
A_{\text{ret}}(r', t) = \frac{v}{c} \varphi_{\text{ret}}(r', t) \hat{z}.
\]

We focus our attention on the Fourier transform of the matrix elements of the perturbation,

\[
V_{\beta\alpha}(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{\hbar} e^{i\omega t} <\phi_\beta|V_1(t) + V_2(t)|\phi_\alpha>
\]

\[
= V_{\beta\alpha}^{(1)}(\omega) + V_{\beta\alpha}^{(2)}(\omega). \tag{7}
\]

Here \( \phi_\alpha \) is an eigenstate of \( H_0 \) corresponding to unperturbed eigenvalue \( E_\alpha \). The first-order Born approximation for the transition between \( \phi_\alpha \) and \( \phi_\beta \) is expressed in terms this matrix element, with \( \omega \) given its on-shell value of \( (E_\beta - E_\alpha) / \hbar \).

### 3 Orders of magnitude

We begin with preliminary comparisons of the relative magnitudes of of matrix elements of \( V_1 \) and \( V_2 \) and of their dependences on bombarding energy and \( \omega \).

**Bombarding energy dependence.** The Fourier transforms of \( V_1 \) and \( V_2 \) require, respectively, the following time integrals:

\[
V_1(|b - \rho'|, z') = \int \frac{dt}{\hbar} e^{i\omega t} \left( \frac{\gamma}{\sqrt{|b - \rho'|^2 + \gamma^2 (vt-z')^2}} \right),
\]

\[
V_2(|b - \rho'|, z') = \int \frac{dt}{\hbar} e^{i\omega t} \left( \frac{\gamma^2}{|b - \rho'|^2 + \gamma^2 (vt-z')^2} \right).
\]

The two \( t \) integrations yield

\[
V_1(|b - \rho'|, z') = \frac{e^{i\frac{\pi}{2} z'}}{h \nu} K_0\left(\frac{\omega}{\gamma \nu} |b - \rho'|\right), \tag{8}
\]

\[
V_2(|b - \rho'|, z') = \frac{e^{i\frac{\pi}{2} z'}}{h \nu} \pi \gamma \frac{e^{-\frac{\pi}{2} \gamma |b - \rho'|}}{|b - \rho'|}. \tag{9}
\]
\(V_{\beta\alpha}^{(1)}(\omega)\) and \(V_{\beta\alpha}^{(2)}(\omega)\) are obtained by calculating the matrix elements of these time integrals between the nuclear states \(\phi_\alpha\) and \(\phi_\beta\).

\(K_0(\frac{\omega}{\gamma} |b - \rho'|)\) in Equation (3.1) diverges like \(\log(\gamma)\) as \(\gamma \to \infty\), whereas \(\gamma e^{-\frac{\omega}{\gamma} |b - \rho'|}\) diverges like \(\gamma\). Thus as the bombarding energy becomes very large, the \(A \cdot A\) matrix element grows relative to the matrix element linear in \((\varphi, A)\). Alternatively, we can say that the extra factor of

\[
\frac{\gamma}{\sqrt{|b - \rho'|^2 + \gamma^2|vt - z'|^2}}
\]

in \(V_2\) increases the effect of retardation, which is most dramatic at large \(\gamma\).

\(\omega\)-dependence. For \(\omega \to \infty\), both expressions (3.1) and (3.2) decay exponentially. However, since \(K_0(x) \to \sqrt{\frac{2}{\pi}} e^{-x}\), there is an extra factor of \(1/\sqrt{\omega}\) in the fall-off of \(K_0(\frac{\omega}{\gamma} |b - \rho'|)\) compared to \(e^{-\frac{\omega}{\gamma} |b - \rho'|}\). Thus at large \(\gamma\), where both matrix elements are appreciable, the \(A \cdot A\) matrix element increases relative to the \((\varphi, A)\) matrix element as \(\omega\) increases. The extra retardation associated with the \(A \cdot A\) term makes the electromagnetic pulse sharper, and so favors higher \(\omega\) values in the Fourier transform of the matrix element.

Relative magnitude. We can get a crude estimate of the relative importance of \(V_2\) and \(V_1\) by considering the ratio

\[
\frac{e^2}{2mc^2} A_{r}^{\text{ret}}(r, t) \cdot A_{t}^{\text{ret}}(r, t) \sim \frac{e^2}{2mc^2} \left( \frac{\varphi_{\text{ret}}(r, t)}{\varphi_{\text{ret}}(r, t)} \right)^2 \sim \frac{e}{2mc^2} \varphi_{\text{ret}}(r, t) (1 - \frac{1}{\gamma^2}).
\]

To make this a little more quantitative, let us give \(\varphi_{\text{ret}}(r, t)\) its maximum value of \(\gamma Z_p e/b\). Then the above ratio is

\[
\gamma Z_p \frac{e^2}{2mc^2 b} (1 - \frac{1}{\gamma^2}).
\]

For \(Z_p = 82\) and \(b = 10 Fm\), this becomes \(\sim 0.0063\gamma (1 - \frac{1}{\gamma})\). This has a value of \(\sim 0.01\) for \(T_p/A = 1\text{ GeV (}\gamma = 2.066\text{)}\), and \(\sim 0.34\) for \(T_p/A = 50\text{ GeV (}\gamma = 54.3\text{)}\). Thus we expect that \(V_1\) can be neglected compared to \(V_1\) at projectile kinetic energies per nucleon of 1 GeV or lower, but at 50 GeV, \(V_2\) may be of comparable importance to \(V_1\). It will be seen below that this expectation is verified by detailed calculations.

### 4 Matrix elements

The evaluation of \(V_{\beta\alpha}^{(1)}(\omega)\) has been fully described in the literature (ref.\[7, 8\]). We therefore turn our attention to \(V_{\beta\alpha}^{(2)}(\omega)\):

\[
V_{\beta\alpha}^{(2)}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{\hbar} e^{i\omega t} < \phi_{M_{\beta}}^{J_{\beta}} | \sum_j \frac{e_j^2}{2mc^2} A_{C}^{\text{ret}}(r'_j, t) \cdot A_{C}^{\text{ret}}(r'_j, t) | \phi_{M_{\alpha}}^{J_{\alpha}} >
\]

\[
= \frac{1}{2\hbar mc^2} \int_{-\infty}^{\infty} dt e^{i\omega t} < \phi_{M_{\beta}}^{J_{\beta}} | \sum_j \frac{e_j^2}{2} [ \varphi_{C}^{\text{ret}}(r'_j, t) ]^2 | \phi_{M_{\alpha}}^{J_{\alpha}} >
\]

\[
= \frac{1}{2\hbar mc^2} \int_{-\infty}^{\infty} dt e^{i\omega t} [ \varphi_{C}^{\text{ret}}(r'_j, t) ]^2 \rho_{\beta\alpha}^{J_{\beta}J_{\alpha}}(r', r).
\]
The transition charge density introduced in the last line of Equation (10) can be expanded in terms of spherical harmonics of \( \hat{\mathbf{r}}' \):

\[
\rho_{J_\beta M_\beta; J_\alpha M_\alpha}^{\mathbf{r}'}(\mathbf{r}') = \langle \phi_{J_\beta M_\beta}^{J_\alpha M_\alpha} | \sum_j e_j^2 (\mathbf{r}' - \mathbf{r}_j') | \phi_{J_\alpha M_\alpha}^{J_\beta M_\beta} \rangle = (-1)^{J_\beta - M_\beta} \sum_L (J_\beta J_\alpha - M_\beta M_\alpha | L M_\alpha - M_\beta ) \rho_L^{J_\alpha M_\alpha} Y_{M_\alpha - M_\beta}^L(\hat{\mathbf{r}}'),
\]

leading to a multipole expansion of our matrix element:

\[
V^{(2)}_{\beta \alpha}(\omega) = (-1)^{J_\beta - M_\beta} \sum_L (J_\beta J_\alpha - M_\beta M_\alpha | L M_\alpha - M_\beta ) V^L_{M_\alpha - M_\beta}^{J_\alpha M_\alpha}(\omega)
\]

\[
V_M^L(\omega) = \frac{v^2}{2 \hbar mc^4} \int d^3r' \rho_L(r') Y_M^L(\hat{\mathbf{r}}') \int_{-\infty}^{\infty} dt e^{i\omega t} \frac{(Ze_\gamma)^2}{|\mathbf{b} - \mathbf{r}'|^2 + \gamma^2(vt - z')^2} = \frac{\pi v}{2 \hbar mc^4} \int d^3r' \rho_L(r') Y_M^L(\hat{\mathbf{r}}') e^{i\omega t} e^{-\frac{1}{2}|\mathbf{b} - \mathbf{r}'|}\hspace{1cm} (11)
\]

We disentangle the \( \mathbf{b} \) and \( \mathbf{r}' \) dependence of \( |\mathbf{b} - \mathbf{r}'| \) by using the expansion

\[
e^{-\frac{1}{2\gamma v}|\mathbf{b} - \mathbf{r}'|} = - \frac{1}{\gamma v} \sum_{\ell=0}^{\infty} \frac{h_{\ell+1}^{(1)}(i\omega/\gamma v) - j_{\ell}(i\omega/\gamma v)}{(\ell - m - 1)!(\ell + m + 1))!!} e^{im(\phi - \phi')} \hspace{1cm} (12)
\]

which is valid when \( \ell' < b \). To complete the evaluation of the \( \mathbf{r}' \) integration in (11), we express \( \rho_L(r') Y_M^L(\hat{\mathbf{r}}') \) in cylindrical coordinates by expanding \( \rho_L(r') Y_M^L(\hat{\mathbf{r}}') \) in terms of 3-dimensional harmonic oscillator eigenfunctions

\[
\rho_L(r') Y_M^L(\hat{\mathbf{r}}') = \sum_n c(n) \psi_M^{nL}(r')
\]

and then performing a unitary transformation to products of 2-dimensional oscillator eigenfunctions \( \psi_{N_\perp M}(r', \varphi') \) and one-dimensional oscillator eigenfunctions \( \psi_{N_z}(z') \):

\[
\psi_M^{nL}(r') = \sum_{N_\perp N_z} < n L M | N_\perp N_z > \psi_{N_\perp M}(r', \varphi') \psi_{N_z}(z')
\]

The coefficients needed for this expansion are given in ref. (11). The \( \mathbf{r}', \varphi', z' \) integrations can now be done in (11), with the final result

\[
V_M^L(\omega) = i^L \sqrt{\frac{2}{\nu mc^2}} \frac{\pi^2}{\hbar} \frac{Ze_\gamma^2}{\nu mc} \sum_{\ell = |M|, |M| + 2, \ldots} (2\ell + 1) h_{\ell+1}^{(1)}(i\omega/\gamma v) (\ell - M - 1)!! (\ell + M - 1)!! (\ell - M)!! (\ell + M)!! \\
\times \sum_{n=0}^{n_{\text{max}}} (-1)^n c(n) \sum_{p=0,1,2,\ldots} (-1)^p \psi_{N_z}(\omega/\gamma v) \sqrt{p!(M + p)!} < nLM | 2p + |M|, 2n + L - |M| - 2p, M > \sum_{j=0}^{p} \sum_{n' = 0}^{\infty} (-1)^j (\frac{\omega}{\gamma v})^{\ell + 2n'} \frac{(M + \ell + 2j + 2n')!!}{j!(M + j)!(p - j)!(2n')!!(2\ell + 2n' + 1)!!}.\hspace{1cm} (14)
\]
In this equation we have set $\phi = \pi/2$ (the projectile trajectory is in the $\hat{y} - \hat{z}$ plane), and assumed that both $\omega$ and $M$ are positive. We get the remaining matrix elements using the symmetry relations

$$V_L^M(-\omega) = (-1)^{L-M}V_L^M(\omega), \quad V_L^M(\omega) = V_L^M(-\omega);$$

$\nu$ in (14) is the harmonic oscillator size parameter $\hbar/(m\omega_{osc})$, which is used for all the oscillator eigenstates. It is chosen to give the best convergence for the expansion (13) of the transition charge density. $\psi_N^l(\hat{r})$ is an oscillator eigenstate with argument $\omega v$, but with size parameter $1/\nu$. In the examples given below, we will consider Coulomb excitation of a giant quadrupole excitation in $^{40}$Ca (ref. [8]). We use the Tassie model ref. [12, 13], which describes the $2^+$ resonance as a one-quantum vibrational oscillation of an incompressible irrotational fluid. Explicit expressions for the transition charge and current densities are given in ref. [8]. In this case, expansion of the transition charge density in terms of harmonic oscillator eigenstates is facilitated by the finite sum

$$(\sqrt{\nu r'})^{L+2\kappa}e^{-\frac{\nu r'^2}{2}}Y_L^M(\hat{r'}) = \frac{\kappa!!(2L + 2\kappa + 1)!!}{2^\kappa} \sqrt{\frac{\nu}{\pi}} \times \sum_{n=0}^{\kappa} \frac{(-1)^n}{(\kappa - n)!} \frac{1}{\sqrt{n!(2L + 2n + 1)!!2^{L+2-n}}} \psi_{nL}^M(\hat{r'}).$$

## 5 Application to the exitation of vibrational states

The $V_2$ matrix elements calculated in Equation (4.5), added to the well-known matrix elements of $V_1$, can be used in a full coupled-channel calculation to yield a complete description of the Coulomb excitation process. Since our object here is only to assess the relative importance of $V_2$, we will limit our attention to the exactly-solvable vibrational model. The main result of this model is that the probability, $P(b, n)$, of exciting a state with $n$ oscillator quanta when the impact parameter is $b$ is given by a Poisson distribution

$$P(b, n) = e^{-q(b)} \frac{q(b)^n}{n!},$$

where $q(b)$ is the square of the on-shell matrix element of the interaction between the $n = 0$ and $n = 1$ states. The Coulomb excitation cross-section for the population of the $n-$quantum state is then obtained from $P(b, n)$ by an integration over $b$:

$$\sigma_n = \int_{b_{\text{min}}}^{\infty} 2\pi b db \ P(b, n).$$

The lower limit $b_{\text{min}}$ is chosen to be large enough to ensure that only electromagnetic interactions contribute to the process. We apply this model to a hypothetical giant quadrupole oscillator band in $^{40}$Ca, with $\hbar\omega = 20$ MeV. The $B(E2, 0^+ \rightarrow 2^+)$ is assumed to have a value of 450 $e^2\text{fm}^4$, which exhausts the energy-weighted sum rule. The projectile is $^{208}$Pb, and we have taken $b_{\text{min}}$ to be 12 fm [14].
Figure 1 shows calculated excitation functions for populating the magnetic substates of the one-quantum $I = 2$ level, using $V_1$ alone. The curves labelled by $M = |1|, |2|$ correspond to the summed cross-sections for $M = \pm 1, \pm 2$, respectively. The strong decrease of the $M = 0$ cross-section at high bombarding energy is a result of cancellation of the contributions of the scalar and vector potentials to $V_1$ of (2.3). At the highest energies, the $M = \pm 1$ cross-section is greatest. This is related to the applicability of the equivalent photon method in this region, since a shower of photons moving in the $+\hat{z}$ direction can only have $M = \pm 1$.

Figure 2 shows calculated excitation functions, including the effects of both $V_1$ and $V_2$. Since the $A \cdot A$ interaction essentially involves the delivery of two photons to the target, arguments limited to one-photon exchange no longer apply. In particular, the $M = 0$ cross-section is no longer suppressed at high bombarding energy. The large cross-section is due to the fact that the strongly-retarded potential can produce $M = 0$ transitions at very large impact parameters.

A polarization-insensitive measurement would yield the cross-section incoherently summed over $M$. This is plotted in Figure 3, for $V_1$ alone, and for the full $V_1 + V_2$ interaction. It is seen that for bombarding energy per nucleon below about 20 GeV, $V_2$ makes a relatively small contribution, but at 80 GeV the full $V_1 + V_2$ interaction yields about twice the cross-section of $V_1$ alone. At higher bombarding energy, the $V_2$ interaction dominates.

6 Discussion

We have seen that qualitative arguments and detailed numerical calculations strongly indicate the importance of the $A \cdot A$ contribution to the response of a target nucleus to the electromagnetic field of a highly-relativistic projectile. It should be emphasized that approximation methods such as the FWW method of virtual quanta provide convenient descriptions of the electromagnetic field, but do not, by themselves, include the effect of $A \cdot A$ which is quadratic in $e_j$.

An uncertainty of our analysis is the applicability of the non-relativistic Hamiltonian (2.2) when the external electromagnetic field is very strong, even when the protons in the target are moving slowly. If we attempt to derive this Hamiltonian by taking the non-relativistic limit of a proton Dirac equation, we need to assume that both the proton kinetic energy and $e\varphi_C^{\text{ret}}$ are small compared to the proton rest energy (see, for example, [13]). But we have seen in Section III that the condition $e\varphi_C^{\text{ret}} <\ll mc^2$ is violated for grazing collisions when the bombarding energy per nucleon exceeds about 50 GeV. Thus, if nucleons are to be regarded as slowly-moving Dirac particles, the discussion of their electromagnetic properties requires something more complicated than Equation (2.2) when we are dealing with the fields encountered in ultra-relativistic collisions.

The $V_2$ contributions to the curves in Figures 2 and 3 suffer from a gauge ambiguity. The on-shell matrix elements of $V_1$ are gauge invariant; the on-shell matrix elements of $V_2$ are not. Of course, if we use $V_2$ in a full coupled-channel calculation, and arrive at exact Coulomb-excitation cross-sections, these will be manifestly gauge invariant. We note that a full coupled-channel calculation using $V_1$ alone will not be gauge invariant, since both $V_1$ and $V_2$ are required in Equation (2.2) in order to yield a gauge-invariant Schrodinger
equation. The vibrational model used in this paper is not fully gauge invariant. The gauge we have used is that implied by the Lienard-Wiechert potential, and is widely employed in studies of Coulomb excitation. Although we cannot claim gauge invariance for our results, we believe that we have shown that, at bombarding energies in excess of about 20 GeV per nucleon, it is important to include the effects of the $\mathbf{A} \cdot \mathbf{A}$ component of the electromagnetic interaction.

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Figure captions

**Fig.1** Coulomb excitation cross sections for one quadrupole phonon for the standard term $V_1$. The curves correspond to different values of magnetic quantum number transfer. More details can be found in the text.
Fig. 2 Coulomb excitation cross sections of one quadrupole phonon for the full interaction $V_1 + V_2$.

Fig. 3 Comparison of the cross sections for $V_1$ and $V_2$ summed over the magnetic quantum numbers.
Figure 1.

Figure 2.

Figure 3.

Bombarding energy per nucleon (GeV)