THE RATIONAL SOLUTIONS OF THE MIXED NONLINEAR SCHRÖDINGER EQUATION

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Abstract. The mixed nonlinear Schrödinger (MNLS) equation is a model for the propagation of the Alfvén wave in plasmas and the ultrashort light pulse in optical fibers with two nonlinear effects of self-steepening and self phase-modulation (SPM), which is also the first non-trivial flow of the integrable Wadati-Konno-Ichikawa (WKI) system. The determinant representation $T_n$ of a n-fold Darboux transformation (DT) for the MNLS equation is presented. The smoothness of the solution $q_{2k}$ generated by $T_{2k}$ is proved for the two cases (non-degeneration and double-degeneration) through the iteration and determinant representation. Starting from a periodic seed (plane wave), rational solutions with two parameters $a$ and $b$ of the MNLS equation are constructed by the DT and the Taylor expansion. Two parameters denote the contributions of two nonlinear effects in solutions. We show an unusual result: for a given value of $a$, the increasing value of $b$ can damage gradually the localization of the rational solution, by analytical forms and figures. A novel two-peak rational solution with variable height and a non-vanishing boundary is also obtained.

Key words: mixed nonlinear Schrödinger equation, rogue wave, rational solution, Darboux transformation.

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1. Introduction

Rogue wave (RW) has been introduced and become an interesting objective in the investigation of oceanography (1, 2) (and references therein), starting with modeling a short-lived large amplitude wave in ocean. Recently, rogue waves have also been observed in photonic crystal fibers (3, 4), in space plasmas (5, 7), in Bose-Einstein condensates (8), in water tanks (9, 11), and so on.

One of widely accepted prototypes of rogue wave in one dimensional space and time is considered as Peregrine soliton (12, 14) of the nonlinear Schrödinger equation (NLS), which usually takes the form of a single dominant peak accompanied by one deep cave at each side in a plane with a nonzero boundary. In other words, the characteristic property of the RW is localization in both space and time directions in a nonzero plane. The existence of this solution is due to modulation instability of the NLS equation (12, 15, 18). In consequence, different patterns of the RW will occur when two or more breathers with different relative phase shifts collide with each other (19, 20). One of the possible generating mechanisms for rogue waves is through the creation of breathers possessing a particular frequency, which is realized theoretically by choosing a special eigenvalue in breathers (26). Recently, by applying Darboux transformation (DT) (27, 30),

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the rogue waves\textsuperscript{31,33} of derivative nonlinear Schrödinger equation (DNLS) are also given in the form of “Peregrine soliton”.

In the field of optics, the nonlinear terms in the NLS and DNLS denote the effects of phase-modulation (SPM) and self-steepening, respectively. So it is natural and worthwhile to look for an integrable equation with these two terms from the points of view of mathematics and physics. There is indeed such an integrable equation—a mixed NLS (MNLS) equation\textsuperscript{34,35}

\[ q_t - i q_{xx} + a(q^* q)_x + ibq^* q^2 = 0, \]  

(1)
in physics. Here \( q \) represents a complex field envelope and asterisk denotes complex conjugation, \( a \) and \( b \) are two non-negative constants, and subscript \( x \) (or \( t \)) denotes partial derivative with respect to \( x \) (or \( t \)). The MNLS equation is used to model the propagations of the Alfvén waves in plasmas\textsuperscript{34} and the ultrashort light pulse in optical fibers\textsuperscript{35}. Moreover, the MNLS equation can also be given by following coupled system

\[ r_t + i r_{xx} - a(r^2 q)_x + ibr^2 q = 0, \]  

(2) 

\[ q_t - i q_{xx} - a(rq^2)_x - ibrq^2 = 0, \]  

(3) 

under a condition \( r = -q^* \). This coupled system is nothing but the first non-trivial flow of the Wadati-Konno-Ichikawa (WKI) system\textsuperscript{36}, and the corresponding Lax pair is given by the WKI spectral problem and a time flow\textsuperscript{36}

\[ \partial_x \psi = (-aJ\lambda^2 + Q_1 \lambda + Q_0)\psi = U\psi, \]  

(4) 

\[ \partial_t \psi = (-2a^2 J\lambda^4 + V_3 \lambda^3 + V_2 \lambda^2 + V_1 \lambda + V_0)\psi = V\psi, \]  

(5) 

with

\[ J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q_1 = \begin{pmatrix} \sqrt{2b}r & aq \\ ar & -\sqrt{2b} \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & i\sqrt{\frac{b}{2}}q \\ i\sqrt{\frac{b}{2}}r & 0 \end{pmatrix}, \]

\[ \psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad V_3 = \begin{pmatrix} 4a\sqrt{2b} & 2a^2 q \\ 2a^2 r & -4a\sqrt{2b} \end{pmatrix}, \quad V_2 = \begin{pmatrix} 4ib - ia^2 rq \\ 3ia\sqrt{2b}r \end{pmatrix} - (4ib - ia^2 rq), \]

\[ V_1 = \begin{pmatrix} \frac{1}{2}ibqr & \sqrt{\frac{1}{2}}b(-q_x + iarq^2) \\ -\frac{1}{2}(r_x + iarq^2) & -\frac{1}{2}ibqr \end{pmatrix}, \quad V_0 = \begin{pmatrix} \frac{1}{2}ibqr & \sqrt{\frac{1}{2}}b(-q_x + iarq^2) \\ -\frac{1}{2}(r_x + iarq^2) & -\frac{1}{2}ibqr \end{pmatrix}. \]

Here \( \lambda \in \mathbb{C} \), is called the eigenvalue (or spectral parameter), and \( \psi \) is called the eigenfunction associated with \( \lambda \) of the WKI system. Equations (2) and (3) are equivalent to the integrability condition \( U_t - V_x + [U, V] = 0 \) of (4) and (5). In addition, the MNLS equation can be generated from other systems (or equations) in the literatures\textsuperscript{37–40}. The MNLS equation is exactly solved by the inverse scattering method under the non-vanishing boundary condition\textsuperscript{41}. Later, periodic solutions of the MNLS equation are be analyzed in terms of the Riemann’s \( \theta \) functions\textsuperscript{42}. Some special solutions, such as breather solutions, of this equation are also discussed in Ref.\textsuperscript{43,44}. At the same time, the solutions of the MNLS equation have been constructed via Backlund or Darboux transformation\textsuperscript{45–48} and the Hirota method\textsuperscript{49,50}. What is more, using the matrix Riemann-Hilbert factorization approach, asymptotic analysis of the MNLS equation is discussed in Ref.\textsuperscript{51,52}. Considering many wave propagation phenomena described by integrable equations in some ideal conditions, the effects of small perturbations on the MNLS equation are study by the direct soliton perturbation theory\textsuperscript{53} and the perturbation
theory based on the inverse scattering transform \cite{54,55}. Recently, the semiclassical analysis of the MNLS has been studied in Ref. \cite{56–58}.

In light of the above results, two questions arise naturally. First, is there a rational solution of the MNLS equation which can be generated from a periodic seed by DT and Taylor expansion? Second, how the localization of this rational solution is affected by the two nonlinear effects through the $a$ and $b$? For the first question, the first order and the second order rational solutions are given explicitly by the determinant representation of the DT and Taylor expansion with respect to the degenerate eigenvalues. To answer the second question, according to a common understanding of the role for the nonlinear effects in wave propagation, one reasonable conjecture is that the localization of this solution will be enhanced because of the appearance of the two nonlinear effects. However we shall show an unusual result: for a given value of $a$, the increasing value of $b$ can damage gradually the localization of the rational solution.

The organization of this paper is as follows. In section 2, we provide a relatively simple approach to DT for the WKI system, and then the expressions of the $q^{[n]}, r^{[n]}$ and $\psi^{[n]}$ of the WKI system are generated by n-fold Darboux transformation. The reduction of DT for the WKI system to the MNLS equation is also discussed by choosing paired eigenvalues and eigenfunctions. In section 3, the smoothness of the solutions $q^{[2k]}$ generated by $T_{2k}$ is proved for the two cases (non-degeneration and double-degeneration). In section 4, we present the rational solutions of the MNLS and discuss its localized properties for a given value of $a$. Finally, we summary our results in section 5.

2. Darboux transformation

Inspired by the results of the DT for the NLS \cite{27,28} and the DNLS \cite{29–32}, the main task of this section is to present a detailed derivation of the Darboux transformation of the MNLS and the determinant representation of the n-fold transformation. It is easy to see that the spectral problem (4) and (5) are transformed to

$$\psi^{[1]}_x = U^{[1]} \psi^{[1]}, \quad U^{[1]} = (T_x + T U)T^{-1}. \quad (6)$$

$$\psi^{[1]}_t = V^{[1]} \psi^{[1]}, \quad V^{[1]} = (T_t + T V)T^{-1}. \quad (7)$$

under a gauge transformation

$$\psi^{[1]} = T \psi. \quad (8)$$

By cross differentiating (6) and (7), we obtain

$$U^{[1]}_t - V^{[1]}_x + [U^{[1]}, V^{[1]}] = T(U_t - V_x + [U, V])T^{-1}. \quad (9)$$

This implies that, in order to make eqs. (2) and eq. (3) invariant under the transformation (8), it is crucial to search a matrix $T$ so that $U^{[1]}, V^{[1]}$ have the same forms as $U, V$. At the same time the old potential (or seed solution) $(q, r)$ in spectral matrices $U, V$ are mapped into new potentials (or new solution) $(q^{[1]}, r^{[1]})$ in transformed spectral matrices $U^{[1]}, V^{[1]}$. Next, it is necessary to parameterize the matrix $T$ by the eigenfunctions associated with the seed solution.

2.1 One-fold Darboux transformation of the WKI system
Without losing any generality, let Darboux matrix \( T \) be the form of

\[
T_1 = T_1(\lambda; \lambda_1) = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} \frac{i\sqrt{2b}}{2a}a_1 & b_0 \\ c_0 & \frac{i\sqrt{2b}}{2a}d_1 \end{pmatrix}.
\]

Here \( a_1, d_1, b_0 \) and \( c_0 \) are undetermined function of \( (x, t) \), which will be parameterized by the eigenfunction associated with \( \lambda_1 \) and seed \( (q, r) \) in the WKI spectral problem. Refer to Appendix I for detail derivation.

First of all, we introduce \( n \) eigenfunctions \( \psi_j \) as

\[
\psi_j = \begin{pmatrix} \phi_j \\ \varphi_j \end{pmatrix}, \quad j = 1, 2, \ldots, n, \phi_j = \phi_j(x, t, \lambda_j), \varphi_j = \varphi_j(x, t, \lambda_j).
\]

**Theorem 1.** The elements of one-fold DT are parameterized by the eigenfunction \( \psi_1 \) associated with \( \lambda_1 \) as

\[
d_1 = \frac{1}{a_1}, \quad a_1 = -\frac{\varphi_1}{\phi_1} \exp(-i(-\frac{b}{a} + \frac{b^2}{a^2} t)),
\]

\[
b_0 = (\lambda_1 + i\frac{\sqrt{2b}}{2a}) \exp(-i(-\frac{b}{a} + \frac{b^2}{a^2} t)), \quad c_0 = (\lambda_1 + i\frac{\sqrt{2b}}{2a}) \exp(i(-\frac{b}{a} + \frac{b^2}{a^2} t)),
\]

then

\[
T_1(\lambda; \lambda_1) = \begin{pmatrix} \frac{-\varphi_1}{\phi_1} \exp(-i(-\frac{b}{a} + \frac{b^2}{a^2} t))(\lambda + i\frac{\sqrt{2b}}{2a}) & (\lambda_1 + i\frac{\sqrt{2b}}{2a}) \exp(-i(-\frac{b}{a} + \frac{b^2}{a^2} t)) \\ (\lambda_1 + i\frac{\sqrt{2b}}{2a}) \exp(i(-\frac{b}{a} + \frac{b^2}{a^2} t)) & \frac{-\varphi_1}{\phi_1} \exp(i(-\frac{b}{a} + \frac{b^2}{a^2} t))(\lambda + i\frac{\sqrt{2b}}{2a}) \end{pmatrix}.
\]

\( T_1 \) implies following new solutions

\[
q^{[1]} = (\frac{\varphi_1}{\phi_1})^2 \exp(-2i(-\frac{b}{a} + \frac{b^2}{a^2} t))q - 2i\frac{\varphi_1}{\phi_1} (\lambda_1 + i\frac{\sqrt{2b}}{2a}) \exp(-2i(-\frac{b}{a} + \frac{b^2}{a^2} t)),
\]

\[
r^{[1]} = (\frac{\varphi_1}{\phi_1})^2 \exp(2i(-\frac{b}{a} + \frac{b^2}{a^2} t))r + 2i\frac{\varphi_1}{\phi_1} (\lambda_1 + i\frac{\sqrt{2b}}{2a}) \exp(2i(-\frac{b}{a} + \frac{b^2}{a^2} t)),
\]

and corresponding new eigenfunction

\[
\psi_j^{[1]} = \begin{pmatrix} \frac{1}{\varphi_1} & -(\lambda_1 + i\frac{\sqrt{2b}}{2a})\phi_j & \varphi_j \\ -(\lambda_1 + i\frac{\sqrt{2b}}{2a})\phi_1 & \varphi_1 \end{pmatrix} \exp(-i(-\frac{b}{a} + \frac{b^2}{a^2} t)).
\]

**Proof.** See appendix II.

It is straightforward to verify that \( T_1 \) annihilate its generating function, i.e., \( \psi_1^{[1]} = 0 \).
2.2 n-fold Darboux transformation for WKI system

The main result in this subsection is the determinant representation of the n-fold DT for WKI system. To this purpose, set \[ D = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \text{ are complex functions of } x \text{ and } t \right\}, \]
\[ A = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \text{ are complex functions of } x \text{ and } t \right\}. \]

According to the form of \( T_1 \) in eq. (10), the n-fold DT should be the form of
\[ T_n = \tilde{T}_n(\tilde{\lambda} ; \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots , \tilde{\lambda}_n) = \sum_{l=0}^{n} P_l \tilde{\lambda}^l, \] (16)
with \( \tilde{\lambda} = \lambda + i \frac{\sqrt{2b}}{2a}, \tilde{\lambda}_j = \lambda_j + i \frac{\sqrt{2b}}{2a}, \lambda_i \neq \lambda_j \) if \( i \neq j \) and
\[ P_n = \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix} \in D, \quad P_{n-1} = \begin{pmatrix} 0 & b_{n-1} \\ c_{n-1} & 0 \end{pmatrix} \in A, \quad P_l \in D \text{ (if } l - n \text{ is even)}, \quad P_l \in A \text{ (if } l - n \text{ is odd)}. \]

In order to get diagonal matrix and anti-matrix coefficients in \( \tilde{T}_n \), we introduce \( \tilde{\lambda} \) and \( \tilde{\lambda}_j \) by a shift. Here \( P_0 \) is a constant matrix, \( P_i (1 \leq i \leq n) \) is the function of \( x \) and \( t \). In particular, \( P_0 \in D \) if \( n \) is even and \( P_0 \in A \) if \( n \) is odd, which leads to a separate discussion on the determinant representation of \( T_n \) in the following by means of its kernel.

Specifically, from algebraic equations,
\[ \psi^{[n]}_k = \tilde{T}_n(\tilde{\lambda}; \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots , \tilde{\lambda}_n)|_{\lambda = \lambda_k} \psi_k = \sum_{l=0}^{n} P_l \tilde{\lambda}^l \psi_k = 0, k = 1, 2, \ldots , n, \] (17)
coefficients \( P_l \) are solved by Cramer’s rule. Thus we get determinant representation of the \( T_n \).

**Theorem 2.** (1) For \( n = 2k(k = 1, 2, 3, \ldots) \), the n-fold DT of the WKI system can be expressed by
\[ T_n = \tilde{T}_n(\tilde{\lambda}; \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots , \tilde{\lambda}_n) = \begin{pmatrix} (T_n)_{11} & (T_n)_{12} \\ W_n & W_n \end{pmatrix} , \] (18)
with
\[ W_n = \begin{vmatrix} \tilde{\lambda}_1^{-n} \phi_1 & \tilde{\lambda}_1^{-n-1} \phi_1 & \cdots & \tilde{\lambda}_1 \phi_1 & \phi_1 \\ \tilde{\lambda}_2^{-n} \phi_2 & \tilde{\lambda}_2^{-n-1} \phi_2 & \cdots & \tilde{\lambda}_2 \phi_2 & \phi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{\lambda}_{n-1}^{-n} \phi_{n-1} & \tilde{\lambda}_{n-1}^{-n-1} \phi_{n-1} & \cdots & \tilde{\lambda}_{n-1} \phi_{n-1} & \phi_{n-1} \\ \tilde{\lambda}_n^{-n} \phi_n & \tilde{\lambda}_n^{-n-1} \phi_n & \cdots & \tilde{\lambda}_n \phi_n & \phi_n \end{vmatrix} , \]
\[
\begin{align*}
\widetilde{(T)_1} &= \begin{pmatrix}
\tilde{\lambda}^n & 0 & \ldots & \tilde{\lambda}^2 & 0 & 1 \\
\tilde{\lambda}_1^n & \tilde{\lambda}_1^{n-1} \varphi_1 & \ldots & \tilde{\lambda}_1^2 \varphi_1 & \tilde{\lambda}_1 \varphi_1 & \varphi_1 \\
\tilde{\lambda}_2^n & \tilde{\lambda}_2^{n-1} \varphi_2 & \ldots & \tilde{\lambda}_2^2 \varphi_2 & \tilde{\lambda}_2 \varphi_2 & \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_{n-1}^n & \tilde{\lambda}_{n-1}^{n-1} \varphi_{n-1} & \ldots & \tilde{\lambda}_{n-1}^2 \varphi_{n-1} & \tilde{\lambda}_{n-1} \varphi_{n-1} & \varphi_{n-1} \\
\tilde{\lambda}_n^n & \tilde{\lambda}_n^{n-1} \varphi_n & \ldots & \tilde{\lambda}_n^2 \varphi_n & \tilde{\lambda}_n \varphi_n & \varphi_n
\end{pmatrix}, \\
\widetilde{(T)_2} &= \begin{pmatrix}
0 & \tilde{\lambda}^{n-1} & \ldots & 0 & \tilde{\lambda} & 0 \\
\tilde{\lambda}_1^n & \tilde{\lambda}_1^{n-1} \varphi_1 & \ldots & \tilde{\lambda}_1^2 \varphi_1 & \tilde{\lambda}_1 \varphi_1 & \varphi_1 \\
\tilde{\lambda}_2^n & \tilde{\lambda}_2^{n-1} \varphi_2 & \ldots & \tilde{\lambda}_2^2 \varphi_2 & \tilde{\lambda}_2 \varphi_2 & \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_{n-1}^n & \tilde{\lambda}_{n-1}^{n-1} \varphi_{n-1} & \ldots & \tilde{\lambda}_{n-1}^2 \varphi_{n-1} & \tilde{\lambda}_{n-1} \varphi_{n-1} & \varphi_{n-1} \\
\tilde{\lambda}_n^n & \tilde{\lambda}_n^{n-1} \varphi_n & \ldots & \tilde{\lambda}_n^2 \varphi_n & \tilde{\lambda}_n \varphi_n & \varphi_n
\end{pmatrix}, \\
\text{and} \\
\widetilde{(T)_3} &= \begin{pmatrix}
0 & \tilde{\lambda}^{n-1} & \ldots & 0 & \tilde{\lambda} & 0 \\
\tilde{\lambda}_1^n & \tilde{\lambda}_1^{n-1} \varphi_1 & \ldots & \tilde{\lambda}_1^2 \varphi_1 & \tilde{\lambda}_1 \varphi_1 & \varphi_1 \\
\tilde{\lambda}_2^n & \tilde{\lambda}_2^{n-1} \varphi_2 & \ldots & \tilde{\lambda}_2^2 \varphi_2 & \tilde{\lambda}_2 \varphi_2 & \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_{n-1}^n & \tilde{\lambda}_{n-1}^{n-1} \varphi_{n-1} & \ldots & \tilde{\lambda}_{n-1}^2 \varphi_{n-1} & \tilde{\lambda}_{n-1} \varphi_{n-1} & \varphi_{n-1} \\
\tilde{\lambda}_n^n & \tilde{\lambda}_n^{n-1} \varphi_n & \ldots & \tilde{\lambda}_n^2 \varphi_n & \tilde{\lambda}_n \varphi_n & \varphi_n
\end{pmatrix}.
\end{align*}
\]

(2) For \( n = 2k + 1(k = 1, 2, 3, \ldots) \),

\[
T_n = \widetilde{T}_n(\tilde{\lambda}; \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n) = \begin{pmatrix}
\widetilde{(T)_1} & \widetilde{(T)_2} \\
\frac{Q_n}{Q_n} & \frac{Q_n}{Q_n}
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
\frac{\widetilde{(T)_1}}{Q_n} \\
\frac{\widetilde{(T)_2}}{Q_n}
\end{pmatrix}.
\]
with

\[
Q_n = \begin{pmatrix}
\tilde{\lambda}_1^{-n-1} & \tilde{\lambda}_1^{-n-2} & \ldots & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & \phi_1 \\
\tilde{\lambda}_2^{-n-1} & \tilde{\lambda}_2^{-n-2} & \ldots & \tilde{\lambda}_2^{-2} & \tilde{\lambda}_2^{-1} & \tilde{\lambda}_2 & \phi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^{-n-1} & \tilde{\lambda}_n^{-n-2} & \ldots & \tilde{\lambda}_n^{-2} & \tilde{\lambda}_n^{-1} & \tilde{\lambda}_n & \phi_n
\end{pmatrix},
\]

\[
(T_n)_{11} = \begin{pmatrix}
\tilde{\lambda}_n & 0 & \ldots & \tilde{\lambda}^3 & 0 & \tilde{\lambda} & 0 \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(-i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(-i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^{-n} & \tilde{\lambda}_n^{-n-1} & \ldots & \tilde{\lambda}_n^{-3} & \tilde{\lambda}_n^{-2} & \tilde{\lambda}_n^{-1} & \tilde{\lambda}_n & -\phi_n \exp(-i\frac{b}{a} x + \frac{b^2}{a^2} t)
\end{pmatrix},
\]

\[
(T_n)_{12} = \begin{pmatrix}
0 & \tilde{\lambda}^{-n-1} & \ldots & 0 & \tilde{\lambda}^2 & 0 & -1 \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(-i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(-i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^{-n} & \tilde{\lambda}_n^{-n-1} & \ldots & \tilde{\lambda}_n^{-3} & \tilde{\lambda}_n^{-2} & \tilde{\lambda}_n^{-1} & \tilde{\lambda}_n & -\phi_n \exp(-i\frac{b}{a} x + \frac{b^2}{a^2} t)
\end{pmatrix},
\]

\[
\tilde{Q}_n = \begin{pmatrix}
\tilde{\lambda}_1^{-n-1} & \tilde{\lambda}_1^{-n-2} & \ldots & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & \phi_1 \\
\tilde{\lambda}_2^{-n-1} & \tilde{\lambda}_2^{-n-2} & \ldots & \tilde{\lambda}_2^{-2} & \tilde{\lambda}_2^{-1} & \tilde{\lambda}_2 & \phi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^{-n-1} & \tilde{\lambda}_n^{-n-2} & \ldots & \tilde{\lambda}_n^{-2} & \tilde{\lambda}_n^{-1} & \tilde{\lambda}_n & \phi_n
\end{pmatrix},
\]

\[
(T_n)_{21} = \begin{pmatrix}
0 & \tilde{\lambda}^{-n-1} & \ldots & 0 & \tilde{\lambda}^2 & 0 & -1 \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^{-n} & \tilde{\lambda}_n^{-n-1} & \ldots & \tilde{\lambda}_n^{-3} & \tilde{\lambda}_n^{-2} & \tilde{\lambda}_n^{-1} & \tilde{\lambda}_n & -\phi_n \exp(i\frac{b}{a} x + \frac{b^2}{a^2} t)
\end{pmatrix},
\]

\[
(T_n)_{22} = \begin{pmatrix}
\tilde{\lambda}^{-n} & 0 & \ldots & \tilde{\lambda}^3 & 0 & \tilde{\lambda} & 0 \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\tilde{\lambda}_1^{-n} & \tilde{\lambda}_1^{-n-1} & \ldots & \tilde{\lambda}_1^{-3} & \tilde{\lambda}_1^{-2} & \tilde{\lambda}_1^{-1} & \tilde{\lambda}_1 & -\phi_1 \exp(i\frac{b}{a} x + \frac{b^2}{a^2} t) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^{-n} & \tilde{\lambda}_n^{-n-1} & \ldots & \tilde{\lambda}_n^{-3} & \tilde{\lambda}_n^{-2} & \tilde{\lambda}_n^{-1} & \tilde{\lambda}_n & -\phi_n \exp(i\frac{b}{a} x + \frac{b^2}{a^2} t)
\end{pmatrix}.
\]
Next, we consider the transformed new solutions \((q^{[n]}, r^{[n]})\) of WKI system corresponding to the n-fold DT. Under covariant requirement of spectral problem of the WKI system, the transformed form should be

\[
\partial_x \psi^{[n]} = (-aJx^2 + Q_1^{[n]} \lambda + Q_0^{[n]})\psi = U^{[n]} \psi,
\]

(20)

with

\[
\psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q_1^{[n]} = \begin{pmatrix} \sqrt{2b} & aq^{[n]} \\ \sqrt{2b} & -\sqrt{2b} \end{pmatrix}, \quad Q_0^{[n]} = \begin{pmatrix} 0 & i\sqrt{\frac{b}{2} q^{[n]}} \\ i\sqrt{\frac{b}{2} r^{[n]}} & 0 \end{pmatrix},
\]

(21)

and then

\[
T_{nx} + T_n U = U^{[n]} T_n.
\]

(22)

Substituting \(T_n\) given by eq.\((16)\) into eq.\((22)\), and then comparing the coefficients of \(\lambda^{n+1}\), it yields

\[
q^{[n]} = \frac{a_n}{d_n} q + 2i \frac{b_{n-1}}{d_n} r, \quad r^{[n]} = \frac{d_n}{a_n} r - 2i \frac{c_{n-1}}{a_n}.
\]

(23)

Furthermore, substitute \(a_n, d_n, b_{n-1}, c_{n-1}\) from eq.\((18)\) for \(n = 2k\) and from eq.\((19)\) for \(n = 2k + 1\), into \((23)\), we get new solutions \((q^{[n]}, r^{[n]})\) of couple system in eq.\((2)\) and eq.\((3)\):

**Theorem 3.** Starting from a seed \(q\), the n-fold DT \(T_n\) in theorem 2 generates new solutions

\[
q^{[n]} = \frac{\Omega_{n1}^2}{\Omega_{n3}^2} q - 2i \frac{\Omega_{n1}\Omega_{n2}}{\Omega_{n3}^2} r, \quad r^{[n]} = \frac{\Omega_{n3}^2}{\Omega_{n1}^2} r + 2i \frac{\Omega_{n3}\Omega_{n4}}{\Omega_{n1}^2}.
\]

(24)

Here, \((1)\) for \(n = 2k\),

\[
\Omega_{n1} = \begin{vmatrix}
\tilde{\lambda}_1^{n-1} \varphi_1 & \tilde{\lambda}_1^{n-2} \varphi_1 & \cdots & \tilde{\lambda}_1 \varphi_1 & \varphi_1 \\
\tilde{\lambda}_2^{n-1} \varphi_2 & \tilde{\lambda}_2^{n-2} \varphi_2 & \cdots & \tilde{\lambda}_2 \varphi_2 & \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{\lambda}^{n-1}_{n-1} \varphi_{n-1} & \tilde{\lambda}^{n-2}_{n-1} \varphi_{n-1} & \cdots & \tilde{\lambda} \varphi_{n-1} & \varphi_{n-1} \\
\lambda^{n-1} \varphi_n & \lambda^{n-2} \varphi_n & \cdots & \lambda \varphi_n & \varphi_n
\end{vmatrix},
\]

(25)

\[
\Omega_{n2} = \begin{vmatrix}
\tilde{\lambda}_1^{n-1} \varphi_1 & \tilde{\lambda}_1^{n-2} \varphi_1 & \cdots & \tilde{\lambda}_1 \varphi_1 & \varphi_1 \\
\tilde{\lambda}_2^{n-1} \varphi_2 & \tilde{\lambda}_2^{n-2} \varphi_2 & \cdots & \tilde{\lambda}_2 \varphi_2 & \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{\lambda}^{n-1}_{n-1} \varphi_{n-1} & \tilde{\lambda}^{n-2}_{n-1} \varphi_{n-1} & \cdots & \tilde{\lambda} \varphi_{n-1} & \varphi_{n-1} \\
\lambda^{n} \varphi_n & \lambda^{n-1} \varphi_n & \cdots & \lambda \varphi_n & \varphi_n
\end{vmatrix},
\]

\[
\Omega_{n3} = \begin{vmatrix}
\tilde{\lambda}_1^{n-1} \varphi_1 & \tilde{\lambda}_1^{n-2} \varphi_1 & \cdots & \tilde{\lambda}_1 \varphi_1 & \varphi_1 \\
\tilde{\lambda}_2^{n-1} \varphi_2 & \tilde{\lambda}_2^{n-2} \varphi_2 & \cdots & \tilde{\lambda}_2 \varphi_2 & \varphi_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{\lambda}^{n-1}_{n-1} \varphi_{n-1} & \tilde{\lambda}^{n-2}_{n-1} \varphi_{n-1} & \cdots & \tilde{\lambda} \varphi_{n-1} & \varphi_{n-1} \\
\lambda^{n} \varphi_n & \lambda^{n-1} \varphi_n & \cdots & \lambda \varphi_n & \varphi_n
\end{vmatrix},
\]
Corollary 4  The determinant representation of transformed eigenfunction $\psi_j^{[n]}$ associated with $q^{[n]}$ and $r^{[n]}$.

In order to consider the smoothness of the solution of MNLS equation in the next section, it is necessary to get the determinant representation of the transformed eigenfunction $\psi_j^{[n]}$ associated with $q^{[n]}$ and $r^{[n]}$.

**Corollary 4** The determinant representation of transformed eigenfunction $\psi_j^{[n]} = (T_n|_{\lambda=\lambda_j})\psi_j (j \geq n + 1)$ is expressed by following formulas.
1. If \( n = 2k \), then

\[
\psi_j^{[n]} = \begin{pmatrix}
\tilde{\lambda}_j^n \phi_j & \tilde{\lambda}_j^{n-1} \varphi_j & \ldots & \tilde{\lambda}_j^2 \phi_j & \tilde{\lambda}_j \varphi_j & \phi_j \\
\tilde{\lambda}_1^n \phi_1 & \tilde{\lambda}_1^{n-1} \varphi_1 & \ldots & \tilde{\lambda}_1^2 \phi_1 & \tilde{\lambda}_1 \varphi_1 & \phi_1 \\
\tilde{\lambda}_2^n \phi_2 & \tilde{\lambda}_2^{n-1} \varphi_2 & \ldots & \tilde{\lambda}_2^2 \phi_2 & \tilde{\lambda}_2 \varphi_2 & \phi_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_{n-1}^{n-1} \phi_{n-1} & \tilde{\lambda}_{n-1}^{n-2} \varphi_{n-1} & \ldots & \tilde{\lambda}_{n-1}^2 \phi_{n-1} & \tilde{\lambda}_{n-1} \varphi_{n-1} & \phi_{n-1} \\
\tilde{\lambda}_n^n \phi_n & \tilde{\lambda}_n^{n-1} \varphi_n & \ldots & \tilde{\lambda}_n^2 \phi_n & \tilde{\lambda}_n \varphi_n & \phi_n \\
\end{pmatrix}
\]

2. If \( n = 2k + 1 \), then

\[
\psi_j^{[n]} = \begin{pmatrix}
\tilde{\lambda}_j^n \phi_j & \tilde{\lambda}_j^{n-1} \varphi_j & \ldots & \tilde{\lambda}_j^3 \phi_j & \tilde{\lambda}_j^2 \varphi_j & \tilde{\lambda}_j \varphi_j & -\varphi_j \exp(-i\frac{b}{a} + \frac{b^2}{a^2} t) \\
\tilde{\lambda}_1^n \phi_1 & \tilde{\lambda}_1^{n-1} \varphi_1 & \ldots & \tilde{\lambda}_1^3 \phi_1 & \tilde{\lambda}_1^2 \varphi_1 & \tilde{\lambda}_1 \varphi_1 & -\varphi_1 \exp(-i\frac{b}{a} + \frac{b^2}{a^2} t) \\
\tilde{\lambda}_2^n \phi_2 & \tilde{\lambda}_2^{n-1} \varphi_2 & \ldots & \tilde{\lambda}_2^3 \phi_2 & \tilde{\lambda}_2^2 \varphi_2 & \tilde{\lambda}_2 \varphi_2 & -\varphi_2 \exp(-i\frac{b}{a} + \frac{b^2}{a^2} t) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_n^n \phi_n & \tilde{\lambda}_n^{n-1} \varphi_n & \ldots & \tilde{\lambda}_n^3 \phi_n & \tilde{\lambda}_n^2 \varphi_n & \tilde{\lambda}_n \varphi_n & -\varphi_n \exp(-i\frac{b}{a} + \frac{b^2}{a^2} t) \\
\end{pmatrix}
\]

2.3 Reduction of the Darboux transformation for WKI system.

The solutions \( q^{[n]} \) and \( r^{[n]} \) in theorem 3 generated by the n-fold DT \( T_n \) of WKI system are solutions of the coupled system in eq.(2) and eq.(3). If it keeps the reduction condition,
\[ q[n] = -(r[n])^*, \] DT of the WKI system in theorem 2 reduces to the DT of the MNLS equation, and then \( q[n] \) in theorem 3 implies automatically a new solution of the MNLS equation. In this subsection, we shall show how to choose the eigenvalues and eigenfunctions in the determinant representations of the \( T_n \) in order to realize the reduction.

Under the reduction condition \( q = -r^* \), the eigenfunction \( \psi_k = \begin{pmatrix} \phi_k \\ \varphi_k \end{pmatrix} \) associated with eigenvalue \( \lambda_k \) has following properties,

(i): \( \phi_k^* = \varphi_k, \lambda_k = -\lambda_k^* \);

(ii): \( \phi_k^* = \varphi_l, \varphi_k^* = \phi_l, \lambda_k^* = -\lambda_l \), where \( k \neq l \).

It is trivial to verify above properties in Lax pair, eq.\((4)\) and eq.\((5)\), of the WKI system by a straightforward calculation. These properties of eigenfunctions for the WKI system provides one kind of possibility for choosing suitable eigenvalues and eigenfunctions such that the reduction condition \( q[n] = -(r[n])^* \) holds in \( n \)-fold DT.

**Lemma 5** Let

\[ \lambda_1 = i\beta_1 (\beta_1 \in R), \quad \psi_1 = \begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix}, \tag{27} \]

in \( T_1 \), then \( q[1] = -(r[1])^* \) and \( T_1 \) reduces to a one-fold DT of the MNLS.

**Proof** According to property (i), it is suitable to let \( \lambda_1 \) and \( \psi_1 \) as eq.\((27)\). Under this choice, \( \varphi_1^* = \phi_1 \) and \( \lambda_1^* = -\lambda_1 \). Substituting this relation and \( q = -r^* \) into eq.\((14)\), then

\[ (q[1])^* = \left( \frac{\phi_1}{\varphi_1} \right)^2 \exp(2i(\frac{b}{a}x + \frac{b^2}{a^2}t))q^* - 2i\frac{\phi_1}{\varphi_1}(\lambda_1 + i\sqrt{2b/2a}) \exp(2i(\frac{b}{a}x + \frac{b^2}{a^2}t)), \]

\[ = -\left( \frac{\phi_1}{\varphi_1} \right)^2 \exp(2i(\frac{b}{a}x + \frac{b^2}{a^2}t))r - 2i\frac{\phi_1}{\varphi_1}(\lambda_1 + i\sqrt{2b/2a}) \exp(2i(\frac{b}{a}x + \frac{b^2}{a^2}t)) = -r[1]. \]

**Lemma 6** Let \( \psi_1 = \begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix} \) be an eigenfunction of \( \lambda_1 \), and

\[ \lambda_2 = -\lambda_1^*, \quad \psi_2 = \begin{pmatrix} \varphi_1^* \\ \phi_1^* \end{pmatrix}, \tag{28} \]

in \( T_2 \) given by eq.\((18)\), then \( q[2] = -(r[2])^* \) and \( T_2 \) reduces to a one-fold DT of the MNLS.

**Proof** According to the above property (ii), it is suitable to choose \( \psi_2 \) and \( \lambda_2 \) as eq.\((28)\). Let \( n = 2 \) in eq.\((24)\), then

\[ q[2] = \frac{\Omega_{21}^2}{\Omega_{23}^2} q - 2i \frac{\Omega_{21}\Omega_{n2}}{\Omega_{23}^2}, \tag{29} \]

\[ r[2] = \frac{\Omega_{23}^2}{\Omega_{21}^2} r + 2i \frac{\Omega_{23}\Omega_{24}}{\Omega_{21}^2}, \tag{30} \]

with \( \Omega_{21} = \tilde{\lambda}_1 \varphi_1 \phi_2 - \tilde{\lambda}_2 \phi_1 \varphi_2, \Omega_{22} = (\tilde{\lambda}_1^2 - \tilde{\lambda}_2^2) \phi_1 \phi_2, \tag{31} \]

\[ \Omega_{23} = \tilde{\lambda}_1 \varphi_1 \varphi_2 - \tilde{\lambda}_2 \varphi_1 \phi_2, \Omega_{24} = (\tilde{\lambda}_1^2 - \tilde{\lambda}_2^2) \varphi_1 \varphi_2. \tag{32} \]

Using the special choice of \( \lambda_2 \) and its eigenfunction in eq.\((28)\), we have \( \Omega_{21} = \Omega_{23}, \Omega_{22} = -\Omega_{24} \) from eq.\((31)\) and eq.\((32)\). These two relations imply \( q[2] = -(r[2])^* \) from eq.\((29)\) and eq.\((30)\) with the help of original reduction condition \( q = -r^* \). Therefore, under the choice in eq.\((28)\), \( T_2 \) reduces to a one-fold DT of the MNLS equation. \( \square \)
Furthermore, by repeatedly iterating $T_2$ as Lemma 6 for $k$ times with paired-eigenvalues and corresponding paired eigenfunctions, we have following results of the $T_{2k}$.

**Theorem 7** Let $\lambda_{2l-1} (l = 1, 2, 3 \ldots, k)$ be $k$ distinct eigenvalues for eq.(4) and eq.(5), $\psi_{2l-1} = \begin{pmatrix} \phi_{2l-1} \\ \varphi_{2l-1} \end{pmatrix}$ be their associated eigenfunctions, and

$$
\lambda_{2l} = -\lambda_{2l-1}^*, \psi_{2l} = \begin{pmatrix} \varphi_{2l-1}^* \\ \phi_{2l-1}^* \end{pmatrix},
$$

(33)
in the $(2k)$-fold DT $T_{2k}$ of WKI system, then $q^{[2k]} = -(r^{[2k]})^*$ in eq.(24), and $T_{2k}$ in eq.(18) reduces to a $k$-fold DT of the MNLS equation.

Thus $q^{[2k]}$ is called $k$-order solution of the MNLS.

Similar to the reduction of the $T_{2k}$ for the WKI system, $T_{2k+1}$ in eq.(19) can also be reduced to the $(k+1)$-fold DT of the MNLS by choosing one pure imaginary $\lambda_{2k+1} = i\beta_{2k+1}$ and $k$ paired-eigenvalues $\lambda_{2l} = -\lambda_{2l-1}^* (l = 1, 2 \ldots, k)$ with corresponding eigenfunctions according to properties (i) and (ii). Of course, there are many other ways to select eigenvalues and eigenfunctions in order to do reduction of $n$-fold DT for the WKI system.

### 3. Smoothness of the solutions $q^{[2k]}$

The smoothness of the $q^{[n]}$ generated by DT is an important property of the solution of the MNLS equation. In this section, we shall study this property for the solution $q^{[2k]}$ through the $T_{2k}$ with the paired eigenfunctions and eigenvalues.

#### 3.1 Non-degenerate case

In this subsection, we shall show the smoothness of the new solution $q^{[2k]} (k = 1, 2, 3 \ldots)$ of the MNLS equation under non-degenerate case: $\psi_j \neq 0$ and $\lambda_i \neq \lambda_j (i \neq j)$. This solution is generated by $T_{2k}$ from a seed solution $q$ with the paired eigenvalues and eigenfunctions in Theorem 7.

**Lemma 8.** Let $\lambda_1 = \alpha_1 + i\beta_1, \alpha_1 \in R$ and $\beta_1 \in R, |\phi_1| + |\varphi_1| > 0$ in the $q^{[2]}$ generated by the $T_2$ of Lemma 6. If $\alpha_1 \neq 0$, then $q^{[2]}$ is a smooth function on whole $(x,t)$-plane.

**Proof** From Lemma 6, by a straightforward calculation, we have

$$
q^{[2]} = \frac{(\tilde{\lambda}_1 \varphi_1 \varphi_1^* + \tilde{\lambda}_1^* \phi_1 \phi_1^*)^2}{(\tilde{\lambda}_1 \phi_1 \phi_1^* + \tilde{\lambda}_1^* \varphi_1 \varphi_1^*)^2} q - 2i(\tilde{\lambda}_1 \phi_1 \phi_1^* + \tilde{\lambda}_1^* \varphi_1 \varphi_1^*)(\lambda_1 \varphi_1 \varphi_1^* + \lambda_1^* \phi_1 \phi_1^*) \phi_1 \phi_1^*. \tag{34}
$$

Note that in the denominator of $q^{[2]}$, $\Omega_{21} = \tilde{\lambda}_1 \phi_1 \phi_1^* + \tilde{\lambda}_1^* \varphi_1 \varphi_1^* = \alpha_1 (|\phi_1|^2 + |\varphi_1|^2) + i(\beta_1 + \sqrt{2b})(|\phi_1|^2 - |\varphi_1|^2). \Omega_{21} \neq 0$ if $\alpha_1 \neq 0$ and $|\phi_1| + |\varphi_1| > 0$. Therefore, $q^{[2]}$ is a smooth function under this condition. □

**Theorem 9** Let $\lambda_{2l-1} = \alpha_{2l-1} + i\beta_{2l-1} (l = 1, 2, 3 \ldots, k)$ be $k$ distinct eigenvalues in the $q^{[2k]}$ generated by $T_{2k}$ of Theorem 7. If $\alpha_{2l-1} \neq 0 (l = 1, 2, 3 \ldots, k)$ and $|\phi_{2l-1}| + |\varphi_{2l-1}| > 0$, then $q^{[2k]}$ is a smooth function on whole $(x,t)$-plane.

**Proof** 1) Using $T_2$, a new eigenfunction associated with $\lambda_3$ is $\psi_3^{[2]} = \begin{pmatrix} \phi_3^{[2]} \\ \varphi_3^{[2]} \end{pmatrix} = T_2(\lambda_3; \tilde{\lambda}_1) \psi_3$.

Furthermore $\psi_3^{[2]} \neq 0$ because $T_2$ is not a degenerate linear transformation and $\psi_3 \neq 0$. Thus $|\phi_3^{[2]}| + |\varphi_3^{[2]}| \neq 0$. 


2) By iteration of $T_2$ once with generating function $\psi_3^{[2]}$, a new solution of the MNLS equation is given by

$$q^{[4]} = \frac{(\tilde{\lambda}_3 \tilde{\varphi}_3^{[2]}(\varphi_3^{[2]})^* + \lambda_3^* \varphi_3^{[2]}(\varphi_3^{[2]})^*)^2}{(\lambda_3 \varphi_3^{[2]}(\varphi_3^{[2]})^* + \tilde{\lambda}_3^* \varphi_3^{[2]}(\varphi_3^{[2]})^*)^2} q^{[2]} - 2i \frac{(\tilde{\lambda}_3^2 - \lambda_3^2)(\tilde{\lambda}_3 \tilde{\varphi}_3^{[2]}(\varphi_3^{[2]})^* + \lambda_3^* \varphi_3^{[2]}(\varphi_3^{[2]})^*)^2}{(\lambda_3 \varphi_3^{[2]}(\varphi_3^{[2]})^* + \tilde{\lambda}_3^* \varphi_3^{[2]}(\varphi_3^{[2]})^*)^2}.$$  

(35)

3) By using Lemma 8, $q^{[4]}$ is a smooth function on $(x,t)$ plane.

4) By this pattern, k-time iteration of $T_2$ with generating functions $\psi_{2l-1}(l = 1, 2, 3, \cdots, k)$ implies $T_{2k}$ and corresponding new solution $q^{[2k]}$ of the MNLS equation. Note that each step of iterations generates smooth solution according to Lemma 8. Thus $q^{[2k]}$, the final solution generated by k-time iteration, is a smooth function on the $(x,t)$-plane. □

It is easy to check that the k-soliton solution is generated by $T_{2k}$ in Theorem 9 from a trivial seed solution–zero solution. It is an interesting problem to study degenerate cases and to apply it to get smooth solutions of the MNLS equations.

3.2 Double degeneration case

It is easy to see from the determinant representations that there are two degenerate cases of $T_{2k}$: 1) degenerate eigenvalues: $\lambda_{2l-1} \rightarrow \lambda_l (l = 1, 2, 3, \cdots, k)$; 2) degenerate eigenfunctions: $\psi_{2l-1}(\lambda_{2l-1}; x, t) = 0 (l = 1, 2, 3, \cdots, k)$ under certain values of parameters. We assume that $\lambda_0$ is only one zero point of the eigenfunction $\psi_1$. In this subsection, we shall apply $T_{2k}$ with double degeneration, i.e., $\lambda_{2l-1} \rightarrow \lambda_0(l = 1, 2, 3, \cdots, k)$ and $\psi_{2l-1}(\lambda_0; x, t) = 0 (l = 1, 2, 3, \cdots, k)$, to get smooth solution of the MNLS equation. Specifically, under the double degeneration, $q^{[2k]}$ generated by $T_{2k}$ is expressed by an indeterminate form $\frac{0}{0}$. Thus it is possible to get a smooth solution of the MNLS by higher order Taylor expansion of it at a special value $\lambda_0$.

To deal with the degeneration of $T_{2k}$, we begin with the $T_2$ under the reduction condition in Lemma 6. Let $\psi_1(\lambda_0)$ be a smooth eigenfunction of Lax pair of the MNLS associated with non-zero eigenvalue $\lambda_0$, which has also continuous dependence on the $\lambda_0$. Under a small shift of eigen-value, it can be expanded as $\psi_1(\lambda_0 + \epsilon; x, t) = \psi_1(\lambda_0; x, t) + \sum_{l=1}^k b_l \epsilon^l + O(\epsilon^k)$. If $\lambda_0$ is the only one zero point of $\psi_1$; i.e., $\psi_1(\lambda_0) = 0$, then it is also a zero point of $W_2$ in $T_2$. Thus $\lambda_0$ is a singularity of $T_2$ and the $q^{[2]}$. However this singularity is removable.

**Lemma 10** Let $\lambda_0$ be the only one zero point of eigenfunction $\psi_1(\lambda_1)$, $\lambda_0 = \alpha_0 + i\beta_0$. If $\alpha_0 \neq 0$, then $q^{[2]}$ in eq.(34) is a smooth solution, and the singular $T_2$ infers a smooth two-fold DT $T'_{2}$.  

**Proof** By Taylor expansion, $\phi(\lambda_0 + \epsilon) = \phi(\lambda_0) + \frac{\partial \phi_1(\lambda_0 + \epsilon)}{\partial \epsilon}|_{\epsilon=0} + O(\epsilon^2) = b_1 \epsilon + O(\epsilon^2)$, and

$$\phi(\lambda_0 + \epsilon) = \phi(\lambda_0) + \frac{\partial \phi_1(\lambda_0 + \epsilon)}{\partial \epsilon}|_{\epsilon=0} + O(\epsilon^2) = b_1 \epsilon + O(\epsilon^2).$$

1) Submitting these expansion and $\lambda_1 = \lambda_0 + \epsilon$ into $q^{[2]}$ in eq. (34), then

$$q^{[2]} = \frac{(\tilde{\lambda}_0|a_1|^2 + \tilde{\lambda}_0^* |b_1|^2)^2 \epsilon^4 + O(\epsilon^5)}{(\tilde{\lambda}_0|b_1|^2 + \tilde{\lambda}_0^* |a_1|^2)^2 \epsilon^4 + O(\epsilon^5)} - 2i \frac{(\tilde{\lambda}_0^2 - \lambda_0^2)(\tilde{\lambda}_0|a_1|^2 + \tilde{\lambda}_0^* |b_1|^2)b_1 a_1^* \epsilon^4 + O(\epsilon^5)}{(\tilde{\lambda}_0|b_1|^2 + \tilde{\lambda}_0^* |a_1|^2)^2 \epsilon^4 + O(\epsilon^5)}.$$  

(36)
This formula provides a smooth solution

\[
q^{[2]} = \frac{(\tilde{\lambda}_0 |a_1|^2 + \tilde{\lambda}_0^* |b_1|^2)^2}{(\tilde{\lambda}_0 |b_1|^2 + \tilde{\lambda}_0^* |a_1|^2)^2}q - 2i(\tilde{\lambda}_0^2 - \tilde{\lambda}_0^* \tilde{\lambda}_0)(\tilde{\lambda}_0 |a_1|^2 + \tilde{\lambda}_0^* |b_1|^2)b_1 a_1^* \frac{2}{(\tilde{\lambda}_0 |b_1|^2 + \tilde{\lambda}_0^* |a_1|^2)^2}
\]

(37)

by setting \( \epsilon \to 0 \) if \(|a_1| + |b_1| \neq 0 \) and \( \alpha_0 \neq 0 \).

2) According to

\[
\psi_j^{[2]} = T_2(\lambda_j; \lambda_1, \lambda_2)\psi_j = \begin{pmatrix}
\tilde{\ldots}
\end{pmatrix}
\]

the following equation

\[
\psi_j^{[2]} = T_2(\lambda_j; \lambda_1, \lambda_2)\psi_j = \begin{pmatrix}
\tilde{\ldots}
\end{pmatrix}
\]

(38)

is singular if \( \phi_1(\lambda_0) = \varphi_1(\lambda_0) = 0 \). Taking above Taylor expansions of \( \phi_1(\lambda_0 + \epsilon) \) and \( \varphi_1(\lambda_0 + \epsilon) \) and \( \lambda_1 = \lambda_0 + \epsilon \) into \( \psi_j^{[2]} \) under the reduction conditions given by eq. (28), then

\[
\psi_j^{[3]} = \begin{pmatrix}
\tilde{\ldots}
\end{pmatrix}
\]

(39)

\[\Delta T_2'(\lambda_j; \lambda_0)\psi_j.\]

Note that the denominator in above formula is a non-zero function of \((x,t)\). This shows \( \psi_j^{[2]} \) and \( T_2' \) are smooth. \( \square \)

**Remark** If \( a_1 = b_1 = 0 \) in \( \psi_1(\lambda_0 + \epsilon) \), we can use next non-zero coefficient of higher order in its expansion to generate smooth \( q^{[2]} \) and \( T_2' \) as the same manner in the above Lemma.

Because \( T_2 \) just includes one eigenvalue \( \lambda_1 \), it does not have double degeneration. The first non-trivial example of double degeneration of DT is \( T_4 \) by setting \( \lambda_3 \to \lambda_0 \) and \( \psi_3 = \psi_1(\lambda_0) = \lambda_0 \).


\[
\begin{pmatrix}
\phi_1(\lambda_0) \\
\varphi_1(\lambda_0)
\end{pmatrix} = 0. \text{ Thus it is easy to find } \psi_3^{[2]} = 0 \text{ from eq.~(39), and we can not do DT again along the eigenvalue } \lambda_0. \text{ But it can be re-obtained by following limit method.}
\]

**Lemma 11** According to lemma 10,
\[
\psi_3^{[2]}|_{\lambda_3 = \lambda_0 + \epsilon} = \begin{pmatrix}
\frac{\partial^2}{\partial \epsilon^2}((\tilde{\lambda}_0 + \epsilon) \phi_1(\lambda_0 + \epsilon))|_{\epsilon = 0} \\
\frac{\partial^2}{\partial \epsilon^2}((\tilde{\lambda}_0 + \epsilon) \varphi_1(\lambda_0 + \epsilon))|_{\epsilon = 0}
\end{pmatrix}
= \begin{pmatrix}
\tilde{\lambda}_0^2 b_1 \\
(\tilde{\lambda}_0^2 a_1)^*
\end{pmatrix}
= 0. \text{ Thus it does not lost any generality by setting that } \lambda_0 = \lambda_0 + \epsilon.
\]

Thus \(|b_2^{[2]}| \text{ and } |a_2^{[2]}| \) are not both zero. Further, \(q^{[4]} \) in eq.~(35) provides a smooth solution of the MNLS equation.

**Proof.** Let \(\lambda_3 = \lambda_0 + \epsilon\) and \(\psi_3 = \psi_1(\lambda_0 + \epsilon)\) in eq.~(39), then \(\psi_3^{[2]}|_{\lambda_3 = \lambda_0 + \epsilon}\) is a non-zero smooth function because \(T_2'(\lambda_0 + \epsilon; \lambda_0)\) is a non-degenerate linear transformation. Thus \(\psi_3^{[2]}|_{\lambda_3 = \lambda_0 + \epsilon}\) can be expanded as following form
\[
\psi_3^{[2]} = \begin{pmatrix}
\tilde{\lambda}_0 b_1 & a_1 \\
(\tilde{\lambda}_0 a_1)^* & b^*_1
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial^2}{\partial \epsilon^2}((\tilde{\lambda}_0 + \epsilon) \phi_1(\lambda_0 + \epsilon))|_{\epsilon = 0} \\
\frac{\partial^2}{\partial \epsilon^2}((\tilde{\lambda}_0 + \epsilon) \varphi_1(\lambda_0 + \epsilon))|_{\epsilon = 0}
\end{pmatrix}
= \begin{pmatrix}
\tilde{\lambda}_0^2 b_1 & a_1 \\
(\tilde{\lambda}_0^2 b_1)^* & b^*_1
\end{pmatrix}
= \begin{pmatrix}
\tilde{\lambda}_0 a_1 & b_1 \\
(\tilde{\lambda}_0 b_1)^* & a^*_1
\end{pmatrix}
= \begin{pmatrix}
b_2^{[2]} \epsilon^2 + O(\epsilon^3) \\
a_2^{[2]} \epsilon^2 + O(\epsilon^3)
\end{pmatrix}.
\]

Here \(b_2^{[2]}\) and \(a_2^{[2]}\) are smooth and are not both zero function. In the expansion of \(\psi_3^{[2]}|_{\lambda_3 = \lambda_0 + \epsilon}\), there exist at least one term \(|b_2^{[2]}| + |a_2^{[2]}| \neq 0 (l \geq 2)\). If all coefficients of expansion are zero, \(\psi_3^{[2]}|_{\lambda_3 = \lambda_0 + \epsilon} = 0\). Thus it does not lost any generality by setting that \(|b_2^{[2]}|\) and \(|a_2^{[2]}|\) are not both zero. Furthermore, this expansion back into eq.~(35) with \(\lambda_3 = \lambda_0 + \epsilon\), then
\[
q^{[4]} = \frac{\lambda_0 b_2^{[2]} a_2^{[2]} + \lambda_0 b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9)}{(\lambda_0 b_2^{[2]} a_2^{[2]} + \lambda_0 b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9) + \epsilon^2)} - \frac{2i}{(\lambda_0 b_2^{[2]} a_2^{[2]} + \lambda_0 b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9))} \frac{\lambda_0^2 - \lambda_0^*}{(\lambda_0^2 b_2^{[2]} a_2^{[2]} + \lambda_0^* b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9))} q^{[2]} - 2i \frac{\lambda_0^2 - \lambda_0^*}{(\lambda_0^2 b_2^{[2]} a_2^{[2]} + \lambda_0^* b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9))} \left[ q^{[2]} - \frac{\lambda_0^2 - \lambda_0^*}{(\lambda_0^2 b_2^{[2]} a_2^{[2]} + \lambda_0^* b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9))} q^{[2]} - 2i \frac{\lambda_0^2 - \lambda_0^*}{(\lambda_0^2 b_2^{[2]} a_2^{[2]} + \lambda_0^* b_2^{[2]} a_2^{[2]} \epsilon^2 + O(\epsilon^9))} \right] \triangleq q^{[4]}_{\text{smooth}}.
\]

which is a smooth solution because \(\alpha_0 \neq 0\) and \(|b_2^{[2]}|\) and \(|a_2^{[2]}|\) are not both zero. Here \(q^{[2]}\) is given by eq.~(37). □
It is trivial to find that $q^{[4]}$ in eq. (42) is generated from $q^{[2]}$ by iteration of $T'_2$ with new generating function $(a_2^2, b_2^2)$. In other words, $q^{[4]}$ is obtained from $q^{[2]}$ through a non-degenerate $T'_2(\lambda_0; a_2^2, b_2^2)$. Thus there exist a non-degenerate four-fold DT $T'_4 = T'_2(\lambda_0; a_2^2, b_2^2)T'_2(\lambda_0; a_1, b_1)$, which gives this smooth $q^{[4]}$ from seed $q$. Therefore we overcome the problem of double degeneration in $T_4$ when $\lambda_3 = \lambda_1$ and $\psi_3(\lambda_0) = \psi_1(\lambda_0) = 0$. Moreover, let $\lambda_3 \mapsto \lambda_1 = \lambda_0 + \epsilon$ and $\psi_3(\lambda_0) = \psi_1(\lambda_0) = 0$ in $\psi^{[4]}_j$ in eq. (27), perform Taylor expansion at the rows associated with $\lambda_1$ and $\lambda_3$, we have

$$
\psi^{[4]}_j|_{\lambda_j=\lambda_0+\epsilon} = \Delta (T'_4(\lambda_j; \lambda_0)\psi_j)|_{\lambda_j=\lambda_0+\epsilon} \quad (43)
$$

which provides a determinant representation of $T'_4$. In the above process, reduction conditions in eq. (33) is used to calculate the elements of the rows associated with $\lambda_2$ and $\lambda_4$. Here

$$
h^l_{m1} = \frac{\partial}{\partial \epsilon}((\tilde{\lambda}_0 + \epsilon)^m \phi_1(\lambda_1 = \lambda_0 + \epsilon))|_{\epsilon=0}, 
$$

$$
h^l_{m2} = \frac{\partial}{\partial \epsilon}((\tilde{\lambda}_0 + \epsilon)^m \varphi_1(\lambda_1 = \lambda_0 + \epsilon))|_{\epsilon=0} \quad (44)
$$

with $m = 0, 1, \cdots, 4, l = 1, 2$. Similarly, let $\lambda_3 = \lambda_1 = \lambda_0 + \epsilon$ in eq. (24), then Taylor expansion with respect to $\epsilon$ gives a representation of $q^{[4]}_{smooth}$ in eq. (42) as follows.

$$
q^{[4]}_{smooth} = \frac{\Omega^{[4]}_{21}}{\Omega^{[4]}_{43}} q - 2i \frac{\Omega^{[4]}_{41} \Omega^{[4]}_{42}}{\Omega^{[4]}_{43}} \quad (45)
$$

and

$$
\Omega_{41} = \begin{vmatrix}
  h_{12} & h_{11} & h_{01} \\
  -h_{12} & h_{11} & h_{01} \\
  h_{22} & h_{11} & h_{02} \\
  -h_{22} & h_{11} & h_{02} \\
\end{vmatrix},
$$

$$
\Omega_{42} = \begin{vmatrix}
  h_{12} & h_{11} & h_{01} \\
  -h_{12} & h_{11} & h_{01} \\
  h_{22} & h_{11} & h_{02} \\
  -h_{22} & h_{11} & h_{02} \\
\end{vmatrix},
$$
\[ \Omega_{43} = \begin{vmatrix}
  h_{31}^1 & h_{12}^1 & h_{11}^1 & h_{12}^1 \\
  -h_{12}^2 & h_{21}^2 & -h_{12}^2 & h_{21}^2 \\
  h_{22}^2 & h_{22}^2 & h_{22}^2 & h_{22}^2 \\
  -h_{22}^2 & h_{22}^2 & -h_{22}^2 & h_{22}^2 
\end{vmatrix}. \]

Note that the smoothness of this solution is analyzed in eq. (42) according to the iteration of non-degenerate \( T'_2 \).

On the one side, by (k-1) times iteration of \( T'_2 \) with a fixed eigenvalue \( \lambda_0 \) but different eigenfunctions, a non-degenerate \((2k-2)\) fold DT \( T'_{2k-2} \) is obtained because each step of iteration is non-degenerate. On the other hand, setting double degeneration, i.e., \( \lambda_i = \lambda_0 \) in \( q^{[2k-2]}_2 \) and \( \psi_i(\lambda_0) = 0 \), and using reduction conditions eq. (33), then this \( T'_{2k-2} \) also can be presented after performing Taylor expansion with respect to \( \epsilon \). The smoothness of \( q^{[2k-2]} \) is provided by the smoothness of each step of iteration as we shown in eq. (42). By the determinant representation of \( \psi^{[2k-2]}(j \geq 2k-1) \) with double degeneration and reduction condition, we have following Lemma.

**Lemma 12** The non-degenerate \( T'_{2k-2} \) generates a smooth eigenfunction of \( \lambda_{2k-1} = \lambda_0 + \epsilon \) from \( \psi_{2k-1}(\lambda_0 + \epsilon) \) as

\[
\psi^{[2k-2]}_{2k-1} |_{\lambda_{2k-1}=\lambda_0+\epsilon} = \left( \begin{array}{c}
b_k^{[2k-2]} e^k + O(\epsilon^{k+1}) \\
a_k^{[2k-2]} e^k + O(\epsilon^{k+1})
\end{array} \right). \tag{46}
\]

Here \( |b_k^{[2k-2]}| \) and \( |a_k^{[2k-2]}| \) are smooth and not both zero.

**Lemma 13** Set \( \psi^{[2k-2]}_{2k-1} |_{\lambda_{2k-1}=\lambda_0+\epsilon} \) be generating function of a two-fold DT \( \hat{T}_2 = T_2(\lambda_0 \pm \epsilon; \psi^{[2k-2]}_{2k-1}) \) with the help of the reduction conditions, then \( q^{[2k]} \), generated from \( q^{[2k-2]} \), is a smooth rational \( k \)-order solution of the MNLS equation.

**Proof** Replace eigenvalue \( \lambda_1 \), eigenfunction \( \psi_1 \) and seed solution \( q \) by \( \lambda_0 + \epsilon, \psi^{[2k-2]}_{2k-1} |_{\lambda_{2k-1}=\lambda_0+\epsilon} \) and \( q^{[2k-2]} \) in the formula of two-fold DT in eq. (34), respectively, then

\[
q^{[2k]} = \frac{\tilde{\lambda}_0 |a_k^{[2k-2]}|^2 + \tilde{\lambda}_0^* |b_k^{[2k-2]}|^2}{\tilde{\lambda}_0 |b_k^{[2k-2]}|^2 + \tilde{\lambda}_0^* |a_k^{[2k-2]}|^2} q^{[2k-2]} - 2i \frac{\tilde{\lambda}_0^2 - \tilde{\lambda}_0^*}{\tilde{\lambda}_0 |b_k^{[2k-2]}|^2 + \tilde{\lambda}_0^* |a_k^{[2k-2]}|^2} q^{[2k-2]} = q_{\text{smooth}}^{[2k]} \tag{47}
\]

which is a smooth solution because \( \alpha_0 \neq 0, |b_k^{[2k-2]}| \) and \( |a_k^{[2k-2]}| \) are not both zero. \[\square\]

Under double degeneration mentioned above, \( T_{2k} \) is degenerate. Thus solution in eq. (24) is singular at \( \lambda_0 \). This lemma shows that its singularity is removable. We shall give a explicit representation of \( q_{\text{smooth}}^{[2k]} \) by Taylor expansion as the same manner of \( q_{\text{smooth}}^{[4]} \). In other words, \( T_{2k} \) gives a non-degenerate \( 2k \)-fold DT \( T'_{2k} \) by Taylor expansion, and \( q_{\text{smooth}}^{[2k]} \) is generated from \( q \) by this non-degenerate DT.

**Theorem 14.** Let \( \psi_1 \) be an smooth eigenfunction of Lax pair of the MNLS, has also continuous...
Substituting $q$ is a periodic solution of the MNLS equation, which will be used as a seed solution of the DT. The eigenfunction with a periodic seed solution, and then present smooth rational $h$-order solutions according to $\text{theorem 14}$. Next, the explicit representations of the first order and the second order rational solutions are constructed. The explicit forms of the higher order rogue waves are presented. We shall analyze and several new patterns of the higher order rogue waves are presented. We shall localize the rational solution.

Let $a_1$ and $c_1$ be two complex constants, then $q = c_1 \exp (i(a_1 x + (-bc_1^2 - a_1^2 - aa_1c_1^2)t))$ is a periodic solution of the MNLS equation, which will be used as a seed solution of the DT. Substituting $q = c_1 \exp (i(a_1 x + (-bc_1^2 - a_1^2 - aa_1c_1^2)t))$ into the spectral problem eq. \((44)\) and eq. \((5)\), and using the method of separation of variables and the superposition principle, the eigenfunction $\psi_{1\text{-order}}$ associated with $\lambda_{2k-1}$ is given by

$$
\phi_{2k-1}(x,t,\lambda_{2k-1}) = C_1 \varphi_1(x,t,\lambda_{2k-1}) \varphi_1(x,t,\lambda_{2k-1})^{*} + C_2 \varphi_2(x,t,\lambda_{2k-1}) \varphi_2(x,t,\lambda_{2k-1})^{*} + C_3 \varphi_3(x,t,\lambda_{2k-1}) \varphi_3(x,t,\lambda_{2k-1})^{*} + C_4 \varphi_4(x,t,\lambda_{2k-1}) \varphi_4(x,t,\lambda_{2k-1})^{*}.
$$

Note again as theorem 7 that $k$-order just denotes the $k$-fold of DT for the MNLS, which is not related to the smoothness of the solution.

4. Rational solutions generated by 2k-fold degenerate Darboux transformation

According to theorem 14, it is a crucial step to find a suitable zero point $\lambda_0$ of eigenfunction such that $h_{m1}^l$ and $h_{m2}^l$ are both polynomials in $x$ and $t$, which will be given from an explicit formula of $\psi_1$ in this section. We shall find explicit forms of the eigenfunction $\psi_1$ associated with a periodic seed solution, and then present smooth rational $k$-order solutions according to theorem 14. Next, the explicit representations of the first order and the second order rational solutions of the MNLS equation are constructed. The explicit forms of the first four order rogue waves are also provided. Furthermore, localization of the first order rational solution is analyzed and several new patterns of the higher order rogue waves are presented. We shall show an unusual result: for a given value of $a$, the increasing value of $b$ can damage gradually the localization of the rational solution.
Here

\[
\left( \begin{array}{c}
\varpi_1(x, t, \lambda_{2k-1})[1] \\
\varpi_1(x, t, \lambda_{2k-1})[2]
\end{array} \right) = \left( \begin{array}{c}
- a_1 + 2a_\lambda_{2k-1}^2 + 2i\sqrt{2}a_\lambda_{2k-1} - \frac{1}{2}i\theta \cdot \exp(-\sqrt{S(\lambda_{2k-1})}P(\lambda_{2k-1}) + \frac{1}{2}i\theta)
\end{array} \right),
\]

\[
\left( \begin{array}{c}
\varpi_2(x, t, \lambda_{2k-1})[1] \\
\varpi_2(x, t, \lambda_{2k-1})[2]
\end{array} \right) = \left( \begin{array}{c}
- a_1 + 2a_\lambda_{2k-1}^2 + 2i\sqrt{2}a_\lambda_{2k-1} + \frac{1}{2}i\theta \cdot \exp(\sqrt{S(\lambda_{2k-1})}P(\lambda_{2k-1}) - \frac{1}{2}i\theta)
\end{array} \right),
\]

\[
\varpi_1(x, t, \lambda_{2k-1}) = \left( \begin{array}{c}
\varpi_1(x, t, \lambda_{2k-1})[1] \\
\varpi_1(x, t, \lambda_{2k-1})[2]
\end{array} \right),
\]

\[
\varpi_2(x, t, \lambda_{2k-1}) = \left( \begin{array}{c}
\varpi_2(x, t, \lambda_{2k-1})[1] \\
\varpi_2(x, t, \lambda_{2k-1})[2]
\end{array} \right),
\]

\[
S(\lambda_{2k-1}) = 4a_\lambda_{2k-1}^4 + 8ia\sqrt{2}a_\lambda_{2k-1}^3 + (4a_\lambda_{2k-1}^2 + 4aa_\lambda_{2k-1} - 8b)\lambda_{2k-1}^2
\]

\[
+ (4i\sqrt{2}a_\lambda_{2k-1}^2 + 4ia\sqrt{2}a_\lambda_{2k-1}^2)\lambda_{2k-1} + a_\lambda_{2k-1}^2 - 2bc_1^2,
\]

\[
P(\lambda_{2k-1}) = \left(4\lambda_{2k-1} - 6i\sqrt{2}a_\lambda_{2k-1}^2 + c_1^2\sqrt{2}b a_i - 4a_\lambda_{2k-1}^3 + \sqrt{2}a_i i + 2a_1 a_\lambda_{2k-1}
\right.

\[
+ 2c_1^2 a_\lambda_{2k-1} t) \left(-2a_\lambda_{2k-1} - \sqrt{2}b i x \right) \left(2a_\lambda_{2k-1}^2 - \sqrt{2}b \right),
\]

\[
\theta = a_1 x + (-bc_1^2 - a_\lambda_{2k-1}^2 - aa_\lambda_{2k-1}^2) t,
\]

and \(a, b, a_1, c_1, x, t \in \mathbb{R}, C_1, C_2, C_3, C_4 \in \mathbb{C}\). Note that \(\varpi_1(x, t, \lambda_{2k-1})\) and \(\varpi_2(x, t, \lambda_{2k-1})\) are two linear independent solutions of the spectral problem eq.(4) and eq.(5). We can only get the trivial solutions through DT of the MNLS equation by setting eigenfunction \(\psi_{2k-1}\) be one of them. This is the reason of setting \(\psi_{2k-1}\) as the linear superposition in eq.(49).

In order to make higher order rational solution of the MNLS, a crucial step is to find the zero point of \(S\) and the eigenfunctions \(\psi_i\) such that exponential functions vanish and the indeterminate form \(\frac{0}{0}\) appear in the \(q^{[2k]}\). By tedious calculation, we have following lemma concerning of this fact.

**Lemma 15** Let

\[
C_1 = -(K_0 + 2) + \exp\left(\frac{1}{2}iS(\lambda_{2k-1}) \sum_{j=0}^{k-1} S_j(\lambda_{2k-1} - \lambda_0)^j\right),
\]

\[
C_2 = -(K_0 + 2) + \exp\left(-\frac{1}{2}iS(\lambda_{2k-1}) \sum_{j=0}^{k-1} S_j(\lambda_{2k-1} - \lambda_0)^j\right),
\]

\[
C_3 = K_0 + \exp\left(\frac{1}{2}iS(\lambda_{2k-1}) \sum_{j=0}^{k-1} L_j(\lambda_{2k-1} - \lambda_0)^j\right),
\]

\[
C_4 = K_0 + \exp\left(-\frac{1}{2}iS(\lambda_{2k-1}) \sum_{j=0}^{k-1} L_j(\lambda_{2k-1} - \lambda_0)^j\right),
\]
then \( \lambda_{2k-1} = \lambda_0 = -i \frac{\sqrt{2b} - \sqrt{2aa_1 + a^2c_1^2 + 2b + ac_1}}{2a} \) is only one zero point of \( S \) and eigenfunction \( \psi_{2k-1} \) in eq.(49). Here \( K_0, S_j, L_j \in \mathbb{C} \).

**Theorem 16** For the eigenfunction \( \psi_1 \) defined by eq.(49) and eq.(50), \( h_{m1}^l \) and \( h_{m2}^l \) are both polynomials in variables \( x \) and \( t \). Furthermore, \( q_{\text{smooth}}^{[2k]} \) generates smooth rational \( k \)-order solutions of the MNLS equation.

**Proof.** Take the eigenfunction \( \psi_{2l-1}(l = 1, 2, 3, \cdots, k) \) defined by eq.(49) and eq.(50) into eq.(24), \( q_{\text{smooth}}^{[2k]} \) provides a solution expressed by an indeterminate form \( 0 \cdot 0 \) of the MNLS equation when \( \lambda_{2k-1} \rightarrow -i \frac{\sqrt{2b} - \sqrt{2aa_1 + a^2c_1^2 + 2b + ac_1}}{2a} \). Obviously, this is a case of double degeneration, and then we can apply Theorem 14 here. Because \( \lambda_0 \) is a zero point of \( S \) and \( \psi_1 \) in eq.(49), \( \exp(S) \) will disappear when \( \lambda_1 = \lambda_0, h_{m1}^l \) and \( h_{m2}^l \) are both polynomials in \( x \) and \( t \). Therefore, take the eigenfunction \( \psi_1 \) defined by eq.(49) and eq.(50) into theorem 14, \( q_{\text{smooth}}^{[2k]} \) are smooth rational solutions of the MNLS. \( \square \)

Note again as theorem 7 that \( k \)-order just denotes the \( k \)-fold of DT for the MNLS, which is not related to the smoothness of the solution or the order of polynomials.

### 4.1 Asymptotic behavior of rational 1-order solution

For simplicity, set \( k = 1, a_1 = -1, c_1 = 1, K_0 = 1, S_0 = L_0 = 0 \) in theorem 16, the rational 1-order solution \( q_{\text{smooth}}^{[2]} \) is given by the form as follows.

\[
q_{\text{rational}}^{[1]} = -\exp((-x - tb - t + a) i) \frac{H_1 H_2}{H_1^2},
\]

(51)

with \( H_1 = 2(a - b)((-2b + 3a^2 - 6a + 4)t^2 - 4(-1 + a)x + x^2) + 1 - 2ia(x - (3a - 2)t), \)
\[
H_2 = 2(a - b)((-2b + 3a^2 - 6a + 4)t^2 - 4(-1 + a)x + x^2) - 3 - 2ia(ax - (-6a + 3a^2 + 4b)t).
\]

By letting \( x \rightarrow \infty, t \rightarrow \infty, |q_{\text{rational}}^{[1]}|^2 \rightarrow 1 \). Moreover, \( q_{\text{rational}}^{[1]} \) is not a traveling wave except \( a = b \). Furthermore, set imaginary part of \( H_1 \) be zero, i.e. \( x = (3a - 2)t \), but its real part \( 4t^2(a - b)^2 + 1 > 0 \), so this solution is smooth on whole plane of \( x \) and \( t \).

In order to show the novel properties, we shall discuss the asymptotic behaviors of the rational 1-order solution.

**Case a.)** Let \( a < b \), there is only one saddle point of the profile for the \( q_{\text{rational}}^{[1]} \) at point \( (0, 0) \). It is a nonlocal solution with two peaks.

**Case b.)** Let \( a = b \), the maximum amplitude of \( |q_{\text{rational}}^{[1]}|^2 \) is equal to 9 and the trajectory is defined by \( x = (-2 + 3b)t \). It is a line soliton with only one peak. Of course it is nonlocal.

**Case c.)** Let \( a > b \geq a - \frac{3}{8}a^2 \), there is only one maximum at \( (0, 0) \) in the profile of \( |q_{\text{rational}}^{[1]}|^2 \), which is 9. This solution has one peak and one vale, and then nonlocal.

**Case d.)** Let \( b < a - \frac{3}{8}a^2 \), there are one maximum at \( (0, 0) \) and and two minima at \( x = \pm \frac{-6a + 3a^2 + 4b}{a - b} \sqrt[3]{\frac{3}{32a - 12a^2 - 32b}, t = \pm a - b \sqrt[3]{\frac{3}{32a - 12a^2 - 32b}} \) in the profile of \( |q_{\text{rational}}^{[1]}|^2 \). The maximum is equal to 9 and the minimum is 0. This solution has one dominant peak and two hollows. It is localized in both \( x \) and \( t \) directions, and then is called the first order rogue wave solution. The dynamical evolution of the \( |q_{\text{rational}}^{[1]}|^2 \) is in accord with the classical "Peregrine soliton" in the NLS equation [12] and DNLS equation [31][32].
This discussion implies an unusual result: for a given value of \( a \), the increasing value of \( b \) can damage gradually the localization of the rational solution. This is in contrast to the usual conjecture that the localization of this solution will be enhanced because of the appearance of the two nonlinear effects represented by \( a \) and \( b \), according to a common understanding of the role for the nonlinear effects in wave propagation.

For a given value \( a = 1 \), the profiles of \( |q_{\text{rational}}[1]|^2 \) in case d), case c) and case a) are given in Figure 1 with \( b = \frac{1}{3}, \frac{3}{4}, 3 \), respectively. This figure shows visually the lost of the localization of the solution due to the increasing value of \( b \). We omit the picture of case b) because it is a standard soliton. The density plots of the solutions of case d), i.e., rogue wave solutions, of the MNLS equation are given in figure 2 with \( b = 0, \frac{1}{3}, \frac{7}{15} \). It is clear to see the diffusion of the peak and hollows of the first order rogue wave when the value of \( b \) is increasing in Figure 2.

The explicit form of the rogue wave in Figure 2(b) is

\[
q_{rw}^{[1]} = \exp\left(-i\frac{3}{2}\right) \frac{(9 - 18ix + 4t^2 + 18it + 12x^2) (27 - 4t^2 + 18ix - 12x^2 + 30it)}{(-9 - 18ix - 4t^2 + 18it - 12x^2)^2},
\]

which is obtained from \( q_{\text{rational}}^{[1]} \) by setting \( a = 1 \) and \( b = 1/3 \). The orders of polynomial in numerator and denominator of \( q_{rw}^{[1]} \) are both 4. Figure 3(a) is plotted for the contour line at height 5 of the rogue wave \( |q_{\text{rational}}[1]|^2 \) with different values of \( b \) in Figure 2. Note that height 5 is half value of the peak over the asymptotic plane. Let \( a = 1 \), then \( d = \frac{1}{2} \sqrt{\frac{3((4b - 3)^2 + 1)}{(b - 1)^2(5 - 8b)}} \) is the distance from the minimum point of the \( |q_{\text{rational}}[1]|^2 \) to the coordinate origin, which is plotted in Figure 3(b). Figure 3 shows the remarkable decrease in localization of the first order rogue wave. In particular, the distance \( d \) goes to infinity when \( b \to \frac{5}{8} \) in Figure 3(b), then \( |q_{\text{rational}}[1]|^2 \) loses completely the localization in \( x \) and \( t \). This fact is consistent with the limit value of \( b \) in case d).

To see the novelty of the two-peak solution for case a), we would like to present its explicit form as

\[
|q_{\text{twopeak}}^{[1]}|^2 = \frac{8 + 96t^2 + 32x^2 - 32t(2x - 2t)}{(20t^2 + 1 - 4x^2)^2 + (2x - 2t)^2} + 1,
\]

which is obtained by setting \( a = 1 \) from eq. (51) and is plotted in Figure 1(c). The approximate trajectory of two peaks in the profile of \( |q_{\text{twopeak}}^{[1]}|^2 \) are two curves defined by \( 20t^2 + 1 - 4x^2 = 0 \) if \( x > \frac{1}{2} \). It is interesting to note that the height of two peaks is gradually increasing (decreasing). The maximum height of peak along the trajectory is \( 21 + 4\sqrt{5} \) and the minimum height is \( 21 - 4\sqrt{5} \). This two-peak solution with a variable height and a non-vanishing boundary of soliton equation has never been discovered, to the best of our knowledge. Besides, two peak soliton in present paper can not be a usual double-soliton because the rational soliton just has one eigenvalue \( \lambda_0 \). However a double-soliton has two eigenvalues in general.

### 4.2 Analytical forms and localization of the higher order rogue wave solutions

Let \( k = 2, a_1 = -1, c_1 = 1, K_0 = 1, S_0 = L_0 = S_1 = L_1 = 0 \) in theorem 16, the rational 2-order solution \( q_{\text{rational}}^{[2]} \) becomes

\[
q_{\text{rational}}^{[2]} = \exp\left(i(-x - bt - t + at)\right)\left(\frac{(iI_1 + R_1)(iI_1 + R_1 + iI_2 + R_2)}{(-R_1 + iI_1)^2}\right).
\]
Here $I_1$, $R_1$, $I_2$ and $R_2$ are given in the appendix III because the formula is complicated. Let $k = 2$, $a_1 = -1$, $c_1 = 1$, $a = 1$, $b = \frac{1}{3}$, $K_0 = 1$, $S_0 = L_0$, $S_1 = L_1$, $L_0 = 1$, $L_1 = 30$ in theorem 16, $q_{\text{smooth}}^{[4]}$ gives following 2-order rogue wave solution (Figure 4(b))

$$q_{rw}^{[2]} = -\exp\left(-\frac{3x + t}{3}\right)\frac{v_{21}v_{22}}{v_{23}}$$

with

$$v_{21} = -v_{23}^*,$$

$$v_{22} = -300348 xt - 204444 x^2 t + 41364 t^2 x - 1080 \sqrt{3} x^4 - 24 t^4 \sqrt{3} + 6756 \sqrt{3} t^3 - 864 x^4 t^2$$

$$- 288 x^2 t^4 - 48348 \sqrt{3} x^3 - 48 t^5 \sqrt{3} - 432 x^5 \sqrt{3} + 55386 x \sqrt{3} + 16110 t \sqrt{3} + 19008 x^3 t$$

$$+ 41040 x^2 t^2 + 6336 x^3 t - 1728 x^3 t^2 - 288 x t^4 + 53262 t^2 \sqrt{3} - 129546 x^2 \sqrt{3}$$

$$- 181440 it - 325782 i t^2 - 864 x^6 - 432 x t^2 \sqrt{3} + 67644 x \sqrt{3} t^2 - 288 x \sqrt{3} t^3$$

$$- 864 \sqrt{3} x^3 t - 288 x^2 \sqrt{3} t^4 - 48 x \sqrt{3} t^4 - 77868 x \sqrt{3} t - 432 x^4 \sqrt{3} t$$

$$- 39204 x^2 \sqrt{3} t + 868482 i x^2 + 720 i t^5 + 562653 i \sqrt{3} + 757836 i x + 67104 i t^3$$

$$+ 9720 i x^4 + 3888 i x^3 + 3888 i x^5 + 216 i t^4 - 288 t^2 \sqrt{3} x^3 - 41472 i x^2 t$$

$$- 68688 i x t^2 - 299268 i t \sqrt{3} - 79110 i \sqrt{3} t^2 - 3216 - 24415111 \sqrt{3} - 1032 t^4$$

$$+ 7992 x^4 + 1272726 x + 1051650 t + 702 x^2 - 2592 x^5 + 20844 x^3 + 29052 t^3$$

$$+ 3888 i \sqrt{3} x^2 t^2 - 8482662 - 382077 i + 1152 i x \sqrt{3} t^3 + 2592 i t^2 x^3 + 432 i x t^4$$

$$+ 3888 i x^2 t^2 + 4320 i x t^3 + 6480 i x^4 t + 936 i \sqrt{3} t^4 + 588384 i x \sqrt{3} t + 576 i \sqrt{3} t^3$$

$$+ 305532 i x t + 12960 i x^3 t + 4320 i x^2 t^3 + 6480 i \sqrt{3} x^3 t + 357858 i \sqrt{3} x^2 t + 3240 i \sqrt{3} x^4 - 364932 i \sqrt{3} x t + 134838 t^2 + 3456 i \sqrt{3} x^3 t + 5184 i \sqrt{3} x^2 t + 3888 i \sqrt{3} x t^2,$$

$$v_{23} = -160380 x t - 251100 x^2 t + 25812 t^2 x - 1080 \sqrt{3} x^4$$

$$- 24 t^4 \sqrt{3} + 1572 \sqrt{3} t^3 - 864 x^4 t^2 - 288 x^2 t^4 - 58716 \sqrt{3} x^3 - 48 t^5 \sqrt{3}$$

$$- 432 x^5 \sqrt{3} - 3582 x \sqrt{3} + 609030 t \sqrt{3} - 12096 x^3 t + 25488 x^2 t^2 - 4032 t^3 x$$

$$- 1728 x^3 t^2 - 288 x t^4 + 48078 t^2 \sqrt{3} - 145098 x^2 \sqrt{3} - 186921 i \sqrt{3} - 161352 i t$$

$$- 24624 i x^3 - 9720 i x^4 - 3888 i x^5 - 216 i t^4 - 150498 i t^2 - 864 x^6$$

$$- 432 x^2 t^2 \sqrt{3} + 57276 x \sqrt{3} t^2 - 288 x \sqrt{3} t^3 - 864 \sqrt{3} x^3 t - 288 x^2 \sqrt{3} t^3$$

$$- 48 x \sqrt{3} t^4 - 93420 x \sqrt{3} t - 432 x^4 \sqrt{3} t - 54756 x^2 \sqrt{3} t + 251100 i x + 50976 i t^3$$

$$+ 432 i t^5 + 498150 i x^2 - 648 i \sqrt{3} x^4 - 2592 i t^2 x^3 - 277688 i t \sqrt{3} - 288 t^2 \sqrt{3} x^3$$

$$- 3888 i x^2 t^2 - 1296 i \sqrt{3} x^3 - 335340 i x t - 432 i x t^4 - 3216 - 2470023 \sqrt{3}$$

$$- 5352 t^4 + 216 x^4 - 929826 x + 1823418 t - 108162 x^2 - 2592 x^5 + 5292 x^3$$

$$+ 23868 t^3 + 1296 i \sqrt{3} x^2 t^2 + 1296 i \sqrt{3} x^2 t^2 + 15552 i x^2 t + 47952 i x t^2 + 2592 i x t^3$$

$$+ 3888 i x^4 t + 504 i \sqrt{3} t^2 + 18792 i x \sqrt{3} + 7776 i x^3 t + 2592 i x^3 t^2 + 35262 i \sqrt{3} t^2$$

$$+ 17334 i \sqrt{3} x^2 - 322380 i \sqrt{3} x t - 103626 t^2 - 9075582 - 43335 i.$$

Furthermore, the explicit formulas, $q_{rw}^{[3]}$ and $q_{rw}^{[4]}$, of the third and fourth rogue waves are obtained from $q_{\text{smooth}}^{[2k]}$ with $k = 3$ and 4, respectively. However only $q_{rw}^{[3]}$ is given in appendix IV, and we have to omit $q_{rw}^{[4]}$ because it is 12 pages long. They are plotted in Figure 5(b) and 6(b), respectively. Of course, they are local solutions. Obviously, the degrees of polynomials
in above $q_{\text{RW}}^{[k]}(k = 2, 3, 4)$ are 12, 24, 40. This fact supports following conjecture: In general, the degree of the polynomial of the denominator for the rational $k$-order solution in theorem 16 is $2k(k + 1)$. It is a double of the corresponding degree $[19]$ of the rogue wave for the NLS equation due to the contribution of the square of $\Omega_\nu$ in theorem 3.

Figures 4, 5 and 6 are plotted for the second order, third order and fourth order rogue waves from $q_{\text{smooth}}^{[k]}(k = 2, 3, 4)$. These figures show that they are localized in both $x$ and $t$ direction and peaks are diffused dramatically when the value of $b$ is increased. This observation is a strong support to show that the localization of the rogue wave for the MNLS equation is decreased remarkably by increasing the value of $b$. This is a unique phenomenon in a rogue wave solution of the MNLS because of the appearance of the two nonlinear terms. However, it can not happen in the rogue wave of the NLS equation.

Our method can also be applied to get other patterns of the rogue wave by selecting different values of the parameters. For example, the fundamental patterns (a simple central highest peak surrounded by several gradually decreasing peaks in two sides) of the second order, the third order and the fourth order rogue wave are plotted in Figure 7 with the help of $|q_{\text{smooth}}^{[k]}|^2$. Similarly, a triangle pattern, a ring-decomposition pattern (a second order rogue wave surrounded by seven first rogue waves) and a pentagon pattern of the fourth order rogue wave are plotted in Figure 8(a), 8(b) and 8(c), respectively. Note that Figures 7-8 are density plots of $|q_{\text{smooth}}^{[k]}|^2$ with $a_1 = -1, c_1 = 1, a = 1, b = \frac{1}{3}$. All figures in the paper are plotted by using the analytical and exact forms of the solutions of the MNLS. The validity of theses exact solutions is verified by symbolic computation with a computer.

5. Conclusions and Discussions

In this paper, the determinant representation of the n-fold DT for the WKI system is given in theorem 2. By choosing paired eigenvalues and paired eigenfunctions in the form $\lambda_l \leftrightarrow \psi_l = \left( \begin{array}{c} \varphi_l \\ \varphi_l^* \end{array} \right)$, and $\lambda_{2l} = -\lambda_{2l-1}^*, \leftrightarrow \psi_{2l} = \left( \begin{array}{c} \varphi_{2l-1}^* \\ \varphi_{2l-1} \end{array} \right)$, the $2k$-fold DT $T_{2k}$ of the MNLS equation are derived in theorem 7. The smoothness of the $q^{[2k]}$ is given in theorem 9 for the non-degenerate case and in theorem 14 for the double degenerate case through the iteration and determinant representation. Furthermore, the smoothness of the rational solutions $q_{\text{smooth}}^{[2k]}$ is given in theorem 16. By a detailed analysis of the localization of the rational solutions and the rogue waves, we get an unusual result: for a given value of $a$, the increasing value of $b$ can damage gradually the localization of the rational solution, and a novel two-peak rational solution with a variable height and a non-vanish boundary in section 4. Note that this two peak rational soliton just has one eigenvalue $\lambda_0$, which can not be a usual double-soliton. We have verified the validity of theses exact solutions by symbolic computation with a computer.

Finally we would like to stress that there is no doubt of the novelty of the rational solutions presented in this paper although there exists a simple gauge transformation, for example, see refs. 43, 59, between the MNLS equation and the DNLS equation. To illustrate this statement clearly, we shall use following form of the DNLS equation of $\tilde{q} = \tilde{q}(X,T)$:

$$i\tilde{q}_T - \tilde{q}_{XX} + i(\tilde{q}^2\tilde{q}^\ast)_X = 0.$$  \hspace{1cm} (56)

The explicit form of this gauge transformation 43, 59 from the MNLS to the DNLS is

$$\tilde{q} = -q(x,t)a^2 \exp(i\frac{abx + b^2t}{a^2}),$$  \hspace{1cm} (57)
with $t = -\frac{T}{a^x}, x = -\frac{a^\frac{5}{2}X + 2bT}{a^\frac{3}{2}}$. The inverse transformation maps the DNLS to the MNLS, which is given by

$$q = -\bar{q}(X,T)a^{-\frac{3}{2}} \exp(-i\frac{-a^\frac{5}{2}bX + b^2T}{a^\frac{12}{5}}), \quad (58)$$

and $T = -a^\frac{2}{5}t, X = -\frac{ax - 2bt}{a^\frac{4}{5}}$. Accordingly, there exists following transformation

$$\begin{pmatrix} |q|_x \\ |q|_t \end{pmatrix} = a^{-\frac{2}{5}} \begin{pmatrix} -a^\frac{3}{2} & 0 \\ -2ba^{-\frac{4}{5}} & -a^\frac{x}{5} \end{pmatrix} \begin{pmatrix} |\bar{q}|_X \\ |\bar{q}|_T \end{pmatrix}, \quad (59)$$

between $\left( |q|_x, |q|_t \right)$ and $\left( |\bar{q}|_X, |\bar{q}|_T \right)$, which shows that invertible transformations in eq.$(57)$ and eq.$(58)$ preserve the numbers of extreme value points for a given $t$ (or $T$) and stationary points in profiles of $|q|$ and $\bar{q}$. In other words, this simple gauge transformation can not change the numbers of peaks or valleys in $|q|$ and $\bar{q}$. Therefore, the first-order RW and the first-order rational soliton of the DNLS [31,60,61] can not be mapped to the two-peak rational solution of the MNLS by this transformation. By the gauge transformation eq.$(58)$, two solutions of the MNLS with $a = 1$ and $b = 3$ are generated from corresponding solutions of the DNLS in eq.$(56)$ with $\alpha_1 = \beta_1 = \frac{1}{2}$ and in eq. $(46)$ with $\beta_1 = \frac{1}{2}$ of Ref. [31], which are plotted in Figure 9. Similarly, two peak rational solution of the DNLS can be generated from a corresponding solution in eq. $(53)$ of the MNLS, which is plotted in Figure 10. The two peak rational solution of the DNLS has only one eigenvalue and a non-vanishing boundary, which has never been reported in literatures.

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Appendix I: The matrix form of the one-fold Darboux matrix

Consider the universality of DT, suppose that a trial Darboux matrix $T$ in eq. (8) is of

$$ T = T(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, $$

(A.1)

Here $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$ are undetermined functions of $x$ and $t$. From

$$ T_x + T U = U^{[1]} T, $$

(A.2)
comparing the coefficients of $\lambda^j, j = 3, 2, 1, 0$, it yields

\[
\lambda^3 : b_1 = 0, \quad c_1 = 0,
\]
\[
\lambda^2 : q \ a_1 + 2 \ d_1 a_1 + r \ d_1 - 2 \ i \ c_0 = 0,
\]
\[
\lambda^1 : a_{1x} + ar \ b_0 - aq^{[1]}c_0 = 0, \quad a_0 aq + \frac{1}{2} ia_1 \sqrt{2bq} - \frac{1}{2} i \sqrt{2bq^{[1]}d_1} - 2b_0 \sqrt{2b} - aq^{[1]}d_0 = 0,
\]
\[
d_{1x} + aqc_0 - ar^{[1]}b_0 = 0, \quad -ar^{[1]}a_0 + 2c_0 \sqrt{2b} + d_0 ar + \frac{1}{2} i d_1 \sqrt{2br} - \frac{1}{2} i \sqrt{2br^{[1]}a_1} = 0,
\]
\[
\lambda^0 : a_{0x} + ib_0 \sqrt{\frac{b}{2}} - i \sqrt{\frac{b}{2}} q^{[1]}c_0 = 0, \quad b_{0x} - i \sqrt{\frac{b}{2}} q^{[1]}d_0 + i \sqrt{\frac{b}{2}} q a_0 = 0,
\]
\[
i \sqrt{\frac{b}{2}} d_0 r + c_{0x} - i \sqrt{\frac{b}{2}} r^{[1]}a_0 = 0, \quad d_{0x} + i \sqrt{\frac{b}{2}} c_0 q - i \sqrt{\frac{b}{2}} r^{[1]}b_0 = 0. \tag{A.3}
\]

Similarly, from

\[
T_t + T \ V = V^{[1]} T, \tag{A.4}
\]

comparing the coefficients of $\lambda^j, j = 4, 3, 2, 1, 0$, it implies

\[
\lambda^4 : q \ a_1 + 2 \ d_1 a_1 + r \ d_1 - 2 \ i \ c_0 = 0,
\]
\[
\lambda^3 : -iaq^2 rq + ia^2 r^{[1]} q^{[1]} a_1 - 2a^2 q^{[1]} c_0 + 2b_0 a^2 r = 0, \quad 2c_0 q + id_1 a^2 rq - 2a^2 r^{[1]} b_0 - ia^2 r^{[1]} q^{[1]} d_1 = 0,
\]
\[
3ia_1 \sqrt{2bq} + 2a_0 a^2 q - 8b_0 a \sqrt{2b} - 3i \sqrt{2bq^{[1]}d_1} - 2a^2 q^{[1]} d_0 = 0,
\]
\[
-2a^2 r^{[1]} a_0 - 3i \sqrt{2bq^{[1]}a_1} + 8c_0 a \sqrt{2b} + 3id_1 \sqrt{2bq^{[1]}d_1} + 2d_0 a^2 r = 0,
\]
\[
\lambda^2 : -\sqrt{2bq^{[1]}a_1} - 3i \sqrt{2bq^{[1]}c_0 + 3ib_0 \sqrt{2b} - ia_0 a^2 rq + ia^2 r^{[1]} q^{[1]} a_0 + a_1 \sqrt{2bq^{[1]}d_1} = 0,
\]
\[
3ic_0 \sqrt{2bq} + id_0 a^2 rq + \sqrt{2bq^{[1]}d_1} - 3i \sqrt{2bq^{[1]}b_0} - d_1 \sqrt{2bq^{[1]}q^{[1]}d_1} = 0,
\]
\[
3ia_1 \sqrt{2bq} + a_1 a^2 rq + ib_0 a^2 rq - 8ib_0 - 3i \sqrt{2bq^{[1]}d_0} - 2a_1 bq + ia_1 a_{xq} + ia^2 r^{[1]} q^{[1]} b_0
\]
\[
-iaq^{[1]} x_1 + 2bq^{[1]} a_1 - a^2 r^{[1]} q^{[1]} d_1 = 0,
\]
\[
2b^{[1]} a_1 - ic_0 a^2 rq + 8ic_0 b - 2d_1 b r - 3i \sqrt{2bq^{[1]}a_0} - id_1 a_{rx} + d_1 a^2 qr^2 - a^2 q^{[1]} r^{[1]} a_1
\]
\[
+iar^{[1]} x_1 - ia^2 r^{[1]} q^{[1]} c_0 + 3id_0 \sqrt{2bq^{[1]}d_1} = 0,
\]
\[
\lambda^1 : -\sqrt{2bq^{[1]}a_0} + a_0 \sqrt{2bq} + a_{tt} + \frac{1}{2} ia_1 bqr - iaq^{[1]} x_0 - 2b_0 br + 2bq^{[1]} c_0 - ib_0 ar
\]
\[
-a^2 r^{[1]} q^{[1]} c_0 - \frac{1}{2} ibq^{[1]} r^{[1]} a_1 + b_0 a^2 qr^2 = 0,
\]
\[
-2a_0 bq + 2b^{[1]} d_0 - i \sqrt{\frac{b}{2}} a^{[1]} q^{[1]} d_1 - a_1 \sqrt{\frac{b}{2}} q_2 + a_0 a^2 rq - a^2 r^{[1]} q^{[1]} d_0 + ia_0 a_{xq}
\]
\[
+i \sqrt{\frac{b}{2}} a_1 arq^2 - iaq^{[1]} x_0 d_0 - b_0 \sqrt{2bq} - \sqrt{2bq^{[1]} q^{[1]} d_1} = 0,
\]
\[
-ia_0 ar + d_0 a^2 qr^2 + iar^{[1]} x_0 + \sqrt{2bq^{[1]} q^{[1]} c_0} - i \sqrt{\frac{b}{2}} aq^{[1]} r^{[1]} a_1 + i \sqrt{\frac{b}{2}} d_1 aqr^2
\]
\[
+ \sqrt{\frac{b}{2}} d_1 r - 2d_0 br - a^2 q^{[1]} r^{[1]} a_0 - \sqrt{\frac{b}{2}} r^{[1]} a + c_0 \sqrt{2bq} + 2b^{[1]} a_0 = 0,
\]
\[
+iar^{[1]} x_0 b_0 - a^2 q^{[1]} r^{[1]} b_0 + \sqrt{2bq^{[1]} q^{[1]} d_0} - d_0 \sqrt{2bq} - 2c_0 b q + d_1 + \frac{1}{2} ibq^{[1]} r^{[1]} d_1
\]
\(-\frac{1}{2} b q r d_1 + 2 b r^{[1]} t_0 + c_0 a^2 q r^2 + i c_0 a q_x = 0,
\)
\[
\lambda^0 : i \sqrt{\frac{b}{2}} b_0 a q r^2 - i \sqrt{\frac{b}{2}} a r^{[1]} q^{[1]} c_0 + b_0 \sqrt{\frac{b}{2}} r_x + \sqrt{\frac{b}{2}} q^{[1]} c_0 + a_0 t + \frac{1}{2} i a_0 b q r - \frac{1}{2} i b q^{[1]} r^{[1]} a_0 = 0,
\]
\[
-\sqrt{\frac{b}{2}} a q_x + i a_0 \sqrt{\frac{b}{2}} a r^2 - i \sqrt{\frac{b}{2}} a r^{[1]} q^{[1]} d_0 - \frac{1}{2} i b q^{[1]} r^{[1]} b_0 + \sqrt{\frac{b}{2}} q^{[1]} d_0 + b_0 t = 0,
\]
\[
c_{0t} - \sqrt{\frac{b}{2}} r^{[1]} a_0 + i d_0 \sqrt{\frac{b}{2}} a r^2 + i \sqrt{\frac{b}{2}} b q^{[1]} r^{[1]} c_0 - i \sqrt{\frac{b}{2}} a q^{[1]} r^{[1]} a_0 + d_0 i \sqrt{\frac{b}{2}} r_x + \frac{1}{2} i c_0 b q r = 0,
\]
\[
d_{0t} - c_0 \sqrt{\frac{b}{2}} q_x - \sqrt{\frac{b}{2}} r^{[1]} b_0 - \frac{1}{2} i d_0 b q r + i \sqrt{\frac{b}{2}} c_0 a r^2 - i \sqrt{\frac{b}{2}} b q^{[1]} r^{[1]} b_0 + \frac{1}{2} i b q^{[1]} r^{[1]} d_0 = 0. \quad (A.5)
\]

We shall construct a basic (or one fold) Darboux transformation matrix \(T\) by solving eqs. (A.3) and eqs. (A.5). Let \(a_1 d_1 a \neq 0\), substituting \(q^{[1]} = q \frac{a_1}{d_1} + \frac{2 i}{a_1} b_0\) from eq. (A.3) into \(a_0 a q + \frac{1}{2} i a_1 \sqrt{2 b q} - \frac{1}{2} i \sqrt{2 b q}[d_1] - 2 b_0 \sqrt{2 b} - a q^{[1]} d_0 = 0\) of eq. (A.3), then
\[
- a_0 a q d_1 + b_0 \sqrt{2 b d_1} + 2 i a d_0 b_0 + a d_0 a_1 q = 0; \quad (A.6)
\]
substituting \(r^{[1]} = r \frac{d_1}{a_1} - \frac{2 i}{a_1} c_0\) from eq. (A.5) into \(-a r^{[1]} a_0 + 2 c_0 \sqrt{2 b} + d_0 a r + \frac{1}{2} i d_1 \sqrt{2 b r} - \frac{1}{2} i \sqrt{2 b r}[a_1] = 0\) of eq. (A.3), then
\[
- a_0 a d_1 r + 2 i a_0 a c_0 + \sqrt{2 b c_0 a_1} + d_0 a r a_1 = 0. \quad (A.7)
\]
Assume \(2 i a d_0 + \sqrt{2 b d_1} \neq 0\ in eq. (A.6) and \(2 i a a_0 + \sqrt{2 b a_1} \neq 0\ in eq. (A.7), we have \(b_0 = -a q(-a q d_1 + a d_0 a_1)\) and \(c_0 = -a r(-a q d_1 + a d_0 a_1)\). Taking these values of \(b_0\) and \(c_0\) into other equations in eq. (A.3) and eq. (A.5), we find that \(a_1, d_1, a_0, d_0\) are only function of \(t\), which gives trivial solutions of the MNLS equation by DT. This means we can choose \(2 i a d_0 + \sqrt{2 b d_1} = 0\) and \(2 i a a_0 + \sqrt{2 b a_1} \neq 0\ in eq. (A.6) and eq. (A.7) without loss any generality, then \(a_0 = i \sqrt{\frac{b}{2}} a\) and \(d_0 = i \sqrt{\frac{b}{2}} d_1\). Furthermore, taking \(q^{[1]}, r^{[1]}, a_0\) and \(d_0\) into eq. (A.3) and eq. (A.5), then
\[
b_{0x} = -\frac{i b b_0}{a}, \quad c_{0x} = \frac{i b c_0}{a}, \quad b_{0t} = -\frac{i b^2 b_0}{a^2}, \quad \text{and} \quad c_{0t} = \frac{i b^2 c_0}{a^2}.
\]

**APPENDIX II: PROOF OF THEOREM 1**

Note that \((a_1 d_1)_x = 0\ is derived from the eq. (A.3) of appendix I, and then we set \(a_1 = \frac{1}{d_1}\) in the following calculation. By transformation eq. (10) and eq. (A.3), new solutions
\[
q^{[1]} = \frac{a_1}{d_1} q + 2 i \frac{b_0}{d_1} r^{[1]} = \frac{d_1}{a_1} r - 2 i \frac{c_0}{a_1}. \quad (A.8)
\]
are generated by \(T_1\ from a seed solution \(q\). We need to parameterize \(T_1\ by the eigenfunctions associated with \(\lambda_1\). This purpose can be realized through a system of equations defined by its kernel, i.e., \(T_1(\lambda)|_{\lambda=\lambda_1} \psi_1 = 0\). Solving this system of algebraic equations on \((a_1, d_1, b_0, c_0)\), eq. (12) is obtained. Next, substituting \((a_1, d_1, b_0, c_0)\) into eq. (A.8), new solutions \(q^{[1]}\) and \(r^{[1]}\.
are given as eq. (14). Further, by using explicit matrix representation eq. (13) of $T_1$, the new eigenfunction $\psi_j^{[1]} = T_1(\lambda_1)\psi_j$ for $j \geq 2$ becomes

$$\psi_j^{[1]} = \begin{pmatrix} \frac{-b}{a} \exp\left(-i\left(\frac{b}{a}x + \frac{b^2}{a^2}t\right)\right) \lambda_1 + i \frac{\sqrt{2b}}{2a} \exp\left(-i\left(\frac{b}{a}x + \frac{b^2}{a^2}t\right)\right) \\ \frac{-b}{a} \exp\left(i\left(\frac{b}{a}x + \frac{b^2}{a^2}t\right)\right) - \frac{\sqrt{2b}}{2a} \exp\left(i\left(\frac{b}{a}x + \frac{b^2}{a^2}t\right)\right) \lambda_1 + i \frac{\sqrt{2b}}{2a} \end{pmatrix} \begin{pmatrix} \phi_j \\ \varphi_j \end{pmatrix}$$

$$= \begin{pmatrix} 1 \lambda_1 + i \frac{\sqrt{2b}}{2a} & \varphi_j \\ 1 \lambda_1 + i \frac{\sqrt{2b}}{2a} & \phi_j \end{pmatrix} \begin{pmatrix} \exp\left(-i\left(\frac{b}{a}x + \frac{b^2}{a^2}t\right)\right) \\ \exp\left(i\left(\frac{b}{a}x + \frac{b^2}{a^2}t\right)\right) \end{pmatrix}.$$  \quad (A.9)

Last, a tedious calculation verifies that $T_1$ in eq. (13) and new solutions indeed satisfy eq. (A.3) and eq. (A.5) in appendix I. In the process of verification, it is crucial to use the fact that $\psi_1$ satisfies eq. (11) and eq. (5) of the Lax pair associated with a seed solution $q$ and eigenvalue $\lambda_1$. So WKI spectral problem is covariant under transformation $T_1$ in eq. (13) and eq. (14). Therefore $T_1$ is the DT of eq. (2) and eq. (3). \quad \Box

**APPENDIX III: THE RATIONAL 2-ORDER SOLUTION**

$$R_1 = (-54 + 54a + 36a^2)x^2 - 12(b - a + 2a^2)(-b + a)x^4 + 8(-b + a)^3x^6$$
$$+ (-216a^3 - 216b + 216a + 36b - 72a^2)ttx$$
$$+ 48(2ab - 2b + 5a^2 + 2a - 6a^2)(-b + a)t^3x - 96(a - 1)(-b + a)^3t^5x^5$$
$$+ (216a - 216b - 450a^2b + 108a^2 + 18a^3 - 360ab + 396b^2 + 324a^4)t^2$$
$$- 72(-b + a)(12a^4 - 23a^3 + 18a^2 + 3a^2b - 12a - 4a + 4b + 2b^2)t^2x^2$$
$$+ 24(-2b + 19a^2 - 38a + 20)(-b + a)^3t^4x^4 + 48(-b + a)$$
$$+ (27a^5 - 72a^4 + 90a^3 - 52a^2 - 42a^2b + 8a + 48ab + 12b^2a - 8b - 12b^2)t^3x$$
$$- 64(a - 1)(17a^2 - 34a + 20 - 6b)(-b + a)^3t^3x^3 - 12(-b + a)(54a^6 - 189a^5$$
$$- 27a^4b + 356a^4 - 136a^3b - 364a^3b + 372a^2b + 144a^2 + 68a^2b^2 - 204b^2a$$
$$- 160ab + 16a + 48b^2 + 16b + 36b^3)t^4$$
$$+ 24(-2b + 3a^2 - 6a + 4)(-2b + 19a^2 - 38a + 20)(-b + a)^3t^4x^2$$
$$- 96(a - 1)(-2b + 3a^2 - 6a + 4)^2(-b + a)^3t^5x$$
$$+ 8(-b + a)^3(-2b + 3a^2 - 6a + 4)^3t^6 + 9.$$  \quad (A.10)

$$I_1 = -54ax^3 - 24(a^2 + a - b)a^x^3 - 24(a - b + a)^2x^5 + 18a(17a - 6)t$$
$$+ 72(-2a - a^2 + 3a^2 + 2b - ab)a^tx^2 + 24a(11a - 10)(-b + a)^2t^4$$
$$- 72(9a^4 - 15a^3 - 2a^2 + 4a - 4b + 8ab + 3a^2b - 2b^2)a^tx^2$$
$$- 48(23a^2 - 42a - 2b + 20)a(-b + a)^2t^2x^3 + 24(27a^5 - 81a^4 + 60a^3$$
\( R_2 = (144 b - 144 a^2 - 144 a) x^2 - 48 (b + a) x^4 + 288 (3 a - 2) (a^2 + a - b) x^6 \\
+ 192 (a - 2) (b^2 + a^2 x^3 + (-864 b^2 + 288 a^2 + 1296 b^2 - 1296 a^4 - 576 a + 432 a^3 + 576 b) t^2 \\
+ 288 (a^2 + 2 a + 2b - 4) (b + a) t^2 x - 192 (9 a^3 - 16 a^2 + 10 a b + 8 - 12 b ) (b + a)^2 x^2 t^3 \\
+ 48 (10 b + 9 a^2 - 10 a - 4) (-2 b + 3 a^2 - 6 a + 4) (b + a)^2 t^4 + 36. \) (A.11)

\( I_2 = 96 a (b + a) x^3 + 144 a x - 24(15 b - 27 a + 18 a^2) t - 288 (b - 3 a + 2a^2) (b + a) x^2 \\
- 96(-b + a)^3 t x^4 + 288(a - 2) (2 b + 3 a^2 - 4 a) (-b + a) t^2 x + 768(a - 1) (-b + a)^3 t^3 x^3 \\
+ 96(9 a^3 - 38 a^2 + 9 a^2 b + 20 a + 16 a b - 2 b^2 - 12 b) (-b + a) t^4 \\
- 192(11 a^2 - 22 a + 12 - 2 b) (-b + a)^3 t^5 x^2 + 768(a - 1) (-2 b + 3 a^2 - 6 a + 4) (-b + a)^3 t^4 x \\
- 96(-2 b + 3 a^2 - 6 a + 4) (-b + a)^3 t^5. \) (A.12)

APPENDIX IV: THE THIRD ORDER AND FOURTH ORDER ROGUE WAVES

Set \( K_0 = 1, S_0 = L_0 = S_1 = L_1 = 0, S_2 = L_2, L_2 = 9000, a_1 = -1, c_1 = 1, a = 1, b = \frac{1}{3} \) in

\( q_{smooth}^{[6]} \) of theorem 16, the third order rogue wave is given by

\[
q_{rw}^{[3]} = \exp \left( \frac{-i \frac{3 x + t}{3}}{v_{31} v_{32}} \right) \frac{v_{31} v_{32}}{v_{33}^2}
\] (A.14)

with

\[
v_{31} = -v_{33*},
\]

\[
v_{32} = 651440377800 x t - 86093442000000 x^2 t
\]

+ 562477154400000 x^3 t + 1408672944000 x^4 t + 32911217600 x^2 t^4

- 130096756800 x^3 t - 90557546400 x^2 t^2 + 713406398400 t^3 x

- 20407344000000 x^3 t^2 - 47617113600 x^3 t^4 + 30611001600 x^4 t

- 10203667200000 x^3 t^3 + 25339106880 x^6 + 110592 i x t^{10} + 1658880 i x^3 t^8

+ 9953280 i x^5 t^6 + 29859840 i x^7 t^4 + 44789760 i x^9 t^2 + 106755868080000 i t x

+ 138364570240000 i x t^3 + 30611001600000 i x^3 t + 314613072000 i x^3 t^2

+ 399856208400 i x t^2 + 606480469200 i t^2 x + 44789760 i t x^{10} + 2764800 i t^9 x^2

+ 1658880 i t^7 x^4 + 3627970560000 i t^4 t^5 + 373248000 i t^7 t^4 + 3287843200 i x^5 t^2

+ 10825125120 i x t^6 + 7936185600 i t^3 x^4 + 150888303600 i x^7 t^2 + 49766400 i t^5 x^6

+ 74649600 i t^3 x^8 + 191545508160 t^6 - 847046095200 t^4 - 88006629600 x^4

- 4761713600000 t^5 + 64570081500000 x + 1627166053800000 t

- 270333407880000 i t^2 - 114791256000000 x^4 - 245674545400 i t^3

- 695984581440 i t^5 - 302330880 i x^9 - 4308215040 i x^7 - 59691453120 i x^5

+ 1394713767933176175 - 387420489000000 i - 619872892886583900 x^2

\]
\[ v_{33} = \frac{774840978000 \cdot x \cdot t + 792059666400000 \cdot x^2 \cdot t + 114791256000000 \cdot x^3 \cdot t^2 - 74543457600 \cdot x^4 \cdot t^2}{-416555265600 \cdot x^2 \cdot t^4 - 191318760000 \cdot x^3 \cdot t^3 + 696400286400 \cdot x^2 \cdot t^2}
\]

\[ v_{33} = \frac{-1089241473600 \cdot x^3 \cdot t - 40814668800000 \cdot x^3 \cdot t^2 - 20407334400000 \cdot x^4 \cdot t^2 + 275499014400000 \cdot x^4 \cdot t + 6122003200000 \cdot x^2 \cdot t^3 - 9580109760 \cdot x^6}{+459165024000000 \cdot i \cdot t^3 + 459165024000000 \cdot i \cdot x^3 \cdot t + 120530818800 \cdot i \cdot t^2 + 1403004240000 \cdot i \cdot t \cdot x^4 + 389439964800 \cdot i \cdot t^3 \cdot x^2 + 327155079600 \cdot i \cdot t^2 \cdot x}
\]

\[ v_{33} = \frac{-26873856 \cdot i \cdot t \cdot x^{10} + 1658880 \cdot i \cdot t^9 \cdot x^2 + 9953280 \cdot i \cdot t^7 \cdot x^4 + 5441955840000 \cdot i \cdot t^5 \cdot x^6 + 403107840 \cdot i \cdot t^7 \cdot x^2 + 67184640 \cdot i \cdot t^5 \cdot x^6 + 253808640 \cdot i \cdot x^3 \cdot t^6 + 2821754880 \cdot i \cdot x^7 \cdot t^2 + 1813985280 \cdot i \cdot x^5 \cdot t^4 + 44971718400 \cdot i \cdot t \cdot x^4 + 28343520000 \cdot i \cdot t^3 \cdot x^4 + 102263420160 \cdot i \cdot t^5 \cdot x^2 + 7482698280 \cdot i \cdot t^9 \cdot x^6 + 29859840 \cdot i \cdot t^5 \cdot x^6 + 44789760 \cdot i \cdot t^3 \cdot x^8 + 68614413120 \cdot x^6 - 394116645600 \cdot t^4 + 29335543200 \cdot x^4 \cdot t^4 - 21087578880000 \cdot t^5 + 43907655420000 \cdot x - 105894933660000 \cdot t}{-4649045878676168025 - 774840978000000 \cdot i - 61987821532704700 \cdot x^2 - 367332019200000 \cdot x^5 + 344373768000000 \cdot x^3 + 631351908000000 \cdot t^3 - 503884800 \cdot t^7 \cdot x - 9943326720 \cdot t^6 \cdot x^2 - 11197440 \cdot t^8 \cdot x^2 + 100776960 \cdot t^5 \cdot x^3 + 77137920 \cdot t^6 \cdot x^4 + 4534963200 \cdot x^5 \cdot t^3 + 5542732800 \cdot t^4 \cdot x^4 - 2015553920 \cdot t^9 \cdot x^9 - 5971968 \cdot t^2 \cdot x^{10} - 2687385600 \cdot t^3 \cdot x^7 - 49766400 \cdot t^8 \cdot x^4 + 4147200 \cdot t^9 \cdot x - 73728 \cdot t^{10} \cdot x^2 + 243792063600 \cdot t^5 \cdot x - 107818750000 \cdot x^5 \cdot t - 2913713856000 \cdot x^3 \cdot t^3 - 12899450880 \cdot x^6 \cdot t^2 + 724101120 \cdot x^6 \cdot t^4 - 5744286720 \cdot x^7 \cdot t + 839808000 \cdot x^8 \cdot t^2 - 2519424000000 \cdot t^6 \cdot x + 1360488960000 \cdot t^5 \cdot x^2 + 2267481600000 \cdot x^4 \cdot t^3 + 7558272000000 \cdot x^3 \cdot t^4 + 4081466880000 \cdot x^5 \cdot t^2 - 6802448400000 \cdot x^6 \cdot t + 10617972480 \cdot i \cdot t^7 + 671528847600000 \cdot i \cdot x^2 + 114877440 \cdot i \cdot t^9 + 184320 \cdot i \cdot t^{11} + 28697814000 \cdot i \cdot x^3 + 293355432000000 \cdot i \cdot t^4 + 929809203732704700 \cdot i \cdot x + 15116544000000 \cdot i \cdot t^6 + 326517350400000 \cdot i \cdot t^9 + 1549682126751993300 \cdot i \cdot t - 206625142301186700 \cdot t^2 - 503884800000 \cdot i \cdot t^7 + 9387187200 \cdot t^8 - 80179200 \cdot t^{10} - 2985925000 \cdot x^{12} - 3966125000 \cdot i \cdot t^8 \cdot x^2 - 3755551640000 \cdot i \cdot x^5 - 48467419200 \cdot i \cdot t \cdot x^4 - 269263440000 \cdot i \cdot t^3 \cdot x^2 - 40814668800000 \cdot i \cdot t^2 \cdot x^4 - 272097792000000 \cdot i \cdot t^4 \cdot x^2 - 33592320 \cdot i \cdot t^8 \cdot x - 39862886400 \cdot i \cdot t^3 \cdot x^6 - 1713208320 \cdot i \cdot x^8 \cdot t - 1052559360 \cdot i \cdot t^4 \cdot x^5 - 671846400 \cdot i \cdot x^3 \cdot t^6 - 4434186240 \cdot i \cdot x^7 \cdot t^2 - 3157678080 \cdot i \cdot x^5 \cdot t^4 - 39932870400 \cdot i \cdot t^4 \cdot x^3 - 18139852800000 \cdot i \cdot x^3 \cdot t^3 - 10425395200 \cdot i \cdot x^6,}
\]
−2211840 t^6 x^6 − 29859840 t^7 x^3 − 552960 t^8 x^4 − 2488320 t^9 x − 73728 t^{10} x^2
−156796352640 t^5 x + 9183300480 x^5 t + 98635449600 x^3 t^3 − 7255941120 x^6 t^2
+425502720 x^6 t^4 − 906992640 x^7 t + 571069440 x^8 t^2 − 251942400000 t^6 x
+1360488960000 t^5 x^2 + 2267481600000 x^4 t^3 + 755827200000 x^3 t^4
+4081466880000 x^5 t^2 − 6802444800000 x^6 t − 440033148000 i x^3
−929809186514016300 i x − 26873856 i x^{11} − 100776960 i x^9
−4761711360 i x^7 − 36223018560 i x^5 − 68874753600000 i x^4
−464904586800000 i x^2 + 1760132592000000 i t^4 − 206624970114302700 t^2
−50388480000 i t^7 − 382579200 t^6 − 3234816 t^{10} − 2985984 x^{12} − 4096 t^{12}
+87340032 x^{10} − 1360488960000 x^7 + 453496320 x^8 + 110592 i t^{11}
+943201486800 i t^3 + 285645994560 i t^5 + 85847040 i t^9
+929809272607458300 i t + 8162933760000 i x^6 + 14016395520 i t^7
+604661760000 i t^6 + 705966224400000 i t^4 − 780580540800000 i t^2 x^2
−103312130400000 i t x − 784265198400 i x t^4 − 103737283200 i x^3 t^2
−13604889600000 i t^4 x^2 − 11197440 i t^8 x − 1209323520 i t^3 x^6
−48977602560000 i x^5 t − 110592 i x t^{10} − 1658880 i x^3 t^8 − 9953280 i x^5 t^6
−29859840 i x^7 t^4 − 44789760 i x^9 t^2 − 302330880 i x^8 t − 12924645120 i x^5 t^2 − 15998342400 i x t^6.
Figure 1. (Color online) Profiles of $|q_{\text{rational}}^{[1]}|^2$ with a given value $a = 1$. From left to right, $b = \frac{1}{3}, b = \frac{3}{4}, b = 3$, which shows visually the lost of the localization of this solution. Note that there are two hollows in Fig(a), and there is a vale in Fig(b).

Figure 2. (Color online) Density plot of the rogue wave $|q_{\text{rational}}^{[1]}|^2$ with $a = 1$. From left to right, $b = 0, \frac{1}{3}, \frac{7}{15}$, which shows the diffusion of the peak and hollows.
Figure 3. (Color online) The decrease in localization of the first order rogue wave $|q_{rational}^{[1]}|^2$ with $a = 1$. (a) Contour line at height 5 with $b = 0$ (inner, red), $\frac{1}{3}$ (middle, green), $\frac{7}{15}$ (outer, blue). (b) The distance from minimum point of $|q_{rational}^{[1]}|^2$ with $a = 1$ to coordinate origin.

Figure 4. (Color online) Diffusion of the second order rogue wave through the density plot of $|q_{smooth}^{[4]}|^2$ with $K_0 = 1, S_0 = L_0, S_1 = L_1, L_0 = 1, L_1 = 30$. From left to right, $b = 0, \frac{1}{3}, \frac{7}{15}$. 
Figure 5. (Color online) Diffusion of the third order rogue wave through the density plot of $|q_{\text{smooth}}|^{2}$ with $K_0 = 1, S_0 = L_0 = S_1 = L_1 = 0, S_2 = L_2, L_2 = 9000$. From left to right, $b = 0, \frac{1}{3}, \frac{7}{15}$.

Figure 6. (Color online) Diffusion of the fourth order rogue wave through the density plot of $|q_{\text{smooth}}|^{2}$ with $K_0 = 1, S_0 = L_0 = 0, S_1 = L_1 = 0, S_2 = L_2 = 500, S_3 = L_3 = 9000$. From left to right, $b = 0, \frac{1}{3}, \frac{7}{15}$.

Figure 7. (Color online) The density plot of the fundamental pattern for rogue waves $|q_{\text{smooth}}|^{2}(k = 2, 3, 4)$. a) The second order: $K_0 = 1, S_0 = L_0 = S_1 = L_1 = 0$; b) The third order: $K_0 = 1, S_0 = L_0 = S_1 = L_1 = S_2 = L_2 = 0$; c) The fourth order: $K_0 = 1, S_0 = L_0 = S_1 = L_1 = S_2 = L_2 = S_3 = L_3 = 0$. 

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Figure 8. (Color online) The density plot of three patterns for the fourth order rogue wave $|q_{smooth}^{[8]}|^2$. a) triangle pattern with $K_0 = 6 - 2i, S_0 = L_0 = 6 + 5i, S_1 = L_1 = 600, S_2 = L_2 = 0, S_3 = L_3 = 0$; b) ring-decomposition pattern with $K_0 = 1, S_0 = L_0 = 0, S_1 = L_1 = 0, S_2 = L_2 = 0, S_3 = L_3 = 9000$; c) pentagon pattern $K_0 = 1, S_0 = L_0 = 0, S_1 = L_1 = 0, S_2 = L_2 = 10000, S_3 = L_3 = 0$.

Figure 9. (Color online) Two solutions of the MNLS generated from corresponding solutions of the DNLS. The left panel is plotted for a RW, the right panel is plotted for a rational soliton.

Figure 10. (Color online) Two peak solution of the DNLS equation generated from a corresponding solution of the MNLS in eq. (53). The right panel is the density plot of the left panel.