Variational homotopy perturbation method for solving the generalized time-space fractional Schrödinger equation

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We suggest and analyze a technique by combining the variational iteration method and the homotopy perturbation method. This method is called the variational homotopy perturbation method. We use this method for solving Generalized Time-space Fractional Schrödinger equation. The fractional derivative is described in Caputo sense. The proposed scheme finds the solution without any discretization, transformation or restrictive assumptions. Several example is given to check the reliability and efficiency of the proposed technique.

Key words: Caputo derivative, variational iteration method, homotopy perturbation method, Schrödinger equation.

INTRODUCTION

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of nonlinear science (Dalir and Bashour, 2010), many important phenomena (Podlubny, 1999), engineering and physics (Miller and Ross, 1993), dielectric polarization (Sun et al., 1984), quantitative finance (Laskin, 2000).

To find explicit solutions of linear and nonlinear fractional differential equations, many powerful methods have been used such as the homotopy perturbation method (Momani and Odibat, 2007; Wang, 2008; Gupta and Singh, 2011), the Adomain decomposition method (Ray, 2009; Herzallah and Gepreel, 2012; Rida et al., 2008), the variational iteration method (He, 2000, 2004, 2007; He and Wang, 2007), the homotopy analysis method (Hemida et al., 2012; Gepreel and Mohamed, 2013; Ganjiani, 2010; Behzadi, 2011), the fractional complex transform (Ghazanfari, 2012; Su et al., 2013), the homotopy perturbation Sumudu transform method (Karbalaie et al., 2014; Mahdy et al., 2015), the local fractional variation iteration method (Yang and Baleanu, 2013), the local fractional Adomain decomposition method (Yang et al., 2013b), the Cantor-type Cylindrical-Coordinate method (Yang et al., 2013c), the variational iteration method with Yang-Laplace (Liu et al., 2013), the Yang-Fourier transform (Yang et al., 2013a), the Yang-Laplace transform (Zhao et al., 2014; Zhang et al., 2014) and variational homotopy perturbation method by (Noor and Mohyud-Din, 2008). The variational homotopy perturbation

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method (VHPM) is a combination of the variational iteration method and homotopy perturbation method. The suggested VHPM provides the solution in a rapid convergent series which may lead the solution in a closed form and is in full agreement with Rida et al. (2008), where similar problems were solved by using the decomposition method. The fact that the proposed technique solves nonlinear problems without using the so-called Adomian’s polynomials is a clear advantage of this algorithm over the decomposition method.

In this paper, we investigate the application of the VHPM for solving the generalized time-space fractional Schrödinger equation with variable coefficients (Rida et al., 2008; Ganjiani, 2010):

\[ i \frac{\partial^{\alpha} u}{\partial t^{\alpha}} + a \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + v(x)u + \gamma |u|^2 u = 0, \]  

(1)

Where \( t > 0 \), \( 0 < \alpha, \beta \leq 1 \) with initial conditions

\[ u(x,0) = u_0(x,t), \]  

(2)

Where \( u = u(x,t) \) is unknown function, \( v(x) \) is the trapping potential, \( 0 < \alpha, \beta \leq 1 \) are parameters describing the order of the fractional Jumaries derivatives (Jumaries, 2007) and \( a, \gamma \) are a real constants, respectively. If we select \( \alpha = \beta = 1 \), \( v(x) = 0 \), this equation turns to the famous nonlinear Schrödinger equation in optical fiber (Hao et al., 2004; Chen and Li, 2008; Li and Chen, 2004). In this paper, notice that Equation (1) is a complex differential equation with complex modulus term \( |u|^2 \), as we all know, a complex function \( u(\zeta) \) can be written as \( u(\zeta) e^{i\theta(\zeta)} \), where \( c(\zeta) \) and \( \theta(\zeta) \) are real functions, noticed that \( |u(\zeta)|^2 = |c(\zeta)|^2 \), assume that \( \lim_{x \to 0} |u|^2 = |u_0|^2 \), we get the VHPM for Equation (1).

**BASIC DEFINITIONS OF FRACTIONAL CALCULUS**

Here, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper.

**Definition 1**

A real function \( f(t), t > 0, f(t) \), is said to be in the space \( C_{m}^{\alpha} \), if there exists a real number \( p > \sigma \) such that \( f(t) = t^p f_1(t) \) where \( f_1(t) \in [0, \infty) \), and it is said to be in the space \( C_{m}^{\alpha} \) if \( f^m \in C_{\sigma} \), \( m \in \mathbb{N} \).

**Definition 2**

The left sided Riemann-Liouville fractional integral of order \( \alpha \geq 0 \), of a function \( f \in C_{\sigma} \), \( \sigma \geq -1 \) is defined as:

\[ J_{-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - \zeta)^{\alpha-1} f(\zeta) \, d\zeta \]  

(3)

where \( \alpha > 0, t > 0 \) and \( \Gamma(\alpha) \) is the Gamma function.

Also one has the following properties:

\[ J_{-}^{\alpha} J_{-}^{\beta} f(x) = J_{-}^{\alpha+\beta} f(x), \]  

\[ J_{-}^{\alpha} J_{-}^{\beta} f(x) = J_{-}^{\beta} J_{-}^{\alpha} f(x), \]  

\[ J_{-}^{\alpha} x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \]  

**Definition 3**

Let \( f \in C_{m}^{n}, n \in \mathbb{N} \cup \{0\} \). The left sided Caputo fractional derivative of \( f \) in the Caputo sense is defined by Podlubny (1999) and He (2014) as follows:

\[ D_{-}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t - \zeta)^{n-\alpha-1} f^{(n)}(\zeta) \, d\zeta, & n-1 < \alpha \leq n, \\ D_{-}^{\alpha} f(t), & \alpha = n, \end{cases} \]  

(5)

Also one has the following properties:

\[ D_{-}^{\alpha} C = 0, \quad (C \text{ is constant}), \]  

\[ D_{-}^{\alpha} x^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\gamma-\alpha}, & \gamma > \alpha - 1, \\ 0, & \gamma \leq \alpha - 1, \end{cases} \]  

\[ J_{-}^{\alpha} D_{-}^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad n-1 < \alpha \leq n, \]  

\[ D_{-}^{\alpha} J_{-}^{\alpha} f(x) = f(x). \]  

(6)

**Definition 4**

The single parameter and the two parameters variants of the Mittag-Leffler function are denoted by \( E_{\alpha}(t) \) and \( E_{\alpha,\beta}(t) \), respectively, which are relevant for their
connection with fractional calculus, and are defined as:

\[ E_{\alpha}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + 1)}, \quad \alpha > 0, \quad t \in C, \]  
\[ E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta > 0, \quad t \in C. \]  

Some special cases of the Mittag-Leffler function are as follows:

\[ E_1(t) = e^t, \quad E_{\alpha,1}(t) = E_{\alpha}(t). \]

Other properties of the Mittag-Leffler functions can be found in Kilbas et al. (2004). These functions are generalizations of the exponential function, because, most linear differential equations of fractional order have solutions that are expressed in terms of these functions.

### VARIATIONAL ITERATION METHOD

To illustrate the basic concept of the technique, we consider the following general differential equation:

\[ L(u) + N(u) - f(x) = 0, \]  

where \( L \) is a linear operator, \( N \) a nonlinear operator, and \( f(x) \) the function term. In the variational iteration method (He, 2000, 2004, 2007; He and Wang, 2007), a correction functional can be constructed as follows:

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + Nu_n(s) - f(s)) \, ds, \]

Where \( \lambda \) is a general Lagrange multiplier (He, 2000, 2004, 2007; He and Wang, 2007), which can be identified optimally via a variational iteration method. The subscripts \( n \) denote the \( n \)th approximation, \( u_n \) is considered as a restricted variation. That is, \( \delta u_n = 0 \); equation (10) is called a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of the variational iteration method and its applicability for various kinds of differential equations are given in (He, 2000, 2004, 2007; He and Wang, 2007). With \( \lambda \) determined then several approximation \( u_{n+1}, \) \( n \geq 0 \) follow immediately. Consequently, the exact solution may be obtained by using \( u = \lim_{n \to +\infty} u_n \).

### HOMOTOPY PERTURBATION METHOD

Consider the following nonlinear differential equation

\[ A(u) - f(x) = 0, \quad x \in \Omega, \]  

Subject to the conditions

\[ B(u, \partial u/\partial m) = 0, \quad x \in \Gamma, \]  

Where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(x) \) is a known an analytical function, \( \Gamma \) is boundary of the domain \( \Omega \) and \( \partial / \partial m \) denotes directional derivative.

The operator \( A \) can be decomposed into a linear operator, denoted by \( L \), and a nonlinear operator, denoted by \( N \). Therefore, Equation (11) can be written as follows:

\[ L(u) + N(u) - f(x) = 0. \]

By the homotopy technique we construct defined as \( v(x, p) : \Omega \times [0, 1] \to R \) with satisfies:

\[ H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(u) - f(x)] = 0, \quad 0 \leq p \leq 1, \]

Which is equivalent to

\[ H(v, 0) = L(v) - L(u_0) = 0, \]

\[ H(v, 1) = L(v) - N(v) - f(x) = A(u) - f(x) = 0, \]

in topology, this changing process is called deformation, and Equations (16) and (17) are called homotopic. If the \( p \)-parameter is considered as small, then the solution of Equations (13) and (14) can be expressed as a power series in \( p \) as follows:

\[ v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots \]

The best approximation for the solution of Equation (11) is

\[ u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \]

It is well known that series (18) is convergent for most of the cases and also the rate of convergence is dependent on \( L(u) \); (Momani and Odibat, 2007; Wang, 2008; Gupta and Singh, 2011). We assume that Equation (19) has a unique solution. The comparisons of like powers of \( p \) give solutions of various orders.

### VARIATIONAL HOMOTOPY PERTURBATION METHOD (VHPM)

To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation:
\[ Lu + Nu = f(x), \]  
(20)

Where \( L \) is the linear operator, \( N \) is the general nonlinear operator and \( f(x) \) the forcing term. According to variational iteration method (He, 2000, 2004, 2007; He and Wang, 2007), we can construct a correct functional as follows:

\[
u_{n+1}(x) = u_n(x) + \int_0^1 \lambda(\zeta)(Lu_n(\zeta) + Nu_n(\zeta) - f(\zeta)) \, d\zeta,
\]
(21)

Where \( \lambda \) is a Lagrange multiplier (He, 2000, 2004, 2007; He and Wang, 2007), which can be identified optimally via variational iteration method. The subscripts \( n \) denote the \( n \)th approximation, \( \tilde{u}_n \) is considered as a restricted variation. That is, \( \delta \tilde{u}_n = 0 \); Equation (21) is called as a correct functional. Now, we apply the homotopy perturbation method.

\[
\sum_{n=0}^{\infty} p^n u_n = u_0(x) + \int_0^1 \lambda(\zeta) \left( \sum_{n=0}^{\infty} p^n L(u_n) + \sum_{n=0}^{\infty} p^n N(\tilde{u}_n) - f(\zeta) \right) \, d\zeta,
\]
(22)

Which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian’s polynomials. A comparison of like powers of \( p \) gives solutions of various orders.

APPLICATIONS

Here, we apply the VHPM developed in Section 5 for solving the Generalized Time-space Fractional Schrödinger Equation with variable coefficients. We develop the correct functional and calculate the Lagrange multipliers optimally via variational theory. The homotopy perturbation method is implemented on the correct functional and finally, comparison of like powers of \( p \) gives solutions of various orders. Numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy. For the sake of comparison, we take the same examples as used in (Herzallah and Gepreel, 2012; Rida et al., 2008; Wazwaz, 2008; Hong and Lu, 2014).

Example 1

We first consider the time-fractional NLS equation:

\[
i \frac{\partial^\alpha u}{\partial t^\alpha} + a \frac{\partial^2 u}{\partial x^2} + \gamma |u|^2 u = 0,
\]
(23)

where \( t > 0 \), \( 0 < \alpha, \beta \leq 1 \) with initial conditions

\[ u(x,0) = A \sec h(x), \]
(24)

The correct functional is given as:

\[ u_{n+1}(x,t) = A \sec h(x) + \int_0^1 \lambda(\zeta) \left( \frac{\partial^\alpha u_n}{\partial t^\alpha} - \partial^2 \frac{\partial \tilde{u}_n}{\partial x^2} - i \gamma |\tilde{u}_n|^2 \right) \, d\zeta, \]
(25)

Where \( \tilde{u}_n \) is considered as restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as \( \lambda = -1 \), which yields the following iteration formula:

\[ u_{n+1}(x,t) = A \sec h(x) - \int_0^1 \lambda(\zeta) \left( \frac{\partial^\alpha u_n}{\partial t^\alpha} - \partial^2 \frac{\partial \tilde{u}_n}{\partial x^2} - i \gamma |\tilde{u}_n|^2 \right) \, d\zeta, \]
(26)

Applying the variational homotopy perturbation method, we have

\[ u_0 = p^0 u_0(x,t) = A \sec h(x), \]

\[ u_1 = p^1 u_1(x,t) = A \sec h(x) - \frac{\partial^\alpha u_0}{\partial t^\alpha} + i \gamma |u_0|^2, \]

\[ u_2 = p^2 u_2(x,t) = A \sec h(x) - \frac{\partial^\alpha u_1}{\partial t^\alpha} + i \gamma |u_1|^2, \]

\[ u_n = p^n u_n(x,t), \]

Comparing the coefficient of like powers of \( p \), we have

\[ p^0 : u_0(x,t) = A \sec h(x), \]

\[ p^1 : u_1(x,t) = A \sec h(x) - 2 \sec h^2(x) \left( \frac{\partial^\alpha u_0}{\partial t^\alpha} + i \gamma |u_0|^2 \right) \frac{t^n}{\Gamma(\alpha + 1)}, \]

\[ p^2 : u_2(x,t) = A \sec h(x) - 2 \sec h^2(x) \left( \frac{\partial^\alpha u_1}{\partial t^\alpha} + i \gamma |u_1|^2 \right) \frac{t^n}{\Gamma(2\alpha + 1)} \]

Thus the solution of Equation (23) is given by

\[ u(x,t) = \lim_{p \to 0} \sum_{n=0}^{\infty} p^n u_n(x,t). \]
(29)

If we put \( \alpha \to 1 \) in Equation (29) or solve Equations (23)
and (24) with $\alpha = 1$, we obtain the exact solution
\[ u(x,t) = \sec h(x) \sum_{n=0}^{\infty} \frac{(ait^{\alpha})^n}{\Gamma(n\alpha + 1)} \]
\[ = \pm \sqrt{\frac{2a}{\gamma}} \sec h(x) e^{iat}. \]

Which is in full agreement with the result in Hong and Lu, (2014)

**Example 2**

We first consider the time-space fractional NLS equation:

\[ i \frac{\partial^\alpha u}{\partial t^\alpha} + a \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + 2a|\dot{u}|^2 u = 0, \tag{30} \]

Where $t > 0$, $0 < \alpha$, $\beta \leq 1$ with initial conditions

\[ u(x,0) = e^{ix}, \tag{31} \]

The correct functional is given as

\[ u_{n+1}(x,t) = e^{ix} + \int_0^1 \lambda(\zeta) \left( \frac{\partial^\alpha u_{n}}{\partial t^\alpha} - ia \frac{\partial^{2\beta} \tilde{u}_n}{\partial x^{2\beta}} - 2ia|\tilde{u}_n|^2 \right) d\zeta. \tag{32} \]

Where $\tilde{u}_n$ is considered as restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

\[ u_{n+1}(x,t) = e^{ix} - J^{\alpha} \left[ \frac{\partial^\alpha u_{n}}{\partial t^\alpha} - ia \frac{\partial^{2\beta} \tilde{u}_{n}}{\partial x^{2\beta}} - 2ia|\tilde{u}_n|^2 \right]. \tag{33} \]

Applying the variational homotopy perturbation method, we have:

\[ u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots = e^{ix} - pJ^{\alpha} \left( \frac{\partial^\alpha u_0}{\partial t^\alpha} - ia \frac{\partial^{2\beta} \tilde{u}_0}{\partial x^{2\beta}} - 2ia|\tilde{u}_0|^2 \right). \tag{34} \]

Comparing the coefficient of like powers of $p$, we have:

\[ p^0 : u_0(x,t) = e^{ix}, \]

\[ p^1 : u_1(x,t) = \left[ aie^{ix} e^{iap\beta} + 2aie^{ix} \right] \frac{at^{\alpha}}{\Gamma(\alpha + 1)} \]
\[ = e^{ix} \left( 2 + e^{iap\beta} \right) \frac{at^{\alpha}}{\Gamma(\alpha + 1)}, \]

\[ p^2 : u_2(x,t) = \left[ e^{ix} \left( 2 + e^{iap\beta} \right) e^{iap\beta} + 2e^{ix} \left( 2 + e^{iap\beta} \right) \right] \frac{(ait^{\alpha})^2}{\Gamma(2\alpha + 1)} \]
\[ = e^{ix} \left( 2 + e^{iap\beta} \right)^2 \frac{(ait^{\alpha})^2}{\Gamma(2\alpha + 1)}, \]

\[ p^3 : u_3(x,t) = e^{ix} \left( 2 + e^{iap\beta} \right)^3 \frac{(ait^{\alpha})^3}{\Gamma(3\alpha + 1)}, \]

\[ p^n : u_n(x,t) = e^{ix} \left( 2 + e^{iap\beta} \right)^n \frac{(ait^{\alpha})^n}{\Gamma(n\alpha + 1)}. \tag{35} \]

Thus the solution of Equation (30) is given by:

\[ u(x,t) = \lim_{n \to \infty} \sum_{i=0}^{n} p^i u_i(x,t) \]
\[ = e^{ix} \left( 1 + (2 + e^{iap\beta}) a \left( a \frac{\alpha p \beta}{\alpha + 1} \right) + (2 + e^{iap\beta})^2 \left( a \frac{\alpha p \beta}{2(\alpha + 1)} \right) + \ldots \right) \]
\[ = e^{ix} \sum_{i=0}^{\infty} (2 + e^{iap\beta})^i \frac{(ait^{\alpha})^i}{\Gamma(i\alpha + 1)} \]
\[ = e^{ix} E_{\alpha}(a t^{2 + e^{iap\beta}}). \tag{36} \]

If we put $\alpha \to 1$ in Equation (36) or solve Equations (30) and (31) with $\alpha = 1$, we obtain the exact solution

\[ u(x,t) = e^{ix} \sum_{n=0}^{\infty} (2 + e^{iap\beta})^n \frac{(ait^{\alpha})^n}{\Gamma(n\alpha + 1)} \]
\[ = e^{ix(1+at)}. \]

Which is in full agreement with the result of Herzallah and Gepreel (2012); Wazwaz (2008) and Hong and Lu (2014)

**Example 3**

We first consider the time-space fractional NLS equation:

\[ i \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2} \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - u \cos^2(x) - |\dot{u}|^2 u = 0, \tag{37} \]

Where $t > 0$, $0 < \alpha$, $\beta \leq 1$ with initial conditions

\[ u(x,0) = \sin(x), \tag{38} \]

The correct functional is given as:
\[ u_{\text{app}}(x, t) = \sin(x) + \int_0^1 \frac{1}{\alpha^2} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + iuv \cos^2(x) + iuv \right] d\zeta, \quad (39) \]

Where \( \tilde{u}_n \) is considered as restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as \( \lambda = -1 \), which yields the following iteration formula:

\[ u_{n+1}(x, t) = \sin(x) - 4 \left[ \frac{\partial^2 u}{\partial x^2} - iuv \cos^2(x) + iuv \right] \left[ \frac{1}{\alpha^2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + p \frac{\partial^2 u}{\partial x^2} + \cdots \right) \right] \quad (40) \]

Applying the variational homotopy perturbation method, we have

\[ u_t + p \frac{\partial u}{\partial x} + p^2 \frac{\partial^2 u}{\partial x^2} + \cdots = \sin(x) - \beta \]

Comparing the coefficient of like powers of \( p \), we have

\[ p^0 : u_0(x, t) = \sin(x), \]

\[ p^1 : u_1(x, t) = \left[ \frac{1}{4} \sin(x + 2\beta) - \sin(x) \right] \frac{it^{\beta}}{\Gamma(\alpha + 1)} \]

\[ p^2 : u_2(x, t) = \left[ \frac{1}{8} \sin(x + 3\beta) - \frac{3}{4} \sin(x + 2\beta) + \frac{3}{2} \sin(x + \beta) \right] \frac{(it^{\beta})^2}{\Gamma(2\alpha + 1)} \]

\[ p^3 : u_3(x, t) = \left[ \frac{1}{16} \sin(x + 4\beta) - \frac{1}{2} \sin(x + 3\beta) - \frac{3}{4} \sin(x + 2\beta) - \frac{5}{4} \sin(x + \beta) \right] \frac{(it^{\beta})^3}{\Gamma(3\alpha + 1)} \]

\[ p^n : u_n(x, t) = C_{\alpha}(x) \frac{(it^{\beta})^n}{\Gamma(n\alpha + 1)}, \quad (42) \]

Where

\[ C_{\alpha}(x) = C_{\alpha,0}(x) \sin(x) + C_{\alpha,1}(x) \sin(x + \alpha \beta) + C_{\alpha,2}(x) \sin(x + 2\alpha \beta) + \cdots + C_{\alpha,n}(x) \sin(x + n\alpha \beta) \]

And where

\[ C_{\alpha,0} = (-1)^n, \quad n \geq 0, \]

\[ C_{\alpha,1} = \frac{1}{2} C_{\alpha,0} - C_{\alpha,-1}, \quad n > 1, \]

\[ C_{\alpha,2} = \frac{1}{2} C_{\alpha,1} - C_{\alpha,-2}, \quad n > 2, \]

\[ \vdots \]

\[ C_{\alpha,n+i} = \frac{1}{2} C_{\alpha,n-i} - C_{\alpha,n+i+1}, \quad i = 0, 1, 2, \ldots \]

\[ C_{\alpha,n} = \frac{1}{2} C_{\alpha,n-1}, \quad n \geq 1. \]

Thus the solution of Equation (37) is given by:

\[ u(x, t) = \lim_{\beta \to \alpha} \sum_{n=0}^{\infty} p^n u_n(x, t) \]

\[ = e^{\alpha} \sum_{n=0}^{\infty} C_{\alpha}(x) \frac{(it^{\beta})^n}{\Gamma(n\alpha + 1)} \]

\[ = \sin \left[ \frac{x^\beta}{\Gamma(\beta + 1)} \right] \exp \left[ -\frac{3it^{\beta}}{2\Gamma(\alpha + 1)} \right] \]

If we put \( \alpha \to 1 \) in Equation (43) or solve Equation (37) and (38) with \( \alpha = 1 \), we obtain the exact solution

\[ u(x, t) = e^t \sum_{n=0}^{\infty} C_{\alpha}(x) \frac{(it)^n}{\Gamma(n\alpha + 1)} \]

\[ = \sin(x) e^{(-3/2)t}. \]

Which is in full agreement with the result of Rida et al. (2008) and Hong and Lu (2014).

Comparisons between the real part of some numerical results and the exact solution (43) are summarized in Tables 1 and 2, and the simulations for \( u_x \), \( u_{abs} \), and \( u \) are plotted in Figures 1 and 2, which shows that VHPM produced a rapidly convergent series.

Comparisons between the imaginary part and the exact solution Equation (43) are plotted in Figures 3 and 4, and the simulations for \( u_x \), \( u_{abs} \), and \( u \), which shows that VHPM produced a rapidly convergent series.
Table 2. Comparison between the real part of $u_4$ and $u$ when $\alpha = 0.7$, $\beta = 0.9$.

| $x$ | $t$ | Approximate $u_{4\text{appr}}$ | Exact solution | Absolute error |
|-----|-----|-------------------------------|----------------|----------------|
| 0.2 | 0.1 | 0.1989280524                 | 0.2288404399   | 0.0299123875   |
| 0.2 | 0.2 | 0.1978265247                 | 0.2080382267   | 0.010211702    |
| 1   | 0.3 | 0.6273816436                 | 0.6355277868   | 0.0081461432   |
| 2   | 0.4 | 0.5092408030                 | 0.6018550035   | 0.0926142005   |

Figure 1. Comparison between the real part of $u_4$ and the exact solution $u$.

Figure 2. Plots of the absolute error $u_{\text{abs}}$ when $\alpha = 0.7$ and $\beta = 0.9$. 
CONCLUSIONS

In this paper, we have introduced a combination of the variational iteration method and homotopy perturbation method for time-space fractional equations. This combination builds a strong method called the VHPM. We used the variational homotopy perturbation method for solving the Generalized Time-space Fractional Schrödinger Equation with variable coefficient. The VHPM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which converge rapidly to exact solutions. The obtained results are compared well with those obtained by VIM, ADM, HAM, MFVIM. Finally, we conclude that the VHPM may be considered as a nice refinement in existing numerical techniques.
Conflict of Interest

The authors have not declared any conflict of interest.

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