Abstract—The holographic transformation is generalized to problems in quantum information science. In this work, the partition function is represented by an inner product of two high-dimensional vectors both of which can be decomposed to tensor products of low-dimensional vectors. On the representation, the holographic transformation is clearly understood by using adjoin linear maps. Furthermore, for the general problem of computing the inner product, the belief propagation and the loop calculus formula are naturally defined and derived, respectively. Finally, an expression of weights in the loop calculus for problems in quantum information science is shown using concepts from quantum information geometry.

I. INTRODUCTION

The computation of the partition function of factor graphs is one of the central problems in statistical physics, information theory, machine learning and computer science [1]. Recently, a general technique transforming a representation of a partition function into a different representation is invented, and called the holographic transformation [2], [3]. Many equalities in broad area can be understood by the holographic transformation, e.g., the high temperature expansion, the MacWilliams identity, the loop calculus, etc. [4]. Especially, the loop calculus, which shows the equality relating the partition function and its approximation obtained by the message-passing algorithm called the belief propagation, is an interesting example [5]. In this paper, the partition function and the holographic transformation are described in an abstract way which allows to generalize the holographic transformation to problems in quantum information science. More specifically, the partition function is represented by an inner product on high-dimensional linear space where each of the two vectors can be decomposed to tensor products of low-dimensional vectors. Some quantity in quantum information science can also be represented by the inner product. On this formulation, the holographic transformation can be clearly understood by using linear maps and their adjoin maps. This understanding is quite clear and also allows to define the belief propagation and to prove the loop calculus for the general problem of computing the inner product. Finally, an expression of weights in the loop calculus for problems in quantum information science is shown using concepts from quantum information geometry. The similar result for the partition function of classical factor graphs on non-binary alphabets was shown in [6].

II. FACTOR GRAPHS AND BIPARTITE NORMAL FACTOR GRAPHS

A factor graph is a bipartite graph defining a probability measure. A factor graph consists of variables, factor nodes and edges between a variable node and a factor node. Let \( V \) be the set of variable nodes and \( F \) be the set of factor nodes. Let \( E \subseteq V \times F \) be the set of edges. For a variable node \( i \in V \), \( \partial_i \subseteq F \) denotes the set of neighborhoods of \( i \). In the same way \( \partial_a \subseteq V \) is defined for \( a \in F \). For each variable node \( i \in V \), there is an associated finite alphabet \( \mathcal{X}_i \) and an associated function \( f_i : \mathcal{X}_i \rightarrow \mathbb{R}_{\geq 0} \). For each factor node \( a \in F \), there is an associated function \( f_a : \prod_{i \in \partial a} \mathcal{X}_i \rightarrow \mathbb{R}_{\geq 0} \). Let \( \mathbf{x}_V \) be \( \prod_{i \in V} \mathcal{X}_i \) be variables corresponding to a subset \( V' \subseteq V \) of variable nodes. Then, the probability measure on \( X := \prod_{i \in V} \mathcal{X}_i \) associated with the factor graph \( G = (V, F, (f_i)_{i \in V}, (f_a)_{a \in F}) \) is defined by

\[
p(x) = \frac{1}{Z(G)} \prod_{a \in F} f_a(x_{\partial a}) \prod_{i \in V} f_i(x_i).
\]

\[
Z(G) := \sum_{\mathbf{x} \in X} \prod_{a \in F} f_a(x_{\partial a}) \prod_{i \in V} f_i(x_i).
\]

Here, the constant \( Z(G) \) for the normalization is called the partition function, which plays an important role in statistical physics, information theory, machine learning and computer science [1].

If all degrees of variable nodes are two, a factor graph is called a normal factor graph. A normal factor graph can be separated into two disjoint sets \( F_1 \) and \( F_2 \) such that \( \partial_a \cap \partial_{a'} = \emptyset \) for \( a \) and \( a' \) both in \( F_1 \) or both in \( F_2 \), the normal factor graph is said to be bipartite. Any factor graph can be transformed to an “equivalent” bipartite normal factor graph, by replacing edges by degree-two variable nodes and by replacing original variable nodes by the equality constraints. For normal factor graphs, a variable can be expressed by an edge since the degrees of variable nodes are two. Let \( (F_1, F_2, E \subseteq F_1 \times F_2) \) be a bipartite graph. For each \((v, w) \in E\), there is an associated finite alphabet \( \mathcal{X}_{v,w} \). For each \( v \in F_1 \) and \( w \in F_2 \), there are associated functions \( f_v : \prod_{w \in \partial v} \mathcal{X}_{v,w} \rightarrow \mathbb{R}_{\geq 0} \) and \( g_w : \prod_{v \in \partial w} \mathcal{X}_{v,w} \rightarrow \mathbb{R}_{\geq 0} \), respectively. A bipartite normal factor graph is denoted by \( (F_1, F_2, E \subseteq F_1 \times F_2, (f_v)_{v \in F_1}, (g_w)_{w \in F_2}) \), whose partition function is

\[
Z(G) := \sum_{\mathbf{x} \in X} \prod_{v \in F_1} f_v(x_{\partial v}) \prod_{w \in F_2} g_w(x_{\partial w}) \tag{1}
\]
where $\mathcal{X} := \prod_{(v,w) \in E} \mathcal{X}_{v,w}$, $x_{\partial v} := (x_{v,w})_{w \in \partial v}$ and $x_{\partial w} := (x_{v,w})_{v \in \partial w}$ for $v \in F_1$ and $w \in F_2$.

III. HOLOGRAPHIC TRANSFORMATION FOR BIPARTITE NORMAL FACTOR GRAPHS

In this section, we briefly introduce the holographic transformation for bipartite normal factor graphs. Let $\phi_{v,w}: \mathcal{X}_{v,w} \times \mathcal{X}_{v,w} \rightarrow \mathbb{R}$ and $\varphi_{v,w}: \mathcal{X}_{v,w} \times \mathcal{X}_{v,w} \rightarrow \mathbb{R}$ be mappings for each $(v,w) \in E$ satisfying

$$
\sum_{y \in \mathcal{X}_{v,w}} \varphi_{v,w}(x,y) \varphi_{w,v}(y,z) = \delta(x,z)
$$

where $\delta(x,z)$ takes 1 if $x = z$ and 0 otherwise. Then, it holds

$$
Z(G) = \sum_{x \in \mathcal{X}} \prod_{v \in F_1} \Phi_v(x_{\partial v}) \prod_{w \in F_2} \varphi_w(x_{\partial w})
$$

$$
= \sum_{x \in \mathcal{X}} \prod_{v \in F_1} \Phi_v(x_{\partial v}) \prod_{w \in F_2} \varphi_w(x_{\partial w}) \prod_{(v,w) \in \mathcal{E}} \delta(x_{v,w}, z_{v,w})
$$

$$
= \sum_{x \in \mathcal{X}} \prod_{v \in F_1} \Phi_v(x_{\partial v}) \prod_{w \in F_2} \varphi_w(x_{\partial w})
$$

$$
\cdot \prod_{(v,w) \in \mathcal{E}} \left( \sum_{y \in \mathcal{X}_{v,w}} \varphi_{v,w}(x,y) \varphi_{w,v}(y,z) \right)
$$

$$
= \sum_{y \in \mathcal{X}} \prod_{v \in F_1} \Phi_v(x_{\partial v}) \prod_{w \in \partial v} \varphi_{v,w}(x_{v,w}, y_{v,w})
$$

$$
\cdot \prod_{w \in F_2} \sum_{z \in \mathcal{X}_{w}} \varphi_{w,v}(y_{v,w}, z_{v,w})
$$

By letting

$$
\hat{f}_v(y_{\partial v}) := \sum_{x_{\partial v}} \Phi_v(x_{\partial v}) \prod_{w \in \partial v} \varphi_{v,w}(x_{v,w}, y_{v,w})
$$

$$
\hat{g}_w(y_{\partial w}) := \sum_{x_{\partial w}} \Phi_w(x_{\partial w}) \prod_{v \in \partial w} \varphi_{w,v}(y_{v,w}, z_{v,w})
$$

one obtains

$$
Z(G) = \sum_{y \in \mathcal{X}} \prod_{v \in F_1} \hat{f}_v(y_{\partial v}) \prod_{w \in F_2} \hat{g}_w(y_{\partial w}).
$$

This equality is called the Holant theorem [2], [3], which explains many known equalities [4].

IV. BIPARTITE MODEL: INNER PRODUCT OF VECTORS DECOMPOSED TO TENSOR PRODUCT

A. Motivation

The partition function (1) of a bipartite normal factor graph can be regarded as an inner product of vectors of dimension $|\mathcal{X}|$ in the following way. Let $\mathcal{V}_{v,w}$ be a linear space on $\mathbb{R}$ of dimension $|\mathcal{X}_{v,w}|$ for $(v,w) \in E$. Let $V_{\partial v} := \otimes_{w \in \partial v} \mathcal{V}_{v,w}$ for $v \in F_1$, $V_{\partial w} := \otimes_{v \in \partial w} \mathcal{V}_{v,w}$ for $w \in F_2$ and $V := \otimes_{(v,w) \in E} \mathcal{V}_{v,w}$ where $\otimes$ denotes the tensor product. Assume that each element of vector of $\mathcal{V}_{v,w}$ is indexed by $x \in \mathcal{X}_{v,w}$ for $(v,w) \in E$. Let $f_v$ and $g_w$ be vectors in $V_{\partial v}$ and $V_{\partial w}$, respectively. A $x_{\partial v} \in \prod_{w \in \partial v} \mathcal{X}_{v,w}$ element of $f_v$ is defined by $f_v(x_{\partial v})$. The vector $g_w$ is also defined in the same way.

Then, the partition function (1) of bipartite normal factor graph is equal to

$$
\left\langle \bigotimes_{v \in F_1} f_v, \bigotimes_{w \in F_2} g_w \right\rangle
$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors. In this paper, the holographic transformation is dealt with on this representation of the partition function. Since the holographic transformation can be defined for the partition function on any field $\mathbb{F}$, and since an inner product can be defined only for linear spaces on $\mathbb{R}$ or $\mathbb{C}$, we have to consider linear spaces on a general field $\mathbb{F}$ and bilinear forms instead of inner products.

B. Bilinear form, adjoint map and tensor product

Let $\mathbb{F}$ be a field. Let $\mathcal{V}$ be a $q$-dimensional linear space on $\mathbb{F}$. Let $\langle \cdot, \cdot \rangle_\mathcal{V}: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{F}$ be a bilinear form. Let $(e_x)_{x=0,...,q-1}$ be an arbitrarily chosen basis for $\mathcal{V}$. A bilinear form is represented by the $q \times q$ coefficient matrix $K_{\mathcal{V}}$ whose $(x,x')$ element is $\langle e_x, e_{x'} \rangle_\mathcal{V}$ for $x, x' \in \{0, ..., q-1\}$. Then, the bilinear form is expressed by $(f, g)_\mathcal{V} = f^T K_{\mathcal{V}} g$ where $f, g \in \mathcal{V}$ are represented by vectors with respect to the basis $(e_x)_x$. In this paper, we always assume that the bilinear form is non-degenerate, i.e., the coefficient matrix $K_{\mathcal{V}}$ is invertible.

Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces with non-degenerate bilinear forms $\langle \cdot, \cdot \rangle_\mathcal{V}$ and $\langle \cdot, \cdot \rangle_\mathcal{W}$, with coefficient matrices $K_{\mathcal{V}}$ and $K_{\mathcal{W}}$, respectively. It holds for any $f \in \mathcal{V}$, $g \in \mathcal{W}$ and a linear map $A: \mathcal{V} \rightarrow \mathcal{W}$ that $\langle Af, g \rangle_\mathcal{W} = \langle f, A^* g \rangle_\mathcal{V}$ where $A^* = K_{\mathcal{V}}^{-1} A^T K_{\mathcal{W}}$ is called the right adjoint of $A$.

Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces on $\mathbb{F}$ with bilinear forms $\langle \cdot, \cdot \rangle_\mathcal{V}$ and $\langle \cdot, \cdot \rangle_\mathcal{W}$, respectively. A bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{V} \otimes \mathcal{W}}$ for the tensor product space $\mathcal{V} \otimes \mathcal{W}$ is defined by

$$
\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{\mathcal{V} \otimes \mathcal{W}} = \langle f_1, f_2 \rangle_\mathcal{V} \langle g_1, g_2 \rangle_\mathcal{W}
$$

for any $f_1, f_2 \in \mathcal{V}$ and $g_1, g_2 \in \mathcal{W}$. It easy to check that $(A_1 \otimes A_2)^* = A_2^* \otimes A_1^*$ where $A_1$ and $A_2$ are arbitrary linear maps on $\mathcal{V}$ and $\mathcal{W}$, respectively.

C. Bipartite model and holographic transformation

Let $(V, W, E \subseteq V \times W)$ be a bipartite graph. Let $\mathbb{F}$ be a field. For each edge $(v,w) \in E$, there is an associated linear space $\mathcal{V}_{v,w}$ of dimension $q_{v,w}$ with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{V}_{v,w}}$. Let $V_{\partial v} := \otimes_{w \in \partial v} \mathcal{V}_{v,w}$, $V_{\partial w} := \otimes_{v \in \partial w} \mathcal{V}_{v,w}$ and $V := \otimes_{(v,w) \in E} \mathcal{V}_{v,w}$. For each $v \in V$, there is a vector $f_v \in V_{\partial v}$. Similarly, for each $w \in W$, there is a vector $g_w \in V_{\partial w}$. The “partition function” of the bipartite model $(V, W, E, (f_v)_{v \in V}, (g_w)_{w \in W})$ is defined by

$$
\left\langle \bigotimes_{v \in V} f_v, \bigotimes_{w \in W} g_w \right\rangle_{\mathcal{V}}.
$$

(2)

The partition function of a bipartite normal factor graph (1) can be expressed in this form. Conversely, (1) can be represented by the partition function of a bipartite normal factor graph on the field $\mathbb{F}$ without any condition on the weights (like the non-negativeness) as follows. Let $(e_{x,w})_{x=0,...,q_{v,w}-1}$ be an
orthonormal basis of $V_{v,w}$ for $(v, w) \in E$. Then, (2) can be expanded into the form

$$\sum_{(v, w) \in \{(0, \ldots, q, v, w-1)\} \in \{(v, w) \in E\}} \langle f, e_x \rangle_{v,w} \langle e_x, g_w \rangle_{v,w} = \sum_{v \in \mathcal{V}} \prod_{w \in \mathcal{W}} \langle f, e_{x,v,w} \rangle_{v,w} \prod_{w \in \mathcal{W}} \langle e_{x,v,w}, g_w \rangle_{v,w}. \quad (3)$$

For the bipartite model, the Holant theorem can be proved as follows.

**Theorem 1** (Holant theorem for the bipartite model). Let $\Phi_{v,w}$ be an invertible linear map on $V_{v,w}$ and $\Phi^*_{v,w}$ be the inverse map of $\Phi_{v,w}$ for $(v, w) \in E$. Then, it holds

$$\langle \bigotimes_{v \in \mathcal{V}} f_v, \bigotimes_{w \in \mathcal{W}} g_w \rangle_{v,w} = \langle \bigotimes_{v \in \mathcal{V}} \hat{f}_v, \bigotimes_{w \in \mathcal{W}} \hat{g}_w \rangle_{v,w} \quad (\text{where})$$

$$\hat{f}_v = \left( \bigotimes_{w \in \partial v} \Phi_{v,w} \right) (f_v), \quad \hat{g}_w = \left( \bigotimes_{v \in \partial w} \Phi^*_{v,w} \right) (g_w).$$

**Proof:**

$$\langle \bigotimes_{v \in \mathcal{V}} f_v, \bigotimes_{w \in \mathcal{W}} g_w \rangle_{v,w} = \langle \left( \bigotimes_{(v, w) \in E} \Phi_{v,w} \right) \left( \bigotimes_{v \in \mathcal{V}} f_v \right), \bigotimes_{w \in \mathcal{W}} g_w \rangle_{v,w}$$

$$= \langle \left( \bigotimes_{(v, w) \in E} \Phi^*_{w,v} \right) \left( \bigotimes_{v \in \mathcal{V}} f_v \right), \bigotimes_{w \in \mathcal{W}} g_w \rangle_{v,w}$$

$$= \langle \left( \bigotimes_{(v, w) \in E} \Phi^*_{v,w} \right) \left( \bigotimes_{(v, w) \in E} \Phi_{w,v} \right) \left( \bigotimes_{v \in \mathcal{V}} f_v \right), \bigotimes_{w \in \mathcal{W}} g_w \rangle_{v,w}. \quad \square$$

**D. Belief propagation for the bipartite model**

In this subsection, we assume $\mathbb{F} = \mathbb{R}$ for defining the belief propagation on the bipartite model. The bilinear forms are assumed to be inner products. Let $C_{v,w}$ be a closed convex cone in the inner product space $V_{v,w}$ for $(v, w) \in E$. For a closed convex cone $C$ in an inner product space $\mathcal{V}$, the dual cone of $C$ is denoted by $C^*$, i.e., $C^* := \{ f \in \mathcal{V} \mid \langle f, g \rangle_{v,w} \geq 0 \ \forall g \in C \}$. Let $\bigotimes_{v \in \partial w} C_{v,w}$ be a closed convex cone generated by $\{ \bigotimes_{v \in \partial w} F_v \mid F_v \in C_{v,w} \}$. Let $C_v := (\bigotimes_{v \in \partial v} C^*_{v,w})$ and $C_w := (\bigotimes_{v \in \partial w} C^*_{v,w})$. Assume that $f_v \in C_v$ and $g_w \in C_w$. Let $f$ and $g$ be vectors in inner product spaces $\mathcal{V} \otimes \mathcal{W}$ and $\mathcal{W} \otimes \mathcal{V}$, respectively, for some inner product space $\mathcal{W}$. Then, the partial inner product $\langle f, g \rangle_{v,w} \in \mathcal{W}$ is defined as the unique vector satisfying $\langle (f, g)_{v,w} \rangle_{v,w} = \langle f \otimes g \rangle_{v,w}$ for any $h \in \mathcal{W}$. Let $\mathcal{V}_{\partial v \setminus \{v\}} := \bigotimes_{w \in \partial v \setminus \{v\}} V_{v,w}$ and $\mathcal{V}_{\partial w \setminus \{w\}} := \bigotimes_{v \in \partial w \setminus \{w\}} V_{v,w}$. For any $(v, w) \in E$ and $x \in \{1, \ldots, q, v, w - 1\}$. These conditions are equivalent to

$$\langle f_v, \Phi^*_{v,w}(e_v^{(x,v,w)} \otimes \dots \otimes \Phi^*_{v,w}(e_w^{(x,w)})), g_w \rangle_{v,w} = 0 \quad (\text{for any } (v, w) \in E \text{ and } x \in \{1, \ldots, q, v, w - 1\}).$$

For any $(v, w) \in E$ and $x \in \{1, \ldots, q, v, w - 1\}$. Furthermore, these conditions are equivalent to

$$\langle f_v, \Phi^*_{v,w}(e_v^{(x,v,w)} \otimes \dots \otimes \Phi^*_{v,w}(e_w^{(x,w)})), \Phi^*_{v,w}(e_v^{(x,v,w)}), g_w \rangle_{v,w} = 0 \quad (\text{for any } (v, w) \in E \text{ and } x \in \{1, \ldots, q, v, w - 1\}).$$

**Definition 2** (Belief propagation). Let $r_v,w$ and $r^*_v,w$ be vectors in interiors of $C_{v,w}$ and $C^*_{v,w}$, respectively. Let $(m_{v,w}^{(t)})_{(v,w) \in E}$ be arbitrarily chosen initial messages. Then, in the belief propagation, the messages are updated according to the following rules

$$m_{v,w}^{(t)} = \frac{1}{Z_{v,w}^{(t)}} \left( \bigotimes_{w' \in \partial v \setminus \{w\}} f_v, \bigotimes_{w' \in \partial v \setminus \{w\}} m_{w'\rightarrow v}^{(t)} \right)_{V_{\partial v \setminus \{w\}}},$$

$$m_{v,w}^{(t)} = \frac{1}{Z_{v,w}^{(t)}} \left( \bigotimes_{w' \in \partial v \setminus \{w\}} g_w, \bigotimes_{w' \in \partial v \setminus \{w\}} m_{w'\rightarrow v}^{(t)} \right)_{V_{\partial v \setminus \{w\}}},$$

for $t = 1, 2, \ldots$ for all $(v, w) \in E$ where the constants $Z_{v,w}^{(t)}$ and $Z_{w,v}^{(t)}$ are chosen such that $\langle m_{v,w}^{(t)}, r_v^w \rangle_{V_{v,w}} = 1$ and $\langle r_{v,w}, m_{w\rightarrow v}^{(t)} \rangle_{V_{v,w}} = 1$, respectively.

Note that it always holds $m_{v\rightarrow w} \in C_{v,w}$ and $m_{w\rightarrow v} \in C^*_{v,w}$ for $(v, w) \in E$.

**Remark 3**. For the standard belief propagation for bipartite normal factor graphs, the closed convex cone $C_{v,w}$ corresponds to the set of non-negative vectors, which is self-dual, i.e., $C^*_{v,w} = C_{v,w}$. The vectors $r_{v,w}$ and $r^*_{v,w}$ correspond to the all-one vector.

Similarly to the standard belief propagation for factor graphs, the general belief propagation defined above gives the exact computation of the partition function of bipartite models on a cycle-free bipartite graph. It will be shown as a corollary of the loop calculus formula in the next section.

**E. Loop calculus for the bipartite model**

In this section, we derive the loop calculus formula for the bipartite model. The derivation in this section is essentially equivalent to that in [3]. Let $(e^{(x,v,w)}_{x=0,\ldots,q,v,w-1})$ be an arbitrarily chosen orthonormal basis for $V_{v,w}$ for $(v, w) \in E$. For the loop calculus, the following additional conditions are required

$$\langle f_v, e^{(v,v,w)}_{x} \otimes \bigotimes_{w' \in \partial v \setminus \{w\}} e^{(v,v,w)}_{x}, g_w \rangle_{v,w} = 0 \quad (4)$$

for any $(v, w) \in E$ and $x \in \{1, \ldots, q, v, w - 1\}$. These conditions are equivalent to

$$\langle f_v, \Phi^*_{v,w}(e^{(v,w)}_{v,w} \otimes \Phi^*_{v,w}(e^{(v,w)}_{0,v,w})), g_w \rangle_{v,w} = 0 \quad (\text{for any } (v, w) \in E \text{ and } x \in \{1, \ldots, q, v, w - 1\}).$$

Furthermore, these conditions are equivalent to

$$\langle f_v, \bigotimes_{w' \in \partial v \setminus \{w\}} \Phi^*_{v,w}(e^{(v,w)}_{v,w}), \Phi^*_{v,w}(e^{(v,w)}_{v,w}) \rangle_{v,w} = 0 \quad (\text{for any } (v, w) \in E \text{ and } x \in \{1, \ldots, q, v, w - 1\}).$$
for any \((v, w) \in E\) and \(x \in \{1, \ldots, q_{v,w} - 1\}\). Since \(\hat{\Phi}_{v,w}(e_{x}^{v,w}) = e_{x}^{v,w}\), it holds
\[
\delta(x, x') = \left\langle \Phi_{v,w}(e_{x}^{v,w}), e_{x'}^{v,w} \right\rangle_{V_{v,w}} = \left\langle \Phi_{v,w}(e_{x}^{v,w}), \hat{\Phi}^{*}_{v,w}(e_{x'}^{v,w}) \right\rangle_{V_{v,w}}
\]
for any \(x, x' \in \{0, 1, \ldots, q_{v,w} - 1\}\) and \((v, w) \in E\). Hence,
\[
\left\langle f_{v}, \bigotimes_{w' \in \partial w \setminus \{v\}} \hat{\Phi}^{*}_{v,w'}(e_{0}^{v,w'}) \right\rangle_{V_{v,w}} = \alpha_{v,w} \left\langle \hat{\Phi}_{v,w}(e_{0}^{v,w}) \right\rangle_{V_{v,w}}
\]
and
\[
\left\langle \bigotimes_{w' \in \partial w \setminus \{v\}} \hat{\Phi}_{v,w'}(e_{0}^{v,w'}) \phi_{w} \right\rangle_{V_{v,w}} = \hat{\alpha}_{v,w} \left\langle \hat{\Phi}_{v,w}(e_{0}^{v,w}) \phi_{w} \right\rangle_{V_{v,w}}
\]
for any \((v, w) \in E\) where \(\alpha_{v,w} := \left\langle f_{v}, \bigotimes_{w' \in \partial w \setminus \{v\}} e_{0}^{v,w'} \phi_{w} \right\rangle_{V_{v,w}}\) and \(\hat{\alpha}_{v,w} := \left\langle \bigotimes_{w' \in \partial w \setminus \{v\}} e_{0}^{v,w'}, \phi_{w} \right\rangle_{V_{v,w}}\). Furthermore, we assume that \(\hat{\Phi}_{v,w}(e_{0}^{v,w}) \in C_{v,w}\) and \(\hat{\Phi}^{*}_{v,w}(e_{0}^{v,w}) \in C^{*}_{v,w}\) for \((v, w) \in E\). Then, without loss of generality, it holds \(\hat{\Phi}_{v,w}(e_{0}^{v,w}) = c_{v,w} m_{v,w} = \hat{c}_{v,w} m_{v,w} \) for some strictly positive constants \(c_{v,w}, \hat{c}_{v,w} \in \mathbb{R}_{+}\) and for some \(m_{v,w}, \hat{m}_{v,w} \in C_{v,w}\) satisfying \(\langle m_{v,w}, \hat{m}_{v,w} \rangle_{V_{v,w}} = 1\) and \(\langle m_{v,w}, m_{v,w} \rangle_{V_{v,w}} = 1\), respectively. Then, it is easy to check that \(m_{v,w} \in E\) and \(m_{v,w} \in E\) have to satisfy the fixed point equation of the belief propagation and \(c_{v,w} \hat{c}_{v,w} = \langle m_{v,w}, m_{v,w} \rangle_{V_{v,w}}\).

In the expansion (3) for the new expression of the inner product with respect to the orthonormal basis, the weight of the all-zero assignment is
\[
\prod_{v \in V} \left\langle f_{v}, m_{v,w} \right\rangle_{V_{v,w}} \prod_{(v, w) \in E} \left\langle m_{v,w}, g_{w} \right\rangle_{V_{v,w}}
\]
This quantity can be regarded as the “Bethe approximation” of the inner product (2) on the chosen fixed point of the belief propagation [6]. In the summation (3), only assignments \(x\) whose non-zero part corresponds to some generalized loop have non-zero weight due to the conditions (4) and (6). Hence, if the bipartite graph is cycle-free, the Bethe approximation is exactly equal to the inner product (2). Choices of \((\Phi_{v,w}(e_{x}^{v,w}))_{x=1,\ldots,q_{v,w}-1}\) and \((\Phi_{v,w}^{*}(e_{x}^{v,w}))_{x=1,\ldots,q_{v,w}-1}\) are arbitrary if they satisfy (5) for any \((v, w) \in E\). For bipartite normal factor graphs, expressions of the remaining degree of freedom using ideas from information geometry was shown in [6]. In Section VI expressions of them for problems in quantum information science are obtained using ideas from quantum information geometry.
\( \mathcal{L}_h(\mathcal{H}_{v_1,z_2} \otimes \mathcal{H}_{v_2,w_2}) \) is a positive semidefinite operator corresponding to outcome of some POVM. Here, \( \mathcal{H}_{v_1,w_1} \) and \( \mathcal{H}_{v_1,z_2} \otimes \mathcal{H}_{v_2,w_2} \) correspond to Bob’s Hilbert space and Alice’s Hilbert space, respectively. More generally, we can deal with larger systems including cycles.

VI. LOOP CALCULUS FOR THE BIPARTITE QUANTUM MODEL

In this section, the expressions of \( (\hat{\Phi}_{v,w}(e_{v,w}^m))_{x=1,\ldots,q_{v,w}}-1 \) and \( (\hat{\Phi}^*_{v,w}(e_{v,w}^m))_{x=1,\ldots,q_{v,w}}-1 \) for the loop calculus on the bipartite quantum model are suggested. For the expression, concepts from quantum information geometry are used similarly to the classical case \([6]\).

**Definition 4** (Quantum exponential family). Let \( d \) be a positive integer. Let \( \mathcal{H} \) be an inner product space on \( C \). Let \( T_1,\ldots,T_d \) be Hermitian operators on \( \mathcal{H} \). Assume that \( T_1,\ldots,T_d,I \) are linearly independent where \( I \) denotes the identity operator on \( \mathcal{H} \). Then, the quantum exponential family is a parametrized family of density operators defined by

\[ \rho_\theta = \exp \left\{ \sum_{k=1}^{d} \theta_k T_k - \psi(\theta) I \right\} \]

where

\[ \psi(\theta) := \log \text{Tr} \left( \exp \left\{ \sum_{k=1}^{d} \theta_k T_k \right\} \right). \]

The parameter \( (\theta_1,\ldots,\theta_d) \in \mathbb{R}^d \) is called a natural parameter.

**Example 5.** Let \( \mathcal{H} \) be an inner product space on \( C \) of dimension \( q \). The family \( \{ \rho \in \mathcal{L}_h(\mathcal{H}) | \rho > 0, \text{Tr} \rho = 1 \} \) can be regarded as \( q^2-1 \) dimensional exponential family.

It holds \( \frac{\partial \psi(\theta)}{\partial \theta} = \text{Tr}(\rho \partial_k T_k) =: \eta_k \). Here, \( (\eta_1,\ldots,\eta_d) \) gives another coordinate system for the parametrized family. The parameter \( (\eta_1,\ldots,\eta_d) \) is called an expectation parameter. In quantum information geometry, the Bogoliubov-Kubo-Mori metric \([9]\) on quantum information manifolds satisfies

\[ \left< \frac{\partial}{\partial \theta_k}, \frac{\partial}{\partial \eta_l} \right>_{\text{BKM}(\rho)} := \int_0^1 \text{Tr} \left( \rho^\lambda \frac{\partial \log \rho}{\partial \theta_k} \rho^{1-\lambda} \frac{\partial \log \rho}{\partial \eta_l} \right) d\lambda = \text{Tr} \left( \frac{\partial \rho}{\partial \eta_l} \log \rho \frac{\partial \rho}{\partial \theta_k} \right) = \delta(k,l). \]

Similarly to the classical case \([6]\), this property is useful for expressing the remaining degree of freedom in the loop calculus for the quantum model. Let \( A \ast B := A^{1/2}BA^{1/2} \) for a positive-semidefinite matrix \( A \) and an Hermitian matrix \( B \). Let \( \hat{b}_{v,w} := (m_{v,w} \ast m_{w,v})/\text{Tr}(m_{v,w} m_{w,v}) \) and \( \hat{b}_{v,w} = (m_{w,v} \ast m_{v,w})/\text{Tr}(m_{w,v} m_{v,w}) \) for \( (v,w) \in E \). Assume that \( \hat{b}_{v,w} \) and \( \hat{b}_{v,w} \) are positive definite and are regarded as members of the quantum exponential family in Example 5.

Fix \( c_{v,w} = 1 \) and

\[ \Phi_{v,w}(e_{v,w}^m) = m_{v,w}^{-1} \ast \left( \frac{\partial \hat{b}_{v,w}}{\partial \theta} \right) \]

for \( x = 1,\ldots,q_{v,w} \) and \( (v,w) \in E \). The above choices satisfy the condition \([5]\). This fact seems to be similar to the equation \([7]\) although we deal with two density matrices \( b_{v,w} \) and \( \hat{b}_{v,w} \) for each \( (v,w) \in E \) in contrast to the classical case \([6]\).

Conversely, the above choices cover all possible choices of \( \Phi_{v,w}(e_{v,w}^m)_{x=1,\ldots,q_{v,w}}-1 \) and \( \Phi^*_{v,w}(e_{v,w}^m)_{x=1,\ldots,q_{v,w}}-1 \) satisfying \([5]\) since both of them have the degrees of freedom represented by a \((q_{v,w}-1) \times (q_{v,w}-1)\) invertible matrix for \( (v,w) \in E \). In the above expression,

\[ m_{v,w}^{-1} \ast \left( \frac{\partial \hat{b}_{v,w}}{\partial \theta} \right) \]

may be regarded as a variant of logarithmic derivatives of \( \hat{b}_{v,w} \). Note that \( \frac{\partial \psi(\theta)}{\partial \theta} \) is independent of \( \eta \). Especially, if the sufficient statistics \( (T_x)_{x=1,\ldots,q_{v,w} \rightarrow 1}; (1/\sqrt{q}) \) is an orthonormal basis for \( \mathcal{L}_h(\mathcal{H}) \), it holds \( \frac{\partial \psi(\theta)}{\partial \theta} = T_x \). On the other hand, it always holds \( \frac{\partial \log \rho}{\partial \theta} = T_x - \eta I \). By summarizing the above, one obtains

\[ \Phi^*_{v,w}(e_{v,w}^m) = \frac{1}{\text{Tr}(m_{v,w} m_{w,v})} \]