Stability and moment bounds under utility-maximising service allocations, with applications to some infinite networks

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Abstract

We study networks of interacting queues governed by utility-maximising service-rate allocations in both discrete and continuous time. For finite networks we establish stability and some steady-state moment bounds under natural conditions and rather weak assumptions on utility functions. These results are obtained using direct applications of Lyapunov-Foster-type criteria, and apply to a wide class of systems, including those for which fluid limit-based approaches are not applicable.

We then establish stability and some steady-state moment bounds for two classes of infinite networks, with single-hop and multi-hop message routes. These results are proved by considering the infinite systems as limits of their truncated finite versions. The uniform moment bounds for the finite networks play the key role in these limit transitions.

1 Introduction

In this paper we consider networks of interacting queues. These models are primarily motivated by wireless systems, where the interference between simultaneous transmissions by different nodes imposes certain constraints. For example, “neighbouring” nodes may not be allowed to transmit simultaneously and/or a node’s effective transmission rate depends on the transmission powers of the node and its neighbours. However, the basic model studied in this paper takes a more abstract point of view, namely, it is an arbitrary network of queues such that individual instantaneous service rates may depend on the state of the entire system. The network state is represented as a set $\bar{X} = (X_i, i \in \mathcal{N})$ of the queue lengths $X_i$ (of jobs, or messages, or customers) at the network nodes $i \in \mathcal{N}$. Each node receives exogenous arrivals of jobs (messages). We consider both discrete-time and continuous-time models, and both finite and infinite networks. Also, in addition to single-hop networks, where each job leaves the system after its service is completed, we consider one special class of multi-hop networks, where a customer may be routed to another node after each service completion.

In discrete-time models time is divided into slots of the same (unit) size, and each job (or message) takes exactly one slot to complete service at a node. A service allocation algorithm (or rule) is any mapping (deterministic or random) of a network state $\bar{X}$ into the set of nodes that serve jobs (transmit messages) in a slot. (See, e.g., [14] for a recent model of an algorithm in discrete time employing a random procedure.) The instantaneous service rate $\mu_i$ of node $i$ in a slot is the probability that it will serve a job. Thus, the deterministic mapping $\psi(\bar{X}) = (\psi_i(\bar{X}), i \in \mathcal{N})$ of a network state $\bar{X}$ into a set of instantaneous service rates $\bar{\mu} = (\mu_i, i \in \mathcal{N})$ is determined by the service algorithm; the mapping $\bar{\psi}(\bar{X})$ is referred to as service rate allocation algorithm (or rule).

In continuous-time models, the instantaneous service rate $\mu_i$ of a node represents the intensity of the Poisson process modelling departures (service completions) of the node. In this case,
the service rate allocation algorithm $\psi(\bar{X})$, mapping a network state $\bar{X}$ into a set of instantaneous service rates $\bar{\mu}$, is all that is needed to specify the service allocation algorithm. (See, e.g., [10] for a recent model of an algorithm in continuous time.)

In this paper we study service allocation algorithms (in both discrete and continuous time), such that the corresponding service rate allocation $\psi(\bar{X})$ maximises some utility function within some set $C$. In some cases, the set $C$ arises naturally as the set of all feasible instantaneous rates $\bar{\mu}$ given the model structure, but not necessarily. Our main goal is to obtain network stability conditions, in terms of set $C$. For example, our main stability results for single-hop networks show that the network is stable when the exogenous arrival rates $\bar{\lambda} = (\lambda_i, i \in \mathbb{N})$ are (“strictly”) within set $C$. In addition to stability, we are able to obtain some steady-state moments bounds. In fact, these moment bounds are key to establishing stability of infinite networks, because they allow a limit transition from finite to infinite networks.

Service rate allocations $\psi(\bar{X})$, under many natural service allocation algorithms, are such that $\psi_i(\bar{X})$ is decreasing in each $X_j$ for $j \neq i$, as for these algorithms a higher load in queue $j$ usually leads to all other queues receiving less service. This property is in fact satisfied by the rates defined by algorithms introduced in [10] and [14] that we will study here as examples.

We would like to emphasise, however, that for our general results we are not going to make this assumption. Our motivation for this stems, again, from wireless networks where there are many competing factors at play and in many situations $\psi_i$ may not be decreasing in $X_j$ for some $j \neq i$ (see, e.g. the model considered in [13], where the authors consider an algorithm designed to ensure avoidance of conflicts which gives advantage to a transmitter if its non-immediate neighbours are transmitting). This leads to a potentially wide range of possible assumptions on the dependence of service rates assigned to different queues on the state of the network.

We are interested in conditions for algorithms’ stability. In finite networks stability, informally speaking, means the ability of all queues to complete service of all jobs, without the number of outstanding jobs building up infinitely. More formally, this means that the Markov chain $\bar{X}(\cdot)$ is positive recurrent. This also implies the existence and uniqueness of a stationary distribution.

In infinite networks, by stability we will understand the existence of a proper invariant distribution. In the cases when the system process is monotone, this implies that the process distribution converges to a proper steady-state (namely, the lower invariant measure), starting from the “empty” initial state, as time goes to infinity.

An important concept, explored extensively in the literature, is that of maximum stability (or throughput optimality). To illustrate this concept, consider a finite network and let $C$ be the set of all feasible long-term rates that can be provided to the nodes, given model constraints. Such set $C$ is convex. Then, an algorithm is called maximally stable (or, throughput-optimal) if it guarantees stability as long as $\bar{\lambda} < \bar{\nu}$ for some $\bar{\nu} \in C$; in other words, essentially, as long as the stability is feasible at all. For a large class of networks, the celebrated MaxWeight algorithm ([17]) and $\alpha$-fair algorithm are known to be maximally stable. (See [6, 7, 8] for introduction of the fair-allocation concepts and [14] for stability proofs.) These algorithms, however, are centralised in that service-rate allocations are given by a solution to an optimisation problem that needs to be found by a certain central entity. There are also decentralised algorithms (where each node regulates its own behaviour according to its queue length) guaranteeing maximal stability (see [5, 11]), but they are known to suffer from large job delays. (This, in particular, prompted the introduction and analysis of algorithms, which are not maximally stable, and instead ensure stability for $\bar{\lambda}$ within a set $C$ which is “smaller” than the set of all feasible long-term rates, and $C$ being not necessarily convex. See, e.g., [16].)

Some maximally stable algorithms are designed in such a way that the average service rates maximise a certain utility function. A notable example is presented by $\alpha$-fair algorithms where
the rates $\psi_i$ are such that

$$
\bar{\psi} \in \arg \max_{\mu \in C} \sum_i X_i \left( \frac{\mu_i}{X_i} \right)^{1-\alpha}, \quad \text{when} \quad \alpha > 0, \, \alpha \neq 1,
$$
or

$$
\bar{\psi} \in \arg \max_{\mu \in C} \sum_i X_i \log(\mu_i/X_i), \quad \text{when} \quad \alpha = 1,
$$

where, recall, the set $C$ is convex. The known stability proofs are based on the fluid-limit approach \cite{9, 2, 15} and, in particular, implicitly use the fact that $\alpha$-fair service-rate allocations are 0-homogeneous (or, asymptotically 0-homogeneous), which allows a relatively simple characterisation of fluid-limit dynamics.

In this paper we consider general utility-optimising algorithms, which, in particular, do not necessarily assign 0-homogeneous rates to queues. We also do not require that the set $C$ is necessarily convex. Our goal is threefold. First, we show that these very general algorithms for finite networks ensure stability when $\bar{\lambda}$ is within $C$. Second, we also find some moment bounds for the stationary queue-length distributions. And finally, we demonstrate how our moment bounds may be used to extend the stability results and moment bounds to some infinite networks.

In the first part of our paper, we consider a class of general utility-optimising algorithms and prove that they are stable when $\bar{\lambda}$ is within $C$. Namely, we study average service-rate allocations $\psi_i$ such that

$$
\bar{\psi} \in \arg \max_{\mu \in C} \sum_i g_i(X_i)h(\mu_i),
$$

with some conditions on the functions $g$ and $h$. Our conditions do not imply that the service-rate allocations are 0-homogeneous, hence the existing stability results, based on fluid limits, do not apply. Moreover, we do not even require that the function $g$ is defined for non-integer values of the argument. Our stability proofs in both discrete- and continuous-time settings are based on the direct application of the Lyapunov-Foster techniques.

In discrete time, for some results we impose a strong additional assumption that the number of arrivals into each queue in a time slot is given by a Bernoulli random variable, while other results only assume a finite third moment of the per-slot number of arrivals. In continuous time however (which is the standard setting for $\alpha$-fair allocations) no additional assumptions are needed. And, again, we do not assume that the set $C$ is convex.

Once stability is established, one is interested in characteristics of the stationary regime. For both discrete- and continuous-time settings, we demonstrate how essentially the same techniques used to prove stability may be employed to establish explicit bounds on the moments of queue states in stationarity.

These bounds are interesting in their own right, especially as very few results are known on the stationary regimes of networks governed by utility-maximising algorithms. We note \cite{12} where an exponential bound has been established for the tail of the total stationary queue length of a system under an $\alpha$-fair algorithm in a Markovian setting, and \cite{3} where sufficient conditions for the existence of finite moments was established for general arrival streams. We note however that the results of both \cite{3} and \cite{12} imply finiteness of some moments of the stationary queue-length distributions but do not imply any bounds on them as the various constants are not explicit. In the second part of our paper, having explicit bounds is crucial for the analysis of some infinite networks.

In the second part of the paper, we apply the moment bounds established in this paper to obtain stability results for infinite networks in discrete and continuous time considered in recent papers \cite{14} and \cite{10}, respectively. The models considered in the two papers are motivated by different wireless networks but share similar service-rate allocations. As our stability and moment analysis is based on service-rate allocations only, it allows us to handle both discrete
and continuous cases, and particular characteristics of the two models (which are very different) – beyond the service-rate allocation – do not play any role in the proofs.

The simplest example of the two networks (results for more general settings are presented in the paper; we focus on a simple example in the introduction only) is given by nodes located on an infinite line \( \mathbb{Z} \) and such that, given the state of the system \( \bar{X} \), the service-rate allocation is given by

\[
\psi_i = \frac{X_i}{X_{i-1} + X_i + X_{i+1}}.
\]

The so-called rate stability (guaranteeing the queue lengths do not grow linearly in time) is demonstrated in both discrete and continuous settings in [14] for arrival rates \( \bar{\lambda} \) within some natural set \( \mathcal{C} \). Authors of [10] considered a continuous-time model where arrival rates into all nodes are the same and equal to \( \lambda \), say. They consider the restrictions of the system dynamics on intervals \((-n, \ldots, n)\) viewed as a circle, with a growing \( n \). These systems are stable for any \( n \), provided \( \lambda < 1/3 \) and one can thus consider their stationary measures. Using the natural monotonicity of the corresponding process, and tightness of these measures, a stationary measure (in fact, the lower invariant measure) is constructed. To establish uniqueness of this stationary measure among those with finite second moments of the queue lengths, one needs a bound on the second moments of stationary measures of the restricted systems, independent of their size. This was not established in [10] and left as a conjecture (Conjecture 1.12).

Our analysis is based on showing that the rates of [10] and [14] are in fact utility-maximising (or 2-fair in the \( \alpha \)-fair terminology) in a certain natural set \( \mathcal{C} \) - a fact already mentioned in [14]. This allows us to use our results on stability and moment bounds for finite systems. In particular, our moment bounds immediately imply a uniform (not depending on the size of the network) bound for second moments. This, in turn, proves [10, Conjecture 1.12] in the case of identical arrival rates, with all its implications, including the uniqueness of the stationary measure constructed there, among stationary measures with finite second moments of the queue lengths.

Our analysis, however, allows to obtain existence of a stationary measure with a finite second moment in far more general settings where arrival rates do not need to be the same at all nodes, but may be periodic (or dominated by periodic).

As our analysis is based on utility maximisation and continuity properties of the processes (see Section 1.1 for the definition of continuity property), it is not specific to the rates considered in [10] and [14] and may be applied to other infinite networks.

Finally, we also consider a multi-hop network of [14]. In a multi-hop network jobs, after being served at one queue, may leave the network or join another queue to be served there. The analysis of multi-hop networks is notoriously difficult. As in [14], we restrict our attention to symmetric routing. We use similar techniques to the ones we applied in the single-hop setting to first obtain moment bounds for finite networks and then apply these bounds to establish stability of an infinite network. Stability in this case is weaker than that obtained in the single-hop case as the multi-hop network lacks monotonicity, which is at the core of the construction of the lower invariant measure in [10].

To summarise, our contributions are the following:

- We provide a proof of stability of utility-maximising algorithms in a general setting, covering cases in which fluid limit technique cannot be applied. In particular, we do not assume that service rate allocations are 0-homogeneous, and thus use more general utility functions compared to the classical \( \alpha \)-fair algorithms. This comes at the expense of additional assumptions on the arrival processes in discrete time. There are, however, no additional assumptions made in the case of a Markovian (driven by Poisson arrivals and departures) continuous-time system.
Using a similar approach, we provide steady-state moment bounds, provided stability conditions hold.

These moment bounds allow us, in particular, to establish stability and moment bounds of some infinite networks.

We use similar techniques to establish moment bounds for a certain finite multi-hop network and use these bounds to establish stability and moment bounds for its infinite version.

The paper is structured as follows. Section 2 is devoted to finite networks. The discrete-time setting is treated in Section 2.1 and continuous-time setting - in Section 2.2, with both sections following the same structure: first the model is described, then assumptions are stated, then stability is established and then moment bounds obtained. Section 3 is devoted to infinite single-hop networks and Section 4 - to an infinite multi-hop network.

1.1 Basic notation, conventions and definitions

We will use the following notation throughout: \( \mathbb{R} \) and \( \mathbb{R}^+ \) are the sets of real and real non-negative numbers, respectively; \( \mathbb{Z}^d \) is the \( d \)-dimensional lattice; \( \mathbb{Z}^+ \) is the set of non-negative integers; \( \bar{y} \) means (finite- or infinite-dimensional) vector \( (y_i) \); for a finite-dimensional vector \( \bar{y} \), \( \|y\| = \sum_i |y_i| \); for a set of functions \( (f_i) \) and a vector \( (y_i) \), \( \bar{f}(\bar{y}) \) denotes the vector \( (f_i(\bar{y})) \); vector inequalities are understood component-wise; we also use the convention that \( 0/0 = 0 \).

Abbreviation \( \text{w.p.1} \) means \( \text{with probability} \ 1 \). The convergence in distribution of random elements is denoted by \( \Rightarrow \). A discrete-time random process \( (Y(k), k = 0, 1, 2, \ldots) \) is often referred to as \( Y(\cdot) \), and similarly for a continuous-time process \( (Y(t), t \geq 0) \).

We will say that a sequence of random processes \( Y^{(m)}(\cdot), m = 1, 2, \ldots, \) and a random process \( Y(\cdot) \) satisfy a continuity property, if the following holds. For any (random) initial state \( Y(0) \), and any sequence of (random) initial states \( Y^{(m)}(0), m = 1, 2, \ldots, \) such that \( Y^{(m)}(0) \Rightarrow Y(0) \), all processes can be coupled (constructed on a common probability space) so that \( Y^{(m)}(k) \to Y(k) \) w.p.1, for any \( k = 0, 1, \ldots \) (or \( Y^{(m)}(t) \to Y(t) \) w.p.1, for any \( t \geq 0 \), for continuous time). This continuity property could be called a generalised Feller-continuity, because in the special case when all \( Y^{(m)}(\cdot) \) are copies of the same process \( Y(\cdot) \), differing only by the initial state, the property defined above is Feller-continuity of \( Y(\cdot) \); we call it continuity for short.

2 Finite single-hop networks: stability analysis and moment bounds

In this Section we consider finite single-hop networks, where a job, after being served at any queue (node) leaves the system. The section is split into discrete- and continuous-time parts. Since the number of nodes is finite, the process describing system evolution is a countable (irreducible) Markov chain (in discrete or continuous time). The finite-network process stability is defined as positive recurrence of the Markov chain, which (due to irreducibility) is equivalent to the existence of unique stationary distribution.

In both the discrete- and continuous-time settings we follow the same structure: first introduce the model and make general assumptions, then prove stability results and then obtain moment bounds on stationary distributions.

2.1 Discrete time

2.1.1 Model and assumptions

Assume that there are \( N \) queues, each having its own arrival stream of jobs, and having an infinite buffer to store outstanding jobs. For models in discrete time, we will assume that all
jobs require service that lasts 1 time unit, time is split into \textit{slots} of length 1, and all arrivals and all service initiations happen at the beginning of a time slot, so that all services are completed by the end of a time slot. These assumptions are motivated mainly by wireless networks.

For convenience we assume that at the beginning of each time slot, first new services are started, and then new arrivals happen. We will denote time slots by $k = 0, 1, \ldots$. We can then write the evolution of the queue of node $i$ as

$$X_i(k + 1) = X_i(k) + \xi_i(k) - \eta_i(k),$$

where $\xi_i(k)$ denotes the number of new job arrivals into queue $i$ at time slot $k$, and $\eta_i(k)$ denotes the number of service completions in queue $i$ at time $k$.

We will assume that for each $i$, the sequence $\xi_i(k), k = 0, 1, 2, \ldots$ consists of i.i.d. random variables such that $\mathbb{E}(\xi_i) = \lambda_i$, where, here and throughout, by $\xi_i$ we denote a random variable with the distribution of any of $\xi_i(k)$. Note that arbitrary dependence between random variables with different values of $i$ is allowed.

We will assume also that random variables $\eta_i(k)$ take values 0 and 1 and are such that, on average, they maximise a global utility function in the following sense. Denote

$$\psi_i(\bar{x}) = \mathbb{E}(\eta_i(k)|\bar{X}(k) = \bar{x})$$

and assume that $\psi_i(\bar{x}) \in [0, 1]$ for all $\bar{x}$, $\psi_i(\bar{x}) = 0$ if $x_i = 0$, and

$$\bar{\psi}(\bar{x}) \in \arg \max_{\bar{\mu} \in \mathcal{C}} \sum_i g(x_i)h(\mu_i),$$

where the set $\mathcal{C}$ is compact and coordinate-convex; we impose, in addition,

\textit{Condition (H)}: the function $h : [0, \infty) \to \mathbb{R}$ is strictly increasing, differentiable and concave; and

\textit{Condition (G)}: the function $g : \mathbb{Z}_+ \to [0, \infty)$ is strictly increasing and such that

$$\frac{g(y)}{\Delta(y)} \to \infty$$

as $y \to \infty$, where $\Delta(y) = g(y+1) - g(y)$. Note that condition \textit{(G)} is equivalent to

$$\frac{g(y+1)}{g(y)} \to 1,$$

as $y \to \infty$.

\textit{Remark 1.} Note that for what is usually referred to as $\alpha$-fair algorithms, $g(y) = y^\alpha$ and $h(y) = \frac{y^{1-\alpha}}{1-\alpha}$ with $\alpha > 0$, so all the above conditions hold.

Throughout the section, we are going to assume that

There exists $\bar{\nu} \in \mathcal{C}$ such that $\bar{\lambda} < \bar{\nu}$. \hfill (5)

We will also denote

$$G(z) = \sum_{y=0}^{z} g(y),$$

and

$$F(y) = \sum_i h'(\nu_i)G(y_i).$$
2.1.2 Stability

In this section we prove that the utility-maximising algorithms described in the previous section are maximally stable. Our proof does not use fluid limits which have been the standard tool for proving stability of algorithms of this type. The advantages and disadvantages of our approach are described in the Introduction.

**Theorem 1.** Consider the discrete-time model in Section 2.1.1 and assume that $\xi_i$ is a Bernoulli random variable with $\mathbb{E}(\xi_i) = \lambda_i$. Assume that the vector $\lambda$ is such that condition (5) holds. Then the Markov chain $\{\bar{X}(k), k = 0, 1, \ldots\}$ is stable.

**Proof of Theorem 1.** We will use the standard Lyapunov-Foster criterion. Fix $\varepsilon > 0$ such that $\lambda_i < \nu_i - \varepsilon$ for all $i$. Note that, due to (2), and the concavity of the function $h$,

$$0 \leq \sum_i g(x_i)(h(\psi(\bar{x})) - h(\nu_i)) \leq \sum_i g(x_i)h'(\nu_i)(\psi(\bar{x}) - \nu_i). \quad (8)$$

We are going to consider

$$\mathbb{E}(F(\bar{x}(1)) - F(\bar{x}(0))|\bar{X}(0) = \bar{x}) = \mathbb{E}(F(\bar{x} + \hat{\xi}(0) - \hat{\eta}(0)|\bar{X}(0) = \bar{x}) - F(\bar{x}).$$

In what follows we are going to assume that $\bar{X}(0) = \bar{x}$ is fixed and will drop the dependence on this event. We will also write $\xi_i$ and $\eta_i$ instead of $\xi_i(0)$ and $\eta_i(0)$, for simplicity. We can write

$$\mathbb{E}(F(\bar{x} + \hat{\xi} - \hat{\eta}) - F(\bar{x})) = \mathbb{E}\left(\sum_i G(x_i + \xi_i - \eta_i)h'(\nu_i)\right) - \sum_i G(x_i)h'(\nu_i)$$

$$= \sum_i h'(\nu_i)\left(\lambda_i\psi(\bar{x})G(x_i) + (1 - \lambda_i)(1 - \psi(\bar{x}))G(x_i)\right)$$

$$+ \lambda_i(1 - \psi(\bar{x}))G(x_i + 1) + (1 - \lambda_i)\psi(\bar{x})G(x_i - 1) - G(x_i)) \right)$$

$$= \sum_i h'(\nu_i)\left(\lambda_i(1 - \psi(\bar{x}))(G(x_i + 1) - G(x_i)) + (1 - \lambda_i)\psi(\bar{x})(G(x_i - 1) - G(x_i))\right)$$

$$= -\sum_i h'(\nu_i)\lambda_i\psi(\bar{x})(G(x_i + 1) + G(x_i - 1) - 2G(x_i))$$

$$+ \sum_i h'(\nu_i)(\lambda_i(G(x_i + 1) - G(x_i)) + \psi(\bar{x})(G(x_i - 1) - G(x_i)))$$

$$= -\sum_i h'(\nu_i)\lambda_i\psi(\bar{x})(g(x_i + 1) - g(x_i)) + \sum_i h'(\nu_i)(\lambda_i g(x_i + 1) - \psi(\bar{x})g(x_i))$$

$$\leq \sum_i h'(\nu_i)(\lambda_i g(x_i + 1) - \psi(\bar{x})g(x_i)) \quad (9)$$

$$= \sum_i h'(\nu_i)(\lambda_i - \psi(\bar{x}))g(x_i) + \sum_i h'(\nu_i)\lambda_i(g(x_i + 1) - g(x_i))$$

$$= \sum_i h'(\nu_i)(\lambda_i - \nu_i)g(x_i) + \sum_i h'(\nu_i)(\nu_i - \psi(\bar{x}))g(x_i) + \sum_i h'(\nu_i)\lambda_i\Delta(x_i)$$

$$\leq -\varepsilon \sum_i h'(\nu_i)g(x_i) + \sum_i h'(\nu_i)\Delta(x_i) = \sum_i h'(\nu_i)(-\varepsilon g(x_i) + \Delta(x_i)),$$

where in the last inequality we used (8).

This, together with (3), implies that if $F(\bar{x})$ is large, then its expected drift may be made arbitrarily small, and certainly negative and bounded away from zero. This completes the proof of Theorem 1. \[\Box\]
2.1.3 Moment bounds

Once stability is established, one can employ arguments similar to those used in the proof of Theorem 1 to obtain bounds on some moments of the stationary distributions of queue states. The stationary regime exists under conditions of Theorem 1 in this section we will write $X$ to represent a random vector with the distribution equal to that of $X(k)$ in the stationary regime.

**Theorem 2.** Assume that all conditions of Theorem 1 hold and fix $\varepsilon > 0$ such that $\lambda_i < \nu_i - \varepsilon$ for each $i$. Then

$$\sum_i h'(\nu_i)\mathbb{E}g(X_i) \leq \frac{1}{\varepsilon} \sum_i h'(\nu_i)\mathbb{E}\Delta(X_i).$$

**Proof of Theorem 2.**

Consider the process with an arbitrary fixed initial state $\bar{X}(0)$. Then, due to the assumptions on the input flows, $\mathbb{E}(F(\bar{X}(k))) < \infty$ for any $k \geq 0$. By Theorem 1 the process is stable, and therefore $\bar{X}(k)$ converges in distribution to $\bar{X}$. We have the following drift estimate (we write $\psi_i$ instead of $\psi_i(\bar{x})$ for convenience):

$$\mathbb{E}(F(\bar{X}(k + 1)) - F(\bar{X}(k)) | \bar{X}(k) = \bar{x})$$

$$= \mathbb{E}(F(\bar{X} + \xi - \eta)) - F(\bar{x}) = \mathbb{E} \left( \sum_i G(x_i + \xi_i - \eta_i)h'(\nu_i) \right) - \sum_i G(x_i)h'(\nu_i)$$

$$= \sum_i h'(\nu_i)\left( \lambda_i\psi_iG(x_i) + (1 - \lambda_i)(1 - \psi_i)G(x_i) + \lambda_i(1 - \psi_i)G(x_i + 1) + (1 - \lambda_i)\psi_iG(x_i - 1) - G(x_i) \right)$$

$$= \sum_i h'(\nu_i)\left( \lambda_i(1 - \psi_i)(G(x_i + 1) - G(x_i)) + (1 - \lambda_i)\psi_i(G(x_i - 1) - G(x_i)) \right)$$

$$= \sum_i h'(\nu_i)\left( \lambda_i\psi_i(g(x_i + 1) - g(x_i)) + \sum_i h'(\nu_i)(\lambda_i g(x_i + 1) - \psi_i g(x_i)) \right)$$

$$\leq \sum_i h'(\nu_i)\left( \lambda_i g(x_i + 1) - \psi_i g(x_i) \right) \quad (10)$$

$$= \sum_i h'(\nu_i)(\lambda_i - \psi_i)g(x_i) + \sum_i h'(\nu_i)\lambda_i(g(x_i + 1) - g(x_i))$$

$$\leq -\varepsilon \sum_i h'(\nu_i)g(x_i) + \sum_i h'(\nu_i)\Delta(x_i) \leq c,$$

where $c$ is a fixed finite constant, and the last inequality follows from the properties of function $g$. Then, we obtain

$$\mathbb{E}[F(\bar{X}(k + 1)) - F(\bar{X}(k))] \leq \mathbb{E} \left[ -\varepsilon \sum_i h'(\nu_i)g(X_i(k)) + \sum_i h'(\nu_i)\Delta(X_i(k)) \right]$$

and then, by Fatou’s Lemma,

$$\limsup_{k \to \infty} \mathbb{E}[F(\bar{X}(k + 1)) - F(\bar{X}(k))] \leq \mathbb{E} \left[ -\varepsilon \sum_i h'(\nu_i)g(X_i) + \sum_i h'(\nu_i)\Delta(X_i) \right].$$
The left-hand side above must be greater or equal to 0 (because otherwise we would have $E F(\bar{X}(k)) \rightarrow -\infty$). This completes the proof of Theorem 12 \(\square\).

In certain particular cases (such as, e.g., when $g(x) = x^\alpha$ with an integer $\alpha$) one can significantly weaken the assumptions on random variables $\xi_i$. We provide the following theorem as an example of this, and also as we will use exactly this choice of functions $g$ and $h$ in section 3 to prove stability of an infinite network.

**Theorem 3.** For a discrete-time model defined in Section 2.1.1, assume that $g(y) = y^2$ and $h(y) = -y^{-1}$. Assume also that $\xi_i$ is a non-negative integer-valued random variable with $E(\xi_i^3) < \infty$ and $E(\xi_i) = \lambda_i$ with the vector $\lambda$ such that condition 3 holds. Then the Markov chain $\{\bar{X}(k), k = 0, 1, \ldots\}$ is stable.

Moreover, fix $\varepsilon > 0$ such that $\lambda_i < \nu_i - \varepsilon$ for all $i$. Then

$$\varepsilon \sum_i \frac{E(\xi_i^2)}{\nu_i^2} \leq A \sum_i \frac{E(\xi_i)}{\nu_i^2} + B,$$

where $\bar{X}$ denotes a random element with the stationary distribution of $X(\cdot)$,

$$A = 3 \sum_i \frac{E(\xi_i^2) + \lambda_i(1 - 2\lambda_i)}{\nu_i^2}$$

and

$$B = \sum_i \frac{E(\xi_i^3) - \lambda_i + 3\lambda_i^2 - 3(1 - 2\lambda_i)(\lambda_i^2 - \lambda_i/2 + E(\xi_i^2)/2).$$

**Proof of Theorem 3.** The specific functions $g(\cdot)$ and $h(\cdot)$ are such that the fluid limits of the process are well defined. We do use this fact to rely on previous results on stability and existence of moments, as will be seen shortly.

Positive recurrence of the Markov chain $X(\cdot)$ under the assumptions (in fact the existence of only the first moments of $\xi_i$’s is sufficient for stability) of the theorem holds due to Lemma 12, where the stability of the corresponding fluid limits is established. (Note that, if one assumed convexity of the set $C$, stability would follow from earlier results, see, e.g. lemma 3; however, the convexity of the set $C$ is not in fact necessary for stability results, which is pointed out in 14.) We will consider the stationary version of the process. The finiteness of the third moment of $\xi_i$ (along with stability of fluid limits) guarantees that $E(\xi_i^3) < \infty$ (see 3).

Due to stationarity, $E(X_i(k + 1)) = E(X_i(k))$ and hence

$$E(\psi_i) = \lambda_i,$$  
(11)

where for simplicity we write $\psi_i$ instead of $\psi_i(\bar{X}).$

Note that

$$E(X_i^l \eta_i) = E(E(X_i^l \eta_i | \bar{X})) = E(X_i^l E(\eta_i | \bar{X})) = E(X_i^l \psi_i)$$  
(12)

for any $l$. Note also that $\eta_i^l = \eta$ a.s. for any $l > 0$. Due to stationarity, we also have $E(X_i^l(k + 1)) = E(X_i^l(k))$, which is equivalent to

$$0 = E(\xi_i^l) + E(\psi_i) - 2E(\xi_i)E(\psi_i) + 2E(\xi_i)E(X_i) - 2E(X_i \psi_i),$$

where we used (11), and hence

$$E(X_i \psi_i) = \lambda_i E(X_i) - \lambda_i^2 + \lambda_i/2 - E(\xi_i^2)/2.$$  
(13)

Assume now that $E(X_i^3) < \infty$ (we will demonstrate how to drop this additional assumption at the end of the proof). Then the equality of the third moments in stationarity implies

$$0 = E(\xi_i^3) - E(\psi_i) + 3E(\xi_i)E(\psi_i) - 3E(\xi_i^2)E(\psi_i) + 3E(X_i \psi_i) - 3E(X_i \psi_i^2) + 3E(X_i^2 \psi_i) + 3E(X_i)E(\xi_i^2)$$
where we used (11) and (13) and where

\[ A_i = 3E(\xi_i^3) - 3\lambda_i(1 - 2\lambda_i) \]

and

\[ B_i = E(\xi_i^3) - \lambda_i + 3\lambda_i^2 - 3(1 - 2\lambda_i)(\lambda_i^2 - \lambda_i/2 + E(\xi_i^3)/2). \]

Due to (3),

\[ 0 \leq \sum_i \frac{\lambda_i}{\nu_i}(\psi_i(\bar{x}) - \nu_i) \]

for any \( \bar{x} \), and hence

\[
\sum_i \frac{\lambda_i E(X_i^2) - E(X_i^2)\psi_i}{\nu_i^2} = \sum_i \left( \frac{\lambda_i - \nu_i}{\nu_i^2} E(X_i^2) \right) + \sum_i \frac{\psi_i(X_i)}{\nu_i^2} 
\leq -\varepsilon \sum_i \frac{E(X_i^2)}{\nu_i^2}.
\]

The statement of the Theorem now follows by dividing (14) by \( \nu_i^2 \) and summing over all \( i \).

We now show that the assumption \( E(X_i^2) < \infty \) can be dropped. Let \( M < \infty \) and consider the system with arrivals given by \( \xi_i(M) = \min\{\xi_i, M\} \) instead of \( \xi_i \). Of course, \( E\xi_i(M) \leq E\xi_i \). Therefore, the system is stable for each \( M \), and let us denote by \( X(M) \) a random element which has its stationary distribution. For each \( M \), \( E((X_i(M))^3) < \infty \) (because \( E((\xi_i(M))^4) < \infty \) and (3)), and the derivations above imply that

\[
\sum_i \frac{E((X_i(M))^2)}{\nu_i^2} \leq A(M) \sum_i \frac{E(X_i(M))}{\nu_i^2} + B(M),
\]

with obvious expressions for \( A(M) \) and \( B(M) \). Since \( E((\xi_i(M))^l) \to E(\xi_i) \) as \( M \to \infty \) for \( l = 1, 2, 3 \), \( A(M) \to A \) and \( B(M) \to B \) as \( M \to \infty \). It is also easy to check that the sequence \( X(M)(\cdot) \) and \( \bar{X}(\cdot) \) satisfy the continuity property. This implies that the stationary versions of \( X(M)(\cdot) \) and \( \bar{X}(\cdot) \) can be coupled so that, w.p.1, \( X(M)(k) \to \bar{X}(k) \) for any \( k \). This, in turn, implies that \( X(M) \Rightarrow \bar{X} \). It remains to rewrite the last display as

\[
\sum_i \frac{1}{\nu_i^2} E[X_i(M)^2 - A(M)^2/2]^2 \leq B(M) + \sum_i \frac{(A(M))^2}{4\nu_i^2},
\]

and apply Fatou’s Lemma to obtain

\[
\sum_i \frac{1}{\nu_i^2} E[X_i - A/2]^2 \leq \liminf_{M \to \infty} \sum_i \frac{1}{\nu_i^2} E[X_i(M)^2 - A(M)^2/2]^2 \leq B + \sum_i \frac{A^2}{4\nu_i^2}.
\]

\[ \square \]

### 2.2 Continuous time

In this Section we present results on stability and moment bounds for utility-maximising algorithms in continuous time in a Markovian setting - a standard assumption for most such problems. The proofs follow the same lines as those in Section 2 with minor changes. Proofs in continuous time are somewhat simpler, as, due to Markovian assumptions, the probability that two or more events happen in a small time interval is negligible.
2.2.1 Model

Assume, as before, that there are $N$ interacting queues. Arrivals into queue $i$ occur according to a Poisson process with a constant rate $\lambda_i$, independent of all the other processes. The instantaneous departure rate from queue $i$ at time $t$, conditioned on the state $\bar{X}(t)$, is $\psi_i(\bar{X}(t))$; more precisely, the number of departures up to time $t$ is $\Pi_i(\int_0^t \psi_i(X(\tau))d\tau)$, where $\Pi_i(\cdot)$ are independent unit-rate Poisson processes.

We assume that all the conditions on the functions $\psi_i(\cdot)$, $i = 1, \ldots, N$ and on the arrival intensities $\bar{\lambda}$ imposed in Section 2.1.1, hold. More precisely, we assume that the functions $\psi_i(\cdot)$, $i = 1, \ldots, N$ satisfy condition (2) with functions $h$ and $g$ satisfying Conditions (H) and (G), respectively; and we assume that the arrival intensities $\bar{\lambda}$ satisfy condition (5). We will also use functions $G$ and $F$ defined in (6) and (7), respectively.

2.2.2 Stability analysis

Theorem 4. For the continuous-time model defined in Section 2.2.1 such that condition (5) holds, the Markov process $\{\bar{X}(t)\}_{t \geq 0}$ is stable.

Proof of Theorem 4. The proof is a simplified version of that of Theorem 1. By the Lyapunov-Foster criterion, in order to show positive recurrence, it is sufficient to show that

$$\sum_i (\lambda_i(F(x_i + 1) - F(x_i) + \psi_i(\bar{x})F(x_i - 1) - F(x_i))) < -\delta,$$

for the function $F$ defined in (7), some $\delta > 0$, and for values of $\bar{x}$ outside of a compact set. This may be found in, e.g. [18]. Alternatively, one can consider the embedded discrete-time Markov chain by looking at transition epochs and apply the standard discrete-time Lyapunov-Foster criterion.

Note that the expression on the LHS of the above may be written as

$$\sum_i h'(\nu_i)(\lambda_i(G(x_i + 1) - G(x_i)) + \psi_i(\bar{x})(G(x_i - 1) - G(x_i))),$$

which is equal to (9), and the rest of the proof of Theorem 1 applies.

2.2.3 Moment bounds

As in discrete time, once stability is established, one can use similar arguments to establish moment bounds.

Since the stationary regime exists under conditions of Theorem 4 in this section we will write $\bar{X}$ to represent a random vector with the distribution equal to that of $\bar{X}(t)$ in stationary regime.

Theorem 5. Assume that all conditions of Theorem 4 hold. Fix $\varepsilon > 0$ such that $\lambda_i < \nu_i - \varepsilon$ for all $i$. Then

$$\sum_i h'(\nu_i)\mathbb{E}(g(X_i)) \leq \frac{1}{\varepsilon} \sum_i h'(\nu_i)\mathbb{E}(\Delta(X_i)).$$

Proof of Theorem 5. Note that condition (4) implies that $g(x) = o(e^{ax})$ as $x \to \infty$ for any $a > 0$. This, in turn, implies that $G(x) = o(e^{ax})$ as $x \to \infty$ for any $a > 0$. Note also that, as arrival flows are given by Poisson processes, for any $t$ and $\bar{x}$, there exists $a > 0$ such that

$$\mathbb{E}(e^{a\bar{X}(t)} | \bar{X}(0) = \bar{x}) < \infty.$$

Then $\mathbb{E}(F(\bar{X}(t)) | \bar{X}(0) = \bar{x}) < \infty$ for any $t$ and any $\bar{x}$.
We consider the following two service algorithms for the discrete-time case. Our results apply to wireless networks), we refer the reader to [10] and [14].

In this section we provide an application of our moment bounds to establishing stability of infinite networks considered in [10] and [14]. The stability of an infinite network we define as the existence of a proper stationary distribution (with all queues finite with probability 1).

As stated above, this is all we need to define the process in the continuous-time case.

For the discretetime system, recall that the arrivals are driven by the set of independent Poisson processes. For each node $i$, the generic departure time is exponentially distributed with rate $\lambda_i$, where $\lambda_i$ is the average arrival rate at node $i$. The departure times of the nodes are independent (given the system state), but we do have to specify the departure (service) mechanism to make sure that the processes we consider satisfy the continuity and/or monotonicity properties. In particular, the continuity will be the key property which we need to make limit transitions from finite systems to infinite ones.

We note that for continuous-time systems, finite or infinite, the average service rates define the service mechanism completely, and so no additional structural assumptions are necessary.

For the motivation of the specific service mechanisms that we consider (in particular related to wireless networks), we refer the reader to [10] and [14].

### 3.1 Model

The queues (or nodes) are assumed to be located on a $d$-dimensional lattice, with the service rates given by

$$\psi_i(x) = \frac{x_i}{\sum_{j \in \mathbb{Z}^d} a_{j-i} x_j},$$

where $a_0 = 1$, $a_i = a_{-i}$ for all $i \in \mathbb{Z}^d$ and $L = \sup \{|i| : a_i > 0\} < \infty$. For each $i$, the nodes $j$ within the finite set $\mathcal{N}_i = \{j \mid a_{j-i} > 0\}$ are called neighbours of $i$. Note that $i \in \mathcal{N}_i$.

As stated above, this is all we need to define the process in the continuous-time case.

For the discrete-time system, recall that the arrivals are driven by the set of independent random variables $\xi_i(k)$, which represent the number of arrivals into node $i$ at time $k$. The sets $\{\xi_i(k)\}$ are i.i.d. across $k$; and for each fixed $i$, $\xi_i(k)$ are i.i.d. across $k$. As before, we denote by $\xi_i$ the generic $\xi_i(k)$, and assume

$$\mathbb{E}\xi_i^3 < \infty.$$

We consider the following two service algorithms for the discrete-time case. Our results apply to both. (Again, see [14] for the motivation of the algorithms.) Recall that $X_i(k)$ are the queue lengths at time $k$.

**Discrete-time service algorithm 1** (D1). The algorithm is driven by the set of i.i.d. (across node indices $i$ and times $k$) random variables $\nu_i(k)$, distributed uniformly in $[0, 1]$. The access priority of node $i$ at time $k$ is $\tau_i(k) = [-\log \nu_i(k)/X_i(k)]$ – it is exponentially distributed with mean $1/X_i(k)$. The smaller the $\tau_i(k)$ the "higher" the priority. Then, node $i$ transmits in slot $k$, if $X_i(k) > 0$ and $\tau_i(k) < \tau_j(k)/a_{j-i}$ for all $j \in \mathcal{N}_i \setminus i$.

Note that the probability of node $i$ transmitting, conditioned on $X(k)$, is exactly $\psi_i(X(k))$, as required by (15). At the same time, the transmissions of the nodes at time $k$, even conditioned...
on $X(k)$ are not independent (except in the degenerate case $N_i = i$). In fact, in the case when all $a_i$ are either 1 or 0, neighbouring nodes can never transmit simultaneously.

*Discrete-time service algorithm 2 (D2).* This algorithm is much simpler—it is a discrete-time version of the continuous-time algorithm. It is also driven by the set of i.i.d. (across node indices $i$ and times $k$) random variables $\nu_i(k)$, distributed uniformly in $[0, 1]$. Node $i$ transmits in slot $k$, if $X_i(k) > 0$ and $\nu_i(k) < \psi_i(\bar{X}(k))$.

In other words, conditioned on $\bar{X}(k)$, the probabilities of nodes transmitting are exactly $\psi_i(\bar{X}(k))$ (as required by (15)), and the transmissions are independent.

### 3.2 Continuity and monotonicity

For the infinite system process, in both continuous and discrete time, we will use continuity (as defined in Section 1.1) and monotonicity properties.

For a continuity property to be well-defined, a topology on the process state space needs to be specified. A state of the process is a set $\bar{X} = \{X_i\}$ of the queue lengths, i.e. a function of $i$. On this state space (which is uncountable for infinite system), we consider the natural topology of component-wise convergence.

We also consider the natural component-wise order relation $\bar{X} \leq \bar{X}^*$ on the state space. With respect to this partial order, it is easy to see that the process for the system defined above has the following *monotonicity* property: two versions of the process, such that $\bar{X}^*(0) \leq \bar{X}(0)$, can be coupled (constructed on a common probability space), so that $\bar{X}^*(k) \leq \bar{X}(k)$ at all times $k \geq 0$ (and analogously for continuous time $t$). We note that this monotonicity only holds for single-hop systems; it does not hold for multi-hop system which we will consider later in Section 4.

### 3.3 Auxiliary system on a finite torus

Denote by $T_n$ the restriction of $\mathbb{Z}^d$ to points at a distance at most $n$ from the origin, seen as a torus. We will consider only the values $n > L$.

Along the lines of [14, Lemma 11], we can show that for any $T_n$, the rates (15) are in fact utility maximising in a certain set. Indeed, denote

$$C = \{\bar{\mu} : \text{there exists } \bar{p} \text{ such that } \bar{\mu} \leq \psi(\bar{p})\}.$$

We can prove the following optimality result.

**Lemma 6.** The rates (15) are utility maximising (they satisfy relation (2)) with the functions $g(y) = y^2$, $h(y) = -y^{-1}$ and the set $C$.

**Remark 2.** Using the standard terminology of $\alpha$-fairness, Lemma states that the rates (15) are 2-fair in the set $C$.

**Proof of Lemma 6.** Indeed, due to the definition of the set $C$, for any $\bar{\mu} \in C$,

$$\sum_i x_i \left( \frac{\mu_i}{x_i} \right)^{-1} \geq \sum_i x_i \left( \frac{\psi_i(\bar{p})}{x_i} \right)^{-1}$$

for the corresponding vector $\bar{p}$. Hence, it is sufficient to show that

$$\sum_i x_i \left( \frac{\psi_i(\bar{X})}{x_i} \right)^{-1} \leq \sum_i x_i \left( \frac{\psi_i(\bar{p})}{x_i} \right)^{-1}$$
for all vectors \( \tilde{p} \). Note that the LHS of the above is equal to
\[
\sum_i x_i \sum_{j \in T_n} a_{j-i} x_j = \sum_i x_i^2 + \sum_i \sum_{j \in T_{n,j} \neq i} a_{j-i} x_i x_j.
\]
Consider now
\[
\sum_i x_i \left( \frac{p_i}{(\sum_{j \in T_n} a_{j-i} p_j) x_i} \right)^{-1} = \sum_i x_i^2 \left( 1 + \sum_{j \in T_{n,j} \neq i} \frac{a_{j-i} p_j}{p_i} \right)
= \sum_i x_i^2 + \frac{1}{2} \sum_i \sum_{j \in T_{n,j} \neq i} \left( x_i^2 \frac{a_{j-i} p_j}{p_i} + x_j^2 \frac{a_{i-j} p_i}{p_j} \right).
\]
For any \( i \) and \( j \),
\[
x_i^2 \frac{a_{j-i} p_j}{p_i} + x_j^2 \frac{a_{i-j} p_i}{p_j} = a_{j-i} \left( x_i^2 \frac{p_j}{p_i} + x_j^2 \frac{p_i}{p_j} \right) \geq 2 a_{j-i} x_i x_j,
\]
where we used the symmetry of the sequence \( a_i \). The equality in the above is possible if and only if \( x_i^2 \frac{p_i}{p_j} = x_j^2 \frac{p_j}{p_i} \), which is equivalent to \( \frac{p_i}{x_i} = \frac{p_j}{x_j} \). Therefore we obtain
\[
\sum_i x_i \left( \frac{p_i}{(\sum_{j \in T_n} a_{j-i} p_j) x_i} \right)^{-1} \geq \sum_i x_i^2 + \sum_{j \in T_{n,j} \neq i} a_{j-i} x_i x_j,
\]
and the equality is possible if and only if \( \frac{p_i}{x_i} = \frac{p_j}{x_j} \) for all \( i \) and \( j \). This implies that \( \frac{p_i}{x_i} \) has to be a constant for each \( i \).

**Lemma 7.** If \( \lambda_i = \lambda \) for each \( i \), then the existence of \( \tilde{\nu} \in C \) such that \( \tilde{\lambda} < \tilde{\nu} \) is equivalent to the inequality \( \lambda < \frac{1}{\sum_{j \in T_n} a_j} = \frac{1}{\sum_{j \in \mathbb{Z}^d} a_j} \) for any \( n > L \).

**Proof.** Indeed, if \( \lambda < \frac{1}{\sum_{j \in T_n} a_j} \), we can take \( \tilde{p} = (1, \ldots, 1) \) and \( \tilde{\nu} = \tilde{\psi}(\tilde{p}) \) - such a vector clearly belongs to \( C \), and it is also clear that \( \tilde{\lambda} < \tilde{\nu} \). In the opposite direction, assume that \( \lambda < \tilde{\nu} \) such that \( \tilde{\nu} \in C \) and fix the corresponding vector \( \tilde{p} \). Then
\[
\frac{1}{\lambda} > \sum_{j \in T_n} a_{j-i} p_j \]
for each \( i \in T_n \). If we add up these inequalities over all \( i \in T_n \), we obtain
\[
\frac{(2n+1)^d}{\lambda} > \sum_{i \in T_n} \sum_{j \in T_n} a_{j-i} p_j \]
\[
= \sum_{i \in T_n} \sum_{j \in T_{n,j} \neq i} \left( a_{j-i} p_j + a_{i-j} p_i \right)
= \sum_{i \in T_n} \sum_{j \in T_{n,j} \neq i} a_{j-i} \left( \frac{p_j}{p_i} + \frac{p_i}{p_j} \right)
\geq (2n+1)^d + (2n+1)^d \sum_{j \in \mathbb{Z}^d,j \neq 0} a_j,
\]
which concludes the proof (recall that \( a_0 = 1 \).  \( \square \)
3.4 Stability analysis

Our results on moment bounds allow us to obtain a stability result, along with a second moment bounds, in both discrete- and continuous-time settings.

We will say that the arrival rates \( \lambda_i \) are \textit{periodic}, if the following holds: (a) the values of \( \lambda_i \) are given for \( i \) within the rectangle \( I = [0, \ldots, C_1 - 1] \times \cdots \times [0, \ldots, C_d - 1] \) with some fixed positive integers \( C_1, \ldots, C_d \); (b) for any \( i \in \mathbb{Z}^d \) and any \( k = 1, \ldots, d \), \( \lambda_i + C_k e_k = \lambda_i \), where \( e_k \) is the \( k \)-th unit coordinate vector (with \( k \)-th entry equal to 1 and all other entries equal to zero).

Similarly, we define periodicity of any other function of \( i \). We will say that random variables \( \xi_i \) are \textit{i.i.d. up to periodicity}, if they are all independent, and \( \xi_{i+C_k e_k} = \xi_i \) have identical distribution for any \( i \) and \( k \).

**Theorem 8.** Consider periodic rates \( \lambda_i \). For a discrete-time system assume in addition that \( \xi_i \) are i.i.d. up to periodicity, and \( \mathbb{E} \xi_i^3 < \infty \) for all \( i \). Assume that there exists a periodic \( \bar{\nu} \) from the set

\[
C = \{ \bar{\mu} : \text{there exists } \bar{p} \text{ such that } \bar{\mu} \leq \bar{\psi}(\bar{p}) \}
\]

such that \( \bar{\lambda} < \bar{\nu} \). Consider an infinite network with arrival rates \( \bar{\lambda} \leq \bar{\lambda} \); and, for the discrete-time system, assume in addition that the per-slot (random) number of arrivals \( \xi_i \) is dominated by \( \xi_i \) w.p.1. \( \xi_i \leq \xi_i \). Then this infinite network is stable in both the discrete- and continuous-time settings, and there exists a stationary regime with finite second moments \( \mathbb{E} X_i^2 \) of the queue lengths.

**Proof.** Due to monotonicity of the process, it suffices to prove the theorem for the periodic arrival rates \( \bar{\lambda} \) and (in discrete time) arrival process \( \xi_i \). Also, to be specific, consider the discrete-time case. (The proof for continuous time is same, almost verbatim.)

If one considers the restriction of the system onto sets \( R_n = \{ i : i_k = -nC_k, \ldots, nC_k - 1 \} \) "wrapped around" to form a torus, then the conditions of the Theorem, along with Lemma 6 imply stability and therefore existence of the (unique) stationary measure for the process on \( R_n \). Lemma 8 and Theorem 3 in discrete case (and Theorem 5 in continuous case), along with the periodicity, imply that

\[
\sum_{i \in I} \frac{\mathbb{E} \left( (X_i^{(n)})^2 \right)}{\nu_i^2} \leq A_1 \sum_{i \in I} \frac{\mathbb{E} X_i^{(n)}}{\nu_i^2} + A_2 \tag{16}
\]

with some constants \( A_1 \) and \( A_2 \), where the upper index \( n \) is used to denote the restriction to \( R_n \). Note that \( A_1 \) and \( A_2 \) do not depend on \( n \). This implies a uniform in \( n \) and \( i \in I \) second moment bound

\[
\mathbb{E} \left( (X_i^{(n)})^2 \right) \leq C < \infty. \tag{17}
\]

Let us view each process \( \bar{X}^{(n)}(\cdot) \) as a process on the entire infinite lattice \( \mathbb{Z}^d \); say, by letting \( X_i(\cdot) \equiv 0 \) for \( i \not\in R_n \). (We note that the node neighbourhood structure remains as that of the torus, and so the process is still as that on the torus.) Correspondingly, we will view the (stationary) distributions of \( \bar{X}^{(n)}(\cdot) \) as distributions on the entire infinite lattice \( \mathbb{Z}^d \); we see from (16) that these distributions are tight (as distributions on \( \mathbb{Z}^d \)). Then there exists a subsequence of (stationary) processes \( \bar{X}^{(n)}(\cdot) \), along which \( \bar{X}^{(n)}(0) \Rightarrow \bar{X}^* \), where \( \bar{X}^* \) is some proper random element (with all components being finite w.p.1), and then \( \bar{X}^{(n)}(k) \Rightarrow \bar{X}^* \) for each \( k \).

It is easy to observe that the sequence of processes \( \bar{X}^{(n)}(\cdot) \) and the process \( \bar{X}(\cdot) \) (which is the "true" infinite system process) satisfy the continuity property (in Section 11). This means the subsequence of (stationary) processes \( \bar{X}^{(n)}(\cdot) \) and the process \( \bar{X}(\cdot) \), with \( X(0) \) distributed as \( \bar{X}^* \), can be coupled in a way such that \( \bar{X}^{(n)}(k) \rightarrow \bar{X}(k) \) w.p.1, for each \( k \geq 0 \). This means that \( \bar{X}(k) \) is equal in distribution to \( \bar{X}^* \) for each \( k \), i.e. we constructed a stationary version of \( \bar{X}(\cdot) \). Since \( X_i^{(n)}(\cdot) \Rightarrow X_i^* \), Fatou’s lemma and (17) imply that \( \mathbb{E} X_i^2 \leq C < \infty. \) \( \square \)
Consider now a special case – a symmetric infinite system; in continuous time this means that \( \lambda_i = \lambda \) for each \( i \), and in discrete time this means that random variables \( \xi_i \) are i.i.d. (In continuous time, this symmetric system was studied in [10].) From Theorem 8 we obtain the following.

**Corollary 9** (In particular, proves Conjecture 1.12 in [10]). Consider the symmetric system and assume

\[
\lambda < \frac{1}{\sum_{j \in \mathbb{Z}^d} a_j}.
\]

(18)

For the discrete-time case, assume additionally that \( \mathbb{E}\xi_i^3 < \infty \). Then the system is stable and its lower invariant measure (i.e., the stationary distribution dominated by any other) is such that \( \mathbb{E}X_i^2 < \infty \).

Indeed, by Lemmas 6 and 7 and Theorem 8, condition (18) ensures stability. By Theorem 8 \( \mathbb{E}X_i^2 < \infty \) holds for some stationary distribution, and therefore it holds for the lower invariant measure as well. Our Corollary 9 proves [10] Conjecture 1.12, with all its implications stated in [10, Section 1.1], in particular the uniqueness of the stationary regime with finite second moments of the queue lengths, for the infinite symmetric network.

**Remark 3.** The simplest case where \( \lambda_i \) are not all the same is when we consider a network on the line such that the rates are periodic with a period 2. Denote the different values by \( \lambda_1 \) and \( \lambda_2 \). From [10], stability only follows if we assume \( \lambda_1, \lambda_2 < 1/3 \). Theorem 8 on the other hand implies stability as long as there exist \( p_1 \) and \( p_2 \) such that

\[
\lambda_1 < \frac{p_1}{p_1 + 2p_2}
\]

and

\[
\lambda_2 < \frac{p_2}{p_2 + 2p_1}.
\]

One can see that by taking, for instance, \( p_1 = 1 \) and \( p_2 = \delta > 0 \), the values of \( \lambda_1 \) for which stability holds may be taken arbitrarily close to 1 (of course at the expense of very low values of \( \lambda_2 \)).

**Remark 4.** The proof of Theorem 8 uses only the utility-maximising properties of the rates and the process continuity properties. It is clear that similar arguments may be used to demonstrate stability of infinite networks in many other cases.

### 4 Stability analysis of an infinite multi-hop network

In this section we demonstrate how techniques similar to the ones we used in the single-hop case may be used to demonstrate the existence of invariant measures for infinite multi-hop networks, where, upon a service completion at a given queue, a job may leave the system or enter the queue of a neighbouring node.

Multi-hop networks have an additional layer of difficulty as the movement of jobs between different queues complicates the dependence structure of the queue states further. Multi-hop networks are notoriously difficult to analyse, and we consider significantly stronger assumptions on the structure of the network, with strictly i.i.d. arrival processes and with symmetric routing (see [14]).

Specifically, the model is as follows. We consider the discrete-time setting only. Just as in Section 3 the nodes (queues) are located on the \( d \)-dimensional lattice. The exogenous arrival processes are strictly i.i.d.; namely, the random variables \( \xi_i \) representing the numbers of new arrivals are i.i.d. with \( \mathbb{E}(\xi_i) = \lambda q \) with a fixed \( q \in (0, 1) \) and with \( \mathbb{E}(\xi_i^2) < \infty \). The service is governed by either algorithm (D1) or (D2), specified in Section 3. Upon a service completion at
any node, a job leaves the system with probability $q$ or joins the queue of a neighbour of node $i$ (i.e., connected by one lattice edge), chosen independently at random (i.e., each neighbour is chosen with probability $(1 - q)/2d$). All the routing decisions are taken independently of everything else. The service rates are given by

$$
\psi_i(\bar{x}) = \frac{x_i}{\sum_{j \in \mathcal{N}_i} x_j},
$$

where $\mathcal{N}_i$ is the neighbourhood of node $i$ on the $\mathbb{Z}^d$ lattice, which by convention includes node $i$ itself. (In other words, the service rates are a special case of those considered in Section 3, with the neighbourhood $\mathcal{N}_i$ of node $i$ including specifically the neighbours in terms of the lattice, and with $a_{j-i} = 1$ for all $j \in \mathcal{N}_i$.)

A restriction of the system to any torus $T_n$ forms a finite $2^d$-regular graph, and [14, Theorem 6] implies that if $\lambda < 1/(2^d + 1)$, then the system is stable. Therefore, there exists a stationary distribution of the number of messages in each queue. Due to symmetry, the (stationary) numbers of messages in any two queues are identically distributed, and we will consider queue 0 for simplicity. Denote the stationary number of messages in queue 0 in the system restricted to $T_n$ by $X(n)$.

We want to emphasise that the described multi-hop process (for both the infinite system and a finite torus) is not monotone (unlike in the single-hop model of Section 3), and this is in fact one of the key challenges of the multi-hop system analysis. Versions of this process, however, do have continuity properties, which we will exploit, just as in the single-hop case.

**Theorem 10.** Consider the multi-hop model on torus $T_n$, described above, and denote by $\xi$ a random variable with the distribution of $\xi_i(k)$ for any $i$ and $k$. Assume that $\mathbb{E}(\xi^2) < \infty$ and $\mathbb{E}(\xi) = \lambda < \frac{1}{2^d + 1}$. Then

$$
\mathbb{E}(X(n)) \leq \frac{\mathbb{E}(\xi^2) + 2^d(1 - q)\lambda + \lambda - 2\lambda^2 q^2}{2q \left( \frac{1}{2^d + 1} - \lambda \right)}. \quad (19)
$$

**Proof of Theorem 10.**

For ease of notation, in this section we are going to write $X_i$ instead of $X_i(n)$ to denote the stationary version of the process restricted to $T_n$. We can describe the evolution of $X_i$ as

$$
X_i(k + 1) = X_i(k) + \xi_i(k) + \sum_{j \in \mathcal{N}_i, j \neq i} I_{ji}(k)\eta_j(k) - \eta_i(k), \quad (20)
$$

where random variables $I_{ji}(s)$ are indicator functions of events that a message potentially leaving node $j$ in time slot $k$ will choose node $i$ as its destination. For ease of notation, as we only consider a single time slot in what follows, we are going to simply write $\xi_i, \eta_i$ and $I_{ji}$.

As in the previous sections, note that random variables $\eta_i$ can only take values 0 and 1 and

$$
\mathbb{E}(\eta_i|\bar{X}) = \mathbb{P}(\eta_i = 1|\bar{X}) = \psi_i(\bar{X}) \quad \text{a.s.}
$$

Note also that

$$
\mathbb{E}(I_{ji}) = (1 - q)\frac{1}{2^d}
$$

for all $j$ and $i$. Due to stationarity of the process $\bar{X}(\cdot)$, $X_i(k)$ and $X_i(k + 1)$ have the same distributions, therefore, in particular, $\mathbb{E}(X_i(k + 1)) = \mathbb{E}(X_i(k))$ and hence, from (20),

$$
0 = \lambda q + \sum_{j \in \mathcal{N}_i, j \neq i} (1 - q)\frac{1}{2^d}\mathbb{E}(\eta_j) - \mathbb{E}(\eta_i), \quad (21)
$$
where we used the fact that $I_{ji}$ and $\eta_j$ are independent. Note now that

$$
\mathbb{E}(\eta_l) = \mathbb{E}(\mathbb{E}(\eta_i | \bar{X})) = \mathbb{E}(\psi_l(\bar{X})) = \mathbb{E} \left( \frac{X_i}{\sum_{j \in \mathcal{N}_i} X_j} \right)
$$

for any $l$, and, due to the symmetry of the model, it does not depend on $l$. Hence, continuing (21),

$$
0 = \lambda q + 2^d (1 - q) \frac{1}{2^d} \mathbb{E}(\eta_i) - \mathbb{E}(\eta_l),
$$

implying

$$
\mathbb{E}(\eta_i) = \lambda
$$

(22)

for any $i$. Denote $A_i = \sum_{j \in \mathcal{N}_i, j \neq i} I_{ji} \eta_j$.

Assume first that $\mathbb{E}X^2 < \infty$. (We will show later in the proof how to get rid of this additional assumption.) Stationarity of the process $\bar{X}(\cdot)$ implies that $\mathbb{E}(X_i^2(k + 1)) = \mathbb{E}(X_i^2(k))$ and hence, from (20),

$$
0 = \mathbb{E}(\xi_i^2) + \mathbb{E}(A_i^2) + \mathbb{E}(\eta_i^2) - 2\mathbb{E}(A_i \eta_i) + 2\mathbb{E}(\xi_i(A_i - \eta_i)) + 2\mathbb{E}(X_i \xi_i) + 2\mathbb{E}(X_i(A_i - \eta_i))
\leq \mathbb{E}(\xi_i^2) + \mathbb{E}(A_i^2) + \lambda + 2\lambda q \mathbb{E}(A_i - \eta_i) + 2\lambda q \mathbb{E}(X_i) + 2\mathbb{E}(X_i(A_i - \eta_i))
= \mathbb{E}(\xi_i^2) + \mathbb{E}(A_i^2) + \lambda - 2\lambda q^2 + 2\lambda q \mathbb{E}(X_i) + 2\mathbb{E}(X_i(A_i - \eta_i)).
$$

(23)

In the derivations above we used the independence of $\xi_i$ from all other random variables, the fact that $\mathbb{E}(\eta_i^2) = \mathbb{E}(\eta_i) = \mathbb{E}(\eta_i = 1)$, equation (22) and finally, in the last equality, a simple calculation of $\mathbb{E}(A_i)$ already performed earlier in this proof (see (21)).

We consider some of the terms above separately. First,

$$
\mathbb{E}(A_i^2) = \mathbb{E} \left( \left( \sum_{j \in \mathcal{N}_i, j \neq i} I_{ji} \eta_j \right)^2 \right) \leq 2^d \mathbb{E} \left( \sum_{j \in \mathcal{N}_i, j \neq i} I_{ji}^2 \eta_j^2 \right) = 2^d \mathbb{E} \left( \sum I_{ji} \eta_j \right) = 2^d (1 - q) \lambda
$$

(24)

where we used convexity of the function $x^2$, independence of $I$‘s and $\eta$‘s, as well as the facts that all the random variables concerned only take values 0 and 1 and therefore are equal to their squares.

Let us now note that

$$
\mathbb{E}(X_i \eta_j) = \mathbb{E}(\mathbb{E}(X_i \eta_j | \bar{X})) = \mathbb{E}(X_i \mathbb{E}(\eta_j | \bar{X})) = \mathbb{E}(X_i \psi_j) = \mathbb{E} \left( X_i \frac{X_j}{\sum_{l \in \mathcal{N}_j} X_l} \right)
$$

for any $i$ and $j$. It is clear that, due to the symmetry of the model, for any $j \in \mathcal{N}_i$, the pairs $(X_i, \eta_j)$ and $(X_j, \eta_i)$ have identical distributions which implies, in particular, that

$$
\mathbb{E}(X_i \eta_j) = \mathbb{E}(X_j \eta_i).
$$

Consider now

$$
\mathbb{E}(X_i(A_i - \eta_i)) = \sum_{j \in \mathcal{N}_i, j \neq i} \mathbb{E}(X_i I_{ji} \eta_j) - \mathbb{E}(X_i \eta_i) = (1 - q) \frac{1}{2^d} \sum_{j \in \mathcal{N}_i, j \neq i} \mathbb{E}(X_i \eta_j) - \mathbb{E}(X_i \eta_i)
\leq (1 - q) \frac{1}{2^d} \sum_{j \in \mathcal{N}_i, j \neq i} \mathbb{E}(X_j \eta_j) - \mathbb{E}(X_i \eta_i) = (1 - q) \frac{1}{2^d} \sum_{j \in \mathcal{N}_i, j \neq i} \mathbb{E}(X_j \psi_j) - \mathbb{E}(X_i \psi_i)
= \mathbb{E} \left( \psi_i \left( 1 - q \frac{1}{2^d} \sum_{j \in \mathcal{N}_i, j \neq i} X_j - X_i \right) \right)
$$

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\[ \lambda < \frac{M}{X} \] for any \( \lambda \).

We further observe that the sequence of processes \( \bar{X} \) allows us to prove a result on an infinite-lattice multi-hop model. By Fatou’s lemma, \( \mathbb{E} \leq \lim \inf_{M \to \infty} \mathbb{E}(M) \) and the limit inf is upper bounded by the RHS of (19). The fact that the bound in (19) does not depend on \( n \) allows us to prove a result on an infinite-lattice multi-hop model.

**Theorem 11.** Consider the multi-hop model of this section defined for the entire lattice \( \mathbb{Z}^d \) and assume that \( \lambda < 1/(2^d + 1) \). Then the process is stable. Moreover, there is a translation-invariant stationary distribution, for which

\[
\mathbb{E}X \leq \frac{\mathbb{E}(\xi^2) + 2^d(1 - q)\lambda + \lambda - 2\lambda^2q^2}{2q \left( \frac{1}{2^d + 1} - \lambda \right)},
\]

where \( X \) has the distribution of \( X_i(k) \) (for any \( i \) and \( k \)) in steady-state.
Remark 5. Since the process is not monotonic, the constructions of [10] cannot be applied. We provide a different construction, based on continuity alone. Note that Theorem [11] does not claim any form of the stationary distribution uniqueness. The uniqueness properties (among the stationary distributions with finite second moments of the queue lengths) derived in [10] and in this paper for the single-hop models, relied in essential way on the process monotonicity.

Proof of Theorem [11]. We already know that for each torus $T_n$ there exists a (unique) stationary distribution of the corresponding process $\bar{X}^{(n)}(\cdot)$. (It is translation-invariant, of course, by symmetry.) We can view this distribution as the distribution on the entire lattice $\mathbb{Z}^d$. Moreover, the uniform in $n$ bound [19] on the expected queue length implies that these distributions (viewed as distributions on the entire lattice) are tight. It is easy to see that the sequence of processes $\bar{X}^{(n)}(\cdot)$ and the process $\bar{X}(\cdot)$ (i.e., the “true” infinite-lattice process) satisfy the continuity property (as in Section [11]). Proceeding analogously to the argument we used in the last two paragraphs of the proof of Theorem [8], we can construct a proper stationary process $\bar{X}(\cdot)$ for the infinite system. The constructed stationary distribution of $\bar{X}(\cdot)$ is a limit of those of $\bar{X}^{(n)}(\cdot)$, and therefore translation-invariant. Finally, (27) follows from (19) and Fatou’s lemma.

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