Effects of entanglement in quantum vortex dynamics

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Abstract

We study the vortex dynamics in the hydrodynamic representation of quantum mechanics for two particles, in which a pointlike vortex comes from a nonintegrable phase of a wavefunction. In particular, we introduce an ansatz for an entangled state of the two particles and discuss how entanglement affects the dynamics of vortex-vortex interaction. For this purpose, we employ the time-dependent quantum variational principle combined with the Rayleigh-Ritz method and elucidate the effects of entanglement on the approximate vortex dynamics.

I Introduction

The hydrodynamic representation of quantum mechanics was originally proposed by Madelung [1], who made a mathematical analogy between the Schrödinger equation and hydrodynamics. The derivation of the hydrodynamic representation is provided by substituting the polar decomposition of a wavefunction for the time-dependent Schrödinger equation in the position representation. This analogy describes virtual flows quantum-mechanically as if it were the theory of fluids in motion. Historically, these works have played a fundamental role in quantum mechanics and continuously attract the interest of physicists [2–9]. One of the attractive properties for the hydrodynamic representation is that it allows to draw some analogies with classical physics and to offer the physically interesting interpretations that differ from the standard formalism of quantum mechanics. In fact, the hydrodynamic representation has been recently discussed from the many viewpoints of, the quantum equilibrium hypothesis [10,11], initial-value problem [12], decomposition of the non-relativistic field velocity [13], Fisher information [14,15], quantum wave packet [16–18], solitary waves [19,20], non-Abelian fluid [21,22], vortex dynamics in quantum trajectories [23–25], convective diffusion in complex hydrodynamics [26], symmetries and conservation laws using the Noether’s theorem [27], quantum teleportation [28], the Klein-Gordon-Einstein equations in a weakly relativistic regime [29], the Navier-Stokes equation for viscous fluid [30], and thermodynamics [31, 32]. In this paper, we provide a quantum-mechanical analogy with dynamics of pointlike vortices in classical two-dimensional flow and further develop the formalism by introducing the concept of quantum entanglement into the vortex dynamics in the hydrodynamic representation.

Entanglement is one of the most remarkable characteristics in quantum mechanics. It appears when two- or multi-particle states are considered, which highlights quantum nonlocality. Its emerging properties have played a vital role as a physical resource in the fields of quantum information and quantum computation. Gaining novel insights into its features is without a doubt of importance in this sense, and perhaps there are undiscovered attractive and exciting aspects of entanglement.

Here, we attempt to find them by establishing a connection between entanglement and vortex dynamics in the hydrodynamic representation, in which a pointlike vortex appears on two-dimensional space when the phase of a wavefunction becomes nonintegrable. In particular, we introduce an idea of a two-particle entangled state describing clockwise and anticlockwise pointlike vortex pairs. This idea raises a nontrivial question of how entanglement affects the dynamics of vortex-vortex interaction. Addressing this question goes beyond the framework for straightforwardly drawing an analogy with classical hydrodynamics, and it is expected to have the interest of shedding further light on the fundamental link between entanglement and quantum-mechanical hydrodynamics. It is, therefore, worthwhile to develop the formalism of vortex dynamics in the hydrodynamic representation, keeping in mind the effects of entanglement.

In this paper, we construct a theory of the dynamics of vortex-vortex interaction for entangled states in the hydrodynamic representation of the two-body Schrödinger theory with nonintegrable phases. For this purpose, we employ the time-dependent variational principle for quantum mechanics developed by Dirac [33,34] combined with the Rayleigh-Ritz method, since a solution of the Schrödinger equation describing such vortex systems cannot be exact. This method allows to treat pointlike vortices as collective coordinates, to derive the
reduced Lagrangian for vortex variables, and to lead to the equations of motion for the coordinates. Based on this variational approach, we discuss the effects of entanglement on the approximate vortex-vortex dynamics. Our main results are to find that entanglement can lead to the nonlinear vortex dynamics and to show that its effects can speed up the dynamical evolution of vortex systems. One of the significant challenges in this paper is to study how entanglement affects the vortex dynamics, whereas several works in the quantum information theory have discussed how quantum dynamics influence entanglement.

II Hydrodynamic representation of quantum mechanics

In this section, we explain an overview of the hydrodynamic representation of quantum mechanics. First, the hydrodynamic representation is based on the polar decomposition of a wavefunction in position representation:

$$\Psi(r, t) = \sqrt{\rho(r, t)} e^{iS(r, t)/\hbar},$$

(1)

where \( r = (x, y, z) \). Here, the amplitude \( \rho(r, t) \) is a positive real function and becomes the probability density that the particle exists at the position \( r \). It is defined by

$$\rho(r, t) = |\Psi(r, t)|^2,$$

(2)

and thus satisfies the normalization condition:

$$\int_{-\infty}^{\infty} d^3 r |\Psi(r, t)|^2 = \int_{-\infty}^{\infty} d^3 r \rho(r, t) = 1.$$

(3)

In addition, the phase \( S(r, t) \) is a real function and has the same dimension as the action. It is defined by

$$S(r, t) = \frac{1}{2i} \log \left( \frac{\Psi(r, t)}{\Psi^*(r, t)} \right).$$

(4)

The wavefunction satisfies the time-dependent Schrödinger equation in position representation

$$i\hbar \frac{\partial \Psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + U(r, t) \Psi(r, t),$$

(5)

where \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \) and \( U(r, t) \) is a time-dependent trapping potential. Henceforth, we consider units where \( \hbar = m = 1 \). Substituting the wavefunction (1) into the Schrödinger equation (5), we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

(6)

$$\frac{\partial S}{\partial t} = -(K + Q + U),$$

(7)

where \( K \equiv \mathbf{u}^2/2 \) and \( Q \equiv -\nabla^2 \sqrt{\rho}/(2\sqrt{\rho}) \) are called the kinetic energy and the quantum potential \( 2 \), respectively. Here, the velocity vector field \( \mathbf{u} \) can be defined by

$$\mathbf{u} \equiv \nabla S.$$

(8)

Note that applying the definition (4) to this velocity vector field yields

$$\mathbf{u} = \nabla \left[ \frac{1}{2i} \log \left( \frac{\Psi}{\Psi^*} \right) \right].$$

(9)

The equation (6) describes the conservation law for probability in quantum mechanics, and the equation (7) is called the quantum Hamilton-Jacobi equation (see 3). Now, taking the gradient of the equation (7), we find

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla (Q + U),$$

(10)

where we use that \( (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla K + (\nabla \times \mathbf{u}) \times \mathbf{u} \) and \( \nabla \times \mathbf{u} = \mathbf{0} \). The obtained two equations (6) and (10) correspond to the mass conservation law and Euler equation in hydrodynamics, respectively.

The hydrodynamic representation of quantum mechanics is interesting for at least four reasons. First, and most important in our opinion, it is strictly based on the position representation of the quantum state. Even if
considering the momentum representation, then we cannot obtain this formulation. Second, the hydrodynamic representation is mathematically equivalent to the time-dependent Schrödinger equation. The equation (10) cannot include the terms that violate the time-reversal invariance of the Schrödinger equation, which leads us to the fact that dissipative terms do not appear there. It follows that the hydrodynamic representation expresses inviscid flow. Third, it describes the virtual fluid that represents the probability current in isolated quantum systems as if it were the theory of fluid dynamics.

Finally, the flow of the velocity vector field is irrotational, i.e., the vorticity natively vanishes:

$$\omega \equiv \nabla \times \mathbf{u} = 0.$$  \hfill (11)

However, this is not the case when the phase of a wavefunction is nonintegrable. Such situations occur at the position where the amplitude of a wavefunction vanishes. These ideas were initially considered by Dirac in his theory of magnetic monopoles [35,36].

### III Wavefunction with nonintegrable phase

In this section, we introduce a pointlike vortex on two-dimensional space into the hydrodynamic representation of the one-body Schrödinger theory. It is well known that the phase of a wavefunction becomes nonintegrable at the position where the amplitude of a wavefunction vanishes. In particular, the position on two-dimensional space is called the nodal point, which can be regarded as a pointlike vortex in the hydrodynamic representation. In this paper, we would like to discuss the dynamics of such pointlike vortices, but it is not sufficient to illustrate it by only using the polar form of the wavefunction. To cope with this situation, we here propose a wavefunction such as having a nodal point that moves in time.

As an example of such a wavefunction, we consider the following wavefunction in the $xy$-plane:

$$\psi(\mathbf{r}, t) \propto x - X(t) + \mathbf{i} \epsilon [y - Y(t)],$$  \hfill (12)

where $\mathbf{r} = (x, y)$, $\epsilon = \pm 1$, and this is not normalized. The amplitude is given by

$$|\psi(\mathbf{r}, t)|^2 \propto [x - X(t)]^2 + [y - Y(t)]^2;$$  \hfill (13)

and therefore vanishes when $x = X(t)$, $y = Y(t)$. Accordingly, this wavefunction has one nodal at $\mathbf{r} = X(t) = (X(t), Y(t))$ and then represents a pointlike vortex. More explicitly, the pointlike vortex can be written as

$$[\nabla \times \mathbf{u}]_x = 2\epsilon \pi \delta^2(\mathbf{r} - X(t)),$$  \hfill (14)

which can be derived by applying to the theory of the Green’s function for Laplace’s equation in two-dimensional space. Here, $\epsilon = \pm 1$ donates the sign of the vortex, which $\epsilon = -1(+1)$ expresses clockwise (anticlockwise). An important consequence of this case is that the pointlike vortex appears at the nodal point in the direction of the virtual $z$-axis perpendicular to the $xy$-plane. Notice that this pointlike vortex is at rest when $X(t) = Y(t) = 0$.

### IV Methods

#### IV.1 Ansatz

In the previous section, we were concerned with the proposal of the wavefunction representing the pointlike vortex in a one-particle quantum system. Here, based on the mentioned wavefunction, we introduce an ansatz for the two-particle entangled state that describes clockwise and anticlockwise pointlike vortex pairs:

$$\Phi_{\text{ansatz}}(\mathbf{r}_1, \mathbf{r}_2, t) = N [\lambda \psi_1 \otimes \phi_2 + (1 - \lambda) \psi_2 \otimes \phi_1],$$  \hfill (15)

where

$$\psi_1(\mathbf{r}_1, t) = \{x_1 - X_1(t) + \mathbf{i} \epsilon_1 [y_1 - Y_1(t)]\} e^{-\alpha(x_1^2 + y_1^2)/2},$$  \hfill (16)

$$\psi_2(\mathbf{r}_1, t) = \{x_1 - X_1(t) + \mathbf{i} \epsilon_2 [y_1 - Y_1(t)]\} e^{-\alpha(x_1^2 + y_1^2)/2},$$  \hfill (17)

$$\phi_1(\mathbf{r}_2, t) = \{x_2 - X_2(t) + \mathbf{i} \gamma_1 [y_2 - Y_2(t)]\} e^{-\alpha(x_2^2 + y_2^2)/2},$$  \hfill (18)

$$\phi_2(\mathbf{r}_2, t) = \{x_2 - X_2(t) + \mathbf{i} \gamma_2 [y_2 - Y_2(t)]\} e^{-\alpha(x_2^2 + y_2^2)/2}. $$  \hfill (19)
Here, the quantum state \( \Phi_{\text{ansatz}} \) defined on Hilbert space \( \mathcal{H}_\psi \otimes \mathcal{H}_\phi \) is entangled if \( \lambda \neq 0 \), and each of unnormalized wavefunctions, \( \psi_1, \phi_1 \), defined on local Hilbert spaces \( \mathcal{H}_\psi, \mathcal{H}_\phi \), is written in the similar form of the wavefunction \( \Phi_{\text{ansatz}} \). In these expressions, \( r_1 = (x_1, y_1), r_2 = (x_2, y_2) \), \( N \) is the normalization factor that will be given below, and
\[
0 \leq \lambda < \frac{1}{2} \tag{20}
\]
donates the entanglement parameter, which we will later precisely explain the reason why we use the symbol of "\(<" rather than the symbol of "\(\leq\". In addition, \( \epsilon_i, \gamma_i = \pm 1, i = 1, 2 \), donate the signs of the pointlike vortices, and moreover, we impose the following condition in order to distinguish between two quantum states:
\[
\epsilon_1 \epsilon_2 = \gamma_1 \gamma_2 = -1. \tag{21}
\]
This condition implies that the rotational directions of the vortices between the wavefunctions \( \psi_1, \psi_2 \) (or \( \phi_1, \phi_2 \)) are opposite. That is, \( \psi_1, \psi_2 \) (or \( \phi_1, \phi_2 \)) are identified by the difference of the signs. Accordingly, quantum entanglement between the two-particle state expressing clockwise and anticlockwise pointlike vortices can be proposed through our ansatz \( \Phi_{\text{ansatz}} \). Furthermore, the factor \( \alpha \) in the Gaussian is a positive real constant and makes the wavefunction normalizable, which is introduced for regularization of the divergent integral. By using the normalization condition
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 r_1 d^2 r_2 |\Phi_{\text{ansatz}}(r_1, r_2, t)|^2 = 1, \tag{22}
\]
we obtain the normalization factor
\[
N = \frac{\alpha/\pi}{\sqrt{\Lambda \left( \frac{1}{\alpha} + X_1^2 + Y_1^2 \right) \left( \frac{1}{\alpha} + X_2^2 + Y_2^2 \right) + \Upsilon \left[ (X_1^2 - Y_1^2) (X_2^2 - Y_2^2) + \mu X_1 Y_1 X_2 Y_2 \right]}} \tag{23}
\]
where
\[
\Lambda = \lambda^2 + (1 - \lambda)^2, \tag{24}
\]
\[
\Upsilon = 2\lambda(1 - \lambda) \tag{25}
\]
\[
\mu = (\epsilon_1 - \epsilon_2)(\gamma_1 - \gamma_2) = \pm 4. \tag{26}
\]
Notice that \( N \) is not a constant because the vortex variables depend on time.

We remark that the form of a wavefunction such as representing pointlike vortices is not unique and would be arbitrary as long as the amplitude vanishes at nodal points. Therefore, several types of quantum-mechanical vortex model in the hydrodynamic representation can be considered. In addition to this, it can be possible to impose some restrictions for three-dimensional position space. However, on the other hand, the main propose in this paper is to make a simple ansatz for the wavefunction describing quantum vortex systems with entangled states, and further find nontrivial results without illustrating such complicated situations. To do so, we require the wavefunction written in the straightforward form such as being able to make analytical calculations.

IV.2 Time-dependent variational approach

Here, based on the ansatz that we previously made, we develop a formulation of vortex-vortex dynamics for two-particle entangled states in the hydrodynamic representation of quantum mechanics with nonintegrable phases. Unfortunately, however, the ansatz cannot be an exact solution of the two-body time-dependent Schrödinger equation describing pointlike vortex systems. The key idea of our study to address this issue is to employ the time-dependent variational principle for quantum mechanics developed by Dirac \[33\] (see \[34\]). This theoretical approach gives us an approximate solution of the Schrödinger equation by proposing a trial wavefunction that represents a physical model intrinsically. It is a powerful and useful tool for our description of the vortex dynamics. Then, we hope that our ansatz is regarded as the trial wavefunction for the Schrödinger equation.

To begin with, let us write the action
\[
I = \int_{t_i}^{t_f} dt \ L, \tag{27}
\]
where \(t_f, t_i\) are the final and initial times, respectively. In the present case, the Lagrangian is given by

\[
L = \int_{-\infty}^{\infty} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \left( i \frac{\partial}{\partial t} - H \right) \Psi^*(\mathbf{r}_1, \mathbf{r}_2, t) \Psi(\mathbf{r}_1, \mathbf{r}_2, t),
\]

where \(H = -\nabla_1^2/2 - \nabla_2^2/2 + U(\mathbf{r}_1, \mathbf{r}_2, t)\), \(U(\mathbf{r}_1, \mathbf{r}_2, t)\) is a time-dependent trapping potential, and \(\nabla_i = (\partial/\partial x_i, \partial/\partial y_i), i = 1, 2\). Here, we emphasize that the two wavefunctions, \(\Psi(\mathbf{r}_1, \mathbf{r}_2, t), \Psi^*(\mathbf{r}_1, \mathbf{r}_2, t)\) are varied independently, and therefore both are different functions each other. It is well known that the time-dependent variational principle for the Schrödinger equation automatically satisfies the normalization constraint and the time boundary conditions.

Let us employ the wavefunction \(\Psi^\dagger(\mathbf{r}_1, \mathbf{r}_2, t)\) with the normalization factor \(\Psi\) as a trial wavefunction in the variational approach combined with the Rayleigh-Ritz method. Substituting it into the above Lagrangian, and performing the straightforward calculations of the integral with respect to \(\mathbf{r}_1, \mathbf{r}_2\) in position space, we arrive at

\[
\mathcal{L} = \frac{E}{\lambda} \left( \frac{1}{\alpha} + X_1^2 + Y_1^2 \right) \left( \dot{X}_1 Y_1 - X_1 \dot{Y}_1 \right) + \frac{\Gamma}{\Lambda} \left( \frac{1}{\alpha} + X_2^2 + Y_2^2 \right) \left( \dot{X}_2 Y_2 - X_2 \dot{Y}_2 \right)
\]

where an irrelevant additive constant has not been included, and the over-dot denotes the time derivative. In this expression, we define

\[
\mathcal{U}(X_1, Y_1, X_2, Y_2, t) = \int_{-\infty}^{\infty} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \Phi^\dagger(\mathbf{r}_1, \mathbf{r}_2, t)U(\mathbf{r}_1, \mathbf{r}_2, t)\Phi(\mathbf{r}_1, \mathbf{r}_2, t),
\]

which is required to be large so that the Gaussian wave packet does not spread.

\[
E = \lambda^2 \epsilon_1 + (1 - \lambda)^2 \epsilon_2,
\]

\[
\Gamma = \lambda^2 \gamma_2 + (1 - \lambda)^2 \gamma_1.
\]

The time-dependent quantum variational approach allows us to treat pointlike vortices as collective coordinates and to obtain the reduced Lagrangian for the vortex coordinates. The advantage of the proposed approach is that it enables us to focus only on the degrees of freedom related to the vortex systems since the others vanish by performing the calculations of the integral. We thus conclude that the Lagrangian form of the Schrödinger equation turns out to be written in the form of, \(L(X_1, Y_1, X_2, Y_2, \dot{X}_1, \dot{Y}_1, \dot{X}_2, \dot{Y}_2)\), by inserting our ansatz as the trial wavefunction into it. What is crucial to understanding this result is that each of the vortex variables, \(X_1, Y_1, X_2, Y_2\), in the trial wavefunction becomes the generalized coordinate in classical phase space. By way of seeing pointlike vortices collectively, we discuss the vortex-vortex dynamics based on the reduced Lagrangian formulation.

Here, it is convenient to assume that the vortex dynamics are localized in the time-dependent trapping potential. To do so, we impose that \(\mathcal{U}\) is approximated to a constant and therefore can be ignored. In addition, the factor \(\alpha\) is required to be large so that the Gaussian wave packet does not spread.

**V Results**

In the previous section, we derived the reduced Lagrangian for the vortex coordinate in two-particle entangled states by using the time-dependent variational principle combined with the Rayleigh-Ritz method. It should be noted now that since the amount of entanglement in the vortex system is represented by the entanglement parameter \(\lambda\), the terms \(\Lambda, E, \Gamma, \) and \(\mathcal{T}\) in the reduced Lagrangian \(\mathcal{L}\) are characterized by entanglement. In this section, based on these terms, we highlight aspects of the influence of entanglement effects on the vortex-vortex dynamics.
V.1 $\lambda \to 1/2$

Here, we focus on a relation between the entanglement parameter and the kinetic term in the reduced Lagrangian. If $\lambda \to 1/2$ [i.e., $E, \Gamma \to 0$], then the kinetic terms natively vanish. This situation raises a question of what physical meanings does this vanishing imply? Making a discussion about such a particular quantum state is important for at least two implications.

The first point to understanding the limit $\lambda \to 1/2$ is from the viewpoint of the variational approach. To begin with, we write the complex conjugate of the wavefunction $\Phi^*_{\text{ansatz}} = N [\lambda \psi_1^* \otimes \phi_2^* + (1 - \lambda) \psi_2^* \otimes \phi_1^*]$. Using the condition (21), we find that $\psi_1 = \psi_2^*$ and $\phi_1 = \phi_2^*$, and therefore have

$$\Phi^*_{\text{ansatz}} = N [\lambda \psi_2 \otimes \phi_1 + (1 - \lambda) \psi_1 \otimes \phi_2]. \quad (33)$$

Thus, if $\lambda = 1/2$, then it becomes a real function:

$$\Phi^*_{\text{ansatz}} = \Phi_{\text{ansatz}}. \quad (34)$$

Since the wavefunction must be complex when we use the time-dependent quantum variational principle, it turns out that the wavefunction (15) is inappropriate as a trial function in the limit $\lambda \to 1/2$. As a result, we cannot employ the time-dependent variational approach in this situation. This statement is the reason why we avoided $\lambda = 1/2$ in the previous section.

The second point is from the von Neumann entropy on the subsystems as an entanglement measure of pure two-particle entangled quantum systems. The von Neumann entropy becomes large as $\lambda$ approaches 1/2 and has a local maximum $\lambda = 1/2$ (see appendix). We thus conclude that one cannot use the time-dependent quantum variational principle when the quantum state in vortex systems is most entangled.

V.2 Nonlinear effects

V.2.1 Hamiltonian formalism

Here, we discuss nonlinear effects on the vortex dynamics in the entangled states. Note now that the terminology of vortex dynamics means the equations of motion for the vortex coordinates and they are obtained by using the Euler-Lagrange equations calculated from the reduced Lagrangian. To analyze the vortex-vortex dynamics, calculating the Euler-Lagrange equations for the vortex variables $X_1, Y_1, X_2,$ and $Y_2$, we, however, can find that they become very complicated nonlinear. On the other hand, if $\lambda = 0$, then they turn out to be simply linear, especially to behave like a harmonic oscillator. From the above discussion, we can find that entanglement can lead to nonlinear vortex-vortex dynamics.

A different way of understanding this nonlinearity is to move on to the Hamiltonian formalism. With the help of the Legendre transformation, the reduced Hamiltonian for the vortex coordinates is written as

$$H = \dot{X}_1 p_{X_1} + \dot{Y}_1 p_{Y_1} + \dot{X}_2 p_{X_2} + \dot{Y}_2 p_{Y_2} - L$$

$$= \frac{1}{2} \Lambda \left( \frac{1}{\alpha} + X_1^2 + Y_1^2 \right) \left( \frac{1}{\beta} + X_2^2 + Y_2^2 \right) + \Gamma \left( (X_1^2 - Y_1^2) (X_2^2 - Y_2^2) + \mu X_1 Y_1 X_2 Y_2 \right) \quad (35)$$

where the canonical momenta are $p_{X_1}, p_{Y_1}, p_{X_2},$ and $p_{Y_2}$, respectively. Now, setting the angular momenta as $s_1 = X_1 p_{Y_1} - Y_1 p_{X_1}$, and $s_2 = X_2 p_{Y_2} - Y_2 p_{X_2}$, we have

$$H = \frac{\alpha \Lambda}{2} (E s_1 + \Gamma s_2 - 2 g s_1 s_2), \quad (36)$$

where a position-dependent coupling coefficient $g$ is defined by

$$g(X_1, Y_1, X_2, Y_2, \lambda)$$

$$= \frac{\Lambda \left( \frac{1}{\alpha} + X_1^2 + Y_1^2 \right) \left( \frac{1}{\beta} + X_2^2 + Y_2^2 \right) + \Gamma \left( (X_1^2 - Y_1^2) (X_2^2 - Y_2^2) + \mu X_1 Y_1 X_2 Y_2 \right)}{E \Gamma (X_1^2 + Y_1^2) (X_2^2 + Y_2^2)}. \quad (37)$$

Note here that this transformation can be not the simple change of variables but the canonical transformation, since the angular momenta $s_1$ and $s_2$ are the canonical variables. In fact, it is known that the angular momenta are the canonical conjugate to the phases $\theta_1 = \tan^{-1}(Y_1/X_1)$ and $\theta_2 = \tan^{-1}(Y_2/X_2)$, respectively. Thus, we can mention that there exists the function $f(s_1, \theta_1, s_2, \theta_2, \lambda)$, such that it is equal to $g(X_1, Y_1, X_2, Y_2, \lambda)$, though it can not explicitly represented.
We now emphasize that the reduced Hamiltonian (36) is in analogy with the Hamiltonian in quantum spin models. In particular, if \( g < 0 \) (\( g > 0 \)) [i.e., \( \epsilon_1 = -\gamma_2 \) (\( \epsilon_1 = +\gamma_2 \))], then the vortex systems in entangled quantum states seem to be characterized as the Ferro-coupling (Antiferro-coupling). However, on the other hand, if \( \lambda = 0 \), then the coupling term \(-2g\xi_1\xi_2\) in the reduced Hamiltonian becomes independent of the vortex coordinates, and therefore, the angular momenta are invariant in time, which is consistent with the result of the harmonically oscillating vortex dynamics. As a result, it turns out that the vortex dynamics strongly depends on the entanglement parameter, and further, we can conclude that the effects of entanglement lead to the nonlinear vortex dynamics.

V.2.2 Relation between \( E \) and \( \Gamma \)

Here, we discuss a relation between the terms \( E \) and \( \Gamma \), where \( E = \lambda^2\epsilon_1 + (1-\lambda)^2\epsilon_2 \) and \( \Gamma = \lambda^2\gamma_2 + (1-\lambda)^2\gamma_1 \). To illustrate the relation as clear as possible, let us perform the transformation from \((X_2, Y_2)\) to \((-X_1, -Y_1)\).

The reduced Lagrangian (29) thus becomes a simple form

\[
L = \frac{\left(\frac{1}{\alpha} + X^2 + Y^2\right)\left[\left(E + \Gamma\right)(\dot{X}Y - XY\dot{Y}) - \Lambda\right]}{\Lambda\left(\frac{1}{\alpha} + X^2 + Y^2\right)^2 + \gamma\left[(X^2 - Y^2)^2 + \mu X^2Y^2\right]},
\]

where we take out the notation of subscript of \((X_1, Y_1)\). If \( E = -\Gamma \) [i.e., \( \epsilon_1 = -\gamma_2 \)], then the kinetic term natively vanishes, and therefore, the vortex dynamics arrives at static systems. Note here that in this case, the trial wavefunction would remain complex unless \( \lambda \to 1/2 \). On the other hand, however, if \( E = +\Gamma \) [i.e., \( \epsilon_1 = +\gamma_2 \)], then the Euler-Lagrange equations can be also nonlinear. From the above discussion, we find the result that the difference between the signs of the vortices also can decide whether the vortex-vortex dynamics becomes nonlinear behavior or not. We remark that since such situations will not happen unless the entangled state is considered, this exciting result also can come from the sort of entanglement effects. Hence, we can conclude that the vortex-vortex dynamics in entangled states can be involved with not only the entanglement parameter but also the signs of the vortices.

V.3 Dirac’s generalized canonical formulation

Here, we develop the theory of vortex dynamics with entangled states by using the Dirac’s generalized canonical formulation. To discuss as explicit as possible, we first fix the vortex coordinate \((X_2, Y_2)\) at the origin \((0, 0)\).

The reduced Lagrangian (29) thus becomes a simple form

\[
L = \frac{E\alpha(\dot{X}Y - XY\dot{Y})}{\Lambda[1 + \alpha(X^2 + Y^2)]} - \frac{\alpha}{2[1 + \alpha(X^2 + Y^2)]},
\]

where we also take out the notation of subscript of \((X_1, Y_1)\). Let us show that both the vortex variables \(X\) and \(Y\) are canonical conjugates to each other. To see this in more detail, we perform the point transformation from \((X, Y)\) to dimensionless variables \((\xi, \eta)\):

\[
\xi = X \sqrt{\frac{2\alpha E}{\Lambda[1 + \alpha(X^2 + Y^2)]}},
\]

\[
\eta = Y \sqrt{\frac{2\alpha E}{\Lambda[1 + \alpha(X^2 + Y^2)]}}.
\]

Now without loss of generality, we determine the signs of the vortices as follows: \( \epsilon_1 = -1 \), \( \epsilon_2 = +1 \), and therefore \( E = 1 - 2\lambda > 0 \). This transformation is regular since the Jacobian does not vanish: \( \partial(\xi, \eta)/\partial(X, Y) = 2\alpha E/[\Lambda\left[1 + \alpha(X^2 + Y^2)\right]^2] \neq 0 \). According to the identical equation, \( 1 + \alpha(X^2 + Y^2) = 2E/[2E - \Lambda(\xi^2 + \eta^2)] \), we have that \( \xi^2 + \eta^2 < 2E/\Lambda \). The reduced Lagrangian (39) is thus transformed to

\[
L = \frac{\dot{\xi}\eta - \dot{\eta}\xi}{2} + \frac{\alpha\Lambda}{4E}(\xi^2 + \eta^2),
\]

where an irrelevant additive constant has been included.
Applying the reduced Lagrangian formulation to the Dirac’s generalized canonical formalism \[38\], we demonstrate the canonical conjugate nature of the canonical coordinates \(\xi\) and \(\eta\) (see appendix):

\[
\{\xi, \eta\}_D = 1,
\]

(43)

where \(\{,\}_D\) is the Dirac bracket. The relation (43) is in analogy with the one for the system of vortices in classical hydrodynamics \[37\]. With the help of the Legendre transformation, the reduced Hamiltonian is written as

\[
\begin{align*}
H &= \dot{\xi} p_\xi + \dot{\eta} p_\eta - L \\
&= -\alpha \Lambda (\xi^2 + \eta^2) / 4E.
\end{align*}
\]

(44)

Since the Hamilton’s equations are written by \(\dot{f} = \{f, H\}_D\), we thus have

\[
\begin{align*}
\dot{\xi} &= -\kappa \alpha \eta, \\
\dot{\eta} &= \kappa \alpha \xi,
\end{align*}
\]

(45) (46)

where \(\kappa = \Lambda / 2E > 0\) is a constant. It is evident that the equations (45) and (46) are the same as the Euler-Lagrange equations calculated from the reduced Lagrangian (42). Let us compare our results with the classical two-dimensional flow. The term \(\xi^2 + \eta^2\) is a constant of motion, which reminds one of the Hamiltonian-like formalism of classical vortex systems \[37\]. In classical hydrodynamics, the Hamiltonian-like function is the logarithm form of the constant of motion, whereas in the present case, the reduced Hamiltonian, \(H \propto \xi^2 + \eta^2\), is not the logarithm of the constant of motion.

Now, we analyze the vortex dynamics with entangled states. From the Euler-Lagrange equations for the vortex variables \(X, Y\) in the reduced Lagrangian (39), we find that the vortex coordinates act as the harmonic oscillator: \(\ddot{X}(t) + \omega^2 X(t) = 0\) and \(\ddot{Y}(t) + \omega^2 Y(t) = 0\), where the angular frequency is define by \(\omega = \kappa \alpha\). Let us mention an interesting relation to the angular frequency. If \(\lambda = 0\), then the angular frequency \(\omega\) is equal to \(\alpha / 2\), whereas if \(\lambda \neq 0\), then it monotonically increases as the entanglement parameter becomes large. This result implies that the effects of entanglement can lead to the rotational speed up in vortex systems, that is, stronger entanglement can yield that the vortex dynamics can evolve faster in time.

VI Conclusion

In this paper, we have considered the hydrodynamic representation of quantum mechanics for two particles with nonintegrable phases and have introduced the ansatz for the wavefunction that represents pointlike vortices in entangled states. Based on the time-dependent variational principle combined with the Rayleigh-Ritz method, we have derived the reduced Lagrangian for the vortex variables and have discussed the role of entanglement on the approximate vortex-vortex dynamics. By considering some limitations, we have discussed that entanglement leads us to nonlinear vortex dynamics, which can contain a seed of potentially interesting problem from mathematically and physically viewpoints. We have also found that that strong entanglement can be related to the rotational speed up in vortex systems, which can go beyond the straightforward analogy with classical hydrodynamics. Moreover, we have developed the formulation of vortex dynamics by using Dirac’s generalized Hamiltonian formalism.

Appendix

The von Neumann entanglement entropy and \(\lambda = 1/2\)

Here, we show that the von Neumann entanglement entropy has a local maximum \(\lambda = 1/2\), which implies that its \(\lambda\)-derivation yields the condition \(\lambda = 1/2\). To show this, we begin by defining the density matrix of the trial wavefunction:

\[
\rho = |\Phi_{\text{ansatz}}\rangle \langle \Phi_{\text{ansatz}}|,
\]

(47)
where the state $|\Phi_{\text{ansatz}}\rangle = N[\lambda|\psi_1\rangle \otimes |\phi_2\rangle + (1-\lambda)|\psi_2\rangle \otimes |\phi_1\rangle]$ becomes the wavefunction (15) when the position representation is considered. Then, the reduced density matrix on the subsystem is given by

$$\rho_\psi = \text{tr}_\phi \rho = a_1 |\psi_1\rangle \langle \psi_1 | + a_2 |\psi_2\rangle \langle \psi_2 | + b |\psi_1\rangle \langle \psi_2 | + b^* |\psi_2\rangle \langle \psi_1 |. \quad (48)$$

where

$$a_1 = \frac{N^2 \lambda^2 \pi}{\alpha} \left( \frac{1}{\alpha} + X_2^2 + Y_2^2 \right), \quad (49)$$

$$a_2 = \frac{N^2 (1-\lambda)^2 \pi}{\alpha} \left( \frac{1}{\alpha} + X_2^2 + Y_2^2 \right), \quad (50)$$

$$b = \frac{N^2 \lambda (1-\lambda) \pi}{\alpha} [X_2^2 - Y_2^2 - i(\gamma_1 - \gamma_2)X_2Y_2]. \quad (51)$$

Considering the inverse unitary diagonalization

$$\rho_\psi = U^\dagger \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix} U, \quad (52)$$

where $U$ is the unitary matrix and

$$p_\pm = \frac{a_1 + a_2 \pm \sqrt{(a_1-a_2)^2 + 4|b|^2}}{2}, \quad (53)$$

we have that

$$S[\rho_\psi] = -\text{tr}_\psi (\rho_\psi \ln \rho_\psi) \quad (54)$$

where the von Neumann entropy is invariant under the unitary transformations. Calculating the $\lambda$-derivation of the entanglement entropy

$$\frac{\partial S[\rho_\psi]}{\partial \lambda} = 0, \quad (55)$$

we immediately obtain the condition $\lambda = 1/2$. Therefore, we find that the entanglement entropy has a local maximum at the most entangled state. Note that the Araki-Lieb formula leads us the following equality:

$$S[\rho_\psi] = S[\rho_\rho]. \quad (56)$$

**Canonical conjugate nature of $\xi$ and $\eta$**

Here, we show the canonical conjugate nature of the canonical coordinates $\xi$ and $\eta$. Now, the canonical momenta are given by $p_\xi = \partial L/\partial \dot{\xi} = \eta/2$, and $p_\eta = \partial L/\partial \dot{\eta} = -\xi/2$. Then they lead to the weak constraints: $\chi_\xi = p_\xi - \eta/2 \approx 0$, $\chi_\eta = p_\eta + \xi/2 \approx 0$, where the symbol “$\approx$” is called the weak equality. It is considered here that, the constraints are the inner ones that come from the structure of the reduced Lagrangian itself, and then, in principle, the dynamics of the constrained vortex systems can be described by the two canonical variables $\xi$, $\eta$. The Poisson bracket is now written as

$$\{A, B\}_P = \sum_{\xi, \eta} \left( \frac{\partial A}{\partial \xi} \frac{\partial B}{\partial p_\xi} - \frac{\partial A}{\partial p_\xi} \frac{\partial B}{\partial \xi} \right), \quad (57)$$

where $A$ and $B$ are the functions of $(\xi, p_\xi, \eta, p_\eta)$. Therefore, we have that $\{\chi_\xi, \chi_\eta\}_P = -1$, which does not vanish. Then, we propose that $\chi_\xi$ and $\chi_\eta$ are the second-class constraints in Dirac’s generalized canonical formalism [58]. In the present case, it is standard to apply not the Poisson bracket but the Dirac bracket defined by

$$\{A, B\}_D = \{A, B\}_P - \sum_{a,b=\xi,\eta} \{A, \chi_a\}_P C^{-1}_{ab} \{\chi_b, B\}_P. \quad (58)$$

Here, $C_{ab}$’s are matrix elements satisfying $C_{ab} = \{\chi_a, \chi_b\}_P$. That is, $C_{\xi \eta} = -C_{\eta \xi} = 0$, and $C_{\xi \xi} = -C_{\eta \eta} = -1$. Note that the second-class constraints are the identical equations in the Dirac bracket. Hence, we have that

$$\{\xi, \eta\}_D = 1. \quad (59)$$
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