Coordination in distributed networks via coded actions with application to power control

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Abstract

This paper investigates the problem of coordinating several agents through their actions. Although the methodology applies to general scenarios, the present work focuses on a situation with an asymmetric observation structure that only involves two agents. More precisely, one of the agents knows the past, present, and future realizations of a state (the system state) that affects the common payoff function of the agents; in contrast, the second agent is assumed either to know the past realizations of the system state or to have no knowledge of it. In both cases, the second agent has access to some strictly causal observations of the first agent’s actions, which enables the two agents to coordinate. These scenarios are applied to the problem of distributed power control; the key idea is that a transmitter may embed information about the wireless channel state into its transmit power levels so that an observation of these levels, e.g., the signal-to-interference plus noise ratio, allows the other transmitter to coordinate its power levels. The main contributions of this paper are twofold. First, we provide a characterization of the set of feasible average payoffs when the agents repeatedly take long sequences of actions and the realizations of the system state are i.i.d.. Second, we exploit these results in the context of distributed power control and introduce the concept of coded power control. We carry out an extensive numerical analysis of the benefits of coded power control over alternative power control policies, and highlight a simple yet non-trivial example of a power control code.

Index Terms

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Channels with state; coding theorems; coordination; distributed power control; distributed resource allocation; game theory; information constraints; interference channel; optimization.

I. INTRODUCTION

The main technical problem studied in this paper is the following. Given an integer $T \geq 1$, three discrete alphabets $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$, and a stage payoff function $w : \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{R}$, one wants to maximize the average payoff

$$W_T(x_0^T, x_1^T, x_2^T) \triangleq \frac{1}{T} \sum_{t=1}^{T} w(x_{0,t}, x_{1,t}, x_{2,t}) \quad (1)$$

with respect to the sequences $x_1^T \triangleq (x_{1,1}, \ldots, x_{1,T}) \in \mathcal{X}_1^T$ and $x_2^T \triangleq (x_{2,1}, \ldots, x_{2,T}) \in \mathcal{X}_2^T$ given the knowledge of $x_0^T \triangleq (x_{0,1}, \ldots, x_{0,T}) \in \mathcal{X}_0^T$. Without further restrictions and with instantaneous knowledge of $x_{0,t}$, solving this optimization problem consists in finding one of the optimal pairs of variables $(x_{1,t}^*, x_{2,t}^*)$ for every $t$. The corresponding maximum value of $W_T$ is then

$$W_T^* = \frac{1}{T} \sum_{t=1}^{T} \max_{x_1,t, x_2} w(x_{0,t}, x_{1,t}, x_{2,t}). \quad (2)$$

In this paper, we introduce the additional restriction that the variable $x_2$ cannot be controlled or optimized directly. As formally described in Section II, the variable $x_2$ is required to result from imperfect observations of $x_0$ through $x_1$, which induces an information constraint in the aforementioned optimization problem. One of the main results of this paper in Section III is to precisely characterize this constraint for large $T$ when $x_0^T$ consists of independent identically distributed (i.i.d.) realizations of a given random variable $X_0$.

The problem at hand is a special case of a distributed optimization problem, in which $K$ agents connected via a given observation structure have the common objective of maximizing the average payoff $W_T$ for large $T$. In this general framework, the variable $x_k$ with $k \in \{1, \ldots, K\}$ is called the action of Agent $k$ and represents the only variable under its control. The variable $x_0$ is outside of the agents’ control and typically represents the realization of a random state (called the system state). The observation structure defines how the agents interact through observations of the random system state and of each other’s actions. The average payoff then measures the degree of coordination between the agents, under

1In this paper, this scenario will be referred to as the costless communication case. In Section VI the corresponding power control scenario will be called costless communication power control (CCPC).

2In other disciplines such as computer science, control, or economics, agents are sometimes called nodes, controllers, or decision-makers.
the observation constraints of the actions imposed by the observation structure. As a concrete example, we apply this framework to power control in Section V in which $x_0$ represents the global wireless channel state information and $x_k$ the power level of Transmitter $k$.

A central question in this general framework is to characterize the possible values of the average payoff $W_T$ when the agents interact many times, i.e., when $T$ is large. Since answering this question in its full generality still appears out of reach, the present paper tackles a special case with $K = 2$ agents. Specifically, we assume that Agent 1 has perfect knowledge of the past, current, and future realizations of the system state sequence $x_T^0$, while Agent 2 obtains imperfect and strictly causal observations of Agent 1’s actions and possesses either strictly causal or no knowledge of the realizations of the system state. In spite of these restricting assumptions, it is possible to extract valuable concepts and insights of practical interest from the present work, which can be applied to the general case of $K \geq 2$ agents and arbitrary observation structures.

A. Related work

In most of the literature on coordination among agents, which includes classical team decision problems [4], the typical assumption is that agents have access to dedicated channels to coordinate their actions. Specifically, these dedicated channels allow the agents to signal or communicate with each other without affecting the objective or payoff function. The works on coordination with dedicated channels that relate most closely to the present work are [5, 6]. Therein, the authors introduce the notions of empirical coordination and strong coordination to measure the ability of agents to coordinate their actions in a network with noiseless dedicated channels. Empirical distribution measures an average coordination behavior over time and requires the joint empirical distribution of the actions to approach a target distribution asymptotically in variational distance; this notion relates to an earlier work on the communication of probability distribution [7], and is analyzed using tools from rate-distortion theory. Strong coordination is more stringent and requires that the distribution of sequences of actions be asymptotically indistinguishable from sequences of actions drawn according to a target distribution, again in terms of variational distance; this notion relates to the concept of channel resolvability [8]. In both cases, the goal is to establish the coordination capacity, which relates the achievable joint distributions of actions to the fixed rate constraints on the noiseless dedicated channels. The results of [5, 6] have been extended to a variety of networks with dedicated channels [9, 10, 11, 12], and optimal codes have been designed for specific settings [13, 14].

In contrast, much less is known about the problem of coordination via the actions of agents in the
absence of dedicated channels, which is the main topic of the present work. The most closely related work is [15], in which the authors analyze the problem of characterizing the set of possible average payoffs in the presence of two agents, assuming that each agent can perfectly monitor the actions of the other agent; the authors establish the set of *implementable distributions*, which are the achievable empirical joint distributions of the actions under the assumed observation structure. In particular, this set is characterized by an information constraint that captures the observation structure between the agents. While [15] largely relies on combinatorial arguments, [16] provides a more traditional information-theoretic approach of coordination via actions under the name of *implicit communication*.

To the best of our knowledge, the present work is the first to apply the framework of coordination through actions to the problem of distributed resource allocation in wireless networks. More specifically, the application developed in this paper is a problem of distributed power control for an interference channel, and for the multiple-access channel as a specific instance. A substantial line of works in the literature of distributed power control exploits game-theoretic tools to design power control schemes and to analyze their performance. One of such schemes is the iterative water-filling algorithm [17], which is a special instance of the best-response dynamics (BRD), and is applied over a time horizon over which the wireless channel state is constant. One of the main drawbacks of the various implementations of the BRD for power control problems, see e.g., [18], [19], [20], is that they tend to converge to Nash-equilibrium power control (NPC) policies. The latter are typically Pareto-inefficient, meaning that there exist some schemes which would allow all the agents to improve their individual utility with respect to (w.r.t.) the NPC policies. Another drawback is that such iterative schemes do not always converge. Only sufficient conditions for convergence are available, see e.g., [21] for the case of multiple input multiple output (MIMO) interference channels, and those conditions are sometimes very restrictive and even met with probability zero for some important special cases such as the parallel multiple-access channels [22]. In contrast, one of the main benefits of *coded power control*, which we develop in Section V is precisely to obtain efficient operating points for the network. This is made possible by exchanging among the transmitters information about the quality of the communication links, and a key observation made in this paper is that this exchange can be achieved through observed quantities such as the signal-to-interference plus noise ratio (SINR). In other words, the SINRs of the different users can be viewed as the outputs of a channel over which transmitters communicate to coordinate their actions, provided they appropriately encode them. Coding appears because a transmitter maps several realizations of the wireless channel state into a sequence of power levels, which then allows other transmitters to exploit the corresponding sequence of their SINRs to select their power levels. Since coding is used, no iterative procedure is
required and convergence issues are therefore avoided. Since this paper focuses on efficiency, NPC will be compared to coded power control in terms of average sum-rate; other aspects such as wireless channel state information availability and complexity should also be considered but are deferred to future work.

B. Main contributions

The main contributions of the present work are as follows.

- First, the results reported in Section III extend [15] by relaxing some assumptions about the observation structure. Specifically, while [15] assumes that Agent 2 perfectly monitors the actions of Agent 1, we consider the case of imperfect monitoring. In addition we analyze both a situation similar to [15] in which Agent 2 has a strictly causal knowledge of the system state (Theorem 8) and a situation in which Agent 2 knows nothing about the system state (Theorem 14).

- Second, we clarify the connections between the game-theoretic formulation of [15] and information-theoretic considerations. In particular, links with coding theorems of the literature on state-dependent channels [23], [24], [25], [26], [27], separation theorems, and contributions on empirical coordination [5], [28] are discussed. In addition, the problem of determining the value of the long-run average payoff is formulated as an optimization problem, which is studied in detail in Section IV and exploited for the application to power control in Section V. This allows us to conduct a thorough numerical analysis for the problem of distributed power control and to effectively assess the potential benefits of the proposed approach.

- Third, we establish a bridge between the problem of coordination via actions and the problem of power control in wireless networks. The present work develops a new perspective on resource allocation and control, in which designing a good resource allocation or control policy with high average common payoff amounts to designing a code. A good resource allocation or control code has to implement a trade-off between sending information about the upcoming realizations of the system state, which play the role of messages and are required to obtain high payoff in the future, and achieving a good value for the current value of the payoff. As an illustration, Section V-E provides the complete description of a power control code for the multiple-access channel.

II. Problem statement

For the reader’s convenience, we provide a non-exhaustive summary of the notation used throughout this paper in Table I.
We now formally introduce the problem studied in the remaining of the paper. We consider \( K = 2 \) agents that have to select their actions repeatedly over \( T \geq 1 \) stages or time-slots and wish to coordinate via their actions in the presence of a random system state and with an observation structure detailed next. At each stage \( t \in \{1, \ldots, T\} \), the action of Agent \( k \in \{1, 2\} \) is \( x_{k,t} \in \mathcal{X}_k \) with \( |\mathcal{X}_k| < \infty \), while the realization of the random system state is \( x_{0,t} \in \mathcal{X}_0 \) with \( |\mathcal{X}_0| < \infty \). The realizations of the system state are assumed to be i.i.d. according to a random variable \( X_0 \) with distribution \( \rho_0 \). The random system state does not depend on the agents’ actions but affects a common agents’ payoff function \( w : \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R} \). Coordination is measured in terms of the average payoff \( W_T(x_0^T, x_1^T, x_2^T) \) as defined in (1). We assume that, at stage \( t \), Agent 2 only has access to imperfect observations \( y_t \in \mathcal{Y} \) of Agent 1’s actions with \( t \in \{1, \ldots, T\} \) and \( |\mathcal{Y}| < \infty \). Specifically, the observations \( y_T = \{y_1, \ldots, y_T\} \) are the output of a memoryless channel with transition probability

\[
P(y^T|x_0^T, x_1^T, x_2^T) = \prod_{t=1}^{T} \Gamma(y_t|x_{0,t}, x_{1,t}, x_{2,t})
\]  

for some conditional probability \( \Gamma \). We consider two asymmetric observation structures that restrict how agents observe the system state and each other’s actions. We characterize these structures through the strategies \( (\sigma_t)_{1 \leq t \leq T} \) and \( (\tau_t)_{1 \leq t \leq T} \) of Agents 1 and 2, respectively, which are sequences of mappings defined for all stage \( t \in \{1, \ldots, T\} \) as follows:

\[
\begin{align*}
\text{case I:} & \quad \begin{cases}
\sigma^I_t & : \mathcal{X}_0^T \rightarrow \mathcal{X}_1 \\
\tau^I_t & : \mathcal{X}_0^{t-1} \times \mathcal{Y}_1^{t-1} \rightarrow \mathcal{X}_2
\end{cases} \\
\text{case II:} & \quad \begin{cases}
\sigma^II_t & : \mathcal{X}_0^T \rightarrow \mathcal{X}_1 \\
\tau^II_t & : \mathcal{Y}_1^{t-1} \rightarrow \mathcal{X}_2
\end{cases}
\end{align*}
\]

(4) and (5).

Our objective is to characterize the set of average payoffs which are feasible asymptotically i.e., the possible values for \( \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} w(x_{0,t}, x_{1,t}, x_{2,t}) \) under the observation structures defined through (4) and (5). The definition of the two corresponding feasible sets is as follows.

**Definition 1** (Feasible sets of payoffs). The feasible set of payoffs in case I is defined as

\[
\Omega^I = \left\{ \omega \in \mathbb{R} : \exists (\sigma^I_t, \tau^I_t)_{1 \leq t \leq T}, \omega = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} w(x_{0,t}, \sigma^I_t(x_0^T), \tau^I_t(x_0^{t-1}, y_1^{t-1})) \right\}.
\]

The feasible set of payoffs in case II is defined as

\[
\Omega^{II} = \left\{ \omega \in \mathbb{R} : \exists (\sigma^II_t, \tau^II_t)_{1 \leq t \leq T}, \omega = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} w(x_{0,t}, \sigma^II_t(x_0^T), \tau^II_t(y_1^{t-1})) \right\}.
\]

(6) and (7).
The feasible sets of payoffs are directly related to the set of implementable empirical joint distributions over
\[ \mathcal{X} \triangleq \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2, \]
deﬁned as follows.

**Deﬁnition 2 (Implementability).** For \( \ell \in \{I, II\} \), the probability distribution \( \overline{Q}(x_0, x_1, x_2) \) is implementable if there exists a pair of strategies \( (\sigma^\ell_t, \tau^\ell_t)_{1 \leq t \leq T} \) inducing at each stage \( t \) a joint distribution
\[ P_{X_{0,t},X_{1,t},X_{2,t},Y_t}(x_0, x_1, x_2, y) \triangleq \Gamma(y|x_0, x_1, x_2)P_{X_{1,t},X_{2,t}|X_{0,t}}(x_1, x_2|x_0)\rho_0(x_0), \]
such that for all \( (x_0, x_1, x_2) \in \mathcal{X} \)
\[ \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{y \in Y} P_{X_{0,t},X_{1,t},X_{2,t},Y_t}(x_0, x_1, x_2, y) = \overline{Q}(x_0, x_1, x_2). \]

Each feasible set of payoffs is the linear image of the corresponding set of implementable distributions under the expectation operator. Therefore, a certain value, say \( \omega \), is feasible (asymptotically) if and only if there exists an implementable distribution \( \overline{Q} \) such that
\[ \omega = E_{\overline{Q}}[w] = \sum_{x_0,x_1,x_2} \overline{Q}(x_0, x_1, x_2)w(x_0, x_1, x_2). \]

In the next section, we focus on the characterization of the implementable distributions rather than the direct characterization of the feasible sets payoffs.

The notion of implementable distribution may also be connected to the notion of achievable empirical distribution \([5]\) upon introducing the type of the sequences of actions.

**Deﬁnition 3 (Type \([29]\)).** Let \( T \geq 1 \). For any sequence of realizations \( z^T \) of the generic random variable \( Z \), the type of \( z^T \), denoted by \( T_{z^T} \), is the probability distribution on \( Z \) deﬁned by
\[ T_{z^T}(z) \triangleq \frac{N(z|z^T)}{T} = \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{z_t = z\}}, \]
where the notation \( N(z|z^T) \) stands for the number of positions of \( z^T \) having the letter \( z \).

By denoting \( T_{X^T} \) the distribution obtained by constructing the histogram of the sequence \( x^T = (x_{0,1}, x_{1,1}, x_{2,1}, \ldots, x_{0,T}, x_{1,T}, x_{2,T}) \), it follows from Deﬁnition \([2]\) that a distribution \( \overline{Q} \in \Delta(\mathcal{X}) \) is implementable if there exists a sequence of strategies \( (\sigma^\ell_t, \tau^\ell_t)_{1 \leq t \leq T}, \ell \in \{I, II\} \), that generate, together with the sequence \( x^T_0 \), the sequence \( x^T \in \mathcal{X} \) such that
\[ \lim_{T \to \infty} ||E(T_{X^T}) - \overline{Q}||_1 = 0, \]
i.e., the average histogram of a sequence of actions is arbitrarily close to the distribution \( \overline{Q} \).

October 28, 2014
Definition 4 (Empirical coordination [5]). For \( \ell \in \{I, II\} \), a distribution \( \overline{Q} \) on \( \mathcal{X} \) is an achievable empirical coordination if there exists a sequence of strategies \((\sigma^\ell_t, \tau^\ell_t)_{1 \leq t \leq T}\) that generate, together with the sequence \(x^T_0\), the sequence \(x^T \in \mathcal{X}\) such that

\[
\forall \epsilon > 0, \quad \lim_{T \to \infty} P(||T_{X^T} - \overline{Q}||_1 > \epsilon) = 0,
\]

i.e., the distance between the histogram of a sequence of actions and \( \overline{Q} \) converges in probability to 0.

Proposition 5. If \( \overline{Q} \in \Delta(\mathcal{X}) \) is an achievable empirical coordination, then it is implementable.

The proof of Proposition 5 is provided in Appendix A.

Before analyzing the considered coordination problem in details in Section III, a few important comments are in order. In [15], the authors assume that the strategies are given by the two sequences of mappings \( \sigma_t : \mathcal{X}^T_0 \times \mathcal{X}^{t-1}_1 \times \mathcal{X}^{t-1}_2 \to \mathcal{X}_1 \) and \( \tau_t : \mathcal{X}^{t-1}_0 \times \mathcal{X}^{t-1}_1 \times \mathcal{X}^{t-1}_2 \to \mathcal{X}_2 \). In particular, this means that Agent 2 perfectly monitors the actions of Agent 1 i.e., \( Y = X_1 \). One of our results is that, under perfect monitoring, there is no loss in terms of feasibility by considering strategies given by [5] instead of those assumed in [15]. However, the set of possible Nash equilibrium points obtained with [5] may not coincide with the one associated with the strategies of [15]. Characterizing the set of possible Nash equilibrium points is relevant when agents have diverging interests. It turns out that when this is the case, the fact that an agent can observe the actions of the other agents matters. In a power control setting for instance, if the transmitters implement a cooperation plan that consists in transmitting at low power as long as no transmitter uses a high power level, see e.g., [30], it matters for the transmitters to be able to check whether the other transmitters effectively use a low power level.

Note that the definition of the strategies are not block strategies, as assumed for conventional coding and decoding. Here, an agent acts at every stage, and the considered strategies can be seen as joint source-channel coding strategies with an online coding and decoding requirement.

Finally, we emphasize that the strategy for Agent 1 assumes a non-causal knowledge of the system state. Such assumptions have been largely used in the information theory literature, going back to the work of Gel’fand and Pinsker [23] on coding over state-dependent channels. For instance, Gel’fand-Pinsker coding is fully relevant for the problem of coding over broadcast channels [31][32] and for watermarking problems [33]. Since one of the main contributions of this paper detailed in Section V is the application of the developed framework to power control, we provide here a few additional practical motivations for such an assumption. First, even if Agent 1 only knows the future realizations for a reduced
time horizon, the network can be coordinated to a degree which offers a significant performance gain compared to conventional approaches, such as implementing single-stage game model Nash equilibrium-type distributed policies, see e.g., [34], [17], [21], [18]. The most typical situation in power control is to assume that two phases are available, a training phase and an action phase, and that one wireless channel state is known in advance to adjust the power level. This special case corresponds to setting $T = 2$ that is, $t \in \{1, 2\}$. In such a scenario, and as illustrated in Fig. 1 a simple coordination strategy when Agent 1 knows the upcoming wireless channel state might be to inform Agent 2 about it on odd time-slots and coordinate their actions on even time-slots. Note that assuming that Agent 1 knows all the realizations of the system state in advance can also be seen as a way of obtaining an upper bound for the performance of scenarios that assume a reduced time horizon for forecasting. It is worth noting that practical scenarios have appeared over the recent years, for which forecasting the wireless channel state over a long time horizon is realistic. For instance, it has become more and more common to exploit the forecast trajectory of a mobile user to optimize the system [35], which also makes our approach relevant when the wireless channel state is interpreted as the path loss. The proposed approach may also be applied to the case where the system state is not i.i.d. from stage to stage, but is i.i.d. from block to block, where a block consists of several stages. Indeed, there exist wireless communication standards that assume the channel to be constant over several time-slots, and the proposed approach suggests that gains can be obtained by varying the power level from time-slot to time-slot even if the channel is constant. At last but not least, it has to be mentioned that the developed framework is general and can be applied to other information structures, including those assuming that only $x_t^0$ is available at Agent 1 on stage $t$.

III. INFORMATION CONSTRAINTS ON IMPLEMENTABLE EMPIRICAL JOINT DISTRIBUTIONS

In this section, we characterize the sets of implementable empirical joint distributions $\overline{Q} \in \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2)$ for the cases described through (4) and (5). We show that these sets consist of distributions in $\Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2)$ subject to an information constraint, which captures the restrictions imposed on the observation structure between the agents. Specifically, Theorem 6 in Section III-A provides a necessary condition for implementability, while Theorem 8 and Theorem 14 in Section III-B provide sufficient conditions for implementability in case I and case II, respectively.

4In a power control problem, knowing only one realization ahead is already very useful.

5For example, Transmitter 1 might use a high (resp. low) power level on an odd time-slot to inform Transmitter 2 that the channel is good (resp. bad) in the next even time-slot.
Fig. 1. This figure illustrates a simple coordination scheme between two transmitters (which are the agents) in a simplified scenario where the alphabets are binary: $X_0 = \{ \text{good for user 1}, \text{good for user 2} \}$, $X_1 = \{ \text{low}, \text{high} \}$, $X_2 = \{ \text{low}, \text{high} \}$. The informed transmitter (i.e., 1) chooses the lowest (resp. highest) transmit power on the current stage $2t' + 1$ if the upcoming wireless channel state on stage $2t' + 2$ is good for user 2 (resp. 1). If Transmitter 2 can perfectly retrieve the power levels of Transmitter 1, it therefore knows the realization of the wireless channel state on stages whose index is even. It transmits at low (resp. high) power if the channel is good for user 1 (resp. 2). For stages whose index is odd, it chooses its power at random.

A. A necessary condition for implementability

**Theorem 6.** Let $\overline{Q}$ be a distribution in $\Delta(X_0 \times X_1 \times X_2)$ such that $\forall x_0 \in X_0$, $\sum_{x_1,x_2} \overline{Q}(x_0, x_1, x_2) = \rho_0(x_0)$. In both case I and case II, a distribution $\overline{Q}$ is implementable if it is the marginal of a distribution $Q \in \Delta(X_0 \times X_1 \times X_2 \times Y)$ factorizing as

$$ Q(x_0, x_1, x_2, y) = \Gamma(y|x_0, x_1, x_2)\overline{Q}(x_0, x_1, x_2), \ (x_0, x_1, x_2, y) \in X_0 \times X_1 \times X_2 \times Y \quad (14) $$

and satisfying the information constraint

$$ I_Q(X_0; X_2) \leq I_Q(X_1; Y|X_0, X_2) \quad (15) $$

where

$$ I_Q(X_0; X_2) = \sum_{x_0,x_2} Q_{X_0,X_2}(x_0, x_2) \log \frac{Q_{X_0,X_2}(x_0, x_2)}{\rho_0(x_0) \sum_{x_0} Q_{X_0,X_2}(x_0, x_2)}, \quad (16) $$

$$ Q_{X_0,X_2}(x_0, x_2) = \sum_{x_1} \overline{Q}(x_0, x_1, x_2), \quad (17) $$
$$I_Q(X_1; Y | X_0, X_2) = \sum_{x_0, x_1, x_2, y} Q(x_0, x_1, x_2, y) \times \left[ \log \frac{Q(x_0, x_1, x_2, y)}{\sum_y Q(x_0, x_1, x_2, y) \sum_{x_1} Q(x_0, x_1, x_2, y)} \right]. \quad (18)$$

**Proof:** Let $\overline{Q} \in \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2)$ be an implementable distribution according to Definition 2. Then for $\ell \in \{I, II\}$, there exists $(\sigma^\ell_t, r^\ell_t)_{1 \leq t \leq T}$ such that for all $(x_0, x_1, x_2) \in \mathcal{X}$

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{y \in \mathcal{Y}} P_{X_0,t, X_1,t, X_2,t, Y_t}(x_0, x_1, x_2, y) = \overline{Q}(x_0, x_1, x_2), \quad (19)$$

where $P_{X_0,t, X_1,t, X_2,t, Y_t}$ is defined in (9). Because of the specific form of (9), this also implies that

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} P_{X_0,t, X_1,t, X_2,t, Y_t}(x_0, x_1, x_2, y) = Q(x_0, x_1, x_2) \quad (20)$$

with $Q$ as in (14). We now establish the information constraint in (15) that must be satisfied by $Q$ through a series of information-theoretic inequalities. We have

$$\sum_{t=1}^{T} I_{P_{X_0,t, X_1,t, X_2,t, Y_t}}(X_0; X_2) = \sum_{t=1}^{T} I(X_{0,t}; X_{2,t}) \quad (21)$$

$$= \sum_{t=1}^{T} H(X_{0,t}) - H(X_{0,t} | X_{2,t}) \quad (22)$$

$$\overset{(a)}{=} H(X_0^T) - T \sum_{t=1}^{T} H(X_{0,t} | X_{2,t})$$

$$= H(X_0^T, Y^T, X_2^T) - H(Y^T | X_2^T) - \sum_{t=1}^{T} H(X_{0,t} | X_{2,t}) \quad (24)$$

$$= H(X_0^T, Y^T, X_2^T) - H(X_2^T | X_0^T) - H(Y^T | X_0^T, X_2^T) - \sum_{t=1}^{T} H(X_{0,t} | X_{2,t}) \quad (25)$$

$$= H(X_0^T, Y^T, X_2^T) - H(X_2^T | X_0^T) - H(Y^T | X_0^T, X_2^T) + H(Y^T | X_0^T, X_1^T, X_2^T)$$

$$\quad - \sum_{t=1}^{T} H(X_{0,t} | X_{2,t}) \quad (26)$$

$$\overset{(b)}{=} \left[ \sum_{t=1}^{T} H(X_{0,t}, Y_t, X_{2,t} | X_0^{t-1}, Y_1^{t-1}, X_2^{t-1}) - H(X_{0,t} | X_{2,t}) - H(Y_t | X_{0,t}, X_1,t, X_{2,t}) \right]$$

$$+ H(Y^T | X_0^T, X_1^T, X_2^T) - H(X_2^T | X_0^T) - H(Y^T | X_0^T, X_1^T, X_2^T) \quad (27)$$

$$= \left[ \sum_{t=1}^{T} H(X_{0,t}, Y_t, X_{2,t} | X_0^{t-1}, Y_1^{t-1}, X_2^{t-1}) - H(X_{0,t} | X_{2,t}) - H(Y_t | X_{0,t}, X_1,t, X_{2,t}) \right]$$

$$+ H(Y_t | X_{0,t}, X_{2,t}) + H(Y_t | X_{0,t}, X_{2,t}) - H(X_2^T | X_0^T)$$
where (a) follows because \((X_{0,t})_{1 \leq t \leq T}\) is i.i.d. and (b) from the fact that the observations \(Y^T\) are the output of a memoryless channel per (3). Note that, for all \(t\), \(B_t \leq 0\) and \(D_t \leq 0\) since conditioning reduces entropy; \(A_t \leq 0\) whether we define \(X_{2,t} = \tau_t^I(X_{0,t}^{t-1}, Y^{t-1})\) or \(X_{2,t} = \tau_t^I(Y^{t-1})\); \(C_t = 0\) whether we define \(X_{2,t} = \tau_t^I(X_{0,t}^{t-1}, Y^{t-1})\) or \(X_{2,t} = \tau_t^I(Y^{t-1})\). Therefore,

\[
\sum_{t=1}^{T} I_{P_{X_{0,t},X_1,t,X_{2,t},Y_t}}(X_0; X_2) \leq \sum_{t=1}^{T} H(Y_t|X_{0,t}, X_{2,t}) - H(Y_t|X_{0,t}, X_{1,t}, X_{2,t})
\]

\[
= \sum_{t=1}^{T} I(X_{1,t};Y_t|X_{0,t}, X_{2,t})
\]

\[
= \sum_{t=1}^{T} I_{P_{X_{0,t},X_1,t,X_{2,t},Y_t}}(X_1;Y|X_0, X_2).
\]

Let us introduce the function \(\Phi^I:\)

\[
\Phi^I : \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) \rightarrow \mathbb{R}
\]

\[
Q \mapsto I_Q(X_0; X_2) - I_Q(X_1;Y|X_0, X_2)
\]  

It follows from (31) and (33) that

\[
\sum_{t=1}^{T} \Phi^I(P_{X_{0,t},X_1,t,X_{2,t},Y_t}) \leq 0.
\]

To conclude the proof, we exploit the following lemma, which proof can be found in Appendix B.
Lemma 7. The function $\Phi^I$ is strictly convex over the set of distributions $Q \in \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y})$ that verify
\[
\forall x_0 \in \mathcal{X}_0, \sum_{x_1, x_2, y} Q(x_0, x_1, x_2, y) = \rho_0(x_0), \tag{36}
\]
and $\forall (x_0, x_1, x_2, y) \in (\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y})$:
\[
Q(x_0, x_1, x_2, y) = \Gamma(y|x_0, x_1, x_2)\rho_0(x_0)Q(x_1, x_2|x_0), \tag{37}
\]
with $\rho_0$ and $\Gamma$ fixed.

Let us define the joint distribution $P_{X_0, X_1, X_2, Y}^{(T)}$ as
\[
P_{X_0, X_1, X_2, Y}^{(T)}(x_0, x_1, x_2, y) \triangleq \frac{1}{T} \sum_{t=1}^{T} \sum_{y \in \mathcal{Y}} P_{X_0,t, X_1,t, X_2,t, Y_t}(x_0, x_1, x_2, y). \tag{38}
\]
Since $\Phi^I$ is convex by Lemma 7, we know that
\[
\Phi^I(P_{X_0, X_1, X_2, Y}^{(T)}) \leq \frac{1}{T} \sum_{t=1}^{T} \Phi^I(P_{X_0,t, X_1,t, X_2,t, Y_t}). \tag{39}
\]
In addition, since $\Phi^I$ is continuous and because of (20), $\forall \varepsilon' > 0$, there exists $T'$ such that $\forall T \geq T'$,
\[
\Phi^I(Q) \leq \Phi^I(P_{X_0, X_1, X_2, Y}^{(T)}) + \varepsilon'. \tag{40}
\]
Therefore, combining (35), (38), and (40), we obtain
\[
\forall \varepsilon' > 0, \quad \Phi^I(Q) \leq 0, \tag{41}
\]
which concludes the proof.

Theorem 6 can be interpreted as follows. Agent 2’s actions, which are represented by $X_2$, correspond to a joint source-channel decoding operation with distortion on the information source, which is represented by $X_0$. To be achievable, the distortion rate has to be less than the transmission rate allowed by the channel, whose input and output are represented by Agent 1’s action $X_1$ and the signal $Y$ observed by Agent 2. Therefore, the pair $S = (X_0, X_2)$ seems to play the same role as the side information in state-dependent channels [32]. Although we exploit this interpretation when establishing sufficient conditions for implementability in Section III-B, the argument seems inappropriate to show that the sufficient conditions are also necessary. In contrast to classical arguments in converse proofs for state-dependent channels [23], [36], in which the transmitted “message” is independent of the channel state, here the role of the message is played by the quantity $X_0^T$, which is not independent of $S^T = (X_0^T, X_2^T)$. This is one of the reasons why the converse proof does not follow from existing results.
B. Sufficient conditions for implementability

We start by characterizing implementable distributions for the observation structure corresponding to case I in (4).

**Theorem 8.** Consider the observation structure in case I. Let \( Q \in \Delta(X_0 \times X_1 \times X_2) \) be such that 
\[
\sum_{(x_1, x_2)} Q(x_0, x_1, x_2) = \rho_0(x_0) \text{ for all } x_0 \in X_0.
\]
If the distribution \( Q \) defined as
\[
Q(x_0, x_1, x_2, y) = \Gamma(y | x_0, x_1, x_2) Q(x_0, x_1, x_2)
\]
for \((x_0, x_1, x_2, y) \in X_0 \times X_1 \times X_2 \times Y\) verifies the constraint 
\[
I_Q(X_0; X_2) < I_Q(X_1; Y | X_0, X_2),
\]
then \( Q \) is implementable in the sense of Definition 2.

**Proof:** Let \( R > 0 \). Consider a joint distribution \( Q_{X_0, X_1, X_2, Y} \in \Delta(X_0 \times X_1 \times X_2 \times Y) \) that factorizes as in (42). We will denote by \( Q_{X_0,X_1,X_2}, Q_{X_1|X_0,X_2}, Q_{X_0|X_2}, \) and \( Q_{X_2} \) the resulting marginal and conditional distributions. The crux of the proof is to design the strategies \( f^1 : (x_0^{(b+1)}, x_1^{(b)}, x_2^{(b)}) \mapsto \hat{x}_1^{(b)} \) and \( g^1 : (x_0^{(b)}, y^{(b)}, x_2^{(b)}) \mapsto \hat{x}_2^{(b+1)} \) from a block-Markov coding scheme that operates over \( B \) blocks of \( n \geq 1 \) actions. As illustrated in Fig. 2 in every block \( 1 \leq b \leq B - 1 \), Agent 1 will communicate to Agent 2 the actions that Agent 2 should play in the next block \( b + 1 \). This is made possible by restricting the sequence of actions played by Agent 2 in each block \( b \) to a codebook of actions \((x_2^{(b)}(i_b))_{1 \leq i_b \leq 2^{nR}}\) with \( i_b \in \{1, \ldots, 2^{nR}\} \), so that Agent 1 has only to communicate the index \( i_b \) to be played in the next block. The problem then essentially reduces to a joint source-channel coding problem over a state-dependent channel, for which in every block \( b \):

![Fig. 2. Illustration of encoding and decoding operations in block b of the proposed block-coding scheme.](image-url)

October 28, 2014 DRAFT
the system state is known non-causally by Agent 1 and causally by Agent 2, to be consistent with
the observation structure defined in (4);

- Agent 1 communicates with Agent 2 over a state-dependent discrete memoryless channel (DMC)
with transition probability \( \Gamma(y|x_0, x_1, x_2) \);

- the actual state of the channel consists of the system state sequence \( x_0^{(b)} \) and the current action
sequence \( x_2^{(b)}(i_{b-1}) \) for Agent 2, \( i_{b-1} \) being the index decoded by Agent 2 from its knowledge at
the end of block \( b-1 \). The actual channel state is perfectly known by Agent 2 while Agent 1
knows the error-free version of it \( (x_0^{(b)}, x_2^{(b)}(i_{b-1})) \). The impact in terms of communication rate
of this difference is proved to be asymptotically negligible a little further;

- the source sequence to be communicated is a sequence \( x_2^{(b+1)}(i_b) \), chosen to be empirically coordi-
nated with the future system state sequence \( x_0^{(b+1)} \) in block \( b+1 \);

- \( i_b \) is encoded in a codeword \( x_1^{(b)} \) chosen to be empirically coordinated with \( (x_0^{(b)}, x_2^{(b)}(i_{b-1})) \).

Intuitively, \( R \) must be sufficiently large so that one may find a codeword \( x_2^{(b)}(i_b) \) coordinated with any
system state sequence \( x_0^{(b)} \); simultaneously, \( R \) must be small enough to ensure that the codeword \( x_2^{(b)}(i_b) \)
can be reliably decoded by Agent 2 after transmission over the channel \( \Gamma(y|x_0, x_1, x_2) \). The formal
analysis of these conditions, which we develop next, will establish the result.

Unlike the block-Markov schemes used, for instance, in relay channels, in which all nodes may agree
on a fixed message in the first block at the expense of a small rate loss, the first block must be dealt
with more carefully in the case of coordination. In fact, Agent 1 must transmit a message defined by
the system state realization \( x_0^{(2)} \), which is unknown to Agent 2. Consequently, we have to account for
an “uncoordinated” transmission in the first block, in which Agent 1 is forced to communicate at rate
\( \widehat{R} \) that differs from the rate \( R \) used in subsequent blocks. To characterize \( \widehat{R} \), we introduce another joint
distribution \( \widehat{Q} \in \Delta(X_0 \times X_1 \times X_2 \times Y) \) that factorizes as
\[
\widehat{Q}(x_0, x_1, x_2, y) = \Gamma(y|x_0, x_1, x_2)\widehat{Q}(x_1|x_0, x_2)\widehat{Q}(x_0)\widehat{Q}(x_2) \tag{44}
\]
Note that \( \widehat{Q} \) differs from \( Q \) in that \( X_0 \) and \( X_2 \) are independent.

Let \( \epsilon > 0 \). Let \( \widehat{R} > 0 \), \( \frac{\epsilon}{2} > \epsilon_2 > \epsilon_1 > 0 \) be real numbers and \( n \geq 1 \) to be specified later, and define
\[
\alpha \triangleq \max \left( \frac{R}{\widehat{R}}, 1 \right) \tag{45}
\]
\[
B \triangleq \left[ 1 + \alpha \left( \frac{4}{\epsilon} - 1 \right) \right] \tag{46}
\]
Intuitively, \( \alpha \) measures the rate penalty suffered from the uncoordinated transmission at rate \( \widehat{R} \) in the
first block. The choice of \( B \) merely ensures that \( \frac{2\alpha}{B-1+\alpha} \leq \frac{\epsilon}{2} \), as exploited later.
Source codebook generation for $b = 1$. Choose a sequence $x_2^{(1)}(i_0) \in \mathcal{T}_{c_1}^n(Q_{X_2})$. This unique sequence is revealed to both agents.

Source codebooks generation for $b \in \{2, \ldots, B + 1\}$. Randomly and independently generate $[2^{nR}]$ sequences according to $\Pi_{t=1}^n Q_{X_2}(x_{2,t})$. Label the sequences $x_2^{(b)}(i)$ with $i \in \{1, \ldots, 2^{nR}\}$. These sequences constitute the source codebook $C_s^{(b)}$, which is revealed to both agents.

Channel codebook generation for $b = 1$. Following [32, Section 7.4.1], generate a codebook $C_c^{(1)}$ of $[2^{nR}]$ codewords and of length $\alpha n$ for a channel with random state $(x_0, x_2)$ available causally at both Agent 1 and Agent 2. With a slight abuse of notation, denote the channel codewords corresponding to a message $i$ and a sequence $(x_0, x_2^{(1)}(i_0))$ as $x_1^{(b)}(i, x_0, i_0)$.

Channel codebooks generation for $b \in \{2, \ldots, B\}$. For every $(x_0, j) \in X_0^n \times \{1, \ldots, 2^{nR}\}$, randomly and independently generate $[2^{nR}]$ sequences according to $\Pi_{t=1}^n Q_{X_1X_0X_2}(x_{1,t}x_{0,t}, x_{2,t}^{(b)}(j))$, where $x_{2,t}^{(b)}(j)$ is the $t$-th component of $x_2^{(b)}(j) \in C_s^{(b)}$. Label the sequences $x_2^{(b)}(i, x_0, j)$ with $i \in \{1, \ldots, 2^{nR}\}$, $x_0 \in X_0^n$, and $j \in \{1, \ldots, 2^{nR}\}$. These sequences constitute the channel codebook $C_c^{(b)}$, which is revealed to both agents.

Source and channel encoding at Agent 1 in block $b \in \{1, \ldots, B\}$. At the beginning of block $b$, Agent 1 uses its non-causal knowledge of the system state $x_0^{(b+1)}$ in the next block $b + 1$ to look for an index $i_b$ such that $(x_0^{(b+1)}, x_2^{(b+1)}(i_b)) \in \mathcal{T}_{c_1}^n(Q_{X_0}X_2)$. If there is more than one such index, it chooses the smallest among them, otherwise it chooses $i_b = 1$. Agent 1 then encodes $i_b$ using its knowledge of the sequence $(x_0^{(b)}, x_2^{(b)}(i_b-1))$ as $x_1^{(b)}(i_b, x_0^{(b)}, i_b-1)$. Note that this is possible in block 1 because of the specific codebook and of the choice of $\alpha$ in [45].

Channel decoding at Agent 2 in block $b = 1$. At the end of the block, Agent 2 uses the same decoding procedure as in [32, Section 7.4.1]; it uses the channel output $y^{(1)}$ and its knowledge of the sequence $(x_0^{(1)}, x_2^{(1)}(i_0))$ to form an estimate $\hat{i}_1$ of $i_1$.

Channel decoding at Agent 2 in block $b \in \{2, \ldots, B\}$. At the end of block $b$, Agent 2 knows the sequence of channel outputs $y^{(b)}$ and the channel state sequence $(x_0^{(b)}, x_2^{(b)}(\hat{i}_{b-1}))$ in block $b$. Agent 2 then looks for an index $\hat{i}_b$ such that

$$
(x_0^{(b)}, x_2^{(b)}(\hat{i}_b), x_2^{(b)}(\hat{i}_{b-1}), y^{(b)}, x_0^{(b)}, x_2^{(b)}(\hat{i}_{b-1})) \in \mathcal{T}_{c_2}^n(Q_{X_0X_1X_2Y}).
$$

(47)

If there is none of more than one such index, Agent 2 sets $\hat{i}_b = 1$.

Source decoding at Agent 2 in block $b \in \{1, \ldots, B\}$. Agent 2 transmits $x_2^{(b)}(\hat{i}_{b-1})$, where $\hat{i}_{b-1}$ is its estimate of the message transmitted by Agent 1 in the previous block $b - 1$, with the convention that $\hat{i}_0 = i_0$. 

October 28, 2014  DRAFT
Analysis. According to Proposition 5 it is sufficient to prove that $\overline{Q}$ is an achievable empirical coordination. We therefore introduce the event

$$E \triangleq \{(X^T_0, X^T, X^T) \notin \mathcal{T}_e^T(\overline{Q})\}$$

(48)

and we proceed to show that $\mathbb{P}(E)$ can be made arbitrarily small for $n$ sufficiently large and a proper choice of the rates $R$ and $\hat{R}$. We start by introducing the following events in each block $b \in \{1, \ldots, B\}$.

$$E_0 \triangleq \{I_1 \neq \hat{I}_1\}$$

$$E_1^{(b)} \triangleq \{(X_0^{(b+1)}_0, X_2^{(b)}(i)) \notin \mathcal{T}_{e_1}^n(Q_{X_0X_2}) \forall i \in \{1, 2, \ldots, 2^{nR}\}\}$$

$$E_2^{(b)} \triangleq \{(X_0^{(b)}, X_1^{(b)}(I_b), X_0^{(b)}, \hat{I}_{b-1}), X_2^{(b)}(\hat{I}_{b-1}), Y^{(b)} \notin \mathcal{T}_{e_2}^n(Q)\}$$

$$E_3^{(b)} \triangleq \{(X_0^{(b)}, X_1^{(b)}(i), X_0^{(b)}, X_1^{(b)}(\hat{I}_{b-1}), X_2^{(b)}(\hat{I}_{b-1}), Y^{(b)} \in \mathcal{T}_{e_2}^n(Q) \text{ for some } i \neq I_b\}.$$  

We start by developing an upper bound for $\|T_{\overline{w}, \overline{w}, \overline{w}} - \overline{Q}\|_1$, whose proof can be found in Appendix C.

**Lemma 9.** We have that

$$\|T_{\overline{w}, \overline{w}, \overline{w}} - \overline{Q}\|_1 \leq \frac{2\alpha}{B - 1 + \alpha} + \frac{1}{B - 1} \sum_{b=2}^{B} \|T_{\overline{w}^{(b)}, \overline{w}^{(b)} - \overline{Q}}\|_1.$$  

(49)

Recalling the choice of $B$ in (46), we therefore have

$$\mathbb{P}(E) = \mathbb{P}\left(\|T_{X^T_0 X^T X^T} - \overline{Q}\|_1 \geq \epsilon\right)$$

(50)

$$\leq \mathbb{P}\left(\frac{1}{B - 1} \sum_{b=2}^{B} \|T_{X_0^{(b)} X_2^{(b)} X_3^{(b)}} - \overline{Q}\|_1 \geq \frac{\epsilon}{2}\right)$$

(51)

$$\leq \mathbb{P}\left(\|T_{X_0^{(b)} X_2^{(b)} X_3^{(b)}} - \overline{Q}\|_1 \geq \frac{\epsilon}{2} \text{ for some } b \in \{2, \ldots, B\}\right)$$

(52)

$$\leq \mathbb{P}\left(E_0 \cup E_1^{(1)} \bigcup_{b=2}^{B} (E_1^{(b)} \cup E_2^{(b)} \cup E_3^{(b)})\right)$$

(53)

$$\leq \mathbb{P}(E_0) + \sum_{b=1}^{B} \mathbb{P}(E_1^{(b)}) + \sum_{b=2}^{B} \left(\mathbb{P}\left(E_2^{(b)} \mid E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0\right)\right)$$

$$+ \sum_{b=2}^{B} \left(\mathbb{P}\left(E_3^{(b)} \mid E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0\right)\right).$$  

(54)

As proved in Appendix C the following lemmas show that all the averages over the random codebooks of the term above vanish as $n \to \infty$.

**Lemma 10.** If $\hat{R} < (1 - \epsilon_2)I_Q(X_1; Y|X_0, X_2) - \delta(\epsilon_2)$, then

$$\lim_{n \to \infty} \mathbb{E}(\mathbb{P}(E_0)) = 0.$$  

(55)
Lemma 11. If $R > I_Q(X_0; X_2) + \delta(\epsilon_1)$, then for any $b \in \{1, \ldots, B\}$,

$$\lim_{n \to \infty} E\left(P(E_1^{(b)})\right) = 0. \quad (56)$$

Lemma 12. For any $b \in \{2, \ldots, B\}$,

$$\lim_{n \to \infty} E\left(P(E_2^{(b)}|E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0^c)\right) = 0. \quad (57)$$

Lemma 13. If $R < I_Q(X_1; Y|X_0, X_2) - \delta(\epsilon_2)$, then for any $b \in \{2, \ldots, B\}$,

$$\lim_{n \to \infty} E\left(P(E_3^{(b)})\right) = 0. \quad (58)$$

Therefore, if $I_\hat{Q}(X_1; Y|X_0, X_2) > 0$ and $I_Q(X_0; X_2) < I_Q(X_1; Y|X_0, X_2)$, we can find $\epsilon_1, \epsilon_2$ small enough such that $R > 0$, $I_Q(X_0; X_2) < R < I_Q(X_1; Y|X_0, X_2)$, and

$$\lim_{n \to \infty} E\left(P(E)\right) = 0. \quad (59)$$

Hence, there must exist at least one set of codes such that $\lim_{n \to \infty} P(E) = 0$. Since $\epsilon > 0$ can be chosen arbitrarily small, $Q$ is an achievable empirical coordination.

A few comments are in order regarding the coding scheme achieving the result in Theorem 8. Note that Agent 1 uses $(x_0^{(b)}, x_2^{(b)}(i_{b-1}))$ as “side information” to correlate its actions during block $b$, while Agent 2 uses $(x_0^{(b)}, x_2^{(b)}(\hat{i}_{b-1}))$. However, under the condition $I_Q(X_0; X_2) < I_Q(X_1; Y|X_0, X_2)$, $\hat{i}_{b-1} = i_{b-1}$ with overwhelming probability as the block length $n$ goes to infinity. Hence, Agent 1 effectively knows the actions of Agent 2 even though it does not directly observe them. Furthermore, note that the system state sequence $x_0^{(b+1)}$, which plays the role of the message to be encoded in block $b$, is independent of the “side information” $(x_0^{(b)}, x_2^{(b)}(i_{b-1}))$; this allows us to reuse classical coding schemes for the transmission of messages over state-dependent channels.

Apart for the strict inequality in (43) instead of the inequality in (15), the information constraints of Theorem 6 and Theorem 8 coincide, hence establishing a necessary and sufficient condition for a joint distribution $Q(x_0, x_1, x_2)$ to be implementable. This also shows that having Agent 1 select the actions played by Agent 2 and separating source and channel encoding operations do not incur any loss of optimality. From Theorem 6 and Theorem 8 one can therefore provide the complete characterization of the set of achievable payoffs for the observation structure defined in (4) and for an i.i.d. random system state. As further explored in Section V, one can apply this result to an interference network with two transmitters and two receivers, in which Transmitter 1 may represent the most informed agent, such as a primary transmitter [37], [38], $\Gamma$ may represent an SINR feedback channel from Receiver 2 to Transmitter 2.

October 28, 2014 DRAFT
While Theorem 8 has been derived for the case of an i.i.d. random system state, the result generalizes to a situation in which the system state is constant over $L \geq 1$ consecutive stages and i.i.d. from one stage to the next. In such case, the information constraint becomes

$$\frac{1}{L} I_Q(X_0; X_2) \leq I_Q(X_1; Y|X_0, X_2). \quad (60)$$

This modified constraint is useful in some setting, such as in wireless communications for which channels can often be assumed to be block i.i.d.. When $L \to \infty$, which correspond to a single realization of the random system state, the information constraint is always satisfied and any joint distribution over $\Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2)$ is implementable.

Finally, we emphasize that the information constraint obtained when coordinating via actions differs from what would be obtained when coordinating using classical communication with a dedicated channel. If Agent 1 could communicate with Agent 2 through a channel with capacity $C$, then all joint distributions $\tilde{Q} \in \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2)$ subject to the information constraint

$$I_{\tilde{Q}}(X_0; X_2) \leq C \quad (61)$$

would be implementable. In contrast, the constraint $I_Q(X_0; X_2) < I_Q(X_1; Y|X_0X_2)$ reflects the following two distinctive characteristics of communication via actions.

1) The input distribution $X_1$ to the “implicit channel” used for communication between Agent 1 and Agent 2 cannot be optimized independently of the actions and of the system state.

2) The output $Y$ of the implicit channel depends not only on $X_1$ but also on $(X_0, X_2)$; essentially, the system state $X_0$ and the actions $X_2$ of Agent 2 act as a state for the implicit channel.

Under specific conditions, the coordination via actions may reduce to coordination with a dedicated channel. For instance, if the payoff function factorizes as $w(x_0, x_1, x_2) \triangleq w_1(x_1)w_2(x_0, x_2)$ and if the observation structure satisfies $(X_0, X_2) - X_1 - Y$, then any joint distribution $\tilde{Q}(x_0, x_1, x_2) \triangleq \tilde{Q}(x_0, x_2)\tilde{Q}(x_1)$ satisfying the information constraint arc

$$I_{\tilde{Q}}(X_0; X_2) \leq I_{\tilde{Q}}(X_1; Y) \quad (62)$$

would be implementable; in particular, one may optimize $\tilde{Q}$ independently. In addition, if $w_1(x_1)$ is independent of $x_1$, the information constraint further simplifies as

$$I_{\tilde{Q}}(X_0; X_2) \leq \max_Q I_Q(X_1; Y), \quad (63)$$

and the implicit communication channel effectively becomes a dedicated channel.
We now characterize implementable distributions for the observation structure corresponding to case II in (5).

**Theorem 14.** Consider the observation structure in case II. Let $U$ be a random variable whose realizations lie in the alphabet $\mathcal{U}$, $|\mathcal{U}| < \infty$. Let $Q \in \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2)$ be such that

$$\sum_{(x_1, x_2)} Q(x_0, x_1, x_2) = \rho_0(x_0)$$

for all $x_0 \in \mathcal{X}_0$. If the distribution $Q$ defined as

$$Q(x_0, x_1, x_2, y, u) = P(u|x_0, x_1, x_2) \Gamma(y|x_0, x_1, x_2) Q(x_0, x_1, x_2)$$

for $(x_0, x_1, x_2, y, u) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y} \times \mathcal{U}$ verifies the constraint

$$I_Q(X_0; X_2) < I_Q(U; Y, X_2) - I_Q(U; X_0, X_2),$$

then $Q$ is implementable in the sense of Definition 2.

**Proof:** Consider a joint distribution $Q_{X_0X_1X_2YU} \in \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y} \times \mathcal{U})$ that factorizes as in (64). We will denote by $Q_{UX_0X_1X_2}$, $Q_{U}$, $Q_{X_0X_1X_2}$, and $Q_{X_2}$ the resulting marginal distributions. As for the proof of Theorem 8 the idea is to construct the strategies $f^{II} : (x_{0}^{(b+1)}, x_{1}^{(b)}, x_{2}^{(b)}) \mapsto x_{1}^{(b)}$ and $g^{II} : (y^{(b)}, x_{2}^{(b)}) \mapsto x_{2}^{(b+1)}$ from a block-Markov coding scheme that operates over $B$ blocks. As illustrated in Fig. 3 Agent 1 will again communicate to Agent 2 during block $b$ the actions that Agent 2 should play in block $b + 1$; however, unlike the situation in Fig. 2, Agent 2 now has only access to noisy versions of the actions of Agent 1 and not to the past realizations of the system state anymore. While coordination in case I relied on coding schemes similar to those for communication over state-dependent channels with channel state information available at the transmitter and receiver, coordination in case II will rely on Gel’fand-Pinsker schemes to deal with the absence of channel state information at the receiver.

![Fig. 3. Illustration of encoding and decoding operations in block $b$ of the proposed block-coding scheme for case II.](image-url)
We will again have to distinguish communication during block 1, which cannot be fully coordinated. We thus introduce another joint distribution \( \hat{Q} \) that factorizes as

\[
\hat{Q}(x_0, x_1, x_2, y, u) = \Gamma(y|x_0, x_1, x_2)\mathcal{P}(u|x_0x_1x_2)\hat{\mathcal{Q}}(x_1|x_0x_2)\hat{Q}(x_0)\hat{Q}(x_2),
\]

(66)

which differs from \( \mathcal{Q} \) by the independence of \( X_0 \) and \( X_2 \). Now, let \( \epsilon > 0 \). Let \( R > 0 \), \( R' > 0 \), \( \hat{R} > 0 \), \( \frac{R'}{2} > \epsilon_3 > \epsilon_2 > \epsilon_1 > 0 \) be real numbers and \( n \geq 1 \) to be specified later. Define \( \alpha \) as in (45) and \( B \) as in (46).

**Source codebook generation for** \( b = 1 \). Choose a sequence \( \hat{x}_2^{(1)}(i_0) \in \mathcal{T}^n_{\epsilon_1}(Q_{X_2}) \). This unique sequence is revealed to both agents.

**Source codebooks generation for** \( b \in \{2, \ldots, B + 1\} \). Randomly and independently generate \( 2^{nR} \) sequences according to \( \Pi_{t=1}^n Q_{X_2}(x_{2,t}) \). Label the sequences \( \hat{x}_2^{(b)}(i) \) with \( i \in \{1, \ldots, 2^{nR}\} \). These sequences constitute the source codebook \( C^{(b)}_n \), which is revealed to both agents.

**Channel codebook generation for** \( b = 1 \). Randomly and independently generate \( 2^{\alpha n(\hat{R} + \hat{R})} \) sequences according to \( \Pi_{t=1}^n \hat{Q}_U(u_t) \). Label the sequences \( \hat{u}^{(1)}(i,j) \) with \( i \in \{1, \ldots, 2^{\alpha nR}\} \) and \( j \in \{1, \ldots, 2^{\alpha nR'}\} \). These sequences constitute the channel codebook for block 1, which is revealed to both agents.

**Channel codebook generation for** \( b \in \{2, \ldots, B\} \). Randomly and independently generate \( 2^{n(R + R')} \) sequences according to \( \Pi_{t=1}^n Q_U(u_t) \). Label the sequences \( \hat{u}^{(b)}(i,j) \) with \( i \in \{1, \ldots, 2^{nR}\} \) and \( j \in \{1, \ldots, 2^{nR'}\} \). These sequences constitute the channel codebook for block \( b \), which is revealed to both agents.

**Source encoding at Agent 1 in block** \( b \in \{1, \ldots, B\} \). At the beginning of block \( b \), Agent 1 uses its non-causal knowledge of the system state \( \hat{x}_0^{(b+1)} \) in the next block \( b+1 \) to look for an index \( i_b \) such that \( (\hat{x}_0^{(b+1)}, \hat{x}_2^{(b+1)}(i_b)) \in \mathcal{T}^n_{\epsilon_1}(Q_{X_0X_2}) \). If there is more than one such index, it chooses the smallest among them, otherwise it chooses \( i_b = 1 \).

**Channel encoding at Agent 1 in block** \( b = 1 \). Agent 1 uses its knowledge of \( (\hat{x}_0^{(1)}, \hat{x}_2^{(1)}(i_0)) \) to look for an index \( j_1 \) such that

\[
(\hat{x}_1^{(1)}(i_1,j_1), \hat{x}_0^{(1)}, \hat{x}_2^{(1)}(i_0)) \in \mathcal{T}^n_{\epsilon_1}(\hat{Q}_{UX_0X_2})
\]

(67)

If there is more than one such index, it chooses the smallest among them, otherwise it chooses \( j_1 = 1 \).

Finally, Agent 1 generates a sequence \( \hat{x}_1^{(1)} \) by passing the sequences \( \hat{x}_1^{(1)}(i_1,j_1), \hat{x}_0^{(1)}, \) and \( \hat{x}_2^{(1)}(i_{b-1}) \) through a DMC with transition probability \( \hat{Q}_{X_1\hat{U}X_0X_2} \); the sequence \( \hat{x}_1^{(1)} \) is transmitted during block 1.
Channel encoding at Agent in block $b \in \{2, \ldots, B\}$. Agent 1 uses its knowledge of $(x_0^{(b)}, x_2^{(b)}(i_{b-1}))$ to look for an index $j_b$ such that
\[
\left( u^{(b)}(i_b,j_b), x_0^{(b)}, y^{(b)}(i_{b-1}) \right) \in \mathcal{T}^n_{e_2}(Q_{UX_0X_2})
\]  
(68)
If there is more than one such index, it chooses the smallest among them, otherwise it chooses $j_b = 1$.

Finally, Agent 1 generates a sequence $x_1^{(b)}$ by passing the sequences $u^{(b)}(i_b,j_b)$, $x_0^{(b)}$, and $x_2^{(b)}(i_{b-1})$ through a DMC with transition probability $Q_{X_1U_{X_0}X_2}$; the sequence $x_1^{(b)}$ is transmitted during block $b$.

Decoding at Agent 2 in block $b = 1$. At the end of block 1, Agent 2 observes the sequence of channel outputs $y^{(1)}$ and knows its sequence of actions $(x_2^{(1)}(i_{b-1}))$ in block 1. Agent 2 then looks for a pair of indices $(\widehat{i}_1, \widehat{j}_1)$ such that
\[
\left( u^{(b)}(\widehat{i}_1, \widehat{j}_1), y^{(1)}, x_2^{(1)}(i_0) \right) \in \mathcal{T}^n_{e_3}(Q_{UYX_2}).
\]  
(69)
If there is none or more than one such index, Agent 2 sets $\widehat{i}_1 = \widehat{j}_1 = 1$.

Channel decoding at Agent 2 in block $b \in \{2, \ldots, B\}$. At the end of block $b$, Agent 2 observes the sequence of channel outputs $y^{(b)}$ and knows its sequence of actions $(x_2^{(b)}(i_{b-1}))$ in block $b$. Agent 2 then looks for a pair of indices $(\widehat{i}_b, \widehat{j}_b)$ such that
\[
\left( u^{(b)}(\widehat{i}_b, \widehat{j}_b), y^{(b)}, x_2^{(b)}(\widehat{i}_{b-1}) \right) \in \mathcal{T}^n_{e_3}(Q_{UYX_2}).
\]  
(70)
If there is none or more than one such index, Agent 2 sets $\widehat{i}_b = \widehat{j}_b = 1$.

Source decoding at Agent 2 in block $b \in \{1, \ldots, B\}$. Agent 2 transmits $x_2^{(b)}(\widehat{i}_{b-1})$, where $\widehat{i}_{b-1}$ is its estimate of the message transmitted by Agent 1 in the previous block $b - 1$, with the convention that $\widehat{i}_0 = i_0$.

Analysis. We follow the same approach as in the proof of Theorem 8. We define again the error event
\[
E \triangleq \{(X_0^T, X_1^T, X_2^T) \not\in \mathcal{T}^n_{e}(Q)\}
\]  
(71)
and we proceed to show that $P(E)$ can be made arbitrarily small for $n$ sufficiently large and a proper choice of the rates $R, R'$, and $\tilde{R}$. We introduce the following events in each block $b \in \{1, \ldots, B\}$.
\[
E_0 \triangleq \{(I_1, J_1) \neq (\widehat{i}_1, \widehat{j}_1)\}
\]
\[
E_1^{(b)} \triangleq \{(x_0^{(b+1)}, x_2^{(b)}(i)) \not\in \mathcal{T}^n_{e_2}(Q_{UX_0X_2}) \forall i \in \{1, \ldots, 2^{nR}\}\}
\]
\[
E_2^{(b)} \triangleq \{(u^{(b)}(I_b, J_b), X_0^{(b)}, x_2^{(b)}(I_b)) \not\in \mathcal{T}^n_{e_2}(Q_{UX_0X_2}) \forall j \{1, \ldots, 2^{nR'}\}\}
\]
\[
E_3^{(b)} \triangleq \left\{(u^{(b)}(I_b, J_b), X_0^{(b)}, X_1^{(b)}, x_2^{(b)}(\widehat{i}_{b-1}), Y^{(b)}) \not\in \mathcal{T}^n_{e_3}(Q_{UX_0X_1X_2Y})\right\}
\]
\[
E_4^{(b)} \triangleq \left\{(u^{(b)}(i, j), x_2^{(b)}(\widehat{i}_{b-1}), Y^{(b)}) \in \mathcal{T}^n_{e_3}(Q) \text{ for some } (i, j) \neq (i_b, j_b)\right\}.
\]
As in the proof of Theorem 8, \( P(E) \) may be upper bounded as

\[
P(E) \leq P(E_0) + \sum_{b=1}^{B} P(E_1^{(b)}) + \sum_{b=2}^{B} P(E_2^{(b)}) + \sum_{b=2}^{B} P(E_3^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_4^{(b-1)c} \cap E_6^{(b-1)c}) + \sum_{b=2}^{B} P(E_3^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_4^{(b-1)c} \cap E_6^{(b-1)c}). \tag{72}
\]

As proved in Appendix D, the following lemmas show that all the averages over the random codebooks of the terms above vanish as \( n \to \infty \).

**Lemma 15.** If \( \hat{R} > I_{\hat{Q}}(U; X_0 X_2) + \delta(\epsilon_2) \) and \( \hat{R} + \hat{R}' < I_{\hat{Q}'}(U; Y X_2) - \delta(\epsilon_3) \), then

\[
\lim_{n \to \infty} E(P(E_0)) = 0. \tag{73}
\]

**Lemma 16.** If \( R > I_Q(X_0; X_2) + \delta(\epsilon_1) \), then for any \( b \in \{1, \ldots, B\} \)

\[
\lim_{n \to \infty} E(P(E_1^{(b)})) = 0. \tag{74}
\]

**Lemma 17.** If \( R' > I_Q(U; X_0, X_2) + \delta(\epsilon_2) \), then for any \( b \in \{2, \ldots, B\} \)

\[
\lim_{n \to \infty} E(P(E_2^{(b)})|E_1^{(b-1)}) = 0. \tag{75}
\]

**Lemma 18.** For any \( b \in \{2, \ldots, B\} \)

\[
\lim_{n \to \infty} E(P(E_3^{(b-1)c} | E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_4^{(b-1)c} \cap E_6^{(b-1)c})) = 0. \tag{76}
\]

**Lemma 19.** If \( R + R' < I_{\hat{Q}}(U; Y, X_2) - \delta(\epsilon_3) \), then for any \( b \in \{2, \ldots, B\} \)

\[
\lim_{n \to \infty} E(P(E_4^{(b)})|E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_4^{(b-1)c} \cap E_6^{(b-1)c}) = 0. \tag{77}
\]

Therefore, if \( I_{\hat{Q}}(U; Y, X_2) - I_{\hat{Q}}(U; X_0, X_2) > 0 \) and \( I_Q(X_0; X_2) < I_{\hat{Q}}(U; Y, X_2) - I_{\hat{Q}}(U; X_0, X_2) \), we can find \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \) small enough such that \( \lim_{n \to \infty} E(P(E)) = 0 \). In particular, there must exists at least one sequence of codes such that \( \lim_{n \to \infty} P(E) = 0 \). Since \( \epsilon > 0 \) can be chosen arbitrarily small, \( \overline{Q} \) is an achievable empirical coordination.

Although the proof of Theorem 14 relies on a separate source channel coding scheme, there is a key difference with respect to the usual Gel’fand-Pinsker coding scheme [23] and its extensions [36]. Indeed, while using the channel decoder’s past outputs does not help in terms of improving the channel capacity, it is useful to achieve coordination. Specifically, a classical Gel’fand-Pinsker coding results would lead
to an information constraint

\[ I_Q(X_0; X_2) < I_Q(U; Y) - I_Q(U; X_0, X_2) \]  \hfill (78)

which is more restrictive than (65).

By specializing to the case in which Agent 2 perfectly monitors Agent 1’s actions, i.e., \( Y = X_1 \), equations (15), (43), (65), and (78) coincide with the information constraint \( I_Q(X_0; X_2) \leq H_Q(X_1 | X_0, X_2) \). This shows that knowing \( X_0 \) strictly causally at Agent 2 as in case I does not bring any performance improvement under perfect monitoring. Not that we have not proved whether the information constraint of Theorem 14 is a necessary condition for implementability in case II. One might be tempted to adopt a side information interpretation of the problem to derive the converse, since (65) resembles the situation of [40]; however, finding the appropriate auxiliary variables does not seem straightforward and is left a refinement of the present analysis.

IV. EXPECTED PAYOFF OPTIMIZATION

In this section, we study the problem of determining the joint distribution(s) \( Q \) that leads to the maximal payoff in case I and case II. We establish two formulations of the problem: one that involves \( Q \) viewed as a function, and one that explicitly involves the vector of probability masses of \( Q \). Although the latter form involves seemingly more complex notation, it is better suited to numerically determine the maximum expected payoff and turns out particularly useful in Section V. While we study the general optimization problem in Section IV-A, we focus on the case of perfect monitoring in Section IV-B for which we are able to gain more insight into the structure of the optimal solution(s).
A. General optimization problem

From the results derived in Section III, the determination of the largest average payoff requires solving the following optimization problem, with $\ell \in \{I, II\}$:

\[
\begin{align*}
\text{minimize} & \quad -\mathbb{E}_Q[w(X_0, X_1, X_2)] = -\sum_{(x_0, x_1, x_2, y, u)} Q(x_0, x_1, x_2, y, u)w(x_0, x_1, x_2) \\
\text{s.t.} & \quad -1 + \sum_{(x_0, x_1, x_2, y, u)} Q(x_0, x_1, x_2, y, u) \sum_{(y,u)} Q(x_0, x_1, x_2, y, u) - \Gamma(y|x_0, x_1, x_2) \overset{(c)}{=} 0 \\
& \quad \forall (x_0, x_1, x_2, y, u) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{U}, \\
& \quad \forall x_0 \in \mathcal{X}_0, \\
& \quad \forall (x_0, x_1, x_2, y, u) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{U}, \\
& \quad -\rho_0(x_0) + \sum_{(x_1, x_2, y, u)} Q(x_0, x_1, x_2, y, u) \overset{(d)}{=} 0 \\
& \quad -Q(x_0, x_1, x_2, y, u) \overset{(e)}{=} 0 \\
& \quad \overset{(f)}{\leq} 0 \\
& \quad \Phi^I(Q) \leq 0 \\
\end{align*}
\]

where in case I $\Phi^I(\cdot)$ is defined in (34) while in case II

\[
\Phi^{II}(Q) \triangleq I_Q(X_0; X_2) - I_Q(U; Y, X_2) + I_Q(U; X_0, X_2).
\]

A first aspect of the optimization problem to consider is its potential convexity [41]. Note that the objective function to be minimized is linear in $Q$. In addition, the constraints (c), (d), (e), and (f) restrict the domain to a convex subset of the unit simplex. Therefore, it suffices to show that the domain resulting from the additional constraint (g) is convex for the optimization problem to be convex. In case I, for which the set $\mathcal{U}$ reduces to a singleton, Lemma 7 already proves that $\Phi^I$ is a convex function of $Q$, which implies that the additional constraint (g) defines a convex domain. In case II, we have not proved that $\Phi^{II}$ is a convex function but, by using a time-sharing argument, it is always possible to make the domain convex. In the remaining of the paper, we assume this convexification is always performed, so that the optimization problem is again convex.

A second aspect to consider is whether Slater’s condition holds, so that the Karush-Kuhn-Tucker (KKT) conditions become necessary conditions for optimality. Since the problem is convex, the KKT conditions would also be sufficient.

**Proposition 20.** Slater’s condition holds in cases I and II for irreducible channel transition probabilities i.e., such that $\forall (x_0, x_1, x_2, y) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}, \Gamma(y|x_0, x_1, x_2) > 0$.

**Proof:** We establish the existence of a strictly feasible point in case II, from which the existence for
case I follows as a special case. Consider a distribution \( Q(x_0, x_1, x_2, y, u) \) such that \( X_0, X_1, \) and \( X_2 \) are independent, and \( U = X_1 \). We can assume without loss of generality that the support of the marginals \( Q_{X_i}, i \in \{0, 1, 2\} \) is full, i.e., \( \forall x_i \in X_i, Q_{X_i}(x_i) > 0 \). If the channel transition probability is irreducible, note that \( Q(x_0, x_1, x_2, y, u) \) is then strictly positive, making the constraint (f) inactive. As for inequality constraint (g), notice that

\[
I_Q(X_0; X_2) - I_Q(U; Y, X_2) + I_Q(U; X_0, X_2) = 0 - I_Q(X_1; Y) - I(X_1; X_2|Y) + I_Q(X_1; X_0, X_2)
\]

\[
= -I_Q(X_1; Y) - I_Q(X_1; X_2|Y)
\]

\[
= -H_Q(X_1) + H_Q(X_1|Y, X_2)
\]

\[
< 0.
\]

Hence, the chosen distribution constitutes a strictly feasible point for the domain defined by constraints (c)-(g), and remains a strictly feasible point after convexification of the domain.

Our objective is now to rewrite the above optimization problem more explicitly in terms of the vector of probability masses that describe \( Q \). This is useful not only to exploit standard numerical solvers in Section [V] but also to apply the KKT conditions in Section [IV-B]. We introduce the following notation. Without loss of generality, the finite sets \( X_k \) for \( k \in \{0, 1, 2\} \) are written in the present section as set of indices \( X_k = \{1, \ldots, n_k\} \); similarly, we write \( U = \{1, \ldots, n_u\} \) and \( Y = \{1, \ldots, n_y\} \). With this convention, we define a bijective mapping \( \psi^\ell : X_0 \times X_1 \times X_2 \times Y \times U \rightarrow \{1, \ldots, n^\ell\} \) as

\[
\psi^\ell(i', j', k', l', m') \triangleq m' + n_u(l' - 1) + n_u n_y(k' - 1) + n_u n_y n_2(j' - 1) + n_u n_y n_2 n_1(i' - 1),
\]

which maps a realization \( (i', j', k', l', m') \in X_0 \times X_1 \times X_2 \times Y \times U \) to a unique index \( \psi^\ell(i', j', k', l', m') \in \{1, \ldots, n^\ell\} \). We also set \( n^I \triangleq n_0 n_1 n_2 n_y \) and \( n^II \triangleq n_0 n_1 n_2 n_y n_u \). This allows us to introduce the vector of probability masses \( q^\ell \triangleq (q_1, q_2, \ldots, q_{n^\ell}) \) for \( \ell \in \{I, II\} \), in which each component \( q_i, i \in \{1, \ldots, n^\ell\} \), is equal to \( Q((\psi^\ell)^{-1}(i)) \), and the vector of payoff values \( w^\ell \triangleq (w_1, w_2, \ldots, w_{n^\ell}) \in \mathbb{R}^{n^\ell} \), in which each component \( w_i \) is the payoff of \( (\psi^\ell)^{-1}(i) \). The relation between the mapping \( Q \) (resp. \( w \)) and the vector \( q^\ell \) (resp. \( w^\ell \)) is summarized in Table [III]

Using the proposed indexing scheme, the optimization problem is written in standard form as follows.
writes in case II (case I follows by specialization with

\[ \forall i \in \{1, \ldots, n_i\}, \quad \frac{q_i}{\Theta_i} - \Gamma_i \equiv 0 \quad (i) \]

\[ \forall i \in \{1, \ldots, n_i\}, \quad -\rho_0(i) + \sum_{j=1+(i-1)n_i}^{i n_i n_u n_u} q_j \equiv 0 \quad (j) \]

\[ \forall i \in \{1, \ldots, n_i\}, \quad -q_i \leq 0 \quad (k) \]

\[ \phi^\ell(q_{n^i}) \leq 0 \quad (\ell) \]

where

\[ \Theta_i = \sum_{j \in \{1, \ldots, n_i n_u\}, k \in \{1, \ldots, n_i\}} q_{(k-1)n_i n_u + j} \cdot 1 \{(k-1)n_i n_u \leq i \leq kn_i n_u - 1\} \quad (87) \]

and \( \forall i \in \{1, \ldots, n_i\}, \rho_0(i) = P(X_0 = i) \) and \( \forall i \in \{1, \ldots, n_i\}, \Gamma_i \) corresponds to the value of \( \Gamma(x_0, x_1, x_2) \), according to Table II. As for the function associated with inequality constraint \((\ell)\), it writes in case II (case I follows by specialization with \(|U| = 1\) as follows:

\[
\phi^\Pi(q_{n^i}) = I_{q^{n_i}}(X_0; X_2) - I_{q^{n_i}}(U; Y, X_2) + I_{q^{n_i}}(U; X_0, X_2)
\]

\[
= H_{q^{n_i}}(X_0) - H_{q^{n_i}}(U, X_0|X_2) + H_{q^{n_i}}(U|Y, X_2)
\]

\[
= H_{q^{n_i}}(X_0) + H_{q^{n_i}}(X_2) - H_{q^{n_i}}(X_0, X_2, U) + H_{q^{n_i}}(X_2, Y, U) - H_{q^{n_i}}(X_2, Y)
\]

(88)

with

\[
H_{q^{n_i}}(X_0) = -\sum_{i=1}^{n_i} \left[ \sum_{j=1+(i-1)n_i n_u n_u}^{i n_i n_u n_u} q_j \log \left( \sum_{j=1+(i-1)n_i n_u n_u}^{i n_i n_u n_u} q_j \right) \right],
\]

(89)

\[
H_{q^{n_i}}(X_2) = -\sum_{i=1}^{n_i} \left[ \sum_{j=1}^{n_i n_1 n_2 n_u} \sum_{k=1}^{n_i n_2 n_u} q_{(i-1)n_u n_u + (j-1)n_2 n_u + k} \log \left( \sum_{j=1}^{n_i n_1 n_2 n_u} \sum_{k=1}^{n_i n_2 n_u} q_{(i-1)n_u n_u + (j-1)n_2 n_u + k} \right) \right],
\]

(90)

\[
H_{q^{n_i}}(X_2, Y, U) = -\sum_{i=1}^{n_i} \left[ \sum_{j=1}^{n_i n_2 n_u} \sum_{k=1}^{n_i n_2 n_u} q_{(j-1)n_2 n_u + k} \log \left( \sum_{j=1}^{n_i n_2 n_u} q_{(j-1)n_2 n_u + k} \right) \right],
\]

(91)
\[ H_{q^n}(X_0, X_2, U) = -\sum_{i=1}^{n_0} \sum_{j=1}^{n_2} \sum_{n_u} \left[ \left( \sum_{l=1}^{n_0} \sum_{m=1}^{n_2} q(i-1)n_1n_2n_u + (j-1)n_2n_u + (l-1)n_2n_u + (m-1)n_u \right) \right. \]
\[
\log \left( \sum_{l=1}^{n_0} \sum_{m=1}^{n_2} q(i-1)n_1n_2n_u + (j-1)n_2n_u + (l-1)n_2n_u + (m-1)n_u \right) \] \tag{92}
\]

and
\[
H_{q^n}(X_2, Y) = -\sum_{i=1}^{n_2} \sum_{n_u} \left[ \left( \sum_{j=1}^{n_0} \sum_{l=1}^{n_2} q(j-1)n_1n_2n_u + (i-1)n_u + (l-1)n_2n_u + (j-1)n_2n_u + (i-1)n_u + (j-1)n_2n_u \right) \right. \]
\[
\log \left( \sum_{j=1}^{n_0} \sum_{l=1}^{n_2} q(j-1)n_1n_2n_u + (i-1)n_u + (j-1)n_2n_u + (i-1)n_u + (j-1)n_2n_u \right) \] \tag{93}
\]

This formulation is directly exploited in Section IV-B and in Section V.

B. Optimization problem for perfect monitoring

In the case of perfect monitoring, for which agent 2 perfectly monitors Agent 1’s actions and \( Y = X_1 \), the information constraints (43) and (65) coincide and
\[
\phi(q^n) \overset{\Delta}{=} \phi_I(q^n) = \phi_{II}(q^n) = H_{q^n}(X_2) - H_{q^n}(X_2 | X_0) - H_{q^n}(X_1 | X_0, X_2) \tag{94}
\]

with \( q^n = (q_1, \ldots, q_n), n = n_0n_1n_2 \). To further analyze the relationship between the vector of payoff values \( w^n \) and an optimal joint distribution \( q^n \), we explicitly express the KKT conditions. The Lagrangian is
\[
\mathcal{L}(q^n, \lambda^n, \mu^n, \nu^n, \lambda_{IC}) = -\sum_{i=1}^{n} w_i q_i + \lambda_i q_i + \mu_0 \left[ -1 + \sum_{i=1}^{n} q_i \right] + \sum_{j=1}^{n_0} \mu_j \left[ -\rho_i + \sum_{i=1}^{j_1n_1n_2} q_i \right] \]
\[
+ \lambda_{IC} \phi(q^n) \tag{95}
\]

where \( \lambda^n = (\lambda_1, \ldots, \lambda_n), \mu^n = (\mu_1, \ldots, \mu_{n_0}), \) and the subscript IC stands for information constraint.

A necessary and sufficient condition for a distribution \( q^n \) to be an optimum point is that it is a solution of the following system:
\[
\forall i \in \{1, \ldots, n\}, \quad \frac{\partial \mathcal{L}}{\partial q_i} = -w_i - \lambda_i + \mu_0 + \sum_{j=1}^{n_0} \mu_j \mathbb{1}_{\{1+n_1n_2(j-1) \leq i \leq j_1n_1n_2\}} + \lambda_{IC} \frac{\partial \phi}{\partial q_i}(q^n) = 0 \tag{97}
\]
\[
q^n \text{ verifies } (h), (i), (j) \tag{98}
\]
\[
\forall i \in \{1, \ldots, n\}, \quad \lambda_i \geq 0 \tag{99}
\]
\[
\lambda_{IC} \geq 0 \tag{100}
\]
\[
\forall i \in \{1, \ldots, n\}, \quad \lambda_i q_i = 0 \tag{101}
\]
\[
\lambda_{IC} \phi(q^n) = 0 \tag{102}
\]
where
\[
\forall i \in \{1, \ldots, n\}, \quad \frac{\partial \phi}{\partial q_i}(q^n) = \left[ -\sum_{k=1}^{n_0} \left( (1 + (k-1)n_1n_2 \leq i \leq kn_1n_2) \sum_{j=1+(k-1)n_1n_2}^{kn_1n_2} q_j \right) \right. \\
\left. -\sum_{k=1}^{n_2} \sum_{j=0}^{n_0n_1-1} q_{k+jn_2} - 1 + \log q_i \right].
\]

(103)

In the following, we assume that there exists a permutation of \( \{1, \ldots, n\} \) such that the vector of payoff values \( w^n \) after permutation of the components is strictly ordered. A couple of observations can then be made by inspecting the KKT conditions above. First, if the expected payoff were only maximized under the constraints \((h)\) and \((k)\), the best joint distribution would be to only assign probability to the greatest element of the vector \( w^n \); in other words the best \( q^n \) would correspond to a vertex of the unit simplex \( \Delta(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2) \). However, as the distribution of the random system state fixed by constraint \((j)\), at least \( n_0 \) components of \( q^n \) have to be positive. It is readily verified that under constraints \((h)\), \((j)\), and \((k)\), the optimal solution is that for each \( x_0 \) the optimal pair \((x_1, x_2)\) is chosen; therefore, \( q^n \) possesses exactly \( n_0 \) positive components. This corresponds to the costless communication scenario. Now, in the presence of the additional information constraint \((\ell)\), the optimal solutions contain in general more than \( n_0 \) positive components. This is due to the fact that optimal communication schemes between the two agents requires several symbols of \( \mathcal{X}_1 \) to be associated with a given realization of the system state. In fact, as shown in the following proposition, there is a unique optimal solution under the assumptions made.

**Proposition 21.** If there exists a permutation such that the payoff vector \( w^n \) is strictly ordered, then the optimization problem (86) has a unique solution.

**Proof:** Assume \( \lambda_{IC} = 0 \) in the Lagrangian, i.e., \( \phi(q^n) < 0 \) and the information constraint is active for optimal solutions. Further assume that a candidate solution of the optimization problem \( q^n \) has two or more positive components in a block of size \( n_1n_2 \) associated with a given realization \( x_0 \) (see Table II). Then, there exist two indices \((i_1, i_2)\) such that \( \lambda_{i_1} = 0 \) and \( \lambda_{i_2} = 0 \). Consequently, the conditions on the gradient \( \frac{\partial \mathcal{L}}{\partial q_i} = 0 \) for \( i \in \{i_1, i_2\} \) imply that \( w_{i_1} = w_{i_2} \), which contradicts the assumption of \( w^n \) being strictly ordered under permutation. Therefore, a candidate solution only possesses a single positive component per block associated with a given realization \( x_0 \), which means that \( X_1 \) and \( X_2 \) are deterministic functions of \( X_0 \). Hence, \( H_{q^n}(X_2|X_0) = H_{q^n}(X_1|X_0X_2) = 0 \) and the information constraint reads \( H_{q^n}(X_2) < 0 \), which is impossible. Hence, \( \lambda_{IC} > 0 \).
From Lemma 7 we know that $\phi(q^n)$ is strictly convex. Since $\lambda_{IC} > 0$ the Lagrangian is then the sum of linear functions and a strictly convex function. Since it is also continuous and optimized over a compact and convex set, there exists a maximum point and it is unique.

Apart from assuming that $w^n$ can be strictly ordered, Proposition 21 does not assume anything on the values of the components of $w^n$. In practice, for a specific problem it will be relevant to exploit the special features of the problem of interest to better characterize the relationship between the payoff function (which is represented by $w^n$) and the optimal joint probability distributions (which are represented by the vector $q^n$). This is one of the purposes of the next section.

V. Coded power control

We now exploit the framework developed in the previous sections to study the problem of power control in interference networks. In this context, the agents are the transmitters and the random system state corresponds to the global wireless channel state, i.e., all the channel gains associated with the different links between transmitters and receivers. Coded power control (CPC) consists in embedding information about the global wireless channel state into transmit power levels themselves rather than using a dedicated signaling channel. Provided that the power levels of a given transmitter can be observed by the other transmitters, the sequence of power levels can be used to coordinate with the other transmitters. Typical mechanisms through which agents may observe power levels include sensing, as in cognitive radio settings, or feedback, as often assumed in interference networks. One of the salient features of coded power control is that interference is directly managed in the radio-frequency domain and does not require baseband detection or decoding, which is very useful in systems such as heterogeneous networks. The main goal of this section is to assess the limiting performance of coded power control and its potential performance gains over other approaches, such as the Nash equilibrium power control policies of a given single-stage non-cooperative game. The corresponding comparison is especially relevant since conventional distributed power control algorithms such as the iterative water-filling algorithm do not exploit the opportunity to exchange information through power levels or vectors to implement a better solution (which e.g., Pareto-dominates the Nash equilibrium power control policies).

A. Coded power control over interference channels

We consider an interference channel with two transmitters and two receivers. We specialize the model to the multiple-access channel in Section V-E for which we develop and analyze an explicit non-trivial power control code.
By denoting $g_{ij}$ the channel gain between Transmitter $i$ and Receiver $j$, each realization of the global wireless channel state is given by

$$x_0 = (g_{11}, g_{12}, g_{21}, g_{22}),$$

(104)

where $g_{ij} \in \mathcal{G}$, $|\mathcal{G}| < \infty$; it is further assumed that the channel gains $g_{ij}$ are independent. Each alphabet $\mathcal{X}_i, |\mathcal{X}_i| < \infty$, $i \in \{1, 2\}$, represents the set of possible power levels for Transmitter $i$. Assuming that the sets are discrete is of practical interest, as there exist wireless communication standards in which the power can only be decreased or increased by step and in which quantized wireless channel state information is used. In addition, the use of discrete power levels may not induce any loss of optimality [42] w.r.t. the continuous case, and we elaborate more on this point in Section V-B.

We consider three stage payoff functions $w^{rate}$, $w^{SINR}$, and $w^{energy}$, which respectively represent the sum-rate, the sum-SINR, and the sum-energy efficiency. Specifically,

$$w^{rate} : \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}_+$$

$$(x_0, x_1, x_2) \mapsto \sum_{i=1}^{2} \log_2 \left( 1 + \frac{g_{ii}x_i}{\sigma^2 + g_{-ii}x_{-i}} \right),$$

(105)

$$w^{SINR} : \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}_+$$

$$(x_0, x_1, x_2) \mapsto \sum_{i=1}^{2} \frac{g_{ii}x_i}{\sigma^2 + g_{-ii}x_{-i}},$$

(106)

$$w^{energy} : \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}_+$$

$$(x_0, x_1, x_2) \mapsto \sum_{i=1}^{2} F \left( 1 + \frac{g_{ii}x_i}{\sigma^2 + g_{-ii}x_{-i}} \right) x_i,$$  

(107)

where $\mathcal{X}_0 = \mathcal{G}^4$; the notation $-i$ stands for the transmitter other than $i$; $\sigma^2$ corresponds to the reception noise level; $F : \mathbb{R}_+ \to [0, 1]$ is a sigmoidal and increasing function which can typically represents the block success rate, see e.g., [43], [44]. The function $F$ is chosen so that $w^{energy}$ is continuous and has a limit when $x_i \to 0$.

Finally, we consider the following three possible observation structures.

- **Perfect monitoring**, in which Agent 2 directly observes the actions of Agent 1, i.e., $Y = X_1$;

- **BSC monitoring**, in which Agent 2 observes the actions of Agent 1 through a binary symmetric channel (BSC), i.e., $Y = X_1 \oplus Z_1$ and $X_1, Y, Z_1$ are Bernoulli random variables and $\oplus$ is the 2−modulo addition;
*Noisy SINR feedback monitoring*, in which Agent 2 observes a noisy version of the SINR of Agent 1 as illustrated in Fig. 4. This corresponds to a scenario in which a feedback channel exists between Receiver 2 and Transmitter 2.

![Diagram of SINR feedback monitoring](image)

Fig. 4. The signal observed by Transmitter 2 lies in an $N$–symbol alphabet i.e., $|\mathcal{Y}| = N$. The symbols correspond to possible values for the SINR at the receiver which is associated with Transmitter 2. Transmitter 2 observes the actual value of its SINR with probability $1 - e$ while there is probability $e \geq 0$ that a neighboring SINR is observed. In the simulations, $e = 0.1$.

The performance of coded power control will be assessed against that of the following three benchmark power control policies.

- *Nash-equilibrium power control (NPC) policy*. In such a policy, each transmitter aims at maximizing an individual stage payoff function $u_i(x_0, x_1, x_2)$. In the sum-rate, sum-SINR, and sum-energy efficiency cases, these individual stage payoff-functions are respectively given by $u_i(x_0, x_1, x_2) = \log_2(1 + \text{SINR}_i)$, $u_i(x_0, x_1, x_2) = \text{SINR}_i$, and $u_i(x_0, x_1, x_2) = \frac{F(\text{SINR}_i)}{x_i}$. In the sum-rate and sum-SINR cases, the unique Nash equilibrium is $(x_1^{\text{NE}}, x_2^{\text{NE}}) = (P_{\max}, P_{\max})$, irrespectively of the value of $x_0$, $P_{\max}$ being the maximal power level for the transmitters. In the sum-energy efficiency case, the unique non-trivial Nash equilibrium may be determined numerically and generally requires some knowledge of $x_0$, depending on how it is implemented (see e.g., [18]).

- *Semi-coordinated power control (SPC) policy*. This policy corresponds to a basic coordination
scheme in which Transmitter 1 optimizes its power knowing that Transmitter 2 transmits at full power; SPC requires the knowledge of the current wireless channel state realization at Transmitter 1. Specifically, \( x_2 = P_{\text{max}}, x_1^* = \arg \max_{x_1} w(x_0, x_1, P_{\text{max}}), \ r \in \{\text{rate, SINR, energy}\}. \)

- **Costless-communication power control (CCPC) policy.** This policy corresponds to the situation in which transmitters may communicate at no cost, so that they may jointly optimize their powers to achieve the maximum of the payoff function \( w^r \) at every stage. In such a case there is no information constraint, and the performance of CCPC provides an upper bound for the performance of all other policies.

The communication signal-to-noise ratio (SNR) is defined as

\[
\text{SNR}(\text{dB}) \triangleq 10 \log_{10} \frac{P_{\text{max}}}{\sigma^2}.
\]  

(108)

### B. Influence of the payoff function

The objective of this subsection is to numerically assess the relative performance gain of CPC over SPC in the case of perfect monitoring. We assume that the channel gains \( g_{ij} \in \{g_{\text{min}}, g_{\text{max}}\} \) are Bernoulli distributed \( g_{ij} \sim \mathcal{B}(p_{ij}) \) with \( p_{ij} \triangleq P(g_{ij} = g_{\text{min}}) \); with our definition of \( X_0 \) in (104), this implies that \(|X_0| = 16\). All numerical results in this subsection are obtained for \( g_{\text{min}} = 0.1, g_{\text{max}} = 2 \) and \((p_{11}, p_{12}, p_{21}, p_{22}) = (0.5, 0.1, 0.1, 0.5)\). The sets of transmit powers \( X_1, X_2 \) are both assumed to be the same alphabet of size four \( \{P_1, P_2, P_3, P_4\} \), with \( P_1 = 0, P_2 = \frac{P_{\text{max}}}{3}, P_3 = \frac{2P_{\text{max}}}{3}, P_4 = P_{\text{max}} \). The quantity \( P_{\text{max}} \) is given by the operating SNR and \( \sigma^2 = 1 \). The function \( F \) is chosen as a typical instance of the efficiency function used in [44], i.e.,

\[
F(x) = \exp \left( -2^{0.9} \frac{1}{x} \right).
\]  

(109)

For all \( r \in \{\text{rate, SINR, energy}\} \), the relative performance gain with respect to the SPC policy is

\[
\text{Relative gain (\%)} = \left( \frac{\mathbb{E}_{Q^*}(w^r)}{\mathbb{E}_{\rho_0} \max_{x_1} w(x_0, x_1, P_{\text{max}})} - 1 \right) \times 100
\]  

(110)

where \( Q^* \) is obtained by solving optimization problem (86) under perfect monitoring. This optimization is numerically performed using the Matlab function \texttt{fmincon}. Fig. 5 illustrates the relative performance gain in % w.r.t. the SPC policy for the sum-energy efficiency, while Fig. 6 illustrates it for the sum-SINR and sum-rate.

As shown in Fig. 5, our simulation results suggest that CPC provides significant performance gains for the sum-energy efficiency. This may not be surprising, as the payoff function (107) is particularly
sensitive to the lack of coordination; in fact, as the transmit power becomes high, \( F(SINR_i) \rightarrow \frac{1}{x_i} \), which means that energy efficiency decreases rapidly. As shown in Fig. [6] the performance gains of CPC for the sum-SINR and the sum-rate are more moderate, with gains as high as 43% for the sum-SINR and 25% for the sum-rate; nevertheless, such gains are still significant, and would be larger if we used NPC instead of SPC as the reference case, as often done in the literature of distributed power control.

We conclude this subsection by providing the marginals

\[
Q^\star_{X_1}(x_1) = \sum_{x_0,x_2} Q^\star(x_0,x_1,x_2),
Q^\star_{X_2}(x_2) = \sum_{x_0,x_1} Q^\star(x_0,x_1,x_2),
\]

and joint distribution

\[
Q^\star_{X_1,X_2}(x_1,x_2) = \sum_{x_0} Q^\star(x_0,x_1,x_2)
\]

of the optimal joint distribution for CPC and CCPC in Table III and Table IV, respectively. In both cases, the results correspond to the maximization of the sum-rate payoff function \( w_{\text{rate}} \) and \( \text{SNR} = 10 \text{ dB} \). Table III shows that, without information constraint, the sum-rate is maximized when the transmitters correlate their power levels so that only three pairs of transmit power levels are used out of 16. This result is consistent with [42], which proves that, for interference channels with two transmitter-receiver pairs, there is no loss of optimality in terms of \( w_{\text{rate}} \) by operating over a binary set \( \{0, P_{\text{max}}\} \) instead of a continuous interval \( [0, P_{\text{max}}] \). Interestingly, as seen in Table IV, the three best configurations of the CCPC policy are exploited 44.3 + 42.9 + 2.1 = 89.3% of the time in the CPC policy, despite the presence of communication constraints between the two transmitters.

![Expected sum energy-efficiency](image.png)

**Fig. 5.** Relative sum-energy gain of coded power control (CPC) with perfect monitoring over semi-coordinated power control (SPC).
C. Influence of the observation structure

In this subsection, we focus on the observation structure defined by case I in (4) and we restrict our attention to the sum-rate payoff function $w_{\text{rate}}$. The set of powers is restricted to a binary set $X_1 = X_2 = \{0, P_{\text{max}}\}$, but unlike the study in Section V-B, we do not limit ourselves to perfect monitoring. Fig. 7 shows the relative performance gain w.r.t. the SPC policy as a function of SNR for three different observation structures. The performance of CPC for BSC monitoring is obtained assuming a probability of error of 5%, i.e., $Z_1 \sim B(0.05)$, $P(Z_1 = 1) = 0.05$. The performance of CPC for noisy SINR feedback monitoring is obtained assuming $e = 0.1$; in this case, it can be checked that the SINR can take one of $N = 7$ distinct values.

Fig. 7 suggests that CPC provides a significant performance gain over SPC over a wide range of operating SNRs irrespective of the observation structure. Interestingly, for SNR = 10 dB, the relative gain of CPC only drops from 22% with perfect monitoring to 18% with BSC monitoring, which suggest that for observation structures with typical noise levels the benefits of CPC are somewhat robust to observation noise. Similar observations can be made for SINR feedback monitoring. Note that one would obtain higher performance gains by considering NPC as the reference policy or by considering scenarios with stronger interference.
Fig. 7. Relative sum-rate gain of costless communication power control (CCPC) and coded power control (CPC) over semi-coordinated power control (SPC) under various monitoring assumptions in the observation structure of Case I.

D. Influence of the wireless channel state knowledge

In this subsection, we restrict our attention to CPC with BSC monitoring with the same parameters as in Section V-C but we consider both Case I and Case II defined in (4) and (5), respectively. The results for Case II are obtained assuming that $|\mathcal{U}| = 10$. While we already know that the performance of CPC is the same in Case I and Case II with perfect monitoring, the results in Fig. 8 suggest that, for typical values of the observation noise, not knowing the past realizations of the global wireless channel state at Transmitter 2 only induces a small performance loss.

E. Influence of the coordination scheme

In this last subsection, we assess the benefits of CPC for an explicit code that operates over blocks of length $n = 3$. To simplify the analysis and clarify the interpretation, several assumptions are made. First, we consider a multiple-access channel, which is a special case of the interference channel studied earlier with two transmitters and a single receiver, so that the global wireless channel state comprises only two components $(g_1, g_2)$. Second, we assume that the global wireless channel state $X_0$ takes values in the binary alphabet $X_0 \in \{ (g_{\min}, g_{\max}), (g_{\max}, g_{\min}) \}$, and is distributed according to Bernoulli random variable $\mathcal{B}(p)$ with $p = P(X_0 = (g_{\min}, g_{\max}))$. In the remaining of this subsection, we identify the
realization \((g_{\min}, g_{\max})\) with “0” and \((g_{\max}, g_{\min})\) with “1”, so that we may write \(x_0 = \{0, 1\}\). Third, we assume that the transmitters may only choose power values in \(\{P_{\min}, P_{\max}\}\), and we identify power \(P_{\min}\) with “0” and power \(P_{\max}\) with “1”, so that we may also write \(x_1 = x_2 = \{0, 1\}\). Finally, we consider the case of perfect monitoring and we restrict our attention to the sum-SINR payoff function \(w^{\text{SINR}}\).

The values of the payoff function used in numerical simulations are provided in Fig. 9 as the entries in a matrix. Each matrix corresponds to a different choice of the wireless channel state \(x_0\); in each matrix, the choice of the row corresponds to the action \(x_1\) of Transmitter 1, which the choice of the column corresponds to the action \(x_2\) of Transmitter 2.

\[
\begin{array}{c|cc}
  & P_{\min} & P_{\max} \\
\hline
P_{\min} & 0 & 20 \\
P_{\max} & 1 & \approx 10
\end{array}
\]

\[
\begin{array}{c|c}
  & P_{\min} & P_{\max} \\
\hline
P_{\min} & 0 & 1 \\
P_{\max} & 20 & \approx 10
\end{array}
\]

Fig. 9. Payoff matrix of \(w^{\text{SINR}}\) for power control over multiple-access channel. Numerical values of the payoff correspond to \(g_{\min} = 0.1, g_{\max} = 2, \sigma^2 = 1, P_{\min} = 0, P_{\max} = 10\).
The coordination code of length 3 that we develop next can be seen as a separate source channel code, which consists of a source code with distortion and a channel code with side information. The source encoder and decoder are defined by the mappings

\[
\begin{align*}
&f_S : \mathcal{X}_0^3 \rightarrow \{m_0, m_1\} \\
&\quad \mathcal{X}_0 \mapsto i \\
\end{align*}
\]

(111)

\[
\begin{align*}
&g_S : \{m_0, m_1\} \rightarrow \mathcal{X}_2^3 \\
&\quad i \mapsto \mathcal{X}_2 \\
\end{align*}
\]

(112)

Note that the chosen source code only uses 2 messages \(\{m_0, m_1\}\) to represent the 8 possible sequences \(\mathcal{X}_0\). The exact choice of \(f_S\) and \(g_S\) is provided after we describe the channel code.

In each block \(b\), Transmitter 1’s channel encoder implements the mapping

\[
\mathcal{X}_1^{(b)} = f_C(\mathcal{X}_0^{(b)}, \mathcal{X}_2^{(b)}, i_{b+1})
\]

(113)

where \(i_{b+1} = f_S(\mathcal{X}_0^{(b+1)})\) is the index associated with the sequence \(\mathcal{X}_2^{(b+1)}\). The idea behind the design of the channel encoder \(f_C\) is the following. If Transmitter 1 did not have to transmit the index \(i_{b+1}\), its optimal encoding would be to exploit its knowledge of \((x_0^3(b), x_2^3(b))\) to choose the sequence \(x_1^3(b)\) resulting in the highest average payoff in block \(b\). However, to communicate the index \(i_{b+1}\), Transmitter 1 will instead choose to transmit the sequence \(x_1^3(b)\) with the highest average payoff in block \(b\) if \(i_{b+1} = m_0\), or the sequence \(x_1^3(b)\) with the second highest average payoff in block \(b\) if \(i_{b+1} = m_1\). Note that Transmitter 2 is able to perfectly decode this encoding given its knowledge of \(\mathcal{X}_0^{(b)}, \mathcal{X}_2^{(b)}, \) and \(\mathcal{X}_1^{(b)}\) at the end of block \(b\). Formally, \(f_C\) is defined as follows. The sequence \(\mathcal{X}_1\) is chosen as

\[
\mathcal{X}_1 = \mathcal{X}_1' \oplus d
\]

(114)

where the modulo-two addition is performed component-wise,

\[
\mathcal{X}_1' \in \arg \max_{\mathcal{X}_1' \in \mathcal{X}_1^3} \sum_{t=1}^{3} w_{\text{SINR}}(x_0^t, x_1^t, x_2^t),
\]

(115)

\[
d = (0, 0, 0) \quad \text{if} \quad i_{b+1} = m_0,
\]

(116)

\[
d^3 \in \arg \max_{d^3 \text{ s.t. } \Omega(d^3) = 1} \sum_{t=1}^{3} w_{\text{SINR}}(x_0^t, x_1^t \oplus d_t, x_2^t) \quad \text{if} \quad i_{b+1} = m_1
\]

(117)

where \(\Omega\) is the Hamming weight function that is, the number of ones in the sequence \(d \in \{0, 1\}^3\). If the argmax set is not a singleton set, we choose the sequence with the smallest Hamming weight.

To complete the construction, we must specify how the source code is designed. Here, we choose the mappings \(f_S\) and \(g_S\) that maximize the expected payoff \(E(w_{\text{SINR}})\) knowing the operation of the channel.
code. The source code resulting from an exhaustive search is given in Table VI and the corresponding channel code is given in Table VII. The detailed expression of the expected payoff required for the search is provided in Appendix E.

The proposed codes admit to an intuitive interpretation. For instance, the first line of Table VII indicates that if the channel is bad for Transmitter 1 for the three stages of block $b$, then Transmitter 1 remains silent over the three stages of the block while Transmitter 2 transmits at all three stages. In contrast, the last line of Table VII shows that if the channel is good for Transmitter 1 for the three stages of block $b$, then Transmitter 1 transmits at all stages while Transmitter 2 remains silent two thirds of the time. While this is suboptimal for this specific global wireless channel state realization, this is required to allow coordination and average optimality of the code.

To conclude this section, we compare the performance of this short code with the best possible performance that would be obtained with infinitely long codes. As illustrated in Fig. 10 while the performance of the short code suffers from a small penalty compared to that of ideal codes with infinite block length, it still offers a significant gain w.r.t. the SPC policy and it outperforms the NPC policy.

![Expected payoff versus SNR for different power control policies.](image)

**VI. CONCLUSION**

In this paper, we adopted the view that distributed control policies or resource allocation policies in a network are joint source-channel codes. Essentially, an agent of a distributed network may convey its
knowledge of the network state by *encoding* it into a sequence of actions, which can then be *decoded* by the agents observing that sequence. As explicitly shown in Section V-E, the purpose of such “coordination codes” is neither to convey information reliably nor to meet a requirement in terms of maximal distortion level, but to provide a high expected payoff. Consequently, coordination codes must implement a trade-off between sending information about the future realizations of the network state, which plays the role of an information source and is required to coordinate future actions, and achieving an acceptable payoff for the current state of the network. Considering the large variety of payoff functions in control and resource allocation problems, an interesting issue is whether universal codes performing well within classes of payoff functions can be designed.

Remarkably, since a distributed control policy or resource allocation policy is interpreted as a code, Shannon theory naturally appears to measure the efficiency of such policies. While the focus of this paper was limited to a small network of two agents, the proposed methodology to derive the best coordination performance in a distributed network is much more general. The assumptions made in this paper are likely to be unsuited to some application scenarios, but provide encouraging preliminary results to further research in this direction. For example, as mentioned in Section II, a detailed comparison between coded power control and iterative water-filling like algorithms would lead to consider a symmetric observation structure while only an asymmetric structure is studied in this paper. The methodology to assess the performance of good coded policies consists in deriving the right information constraint(s) by building the proof on Shannon theory for the problem of multi-source coding with distortion over multi-user channels wide side information and then to use this constraint to find an information-constrained maximum of the payoff (common payoff case) or the set of Nash equilibrium points which are compatible with the constraint (non-cooperative game case). As a key observation of this paper, the observation structure of a multi-person decision-making problem corresponds in fact to a multiuser channel. Therefore, multi-terminal Shannon theory is not only relevant for pure communication problems but also for any multi-person decision-making problem. The above observation also opens new challenges for Shannon-theorists since decision-making problems define new communication scenarios.


APPENDIX A

PROOF OF PROPOSITION [5]

Assume $\overline{Q}$ is an achievable empirical coordination. Then, for any $\epsilon > 0$,

$$||E(T_X^T) - \overline{Q}||_1 \leq E(||T_X^T - \overline{Q}||_1)$$

(118)

$$= E(||T_X^T - \overline{Q}||_1 | ||T_X^T - \overline{Q}||_1 \geq \epsilon)P(||T_X^T - \overline{Q}||_1 \geq \epsilon) + E(||T_X^T - \overline{Q}||_1 | ||T_X^T - \overline{Q}||_1 < \epsilon)P(||T_X^T - \overline{Q}||_1 < \epsilon)$$

(119)

$$\leq 2P(||T_X^T - \overline{Q}||_1 \geq \epsilon) + \epsilon.$$  

(120)

Hence, $\forall \epsilon > 0 \lim_{T \to \infty} ||E(T_X^T) - \overline{Q}||_1 \leq \epsilon$, which means that $\overline{Q}$ is implementable.

APPENDIX B

PROOF OF LEMMA [7]

The function $\Phi = \Phi^1$ can be rewritten as $\Phi(Q) = H_Q(X_0) - H_Q(Y, X_0|X_2) + H_Q(Y|X_0, X_2, X_1)$. The first term $H_Q(X_0) = -\sum_{x_0} \rho_0(x_0) \log \rho_0(x_0)$ is a constant w.r.t. $Q$. The third term is linear w.r.t. $Q$ since, with $\Gamma$ fixed,

$$H_Q(Y|X_0, X_2, X_1) = - \sum_{x_0, x_1, x_2, y} Q(x_0, x_1, x_2, y) \log P(y|x_0, x_1, x_2)$$

(121)

$$= - \sum_{x_0, x_1, x_2, y} Q(x_0, x_1, x_2, y) \log \Gamma(y|x_0, x_1, x_2).$$

(122)

It is therefore sufficient to prove that $H_Q(Y, X_0|X_2)$ is concave. Let $\lambda_1 \in [0, 1]$, $\lambda_2 = 1 - \lambda_1$, $(Q_1, Q_2) \in \Delta^2(X_0 \times X_1 \times X_2 \times Y)$ and $Q = \lambda_1 Q_1 + \lambda_2 Q_2$. By using the standard notation $Z^0 = \emptyset$, $Z^n = (Z_1, \ldots, Z_n)$, $Z$ being a generic sequence of random variables, we have that:

$$H_Q(Y, X_0|X_2) = - \sum_{x_0, x_2, y} \left( \sum_{x_1} \lambda_i Q_i(x_0, x_1, x_2, y) \right) \log \left( \frac{\sum_{x_1, i} \lambda_i Q_i(x_0, x_1, x_2, y)}{\sum_i \lambda_i P^Q_{X_2}(x_2)} \right)$$

(123)

$$= - \sum_{x_0, x_2, y} \left( \sum_i \lambda_i \sum_{x_1} Q_i(x_0, x_1, x_2, y) \right) \log \left( \frac{\sum_i \lambda_i \sum_{x_1, i} Q_i(x_0, x_1, x_2, y)}{\sum_i \lambda_i P^Q_{X_2}(x_2)} \right)$$

(124)

$$> - \sum_i \lambda_i \sum_{x_0, x_2, y} \left( \sum_{x_1} Q_i(x_0, x_1, x_2, y) \right) \log \left( \frac{\lambda_i \sum_{x_1} Q_i(x_0, x_1, x_2, y)}{\lambda_i P^Q_{X_2}(x_2)} \right)$$

(125)

$$= - \sum_i \lambda_i \sum_{x_0, x_2, y} \left( \sum_{x_1} Q_i(x_0, x_1, x_2, y) \right) \log \left( \frac{\sum_{x_1} Q_i(x_0, x_1, x_2, y)}{P^Q_{X_2}(x_2)} \right)$$

(126)

$$= \lambda_1 H_{Q_1}(Y, X_0|X_2) + \lambda_2 H_{Q_2}(Y, X_0|X_2)$$

(127)
where the strict inequality comes from the log sum inequality \cite{29}, with:

\begin{equation}
    a_i = \lambda_i Q^i(x_0, x_1, x_2)
\end{equation}

and

\begin{equation}
    b_i = \lambda_i Q^i_{X_2}(x_2)
\end{equation}

for \( i \in \{1, 2\} \) and for all \((x_0, x_1, x_2)\) such that \( Q^i_{X_2}(x_2) > 0 \).

**APPENDIX C**

**LEMMAS USED IN PROOF OF THEOREM 8**

**A. Proof of Lemma 9**

Recall that \( T = \alpha n + (B - 1)n \) with our coding scheme. Note that

\begin{equation}
\| T_{\hat{x}^1_x, \hat{x}^2_x} - \mathcal{Q} \|_1 = \sum_{x_0, x_1, x_2} \frac{1}{T} \sum_{t=1}^{\alpha n} \left| \frac{1}{\alpha n} \sum_{\{x_0^{(t)}_t, x_1^{(t)}_t, x_2^{(t)}_t\} = (x_0, x_1, x_2)} - \mathcal{Q}(x_0, x_1, x_2) \right| \tag{130} \end{equation}

\begin{equation}
= \sum_{x_0, x_1, x_2} \frac{1}{T} \sum_{t=1}^{\alpha n} \left| \frac{1}{\alpha n} \sum_{\{x_0^{(t)}_t, x_1^{(t)}_t, x_2^{(t)}_t\} = (x_0, x_1, x_2)} - \mathcal{Q}(x_0, x_1, x_2) \right|
+ \sum_{b=2}^{B} \sum_{t=1}^{n(B - 1)} \frac{1}{B - 1} \sum_{x_0, x_1, x_2} \left| \sum_{t=1}^{n(B - 1)} \left| \frac{1}{\alpha n} \sum_{\{x_0^{(t)}_t, x_1^{(t)}_t, x_2^{(t)}_t\} = (x_0, x_1, x_2)} - \mathcal{Q}(x_0, x_1, x_2) \right| \right| \tag{131} \end{equation}

\begin{equation}
\leq \frac{2\alpha}{B - 1 + \alpha} + \frac{1}{B - 1} \sum_{b=2}^{B} \| T_{\hat{x}^b_x, \hat{x}^b_x} - \mathcal{Q} \|_1, \tag{132} \end{equation}

where (132) follows from the sub-additivity of \( \| \|_1 \) and (133) follows from \( \frac{n(B - 1)}{T} \leq 1 \) and \( P - \mathcal{Q} \|_1 \leq 2 \) for \((P, Q) \in \Delta^2(\mathcal{X})\).

**B. Proof of Lemma 10**

Since \( \hat{x}^{(1)}_2(i_0) \) and \( \hat{x}^{(1)}_0 \) are known to the transmitter and the receiver, the channel can multiplexed and demultiplexed based on the realizations \( x^{(1)}_{2,t} \) and \( x^{(1)}_{0,t} \) for \( t \in \{1, \ldots, \alpha n\} \). Since \( \hat{x}^{(1)}_0 \) is drawn according to the i.i.d. distribution \( Q_{X_0} \), and \( \hat{x}^{(1)}_2(i_0) \in \mathcal{T}^{n}_{e_1}(Q_{X_2}) \), then by the conditional typicality lemma \( (\hat{x}^{(1)}_0, \hat{x}^{(1)}_2(i_0)) \in \mathcal{T}^{n}_{e_2}(Q_{X_0}, X_2) \) with overwhelming probability as \( n \to \infty \). Following \cite{32} Section 7.4, the probability of decoding error goes to zero as \( n \to \infty \) if \( \hat{R} < (1 - \varepsilon_2) I_{\hat{Q}}(X_1; Y|X_0,X_2) - \delta(\varepsilon_2) \).
C. Proof of Lemma [11]

Note that
\[ E \left( P(E_1^{(b)}) \right) = P \left( \left( X_0^{(b+1)}, X_2^{(b)}(i) \right) \notin T_{c_1}(Q_{X_0X_2}) \text{ for all } i \in \{1, \ldots, 2^{nR} \} \right) \]
(134)
with \( X_0^{(b)} \) distributed according to \( \prod_{i=1}^{n} Q_{x_i} \), and \( X_2^{(b)}(i) \) independent of each other distributed according to \( \prod_{b=1}^{n} Q_{x_2} \). Hence, the result directly follows from the covering lemma [32, Lemma 3.3], with the following choice of parameters.

\[ U \leftarrow \emptyset \quad X^n \leftarrow X_0^{(b+1)} \quad \hat{X}^n(m) \text{ with } m \in A \leftarrow X_2^{(b)}(i) \text{ with } i \in \{1, \ldots, 2^{nR} \}. \]

D. Proof of Lemma [12]

The result follows from a careful application of the conditional typicality lemma. Note that conditioning on \( E_1^{(b-1)c} \) ensures that \( (X_0^{(b)}, \hat{X}_2^{(b)}(I_{b-1})) \in T_{c_1}(Q_{X_0X_2}) \), while conditioning on \( E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0^c \) guarantees that \( \hat{I}_{b-1} = I_{b-1} \). Consequently,

\[ P \left( E_2^{(b)} | E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0^c \right) \]
\[ = P \left( (X_0^{(b)}, \hat{X}_2^{(b)}(I_{b-1}), \hat{X}_1^{(b)}(I_{b-1}), \hat{Y}^{(b)}) \notin T_{c_1}(Q_{X_0X_2Y}) \left| (X_0^{(b)}, \hat{X}_2^{(b)}(I_{b-1})) \in T_{c_1}(Q_{X_0X_2}) \cap \hat{I}_{b-1} = I_{b-1} \right. \right) \]
(135)

\[ = \sum_{i,j,Y} p_{\hat{I}_{b-1}, I_b, X_0^{(b)}(i, j, Y)} \sum_{y} \Gamma(y \mid x_0, x_1^{(b)}(j, y_0, i), x_2^{(b)}(i)) \mathbb{1} \{ (x_0, x_2(i, y_0, i), y) \notin T_{c_1}(Q_{X_0X_2Y}) \}, \]
(136)

where \( p_{\hat{I}_{b-1}, I_b, X_0^{(b)}} \) denotes the joint distribution of \( \hat{I}_{b-1}, I_b, X_0^{(b)} \) given \( X_0^{(b)}, X_2^{(b)}(I_{b-1}) \) \( \in T_{c_1}(Q_{X_0X_2}) \) and \( \hat{I}_{b-1} = I_{b-1} \). Upon taking the average over the random codebooks, we obtain

\[ E \left( P \left( E_2^{(b)} \mid E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0^c \right) \right) \]
\[ = \sum_{i,j,Y} E \left( p_{\hat{I}_{b-1}, I_b, X_0^{(b)}(i, j, Y)} \mathbb{E} \left( \sum_{y} \Gamma(y \mid x_0, x_1^{(b)}(j, y_0, i), x_2^{(b)}(i)) \mathbb{1} \{ (x_0, x_2(i, y_0, i), y) \notin T_{c_1}(Q_{X_0X_2Y}) \} \right| (X_0^{(b)}, \hat{X}_2^{(b)}(i)) \right) \). \]
(137)

The inner expectation is therefore

\[ P \left( (x_0, x_1, x_2, y) \notin T_{c_1}(Q_{X_0X_2Y}) \right), \]
(138)

where \( (X_1, Y) \) is distributed according to \( \prod_{i=1}^{n} Q_{X_0X_2Y}(x_1, y_0, i, x_2, y_0, i) \Gamma(y \mid x_0, x_1, x_2, y_0, i) \), and \( (x_0, y_0, i) \in T_{c_1}(Q_{X_0X_2}) \). The conditional typicality lemma [32, p. 27] guarantees that (138) vanishes as \( n \rightarrow \infty \).
E. Proof of Lemma 13

The result is a consequence of the packing lemma. Note that

\[
P \left( E_3^{(b)} | E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0 \right) = P \left( (X_1^{(b)}, X_2^{(b)}, \widehat{I}_0, X_2^{(b)}) \in T_{c_3}^{n}(Q) \text{ for some } i \neq I_b \right)
\]

The result is a consequence of the packing lemma. Note that

\[
\sum_{k,j,x} P_{I_n-1,I_b} X_1^{(b)}(k,j,x_0) \sum_{y} \Gamma(y|\hat{z}_0, X_1^{(b)}(j, \hat{z}_0, k), X_2^{(b)}(k)) \mathbb{1} \{ (\hat{z}_0, X_1^{(b)}(i, \hat{z}_0, k), X_2^{(b)}(k)) \in T_{c_3}^{n}(Q) \text{ for some } i \neq j \}
\]

Hence,

\[
\mathbb{E} \left( P \left( E_3^{(b)} | E_1^{(b-1)c} \cap E_2^{(b-1)c} \cap E_3^{(b-1)c} \cap E_0 \right) \right)
\]

Note that \(|\{ i : i \neq i_b \}| = 2^{nR} - 1\), and the random sequences \(X_1^{(b)}(i)\) are generated according to \(\prod_{i=1}^{b} Q_{X_i|X_0X_2}\), and are conditionally independent of \(Y\) given \((X_0, X_2)\). Hence, from [32, Lemma 3.1], we know that if \(R < I_Q(X_1; Y|X_0X_2) - \delta(\epsilon_2)\), then \(\mathbb{E}(P(E_3))\) vanishes as \(n \to \infty\).

APPENDIX D

LEMMAS USED IN PROOF OF THEOREM 14

A. Proof of Lemma 15

The proof of this result is similar to that of the following Lemmas, using the “uncoordinated” distribution \(\hat{Q}\) instead of \(Q\). For brevity, we omit the proof.

B. Proof of Lemma 16

The result follows from the covering lemma as in the proof of Lemma 11 given in Appendix C.

C. Proof of Lemma 17

Note that

\[
\mathbb{E} \left( P(E_2^{(b)} | E_1^{(b-1)}) = P(\{ U_1^{(b)}(I_b, j), X_1^{(b)}, X_2^{(b)}(I_{b-1}) \} \notin T_{c_2}^{n}(QU_{X_0X_2}) \right)
\]

October 28, 2014
where \((X_0^{(b)}, X_2^{(b)}(I_{b-1})) \in T_{c_1}^n(Q_{X_0}X_2),\) and \(U^{(b)}(I_b, j)\) are generated independently of each other according to \(\prod_{t=1}^n Q_U.\) Hence, the result follows directly from the covering lemma \([32, \text{Lemma } 3.3]\) with the following choice of parameters.

\[
U \leftarrow \emptyset \quad X^n \leftarrow X_0^{(b)}, X_2^{(b)}(I_{b-1}) \quad \hat{X}^n(m) \text{ with } m \in \mathcal{A} \leftarrow U^{(b)}(I_b, j) \text{ with } j \in \{1, \ldots, 2^{nR'}\}.
\]

D. Proof of Lemma \[18\]

As in the proof of Lemma \[12\] the result follows from a careful application of the conditional typicality lemma. We omit the details, which are essentially the same as those in Appendix C-D, with the necessary modifications to introduce the codewords \(\underline{x}^{(b)}.\)

E. Proof of Lemma \[19\]

The results follows from an analysis essentially identical to that of Appendix C-E with \(U\) in place of \(X_1.\)

**Appendix E**

**Expression of the Expected Payoff \(W_3\) Which Allows the Best Mappings \(f_S\) and \(g_S\) to Be Selected**

We introduce the composite mapping \(\chi_S = g_S \circ f_S.\) For the channel code defined in Section V-E, the expected payoff only depends on the mappings \(f_S\) and \(\chi_S,\) and we denote it by \(W_3(f_S, \chi_S).\) The following notation is used below: \(\chi_S(x_0^3) = (\chi_1(x_0^3), \chi_2(x_0^3), \chi_3(x_0^3))\) to stand for the three components of \(\chi_S.\)

It can be checked that

\[
W_3(f_S, \chi_S) = \sum_{(i,j,k) \in \{0,1\}^3} W_{ijk}(f_S, \chi_S)
\]

where:

\[
W_{000}(f_S, \chi_S) = P_{x_0^3}[x_0^3[0]=0,0,0] \left[ \frac{P_0(f_S)}{P_{[i_t=1]=m_0}} \max_{x_0^3 \in \{0,1\}^3} \left\{ \sum_{t=1}^3 w(0, x_{1,t}, \chi_t((0,0,0))) \right\} \right]
\]

\[
+ (1 - P_0(f_S)) \times \max_{x_0^3 \in \{0,1\}^3} \left\{ \sum_{t=1}^3 w(0, x_{1,t}, \chi_t((0,0,0))) \right\},
\]

October 28, 2014

DRAFT
\[
\lambda^000_1 = \arg \max_{x^3_1 \in \{0, 1\}^3} \left\{ \sum_{t=1}^{3} w(0, x_{1,t}, \chi_t((0, 0, 0))) \right\},
\]

\[
W_{001}(f_S, \chi_S) = p^2(1 - p) \left[ P_0(f_S) \times \left( \max_{x^3_1 \in \{0, 1\}^3} \left\{ w(0, x_{1,1}, \chi_1((0, 0, 1))) + w(0, x_{1,2}, \chi_2((0, 0, 1))) \\
+ w(1, x_{1,3}, \chi_3((0, 0, 1))) \right\} \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x^3_1 \in \{0, 1\}^3 \setminus X^001_1} \left\{ w(0, x_{1,1}, \chi_1((0, 0, 1))) + w(0, x_{1,2}, \chi_2((0, 0, 1))) \\
+ w(1, x_{1,3}, \chi_3((0, 0, 1))) \right\} \right) \right],
\]

\[
\lambda^001_1 = \arg \max_{x^3_1 \in \{0, 1\}^3} \left\{ w(0, x_{1,1}, \chi_1((0, 0, 1))) + w(0, x_{1,2}, \chi_2((0, 0, 1))) + w(1, x_{1,3}, \chi_3((0, 0, 1))) \right\},
\]

\[
W_{010}(f_S, \chi_S) = p^2(1 - p) \left[ P_0(f_S) \times \left( \max_{x^3_1 \in \{0, 1\}^3} \left\{ w(0, x_{1,1}, \chi_1(0, 1, 0)) + w(1, x_{1,2}, \chi_2(0, 1, 0)) \\
+ w(0, x_{1,3}, \chi_3(0, 1, 0)) \right\} \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x^3_1 \in \{0, 1\}^3 \setminus X^010_1} \left\{ w(0, x_{1,1}, \chi_1(0, 1, 0)) + w(1, x_{1,2}, \chi_2(0, 1, 0)) + \\
+ w(0, x_{1,3}, \chi_3(0, 1, 0)) \right\} \right) \right],
\]

\[
\lambda^010_1 = \arg \max_{x^3_1 \in \{0, 1\}^3} \left\{ w(0, x_{1,1}, \chi_1(0, 1, 0)) + w(1, x_{1,2}, \chi_2(0, 1, 0)) + w(0, x_{1,3}, \chi_3(0, 1, 0)) \right\},
\]

\[
W_{100}(f_S, \chi_S) = p^2(1 - p) \left[ P_0(f_S) \times \left( \max_{x^3_1 \in \{0, 1\}^3} \left\{ w(1, x_{1,1}, \chi_1((1, 0, 0))) + w(0, x_{1,2}, \chi_2((1, 0, 0))) \\
+ w(0, x_{1,3}, \chi_3((1, 0, 0))) \right\} \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x^3_1 \in \{0, 1\}^3 \setminus X^{100}_1} \left\{ w(1, x_{1,1}, \chi_1((1, 0, 0))) + w(0, x_{1,2}, \chi_2((1, 0, 0))) + \\
+ w(0, x_{1,3}, \chi_3((1, 0, 0))) \right\} \right) \right],
\]

\[
\lambda^{100}_1 = \arg \max_{x^3_1 \in \{0, 1\}^3} \left\{ w(1, x_{1,1}, \chi_1((1, 0, 0))) + w(0, x_{1,2}, \chi_2((1, 0, 0))) + w(0, x_{1,3}, \chi_3((1, 0, 0))) \right\},
\]
\[ W_{111}(f_S, \chi_S) = (1 - p)^2 \left[ P_0(f_S) \times \left( \max_{x_1^t \in \{0, 1\}^3} \left\{ \sum_{t=1}^{3} w(1, x_{1,t}, \chi_t((1, 1, 1))) \right\} \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x_1^t \in \{0, 1\}^3 \setminus X_{111}^t} \left\{ \sum_{t=1}^{3} w(1, x_{1,t}, \chi_t((1, 1, 1))) \right\} \right) \right], \quad (154) \]

\[ X_{111}^t = \arg \max_{x_1^t \in \{0, 1\}^3} \left\{ \sum_{t=1}^{3} w(1, x_{1,t}, \chi_t((1, 1, 1))) \right\}, \quad (155) \]

\[ W_{011}(f_S, \chi_S) = p(1 - p)^2 \left[ P_0(f_S) \times \left( \max_{x_1^t \in \{0, 1\}^3} \left\{ w(0, x_{1,1}, \chi_1((0, 1, 1))) + w(1, x_{1,2}, \chi_2((0, 1, 1))) \right\} \right) \\
+ w(1, x_{1,3}, \chi_3((0, 1, 1))) \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x_1^t \in \{0, 1\}^3 \setminus X_{011}^t} \left\{ w(0, x_{1,1}, \chi_1((0, 1, 1))) + w(1, x_{1,2}, \chi_2((0, 1, 1)) \right. \right) \\
+ w(1, x_{1,3}, \chi_3((0, 1, 1))) \left. \right\} \right), \quad (156) \]

\[ X_{011}^t = \arg \max_{x_1^t \in \{0, 1\}^3} \left\{ w(0, x_{1,1}, \chi_1((0, 1, 1))) + w(1, x_{1,2}, \chi_2((0, 1, 1))) + w(1, x_{1,3}, \chi_3((0, 1, 1))) \right\}, \quad (157) \]

\[ W_{101}(f_S, \chi_S) = p(1 - p)^2 \left[ P_0(f_S) \times \left( \max_{x_1^t \in \{0, 1\}^3} \left\{ w(1, x_{1,1}, \chi_1((1, 0, 1))) + w(0, x_{1,2}, \chi_2((1, 0, 1))) \right\} \right) \\
+ w(1, x_{1,3}, \chi_3((1, 0, 1))) \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x_1^t \in \{0, 1\}^3 \setminus X_{101}^t} \left\{ w(1, x_{1,1}, \chi_1((1, 0, 1))) + w(0, x_{1,2}, \chi_2((1, 0, 1))) \right. \right) \\
+ w(1, x_{1,3}, \chi_3((1, 0, 1))) \left. \right\} \right], \quad (158) \]

\[ X_{101}^t = \arg \max_{x_1^t \in \{0, 1\}^3} \left\{ w(1, x_{1,1}, \chi_1((1, 0, 1))) + w(0, x_{1,2}, \chi_2((1, 0, 1))) + w(1, x_{1,3}, \chi_3((1, 0, 1))) \right\}, \quad (159) \]
\[ W_{110}(f_S, \chi_S) = p(1 - p)^2 \left[ P_0(f_S) \times \left( \max_{x_3 \in \{0,1\}^3} \left\{ w(1, x_{1,1}, \chi_1((1, 1, 0))) + w(1, x_{1,2}, \chi_2((1, 1, 0))) \\
\quad + w(0, x_{1,3}, \chi_3((1, 1, 0))) \right\} \right) \\
+ (1 - P_0(f_S)) \times \left( \max_{x_3 \in \{0,1\}^3, x_1^{110} \in \{0,1\}} \left\{ w(1, x_{1,1}, \chi_1((1, 1, 0))) + w(1, x_{1,2}, \chi_2((1, 1, 0))) \\
\quad + w(0, x_{1,3}, \chi_3((1, 1, 0))) \right\} \right) \right], \tag{160} \]

\[ X_1^{110} = \arg \max_{x_3 \in \{0,1\}^3} \left\{ w(1, x_{1,1}, \chi_1((1, 1, 0))) + w(1, x_{1,2}, \chi_2((1, 1, 0))) + w(0, x_{1,3}, \chi_3((1, 1, 0))) \right\}. \tag{161} \]

In the case of Table VI, \( P_0(f_S) \) is given by

\[ P_0(f_S) = P[x_0^3 = (0, 0, 0)] + P[x_0^3 = (0, 0, 1)] + P[x_0^3 = (0, 1, 0)] + P[x_0^3 = (0, 1, 1)] + P[x_0^3 = (1, 0, 0)] + P[x_0^3 = (1, 0, 1)] \tag{162} \]

\[ = p(2 - p) \tag{163} \]

\[ p = \frac{3}{4} \tag{164} \]

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| Symbol | Meaning |
|--------|---------|
| $Z$    | A generic random variable |
| $Z_j$ | Sequence of random variables $(Z_i, \ldots, Z_j)$, $j \geq i$ |
| $Z^n$ or $\mathcal{Z}$ | $Z_i^j$ when $i = 1$ and $j = n$ |
| $\mathcal{Z}$ | Alphabet of $Z$ |
| $|\mathcal{Z}|$ | Cardinality of $\mathcal{Z}$ |
| $\Delta(\mathcal{Z})$ | Unit simplex over $\mathcal{Z}$ |
| $z$ | Realization of $\mathcal{Z}$ |
| $z^n$ or $\underline{z}$ | Sequence or vector $(z_1, \ldots, z_n)$ |
| $\mathbb{E}_P$ | Expectation operator under the probability $P$ |
| $H(Z)$ | Entropy of $Z$ |
| $I(Y;Z)$ | Mutual information between $Y$ and $Z$ |
| $Z_1 - Z_2 - Z_3$ | Markov chain $P(z_1|z_2, z_3) = P(z_1|z_2)$ |
| $1_{\{\cdot\}}$ | Indicator function |
| $\oplus$ | Modulo–2 addition |
| $\mathbb{R}_+$ | $[0, +\infty)$ |
| $T_{\epsilon}^n(Q)$ | $\{z^n \in \mathcal{Z}^n : \|T_{z^n} - Q\|_1 < \epsilon\}$ |
| $T_{z^n}$ | Type of the sequence $z^n$ |
TABLE II

Chosen indexation for the payoff vector \( w \) and probability distribution vector \( q \). Bold lines delineate blocks of size \( n_1 n_2 n_y n_u \) and each block corresponds to a given value of the random system state \( X_0 \). The 5-uplets are sorted according to a lexicographic order.

| Index of \( q_i \) | \( X_0 \) | \( X_1 \) | \( X_2 \) | \( Y \) | \( U \) |
|-------------------|--------|--------|--------|------|------|
| 1                 | 1      | 1      | 1      | 1    | 1    |
| 2                 | 1      | 1      | 1      | 1    | 2    |
| \( \vdots \)      | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( n_u \)         | 1      | 1      | 1      | 1    | \( n_u \) |
| \( n_u + 1 \)     | 1      | 1      | 1      | 2    | 1    |
| \( \vdots \)      | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( 2n_u \)        | 1      | 1      | 1      | 2    | \( n_u \) |
| \( \vdots \)      | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( n_1 n_2 n_y n_u - n_u + 1 \) | 1 | \( n_1 \) | \( n_2 \) | \( n_y \) | 1 |
| \( \vdots \)      | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( n_1 n_2 n_y n_u \) | 1 | \( n_1 \) | \( n_2 \) | \( n_y \) | \( n_u \) |
| \( \vdots \)      | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( (n_0 - 1)n_1 n_2 n_y n_u + 1 \) | \( n_0 \) | 1 | 1 | 1 | 1 |
| \( \vdots \)      | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( n_0 n_1 n_2 n_y n_u \) | \( n_0 \) | \( n_1 \) | \( n_2 \) | \( n_y \) | \( n_u \) |

TABLE III

Optimal marginal and joint distributions (expressed in \% ) for the sum-rate payoff function of the CCPC policy, with SNR = 10 dB and with four possible transmit power levels \( \{0, \frac{14}{3}, \frac{28}{3}, 10\} \).

| \( (\mathcal{Q}_{X_1}(x_1), \mathcal{Q}_{X_2}(x_2), \mathcal{Q}_{X_1 X_2}(x_1,x_2)) \) | \( x_1 = 0 \) | \( x_1 = \frac{10}{3} \) | \( x_1 = \frac{20}{3} \) | \( x_1 = 10 \) |
|------------------------------------------------|--------------|--------------|--------------|--------------|
| \( x_2 = 0 \) | (47.5,47.5,00.0) | (00.0,47.5,00.0) | (00.0,47.5,00.0) | (52.5,47.5,47.5) |
| \( x_2 = \frac{10}{3} \) | (47.5,00.0,00.0) | (00.0,00.0,00.0) | (00.0,00.0,00.0) | (52.5,00.0,00.0) |
| \( x_2 = \frac{20}{3} \) | (47.5,00.0,00.0) | (00.0,00.0,00.0) | (00.0,00.0,00.0) | (52.5,00.0,00.0) |
| \( x_2 = 10 \) | (47.5,52.5,47.5) | (00.0,52.5,00.0) | (00.0,52.5,00.0) | (52.5,52.5,05.5) |
TABLE IV

Optimal marginal and joint distributions (expressed in %) for the sum-rate payoff function of the CPC policy, with SNR = 10 dB and with four possible transmit power levels \( \{0, \frac{10}{3}, \frac{20}{3}, 10\} \).

| \( (Q_{X_1}(x_1), Q_{X_2}(x_2), Q_{X_1,X_2}(x_1,x_2)) \) | \( x_1 = 0 \) | \( x_1 = \frac{10}{3} \) | \( x_1 = \frac{20}{3} \) | \( x_1 = 10 \) |
|-----------------|----------|-------------|-------------|-------------|
| \( x_2 = 0 \)   | (44.4, 50.4, 00.1) | (02.6, 50.4, 00.9) | (08.0, 50.4, 06.5) | (45.0, 50.4, 42.9) |
| \( x_2 = \frac{10}{3} \) | (44.4, 00.0, 00.0) | (02.6, 00.0, 00.0) | (08.0, 00.0, 00.0) | (45.0, 00.0, 00.0) |
| \( x_2 = \frac{20}{3} \) | (44.4, 00.0, 00.0) | (02.6, 00.0, 00.0) | (08.0, 00.0, 00.0) | (45.0, 00.0, 00.0) |
| \( x_2 = 10 \) | (44.4, 49.6, 44.3) | (02.6, 49.6, 01.7) | (08.0, 49.6, 01.5) | (45.0, 49.6, 02.1) |

TABLE V

Proposed source coding and decoding for \( p = \frac{1}{2} \).

| \( x_0^3 \) | \( \text{Index } i = f_S(x_0^3) \) | \( g_S(i) \) |
|----------|----------------|-------------|
| (0,0,0)  | \( m_0 \)          | (1,1,1)     |
| (0,0,1)  | \( m_0 \)          | (1,1,1)     |
| (0,1,0)  | \( m_0 \)          | (1,1,1)     |
| (0,1,1)  | \( m_0 \)          | (1,1,1)     |
| (1,0,0)  | \( m_0 \)          | (1,1,1)     |
| (1,0,1)  | \( m_0 \)          | (1,1,1)     |
| (1,1,0)  | \( m_1 \)          | (0,0,1)     |
| (1,1,1)  | \( m_1 \)          | (0,0,1)     |
TABLE VI

PROPOSED CHANNEL CODING FOR $p = \frac{1}{2}$.

| $x_0^3(b)$ | $x_2^3(b)$ | $i_{b+1}$ | $x_1^3(b)$ |
|------------|------------|------------|------------|
| (0,0,0)    | (1,1,1)    | $m_0$      | (0,0,0)    |
|            |            | $m_1$      | (0,0,1)    |
| (0,0,1)    | (1,1,1)    | $m_0$      | (0,0,1)    |
|            |            | $m_1$      | (0,0,0)    |
| (0,1,0)    | (1,1,1)    | $m_0$      | (0,1,0)    |
|            |            | $m_1$      | (0,0,0)    |
| (0,1,1)    | (1,1,1)    | $m_0$      | (0,1,1)    |
|            |            | $m_1$      | (0,0,1)    |
| (1,0,0)    | (1,1,1)    | $m_0$      | (1,0,0)    |
|            |            | $m_1$      | (0,0,0)    |
| (1,0,1)    | (1,1,1)    | $m_0$      | (1,0,1)    |
|            |            | $m_1$      | (0,0,1)    |
| (1,1,0)    | (0,0,1)    | $m_0$      | (1,1,0)    |
|            |            | $m_1$      | (1,1,1)    |
| (1,1,1)    | (0,0,1)    | $m_0$      | (1,1,1)    |
|            |            | $m_1$      | (1,1,0)    |