Tetracritical behaviour in a spin-$3/2$ quantum chain

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Abstract

Finite-Size-Scaling and Conformal Invariance are used in order to find the phase diagram and critical exponents of a quantum spin chain with spin $S = 3/2$. The model has a tetracritical point besides critical lines. The conformal anomaly and anomalous dimensions of some primary operators are calculated at the tetracritical point.

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Conformal invariance enable us to classify a large number of field theories in two dimensions \[1, 2\], however it is not obvious that there exist Statistical Mechanics models realizing such field theories. Meanwhile several examples of Statistical Mechanics models described by such field theories are known \[3\]. In particular Andrews, Baxter, Forrester \[4\] and Huse \[5\] introduced the RSOS model which gives a realization of the minimal series \[2\] with central charge

\[ c = 1 - \frac{6}{(K + 2)(K + 3)} \quad K = 1, 2, 3, \ldots \]  

(1)

However the quantum Hamiltonian associated to the RSOS model are rather complicated.

Recently Sen \[6\] introduced a series of simple spin-$S$ Hamiltonian. Based in a Zamolodchikov’s Landau-Ginzburg theory he conjectured that, at special multicritical points, these models also give realizations of the minimal series in Eq. (1). These models are defined by the Hamiltonians

\[ H[S] = \sum_{i=1}^{L} \left[ \frac{1}{2} \left( S_i^z - S_{i+1}^z \right) + \gamma S_i^x + \sum_{n=1}^{[S]} a_{2n} (S_i^z)^{2n} \right] \]  

(2)

where $S_i^x, S_i^y$ and $S_i^z$ are spin-$S$ representations of $SU(2)$ algebra acting on site $i$ of an $L$ sites chain and periodic boundary conditions (PBC) are imposed. The integer $K$ in Eq. (1) are related to the spin $S$ in Eq. (2) by
\( K = 2S \). The degree of the polynomial is \([S]\), the largest integer less than or equal \( S \). At a particular point such Hamiltonian is expected \([6]\) to present a multicritical point where \( K + 1 \) phases are indistinguishable. If \( S = 1/2 \) then \( K = 1 \ (c = 1/2) \) and \( H[S] \) [Eq. (2)] reduces to the Ising Hamiltonian.

In the case \( S = 1 \) we have \( K = 2 \) and \( H[S] \) is given by

\[
H_{[1]} = H_{[3/2]} = -\sum_{i=1}^{L} \left[ S_i^z S_{i+1}^z - (1 + a_2) (S_i^z)^2 - \gamma S_i^x \right]
\]  

which is a particular case of the Blume-Emery-Griffths model \([7]\), having a tricritical point governed by a Conformal Field Theory (CFT) with \( c = 7/10 \).

In case \( S = 3/2 \) the Hamiltonian is the same as in Eq. (3) and a tetracritical point is expected with \( c = 4/5 \) \([6]\). This tetracritical point is the meeting point where two second order lines end up in a first order transition line.

In this work by using Finite-Size-Scaling \([8]\) (FSS) and the Conformal Invariance (CI) predictions for finite systems \([8, 10, 11]\) we studied the Hamiltonian \( H[3/2] \) [Eq. (3)] for \( S = 3/2 \) to test the conjectures of reference \([6]\).

Firstly, in order to calculate the phase diagram we locate the second order lines by using FSS. This lines can be found by extrapolating the sequence of
curves obtained from the relation

\[ L G_L(\gamma, a_2) = L' G_{L'}(\gamma, a_2) \]  

(4)

where \( L \) and \( L' \) are the sizes of two finite chains with \( L' = L + 1 \) and \( G_L(\gamma, a_2) \) is the mass gap of the Hamiltonian \( H[3/2] \) evaluated at \( (\gamma, a_2) \) for the finite chain of size \( L \).

The spectral calculations were done numerically using the Lanczos algorithm [12] for lattices sizes up to \( L = 11 \). Beyond the translation invariance the Hamiltonian \( H[3/2] \) commutates with the parity operator

\[ P = \prod_{i=1}^{L} \left[ (S^x_i)^2 - 1 \right] \]  

(5)

and consequently we can separate the Hilbert space into disjoint sectors labelled according to the momentum \( q = \frac{2\pi}{L} l \), \( l = 0, 1, 2, ..., L - 1 \), and parity \( p = \pm 1 \) eigenvalues.

The ground state (GS) is a zero-momentum state with parity \(+ (-)\) for lattices sizes \( L \) even (odd), and the first excited state is also a zero-momentum state but with opposite parity.

In Figure 1 we show in the parameter space of \( \gamma \) and \( a_2 \) the curves satisfying Eq. (3) for some values of \( L \). We also show in a large scale the
region where the intersections of these curves occur. We promptly verify the qualitative agreement of Fig. 1 with the phase diagram proposed in ref. [6].

The tetracritical point \( P \) (see Fig.1) where a conformal anomaly \( c = 4/5 \) is expected [6] is obtained by the intersection of the two second order phase transition lines. Unfortunately, as we can see in Fig. 1, in the region where these intersections occur, the second order lines are almost parallel, which make difficult a high precision determination of the finite-size estimative of the point \( P \). This difficulty obviously will be reflected in the convergence of the numerical sequence used for the bulk limit \( (L \rightarrow \infty) \) evaluation of \( P \).

An alternative way [13], which give us better estimatives of the tetracritical point \( P \), is obtained by using the sequence of points calculated by a generalization of Eq. (4)

\[
L G_L(\gamma, a_2) = L' G_L'(\gamma, a_2) = L'' G_L''(\gamma, a_2)
\]  

where \( L'' = L' + 1 = L + 2 \). This condition is satisfied at the crossing point obtained by using Eq. (4). These crossing points are denoted in Fig. 1 and we show their coordinates in Table 1. The tetracritical point is obtained by extrapolating the coordinates of these estimators. Unfortunately, due to the small number of points these extrapolations are poor, specially for the
sequence related to the coordinate $\gamma$. The best estimatives we get for these extrapolations are $\gamma = 0.36 \pm 0.01$ and $a_2 = 0.065 \pm 0.002$, where the errors estimates are personal.

A further analysis of the data of table 1 show us that these points obey, with a very good fitting, the following relation:

$$a_2(\gamma) = 0.036883 - 0.23522\gamma - 0.1159\gamma^2$$

with squares mean deviations 3\% and 8\% for $a_2$ and $\gamma$ respectively. On the other hand in ref. [6] based in a pertubation expansion of $H[3/2]$ up to order $\gamma^4$, the point P should also lay in the curve

$$a_2(\gamma) = -\frac{1}{2}\gamma^2 + \frac{25}{192}\gamma^4.$$  \hspace{1cm} (8)

An estimative of P is then obtained by equating Eqs. (7) and (8). Apart from spurius solutions we obtain $P = (0.3702, -0.0661)$, which we believe, considering the small number of data we have, to be our best estimative for this tetracritical point.

Once the point P is calculated the next step towards the verification of the conjecture of ref. [6] is the calculation of the conformal anomaly and dimensions of the operators of the underling CFT governing the large physics at this point. This is done by exploiting at point P the consequences of CI in
the finite-size-corrections of the eigenspectra. The conformal anomaly $c$ can be obtained, for periodic chains, from the finite-size-corrections of the GS energy $E_0(L)$, i.e.,

$$\frac{E_0(L)}{L} = e_\infty - \frac{\pi c \zeta}{6L^2} + o(L^{-2}) \quad (9)$$

where $e_\infty$ is the bulk limit of the GS energy per particle. In Eq. (9) $\zeta$ is the sound velocity (non-universal) and can be calculated from the mass gap associated to the first excited state with momentum $\frac{2\pi}{L}$ in the sector with the same parity of the GS, i.e.,

$$\zeta_L = \frac{L}{2\pi} (E_0(L) - E_{1,0}) \xrightarrow{L \to \infty} \zeta \quad (10)$$

Using the Eqs. (9) and (10) we obtain for several points in the second order phase transitions lines of Fig. 1 a value close to $c = 1/2$ (Ising). As we move in Fig. 1 along the second-order transition line, we verify around the point P a clear cross-over (finite-size-effect) of our estimatives of the conformal anomaly, and at the point P we obtain a value $e_\infty = -0.2512$, $\zeta = 0.67$ and $c = 0.76$. Clearly this is very different of $c = 0.5$ of the Ising line and due to numerical instability in our extrapolations, is also compatible with the conjectured value $[6]$ of $c = 0.8$. A better numerical test of the universality class of the point P is obtained by calculating the anomalous dimensions of
the field theory associated to this point. Associated to the dimension \( x_\phi \) of a given primary operator of the theory \( \phi \), with spin \( s_\phi \), there exist tower of eigenstates with energies and momentum given by

\[
E_{m,m'}(L) - E_0(L) = \frac{2\pi \zeta}{L}(x_\phi + m + m') + o(L^{-1})
\]

\[
P_{m,m'}(L) = \frac{2\pi}{L}(s_\phi + m - m') \quad m, m' = 0, 1, 2, ... .
\]

Using these expressions we obtained finite-size estimators for the several dimensions associated to the eigenenergies of \( H[3/2] \). In particular the lowest gap (with opposite parity of the GS) gives the dimension \( x_{\phi_1} = 0.052 \), and in the sector with the parity of the GS the lowest gap give us \( x_{\phi_2} = 0.130 \). These values are much smaller than the values along the second-order transitions lines \((c = 0.5)\), which are \( x_{\phi_1} = 0.125 \) and \( x_{\phi_2} = 0.5 \). According to the conjecture of ref. [6] these gaps should be compared with the two magnetic operators of the modular invariant serie \([10, 14] A(5)\), with dimensions \( x_{\phi_1} = \frac{1}{20} = 0.05 \) and \( x_{\phi_1} = \frac{2}{15} = 0.1333... \). As we can see the agreement with our numbers, taking into account the small number of lattices, is good.

In conclusion, our results for spin 3/2 and early results for \( S = 1 \) are clear indications in favour of the conjecture of ref. [6], namely, the Hamiltonian \( H[S] \) [Eq. (2)] gives us statistical mechanics models with multicritical points.
governed by the modular invariance series $A(K + 2)$ of CFT, having central charge given by Eq. (1) with $K = 2S$. After the completion of our calculations we became aware of a preprint\cite{15} with results in agreement with our conclusions.

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Figure Captions

Fig. 1 - Curves satisfying Eq. (4) for some values of $L$ and, in a large scale, the region where the intersections of curves occur. Three lattices crossing points satisfying Eq. (5) and tetracritical point $P = (0.3702, -0.0661)$ is also showed.

Table Captions

Table 1 - Three lattices crossing points satisfying Eq. (5) and extrapolated values.
Table 1

| $L, L', L''$ | $\gamma$   | $a_2$   |
|--------------|------------|---------|
| 2,3,4        | 0.61867    | -0.153  |
| 3,4,5        | 0.473      | -0.10033|
| 4,5,6        | 0.4266     | -0.084526|
| 5,6,7        | 0.404      | -0.077047|
| 6,7,8        | 0.39166    | -0.073026|
| 7,8,9        | 0.384      | -0.070553|
| extrap.      | 0.36 ±0.01 | -0.065 ±0.002|