Whitham Deformations of Seiberg-Witten Curves for Classical Gauge Groups

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Abstract

Gorsky et al. presented an explicit construction of Whitham deformations of the Seiberg-Witten curve for the $SU(N+1)$ $\mathcal{N} = 2$ SUSY Yang-Mills theory. We extend their result to all classical gauge groups and some other cases such as the spectral curve of the $A_{2N}^{(2)}$ affine Toda system. Our construction, too, uses fractional powers of the superpotential $W(x)$ that characterizes the curve. We also consider the $u$-plane integral of topologically twisted theories on four-dimensional manifolds $X$ with $b_2^+(X) = 1$ in the language of these explicitly constructed Whitham deformations and an integrable hierarchy of the KdV type hidden behind.
1 Introduction

The geometry of Seiberg and Witten’s low energy effective theory of the four-dimensional $\mathcal{N} = 2$ SUSY Yang-Mills theories (with and without matters) \[1\] are based on complex algebraic curves now generally called the “Seiberg-Witten curves”. Coordinates of the moduli space (Coulomb branch) $\mathcal{U}$ of these curves are given by the Casimirs $u_j (j = 1, \cdots, N = \text{rank}(G))$ of the scalar field $\phi$ of the $\mathcal{N} = 2$ vector multiplet. Each curve carries a special meromorphic differential (the “Seiberg-Witten differential”) $dS_{SW}$. Given a suitable set of cycles $A_j, B_j (j = 1, \cdots, N)$, this differential $dS_{SW}$ induces a “special geometry” on the moduli space $\mathcal{U}$. The “special coordinates” are given by the period integrals $a_j$ and $a_j^D$ of $dS_{SW}$ along these cycles. The prepotential $\mathcal{F}$ of the low energy effective theory is determined by this special geometry.

Gorsky et al. \[2\] and Martinec and Warner \[3\] discovered that an integrable system, typically an affine Toda system, is hidden behind this geometric setup. This point of view soon turned out to be very useful not only for studying various four-dimensional $\mathcal{N} = 2$ SUSY gauge theories \[4, 5, 6, 7, 8, 9, 10, 11, 12, 13\] but also for elucidating a universal mathematical structure underlying those examples \[14, 15\]. In fact, all the building blocks of the Seiberg-Witten theory fit to the language of integrable systems. The Seiberg-Witten curve is nothing but the spectral curve of the integrable system. The differential $dS_{SW}$ is related to its complex symplectic structure. The special coordinates $a_j$ are identified with the action variables; the angle variables live on an Abelian subvariety (called the “special” or “distinguished” Prym variety) of the Jacobi variety of the Seiberg-Witten curve. The Casimirs $u_j$ give an involutive set of Hamiltonians. The commuting flows generated by these Hamiltonians sweep out the aforementioned Prym variety. In a more geometric language, the phase space $\mathcal{X}$ has a Lagrangian fibration $\mathcal{X} \to \mathcal{U}$ by these $N$-dimensional complex “Lieouville tori”.

This link with integrable systems implies the existence of another kind of deformations of the Seiberg-Witten curves: Whitham deformations \[2, 3, 4, 9, 10\] The differential equations that characterize these deformations, are called “Whitham equations”. These deformations are parametrized by an extra set of variables $T_n (n = 1, 2, \cdots)$. These variables are referred to as “slow variables” in the theory of Whitham equations; “fast variables” $t_n$ are the time variables of commuting Hamiltonian flows in the integrable
Several interesting physical interpretations of these Whitham deformations have been proposed. Deformations by $T_1$ are identified to be the renormalization group flows \[16, 17, 6, 18, 13\]. In this sense, the other Whitham deformations may be thought of as generalized RG flows. Gorsky et al. \[19\] constructed an explicit solution of the Whitham equations for the $SU(N+1)$ Seiberg-Witten curve, and argued its relation to topologically twisted gauge theories. Edelstein et al. \[20\] interpreted the Whitham deformations of Gorsky et al. as soft breaking of $\mathcal{N} = 2$ SUSY by spurion fields.

In this paper we generalize the construction of Gorsky et al. \[19\] to other classical gauge groups. We consider the $\mathcal{N} = 2$ SUSY Yang-Mills theories in the vector representation of classical gauge groups. As Martinec and Warner pointed out \[3\], the Seiberg-Witten curves for these theories can be written in a unified form, see \[2, 1\], which is essentially the spectral curves of affine Toda systems. This expression contains a function $W(x)$ (called the “superpotential” in analogy with topological Landau-Ginzburg theories of A-D-E singularities). In the case of the $SU(N+1)$ Seiberg-Witten curve, $W(x)$ is a polynomial. Gorsky et al. use fractional powers of $W(x)$ very ingeniously. Actually, their method work for a general rational superpotential. We demonstrate it in the case of the Seiberg-Witten curves for other classical gauge groups and some other Toda spectral curves.

We also address another issue that is raised in our previous paper \[21\]. As discussed therein, an interesting interplay of the “slow variables” $T_n$ and the “fast variables” $t_n$ can be observed in the $u$-plane integral of the topologically twisted theories on four-dimensional manifolds $X$ with $b_2^+(X) = 1$ \[22, 23, 24\]. ($b_2^+(X)$ is the self-dual part of the 2nd Betti number.) Our consideration in the previous paper was limited to the case of $SU(N+1)$. We revisit this issue, now armed with the explicit construction of Whitham deformations for all classical gauge groups.

This paper is organized as follows. Section 2 is a collection of basic mathematical notions concerning Seiberg-Witten curves. Particularly important are the Prym varieties and related differentials that are crucial in handling the cases other than $SU(N+1)$. Section 3 is a review of the construction of Gorsky et al. We present all technical details, which are used in the subsequent sections. In Section 4 we consider the case of the $SO(2N)$
Seiberg-Witten curve in detail. Section 5 deals with the case of $SO(2N+1)$ and $Sp(2N)$, along with some other cases that are not directly related to $\mathcal{N} = 2$ SUSY Yang-Mills theories but can be treated similarly. In Section 6 we consider the $u$-plane integral of the topologically twisted theories. Section 7 is devoted to discussions. Appendix is added to show a precise form of the spectral curves of the affine Toda systems in the usual Lax formalism.

2 Curves, Differentials, and Prym Varieties

2.1 Various complex algebraic curves

The Seiberg-Witten curves of $\mathcal{N} = 2$ SUSY Yang-Mills theories in the vector representation of classical gauge groups can be written in the following common form [3]:

$$z + \frac{\mu^2}{z} = W(x).$$

(2.1)

Here $\mu$ is some power of the renormalization group parameter $\Lambda$, and $W(x)$ (the “super-potential”) a polynomial or a Laurent polynomial of the following form:

- $SU(N+1)$ : $W(x) = x^{N+1} - \sum_{j=2}^{N+1} u_j x^{N+1-j}$.
- $SO(2N+1)$ : $W(x) = x^{-1} \left( x^{2N} - \sum_{j=1}^{N} u_j x^{2N-2j} \right)$.
- $Sp(2N)$ : $W(x) = x^2 \left( x^{2N} - \sum_{j=1}^{N} u_j x^{2N-2j} \right) + 2\mu$.
- $SO(2N)$ : $W(x) = x^{-2} \left( x^{2N} - \sum_{j=1}^{N} u_j x^{2N-2j} \right)$.

The polynomials

$$P(x) = x^{N+1} - \sum_{j=2}^{N+1} u_j x^{N+1-j}$$

(2.2)

for the $SU(N+1)$ gauge group and

$$Q(x^2) = x^{2N} - \sum_{j=1}^{N} u_j x^{2N-2j}$$

(2.3)
for the other gauge groups may be identified with the characteristic polynomial \( \det(xI - \phi) \) of the scalar field \( \phi \) of the \( \mathcal{N} = 2 \) SUSY vector multiplet. The coefficients \( u_j \) are accordingly the expectation value of the Casimirs of \( \phi \). In the case of \( SU(N + 1) \) and \( SO(2N) \), \( W(x) \) coincides with the superpotential of the topological Landau-Ginzburg theories (or \( d < 1 \) strings) of singularities of the A and D type \[24, 20\]. Because of this, \( W(x) \) is referred to as the “superpotential”.

As Martinec and Warner pointed out \[\![3]\] these curves coincide with the spectral curves of the affine Toda system of the type \( g^{(1)\vee} \) dual to the the untwisted affine Lie algebra \( g^{(1)} \) of the gauge group \( G \). In particular, the affine Toda systems for the non-simply-laced gauge groups \( SO(2N + 1) \) and \( Sp(2N) \) are of the twisted affine type \( B_{N+1}^{(1)} = A_{2N-1}^{(2)} \) and \( C_N^{(1)\vee} = D_{N+1}^{(2)} \). The affine Toda systems for the other classical affine Lie algebras, too, have spectral curves of the above form:

\[
B_{N}^{(1)}, C_{N}^{(1)}-\text{Toda} : \quad W(x) = x^{2N} - \sum_{j=1}^{N} u_j x^{2N-2j},
\]

\[
A_{2N}^{(2)}-\text{Toda} : \quad W(x) = x \left( x^{2N} - \sum_{j=1}^{N} u_j x^{2N-2j} \right).
\]

All these curves are hyperelliptic. The following is an equivalent expression in the usual expression \( y^2 = R(x) \) of hyperelliptic curves. Actually, it is in this form (or a quotient curve discussed later on) that the Seiberg-Witten curves for classical gauge groups were first derived \[28, 29, 30, 31\].

1. The Seiberg-Witten curves:

\[
SU(N + 1) : \quad y^2 = P(x)^2 - 4\mu^2, \quad z = (P(x) + y)/2.
\]

\[
SO(2N + 1) : \quad y^2 = Q(x^2)^2 - 4\mu^2 x^2, \quad z = (Q(x^2) + y)/2x.
\]

\[
Sp(2N) : \quad y^2 = Q(x^2)(x^2Q(x^2) + 4\mu), \quad z = (2\mu + x^2Q(x^2) + xy)/2.
\]

\[
SO(2N) : \quad y^2 = Q(x^2)^2 - 4\mu^2 x^4, \quad z = (Q(x^2) + y)/2.
\]

2. Other affine Toda spectral curves:

\[
B_{N}^{(1)}, C_{N}^{(1)} : \quad y^2 = Q(x^2)^2 - 4\mu^2, \quad z = (Q(x^2) + y)/2.
\]

\[
A_{2N}^{(2)} : \quad y^2 = x^2 Q(x^2)^2 - 4\mu^2, \quad z = (xQ(x^2) + y)/2.
\]
The curves other than the $SU(N + 1)$ Seiberg-Witten curve can be classified into two groups:

- A: The $Sp(2N)$ Seiberg-Witten curve and the $A_{2N}^{(2)}$-Toda curve.
- B: The other curves.

We shall show that the curves in the two groups exhibit different properties in many aspects. The first aspect that we now point out, is their genera:

- Case A: The curve has genus $2N$.
- Case B: The curve has genus $2N - 1$.

2.2 Involutions and Prym varieties

The above curves, which we denote by $C$, have several involutions. Common to all are the hyperelliptic involution

$$\sigma_1 : (x, y) \mapsto (x, -y), \quad (x, z) \mapsto (x, \mu^2/z).$$

(2.4)

All the curves other than the $SU(N + 1)$ curves have the second involution

$$\sigma_2 : (x, y) \mapsto (-x, y), \quad (x, z) \mapsto (-x, z).$$

(2.5)

The quotient $C_2 = C/\sigma_2$ by the second involution is also a hyperelliptic curve. It has different properties in accordance with the above classification:

- Case A: $C_2$ has genus $N$, and the covering map $C \to C_2$ is unramified.
- Case B: $C_2$ has genus $N - 1$, and the covering map $C \to C_2$ is ramified.

The double covering $C \to C_2$ determines the Prym variety $\text{Prym}(C/C_2)$. This is an $N$-dimensional Abelian variety, which plays the role of the Jacobi variety $\text{Jac}(C)$ for the $SU(N + 1)$ Yang-Mills theory. (Fay’s book provides us with useful information on this kind of Prym varieties.) Following Fay’s book, let us specify the structure of this Prym variety in more detail.

The algebro-geometric definition of this Prym variety is based on an automorphism $\sigma_2 : \text{Jac}(C) \to \text{Jac}(C)$ induced by the involution $\sigma_2$. Consider the Jacobi variety as
the set of the linear equivalence classes of divisors \( \mathcal{D} \) of degree zero. The Prym variety \( \text{Prym}(C/C_2) \), by definition, is the image of \( \text{id} - \sigma_2 \) (where \( \text{id} \) is the identity map), namely, consists of the linear equivalence classes of divisors of the form \( \mathcal{D} - \sigma_2(\mathcal{D}) \).

An equivalent complex analytic expression is the complex torus
\[
\text{Prym}(C/C_2) \simeq \mathbb{C}^N/(\Delta \mathbb{Z}^N + 2\mathcal{P} \mathbb{Z}^N),
\]
where \( \Delta \) is a diagonal matrix \( \Delta = \text{diag}(d_1, \ldots, d_N) \) with positive integers on the diagonal line, and \( \mathcal{P} \) is a complex symmetric matrix \( (\mathcal{P}_{jk}) \) with positive definite imaginary part. This Prym variety is thus a polarized Abelian variety with the following polarization \((d_1, \ldots, d_N)\):

- Case A: \((d_1, \ldots, d_N) = (2, \ldots, 2, 2)\).
- Case B: \((d_1, \ldots, d_N) = (2, \ldots, 2, 1)\).

In particular, Case A is substantially a principally polarized Abelian variety with period matrix \( \mathcal{P} \); the above expression is simply for dealing with the two cases in a unified way.

The matrix elements of \( \mathcal{P} \) are period integrals of holomorphic differentials \( d\omega_j \) \((j = 1, \ldots, N)\) that are “odd” under the action of \( \sigma_2 \):
\[
\sigma_2^* d\omega_j = -d\omega_j.
\]

These differentials are uniquely determined by the normalization condition
\[
\oint_{A_j} d\omega_k = \delta_{jk},
\]
and the matrix elements \( \mathcal{P}_{jk} \) of \( \mathcal{P} \) are given by
\[
\mathcal{P}_{jk} = \frac{d_j}{2} \oint_{B_j} d\omega_k.
\]

The \( 2N \) cycles \( A_j, B_j \) \((j = 1, \ldots, N)\) in these period integrals have to be chosen as follows:

- Case A: The \( 4N \) cycles \( A_j, -\sigma_2(A_j), B_j, -\sigma_2(B_j) \) \((j = 1, \ldots, N)\) form a symplectic basis of cycles of \( C \).
- Case B: The homology classes \([A_N]\) and \([B_N]\) are “odd” under the action of \( \sigma_2 \), i.e., \( \sigma_2([A_N]) = -[A_N] \) and \( \sigma_2([B_N]) = -[B_N] \). The \( 4N-2 \) cycles \( A_j, -\sigma_2(A_j), B_j, -\sigma_2(B_j) \) \((j = 1, \ldots, N-1)\) and \( A_N, B_N \) altogether form a symplectic basis of cycles of \( C \).

In particular, these cycles have the intersection numbers \( A_j \cdot A_k = B_j \cdot B_k = 0 \) and \( A_j \cdot B_k = \delta_{jk} \).
2.3 Seiberg-Witten differential

The Seiberg-Witten differential is given by

\[ dS_{SW} = \frac{\partial z}{z} = \frac{x W'(x) dx}{\sqrt{W(x)^2 - 4\mu^2}} \quad (2.10) \]

The following list shows a more explicit form of this differential.

1. For the Seiberg-Witten curves:
   
   \[ \begin{align*}
   SU(N + 1) & : dS_{SW} = x P'(x) \frac{dx}{y}, \\
   SO(2N + 1) & : dS_{SW} = \left(2Q'(x^2)x^2 - Q(x^2)\right) \frac{dx}{y}, \\
   Sp(2N) & : dS_{SW} = x \left(2Q'(x^2)x^2 + 2Q(x^2)\right) \frac{dx}{y}. \\
   SO(2N) & : dS_{SW} = \left(2Q'(x^2)x^2 - 2Q(x^2)\right) \frac{dx}{y}.
   \end{align*} \]

2. For the other Toda curves:
   
   \[ \begin{align*}
   B_N^{(1)}, C_N^{(1)} & : dS_{SW} = 2Q'(x^2)x^2 \frac{dx}{y}, \\
   A_{2N}^{(2)} & : dS_{SW} = x \left(2Q'(x^2)x^2 + Q(x^2)\right) \frac{dx}{y}.
   \end{align*} \]

Here the prime means differentiating by \( x \), i.e., \( P'(x) = dP(x)/dx \), etc.

A fundamental property of this differential is that it generates holomorphic differentials on \( C \) as follows. Let \( \left( \frac{\partial}{\partial u_j}\right)_z \cdot \cdot \cdot \mid_{z=\text{const.}} \) denote differentiating the quantity inside by \( u_j \) while keeping \( z \) constant. For the Seiberg-Witten differential, this gives

\[ \frac{\partial}{\partial u_j} dS_{SW}\mid_{z=\text{const.}} = \frac{\partial x}{\partial u_j}\mid_{z=\text{const.}} \frac{dz}{z} = -\frac{\partial W(x)/\partial u_j}{\partial W(x)/\partial x} \frac{x W'(x) dx}{\sqrt{W(x)^2 - 4\mu^2}}. \quad (2.11) \]

Here we have also used the relation

\[ \frac{\partial W(x)}{\partial u_j} + \frac{\partial W(x)}{\partial x} \frac{\partial x}{\partial u_j}\mid_{z=\text{const.}} = 0, \quad (2.12) \]

which follows from the equation of the curve \( C \). One can verify, for each case presented above, that all these differentials for \( j = 1, \cdot \cdot \cdot, N \) are holomorphic differentials. In the case of the \( SU(N+1) \) Seiberg-Witten curve, these \( N \) holomorphic differentials form a basis
of the space of holomorphic differentials on $C$. In the other cases, these differentials are also linearly independent, but “odd” under the action of $\sigma_2$, because the Seiberg-Witten differential itself is also “odd”:

$$\sigma_2^*dS_{SW} = -dS_{SW}. \quad (2.13)$$

The $u_j$-derivatives (2.11) give a basis of “odd” holomorphic differentials (also called “Prym differentials”).

Given a set of cycles $A_j$ and $B_j$ mentioned above, one can define the special coordinates $a_j$ and their duals $a_j^D$ on the moduli space $U$ as follows:

$$a_j = \oint_{A_j} dS_{SW}, \quad a_j^D = \oint_{B_j} dS_{SW}. \quad (2.14)$$

The $N$ functions $a_j$ ($j = 1, \cdots, N$), as well as the $a_j^D$’s, are functionally independent and give a local coordinate system on $U$. Differentiating the Seiberg-Witten differential now by $a_j$’s give the normalized holomorphic differentials $d\omega_j$:

$$\frac{\partial}{\partial a_j} dS_{SW} \bigg|_{z=\text{const.}} = d\omega_j. \quad (2.15)$$

The prepotential $F = F(a_1, \cdots, a_N)$ is defined by the differential equations

$$\frac{\partial F}{\partial a_j} = a_j^D. \quad (2.16)$$

The matrix elements $P_{jk}$ of $P$ can be expressed as second derivatives of the prepotential:

$$P_{jk} = \frac{\partial^2 F}{\partial a_j \partial a_k}. \quad (2.17)$$

### 2.4 Quotient curve of genus $N$

The Prym variety Prym($C/C_2$) is isogenous to the Jacobi variety Jac($C'$) of the quotient curve $C' = C/\sigma'$ by the following involution $\sigma'$:

- Case A: $\sigma' = \sigma_2$.
- Case B: $\sigma' = \sigma_1\sigma_2$. 


The quotient curve $C'$ is also hyperelliptic and has genus $N$. The matrix $\mathcal{P}$ is actually the period matrix of the Jacobi variety $\text{Jac}(C')$:

$$\text{Jac}(C') \simeq \mathbb{C}^N/(\mathbb{Z}^N + \mathcal{P}\mathbb{Z}^N).$$

(2.18)

The differentials $dS_{SW}$ and $d\omega_j$ are “even” (i.e., invariant) under the action of $\sigma'$, so that they are the pull-back of differentials on $C'$. On the other hand, the homology classeses $[A_j]$ and $[B_j]$ are mapped by the projection $p' : C \to C'$ to an integer multiple of the homology classes $[A'_j]$ and $[B'_j]$ of a symplectic basis on $C'$. More precisely,

$$p'([A_j]) = [A'_j], \quad p'([B_j]) = \frac{2}{d_j}[B'_j].$$

(2.19)

This is the origin of the factor $d_j/2$ in the period integral representation of the matrix elements of $\mathcal{P}$.

The equation of the quotient curve $C'$ can be written out in terms of the following two invariants $\xi$ and $\eta$ of $\sigma'$:

- Case A: $\xi = x^2$, $\eta = y$.
- Case B: $\xi = x^2$, $\eta = xy$.

The following equations of the quotient curves $C'$ are thus derived.

1. For Seiberg-Witten curves:

   $$SO(2N + 1) : \quad \eta^2 = \xi \left(Q(\xi^2) - 4\mu^2\xi\right).$$

   $$Sp(2N) : \quad \eta^2 = Q(\xi) \left(\xi Q(\xi) + 4\mu\right).$$

   $$SO(2N) : \quad \eta^2 = \xi \left(Q(\xi)^2 - 4\mu^2\xi^2\right).$$

2. For the other Toda curves:

   $$B_N^{(1)}, C_N^{(1)} : \quad \eta^2 = \xi \left(Q(\xi)^2 - 4\mu^2\right).$$

   $$A_{2N}^{(2)} : \quad \eta^2 = \xi Q(\xi)^2 - 4\mu^2.$$

Note that the curves $C$ and $C'$ exhibit somewhat different characteristics. The curve $C$, viewed as a double covering of the $x$-sphere, has two points $P_\infty$ and $Q_\infty$ at infinity.
above \( x = \infty \). These two points correspond to \( z = \infty \) and \( z = 0 \), and mapped to each other by the hyperelliptic involution \( \sigma_1 \). This is a typical characteristic of the spectral curves of affine Toda systems \([33, 34, 35]\). The curve \( C' \), in contrast, has a single point at infinity above the \( \xi \)-shere. In particular, \( C' \) is branched over \( \xi = \infty \). Hyperelliptic curves of this type arise in the KdV hierarchy \([33, 36]\). As well known, the KdV hierarchy is a special case of the KP hierarchy with only the “odd” time variables \( t_{2n+1} \) being left nontrivial \([36]\).

The \( SU(2) \) curve is however exceptional: It has the involution \( \sigma_2 \). By this accidental symmetry, one can construct the quotient curve \( C' = C/\sigma' \) with \( \sigma' = \sigma_1\sigma_2 \), which can be written

\[
\eta^2 = \xi \left( (\xi - u_2)^2 - 4\mu^2 \right) \tag{2.20}
\]

in terms of the invariants \( \xi = x^2 \) and \( \eta = xy \). By shifting \( \xi \mapsto \xi - u_2 \), this turns into the substantially the same curve

\[
\eta^2 = (\xi + u_2)(\xi - 2\mu)(\xi + 2\mu) \tag{2.21}
\]

as Seiberg and Witten first derived \([1]\). As Gorsky et al. noted \([2]\), this curve appears in a classical study on modulations of elliptic solutions of the KdV equation.

3 Whitham Deformations of \( SU(N+1) \) curve

3.1 Setup and results of Gorsky et al.

The Whitham deformations for the \( SU(N+1) \) Seiberg-Witten curve takes the form

\[
\frac{\partial}{\partial a_j} dS \bigg|_{z=\text{const.}} = d\omega_j, \quad \frac{\partial}{\partial T_n} dS \bigg|_{z=\text{const.}} = d\Omega_n. \tag{3.1}
\]

The precise setup is as follows:

1. The moduli \( \vec{u} = (u_1, \ldots, u_N) \) are now understood to be functions \( u_j(\vec{a}, \vec{T}) \) of \( \vec{a} = (a_1, \ldots, a_N) \) and \( \vec{T} = (T_1, T_2, \ldots) \). The new parameters \( T_n \) are the “slow variables” in the theory of Whitham equations. These deformed moduli \( u_j = u_j(\vec{a}, \vec{T}) \) are required to reduce to the Seiberg-Witten moduli \( u_j = u_j(\vec{a}) \) (i.e., the inverse of the period map \( \vec{u} \to \vec{a} \) from the \( \vec{u} \)-space to the \( \vec{a} \)-space) at \( \vec{T} = (1, 0, 0, \ldots) \).
2. $d\Omega_n$ are meromorphic differentials of the second kind with poles at the two points $P_\infty$ and $Q_\infty$ only, and normalized to have zero $A_j$ periods:

$$\oint_{A_j} d\Omega_n = 0 \quad (j = 1, \cdots, N). \quad (3.2)$$

3. $d\omega_j$ are the normalized holomorphic differentials on the curve $C$.

4. The differential $dS$ is a linear combination of these differentials of the form

$$dS = \sum_{n \geq 1} T_n d\hat{\Omega}_n + \sum_{j=1}^{N} a_j d\omega_j, \quad (3.3)$$

and required to reduce to the Seiberg-Witten differential $dS_{SW}$ at the point $\vec{T} = (1, 0, 0, \cdots)$.

The solution of Gorsky et al. [19] for the $SU(N+1)$ Seiberg-Witten curve is constructed by the following steps.

1. Consider the meromorphic differentials

$$d\hat{\Omega}_n = R_n(x) \frac{dz}{z}, \quad R_n(x) = \left( P(x)^{n/(N+1)} \right)_+. \quad (3.4)$$

Here $(\cdots)_+$ denotes the polynomial part of a Laurent series of $x$. The fractional power of $P(x)$ is understood to be a Laurent series of the form $x^n + \cdots$ at $x = \infty$. Since $R_1(x) = x$, $d\hat{\Omega}_1$ is nothing but the Seiberg-Witten differential. As in the case of the Seiberg-Witten differential (2.11), the $u$-derivatives of these meromorphic differentials turn out to be holomorphic differentials.

2. Consider the differential

$$dS = \sum_{n \geq 1} T_n d\hat{\Omega}_n \quad (3.5)$$

and its period integrals

$$a_j = \oint_{A_j} dS = \sum_{n \geq 1} T_n \oint_{A_j} d\hat{\Omega}_n. \quad (3.6)$$

These period integrals are functions of the moduli $u_j$ and the deformation parameters $T_n$. They determine a family of deformations of the Seiberg-Witten period map $\vec{u} \mapsto \vec{a}$ with parameters $T_n$. 12
3. The period map \( \tilde{u} \mapsto \tilde{a} \) from the \( \tilde{u} \)-space to the \( \tilde{a} \)-space is invertible if \( \tilde{T} \) is close to \((1, 0, 0, \cdots)\), because the Seiberg-Witten period map at this point is invertible. The inverse map \( \tilde{a} \mapsto \tilde{u} = \left(u_1(\tilde{a}, \tilde{T}), \cdots, u_N(\tilde{a}, \tilde{T})\right) \) gives deformations of the Seiberg-Witten moduli \( u_j = u_j(\tilde{a}) \), hence of the curve \( C \), with parameters \( T_n \).

4. The differentials

\[
d\Omega_n = d\hat{\Omega}_n - \sum_{j=1}^{N} c_j^{(n)} d\omega_j, \quad c_j^{(n)} = \oint_{A_j} d\hat{\Omega}_n \tag{3.7}
\]

satisfy the required normalization condition.

5. \( dS \) is now a linear combination of \( d\Omega_n \) and \( d\omega_j \) of the required form.

The outcome is the following theorem:

**Theorem 1** The differential \( dS \) satisfies the Whitham equations (3.1) under the deformations of the curve \( C \) thus constructed.

We review the proof of this result in the rest of this section.

### 3.2 Differentiating \( d\hat{\Omega}_n \) by moduli

The first, and most essential step is to derive the following property of \( d\hat{\Omega}_n \). Note that \( u_j \)'s and \( T_n \)'s are now understood to be independent variables.

**Lemma 1** \((\partial/\partial u_j)d\hat{\Omega}_n|_{z=\text{const.}}\) are holomorphic differentials on \( C \).

**Proof.** We first derive a set of conditions that the polynomial \( R_n(x) \) should satisfy, and verify that they are indeed fulfilled. Differentiating the equation of the curve \( C \) gives the relation

\[
\frac{\partial P(x)}{\partial u_j} + \frac{\partial P(x)}{\partial x} \frac{\partial x}{\partial u_j} \bigg|_{z=\text{const.}} = 0. \tag{3.8}
\]

Therefore, recalling that \( dz/z = P'(x)dx/y \), one can rewrite the \( u_j \)-derivative of \( d\hat{\Omega}_n \) as follows:

\[
\frac{\partial}{\partial u_j} d\hat{\Omega}_n \bigg|_{z=\text{const.}} = \left( \frac{\partial R_n(x)}{\partial u_j} + R_n'(x) \frac{\partial x}{\partial u_j} \bigg|_{z=\text{const.}} \right) \frac{dz}{z} = \left( \frac{\partial R_n(x)}{\partial u_j} + R_n'(x) \frac{x^{N+1-j}}{P'(x)} \right) \frac{P'(x)dx}{y} = \left( \frac{\partial R_n(x)}{\partial u_j} P'(x) + R_n'(x)x^{N+1-j} \right) \frac{dx}{y}. \tag{3.9}
\]
For this differential to be a holomorphic differential on $C$, therefore, the polynomial $R_n(x)$ has to satisfy the following condition:

\[
\text{deg} \left( \frac{\partial R_n(x)}{\partial u_j} P'(x) + R'_n(x)x^{N+1-j} \right) \leq N - 1. \tag{3.10}
\]

Let us confirm that $R_n(x) = \left( P(x)^{n/(N+1)} \right)_+$ has this property. By the definition of $(\cdots)_+$, $R_n(x)$ can be written

\[
R_n(x) = P(x)^{n/(N+1)} + O(x^{-1}).
\]

Differentiating this relation by $x$ and $u_j$, respectively, gives

\[
R'_n(x) = \frac{n}{N + 1} P(x)^{(n-N-1)/(N+1)} P'(x) + O(x^{-2}),
\]

\[
\frac{\partial R_n(x)}{\partial u_j} = -\frac{n}{N + 1} P(x)^{(n-N-1)/(N+1)} x^{N+1-j} + O(x^{-1}). \tag{3.11}
\]

From these relations, one can easily see that the above condition is certainly satisfied. 

**Q.E.D.**

### 3.3 Deriving Whitham equations

Once $d\hat{\Omega}_n$ turns out to have the aforementioned property, deriving the Whitham equations (3.1) is rather straightforward. Let us present this calculation following the work of Itoyama and Morozov [9].

First, since that $d\omega_j$ ($j = 1, \cdots, N$) give a basis of the space of holomorphic differentials, the holomorphic differentials in the above Lemma can be written

\[
\frac{\partial}{\partial u_k} d\hat{\Omega}_n \bigg|_{z=\text{const.}} = \sum_{j=1}^{N} \sigma_{kj}^{(n)} d\omega_j. \tag{3.12}
\]

The coefficients are determined by integrating the both hand sides along $A_j$:

\[
\sigma_{kj}^{(n)} = \oint_{A_j} \frac{\partial}{\partial u_k} d\hat{\Omega}_n \bigg|_{z=\text{const.}} = \int_{A_j} \frac{\partial}{\partial u_k} d\hat{\Omega}_n = \int_{A_j} \frac{\partial c_{j}^{(n)}}{\partial u_k}. \tag{3.13}
\]

Second, the derivatives of $a_j = a_j(\vec{u}, \vec{T})$ turn out to be written

\[
\frac{\partial a_j}{\partial u_k} = \sum_{n \geq 1} T_n \sigma_{kj}^{(n)}, \quad \frac{\partial a_j}{\partial T_n} = c_{j}^{(n)}. \tag{3.14}
\]
This can be readily verified by directly differentiating

\[ a_j = \oint_{A_j} dS = \sum_{n \geq 1} T_n c_j^{(n)} \]  

(3.15)

and using (3.13).

Now change the independent variables from \((\bar{u}, \bar{T})\) to \((\bar{a}, \bar{T})\). These two systems of coordinates are connected by the functions \(a_j = a_j(\bar{u}, \bar{T})\) and \(u_k = u_k(\bar{a}, \bar{T})\). In this setup, the following identities are satisfied:

\[ \sum_{k=1}^N \frac{\partial u_k}{\partial a_i} \frac{\partial a_j}{\partial u_k} = \delta_{ij}, \quad \sum_{k=1}^N \frac{\partial u_k}{\partial T_m} \frac{\partial a_j}{\partial u_k} = -\frac{\partial a_j}{\partial T_m}. \]  

(3.16)

The first relation is obvious from the chain rule. The second one is rather confusing; this is obtained by differentiating the identity

\[ a_j = a_j(u_1(\bar{a}, \bar{T}), \ldots, u_N(\bar{a}, \bar{T})), \bar{T}) \]

by \(T_m\).

Using the chain rule along with these relations (3.12), (3.14) and (3.16), one can verify the Whitham equations (3.1) as follows.

1. The first equation of (3.1):

\[
\frac{\partial}{\partial a_i} dS \bigg|_{z=\text{const.}} = \sum_{n} T_n \frac{\partial}{\partial a_i} d\Omega_n \bigg|_{z=\text{const.}} = \sum_{n,k} T_n \frac{\partial u_k}{\partial a_i} \frac{\partial}{\partial u_k} d\Omega_n \bigg|_{z=\text{const.}} = \sum_{j,k} \frac{\partial u_k}{\partial a_i} \frac{\partial a_j}{\partial u_k} d\omega_j = d\omega_i.
\]

2. The second equation of (3.1):

\[
\frac{\partial}{\partial T_m} dS \bigg|_{z=\text{const.}} = \frac{d\hat{\Omega}_m}{dS} + \sum_{n} T_n \frac{\partial}{\partial T_m} d\hat{\Omega}_n \bigg|_{z=\text{const.}} = \frac{d\hat{\Omega}_m}{dS} + \sum_{n,k} T_n \frac{\partial u_k}{\partial T_m} \frac{\partial}{\partial u_k} d\hat{\Omega}_n \bigg|_{z=\text{const.}} = \frac{d\hat{\Omega}_m}{dS} + \sum_{n,k} T_n \frac{\partial u_k}{\partial a_i} \frac{\partial}{\partial u_k} d\hat{\Omega}_n \bigg|_{z=\text{const.}} = d\omega_i.
\]
\[ d\Omega_m + \sum_{j,k} \frac{\partial u_k}{\partial T_m} \cdot \sum_n T_n \sigma_{kj}^{(n)} \cdot d\omega_j \]

\[ = d\Omega_m + \sum_{j,k} \frac{\partial u_k}{\partial T_m} \frac{\partial a_j}{\partial u_k} d\omega_j \]

\[ = d\Omega_m - \sum_j c_j^{(m)} d\omega_j \]

\[ = d\Omega_m. \]

This completes the proof of the Theorem.

### 3.4 Prepotential and homogeneity

The prepotential \( F = F(a, T) \) is defined by the equations

\[
\frac{\partial F}{\partial a_j} = \oint_{B_j} dS, \quad \frac{\partial F}{\partial T_n} = -\oint_{P_\infty} f_n(z) dS - \oint_{Q_\infty} g_n(z) dS. \tag{3.17}
\]

\( P_\infty \) and \( Q_\infty \) are the two points at infinity \((z = \infty \text{ and } z = 0)\); \( \oint_{P_\infty} \) and \( \oint_{Q_\infty} \) are integrals along a small closed path encircling the indicated point once in the anti-clockwise direction. \( f_n(z) \) and \( g_n(z) \) are Laurent polynomials that represent the singular part of \( d\Omega_n \) at \( P_\infty \) and \( Q_\infty \):

\[
d\Omega_n = df_n(z) + \text{holomorphic} \quad (P \to P_\infty), \\
d\Omega_n = dg_n(z) + \text{holomorphic} \quad (P \to Q_\infty). \tag{3.18}
\]

These Laurent polynomials have constant coefficients for all \( n \).

\[ \text{To see this, let us note that the fractional power of } P(x) \text{ in } R_n(x) \text{ can be written} \]

\[
P(x)^{n/(N+1)} = \left( z + \frac{\mu^2}{z} \right)^{n/(N+1)}. \tag{3.19}
\]

The singular part of Laurent expansion of the right hand side at \( z = \infty \) or \( z = 0 \) determines the Laurent polynomials \( f_n(z) \) and \( g_n(z) \). Obviously the singular part is a Laurent polynomial with constant coefficients, and accordingly \( f_n(z) \) and \( g_n(z) \), too, turn out to have constant coefficients. The compatibility of the above defining equations for

\[ ^1 \text{Our previous paper [21] contains a wrong comment on this fact.} \]
\( \mathcal{F} \) is ensured by Riemann’s bilinear relations. Second derivatives of the prepotential are also related to period integrals:

\[
\frac{\partial^2 \mathcal{F}}{\partial a_j \partial a_k} = \oint_{B_j} d\omega_k, \quad \frac{\partial^2 \mathcal{F}}{\partial a_j \partial T_n} = \oint_{B_j} d\Omega_n,
\]

\[
\frac{\partial^2 \mathcal{F}}{\partial T_m \partial T_n} = -\oint_{P\infty} f_m(z) d\Omega_n - \oint_{Q\infty} g_m(z) d\Omega_n.
\] (3.20)

The construction of the Whitham deformations also ensures the homogeneity

\[
\sum_{n \geq 1} T_n \frac{\partial \mathcal{F}}{\partial T_n} + \sum_{j=1}^{N} a_j \frac{\partial \mathcal{F}}{\partial a_j} = 2\mathcal{F}. \tag{3.21}
\]

To see this, first note that the period integrals \( a_j = a_j(\vec{a}, \vec{T}) \) have the obvious homogeneity

\[
a_j(\vec{u}, \lambda \vec{T}) = \lambda a_j(\vec{u}, \vec{T}). \tag{3.22}
\]

Accordingly, \( u_j = u_j(\vec{a}, \vec{T}) \) are invariant under the rescaling of \( a_j \) and \( T_n \), i.e., they are homogeneous functions of degree zero:

\[
u_j(\lambda \vec{u}, \lambda \vec{T}) = u_j(\vec{u}, \vec{T}). \quad \tag{3.23}
\]

One can see, from these fact, that the period integrals on the right hand side of (3.17) are homogeneous functions of degree one, so that the prepotential becomes homogeneous of degree two (upon suitably normalizing the linear part).

4 **Whitham Deformations of \( SO(2N) \) curve**

4.1 **Setup and results**

As warm-up, let us examine some fundamental properties of the Seiberg-Witten differential \( dS_{SW} \). The \( u_j \)-derivatives of \( dS_{SW} \) can be written

\[
\frac{\partial}{\partial u_j} dS_{SW} \bigg|_{z=\text{const.}} = \frac{x^{2N-2j-2} dx}{\sqrt{x^{-4}Q(x^2)^2 - 4\mu^2}} = \frac{x^{2N-2j} dx}{y} \quad (j = 1, \ldots, N). \tag{4.1}
\]

As expected, they are holomorphic differentials on \( C \). Furthermore, like \( dS_{SW} \) itself, they are invariant under the involution \( \sigma' : (x, y) \mapsto (-x, -y) \), and can be identified with
holomorphic differentials on the quotient curve $C' = C/\sigma'$ with coordinates $\xi = x^2$ and $\eta = xy$:

$$\frac{\partial}{\partial u_j} dS_{SW} \bigg|_{z=\text{const.}} = \frac{\xi^{N-j} d\xi}{2\eta} (j = 1, \ldots, N).$$

(4.2)

Following the method of Gorsky et al., we now seek for a series of meromorphic differentials $d\hat{\Omega}_n$ of the form

$$d\hat{\Omega}_n = R_n(x)\frac{dz}{z}, \quad R_n(x) = \text{polynomial},$$

(4.3)

with the same properties. As one can see by careful inspection of the proof in the $SU(N+1)$ Seiberg-Witten curve, the fractional power construction persists to be meaningful even if the polynomial $P(x)$ is replaced by a rational (or Laurent series) superpotential $W(x)$. A new feature in the present setup is the parity; for the differential to be invariant under $\sigma'$, the polynomial $R_n(x)$ has to be odd,

$$R_n(-x) = -R_n(x).$$

(4.4)

The superpotential $W(x)$ is even, and the leading term is $x^{2N-2}$. We are thus led to the following:

$$R_n(x) = \left(W(x)^{(2n-1)/(2N-2)}\right)_+ (n = 1, 2, \ldots).$$

(4.5)

Note that these polynomials are indeed odd polynomials. Furthermore, $R_1(x) = x$, so that in this case, too, $d\hat{\Omega}_1$ is equal to the Seiberg-Witten differential.

**Lemma 2** $(\partial/\partial u_j)d\hat{\Omega}_n|_{z=\text{const.}}$ are holomorphic differentials on $C$, and invariant under the involution $\sigma'$.

**Proof.** We can proceed in mostly the same way as the case of the $SU(N+1)$ Seiberg-Witten curve. The $u_j$-derivatives of $d\hat{\Omega}_n$ can be written

$$\frac{\partial}{\partial u_j}d\hat{\Omega}_n \bigg|_{z=\text{const.}} = \left(x^2 \frac{\partial R_n(x)}{\partial u_j} W'(x) - x^2 R_n'(x) \frac{\partial W(x)}{\partial u_j}\right) \frac{dx}{y}.$$

(4.6)

The proof now boils down to verifying the following two statements:

1. The factor in front of $dx/y$ is a polynomial.
2. The degree of this polynomial does not exceed \(2N - 2\).

Let us consider the first statement. The problem is that \(\partial W(x)/\partial u_j\) and \(W'(x)\) respectively have a pole of second and third order at \(x = 0\). \(\partial R_n(x)/\partial u_j\) and \(R_n'(x)\), however, have a zero at \(x = 0\), because \(R_n(x)\) is an odd polynomial and has no constant term. Therefore, along with the factor \(x^2\), they cancel the pole of at most third order of the other two functions. The second statement can be verified by the same reasoning as the proof for the the \(SU(N + 1)\) Seiberg-Witten curve. \(Q.E.D.\)

### 4.2 Whitham deformations

Having obtained the meromorphic differentials \(d\hat{\Omega}_n\) with the aforementioned properties, we can now construct a family of Whitham deformations.

The construction is parallel to the case of the \(SU(N + 1)\) Seiberg-Witten curve:

1. Consider the differential

\[
    dS = \sum_{n \geq 1} T_n d\hat{\Omega}_n. \tag{4.7}
\]

and its period integrals

\[
    a_j = \oint_{A_j} dS = \sum_{n \geq 1} T_n \oint_{A_j} d\hat{\Omega}_n. \tag{4.8}
\]

The period integrals \(a_j = a_j(\vec{u}, \vec{T})\) determine a period map \(\vec{u} \mapsto \vec{a}\) from the \(\vec{u}\)-space to the \(\vec{a}\)-space.

2. The period map \(\vec{u} \mapsto \vec{a}\) is invertible if \(\vec{T}\) is close to \((1, 0, 0, \cdots)\). The inverse map \(\vec{a} \mapsto \vec{u}\) gives deformations of the curve \(C\) with parameters \(\vec{T}\).

3. The differentials

\[
    d\Omega_n = d\hat{\Omega}_n - \sum_{j=1}^{N} c_j^{(n)} d\omega_j, \quad c_j^{(n)} = \oint_{A_j} d\hat{\Omega}_n, \tag{4.9}
\]

satisfy the normalization condition

\[
    \oint_{A_j} d\Omega_n = 0 \quad (j = 1, \cdots, N). \tag{4.10}
\]
4. Eventually, the differential \( dS \) can be written

\[
dS = \sum_{n \geq 1} T_n d\Omega_n + \sum_{j=1}^{N} a_j d\omega_j. \tag{4.11}
\]

The following can be proven in exactly the same way as the proof for the \( SU(N + 1) \) Seiberg-Witten curve:

**Theorem 2** The differential \( dS \) satisfies the Whitham equations

\[
\left. \frac{\partial}{\partial a_j} dS \right|_{z=\text{const.}} = d\omega_j, \quad \left. \frac{\partial}{\partial T_n} dS \right|_{z=\text{const.}} = d\Omega_n, \tag{4.12}
\]

under the deformations of the curve \( C \) thus constructed.

### 4.3 Relation to KdV hierarchy

By construction, the differentials \( dS, d\bar{\Omega}_n, d\Omega_n \) and \( d\omega_j \) are all invariant under the involution \( \sigma' \). Accordingly, they actually descend to (or, equivalently, are the pull-back of) differentials on the quotient curve \( C' = C/\sigma' \). In particular, the counterpart of \( d\omega_j \) are also holomorphic, and, as already mentioned, form a normalized basis of holomorphic differentials on \( C' \).

The meromorphic differentials \( d\Omega_n \) possess an even more interesting interpretation. They correspond to meromorphic differentials on \( C' \) with a single pole at \( \xi = \infty \) (which is the image of \( P_\infty \) and \( Q_\infty \)). Recall that this is a branch point of the covering map of \( C' \) over the \( \xi \)-sphere. This is substantially the same setup that emerges in hyperelliptic solutions of the KdV hierarchy [36, 37]. Those hyperelliptic solutions are constructed from a theta function and a series of meromorphic differentials \( d\Omega_n^{KdV} \) \( (n = 1, 2, \cdots) \) with a single pole (of order \( 2n \)) at a fixed branch point (such as the point \( \xi = \infty \)) of the hyperelliptic curve.

In this respect, we should have numbered the deformation variables as \( T_1, T_3, T_5, \cdots \) rather than \( T_1, T_2, T_3, \cdots \), following the usual numbering of the time variables in the KdV hierarchy. Note that the odd indices correspond to the degrees \( 2n - 1 \) of powers of \( W(x)^{1/(2N-2)} \) in the definition of \( R_n(x) \).

We must, however, also add that our meromorphic differentials are not exactly the same as those in the standard formulation of the KdV hierarchy. The meromorphic
differentials $d\Omega_{n}^{KdV}$ in the standard setup are normalized as

$$d\Omega_{n}^{KdV} = d\xi^{n-1/2} + \text{holomorphic} \quad (4.13)$$

at $\xi = \infty$. Our meromorphic differentials $d\Omega_{n}$ are not of this form; they are a linear combination of $d\Omega_{n}^{KdV}$. Accordingly, the “fast” and “slow” time variables are also a linear combination of the standard ones.

These remarks also apply to the other cases where the essential part of the theory is described by the quotient curve $C'$ of the KdV type.

5 Non-simply-Laced Gauge Groups and Some Other Cases

5.1 $SO(2N + 1)$ and $Sp(2N)$

The case of the non-simply-laced gauge groups $SO(2N + 1)$ and $Sp(2N)$ can be treated in almost the same way as the case of $SO(2N)$. The curve $C$ has the involution $\sigma'$, and the quotient curve $C' = C/\sigma'$ has genus $N$. The Seiberg-Witten differential $dS_{SW}$ is invariant under this involution. Our first task is to construct meromorphic differentials

$$d\hat{\Omega}_{n} = R_{n}(x) \frac{dz}{z}, \quad R_{n}(x) = \text{odd polynomial}, \quad (5.1)$$

whose $u_{j}$-derivatives are holomorphic differentials on $C$.

The polynomials $R_{n}(x)$ are again given by the polynomial part of fractional powers of the superpotential $W(x)$:

$$SO(2N + 1) : R_{n}(x) = \left(W(x)^{(2n-1)/(2N-1)}\right)_{+}, \quad (5.2)$$

$$Sp(2N) : R_{n}(x) = \left(W(x)^{(2n-1)/(2N+2)}\right)_{+}. \quad (5.3)$$

The $u_{j}$-derivatives can be written as follows:

1. For $SO(2N + 1)$,

$$\left. \frac{\partial}{\partial u_{j}} d\hat{\Omega}_{n} \right|_{z=\text{const.}} = \left( x \frac{\partial R_{n}(x)}{\partial u_{j}} W'(x) - x R_{n}'(x) \frac{\partial W(x)}{\partial u_{j}} \right) \frac{dx}{y}. \quad (5.4)$$
2. For $Sp(2N)$,
\[
\left. \frac{\partial}{\partial u_j} d\hat{\Omega}_n \right|_{z=\text{const.}} = \left( x^{-1} \frac{\partial R_n(x)}{\partial u_j} W'(x) - x^{-1} R_n'(x) \frac{\partial W(x)}{\partial u_j} \right) \frac{dx}{y}. \tag{5.5}
\]

The prefactor of $dx/y$ turns out to be a polynomial of degree $\leq 2N - 2$ and $\leq 2N - 1$ for $SO(2N + 1)$ and $Sp(2N)$, respectively. Therefore the above differentials are holomorphic differentials on $C$.

The Whitham deformations of $C$ are given by the inverse period map $\vec{a} \mapsto \vec{u}$ of the period integrals
\[
a_j = \oint_{A_j} ds = \sum_{n \geq 1} T_n \int_{A_j} d\hat{\Omega}_n. \tag{5.6}
\]
of the differential
\[
dS = \sum_{n \geq 1} T_n d\hat{\Omega}_n. \tag{5.7}
\]
The normalized meromorphic differentials $d\Omega_n$ are given by
\[
d\Omega_n = d\hat{\Omega}_n - \sum_{j=1}^n c_j^{(n)} d\omega_j, \quad c_j^{(n)} = \int_{A_j} d\hat{\Omega}_n. \tag{5.8}
\]
Now $dS$ can be written
\[
dS = \sum_{n \geq 1} T_n d\Omega_n + \sum_{j=1}^N a_j d\omega_j, \tag{5.9}
\]
and satisfy the Whitham equations
\[
\left. \frac{\partial}{\partial u_j} dS \right|_{z=\text{const.}} = d\omega_j, \quad \left. \frac{\partial}{\partial T_n} dS \right|_{z=\text{const.}} = d\Omega_n \tag{5.10}
\]
under the deformations of the curve $C$ by the inverse period map $\vec{a} \mapsto \vec{u}$.

### 5.2 Some other cases

The same fractional power construction of Whitham deformations also applies to the Toda curves for other classical affine Lie algebras, namely, $B_N^{(1)}$, $C_N^{(1)}$ and $A_{2N}^{(2)}$. The
meromorphic differentials $d\Omega_n$ are obtained in the same form as the other cases. The polynomials $R_n(x)$ are given by the following:

\begin{align*}
B_N^{(1)}, C_N^{(1)} : & \quad W(x) = \left( W(x)^{(2n-1)/(2N)} \right)_+, \\
A_{2N}^{(2)} : \quad W(x) = \left( W(x)^{(2n-1)/(2N+1)} \right)_+.
\end{align*}

The case of the $A_{2N}^{(2)}$ (also called $BC_N$) Toda curve, seems to be particularly interesting and mysterious. Although this does not appear in the list of ordinary $\mathcal{N} = 2$ SUSY Yang-Mills theories, an M-theoretical interpretation \cite{41} might be possible.

Actually, the same construction works with no problem for a more general rational superpotential, e.g.,

\begin{equation}
W(x) = P(x) + \sum_{k=1}^{M} \frac{v_k}{x - c_k}.
\end{equation}

Note that the curve $C$ is still hyperelliptic. This is a special case of M5-branes with semi-infinite D4 branes (arising from the poles of $W(x)$) on both sides of the stack of two NS 5-branes.

### 5.3 Yang-Mills theories with fundamental matters

The Seiberg-Witten curve does not take the previous form (2.1) if fundamental matters (hypermultiplets in the fundamental representation) exist. One can, however, rewrite the curve in this form if $W(x)$ can be an irrational function. Let us examine this prescription in the case of the $SU(N + 1)$ Yang-Mills theory with fundamental matters.

If the theory contains $N_f$ ($< 2N$) fundamental matters, the Seiberg-Witten curve takes the form

\begin{equation}
z + \frac{\mu^2 R(x)}{z} = P(x),
\end{equation}

where $R(x)$ is the polynomial

\begin{equation}
R(x) = \prod_{i=1}^{N_f} (x + m_i).
\end{equation}

This curve can be formally converted into the form of (2.1) by changing coordinates from $x$ and $z$ to $x$ and

\begin{equation}
w = z/\sqrt{R(x)}.
\end{equation}
The converted curve can be written
\[ w + \frac{\mu^2}{w} = W(x) = \frac{P(x)}{\sqrt{R(x)}}, \]  
(5.16)
thus an irrational superpotential arises. The role of \( z \) is now played by \( w \). For instance, the Seiberg-Witten differential can be written
\[ dS_{SW} = x \frac{dw}{w}. \]  
(5.17)

Since the superpotential itself is multi-valued, The fractional power construction requires a more careful handling. This multi-valuedness is however relatively harmless, simply affecting an overall multiplicative constant of the meromorphic differentials \( d\hat{\Omega}_n \).

## 6 Topologically Twisted Gauge Theories

### 6.1 Donaldson-Witten function and \( u \)-plane integral

Correlation functions of topologically twisted gauge theories on a four-dimensional manifold \( X \) can be collected into a generating function. This generating function is called the Donaldson-Witten function. For instance, consider a 2-cycle observable \( I(S) \), \( S \in H_2(X, \mathbb{Z}) \), and a 0-cycle observable \( \mathcal{O}(P) \), \( P \in H_0(X, \mathbb{Z}) \), of the form
\[ I(S) = \text{const.} \int_S G^2 \text{Tr} \phi^2, \quad \mathcal{O}(P) = \sum_k c_k \text{Tr} \phi^k(P). \]  
(6.1)

\( G \) is a transformation that generates a standard solution of the descent equations for observables \cite{22, 23}. The Donaldson-Witten function of these observables is the path integral
\[ Z_{DW} = \left\langle \exp \left( I(S) + \mathcal{O}(P) \right) \right\rangle. \]  
(6.2)

In the case with \( b_2^+(X) = 1 \), the Donaldson-Witten function becomes a sum of two pieces:
\[ Z_{DW} = Z_{SW} + Z_u. \]  
(6.3)
The first piece \( Z_{SW} \) is the contributions from the strong-coupling singularities of the moduli space \( U \) (which is called the “\( u \)-plane” by abuse of the terminology for the \( SU(2) \)
and $SO(3)$ gauge groups). The second piece $Z_u$ is called the $u$-plane integral, which is absent if $b_2^+(X) > 1$. This is the contributions from the whole moduli space $U$.

According to Moore and Witten [22] (for $SU(2), SO(3)$ gauge groups) and Mariño and Moore [23] (for other gauge groups), the $u$-plane integral for the above Donaldson-Witten function $Z_{DW}$ can be written

$$Z_u = \int_U da d\bar{a} A^x B^\sigma \exp(U + S^2 T) \Psi, \quad (6.4)$$

where $\chi$ and $\sigma$ are the Euler number and the signature of $X$; $A$ and $B$ are modular forms on $U$; $U$ is a contribution from $O(P)$ only; $T$ is a “contact term” which is induced by the intersection of the 2-cycle $S$ with itself, accordingly multiplied by the self-intersection number $S^2$; $\Psi$ is a lattice sum collecting the contributions of abelianized gauge filed s and other fermionic degrees of freedom.

### 6.2 Blowup formula and tau function

Mariño and Moore [23] pointed out that the blowup formula of the $u$-plane integral contains a factor that can be interpreted as a special “tau function” of the affine Toda system (or, more precisely, an underlying integrable hierarchy). It should however be noted that this is the interpretation in the case of the $SU(N + 1)$ gauge group.

The blowup formula connects the manifold $X$ and its blowup $\tilde{X}$ at a point $Q$. $X$ is assumed to be a complex algebraic surface (e.g., $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, del Pezzo surfaces, etc.). Let $B$ denote the exceptional divisor (i.e., inverse image of $Q$) in $\tilde{X}$, and consider the following Donaldson-Witten function of $\tilde{X}$:

$$\tilde{Z}_{DW} = \left\langle \exp\left(tI(B) + I(S) + O(P)\right) \right\rangle. \quad (6.5)$$

Note that this path integral is over the fields on $\tilde{X}$; the pull-back of $I(S)$ and $O(P)$ to $\tilde{X}$ are denoted by the same notations. The new observable $I(B)$ with support on $B$ is inserted with the coupling constant $t$. The blowup formula then shows that the integrand of the $u$-plane integral for $\tilde{Z}_{DW}$ is obtained by replacing

$$e^U \rightarrow e^{U \frac{\alpha}{\beta}} \det \left( \frac{\partial u_k}{\partial a_j} \right)^{1/2} \Delta^{-1/8} e^{-t^2 T} \Theta_{\gamma, \delta} \left( \frac{t}{2\pi} \vec{V} \mid P \right) \quad (6.6)$$

in the $u$-plane integral for $Z_{DW}$. Various terms on the right hand side of this rule have the following meaning: $\alpha$ and $\beta$ are some numerical constants; $\Delta$ is the discriminant of the
family of the curves $C$ over $U$; $\Theta_{\gamma,\delta}(Z \mid P)$ is the ordinary $N$-dimensional theta function with characteristic $(\gamma, \delta)$ and period matrix $P$; $\mathbf{V}$ is the gradient vector

$$\mathbf{V} = \left( \frac{\partial V}{\partial a_j} \right)$$

(6.7)
of the gauge invariant potential $V$ from which the integrand $G^2V$ of $I(S)$ and $I(B)$ are constructed (e.g., $V = \text{Tr} \phi^2$ in the aforementioned usual setup of topological gauge theories). For the $SU(N + 1)$ Seiberg-Witten curve, the matrix $P$ is the period matrix of $\text{Jac}(C)$; for the other classical gauge groups, the period matrix of $\text{Prym}(C/C_2)$ (or $\text{Jac}(C')$) appears. The characteristic $(\gamma, \delta)$ is determined by the physical setup; typically, $\gamma = (0, \cdots, 0)$ and $\delta = (1/2, \cdots, 1/2)$.

It is the product of the last two terms in (6.4) that Mariño and Moore, in the case of the $SU(N+1)$ gauge group, identified to be the tau function of the affine Toda system:

$$\tau_{\gamma,\delta}(t) = e^{-t^2 T \Theta_{\gamma,\delta} \left( \frac{t}{2\pi} \mathbf{V} \mid P \right)}.$$  

(6.8)

Thus, the coupling constant $t$ plays the role of a time variable in the $A_{N}^{(1)}$ Toda system. For the other classical gauge groups, however, the relation to the affine Toda systems is slightly more complicated, as we discuss later on.

### 6.3 Multi-time tau function as blowup factor

Our previous paper [21] proposes a “multi-time” analogue of the single-time tau function $\tau_{\gamma,\delta}(t)$ above. In the present setup including all classical gauge groups, the multi-time analogue can be written

$$\tau_{\gamma,\delta}(t_1, t_2, \cdots) = \exp \left( \frac{1}{2} \sum_{m,n \geq 1} q_{mn} t_m t_n \right) \Theta_{\gamma,\delta} \left( \sum_{n \geq 1} t_n \mathbf{V}^{(n)} \mid P \right).$$

(6.9)
The coefficients $q_{mn}$ of the Gaussian factor and the components of the directional vectors $\mathbf{V}^{(n)} = \left( V_j^{(n)} \right)$ are written in terms of period integrals of the meromorphic differentials $d\Omega_n$ that we have considered:

$$q_{mn} = -\frac{1}{2\pi i} \oint_{P_{\infty}} f_n(z) d\Omega_m - \frac{1}{2\pi i} \oint_{Q_{\infty}} g_n(z) d\Omega_m,$n$$

$$V_j^{(n)} = -\frac{1}{2\pi i} \oint_{P_{\infty}} f_n(z) d\omega_j - \frac{1}{2\pi i} \oint_{Q_{\infty}} g_n(z) d\omega_j.$$  

(6.10)
By Riemann’s bilinear relation, \( V^{(n)}_j \) can also be written
\[ V^{(n)}_j = \frac{1}{2\pi i} \oint_{B_j} d\Omega_n. \] (6.11)

Our proposal in the previous paper (for the \( SU(N+1) \) topological gauge theory) is to interpret this tau function as the counterpart of Mariño and Moore’s blowup factor \( \tau_{\gamma,\delta}(t) \) for the Donaldson-Witten function
\[ \tilde{Z}_{DW} = \left\langle \exp \left( \sum_{n \geq 1} t_n I_n(B) + I(S) + O(P) \right) \right\rangle. \] (6.12)

with many 2-cycles observables \( I_n(B) \) inserted. The observables \( I_n(B) \) are of the form
\[ I_n(B) = \text{const.} \int_B G^2 V^{(n)}, \] (6.13)
and the directional vector \( \vec{V}^{(n)} \) is the gradient of the gauge invariant potential \( V^{(n)} \),
\[ \vec{V}^{(n)} = \left( \frac{\partial V^{(n)}}{\partial a_j} \right). \] (6.14)

This leads to the identification of the coefficients \( q_{mn} \) as the “contact terms” \( C \left( V^{(m)}, V^{(n)} \right) \) of higher Casimir observables in the sense of Losev et al. \[24\].

Strong evidence supporting our proposal is that the above multi-time tau function has a good modular property under the symplectic transformations
\[ B_j \rightarrow A_{jk}B_k + B_{jk}A_k, \quad A_j \rightarrow C_{jk}B_k + D_{jk}A_k, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2N, \mathbb{Z}). \] (6.15)

of the cycles \( A_j, B_j \). This is crucial for ensuring the correct modular property of the integrand of the \( u \)-plane integral. Under this symplectic transformation, indeed, the tau function \( \tau_{\gamma,\delta} \) transforms as
\[ \tau_{\gamma,\delta}(t_1, t_2, \cdots) \rightarrow \epsilon \det(CP + D)^{1/2} \tau'_{\gamma',\delta'}(t_1, t_2, \cdots), \] (6.16)
where \( \epsilon \) (an 8th root of unity), \( \gamma' \) and \( \delta' \) are determined by the \( Sp(2N, \mathbb{Z}) \) matrix. This fact can be confirmed in the same way as the proof in the case of the \( SU(N+1) \) topological gauge theory \[21\]. It should be also mentioned that this modular property of the tau function has been known for years \[12, 13\]. Now the point is that the modular property of \( \tau_{\gamma,\delta} \) is independent of \( t_1, t_2, \cdots \). In particular, the modular invariance of the \( u \)-plane integral in Mariño and Moore’s setup at \( t_1 = t \) and \( t_n = 0 \ (n > 1) \) is retained even if the higher coupling constants \( t_n \) are turned on.
6.4 KdV again

What is new in the case of the orthogonal and symplectic groups is that the theta function in the tau function is not a theta function on Jac(C); this is a theta function on the Prym variety Prym(C/C_2) or, up to an isogeny, the Jacobi variety Jac(C') of the quotient curve C'. As remarked in the previous sections, the quotient curve C' and the meromorphic differentials dΩ_n are of the KdV type. The tau function τ_{γ,δ} for the orthogonal and symplectic gauge groups thus turns out to be a tau function of the KdV hierarchy (which is a special case of the algebro-geometric tau functions of the KP hierarchy\cite{39, 40}) rather than of the affine Toda system.

This conclusion might cause confusion, but here is no contradiction. It is well known that the affine Toda system can be mapped to linear flows on the Prym variety \cite{35}. Apart from the case of A_N^{(1)}, however, this does not imply that its solutions and tau functions can be written in terms of theta functions on the Prym variety. Actually, the flows are first linearized on the Jacobi variety Jac(C) of the affine Toda spectral curve itself, and then shown to be confined to a subspace that is parallel (but not identical) to the Prym variety embedded therein. All that one can expect is, accordingly, an expression in terms of theta functions on Jac(C). (Surprisingly, however, very few is known about such an explicit expression of solutions of the affine Toda systems other than the A_N^{(1)} and C_N^{(1)}.) Thus, our tau function τ_{γ,δ} is something different from tau functions of the affine Toda systems, and our interpretation is that it is a special tau function of the KdV type.

6.5 Whitham deformations and prepotential

Let us now turn on the Whitham deformations with “slow variables” T_n. The Seiberg-Witten prepotential F is also deformed and becomes a function F(\vec{a}, \vec{T}) of both \vec{a} = (a_1, \ldots, a_N) and \vec{T} = (T_1, T_2, \ldots). More precisely, F is defined by (3.17) for all cases considered in the preceding sections. As one can immediately see by comparing these equations with the definition of q_{mn}, V_j^{(n)} and P_{jk} in the form of period integrals, these fundamental quantities in our interpretation of the blowup formula can be written as second order derivatives of the prepotential:

\[ V_{jn} = \frac{1}{2\pi i} \frac{\partial^2 F}{\partial a_j \partial T_n}, \quad q_{mn} = \frac{1}{2\pi i} \frac{\partial^2 F}{\partial T_m \partial T_n}, \quad P_{jk} = \frac{\partial^2 F}{\partial a_j \partial a_k}. \] (6.17)
In particular, the potential $V^{(n)}$ of the observables $I_n(B)$ turn out to be first order derivatives of $F$:

$$V^{(n)} = \frac{1}{2\pi i} \frac{\partial F}{\partial T_n}. \quad (6.18)$$

Of course the $a_j$-derivatives give the dual special coordinates $a_j^D = \oint_{B_j} dS$:

$$a_j^D = \frac{\partial F}{\partial a_j}. \quad (6.19)$$

Thus the prepotential $F = F(\tilde{a}, \tilde{T})$ in the Whitham deformations, too, is a kind of “generating function”.

Unlike the “fast variables” $t_n$, the role of the “slow variables” $T_n$ is to deform the period map $\tilde{u} \mapsto \tilde{a}$ that connects the $u$-space and the $a$-space. Presumably, this will be a kind of deformations of “background geometry” in the sense of string theory, but the precise meaning is still beyond our scope.

### 7 Discussions

We have seen that the construction of Whitham deformations by Gorsky et al. [19] can be extended to the Seiberg-Witten curves of all the classical gauge groups and some other complex algebraic curves. Although the construction is based on the somewhat special form (2.1) of the curves, the only requirement seems to be that $W(x)$ be a rational function with a polynomial leading part. Actually, we have obtained partial evidence that an irrational superpotential might be allowed for at least in some special cases.

We have also extended our proposal in the previous paper [21] on the $u$-plane integral of the $SU(N+1)$ topological gauge theory to all other classical gauge groups. A byproduct of the construction of Whitham deformations is to determine which flows of the underlying integrable hierarchy (the Toda hierarchy for the case of $SU(N+1)$ and the KdV hierarchy for the other classical gauge groups) should be extracted; appropriate flows are those generated by the meromorphic differentials $d\Omega_n$ that arise in the construction of Whitham deformations. This enables us to express the relevant quantities $q_{mn}$ etc. as derivatives of the prepotential $F$.

Let us conclude this paper with discussions on possible extensions and implications of these results.
Non-hyperelliptic curves

The first nontrivial step beyond rational superpotentials is irrational superpotentials of the form

\[ W(x) = R_1(x) + R_2(x)\sqrt{R_3(x)}, \tag{7.1} \]

where \( R_i(x) \)'s are polynomial or rational functions of \( x \). This means that the curve \( C \) is no longer hyperelliptic, but can be written in a special quartic polynomial in \( z \) with rational coefficients. Well known examples of curves of this form are the Seiberg-Witten curves of \( SU(5) \) (\( A_4 \)) in the 10-dimensional anti-symmetric representation, \( E_6 \) in the 27-dimensional minimal representation, and \( G_2 \) in the 7-dimensional minimal representation.

An obvious difficulty is that \( W(x) \) itself is multi-valued, so that the fractional powers of \( W(x) \) requires a more careful treatment. This difficulty, however, might be easily overcome, because the same fractional powers are used in the work of Eguchi and Yang on the topological Landau-Ginzburg of the \( E_6 \) singularity. This work also predicts an interesting phenomena if the fractional power construction really works for the case of \( E_6 \). Namely, as they observed in the topological Landau-Ginzburg theory, the admissible Whitham deformations will be limited to those associated with the fractional powers \( W(x)^{n/12} \) with

\[ n \equiv 1, 4, 5, 7, 8, 11 \mod 12. \tag{7.2} \]

The numbers on the right hand side are the exponents of \( E_6 \).

For more general cases, however, an entirely new approach will be necessary. For instance, Witten’s M5-brane construction yields a non-hyperelliptic curve of the form

\[ z^{k+1} + g_1(x)z^k + \cdots + g_k(x)z + 1 = 0 \tag{7.3} \]

for the \( \mathcal{N} = 2 \) SUSY gauge theory (coupled to bifundamental matters) with the product gauge group \( SU(N_1) \times \cdots \times SU(N_k) \). The Seiberg-Witten differential is given by

\[ dS_{SW} = x \frac{dz}{z}. \tag{7.4} \]

A natural ansatz for the meromorphic differentials \( d\hat{\Omega}_n \) of Whitham deformations is to seek for them in the form

\[ d\hat{\Omega}_n = R_n(x, z)\frac{dz}{z}, \quad R_n(x, z) = \text{polynomial}. \tag{7.5} \]
We do not know how to construct the polynomials $R_n(x, z)$. The problem becomes even harder for the elliptic models of M5-branes.

### Relation to topological Landau-Ginzburg theories

The fractional power construction strongly suggests a direct link with topological Landau-Ginzburg theories of A-D-E singularities coupled to gravity or, equivalently, $d < 1$ topological strings \[^{25, 26}\]. The relation between the Seiberg-Witten theory and $d < 1$ topological strings has been studied from several aspects, such as the WDVV equations \[^{46, 47}\], flat coordinates and Gauss-Manin systems \[^{48, 49}\], etc. Of course the very notion of prepotentials itself is a bridge connecting the two worlds. In the $d < 1$ topological strings, the role of the Whitham equations is played by the dispersionless limit of integrable hierarchies \[^{50, 51, 52}\]. The fractional powers of the superpotential are fundamental building blocks of the Lax representation therein. Nevertheless, the emergence of fractional powers in the Seiberg-Witten theory is quite surprising.

An interesting outcome of our Whitham deformations is that they have an exotic limit as $\mu \to 0$. In this limit, the Seiberg-Witten curve reduces to the rational curve

$$z = W(x),$$

and the Seiberg-Witten differential turns into the rational differential

$$dS_{SW} = x \frac{W'(x)dx}{W(x)}. \quad (7.7)$$

As we shall show below, the Whitham equations, too, have a well defined limit. Furthermore, these differential equations are similar, but not identical, to the following counterpart in $d < 1$ topological strings \[^{50, 51, 52}\]:

$$\left. \frac{\partial}{\partial T_n} \sum_{m \geq 1} T_m R_m(x) \right|_{W(x) = \text{const.}} = R_n(x). \quad (7.8)$$

This difference stems from the difference of the two theories as Landau-Ginzburg models. Namely, whereas the Whitham deformations at $\mu = 0$ is still related to a curve defined by \((7.8)\), the Landau-Ginzburg description of $d < 1$ topological strings is based on a 0-dimensional manifold defined by the equation

$$W(x) = 0. \quad (7.9)$$
Now, let us present the Whitham equations at $\mu = 0$. For simplicity, we consider the case of the $SU(N + 1)$ curve where $W(x) = P(x)$; the other case can be treated similarly. Suppose, as usual, that the cycles $A_j$ are chosen to encircle the cuts between two neighboring roots $e_j^\pm$ of $P(x) - 4\mu^2$. As $\mu \to 0$, the $j$-th cuts shrink to a point at the $j$-th root $e_j$ of $P(x) = \prod_{j=1}^{N+1} (x - e_j)$. The period integrals $a_j = \oint_{A_j} dS$ then reduce to residue integrals, which can be readily calculated:

$$a_j = 2\pi i \sum_{n \geq 1} T_n R_n(e_j) \quad (n = 1, \cdots, N). \quad (7.10)$$

This defines a map $\vec{u} \mapsto \vec{a}$ from the $\vec{u}$-space to the $\vec{a}$-space with deformation parameters $T_n$, and this map is invertible if $\vec{T}$ is close to $(1, 0, 0, \cdots)$. (Note that the $N + 1$-th root $e_{N+1}$ of $P(x)$ is not independent; the roots of $P(x)$ obeys the constraint $\sum_{j=1}^{N+1} e_j = 0$.) The inverse map determines, as in the case with $\mu \neq 0$, a family of deformations of the rational curve $z = P(x)$. Under these deformations, the following equations can be eventually derived:

$$\frac{\partial}{\partial T_n} \sum_{m \geq 1} T_m R_m(x) \bigg|_{P(x) = \text{const.}} = R_n(x) - \sum_{k=1}^{N+1} \frac{R_n(e_k)P(x)}{(x - e_k)P'(x)}$$

$$\frac{\partial}{\partial a_j} \sum_{m \geq 1} T_m R_m(x) \bigg|_{P(x) = \text{const.}} = \left( \frac{1}{x - e_j} - \frac{1}{x - e_{N+1}} \right) \frac{P(x)}{2\pi i P'(x)} \quad (7.11)$$

We omit the proof of these equations, but the following comment would be enough for understanding: These equations can be derived from the Whitham equations (3.1) if $d\Omega_n$ and $d\omega_j$ are interpreted as follows:

$$d\Omega_n = R_n(x) \frac{P'(x)dx}{P(x)} - \sum_{j=1}^{N} 2\pi i R_n(e_j)d\omega_j,$$

$$d\omega_j = \left( \frac{1}{x - e_j} - \frac{1}{x - e_{N+1}} \right) \frac{dx}{2\pi i}. \quad (7.12)$$

In fact, they give a correct limit, as $\mu \to 0$, of the differentials on the $\mu \neq 0$ curve.

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A Spectral Curves of Affine Toda Systems

Here we present a list of the Toda spectral curves
\[ \det \left( L(z) - xI \right) = 0 \]  \hspace{1cm} (A.1)

used in the text, along with the \( L \)-matrices \( L(z) \). The \( L \)-matrices are realized in a representation of minimal dimensions. The \( L \)-matrices for \( A_N^{(1)}, B_N^{(1)}, C_N^{(1)} \) and \( D_N^{(1)} \) are borrowed from the work of Adler and van Moerbeke \[35\]. The other cases associated with the twisted affine algebras \( A_N^{(2)}, B_N^{(1)}, C_N^{(2)} \) and \( D_N^{(2)} \) are derived by the following “folding” procedure:
\[ D_{2N}^{(1)} \mapsto A_{2N-1}^{(2)}, \quad D_{N+2}^{(1)} \mapsto D_{N+1}^{(2)}, \quad D_{2N+2}^{(1)} \mapsto A_{2N}^{(2)}. \]  \hspace{1cm} (A.2)

The \( L \)-matrices in the following list have a “symmetric” form, i.e., \( L(z)^T = L(z^{-1}) \), as opposed to the \( L \)-matrices of Martinec and Warner \[3\]. Accordingly, the actual form of the spectral curves becomes
\[ \mu(z + z^{-1}) = W(x). \]  \hspace{1cm} (A.3)

upon removing an overall constant or a non-dynamical factor (the factor \( x \) the case of \( B_N^{(1)} \)). This curve, however, can be readily converted to the form of Martinec and Warner by recalling \( z \to z/\mu \).

Some comments on the notations in the list are in order. \( a_j \) and \( b_j \) (\( j = 1, \ldots, N \)) are related to the canonical variables \( q \) and \( p \) (both in the Cartan subalgebra of the classical part of the affine algebra) of the affine Toda system as follows:
\[ a_j = c_j g e^{\alpha_j q}, \quad b_j = p_j = h_j \cdot p. \]  \hspace{1cm} (A.4)

Here \( \alpha_j \) (\( j = 1, \ldots, N \)) are the simple roots and \( \alpha_0 \) is the affine root. (The null root is ignored.) \( c_j \) are numerical constants related to the root system, and \( g \) the coupling constant. (The \( a_j \)’s should not be confused with the special coordinates \( a_j \)’s in the Seiberg-Witten theory.) \( E_{jk} \) denotes the matrix with the only non-vanishing elements equal to 1 at \( (j, k) \):
\[ (E_{jk})_{mn} = \delta_{jm} \delta_{kn}. \]  \hspace{1cm} (A.5)
1. $A_N^{(1)}$: The L-matrix is $(N+1) \times (N+1)$.

$$L(x) = \sum_{j=1}^{N+1} a_j E_{j,j+1} + a_0 z E_{N1} + \sum_{j=1}^{N+1} b_j E_{jj} + \sum_{j=1}^{N+1} a_j E_{j+1,j} + a_0 z^{-1} E_{1N}.$$

$$\det(L(z) - xI) = (-1)^N A(z + z^{-1}) + (-1)^{N+1} P(x),$$

$$A = a_0 a_1 \cdots a_N, \quad \mu = A.$$

2. $B_N^{(1)}$: The L-matrix is $(2N+1) \times (2N+1)$.

$$L(x) = \sum_{j=1}^{N} a_j (E_{j,j+1} - E_{2N+1-j,2N+2-j}) + a_0 (z E_{2N+1,2} - z E_{2N,1})$$

$$+ \sum_{j=1}^{N} b_j (E_{jj} - E_{2N+1-j,2N+2-j})$$

$$+ \sum_{j=1}^{N} a_j (E_{j+1,j} - E_{2N+2-j,2N+1-j}) + a_0 (z^{-1} E_{2,2N+1} - z^{-1} E_{1,2N}).$$

$$\det(L(z) - xI) = x \left(2(-1)^N A(z + z^{-1}) - Q(x^2)\right),$$

$$A = a_0 a_1^2 \cdots a_N^2, \quad \mu = 2(-1)^N A.$$

3. $C_N^{(1)}$: The L-matrix is $2N \times 2N$.

$$L(x) = \sum_{j=1}^{N-1} a_j (E_{j,j+1} - E_{2N-j,2N+1-j}) + a_N E_{N,N+1} + a_0 z E_{2N,1}$$

$$+ \sum_{j=1}^{N} b_j (E_{jj} - E_{2N+1-j,2N+1-j})$$

$$+ \sum_{j=1}^{N-1} a_j (E_{j+1,j} - E_{2N+1-j,2N-j}) + a_N E_{N+1,N} + a_0 z^{-1} E_{1,2N}.$$

$$\det(L(z) - xI) = (-1)^N A(z + z^{-1}) + Q(x^2),$$

$$A = a_0 a_1^2 \cdots a_{N-1}^2 a_N, \quad \mu = (-1)^N A.$$

4. $D_N^{(1)}$: The L-matrix is $2N \times 2N$.

$$L(z) = \sum_{j=1}^{N-1} a_j (E_{j,j+1} - E_{2N-j,2N+1-j}) + a_N (E_{N,N+2} - E_{N-1,N+1})$$

$$+ a_0 (z E_{2N,2} - z E_{2N-1,1}) + \sum_{j=1}^{N} b_j (E_{jj} - E_{2N+1-j,2N+1-j})$$

$$+ \sum_{j=1}^{N-1} a_j (E_{j+1,j} - E_{2N+1-j,2N-j}) + a_N (E_{N+2,N} - E_{N+1,N-1})$$

$$+ a_0 (z^{-1} E_{2,2N} - z^{-1} E_{1,2N-1}).$$

$$\det(L(z) - xI) = -4(-1)^N A x^2 (z + z^{-1}) + Q(x^2),$$

$$A = a_0 a_1^2 \cdots a_{N-2}^2 a_{N-1} a_N, \quad \mu = 4(-1)^N A.$$
5. $A_{2N-1}^{(2)}$: The $L$-matrix is $2N \times 2N$.

$$
L(z) = \sum_{j=1}^{N-1} a_j (E_{j,j+1} - E_{2N-j,2N+1-j}) + a_N E_{N,N+1} \\
+ a_0 (zE_{2N,2} - zE_{2N-1,1}) + \sum_{j=1}^{N} b_j (E_{jj} - E_{2N+1-j,2N+1-j}) \\
+ \sum_{j=1}^{N-1} a_j (E_{j+1,j} - E_{2N+1-j,2N-j}) + a_N E_{N+1,N} \\
+ a_0 (z^{-1}E_{2,2N} - z^{-1}E_{1,2N-1}).
$$

$$
\det \left( L(z) - xI \right) = 2(-1)^N A x (z + z^{-1}) + Q(x^2).
A = a_0 a_1^2 \cdots a_{N-1}^2 a_N, \quad \mu = -2(-1)^N A.
$$

6. $D_{N+1}^{(2)}$: The $L$-matrix is $(2N + 2) \times (2N + 2)$.

$$
L(z) = \sum_{j=1}^{N} a_j (E_{j+1,j+2} - E_{2N+2-j,2N+3-j}) + a_0 (zE_{2N+2,1} - E_{12}) \\
+ \sum_{j=1}^{N} b_j (E_{j+1,j+1} - E_{2N+3-j,2N+3-j}) \\
+ \sum_{j=1}^{N} a_j (E_{j+2,j+1} - E_{2N+2-j,2N+2-j}) + a_0 (z^{-1}E_{1,2N+2} - E_{21}).
$$

$$
\det \left( L(z) - xI \right) = (-1)^N A (z + z^{-1} - 2) + x^2 Q(x^2).
A = a_0 a_1^2 \cdots a_N^2, \quad \mu = (-1)^N A.
$$

7. $A_2^{(2)}$: The $L$-matrix is $(2N + 1) \times (2N + 1)$.

$$
L(z) = \sum_{j=1}^{N-1} a_j (E_{j+1,j+2} - E_{2N+1-j,2N+2-j}) + a_N E_{N+1,N+2} \\
+ a_0 (zE_{2N+1,1} - E_{12}) + \sum_{j=0}^{N} b_j (E_{j+1,j+1} - E_{2N+2-j,2N+2-j}) \\
+ \sum_{j=1}^{N-1} a_j (E_{j+2,j+1} - E_{2N+2-j,2N+1-j}) + a_N E_{N+2,N+1} \\
+ a_0 (z^{-1}E_{1,2N+1} - E_{21}).
$$

$$
\det \left( L(z) - xI \right) = (-1)^N A (z + z^{-1}) + xQ(x^2).
A = a_0^2 a_1^2 \cdots a_{N-1}^2 a_N, \quad \mu = (-1)^N A.
$$
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