Undrained solution of cylindrical cavity expansion for unsaturated soils with modified Cam-clay model

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Abstract. This paper develops a new semi-analytical approach for undrained cylindrical cavity expansions in unsaturated soils. The unsaturated soil behavior is described within a simple, yet effective, critical state-based elastoplastic framework with suction being treated as a hardening parameter. Different from its saturated counterpart, the cavity expansion problem in unsaturated soils involves five basic variables, namely, the three stress components, specific volume, and suction. A constitutive equation for the water phase, in addition to the soil elastoplastic constitutive relationship plus the radial equilibrium condition, is therefore required to construct a set of coupled, self-contained differential equations for the solution of the cavity boundary value problem. With the introduction of an auxiliary variable, all these governing equations can be expressed in terms of the Lagrangian formulation and then simultaneously solved through the standard numerical procedure. The numerical example results show that the suction has significant hardening effects on the overall cavity expansion responses and, in particular, the stress trajectories calculated for a representative soil element at the cavity surface. It is also interesting to observe that the shear dilation phenomenon, developed during the course of cavity expansion, usually occurs in relatively dense unsaturated soils with high unsaturated suction.

1. Introduction

The analysis of cylindrical cavity expansion in soil provides a versatile and accurate geomechanical approach for the study of important problems in geotechnical engineering, such as the interpretation of pressuremeter tests (PMTs) [1,2], stress-strain response around installed piles [3,4], and tunnel-pile interaction [5]. Over the past few decades, many analytical and semianalytical solutions have been developed for cylindrical cavity expansion problems in elastoplastic materials. The early solutions of the cylindrical cavity expansion problem in saturated soils were presented based on elastic-perfectly plastic soil models, mainly using the Mohr-Coulomb yield criterion [6,7]. As far as sophisticated constitutive models are considered, the solutions of cavity expansion in critical state soils have been developed [8-11]. To investigate the influences of K0 consolidation anisotropy, analytical solutions for cavity expansion in anisotropic critical state soils were presented [12]. However, these solutions are only for the case of a saturated or a fully dry soil, irrespective of the fact that most natural as well as engineered soils widely encountered are in an unsaturated state [13,14].

Unsaturated soils exhibit a complicated and nonunique relationship among suction, moisture content and void ratio. Therefore, unlike in the drained cavity expansion in saturated soil, the suction, as an
additional stress variable, needs to be considered in the description of the stress-strain behavior of unsaturated soils. Nevertheless, to date, analytical investigations into the cavity expansion problem in unsaturated soils are a rarity, with a few exceptions reported notably by Russell and his collaborators. For example, Russell and Khalili [15] presented solutions for spherical cavity expansion problems in unsaturated soils whose behaviors are described by a revised form of the modified Cam Clay (MCC) model. Later, Russell and Khalili [16] further derived a unified semi-analytical solution for the cavity expansion in unsaturated soils by employing the bounding surface plastic model, and taking into the account the effects of constant suction and constant moisture content conditions. More recently, a cavity expansion analysis is presented by Yang et al [17] for unsaturated soils, where three drainage conditions and hydraulic hysteresis are considered. Cheng et al [18] presented a new cavity expansion solution in unsaturated soil of finite radial extent with the aids of the auxiliary variable defined by Chen and Abousleiman [10]. Through their theoretical methods, the boundary size effect of calibration chamber on CPT or model pile tests can be considered. Among these investigations, the similarity solution technique proposed by Collins and Stimpson [8] is used to analyze the expansion of created cavities from zero initial radii, since it allows for the determination of the limiting cavity pressure with relative ease. However, there are more or less simplified rather than rigorous definitions have been used for the deviator and mean effective stresses in the existing semi-analytical solutions for cavity expansions in unsaturated soils. This paper, extending the approach proposed by Chen and Abousleiman [10] to unsaturated soil, is concerned with the influence of the suction of unsaturated soils on the undrained cylindrical cavity expansion process. Without resorting to the similarity technique, the cylindrical cavity undrained expansion problem in unsaturated soils is formulated as solving a set of first-order ordinary differential equations, with the radial, tangential and vertical effective stresses, the specific volume as well as the suction, being the five basic unknowns. Compared with the existing solutions, rigorous definitions have been used for the deviator and mean effective stresses without the imposition of any approximation. In addition, a brief unsaturated constitutive model was introduced to increase the availability of the solutions in related engineering.

2. Constitutive Relationship in Unsaturated Soil

2.1. Soil-Water Characteristic Curve
The soil-water characteristic curve (SWCC) is a fundamental constitutive relationship in unsaturated soil mechanics, which describes the relationship between suction $s$ and the soil water content (or saturation degree $S_r$).

Herein the wetting paths SWCC model proposed by Russell and Buzzi [20], which incorporates the influences from the soil volume changes on the SWCC, is defined as follows:

$$S_r = \begin{cases} 
1 & \text{for } s \leq s_{ex} \\
\left(\frac{s}{s_{ex}}\right)^\alpha & \text{for } s \geq s_{ex}
\end{cases}$$

(1)

where $\alpha$ is the material parameter and $s_{ex}$ is the suction associated with air expulsion in soils, given by

$$s_{ex} = C e^{-\beta}$$

(2)

where $C$ is a positive dimensionless constant, $e$ is the void ratio, and $\beta$ is another dimensionless constant.

2.2. Stress States and Yield Function
The effective stress equation for unsaturated soils was proposed by Bishop [21] as follows:

$$\sigma'_\theta = \sigma''_\theta + \chi s \delta_{ij}$$

(3)

where $\sigma''_\theta$ is the net stress tensor with $\sigma''_\theta = \sigma''_\theta - u_i$, $\sigma''_\theta$ is the total stress tensor, $s$ equals $u_a + u_w$ ($u_a$ and $u_w$ are the pore air and pore water pressures, respectively), $\delta_{ij}$ is the Kronecker delta, and $\chi$ is the
effective stress parameter attaining a value of 1 for fully saturated soils and 0 for completely dry soils. For unsaturated soils, $\chi$ is of the form:

$$
\chi = \begin{cases} 
1 & \text{for } \frac{s}{s_{cs}} \leq 1 \\
\left(\frac{s}{s_{cs}}\right)^{-\gamma} & \text{for } \frac{s}{s_{cs}} \geq 1
\end{cases}
$$

(4)

where $\gamma$ is the material parameter with a best-fit value of 0.55 [22].

Assuming the unsaturated soil behavior after yielding can be described by the modified Cam-Clay (MCC) model, the yield surface in the effective mean-stress $p'$ and shear-stress $q$ plane is elliptical, which can be expressed as

$$
f' (p', q, p'_c) = \frac{q^2}{M^2} p' + p' - p'_c = 0
$$

(5)

where $M$, the slope of the critical state line (CSL) in the $p'$-$q$ plane, is equal to $6\sin\phi'_{cs}/(3-\sin\phi'_{cs})$. Since the critical state angle $\phi'_{cs}$ is found to be unique for saturated and unsaturated conditions, $M$ can be taken as a material parameter independent of the suction. As a result, the CSL seems to be consistent for both saturated and unsaturated soils in the $p'$-$q$ plane [22].

$p'$ (equals to $ij_{ij} 3 \sigma_3^{-}$) and $q$ (equals to $\sqrt{ij_{ij} 3 2}$) are the mean effective skeleton stress $p'$ and the deviator skeleton stress, respectively; $ij$ is the deviatoric tensor. The parameter $c_p'$, as an effective preconsolidation stress of the unsaturated soils, can be read from the size of the yield surface. According to equation (5), the incremental form of $p'_c$ can be obtained as follows:

$$
Dp'_c = \frac{1}{M^2} \left[ (M^2 - \eta^2) Dp' + 2 \eta Dq \right]
$$

(6)

where $\eta = q/p'$ is the stress ratio.

The variation in specific volume $\nu$ with the isotropic consolidation curves in the $\nu$–$\ln p'$ plane is also influenced by the suction, which can be attributed to suction hardening. In detail, the relationship between $\nu$ and $p'$ can be defined as follows:

$$
\nu = N(s) - \lambda(s) \ln(p'/p'_s)
$$

(7)

where $\nu$ is the specific volume, equal to $e+1$, $\lambda(s)$ is the slope of the unsaturated normal consolidation line, $N(s)$ is the specific volume at $p' = 1$ kPa, and $p'_s$ is a reference pressure which is taken as 1 kPa.

As a result, the relationship between $p'_s, s$ and corresponding $\nu$ is given by the incremental form of equation (7):

$$
D\nu = \left[ \frac{\partial N(s)}{\partial s} - \frac{\partial \lambda(s)}{\partial s} \ln \frac{p'_c}{p'_s} \right] Ds - \lambda(s) \frac{Dp'_c}{p'_s}
$$

(8)

The elastic volumetric strain increment, $D\varepsilon^e_p$ in the $\nu$–$\ln p'$ plane is shown as follows:

$$
D\varepsilon^e_p = \frac{\kappa}{\nu p'} Dp'
$$

(9)

where $\kappa$ is the slope of the loading-reloading line in the $\nu$–$\ln p'$ plane.

Since the total volumetric strain increment is defined as $D\varepsilon_p = -\frac{D\nu}{\nu} = D\varepsilon^e_p + D\varepsilon^p_p$, the equation (8), combing with equation (9), can be rewritten as:

$$
-\nu \left( \frac{\kappa}{\nu p'_s} Dp'_c + D\varepsilon^p_p \right) = \left[ \frac{\partial N(s)}{\partial s} - \frac{\partial \lambda(s)}{\partial s} \ln \frac{p'_c}{p'_s} \right] Ds - \lambda(s) \frac{Dp'_c}{p'_s}
$$

(10)

As a result, $Dp'_c$, in addition, can be obtained as follows:
From equation (11), it can be found that $p'_c$, different from its saturated counterpart, incorporates not only the effect of strain hardening, which is typically denoted by plastic volumetric strain for soils, but also the effect of strengthening by suction.

3. Definition of the Cavity Expansion Problem in unsaturated soil

The cylindrical cavity expansion problem with initial radius $a_0$ in an infinite unsaturated soil is schematically shown in Figure 1. Compressive stress and strains are taken as positive. Assuming the soil is isotropic, the internal cavity pressure increases gradually from its initial value $\sigma'_0$, initial yielding first occurs at the cavity wall. The internal pressure will reach $\sigma'_a$ when the cavity radius expands to $a$. As a result, a plastic zone with a radius of $r_p$ will be formed around the cavity. The soil beyond the elastic/plastic interface remains in an elastic state. The symbol $r$ refers to any arbitrary point located in the plastic zone with its initial position denoted by $r_0$.

Figure 1 Expansion of a cylindrical cavity.

Using a cylindrical polar coordinate system, the initial position of a soil element around a cylindrical cavity can be described as $(r, \theta, z)$, and the effective principal stresses are $\sigma'_r$, $\sigma'_\theta$ and $\sigma'_z$. Considering an element at a radial distance $r$ from the center of the cavity, the equilibrium equation for saturated soils is modified to include $s$ in unsaturated soils and is of the form

$$\frac{\partial \sigma'_r}{\partial r} + \frac{\sigma'_r - \sigma'_\theta}{r} - \frac{\partial (\chi s)}{\partial r} = 0$$ (12)

where $\chi$ is a function of $s_{ex}$ and $s$, as shown in equation (4). For fully saturated soil, the $\chi$ is 1, and $s$ is a constant less than $s_{ex}$, the equation (12) will reduce to the effective form of equilibrium equation in saturated soils.

Combining equations (2), (4) and (12), the equilibrium equation can be rewritten as

$$\frac{\partial \sigma'_r}{\partial r} + \frac{\sigma'_r - \sigma'_\theta}{r} - \psi \frac{\partial s}{\partial r} + \beta \frac{\chi s}{\partial \theta} = 0$$ (13)

where

$$\psi = \frac{d(\chi s)}{ds}.$$ 

To simply this problem and drive the analytical solution, it is assumed that the cylinder cavity extends infinitely in the axis direction, thus the problem is taken as a plane-strain problem.

4. Solution in Elastic Region

The soil in the elastic zone is assumed to obey the generalized Hooke’s law. Therefore, the elastic stress-strain relationship can be expressed in terms of the effective stresses as [23]

$$\begin{bmatrix} D\varepsilon'_{e} \\ D\varepsilon'_0 \\ D\varepsilon'_z \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} D\sigma'_r \\ D\sigma'_0 \\ D\sigma'_z \end{bmatrix}$$ (14)
where $De_r^e$, $De_\theta^e$, and $De_z^e$ are the elastic strain increments of a given material particle in the radial, tangential and vertical directions, respectively; $\nu$ is the Poisson’s ratio; and $E$ is the Young’s modulus, given by [24]

$$E = \frac{3(1-2\nu)\mu\rho'}{\kappa}$$

Since $\rho'$ and $\nu$ are constant, $E$ always remains unchanged in the external elastic region and is equal to the initial value $E_0$. The elastic solutions for stresses can be shown as follows [23]:

$$\sigma_r' = \sigma_0' + (\sigma_r' - \sigma_0') \left(\frac{r_p}{r}\right)^2$$

$$\sigma_\theta' = \sigma_0' - (\sigma_\theta' - \sigma_0') \left(\frac{r_p}{r}\right)^2$$

$$\sigma_z' = \sigma_0'$$

where $\sigma_p'$ is the radial stress on the elastic-plastic boundary and $r_p$ denotes the position of the elastic-plastic boundary.

During elastic deformation, the suction and water content remain constant in the elastic region.

5. Elastoplastic Analysis in the Plastic Region

The yielding of soil is described by the MCC model, as defined by equation (5). An associative plastic flow rule in the effective stress-space is adopted, namely, the yield function $f$ is assumed to be the plastic potential. As a result, the plastic strain increases are obtained by

$$\dot{e}_r^p = \Lambda \frac{\partial f}{\partial \sigma_r'} = \Lambda \left\{ \frac{p'(M^2 - \eta^2)}{3} + 3(\sigma_r' - \rho') \right\}$$ (19a)

$$\dot{e}_\theta^p = \Lambda \frac{\partial f}{\partial \sigma_\theta'} = \Lambda \left\{ \frac{p'(M^2 - \eta^2)}{3} + 3(\sigma_\theta' - \rho') \right\}$$ (19b)

$$\dot{e}_z^p = \Lambda \frac{\partial f}{\partial \sigma_z'} = \Lambda \left\{ \frac{p'(M^2 - \eta^2)}{3} + 3(\sigma_z' - \rho') \right\}$$ (19c)

where $\dot{e}_r^p$, $\dot{e}_\theta^p$, and $\dot{e}_z^p$ are the plastic strain increases in the $r$, $\theta$ and $z$ directions; $\Lambda$ is a scalar multiplier, and $\eta=q/\rho'$ is the stress ratio.

Combining equations (6) and (10), the plastic volumetric strain increase $\dot{e}_v^p$ is given by

$$\dot{e}_v^p = \frac{\lambda(s)-\kappa}{\nu \rho'^2(M^2 + \eta^2)} \left[ \frac{p'(M^2 - \eta^2)}{M^2 - \eta^2} Dq' + \frac{2\eta}{M^2 - \eta^2} \left[ \frac{p'(M^2 - \eta^2)}{M^2 - \eta^2} \right] Dq \right]$$

(20)

Since $\dot{e}_v^p = \dot{e}_r^p + \dot{e}_\theta^p + \dot{e}_z^p$, it follows from equations (19a)-(19c) and (20) that

$$\Lambda = \frac{\lambda(s)-\kappa}{\nu \rho'^2(M^2 + \eta^2)} \left[ \frac{Dp'}{M^2 - \eta^2} Dq' - \frac{2\eta}{M^2 - \eta^2} \left[ \frac{Dp'}{M^2 - \eta^2} \right] Dq \right]$$

(21)

For unsaturated soil, the specific volume $\nu$ in the plastic region does not remain constant during cavity expansion since the void air will be rapidly expelled and the soil can be compressed, regardless of the drainage conditions. Thus, there are five unknown variables that must be determined in order to solve the cavity expansion problem in unsaturated soil under undrained conditions,
namely $\sigma'_r$, $\sigma'_\theta$, $\sigma'_z$, $\nu$ and $s$. However, only equations 19(a)-19(c) are available. For this reason, the constitutive equation of the water phase and equilibrium equation (13) with a revised form are introduced, as will be detailed below.

The volume of water, $v_w$, is known as the product of the saturation degree $S_r$ and the volume of void ratio $e$ (assuming the volume of the soil solid $v_s$ is 1), namely, $v_w = S_r e$. Hence, its incremental form is

$$ Dv_w = eDS_r + S_r De $$

(22)

Since $S_r$ is a function of suction $s$ and void ratio $e$, the incremental form of $S_r$ can be expressed as follows:

$$ DS_r = \frac{\partial S_r}{\partial S} DS + \frac{\partial S_r}{\partial e} De $$

(23)

Where $\frac{\partial S_r}{\partial e}$ represents the slope of the SWCC.

Combining equations (1), (22) and (23), the water phase constitutive equation, which links changes in pore water volume with changes in suction, can be expressed as follows:

$$ \frac{\alpha(\nu - 1)}{s} DS + (\alpha \beta + 1) S_r Du = Dv_w $$

(24)

Under the undrained condition, the water content is constant, i.e. $Dv_w = 0$. Therefore, equation (24) can be written as:

$$ Du = \frac{\alpha(1-\nu)}{(\alpha \beta + 1)s} Ds $$

(25)

Substituting equation (21) into equations (19), the solid phase constitutive equations can be obtained. Combing with the water phase constitutive equation (25) in the plastic zone, and introducing an auxiliary variable $\xi (=u_r/r = (r - r_0)/r)$ as Chen and Abousleiman [10], the differential equations for stress variables in the elastic-plastic region can be expressed as

$$ \frac{D\sigma'_r}{D\xi} = b_{11} \frac{\sigma}{\Delta \nu} - b_{12} - b_{11} + b_{14} $$

(26a)

$$ \frac{D\sigma'_\theta}{D\xi} = b_{21} \frac{\sigma}{\Delta \nu} - b_{22} - b_{21} + b_{24} $$

(26b)

$$ \frac{D\sigma'_z}{D\xi} = b_{31} \frac{\sigma}{\Delta \nu} - b_{32} - b_{31} + b_{34} $$

(26c)

$$ \frac{De}{D\xi} = \frac{b_{41}}{\Delta} $$

(26d)

$$ \frac{D\nu}{D\xi} = \frac{\sigma}{\Delta} $$

(26e)

where $D\sigma'_r$, $D\sigma'_\theta$, $D\sigma'_z$ and $De$, $D\alpha_h$, $D\alpha_v$, are the stress and strain increases in a given material particle in the radial, tangential and vertical directions, respectively. $b_{ij}$ and $\Delta$ can be expressed as following (Note $b_{12} = b_{21}, b_{13} = b_{31}, b_{14} = b_{41}$):

$$ b_{11} = \frac{1}{E} \left( 1 - \nu^2 + E \alpha_h^2 y + 2E\alpha_v a_y + Ea_z^2 y \right) $$

(27a)

$$ b_{12} = \frac{1}{E} \left[ -E\alpha_h (a_h + v a_z) y + v(1 + \nu - E\alpha_h a_y + Ea_z^2 y) \right] $$

(27b)

$$ b_{13} = \frac{1}{E} \left[ -E\alpha_v (va_y + a_z) y + v(1 + \nu + Ea_z^2 y - E\alpha_h a_y) \right] $$

(27c)

$$ b_{14} = \frac{1}{E} \left[ -1 + v \right] a_y - v(a_h + a_z) $$

(27d)
To obtain the expression of $\sigma$, the equilibrium equation (13) should be used. By introducing the auxiliary variable $\xi$, the equilibrium equation is first converted to the Lagrangian form as follows:

$$\frac{D\sigma'}{D\xi} - \psi \frac{D\xi}{D\xi} + \beta \psi \frac{\chi \xi}{\nu - 1} \frac{D\sigma'}{D\xi} \left(1 - \xi - \frac{\nu_0}{\nu(1 - \xi)}\right) + \sigma' - \sigma_o = 0$$

where $D/D\xi$ denotes the material derivative with respect to $\xi$ following a given particle (Lagrangian description).

Substituting equation (28) into equation (26a), it can be obtained

$$\sigma' = \left[\frac{\sigma' - \sigma_o}{1 - \xi - \nu_0 / \left(\nu(1 - \xi)\right)} - b_{11} - b_{11} \frac{b_{12} - b_{11}}{\Delta(1 - \xi)}\right] + \frac{b_{11} - b_{14} + \psi b_{15} - \beta \psi \chi \xi}{\Delta(1 - \xi)} - \nu - 1$$

6. Initial Stress Conditions and the Elastic-plastic Boundary

Equations (26a)-(26e) present a system of simultaneous differential equations for cavity expansion under undrained conditions. The corresponding values at the elastic-plastic boundary need to be determined beforehand as the initial values for solving these equations.

In the elastic zone, the change in the mean effective stress is zero, i.e. $\Delta p = 0$, the mean effective stress at the elastic-plastic boundary $r_p$ therefore is

$$P'\left(s_p\right) = P'_o$$
where $p'_0$ is the initial mean effective stress, and $\xi_p = \left( \frac{u_r}{r} \right)_{r=r_p}$.

Substituting equation (30) into equation (5), the deviator stress at the elastic-plastic boundary is

$$q(\xi_p) = M \left\{ \left[ \frac{p'_0}{1 + \frac{\xi_p^2}{3}} \right]^{\frac{1}{2}} \left[ \frac{p'_0 - p'(\xi_p)}{1 + \frac{\xi_p^2}{3}} \right] \right\}^{\frac{1}{2}}$$  \hspace{1cm} (31)

According to definitions of $p'$ and $q$, as well as equations (16), (17) and (18), the effective stress components at the elastic-plastic boundary are

$$\sigma'_c(\xi_p) = p'_0 + \left[ \frac{q(\xi_p)}{3} \right]^{\frac{1}{2}}$$ \hspace{1cm} (32a)

$$\sigma'_p(\xi_p) = p'_0 - \left[ \frac{q(\xi_p)}{3} \right]^{\frac{1}{2}}$$ \hspace{1cm} (32b)

$$\sigma'_s(\xi_p) = \sigma'_0$$ \hspace{1cm} (32c)

The specific volume and suction at the elastic-plastic boundary are equal to their far-field values, namely, \(v(\xi_p) = v_0, \ s(\xi_p) = s_0\), where $v_0$ and $s_0$ are the initial specific volume and suction, respectively.

Thus far, the governing equation (26) and the stress conditions (equation 32) are all expressed in terms of the auxiliary variable $\xi$. To complete the solutions, it is necessary to transfer the variable $\xi$ back to the radial particle positions $r$, which can be achieved as follows:

$$\frac{r}{a} = \exp \left( \int_{\xi=0}^{\xi_p} \frac{d\xi}{\sqrt{1 - \left( \frac{v(\xi)(1-\xi)}{v(\xi_0)} \right)}} \right)$$ \hspace{1cm} (33)

where $\xi(a)$ corresponds to the value of $u_r/r$ of a specific particle at the cavity wall.

### 7. Results and Discussion

#### 7.1. Effects of Suction on Cavity Expansion in Unsaturated Soils

Following Estabragh et al. [25], the consolidated parameters of compacted silty soils with different initial specific volumes $\nu_0$ ($\nu_0=1.85$ and 1.65) are listed in Table 1, where $N(s)$ and $\lambda(s)$ for intermediate suction values can be obtained by linear interpolation from 100kPa to 300kPa intervals. The initial net mean stress $p_n (p-u_a)$ is 50kPa, and the constitutive properties of this soil are $M=1.2$ (or $\phi'_c=30^\circ$), $\kappa=0.012$, and $\nu=0.3$. The values of $\alpha$ in equation (1) are assumed to be -0.28 for loose soil ($\nu_0=1.85$) and -0.23 for dense soil ($\nu_0=1.65$). In addition, the other parameters (in equations 1 and 2) related to the SWCC are $C=30$ and $\beta=2.75$. By taking initial suction $s_0=100kPa$, 200kPa and 300kPa, the suction effects on the distributions of stress components $e^-lnp'$ and $p'-q$ stress paths, and on the cavity expansion pressures are investigated.

| Initial specific volume | Parameters | Initial suction (kPa) |
|-------------------------|------------|-----------------------|
| $\nu_0=1.85$            | $N(s)$     | $s_0=0$ 2.160 2.956 3.066 3.113 |
|                         | $\lambda(s)$ | $s_0=100$ 0.095 0.203 0.203 0.201 |
| $\nu_0=1.65$            | $N(s)$     | $s_0=200$ 1.961 2.128 2.164 2.170 |
|                         | $\lambda(s)$ | $s_0=300$ 0.066 0.076 0.076 0.074 |

The trajectories in the $(lnp', \upsilon)$ plane are shown in Figures 2(a) and 2(b) with initial specific volumes $\nu_0$ of 1.85 and 1.65, respectively. Both $p'$ and $\upsilon$ are constant in the elastic region; therefore, the outer
elastic region is mapped into the initial point of the elastic-plastic trajectory. For the loose soil ($\nu_0=1.85$), the specific volume in the plastic region monotonically decreases to the critical state line (see Figure 2(a)), indicating that there is a volumetric contraction of the soil during cavity expansion. In detail, the decrease in specific volumes is gentle in the first stage and then tends to be rapid as the mean effective stress continues to increase. As $s_0$ increases from 100kPa to 300kPa, the consolidated curves gradually move to the right, which implies that the soil tends to be stronger and stiffer with increasing suction. For the dense soil ($\nu_0=1.65$), the consolidated curve under $s_0=100$kPa has a shape similar to that of the loose soil, i.e. $\nu$ monotonically decreases with $p'$. By contrast, as $s_0$ increases to 200kPa and 300kPa, the specific volumes first increase and then decrease rapidly as $p'$ increases continually. This shows that the unsaturated soil tends to be dilatant at the beginning of the cavity expansion and then compressed as the cavity pressure continues to increase.

Figure 2  Cavity expansion results for unsaturated soil in the $\nu$-$\ln p'$ plane for undrained conditions: (a) $\nu_0=1.85$ and (b) $\nu_0=1.65$.

Figures 3 show the distributions of $\sigma'_r$, $\sigma'_z$ and $\sigma'_\theta$, as well as $\nu$, along the normalized radial distance $r/a$ corresponding to an expanded cavity radius of $a/a_0=4$ for the various values of $\nu_0$ (1.85 and 1.65) considered. From Figure 3, it can be seen that the specific volume $\nu$ and effective mean stress $p'$ (equal to ($\sigma'_r + \sigma'_z + \sigma'_\theta$)/3 ) remain constant in the elastic region. In the plastic region, the soil will be compressed as the cavity expands. As a consequence, the stress components $\sigma'_r$, $\sigma'_z$ and $\sigma'_\theta$ increase while the specific volume $\nu$ monotonically decreases.

Figure 3  Distributions of stress components and specific volume around the cavity in unsaturated soils with (a) $\nu_0=1.85$ and (b) $\nu_0=1.65$. 
Comparing the stress curves in with different suctions, it can be found that the stresses at the cavity wall with high suction are larger than those with low suction, which indicates that suction has a hardening effect on the soil response. From Figure 3(b) \((\nu_0=1.65)\), it is interesting to observe that \(\nu\) increases near the elastic-plastic boundary, which shows that shear dilation occurs during cavity expansion in dense soil. The larger the suction is, the more obvious the shear dilation. In addition, the locations of the elastic-plastic boundaries in Figure 3(b), which are different from those in Figure 3(a), gradually decrease as the suctions increase. The reasons may be attributed to the soil heave around the expanding cavities in dense soil, which reduce the plastic region in the radial distance.

The variations in \(s\) through the plastic annulus are shown in Figure 4. For the loose unsaturated soil \((\nu_0=1.85)\), \(s\) in the elastic region remains constant \((s_0)\) since there are no void volume or water content changes during cavity expansion. In the plastic region, the void volume \(e\) is compressed while \(v_w\) remains constant under an undrained condition. Thus, the soil saturation degree \(S_r\), which is the ratio of \(v_w\) over \(e\), increases as the cavity expands. The suction \(s\) in the plastic region, having a negative correlation with \(S_r\), decreases monotonically and reaches the minimum value at the cavity wall, as shown in Figure 4(a). For the dense soil \((\nu_0=1.65)\), the variation in \(s\) with \(r/r_p\) is similar to that in the loose soil. However, as the initial suction increases to 200kPa and 300kPa, the shear dilation occurs in the plastic region at the beginning of cavity expansion. As a result, \(s\) increases, since the void volume expands while the water content always remains constant.

![Figure 4](image1.png)

**Figure 4** Variation in suction through the plastic annulus of an expanding cylindrical cavity: (a) \(\nu_0=1.85\) and (b) \(\nu_0=1.65\).

![Figure 5](image2.png)

**Figure 5** Variation in internal cavity pressure with cavity radius: (a) \(\nu_0=1.85\) and (b) \(\nu_0=1.65\).

The cavity pressure \(\sigma_a\) is plotted against the normalized cavity radius \(a/a_0\) in Figure 5. In general, the cavity pressure increases rapidly in the range \(1\leq a/a_0\leq 3\) but then increases slowly with a further increase in \(a/a_0\) until the pressure reaches a steady value. For soils with the same \(\nu_0\), the cavity pressures required to expand the cavity increase with suction, which can be attributed to suction hardening effects on
unsaturated soils. Comparing the cavity pressures in unsaturated soils with the same \( s_0 \) but different \( \nu_0 \), as shown in Figure 5(a) and Figure 5(b), it is clearly seen that a decrease in \( \nu_0 \) results in an increase in cavity pressure \( \sigma_a \) because the soil becomes denser as \( \nu_0 \) increases.

7.2. Comparison with the Similarity Solution Technique

Comparisons between the present solution and the traditional similarity solution [15] are shown in Figures 4-5, where the calculated parameters (Table 1), and initial stress \( (p_0' \text{ and } q_0) \) adopted are all the same. From Figure 4, it can be seen that the suction change \( (s-s_0) \) obtained from the similarity technique is smaller than those obtained from the presented method. The comparisons of the cavity pressures calculated by the two methods are also shown in Figure 5. The cavity pressures computed by the similarity technique are all smaller than the values calculated by the presented method. As shown in Figure 5(a) and Figure 5(b), the differences grow as the suction increases. It is inferred that the reason for these deviations is mainly because of the difference in the \( p' \text{ and } q \) definitions adopted in the two approaches. In the similarity technique, \( p' \text{ and } q \) are simplified and do not depend on the axial stress \( \sigma_z' \). In the present method, \( p' \text{ and } q \) are defined as functions of the three stress components \( \sigma_r', \sigma_\theta' \text{ and } \sigma_z' \) without any approximation. In addition, the same values of \( p_0' \text{ and } q_0 \) were adopted in the two calculations. As a result, the stress components calculated from the two approaches are obviously different from each other, although the stress paths in the \( p'-q \) plane are similar.

7.3. Conclusions

A semianalytical elastoplastic solution for the undrained expansion of a cylindrical cavity in unsaturated soils is developed. With the suction being treated as a hardening parameter, the unsaturated soil behavior is described within a simple, yet effective, critical state-based elastoplastic framework. Following the novel solution scheme proposed earlier by Chen and Abousleiman [10], an additional auxiliary variable has been introduced to establish a link between the Eulerian and Lagrangian formulations of the radial equilibrium condition. The plastic zone solution can eventually be obtained by numerically solving a system of five partial differential equations, which essentially involves the three stress components (in the radial, tangential, and vertical directions), specific volume, and suction in saturated soils.

The numerical results demonstrate that suction has significant hardening effects on unsaturated soil responses during cavity expansion. For loose unsaturated soils, the suction will decrease as the soil saturation increases due to soil void compression under undrained conditions. For relatively dense unsaturated soils, the shear dilation phenomenon develops during the course of cavity expansion, and the suction will increase correspondingly. The proposed analytical solution provides a theoretical basis for interpretation of the pressuremeter tests.

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References

[1] Bolton M. D., and Whittle R. W. 1999. A non-linear elastic/perfectly plastic analysis for plane strain undrained expansion tests, Geotechnique. 49(1):133-141.
[2] Chang M. F., Cao L. F., and Teh C. I. 2001. Undrained cavity expansion in modified Cam clay II: Application to the interpretation of cone penetration tests. Geotechnique. 51(4): 331-346.
[3] Randolph M. F., Carter J. P., and Wroth C. P. 1979. Driven piles in clay-the effects of installation and subsequent consolidation. Geotechnique 29(4): 361-393.
[4] Lee F. H., Juneja A., and Tan T. S. 2004. Stress and pore pressure changes due to sand compaction pile installation in soft clay. Geotechnique 54: 1-16.
[5] Marshall A. M. 2012. Tunnel-pile interaction analysis using cavity expansion methods. *J Geotech Geoenviron Eng*. **138**(10): 1237-1246.

[6] Carter J. P., Randolph M. F., and Wroth C. P. 1979. Stress and pore pressure changes in clay during and after the expansion of a cylindrical cavity. *Int J Numer Anal Meth Geomech*. **3**(4): 305-322.

[7] Yu H. S., and Houlsby G. T. 1991. Finite cavity expansion in dilatants soils: loading analysis. *Geotechnique* **41**(2): 173-183.

[8] Collins I. F., and Stimpson J. R. 1994. Similarity solutions for drained and undrained cavity expansion in soils. *Geotechnique* **44**(1): 21-34.

[9] Cao L. F., Teh C. I., and Chang, M. F. 2001. Undrained cavity expansion in modified Cam clay I: Theoretical analysis. *Geotechnique*. **51**(4): 323-334.

[10] Chen S. L., and Abousleiman N. Y. 2013. Exact drained solution for cylindrical cavity expansion in modified Cam Clay soil. *Geotechnique*. **63**(6):510-517.

[11] Li L., Chen H. H., Li J. P., and Sun D. A. 2021. An elastoplastic solution to undrained expansion of a cylindrical cavity in SANICLAY under plane stress condition. Comput. *Geotech*. **132**, 103990.

[12] Li L., Li J. P., and Sun D. A. 2016. Anisotropically elasto-plastic solution to undrained cylindrical cavity expansion in K0-consolidated clay. *Comput. Geotech*. **73**, 83-90.

[13] Lehane B. M., Ismail M. A., and Fahey M. 2004. Seasonal dependence of in situ test parameters in sand above the water table. *Geotechnique*. **54**(3): 215-218.

[14] Pournaghiazar M., Russell A. R. and Khalili N. 2013. The cone penetration test in unsaturated sands. *Geotechnique* **63**(14): 1209-1220.

[15] Russell R., and Khalili N. 2002. Cavity expansion in unsaturated soils. *Proceedings of the 3rd International Conference on Unsaturated soils*, Edited by Juca JFT, de Campos and Marinho FAM. pp 233-238.

[16] Russell R., and Khalili N. 2006. On the problem of cavity expansion in unsaturated soils. *Comput Mech*. **37**(4): 311-330.

[17] Yang H. W., and Russell A. R. 2015. Cavity expansion in unsaturated soils exhibiting hydraulic hysteresis considering three drainage conditions. *Int J Numer Anal Meth Geomech*. **39**:1975-2016.

[18] Cheng Y., Yang H. W., and Sun D. A. 2018. Cavity expansion in unsaturated soils of finite radial extent. *Computers and Geotechnics*. **102**:216-228.

[19] Lu N., and Likos W. J. 2004. Unsaturated soil mechanics, John Wiley and Sons, New Jersey, U.S.

[20] Russell A. R., and Buzzi O. 2012. A fractal basis for soil-water characteristics curves with hydraulic hysteresis. *Geotechnique* **62**(3): 269-274.

[21] Bishop A. W. 1959. The principle of effective stress. *Teknisk Ukeblad*. **106**(39): 859-863.

[22] Loret B., and Khalili N. 2002. An effective stress elastic-plastic model for unsaturated porous media. *Mechanics of Materials*. **34**: 97-116.

[23] Yu H. S. 2000. Cavity expansion methods in geomechanics. Dordrecht, *Kluwer Academic*.

[24] Wood, D. M. 1990. Soil behaviour and critical state soil mechanics. Cambridge UK, Cambridge University Press.

[25] Estabragh A. R. Javadi A. A. and Boot J. C. 2004. Effect of compaction pressure on consolidation behavior of unsaturated silty soil. *Can Geotech J*. **41**:540-550.