Kähler Finsler manifolds with curvatures bounded from below

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Abstract

We obtain a partial parallelism of the complex structure on Kähler Finsler manifolds. As applications, we prove Synge-Tsukamoto theorem and Bonnet-Myers theorem for positively curved Kähler Finsler manifolds. Moreover, we generalize a comparison theorem due to Ni-Zheng by introducing the notion of orthogonal Ricci curvature to Kähler Finsler geometry.

Keywords. Kähler Finsler metric, comparison theorem, Ricci curvature, holomorphic curvature.

MSC2010: 53C60, 53B40.

1 Introduction

Kähler Finsler geometry, a natural generalization of Kähler geometry, was initiated by Abate-Patrizio in [1], where the Kobayashi metric is shown to be weakly Kähler. Recently, there has been a surge of interest in Kähler Finsler geometry, especially in its global and analytic aspects. We attend to study more global properties of Kähler Finsler manifolds.

The classical Synge's theorem gives simply connectedness of Riemannian manifolds with positive sectional curvature. Tsukamoto [13] proved a Kähler version under the assumption of positive holomorphic sectional curvature. The Finsler version of Synge's theorem was first derived by Auslander [2], where the notion of flag curvature is adopted. Won introduced the notion of pseudo-Kähler Finsler metrics and proved a Synge type theorem. Adopting the Kähler notion of Abate and Patrizio, we obtain a Synge type theorem for weakly Kähler Finsler manifolds.

Theorem 1.1. Let \( (M, G) \) be a strongly convex weakly Kähler Finsler manifold. Suppose it is complete and the holomorphic curvature \( H \geq \lambda > 0 \) is bounded below uniformly by a positive constant. Then \( M \) is compact and simply connected.

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The compactness result in the above theorem can be considered as a version of Myers's theorem. To deduce the Myers's theorem in real Finsler case, the positivity of Ricci curvature is assumed. For Kähler Finsler manifolds, Yin-Zhang [19] verified a Myers's theorem for manifolds of positive bisectional curvature. In classical Kähler geometry, by introducing a notion of curvature, namely the so-called orthogonal Ricci curvature, Ni-Zheng [9] proved a Myers type theorem and some comparison results. We generalize the notion of orthogonal Ricci curvature (cf. §6) to Kähler Finsler geometry and verify a Myers type theorem.

**Theorem 1.2.** Let $(M, F)$ be a complete strongly convex weakly Kähler Finsler manifold of complex dimension $n$. Suppose the orthogonal Ricci curvature $\text{Ric}^\perp \geq (2n - 2)\lambda > 0$, then the diameter of $M$ is at most $\pi/\sqrt{\lambda}$.

The comparison technique is widely used in Riemannian geometry. In the real Finsler setting, Shen [11] first extended comparison theorems to Finsler geometry. Later, Wu-Xin [15] proved Hessian and Laplacian comparison theorems under various curvature conditions. For more generalizations, we can refer to [12, 10] and references therein. In the complex Finsler realm, by defining the Hodge Laplacian, Zhong-Zhong [18] and Xiao-Zhong-Qiu [17] obtained some vanishing theorems and Laplacian comparison theorems on the tangent bundle. Yin-Zhang [19] discussed the comparison theorems for the nonlinear complex Hessian. Later, Li-Qiu [6] applied the comparison theorem to show certain Kähler Finsler manifold is Stein. One of the central estimates in these comparison theorems is the Laplacian comparison. The present paper continues investigations in this direction. We introduce the orthogonal Laplacian and derive the following result.

**Theorem 1.3.** Let $(M, G)$ be a complete strongly convex Kähler Finsler manifold of complex dimension $n$. We have the followings whenever the distance function $r$ is smooth:

(i) If the orthogonal Ricci curvature $\text{Ric}^\perp \geq (2n - 2)\lambda$, then

$$\Box^\perp r \leq (2n - 2)ct_\lambda(r).$$

The precise definition of $\Box^\perp r$ and $ct_\lambda$ can be found in §7.

(ii) If the holomorphic curvature $H \geq 4\lambda$, then

$$H(r)(J(\nabla r), J(\nabla r)) \leq 2ct_\lambda(2r)$$

where $H(r)$ is the Hessian of $r$ and $J$ is the complex structure.

The above results provide a Finsler generalization of the results of Liu [8] and Ni-Zheng [9]. If both assumptions $\text{Ric}^\perp \geq (2n - 2)\lambda$ and $H \geq 4\lambda$ are satisfied, the above theorem implies a volume comparison and a eigenvalue comparison.

**Corollary 1.1.** Let $(M, G, \mu)$ be a strongly convex Kähler Finsler $n$-manifold with vanishing Shen curvature. Assume $\text{Ric}^\perp \geq (2n - 2)K$ and $H \geq 4K$ where $K$ is either $+1, 0$ or $-1$. 


(i) For $0 \leq r \leq R$, it holds
\[
\frac{\text{Vol}_G(B_p(R))}{\text{Vol}_G(B_p(r))} \leq \frac{V_K(R)}{V_K(r)}
\]
where $B_p(r)$ is the geodesic ball in $M$ centered at $p$ with radius $r$, and $V_K(r)$ is the volume of the geodesic ball of radius $r$ in the complex space form.

(ii) The first Dirichlet eigenvalue of the geodesic ball of radius $r$ centered at $p$ is bounded above by
\[
\lambda_1(B_p(r)) \leq \lambda_1(B(r, K))
\]
where $\lambda_1(B(r, K))$ is the first Dirichlet eigenvalue of the geodesic ball of radius $r$ on the complex space form.

The paper is organized as follows. In §2 and §3, some fundamental notions of complex Finsler metrics are introduced. In §4, we verify the partial parallelism of the complex structure. In §5 and §6, a Synge type theorem and a Myers type theorem are proved. In §7, the Laplacian comparison is investigated.

2 Strongly convex complex Finsler metrics

Let $(M, J)$ be an $n$-dimensional complex manifold with the complex structure $J$. Suppose $\{z^\alpha\}_{\alpha=1}^n$ be a set of local complex coordinates with $z^\alpha = x^\alpha + \sqrt{-1}x^{\alpha+n}$, then $\{x^\alpha, x^{\alpha+n}\}_{\alpha=1}^n$ forms a local real coordinate system. In this coordinate system, the complex structure has the form
\[
J = J^k_i dx^k \otimes \frac{\partial}{\partial x^i}
\]
(2.1)
where
\[
J^k_i = \begin{cases} 
\delta^k_{i+n}, & 1 \leq k \leq n, \\
-\delta^k_{i-n}, & n+1 \leq k \leq 2n.
\end{cases}
\]
(2.2)

Unless stated otherwise, we always assume that lowercase Greek letters run from 1 to $n$ and lowercase Latin letters run from 1 to $2n$, and the Einstein summation convention is assumed throughout this paper.

Let $T_R M$ and $T^{1,0}M$ be the real tangent bundle and holomorphic tangent bundle of $M$ respectively. A vector in $T_R M$ is denoted by $y = y^i \partial/\partial x^i$, while a vector in $T^{1,0}M$ is denoted by $v = v^\alpha \partial/\partial z^\alpha$. As well known, the bundles $T^{1,0}M$ and $T_R M$ are isomorphic according to the bundle map
\[
o : T^{1,0}M \to T_R M, \quad v^o = v + \bar{v}
\]
and the inverse map
\[
o : T_R M \to T^{1,0}M, \quad y_o = \frac{1}{2} (y - \sqrt{-1}Jy).
\]
Customarily, we denote $v^o = y$, $y_o = v$, and have the relation

$$v^o = y^o + \sqrt{-1}y^{\alpha+n} \tag{2.3}$$

By the isomorphism, the split tangent bundles $T^{1,0}M \setminus \{0\}$ and $T_{\mathbb{R}}M \setminus \{0\}$ can be considered as the same manifold, denoted by $\tilde{M}$. On the split bundle $\tilde{M}$, $\{z^\alpha, v^\beta\}$ is the induced complex coordinates while $\{x^i, y^k\}$ is the local real coordinates. Thus, the complex tangent frame of $\tilde{M}$ can be given as

$$\tilde{\partial}_\alpha := \frac{\partial}{\partial z^\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} - \sqrt{-1} \frac{\partial}{\partial y^{\alpha+n}} \right), \quad \tilde{\partial}_\beta := \frac{\partial}{\partial v^\beta} = \frac{1}{2} \left( \frac{\partial}{\partial y^\beta} + \sqrt{-1} \frac{\partial}{\partial y^{\beta+n}} \right),$$

$$\tilde{\partial}_\alpha := \frac{\partial}{\partial y^\alpha} = \frac{1}{2} \left( \frac{\partial}{\partial y^\alpha} - \sqrt{-1} \frac{\partial}{\partial y^{\alpha+n}} \right), \quad \tilde{\partial}_\beta := \frac{\partial}{\partial v^\beta} = \frac{1}{2} \left( \frac{\partial}{\partial y^\beta} + \sqrt{-1} \frac{\partial}{\partial y^{\beta+n}} \right).$$

**Definition 2.1 (II).** A complex strongly pseudoconvex Finsler metric $F$ on a complex manifold $M$ is a continuous function $F : T^{1,0}M \rightarrow \mathbb{R}_{\geq 0}$ satisfying:

(i) $G = F^2$ is smooth on $\tilde{M}$;

(ii) $F(v) > 0$ for all $v \in \tilde{M}$;

(iii) $F(\zeta v) = |\zeta|F(v)$ for all $v \in T^{1,0}M$ and $\zeta \in \mathbb{C}$;

(iv) the Levi matrix $(G_{\alpha\bar{\beta}})$ is positive-definite on $\tilde{M}$, where

$$(G_{\alpha\bar{\beta}}) = \left( \frac{\partial^2 G}{\partial v^\alpha \partial v^\beta} \right). \tag{2.4}$$

**Definition 2.2 (III).** A real Finsler metric on a manifold $M$ is a function $F^o : T_{\mathbb{R}}M \rightarrow \mathbb{R}^+$ that satisfies the following properties:

(i) $G^o = F^{o2}$ is smooth on $\tilde{M}$;

(ii) $F^o(y) > 0$ for all $y \in \tilde{M}$;

(iii) $F^o(\lambda y) = \lambda F^o(y)$ for all $y \in T_{\mathbb{R}}M$ and $\lambda \in \mathbb{R}^+$;

(iv) the matrix $(g_{ij})$ is positive-definite on $\tilde{M}$, where

$$(g_{ij}) = \frac{1}{2} \left( \frac{\partial^2 G^o}{\partial y^i \partial y^j} \right). \tag{2.5}$$

Abate and Patrizio introduce the notion of strongly convex complex Finsler metrics.

**Definition 2.3 (II).** Let $F : T^{1,0}M \rightarrow \mathbb{R}^+$ be a strongly pseudoconvex complex Finsler metric. We say $F$ is strongly convex if the associated function $F^o(y) := F(y_o)$ is a real Finsler metric.

Throughout this paper, we always assume $G = F^2$ is a strongly convex complex Finsler metric. For convenience, we use the same symbol $F$ (resp. $G$) to denote the associated real Finsler metric $F^o$ (resp. $G^o$). In other words, we shall consider $F$ or $G$ as a function of $\{z^\alpha, v^\beta\}$ and also a function of $\{x^i, y^k\}$.

To start working, we need a few notations. In complex case, we shall denote by indices like $\alpha, \bar{\beta}$ and so on the derivatives with respect to the $v$-coordinates. The derivatives with respect to the $z$-coordinates will be denoted by indices...
after a semicolon. In the real case, the notation is similar by adopting the Latin letters. For instance, some derivatives of $G$ are denoted as follows

$$ G_{\alpha\beta} = \frac{\partial^2 G}{\partial v^\alpha \partial v^\beta}, \quad G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j}, \quad G_{i,\dot{k}} = \frac{\partial^2 G}{\partial x^i \partial \dot{x}^k}, \quad G_{i,k} = \frac{\partial^2 G}{\partial y^i \partial x^k}. $$

From now on, let us denote $u = Jy$. Locally, we have

$$ u = u^i \frac{\partial}{\partial x^i} = J^k_i y^k \frac{\partial}{\partial x^i}. \quad (2.6) $$

We may state the complex Euler Theorem in the real coordinates.

**Lemma 2.1.** Let $H(v)$ be a complex-valued function on $\mathbb{T}^{1,0}M(\cong T_yM)$. Set

$$ H(v) = H(y_o) = R(y) + \sqrt{-1}I(y), \quad (2.7) $$

where $R(y)$ and $I(y)$ are real-valued functions. Then the following four conditions are equivalent.

(i) $H(v)$ is of $(p, q)$-homogeneous;
(ii) $H(\zeta v) = \zeta^p \zeta^q H(v), \forall \zeta \in \mathbb{C};$
(iii) $H_{\alpha} v^\alpha = pH, H_{\dot{\alpha}} v^\alpha = qH;$
(iv) $R_k y^k = (p + q)R, I_k y^k = (p + q)I, R_k u^k = (q - p)I, I_k u^k = (p - q)R.$

Here we adopt the abbreviations $H_{\alpha} = \partial H/\partial v^\alpha$ and $I_k = \partial I/\partial y^k$ and etc.

**Proof.** The equivalence of (i)-(iii) are well known. We shall transform (iii) into (iv). It is easy to find

$$ v^\alpha \frac{\partial}{\partial v^\alpha} = \frac{1}{2} \left( y^i \frac{\partial}{\partial y^i} - \sqrt{-1} u^k \frac{\partial}{\partial y^k} \right). \quad (2.8) $$

Thus

$$ v^\alpha H_{\alpha} = \frac{1}{2} \left( y^i \frac{\partial}{\partial y^i} - \sqrt{-1} u^k \frac{\partial}{\partial y^k} \right) \left( R + \sqrt{-1}I \right) $$
$$ = \frac{1}{2} \left( y^i R^i + u^k I_k \right) + \frac{\sqrt{-1}}{2} \left( y^i I_i - u^j R_k \right). \quad (2.9) $$

Hence, $H_{\alpha} v^\alpha = pH = pR + \sqrt{-1}pI$ is equivalent to

$$ y^i R^i + u^k I_k = 2pR, \quad y^i I_i - u^j R_k = 2pI. \quad (2.10) $$

Similarly, $H_{\dot{\alpha}} v^\alpha = qH = qR + \sqrt{-1}qI$ is equivalent to

$$ y^i R^i - u^k I_k = 2qR, \quad u^k R_k + y^i I_i = 2qI. \quad (2.11) $$

It is clear that (2.10) together with (2.11) is equivalent to (iv).
With the help of the above lemma, we can study the $J$-invariance of the real fundamental tensor

$$ g = g_{ij}(y)dx^i \otimes dx^j. \quad (2.12) $$

This fundamental tensor can be considered as an inner product on the pullback bundle $\pi^*T_R M$ where $\pi : \tilde{M} \to M$. In other words, the inner product at $y$ can be defined as

$$ g_y(X, Y) = g_{ij}(y)X^i Y^j \quad (2.13) $$

where $X = X^i \partial / \partial x^i$ and $Y = Y^i \partial / \partial x^i$. Note that $J$ can naturally act on $\pi^*T_R M$. Thus, we can consider the $J$-invariance of $g_y$.

**Lemma 2.2.** Let $G$ be a strongly convex complex Finsler metric. Then

$$ g_y(Jy, JX) = g_y(y, X), \quad (2.14) $$

for any $X \in \pi^*T_{\pi(y)} M$. Particularly, it holds

$$ g_y(y, y) = g_y(Jy, Jy), \quad g_y(y, Jy) = 0. \quad (2.15) $$

**Proof.** Since $G(v)$ is a real-valued function $G(v) = G(v) + \sqrt{-1} \cdot 0$ and is of $(1,1)$-homogeneous, by Lemma 2.1 (iv), we have

$$ G_{ik}u^k = 0 \quad (2.16) $$

where $u^k = J^k_s y^s$. Differentiate the above formula with $y^i$, we have

$$ G_{ik}u^k + G_{k-i}J^k = 0. \quad (2.17) $$

Noting $G_{ik} = 2g_{ik}$ and $G_{k} = 2g_{jk}y^j$, one has

$$ 0 = g_{ik}u^k + g_{jk}y^j J^k_i = g_{ik}J^k_j y^j + g_{jk}y^j J^k_i \quad (2.18) $$

which is equivalent to $g_y(Jy, JX) = g_y(y, X)$. Taking $X = y$ and $X = u$, one shall get $g_y(y, y) = g_y(u, u)$ and $g_y(y, u) = 0$. ☐

The above Lemma present the partial $J$-invariance of $g$. The following rigid result tells us $g$ is $J$-invariant if and only if the metric is Hermitian.

**Proposition 2.1.** Let $G$ be a strongly convex complex Finsler metric. Then $G$ is a Hermitian metric if

$$ g_y(JX, JY) = g_y(X, Y) $$

for any $X, Y \in \pi^*T_{\pi(y)} M$.

**Proof.** Assume $g_y(JY, JX) = g_y(Y, X)$ for any $X, Y$. In local coordinates, we have

$$ g_{ij}J^i_p J^j_q = g_{pq}. \quad (2.19) $$

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Differentiating it with $y^i$, we have
\begin{equation}
C_{ijs}J^i_p J^j_q = C_{pqrs},
\end{equation}
where $C_{ijs}$ are the component of the Cartan torsion. Since $C$ is symmetric, it turns out
\begin{equation}
C_{ijs}J^i_p J^j_q = C_{ps} = C_{pis}J^i_p J^j_q.
\end{equation}
Hence
\begin{equation}
C_{ips}J^i_p J^j_q J^p_s J^q_b = C_{ijs}J^i_p J^j_q J^p_s J^q_b
\end{equation}
that is
\begin{equation}
-C_{ipb} J^i_s J^j_p = C_{abs}.
\end{equation}
Comparing (2.20) and (2.23), one can find $C_{abs} = 0$ which implies $G$ is a Riemannian metric.

\section{Connections and curvatures}

Let $(M, G)$ be a strongly convex complex Finsler manifold. We shall recall the connections and curvatures of $G$ in both real and complex realm.

Denote $\hat{G}^i$ the \textit{real spray coefficients} which is
\begin{equation}
\hat{G}^i = \frac{1}{4} g^{il}(G_{l;k} y^k - G_{i;l})
\end{equation}
where $(g^{ik}) = (g_{ij})^{-1}$. The \textit{nonlinear connection coefficients} are defined as
\begin{equation}
\hat{\omega}^i = \frac{\partial \hat{G}^k}{\partial y^i}.
\end{equation}
Then the real horizontal frame and vertical coframe can be given by
\begin{equation}
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \hat{G}^i_k \frac{\partial}{\partial y^k}, \quad \frac{\delta}{\delta y^j} = dy^j + \hat{G}^j_k dx^k.
\end{equation}
The Berwald connection 1-forms $\hat{\omega}^i_j$ are
\begin{equation}
\hat{\omega}^i_j = \hat{G}^i_j dx^k
\end{equation}
where
\begin{equation}
\hat{G}^i_{jk} = \frac{\partial^2 \hat{G}^i}{\partial y^j \partial y^k} = \frac{\partial^2 \hat{G}^i}{\partial y^i \partial y^j}.
\end{equation}
The following give the curvature forms of the real Berwald connection
\begin{equation}
\Omega^i_j = \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + B^i_{jkl} dy^l \wedge dx^k
\end{equation}
where
\begin{equation}
R^i_{jkl} = \frac{\delta \hat{G}^i}{\delta x^j} - \frac{\delta \hat{G}^i}{\delta x^j} + \hat{G}^i_{ks} \hat{G}^s_{jl} - \hat{G}^i_{js} \hat{G}^s_{kl}.
\end{equation}
is the Riemann curvature and
\[ B^i_{jk} = \hat{\nabla}^i_{jk} = \frac{\partial^3 \hat{\varphi}^i}{\partial y^j \partial y^k \partial y^l} \] (3.8)
is called the Berwald curvature.

Setting \( R_{ik} = g_{st} R^s_{jkl} y^j y^l \), the flag curvature is defined by
\[ K(y, V) = \frac{R_{ik} V^i V^k}{(g_{ij} - g_{ik} y^j V^j V^k)} \] (3.9)
where \( V = V^i \partial/\partial x^i \), and the Ricci curvature is given by
\[ \text{Ric}(y) = \frac{g^{ik} R_{ik} G(y)}{G(y)} = \sum_{j=1}^{2n-1} K(y, e_i) \] (3.10)
where \( \{e_i\} \) is a \( g_y \)-orthonormal frame with \( e_{2n} = y/F \).

Now let us consider \((M, G)\) as a complex metric. The Chern-Finsler nonlinear connection coefficient \( \Gamma^\alpha_{\beta \gamma} \) is given by
\[ \Gamma^\alpha_{\beta \gamma} := G^\alpha_{\beta \gamma} G_{\bar{\tau} \bar{\sigma}}, \] (3.11)
where \( (G^\alpha) = (G_{\bar{\beta} \bar{\gamma}})^{-1} \). The complex spray coefficients \( G^\alpha \)'s are
\[ G^\alpha = \frac{1}{2} \Gamma^\alpha_{\beta \gamma} v^\beta. \] (3.12)
The complex horizontal frame and vertical coframe are defined as
\[ \delta_{\mu} := \partial_{\mu} - \Gamma^\alpha_{\mu \alpha} \hat{\varphi}^\alpha, \quad \delta v^\alpha = d v^\alpha + \Gamma^\alpha_{\mu} d z^\mu. \] (3.13)
The Chern-Finsler connection is defined by the following 1-forms
\[ \omega^\alpha_{\beta} = \Gamma^\alpha_{\beta \mu} d z^\mu + C^\alpha_{\beta \gamma} \delta v^\gamma, \] (3.14)
where
\[ \Gamma^\alpha_{\beta \mu} := G^\alpha_{\beta \mu} \delta \mu, \quad C^\alpha_{\beta \gamma} := G^\alpha_{\bar{\beta} \bar{\gamma}} G_{\bar{\tau} \bar{\sigma}}. \] (3.15)
The curvature form of of Chern-Finsler connection is
\[ \Omega^\alpha_{\beta} = \bar{\partial} \omega^\alpha_{\beta} \] (3.16)
which has four parts
\[ \Omega^\alpha_{\beta} = R^\alpha_{\beta \mu \nu} dz^\mu \wedge dz^\nu + S^\alpha_{\beta \mu \nu} \delta v^\mu \wedge dz^\nu + P^\alpha_{\beta \mu \nu} dz^\mu \wedge \delta v^\nu + Q^\alpha_{\beta \mu \nu} \delta v^\mu \wedge \delta v^\nu. \] (3.17)
The holomorphic curvature \( H \) is defined as
\[ H(v) = \frac{2}{G^2} R^\alpha_{\beta \bar{\gamma} \bar{\mu}} v^\alpha \bar{v}^\beta \bar{v}^\mu v^\nu \] (3.18)
where \( R^\alpha_{\beta \bar{\gamma} \bar{\mu}} = G_{\bar{\beta} \bar{\gamma}} R^\alpha_{\beta \mu \bar{\nu}}. \)

As the end of this section, we state the Kähler conditions defined by Abate-Patrizio.
Definition 3.1 ([1], [4]). Let $G$ be a strongly pseudoconvex complex Finsler metric.

(i) $G$ is called a Kähler Finsler metric if and only if $\Gamma^\alpha_{\beta,\mu} = \Gamma^\alpha_{\mu,\beta}$;

(ii) $G$ is called a weakly Kähler Finsler metric if $G \left( \Gamma^\alpha_{\beta,\mu} - \Gamma^\alpha_{\mu,\beta} \right)^v = 0$.

For a weakly Kähler Finsler metric, Li-Qiu-Zhong discovered that the holomorphic curvature coincides with the flag curvature of the holomorphic flag.

Theorem 3.1 ([7]). Let $G$ be is a strongly convex weakly Kähler Finsler metric, then $K(y, Jy) = H(v)$ where $v = y_o$.

We shall point out that the holomorphic curvature defined in this paper is twice of that in [7]. Thus the above theorem is slightly different in [7].

4 Parallelism of the complex structure

For a Kähler Hermitian metric, the complex structure is parallel with respect to the Riemann connection. Let us consider the weakly Kähler Finsler case in this section.

According to Proposition 2.6.2 in [1] or Lemma 3.1 in [16], a strongly convex complex Finsler metric is a weakly Kähler if and only if the complex spray and real spray satisfy

$$G^\beta = \hat{G}^\beta + \sqrt{-1} \hat{G}^{\beta+n}$$

in the complex coordinate system. We shall calculate the covariant derivative of $J$ in this coordinate system.

Since $G^\alpha$ are of $(2,0)$-homogeneous, by (2.2) in Lemma 2.1 we have

$$\hat{G}^\alpha_{k} u^k = -2\hat{G}^{\alpha+n}$$ and $$\hat{G}^{\alpha+n}_{k} u^k = 2\hat{G}^\alpha.$$ (4.2)

Recall that $J = J^i_k dx^k \otimes \frac{\partial}{\partial x^i}$ has the form

$$\begin{cases}
J^i_k &= \hat{\delta}^i_{\alpha+n} = \delta^i_{\alpha+n} = -\delta^i_{\alpha} \quad \text{(4.3)}
\end{cases}
$$

in this coordinate system. Thus, (4.2) can be read as

$$\hat{G}^i_{k} u^k = 2J^i_k \hat{G}^k.$$ (4.4)

Taking the derivatives with respect to $y^s$, one has

$$\hat{G}^i_{sk} u^k + \hat{G}^i_{k} J^k_s = 2J^i_k \hat{G}^k.$$ (4.5)

The covariant derivative of $J$ with respect to the real Berwald connection is defined by

$$J^i_{k|l} := \frac{\delta J^i_k}{\delta x^l} + J^s_k \hat{G}^i_{sl} - J^i_s \hat{G}^s_{kl}.$$ (4.6)
Since $J^j_k$ are constant in the complex coordinate system, locally it holds
\[ J^j_k l = J^j_k \hat{G}^i_{sl} - J^j_s \hat{G}^i_{kl}. \]  
(4.7)

Contracting with $y^k$ and $y^l$, we reach
\[ J^j_k l y^k y^l = J^j_k \hat{G}^i_{sl} y^l y^k - J^j_s \hat{G}^i_{kl} y^l y^k \]
\[ = J^j_k \hat{G}^i_{sy} y^k - J^j_s \hat{G}^i_{ks} y^k = \hat{G}^i_{su} y^s - 2J^j_s \hat{G}^s. \]  
(4.8)

By (4.4) and $y^k l = 0$, it turns out
Lemma 4.1. On a strongly convex weakly Kähler Finsler manifold, it holds
\[ (\nabla y^\mu J) y = \nabla y^\mu (J y) = 0. \]  
(4.9)

where $y^\mu = y^\delta / \delta x^\delta$ and $\nabla$ is the covariant differential with respect to the Berwald connection.

Since the real Berwald connection and real Chern connection only differ by a term of Landsberg curvature, the above lemma is also true for the real Chern connection.

5 Synge-Tsukamoto theorem

In this section, we shall prove the Synge-Tsukamoto theorem for weakly Kähler Finsler manifolds.

Let us recall the theory of geodesics in real Finsler geometry (cf. [3]). Consider any unit speed geodesic $\gamma(t)(0 \leq t \leq d)$ with velocity field $T = \dot{\gamma}$. Set the linear covariant derivative with reference vector $T$ as
\[ D^T_{\frac{\partial}{\partial x^i}} = \hat{G}^i_{lj}(T) \frac{\partial}{\partial x^j}. \]  
(5.1)

Then $D^T_0 T = 0$. Suppose $U$ is a variation field along $\gamma$, then the second variation of arc length is
\[ L'' = I_\gamma(U_{\perp}, U_{\perp}) + g_T(D^T_0 U, T)|^d_0. \]  
(5.2)
where $U_{\perp} = U - g_T(U, T)T$ is the $g_T$-orthogonal component with respect to $T$, and $I_\gamma$ is the index form of $\gamma$
\[ I_\gamma(W, W) = \int_0^d \left[ g_T(D^T_0 W, D^T_0 W) - g_T(R(W, T)T, W) \right] dt \]  
(5.3)
where $W$ is any vector field along $\gamma$. Particularly, if $g_T(T, W) = 0$, we have
\[ I_\gamma(W, W) = \int_0^d \left[ g_T(D^T_0 W, D^T_0 W) - g_T(W, W) \cdot K(T, W) \right] dt \]  
(5.4)

Denote $V = JT$. According to Lemma 2.2 and Lemma 4.1 we have
\[ g_T(T, V) = 0, \quad g_T(V, V) = 1, \quad D^T_0 V = 0. \]  
(5.5)
Theorem 5.1. Let \((M, G)\) be a strongly convex weakly Kähler Finsler manifold. Suppose it is complete and the holomorphic curvature \(H \geq \lambda > 0\) is bounded below uniformly by a positive constant. Then \(M\) is compact and simply connected.

Proof. Let us first verify the compactness of \(M\). Consider any unit speed geodesic \(\gamma(t), 0 \leq t \leq d\) with length \(d = \pi/\sqrt{\lambda}\). Set

\[s_\lambda(t) = \sin(\sqrt{\lambda} t)\]  

and

\[W(\gamma(t)) = s_\lambda(t)V(\gamma(t)).\]  

Then

\[D_{\dot{T}}^2 W = s'_\lambda(t)V = \sqrt{\lambda} \cos(\sqrt{\lambda} t)V.\]  

According to Theorem 3.1, we have

\[K(T, W) = K(T, V) = H(V_0).\]  

Thus, it holds

\[I_\gamma(W, W) = \int_0^d [g_T(D_{\dot{T}}^2 W, D_{\dot{T}}^2 W) - g_T(W, W)K(T, W)] dt\]

\[= \int_0^d [g_T(D_{\dot{T}}^2 W, D_{\dot{T}}^2 W) - g_T(W, W)H(T_0)] dt\]

\[\leq \int_0^d [(s'_\lambda(t))^2 - \lambda(s_\lambda(t))^2] dt = \frac{1}{2} \sqrt{\lambda} \sin(2\sqrt{\lambda} d) = 0.\]  

A standard argument shows that the geodesic \(\gamma\) must contain conjugate points, which implies the diameter of \(M\) is bounded from above

\[\text{diam}(M) \leq d = \frac{\pi}{\sqrt{\lambda}}.\]  

Therefore, \(M\) is compact.

Next, we show that \(M\) is simply connected. Suppose the fundamental group \(\pi(M)\) is nontrivial. Picking a nontrivial free homotopy class, according to Theorem 8.7.1 in [3], there exists a shortest smooth closed geodesic \(\gamma(t)(0 \leq t \leq l)\) in this class. The definition of vector fields \(T\) and \(V\) are the same as above. Produce a variation of \(\gamma(t)\) such that the resulting variation vector field is \(V(\gamma(t))\). Hence, the second variation of arc length reduces to

\[L'' = -\int_0^l H(T_0) dt \leq \lambda l < 0\]  

which implies \(\gamma\) is not minimal. This leads a contradiction. \(\Box\)

Following the argument in [3], we have the following result.
Theorem 5.2. Let \((M, G)\) be a compact strongly convex weakly Kähler Finsler manifold with positive holomorphic curvature. Then any holomorphic isometry of \(M\) must have at least one fixed point.

**Proof.** Let \(d(\cdot, \cdot)\) be the distance function of \(G\). Assume that there exists such a holomorphic isometry \(f : M \to M\) with no fixed point. Then the continuous function \(d(x, f(x))\) can attain its positive minimum, saying

\[
d(p, f(p)) = \min d(x, f(x)) = l > 0.
\]

Let \(\gamma\) be a minimal geodesic joining \(p\) to \(f(p)\) with \(\gamma(0) = p\) and \(\gamma(l) = f(p)\).

Since \(f\) is an isometry and \(\gamma\) is a minimal geodesic, we have

\[
l \leq d(\gamma(t), f(\gamma(t))) \leq d(\gamma(t), f(p)) + d(f(p), f(\gamma(t))).
\]

Thus \(\gamma \cup f(\gamma)\) is a geodesic which is smooth at \(f(p)\) and hence \(df(\dot{\gamma}(0)) = \dot{\gamma}(l)\).

Again, set \(T = \dot{\gamma}\) and \(V = JT\). Since \(f\) is holomorphic, we have \(df(V) = J(df(T))\) and hence

\[
V(l) = J(T(l)) = J(df(T(0))) = df(J(T(0))) = df(V(0)). \tag{5.13}
\]

Consider the variation \(\tilde{\gamma}(t, s) = \exp_{\gamma(t)}(sV(t))\). Denoting \(\tilde{\gamma}_s\) the variational field, we have \(\tilde{\gamma}_s(t, 0) = V(t)\). Since \(f\) is an isometry, \(f(\tilde{\gamma}(0, s))\) is also a geodesic with the initial velocity

\[
df(\tilde{\gamma}_s(0, 0)) = df(V(0)) = V(l) = \tilde{\gamma}_s(l, 0). \tag{5.14}
\]

It implies the geodesic \(\tilde{\gamma}(l, s)\) is just \(f(\tilde{\gamma}(0, s))\) and hence \(\tilde{\gamma}_s(l, s) = df(\tilde{\gamma}_s(0, s))\).

Since \(f\) is an isometry, together with \(df(T(0)) = T(l)\) and \(df(\tilde{\gamma}_s(0, s)) = \tilde{\gamma}_s(l, s)\), it must hold

\[
gr(D^T_{\tilde{\gamma}_s} \tilde{\gamma}_s, T)^l_{|t=0} = 0. \tag{5.15}
\]

Thus, the second variation of arc length \(L''\) becomes

\[
L'' = L_2(V, V) = -\int_0^l K(T, V)dt = -\int_0^l H(V_0)dt < 0. \tag{5.16}
\]

This contradicts to

\[
d(\tilde{\gamma}(0, s), \tilde{\gamma}(l, s)) = d(\tilde{\gamma}(0, s), f(\tilde{\gamma}(0, s))) \geq l. \tag{5.17}
\]

and \(\gamma(t) = \tilde{\gamma}(t, 0)\) is minimizing. \(\square\)

6 Bonnet-Myers theorem

Following Ni-Zheng, we introduce the *orthogonal Ricci curvature* for strongly convex weakly Kähler Finsler metrics as

\[
\text{Ric}^+(y) = \text{Ric}(y) - K(y, Jy) = \text{Ric}(y) - H(y_0). \tag{6.1}
\]
Theorem 6.1. Let $(M,F)$ be a complete strongly convex weakly Kähler Finsler metric of complex dimension $n$. Suppose $\text{Ric}^\perp \geq (2n - 2)\lambda > 0$, then the diameter of $M$ is at most $\pi/\sqrt{\lambda}$.

Proof. The proof is similar to the first part of Theorem 5.1. Let $\gamma$ be a unit-speed geodesic such that $\gamma(0) = p$ and $\gamma(d) = q$ where $d = \pi/\sqrt{\lambda}$ is the length of the geodesic. Set $T = \dot{\gamma}$ and $V = JT$. Pick a $g_T$-orthonormal frame $\{E_i\}$ at $q$ such that

$$E_{2n} = T|_q, \quad E_{2n-1} = V|_q.$$ 

Let $\{E_i(t)\}$ be the parallel transportation of $E_i$ along $\gamma$, i.e. $D_T^T E_i = 0$. By (5.5), we have $E_{2n}(t) = T(t)$ and $E_{2n-1}(t) = V(t)$. Moreover $E_i(t)(i \leq 2n - 2)$ are $g_T$-orthogonal to $T$ and $V$. Set $W_i = s_\lambda(t)E_i(t) = \sin(\sqrt{\lambda}t)E_i(t)$.

We have

$$\sum_{i=1}^{2n-2} I_\gamma(W_i, W_i) = \sum_{i=1}^{2n-2} \int_0^d [g_T(D_T^T W_i, D_T^T W_i) - g_T(W_i, W_i)K(T, W_i)] dt$$

$$= \sum_{i=1}^{2n-2} \int_0^d [(s_\lambda(t))^2 - (s_\lambda(t))^2K(T, E_i)] dt$$

$$= \int_0^d [(2n - 2)(s_\lambda(t))^2 - \text{Ric}^\perp(T)(s_\lambda(t))^2] dt$$

$$\leq (2n - 2) \int_0^d [(s_\lambda(t))^2 - \lambda(s_\lambda(t))^2] dt = 0.$$ 

Then, there exists $i_0$ such that $I_\gamma(W_{i_0}, W_{i_0}) \leq 0$. A standard argument shows that $\gamma$ contains conjugate points. Therefore, $\text{diam}(M) \leq d = \pi/\sqrt{\lambda}$.

7 Laplacian comparison

Let $(M,G)$ be a complete strongly convex weakly Kähler Finsler manifold of complex dimension $n$. We shall consider the Laplacian of the distance function in this section.

For any smooth function $f$ on $M$, its gradient is defined by

$$\nabla f = \mathcal{L}^{-1}(df)$$

(7.1)

where $\mathcal{L}^{-1}$ is the inverse of Legendre transformation

$$\mathcal{L}(X) = \begin{cases} g_X(X, \cdot), & X \neq 0, \\ 0, & X = 0. \end{cases}$$

(7.2)

Let $\mathcal{U}_f = \{x \in M | df(x) \neq 0\}$. As $D^T$ introduced in §5, we define a linear connection $D^\nabla f$ on $\mathcal{U}_f$ by setting

$$D^\nabla f\frac{\partial}{\partial x^i} = \hat{\gamma}^{ik}_{ij}(\nabla f)\frac{\partial}{\partial x^k}.$$ 

(7.3)
The Hessian of $f$ can be defined as (cf. [15])

$$H(f)(X,Y) = XY(f) - D^\nabla f_X Y(f) = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \hat{G}^k_{ij} (\nabla f) \frac{\partial f}{\partial x^k} \right) X^i Y^j. \tag{7.4}$$

It should be noted that the connection $D^\nabla f$ defined here is different from that in [15]. In that case the Chern connection is used. Fortunately, the Hessian defined here is the same as that in [15]. In fact, the difference between the real Berwald connection and real Chern connection is the Landsberg curvature $L^i_{jk}$, and hence $L^k_{ij} (\nabla f) \frac{\partial f}{\partial x^k} = 0$. Moreover, the following identity is also true:

$$H(f)(X,Y) = g(\nabla f(D^\nabla f_X \nabla f, Y), \forall X, Y \in T_k M | U_f. \tag{7.5}$$

The $g\nabla f$-trace of $H(f)$ can be considered as some second order differential operator acting on $f$. Let us denote the trace by

$$\square f := \text{tr}_{g\nabla f} H(f) = \sum_{i=1}^{2n} H(f)(E_i, E_i) \tag{7.6}$$

where $\{E_i\}$ is a local $g\nabla f$-orthonormal frame on $U_f$.

Denote $r(x) = d(p, x)$ be the distance function from a fixed point $p \in M$. It is well known that $r$ is smooth on $M \setminus \{p\}$ away from the cut points of $p$, and $G(\nabla r) = 1$ (cf. [11] or [15]). Following Ni and Zheng ([3]), we define the $\square^\perp$-operator as

$$\square^\perp := \square r - H(r)(\nabla r, \nabla r) - H(r)(J(\nabla r), J(\nabla r)). \tag{7.7}$$

Denote $\nabla r = T$ and $J(\nabla r) = V$ for abbreviation. We shall estimate $\square r$.

Assume $\gamma$ is a unit-speed geodesic without a conjugate point up to distance $r$ from $p$ and $\gamma(r) = q$. Then $\gamma' = T|_\gamma$. Thus

$$H(r)(T, T) = g(T, \nabla T) = 0, \quad H(r)(T, V) = g(T, V) = 0. \tag{7.8}$$

Let $\{E_i\}$ be a $g\nabla r$-orthonormal frame at $q$ such that

$$E_{2n} = T|_q, \quad E_{2n-1} = J T|_q = V|_q.$$

Let $\{J_i\}$ be the Jacobi fields along $\gamma$ with

$$J_i(0) = 0, J_i(r) = E_i.$$

According to (4.1) in [15], we have

$$H(r)(E_i, E_i)|_q = I_\gamma(J_i, J_i) = I_\gamma(J_i^\perp, J_i^\perp)$$

where $J_i^\perp = J_i - g(T, J_i)$. 

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Let $E_i(t)$ be the parallel transportation of $E_i$ along $\gamma$. Set

$$s_\lambda(t) = \begin{cases} \sin(\sqrt{\lambda}t), & \lambda > 0, \\ t, & \lambda = 0, \\ \sinh(\sqrt{-\lambda}t), & \lambda < 0. \end{cases} \quad (7.9)$$

Produce

$$W_i = \frac{s_\lambda(t)}{s_\lambda(r)} E_i(t).$$

Since $J_i(t)$ and $W_i(t)$ have the same boundary value, by the index lemma, we have

$$\square r|_q = \sum_{i=1}^{2n-2} H(r)(E_i, E_i)|_q = \sum_{i=1}^{2n-2} I_\gamma(J_i, J_i) \leq \sum_{i=1}^{2n-2} I_\gamma(W_i, W_i) = 15.$$
Recalling (4.11), we have
\[ G^\alpha(T_o) = \hat{G}^\alpha(T) + \sqrt{-1} \hat{G}^{\alpha+n}(T), \]
thus
\[ 2G^\alpha(T_o) \frac{\partial r}{\partial z^\alpha} = (\hat{G}^\alpha(T) + \sqrt{-1} \hat{G}^{\alpha+n}(T)) \left( \frac{\partial r}{\partial x^\alpha} - \sqrt{-1} \frac{\partial r}{\partial x^{\alpha+n}} \right) \]
\[ = \hat{G}^k(T) \frac{\partial r}{\partial x^k} - \sqrt{-1} \hat{G}^i(T) J^k_i \frac{\partial r}{\partial x^k}. \quad (7.14) \]
One can immediately get
\[ T^\alpha T^\beta = (T^\alpha + \sqrt{-1} T^{\alpha+n})(T^\beta + \sqrt{-1} T^{\beta+n}) \]
\[ = T^\alpha T^\beta - T^{\alpha+n} T^{\beta+n} + \sqrt{-1} (T^{\alpha+n} T^\beta + T^\alpha T^{\beta+n}) \]
and
\[ 4 \frac{\partial^2 r}{\partial z^\alpha \partial z^\beta} = \frac{\partial^2 r}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 r}{\partial x^{\alpha+n} \partial x^{\beta+n}} - \frac{\partial^2 r}{\partial x^\alpha \partial x^{\beta+n}}. \]
Thus
\[ 4 T^\alpha T^\beta \frac{\partial^2 r}{\partial z^\alpha \partial z^\beta} = T^i T^j \frac{\partial^2 r}{\partial x^i \partial x^j} - V^i V^j \frac{\partial^2 r}{\partial x^i \partial x^j} - 2\sqrt{-1} V^i T^j \frac{\partial^2 r}{\partial x^i \partial x^j}. \quad (7.15) \]
Hence
\[ 4 r_{11} = 4 T^\alpha T^\beta \frac{\partial^2 r}{\partial z^\alpha \partial z^\beta} - 8 G^\alpha(T_o) \frac{\partial r}{\partial z^\alpha} \]
\[ = T^i T^j \frac{\partial^2 r}{\partial x^i \partial x^j} - V^i V^j \frac{\partial^2 r}{\partial x^i \partial x^j} - 4 \hat{G}^k(T) \frac{\partial r}{\partial x^k} \]
\[ - 2\sqrt{-1} \left( V^i T^j \frac{\partial^2 r}{\partial x^i \partial x^j} - 2 \hat{G}^i(T) J^k_i \frac{\partial r}{\partial x^k} \right) \]
\[ = H(r)(T,T) - H(r)(V,V) + (T^i T^j \hat{G}^k_{ij}(T) - V^i V^j \hat{G}^k_{ij}(T) - 4 \hat{G}^k(T) J^k_i \frac{\partial r}{\partial x^k} \]
\[ - 2\sqrt{-1} \left( H(r)(T,V) + (T^i V^j \hat{G}^k_{ij}(T) - 2 \hat{G}^i(T) J^k_i \frac{\partial r}{\partial x^k} \right). \quad (7.16) \]
By \( T^i \hat{G}^k_{ij}(T) = \hat{G}^k_j(T) \) and (4.11), we have
\[ T^i V^j \hat{G}^k_{ij}(T) = V^j \hat{G}^k_j(T) = 2 \hat{G}^j(T) J^k_j. \quad (7.17) \]
According to (4.11), we have
\[ V^i V^j \hat{G}^k_{ij}(T) = 2 J^k_i \hat{G}^j_j(T) V^j - \hat{G}^k_j(T) J^j_i V^j \]
\[ = 4 J^k_i \hat{G}^j_j(T) J^j_i + \hat{G}^k_j(T) T^i \]
\[ = -4 \hat{G}^k(T) + 2 \hat{G}^k(T) = -2 \hat{G}^k(T) = -T^i T^j \hat{G}^k_{ij}(T). \quad (7.18) \]
Substituting (7.17), (7.18) and \( H(r)(T,T) = H(r)(T,V) = 0 \) into (7.16), we reach
\[ 4 r_{11} = -H(r)(V,V). \]

Theorem 7.2. Let $(M, G)$ be a complete strongly convex Kähler Finsler manifold of complex dimension $n$. If $H \geq 4\lambda$, then we have the following inequality whenever $r$ is smooth

$$H(r)(J(\nabla r), J(\nabla r)) \leq 2ct\lambda(2r).$$

Proof. Define $f(t) = -r_{11}(\gamma(t))$ along the radial geodesic $\gamma$. According to the proof of Theorem 4.4 in [19], it holds

$$4f^2(t) + f'(t) \leq -\frac{1}{4}H(T_o) \tag{7.19}$$

and

$$\lim_{t \to 0^+} tf(t) = \frac{1}{4} \tag{7.20}$$

since $G$ is a Kähler Finsler metric. By the assumption $H \geq 4\lambda$, one can get

$$f(t) \leq \frac{1}{2}ct\lambda(2t). \tag{7.21}$$

Therefore, $H(f)(V, V)|_{\gamma(t)} = 4f \leq 2ct\lambda(2t)$. \hfill \Box

Corollary 7.1. Let $(M, G)$ be a complete strongly convex Kähler Finsler manifold of complex dimension $n$. If $\text{Ric}^\perp \geq (2n - 2)\lambda$ and $H \geq 4\lambda$, then we have the following inequality whenever $r$ is smooth

$$\Box r \leq (2n - 2)ct\lambda(r) + 2ct\lambda(2r). \tag{7.22}$$

Using the above Laplacian comparison, one can follow the arguments in Section 5 of [19] and obtain the following volume comparison and eigenvalue comparison.

Corollary 7.2. Let $(M, G)$ be a complete strongly convex Kähler Finsler $n$-manifold with arbitrary measure $\mu$. Assume $\text{Ric}^\perp \geq (2n - 2)K$ and $H \geq 4K$ where $K$ is either $+1$, $0$ or $-1$. If the Shen curvature vanishes, then for $0 \leq r \leq R$ it holds

$$\frac{\text{Vol}^n_{C}(B_p(R))}{\text{Vol}^n_{C}(B_p(r))} \leq \frac{V_K(R)}{V_K(r)}$$

where $B_p(r)$ is the geodesic ball centered at $p$ with radius $r$, and $V_K(r)$ is the volume of the geodesic ball of radius $r$ in the complex space form.

Corollary 7.3. Let $(M, G, \mu)$ be a strongly convex Kähler Finsler $n$-manifold with vanishing Shen curvature. Assume $\text{Ric}^\perp \geq (2n - 2)K$ and $H \geq 4K$ where $K$ is either $+1$, $0$ or $-1$. Then the first Dirichlet eigenvalue of the geodesic ball of radius $r$ centered at $p$ is bounded above by

$$\lambda_1(B_p(r)) \leq \lambda_1(B(r, K))$$

where $\lambda_1(B(r, K))$ is the first Dirichlet eigenvalue of the geodesic ball of radius $r$ on the complex space form.
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