Time averaging for nonautonomous/random linear parabolic equations

Janusz Mierczyński †
Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
PL-50-370 Wrocław
Poland

and

Wenxian Shen ‡
Department of Mathematics
Auburn University
Auburn University, AL 36849
USA

Abstract

Linear nonautonomous/random parabolic partial differential equations are considered under the Dirichlet, Neumann or Robin boundary conditions, where both the zero order coefficients in the equation and the coefficients in the boundary conditions are allowed to depend on time. The theory of the principal spectrum/principal Lyapunov exponents is shown to apply to those equations. In the nonautonomous case, the main result states that the principal eigenvalue of any time-averaged equation is not larger than the supremum of the principal spectrum and that there is a time-averaged equation whose principal eigenvalue is not larger than the infimum of the principal spectrum. In the random case, the main result states that the principal eigenvalue of the time-averaged equation is not larger than the principal Lyapunov exponent.

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1 Introduction

It is well known that parabolic equations can be used to model many evolution processes in science and engineering. Parabolic equations with general time dependence are gaining more and more attention since they can take various time variations of the underlying processes into account in modeling the processes. A great amount of research work has been carried out toward the existence, uniqueness, and regularity of solutions of general linear, semilinear, quasilinear parabolic equations (see [2], [3], [11], [12], [14], [26], [27], [39], etc.). As a basic tool for nonlinear problems, it is of great significance to study the spectral theory for linear parabolic equations.

Spectral theory, in particular, principal spectrum theory (i.e., principal eigenvalues and principal eigenfunctions theory) for time independent and time periodic parabolic equations is well understood (see, for example, [16]). For such an equation, its principal eigenvalue provides the growth rate of the evolution operator and hence a least upper bound of the growth rates of all the solutions. Recently much effort has been devoted to the extension of principal eigenvalue and principal eigenfunction theory of time independent and periodic parabolic equations to general time dependent and random parabolic equations. See, for example, [17], [18], [19], [20], [21], [22], [28], [29], [31], [35], [36], [37], etc.

In the current paper, we focus on time dependent parabolic equations of the form

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + c(t,x)u, \quad t > 0, \ x \in D, \\
B(t)u &= 0, \quad t > 0, \ x \in \partial D,
\end{aligned}
\]

(1.1)

where \( D \subset \mathbb{R}^N \),

\[
B(t)u = \begin{cases}
0 & \text{(Dirichlet)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} & \text{(Neumann)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + d(t,x)u & \text{(Robin)},
\end{cases}
\]

and random parabolic equations of the form

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + c(\theta_t \omega, x)u, \quad t > 0, \ x \in D, \\
B(\theta_t \omega)u &= 0, \quad t > 0, \ x \in \partial D,
\end{aligned}
\]

(1.2)

where

\[
B(\theta_t \omega)u = \begin{cases}
0 & \text{(Dirichlet)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} & \text{(Neumann)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + d(\theta_t \omega, x)u & \text{(Robin)},
\end{cases}
\]
The functions $D$ and $\alpha > 0$.

(A2) The functions $a_{ij}$, $a_i$ belong to $C^2(D)$ and the functions $b_i$ belong to $C^2(\partial D)$.

(A3) (a) $c \in C^{2+\alpha, 1+\alpha}(\mathbb{R} \times \bar{D})$ (in the case of (1.1)),

(b) $c^\omega \in C^{2+\alpha, 1+\alpha}(\mathbb{R} \times \bar{D})$ for all $\omega \in \Omega$, with the $C^{2+\alpha, 1+\alpha}(\mathbb{R} \times \bar{D})$-norm bounded uniformly in $\omega \in \Omega$ (in the case of (1.2)).

(A4) (a) $d \in C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \partial D)$ (in the case of (1.1)),

(b) $d^\omega \in C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \partial D)$ for all $\omega \in \Omega$, with the $C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \partial D)$-norm bounded uniformly in $\omega \in \Omega$ (in the case of (1.2)).

We also assume the following uniform ellipticity condition and the complementing boundary condition:
(A5) \( a_{ij}(x) = a_{ji}(x) \) for \( i, j = 1, 2, \ldots, N \) and \( x \in \bar{D} \), and there is \( \alpha_0 > 0 \) such that
\[
\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^{N} \xi_i^2, \quad x \in \bar{D}, \quad \xi \in \mathbb{R}^N.
\]

(A6) There is \( \alpha_1 > 0 \) such that
\[
\sum_{i=1}^{N} b_i(x) \nu_i(x) \geq \alpha_1, \quad x \in \partial D,
\]
where \( \nu(x) = (\nu_1(x), \nu_2(x), \cdots, \nu_N(x)) \) is the unit outer normal vector of \( \partial D \) at \( x \in \partial D \).

In the case of (1.1) let
\[
Y(c, d) := \text{cl}\{ (c, d) \cdot t : t \in \mathbb{R} \}, \quad (1.3)
\]
be equipped with the open-compact topology, where \( ((c, d) \cdot t)(s, x) := (c(s + t, x), d(s + t, x)) \), \( s \in \mathbb{R}, x \in \partial D \), and the closure is taken in the open-compact topology of \( \mathbb{R} \times \bar{D} \).

In the case of (1.2) let
\[
Y(\Omega) := \text{cl}\{ (c^\omega, d^\omega) : \omega \in \Omega \} \quad (1.4)
\]
be equipped with the open-compact topology, where the closure is also taken in the open-compact topology. We will write \( Y \) instead of \( Y(c, d) \) (for the case of (1.1)) or instead of \( Y(\Omega) \) (for the case of (1.2)).

For given \((\tilde{c}, \tilde{d}) \in Y \) and \( u_0 \in L_p(D) \), consider
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + \tilde{c}(t, x) u, \quad t > 0, x \in D, \\
\tilde{B}(t)u &= 0, \quad t > 0, x \in \partial D,
\end{aligned}
\]
where
\[
\tilde{B}(t)u = \begin{cases} 
  u & \text{(Dirichlet)} \\
  \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} & \text{(Neumann)} \\
  \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + \tilde{d}(t, x) u & \text{(Robin)},
\end{cases}
\]
with the initial condition
\[
u(0, x) = u_0(x), \quad x \in D. \quad (1.6)
\]

Applying the theory presented by H. Amann in [2], we have that (1.5)+(1.6) has a unique \( L_p(D) \)-solution \( U(\tilde{c}, \tilde{d}, \cdot, p)(\cdot, 0)u_0 : [0, \infty) \rightarrow L_p(D) \) \( (p > 1) \) (see Proposition 3.3).

Note that \( U(\tilde{c}, \tilde{d}, \cdot, p)(\cdot, 0)u_0 \) is also a classical solution of (1.5)+(1.6) (see Section 3 for more detail). We may therefore write \( U(\tilde{c}, \tilde{d}, t, 0)u_0 \) as \( U(\tilde{c}, \tilde{d})(t, 0)u_0 \) for \( u_0 \in L_p(D) \). In the present paper we further assume the following continuous dependence.
For any \( T > 0 \) the mapping
\[
[Y \ni (\tilde{c}, \tilde{d}) \mapsto [0, T] \ni t \mapsto U(\tilde{c}, \tilde{d})(t, 0)] \in B([0, T], \mathcal{L}(L_2(D), L_2(D)))
\]
is continuous, where \( \mathcal{L}(L_2(D), L_2(D)) \) represents the space of all bounded linear operators from \( L_2(D) \) into itself, endowed with the norm topology, and \( B(\cdot, \cdot) \) stands for the Banach space of bounded functions, endowed with the supremum norm.

It should be pointed out that in [4] and [34] conditions, for some special cases (for example, the Dirichlet boundary condition case and the case with infinitely differentiable coefficients), are given that guarantee the continuous dependence of \([0, T] \ni t \mapsto U(\tilde{c}, \tilde{d})(t, 0)] \in B([0, T], \mathcal{L}(L_2(D), L_2(D)))\) on the coefficients. For the general case, the continuous dependence of \([0, T] \ni t \mapsto U(\tilde{c}, \tilde{d})(t, 0)] \in B([0, T], \mathcal{L}(L_2(D), L_2(D)))\) on the coefficients is not covered in [4] and [34]. We will not investigate the conditions under which (A7) is satisfied in this paper.

Then (1.1) ((1.2)) generates the following skew-product semiflow (see Section 3 for detail)
\[
\Pi_t : X \times Y \to X \times Y,
\]
\[
\Pi_t(u_0, (\tilde{c}, \tilde{d})) = (U(\tilde{c}, \tilde{d})(t, 0)u_0, (\tilde{c}, \tilde{d}) \cdot t),
\]
where
\[
X := \begin{cases} 
\overset{\circ}{C}^1(\bar{D}) & \text{(Dirichlet)} \\
C^1(\bar{D}) & \text{(Neumann or Robin)}
\end{cases}
\]
\[
\overset{\circ}{C}^1(\bar{D}) := \{ u \in C^1(\bar{D}) : u(x) = 0 \text{ for each } x \in \partial D \}.
\]

Throughout the paper, we denote \( \|\cdot\| \) as the norm in \( L_2(D) \) (see Section 2 for other notations).

Among others, we prove
1) \( \Pi_t \) is strongly monotone (see Theorem 4.1).
2) (1.5) has a unique (up to multiplication by positive scalars) globally positive solution \( v(t, x; \tilde{c}, \tilde{d}) \) (which is an analog of a principal eigenfunction) (see Theorem 4.2 (we denote \( v((\tilde{c}, \tilde{d}))(\cdot) \) as \( v(0, \cdot; \tilde{c}, \tilde{d})/\|v(0, \cdot; \tilde{c}, \tilde{d})\|)\).
3) Consider (1.1). Then the set \( \Sigma(c, d) \) consisting of all limits
\[
\lim_{n \to \infty} \frac{\ln\|U_{(c,d)} S_n(T_n - S_n, 0)v((c,d) \cdot S_n)\|}{T_n - S_n}
\]
where \( T_n - S_n \to \infty \) as \( n \to \infty \), is a compact interval (see Theorem 5.1).
4) Consider (1.2). Then for a.e. \( \omega \in \Omega \)
\[
\lim_{T \to \infty} \frac{\ln\|U_\omega(T,0)v(\omega)\|}{T} = \text{const}
\]
where \( U_\omega(t,0) = U_{(c^\omega,d^\omega)}(t,0) \) and \( v(\omega) = v((c^\omega,d^\omega)) \) (see Theorem 5.3).
Denote the compact interval in 3) by \([\lambda_{\text{inf}}(c,d), \lambda_{\text{sup}}(c,d)]\) and the constant in 4) by \(\lambda(c,d)\). We call \([\lambda_{\text{inf}}(c,d), \lambda_{\text{sup}}(c,d)]\) the principal spectrum of (1.1) (see Definition 5.1) and call \(\lambda(c,d)\) the principal Lyapunov exponent of (1.2) (see Definition 5.2).

Observe that if \(c(t,x)\) and \(d(t,x)\) in (1.1) are independent of \(t\) or are periodic in \(t\), then \(\lambda_{\text{inf}}(c,d)(= \lambda_{\text{sup}}(c,d))\) is the principal eigenvalue of (1.1) and \(v(t,\cdot;c,d)\) is an eigenfunction associated with \(\lambda_{\text{inf}}(c,d)\) (called a principal eigenfunction). As in the time independent and periodic cases, the principal spectrum of (1.1) and principal Lyapunov exponent of (1.2) provide upper bounds of growth rates of the solutions of (1.1) and (1.2), respectively. This can indeed be easily seen from the fact that

\[
\lim_{n \to \infty} \frac{\ln \| U_{(c,d)} s_n(T_n - S_n,0)v((c,d) \cdot S_n) \|}{T_n - S_n} = \lim_{n \to \infty} \frac{\ln \| U_{(c,d)} s_n(T_n - S_n,0) \|}{T_n - S_n} = \lim_{n \to \infty} \frac{\ln \| U_{(c,d)} s_n(T_n - S_n,0)u_0 \|}{T_n - S_n} \\
\]

for any nontrivial \(u_0 \in X\) with \(u_0(x) \geq 0\) for \(x \in D\) as long as the limits exist (the existence of one of the limits implies the existence of the others), and

\[
\lim_{T \to \infty} \frac{\ln \| U_{\omega}(T,0)v(\omega) \|}{T} = \lim_{T \to \infty} \frac{\ln \| U_{\omega}(T,0) \|}{T} = \lim_{T \to \infty} \frac{\ln \| U_{\omega}(T,0)u_0 \|}{T} \\
\]

for any nontrivial \(u_0 \in X\) with \(u_0(x) \geq 0\) for \(x \in D\) as long as the limits exist (again the existence of one of the limits implies the existence of the others) (this fact follows from Theorem 1.2).

We remark that the existence and uniqueness of globally positive solutions to nonautonomous parabolic equations with time independent boundary conditions were studied in [28], [29], [35]. In [17] the author studied the uniqueness of globally positive solutions to nonautonomous parabolic equations with time dependent boundary conditions. When the boundary conditions are time independent, the results 3) and 4) are proved in [31]. The results 3), 4), and the existence part of 2) for time dependent boundary conditions are new. The strong monotonicity result 1) basically follows from [5, Theorem 11.6] and strongly maximum principal and the Hopf boundary point principle for classical solutions of parabolic equations.

We now consider the averaged equations of (1.1) and (1.2) in the following sense:

In the case of (1.1) we call \((\hat{c}(), \hat{d}())\) an averaged function of \((c,d)\) if

\[
\hat{c}(x) = \lim_{n \to \infty} \frac{1}{T_n - S_n} \int_{S_n}^{T_n} c(t,x) \, dt \quad \text{for } x \in D
\]

and

\[
\hat{d}(x) = \lim_{n \to \infty} \frac{1}{T_n - S_n} \int_{S_n}^{T_n} d(t,x) \, dt \quad \text{for } x \in \partial D
\]

for some \(T_n - S_n \to \infty\), where the limit is uniform in \(x \in \bar{D}\) (resp. in \(x \in \partial D\)).
In the case of (1.2) we call $(\hat{c}(\cdot), \hat{d}(\cdot))$ the averaged function of $(c,d)$ if

$$\hat{c}(x) = \int_{\Omega} c(\omega, x) \, d\mathbb{P}(\omega) \quad \text{for } x \in D$$

and

$$\hat{d}(x) = \int_{\Omega} d(\omega, x) \, d\mathbb{P}(\omega) \quad \text{for } x \in \partial D.$$ 

The equation

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + \hat{c}(x)u, \quad x \in D, \\
\hat{B}u &= 0, \quad x \in \partial D,
\end{aligned}$$

(1.10)

where

$$\hat{B}u = \begin{cases} 
\sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} & \text{(Neumann)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + \hat{d}(x)u & \text{(Robin)},
\end{cases}$$

is called an averaged equation of (1.1) (the averaged equation of (1.2) if $(\hat{c}, \hat{d})$ is an averaged function of $(c,d)$ (the averaged function of $(c,d)$).

Denote $\lambda(\hat{c}, \hat{d})$ to be the principal eigenvalue of (1.10). We then have the following main results of the paper.

5) Consider (1.1). Then $\lambda_{\inf}(c,d) \geq \lambda(\hat{c}, \hat{d})$ for some averaged function $(\hat{c}, \hat{d})$ of $(c,d)$ and $\lambda_{\sup}(c,d) \geq \lambda(\hat{c}, \hat{d})$ for any averaged function $(\hat{c}, \hat{d})$ of $(c,d)$ (see Theorem 6.1(1)). Moreover, if $(c,d)$ is uniquely ergodic and minimal, then $\lambda_{\inf}(c,d) = \lambda_{\sup}(c,d) = \lambda(\hat{c}, \hat{d})$ for the (necessarily unique) averaged function $(\hat{c}, \hat{d})$ of $(c,d)$ if and only if $c(t,x) = c_1(x) + c_2(t)$ and $d(t,x) = d(x)$ (see Theorem 6.1(1)).

6) Consider (1.2). Then $\lambda(c,d) \geq \lambda(\hat{c}, \hat{d})$ (see Theorem 6.1(2)). Further, $\lambda(c,d) = \lambda(\hat{c}, \hat{d})$ if and only if there is $\Omega^* \subset \Omega$ with $\mathbb{P}(\Omega^*) = 1$ such that $c(\theta_\omega, x) = c_1(x) + c_2(\theta_\omega)$ for any $\omega \in \Omega^*$, $t \in \mathbb{R}$ and $x \in \partial D$, and $d(\theta_\omega, x) = d(x)$ for any $\omega \in \Omega^*$, $t \in \mathbb{R}$ and $x \in \partial D$ (see Theorem 6.1(2)).

Hence time variations cannot reduce the principal spectrum and principal Lyapunov exponent (or the principal eigenvalues of the time averaged equations give lower bounds of principal spectrum and principal Lyapunov exponent of non-averaged equations). Indeed, the time variations increase the principal spectrum and principal Lyapunov exponents except in the degenerate cases. In the biological context these results mean that invasion by a new species (see [10], p. 220) is always easier in the time-dependent case or that time variations favor persistence (viewing both (1.1) and (1.10) as linear population growth models, then by 5), positive solutions of all averaged equations (1.10) of (1.1) bounded.
away from zero implies positive solutions of (1.1) also bounded away from zero, but not vice versa in general).

It should be pointed out that the results 5), 6) have been proved in [22] and [31] when the boundary conditions are time independent. They are new when the boundary conditions are time dependent and the proof presented in this paper is not the same as those in [22] and [31].

It should be also pointed out that the results 1)–4) apply to fully time dependent/random parabolic equations (i.e., equations in which all the coefficients can depend on \( t/\theta t, \omega \)). But 5) and 6) are mainly for equations of form (1.1) and (1.2), respectively.

The rest of the paper is organized as follows. In Section 2, we collect several elementary lemmas and introduce some standing notations for future reference. We review some existence and regularity theorems and construct the skew-product semiflow generated by (1.1) and (1.2) in Section 3. Section 4 is devoted to the study of the monotonicity of the skew-product semiflow constructed in Section 5. We prove the time averaging results in Section 6. The authors are grateful to the referees for their remarks.

2 Elementary Lemmas and notations

We collect first, for further reference, some elementary results.

First of all, let \( Z \) be a compact metric space and \( \mathcal{B}(Z) \) be the Borel \( \sigma \)-algebra of \( Z \). \((Z, \mathcal{B}, \mu) := (Z, \{\sigma_t\} \in \mathbb{R})\) is called a compact flow if \( \sigma_t: Z \to Z \) \((t \in \mathbb{R})\) satisfies: \([ (t, z) \mapsto \sigma_t(z) \] is jointly continuous in \((t, z) \in \mathbb{R} \times Z \), \( \sigma_0 = \text{id} \), and \( \sigma_s \circ \sigma_t = \sigma_{s+t} \) for any \( s, t \in \mathbb{R} \). We may write \( z \cdot t \) or \((z, t)\) for \( \sigma_t z \). A probability measure \( \mu \) on \((Z, \mathcal{B}(Z))\) is called an invariant measure for \((Z, \{\sigma_t\} \in \mathbb{R})\) if for any \( E \in \mathcal{B}(Z) \) and any \( t \in \mathbb{R} \), \( \mu(\sigma_t(E)) = \mu(E) \). An invariant measure \( \mu \) for \((Z, \{\sigma_t\} \in \mathbb{R})\) is said to be ergodic if for any \( E \in \mathcal{B}(Z) \) satisfying \( \mu(\sigma_t^{-1}(E) \triangle E) = 0 \) for all \( t \in \mathbb{R} \), \( \mu(E) = 1 \) or \( \mu(E) = 0 \). The compact flow \((Z, \{\sigma_t\} \in \mathbb{R})\) is said to be uniquely ergodic if it has a unique invariant measure (in such case, the unique invariant measure is necessarily ergodic). We say that \((Z, \{\sigma_t\} \in \mathbb{R})\) is minimal or recurrent if for any \( z \in Z \), the orbit \( \{\sigma_t z : t \in \mathbb{R}\} \) is dense in \( Z \).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \( \{\theta_t\} \in \mathbb{R} \) be a family of \( \mathbb{P} \)-preserving transformations (i.e., \( \mathbb{P}(\theta_t^{-1}(F)) = \mathbb{P}(F) \) for any \( F \in \mathcal{F} \) and \( t \in \mathbb{R} \)) such that \((t, \omega) \mapsto \theta_t \omega \) is measurable, \( \theta_0 = \text{id} \), and \( \theta_{t+s} = \theta_t \circ \theta_s \) for all \( t, s \in \mathbb{R} \). Thus \( \{\theta_t\} \in \mathbb{R} \) is a flow on \( \Omega \) and \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\} \in \mathbb{R})\) is called a metric dynamical system. \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\} \in \mathbb{R})\) is said to be ergodic if for any \( F \in \mathcal{F} \) satisfying \( \mathbb{P}(\theta_t^{-1}(F) \triangle F) = 0 \) for any \( t \in \mathbb{R} \), \( \mathbb{P}(F) = 1 \) or \( \mathbb{P}(F) = 0 \).

In the following, we assume that \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\} \in \mathbb{R})\) is an ergodic metric dynamical system.

**Lemma 2.1.** (1) Let \( h_i: [0, T] \times D \to \mathbb{R} \) \((i = 1, 2, \ldots, N)\) be square-integrable in \( t \in [0, T] \) and \( a_{ij} = a_{ji}: D \to \mathbb{R} \) \((i, j = 1, 2, \ldots, N)\) satisfy

\[
\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq a_0 \sum_{i=1}^{N} \xi_i^2
\]
for some $\alpha_0 > 0$ and any $x \in \bar{D}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \in \mathbb{R}^N$. Then for any $x \in D$,

$$
\sum_{i,j=1}^{N} a_{ij}(x) \frac{1}{T} \int_0^T h_i(t,x) \, dt \frac{1}{T} \int_0^T h_j(t,x) \, dt
\leq \sum_{i,j=1}^{N} a_{ij}(x) \frac{1}{T} \int_0^T h_i(t,x) h_j(t,x) \, dt.
$$

Moreover, the equality holds at some $x_0 \in D$ if and only if $h_i(t,x_0) = \tilde{h}_i(x_0)$ for some $\tilde{h}_i(x_0)$ ($i = 1, 2, \ldots, N$) and a.e. $t \in [0,T]$.

(2) Let $h_i: \Omega \times D \to \mathbb{R}$ ($i = 1, 2, \ldots, N$) be square-integrable in $\omega \in \Omega$ and $a_{ij} = a_{ji}: D \to \mathbb{R}$ ($i,j = 1, 2, \ldots, N$) satisfy

$$
\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^{N} \xi_i^2
$$

for some $\alpha_0 > 0$ and any $x \in \bar{D}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \in \mathbb{R}^N$. Then for any $x \in D$,

$$
\sum_{i,j=1}^{N} a_{ij}(x) \int_{\Omega} h_i(\omega,x) \, d\mathbb{P}(\omega) \int_{\Omega} h_j(\omega,x) \, d\mathbb{P}(\omega)
\leq \sum_{i,j=1}^{N} a_{ij}(x) \int_{\Omega} h_i(\omega,x) h_j(\omega,x) \, d\mathbb{P}(\omega).
$$

Moreover, the equality holds at some $x_0 \in D$ if and only if $h_i(\omega,x_0) = \tilde{h}_i(x_0)$ for some $\tilde{h}_i(x_0)$ ($i = 1, 2, \ldots, N$) and a.e. $\omega \in \Omega$.

Proof. See [22, Lemma 2.2] for (1) and [31, Lemma 3.5] for (2). □

**Lemma 2.2** (Birkhoff’s Ergodic Theorem). Let $h \in L^1(\Omega, F, \mathbb{P})$. Then there is an invariant measurable set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T h(\theta_t \omega) \, dt = \int_{\Omega} h(\cdot) \, d\mathbb{P}(\cdot)
$$

for any $\omega \in \Omega_0$.

Proof. See [6] or references therein. □

**Lemma 2.3.** Assume that $h: \Omega \times D \to \mathbb{R}$ (resp. $h: \Omega \times \bar{D} \to \mathbb{R}$) has the following properties:

(i) $h(\cdot, x)$ belongs to $L^1(\Omega)$, for each $x \in D$,

(ii) for each $x \in D$ (resp. $x \in \bar{D}$) and each $\epsilon > 0$ there is $\delta > 0$ such that if $y \in D$ (resp. $y \in \bar{D}$), $\omega \in \Omega$ and $|x-y| < \delta$ then $|h(\omega,x) - h(\omega,y)| < \epsilon$, where $|\cdot|$ stands for the norm in $\mathbb{R}^N$ or the absolute value, depending on the context.
Denote, for each $x \in D$ (resp. $x \in \bar{D}$),

$$
\hat{h}(x) := \int_{\Omega} h(\omega, x) \, d\mathbb{P}(\omega).
$$

Then

(a) for any $x \in D$ (resp. $x \in \bar{D}$) and any $\epsilon > 0$ there is $\delta > 0$ (the same as in (ii)) such that if $y \in D$ (resp. $y \in \bar{D}$), $\omega \in \Omega$ and $|x - y| < \delta$ then $|\hat{h}(x) - \hat{h}(y)| < \epsilon$,

(b) there is a measurable $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that

$$
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} h(\theta_t \omega, x) \, dt = \hat{h}(x)
$$

for all $\omega \in \Omega'$ and all $x \in D$ (resp. $x \in \bar{D}$). Moreover the convergence is uniform in $x \in D_0$, for any compact $D_0 \subset D$ (resp. uniform in $x \in \bar{D}$).

Proof. Part (a) follows easily by the fact that the continuity is uniform in $\omega \in \Omega$.

To prove (b), take a countable dense set $\{x_l\}_{l=1}^{\infty}$ in $D$.

By Birkhoff’s Ergodic Theorem (Lemma 2.2) for each $l \in \mathbb{N}$ there is a measurable $\Omega_l \subset \Omega$ with $\mathbb{P}(\Omega_l) = 1$ such that

$$
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} h(\theta_t \omega, x_l) \, dt = \hat{h}(x_l)
$$

for each $\omega \in \Omega_l$. Take $\Omega' := \bigcap_{l=1}^{\infty} \Omega_l$.

Fix $x \in D$ (resp. $x \in \bar{D}$). For $\epsilon > 0$ take $\delta > 0$ such that if $|x - y| < \delta$ then $|h(\omega, x) - h(\omega, y)| < \epsilon/3$ and $|\hat{h}(x) - \hat{h}(y)| < \epsilon/3$. Let $x_l$ be such that $|x - x_l| < \delta$, and let $T_0 > 0$ be such that

$$
\left| \frac{1}{T} \int_{0}^{T} h(\theta_t \omega, x_l) \, dt - \hat{h}(x_l) \right| < \frac{\epsilon}{3}
$$

for all $T > T_0$. Then

$$
\left| \frac{1}{T} \int_{0}^{T} h(\theta_t \omega, x) \, dt - \hat{h}(x) \right| < \epsilon
$$

for all $T > T_0$. (b) then follows.

\[\square\]

**Lemma 2.4.** Assume that $h: \Omega \times D \to \mathbb{R}$ (resp. $h: \Omega \times \bar{D} \to \mathbb{R}$) has the following properties:

(i) $h(\cdot, x)$ belongs to $L^1(\Omega)$, for each $x \in D$,

(ii) $(\partial h/\partial x_i)(\omega, x)$ exists for each $\omega \in \Omega$ and each $x \in D$ (resp. each $x \in \bar{D}$); further, $(\partial h/\partial x_i)(\cdot, x)$ belongs to $L^1(\Omega)$, for each $x \in D$.  


(iii) there exists $\alpha \in (0, 1]$ such that for each $x \in D$ (resp. each $x \in \bar{D}$) there are $L > 0$ and $\delta_0 > 0$ with the property that
\[
\left| \frac{\partial h}{\partial x_i}(\omega, x) - \frac{\partial h}{\partial x_i}(\omega, y) \right| \leq L|x - y|^\alpha
\]
for any $\omega \in \Omega$ and any $y \in D$ (resp. any $y \in \bar{D}$) with $|x - y| < \delta_0$.

Denote, for each $x \in D$ (resp. $x \in \bar{D}$),
\[
\tilde{h}(x) := \int_\Omega h(\omega, x) \, d\mathbb{P}(\omega).
\]
Then
(a) for each $x \in D$ (resp. each $x \in \bar{D}$) the derivative $(\partial \tilde{h}/\partial x_i)(x)$ exists, and the equality
\[
\frac{\partial \tilde{h}}{\partial x_i}(x) = \int_\Omega \frac{\partial h}{\partial x_i}(\omega, x) \, d\mathbb{P}(\omega)
\]
holds,
(b) for each $x \in D$ (resp. each $x \in \bar{D}$) there are $L > 0$ and $\delta_0 > 0$ (the same as in (ii)) with the property that
\[
\left| \frac{\partial \tilde{h}}{\partial x_i}(x) - \frac{\partial \tilde{h}}{\partial x_i}(y) \right| \leq L|x - y|^\alpha
\]
for any $y \in D$ (resp. $y \in \bar{D}$) with $|x - y| < \delta_0$,
(c) there is a measurable $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that
\[
\frac{\partial \tilde{h}}{\partial x_i}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\partial h}{\partial x_i}(\theta_t \omega, x) \, dt
\]
for all $\omega \in \Omega'$ and all $x \in D$ (resp. $x \in \bar{D}$). Moreover, the convergence is uniform in $x \in D_0$, for any compact $D_0 \subset D$ (resp. uniform in $x \in \bar{D}$).

Proof. Parts (a) and (b) follow in a standard way. Part (c) follows by an application of Lemma 3.3 b) to the function $(\partial \tilde{h}/\partial x_i)(\omega, x)$.

From now on we assume that (A1)–(A6) are satisfied.

Consider the space $H$ consisting of $(\tilde{c}(\cdot, \cdot), \tilde{d}(\cdot, \cdot))$, where $\tilde{c} : \mathbb{R} \times \bar{D} \to \mathbb{R}$ and
\[
\tilde{d} : \mathbb{R} \times \partial D \to \mathbb{R}
\]
are bounded continuous. The set $H$ endowed with the topology of uniform convergence on compact sets (the open-compact topology) becomes a Fréchet space.

For $(\tilde{c}, \tilde{d}) \in H$ and $t \in \mathbb{R}$ we define the time-translate as $(\tilde{c}, \tilde{d}) \cdot t := ((\tilde{c} \cdot t)(\cdot, s), (\tilde{d} \cdot t)(\cdot, s))$, where $(\tilde{c} \cdot t)(s, x) := \tilde{c}(s + t, x)$, $s \in \mathbb{R}$, $x \in \bar{D}$, and $(\tilde{d} \cdot t)(s, x) := \tilde{d}(s + t, x)$, $s \in \mathbb{R}$, $x \in \partial D$. It is well known that $(\tilde{c}, \tilde{d}) \cdot t \in H$ whenever $(\tilde{c}, \tilde{d}) \in H$ and $t \in \mathbb{R}$, and that the mapping $[\mathbb{R} \times H \ni (t, (\tilde{c}, \tilde{d})) \mapsto (\tilde{c}, \tilde{d}) \cdot t \in H]$ is continuous.

In the case of (1.1) let $Y = Y(\cdot, d) := \text{cl}\{ (c, d) \cdot t : t \in \mathbb{R} \}$. In the case of (1.2) let $Y = Y(\Omega) := \text{cl}\{ (c^\omega, d^\omega) : \omega \in \Omega \}$ (see Section 1 for detail).

The following result is a consequence of the Ascoli–Arzelà theorem.
Lemma 2.5.  
(i) $Y$ is a compact subset of $H$.
(ii) For any $(\hat{c}, \hat{d}) \in Y$ and any $t \in \mathbb{R}$ there holds $(\hat{c}, \hat{d}) \cdot t \in Y$.
(iii) For any $(\hat{c}, \hat{d}) \in Y$, $\hat{c} \in C^{2+\alpha, 1+\alpha}(\mathbb{R} \times \hat{D})$. Moreover, the $C^{2+\alpha, 1+\alpha}(\mathbb{R} \times \hat{D})$-norms are bounded uniformly in $Y$ by the same bound as in (A3).
(iv) For any $(\hat{c}, \hat{d}) \in Y$, $\hat{d} \in C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \partial D)$. Moreover, the $C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \partial D)$-norms are bounded uniformly in $Y$ by the same bound as in (A4).
(v) For a sequence $(\hat{c}^{(n)}, \hat{d}^{(n)}) \to (\hat{c}, \hat{d})$ in $Y$, the mixed derivatives of $\hat{c}^{(n)}$ of order up to 2 in $t$ and up to 1 in $x$ converge to the respective derivatives of $\hat{c}$, uniformly on compact subsets of $\mathbb{R} \times \hat{D}$.
(vi) For a sequence $(\hat{c}^{(n)}, \hat{d}^{(n)}) \to (\hat{c}, \hat{d})$ in $Y$, the mixed derivatives of $\hat{d}^{(n)}$ of order up to 2 in $t$ and up to 3 in $x$ converge to the respective derivatives of $\hat{d}$, uniformly on compact subsets of $\mathbb{R} \times \partial D$.

We write $\sigma_t(\hat{c}, \hat{d})$ for $(\hat{c}, \hat{d}) \cdot t \in Y$. We will denote by $(Y, \mathbb{R})$ the compact flow $(Y, (\sigma_t)_{t \in \mathbb{R}})$.

Consider (1.1). For $x \in \hat{D}$ and $S < T$ we denote

$$\hat{c}(x; S, T) := \frac{1}{T - S} \int_{S}^{T} c(t, x) \, dt.$$ 

Similarly, for $x \in \partial D$ and $S < T$ we denote

$$\hat{d}(x; S, T) := \frac{1}{T - S} \int_{S}^{T} d(t, x) \, dt.$$ 

Let

$$\hat{Y}(c, d) := \{(\hat{c}, \hat{d}) : \exists S_n < T_n \text{ with } T_n - S_n \to \infty \text{ such that } (\hat{c}, \hat{d}) = \lim_{n \to \infty} (\hat{c}(\cdot; S_n, T_n), \hat{d}(\cdot; S_n, T_n))\}, \quad (2.1)$$

where the convergence is in $C(\hat{D}) \times C(\partial D)$.

The following result is a consequence of the Ascoli–Arzelà theorem (compare Lemma 2.5).

Lemma 2.6.  
(i) $\hat{Y}(c, d)$ is a nonempty compact subset of $C(\hat{D}) \times C(\partial D)$.
(ii) For any $(\hat{c}, \hat{d}) \in \hat{Y}(c, d)$, $\hat{c} \in C^1(\hat{D})$. Moreover, the $C^1(\hat{D})$-norms are bounded uniformly in $\hat{Y}(c, d)$.
(iii) For any $(\hat{c}, \hat{d}) \in \hat{Y}(c, d)$, $\hat{d} \in C^3(\partial D)$. Moreover, the $C^3(\partial D)$-norms are bounded uniformly in $\hat{Y}(c, d)$.

Definition 2.1.  
(1) Let $a$ be as in (1.1). We say $(c, d)$ is uniquely ergodic if the compact flow $(Y(c, d), \mathbb{R})$ is uniquely ergodic.
(2) Let $a$ be as in (1.1). We say $(c, d)$ is minimal or recurrent if $(Y(c, d), \mathbb{R})$ is minimal.
Remark 2.1.  (1) If \(c(t, x)\) and \(d(t, x)\) are almost periodic in \(t\) uniformly with respect to \(x \in \bar{D}\) and \(x \in \partial D\), respectively, then \((c, d)\) is both uniquely ergodic and minimal.

(2) If \(c(t, x)\) and \(d(t, x)\) are almost automorphic in \(t\) uniformly with respect to \(x \in \bar{D}\) and \(x \in \partial D\), respectively, then \((c, d)\) is minimal, but it may not be uniquely ergodic (see \([23]\) for examples).

(3) There is \((c, d)\) which is neither uniquely ergodic nor minimal. For example, let \(c(t, x) = \tan^{-1}(t)\) and \(d(t, x) \equiv 1\), then \(\{(\pi/2, 1)\} \text{ and } \{(-\pi/2, 1)\}\) are two minimal invariant subsets of \(Y(c, d)\), and hence \(Y(c, d)\) is neither uniquely ergodic nor minimal.

Lemma 2.7. Consider \((c, d)\) uniquely ergodic, \(\mu\) being the unique ergodic measure. For \((\tilde{c}, \tilde{d}) \in Y(c, d)\) put \(\tilde{c}_0(x) := \tilde{c}(0, x)\) and \(\tilde{d}_0(x) := \tilde{d}(0, x)\). Then

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T c(t, x) \, dt = \int_{Y(c, d)} \tilde{c}_0(x) \, d\mu((\tilde{c}, \tilde{d})) \quad (2.2)
\]

uniformly for \(x \in \bar{D}\), and

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T d(t, x) \, dt = \int_{Y(c, d)} \tilde{d}_0(x) \, d\mu((\tilde{c}, \tilde{d})) \quad (2.3)
\]

uniformly for \(x \in \partial D\).

**Proof.** We prove only (2.2), the other proof being similar. It follows via the Ascoli–Arzelà theorem that the set \(\{(1/T) \int_0^T c(t, \cdot) \, dt : T > 0\} \) has compact closure in \(C(\bar{D})\), consequently from any sequence \((T_n)\) with \(\lim_{n \to \infty} T_n = \infty\) one can extract a subsequence \((T_{n_k})\) such that \(\tilde{c}(\cdot; 0, T_{n_k})\) converges uniformly in \(x \in \bar{D}\) to some \(\tilde{c}\) (depending perhaps on the subsequence).

On the other hand, as \((Y(c, d), R)\) is uniquely ergodic, for each continuous \(g: Y(c, d) \to R\) there holds

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T g((c, d) \cdot t) \, dt = \int_{Y(c, d)} g(\cdot) \, d\mu(\cdot)
\]

(compare, e.g., Oxtoby \([23]\)). Fix \(x \in \bar{D}\) and take \(g((\tilde{c}, \tilde{d})) := \tilde{c}_0(x)\). We have thus obtained that if \(\tilde{c}(x; 0, T_n)\) converges, for some \(T_n \to \infty\), uniformly in \(x \in \bar{D}\), then the limit is always equal to \(\tilde{c}(x) = \int_{Y(c, d)} \tilde{c}_0(x) \, d\mu\).

We introduce the following standing notations \((X_1, X_2)\) are Banach spaces):

\(L(X_1, X_2)\) represents the space of all bounded linear operators from \(X_1\) to \(X_2\), endowed with the norm topology;

\(\|\cdot\|_{X_1}\) denotes the norm in \(X_1\);

\(X_1^*\) denotes the Banach space dual to \(X_1\);

\((\cdot, \cdot)_{X_1, X_2^*}\) stands for the duality pairing between \(X_1\) and \(X_2^*\);

\(\|\cdot\|\) denotes the norm in \(L_2(\bar{D})\) or the norm in \(L(L_2(\bar{D}), L_2(\bar{D}))\);

\((\cdot, \cdot)\) stands for the standard inner product in \(L_2(\bar{D})\);
∥·∥_{X_1,X_2} indicates the norm in \( L(X_1,X_2) \);
\([\cdot,\cdot]_\theta\) is a complex interpolation functor;
\([\cdot,\cdot]_{\theta,p}\) is a real interpolation functor (see [9], [38] for more detail);
\(\mathbb{Z}\) denotes the set of integers;
\(\mathbb{N}\) denotes the set of nonnegative integers.

3 Skew-product semiflows

We construct in this section a linear skew-product semiflow on \( X \) generated by (1.1) or by (1.2), where \( X \) is as in (1.7).

To do so, we first use the theory presented by H. Amann in [2] to consider the existence of solution of (1.5)+(1.6) for any \((\tilde{c},\tilde{d})\) \(\in\) \(Y\) and any \(u_0 \in L^p(D)\).

Recall that we assume (A1)–(A6) throughout.

Let \(A(\tilde{c})\) denote the operator given by
\[
A(\tilde{c})u = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + \tilde{c}(0,x)u, \quad x \in D,
\]
and let \(B(\tilde{d})\) denote the boundary operator given by
\[
B(\tilde{d})u = \begin{cases} 
  u & \text{for } x \in \partial D \quad \text{(Dirichlet)} \\
  \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} & \text{for } x \in \partial D \quad \text{(Neumann)} \\
  \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + \tilde{d}(0,x)u & \text{for } x \in \partial D \quad \text{(Robin)}.
\end{cases}
\]

Let
\[
V^1_{\theta}(\tilde{d}) := \{ u \in W^2_p(D) : B(\tilde{d})u = 0 \}.
\]

For given \(0 < \theta < 1\) and \(1 < p < \infty\), let
\[
V^\theta_p := \begin{cases} 
  (L_p(D),W^2_p(D))_{\theta,p} & \text{if } 2\theta \notin \mathbb{N} \\
  [L_p(D),W^2_p(D)]_\theta & \text{if } 2\theta \in \mathbb{N}
\end{cases}
\]
and
\[
V^\theta_p(\tilde{d}) := \begin{cases} 
  (L_p(D),V^1_{\theta}(\tilde{d}))_{\theta,p} & \text{if } 2\theta \notin \mathbb{N} \\
  [L_p(D),V^1_{\theta}(\tilde{d})]_\theta & \text{if } 2\theta \in \mathbb{N}.
\end{cases}
\]

**Proposition 3.1.**

1. \(V^\theta_p = W^{2\theta}_p\).
2. If \(2\theta - \frac{1}{p} \neq 0,1\) then \(V^\theta_p(\tilde{d})\) is a closed subspace of \(V^\theta_p\).

**Proof.** (1) follows from [2] Theorem 11.6.
(2) follows from [2] Lemma 14.4. \(\square\)
Recall the following compact embedding:

\[ W^{j+m}_p(D) \hookrightarrow C^{j,\lambda}(\bar{D}) \tag{3.1} \]

if \( mp > N > (m-1)p \) and \( 0 < \lambda < m - (N/p) \), and

\[
\begin{align*}
W^1_p(D) & \hookrightarrow V^\theta_p \\
V^1_p(\bar{d}) & \hookrightarrow V^\theta_p(\bar{d})
\end{align*}
\tag{3.2}
\]

for any \( 0 \leq \theta < 1 \) and \( \bar{a} \in Y \), where \( V^\theta_p, V^\theta_p(\bar{d}) = L_p(D) \).

Let

\[ A_{(\bar{c}, \bar{d}), \theta}(t) := A(\bar{c} \cdot t)|_{V^\theta_p(\bar{d}, t)} \]

Then \((1.5)+(1.6)\) can be written as

\[
\begin{cases}
u_t = A_{(\bar{c}, \bar{d}), \theta}(t)u \\
u(0) = u_0.
\end{cases}
\tag{3.4}
\]

**Definition 3.1.** \( u = u(t, x) \) is called an \( L_p \)-solution of \((1.5)+(1.6)\) if it is a solution of the evolution equation \((3.4)\) in \( L_p(D) \).

**Definition 3.2.** \( u = u(t, x) \) defined on \((t_0, t_1) \times \bar{D}, t_0 < t_1\), is a classical solution of \((1.5)\) on \((t_0, t_1)\) if it is continuous on \((t_0, t_2) \times \bar{D}\), it satisfies the differential equation in \((1.5)\) for all \( t \in (t_0, t_1)\) and all \( x \in D\), and it satisfies the boundary conditions for all \( t \in (t_0, t_1)\) and all \( x \in \partial D\).

The following existence result follows from [2, Theorem 15.1].

**Proposition 3.2.** For each \((\bar{c}, \bar{d}) \in Y \) and each \( u_0 \in L_p(D) \) there exists a unique \( L_p(D) \)-solution \( U_{(\bar{c}, \bar{d}), \theta}(\cdot, t_0)u_0: [0, \infty) \rightarrow L_p(D) \) of \((1.5)+(1.6)\).

It follows from the uniqueness of \( L_p \)-solutions that the following cocycle property for the solution operator holds:

\[ U_{(\bar{c}, \bar{d}), \theta}(t + s, 0) = U_{(\bar{c}, \bar{d}), \theta,s,p}(t, 0)U_{(\bar{c}, \bar{d}), \theta,p}(s, 0) \quad \text{for any } (\bar{c}, \bar{d}) \in Y, s, t \geq 0. \tag{3.5} \]

We collect now the regularity properties of the \( L_p(D) \)-solutions which will be useful in the sequel.

**Proposition 3.3.** For any \( 1 < p < \infty \), \((\bar{c}, \bar{d}) \in Y \) and \( u_0 \in L_2(D) \) there holds \( U_{(\bar{c}, \bar{d}), \theta}(t, 0)u_0 \in V^1_p(\bar{d}, t) \) for \( t > 0 \). Moreover, for any fixed \( 0 < t_1 \leq t_2 \) there is \( C_p = C_p(t_1, t_2) > 0 \) such that

\[ ||U_{(\bar{c}, \bar{d}), \theta}(t, 0)||_{L_2(D), W^2_p(D)} \leq C_p \]

for all \((\bar{c}, \bar{d}) \in Y \) and \( t_1 \leq t \leq t_2 \).

**Proof.** First of all, by [2, Lemma 6.1 and Theorem 14.5], for any \( 1 < p < \infty \), any \((\bar{c}, \bar{d}) \in Y \), and any \( u_0 \in L_p(D) \),

\[ U_{(\bar{c}, \bar{d}), \theta}(t, 0)u_0 \in V^1_p(\bar{d}, t) \quad \text{for } t > 0. \tag{3.6} \]
Moreover, for any $t_2 > 0$, there is $C_p = C_p(t_2) > 0$ such that
\[ \|U_{(\bar{c}, \bar{d}), p}(t, 0)\|_{L_p(D), W^2_p(D)} \leq \frac{C_p}{t} \]  
(3.7)
for all $(\bar{c}, \bar{d}) \in Y$ and $0 < t \leq t_2$.

Next, note that if $1 < p \leq 2$, then we have $L_2(D) \subset L_p(D)$, $V^1_p(\bar{d} \cdot t) \subset V^1_p(\bar{d} \cdot t)$, and $W^2_p(D) \subset W^2_p(D)$. The proposition then follows from (3.6) and (3.7).

Now, assume $2 < p < \infty$. If $4 \geq N$, then by Sobolev embeddings (see [1, Theorem 6.2]), we have
\[ W^2_p(D) \hookrightarrow C(\bar{D}). \]
(3.8)
Then it follows with the help of (3.7) that $U_{(\bar{c}, \bar{d}), 2}(t/2, 0)u_0 \in L_p(D)$ for all $t > 0$. (3.6) gives that $U_{(\bar{c}, \bar{d}), 2}(t, 0)u_0 \in V^1_p(\bar{d} \cdot t)$ for all $t > 0$. We estimate
\[
\|U_{(\bar{c}, \bar{d}), 2}(t, 0)\|_{L_p(D), W^2_p(D)} \\ \leq \bar{C} \|U_{(\bar{c}, \bar{d}), 2}(t, 0)\|_{L_p(D), W^2_p(D)} \cdot \|U_{(\bar{c}, \bar{d}), 2}(t, 0)\|_{L_p(D), W^2_p(D)} \\
\leq \bar{C} \cdot \frac{C_p(t - t_1/2)}{t - t_1/2} \cdot \frac{C_2(t_1/2)}{t_1/2}
\]
for all $(\bar{c}, \bar{d}) \in Y$ and $t_1 \leq t \leq t_2$, where $\bar{C}$ denotes the norm of the embedding $W^2_p(D) \hookrightarrow L_p(D)$. Hence the proposition also holds.

Finally, assume $p > 2$ and $N > 4$. There are $l \in \mathbb{N}$ and $p_0 = 2 < p_1 < p_2 < \cdots < p_l$ such that $p_{i-1} < p_i < \frac{Np_{i-1}}{2p_{i-1}}$ for $i = 1, 2, \ldots, l$, and $2p_l > N$. For any $\delta > 0$, let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_l = \frac{\delta}{2}$. By (3.7),
\[
\|U_{(\bar{c}, \bar{d}), \tau_{i+1}, p_i}(\tau_{i+1} - \tau_i, 0)\|_{L_{p_i}(D), W^2_{p_i}(D)} \leq \frac{C_{p_i}(\tau_{i+1} - \tau_i)}{\tau_{i+1} - \tau_i},
\]
for $i = 0, 1, 2, \ldots, l - 1$. By Sobolev embeddings (see [1, Theorem 6.2]),
\[ W^2_{p_i}(D) \hookrightarrow L_{p_{i+1}}(D) \]
for $i = 0, 1, 2, \ldots, l - 1$. We then have
\[
\|U_{(\bar{c}, \bar{d}), \tau_{i+1}, p_i}(\tau_{i+1} - \tau_i, 0)\|_{L_{p_i}(D), L_{p_{i+1}}(D)} \leq \bar{C}_{p_i},
\]
(3.9)
for some $\bar{C}_{p_i} > 0$. Further, since $p_l > \frac{N}{2}$, by Sobolev embeddings (see [1, Theorem 6.2]),
\[ W^2_{p_l}(D) \hookrightarrow C(\bar{D}). \]
Consequently, we have an embedding $W^2_{p_l}(D) \hookrightarrow L_p(D)$ (denote its norm by $\bar{C}$). It then follows that for any $u_0 \in L_2(D)$,
\[
U_{(\bar{c}, \bar{d}), 2}(\tau_0, 0)u_0 = U_{(\bar{c}, \bar{d}), \tau_1, p_1}(\tau_1, 0)U_{(\bar{c}, \bar{d}), 2}(\tau_1, 0)u_0 \\
= U_{(\bar{c}, \bar{d}), \tau_{i+1}, p_i}(\tau_{i+1} - \tau_i, 0)U_{(\bar{c}, \bar{d}), \tau_{i+1}, p_{i+1}}(\tau_{i+1} - \tau_{i+1}, 0) \\
\cdots U_{(\bar{c}, \bar{d}), \tau_2, p_1}(\tau_2, 0)U_{(\bar{c}, \bar{d}), 2}(\tau_2, 0)u_0
\in L_p(D).
\]
This implies, via (3.6), that
\[
U_{(\tilde{c},\tilde{d}),2}(t,0)u_0 \in V_p^0(\tilde{d} \cdot t)
\]
for any \( t \geq \delta \) (and hence for any \( t > 0 \), since \( \delta > 0 \) is arbitrary). Now we take \( \delta = t_1 \). It follows from (3.3) and (3.7) that
\[
\|U_{(\tilde{c},\tilde{d}),2}(t,0)\|_{L^2(D),\omega^2(D)} \leq \tilde{C} \tilde{C}_{p_0} \cdots \tilde{C}_{p_{t-1}} \frac{C_p(t_2 - t_1/2)}{t_1/2}
\]
for all \((\tilde{c},\tilde{d}) \in Y\) and \( t_1 \leq t \leq t_2 \).

**Proposition 3.4.** Suppose that \( 2\theta - 1/p \notin \mathbb{N} \). Then for any \( t \geq 0 \) and \( u_0 \in V_p^0(d) \) there holds \( U_{(\tilde{c},\tilde{d}),p}(t,0)u_0 \in V_p^0(\tilde{d} \cdot t) \). Moreover, for any \( T > 0 \) there is \( C_{p,\theta} = C_{p,\theta}(T) > 0 \) such that
\[
\|U_{(\tilde{c},\tilde{d}),p}(t,0)u_0\|_{V_p^0} \leq C_{p,\theta}\|u_0\|_{V_p^0}
\]
for any \((\tilde{c},\tilde{d}) \in Y\), \( 0 \leq t \leq T \), and \( u_0 \in V_p^0(\tilde{d}) \).

**Proof.** See [2, Theorems 7.1 and 14.5].

**Proposition 3.5.** For any \( u_0 \in L_p(D) \), \( U_{(\tilde{c},\tilde{d}),p}(t,0)u_0 \) is a classical solution of (1.5) on \((0, \infty)\).

**Proof.** It follows from Proposition 3.3 and [2, Corollary 15.3]).

Proposition 3.5 allows us to write \( U_{(\tilde{c},\tilde{d})}(t,0)u_0 \) \((t > 0)\) instead of \( U_{(\tilde{c},\tilde{d}),p}(t,0)u_0 \).

In case of (1.2) we write \( U_{\omega}(t,0) \) instead of \( U_{(\tilde{c},\tilde{d}),\omega}(t,0) \).

**Definition 3.3.** A global solution of (1.5) is a classical solution of (1.5) on \((-\infty, \infty)\).

Observe that \( v = v(t, x) \) is a global solution of (1.5) if and only if
\[
U_{(\tilde{c},\tilde{d})}(s,0)v(t,\cdot) = v(t+s,\cdot) \quad \text{for any } t \in \mathbb{R} \text{ and any } s \geq 0.
\]

From now on, we assume (A7). For any sequence \((\hat{z}^{(n)},\hat{d}^{(n)})\) in \( Y \), we write \( \lim_{n \to \infty}(\hat{z}^{(n)},\hat{d}^{(n)}) = (\tilde{c},\tilde{d}) \) if \((\hat{z}^{(n)},\hat{d}^{(n)})\) converges to \((\tilde{c},\tilde{d})\) in \( Y \) as \( n \to \infty \) (here the convergence is uniform in the space variable and uniform on compact sets in the time variable). We then present various continuous dependence propositions.

**Proposition 3.6 (Joint continuity).** For any sequence \((\hat{z}^{(n)},\hat{d}^{(n)})\) in \( Y \), any sequence \((t_n)\) in \((0, \infty)\), and any sequence \((u_n)\) in \( L_2(D) \), if \( \lim_{n \to \infty}(\hat{z}^{(n)},\hat{d}^{(n)}) = (\tilde{c},\tilde{d}) \), \( \lim_{n \to \infty}t_n = t \), where \( t > 0 \), and \( \lim_{n \to \infty}u_n = u_0 \) in \( L_2(D) \), then the following holds.

1. \( U_{(\hat{z}^{(n)},\hat{d}^{(n)})(t_n,0)u_n} \) converges in \( V_p^0 \) to \( U_{(\tilde{c},\tilde{d})}(t,0)u_0 \), where \( 0 \leq \theta < 1 \) and \( 1 < p < \infty \) with \( 2\theta - 1/p \notin \mathbb{N} \).

2. \( U_{(\hat{z}^{(n)},\hat{d}^{(n)})(t_n,0)u_n} \) converges in \( C^1(D) \) to \( U_{(\tilde{c},\tilde{d})}(t,0)u_0 \).
Proof. (1) Proposition 3.3 and Eq. (3.2) imply that there is a subsequence \((n_k)_{k=1}^\infty\) such that \(U_{\widetilde{c}(n_k), \widetilde{d}(n_k)}(t_{n_k}, 0)u_{n_k}\) converges, as \(k \to \infty\), in \(V_p^\theta\) to some \(u^*\). Note that

\[
\|U_{\widetilde{c}, \widetilde{d}}(t_n, 0)u_0 - U_{\widetilde{c}, \widetilde{d}}(t, 0)u_0\| \to 0
\]

and

\[
\|U_{\widetilde{c}, \widetilde{d}}(t_n, 0)u_n - U_{\widetilde{c}, \widetilde{d}}(t_n, 0)u_0\| \to 0.
\]

By (A7) we have that

\[
\|U_{\widetilde{c}_n, \widetilde{d}_n}(t_n, 0)u_n - U_{\widetilde{c}, \widetilde{d}}(t_n, 0)u_n\| \to 0
\]

and hence

\[
\|U_{\widetilde{c}_n, \widetilde{d}_n}(t_n, 0)u_n - U_{\widetilde{c}, \widetilde{d}}(t, 0)u_0\| \to 0
\]
as \(n \to \infty\). As \(V_p^\theta\) embeds continuously in \(L_2(D)\), we must have \(u^* = U_{\widetilde{c}, \widetilde{d}}(t, 0)u_0\) and the sequence \(U_{\widetilde{c}(n), \widetilde{d}(n)}(t_n, 0)u_n\) converges, as \(n \to \infty\), in \(V_p^\theta\), to \(U_{\widetilde{c}, \widetilde{d}}(t, 0)u_0\).

(2) It follows by (1) and Eq. (3.1).

Proposition 3.7 (Norm continuity). (1) Let \(1 < p < \infty\) and \(2\theta - 1/p \not\in \mathbb{N}\).

The mapping

\[
[Y \times (0, \infty) \ni ((\widetilde{c}, \widetilde{d}), t) \mapsto U_{\widetilde{c}, \widetilde{d}}(t, 0) \in \mathcal{L}(L_2(D), V_p^\theta)]
\]
is continuous.

(2) The mapping

\[
[Y \times (0, \infty) \ni ((\widetilde{c}, \widetilde{d}), t) \mapsto U_{\widetilde{c}, \widetilde{d}}(t, 0) \in \mathcal{L}(L_2(D), C^1(D))]\]
is continuous. Moreover, for any \(t > 0\) and any \((\widetilde{c}, \widetilde{d}) \in Y\) the linear operator \(U_{\widetilde{c}, \widetilde{d}}(t, 0)\) is compact (completely continuous).

Proof. (1) Assume that \((\widetilde{c}(n), \widetilde{d}(n))\) converges to \((\widetilde{c}, \widetilde{d})\) in \(Y\) and that \(t_n\) converges to \(t > 0\). Suppose to the contrary that

\[
\|U_{\widetilde{c}(n), \widetilde{d}(n)}(t_n, 0) - U_{\widetilde{c}, \widetilde{d}}(t, 0)\|_{L_2(D), V_p^\theta} \neq 0
\]
as \(n \to \infty\). Then there are \(\epsilon_0 > 0\) and a sequence \((u_n)_{n=1}^\infty \subset L_2(D)\) with \(\|u_n\| = 1\) such that

\[
\|U_{\widetilde{c}(n), \widetilde{d}(n)}(t_n, 0)u_n - U_{\widetilde{c}, \widetilde{d}}(t, 0)u_n\|_{V_p^\theta} \geq \epsilon_0
\]
for all \(n\). By Proposition 3.3 there are \(u^*, u^{**} \in V_p^\theta\) such that (after possibly extracting a subsequence)

\[
U_{\widetilde{c}(n), \widetilde{d}(n)}(t_n, 0)u_n \to u^*
\]
and

\[
U_{\widetilde{c}, \widetilde{d}}(t, 0)u_n \to u^{**}
\]
in \(V_p^\theta\), as \(n \to \infty\). Without loss of generality, we may assume that there is \(\tilde{u}^* \in V_p^\theta\) such that

\[
U_{\widetilde{c}, \widetilde{d}}(t/2, 0)u_n \to \tilde{u}^*
\]
in $V^\theta_p$ as $n \to \infty$. Then by Proposition 3.6 we have

$$
\| U_{(\bar{c},\bar{d})}(t_n,0) u_n - U_{(\bar{c},\bar{d})}(t,0) u_n \| = \| U_{(\bar{c},\bar{d})}^{t/2}(t_n - t/2,0) U_{(\bar{c},\bar{d})}^{t} U_{(\bar{c},\bar{d})}^{-t/2}(t/2,0) u_n - U_{(\bar{c},\bar{d})}^{t/2}(t/2,0) \| = 0
$$

as $n \to \infty$. By the property (A7) we have

$$
\| U_{(\bar{c},\bar{d})}(t_n,0) - U_{(\bar{c},\bar{d})}(t,0) \| \to 0
$$

as $n \to \infty$. Then we must have $u^*_n = u^*$, hence

$$
\| U_{(\bar{c},\bar{d})}(t_n,0) u_n - U_{(\bar{c},\bar{d})}(t,0) u_n \| \to 0
$$

as $n \to \infty$, a contradiction.}

We are now ready to construct the skew-product semiflow on $X$ (as in (1.7)) generated by (1.1) or (1.2). For $t \geq 0$, $(\bar{c}, \bar{d}) \in Y$, $u_0 \in X$, put

$$
\Pi_t(u_0,(\bar{c}, \bar{d})) = \Pi(t; u_0, (\bar{c}, \bar{d})) := (U_{(\bar{c},\bar{d})}(t,0) u_0, (\bar{c}, \bar{d}) \cdot t).
$$

(3.10)

$\Pi = \{ \Pi_t \}_{t \geq 0}$ satisfies the usual algebraic properties of a semiflow on $X$ : $\Pi_0$ equals the identity on $X$, and $\Pi_s \circ \Pi_t = \Pi_{s+t}$ for any $s, t \geq 0$. Moreover, the continuity of $\Pi$ restricted to $(0, \infty) \times X \times Y$ follows by Proposition 3.6 and the embedding $X \hookrightarrow L_2(D)$. (However, we need not have continuity at $t = 0$.)

Sometimes we write $U_{(\bar{c},\bar{d})}(t,s)$ instead of $U_{(\bar{c},\bar{d})}(t-s,0)$, $s \leq t$. The semigroup property $\Pi_s \circ \Pi_t = \Pi_{s+t}$ takes in that notation the following form (see the cocycle property (3.9)):

$$
U_{(\bar{c},\bar{d})}(t,r) = U_{(\bar{c},\bar{d})}(t,s) U_{(\bar{c},\bar{d})}(s,r), \quad r \leq s \leq t.
$$

(3.11)

**Proposition 3.8** (Continuity in $C(D)$ at $t = 0$). Let $\theta \in (1/2, 1)$ and $p > 1$ be such that $2\theta - p \notin \mathbb{N}$ and $V^\theta_p \to C(D)$. Then for any $(\bar{c}, \bar{d}) \in Y$ and $u_0 \in V^\theta_p(\bar{d})$,

$$
\| U_{(\bar{c},\bar{d})}(t,0) u_0 - u_0 \|_{C(D)} \to 0
$$

as $t \to 0$.

**Proof.** It follows from [2] Theorem 15.1 and Eq. (3.1). 

Throughout the rest of this paper, we assume (A1)–(A7).

### 4 Strong monotonicity and globally positive solutions

In this section, we first show that the skew-product semiflow $\Pi_t$ constructed in the previous section is strongly monotone and then show that (1.5) has a unique globally positive solution, which will be used in next section to define the principal spectrum and principal Lyapunov exponent of (1.1) and (1.2).
Let $X$ be as in \eqref{1.7}. The Banach space $X$ is ordered by the standard cone

$$X^+ := \{ u \in X : u(x) \geq 0 \text{ for each } x \in D \}.$$  

The interior $X^{++}$ of $X^+$ is nonempty, where

$$X^{++} = \{ u \in X : u(x) > 0 \text{ for } x \in D \text{ and } (\partial u/\partial \nu)(x) < 0 \text{ for } x \in \partial D \}$$

for the Dirichlet boundary conditions, and

$$X^{++} = \{ u \in X : u(x) > 0 \text{ for } x \in \bar{D} \}$$

for the Neumann or Robin boundary conditions.

For $u_1, u_2 \in X$, we write $u_1 \leq u_2$ if $u_2 - u_1 \in X^+$, $u_1 < u_2$ if $u_1 \leq u_2$ and $u_1 \neq u_2$, and $u_1 \ll u_2$ if $u_2 - u_1 \in X^{++}$. The symbols $\geq$, $>$ and $\gg$ are used in the standard way.

We proceed now to investigate the strong monotonicity property of the solution operator $U_{(\tilde{c},\tilde{d})}(t,0)$. When the equations \eqref{1.1} and \eqref{1.2} are in divergence form, the monotonicity of $U_{(\tilde{c},\tilde{d})}(t,0)$ follows from \cite[Theorem 11.6]{5}. But the strong monotonicity is not included in \cite[Theorem 11.6]{5}. Though the monotonicity for equations in non-divergence form can also be proved by \cite[Theorem 11.6]{5} after verifying certain conditions, however for convenience we will give a proof for the monotonicity directly. We will prove the strong monotonicity by using the strong maximum principle and the Hopf boundary point principle for classical solutions. But before we do that we have to analyze whether the existing theory (as presented, e.g., in \cite{14}) can be applied: notice that in the Robin case $\theta$ may change sign. We show that coefficient can be made nonnegative by an appropriate change of variables.

Indeed, consider

\begin{equation}
\begin{aligned}
\frac{\partial u^*}{\partial t} &= \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u^*}{\partial x_i \partial x_j}, & t > -1, \ x \in D, \\
\sum_{i=1}^{N} b_i(x) \frac{\partial u^*}{\partial x_i} + u^* &= 0, & t > -1, \ x \in \partial D.
\end{aligned}
\tag{4.1}
\end{equation}

Let $p > 1$ and $\theta \in (1/2, 1)$ be as in Proposition \ref{3.8}. By the $C^\infty$ Urysohn Lemma (see \cite[Lemma 8.18]{13}), there is a nonzero $C^\infty$ function $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $0 \leq u_0 \leq 1$ on $D$ and supp $u_0 \Subset D$. Then $u_0 \in V_p^\theta(1)$. Let $u^*(t,x)$ be the solution of \eqref{4.1} with $u^*(-1,x) = u_0(x)$. By Proposition \ref{3.8}

$$\|u^*(t, \cdot) - u_0\|_{C^\infty(D)} \to 0 \quad \text{as} \quad t \to -1^+.$$ 

Hence, the function $u^*$ is continuous on $[-1, \infty) \times \bar{D}$ and satisfies, by Proposition \ref{3.5}, the equation in \eqref{4.1} pointwise on $(-1, \infty) \times D$ and the boundary condition in \eqref{4.1} pointwise on $(-1, \infty) \times \partial D$. Consequently, it follows from the strong maximum principle and the Hopf boundary point principle for parabolic equations that $u^*(t,x) > 0$ for all $t > -1$ and all $x \in \bar{D}$.

Now, let $v(t,x) := e^{M\tau^*(t,x)} u(t,x)$, where $M$ is a positive constant (to be
determined later). Then (1.5) becomes
\[
\begin{cases}
\frac{\partial v}{\partial t} = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial v}{\partial x_i} + \bar{c}(t,x)v, & t > 0, x \in D, \\
\sum_{i=1}^{N} b_i(x) \frac{\partial v}{\partial x_i} + \bar{d}(t,x)v = 0 & t > 0, x \in \partial D,
\end{cases}
\]

where
\[
\bar{a}_i(x) := a_i(x) - M \left( \sum_{j=1}^{N} \left( a_{ij}(x) \frac{\partial u^*}{\partial x_j} + a_{ii}(x) \frac{\partial u^*}{\partial x_i} \right) \right),
\]
\[
\bar{c}(t,x) := \bar{c}(t,x) - M \sum_{i=1}^{N} a_i(t,x) \frac{\partial u^*}{\partial x_i} + M^2 \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u^*}{\partial x_i} \frac{\partial u^*}{\partial x_j},
\]
\[
\bar{d}(t,x) := \bar{d}(t,x) + M u^*(t,x).
\]

We see that for any \((\bar{c}, \bar{d}) \in Y\) and any \(T > 0\), there is \(M = M(T) > 0\) such that \(\bar{d}(t,x) > 0\) for \(t \in [0,T]\) and \(x \in D\). Observe that, since the mapping \([0, \infty) \ni t \mapsto u^*(t, \cdot) \in C^1(\bar{D})\) is continuous by Proposition 3.6, the coefficients \(\bar{a}_i\) and \(\bar{c}\) are bounded on \([0, T] \times \bar{D}\) and the coefficient \(\bar{d}\) is bounded on \([0, T] \times \partial D\).

Consequently, we have the following result.

**Theorem 4.1 (Strong monotonicity).** Let \(u_1, u_2 \in L_2(D)\). If \(u_1 \neq u_2\) and \(u_1(x) \leq u_2(x)\) for a.e. \(x \in D\), then

(i) \(\langle U(\bar{c}, \bar{d})(t,0)u_1(x) \rangle < \langle U(\bar{c}, \bar{d})(t,0)u_2(x) \rangle\) for \((\bar{c}, \bar{d}) \in Y\), \(t > 0\) and \(x \in D\) and

\[
\frac{\partial}{\partial \nu}(U(\bar{c}, \bar{d})(t,0)u_1(x)) > \frac{\partial}{\partial \nu}(U(\bar{c}, \bar{d})(t,0)u_2(x))\]

in the Dirichlet case,

(ii) \(\langle U(\bar{c}, \bar{d})(t,0)u_1(x) \rangle < \langle U(\bar{c}, \bar{d})(t,0)u_2(x) \rangle\) for \((\bar{c}, \bar{d}) \in Y\), \(t > 0\) and \(x \in D\)

in the Neumann or Robin case.

**Proof.** Fix \((\bar{c}, \bar{d}) \in Y\). Assume that \(u_1, u_2 \in X\) and \(u_1 < u_2\). For any given \(T > 0\), in the case of the Robin boundary conditions let \(M > 0\) be such that \(\bar{d}(t,x) > 0\) for all \(t \in [0,T]\) and all \(x \in D\), where \(\bar{d}\) is as in the reasoning above the statement of the present proposition (in the case of the Dirichlet or Neumann boundary conditions put \(M = 0\)). Define \(v_0(x) := e^{Mu^*(0,x)}(u_2(x) - u_1(x))\), \(x \in D\).

Let \(\theta \in (1/2, 1)\) and \(p > 1\) be as in Proposition 3.8. We claim that there is a sequence \((v^{(n)})_{n=1}^{\infty} \subset V_p(\bar{d})\) such that \(v^{(n)}(x) \geq 0\) for all \(n = 1, 2, \ldots, x \in D\), \(v^{(n)} \neq 0\) for all \(n = 1, 2, \ldots,\) and \(\lim_{n \to \infty} \|v^{(n)} - v_0\| = 0\).
First note that there is a sequence \((v_0^{(n)})_{n=1}^\infty\) of simple functions such that
\[
0 \leq v_0^{(1)}(x) \leq v_0^{(2)}(x) \leq \cdots \leq v_0(x) \quad \text{for a.e. } x \in D,
\]
and
\[
v_0^{(n)}(x) \to v_0(x) \quad \text{as } n \to \infty, \quad \text{for a.e. } x \in D,
\]
and \(v_0^{(n)} \to v_0\) uniformly on any set on which \(v_0\) is bounded. It is therefore sufficient to prove the claim for the case that \(v_0 = \chi_E\), where \(E \subset D\) is a Lebesgue measurable set.

Now assume \(v_0 = \chi_E\), where \(E \subset D\) is a Lebesgue measurable set. For \(\epsilon_n := \frac{1}{2^n}\), choose a compact set \(K \subset E\) and an open set \(U \supset K\) such that \(U \subset D\), \(|E \setminus K| < \epsilon_n\) and \(|U \setminus K| < \epsilon_n\), where here \(|\cdot|\) denotes the Lebesgue measure of a set. Then, by the \(C^\infty\) Urysohn Lemma (see [13, Lemma 8.18]), there is a \(C^\infty\) function \(v^{(n)}: \mathbb{R}^N \to \mathbb{R}\) such that \(0 \leq v^{(n)} \leq 1\) on \(D\), \(v^{(n)} \equiv 1\) on \(K\) and \(\text{supp } v^{(n)} \subset U\). It then follows that
\[
\|v^{(n)} - v_0\| \leq |U \setminus K|^{1/2} + |E \setminus K|^{1/2} < \frac{1}{n} \to 0
\]
as \(n \to \infty\). Moreover, since \(\text{supp } v^{(n)} \subset U \subset D\), we also have \(v^{(n)} \in V^p_D\). The claim is thus proved. Denote by \(v(t, \cdot; v_0)\) and \(v(t, \cdot; v^{(n)})\) the solutions of (4.2) with \(v(0, \cdot; v_0) = v_0(\cdot)\) and \(v(0, \cdot; v^{(n)}) = v^{(n)}(\cdot) (n = 1, 2, \ldots)\), respectively.

By Proposition 3.8
\[
\|v(t, \cdot; v^{(n)}) - v^{(n)}\|_{C(D)} \to 0
\]
as \(t \to 0^+\). We can thus apply the strong comparison principle for parabolic equations to conclude that
\[
v(t, x; v^{(n)}) > 0 \quad \text{for } t \in (0, T], \ x \in D, \ n = 1, 2, \ldots
\]
This together with Proposition 3.7 implies that
\[
v(t, x; v_0) \geq 0 \quad \text{for } t \in (0, T], \ x \in D.
\]

By Proposition 3.5 for any \(n = 2, 3, \ldots\) the function \(v(\cdot, \cdot; v_0)\) is continuous on \([T/n, T]\), satisfies the equation in (1.2) pointwise on \((T/n, T] \times D\) and satisfies the boundary condition in (1.2) pointwise on \((T/n, T] \times \partial D\). Further, from Proposition 3.2 and the nonnegativity of \(v\) it follows that for \(n\) sufficiently large there is \(x_n \in D\) such that
\[
v(T/n, x_n; v_0) > 0.
\]
An application of the strong maximum principle for parabolic equations gives \(v(t, x; v_0) > 0\) for each \(t \in (0, T]\) and each \(x \in D\).

In the Dirichlet boundary condition case, suppose to the contrary that there are \(t^* \in (0, T]\) and \(x^* \in \partial D\) such that \(\partial赧\)(\(v(t^*, x^*; v_0)\)) = 0. But this contradicts the Hopf boundary point principle applied to \(v\) restricted to \([t^/^2, t^*] \times D\). Hence \(\partial赧\)(\(v(t, x; v_0)\)) < 0 for any \(t \in (0, T]\) and any \(x \in \partial D\). This completes the proof in that case, since \(v(t, x; v_0) = (U_{\neq_0}(t, 0)u_2)(x) - (U_{\neq_0}(t, 0)u_1)(x)\) for any \(t \in (0, T]\) and \(x \in D\).

Suppose to the contrary that, in the Neumann or Robin boundary condition case, there are \(t^* \in (0, T]\) and \(x^* \in \partial D\) such that \(v(t^*, x^*; v_0) = 0\). It follows from the Hopf boundary point principle (applied to \(v\) restricted to
and $x_0$, so $\tilde{Y}$ cases. This should not cause any misunderstanding.

For each $\tilde{c}, \tilde{d} \in Y$, $u_1, u_2 \in X$ and $t > 0$, if $u_1 < u_2$ then $U_{(\tilde{c},\tilde{d})}(t,0)u_1 \leq U_{(\tilde{c},\tilde{d})}(t,0)u_2$. The theory of existence and uniqueness of globally positive solutions can then be extended to our case. Below, we collect its basic concepts and facts.

**Theorem 4.2.** There are constants $\beta > 0$ such that for any $\tilde{c}, \tilde{d} \in Y$, $\tilde{w} \in X$, and each $t \in \mathbb{R}$, $w((\tilde{c}, \tilde{d}))(x) > 0$ for a.e. $x \in D$.

We shall consider now the problem of existence of globally positive solutions.

**Theorem 4.2.** There exist

- a continuous function $w : Y \rightarrow X^+$, $\|w((\tilde{c}, \tilde{d}))\| = 1$ for each $(\tilde{c}, \tilde{d}) \in Y$, and
- a continuous function $w^* : Y \rightarrow L_2(D)$, $\|w^*((\tilde{c}, \tilde{d}))\| = 1$ for each $(\tilde{c}, \tilde{d}) \in Y$, and such that for each $(\tilde{c}, \tilde{d}) \in Y$, $w^*((\tilde{c}, \tilde{d}))(x) > 0$ for a.e. $x \in D$;

having the following properties:

(i) For each $(\tilde{c}, \tilde{d}) \in Y$ the function $v_{(\tilde{c}, \tilde{d})} = v(t, x; \tilde{c}, \tilde{d})$ given by

$$v(t, \cdot; \tilde{c}, \tilde{d}) := \begin{cases} U_{(\tilde{c}, \tilde{d})}(t,0)w((\tilde{c}, \tilde{d})) & \text{for } t \geq 0, \\ \frac{w((\tilde{c}, \tilde{d}) \cdot t)}{\|U_{(\tilde{c}, \tilde{d})}(t,0)w((\tilde{c}, \tilde{d}) \cdot t)\|} & \text{for } t < 0, \end{cases}$$  \tag{4.3}

is a globally positive solution of \eqref{1.3}.

(ii) Let, for some $(\tilde{c}, \tilde{d}) \in Y$, $v = v(t, x)$ be a globally positive solution of \eqref{1.3}. Then there exists a constant $\beta > 0$ such that $v(t, x) = \beta v(t, x; \tilde{c}, \tilde{d})$ for each $t \in \mathbb{R}$ and each $x \in D$.

(iii) There are constants $C > 0$ and $\mu > 0$ such that

$$\|U_{(\tilde{c}, \tilde{d})}(t,0)w_{0}\| \leq Ce^{-\mu t}\|U_{(\tilde{c}, \tilde{d})}(t,0)w((\tilde{c}, \tilde{d}))\|$$  \tag{4.4}

for any $(\tilde{c}, \tilde{d}) \in Y$, $t > 0$ and $w_{0} \in L_2(D)$ with $\|w_{0}\| = 1$ and $\langle w_{0}, w^*((\tilde{c}, \tilde{d})) \rangle = 0$.

(iv) There are constants $C' > 0$ and $\mu > 0$ such that

$$\|U_{(\tilde{c}, \tilde{d})}(t,0)w_{0}\| x \leq C'e^{-\mu t}\|U_{(\tilde{c}, \tilde{d})}(t,0)w((\tilde{c}, \tilde{d}))\| x$$  \tag{4.5}

for any $(\tilde{c}, \tilde{d}) \in Y$, $t \geq 1$ and $w_{0} \in L_2(D)$ with $\|w_{0}\| = 1$ and $\langle w_{0}, w^*((\tilde{c}, \tilde{d})) \rangle = 0$.  

---

\(23\)
Proof. We start by considering a discrete-time dynamical system on the product bundle $X \times Y$ (where $X$ is a fiber, $Y$ is the base space):

$$\Pi_0(u_0,(\tilde{c}, \tilde{d})) := (U_{(\tilde{c}, \tilde{d})}(n,0)u_0,(\tilde{c}, \tilde{d}) \cdot n), \quad u_0 \in X, (\tilde{c}, \tilde{d}) \in Y, \ n = 1, 2, 3, \ldots \tag{4.6}$$

Proposition [57] and Theorem [41] allow us to use the results contained in [39] to conclude that there are continuous functions $\tilde{w}: Y \to X$, $\tilde{w}^*: Y \to X^*$, $\|\tilde{w}((\tilde{c}, \tilde{d}))\|_X = \|\tilde{w}^*((\tilde{c}, \tilde{d}))\|_{X^*} = 1$ for each $\tilde{a} \in Y$, such that (we write $X_1((\tilde{c}, \tilde{d})) := \text{span } \tilde{w}((\tilde{c}, \tilde{d}))$, $X_2((\tilde{c}, \tilde{d})) := N(\tilde{w}^*((\tilde{c}, \tilde{d})))$, where $N$ stands for the nullspace of an element of $X^*$)

(a) $\tilde{w}((\tilde{c}, \tilde{d})) \in X^{++}$, for each $(\tilde{c}, \tilde{d}) \in Y$.

(b) $(v, \tilde{w}^*((\tilde{c}, \tilde{d})))_{X \cdot X^*} > 0$ for each $(\tilde{c}, \tilde{d}) \in X$ and each nonzero $v \in X^*$. It follows that $X_2((\tilde{c}, \tilde{d}) \cap X^* = \{0\}$, for each $(\tilde{c}, \tilde{d}) \in Y$.

(c) For each $(\tilde{c}, \tilde{d}) \in Y$ there is $d_1 = d_1((\tilde{c}, \tilde{d})) > 0$ such that $U_{(\tilde{c}, \tilde{d})}(1,0)\tilde{w}((\tilde{c}, \tilde{d})) = (\tilde{c}, \tilde{d}) \cdot 1$. It follows that $U_{(\tilde{c}, \tilde{d})}(1,0)X_1((\tilde{c}, \tilde{d})) = X_1((\tilde{c}, \tilde{d}) \cdot 1)$.

(d) For each $(\tilde{c}, \tilde{d}) \in Y$ there is $d_1^* = d_1^*((\tilde{c}, \tilde{d})) > 0$ such that 

$$\langle U_{(\tilde{c}, \tilde{d})}(1,0)^*\tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1) = d_1^*\tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1), \quad \text{where } (U_{(\tilde{c}, \tilde{d})}(1,0))^*: X^* \to X^* \text{ stands for the linear operator dual to } U_{(\tilde{c}, \tilde{d})}(1,0).$$

It follows that $X_2((\tilde{c}, \tilde{d})) \subset X_2((\tilde{c}, \tilde{d}) \cdot 1)$, for any $(\tilde{c}, \tilde{d}) \in Y$.

(e) There are constants $\tilde{C} > 0$ and $0 < \gamma < 1$ such that

$$\|U_{(\tilde{c}, \tilde{d})}(n,0)u_0\|_X \leq \tilde{C}\gamma^n\|U_{(\tilde{c}, \tilde{d})}(n,0)\tilde{w}((\tilde{c}, \tilde{d}))\|_X \tag{4.7}$$

for any $(\tilde{c}, \tilde{d}) \in Y$, any $u_0 \in X_2((\tilde{c}, \tilde{d}) \cdot 1)$, for any $n, u_0 \in X_2((\tilde{c}, \tilde{d}) \cdot 1)$ with $\|u_0\|_X = 1$ and any $n \in \mathbb{N}$.

Put $w((\tilde{c}, \tilde{d})) := \tilde{w}((\tilde{c}, \tilde{d}))/\|\tilde{w}((\tilde{c}, \tilde{d}))\|$, $(\tilde{c}, \tilde{d}) \in Y$. As $X$ embeds continuously in $L_2(D)$, the function $w: Y \to X$ is continuous. Further, put $w^*((\tilde{c}, \tilde{d})) := \tilde{w}^*((\tilde{c}, \tilde{d}))/\|\tilde{w}^*((\tilde{c}, \tilde{d}))\|$, $(\tilde{c}, \tilde{d}) \in Y$. From Proposition [57] it follows that the mapping $\{Y \ni (\tilde{c}, \tilde{d}) \to (U_{(\tilde{c}, \tilde{d})}(1,0))^* \in L(X^*, L_2(D))\}$ is continuous, too, so we obtain with the help of (d) that $w^*: Y \to L_2(D)$ is well defined and continuous.

By the definition of the dual operator,

$$d_1^*((\tilde{c}, \tilde{d})) \cdot (v, \tilde{w}^*((\tilde{c}, \tilde{d})))_{X \cdot X^*} = (v, (U_{(\tilde{c}, \tilde{d})}(1,0))^*\tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1))_{X \cdot X^*} = (U_{(\tilde{c}, \tilde{d})}(1,0)v, \tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1))_{X \cdot X^*}$$

for each $(\tilde{c}, \tilde{d}) \in Y$ and each $v \in X$. As $\tilde{w}^*((\tilde{c}, \tilde{d}))$ is a bounded linear functional on $L_2(D)$ and $X$ is dense in $L_2(D)$, we conclude that

$$d_1^*((\tilde{c}, \tilde{d})) \cdot (v, \tilde{w}^*((\tilde{c}, \tilde{d}))) = (v, (U_{(\tilde{c}, \tilde{d})}(1,0))^*\tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1)) = (U_{(\tilde{c}, \tilde{d})}(1,0)v, \tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1))$$

for each $(\tilde{c}, \tilde{d}) \in Y$ and each $v \in L_2(D)$.

We prove now that $w^*((\tilde{c}, \tilde{d}))(x) > 0$ for a.e. $x \in D$, or, which is equivalent, that $\tilde{w}^*((\tilde{c}, \tilde{d}))(x) > 0$ for a.e. $x \in D$. Suppose first that for some $(\tilde{c}, \tilde{d}) \in X$ there are $D_+, D_- \subset D$ of positive Lebesgue measure such that
\[ \tilde{w}^*((\tilde{c}, \tilde{d}))(x) > 0 \text{ for } x \in D_+, \quad \tilde{w}^*((\tilde{c}, \tilde{d}))(x) < 0 \text{ for } x \in D_- \text{, and } \tilde{w}^*((\tilde{c}, \tilde{d}))(x) = 0 \text{ for } x \in D \setminus (D_+ \cup D_-). \]

Define \( v \in L_2(D) \) to be the simple function equal to \( 1/\int_{D_+} \tilde{w}^*((\tilde{c}, \tilde{d}))(x) \, dx \) on \( D_+ \), equal to \( -1/\int_{D_-} \tilde{w}^*((\tilde{c}, \tilde{d}))(x) \, dx \) on \( D_- \), and equal to zero elsewhere. We have

\[
0 = d_1^*((\tilde{c}, \tilde{d})) \cdot \langle v, \tilde{w}^*((\tilde{c}, \tilde{d})) \rangle = \langle v, (U_{(\tilde{c}, \tilde{d})}(1, 0))^* \tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1) \rangle = \langle U_{(\tilde{c}, \tilde{d})}(1, 0)v, \tilde{w}^*((\tilde{c}, \tilde{d}) \cdot 1) \rangle_{X,X^*}.
\]

By Theorem 4.11, \( U_{(\tilde{c}, \tilde{d})}(1, 0)v \in X^{++} \). This contradicts (b). Suppose now that for some \((\tilde{c}, \tilde{d}) \in X \) there are \( D_+, D_0 \subset D \) of positive Lebesgue measure such that \( \tilde{w}^*((\tilde{c}, \tilde{d}))(x) > 0 \) for \( x \in D_+ \) and \( \tilde{w}^*((\tilde{c}, \tilde{d}))(x) = 0 \) for \( x \in D_0 \), and the complement of the union \( D_+ \cup D_0 \) in \( D \) has Lebesgue measure zero. We repeat the above construction, this time with \( v \) equal to zero on \( D_+ \) and equal to one on \( D_0 \).

Fix \((\tilde{c}, \tilde{d}) \in Y \). The fact that if there exists a globally positive solution of (2.5) then it is unique up to multiplication by a positive constant is proved for the Dirichlet case in [21], and for the Neumann and Robin case in [17]. We proceed now to the construction of a globally positive solution.

We define first the trace of a positive solution \( v(t, x; \tilde{c}, \tilde{d}) \) on \( Z \):

\[
v(k, \cdot; \tilde{c}, \tilde{d}) := \begin{cases} 
U_{(\tilde{c}, \tilde{d})}(k, 0) w((\tilde{c}, \tilde{d})) & \text{for } k = 0, 1, 2, 3, \ldots, \\
\|U_{(\tilde{c}, \tilde{d})}(k, 0) w((\tilde{c}, \tilde{d}) \cdot k)\| & \text{for } k = \ldots, -3, -2, -1.
\end{cases}
\]

It follows from (a) and (c) that

\[
U_{(\tilde{c}, \tilde{d})}(l + k, k) v(k, \cdot; \tilde{c}, \tilde{d}) = v(k + l, \cdot; \tilde{c}, \tilde{d}) \tag{4.8}
\]

for any \( k \in Z \) and any nonnegative integer \( l \). Also, \( \|v(0, \cdot; \tilde{c}, \tilde{d})\| = 1 \). We extend \( v \) to a function defined on \((-\infty, \infty)\) by putting

\[
v(t, \cdot; \tilde{c}, \tilde{d}) := U_{(\tilde{c}, \tilde{d})}(t, [t]) v([t], \cdot; \tilde{c}, \tilde{d}), \quad t \in \mathbb{R} \setminus \mathbb{Z}, \tag{4.9}
\]

where \([t]\) denotes the greatest integer less than or equal to \( t \). To check that the function so defined is indeed a global solution we need to show that

\[
v(s + t, \cdot; \tilde{c}, \tilde{d}) = U_{(\tilde{c}, \tilde{d})}(s + t, t) v(t, \cdot; \tilde{c}, \tilde{d}) \quad \text{for any } t \in \mathbb{R} \text{ and any } s \geq 0 \tag{4.10}
\]

(see Definition 4.3 and Eq. 4.11). We write

\[
v(s + t, \cdot; \tilde{c}, \tilde{d}) = U_{(\tilde{c}, \tilde{d})}(s + t, [s + t]) v([s + t], \cdot; \tilde{c}, \tilde{d}) \quad \text{by (4.10)}
\]

\[
= U_{(\tilde{c}, \tilde{d})}(s + t, [s + t]) U_{(\tilde{c}, \tilde{d})}([s + t], [t]) v([t], \cdot; \tilde{c}, \tilde{d}) \quad \text{by (4.3)}
\]

\[
= U_{(\tilde{c}, \tilde{d})}(s + t, t) U_{(\tilde{c}, \tilde{d})}(t, [t]) v([t], \cdot; \tilde{c}, \tilde{d}) \quad \text{by (5.11)}
\]

\[
= U_{(\tilde{c}, \tilde{d})}(s + t, t) v(t, \cdot; \tilde{c}, \tilde{d}) \quad \text{by (4.9)}
\]

The fact that \((v(t, \cdot; \tilde{c}, \tilde{d}) \in X^{++} \) for each \( t \in \mathbb{R} \) is a consequence of the construction of \( v \) and of Theorem 4.11.

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Formula (4.3) for \( t > 0 \) is straightforward. It follows from the uniqueness of globally positive solutions that

\[
v(t, \cdot; \bar{c}, \bar{d}) = \|v(t, \cdot; \bar{c}, \bar{d})\| w((\bar{c}, \bar{d}) \cdot t), \quad t \in (-\infty, \infty).
\]

From (4.10) we obtain, for any \( t < 0 \), that

\[
1 = \|v(0, \cdot; \bar{c}, \bar{d})\| = \|U_{(\bar{c}, \bar{d})}(-t, 0)v(t, \cdot; \bar{c}, \bar{d})\| = \|v(t, \cdot; \bar{c}, \bar{d})\| \|U_{(\bar{c}, \bar{d})}(-t, 0)w((\bar{c}, \bar{d}) \cdot t)\|,
\]

which concludes the proof of formula (4.3).

We proceed now to the proof of part (iii). Denote by \( M_1 \) the norm of the embedding \( X \hookrightarrow L_2(D) \). Moreover, by the compactness of \( Y \) and the continuity of \( \tilde{w} \) there is \( M_2 > 0 \) such that \( \|\tilde{w}(\bar{c}, \bar{d})\|_X \leq M_2 \|w((\bar{c}, \bar{d}))\| \) for all \( (\bar{c}, \bar{d}) \in Y \).

Take \( u_0 \in L_2(D) \) such that \( \|u_0\| = 1 \) and \( \langle u_0, w^*((\bar{c}, \bar{d})) \rangle = 0 \). It follows from (d) that \( \langle U_{(\bar{c}, \bar{d})}(1, 0)u_0, w^*((\bar{c}, \bar{d}) \cdot 1) \rangle = 0 \). As \( U_{(\bar{c}, \bar{d})}(1, 0)u_0 \in X \), one has \( U_{(\bar{c}, \bar{d})}(1, 0)u_0 \in X_2(\tilde{a} \cdot 1) \). This allows us to estimate, for \( n = 2, 3, 4, \ldots \),

\[
\|U_{(\bar{c}, \bar{d})}(n, 0)u_0\| \leq M_1 \|U_{(\bar{c}, \bar{d})}(n, 1)(U_{(\bar{c}, \bar{d})}(1, 0)u_0)\|_X \leq M_1 \bar{C} \gamma^{n-1}\|U_{(\bar{c}, \bar{d})}(n, 1)\tilde{w}((\bar{c}, \bar{d}) \cdot 1)\|_X \|U_{(\bar{c}, \bar{d})}(1, 0)u_0\|_X \leq M_1 M_2 D_1 \bar{C} \gamma^n \|U_{(\bar{c}, \bar{d})}(n, 0)w((\bar{c}, \bar{d}))\|,
\]

where \( D_1 := \sup \{\|U_{(\bar{c}, \bar{d})}(1, 0)\|_{L_2(D), X} : (\bar{c}, \bar{d}) \in Y \} < \infty \), \( D_2 := \inf \{\|U_{(\bar{c}, \bar{d})}(1, 0)\|_{X} : (\bar{c}, \bar{d}) \in Y \} > 0 \). Clearly, \( \|U_{(\bar{c}, \bar{d})}(1, 0)u_0\| \leq \frac{M_1 M_2 D_2}{D_2} \|U_{(\bar{c}, \bar{d})}(1, 0)w((\bar{c}, \bar{d}))\| \) for all \( (\bar{c}, \bar{d}) \in Y \) and all \( u_0 \in L_2(D) \) with \( \|u_0\| = 1 \) and \( \langle u_0, w^*((\bar{c}, \bar{d})) \rangle = 0 \).

As a consequence we obtain the existence of \( \bar{C} = \frac{M_1 M_2 D_2}{D_2} \max \{\bar{C}, 1\} \) such that

\[
\|U_{(\bar{c}, \bar{d})}(n, 0)u_0\| \leq \bar{C} \gamma^n \|U_{(\bar{c}, \bar{d})}(n, 0)w((\bar{c}, \bar{d}))\| \quad \text{for any } (\bar{c}, \bar{d}) \in Y, \text{ any } n \in \mathbb{N} \text{ and any } u_0 \in L_2(D) \text{ satisfying } \|u_0\| = 1 \text{ and } \langle u_0, w^*((\bar{c}, \bar{d})) \rangle = 0.
\]

To show (4.4) we notice that

\[
\|U_{(\bar{c}, \bar{d})}(t, 0)u_0\| = \|U_{(\bar{c}, \bar{d})}(t, [t])(U_{(\bar{c}, \bar{d})}([t], 0)u_0)\| \leq D_3 \|U_{(\bar{c}, \bar{d})}([t], 0)u_0\| \leq D_3 \bar{C} \gamma^{|t|} \|U_{(\bar{c}, \bar{d})}([t], 0)w((\bar{c}, \bar{d}))\| \leq \frac{D_3 \bar{C}}{\gamma D_4} \|U_{(\bar{c}, \bar{d})}(1, 0)w((\bar{c}, \bar{d}))\| \quad \text{by (4.11)}
\]

for any \( (\bar{c}, \bar{d}) \in Y, t \geq 1 \) and any \( u_0 \in L_2(D) \) with \( \|u_0\| = 1 \) and \( \langle u_0, w^*((\bar{c}, \bar{d})) \rangle = 0 \), where \( D_3 := \sup \{\|U_{(\bar{c}, \bar{d})}(t, 0)\| : t \in [0, 1], (\bar{c}, \bar{d}) \in Y \} < \infty \) and \( D_4 := \inf \{\|U_{(\bar{c}, \bar{d})}(t, 0)\| : t \in [0, 1], (\bar{c}, \bar{d}) \in Y \} > 0 \).
Clearly, \( \|U_{\tilde{c},\tilde{d}}(t,0)u_0\| \leq \frac{D_2}{D_1^c} \|U_{\tilde{c},\tilde{d}}(t,0)w((\tilde{c},\tilde{d}))\| \) for all \((\tilde{c},\tilde{d}) \in Y\), all \(t \in [0,1]\) and all \(u_0 \in L_2(D)\) with \(\|u_0\| = 1\) and \(\langle u_0, w^*((\tilde{c},\tilde{d})) \rangle = 0\).

This proves (4.3), with \(C = \frac{D_2}{D_1^c}\) max\{\(C, 1\)\} and \(\mu = -\ln \lambda\).

To prove (4.5) we estimate, for \(u_0 \in L_2(D)\) with \(\|u_0\| = 1\) and \(\langle u_0, w^*((\tilde{c},\tilde{d})) \rangle = 0,\) and \(t \geq 1,
\[
\|U_{\tilde{c},\tilde{d}}(t,0)u_0\|_X = \|U_{\tilde{c},\tilde{d}}(t, t-1)(U_{\tilde{c},\tilde{d}}(t-1,0)u_0)\|_X \\
\leq D_1\|U_{\tilde{c},\tilde{d}}(t-1,0)u_0\|, \\
\leq D_1Ce^{-\mu(t-1)}\|U_0(t-1,0)w((\tilde{c},\tilde{d}))\| \\
\leq \frac{D_1}{D_5}Ce^{\mu}e^{-\mu t}\|U_{\tilde{c},\tilde{d}}(t,0)w((\tilde{c},\tilde{d}))\|_X ,
\]
where \(M_1 := \sup\{\|u\| : u \in X, \|u\|_X \leq 1\}\), \(D_1 := \sup\{\|U_{\tilde{c},\tilde{d}}(1,0)\|_{L_2(D)} : (\tilde{c},\tilde{d}) \in Y\}\) and \(D_5 := \inf\{\|U_{\tilde{c},\tilde{d}}(1,0)w((\tilde{c},\tilde{d}))\| : (\tilde{c},\tilde{d}) \in Y\}\).

For other approaches to the question of existence and/or uniqueness of globally positive solutions the reader can consult also [18], [19], [20], [28], [29], [35].

**Theorem 4.3.** (1) In the Dirichlet boundary condition case, there is \(M > 0\) such that

\[ w((\tilde{c},\tilde{d}))(x) \leq M \quad \text{for any } x \in D \text{ and any } (\tilde{c},\tilde{d}) \in Y. \]

Further, for each compact \(D_0 \subset D\) there is \(m = m(D_0) > 0\) such that

\[ w((\tilde{c},\tilde{d}))(x) \geq m(D_0) \quad \text{for any } x \in D_0 \text{ and any } (\tilde{c},\tilde{d}) \in Y. \]

(2) In the Neumann or Robin boundary condition case, there are \(M, m > 0\) such that

\[ m \leq w((\tilde{c},\tilde{d}))(x) \leq M \quad \text{for any } x \in D \text{ and any } (\tilde{c},\tilde{d}) \in Y. \]

**Proof.** It follows in a standard way from the compactness of \(Y\) and from the fact that \(w((\tilde{c},\tilde{d})) \in X^{++}\) for each \((\tilde{c},\tilde{d}) \in Y\).

**Theorem 4.4.** The first order derivatives of \(w\) are Hölder in \(x\) uniformly in \((\tilde{c},\tilde{d}) \in Y\) and in \(x \in D\), and the second order derivatives of \(w\) are Hölder in \(x\) uniformly in \((\tilde{c},\tilde{d}) \in Y\) and locally uniformly in \(x \in D\).

**Proof.** By Theorem 4.2 for each \((\tilde{c},\tilde{d}) \in Y\) there holds

\[ w((\tilde{c},\tilde{d})) = \frac{U_{\tilde{c},\tilde{d}}(-1)(1,0)w((\tilde{c},\tilde{d}) \cdot (-1))}{\|U_{\tilde{c},\tilde{d}}(-1)(1,0)w((\tilde{c},\tilde{d}) \cdot (-1))\|}. \]

It is a consequence of the continuity of \(w\), the compactness of \(Y\) and Proposition 5.1 that the denominators on the right-hand side are positive and bounded away from zero, uniformly in \(Y\). Now we apply the parabolic regularity estimates [14], Chapter 3.
5 Principal spectrum and principal Lyapunov exponent

In this section, we collect the basic concepts and facts about the principal spectrum and principal Lyapunov exponent of (1.1) and (1.2).

Definition 5.1. In case of (1.1) we define its principal spectrum to be the set of all limits

$$\lim_{n \to \infty} \ln \frac{\|U_{c,d}(T_n - S_n, 0)w((c, d) \cdot S_n)\|}{T_n - S_n},$$

where $T_n - S_n \to \infty$ as $n \to \infty$.

The following proposition follows from the results contained in [24] (cp., e.g., [30, Thm. 2.10]).

Theorem 5.1. The principal spectrum of (1.1) is a compact interval $[\lambda_{\text{inf}}(c, d), \lambda_{\text{sup}}(c, d)]$. Moreover, if $(c, d)$ is uniquely ergodic and minimal then $\lambda_{\text{inf}}(c, d) = \lambda_{\text{sup}}(c, d)$.

In the case of (1.2), for $\omega \in \Omega$ we write $U_\omega(t, 0)$ for $U_{(c, d)\cdot T_n}(t, 0)$ and $w(\omega)$ for $w((c, d))$.

Theorem 5.2. For (1.2), there exists $\lambda(c, d) \in \mathbb{R}$ such that

$$\lambda(c, d) = \lim_{T \to \infty} \frac{\ln \|U_\omega(T, 0)w(\omega)\|}{T}$$

for a.e. $\omega \in \Omega$.

Proof. It follows from subadditive ergodic theorems (see [25]). \qed

Definition 5.2. The $\lambda(c, d)$ as in Theorem 5.2 is called the principal Lyapunov exponent of (1.2).

Remark 5.1. In the existing literature, the principal spectrum is either defined precisely as in Definition 5.1 (see [30]) or with the $L_2(D)$-norm replaced by the norm in some fractional power space that embeds continuously into $C^1(\bar{D})$ (see, e.g., [31]). In our setting, as $X_1$ is a one-dimensional invariant subbundle spanned by a continuous function from $Y$ into $X$, we can replace the $L_2(D)$-norm in Definition 5.1 with the $X$-norm.

Remark 5.2. Similarly, in the Definition 5.2 the $L_2(D)$-norm can be replaced with the $X$-norm. Further, in [31] the principal Lyapunov exponent was introduced as the (a.e. constant) limit

$$\lim_{T \to \infty} \frac{\ln \|U_\omega(T, 0)\|_{X, X}}{T},$$

where $X$ is some fractional power space that embeds continuously into $C^1(\bar{D})$. With the help of (4.4) one can prove that for those $\omega \in \Omega$ for which $\lambda(c, d) = \lim_{T \to \infty} \frac{\ln \|U_\omega(T, 0)w(\omega)\|}{T}$ there holds also $\lambda(c, d) = \lim_{T \to \infty} \frac{\ln \|U_\omega(T, 0)\|}{T}$ (see the proof of [28 Thm. 3.2(2)]).
Remark 5.3. For the $L_2(D)$-theory of the principal spectrum and principal Lyapunov exponents see the upcoming monograph [32].

We introduce now a useful concept. For $(\tilde{c}, \tilde{d}) \in Y$ put

$$
\kappa((\tilde{c}, \tilde{d})) := \int \left( \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 w((\tilde{c}, \tilde{d}))(x)}{\partial x_i \partial x_j} \right) w((\tilde{c}, \tilde{d}))(x) \, dx
$$

$$
+ \int \left( \sum_{i=1}^{N} a_i(x) \frac{\partial w((\tilde{c}, \tilde{d}))(x)}{\partial x_i} + \tilde{c}(0, x)w((\tilde{c}, \tilde{d}))(x) \right) w((\tilde{c}, \tilde{d}))(x) \, dx.
$$

(5.1)

By Proposition 5.3, $w((\tilde{c}, \tilde{d})) \in W_2^2(D)$, so $\kappa((\tilde{c}, \tilde{d}))$ is well defined.

The function $\kappa: Y \to \mathbb{R}$ is continuous. Indeed, notice that applying integration by parts we can write

$$
\kappa((\tilde{c}, \tilde{d})) = -\int \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial w((\tilde{c}, \tilde{d}))(x)}{\partial x_i} \frac{\partial w((\tilde{c}, \tilde{d}))(x)}{\partial x_j} \, dx
$$

$$
- \int \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_i} \frac{\partial w((\tilde{c}, \tilde{d}))(x)}{\partial x_j} \right) w((\tilde{c}, \tilde{d}))(x) \, dx
$$

$$
+ \int \left( \sum_{i=1}^{N} a_i(x) \frac{\partial w((\tilde{c}, \tilde{d}))(x)}{\partial x_i} + \tilde{c}(0, x)w((\tilde{c}, \tilde{d}))(x) \right) w((\tilde{c}, \tilde{d}))(x) \, dx
$$

$$
+ \int \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(x) \frac{\partial w((\tilde{c}, \tilde{d}))(x)}{\partial x_j} \, dx((\tilde{c}, \tilde{d}))(x) \nu_i(x) \, dS. \quad (5.2)
$$

As $w: Y \to X$ is continuous, the above expression depends continuously on $(\tilde{c}, \tilde{d})$, too.

We point out that the function $\kappa((\tilde{c}, \tilde{d}))$ introduced in (5.1) is a very useful quantity in the investigation of various properties of principal spectrum and principal Lyapunov exponents. This quantity will be heavily used in next section. In the rest of this section, we discuss how to use the function $\kappa$ to characterize the principal spectrum and principal Lyapunov exponents.

Let $\eta((\tilde{c}, \tilde{d}))(t) := \|U((\tilde{c}, \tilde{d}))(t, 0)w((\tilde{c}, \tilde{d}))\| > 0$. Then $\eta((\tilde{c}, \tilde{d}))(t)$ is differentiable and $U((\tilde{c}, \tilde{d}))(t, 0)w((\tilde{c}, \tilde{d})) = \eta((\tilde{c}, \tilde{d}))(t)w((\tilde{c}, \tilde{d}) \cdot t)$. Hence $w((\tilde{c}, \tilde{d}) \cdot t)$ is also differentiable in $t$. By (5.1), we have

$$
\frac{d}{dt} \eta((\tilde{c}, \tilde{d}))(t) w((\tilde{c}, \tilde{d}) \cdot t) + \eta((\tilde{c}, \tilde{d}))(t) \frac{\partial}{\partial t} w((\tilde{c}, \tilde{d}) \cdot t)
$$

$$
= \sum_{i,j=1}^{N} a_{ij}(x) \eta((\tilde{c}, \tilde{d}))(t) \frac{\partial^2 w((\tilde{c}, \tilde{d}) \cdot t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \eta((\tilde{c}, \tilde{d}))(t) \frac{\partial w((\tilde{c}, \tilde{d}) \cdot t)}{\partial x_i}
$$

$$
+ \tilde{c}(t, x) \eta((\tilde{c}, \tilde{d}))(t) w((\tilde{c}, \tilde{d}) \cdot t).
$$

Taking the inner product of the above equation with $w((\tilde{c}, \tilde{d}) \cdot t)$ and observing that $\langle w((\tilde{c}, \tilde{d}) \cdot t), w((\tilde{c}, \tilde{d}) \cdot t) \rangle \equiv 1$ and $(\frac{d}{dt} w((\tilde{c}, \tilde{d}) \cdot t), w((\tilde{c}, \tilde{d}) \cdot t)) \equiv 0$ we get
\[ \dot{\eta}(\hat{c}, \hat{d})(t) = \kappa((\hat{c}, \hat{d}) \cdot t)\eta(\hat{c}, \hat{d})(t), \]
that is,
\[
\frac{d}{dt} \| U(\hat{c}, \hat{d})(t, 0)w((\hat{c}, \hat{d})) \| = \kappa((\hat{c}, \hat{d}) \cdot t)\| U(\hat{c}, \hat{d})(t, 0)w((\hat{c}, \hat{d})) \| \tag{5.3}
\]
for any \((\hat{c}, \hat{d}) \in Y\) and any \(t \geq 0\).

By (5.3), we have
\[
\ln \| U(\hat{c}, \hat{d}) \cdot S(T - S, 0)w((\hat{c}, \hat{d}) \cdot S) \| = \int_{S}^{T} \kappa((\hat{c}, \hat{d}) \cdot t) dt \tag{5.4}
\]
for any \((\hat{c}, \hat{d}) \in Y\) and \(S < T\). Then following from Definition 5.1 we have

**Theorem 5.3.** Let \([\lambda_{\inf}(c, d), \lambda_{\sup}(c, d)]\) be the principal spectrum interval of (1.1). Then
\[
\lambda_{\inf}(c, d) = \liminf_{T - S \to \infty} \frac{1}{T - S} \int_{S}^{T} \kappa((c, d) \cdot t) dt \tag{5.5}
\]
and
\[
\lambda_{\sup}(c, d) = \limsup_{T - S \to \infty} \frac{1}{T - S} \int_{S}^{T} \kappa((c, d) \cdot t) dt. \tag{5.6}
\]

In the case of (1.2) we write \(\kappa(\omega)\) instead of \(\kappa((c^\omega, d^\omega))\). We have

**Theorem 5.4.** Consider (1.2). Then
\[
\lambda = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \kappa(\theta_{t} \omega) dt = \int_{\Omega} \kappa(\cdot) d\mathbb{P}(\cdot)
\]
for a.e. \(\omega \in \Omega\).

**Proof.** By the arguments of Lemma 3.4 in [31], the map \([\Omega \ni \omega \mapsto (c^\omega, d^\omega) \in Y]\) is measurable. The theorem is then a consequence of Theorem 5.2, Eq. (5.4) and Birkhoff’s Ergodic Theorem (Lemma 2.2).

We remark that if \(c(t, x)\) and \(d(t, x)\) are independent of \(t\), then \((c, d) = (\hat{c}, \hat{d})\) and \(\lambda_{\inf}(c, d) = \lambda_{\sup}(c, d) = \lambda(c, d)\). Moreover, we have the following easy theorem about the continuous dependence of \(\lambda(c, d)\) on \((c, d)\).

**Theorem 5.5.** If \(c^{(n)}\) converges in \(C(\overline{D})\) to \(c\) and \(d^{(n)}\) converges in \(C(\partial D)\) to \(d\) then \(\lambda(c^{(n)}, d^{(n)}) \to \lambda(c, d)\).

### 6 Time averaging

In this section we state and prove our results on the influence of time variations on principal spectrum and principal Lyapunov exponent of (1.1) and (1.2).

Consider (1.1). Let \(\Sigma(c, d) := [\lambda_{\inf}(c, d), \lambda_{\sup}(c, d)]\) be the principal spectrum interval of (1.1). For \((\hat{c}, \hat{d}) \in \hat{Y}(c, d)\) let \(\hat{\lambda}(\hat{c}, \hat{d})\) denote the principal eigenvalue of an averaged equation (1.10). Recall that
\[
\hat{Y}(c, d) = \{ (\hat{c}, \hat{d}) : \exists S_n < T_n \text{ with } T_n - S_n \to \infty \text{ such that } \\
(\hat{c}, \hat{d}) = \lim_{n \to \infty} (\hat{c}(\cdot; S_n, T_n), d(\cdot; S_n, T_n)) \}
\]
where \( \hat{c}(x; S_n, T_n) := \frac{1}{T_n - s_n} \int_{s_n}^{T_n} c(t, x) \, dt \), \( \hat{d}(x; S_n, T_n) := \frac{1}{T_n - s_n} \int_{s_n}^{T_n} d(t, x) \, dt \), and the convergence is in \( C(D) \times C(\partial D) \).

Consider (12). Let \( \hat{\lambda}(c, d) \) be the principal Lyapunov exponent. Let

\[
\hat{c}(x) := \int_{\Omega} c(\omega, x) \, d\mathbb{P}(\omega), \quad \hat{d}(x) = \int_{\Omega} d(\omega, x) \, d\mathbb{P}(\omega).
\]

Let \( \lambda(\hat{c}, \hat{d}) \) be the principal eigenvalue of the averaged equation (1.10). Then we have

**Theorem 6.1.** (1) Consider (11). There is \( (\hat{c}, \hat{d}) \in \hat{Y}(c, d) \) such that \( \lambda_{\text{inf}}(c, d) \geq \lambda(\hat{c}, \hat{d}) \) and \( \lambda_{\text{sup}}(c, d) \geq \lambda(\hat{c}, \hat{d}) \) for any \( (\hat{c}, \hat{d}) \in \hat{Y}(c, d) \).

(2) Consider (12). \( \lambda(c, d) \geq \lambda(\hat{c}, \hat{d}) \).

**Theorem 6.2.** (1) Consider (11). If \( (c, d) \) is uniquely ergodic and minimal, then \( \lambda_{\text{inf}}(c, d) = \lambda_{\text{sup}}(c, d) = \lambda(\hat{c}, \hat{d}) \) for \( (\hat{c}, \hat{d}) \in \hat{Y}(c, d) \) \((\hat{Y}(c, d) \text{ is necessarily a singleton})\) if and only if \( c(t, x) = c_1(x) + c_2(t) \) and \( d(t, x) = d(x) \).

(2) Consider (12). \( \lambda(c, d) = \lambda(\hat{c}, \hat{d}) \) if and only if there is \( \Omega^* \subset \Omega \) with \( \mathbb{P}(\Omega^*|t = 1) = 1 \) such that \( \hat{c}(\theta_t \omega, x) = c_1(x) + c_2(\theta_t \omega) \) for any \( \omega \in \Omega^* \), \( t \in \mathbb{R} \) and \( x \in D \), and \( d(\theta_t \omega, x) = d(x) \) for any \( \omega \in \Omega^* \), \( t \in \mathbb{R} \) and \( x \in \partial D \).

In the case that the boundary condition is of the Dirichlet or Neumann type or of the Robin type with \( d \) independent of \( t \), the above theorems have been proved in [31]. For completeness, we will provide proofs of the theorems including the case that the boundary condition is of the Robin type with \( d \) depending on \( t \). We note that the proof in the following for Theorem 6.1 is not the same as that in [31] even in the case \( d \) is independent of \( t \).

**Proof of Theorem 6.1.** First of all, let \( (\hat{c}, \hat{d}) = (c, d) \) in the case of (11) and \( (\hat{c}, \hat{d}) = (c^*, d^*) \) in the case of (12) for some given \( \omega \in \Omega \). For given \( S \) and \( T > 0 \), let

\[
\eta(t; \hat{c}, \hat{d}, S) := \| U_{(\hat{c}, \hat{d}), S}(t, 0) w((\hat{c}, \hat{d}) \cdot S) \|, \quad t \geq 0,
\]

and

\[
\hat{w}(x; \hat{c}, \hat{d}, S, T) := \exp \left( \frac{1}{T} \int_0^T \ln w((\hat{c}, \hat{d}) \cdot (t + S))(x) \, dt \right)
\]

for \( x \in D \) and

\[
\hat{w}(x; \hat{c}, \hat{d}, S, T) = 0
\]

for \( x \in \partial D \) in the Dirichlet boundary condition case, and

\[
\hat{w}(x; \hat{c}, \hat{d}, S, T) := \exp \left( \frac{1}{T} \int_0^T \ln w((\hat{c}, \hat{d}) \cdot (t + S))(x) \, dt \right)
\]

for \( x \in \hat{D} \) in the Neumann and Robin boundary conditions cases. Note that \( \hat{w}(x; \hat{c}, \hat{d}, S, T) \in C(D) \).

Let \( \hat{v}(t; x; \hat{c}, \hat{d}, S) := w((\hat{c}, \hat{d}) \cdot (t + S))(x) \). We have that \( \eta(t; \hat{c}, \hat{d}, S) \) satisfies

\[
\eta_h(t; \hat{c}, \hat{d}, S) = \kappa((\hat{c}, \hat{d}) \cdot (t + S)) \eta(t; \hat{c}, \hat{d}, S), \quad (6.1)
\]
and \( \bar{v}(t, x; \tilde{c}, \tilde{d}, S) \) satisfies

\[
\begin{align*}
\frac{\partial \bar{v}}{\partial t} &= \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 \bar{v}}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_{i}(x) \frac{\partial \bar{v}}{\partial x_i} \\
&\quad + \tilde{c}(t + S, x) \bar{v} - \kappa((\tilde{c}, \tilde{d}) \cdot (t + S)) \bar{v}, \quad x \in D \tag{6.2}
\end{align*}
\]

where \( \tilde{B}(\cdot) \) is as in (1.5). Theorem 4.4 allows us to differentiate sufficiently many times to obtain that for any \( x \in D \) (\( x \) can also be in \( \bar{D} \) in the Neumann and Robin boundary conditions cases) we have

\[
\frac{\partial \hat{w}}{\partial x_i}(x; \tilde{c}, \tilde{d}, S, T) = \hat{w}(x; \tilde{c}, \tilde{d}, S, T) \frac{1}{T} \int_{0}^{T} \left( \frac{1}{w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)} \frac{\partial w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)}{\partial x_i} \right) dt, \tag{6.3}
\]

and that for any \( x \in D \) we have

\[
\begin{align*}
\frac{\partial^2 \hat{w}}{\partial x_i \partial x_j} &= \hat{w}(x; \tilde{c}, \tilde{d}, S, T) \left( \frac{1}{T^2} \int_{0}^{T} \left( \frac{1}{w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)} \frac{\partial w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)}{\partial x_i} \right) \frac{\partial w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)}{\partial x_j} dt \right) \\
&\quad + \hat{w}(x; \tilde{c}, \tilde{d}, S, T) \frac{1}{T} \int_{0}^{T} \left( \frac{1}{w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)} \frac{\partial^2 w((\tilde{c}, \tilde{d}) \cdot (t + S))(x)}{\partial x_i \partial x_j} \right) dt \tag{6.4}
\end{align*}
\]
Then by \((6.2)\), \(\hat{w} = \hat{w}(x; \check{c}, \check{d}, S, T)\) satisfies

\[
\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 \hat{w}}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial \hat{w}}{\partial x_i} = \left( \frac{1}{T} \int_0^T \frac{1}{\breve{w}} \frac{\partial \hat{w}}{\partial t}(t, x; \check{c}, \check{d}, S) \, dt \right) \hat{w} 
\]

\+
\[
\frac{1}{T} \int_0^T \kappa((\check{c}, \check{d}) \cdot (t + S)) \, dt - \frac{1}{T} \int_0^T \breve{e}(t + S, x) \, dt \right) \hat{w}
\]

\+
\[
\hat{w} \sum_{i,j=1}^{N} a_{ij}(x) \left( \frac{1}{T} \int_0^T \frac{1}{w((\check{c}, \check{d}) \cdot (t + S))} \frac{\partial w((\check{c}, \check{d}) \cdot (t + S))}{\partial x_i} \, dt \right)
\]

\-
\[
\hat{w} \sum_{i,j=1}^{N} a_{ij}(x) \left( \frac{1}{T} \int_0^T \frac{1}{w((\check{c}, \check{d}) \cdot (t + S))} \frac{\partial w((\check{c}, \check{d}) \cdot (t + S))}{\partial x_j} \, dt \right)
\]

for \(x \in D\), and \(B_{S,T}\hat{w} = 0\) for \(x \in \partial D\), where

\[
B_{S,T}\hat{w} := \begin{cases} 
\hat{w} & \text{(Dirichlet)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial \hat{w}}{\partial x_i} & \text{(Neumann)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial \hat{w}}{\partial x_i} + \left( \frac{1}{T} \int_0^T \breve{d}(t + S, x) \, dt \right) \hat{w} & \text{(Robin)}. 
\end{cases}
\]  

(6.6)

By Lemma \(2.1\),

\[
\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 \hat{w}}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial \hat{w}}{\partial x_i} \leq \left( \frac{1}{T} \int_0^T \frac{1}{\breve{w}} \frac{\partial \hat{w}}{\partial t}(t, x; \check{c}, \check{d}, S) \, dt \right) \hat{w} 
\]

\+
\[
\frac{1}{T} \int_0^T \kappa((\check{c}, \check{d}) \cdot (t + S)) \, dt - \frac{1}{T} \int_0^T \breve{e}(t + S, x) \, dt \right) \hat{w}. 
\]  

(6.7)

Note that \(\breve{v}(t, x; \check{c}, \check{d}, S) = w((\check{c}, \check{d}) \cdot (t + S))(x)\) and by Theorem \(4.3\) for a fixed compact \(D_0 \Subset D\) there are \(0 < m(D_0) < M\) such that \(m(D_0) \leq \breve{v}(t, x; \check{c}, \check{d}, S) \leq M\).
for any \((\tilde{c}, \tilde{d})\) \(\in Y, t, S \in \mathbb{R}\), and \(x \in D_0\). Hence

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{\tilde{v}} \frac{\partial \tilde{v}}{\partial t}(t, \tilde{c}, \tilde{d}, S) \, dt = \lim_{T \to \infty} \frac{1}{T} (\ln \tilde{v}(T, \tilde{c}, \tilde{d}, S) - \ln \tilde{v}(0, \tilde{c}, \tilde{d}, S)) = 0 \tag{6.8}
\]

for any \((\tilde{c}, \tilde{d})\) \(\in Y, S \in \mathbb{R}\), and \(x \in D\). Moreover, the limits are uniform in \((x, S) \in D_0 \times \mathbb{R}\) for any compact \(D_0 \Subset D\).

(1) We first prove that \(\lambda_{\text{inf}}(c, d) \geq \lambda(\tilde{c}, \tilde{d})\) for some \((\tilde{c}, \tilde{d}) \in \tilde{Y}(c, d)\).

Note that for given \(S, T > 0\),

\[
\eta(t; c, d, S) = \|U_{(c, d), S}(t, 0)w((c, d) \cdot S)\|,
\]

\[
\dot{w}(x; c, d, S, T) = \exp \left( \frac{1}{T} \int_S^{T+S} \ln w((c, d) \cdot t)(x) \, dt \right),
\]

\[
\frac{1}{T} \int_0^T \kappa((c, d) \cdot (t + S)) \, dt = \frac{1}{T} \int_S^{T+S} \kappa((c, d) \cdot t) \, dt,
\]

and

\[
\frac{1}{T} \int_0^T c(t + S, x) \, dt = \frac{1}{T} \int_S^{T+S} c(t, x) \, dt.
\]

By Theorem 5.3 there are \((S_n), (T_n)\) with \(T_n \to \infty\) such that

\[
\frac{1}{T_n} \int_{S_n}^{T_n+S_n} \kappa((c, d) \cdot t) \, dt \to \frac{\ln \eta(T_n; c, d, S_n)}{T_n} \to \lambda_{\text{inf}}(c, d).
\]

Without loss of generality we may assume that the limits

\[
\lim_{n \to \infty} \frac{1}{T_n} \int_{S_n}^{T_n+S_n} c(t, x) \, dt \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{T_n} \int_{S_n}^{T_n+S_n} d(t, x) \, dt,
\]

exist, uniformly in \(x \in D\) (resp. \(x \in \partial D\)). Denote these limits by \((\hat{c}, \hat{d})\).

In the Dirichlet case, it is a consequence of Theorems 4.3 and 4.4 that for each compact \(D_0 \Subset D\) the sets

\[
\{ \dot{w}(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \}, \{ (\partial \dot{w}/\partial x_i)(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \} \quad \text{and} \quad \{ \partial^2 \dot{w}/\partial x_i \partial x_j)(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \},
\]

and

\[
\dot{w}(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \}, \quad \text{have compact closures in} \ C(D_0).
\]

In the Neumann and Robin cases it is a consequence of Theorems 4.3 and 4.4 that the sets

\[
\{ \dot{w}(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \} \quad \text{and} \quad \{ \partial^2 \dot{w}/\partial x_i \partial x_j)(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \},
\]

have compact closures in \(C(\hat{D})\), and that for each compact \(D_0 \Subset D\) the sets

\[
\{ \partial^2 \dot{w}/\partial x_i \partial x_j)(; c, d, S_n, T_n) \mid D_0 : n = 1, 2, \ldots \},
\]

have compact closures in \(C(D_0)\).

We may thus assume that there is \(w^* = w^*(x)\) such that

\[
\lim_{n \to \infty} \dot{w}(x; c, d, S_n, T_n) = w^*(x) \tag{6.9}
\]

\[
\lim_{n \to \infty} \frac{\partial \dot{w}(x; c, d, S_n, T_n)}{\partial x_i} = \frac{\partial w^*(x)}{\partial x_i} \tag{6.10}
\]

\[
\lim_{n \to \infty} \frac{\partial^2 \dot{w}(x; c, d, S_n, T_n)}{\partial x_i \partial x_j} = \frac{\partial^2 w^*(x)}{\partial x_i \partial x_j} \tag{6.11}
\]
for \(i, j = 1, 2, \ldots, N\) and \(x \in D\). In the Dirichlet boundary conditions case, it follows from Theorems 4.3 and 4.4 that \(w^*\) can be extended to a function continuous on \(D\) by putting \(w^*(x) = 0\) for \(x \in \partial D\). Moreover, by Theorem 4.3 \(w^*(x) > 0\) for \(x \in D\).

Regarding the uniformity of convergence, in the Dirichlet case, the limit in (6.10) is uniform for \(x \in D\) and the limits in (6.10) and (6.11) are uniform for \(x \in D\) and the limit (6.11) is uniform for \(x \in D\) in any compact subset \(D_0 \Subset D\), and in the Neumann and Robin cases, the limits in (6.9) and (6.10) are uniform for \(x \in D\) and the limit (6.11) is uniform for \(x \in D\) in any compact subset \(D_0 \Subset D\).

We claim that \(\lambda_{\text{inf}}(c, d) \geq \lambda(\hat{c}, \hat{d})\). In fact, by (6.7)–(6.11),

\[
\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 w^*}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial w^*}{\partial x_i} + (\hat{c}(x) - \lambda_{\text{min}}(c, d))w^* \leq 0, \quad x \in D, \\
\hat{B}w^* = 0, \quad x \in \partial D,
\]

where

\[
\hat{B}w^* := \begin{cases} 
\frac{\partial w^*}{\partial x_i} & \text{(Dirichlet)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial w^*}{\partial x_i} & \text{(Neumann)} \\
\sum_{i=1}^{N} b_i(x) \frac{\partial w^*}{\partial x_i} + \hat{d}(x)w^* & \text{(Robin)}.
\end{cases}
\]

This implies that \(w(t, x) = w^*(x)\) is a supersolution of

\[
\begin{cases} 
w_t = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial w}{\partial x_i} \\
+ (\hat{c}(x) - \lambda_{\text{inf}}(c, d))w, & x \in D, \\
\hat{B}w = 0, & x \in \partial D.
\end{cases}
\]

Let \(w(t, x; \hat{w})\) be the solution of (6.12) with initial condition \(w(0, x; \hat{w}) = w^*(x)\). Then we have

\[
w(t, x; \hat{w}) \leq w^*(x) \quad (6.13)
\]

for \(x \in D\) and \(t \geq 0\). Note that \(\lambda(\hat{c}, \hat{d}) - \lambda_{\text{inf}}(c, d)\) is the principal eigenvalue of (6.10) with \((\hat{c}, \hat{d})\) being replaced by \((\hat{c} - \lambda_{\text{inf}}(c, d), \hat{d})\). It then follows from (6.13) together with the positivity of \(w^*(x)\) that

\[
\lambda(\hat{c}, \hat{d}) - \lambda_{\text{inf}}(c, d) \leq 0. \quad (6.14)
\]

This implies that

\[
\lambda(\hat{c}, \hat{d}) \leq \lambda_{\text{inf}}(c, d).
\]

Next, we prove \(\lambda_{\text{sup}}(c, d) \geq \lambda(\hat{c}, \hat{d})\) for any \((\hat{c}, \hat{d}) \in \hat{Y}(c, d)\). For any \((\hat{c}, \hat{d}) \in \hat{Y}(c, d)\) there are \((S_n), (T_n)\) with \(T_n \to \infty\) such that

\[
\frac{1}{T_n} \int_{S_n}^{T_n+S_n} c(t, x) \, dt \to \hat{c}(x)
\]
and
\[ \frac{1}{T_n} \int_{S_n}^{T_n+S_n} d(t, x) \, dt \to \hat{d}(x), \]
uniformly in \( x \in \bar{D} \) (resp. uniformly in \( x \in \partial D \)). Without loss of generality, assume that
\[ \frac{1}{T_n} \int_{S_n}^{T_n+S_n} \kappa((c, d), t) \, dt \to \lambda_0. \]
By arguments similar to the above, \( \lambda_0 \geq \lambda(\hat{c}, \hat{d}) \). Note that \( \lambda_{\sup}(c, d) \geq \lambda_0 \).

Then we have \( \lambda_{\sup}(c, d) \geq \lambda(\hat{c}, \hat{d}) \).

(2) By Lemma 2.3 there is \( \Omega_1 \subset \Omega \) with \( \mathbb{P}(\Omega_1) = 1 \) such that
\[ \hat{c}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T c(\theta, t, x) \, dt \]
for any \( \omega \in \Omega_1 \) and any \( x \in \bar{D} \), uniformly in \( \bar{D} \), and
\[ \hat{d}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T d(\theta, t, x) \, dt \]
for any \( \omega \in \Omega_1 \) and any \( x \in \partial D \), uniformly in \( \partial D \).

By Theorem 5.2 there is \( \Omega_2 \subset \Omega \) with \( \mathbb{P}(\Omega_2) = 1 \) such that
\[ \lambda = \lim_{T \to \infty} \frac{\ln\|U_\omega(T, 0)w[\omega]\|}{T} \]
for any \( \omega \in \Omega_2 \).

Take an \( \omega \in \Omega_1 \cap \Omega_2 \). Then for any \( T_n \to \infty \),
\[ \frac{1}{T_n} \int_0^{T_n} c(\theta, t, x) \, dt = \frac{1}{T_n} \int_0^{T_n} c^\omega(t, x) \, dt \to \hat{c}(x) \]
uniformly for \( x \in D \),
\[ \frac{1}{T_n} \int_0^{T_n} d(\theta, t, x) \, dt = \frac{1}{T_n} \int_0^{T_n} d^\omega(t, x) \, dt \to \hat{d}(x) \]
uniformly for \( x \in \partial D \),
and
\[ \frac{\ln \eta(T_n; c^\omega, d^\omega, 0)}{T_n} \to \lambda. \]

By arguments as in the proof of Part (1), we must have \( \lambda \geq \lambda_0 \).

\[ \square \]

**Proof of Theorem 6.2** We first prove (2) for the reason that (2) will be used in the proof of (1).

First, suppose that \( c(\omega, x) = c_1(x) + c_2(\theta, \omega) \) for any \( x \in \bar{D} \), any \( t \in \mathbb{R} \) and any \( \omega \in \Omega^* \). Without loss of generality, we may assume \( \int_{\Omega^*} c_2(\omega) \, d\mathbb{P}(\omega) = 0 \) and \( \mathbb{P}(\Omega^*) = 1 \) (for otherwise, we change \( c_1(x) \) to \( c_1(x) + \int_{\Omega^*} c_2(\omega) \, d\mathbb{P}(\omega) \) and change \( c_2(\omega) \) to \( c_2(\omega) - \int_{\Omega^*} c_2(\omega) \, d\mathbb{P}(\omega) \)). Suppose also that \( d(\theta, t, x) = d(x) \). One has \( \hat{c}(x) = c_1(x) \) for \( x \in \bar{D} \), and \( \hat{d}(x) = d(x) \) for \( x \in \partial D \). Let \( u(x) \) be the positive principal eigenfunction of \( L \) normalized so that its \( L_2(D) \)-norm equals 1, and let
\[ v(t, x; \omega) := u(x) \exp \left( \lambda t + \int_0^t c_2(\theta, \omega) \, ds \right) \]
for \( t \in \mathbb{R}, x \in \bar{D} \) and \( \omega \in \Omega^* \). It is then not difficult to see that for any \( \omega \in \Omega^* \) the function \([ \mathbb{R} \ni t \mapsto \nu(t, \cdot; \omega) \in L_2(D) \] is the (necessarily unique) normalized globally positive solution of \((1.2)\). For a.e. \( \omega \in \Omega \), \( \lambda = \lim_{t \to \pm \infty} (1/t) \ln \|\nu(t, \cdot; \omega)\| \). It follows with the help of Birkhoff’s Ergodic Theorem (Lemma 2.2) that the last term equals \( \hat{\lambda} \) for a.e. \( \omega \in \Omega^* \). Consequently, \( \lambda = \hat{\lambda} \).

Conversely, let \( \Omega_1 \) and \( \Omega_2 \) be as in the proof of Theorem 6.1(2). We write \( \eta(t; \omega) \) for \( \eta(t; c^*, d^*, 0) \) and \( \check{w}(x; \omega, T) \) for \( \check{w}(x; c^*, d^*, 0, T) \), respectively. Then

\[
\eta(t; \omega) = \|U_\omega(t, 0)w(\omega)\|
\]

and

\[
\check{w}(x; \omega, T) = \exp \left( \frac{1}{T} \int_0^T \ln \theta_t(\omega)(x) \, dt \right).
\]

Let

\[
\phi(x) := \exp \int_\Omega \ln w(\omega)(x) \, d\bar{\mathbb{P}}(\omega) \quad \text{for} \quad x \in \bar{D}
\]

in the case of Neumann or Robin boundary condition, and

\[
\phi(x) := \begin{cases} 
\exp \int_\Omega \ln w(\omega)(x) \, d\bar{\mathbb{P}}(\omega) & \text{for} \quad x \in D \\
0 & \text{for} \quad x \in \partial D
\end{cases}
\]

in the case of Dirichlet boundary condition. By Lemma 2.3 there is \( \Omega_3 \subset \Omega \) with \( \bar{\mathbb{P}}(\Omega_3) = 1 \) such that

\[
\phi(x) = \lim_{T \to \infty} \exp \left( \frac{1}{T} \int_0^T \ln \theta_t(\omega)(x) \, dt \right) = \lim_{T \to \infty} \check{w}(x; \omega, T)
\]

for any \( \omega \in \Omega_3 \) and \( x \in D \). Clearly, \( \phi(x) > 0 \) for \( x \in D \).

Observe that by Theorems 4.3 and 4.4, \( \frac{\partial w(\omega)(x)}{\partial x_i} (i = 1, 2, \ldots, N) \frac{\partial^2 w(\omega)(x)}{\partial x_i \partial x_j}, i, j = 1, 2, \ldots, N \) are locally Hölder continuous in \( x \in \bar{D} (x \in D) \) uniformly in \( \omega \in \Omega \), and for a fixed \( x \in \bar{D} (x \in D) \) they are bounded in \( \omega \in \Omega \). Hence, Lemma 2.4 together with Eqs. (6.3) and (6.4) gives us the existence of \( \Omega_4 \subset \Omega \) with \( \bar{\mathbb{P}}(\Omega_4) = 1 \) such that

\[
\frac{\partial \phi}{\partial x_i}(x) = \lim_{T \to \infty} \frac{\partial \check{w}(x; \omega, T)}{\partial x_i} = \phi(x) \int_{\Omega} \left( \frac{1}{w(\cdot)(x)} \frac{\partial w(\cdot)(x)}{\partial x_i} \right) d\bar{\mathbb{P}}(\cdot),
\]

and

\[
\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \lim_{T \to \infty} \frac{\partial^2 \check{w}(x; \omega, T)}{\partial x_i \partial x_j} = \phi(x) \int_{\Omega} \left( \frac{1}{w(\cdot)(x)} \frac{\partial w(\cdot)(x)}{\partial x_i} \right) d\bar{\mathbb{P}}(\cdot) \int_{\Omega} \left( \frac{1}{w(\cdot)(x)} \frac{\partial w(\cdot)(x)}{\partial x_j} \right) d\bar{\mathbb{P}}(\cdot)
\]

\[
+ \phi(x) \int_{\Omega} \left( \frac{1}{w(\cdot)(x)} \frac{\partial^2 w(\cdot)(x)}{\partial x_i \partial x_j} - \frac{1}{w^2(\cdot)(x)} \frac{\partial w(\cdot)(x)}{\partial x_i} \frac{\partial w(\cdot)(x)}{\partial x_j} \right) d\bar{\mathbb{P}}(\cdot)
\]

(6.17)
and
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{w(\theta_i \omega)(x)} \frac{\partial w(\theta_i \omega)(x)}{\partial x_i} \, dt = \int_{\Omega} \frac{1}{w(\omega)(x)} \frac{\partial w(\omega)(x)}{\partial x_i} \, d\mathbb{P}(-),
\]
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{w^2(\theta_i \omega)(x)} \frac{\partial w(\theta_i \omega)(x)}{\partial x_i} \frac{\partial w(\theta_i \omega)(x)}{\partial x_j} \, dt = \int_{\Omega} \frac{1}{w(\omega)(x)} \frac{\partial w(\omega)(x)}{\partial x_i} \frac{\partial w(\omega)(x)}{\partial x_j} \, d\mathbb{P}(-)
\]
for \( \omega \in \Omega_4, x \in D, \) and \( \mathcal{B}\phi = 0 \)
for \( \omega \in \Omega_4, x \in \partial D, \) where
\[
\mathcal{B}\phi := \begin{cases} 
\phi & \text{(Dirichlet)} \\
\sum_{i=1}^N b_i(x) \frac{\partial \phi}{\partial x_i} & \text{(Neumann)} (6.18) \\
\sum_{i=1}^N b_i(x) \frac{\partial \phi}{\partial x_i} + \hat{d}(x)\phi & \text{(Robin)}.
\end{cases}
\]

Let \( \Omega_0 := \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4. \) Then (6.15)–(6.18) hold for any \( \omega \in \Omega_0. \)
Put \( \hat{v}(t, x; \omega) := w(\theta_i \omega)(x). \) Since, by Theorem 4.3, for a fixed \( x \in D \) there are \( 0 < m < M \) such that \( m \leq w(\omega)(x) \leq M \) for any \( \omega \in \Omega, \) we have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{v} \frac{\partial \hat{v}}{\partial t}(t, x; \omega) \, ds = \lim_{T \to \infty} \frac{1}{T} (\ln w(\theta T \omega)(x) - \ln w(\omega)(x)) = 0
\]
for any \( \omega \in \Omega_0 \) and \( x \in D. \)

Consequently, by (6.5) and (6.6) we have
\[
\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^N a_i(x) \frac{\partial \phi}{\partial x_i}
= (\lambda - \hat{c}(x))\phi
+ \phi \sum_{i,j=1}^N a_{ij}(x) \int_{\Omega} \left( \frac{1}{w(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \right) \, d\mathbb{P}(\cdot) \int_{\Omega} \left( \frac{1}{w(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \right) \, d\mathbb{P}(\cdot)
- \phi \sum_{i,j=1}^N a_{ij}(x) \int_{\Omega} \left( \frac{1}{w^2(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \frac{\partial w(\cdot)}{\partial x_j} \right) \, d\mathbb{P}(\cdot)
\]
for \( x \in D, \) and \( \mathcal{B}\phi = 0 \) for \( x \in \partial D. \)

Suppose that \( \lambda = \check{\lambda}. \) Consider
\[
\begin{cases} 
u_t = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N a_i(x) \frac{\partial u}{\partial x_i} + (\hat{c}(x) - \lambda)u, & x \in D, \\
\mathcal{B}u = 0.
\end{cases} (6.21)
\]
We have that 0 is the principal eigenvalue of (6.21). Let \( \hat{\phi} \) be a positive principal eigenfunction of (6.21). Let \( u(t, x; \phi) \) be the solution of (6.21) with initial condition \( u(0, x; \phi) = \phi(x) \). By Lemma 2.1(2),

\[
\sum_{i,j=1}^{N} a_{ij}(x) \int_{\Omega} \left( \frac{1}{w(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \right) dP(\cdot) \int_{\Omega} \left( \frac{1}{w(\cdot)} \frac{\partial w(\cdot)}{\partial x_j} \right) dP(\cdot) 
- \sum_{i,j=1}^{N} a_{ij}(x) \int_{\Omega} \left( \frac{1}{w^2(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \frac{\partial w(\cdot)}{\partial x_j} \right) dP(\cdot) \leq 0
\]

for all \( x \in D \). This together with (6.20) implies that \( \phi(x) \) is a supersolution of (6.21) and hence

\[
u(t, x; \phi) \leq \phi(x) \quad \text{for} \quad x \in D, \quad t \geq 0. \tag{6.22}\]

We apply now Theorem 4.2 to the autonomous problem (6.21). In this case, \( Y \) is a singleton, \( w = \hat{\phi} \), and \( w^*(x) > 0 \) for a.e. \( x \in D \). It follows then that \( \langle \hat{\phi}, w^* \rangle > 0 \) and \( \langle \hat{\phi}, w^* \rangle > 0 \). By taking \( \alpha := \langle \phi, w^* \rangle / \langle \hat{\phi}, w^* \rangle > 0 \) we see that

\[
\phi = \alpha \hat{\phi} + \hat{\psi},
\]

where \( \hat{\psi} \in X \) is such that \( \langle \hat{\psi}, w^* \rangle = 0 \). Note that \( u(t, x; \phi) = \alpha \hat{\phi}(x) + u(t, x; \hat{\psi}) \), where \( u(t, x; \hat{\psi}) \) is the solution of (6.21) with \( u(0, x; \hat{\psi}) = \hat{\psi}(x) \). Theorem 4.2(iii) gives that \( \| u(t, \cdot; \hat{\psi}) \| \to 0 \) as \( t \to \infty \). It then follows from (6.22) that

\[
\alpha \hat{\phi}(x) \leq \phi(x) \quad \text{for} \quad x \in D,
\]

and then

\[
\hat{\psi}(x) \geq 0 \quad \text{for} \quad x \in D.
\]

This implies that

\[
\hat{\psi}(x) = 0 \quad \text{for} \quad x \in D,
\]

hence

\[
\alpha \hat{\phi}(x) = \phi(x) \quad \text{for} \quad x \in D.
\]

Therefore we must have

\[
\sum_{i,j=1}^{N} a_{ij}(x) \int_{\Omega} \left( \frac{1}{w(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \right) dP(\cdot) \int_{\Omega} \left( \frac{1}{w(\cdot)} \frac{\partial w(\cdot)}{\partial x_j} \right) dP(\cdot) 
= \sum_{i,j=1}^{N} a_{ij}(x) \int_{\Omega} \left( \frac{1}{w^2(\cdot)} \frac{\partial w(\cdot)}{\partial x_i} \frac{\partial w(\cdot)}{\partial x_j} \right) dP(\cdot)
\]

for all \( x \in D \).

Let \( \{ x^{(n)} : n \in \mathbb{N} \} \) be a countable dense subset of \( D \). By Lemma 2.1(2), for each \( n \in \mathbb{N} \) there is \( \Omega^{(n)} \) with \( P(\Omega^{(n)}) = 1 \) such that \( \frac{1}{w(\omega)(x^{(n)})} \frac{\partial w(\omega)(x^{(n)})}{\partial x_i} \) is independent of \( \omega \in \Omega^{(n)} \). Consequently, from the continuity of \( \frac{1}{w(\omega)(x)} \frac{\partial w(\omega)(x)}{\partial x_i} \).
for a fixed \(\omega \in \Omega\), in \(x \in D\), there are \(\Omega_0 \subset \Omega_\), \(\Omega_\) := \(\Omega_0 \cap \bigcap_{5=1}^{\infty} \Omega(n)\), with \(P(\Omega_\) = 1, and functions \(f_i(x)\) such that
\[
\frac{1}{w(\omega)(x)} \frac{\partial w(\omega)}{\partial x_i} = f_i(x)
\]
for \(i = 1, 2, \ldots, N, \omega \in \Omega_\) and \(x \in D\). Hence
\[
\nabla \ln w(\omega)(x) = (f_1(x), f_2(x), \ldots, f_N(x))^T
\]
for \(\omega \in \Omega_\) and \(x \in D\). This implies that \(w(\omega)(x) = F(x)G(\omega)\) for some continuous \(F(x) > 0\), measurable \(G(\omega) > 0\) and any \(\omega \in \Omega_\), \(x \in D\). Let \(\Omega^* := \bigcap_{t \in Q} \theta_\Omega_\), where \(Q\) is the set of all rational numbers. Clearly, \(P(\Omega^*) = 1\) and \(w(\theta_\omega)(x) = F(x)G(\theta_\omega)\) for \(t \in Q, \omega \in \Omega^*\) and \(x \in D\). The continuity of \(w(\theta_\omega)(x)\) in \(t \in \mathbb{R}\) then implies that the function \([\mathbb{R} \ni t \mapsto w(\theta_\omega)(x)/F(x) \in \mathbb{R}]\) is continuous. Hence, for each \(\omega \in \Omega^*\) and each \(t \in \mathbb{R}\) we can safely write \(G(\theta_\omega)\) for \(w(\theta_\omega)(x)/F(x)\). Therefore, by \((6.2)\),
\[
F(x) \frac{dG(\theta_\omega)}{dt} = \left( \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial F}{\partial x_i} + c(\theta_\omega, x) F - \kappa(\theta_\omega) F \right) G(\theta_\omega)(6.23)
\]
for \(t \in \mathbb{R}, \omega \in \Omega^*\) and \(x \in D\), and
\[
\mathcal{B}(\theta_\omega)F = 0
\]
for \(t \in \mathbb{R}, \omega \in \Omega^*\) and \(x \in \partial D\). By dividing both sides of \((6.23)\) by \(F(x)G(\theta_\omega)\) we obtain
\[
c(\theta_\omega, x) = \frac{dG(\theta_\omega)}{dt} G(\theta_\omega) + \kappa(\theta_\omega) F - \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 F}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} a_i(x) \frac{\partial F}{\partial x_i}(x)
\]
for \(t \in \mathbb{R}, \omega \in \Omega^*\) and \(x \in \partial D\). We can write \(c(\theta_\omega, x) = c_1(x) + c_2(\theta_\omega)\) for some integrable \(c_2(\omega)\) with \(\int_{\Omega} c_2(\cdot) d\mathbb{P}(\cdot) = 0\), any \(x \in D, t \in \mathbb{R}\) and \(\omega \in \Omega^*\). Similarly, by taking the boundary condition \(\mathcal{B}(\theta_\omega)F = 0\) we obtain that
\[
d(\theta_\omega, x) = -\sum_{i=1}^{N} b_i(x) \frac{\partial F}{\partial x_i}(x)
\]
for \(t \in \mathbb{R}, \omega \in \Omega^*\) and \(x \in \partial D\), that is, \(d(\theta_\omega, x) = d(x)\) for each \(x \in \partial D\), each \(t \in \mathbb{R}\) and each \(\omega \in \Omega^*\).

1. Let \(\mathbb{P}\) be the unique ergodic measure on \(Y(c, d)\). By Lemma 2.7
\[
\hat{c}(x) := \lim_{t \to \infty} \frac{1}{t} \int_0^t c(s, x) \, ds
\]
exists for \(x \in D\), and
\[
\hat{d}(x) := \lim_{t \to \infty} \frac{1}{t} \int_0^t d(s, x) \, ds
\]
exists for \(x \in \partial D\).
Assume that the equality $\lambda_{\inf}(c, d) = \lambda(\hat{c}, \hat{d})$ holds. By Theorem 6.2(2), there is $Y_0(c, d) \subset Y(c, d)$ with $\mathbb{P}(Y_0(c, d)) = 1$ such that for any $(\hat{c}, \hat{d}) \in Y_0(c, d)$, 
\[
\hat{c}(t, x) = c_1(x) + \tilde{c}_2((\hat{c}, \hat{d}) \cdot t)
\]
for some $\mathbb{P}$-integrable $\tilde{c}_2$ with $\int_{Y(c, d)} \tilde{c}_2(\hat{c}, \hat{d}) \, d\mathbb{P}(\hat{c}, \hat{d}) = 0$, any $t \in \mathbb{R}$, $x \in D$, and $\hat{d}(t, x) = d(x)$ for $x \in \partial D$. Take a $(\hat{c}, \hat{d}) \in Y_0(c, d)$. Since $Y(c, d)$ is minimal, there is a sequence $(s_n)$ such that $(\hat{c}, \hat{d}) - s_n$ converges in $Y(c, d)$ to $(c, d)$ as $n \to \infty$. This implies that $c(t, x) = c_1(x) + c_2(t)$ and $d(t, x) = d(x)$. By unique ergodicity, $\lim_{t \to \infty} \frac{1}{t} \int_0^t c_2(s) \, ds = \int_{Y(c, d)} \tilde{c}_2(\hat{c}, \hat{d}) \, d\mathbb{P}(\hat{c}, \hat{d}) = 0$.

If $c(t, x) = c_1(x) + c_2(t)$ and $d(t, x) = d(x)$, then the equality $\lambda_{\inf}(c, d) = \bar{\lambda}$ follows clearly.

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