Peak effect at the weak- to strong pinning crossover

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Abstract

In type-II superconductors, the magnetic field enters in the form of vortices; their flow under application of a current introduces dissipation and thus destroys the defining property of a superconductor. Vortices get immobilized by pinning through material defects, thus resurrecting the supercurrent. In weak collective pinning, defects compete and only fluctuations in the defect density produce pinning. On the contrary, strong pins deform the lattice and induce metastabilities. Here, we focus on the crossover from weak- to strong bulk pinning, which is triggered either by increasing the strength $f_p$ of the defect potential or by decreasing the effective elasticity of the lattice (which is parametrized by the Labusch force $f_{Lab}$). With an appropriate Landau expansion of the free energy we obtain a peak effect with a sharp rise in the critical current density $j_c \sim j_0 (a_0 \xi^2 n_p) (\xi^2 / a_0^2) (f_p / f_{Lab} - 1)^2$.

Key words: superconductivity, vortex pinning, peak effect

Pinning of vortices by material defects is crucial in establishing the dissipation-free flow of a supercurrent. In recent years, the focus has been on weak collective pinning theory [1] describing the action of many competing defects; pinning then is due to fluctuations. The critical current is determined by the statistical summation of the competing pins [2] and is characterized by a quadratic dependence on the defect density. On the contrary, first attempts to describe vortex pinning go back to Labusch [3], who studied the action of strong individual pins. Strong pins deform the lattice [4,5,6,7] and induce metastabilities; the pinning energy landscape becomes multi-valued, producing a non-zero average of the pinning force and a critical current which is linear in the defect density. These two approaches have been related in a recent work, where we have studied the pinning diagram exhibiting crossovers between various regimes when varying the defect density $n_p$ measured with respect to the vortex density $1 / a_0^2$ and the pinning force $f_p$ measured with respect to the elasticity, cf. Fig. 1. At low defect density, vortex lattice (bulk) pinning is relevant; for weak defect strength pinning is collective, whereas defects with a force larger than the Labusch force $f_{Lab}$ (see below) pin the lattice individually. At higher defect densities we have established two regimes where pinning acts on individual vortex lines.

In this work, we focus on the crossover from weak- to strong bulk pinning, which is triggered either by increasing the defect strength $\kappa \sim f_p / \xi$ ($\xi$ is the coherence length of the superconductor) or by decreasing the effective elasticity $C$ of the lattice; the critical defect force at the crossover is the...
Labusch force \( f_{\text{Lab}} = \bar{C} \xi \). The crossover is structurally related to the Landau theory of first order phase transitions and can be analyzed with an appropriate Landau expansion of the free energy. As a result, we obtain a strong increase in the critical current density at the crossover,

\[
j_c \sim j_0 (a_0 \xi^2 n_p) \frac{\xi^2}{a_0^2} \left( \frac{f_p}{f_{\text{Lab}}} - 1 \right)^2,
\]

where \( j_0 \) is the depairing current density and \( a_0 = (\Phi_0 / B)^{1/2} \) is the mean vortex spacing (\( \Phi_0 \) is the superconducting flux unit); see below for the precise definition of \( f_p \) and \( f_{\text{Lab}} \). The crossover from weak- to strong pinning can be triggered by reducing the effective elasticity \( \bar{C} \propto f_{\text{Lab}} \), producing a peak in the current density [1].

Below, we will describe the strong pinning situation in more detail and use a Landau type of expansion to find the jumps in the free energy landscape. These jumps are then used to derive the result (1) for the critical current density.

In order to derive a quantitative criterion for strong pinning, we consider the effect of a single (strong) defect located at the origin with pinning potential \( e_p(r) \) producing the pinning contribution

\[
E_p(r, \mathbf{u}) = \sum_{\nu} e_p(r) \delta^2 (R - R_\nu - \mathbf{u}(R_\nu, z))
\]

to the free energy density of the vortex system; \( \mathbf{r} = (R, z) \) and the vortices are positioned at \( R_\nu - \mathbf{u}(R_\nu, z) \), with \( R_\nu \) the equilibrium positions and \( \mathbf{u} \) the displacement field. The elastic part of the vortex lattice free energy reads [2]

\[
\mathcal{F}_{\text{el}} = \int \frac{d^3 k}{(2\pi)^3} \left[ G^{\alpha \beta}(k) \right]^{-1} u_{\nu}^{\alpha}(\mathbf{k}) \left[ G^{\alpha \beta}(k) \right]^{-1} u_{\nu}^{\beta}(k),
\]

with \( G^{\alpha \beta}(k) \) the Fourier transform of the elastic Green function \( G^{\alpha \beta}(r) \); Greek indices denote the in-plane components \( (x, y) \) and we sum over double indices.

The displacement field \( \mathbf{u} \) is obtained from variation of the total energy (elastic and pinning),

\[
u^{\alpha}(r_\nu) = -\int d^3 r' \left[ G^{\alpha \beta}(r_\nu - r') \partial u^\beta E_p(r', \mathbf{u}) \right] (r_\nu, \mathbf{u})
\]

\[
= G^{\alpha \beta}(R_\nu - R_d, z) f_0^\beta (R_d + u(R_d, 0), 0),
\]

with \( R_d \) the distance to the vortex closest to the defect. Evaluating (3) for \( r_\nu = (R_d, z) \) and \( f_p = -\nabla e_p(\mathbf{u}) \) the pinning force of the defect. In the last equation, we have assumed a defect of range much smaller than \( a_0 \) pinning at most one vortex; we further have chosen \( R_d \) as the distance to the vortex closest to the defect. Evaluating (3) for \( r_\nu = (R_d, 0) \), we arrive at the self-consistency equation

\[
\left[ G^{\alpha \beta}(k) \right]^{-1} u_{\nu}^{\alpha}(k) = \bar{C}^{-1} f_p^\alpha [R + \mathbf{u}(R, 0), 0]
\]

with the effective elastic constant

\[
\int \frac{d^3 k}{(2\pi)^3} G^{\alpha \beta}(k) = \bar{C}^{-1} \delta^{\alpha \beta}.
\]

For \( a_0 < \lambda \) the effective elasticity is \( \bar{C} \sim \varepsilon_0 / a_0 \) with \( \varepsilon_0 = (\Phi_0 / 4\pi \lambda)^2 \) the vortex line energy. The solution at \( (R, 0) \) allows for the calculation of the whole displacement field \( \mathbf{u}(R_\nu, z) \),

\[
u(R_\nu, z) = G^{\alpha \beta}(R_\nu - R, z) \bar{C} \mathbf{u}(R, 0),
\]

and we can calculate the total free energy \( e_{\text{pin}}(\mathbf{u}, R) \) (containing both, the contribution from elasticity and disorder) of the vortex system as a function of \( \mathbf{u}(R, 0) \),

\[
e_{\text{pin}} = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[ G^{\alpha \gamma}(k) \left[ G^{\alpha \beta}(k) \right]^{-1} C^{\beta \gamma}(k) \right] \times u^\gamma(R, 0) u^\delta(R, 0) + e_p(R + \mathbf{u}(R, 0))
\]
This (Labusch) criterion \[3\] for strong pinning involves the maximal negative curvature above the inflection point; it tests an individual isolated pin and classifies it as weak or strong.

In a next step, we calculate the average pinning force in the strong pinning case. Using the self-consistency equation (4), we calculate the derivative of the pinning energy \( e_{\text{pin}}(x, y) = e_{\text{pin}}(u(R), R) \) with respect to the drag parameter \( x \),

\[
\partial_x e_{\text{pin}}(x, y) = \bar{C}u\partial_x u - f_p(R + u)[\dot{x} + \partial_x u] = -f_p(R + u); \tag{9}
\]

this equation relates the force along \( x \) exerted by the pin at the vortex position \( R + u(R, 0) \) to the free energy landscape containing both, the contribution from pinning and elasticity. The pinning force has to be averaged over defect locations and, due to the bistabilities, depends on the preparation of the system; here, we focus on the critical current density and therefore search for the maximal force against drag. Averaging the accumulated drag force over the ‘impact parameter’ \( y \), we obtain

\[
\langle f_{\text{pin}} \rangle = \int_0^{L_x} dx \int_0^{L_y} dy \frac{f_p^2(R + u)}{L_x L_y} \tag{10}
\]

the integral over \( x \) can equally well be interpreted as an integrated drag force or as an average over pin locations (maximized with respect to the bistable branches). We express the average along the \( x \)-axis through the jump \( \Delta e_{\text{pin}}(y) \) in the pinning energy,

\[
\langle f_{\text{pin}} \rangle = -\int_{-a_0/2}^{a_0/2} dy \frac{\Delta e_{\text{pin}}(y)}{\tilde{a}(y)} = -\frac{t_{\perp}}{a_0} \Delta e_{\text{pin}}, \tag{11}
\]

where we have assumed a maximal trapping distance \( t_{\perp} \) along the \( y \) axis and require the pin not to overdrag the vortex; then the periodicity of \( e_{\text{pin}} \) is the same as the lattice periodicity and \( \tilde{a} = a_0 \). The remaining task is the determination of the jump \( \Delta e_{\text{pin}} \); as we will see below, it does not depend on the impact parameter \( y \).
for different values of $\bar{u}$, the displacement is shown in Fig. 3, where we have plotted the defect potential (realized at $x = 0$) versus drag $x$. The inset shows the geometry defining the drag parameter $x$ and the displacement $u$.

In a first step, we discuss the solutions of the self-consistency equation (4) when dragging the vortex lattice through the defect center along the positive $x$ direction and calculate the displacement $u$ numerically for a Lorentzian defect potential

$$e_{\text{pin},\text{L}}(u) = -\frac{e_0}{1 + u^2/\xi^2}.$$  

The result is shown in Fig. 3, where we have plotted the displacement $u$ and the pinning energy $e_{\text{pin}}$ for different values of $C/\kappa \approx f_{\text{L}}/f_p$, where $-\kappa = -e_0/2\xi^2$ is the maximal negative curvature of the defect potential (realized at $u = u_\kappa = \xi$), which is the relevant curvature in the Labusch criterion (8). The branch $u_- (u_+)$ corresponds to the pinned (unpinned) vortex and becomes unstable at $x_- (x_+)$; the dashed lines denote unstable branches.

The solutions $u(x)$ of the self-consistency equation (4) correspond to the minima of the functional $e_{\text{pin}}[u,x]$ for fixed $x$, cf. Fig. 4. Starting far from the defect with a negative drag parameter $x$, there is only one solution $u_+$ corresponding to the unpinned vortex system, cf. Figs. 2 and 4. With the vortex approaching the defect a second minimum appears, but is separated from the first one by a barrier. The relevant solution therefore remains the unpinned one, although at some point the pinned solution becomes lower in energy. For $x = -x_+$ the unpinned solution becomes unstable and the vortex gets trapped, as shown in Fig. 4. It remains so until at $x = x_-$ the pinned solution $u_-$ turns unstable and the defect releases the vortex. The corresponding jumps in the free energy are shown in Fig. 2.

In order to attain a more quantitative description of the different branches and their corresponding jumps, we perform a Landau type expansion of the free energy close to the Labusch condition $\kappa = C$ and use the analogy to the Landau theory of first order phase transitions. We denote the location of the maximal negative curvature $-\kappa$ by $u_\kappa$ and expand the pinning potential $e_p(x + u)$ around this point,

$$e_p(x + u) \approx -\epsilon + \nu(x + u - u_\kappa) - \frac{\kappa}{2}(x + u - u_\kappa)^2 + \frac{\alpha}{24}(x + u - u_\kappa)^4.$$  

For the Lorentzian potential the expansion parameters read $u_\kappa = \xi$, $\epsilon = e_0/2$, $\nu = e_0/2\xi$, $\kappa = e_0/2\xi^2$, $\alpha = e_0^2/24\xi^4$. 

Fig. 3. Displacement $u$ and energy $e_{\text{pin}}$ versus drag $x$ of the vortex lattice relative to a defect producing a Lorentzian pinning potential $e_{\text{pin},\text{L}} = -e_0/(1 + u^2/\xi^2)$ (we measure units of length in $\xi$ and units of energy in $e_0$). The evolution of the bistability appearing above the Labusch criterium is shown for pinning centers of increasing strength with $C/\kappa = 0.85, 0.75, 0.65, 0.55$; dashed lines denote unstable branches. The inset shows the geometry defining the drag parameter $x$ and the displacement $u$.

Fig. 4. Schematic plot of the pinning energy as a function of the displacement $u$ for different (negative) values of the drag $x$. The sketch shows the trapping process of an unpinned vortex at the point of instability $-x_+$ and the appearance of the jump $\Delta e_{\text{pin}}$. 

$\bar{u}$
and $\alpha = 3\kappa / \xi^4$. We insert this expansion into the pinning energy $e_{\text{pin}} = Cu^2/2 + e_p(R + u)$; the latter maps to the free energy of a one-component (Ising) magnet in a magnetic field \[8\] if we define the order parameter $\phi = x - u - u_\kappa$, the reduced ‘temperature’ $\tau = \bar{C} - \kappa$, and the ‘magnetic’ field $h = \bar{C}(x - u_\kappa - \nu / \bar{C})$,\[ e_{\text{pin}}[u, x] = e_{\text{mag}}[\phi, h] = \frac{\tau}{2} \phi^2 + \frac{\alpha}{24} \phi^4 - h \phi, \tag{13} \]

where we have dropped the constant term. Consider the zero field case first: In the paramagnetic phase at high-temperatures $\tau > 0$ the order parameter $\phi$ vanishes, while $\phi \propto \pm |\tau|^{1/2}$ in the low temperature ($\tau < 0$) ferromagnetic phase. The two ferromagnetic states ($\phi > 0$ and $\phi < 0$) are separated by an energy barrier increasing as $\Delta e_{\text{mag}} \propto \tau^2$. In the low temperature phase $\tau < 0$, a field $h$ induces a first order transition across $h = 0$ between the two ferromagnetic states. Metastable/hysteretic behavior appears within the field regime $|h| < h^*$, with $h^* \propto |\tau|^{3/2}$ following from comparing the barrier $\Delta e_{\text{mag}}$ and the field energy $h \phi$.

The above considerations translate to the pinning problem in the following way: the high ‘temperature’ phase $\tau > 0$ describes weak pinning. The two low temperature states stand for the pinned ($\phi < 0$) and unpinned ($\phi > 0$) configurations, which transform into one another via the first order transition. The hysteretic regime translates to the bistable pinning domain bounded by

$$x_- = u_\kappa + (\nu + h^*) / \bar{C},$$

with $h^* = (2/3\bar{C}) \sqrt{2/\alpha} |\tau|^{3/2}$. At the instability $x = x_- (h = h^*)$ the order parameter jumps from the pinned solution $u_- = \phi_- + u_\kappa - x_-$ to the unpinned state $u_+ = \phi_+ + u_\kappa - x_-$, where $\phi_- = -\sqrt{2/\alpha} |\tau|^{1/2}$ and $\phi_+ = 2\sqrt{2/\alpha} |\tau|^{1/2}$. As a result, we find the jump $\Delta e = e_{\text{pin}}[u_-, x_-] - e_{\text{pin}}[u_+, x_-] = (9/2\alpha) \tau^2$ in the free energy; a jump of equal magnitude is found at $x_+$. The total jump $\Delta e_{\text{pin}}$ in the pinning energy then reads

$$\Delta e_{\text{pin}} = 2\Delta e = (9/\alpha) (\bar{C} - \kappa)^2.$$ 

Within this Landau description, the Labusch condition $\bar{C} = \kappa$ corresponds to the critical end-point $T = T_c$ terminating a line of first order transitions.

Going beyond this one-dimensional analysis, we have to account for a finite ‘impact parameter’ $y$ of the vortex with respect to the defect ($\mathbf{R}_d = (x, y)$) and determine the transverse trapping distance $t_\perp$ for a vortex to get trapped when dragged along $x$, cf. Fig. 5. Making use of the rotational symmetry of the problem, we find circles with radius $x_\perp$ limiting the bistable regions. The jumps then occur when a solution is dragged through its instability line: The unpinned vortex gets pinned when $x = -(x_\perp^2 - y^2)^{1/2}$; this only happens for impact parameters $|y| < x_\perp$, which leads to a trapping distance $t_\perp = 2x_\perp$ (note, that $t_\perp \approx u_\kappa + \nu / \bar{C}$ remains finite at the transition to weak pinning). The pinned vortex is released when $x = (x_\perp^2 - y^2)^{1/2}$. Note, that the corresponding magnitudes of the jumps do not depend on $y$ because of the planar symmetry.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{vortex_trajectory.png}
\caption{Trapping area around the defect for a circular symmetric situation with pinned, unpinned, and bistable regions; the dark uncompensated area produces the net pinning force. A vortex trajectory for finite impact parameter $y < 0$ is shown.}
\end{figure}

We insert the value for the jump $\Delta e_{\text{pin}}$ into (11), use $t_\perp = 2x_\perp$ and obtain the pinning force $f_{\text{pin}} = -2x_\perp \Delta e_{\text{pin}} / a_0^2$. Adding the contributions of independent strong defects with density $n_p$, the total pinning force density reads $F_{\text{pin}} = n_p f_{\text{pin}}$ and the critical current density is obtained as $j_c \sim -c F_{\text{pin}} / B$. Close to the Labusch condition, we finally obtain the critical current density

$$j_c \sim j_0 a_0 \xi^2 n_p \frac{\xi^2}{a_0^2} \left( \frac{F_p}{f_{\text{Lab}}} - 1 \right)^2,$$ \tag{14}
where we have used a defect size $\xi$, implying $t_\perp \sim \xi$; we define the relevant defect strength $f_p = \xi\kappa = \xi \max_u [f'_p(u)]$ in terms of the maximal curvature of the defect potential. The Labusch force then takes the form $f_{\text{Lab}} = \xi C \sim \varepsilon_0 \xi / a_0$. Interpolating with the weak collective pinning result [2]

$$j_{\text{wcp}} \sim j_0 (a_0 \xi^2 n_p)^{2} \frac{\xi^2}{\lambda^2} \left( \frac{f_p}{f_{\text{ Lab}}} \right)^4$$

(we assume pinned bundles of size larger than $\lambda$), we observe a sharp rise in the critical current density once the strong pinning force overcomes the weak pinning result; this peak effect [1] occurs close to the Labusch condition at $f_p / f_{\text{Lab}} = 1 + (a_0 / \lambda) \sqrt{a_0 \xi^2 n_p}$.

The crossover from weak- to strong pinning can be realized in experiments: increasing the magnetic field and approaching the upper critical field $H_{c2}$ leads to a marked softening of the elastic moduli, thereby reducing $f_{\text{Lab}} \propto C$; the resulting crossover into the strong pinning regime then is observed as a peak [1,9] in the current density. Whether such a crossover from weak- to strong pinning can explain the recently observed peak effects (see, e.g., [10,11] and Refs. therein) remains to be seen.

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