The early exercise premium representation for American options on multiply assets

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Abstract In the paper we consider the problem of valuation of American options written on dividend-paying assets whose price dynamics follow the classical multidimensional Black and Scholes model. We provide a general early exercise premium representation formula for options with payoff functions which are convex or satisfy mild regularity assumptions. Examples include index options, spread options, call on max options, put on min options, multiply strike options and power-product options. In the proof of the formula we exploit close connections between the optimal stopping problems associated with valuation of American options, obstacle problems and reflected backward stochastic differential equations.

Key words American option, multiply assets, early exercise premium, backward stochastic differential equation, optimal stopping, obstacle problem.

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1 Introduction

In the paper we study American options written on dividend-paying assets. We assume that the underlying assets dynamics follow the classical multidimensional Black and Scholes model. It is now well known that the arbitrage-free value of American options can be expressed in terms of the optimal stopping problem (Bensoussan [3], Karatzas [10]; see also [17] for nice exposition and additional references), in terms of variational inequalities (Jaillet, Lamberton and Lapeyre [15]) and in terms of solutions of reflected BSDEs (El Karoui and Quenez [10]). Although these approaches provide complete characterization of the option value (see Section 2 for a short review), the paper by Broadie and Detemple [5] shows that it is of interest to provide alternative representation, which expresses the value of an American option as the value of the corresponding European option plus the gain from early exercise. The main reason is that the representation, called the early exercise premium formula, gives useful information on the determinants of the option value. The formula was proved first by Kim [13] in the case of standard American put option on a single asset. Another important contributions in the case of single asset include El Karoui and Karatzas [9] and Jacka [14] (see also Section 2.7 in [17] and the references therein). The case of options on multiply dividend-paying assets is more difficult and has received rather little attention
in the literature. In the important paper \[5\] and next in Detemple, Feng and Tian \[7\] the early exercise premium formula was established for concrete classes of options on multiply assets.

In the present paper we provide a unified way of treating a wide variety of seemingly disparate examples. It allows us to prove a general exercise premium formula for options with convex payoff functions satisfying the polynomial growth condition or payoff function satisfying quite general condition considered in Laurence and Salsa \[21\]. Verifying the last condition requires knowledge of the payoff function and the structure of the exercise set. Therefore it is a complicated task in general. Fortunately, in most interesting cases one can easily check convexity of the payoff function or check some simpler condition implying the general condition from \[21\]. The class of options covered by our formula includes index options, spread options, call on max options, put on min options, multiply strike options, power-product options and others.

In the proof of the exercise premium formula we rely on some results on reflected BSDEs and their links with optimal stopping problems (see \[10\]) and with parabolic variational inequalities established in Bally, Caballero, Fernandez and El Karoui \[2\]. We also use classical results on regularity of the solution of the Cauchy problem for parabolic operator with constant coefficients. Perhaps it is worth mentioning that we do not use any regularity results on the free boundary problem for an American option. The basic idea of the proof comes from our earlier paper \[20\] devoted to standard American call and put options on single asset.

2 Preliminaries

We will assume that under the risk-neutral probability measure the underlying assets prices \(X^{s,x,1}, \ldots, X^{s,x,n}\) evolve on the time interval \([s,T]\) according to stochastic differential equation of the form

\[X^{s,x,i}_t = x_i + \int_s^t (r - d_i) X^{s,x,i}_\theta d\theta + \sum_{j=1}^n \int_s^t \sigma_{ij} X^{s,x,i}_\theta dW^j_\theta, \quad t \in [s,T].\]  

Here \(r \geq 0\) denotes the rate of interest, \(d_i \geq 0\) is the dividend rate of the asset \(i\), \(\sigma = \{\sigma_{ij}\}\) is the \(n\)-dimensional volatility matrix and \(W = (W^1, \ldots, W^n)\) is a standard \(n\)-dimensional Wiener process. As for the payoff function \(\psi\) we will assume that it satisfies the assumptions:

(A1) \(\psi\) is a nonnegative continuous function on \(\mathbb{R}^n\) with polynomial growth,

(A2) For every \(t \in (0, T)\), \(\psi\) is a smooth function on \(\{\psi = u\} \cap Q_t\) i.e. there exists an open set \(U \subset \mathbb{R}^n\) such that \(\{u = \psi\} \cap Q_t \subset [0, t] \times U\) and \(\psi\) is smooth on \(U\) (Here \(Q_t \equiv [0, t] \times \mathbb{R}^n\) and \(Q_t \equiv [0, t] \times \mathbb{R}^n\))

or

(A3) \(\psi\) is a nonnegative convex function on \(\mathbb{R}^n\) with polynomial growth.

Note that convex functions are locally Lipschitz, so assumption (A3) implies (A1). Assumption (A2) is considered in \[21\]. It is satisfied for instance if
(A2') The region where $\psi$ is strictly positive is the union of several connected components in which $\psi$ is smooth.

Following [21] let us also note that unlike (A2') or (A3), condition (A2) cannot be verified by appealing to the structure of the payoff alone. Verifying (A2) requires additional knowledge of the structure of the exercise set $\{u = \psi\}$.

Let $\Omega = C([0, T]; \mathbb{R}^n)$ and let $X$ be the canonical process on $\Omega$. For $(s, x) \in Q_T$ let $P_{s,x}$ denote the law of the process $X^{s,x} = (X^{s,x,1}, \ldots, X^{s,x,n})$ defined by (1) and let $\mathcal{F}_t$ denote the completion of $\sigma(X_{st}; \theta \in [s, t])$ with respect to the family $\{P_{s,\mu}; \mu \text{ a finite measure on } \mathcal{B}(\mathbb{R}^n)\}$, where $P_{s,\mu}(\cdot) = \int_{\mathbb{R}^n} P_{s,x}(\cdot) \mu(dx)$. Then for each $s \in [0, T)$, $\mathcal{X} = (\Omega, (\mathcal{F}_t)_{t \in [s, T]}, X, P_{s,x})$ is a Markov process on $[0, T]$.

In Bensoussan [3] and Karatzas [16] (see also Section 2.5 in [17]) it is shown that under (A1) the arbitrage-free value $V$ of an American option with payoff function $\psi$ and expiration time $T$ is given by the solution of the stopping problem

$$V(s, x) = \sup_{\tau \in \mathcal{T}_s} E_{s,x}(e^{-r(T-s)}\psi(X_\tau)),$$

where the supremum is taken over the set $\mathcal{T}_s$ of all $\{\mathcal{F}_t\}$-stopping times $\tau$ with values in $[s, T]$.

From the results proved in [8] it follows that under (A1) for every $(s, x)$ there exists a unique solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$, on the space $(\Omega, \mathcal{F}_T, P_{s,x})$, to the reflected BSDE with terminal condition $\psi(X_T)$, coefficient $f: \mathbb{R}^d \rightarrow \mathbb{R}$ defined as $f(y) = -ry$, $y \in \mathbb{R}^n$, and barrier $\psi(X)$ (RBSDE$_{s,x} (\psi, -ry, \psi)$ for short). This means that the processes $Y^{s,x}, Z^{s,x}, K^{s,x}$ are $\{\mathcal{F}_t\}$-progressively measurable, satisfy some integrability conditions and $P_{s,x}$-a.s.,

$$\begin{align*}
Y_t^{s,x} &= \psi(X_T) - \int_t^T rY_s^{s,x} \, d\theta + K_t^{s,x} - \int_t^T Z_s^{s,x} \, dB_{s,\theta}, \quad t \in [s, T], \\
Y_t^{s,x} &\geq \psi(X_t), \quad t \in [s, T], \\
K_t^{s,x} &\text{ is increasing, continuous, } K_s^{s,x} = 0, \quad \int_s^T (Y_t^{s,x} - \psi(X_t)) \, dK_t^{s,x} = 0,
\end{align*}$$

where $B_s$ is some standard Wiener process on $[s, T]$ under $P_{s,x}$. In [8] it is also proved that for every $(s, x) \in Q_T$,

$$Y_t^{s,x} = u(t, X_t), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.,}$$

where $u$ is a viscosity solution to the obstacle problem

$$\begin{align*}
&\begin{cases}
\min(u(s, x) - \psi(x), -u_s - L_{BS}u(s, x) + ru(s, x)) = 0, & (s, x) \in Q_T, \\
u(T, x) = \psi(x),
\end{cases} \\
&\text{with } \quad L_{BS}u = \sum_{i=1}^n (r - d_i)x_i u_{x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}x_ix_j u_{x_i x_j}, \quad (a = \sigma \sigma^*).
\end{align*}$$

From [8, 10] we know that $V$ defined by (2) is equal to $Y_s^{s,x}$. Hence

$$V(s, x) = Y_s^{s,x} = u(s, x), \quad (s, x) \in [0, T] \times \mathbb{R}^n. \quad (6)$$

In the next section we analyze $V$ via (6) but as a matter of fact instead of viscosity solutions of (5) we consider variational solutions which provide more information on the value function $V$. 

3
3 Obstacle problem for the Black and Scholes equation

Assume that \( \psi : \mathbb{R}^n \to \mathbb{R}_+ \) is continuous and satisfies the polynomial growth condition. Let \( H^1_\varrho = \{ u \in L^2(\mathbb{R}^n; \varrho^2 \, dx), \sum_{j=1}^n \sigma_{ij} x_i u_{x_j} \in L^2(\mathbb{R}^n; \varrho^2 \, dx), i = 1, \ldots, n \} \), where \( u_{x_i} \) denotes the partial derivative in the distribution sense, \( \varrho(x) = (1 + |x|^2)^{-\gamma} \) and \( \gamma > 0 \) is chosen so that \( \int_{\mathbb{R}^n} \varrho^2(x) \, dx < \infty \) and \( \int_{\mathbb{R}^n} \psi^2(x) \varrho^2(x) \, dx < \infty \). Following [2, 20] we adopt the following definition.

**Definition.** (a) A pair \((u, \mu)\) consisting of \( u \in L^2(0, T; H^1_\varrho) \cap C(\bar{Q}_T) \) and a Radon measure \( \mu \) on \( Q_T \) is a variational solution to (5) if

\[
L_{BS}u = ru - \mu \text{ satisfies the strong sense, i.e. for every } \eta \in C^\infty_0(Q_T),
\]

\[
\langle u, \eta \rangle_{\mathcal{D}, T} + \langle L_{BS} u, \eta \rangle_{\mathcal{D}, T} = r \langle u, \eta \rangle_{2, \mathcal{D}, T} - \int_{Q_T} \eta \varrho^2 \, d\mu,
\]

where

\[
\langle L_{BS} u, \eta \rangle_{\mathcal{D}, T} = \sum_{i=1}^n \left\langle (r - d_i) x_i u_{x_i}, \eta \right\rangle_{2, \mathcal{D}, T} - \frac{1}{2} \sum_{i,j=1}^n a_{ij} \langle u_{x_i}, (x, x_j \varrho^2) \rangle_{2, \mathcal{D}, T},
\]

Here \( \langle \cdot, \cdot \rangle_{\mathcal{D}, T} \) stands for the duality pairing between \( L^2(0, T; H^1_\varrho) \) and \( L^2(0, T; H^{-1}_\varrho) \), \( \langle \cdot, \cdot \rangle_{2, \mathcal{D}, T} \) is the scalar product in \( L^2(0, T; H^1_\varrho) \) and \( \langle \cdot, \cdot \rangle_{2, T} = \langle \cdot, \cdot \rangle_{2, \mathcal{D}, T} \) with \( \varrho \equiv 1 \).

(b) If \( \mu \) in the above definition admits a density (with respect to the Lebesgue measure) of the form \( \Phi_t(t, x) = \Phi(t, x, u(t, x)) \) for some measurable \( \Phi : \bar{Q}_T \times \mathbb{R} \to \mathbb{R}_+ \), then we say that \( u \) is a variational solution to the semilinear problem

\[
u_t + L_{BS}u = \varrho u - \Phi_u, \quad u(T, \cdot) = \psi, \quad u \geq \psi. \tag{7}
\]

In our main theorems below we show that if \( \psi \) satisfies (A1) and (A2) or (A3) then the measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure and its density has the form \( \Phi_u(t, x) = 1_{\{u(t, x) = \psi(x)\}} \Psi_u(x) \), where \( \Psi_u = \max\{-\Psi, 0\} \) and \( \Psi \) is determined by \( \psi \) and the parameters \( r, d, a \). In the next section we compute \( \Psi \) for some concrete options.

### 3.1 Payoffs satisfying (A1), (A2)

We will need the following notation. \( I = \{0,1\}^n \). For \( i = (i_1, \ldots, i_n) \in I \) we set

\[
D_i = \{ x \in \mathbb{R}^n; (-1)^{i_k} x_k > 0, k = 1, \ldots, n \}, \quad P = \bigcup_{i \in I} D_i, \quad P_T = [0, T) \times P.
\]

**Remark.** One can check that if \( u \) is a solution to (7) then \( v \) defined as

\[
v(t, x) = u(T - t, (-1)^{i_1} e^{x_1}, \ldots, (-1)^{i_n} e^{x_n}) \equiv u(T - t, e^x)
\]
for \( t \in [0, T] \), \( x = (x_1, \ldots, x_n) \in D_t \), \( t \in I \), is a variational solution of the Cauchy problem

\[
v_t - L v = -rv + \Phi, \quad v \geq \bar{\psi}, \quad v(0, \cdot) = \bar{\psi},
\]

where

\[
L v = \sum_{i=1}^{n} (r - d_i - \frac{1}{2} \sigma_i^2) v_{x_i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} v_{x_{ix_j}}
\]

and \( \hat{\Phi}(t, x) = \Phi_u(T - t, e^x) \), \( \bar{\psi}(t, x) = \psi(T - t, e^x) \). Furthermore, a simple calculation shows that if \( \eta \) is a smooth function on \( \mathbb{R}^n \) with compact support and \( U \subset \mathbb{R}^n \) is a bounded open set such that \( \text{supp}[\eta] \subset U \) then \( \tilde{v} = v\eta \) is a solution of the Cauchy-

Dirichlet problem

\[
\tilde{v}_t - \tilde{L} \tilde{v} = -r \tilde{v} + f, \quad \tilde{v}(0, \cdot) = \bar{\psi}, \quad \tilde{v}|_{[0,T] \times \partial U} = 0,
\]

where \( \tilde{\psi} = \bar{\psi} \eta \), \( \tilde{L} \) is some uniformly elliptic operator with smooth coefficients not depending on \( t \) and \( f \in L^2(0, T; L^2(U)) \). By classical regularity results (see, e.g., Theorem 5 in §7.1 in [11]), \( \tilde{v} \in L^2(0, T; H^2(U)) \cap L^\infty(0, T; H^1(U)) \) and \( \tilde{v}_t \in L^2(0, T; L^2(U)) \). From this and the construction of \( \tilde{v} \) we infer that the regularity properties of \( \tilde{v} \) are retained by \( u \). It follows in particular that

\[
u_t + L_{BS} u = ru - \Phi_u \quad \text{a.e. on } P_T. \tag{8}
\]

**Theorem 1.** Assume (A1), (A2).

(i) \( u \) defined by (6) is a variational solution of the semilinear Cauchy problem

\[
u_t + L_{BS} u = ru - \Phi^-_u, \quad u(T, \cdot) = \psi \tag{9}
\]

with

\[
\Phi_u(t, x) = 1_{\{u(t, x) = \psi(x)\}} \Psi(x), \quad (t, x) \in Q_T,
\]

where for \( x \in \mathbb{R}^n \) such that \( (t, x) \in \{u = \psi\} \),

\[
\Psi(x) = -r \psi(x) + L_{BS} \psi(x).
\]

(ii) Set \( \sigma(x) = \{\sigma_{ij} x_i\}_{i,j=1,\ldots,n} \) and

\[
K_{s,t} = \int_s^t \Phi^-_{\sigma}(\theta, X_\theta) \, d\theta, \quad t \in [s, T]. \tag{10}
\]

Then for every \( (s, x) \in P_T \) the triple \((u(\cdot, X), \sigma(X)u_x(\cdot, X), K_{s,\cdot})\) is a unique solution of \( RBSDE_{s,x}(\psi, -r\psi, \psi) \).

**Proof.** Fix \((s, x) \in P_T\). Let \((Y^{s,x}, Z^{s,x}, K^{s,x})\) be a solution of \( RBSDE_{s,x}(\psi, -r\psi, \psi) \) and let \( u \) be a viscosity solution of (5). For \( t_0 \in (s, T) \) let \( U \subset \mathbb{R}^n \) be an open set of assumption (A2). Then there exists \( \eta \in C^\infty(\mathbb{R}^n) \) such that \( \eta \geq 0, \eta \equiv 1 \) on \( \{u = \psi\} \cap Q_{t_0} \) and \( \eta \equiv 0 \) on \( U^c \) (we make the convention that \( \eta(t, x) = \eta(x) \)). Of course \((Y^{s,x}, Z^{s,x}, K^{s,x})\) is a solution of \( RBSDE_{s,x}(Y^{s,x}_{t_0}, -r\psi, \psi) \) on \([0, t_0]\). It is also a
solution of RBSDE\(_{s,x}(Y_{0}^{s,x},-r_{y},\tilde{\psi})\) on \([0,t_{0}]\) with \(\tilde{\psi}(x) = \eta(x)\psi(x)\), because \(\tilde{\psi} \leq \psi\) and by (3) and (4),
\[
\int_{s}^{t_{0}} (Y_{t}^{s,x} - \tilde{\psi}(X_{t})) \, dK_{t}^{s,x} = \int_{s}^{t_{0}} (u(t, X_{t}) - \tilde{\psi}(X_{t})) 1_{\{u(t, X_{t}) = \psi(X_{t})\}} \, dK_{t}^{s,x} \\
= \int_{s}^{t_{0}} (u(t, X_{t}) - \psi(X_{t})) \, dK_{t}^{s,x} = 0.
\]
Since \(\tilde{\psi}\) is smooth, applying Itô’s formula yields
\[
\tilde{\psi}(X_{t}) = \tilde{\psi}(X_{0}) + \sum_{i=1}^{n} \int_{0}^{t} \tilde{\psi}_{x_{i}}(X_{s}) \, dX_{s}^{i} + \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} a_{ij}X_{s}^{i}X_{s}^{j}\tilde{\psi}_{x_{i}x_{j}}(X_{s}) \, d\theta.
\]
Hence, by [8, Remark 4.3] and (4),
\[
dK_{t}^{s,x} = \alpha_{t}^{s,x} 1_{\{u = \psi\}}(X_{t}) \left[ - r\tilde{\psi}(X_{t}) + \sum_{i=1}^{n} (r - d_{i})X_{i}^{i}\tilde{\psi}_{x_{i}}(X_{t}) \right.
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}X_{i}^{i}X_{j}^{j}\tilde{\psi}_{x_{i}x_{j}}(X_{t}) \left. \right] \, dt
\]
on \([0,t_{0}]\). Thus
\[
dK_{t}^{s,x} = \alpha_{t}^{s,x} 1_{\{u(t, X_{t}) = \psi(X_{t})\}} \Psi^{-}(X_{t}) \, dt \quad (11)
on \([0,t_{0}]\) for every \(t_{0} \in [0,T]\). Consequently, the above equation is satisfied on \([0,T]\). By [2, Theorem 3] there exists a function \(\alpha\) on \(Q_{T}\) such that \(0 \leq \alpha \leq 1\) a.e. and for a.e. \((s,x) \in Q_{T}\),
\[
\alpha_{t}^{s,x} = \alpha(t, X_{t}), \quad dt \otimes P_{s,x}\text{-a.s.} \quad (12)
\]
Moreover, from [2, Theorem 3] it follows that \(u \in L^{2}(0, T; H_{0}^{1}) \cap C(\bar{Q}_{T})\) and \(u\) is a variational solution of the Cauchy problem
\[
u_{t} + \mathcal{L}_{BS}u = ru - \alpha 1_{\{u = \psi\}} \Psi^{-} \quad u(T, \cdot) = \psi.
\]
By the above and [8],
\[
\nu_{t} + \mathcal{L}_{BS}u = ru - \alpha 1_{\{u = \psi\}} \Psi^{-} \quad \text{a.e. on } Q_{T}.
\]
Hence
\[
\nu_{t} + \mathcal{L}_{BS}\psi = r\psi - \alpha \Psi^{-} \quad \text{a.e. on } \{u = \psi\}.
\]
On the other hand, by the definition of \(\Psi\),
\[
\nu_{t} + \mathcal{L}_{BS}\psi = \mathcal{L}_{BS}\psi = r\psi + \Psi \quad \text{on } \{u = \psi\}.
\]
Thus \(\Psi = -\alpha \Psi^{-}\) a.e., which implies that \(\alpha \Psi = \Psi\) a.e. on \(\{u = \psi\}\) and hence that
\[
1_{\{u = \psi\}} \alpha \Psi^{-} = 1_{\{u = \psi\}} \Psi^{-} \quad \text{a.e.} \quad (13)
\]
Accordingly (9) is satisfied. By Itô’s formula,
\[
X_{t}^{s,x,i} = x^{i} \exp \left( (r - d_{i} - a_{ii})(t - s) + \sum_{j=1}^{n} \sigma_{ij}(W_{t}^{j} - W_{s}^{j}) \right), \quad t \in [s,T].
\]
Therefore if $s \in [0, T)$ and $x \in D$, for some $t \in I$ then $P_{s,x}(X_t \in D_t, t \geq s) = 1$ and for each $t \in (s, T]$ the random variable $X_t$ has strictly positive density on $D_t$ under $P_{s,x}$.

From this and (13) it follows that

$$\mathbf{1}_{\{u = \psi\}}(t, X_t) \alpha(t, X_t) \Psi^-(X_t) = \mathbf{1}_{\{u = \psi\}}(t, X_t) \Psi^-(X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \quad (14)$$

for every $(s, x) \in P_T$. In [19] it is proved that the function $\mathbf{1}_{\{u = \psi\}} \alpha$ is a weak limit in $L^2(Q_T)$ of some sequence $\{\alpha_n\}$ of nonnegative functions bounded by 1 and such that $\alpha_n(t, X_t) \rightarrow \alpha^s_{t,x}$ weakly in $L^2([0, T] \times \Omega; dt \otimes P_{s,x})$ for every $(s, x) \in Q_T$. Therefore using once again the fact that for every $(s, x) \in P_T$ the process $X$ has a strictly positive transition density under $P_{s,x}$ we conclude that (12) holds for every $(s, x) \in P_T$, which when combined with (13) implies (10). What is left is to prove that for every $(s, x) \in P_T$,

$$Z_t^{s,x} = \sigma(X_t)u_x(t, X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \quad (15)$$

From the results proved in [8, Section 6] it follows that for every $(s, x) \in Q_T$,

$$E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x,n} - Y_t^{s,x}|^2 + E_{s,x} \int_s^T |Z_t^{s,x,n} - Z_t^{s,x}|^2 dt \rightarrow 0, \quad (16)$$

where $(Y^{s,x,n}, Z^{s,x,n})$ is a solution of the BSDE

$$Y_t^{s,x,n} = \psi(X_T) - \int_t^T r Y_{\theta}^{s,x,n} d\theta + \int_t^T n(Y_{\theta}^{s,x,n} - \psi(X_\theta))^- d\theta - \int_t^T Z_{\theta}^{s,x,n} dB_{\theta}. \quad (17)$$

It is known (see [23]) that

$$Y_t^{s,x,n} = u_n(t, X_t), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \quad (17)$$

where $u_n$ is a viscosity solution of the Cauchy problem

$$(u_n)_t + L_B u_n = -ru_n + n(u_n - \psi)^-, \quad u_n(T, \cdot) = \psi. \quad (18)$$

We know that $P_{s,x}(X_t \in D_t, t \geq s) = 1$ if $x \in D$. Moreover, by classical regularity results (see, e.g., [13, Theorem 1.5.9] and Remark preceding Theorem [1], $u_n \in C^{1,2}(P_T)$.

From this, the fact that $X^{s,x,n}$ is a solution of SDE with smooth coefficients and Proposition 1.2.3 and Theorem 2.2.1 in [22] it follows that $Y_t^{s,x,n} \in \mathbb{D}^{1,2}$ for every $(s, x) \in P_T$, where $\mathbb{D}^{1,2}$ is the domain of the derivative operator $D_t$ in $L^2(\Omega)$ (see [22, Section 1.2] for the definition). Therefore as in the proof of [20, Theorem 2.3] one can show that for every $t \in (s, T]$ and $x \in P$, $D_t Y_t^{s,x,n} = Z_t^{s,x,n}, P_{s,x}\text{-a.s.},$ and on the other hand, that $D_t Y_t^{s,x,n} = \sigma(X_t)(u_n)_x(t, X_t), P_{s,x}\text{-a.s.}$ Hence, by (17),

$$Z_t^{s,x,n} = \sigma(X_t)(u_n)_x(t, X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \quad (18)$$

for every $(s, x) \in P_T$. By (16) and (17), $u_n \rightarrow u$ pointwise in $Q_T$. Moreover, from (17), (18) and standard estimates for solutions of BSDEs (see, e.g., [8, Section 6]) it follows that there is $C > 0$ such that for any $(s, x) \in P_T$,

$$E_{s,x} \sup_{s \leq t \leq T} |u_n(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma(X_t)(u_n)_x(t, X_t)|^2 dt \leq CE_{s,x} \sup_{s \leq t \leq T} |\psi(X_t)|^2, \quad (19)$$
while from (16), (18) it follows that

\[ E_{s,x} \sup_{s \leq t \leq T} \int_s^T |\sigma(X_t)((u_n)_x - (u_m)_x)(t, X_t)|^2 \, dt \to 0 \]  

(20)

as \( n, m \to \infty \). From (19) one can deduce that \( u_n \in L^2(0, T; H) \) and then, by using (20), that \( u_n \to u \) in \( L^2(0, T; H) \) (see the arguments following (2.12) in the proof of [20, Theorem 2.3]). From the last convergence and (16), (18) it may be concluded that

\[ E_{s,x} \int_s^T |\sigma(X_t)(u_n)_x(t, X_t) - Z_t^{s,x}|^2 \, dt = 0 \]

for \((s, x) \in P_T\), which implies (15).

3.2 Convex payoffs

Assume that \( \psi : \mathbb{R}^n \to \mathbb{R} \) is convex. Let \( m \) denote the Lebesgue measure on \( \mathbb{R}^n \), \( \nabla_i \psi \) denote the usual partial derivative with respect to \( x_i \), \( i = 1, \ldots, n \), and let \( E \) be set of all \( x \in \mathbb{R}^n \) for which the gradient

\[ \nabla \psi(x) = (\nabla_1 \psi(x), \ldots, \nabla_n \psi(x)) \]

exists. Since \( \psi \) is locally Lipschitz function, \( m(E^c) = 0 \) and \( \nabla \psi = (\psi_{x_1}, \ldots, \psi_{x_n}) \) a.e.

Moreover, for a.e. \( x \in E \) there exists matrix \( \nabla^2 \psi(x) = \{\nabla^2_{ij} \psi(x)\} \) such that

\[ \lim_{E \ni y \to x} \frac{\nabla \psi(y) - \nabla \psi(x) - \nabla \psi(x)(y - x)}{|y - x|} = 0, \]

(21)

i.e. \( \nabla^2_{ij} \psi \) are defined as limits through the set where \( \nabla_i \psi \) exists (see, e.g., [1, Section 7.9]). By Alexandrov’s theorem (see, e.g., [1, Theorem 7.10], if \( \psi \) is a locally Lipschitz function, \( \nabla \psi \) exists, and for each Borel set \( B \), \( \{\mu_{ij}(B)\} \) is a nonnegative definite matrix (see, e.g., [12, Section 6.3]). Let \( \mu_{ij} = \mu^a_{ij} + \mu^s_{ij} \) be the Lebesgue decomposition of \( \mu_{ij} \) into the absolutely continuous and singular parts with respect to \( m \). By Theorem 1 in Section 6.4 in [12],

\[ \mu^a_{ij}(dx) = \nabla^2_{ij} \psi(x) \, dx. \]  

(22)

For \( R > 0 \) set \( D_R = P \cap \{x \in \mathbb{R}^n : |x| < R\} \) and \( \tau_R = \inf\{t \geq s : X_t \notin D_R\} \).

Let \( L_{BS} \) denote the operator formally adjoint to \( L_{BS} \). By [21, Theorem 4.2.5] for a sufficiently large \( \alpha > 0 \) there exist the Green’s functions \( G^0_R, \ G^a_R \) for \( \alpha - L_{BS} \) and \( \alpha - L_{BS} \) on \( D_R \). Let \( A \) be a continuous additive functional of \( \mathcal{F} \) and let \( \nu \) denote the Revuz measure of \( A \) (see, e.g., [25]). By the theorem proved in Section V.5 of [25], for every nonnegative \( f \in C_0(\mathbb{R}^d)\),

\[ E_{s,x} \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA^\nu_t = \int_{\mathbb{R}^n} G^0_R(x, y) f(y) \, \nu(dy). \]
Theorem 4.2.5]).

Note that if $P$ is not identically equal to zero then $\tilde{G}_R^\alpha g$ is strictly positive (see [24, Theorem 4.2.5]).

Set

$$\mathcal{L}_{BS} = \sum_{i=1}^{n} (r - d_i) x_i \nabla_i + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j \nabla_{ij}^2 .$$

**Theorem 2.** Assume (A3). Then assertions (i), (ii) of Theorem [7] hold true with $L_{BS}$ replaced by $\mathcal{L}_{BS}$.

**Proof.** We use the notation of Theorem [1]. Fix $s \in [0, T)$. Since $\psi$ is a continuous convex function, from Itô's formula proved in [3] it follows that there exists a continuous increasing process $A$ such that for $x \in \mathbb{R}^n$,

$$\psi(X_t) = \psi(X_s) + A_t + \int_{s}^{t} \nabla \psi(X_{\theta}) \, dX_{\theta}, \quad t \in [s, T], \quad P_{s,x}-a.s. \quad (24)$$

From [21] it follows that $A$ is a positive continuous additive functional (PCAF for short) of $X$. Let $\nu$ denote the Revuz measure of $A$. We are going to show that $1_P \cdot \nu = 1_P \cdot \mu$ where $\mu$ is the measure on $\mathbb{R}^n$ defined as

$$\mu(dx) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \mu_{ij}(dx).$$

To this end, let us set

$$\mu^{\varepsilon}_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} \ast \rho_{\varepsilon}, \quad \mu^\varepsilon(dx) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \mu^{\varepsilon}_{ij}(dx),$$

where $\{\rho_{\varepsilon}\}_{\varepsilon>0}$ is some family of mollifiers. Fix a nonnegative $g \in C_0(D_R)$ such that $g(x) > 0$ for some $x \in D_R$ and denote by $A^\varepsilon$ the PCAF of $X$ in Revuz correspondence with $\mu_{\varepsilon}$. Then for a sufficiently large $\alpha > 0$,

$$E_{s,g,m} \int_{s}^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t^\varepsilon = \int_{\mathbb{R}^n} \tilde{G}_R^\alpha g(y) f(y) \, \mu^\varepsilon(dy) \quad (25)$$

for all nonnegative $f \in C_0(\mathbb{R}^d)$. By [3, Theorem 2], $E_{s,x} \sup_{t \geq s} |A_{t \wedge \tau_R}^\varepsilon - A_{t \wedge \tau_R}| \rightarrow 0$ as $\varepsilon \downarrow 0$ for every $x \in \mathbb{R}^d$. Hence $\int_{s}^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t^\varepsilon \rightarrow \int_{s}^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t$ weakly under $P_{s,x}$ for $x \in \mathbb{R}^d$. Since

$$A_{t \wedge \tau_R}^\varepsilon = \psi_\varepsilon(X_{t \wedge \tau_R}) - \psi_\varepsilon(X_s) - \int_{s}^{t \wedge \tau_R} \nabla \psi_\varepsilon(X_{\theta}) \, dX_{\theta}$$
and $\sup_{\varepsilon>0} \sup_{|x| \leq R} |\nabla \psi_{\varepsilon}(x)| \leq C(R) < \infty$ by Lemma in [6], it follows that for every compact subset $K \subset \mathbb{R}^n$, $\sup_{x \in K} \sup_{\varepsilon>0} E_{s,x}|A_t^{\varepsilon}|^2 < \infty$. Therefore

$$E_{s,g,m} \int_s^{r}\ e^{-at} f(X_t) \ dA_t \rightarrow E_{s,g,m} \int_s^{r}\ e^{-at} f(X_t) \ dA_t$$

as $\varepsilon \downarrow 0$. On the other hand, since $\mu_t^{\varepsilon} \rightarrow \mu_{ij}$ weakly* for $i, j = 1, \ldots, n$ and, by [24, Theorem 4.2.5], $fG_R^\alpha g \in C_0(G_R)$, we have

$$\sum_{i,j=1}^n \int_{\mathbb{R}^n} G_R^\alpha g(y) f(y) a_{ij} y_i y_j \mu_t^{\varepsilon}(dy) \rightarrow \sum_{i,j=1}^n \int_{\mathbb{R}^n} G_R^\alpha g(y) f(y) a_{ij} y_i y_j \mu_{ij}(dy).$$

Combining this with (23), (25), (26) we see that for every $f \in C_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} G_R^\alpha g(y) f(y) \mu(dy) = \int_{\mathbb{R}^n} G_R^\alpha g(y) f(y) \nu(dy).$$

Since $G_R^\alpha g$ is strictly positive on $D_R$, we conclude from the above that $\mu = \nu$ on $D_R$ for each $R > 0$. Consequently, $\mu = \nu$ on $P$. For $x \in P$, $P_{s,x}(X_t \in \mathbb{R}^n \setminus P) = 0$ for $t \geq s$. Hence

$$A_t^{1(1)} = \int_s^t 1\ P(X_s) \ dA_s = A_t^{1(1, \nu)} = A_t^{1(1, \mu)} \quad t \geq s, \quad P_{s,x}-a.s.$$ (27)

for $x \in P$. Let $ \mu^a$ denote the absolutely continuous part in the Lebesgue decomposition of $1\ P \cdot \mu$. By (22), $\mu^a(dx) = \sum_{i,j=1}^n 1\ P(x) a_{ij} x_i x_j \nabla^2_{ij} \psi(x) dx$. Hence

$$A_t^{\mu^a} = \sum_{i,j=1}^n \int_{s}^{t} a_{ij} X^i_{\theta} X^j_{\theta} \nabla^2_{ij} \psi(X_t) \ d\theta, \quad t \geq s, \quad P_{s,x}-a.s.$$ (28)

for $x \in P$. From (24), (27), (28) and [8, Remark 4.3] it follows that

$$dK_t^{s,x} = \alpha_t^{s,x} 1\{u=\psi\}(X_t) \left( -r \psi(X_t) + \sum_{i=1}^n (r - d_i) X^i_{t} \nabla X^i_{t} \psi(X_t) \right)$$

$$+ \frac{1}{2} \sum_{i,j=1}^n a_{ij} X^i_{\theta} X^j_{\theta} \nabla^2_{ij} \psi(X_t) \ dt.$$ (29)

Let $u$ be a viscosity solution of (16). From the above and the results proved in [2] (see the reasoning following (11)) we conclude that $u \in L^2(0, T; H^1_0) \cap C(Q_T)$ and there is a function $\alpha$ on $Q_T$ such that $0 \leq \alpha \leq 1$ a.e., (12) is satisfied and $u$ is a variational solution of the Cauchy problem

$$u_t + L_{BS} u = ru - \alpha 1\{u=\psi\} \Psi^-, \quad u(T, \cdot) = \psi$$ (29)

with

$$\Psi = -r \psi + L_{BS} \psi \quad \text{on} \ {u = \psi}.$$ (30)

By Remark preceding Theorem 1, $u(t, \cdot) \in H^2_{loc}(\mathbb{R}^n)$. Therefore by Remark (ii) following Theorem 4 in Section 6.1 in [12] the distributional derivatives $u_{x_i}$, $u_{x_ix_j}$ are a.e. equal to the approximate derivatives $\nabla^a_{ij} u$, $(\nabla^a_{ij})^2 u$. Let $L^a_{BS}$ denote the operator defined
as $\mathcal{L}_{BS}$ but with $\nabla_i$, $\nabla_{ij}$ replaced by $\nabla^\text{ap}_i$, $(\nabla^\text{ap})^2_{ij}$. Then $u$ is a variational solution of (29) with $\mathcal{L}_{BS}$ replaced by $\mathcal{L}_{BS}^{ap}$. Hence $u_t + \mathcal{L}_{BS}^{ap}u = ru - \alpha 1_{\{u=\psi\}}\Psi^-$ a.e. on $QT$. Consequently,

$$\mathcal{L}_{BS}^{ap}\psi = r\psi - \alpha\Psi^- \quad \text{a.e. on } \{u = \psi\}. \quad (31)$$

Since $\psi$ is convex, $\mathcal{L}_{BS}\psi = \mathcal{L}_{BS}^{ap}\psi$ a.e. on $\mathbb{R}^n$ by Remark (i) following Theorem 4 in Section 6.1 in [12]. Therefore combining (30) with (31) we see that $\Psi = -\alpha\Psi^-$ a.e. on $\{u = \psi\}$. To complete the proof it suffices now to repeat step by step the arguments following (14) in the proof of Theorem 1.

4 The early exercise premium representation

Let $\xi$ denote the payoff process for an American option with payoff function $\psi$, i.e.

$$\xi_t = e^{-r(t-s)}\psi(X_t), \quad t \in [s,T],$$

and let $\eta$ denote the Snell envelope for $\xi$, i.e. the smallest supermartingale which dominates $\xi$. It is known (see, e.g., Section 2.5 in [17]) that

$$\eta_t = e^{-r(t-s)}V(t, X_t), \quad t \in [s,T].$$

Assume (A1), (A2) or (A3). Applying Itô’s formula and using Theorem 1 or 2 we get

$$\eta_t = e^{-r(t-s)}Y_t^{s,x} = e^{-r(T-s)}\psi(X_T) + \int_t^T e^{-r(\theta-s)}\Phi^-(X_\theta, Y^{s,x}_\theta) \, d\theta$$

$$- \int_t^T e^{-r(\theta-s)}Z^{s,x}_\theta \, dW_\theta, \quad t \in [s,T], \quad \mathbb{P}_{s,x}\text{-a.s.},$$

which leads to the following corollary.

**Corollary 3.** For every $(s, x) \in QT$ the Snell envelope admits the representation

$$\eta_t = E_{s,x}\left( e^{-r(T-s)}\psi(X_T) + \int_s^T e^{-r(\theta-s)}\Phi^-(X_\theta, Y^{s,x}_\theta) \, d\theta \mid \mathcal{F}_t \right), \quad t \in [s,T]. \quad (32)$$

Taking $t = s$ in (32) and using (14) we get the early exercise premium representation for the value function.

**Corollary 4.** For every $(s, x) \in QT$ the value function $V$ admits the representation

$$V(s, x) = V_E(s, x) + E_{s,x} \int_s^T e^{-r(t-s)}1_{\{V(t, X_t) = \psi(X_t)\}}\Psi^-(X_t) \, dt,$$

where

$$V_E(s, x) = E_{s,x}(e^{-r(T-s)}\psi(X_T))$$

is the value of the European option with payoff function $\psi$ and expiration time $T$. 

In closing this section we show by examples that for many options $\Psi^-$ can be explicitly computed. Using results of §4 and §5 in [26] one can check that the payoff functions $\psi$ in examples 1–4 below satisfy (A3). It is also easy to see that the payoff function $\psi$ in example 5 satisfies $(A2')$. Note that the payoff function in example 1 also satisfies $(A2')$ and, by [5, 21], the payoff functions in examples 2–4 satisfy (A2). We would like to stress that the last assertion is by no means evident. On the other hand, the convexity of $\psi$ in examples 2–4 is readily checked.

In all the examples we have computed the corresponding functions $\Psi^-$ on the region \{\(u = \psi\}\}. When computing $\Psi$ we keep in mind that \(\{u = \psi\} \subset [0, T] \times \{\psi > 0\}\).

1. **Index options and spread options**

\[
\psi(x) = \left( \sum_{i=1}^{n} w_i x_i - K \right)^+, \quad \Psi^-(x) = \left( \sum_{i=1}^{n} w_i d_i x_i - rK \right)^+ \quad \text{(call)}
\]

\[
\psi(x) = \left( K - \sum_{i=1}^{n} w_i x_i \right)^+, \quad \Psi^-(x) = \left( rK - \sum_{i=1}^{n} w_i d_i x_i \right)^+ \quad \text{(put)}
\]

(Here $w_i \in \mathbb{R}$ for $i = 1, \ldots, n$).

2. **Max options**

\[
\psi(x) = (\max\{x_1, \ldots, x_n\} - K)^+ \quad \text{(call on max)}
\]

\[
\Psi^-(x) = \left( \sum_{i=1}^{n} d_i 1_{B_i}(x) x_i - rK \right)^+, \quad \text{where } B_i = \{x \in \mathbb{R}^n; x_i > x_j, j \neq i\}.
\]

3. **Min options**

\[
\psi(x) = (K - \min\{x_1, \ldots, x_n\})^+ \quad \text{(put on min)}
\]

\[
\Psi^-(x) = (rK - \sum_{i=1}^{n} d_i 1_{C_i}(x) x_i)^+, \quad \text{where } C_i = \{x \in \mathbb{R}^n; x_i < x_j, j \neq i\}.
\]

4. **Multiple strike options**

\[
\psi(x) = (\max\{x_1 - K_1, \ldots, x_n - K_n\})^+.
\]

\[
\Psi^-(x) = \left( \sum_{i=1}^{n} 1_{B_i}(x - K)(d_i x_i - rK_i) \right)^+,
\]

where $K = (K_1, \ldots, K_n)$.

5. **Power-product options**

\[
\psi(x) = (|x_1 \cdot \ldots \cdot x_n|^\gamma - K)^+ \quad \text{for some } \gamma > 0.
\]

If $x \in D_i$ with $i = (i_1, \ldots, i_n) \in \{0, 1\}^n$ then

\[
\Psi^-(x) = (r - \gamma \sum_{i=1}^{n} (r - d_i - a_{ii}) - \gamma^2 \sum_{i,j=1}^{n} a_{ij} f(x) - rK)^+,
\]

where $f(x) = ((-1)^{i_1} x_1 \cdot \ldots \cdot x_n)^\gamma$ and $|i| = i_1 + \ldots + i_n$. 12
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