An $xp$ model on AdS$_2$ spacetime

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Abstract

In this paper we formulate the $xp$ model on the AdS$_2$ spacetime. We find that the spectrum of the Hamiltonian has positive and negative eigenvalues, whose absolute values are given by a harmonic oscillator spectrum, which in turn coincides with that of a massive Dirac fermion in AdS$_2$. We extend this result to generic $xp$ models which are shown to be equivalent to a massive Dirac fermion on spacetimes whose metric depend of the $xp$ Hamiltonian. Finally, we construct the generators of the isometry group $SO(2,1)$ of the AdS$_2$ spacetime, and discuss the relation with conformal quantum mechanics.

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1. Introduction

In 1999 Berry, Keating and Connes suggested that a spectral realization of the Riemann zeros could be achieved by quantizing the classical Hamiltonian $H = xp$, where $x$ and $p$ are the position and momenta of a particle moving in one dimension [1,2]. Several recent works have been devoted to clarify the possible relation of the $xp$ model with the Riemann zeros [3-9]. In references [1,5,8] it was advocated that one needs to modify the $xp$ Hamiltonian in order to have a discrete spectrum, since the standard quantization of $xp$ yields a continuum [10,11]. The new Hamiltonians take the form $H = w(x)(p + \ell_p^2/p)$, where $w(x) = x \ (x \geq \ell_x)$ and $w(x) = x + \ell_x^2/x \ (x \geq 0)$, with $\ell_x \ell_p = 2\pi \hbar$ [1,5,8].

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The latter results led to the study of the general Hamiltonian \( H = w(x)(p + \ell_p^2/p) \) for arbitrary positive functions \( w(x) \) \cite{8}. It turns out that these models describe the motion of a relativistic particle moving in a 1+1 spacetime whose metric is determined by \( w(x) \). The Riemann scalar curvature vanishes identically for the linear potential \( w(x) = x \), and asymptotically for \( w(x) = x + \ell_p^2/x \). On the other hand, the potential \( w(x) = w_0 \cosh(x/R) \) yields a spacetime with constant negative curvature \( \mathcal{R} = -2/R^2 \), which corresponds to an anti-de-Sitter spacetime (AdS) \cite{8}. Here though, the semiclassical spectrum has positive and negative branches each one given by a harmonic oscillator spectrum, so it is not connected with the spectral interpretation of the Riemann zeros \cite{8}.

In this paper we shall focus on the \( xp \) model whose underlying spacetime is AdS. We shall first show that the previous semiclassical spectrum is exact in the quantum theory. In the course of our investigation we found out that this spectrum coincides with the one of a massive Dirac field in AdS. This connection is not special to AdS but has a wider range of application. The general \( xp \) model formulated in reference \cite{8} is actually equivalent to a massive Dirac fermion on a space-time whose metric can be constructed out from the function \( w(x) \) defining the \( xp \) models. In this paper we shall give a proof of this unexpected connection between two a priori different theories. The AdS metric has an isometry group given by \( SO(2,1) \), which led us to construct the generators of this group in the \( xp \) model. \( SO(2,1) \) is also the conformal group in 0 + 1 dimensions and is therefore the symmetry group of conformal quantum mechanics \cite{12}-\cite{15}. This suggests a connection between field theories in AdS and a conformal quantum mechanics living on the one-dimensional boundary of AdS. However, to find explicit realizations of this correspondence has shown to be rather elusive, motivating several works to clarify the AdS/CFT correspondence \cite{16}-\cite{21}. At the end of the paper we shall present some remarks concerning the conformal quantum mechanics formulated by Alfaro-Fubini-Furlan \cite{13} and related studies by Jackiw and collaborators \cite{17,18}. We have also included three appendices with complementary material.
2. The $xp$-AdS$_2$ model

The general $xp$ Hamiltonians are defined by [8]

$$H = pU(x) + \frac{\ell_p^2 V(x)}{p}, \quad x \in R,$$  \hspace{0.5cm} (1)

where $x$ and $p$ are the position and momentum of a particle moving in the real line $R$, $\ell_p$ is a parameter with the dimension of momenta, and $U(x)$ and $V(x)$ are positive functions. In references [4, 8], the Hamiltonian (1) was defined on a half-line, but this study we shall consider the whole line in order to discuss the AdS$_2$ model.

The Hamiltonian (1) breaks time reversal symmetry, under which $H \to -H$. Using the Hamiltonian equations of motion, the momentum $p$ can be expressed in terms of the velocity $\dot{x}$ as

$$p = \ell_p \eta \sqrt{\frac{V(x)}{U(x) - \dot{x}}},$$  \hspace{0.5cm} (2)

where $\eta = \pm 1$ is the sign of the momentum and the energy, which are conserved quantities. Substituting (2) back into the Lagrangian $L = p \dot{x} - H$ one obtains the action

$$S = \int dt L = -\ell_p \eta \int \sqrt{-ds^2},$$  \hspace{0.5cm} (3)

which, for either sign of $\eta$, is the action of a relativistic massive particle moving in a spacetime with metric

$$ds^2 = 4V(x)(-U(x)dt^2 + dt dx).$$  \hspace{0.5cm} (4)

The momenta $\ell_p$ plays the role of $mc$, where $m$ is a mass and $c$ the speed of light. Note the special form of the metric where the component $g_{xx}$ vanishes. From eq.(1) it is clear that the existence of positive and negative momentum implies that the energy also has two possible signs, so that classically the spectrum consists of two branches of energies. Consequently, the semiclassical and quantum eigenergies will be positive and negative as well.

The Riemann scalar curvature is

$$\mathcal{R}(x) = -\frac{1}{V(x)} \partial_x \left[ \frac{\partial_x(U(x)V(x))}{V(x)} \right] ,$$  \hspace{0.5cm} (5)
such that choosing $U$ and $V$ as

\[ U(x) = V(x) \equiv w(x) = w_0 \cosh \frac{x}{R} \to \mathcal{R}(x) = -\frac{2}{R^2}, \quad x \in (-\infty, \infty). \quad (6) \]

gives a constant negative scalar curvature, that corresponds to a AdS$_2$ spacetime of radius $R$. This fact implies that the classical trajectories generated by the classical Hamiltonian (1), can be seen as the geodesics of the AdS$_2$ spacetime (see Appendix B). The choice (6) gives

\[ \frac{1}{4} ds^2 = -w_0^2 \cosh^2 \left( \frac{x}{R} \right) dt^2 + w_0 \cosh \left( \frac{x}{R} \right) dt \, dx, \quad (7) \]

and making the change of variables

\[ \sinh \frac{x}{R} = \tan \theta, \quad t = \frac{R}{2w_0} (\tau + \theta), \quad \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \quad \tau \in (-\infty, \infty) \quad (8) \]

brings (7) into a standard form of the AdS$_2$ metric (see Appendix A)

\[ ds^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2). \quad (9) \]

3. Semiclassical spectrum

The general covariance of the action (3) allows one to choose a symmetric gauge where $U(x) = V(x) \equiv w(x)$ [8]. Let us further assume that $w(x)$ is an even function as in eq.(6). In these cases the number of semiclassical energy levels, $n(E)$, between 0 and $E > 0$, is given by [8]

\[ n(E) + \frac{1}{2} = \frac{1}{2\pi \hbar} \int_{-x_M}^{x_M} dx \frac{w(x)}{w(x)} \sqrt{E^2 - 4\ell_p^2 w^2(x)} \quad (10) \]

where $x_M$ is the turning point of the classical trajectories, i.e. $E = 2\ell_p w(x_M)$. The constant $\frac{1}{2}$ has been included to account for the Maslov phase associated to the classical trajectories (see Appendix B). Using the variable $\theta$ defined in eq.(8) one finds

\[ n + \frac{1}{2} = \frac{2R\ell_p}{\pi \hbar} \int_0^{\theta_M} d\theta \sqrt{\varepsilon^2 - \frac{1}{\cos^2 \theta}} = \frac{R\ell_p}{\hbar} (\varepsilon - 1), \quad (11) \]
where

$$\varepsilon = \frac{E}{2w_0\ell_p} = \frac{1}{\cos \theta_M}. \quad (12)$$

Hence the semiclassical spectrum is given by

$$E_n = \frac{2\hbar w_0}{R} (n + \frac{R\ell_p}{\hbar} + \frac{1}{2}), \quad n = 0, 1, \ldots \quad (13)$$

which coincides with the harmonic oscillator spectrum with a zero point energy that depends on the dimensionless constant

$$\kappa = \frac{R\ell_p}{\hbar}, \quad (14)$$

that may take any positive value. Note that the spectrum also contains negative eigenenergies whose absolute value is given by eq.(13).

4. Quantum spectrum

The classical Hamiltonian (1) can be quantized in terms of the following normal ordered operator [8]

$$\hat{H} = \sqrt{U(x)} \hat{p} \sqrt{U(x)} + \ell_p^2 \sqrt{V(x)} \hat{p}^{-1} \sqrt{V(x)}. \quad (15)$$

where $\hat{p} = -i\hbar d/dx$ and $\hat{p}^{-1}$ is the pseudo-differential operator that acts of wave functions as

$$(\hat{p}^{-1}\psi)(x) = -\frac{i}{\hbar} \int_{x}^{\infty} dy \psi(y). \quad (16)$$

Note that $\hat{p} \hat{p}^{-1} = \hat{p}^{-1} \hat{p} = 1$ on wave functions that vanish at $x = \infty$. The normal ordering chosen in (15) is not unique, but it leads to an hermitean operator under the conditions to be discussed below. In the case where $U(x) = x$ and $V(x) = 0$, equation (15) yields $\hat{H} = x^{1/2} \hat{p} x^{1/2} = \frac{1}{2}(x \hat{p} + \hat{p} x)$, which coincides with the well known Berry-Keating-Connes Hamiltonian [1] [2]. Using the previous definitions the action of (15) is given by

$$(\hat{H}\psi)(x) = -i\hbar \sqrt{U(x)} \frac{d}{dx} \{\sqrt{U(x)}\psi(x)\} - \frac{i\ell_p^2}{\hbar} \int_{x}^{\infty} dy \sqrt{V(x)V(y)}\psi(y). \quad (17)$$
\( \hat{H} \) is a symmetric operator, i.e.

\[ \langle \psi_1 | \hat{H} | \psi_2 \rangle = \langle \hat{H} \psi_1 | \psi_2 \rangle = 0, \]  

(18)

acting on wave functions that satisfy the conditions

\[ \lim_{x \to \pm\infty} \sqrt{U(x)} \psi(x) = 0, \quad \int_{-\infty}^{\infty} dx \sqrt{V(x)} \psi(x) = 0. \]  

(19)

Choosing the symmetric gauge and defining the function

\[ u(x) = \sqrt{w(x)}, \]  

(20)

one can write the Schroedinger equation as

\[ -i\hbar u(x) \frac{d}{dx} \{u(x) \psi(x)\} - \frac{i\ell^2_p}{\hbar} \int_x^\infty dy u(x) u(y) \psi(y) = E \psi(x), \]  

(21)

and similarly

\[ \hbar^2 \frac{d}{dx} \{u(x) \psi(x)\} + \ell^2_p \int_x^\infty dy u(y) \psi(y) - iE \hbar \frac{\psi(x)}{u(x)} = 0. \]  

(22)

In terms of the function

\[ \phi(x) = u(x) \psi(x), \]  

(23)

eq. (22) reads

\[ \hbar^2 \frac{d\phi(x)}{dx} + \ell^2_p \int_x^\infty dy \phi(y) - iE \hbar \frac{\phi(x)}{w(x)} = 0, \]  

(24)

and the conditions \eqref{19},

\[ \lim_{x \to \pm\infty} \phi(x) = 0, \quad \int_{-\infty}^{\infty} dx \, \phi(x) = 0. \]  

(25)

To solve eq. (24) one takes a derivative obtaining

\[ \hbar^2 \frac{d^2\phi(x)}{dx^2} - iE \hbar \frac{d}{dx} \left( \frac{\phi(x)}{w(x)} \right) - \ell^2_p \phi(x) = 0. \]  

(26)

Equations (26) and (25) imply (24) for functions \( \phi(x) \) which decay sufficiently fast at \( \pm\infty \). For the function
\[ w(x) = w_0 \cosh \frac{x}{R}, \quad (27) \]

one gets

\[ \partial_x^2 \phi(x) - \frac{iE}{\hbar w_0 \cosh(x/R)} \partial_x \phi(x) + \left( \frac{iE \sinh(x/R)}{\hbar w_0 R \cosh^2(x/R)} - \frac{\ell^2}{\hbar^2} \right) \phi(x) = 0. \quad (28) \]

In the limit \( |x| \to \infty \) this equation becomes

\[ \partial_x^2 \phi(x) - \frac{\ell^2}{\hbar^2} \phi(x) \sim 0, \quad (29) \]

so that the wave function decays asymptotically as

\[ \phi(x) \to e^{-|x|\ell \sqrt{\frac{\alpha}{\hbar^2}}} = e^{-|x|\frac{\ell p}{\hbar}}, \quad |x| \to \infty \quad (30) \]

To solve eq. (28) we first write it in terms of the angle variable \( \theta \),

\[ \cos^2 \theta \partial_{\theta}^2 \phi - \cos \theta (\sin \theta + i \alpha \cos \theta) \partial_\theta \phi + (i \alpha \sin \theta \cos \theta - \kappa^2) \phi = 0 \quad (31) \]

where

\[ \alpha = \frac{ER}{\hbar w_0}. \quad (32) \]

Notice that the semiclassical result (13) implies for \( \alpha \)

\[ \alpha_n^{(sc)} = 2n + 2\kappa + 1, \quad n = 0, 1, \ldots \quad (33) \]

together with the negative values \(-\alpha_n^{(sc)}\). Next we define the function \( f(\theta) \) as

\[ f(\theta) = 2 \cos \theta \phi(\theta), \quad (34) \]

which satisfies

\[ \cos^2 \theta \partial_{\theta}^2 f + \cos \theta (\sin \theta - i \alpha \cos \theta) \partial_\theta f + (1 - \kappa^2) f = 0. \quad (35) \]

Defining the complex variable

\[ z = e^{2i\theta}, \quad (36) \]
eq. (35) becomes

\[ z(z+1)^2 \partial_z^2 f + \frac{1}{2}(z+1)((1-\alpha)z+3-\alpha)\partial_z f + (\kappa^2 - 1)f = 0, \quad (37) \]

One can verify that if \( f(z) \) is a solution of this equation then \( f(z^{-1}) \) is a solution with \( \alpha \) replaced by \( -\alpha \). In this manner one gets the negative energy solutions from the positive ones. The conditions (25) read

\[ \lim_{z \to e^{\pm i\pi}} \frac{f(z)}{(1+z)} = 0, \quad (38) \]

\[ \int_{C} dz \frac{f(z)}{(1+z)^{3/2}} = 0. \quad (39) \]

where \( C \) is the unit circle \(|z| = 1\). Choosing \( \kappa = 1 \), the solution of (37) can be easily be found

\[ f(z) = A z^{\frac{\alpha+1}{2}} \left( \frac{z}{\alpha + 1} + \frac{1}{\alpha - 1} \right) + B, \quad \alpha \neq 1, -1. \quad (40) \]

where \( A \) and \( B \) are integration constants. The values \( \alpha = 1, -1 \) are excluded because the associated \( f(z) \) functions contain \( \log z \) such that the condition (38) is not satisfied. From (38) and (40) one finds

\[ f(e^{\pm i\pi}) = 0 \rightarrow B = \frac{2A}{1-\alpha^2} e^{\pm i\pi (\alpha - 1)} \rightarrow e^{i\pi \alpha} = -1 \quad (41) \]

which yields the quantization conditions

\[ \alpha_{n,\pm} = \begin{cases} (2n+3), & n = 0, 1, \ldots, \alpha > 0 \\ (2n+3), & n = -3, -4, \ldots, \alpha < 0 \end{cases} \quad (42) \]

that coincide with the semiclassical result (33) for \( \kappa = 1 \). The expression (40) for the positive energy solutions is

\[ f_{\kappa = 1, n, +}(z) = (-1)^n [(n+1)z^{n+2} + (n+2)z^{n+1} + (-1)^n] \quad (43) \]

and for the negative energy solutions

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\[ f_{\kappa=1,-n-3,-}(z) = (-1)^n \frac{n+1}{n} f_{\kappa=1,n,+}(z^{-1}), \quad n = 0, 1, \ldots \quad (44) \]

The factor multiplying \( f_{\kappa=1,n,+}(z^{-1}) \) will be explained below. Finally, to verify the condition (39), we express (43) as

\[ f_{\kappa=1,n,+}(z) = (z + 1)^2 \sum_{r=0}^{n} (r + 1)(-z)^r. \quad (45) \]

The vanishing of the integral follows from the Cauchy theorem. The solutions \( f_{\kappa=1,n,-}(z) \) also satisfy eq. (39), which can be written as

\[ \int_{C} dz f(z^{-1}) = 0. \quad (46) \]

Let us next consider generic values of \( \kappa \). Eq. (37) has two linear independent solutions given by

\[ f_{\kappa,n,+}(z) = (z + 1)^{\kappa+1} F(\kappa + 1, -n, -\kappa - n + 1, -z), \quad (47) \]
\[ f_{\kappa,n,-}(z) = (z + 1)^{\kappa+1} z^{\kappa+n} F(\kappa, 2\kappa + n + 1, \kappa + n + 1, -z) \]

where \( F \) is the hypergeometric function of type \( F_{2,1} \), and \( \alpha \) has been parametrized as

\[ \alpha = 2n + 2\kappa + 1, \quad (48) \]

with \( n \) an arbitrary number. One can transform these eqs. into \( F_{2,1} \)

\[ f_{\kappa,n,+}(z) = (z + 1)^{1-\kappa} F(-2\kappa - n, -\kappa + 1, -\kappa - n + 1, -z), \quad (49) \]
\[ f_{\kappa,n,-}(z) = (z + 1)^{1-\kappa} z^{\kappa+n} F(n + 1, -\kappa, \kappa + n + 1, -z) \]

The Gauss series defining these hypergeometric functions are absolutely convergent in the unit cycle \(|z| = 1\). The condition (38) implies

\[ F(-2\kappa - n, -\kappa + 1, -\kappa - n + 1, 1) = \frac{\Gamma(2\kappa)\Gamma(-\kappa - n + 1)}{\Gamma(\kappa + 1)\Gamma(-n)} = 0, \quad (50) \]
\[ F(n + 1, -\kappa, \kappa + n + 1, 1) = \frac{\Gamma(2\kappa)\Gamma(\kappa + n + 1)}{\Gamma(\kappa)\Gamma(2\kappa + n + 1)} = 0. \]
For generic values of $\kappa$, the first of these eqs. implies that $n = 0, 1, \ldots$, and correspond to the positive energy solutions, while the second eq. implies $n = -(2\kappa + p + 1)$ ($p = 0, 1, \ldots$), and correspond to the negative energy solutions $\alpha = -(2p + 2\kappa + 1)$. The relation between both solutions is given by

$$f_{\kappa,-n-2\kappa-1,-}(z) = (-1)^n \frac{\kappa + n}{\kappa} f_{\kappa,n,+}(z^{-1}), \quad n = 0, 1, \ldots$$

and can be proved using hypergeometric identities [22]. The case $\kappa = 1$ reproduces eq. (44). As follows from (47), the functions $f_{\kappa,n,+}(z)$ are the product of a polynomial of degree $n$ times the factor $(z + 1)^{\kappa+1}$. Some examples are

$$f_{\kappa,0,+}(z) = (z + 1)^{\kappa+1}$$
$$f_{\kappa,1,+}(z) = (z + 1)^{\kappa+1}(1 - (1 + \kappa^{-1})z)$$
$$f_{\kappa,2,+}(z) = (z + 1)^{\kappa+1}(1 - 2z + (1 + 2\kappa^{-1})z^2)$$

which for $\kappa = 1$, reproduces eqs. (45). Finally, the condition (39) is also satisfied by the Cauchy theorem.

The wave functions $\psi(x)$ can be written as (recall eqs. (23) and (34))

$$\psi = (\cos \theta)^{1/2} \phi(\theta) = \frac{f(\theta)}{2(\cos \theta)^{1/2}},$$

and the scalar product as

$$\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} dx \psi_1^*(x)\psi_2(x) = R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \phi_1^*(\theta)\phi_2(\theta)$$
$$= \frac{R}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta (\cos \theta)^2 f_1^*(\theta)f_2(\theta) = \frac{R}{2i} \int_C dz \frac{f_1^*(z)f_2(z)}{(z + 1)^2}.$$

Using this formula one can compute the norm of the wave functions. For example for $\kappa = 1$ one gets

$$\langle \psi_{\kappa=1,n,+} | \psi_{\kappa=1,n,+} \rangle = \pi R(n + 1)(n + 2), \quad n = 0, 1, \ldots$$
5. The general $xp$ model as a massive Dirac fermion in space-time

The spectrum obtained above turns out to coincide with the spectrum of a massive Dirac fermion in AdS$_2$, which we reproduce for completeness in Appendix C (see [25, 26]). This result may look surprising given that the Dirac fermion in 1+1 dimensions is a two component spinor, $\psi_{\pm}$, while our $xp$ model involves a single wave function $\psi$. The relation between the $xp$ model and the massive Dirac fermion in AdS$_2$ is indeed more general and not specific to the AdS$_2$ space-time. In this section we shall derive this relation in detail which comes from the special form of the spacetime metric underlying the $xp$ model.

Let us start from the massive Dirac equation in 1+1 spacetime dimensions

$$ (e^\mu_\alpha \gamma^a D_\mu + \frac{imc}{\hbar}) \psi = 0, $$

where $\psi^t = (\psi_-, \psi_+)$ is a two component spinor, $e^\mu_\alpha$ is the inverse of the vielbein $e^\alpha_\mu$, $\gamma^a$ are given in terms of the Pauli matrices as $\gamma^0 = \sigma^x$, $\gamma^1 = -i\sigma^y$, $D_\mu$ is the covariant derivative,

$$ D_\mu \psi = (\partial_\mu - \frac{1}{4} \omega^a_{\mu ab} \gamma^a) \psi, $$

where $\omega^a_{\mu ab}$ is the spin connection

$$ \omega^a_{\mu ab} = e^a_\nu \partial_\mu e^{b\nu} + e^a_\nu e^{b\lambda} \Gamma^{\nu}_{\lambda\mu}, $$

with $\Gamma^{\nu}_{\lambda\mu}$ the Christoffel symbols, and $\gamma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b]$. The vielbein $e^a_\mu$ corresponding to the metric (4), that is $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ with diag $\eta_{ab} = (-1, 1)$, can be chosen as

$$ e^0_0 = V + U, \quad e^1_0 = V - U, \quad e^1_1 = -e^0_1 = 1, $$

which yields the spin connection

$$ \omega^0_{01} = \frac{1}{V} \partial_x (UV), \quad \omega^0_{11} = -\frac{1}{V} \partial_x V. $$

Plugging these expressions into the Dirac equation (56) yields
\[
\frac{1}{\sqrt{V}} \partial_x \left( \sqrt{V} \psi_+ \right) - \frac{imc}{\hbar} \psi_- = 0, \quad (61)
\]
\[
\partial_t \psi_- + \sqrt{U} \partial_x \left( \sqrt{U} \psi_- \right) + \frac{imc}{\hbar} V \psi_+ = 0, \quad (62)
\]
which do not contain the time derivative of \( \psi_+ \). This feature implies that the field \( \psi_+ \) can be expressed in terms of \( \psi_- \) as
\[
\psi_+(x, t) = -\frac{imc/\hbar}{\sqrt{V(x)}} \int_x^\infty dy \sqrt{V(y)} \psi_-(y, t), \quad (63)
\]
under the assumption \( \lim_{x \to -\infty} \psi_+(t, x) = 0 \). Replacing (63) into (62) yields the equation of motion of \( \psi_- \)
\[
i\partial_t \psi_- = -i\sqrt{U} \partial_x \left( \sqrt{U} \psi_- \right) - i \left( \frac{mc}{\hbar} \right)^2 \sqrt{V(x)} \int_x^\infty dy \sqrt{V(y)} \psi_-(y, t) \quad (64)
\]
which is nothing but a Schroedinger equation for \( \psi_- \) with the Hamiltonian (17) under the identifications \( \psi_- = \psi \) and \( \ell_p = mc \).

Thus the origin of the non local term \( \hat{p}^{-1} \) in (17), comes from the non dynamical nature of the field \( \psi_+ \), which in turn is due to the special form of the metric (4). It would be interesting to investigate which are the spacetimes admitting a \( xp \) version along the lines shown above. A condition would be space-times with time like killing vectors, that will give rise to a Hamiltonian evolution.

6. \textit{SO}(2, 1) symmetry

Let us now return to the AdS\(_2\) model. Here we notice that the AdS\(_2\) metric has the isometry group \textit{SO}(2, 1). We then expect that this symmetry group can be realized in the \( xp \)-AdS\(_2\) model. In this section we shall construct the generators of this group, first in the classical theory and then in quantum mechanics. The group \textit{SO}(2, 1) is locally isomorphic to \textit{SU}(1, 1) whose generators, \( L_n \ (n = 0, \pm 1) \), satisfy the Poisson brackets
\{L_0, L_{\pm 1}\} = \pm i L_{\pm 1}, \quad \{L_1, L_{-1}\} = -2i L_0. \tag{66}

Choosing \(L_0\) to be proportional to the classical Hamiltonian (1), i.e.

\[ L_0 = \frac{R}{2w_0} H, \tag{67} \]

one finds

\begin{align*}
L_0 &= \frac{R}{2} \cosh \left( \frac{x}{R} \right) \left( p + \frac{\ell^2}{\bar{p}} \right), \\
L_{\pm 1} &= \frac{R}{2} \cosh \left( \frac{x}{R} \right) \left( p e^{\mp 2i \tan^{-1}(\sinh(x/R))} - \frac{\ell^2}{\bar{p}} \right).
\end{align*} \tag{68}

Upon quantization the algebra (66) becomes

\[ [\hat{L}_0, \hat{L}_{\pm 1}] = \mp \hbar \hat{L}_{\pm 1}, \quad [\hat{L}_1, \hat{L}_{-1}] = 2\hbar \hat{L}_0, \tag{69} \]

which holds by a normal ordered version of (68)

\begin{align*}
\hat{L}_0 &= \frac{R}{2} u(x) \left( \hat{p} + \frac{\ell^2}{\bar{p}} \right) u(x), \quad u(x) = (\cosh(x/R))^{1/2}, \\
\hat{L}_{\pm 1} &= \frac{R}{2} u(x) \left( e^{\mp 2i \tan^{-1}(\sinh(x/R))} \hat{p} e^{\mp i \tan^{-1}(\sinh(x/R))} - \frac{\ell^2}{\bar{p}} \right) u(x).
\end{align*} \tag{70}

To verify eqs.(69), it is convenient to perform the similarity transformation

\[ \hat{A}' = u \hat{A} u^{-1}, \tag{71} \]

which amounts to act on the wave function \(\phi\) related to \(\psi\) by eq.(23), i.e.

\[ \hat{A} \psi = u^{-1} \hat{A}' \phi. \tag{72} \]

The transformation of \(\hat{L}_n\) under (71) is given in the variable \(\theta\) by

\begin{align*}
(\hat{L}_0'\phi)(\theta) &= -\frac{i\hbar}{2} \partial_\theta \phi(\theta) - \frac{i\hbar\kappa^2}{2} \frac{1}{\cos \theta} \int_0^{\pi/2} \frac{d\theta'}{\cos \theta'} \phi(\theta'), \\
(\hat{L}_{\pm 1}'\phi)(\theta) &= -\frac{i\hbar}{2} e^{\mp 2i\theta} (\partial_\theta \mp i) \phi(\theta) + \frac{i\hbar\kappa^2}{2} \frac{1}{\cos \theta} \int_0^{\pi/2} \frac{d\theta'}{\cos \theta'} \phi(\theta'). \tag{73}
\end{align*}
The eigenfunctions of the Hamiltonian $\hat{H}$ imply
\[
\hat{L}_0 \phi_{\kappa,n,\pm} = \pm \hbar(n + \kappa + \frac{1}{2})\phi_{\kappa,n,\pm}, \quad n = 0, 1, \ldots
\] (74)
Moreover, the positive and negative energy solutions satisfy
\[
\hat{L}'_{\pm 1} \phi_{\kappa,0,\pm} = 0.
\] (75)
so that the positive energy eigenfunctions can be obtained acting on $\phi_{\kappa,0,+}$ with the raising operator $(\hat{L}'_{-1})^n$. Similarly, the negative energy eigenfunctions can be constructed acting on $\phi_{\kappa,0,-}$ with $(\hat{L}'_{+1})^n$. These two infinite dimensional representations of $SO(2,1)$ are related by complex conjugation.

7. The $xp$-AdS$_2$ model and conformal quantum mechanics

The group $SO(2,1)$ describes the symmetry of conformal quantum mechanics ($\text{CFT}_1$). The most studied model of a $\text{CFT}_1$ was introduced by Jackiw in 1972 [12], and its properties were analyzed in great detail by de Alfaro, Fubini and Furlan (dAFF) [13]. The dAFF model has been recently discussed in the framework of the AdS$_2$/CFT$_1$ correspondence [17, 18, 16] and can be considered as the conformal limit of the matrix model describing flux backgrounds of 2D type 0A string theory [24]. It also has been shown that the spectrum of this limit is similar to that of a free fermion on AdS$_2$ [25].

In this section we shall discuss the relation of these works with the $xp$-AdS$_2$ model, which in turn may shed new light into the AdS$_2$/CFT$_1$ correspondence.

The dAFF Hamiltonian describes a particle on a half-line subject to an inverse square interaction potential, i.e.
\[
H = \frac{1}{2}(p^2 + \frac{g}{x^2}), \quad x > 0, \quad g > 0,
\] (76)
which together with the operators $D$ and $K$, defined as
\[
D = tH - \frac{1}{4}(xp + px),
\] (77)
\[
K = -t^2H + 2tD + \frac{1}{2}x^2,
\]
generate the $SO(2, 1)$ algebra \[13, 17\]

\[
i[D, H] = H, \quad i[D, K] = -K, \quad i[K, H] = 2D.
\] (78)

The Hamiltonian $H$ is essentially self-adjoint for $g \geq 3/4$ with no discrete spectrum, while for $-\frac{1}{4} < g < \frac{3}{4}$ it admits self-adjoint extension and has a bound state of negative energy (see \[27, 28\] and references therein). Herethough we shall consider not the spectrum of $H$ but that of an operator belonging to the $SO(2, 1)$ algebra which in the Cartan basis \[69\] is given by \[13, 17\]

\[
L_0 = \frac{1}{2} \left( \frac{K}{a} + aH \right), \\
L_{\pm 1} = \frac{1}{2} \left( \frac{K}{a} - aH \right) \mp iD,
\] (79)

where $ha$ has dimensions of $(\text{length})^2$, and such that $L_0$ has a discrete spectrum, i.e.

\[
L_0 |n\rangle = \hbar r_n |n\rangle, \\
r_n = n + r_0, \quad n = 0, 1, \ldots, \quad r_0 > 0, \\
\langle n|n'\rangle = \delta_{n,n'}.
\] (80)

The action of the operators $L_{\pm 1}$ acting in this basis is

\[
L_{\pm 1} |n\rangle = \hbar \sqrt{r_n(r_n \mp 1) - r_0(r_0 - 1)} |n \mp 1\rangle.
\] (81)

The parameter $r_0$ labels the representation of the algebra and is related to the Casimir $L^2$ of $SO(2, 1)$ as

\[
L^2 |n\rangle = \hbar^2 r_0(r_0 - 1) |n\rangle, \quad L^2 = L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1).
\] (82)

In the dAFF model, $r_0$ is given by

\[
r_0 = \frac{1}{2} \left( 1 + \sqrt{g + \frac{1}{4}} \right).
\] (83)

On the other hand, the value of $r_0$ in the $xp$-AdS$_2$ model is given by (see \[74\])

15
\[ r_0 = \kappa + \frac{1}{2}. \] (84)

Although the dAFF and the \( xp \)-AdS\(_2\) are two different models, one can establish a correspondence between their parameters based on dimensional arguments and their spectrum

\[ \frac{1}{2} \sqrt{g + \frac{1}{4}} \leftrightarrow \frac{R \ell_p}{\hbar}, \quad \hbar a \leftrightarrow R^2, \] (85)

which links \( a \) to the radius of the AdS\(_2\) space, and \( g \) to that radius measured in units of the length \( \hbar / \ell_p \). However, unlike the dAFF model, where \( g \) and \( a \) does not make direct reference to the AdS\(_2\) space-time, the parameters of the \( xp \) model do: \( R \) is the radius of AdS\(_2\) and \( \ell_p / c \) is the mass of the fermion in the Dirac formulation of the \( xp \) model. We hope that the results presented in this article may shed new insights on the AdS\(_2\)/CFT\(_1\) correspondence.

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Appendix A. Short review on the AdS\(_2\) spacetime

The AdS\(_2\) spacetime is the locus of the one-sheeted hyperboloid [23]

\[ X_0^2 + X_2^2 - X_1^2 = R^2, \] (A.1)

where \( R \) is a length denoted the radius of AdS. The spacetime metric is inherited from the ambient Minkowski spacetime,

\[ (ds)^2 = -dX_0^2 - dX_2^2 + dX_1^2, \] (A.2)

which shows that \( SO(2,1) \) is the isometry group, whose compact subgroup \( SO(2) \) can be identified with time. The hyperboloid (A.1) can be described in global coordinates as
\[ \begin{align*}
X_0 &= R \cosh \rho \cos \tau, \\
X_2 &= R \cosh \rho \sin \tau, \\
X_1 &= R \sinh \rho,
\end{align*} \tag{A.3} \]

where \( \rho \in (-\infty, \infty) \) and \( \tau \in [0, 2\pi) \). The universal covering of AdS_2 spacetime is obtained letting \( \tau \) to take any real value. The metric in these coordinates is

\[ ds^2 = R^2(-\cosh^2 \rho \, d\tau^2 + d\rho^2). \tag{A.4} \]

The boundary at infinity, \( \rho = \pm \infty \), consists of two disconnected worldlines parameterized by the time coordinate \( \tau \in (-\infty, \infty) \). The coordinate \( \rho \) is identified with \( x/R \) in the \( xp \)-AdS_2 model. Throughout this paper we have used the variable \( \theta \) defined as

\[ \sinh \rho = \tan \theta, \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{A.5} \]

The boundaries of the spacetime are at \( \theta = \pm \pi/2 \). This coordinate displays clearly the causal structure of the metric \( (A.4) \), i.e.

\[ ds^2 = \frac{R^2}{\cos^2 \theta}(-d\tau^2 + d\theta^2). \tag{A.6} \]

**Appendix B. Classical trajectories and geodesics**

The equations of motion associated to the Hamiltonian \( (1) \) are \( (8) \)

\[ \begin{align*}
\dot{x} &= w(x)(1 - \frac{\ell^2}{p^2}), \\
\dot{p} &= -w'(x)(p + \frac{\ell^2}{p}).
\end{align*} \tag{B.1} \]

The energy \( E \) is a conserved quantity,

\[ E = w(x)(p + \frac{\ell^2}{p}). \tag{B.2} \]
as well as the sign of the momenta $p$, which coincides with the sign of $E$ since $w(x) > 0$, $\forall x$. We shall assume below that $E, p > 0$. For each position $x$ there are two possible values of the momenta,

$$p_\eta(x, E) = \frac{1}{2w(x)}(E + \eta\sqrt{E^2 - (2\ell_p w(x))^2}), \quad \eta = \pm 1,$$

(B.3)

related by

$$p_-(x, E) = \frac{\ell_p^2}{p_+(x, E)}.$$  

(B.4)

The classical allowed region is given by

$$E \geq 2\ell_p w(x).$$

(B.5)

Replacing (B.3) into (B.1) yields

$$\dot{x} = \frac{1}{2\ell_p^2 w(x)}(2\ell_p w(x))^2 - E^2 + \eta E\sqrt{E^2 - (2\ell_p w(x))^2},$$

(B.6)

so that the classical trajectories are given by

$$\int_{x_0}^{x} dx \frac{2\ell_p^2 w(x)}{(2\ell_p w(x))^2 - E^2 + \eta E\sqrt{E^2 - (2\ell_p w(x))^2}} = t - t_0.$$  

(B.7)

To integrate this eq. we use the variable $\theta$ defined in (8) or (A.5) and

$$w(x) = w_0 \cosh \frac{x}{R} = \frac{w_0}{\cos \theta}. $$

(B.8)

After some algebra (B.7) becomes (we choose $x_0 = t_0$ so that $\theta_0 = 0$)

$$\frac{R}{2w_0} \int_{0}^{\theta} d\theta \left[ 1 + \frac{\eta \varepsilon \cos \theta}{\sqrt{\varepsilon^2 \cos^2 \theta - 1}} \right] = t,$$

(B.9)

where we have defined (recall (12))

$$\varepsilon = \frac{E}{2w_0 \ell_p} \geq 1.$$  

(B.10)
Performing the integral one finds

$$\theta + \eta \tan^{-1}\left[ \frac{\varepsilon \sin \theta}{\sqrt{\varepsilon^2 \cos^2 \theta - 1}} \right] = \frac{2w_0 t}{R} = \tau + \theta. \quad (B.11)$$

Hence, the classical trajectories in the global coordinates are

$$\tau = \eta \tan^{-1}\left[ \frac{\varepsilon \sin \theta}{\sqrt{\varepsilon^2 \cos^2 \theta - 1}} \right], \quad (B.12)$$

whose inverse

$$\theta(\tau) = \text{sign}(\pi - \tau) \cos^{-1}\sqrt{\cos^2 \tau + \frac{1}{\varepsilon^2} \sin^2 \tau}, \quad 0 \leq \tau \leq 2\pi, \quad (B.13)$$

shows that they are periodic with the same period 2\pi for all energies. Finally, the position and momenta are given by

$$x(\tau) = \sinh^{-1}(\tan \theta(\tau)), \quad 0 \leq \tau \leq 2\pi, \quad (B.14)$$

$$p(\tau) = \ell_p(\varepsilon \cos \theta(\tau) + \sqrt{\varepsilon^2 - 1} \cos \tau).$$

Fig. 1 shows an example of a classical trajectory. The phase space contour is traversed in counterclockwise so that the Maslov phase is $-2\pi$, as for the standard harmonic oscillator. This justifies the constant $\frac{1}{2}$ in the semiclassical formula (10) for the energy levels.

**Appendix C. Massive Dirac fermion in AdS$_2$**

The Dirac equation for a massive fermion in 1+1 spacetime dimensions was given in eq.(56). In the case of AdS metric (A.4), the vielbein and spin connection are given by

$$e_0^0 = R \cosh \rho, \quad e_1^1 = R^2, \quad \omega_0^0 = -\omega_0^1 = \sinh \rho, \quad (C.1)$$

which plugged into the eq.(56) yield

$$\left( \frac{i\lambda}{\cosh \rho} + \frac{1}{2} \tanh \rho + \partial_\rho \right) \psi_+ - i\kappa \psi_- = 0, \quad (C.2)$$

$$\left( \frac{-i\lambda}{\cosh \rho} + \frac{1}{2} \tanh \rho + \partial_\rho \right) \psi_- + i\kappa \psi_+ = 0.$$
Fig.1-Left: Position (red line) and momentum (blue line) of a classical trajectory with $E = 3$ in units $w_0 = l_p = R = 1$. Right: Same trajectory in phase-space.

where we have assumed the time dependence $e^{-i\lambda t}$ for $\psi_\pm$. We are working in units $\hbar = c = 1$ so that $\kappa = mR$ according to eq. (14). Combining these eqs. one gets

$$\left( \partial_\rho^2 + \tanh \rho \partial_\rho + \frac{1}{4} - \kappa^2 + \frac{i\lambda \sinh \rho}{\cosh^2 \rho} + \frac{\lambda^2 + 1/4}{\cosh^2 \rho} \right) \psi_- = 0, \quad (C.3)$$

which for the function

$$\phi(\rho) = (\cosh \rho)^{1/2} \psi_-(\rho) \quad (C.4)$$

implies

$$\left( \partial_\rho^2 - \kappa^2 + \frac{i\lambda \sinh \rho}{\cosh^2 \rho} + \frac{\lambda^2}{\cosh^2 \rho} \right) \phi = 0, \quad (C.5)$$

or in terms of the angle $\theta$ defined in (8)

$$\left( \cos^2 \theta \partial_\theta^2 - \sin \theta \cos \theta \partial_\theta - \kappa^2 + i\lambda \sin \theta \cos \theta + \lambda^2 \cos^2 \theta \right) \phi = 0, \quad (C.6)$$

and in terms of $z = e^{2i\theta}$

$$\left( z(z+1)^2 \partial_z^2 + \frac{(z+1)(3z+1)}{2} \partial_z + \kappa^2 - \frac{\lambda^2}{2} - \frac{\lambda + \lambda^2}{4} z + \frac{\lambda - \lambda^2}{4} z^{-1} \right) \phi = 0. \quad (C.7)$$
The solutions of this equation in terms of hypergeometric functions regular at \( z = 0 \), are given by

\[
\phi^+(z) = z^{\frac{1}{2}-\frac{3}{2}}(1+z)^{-\kappa} F(1-\kappa, \frac{1}{2}-\kappa - \lambda, \frac{3}{2} - \lambda, -z), \quad (C.8)
\]
\[
\phi^-(z) = z^{\frac{1}{2}}(1+z)^{-\kappa} F(\frac{1}{2} - \kappa + \lambda, -\kappa, \frac{1}{2} + \lambda, -z)
\]

which using eq. (C.4) implies for the component \( \psi^- \)

\[
\psi^+(z) = z^{\frac{1}{2} - \frac{3}{2}}(1+z)^{\frac{1}{2} - \kappa} F(1-\kappa, \frac{1}{2} - \kappa - \lambda, \frac{3}{2} - \lambda, -z), \quad (C.9)
\]
\[
\psi^-(z) = z^{-\frac{1}{2} + \frac{3}{2}}(1+z)^{\frac{1}{2} - \kappa} F(\frac{1}{2} - \kappa + \lambda, -\kappa, \frac{1}{2} + \lambda, -z)
\]

The relation between \( \lambda \) and the eigenenergies \( E \) of the \( xp \) model can be found from the eqs. (8), (32) and (36) (in units \( \hbar = 1 \))

\[
e^{-iEt} = e^{-i\frac{AE}{2m_0}(\tau + \theta)} = e^{-i\frac{A}{2}\tau}z^{-\frac{A}{2}} = e^{-i\lambda \tau}z^{-\frac{A}{2}} \rightarrow \lambda = \alpha \quad (C.10)
\]

Using eq. (33) for \( \alpha \) one can then easily see that eqs. (C.9), correspond to the solutions (49) found for the \( xp \) model. More generally, requiring that for all values of \( \kappa = mR \) (including zero) the solutions of eq. (C.7) to be single-valued leads to impose eq. (33) (see also [25]). However, recalling the mapping to the \( xp \) model, we have not considered here the case \( \kappa = 0 \) i.e \( \ell_p = 0 \) due to in this limit the spectrum of the \( xp \) model is continuous. Our result is an example of the relation between the \( xp \) model and the massive Dirac equation found in section 5.

Finally, we would like to mention that equation (C.5), for the value \( \lambda = 2 \), coincides with the Schroedinger equation of the Hamiltonian of the PT-symmetric complexified Scarf II potential which is reflectionless and possess a hidden nonlinear supersymmetry [29]. It turns out that this model is also related to the Dirac equation of a spin 1/2 particle in de Sitter space [30]. It would be worthwhile to explore these suggestive connections in further detail.
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