Phase transition, entanglement and squeezing in a triple-well condensate

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Abstract – We provide an in-depth characterization of a three-mode Bose-Einstein condensate trapped in a symmetric circular triple-well potential. We show how the purity related to the $su(3)$ algebra scales for increasing number of atoms and signals the quantum phase transition between two dynamical regimes in a specific configuration. This measure, which is intrinsically related to particle entanglement, also depicts if some squeezing is occurring when we consider the system’s ground state. Unlike the well-known double-well model, the triple-well model exhibits a first-order quantum phase transition, which could be investigated with the current trapping technology.

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Introduction. – Entanglement has played an important role for the understanding of quantum many-body aspects [1] that traditionally belonged to statistical mechanics and quantum field theory. Several investigations in quantum critical models at zero temperature have shown that complex entangled ground state contains all the important correlations that give rise to different phases known to exist in several systems [2]. Characterization of quantum phase transition (QPT) [3] via pairwise and multipartite entanglement has been given in a very conclusive way in refs. [4], dependent on the specific partition employed. To investigate subsystem independent entanglement in many-particle systems one has to employ the generalized purity associated to the pertinent algebra [5]. This measure, known to qualify the semiclassical approximation taken [6], is good as well to indicate the squeezing of moments of the generators of the pertinent algebra. Recently both spin squeezing and entanglement have been demonstrated for a $^{87}$Rb condensate trapped in double and multiple wells of an optical lattice [7], and particle entanglement theoretically investigated in ref. [6] for a double-well trapped condensate. Although independent, those results are profoundly complementary since they relate QPT, entanglement and squeezing, for a system which is a particular realization of the Lipkin-Meshkov-Glick model [8,9]. The interplay between entanglement and squeezing has been investigated previously in many instances [10], and now seems to play an important role in QPT involving many bosons as well.

In this letter we investigate in detail a BEC of interacting neutral atoms trapped in a symmetric triple well in a three-mode approximation and show that the ground state of the model undergoes a QPT. A time-dependent variational principle using the $SU(3)$ coherent state allows for a system of semiclassical equations that enables one to find the fixed points of the model and to investigate how the lowest-energy fixed points change as the collision parameters of the model are varied. Since the lowest-energy state in this system corresponds to a twin condensate fixed point, where effectively the system behaves as if composed of two wells (although coherences between the twin-modes are still present), the pertinent $SU(3)$ coherent states reduce to $SU(2)$ coherent states while in this regime. The non-linear components of the Hamiltonian, provided by the bosons interactions, lead to squeezing for the ground state as we vary the scattering parameter. The interactions also produce non-trivial phenomena such as particle entanglement, which we relate to QPT and to distinct dynamical regimes.

The model. – A Bose-Einstein condensate trapped in a triple-well potential has already been modeled in many distinct configurations [11,12]. Considering that
the coupling between the potential wells is sufficiently weak, we shall use the usual local-modes approximation [13] associated to the ground states $|u_j\rangle$ ($j = 1, 2, 3$) of harmonic approximations around each minimum of the trap. In order to consider this approximation to be valid even when the total number of trapped bosons $N$ is large, the overlap $\varepsilon \equiv \langle u_i | u_j \rangle$ ($i \neq j$) must be very small. Keeping terms up to $O(\varepsilon^2)$ and using the fact that the total number of trapped particles $N$ is conserved we can write, similarly as in [14] for a double well, the following Hamiltonian:

$$H = \Omega \sum_{i \neq j} a_i^\dagger a_j + \kappa \sum_i a_i^2 a_i^\dagger - 2\Lambda \sum_{i \neq j \neq k} a_i^\dagger a_j a_k,$$

where $\Omega = \Omega + 2\Lambda(N - 1)$, and the operator $a_j^\dagger(a_j)$, $j = 1, 2, 3$, annihilates (creates) a boson in the state $|u_j\rangle$. Here $\Omega$ is the tunneling rate, $\Omega'$ is the effective tunneling rate, $\kappa$ is the self-collision parameter and $\Lambda = \kappa \varepsilon^2 \ll \kappa$ is the cross-collision rate, which is proportional to the interaction frequency between bosons in different sites. Although $\Lambda$ is a lower-order parameter of the model, we see that its presence in $\Omega'$ is relevant for $N \gg 1$.

Since the Hamiltonian (1) preserves $N$, we can take advantage of the homomorphism between the commutation relations of $SU(3)$ generator combinations and bilinear combinations of creation and annihilation operators as

$$Q_1 \equiv \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \quad Q_2 \equiv \frac{1}{3}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3);$$

$$J_k \equiv i(a_k^\dagger a_j - a_j^\dagger a_k), \quad P_k \equiv a_k^\dagger a_j + a_j^\dagger a_k;$$

for $k = 1, 2, 3$ and $j = (k + 1) \mod 3 + 1$. Using these eight generators we can rewrite the Hamiltonian

$$H = \left(\Omega' - 2\Lambda\frac{N}{3}\right)(P_1 + P_2 + P_3) + \kappa\left(\frac{1}{2}(4Q_1^2 + 3Q_2^2)\right)$$

$$+ \Lambda\left[2Q_1(P_1 - P_3) + Q_2(2P_2 - P_1 - P_3)\right],$$

showing that $SU(3)$ is the pertinent dynamical group. Note that the tunneling term is linear in the generators, while the collision terms are quadratic.

**Semiclassical dynamics.** – A semiclassical dynamics of the condensate can be developed by using the time-dependent variational principle [15] with coherent states of the group $SU(3)$ [16] as test functions. In this context, the semiclassical Hamiltonian is given by

$$\mathcal{H}(\vec{w}, \vec{\omega}) \equiv \langle N; \vec{w}| H| N; \vec{\omega}\rangle,$$

where $\vec{w}$ and $\vec{\omega}$ represent the non-linear subspace composed of coherent states and represents a point in the phase space associated with the classical analogue system [17]. The semiclassical equations of motion,

$$i\hbar \dot{w}_j = \Omega'\sum_{k \neq j} (w_j + w_k + 1) + \frac{2(N - 1)}{|w_j|^2 + |w_k|^2 + 1}$$

$$\times \left\{ \kappa w_j(|w_j|^2 - 1) + \Lambda \left[ (1 + w_j)w_k(|w_j|^2 - 1) \right. \right.$$

$$\left. - (1 - w_j)w_k^* \left( w_j + w_k + w_j w_k \right) \right\},$$

with $j, k = 1, 2$ and $j \neq k$, are brought to the Hamilton's canonical form by assigning $w_j = \sqrt{I_j}/(N - I_1 - I_2)e^{-i\phi_j}$, where $I_j$ is the average occupancy in the $j$-th local mode, and $\phi_j$ is the phase difference between the condensates located in the $j$-th and the third wells. The semiclassical approximation with the coherent states (4) becomes exact if the Hamiltonian is linear in the generators of the group or in the classical limit of the system, which in this case coincides with the macroscopic limit $N \to \infty$, because $N$ behaves equivalently to the reciprocal of $\hbar$ [18,19].

The position and stability of the equations of motion equilibrium points depend only on the parameters $\chi \equiv \frac{\kappa(N - 1)}{\Omega} \quad \text{and} \quad \mu \equiv \frac{\Lambda(N - 1)}{\Omega}.$

There are fixed-point solutions of (5) given by the three equivalent conditions $w_1 = w_2$, $w_1 = 1$ or $w_2 = 1$, which correspond to configurations where two localized condensates are in phase and have the same occupation average, known as twin condensates [12]. Many of the fixed points of the semiclassical dynamics are contained in this subregime under the additional condition $\phi_1, \phi_2 = 0, \pi$ (implying in $\vec{w} \in \mathbb{R}^2$). Without loss of generality, we consider only $w = w_1 = w_2$, in which case the $SU(3)$ coherent states reduce to the set of coherent states of $SU(2)$, which thus brings many features observed for a two-mode condensate, such as the Rabi Oscillation (RO) of population and the macroscopic self-trapping (MST) of population [14]. The phase space associated with the integrable subregime of twin condensates is isomorphic to $S^2$, a space that parametrizes the set of $SU(2)$ coherent states $|J = \frac{N}{2}, \tau = \sqrt{2\hbar} \tan \frac{\theta}{2} e^{-i\phi}\rangle$. An interesting quantity for the dynamical analysis in the twin condensates subregime is the population balance $I_2 \equiv -\cos \theta = \frac{I_3 - N}{N}$. The point $w_1 = w_2 = 1$ is a solution of the fixed-point equations, independent of the values of $\chi$ and $\mu$, which we call $1^\ast$. Considering always $\mu \ll \chi$, we have another solution if $\chi < \chi_+ (\mu)$, which we call $2^\ast$; whereas if $\chi > \chi_+ (\mu)$ two further solutions of the fixed-points equations exist. In other words, a saddle-node bifurcation with critical parameter $\chi_+ (\mu)$ is responsible for the appearance of the fixed points $3^\ast$ and $4^\ast$. Figure 1(a) shows the semiclassical dynamics of twin condensates in the absence of these fixed points $3^\ast$ and $4^\ast$. We see that all trajectories are around the fixed point $1^\ast$ and therefore vary around the value $I_2 = \frac{1}{2}$, which represents same average occupancy in the three local modes. This is exactly the RO dynamical regime, where the system does not show any preferential mode occupation. In fig. 1(b), where $\chi$ exceeds the critical value of bifurcation, we observe the emergence of new types of orbits in the system,
especially around the new stable equilibrium point $4^\pm$. These new trajectories oscillate around negative values of $I_z$ and show a suppression of tunneling between the twin condensates and the solitary one, resulting in the preferential occupation of the third mode, also known as MST, similarly to the case of two modes [13,20]. As $\chi$ is increased, the region of phase space occupied by orbits associated with the MST regime also grows larger (Fig. 1(c)).

**Generalized purity.** – A quantitative error in the semiclassical approximation is originated by forcing the state to evolve preserving minimal uncertainty on the phase space, i.e., obligating the system to remain as a coherent state. If the system dynamics leads to an increase of the state uncertainty, we cannot expect a high quantitative accuracy in the semiclassical treatment. This includes the situation where squeezing or spreading over the phase space occurs, leading thus to a large deviation of the coherent state. An effective measure of the quality of the semiclassical approximation is the generalized purity associated with the $su(3)$ algebra [5]:

$$P_{su(3)}(\psi) = \frac{9}{N^2} \left( \frac{\langle \psi | Q_1 | \psi \rangle^2}{3} + \frac{\langle \psi | Q_2 | \psi \rangle^2}{4} \right)$$

$$+ \sum_{j=1}^{3} \frac{\langle \psi | P_j | \psi \rangle^2}{12} + \sum_{k=1}^{3} \frac{\langle \psi | J_k | \psi \rangle^2}{12}. \quad (6)$$

This measure is derived from the total uncertainty of the algebra [21] and has maximum value $P_{su(3)}(\psi) = 1$ only if $|\psi\rangle$ is a coherent state of $SU(3)$, corresponding to the state of minimum total uncertainty. In contrast, the purity presents a minimum value $P_{su(3)}(\psi) = 0$ for states of the largest uncertainty. Therefore, $P_{su(3)}$ can be used to measure the “distance” of a particular state to the subspace formed only by the coherent states. That is, the larger the uncertainty of a state during its evolution, the lower is $P_{su(3)}$ and less accurate is the semiclassical approximation using only coherent states [6].

In addition the generalized purity is also a genuine measure of separability in systems of many identical particles. Under a basis transformation of the single-particle Hilbert space, every coherent state of the type (4) can be rewritten as a separable state in each boson,

$$|N; w\rangle = \frac{1}{\sqrt{N!}} \left[ w_1 a_1^\dagger + w_2 a_2^\dagger + a_3^\dagger \right]^N |0\rangle$$

$$= \frac{(a_2^\dagger)^N}{\sqrt{N!}} |0\rangle = |N\psi\rangle = \bigotimes_{i=1}^{N} |\psi^{(i)}_w\rangle, \quad (7)$$

where $|\psi_w\rangle = \frac{w_1|u_1\rangle + w_2|u_2\rangle + |u_3\rangle}{(|w_1|^2 + |w_2|^2 + 1)^{\frac{1}{2}}}$ is an arbitrary state of the single-particle Hilbert space$^1$ with associated annihilation operator $a_2$. Due to the symmetrization principle, (7) are the only possible completely separable states in each boson. Therefore, coherence is equivalent to separability in the Hilbert space considered, i.e., the states with $P_{su(3)}(\psi) = 1$ represent the separable states of $N$ bosons, while the decrease of purity from 1 indicates the entanglement among the particles$^2$. Thus, the semiclassical approximation is accurate only if the state remains separable in each boson, and we see that the entanglement is the fundamental quantum feature responsible for breaking-off the classicality of the system. We can also conclude that a state is entangled, $P_{su(3)}(\psi) < 1$, whenever it is less localized than a coherent state in the phase space. Several dynamical processes can be responsible for the loss of purity of an initially coherent state [14]. One such quantum processes is the squeezing of the state, which is related to regular dynamical regimes of the semiclassical model.

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$^1$Except by a choice of global phase and normalization.

$^2$\(P_{su(3)}\) can be written as the trace of a squared density operator reduced with respect to the algebra. If $\rho \equiv |\psi\rangle\langle\psi|$, we have $P_{su(3)}(\psi) = K \text{Tr}(\rho_{su(3)})$, where $K$ is a factor dependent on $N$, $\rho_{su(3)} = \sum_j \text{Tr}(\rho A_j) A_j$ is the density operator reduced in $su(3)$ and $\{A_j\}$ is a basis of $su(3)$ orthonormalized in relation to the trace in the Fock space of $N$ particles. Thus, pure (mixed) states in the algebra are equivalent to separable (entangled) states in each boson.
Quantum phase transition. − QPT results from non-analyticities in the ground-state energy as a function of a real parameter of the Hamiltonian, characterized only in the macroscopic limit $N \to \infty$ and at zero temperature [3]. However, as stressed earlier, the macroscopic limit of the model is equivalent to its classical limit [18], so that the minimum of the semiclassical energy per particle $\frac{\mathcal{H}}{N}$, which is independent of $N$, can be used exactly to study the QPT, i.e., the minimum of $\mathcal{H}$ displays exactly the same behavior of the ground state energy in the macroscopic limit [19]. In fig. 2 we show $\mathcal{H}$ calculated in some of its extremes, which are the equilibrium points, as a function of $\chi$, with $N = 30$, $\Omega = -1$ and $\mu = 0$. The occurrence of a first-order QPT is evident at the critical value $\chi_c = 2$, since there is a discontinuity of the first derivative of the ground-state energy, resulting from the energy level crossing of the fixed points $1^+$ and $4^+$. As already shown, the fixed point $1^+$ is the center of the orbits related to the RO dynamical regime, characterized by no preferential occupation of the local modes. The three equivalent equilibrium points $4^+$, one in each subregime of twin condensates, are responsible for the emergence of semiclassical trajectories associated with the MST regime in the solitary mode. Therefore, we expect the ground state to display characteristics of the RO (MST) regime for $\chi < \chi_c$ ($\chi > \chi_c$). Furthermore, the behavior of the low-energy equilibrium points suggests a non-degenerate (triply degenerate) ground level when the self-collision parameter is lower (higher) than $\chi_c$ in the limit $N \to \infty$.

The phase space projections of the ground-state signal the quantum phase transition for finite $N$, as shown in fig. 3. The population Husimi function $Q_I(I_1, I_2) \equiv \int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 \langle \langle N; |w| \psi) |^2$ is a quasi-probability distribution of the average occupations in the three local modes, while the function $\Phi(\phi_1, \phi_2) \equiv |\langle \phi_1, \phi_2 | \psi) |^2$, with $|\phi_1, \phi_2) \equiv \sum_{n_1, n_2 + n_3 = N} e^{i \phi_1 n_1 - i \phi_2 n_2} |n_1, n_2, n_3)$, represents the probability distribution of collective phase differences [22] between the local condensates. Considering $\chi = \mu = 0$ in fig. 3(a), the exact ground state is the coherent state $|N; w_1 = w_2 = 1)$, centered at the equilibrium point $1^+$, showing the non-preferential occupation of the local modes. For increasing value of $\chi$ in fig. 3(b), we observe the state expansion in the population subspace, accompanied by squeezing in the conjugate subspace, related thus to phase squeezing. Note that the coherent states are the most localized states in the phase space, but not necessarily in a subspace. When $\chi = \chi_c$ there is no abrupt change in the representations of the ground state, since such changes should happen in a continuous way for finite $N$. When the self-collision parameter takes values slightly larger than $\chi_c$ we notice a profound change in the ground state (fig. 3(c)) as a trifurcation of the occupational distribution, being each of the components more squeezed than the original coherent state. This is accompanied by the emergence of an interference pattern in the conjugate subspace. The fragmentation of $\Phi$ represents the superposition in states of preferential occupancy in each local mode, also visible in the behavior of $Q_I$. Notice that the ground-state representations in phase space display a smooth transition from the RO to MST regime, in contrast to the abrupt transition shown in the macroscopic limit.

Now we return to the discussion of particle entanglement, as given by the generalized purity, and the signaling
of the QPT [4]. In fig. 4 we show \( \frac{d\mathcal{P}_{su}(N)}{d\chi} \) calculated at the ground state as a function of \( \chi \), suggesting a scaling property between purity is faster for a larger number of trapped bosons, associated with the tunneling suppression. The loss of purity is equal to 1. For \( \chi < \chi_c \), the region related to the expansion-compression of the ground-state representations, we see that \( \mathcal{P}_{su}(N) \) decreases slowly with increasing \( \chi \). However, for values of \( \chi \) slightly above \( \chi_c \) we observe a rapid decay of \( \mathcal{P}_{su}(N) \) caused by the ground-state fragmentation associated with the tunneling suppression. The loss of purity is faster for a larger number of trapped bosons, suggesting a scaling property between \( \frac{d\mathcal{P}_{su}(N)}{d\chi} \) and \( N \). We define the scalable quantum critical parameter \( \chi_2(N) \) as the self-collapse parameter value that minimizes \( \frac{d\mathcal{P}_{su}(N)}{d\chi} \) for a specific number of particles. We observe that the minimum value of \( \frac{d\mathcal{P}_{su}(N)}{d\chi} \) becomes more pronounced for increasing \( N \), while \( \chi_2(N) \) moves to the left, toward \( \chi_c \). The values of \( \chi_2(N) \) obtained from fig. 4 suggest a linear relationship between \( \ln[\chi_2(N) - \chi_c] \) and \( \ln(N) \). A linear interpolation of the data provides the following power law: \( \chi_2(N) - \chi_c = c_1 \cdot 2^{d-0.99} \). Therefore, the convergence law of \( \chi_2(N) \) to \( \chi_c = 2 \) as \( N \to \infty \) is characterized. Also \( \mathcal{P}_{su}(3) \) signals the QPT correctly, considering the scaling behavior for finite \( N \) (see footnote 3). Although the critical bifurcation parameter \( \chi_2 \) is distinct from the critical transition parameter \( \chi_c \), the emergence of new fixed points in the semiclassical dynamics shows a clear relation with the occurrence of the QPT. This same characteristic has been observed in the two-local-modes model [23], but with some important differences. The two-mode model presents a second-order QPT accompanied by a pitchfork bifurcation with identical critical parameters \( \chi^TM = \chi^TM = \frac{1}{2} \) [14], in the absence of cross-collisions.

In the classical-macroscopic limit \( \mathcal{P}_{su}(3) \) is not well defined as a signal of QPT because the ground level is triply degenerate after the transition and there is no unique choice of state to calculate the purity, giving different results for different linear combinations of degenerate ground states. This difficulty does not exist for finite \( N \), as the ground state is unique.

However, in both models the equilibrium point that minimizes the semiclassical energy \( \mathcal{H} \) can be described by a single angle \( \theta \) on \( S^2 \), considered as a function of \( \chi \). For example, the restriction \( W_1 = W_2 \in \mathbb{R} \) determines the undimensional manifold of the phase space that contains the minimum-energy fixed point. Therefore, we generally consider \( \mathcal{H} \) restricted to only one dimension in the analysis of the QPT: \( \mathcal{H}(\theta; \chi) = \mathcal{H}_0(\theta) + \chi \mathcal{H}_1(\theta) \); where \( \mathcal{H}_0(\mathcal{H}_1) \) is the portion of the semiclassical Hamiltonian obtained from the linear (quadratic) terms of \( H \) in the generators of the dynamical group. The position of the semiclassical ground state \( \theta_{\text{min}} \equiv \theta_{\text{min}}(\chi) \), which satisfies the necessary conditions \( \frac{\partial\mathcal{H}}{\partial\theta} |_{\theta=\theta_{\text{min}}} = 0 \) and \( \frac{\partial^2\mathcal{H}}{\partial\theta^2} |_{\theta=\theta_{\text{min}}} > 0 \), and the minimum-energy curve \( \mathcal{H}_{\text{min}}(\chi) \equiv \mathcal{H}(\theta_{\text{min}}(\chi), \chi) \) are shown in fig. 5 for the two- and three-mode models4. The function \( \theta_{\text{min}}(\chi) \) possesses a discontinuous first derivative at \( \chi^TM \) for the two-mode model, while for the three-mode model we observe a discontinuity in the curve of \( \theta_{\text{min}}(\chi) \) itself at the critical value \( \chi_c \). The different types of bifurcation are responsible for the distinct behavior of the curve \( \theta_{\text{min}}(\chi) \) in both models, resulting also in different orders of transition. The first derivative of \( \mathcal{H}_{\text{min}} \) with respect to the parameter \( \chi \) is given by

\[
\frac{d\mathcal{H}_{\text{min}}}{d\chi} = \frac{\partial\mathcal{H}}{\partial\theta} \bigg|_{\theta=\theta_{\text{min}}} \frac{d\theta_{\text{min}}}{d\chi} + \frac{\partial\mathcal{H}}{\partial\chi} \bigg|_{\theta=\theta_{\text{min}}} = \mathcal{H}_1(\theta_{\text{min}}).
\]

Therefore, the continuity of \( \theta_{\text{min}} \) implies that \( \frac{d\mathcal{H}_{\text{min}}}{d\chi} \) is also continuous, since \( \mathcal{H}_1(\theta_{\text{min}}) \) is a continuous function of \( \theta_{\text{min}} \). In general, the reciprocal is also true, because \( \mathcal{H}_1(\theta_{\text{min}}) \) is discontinuous in the case of \( \theta_{\text{min}} \) also discontinuous, resulting in a first-order QPT. The discontinuity

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Footnote 3: In the classical-macroscopic limit \( \mathcal{P}_{su}(3) \) is not well defined as a signal of QPT because the ground level is triply degenerate after the transition and there is no unique choice of state to calculate the purity, giving different results for different linear combinations of degenerate ground states. This difficulty does not exist for finite \( N \), as the ground state is unique.

Footnote 4: The minimum-energy level is doubly (triply) degenerated after the transition in the two- (three-) local-modes model, but we can restrict the phase space so that the choice of \( \theta_{\text{min}}(\chi) \) is unique. There is no loss of generality, because the minima are equivalent.
in the derivative of $\theta_{\text{min}}$ can be identified only in the second derivative of $\mathcal{H}_{\text{min}}$: $\frac{d^2\mathcal{H}_{\text{min}}}{d\theta^2} |_{\theta=\theta_{\text{min}}} \neq 0$. So, in general, the discontinuity of $\frac{d\theta_{\text{min}}}{d\theta}$ implies discontinuity only for the second derivative of $\mathcal{H}_{\text{min}}$, resulting in a second-order QPT.

Conclusions. – We have described the dynamics of a BEC trapped in a symmetric triple-well potential in a three-mode approximation and shown in detail how the ground state of the model undergoes a QPT. A time-dependent variational principle using the SU(3) coherent state allows for a system of semiclassical equations that enables one to find the fixed points of the model and to investigate how the lowest-energy fixed points change as the collision parameters of the model are varied. We have shown that the increase in the entanglement of the ground state is associated to phase compression (Rabi oscillation) and to number compression (self-trapping). In this way we relate the loss of coherence in the fundamental state to squeezing, which is not the only effect that leads to the decrease of the generalized purity. Whenever there is a departure from a coherent state the purity associated to the pertinent algebra decreases, and this behavior unambiguously describes particle entanglement in the system, which correctly signals the presence of the QPT as one crosses distinct dynamical regimes. We remark that this three-symmetric-well model allows for a better characterization of QPT than the case for double well, providing not only the dynamical transitions but signaling the presence of squeezing and thus could promote experimental investigation with the existent trapping technology for this phenomenon. This amount of squeezing and entanglement, if controllable, could provide a new scenario for testing quantum physics fundamental questions.

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REFERENCES

[1] AMICO L., FAZIO R., OSTERLOCH A. and VEDRAL V., Rev. Mod. Phys., 80 (2008) 517.
[2] OSBORNE T. and NIELSEN M., Phys. Rev. A, 66 (2002) 032110; OSTERLOH A., AMICO L., FALCI G. and FAZIO R., Nature (London), 416 (2002) 608; VIDAL G., LATORRE J., RICO E. and KITAEV A., Phys. Rev. Lett., 90 (2003) 227902.
[3] SACHDEV S., Quantum Phase Transitions (Cambridge University Press, Cambridge) 1999; SONDHI S. L., GIRVIN S. M., CARINI J. P. and SHAHAR D., Rev. Mod. Phys., 69 (1997) 315.
[4] WU L. A., SARANDY M. S. and LIDAR D. A., Phys. Rev. Lett., 93 (2004) 250404; DE OLIVEIRA T. R., RIGOLIN G., DE OLIVEIRA M. C. and MIRANDA E., Phys. Rev. Lett., 97 (2006) 170401.
[5] BARNUM H., KNILL E., ORTIZ G. and VIOLA L., Phys. Rev. A, 68 (2003) 032308; Phys. Rev. Lett., 92 (2004) 107902; SOMMA R., ORTIZ G., BARNUM H., KNILL E. and VIOLA L., Phys. Rev. A, 70 (2004) 042311; KLYACHKO A. A., arXiv:quant-ph/020601v1; KLYACHKO A. A., OZTOP B. and SHUMOVSKY A. S., Phys. Rev. A, 75 (2007) 032315.
[6] VISCONDI T. F., FURUYA K. and DE OLIVEIRA M. C., Phys. Rev. A, 80 (2009) 013610.
[7] ESTÈVE J., GROSS C., WELLER A., GIOVANAZZI S. and OBERTHALER M. K., Nature, 455 (2008) 1216.
[8] LIPKIN H. J., MESHKOV N. and GLICK A. J., Nucl. Phys., 62 (1965) 188.
[9] VIDAL J., PALACIOS G. and ASLANGUL C., Phys. Rev. A, 70 (2004) 062304.
[10] SORESEN A., DUAN L., CIRAC J. I. and ZOLLER P., Nature, 409 (2001) 63; WANG X. and SANDERS B., Phys. Rev. A, 68 (2003) 012101; KORBICZ J. K., CIRAC J. I. and LEWENSTEIN M., Phys. Rev. Lett., 95 (2005) 120502.
[11] NEMOTO K., HOLMES C. A., MILBURN G. J. and MUNRO W. J., Phys. Rev. A, 63 (2000) 013604; FRANZOSI R. and PENNA V., Phys. Rev. A, 65 (2001) 013601; BUONSANTE P., FRANZOSI R. and PENNA V., Phys. Rev. Lett., 90 (2003) 050404; FRANZOSI R. and PENNA V., Phys. Rev. E, 67 (2003) 046227; MOSSMANN S. and JUNG C., Phys. Rev. A, 74 (2006) 033601; LIU B., FU L.-B., YANG S.-P. and LIU J., Phys. Rev. A, 75 (2007) 033601.
[12] BUONSANTE P., FRANZOSI R. and PENNA V., Laser Phys., 14 (2004) 556.
[13] MILBURN G. J., CORNEY J., WRIGHT E. M. and WALLS D. F., Phys. Rev. A, 55 (1997) 4318.
[14] VISCONDI T. F., FURUYA K. and DE OLIVEIRA M. C., Ann. Phys., 324 (2009) 1837, arXiv:quant-ph/0811.2139.
[15] SARACENO M. and KRAMER P., Geometry of the Time-Dependent Variational Principle in Quantum Mechanics, Lect. Notes Phys., Vol. 140 (Springer-Verlag, New York) 1981.
[16] PERELOMOV A. M., Generalized Coherent States and their Applications (Springer-Verlag, Berlin) 1986; RAGHUNATHAN K., SEETHARAMAN M. and VASAN S. S., J. Phys. A: Math. Gen., 22 (1989) L1098; ZHANG W.-M., FENG D. H. and GILMORE R., Rev. Mod. Phys., 62 (1990) 867; MATHUR M. and SEN D., arXiv:quant-ph/0012099v1.
[17] ZHANG W.-M., FENG D. H., YUAN J.-M. and WANG S.-J., Phys. Rev. A, 40 (1989) 438; 42 (1990) 7125.
[18] YAFFE L. G., Rev. Mod. Phys., 54 (1982) 407.
[19] ZHANG W.-M., FENG D. H. and GILMORE R., Rev. Mod. Phys., 62 (1990) 867.
[20] SMERZI A., FANTONI S., GIOVANAZZI S. and SHENOY S. R., Phys. Rev. Lett., 79 (1997) 4950.
[21] DELBOURGO R., J. Phys. A: Math. Gen., 10 (1977) 1837; DELBOURGO R. and FOX J. R., J. Phys. A: Math. Gen., 10 (1977) L233.
[22] BARNETT S. M. and PEGG D. T., J. Phys. A: Math. Gen., 19 (1986) 3849; PEGG D. T. and BARNETT S. M., J. Mod. Opt., 44 (1997) 225.
[23] HINES A. P., MCKENZIE R. H. and MILBURN G. J., Phys. Rev. A, 67 (2003) 013609; 71 (2005) 042303.