The quantum effects in the undulator of infinite length

V.G. Bagrov\textsuperscript{1}, V.V. Belov\textsuperscript{2}, M.M. Nikitin\textsuperscript{3}, and A.Yu. Trifonov\textsuperscript{3}

\textsuperscript{1}High Current Electronics Institute, Siberian Division Russian Academy of Science
\textsuperscript{4} Akademichesky Ave., 634055 Tomsk Russia
\textsuperscript{2}Department of Applied Mathematics
Moscow Institute of Electronic and Mathematics,
B. Vusovsky 3/12, 109028 Moscow, Russia
\textsuperscript{3}Department of Mathematical Physics,
Tomsk Polytechnical University, 30 Lenin Ave., 634034 Tomsk, Russia

Abstract

The first order quantum correction to the power of spontaneous radiation of electrons in an arbitrary two-component periodic magnetic field was obtained. The phenomenon of self-polarization of the spin of electrons in a process of spontaneous radiation was also studied. By electron’s motion in a spiral magnetic undulator, the quantitative characteristics of self-polarization (the polarization degree and the relaxation time) are different from corresponding ones in synchrotron radiation. The limiting cases of near-axis and ultrarelativistic approximation were considered.

Introduction

The investigation of spontaneous radiation of electrons moving in periodic structures (undulators) is now an important and rather well developed branch of modern physics. A detailed bibliography about this problem can be found, for example, in reviews [1–2] and monographs [3–5].

In the majority of studies, the radiation was investigated by methods of classical electrodynamics [1–5]. Quantum-electrodynamical considerations were applied in few studies [1, 6–15], where the character of quantum corrections to the radiation of ultrarelativistic particles ($\gamma \gg 1$) was calculated and, in particular, it was shown that these corrections are small in actual undulators.

Nevertheless, because of wide usage in theoretical and experimental research of magnetic undulators, it is important to give a detailed analysis of the character of quantum corrections to the radiation over the whole range of electron energy allowed by the undulator regime. This range also includes the nonrelativistic energies, for which the effects caused by particles motion are significant.

In the present paper we considered the first quantum correction to the characteristic of undulator radiation of charged spinor particles in a magnetic undulator. This consideration is limited to the case of a vector potential of an arbitrary magnetic field depending only on one spatial coordinate (along the particle drift). This particular model of an external magnetic field allows us to consider typical regimes of partial motion in magnetic undulators: along the plane periodic trajectory for a plane undulator, and along
the helical trajectory for a spiral undulator. On the other hand, this model allows us to effectively calculate quantum-electrodynamical characteristics by using Bloch's stationary wave functions obtained in [16] by the one-dimensional WKB method [17].

Let us give some general conclusions. The results of our calculations [9–15] which are of certain theoretical interest for the problems of motion and radiation of particles in magnetic undulators of all energies when the quantum (spinor) properties are taken into account.

1. If the characteristics of spontaneous radiation of electrons in a periodic magnetic field (calculated by methods of quantum theory in the first order over the radiation field when the motion is semiclassical) are represented in the form of formal (and generally asymptotic) Taylor series in the Planck constant $\hbar$ as $\hbar \to 0$, then each term of this series can be represented as a functional of the classical particle trajectory, the external field, and derivatives of this field on the classical trajectory. In particular, the $n$ quantum correction contains the $n+2$-derivative of velocity, i.e., $n+1$-derivative of the vector potential. In the case of $n = 1$ [18], we calculated boson radiation in an arbitrary two-component periodical magnetic field, in which the helical motion is realized.

It should be mentioned that the dependence of the characteristics of spontaneous radiation (calculated in the semi-classical approximation as $\hbar \to 0$ with relativistic accuracy to the first order of $\hbar$) on the parameters of classical trajectory was first shown in [19–20] for the case of electron energies in the relativistic range.

2. The quantum corrections to the radiation characteristics and the influence of different quantum effects (in the whole energy range allowed by the undulator regime of electron motion) depend on the specific form of the potential of the external field. For electrons moving along the same classical trajectory in different external fields, the quantum corrections (in comparison with the classical term) in general will differ.

This conclusion is illustrated here by calculations of the characteristics of radiation of spinor particles moving in magnetic undulators. For example:

1) The explicit dependence of quantum corrections to the frequency of the photon radiation on the specific form of potential of the external periodical magnetic field is shown (see Eq. (2.9)).

2) The quantum expansion parameter of the total power of undulator radiation (in the on-axis approximation) qualitatively differs from the corresponding parameter in synchrotron radiation and depends explicitly on the character of magnetic field and its first derivative on the classical trajectory. It is interesting that, for a specific form of the field in which the particle moves along the spiral trajectory, in the ultrarelativistic limit the first quantum correction to the full radiation power coincides with the expression of the quantum term in the power of synchrotron radiation of the charge moving along the spiral in a constant and homogeneous magnetic field [21].

3) The quantitative characteristics of the effect of radiational self-polarization of electrons in a helical undulator (the degree of polarization and the relaxation time) differ from those in the theory of synchrotron radiation [22] and also from similar results in a field of a flat electromagnetic wave [23] and in a axisymmetric focusing electric field [24], though in all considered (examined) cases a electron movement on a spiral is realized. This fact proves that the structure of the external field but not the classical electron trajectory plays a significant role in the phenomena of self-polarization. It should be mentioned that, in the ultrarelativistic limit ($\gamma \to 0$), the characteristics of the process of self-polarization calculated in Sec. 6 coincide (with an accuracy of $\sim \gamma^6$) with those obtained earlier in [7] on the basis of an operator semi-classical method of calculation.

\[1\] In the ultrarelativistic case ($\gamma \to \infty$), the dependence on the structure of an external field can be neglected.
1 Semi-classical electron wave function

Let us consider an electron moving in a stationary magnetic field with the vector potential:

\[ \vec{A}(z) = (A_1(z), A_2(z), 0), \]
\[ \vec{A}(z + l) = \vec{A}(z), \]

(1.1)

where \(A_i(z)\) are arbitrary smooth periodic functions, \(l\) is the field period. Let us describe the electron motion on the basis of the relativistic Dirac equation:

\[ \hat{H}_D\Psi = E\Psi, \quad \hat{H}_D = c(\vec{\alpha}, \vec{\beta}) + \rho_3 m_0 c^2, \]
\[ 0 \quad \vec{\beta} = (-i\hbar \nabla) - \frac{e}{c} \vec{A}(z), \]
\[ \vec{\alpha} = \left( \begin{array}{cc} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{array} \right), \quad \rho_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \]

(1.2)

where \(\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\) are Pauli matrices, \(E\) is the electron energy and \(e = -e_0\) is the electron charge. Let us separate the solutions of equation (1.2) over a polarization states by condition:

\[ c \quad \hat{S}_t\Psi = \zeta \lambda \Psi, \quad \hat{S}_t = (\vec{\Sigma}', \vec{\beta}), \]
\[ 0 \quad \zeta = \pm 1, \quad \lambda^2 = c^{-2}(E^2 - m_0^2 c^4), \]
\[ \vec{\Sigma}' = \left( \begin{array}{cc} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{array} \right). \]

(1.3)

Here the parameter \(\zeta\) describes the spin orientation with reference to the direction of motion (longitudinal polarization) either along the direction of motion \((\zeta = +1)\) or in the opposite direction \((\zeta = -1)\) [25].

Taking into account of quantum integrals of motion

\[ \hat{p}_1\Psi = p_1\Psi, \quad \hat{p}_2\Psi = p_2\Psi, \quad \hat{p} = -i\hbar \nabla, \]

the solution of system (1.2) and (1.3) will be represented by:

\[ \Psi_{E,\vec{\rho}_\perp,\zeta} = N \exp \left\{ \frac{i}{\hbar} (\vec{\rho}_\perp, \vec{r}) \right\} \phi_{E,\vec{\rho}_\perp,\zeta}(z, h), \]

(1.4)

where

\[ \phi_{E,\vec{\rho}_\perp,\zeta}(z, h) = (g_{E,\vec{\rho}_\perp,\zeta}(z, h), \chi_{E,\vec{\rho}_\perp,\zeta}(z, h))^t, \quad \vec{\rho}_\perp = (p_3, 0), \]
\[ \chi_{E,\vec{\rho}_\perp,\zeta}(z, h) = \frac{\zeta c\lambda}{E + m_0 c^2} g_{E,\vec{\rho}_\perp,\zeta}(z, h), \quad \vec{r} = (x, y, z), \]

(1.5)

\(N\) is a normalization constant. Spinor \(g_{E,\vec{\rho}_\perp,\zeta}(z, h)\) satisfies the condition

\[ \left\{ (\vec{\sigma}, \vec{\rho}_\perp) + \sigma_3 \rho_3 \right\} g_{E,\vec{\rho}_\perp,\zeta}(z, h) = \zeta \lambda g_{E,\vec{\rho}_\perp,\zeta}(z, h), \]
\[ \vec{\rho}_\perp = (\vec{\rho}_1, \vec{\rho}_2, 0), \]
\[ \vec{\rho}_i = p_i + \frac{e_0}{c} A_i(z), \quad i = 1, 2 \]

(1.6)

or in the equivalent form \(g_{E,\vec{\rho}_\perp,\zeta}(z, h) = (g_+(z, h, E, \vec{\rho}_\perp, \zeta), g_-(z, h, E, \vec{\rho}_\perp, \zeta))^t\)

\[ \left\{ \left( \rho_3 + \zeta \lambda \right)(\vec{P}_1 - i\vec{P}_2)^{-1}(\rho_3 - \zeta \lambda) + \vec{P}_1 + i\vec{P}_2 \right\} g_+(z, h) = 0, \]
\[ g_-(z, h) = (\vec{P}_1 - i\vec{P}_2)^{-1} (\zeta \lambda - \rho_3) g_+(z, h). \]

(1.7)

\(^2\)In case this causes no misunderstanding the indeces \(z, h, E, \vec{\rho}_\perp, \zeta\) may be omitted.
In the absence of turning points
\[ p^2(z) = \lambda^2 - (\mathcal{P}_1, \mathcal{P}_\perp) > 0 \] (1.8)
the WKB-solution of system (1.6) has the form [26]:
\[ g_+(z, h) = f_+(z, h) \exp \left\{ \frac{i}{\hbar} S(z) \right\}, \] (1.9)
where \( f_+(z, h) = f_+^{(0)}(z, h) + \hbar f_+^{(1)}(z, h) \) is a regular series over \( \hbar \to 0 \). Here, the actual measureless expansion parameter is the ratio of the de'Brojlie electron wavelength to the period of changing of field \( \lambda \approx \frac{\hbar \omega_0}{E \beta_0} \), where [19]
\[ T \omega_0 = 2\pi, \quad c\beta_0 T = l, \quad T = \frac{E}{c^2} \int_0^l \frac{dz}{p(z)}. \]
Substituting (1.9) into (1.7) we obtain:
\[ S(z) = \int_0^z p(z)dz, \] (1.10)
\[ f_+^{(0)}(z) = \exp \left\{ - \int_0^z dz \frac{(\mathcal{P}_1 - i\mathcal{P}_\perp) \cdot (p(z) - \zeta \lambda)}{2p(z)} \right\}, \]
\[ f_+^{(1)}(z) = if_+^{(0)}(z) \int_0^z dz \frac{(\mathcal{P}_1 + i\mathcal{P}_\perp) \cdot f_+^{(0)}(z)}{2p(z)f_+^{(0)}(z) \cdot d\frac{f_+^{(0)}(z)}{dz}}(\mathcal{P}_1 - i\mathcal{P}_\perp). \]
Here, the dot defines the differentiation with respect to the parameter \( z (\zeta = \frac{dz}{dt}) \). By condition (1.8), the spinor \( g(z, h) = (g_+(z, h), g_-(z, h))^t \), where
\[ g_\pm(z, h) = \mu_\pm \left\{ \frac{(\mathcal{P}_1 \mp i\mathcal{P}_\perp)(\lambda \mp \zeta \mathcal{P}_\perp)}{((\mathcal{P}_1, \mathcal{P}_\perp))^{1/2}} \right\}^{1/2} \exp \left\{ \frac{i}{\hbar} \int_0^z \left( \frac{\lambda}{p} \frac{\mathcal{P}_1 + i\mathcal{P}_\perp}{p} \frac{\mathcal{P}_1 - i\mathcal{P}_\perp}{p^2} \right) dz \right\}, \]
\[ \mu_+ = 1, \quad \mu_- = \zeta, \quad p = p(z) \]
satisfies (1.6) with accuracy up to \( O(h^2) \). From here and from the condition
\[ g_{E, \mathcal{P}_\perp, \zeta}(z + l, h) = \exp \left\{ \frac{i}{\hbar} q(E, \mathcal{P}_\perp, \zeta)l \right\} g_{E, \mathcal{P}_\perp, \zeta}(z, h) \]
we find the quasi-momentum of an electron:
\[ q(E, \mathcal{P}_\perp, \zeta) = \frac{1}{l} \int_0^l \left\{ p + \zeta \frac{h\lambda}{2p} \mathcal{P}_1 - \mathcal{P}_\perp \mathcal{P}_\perp - \frac{h^2}{8p^3} \mathcal{P}_1 - \mathcal{P}_\perp \right\} dz. \] (1.12)
In the absence of turning points \( p^2(z) > 0 \), the normalization condition on the electron charge for the semiclassical stationary solutions (1.4) of the Dirac equation
\[ \int \Psi_0^+ \Psi_0 d^3x = \delta_{\nu, b}, \quad b = (E, \mathcal{P}_\perp, \zeta) \]
is equivalent to the condition
\[ 4L^2NN'[1 + \frac{\zeta' e^2 \lambda'}{(E + m_0c^2)(E' + m_0c^2)}] \int L^{-L} dz g^+_{E', \vec{\alpha}_{L}, \zeta'} g_{E, \vec{\alpha}_{L}, \zeta} = \delta_{E', E} \delta_{\zeta', \zeta}. \]

By analogy with the case without spin [12], we obtain
\[ \int L^{-L} dz 2L \delta_{qq'} \frac{1}{1} \int L^{-L} dz g^+_{E', \vec{\alpha}_{L}, \zeta'} g_{E, \vec{\alpha}_{L}, \zeta}, \]
\[ \delta_{qq'} = \left\{ \left[ \frac{\partial(q - q')}{\partial(E - E')} \right]_{q = q'} \right\}^{-1} \delta_{E, E'}, \]
\[ g^+_{E, \vec{\alpha}_{L}, \zeta} g_{E, \vec{\alpha}_{L}, -\zeta} = O(h). \]

Here, we used the fact that the quasi-momentum is defined with the accuracy up to \( 2\pi \hbar \):\[
\int L^{-L} dz 2LN^2 g^+_{E', \vec{\alpha}_{L}, \zeta'}(z, h) g_{E, \vec{\alpha}_{L}, \zeta}(z, h) \left( 1 + \frac{\zeta' e^2 \lambda'}{(E + m_0c^2)(E' + m_0c^2)} \right) = \frac{4L^2N^2}{E + m_0c^2} \left\{ \frac{1}{1} \int L^{-L} dz g^+_{E', \vec{\alpha}_{L}, \zeta'} g_{E, \vec{\alpha}_{L}, \zeta} \right\}^{-1} \delta_{E, E'} \delta_{\zeta', \zeta'}, \]
\[ \frac{1}{1} \int L^{-L} dz g^+_{E, \vec{\alpha}_{L}, \zeta'}(z) g_{E, \vec{\alpha}_{L}, \zeta}(z) = \frac{1}{1} \int L^{-L} dz \left\{ 2L - \zeta \frac{\hat{P}_1 \hat{P}_2 - \hat{P}_2 \hat{P}_1}{p^2} \right\}. \]

As a result, for the normalizing factor we obtain
\[ N^2 = \frac{E + m_0c^2}{4c \lambda'(2L)^3}. \]

2 The spectral-angular distribution of power

The power radiated to the element of solid angle \( d\Omega = \sin \theta d\theta d\varphi \) at the electronic transition from the state \( \Psi_b \) to the state \( \Psi_{b'} \) with the radiation of photon can be found by usual electrodynamical methods and is equal to [21]:

\[ \frac{dW_{b,b'}}{d\Omega} = \frac{c E^2}{2\pi} \int_0^\infty d\kappa \kappa^2 \delta(\kappa - \frac{E - E'}{\hbar}) \{ |\alpha_{\pi}|^2 + |\alpha_{\sigma}|^2 \}, \]  
(2.1)
\[ \alpha_{\pi} = (\vec{e}_{\pi}, \vec{B}_{b,b'}), \quad \alpha_{\sigma} = (\vec{e}_{\sigma}, \vec{B}_{b,b'}); \]
\[ \vec{B}_{b,b'} = \int \Psi_{b'}^* \vec{\alpha} \Psi_b \exp\{-iz(\vec{n}, \vec{r})\} d\vec{r}, \]
\[ \vec{e}_{\pi} = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, -\sin \theta), \]
\[ \vec{e}_{\sigma} = (\sin \varphi, -\cos \varphi, 0), \]
\[ \vec{n} = (e_1, e_2, e_3), \]
\[ e_1 = \cos \varphi \sin \theta, \quad e_2 = \sin \varphi \sin \theta, \quad e_3 = \cos \theta, \]
where \( \vec{\alpha} \) are the Dirac matrices and \( h \kappa \vec{n} \) is the momentum of the radiated photon. We assumed, that the average electron drift along the axes \( Ox \) and \( Oy \) is absent:
\[ \int_0^L \frac{\hat{P}_1(z)}{p(z)} dz = \int_0^L \frac{\hat{P}_2(z)}{p(z)} dz = 0. \]  
(2.3)
The spectral-angular distribution of the total radiation power can be obtained from (2.1) by summation over all final states $E', \mathbf{p}', \mathbf{B}'$:

$$\frac{dW(\zeta, \zeta')}{d\Omega} = \sum_{E', \mathbf{p}', \mathbf{B}'} \frac{dW_{E', \mathbf{p}', \mathbf{B}'}}{d\Omega}. $$

By calculating the matrix elements (2.2) the wave functions (1.4) are represented in a form of Bloch functions and their periodical parts are expanded in Fourier series. After integration over coordinates, this leads to the following conservation laws:

$$\begin{align*}
\{ & \mathbf{p}_\perp - \mathbf{p}_\perp' = h\mathbf{e}_\perp \alpha, \\
& q - q' = e_3 h\alpha + 2\pi h n/l. \tag{2.4} \}
\end{align*}$$

For the spectral-angular distribution of radiation power we obtain:

$$\frac{dW(\zeta, \zeta')}{d\Omega} = c_1\sum_{n=1}^{\infty} \frac{1}{2^\alpha} \left\{ \frac{\partial \Phi(x)}{\partial x}\right\}_{x=0} \left\{ ||c_2, \mathbf{B}(n, \zeta, \zeta')||^2 + ||c_1, \mathbf{B}(n, \zeta, \zeta')||^2 \right\}, \tag{2.5}$$

$$\mathbf{B}(n, \zeta, \zeta') = (2L)^3 N \sum_{l} \int dz e^{-i\omega n z} e_{n, \zeta, \zeta'}^{\perp} P_{n, \zeta, \zeta'}^\perp(z) \Phi_{E, \mathbf{p}, \mathbf{B}, \zeta, \zeta'}(z), \tag{2.6}$$

$$\Phi(x) = h^{-1}(q - q') - x e_3 - 2\pi n/l.$$ 

The radiation frequency $\omega(n, h)$ can be defined from the system of conservation laws (2.4) and the conservation law of energy $E - E' = ch\alpha$. For this, we find the increment of quasi-momentum $\Delta q = q - q'$, which, according to (1.11), is equal to

$$\Delta q = h\omega_0(l) + \frac{1}{2}(\zeta - \zeta') h\rho_3(l) + \frac{1}{2}h^2 \omega^2 \rho_1(l) + \frac{1}{2}h^2 \omega\zeta' \rho_2(l), \tag{2.7}$$

where

$$\begin{align*}
\rho_0(x) &= \frac{1}{c x} \int_0^x \left( \frac{E}{c} - (\mathbf{e}_\perp, \mathbf{p}_\perp) \right) \frac{dz}{p(z)}, \\
\rho_1(x) &= \frac{1}{c^2 x} \int_0^x \left\{ \left( \frac{E}{c} - (\mathbf{e}_\perp, \mathbf{p}_\perp) \right)^2 - p^2 e_3^2 \right\} \frac{dz}{p^3(z)}, \\
\rho_2(x) &= \frac{\lambda}{c x} \int_0^x \left\{ \frac{\rho_2 e_1 - \rho_1 e_2}{P_1^2 + P_2^2} - \frac{\rho_3 P_1 - \rho_1 P_2}{P_1^2 + P_2^2} \times \right. \\
&\left. \times \left( \frac{E (\mathbf{p}_\perp, \mathbf{p}_\perp)}{c^2} - \frac{e_3 (\mathbf{p}_\perp, \mathbf{p}_\perp)}{(P_1^2 + P_2^2)p^2} \right) \right\} \frac{dz}{p(z)}, \\
\rho_3(x) &= \frac{\lambda}{x} \int_0^x \frac{\rho_2 P_1 - \rho_1 P_2}{(P_1, \mathbf{p}_\perp)} dz.
\end{align*} \tag{2.8}$$

Taking into account the condition (2.3), we will get $\rho_0(l) = 1/(c_1 e_3)$. Substituting (2.7) into (2.4), we find the radiation frequency (up to $O(h^2)$) in the form:

$$\begin{align*}
\omega(n, h) &= \omega_{cl} \left\{ 1 - \frac{h c \beta}{2 \psi_0} (\omega_0 \rho_1(l) + \zeta \rho_2(l)) \right\}, \\
\omega_{cl} &= \frac{\beta c}{\psi_0} \left\{ \omega_n - \frac{1}{2} (\zeta - \zeta') \rho_3(l) \right\}, \\
\psi_0 &= 1 - \beta e_3, \quad \omega_n = 2\pi n/l.
\end{align*} \tag{2.10}$$
By analogy with (2.7), we obtain
\[ \left. \frac{\partial \Phi}{\partial \lambda} \right|_{\Phi(\lambda) = 0} = \psi_0 \beta_\parallel \left\{ 1 + \frac{hc\beta_\parallel}{\psi_0} (\omega \rho_1 (l) + \frac{\zeta}{2} \rho_2 (l)) \right\}. \] (2.12)

It should be noted, that the expression for the frequency of classical radiation (2.9) thus obtained differs from the corresponding expression in [12] by a summand, which is proportional to \((\zeta - \zeta') p_3 (l)\) and corresponds to transitions with spin flip. The probability of these transitions is proportional to \(h\). Therefore, at \(h = 0\), the frequency (1.9) differs from the frequency of classical radiation in the field (1.1) to harmonics whose the probability of radiation is equal to zero. This means that they just coincide.

3 The matrix element of the transition currents

Substituting (1.5) into (2.6), we obtain
\[ \tilde{B} (n, \zeta, \zeta') = \frac{1}{2} (1 + \zeta \zeta') \tilde{B}^\dagger (n, \zeta) + \frac{1}{2} (1 - \zeta \zeta') \tilde{B}^\dagger (n, \zeta), \] (3.1)
\[ \tilde{B}^\dagger (n, \zeta) = \frac{\zeta}{2l} \int_0^l dz \exp \left\{-i \xi \omega_3 z \right\} g_{E', \vec{p}'_\perp, \zeta}^+ (z) \tilde{g}_{E, \vec{p}_\perp, \zeta} (z), \] (3.2)
\[ \tilde{B}^\dagger (n, \zeta) = \frac{\xi m \xi \omega_3 \zeta}{4 \lambda^2} \int_0^l dz \exp \left\{-i \xi \omega_3 z \right\} \tilde{g}_{E', \vec{p}'_\perp, -\zeta}^+ (z) \tilde{g}_{E, \vec{p}_\perp, \zeta} (z). \] (3.3)

Let us consider the matrix elements without spin flip
\[ \left\{ \begin{array}{l} g^+_\pm (z, E', \vec{p}'_\perp, \zeta) \\
\phantom{g^+_\pm (z, E', \vec{p}'_\perp, \zeta)} g^- (z, E, \vec{p}_\perp, \zeta) \\
g^- (z, E', \vec{p}'_\perp, \zeta) \\
g^+_\pm (z, E, \vec{p}_\perp, \zeta) \end{array} \right\} = \left\{ \begin{array}{l} f^{(0)}_\pm (z, E', \vec{p}'_\perp, \zeta) f^{(0)}_\pm (z, E, \vec{p}_\perp, \zeta) \left( 1 + \frac{h \xi P_2 - \xi P_1 P_2}{2 (P_1^2 + P_2^2) \xi} (p \mp \zeta \lambda) \right) \\
\phantom{f^{(0)}_\pm (z, E', \vec{p}'_\perp, \zeta)} f^{(0)}_\pm (z, E, \vec{p}_\perp, \zeta) \left( 1 + i \xi h \frac{\lambda \xi P_1 \mp i \xi P_2}{P_1 + i P_2} \right) \right\} \right. \times \exp \left\{ \frac{i}{h} \int_0^z dz \left[ p(z, E, \vec{p}_\perp) - p(z, E', \vec{p}'_\perp) \right] \right\} ; \]
\[ f^{(0)}_\pm (z, E', \vec{p}'_\perp, \zeta) f^{(0)}_\pm (z, E, \vec{p}_\perp, \zeta) = \left[ \frac{\lambda + \zeta \xi p_1}{P_1} - h \frac{2 \lambda + \zeta \xi p_1}{P_2^2 + P_2^2} \left( \frac{\zeta \xi P_1 \mp i \xi P_2}{P_1 + i P_2} \right) \right] \times \exp \left\{ i \xi h \xi z \omega_2 (z)/2 \right\} ; \]
\[ f^{(0)}_\pm (z, E', \vec{p}'_\perp, \zeta) f^{(0)}_\pm (z, E, \vec{p}_\perp, \zeta) = \frac{\zeta}{P_1 \pm i P_2 - \frac{1}{2} h \xi \left\{ e_1 \pm i e_2 + (P_1 \pm i P_2) \right\} \times \frac{1}{P_2^2} \left( \zeta \xi P_1 \mp i \xi P_2 \left( \frac{E_\perp}{P_1} + \frac{(E_\perp)}{P_2^2 + P_2^2} \right) - \frac{E}{c} \right) \right\} \times \exp \left\{ i \xi h \xi z \omega_2 (z)/2 \right\} \]
and by analogy with the scalar case, we have
\[ \int_0^z \left[ p(z, E, \vec{p}_\perp) - p(z, E', \vec{p}'_\perp) \right] dz = \omega z \rho_0 (z) + \frac{1}{2} h \omega^2 z \rho_1 (z). \] (3.4)
So, within an accuracy up to the quantum correction of the first order over $h \to 0$, we obtain:

$$B_{1 \uparrow \uparrow}^{\downarrow}(n, \zeta) = \frac{1}{l} \int_0^l \frac{dz}{p} e^{i\omega z} \left[ \mathcal{P}_1 + \frac{\hbar \kappa}{2} \left\{ \mathcal{P}_1 \frac{E/c - (\vec{e}_1, \vec{P}_\perp)}{p^2} - \right. \right. \right.$$

$$\left. \left. - e_1 - \zeta \lambda \frac{E}{c \lambda^2} \left( \frac{\mathcal{P}_2}{p^2} - \frac{p}{\vec{P}_1 + \vec{P}_2} \right) + \zeta \lambda \frac{p_1}{p^2} \right\} \right],$$

$$B_{2 \downarrow \downarrow}^{\uparrow}(n, \zeta) = \frac{1}{l} \int_0^l \frac{dz}{p} e^{i\omega z} \left[ \mathcal{P}_2 + \frac{\hbar \kappa}{2} \left\{ \mathcal{P}_2 \frac{E/c - (\vec{e}_1, \vec{P}_\perp)}{p^2} - e_2 - \right. \right. \right.$$

$$\left. \left. - \zeta \lambda \frac{E}{c \lambda^2} \left( \frac{\mathcal{P}_1}{p^2} - \frac{p}{\vec{P}_1 + \vec{P}_2} \right) + \zeta \lambda \frac{p_2}{p^2} \right\} \right],$$

$$B_{3 \uparrow \downarrow}^{\downarrow}(n, \zeta) = \frac{1}{l} \int_0^l \frac{dz}{p} e^{i\omega z} \left[ p + \frac{\hbar \kappa}{2} \left\{ E/c - (\vec{e}_1, \vec{P}_\perp) + \right. \right. \right.$$

$$\left. \left. \right. + i \frac{\dot{p}}{\kappa p} - e_3 + i \zeta \lambda \frac{\mathcal{P}_2}{p^2} - e_3 \frac{\mathcal{P}_1}{p^2} \right\} \right],$$

where

$$\psi_1(z, n, \zeta, h) = \omega z \rho_0(z) - \omega z \frac{e_3}{c} + \frac{h}{2} \omega^2 z \rho_1(z) + \frac{\hbar}{2} \omega z \rho_2(z)$$

and $\rho_i(z)$ are defined in (2.7). The matrix elements (3.4)–(3.6) have the structure

$$\vec{B}^{\uparrow \downarrow}(n, \zeta) = \vec{B}^{\uparrow \downarrow}(n, \zeta) - \frac{\hbar \kappa}{2} \frac{\dot{\mathcal{P}}_1}{\dot{\mathcal{P}}_2} \frac{1}{l} \int_0^l \frac{dz}{p} \exp \left\{ i \omega z \left( \rho_0(z) - \frac{e_3}{c} \right) \right\}.$$  

The last term does not influence the radiation power. So we will consider $\vec{B}^{\uparrow \downarrow}(n, \zeta) = \vec{B}^{\uparrow \downarrow}(n, \zeta)$. For the expressions with the spin flip, we perform analogous calculations:

$$f^{(0)\downarrow}(z, E', \vec{P}'_\perp, -\zeta) f^{(0)\downarrow}(z, E, \vec{P}_\perp, \zeta) = \pm (\vec{P}'_\perp, \vec{P}_\perp)^{1/2} \exp \{ i(\psi_2(z, \zeta) - \pi e_3 z) \},$$

$$f^{(0)\uparrow}(z, E', \vec{P}'_\perp, -\zeta) f^{(0)\uparrow}(z, E, \vec{P}_\perp, \zeta) = \pm \zeta \frac{(\mathcal{P}_1 \pm i \mathcal{P}_2)/(\lambda \pm \zeta p)}{(\mathcal{P}_1^2 + \mathcal{P}_2^2)^{1/2}} \exp \{ i(\psi_2(z, \zeta) - \pi e_3 z) \},$$

where

$$\psi_2(z, \zeta) = \omega z \rho_0(z) - \frac{e_3}{c} + \zeta \rho_3(z) = \omega z \rho_0(z) + \zeta \rho_3(z).$$

We obtain:

$$B_{1 \uparrow \downarrow}^{\uparrow}(n, \zeta) = \frac{\hbar \kappa m_0 c}{2} \frac{1}{\lambda^2 \lambda^2} \frac{1}{l} \int_0^l \frac{dz}{p} \exp \{ i \psi_2(z, \zeta) \} dz,$$

$$B_{2 \downarrow \downarrow}^{\uparrow}(n, \zeta) = \frac{\hbar \kappa m_0 c}{2} \frac{1}{\lambda^2 \lambda^2} \frac{1}{l} \int_0^l \frac{dz}{p} \exp \{ i \psi_2(z, \zeta) \} dz,$$

$$B_{3 \uparrow \downarrow}^{\downarrow}(n, \zeta) = \frac{\hbar \kappa m_0 c}{2} \frac{1}{\lambda^2 \lambda^2} \frac{1}{l} \int_0^l \frac{dz}{p} \exp \{ i \psi_2(z, \zeta) \} dz.$$

The obtained expressions (2.5), (2.8), (2.10), (3.1)–(3.13), in principle solve the problem of taking into account the quantum corrections caused by quantum effects of motion itself $\hbar \omega_0/(E \beta ||)$, and by output of radiated photon $\hbar \omega/E$. 

8
4 The full radiation power in the on-axis approximation

To integrate the spectral-angular distribution over all angles and to sum over the spectrum in order to obtain the total radiation power with an accuracy to the first quantum correction is possible only in the near-axis approximation. It can be characterized by the classical parameter

\[ \mu = \max_{t \in [0, T]} \left\{ \left| \frac{\beta_1}{\beta_3} \right|, \left| \frac{\beta_2}{\beta_3} \right| \right\} \]  \hspace{1cm} (4.1)

which is the maximum deflection angle of the electron velocity from the undulator axis. Here \( c \beta_j = c^2 P_j / E \) is the velocity along the \( j \) axis, \( T \) is the period of motion over trajectories

\[ T = \frac{E}{c^2} \int_0^t \frac{dz}{p(z)} \]

which is related to the period of field \( l \) by \( l = c \beta_\parallel T \), where \( c \beta_\parallel = [\rho_1(t)]^{-1} \) is an average drift along the axis \( Oz \), and \( \vec{p} = (P_1, P_2, P) \) is a kinetic momentum.

Summing the spectral-angular distribution of the radiation power over a spin and integrating over the time in (2.8), (3.1) and (3.4)-(3.6), we use the following relations:

\[ \frac{\partial \chi}{\partial z} = \dot{\chi}, \quad \frac{\partial \chi}{\partial t} = \chi', \quad x'_i = c \beta_i, \quad i = 1, 3. \]  \hspace{1cm} (4.2)

Then

\[ P_i = \frac{E}{c} \beta_i, \quad \int_0^{z(t)} F(z) dz = \int_0^{F(z(t))c \beta_3 dt}, \]

\[ \dot{\chi} = \frac{E}{c^2} \beta_3 \chi', \quad \int_0^{\chi(t)} F(z) dz = \int_0^{E(\beta_\perp, \beta_\parallel)} \frac{E}{c} (1 - (\beta_\perp, \beta_\parallel)), \]

\[ z \rho_0(z) - e_3 z = ct - (\vec{r}_c(t), \vec{n}) = F_0(t), \quad z \rho_1(z) = \frac{c^2}{E} \rho_1(t); \]

\[ \frac{\partial \Phi}{\partial x} \bigg|_{\Phi(\infty)} = \frac{1}{\beta_\parallel} \left\{ \psi_0 + \frac{h \omega c \rho_1(T)}{E T} \right\} = \frac{\psi_0}{\beta_\parallel} \frac{\omega^2}{\beta^2} + O(h^2), \]

\[ \omega = \omega_c \left\{ 1 - \frac{h}{2E} \psi_0 F_0(t) \right\}, \quad \psi_1(t) = \omega_c \left\{ F_0(t) + \frac{h \omega c \rho_1(T)}{2E} F_1(t) \right\}, \]

\[ \omega_c = \frac{n \omega_0}{\psi_0}, \quad \omega_0 = \frac{2\pi}{T}, \]

where

\[ \rho_1(t) = \int_0^t \left[ 1 - (\beta_\perp, \beta_\parallel) \right]^2 - \beta_3^2 \psi_0^2 \frac{dt}{\beta_3}, \]

\[ F_1(t) = \rho_1(t) - \frac{\rho_1(T)}{T \psi_0} F_0, \]

and for the matrix elements of transition currents

\[ \vec{B}(n) = \frac{1}{\beta_\parallel T} \int_0^T dt e^{i \omega_0} \left\{ \beta \left( 1 + \frac{h \omega c_0}{2E} \beta^2 \left( 1 - (\beta_\perp, \beta_\parallel) \right) \right) + k_3 \frac{h \beta_3^2}{2E} \right\}, \]

\[ k = (0, 0, 1), \quad \vec{e}_\perp = (e_1, e_2, 0), \quad \vec{\beta}_\perp = (\beta_1, \beta_2, 0), \]

\[ e_1 = \sin \theta \cos \varphi, \quad e_2 = \sin \theta \sin \varphi, \quad e_3 = \cos \theta. \]
The final result for the radiation power into the element of solid angle \(d\Omega = \sin \theta \, d\theta d\varphi\) is given by

\[
\frac{dW}{d\Omega} = \frac{c^2 \beta^2}{2\pi c} \sum_{n=1}^{\infty} \frac{\omega^4}{\omega_{cl}^4 \psi_0} \left( |\alpha_\pi(n)|^2 + |\alpha_\sigma(n)|^2 \right), \tag{4.6}
\]

where \(\beta = \beta_0 = \beta_1 \approx \beta_\parallel\) and \(\omega_{cl}^2 = \omega_0^2 \cos \theta\).

The expressions (2.1), (4.2)–(4.4) obtained above solve the problem of including the first quantum correction into the radiation of a spinless particle. It should be noted, that the same result can be derived by using the solution of the Klein-Gordon equation in the field (1.1) [12].

Using the properties of conjugate Fourier series [27] in expression (4.5) it is possible to sum over \(n\) and to obtain the spectral-angular distribution of the radiation power considering the first quantum correction in the form:

\[
W = \frac{c^2_0}{2\pi c} \int \frac{d\Omega}{2T} \left[ \int_0^t \frac{A^2(t)}{\psi} \, dt + \frac{\hbar}{E_T} \int_0^t \int_0^t dt_2 A(t_1) D(t_2) \cot \Xi \right] + O(\hbar^2), \tag{4.8}
\]

where the double integral is understood in the sense of the principal value, and the functions \(D(t), A(t)\) have the form:

\[
A^2(t) = (\tilde{A}(t), \varepsilon_\varphi)^2 + (\tilde{A}(t), \varepsilon_\theta)^2; \quad \tilde{A}(t) = [\beta_1(n, \beta_3)] / \psi^2;
\]

\[
A(t_1) D(t_2) = (\tilde{A}(t_1), \varepsilon_\sigma)(\tilde{B}(t_2), \varepsilon_\sigma) + (\tilde{A}(t_1), \varepsilon_\pi)(\tilde{B}(t_2), \varepsilon_\pi);
\]

\[
\tilde{B}(t) = \frac{d}{dt} \tilde{B}(t), \quad \psi = F_0 = 1 - (\vec{n}, \vec{\beta});
\]

\[
\tilde{B}(t) = \left\{ (1 - (\vec{n}_\perp, \vec{n}_\perp)) \beta_3 \tilde{A}(t) \psi^2 + [2 \beta_3 \vec{\beta} \psi + 2 \beta_3 \vec{\beta} (\vec{n}, \vec{\beta}) - \beta_3 \psi \beta_3] \frac{\vec{e}_3}{\beta_3} + \frac{F_1(t) \vec{g}}{\psi^2} - \frac{\beta_3}{\beta_3^2} \psi^2 \frac{\vec{e}_3}{\beta_3} \right\} \frac{1}{\psi^2};
\]

\[
F_1(t) = \int_0^t \left[ \frac{[1 - (\vec{n}_\perp, \vec{n}_\perp)]^2}{\beta_3^2} \right] \, dt - \frac{F_0}{\omega_{cl} \psi_0};
\]

\[
\vec{g} = \vec{\beta} \psi^2 + \psi (\vec{n}, \vec{\beta}'), + 3(\vec{n}, \vec{\beta}) [\psi \vec{\beta}' + \vec{\beta} (\vec{n}, \vec{\beta})].
\]

In the classical term of (4.6), one can integrate explicitly over angles, however, in the quantum term, this integration is impossible.

Complete calculation of radiation characteristics can be done in the near-axis approximation \(\mu \to 0\) (4.1). The terms of first order in \(\mu \to 0\) in (4.2)–(4.5) imply

\[
\beta_3 \approx \beta \approx \beta_\parallel, \quad \beta_1 \approx \beta_2 \approx \mu \beta_\parallel,
\]

\[
F_0(t) = \psi_0 t - \int_0^t (\vec{n}_\perp, \vec{n}_\perp) \, dt + O_\mu(\mu^2);
\]

\[
\rho_1(t) = \beta_\parallel^{-2} \left[ (1 - \beta_\parallel^2 \vec{e}_3^2) t - 2 \int_0^t (\vec{n}_\perp, \vec{n}_\perp) \, dt \right] + O_\mu(\mu^2),
\]

where \(\sum_{n=1}^{\infty} \alpha'_\pi(n) = 0\).
where \( dW \)

For the spectral-angular distribution of the radiation power, we obtain

After integration over angles, we find the spectral distribution of radiated power in

For summation over the spectrum we use the Parseval equality and the properties of

where \( \xi_0 = \hbar \omega / (E \beta^2 ||) \).

For the spectral-angular distribution of the radiation power, we obtain

where

\[
S_\pi(\theta, \varphi) = f_\pi(\theta) \left| \frac{1}{T} \int_0^T (\beta_1 \sin \varphi - \beta_2 \cos \varphi) e^{i \omega_0 nt} dt \right|^2,
\]

\[
S_\sigma(\theta, \varphi) = f_\sigma(\theta) \left| \frac{1}{T} \int_0^T (\beta_1 \cos \varphi - \beta_2 \sin \varphi) e^{i \omega_0 nt} dt \right|^2,
\]

After integration over angles, we find the spectral distribution of radiated power in

\[
W = \frac{e^2 \omega_0^2}{c} \sum_{n=1}^{\infty} n^2 (S_\pi + S_\sigma) \sum_{k=1}^{2} \left| \frac{1}{T} \int_0^T e^{i \omega_0 nt} \beta_k dt \right|^2 + O(\mu^2) + O(h^2) =
\]

\[
= \frac{e^2 \omega_0^2 \beta^2 ||}{c} \sum_{n=1}^{\infty} n^2 (S_\pi + S_\sigma) \sum_{k=1}^{2} \left| \frac{1}{T} \int_0^T H_k(z) e^{i \omega_n z} dz \right|^2 + O(\mu^2) + O(h^2),
\]

\[
S_\pi = \frac{1}{3 (1 - \beta^2 ||)^2} \left[ 1 - \frac{1}{5} \xi_0 n^2 \frac{5 + 19 \beta^2 ||}{1 - \beta^2 ||} \right],
\]

\[
S_\sigma = \frac{1}{(1 - \beta^2 ||)^2} \left[ 1 - \xi_0 n \frac{1 + 3 \beta^2 ||}{1 - \beta^2 ||} \right],
\]

where \( \vec{H}(z) = (-\hat{A}_2(z), \hat{A}_1(z), 0) \) is the magnetic field and \( \omega_n = 2 \pi n / l \).

For summation over the spectrum we use the Parseval equality and the properties of conjugate Fourier series [27] were used:

\[
\sum_{l=1}^{2} \sum_{n=1}^{\infty} n^2 \left| \frac{1}{T} \int_0^T e^{i \omega_0 nt} \beta_l dt \right|^2 = \frac{1}{\omega_0^2} \frac{1}{2T} \int_0^T (\beta_+^2) dt = \frac{1}{2 \omega_0^2} \delta_0,
\]
\[
2 \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} n^3 \left| \frac{1}{T} \int_0^T e^{i\omega_0 l t} \beta_l dt \right|^2 = \frac{1}{\omega_0^2} \frac{1}{2T^2} \int_0^T dt \int_0^T d\tau (\bar{\beta}_l^2 (t), \bar{\beta}_l^2 (\tau)) \cot \left( \frac{t - \tau}{T} \pi \right) = \frac{1}{2\omega_0^2} \delta_1.
\]

The last integral is understood in the sense of the principle value. So, the total radiation power with account of the first quantum correction is

\[
W = W_{cl}(I_\pi + I_\sigma), \quad (4.15)
\]

\[
I_\pi = \frac{1}{4} - \frac{h \delta_1}{E \beta_\parallel} \frac{5 + 19 \beta_\parallel^2}{20(1 - \beta_\parallel^2)} \quad (4.16)
\]

\[
I_\sigma = \frac{3}{4} - \frac{h \delta_1}{E \beta_\parallel} \frac{3 + 9 \beta_\parallel^2}{4(1 - \beta_\parallel^2)},
\]

\[
W_{cl} = \frac{2}{3} \frac{e_0^2}{c(1 - \beta_\parallel^2)^2} \frac{1}{T} \int_0^T (\bar{\beta}_l^2)^2 = \frac{2}{3} \frac{e_0^4 \beta_\parallel^2}{m_0 c^3(1 - \beta_\parallel^2)^2} \frac{1}{l} \int_0^l (\bar{H}(z))^2 dz.
\]

If a component of the magnetic field is equal to zero, then we have the radiation power in the plane undulator regime [9, 10].

**Remark.** From a ratio (4.11) it follows that in on-axis approximation the maximum in distribution on harmonics is necessary that number \(n\) for which factor \(H(n)\) of expansion in a Fourier series for a magnetic field force is maximum. In first, on visible, this conclusion verifying the qualitative Motz results [28] (in the respect that not without fail maximum on the module is first \((n = 1)\) Fourier factor) was made in work [29]. In particular, from (4.11) it follows that of magnetic fields, for which the influence of a harmonic \(H(n_{cr})\) is essential, where \(n_{cr} \sim (1 - \beta_\parallel^2)/\xi_0\), the semiclassical expansion of (1.9) type is not true, as the quantum amendment becomes comparable with classical summand.

## 5 The radiation power in the helical undulator

In general, it is impossible to integrate the spectral angular distribution over all angles and to sum it over the whole spectrum. The total radiation power including the first quantum correction can be obtained in the ultrarelativistic case only for special magnetic field, in which the electron moves along the helical trajectory. Let us consider the periodical magnetic field

\[
\bar{H} = \{-H_0 \cos az, -H_0 \sin az, 0\}
\]

with the period \(l = 2\pi/a\). The relativistic equation of motion is

\[
\begin{align*}
x'' &= z' \omega_0 \sin az, & y'' &= -z' \omega_0 \cos az, \\
z'' &= \omega_0 (x' \sin az + y' \cos az), & \omega_0 &= e_0 c H_0 / E, \quad e_0 = |e|.
\end{align*}
\]

In the general case (with the arbitrary initial conditions), the solution of system (5.2) is expressed by the elliptical integral. However, in the case of interest, when the condition of absence of the average drift along the axis \( Ox \) and \( Oy \) is fulfilled and the projection of initial momentum on this axis is zero, the dependence of coordinates on time is of the form

\[
\bar{r}(t) = \{R \sin \omega_0 t, -R \cos \omega_0 t, c \beta_\parallel t\}.
\]

\[12]
The system (5.3) describes the motion of a particle along the helix with frequency of spiral motion \( \omega_0 = 2\pi/T = c\beta || a \) and the radius of spiral \( R = \beta _{\perp}/(\beta || a) \), \( \beta _{\perp} = e_0H/(Ea) \).

In the field (5.1), the spectral-angular distribution of radiation power is defined by the general formulae (4.4) and (4.5) and is of the form (see Appendix A1):

\[
W = \frac{e_0^2}{c\omega_0^2} \sum\limits_{n=1}^{\infty} \int_0^{\pi} \frac{\sin \theta d\theta}{(1 - \beta || \cos \theta)^2} [|\alpha_{\pi}(n)|^2 + |\alpha_{\sigma}(n)|^2],
\]

(5.4)

\[
|\alpha_{\pi}(n)|^2 = \left( \frac{\cos \theta - \beta ||}{\sin \theta} \right)^2 \left\{ J_n^2(z) - \frac{h\omega_0 n}{E\beta ||} \left[ J_n^2(z) \left( \frac{3 + \beta || \cos \theta}{2} - \frac{\beta _{\perp} \cos \theta}{\beta || \cos \theta} + \frac{z^2}{n^2} \right) - \frac{z}{2} \left( 1 - \frac{z^2}{n^2} \right) J_n(z) \right] \right\},
\]

(5.5)

\[
|\alpha_{\sigma}(n)|^2 = \beta _{\perp} \left\{ J_n^2(z) - \frac{h\omega_0 n}{E\beta ||} \left[ J_n^2(z) \left( \frac{1 + 2 \beta || \cos \theta}{1 - \beta || \cos \theta} + \frac{z^2}{n^2} \right) - \frac{z}{2} \left( 1 - \frac{z^2}{n^2} \right) J_n(z) \right] \right\},
\]

where

\[
z = \frac{n\beta _{\perp} \sin \theta}{1 - \beta || \cos \theta}, \quad J_n(z) = \frac{d}{dz} J_n(z),
\]

and \( J_n(z) \) are the Bessel functions [30]. The expression (5.4) as \( h \to 0 \) coincides with the well-known formula for radiation power for the motion along the helix in a homogeneous magnetic field [21].

The integration over angles and the summation over all frequencies in the quantum term of expression (5.4) can be done only in the ultrarelativistic limit \( \beta \to 1, \beta ^2 = \beta ^2 || + \beta ^2 _{\perp} \). We change to the system of reference, which moves along the initial axis \( Oz \) with velocity \( \beta || [21] \), and we use the asymptotic representation of the Bessel function \( J_n(z) \) and its derivatives as the argument \( z \to \nu - 0 [21] \). The result for the total radiation power with account of the first quantum corrections is

\[
W = W_{cl}(f_\pi + f_\sigma) = W_{cl} \left( 1 - \frac{55\sqrt{3}}{16} \frac{h}{m_0cR} \left( \frac{E}{m_0c^2} \right)^2 (1 - \beta ^2 ||) \right),
\]

(5.6)

\[
W_{cl} = \frac{2 e_0^2 c}{3 R^2} \left( \frac{E}{m_0c^2} \right)^4,
\]

and the \( f_\pi, f_\sigma \) are \( \pi \)- and \( \sigma \)-components of polarization which are equal to:

\[
\begin{align*}
f_\pi &= \frac{1}{8} - \frac{5\sqrt{3}}{16} \frac{h}{m_0cR} \left( \frac{E}{m_0c^2} \right)^2 (1 - \beta ^2 ||), \\
f_\sigma &= \frac{7}{8} - \frac{50\sqrt{3}}{16} \frac{h}{m_0cR} \left( \frac{E}{m_0c^2} \right)^2 (1 - \beta ^2 ||).
\end{align*}
\]

### 6 The probability of transition with the spin flip

The probability of transition \( w(\zeta, \zeta') \) with the spin flip can be found by analogy with the expression (2.5):

\[
w(\zeta, \zeta') = \frac{e_0^2}{2\pi \hbar} \int_0^{2\pi} d\phi \int_0^\pi \frac{\sin \theta d\theta}{\cos \phi_0} \sum\limits_{n=|n_0|}^{\infty} \times \{ |\alpha_{\pi}^\dagger(n, \zeta)|^2 + |\alpha_{\sigma}^\dagger(n, \zeta)|^2 \},
\]

(6.1)

\[
\alpha_{\pi}^\dagger(n, \zeta) = (e_\pi, B^{\dagger}(n, \zeta)), \quad \alpha_{\sigma}^\dagger(n, \zeta) = (e_\sigma, B^{\dagger}(n, \zeta)),
\]
where \( \nu_0 \) is determined by the condition \( \kappa(\nu_0) = 0 \). In matrix elements \( B^{\pm\pm}(n, \zeta) \) it is convenient to change the integration variables \( dz = (c^2 p/E)dt \). The result is

\[
\begin{aligned}
\left\{ \begin{array}{c}
B^{++}_1(n, \zeta) \\
B^{++}_2(n, \zeta) \\
B^{++}_3(n, \zeta)
\end{array} \right. = \frac{\hbar \omega m_0^2}{2E^2 \beta^2} \frac{1}{\beta || T} \int_0^T e^{i\psi_2(t)} dt \left\{ \begin{array}{c}
i\beta_2 - \zeta \beta_1 \\
-i\beta_1 - \zeta \beta_2 \\
\zeta (\beta_\perp, \beta_\parallel)
\end{array} \right. ,
\end{aligned}
\] (6.2)

\[
\psi_2(t) = \omega F_0(t) + \zeta \rho_3(t), \quad F_0(t) = \frac{1}{c}[ct - (\vec{r}_c(t), \vec{n})],
\] (6.3)

\[
\rho_3(t) = \beta \int_0^t \frac{\beta'_2 \beta_1 - \beta'_1 \beta_2}{\beta_3 (\beta_1^2 + \beta_2^2)} dt, \quad \beta'_i = \frac{d\beta_i}{dt},
\]

\[
\beta^2 = (\beta_\perp, \beta_\parallel), \quad e\beta = \frac{d\vec{r}_c(t)}{dt}, \quad \bar{\beta}_\perp = (\beta_1, \beta_2, 0),
\]

\[
\omega = \frac{1}{\omega_0} \left( \omega_0 n - \zeta \rho_3(T) \right), \quad \omega_0 = \frac{2\pi}{T}.
\]

Now let us consider the model of a helical undulator (5.1), which is characterized by a helical electron trajectory:

\[
\vec{r}_c(t) = \frac{c}{\omega_0} \{ \beta_\perp \sin \omega_0 t, -\beta_\perp \cos \omega_0 t, \omega_0 \beta_\parallel t \}.
\] (6.4)

In this case

\[
\psi_2(t) = n\omega_0 t - \zeta \sin(\varphi - \omega_0 t), \quad \rho_3(t) = \frac{\beta}{\beta_\parallel} \omega_0 t;
\]

\[
\beta_\perp = \frac{e_0 H}{E_a} = \frac{e_0 H l}{2\pi E} = \frac{e_0 H l}{2\pi m_0 c^2 \gamma};
\]

\[
z = z_0 \left( n - \frac{\zeta \beta}{\beta_\parallel} \right), \quad z_0 = \frac{\beta_\perp \cos \theta}{\nu_0}
\]

and it is possible to integrate over the time [13]

\[
\alpha_\pi(n, \zeta) = \frac{\hbar \omega}{2E^2 \beta_\parallel} (1 - \beta^2)^{1/2} e^{-i\varphi} \left\{ \zeta \beta_\parallel \mathcal{J}_n(z) + \beta \frac{n}{z} \mathcal{J}_n(z) \right\},
\] (6.5)

\[
\alpha_\sigma(n, \zeta) = -\frac{\hbar \omega}{2E^2 \beta_\parallel} (1 - \beta^2)^{1/2} e^{-i\varphi} \left\{ \left( \beta \mathcal{J}_n(z) + \zeta \beta_\parallel \frac{n}{z} \mathcal{J}_n(z) \right) \cos \theta + \zeta \beta_\perp \mathcal{J}_n(z) \sin(\theta) \right\}
\]

Integration of the expressions over angles and their summation over the spectrum is possible in the case of the near-axis approximation, which is characterized by the maximal deflection angle of electron velocity from the undulator axis \( \mu = \beta_\perp/\beta_\parallel \) (4.1). In the case of helical undulator this means that the parameter \( z_0 = (\beta_\perp \sin \theta)/\nu_0^2 \) is very small. Keeping in (6.4), (6.5) only terms, which are linier over \( z_0 \ll 1 \), we obtain

\[
\beta \sim \beta_3 \sim \beta_\parallel, \quad \omega = \omega_0 (n - \zeta)/\psi_0;
\]

\[
\alpha_\pi(n, \zeta) = \frac{\hbar \omega}{2E^2 \beta_\parallel} (1 - \beta^2)^{1/2} e^{-i\varphi} \left\{ \frac{1}{2} \beta (1 - \zeta) \delta_{n,-1} - \frac{1}{2} \zeta \beta_\parallel z \delta_{n,0} + \frac{1}{2} \beta (1 + \zeta) \delta_{n,1} + \frac{1}{4} \beta (1 - \zeta) z \delta_{n,2} \right\};
\]

\[
\alpha_\sigma(n, \zeta) = -\frac{\hbar \omega}{4E^2 \beta_\parallel} (1 - \beta^2)^{1/2} e^{-i\varphi} \left\{ - (\beta (1 + \zeta) \cos \theta + \zeta \beta_\perp \sin \theta) \delta_{n,-1} - (z \beta \cos \theta - 2 \zeta \beta_\perp \sin \theta) \delta_{n,0} + (\beta (1 + \zeta) \cos \theta + \zeta \beta_\perp \sin \theta) \delta_{n,1} + \frac{z}{2} \beta (1 + \zeta) \cos \theta \delta_{n,2} \right\}.
\]
Here, we used the following asymptotic representation of Bessel functions [30]:
\[ J_n(z) \approx \frac{z^n}{\sqrt{2\pi n}} \quad \text{as} \quad z \to 0, \quad J_{-n}(z) = (-1)^n J_n(z). \]

After summation over the spectrum we have
\[ w(\zeta, -\zeta) = \frac{\hbar e^2 \omega_0^3}{16cE^2 \beta^4} \beta_1^2 \{ f_\pi + f_\sigma \}, \quad (6.7) \]
\[ f_\sigma = \int_0^\pi \frac{\beta^2 (1 - \beta^2)}{\psi_0^6} \sin \theta d\theta, \quad (6.8) \]
\[ f_\pi = \int_0^\pi \frac{(\psi_0 - \zeta)^2 (1 - \beta^2)}{\psi_0^6} \sin^3 \theta d\theta. \]

Finally, we have:
\[ w(\zeta, -\zeta) = \frac{\hbar e^2 \omega_0^3}{30E^2 \beta^4 c} (5 - 2\beta^2 + 3\beta^4) \left\{ 1 - \zeta \frac{5 - 5\beta^2}{5 - 2\beta^2 + 3\beta^4} \right\}, \quad (6.9) \]
\[ w(\zeta, -\zeta) = \frac{1}{2\tau} \{ 1 - \zeta \Gamma(\beta) \}, \quad (6.10) \]
\[ \tau = \frac{15E^2 \beta^4 (1 - \beta^2)^3 c}{\hbar \omega_0^3 \beta_1^2 (5 - 2\beta^2 + 3\beta^4)}. \]

The function
\[ \Gamma(\beta) = \frac{5 - 6\beta^2}{5 - 2\beta^2 + 3\beta^4} \]
is a monotonic function within the limits:
\[ 1 = \Gamma(0) \geq \Gamma(\beta) \geq \Gamma(1) = 0. \quad (6.11) \]

It follows from (6.6) that the probability of the electron radiation with spin flip is a function of the initial orientation of the electron spin. This fact causes the effect of radiational selfpolarization of spin in a bunch of electrons [21, 22]. This means that, under certain condition
\[ \frac{\beta_1}{\beta (1 - \beta^2)^{1/2}} \ll 1, \quad \beta_1 = \frac{e_0 H}{E}, \quad \beta^2 = 1 - \left(\frac{m_0 c^2}{E}\right)^2 \quad (6.12) \]
individually of the initial orientation of the spin of particles in a bunch, the primary spin orientation along the electrons motion will be determined. It follows from (6.7) that the polarization is absent, as \( \beta \to 1 \), but it is complete as \( \beta \to 0 \). However, the condition (6.9) does not allow us to consider these limiting cases.

Let us consider in (6.6) the relativistic particles with \( \beta \to 1 \). Then, we find the relaxation time \( \tau \) and asymptotic degree of polarization \( p \):
\[ \tau = \frac{5}{2} \frac{m^2 c^5}{\omega_0^3 \beta_1^2 e_0 \hbar} (1 - \beta^2)^2, \quad p = \frac{5}{6} (1 - \beta^2), \quad \beta^2 = 1 - \gamma^2. \quad (6.13) \]

These expressions within an accuracy up to \( O(\gamma^{-6}) \), \( \gamma \gg 1 \) coincide with expressions previously obtained in [7].

**Acknowledgments**

This work was partially supported by Russian Foundation for Basic Research (Grants 96-01-04578 and 97-02-162179). MMN was supported by International Science Foundation (Grant 539-p). AYuT, whose work is carried out within the research program of International Center for Fundamental Physics in Moscow, is a fellow of INTAS Grant 93-2492.
Appendix. The first quantum correction of the boson radiation in a helical undulator

After the summation over the spin, the spectral-angular distribution of the radiation intensity of the electron moving in field (5.1) with the first quantum correction has the form

\[
W = \frac{e^2}{c} \omega_0^2 \sum_1^\infty n^2 \int_0^\pi \frac{\sin \theta d\theta}{(1 - \beta_0 \cos \theta)} \left(1 - \frac{\hbar \omega_0}{E \beta_0^2} n(2 + \frac{z^2}{n^2})\right) (|\alpha_\pi(n)|^2 + |\alpha_\sigma(n)|^2),
\]

where

\[
\omega_0 = \frac{2\pi}{T}, \quad z = \frac{n\beta_\perp \sin \theta}{1 - \beta_0 \cos \theta},
\]

and

\[
|\alpha_\pi(n)|^2 = |B_1(n) \cos \theta - B_3(n) \sin \theta|^2, \quad |\alpha_\sigma(n)|^2 = |B_2(n)|^2;
\]

\[
\bar{B}(n) = \frac{1}{2\pi} \int_0^{2\pi} d\eta \exp \left\{ i \left( n\eta - z_1 \sin \theta - \frac{\hbar \omega_0 n}{E \beta_0^2 (1 - \beta_0 \cos \theta)} z_2 \sin 2\eta \right) \right\} \times
\]

\[
\times \beta \left\{ 1 - \frac{\hbar \omega_0}{2E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} (1 - \beta_\perp \sin \theta \cos \theta) \right\} ;
\]

\[
z_1 = z \left( 1 - \frac{\hbar \omega_0 n}{2E \beta_0^2} \left( 1 + \frac{1}{2} \frac{\xi^2}{n^2} \right) \right), \quad z_2 = \frac{1}{8} \beta_\perp \sin \theta.
\]

We will integrate (A.2) over \( \eta \) using the following relations:

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} d\eta = J_n(\xi);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \sin \eta d\eta = iJ_n(\xi);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \cos \eta d\eta = \frac{n}{\xi} J_n(\xi);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \sin^2 \eta d\eta = -i\hat{J}_n(\xi);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \cos^2 \eta d\eta = \frac{n^2}{\xi^2} J_n(\xi) - \frac{1}{\xi} \hat{J}_n(\xi);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \sin 2\eta d\eta = i\frac{2n}{\xi} \left( \hat{J}_n(\xi) - \frac{1}{\xi} J_n(\xi) \right);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \sin \eta \cos \eta d\eta = i\frac{n}{\xi} \left( \hat{J}_n(\xi) - \frac{n}{\xi} J_n(\xi) \right);
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \sin 2\eta \sin \eta d\eta = 2i \left\{ -\frac{2n^2}{\xi^2} J_n(\xi) + \frac{n^2}{\xi^2} \hat{J}_n(\xi) - \frac{1}{\xi} \hat{J}_n(\xi) \right\};
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\eta - \xi \sin \eta)} \sin 2\eta \sin \eta d\eta = -2n \left\{ \frac{1}{\xi} \hat{J}_n(\xi) - \frac{2}{\xi^2} \hat{J}_n(\xi) + \frac{2}{\xi^2} J_n(\xi) \right\}.
\]
and
\[ \vec{\beta} = \{ \beta_\perp \cos \eta, -\beta_\perp \sin \eta, \beta_0 \} \).

As a result for the matrix elements we obtain
\[
B_1(n) = \beta_\perp \left\{ \frac{n}{z_1} J_n(z_1) \left(1 - \frac{\hbar \omega_0}{2E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} + \frac{\hbar \omega_0}{E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} \times \right. \right.
\]
\[
\times \left[ \frac{1}{2} \beta_\perp \sin \theta \left( \frac{n^2}{z^2} J_n(z_1) - \frac{1}{z} \dot{J}_n(z_1) \right) + \right.
\]
\[
+ 2z_2 \left( \frac{-2n^2}{z^2} J_n(z_1) + \frac{n^2 + 1}{z^2} \dot{J}_n(z_1) - \frac{1}{z} \ddot{J}_n(z_1) \right) \left\} \right\};
\]
\[
B_2(n) = -i \beta_\perp \left\{ \left(1 - \frac{\hbar \omega_0}{2E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} \right) J_n(z_1) + \frac{\hbar \omega_0}{E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} \times \right.
\]
\[
\times \left[ \frac{n}{2z} \beta_\perp \sin \theta \left( \dot{J}_n(z_1) - \frac{1}{z} \ddot{J}_n(z_1) \right) + \right.
\]
\[
+ 2z_2 \left( \frac{-n \ddot{J}_n(z_1) - 2n^2}{z^2} \dot{J}_n(z_1) + \frac{2n}{z^2} J_n(z_1) \right) \left\} \right\};
\]
\[
B_3(n) = \beta_0 \left\{ \left(1 - \frac{\hbar \omega_0}{2E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} \right) J_n(z_1) + \frac{\hbar \omega_0}{E \beta_0^2} \frac{n}{1 - \beta_0 \cos \theta} \times \right.
\]
\[
\times \left[ \frac{n}{2z} \beta_\perp \sin \theta \frac{\dot{J}_n(z_1)}{z^2} + 2z_2 \left( \frac{n \ddot{J}_n(z_1) - n^2}{z^2} J_n(z_1) \right) \right] \left\} \right\}.
\]

Substituting (A.4) to (A.1) we obtain (5.4).

References

[1] D.F. Alferov, Yu.A. Bashmakov, and E.G. Bessonov, Trudy FIAN SSSR, 80 (1975) 100; English transl. in Proc. of Lebedev Phis. Inst., Novo Science Publ. 80 (1975).
[2] I.M. Ternov, V.R. Khalilov, V.G. Bagrov, and M.M. Nikitin, Izv. Vyssh. Uchebn. Zav. Fizika 24:2 (1990) 5-31; English transl. in: J. Sov. Phys. 24:2 (1980).
[3] V.N. Baier, V.M. Katkov, and V.M. Strakhovenko, Electromagnetic Processes in Oriented Monocrystals under High Energies, Nauka, Novosibirsk, 1989.
[4] M.M. Nikitin and V.Ya. Epp, Undulator Radiation, Energoatomizdat, Moscow, 1988.
[5] I.M. Ternov, V.V. Mikhailin, and V.R. Khalilov, Synchrotron Radiation and It’s Application, Harwood Publ., New-York, London, 1985.
[6] V.N. Baier, V.M. Katkov, and V.M. Strakhovenko, Zh. Eksp. Teor. Phys., 63 (1973) 2121-2130; Engl. transl. in: Sov. Phys.-JETP 36 (1973); Zh. Eksp. Teor Fiz. 80 (1981) 1348-1360; Engl. transl. in: Sov. Phys.- JETP 53 (1982).
[7] Yu.G. Pavlenko and A.Kh. Mussa, Vest. MGU, Ser. Fiz. and Astr. 2 (1977) 57-60.
[8] B.V. Kholomai, Teor. Mat. Fiz. 51:2 (1982) 211; Engl. transl. in: Theor. Math. Phys. (USA) 51:2 (1982).
[9] V.V. Belov, B.V. Kholomai, and A.D. Zhukov-Khovanskii, Izv. Vyssh. Uchebn. Zav. Fizika 29:4 (1985) 50-53; English transl. in J. Sov. Phys. 29:4 (1985).
[10] V.G. Bagrov, V.V. Belov, I.M. Ternov, et al., Dep. VINITI 01.08.84 No. 5622-84 Dep.
[11] V.G. Bagrov and V.V. Belov, Teor. Mat. Fiz. 70:3 (1987) 469-476; English transl.: in Teor. Math. Phys. (USA) 70:3 (1987).
[12] V.V. Belov, A.Yu. Trifonov, and S.N. Khozin, Izv. Vyssh. Uchebn. Zav. Fizika 33:9 (1989) 27-32, 33:10 (1989) 54-58; English transl. in: J. Sov. Phys. 33:9 (1989), 33:10 (1989).
[13] V.G. Bagrov, V.V. Belov, I.M. Ternov, and A.Yu. Trifonov, Vestnik MGU, Ser. Fiz. and Astr. 30:4 (1989) 80-82.
[14] V.G. Bagrov, V.V. Belov, and A.Yu. Trifonov, J. Phys. A. Math. Gen. 26 (1993), 6431.
[15] V.V. Belov, D.V. Boltovskiy, and A.Yu. Trifonov, Int. J. Mod. Phis. B. 8 (1994) (to appear).
[16] V.V. Belov, A.Yu. Trifonov, and S.N. Khozin, Izv. Vyssh. Uchebn. Zav. Fizika 31:11 (1987) 118-119.
[17] J. Heading, An Introduction to Phase-Integral Methods, Methuen, London, 1962.
[18] V.V. Belov, A.Yu. Trifonov, and S.N. Khozin, Dep. VINITI, No. 7554B87 Dep.
[19] V.N. Baier, V.M. Katkov, and S.V. Fadin, Radiation of Relativistic Electrons. Atomizdat, Moscow, 1973.
[20] V.N. Baier and V.M. Katkov, Zh. Eksp. Teor. Fiz. 55:4 (1968) 1542-1554; English transl. in: Sov. Phys. - JETP 28 (1969).
[21] A.A. Sokolov and I.M. Ternov, Relativistic Electron, Nauka, Moscow, 1983.
[22] A.A. Sokolov and I.M. Ternov, Dokl. AN SSSR 153 (1963) 1052-1054; English Transl. in: Sov. Phys. Dokl.
[23] V.G. Bagrov, N.I. Fedosov, G.F. Kopytov, S.S. Oxsyzyan, and V.B. Tlyachev, Nuovo Cim. B, 103:5 (1989) 549-590.
[24] V.G. Bagrov, I.M. Ternov, and B.V. Kholomay, Radiat. Eff. Lett. 85:1 (1984) 7-11.
[25] V.G. Bagrov and D.M. Gitman, it Exact Solutions of Relativistic Wave Equations, Kluwer, Dordrecht, 1990.
[26] V.P. Maslov and M.V. Fedoriuk, Semiclassical Approximation in Quantum Mechanics, Reidel, Boston, 1981.
[27] R.E. Edvards, Fourier series. A Modern Introduction. Vol. 2, Springer-Verlag, Berlin, 1982.
[28] H. Motz, J. Appl. Phys. 22:5 (1951) 527-535.
[29] V.G. Bagrov, D.M. Gitman, A.A. Sokolov, et al. Zh. Tekh. Fiz. 45:9 (1975) 1948-1953.
[30] H. Bateman and A. Erdelyi, Higher Transcendental Functions, Vol. 2, McGraw-Hill, London, 1953.