Gauge Theories with Cayley-Klein \( SO(2; j) \)
and \( SO(3; j) \) Gauge Groups

N.A. Gromov
Department of Mathematics,
Komi Science Center UrD RAS
Kommunistitcheskaya st., 24, Syktyvkar, 167982, Russia
E-mail: gromov@dm.komisc.ru

Abstract

Gauge theories with the orthogonal Cayley-Klein gauge groups \( SO(2; j) \)
and \( SO(3; j) \) are regarded. For nilpotent values of the contraction parameters
\( j \) these groups are isomorphic to the non-semisimple Euclid, Newton, Galilei
groups and corresponding matter spaces are fiber spaces with degenerate metrics. It is shown that the contracted gauge field theories describe the same
set of fields and particle mass as \( SO(2) \), \( SO(3) \) gauge theories, if Lagrangians
in the base and in the fibers all are taken into account. Such theories based
on non-semisimple contracted group provide more simple field interactions
as compared with the initial ones.

1 Introduction

Gauge field theory was suggested by Yang and Mills [1] and is regarded now
as most powerfull method for unified description of fundamental interactions
in particle physics, where the compact semisimple Lie groups seem to play
the most fundamental roles. For example, in the standard Weinberg-Salam
model [2], [3] of electroweak theory, the gauge group is \( SU(2) \times U(1) \).

It was realized by Nappi and Witten [4] that one can also construct gauge
theories for some non-semisimple groups (which admit a nondegenerate in-
variant bilinear form ) and such theories have much simpler structure than
the standard theories with semisimple groups. Later the gauge theories, \( \sigma \-
models and solitonic hierarchies for different non-semisimple groups was in-
vestigated [5–8]. Contraction of the standard electroweak Weinberg-Salam
model to the gauge group \( SU(2; \iota) \times U(1) \) was described in [9].

In this work we discuss gauge theories based on non-semisimple orthog-
onal Cayley-Klein groups which can be obtained from the classical simple
groups by contractions. The spaces of the fundamental representations of
such groups are fiber spaces with degenerate metrics.
2 Gauge theory for $SO(2)$ group

Let $R_4$ is Minkowski space-time: $x_\mu x_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$, $\mu = 0, 1, 2, 3$ and $\Phi_2$ is the target space, i.e. the space of fundamental representation of $SO(2)$ group, that elements named matter fields depend on $x \in R_4$. Gauge transformations: $\phi'(x) = \omega(\alpha(x))\phi(x)$, $\omega(\alpha(x)) \in SO(2)$ or

$$
\begin{pmatrix}
\phi'^1_1(x) \\
\phi'^2_2(x)
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha(x) & \sin \alpha(x) \\
-\sin \alpha(x) & \cos \alpha(x)
\end{pmatrix}
\begin{pmatrix}
\phi_1(x) \\
\phi_2(x)
\end{pmatrix}
$$

leave invariant the form $\phi^t \phi = \phi_1^2 + \phi_2^2$ and define Euclid metrics in $\Phi_2$.

The Lagrangian is written as $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^t D_\mu \phi + \frac{\mu^2}{2} \phi^t \phi - \frac{\lambda}{4} (\phi^t \phi)^2$, (2)

where covariant derivatives are

$$
D_\mu \phi_1 = \partial_\mu \phi_1 + e A_\mu \phi_2, \quad D_\mu \phi_2 = \partial_\mu \phi_2 - e A_\mu \phi_1.
$$

Here $e$ is the coupling constant, $A_\mu(x)$ is the gauge field and the field tensor is defined in the standard way $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Higgs mechanism is the method of generation mass for gauge fields. A Lagrangian ground state is such configuration of fields $A_\mu, \phi_1, \phi_2$, that minimize theirs energy. There are a set of ground states

$$
(\phi_1^\text{vac})^2 + (\phi_2^\text{vac})^2 = \phi_0^2, \quad A_\mu^\text{vac} = \partial_\mu \phi_0, \quad \phi_0 = \frac{\mu}{\sqrt{\lambda}},
$$

which can be obtained by gauge transformations from one of them:

$$
A_\mu^\text{vac} = 0, \quad \phi^\text{vac} = \begin{pmatrix}
\phi_0 \\
0
\end{pmatrix}, \quad \phi_0 = \frac{\mu}{\sqrt{\lambda}},
$$

as it is shown on Fig. 1.

![Figure 1: Lagrangian ground states for $SO(2)$ gauge theory.](image-url)
For small (linear) field excitations with respect of vacuum $A_\mu(x)$, $\phi_1(x) = \phi_0 + \chi(x)$, $\phi_2(x)$ Lagrangian (2) can be written as

$$L = L^{(2)} + L^{(3)} + L^{(4)},$$

where quadratic in fields $A_\mu, \chi, \phi_2$ Lagrangian

$$L^{(2)} = -\frac{1}{4} B_{\mu\nu} B_{\mu\nu} + \frac{e^2}{2} \phi_0^2 B_{\mu \nu} B_{\mu \nu} + \frac{1}{2} (\partial_\mu \chi)^2 - \mu^2 \chi^2,$$

$$B_\mu = A_\mu - \frac{1}{e \phi_0} \partial_\mu \phi_2, \quad F_{\mu\nu} = B_{\mu\nu}$$

describe massive vector field $B_\mu$, $m_\gamma = e \phi_0 = \frac{e \mu}{\lambda}$ — gauge field and massive scalar field $\chi$, $m_\chi = \sqrt{2} \mu$ — matter field (Higgs boson). Field interactions are given by $L^{(3)}$ and $L^{(4)}$

$$L^{(3)} = e A_\mu (\phi_2 \partial_\mu \chi - \chi \partial_\mu \phi_2) + \phi_0 \chi \left[ e^2 A_{\mu}^2 - \lambda \left( \chi^2 + \phi_2^2 \right) \right],$$

$$L^{(4)} = \frac{1}{2} \left( \chi^2 + \phi_2^2 \right) \left[ e^2 A_\mu^2 - \frac{\lambda}{2} \left( \chi^2 + \phi_2^2 \right) \right],$$

which include terms of third and fourth order in fields.

### 3 Gauge theory for Galilei group.

#### 3.1 Galilei group and Galilei geometry

Galilei space $\Phi_2(\iota)$ and Galilei group $G_2 = SO(2; \iota)$ can be obtained from $\Phi_2$ and $SO(2)$ by substitution: $\phi_2 \rightarrow j\phi_2$, $\alpha \rightarrow j\alpha$, where contraction parameter takes two values $j = 1, \iota$, $\iota^2 = 0, \iota/\iota = 1$. Gauge transformations

$$\begin{pmatrix} \phi'_1(x) \\ j\phi'_2(x) \end{pmatrix} = \begin{pmatrix} \cos j\alpha(x) & \sin j\alpha(x) \\ -\sin j\alpha(x) & \cos j\alpha(x) \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ j\phi_2(x) \end{pmatrix}$$

leave invariant the form $\phi'(j)\phi(j) = \phi_1^2 + j^2 \phi_2^2$, which for $j = 1$ define Euclid metrics in $\Phi_2$.

For $j = \iota$ Galilei (degenerate) metrics $\phi'(\iota)\phi(\iota) = \phi_1^2 + \iota^2 \phi_2^2$ in the 2-dim fiber space $\Phi_2(\iota)$ is obtained, where $\{\phi_1\}$ is 1-dim base and $\{\phi_2\}$ is 1-dim fiber. There are two invariants: $\text{inv}_1 = \phi_1^2$ under the general transformations $\phi'(\iota) = \omega(\iota \alpha)\phi(\iota)$, where

$$SO(2; \iota) \ni \omega(\iota \alpha) = \begin{pmatrix} 1 & \iota \alpha \\ -\iota \alpha & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \quad \omega'(\iota \alpha)\omega(\iota \alpha) = 1$$
and \( \text{inv}_2 = \phi_2^2 \) under transformations in the fiber \((\phi_1 = 0)\). Therefore there are two metrics: one in the base and another in the fiber.

A bundle of lines through a point on this two planes has different properties relative to the plane automorphism \([12]\). On Euclid plane, any two lines of the bundle are transformed to each other by rotation around the point (see Fig. 2).

![Figure 2: A bundle of lines on Euclid \( E_2 \) plane.](image2)

On Galilei plane, there is one isolated line that do not superposed with any other line of the bundle by Galilei boost (see Fig. 3).

![Figure 3: A bundle of lines on Galilei \( G_2 \) plane.](image3)

If one interpret these planes in some physical context, then on Euclid plane all lines must have the same physical dimension \([\phi_1] = [\phi_2]\). On Galilei plane, there are infinite many lines with physical dimension identical with dimension of the base \([\phi_1]\) and one isolated line in the fiber with some different physical dimension \([\phi_2] \neq [\phi_1]\) (see \([13]\) for details).

### 3.2 Gauge theory for Galilei group \( G_2 \).

Gauge theory for \( SO(2; \iota) = G_2 \) can be obtained from thouse for \( SO(2) \) by the substitution

\[
\phi_1 \rightarrow \phi_1, \quad \phi_2 \rightarrow j\phi_2, \quad A_\mu \rightarrow jA_\mu, \quad F_{\mu\nu} \rightarrow jF_{\mu\nu}, \quad \text{with} \quad j = \iota. \tag{11}
\]
Full Lagrangian is split on the Lagrangian in the base

\[ L_b = \frac{1}{2} (\partial_{\mu} \phi_1)^2 + \frac{\mu^2}{2} \phi_1^2 - \frac{\lambda}{4} \phi_1^4, \]  

(12)

the Lagrangian in the fiber \( \approx j^2 \)

\[ L_f = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_{\mu} \phi_2)^2 + \frac{\mu^2}{2} \phi_2^2 + \frac{1}{2} \phi_2^2 (e^2 A_\mu^2 - \lambda \phi_2^2) + e A_\mu (\phi_2 \partial_{\mu} \phi_1 - \phi_1 \partial_{\mu} \phi_2), \]  

(13)

and higher order part \( \approx j^4 \)

\[ L_h = \frac{1}{2} \phi_2^2 \left( e^2 A_\mu^2 - \frac{\lambda}{2} \phi_2^2 \right), \]  

(14)

which disappear for \( j = \iota \).

Higgs mechanism is realized in three steps:

(i) the Lagrangian in the base \( L_b \) is maximal and the Lagrangian in the fiber is equal to zero \( L_f = 0 \) at

\[ \phi_1 = \phi_0, \quad \phi_2 = 0, \quad A_{\mu} = \frac{1}{e} \partial_{\mu} \alpha, \quad F_{\mu\nu} = 0, \quad \lambda \phi_0^2 = \mu^2, \]  

(15)

where point \( M(\phi_0, 0) \in \Phi_2(\iota) \) is one of the ground states;

(ii) gauge transformations

\[ \phi'_1 = \phi_1, \quad \phi'_2 = \phi_2 + \alpha \phi_1, \quad A'_{\mu} = A_{\mu} + \frac{1}{e} \partial_{\mu} \alpha, \quad F'_{\mu\nu} = F_{\mu\nu}, \]  

(16)

applied to \( M \) define the set of ground states \( \{ \phi_1^2 = \phi_0^2, \quad \phi_2 \in \mathbb{R}, \quad A'_\mu = \frac{1}{e} \partial_{\mu} \alpha \} \) as sphere in \( \Phi_2(\iota) \) (see Fig. 4);

(iii) field excitations around ground state \( M \) are (see Fig. 5)

\[ \phi_1(x) = \phi_0 + \chi(x), \quad \phi_2(x), \quad A_{\mu}(x). \]  

(17)
As a result we obtain the Lagrangian in the base

\[ L_b = \frac{1}{2} (\partial_\mu \chi)^2 - \mu^2 \chi^2 + L_b^{(3)} + L_b^{(4)}, \]

\[ L_b^{(3)} = -λφ_0 \chi^3, \quad L_b^{(4)} = -\frac{λ}{4} \chi^4, \tag{18} \]

which describe massive scalar field \( \chi, m_\chi = \sqrt{2\mu} \) (Higgs boson), its self-action \( L_b^{(3)}, L_b^{(4)} \) and the Lagrangian in the fiber

\[ L_f = -\frac{1}{4} B_{\mu\nu}^2 + \frac{e^2 φ_0^2}{2} B_\mu^2 + L_f^{(3)} + L_f^{(4)}, \]

\[ L_f^{(3)} = eA_\mu (φ_2 \partial_\mu \chi - \chi \partial_\mu φ_2) + φ_0 \chi \left( e^2 A_\mu^2 - λφ_2^2 \right), \]

\[ L_f^{(4)} = \frac{1}{2} \chi^2 \left( e^2 A_\mu^2 - λφ_2^2 \right), \quad B_\mu = A_\mu - \frac{1}{eφ_0} \partial_\mu φ_0, \tag{19} \]

which describe massive vector gauge field \( B_\mu, m_V = eφ_0 = \frac{eμ}{\sqrt{λ}} \) and field interactions.

So in the theory with Galilei gauge group \( G_2 \) matter field (Higgs boson) \( \chi \) in the base and gauge field \( B_\mu \) in the fiber have different physical dimensions. Nevertheless, the mass dimension of Higgs boson \( m_\chi = \sqrt{2\mu} \) and vector boson \( m_V = eφ_0 = \frac{eμ}{\sqrt{λ}} \) are identical and are the same as for \( SO(2) \) gauge theory. More simple field interactions are provided by Galilei gauge theory as compared to \( SO(2) \) one.

### 4 Gauge theories for \( SO(3; j) \) groups

#### 4.1 Unified gauge model

The orthogonal Cayley-Klein group \( SO(3; j) \), \( j = (j_1, j_2) \) is defined [14] as the transformation group of the real space \( \Phi_3(j) \), which leave invariant the quadratic form

\[ φ^t(j)φ(j) = φ_1^2 + j_1 φ_2^2 + j_1 j_2 φ_3^2. \tag{20} \]
Stress tensor is given by
\[ T_1 = \begin{pmatrix} 0 & -j_1 & 0 \\ j_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -j_1j_2 \\ 0 & 0 & 0 \\ j_1j_2 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -j_2 \\ 0 & j_2 & 0 \end{pmatrix} \]
are subject of the commutation relations
\[ [T_1, T_2] = j_1^2 T_3, \quad [T_2, T_3] = j_2^2 T_1, \quad [T_3, T_1] = T_2. \] (21)

Gauge fields depend on \( x \in \mathbb{R}_4 \) and their values are in the algebra \( so(3; j) \)
\[ A_\mu(x) = g T^a A^a_\mu(x) = g \begin{pmatrix} 0 & -j_1 A^1_\mu & -j_1 j_2 A^2_\mu \\ j_1 A^1_\mu & 0 & -j_2 A^3_\mu \\ j_1 j_2 A^2_\mu & j_2 A^3_\mu & 0 \end{pmatrix}. \] (23)

Stress tensor is given by
\[ F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] = g T^a F^a_{\mu\nu}(x) \] (24)
and its components are as follows
\[ F^1_{\mu\nu} = \partial_\mu A^1_\nu - \partial_\nu A^1_\mu + j_2^2 g (A^2_\mu A^3_\nu - A^3_\mu A^2_\nu), \]
\[ F^2_{\mu\nu} = \partial_\mu A^2_\nu - \partial_\nu A^2_\mu + g (A^3_\mu A^1_\nu - A^1_\mu A^3_\nu), \]
\[ F^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + j_2^2 g (A^1_\mu A^2_\nu - A^2_\mu A^1_\nu). \] (25)

Covariant derivative \( D_\mu \phi(j) = [\partial_\mu + g T(A_\mu)] \phi(j) \) or in the matrices form
\[
\begin{pmatrix} D_\mu \phi_1 \\ j_1 D_\mu \phi_2 \\ j_1 j_2 D_\mu \phi_3 \end{pmatrix} = \begin{pmatrix} \partial_\mu & -j_1 g A^1_\mu & -j_1 j_2 g A^2_\mu \\ j_1 g A^1_\mu & \partial_\mu & -j_2 g A^3_\mu \\ j_1 j_2 g A^2_\mu & j_2 g A^3_\mu & \partial_\mu \end{pmatrix} \begin{pmatrix} \phi_1 \\ j_1 \phi_2 \\ j_1 j_2 \phi_3 \end{pmatrix}
\] (26)
acts on the field components \( \phi(j) \) as
\[ D_\mu \phi_1 = \partial_\mu \phi_1 - g j_1^2 (A^1_\mu \phi_2 + j_2^2 A^2_\mu \phi_3), \quad D_\mu \phi_2 = \partial_\mu \phi_2 + g (A^1_\mu \phi_1 - j_2^2 A^3_\mu \phi_3), \quad D_\mu \phi_3 = \partial_\mu \phi_3 + g (A^2_\mu \phi_1 + A^3_\mu \phi_2). \] (27)

The complete Lagrangian \( L(j) = L_A(j) + L_\phi(j) \) of the model is defined as the sum of the gauge fields Lagrangian
\[ L_A(j) = \frac{1}{8 g^2} \text{Tr}(F_{\mu\nu}(j))^2 = -\frac{1}{4} [j_1^2 (F^1_{\mu\nu})^2 + j_1 j_2^2 (F^2_{\mu\nu})^2 + j_2^2 (F^3_{\mu\nu})^2] \] (28)
and the matter fields Lagrangian

\[ L_\phi(j) = \frac{1}{2} (D_\mu \phi(j))^t (D_\mu \phi(j)) - V(\phi; j), \]  

(29)

where potential is taken in the standard form

\[ V(\phi; j) = \left( \frac{\sqrt{\lambda}}{2} \phi^t(j) \phi(j) - \frac{\mu^2}{2 \sqrt{\lambda}} \right)^2. \]  

(30)

In an explicit form the gauge fields Lagrangian is given by

\[ L_A(j) = -j_1^2 (F_{\mu\nu}^1)^2 + j_2^2 (F_{\mu\nu}^2)^2 + j_3^2 (F_{\mu\nu}^3)^2 - j_1^2 j_2^2 [L_A^{(3)}(j) + L_A^{(4)}(j)]. \]  

(31)

The terms of the third and fourth order in fields are equal to

\[ L_A^{(3)}(j) = 2g [\mathcal{F}_{\mu\nu}^1 (A_\mu \phi_3 - A_3 \phi_\mu) + \mathcal{F}_{\mu\nu}^2 (A_\mu \phi_1 - A_1 \phi_\mu) + \mathcal{F}_{\mu\nu}^3 (A_\mu \phi_2 - A_2 \phi_\mu)], \]

\[ L_A^{(4)}(j) = g^2 [j_2^2 (A_\mu \phi_3 - A_3 \phi_\mu)^2 + (A_\mu \phi_1 - A_1 \phi_\mu)^2 + j_1^2 (A_\mu \phi_2 - A_2 \phi_\mu)^2], \]  

(32)

where \( \mathcal{F}_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \) is stress tensor in a flat space.

Ground states of \( L(j) = L_A(j) + L_\phi(j) \) are such fields configuration, that \( L_A(j) = 0 \) and potential \( (30) \) is minimal. The ground states are realized by the fields

\[ A_\mu(x; j) = \omega(x; j) \partial_\mu \omega^{-1}(x; j), \quad \omega(x; j) \in SO(3; j), \]

\[ \phi^t(j) \phi(j) = \phi_1^2 + j_1^2 \phi_2 + j_2^2 \phi_3 = \phi_0^2, \quad \phi_0 = \frac{\mu}{\sqrt{\lambda}} \]  

(33)

which for \( A_\mu(x; j) = 0 \) belong to the sphere in the matter field space of the radius \( \phi_0 \) and can be obtained by the gauge transformations

\[ \phi(x; j) = \omega(x; j) \phi^{\text{vac}}, \]

\[ A_\mu'(x; j) = \omega(x; j) A_\mu(x; j) \omega^{-1}(x; j) + \omega(x; j) \partial_\mu \omega^{-1}(x; j) \]  

(34)

applied to one of the ground states \( M \). For simplicity this ground state \( M \) is taken in the form

\[ A_\mu(x; j) = 0, \quad (\phi^{\text{vac}})^t = (\phi_0, 0, 0)^t. \]  

(35)

Next step in Higgs mechanism is to introduce small field excitations around the ground state \( M \)

\[ A_\mu^a(x), \quad \phi_1(x) = \phi_0 + \chi(x), \quad \phi_2(x), \quad \phi_3(x) \]  

(36)
and to write down the complete Lagrangian for the new fields
\[ \mathcal{L}(j) = \mathcal{L}^{(2)}(j) + \mathcal{L}^{(3)}(j) + \mathcal{L}^{(4)}(j). \]

The second order in fields Lagrangian
\[ \mathcal{L}^{(2)}(j) = \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2 + j_1^2 \left[ -\frac{1}{4}(B_{\mu\nu}^1)^2 + \frac{g^2 \phi_0^2}{2} (B_{\mu}^1)^2 \right] + j_2^2 \left[ -\frac{1}{2}(F_{\mu\nu}^3)^2 \right] + j_1^2 j_2^2 \left[ -\frac{1}{4}(B_{\mu\nu}^2)^2 + \frac{g^2 \phi_0^2}{2} (B_{\mu}^2)^2 \right], \]
where the new fields are introduced
\[ B_{\mu}^1 = A_{\mu}^1 + \frac{1}{g \phi_0} \partial_\mu \phi_2, \quad B_{\mu}^2 = A_{\mu}^2 + \frac{1}{g \phi_0} \partial_\mu \phi_3, \]
describe one massive scalar matter field \( \chi, m_\chi = \sqrt{2} \mu \) (Higgs boson), two massive vector fields \( B_{\mu}^k, k = 1, 2 \) with the identical masses \( m_V = g \phi_0 = \frac{g \mu}{\sqrt{\lambda}} \) and one massless field \( A_{\mu}^3 \). Field interactions are described by the terms of the third \( \mathcal{L}^{(3)}(j) \) and fourth \( \mathcal{L}^{(4)}(j) \) order in fields
\[ \mathcal{L}^{(3)}(j) = -\lambda \phi_0 \chi^3 + j_1^2 \left\{ -\lambda \phi_0 \chi (\phi_2^2 + j_2^2 \phi_3^2) + g \left[ A_{\mu}^1 (\chi \partial_\mu \phi_2 - j_2 \phi_3) \right] \right. + j_2^2 A_{\mu}^2 (\chi \partial_\mu \phi_3 - j_2 \phi_2) \right\} + g^2 \phi_0 \left[ A_{\mu}^1 \left( A_{\mu}^2 \chi - j_2^2 A_{\mu}^3 \phi_3 \right) + j_2^2 A_{\mu}^2 \left( A_{\mu}^3 \chi + A_{\mu}^3 \phi_2 \right) \right] - j_1 j_2 g \frac{g}{2} \left[ F_{\mu\nu}^1 \left( A_{\mu}^1 A_{\nu}^2 - A_{\mu}^2 A_{\nu}^1 \right) + F_{\mu\nu}^2 \left( A_{\mu}^3 A_{\nu}^1 - A_{\mu}^1 A_{\nu}^3 \right) + F_{\mu\nu}^3 \left( A_{\mu}^1 A_{\nu}^2 - A_{\mu}^2 A_{\nu}^1 \right) \right] \]
\[ \mathcal{L}^{(4)}(j) = -\lambda^4 \chi^4 + j_1^2 \left\{ -\lambda \chi^2 \left( \phi_2^2 + j_2 \phi_3^2 \right) - j_1^2 \lambda \left( \phi_2^2 + j_2 \phi_3^2 \right)^2 \right\} + g^2 \left[ j_1^2 \left( A_{\mu}^1 \phi_2 + j_2 A_{\mu}^2 \phi_3 \right)^2 + \left( A_{\mu}^2 \chi - j_2 A_{\mu}^3 \phi_3 \right)^2 + j_2^2 \left( A_{\mu}^3 \chi + A_{\mu}^3 \phi_2 \right)^2 \right] - j_2^2 g^2 \left[ j_2^2 \left( A_{\mu}^1 A_{\nu}^2 - A_{\mu}^2 A_{\nu}^1 \right)^2 + \left( A_{\mu}^3 A_{\nu}^1 - A_{\mu}^1 A_{\nu}^3 \right)^2 + j_1^2 \left( A_{\mu}^1 A_{\nu}^2 - A_{\mu}^2 A_{\nu}^1 \right)^2 \right]. \]

It is worth to note that gauge theories with Cayley-Klein gauge group \( SO(3; j) \) can be obtained from \( SO(3) \) gauge theory by the substitution
\[ \chi \rightarrow \chi, \quad \phi_2 \rightarrow j_1 \phi_2, \quad \phi_3 \rightarrow j_1 j_2 \phi_3, \quad A_{\mu}^1 \rightarrow j_1 A_{\mu}^1, \quad A_{\mu}^2 \rightarrow j_1 j_2 A_{\mu}^2, \quad A_{\mu}^3 \rightarrow j_2 A_{\mu}^3, \]
or
\[ B_{\mu}^1 \rightarrow j_1 B_{\mu}^1, \quad B_{\mu}^2 \rightarrow j_1 j_2 B_{\mu}^2, \quad F_{\mu\nu}^3 \rightarrow j_2 F_{\mu\nu}^3, \]
for the new fields.
4.2 Gauge model for Euclid group $E_3$

For nilpotent value of the first parameter $j_1 = \iota_1$ rotation group $SO(3)$ is contracted to the non-semisimple group $SO(3; \iota_1, j_2)$, which is isomorphic to the Euclid group $E_3$. The matter space metrics is degenerated $\phi'(\iota_1)\phi(\iota_1) = \phi_1^2 + j_2^2(\phi_2^2 + j_2^2\phi_3^2)$ and $\Phi_3(\iota_1)$ becomes the fiber space with one dimensional base $\{\phi_1\}$ and two dimensional fiber $\{\phi_2, \phi_3\}$. Accordingly two Lagrangians are appeared: first in the base

$$\mathcal{L}_b(\iota_1) = \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2 - \lambda \phi_0 \chi^3 - \frac{\lambda}{4} \chi^4 - j_2^2 \frac{1}{2}(\mathcal{F}_{\mu\nu}^3)^2, \tag{43}$$

which describe Higgs boson $\chi$ with the standard mass $m_\chi = \sqrt{2}\mu$, its self-action and the massless field $A_\mu^3$; second Lagrangian ($\approx j_1^4$) in the fiber

$$\mathcal{L}_f(\iota_1) = -\frac{1}{4}(B_{\mu\nu}^1)^2 + \frac{g^2 \phi_0^2}{2}(B_{\mu\nu}^1)^2 + j_2^2 \left[ -\frac{1}{4}(B_{\mu\nu}^2)^2 + \frac{g^2 \phi_0^2}{2}(B_{\mu\nu}^2)^2 \right] + \mathcal{L}_f^{(3)}(\iota_1) + \mathcal{L}_f^{(4)}(\iota_1), \tag{44}$$

which describe two massive fields $B_{\mu}^1, B_{\mu}^2$ with the identical masses $m_\nu = g\phi_0$. Field interactions in the fiber are given by the terms ($\approx j_1^4$) of the third $\mathcal{L}_f^{(3)}(\iota_1)$ and forth $\mathcal{L}_f^{(4)}(\iota_1)$ order in fields, where

$$\mathcal{L}_f^{(4)}(\iota_1) = \frac{1}{2} \left\{ -\lambda \chi^2 \left( \phi_2^2 + j_2^2 \phi_3^2 \right) + g^2 \left[ \left( A_\mu^1 \chi - j_2^2 A_\mu^3 \phi_3 \right)^2 + j_2^2 \left( A_\mu^2 \chi + A_\mu^3 \phi_2 \right)^2 \right] - j_2^2 g^2 \left[ j_2^2 \left( A_\mu^2 A_\nu^3 - A_\mu^3 A_\nu^2 \right)^2 + \left( A_\mu^3 A_\mu^1 - A_\mu^1 A_\mu^3 \right)^2 \right] \right\}, \tag{45}$$

and $\mathcal{L}_f^{(3)}(\iota_1)$ is given by (41) without the term $-\lambda \phi_0 \chi^3$ and total multiplier $j_1^4$. The proportional to $j_1^4$ forth order terms are equal to zero, i.e. field interactions are more simple as compared with $SO(3)$ model, whereas the fields themselves and their masses are the same as for $SO(3)$ gauge model.

4.3 Gauge model for Newton group $N_3$

For nilpotent value of the second parameter $j_2 = \iota_2$ rotation group $SO(3)$ is contracted to the non-semisimple group $SO(3; \iota_1, j_2)$, which is isomorphic to the Newton group $N_3$. The matter space metrics is degenerated $\phi'(\iota_2)\phi(\iota_2) = \phi_1^2 + j_2^2\phi_2^2 + j_2^2j_1^2\phi_3^2$ and $\Phi_3(\iota_2)$ becomes the fiber space with two dimensional base $\{\phi_1, \phi_2\}$ and one dimensional fiber $\{\phi_3\}$. Lagrangian in the base

$$\mathcal{L}_b(\iota_2) = \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2 - \frac{1}{4}(B_{\mu\nu}^1)^2 + \frac{g^2 \phi_0^2}{2}(B_{\mu\nu}^1)^2 + \mathcal{L}_b^{(3)}(\iota_2) + \mathcal{L}_b^{(4)}(\iota_2) \tag{46}$$
describe Higgs boson $\chi$, massive boson $B^4_\mu$ and interactions in the form
\[
L_b^{(3)}(\tau_2) = -\lambda \phi_0 \chi \left( \chi^2 + j_1^2 \phi_2^2 \right) + j_1^2 \left[ g A^1_\mu (\chi \partial_\mu \phi_2 - \phi_2 \partial_\mu \chi) \right],
\]
\[
L_b^{(3)}(\tau_2) = -\frac{\lambda}{4} \chi^4 + j_1^2 \frac{1}{2} \left[ \left( \chi^2 + j_1^2 \phi_2^2 \right) \left( g^2 (A^1_\mu)^2 - \lambda \phi_2^2 \right) + j_1^2 \frac{\lambda}{2} \phi_2^4 \right].
\]
(47)

Lagrangian in the fiber takes the form
\[
L_f(\tau_2) = -\frac{1}{4} (\mathcal{F}^3_{\mu
u})^2 + j_1^2 \left[ -\frac{1}{4} (B^2_{\mu
u})^2 + \frac{g^2 \phi_0^2}{2} (B^2_\mu)^2 \right] + L_f^{(3)}(\tau_2) + L_f^{(4)}(\tau_2),
\]
(48)
where
\[
L_f^{(3)}(\tau_2) = j_1^2 \left\{ -\lambda \phi_0 \chi \phi_3^2 +
+ g \left[ A^2_\mu (\chi \partial_\mu \phi_3 - \phi_3 \partial_\mu \chi) + A^3_\mu (\phi_2 \partial_\mu \phi_3 - \phi_3 \partial_\mu \phi_2) \right] + g^2 \phi_0 A^2_\mu \left( A^2_\mu \chi + A^3_\mu \phi_2 \right) -
- \frac{g}{2} \left[ \mathcal{F}^1_{\mu
u} \left( A^2_\mu A^3_{\nu} - A^3_\mu A^2_{\nu} \right) + \mathcal{F}^2_{\mu
u} \left( A^1_\mu A^3_{\nu} - A^3_\mu A^1_{\nu} \right) + \mathcal{F}^3_{\mu
u} \left( A^1_\mu A^2_{\nu} - A^2_\mu A^1_{\nu} \right) \right] \right\},
\]
\[
L_f^{(4)}(\tau_2) = j_1^2 \frac{1}{2} \left\{ -\lambda \phi_0 \chi \phi_3^2 +
+ g^2 \left[ (A^2_\mu \chi + A^3_\mu \phi_2)^2 + 2j_1^2 A^1_\mu A^2_\mu \phi_2 \phi_3 - 2A^1_\mu A^3_\mu \chi \phi_3 \right] -
- g^2 \left[ (A^3_\mu A^1_{\nu} - A^1_\mu A^3_{\nu})^2 + j_1^2 \left( A^1_\mu A^2_{\nu} - A^2_\mu A^1_{\nu} \right)^2 \right] \right\}.
\]
(49)

So Lagrangian in the fiber describe the massless field $A^3_\mu$, massive boson $B^2_\mu$ and field interactions.

### 4.4 Gauge model for Galilei group $G_3$

For $j_1 = \tau_1$, $j_2 = \tau_2$ the group $SO(3; \tau)$ is isomorphic to Galilei group $G_3$. The space $\Phi_3(\tau_1)$ becomes twice fiber space, since two dimensional fiber $\{\phi_2, \phi_3\}$ in its part is the fiber space with the base $\{\phi_2\}$ and the fiber $\{\phi_3\}$. There are three invariants: $\text{inv}_1 = \phi_1^2$, $\text{inv}_2 = \phi_2^2$, $\text{inv}_3 = \phi_3^2$. Therefore we have Lagrangian in the base
\[
L_b(\tau) = \frac{1}{2} (\partial_\mu \chi)^2 - \mu^2 \chi^2 - \lambda \phi_0 \chi^3 - \frac{\lambda}{4} \chi^4,
\]
(50)

Lagrangian in the first fiber ($\approx j_1^2$)
\[
L_{f_1}(\tau) = -\frac{1}{4} (B^1_{\mu\nu})^2 + \frac{g^2 \phi_0^2}{2} (B^1_\mu)^2 + L_{f_1}^{(3)}(\tau) + L_{f_1}^{(4)}(\tau),
\]
(51)
where
\[
L^{(3)}(\iota) = \phi_0 \chi \left[ g^2 \left( A^1_{\mu} \right)^2 - \lambda \phi_2^3 \right] + g A^1_{\mu} (\chi \partial_\mu \phi_2 - \phi_2 \partial_\mu),
\]
\[
L^{(4)}(\iota) = \chi^2 \left[ g^2 \left( A^1_{\mu} \right)^2 - \frac{\lambda}{2} \phi_2^2 \right],
\] (52)

Lagrangian in the second fiber \((\approx j_1^2 j_2^2)\)
\[
L_{f_2}(\iota) = \frac{-1}{4} (B^2_{\mu\nu})^2 + \frac{g^2 \phi_2^3}{2} (B^2_{\mu})^2 + L^{(3)}_{f_2}(\iota) + L^{(4)}_{f_2}(\iota),
\] (53)

where
\[
L^{(3)}_{f_2}(\iota) = g^2 \phi_0 A^2_{\mu} (A^2_{\mu} \chi + A^3_{\mu} \phi_2) +
+ g \left[ A^2_{\mu} (\chi \partial_\mu \phi_3 - \phi_3 \partial_\mu \chi) + A^3_{\mu} (\phi_2 \partial_\mu \phi_3 - \phi_3 \partial_\mu \phi_2) \right],
\]
\[
L^{(4)}_{f_2}(\iota) = \frac{1}{2} \left\{ -\lambda \chi^2 \phi_3^2 + g^2 \left( A^2_{\mu} \chi + A^3_{\mu} \phi_2 \right)^2 - g^2 \left( A^3_{\mu} A^4_{\nu} - A^1_{\mu} A^3_{\nu} \right)^2 \right\}. \] (54)

For the two parameter contraction, unlike one parameter ones, the fibered in the gauge field space do not coincide with the fibering in the matter field space, therefore in gauge field Lagrangian \((28)\) there is the term \((\approx j_1^2),\) which it is necessary to regard as one more Lagrangian
\[
L_{g}(\iota) = -\frac{1}{4} (\mathcal{F}_{\mu\nu})^2. \] (55)

Thus, for completely contracted Galilei group \(G_3\) each field is described by own Lagrangian: Higgs boson — by Lagrangian in the base \(L_{b}(\iota),\) massive bosons — by Lagrangian in the first \(L_{f_1}(\iota)\) and in the second \(L_{f_2}(\iota)\) fibers of the matter field space \(\Phi_3(\iota),\) massless field \(A_{\mu}^a — by\) Lagrangian \(L_{g}(\iota)\) in the fiber of the gauge field space, which do not coincides with the fibers of \(\Phi_3(\iota).\)

Nevertheless one can speak about unified description of all fields because all fields are generated by one gauge group.

5 Conclusion

The contracted Cayley-Klein group is the motion group of its fundamental representation space, which has degenerate metrics and some set of invariants with respect of this group \([14]\). This means that in gauge theories with contracted gauge groups the matter field spaces and the gauge field spaces are the fiber spaces. For the complete description of a physical system in such space it is necessary to regard the complete set of Lagrangians: in the base and in the all fibers \([13]\). Only in this case the gauge field theory with
degenerate metrics in target (matter) field space describe the same set of fields and particle mass as non-contracted one.

Although in the case of degenerate metrics there are several Lagrangians nevertheless one can speak about unified description of all fields because all fields are generated by the one contracted non-semisimple gauge group. In other words, unified description of gauge fields means one gauge group rather than one Lagrangian.

Since it is the structure constants that determine the interactions and since under contractions of Lie group some structure constants of its algebra turn to zero, the gauge field theory based on non-semisimple contracted group provide more simple field interactions as compared with initial one.

This work was partially supported by Russian Foundation for Basic Research under grant 07-01-00374.

References

[1] C.N. Yang, R.L. Mills. Phys. Rev., 1954, v. 96, 191.
[2] S. Weinberg. Phys. Rev. Lett., 1967, v. 19, 1264.
[3] A. Salam. *Elementary Particle Theory*. Ed. N. Svartholm, Almquist Forlag AB, 1968. A.
[4] C.R. Nappi, E. Witten. A WZW model based on non-semi-simple group, [hep-th/9310112](http://arxiv.org/abs/hep-th/9310112).
[5] A.A. Tseytlin. On gauge theories for non-semisimple groups, [hep-th/9505129](http://arxiv.org/abs/hep-th/9505129).
[6] J. Nuyts, T.T. Wu. Yang-Mills theory for non-semisimple groups, [hep-th/0210214](http://arxiv.org/abs/hep-th/0210214).
[7] L. Andrianopoli, S. Ferrara, M.A. Lledo, O. Macia. Integration of massive states as contractions of non-linear $\sigma$-models, [hep-th/0503196](http://arxiv.org/abs/hep-th/0503196).
[8] S.I. Vacaru. Curve flows and solitonic hierarchies generated by (semi) Riemannian metrics, [math-ph/0608024](http://arxiv.org/abs/math-ph/0608024).
[9] N.A. Gromov. Gauge theories for target spaces with degenerate metrics, [hep-th/0611079](http://arxiv.org/abs/hep-th/0611079).
[10] V.A. Rubakov. *Classical gauge fields*. Editorial URSS, Moscow, 1999 (in Russian).
[11] P.W. Higgs. Phys. Lett., 1964, v. 12, 321.

[12] R.I. Pimenov. Unified axiomatics of spaces with maximal motion group. Litovskij Matem. Sbornik, 1965, v. 5, N. 3, 457–486 (in Russian).

[13] N.A. Gromov. Linear harmonic oscillator in spaces with degenerate metrics, math-ph/0603004.

[14] N.A. Gromov. Contraction and analytical continuations of classical groups. Unified approach. Komi Science Center, Syktyvkar, 1990 (in Russian).