Connectivity of direct products by $K_2$

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Abstract

Let $\kappa(G)$ be the connectivity of $G$. The Kronecker product $G_1 \times G_2$ of graphs $G_1$ and $G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. In this paper, we prove that $\kappa(G \times K_2) = \min\{2\kappa(G), \min\{|X| + 2|Y|\}\}$, where the second minimum is taken over all disjoint sets $X, Y \subseteq V(G)$ satisfying (1) $G - (X \cup Y)$ has a bipartite component $C$, and (2) $G[V(C) \cup \{x\}]$ is also bipartite for each $x \in X$.

Keywords: connectivity, Kronecker product, separating set

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1. Introduction

Throughout this paper, a graph $G$ always means a finite undirected graph without loops or multiple edges. It is well known that a network is often modeled as a graph and the classical measure of the reliability is the connectivity and the edge connectivity.

The connectivity of a simple graph $G = (V(G), E(G))$ is the number, denoted by $\kappa(G)$, equal to the fewest number of vertices whose removal from $G$ results in a disconnected or trivial graph. A set $S \subseteq V(G)$ is a separating set of $G$, if either $G - S$ is disconnected or has only one vertex. Let $G_1$ and $G_2$ be two graphs, the Kronecker product $G_1 \times G_2$ is the graph defined on the Cartesian product of vertex sets of the factors, with two vertices $(u_1, v_1)$ and $(u_2, v_2)$ adjacent if and only if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$. This product is one of the four standard graph products [1] and is known under many

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different names, for instance as the direct product, the cross product and conjunction. Besides the famous Hedetniemi’s conjecture on chromatic number of Kronecker product of two graphs, many different properties of Kronecker product have been investigated, and this product has several applications.

Recently, Brešar and Špacapan [2] obtained an upper bound and a low bound on the edge connectivity of Kronecker products with some exceptions; they also obtained several upper bounds on the vertex connectivity of the Kronecker product of graphs. Mamut and Vumar [3] determined the connectivity of Kronecker product of two complete graphs. Guji and Vumar [4] obtained the connectivity of Kronecker product of a bipartite graph and a complete graph. These two results are generalized in [5], where the author proved a formula for the connectivity of Kronecker product of an arbitrary graph and a complete graph of order $\geq 3$, which was conjectured in [4]. We mention that a different proof of the same result can be found in [6]. For the left case $G \times K_2$, Yang [7] determined an explicit formula for its edge connectivity. Bottreau and Métivier [8] devised a criterion for the existence of a cut vertex of $G \times K_2$, see also [9]. In this paper, based on a similar argument of [6], we determine a formula for $\kappa(G \times K_2)$. For more study on the connectivity of Kronecker product graphs, we refer to [10, 11].

Let $X, Y \subseteq V(G)$ be two disjoint sets with $V(G) - (X \cup Y) \neq \emptyset$. We shall call $(X, Y)$ a $b$-pair of $G$ if it satisfies:

1. $G - (X \cup Y)$ has a bipartite component $C$, and
2. $G[V(C) \cup \{x\}]$ is also bipartite for each $x \in X$.

Denote $b(G) = \min\{|X| + 2|Y| : (X, Y) \text{ is a } b\text{-pair of } G\}$. Our main result is the following

**Theorem 1.1.** $\kappa(G \times K_2) = \min\{2\kappa(G), b(G)\}$.

We end this section by giving some useful properties of the new defined graph invariant $b(G)$. Let $v \in V(G)$, we use $N(v)$, $d(v)$ and $\delta(G)$ to denote the neighbor set of $v$, the degree of $v$, and the minimum degree of $G$, respectively.

**Lemma 1.1.** Let $m = |G| \geq 2$ and $u$ be any vertex of $G$. Then

1. $b(G) = 0$, if $G$ is bipartite.
2. $b(G) \leq \delta(G)$.
3. $b(G) \leq b(G - u) + 2$.

**Proof.** Part (1) is clear since $(\emptyset, \emptyset)$ is a $b$-pair of any bipartite graph by the definition of $b$-pairs. For each $v \in V(G)$, $(N(v), \emptyset)$ is a $b$-pair of $G$(Take
the isolated vertex \( v \) in \( G - N(v) \) as the bipartite component \( C \). Therefore \( b(G) \leq d(v) \) and part (2) is verified. Similarly, Let \( (X', Y') \) be any b-pair of \( G - u \). It is straightforward to show that \( (X', Y' \cup \{u\}) \) is a b-pair of \( G \). Therefore, \( b(G) \leq |X'| + 2|Y'| + 2 \) and part (3) is verified.

2. Proof of the main result

We first recall some basic results on the connectivity of Kronecker product of graphs [12], see also [1].

**Lemma 2.1.** The Kronecker product of two nontrivial graphs is connected if and only if both factors are connected and at least one factor is nonbipartite. In particular, \( G \times K_2 \) is connected if and only if \( G \) is a connected nonbipartite graph.

**Lemma 2.2.** Let \( G \) be a connected bipartite graph with bipartition \( (P, Q) \) and \( V(K_2) = \{a, b\} \). Then \( G \times K_2 \) has exactly two connected components isomorphic to \( G \), with bipartitions \( (P \times \{a\}, Q \times \{b\}) \) and \( (P \times \{b\}, Q \times \{a\}) \), respectively.

From Lemma 1.1(1) and Lemma 2.1, we only need to proof Theorem 1.1 for connected and nonbipartite graphs. For each \( u \in V(G) \), set \( S_u = \{u\} \times V(K_2) = \{(u, a), (u, b)\} \). Let \( S \subseteq V(G \times K_2) \) satisfy the following two assumptions:

**Assumption 1.** \( |S| < \min\{2\kappa(G), b(G)\} \), and

**Assumption 2.** \( S'_v := S_u - S \neq \emptyset \) for each \( u \in V(G) \).

Let \( G^* \) be the graph whose vertices are the classes \( S'_v \) for all \( u \in V(G) \) and in which two different vertices \( S'_u \) and \( S'_v \) are adjacent if \( G \times K_2 - S \) contains an \( (S'_u - S'_v) \) edge, that is, an edge with one end in \( S'_u \) and the other one in \( S'_v \).

Under the two assumptions on \( S \subseteq V(G \times K_2) \), the connectedness of \( G \times K_2 - S \) is verified by the following two lemmas.

**Lemma 2.3.** If \( G \) is a connected nonbipartite graph then \( G^* \) is connected.

Proof. Suppose to the contrary that \( G^* \) is disconnected. Then the vertices of \( G^* \) can be partitioned into two nonempty parts, \( U^* \) and \( V^* \), such that there are no \( (U^* - V^*) \) edges. Let \( U = \{u \in V(G) : S'_u \in U^*\}, V = \{v \in V(G) : S'_v \in V^*\} \) and \( Z \) be the collection of ends of all \( (U - V) \) edges. Let \( Z^* = \{S'_u : u \in V(G), |S'_u| = 1\} \). For any \( u \in Z \), there exists an edge
have vertex in $U$. Lemma 1.1(2), we have $G$ set of $S$ containing $S' \subseteq Z$. Therefore, by the definition of $b$-pair $s$, there is contrary to the fact that there are no $(U^* - V^*)$ edges. Therefore, $S'_u \in Z^*$ and we have $|Z| \leq |Z^*|$ by the arbitrariness of $u$ in $Z$.

**Case 1:** Either $U \subseteq Z$ or $V \subseteq Z$. We may assume $U \subseteq Z$. Let $u$ be any vertex in $U$, then $d(u) \leq |Z| - 1$, and hence $\delta(G) \leq |Z| - 1$. Therefore, by Lemma 1.1(2), we have $|S| = |Z^*| \geq |Z| > \delta(G) \geq b(G)$, a contradiction.

**Case 2:** $U \not\subseteq Z$ and $V \not\subseteq Z$. Either of $U \cap Z$ and $V \cap Z$ is a separating set of $G$. Therefore, $\kappa(G) \leq \min\{|U \cap Z|, |V \cap Z|\} \leq |Z|/2$. Similarly, we have $|S| = |Z^*| \geq |Z| \geq 2\kappa(G)$, again a contradiction.

**Lemma 2.4.** Any vertex $S'_w$ of $G^*$, as a subset of $V(G \times K_2 - S)$ is contained in the vertex set of some component of $G \times K_2 - S$.

**Proof.** If $|S'_w| = 1$, then the assertion holds trivially. Now assume $|S'_w| = 2$. Let $U = \{u \in V(G) : |S'_u| = 2\}$, $V = \{v \in V(G) : |S'_v| = 1\}$ be the partitions of $V(G)$ and $C$ the component of $G - V$ containing $w \in U$.

Since $|V| = |S| < b(G)$ by Assumption 1, it follows that $(V, \emptyset)$ is not a $b$-pair of $G$. Note $S'_w \subseteq V(C \times K_2)$. We may assume that the component $C$ containing $w$ is bipartite since otherwise $C \times K_2$ is connected by Lemma 2.1 and hence the result follows. Therefore, by the definition of $b$-pairs, there exists a vertex $v \in V$ such that $G[V(C) \cup \{v\}]$ is nonbipartite.

Let $(P, Q)$ be the bipartition of $C$ and $V(K_2) = \{a, b\}$. Then, by Lemma 2.2 $C \times K_2$ has exactly two components $C_1$ and $C_2$ isomorphic to $C$, with bipartition $(P \times \{a\}, Q \times \{b\})$ and $(P \times \{b\}, Q \times \{a\})$, respectively. The nonbipartiteness of $G[V(C) \cup \{v\}]$ implies that $v$ is a common neighbor of $P$ and $Q$. By symmetry, we may assume $S'_w = \{(v, a)\}$. It is easy to see that the subgraph induced by $V(C \times K_2) \cup S'_w$ is connected since $(v, a)$ is a common neighbor of $C_1$ and $C_2$.

**Proof of Theorem 1.1** We apply induction on $m = |V(G)|$. It trivially holds when $m = 1$. We therefore assume $m \geq 2$ and the result holds for all graphs of order $m - 1$.

Let $S_0$ be a minimum separating set of $G$ and $S = S_0 \times V(K_2) = \{(u, a), (u, b) : u \in S_0\}$. Then $G \times K_2 - S \cong (G - S_0) \times K_2$ is disconnected by Lemma 2.1. Therefore, $\kappa(G \times K_2) \leq 2\kappa(G)$.

Let $(X, Y)$ be a $b$-pair of $G$ with $|X| + 2|Y| = b(G)$. Let $C$ be a bipartite component of $G - (X \cup Y)$ with bipartition $(P, Q)$ such that $G[V(C) \cup \{x\}]$ is also bipartite for each $x \in X$. Let $C_1$ and $C_2$ be the two components of
$C \times K_2$ with bipartition $(P \times \{a\}, Q \times \{b\})$ and $(P \times \{b\}, Q \times \{a\})$, respectively. Define an injection $\varphi : X \to V(G \times K_2)$ as follows:

$$
\varphi(x) = \begin{cases} 
(x, b), & \text{if } x \text{ has a neighbor in } P, \\
(x, a), & \text{otherwise}.
\end{cases}
$$

Let $S' = \varphi(X)$ and $S'' = \{(u, a), (u, b) : u \in Y\}$. Then $S' \cup S''$ is a separating set since $C_1$ is a component of $G \times K_2 - (S' \cup S'')$, which implies $\kappa(G \times K_2) \leq |S' \cup S''| = |X| + 2|Y| = b(G)$.

To show the other equality, we may assume $G$ is a connected nonbipartite graph. Let $S \subseteq V(G \times K_2)$ satisfy Assumption 1, i.e., $|S| < \min\{2\kappa(G), b(G)\}$.

**case 1:** $S$ satisfies Assumption 2. It follows by Lemma 2.3 and Lemma 2.4 that $G \times K_2 - S$ is connected.

**case 2:** $S$ does not satisfy Assumption 2, i.e., there exists a vertex $u \in V(G)$ such that $S_u = \{(u, a), (u, b)\} \subseteq S$. Therefore,

$$
|S - S_u| = |S| - 2
< \min\{2\kappa(G), b(G)\} - 2
= \min\{2(\kappa(G) - 1), b(G) - 2\}
\leq \min\{2\kappa(G - u), b(G - u)\},
$$

where the last inequality above follows from Lemma 1.1(3).

By the induction assumption,

$$
\kappa((G - u) \times K_2) = \min\{2\kappa(G - u), b(G - u)\}.
$$

Hence, $(G - u) \times K_2 - (S - S_u)$ is connected. It follows by isomorphism that $G \times K_2 - S$ is connected.

Either of the two cases implies that $(G \times K_2 - S)$ is connected. Thus, $\kappa(G \times K_2) \geq \min\{2\kappa(G), b(G)\}$.

The proof of the theorem is complete by induction. \qed

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