Spectral function and quasi-particle damping of interacting bosons in two dimensions

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We employ the functional renormalization group to study dynamical properties of the two-dimensional Bose gas. Our approach is free of infrared divergences, which plague the usual diagrammatic approaches, and is consistent with the exact Nepomnyashchy identity, which states that the anomalous self-energy vanishes at zero frequency and momentum. We recover the correct infrared behavior of the propagators and present explicit results for the spectral line-shape, from which we extract the quasi-particle dispersion and damping.

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The excitation spectrum of the weakly interacting Bose gas has been studied with field theoretical methods for more than half a century \cite{Singer} and is qualitatively well understood: at high energies the excitations resemble those of free bosons whereas at low energies the excitations are collective Goldstone modes which result from the symmetry broken ground state \cite{review}. In $D = 3$, this picture has recently been confirmed experimentally using the Bragg spectroscopy technique on cold atoms \cite{Bragg}. However, while perturbative approaches (based on a renormalized $T$-matrix) to the weakly interacting Bose gas have been successfully applied to study the excitation spectra and the damping of quasi-particles \cite{pert}, the self-energies in these approaches violate the exact Nepomnyashchy identity \cite{Nepomnyashchy}, which states that the anomalous self-energy at zero frequency and momentum vanishes. Moreover, the perturbative approach is plagued by infrared (IR) divergences \cite{IR}, which makes a numerical evaluation non-trivial already at second order in the effective coupling constant. Modern renormalization group (RG) methods \cite{RG} have resolved this problem and recovered the correct low-energy structure of the self-energies which are in accordance with the Goldstone character of the excitations and the Nepomnyashchy identity. The calculations in Refs. \cite{RG} were however limited to the asymptotic regime near zero energy and are thus unable to describe both the damping of quasi-particles and the crossover from the low energy Goldstone-like dispersion to the free particle dispersion at higher energies.

There is a renewed interest in the physics of interacting bosons since the transition from weak to strong interaction can now be studied experimentally via the Feshbach resonance technique \cite{resonance}. The excitations of the weakly interacting gas are qualitatively different from those of strongly interacting liquids such as $^4$He and the connection between the two remains elusive. A consistent strong coupling approach, which would allow a description of the strongly interacting gas starting from a microscopic model is at present not available. In two dimensions, the need to go beyond mean field theory is more urgent, since the $s$-wave scattering length vanishes and the usual expansion parameter, the gas parameter, is replaced by a new emergent parameter \cite{Kopietz}. In this Letter we show how the functional RG (FRG) can be employed to calculate the single-particle spectral density of the interacting Bose gas in $D$ dimensions and present numerical results for $D = 2$. Our approach can be extended to include arbitrarily strong local correlations and might offer a possible route to the physics of strongly interacting bosons.

We consider bosons with mass $m$, chemical potential $\mu$ and a repulsive contact interaction $u_0$ at zero temperature. The bare Euclidean action is

$$S(\bar{\psi}, \psi) = \int d^D x d\tau \left[ \bar{\psi}(\partial_\tau - \nabla^2/2m - \mu)\psi + \frac{u_0}{2} (\bar{\psi}\psi)^2 \right],$$

where the spatial integrals should be regularized by means of a short-distance cutoff $\Lambda_0^{-1}$, which is related to the finite extent of the interaction or, for the hard core bosons, to the size of particles. Note that the model \cite{model} depends on two dimensionless parameters $\bar{\mu}_0 = 2m\mu\Lambda_0^{-2}$ and $\bar{u}_0 = 2mu_0\Lambda_0^{-2}$, which give the chemical potential and interaction energy in units of $\Lambda_0^{-2}/2m$. The FRG is conveniently formulated in terms of the generating functional of irreducible vertex functions $\Gamma_\Lambda[\bar{\phi}, \phi]$, depending on an IR cutoff $\Lambda$ which eventually will be removed \cite{FRG}. The functional $\Gamma_\Lambda$ is defined through $\Gamma_\Lambda[\bar{\phi}, \phi] = \mathcal{L}_\Lambda[\bar{\phi}, \phi] + \int_K \delta K \Gamma_{0,\Lambda}(K) \phi_K$ where $K = (k, i\omega)$ and $\int_K = (2\pi)^{-(D+1)} \int d^D k d\omega$ is the integration over momenta and Matsubara frequencies. Here, $\mathcal{L}_\Lambda[\bar{\phi}, \phi]$ is the Legendre transform of the generating functional of connected Green functions and $\Gamma_{0,\Lambda}^{-1} = i\omega - \epsilon_k + \mu - R_{\Lambda}(k)$ is the inverse free propagator with dispersion $\epsilon_k = k^2/2m$. The regulator function $R_{\Lambda}(k)$ will be specified below. We shall use the following non-local potential approximation for $\Gamma_\Lambda$,

$$\Gamma_\Lambda = \int_K \hat{\delta}_K \sigma_\Lambda(K) \phi_K + \frac{1}{2} \int_K \delta \rho_K u_\Lambda(K) \delta \rho_{-K}.$$


The quantity $\delta \rho_K = \int_Q \delta \phi_K \phi_{K+Q} - \delta K \rho_0^A$ is the Fourier-transform of $\rho_X - \rho_X^0$ where $X = (x, \tau)$. Here, $\rho_X = |\phi_X|^2$ is the local density and $\rho_X^0$ is the flowing condensate density, which has a finite limit for $\Lambda \to 0$ in the symmetry broken state. The non-local character of the potential coupling function $u_A(K)$, which was not included in previous FRG approaches,[10, 11], is essential to describe correctly the low energy sector of the theory. Although we will in the end calculate the complete momentum and frequency dependence of $\sigma_A(K)$ and $u_A(K)$, we shall also employ the following low energy expansion,

$$
\sigma_A(K) \approx \mu(1 - X_A) + i\omega(1 - Y_A) + \epsilon_k(Z_A^{-1} - 1) - (i\omega)^2 V_A, \\
u_A(K) \approx \mu A_0 \equiv u_A,
$$

with $X_A = \epsilon_A^0/\mu$. The flow of the parameters $\Sigma A, \Lambda A, \Lambda A, \Lambda A, \Lambda A, \Lambda A, \Lambda A, \Lambda A$ will be determined self-consistently. The initial values can be read off from the action[14] and are given by $X_A = Y_A = Z_A = 0$, $V_A = 0$ and $u_A = u_0$. The initial condition for the condensate density is given by the mean field result $\epsilon_A^0 = \mu/\omega$. The usual Beliaev Green functions are defined via the matrix equation

$$G_{\Lambda A}^{-1}(K) = G_{\Lambda A}^{-1}(K) - \Sigma A(K),$$

where

$$G_{\Lambda A}(K) = \begin{pmatrix} G_{\Lambda A}(K) & G_{\Lambda A}(K) \\ G_{\Lambda A}^*(K) & G_{\Lambda A}^*(K) \end{pmatrix},$$

$$G_{0,\Lambda A}(K) = \begin{pmatrix} G_{0,\Lambda A}(K) & 0 \\ 0 & G_{0,\Lambda A}^*(K) \end{pmatrix},$$

$$\Sigma A(K) = \begin{pmatrix} \Sigma A(K) & \Sigma A(K) \\ \Sigma A^*(K) & \Sigma A^*(K) \end{pmatrix}.$$
of the self-energy, the truncations \( \text{(3a,3b)} \) are however not sufficient. Moreover, within these truncations one obtains \( G^4(K) = \lim_{\Lambda \to 0} G^4_{\Lambda}(K) \equiv 0 \) which violates the asymptotic \( K \to 0 \) relation \( \text{(4)} \).

\[
G^N(K) \sim -G^A(K) \sim \frac{m \rho^0 c^2}{\rho} \frac{1}{(\omega^2 + c^2 k^2)}.
\]

where \( \rho^0 \) is the condensate density, \( \rho \) the boson density and \( c \) the thermodynamic sound velocity. This relation can be satisfied only within an approach which takes account of the non-analytic structure of \( u_{\Lambda}(K) \). We further note that the crossover energy scale \( \Delta_c = 4\pi \rho^0 / [m \ln(\rho^0 a^2)] \), obtained by Schick \( \text{(12)} \) for hard core bosons of diameter \( a \), emerges from the \( z = 2 \) regime of Eqs. \( \text{(3a,3b)} \) in the dilute limit and for large \( u_{\Lambda_0} \).

We now proceed with the evaluation of the self-energies. Truncating the momentum dependence of the irreducible vertices on the right-hand-side of the flow equations in accordance with the truncation of \( \Gamma_{\Lambda} \) in Eqs. \( \text{(2)} \) and \( \text{(3a,3b)} \), we find the flow equation of the self-energies (the single scale propagators \( G^A_{\Lambda,N}(K) \) are defined via \( \tilde{G}^A_{\Lambda}(K) = -G^A_{\Lambda}(K) \partial_{\Lambda} G^0_{\Lambda,N}(K) G^A_{\Lambda}(K) \)):

\[
\partial_{\Lambda} \Sigma^N_{\Lambda}(K) = 2u_{\Lambda} \int_Q \left\{ \tilde{G}^N_{\Lambda}(Q) + \tilde{G}^A_{\Lambda}(Q) \right\} - 4u_{\Lambda}^2 \rho^0 \int_Q \left\{ \tilde{G}^N_{\Lambda}(Q) \left[ G^N_{\Lambda}(Q + K) + G^N_{\Lambda}(Q - K) + G^N_{\Lambda}(-Q + K) \right] + 2G^A_{\Lambda}(Q - K) + 2\tilde{G}^A_{\Lambda}(Q) \left[ G^A_{\Lambda}(Q + K) + G^A_{\Lambda}(Q + K) \right] \right\},
\]

\[
\partial_{\Lambda} \Sigma^A_{\Lambda}(K) = 2u_{\Lambda} \int_Q \tilde{G}^N_{\Lambda}(Q) - 4u_{\Lambda}^2 \rho^0 \int_Q \left\{ \tilde{G}^N_{\Lambda}(Q) \left[ G^N_{\Lambda}(Q + K) + G^N_{\Lambda}(Q - K) + G^A_{\Lambda}(Q + K) + G^A_{\Lambda}(Q - K) + 3G^A_{\Lambda}(Q + K) \right] \right\}.
\]

To solve Eqs. \( \text{(11a,11b)} \), we adopt a technique we applied previously to the classical \( \phi^4 \)-model \( \text{(15)} \) (see \( \text{(18, 19)} \) for approaches to classical models in their symmetric phase). Instead of attempting to calculate a completely self-consistent solution, we use a non-self-consistent approach where on the right-hand-side we approximate the self-energies entering the Green functions using Eqs. \( \text{(3a,3b)} \) and \( \text{(7a,7b)} \). The full solution is expected to appreciably deviate from the non-self-consistent solution only for large momenta (at the order of \( \Lambda_0 \)) or at strong coupling. With Eqs. \( \text{(3a,3b)} \) and the already determined flows of \( u_{\Lambda}, \rho^0_{\Lambda}, Y_{\Lambda}, Z_{\Lambda} \) and \( V_{\Lambda} \), we can simply integrate Eqs. \( \text{(11a,11b)} \) to obtain the complete momentum and frequency dependence of the self-energies \( \Sigma^{N/A}_{\Lambda}(K) = \lim_{\Lambda \to 0} \Sigma^{N/A}_{\Lambda}(K) \). The resulting expressions for \( \Sigma^{N/A}_{\Lambda}(K) \) are free from IR divergences since the coupling constant \( u_{\Lambda} \) vanishes in the IR limit.

An important result of this approach is the correct description of the non-analytic structure of \( u_{\Lambda}(K) \) \( \text{(7)} \) with leading terms linear in \( |\omega| \) and \( |k| \) (for \( D = 2 \)). The Green functions \( G^N(K) \) and \( G^A(K) \) are obtained from \( G^{-1}(K) = G_0^{-1}(K) - \Sigma(K) \text{(2)} \) and one can check that since \( \rho^0_{\Lambda \to 0}(K) \gg |\omega^2 V + e_k / Z| \) for small \( K \), \( G^N(K) \) and \( G^A(K) \) have the limiting behavior given in Eq. \( \text{(10)} \) if we identify \( \text{(2)} \) \( \rho^0 / \rho = Z \) and \( c = (2mVZ)^{-1/2} \).

The single-particle spectral density \( A(k,\omega) \) is obtained from the imaginary part of the normal propagator via \( A(k,\omega) = -2\text{Im}G^N(k,i\omega \to \omega + i0) \) and requires analytic continuation to real frequencies. We used the standard Padé technique \( \text{(20)} \) with 200 points. In Fig. \( \text{2} \) the spectral density is plotted as a function of \( \omega > 0 \) for different momenta. One clearly observes a finite peak broadening, which grows with increasing momenta. The broadening arises from Beliaev damping \( \text{(6)} \). Because of the upward curvature of the dispersion, a momentum- and energy-conserving decay of quasi-particles into pairs of quasi-particles is allowed. A second order perturbative analysis \( \text{(21)} \) shows that in \( D \) dimensions this damping at
small momenta and for weak coupling has the form
\[ \gamma_k^{(2)} = I_D(2m\rho)^{-1}k_0^{3-D}k^{2D-1}, \]  
(12)
where \( I_D = 2^{-3+\frac{D-1}{2}}K_{D-1} \int_0^1 dx(x-x^2)^{D-1}, \)
\( k_0 = 2mc_0, \) and \( c_0 = \sqrt{\rho u_0} = m \) is the mean field velocity. While in \( D = 3 \) the perturbative expression is independent of \( c_0 \) and reproduces the Beliaev result \( \gamma_k^{(2)} = 3k^5/(64\pi\rho m^3) \), in \( D = 2 \) the damping depends explicitly on \( c_0 \) and is thus expected to be renormalized. An analysis of the data in Fig. 2 reproduces the \( k^3 \)-behavior for small momenta as predicted by Eq. 12 and shown in Fig. 3. However, the prefactor \( \alpha_0 = I_D(2m\rho)^{-1}k_0^{3-D} \) in Eq. 12 should be replaced by a function of \( \alpha_0, v_0 \) of the model 11. For example for \( \mu_0 = 0.4 \) and \( \bar{v}_0 = 4 \) we obtain \( c/c_0 \approx 1.01 \) and \( \alpha/\alpha_0 \approx 0.915 \), while the results for \( \mu_0 = 0.15 \) and \( \bar{v}_0 = 15 \) are \( c/c_0 \approx 0.669 \) and \( \alpha/\alpha_0 \approx 0.526 \). The inset of Fig. 2 shows the extracted quasi-particle dispersion, which is well fitted by \( E_k^2 = c_k^2 + c^2k^2 \) with renormalized velocity \( c \). The deviation of the spectrum from linearity occurs at the same momentum where the damping deviates from the cubic asymptote.

In conclusion, we have presented FRG results for the spectral function of interacting bosons in \( D = 2 \) in an approach which is consistent with the Nepomnyashchyi identity \( \Sigma^A(0) = 0 \) and the Hugenholtz-Pines relation \( \Sigma_N(0) - \Sigma^A(0) = \mu \). While previous RG calculations 8, 9, 10 were limited to the Goldstone regime, our approach allows calculating the entire spectral line-shape, including the dispersion and the damping of the quasi-particles. Our truncation captures the non-analytic structure of the self-energies, described by the non-local potential coupling \( u_0(K) \), which is essential for a correct description of the low-energy physics. We are currently investigating extensions of our method to include arbitrarily strong local correlations. This would possibly provide a non-perturbative access to strongly interacting bosons.

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Abstract

We employ the functional renormalization group to study dynamical properties of the two-dimensional Bose gas. Our approach is free of infrared divergences, which plague the usual diagrammatic approaches, and is consistent with the exact Nepomnyashchy identity, which states that the anomalous self-energy vanishes at zero frequency and momentum. We recover the correct infrared behavior of the propagators and present explicit results for the spectral line-shape, from which we extract the quasi-particle dispersion and damping.

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The excitation spectrum of the weakly interacting Bose gas has been studied with field theoretical methods for more than half a century [1] and is qualitatively well understood: at high energies the excitations resemble those of free bosons whereas at low energies the excitations are collective Goldstone modes which result from the symmetry broken ground state [2, 3, 4]. In $D = 3$, this picture has recently been confirmed experimentally using the Bragg spectroscopy technique on cold atoms [5]. However, while perturbative approaches (based on a renormalized $T$-matrix) to the weakly interacting Bose gas have been successfully applied to study the excitation spectra and the damping of quasi-particles [6], the self-energies in these approaches violate the exact Nepomnyashchya identity [7], which states that the anomalous self-energy at zero frequency and momentum vanishes. Moreover, the perturbative approach is plagued by infrared (IR) divergences [2, 3], which makes a numerical evaluation non-trivial already at second order in the effective coupling constant. Modern renormalization group (RG) methods [8, 9, 10] have resolved this problem and recovered the correct low-energy structure of the self-energies which are in accordance with the Goldstone character of the excitations and the Nepomnyashchya identity. The calculations in Refs. [8, 9, 10] were however limited to the asymptotic regime near zero energy and are thus unable to describe both the damping of quasi-particles and the crossover from the low energy Goldstone-like dispersion to the free particle dispersion at higher energies.

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We consider bosons with mass $m$, chemical potential $\mu$ and a repulsive contact interaction
$u_0$ at zero temperature. The bare Euclidean action is

$$S[\bar{\psi}, \psi] = \int d^D x d\tau \left[ \bar{\psi} \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi + \frac{u_0}{2} (\bar{\psi} \psi)^2 \right],$$  

(1)

where the spatial integrals should be regularized by means of a short-distance cutoff $\Lambda_0^{-1}$, which is related to the finite extent of the interaction or, for the hard core bosons, to the size of particles. Note that the model (1) depends on two dimensionless parameters $\tilde{\mu}_0 = 2m\mu\Lambda_0^{-2}$ and $\tilde{u}_0 = 2mu_0\Lambda_0^{-D-2}$, which give the chemical potential and interaction energy in units of $\Lambda_0^2/2m$. The FRG is conveniently formulated in terms of the generating functional of irreducible vertex functions $\Gamma_{\Lambda}[\bar{\phi}, \phi]$, depending on an IR cutoff $\Lambda$ which eventually will be removed [13]. The functional $\Gamma_{\Lambda}$ is defined through $\Gamma_{\Lambda}[\bar{\phi}, \phi] = L_{\Lambda}[\bar{\phi}, \phi] + \int_K \tilde{\phi}_K G_{0,\Lambda}^{-1}(K)\phi_K$ where $K = (k, i\omega)$ and $L_{\Lambda}[\bar{\phi}, \phi] = \delta\rho_{K} u_{\Lambda}(K) \delta\rho_{-K}$ is the integration over momenta and Matsubara frequencies. Here, $L_{\Lambda}[\bar{\phi}, \phi]$ is the Legendre transform of the generating functional of connected Green functions and $G_{0,\Lambda}^{-1} = i\omega - \epsilon_k + \mu - R_{\Lambda}(k)$ is the inverse free propagator with dispersion $\epsilon_k = k^2/2m$. The regulator function $R_{\Lambda}(k)$ will be specified below. We shall use the following non-local potential approximation for $\Gamma_{\Lambda}$,

$$\Gamma_{\Lambda} = \int_K \tilde{\phi}_K \sigma_{\Lambda}(K) \phi_K + \frac{1}{2} \int_K \delta\rho_K u_{\Lambda}(K) \delta\rho_{-K}.$$  

(2)

The quantity $\delta\rho_K = \int_Q \tilde{\phi}_Q \phi_{K+Q} - \delta_{K,0} \rho^0_{\Lambda}$ is the Fourier-transform of $\rho_X - \rho^0_{\Lambda}$ where $X = (x, \tau)$. Here, $\rho_X = |\phi_X|^2$ is the local density and $\rho^0_{\Lambda}$ is the flowing condensate density, which has a finite limit for $\Lambda \to 0$ in the symmetry broken state. The non-local character of the potential coupling function $u_{\Lambda}(K)$, which was not included in previous FRG approaches [9, 10], is essential to describe correctly the low energy sector of the theory. Although we will in the end calculate the complete momentum and frequency dependence of $\sigma_{\Lambda}(K)$ and $u_{\Lambda}(K)$, we shall also employ the following low energy expansion,

$$\sigma_{\Lambda}(K) \approx \mu(1 - X_{\Lambda}) + i\omega(1 - Y_{\Lambda}) + \epsilon_k(Z_{\Lambda}^{-1} - 1) - (i\omega)^2 V_{\Lambda},$$  

(3a)

$$u_{\Lambda}(K) \approx u_{\Lambda}(0) \equiv u_{\Lambda},$$  

(3b)

with $X_{\Lambda} = \rho^0_{\Lambda} u_{\Lambda}/\mu$. The flow of the parameters $Y_{\Lambda}, Z_{\Lambda}, V_{\Lambda}, \rho^0_{\Lambda}$, and $u_{\Lambda}$ will be determined self-consistently. The initial values can be read off from the action (1) and are given by $X_{\Lambda_0} = Y_{\Lambda_0} = Z_{\Lambda_0} = 1, V_{\Lambda_0} = 0$ and $u_{\Lambda_0} = u_0$. The initial condition for the condensate density
is given by the mean field result $\rho_{\Lambda_0}^0 = \mu/u_0$. The usual Beliaev Green functions are defined via the matrix equation $G_{\Lambda}^{-1}(K) = G_{0,\Lambda}^{-1}(K) - \Sigma_{\Lambda}(K)$, where

$$G_{\Lambda}(K) = \begin{pmatrix} G_{\Lambda}^N(K) & G_{\Lambda}^A(K) \\ G_{\Lambda}(K)^* & G_{\Lambda}^N(-K) \end{pmatrix},$$  

(4)

$$G_{0,\Lambda}(K) = \begin{pmatrix} G_{0,\Lambda}(K) & 0 \\ 0 & G_{0,\Lambda}(-K) \end{pmatrix},$$  

(5)

$$\Sigma_{\Lambda}(K) = \begin{pmatrix} \Sigma_{\Lambda}^N(K) & \Sigma_{\Lambda}^A(K) \\ \Sigma_{\Lambda}(K)^* & \Sigma_{\Lambda}^N(-K) \end{pmatrix}.$$  

(6)

The normal ($\Sigma^N$) and anomalous ($\Sigma^A$) self-energies are obtained from an expansion of Eq. (2) in $\delta \phi_X = \phi_X - \phi_X^0$ [14-15], where we assume a real valued order parameter such that $\rho_{\Lambda}^0 = \bar{\phi}_{\Lambda}^0 \phi_{\Lambda}^0 = (\phi_{\Lambda}^0)^2$. The self-energies are obtained from $\sigma_{\Lambda}(K)$ and $u_{\Lambda}(K)$ via

$$\Sigma_{\Lambda}^N(K) = \sigma_{\Lambda}(K) + \rho_{\Lambda}^0[u_{\Lambda}(0) + u_{\Lambda}(K)],$$  

(7a)

$$\Sigma_{\Lambda}^A(K) = \rho_{\Lambda}^0 u_{\Lambda}(K).$$  

(7b)

Conversely, we may determine $\sigma_{\Lambda}(K)$, which at $T = 0$ has a well defined low-frequency and momentum expansion [7], and the non-local coupling function $u_{\Lambda}(K)$, which will become non-analytic for $\Lambda \to 0$, from a calculation of the self-energies for all frequencies and momenta. The effective action [2] elegantly divides the well behaved parts of the self-energies from the non-analytic ones. It further guarantees, since only the flows of the $U(1)$-invariants $u_{\Lambda}$, $\rho_{\Lambda}^0$ and $\sigma_{\Lambda}$ are retained, that the Hugenholtz-Pines relation $\Sigma_{\Lambda}^N(0) - \Sigma_{\Lambda}^A(0) = \mu$ [4] is obeyed for all $\Lambda$. We use the Litim cutoff function $R_{\Lambda}(k) = (1 - \delta_{k,0})(2mZ_{\Lambda})^{-1}(\Lambda^2 - k^2)\Theta(\Lambda^2 - k^2)$, which has good convergence properties [16] and yields simple momentum integrals, to derive the flow equations for the coupling parameters [9, 10]. We define $\kappa_{\Lambda} = K_D[1 - \eta_{\Lambda}^z/(D + 2)]/D$ with $K_D = 2^{1-D}\pi^{-D/2}/\Gamma[D/2]$, where $\eta_{\Lambda}^z = \Lambda \partial_{\Lambda} \ln Z_{\Lambda}$ is the anomalous dimension of the fields, and $\Delta_{\Lambda} = u_{\Lambda}\rho_{\Lambda}^0$. We arrive at [9]

$$\partial_{\Lambda}\rho_{\Lambda}^0 = \frac{4\Lambda^{D+1}\kappa_{\Lambda}}{2mZ_{\Lambda}} \int \frac{d\omega}{2\pi} \sum_{n=0}^3 \frac{c_n^{(\rho)} \omega^{2n}}{D_{\Lambda}^2(i\omega)},$$  

(8a)

$$\partial_{\Lambda}u_{\Lambda} = 4u_{\Lambda}^2 \frac{\Lambda^{D+1}\kappa_{\Lambda}}{2mZ_{\Lambda}} \int \frac{d\omega}{2\pi} \sum_{n=0}^3 \frac{c_n^{(u)} \omega^{2n}}{D_{\Lambda}^3(i\omega)},$$  

(8b)
where $D_\Lambda(i\omega) = Y^2_\Lambda \omega^2 + [\bar{\epsilon}_\Lambda + V_\Lambda \omega^2][\bar{\epsilon}_\Lambda + V_\Lambda \omega^2 + 2\Delta_\Lambda]$, and $\bar{\epsilon}_\Lambda = \epsilon_\Lambda/Z_\Lambda$. Here we have defined $c_0^{(u)} = \bar{\epsilon}_\Lambda^2 + \bar{\epsilon}_\Lambda \Delta_\Lambda + \Delta_\Lambda^2$, $c_1^{(u)} = V_\Lambda(2\bar{\epsilon}_\Lambda + \Delta_\Lambda) - Y^2_\Lambda$, and $c_2^{(u)} = V^2_\Lambda$. The flow of $\rho_\Lambda^0$ is chosen such that in the expansion of $\Gamma_\Lambda$ in the field $\delta \phi_X = \phi_X - \phi_\Lambda^0$ no terms linear in $\delta \phi_X$ appear, i. e., one always expands around the flowing minimum of $\Gamma_\Lambda$. For the coefficients entering Eq. (8b) we have $c_0^{(u)} = 5\bar{\epsilon}_\Lambda^3 + 3\bar{\epsilon}_\Lambda^2 \Delta_\Lambda + 6\bar{\epsilon}_\Lambda \Delta_\Lambda^2 + 4\Delta_\Lambda^3$, $c_1^{(u)} = 3V_\Lambda(5\bar{\epsilon}_\Lambda^2 + 2\bar{\epsilon}_\Lambda \Delta_\Lambda + \Delta_\Lambda^2) - Y^2_\Lambda(7\Delta_\Lambda + 11\bar{\epsilon}_\Lambda)$, $c_2^{(u)} = V_\Lambda[3V_\Lambda(5\bar{\epsilon}_\Lambda + \Delta_\Lambda) - 11Y^2_\Lambda]$, and $c_3^{(u)} = 5V^3_\Lambda$. The flow equations of the derivative terms $Y_\Lambda$, $Z_\Lambda$ and $V_\Lambda$ are extracted from the flow equation of $\Sigma^N_\Lambda(K) - \Sigma^A_\Lambda(K) = \rho_\Lambda^0 u_\Lambda + \sigma_\Lambda(K)$. We find

$$ \partial_\Lambda Y_\Lambda = -8\rho_\Lambda^0 u_\Lambda^2 Y_\Lambda \frac{\Lambda^{D+1} K}{2m Z_\Lambda} \int \frac{d\omega}{2\pi} \sum_{n=0}^2 c_n^{(u)}(\omega) \omega^{2n}, $$

$$ \partial_\Lambda Z_\Lambda = 4\rho_\Lambda^0 u_\Lambda^2 Z_\Lambda \frac{\Lambda^{D+1} K}{2m D} \int \frac{d\omega}{2\pi} \frac{1}{D_\Lambda^2(i\omega)}, $$

$$ \partial_\Lambda V_\Lambda = 8\rho_\Lambda^0 u_\Lambda^2 V_\Lambda \frac{\Lambda^{D+1} K}{2m A} \int \frac{d\omega}{2\pi} \sum_{n=0}^2 c_n^{(v)}(\omega) \omega^{2n}, $$

where $c_0^{(v)} = \bar{\epsilon}_\Lambda^2 - 2\bar{\epsilon}_\Lambda \Delta_\Lambda - 2\Delta_\Lambda^2$, $c_1^{(v)} = Y^2_\Lambda + 2(\bar{\epsilon}_\Lambda - \Delta_\Lambda)V_\Lambda$, $c_2^{(v)} = V^2_\Lambda$, and $c_0^{(v)} = -Y^2_\Lambda(\bar{\epsilon}_\Lambda + \Delta_\Lambda) - \bar{\epsilon}_\Lambda(\bar{\epsilon}_\Lambda + 2\Delta_\Lambda)V_\Lambda$, $c_1^{(v)} = 2V_\Lambda[Y^2_\Lambda + V_\Lambda(\bar{\epsilon}_\Lambda + \Delta_\Lambda)]$, $c_2^{(v)} = 3V^3_\Lambda$.

FIG. 1: Typical flows of the parameters $X_\Lambda$, $Y_\Lambda$, $Z_\Lambda$, $V_\Lambda$, $\rho_\Lambda^0$ and dynamical exponent $z_\Lambda = 2 - \eta_\Lambda^x - \eta_\Lambda^y$. The curves are calculated for $\bar{\mu}_0 = 2m\mu\Lambda_0^{-2} = 0.4$ and $\bar{u}_0 = 2mu_0 = 4$. The arrows point to the relevant scales.

Similar equations were already discussed in Ref. [9, 10] and we briefly summarize the results in two dimensions. At $T = 0$ the condensate density $\rho_\Lambda^0$ flows to some finite limit $\rho_\Lambda^0_{\Lambda \to 0} > 0$, while the coupling constant $u_\Lambda$ vanishes for $\Lambda \to 0$ which ensures $\Sigma^N_\Lambda_{\Lambda \to 0}(K = 0) = \mu$ and $\Sigma^A_\Lambda_{\Lambda \to 0}(K = 0) = 0$, in accordance with both the Hugenholtz-Pines relation [4] and the Nepomnyashchyi identity [7]. Furthermore, $Y_\Lambda$ vanishes for $\Lambda \to 0$, again in accordance with exact results [7]. Since $V_\Lambda$ acquires a finite value and $Z_\Lambda$ remains finite for $\Lambda \to 0$, the low energy modes are phonons with a linear dispersion and velocity $c =$
with $V_c = V_{\Lambda = 0}$ and $Z_c = Z_{\Lambda = 0}$. In Fig. [1] we show flows of the coupling parameters and of the dynamical exponent $z = 2 - \eta_0^\nu - \eta_L^\nu$ where $\eta_0^\nu = \Lambda \partial_\Lambda \ln Y_\Lambda$ and $\eta_L^\nu = \Lambda \partial_\Lambda \ln Z_\Lambda$, which conveniently illustrates the crossover from the free particle regime with $z = 2$ to the Goldstone regime, where $z = 1$. To work out how the crossover appears in the momentum dependence of the self-energy, the truncations (3a,3b) are however not sufficient. Moreover, within these truncations one obtains $G^A(K) = \lim_{\Lambda \to 0} G^A_\Lambda(K) \equiv 0$ which violates the asymptotic $K \to 0$ relation

$$G^N(K) \sim -G^A(K) \sim \frac{m \rho c^2}{\rho} \frac{1}{(\omega^2 + c^2 k^2)},$$

(10)

where $\rho^0$ is the condensate density, $\rho$ the boson density and $c$ the thermodynamic sound velocity. This relation can be satisfied only within an approach which takes account of the non-analytic structure of $u_\Lambda(K)$. We further note that the crossover energy scale $\Delta_c = 4\pi \rho^0/[m \ln(\rho^0 a^2)]$, obtained by Schick [12] for hard core bosons of diameter $a$, emerges from the $z = 2$ regime of Eqs. (8b,8a) in the dilute limit and for large $u_{\Lambda_0}$.

We now proceed with the evaluation of the self-energies. Truncating the momentum dependence of the irreducible vertices on the right-hand-side of the flow equations in accordance with the truncation of $\Gamma_\Lambda$ in Eqs. (2) and (3a,3b), we find the flow equation of the self-energies (the single scale propagators $G^A_N(K)$ are defined via $\dot{G}_\Lambda(K) = -G_A(K) \partial_\Lambda G_{0,A}^{-1}(K) G_A(K)$),

$$\partial_\Lambda \Sigma^N_\Lambda(K) = 2u_\Lambda \int_Q \{ \dot{G}^N_\Lambda(Q) + \dot{G}^A_\Lambda(Q) \} - 4u_\Lambda \rho^0 \int_Q \{ \dot{G}^N_\Lambda(Q) [G^N_\Lambda(Q + K) + G^N_\Lambda(Q - K) + G^N_\Lambda(-Q + K)] + 2G^A_\Lambda(Q - K)] + 2\dot{G}^A_\Lambda(Q) [G^A_\Lambda(Q + K) + G^A_\Lambda(Q + K)] \},$$

(11a)

$$\partial_\Lambda \Sigma^A_\Lambda(K) = 2u_\Lambda \int_Q \dot{G}^N_\Lambda(Q) - 4u_\Lambda \rho^0 \int_Q \{ \dot{G}^N_\Lambda(Q) [G^N_\Lambda(Q + K) + G^N_\Lambda(Q - K) + G^A_\Lambda(Q + K) + G^A_\Lambda(Q - K)] + \dot{G}^A_\Lambda(Q) [G^A_\Lambda(Q + K) + G^A_\Lambda(Q - K) + 3G^A_\Lambda(Q + K)] \}.$$  

(11b)

To solve Eqs. (11a,11b), we adopt a technique we applied previously to the classical $\phi^4$-model [15] (see [18,19] for approaches to classical models in their symmetric phase). Instead of attempting to calculate a completely self-consistent solution, we use a non-self-consistent approach where on the right-hand-side we approximate the self-energies entering the Green functions using Eqs. (3a,3b) and (7a,7b). The full solution is expected to appreciably deviate from the non-self-consistent solution only for large momenta (at the order of $\Lambda_0$) or at strong
coupling. With Eqs. (3a,3b) and the already determined flows of $u_\Lambda, \rho_0^0, Y_\Lambda, Z_\Lambda$ and $V_\Lambda$, we can simply integrate Eqs. (11a,11b) to obtain the complete momentum and frequency dependence of the self-energies $\Sigma^{N/A}(K) = \lim_{\Lambda \to 0} \Sigma^{N/A}_\Lambda(K)$. The resulting expressions for $\Sigma^{N/A}(K)$ are free from IR divergences since the coupling constant $u_\Lambda$ vanishes in the IR limit.

An important result of this approach is the correct description of the non-analytic structure of $u_\Lambda(K)$ [7] with leading terms linear in $|\omega|$ and $|k|$ (for $D = 2$). The Green functions $G^N(K)$ and $G^A(K)$ are obtained from $G^{-1}(K) = G^{-1}_0(K) - \Sigma(K)$ [2] and one can check that since $\rho^0|u_{\Lambda=0}(K)| \gg |\omega^2 V_* + \epsilon_k/Z_*|$ for small $K$, $G^N(K)$ and $G^A(K)$ have the limiting behavior given in Eq. (10) if we identify [7] $\rho^0/\rho = Z_*$ and $c = (2mV_*Z_*)^{-1/2}$.

![FIG. 2: Single-particle spectral density $A(k, \omega)$ as a function of $\omega$, for different values of $k$, calculated for $\tilde{\mu}_0 = 0.15$ and $\tilde{u}_0 = 15$. The inset shows the quasi-particle dispersion $E_k$ which deviates at large $k$ from linearity but is well described by the Bogoliubov form $E_k^2 = \epsilon_k^2 + c^2k^2$ with renormalized velocity $c$ (black dots). The peaks (from left to right) correspond to $k/k_c = 0.2, 0.3, 0.4, 0.5$ and 0.6, where $k_c = 2mc$.](image)

The single-particle spectral density $A(k, \omega)$ is obtained from the imaginary part of the normal propagator via $A(k, \omega) = -2\text{Im}G^N(k, i\omega \to \omega + i0)$ and requires analytic contin-
uation to real frequencies. We used the standard Padé technique \[20\] with 200 points. In Fig. 2 the spectral density is plotted as a function of $\omega > 0$ for different momenta. One clearly observes a finite peak broadening, which grows with increasing momenta. The broadening arises from Beliaev damping \[6\]. Because of the upward curvature of the dispersion, a momentum- and energy-conserving decay of quasi-particles into pairs of quasi-particles is allowed. A second order perturbative analysis \[21\] shows that in $D$ dimensions this damping at small momenta and for weak coupling has the form

$$
\gamma_k^{(2)} \approx I_D(2m\rho^0)^{-1}k_0^{3-D}k^{2D-1},
$$

(12)

where $I_D = 2^{-4}3^{1+D/2}K_{D-1} \int_0^1 dx(x-x^2)^{D-1}$, $k_0 = 2mc_0$, and $c_0 = \sqrt{\rho^0u_0/m}$ is the mean field velocity. While in $D = 3$ the perturbative expression is independent of $c_0$ and reproduces the Beliaev result $\gamma_k^{(2)} = 3k^5/(640\pi m\rho^0)$, in $D = 2$ the damping depends explicitly on $c_0$ and is thus expected to be renormalized. An analysis of the data in Fig. 2 reproduces the $k^3$-behavior for small momenta as predicted by Eq. (12) and shown in Fig. 3. However, the prefactor $\alpha_0 = I_D(2m\rho^0)^{-1}k_0^{3-D}$ in Eq. (12) should be replaced by a function $\alpha(\tilde{\mu}_0, \tilde{u}_0)$ of the relevant dimensionless parameters $\tilde{\mu}_0$ and $\tilde{u}_0$ of the model (1). For example for $\tilde{\mu}_0 = 0.4$ and $\tilde{u}_0 = 4$ we obtain $c/c_0 \approx 1.01$ and $\alpha/\alpha_0 \approx 0.915$, while the results for $\tilde{\mu}_0 = 0.15$ and $\tilde{u}_0 = 15$ are $c/c_0 \approx 0.669$ and $\alpha/\alpha_0 \approx 0.526$. The inset of Fig. 2 shows the extracted quasi-particle dispersion, which is well fitted by $E_k^2 = \epsilon_k^2 + c^2k^2$ with renormalized velocity $c$. The deviation of the spectrum from linearity occurs at the same momentum where the damping deviates from the cubic asymptote.

In conclusion, we have presented FRG results for the spectral function of interacting bosons in $D = 2$ in an approach which is consistent with the Nepomnyashchy identity $\Sigma^A(0) = 0$ and the Hugenholtz-Pines relation $\Sigma^N(0) - \Sigma^A(0) = \mu$. While previous RG calculations \[8, 9, 10\] were limited to the Goldstone regime, our approach allows calculating the entire spectral line-shape, including the dispersion and the damping of the quasi-particles. Our truncation captures the non-analytic structure of the self-energies, described by the non-local potential coupling $u_\Lambda(K)$, which is essential for a correct description of the low-energy physics. We are currently investigating extensions of our method to include arbitrarily strong local correlations. This would possibly provide a non-perturbative access to strongly interacting bosons.

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