Abstract and Dedication

This paper is an exploration of simple four-regular graphs in the plane (i.e. loop-free and with no more than one edge between any two nodes). Such graphs are fundamental to the theory of knots and links in three dimensional space, and their planar diagrams. We dedicate this paper to Frank Harary (1921 – 2005), whose fascination with graphs of knots inspired this work, and with whom we had the pleasure of developing this paper.

1 Introduction

This paper is an exploration of four-regular graphs in the plane that have no more than one edge between any two nodes. We call such graphs “lune-free”
since the appearance of two edges from one node to another makes a lune-
shape in the plane. Such graphs are fundamental to the theory of knots and
links in three dimensional space, and their planar diagrams.

The reader interested in the relationship of these graphs with knot theory
should consult the fundamental paper of John H. Conway [2]. In this paper,
Conway uses a small collection of lune-free graphs to construct and enumerate
all knots (knot diagrams up to topological equivalence) with no more than
ten crossings. There is a potential for using the full class of lune-free diagrams
to enable deeper tabulations of knots and links. See [4] for recent work along
these lines.

We became interested in these graphs for their own sake, motivated by
their possible use in the theory of knots and links. In particular, seeing that
the first lune-free link projection is the familiar graph of the Venn diagram for
three circles, and that the first single component (in the sense of knot theory)
lune-free graph had eight nodes, we wanted to prove that beyond eight nodes,
there were lune-free graphs of one component for every choice of the number
of nodes. This conjecture turned out to be true. It is a consequence of some
interesting inductive constructions that we produce in the third section of the
paper. These constructions depend upon an admissibility criterion described
below. It was then natural to ask about the existence of inadmissible lune-
free graphs, and the remainder of the paper is devoted to theorems describing
when they can be constructed. These results are new, and they are answers
to questions that arise naturally in this subject.

Here is an outline of the contents of the paper. We first show that, for
projected knot diagrams, there are no lune-free diagrams with less than 8
nodes, and that from then on there is at least one such diagram with \( v \)
nodes for every \( v \geq 8 \). This result is proved by using the Euler formula for
plane graphs and some graphical recursions in Sections 2 and 3. A lune-
free knot graph is said to be admissible if it has two adjacent regions with
both regions having at least 4 sides. The point about an admissible knot
graph is that it admits a certain recursive procedure, described in Section 3,
that allows us to make infinitely many lune-free knot graphs from it. In the
remaining sections of the paper we explore the existence question of tight,
i.e. inadmissible, lune-free knot and link graphs. The basic question, that
we shall completely answer here, is the following: for which integers \( v \) does there exist a tight lune-free knot or link graph, respectively? For links, the answer is: all integers \( v \geq 8 \) such that \( v \not\equiv \pm 1 \mod 6 \) can be realized as the number of crossings of such graphs. The answer is the same for knots as for links, with the sole exception of \( v = 12 \), which cannot be realized by a tight lune-free knot graph. (However, there is a tight lune-free link graph with 12 crossings and \( \mu = 3 \) components.) This is accomplished in Section 4. Finally, in Section 5, we construct knot graphs with a fixed number of lunes and with any sufficiently large number of vertices.

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## 2 Applying the Euler Formula

We shall sometimes use the knot theory terminology *universe* for a 4-regular connected plane multi-graph. This paper is concerned with those universes that are simple graphs in the sense that they have no more than one edge between two nodes, and have no loops (a loop is an edge whose endpoints are identical). We shall call a simple and connected universe a *lune-free graph*. There are other terminologies for this concept. Such graphs are called *Conway polyhedra* [2]. Graphs of this type can be enumerated by methods of Tutte [6].

In knot theory, each node of a 4-regular plane graph is endowed with extra structure so that the graph can be seen to represent a projection of a curve that is embedded in three-dimensional space. We refer to [5, 3] for this terminology. The present paper will be concerned only with the plane graphs themselves, but we shall call a graph that could be the shadow of a knot or link a *knot graph* or a *link graph*, respectively. A knot graph is a 4-regular plane graph such that one can traverse the entire graph by entering and leaving each node along edges that are *not* adjacent in the cyclic order of the local edges at this node. In a link graph there will be several such circuits, corresponding to the components of the link’s embedding in three-dimensional space.
We shall use the words *vertex*, *node* and *crossing* as synonyms in the discussion that follows.

Let $G$ be a connected plane graph. Then $G$ intrinsically has $v = v(G)$ nodes, $e = e(G)$ edges and, from its embedding in the plane, $f = f(G)$ faces. The faces of $G$ are the connected regions into which the complement of $G$ in the plane (or in the surface of the sphere) is divided. Euler proved the fundamental result:

**Theorem 1.** Let $G$ be a connected plane graph. Then

\[ v(G) - e(G) + f(G) = 2. \]

**Proof.** See [1]. \qed

Let $G$ be a lune-free graph (by which we mean a simple 4-regular connected plane graph). Four-regularity implies that $4v(G) = 2e(G)$. Hence,

\[ 2v(G) = e(G). \]

Substituting this equation into Euler’s formula, we obtain the equation

\[ f(G) = 2 + v(G) \]

when $G$ is a connected lune-free graph.

If $G$ is any plane graph, we let $f_k(G)$ denote the number of faces of $G$ that have $k$ edges on the boundary. Then we can write

\[ f(G) = f_3(G) + f_4(G) + f_5(G) + \cdots \]

when $G$ is simple and lune-free, since in this case $f_1(G) = f_2(G) = 0$.

**Theorem 2.** Let $G$ be a connected lune-free link graph. Then

\[ f_3(G) = 8 + f_5(G) + 2f_6(G) + 3f_7(G) + \cdots. \]

Thus any lune-free graph must have at least eight three-sided regions.
Proof. Let $f = f(G), f_k = f_k(G), v = v(G), e = e(G)$. We have

\[ f = f_3 + f_4 + f_5 + \cdots, \]
\[ 2e = 3f_3 + 4f_4 + 5f_5 + \cdots, \]
\[ 4f = 2e + 8. \]

Hence

\[(4f_3 + 4f_4 + 4f_5 + \cdots) = (3f_3 + 4f_4 + 5f_5 + \cdots) + 8,\]

from which the theorem follows at once.

It follows from Theorem 2 that the smallest (meaning the smallest number of nodes) possible lune-free link graph has $f_3 = 8$ and $f_k = 0$ for all $k$ greater than three. See Figure 1. This diagram is the familiar three-circle Venn diagram, and it is the unique minimal lune-free link graph. Note that the number of link components of this diagram is three. We are independently interested in the number of lune-free link graphs with a given number of link components.

\[ \text{Figure 1 - The Simplest Lune-Free Link Graph} \]
The next simplest link graph has $f_3 = 8$ but $f_4 = 2$ and (necessarily) $f_k = 0$ for all $k > 4$. To see this, start with a four sided region and begin building three sided regions adjacent to it. Four more three sided regions are then forced, and one obtains closure with one extra four sided region on the outside. This graph is minimal and unique with specifications as above. It is an eight crossing knot graph with one knot-theoretic component, hence a lune-free knot graph with eight vertices. See Figure 2.

In Figures 2 and 3 we give tables of lune-free knot graphs with nine through twelve crossings.

Here is a sketch of the rest of this section and the next section: In the remainder of this section we give various examples including link graphs with eight triangle faces and all other faces of order four. We begin by giving a bit more information about the Venn diagram and the graph $G_8$ of Figure 5. There are infinitely many such examples. In Section 3 we show that there exist lune-free knot graphs with $v$ any integer greater than or equal to 8.

Example 1. A smallest lune-free graph must have $f_3(G) = 8$. Witness the standard 3-circle Venn diagram as shown in Figure 1. This diagram is the
unique lune-free graph with $f_3 = 8$, as is easy to see from the construction in Figure 4.

![Figure 4 - Creating the Venn Diagram](image)

**Figure 4 - Creating the Venn Diagram**

![Figure 5 - The Graph $G_8$](image)

**Figure 5 - The Graph $G_8$**

**Example 2.** Let $G_8$ denote the graph shown in Figure 5.

Here we have $f_3 = 8$, $f_4 = 2$, $f = 10$, $v = 8$, $e = 16$, and $G_8$ is a knot projection (the projection of a single component link). This graph is the next largest (in size the number of nodes $v(G)$) lune-free graph after the Venn diagram of Example 1. To see this, note that any lune-free graph with an $n$-sided region must have at least $2n$ nodes. Thus a smallest possible lune-free graph with $f > f_3$ must have at least 8 nodes. $G_8$ is the smallest such example. Note also that the Venn diagram is the unique lune-free graph with $f = f_3$. 
Example 3. $G_8$ is the smallest member of an infinite family of lune-free graphs, as illustrated in Figure 6.

In this family, the graphs $G^n$ have a single knot theoretic component for $n$ odd. Thus in the Figure $G^1 = G_8, G^3, G^5$ are knot graphs. The graph $G^n$ has $4n + 4$ nodes.

Example 4. We can regard the family $\{G^n\}$ as generated from a 4-gon (the central square in each figure). A similar family is generated from any $n$-gon. For example, if we start with $n = 5$, the smallest example is the graph shown in Figure 7. Note that this pentagonal example has ten nodes.
In the same vein, we can start with a triangle and look at the family that starts with the Venn diagram. This family is illustrated in Figure 8. In this case, we obtain knot graphs for 9 nodes, 12 nodes and 18 nodes. For this series we have the number of nodes $v = 3n$ for $n = 1, 2, 3, 4, \ldots$ where the Venn diagram is the case $n = 2$. (In this mode of counting, $n = 1$ is the trefoil knot graph. Since the trefoil has lunes, we start this series at $n = 2$.)

Let $\mu(G)$ denote the number of knot theoretic components of a knot graph $G$. Let $\mu(n)$ denote the number of knot theoretic components of the $n$-th graph in the trefoil series. The component count for this series is given by the formula $\mu(n) = 3$ if $n = 2 + 3k$ and $\mu(n) = 1$ if $n = 3k$ or $n = 3k + 1$. The result is easily verified by induction.
Example 5. At this point we have constructed lune-free knot graphs \((\mu = 1)\) for nodes \(v = 8, 9, 10, 12\) (and some higher order nodes). An 11 node knot graph appears as the modification of the graph for \(v = 10\), as shown in Figure 9. This device produces new lune-free graphs from lune-free graphs, but does not always preserve the knot component count \(\mu\).
Example 6. At this stage we are prepared to exhibit, in Figures 2 and 3, a complete list of all \( \mu = 1 \) lune-free graphs \( G \) with less than or equal to twelve nodes. It is interesting to note that the graph of 9 nodes in Figure 2 can be obtained from the graph of 8 nodes in Figure 2, by the method of Figure 9. There are three distinct lune-free graphs with \( \mu = 1 \) and \( v = 12 \) nodes. Note that in Figure 3, the three distinct lune-free graphs with 12 nodes are labeled \( A \), \( B \) and \( C \). Graphs \( A \) and \( B \) each have only 3 and 4 sided regions, but are not isomorphic graphs. Graph \( C \) has a five-sided region, making it distinct from the other two.

In Figure 10, we show the complete list of lune-free graphs with 13 nodes. We thank Slavik Jablan for the use of his tables of Conway polyhedra [4] for this information.

![Figure 10 - All Lune-Free Graphs with Thirteen Nodes](image-url)
3  Lune-Free Graphs of Every Order

We now turn to the following question: Is there a lune-free knot graph for each number of nodes $v \geq 8$? We shall prove that the answer to this question is yes by an inductive construction.

**Theorem 3.** Let $v \geq 8$. Then there exists at least one lune-free knot graph with $v$ nodes.

**Proof.** First note the inductive construction indicated in Figure 11. In Figure 11, the letters $a, b, c, d, e, f$ denote the number of edges in the boundary of the regions in which they reside. The graph $G'$ is obtained from the graph $G$ as illustrated. In $G'$ the $a, b, e, f$ regions are incremented to $a + 1, b + 1, e + 1, f + 1$ respectively, while the $c$ and $d$ regions are decremented to $c - 1$ and $d - 1$. Two new three-sided regions are produced in the process. Thus the graph $G'$ is lune-free exactly when $c \geq 4$ and $d \geq 4$. In other words, the process $G \rightarrow G'$ induces a lune-free $G'$ whenever $G$ is lune-free and has two adjacent regions with each region having at least 4 sides. When $G$ satisfies this condition, we shall call $G$ admissible. It follows from this construction that if $G$ is admissible, then $G'$ is admissible. If $v(G) = v$, then $v(G') = v + 2$. Hence, for any lune-free admissible $G$ with $v(G) = v$, we obtain an infinite family of lune-free graphs $\{G', G'', G''', \ldots, G^{(n)}, \ldots\}$ with $v(G^{(n)}) = v + 2n$.

Now examine the library of lune-free graphs in Figures 2 and 3. The graphs for $v = 8, 9, 10$ are not admissible. The graph for $v = 11$ is admissible, and each of the graphs for $v = 12$ is admissible. Hence the two inductive processes starting from 11 nodes and from 12 nodes together produce lune-free graphs of all orders greater than 12. Since we have already given examples from order 8 to order 12, this completes the proof of the theorem. □
Figure 11 - Inductive Construction for Increasing Nodes by Two

Figure 12 - Illustration of Inductive Process

Remark. A series of lune-free graphs of orders 11, 13, 15, 17, \ldots obtained by the $G \rightarrow G'$ process is illustrated in Figure 12.

Remark. Other inductive constructions are available that can produce lune-free knot projections from given ones. Some constructions do not depend upon any admissibility condition. One example of such a construction is shown in Figure 13. In this construction, the new graph has order $v + 9$ where $v$ is the number of nodes in the original graph.
Remark. The referee reported the following suggestion, from a colleague, of a possibly simpler proof of Theorem 3: just consider the shadows of some explicit 3-braid closures, namely

\[(\sigma_1\sigma_2\sigma_3)^{4k-1}(\sigma_2\sigma_1)^m\sigma_l^1,\]

with \(k, m, l\) integers satisfying \(k \geq 1, 0 \leq m \leq 5\) and \(0 \leq l \leq 1\).

This is an interesting attempt, but it doesn’t fully work as is. Indeed, denote by \(S(k, m, l)\) the corresponding link graph. A careful analysis reveals that \(S(k, m, l)\) is not a knot graph if \(m = 2, m = 5\) or \(m \equiv l \equiv 1 \mod 3\).

In summary, the proposed construction only gives lune-free knot graphs on \(v = 3(4k - 1) + 2m + l\) nodes if \(v \equiv 3, 4, 5, 9, 10\) or \(11 \mod 12\). Moreover, the graph \(S(k, m, l)\) is tight (see below) only if \(m = l = 0\), i.e. if \(v = 12k - 3\).

Definition. A 4-regular plane graph is said to be tight if for every pair of adjacent faces, one face has 3 edges. Thus tight is synonymous with inadmissible in the terminology above. In the next section, we shall consider questions about tight lune-free knot projections.
4 Tight lune-free knot graphs

In the preceding section, we have shown that there is an admissible lune-free knot graph with \( v \) crossings for every \( v \geq 11 \). Here we address the existence question of tight, i.e. inadmissible, lune-free knot and link graphs. In the above tables, we have already encountered such graphs for \( v = 8, 9 \) and 10. The basic question, that we shall completely answer here, is the following: for which integers \( v \) does there exist a tight lune-free knot or link graph, respectively? For links, the answer is: all integers \( v \geq 8 \) such that \( v \not\equiv \pm 1 \pmod{6} \) can be realized as the number of crossings of such graphs. The answer for knots is the same as for links, with the sole exception of \( v = 12 \), which cannot be realized by a tight lune-free knot graph. However, here is a tight lune-free link diagram with 12 crossings and \( \mu = 3 \) components.

![Figure 14 - A Symmetrical Example with \( v = 12 \).](image)

The aim of this section is to prove the following result.

**Theorem 4.** (1) There exists a tight lune-free link graph with \( v \) crossings if and only if \( v \geq 8 \) and \( v \not\equiv \pm 1 \pmod{6} \). (2) There exists a tight lune-free knot graph with \( v \) crossings if and only if \( v \geq 8, v \not\equiv \pm 1 \pmod{6} \), and \( v \neq 12 \).

For the proof, which will occupy the rest of the section, we shall need several preliminaries. We start by recalling a few things about the medial graph \( D = m(G) \) of a plane graph \( G \). The vertices of \( D \) correspond to the edges of \( G \), and an edge in \( D \) corresponds to a pair of edges in \( G \) which lie
on a common face. Thus, \( D = m(G) \) is a 4-regular plane graph, i.e. a link graph. Figure 15 gives an example. Conversely, every link graph \( D \) arises as the medial graph of a suitable plane graph \( G \), obtained as follows. Color the faces of \( D \) in black and white in checkerboard manner. The vertices of \( G \) correspond to the black faces of \( D \), and there is an edge between two vertices in \( G \) for each common vertex of the corresponding two black faces in \( D \).

Note that the black faces of \( D \) correspond to vertices of \( G \), while the white faces of \( D \) correspond to faces of \( G \). Thus, \( m(G) = m(G^*) \), where \( G^* \) denotes the dual of \( G \). From this description, it follows that a pair of adjacent faces in \( D = m(G) \) corresponds to a pair formed by one vertex and one face incident to it in \( G \). In other words, the faces of \( D = m(G) \) are in one-to-one correspondence with the vertices and faces of the original plane graph \( G \). See Figure 16.
For our purposes here, we do need to address the following two questions. Let $G$ be a simple connected plane graph, and let $D = m(G)$ be its medial graph. Under what conditions on $G$ can one guarantee that:

1. $D$ is a knot graph, rather than a many-component link graph?

2. $D$ is a tight lune-free graph?

In order to answer question (1), we shall make the following definitions. Let $G$ be a simple connected plane graph.

**Definition.** An **angle** in $G$ is a pair $\alpha = (v, F)$ where $v$ is a vertex of $G$ and $F$ is a face of $G$ containing $v$. See Figure 17.
DEFINITION. Two angles $\alpha_1 = (v_1, F_1)$ and $\alpha_2 = (v_2, F_2)$ are adjacent if $v_1, v_2$ are connected by an edge $e$, with $e = F_1 \cap F_2$. Figure 18 is an illustration of adjacency.

One may thus form the *graph $A(G)$ of angles of $G$*. Clearly, each angle is adjacent to exactly two other angles in $G$. It follows that the graph $A(G)$ is regular of degree 2, *i.e.* $A(G)$ is a disjoint union of circuits. We may now give an answer to question (1) above.

**Proposition 5.** Let $G$ be a simple connected plane graph, and let $D = m(G)$ be its medial graph. Then the number of components of any link represented by the link graph $D$ is equal to the number of connected components of the graph of angles $A(G)$.

The proof of this proposition is easy, and will be left to the reader. It suffices to think carefully about the way the medial graph is formed, which essentially amounts to connect adjacent angles by strands. Note that this result was independently observed by Isidoro Gitler as well as Sostenes Lins.

The referee points out that the Tutte polynomial gives another way to compute the number of components $c$ of a link represented by the medial graph of $G$. However, for practical purposes, Proposition 5 is a very efficient way to compute $c$ and actually guided us for the constructions in our proof below of Theorem 4.

Figure 19 illustrates an example of a graph $G$ whose medial graph $D = m(G)$ represents 3-component links. In the Figure, the 3 different sequences of adjacent angles are denoted by $a, b, c$, respectively.
We shall now address question (2). The following definitions will be helpful for this purpose.

**Definition.** Let $F$ be a face in a given plane graph $G$. The *degree* of $F$, denoted $\deg(F)$, is the number of edges of $G$ which are incident to $F$.

**Definition.** A connected plane graph $G$ is *special* if

1. all vertices and faces of $G$ have degree $\geq 3$
2. if a vertex $x$ is incident with a face $F$, then either $\deg(x) = 3$ or $\deg(F) = 3$.

**Lemma 6.** Let $G$ be a simple connected plane graph, and let $D = m(G)$ be its medial graph. Then $D$ is a tight lune-free link graph if and only if $G$ is special in the above sense.

**Proof.** Clearly, the lune-free condition for $D$ corresponds to condition (1) on $G$ as a special graph. Similarly, condition (2) on $G$ exactly means that $D = m(G)$ cannot have two adjacent faces of degree $\geq 4$ each. □

Examples of special graphs include the connected plane cubic graphs, the sphere triangulations (which are the duals of plane cubic graphs), and the wheels. All these graphs are easily seen to have a medial graph which is a tight lune-free link graph. We shall now show that there are no other special graphs.
Theorem 7. Let $G$ be a simple connected plane graph. Then $G$ is special if and only if $G$ is either cubic, or a triangulation, or a wheel.

Proof. We have already observed that cubic graphs, sphere triangulations and wheels are special graphs. Conversely, assume that $G$ is special, and that it is neither cubic nor a triangulation. We shall prove that $G$ is then necessarily a wheel.

Let $L$ (for large) be the set of vertices of degree $\geq 4$, and $S$ the set of vertices which are adjacent to a face of degree $\geq 4$. Neither $L$ nor $S$ is empty, as $G$ is neither cubic nor a triangulation. Moreover, since $G$ is special, all vertices in $S$ have degree exactly 3, i.e. $L \cap S = \emptyset$. Let $d \geq 1$ be the minimal distance between vertices in $L$ and vertices in $S$, and let $x_0 \in L$, $y_0 \in S$ at distance exactly $d$. Let $W$ be the subgraph induced by $x_0$ and its neighbors. Then $W$ is a wheel with center $x_0$, because all faces adjacent to $x_0$ must have degree 3. We will show that $G = W$, thereby completing the proof of the theorem.

Let $x_0, x_1, \ldots, x_{d-1}, x_d = y_0$ be a path in $G$ of length $d$ joining $x_0$ to $y_0$. Of course, $x_1$ belongs to $W$. We must have $\text{deg}(x_1) = 3$, for otherwise $x_1$ would belong to $L$, contradicting the minimality of $d$. Note that all three neighbors of $x_1$ lie in $W$. We will call $z_1, z_2$ the two neighbors of $x_1$ which are distinct from $x_0$. Assume for a contradiction that $d \geq 2$. Then $x_2$ does not belong to $W$, for otherwise there would be a shorter path from $x_0$ to $y_0$. This contradicts the fact that all neighbors of $x_1$ belong to $W$. Therefore, we must have $d = 1$, i.e. $x_1 = y_0$. Let $F$ be the face of degree $\geq 4$ to which $y_0$ is adjacent, and let $C$ be the cycle bounding $F$. Since $G$ is special, all vertices of $C$ have degree exactly 3. Now of course $x_1, z_1$ and $z_2$ belong to $C$. Hence all neighbors of $z_1$ or $z_2$ already lie in $W$. As easily seen, this implies that $C$ is entirely contained in $W$, and actually coincides with the boundary of $W$. Thus, all vertices of $W$ except its center $x_0$ have degree exactly three. Therefore $G = W$, for otherwise, since $G$ is connected, there would be vertices in the boundary of $W$ of degree higher than 3. It follows that $G$ is a wheel, as claimed. Figure 20 illustrates the path combinatorics used in this proof. \[\square\]
Corollary 8. Let $D$ be a tight lune-free link graph. Let $G$ be a simple connected plane graph whose medial graph $m(G)$ is equal to $D$. Then $G$ is either a cubic graph, a sphere triangulation, or a wheel. \(\square\)

We shall now turn to the proof of the main theorem.

Proof of Theorem 4.

(a) The case $v \equiv \pm 1 \pmod{6}$. Our first step will be to prove that there cannot be a tight lune-free link graph with $v$ crossings if $v \equiv \pm 1 \pmod{6}$. Assume the contrary, and let $D$ be such a graph. Let $G$ be a plane graph such that $m(G) = D$. From the hypotheses on $D$ and the above lemma, it follows that $G$ is special graph. Now, by the above theorem, $G$ must be either cubic, or a triangulation, or a wheel. On the one hand, if $G$ is cubic or a triangulation, then the number of edges in $G$ is a multiple of 3. Thus $v \equiv 0$ or $3 \pmod{6}$ in these two cases. On the other hand, if $G$ is a wheel, then it has an even number of edges, whence $v \equiv 0, 2$ or $4 \pmod{6}$ in this third and last case.

(b) The other cases. We shall now construct tight lune-free knot graphs with $v$ crossings, whenever $v \geq 8$, $v \equiv 0, 2, 3$ or $4 \pmod{6}$, and $v \neq 12$. This will be done by induction, and the basic induction step is provided by the following result.

Proposition 9. If there is a tight knot graph with $v$ crossings, then there is a tight knot graph with $v + 12$ crossings.

Proof. Let $G$ be a plane graph with $v$ edges, such that its medial graph $D = m(G)$ is a tight knot graph with $v$ crossings. In particular, $G$ is a special
Since $D$ has at least one (in fact at least 8) 3-face, we may assume, taking the dual of $G$ if necessary, that $G$ has a vertex of degree 3. Now do the move shown in Figure 21 on $G$ near this vertex, calling $H$ the resulting graph.

Clearly, $H$ has $v + 12$ edges, and it is obvious that $H$ is special since $G$ is. Thus, $m(H)$ is a tight link graph with $v + 12$ vertices.

It remains to show that $m(H)$ is in fact a knot graph. To do this, it suffices to compare the paths of adjacent angles in $G$ and in $H$, and observe that the configurations are completely similar. (See Figures 22 and 23.) Since $G$ yields a knot, so will $H$.

In $G$, we picture three partial paths of adjacent angles, namely $(a_1, a_2, a_3)$, $(b_1, b_2)$ and $(c_2, c_3)$. (See Figure 22.)
In $H$, the corresponding partial paths are $(a_1, a_2, \ldots, a_{15})$, $(b_1, \ldots, b_8)$ and $(c_2, \ldots, c_9)$. (See Figure 23.) Thus, $m(H)$ is a knot graph, as asserted. □
We shall now complete the proof of Theorem 4.

(1) The case \( v \equiv 2, 4 \mod 6 \). Let \( n = v/2 \) and \( G = W_n \) be the wheel with \( n + 1 \) vertices. Then \( m(G) \) is a tight lune-free link graph with \( v = 2n \) vertices, and is in fact a knot graph as \( n \) is not divisible by 3.

(2) The case \( v \equiv 0, 3 \mod 6 \). Using the \( v \mapsto v + 12 \) inductive construction in the above proposition, it suffices to exhibit tight lune-free knot graphs with \( v = 9, 15, 18 \) and 24 crossings respectively. These are shown in Figure 25. \( \square \)

![Figure 25 - Plane graphs whose medials are tight lune-free knot graphs with \( v = 9, 15, 18 \) and 24 crossings.](image)

5 Knot graphs with exactly \( k \) lunes.

Let \( k \geq 0 \) be an integer. A knot graph is \( k \)-lune if it has exactly \( k \) lunes. We will show that if \( v \) is large enough with respect to \( k \), then there is a \( k \)-lune knot graph with \( v \) crossings. The case \( k = 0 \) is equivalent to being lune-free.

**Proposition 10.** Let \( k \geq 0 \) be an integer. If \( k \) is even and \( v \geq k + 8 \), or if \( k \) is odd and \( v \geq k + 9 \), then there is a \( k \)-lune knot graph with \( v \) crossings.

**Proof.** The case \( k = 0 \), i.e. the lune-free case, has been dealt with in an earlier section.
(0) Let \( k \) even, \( k \geq 2 \). Let \( v \geq k + 8 \), and let \( D \) be a lune-free knot graph with \( v - k \) crossings, which exists as we already know. Necessarily, \( D \) has a 3-face (in fact at least 8, by Euler’s formula and Theorem 2). Now perform the deformation of this 3-face as in Figure 26.

\[ \text{Figure 26} - \text{3-face deformation} \]

Add \( k \) new crossings to \( D \), with \( k/2 \) lunes above and \( k/2 \) lunes below the desired edge. The new knot graph has exactly \( k \) lunes and \( v \) crossings.

(1) The case \( k = 1 \). Let \( v \geq 10 \) and let \( D \) be a lune-free knot graph with \( v - 2 \) crossings. Then \( D \) admits a large face, say \( F \). Take two non-adjacent edges bounding \( F \), and do the move shown in Figure 27.

\[ \text{Figure 27} - \text{Move on Two Non-Adjacent Edges} \]

The new knot graph on the right has exactly 1 lune and \( v \) crossings, as desired.

(2) Let finally \( k \) be odd, with \( k \geq 3 \). Let \( v \geq k + 9 \), and let \( D \) be a knot graph with \( v - k + 1 \) crossings and exactly one lune, which exists by the above case (1) and the fact that \( v - k + 1 \geq 10 \). Then \( D \) has a 3-face not adjacent to its unique lune. The same modification of this 3-face as in case (0) above, adding \( k - 1 \) new crossings, \( (k - 1)/2 \) lunes above and \( (k - 1)/2 \) lunes below an edge of this 3-face, will produce a knot graph with \( k \) lunes and \( v \) crossings. \( \square \)
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