Improved algorithms for Boolean matrix multiplication via opportunistic matrix multiplication

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Abstract

Karppa & Kaski (2019) proposed a novel type of “broken” or “opportunistic” multiplication algorithm, based on a variant of Strassen’s algorithm, and used this to develop new algorithms for Boolean matrix multiplication, among other tasks. For instance, their algorithm can compute Boolean matrix multiplication in $O(n^{\log_2(6+6/7) \log n}) = O(n^{2.778})$ time. While faster matrix multiplication algorithms exist asymptotically, in practice most such algorithms are infeasible for practical problems. Their opportunistic algorithm is a slight variant of Strassen’s algorithm, so hopefully it should yield practical as well as asymptotic improvements to it.

In this note, we describe a more efficient way to use the broken matrix multiplication algorithm to solve Boolean matrix multiplication. In brief, instead of running multiple iterations of the broken algorithm on the original input matrix, we form a new larger matrix by sampling and run a single iteration of the broken algorithm on it. The resulting algorithm has runtime $O(n^{3 \log \frac{6}{\log 7} (\log n)^{\frac{1}{\log 7}}}) \leq O(n^{2.763})$. We also describe an extension to witnessing Boolean matrix multiplication, as well as extensions to non-square matrices.

The new algorithm is simple and has reasonable constants. We hope it may lead to improved practical algorithms

1 Introduction

In the Boolean matrix multiplication (BMM) problem, we are given matrices $A, B$ of dimension $d_1 \times d_3$ and $d_3 \times d_2$ respectively, whose entries from the Boolean algebra $\{0, 1\}$, and our goal is to compute the $d_1 \times d_2$ matrix $C$ given by

$$C_{ij} = \bigvee_k A_{ik} B_{kj}$$

This problem along with some variants, is an important primitive used in algorithms for transitive closure, parsing context-free grammars, and shortest path problems in unweighted graphs, among other applications. When the matrices are square, we write $n = d_1 = d_2 = d_3$. There is an obvious $O(d_1 d_2 d_3)$ algorithm (the so called “naive algorithm”), which simply iterates over all values $i, j, k$. There are a number of other specialized algorithms based on various combinatorial optimizations; most recently, [14] described an algorithm with runtime roughly $O(n^3 / \log^4 n)$.

There is a closely related problem of Witnessing Boolean Matrix Multiplication (WBMM); here, the goal is not only to determine if there exists $k$ with $A_{ik} = B_{kj} = 1$, but to actually find such $k$. Formally, we wish to generate a “witness matrix” $W$, where $W_{ij}$ should be some (arbitrarily chosen) index $k$ with $A_{ik} = B_{kj} = 1$; if there is no such value then we should $W_{ij} = \emptyset$. This is useful for problems such as finding shortest paths in graphs.

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There is a standard reduction from BMM to standard (integer) matrix multiplication: we compute the matrix product $C = AB$ over the integers, obtaining $C_{ij} = \sum_k A_{ik}B_{kj}$. We then set $C_{ij} = 1$ iff $C_{ij} > 0$.

Alternatively, there is a randomized reduction from BMM to matrix multiplication over the finite field $GF(2)$: each entry of $B$ with $B_{kj} = 1$ is set to zero with probability $1/2$. Then, we compute the matrix product $C = AB$ over $GF(2)$, and we have that $C_{ij} = 1$ with probability $1/2$ whenever $C_{ij} = 1$, whereas $C_{ij} = 0$ with probability one whenever $C_{ij} = 0$. To overcome the randomization step, and obtain the correct matrix $C$ with high probability, we must this procedure for $O(\log(d_1d_2))$ times, and take the disjunction over all runs.

There is also a randomized reduction from WBMM to standard integer multiplication \[13\]; it can be derandomized \[1\], with only small asymptotic overheads in the running time.

The main advantage of reducing BMM to matrix multiplication (over $\mathbb{Z}$ or $GF(2)$) is that, asymptotically, there are a number of very efficient algorithms for the latter problem. These are the so-called “fast matrix multiplication” algorithms. If the matrices are square, then the runtime is $n^{\omega+o(1)}$, where $\omega$ is the linear algebra constant. Currently, the best bound \[7\] is $\omega \leq 2.38$, coming from a variant of Coppersmith-Winograd's algorithm \[4\]. There are also fast algorithms for rectangular matrix multiplication, although less is known about the runtime bounds \[5, 6\].

Unfortunately, nearly all of the fast matrix multiplication algorithms are completely impractical, since the hidden constants are enormous. There are a small handful of algorithms which are efficient in practice; by far the most important of these is Strassen’s algorithm and its variants, which have runtime $O(n\log_2^7)$. Depending on the matrix shape, a few rectangular algorithms may also be practical \[8\]. There is an extensive literature of optimizing and implementing Strassen’s algorithm in various computational platforms, see e.g. \[8, 12, 2\].

For computation over $GF(2)$, a host of additional optimizations and data structures are possible; for example, the famous method of Four Russians. These can often be combined with asymptotically fast algorithms such as Strassen. For example, \[9\] provides an optimized bitsliced algorithm using low-level optimizations at the base levels and Strassen’s algorithm for the high-level iterations.

It seems difficult to make progress on better practical algorithms for matrix multiplication. In \[11\], Karppa & Kaski proposed a novel type of algorithm to break this impasse. Instead of reducing BMM directly to matrix multiplication, one reduces to a randomized “broken” or “opportunistic” form of matrix multiplication. For brevity, we refer to this as pseudo-multiplication. This computes only a subset of the $n^3$ terms present in the full matrix multiplication formula. This pseudo-multiplication can be computed more efficiently via a variant of Strassen’s algorithm. The idea is to compute $L$ independent pseudo-multiplications over the integers, obtaining matrices

$$\tilde{C}^{(\ell)}_{ij} \approx \sum_k A_{ik}B_{kj}$$

for $\ell = 1, \ldots, L$. Each matrix $\tilde{C}^{(\ell)}_{ij}$ only has a small subset of the terms in the full BMM product. We then set $C_{ij} = 1$ iff $C^{(\ell)}_{ij} > 0$ for some value $\ell$. With appropriate choice of $L = \text{poly}(n)$, the runtime is $O(n\log_2^7(6+6/7) \log n) \approx n^{2.776}$ and the estimates are all correct with high probability. See also \[12\] for further details.

In this note, we describe an alternative way to use the pseudo-multiplication algorithm of \[11\]. The idea is that, instead of executing multiple independent pseudo-multiplications, we combine them all into a single larger randomized pseudo-multiplication. We will show the following main results:

**Theorem 1.** For $n$-dimensional square matrices, BMM can be computed in $O((n^3 \log n)^{\frac{\log_6 6}{\log_6 7}}) \leq O(n^{2.763})$ bit operations with high probability, and WBMM can be computed in $O((n^3 \log n)^{\frac{\log_6 6}{\log_6 7}}) \leq O(n^{2.763})$ integer operations with high probability.

We want to emphasize for the asymptotic analysis that the only matrix multiplication algorithms are naive multiplication, Strassen multiplication, and broken Strassen multiplication.
In addition, all the usual optimizations for $GF(2)$ multiplication, including the Four Russians method and bit-slicing, can be used. We have tried throughout to give reasonably tight values for constants and parameters, while also achieving clean and optimal asymptotic bounds.

The new algorithm for BMM is quite simple, and also has a number of advantages from the viewpoint of practical implementations. First, it lends itself in a more straightforward way to non-square input matrices; we will show a more general theorem which describes the runtime scaling for matrices of close-to-square shape. Specifically, the matrix dimensions are allowed to vary by small polynomial factors from each other, with no loss of efficiency. Second, and related to the first point, is that even if the input matrices are non-square, the algorithm itself works with square matrices, and there is considerable freedom to adjust the dimension of these matrices, e.g. to powers of 2. This avoids a number of tedious issues with implementing padding or choosing cut-off points for the recursion. We expect that our new algorithm will be more efficient in practice than Strassen’s algorithm or the original algorithm of [11].

There is one critical difference between our analysis and that of [11]. In [11], the only relevant factor was the gross count of the number of terms computed by the pseudo-multiplication algorithm. In our algorithm, by contrast, we need to analyze the distribution of terms much more carefully. The distribution of the terms is far from uniform, and can lead to significant correlation and skewing the relevant statistics. It will require quite precise and tedious analysis to handle these correlations.

Matrix multiplication obviously has a vast array of applications in computer science. In [11], it was suggested that the pseudo-multiplication could be used for problems such as estimating triangle counts in graphs. We believe that our method could also work here. However, because of the complexity of the statistics, this likely cannot be done in a “black box” way. We leave such applications for future work.

**Discussion: role of randomization.** There is an inherent synergy in combining the randomization inherent in the pseudo-multiplication algorithm with the existing randomized algorithms for BMM or WBMM. When we go from a deterministic to a randomized algorithm, we need to run for more iterations to ensure that the overall output matrix is computed correctly with high probability. This typically causes an additional factor of $\log n$ in the runtime.

The pseudo-multiplication can be viewed as a highly structured, randomized subsampling. The main effect of this is to give better polynomial factors in the runtime. But as a side effect, it can also be used to achieve some of the other types of subsampling and repetition needed for the randomized algorithm, essentially for free. We do not need a separate “isolation” step for WBMM (at least in certain parameter regimes); the solutions are automatically isolated with good probability. We do not need to repeat the Boolean-to-$GF(2)$ sketching multiple times; the matrix already contains multiple trials. (The latter observation was also partially exploited in [11].) Furthermore, the cost of these repetition steps are inherently sublinear in $\log n$, just like fast matrix multiplication algorithms have sublinear cost compared to naive multiplication.

Thus, in addition to saving polynomial factors in the main runtime component, we also save logarithmic factors. Since our algorithm is targeted for the “medium asymptotic” regime (where $n$ is large, but not large enough for algorithms such as Coppersmith-Winograd), the logarithmic terms cannot necessarily be considered negligible compared to the polynomial terms.

## 2 Broken matrix multiplication

To describe our BMM algorithm, we need to discuss the pseudo-multiplication algorithm in depth. We begin by quoting the algorithmic result of [11].

**Theorem 2.** Consider $2 \times 2$ matrices $A, B$ over a ring $R$. Using 14 additions and 6 multipli-
Theorem 3.

In $R$, we can compute the following quantities:

$$
C_{11} = A_{12}B_{21}
$$
$$
C_{12} = A_{11}B_{12} + A_{12}B_{22}
$$
$$
C_{21} = A_{21}B_{11} + A_{22}B_{21}
$$
$$
C_{22} = A_{21}B_{12} + A_{22}B_{22}
$$

This formula differs from the computation of $C = AB$, in that the term $A_{11}B_{11}$ is missing the summand $A_{11}B_{11}$. This computation is achieved by a variant of a Strassen step, except that one of the 7 multiplications is omitted.

We next describe how to iterate pseudo-multiplication. Consider integer matrices $A, B$ of dimension $n = 2^s$. We can identify the integers in the range $[n] = \{0, \ldots, n−1\}$ with binary vectors $\{0, 1\}^s$; thus, we may write $A_{xy}$ for vectors $x, y \in \{0, 1\}^s$. We write $z = x \lor y$ where $z_i$ is zero iff $x_i = y_i = 0$, i.e. the disjunction is done coordinate-wise. Also, we let $|x|$ denote the Hamming weight of a binary vector $x$.

By iterating the computation in Theorem 2 for $s$ levels, we get the following:

**Theorem 3.** Using $7(6^s - 4^s)$ additions and $6^s = n \log_2 6$ multiplications, we can compute the matrix pseudo-product $C = A \otimes B$ defined as follows:

$$
C_{xy} = \sum_{x \lor y \lor z = \bar{1}^s} A_{xz}B_{zy}
$$

where $\bar{1}^s$ denotes the vector of dimension $s$ whose entries are all equal to 1. The computation of $C$ can be performed over any ring.

**Proof.** The arithmetic cost was shown already in [11]. We show the formula for $C$ by induction on $s$; the base case $s = 0$ holds vacuously.

For the induction step, let us represent $A, B, C$ as $2 \times 2$ block matrices $\bar{A}, \bar{B}, \bar{C}$, whose entries are matrices of dimension $2^{s-1}$. By iterating Theorem 2 we have

$$
\bar{C}_{11} = \bar{A}_{12} \otimes \bar{B}_{21}
$$
$$
\bar{C}_{12} = \bar{A}_{11} \otimes \bar{B}_{12} + \bar{A}_{12} \otimes \bar{B}_{22}
$$
$$
\bar{C}_{21} = \bar{A}_{21} \otimes \bar{B}_{11} + \bar{A}_{22} \otimes \bar{B}_{21}
$$
$$
\bar{C}_{22} = \bar{A}_{21} \otimes \bar{B}_{12} + \bar{A}_{22} \otimes \bar{B}_{22}
$$

Now, consider entries $x, y$ whose first coordinate satisfies $x_1 = 1$ or $y_1 = 1$. These correspond to submatrices $\bar{C}_{12}$ or $\bar{C}_{21}$ or $\bar{C}_{22}$. Let us suppose that $x_1 = 1, y_1 = 0$, corresponding to $\bar{C}_{21}$; the other cases are completely analogous. We can write $x = 1x', y = 0y'$ where $x', y' \in \{0, 1\}^{s-1}$ and the notation $1x'$ means the $s$-dimensional vector of 1 followed by $x'$, etc. By induction hypothesis, we have

$$
\bar{A}_{21} \otimes \bar{B}_{11} = \sum_{x' \lor y' \lor z' = \bar{1}^{s-1}} (\bar{A}_{21})_{x'z'}(\bar{B}_{11})_{yz'} = \sum_{x' \lor y' \lor z' = \bar{1}^{s-1}} A_{(1x')(0z')(0y')}.
$$

A similar formula holds for the other term $\bar{A}_{22} \otimes \bar{B}_{21}$, and so

$$
C_{xy} = \sum_{x' \lor y' \lor z' = \bar{1}^{s-1}} A_{(1x')(0z')(0y')} + \sum_{x' \lor y' \lor z' = \bar{1}^{s-1}} A_{(1x')(1z')(0y')}.
$$

Now note that the first coordinate of $x \lor y \lor z$ is always equal to one, hence $x' \lor y' \lor z' = \bar{1}^{s-1}$ iff $x \lor y \lor z = \bar{1}^s$. Thus, this is precisely

$$
C_{xy} = \sum_{x \lor y \lor z = \bar{1}^s} A_{xz}B_{zy}.
$$
as claimed.

Next, let us consider the case $x_1 = y_1 = 0$, corresponding to $C_{11}$. We can write $x = 0x', y = 0y'$. By induction hypothesis, we have

$$A_{11} \otimes B_{11} = \sum_{x' \vee y' \vee z' = \Gamma - 1} (A_{11})_{x'z'} (B_{11})_{z'y'} = \sum_{x' \vee y' \vee z' = \Gamma - 1} A_{(0x')(1z')B(1z')(0y')}$$

So

$$C_{xy} = \sum_{x' \vee y' \vee z' = \Gamma - 1} A_{x(1z')} B_{(1z')y}$$

Now consider a vector $z \in \{0, 1\}^s$. The term $A_{xz} B_{zy}$ appears here precisely if $z = 1z'$ and $x \vee y \vee z = \bar{1}^s$. (In order for the first coordinate of $x \vee y \bar{z}$ to be equal to one, we must have $z_1 = 1$.) So again we have

$$C_{xy} = \sum_{x \vee y \vee z} = \bar{1}^s$$

as desired.  

In particular, this implies the following result (which was the only fact directly needed in the analysis of [11]):

**Corollary 4.** The pseudo-product $C = A \otimes B$ contains $(7/8)^s$ of the summands in the full matrix product $C = AB$.

**Proof.** There are precisely $7^s$ triples of vectors $x, y, z \in \{0, 1\}^s$ with $x \vee y \vee z = \bar{1}^s$.  

Note that [11] includes an additional randomization step, where the entries of the matrices are randomly permuted at each level. We omit this step, since we will later include more extensive randomization in the overall algorithm.

In implementing fast matrix multiplication algorithms, it is typically more efficient to use naive multiplication (or some other method below some “base case”). For concreteness, let us suppose that our matrix corresponds to square matrices of dimension $n = b2^s$. That is, given square matrices of dimension $n = b2^s$, we apply $s$ iterations of pseudo-multiplication and switch to ordinary matrix multiplication on the resulting $b \times b$ submatrices. Equivalently, we apply Theorem 3 over the ring $R$ of $b \times b$ matrices. (It would also be possible to use non-square matrices in the base case; for simplicity, we do not discuss this.)

If we are working in the finite field $GF(2)$, then the base case would typically involve switching to a Four Russians method and/or bitwise operations. These effects are very large in practice, possibly more important than the asymptotic gains from a fast matrix multiplication. From a theoretical point of view, $b$ is typically viewed as a constant. However, in practical implementations, it may be very large. For instance, in the implementation of [9], they choose $b \approx 2^{11}$ roughly matching the L2 cache size. We will explicitly keep track of dependence on $b$ in our parameter choices.

In analyzing the pseudo-multiplication algorithm, this base case does not just affect the computational cost, but also affects the resulting matrix computed. Hence we will need to keep track of it more carefully. For matrices of dimension $n = b2^s$, we can view each integer $x \in [n]$ as equivalent to an ordered pair $(x', x'')$ where $x' \in \{0, 1\}^s$ and $x'' \in [b]$. When we compute the resulting matrix $C = A \otimes_b B$, then it satisfies

$$C_{xy} = \sum_{z=(x',x'')} A_{xz} B_{zy}$$

Since this comes up repeatedly in the calculations, we write simply $C = A \otimes B$ (with dependence on $n$ and $b$ understood). For integers $x, y, z \in \{1, \ldots, n\}$, we write compactly $x \vee y \vee z = \bar{1}$ iff $x' \vee y' \vee z' = \bar{1}^s$. Thus, for instance, we can write more compactly

$$C_{xy} = \sum_{x \vee y \vee z = \bar{1}} A_{xz} B_{zy}$$
3 Boolean matrix multiplication

Let us suppose we are given Boolean matrices $A, B$ of dimensions $d_1 \times d_3$ and $d_3 \times d_2$. Let us define $\psi_1 = d_1 + d_2 + d_3, \psi_2 = d_1d_2 + d_1d_3 + d_2d_3, \psi_3 = d_1d_2d_3$. Our alternative algorithm can be described as follows:

**Algorithm 1:** Algorithm to estimate $C = AB$

1. Choose some parameter $m = 2^b$, where $b$ is the base case of the pseudo-multiplication algorithm
2. Draw random functions $f_i : [m] \to [d_i]$ for $i = 1, 2, 3$
3. Draw a uniformly random $m \times m$ binary matrix $D$
4. Form $m \times m$ matrix $A$ given by $A_{xz} = A_{f_1(x)f_2(z)}$
5. Form $m \times m$ matrix $B$ given by $B_{zy} = B_{f_3(z)f_4(y)}D_{zy}$.
6. Compute pseudo-product $C = A \otimes B$ over field $GF(2)$
7. Return estimated matrix $\hat{C}$ given by

$$\hat{C}_{ij} = \bigvee_{x \in f^{-1}(i), y \in f_2^{-1}(j)} \bar{C}_{xy}$$

The runtime for this process is $O(b^36^s)$ bit operations, where the constant term includes all optimizations for the base case. The values of $s$ and $B$ can be chosen so that the matrix dimension $m$ has good padding properties, e.g., $m$ is highly divisible by 2. The algorithm here is very simple and straightforward, but the analysis of correctness, and the choice of parameter $m$, will require some more involved analysis.

We begin with the following easy observation.

**Proposition 5.** If $C_{ij} = 0$, then $\hat{C}_{ij} = 0$ with probability one.

**Proof.** Consider an entry $\bar{C}_{xy}$ for $x \in f_1^{-1}(i), y \in f_2^{-1}(j)$. We have

$$\bar{C}_{xy} = \sum_{z : x \lor y \lor z = \bar{1}} \bar{A}_{xz} \bar{B}_{zy} D_{zy}$$

For each summand $z$, we have $\bar{A}_{xz} \bar{B}_{zy} = A_{ik}B_{kj}$ where $k = f_3(z)$. Since $C_{ij} = 0$, all such terms are zero. Hence $\bar{C}_{xy} = 0$. Since this holds for all such $x, y$, we have $\hat{C}_{ij} = 0$ as well. \hfill $\square$

As an initial guess, we may guess that for any triple $i, j, k$, there are roughly $m^3/\psi_3$ entries $(x, y, z)$ which are preserved in the pseudo-multiplication and which get mapped to $i, j, k$ in forming $\bar{C}$, i.e. $f_1(x) = i, f_2(y) = j, f_3(z) = k$. Thus, as long as $m^3 \gg \psi_3$, it would seem that $\bar{C}$ should contain all the terms in $C$ and hence it should be a good approximation, i.e. $C = \bar{C}$ with high probability. However, to analyze it formally, we need to take account of some potentially problematic dependencies between the entries in the matrix. This was not an issue encountered in the original analysis of [1], which only worked expectation-wise.

For any pair $i, j$ let us denote by $K_{ij}$ the set of values $k \in [d_3]$ with $A_{ik}B_{kj} = 1$. We consider some pairs $i^*, j^*$ with $C_{i^*, j^*} = 1$, so that $K_{i^*, j^*} \neq \emptyset$. For brevity let us write $K^* = K_{i^*, j^*}$ for this fixed pair $i^*, j^*$ and we also fix some arbitrary element $k^* \in K^*$. Let us say that a triple $(x, y, z) \in [m]^3$ is good if the following conditions hold:

(C1) $f_1(x) = i^*$ and $f_2(y) = j^*$ and $f_3(z) = k^*$
(C2) $x \lor y \lor z = \bar{1}$

To show the existence of a good triple, we will use Janson’s famous inequality in probability theory [1]. We summarize it as follows.

**Theorem 6 ([1]).** Suppose that $X_1, \ldots, X_N$ are independent Bernoulli variables, and suppose $E$ is a collection of “principal events”, i.e. events of the form $E \equiv X_{i_1} = 1 \land X_{i_2} = 1 \land \cdots \land X_{i_3} = 1$ for given values $i_1, \ldots, i_3 \in [N]$. For events $E, E' \in E$, we write $E \sim E'$ if $E \neq E'$ and $E$ and $E'$ are mutually dependent, i.e. they both depend on some common variable $X_i$. 
Let $\mu = \sum_{E \in \mathcal{E}} \Pr(E)$ and let $\Delta = \sum_{E \in \mathcal{E}, E' \in \mathcal{E}, E \sim E'} \Pr(E \cap E')$. Then
\[
\Pr\left( \bigcap_{E \in \mathcal{E}} E \right) \leq e^{-\mu + \Delta/2}
\]

To apply Theorem 6 for each tuple $u = (x, y, z)$, we have an associated event $E_u$ that property (C1) holds, i.e. $f_1(x) = i^*, f_2(y) = j^*$, $f_3(z) = k^*$. Since we have fixed $i^*, j^*, k^*$, we can define Bernoulli variables $X_{(1,x)}, X_{(2,y)}, X_{(3,z)}$ for each value $i, j, k$, namely $X_{(1,i)} = 1$ iff $f_1(x) = i^*$, and so on. The events $E_u$ are all principal events on the underlying Bernoulli variables, where the ground set $N$ has size $3m$. Also, for tuples $u = (x, y, z), u' = (x', y', z')$, note that $E_u \sim E_{u'}$ iff $x = x'$ or $y = y'$ or $z = z'$.

**Proposition 7.** In the language of Theorem 6 there holds
\[
\mu = \frac{b^37^s}{\psi_3}, \quad \Delta \leq \psi_1 b^525^s + \psi_2 b^413^s \psi_3^{-1}
\]

**Proof.** We have
\[
\mu = \sum_u \Pr(E_u), \quad \Delta = \sum_u \sum_{u' \sim u} \Pr(E_u \cap E_{u'})
\]

where $u, u'$ range over triples $(x, y, z)$ satisfying property (C2). There are precisely $m^3 = b^37^s$ such triples $(x, y, z)$, and each satisfies (C1) with probability $\frac{1}{d_1d_2d_3} = 1/\psi_3$. This shows the formula for $\mu$. To calculate the bound on $\Delta$, we consider a number of cases. Let $u' = (x', y', z')$, and let us denote $r = 1/\psi_3$ throughout.

1. $x = x', y \neq y', z \neq z'$. In this case, the probability that $E_u$ and $E_{u'}$ both hold is precisely $1/(d_1d_2d_3) = d_1r$. To enumerate the relevant tuples $u, u'$, we claim that, for each coordinate $i \leq s$, there are 25 choices for the bits $x_i, y_i, y'_i, z_i, z'_i$. For, if $x_i = 0$, then there are 3 choices for $(y_i, z_i)$ and likewise 3 choices for $(y'_i, z'_i)$; otherwise, if $x_i = 1$, then we may select bits $y_i, y'_i, z_i, z'_i$ arbitrarily. So, there are $b^525^s$ choices for $u, u'$ altogether. The total contribution to $\Delta$ is $b^525^sd_1r$.

2. $x \neq x', y = y', z \neq z'$ or $x = x', y \neq y', z = z'$. These are completely analogous to the previous case; they contribute $b^525^sd_2r$ and $b^525^sd_3r$ respectively.

3. $x = x', y = y', z \neq z'$. In this case, the probability that $E_u$ and $E_{u'}$ both hold is precisely $1/(d_1d_2d_3) = d_1d_2r$. To enumerate the relevant tuples $u, u'$, we can check that for each coordinate $i \leq s$, there are precisely 13 choices for the bits $x_i, y_i, z_i, z'_i$. So there are $b^413^s$ choices for $u, u'$ and their total contributes to $\Delta$ is $b^413^sd_1d_2r$.

4. $x = x', y \neq y', z = z'$ or $x \neq x', y = y', z = z'$. Again, these are analogous to the previous case, contributing $b^413^sd_1d_3r$ and $b^413^sd_2d_3r$ to $\Delta$. \hfill \square

We obtain the following important lemma on the number of good triples, which we state in a somewhat general form.

**Lemma 8.** Let $\kappa > 0$ be a given parameter, where $7^sb^3 \geq (1 + o(1))\psi_3\kappa$ and $(25/7)^sb^2 \ll \psi_3/\psi_1$ and $(13/7)^sb \ll \psi_3/\psi_2$. Then for any integer $r \leq \kappa$, there is a probability of at least
\[
1 - e^{-r^r(\kappa/r)^r}
\]

that there exist $r$ good tuples $(x_1, y_1, z_1), \ldots, (x_r, y_r, z_r)$ where values $y_1, \ldots, y_r$ are all distinct.

**Proof.** In light of Proposition 7 our assumptions ensure that if we applied Janson’s inequality to the events $E_u$, then we would have $\Delta \ll \mu$ and hence $\mu - \Delta/2 \geq (1 - o(1))\mu \geq \kappa$.

Our plan is to apply Janson’s inequality, not the events $E_u$ themselves, but to events $E_u'$ determined by an attenuation step. Let us consider the following additional random process: we mark each $y \in [m]$ independently with some probability $p$ (to be determined), and we only consider candidates $(x, y, z)$ where $y$ is marked. Note that we are already using the hash
functions $f_i : [d_i] \to [m]$ to select certain values $x, y, z \in [m]$, and so this effectively changes the marking probability from $1/d_2$ to $p/d_2$ for the $y$ coordinates. The result is to change $\mu$ and $\Delta$ to $\mu' = p\mu$ and $\Delta' \leq p\Delta$.

Let us define event $B$ that there are no good triples $(x, y, z)$ where $y$ is marked. By Janson’s inequality applied to the attenuated events, $B$ has probability at most $e^{-\mu' + \Delta'/2} \leq e^{-p(\mu + \Delta/2)}$.

By hypotheses of the lemma, we get $\Pr(B) \leq e^{-p\kappa}$.

On the other hand, let us define $B'$ to be the event that, among all good triples, there are fewer than $r$ distinct values $y$. Note that $B'$ does not depend on the randomness involved in the attenuation step. Whenever $B'$ holds, the event $B$ will also hold as long as all the $y$ values involved in the good triples get unmarked. Since there are fewer than $r$ of them, this has probability at least $(1 - p)^r$. Overall, we have $\Pr(B \mid B') \geq (1 - p)^r$.

Combining these estimates, we have

$$\Pr(B')(1 - p)^r \leq \Pr(B) \leq e^{-p\kappa}$$

Setting $p = 1 - r/\kappa \in [0, 1]$, this implies

$$\Pr(B') \leq e^{-\kappa}(1 - p)^r = e^{-\kappa}(\kappa/r)^r$$

**Proposition 9.** Let $\delta > 0$ be sufficiently small. Suppose that $7^3b^3 \geq 3\psi_3 \log(1/\delta)$ and $(25/7)b^2 \ll \psi_1/\psi_1$ and $(13/7)b^2 \ll \psi_3/\psi_2$. Then, with probability at least $1 - \delta$, there holds $C_{x_jy_i} = 1$

**Proof.** Let the random variable $R$ denote the number of distinct $y$ values with good triples, i.e. there are good triples $(x_1, y_1, z_1), \ldots, (x_R, y_R, z_R)$ with $y_1, \ldots, y_R$ distinct. Suppose we condition on all the randomness involved in the functions $f$, so that $R$ is also revealed, and $(x_1, y_1, z_1), \ldots, (x_R, y_R, z_R)$ are good. At this point, let us reveal the randomness in the matrix $D$. We claim that, for each $\ell = 1, \ldots, R$, there is a probability of at least $1/2$ that $C_{x_\ell y_\ell} = 1$, and this holds independently for each $\ell$.

For, in forming $C_{x_\ell y_\ell}$, we are adding in $D_{x_\ell y_\ell}$ as well as potentially other entries $D_{x_i y_i}$. These are all independent random bits, and hence their sum over $GF(2)$ is equal to one with probability precisely $1/2$. Furthermore, since $y_1, \ldots, y_R$ are distinct, these all involve distinct entries of $D$.

Also, if $C_{x_\ell y_\ell} = 1$ for any such $\ell$, then since $f_1(x_\ell) = i^*$ and $f_2(y_\ell) = j^*$, this yields $C_{i^* j^*} = 1$. Thus, the probability that $C_{i^* j^*} = 0$, conditional on $R$, is at most $2^{-R}$. Taking expectations over random variable $R$ and using integration by parts, we get for any $\kappa > 0$:

$$\Pr(C_{i^* j^*} = 0) \leq E[2^{-R}] = 2^{-\kappa} + \sum_{r=1}^{\kappa} \Pr(R < r)2^{-r}/2.$$  

By applying Lemma 9 with $\kappa = 3\log(1/\delta)$ for $r \leq \kappa$, we get

$$\Pr(C_{i^* j^*} = 0) \leq 2^{-\kappa} + \frac{1}{2} \sum_{r=1}^{\kappa} e^{r-\kappa}(\frac{\kappa}{2r})^r.$$  

Here, the first term $2^{-\kappa}$ is $o(\delta)$. Simple calculus shows that the summand is maximized at $r = \kappa/2$, and the overall sum is at most $\kappa e^{-\kappa/2} = o(\delta)$.

As we have seen, naive matrix multiplication has runtime $O(\psi_3)$. By comparison, we get the following result:

**Theorem 10.** Suppose that $\psi_3^{0.345} \geq \psi_1$ and $\psi_3^{0.681} \geq \psi_2$ and $b$ is constant. Then with appropriate choice of parameters, we can compute $C = AB$ correctly with probability $1 - 1/poly(\psi_3)$ in runtime $O((\psi_3 \log \psi_3)^{\log 2/\log 5})$.  

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Thus, we say that entry \( C_{ij} \) analysis in all cases will be the same, so we do not discuss this further.

Square matrices satisfy \( \psi_1 = \Theta(\psi_3^{1/3}) = \Theta(n) \) and \( \psi_2 = \Theta(\psi_3^{2/3}) = \Theta(n^2) \). In particular, for square matrices, then we can compute \( C \) with probability \( 1 - 1/\text{poly}(n) \) in time \( O(n^3 \log n) \). As long as the matrices are close enough to square (the dimensions differ by small enough polynomial factors), our algorithm can still be used to compute \( C \) efficiently.

The conditions on \( \psi_1 \) and \( \psi_2 \) come from potential correlations in the statistics; the bounds in Theorem 10 are not tight. It appears to be quite difficult to obtain optimal estimates which are also useful for medium-scale problems (where we cannot assume that only leading-order asymptotic terms are relevant). In Appendix A we use some more advanced analysis to obtain tighter polynomial bounds on \( \psi_1 \) and \( \psi_2 \); however, due to potentially large hidden constant factors, they may not be too useful in practice.

4 Witnessing Boolean matrix multiplication

Again, consider given Boolean matrices \( A, B \) of dimensions \( d_1 \times d_3 \) and \( d_3 \times d_2 \). Our goal now is to generate a witness matrix \( W \) for the product \( AB \). We will do this in two stages. First, we compute a pseudo-product which provides an estimate of \( W \) with some moderate error and which only works in a limited parameter range; we then discuss how to convert this into a full algorithm.

As a starting point, let us consider the following procedure for given parameters \( s, b \):

**Algorithm 2:** Algorithm to estimate witness matrix

1. Initialize \( W_{ij} \leftarrow \emptyset \) for all \( i, j \).
2. Draw random functions \( f_i : [m] \to [d_i] \) for \( i = 1, 2, 3 \).
3. Form \( m \times m \) matrix \( A \) given by \( A_{x,y} = A_{f_i(x)f_j(z)} \).
4. Form \( m \times m \) matrix \( B \) given by \( B_{x,y} = B_{f_i(x)f_j(z)}D_{x,y} \).
5. Compute pseudo-product \( C = A \otimes B \).
6. For each pair \( x, y \in [m] \) do:
   7. Let \( i = f_1(x), j = f_2(y), k = C_{xy} \).
   8. If \( A_{ik}B_{kj} = 1 \) then set \( W_{ij} = k \).

The pseudo-multiplication here can be performed either over the integers, over the ring \( \mathbb{Z}_m \), or over the finite field \( GF(2^{\log_2 d_3}) \). Due to the form of the matrices \( A, B \), the latter would amount to \( \log_2 d_3 \) parallel multiplications over \( GF(2) \), and is probably the most efficient. The analysis in all cases will be the same, so we do not discuss this further.

It is clear that if \( K_{ij} = \emptyset \) the algorithm will correctly set \( W_{ij} \leftarrow \emptyset \) with probability one. Thus, we say that entry \( i, j \) is valid if either \( K_{ij} = \emptyset \) or \( W_{ij} \) is set to a value \( k \in K_{ij} \); this will occur precisely when \( C_{xy} \in K_{ij} \) where \( f_1(x) = i, f_2(y) = j \). We show next that \( i, j \) is valid with high probability as long as \( |K_{ij}| \) is sufficiently small. Then, we use an additional subsampling step to handle larger value of \( |K_{ij}| \).

**Proposition 11.** Consider any fixed entry \( i, j \), and let \( \Gamma = |K_{ij}| \). Suppose that the following inequality holds:

\[
b^2(25/7)^\Gamma \left( \Gamma(d_1 + d_2) + d_3 \right) + b(13/7)^\Gamma \left( \Gamma d_1 d_2 + d_3(d_1 + d_2) \right) \ll \psi_3
\]
Then $i, j$ is valid with probability at least

$$(1 - e^{-(1/2-o(1))7^3b^3\Gamma/\psi_3})(1 - \Gamma/d_3)b^{26\epsilon/7}$$

Proof. First, let us estimate the probability that there is at least one triple $(x, y, z)$ satisfying the following three properties:

(B1) $f_1(x) = i, f_2(y) = j, f_3(z) \in K_{ij}$

(B2) $x \lor y \lor z = \bar{1}$

(B3) $|x \lor y| \leq 6s/7$

In this proof, let us write $K = K_{ij}$ for brevity. We use Janson’s inequality to estimate the probability. The calculations are very similar to Proposition 7 with two changes. First, instead of requiring that $f_3(z) = k^*$ for some fixed value $k^*$, we allow any value of $k \in K$. Thus, the probability that any given triple satisfies (B1) is $\Gamma/\psi_3$ instead of $1/\psi_3$. Second, to estimate $\mu$, we observe that if we choose a triple $(x, y, z)$ satisfying (B1) uniformly at random, then its weight $|x \lor y|$ is a Binomial random variable with mean $6s/7$; hence, given that property (B2) is satisfied, property (B3) is also satisfied with probability tending to $1/2$. So

$$\mu \geq (1/2 - o(1))b^{7s\epsilon/3}\Gamma/\psi_3$$

Likewise, to estimate $\Delta$, we use a slightly crude bound (ignoring property (B3)), getting

$$\Delta \leq \frac{b^25^s(\Gamma^2d_1 + \Gamma^2d_2 + \Gamma d_3) + b^413^s(\Gamma^2d_1d_2 + \Gamma d_2d_3 + \Gamma d_1d_3)}{\psi^2_3}$$

Our hypothesis ensures that $\Delta \ll \mu$, and so the probability of at least one triple is at least $e^{-\mu + \Delta/2} \geq e^{-\mu(1-o(1))} = e^{-(1/2-o(1))7^3b^3\Gamma/\psi_3}$.

Next, we argue that if there is such a triple $(x, y, z)$, then with good probability there is no other triple of the form $(x, y, z')$. To show this, let us first reveal all the randomness in the functions $f_1, f_2$, and slowly reveal the random values $f_3(z)$ for increasing values of $z$. Suppose that, at some point in this process, there first holds $f_1(x) = i, f_2(y) = j, f_3(z) \in K$ for some values $x, y, z$ satisfying (B2), (B3). So $f_3(z') \in K$ for all the previously revealed values $z'$.

At this point, the pair $x, y$ will satisfy the desired condition as long as $f_3(z'') \in K$ for the unrevealed values $z''$ which also satisfy $x \lor y \lor z'' = \bar{1}$. Since $|x \lor y| \leq 6s/7$, there are at most $b^{26s/7}$ such values $z''$. The probability that none of them lie in $K$ is then at least

$$(1 - \Gamma/d_3)b^{26\epsilon/7}$$

Assuming these events hold, the pair $x, y$ has precisely one value $z$ with $x \lor y \lor z = \bar{1}$, such that $f_1(x) = i, f_2(y) = j$ and $f_3(z) = k$ for $k \in K$. So $\bar{C}_{xy} = k$. So $W_{ij}$ is set to $k$, unless it was set to some other valid value at some other pair $x', y'$.

 Proposition 12. Suppose that $\psi_3^{0.345} \geq \psi_1$ and $\psi_3^{0.681} \geq \psi_2$ and $b$ is constant and $\psi_3$ is sufficiently large. For an entry $i, j$ define

$$s_{ij} = \lfloor \log_7(b^{-3}\psi_3 \log \psi_3) - 2 \rfloor - \lceil (7/6) \log_2 \Gamma \rceil$$

If $s_{ij} \geq 0$ and we run Algorithm 1 with parameter $s = s_{ij}$, then $i, j$ is valid with probability at least

$$(1 - e^{-(1/2-o(1))7^3b^3\Gamma/\psi_3})(1 - \psi_3^{-0.027})$$

Proof. Let us first verify the inequalities of Proposition 11 hold for these parameters. For this purpose, we may ignore the dependence on $b$ as well as the ceiling functions used to define $s_{ij}$, since both of these only affect constant factors. With this proviso, we get

$$s = \log_7(\psi_3 \log \psi_3) - (7/6) \log_2 \Gamma$$
and so
\[
\frac{b^2(25/7)^s\left(\Gamma(d_1 + d_2) + d_3\right) + b(13/7)^s\left(\Gamma d_1 d_2 + d_3(d_1 + d_2)\right)}{\psi_3} \\
\leq \left((25/7)^s \psi_1 + (13/7)^s \psi_2\right) \cdot O(\Gamma/\psi_3) \\
\leq \left(\psi_3 \log \psi_3\right)^0.6542 \psi_1 \Gamma^{-2.14} + (\psi_3 \log \psi_3)^0.3182 \psi_2 \Gamma^{-1.04} \cdot O(\Gamma/\psi_3)
\]

Our condition on $\psi_1, \psi_2$ ensure that this is $o(1)$ for $\psi_3$ sufficiently large. Thus, the probability of success is

\[
(1 - e^{-1/(2-o(1))7^{s/3}\Gamma/\psi_3})(1 - \Gamma/d_3)^{2b^{6s}/7}
\]

Let us define $t = \Gamma/d_3 \cdot b^{26s}/7$. Because of the ceiling function used to define $\hat{s}_{ij}$, we have

\[
t \leq \Gamma/d_3 \cdot b^{26s} / (b^{-3} \psi_3 \log \psi_3) - (7/6) \log_3 \Gamma \leq b^{0.09} (\psi_3 \log \psi_3)^{0.30534} / (2d_3)
\]

In particular, this expression does not depend on $\Gamma$. Since $d_3 \geq \psi_3^{1/3}$, we have $t \ll \psi_3^{-0.027}$, and hence $(1 - \Gamma/d_3)^{2b^{6s}/7} \geq 1 - \psi_3^{-0.027}$ for large $\psi_3$.

We are now ready to describe our main algorithm for WBMM:

**Algorithm 3:** Algorithm for WBMM

1. Run the algorithm for BMM, obtaining matrix $\hat{C} = AB$
2. Initialize $W \leftarrow \emptyset$
3. for $s = 0, \ldots, s_{\text{max}} = \lceil \log_7 b^{-3} \psi_3 \log \psi_3 \rceil$ do
4. Run $t = 20 \cdot 4^{s_{\text{max}} - s}$ independent trials of Algorithm 2 obtaining matrices $\hat{W}^1, \ldots, \hat{W}^t$
5. For each value $t$ and entry $i, j$ with $\hat{W}^t_{ij} \neq \emptyset$ set $W_{ij} \leftarrow \hat{W}^t_{ij}$
6. Choose a random permutation $\pi : [d_3] \rightarrow [d_3]$
7. foreach pair $i, j$ with $\hat{C}_{ij} = 1$ and $W_{ij} = \emptyset$ do
8. for $k = 1, \ldots, d_3$ do
9. if $A_{\pi(k)} B_{\pi(k)} = 1$ then set $W_{ij} \leftarrow \pi(k)$ and break.

**Theorem 13.** Suppose that $\psi_3^{0.345} \geq \psi_1$ and $\psi_3^{0.681} \geq \psi_2$ and $b$ is constant. Then Algorithm 3 computes $W$ correctly with high probability. Its expected runtime is

\[
O\left(\left(\psi_3 \log \psi_3\right)^{\frac{\log 6}{\log 7}}\right)
\]

**Proof.** Let us denote $\alpha = \psi_3 \log \psi_3$ for brevity.

By Theorem 10 the matrix $\hat{C}$ is computed correctly with probability $1 - 1 / \text{poly}(\psi_3)$. Then this occurs, then $W$ is correct with probability one. Foir, any entry with $C_{ij} = 1$ is either set to a valid value during some iteration of the loop at line 5, or is set to a valid value later at line 9.

Now let us analyze the runtime. By Theorem 10 the runtime of line 1 is $O(\alpha^{\frac{\log 6}{\log 7}})$. Each iteration $s$ of the loop at line 4 has runtime $O(6^s \cdot 4^{s_{\text{max}} - s})$; overall, these sum to $O(6^{s_{\text{max}}}) = O(\alpha^{\frac{\log 6}{\log 7}})$. To analyze lines 7–9, consider any pair $i, j$ with $\hat{C}_{ij} = 1$ and let $\Gamma = |K_{ij}|$. Since the BMM algorithm has one-sided error, necessarily $\hat{C}_{ij} = 1$ and so $\Gamma > 0$. Let $p$ denote the probability that $W_{ij} = \emptyset$ after lines 3–5. Since permutation $\pi$ is chosen uniformly at random, the expected time until it first finds some value $k \in K_{ij}$ is $d_3 / \Gamma$, and hence the overall expected runtime for $i, j$ is $O(T)$ where we define

\[
T = pd_3 / \Gamma
\]

There are a few cases to consider. If $\hat{s}_{ij} < 0$, then we must have

\[
\Gamma \geq \Omega(\alpha^{0.305} / b^{0.916})
\]

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and hence $T \leq O(d_3/\psi_3^{0.305})$.

Otherwise, if $\hat{s}_{ij} > 0$, then each trial at line 4 of iteration $s = \hat{s}_{ij}$ correctly sets $W_{ij}$ with probability at least $(1 - e^{-(1/2-o(1))(7^s\Gamma)/\psi_3})(1 - \psi_3^{-0.027})$. The probability that it fails in $t' = 4^{\max-s}$ iterations is then at most

$$(e^{-(1/2-o(1))(7^s t'\Gamma)/\psi_3} + \psi_3^{-0.027})^{t'} \leq e^{-(1/2-o(1))(7^s t'\Gamma)/\psi_3}(1 - \psi_3^{-0.027}) + \psi_3^{-0.027}$$

We estimate here

$$7^s b^3 t' \Gamma/\psi_3 = 7^s \max b^3/\psi_3 \cdot (4/7)^{\max-s} \Gamma$$

Now because of the ceiling function used to define $s_{\max}$, we have $7^s \max b^3/\psi_3 \geq (1/49) \log \psi_3$. Also, by the definition of $\hat{s}$, we have $s_{\max} - s \leq (7/6) \log_2 \Gamma$, and hence

$$\Gamma (4/7)^{\max-s} \geq (4/7)^{7/6 \log_2 \Gamma} \cdot \Gamma \geq (1/49) \log \psi_3 \geq 1$$

In particular, we have

$$e^{-(1/2-o(1))(7^s t'\Gamma)/\psi_3} \geq 0.01 \log \psi_3$$

for large enough $\psi_3$. Overall, the failure probability in $t'$ trials is at most

$$\psi_3^{-0.027} + \psi_3^{-0.01} (1 - \psi_3^{-0.027})$$

So in $t = 20t'$ trials, the overall failure probability $p$ gets reduced to $p \leq \psi_3^{-0.1}$. In this case, we have $T \leq \psi_3^{-1} d_3/\Gamma$

Putting the two cases together, we see that

$$T \leq \psi_3^{-0.1} d_3$$

holds for every pair $i, j$. Summing over all $i, j$, the overall runtime is at most $O(d_1 d_2 \cdot \psi_3^{-0.1} d_3) = \psi_3^{-0.1}$. This in turn is at most $O(\alpha^{12/7})$ since $\frac{\log 3}{\log 7} \geq 0.92$. \qed

5 Acknowledgments

Thanks to Richard Stong for suggesting the proof of Lemma 14.

A Improved asymptotic analysis for non-squareness

Because we are mostly interested in asymptotics here, we assume that $b = 1$ throughout for simplicity. Let us again fix triple $i^*, j^*, k^*$ with $A_{i^* k^*} = B_{k^* i^*} = 1$ and we want to estimate the probability of a good triple, satisfying properties (C1), (C2). To get together bounds, we restrict our attention to a limited set of triples, namely, those satisfying two additional conditions:

(C3) $|x| \leq 4s/7$ and $|y| \leq 4s/7$ and $|z| \leq 4s/7$

(C4) $|x \lor y| \leq 6s/7$ and $|x \lor z| \leq 6s/7$ and $|y \lor z| \leq 6s/7$

We say that a triple $(x, y, z)$ satisfying conditions (C2), (C3), (C4) is a candidate; this does not depend on the randomness, in particular, the value of the functions $f$.

Lemma 14. There are $\Omega(7^s)$ candidates

Proof. There are precisely $7^s$ triples $(x, y, z)$ satisfying (C2). Let us consider the probability space wherein these triples are chosen uniformly at random. Equivalently, for each coordinate $i$, the values $(x_i, y_i, z_i)$ are chosen uniformly at random among the 7 non-zero values. We need to show that conditions (C3), (C4) are then satisfied with probability $\Omega(1)$.

For each coordinate $i$, consider the 6-dimensional random vector $V_i = (x_i, y_i, z_i, x_i \lor y_i, x_i \lor z_i, y_i \lor z_i)$ and let $V = \sum V_i$. We need to show that $V - 3b\zeta$ has non-negative entries where
Since the variables $V_i$ are i.i.d. and each has mean $\beta$, the scaled value

$$\sqrt{s}(V/s - \beta)$$

approaches a 6-dimensional normal distribution $N(0, \Sigma)$ with covariance matrix given by

$$\Sigma = \frac{1}{49} \begin{bmatrix} 12 & -2 & -2 & 4 & 4 & -3 \\ -2 & 12 & -2 & 4 & -3 & 4 \\ -2 & -2 & 12 & -3 & 4 & 4 \\ 4 & 4 & -3 & 6 & -1 & -1 \\ 4 & -3 & 4 & -1 & 6 & -1 \\ -3 & 4 & 4 & -1 & -1 & 6 \end{bmatrix}$$

as $s \to \infty$.

Note that $\Sigma$ is non-singular. Hence, for the corresponding distribution $N(0, \Sigma)$ there is positive probability that all six coordinates are negative. For large enough $s$, the probability that $V - \beta s$ has negative entries converges; in particular, it approaches to some constant value as $s \to \infty$. (The lemma holds vacuously when $s = O(1)$.)

We will next apply Janson’s with respect to candidates; this gives the following:

**Proposition 15.** In the language of Theorem 6, there holds

$$\mu \geq \Omega\left(\frac{7^s}{\psi_3}\right), \quad \Delta \leq O\left(\frac{\psi_1 \alpha_1^4 + \psi_2 \alpha_2^2}{\psi_3^2 \sqrt{s}}\right)$$

for constants

$$\alpha_1 = 14 \cdot 541^{1/7} \approx 24.75, \quad \alpha_2 = 7 \cdot 2^{6/7} \approx 12.68$$

**Proof.** First, each candidate $u = (x, y, z)$ satisfies (C1) with probability $1/\psi_3$. The bound on $\mu$ then follows from Lemma 8. For the bound on $\Delta$, there are a number of different cases. Let us write $u' = (x', y', z')$ throughout.

1. $x = x', y \neq y', z \neq z'$. In this case, the probability that $E_u$ and $E_{u'}$ both hold is precisely

$$1/(d_1 d_2 d_3^2) = d_1 / \psi_3^2.$$ To enumerate the relevant tuples $u, u'$, let us suppose that $|x| = \ell$; since $u$ is a candidate, necessarily $\ell \leq 4s/7$. For the coordinates where $x_i = 1$, there are 16 choices for $y_i, y_i', z_i, z_i'$; likewise, for the coordinates where $x_i = 0$, there are 9 choices for $y_i, z_i, y_i', z_i'$. Overall, we have $16^4 9^{s-\ell}$ choices for $y, z, y', z'$. Summing over $\ell$, the total contribution from this case is

$$\frac{d_1}{\psi_3^2} \sum_{\ell=0}^{\lfloor 4s/7 \rfloor} \binom{s}{\ell} 16^4 9^{s-\ell}$$

Let us denote by $g(\ell)$ the summand correspond to value $\ell$. We have $g(\ell) / g(\ell - 1) = 16(s + 1 - \ell) / (9\ell)$. In particular, for $\ell \leq 4s/7$, this ratio is at least 4/3. Hence, the entire sum over $\sum_{\ell=0}^{\lfloor 4s/7 \rfloor} g(\ell)$ is within a constant value of $g(\lfloor 4s/7 \rfloor)$. In turn, by Stirling’s formula, we have

$$g(\lfloor 4s/7 \rfloor) \leq O(\alpha_1^4 / \sqrt{s})$$

and thus the total contribution to $\Delta$ in this case is at most $O\left(\frac{d_1 \alpha_1^4}{\psi_3^2 \sqrt{s}}\right)$.

2. $x \neq x', y = y', z \neq z'$ or $x \neq x', y \neq y', z = z'$. These are precisely analogous to the previous case; they contribute $O\left(\frac{d_2 \alpha_1^4}{\psi_3^2 \sqrt{s}}\right)$ and $O\left(\frac{d_3 \alpha_1^4}{\psi_3^2 \sqrt{s}}\right)$ respectively.

3. $x = x', y = y', z \neq z'$. In this case, the probability that $E_u$ and $E_{u'}$ both hold is precisely

$$1/(d_1 d_2 d_3^2) = d_1 d_2 / \psi_3^3.$$ To enumerate the relevant tuples $u, u'$, let us suppose that $|x \lor y| = \ell$; as before, we must have $\ell \leq 6s/7$. For each coordinate where $x_i \lor y_i = 1$, we have 4 choices for $z_i, z_i'$ and 3 choices for the specific value of $x_i, y_i$; for the other coordinates, we
have just one choice \( z_i = z_i' = 1, x_i = y_i = 0 \). Overall, there are \( 12^\ell \) choices for \( u, u' \) for a given value of \( \ell \). Summing over \( \ell \), the total contribution is

\[
\frac{d_1 d_2}{\psi_3^2} \sum_{\ell=0}^{[6s/7]} \binom{s}{\ell} 12^\ell
\]

Let \( g(\ell) = \binom{s}{\ell} 12^\ell \) denote the summand corresponding to value \( \ell \). We have \( g(\ell)/g(\ell - 1) = \frac{12(s+1-\ell)}{\ell} \); in particular, for \( \ell \leq 6s/7 \), this ratio is at least 2. Hence, the entire sum is within a constant factor of \( g([6s/7]) \). In turn, by Stirling’s formula, we have

\[
g([6s/7]) \leq O\left(\frac{\alpha_2 s}{\sqrt{s}}\right)
\]

and hence the total contribution to \( \Delta \) in this case is \( O\left(\frac{d_1 d_2 \alpha_2 s^2}{\psi_3^2 \sqrt{s}}\right) \).

4. \( x = x', y \neq y', z = z' \) or \( x \neq x', y = y', z = z' \). These are precisely analogous to the previous case, contributing \( O\left(\frac{d_1 d_3 \alpha_2 s^2}{\psi_3^2 \sqrt{s}}\right) \) and \( O\left(\frac{d_2 d_3 \alpha_2 s^2}{\psi_3^2 \sqrt{s}}\right) \) respectively. \( \square \)

With these estimates, we can adapt the proofs of Proposition 9 and Lemma 8 to show the following:

**Proposition 16.** Suppose that \( 7^s \geq \Omega(\psi_3 \log(1/\delta)) \) for some \( \delta \in (0,1) \), and \( (\alpha_1/7)^s/\sqrt{s} \ll \psi_3/\psi_1 \) and \( (\alpha_2/7)^s/\sqrt{s} \ll \psi_3/\psi_2 \). Then, with probability at least \( 1 - \delta \), there holds \( C_{i,j} = 1 \)

The only difference in the proof is that we show that there are sufficiently many good candidates, as opposed to merely good triples. This, in turn, gives the following estimate:

**Theorem 17.** Suppose that \( \psi_3^{0.350} \geq \psi_1 \) and \( \psi_3^{0.694} \). Then, with appropriate choice of parameters, we can compute \( C \) correctly with probability \( 1 - 1/\text{poly}(\psi_3) \) in runtime

\[
O\left(\frac{\psi_3 \log \psi_3}{\log^6 \psi_3}\right)
\]

To achieve Theorem 17 we should take \( s = \log_7 \psi_3 + \Theta(1) \); however, the hidden constants may be large.

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