Topological Forcing Semantics with Settling

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Abstract

It was realized early on that topologies can model constructive systems, as the open sets form a Heyting algebra. After the development of forcing, in the form of Boolean-valued models, it became clear that, just as over ZF any Boolean-valued model also satisfies ZF, so do Heyting-valued models satisfy IZF, which stands for Intuitionistic ZF, the most direct constructive re-working of the ZF axioms. In this paper, we return to topologies, and introduce a variant model, along with a correspondingly revised forcing or satisfaction relation. The purpose is to prove independence results related to weakenings of the Power Set axiom. The original motivation is the second model of [9], based on $\mathbb{R}$, which shows that Exponentiation, in the context of CZF minus Subset Collection, does not suffice to prove that the Dedekind reals form a set. The current semantics is the generalization of that model from $\mathbb{R}$ to an arbitrary topological space. It is investigated which set-theoretic principles hold in such models in general. In addition, natural properties of the underlying topological space are shown to imply the validity of stronger such principles.

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1. Introduction

Topological interpretations of constructive systems were first studied by Stone [15] and Tarski [16], who independently provided such for propositional logic. This was later extended by Mostowski [12] to predicate logic. The first application of this to any sort of higher-order system was Scott’s interpretation of analysis [13, 14]. Grayson [6, 7] then generalized the latter to the whole set-theoretic universe, to provide a model of IZF, Intuitionistic Zermelo-Fraenkel Set Theory. Although not directly relevant to our concerns, it was soon realized that topological semantics could be unified with Kripke and Beth models, and all generalized, via categorical semantics; see [5] and [10] for good introductions. Here is introduced, not a generalized, but rather an alternative semantics. (In-
cidentally, this semantics can also be understood categorically, as determined by Streicher (unpublished).

An instance of this semantics was already applied in \cite{9} to the reals $\mathbb{R}$. The context there was CZF, Constructive ZF, introduced in \cite{1, 2, 3} and exposted in \cite{4}. CZF is currently the most studied system of constructive set theory, because of the modesty of its proof-theoretic strength coupled with its implicational power. For instance, even though CZF does not have the proof-theoretically very strong Power Set Axiom, its substitute, Aczel’s Subset Collection, suffices to prove that the Dedekind reals form a set. What was not clear was whether a further weakening of Power Set from Subset Collection to Exponentiation, or the existence of function spaces, would also prove the same. In the context of the other CZF axioms, Subset Collection and Exponentiation are proof-theoretically equivalent, so proof-theoretic analyses would not be able to answer this question. In contrast, one of the models of \cite{9} satisfies CZF\(_{Exp}\), yet the Dedekind cuts do not form a set, thereby showing the necessity of Subset Collection, or at least of something more than Exponentiation.

The essence of the construction there is that, as in a traditional topological model, the truth value of set membership ($\sigma \in \tau$, where $\sigma$ and $\tau$ are terms) is an open set of $\mathbb{R}$, but at any moment the terms under consideration can collapse to ground model terms. (A ground model term is the canonical image of a ground model set – think of the standard embedding of $V$ into $V[G]$ in classical forcing.) Such a collapse does not make the variable sets disappear, though. So no set could be the Dedekind cuts: any such candidate could at any time collapse to a ground model set, but then it wouldn’t contain the canonical generic because that’s a variable set, and this generic, over $\mathbb{R}$, is a Dedekind cut.\footnote{For those already familiar with a similar-sounding construction by Joyal, this is exactly what distinguishes the two. Joyal started with a topological space $T$, and took the union of $T$ with a second copy of $T$, the latter carrying the discrete topology (i.e. every subset is open). So by Joyal, you could specialize at a point, but then every set is also specialized there. Here, you can specialize every set you’re looking at at a point, but that won’t make the ambient variable sets disappear. Alternatively, the whole universe will specialize, but at the same time be reborn. For an exposition of Joyal’s argument in print, see either \cite{7} or \cite{17} p. 805-807.}

This process of collapsing to a ground model set we call settling down. Our purpose is to show how this settling semantics works in an arbitrary topological space, not just $\mathbb{R}$. This extension is not completely straightforward. Certain uniformities of $\mathbb{R}$ allowed for simplifications in the definition of forcing ($\Vdash$) and for proofs of stronger set-theoretic axioms, most notably Full Separation and Exponentiation. In the next section, we prove as much as we can making no assumptions on the topological space $T$ being worked over; in the following section, natural and appropriately modest assumptions are made on $T$ so that Separation and Exponentiation can be proven.

The greatest weakness in what can be proven in the general case is in the family of Power Set-like axioms. This is no surprise, as the semantics was developed for a purpose which necessitated the failure of Subset Collection (and hence of Power Set itself). That Exponentiation ended up holding is thanks to
the particularities of \( \mathbb{R} \), not to settling semantics. Rather, what does hold in general is a weakened version of all of these Power Set-like axioms. The reason that Power Set fails, like the non-existence of the set of Dedekind cuts above, is that any candidate for the power set of \( X \) might collapse to a ground model set, and so would then no longer contain any variable subset of \( X \). However, that variable subset might itself collapse, and then would be in the classical power set of \( X \). So while the subset in question, before the collapse, might not equal a member of the classical power set, it cannot be different from every such member. That is the form of Power Set which holds in the settling semantics:

**Eventual Power Set:**

\[
\forall X \exists C (\forall Y \in C \ Y \subseteq X) \land (\forall Y \subseteq X \neg \forall Z \in C Y \neq Z).
\]

Although we will not need them, there are comparable weakenings of Subset Collection (or Fullness) and Exponentiation:

**Eventual Fullness:**

\[
\forall X,Y \exists C (\forall Z \in C Z \text{ is a total relation from } X \text{ to } Y) \land (\forall R \text{ if } R \text{ is a total relation from } X \text{ to } Y \text{ then } \neg \forall Z \in C Z \not\subseteq R).
\]

**Eventual Exponentiation:**

\[
\forall X,Y \exists C \forall F \text{ if } F \text{ is a total function from } X \text{ to } Y \text{ then } \neg \forall Z \in C F \neq Z.
\]

It is easy to see that Power Set implies Fullness, which itself implies Exponentiation. Essentially the same arguments will prove:

**Proposition 1.1.**  
**Eventual Power Set implies Eventual Fullness, which in turn implies Eventual Exponentiation.**

As already stated, the original motivation of this work was to generalize an extant construction from one to all topologies. Now that it is done, other uses can be imagined. The theory identified here is incomparable with CZF, so its proof-theoretic strength is unclear. If it turns out to be weak, perhaps it could be combined with CZF to provide a slight strengthening of the latter while maintaining a similar proof theory. In any case, the model-theoretic construction might be useful for further independence results, the purpose of the first, motivating model. A long-term project is some kind of classification of models, topological or otherwise; having this unconventional example might help find other yet-to-be-discovered constructions. A question raised by van den Berg is how the model would have to be expanded in order to get a model of IZF. He observed that the recursive realizability model based on (definable subsets of) the natural numbers [8] (also discovered independently by Streicher in unpublished work), which satisfies CZF + Full Separation (and necessarily not Power Set), is essentially just the collection of subcountable sets from the full recursive realizability model [11], and hence is naturally extendable to an IZF model. It is at best unclear how the current model could be so extended. Somewhat speculatively, applications to computer science are also conceivable, wherever such modeling might be natural. For instance, constructive logic can naturally be used to model computation when objects are viewed as having properties only partially determined at any stage: if in addition parallel computation is part of the programming paradigm, it could be that a variable is passed to several parallel sub-computations, which specify the variable more and in incompatible ways. This is similar to the current construction, where there are two tran-
sition functions, both leading to the same future but under one function the variable/generic is fully specified and under the other it’s not.

2. The General Case

First we define the term structure of the topological model with settling, then truth in the model (the forcing semantics), and then we prove that the model satisfies some standard set-theoretic axioms.

Definition 2.1. For a topological space $T$, a term is a set of the form $\{\langle \sigma_i, J_i \rangle \mid i \in I \} \cup \{\langle \sigma_h, r_h \rangle \mid h \in H \}$, where each $\sigma$ is (inductively) a term, each $J$ an open set, each $r$ is a member of $T$, and $H$ and $I$ index sets.

The first part of each term is as usual. It suffices for the embedding $x \mapsto \hat{x}$ of the ground model into the topological model:

Definition 2.2. $\hat{x} = \{\langle \hat{y}, T \rangle \mid y \in x\}$. Any term of the form $\hat{x}$ is called a ground model term.

For $\phi$ a formula in the language of set theory with (set, not term) parameters $x_0, x_1, ..., x_n$, then $\hat{\phi}$ is the formula in the term language obtained from $\phi$ by replacing each $x_i$ with $\hat{x}_i$.

The second part of the definition of a term plays a role only when we decide to have the term settle down and stop changing. This settling down is described as follows.

Definition 2.3. For a term $\sigma$ and $r \in T$, $\sigma^r$ is defined inductively on the terms as $\{\langle \sigma^r_i, T \rangle \mid \langle \sigma_i, J_i \rangle \in \sigma \wedge r \in J_i \} \cup \{\langle \sigma^r_h, T \rangle \mid \langle \sigma_h, r \rangle \in \sigma \}$.

Note that $\sigma^r$ is a ground model term. It bears observation that $(\sigma^r)^r = \sigma^r$.

Definition 2.4. For $\phi = \phi(\sigma_0, ..., \sigma_i)$ a formula with parameters $\sigma_0, ..., \sigma_i$, $\phi^r$ is $\phi(\sigma_0^r, ..., \sigma_i^r)$.

We define a forcing relation $J \models \phi$, with $J$ an open subset of $T$ and $\phi$ a formula.

Definition 2.5. $J \models \phi$ is defined inductively on $\phi$:

$J \models \sigma = \tau$ iff for all $\langle \sigma_i, J_i \rangle \in \sigma \wedge J \cap J_i \models \sigma_i \in \tau$ and vice versa, and for all $r \in J$, $\sigma^r = \tau^r$.

$J \models \sigma \in \tau$ iff for all $r \in J$ there is a $\langle \tau_i, J_i \rangle \in \tau$ and $J_r \subseteq J_i$ containing $r$ such that $J_r \models \sigma = \tau_i$.

$J \models \phi \land \psi$ iff $J \models \phi$ and $J \models \psi$.

$J \models \phi \lor \psi$ iff for all $r \in J$ there is a $J_r \subseteq J$ containing $r$ such that $J_r \models \phi$ or $J_r \models \psi$. 

\[ J \models \phi \rightarrow \psi \text{ iff for all } J' \subseteq J \text{ if } J' \models \phi \text{ then } J' \models \psi, \text{ and, for all } r \in J, \text{ there is a } J_r \subseteq J \text{ containing } r \text{ such that, for all } K \subseteq J_r, \text{ if } K \models \phi^r \text{ then } K \models \psi^r. \]

\[ J \models \exists x \, \phi(x) \text{ iff for all } r \in J \text{ there is a } J_r \subseteq J \text{ containing } r \text{ and a } \sigma \text{ such that } J_r \models \phi(\sigma). \]

\[ J \models \forall x \, \phi(x) \text{ iff for all } \sigma \, J \models \phi(\sigma), \text{ and for all } r \in J \text{ there is a } J_r \subseteq J \text{ containing } r \text{ such that for all } \sigma \, J_r \models \phi^r(\sigma). \]

(Notice that in the last clause, \( \sigma \) is not interpreted as \( \sigma^r \).)

**Lemma 2.6.** \( \vdash \) is sound for constructive logic.

**Lemma 2.7.** \( T \) forces the equality axioms, to wit:

1. \( \forall x \, x = x \)
2. \( \forall x, y \, x = y \rightarrow y = x \)
3. \( \forall x, y, z \, x = y \land y = z \rightarrow x = z \)
4. \( \forall x, y, z \, x = y \land x \in z \rightarrow y \in z \)
5. \( \forall x, y, z \, x = y \land z \in x \rightarrow z \in y. \)

**proof:**

1: It is trivial to show via a simultaneous induction that, for all \( J \) and \( \sigma, J \models \sigma = \alpha \), and, for all \( \langle \sigma_i, J_i \rangle \in \sigma, J \cap J_i \models \sigma_i \in \alpha. \)
2: Trivial because the definition of \( J \models \sigma = M \, \tau \) is itself symmetric.
3: For this and the subsequent parts, we need a lemma.

**Lemma 2.8.** If \( J' \subseteq J \models \sigma = \tau \) then \( J' \models \sigma = \tau \), and similarly for \( \varepsilon. \)

**proof:** By induction on \( \sigma \) and \( \tau. \)

Returning to the main lemma, we show that if \( J \models \rho = \sigma \) and \( J \models \sigma = \tau \) then \( J \models \rho = \tau \), which suffices. This will be done by induction on terms for all opens \( J \) simultaneously.

For the second clause in \( J \models \rho = \tau \), let \( r \in J. \) By the hypotheses, second clauses, \( \rho^r = \sigma^r \) and \( \sigma^r = \tau^r \), so \( \rho^r = \tau^r. \)

The first clause of the definition of forcing equality follows by induction on terms. Starting with \( \langle \rho_i, I_i \rangle \in \rho \), we need to show that \( J \cap J_i \models \rho_i \in \tau. \) We have \( J \cap J_i \models \rho_i \in \sigma. \) For a fixed, arbitrary \( r \in J \cap J_i \) let \( \langle \sigma_j, J_j \rangle \in \sigma \) and \( J' \subseteq J \cap J_i \) be such that \( r \in J' \cap J_j \models \rho_i = \sigma_j. \) By hypothesis, \( J \cap J_j \models \sigma_j \in \tau. \) So let \( \langle \tau_k, J_k \rangle \in \tau \) and \( J \subseteq J \cap J_j \) be such that \( r \in J \cap J_k \models \sigma_j = \tau_k. \) Let \( J \) be \( J' \cap J \cap J_j. \) Note that \( J \subseteq J \cap J_i, \) and that \( r \in J \cap J_k. \) We want to show that \( J \cap J_k \models \rho_i = \tau_k. \) Observing that \( J \cap J_k \subseteq J' \cap J_j, J \cap J_k, \) it follows by the previous lemma that \( J \cap J_k \models \rho_i = \sigma_j, J, \sigma_j = M \, \tau_k. \) From which the desired conclusion follows by the induction. So \( r \in J \cap J_k \models \rho_i \in \tau. \) Since \( r \in J \cap J_i \) was arbitrary, \( J \cap J_i \models \rho_i \in \tau. \)

4: It suffices to show that if \( J \models \rho = \sigma \) and \( J \models \rho \in \tau \) then \( J \models \sigma \in \tau. \) Let \( r \in J. \) By hypothesis, let \( \langle \tau_i, J_i \rangle \in \tau, J_r \subseteq J_i \) be such that \( r \in J_r \models \rho = \tau_i; \)
without loss of generality \( J_r \subseteq J \). By the previous lemma, \( J_r \models \rho = \sigma \), and by the previous part of the current lemma, \( J_r \models \sigma = \tau_i \). Hence \( J_r \models \sigma \in \tau \). Since \( r \in J \) was arbitrary, we are done.

5: Similar, and left to the reader. \( \square \)

**Lemma 2.9.** 1. For all \( \phi \not\models \phi \).

2. If \( J' \subseteq J \models \phi \) then \( J' \models \phi \).

3. If \( J_i \models \phi \) for all \( i \) then \( \bigcup_i J_i \models \phi \).

4. \( J \models \phi \) iff for all \( r \in J \) there is a \( J_r \subseteq J \) containing \( r \) such that \( J_r \models \phi \).

5. For all \( \phi, J_r \models \phi \) then for all \( r \in J \) there is a neighborhood \( J_r \) of \( r \) such that \( J_r \models \phi_r \).

6. For \( \phi \) bounded (i.e. \( \Delta_0 \)) and having only ground model terms as parameters, \( T \models \phi \) iff \( \check{\phi} \) (i.e. \( V \models \phi \)).

**proof:**

1. Trivial induction. This part is not used later, and is mentioned here only to flesh out the picture.

2. Again, a trivial induction. The base cases, = and \( \in \), are proven by induction on terms, as mentioned just above.

3. By induction. For the case of \( \to \), you need to invoke the previous part of this lemma. All other cases are straightforward.

4. Trivial, using 3.

5. By induction on \( \phi \).

6. For \( \phi \) bounded (i.e. \( \Delta_0 \)) and having only ground model terms as parameters, \( T \models \phi \) iff \( \check{\phi} \) (i.e. \( V \models \phi \)).

At this point, we are ready to show what is in general forced under this semantics.
Theorem 2.10. \( T \) forces:

Infinity

Pairing

Union

Extensionality

Set Induction

Eventual Power Set

Bounded \((\Delta_0)\) Separation

Collection

Some comments on this choice of axioms are in order. The first five are unremarkable. The role of Eventual Power Set was discussed in the Introduction. The restriction of Separation to the \( \Delta_0 \) case should be familiar, as that is also the case in CZF and KP. By way of compensation, the version of Collection in CZF is Strong Collection: not only does every total relation with domain a set have a bounding set (regular Collection), but that bounding set can be chosen so that it contains only elements related to something in the domain (the strong version). In the presence of full Separation, these are equivalent, as an appropriate subset of any bounding set can always be taken. Unfortunately, even the additional hypotheses provided by Collection are not enough in the current context to yield even this modest fragment of Separation, as will actually be shown at the beginning of the next section. In fact, even Replacement fails, as we will see.

proof:

- Infinity: \( \hat{\omega} \) will do. (Recall that the canonical name \( \hat{x} \) of any set \( x \) from the ground model is defined inductively as \( \langle \hat{y},T \rangle \mid y \in x \).)

- Pairing: Given \( \sigma \) and \( \tau \), \( \{ \langle \sigma, T \rangle, \langle \tau, T \rangle \} \) will do.

- Union: Given \( \sigma \), the union of the following four terms will do:
  
  - \( \{ \langle \tau, J \cap J_i \rangle \mid \text{for some } \sigma_i, \langle \tau, J \rangle \in \sigma_i \text{ and } \langle \sigma_i, J_i \rangle \in \sigma \} \)
  
  - \( \{ \langle \tau, r \rangle \mid \text{for some } \sigma_i, \langle \tau, r \rangle \in \sigma_i \text{ and } \langle \sigma_i, r \rangle \in \sigma \} \)
  
  - \( \{ \langle \tau, r \rangle \mid \text{for some } \sigma_i \text{ and } K, \langle \tau, K \rangle \in \sigma_i, r \in K, \text{ and } \langle \sigma_i, r \rangle \in \sigma \} \)
  
  - \( \{ \langle \tau, r \rangle \mid \text{for some } \sigma_i \text{ and } K, \langle \tau, r \rangle \in \sigma_i, r \in K, \text{ and } \langle \sigma_i, K \rangle \in \sigma \} \).

- Extensionality: We need to show that

\[
T \models \forall x \forall y [ \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y].
\] (1)
It suffices to show that for any terms $\sigma$ and $\tau$,

$$T \models \forall z (z \in \sigma \leftrightarrow z \in \tau) \to \sigma = \tau. \quad (2)$$

(Although that is only the first clause in forcing $\forall$, it subsumes the second, because $\sigma$ and $\tau$ could have been chosen as ground model terms in the first place.) To show that, for the second clause in forcing $\to$, it suffices to show that

$$T \models \forall z (z \in \sigma^r \leftrightarrow z \in \tau^r) \to \sigma^r = \tau^r. \quad (3)$$

But, as before, this is already subsumed by choosing $\sigma$ and $\tau$ to be ground model terms in the first place. Hence it suffices to check the first clause in forcing $\to$:

$$\forall J [J \models \forall z (z \in \sigma \leftrightarrow z \in \tau)] \to [J \models \sigma = \tau]. \quad (4)$$

To this end, let $\langle \sigma_i, J_i \rangle$ be in $\sigma$; we need to show that $J \cap J_i \models \sigma_i \in \tau$. By the choice of $J_i$, $J_i \models \sigma_i \leftrightarrow \sigma_i \in \tau$. In particular, $J \cap J_i \models \sigma_i \in \sigma \to \sigma_i \in \tau$. Since $J \cap J_i \models \sigma_i \in \sigma$ (proof of \ref{2.7} part 1)), $J \cap J_i \models \sigma_i \in \tau$. Symmetrically for $\langle \tau_i, J_i \rangle \in \tau$.

Also, let $r \in J$. If $\sigma^r \neq \tau^r$, let $\langle \rho, T \rangle$ be in their symmetric difference. By the choice of $J$, for some neighborhood $J_r$ of $r$, $J_r \models \rho \in \sigma^r \leftrightarrow \rho \in \tau^r$.

This contradicts the choice of $\rho$. So $\sigma^r = \tau^r$.

- Set Induction (Schema): We need to show that

$$T \models \forall x ((\forall y \in x \phi(y)) \to \phi(x)) \to \forall x \phi(x). \quad (5)$$

The statement in question is an implication. The definition of forcing $\to$ contains two clauses.

The first clause is that, for any open set $J$ and formula $\phi$, if $J \models \forall x (\forall y \in x \phi(y) \to \phi(x))$ then $J \models \forall x \phi(x)$. By way of proving that, suppose not. Let $J$ and $\phi$ provide a counter-example. By hypothesis,

$$\forall \sigma J \models \forall y \in \sigma \phi(y) \to \phi(\sigma) \quad (6)$$

and

$$\forall r \in J \exists J' \ni r \forall \sigma' J' \models \forall y \in \sigma' \phi'(y) \to \phi'(\sigma'). \quad (7)$$

Since $J \not\models \forall x \phi(x)$, either

$$\exists \sigma J \not\models \phi(\sigma) \quad (8)$$

or

$$\exists r \in J \forall J' \ni r \exists \sigma' J' \not\models \phi'(\sigma'). \quad (9)$$

If (8) holds, let $r$ as given by (8), and then let $J'$ be as given by (7) for that $r$. By (9), $\exists \sigma' J' \not\models \phi'(\sigma')$; let $\sigma$ be such a $\sigma'$ – so $J' \not\models \phi'(\sigma')$ of minimal V-rank. By (4), we have $J' \models \forall y \in \sigma \phi'(y) \to \phi'(\sigma)$. If we can
show that \( J' \vDash \forall y \in \sigma \phi^r(y) \), then (by the definition of forcing \( \rightarrow \)) we will have a contradiction, showing that (9) must fail.

To that end, we must show, unpacking the abbreviation, that \( J' \vDash \forall y(y \in \sigma \rightarrow \phi^r(y)) \); that is,

\[
\forall \tau J' \vDash \tau \in \sigma \rightarrow \phi^r(\tau)
\]

(10)

and

\[
\forall s \in J' \exists K \ni s \forall \tau K \vDash \tau \in \sigma^s \rightarrow \phi^r(\tau),
\]

(11)

the latter because \((\phi^r)^s = \phi^r\).

By way of showing (10), suppose \( J' \supseteq K \vDash \tau \in \sigma \). Then \( K \) can be covered with open sets \( K_i \) such that \( K_i \vDash \tau = \sigma_i \) and \( K_i \subseteq J_i \) where \( \langle \sigma_i, J_i \rangle \in \sigma \). Since \( \sigma \) has strictly lower V-rank than \( \sigma', J' \vDash \phi^r(\sigma_i) \). Hence \( K_i \vDash \phi^r(\tau) \). Since the \( K_i \)s cover \( K \) (by lemma 2.9 part 3) \( K \) forces the same. We still have to show that for all \( s \in J' \) there is a \( K \supseteq s \) such that for all \( K' \subseteq K \) if \( K' \vDash \tau^s \in \sigma^s \) then \( K' \vDash \phi^r(\tau^s) \). In fact, \( J' \) suffices for \( K \); if \( J' \supseteq K' \vDash \tau^s \in \sigma^s \) then \( K' \vDash \phi^r(\tau^s) \). Moreover, this is the same argument as the one just completed, with \( \sigma^s \) in place of \( \sigma \).

To show (11), we claim that \( J' \) suffices for the choice of \( K \): \( J' \vDash \tau \in \sigma^s \rightarrow \phi^r(\tau) \). Once more, this is just (10), with \( \sigma^s \) in place of \( \sigma \).

This completes the proof that (9) must fail. Hence we have that the negation of (9) must hold, namely

\[
\forall r \in J \exists J' \ni r \forall \sigma' J' \vDash \phi^r(\sigma'),
\]

(12)

as well as (8). Let \( \sigma \) be of minimal V-rank such that \( J \not\vDash \phi(\sigma) \). If we can show that \( J \vDash \forall y \in \sigma \phi(y) \), then by (10) we will have a contradiction, completing the proof of the first clause.

What we need to show are

\[
\forall \tau J \vDash \tau \in \sigma \rightarrow \phi(\tau)
\]

(13)

and

\[
\forall r \in J \exists J' \ni r \forall \tau J' \vDash \tau \in \sigma^r \rightarrow \phi^r(\tau).
\]

(14)

By way of showing (13), suppose \( J \supseteq K \vDash \tau \in \sigma \); we need to show that \( K \vDash \phi(\tau) \). This is the same argument, based on the minimality of \( \sigma \), as in the proof of (10). The other part of showing (13) is

\[
\forall r \in J \exists J' \ni r \forall K \subseteq J' (K \vDash \tau^r \in \sigma^r \Rightarrow K \vDash \phi^r(\tau^r)).
\]

(15)

Both (14) and (15) are special cases of (12).

This completes the proof of the first clause.
The second clause is that for all \( r \in T \) there is a \( J \ni r \) such that for all \( K \subseteq J \) if \( K \models \forall x \ ((\forall y \in x \phi^r(y)) \rightarrow \phi^r(x)) \) then \( K \models \forall x \phi^r(x) \). For any \( r \), let \( J \) be \( T \). Then what remains of the claim has exactly the same form as the first clause, with \( K \) and \( \phi^r \) for \( J \) and \( \phi \) respectively. Since the validity of this first clause was already shown for all choices of \( J \) and \( \phi \), we are done.

- Eventual Power Set: We need to show that

\[
T \models \forall X \exists C \forall Y \ (Y \subseteq X \rightarrow \forall Z (Z \in C \rightarrow Y \neq Z)).
\]  

(Actually, we must also produce a \( C \) that contains only subsets of \( X \). However, to extract such a sub-collection from any \( C \) as above is an instance of Bounded Separation, the proof of which below does not rely on the current proof. So we will make our lives a little easier and prove the version of EPS as stated.) Since the sentence forced has no parameters, the second clause in forcing \( \forall \) is subsumed by the first, so all we must show is that, for any term \( \sigma \),

\[
T \models \exists C \forall Y \ (Y \subseteq \sigma \rightarrow \forall Z (Z \in C \rightarrow Y \neq Z)).
\]  

Let \( \tau = \{ \langle \hat{x}, r \rangle \mid \sigma^r = \hat{s} \land x \subseteq s \} \). This is the desired \( C \). It suffices to show that

\[
T \models \forall Y \ (Y \subseteq \sigma \rightarrow \forall Z (Z \in \tau \rightarrow Y \neq Z)).
\]  

For the first clause in forcing \( \forall \), we need to show that

\[
T \models \rho \subseteq \sigma \rightarrow \forall Z (Z \in \tau \rightarrow \rho \neq Z).
\]  

To do that, first suppose \( T \supseteq J \models \rho \subseteq \sigma \). (Note that that implies that for all \( s \in J \ T \models \rho^s \subseteq \sigma^s \), so that \( \langle \rho^s, s \rangle \in \tau \), and \( T \models \rho^s \in \tau^s \).) We must show that

\[
J \models \forall Z (Z \in \tau \rightarrow \rho \neq Z).
\]  

It suffices to show that no non-empty subset \( K \) of \( J \) forces

\[
\forall Z (Z \in \tau \rightarrow \rho \neq Z)
\]  

or

\[
\forall Z (Z \in \tau^r \rightarrow \rho^r \neq Z)
\]  

(\( r \in J \)). For the former, we will show that \( K \) must violate the second clause in forcing \( \forall \). Let \( s \in K \). Letting \( Z \) be \( \rho^s \), as just observed, all of \( T \) will force \( Z \in \tau^s \) but nothing will force \( \rho^s \neq Z \). Similarly for the latter, by choosing \( Z \) to be \( \rho^r \). To finish forcing the implication, it suffices to show that for all \( r \)

\[
T \models \rho^r \subseteq \sigma^r \rightarrow \forall Z (Z \in \tau^r \rightarrow \rho^r \neq Z).
\]  

Again, it suffices to let \( Z \) be \( \rho^r \).
For the second clause in forcing \( \forall \), for \( r \) in \( T \) and \( \rho \) a term, it suffices to show that
\[
T \models \rho \subseteq \sigma^r \rightarrow \neg \forall Z (Z \in \tau^r \rightarrow \rho \neq Z). \tag{24}
\]
This time letting \( Z \) by any \( \rho^s \) suffices.

- **Bounded Separation:** The important point here is that, for \( \phi \) bounded \((\Delta_0)\) with only ground model terms, \( J \models \phi \) iff \( T \models \phi \) iff \( V \models \phi \) \[(2.9)\] part 6).

We need to show that
\[
T \models \forall X \exists Y \forall Z (Z \in Y \leftrightarrow Z \in X \land \phi(Z)). \tag{25}
\]
This means, first, that for any \( \sigma \),
\[
T \models \exists Y \forall Z (Z \in Y \leftrightarrow Z \in \sigma \land \phi(Z)), \tag{26}
\]
and, second, for any \( r \) in \( T \) there is a \( J \models r \) such that, for any \( \sigma \),
\[
J \models \exists Y \forall Z (Z \in Y \leftrightarrow Z \in \sigma \land \phi^r(Z)). \tag{27}
\]
In the second part, choosing \( J \) to be \( T \), we have an instance of the first part, so it suffices to prove the first only.

Let \( \tau \) be \( \{ (\sigma_i, J \cap J_i) \mid (\sigma_i, J_i) \in \sigma \text{ and } J \models \phi(\sigma_i) \} \cup \{ (\hat{x}, r) \mid (\hat{x}, T) \in \sigma^r \text{ and } T \models \phi^r(\hat{x}) \} \). We claim that \( \tau \) suffices: \( T \models \forall Z (Z \in \tau \leftrightarrow Z \in \sigma \land \phi(Z)) \).

First, let \( \rho \) be a term. We need to show that \( T \models \rho \in \tau \leftrightarrow \rho \in \sigma \land \phi(\rho) \).
Unraveling the bi-implication and the definition of forcing an implication, that becomes \( J \models \rho \in \tau \) iff \( J \models \rho \in \sigma \land \phi(\rho) \), and \( J \models \rho^r \in \tau^r \) iff \( J \models \rho^r \in \sigma^r \land \phi^r(\rho^r) \). The first iff should be clear from the first part of the definition of \( \tau \) and the second iff from the second part of the definition, along with the observation that forcing \( \phi^r(\rho^r) \) is independent of \( J \).

We also need, for each \( r \) in \( T \), a \( J \models r \) such that for all \( \rho J \models \rho \in \tau^r \leftrightarrow \rho \in \sigma^r \land \phi^r(\rho) \). Choosing \( J \) to be \( T \) and unraveling as above (recycling the variable \( J \)) yields \( J \models \rho \in \tau^r \) iff \( J \models \rho \in \sigma^r \land \phi^r(\rho) \), and \( J \models \rho^s \in \tau^r \) iff \( J \models \rho^s \in \sigma^r \land \phi^r(\rho^s) \). These hold because the only things that can be forced to be in \( \tau^r \) or \( \sigma^r \) are (locally) images of ground model terms, and the truth of \( \phi^r \) evaluated at such a term is independent of \( J \).

- **Collection:** Since only regular, not strong, Collection is true here, it would be easiest to his this with a sledgehammer: reflect \( V \) to some set \( M \) large enough to contain all the parameters and capture the truth of the assertion in question; the term consisting of the whole universe according to \( M \) will be more than enough. It is more informative, though, to follow through the natural construction of a bounding set, so we can highlight in the next section just what goes wrong with the proof of Strong Collection.

We need
\[
T \models \forall x \in \sigma \exists y \phi(x, y) \rightarrow \exists z \forall x \in \sigma \exists y \in z \phi(x, y). \tag{28}
\]
It suffices to show that for any $J$

$$(J \vDash \forall x \in \sigma \exists y \phi(x, y)) \rightarrow (J \vDash \exists z \forall x \in \sigma \exists y \in z \phi(x, y)), \quad (29)$$

and the same relativized to $r$. The latter is a special case of the former, so it suffices to show just the former.

By hypothesis, for each $\langle \sigma_i, J_i \rangle \in \sigma$ and $r \in J_i \cap J$, there are $\tau_{ir}$ and $J_{ir} \subseteq J_i \cap J$, $\exists r$ such that $J_{ir} \vDash \phi(\sigma_i, \tau_{ir})$. Also, for all $r \in J$ there is a $J_r \exists r$ such that, for all $\langle \hat{x}, T \rangle \in \sigma^r$, $J_r \vDash \exists y \phi^r'(\hat{x}, y)$. For each $s \in J_r$, let $\tau_{r,s}$ and $K \exists s$ be such that $K \vDash \phi^r'(\hat{x}, \tau_{r,s})$. By (29) part 5), $K \vDash \phi^r'(\hat{x}, \tau_{r,s})$.

We claim that

$$\tau = \{\langle \tau_{ir}, J_{ir} \rangle \mid i \in I, r \in J_i \cap J\} \cup \{\langle \tau_{r,s}, r \rangle \mid r \in J, \langle \hat{x}, T \rangle \in \sigma^r, s \in J_r\} \quad (30)$$

suffices: $J \vDash \forall x \in \sigma \exists y \in \tau \phi(x, y)$.

Forcing a universal has two parts. The first is that for all $\rho$,

$$J \vDash \rho \in \sigma \rightarrow \exists y \in \tau \phi(\rho, y). \quad (31)$$

For the second, it suffices to show that for all $r \in J$ and terms $\rho$

$$J_r \vDash \rho \in \sigma^r \rightarrow \exists y \in \tau^r \phi^r(\rho, y). \quad (32)$$

For the former, first suppose $J \supseteq K \vDash \rho \in \sigma$. It should be clear that the first part of $\tau$ covers this case. For the other part of forcing that implication, for each $r \in J$, it suffices to show that $J_r$ as desired: for all $K \subseteq J_r$, if $K \vDash \rho^s \in \sigma^r$ then $K \vDash \exists y \in \tau^r \phi^r(\rho^s, y)$. This is subsumed by the second implication from above, to which we now turn.

To show $J_r \vDash \rho \in \sigma^r \rightarrow \exists y \in \tau^r \phi^r(\rho, y)$, we need to show first that if $J_r \supseteq K \vDash \rho \in \sigma^r$ then $K \vDash \exists y \in \tau^r \phi^r(\rho, y)$, and second that for all $s \in J_r$ there is a $K \exists s$ such that if $K \supseteq L \vDash \rho^s \in \sigma^r$ then $L \vDash \exists y \in \tau^r \phi^r(\rho^s, y)$. By choosing $K$ to be $J_r$, the second is subsumed by the first. For that, it should be clear that the second part of $\tau$ covers this case. In a bit more detail, it suffices to work locally. (That is, it suffices to find a neighborhood of $s \in K$ forcing what we want, by lemma (2.9).) Locally, $\rho$ is forced equal to some $\hat{x}$, where $\langle \hat{x}, T \rangle \in \sigma^r$. As already shown, some neighborhood of $s$ forces $\phi^r'(\hat{x}, \tau_{r,s}^r)$, and $\langle \tau_{r,s}^r, T \rangle \in \tau^r$ by the second part of $\tau$. 

□
3. Exponentiation, Separation, and Replacement

The second model of [9] is the topological semantics of the current paper applied to \( \mathbb{R} \) (with the standard topology). There it was shown that the model satisfies not just the axioms proven here but also Exponentiation and Separation, and hence, in the presence of Collection, Replacement too. The reason those extra properties hold in that case is that \( \mathbb{R} \) is a “nice” space. It is the purpose of this section to explore just what makes \( \mathbb{R} \) nice and why such niceness is necessary for these additional properties.

3.1. Exponentiation

We can identify exactly the property of \( T \) that would make Exponentiation hold.

**Theorem 3.1.** \( T \vDash \text{Exponentiation iff } T \) is locally connected.

**Proof:** First we do the right-to-left direction. So suppose \( T \) is locally connected.

Given terms \( \sigma \) and \( \chi \), let \( \tau \) be \{\( \langle \rho, J \rangle \mid J \vDash \rho \) is a function from \( \sigma \) to \( \chi \}\} \cup \{\( \langle \hat{x}, r \rangle \mid x \) is a function from \( \sigma^r \) to \( \chi^r \}\}. \( \tau \) can be arranged to be set-sized by requiring that \( \rho \) be hereditarily empty outside of \( J \). It suffices to show that \( T \vDash \forall z \left( z \in \tau \leftrightarrow z \text{ is a function from } \sigma \text{ to } \chi \right) \).

The first clause in forcing \( \forall \) is that, for any term \( \rho \), \( T \vDash \rho \in \tau \leftrightarrow \rho \) is a function from \( \sigma \) to \( \chi \). That \( J \vDash \rho \in \tau \leftrightarrow J \vDash \text{“} \rho \text{ is a function from } \sigma \text{ to } \chi \text{”} \) is immediate from the first part of \( \tau \). As for \( J \vDash \rho^r \in \tau^r \iff J \vDash \text{“} \rho^r \text{ is a function from } \sigma^r \text{ to } \chi^r \text{”} \), by 2.9 part 6), both of those statements are independent of \( J \), and the iff holds because of the second part of \( \tau \).

The crux of the matter is the second clause in forcing \( \forall \): \( J \vDash \rho \in \tau^r \iff J \vDash \text{“} \rho \text{ is a function from } \sigma^r \text{ to } \chi^r \text{”} \). Why can only ground model functions be forced (locally) to be functions? For \( s \in J \), let \( K_s \subseteq J \) be a connected neighborhood of \( s \). For each \( \langle \sigma_i, T \rangle \in \sigma^r \), pick a \( \langle \chi_i, T \rangle \in \chi \) such that the value of (i.e. the largest subset of \( K_s \) forcing) \( \text{“} \rho(\sigma_i) = \chi_i \text{”} \) is non-empty. That set, along with the value of \( \text{“} \rho^r(\sigma_i) \neq \chi_i^r \text{”} \), is a disjoint open cover of \( K_s \). Since \( K_s \) is connected, the latter set is empty. So all of the values of \( \rho \) are determined by \( K_s \), so \( K_s \) forces \( \rho \) to equal a ground model term. Since \( J \) is covered by such sets, \( J \) also forces \( \rho \) to be a ground model term.

Now for the converse, suppose \( T \) is not locally connected. Assuming that \( T \) still forces Exponentiation, we will come up with a contradiction. Let \( x \in J \subseteq T \) be such that no neighborhood of \( x \) which is a subset of \( J \) is connected. Working within \( J \) as a subspace of \( T \), \( J \) is itself not connected, and so can be partitioned into two clopen subsets, \( K_0 \) and \( J_0 \ni x \). Inductively, given \( x \in J_n \) clopen, partition \( J_n \) into clopen \( K_{n+1} \) and \( J_{n+1} \ni x \). Let \( O \) consist of all of the \( J_n \)s. By Exponentiation, \( J \) is covered by sets \( I \) forcing “\( Z_I \) is the function space from \( O \) to \( 2 \)”.

In particular, for each \( r \in J \) there is an \( I_r \subseteq I_r \subseteq I \) such that, for each \( \sigma, I_r \vDash \text{“} \sigma \in Z_I^r \leftrightarrow \sigma \text{ is a function from } O \text{ to } 2 \text{”} \). Notice that each \( Z_I^r \) consists only of ground model terms standing for ground model functions, and each ground model function from \( O \) to \( 2 \) is forced by \( J \) to be such a function.
from \( \hat{O} \) to \( \hat{2} \), so each \( Z^I_r \) equals \( O^2 \). In short, for each \( \sigma \), \( J \Vdash \sigma \in O^2 \leftrightarrow \sigma \) is a function from \( \hat{O} \) to \( \hat{2} \).

Now let \( f \) be the term such that \( J_i \Vdash f(\hat{J}_i) = 1 \) and \( K_i \Vdash f(\hat{J}_i) = 0 \).” \( J \Vdash f \) is a function from \( \hat{O} \) to \( \hat{2} \), so \( J \Vdash f \in O^2 \).

Each point in \( J_\omega := \bigcap_i J_i \) has a neighborhood, necessarily a subset of \( J_\omega \), forcing \( f \) to be the constant function 1, so \( J_\omega \) is open. As an intersection of closed sets, \( J_\omega \) is also closed, and contains \( x \) to boot. This construction can continue indefinitely: at stage \( \alpha + 1 \), split the clopen \( J_\alpha \ni x \) into clopen \( K_{\alpha+1} \) and \( J_{\alpha+1} \ni x \); at stage \( \lambda \) a limit, consider the function space from \( \{ J_\alpha | \alpha < \lambda \} \) to \( \hat{2} \). This is a contradiction, because it produces class-many subsets of \( J \). \( \square \)

An example of a non-locally connected space is Cantor space. Forcing a random 0-1 sequence, which is a function from \( \mathbb{N} \) to \( \hat{2} \). So the canonical generic is in a function space, but cannot be captured by any ground model set.

3.2. Separation

The situation here seems more difficult than for Exponentiation, because we have not yet been able to find a property on \( T \) equivalent to Separation. Indeed, it is questionable whether there is any such nice property, as discussed at the end of this sub-section. Nevertheless, we still have a theorem and some examples.

It is instructive to see why, in the proof of Separation in the main theorem, Full Separation did not hold, only Bounded. The problem came with the setting. Given \( \sigma \) and \( \langle \hat{x}, T \rangle \in \sigma^r \), we need to know whether to put \( \hat{x} \) into the subset \( \tau \) of \( \sigma \) defined by \( \phi \). We can certainly look for a neighborhood forcing \( \phi^r(\hat{x}) \) or its negation. But when forcing a universal, we need a fixed neighborhood \( J_r \) of \( r \) deciding each \( \phi^r(\hat{x}) \) simultaneously, and cannot afford to use a separate \( J_{\hat{x}^2} \) for each different \( \hat{x} \). Since all the parameters in that formula are ground model terms, it is not their meanings that could change over different open sets, but rather only the topology itself and what it makes true and false. So the natural hypothesis to say that this doesn’t happen is that, locally, all points look alike.

**Definition 3.2.** \( T \) is locally homogeneous around \( r, s \in T \) if there are neighborhoods \( J_r, J_s \) of \( r \) and \( s \) respectively and a homeomorphism of \( J_r \) to \( J_s \) sending \( r \) to \( s \).

An open set \( U \) is homogeneous if it is locally homogeneous around all \( r, s \in U \).

\( T \) is locally homogeneous if every \( r \in T \) has a homogeneous neighborhood.

**Lemma 3.3.** If \( U \) is homogeneous, \( \phi \) contains only ground model terms, and \( U \supseteq V \Vdash \phi \) (\( V \) non-empty), then \( U \Vdash \phi \).

**proof:** Let \( r \in V \). For \( s \in U \), let \( V_r \) and \( V_s \) be the neighborhoods \( f \) the homeomorphism given by the homogeneity of \( U \). \( f(\sigma) \) can be defined inductively
on terms \( \sigma \). (Briefly, hereditarily restrict \( \sigma \) to \( V_r \) and apply \( f \) to the second parts of the pairs in the terms.) \( f(\psi) \) is then \( \psi \) with \( f \) applied to the parameters. It is easy to show inductively on formulas that \( V_r \models \psi \) iff \( V_u \models f(\psi) \).

If \( \phi \) contains only ground model terms, then \( f(\phi) = \phi \). So \( U \) is covered by open sets that force \( \phi \). Hence \( U \models \phi \). \( \square \)

**Theorem 3.4.** If \( T \) is locally homogeneous then \( T \models \text{FullSeparation} \).

**proof:** As in the proof of Bounded Separation from the previous section, we have to show that, for any \( \sigma, T \models \exists Y \forall Z \ (Z \in Y \leftrightarrow Z \in \sigma \land \phi(Z)) \), only this time with no restriction on \( \phi \). The choice of witness \( Y \) is slightly different. For each \( r \) let \( K_r \ni r \) be homogeneous. Let \( \tau \) be \( \{\langle \sigma_i, J \cap J_i \rangle \mid \langle \sigma_i, J_i \rangle \in \sigma \) and \( J \models \phi(\sigma_i) \} \cup \{\langle \hat{x}, r \rangle \mid \langle \hat{x}, T \rangle \in \sigma^r \) and \( K_r \models \phi'(\hat{x}) \} \). The difference from before is that in the latter part of \( \tau \) membership is determined by what’s forced by \( K_r \) instead of by \( T \). We claim that \( \tau \) suffices: \( T \models \forall Z \ (Z \in \tau \leftrightarrow Z \in \sigma \land \phi(Z)) \).

For the first clause in forcing \( \forall \), let \( \rho \) be a term. We need to show \( T \models \rho \in \tau \leftrightarrow \rho \in \sigma \land \phi(\rho) \). By the first clause in forcing \( \rightarrow \), we have to show that for all \( J \) \( J \models \rho \in \tau \) iff \( J \models \rho \in \sigma \land \phi(\rho) \), which should be clear from the first part of \( \tau \). For the second clause in \( \rightarrow \) it suffices to show that for all \( J \subseteq K_r \) \( J \models \rho^r \in \tau^r \) iff \( J \models \rho^r \in \sigma^r \land \phi'(\rho^r) \). Regarding forcing membership, all of the terms here are ground model terms, so membership is absolute (does not depend on the choice of \( J \)). If \( \rho^r \) enters \( \tau^r \) because of the first part of \( \tau \)’s definition, then we have \( \sigma_i^r = \rho^r \), \( r \in J \models \phi(\sigma_i) \), \( r \in J_i \), and \( \langle \sigma_i, J_i \rangle \in \sigma \). By \( \text{(14) part 5) \) , some neighborhood \( J_r \) of \( r \) forces \( \phi'(\sigma_i^r) \). By the lemma just above (applied to \( K_r \cap J_r \), \( K_r \) forces the same. Hence we can restrict our attention to terms \( \rho^r \) which enter \( \tau \) because of \( \tau \)’s definition’s second part. Again by the preceding lemma, for \( J \) non-empty, \( J \models \phi'(\rho^r) \) iff \( K_r \models \phi'(\rho^r) \), which suffices. (For \( J \) empty, \( J \) forces everything.)

For the second clause in forcing \( \forall \), it suffices to show that \( K_r \models \rho \in \tau^r \leftrightarrow \rho \in \sigma^r \land \phi'(\rho) \). If any \( J \subseteq K_r \) forces \( \rho \in \tau^r \) or \( \rho \in \sigma^r \), then locally \( \rho \) is forced to be some ground model term, and we’re in the same situation as in the previous paragraph. \( \square \)

We would like to see to what degree we can turn the previous theorem into an iff. Toward that end, suppose \( T \) is not locally homogeneous. So we can choose \( r \in T \) which has no homogeneous neighborhood. That means every neighborhood of \( r \) contains two points, say \( s \) and \( t \), with no local homeomorphism sending \( s \) to \( t \). If there were local homeomorphisms from \( r \) to both \( s \) and \( t \), they could be composed to get one from \( s \) to \( t \). So one of \( s \) and \( t \) can be chosen to be \( r \).

Example 1: It is possible that there is a fixed \( s \) that can be chosen as a witness to \( r \)’s non-homogeneity from every neighborhood of \( r \). In particular, every open containing \( r \) also contains \( s \). If every open set containing \( r \) also contains \( s \) also
contained \( r \), then the function interchanging \( r \) and \( s \) and leaving everything else alone would be a homeomorphism contradicting the assumptions, so some neighborhood of \( s \) does not contain \( r \). The smallest example of such a space is the two-element space \( T = \{ r, s \} \), with opens \( T, \emptyset, \{ s \} \). Let \( \sigma \) be \( \{ 0, r \} \); that is, 0 gets into \( \sigma \) when we settle at \( r \). Let \( \phi \) be \( \forall x, y \neq y \lor \neg x \neq y \). Suppose \( Z \) were \( \{ x \in \sigma \mid \phi \} \):

\[
T \models \forall x \in Z \leftrightarrow x \in \sigma \land \phi.
\]  

In particular, settling at \( r \), we get

\[
T \models \forall x \in Z^r \leftrightarrow x = 0 \land \phi^r.
\]

or, more simply,

\[
T \models 0 \in Z^r \leftrightarrow \phi.
\]

As \( Z^r \) is a ground model term, \( T \) decides \( 0 \in Z^r \). But \( T \) does not decide \( \phi \); \( T \) does not force \( "\sigma \neq 1 \lor \neg \sigma \neq 1" \), but \( \{ s \} \) forces \( \phi \), both of which can be calculated by cranking through the definitions. Hence \( T \) does not force Separation.

Example 2: An example of the opposite kind is where there is no \( s \) in every open neighborhood of \( r \). Here let \( T \) be \( R^\geq 0 \). Let \( \sigma \) be \( \{ \langle 0, 0 \rangle \} \). Let \( \phi \) be

\[
\forall x \subseteq 1 \exists y ((0 \in x \rightarrow y = 0 \lor y = 1) \land (\neg \neg y = 0 \lor \neg \neg y = 1 \rightarrow 0 \in x)).
\]

If Separation held, we could form \( \{ x \in \sigma \mid \phi \} \). Settling at 0, we would have \( \{ 0 \mid \phi \} \) as a ground model set. But \( \phi \) is not decided in any neighborhood of 0. That’s because \( R^>0 \models \phi \), since \( y \) can be chosen to alternate between 0 and 1 on the disjoint intervals that constitute the support of \( x \). But on any neighborhood containing 0, instantiating \( x \) with \( \{ \langle 0, R^\geq 0 \rangle \} \) forces any such \( y \) to be the constant 0 or 1 on the positives, hence not not equal to 0 or not not equal to 1, but does not force \( x \) to be inhabited.

What these examples indicate is less that the failure of homogeneity leads to the failure of Separation, but rather that what is needed in such a construction is the transferability of a property of the underlying topology into the internal language of the set theory. More explicitly, all the proof of Separation needed was a \( K_r \ni r \), which may well depend on the choice of \( \phi \) and \( \sigma \), such that, for all \( \langle \hat{x}, T \rangle \in \sigma^r \), \( K_r \ni \phi^r(\hat{x}) \) or \( K_r \ni \neg \phi^r(\hat{x}) \). It is easy to see that the existence of \( r, \phi \), and \( \sigma \) for which there is no such \( K_r \) immediately provides a counter-example to Separation. This is apparently less than the homeomorphisms needed for local homogeneity. We find it unlikely that there would be a direct correspondence between any natural topological property and this feature of the forcing, so closely tied to set theory and its language. In contrast, there could well be such a topological property for a certain formula or class of formulas. It would be interesting to know to what degree this is possible, and why, even when the answer is simply “not at all.”

3.3. Replacement and Strong Collection

If Separation were to hold (in the presence of the other axioms from above), then Strong Collection would follow, which itself implies Replacement. Hence a
A powerful way to show that Separation is not forced is to give an example in which even Replacement fails. In the example below, the offending formula is a Boolean combination of $\Sigma_1$ formulas. This is about as strong a result as one could hope for, as further restrictions on the formula render Replacement provable.

Suppose, for instance, the function were $\Sigma_1$ definable: $\forall x \in A \exists y \exists z \phi(x,y,z)$, where $\phi$ is $\Delta_0$. By Collection, there is a bounding sets for triples $(x,y,z)$ with $\phi(x,y,z)$, as $x$ runs through $A$. By $\Delta_0$ Separation, we can restrict that bounding set to only triples where the first component is in $A$ and the triple satisfies $\phi$. Then by $\Delta_0$ Separation again, we can project onto the second coordinate. Perhaps there is still some wiggle room between $\Sigma_1$ and Boolean combinations thereof, such as in the number and use of implications (including negations), if one wanted to fine tune this result, but there’s certainly not a lot.

Example 3: Let $T_n$ ($n > 0$) be the standard space for collapsing $\aleph_n$ to be countable: elements are injections from $\aleph_0$ to $\aleph_n$, an open set is given by a finite partial function of the same type, an element is in that open set if it is compatible with the partial function. Let $T$ be the disjoint union of the $T_n$s adjoined with an extra element $\infty$: $\bigcup_n T_n \cup \{\infty\}$. A basis for the topology is given by all the open subsets of each $T_n$, plus the basic open neighborhoods of $\infty$, which are all of the form $\bigcup_{n \geq N} T_n \cup \{\infty\}$ for some $N$.

This $T$ falsifies Replacement. To state the instance claimed to be falsified, we need several parameters. One is $\{\langle n, \infty \rangle \mid n \in \omega\}$, which we will call $\omega^-$. Another is the internalization of the function $n \mapsto \aleph_n$ ($n \in \omega$), which we will refer to via the free use of the notation $\aleph_n$, even when $n$ is just a variable. Finally, we will implicitly need $\omega$ in the assertion “$X$ is countable,” which is the abbreviation for what you think (i.e. the existence of a bijection with $\omega$). Note that “$X$ is uncountable” is taken as the negation of “$X$ is countable.”

**Definition 3.5.** Let $\phi(x,z)$ be the conjunction of:

1. $z = 0 \lor z = 1$
2. $z = 0 \leftrightarrow \aleph_z$ is uncountable
3. $z = 1 \leftrightarrow \neg \aleph_z$ is countable.

**Proposition 3.6.** $T \not\models \forall x \in \omega^- \exists y \phi(x,y) \rightarrow \exists f \forall x \in \omega^- \phi(x,f(x))$.

**proof:** First we show that $T$ forces the antecedent $\forall x (x \in \omega^- \rightarrow \exists y \phi(x,y))$.

For the first clause in forcing $\forall$, we need to show that for all $\sigma$ $T \models \sigma \in \omega^- \rightarrow \exists y \phi(\sigma,y)$. The first clause in forcing that implication is vacuous, as no open set will force $\sigma \in \omega^-$. The second clause is vacuous for all choices of $r$ except $\infty$, as then $\omega^- \models \phi(x,y)$ is empty. Finally, for $r = \infty$, it suffices to show that $T \models \exists y [(y = 0 \lor y = 1) \land (y = 0 \leftrightarrow \aleph_y$ is uncountable) $\land (y = 1 \leftrightarrow \neg \aleph_y$ is countable)]. The term which is 0 on $\bigcup_{0 < \alpha < \omega} T_\alpha$ and 1 on the rest of $T$ suffices.

The second clause in forcing $\forall$ is similar.

Since $T$ forces the antecedent of the conditional, it suffices to show that $T$ does not force the consequent: $T \not\models \exists f \forall x \in \omega^- \phi(x,f(x))$. If that were not the case, there would be a term (we will ambiguously refer to as $f$) and a neighborhood $J$ of $\omega$ such that $J \models \forall x \in \omega^- \phi(x,f(x))$. By lemma 2.4 part 5), there
would be a $K \ni \infty$ such that $K \vDash \forall x \in \omega \phi(x, f^\infty(x))$. $K$, being open, contains a set of the form $\bigsqcup_{n > N} T_n$. Let $M$ be $N + 1$. So $K \vDash \phi(M, f^\infty(M))$. But $f^\infty(M)$ is a ground model term, and so is (forced by $K$ to be) equal to $0$ or $1$. Hence either $K \vDash \check{\aleph}_M$ is uncountable or $K \vDash \neg\neg\check{\aleph}_M$ is countable. But neither is the case, since $K \supseteq T_N \vDash \check{\aleph}_M$ is uncountable and $K \supseteq \bigsqcup_{n > N} T_n \vDash \check{\aleph}_M$ is countable. \hfill $\square$

Since the preceding is an example where Separation fails, by the results of the previous sub-section, local homogeneity must fail too. Arguably, though, that’s not the essence of the construction. In this case, what determined the choice of $y$ was a different open set for each $x$. No given neighborhood of $\infty$ sufficed, because it would have been split for some $x$ into a sub-neighborhood forcing $0$ and another forcing $1$. So it seems to be a matter of connectedness.

**Theorem 3.7.** If $T$ is locally connected then $T \vDash$ Replacement.

**proof:** Sketch of proof: When showing the existence of a good bounding set (cf. the proof of Collection in the main theorem), work in a connected neighborhood $J_r$ of any given point $r$. For every $J \vDash \sigma_i \in \sigma \land \phi(\sigma_i, \tau_i)$, include $\langle \tau_i, J \cap J_r \rangle$ in the bounding set $\tau$. As for the settling, for any $\langle \check{x}, T \rangle \in \sigma'$, by the totality of $\phi$, $J_r$ is covered by sets $K$ forcing $\phi(\check{x}, \tau_K)$, for some choice of $\tau_K$. By settling, $\tau_K$ can be taken to be ground model terms $\check{y}_K$. By uniqueness, the $\check{y}_K$s have to agree wherever the $K$s overlap. Since they’re ground model terms, they don’t vary, so are the same ground model terms on all $K$s that overlap. By the connectedness of $J_r$, all the $\check{y}_K$s are equal, say $\check{y}$. Include $\langle \check{y}, r \rangle$ in $\tau$. \hfill $\square$

Analogously with Separation, we do not believe that there is an equivalence between local connectedness and Replacement. Rather, it’s likely that what’s at stake is some kind of definitional connectedness, whether an open set can be split into disjoint clopens that are the truth values of different statements. It would be nice to see a nice topological equivalent of that property or any interesting fragment of it.

Finally, it would be of interest to see any examples or theorems along these lines pertaining to Strong Collection that cannot be reduced to ones about Replacement or Separation.

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