ABSTRACT

Consider a social network where only a few nodes (agents) have meaningful interactions in the sense that the conditional dependency graph over node attribute variables (behaviors) is sparse. A company that can only observe the interactions between its own customers will generally not be able to accurately estimate its customers’ dependency subgraph: it is blinded to any external interactions of its customers and this blindness creates false edges in its subgraph. In this paper we address the semiblind scenario where the company has access to a noisy summary of the complementary subgraph connecting external agents, e.g., provided by a consolidator. The proposed framework applies to other applications as well, including field estimation from a network of awake and sleeping sensors and privacy-constrained information sharing over social subnetworks. We propose a penalized likelihood approach in the context of a graph signal obeying a Gaussian graphical models (GGM). We use a convex-concave iterative optimization algorithm to maximize the penalized likelihood. The effectiveness of our approach is demonstrated through numerical experiments and comparison with state-of-the-art GGM and latent-variable (LV-GGM) methods.

Index Terms— Network topology inference, Gaussian graphical model, data privacy, convex-concave procedure, alternating direction methods of multipliers

1. INTRODUCTION

Learning a dependency graph given relational data is an important task for sensor network analysis [1, 2] and social network analysis [3]. In many situations, however, a learner may only have access to data on a subgraph: the learner is blinded to the rest of the graph, e.g., due to energy constraints or privacy concerns. For instance, in a sensor network with limited power budget [1, 2], a subset of sensors that were actively collecting data in the recent past may have gone into sleeping mode at the current time. As a result, the fusion center only acquires measurements from the active sensors at the current time. In this scenario, without any information regarding the unobserved external data, confounding marginal correlations may exist between observed variables that are conditionally uncorrelated. In this paper, we consider the semiblind scenario where, in addition to the observed internal data (e.g., measurements of active sensors), the learner receives a noisy summary about partial correlations of data from unobserved external nodes (e.g., previously measured spatial correlation between sleeping sensors). We call this the semiblind scenario. The goal of this paper is to learn the network topology using the observed internal data as well as the partially shared information from external sources.

We consider a random graph signal that follows Gaussian graphical model (GGM) [4]. The GGM can be learned efficiently via sparse inverse covariance estimation [5, 6, 7, 8]. However, due to effects of marginalization [4] over latent factors (the unobserved external nodes), these aforementioned methods suffer from a significant loss in terms of estimation accuracy. In [9], a latent variable Gaussian graphical model (LV-GGM) was proposed to learn the sparse target subnetwork via sparse and low-rank matrix separation. Theoretical analysis [9, 10] shows that the maximum likelihood estimator of LV-GGM is unbiased under some mild conditions. The LV-GGM proposed in [9, 10] assumes a blind scenario where no information about the latent variable subgraph is available. Here we treat the semiblind scenario where the learner has access to noisy information about this subgraph and its associated dependency matrix.

Specifically, we propose a Decayed-influence Latent variable Gaussian Graphical Model (DiLat-GGM) that in this model the influence of each latent variable decays over the underlying network. Examples include, but are not limited to, spatial correlation of sensor networks [11] and propagation of social influence [12]. In contrast with LV-GGM, DiLat-GGM leads to a special non-convex optimization problem, which is identified with a difference of convex (DC) program. An efficient algorithm based on convex-concave procedure (CCP) [13, 14] and an alternating direction method of multipliers (ADMM) [15] is employed to find a locally optimal solution. Extensive experiments are provided to show the superiority of DiLat-GGM over existing methods in terms of estimation accuracy.
accuracy in the semi-blind scenario.

2. PRELIMINARIES AND PROBLEM FORMULATION

We begin by introducing some notations. Let \( G = (V, E) \) be an undirected unweighted graph, where \( V \) is the vertex set with cardinality \( |V| = n \), and \( E \) is the edge set. Let \( x = [x_1, \ldots, x_n] \) be a random vector over vertices in \( V \), following the multivariate Gaussian distribution \( x \sim \mathcal{N}(0, \Theta^{-1}) \), where \( \Theta := \Sigma^{-1} \in \mathbb{R}^{n \times n} \) is the precision matrix, namely, the inverse of covariance matrix \( \Sigma \). Assume that \( x \) obeys the Markov property with respect to \( G \), i.e., \( \Theta_{ij} = 0 \) for all \( i \neq j \) and \( (i, j) \notin E \), where \( \Theta_{ij} \) denotes the \((i, j)\)-th entry of \( \Theta \).

2.1. Problem Statement

Without loss of generality, the vertex set of \( G \) is partitioned into two nonoverlapping subsets \( V_1 \) and \( V_2 \) with \( |V_1| = n_1 \) and \( |V_2| = n_2 \), respectively. Here the sub-graph \( G_1 = (V_1, E \cap (V_1 \times V_1)) \) is called the target sub-network associated with the precision matrix \( \Theta_1 \), and \( G_2 = (V_2, E \cap (V_2 \times V_2)) \) is called the external network with \( \Theta_2 \), where \( \Theta_i \in \mathbb{R}^{n_i \times n_i} \) denotes the submatrix of \( \Theta \) indexed by \( V_i \) for \( i = 1 \) and \( 2 \). We assume that only an internal dataset \( X_1 \in \mathbb{R}^{n_1 \times m} \) (namely, \( m \) i.i.d samples of \( x_{V_1} \)) and a summary information \( \Theta_2 \) (namely, an estimate of \( \Theta_2 \)) are available to the learner. Our task is to estimate the network topology of \( G_1 \) (in terms of \( \Theta_1 \)) only based on data \( X_1 \) and the shared summary information \( \Theta_2 \) from the external source \( G_2 \). We emphasize that data samples of \( x_{V_2} \) (latent variables) over \( G_2 \) are not shared with \( G_1 \) due to the energy or privacy concerns. Figure 1 provides an overview of the studied problem in this paper.

2.2. Blind Sub-network Inference via LV-GGM

In the blind scenario, the learner only has access to \( X_1 \), from which the marginal covariance matrix \( \Sigma_1 := X_1 X_1^T / m \) can be constructed. To recover the sub-network \( G_1 \), the following partitioned matrix inverse identity specifies the relation between the inverse of the mean of \( \Sigma_1 \) (the ensemble marginal covariance) and the associated block \( \Theta_1 \) of \( \Theta \) \[10\]

\[
\hat{\Theta}_1 := (\Sigma_1)^{-1} = \Theta_1 - \Theta_{12} (\Theta_2)^{-1} \Theta_{21} := C - M \quad (1)
\]

where \( \hat{\Theta}_1 \) is the marginal precision matrix over \( x_1 \), recalling that \( \Theta_1 \) and \( \Theta_2 \) are submatrices of the global precision matrix \( \Theta \) over \( V_1 \) and \( V_2 \), and \( \Theta_{21} \in \mathbb{R}^{n_2 \times n_1} \) is the partial cross-covariance matrix between \( V_1 \) and \( V_2 \). As noted in \[9\] \[10\], the marginal precision matrix \( \hat{\Theta}_1 \) can be decomposed into a sparse matrix \( \Theta_1 \), which is associated with \( G_1 \), plus a low-rank matrix that characterizes the effect of marginalization.

In \[9\], the latent variable Gaussian graphical model (LV-GGM) was introduced to find the separation \( (C, M) \) in \[1\] by maximizing the regularized marginal log-likelihood objective function

\[
\text{minimize}_{C,M} - \log \det(C - M) + \text{tr} \left( \hat{\Sigma}_1 (C - M) \right) + \alpha \|C\|_1 + \beta \|M\|_1
\]

subject to \( C - M \succeq 0, \quad M \succeq 0, \) where \( \|M\|_1 \) is the nuclear norm of \( M \), \( \text{tr} (\cdot) \) and \( \log \det (\cdot) \) are trace and log-determinant operator, \( A \succeq 0 \) means that \( A \) is positive semidefinite, \( \hat{\Sigma}_1 \) is the sample covariance, \( \alpha \) and \( \beta \) are regularization parameters for the \( \ell_1 \)-norm and the nuclear-norm, respectively. Asymptotic analysis \[9\] \[10\] shows that the resulting estimator of LV-GGM is unbiased under some mild conditions.

2.3. Semiblind Sub-network Inference under Decayed Influence

In this semiblind scenario the learner has access to both the marginal sample covariance \( \hat{\Sigma}_1 \) and a noisy version of \( \Theta_2 \). It is notable that \( \Theta_2 \) can be determined without knowledge of the full covariance matrix \( \Sigma \), all that is required is the covariance of the buffered external nodes \[16\], defined by first-order neighboring nodes of vertices in \( V_2 \). Compared with LV-GGM, we consider a decayed influence model: the influence of each node decays while propagating along the graph. Therefore, latent variables that are at large hop distance from the target network \( G_1 \) have little impact on the inference of \( G_1 \). This implies that \( \Theta_{21} \) in \[1\] maintains row-sparsity structure. Specifically, let \( B := \Theta_{21} \Theta_1^{-1} \in \mathbb{R}^{n_1 \times n_2} \) and \( \mu_{21} := B^T x_1 \) be the conditional mean of \( x_{V_2} \) given \( x_{V_1} \). The low rank term in \[1\] can be reparameterized as \( M := B \Theta_2 B^T \). Motivated by \( 2 \), Decayed-influence Latent variable Gaussian Graphical Model (DiLat-GGM) is formulated as

\[
\text{minimize}_{C,B} - \log \det(C - B \Theta_2 B^T) + \alpha \|C\|_1 + \text{tr} \left( \hat{\Sigma}_1 (C - B \Theta_2 B^T) \right) + \beta \|\Theta_2 B^T\|_{2,1}
\]

subject to \( C - B \Theta_2 B^T \succeq 0, \)

where \( \alpha \) and \( \beta \) are positive regularization parameters, \( \hat{\Theta}_2 \succeq 0 \) is the available summary information about \( \Theta_2 \) and
where \( \ell_{2,1} \) norm that induces row sparsity of \( \hat{\Theta}_{21} \).

The differences between LV-GGM and the proposed DiLat-GGM are three-fold. First, compared to the blind LV-GGM, the semiblind DiLat-GGM utilizes the information about external network structure and takes into account their influence on the target network. Second, DiLat-GGM explicitly learns the linear mapping \( B \), which enables us to estimate the hidden variables via the conditional mean \( \mu_{21} \). Thus, it can be used to recover graph signals on \( \mathcal{V}_2 \) under GGM. Third, instead of inducing low rank (via nuclear norm) in (2), a different row-sparsity promoting strategy is imposed through \( \ell_{2,1} \) norm, which explicitly drops irrelevant features during the training. However, unlike LV-GGM, DiLat-GGM does not lead to a convex optimization problem. In the next section, we propose an efficient optimization method to solve problem (3).

### 3. Optimization Method

We begin by reformulating problem (3) as below

\[
\begin{align*}
\text{minimize}_{C,B} & \quad - \log \det R(C, B; \hat{\Theta}_2) + \alpha \left\| C \right\|_1 \\
+ & \beta \left\| \hat{\Theta}_2 B^T \right\|_{2,1} \\
\text{subject to} & \quad R(C, B; \hat{\Theta}_2) \succeq 0,
\end{align*}
\]

where \( R(C, B; \hat{\Theta}_2) := \left[ \frac{C}{B^T \hat{\Theta}_2} \right] \) is linear in \((C, B)\), and we have used the fact that \( R(C, B; \hat{\Theta}_2) \succeq 0 \) is equivalent to \( C - \hat{\Theta}_2 B^T \succeq 0 \) under \( \hat{\Theta}_2 \succeq 0 \), as provided by the Shur complement theorem [17]. Note that problem (4) is non-convex, since \( \log \det R(C, B; \hat{\Theta}_2) \) is a difference of convex (DC) functions \( \log \det(\Sigma; C) \) and \( g(B) := \log \det(\Sigma; B^T \hat{\Theta}_2 B^T) \), where the former is the linear with respect to \( C \), and the latter is quadratic with respect to \( B \).

Due to the DC-type nonconvexity, (4) can be solved using the convex-concave procedure (CCP) [13, 14]. Specifically, at each iteration of CCP, it convexifies the concave function \( -g(B) \) through linearization

\[
g(B; t) = g(B_t) + \nabla g(B_t)^T (B - B_t),
\]

where \( g(B) = \log \det(\Sigma; B^T \hat{\Theta}_2 B^T) \) in (4), \( t \) is the iteration index of CCP, and \( \nabla g(B) = 2 \Sigma B^T \hat{\Theta}_2 B^T \) yields the gradient of \( g(B) \) at point \( B_t \). Upon defining \( D_t := B_t \hat{\Theta}_2 \) and substituting (5) into (4), CCP iteratively solves the convex program,

\[
\begin{align*}
\text{minimize}_{C,B} & \quad - \log \det R(C, B; \hat{\Theta}_2) + \alpha \left\| C \right\|_1 \\
+ & \beta \left\| \hat{\Theta}_2 B^T \right\|_{2,1} \\
\text{subject to} & \quad R(C, B; \hat{\Theta}_2) \succeq 0.
\end{align*}
\]

### Algorithm 1 DiLat-GGM via Convex-concave procedure

**Require:** Marginal covariance \( \hat{\Sigma}_1 \) of \( x_1 \in \mathbb{R}^{m_1} \). The parameters \( \alpha, \beta > 0 \). A noisy summary matrix \( \hat{\Theta}_2 > 0 \in \mathbb{R}^{n_2 \times n_2} \) of \( x_2 \).

1. **Initialize:** Random initialization or by heuristic according to [20]. Return \((C_0, B_0)\).
2. **for** \( t = 1, \ldots, T \) or until converge **do**
3. Construct matrix \( D_{t-1} := B_{t-1} \hat{\Theta}_2 \)
4. **Solve the subproblem** (7) via ADMM. Return \((C_t, B_t)\).
5. **end for**

**Ensure:** Output \( C_T := [R_T]_{V_1 \times V_1} \) and \( B_T := [R_T]_{V_1 \times V_2} \).

By introducing auxiliary variables \( R := R(C, B; \hat{\Theta}_2), P := [P_{21}, P_{22}] = R \) and \( W := \hat{\Theta}_2 P_{21} \), we rewrite problem (6) as

\[
\begin{align*}
\text{minimize}_{R,P,W} & \quad - \log \det R + \alpha \left\| P_1 \right\|_1 \\
+ & \beta \left\| W \right\|_{2,1} + 1 \{ R \succeq 0 \}
\end{align*}
\]

subject to \( R = P, P_2 = \hat{\Theta}_2^{-1}, W = \hat{\Theta}_2 P_{21} \), where \( S := \left[ \frac{\hat{\Theta}_2}{-\hat{\Theta}_2}, \frac{-\hat{\Theta}_2^T}{\hat{\Theta}_2} \right], 1 \{ A \} \) is an indicator function of set \( A \). Problem (7) now fits the standard form of ADMM, a convex program with equality constraints. Based on the augmented Lagrangian [7], ADMM leads to three subproblems with respect to \( R, P \) and \( W \), respectively. The proposed ADMM-based algorithm yields the complexity \( O(n^3) \) due to the DSD step while solving the subproblem with respect \( R \) [15]. It is also shown in [19] that CCP converges to a local stationary point regardless of choice of initial points. Here we choose the initial point \((C_0, B_0)\) heuristically according to [20]. We summarize the CCP-based algorithm to solve DiLat-GGM in Algorithm 1.

### 4. Experiments

In this section, we compare the performance of the semiblind DiLat-GGM with three blind graph topology learning algorithms: the graphical Lasso (GLasso) [6]; the latent variable Gaussian graphical model (LV-GGM) [9] and the generalized Laplacian learning (GLap) [21] which is a variant of GLasso.

In the following experiments, we generate a network \( G = (V, \mathcal{E}) \) with \( |V| = n \) and then compute the normalized Laplacian matrix \( L \). The random graph signal \( x \in \mathbb{R}^n \) is drawn from \( \mathcal{N}(0, L + \epsilon I)^{-1} \) where \( \epsilon = 10^{-3} \). We compare over different graph topologies, including the complete binary tree with height \( h \), the grid network with width \( w \) and height \( h \) and the Erdős-Rényi graph with size \( n \) and edge probability \( p \). The vertex set \( V_1 \) of sub-network \( G_1 \) is sampled randomly from \( V \) with \( |V_1| = n_1 \). The edge set \( \mathcal{E}_1 = \mathcal{E} \cap (V_1 \times V_1) \). The data sample matrix \( X \) is composed of \( m \) i.i.d realizations of \( x \), where we choose \( m = 500 \).

To measure the accuracy of semiblind to blind topology inference (as compared with ground truth), we consider the Jaccard distance [22] between two sets \( A, B dist_J(A, B) = \)

\[
\frac{|A \cap B|}{|A \cup B|}.
\]
GLasso exploits more hidden structures in topology estimation, and thus yields a better performance.

In Figure 2(a)-(d), we show the learned network resulting from GLasso, LV-GGM and DiLat-GGM under different choices of optimal parameters $\alpha, \beta$. As we can see, DiLat-GGM edge estimates have lower miss rate and false positive rate compared to GLasso and LV-GGM. The GLasso, however, has a higher false positive rate in boundary vertices due to the effect of marginalization bias.

In Figure 2(e), we demonstrate the sensitivity of the DiLat-GGM model under different choices of regularization parameter $\alpha$ and $\beta$. The underlying network is shown in Figure 2(a). For DiLat-GGM, $\Theta_2$ is the same as above. We observe that when $\alpha$ increases, the learned graph becomes overly sparse and Jaccard distance error increases. The choice of $\beta$ controls the row sparsity of the conditional cross precision $\Theta_{21}$, if it is too small, the DiLat-GGM cannot capture the local effect of the latent variables, which decreases its performance in sub-network learning. In Figure 2(f), we evaluate the robustness of DiLat-GGM when the pre-defined matrix $\Theta_2$ is corrupted by noise of different levels. As expected, when the Signal-to-Noise Ratio (SNR) $\|L_2\|_F^2/\sigma^2_L$ decreases, the performance of DiLat-GGM decreases. However, DiLat-GGM is robust for a wide range of choice of $\sigma_L$.

5. CONCLUSION
In this paper, we proposed a semiblind subgraph estimation algorithm called DiLat-GGM that learns a sparse sub-network topology with noisy information about the network topology external to the subgraph. We show that DiLat-GGM leads to a DC-type nonconvex optimization problem, whose local optimal solution was computed via an efficient CCP-based algorithm. Extensive numerical results show that the proposed semiblind DiLat-GGM outperforms the state-of-the-art blind sparse GGMs in terms of topology estimation accuracy. In the future, we will apply the proposed algorithm to larger scale datasets and develop a strategy to estimate both $G_1$ and $G_2$.
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7. APPENDIX

7.1. Solving subproblem using ADMM

Following the ADMM procedure, we form an augmented Lagrangian for problem as

\[ \mathcal{L}(R, P) = -\log \det R + \text{tr}(SR) + \alpha \|P_1\|_1 + \beta \|W\|_{2,1} + \frac{1}{2} \{R \succeq 0\} \]

\[ + \text{tr} \left( \Lambda^T (R - P) \right) + \frac{\gamma}{2} \|R - P\|_F^2 \]

where \( \Lambda \in \mathbb{R}^{n \times n} \) and \( \Lambda_w \) form dual matrices. ADMM minimizes the augmented Lagrangian via block coordinate descent. In specific, it solves two separable problems: minimize the objective function \( \alpha \|P_1\|_1 + \frac{\gamma}{2} \|R - P\|_F^2 \)

\[ + \beta \|W\|_{2,1} + \text{tr} \left( \Lambda^T (W - \tilde{\Theta}_2 P_{21}) \right) \]

where \( \tilde{\Theta}_2 \) is the eigen-decomposition. This is an equivalent to

[10]

\[ \text{Prox}_{\mathcal{P}_{21}}(Z, \xi) := \min_{Z, \xi} \frac{1}{2\xi} \|P_{21} - Z\|_F^2 + \frac{1}{2\xi_w} \|\tilde{\Theta}_2 P_{21} - Z\|_F^2 \]

and \( P_2 = T \). It involves three proximal operators: first,

\[ \text{Prox}_{\mathcal{P}_{1}}(Z, \xi) := \min_{Z, \xi} \frac{1}{2\xi} \|P_1 - Z\|_F^2 + \alpha \|P_1\|_1 \]

which is equivalent to

[12]

Second,

\[ \text{Prox}_{\mathcal{P}_{21}}(Z, \xi, \xi_w) := \min_{Z, \xi, \xi_w} \frac{1}{2\xi} \|P_{21} - Z\|_F^2 + \frac{1}{2\xi_w} \|\tilde{\Theta}_2 P_{21} - Z\|_F^2 \]

\[ := \text{soft-threshold}(Z, \xi, \xi_w) \]

where \( \tilde{\Theta}_2 = U \text{diag} \left( \lambda_i \right) U^T \) is the eigen-decomposition. And the proximal operator

[13]

\[ \text{Prox}_{\mathcal{P}_{21}}(Z, \xi, \xi_w) := \min_{Z, \xi, \xi_w} \frac{1}{2\xi} \|P_{21} - Z\|_F^2 + \beta \|W\|_{2,1} \]

which has optimal solution \( W \) with \( i \)-th row

\[ W_i = \left( 1 - \frac{\beta \xi}{\|Z\|_2} \right) Z_i, \quad i = 1, \ldots, n_2 \]

Finally, we have the dual updates

\[ \Lambda^{(t)} := \Lambda^{(t-1)} - \rho (R - P) \]

\[ \Lambda_w^{(t)} := \Lambda_w^{(t-1)} + \rho_w (W - \tilde{\Theta}_2 P_{21}) \]

The algorithm of ADMM is summarized in Algorithm 2.
Algorithm 2 DiLat-GGM subproblem via ADMM

Require: Positive definite matrix $S \succ 0$ and $S \in \mathbb{R}^{n \times n}$. The nonnegative regularization parameter $\alpha, \beta > 0$. The pre-defined nonegative definite matrix $\tilde{\Theta}_2 \succeq 0$ and $\tilde{\Theta}_2 \in \mathbb{R}^{n_2 \times n_2}$. Let $T = \tilde{\Theta}_2^{-1}$. Let $n_1 = n - n_2$. Dual update parameter $\mu, \mu_w > 0$.

1: Initialize: Choose an random matrix $R^{(0)} = \begin{bmatrix} R_1^{(0)} & R_2^{(0)} \\ R_2^{(0)} & R_1^{(0)} \end{bmatrix} \in \mathbb{R}^{n \times n}$ and $R^{(0)} \succ 0$. $\Lambda^{(0)} = 0 \in \mathbb{R}^{n \times n} = \begin{bmatrix} \Lambda_1^{(0)} & \Lambda_2^{(0)} \\ \Lambda_2^{(0)} & \Lambda_1^{(0)} \end{bmatrix}$.

$\hat{\Lambda}^{(0)} = 0 \in \mathbb{R}^{n_2 \times n_1}$. Let $P^{(0)} = \begin{bmatrix} P_1^{(0)} & P_2^{(0)} \\ P_2^{(0)} & P_1^{(0)} \end{bmatrix} = R \in \mathbb{R}^{n \times n}$. Choose $W^{(0)} = \hat{\Theta}_2 P^{(0)}$.

2: for $t = 1, \ldots, T$ or until converge do

3: Find $P_1^{(t)} \in \mathbb{R}^{n_1 \times n_1}$ via $P_1^{(t)} = \text{Prox}_{P_1, \alpha} \left( R_1^{(t-1)} + \mu \Lambda_1^{(t-1)} \right)$ as in [12];

4: if $\tilde{\Theta}_2 := \text{diag} \left( \tilde{\Theta}_2 \right)$ then

5: Find $P_2^{(t)} \in \mathbb{R}^{n_2 \times n_2}$ via $P_2^{(t)} = \text{Prox}_{P_2^t, \beta} \left( \tilde{\Theta}_2 P_2^{(t-1)} + \mu \Lambda_2^{(t-1)} \right)$ as in [?];

6: else

7: Find $W^{(t)} \in \mathbb{R}^{n_2 \times n_1}$ via $W^{(t)} = \text{Prox}_{W, \beta} \left( \tilde{\Theta}_2 P_2^{(t-1)} - \mu_w \Lambda_2^{(t-1)} \right)$ as in [14];

8: Find $P_2^{(t)} = \text{Prox}_{P_2^t} \left( R_2^{(t-1)} + \mu \Lambda_2^{(t-1)}, W^{(t)} + \mu_w \Lambda_2^{(t-1)}, \mu, \mu_w \right)$ as in [13];

9: Update dual variables $\Lambda$.

$$\Lambda^{(t)} = \Lambda^{(t-1)} + \frac{1}{\mu_w} \left( \tilde{\Theta}_2 P_2^{(t)} - W^{(t)} \right)$$

10: end if

11: Set $P_2^{(t+1)} = T$ and $P_1^{(t+1)} = \left( P_2^{(t+1)} \right)^T$. Construct $P^{(t)}$.

12: Find $R^{(t)} \in \mathbb{R}^{n \times n}$ via $R^{(t)} = \text{Prox}_{R, \alpha} \left( P^{(t)} - \mu \Lambda^{(t-1)} \right)$ as in [10];

13: Update dual variables $\Lambda$.

$$\Lambda^{(t)} = \Lambda^{(t-1)} + \frac{1}{\mu} \left( R^{(t)} - P^{(t)} \right)$$

14: end for

Ensure: Output $(R^{(T)}, P^{(T)})$ if $\tilde{\Theta}_2$ is diagonal and $(R^{(T)}, P^{(T)}, W^{(T)})$ otherwise.