STABILITY OF THE WAVE EQUATION WITH LOCALIZED KELVIN-VOIGT DAMPING AND BOUNDARY DELAY FEEDBACK

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ABSTRACT. We study the stabilization problem for the wave equation with localized Kelvin–Voigt damping and mixed boundary condition with time delay. By using a frequency domain approach we show that, under an appropriate condition between the internal damping and the boundary feedback, an exponential stability result holds. In this sense, this extends the result of [19] where, in a more general setting, the case of distributed structural damping is considered.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a boundary $\Gamma$ of class $C^2$. We assume that $\Gamma$ is divided into two open parts $\Gamma_0$ and $\Gamma_1$, i.e. $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\text{meas} \Gamma_i \neq 0$, $i = 0, 1$.

In this domain $\Omega$, we consider the initial boundary value problem

\begin{align*}
    u_{tt}(x,t) - \Delta u(x,t) - \text{div} \left( a(x) \nabla u_t(x,t) \right) &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \quad (1.1) \\
    u(x,t) &= 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \quad (1.2) \\
    \frac{\partial u}{\partial \nu}(x,t) &= -a(x) \frac{\partial u_t}{\partial \nu}(x,t) - ku_t(x,t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \quad (1.3) \\
    u(x,0) &= u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \quad (1.4) \\
    u_t(x,t) &= f_0(x,t) \quad \text{in} \quad \Gamma_1 \times (-\tau,0), \quad (1.5)
\end{align*}

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative of $u$. Moreover, $\tau > 0$ is the time delay, $k$ is a real number and $a(x) \in L^\infty(\Omega)$ satisfies

\[ a(x) \geq 0 \quad \text{a.e.} \quad \Omega, \quad a(x) \geq a_0 > 0 \quad \text{a.e.} \quad \omega, \]

where $\omega \subset \Omega$ is an open neighborhood of the part $\Gamma_1$ of the boundary that is supposed to be connected and such that $\text{meas} (\omega \cap \Gamma_0) > 0$. The initial datum $(u_0, u_1, f_0)$ belongs to a suitable space.

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We are interested in giving an exponential stability result for such a problem under a suitable relation between the function \( a(x) \) and the constant \( k \).

We will show that, under some geometrical assumptions described below and the condition

\[
a_0 > |k| C_P, \tag{1.6}
\]

where \( C_P \) is a sort of Poincaré constant, the energy of the solutions of system (1.1)–(1.5) satisfies a uniform exponential decay estimate.

More precisely, \( C_P \) is the smallest positive constant such that

\[
\int_{\Gamma_1} |v|^2 d\Gamma \leq C_P \int_\Omega |\nabla v|^2 dx, \forall v \in H^1_{0_0}(\Omega), \tag{1.7}
\]

where, as usual,

\[
H^1_{0_0}(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \}. \]

The stability problem for the wave equation with local Kelvin-Voigt damping and null Dirichlet boundary condition has been first considered by Liu and Rao [16], without any delay term. They proved an exponential stability result under suitable assumptions. We also refer to [3, 4, 12, 13, 14, 9, 10, 11, 15, 22] for stability results, in absence of time delays (i.e. \( \tau = 0 \)) and for \( k > 0 \), in the most studied case \( a \equiv 0 \), when \( \Gamma_1 \) satisfies a control geometric property.

Time delay effects are often present in applications and physical models and it is well-known that a delay arbitrarily small may induce instability phenomena in several evolution problems, which are uniformly stable in absence of delay. In particular, this is the case of wave type equations (see e.g. [5, 6]). Therefore, it is important to look for feedback laws which are robust with respect to (small) delays. We refer to [18, 21] for stability results for wave equations with delay in the case of frictional damping or standard boundary dissipative damping.

The problem (1.1)–(1.5), in the particular case of a distributed damping, i.e.

\[
a(x) = a_0 \text{ a.e. } x \in \Omega, \text{ with } a_0 > 0,
\]

has been investigated in [19] (see also Morgul [17] who proposed a class of dynamic boundary controllers in the 1-d case). More precisely, there we consider abstract second-order evolution equations with (dynamic) boundary feedback laws with delay and distributed structural damping and prove an exponential stability result under a suitable condition between the internal damping and the boundary conditions. The problem at hand enters in that abstract framework in the case of \( a(x) \) constant. We refer also to [1] and [2] for the analysis of related problems for wave equations with distributed Kelvin–Voigt damping and time delay effects.

The proof in [19] relies mainly on multipliers arguments allowing to obtain appropriate estimates for suitable Lyapunov functionals. The case of a local structural damping is more difficult to deal with, due to the unboundedness of the damping. Then, the analysis of [19] cannot be performed here. Our stability result will be now obtained by using a frequency domain approach introduced by Huang [8] and Prüss [20].

First of all we still consider the problem without time delay and extend the analysis of Liu and Rao [16] to the case of a mixed boundary condition, Dirichlet on the part \( \Gamma_0 \) of the boundary and dynamic boundary condition on \( \Gamma_1 \). Then, we use the stability result for the undelayed problem to obtain the exponential stability of the problem with delay, under the condition (1.6). Note that, as suggested from some counterexamples of [19], such a condition seems to be optimal in order to have stability.
The paper is organized as follows. In sect. 2 we give a well–posedness result by using semigroup theory; in sect. 3 we prove the stability for the delay problem by using the stability of the undelayed one which will be proved in sect. 4.

2. Well-posedness of the problem. In this section we will give well–posedness results for problem (1.1)–(1.5) using semigroup theory.

Like in [18] we introduce the auxiliary unknown
\[ z(x, \rho, t) = u_t(x, t - \tau \rho), \quad x \in \Gamma_1, \; \rho \in (0, 1), \; t > 0. \]  
(2.1)

Then, problem (1.1)–(1.5) is equivalent to
\[ u_{tt}(x, t) - \Delta u(x, t) - \text{div} (a(x)\nabla u_t(x, t)) = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \]  
(2.2)
\[ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \quad \text{in} \quad \Gamma_1 \times (0, 1) \times (0, +\infty), \]  
(2.3)
\[ u(x, t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \]  
(2.4)
\[ \frac{\partial u}{\partial \nu}(x, t) + a \frac{\partial u_t}{\partial \nu}(x, t) = -k z(x, 1, t) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \]  
(2.5)
\[ z(x, 0, t) = u_t(x, t) \quad \text{on} \quad \Gamma_1 \times (0, \infty), \]  
(2.6)
\[ u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega, \]  
(2.7)
\[ z(x, \rho, 0) = f_0(x, -\rho \tau) \quad \text{in} \quad \Gamma_1 \times (0, 1). \]  
(2.8)

If we denote
\[ U := (u, u_t, z)^\top, \]  
then
\[ U' := (u_t, u_{tt}, z_t)^\top = (u_t, \Delta u + \text{div} (a\nabla u_t), -\tau^{-1} z_\rho)^\top. \]

Therefore, problem (2.2)–(2.8) can be rewritten as
\[
\begin{aligned}
U' &= A U, \\
U(0) &= (u_0, u_1, f_0(\cdot, -\tau))^\top,
\end{aligned}
\]  
(2.9)
where the operator \( A \) is defined by
\[
A \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \text{div} (\nabla u + a\nabla v) \\ -\tau^{-1} z_\rho \end{pmatrix},
\]
with domain
\[
\mathcal{D}(A) := \left\{ (u, v, z)^\top \in H_{L_0}^{1}(\Omega)^2 \times L^2(\Gamma_1; H^1(0, 1)) : \right. \\
& \text{div} (\nabla u + a\nabla v) \in L^2(\Omega), \\
& \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} = -k z(\cdot, 1) \mod \Gamma_1; \quad v = z(\cdot, 0) \mod \Gamma_1 \left\}. \right.
\]  
(2.10)

Recall that for a vector field \( v \in H(\text{div}, \Omega) = \{ v \in L^2(\Omega)^n : \text{div} v \in L^2(\Omega) \} \), \( v \cdot \nu \) belongs to \( H^{-1/2}(\Gamma_1) \), the dual space of \( H^{1/2}(\Gamma_1) \) (the space of functions \( u \) in \( H^{1/2}(\Gamma_1) \) such that its extension \( \tilde{u} \) by zero outside \( \Gamma_1 \) belongs to \( H^{1/2}(\Gamma) \)) and the next Green formula is valid (see the identity (1.2.17) and Theorems I.2.4 and I.2.5 of [7])
\[
\int_\Omega v \cdot \nabla w dx = - \int_\Omega \text{div} vwdx + (v \cdot \nu; w)_{\Gamma_1} \forall w \in H_{L_0}^1(\Omega),
\]  
(2.11)
where \( \langle \cdot ; \cdot \rangle_{\Gamma_1} \) means the duality pairing between \( H^{-1/2}(\Gamma_1) \) and \( H^{1/2}(\Gamma_1) \).

Note further that for \( (u, v, z)^\top \in \mathcal{D}(A), \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} \) belongs to \( L^2(\Gamma_1) \) since \( z(\cdot, 1) \) is in \( L^2(\Gamma_1) \).
Denote by $\mathcal{H}$ the Hilbert space
\[ \mathcal{H} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1 \times (0, 1)). \] (2.12)

Assuming that (compare with (1.6))
\[ |k| \leq \frac{a_0}{C_P}, \] (2.13)

we will show that $A$ generates a $C_0$ semigroup on $\mathcal{H}$.

Let $\xi$ be a positive real number such that
\[ |k| \leq \xi \tau \leq 2a_0 C_P - |k|. \] (2.14)

Note that, from (2.13), such a constant $\xi$ exists.

Let us define on the Hilbert space $\mathcal{H}$ the inner product
\[ \langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{z} \end{pmatrix} \rangle_{\mathcal{H}} := \int_{\Omega} \{ \nabla u(x) \cdot \nabla \bar{u}(x) + v(x) \bar{v}(x) \} \, dx \\ + \xi \int_{\Gamma_1} \int_0^1 z(x, \rho) \bar{z}(x, \rho) \, d\rho \, d\Gamma. \] (2.15)

Theorem 2.1. Assume that (2.13) holds. Then for any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution $U \in C([0, +\infty), \mathcal{H})$ of problem (2.9). Moreover, if $U_0 \in D(A)$, then
\[ U \in C([0, +\infty), D(A)) \cap C^1([0, +\infty), \mathcal{H}). \]

Proof. Take $U = (u, v, z)^\top \in D(A)$. Then
\[ \langle A U, U \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} \frac{v}{\Delta u + \text{div}(a \nabla v)} \\ -\tau^{-1} z \rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_{\Omega} \{ \nabla v(x) \nabla \bar{u}(x) + \bar{v}(x)(\Delta u + \text{div}(a \nabla v))(x) \} \, dx \\ -\xi \tau^{-1} \int_{\Gamma_1} \int_0^1 z(x, \rho) \bar{z}(x, \rho) \, d\rho \, d\Gamma. \] (2.16)

So, by Green's formula and using the definition of $D(A)$ we get
\[ \Re \langle A U, U \rangle_{\mathcal{H}} = -\int_{\Omega} a(x)|\nabla v|^2 \, dx - k \Re \int_{\Gamma_1} z(x, 1) \bar{v}(x) \, d\Gamma \\ - \frac{\xi}{2\tau} \int_{\Gamma_1} |z(x, 1)|^2 - |v(x)|^2 \, d\Gamma. \] (2.16)

Using the Cauchy–Schwarz and the Young inequalities we find
\[ \Re \langle A U, U \rangle_{\mathcal{H}} \leq -\int_{\Omega} a(x)|\nabla v|^2 \, dx + \left( \frac{|k|}{2} + \frac{\xi}{2\tau} \right) \int_{\Gamma_1} |v(x)|^2 \, d\Gamma \\ + \left( \frac{1}{2} - \frac{\xi}{2\tau} \right) \int_{\Gamma_1} |z(x, 1)|^2 \, d\Gamma. \] (2.17)
Using the definition (1.7) of $C_P$, we deduce that
\[
\Re (\langle AU, U \rangle_H) \leq \left( -a_0 + C_P \left( \frac{|k|}{2} + \frac{\xi}{2r} \right) \right) \int_\omega |\nabla v|^2 \, dx + \left( \frac{|k|}{2} - \frac{\xi}{2r} \right) \int_{\Gamma_1} |z(x,1)|^2 \, d\Gamma \\
- \int_{\Omega \setminus \omega} a(x)|\nabla v|^2 \, dx.
\]
(2.18)

Now, observing that from (2.14),
\[
- a_0 + C_P \left( \frac{|k|}{2} + \frac{\xi}{2r} \right) \leq 0,
\]
\[
|k|^2 - \xi^2 \leq 0,
\]
(2.19)
we obtain
\[
\langle A U, U \rangle_H \leq 0,
\]
which means that the operator $A$ is dissipative.

Now, we will show that $A$ is surjective. Given $(f, g, h)^\top \in \mathcal{H}$, we seek a $U = (u, v, z)^\top \in D(A)$ solution of
\[
A \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},
\]
that is, verifying
\[
\begin{cases}
v = f, \\
\Delta u + \text{div} (a \nabla v) = g, \\
- \tau^{-1} z = h.
\end{cases}
\]
(2.20)
The first equation of (2.20) gives $v$. From the third equation we deduce
\[
z(x, \rho) = f(x) - \tau \int_0^\rho h(x, \sigma) \, d\sigma \quad \text{on } \Gamma_1 \times (0, 1),
\]
(2.21)
and, in particular,
\[
z(x, 1) = z_0(x) = f(x) - \tau \int_0^1 h(x, \sigma) \, d\sigma, \quad x \in \Gamma_1.
\]
(2.22)
The second equation of (2.20) can be reformulated as
\[
\int_\Omega [\Delta u + \text{div}(a \nabla f)] \, w \, dx = \int_\Omega g \, w \, dx, \quad \forall \ w \in H^1_{\Gamma_0}(\Omega).
\]
(2.23)
Then, integrating by parts,
\[
\int_\Omega [\nabla u \nabla \bar{w} + a \nabla f \nabla w] \, dx - \int_{\Gamma_1} \left( \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} \right) \bar{w} \, d\Gamma \\
= - \int_\Omega g \, w \, dx, \quad \forall \ w \in H^1_{\Gamma_0}(\Omega),
\]
and so, from (2.5),
\[
\int_\Omega \nabla u \nabla \bar{w} \, dx = - \int_\Omega (a \nabla f \nabla \bar{w} + g \bar{w}) \, dx - k \int_{\Gamma_1} z_0 \bar{w} \, d\Gamma, \quad \forall \ w \in H^1_{\Gamma_0}(\Omega).
\]
(2.24)
Therefore, the Lax-Milgram Theorem ensures the existence of a unique solution $u \in H^1_{\Gamma_0}(\Omega)$ of (2.24). If we consider $w \in \mathcal{D}(\Omega)$, then from (2.24) we deduce
\[
\nabla u + a \nabla v = \nabla u + a \nabla f \in H(\text{div}, \Omega).
\]
Moreover, using the Green formula (2.11) in (2.24), we obtain
\[
\frac{\partial u}{\partial \nu} + a \frac{\partial u_t}{\partial \nu} = -k z(x, 1).
\]
So, we have found a triple $(u, v, z)^\top \in \mathcal{D}(A)$ solution of (2.20).
From (2.20), (2.23) and (2.24) we easily deduce
\[ \| (u, v, z) \|_H \leq C \| (f, g, h) \|_H. \]
Then, \( 0 \in \rho(A) \). Therefore, by the contraction principle we obtain \( R(\lambda I - A) = H \) for \( \lambda > 0 \) sufficiently small.

Thus, applying the Lumer-Phillips Theorem we conclude that the operator \( A \) generates a \( C_0 \) semigroup of contractions on \( H \).

Let us introduce the energy of the system
\[
E(t) := E(u(t)) = \frac{1}{2} \int_\Omega \left( u_t^2(x, t) + |\nabla u(x, t)|^2 \right) dx
+ \frac{\xi}{2\tau} \int_{t-\tau}^t \int_{\Gamma_1} |u_t(x, s)|^2 d\Gamma ds,
\]
which is the standard energy for wave equation plus an integral term due to the presence of a time delay, where \( \xi > 0 \) is the parameter fixed above.

**Proposition 2.2.** For any regular solution \( U = (u, u_t, z) \) of problem (1.1)–(1.5) the energy is decreasing and there exists a positive constant \( C \) such that
\[
E'(t) \leq -C \left\{ \int_\Omega |\nabla u_t(x, t)|^2 dx + \int_{\Gamma_1} |u_t(x, t-\tau)|^2 d\Gamma \right\} - \int_{\Omega\setminus\omega} a(x)|\nabla u_t(x, t)|^2 dx.
\]

**Proof.** It suffices to notice that
\[ 2E(t) = \| U \|^2_H, \]
hence
\[ E'(t) = \Re \langle U', U \rangle_H = \Re \langle AU, U \rangle_H. \]
We then conclude owing to (2.18) and the assumption on \( \xi \).

3. **Stability of the delay problem.**

In this section, we will prove an exponential stability result for problem (1.1)–(1.5) under the assumption (1.6) and some geometrical assumptions described below.

Our stability result is based on a frequency domain approach, namely the exponential decay of the energy is obtained by using the following result (see [20] or [8]):

**Lemma 3.1.** A \( C_0 \) semigroup \( (e^{tA})_{t \geq 0} \) of contractions on a Hilbert space \( H \) is exponentially stable, i.e., satisfies
\[ \| e^{tA}U_0 \|_H \leq C e^{-\omega t} \| U_0 \|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0, \]
for some positive constants \( C \) and \( \omega \) if and only if
\[ \rho(A) \ni \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R}, \]
and
\[ \sup_{\beta \in \mathbb{R}} \| (i\beta - A)^{-1} \|_{\mathcal{L}(H)} < \infty, \]
where \( \rho(A) \) denotes the resolvent set of the operator \( A \).

From this result, we are reduced to check that the imaginary axis is included in the resolvent set (condition (3.1)) and to analyze the behaviour of the resolvent on the imaginary axis.
3.1. **Spectrum on the imaginary axis.** Since the resolvent of $\mathcal{A}$ is not compact, we check condition (3.1) by showing that $\ker(i\beta I - \mathcal{A}) = \{0\}$, and $R(i\beta I - \mathcal{A}) = \mathcal{H}$, for all $\beta \in \mathbb{R}$.

**Lemma 3.2.** For all $\beta \in \mathbb{R}$, one has

$$\ker(i\beta I - \mathcal{A}) = \{0\}.$$  

**Proof.** Let $(u, v, z)^T \in \mathcal{D}(\mathcal{A})$ be such that

$$\mathcal{A}(u, v, z) = i\beta(u, v, z).$$  

(3.3)

Recall that (2.18) means that

$$\Re \langle \mathcal{A}(u, v, z), (u, v, z) \rangle_{\mathcal{H}} \leq -(a_0 - \frac{|k|}{2} C_P - \frac{\xi \tau^{-1}}{2} C_P) \int_{\Omega} |\nabla v|^2 dx$$

$$- \left( \frac{\xi \tau^{-1}}{2} - \frac{|k|}{2} \right) \int_{\Gamma_1} |z(x, 1)|^2 d\Gamma - \int_{\Omega \setminus \omega} a(x) |\nabla v|^2 dx,$$  

(3.4)

where $C_P$ is the constant in (1.7). Then, recalling (3.3) and (2.19), from (3.4) we deduce

$$\int_{\omega} |\nabla v|^2 dx = 0, \quad \int_{\Gamma_1} |z(x, 1)|^2 d\Gamma = 0, \quad \text{and} \quad \int_{\Omega \setminus \omega} a(x) |\nabla v|^2 dx = 0.$$  

(3.5)

Therefore, from (3.5) we have that

$$a \nabla v = 0 \quad \text{in} \quad \Omega,$$  

(3.6)

and so, since $a \geq a_0 > 0$ in $\omega$,

$$\nabla v = 0 \quad \text{in} \quad \omega,$$  

(3.7)

and then

$$v = c \quad \text{in} \quad \omega,$$  

(3.8)

for some real constant $c$. Moreover, from (3.5), we deduce

$$z(x, 1) = 0 \quad \text{on} \quad \Gamma_1.$$  

(3.9)

Now, note that (3.3) can be rewritten as

$$\begin{cases}
    i\beta u - v = 0 & \text{in} \ \Omega, \\
    \Delta u + \text{div}(a \nabla v) = i\beta v & \text{in} \ \Omega, \\
    \tau^{-1} z_\rho + i\tau \beta z = 0 & \text{in} \ \Gamma_1 \times (0, 1).
\end{cases}$$  

(3.10)

Using (3.6) in (3.10) we have

$$\begin{cases}
    i\beta u - v = 0 & \text{in} \ \Omega, \\
    \beta^2 u + \Delta u = 0 & \text{in} \ \Omega, \\
    z_\rho + i\tau \beta z = 0 & \text{in} \ \Gamma_1 \times (0, 1).
\end{cases}$$  

(3.11)

Then, from the third equation in (3.11) and (3.9) we can obtain

$$z(x, \rho) = 0 \quad \text{in} \ \Gamma_1 \times (0, 1),$$  

(3.12)

and thus

$$v(x) = z(x, 0) = 0, \quad x \in \Gamma_1.$$  

(3.13)

Therefore, from (3.13) and (3.8), we deduce

$$v = 0, \quad \text{in} \ \omega.$$  

(3.14)

Now, using the first equation of (3.11), we have also

$$u = 0, \quad \text{in} \ \omega.$$
Since $u$ satisfies the second equation of (3.11) a unique continuation result allows to deduce
\[ u = 0, \quad \text{in} \quad \Omega. \tag{3.15} \]
Therefore, $(u, v, z) = (0, 0, 0)$. \hfill \Box

**Proposition 3.3.** Assume that $a \in C^{1,1}(\bar{\Omega})$. Then for all $\beta \in \mathbb{R}$, one has
\[ R(i\beta I - A) = \mathcal{H}. \]

**Proof.** Since we have already shown in Theorem 2.1 that $R(A) = \mathcal{H}$, we only need to prove that $R(i\beta I - A) = \mathcal{H}$, for all $\beta \in \mathbb{R}$, $\beta \neq 0$.

Given $(f, g, h) \in \mathcal{H}$, we have to find $(u, v, z) \in D(A)$ such that
\[ (i\beta I - A)(u, v, z) = (f, g, h), \tag{3.16} \]
that is
\[
\begin{aligned}
  i\beta u - v &= f \quad \text{in} \quad \Omega, \\
  i\beta v - \Delta u - \text{div} (a \nabla v) &= g \quad \text{in} \quad \Omega, \\
  i\beta z + \tau^{-1}z_{\rho} &= h \quad \text{on} \quad \Gamma_{1} \times (0,1). 
\end{aligned} \tag{3.17}
\]
The first identity of (3.17) gives
\[ v = i\beta u - f. \tag{3.18} \]

Now, observe that, from the third identity in (3.17),
\[ z(x, \rho) = e^{-i\tau \beta \rho}(v + \tau \int_{0}^{\rho} e^{i\tau \beta \sigma} h(x, \sigma) d\sigma), \tag{3.19} \]
and then
\[ z(x, 1) = e^{-i\beta v} + \tau e^{-i\beta} \int_{0}^{1} e^{i\tau \beta \sigma} h(x, \sigma) d\sigma. \tag{3.20} \]
Let us now introduce the auxiliary variable
\[ \tilde{u} = u + av. \tag{3.21} \]

Since, by (3.18), $f = i\beta u - v$, we find
\[
\begin{aligned}
  u &= \frac{\tilde{u} + af}{1 + i\beta a}; \\
  v &= \frac{1 + i\beta a}{1 + i\beta a} \tilde{u} - \frac{1}{1 + i\beta a} f. 
\end{aligned} \tag{3.22}
\]

Now, observe that
\[ \nabla u + a \nabla v = \nabla (u + av) - v \nabla a = \nabla \tilde{u} - v \nabla a, \]
and so
\[- \text{div} (\nabla u + a \nabla v) = -\Delta \tilde{u} + \text{div} (v \nabla a). \]
Then, the second identity of (3.17),
\[ i\beta v - \text{div} (\nabla u + a \nabla v) = g, \]
can be rewritten as
\[ i\beta v - \Delta \tilde{u} + \text{div} (v \nabla a) = g. \tag{3.23} \]

Multiplying (3.23) by $\varpi \in H^{1}_{\omega}(\Omega)$ and integrating in $\Omega$, we find
\[ \int_{\Omega} (i\beta v - \Delta \tilde{u} + \text{div} (v \nabla a)) \varpi dx = \int_{\Omega} g \varpi dx, \]
and then, after integration by parts,
\[
\int_{\Omega} (i\beta v \overline{w} + \nabla \overline{u} \cdot \nabla w + \text{div} (v \nabla a) \overline{w}) \, dx - \int_{\Gamma_1} \frac{\partial \overline{u}}{\partial \nu} \overline{w} \, d\Gamma = \int_{\Omega} g \overline{w} \, dx. \tag{3.24}
\]
Now observe that, from the boundary condition on \(\Gamma_1\),
\[
\frac{\partial \overline{u}}{\partial \nu} = \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} + \frac{\partial a}{\partial \nu} v = -kz(\cdot, 1) + \frac{\partial a}{\partial \nu} v.
\]
Hence by (3.20), we get
\[
\frac{\partial \overline{u}}{\partial \nu} = \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} + \frac{\partial a}{\partial \nu} v = -\left(ke^{-i\tau \beta} - \frac{\partial a}{\partial \nu} v - k\tau e^{-i\tau \beta} \int_0^1 e^{i\tau \beta \sigma} h(x, \sigma) d\sigma\right). \tag{3.25}
\]
This identity and (3.22) allow to transform (3.24) as
\[
\int_{\Omega} \left\{ i\beta \left( i\beta (i\beta - 1) + i\beta a \right) \overline{w} + \nabla \overline{u} \cdot \nabla w \right. \\
\left. + \text{div} \left( \left( i\beta (i\beta - 1) + i\beta a \right) \nabla a \right) \overline{w} \right\} \, dx \\
+ \int_{\Gamma_1} (ke^{-i\tau \beta} - \frac{\partial a}{\partial \nu}) \left( i\beta (i\beta - 1) + i\beta a \right) \overline{u} \, d\Gamma \\
= \int_{\Omega} g \overline{w} \, dx + \int_{\Gamma_1} k\tau e^{-i\tau \beta} \left( \int_0^1 e^{i\tau \beta \sigma} h(x, \sigma) d\sigma \right) \overline{w} \, d\Gamma. \tag{3.26}
\]
Then,
\[
\int_{\Omega} \left\{ -\beta^2 \overline{u} w + \nabla \overline{u} \cdot \nabla w + i\beta \text{div} \left( \frac{\overline{u}}{1 + i\beta a} \nabla a \right) w \right\} \, dx \\
+ i\beta \int_{\Gamma_1} (ke^{-i\tau \beta} - \frac{\partial a}{\partial \nu}) \frac{\overline{u}}{1 + i\beta a} \overline{w} \, d\Gamma \\
= \int_{\Omega} g \overline{w} \, dx + \int_{\Omega} \frac{i\beta f}{1 + i\beta a} \overline{w} \, dx + \int_{\Omega} \text{div} \left( f \nabla a \right) \overline{w} \, dx \\
+ \int_{\Gamma_1} (ke^{-i\tau \beta} - \frac{\partial a}{\partial \nu}) \frac{f}{1 + i\beta a} \overline{w} \, d\Gamma \\
+ \int_{\Gamma_1} k\tau e^{-i\tau \beta} \left( \int_0^1 e^{i\tau \beta \sigma} h(x, \sigma) d\sigma \right) \overline{w} \, d\Gamma. \tag{3.27}
\]
So we can rewrite identity (3.27) as
\[
a_{\tau, \beta}(\overline{u}, w) = F_{\tau, \beta}(w), \quad \forall w \in H^1_{\Gamma_0}(\Omega). \tag{3.28}
\]
Let us denote
\[
V := H^1_{\Gamma_0}(\Omega)
\]
and let \(V'\) its dual space. We introduce the operator
\[
A_{\tau, \beta} : V \to V' : \overline{u} \mapsto A_{\tau, \beta} \overline{u}
\]
with
\[
(A_{\tau, \beta} \overline{u})(w) = a_{\tau, \beta}(\overline{u}, w), \quad \forall \ w \in V.
\]
Then, we rewrite identity (3.28) as
\[
A_{\tau, \beta} \overline{u} = F_{\tau, \beta} \text{ in } V'. \tag{3.30}
\]
Let us denote, for \( s > \frac{1}{2}, \)
\[
H^s_{\Gamma_0}(\Omega) = \{ u \in H^s(\Omega) : u = 0 \text{ on } \Gamma_0 \}.
\]
To conclude the proof of Proposition 3.3 we need some preliminary results.

**Proposition 3.4.** For \( \tau \geq 0 \) and for every \( \beta, \beta' \in \mathbb{R} \), the operator \( A_{\tau, \beta} - A_{\tau, \beta'} \) is compact.

**Proof.** From (3.27) we have

\[
\begin{align*}
\nonumber a_{\tau, \beta}(\tilde{u}, w) - a_{\tau, \beta'}(\tilde{u}, w) &= \int \Omega \left\{ \left( \frac{\beta'^2}{1+ia\beta'} - \frac{\beta^2}{1+ia\beta} \right) \tilde{u} w \right. \\
\nonumber &+ \text{div} \left( \frac{i\beta'}{1+ia\beta'} \tilde{u} \nabla a \right) w \} \, dx + \int_{\Gamma_1} \frac{\partial a}{\partial \nu} \left( \frac{i\beta'}{1+ia\beta'} - \frac{i\beta}{1+ia\beta} \right) \tilde{u} w d\Gamma \\
\nonumber &+ \int_{\Gamma_2} k \left( \frac{i\beta e^{-t\tau}}{1+ia\beta} - \frac{i\beta' e^{-t\tau}}{1+ia\beta'} \right) \tilde{u} w d\Gamma.
\end{align*}
\]

Therefore,

\[
|a_{\tau, \beta}(\tilde{u}, w) - a_{\tau, \beta'}(\tilde{u}, w)| \leq C \|\tilde{u}\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)} + C \|\tilde{u}\|_{L^2(\Gamma_1)} \|w\|_{L^2(\Gamma_1)},
\]

for a suitable positive constant \( C \). And so,

\[
|a_{\tau, \beta}(\tilde{u}, w) - a_{\tau, \beta'}(\tilde{u}, w)| \leq C \|\tilde{u}\|_{H^s(\Omega)} \|w\|_{H^{-s}(\Omega)},
\]

for all \( s \in (\frac{1}{2}, 1) \), for a suitable constant \( C \).

This proves that \( A_{\tau, \beta} - A_{\tau, \beta'} \in L(V, (H^s_{\Gamma_0}(\Omega))^\prime) \).

Since from \( H^1_{\Gamma_0}(\Omega) \hookrightarrow C H^s_{\Gamma_0}(\Omega) \) we deduce \( (H^s_{\Gamma_0}(\Omega))' \hookrightarrow C (H^1_{\Gamma_0}(\Omega))' \), and the claim follows. \qed

**Corollary 3.5.** For any \( \tau \geq 0 \), if \( A_{\tau, 0} \) is an isomorphism, then \( A_{\tau, \beta} \) is a Fredholm operator of index 0, for all \( \beta \in \mathbb{R} \).

**Corollary 3.6.** For any \( \tau \geq 0 \), if \( A_{\tau, 0} \) is an isomorphism and \( \ker A_{\tau, \beta} = \{0\} \), then \( A_{\tau, \beta} \) is an isomorphism, for all \( \beta \in \mathbb{R} \).

**Lemma 3.7.** For any \( \tau \geq 0 \), \( A_{\tau, 0} \) is an isomorphism.

**Proof.** For \( \beta = 0 \), one has

\[
a_{\tau, 0}(\tilde{u}, w) = \int \Omega \nabla \tilde{u} \nabla w \, dx.
\]

Hence, the statement holds owing to Poincaré inequality. \qed

**Lemma 3.8.** For any \( \tau > 0 \), and for all \( \beta \in \mathbb{R} \) it holds

\[
\ker A_{\tau, \beta} = \{0\}.
\]

**Proof.** Let \( \tilde{u} \in \ker A_{\tau, \beta} \), i.e.

\[
a_{\tau, \beta}(\tilde{u}, w) = 0, \quad \forall w \in V.
\]

Then from (3.27) we find that

\[
-\frac{\beta^2}{1+ia\beta} \tilde{u} - \text{div} \nabla \tilde{u} + i\beta \text{div} \left( \frac{\tilde{u}}{1+ia\beta} \nabla a \right) = 0 \quad \text{in} \; \mathcal{D}'(\Omega) \tag{3.31}
\]

and

\[
\frac{\partial \tilde{u}}{\partial \nu} + \left( ke^{-t\tau} - \frac{\partial a}{\partial \nu} \right) \frac{1}{1+ia\beta} \tilde{u} = 0 \quad \text{on} \; \Gamma_1. \tag{3.32}
\]
Thus, if we set
\[ u = \frac{\tilde{u}}{1 + i\beta a} \quad \text{and} \quad v = \frac{i\beta}{1 + i\beta a} \tilde{u} , \]
and define (compare with (3.19))
\[ z(x, \rho) = e^{-i\tau \beta \rho} v , \]
we find that \((u, v, z)\) belongs to \(D(A)\) and satisfies
\[
(i\beta - A) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} i\beta u - v \\ i\beta v - \Delta u - \text{div} \left( a \nabla v \right) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i\beta v - \text{div} \left( \nabla u + a \nabla v \right) \\ 0 \end{pmatrix} .
\]
(3.33)

Now, observe that
\[
\nabla u + a \nabla v = \nabla \left( \frac{\tilde{u}}{1 + i\beta a} \right) + ai\beta \nabla \left( \frac{\tilde{u}}{1 + i\beta a} \right)
\]
\[
= (1 + i\beta a) \left( \frac{\nabla \tilde{u}}{1 + i\beta a} \right) + (1 + i\beta a) \tilde{u} \nabla \left( \frac{1}{1 + i\beta a} \right)
\]
\[
= \nabla \tilde{u} - \frac{1}{1 + i\beta a} \tilde{u} (i\beta a) .
\]
(3.34)

Hence,
\[
i\beta v - \text{div} \left( \nabla u + a \nabla v \right) = \frac{-\beta^2 \tilde{u}}{1 + i\beta a} - \text{div} \nabla \tilde{u} + i\beta \text{div} \left( \frac{\tilde{u}}{1 + i\beta a} \nabla a \right) = 0 ,
\]
and therefore, from (3.33), we deduce
\[
(i\beta - A) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .
\]
(3.35)

This implies \(u = v = z = 0\), then \(\tilde{u} = 0\). \(\square\)

**End of the proof of Proposition 3.3.**

Given \((f, g, h) \in \mathcal{H}\), the previous considerations yield a unique \(\tilde{u} \in H^1_{\Gamma_0}(\Omega)\) solution of (3.28). Defining \(u, v\) by (3.22) and \(z\) by (3.19), we get \((u, v, z) \in D(A)\) solution of (3.16). \(\square\)

### 3.2. Boundedness of the resolvent on the imaginary axis.

Inspired from Liu and Rao [16], we assume

(H) \(\text{meas } \Gamma_1 > 0\);

(A1) \(\exists \delta > 0 \quad : \quad a(x) \geq a_0 > 0 \quad \forall x \in \mathcal{O}_\delta\), where
\[
\mathcal{O}_\delta = \{ x \in \Omega : |x - y| \leq \delta, \forall y \in \Gamma_1 \} ;
\]

(A2) \(a \in C^{1,1}(\overline{\Omega})\), \(\Delta a \in L^\infty(\Omega)\).

Also, we assume the following conditions.

There exists a function \(q \in C^1(\Omega; \mathbb{R}^n)\) and constants \(0 < \alpha < \beta < \delta\) such that

(Q1) \(\partial_j q_k = \partial_k q_j\), \(\text{div} \ q \in C^1(\Omega_\delta)\) and \(q \equiv 0\) on \(\mathcal{O}_\alpha\), where \(\Omega_\beta = \Omega \setminus \mathcal{O}_\beta\);

(Q2) there exists a constant \(\sigma > 0\) such that
\[
(\partial_j q_k)_{1 \leq k, j \leq n} \geq \sigma I , \quad \forall \ x \in \Omega_\beta ;
\]
there exists a constant $C > 0$ such that for all $v \in H^1_0(\Omega)$ we have
\[ |(q \cdot \nabla)v - (g \cdot \nabla)\nabla a| \leq C \sqrt{a} |\nabla v|, \ \forall \ x \in \Omega \beta; \]

(Q4) $q(x) \cdot \nu(x) \leq 0 \ \forall \ x \in \Gamma_0$.

**Remark 3.9.** For examples and simple situations where (Q1)–(Q4) are satisfied we refer to Remark 3.1 in [16].

Introduce
\[ H_0 = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \] (3.36)
equipped with the inner product
\[ \langle (u, v)^\top, (\tilde{u}, \tilde{v})^\top \rangle_{H_0} := \int_{\Omega} \{ \nabla u(x) \cdot \nabla \tilde{u}(x) + v(x)\tilde{v}(x) \} \ dx, \]
and let $A_0$ be the operator corresponding to $\tau = 0$ and $k = 1$, that is
\[ A_0 : D(A_0) \to H_0 : (u, v)^\top \to (v, \text{div} (\nabla u + a \nabla v))^\top, \]
with
\[ D(A_0) = \left\{ (u, v) \in H^1_{\Gamma_0}(\Omega)^2 : \text{div} (\nabla u + a \nabla v) \in L^2(\Omega), \ \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} = -v \text{ on } \Gamma_1 \right\}. \]

We will prove in the next section that the above assumptions imply that $A_0$ generates an exponentially stable semigroup. Then, from [8], we know that $i \mathbb{R} \subset \rho(A_0)$ and that there exists $C > 0$ such that
\[ \| (i \xi - A_0)^{-1} \|_{L(H_0)} \leq C, \ \forall \ \xi \in \mathbb{R}. \] (3.37)

From this estimate, we can deduce the estimate (3.42) below that will be useful in Proposition 3.10. Indeed, from (3.37), for every $F_0 \in H_0$, the solution $(u^*, v^*)^\top \in D(A_0)$ of
\[ (i \xi - A_0) \left( \begin{array}{c} u^* \\ v^* \end{array} \right) = F_0, \] (3.38)
satisfies
\[ \left\| \left( \begin{array}{c} u^* \\ v^* \end{array} \right) \right\|_{H_0} \leq C \| F_0 \|_{H_0}, \] (3.39)
which is equivalent to
\[ \| u^* \|_{H^1_{\Gamma_0}(\Omega)} + \| v^* \|_{L^2(\Omega)} \leq C \| F_0 \|_{H_0}. \] (3.40)
Moreover,
\[ \Re \left\langle A_0 \left( \begin{array}{c} u^* \\ v^* \end{array} \right), \left( \begin{array}{c} u^* \\ v^* \end{array} \right) \right\rangle_{H_0} = - \int_{\Omega} a |\nabla v^*|^2 \ d x - \int_{\Gamma_1} |v^*|^2 \ d \Gamma. \]

Then,
\[ \int_{\Gamma_1} |v^*|^2 \ d \Gamma + \int_{\Omega} a |\nabla v^*|^2 \ d x = \Re \left\langle (i \xi - A_0) \left( \begin{array}{c} u^* \\ v^* \end{array} \right), \left( \begin{array}{c} u^* \\ v^* \end{array} \right) \right\rangle_{H_0} = \Re \left\langle F_0, \left( \begin{array}{c} u^* \\ v^* \end{array} \right) \right\rangle_{H_0} \leq \| F_0 \|_{H_0} \left\| \left( \begin{array}{c} u^* \\ v^* \end{array} \right) \right\|_{H_0} \leq C \| F_0 \|^2_{H_0} \] (3.41)
and so
\[ \int_{\Gamma_1} |v|^2 \ d \Gamma \leq C \| F_0 \|^2_{H_0}. \] (3.42)
Proposition 3.10. Under the assumptions (1.6), (H), (A1), (A2), (Q1)−(Q4), the operator $A$ satisfies
\[ \sup_{\beta \in \mathbb{R}} \|(i\beta I - A)^{-1}\|_{\mathcal{H}} < +\infty. \]

Proof. For $F \in \mathcal{H}$ and $\beta \in \mathbb{R}$, let $U \in \mathcal{D}(A)$ be a solution of
\[ (i\beta - A)U = F = (f, g, h)^\top, \]
that is
\[ \begin{cases} 
    i\beta u - v = f & \text{in } \Omega, \\
    i\beta v - \Delta u - \text{div} \; (a\nabla v) = g & \text{in } \Omega, \\
    i\beta z + \tau^{-1}z_\rho = h & \text{on } \Gamma_1.
\end{cases} \]  

The first identity of (3.44) gives
\[ v = i\beta u - f. \]

Now, observe that, from the identity (3.20),
\[ v(x) = e^{i\tau \beta}z(x, 1) - \tau \int_0^1 e^{i\tau \beta \sigma}h(x, \sigma)d\sigma, \]
and so,
\[ \|v\|_{L^2(\Gamma_1)} \leq \|z(\cdot, 1)\|_{L^2(\Gamma_1)} + C\|h\|_{L^2(\Gamma_1 \times (0, 1))}. \]  

Moreover, from (2.18), we have
\[ C \int_{\Gamma_1} |z(x, 1)|^2d\Gamma \leq -\Re \langle AU, U \rangle_{\mathcal{H}}. \]

Then,
\[ C \int_{\Gamma_1} |z(x, 1)|^2d\Gamma \leq \Re \langle F, U \rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \]  

From (3.47) and (3.46) we deduce
\[ \|v\|_{L^2(\Gamma_1)}^2 \leq C(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}). \]  

From (3.20) we have also
\[ \|z\|_{L^2(\Gamma_1 \times (0, 1))} \leq C(\|v\|_{L^2(\Gamma_1)} + \|h\|_{L^2(\Gamma_1 \times (0, 1))}). \]  

By using (3.49) in (3.48) we obtain
\[ \|v\|_{L^2(\Gamma_1)}^2 \leq C(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}(\|u, v\|_{\mathcal{H}_0} + \|v\|_{L^2(\Gamma_1)} + \|h\|_{L^2(\Gamma_1 \times (0, 1))})), \]
and therefore
\[ \|v\|_{L^2(\Gamma_1)}^2 \leq C(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}(\|u, v\|_{\mathcal{H}_0} + \|v\|_{L^2(\Gamma_1)}), \]
from which follows, by using Young’s inequality,
\[ \|v\|_{L^2(\Gamma_1)}^2 \leq C(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}(\|u, v\|_{\mathcal{H}_0})). \]  

Estimates (3.47), (3.49) and (3.50) imply
\[ \|z(\cdot, 1)\|_{L^2(\Gamma_1)}^2 \leq C(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}(\|u, v\|_{\mathcal{H}_0})). \]

We have now to estimate $\|(u, v)\|_{\mathcal{H}_0}$. For this, let $(u^*, v^*) \in \mathcal{D}(A_0)$ be the solution of
\[ (-i\beta - A_0) \begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix}, \]  

STABILITY OF THE WAVE EQUATION 803
which is equivalent to
\[
\begin{aligned}
-\beta u^* - v^* &= u \quad \text{in } \Omega, \\
-\beta v^* - \text{div} (\nabla u^* + a \nabla v^*) &= -v \quad \text{in } \Omega, \\
\frac{\partial u^*}{\partial \nu} + a \frac{\partial v^*}{\partial \nu} &= -v^* \quad \text{on } \Gamma_1.
\end{aligned}
\] (3.53)

We have then
\[
\langle (i\beta - A)U, \begin{pmatrix} u^* \\ -v^* \\ 0 \end{pmatrix} \rangle_H = \int_\Omega \left\{ \nabla (i\beta u - v) \cdot \nabla \bar{u}^* - (i\beta v - \text{div}(\nabla u + a \nabla v)) \cdot \bar{v}^* \right\} dx
- \int_{\Gamma_1} k z(x,1) \bar{v}^* d\Gamma
= \int_\Omega \left\{ -\nabla u \nabla (i\beta u^* - \nabla v^*) - i\beta v \cdot \bar{v}^* - \nabla v (\nabla \bar{u}^* + a \nabla \bar{v}^*) \right\} dx
- \int_{\Gamma_1} k z(x,1) \bar{v}^* d\Gamma
= \int_\Omega \left\{ -\nabla u \nabla v + v^* - i\beta v \cdot v^* \quad \text{in } \Omega, \\
\frac{\partial u^*}{\partial \nu} + a \frac{\partial v^*}{\partial \nu} &= -v^* \quad \text{on } \Gamma_1.
\end{aligned}
\]

Thus, using the boundary condition on \( \Gamma_1 \),
\[
\langle (i\beta - A)U, \begin{pmatrix} u^* \\ -v^* \\ 0 \end{pmatrix} \rangle_H = \int_\Omega \left\{ \nabla u \nabla (i\beta u^* - \nabla v^*) - i\beta v \cdot \bar{v}^* - \nabla v (\nabla \bar{u}^* + a \nabla \bar{v}^*) \right\} dx
- \int_{\Gamma_1} k z(x,1) \bar{v}^* d\Gamma
= \int_\Omega \left\{ -\nabla u \nabla v + v^* - i\beta v \cdot v^* \quad \text{in } \Omega, \\
\frac{\partial u^*}{\partial \nu} + a \frac{\partial v^*}{\partial \nu} &= -v^* \quad \text{on } \Gamma_1.
\end{aligned}
\]

Then, from (3.53), we have
\[
\langle (i\beta - A)U, \begin{pmatrix} u^* \\ -v^* \\ 0 \end{pmatrix} \rangle_H = \int_\Omega (|\nabla u|^2 + |v|^2) dx - \int_{\Gamma_1} k z(x,1) \bar{v}^* d\Gamma + \int_{\Gamma_1} v v^* d\Gamma,
\]
and so
\[
\int_\Omega (|\nabla u|^2 + |v|^2) dx = \int_{\Gamma_1} (k z(x,1) - v) \bar{v}^* d\Gamma,
\]
from which follows, by using Cauchy–Schwarz inequality,
\[
\| (u, v) \|^2_{\mathcal{H}_0} \leq \| F \|_{\mathcal{H}} \| (u^*, v^*) \|_{\mathcal{H}_0} + (|k| \|z(\cdot, 1)\|_{L^2(\Gamma_1)} + \|v\|_{L^2(\Gamma_1)}) \|v^*\|_{L^2(\Gamma_1)}. \] (3.54)

Now, observe that from the estimates (3.39) and (3.42) we have
\[
\| (u^*, v^*) \|_{\mathcal{H}_0} \leq C \| (u, v) \|_{\mathcal{H}_0},
\]
and
\[
\| v^* \|_{L^2(\Gamma_1)} \leq C \| (u, v) \|_{\mathcal{H}_0}.
\]
Using these last inequalities in (3.54), we obtain
\[ \| (u,v) \|_{H_0} \leq C \left( \| F \|_{H} + \| z(x,1) \|_{L^2(\Gamma_1)} + \| v \|_{L^2(\Gamma_1)} \right). \]

By recalling (3.50) and (3.51) we have then
\[ \| (u,v) \|_{H_0}^2 \leq C \left( \| F \|_{H}^2 + \| F \|_{H} \| (u,v) \|_{H_0} \right), \]
and so, using Young’s inequality,
\[ \| (u,v) \|_{H_0} \leq C \| F \|_{H_0}. \tag{3.55} \]

Using (3.55) in (3.50), we obtain
\[ \| v \|_{L^2(\Gamma_1)} \leq C \| F \|_{H} \]
and then, from (3.49),
\[ \| z \|_{L^2(\Gamma_1 \times (0,1))} \leq C \| F \|_{H}. \]
This proves that the resolvent of \( A \) is uniformly bounded on the imaginary axis. \( \square \)

The above resolvent analysis and Lemma 3.1 allow to state our stability result.

**Theorem 3.11.** Under the assumptions (H), (A1), (A2), (Q1) – (Q4), if condition (1.6) is satisfied, then there are two constants \( M > 0, \omega > 0 \), such that for all initial data \( U_0 \in \mathcal{H} \) the solution \( U := (u,u_t,z) \) of problem (1.1) – (1.5) satisfies the following uniform exponential decay estimate
\[ \| U(t) \|_{\mathcal{H}} \leq Me^{-\omega t} \| U_0 \|_{\mathcal{H}}. \]

4. Stability of the undelayed problem.

As in section 2, we can prove that \( A_0 \) generates a \( C_0 \)-semigroup of contractions in \( \mathcal{H}_0 \). Hence in order to prove that this semigroup is exponentially decaying, we again use Lemma 3.1.

As before we start with condition (3.1).

**Lemma 4.1.** For all \( \beta \in \mathbb{R} \), one has
\[ \ker(i\beta I - A_0) = \{0\}. \]

**Proof.** Let \( U = (u,v)^T \in \mathcal{D}(A_0) \) be such that
\[ A_0(u,v)^T = i\beta(u,v)^T. \]
As
\[ \Re(A_0U, U)_{H_0} = - \int_{\Omega} a(x)|\nabla v|^2 \, dx - \int_{\Gamma_1} |v(x)|^2 \, d\Gamma, \]
we deduce that
\[ v = 0, \quad \text{in } \omega. \tag{4.1} \]
As \( (u,v)^T \) also satisfies
\[ \begin{cases} i\beta u - v = 0 & \text{in } \Omega, \\ \beta^2 u + \Delta u = 0 & \text{in } \Omega, \end{cases} \]
u is also equal to 0 in \( \omega \) and a unique continuation result allows to deduce \( u = 0 \) and hence \( v = 0 \). \( \square \)

**Lemma 4.2.** If \( a \in C^{1,1}(\bar{\Omega}) \), then for all \( \beta \in \mathbb{R} \), one has
\[ R(i\beta I - A_0) = \mathcal{H}_0. \]
Proof. Given \((f, g) \in \mathcal{H}\), we have to find \((u, v) \in \mathcal{D}(\mathcal{A})\) such that
\[
(i \beta I - \mathcal{A}_0)(u, v) = (f, g),
\]
that is
\[
\begin{aligned}
 i \beta u - v &= f \quad \text{in } \Omega, \\
 i \beta v - \Delta u - \text{div} \, (a \nabla v) &= g \quad \text{in } \Omega.
\end{aligned}
\]
We denote
\[
\tilde{u} = u + av.
\]
Since, by (4.3), \(f = i \beta u - v\), we find
\[
\begin{aligned}
 u &= \frac{\tilde{u} + af}{1 + i \beta a}; \\
 v &= \frac{i \beta}{1 + i \beta a} \tilde{u} - \frac{1}{1 + i \beta a} f.
\end{aligned}
\]
Now, observe that
\[
\nabla u + a \nabla v = \nabla (u + av) - v \nabla a = \nabla \tilde{u} - v \nabla a,
\]
and so
\[
-\text{div} \, (\nabla u + a \nabla v) = -\Delta \tilde{u} + \text{div} \, (v \nabla a).
\]
Then, the second identity of (4.3),
\[
i \beta v - \text{div} \, (\nabla u + a \nabla v) = g,
\]
can be rewritten as
\[
i \beta v - \Delta \tilde{u} + \text{div} \, (v \nabla a) = g.
\]
Multiplying (4.6) by \(\overline{w} \in H^1_{\Gamma_0}(\Omega)\) and integrating in \(\Omega\), we find
\[
\int_{\Omega} (i \beta v - \Delta \tilde{u} + \text{div} \, (v \nabla a)) \overline{w} \, dx = \int_{\Omega} g \overline{w} \, dx,
\]
and then, after integration by parts,
\[
\int_{\Omega} (i \beta v \overline{w} + \nabla \tilde{u} \cdot \nabla \overline{w} + \text{div} \, (v \nabla a) \overline{w}) \, dx - \int_{\Gamma_1} \frac{\partial \tilde{u}}{\partial \nu} \overline{w} d\Gamma = \int_{\Omega} g \overline{w} \, dx.
\]
Now observe that, from the boundary condition on \(\Gamma_1\),
\[
\frac{\partial \tilde{u}}{\partial \nu} - \frac{\partial u}{\partial \nu} + a \frac{\partial v}{\partial \nu} + \frac{\partial a}{\partial v} v = -(1 - \frac{\partial a}{\partial v}) v.
\]
Therefore, (4.7) becomes
\[
\int_{\Omega} \left\{ i \beta \left( \frac{i \beta}{1 + i \beta a} \tilde{u} - \frac{1}{1 + i \beta a} f \right) \overline{w} + \nabla \tilde{u} \cdot \nabla \overline{w} \\
+ \text{div} \, \left( \left( \frac{i \beta}{1 + i \beta a} \tilde{u} - \frac{1}{1 + i \beta a} f \right) \nabla a \right) \overline{w} \right\} \, dx + \int_{\Gamma_1} (1 - \frac{\partial a}{\partial v}) \left( \frac{i \beta}{1 + i \beta a} \tilde{u} - \frac{1}{1 + i \beta a} f \right) \overline{w} d\Gamma = \int_{\Omega} g \overline{w} \, dx.
\]
Then,
\[
\int_{\Omega} \left\{ \frac{-\beta^2}{1 + ia\beta} \hat{u}w + \nabla \hat{u} \nabla w + i\beta \text{div} \left( \frac{\hat{u}}{1 + ia\beta} \nabla a \right) w \right\} dx + i\beta \int_{\Gamma_1} \left( 1 - \frac{\partial a}{\partial \nu} \right) \frac{1}{1 + ia\beta} \hat{u} w d\Gamma \\
= \int_{\Omega} g w dx + \int_{\Omega} i\beta f \frac{w}{1 + ia\beta} dx + \int_{\Omega} \text{div} \left( \frac{f \nabla a}{1 + ia\beta} \right) \frac{w}{1 + ia\beta} dx + \int_{\Gamma_1} \left( 1 - \frac{\partial a}{\partial \nu} \right) \frac{f}{1 + ia\beta} w d\Gamma.
\]
(4.10)

With the notations introduced before, (4.10) is equivalent to
\[
a_{0,\beta}(\hat{u}, w) = F_{0,\beta}(w), \quad \forall w \in H^1_{\Gamma_0}(\Omega),
\]
(4.11)
or equivalently
\[
A_{0,\beta} \hat{u} = F_{0,\beta} \text{ in } V'.
\]
(4.12)

Since we have shown in the previous section that $A_{0,\beta}$ is a Fredholm operator of index 0, for all $\beta \in \mathbb{R}$, then the result follows from Lemma 4.3 below.

**Lemma 4.3.** For all $\beta \in \mathbb{R}$ it holds
\[
\ker A_{0,\beta} = \{0\}.
\]

**Proof.** Let $\hat{u} \in \ker A_{0,\beta}$, i.e.
\[
a_{\beta}(\hat{u}, w) = 0, \quad \forall w \in V.
\]
Then from (4.10) we find that $\hat{u}$ satisfies (3.31) and (3.32) with $\tau = 0$. Thus, if we set
\[
u = \frac{i\beta}{1 + i\beta a} \text{ and } v = \frac{i\beta}{1 + i\beta a} \hat{u},
\]
we find that $(u, v) \in D(A_0)$ and that
\[
(i\beta - A_0) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
This implies $u = v = 0$, then $\hat{u} = 0$.

**Lemma 4.4.** Under the assumptions (H), (A1), (A2) and (Q1)–(Q4), $A_0$ satisfies
\[
\sup_{\beta \in \mathbb{R}} \| (i\beta I - A_0)^{-1} \|_{\mathcal{L}(H_0)} < +\infty.
\]
(4.13)

**Proof.** Suppose that (4.13) does not hold. Then, there exist a sequence $\{\beta_n\} \in \mathbb{R}$ and a sequence $\{(u_n, v_n)\} \in D(A_0)$ such that
\[
|\beta_n| \to +\infty,
\]
(4.14)
\[
\|u_n\|_V^2 + \|v_n\|_{\tilde{H}}^2 = 1,
\]
(4.15)
\[
i\beta_n(u_n, v_n) - A(u_n, v_n) := (f_n, g_n) \to 0 \text{ in } \mathcal{H}.
\]
(4.16)
Then,
\[
i\beta_n u_n - v_n := f_n \to 0 \text{ in } V,
\]
(4.17)
\[
i\beta_n v_n - \text{div} (\nabla u_n + a \nabla v_n) := g_n \to 0 \text{ in } H = L^2(\Omega).
\]
(4.18)
Taking the inner product of (4.17) with $u_n$ in $V$ and of (4.18) with $v_n$ in $H$, we obtain

$$i\beta_n \|u_n\|_V^2 - \langle u_n, v_n \rangle_V = o(1),$$

$$i\beta_n \|v_n\|_H^2 + \langle u_n, v_n \rangle_V + \int_\Omega a\nabla v_n^2 dx + \int_{\Gamma_1} |v_n|^2 d\Gamma = o(1). \quad (4.19)$$

Summing the above identities and taking the real part, we deduce

$$\int_\Omega a|\nabla v_n|^2 dx + \int_{\Gamma_1} |v_n|^2 d\Gamma = o(1). \quad (4.20)$$

Now, taking the inner product in $H$ of (4.17) with $v_n$ and (4.18) with $u_n$, we have

$$i\beta_n \langle u_n, v_n \rangle_H - \|v_n\|_H^2 = o(1), \quad (4.21)$$

and

$$i\beta_n \langle v_n, u_n \rangle_H + \|u_n\|_V^2 + \int_\Omega a\nabla v_n \nabla \bar{u}_n dx + \int_{\Gamma_1} v_n \bar{u}_n d\Gamma = o(1). \quad (4.22)$$

Thus, summing (4.21) and (4.22) and taking the real part, using also (4.20), we deduce

$$\|u_n\|_V^2 \sim \|v_n\|_H^2. \quad (4.23)$$

Now, we want to prove that

$$\|u_n\|_V = o(1), \quad (4.24)$$

arriving in this way, by (4.23), to a contradiction with (4.15). From (4.20), by using the assumption (A1) and Poincaré inequality, we find

$$\int_{\Omega_\delta} |v_n|^2 dx = o(1). \quad (4.25)$$

Now we introduce a cut-off function $\eta \in C^1(\Omega)$ such that

$$\eta(x) = \begin{cases} 
1 & \text{if } x \in \Omega_{\delta-\epsilon}, \\
0 & \text{if } x \in \Omega_\delta, \\
\eta(x) \in [0,1] & \text{elsewhere}.
\end{cases} \quad (4.26)$$

Multiplying (4.18) by $\eta u_n$, we obtain, after integration by parts,

$$\int_{\Omega_\delta} \eta |u_n|^2 dx + \int_{\Omega_\delta} (i\beta_n v_n \eta \bar{u}_n + \bar{u}_n \nabla u_n \cdot \nabla \eta + a \nabla v_n \cdot \nabla (\eta \bar{u}_n)) dx = o(1), \quad (4.27)$$

where we have used the fact that

$$\int_{\Gamma_1} \left( \frac{\partial u_n}{\partial \nu} + a \frac{\partial v_n}{\partial \nu} \right) \eta \bar{u}_n d\Gamma = \int_{\Gamma_1} v_n \bar{u}_n d\Gamma = o(1). \quad (4.28)$$

Now, from (4.15) and (4.17), we deduce that $\|\beta_n u_n\|_H \leq \|v_n\|_H + \|f_n\|_V$ is uniformly bounded for $n \to \infty$. Then, from (4.20) and (4.25), we easily deduce

$$\int_{\Omega_\delta} |\nabla u_n|^2 dx = o(1). \quad (4.29)$$

Multiplying (4.18) by $i\beta_n a v_n$, and integrating by parts we have

$$\int_{\Omega} a |\beta_n v_n|^2 dx = \int_{\Omega} i\beta_n (\nabla u_n + a \nabla v_n) \nabla (a \bar{v}_n) dx$$

\[ - \int_{\Omega} i\beta_n \bar{v}_n g_n dx + i\beta_n \int_{\Gamma_1} \left( \frac{\partial u_n}{\partial \nu} + a \frac{\partial v_n}{\partial \nu} \right) a \bar{v}_n d\Gamma. \]
Since
\[ i\beta_n \int_{\Gamma_1} \left( \frac{\partial u_n}{\partial \nu} + a \frac{\partial v_n}{\partial \nu} \right) a v_n d\Gamma = i\beta_n \int_{\Gamma_1} |v_n|^2 d\Gamma \]
then
\[ \Re \left( i\beta_n \int_{\Gamma_1} \left( \frac{\partial u_n}{\partial \nu} + a \frac{\partial v_n}{\partial \nu} \right) a v_n d\Gamma \right) = 0. \]
Thus, arguing like Liu and Rao ([16], p. 427), we deduce
\[ \Re \left( i\beta_n \int_{\Gamma_1} \left( \frac{\partial u_n}{\partial \nu} + a \frac{\partial v_n}{\partial \nu} \right) a v_n d\Gamma \right) = 0. \]
and so, from (A2) and (4.15),
\[ \int_{\Omega} a|\beta_n v_n|^2 dx \leq c \|\Delta a\|_{\infty} \int_{\Omega} |v_n|^2 dx + o(1), \quad (4.30) \]
and so, from (A2) and (4.15),
\[ \int_{\Omega} a|\beta_n v_n|^2 dx \leq c \|\Delta a\|_{\infty} + o(1) \leq C, \quad (4.31) \]
for a suitable positive constant C.

Now observe that
\[ \text{div} \ ( \nabla u_n + a \nabla v_n ) \in L^2(\Omega), \]
and
\[ \text{div} \ ( \nabla u_n + a \nabla v_n ) \in L^2(\Omega) = \text{div} \ ( \nabla (u_n + av_n) - \nabla av_n ). \]
Then,
\[ \begin{cases} \Delta (u_n + av_n) \in L^2(\Omega), \\ u_n + av_n = 0 \text{ on } \Gamma_0, \end{cases} \]
which implies
\[ u_n + av_n \in H^2(\Omega \setminus W) \]
where W is any neighborhood of \( \Gamma_1 \). Let
\[ M_n = \nabla u_n + a \nabla v_n = \nabla u_n + o(1). \quad (4.32) \]
Then \( q \cdot M_n \in H^1(\Omega) \). Now, observe that by (4.18), taking the inner product with \( q \cdot M_n \), it results
\[ \Re \int_{\Omega} i\beta_n v_n q \cdot \tilde{M}_n dx - \Re \int_{\Omega} \text{div} \ M_n q \cdot \tilde{M}_n dx = o(1). \quad (4.33) \]
Further using Green’s formula, we have
\[ \int_{\Omega} \text{div} \ M_n q \cdot \tilde{M}_n dx = \int_{\partial \Omega} M_n \cdot \nu q \cdot \tilde{M}_n d\Gamma - \int_{\Omega} M_n \nabla (q \cdot \tilde{M}_n) dx \]
\[ = \int_{\Gamma_0} M_n \cdot \nu q \cdot M_n d\Gamma - \int_{\Omega} M_n \nabla (q \cdot M_n) dx. \quad (4.34) \]
From (4.34), using the definition (4.32) of \( M_n \) and the boundary condition \( u = 0 \) on \( \Gamma_0 \), we deduce
\[ -\int_{\Omega} \text{div} \ M_n q \cdot \tilde{M}_n dx = -\int_{\Gamma_0} q \cdot \nu |M_n \cdot \nu|^2 d\Gamma \]
\[ + \int_{\Omega} (M_{n,j} \partial_j q_k \tilde{M}_{n,k} + M_{n,j} q_k \partial_k \tilde{M}_{n,j}) dx \]
\[ = -\int_{\Gamma_0} q \cdot \nu |M_n \cdot \nu|^2 d\Gamma + \int_{\Omega} (M_{n,j} \partial_j q_k \tilde{M}_{n,k} + M_{n,j} q_k \partial_k \tilde{M}_{n,j}) dx \]
\[ \int_{\Omega} M_{n,j} q_k (\partial_j \tilde{M}_{n,k} - \partial_k \tilde{M}_{n,j}) dx. \quad (4.35) \]
Let us denote
\[ I = \int_\Omega M_{n,j} q_k \partial_k \tilde{M}_{n,j} dx. \] (4.36)

Integrating by parts,
\[ I = -\int_\Omega \partial_k ( M_{n,j} q_k ) \tilde{M}_{n,j} dx + \int_{\Gamma_0^\beta} q \cdot \nu |M_n|^2 d\Gamma \\
- \tilde{I} - \int_\Omega \text{div} q |M_n|^2 dx + \int_{\Gamma_0} q \cdot \nu |M_n|^2 d\Gamma. \] (4.37)

Then,
\[ \Re I = -\frac{1}{2} \int_\Omega \text{div} M_n q \cdot \tilde{M}_n dx + \int_{\Gamma_0} q \cdot \nu |M_n|^2 d\Gamma. \] (4.38)

Using (4.38) in (4.35), we obtain
\[ -\Re \int_\Omega \text{div} M_n q \cdot \tilde{M}_n dx = -\frac{1}{2} \int_{\Gamma_0} q \cdot \nu |M_n|^2 d\Gamma \\
+ \Re \int_\Omega ( M_{n,j} \partial_j q_k \tilde{M}_{n,k} - \frac{1}{2} \text{div} q |M_n|^2 ) dx \\
+ \Re \int_\Omega M_{n,j} q_k ( \partial_j \tilde{M}_{n,k} - \partial_k \tilde{M}_{n,j} ) dx. \] (4.39)

Note that
\[ \partial_j \tilde{M}_{n,k} - \partial_k \tilde{M}_{n,j} = \partial_j ( \partial_k u_n + a \partial_k v_n ) - \partial_k ( \partial_j u_n + a \partial_j v_n ) \\
= \partial_j a \partial_k v_n - \partial_k a \partial_j v_n. \]

Then,
\[ \Re \int_\Omega M_{n,j} q_k ( \partial_j \tilde{M}_{n,k} - \partial_k \tilde{M}_{n,j} ) dx = \Re \int_\Omega M_{n,j} q_k ( \partial_j a \partial_k \tilde{v}_n - \partial_k a \partial_j \tilde{v}_n ) dx \\
= \Re \int_\Omega ( M_n \cdot \nabla a q \cdot \nabla \tilde{v} - M_n \cdot \nabla \tilde{v} q \cdot \nabla a ) dx. \] (4.40)

Using (4.40) in (4.39), we have
\[ -\Re \int_\Omega \text{div} M_n q \cdot \tilde{M}_n dx = -\frac{1}{2} \int_{\Gamma_0} q \cdot \nu |M_n|^2 d\Gamma \\
+ \Re \int_\Omega ( M_{n,j} \partial_j q_k \tilde{M}_{n,k} - \frac{1}{2} \text{div} q |M_n|^2 ) dx \\
+ \Re \int_\Omega ( M_n \cdot \nabla a q \cdot \nabla \tilde{v} - M_n \cdot \nabla \tilde{v} q \cdot \nabla a ) dx. \] (4.41)

Now, observe that
\[ \Re \int_\Omega i \beta_n v_n q \cdot M_n dx = \Re \int_\Omega i \beta_n v_n q \cdot \nabla \tilde{u}_n dx + o(1) \\
= -\Re \int_\Omega v_n q \cdot \nabla ( \tilde{v}_n + \tilde{f}_n ) dx + o(1) = -\Re \int_\Omega v_n q \cdot \nabla \tilde{v}_n dx + o(1). \] (4.42)

We have
\[ \int_\Omega v_n q \cdot \nabla \tilde{v}_n dx = \int_\Omega ( -|v_n|^2 \text{div} q + q \cdot \nabla v_n \tilde{v}_n ) dx, \]
and then
\[ \Re \int_\Omega v_n q \cdot \nabla \tilde{v}_n dx = -\frac{1}{2} \int_\Omega |v_n|^2 \text{div} q dx. \] (4.43)
From (4.42) and (4.43), we have
\[ \Re \int_\Omega i \beta_n v_n q \cdot \bar{M}_n dx = \frac{1}{2} \int_\Omega |v_n|^2 \text{div } q dx + o(1). \] (4.44)

Now, let \( h \in C^0(\Omega) \). By multiplying (4.17) by \( h \bar{v}_n \) and (4.18) by \( h \bar{\bar{u}}_n \), we have
\[ \int_\Omega i \beta_n h u_n \bar{v}_n dx - \int_\Omega h |v_n|^2 dx = o(1), \] (4.45)
\[ \int_\Omega i \beta_n h v_n \bar{u}_n dx + \int_\Omega (\nabla u_n + a \nabla v_n) \cdot \nabla (h \bar{\bar{u}}_n) dx \]
\[ + \int_{\Gamma_1} v_n h \bar{\bar{u}}_n d\Gamma = o(1). \] (4.46)

Recalling (4.15), (4.20) and (4.27), we deduce from (4.46)
\[ \int_\Omega i \beta_n h v_n \bar{u}_n dx + \int_\Omega (h |\nabla u_n|^2 + \nabla u_n \cdot \nabla h \bar{\bar{u}}_n) dx = o(1). \] (4.47)

Taking the real part of the sum of (4.45) and (4.47) we have
\[ \int_\Omega h |\nabla u_n|^2 dx = \int_\Omega |v_n|^2 dx - \int_\Omega \bar{u}_n \nabla u_n \cdot \nabla h dx + o(1). \] (4.48)

Moreover, using (4.17),
\[ \int_\Omega \bar{u}_n \nabla u_n \cdot \nabla h dx = - \frac{i}{\beta_n} \int_\Omega (\bar{v}_n + \bar{f}_n) \nabla u_n \cdot \nabla h dx = o(1). \] (4.49)

From (4.48) and (4.49) we get
\[ \int_\Omega h |v_n|^2 dx = \int_\Omega h |\nabla u_n|^2 dx + o(1). \] (4.50)

Choosing \( h = \text{div } q \) in (4.50) and using it in (4.44), we deduce
\[ \Re \int_\Omega i \beta_n v_n q \cdot \bar{M}_n dx = \int_\Omega \frac{1}{2} \text{div } q |\nabla u_n|^2 dx + o(1). \] (4.51)

By using the definition (4.32) of \( M_n \), we can estimate the second integral in the right–hand side of (4.41) as
\[ \int_\Omega \left( M_{n,j} \partial_j q_k \bar{M}_{n,k} - \frac{1}{2} \text{div } q |M_n|^2 \right) dx \]
\[ = \int_\Omega \left( \partial_j u_n \partial_j q_k \partial_k \bar{u}_n - \frac{1}{2} \text{div } q |\nabla u_n|^2 \right) dx + o(1). \] (4.52)

From (Q3), (4.20) and the fact that \( M_n \) is bounded in \( L^2(\Omega) \), we can estimate the third integral in the right–hand side of (4.41) as
\[ \int_\Omega |M_n| |q \cdot \nabla \bar{v}_n| \nabla a - (q \cdot \nabla a) \nabla \bar{v}_n| dx \leq C \int_\Omega |M_n|^2 dx \int_\Omega |\nabla v_n|^2 dx = o(1). \] (4.53)

Coming back to (4.33), by substituting (4.41) (taking into account (4.52) and (4.53)) and (4.51) we deduce
\[ \int_\Omega \partial_j q_k \partial_j u_n \partial_k \bar{u}_n dx - \frac{1}{2} \int_{\Gamma_0} q \cdot \nu |M_n \cdot \nu|^2 d\Gamma = o(1), \] (4.54)
and so, recalling (Q3) and (4.28)
\[ C \int_{\Omega_\beta} |\nabla u_n|^2 dx - \frac{1}{2} \int_{\Gamma_0} q \cdot \nu |M_n \cdot \nu|^2 d\Gamma = o(1), \]
for some $C > 0$. By the assumption (Q4) we then get

$$\int_{\Omega_2} |\nabla u_n|^2 \, dx = o(1).$$

This property and (4.28) lead to (4.24). Hence the contradiction by (4.23).

In summary we have obtained the next stability result.

**Theorem 4.5.** Under the assumptions (H), (A1), (A2), (Q1) – (Q4), if condition (1.6) is satisfied, then $A_0$ generates a $C_0$-semigroup of contraction $(T_0(t))_{t \geq 0}$ that is exponentially stable, namely there exist two constants $M_0 > 0$ and $\omega_0 > 0$, such that for all $U_0 \in H_0$ we have

$$\|T_0(t)U_0\|_{H_0} \leq M_0 e^{-\omega_0 t}\|U_0\|_{H_0}, \forall t \geq 0.$$

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**REFERENCES**

[1] K. Ammari and S. Gerbi, Interior feedback stabilization of wave equations with dynamic boundary delay, arXiv:1405.6865.

[2] K. Ammari, S. Nicaise and C. Pignotti, Stability of abstract-wave equation with delay and a Kelvin–Voigt damping, *Asymptot. Anal.*, 95 (2015), 21–38.

[3] G. Chen, Control and stabilization for the wave equation in a bounded domain I, *SIAM J. Control Optim.*, 17 (1979), 66–81.

[4] G. Chen, Control and stabilization for the wave equation in a bounded domain II, *SIAM J. Control Optim.*, 19 (1981), 114–122.

[5] R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, *SIAM J. Control Optim.*, 26 (1988), 697–713.

[6] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.*, 24 (1986), 152–156.

[7] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer Series in Computational Mathematics, 5 Springer, Berlin, 1986.

[8] F. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differential Equations*, 1 (1985), 43–56.

[9] V. Komornik, Rapid boundary stabilization of the wave equation, *SIAM J. Control Optim.*, 29 (1991), 197–208.

[10] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, RAM: Research in Applied Mathematics, 36, Masson, Paris, 1994.

[11] V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation, *J. Math. Pures Appl.*, 69 (1990), 33–54.

[12] J. Lagnese, Decay of solutions of wave equation in a bounded region with boundary dissipation, *J. Differential Equations*, 50 (1983), 163–182.

[13] J. Lagnese, Note on boundary stabilization of wave equations, *SIAM J. Control and Optim.*, 26 (1988), 1250–1256.

[14] I. Lasiecka and R. Triggiani, Uniform exponential energy decay of wave equations in a bounded region with $L_2(0, T; L_2(\Sigma))$-feedback control in the Dirichlet boundary conditions, *J. Differential Equations*, 66 (1987), 340–390.

[15] J. L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués. Tome 1*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics] Masson, Paris, 1988.

[16] K. Liu and B. Rao, Exponential stability for the wave equations with local Kelvin–Voigt damping, *Z. angew. Math. Phys.*, 57 (2006), 419–432.

[17] Ö. Mångul, On the stabilization and stability robustness against small delays of some damped wave equations, *IEEE Trans. Automat. Control.*, 40 (1995), 1626–1630.

[18] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, 45 (2006), 1561–1585.
[19] S. Nicaise and C. Pignotti, Exponential stability of second-order evolution equations with structural damping and dynamic boundary delay feedback, *IMA J. Math. Control Inform.*, 28 (2011), 417–446.

[20] J. Prüss, On the spectrum of $C_0$–semigroups, *Trans. Amer. Math. Soc.*, 284 (1984), 847–857.

[21] G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of wave systems with input delay in the boundary control, *ESAIM: Control Optim. Calc. Var.*, 12 (2006), 770–785.

[22] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, *Comm. Partial Differential Equations*, 15 (1990), 205–235.

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