ON PERPETUITIES RELATED TO
THE SIZE-BIASED DISTRIBUTIONS

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Abstract. We study perpetuities of a special type related to the size-biased distributions. Necessary and sufficient conditions of their existence and uniqueness are obtained. A crucial point in proving all results is a close connection between perpetuities treated in the paper and fixed points of so-called Poisson shot noise transforms.

Introduction.

Let $\mathcal{P}^+$ be the set of all probability measures on the Borel subsets of $\mathbb{R}^+ := [0, \infty)$ and, for fixed $m > 0$, $\mathcal{P}^+_m := \{\nu \in \mathcal{P}^+ : \int_0^\infty x \nu(dx) = m\}$. Recall that given distribution $\nu \in \mathcal{P}^+_m$, a distribution $\nu_{sb}$ is said to be the size-biased distribution corresponding to $\nu$, if

$$\nu_{sb}(dx) = m^{-1} x \nu(dx).$$

Throughout this paper the symbols $\mathcal{L}(\cdot)$ and $\mathcal{L}_{sb}(\cdot)$ stand for the probability distribution and the size-biased distribution of a random variable (rv) in question, respectively.

Consider the distributional equality

(1) \[ X \overset{d}{=} AX + B, \]

where a random pair $(A, B)$ is independent of an rv $X$, and $\overset{d}{=}$ means "equality of distributions". In the recent literature it is customary to say that the rv $X$ is a perpetuity. Cf. Embrechts, Goldie (1994) for more details.

In this paper we treat a quite special type of equality (1) in which $\mathcal{L}(B) = \mu$, $\mathcal{L}(X) = \mu_{sb}$, and $A, B$ are independent rv’s. It is convenient to put $B \equiv \eta$, $X \equiv \eta_{sb}$ and rewrite (1) as follows

(2) \[ \eta_{sb} \overset{d}{=} A \eta_{sb} + \eta. \]

Pitman, Yor (2000, p.35) mention the following problem: "given a distribution of $A$...whether there exists such a distribution of $\eta$".

In all Propositions stated below we assume that $\mathbb{P}(A = 0) = 0$ and keep the following notations $\mu := \mathcal{L}(\eta)$, $\rho := \mathcal{L}(A)$ where $\eta$ and $A$ satisfy (2). First we provide necessary and sufficient conditions of the existence of non-zero $\mu$ and reveal that when exists this distribution is unique up to the scale.

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Proposition 1.1 (Existence and uniqueness).
Assume that $E \log A$ exists, finite or infinite. Then non-zero $\mu$ exists iff
$$E \log A < 0.$$  
(3)

Given $m > 0$ there exists the unique distribution $\mu$ of mean $m$.

In fact the above Proposition is essentially based upon a close relation between solutions to (2) and fixed points of so-called Poisson shot-noise transform (see Section 2 for definition and some properties). This is the content of Proposition 2.1.

Given $\rho = \mathcal{L}(A) \in \mathcal{P}^+$ define a random map $U_\rho : \mathcal{P}_{m}^+ \to \mathcal{P}_{m}^+$ as follows:

$$\mathcal{L}_{sb}(U_\rho \theta) = \mathcal{L}_m \mathcal{L}(AX_{sb}) ,$$

where $\theta = \mathcal{L}(X) \in \mathcal{P}_{m}^+$, $\mathcal{L}(X_{sb}) = \theta_{sb}$, $A$ and $X_{sb}$ are independent rv’s and "*" stands for the convolution of measures. To be sure that this map is well-defined it suffices to verify that $U_\rho \theta$ is of finite mean. To this end consider the measure $N$ given by the equality $xN(dx) := \mathcal{L}(AX_{sb})(dx)$. Clearly, it is the Lévy measure of some infinitely divisible (ID) distribution $\kappa$, say. Since $\int_0^\infty xN(dx) < \infty$ then $\int_0^\infty x\kappa(dx) < \infty$, and now it is obvious that $\kappa = U_\rho \theta$.

In the proof of Proposition 1.1 it will be shown that $\mu$ is a weak limit of iterates $U^n_\rho = U_\rho(U^{n-1}_\rho)$, $n \in \mathbb{N}$, and therefore is a fixed point of $U_\rho$. As it turned out, by putting $\rho$ some additional moment restrictions one may prove that $U_\rho$ is a strict contraction acting on some complete metric space $(T, r)$. Using the Banach Fixed Point Theorem allows us to assert that the iterates of $U_\rho \theta$ converge exponentially fast to a unique fixed point of $U_\rho$, for every $\theta \in T$.

For fixed $1 < \Delta < 2$, $m > 0$ consider the set of probability measures

$$P_m^+(\Delta) := \{ \nu \in \mathcal{P}_{m}^+: \int_0^\infty x^\Delta \nu(dx) < \infty \}.$$  

In view of Lemma 3.1 by Baringhaus, Grübel (1997) the quantity

$$r_\Delta := r_\Delta(\nu_1, \nu_2) =$$

$$= \int_0^\infty s^{-\Delta-1} \left| \int_0^\infty \exp(isx)\nu_1(dx) - \int_0^\infty \exp(isx)\nu_2(dx) \right| ds ,$$

defined for all $\nu_1, \nu_2 \in \mathcal{P}_m^+(\Delta)$, is a metric on $\mathcal{P}_m^+(\Delta)$, and $(\mathcal{P}_m^+(\Delta), r_\Delta)$ is a complete metric space.

In the next Proposition we study the case of $\rho$ having some finite moments, so the function $g(x) := E A^x$ is finite (and log-convex) at least on some neighbourhood of the origin.

Proposition 1.2 (Contraction properties and results on moments).

a) Let

$$g(p) < 1, \text{ for some } p > 0 ,$$  

and $q \in (1, 2)$ be any fixed number such that $g(q-1) < 1$. Then

1) $U_\rho$ defined on a complete metric space $(\mathcal{P}_m^+(q), r_q)$ is a strict contraction, and therefore for every $\theta \in \mathcal{P}_m^+(q)$ the sequence $U^n_\rho \theta$, $n = 1, 2, ...$ converges in $r_q$-metric
(hence, weakly) at an exponential rate to a unique fixed point \( \mu \) of \( \mathbb{U}_\rho \);
2) the rv \( \eta \) with \( \mathcal{L}(\eta) = \mu \) solves (2), \( \mathbb{E}\eta = m \) and \( \mathbb{E}\eta^{1+p} < \infty \).
b) If there exists non-zero \( \eta \) satisfying (2) and \( \mathbb{E}\eta^{p+1} < \infty \), for some \( p > 0 \), then \( \mathbb{E}\eta^{p} < 1 \).

It is easily seen that all \( \mu \)'s are ID. This follows from the representation of non-negative ID distributions due to Steutel (see Sato (1999, Theorem 51.1) for a detailed proof). In the next assertion we study the structure of \( \mu \) as an ID distribution more carefully.

**Proposition 1.3 (Infinite divisibility).** All non-zero \( \mu \)'s are ID with the shift 0 and the Lévy measure \( M(dx) = x^{-1}L(A\xi_{sb})(dx) \). Furthermore, \( \mu \)'s are compound Poisson provided \( x^{-1}\rho(dx) \) is integrable at the neighbourhood of zero.

Two other results of the paper deal with the tail behaviour of \( \mu \). This is changed considerably according to whether the rv \( A \) may take values greater than 1, or not; if not, then whether esssup \( \rho = 1 \) or not.

**Proposition 1.4 (Exponential moments).**
a) \( \mu \)'s have finite exponential moment iff esssup \( \rho \leq 1 \);
b) if esssup \( \rho < 1 \) then \( \mu \)'s have entire characteristic functions.

We also give a very simple independent proof of the next result one implication of which follows by Proposition 1.4(a). Corollary 4.2 of Goldie, Gründel (1996) contains a result concerning general perpetuities in the spirit of the converse part of Proposition 1.5.

**Proposition 1.5.** Condition "\( \rho \) is concentrated on \((0,1]\)" is necessary and sufficient to ensure the existence of the solution \( \eta \) to (2) which is completely determined by its moments.

**Notations and convention.** "LT" ("LST")-Laplace (Stieltjes) transform, "rv" - random variable, "ID" - infinitely divisible, "a.s." - almost surely, "w.l.o.g." - without loss of generality; we always take distribution functions to be right-continuous.

**Connection to fixed points of shot noise transforms.**

Throughout this Section we assume that all rv's involved live on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \( \xi, \xi_1, \xi_2, \ldots \) be non-negative iid rv's, independent of the Poisson flow \( \{\tau_i\}, i \in \mathbb{N} \) with the intensity \( 0 < \lambda < \infty \). Given \( \mathcal{L}(\xi_i) \) fix a Borel measurable function \( h : (0, \infty) \to (0, \infty) \) enjoying the property \( \int_0^\infty \mathbb{E}[1 \wedge h(s)\xi]ds < \infty \). Recall that under the above assumptions random series \( \sum_{i=1}^\infty \xi_i h(\tau_i) \) converges a.s., and its distribution is called a (Poisson) shot noise distribution. The function \( h \) is said to be the response function. We refer to Vervaat (1979) and Bondesson (1992, Section 10) for some additional information regarding the shot noise distributions.

For a fixed \( \lambda \), consider a Poisson shot noise transform (SNT, in short) \( T_{h,\lambda} \) acting on the set (domain)

\[
\mathcal{P}_h^+ := \{ \nu \in \mathcal{P}^+ : \int_0^\infty \int_0^\infty [1 \wedge h(s)y]d\nu(dy) < \infty \}
\]

with values in \( \mathcal{P}^+ \) as follows

\[
T_{h,\lambda}(\mathcal{L}(\xi)) := \mathcal{L} \left( \sum_{i=1}^\infty \xi_i h(\tau_i) \right).
\]
Thus the domain $P_h^+$ is the set of possible distributions for rv $\xi$ that would ensure the well-definedness of shot noise distribution. By fixed points of the SNT we mean non-zero distributional solutions to the equation

$$\mu^* = T_{h,\lambda}(\mu^*),$$

where $\mu^* = \mathcal{L}(\xi)$. We will essentially make use of an equivalent definition of fixed points given via the LST $\varphi(s) = \int_{0}^{\infty} e^{-sx}\mu^*(dx)$. Namely $\mu^*$ is a fixed point of the SNT $T_{h,\lambda}$ iff

$$\varphi(s) = \exp\left(\lambda \int_{0}^{\infty} (\varphi(sh(u)) - 1)du\right).$$

In what follows it is assumed that

$$\text{(6) the response function } h \text{ is right-continuous and non-increasing.}$$

This assumption permits to define the right-continuous and non-increasing generalized inverse of $h$ given as follows

$$h^-(x) = \int_{x}^{b} z^{-1}\rho(dz), x \in (a,b).$$

Therefore $h$ is right-continuous and non-increasing with $\int_{0}^{\infty} h(z)dz = 1$ and $\int_{0}^{\infty} h(z) \log h(z)dz < 0$. 

b) Conversely, if $\mu^*$ is a fixed point of the SNT $T_{h,\lambda}$ with $h$ satisfying (6) and

$$\lambda \int_{0}^{\infty} h(z)dz = 1, \int_{0}^{\infty} h(z) \log h(z)dz < 0 \text{ and } \lim_{z \to 0^+} h(z) = b \in (0,\infty]$$

then an rv $\eta$ such that $\mathcal{L}(\eta) = \mu^*$ verifies (2) with an rv $A$ whose distribution $\rho(dx) = -\lambda x h^-(dx)$ is concentrated on $[a,b]$.

The proofs.

The first Lemma of this Section is implicit in Athreya (1969, Theorem 1).

**Lemma 3.1.** Let $\varphi_1(s)$ and $\varphi_2(s)$ be the LST’s of probability measures of the same (finite) mean. If for large enough positive integers $n$ the function $\psi(s) = \frac{|\varphi_1(s) - \varphi_2(s)|}{s}$ satisfies the inequality $\psi(s) \leq \mathbb{E}\psi(C_n s)$, where $\{C_n\}$ is a sequence of rv’s tending to 0 a.s., then $\varphi_1(s) \equiv \varphi_2(s)$.

The next Lemma is both an existence and uniqueness result concerning fixed points of the SNT. In fact, the main part of Proposition 1.1 is merely a combination of Lemma 3.2 and Proposition 2.1.

**The proofs.**
Lemma 3.2. Let $h$ satisfies (6), $\lambda \int_0^\infty h(z)dz = 1$ and $\int_0^\infty h(z) \log h(z)dz < 0$. Then given $m \in (0, \infty)$ $T_{h, \lambda}$ has a unique fixed point $\mu^*$ with $\int_0^\infty x\mu^*(dx) = m$.

Proof. For fixed $m > 0$ consider the set of probability measures $P^+_{h,m} := \{v \in P^+_h : \int_0^\infty xv(dx) = m\}$. Starting with $\mu_0 = \delta_m$, construct the sequence

$$
\mu_n := T_{h, \lambda}\mu_{n-1} := T_{h, \lambda}\mu_0, n = 1, 2, ...
$$

which is trivially well-defined on $P^+_{h,m}$ provided $\int_0^\infty h(z)dz < \infty$. The corresponding LST’s $\varphi_n^{(1)}(s) = \int_0^\infty e^{-sx}\mu_n(dx), n = 0, 1, ...$ satisfy equations

$$
(9) \quad \varphi_0^{(1)}(s) = e^{-ms}, \varphi_n^{(1)}(s) = \exp\{-\lambda \int_0^\infty (1 - \varphi_{n-1}(sh(u)))du\}, n = 1, 2, ...
$$

Similarly, for fixed $s_0 > 0$, let us define yet another sequence $\{\varphi_n^{(2)}\}$ that satisfies (9) for $n = 1, 2, ...$, but $1 - \varphi_0^{(2)}(s) = (1 - e^{-s})1_{s \in [0, s_0]}$, where $1_A$ is the indicator of set $A$.

Let us verify that the weak limit of $\mu_n$, as $n \to \infty$, exists and has mean $m$. As it is well-known, this will mean that $T_{h, \lambda}$ has a fixed point (which is this weak limit) on $P^+_{h,m}$.

In what follows we use some ideas of Durrett, Liggett (1983, the proof of Theorem 2.7). From (9) one gets, $\varphi_1^{(i)}(s) \geq \varphi_0^{(i)}(s), i = 1, 2$ that implies

$$
\varphi_n^{(i)}(s) \geq \varphi_{n-1}^{(i)}(s), n = 1, 2, ..., s \geq 0, i = 1, 2.
$$

Thus the monotone and bounded sequence $\{\varphi_n^{(i)}\}, n = 1, 2, ...$ has a unique limit $\varphi^{(i)}, i = 1, 2, ...$

$$
(10) \quad \limsup_{s \to +0}^{-1}(1 - \varphi^{(i)}(s)) \leq m, i = 1, 2.
$$

For what follows it is essential that $\varphi_0^{(2)}(s) \geq \varphi_0^{(1)}(s)$ implies

$$
(11) \quad s^{-1}(1 - \varphi^{(2)}(s)) \leq s^{-1}(1 - \varphi^{(1)}(s)), s \geq 0.
$$

Note $\varphi^{(1)}(s)$ is the LST of a probability measure $\mu^{(1)}$, say. Since $\int_0^\infty h(z)dz < \infty$ then by dominated convergence it is easily seen that $\varphi^{(1)}(s)$ satisfies the fixed point equation (5) or equivalently $\mu^{(1)}$ is a (possibly degenerate at 0) fixed point of the SNT. It remains to check that $\mu^{(1)} \in P^+_{h,m}$.

To this end for $n = 0, 1, ...$ put $\Phi_n(s) := e^s(1 - \varphi_n^{(2)}(e^{-s})), \Psi_n(s) := e^s(1 - \varphi_n^{(2)}(e^{-s}) + \log \varphi_n^{(2)}(e^{-s})).$ In view of assumptions of the Lemma $\pi(dx) := -\lambda xh^{+\ast}(dx)$ is a probability distribution. Let $\theta_1, \theta_2, ...$ be independent rv’s with this distribution. Under these notations one obtains from (9) by change of variable and monotonicity of $\Psi_n$

$$
(12) \quad \Phi_{n+1}(s) = E\Phi_n(s - \log \theta) - \Psi_n(s) \geq E\Phi_n(s - \log \theta) - \Psi_0(s), n = 1, 2, ...
$$
Consider the random walk \( S_0 = 0, S_n = -\sum_{i=1}^{n} \log \theta_i, n = 1, 2, \ldots \) On iterating (12) one gets

\[
\Phi_n(s) \geq \mathbb{E}\Phi_0(s + S_n) - \mathbb{E} \sum_{i=0}^{n-1} \Psi_0(s + S_i), n = 1, 2, \ldots
\]

Note that \( \Phi_0(s) := e^s(1 - \exp(-e^s))1\{s \geq \log s_0\} \) and \( \Psi_0(s) := e^s(e^{-s} - (1 - e^{-s}))1\{s \geq \log s_0\}. \) Since \( \mathbb{E}\log \theta_i = \lambda \int_{0}^{\infty} h(z) \log h(z)dz \leq 0 \) then by the strong law of large numbers \( S_n \rightarrow +\infty \) a.s., as \( n \rightarrow \infty. \) Consequently by dominated convergence

\[
\lim_{n \rightarrow \infty} \mathbb{E}\Phi_0(s + S_n) = m.
\]

Let us verify that \( \Psi_0(s) \) is directly Riemann integrable (dRi). This together with the renewal theorem on the whole line (Feller (1966), Theorem XI.1, p. 368) will imply

\[
\lim_{s \rightarrow +\infty} \mathbb{E} \sum_{i=0}^{\infty} \Psi_0(s + S_i) = 0.
\]

Since \( e^{-s}\Psi_0(s) \) is a decreasing function in \( s \) and

\[
\int_{-\infty}^{\infty} \Psi_0(s)ds = \int_{0}^{s_0} s^{-2}(s - (1 - e^{-s}))ds < \infty \text{ (the latter integrand is equivalent to a constant near 0)} \text{ then } \Psi_0(s) \text{ is indeed dRi. Now according to (13), (14)}

\[
m \leq \liminf_{s \rightarrow +\infty \rightarrow \infty} \Phi_n(s) \leq \liminf_{s \rightarrow +\infty \rightarrow 0} s^{-1}(1 - \varphi(2)(s)) \leq \lim_{s \rightarrow +\infty} s^{-1}(1 - \varphi(1)(s))
\]

whence from (10) \( \lim_{s \rightarrow +\infty} s^{-1}(1 - \varphi(1)(s)) = m \) or equivalently \( \mu(1) = \mu^* \in \mathcal{P}^+_{h,m}. \)

To prove uniqueness let us assume on the contrary that there exists another LST \( \tilde{\varphi}(s) \) with \( \lim_{s \rightarrow +\infty} s^{-1}(1 - \tilde{\varphi}(s)) = m \) that satisfies (5). Choose in Lemma 3.1 \( \varphi_1 := \tilde{\varphi}, \varphi_2 := \varphi(1), \) and for each positive integer \( n, C_n := \theta_1 \ldots \theta_n. \) Note that \( C_n = \exp(-S_n) \rightarrow 0 \) a.s, as \( n \rightarrow \infty. \) By using an elementary inequality \( |e^{-x} - e^{-y}| \leq |x - y|, x, y \in \mathbb{R} \) one concludes from (5) \( \psi(s) \leq \mathbb{E} \psi(\theta s), s \geq 0. \) On iterating the latter inequality \( n \) times one gets \( \psi(s) \leq \mathbb{E} \psi(C_n s) \) which in turn implies that Lemma 3.1 does apply. Hence \( \tilde{\varphi} \equiv \varphi(1) \) which completes the proof. \( \square \)

Now we are ready to prove Proposition 2.1.

**Proof of Proposition 2.1.** We will prove both parts of the Proposition simultaneously. First of all note that the statement "\( \rho \) is a probability measure" and condition (6) with \( \lambda \int_{0}^{\infty} h(z)dz = 1 \) are equivalent.

Further suppose that the conditions of part (b) are in force and the SNT \( \mathcal{T}_{h,\lambda} \) has a fixed point \( \mu^*. \) Hence \( \varphi^*(s) = \int_{0}^{\infty} e^{-sz} \mu^*(dz) \) satisfies (5), that is, \( \varphi^*(s) = \)

\[
\exp\{-\lambda \int_{0}^{\infty} (1 - \varphi^*(sh(u)))du\} = \exp\{\lambda \int_{a}^{b} (1 - \varphi^*(sz))h^+(dz)\} = \exp\{-\lambda \int_{a}^{b} (1 - \varphi^*(sz))z^{-1}\rho(dz)\},
\]

\[
= \exp\{-\lambda \int_{0}^{\infty} (1 - \varphi^*(sh(u)))du\} = \exp\{\lambda \int_{a}^{b} (1 - \varphi^*(sz))h^+(dz)\} = \exp\{-\lambda \int_{a}^{b} (1 - \varphi^*(sz))z^{-1}\rho(dz)\},
\]
where \( \rho(dz) = -\lambda z h^-(dz) \) is a probability measure. The latter follows from (6) and (8). In view of Lemma 3.2 condition (8) implies \( m = \int_0^\infty x \mu^*(dx) < \infty \), for some \( m > 0 \). W.l.o.g. we may and do assume \( m = 1 \), and therefore \( \lim_{s \to 0^+} (1 - \varphi(s)) = 1 \).

On the other hand, assume that an rv \( \eta \) with \( \mathbb{E}\eta = 1 \) satisfies (2) and the conditions of part (a) hold. Then the LT \( \varphi(s) = \mathbb{E}e^{-s\eta} \) solves

\[
\varphi'(s) = \varphi(s) \int_0^\infty \varphi'(sz) \rho(dz).
\]

Note that \( -\varphi'(s) \) is the LST of probability measure \( \mu_{sb}(dx) := x \mu(dx) \). By using Fubini’s Theorem one has \( \log \varphi(s) = \int_0^s [\log \varphi(u)] du = \int_0^s \varphi'(uz) \rho(dz) du = \int_0^\infty z^{-1} \rho(dz)(\varphi(sz) - 1) \) or equivalently

\[
\varphi(s) = \exp\{-\int_0^\infty (1 - \varphi(sz)) z^{-1} \rho(dz)\}.
\]

Consider the (possibly \( \sigma \)-finite) measure \( \nu(dz) = (\lambda z)^{-1} \rho(dz) \) and the (right-continuous and non-increasing) function \( V(x) = -\int_x^b \nu(dz) \). Now define the response function \( h \) via its generalized inverse \( h^+(x) = V(x) \). This implies \( \rho(dz) = -\lambda h^+(dz) \) or equivalently (7).

It remains to verify that \( \varphi^*(s) = \varphi(s) \) which can be achieved in the same manner as in the proof of Lemma 3.2 (use Lemma 3.1). The proof is completed. \( \Box \)

Proof of Proposition 1.1. Assume that \( \mathbb{P}\{A = 0\} = 0 \) and there exists non-zero \( \mu \) solving (2), but \( \mathbb{E}\log A \geq 0 \). By Theorem 1.6(a) of Vervaat (1979) the latter implies that \( \mathcal{L}(\eta_{sb}) = \delta_c \), for some \( c \in \mathbb{R} \). This is possible if \( c = 0 \), the case excluded by us. A contradiction.

Assume now that (3) holds and \( \mathbb{P}\{A = 0\} = 0 \). Consider the SNT \( \mathcal{T}_{h,1} \) with the response function \( h \) defined via its generalized inverse \( h^+(z) \) as in (7). Since \( \mathbb{E}\log A < 0 \) implies \( \int_0^\infty h(z) \log h(z)dz < 0 \), Lemma 3.2 applies that allows us to conclude that given \( m \in (0, \infty) \) \( \mathcal{T}_{h,1} \) has a unique fixed point \( \mu^* \) with \( m = \int_0^\infty x \mu^*(dx) \). It remains to appeal to Proposition 2.1(b). This finishes the proof of Proposition 1.1. \( \Box \)

To prove Proposition 1.2 we need the following result.

**Lemma 3.3.** Let \( h \) be a positive measurable function and for some \( p > 0 \)

\[
\int_0^\infty h^p(u) du < \infty. \text{ If } \mathbb{E}\xi_1^p < \infty \text{ then } \mathbb{E}(\sum_{i=1}^\infty \xi_i h(\tau_i))^p < \infty.
\]

**Proof.** Shot noise distributions are ID, and therefore as it is well-known,

\[
\mathbb{E}(\sum_{i=1}^\infty \xi_i h(\tau_i))^p < \infty \text{ iff } \int_0^\infty x^p M(dx) < \infty \text{ where } M \text{ is the Lévy measure. Since } \int_0^\infty x^p M(dx) = \mathbb{E}\xi_1^p \int_0^\infty h^p(u) du < \infty, \text{ the assertion follows.} \]

**Proof of Proposition 1.2(a).** 1) Given the rv \( A \) with \( \mathbb{P}\{A = 0\} = 0 \) and \( \mathbb{E}A^{q-1} < 1, \ q \in (1,2) \), define the SNT \( \mathcal{T}_{h,1} \) with the response function \( h \) whose generalized inverse \( h^+ \) is defined by (7). By Proposition 2.1(a), first, \( \int_0^\infty h(u) du = 1 \) from which one concludes that the restriction of \( \mathcal{T}_{h,1} \) to the set \( \mathcal{P}_m^+(q) \) is well-defined and for any \( \theta \in \mathcal{P}_m^+(q) \), \( \int_0^\infty x \mathcal{T}_{h,1}\theta(dx) = m \); second, \( \int_0^\infty h^\theta(u) du < 1 \) that in conjunction with Lemma 3.3 imply \( \int_0^\infty x^\theta(\mathcal{T}_{h,1}\theta)(dx) < \infty \). Thus \( \mathcal{T}_{h,1} \) maps \( \mathcal{P}_m^+(q) \)
into itself. The key observation for what follows is that for any \( \theta \in \mathcal{P}_{m}^{+}(q) \) and \( \rho = \mathcal{L}(A) \) where \( A \) satisfies the above conditions, the map \( \cup_{\rho} \) is well-defined and moreover

\[ \cup_{\rho} \theta = T_{h,1}\theta. \]

Formally this can be verified by recording the corresponding LST’s. To prove 1), it remains to add that according to Lemma 3.5 of Iksanov, Jurek (2002) the restriction of \( T_{h,1} \) defined on a complete metric space \( (\mathcal{P}_{m}^{+}(q), r_{q}) \) is a strict contraction, and apply the Banach Fixed Point Theorem.

2) The preceding display and Proposition 2.1(b) imply that \( \mu \) solves (2). Assume now (4) holds and at once note that the assertion \( \mathbb{E}\eta = m \) is obvious. We intend to verify that (4) implies \( \mathbb{E}\eta^{1+p} < \infty \). W.l.o.g. set \( m = 1 \) and consider two cases:

1) \( p \in \mathbb{N} \): if \( p = 1 \) then (2) implies \( \mathbb{E}\eta_{sb} = \mathbb{E}\eta/(1 - \mathbb{E}A) < \infty \) since by the definition of the size-biased distribution \( \mathbb{E}\eta < \infty \). Further we proceed by induction. Assume it is already known that for some \( 1 < k < p, k \in \mathbb{N} \)

\[ \mathbb{E}\eta^{1+k} = \mathbb{E}\eta_{sb}^{1+k} < \infty. \]

By assumption (4) the function \( g(x) \) is log-convex on \((0, p)\) which implies \( g(l) = \mathbb{E}A^{l} < 1 \) for all \( l \in (0, p) \). Therefore \( \mathbb{E}A^{k+1} < 1 \). Now an appeal to (2) allows us to write formally

\[ \mathbb{E}\eta^{2+k} = \mathbb{E}\eta_{sb}^{1+k} = \mathbb{E}(A\eta_{sb} + \eta)^{1+k} = \]

\[ = \sum_{i=0}^{1+k} \binom{1+k}{i} \mathbb{E}A^{i}\mathbb{E}\eta^{i+1}\mathbb{E}\eta^{1+k-i} = \]

\[ = \mathbb{E}A^{1+k}\mathbb{E}\eta_{sb}^{1+k} + \sum_{i=0}^{k} \binom{1+k}{i} \mathbb{E}A^{i}\mathbb{E}\eta^{i+1}\mathbb{E}\eta^{1+k-i} := \]

\[ := \mathbb{E}A^{1+k}\mathbb{E}\eta^{2+k} + t_{k}, \]

which yields \( \mathbb{E}\eta^{2+k} = t_{k}/(1 - \mathbb{E}A^{1+k}) < \infty \) by noting that \( t_{k} \) consists of the finite number of finite terms. This completes the study of this case.

2) \( p \notin \mathbb{N} \): denote by \( \alpha \) the fractional part of \( p \) then there exists \( n \in \mathbb{N} \cup \{0\} \) such that \( p = n + \alpha \). If \( n = 0 \) the assertion follows from the first part of the Proposition where in fact was shown that if \( \mathbb{E}A^{n} < 1 \) then \( \mu = \mathcal{L}(\eta) \in \mathcal{P}_{m}^{+}(1 + \alpha) \). Suppose we have already verified that for some \( 0 < k < n, k \in \mathbb{N} \)

\[ \mathbb{E}\eta^{1+k+\alpha} = \mathbb{E}\eta_{sb}^{k+\alpha} < \infty. \]

Set \( k_{\alpha} := 1 + k + \alpha \). Now (2) implies that one may write formally

\[ (\mathbb{E}(\eta^{k+1}))^{k-1} = (\mathbb{E}\eta_{sb}^{k-1})^{k-1} = \]

\[ = (\mathbb{E}(A\eta_{sb} + \eta))^{k-1} \leq \]

\[ \leq (\mathbb{E}A^{k})^{k-1}(\mathbb{E}\eta^{k+1})^{k-1} + (\mathbb{E}\eta^{k})^{k-1}, \]

the inequality being implied by the triangle inequality in the space \( L_{k_{\alpha}} \). Now the latter inequality can be rewritten as follows

\[ \mathbb{E}\eta^{k+1} \leq \mathbb{E}\eta^{k+1}/(1 - (\mathbb{E}A^{k})^{k-1})^{k} < \infty \]
where finiteness is implied by (15) and log-convexity of the function \( g(x) \) that guarantees \( 1 - (EA^k)_\omega > 0 \). A usual inductive argument completes the proof. \( \square \)

**Proof of Proposition 1.2(b).** While for \( p > 1 \) we use an elementary inequality
\[
(x + y)^p \geq x^p + y^p, \quad \text{for } x, y \geq 0, \text{ to obtain } E\eta_{sb}^p = E(\eta + A\eta_{sb})^p \geq E\eta^p + EA^p E\eta_{sb}^p > EA^p E\eta_{sb}^p; \quad \text{if } p \in (0, 1), \text{ then by a variant of Minkowski inequality } (E\eta_{sb}^p)^{1/p} = (E(\eta + A\eta_{sb})^p)^{1/p} \geq (E\eta^p)^{1/p} + (EA^p)^{1/p}(E\eta_{sb}^p)^{1/p} > (EA^p)^{1/p}(E\eta_{sb}^p)^{1/p}. \text{ Either of them implies the desired. The case } p = 1 \text{ is trivial. } \square
\]

**Proof of Proposition 1.3.** Clearly, the proof could be done by using the connection to the shot noise distributions (Proposition 2.1). However, we choose to use a more direct way. Let \( \omega \) be a non-negative ID distribution with the Lévy measure \( N \) and shift \( \gamma \geq 0 \). The following representation due to Steutel (1970, p.86) is well-known
\[
\int_0^x y\omega(dy) = \int_0^x \omega[0, x - y]yN(dy) + \gamma\omega[0, x]. \tag{16}
\]
Note that \( \gamma \) is merely a mass of an atom at zero of the measure \( R(dx) := xN(dx) \).
On the other hand, each distribution \( \omega \) satisfying (16) with appropriate \( N \) and \( \gamma \) is ID.

Putting \( m := 1 \) let us now rewrite equality (2) in terms of distribution functions to obtain
\[
\mu_{sb}[0, x] = \int_0^x y\mu(dy) = \int_0^x \mu[0, x - y]yM(dy), \tag{17}
\]
where the measure \( M \) satisfies the relation
\[
\int_0^x yM(dy) = \int_{c}^{b} \mu_{sb}[0, x/y] \rho(dy), \tag{18}
\]
or equivalently \( M(dx) = x^{-1}L(A\eta_{sb})(dx) \). By comparing (18), (17) with (16) one immediately concludes that \( \mu \) is ID with the Lévy measure \( M \). Its shift is \( 0 \) because \( \mathbb{P}\{A = 0\} = 0 \) which implies that the measure \( S(dx) := xM(dx) \) is atomless at zero.
To complete the proof recall that finiteness of the Lévy measure is necessary and sufficient condition for a distribution to be compound Poisson. In our case this requirement reduces to the integrability of the measure \( x^{-1}\rho(dx) \) at the origin. \( \square \)

**Proof of Proposition 1.4.** The part (a) follows from Theorem 1.1(b) of Iksanov, Jurek (2002) taking into account Proposition 2.1. Turn now to the proof of part (b). W.l.o.g. we only consider the case \( m = 1 \). By Proposition 1.3 \( \mu \) is ID with the Lévy measure \( M(dx) = x^{-1}L(A\eta_{sb})(dx) \).
By Corollary 25.8 of Sato (1999)
\[
\mathbb{E} e^{s\eta_{sb}} = \int_0^\infty e^{sx} \mu(dx) < \infty \iff \int_1^\infty e^{ax} xM(dx) < \infty. \tag{19}
\]
Put \( c = \text{esssup } \rho \in (0, 1) \). It suffices to verify that if \( \mathbb{E} e^{s\eta_{sb}} < \infty \) then \( \mathbb{E} e^{(s/c)\eta_{sb}} < \infty \) as well. In view of part (a), there exists \( s_0 = s_0(c) > 0 \) such that \( \mathbb{E} \exp(s_0 c\eta_{sb}) < \infty \). Then \( \mathbb{E} \exp(s_0 A\eta_{sb}) < \infty \) and consequently \( \int_1^\infty e^{s_0 x} xM(dx) < \infty \). It remains to use (19) which finishes the proof. \( \square \)
Proof of Proposition 1.5. Suppose $A \in (0, 1]$ a.s. Then $EA^p < 1$, for all $p > 0$. Consequently, by Proposition 1.2 (a) $\eta^p < \infty$, for all $p > 0$. In the sequel ”$\rightarrow$“ means ”uniquely determines“. Let us imagine the chain and prove its validity 
\{
\eta^n\}_{n \in \mathbb{N}} \rightarrow \{EA^n\}_{n \in \mathbb{N}} \rightarrow \mathcal{L}(A) \rightarrow \mathcal{L}(\eta),
\] where $\eta$ is the rv with fixed mean $m < \infty$.
1) $\{\eta^n\}_{n \in \mathbb{N}} \rightarrow \{EA^n\}_{n \in \mathbb{N}}$ via the relation
\[E\eta^{n+1} = \sum_{k=0}^{n} \binom{n}{k} E\eta^{k+1} E\eta^{n-k}.
\]
2) $\{EA^n\}_{n \in \mathbb{N}} \rightarrow \mathcal{L}(A)$ since $A \in (0, 1]$ a.s. (the moments problem).
3) $\mathcal{L}(A) \rightarrow \mathcal{L}(\eta)$ which is obvious by Proposition 1.1(a) (or 1.2 (a)).
Conversely, if $\mathcal{L}(\eta)$ is completely determined by its moments then $E\eta^p < \infty$, for all $p > 0$ which in view of Theorem 1.2(b) implies $EA^p < 1$, for all $p > 0$. Clearly, this makes $A \in (0, 1]$ a.s. and completes the proof. □

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