Complex structures, moment maps, and the Ricci form

Oscar García-Prada  Dietmar A. Salamon
ICMAT Madrid  ETH Zürich

Samuel Trautwein
ETH Zürich

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Abstract

The Ricci form is a moment map for the action of the group of exact volume preserving diffeomorphisms on the space of almost complex structures. This observation yields a new approach to the Weil–Petersson symplectic form on the Teichmüller space of isotopy classes of complex structures with real first Chern class zero and nonempty Kähler cone.

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1 Introduction

This paper is based on a remark by Simon Donaldson. The remark is that the space of linear complex structures on $\mathbb{R}^{2n}$ can be viewed as a co-adjoint $\text{SL}(2n,\mathbb{R})$-orbit and hence is equipped with a canonical symplectic form and a Hamiltonian $\text{SL}(2n,\mathbb{R})$-action. Thus, for any volume form $\rho$ on a closed oriented $2n$-manifold $M$, the space $\mathcal{J}(M)$ of almost complex structures on $M$ that induce the same orientation as $\rho$ carries a natural symplectic structure.

Following [12], one can then deduce that the action of the group $\text{Diff}^{\text{ex}}(M, \rho)$ of exact volume preserving diffeomorphisms on the space $\mathcal{J}(M)$ is a Hamiltonian group action with the Ricci form as a moment map. In the integrable case this picture yields a new approach to the Weil–Petersson symplectic form on the Teichmüller space of isotopy classes of complex structures with real first Chern class zero and nonempty Kähler cone. Here are the details.

Fix a closed oriented $2n$-manifold $M$ and a positive volume form $\rho$ and denote by $\mathcal{J}(M)$ the space of almost complex structures compatible with the orientation. This space is equipped with a natural symplectic form $\Omega_{\rho,J}(\tilde{J}_1, \tilde{J}_2) := \frac{1}{2} \int_M \text{trace}\left( \tilde{J}_1 J \tilde{J}_2 \right) \rho$ for $\tilde{J}_1, \tilde{J}_2 \in \Omega^1(M, TM)$. (1.1)

The **Ricci form** $\text{Ric}_{\rho,J} \in \Omega^2(M)$ associated to $\rho$ and $J$ is defined by

$$\text{Ric}_{\rho,J}(u, v) := \frac{1}{4} \text{trace}\left( (\nabla u) J (\nabla v) J \right) + \frac{1}{2} \text{trace}\left( J R^\nabla(\nabla(u, v)) \right) + \frac{1}{2} d\lambda^\nabla_J(u, v)$$

for $u, v \in \text{Vect}(M)$, where $\nabla$ is a torsion-free $\rho$-connection and the 1-form $\lambda^\nabla_J$ on $M$ is defined by $\lambda^\nabla_J(u) := \text{trace}\left( (\nabla J) u \right)$ for $u \in \text{Vect}(M)$.

**Theorem A (The Ricci Form).** The Ricci form is independent of the torsion-free $\rho$-connection $\nabla$ used to define it. It is closed, represents the cohomology class $2\pi c_1(TM, J)$, satisfies $\phi^*\text{Ric}_{\rho,J} = \text{Ric}_{\phi^*\rho, \phi^*J}$ for every diffeomorphism $\phi$, and $\text{Ric}_{\phi^*\rho, J} = \text{Ric}_{\rho,J} + \frac{1}{2} d(\text{df} \circ J)$ for all $f \in \Omega^0(M)$, so there is at most one volume form $\rho$ up to scaling such that $\text{Ric}_{\rho,J} = 0$. Moreover, the map $J \mapsto 2\text{Ric}_{\rho,J}$ is a moment map for the action of the group $\text{Diff}^{\text{ex}}(M, \rho)$ of exact volume preserving diffeomorphisms on $\mathcal{J}(M)$, i.e. if $t \mapsto J_t$ is a smooth path of almost complex structures on $M$, then

$$\frac{d}{dt} \int_M 2\text{Ric}_{\rho,J_t} \wedge \alpha = \frac{1}{2} \int_M \text{trace}\left( (\partial_t J_t) J_t (\mathcal{L}_{v_\alpha} J_t) \right) \rho$$

(1.2)

for $t \in \mathbb{R}$ and $\alpha \in \Omega^{2n-2}(M)$, where $v_\alpha \in \text{Vect}(M)$ is defined by $\iota(v_\alpha) \rho = d\alpha$.

**Proof.** See Theorem 2.6. \qed
The proof of Theorem A is based on the aforementioned observation that the space of linear complex structures is a co-adjoint $\text{SL}(2n, \mathbb{R})$-orbit. Theorem A can then be derived from a general result of Donaldson [12] about the action of the group $\text{Diff}^\infty(M, \rho)$ on a suitable space of sections of a fibration over $M$. In Section 2 we give a direct proof which does not rely on [12].

Equation (1.2) extends to an identity that holds for all vector fields $v$. This identity takes the form

$$\int_M \Lambda_{\rho}(J, \hat{J}) \wedge \iota(v)\rho = \frac{1}{2} \int_M \text{trace}\left(\hat{J} J \mathcal{L}_v J\right) \rho$$

(1.3)

for all $\hat{J} \in \Omega^0_1(M, TM)$ and $v \in \text{Vect}(M)$, where $\Lambda_{\rho} \in \Omega^1(\mathcal{J}(M), \Omega^1(M))$ is a 1-form on the infinite-dimensional manifold $\mathcal{J}(M)$ with values in the space $\Omega^1(M)$ of 1-forms on $M$, defined by

$$(\Lambda_{\rho}(J, \hat{J}))(u) := \text{trace}\left(\nabla u + \frac{1}{2} \hat{J} J \nabla u\right)$$

(1.4)

for $u \in \text{Vect}(M)$. In the symplectic nonKähler case one obtains the formula

$$\text{Ric}_{\rho, J} = \frac{1}{2} \text{trace}(JR^\nabla), \quad \nabla := \nabla - \frac{1}{2} J \nabla J,$$

(1.5)

whenever $\omega$ is a symplectic form with $\omega^n/n! = \rho$, $J$ is an $\omega$-compatible almost complex structure, and $\nabla$ is the Levi-Civita connection of the Riemannian metric $\langle \cdot, \cdot \rangle := \omega(\cdot, J \cdot)$ (see part (vii) of Theorem 2.6). In the integrable nonKähler case one can choose the torsion-free $\rho$-connection $\nabla$ such that $\nabla J = 0$ (see Lemma A.1) and then one obtains the familiar formula $\text{Ric}_{\rho, J} = \frac{1}{2} \text{trace}(JR^\nabla)$.

**Theorem B (The Integrable Case).** Assume $J$ is integrable and fix a positive volume form $\rho$. Then there exists a diffeomorphism $\phi \in \text{Diff}_0(M)$ such that $\text{Ric}_{\rho, \phi^* J} = 0$ if and only if the first Bott–Chern class of $(TM, J)$ vanishes. Moreover, if $\text{Ric}_{\rho, J} = \text{Ric}_{\rho, \phi^* J} = 0$ for some orientation preserving diffeomorphism $\phi$ then $\phi^* \rho = \rho$. Also, for any vector field $v \in \text{Vect}(M)$ the 1-form $\Lambda_{\rho}(J, \mathcal{L}_v J) \in \Omega^1(M)$ is given by

$$\Lambda_{\rho}(J, \mathcal{L}_v J) = 2t(v)\text{Ric}_{\rho, J} - df_v \circ J + df_{Jv},$$

(1.6)

where $f_v := df(v)\rho$.

**Proof.** See Theorem 3.2 and Lemma 3.5.
The space $\mathcal{J}(M)$ carries a complex structure $\mathcal{J} \mapsto -J\tilde{J}$ and the symplectic form $(1.1)$ is of type $(1,1)$. However, it is not Kähler because the symmetric pairing $\langle \tilde{J}_1, \tilde{J}_2 \rangle = \frac{1}{2} \int_M \text{trace}(\tilde{J}_1 \tilde{J}_2)\rho$ is indefinite in general. Thus complex submanifolds of $\mathcal{J}(M)$ need not be symplectic. The space $\mathcal{J}_{\text{int}}(M)$ of integrable almost complex structures is an example. Its tangent space at $J$ is the kernel of $\bar{\partial}_J : \Omega^0,1(M, TM) \rightarrow \Omega^0,2(M, TM)$. If $\text{Ric}_{\rho,J} = 0$ and $\bar{\partial}_J \tilde{J} = 0$ then there are smooth functions $f = f_{\tilde{J}}$ and $g = f_{\tilde{J}}$ such that

$$\Lambda_\rho(J, \tilde{J}) = -df \circ J + dg$$

and so the restriction of the 2-form $\Omega_{\rho,J}$ to $\ker \bar{\partial}_J$ vanishes on the subspace of all $L_i$ such that $f_i = f_{i\nu} = 0$. This is in fact precisely the kernel of $\Omega_{\rho,J}$ and hence $\Omega_\rho$ descends to a symplectic form on the Teichmüller space

$$\mathcal{T}_0(M, \rho) := \left\{ J \in \mathcal{J}_{\text{int}}(M) \mid \text{Ric}_{\rho,J} = 0 \text{ and } J \text{ admits a Kähler form} \right\} / \text{Diff}_0(M, \rho).$$

The symplectic form is independent of $\rho$ and can be defined on the Teichmüller space $\mathcal{T}_0(M)$ of all complex structures with real first Chern class zero and nonempty Kähler cone modulo the action of $\text{Diff}_0(M)$. For every such $J$ let $\rho_J$ be the unique volume form with $\text{Ric}_{\rho,J} = 0$ and $\int_M \rho_J = V$.

**Theorem C (Teichmüller Space).** The formula

$$\Omega_J(\tilde{J}_1, \tilde{J}_2) := \int_M \left( \frac{1}{2} \text{trace}(\tilde{J}_1 J \tilde{J}_2) - f_i g_2 + f_2 g_1 \right) \rho_J,$$

for $J \in \mathcal{J}_{\text{int}}(M)$ with real first Chern class zero and nonempty Kähler cone and $\tilde{J}_i \in \Omega^0,1(M, TM)$ with $\bar{\partial}_J \tilde{J}_i = 0$ and $f_i, g_i$ as in $(1.7)$, defines a symplectic form on the Teichmüller space $\mathcal{T}_0(M)$. It satisfies the naturality condition $\Omega_{\phi,J}(\phi^* \tilde{J}_1, \phi^* \tilde{J}_2) = \phi^* \Omega_J(\tilde{J}_1, \tilde{J}_2)$ for every $\phi \in \text{Diff}^+(M)$ and thus the mapping class group acts on $\mathcal{T}_0(M)$ by symplectomorphisms.

**Proof.** See Theorem 4.2.

Theorem C gives an alternative construction of the Weil–Petersson symplectic form on Calabi–Yau Teichmüller spaces (see $[20, 24, 29, 30]$ for the polarized case and $[14, \text{Ch 16}]$ for the symplectic form on $\mathcal{T}_0(M)$ for the K3 surface). The proof relies on Yau’s theorem and the observations, for Ricci-flat Kähler manifolds $(M, \omega, J)$, that a vector field $v$ is holomorphic if and only if $i(v)\omega$ is harmonic (Lemma 3.7), and that the space of $\bar{\partial}_J$-harmonic 1-forms $\tilde{J} \in \Omega^0,1(M, TM)$ is invariant under the map $\tilde{J} \mapsto \tilde{J}^*$ (Lemma 3.10).

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Associated to the symplectic form (1.8) on $T^0(M)$ and the complex structure $\hat{J} \mapsto -\hat{J}$ is the symmetric bilinear form
\[ \langle \hat{J}_1, \hat{J}_2 \rangle = \int_M \left( \frac{1}{2} \text{trace}(\hat{J}_1 \hat{J}_2) - f_1 f_2 - g_1 g_2 \right) \rho_J. \tag{1.9} \]
This is indefinite in general, so $T^0(M)$ is not Kähler. If $\omega$ is a Kähler form with $\omega^n/n! = \rho_J$, then the subspace of self-adjoint harmonic endomorphisms $\hat{J} = \hat{J}^\ast \in \Omega^{0,1}_J(M, TM)$ is positive for (1.9) (and tangent to the Teichmüller space of $\omega$-compatible complex structures). Its symplectic complement is the negative subspace of skew-adjoint harmonic endomorphisms. The 2-form (1.8) defines a symplectic connection on the space $E_0(M)$ of isotopy classes of Ricci-flat Kähler structures, fibered over the space $B_0(M)$ of isotopy classes of symplectic forms with real first Chern class zero.

**Theorem D (A Connection).** The projection $E_0(M) \to B_0(M)$ is a submersion with fibers $T^0(M, \omega)$ (the spaces of $\omega$-compatible complex structures $J$ with $\text{Ric}_{\omega,J} = 0$ modulo $\text{Symp}(M, \omega) \cap \text{Diff}_0(M)$), the formula (1.8) defines a symplectic connection form on $E_0(M)$, and the connection 1-form $\mathcal{A}$ assigns to each Ricci-flat Kähler structure $(\omega, J)$ and each closed 2-form $\hat{\omega}$ the unique element $\hat{J} = \mathcal{A}_{\omega,J}(\hat{\omega}) \in \Omega^{0,1}_J(M, TM)$ such that $\Omega_J(\hat{J}, \hat{J}') = 0$ for all $\hat{J}' \in \Omega^{0,1}_J(M, TM)$ with $\bar{\partial}_J \hat{J}' = 0$ and $\hat{J}' = (\hat{J}')^\ast$, and
\[ \bar{\partial}_J \hat{J} = 0, \quad \Lambda^\rho(J, \hat{J}) = -d \langle \hat{\omega}, \omega \rangle \circ J, \quad \hat{\omega}(\cdot, \cdot) - \hat{\omega}(J \cdot, J \cdot) = \langle (\hat{J} - \hat{J}^\ast) \cdot, \cdot \rangle. \]

This connection is $\text{Diff}^+(M)$-equivariant and satisfies $\mathcal{A}_{\omega,J}(d\iota(v)\omega) = L_v J$ for all $v \in \text{Vect}(M)$ with $d\iota(Jv)\rho = 0$.

**Proof.** See Theorem 4.3.

The Weil–Petersson metric on the fiber $T^0(M, \omega)$ in Theorem D is Kähler and has been studied by many authors (see e.g. [6], [15], [16], [24], [27], [29]–[31], [32], [33], [34], [35], [36], [39], [41] and the references therein). A key step in the proof of Theorem D is the observation that closed $(0, 2)$-forms on Ricci-flat Kähler manifolds are parallel (see [21] and Lemma 3.11). Two appendices deal with torsion-free connections (Appendix A) and holomorphic n-forms (Appendix B). The results of Appendix B are used in the proofs of Lemma 3.10 and Lemma 3.11.

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2 The Ricci form

Linear complex structures

The standard orientation of \( \mathbb{R}^{2n} \) with the coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_n \) is determined by the volume form \( dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \). The space of linear complex structures on \( \mathbb{R}^{2n} \) compatible with the orientation is given by

\[
\mathcal{J}_n = \left\{ g J_0 g^{-1} \mid g \in \text{SL}(2n, \mathbb{R}) \right\}, \quad J_0 := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.
\]

(2.1)

This is a co-adjoint orbit equipped with a Hamiltonian \( \text{SL}(2n, \mathbb{R}) \)-action. Abbreviate \( G := \text{SL}(2n, \mathbb{R}) \) and \( \mathfrak{g} := \text{Lie}(G) = \mathfrak{sl}(2n, \mathbb{R}) \) and note that \( \mathcal{J}_n \subset \mathfrak{g} \).

Lemma 2.1. The set \( \mathcal{J}_n \subset \mathbb{R}^{2n \times 2n} \) is a connected \( 2n^2 \)-dimensional submanifold and its tangent space at \( J \in \mathcal{J}_n \) is given by

\[
T_J \mathcal{J}_n = \left\{ \widehat{J} \in \mathbb{R}^{2n \times 2n} \mid \widehat{J} J + J \widehat{J} = 0 \right\} = \{ [\xi, J] \mid \xi \in \mathfrak{g} \}.
\]

(2.2)

The formula \( \mathcal{J} \mapsto -J \widehat{J} \) defines a complex structure on \( \mathcal{J}_n \) and the formula

\[
\tau_J(\widehat{J}_1, \widehat{J}_2) := \frac{1}{2} \text{trace}(\widehat{J}_1 J \widehat{J}_2) = -\text{trace}([\xi_1, \xi_2], J).
\]

(2.3)

for \( \xi_i \in \mathfrak{g} \) and \( \widehat{J}_i := [\xi_i, J] \) defines a symplectic form \( \tau \in \Omega^2(\mathcal{J}_n) \). The \( G \)-action \( G \times \mathcal{J}_n \to \mathcal{J}_n : (g, J) \mapsto g J g^{-1} \) is Hamiltonian and is generated by the \( G \)-equivariant moment map \( \mu : \mathcal{J}_n \to \mathfrak{g} \) given by \( \mu(J) = -J \) for \( J \in \mathcal{J}_n \).

Proof. The set \( H := \{ h \in \text{SL}(2n, \mathbb{R}) \mid h J_0 = J_0 h \} \) is a Lie subgroup of \( G \) and is isomorphic to the group of complex \( n \times n \)-matrices with determinant in the unit circle. So \( \dim H = 2n^2 - 1 \) and \( \dim G = 4n^2 - 1 \) and thus the homogeneous space \( G/H \) is a manifold of dimension \( 2n^2 \). Since \( G \) is connected, so is \( G/H \). Next we claim that the map \( G \to \mathbb{R}^{2n \times 2n} : g \mapsto g J_0 g^{-1} \) descends to a proper injective immersion \( \iota : G/H \to \mathbb{R}^{2n \times 2n} \). It is injective by definitions. To see that \( \iota \) is an immersion, observe that \( T_{[g]} G/H \cong g \mathfrak{g} / g \mathfrak{h} \) and \( d\iota([g])[g\xi] = g[\xi, J_0] g^{-1} \) for \( g \in G \) and \( \xi \in \mathfrak{g} \). To prove that \( \iota \) is proper, choose \( g_k \in G \) such that the sequence \( J_k := g_k J_0 g_k^{-1} \) converges to \( J_0 \), and define \( h_k := g_k^{-1} [e_1 \cdots e_n J_k e_1 \cdots J_k e_n] \), where the vectors \( e_1, \ldots, e_n \in \mathbb{R}^{2n} \) form the standard basis of \( \mathbb{R}^n \times \{0\} \). Then \( h_k \in H \) for all \( k \) and \( \lim_{k \to \infty} g_k h_k = \text{Id} \).

This shows that the map \( \iota : G/H \to \mathbb{R}^{2n \times 2n} \) is a proper injective immersion. Hence its image \( \mathcal{J}_n = \iota(G/H) \) is a connected \( 2n^2 \)-dimensional submanifold of \( \mathbb{R}^{2n \times 2n} \).
Now let $J \in \mathcal{J}_n$. Then $g J g^{-1} \in \mathcal{J}_n$ for all $g \in G$ and so $[\xi, J] \in T_J \mathcal{J}_n$ for all $\xi \in \mathfrak{g}$. Thus $\{[\xi, J] | \xi \in \mathfrak{g}\} \subset T_J \mathcal{J}_n \subset \{\hat{J} \in \mathbb{R}^{2n \times 2n} | \hat{J} + J \hat{J} = 0\}$. Since all three spaces have dimension $2n^2$, equality holds and this proves (2.2). The formula (2.3) follows by direct calculation. To show that the 2-form $\tau$ is closed and the complex structure $\hat{J}$ is integrable by Lemma [A.1], as both structures are preserved by the torsion-free connection

$$\nabla_\xi \hat{J} = \frac{\partial}{\partial \xi} \hat{J} + \frac{1}{2} \hat{J} \hat{J} J + \frac{1}{2} J \hat{J} \hat{J}.$$

The map $\mathcal{J}_n \rightarrow \mathfrak{g} : J \mapsto \mu(J) := -J$ is a moment map for the $G$-action because $\tau_J([\xi, J], \hat{J}) = -\text{trace}(\hat{J} \hat{J}) = \text{trace}((d\mu(J) \hat{J}) \xi)$ for $J \in \mathcal{J}_n$, $\hat{J} \in T_J \mathcal{J}_n$, and $\xi \in \mathfrak{g}$. This proves Lemma 2.1. \hfill $\square$

Remark 2.2. The symplectic form $\tau$ in (2.3) is a (1, 1)-form with respect to the complex structure $\hat{J} \mapsto -J \hat{J}$. However, it is not a Kähler form because the bilinear form $\langle \hat{J}_1, \hat{J}_2 \rangle = \frac{1}{2} \text{trace}(\hat{J}_1 \hat{J}_2)$ is indefinite on each tangent space.

Remark 2.3. Let $\omega_0 := \sum_{i=1}^n dx_i \wedge dy_i$ denote the standard symplectic form on $\mathbb{R}^{2n}$ and consider the space of $\omega_0$-compatible linear complex structures

$$\mathcal{J}_{n,0} := \left\{ J \in \mathcal{J}_n \left| J^* \omega_0 = \omega_0 \text{ and } \omega_0(\zeta, J\zeta) > 0 \right. \right\}.$$

(2.4)

This is a complex submanifold of $\mathcal{J}_n$ of real dimension $n^2 + n$ and the symplectic form (2.3) restricts to a Kähler form on $\mathcal{J}_{n,0}$. The symplectic linear group $\text{Sp}(2n)$ acts on $\mathcal{J}_{n,0}$ by Kähler isometries and a moment map $\mu : \mathcal{J}_{n,0} \rightarrow \mathfrak{sp}(2n)$ for this action is again given by $\mu(J) = -J$.

Remark 2.4. The group $\text{Sp}(2n)$ acts on Siegel upper half space $\mathcal{J}_n \subset \mathbb{C}^{n \times n}$ of symmetric matrices with positive definite imaginary part via

$$g_* Z := (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for $g \in \text{Sp}(2n)$ and $Z \in \mathcal{J}_n$. There is a unique $\text{Sp}(2n)$-equivariant diffeomorphism from $\mathcal{J}_n$ to $\mathcal{J}_{n,0}$ that sends $i1 \in \mathcal{J}_n$ to $J_0 \in \mathcal{J}_{n,0}$. It is given by

$$J(Z) = \begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1} \end{pmatrix} \in \mathcal{J}_{n,0}, \quad Z = X + iY \in \mathcal{J}_n.$$

This diffeomorphism is a Kähler isometry with respect to the Kähler metric on $\mathcal{J}_n$ given by $|\hat{Z}|^2 = \text{trace}((Y^{-1} \hat{X})^2 + (Y^{-1} \hat{Y})^2)$ for $\hat{X} + i\hat{Y} \in T_Z \mathcal{J}_n$. 

7
Ricci form

By Lemma 2.1 the space $\mathcal{J}$ fits as a fiber into the general framework developed by Donaldson [12]. Starting from this observation we will show that the action of the group of exact volume preserving diffeomorphisms on the space of almost complex structures is a Hamiltonian group action with twice the Ricci form as a moment map. Let $M$ be a closed oriented $2n$-manifold. Assume $M$ admits an almost complex structure compatible with the orientation and denote the space of such almost complex structures by

$$\mathcal{J}(M) := \left\{ J \in \Omega^0(M, \text{End}(TM)) \mid J^2 = -\mathbb{1} \text{ and } J \text{ is compatible with the orientation of } M \right\}. \quad (2.5)$$

Thus $\mathcal{J}(M)$ is the space of sections of a bundle each of whose fibers is equipped with a natural symplectic form by Lemma 2.1. It can be viewed formally as an infinite-dimensional manifold whose tangent space at $J$ is the space $T_J \mathcal{J}(M) = \{ \widehat{J} \in \Omega^0(M, \text{End}(TM)) \mid \widehat{J}J + J\widehat{J} = 0 \} = \Omega^{0,1}(M, TM)$ of complex anti-linear 1-forms on $M$ with values in $TM$. Every positive volume form $\rho \in \Omega^{2n}(M)$ determines a symplectic form $\Omega\rho$ on $\mathcal{J}(M)$ defined by

$$\Omega\rho,J(u, v) := \frac{1}{2} \int_M \text{trace} \left( \widehat{J}_1 J \widehat{J}_2 \right) \rho \quad (2.6)$$

for $J \in \mathcal{J}(M)$ and $\widehat{J}_1, \widehat{J}_2 \in T_J \mathcal{J}(M)$. The group $G = \text{Diff}(M, \rho)$ of volume preserving diffeomorphisms acts on $\mathcal{J}(M)$ contravariantly by $J \mapsto \phi^*J$ for $\phi \in G$ and $J \in \mathcal{J}(M)$. This action preserves the symplectic form $\Omega\rho$.

**Definition 2.5 (Ricci Form).** Fix a torsion-free $\rho$-connection $\nabla$ on $M$ and an almost complex structure $J \in \mathcal{J}(M)$. The **Ricci form** of the pair $(\rho, J)$ is the 2-form $\text{Ric}_{\rho,J} \in \Omega^2(M)$ defined by

$$\text{Ric}_{\rho,J}(u, v) := \frac{1}{4} \text{trace}((\nabla_u J)(\nabla_v J)) + \frac{1}{2} \text{trace}(J R^\nabla(u, v))$$

$$+ \frac{1}{2} d\lambda^\nabla_J(u, v) \quad (2.7)$$

for $u, v \in \text{Vect}(M)$, where $\lambda^\nabla_J \in \Omega^1(M)$ is defined by

$$\lambda^\nabla_J(u) := \text{trace}((\nabla J)u) \quad \text{for } u \in \text{Vect}(M). \quad (2.8)$$

For $J \in \mathcal{J}(M)$ and $\widehat{J} \in \Omega^{0,1}(M, TM)$ define $\Lambda_{\rho}(J, \widehat{J}) \in \Omega^1(M)$ by

$$(\Lambda_{\rho}(J, \widehat{J}))(u) := \text{trace}((\nabla \widehat{J})u + \frac{1}{2} J \widehat{J} \nabla_u J) \quad \text{for } u \in \text{Vect}(M). \quad (2.9)$$
The next theorem is the main result of this section. It asserts that the action of the subgroup
\[
\mathcal{G}^{\text{ex}} := \left\{ \phi \in \text{Diff}(M) \right\}
\]
there exists a smooth isotopy
\[ [0, 1] \times \text{Diff}(M) : t \mapsto \phi_t \]
and a smooth family of vector fields
\[ [0, 1] \rightarrow \text{Vect}(M) : t \mapsto v_t \]
such that \( \iota(v_t)\rho \) is exact for all \( t \)
and \( \partial_t \phi_t = v_t \circ \phi_t \) for all \( t \)
and \( \phi_0 = \text{id} \) and \( \phi_1 = \phi \)
of exact volume preserving diffeomorphisms on \( \mathcal{J}(M) \) is a Hamiltonian group action and is generated by the \( \mathcal{G} \)-equivariant moment map which assigns to each \( J \in \mathcal{J}(M) \) twice the Ricci form \( \text{Ric}_{\rho,J} \). The moment map must take values in the dual space of the Lie algebra
\[
\text{Lie}(\mathcal{G}^{\text{ex}}) = \text{Vect}^{\text{ex}}(M, \rho) = \{ v \in \text{Vect}(M) \mid \iota(v)\rho \text{ is exact} \}.
\]
Every \((2n-2)\)-form \( \alpha \in \Omega^{2n-2}(M) \) determines an exact divergence-free vector field \( v_\alpha \in \text{Vect}^{\text{ex}}(M, \rho) \) via
\[
\iota(v_\alpha)\rho = d\alpha.
\]
Thus \( \text{Vect}^{\text{ex}}(M, \rho) \) can be identified with the quotient of the space \( \Omega^{2n-2}(M) \) by the space of closed \((2n-2)\)-forms on \( M \). Its dual space can be viewed formally as the space of exact \(2\)-forms on \( M \), in that every exact \(2\)-form \( \tau \) on \( M \) determines a linear functional
\[
\text{Vect}^{\text{ex}}(M, \rho) \rightarrow \mathbb{R} : v_\alpha \mapsto \int_M \tau \wedge \alpha.
\]
With this understood, equation (2.14) in the following theorem is the assertion that the map \( J \mapsto 2\text{Ric}_{\rho,J} \) is a moment map for the action of \( \mathcal{G}^{\text{ex}} \) on \( \mathcal{J}(M) \). In general, however, the Ricci form is only closed and not exact; only its differential in the direction of an infinitesimal almost complex structure is always exact. Thus the map \( J \mapsto 2\text{Ric}_{\rho,J} \) is only a moment in the strict sense of the word when restricted to the space of almost complex structures with real first Chern class zero. One could attempt to rectify this situation by subtracting a closed 2-form in the appropriate cohomology class from the Ricci form, however such a modification would destroy the \( \mathcal{G}^{\text{ex}} \)-equivariance of the moment map unless \( M \) has real dimension two.
Theorem 2.6 (The Ricci Form as a Moment Map). Let $\rho \in \Omega^2(M)$ be a positive volume form, let $J \in \mathcal{J}(M)$, and let $\tilde{J} \in \Omega^0_J(M, TM)$. Then the following holds.

(i) The Ricci form $\text{Ric}_{\rho, J}$ and the 1-form $\Lambda_{\rho}(J, \tilde{J})$ are independent of the choice of the torsion-free $\rho$-connection $\nabla$ used to define them. Moreover,

$$\text{Ric}_{e^f \rho, J} = \text{Ric}_{\rho, J} + \frac{1}{2} d(df \circ J)$$

for every $f \in \Omega^0(M)$.

(ii) Every vector field $v \in \text{Vect}(M)$ satisfies the equation

$$\int_M \Lambda_{\rho}(J, \tilde{J}) \wedge \iota(v) \rho = \frac{1}{2} \int_M \text{trace}((\tilde{J} \mathcal{L}_v J) \rho).$$

(iii) If $\mathbb{R} \to \mathcal{J}(M) : t \mapsto J_t$ is a smooth path of almost complex structures satisfying $J_0 = J$ and $\frac{d}{dt}{\big|}_{t=0} J_t = \tilde{J}$, then

$$\text{Ric}_{\rho, J} := \frac{d}{dt}{\big|}_{t=0} \text{Ric}_{\rho, J_t} = \frac{1}{2} d(\Lambda_{\rho}(J, \tilde{J})).$$

(iv) If $\alpha \in \Omega^{2n-2}(M)$ and $v_\alpha \in \text{Vect}(M)$ is defined by $\iota(v_\alpha) \rho = d\alpha$, then

$$\int_M 2\text{Ric}_{\rho, \tilde{J}} \wedge \alpha = \frac{1}{2} \int_M \text{trace}((\tilde{J} \mathcal{L}_{v_\alpha} J) \rho).$$

(v) The Ricci form and the 1-form $\Lambda_{\rho}(J, \tilde{J})$ satisfy the naturality condition

$$\phi^* \text{Ric}_{\rho, J} = \text{Ric}_{\phi^* \rho, \phi^* J}, \quad \phi^* \Lambda_{\rho}(J, \tilde{J}) = \Lambda_{\phi^* \rho}(\phi^* J, \phi^* \tilde{J})$$

for all $\phi \in \text{Diff}(M)$.

(vi) Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form compatible with $J$ such that $\omega^n/n! = \rho$, let $\nabla$ be the Levi-Civita connection of the Riemannian metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J \cdot)$, and define $\tilde{\nabla} := \nabla - \frac{1}{2} J \nabla J$. Then $\tilde{\nabla}$ preserves $\rho$, $J$, and the metric, and the Ricci form of $(\rho, J)$ is given by

$$\text{Ric}_{\rho, J} = \frac{1}{2} (\text{trace}(JR^{\tilde{\nabla}}) + d\lambda_J^\tilde{\nabla}).$$

Thus $\text{Ric}_{\rho, J}$ is closed and represents the cohomology class $2\pi c_1(TM, J)$.

(vii) If the 2-form $\omega$ in (vi) is closed then $\lambda_J^\tilde{\nabla} = 0$ and $\text{Ric}_{\rho, J} = \frac{1}{2} \text{trace}(JR^{\tilde{\nabla}})$. 

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Proof. We prove (i) and (ii). Choose a smooth path $\mathbb{R} \to \Omega^{2n}(M) : t \mapsto \rho_t$ of positive volume forms and a smooth path of torsion-free connections $\nabla_t$ on $TM$ such that $\nabla_t \rho_t = 0$ for all $t \in \mathbb{R}$. Define

$$\rho := \rho_0, \quad \nabla := \nabla_0, \quad \hat{\rho} := \frac{d}{dt}|_{t=0} \rho_t, \quad A := \frac{d}{dt}|_{t=0} \nabla_t.$$ 

Then the endomorphism valued 1-form $A \in \Omega^1(M, \text{End}(TM))$ satisfies

$$A(u)v = A(v)u, \quad \text{trace}(A(u)) = \mathcal{L}_u (\hat{\rho}/\rho)$$

for all $u, v \in \text{Vect}(M)$. Define $\alpha \in \Omega^1(M)$ and $\tau^\nabla_{J} \in \Omega^2(M)$ by

$$\alpha(u) := \text{trace}(JA(u)), \quad \tau^\nabla_{J}(u, v) := \frac{1}{2}\text{trace}((\nabla_{u}J)(\nabla_{v}J)) + \text{trace}(JR^\nabla)$$

for $u, v \in \text{Vect}(M)$ and $t \in \mathbb{R}$. Since

$$\frac{d}{dt}|_{t=0} \nabla_{t,u}J = [A(u), J], \quad \frac{d}{dt}|_{t=0} R^\nabla = d^\nabla A,$$

it follows from (2.18) that

$$\frac{d}{dt}|_{t=0} \tau^\nabla_{J}(u, v) = \text{trace}((\nabla_{u}J)(A(v) - (\nabla_{v}J)(A(u)) + \text{trace}(J d^\nabla A(u, v))))$$

$$= \text{trace}(d^\nabla (JA)(u, v)))$$

$$= d\alpha(u, v)$$

for all $u, v \in \text{Vect}(M)$. Moreover, it follows from (2.17) that

$$\frac{d}{dt}|_{t=0} \lambda^\nabla_{J}(u) = \text{trace}([A, J]u)$$

$$= \text{trace}(A(Ju)) - \text{trace}(JA(u))$$

$$= \mathcal{L}_{Ju} (\hat{\rho}/\rho) - \alpha(u)$$

for all $u \in \text{Vect}(M)$. Hence

$$\frac{d}{dt}|_{t=0} (\tau^\nabla_{J} - d\lambda^\nabla_{J}) = d (d (\hat{\rho}/\rho) \circ J).$$

Integrate this equation to obtain (2.11) and consider the case where $\rho_t = \rho$ is independent of $t$ to deduce that the 2-form $\text{Ric}_{\rho, J}$ is independent of the choice of the connection $\nabla$ used to define it. That $\Lambda_{\rho}(J, \hat{J})$ is also independent of $\nabla$ will follow from equation (2.12) in part (ii), which we prove next.
To prove (2.12), we use the formulas
\[
\text{trace}(\nabla u)\rho = dt(u)\rho, \quad (2.19)
\]
\[
(\mathcal{L}_v J)u = J\nabla_u v - \nabla\nabla_v u + (\nabla_v J)u \quad (2.20)
\]
for \(u, v \in \text{Vect}(M)\). By (2.20), we have
\[
\text{trace}(\hat{J}\mathcal{L}_v J) = \text{trace}(-\hat{J}\nabla v - \hat{J}J\nabla J v + \hat{J}J\nabla_v J)
\]
\[
= \text{trace}(-2\hat{J}\nabla v + \hat{J}J\nabla_v J)
\]
for all \(u, v \in \text{Vect}(M)\). Here the second equality holds because two endomorphisms \(\Phi\) and \(-J\Phi J\) are conjugate and so have the same trace. Thus
\[
\Lambda_{\rho}(J, \hat{J})(v) = \text{trace}((\nabla\hat{J})v + \frac{1}{2}\hat{J}J\nabla_v J)
\]
\[
= \text{trace}(\nabla(\hat{J}v) - \hat{J}J\nabla v + \frac{1}{2}\hat{J}J\nabla_v J)
\]
\[
= \text{trace}(\nabla(\hat{J}v)) + \frac{1}{2}\text{trace}(\hat{J}J\mathcal{L}_v J)
\]
for all \(v \in \text{Vect}(M)\). Hence it follows from (2.19) with \(u = \hat{J}v\) that
\[
\int_M \Lambda_{\rho}(J, \hat{J}) \land \iota(v)\rho = \int_M \Lambda_{\rho}(J, \hat{J})(v)\rho = \frac{1}{2} \int_M \text{trace}(\hat{J}J\mathcal{L}_v J)\rho
\]
for all \(v \in \text{Vect}(M)\). This proves (2.12) and parts (i) and (ii).

We prove part (iii). Fix a torsion-free \(\rho\)-connection \(\nabla\), let \(\tau_{\rho,J}^v \in \Omega^2(M)\) be as in (2.18), and abbreviate
\[
\tilde{\lambda}(u) := \text{trace}((\nabla\hat{J})u) = \frac{d}{dt}|_{t=0} \lambda_{\rho,J}^u(u), \quad \tilde{\beta}(u) := \frac{1}{2}\text{trace}(\hat{J}J\nabla_v J)
\]
for \(u \in \text{Vect}(M)\). Then \(\Lambda_{\rho}(J, \hat{J}) = \tilde{\lambda} + \tilde{\beta}\) and
\[
d\tilde{\beta}(u, v)
\]
\[
= \frac{1}{2}\mathcal{L}_u \text{trace}(\hat{J}J\nabla_v J) - \frac{1}{2}\mathcal{L}_u \text{trace}(\hat{J}J\nabla_v J) + \frac{1}{2}\text{trace}(\hat{J}J\nabla_{[u,v]} J)
\]
\[
= \frac{1}{2}\text{trace}((\nabla_u(\hat{J}J))(\nabla_v J)) - \frac{1}{2}\text{trace}((\nabla_v(\hat{J}J))(\nabla_u J)) + \frac{1}{2}\text{trace}(\hat{J}J(\nabla_u J - \nabla_v J + \nabla_{[u,v]} J))
\]
\[
= \text{trace}((\nabla_u \hat{J}J)(\nabla_v J)) - \frac{1}{2}\text{trace}((\nabla_v \hat{J}J)(\nabla_u J)) + \frac{1}{2}\text{trace}(\hat{J}J[R\nabla J u, v])
\]
\[
= \frac{1}{2}\text{trace}((\nabla_u \hat{J}J)(\nabla_v J)) + \frac{1}{2}\text{trace}((\nabla_v \hat{J}J)(\nabla_u J)) + \text{trace}(\hat{J}R\nabla J u, v))
\]
\[
= \frac{d}{dt}|_{t=0} \tau_{\rho,J}^{\nabla J}(u, v)
\]
for all \(u, v \in \text{Vect}(M)\). Since \(\text{Ric}_{\rho,J}^J = \frac{1}{2}(\tau_{\rho,J}^{\nabla J} + d\lambda_{\rho,J}^J)\) this proves part (iii).

Part (iv) follows from (ii), (iii), and Stokes’ theorem and part (v) follows from (i) and the definitions.
We prove part (vi). The connection $\nabla$ in (vi) will in general no longer be torsion-free. However, since the endomorphism $J\nabla_u J$ is skew-adjoint for all $u \in \text{Vect}(M)$, it preserves the Riemannian metric on $M$ and the volume form $\rho$. In addition it preserves the almost complex structure $J$ because

$$\nabla_u J = \nabla_u J - \frac{1}{2}[J\nabla_u J, J] = \nabla_u J - \frac{1}{2}J(\nabla_u J)J + \frac{1}{2}JJ\nabla_u J = 0$$

for all $u \in \text{Vect}(M)$. Next we compute the curvature tensor of $\nabla$. Fix three vector fields $u, v, w \in \text{Vect}(M)$. Then $\tilde{\nabla}_v w = \nabla_v w - \frac{1}{2}J(\nabla_v J)w$, hence

$$\tilde{\nabla}_v \tilde{\nabla}_w = \tilde{\nabla}_v (\nabla_w - \frac{1}{2}J(\nabla_w J)w) = \tilde{\nabla}_v \nabla_w w - \frac{1}{2}J \tilde{\nabla}_v ((\nabla_w J)w)$$

$$= \nabla_v \nabla_w w - \frac{1}{2}J(\nabla_v \nabla_w J)w - \frac{1}{2}J(\nabla_v J)(\nabla_w J)w$$

$$- \frac{1}{2}J(\nabla_v J)\nabla_w w - \frac{1}{2}J(\nabla_v J)\nabla_w w.$$ Hence

$$R_{\nabla} (u, v) w = \tilde{\nabla}_u \tilde{\nabla}_v w - \tilde{\nabla}_u \tilde{\nabla}_v w - \tilde{\nabla}_{[u,v]} w$$

$$= \nabla_u \nabla_v w - \frac{1}{2}J(\nabla_u \nabla_v J)w - \frac{1}{4}(\nabla_u J)(\nabla_v J)w$$

$$- \nabla_v \nabla_u w + \frac{1}{2}J(\nabla_v \nabla_u J)w + \frac{1}{4}(\nabla_v J)(\nabla_u J)w$$

$$+ \nabla_{[u,v]} w - \frac{1}{2}J(\nabla_{[u,v]} J)w$$

$$= R_{\nabla} (u, v) w - \frac{1}{4}(\nabla_u J)(\nabla_v J)w + \frac{1}{4}(\nabla_v J)(\nabla_u J)w$$

$$- \frac{1}{2}J(\nabla_u \nabla_v J) - \nabla_v (\nabla_u J) + \nabla_{[u,v]} J)w.$$

This implies

$$JR_{\nabla} (u, v) w = JR_{\nabla} (u, v) w + \frac{1}{4}(\nabla_u J)(\nabla_v J)w + \frac{1}{4}(\nabla_v J)(\nabla_u J)w$$

$$+ \frac{1}{2}(\nabla_u \nabla_v J) - \nabla_v (\nabla_u J) + \nabla_{[u,v]} J)w.$$ (2.21)

The endomorphism $\nabla_u \nabla_v J$ has trace zero because $\text{trace}(\nabla_u \Phi) = L_u \text{trace}(\Phi)$ for all $u \in \text{Vect}(M)$ and all $\Phi \in \Omega^0(M, \text{End}(TM))$. Hence

$$\text{trace}(JR_{\nabla} (u, v)) = \text{trace}(JR_{\nabla} (u, v)) + \frac{1}{4}\text{trace}(\nabla_u J)(\nabla_v J).$$ (2.22)

Thus $\text{trace}(JR_{\nabla}) = \tau_J^2$ and this proves (2.16). Since $\nabla$ is a Hermitian connection, the 2-form $\text{trace}(\frac{1}{4\pi}JR_{\nabla}) = \text{trace}(\frac{1}{4\pi}JR_{\nabla}) \in \Omega^2(M)$ is closed and represents the first Chern class of $(TM, J)$. This proves (vi).

We prove part (vii). If $\omega$ is closed then $\nabla_v J = -J(\nabla_v J)$ for every vector field $v \in \text{Vect}(M)$ by [28] Lemma 4.1.14, so the endomorphism $v \mapsto (\nabla_v J)u$ anti-commutes with $J$ and therefore has trace zero. Hence $\lambda_J^2 = 0$. This proves part (vii) and Theorem 2.6.
Symplectic complements

The next lemma examines symplectic complements in $T_J\mathcal{J}(M)$. Part (iv) shows that there is a genuine Marsden–Weinstein quotient in this setting.

**Lemma 2.7 (Complements).** Let $\rho \in \Omega^{2n}(M)$ be a positive volume form and $J \in \mathcal{J}(M)$, $\mathcal{J} \in \Omega^{0,1}(M, TM)$, $\lambda \in \Omega^1(M)$. Then the following holds.

(i) Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form that is compatible with $J$ and satisfies $\omega^n/n! = \rho$, let $\nabla$ be the Levi-Civita connection of the Riemannian metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$, and define $D_J : \Omega^0(M, TM) \to \Omega^{0,1}(M, TM)$ by

\[
(D_Jv)u = -\frac{1}{2} J(L_v J)u = \frac{1}{2} (\nabla_u v + J\nabla_J u - J(\nabla_v J)u) \tag{2.23}
\]

for $u, v \in \text{Vect}(M)$ (see (2.20)). Then $\Lambda_\rho(J, \mathcal{J}) = \iota(JD_J^* \mathcal{J}^*)\omega$ and

\[
\int_M \Lambda_\rho(J, \mathcal{J}) \wedge \iota(v)\rho = -\langle D_J^* \mathcal{J}^*, v \rangle_{L^2}. \tag{2.24}
\]

for all $\mathcal{J} \in \Omega^{0,1}_{M}(M, TM)$ and all $v \in \text{Vect}(M)$.

(ii) There exists a $\mathcal{J}' \in \Omega^{0,1}_{M}(M, TM)$ such that $\Lambda_\rho(J, \mathcal{J}') = \mathcal{J}$ if and only if $\int_M \mathcal{J} \wedge \iota(v)\rho = 0$ for all $v \in \text{Vect}(M)$ with $L_v J = 0$.

(iii) There exists a $v \in \text{Vect}(M)$ with $L_v J = \mathcal{J}$ if and only if $\Omega_{\rho,J}(\mathcal{J}, \mathcal{J}') = 0$ for all $\mathcal{J}' \in \Omega^{0,1}_{M}(M, TM)$ with $\Lambda_\rho(J, \mathcal{J}') = 0$.

(iv) There exists a $v \in \text{Vect}(M)$ such that $L_v J = \mathcal{J}$ and $\iota(v)\rho$ is exact if and only if $\Omega_{\rho,J}(\mathcal{J}, \mathcal{J}') = 0$ for all $\mathcal{J}' \in \Omega^{0,1}_{M}(M, TM)$ such that $\text{Ric}_\rho(J, \mathcal{J}') = 0$.

**Proof.** Part (i) follows from the equation

\[
\int_M \Lambda_\rho(J, \mathcal{J}) \wedge \iota(v)\rho = \frac{1}{2} \int_M \text{trace}(\mathcal{J}L_v J)\rho = -\langle \mathcal{J}^* D_J v, v \rangle_{L^2}.
\]

We prove part (ii). The condition is necessary by equation (2.24). Conversely, let $\lambda \in \Omega^1(M)$ such that $\int_M \lambda \wedge \iota(v)\rho = 0$ for all $v \in \text{Vect}(M)$ with $L_v J = 0$.

Choose $\omega$ as in (i) and define $u \in \text{Vect}(M)$ by $\iota(Ju)\omega = \lambda$. Then

\[
\langle u, v \rangle_{L^2} = \int_M \omega(u, Jv)\rho = -\int_M \tilde{\lambda}(v)\rho = -\int_M \lambda \wedge \iota(v)\rho = 0 \tag{2.25}
\]

for all $v \in \text{ker} D_J$. Thus there exists a $\mathcal{J}' \in \Omega^{0,1}_{M}(M, TM)$ with $D_J^*(\mathcal{J}') = u$.

Hence $\int_M \lambda \wedge \iota(v)\rho = -\langle D_J^*(\mathcal{J})^*, v \rangle_{L^2} = \int_M \Lambda_\rho(J, \mathcal{J}) \wedge \iota(v)\rho$ for every vector field $v$ by (2.24) and (2.25). Thus $\Lambda_\rho(J, \mathcal{J}) = \lambda$ and this proves (ii).
We prove part (iii). The condition is necessary by (2.12). Conversely, assume $\Omega_{\rho,J}(\hat{J}, \hat{J}^\prime) = 0$ for all $\hat{J} \in \Omega_\rho^{1,1}(M, TM)$ that satisfy $\Lambda_\rho(J, \hat{J}^\prime) = 0$. Let $v \in \text{Vect}(M)$ with $D_v^\rho(D_Jv + \frac{1}{2}J\hat{J}) = 0$ and define $\hat{J}^\prime := (D_Jv + \frac{1}{2}J\hat{J})^\ast$. Then $D_v^\rho(\hat{J}^\prime)^\ast = 0$, hence $\Lambda_\rho(J, \hat{J}^\prime) = 0$ by (2.24), and hence $\Omega_{\rho,J}(\hat{J}, \hat{J}^\prime) = 0$. This implies
\[
\int_M |\hat{J}|^2 \rho = \int_M \text{trace}(\hat{J}^\ast (D_Jv + \frac{1}{2}J\hat{J})) \rho = \langle (\hat{J}^\ast)^\ast, D_Jv \rangle_{L^2} = 0.
\]
Thus $\hat{J}^\prime = 0$ and so $\hat{J} = 2JD_Jv = \mathcal{L}_vJ$ by (2.23). This proves (iii).

We prove part (iv). The condition is necessary by (2.14). Conversely, assume $\Omega_{\rho,J}(\hat{J}, \hat{J}^\prime) = 0$ for all $\hat{J} \in \Omega_\rho^{0,1}(M, TM)$ such that $\text{Ric}_\rho(J, \hat{J}^\prime) = 0$. Then by (iii) there is a $v \in \text{Vect}(M)$ with $\mathcal{L}_vJ = \hat{J}$. Choose a basis $u_1, \ldots, u_\ell$ of the vector space $V := \{u \in \text{Vect}(M) \mid \mathcal{L}_uJ = 0\}$ such that $u_{k+1}, \ldots, u_\ell$ form a basis of the subspace $\{u \in V \mid \iota(u) \rho \in \text{im}d\}$. Then $\iota(u_1) \rho, \ldots, \iota(u_\ell) \rho$ are linearly independent in the quotient $\Omega^{2n-1}(M)/\text{im}d$. Hence, by Poincaré duality, there exist closed 1-forms $\lambda_1, \ldots, \lambda_k \in \Omega^1(M)$ such that $\int_M \lambda_i \wedge \iota(u_j) \rho = \delta_{ij}$ for $i, j = 1, \ldots, k$. Define
\[
v_0 := v - \sum_{i=1}^k x_i u_i, \quad x_i := \int_M \lambda_i \wedge \iota(v) \rho.
\]
Then $\mathcal{L}_{v_0}J = \hat{J}$. Let $\hat{\lambda} \in \Omega^1(M)$ be any closed 1-form and define
\[
\hat{\lambda}^\prime := \hat{\lambda} - \sum_{i=1}^k y_i \lambda_i, \quad y_i := \int_M \hat{\lambda} \wedge \iota(u_i) \rho.
\]
Then $\int_M \hat{\lambda}^\prime \wedge \iota(u_j) \rho = 0$ for $j = 1, \ldots, \ell$. Hence there is a $\hat{J} \in \Omega_\rho^{0,1}(M, TM)$ with $\Lambda_\rho(J, \hat{J}^\prime) = \hat{\lambda}^\prime$ by (ii). Thus $\text{Ric}_\rho(J, \hat{J}^\prime) = 0$, so $\Omega_{\rho,J}(\hat{J}, \hat{J}^\prime) = 0$ and
\[
\int_M \hat{\lambda} \wedge \iota(v_0) \rho = \int_M \hat{\lambda} \wedge \iota(v) \rho - \sum_{i=1}^k x_i \int_M \hat{\lambda} \wedge \iota(u_i) \rho
\]
\[
= \int_M \hat{\lambda} \wedge \iota(v) \rho - \sum_{i=1}^k y_i \int_M \lambda_i \wedge \iota(v) \rho
\]
\[
= \int_M \Lambda_\rho(J, \hat{J}^\prime) \wedge \iota(v) \rho
\]
\[
= \Omega_{\rho,J}(\hat{J}^\prime, \hat{J}) = 0.
\]
Hence $\iota(v_0) \rho$ is exact and this proves Lemma 2.7. 

\[\square\]
Scalar curvature

Let \((M, \omega)\) be a 2n-dimensional closed symplectic manifold and denote by

\[
\mathcal{J}(M, \omega) := \left\{ J \in \Omega^0(M, \text{End}(TM)) \mid \begin{array}{l}
J^2 = -\mathbb{1} \text{ and } J^*\omega = \omega \\
\text{and } \omega(\hat{m}, J\hat{m}) > 0 \\
\text{for all } \hat{m} \in T_{\hat{m}}M \setminus \{0\}
\end{array} \right\}
\]

the space of all almost complex structures that are compatible with \(\omega\). This is an infinite-dimensional Kähler submanifold of \(\mathcal{J}(M)\) with the tangent spaces \(T_J \mathcal{J}(M, \omega) = \{ \tilde{J} \in \Omega^{0,1}(M, TM) \mid \omega(\tilde{J}, \cdot) + \omega(\cdot, \tilde{J}) = 0 \}\), the symplectic form \(\Omega_\omega\) in (2.6), and the complex structure \(\tilde{J} \mapsto -J\tilde{J}\).

**Definition 2.8 (Scalar Curvature).** Let \(\omega\) be a symplectic form on \(M\), let \(J\) be a \(\omega\)-compatible almost complex structure on \(M\), let \(\nabla\) be the Levi-Civita connection of the metric \(\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)\), and let \(\tilde{\nabla} := \nabla - \frac{1}{2} J(\nabla J)\). Define the **Ricci form** of \((\omega, J)\) by \(\text{Ric}_{\omega,J} := \text{Ric}_{\omega,J}^\rho = \frac{1}{2} \text{trace}(JR\tilde{\nabla})\) and define the **scalar curvature** by

\[
S_{\omega,J} := 2\text{Ric}_{\omega,J} := \frac{2\text{Ric}_{\omega,J} \wedge \omega^{n-1}/(n-1)!}{\omega^n/n!} \in \Omega^0(M). \tag{2.26}
\]

By part (iv) of Theorem 2.6 the scalar curvature \(S_{\omega,J}\) in (2.26) satisfies \(\int_M S_{\omega,J} \frac{\omega^n}{n!} = 4\pi c_1(TM, J) \sim \frac{\omega^{n-1}}{(n-1)!} [M]\) and \(\phi^*S_{\omega,J} = S_{\phi^*\omega, \phi^*J}\) for every diffeomorphism \(\phi : M \to M\). The following result was proved by Donaldson [11], and independently by Fujiki [15] (in the integrable case) and Quillen (for Riemann surfaces).

**Corollary 2.9 (Fujiki–Quillen–Donaldson).** The map \(J \mapsto S_{\omega,J}\) is an equivariant moment map for the action of \(\text{Ham}(M, \omega)\) on \(\mathcal{J}(M, \omega)\), i.e. if \(H \in \Omega^0(M)\) and \(v_H \in \text{Vect}(M)\) is the Hamiltonian vector field defined by \(\iota(v_H)\omega = dH\), then every smooth path \(\mathbb{R} \to \mathcal{J}(M, \omega) : t \mapsto J_t\) satisfies

\[
\frac{d}{dt} \int_M S_{\omega,J_t} H \frac{\omega^n}{n!} = \frac{1}{2} \int_M \text{trace}\left((\partial_t J_t)(\mathcal{L}_{v_H} J_t)\right) \frac{\omega^n}{n!}. \tag{2.27}
\]

**Proof.** Let \(J := J_0, \quad \tilde{J} := \frac{d}{dt} \big|_{t=0} J_t, \quad \rho := \omega^n/n!\). Then

\[
\frac{d}{dt} \bigg|_{t=0} \int_M S_{\omega,J_t} H \frac{\omega^n}{n!} = \int_M 2H \tilde{\text{Ric}}_\rho(J, \tilde{J}) \wedge \frac{\omega^{n-1}}{(n-1)!}
\]

\[
= \int_M H d\Lambda_\rho(J, \tilde{J}) \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_M \Lambda_\rho(J, \tilde{J}) \wedge \iota(v_H)\rho.
\]

Hence the assertion follows from Theorem 2.6 \(\square\)

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3 The integrable case

Let $M$ be a closed oriented smooth $2n$-manifold. In this section we restrict attention to (integrable) complex structures that are compatible with the orientation. Denote the space of such complex structures by $\mathcal{J}_{\text{int}}(M)$.

The Ricci form in the integrable case

Let $J \in \mathcal{J}_{\text{int}}(M)$ and fix a torsion-free connection $\nabla$ on $TM$ with $\nabla J = 0$. Then $(TM, J)$ is a holomorphic vector bundle, the Lie derivative of $J$ in the direction of a vector field $v \in \text{Vect}(M)$ is given by

$$(\mathcal{L}_v J)u = J\nabla_u v - \nabla_{Ju} v = 2J(\bar{\partial}_J v)u,$$

and the derivative of the Nijenhuis tensor along a path $\mathbb{R} \to \mathcal{J}(M) : t \mapsto J_t$ with $J_0 = J$ and $\tilde{J} := \frac{d}{dt}|_{t=0} N_{J_t} = -2J\bar{\partial}_J \tilde{J}$, i.e.

$$\frac{d}{dt}|_{t=0} N_{J_t}(u, v) = -J(\nabla_u \tilde{J})v + J(\nabla_v \tilde{J})u + J(\bar{\partial}_J u \tilde{J})v - J(\bar{\partial}_J v \tilde{J})u,$$

for $u, v \in \text{Vect}(M)$ (differentiate equation (A.2)).

Remark 3.1. Since $d(df \circ J)(u, v) - d(df \circ J)(Ju, Jv) = df(JN_J(u, v))$, an almost complex structure $J$ is integrable if and only if the 2-form $d(df \circ J)$ is of type $(1, 1)$ for all $f \in \Omega^0(M)$. Theorem 3.2 uses the Bott-Chern cohomology group $H_{BC}^{1,1}(M, J) := (\ker d \cap \Omega^{1,1}_J(M, TM))/\{d(df \circ J) | J \in \Omega^0(M)\}$ [1, 2, 3, 5]. The theorem shows that our formula for the Ricci form of an almost complex structure gives the standard formula in the integrable case.

Theorem 3.2. Let $\rho \in \Omega^{2n}(M)$ be a positive volume form, let $J \in \mathcal{J}_{\text{int}}(M)$, and let $\nabla$ be a torsion-free $\rho$-connection with $\nabla J = 0$. The following holds.

(i) $\text{Ric}_{\rho, J} = \frac{1}{2} \text{trace}(JR)\nabla$ is a closed $(1, 1)$-form.

(ii) The $(1, 1)$-form $\frac{1}{2\rho} \text{Ric}_{\rho, J}$ represents the first Bott–Chern class of the holomorphic tangent bundle $(TM, J)$.

(iii) There exists a diffeomorphism $\phi \in \text{Diff}_0(M)$ such that

$$\text{Ric}_{\rho, \phi^* J} = 0$$

if and only if the first Bott–Chern class of $(TM, J)$ vanishes.

(iv) Let $\phi : M \to M$ be an orientation preserving diffeomorphism. If

$$\text{Ric}_{\rho, J} = \text{Ric}_{\rho, \phi^* J} = 0,$$

then $\phi^* \rho = \rho$. 

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Proof. The formula \( \text{Ric}_{\rho,J} = \frac{1}{2} \text{tr}(JR^\nabla) \) follows from Definition 2.4. Moreover, \( \text{Ric}_{\rho,J} \) is closed and independent of the choice of \( \nabla \) by part (i) of Theorem 2.6; it is a \((1,1)\)-form by Lemma A.2 and represents the cohomology class \( 2\pi c_1(TM, J) \in H^2(M; \mathbb{R}) \) by part (vi) of Theorem 2.6. This proves (i).

To prove part (ii), choose a nondegenerate 2-form \( \omega \in \Omega^2(M) \), compatible with \( J \), such that \( \rho \) is the volume form of the metric \( \langle \cdot, \cdot \rangle = \omega(\cdot, J \cdot) \). Let \( \nabla \) be the Levi–Civita connection of this metric and define

\[
\tilde{\nabla} := \nabla - \frac{1}{2} J \nabla J, \quad \hat{\nabla} := \tilde{\nabla} - \frac{1}{4}(A - A^*),
\]

where \( A \in \Omega^1(M, \text{End}(TM)) \) is the endomorphism valued 1-form defined by

\[
A(u) v := J(\nabla_v J) u + (\nabla_{Jv} J) u \quad (3.4)
\]

for \( u, v \in \text{Vect}(M) \). Then, for all \( u \in \text{Vect}(M) \),

\[
A(u) J = JA(u) = -A(Ju), \quad A(u)^* J = JA(u)^* = A(Ju)^*. \quad (3.5)
\]

Moreover, the connection \( \tilde{\nabla} - \frac{1}{4} A \) is torsion-free and preserves \( J \). Hence

\[
\bar{\partial}_u v = \hat{\partial}_u v = \hat{\partial}_u - \frac{1}{4} A v = \frac{1}{2} \left( \nabla_u v + J \nabla_{Ju} v - J(\nabla_v J) u \right) \quad (3.6)
\]

for all \( u, v \in \text{Vect}(M) \). Here we have used (3.5) and (A.2). Thus \( \hat{\nabla} \) is the unique Hermitian connection on \( TM \) with \( \hat{\partial} \hat{\nabla} = \bar{\partial} \).

The curvature tensor of \( \hat{\nabla} \) is given by

\[
\hat{\mathcal{R}} = \mathcal{R} + \frac{1}{4} d \hat{\nabla}(A^* - A) + \frac{1}{32} [ (A^* - A) \wedge (A^* - A) ]. \quad (3.7)
\]

Since \( J \) commutes with \( A^* - A \) by (3.5), we obtain

\[
\text{tr}(J \hat{\mathcal{R}}) = \text{tr}(J \mathcal{R}) + \frac{1}{4} \text{tr}(J d \hat{\nabla}(A^* - A)) = \text{tr}(J \mathcal{R}) + \frac{1}{4} \text{tr}(d \hat{\nabla}(JA^* - JA)) = \text{tr}(J \mathcal{R}) + \frac{1}{2} d \left( \text{tr}(A \circ J) \right) = \text{tr}(J \mathcal{R}) + d \lambda^J \mathcal{R} = 2 \text{Ric}_{\rho,J}.
\]

Here the third equality follows from (3.5) and the fact that the endomorphisms \( A(Ju) \) and \( A(Ju)^* \) have the same trace, the fourth equality uses the fact that the two summands in \( v \mapsto A(Ju) v = (\nabla_v J) u + (\nabla_{Jv} J) Ju \) have the same trace, both equal to \( \lambda^J v(u) \) (see equation (2.3)), and the last equality follows from part (vi) of Theorem 2.6. This proves (ii).
We prove part (iii). Let \( \phi \in \text{Diff}_0(M) \) such that \( \text{Ric}_{\rho, \phi^*J} = 0 \). Then we have \( \text{Ric}_{\phi_*\rho, J} = \phi_*\text{Ric}_{\rho, \phi^*J} = 0 \) by part (ii) of Theorem 2.6. Define the function \( f \in \Omega^0(M) \) by \( e^{-f}\rho := \phi_*\rho \). Then

\[
\text{Ric}_{\rho, J} = \text{Ric}_{\rho, J} - \text{Ric}_{\phi_*\rho, J} = \text{Ric}_{\rho, J} - \text{Ric}_{e^{-f}\rho, J} = \frac{1}{2}d(df \circ J).
\]

Here the last equality uses (2.11). Since \( \text{Ric}_{\rho, J} \) represents \( 2\pi \) times the first Bott–Chern class of \((TM, J)\) by (ii), this shows that \( c_{1,BC}(TM, J) = 0 \).

Conversely, assume \( c_{1,BC}(TM, J) = 0 \). Then, by part (iii), there exists a smooth function \( f : M \to \mathbb{R} \) such that \( \text{Ric}_{\rho, J} = \frac{1}{2}d(df \circ J) \). Choose \( c \in \mathbb{R} \) such that \( e^c \int_M \rho = \int_M e^{-f}\rho \) and replace \( f \) by \( f + c \) to obtain \( \int_M e^{-f}\rho = \int_M \rho \).

Then by Moser isotopy there exists a smooth isotopy \( \{ \phi_t \}_{0 \leq t \leq 1} \) of \( M \) such that \( \phi_0 = \text{id} \) and \( \phi_t^*((1-t)\rho + te^{-f}\rho) = \rho \) for \( 0 \leq t \leq 1 \). Thus the diffeomorphism \( \phi := \phi_1 \) is isotopic to the identity and satisfies \( \phi^*(e^{-f}\rho) = \rho \). Hence

\[
\text{Ric}_{\rho, \phi^*J} = \frac{1}{2}d(df \circ J) = 0.
\]

This proves (iii).

We prove part (iv). Let \( \phi \in \text{Diff}(M) \) be orientation preserving, assume that \( \text{Ric}_{\rho, \phi^*J} = \text{Ric}_{\rho, J} = 0 \), and define \( f \in \Omega^0(M) \) by \( e^{-f}\rho := \phi_*\rho \). Then

\[
\frac{1}{2}d(df \circ J) = \text{Ric}_{\rho, J} - \text{Ric}_{e^{-f}\rho, J} = -\text{Ric}_{\phi_*\rho, J} = -\phi_*\text{Ric}_{\rho, \phi^*J} = 0.
\]

Thus \( f \) is constant. Since \( \int_M e^{-f}\rho = \int_M \phi_*\rho = \int_M \rho \), it follows that \( f = 0 \) and so \( \phi_*\rho = \rho \). This proves part (iv) and Theorem 3.2.

Example 3.3. Assume \( n = 1 \), suppose \( M \) has genus \( g \geq 1 \), define \( V := \int_M \rho \), let \( K_{\rho, J} \) be the Gaussian curvature, and define \( c := 2\pi(2 - 2g)V^{-1} \leq 0 \). Then the moment map

\[
\mathcal{J}(M) \to \Omega^2(M) : J \mapsto 2(\text{Ric}_{\rho, J} - c\rho) = 2(K_{\rho, J} - c)\rho
\]

is \( \mathcal{G} \)-equivariant and takes values in the space of exact 2-forms. The uniformization theorem for Riemann surfaces asserts that for every \( J \in \mathcal{J}(M) \) there exists a diffeomorphism \( \phi \in \text{Diff}_0(M) \) such that \( K_{\phi_*\rho, J} = c \) and therefore \( \text{Ric}_{\rho, \phi^*J} = c\rho \). Moreover, if \( \text{Ric}_{\rho, J} = \text{Ric}_{\rho, \phi^*J} = c\rho \) for some orientation preserving diffeomorphism \( \phi \) and \( \phi_*\rho =:\ e^f\rho \), then \( \frac{1}{2}d(df \circ J) = c(e^f - 1)\rho \).

Hence \( d^*df = 2c(e^f - 1) \) and this implies \( \int_M |df|^2 \rho = 2c \int_M f(e^f - 1)\rho \leq 0 \). Thus \( f \) is constant and \( \int_M e^f\rho = \int_M \phi_*\rho = \int_M \rho \), so \( f \equiv 0 \) and \( \phi^*\rho = \rho \).
For a Kähler potential $h : M \to \mathbb{R}$ (with mean value zero) denote by $\omega_h := \omega + i \partial \bar{\partial} h$ the associated symplectic form and let $\rho_h := \omega_h^n / n!$. The Calabi conjecture asserts that the map $h \mapsto \text{Ric}_{\rho_h,J}$ is a bijection onto the space of closed $(1,1)$-forms representing the cohomology class $2\pi c_1(TM,J)$. Injectivity was proved by Calabi [7, 8] and surjectivity by Yau [42, 43].

**Corollary 3.4.** Let $(M, \omega, J)$ be a closed connected Kähler manifold and let $\rho$ be a positive volume form with $\int_M \rho = \int_M \omega^n / n!$. Then the following holds.

(i) There exists a unique Kähler potential $h : M \to \mathbb{R}$ such that $\rho_h = \rho$.

(ii) Assume 
\[
\frac{\omega^n}{n!} = \rho, \quad c_1(TM,J) = 0 \in H^2(M; \mathbb{R}).
\]
Then there exists a diffeomorphism $\phi \in \text{Diff}_0(M)$ such that 
\[
\text{Ric}_{\rho,\phi^*J} = 0 \quad \text{and} \quad \phi^*J \text{ is compatible with } \omega. \tag{3.8}
\]

(iii) Assume 
\[
\frac{\omega^n}{n!} = \rho, \quad \text{Ric}_{\rho,J} = 0.
\]
If $\phi \in \text{Diff}(M)$ satisfies (3.8) and $\phi^*\omega - \omega$ is exact, then $\phi^*\omega = \omega$.

**Proof.** We prove part (i). By part (i) of Theorem 3.2, $\text{Ric}_{\rho,J}$ is a closed $(1,1)$-form representing the cohomology class $2\pi c_1(TM,J)$. Hence, by Yau’s existence theorem [42, 43] and Calabi’s uniqueness theorem [7, 8], there exists a unique Kähler potential $h$ such that $\text{Ric}_{\rho_h,J} = \text{Ric}_{\rho,J}$. Since $\int_M \rho_h = \int_M \rho$ by assumption, this implies $\rho_h = \rho$ by equation (2.11) in part (i) of Theorem 2.6.

We prove part (ii). By assumption and part (i) of Theorem 3.2 $\text{Ric}_{\rho,J}$ is an exact $(1,1)$-form. Since $J$ admits a compatible Kähler form, this implies that there exists a function $f \in \Omega^0(M)$ such that 
\[
\text{Ric}_{\rho,J} = \frac{1}{2} d(df \circ J), \quad \int_M e^{-f} \rho = \int_M \rho.
\]
Hence $\text{Ric}_{e^{-f}\rho,J} = 0$ by part (i) of Theorem 2.6. Now it follows from (i) that there exists a Kähler potential $h$ such that $\rho_h = e^{-f} \rho$. Since $\omega_h$ and $\omega$ are compatible with $J$, Moser isotopy yields a diffeomorphism $\phi \in \text{Diff}_0(M)$ with $\phi^*\omega_h = \omega$. Thus $\phi^*J$ is compatible with $\omega$ and $\phi^*\rho_h = \rho$. This implies $\text{Ric}_{\rho,\phi^*J} = \phi^*\text{Ric}_{\rho_h,J} = 0$ by part (v) of Theorem 2.6.

To prove (iii), note that $(\phi^{-1})^*\omega$ is compatible with $J$ and represents the cohomology class of $\omega$. Thus there is a Kähler potential $h$ with $\omega_h = (\phi^{-1})^*\omega$. Hence $\phi^*\rho_h = \rho$ and $\phi^*\text{Ric}_{\rho_h,J} = \text{Ric}_{\rho,\phi^*J} = 0$ by part (v) of Theorem 2.6. Thus $h = 0$ by Calabi uniqueness, so $\phi^*\omega = \omega$. This proves Corollary 3.4. □
The 1-form $\Lambda_\rho(J, \hat{J})$ in the integrable case

For $v \in \text{Vect}(M)$ define $f_v \in \Omega^0(M)$ by $f_v \rho := d\iota(v) \rho$. Then, by (2.13) and part (i) of Theorem 2.6 we have

$$\mathcal{L}_v \text{Ric}_\rho, J = \hat{\text{Ric}}_\rho(J, \mathcal{L}_v J) + \frac{1}{2} d(\text{df}_v \circ J)$$

(3.9)

for all $v \in \text{Vect}(M)$ and all $J \in \mathcal{J}_\text{int}(M)$. This equation can also be obtained by taking the differential on both sides of equation (3.10) below.

**Lemma 3.5.** Let $\rho \in \Omega^{2n}(M)$ be a positive volume form, let $J \in \mathcal{J}_\text{int}(M)$, and let $v \in \text{Vect}(M)$. Then

$$\Lambda_\rho(J, \mathcal{L}_v J) = 2\iota(v) \text{Ric}_\rho, J - \text{df}_v \circ J + \text{df}_J v.$$  

(3.10)

**Proof.** Let $\hat{J}_v := \mathcal{L}_v J$ and $\hat{\Lambda}_\rho(J, \mathcal{L}_v J)$. Then $\hat{J}_v u = J\nabla_u v - \nabla_{Ju} v$ for all $u, v \in \text{Vect}(M)$ by (3.1) and hence

$$((\nabla \hat{J}_v) u) w = (\nabla_w \hat{J}_v) u = \nabla_w (\hat{J}_v u) - \hat{J}_v \nabla_w u$$

$$= J\nabla_w \nabla_u v - \nabla_w \nabla_{Ju} v - J\nabla_{\nabla_w u} v + \nabla_{J\nabla_w u} v$$

(3.11)

for all $u, v, w \in \text{Vect}(M)$. Since the endomorphism $\nabla_w \hat{J}_v$ is complex anti-linear its trace vanishes. Hence the trace of $(\nabla \hat{J}_v) u$ agrees with the trace of the endomorphism $\Phi(v, u) \in \Omega^0(M, \text{End}(TM))$, defined by

$$\Phi(v, u) w := (\nabla_w \hat{J}_v) u - (\nabla_u \hat{J}_v) w$$

$$= J\nabla_w \nabla_u v - \nabla_w \nabla_{Ju} v - J\nabla_{\nabla_w u} v + \nabla_{J\nabla_w u} v$$

$$- J\nabla_u \nabla_w v + \nabla_u \nabla_{Jw} v + J\nabla_{\nabla_w u} v - \nabla_{J\nabla_w u} v$$

(3.12)

$$= -JR^\nabla(v, u) w - \nabla_w \nabla_{Ju} v + \nabla_u \nabla_{Jw} v + \nabla_{[u, w]} v$$

$$= JR^\nabla(v, u) w + \Psi(v, u) w$$

for $w \in \text{Vect}(M)$, where

$$\Psi(v, u) w := JR^\nabla(v, u) w - \nabla_w \nabla_{Ju} v + \nabla_u \nabla_{Jw} v + \nabla_{[u, w]} v$$

$$= R^\nabla(v, u) J u + R^\nabla(Ju, w) v + R^\nabla(u, Jw) v$$

$$- \nabla_{Ju} \nabla_w v + \nabla_{Ju} \nabla_u v + \nabla_{J[u, w]} v - \nabla_{Ju} \nabla_{[u, w]} v + \nabla_{Ju} \nabla_{[u, w]} v - \nabla_{Ju} \nabla_{[u, w]} v$$

(3.13)
Since $\nabla$ is a $\rho$-connection, the endomorphism $R^\nabla(u, Jv)$ has trace zero. Hence $\Psi(u, v)$ has the same trace as the endomorphism $\Psi'(u, v)$ defined by

$$\Psi'(u, v)w := R^\nabla(u, Jw)v + \nabla_{Jw} \nabla_u v - \nabla_{Ju} \nabla_w v - \nabla_{[Ju, Jw]} v$$

$$= \nabla_u Jw v + \nabla_{[u, Jw]} v - \nabla_{Ju} \nabla_w v + \nabla_{J[Ju, Jw]} v$$

$$= \nabla_u \nabla_{Ju,v} - \nabla_{\nabla_u (Jw)} v - \nabla_{Ju} \nabla_v + \nabla_{\nabla_Ju, Jw} v.$$  \hspace{1cm} (3.14)

It is convenient to conjugate the first two summands in this expression by $J$. This operation does not change the trace. Thus $\Psi'(u, v)$ has the same trace as the endomorphism $\Psi''(u, v)$ defined by

$$\Psi''(u, v)w := J\nabla_u \nabla_v - J\nabla_{\nabla_u (Jw)} v - \nabla_{Ju} \nabla_w v + \nabla_{\nabla_Ju, Jw} v$$

$$= \nabla_u \nabla_v (Jw) - \nabla_{\nabla_u (Jw)} v - \nabla_{Ju} \nabla_w v + \nabla_{\nabla_Ju, Jw} v.$$  \hspace{1cm} (3.15)

for $w \in \text{Vect}(M)$. Differentiating the function $f_v := \text{trace}(\nabla v)$ yields

$$df_v(Ju) = \text{trace}(\nabla Ju \nabla v) = \text{trace}\left(w \mapsto (\nabla Ju \nabla v)w = \nabla_{Ju} \nabla_w v - \nabla_{\nabla_Ju, Jw} v\right)$$

and so $\text{trace}(\Psi''(u, v)) = -df_v(Ju) + df_{Ju}(u)$ by (3.15). Hence

$$\hat{\lambda}_v(u) = \text{trace}(JR^\nabla(v, u) + \Psi''(u, v)) = 2\text{Ric}_{\rho, J}(v, u) - df_v(Ju) + df_{Ju}(u).$$

This proves equation (3.10) and Lemma 3.5. \hfill \Box

The next lemma examines vector fields $v$ on a Kähler manifold $(M, \omega, J)$ such that the section $\mathcal{L}_v J$ of the endomorphism bundle is self-adjoint with respect to the Kähler metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J \cdot)$. For $H \in \Omega^0(M)$ the Hamiltonian and gradient vector fields are given by $\iota(v_H)\omega = dH$ and $\nabla H = Jv_H$.

**Lemma 3.6.** Let $(M, \omega, J)$ be a closed connected $2n$-dimensional Kähler manifold, let $\rho := \omega^n/n!$, and let $v \in \text{Vect}(M)$. Then the following holds.

(i) Define $\tilde{\omega} := d\iota(v)\omega$. Then, for all $u, u' \in \text{Vect}(M)$, we have

$$\tilde{\omega}(u, u') - \tilde{\omega}(Ju, Ju') = \langle (\mathcal{L}_v J)u - (\mathcal{L}_v J)^* u, u' \rangle.$$  \hspace{1cm} (3.16)

(ii) $\mathcal{L}_v J$ is self-adjoint if and only if $d\iota(v)\omega \in \Omega^1(M)$ if and only if there exists a function $F \in \Omega^0(M)$ such that $d\iota(v + \nabla F)\omega = 0$.

(iii) The 1-form $\iota(v)\omega$ is harmonic if and only if $d\iota(v)\omega = d\iota(Jv)\omega = 0$ if and only if $\iota(Jv)\omega$ is harmonic.

(iv) Let $H \in \Omega^0(M)$. Then $f_{vH} = 0$ and $f_{\nabla H} = -d^* dH$. 

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Proof. We prove (i) and (ii). Let $\nabla$ the Levi-Civita connection and note that $(\iota(v)\omega)(u) = \langle Ju, u \rangle$ for all $u \in \text{Vect}(M)$. Hence

$$\tilde{\omega}(u, u') = L_u \langle Ju, u' \rangle - L_{u'} \langle Ju, u \rangle + \langle Ju, [u, u'] \rangle = \langle J\nabla_u v, u' \rangle - \langle J\nabla_{u'} v, u \rangle$$

and so $\tilde{\omega}(Ju, Ju') = \langle \nabla_{Ju} v, u' \rangle - \langle \nabla_{Ju'} v, u \rangle$ for all $u, u' \in \text{Vect}(M)$. Thus

$$\tilde{\omega}(u, u') - \tilde{\omega}(Ju, Ju') = \langle J\nabla_u v - \nabla_{Ju} v, u' \rangle - \langle J\nabla_{u'} v - \nabla_{Ju'} v, u \rangle$$

for all $u, u' \in \text{Vect}(M)$ and so (3.16) follows from (3.1). It follows from (3.16) that $\mathcal{L}_v J = (\mathcal{L}_v J)^*$ if and only if $d\iota(v)\omega \in \Omega_{F}^{1,1}(M)$. If $d\iota(v)\omega \in \Omega_{F}^{1,1}(M)$ then there exists an $F \in \Omega^0(M)$ with

$$d\iota(v)\omega = d(df \circ J) = -d(\nabla F)\omega.$$ 

Conversely, the covariant Hessian of $F$ is self-adjoint and hence

$$\langle u, (\mathcal{L}_{\nabla F} J)v \rangle = \langle u, J\nabla_v \nabla F - \nabla_{Ju} \nabla F \rangle = -\langle Ju, \nabla_v \nabla F \rangle - \langle u, \nabla_{Ju} \nabla F \rangle$$

$$= -\langle \nabla_{Ju} \nabla F, v \rangle - \langle \nabla_{Ju} \nabla F, Ju \rangle = \langle J\nabla_u \nabla F - \nabla_{Ju} \nabla F, v \rangle$$

$$= \langle (\mathcal{L}_{\nabla F} J)u, v \rangle$$

for all $u, v \in \text{Vect}(M)$. Thus $\mathcal{L}_{\nabla F} J$ is self-adjoint and so is $\mathcal{L}_{\nabla^* F} J = J\mathcal{L}_{\nabla F} J$. Since every symplectic vector field is locally Hamiltonian, this proves (ii).

We prove (iii) and (iv). Assume $d\iota(v)\omega = 0$ and $d* \iota(v)\omega = 0$. Since every 1-form $\lambda$ satisfies $*\lambda = -\lambda \wedge \omega^{n-1}/(n-1)!$, we have

$$*\iota(v)\omega = -(\iota(v)\omega \circ J) \wedge \frac{\omega^{n-1}/(n-1)!}{(n-1)!} = \iota(Jv)\omega \wedge \frac{\omega^{n-1}}{(n-1)!} = \iota(Jv)\rho$$

(3.17)

and so $d\iota(Jv)\rho = 0$. Also $\mathcal{L}_v J$ is self-adjoint by (ii) and so is $\mathcal{L}_{\nabla v} J = J\mathcal{L}_{\nabla v} J$. Hence by (ii) there exists a function $G \in \Omega^0(M)$ such that

$$d\iota(Jv + \nabla G)\omega = 0.$$ 

This implies $d(df \circ J) = d(Jv)\omega$ and hence

$$d(df \circ J) \wedge \frac{\omega^{(n-1)/(n-1)!}}{(n-1)!} = d(Jv)\rho = 0.$$ 

Thus $G$ is constant and $d(Jv)\omega = 0$. This proves (iii). Moreover, by (3.17) we have

$$f_{\nabla H} \rho = d(Jv_H)\rho = d*dH = -(d^*dH)\rho.$$ 

This proves Lemma 3.6 \hfill \Box
Ricci-flat Kähler manifolds

The next lemma uses the identity (see part (i) of Lemma 2.7)

\[
\iota(J\tilde{\partial}_J^*\tilde{\partial}_J^*\omega) = \Lambda_\rho(J, \tilde{J}).
\] (3.18)

Lemma 3.7. Let \((M, \omega, J)\) be a closed connected 2n-dimensional Kähler manifold, let \(\rho := \omega^n/n!\), and assume \(\text{Ric}_\rho = 0\). Then the following holds.

(i) Let \(v \in \text{Vect}(M)\). Then \(\mathcal{L}_v J = 0\) if and only if \(\iota(v)\omega\) is harmonic. If \(\iota(v)\rho\) is closed, there is a \(v_0 \in \text{Vect}(M)\) such that \(\iota(v_0)\rho\) is exact and \(\mathcal{L}_{v_0} J = \mathcal{L}_v J\).

(ii) If \(\tilde{J} \in \Omega^{0,1}_J(M, TM)\) satisfies \(\partial_J^* \tilde{J} = 0\) then \(\tilde{\text{Ric}}_\rho(J, \tilde{J}) \in \Omega^{1,1}_J(M)\).

(iii) If \(\tilde{J} \in \Omega^{0,1}_J(M, TM)\) satisfies \(\partial_J^* \tilde{J} = 0\) and \(\tilde{d}_J^* \tilde{J} = 0\) then \(\Lambda_\rho(J, \tilde{J}) = 0\).

(iv) If \(\tilde{J} \in \Omega^{0,1}_J(M, TM)\) satisfies \(\partial_J^* \tilde{J} = 0\) then there exist unique smooth functions \(f = f_J : M \to \mathbb{R}\) and \(g = g_J : M \to \mathbb{R}\) such that

\[
\Lambda_\rho(J, \tilde{J}) = -df \circ J + dg, \quad \int_M f\rho = \int_M g\rho = 0. \quad (3.19)
\]

Proof. We prove part (i). If \(\iota(v)\omega\) is harmonic then \((\mathcal{L}_v J)^* = \mathcal{L}_v J\) by part (ii) of Lemma 3.6 and hence, by (3.1) and (3.18), we have

\[
\iota(\tilde{\partial}_J^* \partial_J^* v)\omega = -\frac{1}{2} \iota( J \tilde{\partial}_J^* \mathcal{L}_v J )\omega = -\frac{1}{2} \iota( J \tilde{\partial}_J^* ( \mathcal{L}_v J )^* )\omega = -\frac{1}{2} \Lambda_\rho(J, \mathcal{L}_v J) = 0.
\]

This proves (i).

Now assume \(\iota(v)\rho = 0\), choose \(\alpha_0 \in \Omega^{2n-2}(M)\) with \(d^* d\alpha_0 = \iota(Jv)\omega\), and choose \(v_0 \in \text{Vect}(M)\) such that \(\iota(v_0)\rho = d\alpha_0\). Then \(\iota(Jv_0)\omega = *\iota(v_0)\rho\) by (3.17), hence \(\iota(Jv_0)\omega = d^* \iota(v_0)\rho = d^* d\alpha_0 = \iota(Jv)\omega\), hence \(\iota(v - v_0)\omega\) is harmonic by part (iii) of Lemma 3.6 and so \(\mathcal{L}_{v_0} J = \mathcal{L}_v J\). This proves (i).

Part (ii) follows from Lemma 3.5 and the holomorphic Poincaré Lemma.

We prove part (iii). Let \(\tilde{J} \in \Omega^{0,1}_J(M, TM)\) with \(\partial_J^* \tilde{J} = 0\) and \(\tilde{d}_J^* \tilde{J} = 0\) and define \(v := \tilde{\partial}_J^* \tilde{\partial}_J^* \in \text{Vect}(M)\). Then \(\iota(v)\omega = -\Lambda_\rho(J, J\tilde{J})\) by (3.18) and so \(\iota(v)\omega = -2\tilde{\text{Ric}}_\rho(J, J\tilde{J})\) is an exact \((1, 1)\)-form by (ii). Thus \(\mathcal{L}_J v\) is self-adjoint by Lemma 3.6. Hence \(\partial_J^* \tilde{\partial}_J^* (\tilde{J}^* - \tilde{J}) = \tilde{d}_J^* \tilde{J}^* v = -\frac{1}{2} J \mathcal{L}_v J\) is self-adjoint and so is \(L^2\) orthogonal to \(\tilde{J}^* - \tilde{J}\). Thus \(\tilde{d}_J^* \tilde{J}^* = \tilde{\partial}_J^* (\tilde{J}^* - \tilde{J}) = 0\) and so \(\Lambda_\rho(J, \tilde{J}) = 0\) by equation (3.18). This proves (iii).
To prove (iv), choose \( v \in \text{Vect}(M) \) such that \( \bar{\partial}_J^* (\tilde{J} - \mathcal{L}_v J) = 0 \). Then by (iii) we have \( \Lambda_\rho(J, \tilde{J}) = \Lambda_\rho(J, \mathcal{L}_v J) \) and so \( f := f_v \) and \( g := f_{Jv} \) satisfy the requirements of part (iv) by Lemma 3.5. This proves Lemma 3.7.

**Corollary 3.8.** Let \((M, \omega, J)\) be a closed connected \(2n\)-dimensional Kähler manifold, define \( \rho := \omega^n/n! \), assume \( \text{Ric}_{\rho, J} = 0 \) and let \( H \in \Omega^0(M) \). Then

\[
\Lambda_\rho(J, \mathcal{L}_v \nabla H) = dd^* dH \circ J, \quad \Lambda_\rho(J, \mathcal{L}_{vH} J) = -dd^* dH, \quad \bar{\partial}_J^* \mathcal{L}_{vH} J = \nabla d^* dH.
\]

**Proof.** The endomorphisms \( \mathcal{L}_{vH} J \) and \( \mathcal{L}_J \nabla H J \) are self-adjoint by part (ii) of Lemma 3.6. Hence the identities in (3.20) follow from Lemma 3.5, part (iv) of Lemma 3.6 and equation (3.18). This proves Corollary 3.8.

**Remark 3.9.** The symplectic form \( \Omega_\rho \) in (2.6) on \( \mathcal{J}(M) \) is a \((1,1)\)-form with respect to the complex structure \( \tilde{J} \mapsto -J\tilde{J} \). However, the resulting symmetric bilinear form \( \langle \tilde{J}_1, \tilde{J}_2 \rangle_{\rho,J} = \frac{1}{2} \int_M \text{trace}(\tilde{J}_1 \tilde{J}_2) \rho \) is indefinite, so \( \mathcal{J}(M) \) is not a Kähler manifold. Thus complex submanifolds of \( \mathcal{J}(M) \) need not be symplectic. An example is the space \( \mathcal{J}_{\int 0}(M) \) of all integrable complex structures with real first Chern class zero and nonempty Kähler cone. Its tangent space is the kernel of the operator \( \bar{\partial}_J : \Omega_{J}^{0,1}(M, TM) \to \Omega_{J}^{0,2}(M, TM) \).

Let \( J \in \mathcal{J}_{\int 0}(M) \) such that \( \text{Ric}_{\rho, J} = 0 \) and choose a vector field \( v \in \text{Vect}(M) \) such that \( v \) and \( Jv \) are divergence-free, i.e. \( dv(v) = d(Jv) \rho = 0 \). Choose any \( \tilde{J} \in \Omega_{J}^{0,1}(M, TM) \) with \( \bar{\partial}_J \tilde{J} = 0 \) and let \( f, g \in \Omega^0(M) \) be as in Lemma 3.7, so that \( \Lambda_\rho(J, \tilde{J}) = -df \circ J + dg \). Then, by Lemma 3.5, we have

\[
\Omega_{\rho,J}(\tilde{J}, \mathcal{L}_v J) = \frac{1}{2} \int_M \text{trace}(\tilde{J} J \mathcal{L}_v J) \rho = \int_M \Lambda_\rho(J, \tilde{J}) \wedge \iota(v) \rho = \int_M (-df \circ J + dg) \wedge \iota(v) \rho = \int_M (f d\iota(Jv) \rho - g d\iota(v) \rho) = 0.
\]

Thus the space \( \{ \mathcal{L}_v J | dv(v) = d(Jv) \rho = 0 \} \) is contained in the kernel of the 2-form \( \Omega_{\rho,J} \) on \( T_J \mathcal{J}_{\int 0}(M) = \ker \bar{\partial}_J \). In the next section we will prove that this subspace is precisely the kernel of \( \Omega_{\rho,J} \) and hence \( \Omega_{\rho,J} \) does induce a symplectic form on the Teichmüller space \( \mathcal{T}_0(M) := \mathcal{J}_{\int 0}(M)/\text{Diff}_0(M) \).
The next lemma is the key to the proof of nondegeneracy in Theorem 4.2.

**Lemma 3.10.** Let \((M, J, \omega)\) be a closed connected \(2n\)-dimensional Kähler manifold such that \(\text{Ric}_{\rho, J} = 0\), where \(\rho := \frac{\omega^n}{n!}\), and let \(\tilde{\mathcal{J}} \in \Omega^{0,1}(M, TM)\) such that \(\bar{\partial}_J \tilde{\mathcal{J}} = 0\) and \(\bar{\partial}^*_J \tilde{\mathcal{J}} = 0\). Then \(\bar{\partial}_J \tilde{\mathcal{J}}^* = 0\) and \(\bar{\partial}^*_J \tilde{\mathcal{J}}^* = 0\).

**Proof.** Choose a Hermitian line bundle \(L\) with \(c_1(L) = c_1(TM, J)\), a Hermitian connection \(\nabla_L\), and an \(n\)-form \(\theta \in \Omega^n_J(M, L)\) that satisfies \(d\nabla_L \theta = 0\) and \(c_n(\theta \wedge \theta) = \rho\) (see equation (B.1) and Lemma B.3). Let \(\beta \in \Omega^{n-1, 1}_J(M, L)\) satisfy (B.4). Then \(\bar{\partial}^*_J \beta = 0\) and \((\bar{\partial}^*_J \beta)^* = 0\) by Lemma B.4 and \(i F^L = 0\) by Lemma B.3. Hence

\[
(\bar{\partial}^*_J \beta)^* \bar{\partial}_J \beta + \bar{\partial}_J (\bar{\partial}^*_J \beta) ^* = 0
\]

by the Akizuki–Nakano Theorem (see [10, page 330]). This implies \(\bar{\partial}_J \tilde{\mathcal{J}} \beta = 0\) and \((\bar{\partial}_J \tilde{\mathcal{J}})^* \beta = 0\) and thus \((\bar{\partial}_J \tilde{\mathcal{J}})^* \beta = 0\) and \(\bar{\partial}_J \tilde{\mathcal{J}}^* = 0\). Hence \(\bar{\partial}^*_J \tilde{\mathcal{J}}^* = 0\) and \(\bar{\partial}_J \tilde{\mathcal{J}}^* = 0\) by Lemma B.2 and Lemma B.3. This proves Lemma 3.10. \(\square\)

The next lemma shows that every closed \((0, 2)\)-form on a Ricci-flat Kähler manifold is parallel. This is used in Theorem 4.3.

**Lemma 3.11.** Let \((M, J, \omega)\) be a closed connected \(2n\)-dimensional Kähler manifold such that \(\text{Ric}_{\rho, J} = 0\), where \(\rho := \frac{\omega^n}{n!}\), and let \(\nabla\) be the Levi-Civita connection of the Kähler metric. Let \(\tilde{\mathcal{J}} \in \Omega^{0,1}_J(M, TM)\) such that \(\tilde{\mathcal{J}} + \tilde{\mathcal{J}}^* = 0\) and define \(\tilde{\omega} \in \Omega^2(M)\) by

\[
\tilde{\omega}(u, v) := \langle \tilde{\mathcal{J}} u, v \rangle
\]

for \(u, v \in \text{Vect}(M)\). Then \(\tilde{\omega}^{1,1} = 0\) and the following are equivalent.

(i) \(\bar{\partial}_J \tilde{\mathcal{J}} = 0\) and \(\bar{\partial}^*_J \tilde{\mathcal{J}} = 0\).

(ii) \(d\tilde{\omega} = 0\)

(iii) \(\tilde{\omega}\) is a harmonic 2-form.

(iv) \(\nabla \tilde{\omega} = 0\).

(v) \(\nabla \tilde{\mathcal{J}} = 0\).
Proof. We have \( \hat{\omega}(u, v) + \tilde{\omega}(Ju, Jv) = 0 \) for \( u, v \in \text{Vect}(M) \) by \((3.22)\) and so \( \hat{\omega}^{1,1}_{J} = 0 \). Now let \( L \to M \) be a Hermitian line bundle with integral first Chern class \( c_1(L) = c_1(TM, J) \in H^2(M; \mathbb{Z}) \), let \( \nabla_L \) be a flat Hermitian connection on \( L \), and let \( \theta \in \Omega^{0,0}_{J}(M, L) \) such that \( d\nabla_L \theta = 0 \) and \( c_n(\theta \wedge \theta) = \rho \) (see \((B.1)\)). Let \( \beta \in \Omega^{n-1,1}_{J}(M, L) \) be the unique \((n - 1, 1)\)-form with values in \( L \) that satisfies \((B.4)\) and define \( \alpha \in \Omega^{n-2,0}_{J}(M, L) \) by

\[
\alpha := \frac{1}{2c_n}(\omega \wedge \beta). \tag{3.23}
\]

We prove that

\[
\omega \wedge \alpha = \beta, \quad \omega \wedge \beta + \hat{\omega} \wedge \theta = 0. \tag{3.24}
\]

Since \( \hat{\omega}^{1,1}_{J} = 0 \) the identity \( \omega \wedge \beta + \hat{\omega} \wedge \theta = 0 \) follows from \((3.21)\) and part (iv) of Lemma \(B.2\). Second, the Hodge \(*\)-operator on the space of \((n - 2, 0)\)-forms is given by \(*\alpha = \frac{i}{2} \omega^2 \wedge \alpha \) for all \( \alpha \in \Omega^{n-2,0}_{J}(M, L) \). By \((3.23)\) we also have \(*\alpha = \frac{i}{2c_n})*(\omega \wedge \beta) = \frac{i}{2c_n}(-1)^n \omega \wedge \beta = \frac{i}{2} \omega \wedge \beta \), and so \( \omega^2 \wedge \alpha = \omega \wedge \beta \). Thus \(*((\beta - \omega \wedge \alpha)\) is equal to both \( c_n(\beta - \omega \wedge \alpha) \) and \( -c_n(\beta - \omega \wedge \alpha) \) by part (iii) of Lemma \(B.2\). Hence \( \omega \wedge \alpha = \beta \) and this proves \((3.24)\).

We prove that \( \nabla \theta = 0 \). By Lemma \(B.3\) there exists a torsion-free connection \( \nabla' = \nabla + A \) on \( TM \) such that \( \nabla'J = 0 \) and \( \nabla' \theta = 0 \). Then \( \nabla' \rho = \nabla \rho = 0 \), so \( A \in \Omega^1(M, \text{End}(TM)) \) has real trace zero. Moreover, \( A(Ju)v = JA(u)v \) for all \( u, v \in \text{Vect}(M) \). Hence \( A \) has complex trace zero and so \( \nabla \theta = 0 \).

We prove that (i) implies (ii). Note first that, by (i) and Lemma \(B.4\) we have \( \partial^{\nabla_L}_{J} \beta = 0 \) and \( (\partial^{\nabla_L}_{J})^* \beta = 0 \). Hence \( d^{\nabla_L} \beta = 0 \) and \( (d^{\nabla_L})^* \beta = 0 \) by Lemma \(3.10\). Now the result in \([17] \) page 115 or \([10] \) Lemma 6.28 carries over to sections of \( L \) because \( \nabla_L \) is flat, so the Lefschetz operator \( \omega \wedge \cdot \) commutes with the Hodge Laplace operator on \( \Omega^*(M, L) \). Hence \( \partial^{\nabla_L}_{J} (\omega \wedge \beta) = 0 \) and \( (\partial^{\nabla_L}_{J})^*(\omega \wedge \beta) = 0 \). Thus

\[
(\partial^{\nabla_L}_{J} \omega^{0,2}) \wedge \theta = -\partial^{\nabla_L}_{J} (\hat{\omega} \wedge \theta) = 0.
\]

Moreover, \( (\sigma \wedge \theta, \tau \wedge \theta) = (\sigma, \tau) \) for all \( \sigma, \tau \in \Omega_{0,q}^0(M) \) and all \( q \). This implies

\[
(\partial^{\nabla_L}_{J} \omega^{0,2}) \wedge \theta = -(\partial^{\nabla_L}_{J})^*(\hat{\omega} \wedge \theta) = 0.
\]

Thus \( \partial^{\nabla_L}_{J} \omega^{0,2} = 0 \) and \( (\partial^{\nabla_L}_{J} \omega^{0,2}) = 0 \) and so \( \omega^{0,2} \) is harmonic. Since \( \hat{\omega}^{1,1}_{J} = 0 \), we have \( \hat{\omega}^{0,2} (u, v) = \hat{\omega}(u, v) - i\hat{\omega}(Ju, v) \) for all \( u, v \in \text{Vect}(M) \). Hence \( \hat{\omega} \) is harmonic and so is closed. This shows that (i) implies (ii).

We prove that (ii) implies (iii). Thus assume \( \hat{\omega} \) is closed. Since \( \hat{\omega}^{1,1}_{J} = 0 \) we have \(*\hat{\omega} = \hat{\omega} \wedge \omega^{0,-2}_{(n-2)!} \). Thus \( d*\hat{\omega} = 0 \) and so \( \hat{\omega} \) is harmonic. This shows that (ii) implies (iii).
We prove that (iii) implies (i). Since $\widetilde{\omega}$ is a harmonic 2-form so is $\widetilde{\omega}_{j}^{0,2}$ and so $\bar{\partial}_{j}\tilde{\omega}_{j}^{0,2} = 0$ and $\tilde{\partial}_{j}^{*}\tilde{\omega}_{j}^{0,2} = 0$. Since $\nabla \theta = 0$, it then follows from (3.24) that

$$\bar{\partial}_{j}\tilde{\omega}_{j}^{0,2} = -\bar{\partial}_{j}^{*}\tilde{\omega}_{j}^{0,2} = 0,$$

This implies $\bar{\partial}_{j}\tilde{\omega}_{j}^{0,2} = 0$ and $(\bar{\partial}_{j}^{*})^{\alpha} = 0$ by the Hard Lefschetz Theorem for sections of $L$ (see [17, page 122] or [40, Theorem 6.25]). Since $\beta = \omega \wedge \alpha$ by (3.24) and the Lefschetz operator $\omega \wedge \cdot$ commutes with the Hodge Laplace operator on sections of $L$, we obtain $\bar{\partial}_{j}^{*}\tilde{\omega}_{j}^{0,2} = 0$ and $(\bar{\partial}_{j}^{*})^{*}\tilde{\omega}_{j}^{0,2} = 0$, and this implies (i) by part (ii) of Lemma 3.11.

Thus we have proved that the assertions (i), (ii), and (iii) are equivalent. The equivalence of (iv) and (v) follows from the definition of $\tilde{\omega}$ in (3.23) and that (iv) implies (ii) follows from the fact that $\nabla$ is torsion-free (see the proof of Lemma A.1).

It remains to prove that (i) implies (v). Assume $\bar{\partial}_{j}\tilde{\omega}_{j} = 0$ and $\bar{\partial}_{j}^{*}\tilde{\omega}_{j} = 0$. Since we already proved that (i) implies (ii), we have

$$0 = d\tilde{\omega}(u, v, w) = \langle(\nabla_{u}\tilde{\omega})v, w\rangle + \langle(\nabla_{v}\tilde{\omega})w, u\rangle + \langle(\nabla_{w}\tilde{\omega})u, v\rangle$$

(3.25) for all $u, v, w \in \text{Vect}(M)$ by (3.22). This implies

$$\langle(\nabla_{u}\tilde{\omega})v, w\rangle = -\langle(\nabla_{v}\tilde{\omega})w, u\rangle - \langle(\nabla_{w}\tilde{\omega})u, v\rangle$$

$$= \langle(\nabla_{u}\tilde{\omega})v, w\rangle - \langle(\nabla_{u}\tilde{\omega})Jv, Ju\rangle$$

$$= -\langle(\nabla_{u}\tilde{\omega})v, w\rangle - \langle(\nabla_{u}\tilde{\omega})Ju, Ju\rangle - \langle(\nabla_{u}\tilde{\omega})Jv, Ju\rangle$$

$$= 2\langle(\nabla_{u}\tilde{\omega})v, w\rangle - \langle(\nabla_{u}\tilde{\omega})Ju, Ju\rangle - \langle(\nabla_{u}\tilde{\omega})Jv, Ju\rangle$$

$$= 2\langle(\nabla_{u}\tilde{\omega})v, w\rangle - \langle(\nabla_{u}\tilde{\omega})Ju, Ju\rangle - \langle(\nabla_{u}\tilde{\omega})Jv, Ju\rangle$$

for all $u, v, w \in \text{Vect}(M)$ and hence $\nabla_{u}\tilde{\omega} = J\nabla_{Ju}\tilde{\omega}$ for all $u \in \text{Vect}(M)$. Moreover, $0 = 2\bar{\partial}_{j}\tilde{\omega}(u, v) = (\nabla_{u}\tilde{\omega})v - (\nabla_{v}\tilde{\omega})u - (\nabla_{Ju}\tilde{\omega})Ju + (\nabla_{Jv}\tilde{\omega})Ju$, hence

$$(\nabla_{u}\tilde{\omega})v - (\nabla_{v}\tilde{\omega})u = -\langle(\nabla_{u}\tilde{\omega})Ju, Ju\rangle + (\nabla_{Jv}\tilde{\omega})Ju = (\nabla_{u}\tilde{\omega})v - (\nabla_{v}\tilde{\omega})u,$$

and hence $(\nabla_{u}\tilde{\omega})u = (\nabla_{v}\tilde{\omega})v$ for all $u, v \in \text{Vect}(M)$. Thus, by (3.24) we have

$$\langle(\nabla_{u}\tilde{\omega})u, v\rangle = -\langle(\nabla_{u}\tilde{\omega})w, u\rangle - \langle(\nabla_{v}\tilde{\omega})v, w\rangle = \langle(\nabla_{u}\tilde{\omega})u, -(\nabla_{u}\tilde{\omega})v, v\rangle = 0$$

for all $u, v, w \in \text{Vect}(M)$ and so $\nabla\tilde{\omega} = 0$. This proves Lemma 3.11. \(\square\)
4 Teichmüller space

The Calabi–Yau Teichmüller space

Consider the Teichmüller space

\[ T_0(M) := \mathcal{J}_{\text{int},0}(M) / \text{Diff}_0(M), \]

\[ \mathcal{J}_{\text{int},0}(M) := \left\{ J \in \mathcal{J}_{\text{int}}(M) \middle| c_1(TM, J) = 0 \in H^2(M; \mathbb{R}) \right\}, \]

(4.1)

of isotopy classes of complex structures with real first Chern class zero and nonempty Kähler cone. For every \( J \in \mathcal{J}_{\text{int},0}(M) \) the space of holomorphic vector fields is isomorphic to \( H^1(M; \mathbb{R}) \) by Lemma 3.7. Moreover, the Bogomolov–Tian–Todorov theorem asserts that the obstruction class vanishes \([4, 34, 35]\). Hence the cohomology of the complex

\[ \Omega^0(M, TM) \xrightarrow{\partial_J} \Omega^{0,1}_J(M, TM) \xrightarrow{\partial_J} \Omega^{0,2}_J(M, TM) \]

(4.2)

has constant dimension. It follows that the Teichmüller space \( T_0(M) \) is a smooth manifold \([9, 22, 23, 25, 26]\) whose tangent space at \( J \in \mathcal{J}_{\text{int},0}(M) \) is the cohomology of the complex (4.2), i.e.

\[ T_J T_0(M) = \ker(\partial_J : \Omega^{0,1}_J(M, TM) \to \Omega^{0,2}_J(M, TM)) / \text{im}(\partial_J : \Omega^0(M, TM) \to \Omega^{0,1}_J(M, TM)). \]

(4.3)

Remark 4.1. The Teichmüller space is in general not Hausdorff, even for the K3 surface \([18, 37]\). Let \((M, J)\) be a K3-surface that admits an embedded holomorphic sphere \( C \subset M \) with self-intersection number \( C \cdot C = -2 \), and let \( \tau : M \to M \) be a Dehn twist about \( C \). Then there exists a smooth family of complex structures \( \{J_t \in \mathcal{J}_{\text{int},0}(M)\}_{t \in \mathbb{C}} \) and a smooth family of diffeomorphisms \( \{\phi_t \in \text{Diff}_0(M)\}_{t \in \mathbb{C} \setminus \{0\}} \) such that \( J_0 = J \) and \( \phi_t^* J_t = \tau^* J_{-t} \) for all \( t \in \mathbb{C} \setminus \{0\} \). Thus \( J_t \) and \( \tau^* J_t \) represent the same class in Teichmüller space, however, their limits \( \lim_{t \to 0} J_t = J_0 \) and \( \lim_{t \to 0} \tau^* J_t = \tau^* J_0 \) do not represent the same class in Teichmüller space because their effective cones differ. Namely, the class \( [C] \in H_2(M; \mathbb{Z}) \) belongs to the effective cone of \( J_0 \) while the class \( -[C] \in H_2(M; \mathbb{Z}) \) belongs to the effective cone of \( \tau^* J_0 \).

For general hyperKähler manifolds the Teichmüller space becomes Hausdorff after identifying inseparable complex structures (see Verbitsky \([37, 38]\)), which are bimeromorphic by a theorem of Huybrechts \([19]\).
A symplectic form

Let $V > 0$. Then by Theorem 2.6 every complex structure $J \in \mathcal{J}_{\text{int},0}(M)$ admits a unique positive volume form $\rho = \rho_J$ such that

$$\text{Ric}_{\rho,J} = 0, \quad \int_M \rho = V. \tag{4.4}$$

**Theorem 4.2 (Weil–Petersson Symplectic Form).** For a complex structure $J \in \mathcal{J}_{\text{int},0}(M)$, for the volume form $\rho = \rho_J \in \Omega^2(M)$ satisfying (4.4), and for $\tilde{J}_1, \tilde{J}_2 \in \Omega^{0,1}(M, TM)$ with $\bar{\partial}_J \tilde{J}_i = 0$ and $f_i, g_i$ as in Lemma 3.3, define

$$\Omega_J(\tilde{J}_1, \tilde{J}_2) := \int_M \left( \frac{1}{2} \text{tr}(\tilde{J}_1 J \tilde{J}_2) - f_1 g_2 + f_2 g_1 \right) \rho_J. \tag{4.5}$$

This bilinear form is skew-symmetric and has the following properties.

(i) The 2-form $\Omega_J$ in (4.5) descends to a nondegenerate 2-form on the quotient space $\mathcal{F}_0(M)$ and defines a symplectic form on $\mathcal{F}_0(M)$.

(ii) If $\phi : M \to M$ is an orientation preserving diffeomorphism then

$$\Omega_{\phi^* J}(\phi^* \tilde{J}_1, \phi^* \tilde{J}_2) = \Omega_J(\tilde{J}_1, \tilde{J}_2) \tag{4.6}$$

for all $\tilde{J}_1, \tilde{J}_2 \in \Omega^{0,1}(M, TM)$ such that $\bar{\partial}_J \tilde{J}_i = 0$. Thus the mapping class group $\Gamma := \text{Diff}^+(M)/\text{Diff}_0(M)$ acts on $\mathcal{F}_0(M)$ by symplectomorphisms.

(iii) Choose a Hermitian line bundle $L$ with $c_1(L) = c_1(TM, J) \in H^2(M; \mathbb{Z})$ a flat Hermitian connection $\nabla_L$ on $L$, and an n-form $\theta \in \Omega^{0,0}(M, L)$ such that $d^\nabla_L \theta = 0$ and $c_n(\theta \wedge \theta) = \rho$ (see Lemma B.1). Let $\tilde{J}_1, \tilde{J}_2 \in \Omega^{0,1}(M, TM)$ such that $\bar{\partial}_J \tilde{J}_i = 0$ and let $\theta_1, \theta_2 \in \Omega^n(M, L)$ be as in Corollary B.3. Then

$$\Omega_J(\tilde{J}_1, \tilde{J}_2) = 4 \text{Im} \left( \int_M c_n(\tilde{\theta}_1 \wedge \tilde{\theta}_2) \right). \tag{4.7}$$

(iv) Let $\omega \in \Omega^2(M)$ be a symplectic form with real first Chern class zero and define $\mathcal{J}_{\text{int}}(M, \omega) := \{ J \in \mathcal{J}_{\text{int}}(M) \mid J \text{ is compatible with } \omega \}$. Then the submanifold $\mathcal{F}(M, \omega) := \mathcal{J}_{\text{int}}(M, \omega)/\sim$ of the Teichmüller space $\mathcal{F}_0(M)$ is Kähler with the symplectic form (4.4) and complex structure $\tilde{J} \mapsto -J \tilde{J}$. Here $J_0 \sim J_1$ iff there is a diffeomorphism $\phi \in \text{Diff}_0(M)$ such that $\phi^* J_0 = J_1$. The symmetric bilinear form

$$\langle \tilde{J}_1, \tilde{J}_2 \rangle_J := \Omega_J(\tilde{J}_1, -J \tilde{J}_2) = \int_M \left( \frac{1}{2} \text{tr}(\tilde{J}_1 J \tilde{J}_2) - f_1 g_2 + f_2 g_1 \right) \rho_J \tag{4.8}$$

is positive on $T_{\tilde{J}_1} \mathcal{F}(M, \omega)$ and negative on its symplectic complement.
The inclusion \( \imath_0 \) cause Diff space of \( \mathcal{T} \). Thus (4.5) defines a nondegenerate 2-form on Teichmüller space.

The derivative of \( \hat{\Omega} \) into (4.3). This implies \( \bar{\Omega} \) generate. Fix an element \( v \in \text{Vect}(M) \) and let \( f = f_J \) and \( g = f_J^\vee \) be as in part (iii) of Lemma 3.7. Then

\[
\Omega_J(\hat{J}, \mathcal{L}_v J) = 0
\]

for all \( v \in \text{Vect}(M) \) and all \( \hat{J} \in \Omega_{J}^{0,1}(M, TM) \) with \( \partial_J \hat{J} = 0 \). To see this, let \( f_v \) and \( f_{Jv} \) be as in Lemma 3.3 and let \( f = f_J^\vee \) and \( g = f_J^\vee \) be as in part (iii) of Lemma 8.7. Then

\[
\Omega_J(\hat{J}, \mathcal{L}_v J) = \frac{1}{2} \int_M \text{trace}(\hat{J} \mathcal{J}_v J) - \int_M (f f_{Jv} - g f_v) \rho + \int_M f df(v) \rho + \int_M g dt(v) \rho = 0
\]

because \( \Lambda^\rho(J, \hat{J}) = -df \circ J + dg \). This proves (4.9) and so the 2-form \( \Omega_J \) in (4.3) descends to the quotient space in (4.3).

We prove that the induced 2-form on the quotient space (4.3) is nondegenerate. Fix an element \( \hat{J} \in \Omega_{J}^{0,1}(M, TM) \) such that \( \partial_J \hat{J} = 0 \) and assume that \( \Omega_J(\hat{J}, \hat{J}^\vee) = 0 \) for all \( \hat{J} \in \Omega_{J}^{0,1}(M, TM) \) with \( \partial_J \hat{J} = 0 \). Choose a vector field \( v \in \text{Vect}(M) \) such that \( \partial_J (\hat{J} - \mathcal{L}_v J) = 0 \). Then \( \partial_J (\hat{J} - \mathcal{L}_v J)^\vee = 0 \) by Lemma 3.10 and \( \partial_J \hat{J} = 0 \) implies \( \Omega_J(\hat{J} - \mathcal{L}_v J, \hat{J}^\vee) = 0 \) by (4.9). This yields the equation \( \langle (\hat{J} - \mathcal{L}_v J)^\vee, \hat{J}\rangle_{L^2} = 0 \) for all \( \hat{J} \in \Omega_{J}^{0,1}(M, TM) \) with \( \partial_J \hat{J} = 0 \). Hence there exists a 2-form \( \sigma \in \Omega_{J}^{0,2}(M, TM) \) such that \( (\hat{J} - \mathcal{L}_v J)^\vee = \partial_J^\vee \sigma \). This implies \( \partial_J \partial_J^\vee \sigma = \partial_J (\hat{J} - \mathcal{L}_v J)^\vee = 0 \), hence \( \partial_J^\vee \sigma = 0 \), and so \( \hat{J} = \mathcal{L}_v J \). Thus (4.3) defines a nondegenerate 2-form on Teichmüller space.

To prove that it is closed, fix a positive volume form \( \rho \) on \( M \) and define

\[
\mathcal{T}_0(M, \rho) := \mathcal{J}_{int,0}(M, \rho)/\text{Diff}_0(M, \rho),
\]

\[
\mathcal{J}_{int,0}(M, \rho) := \{ J \in \mathcal{J}_{int,0}(M) | \overline{\text{Ric}}_{\rho, J} = 0 \}.
\]

The inclusion \( \iota_\rho : \mathcal{T}_0(M, \rho) \to \mathcal{T}_0(M) \) is bijective by Theorem 3.2 and because \( \text{Diff}_0(M, \rho) = \text{Diff}(M, \rho) \cap \text{Diff}_0(M) \) by Moser isotopy. The tangent space of \( \mathcal{T}_0(M, \rho) \) at \( J \in \mathcal{J}(M, \rho) \) is the quotient

\[
T_J \mathcal{T}_0(M, \rho) = \left\{ \hat{J} \in \Omega_{J}^{0,1}(M, TM) | \partial_J \hat{J} = 0, \overline{\text{Ric}}_\rho(J, \hat{J}) = 0 \right\} / \{ \mathcal{L}_v J | v \in \text{Vect}(M), dt(v) = 0 \}.
\]

The derivative of \( \iota_\rho \) at \( J \) is the obvious inclusion of the quotient (4.10) into (4.3). This inclusion is injective because \( \overline{\text{Ric}}_\rho(J, \mathcal{L}_v J) = -\frac{1}{2} d(df_v \circ J) \),
hence $\hat{\text{Ric}}_\rho(J, \mathcal{L}_vJ) = 0$ implies $f_v = 0$ and thus $dt(v)\rho = 0$. It is surjective because, if an element $\hat{J} \in \Omega^{0,1}_J(M, TM)$ satisfies $\bar{\partial}_J \hat{J} = 0$, then the unique solution $F : M \to \mathbb{R}$ of the equation $d^*dF = f_\hat{J}$ with mean value zero satisfies

$$\Lambda_\rho(J, L(\nabla F)_J) = 0$$

by Corollary 3.8 and hence $\hat{\text{Ric}}_\rho(J, \hat{J} + \mathcal{L}_\nabla F_J) = 0$. Thus $t_\rho$ is a diffeomorphism. The pullback of the 2-form (4.5) on $\mathcal{T}_0(M)$ under $t_\rho$ is the restriction of the standard symplectic form $\Omega_\rho$ on $\mathcal{J}(M)$ in (2.6) to the subquotient $\mathcal{T}_0(M, \rho)$. Hence it is closed and this proves part (i). Part (ii) follows directly from the definitions, part (iii) follows from Corollary B.5, and part (iv) holds because $T[\mathcal{J}, T](M, \omega)$ is the quotient of the space of self-adjoint endomorphisms $\hat{J} = \hat{J}^* \in \Omega^{0,1}_J(M, TM)$ with $\bar{\partial}_J \hat{J} = 0$ modulo those generated by Hamiltonian and gradient vector fields. This proves Theorem 4.2.

A symplectic connection

Fix a closed connected oriented $2n$-manifold $M$ and consider the maps

$$\mathcal{K}_0(M, \omega) \hookrightarrow \mathcal{E}_0(M) \rightarrow \mathcal{B}_0(M),$$

where $\mathcal{B}_0(M)$ denotes the space of isotopy classes of symplectic forms with real first Chern class zero which admit compatible complex structures, i.e.

$$\mathcal{B}_0(M) := \mathcal{K}_0(M)/\text{Diff}_0(M),$$

$$\mathcal{K}_0(M) := \left\{ \omega \in \Omega^2(M) \left| \begin{array}{l} d\omega = 0, \omega^n > 0, \\ c_1(\omega) = 0 \in H^2(M; \mathbb{R}), \\ \mathcal{J}_{\text{int}}(M, \omega) \neq \emptyset \end{array} \right. \right\},$$

(4.11)

and $\mathcal{E}_0(M)$ denotes the space of isotopy classes of Ricci-flat Kähler structures $(\omega, J)$ on $M$, i.e.

$$\mathcal{E}_0(M) := \mathcal{K}_0(M)/\text{Diff}_0(M),$$

$$\mathcal{K}_0(M) := \left\{ (\omega, J) \left| \begin{array}{l} \omega \in \mathcal{K}_0(M), J \in \mathcal{J}_{\text{int}}(M), \\ J \text{ is compatible with } \omega, \\ \text{and } \text{Ric}_{\omega, J} = 0 \end{array} \right. \right\}.$$  

(4.12)

The spaces $\mathcal{E}_0(M)$ and $\mathcal{B}_0(M)$ are finite-dimensional manifolds and the projection $\mathcal{E}_0(M) \rightarrow \mathcal{B}_0(M)$ is a surjective submersion with fibers $\mathcal{T}_0(M, \omega)$. The symplectic form in Theorem 4.2 gives rise to a closed 2-form on $\mathcal{E}_0(M)$ which restricts to the canonical Kähler form on each fiber, and hence gives rise to a symplectic connection on $\mathcal{E}_0(M)$ as in [28, Chapter 6].
Theorem 4.3 (Symplectic Connection). Let $(\omega, J) \in \mathcal{H}_0(M)$ be a Ricci-flat Kähler structure and let $\rho := \omega^n/n!$. Then the following holds.

(i) There exists a unique linear map 

$$\mathcal{A}_{\omega,J} : \Omega^2(M) \ni \omega \mapsto \Omega^{0,1}_J(M,TM)$$

(4.13)

which assigns to every closed real valued 2-form $\tilde{\omega} \in \Omega^2(M)$ an infinitesimal almost complex structure $\tilde{J} = \mathcal{A}_{\omega,J}(\tilde{\omega}) \in \Omega^0_{J} (M,TM)$ that satisfies

$$\partial J = 0, \quad \Lambda_\rho(J, \tilde{J}) = -\tilde{d}(\tilde{\omega}, \omega) \circ J,$$  

(4.14)

$$\tilde{\omega}(u, u') - \tilde{\omega}(Ju, Ju') = \omega(\tilde{J}u, Ju') + \omega(Ju, \tilde{J}u')$$

(4.15)

for all $u, u' \in \text{Vect}(M)$, and

$$\tilde{J}^* = (\tilde{J})^*, \quad \partial J = 0 \implies \Omega_J(J, \tilde{J}) = 0$$

(4.16)

for all $\tilde{J} \in \Omega^0_{J} (M,TM)$.

(ii) If $v \in \text{Vect}(M)$ satisfies $d\iota(Jv)\rho = 0$ then

$$\mathcal{A}_{\omega,J}(d\iota(v)\omega) = \mathcal{L}_v J$$

(4.17)

(iii) The 1-form $\mathcal{A}_{\omega,J}$ in (4.13) is $\text{Diff}(M)$-equivariant in that

$$\phi^* \mathcal{A}_{\omega,J}(\tilde{\omega}) = \mathcal{A}_{\phi^* \omega, \phi^* J}(\phi^* \tilde{\omega})$$

for all $(\omega, J) \in \mathcal{H}_0(M)$, every closed 2-form $\tilde{\omega} \in \Omega^2(M)$, and every orientation preserving diffeomorphism $\phi : M \to M$.

(iv) The curvature of the connection $\mathcal{A}$ is a $\text{Diff}_0(M)$-equivariant 2-form on $\mathcal{H}_0(M)$ with values in the space of smooth functions on the fiber $\mathcal{F}_0(M, \omega)$. It assigns to every $\omega \in \mathcal{H}_0(M)$ and every pair $\tilde{\omega}_1, \tilde{\omega}_2$ of closed 2-forms on $M$ the Hamiltonian function $\mathcal{H}_{\omega, \tilde{\omega}_1, \tilde{\omega}_2} : \mathcal{F}_0(M, \omega) \to \mathbb{R}$ given by

$$\mathcal{H}_{\omega, \tilde{\omega}_1, \tilde{\omega}_2} : \mathcal{F}_0(M, \omega) \to \mathbb{R}$$

$$= -\Omega_J(\mathcal{A}_{\omega,J}(\tilde{\omega}_1), \mathcal{A}_{\omega,J}(\tilde{\omega}_2))$$

$$= \frac{1}{2} \int_M (\iota(J)(\tilde{\omega}_1 - d\tilde{\lambda}_1)) \wedge \tilde{\omega}_2 \wedge \omega^{n-2} (n-2)!$$

(4.18)

for $J \in \mathcal{F}_{\text{int}}(M, \omega)$ with $\text{Ric}_{\omega,J} = 0$, where the 1-form $\tilde{\lambda}_1 \in \Omega^1(M)$ is chosen such that $d\star_J(\tilde{\omega}_1 - d\tilde{\lambda}_1) = 0$. (See the proof for the notations $\star_J$ and $\iota(J)\tilde{\omega}$.) The Hamiltonian vector field on $\mathcal{F}_0(M, \omega)$ generated by this function is the vertical part of the Lie bracket of the horizontal lifts of two vector fields on $\mathcal{F}_0(M)$ that take the values $\tilde{\omega}_i$ at $\omega$ (see 28 Lemma 6.4.8).
Proof. We prove uniqueness. Thus assume that \( \hat{J} \in \Omega^{0,1}_J(M, TM) \) satisfies (4.14), (4.15), and (4.16) with \( \hat{\omega} = 0 \). Then

\[
\bar{\partial}_J \hat{J} = 0, \quad \Lambda_{\rho}(J, \hat{J}) = 0, \quad \hat{J} = \hat{J}^*.
\]

Now let \( \check{J}' \in \Omega^{0,1}_J(M, TM) \) be any infinitesimal almost complex structure such that \( \bar{\partial}_J \check{J}' = 0 \). Then there exists a vector field \( v' \in \text{Vect}(M) \) such that

\[
\bar{\partial}_J^{*}(\check{J}' - \mathcal{L}_{v'} J) = 0.
\]

Thus the section

\[
(\check{J}' - \mathcal{L}_{v'} J)^{+} := \frac{1}{2}(\check{J}' - \mathcal{L}_{v'} J) + \frac{i}{2}(\check{J}' - \mathcal{L}_{v'} J)^{*}
\]

of the endomorphism bundle is self-adjoint and satisfies \( \bar{\partial}_J(\check{J}' - \mathcal{L}_{v'} J)^{+} = 0 \) by Lemma 3.10 Hence \( \Omega_J(\check{J}', (\check{J}' - \mathcal{L}_{v'} J)^{+}) = 0 \) by (4.16). By Theorem 4.2 this implies

\[
\Omega_J(\check{J}, \check{J}') = \Omega_J(\check{J}, \check{J}' - \mathcal{L}_{v'} J) = \Omega_J(\check{J}, (\check{J}' - \mathcal{L}_{v'} J)^{+}) = 0.
\]

Here the third equality holds because \( \check{J} = \check{J}^* \). Since \( \Omega_J \) descends to a non-degenerate 2-form on the quotient space \( \text{ker} \bar{\partial}_J / \text{im} \bar{\partial}_J \), there exists a vector field \( v_0 \in \text{Vect}(M) \) such that

\[
\mathcal{L}_{v_0} J = 0.
\]

Thus

\[
d d^* F \circ J + d d^* G = \Lambda_{\rho}(J, \mathcal{L}_{v F + Jv G} J) = \Lambda_{\rho}(J, \mathcal{L}_{v} J) = \Lambda_{\rho}(J, \check{J}) = 0
\]

by Corollary 3.8 hence \( F \) and \( G \) are constant, so \( \check{J} = \mathcal{L}_{v_0} J = 0 \). This proves uniqueness.

To prove existence, assume first that \( \hat{\omega} = d_l (v) \omega \), where \( v \in \text{Vect}(M) \) satisfies \( d_l (Jv) \rho = 0 \). Define \( \check{J} := \mathcal{L}_v J \). Then \( \bar{\partial}_J \check{J} = 0 \) and \( \Lambda_{\rho}(J, \check{J}) = -d f_v \circ J \) by Lemma 3.5 Since

\[
\langle \hat{\omega}, \omega \rangle = \hat{\omega} \wedge \frac{\omega^{n-1}}{(n-1)!} = d_l(v) \rho = f_v \rho,
\]

this shows that \( \check{J} \) satisfies (4.14). Moreover, it follows from Theorem 4.2 that \( \Omega_J(\check{J}, \check{J}') = 0 \) for all \( \check{J}' \in \Omega^{0,1}_J(M, TM) \) with \( \bar{\partial}_J \check{J}' = 0 \). Thus \( \check{J} \) satisfies (4.16). That \( \check{J} \) also satisfies (4.15) follows from part (i) of Lemma 3.6.
exists a unique 1-form solution of (4.14), (4.15), and (4.16). Thus \( \bar{\omega} = 0 \) is the unique solution of (4.14), (4.15), and (4.16).

Third, assume \( \hat{\omega} \) is a closed 2-form such that \( \hat{\omega}^{1,1} = 0 \). Then there exists a unique 1-form \( \hat{\omega} \in \Omega^1(M) \) such that \( (Ju, u') = \hat{\omega}(u, u') \) for all \( u, u' \in \text{Vect}(M) \). Thus \( \bar{\omega} = 0 \) and so \( \bar{\omega} \) satisfies (4.15) and (4.16). Moreover \( \nabla \bar{\omega} = 0 \) by Lemma 3.11 and so \( \bar{\omega} = 0 \). Thus \( \bar{\omega} \) also satisfies (4.14).

To prove existence in general, let \( \hat{\omega} \in \Omega^2(M) \) be any closed 2-form, choose a 1-form \( \hat{\lambda} \in \Omega^1(M) \) such that \( d^* \hat{\lambda} = 0 \) and \( d^* (\hat{\omega} - d \hat{\lambda}) = 0 \), and choose a vector field \( v \in \text{Vect}(M) \) such that \( \iota(v) \omega = \hat{\lambda} \). Then \( d \iota(Jv) = d^* \iota(v) \omega = 0 \) by (3.17). Moreover \( \hat{\omega} - d \hat{\lambda} \) is harmonic and so is \( \hat{\omega}_0 := \hat{\omega} - d \hat{\lambda} - (\hat{\omega} - d \hat{\lambda})^{1,1} \). Thus there exists a unique \( \hat{\omega}_0 \in \Omega^0(M) \) such that \( (J_0 u, u') = \hat{\omega}_0(u, u') \) for all \( u, u' \in \text{Vect}(M) \), and it follows from the above that \( \hat{\omega} := \hat{\omega}_0 + \mathcal{L}_v J \) satisfies (4.14), (4.15), and (4.16). This proves parts (i) and (ii). Part (iii) follows by combining the uniqueness statement in part (i) with part (v) of Theorem 2.6 and part (ii) of Theorem 4.2.

To prove part (iv) we must verify the second equality in (4.18). Fix a symplectic form \( \omega \in \Omega^2(M) \) with real first Chern class zero, denote by \( \rho := \omega^n/n! \) its volume form, and let \( \hat{\omega}_1, \hat{\omega}_2 \in \Omega^2(M) \) be closed. Let \( J \) be a complex structure on \( M \) that is compatible with \( \omega \) and satisfies \( \text{Ric}_{\rho, J} = 0 \), and denote by \( *_{\rho, J} \) the Hodge \( * \)-operator associated to the Kähler metric \( \langle \cdot, \cdot \rangle = \omega(\cdot, J \cdot) \). Then there exist 1-forms \( \hat{\lambda}_1, \hat{\lambda}_2 \in \Omega^1(M) \) such that

\[
d *_{\rho, J} (\hat{\omega}_i - d \hat{\lambda}_i) = 0, \quad d *_{\rho, J} \hat{\lambda}_i = 0 \tag{4.19}
\]

for \( i = 1, 2 \). Choose vector fields \( v_1, v_2 \in \text{Vect}(M) \) such that \( \iota(v_i) \omega = \hat{\lambda}_i \) for \( i = 1, 2 \). Then \( d \iota(Jv_i) = 0 \) by (3.17) and the second equation in (4.19). Hence it follows from the explicit formula in the proof of part (i) that

\[
\mathcal{L}_v J = \hat{\omega}_i, \quad 2 \langle \hat{\omega}_i - d \hat{\lambda}_i \rangle = (\hat{\omega}_i - d \hat{\lambda}_i)(u, u') - (\hat{\omega}_i - d \hat{\lambda}_i)(Ju, Ju'), \tag{4.20}
\]

for \( i = 1, 2 \) and \( u, u' \in \text{Vect}(M) \). We will use the equation

\[
\hat{\omega}_i^{2,0} = \epsilon (\hat{\omega} - J^* \hat{\omega}) - \frac{1}{4} \epsilon(J \hat{\omega}), \quad \epsilon(J \hat{\omega})(u, u') := \hat{\omega}(Ju, Ju'), \quad \epsilon(\iota(J \hat{\omega}))(u, u') := \hat{\omega}(Ju, u') + \hat{\omega}(u, Ju'). \tag{4.21}
\]

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Since $\Lambda_p(J, \tilde{J}_t) = 0$ and $f_{Jv_t} = 0$, equation (4.20) yields
\[
\mathcal{H}_{\omega_1, \omega_2}(J) = -\Omega(J + L_{v_1} J, \tilde{J}_2 + L_{v_2} J) = -\frac{1}{2} \int_M \text{trace}(\tilde{J}_1 J \tilde{J}_2) \rho = -4\text{Im} \left( \int_M c_n(\beta_1 \wedge \beta_2) \right). \tag{4.22}
\]
Here we choose a Hermitian line bundle $L \to M$, a Hermitian connection $\nabla_L$ on $L$, and a section $\theta \in \Omega^0_{\mathbb{C}}(M, L)$ satisfying $d\nabla_L \theta = 0$ and $c_n(\theta \wedge \theta) = \rho$, and choose $\beta_i \in \Omega^{n-1,1}_\mathbb{C}(M, L)$ such that the pair $(\tilde{J}_i, \beta_i)$ satisfies (B.3). Then the last equality in (4.22) follows from part (iii) of Theorem 4.2.

Since $\tilde{J}_i$ is skew-adjoint, part (iii) of Lemma B.2 asserts that there exists an $\alpha_i \in \Omega^{n-2,0}_\mathbb{C}(M, L)$ such that $\omega \wedge \alpha_i = \beta_i$. Then $c_n \alpha_i = \frac{1}{2} * (\omega \wedge \beta_i)$ and $\omega \wedge \beta_i + (\tilde{\omega}_i - \tilde{d}\tilde{\lambda}_i) \wedge \theta = 0$ by Lemma 3.11. Hence
\[
c_n(\beta_1 \wedge \beta_2) = \frac{1}{2} ((\omega \wedge \beta_1) \wedge * (\omega \wedge \beta_2)) = \frac{1}{2} (\tilde{\omega}_1 - \tilde{d}\tilde{\lambda}_1)_{\mathbf{J}}^{2.0} \wedge * (\tilde{\omega}_2 - \tilde{d}\tilde{\lambda}_2)_{\mathbf{J}}^{0.2} = \frac{1}{2} (\tilde{\omega}_1 - \tilde{d}\tilde{\lambda}_1)_{\mathbf{J}}^{2.0} \wedge (\tilde{\omega}_2 - \tilde{d}\tilde{\lambda}_2)_{\mathbf{J}}^{0.2} \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]
Since $(\tilde{\omega}_1 - \tilde{d}\tilde{\lambda}_1)_{\mathbf{J}}^{2.0}$ is closed, it now follows from (4.21) and (4.22) that
\[
\mathcal{H}_{\omega_1, \omega_2}(J) = -2\text{Im} \left( \int_M (\tilde{\omega}_1 - \tilde{d}\tilde{\lambda}_1)_{\mathbf{J}}^{2.0} \wedge (\tilde{\omega}_2 - \tilde{d}\tilde{\lambda}_2)_{\mathbf{J}}^{0.2} \wedge \frac{\omega^{n-2}}{(n-2)!} \right)
\]
\[
= \frac{1}{2} \int_M (\iota(J)(\tilde{\omega}_1 - \tilde{d}\tilde{\lambda}_1)) \wedge \tilde{\omega}_2 \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]
This proves (4.18). The right hand side of (4.18) depends only on the cohomology classes of $\tilde{\omega}_1$ and $\tilde{\omega}_2$. Hence it is invariant under the action of $\text{Diff}_0(M) \cap \text{Symp}(M, \omega)$ on $J$, because $\phi^* \tilde{\omega}_i - \tilde{\omega}_i$ is exact for $\phi \in \text{Diff}_0(M)$. Thus it descends to a function on $\mathcal{F}_0(M, \omega)$. This proves Theorem 4.3.

The connection on $\mathcal{F}_0(M)$ is determined by the differential equation
\[
\partial_t J_t = \mathcal{K}_{\omega_t, J_t}(\partial_t \omega_t). \tag{4.23}
\]
The solutions of (4.23) are $\text{Diff}(M)$-equivariant in the sense that, if $t \mapsto \phi_t$ is a smooth isotopy of $M$ and $t \mapsto J_t$ is a horizontal lift of a path $t \mapsto \omega_t$ in $\mathcal{F}_0(M)$, then $t \mapsto \phi^*_t J_t$ is a horizontal lift of the path $t \mapsto \phi^*_t \omega_t$. The solutions of (4.23) may not exist for all time, because the fibers $\mathcal{F}_0(M, \omega_t)$ are noncompact. Wherever defined, they determine symplectomorphisms between the fibers along the path $t \mapsto \omega_t$ by [28, Lemma 6.3.5].
A Torsion-free connections

Let $M$ be an oriented $2n$-manifold. We prove that a nondegenerate 2-form on $M$ is preserved by a torsion-free connection if and only if it is closed, and that an almost complex structure on $M$ is preserved by a torsion-free connection if and only if it is integrable. We use the sign conventions

$$[\mathcal{L}_u, \mathcal{L}_v] + \mathcal{L}_{[u,v]} = 0$$

for the Lie bracket and

$$N_J(u, v) = [u, v] + J[J u, v] + J[u, J v] - [J u, J v] \quad (A.1)$$

for the Nijenhuis tensor. If $\nabla$ is a torsion-free connection on $TM$ then

$$N_J(u, v) = (\nabla_u J) v + (\nabla_J u) v - (\nabla_v J) J u - (\nabla_J v) J u. \quad (A.2)$$

Lemma A.1. Let $M$ be a $2n$-manifold.

(i) An almost complex structure $J$ is integrable if and only if there exists a torsion-free connection $\nabla$ on $TM$ such that $\nabla J = 0$. If $J$ is integrable and $\rho \in \Omega^{2n}(M)$ is a volume form inducing the same orientation as $J$ then there exists a torsion-free connection $\nabla$ on $TM$ such that $\nabla \rho = 0$ and $\nabla J = 0$.

(ii) A nondegenerate 2-form $\omega \in \Omega^2(M)$ is closed if and only if there exists a torsion-free connection $\nabla$ on $TM$ such that $\nabla \omega = 0$.

Proof. We prove part (i). If $\nabla$ is a torsion-free connection with $\nabla J = 0$ it follows directly from (A.2) that $N_J = 0$. Conversely suppose $J$ is integrable and let $\rho$ be a volume form on $M$ inducing the same orientation as $J$. Fix a background metric $g$ on $M$. Then $g_J := g + J^* g$ is a metric with respect to which $J$ is skew-adjoint, and if $d\text{vol}_J \in \Omega^{2n}(M)$ is the volume form of this metric then the metric $g_{\rho, J} := (\rho / d\text{vol}_J)^{1/n} g_J$ has the volume form $\rho$. Let $\widehat{\nabla}$ be the Levi-Civita connection of the metric $g_{\rho, J}$. Then $\widehat{\nabla}$ is torsion-free and $\widehat{\nabla} \rho = 0$. Let $\alpha(u) := \frac{1}{2} \text{trace}(J(\nabla J)u)$ and define

$$\widehat{\nabla}_u v := \nabla_u v - \frac{1}{2} J(\nabla_u J) v - \frac{1}{4} J(\nabla_v J) u - \frac{1}{4}(\nabla_{J v} J) u + \frac{\alpha(u) v + \alpha(v) u - \alpha(J u) J v - \alpha(J v) J u}{2n + 2}. \quad (A.3)$$

Then $\widehat{\nabla} \rho = 0$, $\widehat{\nabla} J = 0$, and a calculation shows that $\text{Tor} \widehat{\nabla} = -\frac{1}{4} N_J$, so $\widehat{\nabla}$ is torsion-free if and only if $J$ is integrable. This proves (i).
We prove part (ii). If $\nabla$ torsion-free and $\nabla \omega = 0$ then
\[
d\omega(u, v, w) = \mathcal{L}_u(\omega(v, w)) + \mathcal{L}_v(\omega(w, u)) + \mathcal{L}_w(\omega(u, v)) \\
+ \omega([v, w], u) + \omega([w, u], v) + \omega([u, v], w) \\
= \omega([v, w] - \nabla_v w + \nabla_w u, u) + \omega([w, u] - \nabla_w u + \nabla_w v, u) \\
+ \omega([u, v] - \nabla_u v + \nabla_u w, v) = 0.
\]

Conversely, suppose $\omega$ is a symplectic form and choose an almost complex structure $J$ on $M$ that is compatible with $\omega$, so $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\nabla$ be its Levi-Civita connection. Then
\[
\langle (\nabla_u J)v, w \rangle + \langle (\nabla_v J)w, u \rangle + \langle (\nabla_w J)u, v \rangle = d\omega(u, v, w) = 0 \quad (A.4)
\]
for all $u, v, w \in \text{Vect}(M)$ by [28, Lemma 4.1.14]. Define
\[
\mathcal{W}_u v := \nabla_u v + A(u)v, \quad A(u)v := -\frac{1}{3} J(\langle \nabla_u J \rangle v + (\nabla_v J)u).
\]
This connection is torsion-free and satisfies $JA(u) + A(u)^* J = \nabla_u J$ for every vector field $u \in \text{Vect}(M)$ by a straight forward calculation. Hence
\[
\omega(\mathcal{W}_u v, w) + \omega(v, \mathcal{W}_u w) = \langle J\nabla_u v + JA(u)v, w \rangle + \langle Jv, \nabla_w A(u)w \rangle \\
= \langle JA(u) + A(u)^* J \rangle v, w \rangle + \langle Jv, \nabla_v w \rangle + \langle Jv, \nabla_w w \rangle \\
= \langle (\nabla_u J)v, w \rangle + \langle J\nabla_v w, v \rangle + \langle Jv, \nabla_w w \rangle \\
= \mathcal{L}_u(\langle Jv, w \rangle) = \mathcal{L}_u(\omega(v, w))
\]
for all $u, v, w \in \text{Vect}(M)$. This proves Lemma A.1. \qed

**Lemma A.2.** Let $M$ be an oriented $2n$-manifold, let $\rho \in \Omega^{2n}(M)$ be a positive volume form, let $J \in \mathcal{J}_{\text{int}}(M)$ be a complex structure compatible with the orientation, and let $\nabla$ be a torsion-free $\rho$-connection such that $\nabla J = 0$. Then $\text{trace}(JR^\nabla)$ is a $(1, 1)$-form.

**Proof.** Since $\nabla$ is torsion-free, $R^\nabla$ satisfies the first Bianchi identity. Thus
\[
R(u, v)w + JR(Ju, v)w + JR(u, Jv)w = R(Ju, Jv)v \\
= R(u, v)w + JR(Ju, v)w + JR(u, Jv)w + R(Jv, w)Ju + R(w, Ju)Jv \\
= R(u, v)w + JR(Ju, v)w + JR(u, Jv)w + JR(u, Jv)w + JR(Jv, w)u \\
= R(u, v)w - JR(v, w)Ju - JR(w, u)Jv \\
= R(u, v)w + R(v, w)u + R(w, u)v = 0
\]
and so $JR(u, v) - R(Ju, v) - R(u, Jv) - JR(Ju, Jv) = 0$. Take the trace to obtain $\text{trace}(JR(u, v)) = \text{trace}(JR(Ju, Jv))$. This proves Lemma A.2. \qed
B Complex structures and n-forms

Fix a closed connected oriented 2\(n\)-manifold \(M\) and a complex line bundle \(L \to M\) with a Hermitian form \(\langle s_1, s_2 \rangle\) for \(s_1, s_2 \in \Omega^0(M, L)\) (complex anti-linear in the first variable and complex linear in the second variable). Define
\[
c_n := (-1)^{\frac{n(n+1)}{2}} i^n = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
-i, & \text{if } n \text{ is odd}.
\end{cases} \tag{B.1}
\]

Lemma B.1. Let \(J \in \mathcal{J}(M)\) be an almost complex structure compatible with the orientation. Then \(c_1(TM, J) = c_1(L) \in H^2(M; \mathbb{Z})\) if and only if there exists a nowhere vanishing \(n\)-form \(\theta \in \Omega^n(M, L)\). If this holds then
\[
\rho := c_n \langle \theta \wedge \theta \rangle \in \Omega^{2n}(M) \tag{B.2}
\]
is a positive volume form on \(M\).

Proof. The first Chern class of \((TM, J)\) agrees with minus the first Chern class of the complex line bundle \(\Lambda^{n,0}_J T^* M\). Hence \(c_1(TM, J) = c_1(L)\) if and only if \(E := \Lambda^{n,0}_J T^* M \otimes L\) admits a trivialization or, equivalently, a nowhere vanishing section, and such a section is an \((n,0)\)-form \(\theta \in \Omega^n(M, L)\).

To show that, for any nowhere vanishing \(n\)-form \(\theta \in \Omega^n(M, L)\), the formula \((B.2)\) defines a positive volume form on \(M\), fix an element \(m \in M\) and choose a complex isomorphism \((\mathbb{C}^n, i) \to (T_m M, J)\). Let \(z_i = x_i + i y_i\) for \(i = 1, \ldots, n\) be the coordinates on \(\mathbb{C}^n\). Then there is an element \(\lambda \in L_m\) (the fiber of \(L\) over \(m\)) such that
\[
\theta_m = \lambda dz_1 \wedge \cdots \wedge dz_n.
\]

Hence
\[
\rho_m = c_n \langle \theta_m \wedge \theta_m \rangle = \frac{(-1)^{\frac{n(n-1)}{2}}}{i^n} |\lambda|^2 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n = 2^n |\lambda|^2 dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.
\]

Thus \(\rho\) is a positive volume form on \(M\) and this proves Lemma B.1. \(\square\)
Lemma B.2. Let $J \in \mathcal{J}(M)$ be an almost complex structure compatible with the orientation, let $\theta \in \Omega^0_J(M, L)$ be a nowhere vanishing $(n,0)$-form, let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form that is compatible with $J$ such that

$$\frac{\omega^n}{n!} = c_n(\theta \wedge \theta) =: \rho, \quad (B.3)$$

and let $*: \Omega^p_J(M, L) \to \Omega^{n-p}_J(M, L)$ be the Hodge $*$-operator of the Riemannian metric $\langle \cdot, \cdot \rangle := \omega(\cdot, J \cdot)$. Then the following holds.

(i) For every $\hat{\beta} \in \Omega^{0,1}_J(M, TM)$ there is a unique $\beta \in \Omega^{n-1,1}_J(M, L)$ such that

$$i_u(\beta) = i(J_u)\beta$$

for all $u \in \text{Vect}(M)$.

(ii) For every $\beta \in \Omega^{n-1,1}_J(M, L)$ there exists a unique $\hat{\beta} \in \Omega^{0,1}_J(M, TM)$ such that $[B.4]$ holds for all $u \in \text{Vect}(M)$.

(iii) Suppose $\beta \in \Omega^{n-1,1}_J(M, L)$ and $\hat{\beta} \in \Omega^{0,1}_J(M, TM)$ satisfy equation $[B.4]$. Then

$$i(u)\hat{\beta} - i(J_u)\beta = -c_n(i(J_u)\theta) \quad (B.5)$$

for all $u \in \text{Vect}(M)$. Moreover,

$$\hat{J} = \hat{\beta}^* \iff \beta = c_n\beta \iff \beta \wedge \omega = 0, \quad (B.6)$$

$$\hat{J} + \hat{\beta}^* = 0 \iff \beta = c_n\beta \iff \beta \in \Omega^{n-2,0}_J(M, L) \wedge \omega. \quad (B.7)$$

(iv) Suppose $\beta \in \Omega^p_J(M, L)$ and $\hat{\beta} \in \Omega^{0,1}_J(M, TM)$ satisfy equation $[B.4]$ and let $\hat{\omega} \in \Omega^2(M)$. Then

$$\omega \wedge \beta + \hat{\omega} \wedge \theta = 0 \iff \hat{\omega}(u, v) - \omega(J_u, Jv) = \langle (\hat{\beta} - \hat{\beta}^*)u, v \rangle$$

for all $u, v \in \text{Vect}(M)$. \quad (B.8)

(v) Let $\beta, \beta' \in \Omega^{n-1,1}_J(M, L)$ and $\hat{\beta}, \hat{\beta}' \in \Omega^{0,1}_J(M, TM)$ be given such that the pairs $(\beta, \hat{\beta})$ and $(\beta', \hat{\beta}')$ satisfy $[B.4]$. Then the pointwise inner product of $\beta$ and $\beta'$ is given by

$$\langle \beta, \beta' \rangle = \text{Re} \left( \frac{\langle \beta \wedge \beta' \rangle}{\rho} \right) = \frac{1}{8} \text{trace}(\hat{\beta} \hat{\beta}') \rho. \quad (B.9)$$

Moreover, we have

$$c_n(\beta \wedge \beta') = -\frac{1}{8} \text{trace}(\hat{\beta} \hat{\beta}') \rho + \frac{1}{8} \text{trace}(\hat{J} \hat{\beta}' \hat{\beta}) \rho. \quad (B.10)$$

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Proof. Define $\beta \in \Omega^n(M, L)$ by

$$
\beta(v_1, \ldots, v_n) := \theta(-\frac{1}{2} J \hat{J} v_1, v_2, \ldots, v_n) \\
+ \theta(v_1, -\frac{1}{2} J \hat{J} v_2, v_3, \ldots, v_n) \\
+ \cdots + \theta(v_1, \ldots, v_{n-1}, -\frac{1}{2} J \hat{J} v_n)
$$

for $v_1, \ldots, v_n \in \text{Vect}(M)$. Then

$$
\beta(Ju, v_2, \ldots, v_n) + \theta(Ju, v_2, \ldots, v_n) = \theta(\frac{1}{2} \hat{J} u, v_2, \ldots, v_n) \\
+ \theta(Ju, \frac{1}{2} \hat{J} v_2, v_3, \ldots, v_n) \\
+ \cdots + \theta(Ju, v_2, \ldots, v_{n-1}, \frac{1}{2} \hat{J} v_n)
$$

for all $u, v_2, \ldots, v_n \in \text{Vect}(M)$. Thus $\beta$ is an $(n-1,1)$-form that satisfies equation (B.4). If $\beta'$ is another $(n-1,1)$-form that satisfies equation (B.4), then $\iota(Ju)(\beta' - \beta) = \iota(u)(\beta' - \beta)$, thus $\beta' - \beta \in \Omega^{n,0}(M, L)$, and so $\beta' = \beta$. This proves (i).

We prove part (ii). Thus let $\beta \in \Omega^{n-1,1}(M, L)$ be given. Then for every vector field $u \in \text{Vect}(M)$ the $(n-1)$-form $\iota(u)\beta - \iota(Ju)\beta$ is of type $(n-1,0)$ and hence can be written in the form $\iota(v)\theta$ for some vector field $v \in \text{Vect}(M)$ that is uniquely determined by $u$. This shows that there exists a unique section $\hat{J} \in \Omega^0(M, \text{End}(TM))$ of the endomorphism bundle that satisfies (B.4) for all $u \in \text{Vect}(M)$. By (B.4) we have

$$
\iota(\hat{J}Ju)\theta = \iota(\hat{J}u)\beta + \iota(u)\beta = -\iota(\hat{J}u)\theta = -\iota(J\hat{J}u)\theta
$$

for all $u \in \text{Vect}(M)$ and thus $\hat{J}J + J\hat{J} = 0$. This proves (ii).

We prove part (iii). It suffices to consider the trivial line bundle and the standard structures on $\mathbb{R}^{2n}$ with the coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. They are given by

$$
J = \left( \begin{array}{cc} 0 & -\mathbb{1} \\
\mathbb{1} & 0 \end{array} \right), \quad \omega = \sum_{i=1}^n dx_i \wedge dy_i, \quad \theta = \frac{dz_1}{\sqrt{2}} \wedge \cdots \wedge \frac{dz_n}{\sqrt{2}};
$$

(B.11)

where $z_i := x_i + iy_i$ for $i = 1, \ldots, n$. A complex anti-linear endomorphism has the form

$$
\hat{J} = \left( \begin{array}{cc} A & B \\
B & -A \end{array} \right), \quad A + iB = (a_{ij})_{i,j=1,\ldots,n} \in \mathbb{C}^{n \times n}.
$$

(B.12)
The corresponding \((n - 1, 1)\)-form \(\beta \in \Omega_{n-1,1}^n(\mathbb{R}^{2n})\) is given by
\[
\beta = \frac{1}{2i} \sum_{i,j=1}^{n} a_{ij} \frac{dz_i}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_{i-1}}{\sqrt{2}} \wedge \frac{dz_j}{\sqrt{2}} \wedge \frac{d\bar{z}_{i+1}}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_n}{\sqrt{2}}. \tag{B.13}
\]

Now \(\hat{J}^*\) is represented by the transposed matrix \(A^T + iB^T = (a_{ji})_{i,j=1,\ldots,n}\) and
\[
*\beta = -\frac{c_n}{4} \sum_{i,j=1}^{n} a_{j,i} \frac{dz_i}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_{i-1}}{\sqrt{2}} \wedge \frac{dz_j}{\sqrt{2}} \wedge \frac{d\bar{z}_{i+1}}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_n}{\sqrt{2}}
\]
This proves (B.5). Now (B.6) and (B.7) follow from (B.5) and the eigenspace decomposition of the Hodge \(*\)-operator on \(\Omega^{n-1,1}(M)\). This proves (iii).

We prove part (iv). Continue the notation in the proof of part (iii), so \(J, \omega, \theta, \hat{J}, \beta\) are as in (B.11), (B.12), and (B.13). Then a 2-form \(\hat{\omega} \in \Omega^2(M)\) satisfies \(\hat{\omega}(u,v) = \hat{J}(Ju,Jv) = \langle J^* - \hat{J}^* \rangle u,v\) for all \(u,v \in \text{Vect}(M)\) and only if its \((0,2)\)-part is given by \(\hat{\omega}^{0,2} = -\frac{1}{4} \sum_{i,j} a_{ij} d\bar{z}_i \wedge d\bar{z}_j\), and this in turn is equivalent to the equation \(\hat{\omega} \wedge \theta = -\omega \wedge \beta\). This proves (iv).

We prove part (v). Continue the notation in the proof of part (iii) and use the same notation for \((\beta', \hat{J}')\) with \(A, B, a_{ij}\) replaced by \(A', B', a'_{ij}\). Then
\[
\overline{\beta} \wedge *\beta' = -\frac{c_n}{4} \sum_{i,j=1}^{n} a_{ij} a'_{jk} \frac{d\bar{z}_1}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_{i-1}}{\sqrt{2}} \wedge \frac{dz_j}{\sqrt{2}} \wedge \frac{d\bar{z}_{i+1}}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_n}{\sqrt{2}}
\]
\[
\wedge \frac{dz_1}{\sqrt{2}} \wedge \cdots \wedge \frac{dz_k-1}{\sqrt{2}} \wedge \frac{dz_k}{\sqrt{2}} \wedge \frac{d\bar{z}_{k+1}}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_n}{\sqrt{2}}
\]
\[
= c_n \sum_{i,j=1}^{n} a_{ij} a'_{ji} \frac{d\bar{z}_1}{\sqrt{2}} \wedge \cdots \wedge \frac{d\bar{z}_{n}}{\sqrt{2}} \wedge \frac{dz_1}{\sqrt{2}} \wedge \cdots \wedge \frac{dz_n}{\sqrt{2}}
\]
\[
= \frac{1}{4} \sum_{i,j=1}^{n} a_{ij} a'_{ji} c_n \bar{\theta} \wedge \theta
\]
\[
= \frac{1}{4} \text{trace}(A - iB)^T (A' + iB') \rho.
\]
Thus \(\text{Re}(\overline{\beta} \wedge *\beta') = \frac{1}{4} \text{trace}(A^T A' + B^T B') \rho = \frac{1}{8} \text{trace}(\hat{J}^T \hat{J}') \rho\) and this proves equation (B.9). Moreover, the pair \((c_n \beta, *\beta)\) satisfies (B.4) by part (iii). Thus \(\text{Re}(c_n \beta \wedge \beta') = \text{Re}(c_n *\beta \wedge \beta') = -\frac{1}{8} \text{trace}(\hat{J}^T \hat{J}') \rho\). This confirms (B.10) for the real part. The formula for the imaginary part holds because both sides of the equation are complex linear in \(\hat{J}'\) with respect to the complex structure \(\hat{J}' \mapsto J\hat{J}'\). This proves (v) and Lemma B.2. \(\square\)
The next lemma adapts an observation by Donaldson in [13, Lemma 1] to the present setting.

**Lemma B.3.** Let \( \rho \) be a positive volume form and let \( J \in J(M) \) be a positive almost complex structure such that \( c_1(TM, J) = c_1(L) \in H^2(M; \mathbb{Z}) \). Then the following are equivalent.

(i) \( J \) is integrable.

(ii) There exists a nowhere vanishing \( n \)-form \( \theta \in \Omega^n(M, L) \) and a Hermitian connection \( \nabla_L \) on \( L \) such that \( d\nabla_L \theta = 0 \) and \( c_n(\theta \wedge \theta) = \rho \).

If (i) holds then the pair \((\nabla_L, \theta)\) in (ii) is uniquely determined by \( J \) up to unitary gauge equivalence. If (ii) holds then

\[
(F^\nabla)^{0,2} = 0, \quad \text{Ric}_{\rho, J} = iF^\nabla, \quad \tag{B.14}
\]

and there exists a torsion-free connection \( \nabla \) on \( TM \) that satisfies \( \nabla \theta = 0 \) and \( \nabla J = 0 \).

**Proof.** We prove that (i) implies (ii). By Lemma B.3 there exists a nowhere vanishing \((n,0)\)-form \( \theta \in \Omega^{n,0}(M, L) \) such that \( c_n(\theta \wedge \theta) = \rho \). Choose any Hermitian connection \( \nabla_0 \) on \( L \). Then \( d\nabla_0 \theta \in \Omega^{n,1}(M) \) because \( J \) is integrable and hence there exists a unique 1-form \( \eta \in \Omega^{0,1}(M) \) such that \( \eta \wedge \theta = d\nabla_0 \theta \). Define the Hermitian connection \( \nabla_L \) by

\[
\nabla_L := \nabla_0 + \eta - \bar{\eta}. \tag{1}
\]

because \( \bar{\eta} \in \Omega^{1,0}_J(M) \). This shows that (i) implies (ii). Moreover, (ii) implies

\[
(F^{\nabla_L})^{0,2} \wedge \theta = F^{\nabla_L} \wedge \theta = d^{\nabla_L}d^{\nabla_L} \theta = 0
\]

and hence \((F^{\nabla_L})^{0,2} = 0\).

We prove uniqueness in (ii). If \((\theta', \nabla_L')\) is any other pair as in (ii) then there exists a unique unitary transformation \( g : M \to S^1 \) such that

\[
\theta' = g^{-1} \theta
\]

Hence the 1-form \( \alpha := \nabla_L' - \nabla_L \in \Omega^1(M, \mathbb{i} \mathbb{R}) \) satisfies

\[
0 = d^{\nabla_L'} \theta' = d^{\nabla_L + \alpha}(g^{-1} \theta) = \alpha \wedge g^{-1} \theta + dg^{-1} \wedge \theta = (\alpha^{0,1} - g^{-1} \bar{\alpha} \bar{g}) \wedge g^{-1} \theta.
\]

Hence \( \alpha^{0,1} = g^{-1} \bar{\alpha} \bar{g} \) and so \( \alpha = g^{-1} \bar{\alpha} \bar{g} - \bar{g}^{-1} \partial_J \bar{g} = g^{-1}dg \) because \( g^{-1}dg \) is a 1-form on \( M \) with values in \( \mathbb{i} \mathbb{R} \). Thus

\[
\nabla_L' = \nabla_L + g^{-1}dg = g^* \nabla_L
\]

and this proves uniqueness up to unitary gauge equivalence.
We prove that (ii) implies (i). If $\theta \in \Omega^{n,0}_J(M)$ and $\nabla_L$ is any complex connection on $L$ then $(d\nabla_L J)^n - 1 = \frac{1}{4}(N_J)\theta$, where

\[
(\iota(N_J)\theta)(v_1, \ldots, v_{n+1}) := \sum_{i<j} (-1)^{i+j-1} \theta(N_J(v_i, v_j), v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{n+1})
\]

(B.15)

for $v_1, \ldots, v_{n+1} \in \text{Vect}(M)$. If $d\nabla_L J = 0$ it follows that $\iota(N_J)\theta = 0$. If $\theta$ vanishes nowhere this implies $N_J = 0$. To see this, fix two vector fields $v_1, v_2$. Then $\iota(N_J(v_1, v_2))\theta$ is a nonzero $(n-1, 0)$-form while the remaining summands on the right in (B.15) are of type $(n-2, 1)$ or $(n-3, 2)$. This implies that $\iota(N_J(v_1, v_2))\theta = 0$ and hence $N_J(v_1, v_2) = 0$ because $\theta$ vanishes nowhere. Thus $N_J = 0$ and therefore $J$ is integrable.

Next we prove that under the assumption (ii) there exists a torsion-free connection $\nabla$ on $M$ that satisfies $\nabla J = 0$ and $\nabla \theta = 0$. To see this, let $\nabla_0$ be any torsion-free connection on $TM$ that satisfies $\nabla_0 J = 0$ and define the 1-form $\alpha \in \Omega^1(M, \mathbb{C})$ by

\[
\alpha(u)\theta(v_1, \ldots, v_n) := (\nabla_{0,u}\theta)(v_1, \ldots, v_n) = \nabla_{L,u}(\theta(v_1, \ldots, v_n)) - \theta(\nabla_u v_1, v_2, \ldots, v_n) - \cdots - \theta(v_2, \ldots, v_{n-1}, \nabla_u v_n)
\]

for $u, v_1, \ldots, v_n \in \text{Vect}(M)$. Define $\beta \in \Omega^1(M, \mathbb{C})$ by

\[
\beta(u) := \frac{(n+2)\alpha(u) + \im \alpha(Ju)}{4n+4}
\]

for $u \in \text{Vect}(M)$ and define $A \in \Omega^1(M, \text{End}(TM))$ by

\[
A(u)v := \beta(u)v + \beta(v)u - \beta(Ju)Jv - \beta(Jv)Ju
\]

for $u, v \in \text{Vect}(M)$. Then $A(u)v = A(v)u$, $A(u)J = JA(u)$, and

\[
\text{trace}^c(A(u)) = (n+2)\beta(u) - \im \beta(Ju) = \alpha(u)
\]

for all $u, v \in \text{Vect}(M)$. Hence the connection $\nabla := \nabla_0 - A$ is torsion-free and satisfies $\nabla J = 0$ and $\nabla \theta = 0$. This implies $\text{trace}^c(R^\nabla) = F^\nabla$ and therefore

\[
\text{Ric}_{\rho,J} = \frac{1}{2}\text{trace}(JR^\nabla) = \im \text{trace}^c(R^\nabla) = \im F^\nabla.
\]

This proves Lemma [B.3].

\[\square\]
Lemma B.4. Let $\rho \in \Omega^{2n}(M)$ be a positive volume form, let $J \in \mathfrak{J}_{\text{int}}(M)$, let $\nabla_L$ be a Hermitian connection on $L$, and let $\theta \in \Omega^{n,0}_J(M,L)$ be nowhere vanishing such that $d\nabla_L \theta = 0$ and $c_n(\theta \wedge \theta) = \rho$. Then the following holds.

(i) Let $v \in \text{Vect}(M)$ and define $\widehat{J} := L_v J$ and $\beta := \partial^{\nabla_L} \nu J(v) \theta \in \Omega^{n-1,1}_J(M,L)$. Then (B.4) holds for all $u \in \text{Vect}(M)$.

(ii) Suppose $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ and $\beta \in \Omega^{n-1,1}_J(M,L)$ satisfy (B.4). Then

$$
\partial^{\nabla_L}_J \widehat{J} = 0 \iff (\partial^{\nabla_L}_J)^* \beta = 0, \quad (B.16)
$$

$$
\partial^{\nabla_L}_J = 0 \iff \partial^{\nabla_L}_J \beta = 0. \quad (B.17)
$$

(iii) Let $\widehat{J}$ and $\beta$ be as in (ii) and let $\Lambda_{\rho}(J,\widehat{J})$ be as in (2.9). Then

$$
i \partial^{\nabla_L}_J \beta + \frac{1}{2} \Lambda_{\rho}(J,\widehat{J}) \wedge \theta = 0. \quad (B.18)
$$

(iv) Let $\widehat{J}$ and $\beta$ be as in (ii) with $\partial^{\nabla_L}_J \widehat{J} = 0$ and let $\text{Ric}_{\rho,J}$ and $\text{Ric}_{\rho,J}(\widehat{J})$ be as in Theorem 2.6. Then $\text{Ric}_{\rho,J} \wedge \beta + \text{Ric}_{\rho,J}(\widehat{J}) \wedge \theta = 0$.

(v) Let $\widehat{J}$ and $\beta$ be as in (ii) with $\partial^{\nabla_L}_J \widehat{J} = 0$ and assume $F^{\nabla_L} = 0$ and $J$ admits a Kähler form. Then there exists a unique function $h \in \Omega^0(M,\C)$ such that

$$d^{\nabla_L} (\beta + h \theta) = 0, \quad \int_M h \rho = 0. \quad (B.19)
$$

Moreover, $h = \frac{1}{2} (f - ig)$ in the notation of part (iv) of Lemma 3.1.

Proof. Fix a torsion-free connection $\nabla$ such that $\nabla J = 0$ and $\nabla \theta = 0$. Next define the covariant Lie derivative of $\alpha \in \Omega^k(M,L)$ in the direction of a vector field $v \in \text{Vect}(M)$ by $L_v^{\nabla_L} \alpha := d^{\nabla_L} \nu J(v) \alpha + \nu(v) d^{\nabla_L} \alpha$. Then

$$(L_v^{\nabla_L} \alpha)(v_1, \ldots, v_k) = \nabla_{L,v} (\alpha(v_1, \ldots, v_k)) - \alpha([v_1, v], v_2, \ldots, v_k) - \cdots - \alpha(v_1, \ldots, v_{k-1}, [v_k, v])$$

for $v, v_1, \ldots, v_k \in \text{Vect}(M)$. Then $L_v^{\nabla_L} \theta = d^{\nabla_L} \nu J(v) \theta$ because $d^{\nabla_L} \theta = 0$. Hence it follows from the Leibniz rule and the equation $\nabla \theta = 0$ that

$$(d^{\nabla_L} \nu J(v) \theta)(v_1, \ldots, v_n) = \theta(\nabla_{v_1} v, v_2, \ldots, v_n) + \cdots + \theta(v_1, \ldots, v_{n-1}, \nabla_{v_n} v)$$

for all $v, v_1, \ldots, v_n \in \text{Vect}(M)$. Since $\theta$ is complex multi-linear this implies

$$i \nu(u) d^{\nabla_L} \nu J(v) \theta - i(Ju) d^{\nabla_L} \nu J(v) \theta = i(J\nabla_u v - \nabla_{Ju} v) \theta = i((L_u J) \theta) \quad (B.20)$$

for all $u, v \in \text{Vect}(M)$. Hence

$$i((L_u J) \theta) = i(u)(d^{\nabla_L} \nu J(v) \theta)^{n-1,1}_J - i(Ju)(d^{\nabla_L} \nu J(v) \theta)^{n-1,1}_J = i(u) \beta - i(Ju) \beta$$

for all $u, v \in \text{Vect}(M)$ and this proves (i).
We prove part (ii). The equivalence in \((B.16)\) follows from the identity
\[
\langle \beta, \delta J \iota(v) \theta \rangle_L^2 = \frac{1}{8} \int_M \text{trace}(\bar{J}^* L_v J) \rho = \frac{1}{4} \langle \bar{J}, J \delta J v \rangle_{L^2} = \frac{1}{4} \langle \delta J J, J v \rangle_{L^2}
\]
for \(v \in \text{Vect}(M)\). Here we have used part (v) of Lemma \((B.2)\) as well as (i). To prove \((B.17)\), define \(\alpha_u \in \Omega^\nu(M, L)\) by
\[
\alpha_u := \iota(u)d_{\hat{\nabla}} \beta - \iota(Ju) d_{\nabla} \beta
\]
for all \(u \in \text{Vect}(M)\). We will prove that, for all \(u, v \in \text{Vect}(M)\),
\[
\iota(v) \alpha_u - \iota(Jv) \alpha_u = \iota(J \delta J \hat{J}(u, v)) \theta.
\]
Equation \((B.22)\) shows that \(\delta J \hat{J} = 0\) if and only if \(\alpha_u \in \Omega^\nu_j(M, L)\) for every vector field \(u \in \text{Vect}(M)\). By \((B.21)\), this is equivalent to the condition \(d_{\hat{\nabla}} \beta \in \Omega_j^0(M, L)\) or, equivalently, to \(d_{\hat{\nabla}} \beta = (d_{\nabla} \beta)^{n-1,2} = 0\).

To prove \((B.22)\), fix a torsion-free connection \(\nabla\) that satisfies \(\nabla J = 0\) and \(\nabla \theta = 0\). Then it follows from \((B.20)\) with \(v\) replaced by \(Jv\) that
\[
\iota(u)d_{\nabla} \iota(\hat{J} v) \theta - \iota(Ju) d_{\nabla} \iota(\hat{J} v) \theta = \iota(J(\nabla_u \hat{J}) v - (\nabla_{Ju} \hat{J}) v) \theta + \iota(J \nabla v - \hat{J} \nabla v) \theta
\]
for all \(u, v \in \text{Vect}(M)\). Moreover,
\[
\alpha_u := \iota(u)d_{\nabla} \beta - \iota(Ju) d_{\nabla} \beta = \iota(\hat{J} u)(\nabla_u \hat{J}) v - (\nabla_{Ju} \hat{J}) v) \theta + \iota(J \nabla v - \hat{J} \nabla v) \theta
\]
for all \(u \in \text{Vect}(M)\). With this understood, we obtain
\[
\iota(v) \alpha_u - \iota(Jv) \alpha_u = -\iota(v) d_{\hat{\nabla}} \beta - \iota(v) \hat{L}_{J_u} \beta - \iota(v) d_{\hat{\nabla}} \hat{J} u \theta
\]
and this equality follows from \((B.24)\), the third from \((B.4)\), and the fourth from \((B.23)\). This proves \((B.22)\) and (ii).
We prove part (iii). Since $i\partial JL\beta \in \Omega^{0,1}_J(M, L)$, there exists a unique $(0, 1)$-form $\eta \in \Omega^{0,1}_J(M)$ such that

$$i\partial JL\beta + \eta \wedge \theta = 0. \quad (B.25)$$

Now let $v \in \text{Vect}(M)$. Then the pair $(L_v J, \partial JL\iota(v)\theta)$ satisfies (B.4) by (i). Hence, by (2.12) and (B.10), we have

$$\frac{1}{4} \int_M \Lambda_{\rho}(J, \bar{J}) \wedge \iota(v)\rho = \frac{1}{4} \int_M \text{trace}(JL_v \rho)$$

$$= \text{Im} \left( \int_M c_n \langle \beta \wedge \partial JL\iota(v)\theta \rangle \right)$$

$$= \text{Re} \left( \int_M c_n \langle i\beta \wedge dJL\iota(v)\theta \rangle \right)$$

$$= (-1)^{n+1} \text{Re} \left( \int_M c_n \langle i\partial JL\beta \wedge \iota(v)\theta \rangle \right)$$

$$= (-1)^{n+1} \text{Re} \left( \int_M c_n \langle (i\partial JL\beta) \wedge \theta \rangle \right)$$

$$= \text{Re} \left( \int_M \eta \langle \theta \wedge \theta \rangle \right)$$

$$= \int_M \text{Re}(\eta) \wedge \iota(v)\rho.$$

Here the penultimate equality follows from (B.25) and the last equality holds because $c_n(\theta \wedge \theta) = \rho$. Thus $\text{Re}(\eta) = \frac{1}{4} \Lambda_{\rho}(J, \bar{J})$ and so $\eta = \frac{1}{2} \Lambda_{\rho}(J, \bar{J})^{0,1}$. Hence equation (B.18) follows from (B.25) and this proves (iii).

We prove part (iv). Since $\partial J\bar{J} = 0$ it follows from part (ii) that $\partial J\bar{J}\beta = 0$. Hence $i\partial J\beta + \frac{1}{2} \Lambda_{\rho}(J, \bar{J}) \wedge \theta = 0$ by (B.18) and so, by Lemma (B.3) we have

$$\text{Ric}_{\rho,J} \wedge \beta = iF^\nabla \wedge \beta = i d^\nabla \beta = -\frac{1}{2} d\Lambda_{\rho}(J, \bar{J}) \wedge \theta = -\text{Ric}_{\rho}(J, \bar{J}) \wedge \theta.$$ 

This proves (iv).

We prove part (v). Since $F^\nabla = 0$ we have $\text{Ric}_{\rho,J} = 0$ by Lemma (B.3). Since $J$ admits a Kähler form, part (iv) of Lemma (B.7) asserts that there is a unique pair of smooth functions $f, g \in \Omega^0(M)$ of mean value zero such that $\Lambda_{\rho}(J, \bar{J}) = df \circ J + dg$. Let $h := \frac{1}{2} (f - ig)$. Then $\partial Jh = -\frac{1}{2} \Lambda_{\rho}(J, \bar{J})^{0,1}$ and so $d^\nabla (\beta + h\theta) = 0$ by (B.18). This proves (v) and Lemma (B.4).
Proof. This follows directly from (B.9) and the definition of \( \tilde{\theta} := \beta + h\theta \in \Omega^n(M, L) \) in part (v) of Lemma B.4 should be thought of as the tangent vector associated to \( \tilde{J} \) in the projective space of closed complex valued \( n \)-forms modulo scaling. Namely, if \( t \mapsto J_t \) is a smooth path of (integrable) complex structures such that \( \partial_t|_{t=0} J_t = \tilde{J} \), and \( t \mapsto \theta_t \in \Omega_{J_t}^{p,0}(M, L) \) is a smooth path of nowhere vanishing closed \( (n,0) \)-forms, then \( \partial_t|_{t=0} \theta_t \in \tilde{\theta} + \mathbb{C}\theta \).

**Corollary B.5.** Let \( M \) be an closed connected oriented \( 2n \)-manifold, let \( J \) be a complex structure on \( M \) with real first Chern class zero and nonempty Kähler cone, let \( L \to M \) be a Hermitian line bundle equipped with a flat connection \( \nabla_L \) such that \( c_1(L) = c_1(TM, J) \in H^2(M; \mathbb{Z}) \), let \( \theta \in \Omega_{J_0}^{p,0}(M, L) \) be a nowhere vanishing \( (n,0) \)-form such that \( d\Omega \theta = 0 \), and define \( \rho := c_n(\theta \wedge \theta) \). For \( i = 1, 2 \) let \( \tilde{J}_i \in \Omega_{J_i}^{0,1}(M, TM) \) such that \( \partial_{\tilde{J}_i} \tilde{J}_i = 0 \), let \( \tilde{\beta}_i \in \Omega_{J_i}^{n-1,1}(M, L) \) satisfy (B.4) for all \( u \in \text{Vect}(M) \) with \( \tilde{J} = \tilde{J}_i \), let \( h_i \in \Omega^0(M, \mathbb{C}) \) be the unique function that satisfies (B.19) with \( \beta = \beta_i \), and define \( \tilde{\theta}_i := \beta_i + h_i\theta \). Then

\[
\begin{align*}
\text{Re} \left( c_n \int_M \langle \tilde{\theta}_1 \wedge \tilde{\theta}_2 \rangle \right) &= -\frac{1}{8} \int_M \text{trace} (\tilde{J}_1 \tilde{J}_2) \rho + \int_M \text{Re}(h_1 h_2) \rho, \\
\text{Im} \left( c_n \int_M \langle \tilde{\theta}_1 \wedge \tilde{\theta}_2 \rangle \right) &= \frac{1}{8} \int_M \text{trace} (\tilde{J}_1 J \tilde{J}_2) \rho + \int_M \text{Im}(h_1 h_2) \rho.
\end{align*}
\]

(B.26)

**Proof.** This follows directly from (B.9) and the definition of \( \tilde{\theta}_i \). □

The discussion in this appendix is inspired by Donaldson’s symplectic form on the space of complex structures on a Fano manifold in [13]. He proved in [13, Theorem 1] in the Fano case that the Hermitian form

\[
(\tilde{J}_1, \tilde{J}_2) \mapsto c_n \int_M \langle \tilde{\theta}_1 \wedge \tilde{\theta}_2 \rangle
\]

is negative definite on the space of complex structures compatible with a fixed symplectic form \( \omega \). In the Calabi–Yau case (with the symplectic form not fixed) this Hermitian form on the kernel of \( \partial_{\tilde{J}} : \Omega_{\tilde{J}}^{0,1}(M, TM) \to \Omega_{\tilde{J}}^{0,2}(M, TM) \) vanishes on the image of the operator \( \partial_{\tilde{J}} : \Omega^0(M, TM) \to \Omega_{\tilde{J}}^{0,1}(M, TM) \) and descends to a well-defined and nondegenerate, but indefinite, Hermitian form on the quotient space \( \ker \partial_{\tilde{J}} / \text{im} \partial_{\tilde{J}} = T_{[\tilde{J}]} \mathcal{Z}_0(M) \). Its imaginary part is the symplectic form on Teichmüller space in Theorem 4.2.
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