RCF 3
Map-Code Interpretation via Closure
∈ U̅

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Abstract

For a (minimal) Arithmetical theory with higher Order Objects, i.e. a (minimal) Cartesian closed arithmetical theory – coming as such with the corresponding closed evaluation – we interpret here map codes, out of [A, B] say, into these maps “themselves”, coming as elements (“names”) of hom-Objects B^A. The interpretation (family) uses a Chain of Universal Objects U_n, one for each Order stratum with respect to “higher” Order of the Objects. Combined with closed, axiomatic evaluation, these interpretation family gives code-self-evaluation. Via the usual diagonal argument, Antinomie RICHARD then can be formalised within our minimal higher Order (Cartesian closed) arithmetical theory, and yields this way inconsistency, for all of its extensions, in particular of set theories as ZF, of the Elementary Theory of (higher Order) Topoi with Natural Numbers Object as considered by FREYD as well as already of the Theory of Cartesian Closed Categories with NNO considered by LAMBEK.

1 Introduction

Starting point is a discussion of CANTOR’s (indirect) argument for uncountability of the real numbers (in the unit interval), i.e. of the set 2^N = P N of (“actual” infinit) sequences a = a(j) : N → 2.

This indirect argument assumes all these a : N → 2 to be enumerated in form a_i = a_i(j) : N → 2, i ∈ N. CANTOR then takes as sequence outside this enumeration of the a_i the sequence ˜a = ˜a(j) = def ¬a_i(i) : N → 2.
But what is this $a_i(i) \in 2$? Let us try to apply CANTOR’s argument to any type of constructive real numbers, where in fact there is an enumeration, $a_i$ of all (finite) texts, (Computer) programs, standing for – “describing” – these constructive real numbers, e.g. the primitive recursive power-series descriptions for $e$ and $\pi$. But if you want to change the diagonal values in this CANTOR’s infinit table $a_i(j)$ of the constructive reals, you must be able to evaluate the $i$th of these – say primitive recursive – programs at $i \in \mathbb{N}$. Now ACKERMANN has shown, that for the case of PR function codes (“programs”, texts) this diagonal evaluation (and then its a posteriori modification) cannot be PR any more: The related (equi-complex) “Ackermann function”, namely diagonal evaluation $\varepsilon(f_n, n) : \mathbb{N} \xrightarrow{\Delta} \mathbb{N} \times \mathbb{N} \xrightarrow{\# \times \text{id}_\mathbb{N}} [\mathbb{N}, \mathbb{N}] \times \mathbb{N} \not\rightarrow \mathbb{N}$, grows faster then any PR function; here $\#(n) : \mathbb{N} \xrightarrow{\cong} [\mathbb{N}, \mathbb{N}]$ is the PR enumeration of all PR map PR codes $f_n$ “from” $\mathbb{N}$ “to” $\mathbb{N}$. The diagonal then says: “apply” n-th PR map to – evaluate n-th PR map code at – argument n.

[Presumably this non-closedness under code-evaluation applies to any constructive class of real numbers and power sets, such real numbers obtained e.g by (iterated) “application” of Intermediate-Value Theorem taken as axiom.]

So the possibility of closed evaluation, here of

$$\varepsilon_{\mathbb{N}, 2} = \varepsilon_{\mathbb{N}, 2}(\chi, n) = \chi(n) = [n \in \chi] : 2^{\mathbb{N}} \times \mathbb{N} = \mathcal{P} \times \mathbb{N} \rightarrow 2$$

is at the basis of classical set theory, with its closure under (iterated) formation of power set (and internal hom sets). This gave rise to investigation of “all” the uncountable cardinalities in set theory, a central branch of this theory proper.

The claim of present investigation is that these uncountabilities, at least a (potentially) infinit ascending chain of uncountabilities, leads to a contradiction. The idea is to interpret the map codes, $\uparrow f \downarrow \in [A, B]$ say, of a (minimally presented) theory $\text{PR} \in \text{PR}$ Arithmetic with (“higher Order”) Cartesian Closure added, into these maps “themselves”, $f \in B^A$, out of internal hom Object $B^A$, in set theory the map set

$$B^A = \{f \in \mathcal{P}(A \times B) | \forall a \in A \exists! b \in B \langle a, b \rangle \in f\}.$$ 

Combined with closed, axiomatic evaluation $\varepsilon_{A, B} : B^A \times A \rightarrow B$, $\varepsilon_{A, B}(f, a) = f(a)$, available in set theory and there needed for (generalisation of) CANTOR’s argument above to establish the strictly ascending hierarchy of cardinals, will give a code-self-evaluation, $\tilde{\varepsilon}_{\mathbb{N}, 2} : [\mathbb{N}, 2] \times \mathbb{N} \rightarrow 2$, and from this – because of the “self” – an (anti-)diagonal predicate $d = d(n) : \mathbb{N} \rightarrow 2 \overset{\sim}{\not\rightarrow} 2$, whence a liar map $\text{liar} = \neg \text{liar} : 1 \rightarrow 2$ establishing the asserted contradiction for (minimal) Cartesian Closed PR Theory $\text{PR} \in$ and its extensions.

We now outline the sections to come and forshadow at this occasion some of the notations to be introduced:
2 Theory Closure by Internal hom and Evaluation:
Here we extend basic (categorical) Theory $\text{PR}_A = \text{PR}$ (abstr) of Primitive Recursion with (virtual) extensions $\{A \mid \chi\}$ of PR predicates (see part RCF1) by Cartesian Closure, this in form of adding just new internal hom Objects, $B^A$, new map constants $\epsilon_{A,B} : \text{closed evaluation}$, and $\land_{A,B}$ for Cartesian Closure front adjuctions, as well as suitable equations for then already available conjugate and coconjugate maps, but no new (meta) operations for maps. Resulting Theory is called $\text{PR}_{\in}$, since its decisive ingredient over Theory $\text{PR}$ is closed evaluation $\epsilon_{A,B} : B^A \times A \to B$ with its characteristic equations.

3 Order Stratification for Closed Arithmetic $\text{PR}_{\in}$
In this section we divide higher Order Theory $\text{PR}_{\in}$ into strata $\text{PR}_{\in} \prec \text{PR}_{\in}$, Cartesian PR theories with Order of Objects up to $n$. Note: $\text{Ord}(C^B)^A \triangleq \text{Ord}C^B \times A < \text{Ord}C + \text{Ord}B + \text{Ord}A = \text{Ord}C^B \triangleq \text{by def} \text{Ord}(C(B^A))$, “since” $(C^B)^A \cong C^B \times A$.

4 An Ascending, Universal Object Chain
Based on Universal Object $X \subset \mathbb{N}$ for Theory $\text{PR}_A$, $X$ made out of all (codes of) singletons $(n)$ and (possibly nested) pairs $(a;b)$ of natural numbers – it contains all Objects $A$ of $\text{PR}_A$ coretractively embedded – we obtain an ascending Chain $U : U_1 = X \bslash U_2 = X \times \cdots$ of Objects and coretractions, each $U_{n \in} \text{universal}$ for its stratum $\text{PR}_{\in} \prec A \supset U_{\in}$ coretractive for each (pointed) Object $A$ of Order up to $n$.

5 Map-Code Interpretation
This section develops the central idea of present investigation: An interpretation map family

$$\text{int}_{\in} = \{\text{int}_{\in} : [A,B]_{\text{PR}_{\in}} \to B^A\}_{A,B}, \quad n \in \mathbb{N}$$

is constructed, stratum by stratum, the $\text{int}_{\in}$ leading into Universal Object (at most) $U_{\in}$.

Technically, these Object-pairs indexed families (must and) can be “derived” from a stratum specific “global” Interpretation $\text{int}_{\in} = \text{Int}_{\in}(u) : V_{\in} \to U_{\in}$; $V_{\in}$ the map code set of (whole) stratum $\text{PR}_{\in}$; reason for considering Universal Objects, here: $U_{\in}$.

What we have to do is to “interprete” code constants and code operations, namely (formal) composition, Cartesian product and iteration of map codes into the objective correspondants, e.g.: plugged into $\text{ZF}$ –

$$\text{g} \circ \text{f} \triangleq \text{g} \circ \text{f} = \text{by def} \text{g} \circ \text{f} \triangleq \text{int}(g) \circ \text{int}f = g \circ f.$$

In our “formally minimal” context, this interpretation is based on the name $\{f\} : 1 \to B^A$ of a map $f : A \to B$, $\{f\}$ easily defined via conjugation, in set theoretical terms: $\{f\} = \{(\emptyset, f)\} : 1 \to B^A$.

Interpretation int works by the correspondence of operations $\circ \triangleq \text{g} \circ \text{f}$, $\times \triangleq \text{g} \times \text{f}$, and $\text{g} \times \text{f}$ on map codes for composition, Cartesian product and iteration one hand, and associated internal closed operations, called $\subseteq \text{g}$, $\subseteq \text{f}$, as well as $\subseteq \text{g} \times \text{f}$ on the other. These latter are all defined out of set theoretically motivated “coconjugated” ones, by conjugation.
Straightforward but technically complicated calculations then give the central **Interpretation Theorem**, saying essentially that (stratum specific) interpretation

\[
\text{int}_{A,B}^n : [A, B]_{PR \in \Pi} \rightarrow B^A \sqsubseteq U_{2^n} \text{ is objective, i.e.}:
\]

\[
\text{int}_{A,B}^n(f^\uparrow) = [f] : 1 \rightarrow B^A, \text{ for } f : A \rightarrow B \text{ in } PR \in \Pi.
\]

**6 Self-Evaluation**

With interpretation properties above it is now easy to give a sound, objective code-self-evaluation for “minimal” Cartesian Closed PR Theory \( PR \in \), namely

\[
\tilde{\varepsilon}_{A,B}(u, a) = \text{def } \varepsilon_{A,B}(\text{int}_{A,B}(u), a) :
\]

\[
[A, B]_{PR \in \Pi} \times A \overset{\text{int}\times A}{\rightarrow} B^A \times A \overset{\varepsilon}{\rightarrow} B, \text{ with}
\]

\[
\tilde{\varepsilon}_{A,B}(f^\uparrow, a) = f(a) : A \rightarrow B.
\]

(Objectivity).

This then gives immediately formalisation of Antinomie Richard for \( PR \in \) by the usual diagonal argument.

Notions and results for basic Theory \( PR_A = PR + (\text{abstr}) \) of Free-Variables (categorical) Theory of Primitive Recursion with schema of predicate abstraction – and its Universal Object – are given in Pfender/Kröplin/Pape 1994 and in Pfender 2008 RCF1, RCFX.

**2 Theory Closure by Internal hom and Evaluation**

We extend here categorical Theory \( PR_A = PR + (\text{abstr}) \) of Primitive Recursion – with predicate abstraction \( \chi \mapsto \{A|\chi\} \) – into a Theory \( PR \in = \text{def } PR_A + (\text{hom}) \), with – in addition – internal hom \( (A, B) \mapsto B^A \) given by axiom, as well as theory internal – axiomatic, closed – evaluation

\[
\varepsilon = [\varepsilon_{A,B} : B^A \times A \rightarrow B]_{A,B \in PR \in}.
\]

This in – logical – contrast to constructive, Ackermann type, formally partial – but still “constructive” – evaluation family

\[
\varepsilon = [\varepsilon_{A,B} : [A, B] \times A \rightarrow B]_{A,B \in PR_A}
\]

for theories \( \pi_{OR} \) (strengthening \( PR_A \)) above – family obtained out of one single (formally partial PR) map

\[
\varepsilon = \varepsilon(u, x) : PR_A \times X = [X, X]_{PR_A} \times X \rightarrow X.
\]

**Comment on Notation:** Closed evaluation reads e.g.

\[
\varepsilon_{A,2}(\chi, a) = \chi(a) = [a \in \{A|\chi\}] : 2^A \times A = PA \times A \rightarrow 2.
\]

This motivates notation for closed evaluation. The “other” use of symbol “\(\varepsilon\)” is – in Cartesian Theories – “\(a \in A\) free”: \(a\) is a (free) variable on
A, categorical meaning: a is (identity of A) or a projection onto A. This legitimates free-variables diagram chase below categorically.

Theories \( \text{PR}_A \) and \( \text{PR} \in \) fixed, we explain now some (known) basic concepts and results, in the language of Primitive Recursion and Higher Order Arithmetic sketched above.

Basic for our Universal Chain of Objects – upwards open (!) – is the First Order Universal Object \( \mathbb{X} \subset \mathbb{N} \) of all (codes of) singletons, \( \langle n \rangle \), and (possibly nested) pairs, \( \langle a; b \rangle \), of natural numbers.

Each fundamental \( \text{PR}-\)Object \( /BD \), \( \mathbb{N} \), \( (\mathbb{N} \times \mathbb{N}) \) etc. is coretractively embedded into \( /CG \), for example \( (\mathbb{N} \times \mathbb{N}) \ni (m, n) \rightarrow (m; n) \in (\mathbb{N} \times \mathbb{N}) \subset /CG \).

Extension of Theory \( \text{PR}_A \) into Cartesian Closed Theory \( \text{PR} \in \) presented equationally – by Horn inferences – via additional (formal) exponential Objects (Object terms) of form \( (B^A) \) for \( A, B \) “already there”, examples: \( \mathbb{N}^\mathbb{N}, /U2 = \mathbb{X}^\mathbb{X}, /U3 = \mathbb{X}^{(\mathbb{X}^\mathbb{X})} \) etc., as well as (additional) families of map constants

\[ \varepsilon_{A,B} : B^A \times A \rightarrow B \text{ (axiomatic, closed evaluation)}, \]

\[ \text{[within set theory: } \varepsilon(f, a) = \text{def } f(a), \text{]} \]

\[ \wedge_{A,B} : A \rightarrow (A \times B)^B \text{, closed front adjunction, } “A \ni a \rightarrow (b \mapsto (a, b))” \].

These two families are to satisfy the adjointness equations for (covariant) Functors, \( A \times B \vdash B^A : \text{PR} \in \rightarrow \text{PR} \in \), \( (A \text{ “fixed”}) \), namely defining conjugation and coconjugation below as mutually inverse (meta) bijections.

These Horn schemata are merged with those of \( \text{PR}_A \), here: with forming Cartesian products of Objects, with iteration schema (and Freyd’s uniqueness of initialised iterated), as well as schema (abstr) of forming (virtual) extensions, cf part RCF 1.

Taken together the above internal hom structure with endo map iteration – and Freyd’s uniqueness of the initialised iterated – as well as with (virtual) predicate abstraction – we arrive at Theory \( \text{PR} \in = \text{PR}_A + (\text{hom}) = \text{PR} + (\text{abstr}) + (\text{hom}), \) of Primitive Recursion with Object exponentiation and closed evaluation: Evaluation within the Theory itself.

[The latter in contrast to availability of “only” – Ackermann type, not PR, (still) constructive – evaluation of Theory \( \text{PR}_A = \text{PR} + (\text{abstr}) \) within “only” Theory \( \text{PR}_A \) – of formally partial PR maps, theory equivalent to Theory \( \mu R \) of (partial) mu-recursive maps, see RCF1.]

Remark: Theory \( \text{Fin} \) of finite (number) sets has internal hom – exponentiation – coming with closed evaluation family \( \varepsilon_{A,B} : B^A \times A \rightarrow B \). But if you want to define this – infinitely indexed family – made out of (finite) maps, you need Primitive Recursive case distinction on \( \mathbb{N} \supset B^A \), and this “global”, mother evaluation

\[ \hat{\varepsilon} : \mathbb{N} \times \mathbb{N} \supset \bigoplus_{A,B} (B^A \times A) \rightarrow B \subset \mathbb{N} \]
is necessarily genuine PR, not finite.

Internal hom – and “closed” evaluation ∈ – give, within Theory PR of
Higher order Arithmetic, cf. Eilenberg & Kelly 1966 for internal
hom structure, as well as Freyd 1972 and Lambek & Scott 1986 for
the combined structure, the following defined map families conjugation
and coconjugation:

Conjugation is given by schema

\[(\text{conj}) \quad f : A \times B \to C \text{ in PR} \]

\[\overline{f} = \text{conj}[f] =_{\text{def}} f^B \circ \triangleleft_{A,B} : A \to (A \times B)^B \to C^B,\]

in set theory conjugate \(\overline{f}\) realised as

\[a \mapsto [b \mapsto (a, b) \mapsto f(a, b) \in C],\]

and coconjugation is introduced by schema

\[(\text{coconj}) \quad g : A \to C^B \text{ in PR} \]

\[\overline{g} = \text{coconj}[g] =_{\text{def}} \in B,C \circ (g \times \text{id}_B) : A \times B \to C^B \times B \to C,\]

in set theory coconjugate \(\overline{g}\) realised as

\[[(a, b) \mapsto (g(a), b) \mapsto \overline{g}(a, b) =_{\text{by def}} (g(a))(b) \in C].\]

These two families are to satisfy – by axiom, and do so (already
within finite) set theory and the Elementary Theory of Topoi ETT in
place of Theory PR around to be “constructed” – the following higher
order meta-bijection equations:

\[(\text{co/conj}) \quad f : A \times B \to C \text{ in PR} \]

\[\text{PR} \vdash f = \text{coconj}[\text{conj}[f]] = f : A \times B \to C\]

as well as

\[(\text{conj/co}) \quad g : A \to C^B \text{ in PR} \]

\[\text{PR} \vdash \overline{g} = \text{conj}[\text{coconj}[g]] = g : A \to C^B.\]

The above data, in particular (axiomatically given) families \(\triangleleft\) and
\(\in\), define the following meta-map, and make it into a covariant functor
hom – the covariant internal hom functor – via the following schema:

\[(\text{hom-co}) \quad A, g : B \to C \text{ in PR} \]

\[g^A =_{\text{def}} g \circ \epsilon_{A,B} : B^A \to C^A\]

Analogous schema defining the contravariant (closed) internal hom
functor:

\[(\text{hom-contr}) \quad A, g : B \to C \text{ in PR} \]

\[\epsilon \circ (A^C \times g) : A^C \times B \xrightarrow{A^C \times g} A^C \times C \xrightarrow{\epsilon} A\]

\[A^g =_{\text{def}} \epsilon \circ (A^C \times g) : A^C \to A^B.\]
All four: *Universal property*, the two *Functor properties*, and *right adjointness*, of covariant closed internal hom $g \mapsto g^A$ – namely right adjointness to *Cylindrification*

$$\langle g : B \to C \rangle \mapsto \langle A \times g : A \times B \to A \times C \rangle, \quad \text{Object } A \text{ fixed},$$

are consequences of the pair conj/coconj above to be a pair of meta-bijections, inverse to each other.

**Remark:** $\triangleleft_{A,B}$ and $\epsilon_{A,B}$ are natural transformations, but we will not rely on these properties here.

### 3 Order Stratification for Closed Arithmetic $\text{PR}\in$

**Definition:** The – formal – Order $\text{Ord } A$ of a higher order Object – of Theory $\text{PR}\in$ – is **defined** externally $\text{PR}$ as follows:

- $\text{Ord } 1, \text{Ord } N =_{\text{def}} 1$,
- $\text{Ord } (A \times B) =_{\text{def}} \text{max}\{\text{Ord } A, \text{Ord } B\}$,
- $\text{Ord } \{A\mid \chi : A \to 2\} =_{\text{def}} \text{Ord } A$,
  in particular $\text{Ord } 2 = \text{Ord } \{n \in N\mid n < 2\} = 1$,
  $\text{Ord } X = \text{Ord } \{N\mid X : N \to 2\} = 1$ ($X$ is a predicative subset of $N$.)

For $B$ in $\text{PR}_A$ and $A$ in $\text{PR}\in$ (Ord $A$ “already known”):

- $\text{Ord } B^A = 1 + \text{Ord } A$;
- finally: for $C \in \text{PR}_A$, $B, C$ in $\text{PR}\in$:
  - $\text{Ord } (C^B)^A =_{\text{def}} \text{Ord } C^{B \times A} =_{\text{by def}} 1 + \text{Ord } (B \times A)$
  - $=_{\text{by def}} 1 + \text{max}(\text{Ord } B, \text{Ord } A)$.

The latter clause takes in account the (canonical) $\text{PR}\in$ reduction isomorphism $(C^B)^A \cong C^{B \times A}$.

With this **definition**, we have in particular $\text{Ord } B^A \leq \text{Ord } B + \text{Ord } A$ for all $\text{PR}\in$ Objects $A, B$, as well as $\text{Ord } U_n = n$, e.g. $\text{Ord } U_3 =_{\text{by def}} \text{Ord } X^{X^X} = \text{Ord } X^{(X^X)} = 3$.

So subSystem $\text{PR}_A$ of Theory $\text{PR}\in$ has all its (presenting) Objects of Order 1, it is our basic, “1st” Order, subSystem of Theory $\text{PR}\in$ – not a priori an (“embedded”) subCategory, since the higher-order axioms of $\text{PR}\in$ may entail – within $\text{PR}\in$ – *new* equations between map terms of $\text{PR}_A$ viewed as map terms of $\text{PR}\in$, in logical terms: The Extension $\text{PR}\in$ of $\text{PR}_A$ may be not conservative.

**Broadening to Theories Extension Chain:** We **define** an exhaustive Chain of subSystems $\text{PR}\in_n \preceq \text{PR}\in$, $n \in \mathbb{N}$, $\text{PR}$ as follows:

- $\text{PR}\in 1 =_{\text{def}} \text{PR}_A$;
- Assume $\text{PR}\in_n \preceq \text{PR}\in$ to be known via its (canonical) presentation:
  Object terms, map terms, schemata for map (term) equations.
Then subSystem $\text{PR} \in n+ = \text{PR} \in [n+1]$ is defined to be the Cartesian-$\text{PR}$-Category Closure of subSystem $\text{PR} \in n$ merged with Closure under formal adjunction of

- all Objects of Order $n+1$
- the canonical isomorphisms $(C^B)^A \cong C^{B \times A}$ given in $\text{PR} \in n$, for $C$ in $\text{PR} \in A$, $A, B$ in $\text{PR} \in n$, and their inverses $C^{B \times A} \cong (C^B)^A$
- $\text{PR} \in$ families $\times_{A,B} : A \rightarrow (A \times B)^B$ as well as $\varepsilon_{A,B} : B^A \times A \rightarrow B$, this for $\text{Ord} A + \text{Ord} B$, $2 \text{Ord} B \leq n+1$, and $\text{Ord} A + \text{Ord} B \leq n+1$ respectively.

Additional (merged) equations come in, for the maps of $\text{PR} \in n+$, via schemata (co/conj) as well as (conj/co) of $\text{PR} \in$ (above), which are to establish the conjugation/coconjugation bijection for all those of their instances, for which all formal ingredients – Object terms and map terms – are enumerated so far within $\text{PR} \in n+$.

**Corollary to this Definition:**

(i) *Conjugation upgrade:*

\[
\text{(upgrade)} \quad f : A \times B \rightarrow C \text{ in } \text{PR} \in n \prec \text{PR} \in, \\
\overline{f} = \text{conj}[f] = f^B \circ \times_{A,B} : A \rightarrow (A \times B)^B \rightarrow C^B \\
\text{lives in } \text{PR} \in 2n \prec \text{PR} \in.
\]

(ii) *Coconjugation upgrade:*

\[
\text{(co-upgrade)} \quad g : A \rightarrow C^B \text{ in } \text{PR} \in n \prec \text{PR} \in, \\
\overline{g} = \text{coconj}(g) = \varepsilon_{B,C} \circ (g \times \text{id}_B) : \\
A \times B \rightarrow C^B \times B \rightarrow C \text{ lives already in } \text{PR} \in n:
\]

Critical exponential Object $C^B$ is presupposed to belong already to Theory $\text{PR} \in n$.

(iii) Theory $\text{PR} \in n$ contains Objects up to Order $n$, and in fact some of its Objects have this Order.

(iv) External ascending “union” of all subSystems $\text{PR} \in n$, $n \in \mathbb{N}$, exhausts Theory $\text{PR} \in$, i.e. gives a – stratified – presentation of Theory $\text{PR} \in$ : Objects, maps, and equations.

### 4 An Ascending, Universal Object Chain

Basic – 1st Order – Arithmetical Theory $\text{PR} \in 1 = \text{PR}_A$ has a Universal Object in itself, a first-Order Universal Object, namely the Object $X \subset \mathbb{N}$ – of (codes of) all singleton (lists) and of pairs, possibly nested: binary bracketed NNO tuples.
X is a Universal Object – of Theory PRₐ and therefore also of its strengthenings, as for example for the full first order subcategory PR∈₁ of PR∈. Object X is universal in the following sense:

X admits – for each PRₐ-Object A, an embedding (here an injective map), even a coretractive map (see below), ⊏ₐ : A ⊑ X, defined externally PR in the obvious way.

All these embeddings ⊏ₐ : A ⊑ X – disjoint as far as fundamental Objects A are concerned, namely binary bracketed powers of N, no genuine abstracted sets – come with canonical retractions ⊐ₐ : X ⊐ A, the latter equally for abstracted Objects \{A | χ\} having a point, a₀ : 1 → \{A | χ\}, as in particular X ⊂ N, coming with “its” zero ⟨0⟩ : 1 → X.

**Graded-Universal-Object Chain:** Each of our Theories PR∈ₙ in the hierarchy – except (!) “roof” Theory PR∈ itself – comes with a canonical Universal Object, Uₙ = X↑ₙ, externally PR defined as follows, as an internal version of a Grothendieck-Universe (?):

\[
\begin{align*}
U_1 &= X^{1} =_{\text{def}} X^1 = X, \\
U_{n+1} &=_{\text{def}} X^{X^n} = X^{X^1} =_{\text{by def}} X^{n+1}
\end{align*}
\]

For opening the possibility that a higher, later Universal Object in the chain is good also as Universal Object for a lower, earlier Theory in the hierarchy, we establish first the Universal Chain U as a chain of embeddings ⊏ = ⊑ₙ : Uₙ → Uₙ₊₁ coming each with a retraction ⊐ = ⊐ₙ : Uₙ₊₁ → Uₙ, as follows:

Universal Chain U begins with (commutative) diagram

Diagram chase in case of set theory:

The general Universal Chain member then is recursively defined by commutativity of diagram
Easy Diagram chase for verifying section/retraction property e.g. in set theory.

Generalising the above to the case of $B^A$ instead of $X^U_n$ we now **define** recursively the (coretractive) embeddings

$$\sqcup = \sqcup_{BA} : B^A \to U_{n+1}, \quad B^A \text{ in } PR \in [n+1]$$

based on the (coretractive) embeddings $\sqcup : B \to X = \mathbb{U}_1$ above, as follows, “but” first only for Object $B$ in $PR \in 1 = PR_A$ :

- Anchor: for $A$ in $PR_A$, (natural) embedding $\sqcup_A : A \to U_1 = \text{def} \ X^1 = X$ has been **defined** above by converting natural numbers $n$ in singleton codes $\langle n \rangle$, and – recursively – pairs in code pairs, out of $PR_A$.

  Universal Object $\mathbb{U} \subset N$.

  Furthermore, a canonical retraction $\sqcup_A : \mathbb{U} \to A$ for the embedding has been mentioned above, for Object $A$ coming with a point, $a_0 : 1 \to A$ say.

- Step: Assume embedding $\sqcup_A : A \to U_n$ to be given, together with retraction $\sqcup_A : U_n \to A$, for “each” Object $A$ of Order $n$ – in $PR \in n$.

  Consider then a (genuine) Object in $PR \in n_+$, of form $B^A$, $A$ in $PR \in n$, $B$ in $PR_A$ (!). Then the **Diagram** below – simplified one of the former one above – **defines** “universal” embedding and retraction for Object $B^A$ into/from $U_{n+1} = \text{def} \ X^U_n$ :

Again easy Diagram chase for verifying section/retraction property in case of set theory.

The general, not normal form case, of a $PR \in n_+$ Object of form $B^A$, $B = D^C$ not basic, not in $PR_A$, is reduced to the above one via (natural) isomorphism $(D^C)^A \cong D^{(C \times A)}$ – such isomorphism possibly applied several times –, to a normal form case Object to be embedded, by a map within $PR \in n_+$ (or lower) into $U_{n+1}$ or lower, by the method above for the case of Object $B$ in $PR_A$. Embedding into $U_{n+1}$ in the latter case then is by composition with embedding $U_m \sqcup \to U_{n+1}$, $m < n + 1$. 

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Taken together the above – including the modification for the non-normal-form case – we have $PR \in \mathbb{N}^+$ embedded all Objects of $PR \in \mathbb{N}^+$ into $U_{\mathbb{N}^+}$, namely all $PR \in \mathbb{N}$ Objects of – up to – Order $\mathbb{N}^+ + 1$. This proves

**Embedding Theorem** for Chain $U$:

(i) Each single of our Theories $PR \in \mathbb{N}$ admits coretractive embeddings $\sqsubseteq_A: A \rightarrow U_{\mathbb{N}}$ for each of its (pointed) Objects $A$, into “its” Universal Object within the section/retraction Chain

$$U: \text{U}_1 = \bigotimes \xrightarrow{\ldots} \text{U}_{\mathbb{N}} \xrightarrow{\ldots} \text{U}_{\mathbb{N}^+}$$

of these “Universal” Objects, the Chain $U$ hosted as an ascending chain in global, higher Order Theory $PR \in \mathbb{N}$.

(ii) By the above discussion of – canonical – natural retractions $\sqsupseteq_{\mathbb{N}}: \text{U}_{\mathbb{N}^+} \rightarrow \text{U}_{\mathbb{N}}$; retractions to embeddings $\sqsubseteq_{\mathbb{N}}: \text{U}_{\mathbb{N}} \rightarrow \text{U}_{\mathbb{N}^+}$, the above coretractive embedding for all Objects of $PR \in \mathbb{N}$, into $U_{\mathbb{N}}$, gives also (canonical) embeddings into later Objects of chain $U$, i.e. if $U_{\mathbb{N}}$ is replaced by $U_m, m > n$, and (coretractive) embedding $A \rightarrow U_m$ is taken as $\sqsubseteq_A: A \rightarrow U_{\mathbb{N}} \rightarrow \ldots \rightarrow U_m$.

5 Map-Code Interpretation

Using Order Stratification above – of higher order Cartesian Closed Theory $PR \in \mathbb{N} = PR_A + (\text{hom})$ – we now define – via PR – a Theory-internal interpretation map family

$$\text{int} = [\text{int}_{A,B}^n : [A, B]_{\mathbb{N}} =_{\text{def}} [A, B]_{PR \in \mathbb{N}} \rightarrow B^A]_{A,B}, \ n \in \mathbb{N}. \ n \in \mathbb{N}^+$$

$A, B$ Objects of stratum $PR \in \mathbb{N}$; interpretation $\text{int}_{A,B}^n$ will be defined inside stratum $PR \in 2^n$. 

**Example:** $\text{int}_{\mathbb{N},2}^1 : [\mathbb{N}, 2]_{PR_A} = [\mathbb{N}, 2]_{PR \in 1} \rightarrow 2^{\mathbb{N}}$ will live inside stratum $PR\in 2$, and higher $\rightarrow$, see discussion in foregoing section.

[Such a stratum is a PR Cartesian Theory, but it is truncated what concerns (exponential) Order of Objects and (axiomatic) evaluation. We will see below – in particular for our interpretation of constructive, PR defined “internal” hom sets $[A, B]$ into closed ones $B^A$, that it is sufficient to climb up to stratum $2^n$ for interpretation of stratum $n$.]

In our present – categorical – context, family $\text{int}_{A,B} = \text{int}_{A,B}^n, n \in \mathbb{N}$ fixed, can and must (?) be defined formally as (a family) derived from one single $PR \in 2^n$ map. So, as one Interpretation for all – on stratum $PR \in \mathbb{N}$ fixed – we are lead to define – PR over $PR \in 2^n$ – this global Interpretation as a $PR \in 2^n$ map, with suitable, universal, Domain and CoDomain.

We start by type-description of this family – to be defined, later, as a family of Domain/CoDomain restrictions of the one single map $\text{Int}_{\mathbb{N}}$ of Theory $PR \in \mathbb{N}$ to be (objectively) PR defined – of following type:
\[ \text{Int}^n = \text{Int}^n(u) : V^n = \bigoplus_{A,B} |A,B|_n \rightarrow U_{2^n}, \text{ where } |A,B|_n \]
is an abbreviation for internal, syntactical \( \text{PR} \in n \)-map code (!) set \([A,B]_{\text{PR} \in n} \subset V \subset \mathbb{N}, \text{ from } A \text{ to } B, \text{ both Objects of } \text{PR} \in n. \)

We turn now our “typifying” proposal (!) above, into a diagram which displays a special – central – countable sum (“disjoint union”), and its (litteral) component-inclusions. This “special” sum-DIAGRAM is available within \( \text{PR} \in n \) – as litteral, disjoint union of predicates, disjoint by definition.

*Global \( \text{PR} \in n \) Interpretation \( \text{Int}^n \), \( n \) fixed, to be defined following actual type-discussion, then will be characterised a posteriori (!) as \( \text{PR} \in 2^n \)-map, induced map out of the (countable) sum, induced by its components \( \text{int}_{A,B}^n : |A,B|_n \rightarrow B^A \subset U_{2^n} \), \( A,B \) in stratum \( \text{PR} \in n. \)

In other words: \( \text{Int}^n \) will be \( \text{PR} \) “constructed” – “over” \( \text{PR} \in n \), “but only” within \( \text{PR} \in 2^n \) – in such a way that it becomes the (unique) \( \text{PR} \in 2^n \)-map out of \( \sum V^n \subset \mathbb{N} \), which makes commute the following (externally) countable diagram, this diagram available within \( \text{PR} \in 2^n \):

\[
\begin{array}{c}
|A,B|_n \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\bigoplus_{A,B} |A,B|_n \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
V^n = \bigoplus |A,B|_n \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
U_{2^n} = \bigoplus |A,B|_n \\
\end{array}
\]

Interpretation map diagram \((A,B \text{ in } \text{PR} \in n)\)

\( \text{PR} \) Construction of \( \text{PR} \in n \)-map \( \text{Int}^n : \mathbb{N} \supset V^n = \bigoplus_{A,B} |A,B| \rightarrow U_{2^n} \)
is recursively merged with that of maps \( \text{int}_{A,B}^n : |A,B|_n \rightarrow B^A \), the latter being (recursively) defined as Domain/Codomain restrictions of universal \( \text{PR} \) defined Interpretation map \( \text{Int}^n \) within \( \text{PR} \in 2^n \), in fact by the followig defining commutative diagram \((B \text{ pointed}):\)

\[
\begin{array}{c}
|A,B|_n \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
B^A \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
|A,B|_n \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
V^n = \bigoplus |A,B|_n \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
U_{2^n} \\
\end{array}
\]

This type of restriction becomes possible – at least easier – by the fact that “all” maps considered come as section/retraction pairs. This is in particular the case for all injections-into-sums embeddings here to be treated.
Constructive Internalisation of meta operations for our Theories \(\mathbb{PR}\in\mathfrak{n}\) and subSystems \(\mathbb{PR}\in\mathfrak{n}\):

Composition \(\circ : T \times T \to T(A, C)\) of \(T\) – Theory \(T\) any (categorical) theory – constructively internalises to

\[\circ = \circ^\land : [B, C]_T \times \langle A, B \rangle_T \to [A, C]_T, \quad (v, u) \mapsto \langle v \circ u \rangle \in [A, C]_T.\]

As Objects \(A, B\) here all Objects of \(T\) are allowed, for \(T := \mathbb{PR}\in\mathfrak{n}\) in particular Object \(U_\mathfrak{n}\) and its (embedded) subobjects.

Analogously Cartesian product “\(\times\)” has as coded version family

\[[A, B]_T \times [C, D]_T \ni (u, v) \mapsto \langle u \times v \rangle \in [A \times C, B \times D]_T,\]

for (arbitrary) \(T\)-Objects \(A, B, C, D\), including in particular Objects \(U_\mathfrak{n}\) in case of theory \(T := \mathbb{PR}\in\mathfrak{n}\).

Analogously for iteration “\(\ast\)” within (Cartesian) \(\mathbb{PR}\) theories in particular “again” for extension \(\mathbb{PR}\in\mathfrak{n}\) of PR Theory \(\mathbb{PR}_\mathfrak{A} = \mathbb{PR}+(\text{abstr}) : [A, A] \ni v \mapsto \langle v \ast \rangle \in [A \times \mathbb{N}, A]\), here e.g. for iteration of \(\mathbb{PR}\in\mathfrak{n}\) endo maps with Domain \(A := U_\mathfrak{n}\) and their internalisations.

Definition: The constructive \(\mathbb{PR}\in\mathfrak{n}\)-codes in \(V_\mathfrak{n}\) are – first – the constructive internal map-constants

\[\land_{A, B} : \mathfrak{1} \to [A, (A \times B)B]\]

and \(\land_{A, B} : \mathfrak{1} \to [B \times A, B]\)

for \(\land_{A, B} \in \mathbb{PR}\in\mathfrak{n}\).

Second: the “derived” Cartesian map constants for the new Objects and their Cartesian products – with the “old” ones and with the new ones –: identities, terminal maps, (left and right) projections, and

Third: “Closure” under composition and cylindrification (Cartesian product with an identity) as well as under iteration of endo maps.

Next we define, for \(f : A \to B\) in \(\mathbb{PR}\in\mathfrak{n}\) – and hence in particular Objects \(A, B\) in \(\mathbb{PR}\in\mathfrak{n}\), the notion name of \(f : A \to B\), symbolised as \([f] = [f : A \to B] : \mathfrak{1} \to B^A\), available in stratum \(2\mathfrak{n}\).

This up-to-\(2\mathfrak{n}\)th Order construct \([f]\) is defined simply by conjugation, as \([f] = f \circ \ast_{1, A}\).

Name \([f]\) of \(f\) represents, meta-bijectively, map \(f : A \to B\) within – as defined element of – closed internal hom set \(B^A\). In set theory:

\([f] = \text{def} \{(0, f)\} : \mathfrak{1} \to B^A \subset \mathcal{P}(A \times B)\).

By its definition via conjugation, \([f]\) has characteristic property \(\in_{A, B}(\langle f \rangle, a) = f(a) = f : A \to B, a \in A\) free. Verification of this closed Objectivity from definition is trivial for set theoretic environment, and straight forward for the general higher Order case.

Definition of global Interpretation \(\text{Int}^\mathfrak{u} : \mathbb{PR} \ni \mathfrak{u} : \mathbb{N} \to U_\mathfrak{u}\) of \(\mathfrak{u}\)-truncated, internal map-code-set \(V_\mathfrak{n}\) of Theory \(\mathbb{PR}\in\mathfrak{n}\) into \(\mathbb{PR}\in\mathfrak{n}\)’s Universal Object \(U_\mathfrak{u}\) – within (the language of) Theory \(\mathbb{PR}\in\mathfrak{n}\) is by recursive case distinction on the structure of the map code \(u \in V_\mathfrak{n}\) to be interpreted. (At beginning we do not typify into types \(A, B\) for \([A, B]_\mathfrak{n} \subset V_\mathfrak{n}\)). This PR case distinction for Definition of Interpretation \(\text{Int}^\mathfrak{u} : V_\mathfrak{n} \to U_\mathfrak{u}\) runs as follows:
– Case of $\mathbf{PR} \in n$ map constants “bas”, namely $0 : \mathbb{1} \rightarrow \mathbb{N}$ and $s : \mathbb{N} \rightarrow \mathbb{N}$ as well as all Cartesian map constants of $\mathbf{PR} \in n$: identities, terminal maps, diagonals, (binary) projections, as well as case of the additional – closed – map constants $\land_{A,B}, \in_{A,B}$ of $\mathbf{PR} \in n$:

For all of these anchor cases, we define $\text{Interpretation} \ Int^\mathfrak{n} = Int^\mathfrak{n} (u) : V_\mathfrak{n} \rightarrow U_{2\mathfrak{n}}$ in the below – PR – by “codes to names:”

$$\text{Int}(⌜bas⌝) = \text{def} \; ⌊bas⌋ : \mathbb{N} \rightarrow U_{2\mathfrak{n}},$$

$$\text{Int}(⌜\ell : A \times B \rightarrow A⌝) = \text{def} \; ⌊\ell : A \times B \rightarrow A⌋ : \mathbb{N} \rightarrow A \times B \subseteq U_{2\mathfrak{n}};$$

Objects $A, B$ in stratum $\mathbf{PR} \in n$.

This gives in particular for the “extra” $\mathbf{PR} \in n$ basic codes, with appropriate $\mathbf{PR} \in n$ Objects as types:

$$\text{Int}(⌜\land_{A,B} : A \rightarrow (A \times B)^B⌝) = \text{def} \; [\land_{A,B}] : \mathbb{N} \rightarrow ((A \times B)^B)^B \subseteq U_{2\mathfrak{n}},$$

$$\text{Int}(⌜\in_{A,B} : B^A \times A \rightarrow B⌝) = \text{def} \; [\in_{A,B}] : \mathbb{N} \rightarrow B^A \times B \subseteq U_{2\mathfrak{n}}.$$

The latter two “inclusions” $\subseteq U_{2\mathfrak{n}}$ are available by the fact that $\land_{A,B}$ and $\in_{A,B}$ were supposed to live “already” within $\mathbf{PR} \in n$, and that conjugation – at the base of name $[f]$ – at most doubles Order of (minimal) “receiving” stratum, here Order $n$.

What we still have to worry about is self-referential (!) Interpretation of family members $\text{int}^\mathfrak{n} A,B : [A, B] \rightarrow B^A$, obtained from $\text{Int}^\mathfrak{n} : V_\mathfrak{n} = \mathbf{PR} \in n \rightarrow U_{2\mathfrak{n}}$ by Domain/CoDomain restriction.

For these injections into sum $V_\mathfrak{n}$, we will obtain (!), out of our PR case-definition of global Interpretation $\text{Int}^\mathfrak{n} : V_\mathfrak{n} \rightarrow U_{2\mathfrak{n}}$, by definition – below – of families

$$\text{Int}^\mathfrak{n} (⌜\text{int}^\mathfrak{n} A,B⌝) = [\text{int}^\mathfrak{n} A,B] : \mathbb{N} \rightarrow (B^A)^{[A,B]} \cong B^A \times [A,B] \text{ within } \mathbf{PR} \in 2\mathfrak{n}.$$

The latter map will lead in fact – Order verification – into $\mathbf{PR} \in 2\mathfrak{n}$ by our definition of

$$\text{Ord} (B^A)^{[A,B]} = \text{Ord} B^A \times [A,B]$$

$$\leq \text{Ord} B + \text{max} (\text{Ord} A, \text{Ord} [A,B])$$

$$= \text{Ord} B + \text{max} (\text{Ord} A, \text{Ord} \mathbb{N})$$

$$\leq \text{Ord} B + \text{Ord} A \leq 2\mathfrak{n},$$

and since the isomorphism pair $(B^A)^{[A,B]} \cong B^A \times [A,B]$ is included in $\mathbf{PR} \in 2\mathfrak{n}$ by definition of stratum $\mathbf{PR} \in n^+ = \mathbf{PR} \in [\mathfrak{n} + 1]$.

Based on the anchor cases above, we define by genuine primitive recursion stratum Interpretation $\text{Int}^\mathfrak{n}$ of (constructively) composed codes, Cartesian “parallelised” as well as of iterated ones, as follows by PR
case distinction on Iteration Domain for PR definition of Int₂ⁿ : V₂ⁿ → U₂ⁿ. PR case distinction on the disjoint components |A, B|₂ⁿ of “syntactic (code) universe” V₂ⁿ ⊆ N, which in turn is a PR defined predicative subObject of N within Theory PR₂ⁿ – in the rôle of (internal) Metamathematics – PR₂ⁿ subsystem of PR∈N ⊏ PR∈∞ [ = “PR∈∞” ].

With – always below – abbreviation

|A, B|₂ⁿ = by def |A, B|PR∈N ⊂ V₂ⁿ = PR∈N ⊂ N, we introduce

PR∈N map (map-family, indexed on n ∈ N)

Int₂ⁿ = Int₂ⁿ(u) : V₂ⁿ = ⋃ₐₐ B |A, B|₂ⁿ → U₂ⁿ.

merged with its Domain/Codomain restrictions, recursively as follows:

Interpretation of constructive internal composition: For A, B, C in stratum PR∈N :

for u ∈ |A, B|₂ⁿ ⊂ V₂ⁿ, v ∈ |B, C|₂ⁿ ⊂ V₂ⁿ

\[ \implies \langle v \circ u \rangle \in |A, C|₂ⁿ \subset V₂ⁿ \]

Int₂ⁿ(v ⊔ u) = Intₐₐ B,C(⟨v ⊔ u⟩) ⊂ V₂ⁿ

= by def Int₂ⁿ(v) ⊔ Int₂ⁿ(u)

= by def ⊔ Intₐₐ B,C(⟨Int₂ⁿ(v), Int₂ⁿ(u)⟩) :

\[ V₂ⁿ \times V₂ⁿ \xrightarrow{\supset} |B, C|₂ⁿ \times |A, B|₂ⁿ \xrightarrow{\text{Int}_B \times \text{Int}_A} C_B \times B_A \xrightarrow{\supset} C_A \subseteq U₂ⁿ. \]

This is a formally defined PR∈N map, in particular since V₂ⁿ × V₂ⁿ ⊗ \[ |B, C|₂ⁿ \times |A, B|₂ⁿ \] is – obviously – a retraction. We recall further that “embedding” \[ C_A \subseteq U₂ⁿ \] also comes with a retraction, U₂ⁿ \[ \supseteq \subseteq \quad C_A \]

Axiomatic internal composition – competing with constructive internal composition ⊗ = \[ \supseteq \quad \text{Int}_A \times \text{Int}_B \] gets a similar symbol, \[ \supseteq \quad \text{Int}_A \times \text{Int}_B \], which may be read Closed internal composition, similarly: \[ \supseteq \quad \text{Int}_A \times \text{Int}_B \] which may be read Closed internal composition, similarly: \[ \supseteq \quad \text{Int}_A \times \text{Int}_B \]

Closed internal composition \[ \supseteq \quad \text{Int}_A \times \text{Int}_B \] is defined via conjugate \[ \supseteq \quad \text{Int}_A \times \text{Int}_B \] of

\[ \supseteq \quad \text{Int}_A \times \text{Int}_B \] of

\[ (C_B \times B_A) \times A \rightarrow C \]

= def \[ \epsilon \circ (C_B \times \epsilon) : C_B \times B_A \times A \xrightarrow{C_B \times \epsilon} C_B \times B \xrightarrow{\epsilon} C \],

with Cartesian associativity (natural) isomorphisms of form

\[ A \times B \times C = \text{def} \ (A \times B) \times C \xrightarrow{\text{ass}} A \times (B \times C) \text{ omitted.} \]
This case of $\text{PR} \in$-map $\text{Int}^n : V^n \xrightarrow{\gamma} |A, C|_n \to C^A \subseteq \mathcal{U}_{2^n}$ describes in fact a $\text{PR} \in 2^n$ map:

In its chain of Objects – and in its Order minimal presentation of maps – it is at most of Order $2^n$ – for Objects $A, B, C$ all of Order at most $n$.

- **Interpretation** of constructive internal product of maps: This is analogous to the above, even easier, since the two components of a Cartesian product are completely independent of each other, “exercise”.

- **Case** of an internally iterated $v^\gamma \in |A \times \mathbb{N}|_n$, $v \in |A, A|_n$ free, Object $A$ in $\text{PR} \in n$. **Define** in this case

$$V^n \supset |A, A|_n \ni v \text{ Int}^n \xrightarrow{\gamma} \text{Int}^n(v^\gamma) \in A^{A \times \mathbb{N}} \subseteq \mathcal{U}_{2^n}$$

Here $\text{PR} \in 2^n$ map $\text{Int}^n : A^A \to A^{A \times \mathbb{N}}$ is defined as conjugate to

$$\text{Int}^n = (\prod_{A^A} \circ \text{int}^n_{A, B}) : |A, B|_n \to \mathcal{U}_{2^n}, \text{ Ord}A \leq n.$$

With the above, in particular with definition of Interpretation map $\text{Int}^n$ on map constants – among them (the codes of) $\wedge$ and $\in$, $\text{Int}^n$ is (PR) defined on all of its arguments, in particular on conjugated and hom-functor values, since these are definable in terms of Composition, Cartesian Product and Iteration out of the basics. Furthermore, the above type insertions show that $\text{PR} \in n$ map

$$\text{Int}^n = \text{Int}^n(u) : |A, C|_n \to \mathcal{U}_{2^n}, \text{ Ord}A \leq n.$$

is – as expected – induced by Object-pair typified family

$$\text{int}^n_{A, B} : |A, B|_n \to B^A \subseteq \mathcal{U}_{2^n}.$$

($n$ still fixed), more precisely: it is the induced out of countable sum:

$$\text{Int}^n = \bigoplus \mathcal{U}_{2^n}.$$
Structure Preservation by Constructive Coding:

Composition: for \( f : A \to B \) and \( g : B \to C \) in \( \text{PR}\in \mathbb{N} \):
\[
g \circ f \Downarrow = \Downarrow (g \circ f) =_{\text{by def}} \Downarrow (\Downarrow g \circ \Downarrow f) : 1 \to |A|_n \rightarrow |B|_n.
\]

Cartesian product: for \( f : A \to C \) and \( g : B \to D \) in \( \text{PR}\in \mathbb{N} \):
\[
(\text{pr} f \times \text{pr} g) = (\Downarrow (\Downarrow f \times \Downarrow g)) : 1 \to |A|_n \times |B|_n,
\]
as well as Iteration: for \( f : A \to A \) in \( \text{PR}\in \mathbb{N} \):
\[
\text{pr} f^\circ = \Downarrow (\Downarrow f \circ \Downarrow f^\circ) : 1 \to |A|_n \times |A|_n.
\]

For closed internalisation we have an analogous result, namely

Structure Preservation by Closed Internalisation: Naming

\[
\langle f : A \to B \rangle \mapsto \langle \Downarrow f : 1 \rightarrow B^A \rangle
\]

preserves Composition, map-Product and iteration into the corresponding closed families \( \Downarrow \circ \Downarrow \text{pr} A,B,C \), \( \Downarrow \times \Downarrow \text{pr} A,B,C,D \), as well as \( \Downarrow \circ \Downarrow A \), in detail:

- Composition: For \( f \xrightarrow{\text{f}} B \xrightarrow{\text{g}} C \) in \( \text{PR}\in \mathbb{N} \) we have:
\[
\Downarrow (\Downarrow g \circ \Downarrow f) : 1 \rightarrow (\Downarrow B) \rightarrow C
\]
it lives within stratum \( \text{PR}\in 2_\mathbb{N} \).

- Cartesian product: For \( f : A \rightarrow C \), and \( g : B \rightarrow D \) in \( \text{PR}\in \mathbb{N} \):
\[
\Downarrow (\text{pr} f \times \text{pr} g) = (\Downarrow f \times \Downarrow g) =_{\text{def}} \Downarrow \times \Downarrow (\Downarrow f \times \Downarrow g) : 1 \rightarrow (\Downarrow C) \times (\Downarrow D) \rightarrow (\Downarrow \times \Downarrow) \rightarrow C
\]
this again lives in stratum \( \text{PR}\in 2_\mathbb{N} \).

- Iteration: For \( f : A \rightarrow A \) in \( \text{PR}\in \mathbb{N} \):
\[
\Downarrow (\Downarrow f \circ \Downarrow f^\circ) = \Downarrow (\Downarrow f \circ \Downarrow f^\circ) : 1 \rightarrow (\Downarrow A) \rightarrow (\Downarrow A) \rightarrow (\Downarrow A) \times \mathbb{N}
\]
it is likewise a \( \text{PR}\in 2_\mathbb{N} \) map.

Proof:

- (Central), Composition case: We consider first coconjugated composition, namely
\[
(\Downarrow f \circ \Downarrow f^\circ) : 1 \times A \rightarrow C^B \times B^A \times A \rightarrow C
\]
\[
=_{\text{by def}} \Downarrow (C^B \times \text{pr} (\Downarrow f \times \Downarrow f^\circ) \circ A) : 1 \times A \rightarrow C^B \times B^A \times A \rightarrow C
\]
\[
\Downarrow \circ \Downarrow (\Downarrow f \circ \Downarrow f^\circ) : 1 \times A \rightarrow B \rightarrow C,
\]

\[(*)\]

The latter equation follows from the evaluation properties of closed evaluation instances \( \varepsilon : B^A \times A \rightarrow B \), and \( \varepsilon : C^B \times B \rightarrow C \), by Free Variable chasing – namely free variable \( a := n_{1,A} : 1 \times A \rightarrow A \).
By *conjugation* of (both sides of) the above equation we get the assertion in the present composition case:

\[
\begin{align*}
\text{(conj)} & \Rightarrow (g[f]) : 1 \times A \to C^B \times B^A \times A \to C \\
& = (g \circ f) : 1 \times A \to A \to C \\
& = [g \circ f] : 1 \to C^A.
\end{align*}
\]

- Case of *Cartesian product*: analogous, “exercise”.
- *Iteration case*: We start again with the conjugate side: For a \( \text{PR} \in n \) endo \( f : A \to A \), we want to show

\[
\begin{align*}
& [f^\#] : 1 \times (A \times \mathbb{N}) \to A \\
& = \text{by def} \quad f^\# \circ \cong : 1 \times (A \times \mathbb{N}) \xrightarrow{\cong} A \times \mathbb{N} \xrightarrow{f^\#} A
\end{align*}
\]

For **Proof** of (**∗∗∗**) we use the **definition** above, of \( \langle r \rangle : A^A \to A^{A \times \mathbb{N}} \), making commute the lower two rectangles of the following diagram:

\[
\begin{array}{c}
1 \times (A \times \mathbb{N}) \\
\xrightarrow{= \cong} \\
(1 \times A) \times \mathbb{N}
\end{array}
\xrightarrow{\cong}

\begin{array}{c}
A \times \mathbb{N} \\
\xrightarrow{f^\#} \\
A
\end{array}
\xrightarrow{\langle r \rangle}

\begin{array}{c}
A^A \times (A \times \mathbb{N}) \\
\xrightarrow{=} \\
(A^A \times A) \times \mathbb{N}
\end{array}
\xrightarrow{=} \\
\left(\ell, n\right) \in \mathbb{N}

\begin{array}{c}
A^A \times A
\end{array}
\]

For showing (**∗∗∗**), we show commutativity of the frame diagram, by free variables diagram chasing, with free variables \( a := \ell_{A,N} \) \( n := r_{A,N} \):

\[
\begin{array}{c}
(0, (a, n)) \\
\xrightarrow{\cong}
\end{array}
\xrightarrow{= \cong}

\begin{array}{c}
(a, n) \\
\xrightarrow{\cong}
\end{array}
\xrightarrow{\ell, n \in \mathbb{N}}

\begin{array}{c}
(f^\#(a, n)) \\
\xrightarrow{\ell, n \in \mathbb{N}}
\end{array}
\]

Remains to show (**→**), i.e. to show:

\[
\begin{align*}
(\ell, n) \in \mathbb{N} & : (f[a], n) = (f^\#, a) : 1 \times A \times \mathbb{N} \\
& \to A^A \times A.
\end{align*}
\]
We show this by **external** Peano Induction, i.e. by uniqueness of the iterated, as follows:

\[(\ell, \in)\hat{=}((|f|, a), 0) = (|f|, a) = (|f|, f^\hat{=} (a, 0))\] (anchor)

as well as

\[(\ell, \in)\hat{=}((|f|, a), n + 1) = (\ell, \in)\hat{=}((|f|, \in_{A,A} (|f|, a), n)) = (\ell, \in)\hat{=}((|f|, f(a)), n)\]

by *evaluation property* of \(\in_{A,A} : A^A \times A \to A\)

\(= (|f|, f^\hat{=} (f(a), n))\) by induction hypothesis on \(n\)

\(= (|f|, f^\hat{=} (a, n + 1)) : A \times \mathbb{N} \to A.\) (step)

This shows (●), i.e. (→) in the diagram: Map \((\ell, e)^\hat{=}\) – diagram – throws in fact \(((|f|, a), n)\) into \((|f|, f^\hat{=} (a, n))\). So assertion (** **) above has been shown. Whence, by conjugation:

\[|f^\hat{=}| = \text{cocon}[f^\hat{=} \circ \cong 1 \times (A \times \mathbb{N}) \to A \times \mathbb{N} \overset{f^\hat{=}}{\to} A]\]

\[= \cup \notin \cdot |f| : 1 \overset{|f|}{\rightarrow} A^A \overset{\cup \notin \cdot}{\rightarrow} A^A \times \mathbb{N},\]

and that proves the remaining case of *Structure Preservation* via Closed Internalisation **q.e.d.**

We now come to our central result, the

**Interpretation Theorem:**

(i) **CoDomain Suitability** of interpretation family: PR defined \(\text{PR} \in 2^n\) interpretation family \(\text{int}^n_{A,B} : |A, B|_n \to \mathcal{U}_2^\mathbb{N}\) – indexed by Object-pairs, stratum (stra) \(\text{PR} \in \mathbb{N}\) (and \(\text{PR} \in 2^n\)) – restricts in its (single) CoDomains to

\(\text{int}^n_{A,B} : |A, B|_n \to B^A [\overset{\subseteq}{\rightarrow} \mathcal{U}_2^\mathbb{N}].\) (\(*\))

within \(\text{PR} \in 2^n\), in form of a commuting diagram, for \(B\) having a point:

![Interpretation Diagram](image)

Interpretation **Diagram**: stratum by stratum, global/individual with respect to map-code sets
(ii) **Objectivity** within one stratum: For \( f : A \to B \) in \( \mathbf{PR} \in \mathbb{N} \subseteq \mathbf{PR} \in \), we have

\[
\mathbf{PR} \in \mathbb{N} \vdash \text{int}_{A,B}^{\mathbb{N}} (\lceil f \rceil) =_{\text{by def}} \text{int}_{A,B}^{\mathbb{N}} \circ \lceil f \rceil = [f] : 1 \to B^A \quad (**) 
\]

Codes “originating from” Objective level are interpreted into names.

(iii) **Stratum-Globalisation** of Interpretation: Stratum-indexed family

\[ [\text{int}_{A,B}^{\mathbb{N}} : |A, B|_\mathbb{N} \to B^A]_{\mathbb{N} \in \mathbb{N}} \]

admits, within Theory \( \mathbf{PR} \in \), Object \( \lceil A, B \rceil \) of \( \mathbf{PR} \in \) again as an ascending Union, written \( \lceil A, B \rceil =_{\text{by def}} \bigcup_{\mathbb{N} \in \mathbb{N}} |A, B|_\mathbb{N} \) predicate-ly, and has the universal property of an inductive limit by \( \mathbf{PR} \) “construction”.

In particular, family \( \text{int}_{A,B}^{\mathbb{N}} : |A, B|_\mathbb{N} \to B^A \) above induces a – unique – strata-global map \( \text{int}_{A,B} : \lceil A, B \rceil \to B^A \) (***) making commute the following diagram:

(iv) **Strata-global Objectivity** of Interpretation, “Codes to names”:

For an arbitrary \( \mathbf{PR} \in \mathbb{N} \) map \( f : A \to B \) we have:

\[
\mathbf{PR} \in \vdash \text{int}_{A,B} (\lceil f \rceil) =_{\text{by def}} \text{int}_{A,B} \circ \lceil f \rceil = [f] : 1 \to B^A \subseteq U_{2\mathbb{N}}. 
\]

**Proof:**

(i) Type control \( V_{\mathbb{N}} \supset |A, B|_\mathbb{N} \ni u \mapsto \text{int}_{A,B}^{\mathbb{N}} (u) \in B^A \subseteq U_{2\mathbb{N}} \):

This is proved by structural induction on \( u \), i.e. on depth(\( u \)) : \( |A, B|_\mathbb{N} \supset |A, B|_{\mathbb{N}} \), \( \mathbb{N} \) “suitable” such that all the finitely many building blocks \( v, w, \ldots \) are in finitely many components of sum

\[
V_{\mathbb{N}} = \bigoplus_{A,B} |A, B|_{\mathbb{N}}. 
\]

This type assertion has been (pre-) discussed already above.
Proof of second assertion (***) on Objectivity of each member of the \( n \in \mathbb{N} \) and Object-pair \( A, B \) indexed family is now as expected, namely by external structural induction on (external) \( \text{depth} [f] \) of map \( f : A \rightarrow B \) in \( \text{PR} \in \) in question, with \( \gamma \in |A, B|_n \), suitable \( n \in \mathbb{N} \): Each such \( f \) comes with such a “suitable” \( n \), since obviously the \( \text{PR} \in \mathbb{N} \), \( n \in \mathbb{N} \), exhaust all of Theory \( \text{PR} \in \) here considered. Now here is the Proof of Interpretation-Objectivity, by structural induction on \( \text{depth} [f : A \rightarrow B] \) “to be interpreted”:

For \( f \) one of the map constants of \( \text{PR} \in = \text{PR}_A + (\text{hom}) \) with \( \text{depth} [f] = 1 \) say, in particular for the members of adjunction map families \( \in_{A, B} \) and \( \triangleleft_{A, B} \), the assertion is trivial, by definition of interpretation \( \text{Int} \), and corresponding \( \text{int} \in_{A, B} \) in these cases. We now consider \( \text{PR} \in \) maps with greater \( \text{depth} : \)

For \( f : A \rightarrow B, g : B \rightarrow C \) in \( \text{PR} \in \)

\[
\text{PR} \in \mathbb{N} \vdash \text{int}^{n}_{A, C} (\gamma g \circ f \gamma) = \text{int}^{n}_{A, C} (\gamma g \gamma \odot \gamma f \gamma)
\]

\[
\text{[} \odot \text{ = } \gamma \odot \text{ constructively internalises } \circ \text{]}
\]

\[
= \text{by def } \text{int}^{n}_{B, C} (\gamma g \gamma) \odot \text{int}^{n}_{A, B} (\gamma f \gamma)
\]

\[
= [g] \odot \text{hyp. on } f \text{ and } g
\]

\[
= [g \circ f] : 1 \rightarrow C^A,
\]

the latter by the composition case of Structure preservation by Axiomatic Internalisation above.

Similar (recursive) Proof for the assertion in case of the other binary meta-operation, the Cartesian product of maps.

- Remains the case of an iterated \( f^\S : A \times N \rightarrow A \), given by the unary meta-operation \( \S \): In this case we have

\[
\text{PR} \in \mathbb{N} \vdash \text{int}^{n}_{A \times N, A} (\gamma f^\S \gamma) = \text{int}^{n}_{A \times N, A} (\gamma f \gamma \gamma^\S)
\]

\[
\text{by definition of constructive code of an iterated}
\]

\[
= \text{by def } \S (\text{int}^{n}_{A, A} (\gamma f \gamma))
\]

\[
(\text{“homomorphic” PR definition of interpretation Int})
\]

\[
= \S \circ \text{hyp. on } \text{depth} [f]
\]

\[
= [f^\S] : 1 \rightarrow A^{A \times N},
\]

the latter, eventually, by the iteration case of Structure Preservation of Closed Internalisation.

The last two assertions of the Theorem – (**) and (\*) – follow straightforward from the former two, by the inductive-limit property of our Universal Chain \( U \).

Comment: The members of family \( \text{Int}^{n}_{A} : V_n \rightarrow U_2 \), are special maps – Objective map terms – of Theory \( \text{PR} \in 2n \), \( \text{PR} \in \), and are therefore covered “themselves” by the – in this regard self-related Interpretation
Theorem above. This is the reason why I have chosen as a Universal Class not a single Object or “super-Object” for Theory $\text{PR} \in$, but an ascending chain of “Universal Objects” $\mathcal{U}_n$, such that Object $\mathcal{U}_2 \subseteq$, hosts in particular interpretation of all map codes of stratum $\text{PR} \in 2\mathbb{N}$: Chain $\mathcal{U}$ is “upwards open”, think at HILBERT’s hotel.

6 Self-Evaluation

Here is the key Consequence of the two last assertions (*** and (●) of the Interpretation Theorem, namely possibility for a constructive self-evaluation of Theory $\text{PR} \in$:

Define code-self-evaluation family for Theory $\text{PR} \in$, called $\tilde{\varepsilon} = \tilde{\varepsilon}_{A,B}$:

$$\tilde{\varepsilon}_{A,B} = \tilde{\varepsilon}_{A,B} (u, a) = \exists_{A,B} (\text{int}_{A,B} (u), a) :$$

$$V \times A \supset [A, B]_{\text{PR} \in} \times A \rightarrow B \times A \subseteq B.$$

Comment: Here we used assertion (*** for availability of suitable Order-global interpretation family $\text{int}_{A,B}$:

$$\text{int}_{A,B} : [A, B]_{\text{PR} \in} = \bigcup_{\mathbb{N}} \uparrow [A, B] \rightarrow B^A.$$

We get further, by last assertion – (●) of the Theorem, objectivity of self-evaluation $\tilde{\varepsilon}$, namely: for (any) $f : A \rightarrow B$ in $\text{PR} \in$

$$\text{PR} \in \vdash \tilde{\varepsilon}_{A,B} (\vec{f} \vec{\neg}, a) = \exists_{A,B} (\text{int}_{A,B} (\vec{f} \vec{\neg}), a)$$

$$= \exists_{A,B} (\lfloor f \rfloor, a) = f(a) : A \rightarrow B. \quad (\ast)$$

For this latter equation see introduction – and discussion – of name of $f$ above, $\lfloor f \rfloor = [f : A \rightarrow B] : 1 \rightarrow B^A$ – in set theory: $\lfloor f \rfloor = \{(\emptyset, f)\} : 1 \rightarrow B^A \subset \mathcal{P}(A \times B)$.

Based on this self-evaluation family of Theory $\text{PR} \in$, we now find within $\text{PR} \in$ the following (anti) diagonal $d = d(n) : \mathbb{N} \rightarrow 2$: $\text{PR} \in$-map $d = d(n) : \mathbb{N} \rightarrow 2$ is defined as

$$d = \text{def} \rightarrow \circ \tilde{\varepsilon}_{\mathbb{N},2} \circ (\#, \text{id}_{\mathbb{N}}) : \mathbb{N} \rightarrow [\mathbb{N}, 2]_{\text{PR} \in} \times \mathbb{N} \xrightarrow{\tilde{\varepsilon}} 2 \xrightarrow{\neg} 2,$$

with $\# = \#(n) : \mathbb{N} \xrightarrow{\cong} [\mathbb{N}, 2]_{\text{PR} \in}$ the – isomorphic – $\text{PR}$ count of all (internal) predicate codes (“Klassenzeichen” in GÖDEL’s sense), of Theory $\text{PR} \in$. As expected in such diagonal argument – Antinomie Richard quoted by GÖDEL – we substitute, within Theory $\text{PR} \in$, the counting index $q = \text{def} \#^{-1} (\vec{r} \vec{\neg}) = \#^{-1} \circ \vec{r} \vec{\neg} : 1 \rightarrow [\mathbb{N}, 2]_{\text{PR} \in} \xrightarrow{\tilde{\varepsilon}} \mathbb{N}$, of $d$’s code into $\text{PR} \in$-map $d : \mathbb{N} \rightarrow 2$ itself, and get a “liar” map liar: $1 \rightarrow 2$, called
liar because it turns out that this map is its own negation, as follows:

\[
\begin{align*}
\text{PR} & \vdash \text{liar} = \text{def } d \circ q : \mathbb{I} \to \mathbb{N} \to 2 \\
& = \text{by def } d \circ \#^{-1} \circ \gamma d^n \\
& = \text{by def } \neg \circ \tilde{\varepsilon}_{\mathbb{N},2} \circ (\# \circ \text{id}_\mathbb{N}) \circ \#^{-1} \circ \gamma d^n \\
& = \neg \circ \tilde{\varepsilon}_{\mathbb{N},2} (\gamma d^n, \#^{-1} \circ \gamma d^n) \\
& = \text{by def } \neg \circ \tilde{\varepsilon}_{\mathbb{N},2} (\gamma d^n, q) \\
& = \neg \circ d(q) = \neg \circ d \circ q \\
& = \text{by def } \neg \text{liar} : \mathbb{I} \to 2 \to 2,
\end{align*}
\]

a contradiction: The argument is equation marked (**), which is a special instance of objectivity equation (*) above, objectivity of self-evaluation \(\tilde{\varepsilon}\), which has been defined within theory \(\text{PR} \in\) out of closed evaluation \(\in\) composed with interpretation family \(\text{int}\), of map codes into names.

**Conclusion:** The argument shows incompatibility of (even just potential) infinity with (formally, axiomatically given) Cartesian Closed “Higher Order” structure of Theory \(\text{PR} \in\).

We obtain this way inconsistency of all extensions of Theory \(\text{PR} \in\), in particular of – higher order – set theories, and also of any type of higher Order Arithmetic, even when given in a categorical setting, as in particular in Lawvere 1963, and then in Freyd’s 1972 setting of (higher Order) Topos Theory with NNO, and in that of Lambek & Scott 1986.

The present argument does not depend on quantification nor on availability of a subobject classifier: the (equality) predicates we rely on here are given by the Cartesian PR Arithmetic of theories considered.

**Disclaimer:** The argument does not apply to Closed Categories in the sense of Eilenberg & Kelly, since there is no NNO required for the theory. In the applications, e.g. Categories of Modules, there is an NNO only downstairs, in a suitably conceived category of sets. But that NNO does not bear (naturally) the structure of an abelian group.

Even if you consider the category of abelian semi-groups which includes semigroup \(\mathbb{N} = \langle \mathbb{N}, 0, + \rangle\): an *iterated* \(f^\# : A \times \mathbb{N} \to A\) will not become linear, even not bilinear, and hence even not linear when converted into a map \(f^\#: A \otimes \mathbb{N} \to A\) from the tensor product into \(A\). So this category cannot have \(\mathbb{N}\) as an NNO in any suitable way.

Analogously, the original Elementary Theory ETT of Topoi seems me to be not concerned, ETT in the sense explained by Wraith 1973 on the base of mainly (?) Lawvere 1970, 1972, and Tierney 1971, as well as more recently explained in Lawvere & Shanduel 1991:

The data and axioms for this genuine Theory of Topoi do not include an NNO. The motivating examples for Topoi are Categories of sheaves over a topological space. Question: Do these – Cartesian Closed – Categories come with an NNO on sheaf level? By the above, they cannot come so, except they are based on an – inconsistent – Cartesian Closed set Theory with NNO.
**Problem:** Diagonal map above is a map within subSystem $\text{PR} \in n$, subSystem of Theory $\text{PR} \in n$, for $n$ from some $n_0$ upwards. Presumably an upper bound for such contradictory Order $n_0$ can be calculated. It would be certainly interesting to know a lower bound $n_0$, making $P^2$ contradictory, incompatible with (potential) infinity, in the sense of availability of a Natural Numbers Object $N$.

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