Universal enveloping Rota—Baxter algebras of preassociative and postassociative algebras
V.Yu. Gubarev

Abstract
Universal enveloping Rota—Baxter algebras of preassociative and postassociative algebras are constructed. The question of Li Guo is answered: the pair of varieties (\(RB_\lambda As, postAs\)) is a PBW-pair and the pair (\(RBAs, preAs\)) is not.

Keywords: Rota—Baxter algebra, universal enveloping algebra, PBW-pair of varieties, preassociative algebra, postassociative algebra.

Introduction

Linear operator \(R\) defined on an algebra \(A\) over the key field \(k\) is called Rota—Baxter operator (RB-operator, for short) of a weight \(\lambda \in k\) if it satisfies the relation

\[
R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \quad x, y \in A.
\] (1)

An algebra with given RB-operator acting on it is called Rota—Baxter algebra (RB-algebra, for short).

G. Baxter defined (commutative) RB-algebra in 1960 [4], solving an analytic problem. The relation (1) with \(\lambda = 0\) appeared as a generalization of by part integration formula. J.-C. Rota and others [31, 9] studied combinatorial properties of RB-operators and RB-algebras. In 1980s, the deep connection between Lie RB-algebras and Young—Baxter equation was found [5, 32]. To the moment, there are a lot of applications of RB-operators in mathematical physics, combinatorics, number theory, and operad theory [11, 12, 13, 20].

There exist different constructions of free commutative RB-algebra, see the articles of J.-C. Rota, P. Cartier, and L. Guo [31, 9, 24]. In 2008, K. Ebrahimi-Fard and L. Guo obtained free associative RB-algebra [16]. In 2010, L.A. Bokut et al [8] got a linear base of free associative RB-algebra with the help of Gröbner—Shirshov technique. Diverse linear bases of free Lie RB-algebra were recently found in [19, 22, 30].

Pre-Lie algebras were introduced in 1960s independently by E.B. Vinberg, M. Gerstenhaber, and J.-L. Koszul [34, 17, 25], they satisfy the identity \((x_1 x_2)x_3 - x_1(x_2 x_3) = (x_2 x_1)x_3 - x_2(x_1 x_3)\).

J.-L. Loday [27] defined the notion of (associative) dendriform dialgebra, we will call it preassociative algebra or associative prealgebra. Preassociative algebra is a vector space with two bilinear operations \(>, <\) satisfying the identities

\[
(x_1 > x_2 + x_1 < x_2) > x_3 = x_1 > (x_2 > x_3), \quad (x_1 > x_2) < x_3 = x_1 > (x_2 < x_3),
\]

\[
x_1 < (x_2 > x_3 + x_2 < x_3) = (x_1 < x_2) < x_3.
\]

In [26], J.-L. Loday also defined Zinbiel algebra (we will call it as precommutative algebra), on which the identity \((x_1 > x_2 + x_2 > x_1) > x_3 = x_1 > (x_2 > x_3)\) holds. Any
preassociative algebra with the identity $x \succ y = y \prec x$ is a precommutative algebra and with respect to the new operation $x \cdot y = x \succ y - y \prec x$ is a pre-Lie algebra. There is an open problem if every pre-Lie algebra injectively embeds into its universal enveloping preassociative algebra (in affirmative case, will be the pair of varieties of pre-Lie and preassociative algebras a PBW-pair [29]).

In [28], there was also defined (associative) dendriform trialgebra, i.e., an algebra with the operations $\prec, \succ, \cdot$ satisfying certain 7 axioms. (We will call such algebra as postassociative algebra or associative postalgebra.) Post-Lie algebra [33] is an algebra with two bilinear operations $[.,.]$ and $\cdot$; moreover, Lie identities with respect to $[.,.]$ hold and the next identities are satisfied:

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) - (y \cdot x) \cdot z + y \cdot (x \cdot z) = [y,x] \cdot z, \quad x \cdot [y,z] = [x \cdot y, z] + [y, x \cdot z].$$

Given a binary operad $\mathcal{P}$, the notion of successor [2] provides the defining identities for pre- and post-$\mathcal{P}$-algebras. Equivalently, one can define the operad of pre- and post-$\mathcal{P}$-algebras as $\mathcal{P} \bullet \text{PreLie}$ and $\mathcal{P} \bullet \text{PostLie}$ respectively. Here PreLie denotes the operad of pre-Lie algebras and PostLie — the operad of post-Lie algebras, $\mathcal{V} \bullet \mathcal{W}$ is the black product of operads $\mathcal{V}, \mathcal{W}$ (see [18] about operads and Manin products). Hereinafter, pre- and postalgebra will denote pre- and post-$\mathcal{P}$-algebra for some variety (operad) $\mathcal{P}$.

In 2000, M. Aguiar [11] remarked that any associative algebra with defined on it a Rota—Baxter operator $R$ of zero weight with respect to the operations $a \prec b = aR(b)$, $a \succ b = R(a)b$ is a preassociative algebra. In 2002, K. Ebrahimi-Fard [13] showed that one can additionally define on an associative RB-algebra of nonzero weight $\lambda$ the third operation $a \cdot b = \lambda ab$ and get the structure of postassociative algebra under the operations $\prec, \succ, \cdot$. In 2007, the notion of universal enveloping RB-algebras of pre- and postassociative algebra was introduced [15].

For free preassociative algebra $C$, injectivity of embedding $C$ into its universal enveloping was proved in [15].

In 2010, with the help of Gröbner—Shirshov bases [8], Yu. Chen and Q. Mo proved that any preassociative algebra over the field of zero characteristic injectively emdebs into its universal enveloping RB-algebra [10].

In 2010, C. Bai et al [3] introduced $O$-operator, a generalization of RB-operator, and stated that any associative pre- and postalgebra injectively embeds into an algebra with $O$-operator.

In [2], the construction of M. Aguiar and K. Ebrahimi-Fard was generalized on the case of arbitrary operad, not only associative.

In [20], given a variety $\text{Var}$, it was proved that any pre-$\text{Var}$-algebra injectively embeds into its universal enveloping $\text{Var}$-$\text{RB}$-algebra of zero weight and any post-$\text{Var}$-algebra injectively embeds into its universal enveloping $\text{Var}$-$\text{RB}$-algebra of weight $\lambda \neq 0$. Based on the last results, we have

**Problem 1.** To construct an universal enveloping $\text{RB}$-algebra of pre- and postalgebra.

In the comments to the head $V$ of the unique monograph on $\text{RB}$-algebras [23], L. Guo actually stated the following

**Problem 2.** To clarify if the pairs of varieties ($\text{RBAs, preAs}$) and ($\text{RB}_\lambda\text{As, postAs}$) for $\lambda \neq 0$ are PBW-pairs [29].
Here RBAs (RB\(_\lambda\)As) denotes the variety of associative algebras endowed with an RB-operator of (non)zero weight \(\lambda\).

In the current article, Problem 1 is solved for associative pre- and postalgebras and Problem 2 is solved completely.

In §1, we state preliminaries about RB-algebras, PBW-pairs, preassociative and post-associative algebras. Universal enveloping RB-algebras of preassociative (§2) and post-associative algebras (§3) are constructed. As a corollary, we obtain that the pair of varieties (RB\(_\lambda\)As, postAs) is a PBW-pair and the pair (RBAs, preAs) is not.

1 Preliminaries

1.1 RB-operator

Let us consider some well-known examples of RB-operators (see, e.g., [23]):

**Example 1.** Given an algebra \(A\) of continuous functions on \(\mathbb{R}\), an integration operator \(R(f)(x) = \int_0^x f(t) \, dt\) is an RB-operator on \(A\) of zero weight.

**Example 2.** Given an invertible derivation \(d\) on an algebra \(A\), \(d^{-1}\) is an RB-operator on \(A\) of zero weight.

**Example 3.** Let \(A = A_1 \oplus A_2\) be a direct sum (as vector space) of its two subalgebras. An operator \(R\) defined as

\[ R|_{A_1} \equiv 0, \quad R|_{A_2} \equiv \lambda \text{id}, \]

where \(\text{id}\) denotes the identical map, is an RB-operator on \(A\) of the weight \(-\lambda\).

Further, unless otherwise specified, RB-operator will mean RB-operator of zero weight.

We denote a free algebra of a variety \(\text{Var}\) generated by a set \(X\) by \(\text{Var}(X)\), and a free RB-algebra of a weight \(\lambda\) respectively by \(\text{RB}_\lambda\text{Var}(X)\). For short, denote \(\text{RB}_0\text{Var}(X)\) by \(\text{RBVar}(X)\).

1.2 PBW-pair of varieties

In 2014, A.A. Mikhalev and I.P. Shestakov introduced the notion of a PBW-pair [29]; it generalizes the relation between the varieties of associative and Lie algebras in the spirit of Poincaré—Birkhoff—Witt theorem.

Given varieties of algebras \(\mathcal{V}\) and \(\mathcal{W}\), let \(\psi: \mathcal{V} \to \mathcal{W}\) be a such functor that maps an algebra \(A \in \mathcal{V}\) to the algebra \(\psi(A) \in \mathcal{W}\) preserving \(A\) as vector space but changing the operations on \(A\). There exists left adjoint functor to the \(\psi\) called universal enveloping algebra and denoted as \(U(A)\). Defining on \(U(A)\) a natural ascending filtration, we get associated graded algebra \(\text{gr} \, U(A)\).

A pair of varieties \((\mathcal{V}, \mathcal{W})\) with the functor \(\psi: \mathcal{V} \to \mathcal{W}\) is called PBW-pair if \(\text{gr} \, U(A) \cong U(\text{Ab} \, A)\). Here \(\text{Ab} \, A\) denotes the vector space \(A\) with trivial multiplicative operations.
1.3 Free associative RB-algebra

In [16], free associative RB-algebra in the terms of rooted forests and trees was constructed. In [8] Gröbner—Shirshov theory of associative RB-algebras was developed. Based on the results [8, 16], one can get by induction linear base of free associative RB-algebra.

Further, we will refer to the constructed base as the (standard) base of RB-algebra.

Given a word \( u \) from the standard base of RB\( _\lambda \text{As}(X) \), the number of appearances of the symbol \( R \) in the notation of \( u \) is called \( R \)-degree of the word \( u \), denotation: \( \deg_R(u) \).

We will call any element of the standard base of RB\( _\lambda \text{As}(X) \) of the form \( R(w) \) as a degree \( R \)-letter. By \( X_\infty \) we denote the union of the set \( X \) and the set of all \( R \)-letters. Let us define a degree \( \deg u \) of the word \( u \) from the standard base as the length of \( u \) in the alphabet \( X_\infty \). For example, given the word \( u = x_1 R(x_2)x_3 R(x_4 x_5) \), we have \( \deg_R(u) = 2 \) and \( \deg u = 4 \), since \( u \) consists of four different letters \( x_1, x_3, R(x_2), R(x_4 x_5) \in X_\infty \).

1.4 Preassociative algebra

A linear space within two bilinear operations \( \succ, \prec \) is called preassociative algebra if the following identities hold on it:

\[
\begin{align*}
(x_1 \succ x_2 + x_1 \prec x_2) \succ x_3 &= x_1 \succ (x_2 \succ x_3), \\
(x_1 \succ x_2) \prec x_3 &= x_1 \succ (x_2 \prec x_3), \\
x_1 \prec (x_2 \succ x_3 + x_2 \prec x_3) &= (x_1 \prec x_2) \prec x_3.
\end{align*}
\]

In [27], free preassociative algebra in the terms of forests was constructed. Let \( A \) be a preassociative algebra, \( a_1, a_2, \ldots, a_k \in A \). Introduce

\[
\begin{align*}
(a_1, a_2, \ldots, a_k)_{\prec} &= a_1 \prec (a_2 \prec \ldots (a_{k-2} \prec (a_{k-1} \prec a_k)) \ldots), \\
(a_1, a_2, \ldots, a_k)_{\succ} &= (\ldots ((a_1 \succ a_2) \prec a_3) \prec \ldots) \prec a_{k-1} \prec a_k.
\end{align*}
\]

**Statement.** The following relation holds in any preassociative algebra \( A \):

\[
((b_1, b_2, \ldots, b_k, a)_{\succ}, c_1, c_2, \ldots, c_l)_{\prec} = (b_1, b_2, \ldots, b_k, (a, c_1, c_2, \ldots, c_l)_{\prec})_{\succ},
\]

for any natural numbers \( k, l \) and \( a, b_1, \ldots, b_k, c_1, \ldots, c_l \in A \).
PROOF. We will prove the statement by induction on \( k \). Let \( k = 1 \). Now we will proceed the induction on \( l \). For \( l = 1 \) the statement follows from (3). Assume that we have proved the formula for all numbers less \( l \). The inductive step on \( l \) follows from the equalities

\[
((b_1, a), c_1, c_2, \ldots, c_l)_\prec = \ldots (((b_1 \succ a) \prec c_1) \prec c_2) \prec \ldots \prec c_l \\
= \ldots (((b_1 \succ (a \prec c_1)) \prec c_2) \prec \ldots \prec c_l = ((b_1, a \prec c_1), c_2, c_3, \ldots, c_l)_\prec \\
= (b_1, (a \prec c_1, c_2, c_3, \ldots, c_l)_\prec)_\prec = (b_1, (a, c_1, c_2, \ldots, c_l)_\prec)_\prec.
\]

Assuming the statement is true for all natural numbers less \( k \), the following equalities prove the inductive step:

\[
((b_1, b_2, \ldots, b_k, a), c_1, c_2, \ldots, c_l)_\prec = ((b_1, b_2, \ldots, b_{k-1}, b_k \succ a), c_1, c_2, \ldots, c_l)_\prec \\
\quad \quad = (b_1, b_2, \ldots, b_{k-1}, (b_k \succ a, c_1, c_2, \ldots, c_l)_\prec)_\prec \\
\quad \quad = (b_1, b_2, \ldots, b_{k-1}, (b_k, (a, c_1, c_2, \ldots, c_l)_\prec)_\prec) \\
\quad \quad \quad \quad = (b_1, b_2, \ldots, b_k, (a, c_1, c_2, \ldots, c_l)_\prec)_\prec.
\]

### 1.5 Postassociative algebra

Postassociative algebra is a linear space with three bilinear operations \( \cdot, \succ, \prec \) satisfying 7 identities:

\[
(x \prec y) \prec z = x \prec (y \succ z + y \succ z + y \cdot z), \quad (x \succ y) \prec z = x \succ (y \prec z), \quad (x \succ y + y \succ x + x \cdot y) \succ z = x \succ (y \succ z), \\
x \succ (y \cdot z) = (x \succ y) \cdot z, \quad (x \prec y) \cdot z = x \cdot (y \prec z), \\
(x \cdot y) \prec z = x \cdot (y \prec z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z).
\]

**Example 4** [28]. Let \( K = \mathbb{k}[y_1, y_2, \ldots, y_k, \ldots] \) be polynomial algebra on countable number of variables \( y_k \). By induction, define the operations \( \cdot, \succ, \prec \) on the augmentation ideal \( I \triangleleft K \):

\[
y_k \omega \cdot y_{k'} \omega' = y_{k+k'}(\omega * \omega'), \quad y_k \omega \succ y_{k'} \omega' = y_{k'}(y_k \omega * \omega'), \quad y_k \omega \prec y_{k'} \omega' = y_k(\omega * y_{k'} \omega'),
\]

where \( a \ast b = a \succ b + a < b + a \cdot b \). The space \( I \) with the defined operations is postassociative algebra.

In [28], free postassociative algebra in the terms of trees was constructed.

### 1.6 Embedding of Loday algebras in RB-algebras

The common definition of the varieties of pre- and post-Var-algebra for a variety \( \text{Var} \) could be found in [3] or [21].
Given an associative algebra $B$ with RB-operator $R$ of zero weight, the space $B$ with respect to the operations

$$x \succ y = R(x)y, \quad x \prec y = xR(y)$$

is a preassociative algebra.

For any RB-operator $R$ on $B$ of weight $\lambda \neq 0$, we have that an operator $R' = \frac{1}{\lambda}R$ is an RB-operator of unit weight. The space $B$ with the operations $x \cdot y = xy$ and (7) defined for $R'$ is a postassociative algebra. Denote the constructed pre- and postassociative algebras as $B^{(R)}_{\lambda}$. For short, we will denote $B^{(R)}_{0}$ as $B^{(R)}$.

Given a preassociative algebra $\langle C, \succ, \prec \rangle$, universal enveloping associative RB-algebra $U$ of $C$ is an universal algebra in the class of all associative RB-algebras of zero weight such that there exists injective homomorphism from $C$ to $U^{(R)}$. Analogously universal enveloping associative RB-algebra of a postassociative algebra is defined. The common denotation of universal enveloping of pre- or postassociative algebra: $U_{RB}(C)$.

**Theorem 1** [20].

a) Any pre-Var-algebra could be embedded into its universal enveloping RB-algebra of the variety Var and zero weight.

b) Any post-Var-algebra could be embedded into its universal enveloping RB-algebra of the variety Var and nonzero weight.

Based on Theorem 1, we have the natural question: What does a linear base of universal enveloping RB-algebra of a pre- or post-Var-algebra look like for an arbitrary variety Var? In the case of associative pre- and postalgebras, the question appeared in [23]. The current article is devoted to answer the question in the associative case.

In the article, the common method to construct universal enveloping is the following. Let $X$ be a linear base of a preassociative algebra $K$. We find a base of universal enveloping $U_{RB}(K)$ as the special subset $E$ of the standard base of $\langle B \rangle$ closed under the action of RB-operator. By induction, we define a product $\ast$ on the linear span of $E$ and prove its associativity. Finally, we state universality of the algebra $\mathbb{k}E$.

In the case of postassociative algebras, as it was mentioned above, we will consider universal enveloping associative RB-algebra of unit weight.

## 2 Universal enveloping Rota—Baxter algebra of preassociative algebra

In the paragraph, we will construct universal enveloping RB-algebra of arbitrary preassociative algebra $\langle C, \succ, \prec \rangle$. Let $B$ be a linear base of $C$.

**Definition.** An element $v = R(x)$ of the standard base $\langle B \rangle$ is

- left-good if $v$ is not of the form $R(b)$ or $R(bR(y))$ for $b \in B$,
- right-good if $v$ is not of the form $R(b)$ or $R(R(y)b)$ for $b \in B$,
- good if $v$ is left- and right-good simultaneously,
- semigood if $v$ is good or of the form

$$v = R(a_1 R(\ldots R(a_{2k-1} R(R(y)a_{2k})) \ldots) a_4)) a_2),$$

where $R(y)$ is good, $a_1, a_2, \ldots, a_{2k} \in B$. 

Remark that any semigood element is right-good and any element from the standard base of the form \( v = R(x), x \notin B \), is left-good or right-good.

Let us consider an associative algebra \( A = \text{As}(B)((a \prec b)c - a(b \succ c), a, b, c \in B) \) with the multiplication \( \cdot \). Here the expressions \( a \prec b \) and \( a \succ b \), \( a, b \in B \), equal the results of the products in the preassociative algebra \( C \). As \( B \) is the linear base of \( C \), the expressions are linear combinations of the elements of \( B \).

Due to Gröbner—Shirshov theory, namely diamond lemma for associative algebras \([6, 7]\), there exists such set \( E_0 \subset S(B) \) that \( \bar{E}_0 \) — the image of \( E_0 \) under the factorization of \( \text{As}(B) \) by the ideal \( \langle (a \prec b)c - a(b \succ c), a, b, c \in B \rangle \) — is a base of \( A \); moreover, for any decomposition of an element \( v \in E_0 \) into a concatenation \( v = v_1v_2 \) we have \( v_1, v_2 \in E_0 \).

Let us define by induction \( \text{Envelope} \)-words (shortly \( E \)-words), a subset of the standard base of \( \text{RBAs}(B) \):

1) elements of \( E_0 \) are \( E \)-words of the type 1;
2) given \( E \)-word \( u \), we define \( R(u) \) as an \( E \)-word of the type 2;
3) the word

\[
v = u_0R(v_1)u_1R(v_2)u_2 \cdots u_{k-1}R(v_k)u_k, \quad \deg v \geq 2, \deg_R(v) \geq 1,
\]

is an \( E \)-word of the type 3, if \( u_1, \ldots, u_{k-1} \in E_0, u_0, u_k \in E_0 \cup \{\emptyset\} \), \( v_1, \ldots, v_k \) are \( E \)-words, \( R(v_2), \ldots, R(v_{k-1}) \) are semigood elements of the standard base, \( R(v_1) \) is right-good. Given \( u_0 \not= \emptyset \), the element \( R(v_1) \) is semigood. Given \( u_k \not= \emptyset \), the element \( R(v_k) \) is semigood, else \( R(v_k) \) is left-good.

**Theorem 2.** The set of all \( E \)-words forms a linear base of universal enveloping associative RB-algebra of \( C \).

**Lemma 1.** Let \( D \) denote a linear span of all \( E \)-words. One can define such bilinear operation \( * \) on the space \( D \) that \((k-l) \) denotes below the condition on the product \( v * u \), where \( v \) is an \( E \)-word of the type \( k \) and \( u \) is an \( E \)-word of the type \( l \).

1–1: given \( v, u \in E_0 \), we have

\[
v * u = v \cdot u.
\]

1–2: given \( v = w'a \in E_0, a \in B, w' \in E_0 \cup \{\emptyset\} \), an \( E \)-word \( u = R(p) \) of the type 2, we have

\[
v * u = \begin{cases} 
  w' \cdot (a \prec b), & p = b \in B, \\
  (w' \cdot (a \prec b))R(x) - wR(R(b) * x), & p = bR(x), b \in B, \\
  wR(p), & R(p) \text{ is left-good}.
\end{cases}
\]

1–3: given \( v = w = w'a \in E_0, a \in B, w' \in E_0 \cup \{\emptyset\} \), an \( E \)-word \( u \) of the type 3, we have

\[
v * u = \begin{cases} 
  (w \cdot u_0)R(x)u', & u = u_0R(x)u', u_0 \in E_0, \\
  (w' \cdot (a \prec b)) \cdot (R(y) * u') & u = R(bR(y))u', b \in B, \\
  -wR(R(b) * y) * u' & R(bR(y)) \text{ is not semigood}, \\
  wR(x)u', & u = R(x)u', R(x) \text{ is semigood}.
\end{cases}
\]
3–1: given \( u = w = aw' \in E_0, a \in B, w' \in E_0 \cup \{\emptyset\} \), an \( E \)-word \( v \) of the type 3, we have

\[
v * u = \begin{cases} 
v'R(x)(v_0 * w), & v = v'R(x)v_0, v_0 \in E_0, \\
(v' * R(y)) * ((b \succ a) * w') & v = v'R(R(y)b), b \in B, \\
-v' * R(y * R(b))w, & v = v'R(x), R(x) \text{ is good.}
\end{cases}
\]

(11)

2–2: given \( E \)-words \( v = R(v'), \ u = R(u') \) of the type 2, we have

\[
v * u = v'R(v') * R(u') = R(R(v') * u' + v' * R(u')).
\]

(12)

2–3: given an \( E \)-word \( v = R(x) \) of the type 2, an \( E \)-word \( u \) of the type 3, we have

\[
v * u = \begin{cases} 
((a \succ b) \cdot u_0)R(y)u', & x = a, u = bu_0R(y)u', u_0 \in E_0, a, b \in B; \\
R(z)((a \succ b) \cdot u_0)R(y)u' & x = R(z)a, u = bu_0R(y)u', u_0 \in E_0, a, b \in B; \\
-R(z * R(a))u, & R(x)u, \\
R(x)u, & u = bu', b \in B, R(x) \text{ is right-good,} \\
R(R(x) * y + x * R(y))u', & u = R(y)u'.
\end{cases}
\]

(13)

3–3: given \( E \)-words \( v, u \) of the type 3, we have

\[
v * u = \begin{cases} 
v'R(R(x) * y + x * R(y))u', & v = v'R(x), u = R(y)u', \\
v'R(x)(w * u), & v = v'R(x)w, u = R(y)u', w \in E_0, \\
(v * w)R(y)u', & v = v'R(x), u = wR(y)u', w \in E_0, \\
v'R(x)(v_0 \cdot u_0)R(y)u', & v = v'R(x)v_0, u = u_0R(y)u', v_0, u_0 \in E_0.
\end{cases}
\]

(14)

The following conditions are also satisfied:

L1) Given left-good \( E \)-word \( R(x), b \in B \), and \( R(b) * x = \sum u_i, \) where \( u_i \) is an \( E \)-word, we have that \( R(u_i) \) is left-good for every \( i \).

L2) Given right-good \( E \)-word \( R(y) \), any \( E \)-word \( x \), and \( R(x) * y = \sum u_i, \) where \( u_i \) is an \( E \)-word, we have that \( R(u_i) \) is right-good for every \( i \).

L3) Given \( E \)-word \( x \notin B, \) \( E \)-word \( u \) of the type 1 or 3, and \( R(x) * u = \sum u_i, \) where \( u_i \) is an \( E \)-word, we have that \( u_i \) is an \( E \)-word of the type 3 and begins with an \( R \)-letter, i.e., has a view \( u_i = R(x_i)u'_i \), for every \( i \).

L4) Given an \( E \)-word \( u = u'a, a \in B, \) of the type 1 or 3, any \( E \)-word \( v \), and \( v * u = \sum u_i, \) where \( u_i \) is an \( E \)-word, we have \( u_i = u'a_i, a_i \in B \).

L5) Given \( E \)-word \( u = aw, a \in B, w \in E_0 \cup \{\emptyset\} \), of the type 1, any \( E \)-word \( v \), and \( v * u = \sum u_i, \) where \( u_i \) is an \( E \)-word, we have that \( u_i \) is one of the following forms:

\[ w' \cdot aw, \quad (c \succ a) \cdot w, \quad v'_iR(x_i)((c_1, c_2, \ldots, c_k, a)_\succ \cdot w), \]

for \( c, c_1, c_2, \ldots, c_k \in B, k \geq 0, w' \in E_0 \). Moreover, the number, view of summands and values of \( v'_iR(x_i) \) and \( c_i \) depend only on \( v \).

L6) Given \( E \)-word \( u = awR(p)u', a \in B, w \in E_0 \cup \{\emptyset\}, \) of the type 3, any \( E \)-word \( v \), and \( v * u = \sum u_i, \) where \( u_i \) is an \( E \)-word, we have that \( u_i \) is one of the following forms:

\[ (w' \cdot aw)R(p)u', \quad ((c \succ a) \cdot w)R(p)u', \quad v'_iR(x_i)((c_1, c_2, \ldots, c_k, a)_\succ \cdot w)R(p)u', \]
where \( c_1, c_2, \ldots, c_k \in B, k \geq 0, w' \in E_0 \). Moreover, the number, view of summands and values of \( v'_i R(x_i) \) and \( c_i \) depend only on \( v \).

L7) Given \( E \)-word \( u = R(s)aw R(t)u', a \in B, w \in E_0 \cup \{ \emptyset \} \), of the type 3, in which empty word could stay instead of \( R(t)u' \), any \( E \)-word \( v \), and \( v \cdot u = \sum u_i \), where \( u_i \) is an \( E \)-word of the form

\[
v'_i R(s_i)((c_1, c_2, \ldots, c_k, a)_x \cdot w) R(t)u',
\]

\( c_1, c_2, \ldots, c_k \in B, k \geq 0 \). Moreover, the number, view of summands and values of \( v'_i R(s_i) \) and \( c_i \) depend only on \( v \) and \( s \).

The relations for the products of \( E \)-words of types 2–1, 3–2 and the conditions R1–R7 are defined analogously to the products of \( E \)-words of types 1–2, 2–3 and the conditions L1–L7 by the inversion of letters, multipliers and operations symbols, wherein \( \succ \) and \( \prec \) turn in each other.

PROOF. Let us define the operation \( \ast \) with the prescribed conditions for \( E \)-words \( v, u \) by induction on \( r = \deg_R(v) + \deg_R(u) \). For \( r = 0 \) define \( v \ast u = v \cdot u, v, u \in E_0, \) which satisfies the condition 1–1. It is easy to see that the conditions L5 and R5 are also fulfilled, all others are true because of \( r = 0 \).

For \( r = 1 \), let us define \( v \ast u \) by induction on \( d = \deg(v) + \deg(u) \). For \( d = 2, v = a \in B, u = R(w), w \in E_0, \) we define

\[
v \ast u = a \ast R(w) = \begin{cases} a \prec b, & w = b \in B, \\ aR(w), & w \in E_0 \setminus B, \end{cases}
\]

which satisfies the condition 1–2. We define the product \( R(w) \ast a \) analogously to (15) up to the inversion. The cases 1–1 and 2–2 are not realizable because of \( r = 1 \). It is correct to write \( aR(w), \) as the element \( R(w) \) is left-good for \( w \in E_0 \setminus B \).

For \( r = 1, d > 2 \), we define \( v \ast u \) for pairs of \( E \)-words of types 1–2, 1–3; the definition for types 2–1, 3–1 is analogous up to the inversion. The cases 2–3, 3–2, and 3–3 do not appear for \( r = 1 \).

Let \( v = w_1 = w'a \in E_0, a \in B, u = R(w_2), w_1, w_2 \in E_0, \) define

\[
v \ast u = \begin{cases} w' \cdot (a \prec b), & w_2 = b \in B, \\ w_1 R(w_2), & w_2 \in E_0 \setminus B, \end{cases}
\]

what is consistent with the condition 1–2. The notation \( w_1 R(w_2) \) is correct because the element \( R(w_2) \) is left-good for \( w_2 \in E_0 \setminus B \).

Let \( v \in E_0, u \) be an \( E \)-word of the type 3, define

\[
v \ast u = \begin{cases} (v \cdot u_0) R(w)u', & u = u_0 R(w)u', u_0 \in E_0, w' \in E_0 \cup \{ \emptyset \}, w \in E_0 \setminus B, \\ v R(w)u', & u = R(w)u', w' \in E_0, w \in E_0 \setminus B, \end{cases}
\]

what is consistent with the condition 1–2. The notations are correct by the same arguments as above.

For \( r = 1 \), clarify that the conditions L1–L7 (and analogously R1–R7) hold. Indeed, L1) \( R(x) \) is left-good, \( \deg_R(x) = 0 \), so \( x \in E_0, \deg(x) \geq 2, x = ax', a \in B \). Hence,
\( R(b) \ast x = (b \succ a) \cdot x' \) is a linear combination of the elements from \( E_0 \setminus B \). L3) Given
left-good \( E \)-word \( R(x) \), we have \( x \in E_0 \setminus B \) and \( R(x) \ast w = R(x)w \) for \( w \in E_0 \). L2) Given
right-good \( E \)-word \( R(y) \), \( y \in E_0 \) we have \( y = ay' \in E_0 \setminus B \), \( a \in B \). So \( R(x) \ast y \) equal
\( R(xy) \) for \( x \in E_0 \setminus B \) or \( (x \succ a) \cdot y' \) for \( x \in B \) is a linear combination of the elements
from \( E_0 \setminus B \). It is easy to check the conditions L4–L7 (R1–R7).

Suppose that the product \( v \ast u \) is yet defined for all pairs of \( E \)-words \( v, u \) such that
\( \deg_{R}(v) + \deg_{R}(u) < r, \ r \geq 2 \), and all conditions of the statement on \( \ast \) are satisfied. Let us
define \( \ast \) on \( E \)-words \( v, u \) with \( \deg_{R}(v) + \deg_{R}(u) = r \) by induction on \( d = \deg(v) + \deg(u) \).
For \( d = 2 \), consider the cases 1–2 and 2–2 (the case 2–1 is analogous to 1–2).

1–2: let \( v = a \in B, \ u = R(p) \) be an \( E \)-word of the type 2, \( \deg_{R}(u) = r \geq 2 \), define
\( v \ast u \) by (9).

2–2: let \( v = R(v'), u = R(u') \) be an \( E \)-word of the type 2, define \( v \ast u \) by (12).

The products \( R(b) \ast x, R(v') \ast u', v' \ast R(u') \) in (9), (12) are defined by induction on \( r \).
The element \( p = bR(x) \) is an \( E \)-word of the type 3, \( R(x) \) is left-good and, therefore,
the concatenation \( (a \prec b)R(x) \) is correct. By the condition L1 for \( R(b) \ast x \) holding by
inductive assumption, the notation \( aR(B(b) \ast x) \) is correct. The conditions L2–L5 (and
also R2–R5) in the cases \( d = 2 \), 2–1 and 2–2 hold; the conditions L1 and R1 are realizable
only in the case 2–2, hence, they are also fulfilled; the conditions L6, L7 (as R6, R7) are
totally not realizable, so they hold.

For \( d > 2 \), define the product \( v \ast u \) for \( E \)-words pairs of the following cases: 1–2, 2–3,
3–3, 1–3, 3–1; the products for the cases 2–1 and 3–2 are defined analogously up to the
inversion.

1–2: let \( v = w = w'a \in E_0, a \in B, \ u = R(p) \) be an \( E \)-word of the type 2, \( \deg_{R}(u) = r \geq 2 \), define
\( v \ast u \) by (9).

The product \( R(b) \ast x \) in (9) is defined by induction on \( r \). The element \( p = B(bR) \) is
\( E \)-word of the type 3, \( R(x) \) is left-good and the concatenation \( (w' \cdot (a \prec b)R(x) \) is
correct. By the condition L1 holding for \( R(b) \ast x \) by inductive assumption, the notation
\( wR(B(b) \ast x) \) is correct. The definition of \( v \ast u \) is consistent with L1–L7 (R1–R7).

2–3: let \( v = R(x) \) be an \( E \)-word of the type 2, \( u \) be an \( E \)-word of the type 3, define
\( v \ast u \) by (13).

The notations \( R(z \ast R(a))u, R(R(x) \ast y + x \ast R(y))u \) in (13) are correct by the
conditions R1, L2, and R3; the products are defined by induction on \( r \). The definition of
\( v \ast u \) is consistent with L1–L7 (R1–R7).

3–3: let \( v, u \) be \( E \)-words of the type 3, define \( v \ast u \) by (14).

The correctness of the products in (14) follows in the first case by the conditions L2,
L3, R2, R3, in the second and third — by L4, R4. The definition of \( v \ast u \) is consistent
with L1–L7 (R1–R7).

1–3: let \( v = w = w'a \in E_0, a \in B, \ u \) be an \( E \)-word of the type 3, define \( v \ast u \) by (10).

3–1: let \( u = w = aw' \in E_0, a \in B, \ v \) be an \( E \)-word of the type 3, define \( v \ast u \) by (11).

The conditions L1 and R1 provide that the notations \( wR(B(b) \ast y), R(y \ast R(b))u \) in
(10), (11) are correct. The definition of the product in the case 1–3 for \( u = R(bR(y))u' \),
where \( R(bR(y)) \) is not semigood, is reduced to the cases 3–1 and 3–3. In the last variant
of the case 3–1, the product is expressed by the one from the case 1–3. We have to prove
that the process of computation \( v \ast u \) in the cases 1–3 and 3–1 is finite. Suppose we have
\( v = w = w'b \in E_0, b \in B, \ u = R(bR(y))u' \), and \( R(bR(y)) \) is not semigood, i.e., has the
form

\[ u = R(a_1 R(R(a_3 R(R(\ldots R(a_{2k+1} R(y)) \ldots) a_4)) a_2))cw'' \]

where \( R(y) \) is good, \( a_1, a_2, \ldots, a_{2k+1}, c \in B \). Provided \( t - p \) is a sum of \( E \)-words or products with summary \( R \)-degree less than \( r \), we will denote it by \( t \equiv p \). Write

\[
v \ast u \equiv -v R(R(a_1) \ast (R(a_3 R(R(\ldots R(a_{2k+1} R(y)) \ldots) a_4)) a_2)) \ast cw''
\]

\[
\equiv -v R(R(a_1) \ast R(a_3 R(R(\ldots R(a_{2k+1} R(y)) \ldots) a_4)) a_2) \ast cw''
\]

\[
\equiv v R(R(a_1 R(R(a_3) \ast (R(a_5 R(R(\ldots R(a_{2k+1} R(y)) \ldots) a_6)) a_4)) a_2) \ast cw''
\]

\[
\equiv -v R(R(a_1 R(R(a_3 \ast R(a_5 R(R(\ldots R(a_{2k+1} R(y)) \ldots) a_6)) a_4)) a_2) \ast cw''). \quad (16)
\]

Continuing on and rewriting analogously the product into the action of the central \( R \)-letter, finally we will have the expression

\[
v \ast u \equiv -v R(R(a_1 R(a_3 R(\ldots R(R(\ldots R(a_{2k-1} R(R(a_{2k+1}) \ast y)) a_{2k}) \ldots) a_4)) a_2) \ast cw'', \quad (17)
\]

in which \( R(R(a_{2k+1}) \ast y) \) is left-good by the condition L1 and right-good by L2. Let

\[
R(a_{2k+1}) \ast y = \sum z_i, \quad (18)
\]

for good \( E \)-words \( R(z_i) \). We will prove that the definition of \( v \ast u \) is correct and has no cycles for any \( i \). By the definition of the product for types 3–1, we have

\[
v R(R(a_1 R(R(a_3 R(\ldots R(R(\ldots R(a_{2k-1} R(z_i)) a_{2k}) \ldots) a_4)) a_2) \ast cw''
\]

\[
\equiv -v R(R(a_1 R(R(a_3 R(\ldots R(R(\ldots R(a_{2k-1} R(z_i)) a_{2k}) \ldots) a_4) a_2)) cw''
\]

\[
\equiv -v R(R(a_1 R(R(a_3 R(\ldots R(R(a_{2k-1} R(z_i)) a_{2k}) \ldots) a_4)) a_2) cw''
\]

\[
\equiv v R(R(a_1 R(R(a_3 R(\ldots R(R(a_{2k-1} R(z_i)) a_{2k}) \ldots) a_4) a_2)) cw''
\]

\[
\equiv -v R(R(a_1 R(\ldots (a_3 R(\ldots R(R(\ldots R(a_{2k-1} R(z_i)) a_{2k}) \ldots) a_4) a_2)) cw''
\]

\[
\equiv \ldots \equiv -v R(a_1 R(\ldots R(R(\ldots R(a_{2k-1} R(z_i)) a_{2k}) \ldots) a_4) a_2)) cw''
\]

\[
= -v R(a_1 R(\ldots R(R(\ldots R(a_{2k-1} R(z_i) a_{2k} + z_i \ast R(a_{2k}) \ldots) a_4)) a_2)) cw'', \quad (19)
\]

\( R(z_i \ast R(a_{2k})) \) is good by R1 and R2, so the last equality is true.

The product \( v \ast u \) of \( E \)-words of types 1–3 and 3–1 satisfies the conditions L1–L7 (R1–R7).

Lemma 2. The space \( D \) with the operations \( \ast, R \) is an RB-algebra.

Proof. It follows from (12).

Lemma 3. The relations \( R(a) \ast b = a \succ b, a \ast R(b) = a \prec b \) hold in \( D \) for every \( a, b \in B \).

Proof. It follows from Lemma 1, the first case of (9), and analogous relation of 2–1.

Lemma 4. Given any \( E \)-word \( w \in E_0, a, b \in B \), the equality \((w, a, R(b)) = (R(b), a, w) = 0 \) is true on \( D \).

Proof. Let us define a map \( \vdash : As(B) \odot kB \to As(B) \) on the base as follows:

\[
w a \vdash b = w(a \prec b), \quad (20)
\]
From (3) and linearity, we have (equality, we obtain
\[ (a < b)c - a(b > c), a, b, c \in B. \] Let
\[ I = \langle (a < b)c - a(b > c), a, b, c \in B \rangle \triangleleft \text{As}(B). \]

From (3) and linearity, we have \((b > c) \vdash d = b > (c < d), b, c, d \in B\). Applying this equality, we obtain
\[
(wa(b > c) - w(a < b)c) \vdash d = wa((b > c) \vdash d) - w(a < b)(c < d)
= wa(b > (c < d)) - w(a < b)(c < d) \in I,
\]
\[
(a(b > c)we - (a < b)cwe) \vdash d = a(b > c)w(e < d) - (a < b)cw(e < d) \in I
\]
for \(a, b, c, d \in B, w \in E_0 \cup \{\emptyset\}\).

Hence, the induced map \(\vdash: A \otimes \mathbb{k}B \rightarrow A\) is well-defined. Notice that
\[
(wa) \ast R(b) = w \cdot (a < b) = wa \vdash b + I, \quad a, b \in B, \quad w \in E_0 \cup \{\emptyset\}, \quad (21)
\]
where \(wa \in E_0, a \in B, w \in E_0 \cup \{\emptyset\}\). By (20) and (21), conclude
\[
(w, a, R(b)) = (w \cdot a) \vdash b - w \cdot (a < b) = 0.
\]

The proof of the equality \((R(b), a, w) = 0\) is analogous.

**Lemma 5.** The operation \(\ast\) on \(D\) is associative.

**Proof.** Given \(E\)-words \(x, y, z\), let us prove associativity
\[
(x, y, z) = (x \ast y) \ast z - x \ast (y \ast z) = 0
\]
by inductions on two parameters: at first, on summary \(R\)-degree \(r\) of the triple \(x, y, z\), at second, on summary degree \(d\) of \(x, y, z\).

For \(r = 0\), we have \(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) = 0, x, y, z \in E_0\), as the product \(\cdot\) is associative in the algebra \(A\).

Let \(r > 0\) and suppose that associativity for all triples of \(E\)-words with the less summary \(R\)-degree is proven.

We prove the statement for the triples \(x, y, z\), in which \(y\) is an \(E\)-word of the type 1 or 3, \(d\) is any. Let \(y\) be an \(E\)-word of the type 1. Consider the case \(y = a, a \in B\). By the condition L5, the product \(x \ast y\) is a sum \(\sum_{i \in I} u_i\), where \(u_i\) has one of the following forms:
\[
w' \cdot a, \quad e \succ a, \quad v_i' R(x_i)(c_1, c_2, \ldots, c_k, a)\succ, e, c_1, \ldots, c_k \in B.
\]

By R5, the product \(u_i \ast z\) equals to \(\sum_{j \in J} t_{ij}\), where \(t_{ij}\) is one of following forms: (The antepenultimate case is written by Lemma 4):
\[
(w' \cdot a) \cdot w'', \quad (e \succ a) \cdot w'', \quad v_i' R(x_i)((c_1, c_2, \ldots, c_k, a)\succ \cdot w''),
\]
\[
w' \cdot (a < f), \quad (e = a) < f, \quad v_i' R(x_i)((c_1, c_2, \ldots, c_k, a)\succ < f),
\]
\[
(w' \cdot (a, d_1, d_2, \ldots, d_l)\prec R(y_i)u'_i, \quad (e = a, d_1, d_2, \ldots, d_l)\prec R(y_i)u'_i,
\]
\[
v_i' R(x_i)((c_1, c_2, \ldots, c_k, a)\succ, d_1, d_2, \ldots, d_l)\prec R(y_i)u'_i.
\]
By the conditions R5 and L5, the product \( x \ast (y \ast z) \) is a sum with the same indexing sets \( I, J \) and summands of the forms

\[
\begin{align*}
w' \cdot (a \cdot w'') &\quad (e \succ a) \cdot w'', \quad v'_i R(x_i)((c_1, c_2, \ldots, c_k, a)_{\succ} \cdot w''), \\
w' \cdot (a < f) &\quad e \succ (a < f), \quad v'_i R(x_i)((c_1, c_2, \ldots, c_k, a < f)_{\succ}), \\
(w' \cdot (a, d_1, d_2, \ldots, d_l)_{\prec}) R(y_i) u'_i &\quad (e \succ (a, d_1, d_2, \ldots, d_l)_{\prec}) R(y_i) u'_i, \\
v'_i R(x_i)((c_1, c_2, \ldots, c_k, a, d_1, d_2, \ldots, d_l)_{\prec}) \ast R(y_i) u'_i.
\end{align*}
\]  

(23)

The corresponding summands in (22) and (23) either coincide or are equal by associativity in the algebra \( A \), the equality (3), and Statement.

For \( y = aw, a \in B, w \in E_0 \), the proof of associativity is analogous to the case \( y = a \).

Let us consider the case when \( y \) is an \( E \)-word of the type 3. Given \( y = aR(p)u', a \in B, u' \neq \emptyset \), by (11), (13), (14), and the conditions L6 and R4, we have \( (x \ast y) \ast z = \sum u_i \), where \( u_i \) is one of the forms:

\[
(w' \cdot a) R(p)(u' \ast z), \quad (e \succ a) R(p)(u' \ast z), \quad u'_i R(x_i)((c_1, c_2, \ldots, c_k, a)_{\prec} \ast R(p)(u' \ast z). \]  

(24)

By (11), (13), (14), and the conditions L6, R4, \( x \ast (y \ast z) = \sum u_i \), where \( u_i \) is one of the forms listed in (24). It proves associativity for the triple \( x, y, z \).

Let \( y = aR(p), a \in B \). By the conditions L6, R7, R3, \( (x \ast y) \ast z = \sum u_i \), where \( u_i \) has one of the following forms:

\[
((w' \cdot a), d_1, \ldots, d_l)_{\prec} R(p_i), \quad ((e \succ a), d_1, \ldots, d_l)_{\prec} R(p_i),

v'_i R(x_i)((c_1, c_2, \ldots, c_k, a)_{\prec}, d_1, \ldots, d_l)_{\prec} R(p_i),
\]  

(25)

\( e, c_1, \ldots, c_k, d_1, \ldots, d_l \in B, k, l \geq 0 \).

By the conditions L6, R7, R4, \( x \ast (y \ast z) = \sum u_i \), where \( u_i \) has one of the following forms:

\[
(w' \cdot (a, d_1, \ldots, d_l)_{\prec}) R(p_i), \quad (e \succ (a, d_1, \ldots, d_l)_{\prec}) R(p_i),

v'_i R(x_i)((c_1, c_2, \ldots, c_k, a, d_1, \ldots, d_l)_{\prec}) \ast R(p_i),
\]  

(26)

The corresponding summands in (25) and (26) are equal by Statement and Lemma 4.

Let \( y = R(s)aR(t), a \in B \). By the conditions L7, R7, R3, \( (x \ast y) \ast z = \sum u_i \), where \( u_i \) has one of the following forms:

\[
v'_i R(s_i)((c_1, c_2, \ldots, c_k, a)_{\prec}, d_1, \ldots, d_l)_{\prec} R(t_i) z'.
\]

By the conditions L7, R7, R4, \( x \ast (y \ast z) = \sum u_i \), where \( u_i \) is one of the following forms:

\[
v'_i R(s_i)((c_1, c_2, \ldots, c_k, a, d_1, \ldots, d_l)_{\prec}) \ast R(t_i) z'.
\]

By Statement, we have associativity.

The cases \( y = awR(p)u', y = R(s)awR(t)u' \), where \( a \in B, w \in E_0 \), are analogous to ones considered above.

Hence, we have to prove associativity only for triples \( x, y, z \), in which \( y \) is an \( E \)-word of the type 2. The definition of \( \ast \) by Lemma 1 is symmetric with respect to the
inversion, except the cases 1–3 and 3–1, so associativity in the triple of types $k$–$2$–$l$ leads to associativity in the triple $l$–$2$–$k$.

Prove associativity for $d = 3$ and $E$-word $y$ of the type 2.

1–2–1. Consider possible cases of $y$. a) For $y = R(b)$, $b \in B$, we have

$$(a \ast R(b)) \ast c - a \ast (R(b) \ast c) = (a \prec b) \cdot c - a \cdot (b \succ c) = 0.$$ b) If $y = R(u)$ is good, then

$$(x \ast y) \ast z - x \ast (y \ast z) = aR(u)c - aR(u)c = 0.$$ c) For $y = R(R(p)b)$, $b \in B$, we have

$$(x \ast y) \ast z - x \ast (y \ast z) = (a \ast R(R(p)b)) \ast c - a \ast (R(R(p)b) \ast c)$$
$$= (a \ast R(p)) \ast (b \succ c) - a \ast R(p \ast R(b))c - a \ast R(p)(b \succ c) + a \ast R(p \ast R(b))c$$
$$= (a, R(p), b \succ c) = 0,$$

the last is true by the induction on $r$.

d) For $y = R(bR(p))$, $b \in B$, we have

$$(x \ast y) \ast z - x \ast (y \ast z) = (a \ast R(bR(p))) \ast c - a \ast (R(bR(p))c)$$
$$= (a \prec b)R(p) \ast c - aR(R(b) \ast p)c - (a \prec b)R(p) \ast c + aR(b) \ast p \ast c = 0.$$ 1–2–2. a) Given left-good $y$, associativity follows from (9), (13).

b) Let $y = R(b)$, $b \in B$. If $z = R(c)$, $c \in B$, then by (14)

$$(a \ast R(b)) \ast R(c) - a \ast (R(b) \ast R(c)) = (a \prec b) \prec c - a \prec (b \succ c + b \prec c) = 0.$$ If $z = R(u)$ is left-good, then

$$a \ast (R(b) \ast R(u)) = a \ast R(R(b) \ast u + bR(u)) = a \ast R(R(b) \ast u) - a \ast R(R(b) \ast u) + (a \prec b)R(u)$$
$$= (a \ast R(b)) \ast R(u).$$ Finally, if $u = cR(p)$, $c \in B$, where $R(p)$ is left-good, then from one hand,

$$(a \ast R(b)) \ast R(cR(p)) = (a \prec b) \ast R(cR(p))$$
$$= -(a \prec b)R(R(c) \ast p) + ((a \prec b) \prec c)R(p). \quad (27)$$ From another hand,

$$a \ast (R(b) \ast R(cR(p))) = a \ast R(R(b) \ast cR(p) + b \ast R(cR(p)))$$
$$= a \ast R((b \succ c)R(p)) + a \ast R(-bR(R(c) \ast p) + (b \prec c)R(p))$$
$$= -aR(R((b \succ c)) \ast p) + (a \prec (b \succ c))R(p)$$
$$+ a \ast R(R(b) \ast (R(c) \ast p)) - (a \prec b)R(R(c) \ast p)$$
$$- aR(R(b \prec c) \ast p) + (a \prec (b \prec c))R(p). \quad (28)$$
Subtracting (28) from (27), by (4) we have

\[ aR(R((b \succ c)) \ast p) - a \ast R(R(b) \ast (R(c) \ast p)) + aR(R(b \prec c) \ast p) = a \ast R((R(b), R(c), p)) = 0, \]

which is true by induction on \( d \).

c) The last case, \( y = R(bR(u)), b \in B; z = R(v) \). From one hand,

\[
(x \ast y) \ast z = (a \ast R(bR(u))) \ast R(v) = (-aR(R(b) \ast u) + (a \prec b)R(u)) \ast R(v)
\]

\[
= -a \ast R(R(bR(b)) \ast u) \ast v + (R(b) \ast u) \ast R(v)) + (a \prec b) \ast R(R(u) \ast v + u \ast R(v)).
\]

From another hand, applying the condition L2, we have

\[
x \ast (y \ast z) = a \ast (R(bR(u)) \ast R(v)) = a \ast R(R(bR(u))) \ast v + bR(u) \ast R(v)
\]

\[
= a \ast R(R(bR(u))) \ast v + a \ast R(bR(R(u)) \ast v + u \ast R(v))
\]

\[
= a \ast R(R(bR(u))) \ast v - a \ast R(R(b) \ast (R(u) \ast v + u \ast R(v)))
\]

\[
+ (a \prec b) \ast R(R(u) \ast v + u \ast R(v)).
\]

Subtracting (30) from (29) we get the expression \( a \ast R(\Delta) \) for

\[
\Delta = -R(R(b) \ast u) \ast v - (R(b) \ast u) \ast R(v) - R(bR(u)) \ast v + R(b) \ast (R(u) \ast v + u \ast R(v))
\]

\[
= -(R(b), u, R(v)) - (R(b), R(u), v) = 0,
\]

the conclusion is true by induction on \( r \).

2–2–2. Given \( x = R(u), y = R(v), z = R(p) \), by induction we have

\[
(R(u), R(v), R(p)) = R((R(u), R(v), p) + (R(u), y, R(p)) + (x, R(v), R(p)) = 0.
\]

Let \( d > 3 \), consider triples \( x, y, z \), in which \( y \) is an \( E \)-word of the type 2.

Show that the cases with \( x \) or \( z \) equal to \( w \in E_0 \) are reduced to the cases with \( w = a \in B \). Indeed, let \( x = w = w'a, a \in B, w' \in E_0 \). By the condition R4 and associativity in the triples \( k-1-l \) and \( k-3-l \), we have

\[
(x \ast y) \ast z = ((w'a) \ast y) \ast z = (w' \ast (a \ast y)) \ast z = w' \ast ((a \ast y) \ast z),
\]

\[
x \ast (y \ast z) = w'a \ast (y \ast z) = w' \ast (a \ast (y \ast z)).
\]

Let at least one \( E \)-word from \( x, z \) be an \( E \)-word of the type 3, e.g., \( z \). For \( z = az' \), \( a \in B \), by the condition L4, associativity in the triples \( k-1-l \), \( k-3-l \), and induction on \( r \), we have

\[
(x \ast y) \ast z \ast x \ast (y \ast z) = (x \ast y) \ast (az') = x \ast (y \ast (az')) = ((x \ast y) \ast a) \ast z' = x \ast ((y \ast a) \ast z')
\]

\[
= ((x \ast y) \ast a) \ast z' - x \ast ((y \ast a) \ast z') = (x, y, a) \ast z' = 0.
\]

If \( z = R(s)az', a \in B, y = R(p) \), then we have

\[
(x \ast y) \ast z \ast x \ast (y \ast z) = (x \ast R(p)) \ast (R(s)az') = x \ast (R(p) \ast (R(s)az'))
\]
\[
= ((x \ast R(p)) \ast R(s)) \ast (az') - x \ast ((R(p) \ast (R(s))) \ast az') \\
= (x \ast (R(p) \ast R(s))) \ast (az') - x \ast ((R(p) \ast (R(s))) \ast az') \\
= (x \ast R(R(p) \ast s + p \ast R(s))) \ast (az') - x \ast (R(R(p) \ast s + p \ast R(s))) \ast az') \\
= (x, R(R(p) \ast s + p \ast R(s)), az') = 0
\]

by associativity in the triple \( k-3-l \) and induction on \( d \).

We have considered all possible cases of triples \( x, y, z \) of \( E \)-words. Lemma 5 is proved.

**Proof of Theorem 2.** Let us prove that the algebra \( D \) is exactly universal enveloping algebra for the preassociative algebra \( C \), i.e., is isomorphic to the algebra

\[ U_{\text{RBAs}}(C) = \text{RBAs}(B|a \succ b = R(a)b, a \prec b = aR(b), a, b \in B) \]

By the construction, the algebra \( D \) is generated by \( B \). Therefore, \( D \) is a homomorphic image of a homomorphism \( \varphi \) from \( U_{\text{RBAs}}(C) \). We will prove that all basic elements of \( U_{\text{RBAs}}(C) \) are linearly expressed by \( D \), this leads to nullity of kernel of \( \varphi \) and \( D \cong U_{\text{RBAs}}(C) \).

As the equality

\[ (a \prec b)c = aR(b)c = a(b \succ c) \]

is satisfied on \( U_{\text{RBAs}}(C) \), the space \( U_{\text{RBAs}}(C) \) is a subspace of \( \text{RBAs}(B|(a \prec b)c = a(b \succ c), a, b, c \in B) \). It is well-known [23] that algebras \( \text{RBA}(B) \) and \( \text{RBA}(\text{As}(B)) \) coincide. Consider the map \( \varphi: B \rightarrow \text{RBA}(\text{As}(B))/I \), where \( I \) is an RB-ideal (i.e., the ideal closed under RB-operator) generated by the set \( \{ (a \prec b)c = a(b \succ c), a, b, c \in B \} \), and the map \( \psi \), the composition of trivial maps \( \psi_1: B \rightarrow A \) and \( \psi_2: A \rightarrow \text{RBA}(A) \), where as above,

\[ A = \text{As}(B|(a \prec b)c - a(b \succ c), a, b, c \in B), \quad \text{RBA}(A) = \text{RBA}(E_0|w_1w_2 = w_1 \cdot w_2). \]

The base of \( \text{RBA}(A) \) [16] could be constructed by induction, it contains the elements

\[ w_1R(u_1)w_2R(u_2) \ldots w_kR(u_k)w_{k+1}, \]

where \( w_i \in E_0, i = 2, \ldots, k, w_1, w_{k+1} \in E_0 \cup \{ \emptyset \} \), and the elements \( u_1, \ldots, u_k \) are constructed on the previous step.

From \( \ker \psi \subseteq \ker \varphi \), we have the injective embedding \( U_{\text{RBAs}}(C) \) into \( \text{RBA}(A) \).

It remains to show, using (7), that the complement \( E' \) of the set of all \( E \)-words in the base of \( \text{RBA}(A) \) is linearly expressed via \( E \)-words. Applying the inductions in \( \text{RBA}(A) \) on the \( R \)-degree and the degree of base words, the relations

\[ R(a)u = \begin{cases} (a \succ b)u', & u = bu', b \in B, \\
R(R(a)t)u' + R(aR(t))u', & u = R(t)u'; \end{cases} \]

\[ aR(bR(u)) = aR(b)R(u) - aR(R(b)u) = (a \prec b)R(u) - aR(R(b)u), \quad a, b \in B, \]

the relations for \( uR(a) \) and \( R(R(u)b)a \) analogous to (31), (32), and the relations derived from (10)–(19) by the removing the symbol \( * \), we prove that the elements of \( E' \) are linearly expressed via \( E \)-words. The theorem is proved.

**Example 5.** Let \( C \) denote a free preassociative algebra generated by a set \( X \). Universal enveloping algebra \( U_{\text{RB}}(C) \) is a free associative RB-algebra generated by the \( X \).
This statement proven in [13] one could theoretically deduce from Theorem 2; but the proof of such corollary does not seem to be obvious.

Given a preassociative algebra $C$, $U_0(C)$ denotes a linear span of all $E$-words of zero $R$-degree in $U_{RB}(C)$.

**Example 6.** Let $D$ be a vector space of square matrices from $M_n(k)$, then $D$ is a preassociative algebra with respect to the operations $a \succ b = ab$ (as in $M_n(k)$), $a \prec b = 0$. From $D^2 = D$ we conclude $U_0(D) = D$.

**Example 7.** Let $K$ be a vector space of the dimension $n^2$ over the field $k$ with the operations $\succ, \prec$ defined as $a \succ b = a \prec b = 0$. We have that $U_0(K)$ equal $\text{As}\langle K \rangle$, a free associative algebra generated by $K$.

**Corollary 1.** The pair of varieties (RBAs, preAs) is not a PBW-pair.

**Proof.** Universal enveloping associative RB-algebras of finite-dimensional preassociative algebras $D$ and $K$ (from Examples 6 and 7) of the same dimension are not isomorphic, else the spaces $U_0(D)$ and $U_0(K)$ were isomorphic as vector spaces. But we have

$$\dim U_0(D) = \dim D = n^2 < \dim U_0(K) = \dim \text{As}\langle K \rangle = \infty.$$

Therefore, the structure of universal enveloping associative RB-algebra of a preassociative algebra $C$ essentially depends on the operations $\succ, \prec$ on $C$.

### 3 Universal enveloping Rota—Baxter algebra of post-associative algebra

In the paragraph, we will construct universal enveloping associative RB-algebra for a postassociative algebra $\langle C, \succ, \prec, \cdot \rangle$. Let $B$ be a linear base of $C$.

Define by induction $E$-words, a subset of the standard base of $\text{RB}_1\text{As}\langle B \rangle$:

1) elements of $B$ are $E$-words of the type 1;

2) given $E$-word $u$, we define $R(u)$ as an $E$-word of the type 2;

3) the word

$$v = u_0R(v_1)u_1R(v_2)u_2\ldots u_{k-1}R(v_k)u_k, \quad \deg v \geq 2, \deg R(v) \geq 1,$$

is $E$-word of the type 3, if $u_1, \ldots, u_{k-1} \in B$, $u_0, u_k \in B \cup \emptyset$, $v_1, \ldots, v_k$ are $E$-words of any type, moreover, $R(v_2), \ldots, R(v_{k-1})$ are semigood elements, $R(v_1)$ is right-good. Given $u_0 \neq \emptyset$, $R(v_1)$ is semigood. Given $u_k \neq \emptyset$, $R(v_k)$ is semigood, else $R(v_k)$ is left-good.

**Theorem 3.** The set of all $E$-words forms a linear base of universal enveloping associative RB-algebra of $C$.

**Lemma 6.** Let $D$ denote a linear span of all $E$-words. One can define such bilinear operation $*$ on the space $D$ that ($k-l$ denotes below the condition on the product $v * u$, where $v$ is an $E$-word of the type $k$ and $u$ is an $E$-word of the type $l$.)

1–1: given $v, u \in B$, we have

$$v * u = v \cdot u.$$  \hfill (33)
1–2: given \( v = a \in B \), an \( E \)-word \( u = R(p) \) of the type 2, we have
\[
v \star u = \begin{cases} 
  a \prec b, & p = b \in B, \\
  (a \prec b) R(x) - a \ast R(R(b) \ast x) - a \ast R(b \ast x), & p = bR(x), b \in B, \\
  aR(p), & R(p) \text{ is left-good.}
\end{cases}
\] (34)

1–3: given \( v = a \in B \), an \( E \)-word \( u \) of the type 3, we have
\[
v \star u = \begin{cases} 
  (a \cdot b) R(x) u', & u = bR(x) u', b \in B, \\
  (a \prec b) \ast (R(y) \ast u') - (a \ast R(R(b) \ast y)) \ast u' & u = R(bR(y)) u', b \in B, \\
  -(a \ast R(b \ast y)) \ast u', & R(bR(y)) \text{ is not semigood,} \\
  aR(x) u', & u = R(x) u', R(x) \text{ is semigood.}
\end{cases}
\] (35)

3–1: given \( u = a \in B \), an \( E \)-word \( v \) of the type 3, we have
\[
v \star u = \begin{cases} 
  v' R(x) (b \cdot a), & v = v' R(x) b, b \in B, \\
  (v' \ast R(y)) \ast (b \succ a) - v' \ast (R(y \ast R(b)) \ast a) & v = v' R(R(y) b), b \in B, \\
  -v' \ast (R(y \ast b) \ast a), & v = v' R(x), R(x) \text{ is good.}
\end{cases}
\] (36)

2–2: given \( E \)-words \( v = R(v') \), \( u = R(u') \) of the type 2, we have
\[
v \star u = R(v') \ast R(u') = R(R(v') \ast u' + v' \ast R(u') + v' \ast u').
\] (37)

2–3: given an \( E \)-word \( v = R(x) \) of the type 2, an \( E \)-word \( u \) of the type 3, we have
\[
v \star u = \begin{cases} 
  (a \succ b) R(y) u', & x = a, u = bR(y) u', a, b \in B, \\
  R(z) (a \succ b) R(y) u' - R(z \ast R(a)) \ast u & x = R(z) a, u = bR(y) u', a, b \in B, \\
  -R(z \ast a) \ast u, & u = bR(y) u', b \in B, R(x) \text{ is right-good,} \\
  R(x) u, & u = bR(y) u', b \in B, R(x) \text{ is right-good,} \\
  R(R(x) \ast y + x \ast R(y) + x \ast y) \ast u', & u = R(y) u'.
\end{cases}
\] (38)

3–3: given \( E \)-words \( v, u \) of the type 3, we have
\[
v \star u = \begin{cases} 
  (v' \ast R(x) \ast y + x \ast R(y))) \ast u' & v = v' R(x), u = R(y) u', \\
  v' R(x) (a \ast u'), & v = v' R(x) a, u = R(y) u', a \in B, \\
  (v \ast a) R(y) u', & v = v' R(x), u = aR(y) u', a \in B, \\
  v' R(x) (a \cdot b) R(y) u', & v = v' R(x) a, u = bR(y) u', a, b \in B.
\end{cases}
\] (39)

The following conditions are also satisfied:
L1) Given left-good E-word $R(x)$, $b \in B$, and $R(b) \ast x = \sum u_i + \sum u_i'$, where $u_i, u_i'$ are E-words, $\deg_{R}(u_i') < \deg_{R}(u_i) = \deg_{R}(x) + 1$, we have that $R(u_i)$ is left-good for every $i$.

L2) Given right-good E-word $R(y)$, any E-word $x$, and $R(x) \ast y = \sum u_i + \sum u_i'$, where $u_i, u_i'$ are E-words, $\deg_{R}(u_i') < \deg_{R}(u_i) = \deg_{R}(x) + \deg_{R}(y) + 1$, we have that $R(u_i)$ is right-good for every $i$.

L3) Given E-word $x \not\in B$, an E-word $u$ of the type 1 or 3, and $R(x) \ast u = \sum u_i + \sum u_i'$, where $u_i, u_i'$ are E-words, $\deg_{R}(u_i') < \deg_{R}(u_i) = \deg_{R}(x) + \deg_{R}(u) + 1$, we have that $u_i$ is an E-word of the type 3 and begins with an $R$-letter, i.e., has a view of $u_i = R(x_i)u_i'$, for every $i$.

L4) Given an E-word $u = u'a$, $a \in B$, of the type 1 or 3, any E-word $v$, and $v \ast u = \sum u_i$, where $u_i$ are E-words, we have that $u_i = u'_ia_i$, $a_i \in B$.

L5) Given $a \in B$, an E-word $u = R(x)bu'$, where the word $bu'$ could be empty, $b \in B$, we have $a \ast u = \sum a_jR(x_j)b_ju' + \sum (a_j \cdot b_j)u'$ with $a_j, b_j \in B$, $j \in J_1 \cup J_2$, of the view $a_j = (a, c_1, c_2, \ldots, c_k)_<$, $b_j = (d_1, d_2, \ldots, d_k)_>$, $c_k, d_k \in B$. Moreover, the number of summands and values of $c_j, d_k, x_j$ depend only on $x$.

The relations for the products of E-words of types 2–1, 3–2 and the conditions R1–R5 are defined analogously to the products of E-words of types 1–2, 2–3 and the conditions L1–L5 by the inversion of letters, multipliers and operations symbols, wherein $>$ and $<$ turn in each other, and $\cdot$ does not change.

**Proof.** Let us define the operation $\ast$ with the prescribed conditions for E-words $v, u$ by induction on $r = \deg_{R}(v) + \deg_{R}(u)$. For $r = 0$, define $v \ast u = v \cdot u$, $v, u \in B$, it satisfies the condition 1–1. It is easy to see that all conditions L1–L5 (R1–R5) hold.

The case $r = 1$ is possible only if $v = a \in B$, $u = R(b)$, $b \in B$ (or $v = R(b)$, $u = a$). Define $a \ast R(b) = a \prec b$, $R(b) \ast a = b \succ a$, this satisfies the condition 1–2. It is clear that all conditions L1–L5 (R1–R5) for $r = 1$ are fulfilled.

Suppose that the product $v \ast u$ is yet defined for all pairs of E-words $v, u$ such that $\deg_{R}(v) + \deg_{R}(u) < r$, $r \geq 2$, and all conditions of the statement on $\ast$ are satisfied. Let us define $\ast$ on E-words $v, u$ with $\deg_{R}(v) + \deg_{R}(u) = r$ by induction on $d = \deg(v) + \deg(u)$.

For $d = 2$, consider the cases 1–2 and 2–2 (the case 2–1 is analogous to 1–2).

1–2: let $v = a \in B$, $u = R(p)$ be an E-word of the type, $\deg_{R}(u) = r \geq 2$, define $v \ast u$ by (34).

2–2: let $v = R(v')$, $u = R(u')$ be an E-word of the type, define $v \ast u$ by (37).

The products $R(b) \ast x$, $b \ast x$, $R(v') \ast u'$, $v' \ast R(u')$, $v' \ast u'$ in (34), (37) are defined by induction on $r$. The multiplication of $a$ on $R(b \ast x)$ is defined, as $\deg_{R}(a) + \deg_{R}(R(b \ast x)) < r$. The multiplication of $a$ on $R(b \ast x)$ is defined by the condition L1 holding by inductive assumption for $R(b) \ast x$. The element $p = bR(x)$ is an E-word of the type 3, so $R(x)$ is left-good and, therefore, the concatenation $(a \prec b)R(x)$ is correct. The conditions L2–L5 (and R2–R5) in the cases $d = 2$, 2–1 and 2–2 hold; the conditions L1 and R1 are realizable only in the case 2–2, hence, they are also fulfilled.

For $d > 2$, define the product $v \ast u$ for E-words pairs of the cases 1–2, 2–3, 3–1, 3–2, 3–3, 1–3, 1–1; the products for the cases 2–1 and 3–2 are defined analogously up to the inversion.

1–2: let $v = a \in B$, $u = R(p)$ be an E-word of the type 2, $\deg_{R}(u) = r \geq 2$, define $v \ast u$ by (34).

The definition is correct by the same reasons as in the case 1–2 for $d = 2$. 

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2–3: let \( v = R(x) \) be an \( E \)-word of the type 2, \( u \) be an \( E \)-word of the type 3, define \( v \ast u \) by (38).

By the conditions R1, L2, R3 and induction on \( r \), the definition of the product in the case 2–3 is correct. The definition of \( v \ast u \) is consistent with L1–L5 (R1–R5).

3–3: let \( v, u \) be \( E \)-words of the type 3, define \( v \ast u \) by (39).

The correctness of products in (39) follows in the first case by the conditions L2, L3, R2, R3, in the second and third — by L4, R4. The definition of \( v \ast u \) is consistent with L1–L5 (R1–R5).

1–3: let \( v = a \in B \), \( u \) be an \( E \)-word of the type 3, define \( v \ast u \) by (35).

3–1: let \( u = a \in B \), \( v \) be an \( E \)-word of the type 3, define \( v \ast u \) by (36).

The conditions L1 and R1 provide that the definition in the cases 1–3 and 3–1 are true. The definition of the product in the case 1–3 for \( u = R(bR(y))u' \), where \( R(bR(y)) \) is not semigood, is reduced to the cases 3–1 and 3–3. In the last variant of the case 3–1, the product is expressed by the one from the case 1–3. The process of computation \( v \ast u \) is finite by the same reasons as in the proof of Lemma 1.

Actually the product in the case \( k \sim l \) from Lemma 6 differs from the one from Lemma 1 in additional summands of less \( R \)-degree.

The definition of \( v \ast u \) in the cases 1–3 and 3–1 is consistent with L1–L5 (R1–R5).

**Lemma 7.** The space \( D \) with the operations \( \ast, R \) is an RB-algebra.

**Proof.** It follows from (37).

**Lemma 8.** The relations \( R(a) \ast b = a \succ b, a \ast R(b) = a \prec b, a \ast b = a \cdot b \) hold in \( D \) for every \( a, b \in B \).

**Proof.** It follows from Lemma 1, equality 1–1 (33), the first case of (34), and analogous relation of 2–1.

**Lemma 9.** The operation \( \ast \) on \( D \) is associative.

**Proof.** Let us prove associativity on \( D \) by inductions on two parameters: at first, on summary \( R \)-degree \( r \) of the \( E \)-words triple \( x, y, z \), at second, on summary degree \( d \) of the triple \( x, y, z \).

For \( r = 0 \), we have \( (x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) = 0, x, y, z \in E_0 \), as the product \( \cdot \) is associative in the algebra \( A \).

Let \( r > 0 \) and suppose that associativity for all triples of \( E \)-words with the less summary \( R \)-degree is proven.

We prove the statement for the triples \( x, y, z \), in which \( y \) is an \( E \)-word of the type 1 or 3, \( d \) is any. We consider the cases 1–1–2, 1–1–3, and 1–3–1, all others could be proven analogously.

1–1–2. a) Given \( x = a, y = b, z = R(c) \), \( a, b, c \in B \), by (6), we compute

\[
(x \ast y) \ast z - x \ast (y \ast z) = (a \cdot b) \prec c - a \cdot (b \prec c) = 0.
\]

b) If \( z = R(x) \) is left-good, associativity is obvious.

c) Given \( x = a, y = b, z = R(cR(t)) \), \( a, b, c \in B \), we have

\[
(x \ast y) \ast z = (a \cdot b) \ast R(cR(t))
\]

\[= ((a \cdot b) \prec c)R(t) - (a \cdot b) \ast R(R(c) \ast t) - (a \cdot b) \ast R(c \ast t); \quad (40)
\]
\[ x \cdot (y \cdot z) = a \cdot ((b < c)R(t) - b \cdot R(R(c) \cdot t) - b \cdot R(c \cdot t)) \]
\[ = (a \cdot (b < c))R(t) - a \cdot (b \cdot R(R(c) \cdot t)) - a \cdot (b \cdot R(c \cdot t)). \quad (41) \]

The first summands of RHS of \((40)\) and \((41)\) equal by \((6)\), the second ones — by the condition L1 and induction on \(r\), the third ones — by induction on \(r\).

1–1–3. a) Given \(x = a, y = b, z = c z', a, b, c \in B\), by \((6)\), we compute
\[ (x \cdot y) \cdot z - x \cdot (y \cdot z) = ((a \cdot b) \cdot c)z' - (a \cdot (b \cdot c))z' = 0. \]

b) If \(z = R(x)z'\) and \(R(x)\) is semigood, associativity is obvious.

c) Given \(x = a, y = b, z = z_0 z', a, b, c \in B\), where an \(R\)-letter \(z_0\) is not semigood. Let us present \(z_0\) as
\[ z_0 = R(a_1 R(a_3 R(\ldots R(a_{2k-1} R(R(a_{2k+1} R(t)) a_{2k})) \ldots) a_4)) a_2), \]
\(R(t)\) is good, \(a_1, \ldots, a_{2k+1} \in B\).

Let \(k = 0, z_0 = R(c R(t))\), \(R(t)\) be good, we have
\[ (x \cdot y) \cdot z = (a \cdot b) \cdot R(c R(t)) z' \]
\[ = ((a \cdot b < c) R(t) \cdot z' - ((a \cdot b) \cdot R(R(c) \cdot t)) \cdot z' - (a \cdot (b \cdot R(c \cdot t)) \cdot z'). \quad (42) \]
\[ x \cdot (y \cdot z) = a \cdot ((b < c) R(t) \cdot z' - (b \cdot R(R(c) \cdot t)) \cdot z' - (b \cdot R(c \cdot t)) \cdot z' \]
\[ = (a \cdot (b < c)) R(t) \cdot z' - a \cdot ((b \cdot R(R(c) \cdot t)) \cdot z') - a \cdot ((b \cdot R(c \cdot t)) \cdot z'). \quad (43) \]

The first summands of RHS of \((42)\) and \((43)\) equal by \((6)\) and induction on \(r\), the third ones — by induction on \(r\), the second ones — by the condition L1, induction on \(r\), and the fact that \((b \cdot R(R(c) \cdot t)) \cdot z' = b R(s_0)) z' + (b \cdot R(s_1)) \cdot z'\) for \(\deg_R(s_1) < \deg_R(s_0) = \deg_R(t) + 1\).

Let \(k > 0\). Presenting \(z_0 = R(c R(R(t) d))\), by \((42), (43)\), and the induction on \(r\), we have
\[ (x, y, z) = -((a \cdot b) \cdot R(R(c) \cdot R(t) \cdot d)) \cdot z' + a \cdot ((b \cdot R(R(c) \cdot R(t) d)) \cdot z') \]
\[ = -(a \cdot b) \cdot (R(R(R(c) \cdot t) \cdot d + R(c \cdot R(t)) \cdot d + R(c \cdot t) \cdot d)) \cdot z' \]
\[ + a \cdot ((b \cdot R(R(c) \cdot t) \cdot d + R(c \cdot R(t)) \cdot d + R(c \cdot t) \cdot d)) \cdot z') \quad (44) \]

As \(z_0\) is not semigood, \(t = c R(p), e \in B\), from \((41)\) we have
\[ (x, y, z) = (a \cdot b) \cdot (R(R(c \cdot R(R(e) \cdot p)) \cdot d) \cdot z') - a \cdot ((b \cdot R(R(c \cdot R(R(e) \cdot p)) \cdot d)) \cdot z') \]
Continuing on the process, we obtain
\[ (x, y, z) = \]
\[ [(a \cdot b) \cdot (R(R(a_1 R(R(a_3 R(\ldots R(a_{2k-1} R(R(a_{2k+1} R(q)) a_{2k})) \ldots) a_4)) \cdot a_2) \cdot z') \]
\[ - a \cdot ((b \cdot R(R(a_1 R(R(a_3 R(\ldots R(a_{2k-1} R(R(a_{2k+1} R(q)) a_{2k})) \ldots)) a_4)) \cdot a_2) \cdot z') \]
\[ (45) \]
As \(R(q)\) is good, the inner product \(a_{2k+1} \ast R(q)\) equals \(s_0 + s_1\) with \(\deg_R(s_1) < \deg_R(s_0) = \deg_R(q) + 1\). \(R(s_0)\) is good. Hence, \((x, y, z) = 0\) by induction on \(r\) and the condition 1–3.

1–3–1. The only case which is not analogous to the cases considered above is the following: \(x = a, y = R(s)bR(t), z = c, a, b, c \in B, R(s)\) is not semigood, \(R(t)\) is not right-good. From one hand,

\[
(a \ast R(s)bR(t) \ast c) = aR(s_0)bR(t) \ast c + \sum_i a_i R(s_i)b_i R(t) \ast c + \sum_{k,i'} (a_k \cdot b_{i'}) R(t) \ast c
\]

\[
= aR(s_0)bR(t_0)c + \sum_j a_i R(s_i)b_j R(t_j)c_j + \sum_{i,i'} a_i R(s_i)(b_{i'})_j R(t_j)c_j + \sum_{i,i',j'} a_i R(s_i)((b_{i'})_{i'} \cdot \tilde{c}_l)
\]

\[
+ \sum_{k,i'} (a_k \cdot b_{i'}) R(t_0)c + \sum_{j,k,i'} (a_k \cdot b_{i'})(b_j)_{i'} R(t_j)c_j + \sum_{k,l,i',j'} ((a_k \cdot b_{i'})(b_j)_{i'} \cdot \tilde{c}_l), \quad (46)
\]

where \(\deg_R(s_i) < \deg_R(s_0) = \deg_R(s), \deg_R((t_j) < \deg_R(t_0) = \deg_R(t)\), all variations of letters of \(a, b, c\) lie in \(B\), and indexes \(i, j, k, l, i', j'\) run over some disjoint finite sets.

From another hand,

\[
a \ast (R(s)bR(t) \ast c) = a \ast R(s)bR(t_0)c + a \ast \sum_j R(s)b_j R(t_j)c_j + a \ast \sum_{i,i'} R(s)(b_{i'} \cdot \tilde{c}_l)
\]

\[
= aR(s_0)bR(t_0)c + \sum_i a_i R(s_i)b_i R(t_0)c + \sum_{k,i'} (a_k \cdot b_{i'}) R(t_0)c
\]

\[
+ \sum_j aR(s_0)b_j R(t_j)c_j + \sum_{i,i'} a_i R(s_i)(b_j)_{i'} R(t_j)c_j + \sum_{j,k,i'} (a_k \cdot (b_j)_{i'})(b_{i'})_{i'} R(t_j)c_j
\]

\[
+ \sum_{l,i',j'} aR(s_0)(b_{i'} \cdot \tilde{c}_l) + \sum_{l,i',j'} a_i R(s_i)(b_{i'} \cdot \tilde{c}_l)_i + \sum_{k,l,i',j'} (a_k \cdot (b_{i'} \cdot \tilde{c}_l)_i). \quad (47)
\]

Comparing (46) and (47), it is enough to state that

\[
\sum_{i,j} a_i R(s_i)(b_j)_{i} R(t_j)c_j = \sum_{i,j} a_i R(s_i)(b_j)_{i} R(t_j)c_j, \quad (48)
\]

\[
\sum_{i,l,j'} a_i R(s_i)((b_j)_{i'} \cdot \tilde{c}_l)_i = \sum_{i,l,j'} a_i R(s_i)((b_j)_{i'} \cdot \tilde{c}_l)_i, \quad (49)
\]

\[
\sum_{j,k,i'} (a_k \cdot (b_j)_{i'})(b_{i'})_{i'} R(t_j)c_j = \sum_{j,k,i'} (a_k \cdot (b_j)_{i'})(b_{i'})_{i'} R(t_j)c_j, \quad (50)
\]

\[
\sum_{k,l,i',j'} ((a_k \cdot b_{i'})(b_j)_{i'} \cdot \tilde{c}_l) = \sum_{k,l,i',j'} ((a_k \cdot b_{i'})(b_j)_{i'} \cdot \tilde{c}_l). \quad (51)
\]

The equality (48) follows from the conditions L5, R5 and Statement. The equality (49) due to the conditions L5, R5 is equivalent to the equality

\[
(d_1, \ldots, d_p, (b, r_1, \ldots, r_q)_\prec) \cdot \tilde{c}_l = ((d_1, \ldots, d_p, b)_\prec, r_1, \ldots, r_q))_\prec \cdot \tilde{c}_l,
\]
which is true by Statement and (6). The proof of (50) and (51) is analogous.

Hence, we have to prove associativity only for triples $x, y, z$, in which $y$ is an $E$-word of the type 2. The definition of $*$ by Lemma 6 is symmetric with respect to the inversion, except the cases 1–3 and 3–1, so associativity in the triple of types $k–2–l$ leads to associativity in the triple $l–2–k$.

Prove associativity for $d = 3$ and $E$-word $y$ of the type 2. We consider only those cases whose proof is not directly analogous to the one from Lemma 5.

1–2–1. Given $x = a, y = R(b), z = c, a, b, c \in B$, by (6), we compute

$$(a * R(b)) * c - a * (R(b) * c) = (a < b) * c - a * (b > c) = 0.$$  

1–2–2. a) Let $x = a, y = R(b), z = R(c), a, b, c \in B$. By (6), we have

$$(a * R(b)) * R(c) - a * (R(b) * R(c)) = (a < b) < c - a < (b > c + b < c + b * c) = 0.$$  

b) $x = a, y = R(b), z = R(cR(t)), a, b, c \in B$.

$$(x * y) * z = (a < b) * R(cR(t))$$  

$$= ((a < b) < c)R(t) - (a < b) * R(R(c) * t) - (a < b) * R(c * t).$$  

Applying inductive assumption, write down the penultimate string of (53):

$$a * R((b < c)R(t) - b * R(R(c) * t) - b * R(c * t))$$

$$= (a < (b < c))R(t) - a * R((b < c) * t) - a * R((b < c) * t)$$

$$+ a * R((b < c)R(t) - b * R(R(c) * t) - b * R(c * t))$$

$$+ (a < (b < c))R(t) - a * R(R(b * c) * t) - a * R((b * c) * t).$$  

Substituting (54) in (53) and subtracting the result from (52), by (6), we have $a * (R(b), R(c), t)$ equal to zero by induction.

c) $x = a, y = R(bR(t)), z = R(u), a, b \in B$. Notice that $(a * R(R(b) * t)) * R(u) = a * (R(R(b) * t) * R(u))$. Indeed, let $R(b) * t = s_1 + s_2$, where $s_1$ denotes a linear combination of $E$-words starting with a $R$-letter and $s_2$ — starting with a letter from $B$. Thus, $\deg_R(s_2) < \deg_R(t) + 1$ and, hence, by induction, we have

$$(a * R(R(b) * t)) * R(u) = (aR(s_1) + a * R(s_2)) * R(u)$$

$$= a * (R(s_1) * R(u)) + a * (R(s_2) * R(u)) = a * (R(s_1 + s_2) * R(u))$$

$$= a * (R(R(b) * t) * R(u)).$$
Applying the result, compute

\[(x \ast y) \ast z = (a \ast R(bR(t)) \ast R(u)) = ((a \prec b)R(t) - a \ast R(R(b) \ast t) - a \ast R(b \ast t)) \ast R(u)\]
\[= (a \prec b) \ast R(R(t) \ast u + t \ast R(u) + t \ast u) - a \ast R(R(b) \ast t \ast R(u) + R(b) \ast t \ast u)
\]
\[= a \ast R(R(b \ast t) \ast u + b \ast t \ast R(u) + b \ast t \ast u). \quad (55)\]

\[x \ast (y \ast z) = a \ast R(R(bR(t)) \ast R(u) + bR(t) \ast u) \]
\[= a \ast R(R(bR(t)) \ast u + bR(t) \ast u) + (a \prec b) \ast R(R(t) \ast u + t \ast R(u) + t \ast u)
\]
\[= a \ast R(R(b) \ast (R(t) \ast u + t \ast R(u) + t \ast u)) - a \ast R(R(b \ast t) \ast u + b \ast t \ast R(u) + b \ast t \ast u)). \quad (56)\]

Subtracting (56) from (55) and using the equality \((R(b), R(t), u) = 0\) holding by induction, we get zero.

For \(d > 3\), one consider other cases analogously to the cases from the proof of Lemma 5. Thus, Lemma 9 is proven.

**Proof of Theorem 3.** Let us prove that the algebra \(D\) is exactly universal enveloping algebra for the preassociative algebra \(C\), i.e., is isomorphic to the algebra

\[U_{RB_1As}(C) = RB_1As\langle B \mid a \prec b = R(a)b, a \prec b = aR(b), a \cdot b = ab, a, b \in B \rangle.\]

By the construction, the algebra \(D\) is generated by \(B\). Therefore, \(D\) is a homomorphic image of a homomorphism \(\varphi\) from \(U_{RB_1As}(C)\). We will prove that all basic elements of \(U_{RB_1As}(C)\) are linearly expressed by \(D\), then \(\ker \varphi = \langle 0 \rangle\) and \(D \cong U_{RB_1As}(C)\).

In \(U_{RB_1As}(C)\), the equality \(ab = a \cdot b\) holds, therefore, \(U_{RB_1As}(C)\) is a subspace of \(RB_1As(B)\). Denote by \(E'\) the complement of the set of all \(E\)-words in the base of \(RB_1As(B)\). Applying the inductions in \(RB_1As(A)\) on the \(R\)-degree and the degree of base words, the equalities \(xy = x \cdot y, x, y \in B\), the relations

\[R(a)u = \begin{cases} (a \prec b)u', & u = bu', b \in B, \\ R(R(a)t)u' + R(aR(t))u' + R(at)u', & u = R(t)u'; \end{cases} \quad (57)\]

\[aR(bR(u)) = (a \prec b)R(u) - aR(R(b)u) - aR(bu), \quad a, b \in B, \quad (58)\]

the relations for \(uR(a)\) and \(R(R(u)b)a\) analogous to (57), (58), and the relations derived from the analogues of (16)–(19) by the removing the symbol \(*\), we prove that the elements of \(E'\) are linearly expressed via \(E\)-words.

**Corollary 2.** The pair of varieties \((RB_1As, postAs)\) is a PBW-pair.

The author expresses his gratitude to P. Kolesnikov for important corrections. The research is supported by RSF (project N 14-21-00065).
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Gubarev Vsevolod
Sobolev Institute of Mathematics of the SB RAS
Acad. Koptyug ave., 4
Novosibirsk State University
Pirogova str., 2
Novosibirsk, Russia, 630090
E-mail: wsewolod89@gmail.com