COMPACTIFIED JACOBIANS OF EXTENDED ADE CURVES AND
LAGRANGIAN FIBRATIONS

ADAM CZAPLIŃSKI, ANDREAS KRUG, MANFRED LEHN, AND SÖNKE ROLLENSKE

Abstract. We observe that general reducible curves in sufficiently positive linear systems on K3 surfaces are of a form that generalises Kodaira’s classification of singular elliptic fibres and thus call them extended ADE curves.

On such a curve $C$, we describe a compactified Jacobian and show that its components reflect the intersection graph of $C$. This extends known results when $C$ is reduced, but new difficulties arise when $C$ is non-reduced. As an application, we get an explicit description of general singular fibres of certain Lagrangian fibrations of Beauville-Mukai type.

1. Introduction

In this article we want to discuss the geometry of compactified Jacobian varieties associated to certain reducible or even non-reduced projective curves over a field of characteristic 0. Compactified Jacobian varieties have a long history. If $C$ is a smooth projective curve, the Picard group $\text{Pic}^0(C)$ parameterising invertible sheaves of degree 0 is an abelian variety of dimension equal to the genus $g(C)$, and Jacobian variety is used as a synonym for any of the connected components of $\text{Pic}(C)$. If $C$ is an integral curve with singularities, the Picard group $\text{Pic}(C)$ is still a group scheme of finite type, but no longer projective, as line bundles can degenerate into torsion free sheaves that fail to be locally free at one or more of the singular points of $C$. As the name suggests, a compactified Jacobian is an appropriate compactification of a component of $\text{Pic}(C)$. For a historical overview and more details we refer to the paper by Altman and Kleiman [AK80] and references therein.

The problem becomes more difficult if one allows $C$ to be a non-integral curve, in particular if the curve has a non-reduced scheme structure. The Picard scheme of such curves has been described in full generality by Bosch, Lütkebohmert and Raynaud [BLR90]. A compactified Jacobian for such curves can be defined as, and this is the view adopted in this article, the moduli space of pure sheaves on $C$ that are semi-stable with respect to an appropriate polarisation and have the same length as $\mathcal{O}_C$ at each generic point of $C$. For non-integral curves, it can happen that (components of) the Picard scheme form an open part of the moduli space of these sheaves, but that this part is no longer dense. We will see that this happens in our context.

In this paper we assume $C$ to be a projective curve in some ambient smooth surface $X$ of the form $C = \sum_i m_i C_i$ with smooth components $C_i$ and certain multiplicities $m_i$. The dual graph $\Gamma$ is assumed to be an affine simply-laced Dynkin graph, see Figure 1, and the multiplicity vector $m = (m_i) \in \mathbb{Z}^\ell$, $\ell = |\Gamma|$, is to be the minimal positive integral null-vector of the semi-negative Cartan matrix $S$ associated to $\Gamma$. We assume that all multiple components are rational with self-intersection $-2$. We call a curve $C$ meeting these requirements an extended ADE-curve of type $\Gamma$. 
Figure 1. The extended Dynkin graphs

\[ \tilde{A}_n: \begin{array}{c}
\udots \\
1 \\
\udots \end{array} \quad \tilde{D}_n: \begin{array}{c}
\udots \\
1 \\
2 \\
\udots \\
1 \\
\udots \\
1 \\
\udots \end{array} \quad \tilde{E}_6: \begin{array}{c}
\udots \\
1 \\
2 \\
\udots \\
3 \\
2 \\
\udots \\
1 \\
\udots \end{array} \quad \tilde{E}_7: \begin{array}{c}
\udots \\
1 \\
2 \\
\udots \\
3 \\
4 \\
3 \\
2 \\
\udots \\
1 \\
\udots \end{array} \quad \tilde{E}_8: \begin{array}{c}
\udots \\
1 \\
2 \\
\udots \\
3 \\
6 \\
5 \\
4 \\
3 \\
2 \\
\udots \\
1 \\
\udots \end{array} \]

If \( \Gamma \) is of type \( \tilde{A}_n \), all multiplicities are 1, hence the curve \( C \) is reduced. In [LM05], López-Martín classifies stable sheaves on reduced singular fibres of elliptic fibrations, in particular on \( \tilde{A}_n \)-configurations of smooth rational curves; for a comparison to our results see Subsection 5.4. In the study of the non-reduced cases \( \tilde{D}_n \) and \( \tilde{E}_n \) several new difficulties arise.

Let \( \mathcal{M}_\chi(C) \) denote the Simpson moduli space of semistable sheaves \( F \) with Euler characteristic \( \chi \) on \( C \) such that \( F \) has length \( m_i \) at the generic points \( \eta_i \in C_i \) (with appropriately chosen polarisation and conditions on \( \chi \) to be discussed in the main text, where a sheaf satisfying our requirement on the lengths at the generic points is called a sheaf of type \( m \)).

The following is a concise reformulation of a large part of the results obtained in Section 3.

**Theorem 1.1.** Let \( H \) be a polarisation as in Assumption 3.12. Then every semi-stable sheaf in the compactified Jacobian \( \mathcal{M}_\chi(C) \) is already stable and of the form

\[ L(x) := \mathcal{H}om(I, L), \]

where \( I \subset O_C \) is the ideal sheaf of a closed point \( x \) and \( L \) is a line bundle in a fixed component of the Picard group, the component depending on \( H \) and \( \chi \).

The sheaf \( L(x) \) is a line bundle if and only if \( x \) is a smooth point of \( C \). Otherwise, \( x \) is the unique point where \( L(x) \) fails to be locally free.

Our second main result describes the geometry of the compactified Jacobian \( \mathcal{M}_\chi(C) \). The following is deduced from a more detailed and technical description in Theorem 4.12.

**Theorem 1.2.** Let \( C = \sum_i m_i C_i \) be an extended ADE curve of arbitrary type \( \Gamma \in \{ \tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \} \), and let \( J := \prod_i \text{Jac}(C_i) \) denote the product of the Jacobians of the components of \( C \). Then \( \mathcal{M}_\chi(C)_{\text{red}} \) consists of components \( \{ Y_v \}_{v \in V} \) which are \( \mathbb{P}^1 \)-bundles over \( J \). They intersect transversally in sections, and their intersection graph is again \( \Gamma \).

For \( \Gamma = \tilde{A}_n \), the compactified Jacobian is reduced, so the above already gives a description of \( \mathcal{M}_\chi(C) \) itself. For \( \Gamma \in \{ \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \} \), those components \( Y_v \) which correspond to multiple components of the curve \( C \) are again non-reduced, but we do not compute their multiplicities. Our expectation, however, is that the multiplicity of \( Y_v \) is equal to the multiplicity of \( C_v \) in \( C \); compare Conjecture 4.4.
The motivation to consider curves of this specific form comes from our interest in determining the structure of generic singular fibres in Lagrangian fibrations $\mathcal{M}_\chi(K3) \to \mathbb{P}^n$ of moduli of semistable sheaves on K3 surfaces. Indeed, extended ADE curves occur as generic non-integral members of linear systems on K3 surfaces; see Subsection 5.1 and Subsection 5.2 for some more details on this. The point of departure was the construction of a K3 surface as a minimal resolution of the singularities of a double cover of the plane branched along a reduced but possibly reducible singular sextic (with at worst ADE singularities) and the explicit classification of the singular fibers of the associated Beauville-Mukai system $\mathcal{M}_\chi(K3) \to \mathbb{P}^5 = |\mathcal{O}(2)|$, see Czapliński [Cza18].

Let us sketch the structure of the article. In Section 2 we discuss some generalities concerning sheaves on curves. The next two sections are the technical heart of the work. In Section 3 we completely classify stable sheaves on an extended ADE curve $C$ and in the following Section 4 we give a description of the moduli space $\mathcal{M}_\chi(C)$. In the last Section 5 we discuss, among other things, some codimension one strata in linear systems on K3 surfaces, Beauville-Mukai systems, characteristic cycles, and work of other authors related to our results.

Acknowledgements. We would like to thank Andreas Knutsen for some advice concerning linear systems on K3 surfaces.

2. Generalities concerning sheaves on curves

2.1. Picard varieties of curves. In this subsection, a curve is a purely one-dimensional scheme of finite type over $\mathbb{C}$. In the later subsections, we will make stronger assumptions on our curves.

Proposition 2.1. Let $C$ be a connected and projective curve with irreducible components $C_1, \ldots, C_\ell$ (here, we mean the reduction of the components). Then, the Picard functor of $C$ is represented by a scheme $\text{Pic}(C)$. Its connected component $\text{Pic}^0(C)$ containing the point corresponding to the trivial line bundle consists of those line bundles whose restriction to every $C_i$ is of degree 0. Let $C'_i \to C_i$ be the normalisation. Then, pull-back along the morphism $\prod_{i=1}^\ell C'_i \to C$ gives an exact sequence of algebraic groups

$$1 \to G \to \text{Pic}^0(C) \to \prod_{i=1}^\ell \text{Pic}^0(C'_i) \to 1,$$

where $G$ is a smooth connected algebraic group of dimension $h^1(\mathcal{O}_C) - \sum_{i=1}^\ell h^1(\mathcal{O}_{C'_i})$.

Proof. See Thm. 8.2.3 and Cor. 9.2.11 & 9.2.13 of [BLR90].

Corollary 2.2. If $h^1(\mathcal{O}_C) = \sum_{i=1}^\ell h^1(\mathcal{O}_{C'_i})$, then the natural map $\text{Pic}^0(C) \to \prod_{i=1}^\ell \text{Pic}^0(C'_i)$ is an isomorphism.

2.2. Duals of ideal sheaves of singular points. In this subsection, let $C$ be a (not necessarily projective) Gorenstein curve. By [Har94, Prop. 1.6] the Gorenstein condition implies that a sheaf $F$ on $C$ is reflexive if and only if it is purely one-dimensional, so we do not lose information by taking duals.

For a point $x \in C$ we denote the dual of its ideal sheaf $\mathcal{I}_x$ by $\mathcal{I}_x'$. Then

$$\mathcal{O}_C(x) := \mathcal{I}_x'.$$
Furthermore, given a line bundle \( L \in \text{Pic}(C) \), we write

\[
L(x) := L \otimes O_C(x) \cong \text{Hom}(I_x, L).
\]

**Lemma 2.3.** 
(i) For every \( x \in C \), the sheaf \( O_C(x) \) is purely one-dimensional, and there is a non-split short exact sequence

\[
0 \to O_C \to O_C(x) \to O_x \to 0.
\]  

(ii) Conversely, every purely one-dimensional sheaf \( F \) fitting into a short exact sequence

\[
0 \to O_C \to F \to O_x \to 0
\]

is already of the form \( F \cong O_C(x) \).

**Proof.** Part (i) is [Har94, Prop. 2.8 & 2.10]. By the purity of \( F \), the sequence of part (ii)

\[
0 \to O_C \to F \to O_x \to 0
\]

is non-splitting. By Serre-duality on a Gorenstein curve, \( \text{ext}^1(O_x, O_C) = 1 \). Hence, there is only one isomorphism class of sheaves which fit into such a non-split exact sequence. Hence, by part (i), \( F \cong O_C(x) \).

**Lemma 2.4.** The sheaf \( O_C(x) \) is a line bundle if and only if \( x \) is a smooth point of \( C \).

**Proof.** A smooth point is a Cartier divisor. Hence, the associated sheaf is a line bundle.

If \( x \) is a singular point, we have \( \text{ext}^1(O_x, O_x) = \dim T_{C,x} > 1 \). Applying \( \text{Hom}(\_, O_x) \) to (1) gives an exact sequence

\[
\mathbb{C} = \text{Hom}(O_C, O_x) \to \text{Ext}^1(O_x, O_x) \to \text{Ext}^1(O_C(x), O_x) \to 0.
\]

This yields \( \text{Ext}^1(O_C(x), O_x) \neq 0 \) which cannot happen for a line bundle.

**Lemma 2.5.** Let \( F \) be a purely one-dimensional sheaf on \( C \). If there is a point \( x \in C_\text{sing} \) such that \( F_{C \setminus \{x\}} \) is a line bundle, and \( F_U \cong O_U(x) \) for some open neighbourhood \( x \in U \subset C \), then there exists an \( L \in \text{Pic}(C) \) such that \( F \cong L(x) \).

**Proof.** This follows from [Har94, Prop. 2.12] noting that \( \text{APic}(C) = \text{Pic}(C) \) for a curve.

For two types of singularities \( x \in C \), we need a more detailed description of \( O_C(x) \), given by the following two lemmas.

**Lemma 2.6.** Let \( m \geq 2 \) and \( C = mD \) where \( D \) is a smooth divisor in a smooth surface \( X \), and let \( x \in C \).

(i) The natural map \( \text{End}_{O_C}(I_x) \to O_C(x) \), given by post-composition of endomorphisms with the embedding \( I_x \to O_C \), is an isomorphism.

(ii) The \( O_C \)-algebra \( \text{End}_{O_C}(I_x) \) is commutative and satisfies \( \text{End}_{O_C}(I_x)_{\text{red}} \cong O_{C_{\text{red}}} \).

(iii) Let \( E \subset X \) be another smooth divisor intersecting \( D \) transversally in \( x \). Let \( \zeta = mD \cap E \) be the scheme-theoretic intersection. Then, there is no surjective \( O_C \)-linear morphism \( O_C(x) \to O_\zeta \).

**Proof.** As the assertions are local, we can assume that \( C = \text{Spec} A \), with \( A = \frac{\mathbb{C}[u,v]}{(u^m)} \), and \( x = V(\mathfrak{m}) \) with \( \mathfrak{m} = (u, v) \). Then \( O_C(x) = \text{Hom}_A(\mathfrak{m}, A) \) is spanned, as an \( A \)-module, by the inclusion \( \text{id}: \mathfrak{m} \to A \) together with the homomorphism

\[
\varphi: \quad u \mapsto v^{m-1}, \quad v \mapsto 0.
\]
Indeed, one checks that \( \varphi \) is well defined, and the sequence \( 0 \to \mathcal{O}_C \to \mathcal{O}_C(x) \to \mathcal{O}_x \to 0 \) shows that any homomorphism which is not a multiple of id suffices to generate \( \mathcal{O}_C(x) \). As both, id and \( \varphi \) map \( m \) to itself, part (i) follows. The relations
\[
v_\varphi = 0 \quad , \quad w_\varphi = v^{m-1} \text{id} \quad , \quad \varphi^2 = 0
\]
give an isomorphism of \( A \)-algebras
\[
\text{End}_A(m) \cong \frac{\mathbb{C}[u,v,\varphi]}{(v^m, v\varphi, u\varphi - v^{m-1}, \varphi^2)} ,
\]
which proves part (ii). For part (iii), we may assume that \( \zeta = \text{Spec}(B) \) with \( B = \frac{\mathbb{C}[u,v]}{(u,v^m)} \). Let \( \alpha : \text{End}_A(m) \to B \) be \( A \)-linear. The relation \( v_\varphi = 0 \) shows that \( \alpha(\varphi) \in (v^{m-1}) \). Furthermore, the relation \( w_\varphi = v^{m-1} \text{id} \) shows that
\[
v^{m-1}\alpha(\text{id}) = u\alpha(\varphi) = 0 ,
\]
hence \( \alpha(\text{id}) \in (v) \). In summary, the image of \( \alpha \) is contained in \( (v) \), which means that \( \alpha \) is not surjective. \( \square \)

**Lemma 2.7.** Let \( D, E \) be two smooth divisors in a smooth surface \( X \) intersecting transversally in a point \( x \in X \). Let \( C := mD + nE \) for some \( m, n \in \mathbb{N} \).

(i) The natural map \( \text{End}_{\mathcal{O}_C}(\mathcal{I}_x) \to \mathcal{O}_C(x) \) is an isomorphism.

(ii) The \( \mathcal{O}_C \)-algebra \( \text{End}_{\mathcal{O}_C}(\mathcal{I}_x) \) is commutative.

(iii) If \( (m,n) \neq (1,1) \), we have \( \text{End}_{\mathcal{O}_C}(\mathcal{I}_x)_{\text{red}} \cong \mathcal{O}_{C_{\text{red}}} \).

(iv) If \( (m,n) = (1,1) \), we have \( \text{End}_{\mathcal{O}_C}(\mathcal{I}_x) \cong \mathcal{O}_D \times \mathcal{O}_E \).

**Proof.** We proceed analogously to the proof of Lemma 2.6. We assume that \( C = \text{Spec} \ A \), with \( A = \frac{\mathbb{C}[u,v]}{(u,v^m)} \), and \( x = V(m) \) with \( m = (u,v) \). Let us assume without loss of generality that \( m \geq n \). Then \( \mathcal{O}_C(x) = \text{Hom}_A(m,A) \) is spanned, as an \( A \)-module, by the inclusion \( \text{id} : m \to A \) together with the homomorphism
\[
\varphi : \quad u \mapsto u^nv^{m-1} \quad , \quad v \mapsto 0 .
\]
As both, id and \( \varphi \) map \( m \) to itself, part (i) follows. As \( \varphi \) commutes with id and all its multiples, we get part (ii).

If \( (m,n) \neq (1,1) \), then \( \varphi \) is nilpotent. This gives part (iii). If \( m = 1 = n \), then \( \varphi \) is not nilpotent, but \( \varphi^2 = \varphi \). We get an isomorphism
\[
\text{End}_A(m) \cong \frac{\mathbb{C}[u,v,\varphi]}{(uv, v\varphi, u\varphi - u, \varphi^2 - \varphi)} = \frac{\mathbb{C}[u,v,\varphi]}{(u, \varphi) \cdot (v, \varphi - 1)}
\]
which proves (iv). \( \square \)

**Remark 2.8.** By [Har94, Prop. 1.6], the sheaf \( \mathcal{I}_x \) is reflexive. It follows that \( \text{End}(\mathcal{I}_x) \cong \text{End}(\mathcal{O}_C(x)) \).

2.3. Pure sheaves on divisorial curves. In this subsection, on a smooth projective surface \( X \) we consider possibly reducible and non-reduced curves. The curves are still assumed to be Gorenstein, hence without embedded points. We identify such a curve \( C \) with the corresponding Weil divisor and write
\[
C = \sum_{i=0}^{\ell} m_i C_i \subset X ,
\]
where $C_0, \ldots, C_\ell$ are the irreducible components of $C$ and $m_0, \ldots, m_\ell$ their multiplicities. We assume that all the components $C_i$ are smooth.

Most of the results in this section are true in a more general context, but we did not feel the need to strive for maximal generality.

The following is an easy but useful observation.

**Lemma 2.9.** For any decomposition $C = A + B$ there are two short exact sequences

$$0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \to \mathcal{O}_B \to 0,$$

$$0 \to \mathcal{O}_B(-A) \to \mathcal{O}_C \to \mathcal{O}_A \to 0.$$  

The second map in each sequence is the restriction, and the two sequences are dual in the sense that they arise from each other by applying $\text{Hom}_{\mathcal{O}_C}(\_\_\_, \mathcal{O}_C)$.

**Definition 2.10.** Let $F$ be a coherent sheaf on a curve with one-dimensional support. We write $\text{pure}(F) := F/(\text{zero-dimensional torsion})$ for the maximal purely one-dimensional quotient of $F$.

**Lemma 2.11.** Let $F$ be a purely one-dimensional sheaf on a curve $C$ which is generically a line bundle. Then, every purely one-dimensional quotient $F \to F''$ is of the form $F'' = \text{pure}(F|_{Z})$ for some subcurve $Z \subset C$.

Here, by a subcurve $Z \subset C$, we mean a subdivisor $Z = \sum_{i=0}^\ell n_i C_i$ with $0 \leq n_i \leq m_i$.

**Proof.** Let $U \subset C$ be a dense open subset with $F|_U \cong \mathcal{O}_U$. Then, there is a subcurve $W \subset U$ such that $F|_U \to F''|_U$ is isomorphic to the restriction map $\mathcal{O}_U \to \mathcal{O}_W$. Let $Z \subset C$ be the unique subdivisor such that $Z \cap U = W$. Let $F' := \ker(F\to F'')$. The composition $F' \to F \to \text{pure}(F|_{Z})$ is zero over the dense open set $U$. As $\text{pure}(F|_{Z})$ is purely one-dimensional, we get that the composition is zero. Hence, we get a morphism of quotients

$$\begin{array}{ccc}
F & \longrightarrow & F'' \\
\downarrow \text{id} & & \downarrow \\
F & \longrightarrow & \text{pure}(F|_{Z}) \longrightarrow 0.
\end{array}$$

The right vertical map is surjective and an isomorphism over $U$. As $F''$ is purely one-dimensional, it is an isomorphism. \hfill \Box

From now on, we also assume that all components $C_i$ of $C$ are smooth, that at most two components of $C$ meet in one point, and that the components of $C$ intersect transversally. That is, at each intersection point $x \in C_i \cap C_j$ the intersection subscheme $\zeta_x := m_i C_i \cap m_j C_j$ satisfies

$$\mathcal{O}_{\zeta_x} \cong \mathbb{C}[u,v]/(u^{m_i}, v^{m_j}).$$

(2)

We will study sheaves on $C$ by means of a canonically associated short exact sequence given as follows.

**Lemma 2.12.** Let $F$ be a sheaf on $C$ which is pure of dimension 1. For each component $C_j$ define

$$F_j := \text{pure}(F|_{m_j C_j}).$$
Then there is an associated exact sequence

$$0 \to F \to \bigoplus_j F_j \to \bigoplus_x T_x \to 0,$$

where the first direct sum runs through all components of $C$, the second direct sum runs through all intersection points $x$ of components of $C$ and $T_x$ is a zero-dimensional sheaf supported in $x$ such that all components $F_j \to T_x$ with $x \in C_j$ are surjective.

Proof. The map $F \to \bigoplus_j F_j$, whose components are the restriction maps, is an isomorphism over $U \subset C$, the complement of the intersection points. Hence, the map is injective, as $F$ is pure, and its cokernel $\bigoplus_x T_x$ is supported in the intersection points.

Let $x$ be one of the intersection points. By our assumption, only two components, say $C_i$ and $C_j$, meet in $x$. Hence, locally near $x$, we have the following square

$$\begin{array}{ccc}
F_i & \to & T_x \\
\uparrow & & \uparrow \\
F & \to & F_j,
\end{array}$$

which is commutative up to a sign. It follows that the images of the two maps $F_i \to T_x$ and $F_j \to T_x$ agree. As the sum of both maps is surjective, it follows that both maps are surjective individually. □

Remark 2.13. Let $L \in \text{Pic}(C)$ be a line bundle, and $F$ a purely one-dimensional sheaf on $C$. Then, for every component $C_j$ of $C$, we have

$$(L \otimes F)_j \cong L_j \otimes F_j.$$}

Furthermore, the sheaves $T_x$ associated to $F$ and $L \otimes F$ are isomorphic.

Let $F$ be a purely one-dimensional sheaf on $C$. We denote by $\mu_j(F)$ the multiplicity of $F$ along the component $C_j$. That means that $\mu_j(F)$ is the length of the stalk of $F$ at the generic point of $C_j$.

For every component $C_j$, there is a natural filtration

$$F_j = F_{j0} \supset F_{j1} \supset \ldots \supset F_{jm_j} = 0,$$

where $F_{jk}$ is the saturation of the subsheaf $O_X(-kC_j): F_j = \mathcal{I}_kC_j \cdot F_j$ in $F_j$. The factors $\text{gr}_k F_j = F_{j(k-1)}/F_{jk}$ are pure and hence locally free sheaves on the (reduced) curve $C_j$ of a certain rank $r_{jk}$. By construction, there are natural maps

$$\text{gr}_k F_j \otimes \mathcal{O}_{C_j}(-C_j) \to \text{gr}_{k+1} F_j$$

that are generically surjective. In particular, the ranks $r_{jk}$ satisfy

$$r_{j1} \geq r_{j2} \geq \ldots .$$

Lemma 2.14. Let $x \in C_i \cap C_j$. Then

$$\chi(T_x) \leq \sum_{k,k'} \min\{r_{ik}, r_{jk'}\}$$

where the sum runs over all pairs $(k,k')$ with $1 \leq k \leq m_i$ and $1 \leq k' \leq m_j$. 

Lemma 2.12: Let $F_i$ and $F_j$ be purely one-dimensional sheaves on $m_iC_i$ and $m_jC_j$ and consider their natural filtrations and rank sequences as defined above. Let $T_x$ be a sheaf supported in $x$ such that there are surjections $F_i \to T_x$ and $F_j \to T_x$. Then $\chi(T_x) \leq \sum_{k,k'} \min\{r_{ik}, r_{jk'}\}$.

Proof. We prove this by induction on the lengths of the filtrations of $F_i$ and $F_j$. Note that we can reformulate the assertion in the following way, which is more useful for our proof; compare their natural filtrations and rank sequences as defined above. Let $T_x$ be a sheaf supported in $x$ such that there are surjections $F_i \to T_x$ and $F_j \to T_x$. Then $\chi(T_x) \leq \sum_{k,k'} \min\{r_{ik}, r_{jk'}\}$.

Assume first that the length of the filtrations on $F_i$ and $F_j$ equal 1, so that $F_i$ and $F_j$ are scheme-theoretically supported on the reduced curves $C_i$ and $C_j$, respectively, and are locally free on these reduced curves. As $T_x$ is a quotient both of $F_i$ and $F_j$, it must be of the form $O_x^{\oplus r}$ with rank $r \leq \text{rk}(F_i) = r_{i1}$ and $r \leq \text{rk}(F_j) = r_{j1}$. So we have $\chi(T_x) \leq \min\{r_{i1}, r_{j1}\}$ and the assertion is clear in this case.

Now, we assume that the length of the filtration on $F_i$ is $\geq 2$. So we are in the situation that $F_{i1} \neq 0$. Let $T' \subset T_x$ denote the image of $F_{i1}$ in $T_x$, and let $F_j'$ denote the preimage of $T'$ under the surjection $F_j \to T_x$. We get the following maps

$$F_{i1} \to T' \leftarrow F_j', \quad \text{gr}_1(F_i) \to T_x/T' \leftarrow F_j$$

all of which are still surjective by construction. As the inclusion $F_j' \hookrightarrow F_j$ is generically an isomorphism, the sheaves $F_j'$ and $F_j$ have the same rank sequences. Hence, applying the induction hypotheses to the left pair of surjections, we obtain

$$\chi(T') \leq \sum_{k,k', k' \geq 2} \min\{r_{jk}, r_{ik'}\}. \quad (3)$$

Furthermore, applying the induction hypotheses to the right pair of surjections, gives

$$\chi(T_x/T') \leq \sum_k \min\{r_{jk}, r_{i1}\}. \quad (4)$$

As $\chi(T_x) = \chi(T') + \chi(T_x/T')$, the assertion follows from combining (3) and (4). $\square$

3. Classification of stable sheaves on an extended ADE curve

3.1. Set-up and Notation. We continue with the assumptions from Subsection 2.3 on our curve $C$: It is embedded as a divisor into a smooth projective surface, its components are smooth, and at most two components intersect transversally in one point.

Associated to $C$ we have the intersection graph $\Gamma = (V(\Gamma), E(\Gamma), m(\Gamma)) = (V, E, m)$. The vertex set $V$ consist of one vertex for each component of $C$. The label $m: V \to \mathbb{N}$ sends the vertex corresponding to a component $C_i$ to its multiplicity $m_i$. There is an edge between the vertices corresponding to $C_i$ and $C_j$ for each intersection point of the two components.

Definition 3.1. The extended ADE graphs are the labelled graphs displayed in Figure 1 in the introduction. Given an extended ADE graph $\Gamma$, we decompose its vertex set as $V(\Gamma) = I(\Gamma) \sqcup O(\Gamma)$, where $O(\Gamma)$ is the set of vertices with label 1, and $I(\Gamma)$ is the set of vertices of label greater than 1. Later, we will often omit the $\Gamma$ from the notation and simply write

$$V := V(\Gamma) \ , \ I := I(\Gamma) \ , \ O := O(\Gamma).$$

Definition 3.2. We call $C$ as above an extended ADE-curve if

(i) its labeled intersection graph $\Gamma$ is one of the extended ADE Dynkin graphs, (ii) all curves $C_i$ with $i \in I(\Gamma)$ are rational $(-2)$-curves.
From now on, $C$ will always denote an extended ADE curve. A priori, the $C_o$ with $o \in O(\Gamma)$ (that means the reduced components of $C$) can be of arbitrary genus. However, the geometry of the ambient surface $X$ can impose strong restrictions on the genera. We will see this later in Subsection 5.1 in the case that $X$ is a K3 surface.

**Remark 3.3.** Let $S = S_p$ be the negative of the extended Cartan matrix of type $\Gamma$. Concretely, this means that $S_{vw} = C_v \cdot C_w$ for $(v, w) \in V \times V$ with $v \neq w$, and $S_{v,v} = -2$ for every $v \in V$. In other words, $S$ is the intersection matrix of $C$ in the case that all $C_v$ (also for $v \in O$) are $(-2)$-curves.

For us, the main property of the matrix $S$ is the following: The integral lattice $(\mathbb{Z}^V, S)$ is even and negative semi-definite and its kernel is generated by the primitive vector $m = (m_v)_{v \in V}$ whose entries are the labels of the vertices of $\Gamma$; see e.g. [BHPV04, Lem. I.2.12].

Another key property of extended ADE diagrams that we will use multiple times in the text is that twice the label of a vertex is the sum of the labels of adjacent vertices:

$$2m_v = \sum_{w \in V \setminus \{v\}, C_v \cap C_w \neq \emptyset} m_w . \quad (5)$$

If all but one component $C_o$ are rational, then $E := C - C_o$ is the fundamental cycle of an ADE singularity.

**Remark 3.4.** Extended ADE curves where all components (including the reduced ones) are rational $(-2)$-curves occur as singular fibers of elliptic fibrations.

**Example 3.5.** Let $\pi: \bar{X} \to \bar{X}$ be the minimal resolution of an ADE singularity $p \in \bar{X}$. If $p \in \bar{C}_0 \subset \bar{X}$ is a general curve which is Cartier near $p$, then $\pi^* C_0$ is an extended ADE-curve.

**Lemma 3.6.** Every extended ADE curve $C$ is numerically 2-connected. This means that, for every decomposition of effective divisors $C = A + B$, we have $A.B \geq 2$.

**Proof.** Since the self-intersections of reduced components of $C$ do not affect the intersection number $A.B$, we can pretend that all components (including the reduced ones) are rational $(-2)$-curves. In this case $C^2 = 0$; see Remark 3.3 or Remark 3.4. By Remark 3.3, we also have $A^2, B^2 \leq -2$. Hence, $0 = C^2 = (A + B)^2 = A^2 + 2A.B + B^2$ implies $A.B \geq 2$. \hfill $\Box$

### 3.2. Sheaves on multiple components.

In this subsection, we study sheaves on the non-reduced components of $C$. Let $B$ be a rational $(-2)$-curve on a smooth surface, and fix some $m \in \mathbb{N}$. Let $G$ be purely one-dimensional sheaf on $mB$. As above in Subsection 2.3, we consider the natural filtration of $G$. More concretely, $G_k \subset G$ denotes the saturation of the subsheaf $\mathcal{O}_X(-kB) \cdot G$ for $0 \leq k \leq m$, and $\text{gr}_k G = G_{k-1}/G_k$. As $G/G_k = \text{pure}(G_{[kB]}$, we can also describe the $\text{gr}_k G$ as the kernels

$$0 \to \text{gr}_k G \to \text{pure}(G_{[kB]} \to \text{pure}(G_{(k-1)B}) \to 0 . \quad (6)$$

**Lemma 3.7.** Let $1 \leq a \leq m$. If $\text{gr}_k G \cong \mathcal{O}_B(2(k - 1))$ for all $1 \leq k \leq a$, then we have $G/G_a \cong \mathcal{O}_{aB}$.\hfill $\Box$

**Proof.** We proof this by induction on $a$. For $a = 1$, this is immediate. For $a > 1$, we have the short exact sequence of $\mathcal{O}_{aB}$-modules

$$0 \to \text{gr}_a G \to G/G_a \to G/G_{a-1} \to 0 .$$

Applying the induction hypothesis to the right term, this sequence takes the form

$$0 \to \mathcal{O}_B(2(a-1)) \to G/G_a \to \mathcal{O}_{(a-1)B} \to 0 . \quad (7)$$
Applying $\text{Hom}_{\mathcal{O}_B}(\_, \mathcal{O}_B(2(a - 1)))$ to the structure sequence
\[ 0 \to \mathcal{O}_B(2(a - 1)) \to \mathcal{O}_B \to \mathcal{O}_{(a-1)B} \to 0 \] (8)
gives
\[
0 \to \text{Hom}(\mathcal{O}_{(a-1)B}, \mathcal{O}_B(2(a - 1))) \xrightarrow{\sim} \text{Hom}(\mathcal{O}_B, \mathcal{O}_B(2(a - 1)))
\]
\[
\to \text{Hom}(\mathcal{O}_B(2(a - 1)), \mathcal{O}_B(2(a - 1))) \to \text{Ext}^1(\mathcal{O}_{(a-1)B}, \mathcal{O}_B(2(a - 1)))
\]
\[
\to H^1(\mathcal{O}_B(2(a - 1))) = 0.
\]
Hence, $\text{ext}^1(\mathcal{O}_{(a-1)B}, \mathcal{O}_B(2(a - 1))) = \text{hom}(\mathcal{O}_B(2(a - 1)), \mathcal{O}_B(2(a - 1))) = 1$. As (7) is non-splitting (otherwise, we would have $\text{gr}_1 G \cong \mathcal{O}_B^2$), the two sequences (7) and (8) agree up to scalar multiplication. In particular, the middle terms agree. □

**Lemma 3.8.** The following three conditions are equivalent:

(i) $\text{gr}_k G \cong \begin{cases} 
\mathcal{O}_B(2(k - 1)) & \text{for } k = 1, \ldots, m - 1 \\
\mathcal{O}_B(2(m - 1) + 1) & \text{for } k = m
\end{cases}$

(ii) There is a non-splitting short exact sequence
\[ 0 \to \mathcal{O}_B(2(m - 1) + 1) \to G \to \mathcal{O}_{(m-1)B} \to 0, \]

(iii) $G \cong \mathcal{O}_{mB}(x)$ for some $x \in B$.

**Proof.** We first note that the implication (i) $\implies$ (ii) follows from Lemma 3.7 (with $a = m - 1$). Also note that the equivalence of the three conditions is obvious for $m = 1$, where $\mathcal{O}_{(m-1)B}$ has to be interpreted as the zero sheaf. Thus, from now on, we assume $m \geq 2$.

For (ii) $\implies$ (iii) we first note that, by a computation analogous to the one in the proof of Lemma 3.7, applying $\text{Hom}_{\mathcal{O}_B}(\_, \mathcal{O}_B(2(m - 1) + 1))$ to the structure sequence
\[ 0 \to \mathcal{O}_B(2(m - 1)) \to \mathcal{O}_B \to \mathcal{O}_{(m-1)B} \to 0 \]
induces an isomorphism
\[
\text{Hom}(\mathcal{O}_B(2(m - 1)), \mathcal{O}_B(2(m - 1) + 1)) \cong \text{Ext}^1(\mathcal{O}_{(m-1)B}, \mathcal{O}_B(2(m - 1) + 1)).
\]
This implies that every sequence as in (ii) admits a morphism of short exact sequences
\[
\begin{array}{cccc}
0 & \to & \mathcal{O}_B(2(m - 1)) & \to & \mathcal{O}_B & \to & \mathcal{O}_{(m-1)B} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_B(2(m - 1) + 1) & \to & G & \to & \mathcal{O}_{(m-1)B} & \to & 0
\end{array}
\]

The cokernel of the left vertical arrow is $\mathcal{O}_x$ for some $x \in X$. Hence, the Snake Lemma gives a short exact sequence $0 \to \mathcal{O}_B \to G \to \mathcal{O}_x \to 0$. By Lemma 2.3, this gives $G \cong \mathcal{O}_{mB}(x)$.

Let us now show (iii) $\implies$ (i). In general, if $F \subset G$ is a purely one-dimensional subsheaf, we have that $\text{gr}_k F \subset \text{gr}_k G$ for all $k$. We can see this from (6), as $\text{pure}(F|_{kB}) \subset \text{pure}(G|_{kB})$ (note that a possible kernel of $F|_{kB} \to G|_{kB}$ zero-dimensional). We apply this to $F = \mathcal{O}_{mB} \subset G = \mathcal{O}_{mB}(x)$. By Lemma 3.7, we already know that $\text{gr}_k \mathcal{O}_{mB} \cong \mathcal{O}_B(2(k - 1))$. The $\text{gr}_k \mathcal{O}_{mB}(x)$ must be line bundles containing $\text{gr}_k \mathcal{O}_{mB}$, hence $\text{gr}_k \mathcal{O}_{mB}(x) \cong \mathcal{O}_B(2(k - 1) + \delta_k)$ for some $\delta_k \geq 0$. By Lemma 2.3, we have $\chi(\mathcal{O}_{mB}(x)) = \chi(\mathcal{O}_{mB}) + 1$. Hence, we must have $\delta_k = 1$. 

for exactly one \( k \in \{1, \ldots, m\} \) and \( \delta_{k'} = 0 \) for all \( k' \neq k \). Since \( \mathcal{O}_B(-B) \cong \mathcal{O}(2) \), we have non-zero multiplication maps

\[
\mathcal{O}(2) \otimes \text{gr}_k \mathcal{O}_{mB}(x) \to \text{gr}_{k+1} \mathcal{O}_{mB}(x)
\]

which shows that the degrees of the graded pieces have to grow by at least 2 in every step. In other words, the series \( \delta_1, \delta_2, \ldots, \delta_m \) is non-decreasing. It follows that the unique \( k \) with \( \delta_k = 1 \) must be \( k = m \).

\[\square\]

3.3. Line bundles on extended ADE curves.

**Lemma 3.9.** We have \( g(C) := h^1(\mathcal{O}_C) = 1 + \sum_{o \in O} g(C_o) \).

**Proof.** Note that, for \( i \in I \), the sheaf \( \mathcal{O}_{m_iC_i} \) has a filtration whose factors are the line bundles

\[
\mathcal{O}_{C_i}, \mathcal{O}_{C_i}(2), \ldots, \mathcal{O}_{C_i}(2(m - 1)) \; ;
\]

see Lemma 3.7. This gives \( h^1(\mathcal{O}_{m_iC_i}) = 0 \) and

\[
h^0(\mathcal{O}_{m_iC_i}) = \sum_{k=0}^{m_i-1} h^0(\mathcal{O}(2k)) = \sum_{k=0}^{m_i-1} (2k + 1) = m_i^2.
\]

Also, \( h^0(\mathcal{O}_{C_o}) = 1 \) for every \( o \in O \). Hence,

\[
h^0(\mathcal{O}_{m_iC_i}) = m_v^2 \quad \text{for all } \; v \in V. \tag{9}
\]

We have a short exact sequence

\[
0 \to \mathcal{O}_C \to \bigoplus_{v \in V} \mathcal{O}_{m_vC_v} \to \bigoplus_x \mathcal{O}_{\xi_x} \to 0,
\]

where the direct sum on the right runs through all intersection points of components of \( C \) (with \( \xi_x = (m_vC_v) \cap (m_wC_w) \) denoting the scheme-theoretic intersection when \( x = C_v \cap C_w \)), and the components of the maps are just the restriction maps; compare Lemma 2.12. Applying the global sections functor to this sequence gives

\[
0 \to H^0(\mathcal{O}_C) \to \bigoplus_{v \in V} H^0(\mathcal{O}_{m_vC_v}) \to \bigoplus_x H^0(\mathcal{O}_{\xi_x}) \to H^1(\mathcal{O}_C) \to \bigoplus_{o \in O} H^1(\mathcal{O}_{C_o}) \to 0.
\]

As \( C \) is a numerically 1-connected Gorenstein curve, we have \( h^0(\mathcal{O}_C) = 1 \); see [CFHR99, Thm. 3.3]. By (9), the second term of the long exact sequence has dimension \( \sum_{v \in V} m_v^2 \). For \( x = C_v \cap C_w \), we have \( h^0(\mathcal{O}_{\xi_x}) = m_vm_w \); compare (2). Hence, it follows from (5) that the third term of the long exact sequence has the same dimension \( \sum_{v \in V} m_v^2 \) as the second term. Taking this information together, we see that the cokernel of the second map is one-dimensional. Hence, we have a short exact sequence

\[
0 \to \mathbb{C} \to H^1(\mathcal{O}_C) \to \bigoplus_{o \in O} H^1(\mathcal{O}_{C_o}) \to 0. \tag{\text{□}}
\]

By Proposition 2.1, we get

**Corollary 3.10.** There is an exact sequence of algebraic groups

\[
1 \to G \to \text{Pic}^0(C) \to \prod_{o \in O} \text{Pic}^0(C_o) \to 1,
\]

where \( G \) is a one-dimensional smooth connected algebraic group.

We will see later that \( G = \mathbb{G}_m \) if \( \Gamma = A_n \), and \( G = \mathbb{G}_a \) if \( \Gamma = D_n, E_6, E_7, E_8 \).
3.4. Definition of Stability. The version of stability that we use is \textit{Gieseker-stability} which uses the reduced Hilbert polynomial. Let us quickly review the relevant notions; see [HL10] for a general reference.

Let \( X \) be a projective noetherian scheme, and fix a polarisation \( H \) on \( X \), i.e., an ample line bundle. The Hilbert polynomial \( P(F) \) is given by

\[
P(F, n) = \chi(F(nH)).
\]

Moreover, the Hilbert polynomial can be uniquely written in the form

\[
P(F, n) = \sum_{i=0}^{\dim(F)} \alpha_i(F) \frac{n^i}{i!},
\]

where \( \alpha_i(F) \) are rational coefficients for \( i = 0, \ldots, d = \dim(F) \). The \textit{reduced Hilbert polynomial}

\[
p(F, n) := \frac{P(F, n)}{\alpha_d(F)}.
\]

For two polynomials \( p \) and \( q \), we write \( p > q \) if \( p(n) > q(n) \) for all \( n \gg 0 \).

\textbf{Definition 3.11.} A coherent sheaf \( F \) of dimension \( d \) is \textit{semi-stable} if \( F \) is pure and for any proper purely \( d \)-dimensional quotient \( F \to Q \) we have \( p(F) \leq p(Q) \). It is called \textit{stable} if \( F \) is semi-stable and the inequality is strict, i.e., \( p(F) < p(Q) \) for any proper purely \( d \)-dimensional quotient \( F \to Q \).

If \( X = C \) is an extended ADE curve, then the reduced Hilbert polynomial is a monic polynomial of degree 1, hence determined by its constant term. More concretely, by Riemann-Roch, we have

\[
P_H(F, n) = \left( \sum_v \mu_v(F)e_v \right)n + \chi(F). \tag{10}
\]

for every purely one-dimensional sheaf \( F \) on \( C \), and hence

\[
p(F, n) = n + \frac{\chi(F)}{\sum_{v \in V} \mu_v(F)e_v},
\]

where \( \mu_v(F) \) denotes the multiplicity of \( F \) along \( m_vC_v \) and \( e_v = \deg(H_{|C_v}) \).

3.5. Condition on the polarisation. From now on, let us fix some positive integer \( \chi > 0 \). We want to classify certain stable sheaves on \( C \) with Euler characteristic \( \chi \). In order to talk about stability, we need to fix some polarisation \( H \) on \( C \). The following assumption is necessary for our classification of stable sheaves of type \( m \) and Euler characteristic \( \chi \).

\textbf{Assumption 3.12.} Let \( e_v := \deg(H_{|C_v}) \) and \( e := \deg(H) = \sum_{v \in V} m_v e_v \). Setting \( b_v := \left\lceil \frac{e_v}{\chi} \right\rceil \) for \( v \in V \), we make the following assumption on our polarisation:

\[
\sum_{o \in O} b_o = \chi + |O| - 1. \tag{11}
\]

This means that \( \sum_{o \in O} b_o \) takes the maximal possible value. Indeed, by definition of the \( b_o \) as the ceiling of \( \frac{e_o}{\chi} \), we have

\[
\sum_{o \in O} b_o < |O| + \sum_{o \in O} \frac{e_o}{\chi}. \tag{12}
\]
Furthermore, setting \( e_I := \sum_{i \in I} m_i e_i \), for any given polarisation, we have

\[
\sum_{o \in O} \frac{e_o}{e} \chi \leq \frac{e_I + \sum_{o \in O} e_o}{e} \chi = \frac{e}{e} \chi = \chi
\]

(13)

with an equality for \( \Gamma = \tilde{A}_n \), where \( I = \emptyset \) hence \( e_I = 0 \), and a proper inequality otherwise. Combining (12) and (13), we see that for every polarisation, we have \( \leq \) in (11).

Under Assumption 3.12, we have \( \frac{e_o}{e} \chi \not\in \mathbb{N} \) for \( o \in O \). Indeed, otherwise the value of the left-hand of (12) would drop. The condition \( \frac{e_o}{e} \chi \not\in \mathbb{N} \) is necessary to prevent properly semi-stable sheaves.

Looking at (13), we also see that the \( e_i \) need to be small relative to the \( e_o \) for our assumption to hold.

The following lemma will be needed in the proof of our classification of stable sheaves in Proposition 3.19.

**Lemma 3.13.** Let \( H \) be a polarisation satisfying Assumption 3.12

(i) For every \( i \in I \), we have \( b_i = 1 \).

(ii) Let \( J \subset O \), and \( 0 \leq f \leq e_I \), and assume that at least one of \( J \not\subseteq O \) or \( f < e_I \) holds. Then

\[
1 - |J| + \sum_{j \in J} b_j > \frac{f + \sum_{j \in J} e_j}{e} \chi.
\]

Proof. Looking at (12) and (13), we see that, for \( \sum_{o \in O} b_o \) to have a chance to take the maximal value \( \chi + |O| - 1 \), we must have \( \sum_{o \in O} \frac{e_o}{e} \chi > \chi - 1 \). Hence, \( \frac{e_o}{e} \chi < 1 \) which gives (i).

Regarding (ii), let us assume for a contradiction that

\[
1 - |J| + \sum_{j \in J} b_j \leq \frac{f + \sum_{j \in J} e_j}{e} \chi.
\]

Noting that \( b_j - 1 < \frac{e_j}{e} \chi \) for every \( j \in O \), we can add \( \sum_{o \in O \setminus J} \frac{e_o}{e} \chi \) to both sides to get

\[
1 - |O| + \sum_{o \in O} b_o = 1 - |J| + \sum_{j \in J} b_j + \sum_{o \in O \setminus J} (b_j - 1) \leq \frac{f + \sum_{j \in J} e_j}{e} \chi + \sum_{o \in O \setminus J} \frac{e_j}{e} \chi
\]

\[
= \frac{f + \sum_{o \in O} e_j}{e} \chi
\]

\[
\leq \frac{e_I + \sum_{o \in O} e_j}{e} \chi
\]

\[
= \chi.
\]

Note that, by Assumption 3.12, not only the right-hand, but also the left-hand side of this chain of inequalities is \( \chi \). If \( J \not\subseteq O \), the first inequality is proper. If \( f < e_I \), the second inequality is proper. Either way, we arrive at the contradiction \( \chi < \chi \). \( \square \)

**Remark 3.14.** If \( \chi = 1 \), we have \( b_v = 1 \) for every \( v \in V \). Hence, Assumption 3.12 is automatically satisfied. Hence, after proving our classification Theorem 1.1 under Assumption 3.12, it follows in particular that stability of sheaves of type \( m \) and Euler characteristic \( \chi = 1 \) does not depend on the chosen polarisation. For higher \( \chi \), however, the stability condition depends on the polarisation; see Subsection 5.4 for some further comments on this.
Note that, given any $\chi > 0$, we can find a polarisation $H$ on $C$ which satisfies Assumption 3.12. The reason is that we can choose the $e_v$ arbitrarily as long as they are sufficiently large.

A more subtle question is whether we can find a polarisation $H$ on the ambient surface $X$ which restricts to a polarisation of $C$ satisfying Assumption 3.12. This is a relevant question as, later in Subsection 5.2, we want to regard the moduli space $\mathcal{M}$ of sheaves of type $m$ and Euler characteristic $\chi$ as the fibre of $N \to |C|$ over $C$, where $N$ is the moduli space of $H$-stable sheaves on the surface $X$ with rank 0, Euler characteristic $\chi$, and first Chern class $c_1 = [C]$. In an important class of examples, this is always possible:

**Proposition 3.15.** Let $C$ be an extended ADE curve such that all but one component $C_o$ are rational $(-2)$-curves. Then, for every $\chi > 0$ there exists a polarisation $H$ of the surface $X$ which satisfies Assumption 3.12.

**Proof.** As $E := C - C_o$ is a configuration of $(-2)$-curves, we can contract it. Let $\pi: X \to \bar{X}$ be the contraction. For $\bar{H}$ an ample divisor on $\bar{X}$ and $\varepsilon > 0$ sufficiently small, the divisor $H := \pi^*\bar{H} - \varepsilon E$ is again ample. It is a priori only a $\mathbb{Q}$-divisor, but, as Assumption 3.12 is stable under replacing $H$ by some multiple $mH$, this is not a problem. By making $\varepsilon$ even smaller, if necessary, we can achieve that $\frac{\varepsilon}{d}\chi > \chi - 1$. Then $b_0 = \chi$ and $b_v = 1$ for all $v \neq o$, which gives (11).

However, there are examples of extended ADE curves $C \hookrightarrow X$ and numbers $\chi > 0$, where it is impossible to find a polarisation on the surface $X$ satisfying Assumption 3.12:

**Example 3.16.** Let $X = \mathbb{P}^2$ and $C = Q + L$ the union of a smooth conic and a general line, which is an extended ADE curve with $\Gamma = \tilde{A}_1$. For $H = \mathcal{O}_{\mathbb{P}^2}(d)$, Assumption 3.12 becomes

$$\left\lfloor \frac{d}{3d} \chi \right\rfloor + \left\lfloor \frac{2d}{3d} \chi \right\rfloor = \chi + 1$$

which is satisfied (for any positive $d$) if and only if $\chi$ is not divisible by 3. We will see a similar example on a K3 surface later in Example 5.10.

At least, given a curve $C \hookrightarrow X$ there are always infinitely many values of $\chi > 0$ for which we can find a polarisation $H$ on the surface $X$ satisfying Assumption 3.12. To see this, note that, as the $C_i$ are rational $(-2)$-curves, we can always contract $E = \sum_{i \in I} m_i C_i$, say via $\pi: X \to \bar{X}$. Then, choose any polarisation $\bar{H}$ on $\bar{X}$, and set $H_0 := \pi^*\bar{H}$.

This $H_0$ is not ample, hence not a polarisation. Anyway, let us note that, numerically, it satisfies Assumption 3.12 for every $\chi$ with $\chi \equiv 1 \mod \deg(H_0)$. Let us further note that Assumption 3.12 is stable under small perturbations of $H$ as an $\mathbb{Q}$-divisor, due to the fact, noted above, that Assumption 3.12 implies $\frac{\varepsilon}{d}\chi \notin \mathbb{N}$. Hence, given some $\chi$ with $\chi \equiv 1 \mod \deg(H_0)$, we can find a sufficiently small $\varepsilon > 0$ such that $H := H_0 - \varepsilon E$ is a polarisation on $X$ which still satisfies Assumption 3.12.

3.6. **Classification of stable sheaves by their components.**

**Definition 3.17.** A coherent sheaf $F$ on $C$ is of type $m$, if it is purely one-dimensional, and $\mu_v(F) = m_v$ for every $v \in V$.

Every purely one-dimensional sheaf $F$ which is generically a line bundle is a sheaf of type $m$. But, if $C$ is non-reduced, i.e. if $\Gamma \in \{ \tilde{D}_6, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \}$, the converse is not true. For
example, consider $F = \bigoplus_{v \in V} O_{C_v}^{\oplus m_v}$. This $F$ has scheme-theoretic support only on $C_{\text{red}}$, but still satisfies $\mu_v(F) = m_v$ for all $v \in V$.

However, our classification of the stable sheaves of type $m$ with respect to a polarisation satisfying Assumption 3.12 which follows below shows in particular that every stable sheaf of type $m$ is generically a line bundle.

Looking at (10), we see that the Hilbert polynomial of a sheaf $F$ of type $m$ is

$$P_H(F, n) = cn + \chi(F).$$

Recall that by Lemma 2.12, for every sheaf $F$ of type $m$, we have an exact sequence

$$0 \rightarrow F \rightarrow \bigoplus_{v \in V} F_v \rightarrow \bigoplus_x T_x \rightarrow 0,$$

where $F_v = \text{pure}(F_{m_vC_v})$, and $x$ runs through all intersection points of components of $C$. Also recall that, for $x = C_u \cap C_w$, we write $\zeta_x := (m_uC_u) \cap (m_wC_w)$, and that, by our transversality assumption, $O_{\zeta_x} \cong \mathbb{C}[x, y]/(x^{m_u}, y^{m_w})$. Hence, its socle $(x^{m_u-1}y^{m_w-1})$ is one-dimensional. Let $\zeta'_x \subset \zeta_x$ be the subscheme with $O_{\zeta'_x} = O_{\zeta_x}/\text{socle}$.

**Definition 3.18.** Let $F$ be a sheaf of type $m$.

(i) For $i \in I$, the sheaf $F_i$ on $m_iC_i$ is

(a) ordinary if $F_i \cong O_{m_iC_i}$.

(b) special if it satisfies the equivalent conditions of Lemma 3.8, which means that $F_i \cong O_{m_iC_i}(x)$ for some $x \in C_i$.

(ii) For $o \in O$, the sheaf $F_o$ on $C_o$ is

(a) ordinary if $F_o$ is a line bundle of Euler characteristic $b_o := \lceil \frac{e}{d} \chi \rceil$, compare Subsection 3.5.

(b) special if $F_o$ is a line bundle of Euler characteristic $b_o + 1$.

(iii) Let $x$ be an intersection point of two components of $C$, the sheaf $T_x$ is

(a) ordinary if $T_x \cong O_{\zeta_x}$.

(b) special if $T_x \cong O_{\zeta'_x}$ where $O_{\zeta'_x} = O_{\zeta_x}/\text{socle}$.

Note that there are many sheaves of type $m$ such that the $F_v$ and $T_x$ are neither special nor ordinary.

**Proposition 3.19.** The following three statements are equivalent for $F$ a sheaf of type $m$:

(i) $F$ is semi-stable,

(ii) $F$ is stable,

(iii) exactly one of the sheaves $F_v$ and $T_x$ occurring in (14) is special, and all the others are ordinary.

**Proof.** We start with the (i) $\implies$ (iii) part. For the following notation related to the natural filtration of $F_i$, see Subsection 2.3. For $i \in I$, all the vector bundles $\text{gr}_k F_i$ on $C_i \cong \mathbb{P}^1$ decompose into line bundles, say

$$\text{gr}_k F_i = O_{C_i}(a_{k1}) \oplus \ldots \oplus O_{C_i}(a_{k_{rk}}).$$

Every line bundle $L$ appearing in the decomposition of $\text{gr}_1 F_i$ is a quotient of $F$. As

$$P(L, n) = e_in + \chi(L), \quad P(F, n) = en + \chi,$$
semi-stability implies $\chi(L) \geq \frac{\delta}{\epsilon} \chi$. Applying Lemma 3.13(i) gives $\chi(L) \geq b_i = 1$. Since $\mathcal{O}_{C_i}(-C_i) \cong \mathcal{O}_{C_i}(2)$, and the natural map $\text{gr}_k F_i \otimes \mathcal{O}_{C_i}(-C_i) \to \text{gr}_{k+1} F_i$ is generically surjective,

$$\min\{a_{k\alpha} \mid \alpha = 1, \ldots, r_i\} + 2 \leq \min\{a_{(k+1)\alpha} \mid \alpha = 1, \ldots, r_{(k+1)}\}.$$  

It follows that $\chi(L) \geq 2k - 1$ for every line bundle $L$ appearing as a direct summand of $\text{gr}_k F_i$. This yields $\chi(\text{gr}_k F_i) \geq (2k - 1)r_{ik}$.

For $r \in \mathbb{N}$, we set $n_{ir} := \max\{k \mid r_{ik} \geq r\}$. Note that the non-increasing sequences $(r_{ik})_k$ and $(n_{ik})_k$ are dual partitions (i.e. they have transposed Young diagrams) of $\mu_i(F) = m_i$. In particular, $\sum_r n_{ik} = m_i = \sum_k r_{ik}$. We get

$$\chi(F_i) = \sum_k \chi(\text{gr}_k F_i) \geq \sum_k r_{ik}(2k - 1) = \sum_r n_{ir}^2$$

where the last equality is due to the general identity $\sum_{k=1}^n (2k - 1) = n^2$. Similarly, as $F_o$ is a quotient of $F$ for every $o \in O$, semi-stability gives $\chi(F_o) \geq b_o$. Taking these inequalities for varying $o \in O$ together, and using Assumption 3.12, we get

$$\chi - \sum_{o \in O} \chi(F_o) \leq 1 - |O|.$$

Let $x = C_v \cap C_w$ be an intersection point of two components. By a general combinatorial consideration, for example using induction on the lengths of the partitions $(r_{vk})_k$ and $(r_{wk'})_{k'}$, we see that

$$\sum \min\{r_{vk}, r_{wk'}\} = \sum r_{wr} n_{vr}.$$

Hence, Lemma 2.14 becomes

$$\chi(T_x) \leq \sum_r n_{vr} n_{wr}.$$  

(17)

Note that, for $o \in O$, we have $n_{o1} = 1$ and $n_{or} = 0$ for $r \geq 2$. Let $S = S_{\Gamma}$ be minus the extended Cartan matrix; compare Remark 3.3. From the exact sequence (14) and the inequalities (15) and (16), and (17), we conclude that

$$0 = \chi(F) - \sum_{v \in V} \chi(F_v) + \sum_x \chi(T_x) \leq 1 - |O| - \sum_{i \in I} \sum_{r} n_{ir}^2 + \sum_{\{u,w\} \subset V, u \neq w} S_{uw} \sum_r n_{ur} n_{wr}$$

$$\leq 1 - \sum_{v \in V} \sum_{r} n_{vr}^2 + \sum_{\{u,w\} \subset V, u \neq w} S_{uw} \sum_r n_{ur} n_{wr}$$

$$= 1 + \frac{1}{2} \sum_r n_{vr} S_{uv} n_{wr}$$

$$= 1 + \frac{1}{2} \sum_r \tilde{n}_r S_{\tilde{n}_r},$$

where $\tilde{n}_r = (n_{vr})_{v \in V} \in \mathbb{Z}^V$. The properties of $S$ mentioned in Remark 3.3 give that any non-trivial vector $0 \neq v \in \mathbb{Z}^V$ is either a multiple of $\tilde{m} = (m_v)_{v \in V}$ or satisfies $v^t S v \leq -2$. Since $\tilde{n}_1 + \tilde{n}_2 + \ldots = \tilde{m}$, the inequality $-2 \leq \sum_r \tilde{n}_r^t S \tilde{n}_r$ shows that we must have $\tilde{n}_1 = \tilde{m}$ and $\tilde{n}_r = 0$ for all $r > 1$. We can infer two facts: every subfactor $\text{gr}_k F_i$, with $i \in I$, is a line
bundle on $C_t$, and, second, as our inequality now reads $0 \leq 1 + \frac{1}{2} \overline{n}_1 S \overline{n}_1 = 1$, that exactly one of the inequalities used in the deduction, namely

$$\chi(\text{gr}_k F_i) \geq 2k - 1 \quad \text{for } i \in I, \ k \leq m_i, \quad (18)$$
$$\chi(F_o) \geq b_o \quad \text{for } o \in O, \quad (19)$$
$$\chi(T_x) \leq m_u m_w \quad \text{for } x = C_u \cap C_w \quad (20)$$

fails to be an equality and by an excess of exactly 1.

If $\chi(\text{gr}_k F_i) > 2k - 1$ for some $k \leq m_i$, we must already have $k = m_i$. Otherwise, as the Euler characteristic of the factors grows by at least 2 in every step, we would also have $\chi(\text{gr}_{k+1} F_i) > 2(k + 1) - 1$ as a second proper inequality. Hence, in this case, Lemma 3.7 and Lemma 3.8 show that $F_i$ is special, and all other $F_o$ and all $T_x$ are ordinary.

If (19) is a proper inequality, then $F_o$ is special, and all other $F_v$ and all $T_x$ are ordinary; see again Lemma 3.7.

Let $\chi(T_x) = m_u m_w - 1$ for $x = C_u \cap C_w$. Then the other two inequalities, (18) and (19), must be equalities, which implies that all $F_v$ are ordinary, hence line bundles. As the maps $F_u \to T_x$ and $F_w \to T_x$ are surjections, we must have that $T_x$ is a quotient of $(F_u)_x \cong \mathcal{O}_{m_u C_u, x}$ and of $(F_w)_x \cong \mathcal{O}_{m_w C_w, x}$, hence of $\mathcal{O}_{\zeta_x}$. The only quotient of $\mathcal{O}_{\zeta_x}$ of dimension $m_u m_w - 1$ is $\mathcal{O}_{\zeta_x}$, which means that $T_x$ is special.

We now come to the proof of the implication (iii) $\implies$ (ii). All purely one-dimensional quotients $F''$ of $F$ are of the form $F'' = \text{pure}(F|Z)$ where $Z = \sum_{v \in V} m_v C_v$ for some $0 \leq m_v'' \leq m_v$; see Lemma 2.11. Hence, for any such proper quotient $F \to F'' = \text{pure}(F|Z)$, we need to show the inequality $p(F, n) < p(F'', n)$. We consider the short exact sequence

$$0 \longrightarrow F'' \longrightarrow \bigoplus_{v \in V} F_v'' \longrightarrow \bigoplus_{x} T_x'' \longrightarrow 0 \quad (21)$$

which is the analogue of (14) for $F''$; see Lemma 2.12. We set

$$J := \{ j \in O \mid m_j'' = 1 \} = \{ j \in O \mid C_j \subset \text{supp}(F'') \} = \{ j \in O \mid F_j'' \neq 0 \}. $$

For $j \in J$, we have $F_j'' = F_j$. Hence,

$$\chi(F_j'') \geq b_j \quad \text{for } j \in J. $$

For $i \in I$, we have $\text{gr}_k F''_i \cong \text{gr}_k F_i \cong \mathcal{O}_{C_i}(2k - 2)$ for $1 \leq k \leq m_i''$ with one possible exception: If $k = m_i'' = m_i$ and $F_i$ is special, we have $\text{gr}_k F''_i \cong \mathcal{O}_{C_i}(2k - 1)$. Anyway, we get

$$\chi(F_i'') \geq \sum_{k=1}^{m_i''} (2k - 1) = (m_i'')^2. $$

Furthermore, for $x = C_u \cap C_w$, we have $T_x'' = \mathcal{O}_{m_u C_u \cap m_w C_w}$, with one possible exception: If $m_u'' = m_u$, $m_w'' = m_w$, and $T_x$ is special, we have $T_x'' = T_x \cong \mathcal{O}_{\zeta_x}$. Anyway, $T_x''$ is a quotient of $\mathcal{O}_{m_u C_u \cap m_w C_w}$, which gives

$$\chi(T_x'') \leq m_u'' m_w'',$$
In summary, using (21), we get
\[
\chi(F'') \geq \sum_{j \in J} b_j + \sum_{i \in I} (m''_i)^2 - \sum_x m''_u m''_w
\]
\[
= \sum_{j \in J} b_j - |J| + \sum_{j \in O} (m''_j)^2 + \sum_{i \in I} (m''_i)^2 - \sum_x m''_u m''_w
\]
\[
= \sum_{j \in J} b_j - |J| - \frac{1}{2}(\vec{m}'')S\vec{m}'
\]
\[
\geq \sum_{j \in J} b_j - |J| + 1,
\]
where the last inequality is due to the fact that \(\vec{m}'' = (m''_v)_v \in V\) is not a multiple of \(\vec{m}\) and Remark 3.3. Combining this with Lemma 3.13(ii) gives
\[
\chi(F'') > \sum_{i \in I} m''_i e_i + \sum_{j \in J} e_j \chi.
\]
This is exactly what we need as, by (10), we have
\[
P(F'', n) = (\sum_{i \in I} m''_i e_i + \sum_{j \in J} e_j) n + \chi(F'').
\]
As the implication (ii) \(\Rightarrow\) (i) is trivial, we finished the proof. \(\square\)

3.7. Presentation of stable sheaves as generalised divisors.

Lemma 3.20. Let \(F\) be a purely one-dimensional sheaf on \(C\), and let \(x = C_v \cap C_w\) be an intersection point of two components. If \(T_x \cong O_\zeta\) is ordinary, and \(F_v\) and \(F_w\) are line bundles in a neighbourhood of \(x\), then \(F\) is a line bundle in a neighbourhood of \(x\).

Proof. Over a sufficiently small open neighbourhood of \(x\), the sequence (14) becomes
\[
0 \to F \to O_{m_vC_v} \oplus O_{m_wC_w} \to O_\zeta \to 0. \tag{22}
\]
As, on the affine neighbourhood, every invertible section of \(O_\zeta\) can be lifted to one of \(O_{m_vC_v}\) and to one of \(O_{m_wC_w}\), the sequence (22) is isomorphic to the sequence where both maps \(O_{m_vC_v} \to O_\zeta\) and \(O_{m_wC_w} \to O_\zeta\) are just the restriction maps. This means that \(F \cong O_C\) on our open affine neighbourhood of \(x\). \(\square\)

Proposition 3.21. Let \(F \in \text{Coh}(X)\) be a stable sheaf of type \(m\).

(i) Let \(x\) be an intersection point of two components of \(C\), and let \(L \in \text{Pic}(C)\) with \(\chi(L|_{C_v}) = b_v\) for all \(v \in V\). Then
\[
F = L(x)
\]
is a stable sheaf with \(T_x\) special. Conversely, every stable sheaf of type \(m\) with \(T_x\) special is of this form.

(ii) Let \(i \in I\), \(t \in C_i\) a point which is not an intersection point with another component of \(C\), and let \(L \in \text{Pic}(C)\) with \(\chi(L|_{C_v}) = b_v\) for all \(v \in V\). Then
\[
F = L(t)
\]
is a stable sheaf with \(F_i\) special. Conversely, every stable sheaf of type \(m\) with \(F_i\) special is of this form.
(iii) Let $o \in O$. Every $F = M \in \text{Pic}(C)$ with $\chi(M_{C_v}) = b_v$ for all $v \in V \setminus \{o\}$ and $\chi(M_{C_v}) = b_o + 1$ is stable with $F_o$ special. Conversely, every stable sheaf of type $m$ with $F_o$ special is of this form.

Proof. Let $x = C_i \cap C_j$ be an intersection point of two components of $C$. We first claim that the sequence of Lemma 2.12 for $O_C(x)$ takes the form

$$0 \rightarrow O_C \rightarrow \bigoplus_{v \in V} O_{m_v C_v} \rightarrow \bigoplus_{y \neq x} O_{\zeta_y} \rightarrow 0,$$

(23)

where all components of the second map are the natural restriction maps. Indeed, let us denote the kernel of (23) by $K$ with the goal to show $K \cong O_C(x)$. There is a morphism of short exact sequences

$$0 \rightarrow O_C \rightarrow K \rightarrow O_x \rightarrow 0. \quad (24)$$

As the kernel of the right vertical map is $O_x$, Snake Lemma gives a short exact sequence $0 \rightarrow O_C \rightarrow K \rightarrow O_x \rightarrow 0$. Hence, by Lemma 2.3, $K \cong O_C(x)$. Now, by Remark 2.13, it follows that $L(x) = L \otimes O_C(x)$, satisfies Proposition 3.19(iii) with $T_x$ special. Hence, it is stable.

Conversely, let $F$ be a stable sheaf with $T_x$ special. Then, by Proposition 3.19, all $F_v$ are ordinary, hence line bundles on $m_v C_v$. Analogously to the proof of Lemma 3.20, we can find an open neighbourhood $x \in U \subset C$ over which (14) is isomorphic to

$$0 \rightarrow F \rightarrow O_{m_i C_i} \oplus O_{m_j C_j} \rightarrow O_{\zeta_x} \rightarrow 0,$$

where $r$ denotes the appropriate restriction maps. Hence, over $U$, we get a morphism of short exact sequences

$$0 \rightarrow O_C \rightarrow O_{m_i C_i} \oplus O_{m_j C_j} \rightarrow O_{\zeta_x} \rightarrow 0.$$

As the kernel of the restriction map $r: O_{\zeta_x} \rightarrow O_{\zeta_x}$ is $O_x$, Snake Lemma gives a short exact sequence

$$0 \rightarrow O_U \rightarrow F_{|U} \rightarrow O_x \rightarrow 0.$$

By Lemma 2.3, this gives $F_{|U} \cong O_U(x)$. Furthermore, by Proposition 3.19, all sheaves $T_{x'}$ for intersection points $x' \neq x$ are ordinary. Hence, by Lemma 3.20, $F_{|C \setminus \{x\}}$ is a line bundle. The assertion now follows by Lemma 2.5.

For part (ii), note that $F = L(t)$ satisfies the assumptions of Proposition 3.19(iii) with $F_i$ special. Hence, it is stable.

Conversely, let $F$ be a stable sheaf with $F_i$ special. Then, by definition, $F_i \cong O_{m_i C_i}(t)$ for some $t \in C_i$. By Lemma 2.6(iii), we get that $t$ cannot be an intersection point of $C_i$ with another component. The assertion that $F$ is of the desired form $L(t)$ follows by Lemma 3.20 together with Lemma 2.5.

Part (iii) follows directly from Lemma 3.20. 

□
Let \( L \) be a line bundle on \( C \) with \( \chi(L|_{C_v}) = b_v \) for all \( v \in V \). Then, for \( o \in O \), and \( x \in C_o \) a point which is not an intersection point with another component of \( C \), the line bundle \( M = L(x) \) satisfies the properties of Proposition 3.21(iii). Hence, we can summarise Proposition 3.21 as follows, which is Corollary 3.22.

**Corollary 3.22.** The stable sheaves of type \( m \) are exactly the sheaves of the form \( F = L(x) \) for some line bundle \( L \) on \( C \) with \( \chi(L|_{C_v}) = b_v \) for all \( v \in V \) and some \( x \in C \).

Note, however, that \( F \) does not determine \( L \) and \( x \), which means that we can have \( L(x) \sim L'(x') \) for \( L \not\sim L' \) and \( x \neq x' \). For singular stable sheaves, however, we will see a result in the direction of uniqueness of the presentation \( L(x) \) in the next subsection.

Note that, combining Corollary 3.22 with the equivalence of the first two points in Proposition 3.19, we have now proved the classification Theorem 1.1 from the introduction.

### 3.8. Uniqueness for singular stable sheaves

Our goal for this subsection is to prove

**Proposition 3.23.** Let \( x, x' \in C_{\text{sing}} \) and \( L, L' \in \text{Pic}(C) \). Then

\[
L(x) \cong L'(x') \iff x = x' \text{ and } L|_{C_v} \cong L'|_{C_v} \text{ for all } v \in V .
\]

We first need some notation and a few lemmas. For \( v \in V \), we write \( C_v : = C - C_v \). By (5), we have

\[
C_v . C_v = 2 . \quad (24)
\]

**Lemma 3.24.** The curve \( C_v \) is numerically 1-connected. This means that, for every decomposition of effective divisors \( C_v = A + B \), we have \( A . B \geq 1 \).

*Proof.* Let \( C_v = A + B \). By (24), we must have \( A . C_v < 2 \) or \( B . C_v < 2 \). Let us assume without loss of generality that \( A . C_v < 2 \). We have \( C = A + (B + C_v) \). As, by Lemma 3.6, the curve \( C \) is numerically 2-connected,

\[
2 \leq A . (B + C_v) = A . B + A . C_v .
\]

Our assumption \( A . C_v < 2 \) now gives \( 1 \leq A . B \), as asserted. \( \square \)

**Lemma 3.25.** Let \( x \in C_{\text{sing}} \cap C_v \) for some \( v \in V \) (if \( v \in I \), then \( C_{\text{sing}} \cap C_v = C_v \), and if \( v \in O \), then \( C_{\text{sing}} \cap C_v \) is the set of intersection points of \( C_v \) with other components). Then, there is an exact sequence

\[
0 \to \mathcal{O}_{C_v}(-C_v + x) \to \mathcal{O}_C(x) \to \mathcal{O}_{C_v} \to 0 .
\]

Note that \( \deg \mathcal{O}_{C_v}(-C_v + x) = -1 \).

*Proof.* We have a chain of inclusions

\[
\mathcal{I}_{C_v \hookrightarrow C} = \mathcal{O}_{C_v}(-C_v) \subset \mathcal{O}_C(-x) \subset \mathcal{O}_C .
\]
Snake lemma gives a commutative diagram with exact columns and rows

\[
\begin{array}{c}
0 \\
\downarrow \\
\mathcal{O}_C(-C_v) \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{O}_C(-x) \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{O}_C(-x) \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{O}_C(x) \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{O}_C(x) \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{O}_C(x) \\
\downarrow \\
0 \\
\downarrow \\
\mathcal{O}_C(x) \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\]

Taking the dual of the left column gives the desired short exact sequence; see Lemma 2.9. \(\square\)

For \(x \in C_{\text{sing}}\), the natural map \(\text{End}_{\mathcal{O}_C}(I_x) \to \mathcal{O}_C(x)\) is an isomorphism, which endows \(\mathcal{O}_C(x)\) with the structure of a commutative \(\mathcal{O}_C\)-algebra; see Lemma 2.6 and Lemma 2.7. We write

\[\nu^x : C^x := \text{Spec}_{\mathcal{O}_C}(\mathcal{O}_C(x)) \to C,\]

which is an isomorphism away from \(x\).

**Lemma 3.26.** Let \(L, L' \in \text{Pic}(C)\) and \(x \in C_{\text{sing}}\). Then

\[L(x) \cong L'(x) \iff \nu^x_* L \cong \nu^x_* L'.\]

**Proof.** By definition, we have \(\mathcal{O}_C(x) \cong \nu^x_* \mathcal{O}_{C^x}\). Hence, projection formula gives

\[L(x) \cong \nu^x_* \nu^x_* L, \quad L'(x) \cong \nu^x_* \nu^x_* L'.\]

By Lemma 2.6, Lemma 2.7, and Remark 2.8, we have \(\mathcal{E}nd(\mathcal{O}_C(x)) \cong \mathcal{O}_C(x)\). Hence,

\[\text{Hom}_{\mathcal{O}_C}(L(x), L'(x)) \cong \text{Hom}_{\mathcal{O}_C}(L, L'(x)) \cong \text{Hom}_{\mathcal{O}_C}(L, \nu^x_* \nu^x_* L') \cong \text{Hom}_{\mathcal{O}_{C^x}}(\nu^x_* L, \nu^x_* L'),\]

which implies the assertion. \(\square\)

By parts (ii) of Lemma 2.6 and Lemma 2.7, we have \((C^x)_{\text{red}} \cong C_{\text{red}}\) except for one case: If \(\Gamma = A_n\) and \(x\) is an intersection point of two components, then \(C_x\) is an \(A_{n+1}\)-chain of curves, and \(\nu^x\) glues the two extremal curves of the chain at \(x\); compare Lemma 2.7(iv). Anyway, \(C\) and \(C^x\) have the same irreducible components, which get identified by \(\nu^x\).

**Lemma 3.27.** The morphism \(\text{Pic}(C^x) \to \prod_{v \in V} \text{Pic}(C_v)\), given by restriction of line bundles to the irreducible components, is an isomorphism.

**Proof.** As, by Lemma 3.24, \(C^v = C - C_v\) is a numerically 1-connected Gorenstein curve, we have \(h^0(\mathcal{O}_{C^v}) = 1\); see [CFHR99, Thm. 3.3]. Hence, taking global sections of the short exact sequence of Lemma 3.25 gives

\[h^0(\mathcal{O}_C(x)) = 1 = h^0(\mathcal{O}_C).\]

On the other hand, the short exact sequence \(0 \to \mathcal{O}_C \to \mathcal{O}_C(x) \to \mathcal{O}_x \to 0\) yields

\[\chi(\mathcal{O}_C(x)) = \chi(\mathcal{O}_C) + 1.\]
Combining this with (25) gives $h^1(\mathcal{O}_C(x)) = h^0(\mathcal{O}_C) - 1$. Together with Lemma 3.9, this implies $h^1(\mathcal{O}_C(x)) = \sum_{o \in O} h^1(\mathcal{O}_{C_o})$, which by Corollary 2.2 proves the assertion.

\begin{proof}[Proof of Proposition 3.23] As $x \in C_{\text{sing}}$ is the unique point where $L(x)$ fails to be a line bundle, $L(x) \cong L'(x')$ implies $x = x'$; see Lemma 2.4. Now, the assertion follows by combining Lemma 3.26 and Lemma 3.27.
\end{proof}

4. Description of the moduli space

Fix some $\chi > 0$. Let $\mathcal{M}$ be the moduli space of stable sheaves of type $m$ and Euler characteristic $\chi$ on $C$. In this section, we describe $\mathcal{M}_{\text{red}}$. The main goal is to prove Theorem 1.2 from the introduction, respectively its more technical cousin Theorem 4.12.

4.1. An overparametrisation of the stable sheaves. Let $\tilde{J} \subset \text{Pic}(\mathcal{C})$ denote the connected component of the Picard scheme parametrising line bundles $L \in \text{Pic}(\mathcal{C})$ with $\chi(L|_{C_v}) = b_v$ for all $v \in V$.

Proposition 2.1 and Corollary 3.10 provide the following description of $\tilde{J}$. Restriction of a line bundle to the components of $C$ gives a morphism

$$\pi: \tilde{J} \to J := \prod_{o \in O} \text{Pic}_{b_o}(C_o), \quad L \mapsto (L|_{C_o})_{o \in O},$$

where $\text{Pic}_{b_o}(C_o)$ is the moduli space of line bundles on $C_o$ of Euler characteristic $b_o$. Note that, for $L \in \tilde{J}$, we have $L|_{C_i} \cong \mathcal{O}_{C_i}$ as $b_i = 1$ and $C_i \cong \mathbb{P}^1$ for every $i \in I$. Let $G$ be the one-dimensional connected algebraic group of line bundles on $C$ whose restriction to every component is trivial. Then $G$ acts freely on $\tilde{J}$ by the tensor product, and $\pi$ is the quotient by this action. In other words, $J = \tilde{J}/G$.

\begin{lemma}
Let $\Delta \subset C \times C$ denote the diagonal. Then, the sheaf

$$\mathcal{O}_{C \times C}(\Delta) := (\mathcal{I}_{\Delta} \to C \times C)^{\vee}$$

is flat over $C$ (via either of the projections).
\end{lemma}

\begin{proof}
By [Har94, Prop. 2.10], we have a short exact sequence,

$$0 \to \mathcal{O}_{C \times C} \to \mathcal{O}_{C \times C}(\Delta) \to \mathcal{L}_{\Delta} \to 0,$$

where $\mathcal{L}_{\Delta}$ is the push-forward of some line bundle on $\Delta$ along the embedding $\Delta \hookrightarrow C \times C$. Since the first and the last term of the sequence are flat over $C$, the same holds for the middle term.
\end{proof}

Let $\mathcal{P}$ be a universal line bundle on $\tilde{J} \times C$. On $\tilde{J} \times C \times C$, we consider the coherent sheaf

$$\mathcal{F} := \text{pr}_{13}^* \mathcal{P}(\tilde{J} \times \Delta) := \text{pr}_{13}^* \mathcal{P} \otimes \text{pr}_{23}^* \mathcal{O}_{C \times C}(\Delta).$$

By Lemma 4.1, $\mathcal{F}$ is a flat family of sheaves on $C$ over $\tilde{J} \times C$, via $\text{pr}_{12}$. For $(L, x) \in \tilde{J} \times C$, its fibre is $\mathcal{F}_{(L,x)} \cong L(x)$. By Corollary 3.22, this means that the fibres of $\mathcal{F}$ are exactly the stable sheaves of type $m$. Hence, we get a surjective classifying morphism

$$\varphi: \tilde{J} \times C \to \mathcal{M}, \quad (L, x) \mapsto L(x).$$
Lemma 4.2. For every \((L, x) \in \tilde{J} \times C\), the kernel of the differential

\[ d\varphi_{(L, x)} : T_{\tilde{J} \times C}(L, x) \to T_M(L(x)) \]

is one-dimensional. If \(x \in C_{\text{sing}}\), the kernel of \(d\varphi_{(L, x)}\) consists exactly of the vectors which are tangent to the \(G\)-orbit of \((L, x)\) (with \(G\) acting trivially on the second factor).

Proof. Note that we have a direct sum decomposition

\[ T_{\tilde{J} \times C}(L, x) = T_{\tilde{J}}(L) \oplus T_C(x). \]  \hspace{1cm} (26)

Let us start with the case that \(\text{lemma} 4.2\).

Hence, the restriction of \(d\varphi\) is one-dimensional for the smooth point \(x\). This means that \(x \in C_o\) for some \(o \in o\), and \(x\) is not an intersection point with another component. Let \(J_o\) denote the component of \(\text{Pic}(C)\) of line bundles \(M\) with \(\chi(M|_{C_o}) = b_o + 1\) and \(\chi(M|_{C_v}) = b_v\) for all \(v \in V \setminus \{o\}\). These are exactly the stable line bundles with \(M_o\) special; see Proposition 3.21.

Hence, \(J_o \subset M\) is an open subscheme. The restriction \(\varphi_x : \tilde{J} \cong \tilde{J} \times \{x\} \to M\) factorises over this open subscheme as the isomorphism

\[ \tilde{J} \to J_o, \quad L \mapsto L(x) \]

Hence, the restriction of \(d\varphi_{(L, x)}\) to the first summand of (26) is an isomorphism. This means that the kernel of \(d\varphi_{(L, x)}\) is isomorphic to the second summand \(T_C(x)\), which is one-dimensional for the smooth point \(x\).

Let now \(x \in C_{\text{sing}}\). Let \(v = (v_1, v_2) \in T_{\tilde{J} \times C}(L, x) = T_{\tilde{J}}(L) \oplus T_C(x)\). This corresponds to a morphism

\[ v = (v_1, v_2) : \eta \to \tilde{J} \times C, \]

where \(\eta := \text{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)\). We assume that \(d\varphi(v) = 0\) which means that

\[ (v \times \text{id}_C)^* \mathcal{F} \cong (q \times \text{id}_C)^*(L(x)), \]  \hspace{1cm} (27)

where \(q : \eta \to \text{Spec} \mathbb{C}\). From this, we want to deduce that \(v \in \text{ker}(d(\pi \times \text{id}_C))\) (with \(v\) regarded as a tangent vector). In other words, that \((\pi \times \text{id}_C) \circ v\) (with \(v\) regarded as a morphism) equals the constant morphism

\[ \eta \xrightarrow{q} \text{Spec} \mathbb{C} \xrightarrow{\pi(L,x)} J \times C. \]

Let \(x \in U \subset C\) be an open affine neighbourhood such that

\[ (v_1 \times \text{id})^* \mathcal{P}_{\eta \times U} \cong \mathcal{O}_{\eta \times U}, \quad L|_U \cong \mathcal{O}_U. \]

Then \((v \times \text{id}_C)^* \mathcal{F}_{\eta \times U} \cong (v_2 \times \text{id}_U)^* \mathcal{O}_{U \times U}(\Delta) \cong \mathcal{O}_{\eta \times U}(\Gamma_{v_2})\) and

\[ (q \times \text{id}_C)^* L(x)_{\eta \times U} \cong (q \times \text{id}_U)^* \mathcal{O}(x) \cong \mathcal{O}_{\eta \times U}(\eta \times x). \]

Hence, by (27), we have \(\mathcal{O}_{\eta \times U}(\Gamma_{v_2}) \cong \mathcal{O}_{\eta \times U}(\eta \times x)\). This means that the generalised divisors \(\Gamma_{v_2}\) and \(\eta \times x\) on \(\eta \times U\) are linearly equivalent; see [Har94, Prop. 2.8(c)].

By parts (i) of Lemma 2.6 and Lemma 2.7, we have that the natural map \(\text{End}(\mathcal{O}_U(-x)) \to \mathcal{O}_U(x)\) is an isomorphism. Hence, the natural map \(\text{End}(\mathcal{O}_{\eta \times U}(-\eta \times x)) \to \mathcal{O}_{\eta \times U}(\eta \times x)\) is an isomorphism too. This implies that \(\text{End}(\mathcal{O}_{\eta \times U}(\eta \times x))\) acts transitively on \(\Gamma(\mathcal{O}_{\eta \times U}(\eta \times x))\). This, in turn, says that the generalised divisor \(\eta \times x\) on \(\eta \times U\) is linearly equivalent only to itself; see [Har07, Rem. 2.9]. Hence, we must have \(\eta \times x = \Gamma_{v_2}\) as subschemes of \(\eta \times U\). This means that \(v_2\) equals the constant map

\[ v_2 : \eta \xrightarrow{q} \text{Spec} \mathbb{C} \xrightarrow{x} C, \]
moduli of stable singular sheaves are attached to each other to form a closed embedding $\text{Conjecture 4.4.}$

component of the moduli space. We think that the following more precise result should hold.

By Lemma 3.27, the morphisms $\pi'$ and $\pi'$ agree up to an isomorphism of their target varieties. Hence, also $\nu_1 \in \ker(d\pi')$.

4.2. Moduli of singular stable sheaves. We consider the restriction

$$\varphi_{\text{sing}} := \varphi_{\bar{J} \times (C_{\text{sing}})_{\text{red}}} : \bar{J} \times (C_{\text{sing}})_{\text{red}} \to \mathcal{M}$$

of the classifying map $\varphi$ from the last subsection, and the free action of $G$ on $\bar{J} \times (C_{\text{sing}})_{\text{red}}$, given by the tensor product of line bundles on the first factor, and by the trivial action on the second factor. By Proposition 3.23, $\varphi_{\text{sing}}$ is invariant under this action. We denote the factorisation over the quotient $(\pi \times \text{id}) : \bar{J} \times (C_{\text{sing}})_{\text{red}} \to J \times (C_{\text{sing}})_{\text{red}}$ by $\psi : J \times (C_{\text{sing}})_{\text{red}} \to \mathcal{M}$.

**Proposition 4.3.** The morphism $\psi : J \times (C_{\text{sing}})_{\text{red}} \to \mathcal{M}$ is a closed embedding.

**Proof.** By Proposition 3.23, $\psi$ is injective, and by Lemma 4.2 its differential in every point is injective.

As the image of $\psi$ consist exactly of the singular stable sheaves of type $m$, we now have a description of the reduced moduli space of singular sheaves of type $m$. We also see that taking the reduction of the moduli space really makes a difference in the cases $\Gamma \in \{\bar{D}_n, \bar{E}_6, \bar{E}_7, \bar{E}_8\}$.

If $C_i$ is a non-reduced component of $C$, then, for any $L \in \bar{J}$ and $x \in C_i$, the tangent space

$$T_{J \times C}(L, x) = T_{\bar{J}}(L) \oplus T_{C}(x)$$

is of dimension $\dim \bar{J} + 2 = \dim J + 3$. Hence, by Lemma 4.2, the dimension of $T_{\mathcal{M}}(L(x))$ is at least $\dim J + 2 = \dim (J \times C_i) + 1$, showing that $J \times C_i \subset J \times (C_{\text{sing}})_{\text{red}}$ is a multiple component of the moduli space. We think that the following more precise result should hold.

**Conjecture 4.4.** Let $\Gamma \in \{\bar{D}_n, \bar{E}_6, \bar{E}_7, \bar{E}_8\}$, which gives $C_{\text{sing}} = \sum_{i \in I} m_i C_i$. Then there is a closed embedding $J \times C_{\text{sing}} \hookrightarrow \mathcal{M}$ whose image is the moduli space of singular stable sheaves of type $m$.

It feels like the proof should be pretty straight-forward considering all the information we already collected. However, there are some technical difficulties which can be avoided by working with reduced schemes as in Proposition 4.3, and the authors did not have the drive to work through these difficulties.

The moduli of stable line bundles of type $m$ is already described by Proposition 3.21 together with Corollary 3.10. It is a disjoint union $\prod_{o \in O} U_o$ where $U_o$ is the moduli space of line bundles $F$ with $F_o$ special. Every $U_o$ is isomorphic to $\bar{J}$, hence a $G$-torsor over $J$.

The question that remains is how exactly the moduli of stable line bundles and the (reduced) moduli of stable singular sheaves are attached to each other to form $\mathcal{M}_{\text{red}}$. The key observation
to answer this question is that, given \( o \in O \), the stable sheaves \( F \) with either \( F_o \) special (in which case \( F \) is a line bundle) and the stable sheaves with \( T_x \) special for some intersection point of \( C_o \) with another component of \( C \) (in which case \( F \) is singular in \( x \)) have something in common: They can be written as an extension

\[
0 \to K \to F \to M \to 0,
\]

where \( K \in \text{Pic}(C_o) \) with \( \chi(K) = b_o - 1 \) and \( M \in \text{Pic}(C^o) = \text{Pic}(C - C_o) \) with \( \chi(M_{|C_o}) = b_v \) for all \( v \in V \setminus \{ o \} \). This will allow us to realise the locus in \( \mathcal{M} \) of stable sheaves with either \( F_o \) or \( T_x \) special as a projectivisation of a relative extension bundle over \( J \). We carry out the details in the following subsections.

4.3. Universal relative extension bundles. Let \( f: X \to S \) be a flat morphism and \( \mathcal{F}, \mathcal{G} \) coherent sheaves on \( X \). For \( i \geq 0 \), the relative extension sheaf is defined by

\[
\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}) := \mathcal{H}^i(Rf_*R\mathcal{H}om(\mathcal{F}, \mathcal{G})).
\]

In the following, we need to assume that \( \mathcal{E}xt^0_X(\mathcal{F}, \mathcal{G}) = 0 \) and that \( s \mapsto \text{ext}^1_{X_s}(\mathcal{F}_s, \mathcal{G}_s) \) is a constant function on \( S \), which implies that \( \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{G}) \) is locally free. Furthermore, we assume that \( S \) is reduced. We consider the \( \mathbb{P} \)-bundle \( \alpha: Y = \mathbb{P}(\mathcal{E}xt^1_X(\mathcal{F}, \mathcal{G})) \to S \) and the cartesian diagram

\[
\begin{array}{ccc}
X_Y & \xrightarrow{\alpha_X} & X \\
\downarrow f_Y & & \downarrow f \\
Y & \xrightarrow{\alpha} & S.
\end{array}
\]

**Theorem 4.5** ([Lan83]). On \( X_Y \), there is a short exact sequence

\[
0 \to \alpha_X^\ast \mathcal{G} \otimes f_Y^\ast \mathcal{O}_o(1) \to \mathcal{E} \to \alpha_X^\ast \mathcal{F} \to 0
\]

(28)

which has the following universal property: Let \( g: T \to S \) be an \( S \)-scheme and consider the cartesian diagram

\[
\begin{array}{ccc}
X_T & \xrightarrow{g_X} & X \\
\downarrow f_T & & \downarrow f \\
T & \xrightarrow{g} & S.
\end{array}
\]

Then, for every \( M \in \text{Pic}(T) \), and every short exact sequence

\[
0 \to g_X^\ast \mathcal{G} \otimes f_T^\ast M \to D \to g_X^\ast \mathcal{F} \to 0
\]

(29)

which is non-splitting over every \( t \in T \), there is a unique classifying \( S \)-morphism \( \beta: T \to Y \) such that the short exact sequences \( \beta^\ast (28) \) and (29) are isomorphic up to the action of \( \mathcal{H}^0(T, \mathcal{O}_T^+) \) on short exact sequences given by multiplication of the second map of the sequences. In particular, \( \beta^\ast \mathcal{E} \cong D \).

4.4. Closure of the moduli of stable line bundles. For \( o \in O \), write \( \zeta_o := C^o \cap C_o \) for the scheme-theoretic intersection. By (24), this is a length 2 scheme, consisting of two reduced points if \( \Gamma = \tilde{A}_n \), and of one fat point if \( \Gamma \in \{ \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \} \). We also write \( \Delta_o \subset C_o \times C_o \subset C_o \times C \) for the diagonal of \( C_o \).

**Lemma 4.6.** For every \( o \in O \), there is a short exact sequence

\[
0 \to \mathcal{O}_{C_o \times C_o}(-(C_o \times \zeta_o) + \Delta_o) \to \mathcal{O}_{C_o \times C}(\Delta_o) \to \mathcal{O}_{C_o \times C^o} \to 0.
\]

(30)
Note that the left term of the sequence is a line bundle on \( C_o \times C_o \) as \( \Delta_0 \) is a Cartier divisor in \( C_o \times C_o \). But the middle term is not a line bundle on \( C_o \times C \) as \( \Delta_o \) is not a Cartier divisor in \( C_o \times C \).

**Proof.** This is just a relative version of Lemma 3.25 with a very similar proof: we have a chain of inclusions

\[
\mathcal{I}_{C_o \times C_o \rightarrow C_o \times C} = \mathcal{O}_{C_o \times C^0}(-C_o \times \zeta_o) \subset \mathcal{O}_{C_o \times C}(-\Delta_o) \subset \mathcal{O}_{C_o \times C}.
\]

Snake lemma gives a commutative diagram with exact columns and rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_{C_o \times C^0}(-C_o \times \zeta_o) & \rightarrow & \mathcal{O}_{C_o \times C^0}(-C_o \times \zeta_o) & \rightarrow & \mathcal{O}_{\Delta_o} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_{C_o \times C}(-\Delta_o) & \rightarrow & \mathcal{O}_{C_o \times C^0} & \rightarrow & \mathcal{O}_{\Delta_o} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_{C_o \times C^0}(-\Delta_o) & \rightarrow & \mathcal{O}_{C_o \times C^0} & \rightarrow & \mathcal{O}_{\Delta_o} & \rightarrow & 0
\end{array}
\]

Taking the dual of the left column gives the desired short exact sequence; see Lemma 2.9. \( \square \)

Let \( F = \text{pr}_{13}^* \mathcal{P}(\tilde{J} \times \Delta) \) be the sheaf on \( \tilde{J} \times C \times C \) considered in Subsection 4.1 to give the classifying morphism \( \varphi: \tilde{J} \times C \rightarrow M \). We write \( F_o := F|_{\tilde{J} \times C_o \times C} \).

**Corollary 4.7.** There is a short exact sequence

\[
0 \rightarrow \text{pr}_{13}^* \mathcal{P}|_{\tilde{J} \times C_o}(-C_o \times \zeta_o) + \Delta_o) \rightarrow F_o \rightarrow \text{pr}_{13}^* \mathcal{P}|_{\tilde{J} \times C^0} \rightarrow 0. \tag{31}
\]

**Proof.** We apply tensor product with the line bundle \( \text{pr}_{13}^* \mathcal{P} \) to \( \text{pr}_{23}^* \mathcal{P} \). \( \square \)

**Lemma 4.8.** For any two line bundles \( K \in \text{Pic}(C_o) \) and \( M \in \text{Pic}(C^0) \), we have

\[
\text{hom}_{\mathcal{O}_C}(M, K) = 0, \quad \text{ext}_{\mathcal{O}_o}^1(M, K) = 2.
\]

**Proof.** As the intersection of the supports is the zero-dimensional scheme \( \zeta_o \), we may assume that both line bundles are trivial: \( K \cong \mathcal{O}_{C_o}, \ M \cong \mathcal{O}_{C^0} \). For support reasons, we have \( \text{Hom}(\mathcal{O}_{C^0}, \mathcal{O}_{C_o}) = 0 \). Considering the exact sequence

\[
0 \rightarrow \mathcal{O}_{C_o}(-\zeta_o) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C^0} \rightarrow 0
\]

of Lemma 2.9 and applying \( \text{Hom}(\_ , \mathcal{O}_{C_o}) \) gives

\[
0 \rightarrow \mathcal{O}_{C_o} \rightarrow \mathcal{O}_{C_o}(\zeta_o) \rightarrow \mathcal{E}xt^1(\mathcal{O}_{C^0}, \mathcal{O}_{C_o}) \rightarrow 0.
\]

Hence, \( \mathcal{E}xt^1(\mathcal{O}_{C^0}, \mathcal{O}_{C_o}) \cong \mathcal{O}_{\zeta_o} \). The local-to-global-Ext spectral sequence now gives the assertion as \( h^0(\mathcal{O}_{\zeta_o}) = 2 \). \( \square \)
Lemma 4.9. Let $K \in \text{Pic}(C_o)$ and $M \in \text{Pic}(C^o)$, and consider two extensions
\begin{align*}
0 & \to K \to F_1 \to M \to 0, \\
0 & \to K \to F_2 \to M \to 0
\end{align*}
of coherent sheaves on $C$. Then, $F_1 \cong F_2$ if and only if (32) and (33) agree up to the action of $\mathbb{C}^* = \text{H}^0(\mathcal{O}_C^*)$ on extensions.

Proof. By definition, the $\mathbb{C}^*$-action only changes the second map of the extensions, but not the middle term. This already gives the if part of the statement.

Let now $\varphi : F_1 \to F_2$ be an isomorphism. We have $\text{pure}(F_1|_{C^o}) \cong M \cong \text{pure}(F_2|_{C^o})$ by Lemma 3.9. Hence, we get an isomorphism of short exact sequences
\begin{align*}
0 \to K & \to F_1 \to M \to 0 \\
0 \to K & \to F_2 \to M \to 0.
\end{align*}
Since $K$ and $M$ are simple (for the latter, this follows by Lemma 3.24), it follows that (32) and (33) agree up to a non-zero scalar. $\blacksquare$

We write $\text{Pic}_b(C^o)$ for the connected component of $\text{Pic}(C^o)$ consisting of line bundles $M$ on $C^o$ with $\chi(M|_{C_v}) = b_v$ for all $v \in V \setminus \{o\}$.

Lemma 4.10. The morphism
$$
\pi_o : \text{Pic}_b(C^o) \to \prod_{v \in V \setminus \{o\}} \text{Pic}_b(C_v) \cong \prod_{j \in O \setminus \{o\}} \text{Pic}_{b_j}(C_j),
$$
given by restriction of line bundles to the irreducible components, is an isomorphism.

Proof. This is similar to the proof of Lemma 3.27. By Corollary 2.2, it is enough to prove that $g(C^o) = \sum_{j \in O \setminus \{o\}} g(C_j)$. By (24), we have
$$
\chi(\mathcal{O}_{C^o}) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_{C_o}) + 2.
$$
As we have seen before, the fact that $C^o$ is a numerically 1-connected Gorenstein curve implies $\text{h}^0(\mathcal{O}_{C^o}) = 1 = \text{h}^0(\mathcal{O}_C) = \text{h}^0(\mathcal{O}_{C_o})$; see Lemma 3.24 and [CFHR99, Thm. 3.3]. Combining this with (34) and Lemma 3.9 gives
$$
g(C^o) = g(C) - g(C_o) - 1 = 1 + \sum_{j \in O} g(C_j) - g(C_o) - 1 = \sum_{j \in O \setminus \{o\}} g(C_j). \quad \blacksquare
$$

We set
$$
J_o := \text{Pic}_{b_o-1}(C_o) \times \text{Pic}_b(C^o).
$$
Note that, by Lemma 4.10, $J_o$ is canonically isomorphic to $\prod_{j \in O \setminus \{o\}} \text{Pic}_{b_j}(C_j)$, hence non-canonically isomorphic to $J = \prod_{j \in O} \text{Pic}_{b_j}(C_j)$. We write the projections as
$$
p_o : J_o \to \text{Pic}_{b_o-1}(C_o), \quad q_o : J_o \to \text{Pic}_b(C^o)
$$
and denote universal families of $\text{Pic}_{b_o-1}(C_o)$ and $\text{Pic}_b(C^o)$ by $\mathcal{P}_o$ and $\mathcal{P}^o$, respectively. We consider the two sheaves
$$
\mathcal{Q}_o := i_* (p_o \times \text{id}_{C_o})^* \mathcal{P}_o, \quad \mathcal{Q}^o := j_* (q_o \times \text{id}_{C^o})^* \mathcal{P}^o
$$
on $J_o \times C$ where $i : J_o \times C_o \to J_o \times C$ and $j : J_o \times C^o \to J_o \times C$ are the embeddings.
For any point \( t \in J_0 \), the fibre \((Q_o)_t\) is a line bundle on \( C_o \) and \((Q^o)_t\) is a line bundle on \( C^o \). Hence, Lemma 4.8 shows that \( \text{ext}^1_{P_1}(Q^o, (Q_o)_t) = 2 \) is a constant function in \( t \in J_0 \), and \( \text{Ext}^1_{P_1}(Q^o, Q_o) = 0 \), where \( f_o: J_0 \times C \rightarrow J_0 \) denotes the projection to the first factor. Hence, the assumptions of Subsection 4.3 are fulfilled, which gives that

\[
\alpha_o: Y_o := \mathbb{P}(\text{ext}^1_{P_1}(Q^o, Q_o)^{\vee}) \rightarrow J_0
\]

is a \( \mathbb{P}^1 \)-bundle satisfying the universal property described in Theorem 4.5.

**Proposition 4.11.** There is a commutative diagram

\[
\begin{array}{ccc}
\tilde{J} \times C_o & \xrightarrow{\beta_o} & Y_o \\
\downarrow & & \downarrow \alpha_o \\
J_o & \xrightarrow{\iota_o} & M
\end{array}
\]

where \( \varphi_o = \varphi_{|\tilde{J} \times C_o} \), \( g_o(L, x) = (L|_{C_o}(-\zeta_o + x), L|_{C^o}) \), \( \beta_o \) is surjective, and \( \iota_o \) is a closed embedding.

**Proof.** The map \( \beta_o \) is the classifying morphism for the extension (31). Let us explain this in more detail. The sheaf \( \mathfrak{p}_{P_1}^{13} \mathcal{P}_{B \times C_o}^{-}(C_o \times \zeta_o + \Delta_o) \) is a family of line bundles on \( C_o \) of Euler characteristic \( b_o - 1 \), parametrised by \( \tilde{J} \times C_o \). Let \( u: \tilde{J} \times C_o \rightarrow \text{Pic}_{b_o-1}(C_o) \) be the classifying morphism. The sheaf \( \mathfrak{p}_{P_1}^{13} \mathcal{P}_{B \times C^o} \) is a family of line bundles on \( C^o \) whose restrictions to \( C^o_j \) for \( j \in O \setminus \{o\} \) are of Euler characteristic \( b_j \), parametrised by \( \tilde{J} \times C_o \). Let \( v: \tilde{J} \times C_o \rightarrow \text{Pic}_b(C^o) \) be the classifying morphism. Let \( g_o = (u, v): \tilde{J} \times C_o \rightarrow J_0 \). On closed points, it is given by \( g_o(L, x) = (L|_{C_o}(-\zeta_o + x), L|_{C^o}) \), as asserted. Furthermore, (31) is isomorphic to

\[
0 \rightarrow (g_o \times \text{id}_C)^* Q_o \rightarrow F_o \rightarrow (g_o \times \text{id}_C)^* Q^o \rightarrow 0.
\]

Hence, Theorem 4.5 gives a classifying morphism \( \beta_o \) making the lower part of diagram (35) commute.

The fibres of \( \alpha_o \) are irreducible projective curves (namely, isomorphic to \( \mathbb{P}^1 \)), and the fibres of \( g_o \) are irreducible (namely, isomorphic to \( G \times C_o \)). Hence, for the surjectivity of \( \beta_o \), it suffices to show that no fibre of \( g_o \) is contracted by \( \beta_o \) to a point. To see this, let \((K, M) \in J_o \), let \( x \in \zeta_o \) be an intersection point of \( C_o \) with another component of \( C \), and let \( L \in \tilde{J} \) be any line bundle with \( L|_{C_o}(-\zeta_o + x) \cong K \) and \( L|_{C^o} \cong M \), hence \( g_o(L, x) = (K, M) \). A second point in the same fibre of \( g_o \) is \((L', y)\) for some \( y \in C_o \setminus \zeta_o \) and some line bundle \( L' \in \tilde{J} \) with \( L'|_{C_o}(-\zeta_o + y) \cong K \) and \( L'|_{C^o} \cong M \). But \( F_o(L, x) \cong L(x) \not\cong L'(y) \cong F_o(L', x) \) as the right side is a line bundle, while the left side is not. Hence, also \( \beta_o(L, x) \neq \beta_o(L', y) \); see the if part of Lemma 4.9.

The surjectivity of \( \beta_o \) implies that every fibre of the middle term of the universal extension on \( Y_o \) occurs as a fibre of \( F_o \). In particular, every fibre of the middle term of the universal extension is a stable sheaf of type \( m \). Hence, we get the morphism \( \iota_o: Y_o \rightarrow M \) as the classifying morphism for this middle term.

The injectivity of \( \iota_o \) is the only if part of Lemma 4.9. By Lemma 4.2, the differential of \( \iota_o \) is injective in every point. Hence, \( \iota_o \) is a closed embedding. \(\Box\)
4.5. **Description of the moduli space.** In this subsection, we collect our results. **Theorem 4.12** stated in the introduction follows immediately from a more precise result that we now formulate.

**Theorem 4.12.** *In the above situation the following hold:*

(i) The locus in $\mathcal{M}_{\text{red}}$ parametrising singular stable sheaves (i.e. sheaves which are not line bundles) is isomorphic to $J \times (\mathcal{C}_{\text{sing}})_{\text{red}}$, with the isomorphism given by $L(x) \mapsto (\mathcal{L}_{|C_o})_{o \in O}, x$.

(ii) The locus in $\mathcal{M}_{\text{red}}$ parametrising stable line bundles consist of $|O|$ connected components $U_o$, each of which is a $G$-torsor over $J$ where $G = \mathbb{G}_m$ if $\Gamma = \hat{A}_n$, and $G = \mathbb{G}_a$ if $\Gamma \in \{\hat{D}_n, \hat{E}_6, \hat{E}_7, \hat{E}_8\}$.

(iii) Let $Y_o$ denote the closure of $U_o$ in $\mathcal{M}_{\text{red}}$. Then, $Y_o$ is a $\mathbb{P}^1$-bundle over $J$ for every $o \in O$. The complement of $U_o$ in $Y_o$ is a disjoint union of sections of this $\mathbb{P}^1$-bundle, one section $S^o_x$ for every intersection point $x$ of $C_o$ with another component of $C$. The sections $S^o_x$ parametrise stable sheaves which are not line bundles in $x$.

(iv) Let $\Gamma = \hat{A}_n$. Then, for an intersection point $x = C_o \cap C_{o'}$, the components $Y_o$ and $Y_{o'}$ of $\mathcal{M}_{\text{red}}$ intersect transversally in the sections $S^o_x$ and $S^{o'}_x$.

(v) Let $\Gamma \in \{\hat{D}_n, \hat{E}_6, \hat{E}_7, \hat{E}_8\}$. Then, for every $o \in O$, there is a single intersection point $x$ with another component $C_i$ which is of multiplicity 2. Then, the component $Y_o$, and the component $Y_i := J \times C_i$ (parametrizing stable sheaves which are singular in some point of $C_i$) intersect transversally in $S^o_x = J \times \{x\}$.

**Proof.** By **Proposition 4.3**, we have a closed embedding

$$\psi: J \times (\mathcal{C}_{\text{sing}})_{\text{red}} \hookrightarrow \mathcal{M}, \quad ((L_o)_{o \in O}, x) \mapsto L(x),$$

where $L \in \text{Pic}(C)$ is any line bundle with $L_{|C_o} = L_0$ (by **Proposition 3.23**, it does not matter which line bundle $L$ with this property we pick). By **Proposition 3.21** its image is exactly the locus of singular stable sheaves. So we proved part (i).

For part (iii), we identify the $\mathbb{P}^1$-bundles

$$Y_o \xrightarrow{\alpha_o} J_o$$

of **Proposition 4.11** with their images under the closed embeddings $\alpha_o: Y_o \hookrightarrow \mathcal{M}$. We can consider $Y_o$ as a $\mathbb{P}^1$-bundle over $J$ by choosing any isomorphism $J_o \cong J$.

By **Proposition 4.11**, we have $Y_o = \varphi(J \times C_o)$. Hence, the locus of singular sheaves in $Y_o$ is the union $\bigsqcup_x \varphi(J \times \{x\})$ over all intersection points $x$ of $C_o$ with another component. Let $x$ be such an intersection point. The restrictions to $J \times \{x\}$ of all three morphisms $\varphi, \beta, g_0$ in the diagram (35) are $G$-invariant, hence factor over the quotient $J \times \{x\}$. The morphism $g_0|J \times \{x\}$ induces the morphism

$$\tilde{g}^x_0: J \times \{x\} \to J_o, \quad (L_j)_{j \in O} \mapsto (L_o, \pi_{o}^{-1}(\sum_{j \in O \setminus \{0\}} (L_j))).$$

This is an isomorphism; see **Lemma 4.10**. Hence, the locus $S^o_x$ of stable sheaves which are singular at $x$ is a section of $Y_o$.

If $\Gamma \in \{\hat{D}_n, \hat{E}_6, \hat{E}_7, \hat{E}_8\}$, there is only one intersection point $x$ of $C_o$ with other components of $C$. Hence, $U_o = Y_o \setminus S^o_x$ is a $\mathbb{G}_a$-bundle over $J$.

If $\Gamma = \hat{A}_n$ there are two intersection points $x, y$ of $C_o$ with other components of $C$. Hence, $U_o = Y_o \setminus (S^o_x \cup S^o_y)$ is a $\mathbb{G}_m$-bundle over $J$. Thus, we have also proved part (ii).
Finally, the transversally statement in parts (iv) and (v) follows from Lemma 4.2. Indeed, let \( x = C_o \cap C_j \), be an intersection point. We have \( Y_o = \varphi(\tilde{J} \times C_o) \). If we are in the \( \Gamma = \tilde{A}_n \) case, where \( j \in O \), also \( Y_j = \varphi(\tilde{J} \times C_j) \). If we are in the \( \Gamma \in \{ \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \} \) case, we still write \( Y_j := \varphi(\tilde{J} \times C_j) = \psi(\tilde{J} \times C_j) \) for the component of \( \mathcal{M}_{\text{red}} \) intersecting \( Y_o \). We have a direct sum decomposition

\[
T_{J \times C}(L,x) \cong T_J(L) \oplus T_C(x) \cong K \oplus T_J(\pi(L)) \oplus T_{C_o}(x) \oplus T_{C_j}(x),
\]

where \( K = \ker(\varphi_{L,x}) \); see Lemma 4.2. It follows that

\[
T_{Y_o}(L(x)) = d\varphi(K \oplus T_J(\pi(L)) \oplus T_{C_o}(x)) = d\varphi(T_J(\pi) \oplus T_{C_o}(x)),
\]

and

\[
T_{Y_j}(L(x)) = d\varphi(K \oplus T_J(\pi(L)) \oplus T_{C_j}(x)) = d\varphi(T_J(\pi) \oplus T_{C_j}(x))
\]

are non-identical subspaces of \( T_{\mathcal{M}_{\text{red}}}(L(x)) \), which means transversality of \( Y_o \) and \( Y_j \).

5. Further Remarks

5.1. Codimension one strata in linear systems on K3 surfaces. Extended ADE curves occur as generic singular members of linear systems on K3 surfaces as we will now explain. Throughout this section, let \( X \) be a K3 surface. Let \( L \) be a line bundle on \( X \) with at least two sections such that the general element in \( |L| \) is a smooth connected curve of genus \( g \).

Lemma 5.1. The following properties hold:

(i) \( L^2 = 2g - 2 \geq 0 \) and \( \dim |L| = g \);
(ii) \( L \) is nef and \( |L| \) has no base points;
(iii) if \( L^2 > 0 \) then \( L \) is big and nef;
(iv) if \( L^2 = 0 \) then \( L \) is an elliptic pencil.

Proof. These are well-known results, all of which can be found in the textbook [Huy16]. More precisely, see Cor. 2.1.5, Lem. 2.2.1, and Rem. 8.2.13 of loc. cit. \( \square \)

Proposition 5.2. Let \( C \to X \) be an extended ADE curve in \( |L| \). Then, in the linear system \( |C| = |L| \), the locus of curves which are still extended ADE curves of the same type (i.e. the same intersection graph \( \Gamma \) and the same genera of the reduced components) as \( C \) is of codimension 1.

Proof. We consider the natural morphism

\[
f: \prod_{o \in O} |C_o| \to |C|,
\]

whose image consists of all curves which arise from \( C \) by moving the reduced components \( C_o \) of \( C \) in their linear systems. This map is generically injective and a general member of its image is still an extended ADE curve of the same type as \( C \). By Lemma 3.9 together with Lemma 5.1(i), the dimension of the domain of \( f \) is \( \dim |C| - 1 \), which proves theclaim. \( \square \)

In the following, we will see that, under a positivity assumption, a kind of converse of Proposition 5.2 holds. Namely, reducible members of the linear system \( |L| \) which occur in codimension 1 must be extended ADE curves, with only one irrational component.

The linear system \( |L| \) is naturally stratified into equisingular strata [DS17, Wah74]. By our assumptions the general element is a smooth curve, so the discriminant \( \Delta \subset |L| \) parametrising singular curves in the linear system is a proper closed subset. We want to investigate, what
kind of curves occur at the general point of a component of the discriminant $V \subset |L|$ of codimension one.

If $L^2 = 0$, then $|L|$ is an elliptic pencil and in codimension one we have the possible singular fibres classified by Kodaira. Thus in the following we assume that $L^2 = 2g - 2 > 0$.

The following was explained to us by Andreas Knutsen:

**Proposition 5.3.** Assume that the integral curve $C$ is general in a codimension one stratum $V \subset |L|$. Then $C$ has a unique node.

*Proof.* Let $\tilde{C} \to C$ be the normalisation. By [DS17, Prop. 4.5], the family of curves of geometric genus $g(\tilde{C})$ is of dimension $g(\tilde{C}) = \dim V = \dim |L| - 1$ and by [DS17, Prop. 4.6] the generality of $C$ implies that $C$ has ordinary singularities, i.e., $\tilde{C} \to X$ is an immersion. Then $C$ can only have one node, as $p_a(C) = g(\tilde{C}) + 1$. □

We were also made aware by Justin Sawon that Proposition 5.3 was already proven in [Saw15, Lemma 2.4]. As the proof above differs a lot from the one in loc. cit., we decided to keep it in the text.

Let us consider an example to illustrate how reducible curves can arise in codimension one strata of $|L|$. This example was the original motivation for the present work.

**Example 5.4.** Let $B \subset \mathbb{P}^2$ be a plane sextic, possibly with ADE singularities. Consider the double cover $\bar{X}$ branched over $B$ and its minimal resolution $\bar{X}$ so that $X$ is a smooth K3 surface and $\bar{X}$ possibly has ADE singularities. Clearly, $X = \bar{X}$ if and only if $B$ is smooth.

Let $\bar{L} = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $L = \eta^*\bar{L}$. By the projection formula we have $|dL| = \eta^*\pi^*|\mathcal{O}_{\mathbb{P}^2}(d)|$ for $d = 1, 2$.

The pullback of a line $M \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ will be reducible in $\bar{X}$ if and only if $B|_M$ is divisible by 2 as a divisor on $M$, i.e. if $M$ is a triple tangent of $B$. Such lines lead to singularities in the dual curve $B^\vee$ and in particular there are only finitely many. Therefore, all but finitely many curves in $|\bar{L}|$ are irreducible.

If the branch locus is smooth, then all strata of codimension one in $|L|$ parametrise curves with one node as in Proposition 5.3. These correspond to tangent lines to $B$ and are thus parametrised by an open subset of the dual curve $B^\vee$.

If $p$ is a singular point of the branch curve $B$, then the pencil of lines through $p$ pulls back to a pencil of curves on $\bar{X}$ passing through the ADE surface singularity. On $X$ we get a pencil of curves containing the exceptional divisor of the singularity as a fixed part. The general such curve is an extended ADE curve where the only non-rational component is the strict transform of the pullback of the line, which is a smooth elliptic curve.

So in this example the discriminant consists of the dual curve $B^\vee$ together with one line for each singular point of $B$.

Now consider the linear system $|2L|$. Here we find three different strata in codimension one:

(i) Pullbacks of conics that are tangent to $B$ in a point give irreducible curves with a node.
(ii) Pullbacks of general conics passing through a singular point of $B$ give curves containing an ADE configuration, where the only non-rational curve is the strict transform of the pullback of the conic, which is a smooth curve of genus 4.

(iii) Pullbacks of the sum of two general lines give an $\tilde{A}_1$-curve where both components are smooth of genus two.

Note that the last type of component occurs even when $B$ is smooth.

We take away the impression that components as in (i) will always exist and components of type (ii) will exist if $L$ is big and nef but not ample. The extra component like in (iii) however needs particular reasons to exist.

**Proposition 5.5.** Let $\tilde{X}$ be a possibly singular K3 surface with ADE singularities, and let $\eta: \tilde{X} \to X$ be the minimal resolution. Let $L$ be a line bundle on $\tilde{X}$ such that

(i) $L$ is very ample;

(ii) $L^2 \geq 12$;

(iii) for any curve $C \subset \tilde{X}$ we have $L.C > 8$.

Then the general curve in a codimension one stratum of $|L| = |\eta^*L|$ is either an integral curve with one node or the pullback of a general curve in $|\tilde{L}|$ passing through one ADE singularity of $\tilde{X}$. The latter are extended ADE-curves where all but one components are $-2$ curves.

We believe the positivity conditions could be sharpened by adapting the techniques of [DS17] to reducible curves and of [Knu01] to singular K3 surfaces.

**Remark 5.6.** Note that, if $L$ itself is ample, we cannot have an extended ADE curve with a $(-2)$-component as a member of $|L|$, as the restriction of $L$ to the $(-2)$-component would be trivial. Hence, for $L$ ample and sufficiently positive, Proposition 5.5 says that singular members of $|L|$ in codimension 1 are integral with one node.

**Proof.** Let $x \in \tilde{X}$ be a smooth point and let $Z_x$ be the subscheme defined by $O_{\tilde{X}}/m_x^2$. By [Knu01, Propositions 3.1 and 3.7] the map $H^0(X, L) \to H^0(X, L \otimes O_{Z_x+y})$ is surjective for any $y \in X \setminus \{x\}$. In other words, the linear system $|m_x^2L| \subset |L|$ has $x$ as its only base point. By Bertini, the general curve in $|L|$ which is singular at $x$ is irreducible and has a unique singular point. By Proposition 5.3 the singularity is a node.

Now consider curves passing through a singular point $\tilde{x} \in \tilde{X}$. Then, since $|L|$ separates points, the linear system $|m_{\tilde{x}}^2L| \subset |L|$ has no other base points. Thus, by Bertini, the general such curve is smooth outside $\tilde{x}$ and irreducible. It is clear, that the pullback of a general hyperplane passing through $\tilde{x}$ is an extended ADE curve on $X$. □

**Example 5.7.** Let $X$ be as in Example 5.4 and $L = \eta^*\pi^*O_{\mathbb{P}^2}(d)$. Then the numerical conditions of Proposition 5.5 are satisfied for $d \geq 5$ by the projection formula.

For $d = 1$ the numerical conditions are not satisfied, but the conclusion of Proposition 5.5 still holds for the linear system $|L|$, while for $d = 2$ this is not the case.

We did not consider the remaining cases $d = 3, 4$ in detail.

**Example 5.8.** Let $\varphi: \tilde{X} \to Q \subset \mathbb{P}^3$ be a double cover of the quadric cone in $\mathbb{P}^3$ branched over a general quartic section. Since no branching occurs at the vertex of the cone, $\tilde{X}$ contains two $A_1$-singularities both mapping to the vertex.

Let $L = \varphi^*O_{\mathbb{P}^3}(1)|_Q$. By the projection formula the elements of $|L|$ are the pullbacks of hyperplane section of $Q$. The general element is a smooth curve of genus three. We now
describe the general members of the subsystem of codimension one coming from hyperplanes containing the vertex.

If $H$ is a general hyperplane through the vertex of the cone, the intersection $H \cap Q$ consists of two lines passing through the vertex. Therefore, the pullback $\bar{X} \supset \varphi^* H = C_1 + C_2$ where $C_i$ is a smooth elliptic curve passing through both singular points. Pulling back further to the minimal resolution $f: X \to \bar{X}$, which is a K3 surface, we have

$$f^* \varphi^* H = E_1 + C_1 + E_2 + C_2,$$

an $\tilde{A}_4$ configuration consisting of two elliptic and two $-2$ curves.

Note that this extended ADE-configuration does not conform to the simple pattern found in Proposition 5.5 (more than one component is non-rational), which does not apply because $L$ is not sufficiently positive.

Example 5.7 shows that the positivity conditions are not necessary for the conclusion of Proposition 5.5 to hold.

But even if the conclusion of Proposition 5.5 does not hold for a linear system $|L|$ on a K3 surfaces, in the examples we looked at it is still true that the general curve in a codimension one stratum is always either irreducible with one node as in Proposition 5.3 or an extended ADE curve, with possibly more than one non-rational component. It is therefore legitimate to speculate that this is always the case.

To apply our description of moduli spaces of sheaves to different curves in a linear system $|L|$ we should pick a global polarisation on the K3 surface $X$.

**Proposition 5.9.** Assume that on a K3 surface $X$ we have a linear system $|L|$ satisfying the conclusion of Proposition 5.5.

Then there exists a polarisation $H$ on $X$ such that, for any reducible curve $C$ which is general in codimension one in $|L|$, the restricted polarisation satisfies Assumption 3.12.

**Proof.** We consider the contraction $\eta: X \to \bar{X}$ from Proposition 5.5 and for every singular point $p \in \bar{X}$ a fundamental cycle $Z_p$, such that $Z_p$ is supported on the preimage of $p$ and $-Z_p \cdot E > 0$ for any curve contracted to $p$.

Now choose any ample line bundle $\bar{H}$ on $\bar{X}$ and consider the $\mathbb{Q}$-divisor

$$H = \eta^* \bar{H} - \sum_{p \in \text{Sing } \bar{X}} \epsilon_p Z_p.$$

If we choose all $\epsilon_p$ sufficiently small, we achieve Assumption 3.12 for all curves of the form $\eta^* D$ where $D \in |\bar{L}|$ is a general curve passing through some $p \in \text{Sing } \bar{X}$ by the same argument as in Proposition 3.15. \qed

**Example 5.10.** We consider the double cover of the plane as in Example 5.4 for a very general sextic branch curve $B$. Then $X$ is smooth and has Picard rank one by a result of Cox [Cox89], so up to multiples the only polarisation is given by $L = \pi^* \mathcal{O}(1)$.

The only reducible singular curves occurring in codimension one in $|2L|$ are the pullbacks of two lines forming an $A_1$ configuration. Assumption 3.12 becomes

$$\left\lceil \frac{1}{2} \chi \right\rceil + \left\lceil \frac{1}{2} \chi \right\rceil = \chi + 1$$

which is satisfied if and only if $\chi$ is odd. In the case that $\chi$ is even, Theorem 1.1 does not hold anymore. Indeed, we then have properly semi-stable line bundles, namely those whose
restriction to one component has Euler characteristic $\frac{1}{2}$ while the restriction to the other component has Euler characteristic $\frac{1}{2} + 2$.

5.2. Beauville–Mukai systems. We now want to let our moduli spaces vary along the linear system $|C| = \mathbb{P}(H^0(\mathcal{O}_X(C)))$. More precisely, let $C \hookrightarrow X$ be an extended ADE curve inside some smooth projective surface $X$, and $H$ a polarisation on $X$. We write $\mathcal{N} := \mathcal{M}_{0,[C],\chi}(X)$ for the moduli space of semi-stable sheaves on $X$ of rank 0, first Chern class $|C|$, and Euler characteristic $\chi$. As every $[F] \in \mathcal{M}_{0,[C],\chi}$ represents a purely one-dimensional sheaf, we have a locally free resolution

$$0 \to E_1 \to E_0 \to F \to 0.$$  

From this, we get an element $Z(\det(s)) \in |C|$ not depending on the chosen resolution. This gives a well-defined morphism $\mathcal{N} \to |C|$, and our moduli space $\mathcal{M} := \mathcal{M}_{m,\chi}(C)$ of stable sheaves of type $m$ and Euler characteristic $\chi$ is the fibre $\mathcal{N}_C$ over $C \in |C|$. Hence, if $H|_C$ satisfies Assumption 3.12, we have a description of this fibre of $\mathcal{N} \to |C|$.

Our main motivation for this work was to study these fibres when $X$ is a K3 surface. In this case, if the Mukai vector $(0, |C|, \chi)$ is primitive, then $\mathcal{N}$ is an irreducible holomorphic symplectic variety, and $\mathcal{N} \to |C|$ is a Lagrangian fibration, called a Beauville–Mukai integrable system.

From now on, until the end of the next subsection, let $X$ be a K3 surface. By Proposition 5.2 together with Proposition 5.9, our main result Theorem 4.12 describes certain generic singular fibres, as this is the case relevant to us. Let $\mathcal{N}$ be a holomorphic Lagrangian fibration. Then, by [HO09, Prop. 3.1], the discriminant locus

$$\Delta = \{ b \in \mathbb{P}^n \mid M_b \text{ is singular} \}$$

is a hypersurface in $\mathbb{P}^n$. Furthermore, [HO09], [HO11] give strong results on the geometry of the general fibres over the components of $\Delta$. We only recall (parts of) the results for reducible generic singular fibres, as this is the case relevant to us. Let $b \in \Delta$ be a general point in some component such that $M_b$ is reducible. Let $\{ Y_w \}_{w \in \mathcal{W}}$ be the family of the irreducible components of $(M_b)_{\text{red}}$. Then the $Y_w$ are smooth of Albanese dimension $n - 1$, and the Albanese map $\alpha_w : Y_w \to \text{Alb}(Y_w)$ is a $\mathbb{P}^1$-bundle.

Now, one can define reduced characteristic cycles on $(M_b)_{\text{red}}$. Namely, let $\sim$ denote the equivalence relation on closed points of $(M_b)_{\text{red}}$ generated by the relation $P \sim Q$ for two points on the same $\mathbb{P}^1$-fibre of the same component $\alpha_w : Y_w \to \text{Alb}(Y_w)$ of $M_b$. A reduced characteristic cycle is then an equivalence class under this equivalence relation.
Intuitively, we form a reduced characteristic cycle by starting at any point $P$ of $M_b$, say with $P \in Y_w$. Then we can walk along the $\mathbb{P}^1$-fibre of $Y_w$ until we reach the intersection locus with some other $\mathbb{P}^1$-bundle component of $M_b$. Then we can walk along the $\mathbb{P}^1$-fibre of the other bundle containing the intersection point, and so on. The configuration of smooth rational curves that we obtain is the reduced characteristic cycle of $(M_b)_{\text{red}}$. One then defines the (not necessarily reduced) characteristic cycle of $M_b$ by counting every $\mathbb{P}^1$-fibre of the reduced characteristic cycle with the multiplicity of the component $Y_w$ it lies in. It is then proven in [HO09], [HO11] that all characteristic cycles of $M_b$ are isomorphic, and of one of the following types:

(i) a reducible fibre of an elliptic fibration, i.e. of type III, type IV, or an extended ADE curve where all components are rational,

(ii) of type $A_\infty$ which means an infinite chain of smooth rational curves.

Let us see how the characteristic cycles look like in the case of our moduli spaces $\mathcal{M}$ of sheaves of type $m$ on extended ADE curves. In the case that $\Gamma \in \{\tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$, it is quite obvious that Theorem 1.2 implies that the reduced characteristic cycles of $\mathcal{M}_{\text{red}}$ are of the same type $\Gamma$ (without multiplicities). Then the results of [HO09], [HO11] say that also the non-reduced characteristic cycles must be of the same type $\Gamma$, but now with multiplicities given by the labels of $\Gamma$. This gives further evidence for Conjecture 4.4. See [Hel21] for the computation of multiplicities of the components of moduli space of sheaves on a curve in a somewhat related special case.

In the case $\Gamma = \tilde{A}_n$, we have $\mathcal{M} = \mathcal{M}_{\text{red}}$. However, there is another subtlety making things complicated. Namely, if we walk around the $\tilde{A}_n$-configuration along the fibres to form the characteristic cycle as described above, in general, after walking one round, we will end up in another fibre than the one in which we started. This means that it can happen that the characteristic cycle is not of type $\tilde{A}_n$ but of type $\tilde{A}_{kn}$ for some $k \in \mathbb{N}$, or even $k = \infty$. This phenomenon is well-known in other examples of Lagrangian fibrations; see e.g. the pictures in the introduction of [Saw22]. In Lemma 8 and the remark following its proof in loc. cit. the characteristic cycles for moduli spaces of rank 1 sheaves over curves with one node are described.

In the following, we describe the characteristic cycles in the case of an extended ADE curve of type $\Gamma = \tilde{A}_n$. We make an identification $V = \{0, 1, \ldots, n\}$ in such a way that $C_{v-1}$ and $C_v$ as well as $C_n$ and $C_0$ intersect. We write the intersection points as $x_v := C_{v-1} \cap C_v$ and $x_0 := x_{n+1} := C_n \cap C_0$. 

![Diagram of characteristic cycles](image-url)
Let \((L_u \in \text{Pic}_{B_u}(C_u))_{u \in V}\) be a collection of line bundles. We set
\[
L'_u := L_u(x_u - x_{u+1}) \quad \text{for } u \in V.
\]
Furthermore, for \(v = 0, \ldots, n+1\), let \(F^v\) be the unique (see Proposition 3.23) stable sheaf on \(C\) which is special in \(x_v\) and satisfies
\[
(F^v)_j = \text{pure}(F^v|_{C_v}) \simeq \begin{cases} L'_j & \text{for } j = 0, \ldots, v - 1, \\ L_j & \text{for } j = v, \ldots, n. \end{cases}
\]
For \(v = 1, \ldots, n\), we can write \(F^v\) as an extension in two different ways:
\[
0 \to L'_{v-1}(-x_{v-1}) \to F^v \to L_{v-1}(L'_0, \ldots, L'_{v-2}, L_v \ldots, L_n) \to 0
\]
\[
0 \to L_{v}(-x_{v+1}) \to F^v \to L_v(L_0', \ldots, L'_{v-1}, L_{v+1} \ldots, L_n) \to 0,
\]
where \(L_v(L'_0, \ldots, L'_{n-1}, L_{n+1} \ldots, L_n)\) denotes the unique line bundle on \(C^u = C - C_v\) whose restriction to \(C_u \) is \(L'_u\) for \(u = 0, \ldots, v - 1\) and \(L_u\) for \(u = v + 1, \ldots, n\). In other words, \(L_v(\_\_)\) is the inverse of the isomorphism \(\pi_v\) of Lemma 4.10. For \(v = 0\), only (38) is valid. For \(v = n + 1\), only (37) is valid.

By (36), the outer terms of the short exact sequence (38) for \(v = u + 1\) agree. This means that \([F^v]\) and \([F^{v+1}]\) are two points of the same \(\mathbb{P}^1\)-fibre of \(Y_v \to J_u\). More precisely, they are the intersection points of this one fibre of \(Y_v\) with the components \(Y_{v-1}\) and \(Y_{v+1}\), respectively. Hence, we see that, when starting in \([F^0] \in Y_n \cap Y_0\) and walking the first lap of the characteristic cycle containing this point, we pass through \([F^1], [F^2], \ldots, [F^n]\) until we reach \([F^{n+1}]\), which is again a point in \(Y_n \cap Y_0\).

Proposition 5.11. Let \(C\) be an extended ADE curve with intersection graph \(\Gamma = \tilde{A}_n\). Then, the characteristic cycles of \(\mathcal{M}\) are configurations of smooth rational curves of type \(\tilde{A}_k\) where \(k\) is the (possibly infinite) order of the translation automorphism
\[
J \xrightarrow{\cong} J \quad (L_0, \ldots, L_n) \mapsto (L'_0, \ldots, L'_n) = (L_0(x_0 - x_1), \ldots, L_n(x_n - x_{n+1})).
\]

The case \(k = 1\) only occurs if all \(C_u\) are rational, i.e. if \(C\) is a singular elliptic fibre. This means that, in all other cases, the characteristic cycle can never close after just one lap. Indeed, for \(g(C_u) \geq 1\), the line bundle \(\mathcal{O}_{C_u}(x_u - x_{u+1})\) is never trivial.

All the cases \(k \geq 2\) actually occur, including the case \(k = \infty\), as shown by the following example. Let \(L \subset \mathbb{P}^2\) be a line in the plane and fix four distinct points \(p_1, \ldots, p_4 \in L\). Let \(f : E \to L\) be the double cover branched over these four points. Choosing, say, the preimage of \(p_1\) as the origin, \(E\) is an elliptic curve, with multiplication by \(-1\), which we denote by \(\ominus\), interchanging the points of the fibres of \(f\).

Let now \(s \in E\) be a point that is not \(2\)-torsion, which is equivalent to \(f(s) \notin \{p_1, p_2, p_3, p_4\}\). Counting parameters, we can choose a sextic \(B \subset \mathbb{P}^2\) which has a node at \(f(s)\), is smooth away from \(f(s)\), and intersects \(L\) transversally in \(p_1, \ldots, p_4\). Now, as in Example 5.8, let \(X \to \mathbb{P}^2\) be the minimal resolution of the double cover branched over \(B\). Then the preimage \(C\) of \(L\) is an extended ADE curve with intersection graph \(\Gamma = \tilde{A}_1\). One component is the exceptional
divisor, hence rational, and the other component is $E$. The two intersection points of the components are $s$ and $\ominus s$. Hence, fixing some $\chi > 0$ and some polarisation $H$ on $C$ satisfying Assumption 3.12, by Proposition 5.11, the characteristic cycle on the moduli space $M$ is of type $\tilde{A}_{kn}$ where $k$ is the order of the line bundle $\mathcal{O}_E(s - \ominus s)$. Now, if $s$ is not torsion, then $\mathcal{O}_E(s - \ominus s)$ is not torsion. If $s$ is $m$-torsion for $m$ odd, then $\mathcal{O}_E(s - \ominus s)$ is $m$-torsion. Finally, if $s$ is $m$-torsion for $m$ even, then $\mathcal{O}_E(s - \ominus s)$ is $\frac{m}{2}$-torsion. Hence, for any $k \geq 2$, we get characteristic cycles of type $\tilde{A}_{kn}$ by this construction.

\section{Relation to work of López-Martín, and other polarisations.}

In [LM05], López-Martín classifies stable sheaves of type $m$ on $\tilde{A}_n$-configurations of smooth rational curves (i.e. extended ADE curves with $\Gamma = \tilde{A}_n$ and $C_o \cong \mathbb{P}^1$ for every $o \in O$), and also on the other two types of reducible but reduced singular fibres of elliptic fibrations, namely types III and IV in Kodaira’s list. Remarkably, this is done for every polarisation, not only those satisfying Assumption 3.12.

We believe that the results of [LM05] generalise in a straightforward way to arbitrary ADE curves of type $\Gamma = \tilde{A}_n$ with almost unchanged proofs. Furthermore, we believe that the classifications for types III and IV can be generalised to configurations of curves whose components intersect as in types III and IV, but may be of higher genera.

Anyway, even if we restrict ourselves to the case that all components of our extended ADE curve of type $\tilde{A}_n$ are rational, the results of [LM05] show that Assumption 3.12 is strictly necessary for our classification result Theorem 1.1 to hold. It is easy to see that we have properly semi-stable sheaves as soon as some $\frac{e_o}{e} \chi$ is an integer for some $o \in O$, i.e. if $b_o = \frac{e_o}{e} \chi$. But even if we choose our $H$ in a way that avoids properly semi-stable sheaves, the class of stable sheaves changes if $\sum_{o \in O} b_o$ has another value than the one required by Assumption 3.12.

For example, let $\Gamma = \tilde{A}_4$, $\chi = 2$, and choose $H$ such that $e_o = 1$ for all $o = 0, 1, 2, 3, 4$. Then, $b_o = 1$ for all $o \in O$, hence $\sum_{o \in O} b_o = 5 = \chi + \vert O \vert - 2$. In this case, the stable line bundles $L_o$ on $C$ are those with $\chi(L_o) = b_o$ for three of the components, and $\chi(L_o) = b_o + 1$ for two of the components, and these two components are not allowed to intersect; compare [LM05, Prop. 5.13]. However, though the class of stable sheaves changes, one can still show in this case that the moduli space of stable sheaves of type $m$ is again a $\tilde{A}_4$-configuration of smooth rational curves.

\section{Negative $\chi$.}

All our results are formulated for $\chi > 0$. However, one can easily deduce analogous results for $\chi < 0$ from them by duality. Indeed, $F$ is a stable sheaf with Euler characteristic $\chi$ if and only if $F^D := \text{Hom}(F, \omega_C)$ is a stable sheaf of Euler characteristic $-\chi$. Hence, duality $(-)^D$ gives an isomorphism between the moduli space of sheaves of type $m$ and Euler characteristic $\chi$ and the moduli space of sheaves of type $m$ and Euler characteristic $-\chi$.

One can also deduce a characterisation of the stable sheaves of type $m$ and negative Euler characteristic $-\chi$ from Theorem 1.1 using duality. Namely, under Assumption 3.12 the stable sheaves are exactly those of the form $F \cong L(-x) := L \otimes \mathcal{I}_{x \rightarrow C}$ for some $x \in C$ and some line bundle $L \in \text{Pic}(C)$ with $\chi(L_{C_x}) = -b_o + 2$.

Note, however, that for $\chi = 0$ properly semi-stable sheaves occur, and the description of the moduli space changes. For example, for $\Gamma = \tilde{A}_n$, the moduli space becomes just one $\mathbb{P}^1$-bundle over $J$ which is glued with itself along two sections; compare [LM06, Thm. 4.1(2)]
(again, the statement in loc. cit. is for singular elliptic fibres, but can be extended to our more general set-up of extended ADE curves).

References

[AK80] Allen B. Altman and Steven L. Kleiman. Compactifying the picard scheme. Advances in Mathematics, 35(1):50–112, 1980.

[BHPV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. Compact complex surfaces, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, second edition, 2004.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[CFHR99] Fabrizio Catanese, Marco Franciosi, Klaus Hulek, and Miles Reid. Embeddings of curves and surfaces. Nagoya Math. J., 154:185–220, 1999.

[Cox89] David A. Cox. Picard numbers of surfaces in 3-dimensional weighted projective spaces. Math. Z., 201(2):183–189, 1989.

[Cza18] Adam M. Czapliński. Lagrangian fibrations with designed singular fibres. PhD thesis, Johannes Gutenberg-Universität Mainz, 2018.

[DS17] Thomas Dedieu and Edoardo Sernesi. Equigeneric and equisingular families of curves on surfaces. Publ. Mat., Barc., 61(1):175–212, 2017.

[Har94] Robin Hartshorne. Generalized divisors on Gorenstein schemes. In Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), volume 8, pages 287–339, 1994.

[Har07] Robin Hartshorne. Generalized divisors and biliaison. Illinois J. Math., 51(1):83–98, 2007.

[Hel21] Isabell Hellmann. The nilpotent cone in the Mukai system of rank two and genus two. Math. Ann., 380(3-4):1687–1711, 2021.

[HL10] Daniel Huybrechts and Manfred Lehn. The Geometry of Moduli Spaces of Sheaves; 2nd ed. Cambridge mathematical library. Cambridge University Press, 2010.

[HO09] Jun-Muk Hwang and Keiji Oguiso. Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration. Amer. J. Math., 131(4):981–1007, 2009.

[HO11] Jun-Muk Hwang and Keiji Oguiso. Multiple fibers of holomorphic Lagrangian fibrations. Commun. Contemp. Math., 13(2):309–329, 2011.

[Huy16] Daniel Huybrechts. Lectures on K3 surfaces, volume 158 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016.

[Knu01] Andreas Leopold Knutsen. On 4th-order embeddings of K3 surfaces and Enriques surfaces. Manuscripta Math., 104(2):211–237, 2001.

[Lan83] Herbert Lange. Universal families of extensions. J. Algebra, 83(1):101–112, 1983.

[LM05] Ana Cristina López-Martín. Simpson Jacobians of reducible curves. J. Reine Angew. Math., 582:1–39, 2005.

[LM06] Ana Cristina López-Martín. Relative Jacobians of elliptic fibrations with reducible fibers. J. Geom. Phys., 56(3):375–385, 2006.

[Mats01] Daisuke Matsushita. On singular fibres of Lagrangian fibrations over holomorphic symplectic manifolds. Math. Ann., 321(4):755–773, 2001.

[Saw08] Justin Sawon. On the discriminant locus of a Lagrangian fibration. Math. Ann., 341(1):201–221, 2008.

[Saw15] Justin Sawon. On Lagrangian fibrations by Jacobians I. J. Reine Angew. Math., 701:127–151, 2015.

[Saw22] Justin Sawon. Singular fibers of very general Lagrangian fibrations. Commun. Contemp. Math., 24(9):Paper No. 2150070, 19, 2022.

[Wah74] Jonathan M. Wahl. Equisingular deformations of plane algebroid curves. Trans. Am. Math. Soc., 193:143–170, 1974.