The Algorithmic Complexity of Bondage and Reinforcement Problems in bipartite graphs

Fu-Tao Hu\textsuperscript{a}, Moo Young Sohn\textsuperscript{b}\textsuperscript{*}

\textsuperscript{a}School of Mathematical Sciences, Anhui University, Hefei, 230601, P.R. China

Email: hufu@mail.ustc.edu.cn

\textsuperscript{b}Mathematics, Changwon National University, Changwon, 641-773, Republic of Korea

Abstract

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a dominating set if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. The bondage number of a nonempty graph $G$ is the smallest number of edges whose removal from $G$ results in a graph with domination number larger than $\gamma(G)$. The reinforcement number of $G$ is the smallest number of edges whose addition to $G$ results in a graph with smaller domination number than $\gamma(G)$. In 2012, Hu and Xu proved that the decision problems for the bondage, the total bondage, the reinforcement and the total reinforcement numbers are all NP-hard in general graphs. In this paper, we improve these results to bipartite graphs.

Key words: Complexity; NP-completeness; NP-hardness; Domination; Bondage; Total bondage; Reinforcement; Total reinforcement

AMS Subject Classification (2010): 05C69, 05C85

1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to Xu \cite{19}. Let $G = (V, E)$ be a finite, undirected and simple graph, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set of $G$. For a vertex $x \in V(G)$, let $N_G(x) = \{y : xy \in E(G)\}$ be the open set of neighbors of $x$ and $N_G[x] = N_G(x) \cup \{x\}$ be the closed set of neighbors of $x$.

A subset $D \subseteq V$ is a dominating set of $G$ if every vertex in $V - D$ has at least one neighbor in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of $G$. A dominating set $D$ is called a $\gamma$-set of $G$ if $|D| = \gamma(G)$. The domination is an important and classic notion that has become one of the most widely researched topics in graph theory and also is used to study property of networks frequently. A thorough study of domination appears in the books \cite{7, 8} by Haynes, Hedetniemi, and Slater. Among various problems related...
to the domination number, some focus on graph alterations and their effects on the domination number. Here, we are concerned with two particular graph modifications, the removal and addition of edges from a graph. The bondage number of $G$, denoted by $b(G)$, is the minimum number of edges whose removal from $G$ results in a graph with a domination number larger than the one of $G$. The reinforcement number of $G$, denoted by $r(G)$, is the smallest number of edges whose addition to $G$ results in a graph with a domination number smaller than the one of $G$. The bondage number and the reinforcement number were introduced by Fink et al. [3] and Kok, Mynhardt [13], respectively, in 1990. The reinforcement number for digraphs has been studied by Huang, Wang and Xu [12]. The bondage number and the reinforcement number are two important parameters for measuring the vulnerability and stability of the network domination under link failure and link addition. Recently, Xu [20] gave a review article on bondage numbers in 2013.

A dominating set $D$ of a graph $G$ without isolated vertices is called a total dominating set if every vertex in $D$ is also adjacent to another vertex in $D$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets of $G$. In this paper, we use the symbol $D_t$ to denote a total dominating set. A total dominating set $D_t$ is called a $\gamma_t$-set of $G$ if $|D_t| = \gamma_t(G)$. The total domination was introduced by Cockayne et al. [2]. Total domination in graphs has been extensively studied in the literature. In 2009, Henning [6] surveyed the recent results on total domination in graphs. The total bondage number of $G$ without isolated vertices, denoted by $b_t(G)$, is the minimum number of edges whose removal from $G$ results in a graph with a total domination number larger than the one of $G$. The total reinforcement number of $G$ without isolated vertices, denoted by $r_t(G)$, is the smallest number of edges whose addition from $G$ results in a graph with a total domination number smaller than the one of $G$. The total bondage number of a graph was first studied by Kulli and Patwari [14] and further studied by Sridharan, Elias, Subramanian [17], Huang and Xu [11]. The total reinforcement number of a graph was first studied by Sridharan, Elias, Subramanian [18] and further studied by Henning, Rad and Raczek [9].

For a graph parameter, knowing whether or not there exists a polynomial-time algorithm to compute its exact value is the essential problem. If the decision problem corresponding to the computation of this parameter is NP-hard or NP-complete, then polynomial-time algorithms for this parameter do not exist unless $NP = P$. The problem of determining the domination number has been proved NP-complete for chordal bipartite graphs [15]. For the total domination number, the problem has been proved NP-complete for bipartite graphs [16]. There are many other complexity results for variations of domination, these results can be found in the two books [1, 8] and the survey [6].

As regards the bondage problem, Hattingh et al. [5] showed that the restrained bondage problem is NP-complete even for bipartite graphs. Hu and Xu [10] have showed that the bondage, the total bondage, the reinforcement and the total reinforcement numbers are all NP-hard for general graphs. We know that even if a problem is known to be NP-hard or NP-complete, it may be possible to find a polynomial-time algorithm for a restricted set of instances from a particular application. The bondage number and reinforcement number in graphs are very interesting research problems in graph theory. There are many results about the bondage number and reinforcement
number in bipartite graphs. Many famous networks are bipartite graphs, such as hypercube graphs, partial cube, grid graphs, median graphs and so on. If we proved these decision problems for the bondage and the reinforcement are all NP-hard, then the studies on the bondage number and reinforcement number in bipartite graphs are more meaningful and we can directly deduce the decision problems for the bondage and the reinforcement are both NP-hard in general graphs. So we should be concerned about the algorithmic complexity of the bondage and reinforcement problems in bipartite graphs.

In this paper, we will show that the decision problems for the bondage, the total bondage, the reinforcement and the total reinforcement numbers are all NP-hard even for bipartite graphs. In other words, there are not polynomial-time algorithms to compute these parameters unless \( P = NP \). The proofs are in Section 3, Section 4 and Section 5, respectively.

We have considered about whether these four problems are belong to NP or not. Since the problem of determining the domination number is NP-complete, and it is not clear that there is a polynomial algorithm to verify \( \gamma(G - B) > \gamma(G) \) (or \( \gamma(G + R) < \gamma(G) \)) for any subset \( B \subseteq E(G) \) (or \( R \subseteq \bar{E}(G) \)), these four problems are not obviously seen to be in NP. We conjecture that they are not in \( NP \). But we can not prove that determining the bondage and the reinforcement are not NP-problems. This will be our work to study further. In this paper, we only present the results that these four problems are all NP-hard in bipartite graphs.

## 2 3-satisfiability problem

In *Computers and Intractability: A Guide to the Theory of NP-Completeness* [4], Garey and Johnson outline three steps to prove a decision problem to be NP-hard. We follow the three steps for proving our four decision problems to be NP-hard. We prove our results by describing a polynomial transformation from the known NP-complete problem: 3-satisfiability problem. To state the 3-satisfiability problem, in this section, we recall some terms.

Let \( U \) be a set of Boolean variables. A **truth assignment** for \( U \) is a mapping \( t : U \rightarrow \{ T, F \} \). If \( t(u) = T \), then \( u \) is said to be “true” under \( t \); if \( t(u) = F \), then \( u \) is said to be “false” under \( t \). If \( u \) is a variable in \( U \), then \( u \) and \( \bar{u} \) are **literals** over \( U \). The literal \( u \) is true under \( t \) if and only if the variable \( u \) is true under \( t \); the literal \( \bar{u} \) is true if and only if the variable \( u \) is false.

A **clause** over \( U \) is a set of literals over \( U \). It represents the disjunction of these literals and is **satisfied** by a truth assignment if and only if at least one of its members is true under that assignment. A collection \( \mathcal{C} \) of clauses over \( U \) is **satisfiable** if and only if there exists some truth assignment for \( U \) that simultaneously satisfies all the clauses in \( \mathcal{C} \). Such a truth assignment is called a **satisfying truth assignment** for \( \mathcal{C} \). The 3-satisfiability problem is specified as follows.
3-satisfiability problem (3SAT):

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set $U$ of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: Is there a truth assignment for $U$ that satisfies all the clauses in $\mathcal{C}$?

Theorem 2.1 (Theorem 3.1 in [4]) The 3-satisfiability problem is NP-complete.

3 NP-hardness of bondage

In this section, we will show that the problem determining the bondage number in bipartite graphs is NP-hard. We first state the problem as the following decision problem.

Bondage problem:

Instance: A nonempty graph $G$ and a positive integer $k$.

Question: Is $b(G) \leq k$?

Theorem 3.1 The bondage problem is NP-hard even when restricted to bipartite graphs and $k = 1$.

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. A graph $G$ will be constructed from the instance of 3SAT, such that $\mathcal{C}$ is satisfiable if and only if $b(G) = 1$. Such a graph $G$ can be constructed as follows.

For each variable $u_i \in U$, create a cycle $H_i = (u_i, v_i, \bar{u}_i, r_i, q_i, p_i, u_i)$. Create a single vertex $c_j$ for each $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$ and add the set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$ to the edge set. Finally, add a path $P = s_1s_2s_3$, and join $s_1$ and $s_3$ to each vertex $c_j$ with $1 \leq j \leq m$.

Figure 1 illustrates this construction when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{\bar{u}_1, u_2, u_4\}$, $C_3 = \{\bar{u}_2, u_3, u_4\}$.

To prove that this is indeed a transformation, it remains to show that $b(G) = 1$ if and only if there is a truth assignment for $U$ that satisfies all the clauses in $\mathcal{C}$. This aim can be fulfilled by proving the following four claims.

Claim 3.1 $\gamma(G) \geq 2n + 1$. Moreover, if $\gamma(G) = 2n + 1$, then for any $\gamma$-set $D$ in $G$, $D \cap V(P) = \{s_2\}$, $|D \cap V(H_i)| = 2$ and $|D \cap \{u_i, \bar{u}_i\}| \leq 1$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D$ for each $j = 1, 2, \ldots, m$.

Proof. Let $D$ be a $\gamma$-set of $G$. By the construction of $G$, since $s_2$ can be dominated only by vertices in $V(P)$, which implies $|D \cap V(P)| \geq 1$; for each $i = 1, 2, \ldots, n$, it is easy to see that $|D \cap N_G[u_i]| \geq 1$ and $|D \cap N_G[\bar{u}_i]| \geq 1$, this implies $|D \cap V(H_i)| \geq 2$. It follows that $\gamma(G) = |D| \geq 2n + 1.$
Figure 1: An instance of the bondage problem. Here $\gamma = 9$, where the set of bold points is a $\gamma$-set.

Suppose that $\gamma(G) = 2n + 1$. Then $|D \cap V(P)| = 1$ and $|D \cap V(H_i)| = 2$ for each $i = 1, 2, \ldots, n$. Consequently, $c_j \notin D$ for each $j = 1, 2, \ldots, m$. Since $q_i$ should be dominated by $D$, $|D \cap \{u_i, \bar{u}_i\}| \leq 1$. Since all vertices in $V(P)$ can be dominated only by $D \cap V(P)$, this implies $D \cap V(P) = \{s_2\}$.

Claim 3.2 $\gamma(G) = 2n + 1$ if and only if $\mathcal{C}$ is satisfiable.

**Proof.** Suppose that $\gamma(G) = 2n + 1$ and let $D$ be a $\gamma$-set of $G$. By Claim 3.1, for each $i = 1, 2, \ldots, n$, $|D \cap \{u_i, \bar{u}_i\}| \leq 1$. Define a mapping $t : U \rightarrow \{T, F\}$ by

$$
t(u_i) = \begin{cases} 
T & \text{if } u_i \in D, \\
F & \text{otherwise, } i = 1, 2, \ldots, n.
\end{cases}$$

(3.1)

Arbitrarily choose a clause $C_j \in \mathcal{C}$ with $1 \leq j \leq m$. There exists some $i$ with $1 \leq i \leq n$ such that $c_j$ is dominated by $u_i \in D$ or $\bar{u}_i \in D$. Suppose without loss of generality that $c_j$ is dominated by $u_i \in D$. Since $u_i$ is adjacent to $c_j$ in $G$ and $u_i \in D$, it follows that $t(u_i) = T$ by (3.1), which implies that the clause $C_j$ is satisfied by $t$. By the arbitrariness of $j$ with $1 \leq j \leq m$, it shows that $t$ satisfies all the clauses in $\mathcal{C}$, that is, $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable, and let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. Construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$, then put the vertex $u_i$ and $r_i$ in $D'$; if $t(u_i) = F$, then put the vertex $\bar{u}_i$ and $p_i$ in $D'$. Clearly, $|D'| = 2n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, \ldots, m$, at least one of the three literals in $C_j$ is true under the assignment $t$. It follows that $c_j$ can be dominated by $D'$. Thus $D' \cup \{s_2\}$ is a dominating set of $G$, and so $\gamma(G) \leq |D' \cup \{s_2\}| = 2n + 1$. By Claim 3.1, $\gamma(G) \geq 2n + 1$, and so $\gamma(G) = 2n + 1$.

Claim 3.3 $\gamma(G - e) \leq 2n + 2$ for any $e \in E(G)$. 
Proof. For every edge $e$ in any 6-cycle $H_i$, we have $\gamma(H_i - e) = 2$. Let $G'$ be the subgraph of $G$ induced by \{c_1, c_2, \ldots, c_n, s_1, s_2, s_3\} of $G$. For any edge $e' \in E(G')$, $\{s_1, s_3\}$ is a dominating set of $G' - e'$. Therefore, $\gamma(G - e) \leq 2n + 2$ for any $e \in E(G)$.

Claim 3.4 $\gamma(G) = 2n + 1$ if and only if $b(G) = 1$.

Proof. Assume $\gamma(G) = 2n + 1$ and consider the edge $e = s_1s_2$. Suppose $\gamma(G) = \gamma(G - e)$. Let $D'$ be a $\gamma$-set in $G - e$. It is clear that $D'$ is also a $\gamma$-set of $G$. By Claim 3.1, we have $c_j \notin D'$ for each $j = 1, 2, \ldots, m$ and $D' \cap V(P) = \{s_2\}$. But then $s_1$ can not be dominated by $D'$, a contradiction. Hence, $\gamma(G) < \gamma(G - e)$, and so $b(G) = 1$.

Now, assume $b(G) = 1$. By Claim 3.1, we have that $\gamma(G) \geq 2n + 1$. Let $e'$ be an edge such that $\gamma(G) < \gamma(G - e')$. By Claim 3.3, we have that $\gamma(G - e') \leq 2n + 2$. Thus, $2n + 1 \leq \gamma(G) < \gamma(G - e') \leq 2n + 2$, which yields $\gamma(G) = 2n + 1$.

By Claim 3.2 and Claim 3.4, we prove that $b(G) = 1$ if and only if there is a truth assignment for $U$ that satisfies all the clauses in $\mathcal{C}$. Since the graph $G$ contains $2n + m + 3$ vertices and $6n + 5m + 2$ edges, this is clearly a polynomial transformation.

4 NP-hardness of total bondage

In this section, we will show that the problem determining the total bondage number in bipartite graphs is NP-hard. We first state it as the following decision problem.

Total bondage problem:

Instance: A nonempty graph $G$ without isolated vertices and a positive integer $k$.

Question: Is $b_t(G) \leq k$?

Theorem 4.1 The total bondage problem is NP-hard even when restricted to bipartite graphs and $k = 1$.

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a graph $G$ such that $\mathcal{C}$ is satisfiable if and only if $b_t(G) = 1$.

Such a graph $G$ can be constructed as follows.

For each $u_i \in U$, create a graph $H_i$ with vertex set $V(H_i) = \{u_i, \bar{u}_i, v_i, p_i, q_i\}$ and edge set $E(H_i) = \{u_iv_i, u_iq_i, \bar{u}_iv_i, v_ip_i, p_iq_i, \bar{u}_iq_i\}$. For each $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex $c_j$ and add the set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$ to the edge set, $1 \leq j \leq m$. Finally, add a graph $T$ with vertex set $V(T) = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and edge set $E(T) = \{s_1s_2, s_1s_4, s_2s_3, s_2s_5, s_3s_4, s_4s_5, s_5s_6\}$, and join $s_1$ and $s_3$ to each vertex $c_j$, $1 \leq j \leq m$.

Figure 2 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{u_1, u_2, u_4\}$ and $C_3 = \{u_2, u_3, u_4\}$.
Besides, $D$ belongs to $\bar{C}$ by proving the following four claims.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{total_bondage_example}
\caption{An instance of the total bondage problem. Here $\gamma_t = 10$, where the set of bold points is a $\gamma_t$-set.}
\end{figure}

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $\mathcal{C}$ is satisfiable if and only if $b_t(G) = 1$. This aim can be fulfilled by proving the following four claims.

\textbf{Claim 4.1} $\gamma_t(G) \geq 2n + 2$. For any $\gamma_t$-set $D_t$ of $G$, $s_5 \in D_t$ and at least one of $v_i$ and $q_i$ belongs to $D_t$ for each $i = 1, 2, \ldots, n$. Moreover, if $\gamma_t(G) = 2n + 2$, then $D_t \cap V(T) = \{s_2, s_5\}$ or $\{s_4, s_5\}$, $|D_t \cap V(H_i)| = 2$ and $|D_t \cap \{u_i, \bar{u}_i\}| \leq 1$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D_t$ for each $j = 1, 2, \ldots, m$.

\textbf{Proof.} Let $D_t$ be a $\gamma_t$-set of $G$. By the construction of $G$, it is clear that at least one of $v_i$ and $q_i$ should be in $D_t$ to dominate $p_i$, and $v_i$ or $q_i$ can be dominated only by another vertex in $H_i$. It follows that at least one of $v_i$ and $q_i$ belongs to $D_t$ and $|D_t \cap V(H_i)| \geq 2$ for each $i = 1, 2, \ldots, n$. It is also clear that $s_5$ is certainly in $D_t$ to dominate $s_6$, and $s_5$ can be dominated only by another vertex in $T$. This fact implies that $s_5 \in D_t$ and $|D_t \cap V(T)| \geq 2$. Thus, $\gamma_t(G) = |D_t| \geq 2n + 2$.

Suppose that $\gamma_t(G) = 2n + 2$. Then $|D_t \cap V(H_i)| = 2$ for each $i = 1, 2, \ldots, n$, and $|D_t \cap V(T)| = 2$. Consequently, $c_j \notin D_t$ for each $j = 1, 2, \ldots, m$. Since $p_i$ should be dominated by $D_t$, we have $|D \cap \{u_i, \bar{u}_i\}| \leq 1$ for each $i = 1, 2, \ldots, n$. Besides, $s_5$ can be dominated only by the vertex $s_3$ or $s_4$ in $T$, that is, at least one of $s_2$ and $s_4$ belongs to $D_t$. Noting $|D_t \cap V(T)| = 2$, we have $D_t \cap V(H) = \{s_2, s_5\}$ or $\{s_4, s_5\}$.

\textbf{Claim 4.2} $\gamma_t(G) = 2n + 2$ if and only if $\mathcal{C}$ is satisfiable.

\textbf{Proof.} Suppose that $\gamma_t(G) = 2n + 2$ and let $D_t$ be a $\gamma_t$-set of $G$. By Claim 4.1,
Define a mapping $t : U \rightarrow \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } u_i \in D_t, \\ F & \text{otherwise, } \end{cases} \quad i = 1, 2, \ldots, n. \quad (4.1)$$

Arbitrarily choose a clause $C_j \in \mathcal{C}$. Since the vertex $c_j$ is not adjacent to any member of $\{s_2, s_4, s_5\} \cup \{v_i, p_i, q_i : 1 \leq i \leq n\}$, there exists some $i$ with $1 \leq i \leq n$ such that $c_j$ is dominated by $u_i \in D_t$ or $\bar{u}_i \in D_t$.

Suppose without loss of generality that $c_j$ is dominated by $\bar{u}_i \in D_t$. Then $\bar{u}_i$ is adjacent to $c_j$ in $G$. Since $u_i \in D_t$ and $|D_t \cap \{u_i, \bar{u}_i\}| \leq 1$, we have $t(\bar{u}_i) = T$ by (4.1), which implies that the clause $C_j$ is satisfied by $t$. Since the arbitrariness of $j$ with $1 \leq j \leq m$, $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable, and let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. Construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$, then put the vertex $u_i$ in $D'$; if $t(u_i) = F$, then put the vertex $\bar{u}_i$ in $D'$. Clearly, $|D'| = n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D'$. Let $D'_t = D' \cup \{s_2, s_5, v_1, \ldots, v_n\}$. Clearly, $D'_t$ is a total dominating set of $G$ and $|D'_t| = 2n + 2$. Hence, $\gamma_t(G) \leq |D'_t| = 2n + 2$. By Claim 4.1, $\gamma_t(G) \geq 2n + 2$. Therefore, $\gamma_t(G) = 2n + 2$.

Claim 4.3 For any $e \in E(G)$, $\gamma_t(G - e) \leq 2n + 3$.

Proof. It is easy to see that for any edge $e \in E(H_i)$ for each $i = 1, 2, \ldots, n$, $\gamma_t(H_i - e) = 2$. Let $G' = G - \{H_1, H_2, \ldots, H_n\}$. For any edge $e' \in E(G')$, it can easily be checked that $\gamma_t(G') \leq 3$. Thus, for any $e \in E(G)$, $\gamma_t(G - e) \leq 2n + 3$.

Claim 4.4 $\gamma_t(G) = 2n + 2$ if and only if $b_t(G) = 1$.

Proof. Assume $\gamma_t(G) = 2n + 2$ and take $e = s_2s_5$. Suppose that $\gamma_t(G - e) = \gamma_t(G)$. Let $D'_t$ be a $\gamma_t$-set of $G - e$. As $D'_t$ is also a $\gamma_t$-set of $G$, by Claim 4.1 $D'_t \cap V(H) = \{s_2, s_5\}$ or $\{s_4, s_5\}$, which contradicts the fact that $s_2$ is dominated by $D'_t$ in $G - e$. This contradiction shows that $\gamma_t(G - e) > \gamma_t(G)$, hence $b_t(G) = 1$.

Now, assume $b_t(G) = 1$. By Claim 4.1, we have that $\gamma_t(G) \geq 2n + 2$. Let $e'$ be an edge such that $\gamma_t(G - e') > \gamma_t(G)$. By Claim 4.3, we have that $\gamma_t(G - e) \leq 2n + 3$. Thus, $2n + 2 \leq \gamma_t(G) < \gamma_t(G - e') \leq 2n + 3$, which yields $\gamma_t(G) = 2n + 2$.

It follows from Claim 4.2 and Claim 4.4 that $b_t(G) = 1$ if and only if $\mathcal{C}$ is satisfiable. The theorem follows.

5 NP-hardness of reinforcement

In this section, we will show that the problems of determining the reinforcement number and total reinforcement number in bipartite graphs are NP-hard. We first state them as the following decision problem.
(Total) Reinforcement problem:

Instance: A graph $G$ and a positive integer $k$.

Question: Is $(r_t(G)) r(G) \leq k$?

Theorem 5.1 The reinforcement problem is NP-hard even when restricted to bipartite graphs and $k = 1$.

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $C = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a graph $G$ such that $C$ is satisfiable if and only if $r(G) = 1$. Such a graph $G$ can be constructed as follows.

For each $u_i \in U$, associate a cycle $H_i = (u_i, v_i, \bar{u}_i, r_i, q_i, p_i, u_i)$. For each $C_j = \{x_j, y_j, z_j\} \in C$, associate a single vertex $c_j$ and add edges $(c_j, x_j)$, $(c_j, y_j)$ and $(c_j, z_j)$, $1 \leq j \leq m$. Finally, add a vertex $s$ and join $s$ to every vertex $c_j$.

Figure 3 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{\bar{u}_1, u_2, u_4\}$, $C_3 = \{\bar{u}_2, u_3, u_4\}$.

![Diagram](image.png)

Figure 3: An instance of the reinforcement problem. Here $\gamma = 5$, where the set of bold points is a $\gamma$-set.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $C$ is satisfiable if and only if $r(G) = 1$. To this aim, we first prove the following two claims.

Claim 5.1.1 $\gamma(G) = 2n + 1$.

Proof. On the one hand, let $D$ be a $\gamma$-set of $G$, then $\gamma(G) = |D| \geq 2n + 1$ since $|D \cap V(H_i)| \geq 2$ and $|D \cap N[s]| \geq 1$. On the other hand, $D' = \{s, u_1, r_1, u_2, r_2, \ldots, u_n, r_n\}$ is a dominating set of $G$, which implies that $\gamma(G) \leq |D'| = 2n + 1$. It follows that $\gamma(G) = 2n + 1$. 

\[ \square \]
Claim 5.1.2 If there exists an edge \( e \in E(\bar{G}) \) such that \( \gamma(G + e) = 2n \), and if \( D_e \) denotes a \( \gamma \)-set of \( G + e \), then \( |D_e \cap V(H_i)| = 2 \) and \( |D_e \cap \{u_i, \bar{u}_i\}| \leq 1 \) for each \( i = 1, 2, \ldots, n \), while \( s \notin D_e \) and \( c_j \notin D_e \) for each \( j = 1, 2, \ldots, m \).

Proof. Suppose to the contrary that \( |D_e \cap V(H_i)| < 2 \) for some \( i_0 \) with \( 1 \leq i_0 \leq n \). Since \( \{v_i, p_i, q_i, r_i\} \) should be dominated by \( D_e \), \( D_e \cap V(H_i) = \{q_i\} \), and then one end-vertex of the edge \( e \) should be \( v_{i_0} \) since \( D_e \) dominates it via the edge \( e \) in \( G + e \), and for every \( i \neq i_0 \), \( |D_e \cap V(H_i)| \geq 2 \) since \( D_e \) dominates \( \{v_i, p_i, q_i, r_i\} \). By the hypotheses, two literals \( u_{i_0} \) and \( \bar{u}_{i_0} \) do not simultaneously appear in the same clause in \( \mathcal{C} \), there is no \( j \) such that vertex \( c_j \) is adjacent to both of them. Since \( u_{i_0} \) and \( \bar{u}_{i_0} \) should be dominated by \( D_e \), there exist two distinct vertices \( c_j, c_l \in D_e \) such that \( c_j \) dominates \( u_{i_0} \) and \( c_l \) dominates \( \bar{u}_{i_0} \). Thus, \( |D_e| \geq 2n + 1 \), a contradiction. Hence, \( |D_e \cap V(H_j)| = 2 \) for each \( i = 1, 2, \ldots, n \), and \( c_j \notin D_e \) for every \( j \) since \( |D_e| = 2n \). Therefore, \( s \) should be dominated by \( D_e \) via the edge \( e \) in \( G + e \). Since \( q_i \) should be dominated by \( D_e \), \( |D_e \cap \{u_i, \bar{u}_i\}| \leq 1 \) for each \( i = 1, 2, \ldots, n \). \( \blacksquare \)

We now show that \( \mathcal{C} \) is satisfiable if and only if \( r(G) = 1 \).

Suppose that \( \mathcal{C} \) is satisfiable, and let \( t : U \to \{T, F\} \) be a satisfying truth assignment for \( \mathcal{C} \). We construct a subset \( D' \subseteq V(G) \) as follows. If \( t(u_i) = T \) then put the vertex \( u_i \) and \( r_i \) in \( D' \); if \( t(u_i) = F \) then put the vertex \( \bar{u}_i \) and \( p_i \) in \( D' \). Then \( |D'| = 2n \). Since \( t \) is a satisfying truth assignment for \( \mathcal{C} \), for each \( j = 1, 2, \ldots, m \), at least one of the three literals in \( C_j \) is true under the assignment \( t \). It follows that the corresponding vertex \( c_j \) in \( G \) is adjacent to at least one vertex in \( D' \) since \( c_j \) is adjacent to each literal in \( C_j \) by the construction of \( G \). Without loss of generality let \( t(u_1) = T \), then \( D' \) is a dominating set of \( G + su_1 \), and hence \( \gamma(G + su_1) \leq |D'| = 2n \). By Claim 5.1.1, we have \( \gamma(G) = 2n + 1 \). It follows that \( \gamma(G + su_1) \leq 2n < 2n + 1 = \gamma(G) \), which implies \( r(G) = 1 \).

Conversely, assume \( r(G) = 1 \). Then there exists an edge \( e \) in \( G \) such that \( \gamma(G + e) = 2n \). Let \( D_e \) be a \( \gamma \)-set of \( G + e \). By Claim 5.1.2, \( |D_e \{u_i, \bar{u}_i\}| \leq 1 \) for each \( i = 1, 2, \ldots, n \), \( s \notin D_e \) and \( c_j \notin D_e \) for each \( j = 1, 2, \ldots, m \). Define \( t : U \to \{T, F\} \) by

\[
t(u_i) = \begin{cases} T & \text{if } u_i \in D_e, \\ F & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, n. 
\]

(5.1)

We will show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \). It is sufficient to show that every clause in \( \mathcal{C} \) is satisfied by \( t \).

Consider arbitrary clause \( C_j \in \mathcal{C} \) with \( 1 \leq j \leq m \). By Claim 5.1.2, the corresponding vertex \( c_j \) in \( G \) is dominated by \( u_i \) or \( \bar{u}_i \) in \( D_e \) for some \( i \). Suppose without loss of generality that \( c_j \) is dominated by \( u_i \in D_e \). Then \( u_i \) is adjacent to \( c_j \) in \( G \), that is, the literal \( u_i \) is in the clause \( C_j \) by the construction of \( G \). Since \( u_i \in D_e \), we have \( t(u_i) = T \) by (5.1), which implies that \( C_j \) is satisfied by \( t \). The arbitrariness of \( j \) with \( 1 \leq j \leq m \) shows that every clause in \( \mathcal{C} \) is satisfied by \( t \), that is, \( \mathcal{C} \) is satisfiable. \( \blacksquare \)

By using an analogous argument as in the proof of Theorem 5.1 we can prove that total reinforcement problem is also NP-hard even when restricted to bipartite graphs and \( k = 1 \). Here we give an outline of the proof, the details are omitted.

Theorem 5.2 The total reinforcement problem is NP-hard even when restricted to bipartite graphs and \( k = 1 \).
Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a graph $G$ such that $\mathcal{C}$ is satisfiable if and only if $r_t(G) = 1$. Such a graph $G$ can be constructed as follows.

For each $u_i \in U$, associate a graph $H_i$ with vertex set $V(H_i) = \{u_i, \bar{u}_i, v_i, p_i, q_i\}$ and edge set $E(H_i) = \{u_i v_i, u_i q_i, \bar{u}_i v_i, v_i p_i, p_i q_i, \bar{u}_i q_i\}$. For each $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex $c_j$ and add an edge set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$, $1 \leq j \leq m$. Finally, add a path $P = s_1 s_2 s_3$ and join $s_1$ to each vertex $c_j$, $1 \leq j \leq m$.

Figure 4 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{u_1, \bar{u}_2, u_4\}$ and $C_3 = \{\bar{u}_2, \bar{u}_3, u_4\}.$

![Figure 4: An instance of the total reinforcement problem. Here $\gamma_t = 10$, where the set of bold points is a $\gamma_t$-set. Add the edge $u_1 s_2$ and remove the vertex $s_1$ to decrease the total domination number.](image-url)

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $\mathcal{C}$ is satisfiable if and only if $r_t(G) = 1$.

Claim 5.2.1 $\gamma_t(G) = 2n + 2$.

Claim 5.2.2 If there exists an edge $e \in E(\bar{G})$ such that $\gamma_t(G + e) < 2n + 2$, and if $D_e$ be a $\gamma_t$-set of $G + e$, then $|D_e \cap V(H_i)| = 2$ and $|D_e \cap \{u_i, \bar{u}_i\}| \leq 1$ for each $i = 1, 2, \ldots, n$, while $s_1 \notin D_e$ and $c_j \notin D_e$ for each $j = 1, 2, \ldots, m$.

We now show that $\mathcal{C}$ is satisfiable if and only if $r_t(G) = 1$.

Suppose that $\mathcal{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$ then put the vertex $u_i$ in $D'$; if $t(u_i) = F$ then put the vertex $\bar{u}_i$ in $D'$. Then $|D'| = n$. Let $D'_t = D' \cup \{v_1, v_2, \ldots, v_n, s_2\}$. Without loss of generality let $u_1 \in D'_t$. We can easily check that $D'_t$ is a total dominating set of $G + s_2 u_1$, and hence $\gamma_t(G + s_2 u_1) \leq |D'_t| = 2n + 1$.

By Claim 5.2.1, we have $\gamma_t(G) = 2n + 2$. It follows that $r_t(G) = 1$. 
Conversely, assume \( r_t(G) = 1 \). Then there exists an edge \( e \) in \( \bar{G} \) such that \( \gamma(G+e) = 2n \). Let \( D_e \) be a \( \gamma_t \)-set of \( G+e \). By Claim 5.1.2, \( |D_e\{u_i, \bar{u}_i\}| \leq 1 \) for each \( i = 1, 2, \ldots, n \), \( s_1 \notin D_e \) and \( c_j \notin D_e \) for each \( j = 1, 2, \ldots, m \). Define \( t : U \to \{T, F\} \) by
\[
t(u_i) = \begin{cases} T & \text{if } u_i \in D_e, \\ F & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, n.
\]

(5.2)

Using the same methods as in Theorem 5.1, we can show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \).

Acknowledgments.

The authors would like to thank the anonymous referees for their kind comments and helpful suggestions on the original manuscript, which resulted in this revised version.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2005115). The first author was supported by the doctoral scientific research startup fund of Anhui University.

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