On the qualitative theory of non-Gamov decay states

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Abstract

Complex potential transformations which add imaginary parts to chosen energy levels are given and qualitatively explained. Unexpected shape similarity of potential perturbations for real and imaginary E-shifts of bound states are exhibited. The imaginary E-shifts in the continuous spectrum lead to a surprising quasi-periodic field raking up initial propagating waves into localized states. Complex periodic potentials without lacunas (!) are constructed. The fission of quasi-bound states when neighbour complex eigenvalues approach one another is demonstrated. E-shift algorithms represent wide classes of exactly solvable quantum models for non-self-adjoint operators.

1 Introduction

The significant success has been achieved last years due to inverse problem (IP) and SUSYQ approach [1-7]. The merit of IP is that the observables are input data in this formalism and their variations correspond to a complete set of IP exactly solvable models (ESM). This inspires an idea to control spectral parameters, which may allow one to unveil new relations between the spectral, scattering, decay data and forces acting in quantum objects, see examples in [8-12]. Such a concept has been realized last years when we have developed a qualitative theory of quantum design [5,6,13-17]. Through the computer visualization, we have established physically clear algorithms of elementary transformations associated with given variations of distinct spectral parameters. We have succeeded in revealing universal potential blocks from which any quantum systems may be engineered, at least mentally. All this is by no means reduced only to content of the distinct images but embraces continuum of all the elementary variations of potential shapes. So one acquires a "mathematical vision" in microworld which is hidden from our eyes.

In this paper we consider the generalization of quantum design to imaginary shifts of energy values leading to non-self-adjoint Schrödinger operator (complex-valued potentials). In the mathematical physics, non-Hermitian operators represent an area of permanent interest. The spectral analysis of non-Hermitian differential operators was considered by Lyance V.E. in Appendix 1 to the 2nd ed. of the book by Naimark [18]. As a widely known example of applying complex potentials in physics, the optical model can be given. This is an effective tool for accounting the processes with particle number nonconservation in a specific channel occurring in many nuclear
(e. g. heavy-ion) reactions. There were some papers devoted to construction of complex potentials with pure real spectrum of eigenvalues, see [19] and references therein. The SUSYQ approach was applied to generate complex potential partners with the same eigenvalues spectra distinguishing in only one eigenenergy (or finite, but given, number of spectrum points), see [20, 21]. Here we present a technique of shifting energy levels into complex energy plane. This makes more complete the set of ESM for non-self-adjoint Schrödinger operators.

2 Formalism

To add imaginary value to the energy of chosen state we shall deal with SUSYQ approach, see, papers [2, 12, 20, 21]. However, we perform SUSYQ transformation twice at different (and complex) energies. In the first step, the SUSYQ procedure results in deletion of state at initial energy level. The second transformation is made at the energy level whose real part coincides with initial energy value whereas the imaginary value is added. As a result a chosen state energy is shifted in a complex ”direction”, at that all the remaining energy levels remain unaltered. This is extension of the results on only creation (or deletion) of complex energy eigenvalue (SUSYQ transformation of non-Hermitian operators).

In the SUSYQ approach the initial Hamiltonian $H_0$, the second-order differential operator, is factorized into first order operators

$$A^\pm = \pm \partial + W(x),$$  
(1)

where $\partial$ is a symbol of the derivative :

$$H_0 = A^+ A^- + \mathcal{E}. \tag{2}$$

Symbol $\mathcal{E}$ denotes a constant factorization energy and superpotential $W(x)$ is determined by the equation

$$A^- \psi_0(x, \mathcal{E}) = \{-\partial + W(x)\} \psi_0(x, \mathcal{E}) = 0. \tag{3}$$

Here $\psi_0(x, \mathcal{E})$ is the solution of Schrödinger equation with the initial potential $V_0(x)$ at the factorization energy $\mathcal{E}$. We have from Eqs. (1) and (3)

$$W(x) = \psi'_0(x, \mathcal{E}) \psi_0(x, \mathcal{E})^{-1}. \tag{4}$$

The SUSYQ transformation consists in permutation of $A^\pm$:

$$H_0 = A^+ A^- + \mathcal{E} \to H_1 = A^- A^+ + \mathcal{E}; \tag{5}$$

$$V_1(x) = V_0(x) - 2W'(x). \tag{6}$$

There is a simple relation between the solutions $\psi_0(x, E)$ and $\psi_1(x, E)$ of the Schrödinger equation at arbitrary energy $E$ with the initial and new Hamiltonians $H_0, H_1$

$$\psi_1(x, E) = A^- \psi_0(x, E) = \psi_0(x, \mathcal{E})^{-1} \theta_E(x),$$

$$\theta_E(x) = \psi'_0(x, \mathcal{E}) \psi_0(x, E) - \psi_0(x, \mathcal{E}) \psi'_0(x, E), \tag{7}$$
where we used Eq. (4). New solution at $\mathcal{E}$ is

$$
\psi_1(x, \mathcal{E}) = \psi_0(x, \mathcal{E})^{-1}.
$$

The final results of this transformation depend on the choice of $\mathcal{E}$ and $\psi_0(x, \mathcal{E})$.

Let us perform SUSYQ transformation once more, only as an initial system we shall take already the transformed one. New factorization energy is $\mathcal{E}' = \mathcal{E} + i\Gamma$.

As an analog of $\psi_0(x, \mathcal{E})$ we must take linearly independent counterpart of $\psi_1(x, \mathcal{E})$ but at the energy shifted to complex plane

$$
\tilde{\psi}_1(x, \mathcal{E}) = A^{-1}\tilde{\psi}_0(x, \mathcal{E}).
$$

Here $\tilde{\psi}_0(x, \mathcal{E})$ stands for the solution which is obtained from linearly independent counterpart of $\psi_0(x, \mathcal{E})$ by adding imaginary value to $\mathcal{E}$. The second step gives the transformed solution at $\mathcal{E}'$ as follows

$$
\psi_2(x, \mathcal{E}') = \tilde{\psi}_1(x, \mathcal{E})^{-1} = \psi_0(x, \mathcal{E})\theta_{\mathcal{E}}(x)^{-1},
$$

where the last equality follows from Eq. (9) and $\theta_{\mathcal{E}}(x)$ is given by

$$
\theta_{\mathcal{E}}(x) = \psi_0'(x, \mathcal{E})\tilde{\psi}_0(x, \mathcal{E}) - \psi_0(x, \mathcal{E})\tilde{\psi}_0'(x, \mathcal{E}).
$$

For our scheme to work the modulus $|\theta_{\mathcal{E}}(x)|$ must be non-zero throughout the range of solution definition, otherwise singularities will appear. Relative to this question, the following reasoning can be adduced. Since $\theta_{\mathcal{E}}(x) = \text{const}$ if $\Gamma = 0$, it is clear that, by continuity, $|\theta_{\mathcal{E}}(x)| > 0$ for $\Gamma$ lying in some neighbourhood of zero. On the other hand, vanishing of $|\theta_{\mathcal{E}}(x)|$ at any fixed $x$ means that logarithmic derivatives of $\psi_0(x, \mathcal{E})$ and $\tilde{\psi}_0(x, \mathcal{E})$ coincide at this point. But this is hardly feasible in practice because the only free parameter we have is $\Gamma$, whereas we must obey two independent requirements. First, we must put imaginary part of $\tilde{\psi}_0'(x, \mathcal{E})/\tilde{\psi}_0(x, \mathcal{E})$ to be zero and, second, equate the real part to $\psi_0'(x, \mathcal{E})/\psi_0(x, \mathcal{E})$. In principle, however, this does not exclude that $|\theta_{\mathcal{E}}(x)|$ might accidentally vanish at some $x$ and $\mathcal{E}$. So, a final solution of the problem depends on computer-based calculations.

The expression for superpotential obtained in the second-step SUSYQ transformation is given by

$$
\tilde{W}(x) = \tilde{\psi}_1'(x, \mathcal{E})\tilde{\psi}_1(x, \mathcal{E})^{-1}
= \{\psi_0(x, \mathcal{E})^{-1}\theta_{\mathcal{E}}(x)\}'\psi_0(x, \mathcal{E})\{\theta_{\mathcal{E}}(x)\}^{-1} = 
-\psi_0'(x, \mathcal{E})\psi_0(x, \mathcal{E})^{-1} + \theta_{\mathcal{E}}'(x)\theta_{\mathcal{E}}(x)^{-1} = 
-\psi_0'(x, \mathcal{E})\psi_0(x, \mathcal{E})^{-1} + (\mathcal{E} - \mathcal{E})\psi_0(x, \mathcal{E})\tilde{\psi}_0(x, \mathcal{E})\theta_{\mathcal{E}}(x)^{-1},
$$

(12)

where the prime denotes the differentiation over the coordinate $x$. Furthermore, we used here a known identity $\theta_{\mathcal{E}}'(x) = (\mathcal{E} - \mathcal{E})\psi_0(x, \mathcal{E})\tilde{\psi}_0(x, \mathcal{E})$. From Eqs. (12), (4) and (8) we get the resulting expression for second-step potential as follows

$$
V_2(x) = V_1(x) - 2\tilde{W}'(x) = V_0(x) - 2(\bar{\mathcal{E}} - \mathcal{E})
\times\{\psi_0(x, \mathcal{E})\tilde{\psi}_0(x, \mathcal{E})\theta_{\mathcal{E}}(x)^{-1}\}',
$$

(13)
By using the identity
\[ \theta_E(x) = (E - \mathcal{E}) \int^x \psi_0(y, \mathcal{E}) \psi_0(y, E) \, dy \]
we can give the expression for unnormalized solution at the arbitrary energy related to the potential \( V_2(x) \):
\[
\psi_2(x, E) = (E - \mathcal{E})^{-1} \left( -\partial + \tilde{W}(x) \right) \psi_0(x, E) \\
\times (E - E)^{-1} \psi_0(x, \mathcal{E}) \psi_0(x, \mathcal{E})^{-1} \theta_E(x) \\
+ \psi_0(x, E) - (E - \mathcal{E})^{-1} \psi_0^\prime(x, \mathcal{E}) \psi_0(x, \mathcal{E})^{-2} \\
\times \theta_E(x) + (E - \mathcal{E})^{-1} \tilde{\psi}_0(x, \mathcal{E}) \theta_E(x) \tilde{\psi}_0(x, \mathcal{E})^{-1} \\
\times \theta_E(x) \tilde{\psi}_0(x, \mathcal{E})^{-1} = \psi_0(x, E) - \tilde{\psi}_0(x, \mathcal{E}) \theta_E(x) \tilde{\psi}_0(x, \mathcal{E})^{-1} \\
\times \int^x \psi_0(y, \mathcal{E}) \psi_0(y, E) \, dy, \tag{14}
\]

The results below are based on these formulae for both point and continuous spectrum.

3 Bound states

The appearance of imaginary parts of \( V \) and \( E \) violates the hermiticity (norm or flux are not conserved) which can be interpreted as an effective coupling with "hidden channels" as in the optical model. The wide manifold of ESM with complete set of spectral parameters allows to control absorption in the constructed quantum systems.

The complex energy shifts require consideration of a system of two coupled Schrödinger equations for \( \text{Re} \Psi(x) \) and \( \text{Im} \Psi(x) \) instead of ordinary one. A one-dimensional example of the system is given below.

\[
- \text{Re} \psi''(x) = (\text{Re} E - \text{Re} V(x)) \text{Re} \psi(x) - \\
- (\text{Im} E - \text{Im} V(x)) \text{Im} \psi(x); \\
- \text{Im} \psi''(x) = (\text{Re} E - \text{Re} V(x)) \text{Im} \psi(x) + \\
+ (\text{Im} E - \text{Im} V(x)) \text{Re} \psi(x). \tag{15}
\]

It was a surprise to reveal that the algorithms of qualitative prediction of potential form for eigenvalue imaginary shift are sometimes the same as for real energy shifts, although equations become much more complicated. It is evident that an addition of a constant imaginary potential produces a parallel imaginary shift of all spectral points. But for imaginary shift of a single spectral point leaving energies of all other states unaltered (elementary transformation) we need to use special shape of potential perturbation. One should take into account the different sensitivity to potential variations in space of the complete set of orthogonal states. In fact, let us consider imaginary shift of ground state in different potentials. The potential
perturbations shapes for the energy shifts $\Delta E = -i$ and $\Delta E = -1$ for the ground state of soliton-like potential are demonstrated in figure 1. It is interesting that for the oscillator potential corresponding picture turns out to be very similar. The ground states are most sensitive to potential perturbations in the middle where the probability distribution has maximum. To compensate the influence of the wells on the remaining spectrum, additional potential bumps on both sides are needed [13]. Particularly, the perturbed soliton-like potential remains reflectionless for waves in continuum spectrum. But the fluxes of incoming and outgoing waves become different. This gives possibility to control degree of the wave absorption.

Only the states shifted into complex $E$-plane become quasi-bound because for them the time dependent factor $\exp(-iEt)$ with negative imaginary part in $E$ is exponentially decreasing in time. The ordinary Gamov decaying states have exponential growth at resonance points of scattering matrix in complex energy plane. Unlike this, the wave functions of quasi-bound states obtained by imaginary shifts of eigenvalues are usually \textit{quadratically integrable} [20]. The absolute values of all other states remain time independent.

It is interesting that changing the sign of imaginary energy shift $\Delta E = i$ results in sign inversion of $\text{Im} \Delta V(x)$. But in the case of real shifts $\Delta E = \pm 1$ upward or downward there is no such exact coincidence of $|\Delta V(x)|$. It is because the real shift of the ground state makes spectrum more dense or rarifies it, respectively. Due to the coupling of equations (15) there appears comparatively small $\text{Re} \Delta V(x)$ beside the main perturbation $\text{Im} \Delta V(x)$.

The form of $\Delta \text{Re} V(x)$ perturbation of the first and the second (third and fourth etc.) energy levels for imaginary shifts upwards or downwards appeared to be of the same qualitative gross-structure (see Fig.2) although the shapes of states differ very much from one another. For a long time we could not explain this fact. Surprisingly, it turns out that the simple rule we have found for shifts of configurational space-localization of bound states without moving energy levels is also valid here. The potential barrier-well (well-barrier) block pushes to the right (to the left) the corresponding wave function bump if their spatial intervals coincide. Each of two first states can be considered as having two bumps. In fact, we can conceive the left and the right parts of the ground state as two ”bumps” (halves). The shapes of $\Delta \text{Re} V(x)$ in figure 2 consist of two blocks. The left one (barrier-well) deflects the left side of the ground state to the right and the right block (well-barrier) pushes the right side of the state to the left.

We can consider the imaginary shift of the lowest level as making the energy spectrum something less dense, compare with [11]. So it is naturally to suppose that it will be followed by spatial compressing ground state to the center in contrast to widening when making spectrum more dense [8].

For real energy shifts we have found the phenomenon of fission of states and moving their parts away from one another when their energy levels approach one another (gradual degeneration). The exact degeneracy leads to effective ”annihilation” of states [8]. We have found that the same phenomenon occurs for two states with the neighbour complex eigenvalues. It is illustrated in figures 3, 4 obtained by using one-step SUSYQ transformation, Eqs. (6) - (8).

This enriches our quantum intuition: if we squeeze space localization of states, they become less dense on the energy scale and vice versa. The more broad is a
potential well, the denser becomes the energy spectrum.

It is surprising that for the infinite initial rectangular well strong potential perturbation slightly change the probability distribution for the shifted state (small difference between absolute values of initial and resulting wave functions). It is despite that the real and imaginary parts of the wave function are drastically changed. Also for all other bound states absolute values of eigenfunctions are approximately conserved under transformation.

In the case of IP approach the imaginary shift of a ground state energy level \( E \) in initial infinite rectangular well to \( E - i \) violates boundary conditions.

Let us now discuss the complex transformations preserving all real physical eigenvalues. Again we shall consider the infinite rectangular potential model. The complete spectrum of physical states corresponds to zero boundary conditions (\( \psi = 0 \) at both walls). We can also consider auxiliary ”non-physical” eigenvalue problem with boundary conditions: \( \frac{d}{dx} \psi = 0 \) and \( \psi = 0 \) at different boundaries. If now we add to one non-physical eigenvalue some imaginary part we shall get by using the SUSYQ formalism a complex potential. There is a theorem of two real spectra (see [1] p. 401, [4] p. 34 and references therein) which have analog for complex-valued eigenvalues. Daskalov suggested algorithms of ESM construction when the control levers are two-spectra parameters which determine potential uniquely in contrast to IP approach where the complete spectral data are eigenvalues and spectral weights (reduced widths). It is possible to vary one eigenvalue keeping all other unchanged (in this case spectral weights are not conserved). So we can get real physical spectrum identical to one of the initial eigenvalue problem but with real potential transformed into complex one. We have checked the biorthogonality of complex eigenfunctions which plays a part of usual orthogonality for self-adjoint operators [18]. There is also a corresponding analog of the completeness relation.

To have a notion about possible shapes of corresponding potential and function transformations, it is instructive to consider, as an especially simple example, one half of a potential with right-left symmetry according to its middle point at the origin. In this case the physical spectrum of the whole potential is a sum of two spectra for the half of potential corresponding to two different boundary conditions at the origin \( (\frac{d}{dx} \psi|_{x=0} = 0 \) or \( \psi|_{x=0} = 0 \). Then the known rules of transformations for shifting odd and even energy levels in the whole potential are simultaneously valid for variations of chosen levels in one of two spectra.

4 Transformation of scattering states

The imaginary shift \( (\mathcal{E} > 0 \to \mathcal{E} + i\Gamma) \) of energy value in the continuous spectrum of free motion by using double SUSYQ transformation leads to quasi-periodic (asymptotically periodic) complex potential perturbation as shown in figure 5. The infinite number of the potential bumps and wells correspond to the infinite bumps of the initial state \( (\sin(kx)) \). There appears quadratically integrable decaying state at \( E + i\Gamma \) drawn by dotted line. A simple explanation of this fact is an open problem. The solutions on the real \( E \)-axis become quasi-Bloch waves. It is interesting that the continuous spectrum remains without lacunas (where solutions diverge) except for a single point \( E = \mathcal{E} \). This point can be considered as a collapsed ’forbidden”
zone of zero width, the pinned-out point (POP). Simultaneously this is the junction point of boundaries of allowed zones. The solution at this point is demonstrated in figure 6. At the neighbour real energy points the solution has beats (figure 7).

For shift \( \mathcal{E} \rightarrow \mathcal{E} - i\Gamma \) in IP-approach we have another result: there appears localized (asymptotically vanishing) complex potential. Waves at real energies below and above POP are reflectionless but with different fluxes on both sides of the potential perturbation. The parameter \( \Gamma \) can serve as a control lever of the wave absorption.

Energy dependence of \( \Delta I(E) = I(x, E)|_{x\rightarrow\infty} - I(x, E)|_{x\rightarrow-\infty} \) (non-conservation of flux \( I \)) has a resonance character. The position of the resonance peak coincides with the real part of the quasi-bound state’s eigenvalue \( \mathcal{E} \), which resembles the case of ordinary resonances.

It would be interesting to consider a possibility to use complete sets of eigenstates related to non-Hermitian operators for R-matrix discrete parametrization of scattering data (open problem).

It turns out that purely periodical potentials can be created by using one-step SUSYQ transformation of free motion system (whole line), Eqs. (6) – (8). As an initial solution at factorisation energy \( \mathcal{E} > 0 \) we take \( \psi_0(x, \mathcal{E}) = \exp(-i\sqrt{\mathcal{E}}x) + c \exp(i\sqrt{\mathcal{E}}x) \). Potential and wave function for \( c = 2 \) and \( \mathcal{E} = 0.5 \) are shown in figures 8 and 9.

## 5 Conclusions

The 100 years old good quantum mechanics acquires additional attractiveness with its visualization. This gives a deeper insight into wave objects (generalization of spectral design). But there still remain open problems, for example, the extention of the qualitative theory to multichannel systems and revealing a lot of physical effects, which can be expected due to more of degrees of freedom.

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Figure Captions

Figure 1. Comparison of real $\Delta V(x)$ (in solid) shifting down the ground state energy level ($\Delta E = -1$) in the soliton-like initial potential well and imaginary part (dashed line) of potential perturbation $Im\Delta V(x)$ (for the same initial potential) shifting the initial ground state energy level ($\Delta E = i\Gamma = -i$) into complex energy plane.

Figure 2. The remarkable qualitative similarity of $\Delta ReV(x)$ shapes for the shifts of the first and second levels in infinite rectangular well: (curve 1) $E_1 = 1 \to 1 - i$; (curve 2) $E_2 = 4 \to 4 - i$). It is ”paradoxical” taking into account that the wave functions of the shifted states are quite different.

Figure 3. Real part (solid line) and imaginary part (dashed line) of the transformed soliton-like potential when creating a new state with the same real part of eigenvalue $E = -1 - i$ as for the neighbour stationary state $E = -1$. Absolute values of the wave functions for both states (dotted line) are splitted in two parts and concentrated mainly inside the deep separated potential wells. This resembles the tendency to effective ”annihilation” while gradual degeneracy of the energy levels in the case of real energy shifts [8].

Figure 4. Real part of the transformed oscillator potential when creating a new state with the same real part of eigenvalue $E = 1 - 0.1i$ (solid line) and $E = 1 - 0.0001i$ (dashed line) as for the neighbour stationary state $E = -1$. Wave functions for both states are splitted as in figure 3 and localized in the narrow potential wells.

Figure 5. a) Asymptotically quasi-periodic potential (solid line, only real part is shown) for the energy value $\mathcal{E} = 1$ of the initial continuous spectrum shifted to $\mathcal{E} = 1 - i$. Absolute value of wave function which became quadratically integrable as a result of this SUSYQ transformation is shown by dashed line.

Figure 6. Real part of the wave function at the pinned out point (POP) $\mathcal{E} = 1$ of the continuum spectrum has a characteristic behaviour of a solution on the boundary of spectral zones with linear increase of amplitude (imaginary part has an analogous shape). In this specific limiting case the upper and the lower boundaries of two allowed zones coincide: the width of the forbidden zone is squeezed to zero.

Figure 7. Beats of the solution at the real energy $\mathcal{E} + 1.1$ near the POP $\mathcal{E} = 1$, see figures 5. The frequency of these beats decreases to zero when the energy approaches POP $\mathcal{E}$, see figure 6.

Figure 8. Real (solid line) and imaginary (dashed line) parts of periodic potential obtained by one-step SUSYQ transformation of free motion system at the point $\mathcal{E} = 0.5$.

Figure 9. Real (solid line) and imaginary (dashed line) parts of periodic wave function at the energy $\mathcal{E} = 0.5$, see figure 8.
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Figure 2 Chabanov V M  Phys Rev A
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