Irreducible representations of wedge products of table algebras and applications to association schemes

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Abstract

In this paper we first determine all irreducible representations of a wedge product of two table algebras in terms of the irreducible representations of two factors involved. Then we give some necessary and sufficient conditions for a table algebra to be a wedge product of two table algebras. Some applications to association schemes are also given.

Key words: table algebra, association scheme, wedge product, representation, character.

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1 Introduction

One of the important ways to construct the new association schemes from old ones, is the wedge product of association schemes. This product is a generalization of the wreath product of association schemes; see [5]. Recently, in [3] the wedge product of table algebras as a generalization of the wreath product of table algebras has been also given and some properties of this product have been presented. Moreover, some applications to association schemes have been studied.

Since the representation theory is a valuable tool for the study of table algebras and association schemes, it is natural to ask what we can say about the representations of the wedge products of table algebras and association schemes in terms of the representations of two factors involved. In this paper we consider the wedge product of two table algebras $(C, D)$ and $(A, B)$ and determine all its irreducible representations in terms of irreducible representations of $(C, D)$ and $(A, B)$. This enables us to give some necessary and sufficient conditions for a table algebra to be a wedge product of two table algebras. Some applications to association schemes are also given.
2 Preliminaries

In this section, we state some necessary definitions and known results about table algebras and association schemes. Throughout this paper, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}^+$ denote the complex numbers, the real numbers and the positive real numbers, respectively.

2.1 Table algebras

We follow from [1] for the definition of table algebras. Hence we deal with table algebra as the following:

**Definition 2.1.** A table algebra $(A, B)$ is a finite dimensional associative algebra $A$ over the complex field $\mathbb{C}$ and a distinguished basis $B = \{b_1 = 1_A, \ldots, b_d\}$ for $A$, where $1_A$ is the identity element of $A$, such that the following properties hold:

(I) The structure constants of $B$ are nonnegative real numbers, i.e., for $a, b \in B$:

$$ab = \sum_{c \in B} \lambda_{abc} c, \quad \lambda_{abc} \in \mathbb{R}^+ \cup \{0\}.$$

(II) There is a semilinear involutory anti-automorphism (denoted by $^*$) of $A$ such that $B^* = B$.

(III) For all $a, b \in B$, $\lambda_{ab1} = 0$ if $b \neq a^*$; and $\lambda_{aa^*1} > 0$.

**Remark 2.2.**

(i) Let $(A, B)$ be a table algebra. Then [1, Theorem 3.11] implies that $A$ is semisimple.

(ii) For any table algebra $(A, B)$, there is a unique algebra homomorphism $|.| : A \to \mathbb{C}$, called the degree map, such that $|b| = |b^*| > 0$ for all $b \in B$ (see [1, Theorem 3.14]).

(iii) If for all $b \in B$, $|b| = \lambda_{bb^*1_A}$, then the table algebra $(A, B)$ is called the standard table algebra.

(iv) Let $(A, B)$ be a table algebra. Let $B' = \{\lambda_b b \mid b \in B\}$, where $\lambda_{1_A} = 1$ and $\lambda_b = \lambda_{b^*} \in \mathbb{R}^+$ for all $b \in B$. Then $(A, B')$ is also a table algebra which is called a rescaling of $(A, B)$ (see [1, Section 3]). If $(A, B')$ is a rescaling of $(A, B)$, then we may simply say that $B'$ is a rescaling of $B$. From [1, Theorem 3.15] one can see that any table algebra can be rescaled to a standard table algebra.

Throughout the paper, we focus on the standard table algebras, although our main results are valid for an arbitrary table algebra.

Let $(A, B)$ be a table algebra. The value $|b|$ is called the degree of the basis element $b$. For an arbitrary element $\sum_{b \in B} x_b b \in A$, we have $|\sum_{b \in B} x_b b| = |\sum_{b \in B} x_b|$. For each $a = \sum_{b \in B} x_b b$, we set $a^* = \sum_{b \in B} \overline{x_b} b^*$, where $\overline{x_b}$ means the complex conjugate of $x_b$. 

For any $x = \sum_{b \in B} x_b b \in A$ we denote by $\text{Supp}(x)$ the set of all basis elements $b \in B$ such that $x_b \neq 0$. If $N, M$ are nonempty closed subsets of $B$, then we set

$$NM = \bigcup_{b \in N, c \in M} \text{Supp}(bc).$$

The set $NM$ is called the complex product of closed subsets $N$ and $M$. If one of the factors in a complex product consists of a single element $b$, then one usually writes $b$ for $\{b\}$. A nonempty subset $N \subseteq B$ is called a closed subset, denoted by $N \leq B$, if $N^* N \subseteq N$, where $N^* = \{b^*|b \in N\}$. If $N$ is a closed subset of $B$, then $(CN, N)$, where $CN$ is the $C$-space spanned by $N$, is a table algebra with respect to the restriction of the automorphism $\ast$.

Let $(A, B)$ be a table algebra. For every closed subset $N$ of $B$, the order of $N$, $o(N)$, is defined by $o(N) = \sum_{b \in N} |b|$, and $C^+$ is defined by $N^+ = \sum_{b \in N} b$.

Let $(A, B)$ be a table algebra. If $N$ is a closed subset of $B$ such that for any $b \in B$, $bN = Nb$, then $N$ is called a normal closed subset of $B$.

Let $(A, B)$ be a table algebra and $N$ be a closed subset of $B$. It follows from [1, Proposition 4.7] that $\{NbN | b \in B\}$ is a partition of $B$. A subset $NbN$ is called a double coset with respect to the closed subset $N$.

**Theorem 2.3.** ([1, Theorem 4.9].) Let $(A, B)$ be a table algebra and let $N$ be a closed subset of $B$. Suppose that $\{b_1 = 1_A, \ldots, b_k\}$ is a complete set of representatives of double cosets with respect to $N$. Then the vector space spanned by the elements $b_i/N, 1 \leq i \leq k$, where

$$b_i/N := o(N)^{-1}(NbN)^+ = o(N)^{-1} \sum_{x \in NbN} x,$$

is a table algebra (which is denoted by $A/N$) with a distinguished basis $B/N = \{b_i/N | 1 \leq i \leq k\}$. The structure constants of this algebra are given by the following formula:

$$\gamma_{ijk} = o(N)^{-1} \sum_{r \in Nb_iN, s \in Nb_jN} \lambda_{rst},$$

where $t \in Nb_kN$ is an arbitrary element.

The table algebra $(A/N, B/N)$ is called the quotient table algebra of $(A, B)$ modulo $N$.

Let $(A, B)$ and $(C, D)$ be two table algebras. A map $\varphi : A \to C$ is called the table algebra homomorphism of $(A, B)$ into $(C, D)$ if

(i) $\varphi : A \to C$ is an algebra homomorphism; and

(ii) $\varphi(B) := \{\varphi(b)|b \in B\}$ consists of positive scalar multiples of elements $D$.

A table algebra homomorphism is called a monomorphism (epimorphism, isomorphism, resp.) if it is injective (surjective, bijective, resp.). Two table algebras $(A, B)$ and $(C, D)$ are called isomorphic, denoted by $(A, B) \cong (C, D)$ or simply $B \cong D$, if there exists a table algebra isomorphism $\varphi : (A, B) \to (C, D)$. 

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Example 2.4. Let \((A, B)\) and \((C, D)\) be two table algebras. Define \(\varphi : (A, B) \to (C, D)\) such that \(\varphi(b) = |b|1_C\). Then \(\varphi\) is a table algebra homomorphism, and is called the trivial table algebra homomorphism.

Example 2.5. Let \((A, B)\) be a table algebra and \(N\) be a normal closed subset of \(B\). It follows from \([7, \text{Theorem 2.1}]\) that, there is a table algebra epimorphism \(\pi : (A, B) \to (A/N, B/N)\) such that

\[
\pi(b) = \frac{|b|}{|b/N|}(b/N), \quad \forall b \in B.
\]

The table algebra epimorphism \(\pi\) is called the canonical epimorphism from \((A, B)\) to \((A/N, B/N)\).

Let \((A, B)\) and \((C, D)\) be two table algebras and \(\varphi : (A, B) \to (C, D)\) be a table algebra homomorphism. Then the set \(\varphi^{-1}(1_A)\) is called the kernel of \(\varphi\) in \(B\) and is denoted by \(\ker_B \varphi\). It follows from \([8, \text{Proposition 3.5}]\) that \(\ker_B \varphi\) is a normal closed subset of \(B\), and \(\varphi\) is injective if and only if \(\ker_B \varphi = \{1_A\}\).

Moreover, \([8, \text{Theorem 4.1}]\) shows that if \(\varphi : (A, B) \to (C, D)\) is a table algebra homomorphism, then \(B/\ker_B \varphi \cong \varphi(B)\).

Let \((A, B)\) be a table algebra and let \(V\) be an \(A\)-module. The kernel of \(V\) in \(B\) is defined by

\[
\ker_B V = \{b \in B \mid bx = |b|x, \forall x \in V\}.
\]

We can see that \(\ker_B V\) is a closed subset in \(B\), and if \(\chi\) is the character of \(A\) afforded by the \(A\)-module \(V\), then \(\ker_B B = \ker_B \chi\), where \(\ker_B \chi = \{b \in B \mid \chi(b) = |b|\chi(1)\}\). Let \(N\) be a normal closed subset of \(B\). If we denote the set of all irreducible characters of \(A\) by \(\text{Irr}(B)\) and the set of irreducible characters of \(A/N\) by \(\text{Irr}(B/N)\), then it follows from \([7, \text{Theorem 3.6}]\) that \(\text{Irr}(B/N) = \{\chi \in \text{Irr}(B) \mid N \subseteq \ker_B \chi\}\).

Let \((A, B)\) be a table algebra. Define a linear function \(\zeta\) on \(A\) by \(\zeta(b) = \delta_{b,1_A} o(B)\), for every \(b \in B\). Then \(\zeta\) is a non-degenerate feasible trace on \(A\), and it follows from \([6]\) that

\[
\zeta = \sum_{\chi \in \text{Irr}(B)} \zeta_\chi \chi,
\]

where \(\zeta_\chi \in \mathbb{C}\) and all \(\zeta_\chi\) are nonzero. The feasible trace \(\zeta\) is called the standard feasible trace and \(\zeta_\chi\) is called the standard feasible multiplicity of the character \(\chi\).

2.2 Association schemes

Here we state some necessary definitions and notations for the association schemes.

Definition 2.6. Let \(X\) be a finite set and \(G\) be a partition of \(X \times X\). Then the pair \((X, G)\) is called an association scheme on \(X\) if the following properties hold:

(I) \(1_X \in G\), where \(1_X := \{(x, x) \mid x \in X\}\).

(II) For every \(g \in G\), \(g^*\) is also in \(G\), where \(g^* := \{(x, y) \mid (y, x) \in g\}\).
(III) For every $g, h, k \in G$, there exists a nonnegative integer $\lambda_{ghk}$ such that for every $(x, y) \in k$, there exist exactly $\lambda_{ghk}$ elements $z \in X$ with $(x, z) \in g$ and $(z, y) \in h$.

Let $(X, G)$ be an association scheme. For each $g \in G$, we call $n_g = \lambda_{gg^{1}x}$ the valency of $g$. For any nonempty subset $H$ of $G$, put $n_H = \sum_{h \in H} n_h$. Clearly $n_G = |X|$. For every $g \in G$, let $A(g)$ be the adjacency matrix of $g$. For every nonempty subset $H$ of $G$, put $A(H) := \{A(h) | h \in H\}$ and let $\mathbb{C}[H]$ denote the $\mathbb{C}$-space spanned by $A(H)$. It is known that

$$\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}A(g),$$

the complex adjacency algebra of $(X, G)$, is a semisimple algebra (see [9, Theorem 4.1.3]). The set of irreducible characters of $G$ is denoted by $\text{Irr}(G)$. The feasible trace $\zeta$ for table algebra $(\mathbb{C}[G], A(G))$ is called the standard character of $(X, G)$ and $\zeta_{\chi}$ is called the multiplicity of character $\chi$ and is denoted by $m_{\chi}$.

Let $(X, G)$ be an association scheme. A nonempty subset $H$ of $G$ is called a closed subset of $G$ if $A(H)$ is a closed subset of $\mathbb{C}[G]$. If $H$ is a closed subset of $G$, then $(\mathbb{C}[H], A(H))$ is a table algebra.

Let $H$ be a closed subset of $G$. For every $h \in H$ and every $x \in X$, we define $xh = \{y \in X | (x, y) \in h\}$. Put $X/H = \{xH | x \in X\}$, where $xH = \bigcup_{h \in H} xh$ and

$$gH = \{(xH, yH) | y \in xHgH\}.$$ Then $(X/H, G/H)$ is an association scheme, called the quotient scheme of $(X, G)$ over $H$. For $x \in X$, the subscheme $(X, G)_{xH}$ induced by $xH$, is an association scheme $(xH, H_{xH})$ where $H_{xH} = \{h_{xH} | h \in H\}$ and $h_{xH} = h \cap xH \times xH$.

Let $(X, G)$ and $(Y, S)$ be two association schemes. A scheme epimorphism is a mapping $\varphi : (X, G) \rightarrow (Y, S)$ such that

(i) $\varphi(X) = Y$ and $\varphi(G) = S$,

(ii) for every $x, y \in X$ and $g \in G$ with $(x, y) \in g$, $(\varphi(x), \varphi(y)) \in \varphi(g)$.

Let $\varphi : (X, G) \rightarrow (Y, S)$ be a scheme epimorphism. The kernel of $\varphi$ is defined by

$$\ker \varphi = \{g \in G | \varphi(g) = 1_Y\}.$$ It is known that $A(\ker \varphi)$ is a closed subset of $A(G)$. If $A(\ker \varphi) \subseteq A(G)$, then the scheme epimorphism $\varphi$ is called the normal scheme epimorphism. A scheme epimorphism with a trivial kernel is called a scheme isomorphism.

An algebraic isomorphism between two association schemes $(X, G)$ and $(Y, S)$ is a bijection $\theta : G \rightarrow S$ such that it preserves the structure constants, that is $\lambda_{ghl} = \lambda_{\theta(g)\theta(h)\theta(l)}$, for every $g, h, l \in G$. It is known that if $\varphi : (X, G) \rightarrow (Y, S)$ is a scheme isomorphism, then $\varphi$ induces an algebraic isomorphism between $G$ and $S$.

2.3 A wedge product of table algebras

Here we have a look at the wedge product of table algebras. We refer the reader to [3], for more details.
Let \((A, B)\) be a table algebra and \(N\) be a closed subset of \(B\). Suppose that \((C, D)\) is a table algebra and
\[
\varphi : (C, D) \to (\mathbb{C}N, N)
\]
is a table algebra epimorphism. Put \(K = \ker_D \varphi = \{d \in D | \text{Supp}(\varphi(d)) = 1_A\}\). Then \(K \subseteq D\) and
\[
(C/K, D/K) \cong (\mathbb{C}N, N).
\]
(1)

For every \(1_A \neq b \in B\), put \(\bar{b} = o(K)b\). Suppose that \((A, \overline{B})\) is a rescaling of \((A, B)\) where
\[
\overline{B} = \{1_A\} \cup \{\bar{b} | b \in B \setminus \{1_A\}\}.
\]
Put \(X = D \cup \overline{B}\) and let \(\tilde{A}\) be the \(\mathbb{C}\)-space spanned by \(X\). Suppose that \(D = \{d_1, d_2, \ldots, d_n\}\), \(B = \{b_1, b_2, \ldots, b_m\}\) and the sets \(\{\lambda_{xyz} | x, y, z \in B\}\) and \(\{\mu_{xyz} | x, y, z \in D\}\) are the structure constants of \((A, B)\) and \((C, D)\), respectively. We define a multiplication 
\[
\cdot
\]
on the elements of \(X\) as follows:

(i) for every \(d_i, d_j \in D\),
\[
d_i \cdot d_j = \sum_{z=1}^{n} \mu_{d_id_jd_z} d_z,
\]
(ii) for every \(\bar{b}_i, \bar{b}_j \in \overline{B}\),
\[
\bar{b}_i \cdot \bar{b}_j = o(K) \sum_{t=1}^{m} \lambda_{\bar{b}_ib_j} \bar{b}_t,
\]
(iii) for every \(d_i \in D\) with \(\text{Supp}(\varphi(d_i)) = h_i\), and \(\bar{b}_j \in \overline{B}\),
\[
d_i \cdot \bar{b}_j = o(K) \varphi(d_i) \bar{b}_j = \frac{|d_i|}{|h_i|} \sum_{t=1}^{m} \lambda_{h_i b_j} \bar{b}_t,
\]
and similarly,
\[
\bar{b}_j \cdot d_i = o(K) b_j \varphi(d_i) = \frac{|d_i|}{|h_i|} \sum_{t=1}^{m} \lambda_{b_j h_i} \bar{b}_t.
\]

If we extend 
\[
\cdot
\]
linearly to all \(\tilde{A}\), then it defines the structure of an associative \(\mathbb{C}\)-algebra on \(\tilde{A}\).

Suppose that \(d \in D\) such that \(\text{Supp}(\varphi(d)) = h\). Since \(\varphi(K^+) = o(K)\), we have
\[
\varphi(dK^+) = o(K) \varphi(d) = \frac{|d|}{|h|} \bar{h}.
\]
Then for every \(\bar{b} \in \overline{B}\),
\[
(dK^+) \cdot \bar{b} = o(K) \varphi(dK^+) \cdot b = \frac{|d|}{|h|} \bar{h} \cdot \bar{b}.
\]
In particular, 
\[(dK^+) \cdot 1_A = \frac{|d|}{|h|} \bar{h}.
\]
If we identify \( (dK^+) \cdot 1_A \) with \( dK^+ \), then we can assume that \( dK^+ = \frac{|d|}{|h|} \bar{h} \). By this identification we can see that \( \tilde{B} = D \cup (\bar{B} \setminus \bar{N}) \) is a base for the algebra \( \tilde{A} \). Moreover, for every \( \bar{b} \in \bar{B} \setminus \bar{N} \), we have \( 1_D \cdot \bar{b} = o(K) \varphi(1_D) b \). But \( \varphi(1_D) = 1_A \). So \( 1_D \cdot \bar{b} = o(K) 1_A b = \bar{b} \).

Similarly, \( b \cdot 1_D = b \). Hence \( 1_D \in \tilde{B} \) is the identity element of \( \tilde{A} \).

Suppose that \( *_1 \) and \( *_2 \) are semilinear involuntary anti-automorphisms of table algebras \( (A, B) \) and \( (C, D) \), respectively. Then we can define a semilinear involuntary anti-automorphism \( * \) on \( \tilde{B} \) as follows:

(i) for every \( d \in D \), \( d^* := d^{*_2} \),
(ii) for every \( \bar{b} \in \bar{B} \), \( \bar{b}^* := b^{*_1} = o(K) b^{*_1} \).

**Theorem 2.7.** (See [3, Theorem 3.2].) With the notation above, the pair \( (\tilde{A}, \tilde{B}) \) is a table algebra.

The table algebra \( (\tilde{A}, \tilde{B}) \) is called the wedge product of table algebras \( (C, D) \) and \( (A, B) \) relative to \( \varphi \).

**Remark 2.8.** Let \( (\tilde{A}, \tilde{B}) \) be the wedge product of table algebras \( (C, D) \) and \( (A, B) \) relative to \( \varphi \). If \( \ker_D \varphi = D \), then for every \( d \in D \) and every \( x \in \tilde{B} \setminus D \) we have \( xd = |d|x = dx \). So it follows from [4, Definition 1.2] that \( (\tilde{A}, \tilde{B}) \) is a wreath product \( (\tilde{B}, D) \). Thus the wreath product of table algebras is a partial case of the wedge product of table algebras whenever \( \varphi \) is the trivial table algebra homomorphism; see Example 2.4.

**Theorem 2.9.** (See [3, Corollary 3.7].) Let \( (A, B) \) be a table algebra and \( K \leq D \) be the closed subsets of \( B \). Then the following are equivalent:

(i) \( K \leq B \), and for every \( b \in B \setminus D \), \( bK^+ = o(K)b = K^+b \),
(ii) \( (A, B) \) is the wedge product of \( (< D >, D) \) and \( (A/\perp K, B/\perp K) \) relative to the canonical epimorphism \( \pi : (\perp D, D) \rightarrow (\perp D/\perp K, D/\perp K) \).

### 3 Irreducible Representations

Let \( (A, B) \) be a table algebra and \( N \leq B \). Suppose \( (C, D) \) be a table algebra with the table algebra epimorphism \( \varphi : (C, D) \rightarrow (CN, N) \). Put \( K = \ker_D \varphi \) and suppose \( (\tilde{A}, \tilde{B}) \) be the wedge product of \( (C, D) \) and \( (A, B) \) relative to \( \varphi \).

Let \( W \) be an \( A \)-module. Define the scalar multiplication \( \cdot \) on \( W \) as the following:

(i) for every \( \bar{b} \in \tilde{B} \) and \( w \in W \), \( \bar{b} \cdot w = o(K)bw \),
Lemma 3.1. With the notation above, the scalar multiplication “·” defines a structure \( \tilde{A} \)-module on \( W \). In particular, if \( W \) is irreducible as an \( A \)-module, then it is irreducible as an \( \tilde{A} \)-module.

Proof. First note that since for every \( d \in D \) with \( \text{Supp}(\varphi(d)) = h \),

\[
(dK^+) \cdot w = o(K)\varphi(d)w = o(K)\frac{|d|}{|h|}hw = \frac{|d|}{|h|}h \cdot w,
\]

it follows that the scalar multiplication “·” on \( \tilde{B} \) is well-defined. Then we see that for every \( b, c \in B \setminus N \) and \( w \in W \),

\[
(bc) \cdot w = b \cdot (c \cdot w).
\]

Furthermore, for every \( \overline{b} \in \overline{B} \setminus \overline{N} \), \( d \in D \) and \( w \in W \),

\[
(\overline{b}d) \cdot w = \overline{b} \cdot (d \cdot w).
\]

Since for every \( w \in W \),

\[
1_{\overline{A}} \cdot w = 1_D \cdot w = \varphi(1_D)w = 1_Aw = w,
\]

we conclude that \( W \) is an \( \overline{A} \)-module. Now suppose that \( W \) is irreducible as an \( A \)-module. If \( W' \) is a nontrivial \( \overline{A} \)-submodule of \( W \), then for every \( b \in B \setminus N \), \( bW' = o(K)^{-1}bW' \subseteq W' \). Moreover, for every \( h \in N \),

\[
hW' = \frac{|h|}{|d|}\varphi(d)W' \subseteq W',
\]

where \( \text{Supp}(\varphi(d)) = h \). So \( W' \) is a nontrivial \( A \)-submodule, which is a contradiction. So \( W \) is also irreducible as an \( \overline{A} \)-module. \( \blacksquare \)

Now let \( M \) be a \( C \)-module such that \( K \nsubseteq \ker M \). We define the scalar multiplication “·” on \( U \) as the following:

(i) for every \( d \in D \) and \( m \in M \), \( d \cdot m = dm \),

(ii) for every \( \overline{b} \in \overline{B} \) and \( m \in M \), \( \overline{b} \cdot m = 0 \).
Lemma 3.2. With the notation above, the scalar multiplication “·” defines a structure $\tilde{A}$-module on $M$. In particular, if $M$ is an irreducible as a $C$-module, then it is irreducible as an $\tilde{A}$-module.

Proof. First note that since $K \not\subseteq \ker M$, it follows from [7, Corollary 3.8] that $eM = 0$ where $e = o(K)^{-1}K^+$. Then $K^+.m = 0$, for every $m \in M$. So
\[
\overline{h}.m = \frac{|h|}{|d|} (dK^+).m = \frac{|h|}{|d|} d(K^+.m) = 0,
\]
and thus the scalar multiplication “·” on $\tilde{B}$ is well-defined. Now by direct calculations, we have the result.

For every irreducible $A$-module $W$ and every irreducible $C$-module $M$, we use the notations $\overline{W}$ and $\overline{M}$ to denote the modules $W$ and $M$ viewed as $\tilde{A}$-modules.

Theorem 3.3. Let $(\tilde{A}, \tilde{B})$ be the wedge product of table algebras $(C, D)$ and $(A, B)$ relative to $\varphi$. Let $W_1, \ldots, W_n$ be all non-isomorphic irreducible $A$-modules and $M_1, \ldots, M_r$ be all non-isomorphic irreducible $C$-modules such that $K \not\subseteq \ker M_i$. Then $\overline{W}_1, \ldots, \overline{W}_n, \overline{M}_1, \ldots, \overline{M}_r$ are all non-isomorphic irreducible $\tilde{A}$-modules.

Proof. We already know that $\overline{W}_1, \ldots, \overline{W}_n, \overline{M}_1, \ldots, \overline{M}_r$ are the non-isomorphic irreducible $\tilde{A}$-modules from Lemmas 3.1 and 3.2.

On the other hand, 
\[
\dim A = \sum_{i=1}^{n} (\dim W_i)^2
\]
and
\[
\dim C = \sum_{i=1}^{s} (\dim M'_i)^2 + \sum_{i=1}^{r} (\dim M_i)^2,
\]
where $M'_1, \ldots, M'_s$ are all non-isomorphic irreducible $C$-modules where $K \subseteq \ker M'_i$. So
\[
\dim \tilde{A} \geq \sum_{i=1}^{n} (\dim W_i)^2 + \sum_{i=1}^{r} (\dim M_i)^2 = \dim A + \dim C - \sum_{i=1}^{s} (\dim M'_i)^2.
\]
But since $D \parallel K \cong N$, we have
\[
\sum_{i=1}^{s} (\dim M'_i)^2 = \dim(C \parallel K) = \dim(CN) = |N|.
\]
Thus we conclude that
\[
\dim \tilde{A} \geq \dim A + \dim C - \sum_{i=1}^{s} (\dim M'_i)^2 = \dim A + \dim C - \dim(CN) = |B| + |D| - |N| = \dim \tilde{A}.
\]
Hence $$\dim \tilde{A} = \sum_{i=1}^{n} (\dim W_i)^2 + \sum_{i=1}^{r} (\dim M_i)^2$$ and, so $$\overline{W_1}, \ldots, \overline{W_n}, \overline{M_1}, \ldots, \overline{M_r}$$ are all non-isomorphic irreducible $$\tilde{A}$$-modules.

**Corollary 3.4.** Let $$(\tilde{A}, \tilde{B})$$ be the wedge product of table algebras $$(C, D)$$ and $$(A, B)$$ relative to $$\varphi$$. Let $$D_1, \ldots, D_n$$ be all non-equivalent irreducible representations of $$A$$ corresponding to the $$A$$-modules $$W_1, \ldots, W_n$$, and $$T_1, \ldots, T_r$$ be all non-equivalent representations of $$D$$ corresponding to the $$D$$-modules $$M_1, \ldots, M_r$$ such that $$K \not\subseteq \ker M_i$$ for every $$1 \leq i \leq r$$. Then $$\overline{D_1}, \ldots, \overline{D_n}, \overline{T_1}, \ldots, \overline{T_r}$$ are all non-equivalent irreducible representations of $$\tilde{A}$$ such that

$$\begin{cases} \overline{D_i}(b) = o(K)D_i(b), \\ \overline{D_i}(d) = \frac{|d|}{|K|}D_i(h), \end{cases}$$

where $$\text{Supp}(\varphi(d)) = h$$, and

$$\begin{cases} \overline{T_i}(b) = 0, \\ \overline{T_i}(d) = T_i(d). \end{cases}$$

**Proof.** The result follows by Theorem 3.3.

Let $$(A, B)$$ be a table algebra and $$H \leq B$$. For every $$b \in B$$, define

$$\text{St}_H(b) = \{x \in H \mid xb = |x|b = bx\},$$

and for every subset $$U \subseteq B$$, put $$\text{St}_H(U) = \bigcap_{b \in U} \text{St}_H(b)$$. Then as a direct consequence of Theorem 2.9 and Theorem 3.4, we have the following corollary.

**Corollary 3.5.** Let $$(A, B)$$ be a table algebra and $$K \leq D$$ be the closed subsets of $$B$$ such that $$K \not\subseteq B$$ and $$K \subseteq \text{St}_B(B \setminus D)$$. Suppose that $$D_1, \ldots, D_n$$ are all non-equivalent irreducible representations of $$(A/\!/K, B/\!/K)$$ and $$T_1, \ldots, T_r$$ are all non-equivalent irreducible representations of $$(CD, D)$$. Define

$$\begin{cases} \overline{D_i}(b) = o(K)D_i(b/\!/K), & b \in B \setminus D \\ \overline{D_i}(d) = \frac{|d|}{|d/\!/K|}D_i(d/\!/K), & d \in D \end{cases}$$

and

$$\begin{cases} \overline{T_i}(b) = 0, & b \in B \setminus D \\ \overline{T_i}(d) = T_i(d), & d \in D. \end{cases}$$

Then $$\{\overline{D_1}, \ldots, \overline{D_n}, \overline{T_1}, \ldots, \overline{T_r}\}$$ is the set of all non-equivalent irreducible representations of $$(A, B)$$.

**Proof.** It follows from Theorem 2.9 that $$(A, B)$$ is the wedge product of $$(C, D)$$ and $$(A/\!/K, B/\!/K)$$ relative to the canonical epimorphism $$\pi : (CD, D) \to (C/\!/K, D/\!/K)$$. Then the result follows by Theorem 3.4.
4 Irreducible characters

Let \((\tilde{A}, \tilde{B})\) be the wedge product of table algebras \((C, D)\) and \((A, B)\) relative to \(\varphi\). Let \(\text{Irr}(B) = \{\chi_1, \ldots, \chi_m\}\) and \(\text{Irr}(D) \setminus \text{Irr}(D/\!/K) = \{\psi_1, \ldots, \psi_r\}\). Define

\[
\begin{align*}
\overline{\chi_i}(b) &= o(K)\chi_i(b), \\
\overline{\chi_i}(d) &= \frac{|d|}{|\varphi(d)|}\chi_i(h),
\end{align*}
\]

where \(\text{Supp}(\varphi(d)) = h\), and

\[
\begin{align*}
\overline{\psi_i}(b) &= 0, & b &\in B \setminus D \\
\overline{\psi_i}(d) &= \psi_i(d), & d &\in D.
\end{align*}
\]

Then from corollary 3.4, we have the following

**Theorem 4.1.** With the notation above, \(\overline{\chi_1}, \ldots, \overline{\chi_m}, \overline{\psi_1}, \ldots, \overline{\psi_r}\) are all irreducible characters of \((\tilde{A}, \tilde{B})\).

As a direct consequence of Corollary 3.5, we have the following

**Corollary 4.2.** Let \((A, B)\) be a table algebra and \(K \leq D\) be the closed subsets of \(B\) such that \(K \subseteq B\) and \(K \subseteq \text{St}_B(B \setminus D)\). If \(\text{Irr}(B/\!/K) = \{\chi_1, \ldots, \chi_m\}\) and \(\text{Irr}(D) \setminus \text{Irr}(D/\!/K) = \{\psi_1, \ldots, \psi_r\}\), then \(\{\overline{\chi_1}, \ldots, \overline{\chi_m}, \overline{\psi_1}, \ldots, \overline{\psi_r}\}\) is the set of irreducible characters of \((A, B)\) where

\[
\begin{align*}
\overline{\chi_i}(b) &= o(K)\chi_i(b/\!/K), & b &\in B \setminus D \\
\overline{\chi_i}(d) &= \frac{|d|}{|\varphi(d)|}\chi_i(d/\!/K), & d &\in D
\end{align*}
\]

and

\[
\begin{align*}
\overline{\psi_i}(b) &= 0, & b &\in B \setminus D \\
\overline{\psi_i}(d) &= \psi_i(d), & d &\in D.
\end{align*}
\]

In the following we give some necessary and sufficient conditions for a table algebra to be a wedge product of two table algebras.

**Theorem 4.3.** Let \((A, B)\) be a table algebra and \(K \leq D\) be the closed subsets of \(B\). Then the following are equivalent:

(i) \(K \subseteq B\) and \(K \subseteq \text{St}_B(B \setminus D)\),

(ii) \((A, B)\) is the wedge product of \((CD, D)\) and \((A/\!/K, B/\!/K)\) relative to the canonical epimorphism \(\pi : (CD, D) \rightarrow (CD/\!/K, D/\!/K)\),

(iii) for every \(\chi \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K)\), \(\chi_D \in \text{Irr}(D)\) and \(\chi(b) = 0\) for every \(b \in B \setminus D\).

(iv) for every \(\chi \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K)\), \(\chi_D = \psi \in \text{Irr}(D)\) and \(\zeta_\chi = \frac{o(B)}{o(D)}\zeta_\psi\).
The equivalence of \((i)\) and \((ii)\) follows from Theorem 2.9. Moreover, for every \(\chi \in \text{Irr}(B) \setminus \text{Irr}(B/K)\), where \(\chi_D = \psi \in \text{Irr}(D)\), we have

\[
\frac{\psi(1)}{\zeta_\psi} = [\chi_D, \chi_D] \leq \frac{o(B)}{o(D)}[\chi, \chi] = \frac{o(B)}{o(D)} \chi(1) \zeta_\chi
\]

with equality iff \(\chi(b) = 0\) for every \(b \in B \setminus D\); see [2]. So the equivalence of \((iii)\) and \((iv)\) follows.

\((i) \Rightarrow (iii)\) follows directly from Corollary 4.2.

\((iii) \Rightarrow (i)\) Let \(\chi \in \text{Irr}(B) \setminus \text{Irr}(B/K)\) and let \(T\) be the representation of \(A\) such that \(\chi\) is afforded by \(T\). Since \(\chi(b) = 0\) for every \(b \in B \setminus D\), it follows from [4, Corollary 2.3] that \(T(b) = 0\) for every \(b \in B \setminus D\). Now let \(k \in K\) and \(b \in B \setminus D\). Put \(x = kb - |k|b\).

For every irreducible representation \(T : A\!/K \rightarrow \text{Mat}_n(\mathbb{C})\), since \(T(k) = |k|I_n\) where \(I_n\) is the \(n \times n\) identity matrix, it follows that

\[
T(x) = T(kb - |k|b) = T(kb) - |k|T(b) = T(k)T(b) - |k|T(b) = |k|T(b) - |k|T(b) = 0.
\]

Moreover, for every irreducible representation \(T : A \rightarrow \text{Mat}_n(\mathbb{C})\) of \(A\) such that \(K \nsubseteq \ker(T)\), we have \(T(x) = T(kb - |k|b) = T(kb) - |k|T(b) = 0\). Then we conclude that \(x \in J(A) = \{0\}\) and so \(x = 0\). Thus \(kb = |k|b\). Similarly, \(bk = |k|b\). This shows that \(K \subseteq \text{St}_B(B \setminus D)\). Furthermore, since \(K \subseteq D\) and \(kb = |k|b = bk\) for every \(k \in K\) and \(b \in B \setminus D\), it follows that \(K \subseteq B\) and \((i)\) holds.

\[\blacksquare\]

### 5 Applications to association schemes

Let \((X, G)\) be an association scheme and \(D \subseteq G\). Suppose that \(X\!/D = \{X_1, \ldots, X_m\}\). Put \(D_i = D_{X_i}\). Consider the bijection \(\varepsilon_i : D \rightarrow D_i\) such that \(\varepsilon_i(d) = d_{X_i}\). Then \(\varepsilon_j \varepsilon_i^{-1} : D_i \rightarrow D_j\) is an algebraic isomorphism between association schemes \((X_i, D_i)\) and \((X_j, D_j)\). Assume that for every \(i\), there exists an association scheme \((Y_i, B_i)\) and a scheme normal epimorphism \(\psi_i : Y_i \cup B_i \rightarrow X_i \cup D_i\). Moreover, assume that there exist algebraic isomorphisms \(\varphi_i : B_i \rightarrow B_1\) such that \(\psi_i \varphi_i = \varepsilon_i \varepsilon_1^{-1} \psi_1\) for every \(i\).

Assume that \(Y_i, 1 \leq i \leq m\), are pairwise disjoint. Put \(Y = Y_1 \cup \cdots \cup Y_m\), \(\bar{\psi} = \psi_1 \cup \cdots \cup \psi_m\), \(G = \{\bar{g} \mid g \in G\}\), where \(\bar{g} = \psi^{-1}(g)\), and for every \(b \in B_1\), \(\tilde{b} = \cup_{i=1}^m \varphi_i(b)\). Let \(\mathbb{C}[\overline{B_1}]\) be the \(\mathbb{C}\)-space spanned by \(\overline{B_1} = \{\bar{b} \mid b \in B_1\}\). Then it follows from [5, Theorem 2.2] that \(U = \mathbb{C}[\overline{G}] + \mathbb{C}[\overline{B_1}]\) is the adjacency algebra of an association scheme \((Y, \overline{B_1} \cup (\overline{G} \setminus \overline{B_1}))\), which is called the wedge product of \((Y, B_i), 1 \leq i \leq m\), and \((X, G)\).

Now consider the scheme epimorphism \(\psi_1 : (Y_1, B_1) \rightarrow (X_1, D_1)\) and the algebra isomorphism \(\varepsilon_1^{-1} : (\mathbb{C}[D_1], A(D_1)) \rightarrow (\mathbb{C}[D], A(D))\).

Then we can define a table algebra epimorphism

\[
\overline{\psi}_1 : (\mathbb{C}[\overline{B_1}], A(B_1)) \rightarrow (\mathbb{C}[D], A(D))
\]
such that
\[ \overline{\psi_1}(A(b)) = \frac{n_b}{n_{\psi_1}(b)} A(d) \]
where \( \text{Supp}(\overline{\psi_1}(A(b))) = A(d) \). Then we can define a linear map
\[ \varphi : (\mathbb{C}[\widehat{B}_1], A(\widehat{B}_1)) \to (\mathbb{C}[D], A(D)) \]
by \( \varphi(\widehat{A}(b)) = \overline{\psi_1}(A(b)) \).

**Theorem 5.1.** (See [3, Theorem 5.4].) Let \( U = \mathbb{C}[G] + \mathbb{C}[\widehat{B}_1] \) be the complex adjacency algebra of the wedge product of \( (Y_i, B_i), 1 \leq i \leq m, \) and \( (X, G) \). Then the table algebra \( (U, V) \), where \( V = \{ A(x) \mid x \in \widehat{B}_1 \cup (\mathbb{C} \setminus D) \} \), is the wedge product of table algebras \( (\mathbb{C}[\widehat{B}_1], A(\widehat{B}_1)) \) and \( (\mathbb{C}[G], A(G)) \) relative to \( \varphi \), where
\[ \varphi : (\mathbb{C}[\widehat{B}_1], A(\widehat{B}_1)) \to (\mathbb{C}[D], A(D)) \]
such that \( \varphi(\widehat{A}(b)) = \overline{\psi_1}(A(b)) \).

By applying Theorem 5.1 and Corollary 3.4, we can give the following result.

**Theorem 5.2.** Let \( U = \mathbb{C}[G] + \mathbb{C}[\widehat{B}_1] \) be the complex adjacency algebra of the wedge product of \( (Y_i, B_i), 1 \leq i \leq m, \) and \( (X, G) \). Let \( D_1, \ldots, D_n \) be all non-equivalent irreducible representations of \( \mathbb{C}[G] \), and \( T_1, \ldots, T_r \) are all non-equivalent irreducible representations of \( \mathbb{C}[B_1] \) such that \( K \not\subset \ker D_i \), where \( K = \ker \psi_1 \). Then \( D_1, \ldots, D_n, T_1, \ldots, T_r \) such that
\[
\begin{aligned}
D_i(A(\overline{g})) &= n_K D_i(A(\overline{g})), \\
D_i(\overline{b}) &= \frac{n_b}{n_{\overline{\psi_1}}(A(\overline{b}))) D_i(A(d)),
\end{aligned}
\]
where \( \text{Supp}(\overline{\psi_1}(A(b))) = A(d) \), and
\[
\begin{aligned}
T_i(A(\overline{g})) &= 0, \\
T_i(\overline{b}) &= T_i(A(b)),
\end{aligned}
\]
are all non-equivalent irreducible representations of \( U \).

The following corollary is immediate.

**Corollary 5.3.** Let \( U = \mathbb{C}[G] + \mathbb{C}[\widehat{B}_1] \) be the complex adjacency algebra of the wedge product of \( (Y_i, B_i), 1 \leq i \leq m, \) and \( (X, G) \). Let \( \text{Irr}(G) = \{ \chi_1, \ldots, \chi_m \} \) and \( \text{Irr}(B_1) \setminus \text{Irr}(B_1/K) = \{ \psi_1, \ldots, \psi_r \} \). Define
\[
\begin{aligned}
\overline{\chi_i}(A(\overline{g})) &= n_K \chi_i(A(g)), \\
\overline{\chi_i}(\overline{b}) &= \frac{n_b}{n_{\overline{\psi_1}}(A(\overline{b}))) \chi_i(A(d)),
\end{aligned}
\]
where \( \text{Supp}(\overline{\psi_1}(A(b))) = A(d) \), and
\[
\begin{aligned}
\overline{\psi_i}(A(\overline{g})) &= 0, \\
\overline{\psi_i}(\overline{b}) &= \psi_i(A(b)).
\end{aligned}
\]
Then \( \overline{\chi_1}, \ldots, \overline{\chi_m}, \overline{\psi_1}, \ldots, \overline{\psi_r} \) are all irreducible characters of \( U \).
Corollary 5.4. Let \((Y, S)\) be an association scheme and \(K \leq B\) the closed subsets of \(S\) such that

(i) for every \(k \in K\) and every \(s \in S \setminus B\), \(A(s)A(k) = n_kA(s) = A(k)A(s)\),

(ii) \(K \trianglelefteq S\).

If \(\text{Irr}(S//K) = \{\chi_1, \ldots, \chi_n\}\) and \(\text{Irr}(B) \setminus \text{Irr}(B//K) = \{\psi_1, \ldots, \psi_r\}\), then \(\text{Irr}(S) = \{\overline{\chi_1}, \ldots, \overline{\chi_n}, \overline{\psi_1}, \ldots, \overline{\psi_r}\}\) such that

\[
\begin{align*}
\overline{\chi_i}(A(s)) &= n_k\chi_i(A(s^K)), & s & \in S \setminus B \\
\overline{\chi_i}(A(s)) &= \frac{n_s}{n_K}\chi_i(A(s^K)), & s & \in B,
\end{align*}
\]

and

\[
\begin{align*}
\overline{\psi_i}(A(s)) &= 0, & s & \in S \setminus B \\
\overline{\psi_i}(A(s)) &= \psi_i(A(s)), & s & \in B.
\end{align*}
\]

Proof. If \((\mathbb{C}[S], A(S))\) is the complex adjacency algebra of \((Y, S)\), then \(A(K) \trianglelefteq A(S)\) and \(A(K) \subseteq \text{St}_{A(S)}(A(S) \setminus A(D))\). Now by applying Corollary 4.2 we get the result.

As a direct consequence of Theorem 4.3 we can give some necessary and sufficient conditions for an association scheme to be a wedge product of two association schemes.

Theorem 5.5. Let \((Y, S)\) be an association scheme and \(K \trianglelefteq B\) be the closed subsets of \(S\). Then the following are equivalent:

(i) \(K \trianglelefteq S\) and for every \(s \in S \setminus B\), \(A(s)A(K) = n_KA(s) = A(K)A(s)\),

(ii) \(S\) is the wedge product of \((Y, B)_{\psi B}\) and \((Y//K, B//K)\),

(iii) for every \(\chi \in \text{Irr}(S) \setminus \text{Irr}(S//K)\), \(\chi_B \in \text{Irr}(B)\) and \(\chi(A(s)) = 0\) for every \(s \in S \setminus B\),

(iv) for every \(\chi \in \text{Irr}(S) \setminus \text{Irr}(S//K)\), \(\chi_B = \psi \in \text{Irr}(B)\) and \(m_\chi = \frac{n_s}{n_B}m_\psi\).

Proof. Let \((\mathbb{C}[S], A(S))\) be the complex adjacency algebra of \((Y, S)\). Then \(A(K) \trianglelefteq A(B)\) are the closed subsets of \(A(B)\). Now the result follows by Theorem 4.3.

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