Circuit QED: implementation of the three-qubit refined Deutsch–Jozsa quantum algorithm

Qi-Ping Su · Chui-Ping Yang

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Abstract We propose a protocol to construct the 35 \( f \)-controlled phase gates of a three-qubit refined Deutsch–Jozsa (DJ) algorithm, by using single-qubit \( \sigma_z \) gates, two-qubit controlled phase gates, and two-target-qubit controlled phase gates. Using this protocol, we discuss how to implement the three-qubit refined DJ algorithm with superconducting transmon qutrits resonantly coupled to a single cavity. Our numerical calculation shows that implementation of this quantum algorithm is feasible within the present circuit QED technique. The experimental realization of this algorithm would be an important step toward more complex quantum computation in circuit QED.

Keywords Deutsch–Jozsa algorithm · \( f \)-controlled phase gates · Superconducting

1 Introduction

As one of the most promising solid-state candidates for quantum information processing [1–4], the physical system composed of circuit cavities and superconducting qubits is of particular interest. This is because: (i) superconducting qubits and microwave resonators (a.k.a. cavities) can be fabricated using modern integrated circuit technology, and their properties can be characterized and adjusted in situ, (ii) superconducting qubits have relatively long decoherence times [5–8], and (iii) a superconducting microwave cavity or resonator plays a role of quantum bus which can mediate long-range and fast interaction between distant superconducting qubits [9–11]. Moreover, the strong coupling between the cavity field and superconducting qubits, which is difficult to implement with atoms in a microwave cavity, was earlier predicted in theory [1,12] and has been experimentally demonstrated [13,14]. Because of these features,
circuit QED has been widely utilized for quantum information processing. Based on circuit QED, many theoretical proposals have been presented for realizing two-qubit gates [9,15–24] and multiple-qubit gates [25–28] with superconducting qubits. Moreover, experimental demonstration of two-qubit gates [10,11,29,30] and three-qubit gates [31–33] with superconducting qubits in circuit QED has been also reported.

The interest in quantum computation is stimulated by the discovery of quantum algorithms [34,35], which can solve problems of significance much more efficiently than their classical counterparts. Among important quantum algorithms, there exist the Deutsch algorithm [36,37], the Deutsch–Jozsa (DJ) algorithm [38], the Shor algorithm [39], the Simon algorithm [40], the quantum Fourier transform algorithm, and the Grover search algorithm [41]. As is well known, the Deutsch algorithm and the DJ algorithm were the first two that make use of the features of quantum mechanics for quantum computation. Compared with other quantum algorithms, these two algorithms are easy to be implemented and thus have been considered as the natural candidates for demonstrating power of quantum computation.

We note that with superconducting qubits coupled to a circuit cavity, a two-qubit Deutsch–Jozsa quantum algorithm and a two-qubit Grover search quantum algorithm were previously demonstrated in experiments [11]. However, after a thorough investigation, we note that how to implement a three-qubit DJ quantum algorithm with superconducting qubits or qutrits in circuit QED has not been reported in both theoretical and experimental aspects. As is known, the experimental realization of the three-qubit DJ algorithm with a cavity-superconducting-device system is important because it would be an important step toward more complex quantum computation in circuit QED.

In this paper, we propose a protocol to construct the $35 \ f$-controlled phase gates of a three-qubit refined DJ algorithm, by using single-qubit $\sigma_z$ gates, two-qubit CP gates, and two-target-qubit CP gates. It should be noted that a two-target-qubit CP gate consists of two sequential controlled $\sigma_z$ gates, which have a common control qubit but a different target qubit. The protocol is quite general and can be applied to implement the three-qubit DJ algorithm in various physical systems. Based on this protocol, we further discuss how to implement the three-qubit refined DJ algorithm with superconducting transmon qutrits resonantly coupled to a single cavity. Our numerical calculation shows that implementation of this quantum algorithm is feasible within the present circuit QED technique.

The paper is organized as follows. In Sect. 2, we review the refined DJ algorithm. In Sect. 3, we present a protocol to construct the $35 \ f$-controlled phase gates. In Sect. 4, we discuss how to implement this DJ algorithm with superconducting transmon qutrits coupled to a cavity via resonant interaction and analyze the experimental feasibility. A concluding summary is given in Sect. 5.

2 Refined Deutsch–Jozsa algorithm

The DJ algorithm is aimed at distinguishing the constant function from the balanced functions on $2^n$ inputs. The function $f(x)$ takes either 0 or 1. For the constant function, the function values are constant (0 or 1) for all $2^n$ inputs. In contrast, for the balanced
function, the function values are equal to 1 for half of all the possible inputs, and 0 for the other half. Using the DJ algorithm, whether the function is constant or balanced can be determined by only one query. However, a classical algorithm would require $2^n - 1 + 1$ queries to answer the same problem, which grows exponentially with input size.

The refined DJ algorithm was proposed by Collins et al. in 2001 [42], which is illustrated in Fig. 1. This refined DJ algorithm is described below:

(i) Each input query qubit is prepared in the initial state $|0\rangle$.

(ii) Perform a Hadamard transformation $H$ on each qubit, resulting in $|0\rangle \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}$ and $|1\rangle \rightarrow (|0\rangle - |1\rangle)/\sqrt{2})$. As a result, the $n$-qubit initial state $|00...0\rangle$ changes to the state $\frac{1}{2^n} \sum_{x=0}^{2^n-1} |x\rangle$ (denoted as $|\psi_1\rangle$).

(iii) Apply the $f$-controlled phase shift $U_f$, described by

$$|x\rangle \rightarrow (-1)^{f(x)} |x\rangle,$$  \hspace{1cm} (1)

which leads the state $|\psi_1\rangle$ to the state $\frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^f |x\rangle$ (denoted as $|\psi_2\rangle$).

(iv) Perform another Hadamard transformation $H$ on each qubit. As a result, the state $|\psi_2\rangle$ becomes $\frac{1}{2^n} \sum_{z=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{x+z+f(x)} |z\rangle$.

(v) Measure the final state of the $n$ qubits. If the $n$ qubits are measured in the state $|00...0\rangle$, then $f(x)$ is constant. However, if they are measured in other $n$-qubit computational basis states, then $f(x)$ is balanced. This is because the amplitude $a_{|00...0\rangle}$ of the state $|00...0\rangle$ is given by $a_{|00...0\rangle} = \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^f(x)$, which is $\pm 1$ for a constant $f(x)$ while 0 for a balanced $f(x)$.

One can see that when compared with the original DJ algorithm [38], this refined DJ algorithm does not need an auxiliary working qubit. Hence, it requires one qubit fewer than the original DJ algorithm. Consequently, its physical implementation requires one fewer two-state system.

### 3 Protocol for construction of the $f$-controlled phase gates

For a $n$-qubit DJ algorithm, there are a total of $C_{2^n}^{2^n-1}$ balanced functions, among which only $C_{2^n}^{2^n-1}/2$ balanced functions are nontrivial if the symmetry is taken into account. For the three-qubit DJ algorithm, i.e., $n = 3$, there thus exist $C_8^4/2 = 35$ nontrivial balanced functions $U_{f_1}, U_{f_2}, \ldots, U_{f_{35}}$ (see Table 1). In this section, we show how to construct the 35 $f$-controlled phase gates, by using single-qubit $\sigma_z$ gates, two-qubit CP gates, and two-target-qubit CP gates.
A single-qubit $\sigma_z$ gate results in the transformation $\sigma_z |0\rangle = |0\rangle$ while $\sigma_z |1\rangle = -|1\rangle$. A two-qubit CP gate $C_{jk}$ on qubits $j$ and $k$ considered here is described as follows

$$ |mn\rangle_{jk} \rightarrow (-1)^{m \times n} |mn\rangle_{jk}; \quad m, n \in \{0, 1\} $$

which implies that if and only if the control qubit $j$ (the first qubit) is in the state $|1\rangle$, a phase flip happens to the state $|1\rangle$ of the target qubit $k$ (the second qubit), but nothing happens otherwise. In addition, a two-target-qubit CP gate $T_{jkl}$ with the control qubit $j$ and the two-target qubits $k$ and $l$ is defined below

$$ |mnr\rangle_{jkl} \rightarrow (-1)^{m \times n} (-1)^{m \times r} |mnr\rangle_{jkl}; \quad m, n, r \in \{0, 1\} $$

which shows that if and only if the control qubit $j$ (the first qubit) is in the state $|1\rangle$, a phase flip happens to the state $|1\rangle$ of the target qubit $k$ (the second qubit) and the state $|1\rangle$ of the target qubit $l$ (the last qubit).

The construction for each of the 35 $f$-controlled phase gates is listed in Table 2. One can see from Table 2 that the 35 $f$-controlled phase gates $U_{f1}, U_{f2}, \ldots, U_{f35}$ are classified into the following four types: (i) type 1 includes seven $f$-controlled phase gates each constructed with single-qubit $\sigma_z$ gates only; (ii) type 2 contains twelve

| Table 1 | List of the 35 balanced functions for a three-qubit refined Deutsch–Jozsa quantum algorithm |
|---------|-----------------------------------------------------------------------------------|
| $U_{f1} : f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ | $U_{f19} : f(x_1) = f(x_3) = f(x_4) = f(x_8) = 0$ |
| $U_{f2} : f(x_1) = f(x_2) = f(x_3) = f(x_5) = 0$ | $U_{f20} : f(x_1) = f(x_3) = f(x_5) = f(x_6) = 0$ |
| $U_{f3} : f(x_1) = f(x_2) = f(x_3) = f(x_6) = 0$ | $U_{f21} : f(x_1) = f(x_3) = f(x_5) = f(x_7) = 0$ |
| $U_{f4} : f(x_1) = f(x_2) = f(x_3) = f(x_7) = 0$ | $U_{f22} : f(x_1) = f(x_3) = f(x_5) = f(x_8) = 0$ |
| $U_{f5} : f(x_1) = f(x_2) = f(x_3) = f(x_8) = 0$ | $U_{f23} : f(x_1) = f(x_3) = f(x_6) = f(x_7) = 0$ |
| $U_{f6} : f(x_1) = f(x_2) = f(x_4) = f(x_5) = 0$ | $U_{f24} : f(x_1) = f(x_3) = f(x_6) = f(x_8) = 0$ |
| $U_{f7} : f(x_1) = f(x_2) = f(x_4) = f(x_6) = 0$ | $U_{f25} : f(x_1) = f(x_3) = f(x_7) = f(x_8) = 0$ |
| $U_{f8} : f(x_1) = f(x_2) = f(x_4) = f(x_7) = 0$ | $U_{f26} : f(x_1) = f(x_4) = f(x_5) = f(x_6) = 0$ |
| $U_{f9} : f(x_1) = f(x_2) = f(x_4) = f(x_8) = 0$ | $U_{f27} : f(x_1) = f(x_4) = f(x_5) = f(x_7) = 0$ |
| $U_{f10} : f(x_1) = f(x_2) = f(x_5) = f(x_6) = 0$ | $U_{f28} : f(x_1) = f(x_4) = f(x_5) = f(x_8) = 0$ |
| $U_{f11} : f(x_1) = f(x_2) = f(x_5) = f(x_7) = 0$ | $U_{f29} : f(x_1) = f(x_4) = f(x_6) = f(x_7) = 0$ |
| $U_{f12} : f(x_1) = f(x_2) = f(x_5) = f(x_8) = 0$ | $U_{f30} : f(x_1) = f(x_4) = f(x_6) = f(x_8) = 0$ |
| $U_{f13} : f(x_1) = f(x_2) = f(x_6) = f(x_7) = 0$ | $U_{f31} : f(x_1) = f(x_4) = f(x_7) = f(x_8) = 0$ |
| $U_{f14} : f(x_1) = f(x_2) = f(x_6) = f(x_8) = 0$ | $U_{f32} : f(x_1) = f(x_5) = f(x_6) = f(x_7) = 0$ |
| $U_{f15} : f(x_1) = f(x_2) = f(x_7) = f(x_8) = 0$ | $U_{f33} : f(x_1) = f(x_5) = f(x_6) = f(x_8) = 0$ |
| $U_{f16} : f(x_1) = f(x_3) = f(x_4) = f(x_5) = 0$ | $U_{f34} : f(x_1) = f(x_5) = f(x_7) = f(x_8) = 0$ |
| $U_{f17} : f(x_1) = f(x_3) = f(x_4) = f(x_6) = 0$ | $U_{f35} : f(x_1) = f(x_6) = f(x_7) = f(x_8) = 0$ |
| $U_{f18} : f(x_1) = f(x_3) = f(x_4) = f(x_7) = 0$ | $U_{f36} : f(x_1) = f(x_6) = f(x_7) = f(x_8) = 0$ |

Here, $x_1 = 000, x_2 = 001, x_3 = 010, x_4 = 011, x_5 = 100, x_6 = 101, x_7 = 110$, and $x_8 = 111$. For simplicity, we only list the function values, which are “0”, for four inputs corresponding to each balanced function. Note that for each balanced function, the function values for the other four inputs (not listed) take a value “1”. For instance, for the balanced function corresponding to $U_{f1}$, the function values for the other four inputs (not listed) are $f(x_5) = f(x_6) = f(x_7) = f(x_8) = 1$.
List of 35 $f$-controlled phase gates for a three-qubit refined Deutsch–Jozsa quantum algorithm

| Type 1       | $U_{f1} = \sigma Z_1$ | $U_{f15} = \sigma Z_1 \sigma Z_2$ | $U_{f29} = \sigma Z_1 \sigma Z_2 \sigma Z_3$ |
|--------------|-----------------------|------------------------------------|-----------------------------------------------|
| $U_{f10} = \sigma Z_2$ | $U_{f24} = \sigma Z_1 \sigma Z_3$ |                                   |                                               |
| $U_{f21} = \sigma Z_3$ | $U_{f28} = \sigma Z_2 \sigma Z_3$ |                                   |                                               |

| Type 2       | $U_{f5} = C_{23}\sigma Z_1$ | $U_{f13} = C_{13}\sigma Z_1 \sigma Z_2$ | $U_{f27} = C_{12}\sigma Z_2 \sigma Z_3$ |
|--------------|-------------------------------|--------------------------------------------|-------------------------------------------|
| $U_{f12} = C_{13}\sigma Z_2$ | $U_{f17} = C_{23}\sigma Z_1 \sigma Z_3$ |                                   |                                             |
| $U_{f22} = C_{12}\sigma Z_3$ | $U_{f23} = C_{12}\sigma Z_1 \sigma Z_3$ |                                   |                                             |
| $U_{f8} = C_{23}\sigma Z_1 \sigma Z_2$ | $U_{f26} = C_{13}\sigma Z_2 \sigma Z_3$ |                                   |                                             |

| Type 3       | $U_{f3} = T_{312}\sigma Z_1$ | $U_{f16} = T_{213}\sigma Z_3$ | $U_{f19} = T_{312}\sigma Z_1 \sigma Z_3$ |
|--------------|-----------------------------|-------------------------------|-------------------------------------------|
| $U_{f4} = T_{213}\sigma Z_1$ | $U_{f20} = T_{123}\sigma Z_3$ |                                   |                                             |
| $U_{f6} = T_{312}\sigma Z_2$ | $U_{f19} = T_{213}\sigma Z_1 \sigma Z_2$ |                                   |                                             |
| $U_{f11} = T_{123}\sigma Z_2$ | $U_{f14} = T_{123}\sigma Z_1 \sigma Z_2$ |                                   |                                             |

| Type 4       | $U_{f2} = T_{123}C_{23}$ | $U_{f18} = T_{123}C_{23}\sigma Z_1 \sigma Z_3$ | $U_{f32} = T_{123}C_{23}\sigma Z_2 \sigma Z_3$ |
|--------------|--------------------------|-------------------------------------------------|-----------------------------------------------|
| $U_{f7} = T_{123}C_{23} \sigma Z_1 \sigma Z_2$ |                                   |                                               |                                             |

Here, $\sigma z_j$ represents a single-qubit $\sigma z_j$ gate on qubit $j$ ($j = 1, 2, 3$); $C_{jk}$ indicates a two-qubit controlled phase gate on qubits $j$ and $k$ ($j, k = 1, 2, 3$), described by Eq. (2); and $T_{jkl}$ is a two-target-qubit CP gate described by Eq. (3).

$f$-controlled phase gates each constructed with single-qubit $\sigma z$ gates and one two-qubit CP gate; (iii) type 3 has twelve $f$-controlled phase gates each constructed by using single-qubit $\sigma z$ gates and a two-target-qubit CP gate; and (iv) type 4 involves four $f$-controlled phase gates each implemented with single-qubit $\sigma z$ gates, a two-qubit CP gate, and a two-target-qubit CP gate at most.

4 Implementation of the three-qubit refined DJ algorithm in circuit QED

Using the protocol, we now discuss how to implement the three-qubit refined DJ algorithm with three superconducting transmon qutrits (1, 2, 3) each having three levels (i.e., the ground $|0\rangle$, the first excited $|1\rangle$, and the second excited level $|2\rangle$). We then estimate the fidelity of the operation, which is performed in a setup composed of three phase qutrits and a one-dimensional coplanar waveguide resonator.

4.1 Implementing the algorithm

Since the $f$-controlled phase gates belonging to type 1 are constructed using single-qubit gates only, their implementation does not require entanglement and thus can be
realized in a classical way. Therefore, in the following, we focus on the \( f \)-controlled phase gates belonging to type 2, type 3, and type 4. Without loss of generality, let us consider the three \( f \)-controlled phase gates \( U_{f30} \) (belonging to type 2), \( U_{f9} \) (belonging to type 3), and \( U_{f7} \) (belonging to type 4). By comparing them with other \( f \)-controlled phase gates in the same types, it can be found that these three unitary gates \( U_{f30}, U_{f9}, \) and \( U_{f7} \) contain the same number of two-qubit CP gates and/or two-target-qubit CP gates but a greater or equal number of single-qubit gates than the other \( f \)-controlled phase gates in the same types. Hence, if the three \( f \)-controlled phase gates \( U_{f30}, U_{f9}, \) and \( U_{f7} \) can be implemented, then other \( f \)-controlled phase gates in the same types can be achieved with a higher or equal fidelity. In this sense, to see how well the proposal works, it would be sufficient to explore the implementation feasibility of the following three joint quantum operations, described by

\[
U_1 = H^3 U_{f30} H^3 = H^3 C_{12} \sigma_z^1 \sigma_z^2 \sigma_z^3 H^3, \\
U_2 = H^3 U_{f9} H^3 = H^3 T_{213} \sigma_z^1 \sigma_z^2 H^3, \\
U_3 = H^3 U_{f7} H^3 = H^3 T_{123} C_{23} \sigma_z^1 \sigma_z^2 H^3,  \tag{4}
\]

where the gate operation sequence is from right to left. The \( U_1, U_2, \) and \( U_3 \) here are constructed, according to Fig. 1, and the decomposition of \( U_{f30}, U_{f9}, \) and \( U_{f7} \) is given in Table 2. From Eq. (4), one can see that \( U_1, U_2, \) and \( U_3 \) are implemented through the single-qubit \( \sigma_z \) and \( H \) gates, two-qubit CP gates, and two-target-qubit CP gates.

4.1.1 Implementing single-qubit gates

The single-qubit Hadamard \( H \) gate or \( \sigma_z \) gate on qutrit \( j \) can be realized by applying a pulse resonant with the \( |0\rangle \leftrightarrow |1\rangle \) transition of qutrit \( j \) \((j = 1, 2, 3)\). To eliminate the leakage into the level \( |2\rangle \), one can employ the DRAG pulse [43,44], which can reduce the gate error by an order of magnitude relative to the state of the art, all based on smooth and feasible pulse shapes [43]. In addition, to shorten the gate time, the three joint Hadamard gates \( H^3 \) in Eq. (4) are performed simultaneously, which can be achieved by turning on and off the pulses applied to the three qutrits at the same time. In the same manner, the three \( \sigma_z \) gates involved in \( U_1 \) (the two \( \sigma_z \) gates in \( U_2 \) and \( U_3 \)) are performed simultaneously.

Detailed discussion of how to implement the \( H \) gate or \( \sigma_z \) gate is omitted here since implementing a single-qubit gate depends on the use of the pulse shapes and is straightforward in experiments.

4.1.2 Implementing a two-qubit CP gate

We define \( g_{j01} \) \((g_{k12})\) as the resonant coupling constant between the cavity mode and the \( |0\rangle \leftrightarrow |1\rangle \) \((|1\rangle \leftrightarrow |2\rangle)\) transition of qutrit \( j \) \((k)\). The cavity is initially in the vacuum state \( |0\rangle_c \). The procedure for realizing \( C_{jk} \) is listed as follows:

Step (i) Adjust the level spacings of qutrit \( j \) such that the \( |0\rangle \leftrightarrow |1\rangle \) transition is on resonance with the cavity (Fig. 2a). After an interaction time \( t_1 = \)
Fig. 2  (Color online) Illustration of qutrit-cavity resonant interaction. a The cavity is resonant with the $|0\rangle \leftrightarrow |1\rangle$ transition of qutrit $j$ with a coupling constant $g_{j01}$. b The cavity is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of qutrit $k$ with a coupling constant $g_{k12}$.

\[ \pi / (2g_{j01}) \], the state $|1\rangle_j |0\rangle_c$ changes to $-i |0\rangle_j |1\rangle_c$ while nothing happens to the state $|0\rangle_j |0\rangle_c$ (e.g., see [25,26]).

Step (ii) Adjust the level spacings of qutrit $k$ such that the $|1\rangle \leftrightarrow |2\rangle$ transition is on resonance with the cavity (Fig. 2b). After an interaction time $t_2 = \pi / g_{k12}$, the state $|1\rangle_k |1\rangle_c$ becomes $-|1\rangle_k |1\rangle_c$ while the states $|0\rangle_k |0\rangle_c$, $|1\rangle_k |0\rangle_c$ and $|0\rangle_k |1\rangle_c$ remain unchanged.

Step (iii) Adjust the level spacings of qutrit $j$ such that the $|0\rangle \leftrightarrow |1\rangle$ transition is on resonance with the cavity (Fig. 2a). After an interaction time $t_3 = 3\pi / (2g_{j01})$, the state $|0\rangle_j |1\rangle_c$ changes to $i |1\rangle_j |0\rangle_c$, while nothing happens to the state $|0\rangle_j |0\rangle_c$.

One can check that the state $|1\rangle_j |0\rangle_k |0\rangle_c$ remains unchanged while the state $|1\rangle_j |1\rangle_k |0\rangle_c$ changes to $-|1\rangle_j |1\rangle_k |0\rangle_c$ after the above operations. On the other hand, the initial states $\{|0\rangle_j |0\rangle_k |0\rangle_c, |0\rangle_j |1\rangle_k |0\rangle_c, |1\rangle_j |0\rangle_c\}$ remain unchanged during the entire operation above. These results show that a two-qubit CP gate $C_{jk}$, described by Eq. (2), was achieved with qutrit $j$ (the control) and qutrit $k$ (the target) after the above process, while the cavity returns to its original vacuum state.

4.1.3 Realizing a two-target-qubit CP gate

By carefully examining the procedure described above for implementing $C_{jk}$, we note that a two-target-qubit CP gate $T_{jkl}$ described by Eq. (3) can be realized using four operational steps only:

Steps (i) and (ii): the operations for these two steps are the same as those for steps (i) and (ii) described above.

Step (iii): Adjust the level spacings of qutrit $l$ such that the $|1\rangle \leftrightarrow |2\rangle$ transition is on resonance with the cavity. After an interaction time $t_3 = \pi / g_{l12}$ (where $g_{l12}$ is the coupling constant between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition of qutrit $l$), the state $|1\rangle_l |1\rangle_c$ becomes $-|1\rangle_l |1\rangle_c$, while the states $|0\rangle_l |0\rangle_c$, $|1\rangle_l |0\rangle_c$ and $|0\rangle_l |1\rangle_c$ remain unchanged.
Step (iv): the operation for this step is the same as that for step (iii).

During performing single-qubit-gate operations, all three superconducting phase qutrits 1, 2, and 3 need to be decoupled from the cavity mode, and during performing a two-qubit CP gate or a two-target-qubit CP gate, irrelevant qutrits need to be decoupled from the cavity mode. This requirement can be met by a prior adjustment of the level spacings of the qutrits. Note that for superconducting qutrits, the level spacings can be rapidly adjusted by varying external control parameters (e.g., magnetic flux applied to phase, transmon, or flux qutrits, see, e.g., [5,45,46]).

As a final note, it should be mentioned that the method described above for implementing a two-qubit CP gate via resonant interaction is not new, which was previously proposed [47,48]. We would like to stress that our focus is to take the resonant interaction as an example to explore the possibility of implementing the three-qubit DJ algorithm with superconducting transmon qutrits coupled to a single cavity, by using the protocol presented in the previous section.

4.2 Fidelity

Let us now study the fidelity of the operation. Since the resonant interaction is used in the implementation of the single-qubit $H$ gates or $\sigma_z$ gates, these basic gates can be completed within a very short time (e.g., by increasing the pulse Rabi frequency). In addition, as mentioned previously, one can apply the DRAG pulses to eliminate the leakage into the level $|2\rangle$. Thus, the single-qubit gate error is negligibly small. In this case, decoherence of the system would have a negative impact on the operation of implementing a two-qubit CP gate as well as a two-target-qubit CP gate, due to the population of the cavity photons during the operation. As discussed above, the implementation of these CP gates involves two basic operations:

(i) The first one requires that during performing $C_{jk}$ and $T_{jkl}$, the cavity mode is resonant with the $|0\rangle \rightarrow |1\rangle$ transition of the control qutrit $j$. In realistic case, the interaction Hamiltonian for this basic operation is given by

$$H_{I,1} = (g_{j01}a^+S_{j01}^- + h.c.) + (g'_{j12}e^{-i\delta_{j12}t}a^+S_{j12}^- + h.c.),$$

where $a^+$ is the photon creation operator of the cavity mode, and the second term represents the unwanted off-resonant coupling between the cavity mode and the $|1\rangle \leftrightarrow |2\rangle$ transition, with a coupling constant $g'_{j12}$ and a detuning $\delta_{j12} = \omega_{j12} - \omega_c < 0$ (Fig. 3a).

(ii) The second one requires that during performing $C_{jk}$ and $T_{jkl}$, the cavity mode is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of the target qutrit $k$. The interaction Hamiltonian for this basic operation is given by

$$H_{I,2} = (g_{k12}a^+S_{k12}^- + h.c.) + (g'_{k01}e^{-i\delta_{k01}t}a^+S_{k01}^- + h.c.),$$

where the second term represents the unwanted off-resonant coupling between the cavity mode and the $|0\rangle \leftrightarrow |1\rangle$ transition, with a coupling constant $g'_{k01}$ and a detuning $\delta_{k01} = \omega_{k01} - \omega_c > 0$ (Fig. 3b).
Fig. 3 (Color online) Illustration of qutrit–cavity interaction during the cavity mode interacting with qutrit \( j \) or qutrit \( k \) \( (j, k = 1, 2, \text{ or } 3) \). For the details, see Sect. 4.2

As discussed previously, the cavity mode needs to be resonant with the \(|1\rangle \rightarrow |2\rangle\) transition of the target qutrit \( l \) during performing \( T_{jkl} \). Note that the Hamiltonian governing this basic operation is the same as \( H_{1,2} \) with a replacement of the index \( k \) by \( l \).

It should be mentioned that the term describing the pulse-induced or the cavity-induced coherent \(|0\rangle \leftrightarrow |2\rangle\) transition for each qutrit is not included in the Hamiltonians \( H_{1,1} \) and \( H_{1,2} \), since this transition is negligible because of \( \omega_c, \omega_{j02}, \omega_{k02} \) \( (j, k = 1, 2, 3) \) (Fig. 3).

For each of the two basic types of operations described above, the dynamics of the lossy system, composed of three qutrits \((1, 2, 3)\) and the cavity, is determined by

\[
\frac{d\rho}{dt} = -i [H_l, \rho] + \kappa \mathcal{L} [a] + \sum_{j=1}^{3} \left\{ \gamma_{j21} \mathcal{L} \left[ S_{j21}^- \right] + \gamma_{j20} \mathcal{L} \left[ S_{j20}^- \right] + \gamma_{j10} \mathcal{L} \left[ S_{j10}^- \right] \right\} + \sum_{j=1}^{3} \gamma_{j,\varphi 2} \left( S_{j22} \rho S_{j22} - S_{j22} \rho / 2 - \rho S_{j22} / 2 \right) + \sum_{j=1}^{3} \gamma_{j,\varphi 1} \left( S_{j11} \rho S_{j11} - S_{j11} \rho / 2 - \rho S_{j11} / 2 \right),
\]

where \( H_l \) is the \( H_{1,1} \) or \( H_{1,2} \) above, \( j \) represents qutrit \( j \) \( (j = 1, 2, 3) \), \( S_{j20}^- = |0\rangle \langle 2| \), \( S_{j22} = |2\rangle \langle 2| \), \( S_{j11} = |1\rangle \langle 1| \), \( \mathcal{L} [a] = \Lambda \rho \Lambda^+ - \Lambda^+ \Lambda \rho / 2 - \rho \Lambda^+ \Lambda / 2 \) with \( \Lambda = a, S_{j21}^-, S_{j20}^-, S_{j10}^- \). In addition, \( \kappa \) is the decay rate of the cavity mode, \( \gamma_{j21}, \gamma_{j20}, \text{ and } \gamma_{j10} \) are, respectively, the energy relaxation rates of the level \(|2\rangle\) of qutrit \( j \) for the decay paths \(|2\rangle \rightarrow |1\rangle, |2\rangle \rightarrow |0\rangle, \text{ and } |1\rangle \rightarrow |0\rangle\), and \( \gamma_{j,\varphi 2} (\gamma_{j,\varphi 1}) \) is the dephasing rate of the level \(|2\rangle (|1\rangle)\) of qutrit \( j \).
The fidelity of the operation is given by Nielsen and Chuang [49]

$$
\mathcal{F} = \sqrt{\langle \psi_{id} | \tilde{\rho} | \psi_{id} \rangle},
$$

(8)

where $|\psi_{id}\rangle$ is the output state of an ideal system (i.e., without dissipation and dephasing) after a joint operation $U_1$, $U_2$, or $U_3$ is performed on the qutrit system initially in the state $|000\rangle$ and the cavity mode initially in the vacuum state $|0\rangle_c$, which is given by

$$
U_1 : |\psi_{id}\rangle = \frac{1}{2} (|001\rangle + |011\rangle + |101\rangle + |111\rangle) \otimes |0\rangle_c
$$

$$
U_2 : |\psi_{id}\rangle = \frac{1}{2} (|001\rangle + |011\rangle + |100\rangle + |110\rangle) \otimes |0\rangle_c
$$

$$
U_3 : |\psi_{id}\rangle = \frac{1}{2} (|001\rangle + |010\rangle + |100\rangle + |111\rangle) \otimes |0\rangle_c,
$$

(9)

while $\tilde{\rho}$ is the final density operator of the whole system when the gate operations are performed in a realistic physical system.

We now numerically calculate the fidelity of the joint operations $U_1$, $U_2$, and $U_3$, for a setup shown in Fig. 4. Without loss of generality, let us consider three identical transmon qutrits. In this case, we can drop off the first subscript $(j, k, l)$ for the detunings, Rabi frequencies, and coupling constants. For simplicity, we assume that $g_{01} \sim g'_{01} = g$. One has $g_{12} \sim g'_{12} \sim \sqrt{2}g$ for the transmon qutrit here [50]. Choose $g/(2\pi) \sim 15$ MHz, which can be reached for a superconducting transmon qutrit coupled to a one-dimensional standing-wave CPW (coplanar waveguide) resonator [51]. Other parameters used in the numerical calculation are as follows: $\gamma_{j,\varphi2}^{-1} = \gamma_{j,\varphi1}^{-1} = 10$ $\mu$s, $\gamma_{21}^{-1} = 15$ $\mu$s, $\gamma_{20}^{-1} = 150$ $\mu$s [52], $\gamma_{10}^{-1} = 20$ $\mu$s, and $\kappa^{-1} = 5$ $\mu$s. Define $b_0 = \delta_{01}/g'_{01}$ and $b_1 = -\delta_{12}/g'_{12}$. For simplicity, we choose $b_1 = 10$. For the parameters chosen above, the fidelity versus $b_0$ is shown in Fig. 5, from which one
Fig. 5 (Color online) Fidelity versus $b_0$. Here, $b_0 = \delta_{01}/g_0^\prime$; the red squares, green diamonds, and blue circles correspond to the joint operations $U_1$, $U_2$, and $U_3$ given in Eq. (4), respectively. Refer to the text for the parameters used in the numerical calculation (Color figure online).

can see that for $b_0 = 24$, a high fidelity $\sim 99.1$, $98.0$, and $97.2\%$ can be achieved for the joint operations $U_1$, $U_2$, and $U_3$, respectively. We remark that the fidelity can be further increased by improving system parameters.

For $b_0 = 24$ and $b_1 = 10$, we have $-\delta_{12} \sim 0.21$ GHz, $\delta_{01} \sim 0.36$ GHz, which is achieved in experiments [53]. $T_1$ and $T_2$ can be made to be on the order of $20-60 \mu$s for state-of-the-art superconducting transmon devices [6–8]. For superconducting transmon qutrits, the typical transition frequency between two neighbor levels is between 4 and 10 GHz [10,29,31,32]. As an example, let us consider a cavity with frequency $\nu_c \sim 6$ GHz. Thus, for the $\kappa^{-1}$ used in the numerical calculation, the required quality factor for the cavity is $Q \sim 1.9 \times 10^5$. Note that superconducting CPW resonators with a loaded quality factor $Q \sim 10^6$ have been experimentally demonstrated [54,55], and planar superconducting resonators with internal quality factors above one million ($Q > 10^6$) have also been reported recently [56]. Our analysis given here demonstrates that implementation of the three-qubit refined DJ algorithm is feasible within the present circuit QED technique.

5 Conclusion

We have proposed a protocol for constructing the 35 $f$-controlled phase gates of a three-qubit refined DJ algorithm, by using single-qubit $\sigma_z$ gates, two-qubit CP gates, and two-target-qubit CP gates. Using this protocol, we have discussed how to implement the three-qubit refined DJ algorithm with superconducting transmon qutrits resonantly coupled to a cavity. Our numerical calculation shows that implementation of this quantum algorithm is feasible for the current circuit QED. Finally, it is noted that this protocol is quite general and can be applied to implement the three-qubit refined DJ algorithm in various of physical systems.

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