Designing Differentially Private Estimators in High Dimensions

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Abstract

We study differentially private mean estimation in a high-dimensional setting. Existing differential privacy techniques applied to large dimensions lead to computationally intractable problems or estimators with excessive privacy loss. Recent work in high-dimensional robust statistics has identified computationally tractable mean estimation algorithms with asymptotic dimension-independent error guarantees. We incorporate these results to develop a strict bound on the global sensitivity of the robust mean estimator. This yields a computationally tractable algorithm for differentially private mean estimation in high dimensions with dimension-independent privacy loss. Finally, we show on synthetic data that our algorithm significantly outperforms classic differential privacy methods, overcoming barriers to high-dimensional differential privacy.

1 Introduction

1.1 Background and Problem Statement

An algorithm for releasing output from a database satisfies $\epsilon$-differential privacy if adding, removing, or changing a record in the database does not result in a significant change in the output of the algorithm, where the allowable change is determined by $\epsilon$. The adoption of differentially private algorithms by the US Census Bureau highlights the importance of differential privacy in, among other high-level applications, ensuring the privacy of data release. However, existing methods of differential privacy becomes complicated when dealing with the dimensionality of the dataset, which often creates excessive dimension-dependent error or makes differential privacy algorithms computationally intractable [14, 15].

It is well known that the Laplace mechanism guarantees differential privacy. So, if we can derive the sensitivity of the function to which we want to add noise in computationally tractable time, we then can make said the output of that function differentially private. Formally, we then consider the following problem for mean estimation: given a function, $f$, and a set of points $x$, find a strict, dimension-independent upper bound on the following objective:

$$\max_{x, x', d(x, x') = 1} ||f(x) - f(x')||$$

The concept of sensitivity in differential privacy has strong similarities to other concepts of statistical stability, notably in robustness. This connection was first investigated in-depth by Dwork and Lei [5]. They proposed a framework known as Propose-Test-Release (PTR) for making robust statistics differentially private.

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The PTR approach was inspired by the fact that robust statistics often have bounded influence functions. PTR enables the use of local sensitivity, which is typically much smaller than global sensitivity but not differentially private when applied naively, in calibrating noise [16]. For instance, under i.i.d. data draws, the sample interquartile range should be $O(1/\sqrt{n})$ away from the distribution interquartile range, leading the authors to use this as the scale for the proposed sensitivity.

Recent work on robust statistics have focused in the high-dimensional case, where many historical approaches either yield dimension-dependent error or are computationally intractable. A common theme in this recent literature is the development of certificates for robustness, which has important implications for differential privacy query sensitivities. However, simply using PTR with these asymptotic error guarantees presents the challenge of unbounded and potentially intractable testing steps. This paper will therefore adapt and augment these error guarantees to bound the global sensitivity for use in designing a differentially private high-dimensional mean estimator. We will specifically investigate the methods by Diakonikolas et al [3, 4], who find a computationally tractable robust mean estimation algorithm with dimension-independent error.

We advance the state-of-the-art in several aspects:

- We derive an upper bound on the global sensitivity of mean estimators by converting the asymptotic bounds found in [4] to strict bounds.
- We design a differentially private algorithm for the robust mean estimation in the presence of adversarial corruption.
- We show that that our algorithm does not incur dimension-dependent privacy loss, nor does it make assumptions that the population mean lie in some known bounded interval, as many popular techniques like the Winsorized mean currently do.
- We show that this has the added benefit of not requiring additional computational complexity to compute the bound once mean estimation is done, so the algorithm to release our statistic is computationally tractable.
- We empirically compare our algorithm with conventional attempts at guaranteeing privacy over large datasets, and show that our algorithm significantly outperforms existing methods for high-dimensional data.

1.2 Related and Prior Work

Robust Statistics Robustness is a field with a long history, pioneered decades ago by John Tukey [18]. The field has come a long way since then [10] and continues to grow. Recent focus has turned to robustness for high-dimensions given the feature richness and scale of modern data analysis tasks. While some techniques fail to hold in the high-dimensional case, recent advancements still provide good results for estimation in large datasets. Kane and Diakonikolas make use of filtering methods to prune corrupted points from a set of data, showing that filtering methods requires computation of the largest eigenvalue of a covariance matrix, yielding a computationally tractable algorithm with dimension-independent guarantees. Lai, Rao, and Vempala similarly use spectral methods for agnostic mean estimation [12]. Hopkins and others achieve similar robustness results for both Gaussian and heavy-tailed distributions, relying on Sum-of-Squares proofs and semi definite programming instead of filtering methods [8, 2, 9]. This field continues to evolve, with recent works applying gradient estimation and descent to the Sum-of-Squares hierarchy to yield similar error guarantees with faster runtime [1].

Differential Privacy Private data analysis enabled by differential privacy is constantly advancing with new techniques and applications [7]. Differential privacy in robust estimators was first detailed by Dwork and Lei [5], who outlined the Propose-Test-Release framework for mapping robustness results to differential privacy. Work since then has included results on differentially private M-estimators [13]. A recent paper also generalizes Propose-Test-Release by applying finite sample breakdown points, a recurring aspect of robust statistics, over discretized bins, increasing the probability of a reply by the Propose-Test-Release algorithm and decreasing the estimation error [1]; thus yielding a high probability bound for differentially private mean estimation without requiring assumptions on the boundedness of the data. The concept of sensitivity has also been studied in great detail and extended by works like [16], which define concepts like local and smooth sensitivity for achieving higher utility while still maintaining privacy.
2 Preliminaries

2.1 Differential Privacy

A line of work known as differential privacy has emerged for providing mathematical guarantees on privacy loss beginning from [6].

**Definition 2.1** (Differential Privacy [7]). A randomized function \( M \) is considered to give \((\epsilon, \delta)\)-differential privacy if for all adjacent data sets \( x, x' \) and all \( S \subseteq \text{Range}(M) \):

\[
P[M(x) \in S] \leq \exp(\epsilon)P[M(x') \in S] + \delta
\]

Intuitively, this is saying that data sets differing by a single individual should yield query results that differed in probability by a multiplicative factor of at most \( \exp(\epsilon) \) and by an additive factor of at most \( \delta \). When \( \delta = 0 \), this is simply referred to as \( \epsilon \)-differential privacy or pure differential privacy.

In order to provide such guarantees, there must be some restriction for how the function differs between these adjacent data sets. This is known as sensitivity.

**Definition 2.2** (Global Sensitivity [16]). The global \( \ell_1 \)-sensitivity of a function \( f \) is:

\[
\Delta f = \max_{x,x':d(x,x')=1} ||f(x) - f(x')||
\]

Different techniques have been developed for performing differentially private data analysis. The most common approach is known as the Laplace mechanism, which adds noise drawn from a Laplace distribution with scale parameter \( b = \text{Lap}(\Delta f/\epsilon) \) to the result of the query function \( f(x) \).

**Fact 2.1.** The Laplace mechanism guarantees \( \epsilon \)-differential privacy [7].

2.2 Robust High-Dimensional Mean Estimation by Filtering

Diakonikolas et al. study the task of robust mean estimation specifically for high-dimensions [3, 4]. The specific adversarial framework they use, known as \( \gamma \)-corruption, is powerful and can easily be generalized to other models.[4]

**Definition 2.3** (\( \gamma \)-corruption [4]). Given \( \gamma > 0 \) and a set of samples of size \( m \), the samples are \( \gamma \)-corrupted if an adversary is allowed to inspect the samples, remove \( m' \sim \text{Bin}(\gamma, m) \) of them, and replace them with arbitrary points.

The authors proceed to make a key observation involving a certificate of robustness. Specifically, bounding the spectral norm of the empirical covariance matrix will provide dimension-independent error guarantees for the mean estimate. They use this certificate to design a novel algorithm with dimension-independent error guarantees.

**Theorem 2.1.** Let \( G \) be a sub-Gaussian distribution on \( \mathbb{R}^d \) with parameter \( \nu = \Theta(1) \), mean \( \mu^G \), covariance matrix \( I \), and \( \gamma > 0 \). Let \( S \) be a \( \gamma \)-corrupted set of samples from \( G \) of size \( \Omega((d/\gamma^2) \log(d/\gamma)) \). There exists an efficient algorithm that, on input \( S \) and \( \gamma > 0 \), returns a mean vector \( \hat{\mu} \) so that with probability at least 9/10 we have \( ||\hat{\mu} - \mu^G||_2 = O(\gamma \sqrt{\log(1/\gamma)}) \) [4].

Note that we assume the uncontaminated data generating process follows a sub-Gaussian distribution, since Diakonikolas et al. provide an algorithm satisfying Theorem [2.1] for the sub-Gaussian case, called Filter-Gaussian-Unknown-Mean [3]. However, Diakonikolas et al. also propose a related algorithm for heavy-tailed distributions that only requires bounded fourth moments, which our work could similarly be extended to.

Each recursive call of the Filter-Gaussian-Unknown-Mean algorithm computes the spectral norm of the empirical covariance and compares the result against a threshold. If \( ||\Sigma||_2 \leq \text{Thresh}(\gamma) \), where \( \text{Thresh}(\gamma) = C \gamma \log(1/\gamma) = O(\gamma \log(1/\gamma)) \) for some constant \( C \), then by the certificate of robustness, the error is appropriately bounded and the empirical mean can be returned. Otherwise, the algorithm will project the data set onto a particular direction and remove data points that are far from the current empirical mean. The algorithm will repeat this procedure on this new filtered set.

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3Diakonikolas et al. use \( \epsilon \) as the corruption parameter, but it has been replaced here with \( \gamma \) to avoid confusion with the \( \epsilon \) privacy loss parameter of differential privacy.
3 Robustness Error Guarantees

In this section, we construct a novel and strict bound on the global sensitivity of the robust estimation procedure, enabling the application of differential privacy. We prove these bounds by decomposing the data into multisets, and then applying various asymptotic inequalities to generate strict bounds on said multisets. We begin by defining $(\gamma, \tau)$-good multisets, extending from their definition in [3,4].

**Definition 3.1** ($(\gamma, \tau)$-good multisets [3]). Let $G$ be an identity covariance Gaussian in $d$ dimensions with mean $\mu^G$, and $\gamma, \tau > 0$. A multiset $S$ is $(\gamma, \tau)$-good with respect to $G$ if:

1. For all $x \in S$, $||x - \mu^G||_2 \leq O(\sqrt{d \log(|S|/\tau)})$
2. For every affine function $L: \mathbb{R}^d \to \mathbb{R}$ such that $L(x) = v \cdot (x - \mu^G) - T$, $||v||_2 = 1$, we have that $P_{X \in u} L(X) \geq 0) - P_{X \in G} L(X) \geq 0) \leq \frac{\gamma}{T^2 \log(d \log(d/\gamma \tau))}$
3. $||\mu^S - \mu^G||_2 \leq \gamma$
4. $||M_S - I||_2 \leq \gamma$

**Lemma 3.1.** Let $G$ be an identity covariance Gaussian with $\gamma, \tau > 0$. If the multiset $S$ is obtained by taking $\Omega(\gamma, \tau)$ independent samples from $G$, then $S$ is $(\gamma, \tau)$-good with respect to $G$ with probability at least $1 - \tau$.

This lemma follows from a proof given in Appendix B of [3]. Using $S$ as defined in Lemma 3.1, there exist disjoint multisets $L, E$ where $L \subseteq S$, such that $S' = (S \setminus L) \cup E$. We treat $S'$ as the final data set produced by the iterative Filter-Gaussian-Unknown-Mean algorithm, such that $\mu^S = \hat{\mu}$ is the mean estimate ultimately returned by the estimation procedure.

The mean of each multiset $S \in \{S, S', L, E\}$ will be represented as $\mu^S$ and the mean of the distribution $G$ as $\mu^G$. Similarly, $M_S$ will denote matrices of the form $E_{X \in u} [(X - \mu^G)(X - \mu^G)^T]$ for $S \in \{S, S', L, E\}$.

**Definition 3.2.** Given finite multisets $S$ and $S'$ we let $\Delta(S, S')$ denote the size of the symmetric difference between $S$ and $S'$ divided by the cardinality of $S$.

**Lemma 3.2.** $(|L| + |E|)/|S| = \Delta(S, S') \leq 2\gamma$

**Lemma 3.3.** $(1 - 2\gamma)|S| \leq |S'| \leq (1 + 2\gamma)|S|$

Lemma 3.2 follows from the definition of $S'$ and the fact that it is a multiset that has been filtered from the original data set, which itself was at most $\gamma$-corrupted with respect to $S$. Lemma 3.3 is a direct algebraic result of Lemma 3.2.

The goal is to derive a bound on $||\mu^S - \mu^G||_2$, which represents the error between the empirical mean of the final filtered data samples and the true distribution mean. To begin, the following is true by definition.

$$|S'|(|\mu^S - \mu^G| = |S|(|\mu^S - \mu^G| - |L|(|\mu^L - \mu^G| + |E|(|\mu^E - \mu^G|$$

$$\implies \mu^S - \mu^G = \frac{|S|}{|S'|}(|\mu^S - \mu^G| - \frac{|L|}{|S'|}(|\mu^L - \mu^G| + \frac{|E|}{|S'|}(|\mu^E - \mu^G|)$$

(1)

We then construct bounds on the terms in (1) yielding the novel upper bounds in Lemmas 3.4, 3.5, 3.6. The full proofs for these lemmas is given in Appendix A and follows from Lemmas 3.1, 3.2 and 3.3.

**Lemma 3.4.** We have that $(|S|/|S'|)||\mu^S - \mu^G||_2 \leq \frac{\gamma}{1 - 2\gamma}$.

**Lemma 3.5.** We have that $(|L|/|S'|)||\mu^L - \mu^G||_2 \leq \sqrt{2\gamma + \sqrt{2\gamma} \kappa}$.

**Lemma 3.6.** Let $\kappa = \frac{\gamma}{1 - 2\gamma} + \sqrt{2\gamma + \sqrt{2\gamma} \kappa}$ and $\lambda^*$ represent the largest eigenvalue of the empirical covariance matrix. Then we have $(|E|/|S'|)||\mu^E - \mu^G||_2 \leq (2 + 2\sqrt{\gamma})\kappa + 2\sqrt{\gamma}\lambda^*$.

Applying our contributions in Lemmas 3.4, 3.5, 3.6 we derive a bound on sensitivity that can be stated in the following theorem.
Theorem 3.7. Let $\kappa = \frac{\gamma}{1-2\gamma} + \sqrt{\frac{\sqrt{\gamma} + 2\gamma}{1-2\gamma}}$ and $\lambda^*$ represent the largest eigenvalue of the empirical covariance matrix. Then we have $||\mu - \mu^G||_2 \leq (3 + 2\sqrt{\gamma})\kappa + 2\gamma\sqrt{C\log(1/\gamma)}$.

Proof. This follows by bounding Equation 1 using Lemmas 3.4, 3.5, and 3.6, and an application of the triangle inequality.

Corollary 3.7.1. Let $\hat{\mu}$ represent the final output of the recursive filtering process. Then we have that:

$||\hat{\mu} - \mu^G||_2 \leq (3 + 2\sqrt{\gamma})\kappa + 2\gamma\sqrt{C\log(1/\gamma)}$

Proof. Recall that $S'$ is analogous to the well-behaved multiset after filtering. The termination condition is the case of the small spectral norm, meaning that $\lambda^*$ is bounded by $C\gamma\log(1/\gamma)$.

4 Differentially Private Robust Mean Estimation

We build on top of our contributions in the previous section by proposing the following differentially private algorithm for robust mean estimation in high dimensions. It uses the work by Diakonikolas et al. [4] to calculate the true robust result. The error guarantees have been adapted in the proofs above to provide an upper bound on global sensitivity, such that noise can then be appropriately added using the Laplace mechanism.

Algorithm 1 DP-Robust-Mean-Estimation

**Require:** data set $S$, corruption level $\gamma$, confidence level $\tau$, privacy loss $\epsilon$, threshold factor $C$

$\hat{\mu} \leftarrow \text{Filter-Gaussian-Unknown-Mean}(S, \gamma, \tau)$

$\hat{\mu}_{DP} \leftarrow \hat{\mu} + \text{Lap}\left[\frac{2}{\epsilon} \left((3 + 2\sqrt{\gamma})\kappa + 2\gamma\sqrt{C\log(1/\gamma)}\right)\right]$

**return** $\hat{\mu}_{DP}$

Theorem 4.1. The DP-Robust-Mean-Estimation algorithm returns a robust mean in polynomial time with $(\epsilon, \tau)$-differential privacy.

Proof. The correctness and polynomial run-time of the mean estimation component is given by [4]. The only additional step is adding the Laplace noise, which takes linear time with respect to the number of dimensions.

The certificate of robustness bounds the sensitivity of the robust mean estimator $f$, which is computed using the Filter-Gaussian-Unknown-Mean algorithm of [4]. From Theorem 3.7, then $||f(S) - \mu||_2 \leq (3 + 2\sqrt{\gamma})\kappa + 2\gamma\sqrt{C\log(1/\gamma)}$. Therefore, for any adjacent data sets $S, S'$, by the triangle inequality $||f(S) - f(S')||_2 \leq \frac{2}{\epsilon} \left((3 + 2\sqrt{\gamma})\kappa + 2\gamma\sqrt{C\log(1/\gamma)}\right)$. The sensitivity of the robust mean estimation task is $\Delta f = (3 + 2\sqrt{\gamma})\kappa + 2\gamma\sqrt{C\log(1/\gamma)}$. Certain elements of the proof hold with probability $1 - \tau$, which provides the additive privacy loss term. Since Laplace noise is added to the resulting estimate with scale of $\Delta f / \epsilon$, it follows by the correctness of the Laplace mechanism that DP-Robust-Mean-Estimation is $(\epsilon, \tau)$-differentially private.

5 Empirical Results

We performed an empirical evaluation of the above algorithm on synthetic data, using the implementation of [4] for the robust mean estimation algorithm. The focus of this evaluation was on statistical accuracy, and we have a series of results for varying parameters $n, \gamma$, where $n$ refers to the number of overall data points and $\gamma$ is, as identified earlier, the level of ‘corruption’ that the robust mean estimation algorithm eliminates. We include $\gamma$ to parametrize the bound on sensitivity, as per Algorithm 1.
In our evaluation, we compare the error loss generated by our differentially private algorithm to the error loss generated by the differentially private Winsorized mean. Winsorization is a statistical technique that clamps data points to a specified interquantile interval by rounding the bottom $\alpha n$ data points up to the $\alpha$-percentile and rounding the top $\alpha n$ data points down to the $(1 - \alpha)$-percentile. It also restricts the global sensitivity of the mean, leading to the design of differentially private algorithms as in [17], which was the reference algorithm used in our empirical evaluations. In all of these experiments, we set the privacy loss parameter $\epsilon = 1$. Code of our implementation is available at [https://github.com/TurboFreeze/dp-robust-filter](https://github.com/TurboFreeze/dp-robust-filter).

We make several observations of note. First, our algorithm significantly outperforms the differentially private Winsorized algorithm in high dimensions across evaluations of $n = \{1000, 10000\}$. In and of itself, this is a strong result, especially given the scale of error in the differentially private Winsorized mean shown in Fig. 1. In modern data analysis, data sets are often of the range that we study here in the thousands or tens of thousands of observations, particularly in the social sciences where data privacy is most relevant.

Second, our algorithm outperforms the differentially private Winsorization in very high dimensions for $n = 100,000$. As shown in Fig. 1c, there is a clear jump in error of the differentially private Winsorization algorithm around $d = 300$ to well above the error loss of our algorithm. We suggest that, while differentially private Winsorized means can initially adjust for high-dimensionality because the computation of the mean includes a $\frac{1}{n}$ term, as $\frac{d}{n}$ increases, the effect of dimensionality outweighs the effect of having lots of data. Importantly, the $\ell_2$-error of our differentially private algorithm does not increase as $n$ decreases. This is obviously by construction; our bounds are all $n$-independent even if dimension-dependent error is still generated. While intuitive, this is a useful result to have: the performance of differentially private Winsorized means sharply deteriorates with lower $n$, and makes our algorithm a far preferable solution, especially when dealing with medium to large as opposed to very large datasets.

Third, our evaluations utilize a rather high $\gamma$ value of 0.1. This is of particular significance: paring down the $\gamma$ value intuitively makes the bound stronger, both by definition of the strict bound that
we have shown, but also by intuition: a robust estimator that is robust to less corruption of the data approaches a differentially private estimator that is only required to be ‘robust’ to one adjusted data point. We examine this in Fig. 2, where we show that low $\gamma$ values allow us to significantly reduce the $\ell_2$-error. In and of itself, the fact that we can still achieve starkly better rates than Winsorized means on our loose sensitivity bound with high $\gamma$ suggests the strength of the algorithm.

6 Discussion

Conventional methods of differential privacy generally have privacy loss parameters that scale with the number of dimensions. Naive truncation, as in the cases of $\alpha$-trimmed or Winsorized means, will thus have dimension-dependent privacy loss. As a result, existing algorithms require databases to be bounded or assume that the parameter in question lies in a bounded and known interval. These are problematic assumptions to require, because it means we cannot construct differentially private estimators over commonly used distributions, including normal and t-distributions. Moreover, this is often infeasible, especially in high dimensions. Having the user define bounds in which the population mean lays for $\alpha$-trimmed and Winsorized means to work requires the user defining the bounds for the large number of dimensions in this setting, which is impractical. Our result makes a significant contribution in this regard: by providing strict dimension-independent error guarantees without relying on any metadata, we can achieve accurate results on large-dimensional datasets without requiring boundedness constraints. This also allows the algorithm to function unsupervised without the user providing bounds in each dimension. The significance of the work done by Diakonikolas et al. in [3, 4] and others like Hopkins and Li [9, 8], which achieve dimension-independent errors, thus yields an effective filtering procedure that underlies this paper. In and of itself, generating dimension-independent privacy loss thus represents a significant step forward to being able to tackle differential privacy problems in high-dimensional settings.

The differentially private algorithm proposed in this paper uses the same effective filtering procedure proposed by Diakonikolas et al., but still requires adding noise in each dimension in order to achieve differential privacy. We make two remarks with regard to this. First, we opt into the use of global sensitivity to calibrate noise as opposed to the local sensitivity across each dimension. There is recent work in this direction [11]. We choose to use global sensitivity because our algorithm gives a computationally tractable bound on said sensitivity; local sensitivity remains a direction for future work. Second, privacy can be lost through any dimension; this proposal should be fairly intuitive. Therefore, differential privacy would inevitably require noise in each dimension. As a result, even if we can entirely eliminate dimensionality as a concern in privacy loss, differentially privacy algorithms would inevitably introduce some dimension-dependent error. This remains a significant issue inherent to any high-dimensional privacy problems, not just robust mean estimation as studied in this paper, and could pose a challenge for imposing differential privacy in modern machine learning or institutional data release for high-dimensional datasets. Our empirical results suggest that our algorithm is a good solution to the conventional ‘curse of dimensionality’, where increasing $\frac{d}{n}$ ratios result in algorithms lacking the information to compute accurate estimates in higher and higher dimensions. We note that by bounding privacy loss, even though we cannot achieve fully dimension-independent error, we can sharply minimize error that would otherwise grow exponentially in $d$.

Finally, we note the importance of spectral methods as used in prior work. These methods remain a powerful mechanism that are gaining traction for designing and validating the robustness of statistics. While the work in this paper is only for one specific technique, it can be adapted to a number of different settings as discussed in related work. The work by Diakonikolas et al., for example, gives the case of bounded second moments in addition to the general sub-Gaussian case studied here; the work by Hopkins, in analyzing both Gaussian and sub-Gaussian deviations, applies here as well.

7 Conclusion

The work here presents a novel algorithm that reduces privacy loss and improves tractability in high dimensional settings, marking a large step forward for analyzing differential privacy in such cases. By concretizing asymptotic bounds yielded from [4], we develop a dimension-independent guarantee for privacy loss, which can be tractably computed in high dimensions. Our resulting empirical work identifies a problem which appears to hold true for differential privacy in the high dimensional setting: because privacy leakage can occur from any dimension, the generation of a differentially private
statistic results in dimension-dependent error even when privacy loss is dimension-independent. We note then that if dimensionality is a permanent barrier to differential privacy in large datasets, our algorithm provides the best possible error guarantee by eliminating dimension-dependent privacy loss and having dimensionality impact only the generation of Laplacian noise. This paper thus provides two important results to understanding differential privacy in high dimensions.

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A Appendix

Deferred proofs for Lemmas 3.4, 3.5, and 3.6 will be given here. These proofs extend and build upon the work in Section 5 of [3] and Appendix A of [4]. Where possible, we refer to their work to minimize redundancy.

A.1 Proof for Lemma 3.4

Proof. From condition (3) of Definition 3.1 on \((\gamma, \tau)\)-good multisets, \(||\mu^S - \mu^G||_2 \leq \gamma\). Furthermore, Lemma 3.3 provides that \(1 - 2\gamma \leq ||S||/|S| \leq 1 + 2\gamma\), so \(||S||/|S'||\||\mu^S - \mu^G||_2 \leq \frac{1}{1 - 2\gamma}\) as desired.

A.2 Proof for Lemma 3.5

Proof. Lemma 5.9 of [3] yields:

\[ |v^T M_L v| = \mathbb{E}_{X \in \mathcal{L}} [v \cdot (X - \mu^G)^2] \ll \log(||S||/|L|) + \gamma \cdot |S||/|L|
\]

This implies:

\[ ||\mu^L - \mu^G||_2 \leq ||M_L||_2 \leq \sqrt{\log(||S||/|L|) + \gamma \cdot |S||/|L|} \leq \sqrt{\log(||S||/|L|) + \sqrt{\gamma |S||/|L|}}
\]

Lemma 3.2 implies both \(|L||/|S| \leq 2\gamma\) and \(|E||/|S| \leq 2\gamma\). Furthermore, Lemma 3.3 implies \((1 - 2\gamma)||S|| \leq |S'| \leq (1 + 2\gamma)||S||\). This allows for the bound to be derived as follows:

\[
\begin{align*}
\frac{|L|}{|S'|} ||\mu^L - \mu^G||_2 &\leq \frac{|L|}{(1 - 2\gamma)||S||} \sqrt{\log(||S||/|L|) + \frac{|L|}{(1 - 2\gamma)||S||}} \sqrt{\gamma |S||/|L|} \\
&\leq \frac{1}{1 - 2\gamma} \left[ \frac{|L|}{|S|} \sqrt{\log(||S||/|L|)} + \sqrt{2\gamma^2} \right] \\
&= \frac{1}{1 - 2\gamma} \left[ \sqrt{2\gamma} + \sqrt{\frac{|L|}{|S|}} \right] \\
&\leq \frac{1}{1 - 2\gamma} \left[ \sqrt{2\gamma} + \sqrt{\frac{\log(x)}{x}} \right] \\
&\leq \frac{\sqrt{2\gamma} + \sqrt{2\gamma}}{1 - 2\gamma}
\end{align*}
\]

where the second line follows from the bound on \(|L||/|S|\) and the fourth line follows from the fact that \(\frac{\log(x)}{x} < 1 \forall x \geq 1\). Because we know \(|S| \geq |L|\) by construction, we have \(|S|/|L| \geq 1\).

A.3 Proof for Lemma 3.6

Proof. With \(\Sigma\) denoting the empirical covariance matrix of \(S'\), then by definition:

\[
\Sigma - I = M_{S'} - I - (\mu^S' - \mu^G)(\mu^S' - \mu^G)^T
\]

\[
\Sigma - I + (\mu^S' - \mu^G)(\mu^S' - \mu^G)^T = M_{S'} - I
\]

\[
||\Sigma - I||_2 + ||\mu^S' - \mu^G||^2 \geq ||M_{S'} - I||_2
\]

Using another bound from Corollary 5.10 of [3]:

\[
||\Sigma - I||_2 + ||\mu^S' - \mu^G||^2 \geq ||M_{S'} - I||_2 \geq ||E||/|S'||M_E||_2
\]
Lemmas 3.4 and 3.5 jointly can provide the bound:

\[ ||\mu^{S'} - \mu^G||_2 \leq \frac{|E|}{|S'|} ||\mu^E - \mu^G||_2 + \kappa \]

where \( \kappa = \frac{\gamma}{1 - 2\gamma} + \frac{\sqrt{2\gamma} + \sqrt{2\gamma}}{1 - 2\gamma} \) by triangle inequality on the two constituent bounds. Then using the fact that \( ||M_E||_2 \geq ||\mu^E - \mu^G||_2 \):

\[ (|E|/|S'|)||M_E||_2 \leq ||\Sigma - I||_2 + \left( \frac{|E|}{|S'|} ||\mu^E - \mu^G||_2 + \kappa \right)^2 \leq ||\Sigma - I||_2 + \left( \frac{|E|}{|S'|} \sqrt{||M_E||_2 + \kappa} \right)^2 \]

Here we let \( a = |E|/|S'| \) and \( u = ||\mu^E - \mu^G||_2 \) to simplify notation.

\[
au^2 \leq ||\Sigma - I||_2 + (au + \kappa)^2
\]

\[
(a - a^2)u^2 - 2a\kappa u \leq \lambda^* + \kappa^2
\]

\[
u^2 - 2\left( \frac{\kappa}{1 - a} \right) u + \left( \frac{\kappa}{1 - a} \right)^2 \leq \frac{\lambda^* + \kappa^2}{a - a^2}
\]

\[
\left( u - \frac{\kappa}{1 - a} \right)^2 \leq \frac{\lambda^* + \kappa^2}{a - a^2} + \left( \frac{\kappa}{1 - a} \right)^2
\]

\[
u \leq \sqrt{\frac{\lambda^* + \kappa^2}{a - a^2}} + \frac{2\kappa}{1 - a}
\]

\[
.: (\mu^E - \mu^G) \leq \sqrt{\frac{\lambda^* + \kappa^2}{a - a^2}} + 4\kappa
\]

where the last line makes use of the fact that \( a = |E|/|S| \leq 0.5 \) by construction (because if more than 0.5|S| terms are drawn from a separate Gaussian, we can no longer produce a robust mean for the initial distribution) and then that \( 1/(1 - x) \leq 2 \) for \( 0 \leq x \leq 0.5 \), which we will apply again below. The term of interest we care about is \( au \), or \( (|E|/|S'|)||\mu^E - \mu^G||_2 \).

\[
\frac{|E|}{|S'|} ||\mu^E - \mu^G||_2 \leq a \sqrt{\frac{\lambda^* + \kappa^2}{a - a^2}} + 4a\kappa
\]

\[
\leq 2\kappa + \sqrt{a(\lambda^* + \kappa^2)} \sqrt{\frac{a}{a - a^2}}
\]

\[
\leq 2\kappa + \sqrt{2\gamma(\lambda^* + \kappa^2)} \sqrt{\frac{1}{1 - a}}
\]

\[
\leq 2\kappa + \sqrt{4\gamma(\lambda^* + \kappa^2)}
\]

\[
\leq (2 + 2\sqrt{\gamma})\kappa + 2\sqrt{\gamma\lambda^*}
\]

\[\square\]