HYDROMAGNETIC AND GRAVITOMIC MAGNETIC CRUST-CORE COUPLING IN A PRECESSING NEUTRON STAR

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ABSTRACT

We consider two types of mechanical coupling between the crust and the core of a precessing neutron star. First, we find that a hydromagnetic (MHD) coupling between the crust and the core strongly modifies the star’s precessional modes when $t_A \lesssim \left( T_{\text{spin}} T_{\text{precess}} \right)^{1/2}$; here $t_A$ is the Alfvén crossing timescale, and $T_{\text{spin}}$ and $T_{\text{precess}}$ are the star’s spin and precession periods, respectively. We argue that in the precessing pulsar PSR B1828–11 the restoring MHD stress prevents a free wobble of the crust relative to the nonprecessing core. Instead, the crust and the proton-electron plasma in the core must precess in unison, and their combined ellipticity determines the period of precession. Link has recently shown that the neutron superfluid vortices in the core of PSR B1828–11 cannot be pinned to the plasma; he has also argued that this lack of pinning is expected if the proton Fermi liquid in the core is a type I superconductor. In this case, the neutron superfluid is dynamically decoupled from the precessing motion. The pulsar’s precession decays as a result of the mutual friction between the neutron superfluid and the plasma in the core. The decay is expected to occur over tens to hundreds of precession periods and may be measurable over a human lifetime. Such a measurement would provide information about the strong neutron-proton interaction in the neutron star core. Second, we consider the effect of gravitomagnetic coupling between the neutron superfluid in the core and the rest of the star and show that this coupling changes the rate of precession by about 10%. The general formalism developed in this paper may be useful for other applications.

Subject headings: dense matter — MHD — stars: neutron

1. INTRODUCTION

The most conclusive evidence for a free precession of an isolated pulsar comes from Stairs et al. (2000, hereafter SLS00); see also Cordes (1993) and Shabanova et al. (2001). Their discovery has shown convincingly that some pulsars are precessing and has opened a new window into the interior of neutron stars (Link & Epstein 2001; Jones & Andersson 2001; Link & Cutler 2002; Cutler et al. 2003; Wasserman 2003; Link 2003b; see also Link 2003a for a review). The pulsar PSR B1828–11, which has been monitored by SLS00 for about a decade, is spinning with a period of 0.4 s and precessing with a period of 500 or 1000 days. The large ratio of the precession to spin periods is difficult to reconcile with the current theoretical ideas about the neutron star’s internal structure. In particular, it has long been argued that the neutron superfluid vortices are pinned to the crystal lattice of the crust; this has been used to explain pulsar glitches (sudden spin-ups of young isolated pulsars). However, as was shown in the pioneering work of Shaham (1977), the perfect crustal pinning leads to rather short precession periods, $T_{\text{precess}} = \left( I_{\text{star}} / I_{\text{superfluid}} \right) T_{\text{spin}}$. Here $T_{\text{spin}}$ and $T_{\text{precess}}$ are the spin and the precession periods, respectively, and $I_{\text{star}}$ and $I_{\text{superfluid}}$ are the moments of inertia of the star and the pinned superfluid, respectively. More recently, the case of imperfect pinning was considered by Sedrakian et al. (1999). They found that even in this case a reasonable phenomenological model predicts a value of the precession period that is not significantly greater than Shaham’s estimate. The expected precession period of PSR B1828–11 would be of the order of 100 s, in sharp contrast with what has been observed. Link & Cutler (2002) have proposed a way out of this contradiction: they argue that the observed precession is so strong that the superfluid vortices are completely unpinned from the crustal lattice. Another possibility, suggested by Jones (1998), is that the vortices in the superfluid in the crust are not pinned to the crustal lattice even when the pulsar is not precessing. We feel that both of these ideas, while potentially viable, require more detailed calculations.

In addition, if the proton-electron plasma in the core participates in the precessing motion and if, as is commonly believed, the protons condense into a type II superconductor (Baym et al. 1969), then the expected strong interaction between the superconductor’s flux tubes and the vortices of the neutron superfluid does not allow slow precession with small damping (Link 2003b). The conflict with observations is avoided if either (1) the proton-electron plasma does not participate in the precessing motion, and the crust alone precesses ("Chandler wobble"); we show that this possibility is excluded because of the MHD crust-core coupling) or (2) the protons in the core do not form a type II superconductor, as is commonly believed, but instead form a type I superconductor (Link 2003b). This is not far fetched, since proton-pairing calculations in the core are uncertain; moreover, recent work by Buckley et al. (2004) argues that the interaction between the proton and neutron condensates may turn the neutron star interior into a type I superconductor even if the proton-pairing calculations favor the type II phase.

While it would be exciting to gain an observational handle on the exotic quantum fluids in the neutron star interior, this is not the main focus of this paper. Here, we concentrate instead on the MHD and gravitomagnetic coupling between the crust and the core; these effects make an impact on the precession dynamics, yet their nature is well understood theoretically and (in the case of MHD) is well tested in laboratory experiments.
We do, however, also discuss the decay of pulsar precession due to the mutual friction between the neutron superfluid and the proton-electron plasma in the core.

This paper is structured as follows. In § 2 we present a toy model for MHD coupling between the crust and the core and solve the Euler precession equations within this model. We show that the precession period is strongly affected once the timescale of magnetic coupling is comparable to the geometric mean between the spin and precession periods. In this section we also consider the damping of precession via electron scattering off the magnetized neutron superfluid vortices in the core; this process is called mutual friction and was first considered in the context of pulsar precession by Alpar & Sauls (1988). The damping timescale is generally tens or hundreds of precession periods, and its exact value is sensitive to the effective proton mass in the core. Thus, by monitoring the precession decay over a few decades one may be able to constrain the strong neutron-proton interactions in the neutron star core.

In § 3 we move away from the toy model and consider the nature of slow MHD waves in a rotating gravitationally stratified neutron star core. Our calculations for this more realistic model generally confirm our toy-model results.

Finally, in § 4 we take into account relativistic frame dragging around a spinning neutron star and find two modes of relativistic precession. The first mode is the Lense-Thirring (LT) precession of the crust in the gravitomagnetic field of the core, first considered by Blandford and Coppi (Blandford 1995). The LT precession is relatively fast; its period is only a few decades shorter than the spin period of the core; this process is called mutual friction and was first considered in the context of pulsar precession by Alpar & Sauls (1988). The damping timescale is generally tens or hundreds of precession periods, and its exact value is sensitive to the effective proton mass in the core. Thus, by monitoring the precession decay over a few decades one may be able to constrain the strong neutron-proton interactions in the neutron star core.

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The second mode is the Eulerian precession, which is modified by inertial frame dragging. This mode is most easily excited by a change in the crustal tensor of inertia, for example, by a sudden deformation of the crust due to magnetic forces. We find that the frame dragging modifies the precession frequency by about 10%.

2. IDEALIZED MODEL FOR THE CRUST-CORE MHD COUPLING

Ohmic dissipation inside the neutron star is very slow compared to the precession period, and therefore, ideal MHD provides an excellent description of the neutron star interior. The magnetic field threads both the crust and the core; in ideal MHD the relative displacement of the crust and the core creates magnetic stresses that oppose this displacement. Thus, the magnetic field lines act as elastic strings (e.g., Blandford & Thorne 2003). Motivated by this, we follow the spirit of the Bondi & Gold (1955), hereafter BG55 analysis of the Chandler wobble and consider the crust and the core as solid bodies coupled by a torque that opposes their relative displacement:

\[ \tau = -\mu \delta \phi. \]  

Here \( \delta \phi \) is the small angular displacement between the crust and the core and \( \mu \) is a constant representing the strength of MHD coupling. While this model is simplistic, it is (1) fully solvable and (2) correct in predicting the main features of the precession with MHD crust-core coupling. We consider a more realistic model in the next section.

For mathematical simplicity, we assume the crust is axi-symmetric and we work in the coordinate system \((e_1, e_2, e_3)\) rigidly attached to the crust so that \(e_3\) is directed along the symmetry axis. We also assume the core to be spherically symmetric. We denote by \((A, B, C)\) and \((D, E, F)\) the crust’s and the core’s three principal moments of inertia, respectively. The dynamics of the system is described by the coupled Euler equations (cf. eq. [1] of BG55):

\[
\begin{align*}
A\dot{\omega}_1 + (C - A)\omega_2\omega_3 &= -\mu\delta \phi_1 = -D(\dot{\Omega}_1 + \omega_2\Omega_3 - \omega_3\Omega_2), \\
A\dot{\omega}_2 - (C - A)\omega_1\omega_3 &= -\mu\delta \phi_2 = -D(\dot{\Omega}_2 - \omega_1\Omega_3 + \omega_3\Omega_1), \\
C\dot{\omega}_3 &= -\mu\delta \phi_3 = -D\dot{\Omega}_3.
\end{align*}
\]

We find that the frame dragging modifies the precession frequency by about 10%.

Here \( \omega = (\omega_1, \omega_2, \omega_3) \) and \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \) are the angular velocity vectors of the crust and the core, respectively, and the sign of \( \delta \phi \) is chosen so that

\[ \frac{d\delta \phi}{dt} = \omega - \Omega. \]

The observed wobble angle of PSR B1828–11 is only \( \sim 3^{\circ} \) (Link & Epstein 2001); motivated by this we restrict our analysis to small-amplitude precession. More precisely, we consider a small periodic perturbation of an equilibrium state in which both the crust and the core are rotating around the crust’s symmetry axis with the angular velocity \( n \). The dynamical quantities are then expressed as

\[
\begin{align*}
\omega_1 &= \tilde{\omega}_1 e^{i\nu}, \\
\omega_2 &= \tilde{\omega}_2 e^{i\nu}, \\
\omega_3 &= n + \tilde{\omega}_3 e^{i\nu},
\end{align*}
\]

and analogously,

\[
\begin{align*}
\Omega_1 &= \tilde{\Omega}_1 e^{i\nu}, \\
\Omega_2 &= \tilde{\Omega}_2 e^{i\nu}, \\
\Omega_3 &= n + \tilde{\Omega}_3 e^{i\nu}.
\end{align*}
\]

Here it is assumed that the complex amplitudes \( \tilde{\omega}_1 \) and \( \tilde{\Omega}_1 \) are small compared to \( n \). It is also convenient to define, in the usual way, the crust’s ellipticity:

\[ \epsilon = \frac{C - A}{A}. \]

In the dynamical equations (eq. [2]) we can neglect the terms that are of second order with respect to \( \tilde{\omega}_1 \) and \( \tilde{\Omega}_1 \) and use equation (3) to eliminate \( \delta \phi \). The linearized equations of motion are

\[
\begin{align*}
i\sigma A\tilde{\omega}_1 + \epsilon A n \tilde{\omega}_2 &= \frac{\mu}{i\sigma} (\tilde{\Omega}_1 - \tilde{\omega}_1) = -iD\sigma \dot{\tilde{\omega}}_1 - Dn(\tilde{\omega}_2 - \tilde{\Omega}_2), \\
i\sigma A\tilde{\omega}_2 - \epsilon A n \tilde{\omega}_1 &= \frac{\mu}{i\sigma} (\tilde{\Omega}_2 - \tilde{\omega}_2) = -iD\sigma \dot{\tilde{\omega}}_2 + Dn(\tilde{\omega}_1 - \tilde{\Omega}_1), \\
i\sigma C\tilde{\omega}_3 &= \frac{\mu}{i\sigma} (\tilde{\Omega}_3 - \tilde{\omega}_3) = -iD\sigma \dot{\tilde{\Omega}}_3.
\end{align*}
\]
The third equation above is decoupled from the first two; it describes the small rotations of the crust and the core around the symmetry axis of the crust. This equation alone gives two frequency eigenvalues:

\[
\sigma_3 = 0,
\]

\[
\sigma_4 = \left[ \frac{\mu(C + D)}{CD} \right]^{1/2}.
\]

The trivial eigenvalue \(\sigma_3\) corresponds to the crust and the core rotating in unison without any relative displacement, whereas the eigenvalue \(\sigma_4\) corresponds to the crust-core oscillations around the rotation axis; these oscillations are not affected by the ellipticity of the crust and the rate of stellar rotation and do not represent precession. The information about precession is contained in the first two lines of equation (7). Following BG55, we can simplify the algebra by considering the sum of the first equation and \(i\) times the second equation and by introducing the new variables \(\omega^* = \omega_1 + i\omega_2\) and \(\Omega^* = \Omega_1 + i\Omega_2\). In the end, we get the following eigenvalue equation:

\[
(\sigma - \epsilon n)(\sigma^2 + n\sigma) = \mu\left( \frac{A + D}{AD} \sigma - \epsilon \frac{n}{D} \right).
\]

This equation has three solutions,

\[
\sigma_0 \approx -n - \sigma_m^2/n, \\
\sigma_{1,2} \approx \left( \sigma_p \pm \sqrt{\sigma_p^2 - 4\sigma_d^2} \right)/2,
\]

where

\[
\sigma_m = \sqrt{\frac{\mu(A + D)}{AD}}
\]

is the frequency of the mode in which the crust and the core of a nonrotating star oscillate differentially, with the restoring force of purely MHD origin, and

\[
\sigma_p = \epsilon n + \sigma_m^2/n, \\
\sigma_d = \sqrt{\frac{\mu\epsilon}{D}}
\]

Since in our case \(A \ll D\), one can show that \(\sigma_p \gg 2\sigma_d\) for all values of \(\epsilon\) and \(\mu\). We therefore have

\[
\sigma_1 \approx \sigma_p = \epsilon n + \sigma_m^2/n, \\
\sigma_2 \approx \frac{A}{A + D} \epsilon n \left( 1 + \frac{\epsilon n^2}{\sigma_m^2} \right)^{-1}.
\]

The frequency \(\sigma_1\) characterizes the differential precession between the crust and the core; in the limit of zero magnetic coupling (i.e., \(\sigma_m = 0\)) its value \(\sigma_1 = \epsilon n c\) is the frequency of a free precession of the crust. By contrast, the frequency \(\sigma_2\) corresponds to the mode in which the crust and the core are trying to precess in unison. In the limit of infinite magnetic coupling (i.e., \(\sigma_m = \infty\)) its value of \(\sigma_2 = \epsilon n A/(A + D)\) is the frequency of precession of the neutron star as a whole: the crust is the source of ellipticity, but the core is rigidly attached to the crust and they precess together.

Let us apply these results to PSR B1828–11. The inferred dipole magnetic field of this pulsar is \(B \approx 5 \times 10^{12}\) G (see SLS00), and the Alfvén speed in the core is

\[
e_v = 10^5 \left( \frac{B}{5 \times 10^{12} \text{ G}} \right) \left( \frac{2 \times 10^{14} \text{ g cm}^{-3}}{\rho} \right)^{1/2} \text{ cm s}^{-1}
\]

when the core is not superconducting and

\[
e_v = 1.4 \times 10^6 \left( \frac{B}{5 \times 10^{12} \text{ G}} \right)^{1/2} \left( \frac{2 \times 10^{14} \text{ g cm}^{-3}}{\rho} \right)^{1/2} \text{ cm s}^{-1}
\]

when the core is superconducting. Here \(\rho\) is the density of the core material that is participating in the Alfvén-wave motion. We estimate the characteristic \(\sigma_m \approx \pi v_A/R\) to be 0.3 s\(^{-1}\) for a nonsuperconducting core and 4.2 s\(^{-1}\) for a superconducting core. In deriving these numbers we have assumed that all of the core is participating in the Alfvén-wave motion; we remark that if the neutrons form a superfluid, then only the charged proton-electron plasma is magnetically coupled to the crust, and the estimates for \(\sigma_m\) should increase by a factor of \(\sim 4\). The spin period of PSR B1828–11 is \(T_{\text{spin}} \approx 0.4\) s, and from equation (14) we see that the period of the crust-core differential precession (Chandler wobble) is

\[
T_1 = 2\pi/\sigma_1 \approx 10^3 \text{ s}
\]

for a nonsuperconducting core and

\[
T_1 \approx 5 \text{ s}
\]

for a superconducting core. In the above estimates, we have assumed zero ellipticity for the star and that all of the core is magnetically coupled to the crust; thus, our estimates are upper limits on \(T_1\). We note that such short-period precession has never been convincingly observed for any pulsar. Since the precession period of PSR B1828–11 is \(\sim 4 \times 10^7\) s, we can say with certainty that the observed precession is not the Chandler wobble of the crust relative to the core. Rather, in agreement with the argument sketched by Link (2003b), the crust and the magnetically coupled part of the core precess in unison and their precession period is found from equation (15):

\[
T_2 \approx \frac{T_{\text{spin}} A + D}{\epsilon A} = 8 \times 10^7 \left( \frac{T_{\text{spin}}}{0.4 \text{ s}} \right) \left( \frac{10^{-7}}{\epsilon} \right) \left( \frac{1 + D/A}{20} \right) \text{ s}.
\]

We note that Link’s original argument stated that the crust and the charged part of the core precess in unison when the precession frequency is \(\omega_{\text{prec}} < \sigma_m\). However, this estimate does not take into account the rotation of the star; we see from our equation (14) that the correct criterion is \(\omega_{\text{prec}} < \sigma_m^2/n\), a more stringent condition.

In deriving equations (15) and (20), we have assumed that there are no extra torques acting on the charged plasma of the core. This assumption breaks down if the core is a type II superconductor and the neutron superfluid vortices interact strongly with the magnetic flux tubes. Link (2003b) has shown that a strong vortex–flux tube interaction is inconsistent with the observed precession of PSR B1828–11, but he has pointed out that the difficulty is alleviated if the core superconductivity is of type I rather than type II. In this case, the
magnetic field is contained not in quantized flux tubes but in larger domains (although these domains probably still densely thread the neutron star interior). Then the relative motion of the plasma and the neutron superfluid is damped by the scattering of the electrons on the magnetized superfluid vortices (Alpar et al. 1984; Alpar & Sauls 1988). This damping is known as “mutual friction,” and its characteristic timescale is

\[ t_{mf} = 10 T_{spin} \left( \frac{m_p}{\rho m_p} \right)^2 \left( m_p^2, m_n^2, \Delta_n, \rho_c, \rho \right); \]  

see equation (32) of Alpar et al. (1984). Here \( m_p/m_n \) and \( m_p^2/m_n^2 \) are the bare and the effective proton/neutron masses, \( \Delta_n \) is the neutron condensate gap, \( \rho_c \) is the density of the proton-electron plasma in the core, and \( f \) is a function that depends weakly on its variables. Alpar & Sauls (1988) give detailed discussion of \( t_{mf} \), and we refer to them for the details.

Here, we only note that the use of equation (21) requires caution since the ratio \( m^2/\rho m \) is generally not constant throughout the star. Therefore, the precise expression for the mutual friction timescale should involve the appropriately weighted averaging over the stellar volume. Here, we use equation (21) as a guide for our estimates and leave the detailed model-dependent calculations for future work.

The precession damping timescale \( \tau_{pr} \) is determined by the following relation (see, e.g., BG55):\(^3\)

\[ \tau_{pr} = T_2 t_{mf}/T_{spin} \sim 15 \left( \frac{m_p}{\rho m_p} \right)^2 \text{yr}. \]  

Thus, the mutual-friction damping of precession may be observable for PSR B1828–11 over the timescale of human life, and its measurement will yield the information about \( \rho m_p/m_p \).

3. ALFVÉN WAVES IN A ROTATING NEUTRON STAR

It is interesting to note that in the absence of the crust ellipticity, the frequency of the neutron star Chandler wobble scales as \( 1/n \) (see eq. [14]). As is seen from this equation, the fast rotation reduces the effectiveness of MHD crust-core coupling. For a neutron star with a fluid core the coupling is mediated by the Alfven waves, which are excited by the precessing crust and propagate into the core. The characteristic timescale for the coupling is \( \sim R/v_{A,B} \), where \( v_{A,B} \) is the speed of these Alfven waves. Thus, we expect that the Alfven waves are slowed down as the star spins faster; our detailed analysis below confirms this expectation.

MHD in rotating fluids has been the subject of extensive research in geophysical fluid dynamics, with applications to the Earth’s fluid core (see Hide et al. 2000 and references therein). There is, however, a significant difference between the Earth and neutron star cores. The Earth interior is approximately isentropic, and the Taylor-Proudman theorem is applicable; thus, the velocity field is almost constant along the lines parallel to the rotation axis. By contrast, the neutron star interior is stably stratified because of the core’s radial composition gradient (Reisenegger & Goldreich 1992). The fluid motion is restricted to equipotential shells, which we assumed to be spherical (this is a good approximation for the slowly spinning PSR B1828–11). The motion is strongly subsonic; therefore, the velocity field is divergence-free.

Under these conditions, the general small fluid displacements can be represented by the radius-dependent stream function \( \psi(r, \theta, \phi) \), so that in spherical coordinates the displacement components are

\[ \zeta_r = 0, \]
\[ \zeta_\phi = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \]
\[ \zeta_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi}. \]  

The radial component of the vorticity of the fluid is

\[ \eta = (1/r^2) (\partial/\partial t) \nabla^2 \psi, \]  

where \( \nabla^2 \) is the Laplacian operator on the unit sphere:

\[ \nabla^2_r = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right]. \]  

Since the fluid inside the neutron star is strongly stratified by gravity (i.e., the Brunt-Väisälä frequency \( N \gg \Omega, \sigma_m \)), the fluid motions on different shells are coupled only via magnetic stresses. One can write down the dynamical equation for the radial component of the absolute vorticity; see equation (5) of Levin & Ushomirsky (2001):

\[ \frac{d}{dt} \left( \eta + 2 \Omega \cos \theta \right) = \dot{\mathbf{r}} \cdot \left( \nabla \times \mathbf{a}_B \right). \]  

Here \( d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \) is the Lagrangian time derivative, \( \dot{\mathbf{r}} \) is the unit radial vector, and \( \mathbf{a}_B \) is the acceleration due to the restoring magnetic stress. Following Kinney & Mendell (2003), we restrict ourselves to the special case of the spherically symmetric radial magnetic field, \( \mathbf{B} = B(r) \mathbf{r} \). The results obtained below should be qualitatively correct for a more general field configuration; however, we have chosen a particularly simple geometry in which the mathematical evaluation of the right-hand side in equation (26) is greatly simplified. The relevant components of \( \mathbf{a}_B \) are given by

\[ \mathbf{a}_B \cdot \mathbf{e}_{\theta, \phi} = \frac{1}{4 \pi \rho r} \frac{\partial}{\partial r} \left[ B^2 r^2 \frac{\partial}{\partial r} \left( \frac{\sigma_{\theta, \phi}}{r} \right) \right]. \]  

We can now write down the linearized equation of motion for the stream function:

\[ \frac{\partial^2}{\partial t^2} \nabla^2 \psi + 2 \Omega \frac{\partial}{\partial t} \frac{\partial}{\partial \phi} \psi = \frac{1}{4 \pi \rho} \frac{\partial}{\partial r} \left[ B^2 r^2 \frac{\partial}{\partial r} \left( \frac{\nabla^2 \psi}{r^2} \right) \right]. \]  

We look for the solution of equation (28) in the following form:

\[ \psi(r, \theta, \phi) = \sum_{l,m} \psi_{lm}(r) Y_{lm}(\theta, \phi) e^{i \omega_{lm} t}. \]  

---

\(^3\) Alpar & Sauls (1988) have erroneously overestimated the precession damping timescale by a factor \( \rho/\rho_c \). They have associated the viscous damping timescale with \( t_{mf}/\rho_c \), since this is the timescale it takes for the neutron superfluid to come to corotation with the charged plasma. However, the neutron superfluid carries most of the star’s moment of inertia, and if the superfluid spins at a different rate than the rest of the star, it is the crust/plasma that is coming to corotation with it. Therefore, one should use \( t_{mf} \) for the viscous damping timescale when evaluating the precession-damping timescale.
Since \( Y_{lm} \) is an eigenfunction of both \( \partial/\partial \phi \) and \( \nabla_r^2 \), equation (28) separates into individual ordinary differential equations for \( \psi_{lm} \):

\[
\left[ \frac{\sigma^2_{lm} - \frac{2m \Omega \sigma_{lm}}{l(l+1)}}{\hbar} \right] \psi_{lm}(r) + \frac{1}{4 \pi \rho} \frac{\partial}{\partial r} \left( B^2 \frac{\partial}{\partial r} \left[ \frac{\psi_{lm}(r)}{r^2} \right] \right) = 0.
\]

(30)

It is instructive to consider the short-wavelength (WKB) approximation for the above equation, and hence derive the following dispersion relation:

\[
k^2 = \frac{1}{v_A^2} \left[ \frac{\sigma^2_{lm} - \frac{2m \Omega \sigma_{lm}}{l(l+1)}}{\hbar} \right],
\]

(31)

where \( k \) is the radial wavevector. The WKB approximation is valid if \(|kR| \gg 2\pi\). The purely toroidal Alfvén waves correspond to the case in which \( m = 0 \) is in the above equation. These waves are not affected by the stellar rotation and have a dispersion relation identical to that of Alfvén waves in a non-rotating star. However, in PSR B1828–11 the Alfvén waves are excited by the slowly precessing crust; since precession is represented by an instantaneous rotation about moving horizontal axis, one should consider the waves with \( l = 1 \) and \( m = \pm 1 \). In this case the second term on the right-hand side of equation (31) is the dominant one, since \( \sigma \ll \Omega \), and the wave is strongly slowed down by the stellar rotation, just as we expected. Intuitively, this happens because the inertial terms in the equation of motion become unimportant compared to the terms representing Coriolis acceleration. The radial wavelength of the excited Alfvén mode is

\[
\lambda_A = \frac{2\pi}{|k|} \approx 2\pi v_A / \sqrt{\sigma \Omega},
\]

(32)

which equals \(~6 \times 10^8\) and \(~8 \times 10^9\) cm for a normal and a superconducting neutron star interior, respectively. When \( m = 1 \), the wavevector \( k \) is imaginary and the Alfvén mode is an evanescent wave inside the neutron star. In both cases, \( \lambda_A \) is more than 2 orders of magnitude greater than the radius of the neutron star, \(~10^6\) cm. Therefore, the part of the neutron star interior that is magnetically coupled to the solid crust will precess in unison with the crust. Even though equation (32) is based on the WKB approximation, which is invalid at long wavelengths, a more rigorous analysis confirms the crust-core coprecession. This conclusion is robust and is in agreement with our toy-model results from the previous section.

4. GRAVITOMAGNETIC COUPLING BETWEEN THE CRUST AND THE CORE

The gravitational redshift at a neutron star surface is \(~0.3\), and therefore relativistic effects, including the dragging of the inertial frames, are strong in and around neutron stars. In this section we analyze how frame dragging affects the relative precession of the crust and the core. Our post-Newtonian calculations rely on the usage of the gravitomagnetic field, \( H \); see Thorne et al. (1986) for the details of this formalism.

4.1. The Gravitomagnetic Coupling Torque

Consider the gravitomagnetic force acting on a small region of the crust of mass \( dm \).\(^4\) In the post-Newtonian approximation, it is given by

\[
dF_{GM} = dm \mathbf{v} \times \mathbf{H} = dm (\mathbf{\omega} \times \mathbf{r}) \times \mathbf{H},
\]

(33)

where \( \mathbf{v} \), \( \mathbf{H} \), \( \mathbf{\omega} \), and \( \mathbf{r} \) are the velocity of the small region, the gravitomagnetic field, the instantaneous angular velocity of the crust, and the radius vector of the region, respectively. The torque acting on this region of the crust is given by

\[
dT = r \times dF_{GM} = dm (\mathbf{H} \times \mathbf{r}) \times \mathbf{r},
\]

(34)

where we have used the vector identity \((\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A}) \mathbf{B} - (\mathbf{C} \cdot \mathbf{B}) \mathbf{A}\). The field \( \mathbf{H} \) is that of a dipole, and

\[
\mathbf{H} \cdot \mathbf{r} = -(4/r^3) \mathbf{J} \cdot \mathbf{r} = -(4/r^3) \Omega \cdot \mathbf{r},
\]

(35)

where \( \mathbf{J} \), \( \Omega \), and \( D \) are the angular momentum, the angular velocity, and the moment of inertia of the spherical core, respectively. (Here we ignore interaction of the crust with its own gravitomagnetic field. It can be shown [Thorne & Gursel 1983] that this self-interaction can be absorbed into the free precession.)

Now, substituting equation (35) into equation (34) and integrating over the crust, we arrive at the following form of the gravitomagnetic torque:

\[
T_{GM} = \mathbf{\omega} \times I_{GM} \Omega,
\]

(36)

where \( I_{GM} \) is the linear operator (represented, generally, by a \( 3 \times 3 \) matrix) defined as

\[
I_{GM} \Omega = -\int d^3r \rho(r) (4D/r^3)(\Omega \cdot \mathbf{r}) \mathbf{r}.
\]

(37)

We use Dirac’s bra and ket notation and express this operator as

\[
I_{GM} = -\int d^3r \rho(r) (4D/r^3) |\mathbf{\hat{r}} \rangle \langle \mathbf{\hat{r}}|,
\]

(38)

where we define the projection operator \( |\mathbf{\hat{r}} \rangle \langle \mathbf{\hat{r}}| \) by its action on an arbitrary vector \( \mathbf{a} \):

\[
|\mathbf{\hat{r}} \rangle \langle \mathbf{\hat{r}}| \mathbf{a} = (\mathbf{r} \cdot \mathbf{a}) \mathbf{a}.
\]

(39)

From equation (38) we see that \( I_{GM} \) is a Hermitian operator: since the integrand \( |\mathbf{\hat{r}} \rangle \langle \mathbf{\hat{r}}| \) in equation (38) is Hermitian, the integral must also be Hermitian. This means that the matrix representing \( I_{GM} \) is symmetric.

If the crust is spherically symmetric, then \( I_{GM} = pl \), where \( p \) is a real number and \( I \) is a unit matrix. In this case, the torque acting on the crust is

\[
T = p \mathbf{\omega} \times \Omega,
\]

(40)

which is the familiar form of the LT torque acting on the gyroscope. In the situation considered here the crust is slightly deformed, so that

\[
I_{GM} = pl + \epsilon p K,
\]

(41)

where \( K \) is a \( 3 \times 3 \) matrix with entries of order 1.

4.2. The Dynamics of Relativistic Precession

Again, we use the BG55 approach to the precession of an interacting crust and core. The equations of motion that
include the gravitomagnetic torque components \( T_1, T_2, \) and \( T_3 \) are (see eq. [1] of BG55):

\[
A\dot{\omega}_1 + (C-A)\omega_2\omega_3 = \dot{\lambda}(\Omega_1 - \omega_1) + T_1 = -D(\Omega_1 + \omega_2\Omega_2 - \omega_3\Omega_2),
\]

\[
A\dot{\omega}_2 - (C-A)\omega_3\omega_1 = \dot{\lambda}(\Omega_2 - \omega_2) + T_2 = -D(\Omega_2 + \omega_3\Omega_2 - \omega_1\Omega_3),
\]

\[
C\dot{\omega}_3 = \dot{\lambda}(\Omega_3 - \omega_3) + T_3 = -D(\Omega_3 + \omega_1\Omega_1 - \omega_2\Omega_2).
\]

(42)

Here we have added the terms \( \dot{\lambda}(\Omega - \omega) \), which represent the viscous torque between the crust and the core (e.g., due to mutual friction between the neutron superfluid and the core plasma coupled to the solid crust). As in the previous sections, we are interested in the small-amplitude precession, when the motion differs only slightly from the rigid motion rotation about the \( z \)-axis, so that \( \omega_1, \omega_2, \omega_3 = \omega - n, \Omega_1, \Omega_2, \) and \( \Omega_3 = \Omega - n \) are all much less than \( n \). Then equation (42) becomes

\[
A\dot{\omega}_1 + (C-A)n\omega_2 = \dot{\lambda}(\Omega_1 - \omega_1) + T_1 = -D[\Omega_1 + n(\omega_2 - \Omega_2)],
\]

\[
A\dot{\omega}_2 - (C-A)n\omega_3 = \dot{\lambda}(\Omega_2 - \omega_2) + T_2 = -D[\Omega_2 + n(\Omega_1 - \omega_1)],
\]

\[
C\dot{\omega}_3 = \dot{\lambda}(\Omega_3 - \omega_3) + T_3 = -D\dot{\lambda},
\]

(43)

The general strategy now is to identify the leading terms in \( T_1, T_2, \) and \( T_3 \), using equation (36), and then solve equation (43). Since \( D \gg A \), we consider a simplified case in which the spherical core has an infinite inertia: \( D \rightarrow \infty \), so that the core’s spin does not change in the inertial frame of reference. Therefore we have, from equation (43),

\[
\dot{\Omega}_1 + n(\omega_2 - \Omega_2) = \dot{\Omega}_2 - n(\omega_1 - \Omega_1) = \dot{\Omega}_3 = 0.
\]

(44)

We look for a mode with the growth rate \( \gamma \), so that \( \dot{\Omega}_1 = \gamma \Omega_1 \), etc. By considering the sum of the first component of eq. (44) plus \( i \) times the second component of eq. (44), we get

\[
\Omega^+ = \frac{i n}{\gamma} \omega^+,
\]

(45)

where \( \Omega^+ = \Omega_1 + i \Omega_2 \) and \( \omega^+ = \omega_1 + i \omega_2 \).

Let us restrict ourselves to the case of the axially symmetric crust. In this case, both the tensor of inertia and \( I_{GM} \) diagonalize in the same basis because of the axial symmetry. We can then write

\[
I_{GM}\Omega = p\Omega_1 e_1 + p\Omega_2 e_2 + p(1 + e)k\Omega_3 e_3,
\]

(46)

where \( e_1, e_2, e_3 \) are the unit vectors along the \( x, y, \) and \( z \)-axes, respectively, with the \( z \)-axis chosen to be the axis of symmetry (as in eq. [43]), and \( k \) is a number of order 1. Then the gravitomagnetic torque in equation (36), to leading order, is

\[
T = np\{[\omega_2(1 + ek) - \Omega_2]e_1 + [\Omega_1 - \omega_1(1 + ek)]e_2\}.
\]

(47)

Now, let us substitute this expression into equation (43), add the first row plus \( i \) times the second row, and ignore the right-hand side with \( D \) in it (we have already taken care of it by setting \( D \rightarrow \infty \)). We get, after dividing by \( A \) and substituting \( \gamma \) instead of the time derivative,

\[
\gamma \omega^+ + i n[1 - (p/A)k] \omega^+ = (1/A)(\lambda + i n p)(\Omega^+ - \omega^+).
\]

(48)

Note that in the above equations the contribution due to the gravitomagnetic terms can be represented by an effective modification of the ellipticity \( \epsilon \rightarrow \epsilon[1 - (p/A)k] \) and of the viscous coupling coefficient \( \dot{\lambda} \rightarrow \dot{\lambda} + i n p \). Therefore, one can consider the precession solution without gravitomagnetic terms and then substitute the ellipticity and coupling in this solution by their modified values. The resulting expressions then represent the precession solution that includes the gravitomagnetic coupling.

The remaining calculation is straightforward. Let \( \bar{\epsilon} = \epsilon[1 - (p/A)k] \) and \( \bar{\lambda} = (1/A)(\lambda + i n p) \). By substituting equation (45) into equation (48), we get the following equation for the growth rate \( \gamma \):

\[
\gamma^2 + [in(1 + \bar{\epsilon}) + \bar{\lambda}]\gamma + n^2\bar{\epsilon} = 0,
\]

(49)

which has two solutions,

\[
\gamma = (1/2)\left\{-[in(1 + \bar{\epsilon}) + \bar{\lambda}] \pm \sqrt{[in(1 + \bar{\epsilon}) + \bar{\lambda}]^2 - 4n^2\bar{\epsilon}}\right\}.
\]

(50)

We now use the fact that \( \bar{\epsilon} \ll 1 \) in equation (50), and we get to the leading order in \( \bar{\epsilon} \) for the “+” solution

\[
\gamma = i \frac{n\bar{\epsilon}}{1 - i(\lambda/n)} \approx in\bar{\epsilon} - \bar{\lambda}\bar{\epsilon}.
\]

(51)

This solution corresponds to the Eulerian precession of the crust, with the frequency

\[
\omega_{pe} = n\bar{\epsilon} - \bar{\epsilon}n(p/A) = n[1 - (p/A)k][1 - (p/A)]
\]

(52)

and the damping rate

\[
1/\tau_{precc} = (\lambda/A)e[1 - (p/A)].
\]

(53)

The contribution of the frame dragging comes in through terms that contain \( p/A \). The frame dragging reduces both the precession frequency and the damping rate by relative order of \( p/A \). Now, from equation (40), we can work out that \( p/A = \omega_{LT}/n, \) where \( \omega_{LT} \) is the frequency of the conventional, gyrosopic LT precession. The calculations of Blandford and Coppi (Blandford and Coppi 1995) show that \( \omega_{LT}/n \sim 1/7 \). Therefore, we expect the dragging of inertial frames to reduce the frequency and the damping rate of precession by \( \sim 10\% \).

What about the second, “–” solution of equation (49)? We have, to the leading order in \( \bar{\epsilon} \),

\[
\gamma = -in - \bar{\lambda} = -in(1 + p/A) - \lambda/A.
\]

(54)

This mode corresponds to the situation in which the spin of the crust is misaligned with that of the core. In the inertial frame of reference (as opposed to the frame attached to the
body), one must take the \(-\) in out of \(\gamma\). The piece that is left is then

\[
\gamma_{\text{inertial}} = -inp/A - \lambda/A.
\]

This corresponds to the LT precession considered by Blandford and Coppi (Blandford 1995), which is damped on a short timescale \(\lambda/A\), i.e., the viscous time on which corotation of the crust and the core is enforced. This timescale is expected to be rather short: Alpar & Sauls (1988) estimate that the crust-core corotation is enforced in 10–100 rotation periods.

5. DISCUSSION

In this paper we have analyzed the effect of crust-core coupling on the Chandler wobble of the neutron star crust. We have found (§4) that gravitomagnetic crust-core coupling does not strongly affect the Chandler wobble but instead modifies its frequency by about 10%. By contrast, we have found that the MHD interaction between the crust and the core of a rotating neutron star dramatically changes the dynamics of the wobble, for typical values of the pulsar spin and magnetic field. In particular, we have shown that the observed precession in PSR B1828–11 cannot be the Chandler wobble of its crust; instead, the crust and the plasma in the core must precess in unison. The precession is damped by the mutual friction in the core. This damping has a timescale of tens or hundreds of precession periods and may be observed over the span of human life. The measurement of the damping timescale would constrain the value of \(\hat{\kappa}'_p/m_p\) and thus provide information about the strong proton-neutron interactions in the neutron star core.

While the immediate astrophysical impact of our paper is modest, it presents some novel analytical techniques. In §3, we have developed the theory of slow Alfvén waves in a gravitationally stratified uniformly rotating fluid (as is applicable for a neutron star). In §4, we have analyzed the precession dynamics of a biaxial rigid body in the presence of a strong gravitomagnetic field. As far as we are aware, both of these technical developments are new, and we envisage their further applications to the dynamics of neutron stars.

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