GRAHAM-WITTEN’S CONFORMAL INVARIANT FOR CLOSED FOUR DIMENSIONAL SUBMANIFOLDS

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Dedicated to Professor A. Chang and P. Yang on the occasion of their 70th birthdays.

Abstract. It was proved by Graham and Witten in 1999 that conformal invariants of submanifolds can be obtained via volume renormalization of minimal surfaces in conformally compact Einstein manifolds. The conformal invariant of a submanifold \( \Sigma \) is contained in the volume expansion of the minimal surface which is asymptotic to \( \Sigma \) when the minimal surface approaches the conformal infinity. In the paper we give the explicit expression of Graham-Witten’s conformal invariant for closed four dimensional submanifolds and find critical points of the conformal invariant in the case of Euclidean ambient spaces.

1. Introduction

In the introduction we give a description of the main result and some related background of the paper. The terminologies used in the introduction will be recalled in the next section.

Let \((X^{d+1}, g_+)\) be a conformally compact Einstein manifold and \((M^d, g_{\text{conf inf}})\) its conformal infinity. A given metric \(g \in [g_{\text{conf inf}}]\) uniquely determines a special defining function \(r\) on a neighborhood of \(M\) in \(X\), upon to the conditions that \(r^2 g_+|_M = \overline{g}\) and \(|dr|^2 g_+ = 1\). We denote \(g_c = r^2 g_+\). With the special defining function \(r\), one can identify \(M \times [0, \varepsilon)\), for some \(\varepsilon > 0\), with a neighborhood of \(M\) in \(X\). We denote the neighborhood by \(X\), and the identification (1.1)

\[ M \times [0, \varepsilon) \cong X \]

is defined as follows: \((p, r) \in M \times [0, \varepsilon)\) corresponds to the point obtained by following the flow of \(\nabla g_c r\) emanating from \(p\) for \(r\) units of time. \(g_c\) on \(M \times [0, \varepsilon)\) takes the form of (1.2)

\[ g_c = dr^2 + \tilde{g}, \]

where \(\tilde{g}\) is a 1-parameter family of metrics on \(M\) with the parameter \(r\). By solving the Einstein equation \(Ric(g_+) = -dg_+\), for \(d\) odd the expansion of \(\tilde{g}\) is of the form (1.3)

\[ \tilde{g} = g^{(0)} + g^{(2)} r^2 + (\text{even powers}) + g^{(d-1)} r^{d-1} + g^{(d)} r^d + \cdots \]

where \(g^{(j)}\) are tensors on \(M\) and the dots stand for terms vanishing to higher order. For \(j\) even and \(0 \leq j \leq d-1\), the tensor \(g^{(j)}\) is locally formally determined by the boundary value \(g^{(0)} = \overline{g}\), but \(g^{(d)}\) is formally undetermined; for \(d\) even the expansion is (1.4)

\[ \tilde{g} = g^{(0)} + g^{(2)} r^2 + (\text{even powers}) + hr^d \log r + g^{(d)} r^d + \cdots \]

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where \( g^{(j)} \) and \( h \) are locally formally determined for \( j \) even and \( 0 \leq j \leq d - 2 \) by \( g^{(0)} = \overline{g} \).

The main object of the paper is a minimal surface in the conformally compact Einstein manifold \((X, g_+)\) with prescribed asymptotic boundary. Let \( \Sigma^n \) be a submanifold of \( M \) and \( Y^{n+1} \hookrightarrow (X, g_+) \) be a minimal surface which is asymptotic to \( \Sigma \). The problem of existence and regularity of such minimal surfaces has been studied by Anderson [3, 4], Hardt-Lin [27], Lin [30, 31, 32], Tonegawa [35], Han-Jiang [25] and Han-Shen-Wang [26]. We denote

\[
(1.5) \quad g = \overline{g}|_{\Sigma}.
\]

The connections with respect to \((M, \overline{g})\) and \((\Sigma, g)\) will be denoted by \( \nabla \) and \( \nabla \) respectively, and the connection of the normal bundle \( T^\perp \Sigma \) of the immersion \( \Sigma \hookrightarrow (M^d, \overline{g}) \) will be denoted by \( \nabla^\perp \).

Graham and Witten [16] have introduced a natural and useful way to reformulate \( Y \). Namely, near the boundary \( M \) they express \( Y \) as a graph over \( \Sigma \times [0, \varepsilon) \) and expand the height functions of the graph in \( r \). Near a point of \( \Sigma^n \), let \((x^i, y^\alpha)\) be a local coordinate chart of \( M^d \), where \( 1 \leq i \leq n \) and \( n + 1 \leq \alpha \leq d \), so that

\[
(1.6) \quad \Sigma = \{ y = 0 \}; \quad \overline{g} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha} \right) = 0 \quad \text{on} \quad \Sigma, \forall \ i, \alpha.
\]

Note that via the identification \((1.1)\), one has an extension of the coordinates \((x^i, y^\alpha)\) into \( X \), which together with \( r \) forms a local coordinate chart of \( \overline{X} \). The minimal surface \( Y \) can be written as a graph \( \{ y^\alpha = u^\alpha(x, r) \} \). That is, near the boundary \( Y = (x^i, u^\alpha(x, r), r) \).

Graham and Witten [16] proved that for \( n \) odd

\[
(1.7) \quad u = u^{(2)} r^2 + (\text{even powers}) + u^{(n+1)} r^{n+1} + u^{(n+2)} r^{n+2} + \cdots,
\]

and for \( n \) even

\[
(1.8) \quad u = u^{(2)} r^2 + (\text{even powers}) + u^{(n)} r^n + w_n r^{n+2} \log r + u^{(n+2)} r^{n+2} + \cdots,
\]

where the \( u^{(k)} = (u^{(k)\alpha})_k \), \( k < n + 2 \), and \( w_n = (w_n^\alpha) \) are functions of \( \Sigma \) and locally determined, while \( u^{(n+2)} \) is not locally determined. They showed that the first non-vanishing coefficient \( u^{(2)} \) is related to the mean curvature of \( \Sigma \hookrightarrow (M, \overline{g}) \) by

\[
(1.9) \quad u^{(2)} = \frac{1}{2n} H.
\]

In the paper, we calculate the coefficient \( u^{(4)} \) for \( n \geq 3 \) by using the graphic minimal surface equation of \( Y \).

**Theorem 1.1.** For \( n \geq 3 \),

\[
8n(n-2)u^{(4)} = \Delta H^\alpha + g^{ij} g^{kl} (A_{ik}, H) A_{jl}^\alpha - \frac{2}{n^2} |H|^2 H^\alpha
\]

\[
+ g^{ij} \overline{g}^{\alpha \beta \gamma} \nabla_i \beta j \gamma + (n-4) \overline{g}^{\alpha \beta \gamma} \overline{P}_{\beta j} H^\alpha - g^{ij} \overline{P}_{ij} H^\alpha + 2 n g^{ik} g^{jl} \overline{P}_{kl} A_{ij}^\alpha
\]

\[
+ n \overline{g}^{\alpha \beta \gamma} g^{ij} (\nabla_i \overline{P}_{ij} - 2 \overline{P}_{ij} \nabla_i) - \frac{n-2}{n} H^\beta H^\gamma \overline{\nabla}_{\beta \gamma} \overline{A},
\]

where \( A_{ij} \) denotes the second fundamental form of \( \Sigma \hookrightarrow (M, \overline{g}) \), \( \overline{\Gamma}, \overline{P} \) and \( \overline{W} \) denote the Christoffel symbol, Schouten and Weyl tensor of \( \overline{g} \) respectively.
For $n = 2$, the calculation of $w_2$ in \((1.8)\) can be carried out in a similar way. We have

\begin{equation}
(1.11) \quad w_2 = -\frac{1}{16} W,
\end{equation}

where

\begin{equation}
W^\alpha = \Delta^\perp H^\alpha + g^{ij} g^{kl} \langle A_{ik}, H \rangle A_{jl}^\alpha - \frac{1}{2} |H|^2 H^\alpha
+ g^{ij} g^{kl} \langle H^\alpha, H \rangle A_{ij}^\alpha
+ 2 g^{\alpha \gamma} g^{ij} (\nabla_\gamma P_{ij} - 2 \nabla_i P_{j\gamma}).
\end{equation}

Note that if $(X, g^+)$ is the Poincaré half-plane model of the hyperbolic space $\mathbb{H}^{d+1}$, for which $(M, g) = \mathbb{R}^d$ and $g^{(2)} = -P = 0$, then for $\Sigma^2$ we have

\begin{equation}
W = \Delta^\perp H + g^{ij} g^{kl} \langle A_{ik}, H \rangle A_{jl}^\alpha - \frac{1}{2} |H|^2 H.
\end{equation}

A surface $\Sigma^2 \hookrightarrow \mathbb{R}^d$ with $W = 0$ is well-known to be a Willmore surface. Han and Jiang [25] proved that in the case of $\Sigma^2 \hookrightarrow \mathbb{R}^3$ the log term vanishes if and only if $\Sigma$ is a Willmore surface.

Our main aim of the paper is to calculate the conformal invariant, introduced by Graham and Witten (for each dimensional submanifold) [16], for closed four dimensional submanifolds. Here the conformal invariant for submanifolds is a functional of submanifolds which is invariant under conformal transformations of the metric of the ambient space. Recently, conformal invariants for hypersurfaces, constructed by using the volume renormalization of solutions to the singular Yamabe problem or general singular volume measures, and related aspects have been extensively studied [11, 14, 18, 19, 20, 21, 22, 36]. Graham-Witten’s conformal invariants are obtained from the renormalization process of the volume of a minimal surface $Y^{n+1}$ in a conformally compact Einstein manifold $(X^{d+1}, g^+)$ which is asymptotic to a submanifold $\Sigma^n$ immersed in the conformal infinity of $X$.

Near the asymptotic boundary, the volume form of $Y$ takes the form of

\begin{equation}
d\mu_Y = r^{-n-1} [v^{(0)} + v^{(2)} r^2 + (\text{even powers}) + v^{(n)} r^n + \cdots] d\mu_{\Sigma} dr,
\end{equation}

where $v^{(j)}$ are locally determined functions of $\Sigma$, $v^{(n)} = 0$ for $n$ odd, and $d\mu_{\Sigma}$ is the volume form of $(\Sigma, g)$. As $\epsilon \to 0$ and for $n$ odd, the volume

\begin{equation}
\text{Vol}_{g^+} (Y \cap \{ r > \epsilon \}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + (\text{odd powers}) + c_{n-1} \epsilon^{-1} + c_n + o(1),
\end{equation}

and for $n$ even

\begin{equation}
\text{Vol}_{g^+} (Y \cap \{ r > \epsilon \}) = c_0 \epsilon^{-n} + (\text{even powers}) + c_{n-2} \epsilon^{-2} + L_n \log \frac{1}{\epsilon} + c_n + o(1),
\end{equation}

where

\begin{equation}
L_n = \int_{\Sigma} v^{(n)} d\mu_{\Sigma}.
\end{equation}

An area law was proposed by Ryu and Takayanagi [33, 34] which holographically identifies the volume in \((1.14)\) or \((1.15)\) with the entanglement entropy in quantum (conformal) field theories. Graham and Witten proved the following

**Theorem 1.2.** (16) If $n$ is odd, then $c_n$ is independent of the choice of special defining function. If $n$ is even, then $L_n$ is independent of the choice of special defining function.
Note that there is the one-to-one correspondence between representative metrics of the conformal class \((M, [g_{\text{conf inf}}])\) and special defining functions. Hence \(c_n\) for \(n\) odd and \(L_n\) for \(n\) even are conformal invariants. \(c_n\) is called the renormalized volume of \(Y\). For \(n = 1\), an explicit expression of the renormalized volume \(c_1\) has been obtained by Alexakis and Mazzeo [2]. For \(n = 2\), Graham and Witten [16] showed that

\[
-8L_2 = \int_{\Sigma} (|H|^2 + 4g^{ij}\mathcal{P}_{ij})d\mu_{\Sigma},
\]

where \(\mathcal{P} = \frac{1}{d-2} (\text{Ric} - \frac{1}{2(d-1)} R\tilde{g})\) is the Schouten tensor of \((M^d, \tilde{g})\). \(L_2\) is closely related to the Willmore functional. For example, when \(\Sigma\) is a closed two dimensional hypersurface in \((M^3, \tilde{g})\), it easily follows from the Gauss formula and Gauss-Bonnet formula that

\[
-8L_2 = 8\pi \chi(\Sigma) + 2 \int_{\Sigma} |\tilde{A}|^2 d\mu_{\Sigma},
\]

where \(\tilde{A}\) denotes the traceless part of the second fundamental form of \(\Sigma\).

Volume renormalization of a conformally compact Einstein manifold had been studied at almost the same time [28, 29, 12], which can be viewed as the extreme case under the setting of the paper: \(\Sigma = M\) and \(Y = X\). The log terms of the volume expansion were calculated explicitly in lower dimensions. For example, in dimension two

\[
L_{d=2} = -\frac{1}{4} \int_{M^2} R\tilde{g} d\mu_{\tilde{g}},
\]

and in dimension four

\[
L_{d=4} = \frac{1}{4} \int_{M^4} \sigma_2(\mathcal{P}) d\mu_{\tilde{g}},
\]

(1.17)

where \(\mathcal{P}\) is the Schouten tensor of \(\tilde{g}\) and \(\sigma_2\) is the second elementary symmetric function. The geometry and topology of closed four dimensional manifolds which admit a metric \(\tilde{g}\) such that \(\int_{M^4} R\tilde{g} d\mu_{\tilde{g}} > 0\) and \(\int_{M^4} \sigma_2(\mathcal{P}) d\mu_{\tilde{g}} > 0\) has been studied by Gursky [23] and Chang-Gursky-Yang [7]. For discussions on the renormalized volume of conformally compact Einstein manifold, see for instance [1, 3, 6, 8, 10, 17]. We believe that the renormalized volume corresponds to the log-determinant of a pseudo-differential operator of orders \(d\) at the quantum field theory side. A likely choice of the pseudo-differential operator could be the linearization of the Dirichlet-to-Neumann operator given by the conformally compact Einstein metric.

In the paper, we calculate \(L_4\) for closed four dimensional submanifolds.

**Theorem 1.3.** Let \(\Sigma^4\) be a closed submanifold in \((M^d, \tilde{g})\), we have

\[
128L_4 = \int_{\Sigma} (|\nabla^\perp H|^2 - g^{ij} g^{kl} (A_{ij}; H) (A_{kl}; H) + \frac{7}{16} |H|^4) d\mu_{\tilde{g}}
\]

(1.18)

\[
+ 16 \int_{\Sigma} (g^{ij} \mathcal{P}_{ij})^2 - g^{ik} g^{jl} \mathcal{P}_{kl} + g^{ij} g^{k\beta} \mathcal{P}_{i\alpha} \mathcal{P}_{j\beta} - \frac{1}{d-4} g^{ij} \mathcal{P}_{ij} d\mu_{\tilde{g}}
\]

\[
+ \int_{\Sigma} (-16 g^{ik} g^{jl} \mathcal{P}_{ij} (A_{kl}; H) + 5g^{ij} \mathcal{P}_{ij} |H|^2 + 8 \mathcal{P}(H, H) - g^{ij} \mathcal{W}_{\alpha i j \beta} H^\alpha H^\beta) d\mu_{\tilde{g}}
\]

\[
- 8 \int_{\Sigma} g^{ij} H^\beta (\nabla_\beta \mathcal{P}_{ij} - 2 \nabla_j \mathcal{P}_{i\beta}) d\mu_{\tilde{g}},
\]
where $\overline{B}_{ij}$ is the Bach tensor of $(M^d, g)$.

When $(X^{d+1}, g_+)$ is the Poincaré half-plane model with $(M^d, g) = \mathbb{R}^d$, $L_4$ is simplified to

$$
\frac{1}{128} \int_\Sigma \left( |\nabla^2 H|^2 - g^{ik} g^{jl} \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16} |H|^4 \right) d\mu_g.
$$

In the last part of the paper, we will consider the functional

$$
\mathcal{L}_4(\Sigma^4) := \frac{1}{2} \int_\Sigma \left( |\nabla^2 H|^2 - g^{ik} g^{jl} \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle + \frac{7}{16} |H|^4 \right) d\mu_g,
$$

for closed four dimensional submanifolds $\Sigma^4 \hookrightarrow \mathbb{R}^d$. Guven [24] has constructed a bending energy for closed four dimensional submanifolds immersed in $\mathbb{R}^5$, which is invariant under special conformal transformations of $\mathbb{R}^5$ and reads

$$
H_2 = \frac{1}{2} \int_\Sigma \left( |\nabla H|^2 - |A|^2 H^2 + \frac{7}{16} |H|^4 \right) d\mu_g.
$$

(A factor of $-2$ had been dropped in the original paper [24], pointed out by Graham and Reichert [15].) Note also that Graham and Reichert [15] prove that up to a multiple the gradient of $L_n$ is given by $w_n$ in (1.8).

In general, the functional $\mathcal{L}_4$ is unbounded from below and above even for closed four dimensional submanifolds with a fixed topology. Let $S^k(r)$ denote the round sphere in $\mathbb{R}^{k+1}$ of radius $r$. Examples of critical points of $\mathcal{L}_4$ include $S^4$, $S^3(1) \times S^1(\frac{1}{\sqrt{3}})$, $S^3(1) \times S^1(\frac{2}{\sqrt{3}})$, $S^2(1) \times S^2(1)$, $S^2(1) \times S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$, $S^2(1) \times S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{3}{\sqrt{5}})$, $S^1(1) \times S^1(1) \times S^1(1)$, and $S^1(1) \times S^1(1) \times S^1(1) \times S^1(1)$. These are all the closed and four dimensional critical points of $\mathcal{L}_4$ which is a product of round spheres.

The calculations in the paper are elementary. In Section 2, we give a brief recall of conformally compact Einstein manifolds and minimal surfaces in conformally compact Einstein manifolds. We identify the constituent components of $u^{(4)}$ for $n \geq 3$ by using the graphic minimal surface equation. In Section 3, we reformulate $u^{(4)}$ in a covariant form and prove Proposition 1.1. In Section 4, we calculate the coefficient $\alpha^{(4)}$ in (1.3) for $n \geq 4$, from which Proposition 1.3 follows so that we obtain Graham-Witten’s conformal invariant $L_4$ for closed four dimensional submanifolds. In the last section, we consider Graham-Witten’s conformal invariant for closed four dimensional submanifolds of Euclidean spaces and find simple critical points of it.

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Notice. After the completion of the manuscript, we learn that, along with other interesting results and discussions, $u^{(4)}$ and $L_4$ have also been calculated by Graham and Reichert [15]. The critical points contained in the paper have also been found in [15].

2. Minimal submanifolds in conformally compact Einstein manifolds

In the first part of this section, referring mainly to [12, 16], we give a brief review to the related aspects of conformally compact Einstein manifolds, and minimal submanifolds in
conformally compact Einstein manifolds. In the second part, we identify the constituent components of \( u^{(4)} \) from the graphic minimal surface equation of \( Y \).

2.1. Minimal surfaces in a conformally compact Einstein manifold. Let \( \overline{X} \) be an \((d+1)\)-dimensional manifold with a \( d\)-dimensional boundary \( M \). We call a function \( r \in C^\infty(\overline{X}) \) a defining function for \( M \) if it satisfies that \( r|_X > 0 \), \( r|_M = 0 \) and \( (dr)|_M \neq 0 \). A metric \( g_+ \) defined on \( X \) is said to be conformally compact if \( r^2 g_+ \) extends to a metric on \( \overline{X} \). Given a conformally compact manifold \( (X, g_+) \), \((r^2 g_+)|_{TM} \) induces a conformal class of metrics as \( r \) runs over the space of defining functions, called the conformal infinity of \( (X, g_+) \) and denoted by \((M, [g_{confinj}]r)\). Let \( g_c = r^2 g_+ \). The sectional curvature of \( (X, g_+) \) is asymptotic to \(-(|dr|^2|_{g_c})/M\), which is independent of the the choices of \( r \), as \( r \to 0 \).

If a conformally compact manifold \( (X, g_+) \) satisfies \( \text{Ric}_{g_+} = -d g_+, (X, g_+) \) is called a conformally compact Einstein manifold. For a conformally compact Einstein metric and any defining function \( r \), \(|dr|^2|_{g_c} = 1 \) on \( M \). Hence conformally compact Einstein manifolds are asymptotic hyperbolic Einstein manifolds. From now on let \((X^{d+1}, g_+)\) be a conformally compact Einstein manifold. For a conformally compact Einstein manifold \((X, g_+)\), as indicated in the introduction, there is a one-to-one correspondence between \( \overline{g} \in [g_{confinj}]r \) and special defining functions defined near \( \partial X \), and the identification \((\text{M, r})\) of \( M \times [0, \epsilon) \) with a neighborhood of \( M \) in \( \overline{X} \) so that \( g_+ = r^{-2}(dr^2 + \overline{g}) \), where an expansion of \( \overline{g} \) is given by \((\text{M, r})\) for \( d \) odd and even respectively. The determined coefficients in these expansions can be calculated and are combinations of \( \overline{g} \) and its derivatives, see \( [9, 13] \). For example, for \( d \geq 3 \) one has

\[
(2.1) \quad g^{(2)} = -\overline{\mathcal{P}} = -\frac{1}{d-2}
\frac{(\text{Ric} - \frac{R}{2(d-1)})\overline{g}},
\]

de \( \overline{\mathcal{P}} \) is the Schouten tensor of \( \overline{g} \). It is well-known that

\[
(2.2) \quad \overline{R}_{ABCD} = \overline{W}_{ABCD} + \overline{P}_{AC} \overline{g}_{BD} + \overline{P}_{BD} \overline{g}_{AC} - \overline{P}_{AD} \overline{g}_{BC} - \overline{P}_{BC} \overline{g}_{AD},
\]

de \( \overline{W} \) is the Weyl tensor of \( \overline{g} \). For \( d \geq 5 \),

\[
(2.3) \quad g^{(4)}_{\alpha\beta} = \frac{1}{4(4-d)} \overline{B}_{\alpha\beta} + \frac{1}{4} \overline{g}^{EF} \overline{P}_{CE} \overline{P}_{DF},
\]

where

\[
(2.4) \quad \overline{B}_{\alpha\beta} = \overline{\Delta} \overline{P}_{\alpha\beta} - \overline{\nabla}^E \overline{\nabla}_D \overline{P}_{CE} + \overline{P}^{EF} \overline{W}_{C\alpha\beta},
\]

is the Bach tensor of \( \overline{g} \).

Let \((X^{d+1}, g_+)\) be a conformally compact Einstein manifold and \( r \) a special defining function such that \( g_c|_M = \overline{g} \). Now we consider a minimal surface \( Y^{n+1}, 0 \leq n \leq d-1 \), of \((X, g_+)\) with the smooth boundary \( \Sigma ^n \hookrightarrow M^d \). Near a point of \( \Sigma \), let \((x^i, y^\alpha)\) be a local coordinate chart as indicated in \((1.6)\), which is extended into a neighborhood of the point in \( \overline{X} \) via the identification \((\text{M, r})\). The second fundamental form of \( \Sigma \) is

\[
(2.5) \quad A_{ij}^\alpha = \overline{g}^{\alpha\beta} \overline{g}(\overline{\nabla}_i \partial_j, \partial_{\beta}) = \frac{1}{2} \overline{g}^{\alpha\beta} \partial_j \overline{g}_{ij} = \overline{\Gamma}_{ij}^\alpha.
\]

We consider minimal surfaces \( Y \) in \((X, g_+)\) which may be written as a graph \( \{y^\alpha = u^\alpha(x, r)\} \). That is near the boundary, \( Y = (x^i, u^\alpha(x, r), r) \). Let

\[
(2.6) \quad h = g_+|_Y, \quad \overline{h} = r^2 h = g_c|_Y = (dr^2 + \overline{g})|_Y,
\]
where \( \bar{g} \) is given by (1.2). Near the boundary, the tangent space of \( Y \) is spanned by

\[
Y_r = \partial_r + u^\gamma_r \partial_\gamma,
\]

and

\[
Y_i = \partial_i + u^\gamma_i \partial_\gamma, \quad i = 1, 2, \ldots, n.
\]

Then

\[
\bar{h}_{rr} = \bar{h}(Y_r, Y_r) = 1 + \bar{g}_{\alpha\beta} u^\alpha_r u^\beta_r,
\]

\[
\bar{h}_{ir} = \bar{h}(Y_i, Y_r) = \bar{g}_{\alpha} u^\alpha_r + \bar{g}_{\beta} u^\beta_r u^\gamma_r,
\]

and

\[
\bar{h}_{ij} = \bar{h}(Y_i, Y_j) = \bar{g}_{ij} + \bar{g}_{\alpha} u^\alpha_i + \bar{g}_{\beta} u^\beta_j + \bar{g}_{\alpha\beta} u^\alpha_i u^\beta_j.
\]

Graham and Witten [16] showed that the graphic minimal surface equation of \( Y \) is

\[
M(u) = 0,
\]

where

\[
M(u)_r = [r \partial_r - (n + 1) + \frac{1}{2} r \partial_r \bar{L}][\bar{h}_{rr} \bar{g}_{\gamma\gamma} u^\gamma_r + \bar{h}_{r} (\bar{g}_{\gamma\gamma} + \bar{g}_{\beta} u^\beta_r)]
\]

\[
+ r[\partial_j + \frac{1}{2} \partial_j \bar{L}][\bar{h}_{ij} \bar{g}_{\gamma\gamma} u^\gamma_r + \bar{h}_{ij} (\bar{g}_{\gamma\gamma} + \bar{g}_{\beta} u^\beta_r)]
\]

\[
- \frac{1}{2} r \bar{h}^{ij} [\partial_i \bar{g}_{ij} + 2 \partial_i \bar{g}_{ji} + \partial_i \bar{g}_{\alpha\beta} u^\alpha_i u^\beta_j]
\]

\[
- r \bar{h}^{ij} [\partial_i \bar{g}_{\alpha} u^\alpha_i + \partial_i \bar{g}_{\beta} u^\beta_i] - \frac{1}{2} r \bar{h}^{rr} \partial_r \bar{g}_{\alpha\beta} u^\alpha_r u^\beta_r,
\]

(2.7)

and

\[
(2.8) \quad \bar{L} = \log \det \bar{h}.
\]

By the choice of the local coordinates \((x^i, y^\alpha)\) on \( M \), we have \( u(x, 0) = 0 \). Graham and Witten [16] found that at \( r = 0 \),

\[
(2.9) \quad u_r = 0,
\]

which means that \( Y \) intersects with \( M \) orthogonally; and for \( n \geq 1 \)

\[
(2.10) \quad u^\alpha_{rr} = -\frac{1}{2n} g^{ij} \bar{g}_{\alpha\beta} \partial_\beta \bar{g}_{ij} = \frac{1}{n} H^\alpha,
\]

here \( H \) is the mean curvature of \( \Sigma \hookrightarrow (M, \bar{g}) \). The expansion of the height functions \( u \) in \( r \), up to a critical order, takes the form of (1.7) for \( n \) odd and of (1.8) for \( n \) even. Namely, for \( n \) odd

\[
u = u^{(2)} r^2 + \text{(even powers)} + u^{(n+1)} r^{n+1} + u^{(n+2)} r^{n+2} + \cdots,
\]

and for \( n \) even

\[
u = u^{(2)} r^2 + \text{(even powers)} + u^{(n)} r^n + w_n r^{n+2} \log r + u^{(n+2)} r^{n+2} + \cdots,
\]

where \( u^{(k)}, k < n + 2 \), and \( w_n \) are locally determined functions of \( \Sigma \).
2.2. Coefficient $u^{(4)}$ in (1.7) and (1.8) for $n \geq 3$. For $n \geq 3$, let

\begin{equation}
(2.11) \quad u = vr^2 + wr^4 + O(r^5),
\end{equation}

where, it follows from (2.10),

\begin{equation}
(2.12) \quad v^\alpha = \frac{1}{2n} H^\alpha.
\end{equation}

It follows from (2.8) that

\[ \partial_r \mathcal{L} = O(r). \]

Note that

\[ \bar{h}_{ir} = O(r^3), \quad \bar{h}^r_r = O(r^3), \]

and the fact that the coefficient $w$ in (2.11) can be derived by tracing the coefficient of $r^3$ in (2.7). First, we have

\[ M(u)_{\gamma} = [r \partial_r - (n+1)]( \bar{r} \bar{g}_{\beta \gamma} u^\beta_r ) + \frac{1}{2} r \partial_r \bar{L} \bar{g}_{\beta \gamma} u^\beta_r \]

\[ + r \partial_j \bar{h}^j \left( \bar{g}_{ij} + \bar{g}_{\beta \gamma} u^\beta_r \right) + \frac{1}{2} r \partial_j \bar{L} \bar{h}^j \left( \bar{g}_{ij} + \bar{g}_{\beta \gamma} u^\beta_r \right) \]

\[ \leq \frac{1}{2} r \bar{h}^j \partial_j \bar{g}_{ij} - r \bar{h}^j \partial_j \bar{g}_{\alpha \beta} u^\beta_r - \frac{1}{2} r \bar{h}^j \partial_j \bar{g}_{\alpha \beta} u^\beta_r + o(r^3). \]

Lemma 2.1. For the right hand side of (2.13), we have

\begin{equation}
(2.16) \quad [r \partial_r - (n+1)]( \bar{r} \bar{g}_{\beta \gamma} u^\beta_r ) + \frac{1}{2} r \partial_r \bar{L} \bar{g}_{\beta \gamma} u^\beta_r \]

\[ \leq -2nr \bar{g}_{\beta \gamma} v^\beta - (n-2)(2g_{\beta \gamma}^{(2)} v^\beta + 2 \partial_\alpha \bar{g}_{\beta \gamma} v^\alpha v^\beta + 4 \bar{g}_{\beta \gamma} w^3)\]

\[ + r^3 \left[ -8|v|^2 \bar{g}_{\beta \gamma} v^\beta + 2g_{\beta \gamma}^{(2)} v^\beta + 2 \partial_\alpha \bar{g}_{\beta \gamma} v^\alpha v^\beta + 4 \bar{g}_{\beta \gamma} w^3 \right] + o(r^3). \]

Proof. Note that

\[ \bar{h}^T = 1 - 4|v|^2 r^2 + o(r^2). \]

Hence

\[ \bar{h}^\alpha \bar{g}_{\beta \gamma} u^\beta_r \]

\[ = (1 - 4|v|^2 r^2) (\bar{g}_{\beta \gamma} + r^2 g_{\beta \gamma}^{(2)} + r^2 \partial_\alpha \bar{g}_{\beta \gamma} v^\alpha) (2r v^\beta + 4r^3 w^3) + o(r^3) \]

\[ = 2r \bar{g}_{\beta \gamma} v^\beta + (-8|v|^2 \bar{g}_{\beta \gamma} v^\beta + 2g_{\beta \gamma}^{(2)} v^\beta + 2 \partial_\alpha \bar{g}_{\beta \gamma} v^\alpha v^\beta + 4 \bar{g}_{\beta \gamma} w^3) r^3 + o(r^3). \]

Therefore,

\[ [r \partial_r - (n+1)]( \bar{r} \bar{g}_{\beta \gamma} u^\beta_r ) = -2nr \bar{g}_{\beta \gamma} v^\beta \]

\[ -(n-2)(-8|v|^2 \bar{g}_{\beta \gamma} v^\beta + 2g_{\beta \gamma}^{(2)} v^\beta + 2 \partial_\alpha \bar{g}_{\beta \gamma} v^\alpha v^\beta + 4 \bar{g}_{\beta \gamma} w^3) r^3 + o(r^3). \]

Note that

\[ \partial_r \mathcal{L} = \bar{h}^\alpha \partial_r \bar{h}^\alpha + \bar{h}^\beta \partial_r \bar{h}_{ij} + o(r) \]

\[ = 8r \bar{g}_{\alpha \beta} v^\alpha v^\beta + g^{ij} (2rg_{ij}^{(2)} + 2 \partial_\alpha \bar{g}_{ij} v^\alpha) + o(r) \]

\[ = r[(8 - 8n)|v|^2 + 2g^{ij} g_{ij}^{(2)}] + o(r), \]
where in the last equality we used (2.25) and (2.12). Hence

\[(2.18)\]  
\[
\frac{1}{2}r \partial_r \mathcal{L}^\alpha r \bar{g}_{\beta \gamma} u^\beta_r = r^3[(8 - 8n)|v|^2 + 2g^{ij} \bar{g}_{\gamma i} (2)\bar{g}_{\gamma j} v^\beta + o(r^3). 
\]

(2.16) then follows from (2.17) and (2.18).

**Lemma 2.2.** For (2.14), we have

\[
\text{Lemma 2.3.} \quad \text{For (2.15), we have}
\]

\[
(2.21)
\]

\[
(2.18)
\]

\[
(2.19)
\]

\[
(2.20)
\]

\[
(2.22)
\]

Proof. It is easy to see that

\[
r \partial_j [\mathcal{L}^i (\bar{g}_{i \gamma} + \bar{g}_{\beta \gamma} u^i_\gamma)] = -r^3 g^{ik} g^j l \partial_j g_{kl} (g_{i \gamma}^2 + \partial_\beta \bar{g}_{i \gamma} v^\beta + \bar{g}_{\beta \gamma} v^\beta) + r^3 g^{ij} \partial_j \bar{g}_{i \gamma} + r^3 g^{ij} (\partial_j \bar{g}_{\beta \gamma} v^\beta + \bar{g}_{\beta \gamma} \partial_j v^\beta) + o(r^3).
\]

It then follows from

\[
\partial_j \bar{g}_{i \gamma} = r^2 (\partial_j g_{i \gamma}^2 + \partial_\beta \partial_\beta \bar{g}_{i \gamma} v^\beta + \partial_\beta \bar{g}_{i \gamma} v^\beta) + o(r^2)
\]

that

\[
r \partial_j [\mathcal{L}^i (\bar{g}_{i \gamma} + \bar{g}_{\beta \gamma} u^i_\gamma)] = -r^3 g^{ik} g^j l \partial_j g_{kl} (g_{i \gamma}^2 + \partial_\beta \bar{g}_{i \gamma} v^\beta + \bar{g}_{\beta \gamma} v^\beta) + r^3 g^{ij} \partial_j \bar{g}_{i \gamma} + r^3 g^{ij} (\partial_j \bar{g}_{\beta \gamma} v^\beta + \bar{g}_{\beta \gamma} \partial_j v^\beta) + o(r^3).
\]

It is easy to see that

\[
\frac{1}{2} r \mathcal{L}^\alpha (\bar{g}_{i \gamma} + \bar{g}_{\beta \gamma} u^i_\gamma)
\]

\[
(2.20)
\]

\[
(2.21)
\]

\[
(2.19)
\]

\[
(2.14)
\]

\[
(2.13)
\]

**Lemma 2.3.** For (2.13), we have

\[
(2.22)
\]

\[
\text{Proof. It is clear that we have}
\]

\[
\bar{h}^{ij} = g^{ij} - g^{ik} g^j l (g_{kl}^2 + \partial_\alpha \bar{g}_{kl} v^\alpha) r^2 + o(r^2),
\]

and

\[
\partial_\gamma \bar{g}_{ij} = \partial_\gamma \bar{g}_{ij} + r^2 (\partial_\gamma g_{ij}^2 + \partial_\beta \partial_\gamma g_{ij} v^\beta) + o(r^2).
\]
Hence by using (2.23), we get

\[- \frac{1}{2} \bar{r}^{ij} \partial_{i} g_{ij} = 2nr\bar{g}_{\beta\gamma}v^{\beta}\]

(2.23)

\[+ \left[ -\frac{1}{2} g^{ij} \partial_{i} g_{ij} - \frac{1}{2} g^{ij} \partial_{j} g_{ij} v^{\beta} - g^{ik} g^{jl} A_{ij}^{\beta} A_{kl}^{\gamma} \right] r^{3} + 2r^{3} g^{kl} A_{kl}^{\beta} \bar{g}_{\gamma} + o(r^{3}).\]

On the other hand, it’s easy to see that

\[- \bar{r}^{ij} \partial_{i} \bar{g}_{\alpha\beta} u_{r}^{\alpha} = - \frac{1}{2} \bar{r}^{ij} \partial_{i} \bar{g}_{\alpha\beta} u_{r}^{\beta}\]

(2.24)

\[= r^{3} \left[ - g^{ij} \partial_{i} \bar{g}_{\alpha\beta} v_{j}^{\alpha} - 2\partial_{r} \bar{g}_{\alpha\beta} v^{\alpha} v^{\beta} \right] + o(r^{3}).\]

(2.22) then follows from (2.23) and (2.24). \(\square\)

Putting (2.16), (2.19) and (2.22) together, we get the following

**Corollary 2.1.** For \(n \geq 3\), we have

\[M(u) = -(n - 2)(2g_{\beta\gamma}v^{\beta} + 2\partial_{\beta} \bar{g}_{\beta\gamma} v^{\alpha} v^{\beta} + 4\bar{g}_{\beta\gamma} v^{\beta}) r^{3}\]

\[+ r^{3} \left[ - 8|v|^{2}\bar{g}_{\beta\gamma} v^{\beta} + 2g^{ij} g^{(2)}_{ij} \bar{g}_{\beta\gamma} v^{\beta} \right] + r^{3} g^{ij} \partial_{i} g_{ij}^{(2)} + \partial_{i} \partial_{j} \bar{g}_{\alpha\beta} v_{i}^{\alpha} v_{j}^{\beta} + \partial_{\beta} \bar{g}_{\alpha\beta} v_{i}^{\alpha} v_{i}^{\beta} + \bar{g}_{\beta\gamma} \partial_{j} v_{i}^{\beta} \]

(2.25)

\[\left[ - g^{kl} \partial_{i} \bar{g}_{ij}^{(2)} + \partial_{i} \partial_{j} \bar{g}_{\alpha\beta} v_{i}^{\alpha} v_{j}^{\beta} - g^{ik} g^{jl} A_{ij}^{\beta} A_{kl}^{\gamma} \right] r^{3} + 2r^{3} g^{kl} A_{kl}^{\beta} \bar{g}_{\gamma} + r^{3} \left[ - g^{ij} \partial_{i} \bar{g}_{\alpha\beta} v_{j}^{\alpha} - 2\partial_{r} \bar{g}_{\alpha\beta} v^{\alpha} v^{\beta} \right] + o(r^{3}).\]

Note that

\[w = u^{(4)},\]

so we have

**Proposition 2.4.** Let \(Y^{n+1}, n \geq 3\), be a minimal surface in \((X^{d+1}, g_{+})\). Then

\[4(n - 2)\bar{g}_{\beta\gamma} u^{(4)\beta} = -2(n - 2)g_{\beta\gamma}v^{\beta} - 8|v|^{2}\bar{g}_{\beta\gamma} v^{\beta} + 2g^{ij} g^{(2)}_{ij} \bar{g}_{\beta\gamma} v^{\beta}\]

(2.26)

\[- g^{ik} g^{jl} A_{ij}^{\beta} \bar{g}_{\beta\gamma} + 2g^{ik} g^{jl} (A_{kl}^{\beta} \bar{g}_{\gamma} + Q_{\gamma}) \]

where

\[Q_{\gamma} = g^{ij} \partial_{j} g_{ij}^{(2)} + \partial_{i} \partial_{j} \bar{g}_{\alpha\beta} v_{i}^{\alpha} v_{j}^{\beta} + \partial_{\beta} \bar{g}_{\alpha\beta} v_{i}^{\alpha} v_{j}^{\beta} \]

\[- g^{kl} \partial_{i} \bar{g}_{ij}^{(2)} + \partial_{i} \partial_{j} \bar{g}_{\alpha\beta} v_{i}^{\alpha} v_{j}^{\beta} \]

(2.27)

\[- 2(n - 2)\partial_{\alpha} \bar{g}_{\beta\gamma} v^{\alpha} v^{\beta} - 2\partial_{r} \bar{g}_{\alpha\beta} v^{\alpha} v^{\beta}.\]
3. Proof of Proposition 1.1

Assume $n \geq 3$. In this section, we will reformulate $Q_\gamma$ to get the expression of $u^{(4)}$, as given by (1.10). Let

$$Q^\alpha = \bar{g}^{\alpha \gamma} Q_\gamma.$$  

It follows from (2.27) that

$$4(n - 2)u^{(4)\alpha} = -2(n - 2)\bar{g}^{\alpha \gamma} g_{\beta \gamma}^{(2)} \nu^\beta - 8|\nu|^2 \nu^\alpha + 2g^{ij} g_{ij}^{(2)} \nu^\alpha$$

(3.1)

$$-g^{jk} g^{il} A_{ij}^{(2)} + 2g^{jk} g^{il} A_{kl}^{(2)} + Q^\alpha,$$

where

$$Q^\alpha = I^\alpha + II^\alpha,$$

and

$$I^\alpha = g^{ij}[\partial_j \nu_i^\alpha - \frac{1}{2}\bar{g}^{\alpha \gamma} \partial_j \partial_\beta \bar{g}_{ij}^\gamma \nu^\beta + \bar{g}^{\alpha \gamma} \partial_j \partial_\beta \bar{g}_{ij}^\gamma \nu^\beta] + 2g^{ij} H_{ij}^{(2)} \nu^\beta$$

(3.2)

$$-2(n - 2)\bar{g}^{\alpha \gamma} \partial_j \eta_i^\beta \nu^\gamma - 2\bar{g}^{\alpha \gamma} \partial_j \eta_i^\beta \nu^\gamma$$

$$-\bar{g}^{\alpha \gamma} g^{kl} \Gamma_{kl}^i \eta_i^\beta (\partial_\beta \bar{g}_{ij}^\gamma \nu^\beta + \bar{g}_{\beta \gamma} \nu^\beta),$$

$$II^\alpha = g^{ij} \bar{g}^{\alpha \gamma} [-\frac{1}{2} \partial_\gamma g_{ij}^{(2)} + \partial_j g_{i\gamma}^{(2)} - \Gamma_{ij}^k g_{k\gamma}].$$

We first compute $II^\alpha$.

Lemma 3.1. We have

(3.3) $$II^\alpha = -\frac{1}{2} \bar{g}^{\alpha \gamma} g^{ij} (\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha \gamma} H_{ij}^{(2)} \nu^\beta.$$

Proof. Let $D$ range from 1 to $d$ so that for $\partial_D = \partial_i$ for $D = i \leq n$ and $\partial_D = \partial_\alpha$ for $D = \alpha \geq n + 1$. We have

$$\partial_\gamma g_{ij}^{(2)} - \partial_j g_{i\gamma}^{(2)} - \partial_i g_{j\gamma}^{(2)}$$

(3.4)

$$= (\bar{\nabla}_\gamma g_{ij}^{(2)} + \Gamma_{ij}^D \nu^D_{ij} + \Gamma_{ij}^D \nu^D_{ij} - (\bar{\nabla}_j g_{i\gamma}^{(2)} + \Gamma_{ji}^D \nu^D_{ji} + \Gamma_{ji}^D \nu^D_{ji})$$

$$- (\bar{\nabla}_\gamma g^{(2)}_{ij} + \Gamma_{ij}^D \nu^D_{ij} + \Gamma_{ij}^D \nu^D_{ij})$$

Hence by using (2.5), we get

$$II^\alpha = g^{ij} \bar{g}^{\alpha \gamma} [-\frac{1}{2} \partial_\gamma g_{ij}^{(2)} + \partial_j g_{i\gamma}^{(2)} - \Gamma_{ij}^k g_{k\gamma}]$$

$$= -\frac{1}{2} \bar{g}^{\alpha \gamma} g^{ij} (\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha \gamma} g^{ij} \Gamma_{ij}^D \nu^D_{ij} - \bar{g}^{\alpha \gamma} g^{ij} \Gamma_{ij}^D \nu^D_{ij}$$

$$= -\frac{1}{2} \bar{g}^{\alpha \gamma} g^{ij} (\bar{\nabla}_\gamma g_{ij}^{(2)} - 2\bar{\nabla}_i g_{j\gamma}^{(2)}) + \bar{g}^{\alpha \gamma} H_{ij}^{(2)} \nu^\beta.$$

We now deal with $I^\alpha$. 

Lemma 3.2. We have

\[ \partial_j v_i^\alpha = \nabla_j^\bot \nabla_i^\bot v^\alpha - g^{\beta\gamma} (A_{ik}, v) A_{jl}^\alpha - v_i^\beta \Gamma_{ij}^\alpha - v_j^\alpha \Gamma_{ij}^\beta + v_k^\alpha \Gamma_{ij}^k \]

(3.5)

On the other hand,

Therefore, (3.5) follows from Lemma 3.3.

We have

\[ g^{ij} [- \frac{1}{2} \bar{g}^{\gamma\delta} \partial_\gamma \partial_\delta \bar{g}_{ij} v^\beta + \bar{g}^{\gamma\delta} \partial_\gamma \partial_\delta \bar{g}_{ij} v^\beta] \]

(3.6)

\[ g^{ij} v^\beta \partial_\beta \Gamma_{ij}^\alpha + g^{ij} \Gamma_{ij}^{\alpha\beta} \bar{g}^{\gamma\delta} \partial_\gamma \partial_\delta \bar{g}_{ij} v^\beta + 2n \bar{g}^{\gamma\delta} v^\beta \partial_\gamma \bar{g}_{ij} v^\delta. \]

Proof. Note that

\[ \Gamma_{ij}^\alpha = \frac{1}{2} \bar{g}^{\gamma\delta} (\partial_i \bar{g}_{j\gamma} + \partial_j \bar{g}_{i\gamma} - \partial_{i\gamma} \bar{g}_{ij} + \frac{1}{2} \bar{g}^{\gamma\delta} (\partial_j \bar{g}_{ik} + \partial_j \bar{g}_{ik} - \partial_k \bar{g}_{ij}), \]

hence

\[ g^{ij} [- \frac{1}{2} \bar{g}^{\gamma\delta} \partial_\gamma \partial_\delta \bar{g}_{ij} v^\beta + \bar{g}^{\gamma\delta} \partial_\gamma \partial_\delta \bar{g}_{ij} v^\beta] \]

\[ = g^{ij} v^\beta (\partial_\beta [\bar{g}^{\gamma\delta} (\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma})] - \partial_\beta \bar{g}^{\gamma\delta} (\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma})) \]

\[ = g^{ij} v^\beta (\partial_\beta [\Gamma_{ij}^\alpha - \frac{1}{2} \bar{g}^{\gamma} (\partial_i \bar{g}_{jk} + \partial_j \bar{g}_{ik} - \partial_k \bar{g}_{ij})] - \partial_\beta \bar{g}^{\gamma\delta} (\frac{1}{2} \partial_\gamma \bar{g}_{ij} + \partial_j \bar{g}_{i\gamma})). \]

\[ \square \]
Note that on $\Sigma^n$

$$\overline{g}^{\alpha k} = 0,$$

hence we have

\[
g^{ij}\left[ -\frac{1}{2}\overline{g}^{\alpha\gamma}\partial_j\partial_{\beta}\overline{g}_{ij}v^\beta + \overline{g}^{\alpha\gamma}\partial_j\partial_{\beta}\overline{g}_{ij}v^\beta \right]
\]

\[
= g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij} + \frac{1}{2}g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij}(\partial_j\overline{g}_{jk} + \partial_j\overline{g}_{ik} - \partial_k\overline{g}_{ij})
\]

\[
- g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij}(-\frac{1}{2}\partial_j\overline{g}_{ij} + \partial_j\overline{g}_{ij})
\]

\[
= g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij} + \frac{1}{2}g^{ij}v^\beta\overline{g}^{\alpha\xi}g^{kl}\partial_{\beta}\overline{g}_{ik}(\partial_j\overline{g}_{jk} + \partial_j\overline{g}_{ik} - \partial_k\overline{g}_{ij})
\]

\[
+ g^{ij}v^\beta\overline{g}^{\alpha\xi}\overline{g}_{ij}\partial_{\beta}\overline{g}_{ij}(\partial_j\overline{g}_{jk} + \partial_j\overline{g}_{ij} - \partial_k\overline{g}_{ij})
\]

\[
= g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij} + g^{ij}v^\beta\overline{g}^{\alpha\xi}\overline{g}_{ij}\partial_{\beta}\overline{g}_{ij} + g^{ij}v^\beta\overline{g}^{\alpha\xi}\overline{g}_{ij}\partial_{\beta}\overline{g}_{ij}.
\]

It then follows from (2.5) and (2.12) that

\[
g^{ij}\left[ -\frac{1}{2}\overline{g}^{\alpha\gamma}\partial_j\partial_{\beta}\overline{g}_{ij}v^\beta + \overline{g}^{\alpha\gamma}\partial_j\partial_{\beta}\overline{g}_{ij}v^\beta \right]
\]

\[
= g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij} + g^{ij}\overline{g}_{ij}\overline{g}^{\alpha\xi}\overline{g}_{ij}\partial_{\beta}\overline{g}_{ij} + 2\overline{g}^{\alpha\xi}v^\beta\partial_{\beta}\overline{g}_{ij}.
\]

That is (3.6). \hfill \Box

**Lemma 3.4.**

(3.7) \[ I^\alpha = \Delta^\perp v^\alpha - g^{ij}g^{kl}(A_{ik}, v)A_{jl}^\alpha + g^{ij}g^{\alpha\gamma}v^\beta\overline{R}_{ij\beta j\gamma} + (4 - 2n)v^\beta\overline{v}\overline{T}_{\beta\eta}^\alpha. \]

Proof. It follows from (3.5), (3.5) and (3.6) that

\[
I^\alpha = \Delta^\perp v^\alpha - g^{ij}g^{kl}(A_{ik}, v)A_{jl}^\alpha
\]

\[
- g^{ij}v^\beta\partial_{\beta}\overline{g}_{ij} + \overline{G}_{ij}^\alpha + \overline{G}_{ij}^\alpha + \overline{G}_{ij}^\alpha - \overline{G}_{ij}^\alpha
\]

\[
+ 2\overline{g}^{\alpha\gamma}\partial_{\beta}\overline{g}_{ij}v^\beta - 2\overline{g}^{\alpha\gamma}\partial_{\beta}\overline{g}_{ij}v^\beta v^\eta.
\]

Note that

\[
\overline{g}^{\alpha\gamma}\overline{R}_{ij\beta\eta} = \overline{g}^{\alpha\gamma}(\overline{\nabla}_{\beta}\overline{\nabla}_{\eta} - \overline{\nabla}_{\eta}\overline{\nabla}_{\beta}, \partial_{\eta})
\]

\[
\overline{g}^{\alpha\gamma}\overline{R}_{ij\beta\eta} = \partial_{\beta}\overline{G}_{ij}^\alpha + \overline{G}_{ij}^\alpha + \overline{G}_{ij}^\alpha - \overline{G}_{ij}^\alpha - \overline{G}_{ij}^\alpha - \overline{G}_{ij}^\alpha + \overline{G}_{ij}^\alpha.
\]

(3.7) then follows from (3.8) and (3.9). \hfill \Box

**Proposition 3.5.** Let $Y^{n+1}$, $n \geq 3$, be a minimal surface in $(X^{d+1}, g_+)$ and of the form $Y = (x, u(x, r), r)$ near the boundary of $X$, where

$$u(x, r) = \frac{1}{2^n}H(x)r^2 + u(4)(x)r^4 + \cdots$$

Then we have

\[
8n(n - 2)u^{(4)}\alpha = \Delta^\perp H^\alpha + g^{ij}g^{kl}(A_{ik}, H)A_{jl}^\alpha - \frac{2}{n^2}H^\beta H^\alpha
\]

\[
+ g^{ij}\overline{g}^{\gamma\beta}H^\beta \overline{R}_{ij\beta j\gamma} + 4\overline{g}^{\gamma\beta}g^{(2)}_{\beta\gamma}H^\beta + 2g^{ij}g^{(2)}_{ij}H^\alpha - 2ng^{ij}g^{(2)}_{ij}A^\alpha_{ij}
\]

\[
- n\overline{g}^{\gamma\beta}g^{ij}(\overline{\nabla}_\gamma g^{(2)}_{ij} - 2\overline{\nabla}_i g^{(2)}_{ij}) - \frac{n - 2}{n}H^\beta H^\alpha T_{\beta\eta}^\alpha.
\]
Proof. It follows from (3.1), (3.7) and (3.3) that

\[
4(n - 2)u^{(4)\alpha} = -2(n - 2)\mathfrak{g}^{\alpha\gamma}g^{(2)}_{\beta\gamma}v^\beta - 8v^2v^\alpha + 2g^{ij}g^{(2)}_{ij}v^\alpha
\]
\[
-\mathfrak{g}^{ik}g^{jl}g^{(2)}_{kl}A_{ij} + 2g^{ik}g^{jl}(A_{kl}, v)A_{ij}
\]
\[
+ \Delta v^\alpha - g^{ij}g^{kl}(A_{ik}, v)A^{(2)}_{jkl} + g^{ij}g^{\alpha\gamma}v^\beta R_{ij\beta\gamma} + (4 - 2n)v^\beta v^\gamma \mathfrak{g}^{\alpha\gamma}g^{(2)}_{\beta\gamma}
\]
\[
- \frac{1}{2}\mathfrak{g}^{\alpha\gamma}g^{ij}(\nabla_\gamma g^{(2)}_{ij} - 2\nabla_i g^{(2)}_{j\gamma}) + \mathfrak{g}^{\alpha\gamma}H^\beta g^{(2)}_{\beta\gamma}.
\]

Using (2.12), we get (3.10).\]

Substituting \(g^{(2)} = -\mathcal{P}\) and using (2.2), one gets (1.10). For \(n = 2\), similar calculations imply the following

**Proposition 3.6.** Let \(Y^3\) be a minimal surface in \((X^{d+1}, g_+)\) and of the form \(Y = (x, u(x, r), r)\) near the boundary of \(X\), where

\[
u(x, r) = \frac{1}{4}H(x)r^2 + w_2(x)r^4 \log r + u^{(4)}(x)r^4 + \ldots
\]

Then we have

\[
-16w_2^\alpha(x) = \Delta \mathfrak{g} + g^{ij}g^{kl}(A_{ik}, H)g^{\alpha}_{jkl} - \frac{1}{2}|H|^2H^\alpha
\]
\[
+ g^{ij}\mathfrak{g}^{\alpha\gamma}H^{\beta\gamma}\nabla_{i\beta}g - 2g^{ij}\mathfrak{g}^{\beta\gamma}H^{\alpha\beta} - g^{ij}(\mathfrak{g}^{\alpha\beta}H - g^{ij}H^\alpha + 4g^{ij}g^{kl}A_{ij})
\]
\[
+ 2g^{ij}\mathfrak{g}^{\alpha\gamma}(\nabla_\gamma \mathfrak{g} - 2\nabla_i \mathfrak{g} + \mathfrak{g}^{\alpha\gamma}H^\beta g^{(2)}_{\beta\gamma}).
\]

4. **Proof of Proposition 1.3**

In this section we prove Proposition 1.3. Namely, we calculate Graham-Witten’s conformal invariant \(L_4\) for closed four dimensional submanifolds. The volume form and the volume expansion of the minimal surface \(Y^{n+1} \hookrightarrow (X^{d+1}, g_+)\) are given by [16]

\[
d\mu_Y = r^{-n-1}\sqrt{\det h}dxdr = r^{-n-1}\left[v(0) + v(2)r^2 + \text{(even powers)} + v(n)r^n + \ldots\right]d\mu_\Sigma dr,
\]

and for \(n\) even

\[
Vol_{g_+}(Y \cap \{r > \epsilon\}) = c_0\epsilon^{-n} + \text{(even powers)} + c_{n-2}\epsilon^{-2} + L_n \log \frac{1}{\epsilon} + c_n + o(1),
\]

where

\[
L_n = \int_{\Sigma} v^{(n)}d\mu_\Sigma
\]

is invariant under conformal transformations of \((M^d, \mathfrak{g})\).

Let \(n \geq 4\) and

\[
u^\alpha(x, r) = v^\alpha(x)r^2 + w^\alpha(x)r^4 + \ldots
\]

where \(v(x) = \frac{1}{2n}H\) and \(w = u^{(4)}\) is given by (3.10). We now compute \(v^{(4)}\) in (4.1). Recall that

\[
\mathcal{H}_{rr} = \mathcal{H}(Y_r, Y_r) = 1 + \tilde{g}_{\alpha\beta}u^\alpha_r u^\beta_r,
\]
\[
\mathcal{H}_{ir} = \mathcal{H}(Y_i, Y_r) = \tilde{g}_{\alpha\beta}u^\alpha_i u^\beta_r,
\]

and

\[
\mathcal{H}_{ij} = \mathcal{H}(Y_i, Y_j) = \tilde{g}_{ij} + \tilde{g}_{\alpha\beta}u^\alpha_i u^\beta_j + \tilde{g}_{\alpha\beta}u^\alpha_j u^\beta_i + \tilde{g}_{\alpha\beta}u^\alpha_i u^\beta_j.
\]
Hence

\[
\overline{h}_{rr} = 1 + u^0_r u^\beta_r \overline{g}_{\alpha\beta}
\]

\[
= 1 + (2rv^\alpha + 4r^3w^\alpha)(2rv^\beta + 4r^3w^\beta)(\overline{g}_{\alpha\beta} + g^{(2)}_{\alpha\beta}r^2 + \partial_\gamma \overline{g}_{\alpha\beta}v^\gamma r^2) + o(r^4)
\]

\[
= 1 + 4r^2|v|^2 + 4r^4(v^\alpha v^\beta \overline{g}_{\alpha\beta} + v^\alpha v^\gamma \partial_\gamma \overline{g}_{\alpha\beta} + 4v^\alpha w^\beta \overline{g}_{\alpha\beta}) + o(r^4),
\]

\[
\overline{h}_{ir} = \overline{g}_{i\alpha} u^\alpha_r + u^\beta_r \overline{g}_{\alpha\beta}
\]

\[
= 2rv^\alpha(g^{(2)}_{i\alpha}r^2 + \partial_\gamma \overline{g}_{i\alpha}v^\gamma r^2) + v^\alpha r^2 2v^\beta r\overline{g}_{\alpha\beta} + o(r^4)
\]

\[
= 2r^3 v^\alpha(g^{(2)}_{i\alpha} + \partial_\gamma \overline{g}_{i\alpha}v^\gamma + v^\beta \overline{g}_{\alpha\beta}) + o(r^4),
\]

and

\[
\overline{h}_{ij} = \overline{g}_{ij} + \overline{g}_{i\alpha} u^\alpha_j + \overline{g}_{j\alpha} u^\alpha_i + \overline{g}_{\alpha\beta} u^\alpha_i u^\beta_j
\]

\[
= g_{ij} + (g^{(2)}_{ij} + \partial_\gamma \overline{g}_{ij}v^\gamma)r^2 + r^4(g^{(4)}_{ij} + \partial_\beta g^{(2)}_{ij}v^\beta + \partial_\gamma \overline{g}_{ij}w^\gamma + \frac{1}{2}\partial_\beta \partial_\gamma \overline{g}_{ij}v^\beta v^\gamma)
\]

\[
+ r^4(g^{(2)}_{ij} + \partial_\gamma \overline{g}_{ij}v^\gamma)v^\alpha + r^4(g^{(2)}_{j\alpha} + \partial_\gamma \overline{g}_{j\alpha}v^\gamma)v^\alpha + r^4 \overline{g}_{\alpha\beta} v^\alpha v^\beta + o(r^4).
\]

We rewrite the above as

\[
\overline{h}_{rr} = 1 + ar^2 + br^4,
\]

\[
\overline{h}_{ir} = a_i r^3,
\]

\[
\overline{h}_{ij} = g_{ij} + a_{ij} r^2 + b_{ij} r^4,
\]

where

\[
a = 4|v|^2,
\]

\[
b = 4(v^\alpha v^\beta \overline{g}_{\alpha\beta}^{(2)} + v^\alpha v^\gamma \partial_\gamma \overline{g}_{\alpha\beta} + 4\langle v, w \rangle),
\]

\[
a_i = 2v^\alpha(g^{(2)}_{i\alpha} + \partial_\gamma \overline{g}_{i\alpha}v^\gamma + v^\beta \overline{g}_{\alpha\beta}),
\]

\[
a_{ij} = g^{(2)}_{ij} - 2\langle A_{ij}, v \rangle,
\]

\[
b_{ij} = g^{(4)}_{ij} - 2\langle A_{ij}, w \rangle + \partial_\beta g^{(2)}_{ij}v^\beta + v^\alpha g^{(2)}_{j\alpha} + v^\alpha g^{(2)}_{i\alpha}
\]

\[
+ \overline{g}_{\alpha\beta} v^\alpha v^\beta + \frac{1}{2}\partial_\beta \partial_\gamma \overline{g}_{ij} v^\beta v^\gamma + \partial_\gamma \overline{g}_{ij} v^\gamma v^\alpha + \partial_\gamma \overline{g}_{j\alpha} v^\gamma v^\alpha.
\]

Let \( A = (a_{ij}), B = (b_{ij}) \). Then

\[
\det \overline{h} = (1 + ar^2 + br^4) \det(\overline{h}_{ij}) + o(r^4)
\]

\[
= \det(\overline{h}_{ij}) + ar^2 \det(g_{ij} + a_{ij} r^2) + br^4 \det(g_{ij}) + o(r^4)
\]

\[
= \det(\overline{h}_{ij}) + ar^2 \det(g_{ij})(1 + g^{ij} a_{ij} r^2) + br^4 \det(g_{ij}) + o(r^4).
\]

Note that

\[
\det(\overline{h}_{ij}) = \det(g_{ij})(1 + g^{ij} a_{ij} r^2 + (g^{ij} b_{ij} + \sigma_2(g^{-1} A)) r^4) + o(r^4).
\]

Therefore,

\[
\det \overline{h} = \det(g_{ij})(1 + Cr^2 + Dr^4) + o(r^4),
\]
where

\[ C = a + g^{ij}a_{ij}, \]
\[ D = a tr(g^{-1}A) + \sigma_2(g^{-1}A) + b + \text{tr}(g^{-1}B). \]

Then

\[
d\mu_Y = r^{-(n+1)}\sqrt{\det h}dxdr
\]

(4.4)

\[
d\mu_Y = r^{-(n+1)}(1 + \frac{C}{2}r^2 + \frac{1}{2}(D - \frac{C^2}{4})r^4 + o(r^4))d\mu_\Sigma dr,
\]

and the coefficient \( v^{(4)} \) in (4.1) is

(4.5)

\[
v^{(4)} = \frac{1}{2}(D - \frac{C^2}{4}).
\]

**Proposition 4.1.** Let \( Y^{n+1}, n \geq 4, \) be a minimal surface in \( (X^{d+1}, g_+) \). Then we have

\[
2v^{(4)} = 4|v|^2(g^{ij}g^{(2)}_{ij} - 4n|v|^2) + \frac{1}{2}(g^{ij}g^{(2)}_{ij} - 4n|v|^2)^2 - \frac{1}{2}g^{ij}_g - 2\langle A_{ij}, v \rangle_g^2
\]

(4.6)

\[
-\frac{1}{4}(g^{ij}g^{(2)}_{ij} - 4(n - 1)|v|^2)^2 + 4g^{ij}_{\alpha\beta}v^\alpha v^\beta - 4(n - 4)(v, w) + g^{ij}_g^{(4)} + I + II,
\]

where

(4.7)

\[
I := g^{ij}_{\alpha\beta}v^\alpha v^\beta + \frac{1}{2}g^{ij}\partial_\beta \partial_\delta g_{ij}v^\alpha v^\beta + 2g^{ij}\partial_\gamma g_{\alpha\beta}v^\gamma v^\alpha + 4v^\alpha v^\beta v^\gamma \partial_\gamma g_{\alpha\beta},
\]

and

(4.8)

\[
II := g^{ij}_{\alpha\beta}v^\alpha v^\beta + 2g^{ij}_{\alpha\beta}v^\alpha.
\]

We first deal with (4.8).

**Lemma 4.2.** We have

(4.9)

\[
II = g^{ij}v^\beta(\nabla_\beta g_{ij}^{(2)} - 2\nabla_j g_{i\beta}^{(2)}) - 4ng_{\alpha\beta}v^\alpha v^\beta + 2g^{ij}\nabla_j(v^\beta g_{ij}^{(2)}).
\]

Proof. Note that

\[
g^{ij}_j(v^\beta g_{ij}^{(2)}) = g^{ij}_j[\partial_j(v^\beta g_{ij}^{(2)}) - \Gamma^k_{ij}v^\beta g_{k\beta}^{(2)}]
\]

\[
= g^{ij}_j[v^\beta g_{ij}^{(2)} + g^{ij}v^\beta \partial_j g_{ij}^{(2)} - g^{ij}v^\beta \nabla_j g_{ij}^{(2)}],
\]

hence

\[
g^{ij}_j(\partial_\beta g_{ij}^{(2)}v^\beta + 2g^{ij}_j v^\beta) - 2g^{ij}_j\nabla_j(v^\beta g_{ij}^{(2)})
\]

\[
= g^{ij}v^\beta(\partial_\beta g_{ij}^{(2)} - 2\partial_j g_{i\beta}^{(2)} + 2\Gamma_k^{ij}g_{k\beta}^{(2)})
\]

\[
= g^{ij}v^\beta(\partial_\beta g_{ij}^{(2)} - \partial_j g_{i\beta}^{(2)} - \partial_\beta g_{ij}^{(2)} + 2\Gamma_k^{ij}g_{k\beta}^{(2)}).
\]

As we have shown in (3.4), we have

\[
\partial_\beta g_{ij}^{(2)} = \partial_j g_{i\beta}^{(2)} - \partial_\beta g_{ij}^{(2)} = \nabla_\beta g_{ij}^{(2)} - \nabla_j g_{i\beta}^{(2)} - \nabla_i g_{j\beta}^{(2)} - 2\Gamma_k^{ij}g_{k\beta}^{(2)}.
\]
Therefore,
\[
g^{ij}(\partial_{\beta} g^{(2)}_{ij}) v^\beta + 2g^{ij} v^\beta \nabla_j (g^{(2)}_{ij}) - 2g^{ij} \nabla_j (v^\beta g^{(2)}_{ij})
\]
\[
= g^{ij} v^\beta (\nabla_{\beta} g^{(2)}_{ij}) - 2\nabla_j g^{(2)}_{ij} (2\Gamma^D_{ij} g^{(2)}_{D\beta} + 2\Gamma^k_{ij} g^{(2)}_{k\beta})
\]
\[
= g^{ij} v^\beta (\nabla_{\beta} g^{(2)}_{ij}) - 2\nabla_j g^{(2)}_{ij} (2\Gamma^D_{ij} g^{(2)}_{D\beta} + 2\Gamma^k_{ij} g^{(2)}_{k\beta})
\]
\[
= g^{ij} v^\beta (\nabla_{\beta} g^{(2)}_{ij}) - 2\nabla_j g^{(2)}_{ij} (2\Gamma^D_{ij} g^{(2)}_{D\beta} + 2\Gamma^k_{ij} g^{(2)}_{k\beta}).
\]

\[\square\]

Lemma 4.3. We have
\[
(4.10) \quad I = -\langle \Delta v, v \rangle + g^{ij} A_{ik} (A_{jl}, v) - g^{ij} v^\alpha v^\gamma \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + g^{ij} \nabla_j (v^\alpha g_{ij} v^\gamma) + 2g^{ij} \nabla_j (v^\alpha \nabla_i v^\gamma) + 4v^\alpha v^\gamma \nabla_i v^\gamma.
\]

Proof. Note that
\[
g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) = g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma).
\]

It then follows from (3.5):
\[
\partial_j v^\alpha = \nabla_j \nabla_i v^\alpha - g^{kl} (A_{ik} v) a_{jl} - v^\alpha \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + 4v^\alpha v^\gamma \nabla_i v^\gamma.
\]

that
\[
I = g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma).
\]

Note that
\[
2g^{ij} \partial_j (g_{ij} v^\gamma) - g^{ij} \partial_j (g_{ij} v^\gamma) + 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) - g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma).
\]

Hence
\[
I = -\langle \Delta v, v \rangle + g^{ij} A_{ik} (A_{jl}, v) + g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + 2g^{ij} \nabla_j (\nabla^\alpha g_{ij} v^\gamma) + 4v^\alpha v^\gamma \nabla_i v^\gamma.
\]

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Recall (3.8):
\[ g^{ij} \left[ - \frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \nabla_j v^\alpha + \bar{g}^\gamma^\beta \partial_j \bar{g}^\gamma v^\beta \right] \]
\[ = g^{ij} v^\alpha \partial_\beta \bar{\Gamma}^\alpha_{ij} + g^{ij} \Gamma^\gamma_{ij} \bar{g}^{\kappa\gamma} \partial_\kappa v^\beta + 2n \bar{g}^{\kappa\gamma} v^\beta v^\gamma \partial_\kappa \bar{g}^{\gamma\eta}, \]

hence
\[ \frac{1}{2} g^{ij} \partial_\beta \partial_\gamma \bar{g}^{\gamma}_{ij} v^\beta v^\gamma - g^{ij} \partial_\gamma \partial_\beta \bar{g}^{\beta}_{ij} v^\gamma \]
\[ = - \bar{g}_{\alpha\gamma} v^\xi g^{ij} \left[ - \frac{1}{2} \bar{g}^{\alpha\gamma} \partial_\gamma \bar{g}^{\xi}_{ij} v^\beta + \bar{g}^\gamma^{\beta\gamma} \partial_\beta \bar{g}^\gamma v^\beta \right] \]
\[ = - g^{ij} \bar{g}_{\alpha\gamma} v^\xi v^\beta \partial_\beta \bar{\Gamma}^\alpha_{ij} - g^{ij} \Gamma^\kappa_{ij} v^\xi v^\beta \partial_\beta \bar{g}^\kappa_{\xi\eta} - 2n v^\xi v^\beta v^\gamma \partial_\kappa \bar{g}^{\gamma\eta}. \]

Then
\[ I = - \langle \Delta v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + g^{ij} \nabla_j \left( \bar{g}_{\alpha\beta} v^\alpha v^\beta + \partial_\gamma \bar{g}_{\alpha\beta} v^\alpha v^\gamma \right) \]
\[ + g^{ij} \bar{g}_{\alpha\gamma} v^\xi v^\beta \left( \partial_j \bar{\Gamma}^\alpha_{ij} + \bar{\Gamma}^\gamma_{ij} \bar{\Gamma}^\alpha_{ij} + \bar{\Gamma}^\alpha_{ij} \bar{\Gamma}^\gamma_{ik} - \partial_\beta \bar{\Gamma}^\alpha_{ij} - \Gamma^\kappa_{ij} \bar{\Gamma}^\alpha_{kj} \right) \]
\[ + (4 - 2n) v^\alpha v^\beta v^\gamma \partial_\kappa \bar{g}_{\alpha\beta}. \]

Recall (3.9):
\[ \bar{g}^{\alpha\beta} \bar{R}_{eta j} = \partial_j \bar{\Gamma}^\alpha_{\beta i} + \bar{\Gamma}^\gamma_{\beta i} \bar{\Gamma}^\alpha_{\gamma j} + \bar{\Gamma}^\kappa_{\beta i} \bar{\Gamma}^\alpha_{\kappa j} - \partial_\beta \bar{\Gamma}^\alpha_{ij} - \Gamma^\kappa_{ij} \bar{\Gamma}^\alpha_{\kappa j} - \bar{\Gamma}^\alpha_{ij} \bar{\Gamma}^\beta_{\gamma j}, \]

and \( \bar{\Gamma}^\alpha_{ij} = A^\alpha_{ij}, H^\gamma = 2 n v^\gamma, \) so we have
\[ I = - \langle \Delta v, v \rangle + g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + g^{ij} \nabla_j \left( \bar{g}_{\alpha\beta} v^\alpha v^\beta + \partial_\gamma \bar{g}_{\alpha\beta} v^\alpha v^\gamma \right) \]
\[ - g^{ij} v^\alpha v^\beta \bar{R}_{\alpha ij} - 2(n - 4) \bar{g}_{\alpha\gamma} v^\xi v^\beta v^\gamma \bar{\Gamma}^\alpha_{\beta \gamma}. \]

It follows from (4.6), (4.10) and (4.9) the following

**Proposition 4.4.** Let \( Y^{n+1}, n \geq 4, \) be a minimal surface in \( (X^{d+1}, g_+) \). The coefficient \( v^{(4)} \) in (4.10) is given by
\[
2 v^{(4)} = - \langle \Delta v, v \rangle - g^{ij} g^{kl} \langle A_{ik}, v \rangle \langle A_{jl}, v \rangle + (4n^2 - 2n - 1) |v|^4
\]
\[
+ 4 (g^{ij} g^{kl}) - \frac{1}{2} [g^{ij}, g^{kl}] + g^{ij} g^{kl}
\]
\[
+ 2 g^{ij} g^{kl} \langle A_{ik}, v \rangle - 2(n - 1) g^{ij} g^{kl} |v|^2 - 4(n - 1) g^{ij} g^{kl} v^\alpha v^\beta - g^{ij} \bar{R}_{\alpha ij} v^\alpha v^\beta
\]
\[
+ g^{ij} v^\beta \left( \bar{\nabla}_{\beta} g_{ij} - 2 \bar{g}_{ij} g_{ij} \right)
\]
\[
- 4(n - 4) \langle v, w \rangle - 2(n - 4) \bar{g}_{\gamma\beta} v^\xi v^\beta v^\gamma \bar{\Gamma}^\alpha_{\beta \gamma}
\]
\[
+ g^{ij} \nabla_j (\bar{g}_{\gamma\beta} v^\gamma v^\beta + \partial_\gamma \bar{g}_{\gamma\beta} v^\gamma v^\beta + 2 \bar{g}_{\gamma\beta} v^\gamma),
\]

where \( v = \frac{1}{2n} H \) and \( w = v^{(4)} \) is given by (3.10).

Notice that the second last line in (4.11) vanishes for \( n = 4 \) and for \( n \geq 5 \) the part involving \( \bar{\Gamma} \) cancels. Note also that the last line is a divergence which is independent of the choice of local coordinates \( (x^i, y^\alpha) \). By integrating (4.11) over a closed submanifold \( \Sigma^d \) and using (2.1), (2.2) and (2.3), we obtain the expression (1.18) of Graham-Witten’s conformal invariant \( L_4 \).
5. $L_4$ for closed submanifolds of Euclidean spaces

In this section, we assume $n = 4, d \geq 5$ and the ambient space is $(M^d, g) = \mathbb{R}^d$. It follows from (1.18) that

$$128L_4 = \int_{\Sigma}(|\nabla^\perp H|^2 - g^{ik}g^{jl}\langle A_{ij}, H \rangle\langle A_{kl}, H \rangle + \frac{7}{16}|H|^4)d\mu_g.$$ 

We now consider the following functional for closed four dimensional submanifolds $F : \Sigma \rightarrow \mathbb{R}^d$

$$\mathcal{L}_4(\Sigma^4) = \frac{1}{2}\int_{\Sigma}(|\nabla^\perp H|^2 - g^{ik}g^{jl}\langle A_{ij}, H \rangle\langle A_{kl}, H \rangle + \frac{7}{16}|H|^4)d\mu_g.$$ 

In the sequel we use local coordinates $(x^i)$ of $\Sigma$ which is normal at the point of consideration with respect to the induced metric

$$g_{ij} := \langle F_i, F_j \rangle.$$ 

**Proposition 5.1.** $F : \Sigma^4 \hookrightarrow \mathbb{R}^d$ is a critical point of $\mathcal{L}_4$ if and only if

$$\mathcal{E} = 0,$$

where

$$\mathcal{E} = \Delta^\perp(\Delta^\perp H + \langle A_{ij}, H \rangle A_{ij} - \frac{7}{8}|H|^2H) + 2\langle A_{ij}, H \rangle\Delta^\perp A_{ij} + 2\langle A_{ij}, \nabla^\perp_i H \rangle\nabla^\perp_j H + 3\langle H, \nabla^\perp_i H \rangle \nabla^\perp_i H$$

$$-\langle A_{ij}, H \rangle\langle A_{ji}, H \rangle A_{ij} - \frac{1}{2}\langle A_{ij}, H \rangle\langle A_{ij}, H \rangle H + \frac{7}{32}|H|^4H$$

(5.2)

$$-2\langle A_{ij}, H \rangle\langle A_{jk}, A_{jl} \rangle A_{kl} + 2\langle A_{ij}, H \rangle\langle A_{ik}, A_{kl} \rangle A_{jl}.$$ 

Proof. Let $X \in T^\perp \Sigma$ be a variation field. We have

$$\delta g_{ij} = -2\langle A_{ij}, X \rangle,$$

$$\delta A_{ij} = \nabla^\perp_i \nabla^\perp_j X - \langle A_{jk}, X \rangle A_{ik} - \langle A_{ij}, \nabla^\perp_k X \rangle F_k,$$

$$\delta H = \Delta^\perp X + \langle A_{kl}, X \rangle A_{kl} - \langle H, \nabla^\perp_k X \rangle F_k,$$

and

$$[\delta \nabla^\perp_i H]^\perp = \nabla^\perp_i (\Delta^\perp X + \langle A_{kl}, X \rangle A_{kl} - \langle H, \nabla^\perp_k X \rangle A_{ik} + \langle A_{ik}, H \rangle \nabla^\perp_k X).$$
Hence

\[ \delta \int_{\Sigma} |\nabla^\perp H|^2 d\mu_g \]

\[ = \int_{\Sigma} \left( 2\langle A_{ij}, X \rangle \langle \nabla_i^\perp H, \nabla_j^\perp H \rangle - |\nabla^\perp H|^2 \langle H, X \rangle \right) d\mu \]

\[ + \int_{\Sigma} \left( 2\langle \nabla_i^\perp (\Delta^\perp X + \langle A_{kl}, X \rangle A_{kl} \rangle - \langle H, \nabla_k^\perp X \rangle A_{ik} + \langle A_{ik}, H \rangle \nabla_k^\perp X, \nabla_i^\perp H \rangle \right) d\mu \]

\[ = \int_{\Sigma} \left( -2(\Delta^\perp)^2 H - 2(\Delta^\perp H, A_{ij}) A_{ij} + 2\langle \nabla_i^\perp H, \nabla_j^\perp H \rangle A_{ij} - |\nabla^\perp H|^2 H, X \rangle \right) d\mu \]

\[ + \int_{\Sigma} \left( 2\langle \nabla_k^\perp (\langle A_{ik}, \nabla_i^\perp H \rangle H - \langle A_{ik}, H \rangle \nabla_i^\perp H \rangle, X \rangle \right) d\mu. \]

We then compute

\[ \delta \int_{\Sigma} (-g^{ik} g^{jl} \langle A_{ij}, H \rangle \langle A_{kl}, H \rangle) d\mu_g \]

\[ = \int_{\Sigma} (-4\langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} + \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H, X \rangle d\mu \]

\[ -2 \int_{\Sigma} \langle A_{ij}, H \rangle \langle \nabla_i^\perp \nabla_j^\perp X - \langle A_{jk}, X \rangle A_{ik}, H \rangle d\mu \]

\[ -2 \int_{\Sigma} \langle A_{ij}, H \rangle \langle A_{ij}, \Delta^\perp X + \langle A_{kl}, X \rangle A_{kl} \rangle d\mu \]

\[ = \int_{\Sigma} (-2\nabla_j^\perp \nabla_i^\perp (\langle A_{ij}, H \rangle H) - 2\Delta^\perp (\langle A_{ij}, H \rangle A_{ij}) , X) d\mu \]

\[ + \int_{\Sigma} (-2\langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} + \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H - 2\langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl}, X \rangle d\mu, \]

and

\[ \delta \int_{\Sigma} \frac{7}{16} |H|^4 d\mu_g = \int_{\Sigma} \left( \frac{7}{4} |H|^2 \langle H, \Delta^\perp X + \langle A_{ij}, X \rangle A_{ij} \rangle d\mu - \int_{\Sigma} \frac{7}{16} |H|^4 \langle H, X \rangle d\mu \right) \]

\[ = \int_{\Sigma} \left( \frac{7}{4} \Delta^\perp (|H|^2 H) + \frac{7}{4} |H|^2 \langle A_{ij}, H \rangle A_{ij} - \frac{7}{16} |H|^4 H \right) d\mu. \]

Therefore

\[ (5.3) \quad \delta \mathcal{L}_4 (X) = - \int_{\Sigma} \langle \mathcal{E}, X \rangle d\mu, \]

where

\[ \mathcal{E} = (\Delta^\perp)^2 H + \langle \Delta^\perp H, A_{ij} \rangle A_{ij} - \langle \nabla_i^\perp H, \nabla_j^\perp H \rangle A_{ij} + \frac{1}{2} |\nabla^\perp H|^2 H \]

\[ - \nabla_k^\perp (\langle A_{ik}, \nabla_i^\perp H \rangle H) + \nabla_k^\perp (\langle A_{ik}, H \rangle \nabla_i^\perp H) \]

\[ + \nabla_i^\perp \nabla_j^\perp (\langle A_{ij}, H \rangle H) + \Delta^\perp (\langle A_{ij}, H \rangle A_{ij}) - \frac{7}{8} \Delta^\perp (|H|^2 H) \]

\[ + \langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} - \frac{1}{2} \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H + \langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl} \]

\[ - \frac{7}{8} |H|^2 \langle A_{ij}, H \rangle A_{ij} + \frac{7}{32} |H|^4 H. \]
Note that
\[ -\nabla^k_i((A_{ik}, \nabla^l_i H)H) + \nabla^k_j((A_{ij}, H)H) + \nabla^k_j((A_{ij}, H)H) \]
\[ = 2(A_{ij}, H)\nabla^l_i \nabla^l_j H + 2(A_{ij}, \nabla^l_i H)\nabla^l_j H + 3(H, \nabla^l_i H)\nabla^l_j H \]
\[ + \Delta^{ij} H, H H + |\nabla^l H|^2 H, \]
and
\[ \nabla^l_i \nabla^l_j H = \Delta^{ij} A_{ij} + \langle A_{ij}, A_{kl} \rangle A_{kl} - \langle A_{ik}, A_{jl} \rangle A_{kl} \]
\[ = \langle A_{ij}, A_{kl} \rangle A_{kl} - \langle A_{ik}, A_{jl} \rangle A_{kl} - \langle A_{ik}, H \rangle A_{jk}, \]

hence
\[ \mathcal{E} = \Delta^{ij} \Delta^{ij} H + \langle A_{ij}, H \rangle A_{ij} - \frac{7}{8} |H|^2 H \]
\[ + \langle A_{ij}, H \rangle A_{ij} + < \Delta^{ij} H, H > - \langle \nabla^l_i H, \nabla^l_j H \rangle A_{ij} + \frac{3}{2} |\nabla^l H|^2 H \]
\[ + 2(A_{ij}, H)\Delta^{ij} A_{ij} + 2(A_{ij}, \nabla^l_i H)\nabla^l_j H + 3(H, \nabla^l_i H)\nabla^l_j H \]
\[ - \langle A_{ij}, H \rangle \langle A_{jk}, H \rangle A_{ik} + \frac{1}{2} \langle A_{ij}, H \rangle \langle A_{ij}, H \rangle H + 3\langle A_{ij}, H \rangle \langle A_{ij}, A_{kl} \rangle A_{kl} \]
\[ - \frac{7}{8} |H|^2 \langle A_{ij}, H \rangle A_{ij} + \frac{7}{32} |H|^4 H \]
\[ + 2\langle A_{ij}, H \rangle (-\langle A_{ik}, A_{jl} \rangle A_{kl} + \langle A_{ik}, A_{kl} \rangle A_{jl}). \]

We search for critical points of \( \mathcal{L}_4 \) of the form
\[(5.5) \quad S^{k_1}(r_1) \times \cdots \times S^{k_m}(r_m), \]
where \( r_i > 0, i = 1, \cdots, m, \) is the radius of a round sphere \( S^{k_i} \hookrightarrow \mathbb{R}^{k_i+1}, \) and \( k_1 \geq k_2 \geq \cdots \geq k_m, \) \( k_1 + \cdots + k_m = 4. \) Submanifolds of the form \((5.5)\) have parallel second fundamental form and parallel mean curvature in the normal bundle. We fix \( r_1 = 1. \)

For \( m = 1, \) using the equation \((5.2)\) we have the critical point of the round sphere and
\[ \mathcal{L}_4(S^4) = 24 Vol(S^4) = 64\pi^2. \]

For \( m = 2 \) and \( k_1 = 3, \) that is to consider \( S^3(1) \times S^1(r_2), \) in the case we have
\[ A = \delta_{11} \nu_1 + \delta_{22} \nu_1 + \delta_{33} \nu_1 + \frac{1}{r_2} \delta_{44} \nu_2, \quad H = 3 \nu_1 + \frac{1}{r_2} \nu_2, \]
where \( \nu_1 \) and \( \nu_2 \) are unit inner normal vector fields. The equation \((5.2)\) now reads
\[ \mathcal{E} = \frac{27}{32} \left( \frac{1}{r_2} \right)^3 \left( \frac{1}{r_2} - 3 \right) \left( \frac{5}{3} \right) \left( -\nu_1 + \frac{1}{r_2} \nu_2 \right), \]
hence we have the following two solutions of the form \((5.5)\) to \((5.1)\)
\[ S^3(1) \times S^1(\frac{1}{\sqrt{3}}), \quad S^3(1) \times S^1(\sqrt{\frac{3}{5}}). \]

The value
\[ \mathcal{L}_4[S^3(1) \times S^1(r_2)] = 2\pi^3 r^2 \left[ \frac{7}{16} (9 + \frac{1}{r_2}^2)^2 - 27 - \frac{1}{r_2} \right]. \]
In particular
\[ \lim_{r_2 \to +\infty} \mathcal{L}_4[S^3(1) \times S^1(r_2)] = +\infty, \quad \lim_{r_2 \to 0} \mathcal{L}_4[S^3(1) \times S^1(r_2)] = -\infty, \]
and
\[ \mathcal{L}_4[S^3(1) \times S^1(\frac{1}{\sqrt{3}})] = 18\sqrt{3}\pi^3, \quad \mathcal{L}_4[S^3(1) \times S^1(\sqrt{\frac{3}{5}})] = 8\sqrt{15\pi^3}. \]

For \( m = 2 \) and \( k_1 = 2 \), that is to consider \( S^2(1) \times S^2(r_2) \), in the case we have
\[ A = \delta_{11}\nu_1 + \delta_{22}\nu_1 + \frac{1}{r_2}\delta_{33}\nu_2 + \frac{1}{r_2}\delta_{44}\nu_2, \quad H = 2\nu_1 + \frac{2}{r_2}\nu_2. \]
The equation (5.2) now reads
\[ \mathcal{E} = \left( \frac{1}{r_2^4} - 1 \right)(-\nu_1 + \frac{1}{r_2}\nu_2), \]
hence we have the following solution of the form (5.5) to (5.1)
\[ S^2(1) \times S^2(1). \]
The value
\[ \mathcal{L}_4[S^2(1) \times S^2(r_2)] = 8\pi^2r_2[7(1 + \frac{1}{r_2})^2 - 8(1 + \frac{1}{r_2})]. \]
In particular
\[ \lim_{r_2 \to +\infty} \mathcal{L}_4[S^2(1) \times S^2(r_2)] = \lim_{r_2 \to 0} \mathcal{L}_4[S^2(1) \times S^2(r_2)] = -\infty \]
and
\[ \mathcal{L}_4[S^2(1) \times S^2(1)] = 96\pi^2. \]
For \( m = 3 \), that is to consider \( S^2(1) \times S^1(r_2) \times S^1(r_3) \), in the case we have
\[ A = \delta_{11}\nu_1 + \delta_{22}\nu_1 + \frac{1}{r_2}\delta_{33}\nu_2 + \frac{1}{r_3}\delta_{44}\nu_3, \quad H = 2\nu_1 + \frac{1}{r_2}\nu_2 + \frac{1}{r_3}\nu_3. \]
Let \( x = \frac{1}{r_2} \) and \( y = \frac{1}{r_3} \). The equation (5.2) now reads
\[ \mathcal{E} = \left[ 16 - (8 + x^2 + y^2) - \frac{7}{2}(4 + x + y) + \frac{7}{16}(4 + x + y)^2 \right] \nu_1 \]
\[ + \left[ 2x^2 - \frac{1}{2}(8 + x^2 + y^2) - \frac{7}{8}(4 + x + y)x + \frac{7}{32}(4 + x + y)^2 \right] \nu_2 \]
\[ + \left[ 2y^2 - \frac{1}{2}(8 + x^2 + y^2) - \frac{7}{8}(4 + x + y)y + \frac{7}{32}(4 + x + y)^2 \right] \nu_3. \]
Then \( \mathcal{E} = 0 \) if \( x = y = 2 \), or \( x = 2, y = \frac{10}{9} \). We have the following two solutions of the form (5.5) to (5.1)
\[ S^2(1) \times S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}), \quad S^2(1) \times S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{3}{\sqrt{10}}). \]
The value
\[ \mathcal{L}_4[S^2(1) \times S^1(r_2) \times S^1(r_3)] = 8\pi^3r_2r_3[\frac{7}{16}(4 + x + y)^2 - (8 + x^2 + y^2)], \]
In particular it is unbounded from above and below, and
\[
\mathcal{L}_4[S^2(1) \times S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})] = 48\pi^3, \quad \mathcal{L}_4[S^2(1) \times S^1(\frac{1}{\sqrt{10}})] = 64\sqrt{5} \cdot \pi^3.
\]
For \( m = 4 \), that is \( S^1(r_1) \times S^1(r_2) \times S^1(r_3) \times S^1(r_4) \), let \( x_i = \frac{1}{r_i} \), in the case we have
\[
\mathcal{E} = \sum_{i=1}^{4} \left[ 2x_i^2 - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2) + \frac{7}{8}(x_1 + x_2 + x_3 + x_4)x_i \right] + \frac{7}{32}(x_1 + x_2 + x_3 + x_4)^2 \frac{\nu_i}{r_i}.
\]
We have in the case only, modulo conformal equivalence, the following two solutions of the form \([3.5]\) to \([5.1]\)
\[
\mathcal{L}_4[S^1(1) \times S^1(1) \times S^1(1) \times S^1(1)] = 24\pi^4, \quad \mathcal{L}_4[S^1(1) \times S^1(1) \times S^1(1) \times S^1(\frac{3}{\sqrt{5}})] = \frac{32\sqrt{5}}{3} \cdot \pi^4.
\]
The value
\[
\mathcal{L}_4[S^1(r_1) \times S^1(r_2) \times S^1(r_3) \times S^1(r_4)] = 8\pi^4 r_1 r_2 r_3 r_4 \left[ \frac{7}{16}(x_1 + x_2 + x_3 + x_4)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2) \right].
\]
In particular it is unbounded from above and below, and
\[
\mathcal{L}_4[S^1(1) \times S^1(1) \times S^1(1) \times S^1(1)] = 24\pi^4, \quad \mathcal{L}_4[S^1(1) \times S^1(1) \times S^1(1) \times S^1(\frac{3}{\sqrt{5}})] = \frac{32\sqrt{5}}{3} \cdot \pi^4.
\]

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