ROBUST TANGENCIES OF LARGE CODIMENSION

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Abstract. We construct $C^2$-robust homoclinic and heterodimensional tangencies of large codimension inside transitive partially hyperbolic sets.

1. Robust cycles, tangencies and transitivity in dynamical systems

The study of uniformly hyperbolic dynamics began with the works of Anosov and Smale in the 60s, and nowadays these systems are well understood from the topological and statistical perspectives (see [25]). Although uniform hyperbolicity was first believed to involve a dense subset of $C^r$-diffeomorphisms of a compact manifold, it soon emerged that this was not true [2, 30, 39]. There are two important mechanisms that yield non-hyperbolic dynamics: heterodimensional cycles and homoclinic tangencies. A diffeomorphism $f$ has a heterodimensional cycle associated with two transitive hyperbolic sets if these sets have different indices (dimension of the stable bundle) and their invariant manifolds meet cyclically. On the other hand, $f$ has a homoclinic tangency if there is a pair of points $P$ and $Q$ belonging to the same transitive hyperbolic set so that the unstable invariant manifold of $P$ and the stable invariant manifold of $Q$ have a non-transverse intersection $Y$. The positive integer $c_T = \dim T_Y W^u(P) \cap T_Y W^s(Q)$ is called the codimension of the tangency (at $Y$) and we say that the tangency is large (or degenerated) if $c_T \geq 2$.

A well-known and important conjecture by Palis [33] claims that these cycles are $C^r$-dense in the complement of hyperbolic systems. In contrast with hyperbolic systems, heterodimensional cycles and homoclinic tangencies are in general not robust. However, important examples of open subsets of non-hyperbolic diffeomorphisms arise from robust cycles, i.e., from heterodimensional cycles or homoclinic tangencies that persist under perturbations. Namely, a $C^r$-diffeomorphism $f$ has a

- $C^r$-robust heterodimensional cycle if there are transitive hyperbolic sets $\Lambda_1$ and $\Lambda_2$ of $f$ with different indices and a $C^r$-neighborhood $\mathcal{U}$ of $f$ such that any $g \in \mathcal{U}$ has a heterodimensional cycle associated with the continuations of $\Lambda_1$ and $\Lambda_2$ for $g$ respectively;

- $C^r$-robust homoclinic tangency (of codimension $c_T > 0$) if there are a transitive hyperbolic set $\Lambda$ and a $C^r$-neighborhood $\mathcal{U}$ of $f$ such that any $g \in \mathcal{U}$ has a homoclinic tangency (of codimension $c_T > 0$) associated with the continuation of $\Lambda$ for $g$.

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In [30, 31], Newhouse constructed surface diffeomorphisms having $C^2$-robust homoclinic tangencies and these results were extended to higher dimensions in [17, 34, 36]. Also in [4, 11] robust homoclinic tangencies are built in the $C^1$-topology and dimension higher than 2. In dimension 2, results by Moreira in [28] imply that there are no $C^1$-robust homoclinic tangencies associated with hyperbolic basic sets. However, in all these constructions the homoclinic tangencies have codimension $c_T = 1$.

The study of bifurcations involving heterodimensional cycles leading to robustly non-hyperbolic dynamics was started in the pioneering papers by Díaz [13, 12]. New examples of robustly non-hyperbolic transitive diffeomorphisms were constructed by Bonatti and Díaz in [10]. The non-hyperbolicity of these examples comes from the existence of a robust heterodimensional cycle and, in particular, from the coexistence of periodic points with different indices inside the same transitive set. The construction of Bonatti and Díaz can be thought as a generalization of previous examples due to Shub and Mañé [38, 27] using a better understanding of the mechanism that provides the robustness of cycles and transitivity. They named this local plug, which is a special transitive hyperbolic set, as blender.

A strong version of the Palis conjecture [11] states that every diffeomorphism can be approximated either by a hyperbolic one or by one with a $C^1$-robust cycle. In view of this, it is important to understand and describe mechanisms generating robust cycles. This is one of the contributions of this paper concerned with two main issues: robust heterodimensional cycles of any co-index (absolute value of the difference of indices of the hyperbolic sets cyclically connected) and robust tangencies of any codimension. Namely, we deal with the construction of $C^2$-robust tangencies (of large codimension) inside transitive partially hyperbolic sets in dimension $d \geq 4$. By a partially hyperbolic set we understand a closed invariant set whose tangent bundle admits a dominated splitting into three non-trivial continuous vector subbundles $E^s \oplus E^c \oplus E^u$ so that $E^s$ is uniformly contracted and $E^u$ is uniformly expanded.

The derivative of the map naturally induces a cocycle on the tangent bundle. Robust heterodimensional cycles constructed for this induced map on the tangent bundle will “project” to robust tangencies for the original diffeomorphism (see Section 2 for a geometrical outline of this idea). To obtain the robust cycle for the tangent bundle dynamics (or namely in Grassmannian bundle manifolds), we will use methods similar to those presented in [6]. We construct blenders (with a higher dimensional center direction) for the induces dynamics which will require the induced map to be $C^1$, and thus the original diffeomorphism to be $C^2$. In contrast, the known examples in higher dimensions with robust tangencies are based on either a dimension reduction using normally-hyperbolic manifolds or on the construction of blenders (with one-dimensional center direction) in the ambient manifold. In addition, in the recent works of Berger [7, 8] robust tangencies were constructed for open sets of parametric families such that the order of the unfolding of the tangency with respect to the parameter is controlled. For this purpose blenders were created in the space of jets. We observe that these are still codimension one tangencies. Our objective in this article is the construction of
higher codimensional tangencies and this involves different ideas such as the dynamics in the tangent bundle.

1.1. Homoclinic tangencies. In the following theorem we obtain robust homoclinic tangencies of large codimension. We observe that the manifold is not necessarily compact, and in this case the set of diffeomorphisms is endowed with the compact open $C^r$-topology.

Theorem A. Every manifold of dimension $d \geq 4$ admits a diffeomorphism with a transitive partially hyperbolic set having a $C^2$-robust homoclinic tangency of codimension $c_T$ which can be chosen to be any integer $0 < c_T \leq \lfloor (d - 2)/2 \rfloor$.

Tangencies in the above result can have large codimension for $d \geq 6$. Notice that in general large homoclinic tangencies require $d \geq 4$. However, if we restrict the dynamics to invariant partially hyperbolic sets, then the above result, on the existence of robust tangencies of large codimension, is actually optimal with respect to the dimension of the manifold. This is because partially hyperbolic sets would require at least two extra dimensions for the stable ($E^s$) and the unstable ($E^u$) directions. Still, the problem of the existence of large tangencies for lower dimensional manifolds (and far from partially hyperbolic sets) remains open.

1.2. Heterodimensional tangencies. The study of non-transverse intersections between stable and unstable manifolds of two hyperbolic sets with different indices led to the notion of a heterodimensional tangency, formally defined below. The dynamical consequences of heterodimensional tangencies were first studied in [14]. But the first examples of robust heterodimensional tangencies associated with a heterodimensional cycle were given later on by Kiriki and Soma in [26].

A $C^r$-diffeomorphism $f$ of a manifold $M$ has a heterodimensional tangency if there are transitive hyperbolic sets $\Lambda_1$ and $\Lambda_2$, points $P \in \Lambda_1$, $Q \in \Lambda_2$ and $Y \in W^u(P) \cap W^s(Q)$ such that

$$\dim T_Y W^u(P) + \dim T_Y W^s(Q) > \dim M \quad \text{and} \quad T_Y M \neq T_Y W^u(P) + T_Y W^s(Q).$$

Observe that the above condition implies $i_1 < i_2$ where $i_1$ and $i_2$ are the indices of $\Lambda_1$ and $\Lambda_2$ respectively. We also call the codimension of the tangency (at $Y$) the positive integer

$$c_T = \dim M - \left[ \dim T_Y W^u(P) + \dim T_Y W^s(Q) - \dim T_Y W^u(P) \cap T_Y W^s(Q) \right].$$

The codimension measures how far the tangential intersection is from a transverse intersection. The above heterodimensional tangency is said to be $C^r$-robust (of codimension $c_T > 0$) associated with $\Lambda_1$ and $\Lambda_2$ if every small enough $C^r$-perturbation $g$ of $f$ has an heterodimensional tangency (of codimension $c_T$) associated with the continuations of $\Lambda_1$ and $\Lambda_2$ for $g$.

The examples in [26] are $C^2$-robust heterodimensional tangencies of codimension $c_T = 1$. The following result provides the first example of robust heterodimensional tangencies of large codimension. We need first some notation. Let us consider a manifold $M$ of dimension $c \geq 3$ and a $C^r$-diffeomorphism $F$ of a manifold $N$ having a hyperbolic set $\Lambda \subset N$ of index $i_F$ conjugated to a full shift with a sufficiently large number of symbols that will depend only on the dimension of $M$. Both of the manifolds, $M$ and $N$, are not necessarily compact.
Theorem B. For any pair of integers \(0 < k < c - 1\) and \(0 < i_<c - k\), there is an arc \(\{f_\varepsilon\}_{\varepsilon \geq 0}\) of \(C^\alpha\)-diffeomorphisms of \(N \times M\) such that \(f_0 = F \times \text{id}\) and for every \(\varepsilon > 0\), any small enough \(C^\beta\)-perturbation \(g\) of \(f_\varepsilon\) has

- a transitive partially hyperbolic set \(\Delta_{\varepsilon} \subset N \times M\) homeomorphic to \(\Lambda \times M\) and
- a heterodimensional tangency (in \(\Delta_{\varepsilon}\)) between basic sets of indices \(i_\varepsilon + i_<c - k\).

The codimension of the tangency can be chosen to be any integer \(0 < c_T \leq \min\{i_<c, c - i_<c - k\}\). Thus we can get tangencies of codimension \(c_T = \lfloor(c - 1)/2\rfloor\) which are large for \(c \geq 5\).

The statement of the above theorem emphasizes the skew-product construction of the maps and the fact that the tangencies can be embedded in a partially hyperbolic invariant set. But tangencies are by themselves local objects and can be created in any manifold (it is enough to take \(N\) and \(M\) of the previous theorem as local charts). These considerations imply as a consequence a similar result as Theorem A for heterodimensional tangencies. In particular, we get that any manifold of dimension \(d \geq 5\) admits a diffeomorphism with a transitive partially hyperbolic set having inside a \(C^\alpha\)-robust heterodimensional tangency of codimension \(c_T = \lfloor(d - 3)/2\rfloor\). Similarly as homoclinic tangencies, heterodimensional tangencies of large codimension need the dimension to be at least 5. This can be checked by considerations on the co-index \(k\) of the transitive hyperbolic sets involved since \(c_T = \dim T_Y W^u(P) \cap T_Y W^s(Q) - k\). However, once again notice that to get a large heterodimensional tangency inside a partially hyperbolic set requires \(d \geq 7\).

1.3. Bundle tangencies. The non-hyperbolicity of a transitive set with homoclinic tangencies or heterodimensional cycles comes, in particular, from the existence of a direction in the tangent space that is exponentially contracted for forward and backward iterations. Let \(f\) be a diffeomorphism of a manifold \(M\) and consider a point \(x \in M\). We call a unitary vector \(v \in T_x M\) a tangent direction if there exist constants \(C > 0\) and \(0 < \lambda < 1\) so that

\[
\|Df^n(x)v\| \leq C\lambda^n
\]

for all \(n \in \mathbb{Z}\).

Recently, in an announcement [20], Gourmelon introduced the notion of a bundle tangency, which allows to unify the different types of tangencies: homoclinic and heterodimensional. Namely, a \(C^\alpha\)-diffeomorphism \(f\) has a bundle tangency between the unstable manifold \(W^u(\Lambda_1)\) and the stable manifold \(W^s(\Lambda_2)\) of transitive hyperbolic sets \(\Lambda_1\) and \(\Lambda_2\) respectively if there are points \(P \in \Lambda_1, Q \in \Lambda_2\) and \(Y \in W^u(P) \cap W^s(Q)\) such that

\[
d_T \overset{\text{def}}{=} \dim T_Y W^u(P) \cap T_Y W^s(Q) > 0 \quad \text{and} \quad T_Y M \neq T_Y W^u(P) + T_Y W^s(Q). \tag{1}
\]

The integer \(d_T\) is called the dimension of the bundle tangency (at \(Y\)). Notice that \(d_T\) is the maximum number of independent tangent directions in \(T_Y M\). Note that (1) is equivalent to

\[
d_T > 0 \quad \text{and} \quad \text{ind}^u(\Lambda_1) + \text{ind}^s(\Lambda_2) - d_T < \dim M.
\]

This forces that \(d_T > \max\{0, i_u - i_s\}\) where \(i_u = \text{ind}^u(\Lambda_2)\) and \(i_s = \text{ind}^s(\Lambda_1)\).

The codimension of the bundle tangency is defined by

\[
c_T \overset{\text{def}}{=} \dim M - [\text{ind}^u(\Lambda_1) + \text{ind}^s(\Lambda_2) - d_T] = d_T - (i_u - i_s).
\]
Homoclinic and heterodimensional tangencies are particular cases of bundle tangencies, where for homoclinic tangencies one just needs to take $\Lambda_1 = \Lambda_2$. In fact, all these cases require that $\text{ind}^u(\Lambda_1) + \text{ind}^s(\Lambda_2) \geq \dim M$. Then it must hold that $i_\omega \leq i_\alpha$ and hence the codimension of tangency is given by $c_T = d_T - k$ where $k = i_\alpha - i_\omega \geq 0$ is the co-index between $\Lambda_1$ and $\Lambda_2$. Having in mind this relation between the dimension of the tangency and the co-index of the involved hyperbolic sets, the robust tangencies of Theorems A and B follow from the next result.

**Theorem C.** For any integer $0 < i_1, i_2 < c$ there is an arc $\{f_\varepsilon\}_{\varepsilon \geq 0}$ of $C^r$-diffeomorphisms of $N \times M$ such that $f_0 = F \times \text{id}$ and for every $\varepsilon > 0$, $f_\varepsilon$ has a

- $C^2$-robust bundle tangency between the unstable and the stable manifold of basic sets $\Gamma^u_1$ and $\Gamma^s_2$
  contained in $\Lambda \times M$ of indices $i_\alpha = i_F + i_1$ and $i_\omega = i_F + i_2$ respectively.

The dimension $d_T$ of the tangency can be chosen to be any integer so that

$$\max\{0, i_2 - i_1\} < d_T \leq \min\{c - i_1, i_2\}.$$  

Moreover, these arcs can be taken in such a way that $f_\varepsilon$ also has a $C^2$-robust bundle tangency between the stable manifold of $\Gamma^u_1$ and the unstable of $\Gamma^s_2$.

### 1.4. Symbolic skew-products.

Besides the construction of robust cycles for partially hyperbolic diffeomorphisms, we introduce blenders and (robust) cycles for bi-Lipschitz skew-product homeomorphisms. Namely we work with skew-products over a shift of finite symbols and whose fiber maps depend Hölder continuously on the base points, that we call **symbolic skew-products**. The above theorems are, in fact, a consequence of the realization, via a straightforward application of known results [18, 19, 24, 6], of a more general construction in this setting (see Theorem 5.12 and Theorem 5.18). Moreover, we give criteria to construct robust heterodimensional cycles (Theorem 4.2) and robust tangencies (Theorem 4.9).

### 1.5. Structure of the paper.

In the next section, we explain the geometric ideas behind the construction of blenders and our new criterion for robust tangencies of large codimension. These constructions will actually be made in the abstract setting of the so-called **symbolic skew products**. Thus, section 3 gives the necessary definitions and properties of symbolic skew-products and defines partial hyperbolicity and blenders in this context. Section 4 provides the criteria for constructing robust cycles and tangencies in symbolic skew-products. In section 5, it is explained how to build an actual example satisfying the previous criteria, but still in the symbolic setting. Finally, in section 6, it is shown how to transfer the examples from the symbolic setting to a realization in smooth manifolds.

### 2. Outline of the method to yield robust tangencies of large codimension

In this section we explain the ideas behind the construction of robust tangencies of large codimension. To accomplish this task, we need first to introduce the class of blenders that we will consider and explain how these local mechanisms are used to construct robust cycles.
2.1. Blenders. In our constructions, we will consider a particular class of blenders obtained by the covering property criterion:

A compact invariant set $\Gamma$ of a diffeomorphism $f$ of a manifold $\mathcal{M}$ is a *cs-blender* if

i) there is an open neighborhood $\mathcal{U}$ of $\Gamma$ such that $\Gamma$ is the maximal invariant set in $\overline{\mathcal{U}}$,

ii) the set $\Gamma$ is transitive, (uniformly) hyperbolic with a dominated splitting

$$T_\Gamma \mathcal{M} = E^{ss} \oplus E^s \oplus E^{cu} \oplus E^{uu},$$

where $E^c = E^{cs} \oplus E^{cu}$ with $\dim E^{cs} > 0$ and $E^{ss} \oplus E^c \oplus E^{cu} \oplus E^{uu}$ being the contracting and the expanding bundle respectively;

iii) there are an open set $\mathcal{B} \subset \mathcal{U}$, called the *superposition domain*, and integers $n_1, \ldots, n_k \in \mathbb{N}$ such that $f^{n_1}(\mathcal{B}) \cup \cdots \cup f^{n_k}(\mathcal{B})$ intersects $\mathcal{B}$ as is shown in Figure 1 and explained below.

The superposition domain in Figure 1 is the set $\mathcal{B} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ where we write $\mathcal{H}_i = H_i \times \mathcal{B}$ so that $H_i$ is an open "rectangle" in $E^{ss} \oplus E^{uu}$ and $\mathcal{B}$ is an open set in $E^c$. Let $B_i$ be

![Figure 1](image-url)

*Figure 1.* This figure shows the case of a cs-blender in dimension 4 with $\dim E^{cs} > 0$, $\dim E^{cu} > 0$. Figure 1(a) represents the superposition domain $\mathcal{B} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ and $f^{n_1}(\mathcal{H}_1) \cup f^{n_2}(\mathcal{H}_2) \cup f^{n_3}(\mathcal{H}_3)$ in the section $E^{ss} \oplus E^{cs} \oplus E^{uu}$ and Figure 1(b) in the section $E^{ss} \oplus E^{cu} \oplus E^{uu}$. Figure 1(c) is the projection onto the central direction, $E^c = E^{cs} \oplus E^{cu}$, showing the corresponding covering property.
the projections of $f^n(H_i)$ onto $E_c$. The covering property is defined by the condition

$$\overline{B} \subset \bigcup_i B_i,$$

where the Lebesgue number of the cover is large enough in relation to the transverse variation of the strong stable leaves (see Theorem 3.8). We will refer to $B$ as the blending region.

Let $\Lambda^u = \Lambda^u(B; f)$ be the set of points $P \in B$ such that $f^{-n_i} \circ \cdots \circ f^{-n_1}(P) \in B$ for some sequence of integers $n_i \in \{n_1, \ldots, n_k\}$. Observe that when all of $n_i = 1$ for all $i = 1, \ldots, k$, then $\Lambda^u$ is simply the maximal forward invariant set in $B$, and furthermore it holds that

$$\Lambda^u \cap W_{\text{loc}}^{\text{ss}}(P) \neq \emptyset$$

for all $P \in \mathcal{B}$.

The main property of a superposition domain $\mathcal{B}$ of a $cs$-blender $\Gamma$ for $f$ is the following:

$$\Lambda^u \cap W_{\text{loc}}^{\text{ss}}(P) \neq \emptyset \quad \text{for all } P \in \mathcal{B}.$$  \hfill (2)

Indeed, the covering property implies the intersection property: given any $P \in \mathcal{B}$, by the covering property (see figure 1(a)), the local strong stable manifold $D^s = W_{\text{loc}}^{\text{ss}}(P)$ of $P$ goes through some $f^{n_i}(H_i\cdot)$. Then the pre-image $f^{-n_i}(D^s)$ contains a new disc $D^s_1$, which is the strong stable manifold of another point in $\mathcal{B}$. This allows us to repeat the process inductively.

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**Figure 2. Criterion for robust cycles**
Figure 3. Robust cycle between a cs and cu-blender as seen on $E^c$. The sets $B_1 \equiv \mathcal{B}_1$ and $B_2 \equiv \mathcal{B}_2$ are, respectively, a cs and cu-blending region. The maps $T_1 \equiv f^j$ and $T_2 \equiv f^k$ are the two transitions.

and get a sequence of discs

$$D_j^s \subset f^{-n_j} \circ \cdots \circ f^{-n_1}(D^s) \quad \text{in} \quad \mathcal{B}.$$ 

Hence, by the hyperbolicity of $f$ on $\mathcal{B}$, the nested sequence $f^{n_1} \circ \cdots \circ f^{n_j}(D^s) \subset D^s$ defines a point in $\Lambda^s \cap W_{ss}^{loc}(P)$. Moreover, since the covering property of a cs-blender persists under $C^1$-perturbations of $f$, the above property (2) is $C^1$-robust.

Remark 2.1 (cu-blender and double-blender). A cu-blender (for $f$) is a cs-blender for $f^{-1}$. Similarly, the cu-covering property implies that $C^1$-robustly

$$\Lambda^s \cap W_{uu}^{loc}(P) \neq \emptyset \quad \text{for all} \quad P \in \mathcal{B} \quad \text{where} \quad \Lambda^s = \Lambda^s(B; f) \overset{\text{def}}{=} \Lambda^u(B; f^{-1}). \quad (3)$$

We say that $\Gamma$ is a double-blender if it is simultaneously both, a cs-blender and a cu-blender for $f$.

2.2. Criterion for robust cycles. Let $f$ have a cs-blender $\Gamma_1$ and a cu-blender $\Gamma_2$ with superposition domains $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Assume there are $P \in \Gamma_2$ and $Q \in \Gamma_1$ so that

$$W_{ss}(P) \text{ and } W_{uu}(Q) \text{ meets } \mathcal{B}_1 \text{ and } \mathcal{B}_2$$

This implies there are $X \in \mathcal{B}_2 \cap W^s(\Gamma_2)$, $Y \in \mathcal{B}_1 \cap W^u(\Gamma_1)$ and integers $j, k \in \mathbb{N}$ so that

$$W_{loc}^{ss}(f^{-i}(X)) \text{ crosses } \mathcal{B}_1 \text{ and } W_{loc}^{uu}(f^k(Y)) \text{ crosses } \mathcal{B}_2,$$ 

Then (2) and (3) imply that

$$W^s(\Gamma_1) \cap W^u(\Gamma_2) \neq \emptyset \quad \text{and} \quad W^u(\Gamma_1) \cap W^s(\Gamma_2) \neq \emptyset \quad C^1\text{-robustly.}$$

In Figure 3 the heterodimensional cycle is re-interpreted via the projection mappings on $E^c$. 
Figure 4. The figure shows the covering properties of the blenders of $\hat{f}$ in the product manifold and the tangent space as seen in the central direction. The superposition domains are $\hat{B}_1 = B_1 \times E_1$.

Condition (T1) implies that the covering property occurs simultaneously for the regions $B_i$ on the manifold, and $E_i$ which are inside the respective cones.

2.3. Criterion for robust tangencies. Let $f$ be as above, having a $cs$-blender $\Gamma_1$ and a $cu$-blender $\Gamma_2$ with superposition domains $B_1$ and $B_2$ respectively. Observe that $f$ induces a map $\hat{f}$ on the set of $\ell$-dimensional subspaces of the tangent bundle given by

$$\hat{f}(x, V) = (f(x), Df(x)V) \quad \text{where } x \in M \text{ and } V \subset T_xM \text{ with dim } V = \ell.$$ 

To create robust tangencies the key idea is to construct a robust cycle as before, but for the map $\hat{f}$. To accomplish this we will need a cone field $C^{uu}$ of dimension $\ell$ on $B_1$ which is $n$-th forward invariant and expanded by $n$-th forward iteration: there are $C > 0$ and $0 < \lambda < 1$ so that for every $n \in \mathbb{N}$ and $x \in B \cap f^{-n}(B)$,

$$Df^n(x)C^{uu}_x \subset \text{int}(C^{uu}_{f^n(x)}) \quad \text{and} \quad \|Df^n(x)v\| \geq C\lambda^n\|v\| \text{ for all } v \in C^{uu}_x.$$ 

Also, assume there is a cone field $C^{ss}$ on $B_2$ of dimension $\ell$ which is $n$-th backward invariant and expanded by $n$-th backward iteration. To construct the cycle for $\hat{f}$, we require a similar criterion as (4), that is, two blenders for $\hat{f}$ and a connection between them along the strong stable manifold, see Figure 4 and compare with Figures 2 and 3.

- **(T1)** There are a $cs$-blender $\hat{\Gamma}_1$ and a $cu$-blender $\hat{\Gamma}_2$ for $\hat{f}$ with superposition domains $\hat{B}_1$ and $\hat{B}_2$ respectively so that $\hat{B}_1 \subset B_1 \times C^{uu}$ and $\hat{B}_2 \subset B_2 \times C^{ss}$, where by $B \times C$ we denote the set of points of the form $(x, v)$ with $x \in B$ and $v \in C_x$.

- **(T2)** There is $X \in \Lambda^{s}(\hat{B}_2)$ and $j \in \mathbb{N}$ so that $W^{ss}_\text{loc}(\hat{f}^{-j}(X))$ crosses $\hat{B}_1$. 
This cycle provides a (robust) tangency as is outlined in the next argument. Condition (T2) implies that $X$ belongs to $W^s(\hat{\Gamma}_2)$ and $\hat{f}^{-i}(X)$ is in the superposition domain $\hat{\Gamma}_1$ of the blender $\hat{\Gamma}$. Using the property (2) of the blender, there is a point $y = (y, V)$ in the intersection between $\Lambda^u(\hat{\Gamma}_1)$ and $W^u_{loc}(\hat{f}^{-i}(X))$. Then $y$ will be a point of tangency between $W^u(\Gamma_1)$ and $W^u(\Gamma_2)$ for the original map $f$, with the tangent vector space $V$. Indeed, by (T1), we have that the projection of $\Lambda^u(\hat{\Gamma}_1)$ on the manifold is contained in $\Lambda^u(\hat{\Gamma}_2)$, and analogously for $\Lambda^s(\hat{\Gamma}_2)$. Also, since $\hat{f}^{i}(y)$ is in the projection of $W^u_{loc}(X) \subset \Lambda^s(\hat{\Gamma}_2)$ then

$$y \in \Lambda^u(\hat{\Gamma}_1) \subset W^u(\Gamma_1), \quad \hat{f}^{i}(y) \in \Lambda^s(\hat{\Gamma}_2) \subset W^s(\Gamma_2)$$

Now the backward iterates of $Y$ by $\hat{f}$ go to $\hat{\Gamma}_1$, and in fact are in $\hat{\Gamma}_1$ for some sequence of iterates $\hat{f}^{-m_i}$, $m_i \to \infty$. Since $\hat{\Gamma}_1 \subset B_1 \times C^{au}$ and $\hat{f}^{-m_i}(Y) = (f^{-m_i}(y), Df^{-m_i}(y)V)$, then

$$f^{-m_i}(y) \in B_1 \cap f^{-m_i}(\hat{\Gamma}_1) \quad \text{and} \quad Df^{-m_i}(y)V \in C^{au}(\hat{f}^{-m_i}(y)).$$

Consequently, for a vector $v \in V$, $Df^{-m_i}(y)v$ belongs to the expanding cone field $C^{au}$ and thus $\|Df^{-m_i}(y)v\| \leq C\lambda^m_i\|v\|$, which implies $\|Df^{-n}(y)v\| \to 0$ as $n \to \infty$. Similarly also $\|Df^n(y)v\| \to 0$ exponentially fast as $n \to \infty$. Therefore $V$ is a tangent vector space and hence we obtain a (robust) tangency of dimension $\dim V = \ell$.

**Remark 2.2.** When $\Gamma_1 = \Gamma_2 = \Gamma$, we conclude a (robust) homoclinic tangency of codimension $\ell$ associated with the double-blender $\Gamma$.

### 3. Preliminaries:

Let $\mathcal{A}$ be a finite set (with at least two points), that we call an alphabet of symbols, and fix $0 < \nu < 1$ and $0 < \alpha \leq 1$. Consider the product space $\Sigma \equiv \Sigma(\mathcal{A}, \nu) \equiv \mathcal{A}^\mathbb{Z}$ of the bi-sequences $\xi = (\xi_i)_{i \in \mathbb{Z}}$ of symbols in $\mathcal{A}$ endowed with the metric

$$d_\Sigma(\xi, \zeta) \overset{\text{def}}{=} \nu^{|i|}, \quad \ell = \min\{i \geq 0 : \xi_i \neq \zeta_i \text{ or } \xi_{-i} \neq \zeta_{-i}\}.$$  

We remark that results in §3 and §4 are valid relaxing the finiteness of the alphabet assuming that $\mathcal{A}$ is a compact metric space and $\Sigma$ is endowed with the metric given in [5, Example 2.1]. Let $M$ be a topological manifold (not necessarily compact and not necessarily boundaryless).

#### 3.1. Symbolic skew-products.

Given a compact set $K$ in $M$, we consider the pseudometric in the set $C^0(M)$ of continuous functions of $M$ given by

$$d_{C^0}(\phi, \psi)_K \overset{\text{def}}{=} \max_{x, y \in K} d(\phi(x), \psi(x)) \quad \text{for any } \phi, \psi \in C^0(M).$$  

Since $M$ is $\sigma$-compact, there is a sequence of relatively compact subsets $K_n$ whose union is $M$ and then we can endow $C^0(M)$ with the weak topology (also called compact-open topology) induced by the family of pseudometrics (5). That is,

$$d_{C^0}(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_{C^0}(\phi, \psi)_{K_n}}{1 + d_{C^0}(\phi, \psi)_{K_n}}, \quad \text{for any } \phi, \psi \in C^0(M).$$
3.1.1. The sets of skew-products. We consider skew-product homeomorphisms of the form

\[ \Phi : \Sigma \times M \to \Sigma \times M, \quad \Phi(\xi, x) = (\tau(\xi), \phi_{\xi}(x)) \]  

where the base map \( \tau : \Sigma \to \Sigma \) is the lateral shift map and the fiber maps \( \phi_{\xi} : M \to M \) are homeomorphisms of \( M \). In order to emphasize the role of the fiber maps we write \( \Phi = \tau \circ \phi_{\xi} \) and call it a \textit{symbolic skew-product}. When no confusion arises we also write \( M = \Sigma \times M \).

For every \( n > 0 \) and \((\xi, x) \in M\) set

\[ \phi^{n}_{\xi}(x) \overset{\text{def}}{=} \phi_{\tau^{-1}(\xi)} \circ \cdots \circ \phi_{\xi}(x) \quad \text{and} \quad \phi^{-n}_{\xi}(x) \overset{\text{def}}{=} \phi_{\tau^{-1}(\xi)}^{-1} \circ \cdots \circ \phi_{\xi}^{-1}(x) \]  

and hence

\[ \Phi^{n}(\xi, x) = (\tau^{n}(\xi), \phi^{n}_{\xi}(x)) \quad \text{for all} \ n \in \mathbb{Z}. \]

We introduce the set of symbolic skew-products with which we will work:

\textbf{Definition 3.1.} Denote by \( S(M) \equiv S_{\alpha, \nu}^{\gamma}(M) \) the set of \( \alpha \)-H"older symbolic skew-product homeomorphisms of \( M \). This is, the set of symbolic skew-products \( \Phi = \tau \circ \phi_{\xi} \) as in (6) such that

\begin{itemize}
  \item \( \phi_{\xi} \) are positive bi-Lipschitz homeomorphisms (uniform in \( \xi \)): there are positive constants \( \gamma \equiv \gamma(\Phi) > 0 \) and \( \gamma \equiv \gamma(\Phi) > 0 \) such that

\[ \gamma d(x, y) < d(\phi_{\xi}(x), \phi_{\xi}(y)) < \gamma^{-1} d(x, y), \quad \text{for all} \ x, y \in M \ \text{and} \ \xi \in \Sigma, \]  

\[ d_{C^{\gamma}}(\phi_{\xi}^{<1}, \phi_{\xi}^{<1}) \leq C_{0} d_{\Sigma}(\xi, \zeta)^{\alpha} \quad \text{for all} \ \xi, \zeta \in \Sigma \ \text{with} \ \xi_{0} = \zeta_{0}. \]  

\end{itemize}

We define in \( S(M) \) the metric

\[ d_{S}(\Phi, \Psi) \overset{\text{def}}{=} d_{0}(\Phi, \Psi) + \text{Lip}_{0}(\Phi, \Psi) + \text{Hol}_{0}(\Phi, \Psi) \]

where the symbolic skew-products \( \Phi = \tau \circ \phi_{\xi} \) and \( \Psi = \tau \circ \psi_{\xi} \) belong to \( S(M) \) and

\[ \text{Lip}_{0}(\Phi, \Psi) \overset{\text{def}}{=} \max_{\xi \in \Sigma} \left| \text{Lip}(\phi_{\xi}^{+1}) - \text{Lip}(\psi_{\xi}^{+1}) \right|, \]

\[ d_{0}(\Phi, \Psi) \overset{\text{def}}{=} \max_{\xi \in \Sigma} d_{C^{\gamma}}(\phi_{\xi}^{+1}, \psi_{\xi}^{+1}) \quad \text{and} \quad \text{Hol}_{0}(\Phi, \Psi) \overset{\text{def}}{=} \left| C_{0}(\Phi) - C_{0}(\Psi) \right| \]

with

\[ \text{Lip}(\phi) = \sup_{x \neq y} \frac{d(\phi(x), \phi(y))}{d(x, y)} \quad \text{being} \ \phi \ \text{a bi-Lipschitz homeomorphism of} \ M. \]

An important class of \( \alpha \)-H"older continuous symbolic skew-products is the following:

\textbf{Definition 3.2.} A symbolic skew-product \( \Phi = \tau \circ \phi_{\xi} \in S(M) \) is partially hyperbolic, if

\[ \nu^{<} < \gamma < 1 < \gamma^{-1} < \nu^{>} \]

where \( \gamma \) and \( \gamma \) are given in (8). We denote by \( \mathcal{PHS}(M) \equiv \mathcal{PHS}_{\alpha, \nu}^{\alpha}(M) \) the set of partially hyperbolic symbolic skew-products. Notice that \( \mathcal{PHS}(M) \) is an open subset of \( S(M) \).
3.1.2. Stable and unstable sets for skew-products. We define the local stable and unstable set of the lateral shift map $\tau : \Sigma \to \Sigma$ at $\xi \in \Sigma$ respectively as

$$W^s_{\text{loc}}(\xi) \equiv W^s_{\text{loc}}(\xi; \tau) \overset{\text{def}}{=} \{ \zeta \in \Sigma : \zeta_i = \xi_i, \ i \geq 0 \},$$

$$W^u_{\text{loc}}(\xi) \equiv W^u_{\text{loc}}(\xi; \tau) \overset{\text{def}}{=} \{ \zeta \in \Sigma : \zeta_i = \xi_i, \ i < 0 \}.$$  

The (global) stable set of the skew-product map $\Phi : M \to M$ at $P \in M$ is defined as

$$W^s(P) \equiv W^s(P; \Phi) \overset{\text{def}}{=} \{ Q \in M : \lim_{n \to \infty} d(\Phi^n(Q), \Phi^n(P)) = 0 \}.$$  

We define the (global) stable set of a compact $\Phi$-invariant set, i.e. so that $\Phi(\Gamma) = \Gamma$, by

$$W^s(\Gamma) \equiv W^s(\Gamma; \Phi) \overset{\text{def}}{=} \{ P \in M : \lim_{n \to \infty} d(\Phi^n(P), \Gamma) = 0 \}$$  

or equivalently as the set of the points of $M$ so that its $\omega$-limit is contained in $\Gamma$. The set $\Gamma$ is called isolated (or maximal invariant set) if there is a compact neighborhood $U$ of $\Gamma$, called the isolating neighborhood for $\Gamma$, such that every invariant subset of $U$ lies in $\Gamma$. In such a case, we introduce the local stable set of $\Gamma$ as the forward invariant set of $\Phi$ in the isolating neighborhood $U$, that is,

$$W^s_{\text{loc}}(\Gamma) \equiv W^s_{\text{loc}}(\Gamma; \Phi) \overset{\text{def}}{=} \{ P \in M : \Phi^n(P) \in U \text{ for } n \geq 0 \} = \bigcap_{n \geq 0} \Phi^n(U).$$

Similarly $W^u_{\text{loc}}(\Gamma) \equiv W^u_{\text{loc}}(\Gamma; \Phi)$ and $W^u(\Gamma) \equiv W^u(\Gamma; \Phi)$ are, respectively, the local unstable set and the global unstable set of $\Gamma$. We have that

$$W^s(\Gamma) = \bigcup_{n \geq 0} \Phi^{-n}(W^s_{\text{loc}}(\Gamma)) \quad \text{and} \quad W^u(\Gamma) = \bigcup_{n \geq 0} \Phi^n(W^u_{\text{loc}}(\Gamma)).$$

Finally, given an $S$-perturbation of $\Phi$, that is a symbolic skew-product $\Psi$ close to $\Phi$ in the metric given in (10), we denote by $\Gamma_\Psi$ the maximal invariant set in $U$ of $\Psi$. Although isolated sets vary, a priori, just upper semicontinuously by an abuse of terminology, we call $\Gamma_\Psi$ the continuation of $\Gamma$ for $\Psi$.

3.1.3. Strong laminations for partially hyperbolic skew-products. Under the global assumption of domination introduced in Definition 3.2, the usual graph transform argument yields a local strong stable $W^{ss}$ and unstable $W^{uu}$ partition for partially hyperbolic symbolic skew-products:

**Proposition 3.3** ([5, 1]). For every $\Phi \in \mathcal{FS}(M)$ there exist unique partitions

$$W^{ss} = \{ W^{ss}_{\text{loc}}(\xi, x) : (\xi, x) \in M \} \quad \text{and} \quad W^{uu} = \{ W^{uu}_{\text{loc}}(\xi, x) : (\xi, x) \in M \}$$

of $M = \Sigma \times M$ such that it holds

i) every leaf $W^{ss}_{\text{loc}}(\xi, x)$ is the graph of an $\alpha$-Hölder function $\gamma^{ss}_{\xi, x} : W^{s}_{\text{loc}}(\xi) \to M$ with $\alpha$-Hölder constant less or equal than $C_0 \cdot (1 - \gamma^{-1})^{-1} \geq 0$;

ii) $W^{ss}_{\text{loc}}(\xi, x)$ varies continuously with respect to $(\xi, x)$, i.e., the map $(\xi, \xi', x) \mapsto \gamma^{ss}_{\xi, x}(\xi')$ is continuous where $(\xi, \xi')$ varies in the space of pairs of points in the same local stable set for $\tau$. Furthermore, it depends continuously on $\Phi$;
iii) $\Phi(W_{\text{loc}}^{ss}(\xi,x)) \subset W_{\text{loc}}^{ss}(\Phi(\xi,x))$ for all $(\xi,x) \in \mathcal{M}$;

iv) $W_{\text{loc}}^{ss}(\xi,x) \subset W^u(\xi,x)$ for all $(\xi,x) \in \mathcal{M}$.

The partition $W_{uu}$ verifies analogous properties.

Each leaf of the partition $W^{ss}$ is called the local strong stable set. We define the (global) strong stable set of $\Phi$ at $P$ as

$$W^{ss}(P) \equiv W^{ss}(P;\Phi) \overset{\text{def}}{=} \bigcup_{n \geq 0} \Phi^{-n}(W_{\text{loc}}^{ss}(\Phi^n(P))) \subset W^s(P).$$

Let $\Gamma$ be an isolated set. We define the local strong stable set of $\Phi$ at $\Gamma$ as

$$W_{\text{loc}}^{ss}(\Gamma) \equiv W_{\text{loc}}^{ss}(\Gamma;\Phi) \overset{\text{def}}{=} \bigcup_{P \in \Gamma} W_{\text{loc}}^{ss}(P).$$

In the same manner, the (global) strong stable set of $\Phi$ at $\Gamma$, $W^{ss}(\Gamma)$, is defined. Also the definitions of local/global strong unstable sets of $\Phi$ at $\Gamma$ are analogous.

3.2. Blenders in symbolic skew-products. Roughly speaking, a blender is a basic (hyperbolic) set of a dynamical system, which provides that a non-transversal intersection between stable/unstable manifolds becomes a robust intersection. In this section, we will first introduce the notion of hyperbolic set for symbolic skew-products homeomorphisms. After that we give the formal definition of blenders and finally we provide a criterion to obtain these local tools.

3.2.1. Hyperbolic sets. The following definitions and results come from the literature on hyperbolic homeomorphisms [3, 32].

Fix $\varepsilon > 0$ small enough. We introduce the local stable set (of size $\varepsilon$) of $\Phi$ at $P = (\xi,x)$ as

$$W^s_{\varepsilon}(P) \equiv W^s_{\varepsilon}(P;\Phi) \overset{\text{def}}{=} \{ Q \in \mathcal{M} : d(\Phi^n(Q),\Phi^n(P)) \leq \varepsilon, \ n \geq 0 \} \subset W^s(\xi) \times \mathcal{M}.$$ 

The local unstable set (of size $\varepsilon$), denoted by $W^u_{\varepsilon}(P)$, is defined analogously.

**Definition 3.4.** A compact invariant set $\Gamma \subset \mathcal{M}$ is hyperbolic (for $\Phi$) if there exist constants $\varepsilon > 0$, $K > 0$, $0 < \theta < 1$ such that

$$d(\Phi^n(P),\Phi^n(Q)) \leq K\theta^n \text{ for all } P,Q \in W^s_{\varepsilon}(P) \text{ and } n \geq 0;$$

$$d(\Phi^{-n}(P),\Phi^{-n}(Q)) \leq K\theta^n \text{ for all } P,Q \in W^u_{\varepsilon}(P) \text{ and } n \geq 0;$$

and there exists $\delta > 0$ such that

$$\#W^s_{\varepsilon}(P) \cap W^u_{\varepsilon}(Q) = 1 \text{ for all } P,Q \in \Gamma \text{ with } d(P,Q) \leq \delta.$$ 

An isolated set is hyperbolic if and only if it is expansive and has the shadowing property (see definitions in [3]). Every isolated hyperbolic set $\Gamma$ for $\Phi$ is topologically stable; i.e., there is an isolating neighborhood $U$ of $\Gamma$ such that for any homeomorphism $\Psi$ which is $C^0$ near $\Phi$, the restriction of $\Psi$ to the maximal invariant set in $U$ (that is, to the continuation $\Gamma_\Psi$ of $\Gamma$), is semiconjugate to the restriction of $\Phi$ to $\Gamma$. 


We define 

$$[P, Q] = W^s_\varepsilon(P) \cap W^u_\varepsilon(Q)$$

for every $P$ and $Q$ in $\Gamma$ with $d(P, Q) \leq \delta$ where $\varepsilon > 0$ and $\delta > 0$ follow from the hyperbolicity of $\Gamma$. The hyperbolic set $\Gamma$ has a local product structure if $[P, Q] \in \Gamma$ or equivalently if $\Gamma$ is an isolated set of $\Phi$. In such a case, the map 

$$[,] : \{\{P, Q\} \in \Gamma \times \Gamma : d(P, Q) \leq \delta\} \to \Gamma$$

is continuous (see [37]). Hence, the map $[,] : M^s_\varepsilon(P) \times M^u_\varepsilon(P) \to M$ is a homeomorphism onto its image for all $P = (\xi, x) \in \Gamma$ where

$$M^s_\varepsilon(P) = W^s_\varepsilon(P) \cap (\{\xi\} \times M) \quad \text{and} \quad M^u_\varepsilon(P) = W^u_\varepsilon(P) \cap (\{\xi\} \times M).$$

In the sequel we will assume that the topological dimension (in the sense of the Lebesgue covering dimension) of $M^s_\varepsilon(P)$ and $M^u_\varepsilon(P)$ depend continuously with respect to $P = (\xi, x) \in \Gamma$.

We will now introduce the notion of index of an isolated transitive hyperbolic set $\Gamma$ in our context. From the above assumption, the dimensions of $M^s_\varepsilon(P)$ and $M^u_\varepsilon(P)$ are locally constant. Hence, being $\Gamma$ transitive, the dimensions remain constant for any $P \in \Gamma$. Thus, we may define the $cs$-index (resp. $cu$-index) of $\Gamma$, denoted by $\text{ind}^{cs}(\Gamma)$ (resp. $\text{ind}^{cu}(\Gamma)$) as this dimension. Notice that $\dim M = \text{ind}^{cs}(\Gamma) + \text{ind}^{cu}(\Gamma)$ and from the topological stability, the $cs$-index remains constant under small $\delta$-perturbations of $\Phi$.

3.2.2. Blenders. In order to introduce the notion of a blender we need first to define families of $s$-discs and $u$-discs, which provide the superposition regions. To do this, we will consider a basic open set $\mathcal{B}$ of $\mathcal{M}$, i.e., a set of the form $V \times B$ where $V$ is an open set of $\Sigma$ and $B$ is an open set of $M$.

**Definition 3.5** (s-discs). A set $\mathcal{D}^s \subset \mathcal{M}$ is called a $s$-disc in $\mathcal{B}$ if there is $\xi \in V$ such that $\mathcal{D}^s$ is a graph of an $\alpha$-Hölder function from $W^s_{\text{loc}}(\xi) \cap V$ to $B$.

We say that two $s$-discs, $\mathcal{D}^s_1, \mathcal{D}^s_2 \subset W^s_{\text{loc}}(\xi) \times M$ are close if they are the graphs of close $\alpha$-Hölder functions. This proximity between discs allows us to introduce the following:

**Definition 3.6** (open set of s-discs). We say that a collection of discs $\mathcal{D}^s$ is an open set of $s$-discs in $\mathcal{B}$ if given $\mathcal{D}^s_0 \in \mathcal{D}^s$, every $s$-disc $\mathcal{D}^s$ close enough to $\mathcal{D}^s_0$ is a $s$-disc contained in $\mathcal{B}$ and belongs to $\mathcal{D}^s$.

Example of $s$-discs are the almost horizontal discs defined as follows: given $\delta > 0$ and a point $(\xi, x) \in \mathcal{B}$, we say that a set $\mathcal{D}^s \equiv \mathcal{D}^s(\xi, x) \subset \mathcal{M}$ is a $\delta$-horizontal disc in $\mathcal{B}$ if

- $\mathcal{D}^s$ is a graph of a $(\alpha, C)$-Hölder function $g : W^s_{\text{loc}}(\xi) \cap V \to B$,
- $d(g(\zeta), x) < \delta$ for all $\zeta \in W^s_{\text{loc}}(\xi) \cap V$,
- $C^\alpha < \delta$.

The set of all $\delta$-horizontal discs in $\mathcal{B}$ is an open set of $s$-discs in $\mathcal{B}$. Similarly we define $u$-discs in $\mathcal{B}$, open set of $u$-discs in $\mathcal{B}$ and we have that the set of almost vertical discs is an example of an open set of $u$-discs.

Following [29, 6], we introduce symbolic $cs$, $cu$ and double-blenders.
Definition 3.7 (blenders). Let $\Phi \in S(M)$ be a symbolic skew-product. A transitive hyperbolic maximal invariant set $\Gamma$ in a relatively compact open set $U \subset M$ of $\Phi$ is called

i) cs-blender if $\text{ind}^{cs}(\Gamma) > 0$ and there exist a basic open set $B \subset U$ and an open set $D^s$ of $s$-discs in $B$ such that for every small enough $s$-perturbation $\Psi$ of $\Phi$,

$$W^u_{\text{loc}}(\Psi) \cap D^s \neq \emptyset \quad \text{for all } D^s \in D^s.$$ 

ii) cu-blender if $\text{ind}^{cu}(\Gamma) > 0$ and there exist a basic open set $B \subset U$ and an open set $D^u$ of $u$-discs in $B$ such that for every small enough $s$-perturbation $\Psi$ of $\Phi$,

$$W^s_{\text{loc}}(\Psi) \cap D^u \neq \emptyset \quad \text{for all } D^u \in D^u.$$ 

iii) double-blender if both (i) and (ii) hold (not necessarily for the same $B$).

The open set $B$ is called a superposition domain and the open sets of discs $D^s$ and $D^u$ are called the superposition regions of the blender. Finally, the cs-blender (resp. cu-blender) with cs-index (resp. cu-index) is equal to $\dim M$ is called a contracting-blender (resp. expanding-blender).

The above condition in the definition of cs-blender (resp. cu-blender) about the positivity of the cs-index (resp. cu-index) of $\Gamma$ is imposed in order to avoid transversal intersections between $s$-discs (resp. $u$-discs) and local unstable (resp. stable) sets of $\Phi$ at points of $\Gamma$. We understand transversality in the sense of [16, Definition 1.9]. In fact, it is more convenient to introduce the notion of the unstable intersection property in Euclidean spaces: two compacta $X$ and $Y$ have unstable intersection in $\mathbb{R}^c$ if every pair of continuous maps $f : X \to \mathbb{R}^c$ and $g : Y \to \mathbb{R}^c$ can be approximated arbitrarily closely by continuous maps $f' : X \to \mathbb{R}^c$ and $g' : Y \to \mathbb{R}^c$ with $f'(X) \cap g'(Y) = \emptyset$. If one of the compacta $X$ and $Y$ is 0-dimensional, then $X$ and $Y$ have unstable intersection if and only if $\dim(X \times Y) < c$ [15]. In our case, we assume that $\text{ind}^{cs}(\Gamma) > 0$ and consider an $s$-disc $D^s \subset W^s_{\text{loc}}(\xi) \times M$ and a local unstable set $W^u_{\text{loc}}(P) \subset W^u_{\text{loc}}(\xi) \times M$ where $P = (\xi, x) \in \Gamma$. Observe that $D^s \cap W^u_{\text{loc}}(P) \subset \{\xi\} \times M$. Set $X$ and $Y$ as $D^s \cap \{\xi\} \times M$ and $M^{cu}(P) = W^u_{\text{loc}}(P) \cap \{\xi\} \times M$ respectively and take $c = \dim M$. Since $X$ is a singleton, it is 0-dimensional and thus

$$\dim(X \times Y) \leq \dim Y = \dim M^{cu}(P) = c - \text{ind}^{cs}(\Gamma) < c.$$ 

This implies that $X$ and $Y$ have unstable intersection in $\mathbb{R}^c$ and thus $D^s$ and $W^u_{\text{loc}}(P)$ have no transversal intersection. Consequently, from the definition of a cs-blender (resp. cu-blender) follows that the dimension of $W^u_{\text{loc}}(\Gamma)$ (resp. $W^s_{\text{loc}}(\Gamma)$) is robustly larger than the cu-index (resp. cs-index) of $\Gamma$. Namely, the dimension of the projection of this local unstable (resp. stable) set on $M$ is equal to $\dim M$. Notice that double-blenders have simultaneously these large dimensional projections.

3.2.3. Covering property as a criterion to yield blenders. In this section we give a criterion that allows us to guarantee that a symbolic skew-product has a blender.

Given $i \in \mathcal{A}$ we define the $i$-horizontal cylinder and $i$-vertical cylinder, respectively, by

$$H_i \overset{\text{def}}{=} \{\xi \in \Sigma : \xi_0 = i\} \quad \text{and} \quad V_i \overset{\text{def}}{=} \{\xi \in \Sigma : \xi_{-1} = i\}.$$
Let $\mathcal{B}$ be a subset of $\mathcal{M}$ and consider a symbolic skew-product $\Phi : \mathcal{M} \to \mathcal{M}$. Set
\[
\Lambda^u(\mathcal{B}; \Phi) \overset{\text{def}}{=} \bigcap_{n \geq 0} \Phi^n(\mathcal{B}) = \{ P \in \mathcal{M} : \Phi^{-n}(P) \in \mathcal{B} \text{ for all } n \geq 0 \}.
\]

**Theorem 3.8** (criterion for blenders). Let $\Phi \in \mathcal{PHS}(\mathcal{M})$ be a partially hyperbolic symbolic skew-product. Assume that the following cs-covering property holds:

There are a finite set $S \subset \mathcal{A}$ and open sets $\mathcal{B} \subset \mathcal{M}$ and $\mathcal{B}_i \subset \mathcal{M}$ for all $i \in S$ such that
\[
V_i \times \overline{B}_i \subset \Phi(H_i \times \mathcal{B}) \quad \text{for all } i \in S, \quad \overline{\mathcal{B}} \subset \bigcup_{i \in S} \mathcal{B}_i \tag{11}
\]
and $C \overset{\text{def}}{=} C_0 \cdot (1 - \gamma^{-1} \nu^a)^{-1} < L$ where $L$ is the Lebesgue number of (11). Then for any $0 < \delta < \gamma L / 2$ and for every small enough $S$-perturbation $\Psi$ of $\Phi$,
\[
\Lambda^u(\mathcal{B}; \Phi) \cap \mathcal{D}^\delta \neq \emptyset, \quad \text{for all } \text{$\delta$-horizontal discs } \mathcal{D}^\delta \text{ in } \mathcal{B} \overset{\text{def}}{=} \Sigma^+_S \times \mathcal{B} \text{ where } \Sigma^+_S \overset{\text{def}}{=} \bigcup_{i \in S} H_i.
\]

In addition, assume that $\mathcal{B}$ is contained in a relatively compact open set $\mathcal{U}$ which is the isolating neighborhood of a transitive hyperbolic set $\Gamma$ of $\Phi$ with $\text{ind}^cs(\Gamma) > 0$. Then $\Gamma$ is a cs-blender of $\Phi$ whose superposition region contains the open set of almost horizontal discs in $\mathcal{B}$.

**Remark 3.9.** Observe that $C \geq 0$ is a Hölder constant of the local strong stable partition of $\Phi$. Thus, if $C \nu^a < \gamma L / 2$ then the superposition region of the cs-blender above contains the family of local strong stable sets in $\mathcal{B}$.

**Remark 3.10.** Analogously, we get a cu-blender $\Gamma$ whose superposition region contains the open set of almost vertical discs in
\[
\mathcal{B} \overset{\text{def}}{=} \Sigma^-_S \times \mathcal{B} \quad \text{where} \quad \Sigma^-_S \overset{\text{def}}{=} \bigcup_{i \in S} V_i
\]
In this case $\mathcal{B}$ is contained in the isolating neighborhood of $\Gamma$, the cu-index of $\Gamma$ is positive and we have the following cu-covering property:
\[
H_i \times \overline{B}_i \subset \Phi^{-1}(V_i \times \mathcal{B}) \quad \text{for all } i \in S, \quad \overline{\mathcal{B}} \subset \bigcup_{i \in S} \mathcal{B}_i \tag{12}
\]
with $\hat{C} \overset{\text{def}}{=} C_0 \cdot (1 - \hat{\gamma}^{-1} \nu^a)^{-1} < L$. Moreover, if $\hat{C} \nu^a < \hat{\gamma} L / 2$ then the superposition region contains the family of local strong unstable sets in $\mathcal{B}$.

**Remark 3.11.** A transitive hyperbolic set satisfying the covering properties (11) and (12) (not necessarily for the same open set $\mathcal{B}$) is a double-blender.

A note on the proof of Theorem 3.8. First notice that the second part of the theorem is a consequence of the first part. Indeed, since $\Gamma$ is the maximal invariant set in $\mathcal{U}$ and $\mathcal{B} \subset \mathcal{U}$ then $\Lambda^u(\mathcal{B}; \Phi) \subset W^u_{\text{int}}(\Gamma)$. This inclusion and the first part of the theorem conclude that $\Gamma$ is a cs-blender of $\Phi$ whose superposition region contains the open set of almost horizontal discs in $\mathcal{B}$. We give the details of the first part of the proof in Appendix A.
3.2.4. *On the definition and criterion for blender.* Blenders are actually a power tool in partially hyperbolic dynamics when the superposition region contains the local strong stable/unstable set in the superposition domain. For this reason, without loss of generality, we will assume the following blender properties:

**Scholium 3.12.** Consider \( \Phi \in \mathcal{PHS}(M) \) and let \( \Gamma \) be a cs-blender with superposition domain \( \mathcal{B} = V \times B \) and superposition region \( \mathcal{D}^s \). There is a \( S \)-neighborhood \( \mathcal{U} \) of \( \Phi \) such that for any \( \Psi \in \mathcal{U} \),

(1) the open set of discs \( \mathcal{D}^s \) contains the family of local strong stable sets of \( \Psi \) in \( \mathcal{B} \):

\[
\text{if } W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B} \quad \text{then} \quad W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \in \mathcal{D}^s.
\]

(2) if \( W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B} \) then

\[
W^s_{\text{loc}}(\Gamma \Psi; \Psi) \cap W^s_{\text{loc}}(P'; \Psi) \neq \emptyset \quad \text{for all } P' \text{ close enough to } P.
\]

(3) for any \( P \in \Lambda^u(\mathcal{B}; \Phi) \), there exists \( P_{\Psi} \) close to \( P \) so that

\[
W^u_{\text{loc}}(P_{\Psi}; \Psi) \cap \mathcal{B} \subset \Lambda^u(\mathcal{B}; \Psi) \subset W^u_{\text{loc}}(\Gamma \Psi; \Psi)
\]

where

\[
\Lambda^u(\mathcal{B}; \Psi) = \bigcap_{n \geq 0} \Psi^n(\mathcal{B}).
\]

Similar conditions are also assumed for cu-blenders of partially hyperbolic skew-products.

We must show that the properties (B2) and (B3) follow from the definition of blender, i.e., Definition 3.7.

**Proof.** First of all, notice that the assumption (B1) and Definition 3.7 imply that

\[
\text{if } W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B} \quad \text{then} \quad W^u_{\text{loc}}(\Gamma; \Phi) \cap W^s_{\text{loc}}(P; \Phi) \neq \emptyset \quad S\text{-robustly.}
\]

A priori, the neighborhood of the \( S \)-perturbation of \( \Phi \) where (13) holds depends on the \( s \)-disc \( W^s_{\text{loc}}(P; \Phi) \). However, this can be taken independent of the disc assuming that the disc belongs to a superposition subdomain \( \mathcal{B}_0 \) of \( \mathcal{B} = V \times B \). That is if \( W^s_{\text{loc}}(P; \Phi) \cap (V \times M) \subset \mathcal{B}_0 \) where \( \mathcal{B}_0 = V \times B_0, B_0 \) is an open set whose closure is contained in \( B \). For this reason, without loss of generality, we can assume that (B2) holds.

With respect to (B3), first notice that \( \Gamma \) is the maximal invariant set in \( \mathcal{U} \). Since \( \mathcal{B} \subset \mathcal{U} \) then clearly \( \Lambda^u(\mathcal{B}; \Phi) \subset W^u_{\text{loc}}(\Gamma; \Phi) \). Let \( \mathcal{B}_1 = V \times B_1 \) where \( B_1 \) an open set containing the closure of \( B \). If \( P \in \Lambda^u(\mathcal{B}; \Phi) \), by the in phase result [32, Prop. 10], there is \( Q \in \Gamma \cap \Lambda^u(\mathcal{B}_1; \Phi) \) such that \( P \in W^u_{\text{loc}}(Q; \Phi) \). Hence, for any \( S \)-perturbation \( \Psi \) of \( \Phi \), there exists a continuation point \( P_{\Psi} \) of \( P \) and \( Q_{\Psi} \) of \( Q \) so that \( P_{\Psi} \in W^u_{\text{loc}}(Q_{\Psi}; \Psi) \) and \( Q_{\Psi} \in \Gamma_{\Psi} \cap \Lambda^u(\mathcal{B}_1; \Psi) \), and so \( P_{\Psi} \in \Lambda^u(\mathcal{B}_1; \Psi) \). This implies that \( W^u_{\text{loc}}(P_{\Psi}; \Psi) \cap \mathcal{B}_1 \subset \Lambda^u(\mathcal{B}_1; \Psi) \).

Taking as above a superposition subdomain \( \mathcal{B}_0 \) of \( \mathcal{B} \) we obtain that the \( S \)-neighborhood of the perturbation is independent of the point \( P \). Therefore, without loss of generality, we can also assume that (B3) holds. \( \square \)
Blenders constructed from covering property have extra useful properties that are not implied directly from the definition. We collect here these properties which will be used later on.

**Scholium 3.13.** Let $\Gamma$ be a cs-blender with a superposition domain $\mathcal{B} = \mathcal{V} \times \mathcal{B}$ for $\Phi \in \mathcal{P}\mathcal{H}\mathcal{S}(M)$, constructed from the criterion of Theorem 3.8.

There exists a $\mathcal{S}$-neighborhood $\mathcal{U}$ of $\Phi$ such that for any $\Psi \in \mathcal{U}$,

(B4) if $W^{\omega}_{\text{loc}}(P; \Psi) \cap (V \times M) \subset \mathcal{B}$ then

$$\Lambda^u(\mathcal{B}; \Psi) \cap W^{\omega}_{\text{loc}}(P'; \Psi) \neq \emptyset \quad \text{for all } P' \text{ close enough to } P;$$

(B5) $\mathcal{B} \subset \mathcal{P}(\Lambda^u(\mathcal{B}; \Psi))$, where $\mathcal{P} : M \to M$ is the standard projection on the fiber-space.

Similar conditions also hold for cu-blenders with respect to

$$\Lambda^s(\mathcal{B}; \Psi) = \bigcap_{n \geq 0} \Psi^{-n}(\mathcal{B}) = \{ P \in M : \Psi^n(P) \in \mathcal{B} \text{ for } n \geq 0 \}.$$  

4. Criteria: robust cycles and robust tangencies

**4.1. Criterion to yield robust cycles.** We explain how blenders can be used to yield robust cyclic intersections between stable and unstable sets.

**Definition 4.1** (cycles). Let $\Phi \in \mathcal{S}(M)$ be a symbolic skew-product with a pair of disjoint isolated invariant sets $\Gamma^1$ and $\Gamma^2$. We say that $\Phi$ has a cycle associated with $\Gamma^1$ and $\Gamma^2$ if their stable and unstable sets meet cyclically, that is,

$$W^s(\Gamma^1) \cap W^u(\Gamma^2) \neq \emptyset \quad \text{and} \quad W^u(\Gamma^1) \cap W^s(\Gamma^2) \neq \emptyset.$$  

Assuming that $\Gamma^1$ and $\Gamma^2$ are also transitive hyperbolic sets, the cycle is said to be $\mathcal{S}$-robust if any small enough $\mathcal{S}$-perturbation $\Psi$ of $\Phi$ has a cycle associated with the continuations $\Gamma^1_\Psi$ and $\Gamma^2_\Psi$ of $\Gamma^1$ and $\Gamma^2$ respectively. We define the co-index of the cycle as the integer $|\text{ind}^{cs}(\Gamma^1) - \text{ind}^{cs}(\Gamma^2)| = c$. If $c > 0$ the cycle is called heterodimensional and otherwise equidimensional.

Firstly, we would like to obtain a robust intersection between the unstable and the stable sets of two different isolated hyperbolic sets. To this end, we use the following criterion:

**Theorem 4.2** (criterion for robust cycles). Let $\Gamma^1$ and $\Gamma^2$ be isolated transitive hyperbolic sets of $\Phi \in \mathcal{P}\mathcal{H}\mathcal{S}(M)$. Assume that $\Gamma^1$ is a cs-blender with a superposition domain $\mathcal{B} = \mathcal{V} \times \mathcal{B}$ such that

(RC1) there is a point $P \in W^s_{\text{loc}}(\Gamma^2)$ such that

$$W^{\omega}_{\text{loc}}(\Phi^{-n}(P)) \cap (V \times M) \subset \mathcal{B} \quad \text{for some integer } n \geq 0.$$  

Then the unstable and stable sets of $\Gamma^1$ and $\Gamma^2$ respectively meet $\mathcal{S}$-robustly. That is, for any small $\mathcal{S}$-perturbation $\Psi \in \mathcal{P}\mathcal{H}\mathcal{S}(M)$ of $\Phi$ it holds $W^u(\Gamma^1_\Psi) \cap W^s(\Gamma^2_\Psi) \neq \emptyset$.

**Proof.** Since $\Gamma^1$ is a cs-blender, then (B1) and (RC1) imply that $W^{\omega}_{\text{loc}}(\Phi^{-n}(P)) \cap (V \times M) \in \mathcal{D}^s$. As $\mathcal{D}^s$ is an open set of $s$-discs, then for every small enough $\mathcal{S}$-perturbation $\Psi$ of $\Phi$,
\( W_{loc}^{ss}(\Psi^{-n}(P')) \cap (V \times M) \in \mathscr{P} \), where \( P' \) is the continuation\(^1\) of \( P \) in the local stable set of \( \Gamma_{\Psi}^2 \). By definition of the cs-blender, \( W_{loc}^{ss}(\Psi^{-n}(P')) \cap W_{loc}^{un}(\Gamma_{\Psi}^1) \neq \emptyset \) and then \( S \)-robustly the global unstable set of \( \Gamma^1 \) meets the global stable set of \( \Gamma^2 \).

Similar conditions guarantee the robustness of the other intersection. Therefore it is possible to construct robust cycles associated with two cs or cu-blenders, or a cs-blender and a cu-blender, or even between two double-blenders.

**Remark 4.3** (cycles between non-hyperbolic sets). The usefulness of the double-blender in this setting is that it permits to construct robust cyclic connections with any invariant set (of saddle type) having a non-empty continuation. For example, the other set may be a non-hyperbolic fixed point, and then we obtain the cyclic intersections between the stable and unstable sets, if the transition condition \((RC1)\) is satisfied.

4.2. **Criterion to yield robust tangencies.** We explain how robust cycles can be used to robust tangencies. In the sequel \( M \) denotes a differentiable manifold of dimension \( c \geq 1 \).

4.2.1. **The set of smooth symbolic skew-products.** Since we will need to work with differentiable fiber maps, it will be useful to extend the sets of Hölder skew-products to this setting.

**Definition 4.4.** For an integer \( r \geq 1 \), \( S'^r(M) \equiv S'^{r+\alpha}(M) \) denotes the set of skew-products \( \Phi = \tau \circ \phi_{\xi} \) on \( M = \Sigma \times M \) such that there are \( \gamma \equiv \gamma(\Phi) > 0, \tilde{\gamma} \equiv \tilde{\gamma}(\Phi) > 0 \) and \( C_\gamma \equiv C_\gamma(\Phi) \geq 0 \) satisfying

- \( d_{C_\gamma}(\phi_{\xi}^{\pm 1}, \phi_{\zeta}^{\pm 1}) \leq C_\gamma \) \( d_\xi(\xi, \zeta)^\alpha \) for all \( \xi, \zeta \in \Sigma \) with \( \xi_0 = \zeta_0 \), and
- \( \phi_{\xi} \) are \( C_\gamma \)-diffeomorphisms of \( M \) with \( D\phi_{\xi}^{\pm 1} \) Lipschitz (with uniform Lipschitz constant)

\[ \gamma < m(D\phi_{\xi}(x)) < ||D\phi_{\xi}(x)|| < \tilde{\gamma}^{-1} \text{ for all } (\xi, x) \in \Sigma \times M. \]

For \( r = 0 \), \( S^0(M) \), we understand as \( \Phi \in S(M) \) with fiber maps \( C^1 \)-diffeomorphisms. In addition,

\[ \mathcal{PH}(S^r(M)) \equiv \mathcal{PH}(S^{r+\alpha}(M)) \equiv \mathcal{PH}(S(M)) \cap S^r(M) \quad \text{for } r \geq 0. \]

That is, the domination conditions \( \nu^\theta < \gamma < 1 < \tilde{\gamma}^{-1} < \nu^{-\theta} \) hold where \( 0 < \alpha \leq \theta \leq 1 \) is the exponent of the Hölder continuity in the \( C^0 \)-metric. Finally, a partially hyperbolic skew-product is said to be fiber bunching if \( \nu^a < \gamma \tilde{\gamma} \).

We endow \( S^r(M) \) for \( r \geq 0 \) with the metric

\[ d_{S^r}(\Phi, \Psi) \equiv d_r(\Phi, \Psi) + \text{Lip}_r(\Phi, \Psi) + \text{Hol}_r(\Phi, \Psi) \]

where

\[ \text{Lip}_r(\Phi, \Psi) \equiv \max_{\xi \in \Sigma} \text{Lip}(D\phi_{\xi}^{\pm 1}) - \text{Lip}(D\phi_{\xi}^{\pm 1}) \]

\[ d_r(\Phi, \Psi) \equiv \max_{\xi \in \Sigma} d_{C_\gamma}(\phi_{\xi}^{\pm 1}, \psi_{\xi}^{\pm 1}) \quad \text{and} \quad \text{Hol}_r(\Phi, \Psi) \equiv |C_\gamma(\Phi) - C_\gamma(\Psi)|. \]

Hence \( \mathcal{PH}(S^r(M)) \) is an open set of \( S^r(M) \) and \( S^{r+1}(M) \subset S^r(M) \subset S(M) \) for \( r \geq 0. \)

\(^1\)The existence of \( P' \) follows from the in phase result [32, Prop. 10] using that \( \Gamma^2 \) is an isolated hyperbolic set.
4.2.2. **Tangencies in symbolic skew-products.** To define the notion of tangency for symbolic skew-products we first need to introduce the notion of a tangent direction.

**Definition 4.5** (tangent direction). Let \( \Phi = \tau \times \phi_\xi \in S'(M) \) be a symbolic skew-product with a pair of transitive hyperbolic sets \( \Gamma^1 \) and \( \Gamma^2 \) and suppose \((\xi, x) \in W^u(\Gamma^1) \cap W^s(\Gamma^2)\). A unitary vector \( v \in T_xM \) is called tangent direction at \((\xi, x)\) if there are \( C > 0 \) and \( 0 < \lambda < 1 \) such that

\[
\|D\phi_\xi^n(x)v\| \leq C|n| \quad \text{for all } n \in \mathbb{Z}.
\]

The maximum number of independent tangent directions at \((\xi, x)\) is denoted by \( d_T \equiv d_T(\xi, x) \).

Now we are ready to give the definition of a tangency.

**Definition 4.6** (tangency). We say that \( \Phi \in S'(M) \) has a (bundle) tangency of dimension \( \ell > 0 \) between \( W^u(\Gamma^1) \) and \( W^s(\Gamma^2) \) if there exists \((\xi, x) \in W^u(\Gamma^1) \cap W^s(\Gamma^2)\) such that

\[
\ell = d_T(\xi, x) \quad \text{and} \quad \text{ind}^u(\Gamma^1) + \text{ind}^s(\Gamma^2) - \ell < c.
\]

If \( \Gamma^1 = \Gamma^2 \), the tangency is called homoclinic, and otherwise heteroclinic. The tangency (of dimension \( \ell \)) is said to be \( S' \)-robust if for any small enough \( S' \)-perturbation \( \Psi \) of \( \Phi \) has a tangency (of dimension \( \ell \)) between the unstable set \( W^u(\Gamma^1_\Psi) \) and the stable set \( W^s(\Gamma^2_\Psi) \).

The codimension of the tangency is defined as \( c_T = c - \left( \text{ind}^u(\Gamma^1) + \text{ind}^s(\Gamma^2) - \ell \right) \).

Assume \( \Phi \) has a \((S)-robust\) equi/heterodimensional cycle associated with \( \Gamma^1 \) and \( \Gamma^2 \) and \( \text{ind}^s(\Gamma^1) \leq \text{ind}^s(\Gamma^2) \). We say that \( \Phi \) has a \((S')-robust\) equi/heterodimensional tangency on the cycle if \( \Phi \) has a \((S')-robust\) tangency between \( W^u(\Gamma^1) \) and \( W^s(\Gamma^2) \).

In what follows, we want to construct symbolic skew-products with tangencies, which will be created locally. For this reason, in order to be clearer, we will work in local coordinates and may assume that \( M = \mathbb{R}^c \).

4.2.3. **Cone fields in symbolic skew-products.** Consider an integer \( 1 \leq \ell \leq c \). A standard \( \ell \)-cone in \( \mathbb{R}^c \) is a set of the form \( C = \{(v, w) \in \mathbb{R}^c : v \in \mathbb{R}^\ell \text{ and } \|v\| \leq \rho\|w\| \text{ for some } \rho > 0\} \). More generally, a \( \ell \)-cone is the image of a standard \( \ell \)-cone under an invertible linear map.

**Definition 4.7** (unstable cone). Let \( \Phi = \tau \times \phi_\xi \in S'(\mathbb{R}^c) \) and consider an open set \( \mathcal{B} \) of \( M = \Sigma \times \mathbb{R}^c \). An \( \ell \)-cone \( \mathcal{C}^{uu} \) in \( \mathbb{R}^c \) is said to be unstable for \( \Phi \) on \( \mathcal{B} \) if there is \( 0 < \lambda < 1 \) such that

\[
D\phi_\xi(x)\mathcal{C}^{uu} \subset \text{int}(\mathcal{C}^{uu}) \quad \text{and} \quad \|D\phi_\xi(x)v\| \geq \lambda^{-1}\|v\| \text{ for all } v \in \mathcal{C}^{uu}, \ (\xi, x) \in \mathcal{B} \cap \Phi^{-1}(\mathcal{B}).
\]

Observe that the unstable \( \ell \)-cone \( \mathcal{C}^{uu} \) is \( S^0 \)-robust: there is a \( S^0 \)-neighborhood \( \mathcal{U} \) of \( \Phi \) so that \( \mathcal{C}^{uu} \) is a unstable \( \ell \)-cone for \( \Psi \) on \( \mathcal{B} \) for any \( \Psi \in \mathcal{U} \). Similarly we define the stable \( \ell \)-cone \( \mathcal{C}^{ss} \) for \( \Phi \) on \( \mathcal{B} \) and \( \mathcal{C}^{ss} \) is \( S^0 \)-robust.

**Lemma 4.8.** Let \( \mathcal{C}^{uu} \) be an unstable \( \ell \)-cone for \( \Phi \) on \( \mathcal{B} \). Then for each \( n \in \mathbb{N} \) it holds that

\[
\|D\phi_\xi^n(x)v\| \leq \lambda^n\|v\| \text{ for } v \in \mathcal{C}^{uu} \text{ and } (\xi, x) \in \mathcal{B} \cap \Phi(\mathcal{B}) \cap \cdots \cap \Phi^n(\mathcal{B}).
\]

**Proof.** It suffices to note that \( \|v\| = \|D\phi_\xi^n(\phi_\xi^{-n}(x))D\phi_\xi^{-n}(x)v\| \geq \lambda^{-n}\|D\phi_\xi^{-n}(x)v\| \). \( \square \)
An $\ell$-dimensional vector subspace of $\mathbb{R}^c$ is called a $\ell$-plane. The Grassmannian manifold $G(\ell, c)$ is defined as the set of $\ell$-planes in $\mathbb{R}^c$. Given $E, F$ in $G(\ell, c)$, we define the distance

$$d_G(E, F) = \inf\{\|A - i_E\| : A : E \to \mathbb{R}^c \text{ is a linear operator such that } A(E) \subset F\}$$

where $i_E : E \to \mathbb{R}^c$ denotes the inclusion and $\| \cdot \|$ is the Euclidian operator norm. It is not difficult to prove that the infimum is attained at the map $A = P_F|E$ (i.e., the orthogonal projection onto $F$ restricted to $E$). Trivially,

$$\|P_F|E - i_E\| = \sup\{\inf_{f \in F}\|f - e\| : e \in E, \|e\| = 1\}$$

which provides another characterization of the above metric in $G(\ell, c)$. It follows that any $\ell$-cone $C$ in $\mathbb{R}^c$ induces an open set in $G(\ell, c)$, which we will continue denoting by $C$. Moreover, any unstable $\ell$-cone $C_{uu}$ for $\Phi$ on $\mathcal{B}$ is strictly invariant by the action of $\mathcal{A} = \{D\phi_x(x) : (\xi, x) \in \mathcal{B} \cap \Phi^{-1}(\mathcal{B})\}$. That is, $\mathcal{A}C_{uu}$ is contained in the interior of $C_{uu}$. Here $\Phi_{uu}$ denotes the image of $C_{uu} \subset G(\ell, c)$ under the action of $\mathcal{A}$, i.e., the subset $\{AE : A \in \mathcal{A}, E \in C_{uu}\}$ of $G(k, c)$. Similar observations hold for stable $\ell$-cones.

4.2.4. Criterion to yield robust tangencies. Given $\Phi = \tau \prec \phi_\xi \in \Phi^0(\mathbb{R}^c)$ and $1 \leq \ell \leq c$, we define the induced $\ell$-th Grassmannian skew-product $\hat{\Phi}$ on

$$\hat{\mathcal{M}} \equiv \Sigma \times \hat{\mathcal{M}}, \quad \hat{\mathcal{M}} \equiv \mathbb{R}^c \times G(\ell, c)$$

as

$$\hat{\Phi} : \hat{\mathcal{M}} \to \hat{\mathcal{M}}, \quad \hat{\Phi}(\xi, (x, E)) = (\tau(\xi), (\phi_x(x), D\phi_x(x)E)).$$

**Theorem 4.9** (criterion for robust tangencies). Let $\Gamma^1$ and $\Gamma^2$ be a cs-blender and a cu-blender of a skew-product $\Phi \in \Phi(S^1(\mathbb{R}^c))$ with, respectively, superposition domains

$$\mathcal{B}_1 = V_1 \times B_1 \quad \text{and} \quad \mathcal{B}_2 = V_2 \times B_2,$$

and cs-indices $0 < i_1 < c$ and $0 < i_2 < c$. Consider an integer $\max\{0, i_2 - i_1\} < \ell \leq \min\{c - i_1, i_2\}$. Assume that

- (T1) the induced $\ell$-th Grassmannian skew-product $\hat{\Phi}$ on $\hat{\mathcal{M}}$ belongs to $\Phi(S(\hat{\mathcal{M}}))$;

and there exist

- (T2) unstable and stable $\ell$-cones $C_{uu}$ and $C_{ss}$ for $\Phi$ on $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively;
- (T3) a cs-blender $\hat{\Gamma}^1$ and a cu-blender $\hat{\Gamma}^2$ of $\hat{\Phi}$ with superposition domains $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{B}}_2$ respectively, such that $\hat{\Gamma}^1$ satisfies (B4) and

$$\hat{\mathcal{B}}_1 = V_1 \times \hat{B}_1, \quad \hat{\mathcal{B}}_2 = V_2 \times \hat{B}_2, \quad \text{with} \quad \hat{\mathcal{B}}_1 \subset B_1 \times C_{uu} \quad \text{and} \quad \hat{\mathcal{B}}_2 \subset B_2 \times C_{ss};$$
- (T4) a point $\hat{\tau} \in \Lambda^s(\hat{\mathcal{B}}_2; \hat{\Phi}) \subset W_{loc}^s(\hat{\Gamma}^2)$ and an integer $n \geq 0$ such that

$$W_{loc}^s(\hat{\Phi}^{-n}(\hat{\tau})) \cap (V_1 \times \hat{\mathcal{M}}) \subset \hat{\mathcal{B}}_1.$$

Then

- i) $W^u(\Gamma^1) \cap W^u(\Gamma^2) \neq \emptyset$ S-robustly (in $\Phi(S(\hat{\mathcal{M}}))$),
- ii) $W^u(\Gamma^1) \cap W^u(\Gamma^2) \neq \emptyset$ S-robustly (in $\Phi(S(\mathbb{R}^c))$), and
- iii) $\Phi$ has a $S^1$-robust tangency of dimension $\ell$ between $W^u(\Gamma^1)$ and $W^u(\Gamma^2)$. 
In order to prove the above theorem, we first need a lemma:

**Lemma 4.10.** Consider $1 \leq \ell \leq c$, $\Phi \in S^1(\mathbb{R}^c)$ and let $\hat{\Phi}$ be the induced $\ell$-th Grassmannian symbolic skew-product on $\hat{\mathcal{M}} = \Sigma \times \hat{M}$ with $\hat{M} = \mathbb{R}^c \times G(\ell, c)$.

i) If $\Phi \in S^1(\mathbb{R}^c)$ then $\hat{\Phi} \in S(\hat{\mathcal{M}})$.

ii) For every $\delta$-neighborhood $\mathcal{U}$ of $\hat{\Phi}$ there exists a $\delta$-neighborhood $\mathcal{V}$ of $\Phi$ such that for $\Psi \in \mathcal{V}$, $\Psi$ belongs to $\mathcal{U}$. Here $\Psi$ denotes the induced $\ell$-th Grassmannian skew-product on $\hat{\mathcal{M}}$ by $\Psi$.

iii) If $\Phi \in \mathcal{P}(S(\hat{\mathcal{M}}))$ then $\Phi \in \mathcal{P}(S(\mathbb{R}^c))$ and

$$
\pi W^{uu}_{\hat{\mathcal{M}}} (\hat{P}) = W^{uu}_{\mathcal{M}} (\pi \hat{P}), \quad \pi W^{uu}_{\hat{\mathcal{M}}} (\hat{P}) = W^{uu}_{\mathcal{M}} (\pi \hat{P}) \quad \text{for all } \hat{P} \in \hat{\mathcal{M}}
$$

where $\pi : \hat{\mathcal{M}} \to \Sigma \times \mathbb{R}^c$ is the standard projection.

**Proof.** Assume that $\Phi = \tau \times \phi_\xi$ belongs to $S^1(\mathbb{R}^c)$ and let $\hat{\Phi} = \tau \times \hat{\phi}_\xi$. Notice that $\hat{\phi}_\xi$ depends $\alpha$-Hölder on the base point $\xi$. Thus, to prove that $\hat{\Phi}$ belongs to $S(\hat{\mathcal{M}})$, it suffices to show that $\hat{\phi}_\xi$ are bi-Lipschitz homeomorphisms (with uniform constants that only depends on $\Phi$).

First we need the following claim:

**Claim 4.11.** Let $T$ be a linear automorphism of $\mathbb{R}^c$. Then the induced transformation on the Grassmannian $G(\ell, c)$ is bi-Lipschitz with constant $||T|| ||T^{-1}||$.

**Proof.** $d_G(TV, TW) = \inf_{BV \subseteq TV} ||B - iTV|| = \inf_{AV \subseteq W} ||T(A - iv)T^{-1} - iTV|| \leq ||T|| ||T^{-1}|| d(V, W)$. □

Now, using the triangular inequality and the above claim

$$
d_{\hat{\mathcal{M}}} (\hat{\phi}_\xi (x, E), \hat{\phi}_\xi (x', E')) = d(\phi_\xi (x), \phi_\xi (x')) + d_G (D\phi_\xi (x) E, D\phi_\xi (x') E') 
\leq \gamma^{-1} d(x, x') + (\gamma \gamma')^{-1} d(E, E') + d_G (D\phi_\xi (x) E', D\phi_\xi (x') E').
$$

By definition of the Grassmannian metric and since $\Phi \in S^1(\mathbb{R}^c)$ it holds that

$$
d_G (D\phi_\xi (x) E', D\phi_\xi (x') E') \leq ||D\phi_\xi (x) - D\phi_\xi (x')|| \leq \gamma^{-1} L d(x, x') \tag{14}
$$

where $L \geq 0$ is a uniform (only depending on $\Phi$) Lipschitz constant for $D\phi_\xi$. Thus,

$$
d_{\hat{\mathcal{M}}} (\hat{\phi}_\xi (x, E), \hat{\phi}_\xi (x', E')) \leq \max \{ \gamma^{-1} + \gamma^{-1} L, (\gamma \gamma')^{-1} \} d_M ((x, E), (x', E')). \tag{15}
$$

Analogously we can prove a similar inequality for $\hat{\phi}_\xi^{-1}$. Therefore, we have obtained that $\hat{\phi}_\xi$ is bi-Lipschitz (with uniform constants on $\xi$).

Let $\Psi = \tau \times \psi_\xi$ be a $S^1$-perturbation of $\Phi$. We will show that $\hat{\Psi} = \tau \times \hat{\psi}_\xi$ is a $S$-perturbation of $\hat{\Phi}$. Indeed, the $S$-distance between $\hat{\Phi}$ and $\hat{\Psi}$ is less or equal than the maximum of

$$
d_S (\hat{\phi}_\xi^{\pm 1}, \hat{\psi}_\xi^{\pm 1}) = \max_{(x, E) \in \hat{\mathcal{M}}} D_G (D\hat{\phi}_\xi (x) E, D\hat{\psi}_\xi (x) E) + ||D\phi_\xi (x) - D\psi_\xi (x)|| + |\text{Lip}(\hat{\phi}_\xi) - \text{Lip}(\hat{\psi}_\xi)| + |C_1(\Phi) - C_1(\Psi)|.
$$

Similar estimates as in (14) and (15) show that $d_G (D\phi_\xi (x) E, D\psi_\xi (x) E) \leq \gamma^{-1} ||D\phi_\xi (x) - D\psi_\xi (x)||$ and $|\text{Lip}(\hat{\phi}_\xi) - \text{Lip}(\hat{\psi}_\xi)| \leq \gamma^{-1} |\text{Lip}(D\phi_\xi) - \text{Lip}(D\psi_\xi)|$. Analogously for $\hat{\phi}_\xi^{-1}$. Thus,

$$
d_S (\hat{\Phi}, \hat{\Psi}) \leq (\gamma \gamma')^{-1} \max_{\xi \in \Sigma} \left( \sup_{\xi \in \Sigma} d_G (\phi_\xi, \psi_\xi) + |\text{Lip}(D\hat{\phi}_\xi) - \text{Lip}(D\hat{\psi}_\xi)| \right)
+ |C_1(\Phi) - C_1(\Psi)| \leq (\gamma \gamma')^{-1} d_S (\Phi, \Psi).
$$
This proves (ii).

Finally, it is clear that if $\hat{\Phi} \in \mathcal{PHS}(\hat{M})$ then $\Phi \in \mathcal{PHS}(\mathbb{R}^c)$. To prove that the strong laminations of $\hat{\Phi}$ project on the strong laminations of $\Phi$, remember that the strong laminations are Hölder graphs over the stable sets of the symbolic shift. Then it suffices to see that the projection of the Hölder graphs in $\Sigma \times \hat{M}$ is as well an invariant lamination of $\Sigma \times \mathbb{R}^c$ (i.e., the leaves are invariant Hölder graphs over the strong set of the shift). But this is clear by the invariance of the strong lamination on $\hat{M}$. □

**Proof of Theorem 4.9.** First of all, notice that (T1), (T3) and (T4) are the assumptions of Theorem 4.2 and hence (i) follows immediately from this result when applied to $\hat{\Phi}$. To obtain (ii), observe that (T3) implies

$$\pi \Lambda^s(\hat{B}_2; \hat{\Phi}) \subset \Lambda^s(B_2; \Phi) \subset W^s_{\text{loc}}(\Gamma^1).$$

Thus, by Lemma 4.10(iii) and (T4) all the assumptions of Theorem 4.2 are satisfied for the symbolic skew-product $\Phi$, and this concludes (ii).

Now we will show (iii). To do this we basically need to reprove (i) but extracting more information. Since $\hat{P}$ belongs to $\Lambda^s(\hat{B}_2; \hat{\Phi})$ where $\hat{B}_2$ is a superposition domain of the $cs$-blender $\hat{\Gamma}_2$ it follows from (B3) that

$$W^s_{\text{loc}}(\hat{P}) \cap \hat{B}_2 \subset \Lambda^s(\hat{B}_2; \hat{\Phi}) \quad \text{S-robustly}. \quad (16)$$

On the other hand, according to (T4) and since $\hat{B}_1$ is a superposition domain of the $cs$-blender $\hat{\Gamma}_1$ which satisfies (B4) then

$$\Lambda^u(\hat{B}_1; \hat{\Phi}) \cap W^s_{\text{loc}}(\hat{\Phi}^{-n}(\hat{P})) \neq \emptyset \quad \text{S-robustly}. \quad (17)$$

Since $\hat{\Phi}^m(\hat{P}) \in \Lambda^s(\hat{B}_2; \hat{\Phi})$ for all $m \geq 0$ then iterating $\hat{P}$ if necessary we can suppose that $n$ is large enough so that

$$\hat{\Phi}^n(W^s_{\text{loc}}(\hat{\Phi}^{-n}(\hat{P})) \subset W^s_{\text{loc}}(\hat{P}) \cap \hat{B}_2.$$

Notice that (16) and (17) implies that $S$-robustly $W^s_{\text{loc}}(\Gamma^1)$ meets $W^u_{\text{loc}}(\Gamma^1)$. Similar expressions can be obtained for $\Phi$ applying Lemma 4.10(iii). However, a priori, the projection on $S(\mathbb{R}^c)$ of the $S$-neighborhood $\mathcal{U}$ of $\hat{\Phi}$ where (16) and (17) hold, is not a $S$-neighborhood of $\Phi$ and thus (ii) cannot follow from this. Anyway, by Lemma 4.10(ii), there is a $S^1$-neighborhood $\mathcal{U}$ of $\Phi$ so that $\Psi \in \mathcal{U}$ for all $\Phi \in \mathcal{U}$. Consider $\Psi = \tau \times \psi$ in $\mathcal{U}$ and the continuation of $\hat{\Phi}$ for $\Psi$ (continue to call $\hat{P}$) satisfying (16). Let $\hat{Q} = (\xi, (x, E))$ be an intersection point of (17) for $\hat{\Psi}$. Then $\hat{\Phi}^{-m}(\hat{Q}) \in \hat{B}_1$ and $\hat{\Psi}^m(\hat{Q}) \in \hat{B}_2$ for all $m \geq 0$. In particular, by (T3), it follows that

$$\Psi^{-m}(\xi, x) \in B_1 \text{ and } D \psi^{-m}(x)E \in C^{mu} \text{ for all } m \geq 0.$$

Since $\Gamma^1$ is a $cs$-blender with superposition domain $B_1$ then $\Lambda^u(B_1; \Psi) \subset W^u_{\text{loc}}(\Gamma^1_{\Psi})$ and hence $(\xi, x) \in W^u_{\text{loc}}(\Gamma^1_{\Psi})$. A similar argument shows that $\Psi^n(\xi, x) \in W^s_{\text{loc}}(\Gamma^2_{\Psi})$, and proves that

$$(\xi, x) \in W^s(\Gamma^1_{\Psi}) \cap W^s(\Gamma^2_{\Psi}).$$
On the other hand, by the $S^0$-robustness of unstable cones, $C^{cs}$ is also an unstable $\ell$-cone for $\Psi$ on $B_1$ and hence, by Lemma 4.8, since $(\xi, x) \in \Lambda^u(B_1; \Psi)$,
\[ \|D\psi^m_\xi(x)v\| \leq \lambda^m \|v\| \quad \text{for all } m \in \mathbb{N} \text{ and } v \in E. \]
Analogous argument proves that $\|D\psi^m_\xi(y)w\| \leq \lambda^m \|w\|$ for all $m \in \mathbb{N}$ and $w \in F = D\psi^m_\xi(x)E$ where $(\xi, y) = \Psi^m(Q)$. Then, there is a constant $C > 0$ such that $\|D\psi^m_\xi(x)v\| \leq C\lambda^m \|v\|$ for all $m \in \mathbb{N}$ and $v \in E$. Adjusting the constants if necessary we get $C > 0$ and $0 < \lambda < 1$ such that
\[ \|D\psi^m_\xi(x)v\| \leq C\lambda^m \quad \text{for all } m \in \mathbb{Z}, \ v \in E \text{ with } \|v\| = 1. \]
Finally, since the dimension of $E$ is equal to $\ell$ and $\ell > i_2 - i_1$ then
\[ \ell = d_\Gamma(\xi, x) \quad \text{and} \quad \text{ind}^{cs}(\Gamma^1) + \text{ind}^{cu}(\Gamma^2) - \ell = c - i_1 + i_2 - \ell < c. \]
Thus $\Phi$ has a tangency of dimension $\ell$ between the unstable set of $\Gamma_1^\psi$ and the stable set of $\Gamma_2^\psi$. This proves (iii) concluding the theorem. \qed

If $\Gamma^1$ is equals to $\Gamma^2$ in Theorem 4.9 then $\Gamma \equiv \Gamma^1 = \Gamma^2$ is a double-blender. In this case the conclusion is that $\Gamma$ has a $S^1$-robust homoclinic tangency. Similarly, the previous theorem allows us to construct a $S^1$-robust tangency on an equi/heterodimensional cycle:

**Corollary 4.12.** Let $\Phi \in \mathcal{P}H(S^1(\mathbb{R}))$. Then,

i) if $\Gamma$ is a double-blender satisfying (T1), (T2), (T3) and (T4) in Theorem 4.9 for $\Phi$ and $\Phi^{-1}$ then $\Phi$ has a $S^1$-robust homoclinic tangency associated with $\Gamma$.

ii) if $\Gamma^1$ and $\Gamma^2$ are a cs-blender and a cu-blender respectively with $\text{ind}^{cs}(\Gamma^1) \leq \text{ind}^{cs}(\Gamma^2)$, satisfying (T1), (T2), (T3) and (T4) in Theorem 4.9 and the transition property (RC1) for $\Phi^{-1}$ then $\Phi$ has a $S^1$-robust tangency on an equi/heterodimensional cycle associated with $\Gamma^1$ and $\Gamma^2$.

iii) if $\Gamma^1$ and $\Gamma^2$ are a pair of double-blenders satisfying (T1), (T2), (T3) and (T4) in Theorem 4.9 for $\Phi$ and $\Phi^{-1}$ then $\Phi$ has a $S^1$-robust tangency between both cyclic intersections of the stable and unstable sets of $\Gamma^1$ and $\Gamma^2$.

5. **Constructions: cycles and tangencies from one-step maps**

From now on the alphabet is $\mathcal{A} = \{1, \ldots, d\}$ with $d \geq 2$ and the fiber space $M$ is a differentiable manifold of dimension $c \geq 1$. We will give the prototypical construction of blenders, robust cycles and robust tangencies from a particular class of skew-products, the so-called, one-step maps:

**Definition 5.1** (one-step skew-product maps). A symbolic skew-product $\Phi = \tau \ltimes \phi_\xi$ is called one-step if the fiber maps $\phi_\xi$ only depend on the coordinate $\xi_0$ of the bi-sequences $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma$. In this case we have $\phi_\xi = \phi_i$ if $\xi_0 = i$ and write $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_d)$.

For a one-step skew-product $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_d)$, one has the underline dynamics given by the semigroup action generated by $\phi_1, \ldots, \phi_d$ (often referred to as the *iterated function system* or simply IFS). In what follows, $(\phi_1, \ldots, \phi_d)^+$ denotes the semigroup generated by these bi-Lipschitz homeomorphisms.
5.1. **Blenders from one-step maps.** The notion of a blending region was introduced simultaneously in [23] and [6] as a local open set with robust minimal dynamics for an IFS. Here we will introduce an extension of this notion.

**Definition 5.2 (blending region).** Let \( \Phi = \tau \ltimes (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}\mathcal{S}(M) \). Consider bounded open sets \( B \) and \( D \) of \( M \) with \( B \subset D \), a subset \( S \subset \mathcal{A} \), and a hyperbolic transitive set

\[
\Gamma \overset{\text{def}}{=} \bigcap_{n \in \mathbb{Z}} \Phi^n(S^Z \times D) = \bigcap_{n \in \mathbb{Z}} \Phi^n(V \times D),
\]

where \( V \) denotes any isolating neighborhood of \( \Sigma^+_S \overset{\text{def}}{=} \{ \xi \in \Sigma : \xi_0 \in S \} \) and \( \Sigma^-_S \overset{\text{def}}{=} \{ \xi \in \Sigma : \xi_{-1} \in S \} \). We say that \( B \) is a cs/cu/double-blending region with respect to \( \{ \phi_i : i \in S \} \) on \( D \) if there exists respectively a

i) cs-cover: \( \{ \phi_i(B) : i \in S \} \) is an open cover of \( \overline{B} \) and \( \text{ind}^{cs}(\Gamma) > 0 \);

ii) cu-cover: \( \{ \phi_i^{-1}(B) : i \in S \} \) is an open cover of \( \overline{B} \) and \( \text{ind}^{cu}(\Gamma) > 0 \);

iii) double-cover: both (i) and (ii) are true.

We call cs-index (resp. cu-index) of the blending region \( B \) the cs-index (resp. cu-index) of \( \Gamma \). As in the case of the blender, if its cs-index (resp. cu-index) is equal to dimension of \( X \) the blending region is called contracting (resp. expanding).

With the above terminology we get the following corollary of Theorem 3.8:

**Corollary 5.3 (blenders from one-step maps).** Let \( \Phi = \tau \ltimes (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}\mathcal{S}(M) \). Assume that there are a finite set \( S \subset \mathcal{A} \) and a

- cs/cu/double-blending region \( B \) with respect to \( \{ \phi_i : i \in S \} \) on \( D \).

Then the maximal invariant set \( \Gamma \) in \( S^Z \times D \) is a cs/cu/double-blender of \( \Phi \) whose superposition region contains the family of almost horizontal discs in \( \Sigma^+_S \times B \) or/and almost vertical discs in \( \Sigma^-_S \times B \). Moreover, it also contains the family of local strong stable/unstable sets, i.e., (B1) holds.

**Proof.** We assume that \( B \) is a cs-blending region. The remainder cases are proved analogously.

From the cs-cover (i), for every \( i \in S \) we find an open set \( B_i \subset M \) such that

\[
\overline{B} \subset \bigcup_{i \in S} B_i \quad \text{and} \quad \overline{B_i} \subset \phi_i(B).
\]

(18)

This implies the covering property (11) of Theorem 3.8. As \( \Phi \) is a one-step map then the Hölder constant \( C_0 = 0 \). Thus, it holds that \( C = C_0 \cdot (1 - \gamma^{-1} \nu^\alpha)^{-1} < L \) and \( C \nu^\alpha < \gamma L / 2 \) where \( L > 0 \) is the Lebesgue number of (18). Therefore according to Theorem 3.8 and Remark 3.9, \( \Gamma \) is a symbolic cs-blender whose superposition region contains the family of almost horizontal discs and local strong sets in \( \Sigma^+_S \times B \). This concludes the proof. \( \Box \)

5.1.1. **Hyperbolicity and the Conley-Moser conditions.** To construct symbolic blenders from one-step maps one needs to know when the invariant set is hyperbolic. We will describe some sufficient conditions on the fiber maps in order to guarantee this requirement.
A simple example of a one-step map with a hyperbolic set where the fiber maps have both expanding and contracting directions is to take a direct product in the following manner.

**Example 5.4.** Let us consider a finite set \( S \subset \mathcal{A} \), the fiber space \( M = \mathbb{R}^c \) and a subset \( D = D_{cs} \times D_{cu} \) of \( M \) where \( D_{cs} \) and \( D_{cu} \) are bounded open sets of \( \mathbb{R}^s \) and \( \mathbb{R}^c \) respectively. We take two one-step symbolic skew-products

\[
\Phi_{cs} = \tau \times (f_1, \ldots, f_d) \in \mathcal{S}(\mathbb{R}^s) \quad \text{and} \quad \Phi_{cu} = \tau \times (h_1, \ldots, h_d) \in \mathcal{S}(\mathbb{R}^c)
\]

such that \( f_i \) and \( h_i^{-1} \) are contracting maps of \( D_{cs} \) and \( D_{cu} \) respectively for all \( i \in S \). Let \( \Gamma_{cs} \) and \( \Gamma_{cu} \) be the maximal invariant sets in the closure of \( \Sigma^+_S \times D_{cs} \) and \( \Sigma^+_S \times D_{cu} \) of \( \Phi_{cs} \) and \( \Phi_{cu} \). These sets are attracting and repelling continuous invariant sections \([21, 6]\). Namely, they are graphs of continuous maps \( g_{cs} : \mathbb{S}^Z \rightarrow D_{cs} \) and \( g_{cu} : \mathbb{S}^Z \rightarrow D_{cu} \) such that

\[
g_{cs} \circ \tau(\xi) = f_{s_0} \circ g_{cs}(\xi) \quad \text{and} \quad g_{cu} \circ \tau(\xi) = h_{s_0} \circ g_{cu}(\xi).
\]

Hence, \( \Phi_{cs} |_{\Gamma_{cs}} \) and \( \Phi_{cu} |_{\Gamma_{cu}} \) are conjugated to the shift map \( \tau : \mathbb{S}^Z \rightarrow \mathbb{S}^Z \). Take the direct product

\[
\Phi = \tau \times (\phi_1, \ldots, \phi_d) \in \mathcal{S}(\mathbb{M}), \quad \phi_i(x_{cs}, x_{cu}) = (f_i(x_{cs}), h_i(x_{cu})) \quad \text{for} \quad i = 1, \ldots, d.
\]

The maximal invariant set \( \Gamma \) of \( \Phi \) in \( \Sigma^+_S \times \overline{D} \) is the graph of the continuous function

\[
g : \mathbb{S}^Z \rightarrow D, \quad g(\xi) = (g_{cs}(\xi), g_{cu}(\xi)) \quad \text{with} \quad g \circ \tau(\xi) = \phi_\xi \circ g(\xi).
\]

Thus, \( \Phi |_{\Gamma} \) is conjugated to the shift map \( \tau : \mathbb{S}^Z \rightarrow \mathbb{S}^Z \) and hence \( \Gamma \) is an isolated hyperbolic transitive set of \( \Phi \) with cs-index equal to the dimension of \( D_{cs} \).

When \( D \) is a bounded open subset of \( \mathbb{R}^c \), \( c \geq 2 \), the topological criteria for a continuous map \( f : \overline{D} \rightarrow \mathbb{R}^c \) to have a hyperbolic set in \( D \) are known as the \textit{Conley-Moser conditions} which are explained in detail in \([41]\). The conditions are given with respect to the so-called horizontal and vertical (contracting and expanding directions) slabs, which are fattened up horizontal and vertical Lipschitz graphs in \( D \).

A modification of the above example using the Conley-Moser conditions is the following.

**Example 5.5.** Let \( \Phi = \tau \times (\phi_1, \ldots, \phi_d) \in \mathcal{S}(\mathbb{R}^c), S \subset \mathcal{A} \) and \( D = D_{cs} \times D_{cu} \subset \mathbb{R}^c \). Assume that \( \phi_i^{-1}(\overline{D}) \cap D \) and \( \phi_i(\overline{D}) \cap D \) are both, respectively, a unique horizontal and vertical slabs \( H_i \) and \( V_i \) in \( D \) for all \( i \in S \). Moreover,

\[
\pi_{cs} \circ \phi_i(\tau, x_{cu}) : D_{cs} \rightarrow D_{cs} \quad \text{is a contracting map for all} \quad x_{cu} \in \overline{D}_{cu},
\]

\[
\pi_{cu} \circ \phi_i^{-1}(x_{cs}, \cdot) : \overline{D}_{cu} \rightarrow D_{cu} \quad \text{is a contracting map for all} \quad x_{cs} \in \overline{D}_{cs},
\]

where \( \pi \) is the projection on the *-coordinates, * \( \in \{cs, cu\} \). Hence \( H_i = H_i \times H_i \) for \( i \in S \) where \( H_i = \{ \xi \in \Sigma : \xi_0 = i \} \) are the horizontal slabs for \( \Phi \) satisfying the Conley-Moser conditions and therefore the maximal invariant set in the closure of \( \Sigma^+_S \times D \) for \( \Phi \) is conjugated to the full shift \( \tau : \mathbb{S}^Z \rightarrow \mathbb{S}^Z \).

Observe that in general, unlike in the above examples, the hyperbolic invariant set given by the Conley-Moser conditions does not have to be a graph over the base. It suffices to modify the above example so that \( \phi_i^{-1}(\overline{D}) \cap D \) and \( \phi_i(\overline{D}) \cap D \) are two disjoints horizontal and vertical slabs respectively.
5.1.2. Construction of blending regions. We will describe now the method to construct a blending region. First we work in local coordinates with $C^r$-diffeomorphisms with $r \geq 0$. A $C^0$-diffeomorphism is understood as a bi-Lipschitz homeomorphism.

**Proposition 5.6.** Consider a $C^r$-diffeomorphism $\phi$ of $\mathbb{R}^c$ with a hyperbolic attracting/repelling/saddle fixed point $x$. Then, there exist an integer $k \equiv k(\phi, c) \geq 2$, arcs of $C^r$-diffeomorphisms of $\mathbb{R}^c$, $\phi_1 \equiv \phi_1(\epsilon), \ldots, \phi_k \equiv \phi_k(\epsilon)$ and bounded open sets $D \equiv D(\epsilon)$, $\epsilon \geq 0$, such that

- $\phi_i(0) = \phi$ for $i = 1, \ldots, k$;
- $\phi_i = T_1 \circ \phi$ where $T_i \equiv T_i(\epsilon)$ is a translation (moreover, one can take $\phi_1 = \phi$);
- $B \equiv B_0(x) \subset D \subset B_2(x)$ for some $\delta \equiv \delta(\epsilon) > 0$;
- $B$ is $cs/cu$-double-blending region with respect to $\{\phi_1, \ldots, \phi_k\}$ on $D$ for all $\epsilon > 0$.

Moreover, the $cs$-index of the blending region is equal to the $s$-index of the hyperbolic fixed point $x$.

**Proof.** Fix $\epsilon > 0$. Assume first that $x$ is an attracting/saddle fixed point of $\phi$. We will construct a $cs$-blending region. Repeating the arguments for $\phi^{-1}$ one constructs $cu$-blending regions and combining both one concludes the proposition.

Let $D = D_{cs} \times D_{cu}$ be a small neighborhood of $x$ with $D_1$ being an open ball of radius $\epsilon > 0$ centered at $\pi_* (x)$ where $* \in \{cs, cu\}$. In the case that $x$ is an attracting fixed point we allow that $D_{cu}$ could be empty, i.e., $D = D_{cs}$. Moreover, assume that

- $\pi_{cs} \circ \phi(x, x_{cu}) : D_{cs} \rightarrow D_{cs}$ is a contracting map for all $x_{cu} \in D_{cu}$.
- $\pi_{cu} \circ \phi^{-1}(x_{cs}, \epsilon) : D_{cu} \rightarrow D_{cu}$ is a contracting map for all $x_{cs} \in D_{cs}$.

Consider an open ball $B \subset D$ of radius $0 < \delta < \epsilon$ centered at $x$. Let us denote by $T_\nu$ the translation map by the vector $\nu$. Applying [6, Lemma 5.6] (see also [29, Proposition 2.3]) for the map $\pi_{cs} \circ \phi(x, x_{cu})$ and since $\pi_{cu} \circ \phi(x_{cs}, \epsilon)$ is expanding on $D_{cu}$, there exist $k \in \mathbb{N}$, vectors $u_i$ in the ball of radius 1 so that

$$B \subset \bigcup_{i=1}^{k} T_{v_i} \circ \phi(B) \quad \text{where } v_i = \delta u_i \text{ for } i = 1, \ldots, k.$$ 

In fact, the number $k$ of translation and the direction of translation $u_1, \ldots, u_k$ only depend on the (uniform) contraction lower bound of the maps $\pi_{cs} \circ \phi(x, x_{cu})$ and the dimension $c$. Having in mind Example 5.5 and taking $\phi_1 = T_{v_1} \circ \phi$ one concludes that $B$ is a $cs$-blending region with respect to $\{\phi_1, \ldots, \phi_k\}$ on $D$.

To conclude the proposition is enough to observe that $v_i(\epsilon)$ tends continuously to zero as $\epsilon \rightarrow 0$. Moreover, we can assume that $\phi_1(\epsilon) = \phi$ for all $\epsilon \geq 0$. Consequently $\phi_1(\epsilon), \ldots, \phi_k(\epsilon)$, $\epsilon \geq 0$ are arcs of $C^r$-diffeomorphisms satisfying the required conditions. \hfill $\square$

For every $x \in M$, by means of a arbitrarily small perturbation of the identity map [22, 23] we can create a map $\phi$ for which $x$ is a hyperbolic fixed point (or periodic of arbitrary large period). Hence, the above result implies the following:

**Corollary 5.7** (blending regions homotopic to the identity). For every $x \in M$, there exist arcs of $C^r$-diffeomorphisms $\phi_1 \equiv \phi_1(\epsilon), \ldots, \phi_k \equiv \phi_k(\epsilon)$ of $M$ (where $k$ only depends on the dimension of $M$), bounded open sets $D \equiv D(\epsilon) \subset B_{2\epsilon}(x)$ and $\delta \equiv \delta(\epsilon) > 0$, for $\epsilon \geq 0$, such that
- \( \phi_i(0) = \text{id} \) for \( i = 1, \ldots, k \) and
- \( B \equiv B_\varepsilon(x) \subset D \) is a blending region with respect to \( \{\phi_1, \ldots, \phi_k\} \) on \( D \) for all \( \varepsilon > 0 \).

Moreover, the blending region can be constructed having any cs-index between 0 and dimension of \( M \).

5.2. Robust cycles from one-step maps. Theorem 4.2 provides a criterion to yield robust cycles using blenders. Now we translate this criterion to the dynamics of a one-step skew-product and show how to construct arcs of one-step maps satisfying this criterion.

5.2.1. Criterion to yield robust cycles from one-step maps. Let \( \Phi = \tau \kappa (\phi_1, \ldots, \phi_d) \in \mathcal{S}(M) \) be a one-step map. We need a cs-blender for \( \Phi \). To accomplish this, from Corollary 5.3, we can assume that there is a cs-blending region \( B \) with respect to \( \{\phi_1, \ldots, \phi_d\} \) on \( D \). Hence the maximal invariant set \( \Gamma^1 \) in the closure of \( \Sigma \times D \) for \( \Phi \) is a cs-blender with superposition domain \( B = \Sigma \times B \). Now, we need to translate the transition property (RC1). To do this, let \( \Gamma^2 \) be another isolated transitive hyperbolic set of \( \Phi \) satisfying the following condition:

(RC2) there are \( x \in \mathcal{H}(W^s(\Gamma^2)) \) and \( T \in \langle \phi_1, \ldots, \phi_d \rangle^+ \) such that \( T^{-1}(x) \in B \).

Take any \( \xi \in \Sigma \) such that \( T^{-1} = \phi_{\tau(n)} \equiv \phi_{\tau(n)}^{-1} \circ \cdots \circ \phi_{\tau(1)}^{-1} \) for some \( n \geq 1 \). Then,

\[
W^s_{\loc}(\Phi^{-n}(\xi, x)) = W^s_{\loc}(\tau^{-n}(\xi)) \times\{T^{-1}(x)\} \subset \Sigma \times B.
\]

Since \( \Gamma^1 \) is a cs-blender satisfying (B1), the above inclusion implies \( W^s(\xi, x) \cap W^u(\Gamma^1) \neq \emptyset \). Since \( x \) belongs to the projection on the fiber space of \( W^s(\Gamma^2) \), one can choose \( \xi \) such that \( W^s(\xi, x) \subset W^s(\Gamma^2) \). Therefore (RC2) is equivalent to (RC1) and by Theorem 4.2 we get that \( S \)-robustly the global stable set \( W^s(\Gamma^2) \) meets the global unstable set \( W^u(\Gamma^1) \).

A similar condition to the above guarantees the robustness of the other intersection.

Definition 5.8 (transition). Let \( A_1 \) and \( A_2 \) be two subsets of \( M \). We say that \( \langle \phi_1, \ldots, \phi_d \rangle^+ \) has transition from \( A_1 \) to \( A_2 \) if there exist \( x \in A_1 \) and \( T \in \langle \phi_1, \ldots, \phi_d \rangle^+ \) such that \( T(x) \in A_2 \).

With this terminology we have obtained the following consequence of Theorem 4.2:

Corollary 5.9 (robust cycles form one-step maps). Let \( \Phi = \tau \kappa (\phi_1, \ldots, \phi_d) \in \mathcal{S}(M) \). Suppose that there are bounded open sets \( B_1 \subset D_1 \) and \( B_2 \subset D_2 \) with \( D_1 \cap D_2 = \emptyset \) such that

- \( B_1 \) is a cs-blending region with respect to \( \{\phi_1, \ldots, \phi_d\} \) on \( D_1 \);
- \( B_2 \) is a cu-blending region with respect to \( \{\phi_1, \ldots, \phi_d\} \) on \( D_2 \);
- \( \langle \phi_1, \ldots, \phi_d \rangle^+ \) has transition from \( B_1 \) to \( B_2 \) and vice-versa.

Then \( \Phi \) has a robust cycle associated with a symbolic cs-blender (the maximal invariant set in \( \Sigma \times \overline{D_1} \)) and a symbolic cu-blender (the maximal invariant set in \( \Sigma \times \overline{D_2} \)).

A point \( x \in X \) is called a periodic point of \( \langle \phi_1, \ldots, \phi_d \rangle^+ \) if there is \( f \in \langle \phi_1, \ldots, \phi_d \rangle^+ \) such that \( f(x) = x \). Observe that the periodic points of \( \langle \phi_1, \ldots, \phi_d \rangle^+ \) are the projection on the fiber space of the periodic points of \( \Phi = \tau \kappa (\phi_1, \ldots, \phi_d) \).
Example 5.10 (Robust cycles with non-hyperbolic periodic points). Suppose $B$ is a double-blending region with respect to $\{\phi_1, \ldots, \phi_d\}$ on $D$ and $x \in M \setminus D$ is a periodic point of $\langle \phi_1, \ldots, \phi_d \rangle^+$ such that there are $T, S \in \langle \phi_1, \ldots, \phi_d \rangle^+$ so that $T^{-1}(x), S(x) \in B$. Then, one gets a symbolic cycle associated with a double-blender and a periodic point. Notice that this criteria allows us to construct a robust cycle between a double-blender and any periodic point that admits continuation under small perturbations.

5.2.2. Construction of blending regions with transition. We will now explain how to construct examples of blending regions with transition on the manifold $M$. To do this, we will work in local coordinates providing a set of $C^r$-diffeomorphisms $\{\phi_1, \ldots, \phi_d\}$ from $\mathbb{R}^c$ to itself, as well as disjoint bounded open sets $D_1$ and $D_2$ of $\mathbb{R}^c$ and open sets $B_1 \subset D_1, B_2 \subset D_2$ such that the conditions in Corollary 5.9 hold.

Let $h$ be a $C^r$-diffeomorphism of $\mathbb{R}^c$ with two different hyperbolic fixed points $p$ and $q$ so that $W^u(p; h) \cap W^s(q; h) \neq \emptyset$. By means of Proposition 5.6 we get arcs of $C^r$-diffeomorphisms, $\phi_2 \equiv \phi_2(\varepsilon), \ldots, \phi_k \equiv \phi_k(\varepsilon)$ and disjoint bounded open sets $D_1 \equiv D_1(\varepsilon), D_2 \equiv D_2(\varepsilon)$ so that $\phi_i(0) = h$ and $B_1 \equiv B_1(\varepsilon), B_2 \equiv B_2(\varepsilon)$ are, respectively, a cs and cu-blending regions for $\{h, \phi_2, \ldots, \phi_k\}$ on $D_1$ and $D_2$. Since the stable set of $p$ and unstable set of $q$ for $h$ have non-empty intersection, there is a point $y$ arbitrarily close to $q$ such that $h^i(y)$ converges to $p$. Hence for any $\varepsilon > 0$ small enough one can assume that $y \in B_1$ and $h^i(y) \in B_2$. This provides a transition from $B_2$ to $B_1$. To give the other transition we consider the map $T(x) = x + q - p$. Since this additional map is a translation so that $T(D_1) \cap D_1 = \emptyset$ one easily gets that

- $B_1$ is a cs-blending region with respect to $\{h, \phi_2, \ldots, \phi_k, T\}$ on $D_1$;
- $B_2$ is a cu-blending region with respect to $\{h, \phi_2, \ldots, \phi_k, T\}$ on $D_2$;
- $\langle h, \phi_2, \ldots, \phi_k, T \rangle^+$ has transition from $B_1$ to $B_2$ and vice-versa.

Notice that the above construction can be done homotopic to the identity taking an arc of $C^r$-diffeomorphisms $h_c : \mathbb{R}^c \to \mathbb{R}^c$ as above with the additional properties that $h_c$ goes to the identity map and $||p(\varepsilon) - q(\varepsilon)|| \to 0$ as $\varepsilon \to 0$. Therefore we have obtained the following.

Proposition 5.11 (blending regions with transition homotopic to the identity). For every $x \in M$, there exist arcs of $C^r$-diffeomorphisms $\phi_1 \equiv \phi_1(\varepsilon), \ldots, \phi_k+1 \equiv \phi_k+1(\varepsilon), \varepsilon \geq 0$ of $M$ where $k$ only depends on the dimension of $M$ and disjoints bounded open sets $D_1 \equiv D_1(\varepsilon)$ and $D_2 \equiv D_2(\varepsilon)$ in a small neighborhood of $x$ such that

- $\phi_i(0) = \text{id}$ for all $i = 1, \ldots, k + 1$,
- there is a cs-blending region $B_1$ with respect to $\{\phi_1, \phi_2, \ldots, \phi_{k+1}\}$ on $D_1$;
- there is a cu-blending region $B_2$ with respect to $\{\phi_1, \phi_2, \ldots, \phi_{k+1}\}$ on $D_2$;
- $\langle \phi_1, \ldots, \phi_{k+1} \rangle^+$ has transition from $B_1$ to $B_2$ and vice-versa.

Moreover, both blending regions can be constructed having any co-index (difference between the cs-indices) between zero and the dimension of $M$.

5.2.3. Arcs of one-step maps with robust cycles. Previous constructions allow us to construct arcs of symbolic skew-product with robust cycles. Moreover, we obtain these cycles for
robust topologically mixing symbolic skew-products. That is, for symbolic skew-products $\Phi$ such that for any small enough $\delta^0$-perturbation $\Psi$ and for every pair of open sets $U$, $V$ of $M$, there is $n_0 > 0$ such that $\Psi^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. In particular topologically mixing implies transitivity.

**Theorem 5.12.** Let $M$ be a differentiable manifold of dimension $c \geq 1$. Then, there are an integer $d \equiv d(c) \geq 3$ and an arc of one-step maps $\Phi_{\varepsilon} = \tau \kappa (\phi_1, \ldots, \phi_d) \in \mathcal{PHS}^1(M)$, $\varepsilon \geq 0$, such that $\Phi_0 = \tau \times \text{id}$ and for every $\varepsilon > 0$,

- $\Phi_{\varepsilon}$ is $\delta^0$-robustly topologically mixing;
- $\Phi_{\varepsilon}$ has a $\delta$-robust (heterodimensional or equidimensional) cycle.

Moreover, the arc can be taken in such a way that $\Phi_{\varepsilon}$ has robust cycles of all intermediate co-index.

**Proof.** Consider a point $x \in M$. By Proposition 5.11 we get arcs of $C^\gamma$-diffeomorphisms $\phi_1 \equiv \phi_1(\varepsilon), \ldots, \phi_{k+1} \equiv \phi_{k+1}(\varepsilon)$ homotopic to the identity as $\varepsilon \rightarrow 0^+$, where $k \geq 2$ and only depends on $c$. We obtain as well the $c_s$ and $c_u$ blending regions $B_1$ and $B_2$ respectively in $B_\varepsilon(x)$ with transitions. Without loss of generality, assume that $B_1$ is a contracting-blending region. We can add to these maps, if necessary, a pair of Morse-Smale diffeomorphisms (homotopic to the identity) without periodic points in common and having respectively an attracting/repelling fixed point in $B_1$ with a dense stable/unstable manifold. The existence of such a map can be constructed by perturbations of time-one maps of gradient-like vector fields. Take $d = k+1+2 \geq 3$, which only depends on $c$, and set $\Phi_{\varepsilon} = \tau \kappa (\phi_1, \ldots, \phi_d) \in \mathcal{PHS}^1(M)$. Hence $\Phi_{0} = \tau \times \text{id}$ and according to [6, Theorem 5.7] and Corollary 5.9, for any $\varepsilon > 0$, $\Phi_{\varepsilon}$ is $\delta^0$-robustly topologically mixing and has a $\delta$-robust symbolic cycle for any $\varepsilon > 0$. Notice that the co-index of the cycle may be chosen between zero and $c$. In fact, repeating the above procedure and adding more maps if necessary, the arcs may be taken in such a way so that $\Phi_{\varepsilon}$ has $\delta$-robust cycles of all intermediate co-indices between zero and $c$. This completes the proof of the theorem. \hfill $\Box$

5.3. **Robust tangencies from one-step maps.** In Theorem 4.9 we gave conditions (T1), (T2), (T3) and (T4) to construct robust tangencies. We will translate these conditions to one-step maps (Corollary 5.15) and then we will construct arcs of IFSs unfolding from the identity satisfying the criterion that imply robust tangencies (Proposition 5.17).

5.3.1. **Criterion to yield robust tangencies from one-step maps.** Let $\Phi = \tau \kappa (\phi_1, \ldots, \phi_d) \in \mathcal{PHS}^1(M)$ be a fiber bunched one-step map. That is,

$$v^a < \gamma < m(D\phi_i) < \|D\phi_i\| < \hat{\gamma}^{-1} < v^{-a} \quad \text{with} \quad \gamma \hat{\gamma} > v^a \quad \text{for all} \quad i = 1, \ldots, d.$$

We take integers $0 < i_1, i_2 < c$ and $\max\{0, i_2 - i_1\} < \ell \leq \min\{c - i_1, i_2\}$.

In order to translate (T1) for one-step maps, we need the following lemma:

**Lemma 5.13.** If $\Phi = \tau \kappa \phi_\varepsilon \in \mathcal{PHS}^1(M)$ is fiber bunched with the uniform Lipschitz constant of $D\phi_\varepsilon$ small enough then the induced $\ell$-th Grassmannian symbolic skew-product $\hat{\Phi}$ on $\hat{M} = \Sigma \times \hat{M}$ belongs to $\mathcal{PHS}(\hat{M})$, where $\hat{M} = \mathbb{R}^c \times G(\ell,c)$, with $1 \leq \ell \leq c$. 


Proof. Clearly Φ is a α-Hölder symbolic skew-product. According to (15), since Φ is fiber bunched (να < γα) and taking the Lipschitz constant L of Dφξ less than γ(να−γα−1), it follows that the Lipschitz constant of ̂φξ satisfies max{γ−1} + γ−1L, (γγ)−1 < να. A similar inequality holds for ̂φξ, implying that Φ = τ ⋙ ̂φξ ∈ P3(S(M)). □

In view of the above lemma, we will assume that the Lipschitz constant of DΦ, i.e., of Dφ^i∈{1,...,d} is small enough for all i = 1, … , d. Then ̂Φ = τ ⋙ (̂φ_1, ..., ̂φ_d) satisfies (T1) where

\[ ̂φ_i(x, E) = (φ_i(x), Dφ_i(x)E), \quad (x, E) ∈ ̂M \overset{def}{=} R^c × G(ℓ, c), \quad \text{for} \ i = 1, … , d. \]

Now, we will suppose that there are cs and cu blending regions B_1 and B_2 for {φ_1, … , φ_d} with cs-indices i_1 and i_2 respectively. According to Corollary 5.3, these conditions provide a cs-blender Γ_1 and a cu-blender Γ_2 with superposition domains B_1 = Σ × B_1 and B_2 = Σ × B_2 respectively. We will also assume that there are an unstable ℓ-cone c^uu and a stable ℓ-cone c^ss for Φ on B_1 and B_2 respectively. Under these assumptions we have that (T2) holds.

To translate property (T3) for the one-step case, it suffices to ask that there are a cs-blending region ̂B_1 and a cu-blending region ̂B_2 with respect to {̂φ_1, … , ̂φ_d} such that ̂B_1 ⊂ B_1 × c^uu and ̂B_2 ⊂ B_2 × c^ss. Notice that property (B4) is satisfied since we are working with blenders constructed from the covering property.

Finally, having into account properties (B5), (RC2) and Definition 5.8, (T4) is equivalent in this case to the existence of a transition for (̂φ_1, … , ̂φ_d)_+ from ̂B_1 to ̂B_2.

Definition 5.14 (blending region with tangency). Let B, D be bounded open sets of R^c and 0 < ℓ < c. We say that a cs-blending region, B, with respect to {φ_1, … , φ_d} on D has a tangency of dimension ℓ if there exist bounded open sets ̂B and ̂D in ̂M = R^c × G(ℓ, c) such that

\[ ̂B \text{ is a cs-blending region with respect to } {̂φ_1, … , ̂φ_d} \text{ on } ̂D \text{ so that } ̂B ⊂ B \times c^uu. \]

Here c^uu is an unstable ℓ-cone for Φ = τ ⋙ (φ_1, … , φ_d) on Σ × B. We say that the set ̂B is a ℓ-tangency (of the blending region B). Similarly, we define a cu-blending region with a tangency of dimension ℓ.

Using this terminology, we summarize Theorem 4.9 in the one-step setting:

Corollary 5.15 (robust tangencies from one-step maps). Let Φ = τ ⋙ (φ_1, … , φ_d) ∈ P3(S^1(R^c)) be fiber bunched one-step map with small enough Lipschitz constant of Dφ^i∈{1,...,d}. Consider integers

\[ 0 < i_1, i_2 < c \quad \text{and} \quad \max\{0, i_2 - i_1\} < ℓ ≤ \min\{c - i_1, i_2\}. \]

Suppose that there are bounded open sets D_1, D_2, B_1, B_2 of R^c such that with respect to {φ_1, … , φ_d},

- B_1 is a cs-blending region on D_1, with cs-index i_1 and a ℓ-tangency ̂B_1;
- B_2 is a cu-blending region on D_2, with cs-index i_2 and a ℓ-tangency ̂B_2;
- (̂φ_1, … , ̂φ_d)_+ has a transition from ̂B_1 to ̂B_2.

Then Φ has a S^1-robust tangency of dimension ℓ between W^u(Γ_1; Φ) and W^u(Γ_2; Φ) where Γ_1 and Γ_2 are the maximal invariant sets in Σ × D_1 and Σ × D_2.
5.3.2. Construction of blending regions with tangencies. We will now explain how to construct examples of blending regions with tangencies.

**Proposition 5.16** (blending region with tangency). Let \( p \) be a hyperbolic fixed point of a \( C^r \)-diffeomorphism \( \phi \) of \( \mathbb{R}^c \) with \( r \geq 2 \) and having a dominated splitting \( E^s \oplus E^{cu} \oplus E^{uu} \) where the unstable direction is \( E^{uu} = E^{iu} \oplus E^{uu} \) and \( 0 < \ell = \dim E^{uu} < c \). Then, there are

i) an unstable \( \ell \)-cone \( C^{uu} \) around \( E^{uu} \),

ii) neighborhoods \( B \) of \( p \) in \( \mathbb{R}^c \) and \( G \subset C^{uu} \) of \( E^{uu} \) in \( G(\ell, c) \), and

iii) a finite set of \( C^0 \)-diffeomorphisms \( \{ \hat{\phi}_1, \ldots, \hat{\phi}_d \} \) on \( \hat{M} = \mathbb{R}^c \times G(\ell, c) \) induced by \( \phi_i = A_i \cdot \phi + c_i \), where \( A_i \) is an orthogonal matrix and \( c_i \in \mathbb{R}^c \),

such that

\[
\overline{B} \subset \bigcup_{i=1}^{d} \hat{\phi}_i(B) \quad \text{where} \quad \hat{B} = B \times G.
\]

That is, \( B \) is a cs-blending region with respect to \( \{ \phi_1, \ldots, \phi_d \} \) of cs-index \( \dim E^s \) and a \( \ell \)-tangency. Moreover, the maps \( \phi_1, \ldots, \phi_d \) can be constructed as an arc of \( C^\ell \)-diffeomorphisms homotopic to \( \phi \).

**Proof.** As in Proposition 5.6, we can take a small neighborhood \( B \) of \( p \) in \( \mathbb{R}^c \) and choose arcs of translations \( f_i = \phi + c_i \) homotopic to \( \phi \) with \( c_i \in \mathbb{R}^c \) in order to have that \( B \) is a cs-blending region with respect to \( \{ f_1, \ldots, f_k \} \). In particular, \( \{ f_i(B) : i = 1, \ldots, k \} \) is an open cover of \( \overline{B} \). Notice that there exists an \( \varepsilon > 0 \), such that the above covering property holds for any \( \varepsilon \)-perturbation \( \phi_i \) of \( f_i \).

On the other hand, \( D\phi(p) \) induces a map \( A \) on \( G(\ell, c) \) given by \( A(E) = D\phi(p)E \). Since \( \phi \) is \( C^2 \), this map has a hyperbolic attracting fixed point \( E^{uu} \). Then, as in Proposition 5.6, there are arcs \( T_1 \equiv T_1(\varepsilon), \ldots, T_{k_2} \equiv T_{k_2}(\varepsilon) \) of translations on \( G(\ell, c) \) homotopic to the identity and a neighborhood \( G \equiv G(\varepsilon) \) of \( E^{uu} \) in \( G(k, c) \), such that \( \{ F_j(G) : j = 1, \ldots, k_2 \} \) is an open cover of \( G \) where \( F_j = T_j \circ A \). Each translation map \( F_j \) in the Grassmannian corresponds to a map of the form \( A_j \cdot D\phi(p) \) in the tangent bundle where \( A_j \) is an orthogonal matrix and \( \| A_j - \text{id} \| \to 0 \) as \( \varepsilon \to 0 \) for all \( j = 1, \ldots, k_2 \). Moreover, we can take \( G \) small enough so that \( G \subset C^{uu} \) where \( C^{uu} \) is an unstable \( \ell \)-cone around \( E^{uu} \) for \( \phi \). Taking

\[
\| A_j - \text{id} \| < \varepsilon / \| \phi \| \quad \text{and} \quad \phi_{ij} = A_j \cdot \phi + c_i
\]

then \( \| \phi_{ij} - f_i \| < \varepsilon \) and so

\[
\overline{B} \subset \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} \phi_{ij}(B).
\]

Finally, by construction, the maps \( \phi_{ij} \) induce a set of maps \( \hat{\phi}_{ij} \) on \( \hat{M} \) so that \( \{ \hat{\phi}_{ij} \} \) is an open cover of the closure of \( \hat{B} = B \times G \). Moreover, since \( A_j \) is an orthogonal matrix then the hyperbolicity of the blending regions still holds. That is, \( B \) is also a cs-blending region with respect to \( \{ \phi_{ij} : i = 1, \ldots, k_1, \ j = 1, \ldots, k_2 \} \) with cs-index equal to \( \dim E^s \) and as well is a \( \ell \)-tangency (the cs-blending region \( \hat{B} \) on \( \hat{M} \)). This completes the proof. \( \square \)
Notice that the above construction can be done homotopic to the identity by means of Corollary 5.7. Also we can construct a transition homotopic to the identity as was done in Proposition 5.11. Therefore we get the following:

**Proposition 5.17** (blending regions with tangencies and transition homotopic to the identity). Consider integers $0 < i_1, i_2 < c$ and $\max\{0, i_2 - i_1\} < \ell \leq \min\{c - i_1, i_2\}$. For every $x \in \mathbb{R}^c$, there are arcs of $C^r$-diffeomorphisms $\phi_1 \equiv \phi_1(x), \ldots, \phi_d \equiv \phi_d(x)$, $\varepsilon \geq 0$ where $r \geq 2$ and $d \equiv d(c) \geq 2$, and bounded open sets $B_1 \equiv B(\varepsilon), B_2 \equiv B_2(\varepsilon)$, $D_1 \equiv D_1(\varepsilon)$ and $D_2 \equiv D_2(\varepsilon)$ in a small neighborhood of $x$ such that with respect to $(\phi_1, \ldots, \phi_d)$,

- $B_1$ is a cs-blending region on $D_1$, with cs-index $i_1$ and a $\ell$-tangency $\hat{B}_1$;
- $B_2$ is a cu-blending region on $D_2$, with cs-index $i_2$ and a $\ell$-tangency $\hat{B}_2$;
- $\langle \hat{\phi}_1, \ldots, \hat{\phi}_d \rangle^+$ has transition between $\hat{B}_1$ and $\hat{B}_2$.

**5.3.3. Arcs of one-step maps with robust tangencies.** We prove Theorem C in the symbolic setting:

**Theorem 5.18.** Let $M$ be a differentiable manifold of dimension $c > 1$. Consider integers

$$0 < i_1, i_2 < c \quad \text{and} \quad \max\{0, i_2 - i_1\} < \ell \leq \min\{c - i_1, i_2\}.$$  

Then, there are an integer $d \equiv d(c) \geq 3$ and arcs of one-step maps $\Phi_\varepsilon = \tau \star (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}\mathcal{S}'(M)$, $\varepsilon > 0$, such that $\Phi_0 = \tau \times \text{id}$ and for every $\varepsilon > 0$,

- there are a cs-blender $\Gamma_1^\varepsilon$ and a cu-blender $\Gamma_2^\varepsilon$ of $\Phi_\varepsilon$ of indices $i_1$ and $i_2$ respectively,
- $\Phi_\varepsilon$ has a $S^1$-robust tangency of dimension $\ell$ between $W^s(\Gamma_1^\varepsilon)$ and $W^u(\Gamma_2^\varepsilon)$.

Moreover, these arcs can be taken in such a way so that $\Phi_\varepsilon$ also has a $S^1$-robust tangency between $W^u(\Gamma_1^\varepsilon)$ and $W^s(\Gamma_2^\varepsilon)$. In this case $\Gamma_1^\varepsilon$ and $\Gamma_2^\varepsilon$ are double-blenders for $\Phi_\varepsilon$. Namely, we can take

- $\Phi_\varepsilon$ having a $S^1$-robust homoclinic tangency associated with $\Gamma_1^\varepsilon = \Gamma_2^\varepsilon$ (hence $i_1 = i_2$);
- $\Phi_\varepsilon$ having a $S^1$-robust equi/heterodimensional tangency on a cycle associated with $\Gamma_1^\varepsilon$ and $\Gamma_2^\varepsilon$.

**Proof.** We will work in local coordinates, and hence it suffices to construct arcs of one-step maps

$$\Phi = \tau \star (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}\mathcal{S}'(\mathbb{R}^c),$$

with $c > 1$ and $r \geq 2$ satisfying the statement of the theorem. Set

$$0 < i_1, i_2 < c \quad \text{and} \quad \max\{0, i_2 - i_1\} < \ell \leq \min\{c - i_1, i_2\}.$$  

By Proposition 5.17 we obtain arcs of $C^r$-diffeomorphisms $\phi_1 \equiv \phi_1(x), \ldots, \phi_d \equiv \phi_d(x)$ homotopic to the identity as $\varepsilon \to 0^+$, where $d \equiv d(c) \geq 2$ only depends on $c$, and having cs and cu-blending regions $B_1$ and $B_2$ of cs-indices $i_1$ and $i_2$ with $\ell$-tangencies $\hat{B}_1$ and $\hat{B}_2$ respectively. Moreover, the IFS generated by the induced maps $\hat{\phi}_1, \ldots, \hat{\phi}_d$ on $\hat{M} = \mathbb{R}^c \times G(\ell, c)$ has transition between $\hat{B}_1$ and $\hat{B}_2$.

Set $\Phi_\varepsilon = \tau \star (\phi_1, \ldots, \phi_d) \in \mathcal{P}\mathcal{H}\mathcal{S}'(M)$. Hence $\Phi_0 = \tau \times \text{id}$ and according to Corollary 5.15, for any $\varepsilon > 0$, $\Phi_\varepsilon$ has a $S^1$-robust tangency of dimension $\ell$ between $W^u(\Gamma_1^\varepsilon)$ and $W^s(\Gamma_2^\varepsilon)$. Here $\Gamma_1^\varepsilon$ and $\Gamma_2^\varepsilon$ are cs and cu-blenders of cs-indices $i_1$ and $i_2$ associated with the blending regions $B_1$ and $B_2$ respectively. This completes the first part of the theorem.
The second part follows similarly by asking that the above IFS is constructed (doing the corresponding modifications in Proposition 5.17 and Corollary 5.15) in such a way that the associated one-step skew-product satisfies the required properties of Corollary 4.12. □

6. Realizations: Proofs of Theorems A, B and C

First of all, we will construct diffeomorphisms satisfying the hypothesis of Theorem 5.12. This smooth realization of robust cycles will done in a similar manner as in [6].

**Theorem 6.1.** For any integer $0 < k \leq \dim M$, there is an arc $\{f_\varepsilon\}_{\varepsilon \in \varepsilon}$ of $C^r$-diffeomorphisms of $N \times M$ such that $f_0 = F \times \text{id}$ and for every $\varepsilon > 0$, any small enough $C^1$-perturbation $g$ of $f_\varepsilon$ has

- a transitive partially hyperbolic set $\Delta_g \subset N \times M$ homeomorphic to $\Lambda \times M$ and
- a $C^1$-robust heterodimensional cycle (in $\Delta_g$) of co-index $k$.

**Proof.** By hypothesis, $F : N \to N$ has a horseshoe $\Lambda \subset N$, which is conjugated to a full shift of $d$ symbols. Here $d \geq 2$ is a sufficiently large integer coming from the number of generators of the IFSs required in the constructions of Theorem 5.12, which only depends on the dimension of $M$. Take $R_1, \ldots, R_d$ to be the rectangles in the manifold $N$ such that $\{R_1 \cap \Lambda, \ldots, R_d \cap \Lambda\}$ is a Markov partition for $F|_\Lambda$.

From Theorem 5.12 we have arcs, $\Phi_\varepsilon = \tau \ast (\phi_1, \ldots, \phi_d) \in \mathcal{PHS}(M)$, $\varepsilon \geq 0$, of $S^0$-robustly topologically mixing symbolic skew-products with a $S$-robust cycle of co-index $0 \leq k \leq c$ where $c = \dim M$. We can modify $f_0 = F \times \text{id}$ in $R_i \times M$ to get a one-parameter family of diffeomorphisms $f_\varepsilon$ satisfying

$$f_\varepsilon|_{R_i \times M} = F \times \phi_i \quad \text{for } i = 1, \ldots, d.$$

Note that the locally constant skew-product diffeomorphism $f_\varepsilon$ restricted to the set $\Lambda \times M$ is conjugated to the one-step symbolic skew-product $\Phi_\varepsilon$. In fact, according to [6, Prop. A.2] (see also, [19, 24, 35]), for every small enough $C^1$-perturbation $g$ of $f_\varepsilon$, $g$ has an invariant set $\Delta_g$ homeomorphic to $\Lambda \times M$ such that $g|_{\Delta_g}$ is conjugated with a $S^0$-perturbation $\Psi_g$ of $\Phi_\varepsilon$. The conjugation will associate with the two hyperbolic sets and their cyclic connections constructed for $\Psi_g$, a cycle for the map $g$ restricted to $\Delta_g$ of the same co-index. Thus, $f_\varepsilon$ restricted to $\Delta = \Lambda \times M$ is $C^1$-robustly transitive partially hyperbolic and has a $C^1$-robust cycle of co-index $0 \leq k \leq c$. This completes the proof. □

Notice that Theorem A and B are an immediate combination of the constructions in Theorems 6.1 and C. Thus, we only need to show Theorem C.

6.1. **Proof of Theorem C.** From Theorem 5.12 we have an arc $\Phi_\varepsilon = \tau \ast (\phi_1, \ldots, \phi_d) \in \mathcal{PHS}(M)$, $\varepsilon \geq 0$, of symbolic skew-products having a $S^1$-robust tangency between the stable and unstable sets of a $cs$-blender and a $cu$-blender respectively. As in the proof of Theorem 6.1, take $f_0 = F \times \text{id}$ in $R_i \times M$ to be a one-parameter family of diffeomorphisms $f_\varepsilon$ satisfying

$$f_\varepsilon|_{R_i \times M} = F \times \phi_i \quad \text{for } i = 1, \ldots, d.$$
For a small $C^2$-perturbation $g$ of $f$, $g$ has an invariant set $\Delta_g$ homeomorphic to $\Lambda \times M$ such that $g|_{\Delta_g}$ is conjugated with an $S^1$-perturbation of $\Psi_g$ of $\Psi_f$ (see [19]). The conjugacy will associate the intersections between the unstable and stable sets of the blenders for $\Psi_g$ with intersections between the unstable and stable manifolds of the corresponding blenders for $g$. We now would like to show that the tangent directions associated with the tangency for $\Psi_g$ will correspond to tangent directions for $g$.

Let $h$ be the homeomorphism $h : \Sigma \times M \to \Delta_g \subset N \times M$, that conjugates $\Psi_g$ and $g$ restricted to $\Delta_g$. Since we are working in the $C^2$ topology, by [19], restricted to the fibers $h$ is $C^2$. That is for a fixed sequence $\xi$, the map $h_\xi : M \to \Delta_g$, given by $h_\xi(x) = h(\xi, x)$, is $C^2$. Moreover by [24] (see [6, Equations A.5–A.6]), the map $h_\xi$ is $C^1$-close to the identity of order $\epsilon$, where $\epsilon$ depends on the size of the initial arc $f_\xi$, but does not depend on the sequence $\xi$.

Going back to $\Psi_g = \tau \circ \psi_\xi$, let $(\zeta, y) \in \Sigma \times M$ be the tangency point and consider an unitary tangential direction $v \in T_y M$. Hence, there are $C > 0$ and $0 < \lambda < 1$ such that

$$\|D\psi^n_\zeta(y)v\| \leq C\lambda^n \quad \text{for all } n \in \mathbb{Z}.$$ 

On the other hand, $Y = h_\xi(y) = h(\zeta, y)$ is a point in the intersection between the stable and unstable manifolds of the blenders for $g$. Consider the vector,

$$w \overset{\text{def}}{=} Dh_\xi(y)v \in T_Y (M \times N).$$

We would like to show that $\|Dg^n(Y)w\|$ goes exponentially fast to zero as $|n| \to \infty$. This would imply that $w$ is a tangential direction at $Y$ for $g$.

Using the conjugation on the fibers, $g^n \circ h_\xi = h_{\tau^n(\xi)} \circ \psi^n_\zeta$, and so

$$\|Dg^n(Y)w\| = \|Dg^n(h_\xi(y))Dh_\xi(y)v\|
= \|D(g^n \circ h_\xi(y))v\| = \|D(h_{\tau^n(\xi)} \circ \psi^n_\zeta(y))v\|
\leq \|Dh_{\tau^n(\xi)}(\psi^n_\zeta(y))\| \cdot \|D\psi^n_\zeta(y)v\| \leq (1 + \epsilon)\|D\psi^n_\zeta(y)v\| \leq (1 + \epsilon)C\lambda^n$$

where we have used the fact that $Dh_{\tau^n(\xi)}$ is $\epsilon$-close to the identity. This proves that that the tangency at $Y$ for $g$ has the same number of independent tangential directions as the tangency $(\zeta, y)$ for $\Psi_g$, completing the proof of Theorem $C$.

**Remark 6.2.** Theorem $C$ allows the construction of $C^2$-robust heterodimensional tangencies on a $C^1$-robust cycle. It is natural to ask, if there exist open sets of diffeomorphisms which have heterodimensional tangencies on heterodimensional cycles of a given co-index $k$. This question was posed in [26], and was solved in the case of $k = d - 2$ where $d$ is the dimension of the manifold. Hence, by Theorem $C$ and the result of [26], every $C^2$-manifold of dimension $d \geq 3$ admits a $C^2$-robust heterodimensional tangency on a cycle of any co-index $1 \leq k \leq d - 4$ and $k = d - 2$. The case of co-index $k = d - 3$ remains open.
Appendix A. Proof of the criterion for blenders

In this section we prove Theorem 3.8. Since blenders are local dynamical tools we can consider only local perturbations. For this reason, we need to extend the set of symbolic skew-products.

Let $M$ be a separable locally compact metric space with finite Lebesgue covering dimension and consider $D \subset M$ and $0 < \lambda < \beta$. A map $\phi: M \to M$ is $(\lambda, \beta)$-Lipschitz on $D$ if

$$\lambda d(x, y) < d(\phi(x), \phi(y)) < \beta d(x, y), \quad \text{for all } x, y \in \overline{D}.$$ 

For a subset $S \subset \mathcal{A}$, set $\Sigma^+_S = \{\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma : \xi_0 \in S\}$. We say that $\Phi = \tau \circ \phi_{\xi}$ is locally $\alpha$-Hölder continuous on $D$ at $S$ if there is $C \geq 0$ such that

$$d(\phi^1_{x}(x), \phi^1_{y}(x)) \leq C d(x, y)^{\alpha} \quad \text{for all } x \in \overline{D} \text{ and } \xi, \zeta \in \Sigma^+_S \text{ with } \xi_0 = \zeta_0.$$  

We denote by $C_0(\Phi, D, S)$ the smallest non-negative constant satisfying (A.1).

**Definition A.1.** Consider $D \subset M$, $0 < \lambda < \beta$ and $S \subset \mathcal{A}$. We define $\mathcal{S}(D, S, \lambda, \beta) \equiv \mathcal{S}(\mathcal{A}, \lambda, \beta)(D, S, \lambda, \beta)$ as the set of symbolic skew-products $\Phi = \tau \circ \phi_{\xi}$ as in (6) such that

- $\phi_{\xi}$ is $(\lambda, \beta)$-Lipschitz on $D$ for all $\xi \in \Sigma^+_S$,
- $\phi_{\xi}$ depends locally $\alpha$-Hölder continuously on $D$ with respect to $\xi \in \Sigma^+_S$.

Assuming $D$ a relatively compact set, the set $\mathcal{S}(D, S, \lambda, \beta)$ is endowed with the pseudometric $d_{\mathcal{S}}(\Phi, \Psi)_{D, S} \equiv d_0(\Phi, \Psi)_{D, S} + \text{Lip}_0(\Phi, \Psi)_{D, S} + \text{Hol}_0(\Phi, \Psi)_{D, S},$

where for $\Phi = \tau \circ \phi_{\xi}$ and $\Psi = \tau \circ \psi_{\xi}$ in $\mathcal{S}(D, S, \lambda, \beta)$

$$d_0(\Phi, \Psi)_{D, S} \equiv \sup_{\xi \in \Sigma^+_S} d_{\overline{D}}(\phi_{\xi}^1, \psi_{\xi}^1), \quad \text{Lip}_0(\Phi, \Psi)_{D, S} \equiv \sup_{\xi \in \Sigma^+_S} \left|L^1(\phi_{\xi}, D) - L^1(\psi_{\xi}, D)\right|$$

$$\text{Hol}_0(\Phi, \Psi)_{D, S} \equiv |C_0(\Phi, D, S) - C_0(\Psi, D, S)|$$

with

$$L(\phi, D) = \max_{x, y \in D, x \neq y} \frac{d(\phi(x), \phi(y))}{d(x, y)} \quad \text{and} \quad L^{-1}(\phi, D) = \min_{x, y \in \overline{D}, x \neq y} \frac{d(\phi^{-1}(x), \phi^{-1}(y))}{d(x, y)}.$$

Given $\bar{\omega} = \omega_{-\ell} \ldots \omega_{-1} \omega_0 \omega_1 \ldots \omega_{j-1}$, where $\ell, j \geq 0$ and $\omega_i \in \{1, \ldots, d\}$, we define the bi-lateral cylinder by

$$C_{\bar{\omega}} \equiv \{\xi \in \Sigma : \xi_i = \omega_i, -\ell \leq i \leq j - 1\}.$$ 

We denote these bi-lateral cylinders by $V_{\bar{\omega}}$ and $H_{\bar{\omega}}$ in the particular cases that the finite word $\bar{\omega}$ has $\ell > 0, j = 0$ and $\ell = 0, j > 0$ respectively. Finally, given a finite set of integers $I = \{n_1, \ldots, n_k\} \subset \mathbb{N}$ and a set $\mathcal{B} \subset \mathcal{M} = \Sigma \times M$, we define

$$\Lambda^I_{\mathcal{B}}(\mathcal{B}; \Phi) = \{P \in \mathcal{M} : \text{ there is } m_i \to \infty \text{ so that } m_{i+1} - m_i \in I \text{ and } \Phi^{-m_i}(P) \in \mathcal{B}\}.$$
Theorem A.2. Let $\Phi \in S(D,S,\lambda,\beta)$ be a symbolic skew-product with $\nu^* < \lambda$. Assume that the following cs-covering property holds: there exist an integer $k \geq 2$, open sets $B,B_i \subset D$, integers $n_i \in \mathbb{N}$ and words $\alpha_i \in S^n$ for all $i = 1,\ldots,k$ such that

$$V_{\alpha_i} \times B_i \subset \Phi^{n_i}(H_{\alpha_i} \times B) \quad \text{for} \quad i = 1,\ldots,k \quad \text{and} \quad B \subset \bigcup_{i=1}^k B_i \quad \text{(A.2)}$$

where $L$ is the Lebesgue number of (A.2). Then for any $0 < \delta < \lambda L/2$ and for every small enough $S$-perturbation $\Psi$ of $\Phi$,

$$\Lambda^u_I(B;\Psi) \cap D^s \neq \emptyset, \quad \text{for all } \delta\text{-horizontal discs } D^s \text{ in } B = V \times B$$

where

$$I = \{n_1,\ldots,n_k\}, \quad \text{and} \quad V = H_{\alpha_1} \cup \cdots \cup H_{\alpha_k}.$$ 

In addition, assuming that the maximal invariant set $\Gamma$ in $\Sigma_{S}^+ \times D$ is a transitive hyperbolic set of $\Phi$ with $\text{ind}^c(\Gamma) > 0$, then $\Gamma$ is a cs-blender of $\Phi$ whose superposition region contains the open set of almost horizontal disks in $V \times B$.

Addendum A.3. Under the assumption of Theorem A.2, if $n_i = \ell \geq 1$ for all $i = 1,\ldots,k$ then for every small enough $S$-perturbation $\Psi$ of $\Phi$,

$$\Lambda^u(B;\Psi^\ell) \cap D^s \neq \emptyset \quad \text{for all } \delta\text{-horizontal discs } D^s \text{ in } B \text{ where } \Lambda^u(B;\Psi^\ell) \overset{\text{def}}{=} \bigcap_{n \geq 0} \Psi^{n\ell}(B).$$

In addition, if the maximal invariant set $\Gamma$ in $\Sigma_{S}^+ \times D$ is a transitive hyperbolic set of $\Phi^\ell$ with $\text{ind}^c(\Gamma) > 0$, then $\Gamma$ is a cs-blender of $\Phi^\ell$.

Analogously conditions yield $cu$-blenders and double-blenders (see Remark 3.10).

Remark A.4. In another approach, Moreira and Silva [40] used the criterion of the recurrent compact set to obtain blenders. Their techniques have certain similarities to the proof below, and in fact the covering property (A.2) implies that the blender-horseshoe constructed satisfies the recurrent compact criterion. Having in mind the recurrent compact set criterion, a new dynamical definition of a blender was given in [9]. The different relationships between these concepts is an interesting problem.

First of all notice that Theorem 3.8 follows from the above result. In fact, similar as it was argued in §3.2.3, since $\Lambda^u_I(B;\Psi) \subset W_{loc}^u(\Gamma)$ the second part of Theorem A.2 is an immediate consequence of the first part. We will now split the proof into three steps.
A.1. Main Lemma. Given $\zeta \in \Sigma$ and $\bar{\omega} = \omega_{-\ell} \cdots \omega_{-1}$, where $\ell \geq 1$ and $\omega_i \in \mathcal{A}$, we define the relative cylinder by
\[ C_{\bar{\omega}}(\zeta) \overset{\text{def}}{=} \mathcal{V}_{\bar{\omega}} \cap W^s_{\text{loc}}(\zeta) \]
and recall the notation (7),
\[ \psi^{\alpha}_\zeta(x) \overset{\text{def}}{=} \psi_{\tau^{e_{i-1}(\zeta)}} \circ \cdots \circ \psi_{\tau^{e_0(\zeta)}}(x) \quad \text{and} \quad \psi^{-\alpha}_\zeta(x) \overset{\text{def}}{=} \psi_{\tau^{-1}(\zeta)} \circ \cdots \circ \psi_{\tau^{-1}(\zeta)}(x). \]

Lemma A.5 ([6]). Consider $\Psi = \tau \times \psi_{\xi} \in \mathcal{S}(D, S, \lambda, \beta)$, a word $\bar{\omega} = \omega_{-n} \cdots \omega_{-1}$; $\omega_0 \cdots \omega_n$ with $\omega_{-j} \in \mathcal{S}$ for all $i > 0$ and a point $x \in \overline{D}$ such that
\[ \psi^{-\alpha}_\zeta(x) \in \overline{D} \quad \text{for all} \quad \zeta \in C_{\bar{\omega}} \quad \text{and} \quad 1 \leq j \leq n. \]
Then for every $1 \leq i \leq n$ and $\xi, \zeta \in C_{\bar{\omega}}$,
\[ d(\psi^{-\alpha}_\zeta(x), \psi^{-\alpha}_\xi(x)) \leq C(\Psi, D, S) V^{-\alpha i} \sum_{j=0}^{i-1} (\lambda^{-1} \nu^\alpha)^j \rho(\xi, \zeta)^\alpha. \]

A.2. Choice of the $\mathcal{S}$-neighborhood. Observe that (11) is robust under $\mathcal{S}$-perturbations. Thus, we take a $\mathcal{S}$-neighborhood $\mathcal{U}$ of $\Phi$ such that if $\Psi \in \mathcal{U}$, then (11) holds for $\Psi$. In particular,
\[ \psi^{-\alpha}_\xi(B_j) \subset D \quad \text{for all} \quad j = 0, \ldots, n_i - 1 \quad \text{and} \quad \psi^{-\alpha}_\xi(B_j) \subset B \quad \text{for all} \quad \xi \in V_{n_i}. \]
By hypothesis, shrinking the neighborhood $\mathcal{U}$ if necessary, we can assume that
\[ C(\Psi, D, S) \cdot (1 - \lambda^\alpha \nu^{-1}) < L \quad \text{for all} \quad \Psi \in \mathcal{U}. \]

A.3. Existence of an intersection point. The following proposition provides the final step.

Proposition A.6. Consider $0 < \delta < \lambda L/2$ and let $D^\xi = D^\xi(\zeta, z)$ be a $\delta$-horizontal disc in $\mathcal{B} = \mathcal{V} \times B$ with $\alpha$-Hölder constant $C \geq 0$. Then for every $\Psi = \tau \times \psi_{\xi} \in \mathcal{U}$ there are an infinite word $\bar{\omega} = \cdots \omega_{-j} \cdots \omega_{-1}$ with $\omega_{-j} = \alpha_{ij}$ and a sequence of nested compact subsets $\{V_n\}$ of $\mathcal{M}$ such that
\begin{enumerate}
  \item $V_n \subset \mathcal{P}(D^\xi \cap (C_{\bar{\omega}^n}(\xi) \times B))$,
  \item $\Psi^{-ij}(C_{\bar{\omega}^n}(\xi) \times V_j) \subset \Omega_B$ for $j = 0, \ldots, m_n - 1$,
  \item $\Psi^{-m_n}(C_{\bar{\omega}^n}(\xi) \times V_{n}) \subset \Omega_B \cap B$,
  \item $\text{diam}(\psi^{-m_n}(V_{n})) \leq C(\lambda^{-1} \nu^\alpha)^{m_n}$ for all $\xi \in C_{\bar{\omega}^n}(\xi)$,
\end{enumerate}
where $\bar{\omega}^n = \omega_{-n} \cdots \omega_{-1}$, $m_n$ is the length of the word $\bar{\omega}^n$, and $\mathcal{P} : \Sigma \times M \to M$ is the projection on the fiber space.

This proposition concludes the first part of Theorem A.2. Indeed, let
\[ \{x\} = \bigcap_{n \in \mathbb{N}} V_n \subset B \quad \text{and} \quad \{\xi\} = \bigcap_{n \in \mathbb{N}} C_{\bar{\omega}^n}(\xi) \subset W^s_{\text{loc}}(\zeta). \]
Observe that $(\xi, x) \in D^\xi$ and $\Psi^{-m_n}(\xi, x) \in \Omega_B \cap B$ for all $n \geq 1$ and $m_n + 1 - m_n \in I \overset{\text{def}}{=} \{n_1, \ldots, n_k\}$. Hence
\[ (\xi, x) \in \Lambda^\xi_{f}(B; \Psi) \cap D^\xi. \]
Proof of Proposition A.6. Fix a skew-product $\Psi = \tau \times \psi_\xi \in \mathcal{U}$ and consider the $(\alpha, \mathcal{C})$-Hölder map $h : W^s_{loc}(\zeta, \tau) \to B$ associated with the $\delta$-horizontal disc $D^s = D^s(\xi, z) \subset V \times B$. The construction of the nested sequence of sets $\{V_n\}$ and the infinite word $\bar{\omega} = \omega \bar{\omega}_j \ldots \bar{\omega}_{-1} \omega$ with $\bar{\omega}_{-j} = \alpha_i$ is done inductively. Let $V = \mathcal{P}(D^s) \subset B$. Note that $\text{diam}(V) \leq 2\delta < L$. By the definition of the Lebesgue number, we have that $V \subset B_{i_j}$ for some $i_j \in \{1, \ldots, k\}$. Consider $\bar{\omega}^1 = \bar{\omega}_{-1} = \alpha_{i_1}$, $m_1 = n_{i_1}$ and $V_1 \overset{\text{def}}{=} \mathcal{P}(D^s \cap (\mathcal{C}_{\omega_1}(\xi) \times V))$.

By construction, $V_1 \subset V$ and

$$\psi^{-j}_{\xi}(V_1) \subset D \quad \text{for all } j = 0, \ldots, m_1 - 1 \quad \text{and} \quad \psi^{-m_1}_{\xi}(V_1) \subset B \quad \text{for all } \xi \in \mathcal{C}_{\omega_1}(\xi).$$

Claim A.7. $\text{diam}(V_1) \leq \mathcal{C} V^{m_1, \alpha}$.

Proof. Given $x$ and $y$ in $V_1$ there are $\xi$ and $\eta$ in $\mathcal{C}_{\omega_1}(\xi)$ such that $x = h(\xi)$ and $y = h(\eta)$. Since $h$ is $(\alpha, \mathcal{C})$-Hölder continuous, $d(x, y) = d(h(\xi), h(\eta)) \leq C d_\xi(\xi, \eta)^\alpha \leq \mathcal{C} V^{m_1, \alpha}$, proving the claim. □

By Claim A.7, setting $\delta_1 = \mathcal{C} V^{m_1, \alpha}$ and since the fiber-maps $\psi_\xi$ are $(\lambda, \beta)$-Lipschitz on $\overline{D}$,

$$\text{diam}(\psi^{-m_1}_{\xi}(V_1)) \leq \lambda^{-m_1} \delta_1 \quad \text{for all } \xi \in \mathcal{C}_{\omega_1}(\xi).$$

Since $\lambda^{-1} V^{\alpha} < 1$ and $D^s$ is a $\delta$-horizontal disc ($\mathcal{C} V^{\alpha} < \delta$), it follows that $\lambda^{-m_1} \delta_1 = C(\lambda^{-1} V^{\alpha})^{m_1} < \lambda^{-1} \delta \leq L/2$. Therefore, the diameter of $\psi^{-j}_\xi(V_1)$ is less or equal than $L/2$.

Arguing inductively, suppose that we have constructed a finite word $\bar{\omega}^j \overset{\text{def}}{=} \bar{\omega}_{-n} \ldots \bar{\omega}_{-1}$ (the word $\bar{\omega}^j$ is obtained by adding $\bar{\omega}_{-j} = \alpha_i$ to the word $\bar{\omega}^{j-1}$). As well we have the closed sets $V_n \subset V_{n-1} \subset \cdots \subset V_1$ with $\text{diam}(V_n) \leq \delta_n = C_\mathcal{C} V^{m_n, \alpha}$, where $m_n$ is the length of the word $\bar{\omega}^n$, such that

$$\psi^{-j}_{\xi}(V_n) \subset \mathcal{D} \quad \text{for all } j = 0, \ldots, m_n - 1, \psi^{-m_n}_{\xi}(V_n) \subset B \quad \text{(A.3)}$$

and

$$\text{diam}(\psi^{-m_n}_{\xi}(V_n)) \leq \lambda^{-m_n} \delta_n \quad \text{for all } \xi \in \mathcal{C}_{\omega_{-m}(\xi)}. \quad \text{(A.4)}$$

We now construct the word $\bar{\omega}^{m_n+1}$ and the closed set $V_{m+1} \subset V_n$ satisfying analogous inclusions and inequalities. By (A.3) and (A.4) we have that

$$A_n \overset{\text{def}}{=} \bigcup_{\xi \in \mathcal{C}_{\omega_n}(\xi)} \psi^{-m_n}_{\xi}(V_n) \subset B.$$

Claim A.8. $\text{diam}(A_n) < L$.

Proof. Given $\bar{x}$ and $\bar{y}$ in $A_n$ there are $x, y \in V_n$ and $\xi, \eta \in \mathcal{C}_{\omega_{-m}(\xi)}$ such that $\bar{x} = \psi^{-m_n}_{\xi}(x)$ and $\bar{y} = \psi^{-m_n}_{\eta}(y)$. Then

$$d(\bar{x}, \bar{y}) = d(\psi^{-m_n}_{\xi}(x), \psi^{-m_n}_{\eta}(y)) \leq C(\psi, D, S) V^{-m_n} \sum_{j=0}^{m_n-1} (\lambda^{-1} \alpha)^j \text{diam}(\xi, \eta)^\alpha + \lambda^{-m_n} \delta_n$$
where the last inequality follows from Lemma A.5 and the induction hypothesis (A.3). Since \( \xi \) and \( \eta \) belong to \( C_{\bar{\omega}}(\bar{\zeta}) \) then \( d_{\bar{\zeta}}(\xi, \eta) \leq r^{m_{n}}. \) Hence
\[
d(\bar{x}, \bar{y}) \leq C(\Psi, D, S) \sum_{j=0}^{m_{n}-1} (\lambda^{-1}v)^{j} + \lambda^{-m_{n}} \delta_{n} \leq L/2 + \lambda^{-m_{n}} \delta_{n}
\]
where the last inequality follows from the fact that \( C(\Psi, D, S) \cdot (1 - v^{(1/\lambda)^{-1}}) < L. \) As \( \lambda^{-m_{n}} \delta_{n} = C (\lambda^{-1}v^{m_{n}}) \leq C \lambda^{-1}v^{a} < \lambda^{-1} \delta < L/2, \)
therefore \( d(\bar{x}, \bar{y}) \leq L \) and thus \( \text{diam}(A_{n}) < L, \) proving the claim. \( \square \)

Since \( L \) is a Lebesgue number of the covering \( \{B_{i}\}, \) the claim implies there is \( i_{n+1} \in \{1, \ldots, k\} \) such that \( \mathcal{P}(A_{n}) \subset B_{i_{n+1}}. \) Set
\[
\bar{\omega}^{n+1} = \alpha_{i_{n+1}} \alpha_{i_{n}} \ldots \alpha_{1} \quad \text{and} \quad V_{n+1} = \mathcal{P}(D^{\delta} \cap (C_{\bar{\omega}^{n+1}}(\bar{\zeta}) \times V_{n})).
\]
By the construction \( V_{n+1} \subset V_{n}, \) and arguing as in Claim A.7
\[
\text{diam}(V_{n+1}) \leq CV^{m_{n+1}} a \overset{\text{def}}{=} \delta_{n+1}
\]
where \( m_{n+1} \) is the length of \( \bar{\omega}^{n}, \) that is \( m_{n} + \ell \) where \( \ell = n_{i_{n+1}} \) is the length of the word \( \alpha_{i_{n+1}}. \) Hence,
\[
\text{diam}(\psi^{-m_{n+1}}(V_{n+1})) \leq \lambda^{-m_{n+1}} \delta_{n+1} \quad \text{for all } \xi \in C_{\bar{\omega}^{n+1}}(\bar{\zeta}).
\]
Since \( V_{n+1} \subset V_{n} \) and \( A_{n} \subset B_{i_{n+1}}, \) it follows that for every \( \xi \in C_{\bar{\omega}^{n+1}}(\bar{\zeta}), \)
\[
\psi^{-m_{n}}(V_{n+1}) \subset \psi_{\bar{\xi}}^{-j} \circ \psi_{\xi}^{-m_{n}}(V_{n}) \subset \psi_{\bar{\xi}}^{-j} \circ \psi_{\xi}^{-m_{n}}(A_{n}) \subset \psi_{\bar{\xi}}^{-j} \circ \psi_{\xi}^{-m_{n}}(B_{i_{n+1}}) \subset D
\]
for all \( j = 0, \ldots, \ell - 1 \) and
\[
\psi_{\bar{\xi}}^{-m_{n+1}}(V_{n+1}) \subset \psi_{\bar{\xi}}^{-\ell} \circ \psi_{\bar{\xi}}^{-m_{n}}(V_{n}) \subset \psi_{\bar{\xi}}^{-\ell} \circ \psi_{\bar{\xi}}^{-m_{n}}(A_{n}) \subset \psi_{\bar{\xi}}^{-\ell} \circ \psi_{\bar{\xi}}^{-m_{n}}(B_{i_{n+1}}) \subset B.
\]
Similarly as above the diameter of these sets is less or equal than \( \lambda^{-(n+1)} \delta_{n+1}. \) Therefore (A.3) and (A.4) hold for the \((n+1)\)th-step and we can continue arguing inductively. This completes the construction of the infinite word \( \bar{\omega} \) and the sequence of nested sets \( \{V_{n}\} \) in the proposition, ending the proof. \( \square \)

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