Geometry, D-Branes and $N = 1$ Duality in Four Dimensions I

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ABSTRACT

We consider $N = 1$ dualities in four dimensional supersymmetric gauge theories as a geometrical realization of wrapping D 6-branes around 3-cycles of Calabi-Yau threefolds in type IIA string theory. By extending the recent work of Ooguri and Vafa to the case of $SU, SO$ and $Sp$ gauge groups with additional fields together with defining fields, we give simple geometrical descriptions of the interrelation between the electric theory and its magnetic dual in terms of the configuration of D 6-branes wrapped 3-cycles.

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1 Introduction

String theory interprets many nontrivial aspects of four dimensional supersymmetric field theory by exploiting T-duality on the local model for the compactification manifold. The compactification of F-theory (defined as type IIB string compactification where the dilaton and the axion are not constant but vary over the internal manifold) on elliptic Calabi-Yau(CY) fourfolds from 12 dimensions leads to $N = 1$ supersymmetric field theories in four dimensions. For the case of pure $SU(N_c)$ Yang-Mills gauge theory, it has been studied in [1] that the gauge symmetry can be obtained in terms of the structure of the D(irichlet) 7-brane worldvolume, that is the product of uncompactified four dimensional spacetime and four dimensional subspace of base threefold of CY fourfolds, over which the elliptic fibration has ADE type of singularity. By adding some numbers of D 3-branes and bringing them near the complex 2-dimensional surface (which is the compact part of D 7-brane worldvolume), it has been shown in [2] that the local string model gives rise to matter hypermultiplets in the fundamental representation with pure $SU(N_c)$ Yang-Mills theory. Moreover, Seiberg’s duality [3] for the $N = 1$ supersymmetric field theory can be mapped to T-duality exchanging the D 3-brane charge and D 7-brane charge even though we are working on F-theory which has no T-duality generically. However this makes sense in the fact that F-theory local model can be realized by D 7-branes of type IIB string theory for which the T-duality symmetry holds. It has been further discussed in [4] that for the extension of $SO(N_c)$ and $Sp(N_c)$ gauge theories [5, 6] coupled to matter the local string models are type IIB orientifolds, for which T-duality symmetry applies, with D 7-branes on a curved orientifold 7-plane.

On the other hand, other simple interpretation of $N = 1$ duality for $SU(N_c)$ gauge theory was given in [7] by D-brane description in the presence of NS 5-branes in a flat geometry according to the approach of [8]. Extension of this to the case of $SO(N_c)$ and $Sp(N_c)$ gauge theories with flavors was presented in [9] by inserting an orientifold 4-plane (See also the relevant papers appeared in [10, 11] along the line of this approach).
Recently, Ooguri and Vafa have considered in [12] that $N = 1$ duality can be embedded into type IIA string theory with D 6-branes, partially wrapped around three cycles of CY threefold, filling four dimensional spacetime. They discussed what happens to the wrapped cycles and studied the relevant field theory results when a transition in the moduli of CY threefolds occurs. Furthermore, they reinterpreted the configuration of D-branes in the presence of NS 5-branes [7] as purely classical geometrical realization.

In this paper, we generalize the approach of [12] to various models, consisting of one or two 2-index tensors and some fields in the defining representation (fundamental representation for $SU(N_c)$ and $Sp(N_c)$, vector representation for $SO(N_c)$), presented earlier by many authors [13, 14, 15, 16] and study its geometric realizations by wrapping D 6-branes about 3-cycles of CY threefolds.

2 Geometrical Realization of $N = 1$ Duality

Let us start with the compactification of type IIB string theory on the CY threefold leading to $N = 2$ supersymmetric field theories in 4 dimensions. Suppose a number of D 3-branes wrapped around a set of three cycles of CY threefold. It is known [17] that whenever the integration of the holomorphic 3-form on the CY threefold around three cycles form parallel vectors in the complex plane, such a D 3-brane configuration give rises to a BPS state. Therefore we obtain type IIA string theory with D 6-branes, partially wrapped around three cycles of CY threefold, filling 4 dimensional spacetime after T-dualizing the 3-spatial directions of three torus $T^3$. We end up with $N = 1$ supersymmetric field theories in 4 dimensions from the BPS states of $N = 2$ string theory. The local model of CY threefold can be described by [18, 12] five complex coordinates $x, y, x', y'$ and $z$ satisfying the following equations:

$$x^2 + y^2 = \prod_i (z - a_i), \quad x'^2 + y'^2 = \prod_j (z - b_j)$$
where each of $C^*$'s is embedded in $(x, y)$-space and $(x', y')$-space respectively over a generic point $z$. More precisely, this describes a family of a product of two copies of one-sheeted hyperboloids in $(x, y)$-space and $(x', y')$-space respectively parameterized by the $z$-coordinates. For a fixed $z$ away from $a_i$ and $b_j$ there exist nontrivial $S^1$'s in each of $C^*$'s corresponding to the waist of the hyperboloids. It is easy to see that when $z = a_i$ or $z = b_j$ the corresponding circles vanish as the waists shrink. Then we regard 3 cycles as the product of $S^1 \times S^1$ cycles over each point on the $z$-plane, with the segments in the $z$-plane ending on $a_i$ or $b_j$. When we go between two $a_i$'s ($b_j$'s) without passing through $b_j$ ($a_i$) the 3 cycles sweep out $S^2 \times S^1$. On the other hand, when we go between $a_i$ and $b_j$ the 3 cycle becomes $S^3$. We will denote the 3-cycle lying over between $a_i$ and $b_j$ by $[a_i, b_j]$ and also denote other cycles in a similar fashion.

From now on we consider particular $N = 1$ supersymmetric field theories and see how their $N = 1$ dualities arise from the transition in moduli space of CY threefolds.

1) $SU(N_c)$ with an adjoint field and $N_f$ fundamental flavors$^{[13, 16]}$:

We study supersymmetric Yang-Mills theory with gauge group $SU(N_c)$, a chiral matter superfield $X$ in the adjoint representation of $SU(N_c)$, $N_f$ fundamental multiplets $Q_i$, and $N_f$ antifundamental multiplets $\tilde{Q}_\tilde{i}$ where $i, \tilde{i} = 1, \cdots, N_f$. The superpotential is $\text{Tr}X^{k+1}$. The theory given by this superpotential has a stable vacuum if and only if $kN_f \geq N_c$. For $k = 1$, the dual theory, which has gauge group $SU(N_f - N_c)$ with adjoint superfield $Y$, $N_f$ flavors of magnetic quarks $q^i, \tilde{q}_{\tilde{i}}$ and magnetic meson since $X$ and $Y$ are massive and can be integrated out, was realized geometrically in$^{[12]}$ by the configuration of points ordered as $(b, a_1, a_2)$ along the real axis with $(N_f - N_c)$ D-branes$^{\ddagger}$ wrapping around $[b, a_1]$ cycle and $N_f$ D-branes wrapping around $[a_1, a_2]$ cycle. Recall that the electric theory corresponds to the configuration of $N_c$ D-branes around the cycle $[a_1, b]$ and $N_f$ D-branes around $[b, a_2]$ cycle after moving $a_1$ to the left side of $b$.

$^{\ddagger}$We denote D 6-branes by D branes for simplicity.
We now proceed the case of \( k = 2 \) and discuss how the \( N = 1 \) duality is realized geometrically by D-brane picture. Let us consider that there are two points \( a_1 \) and \( a_2 \) along the real part of \( z \)-plane where the first \( C^* \) degenerates and two points \( b_1 \) and \( b_2 \) along the real axis between \( a_1 \) and \( a_2 \) where the second \( C^* \) degenerates. Suppose we have four ordered special points \( (a_1, b_1, b_2, a_2) \) along the real axis. Then the three cycle \([a_1, b_1]\) lying between \( a_1 \) and \( b_1 \) is \( S^3 \) and the three cycle \([b_1, b_2]\) lying between \( b_1 \) and \( b_2 \) is \( S^1 \times S^2 \). Thus the three cycle \([a_1, b_2]\) is a bouquet of \( S^3 \) and \( S^1 \times S^2 \) joined together at \( z = b_1 \). Then we wrap \( N_1 \) D-branes around the three cycle \([a_1, b_1]\) and \( N_2 \) D-branes around three cycle \([a_1, b_2]\) such that \( N_1 + N_2 = N_c \) (because in the limit of \( b_1 \to b_2 \) this system should be consistent with \( N_c \) D-branes around the cycle \([a_1, b_2]\)) and \( N_f \) D-branes around the three cycle \([b_1, a_2]\) and \( N_f \) D-branes around the three cycle \([b_2, a_2]\) where we assume \( 2N_f \geq N_c \).

Now we want to move to other point in the moduli of CY threefolds and end up with the configuration in which the degeneration points are along the real \( z \)-axis except that the orders are changed from \( (a_1, b_1, b_2, a_2) \) to \( (b_1, b_2, a_1, a_2) \). As done in [12], we push the point \( b_1 \) up along the imaginary direction since we have the freedom to turn on a Fayet-Iliopoulos (FI) D term. Then \((N_1 + N_2)\) of D-branes connect directly between \((a_1, b_2)\) and \((N_2 + N_f - N_1 - N_2)\) of D-branes go between \((b_1, b_2)\). We continue to move \( b_1 \) along the negative real axis and pass the \( x \)-coordinate of \( a_1 \) and push down it to the real axis. At this moment, the \((N_f - N_1)\) D-branes which were between \((b_1, b_2)\) decompose to \((N_f - N_1)\) D-branes between \((b_1, a_1)\) and \((N_f - N_1)\) D-branes between \((a_1, b_2)\) which amounts to the decomposition of the three cycle \([b_1, b_2]\) into a bouquet of two 3-cycles of \( S^3 \). The \((N_1 + N_2)\) D-branes which were going between \((a_1, b_2)\) will recombine with the newly generated \((N_f - N_1)\) D-branes ending up with the total of \((N_f + N_2)\) D-branes along \([a_1, b_2]\) cycle. Similarly we do push the point \( b_2 \) in turn and move it between \( b_1 \) and \( a_1 \) using the above operation. Then \((N_f + N_2)\) of D-branes connect directly between \((a_1, a_2)\) and \((2N_f - N_2 - N_f)\) of D-branes go between \((b_2, a_2)\). We can see that the \((N_f - N_2)\) D-branes which were between \((b_2, a_2)\) decompose to \((N_f - N_2)\) D-branes
between \((b_2, a_1)\) and \((N_f - N_2)\) D-branes between \((a_1, a_2)\). The \((N_f + N_2)\) D-branes which were going between \((a_1, a_2)\) will recombine with the new \((N_f - N_2)\) D-branes ending up with the total of \(2N_f\) D-branes along \([a_1, a_2]\) cycle. Therefore the final configuration is a configuration of points ordered as \((b_1, b_2, a_1, a_2)\) with \((N_f - N_1)\) D-branes wrapped around \([b_1, b_2]\) and \((2N_f - N_1 - N_2)\) D-branes wrapped around \([b_2, a_1]\) and \(2N_f\) D-branes wrapped around \([a_1, a_2]\). Notice that the number of D-branes along the cycle \([b_2, a_2]\) in the original configuration are the same of those along the cycle \([a_1, a_2]\) after we moved the points \(b_1\) and \(b_2\). In the limit \(b_1 \to b_2\), we note that this is the magnetic description of the original theory. The gauge group \([13]\) is \(SU(\tilde{N}_c) = SU(2N_f - N_c)\) due to the fact that \(N_1 + N_2 = N_c\). In addition to the dual quarks \(q^i, \bar{q}^i\) and adjoint field \(Y\) we have two singlet chiral superfields \(M_1, M_2\) which interact with the dual quarks through the superpotential in the magnetic theory.

We expect that for general value of \(k\), the above procedure can be done similarly. Suppose we wrap \(N_1\) D-branes around the three cycle \([a_1, b_1]\) and \(N_2\) D-branes around the three cycle \([a_1, b_2]\) and so on \(N_k\) D-branes around the three cycle \([a_1, b_k]\) such that \(N_1 + N_2 + \cdots + N_k = N_c\) and \(N_f\) D-branes around the three cycle \([b_1, a_2]\) and \(N_f\) D-branes around the three cycle \([b_2, a_2]\) and so on \(N_f\) D-branes around the three cycle \([b_k, a_2]\). Therefore the final configuration after all the \(b_i's\) are moved to the left of \(a_1\) in the configuration of points ordered as \((a_1, b_1, \cdots, b_k, a_2)\) we get is a configuration of points ordered as \((b_1, b_2, \cdots, b_k, a_1, a_2)\) with \((N_f - N_1)\) D-branes wrapped around \([b_1, b_2]\), \((2N_f - N_1 - N_2)\) D-branes wrapped around \([b_2, b_3]\) and so on \(((k - 1)N_f - N_1 - N_2 - \cdots - N_{k-1})\) D-branes around \([b_{k-1}, b_k]\) and \((kN_f - N_1 - N_2 - \cdots - N_k)\) D-branes wrapped around \([b_k, a_1]\), \(kN_f\) D-branes wrapped around \([a_1, a_2]\). In the limit \(b_i(i = 1, 2, \cdots, k - 1) \to b_k\), the gauge group \(SU(\tilde{N}_c) = SU(kN_f - N_c)\) appears. In this case there are singlet fields, \(M_i(i = 1, \cdots, k)\) coupled to the magnetic quarks.

2) \(SO(N_c)\) with a traceless symmetric tensor and \(N_f\) vectors \((Sp(N_c)\) with a traceless antisymmetric tensor and \(N_f\) flavors) \([14, 16]\):

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In order to study $SO(N_c)$ and $Sp(N_c)$ theories, as done in [12] consider the local model of the CY threefold given by

$$x^2 + y^2 = -\prod_i (z - a_i)(z - a'), \quad x'^2 + y'^2 = -z$$

where $a_i$'s and $a'$ are real numbers with $a_1 < a_2 < \cdots < a_k < 0 < a'$. Observe that the $S^2 \times S^1$ associated with $[a_{i-1}, a_i]$ for $i < k$ is realized either by real values for $x, y, x', y', z$ or by purely imaginary values for $x, y, x', y'$ but real values for $z$. Also note that the $S^3$ associated with $[a_k, 0]$ is realized by taking the real values of $x, y, x', y'$ and $z$ while the $S^3$ associated with $[0, a']$ is realized by taking the imaginary values of $x, y, x'$ and $y'$ and real value for $z$.

We discuss supersymmetric Yang-Mills theory with gauge group $SO(N_c)$ where the field $X$ is in the $\frac{N_c(N_c+1)}{2} - 1$ traceless symmetric tensor representation of $SO(N_c)$ ($X_{ab} = X_{ba}$ and $X_{ab}\delta^{ab} = 0$), $N_f$ fields $Q^i$ are in the $N_c$ dimensional vector representation of $SO(N_c)$ ($i = 1, \cdots, N_f$). The superpotential is $\text{Tr}X^{k+1}$. The theory which has this superpotential has a stable vacuum provided $N_f \geq \frac{N_c}{k} - 4$. For clear understanding let us first analyze the simplest case for the case of $k = 2$ (Of course, for $k = 1$ case we have already seen in [12] after integrating out massive $X$ and $Y$ that magnetic dual can be described in terms of the configuration of $(\frac{N_f}{2} - \frac{N_c}{2} + 2)$ D-brane charge on the cycle $[0, a]$ and $\frac{N_f}{2}$ D-brane charge on $[a, a']$). We wrap $N_1$ D-branes around $[a_1, 0]$ and $N_2$ D-branes around $[a_2, 0]$ and each of them has $N_f$ D-branes around $[0, a']$ such that $N_1 + N_2 = N_c$ which can be understood that the number of D-branes on the cycle $[a_2, 0]$ should be $N_c$ as $a_1$ gets close to $a_2$. Now the conjugation

$$(x, y, x', y') \mapsto (x^*, y^*, x'^*, y'^*, z^*),$$

together with exchange of left- and right-movers on the world sheet, will produce an orientifolding of the above configuration. The conjugation preserves the above equation for $a_i$ and $a'$ real. In view of type I' theory, these D-branes must be counted as $\frac{N_f}{2}, \frac{N_c}{2}$ and $\frac{N_f}{2}$ D-branes after orientifolding since the orientifolding leaves these cycles invariant.
We get D-brane charge of the \([a_2, 0]\) cycle of \(\frac{N_f}{2} \mp 2\). Here the factor \(\mp 2\) is due to a contribution from the orientifold plane in addition to the physical D-branes, the upper sign corresponds to the \(SO(N_c)\) gauge group and the lower one does the \(Sp(N_c)\) gauge group. The D-brane charge of the \([0, a']\) cycle is \(\frac{N_f}{2}\) after the action of the orientifolding on the D-branes. By passing the point \(a_2\) through the point 0 directly along the real axis due to the fact that this operation should keep the orientifolding, the D-brane charge gets changed to the value of \(\frac{N_f}{2} - \frac{N_c}{2} \pm 2\) on \([0, a_2]\) where we assumed \(N_f \geq N_c \mp 4\).

Next we move the point \(a_1\) to the positive real axis. The final configurations are given by the \((-\frac{N_1}{2} \pm 2)\) D-brane charge on \([0, a_1]\), \((\frac{N_f}{2} - \frac{N_1}{2} \pm 2)\) D-brane charge around \([0, a_2]\), \(\frac{N_f}{2}\) D-brane charge on \([a_2, a']\) and \(\frac{N_f}{2}\) D-brane charge on \([0, a']\). Recombining the last \(\frac{N_f}{2}\) D-branes into the D-branes on \([0, a_2]\) cycle and taking the limit of \(a_1 \to a_2\), it leads to \(N_f - \frac{N_1}{2} - \frac{N_c}{2} + 4\) that shows our expression for the magnetic dual group for \(k = 2\) case, \(SO(\tilde{N}_c) = SO(2N_f - N_c + 8)\) since there is no orientifold plane for \(a_1, a_2 > 0\) and all these D-brane charges are physical D-branes. For the case of \(Sp\) group with a traceless antisymmetric tensor and \(N_f\) flavors, by recognizing that in the convention of \([16]\) the symplectic group whose fundamental representation is \(2N_c\) dimensional as \(Sp(N_c)\) and a flavor of it has two fields in the fundamental representation therefore \(2N_f\) fields we obtain \(2\tilde{N}_c = 2(2N_f) - 2N_c - 8\) which gives rise to the following dual description \(Sp(\tilde{N}_c) = Sp(2N_f - N_c - 4)\).

For general value of \(k\), we get D-brane charges on the \([a_i, 0]\) cycle of \(\frac{N_i}{2} \mp 2\) and D-brane charges on the \([0, a']\) cycle of \(\frac{N_f}{2}\) for each \(i = 1, 2, \ldots, k\) after the action of the orientifolding on the D-branes where we have \(N_1 + N_2 + \cdots + N_k = N_c\). The final configurations, after we moved all the \(a_i\)’s to the right of the position of zero by successively doing similar things for the previous case, are given by the \((-\frac{N_1}{2} \pm 2)\) D-brane charge on \([0, a_1]\), the \((-\frac{N_i}{2} \pm 2)\) D-brane charge on \([0, a_2]\) and so on the \((-\frac{N_{k-1}}{2} \pm 2)\) D-brane charge on \([0, a_{k-1}]\) and \((\frac{kN_f}{2} - \frac{N_1 + N_2 + \cdots + N_k}{2} \pm 2k)\) D-brane charge around \([0, a_k]\), \(\frac{kN_f}{2}\) D-brane charge on \([a_k, a']\). In the limit of \(a_i (i = 1, 2, \cdots, k - 1) \to a_k\), we get the magnetic dual group, \(SO(\tilde{N}_c) = SO(k(N_f + 4) - N_c)\). By similar reasoning for
the case of $k = 2$, we obtain the magnetic dual gauge group for symplectic group as $Sp(\tilde{N}_c) = Sp((k - 2)N_f - N_c)$.

3) $SO(N_c)$ with an adjoint field and $N_f$ vectors ($Sp(N_c)$ with an adjoint field and $N_f$ vectors) [15, 16]:

We analyze supersymmetric Yang-Mills theory with gauge group $SO(N_c)$ where the adjoint field $X$ is in the $\frac{N_c(N_c - 1)}{2}$ dimensional antisymmetric tensor of $SO(N_c)$ ($X_{ab} = -X_{ba}$), $N_f$ flavors $Q^i$ are in the $N_c$ dimensional vector representation of $SO(N_c)$ ($i = 1, \cdots , N_f$). The superpotential is $\text{Tr} X^{2(k+1)}$. The theory given by this superpotential has a stable vacuum for $N_f \geq \frac{N_c - 4}{2k+1}$ For simplicity, we will start with the case of $k = 1$. Let us consider the case of wrapping $N_0$ D-branes around $[a, 0]$ and $N_f$ D-branes around $[0, a']$. After orientifolding, the net D-brane charge of $[a, 0]$ cycle becomes $\frac{N_0}{2} \mp 2$ and that of $[0, a']$ cycle is $\frac{N_f}{2}$. Now we bring other D-branes to the left hand side of the point $a$ in the real axis whose D-brane charge of $[b_1, 0]$ cycle is $N_1$ and that of $[0, a']$ cycle is $N_f$. In order to be consistent with the number of D-branes when we take the limit of $b \to a$, we should have $N_0 + 2N_1 = N_c$. We push the point $a$ along the real axis to the right and pass the point 0. In order to count the number of D-branes wrapped around $[0, a]$ and $[a, a']$ we use the D-brane charge conservation and the orientation of the D-branes. Then we have ($\frac{N_f}{2} - \frac{N_0}{2} \pm 2$) D-brane charge on $[0, a]$ and $\frac{N_f}{2}$ D-branes on $[a, a']$. Next we take the point $b$ along the real axis from negative to positive values. The D-brane charge on $[0, b_1]$ is $-N_1$ due to the orientation. The $N_f$ D-branes which were going between $(0, a')$ can be decomposed into $N_f$ D-branes between $(0, a)$ and $N_f$ D-branes between $(a, a')$. Therefore the final picture we end up with is that there are $-N_1$ D-brane charge on $(0, b_1)$ and ($\frac{3N_f}{2} - \frac{N_0}{2} \pm 2$) on $(0, a)$ and $\frac{3N_f}{2}$ on $(a, a')$. In the limit of $b_1 \to a$, the magnetic dual group can be written as $SO(\tilde{N}_c) = SO(3N_f - N_c + 4)$ by adding the two contributions. By twicing the $N_f$ and $N_c$ and dividing by two which leads to $\frac{3(2N_f - 2N_f - 4)}{2}$, we get $Sp(\tilde{N}_c) = Sp(3N_f - N_c - 2)$ for the symplectic group.

For the general value of $k \geq 2$, suppose we have the following picture: after ori-
entifolding, the net D-brane charge of \([a, 0]\) cycle becomes \(\frac{N_0}{2} \mp 2\) and that of \([0, a']\) cycle is \(\frac{N_f}{2}\) and we bring other D-branes to the left hand side of the point \(a\) in the real axis whose D-brane charges of \([b_i, 0]\) cycle are \(N_i\) and that of \([0, a']\) cycle are \(N_f\) for each \(i = 1, 2, \ldots, k\). In this case we also have the following relation, \(N_0 + 2(N_1 + \cdots + N_k) = N_c\). Then the final configuration is that there are \(-N_i\) D-brane charges on \((0, b_i)\) for each \(i = 1, 2, \ldots, k\) and \(\frac{(2k+1)N_f}{2} - \frac{N_0}{2} \pm 2 - N_1 - N_2 - \cdots - N_k\) on \((0, a)\) and \(\frac{(2k+1)N_f}{2}\) on \((a, a')\). In the limit of \(b_i(i = 1, 2, \ldots, k) \to a\), the dual theory has gauge group, \(SO(\tilde{N}_c) = SO((2k + 1)N_f - N_c + 4)\) indicating \((2k + 1)\) mesons coupled dual quarks and \(Sp(\tilde{N}_c) = Sp((2k + 1)N_f - N_c - 2)\).

4) \(SU(N_c)\) with a symmetric flavor and \(N_f\) fundamental flavors \((SU(N_c)\) with an antisymmetric flavor and \(N_f\) fundamental flavors) \[16\]:

The fields \(X\) and \(\tilde{X}\) are a flavor of symmetric tensor representations of \(SU(N_c)\) and there are \(N_f\) fundamental multiplets \(Q^i\), and \(N_f\) antifundamental multiplets \(\tilde{Q}_{\bar{i}}\) where \(i, \bar{i} = 1, \ldots, N_f\). The superpotential is \(\text{Tr}(X\tilde{X})^{k+1}\). Now we continue to repeat the procedure we have done so far for the case of ordered configuration as \((b, a_1, a_2, 0, a')\) when we consider \(k = 2\) case (Recall that when \(k = 1\), the configuration was \(-N_0\) D-brane charge on \([0, b]\) and \(N_f - \frac{N_f}{2} \pm 2\) on \([0, a_1]\) and \(N_f\) on \([0, a']\)). After orientifolding, D-brane charges are \(N_0\) around \([b, 0]\), \(\frac{N_f}{2} \mp 2\) around \([a_1, 0]\), \(\frac{N_f}{2} \mp 2\) around \([a_2, 0]\). Each of them produces \(N_f\) D-brane charge around \([0, a']\) where \(N_0 + N_1 + N_2 = N_c\). We have seen already ordered configuration as \((a_1, a_2, 0, a')\) in the previous subsection. The final configuration is given by the \((-\frac{N_f}{2} \pm 2)\) D-brane charge on \([0, a_1]\), \((N_f - \frac{N_f}{2} \pm 2)\) D-brane charge around \([0, a_2]\), \(N_f\) D-brane charge on \([a_2, a']\) and \(N_f\) D-brane charge on \([0, a']\). Next we move the point \(b\) along the real axis from negative to positive values. The D-brane charge on \([0, b]\) is \(-N_0\) due to the orientation. Therefore the final configuration is given by the \(-N_0\) D-brane charge on \([0, b]\), the \((-\frac{N_f}{2} \pm 2)\) D-brane charge on \([0, a_1]\), \((N_f - \frac{N_f}{2} \pm 2)\) D-brane charge around \([0, a_2]\), \(N_f\) D-brane charge on \([a_2, a']\) and \(2N_f\) D-brane charges on \([0, a']\). Recombining one of \(N_f\)'s D-branes into the D-branes on \([0, a_2]\)
cycle and taking the limit of \( a_1 \to a_2 \), we obtain \( 2N_f - \frac{N_1}{2} - \frac{N_2}{2} \pm 4 \). Finally we see that the dual theory has the gauge group \( SU(\tilde{N}_c) = SU(5N_f - N_c + 8) \) by taking into account of the contributions from \( -N_0 \) D-brane charge on \([0, b]\) and \( N_f \) on \([0, a']\). On the other hand, when we consider antisymmetric flavor instead of symmetric one we get \( SU(\tilde{N}_c) = SU(5N_f - N_c - 8) \).

For the general value of \( k \), after orientifolding D-brane charges are \( N_0 \) around \([b, 0]\), \( \frac{N_i}{2} \mp 2 \) around \([a_i, 0]\) for each \( i = 1, 2, \ldots, k \) and each of them has \( N_f \) D-brane charges around \([0, a']\) respectively where \( N_0 + N_1 + N_2 + \cdots + N_k = N_c \). Since \( 2(kN_f - \frac{1}{2}(N_1 + \cdots + N_k) \pm 2k) - N_0 + N_f = (2k + 1)N_f - N_c \pm 4k \), we arrive at the magnetic dual group for \( SU, SU(\tilde{N}_c) = SU((2k + 1)N_f + 4k - N_c) \) and that for \( SU(N_c) \) with an antisymmetric flavor and \( N_f \) fundamental flavors is \( SU(\tilde{N}_c) = SU((2k + 1)N_f - 4k - N_c) \).

5) \( SU(N_c) \) with an antisymmetric tensor and a symmetric tensor \((\text{16})\):

In this case the field \( X \) is in the \( \frac{N_c(N_c - 1)}{2} \) representation, the field \( \tilde{X} \) in the \( \frac{N_c(N_c + 1)}{2} \) representation, \( m_f(\bar{m}_f) \) fields \( Q^i(\bar{Q}^i) \) in the (anti)fundamental representation. The superpotential is given by \( \text{Tr}(X\tilde{X})^{2(k+1)} \). We expect that the above procedure of case 1) can be applied similarly, for example, \( k = 1 \). We wrap \( N_0 \) D-branes around the three cycle \([a_1, b_0]\) and \( 2N_1 \) D-branes around the three cycle \([a_1, b_1]\) such that \( N_0 + 2N_1 = N_c \) and \( \frac{3(m_f + \bar{m}_f)}{2} \) D-branes around the cycle \([b_0, a_2]\) and \( 2(m_f + \bar{m}_f) \) D-branes around the cycle \([b_1, a_2]\). The final configuration after we move \( b_0 \) and \( b_1 \) to the left of \( a_1 \) in the configuration of point ordered as \((a_1, b_0, b_1, a_2)\) we get is a configuration of points ordered as \((b_0, b_1, a_1, a_2)\) with \( \frac{3(m_f + \bar{m}_f)}{2} - N_0 \) D-branes wrapped around \([b_0, b_1]\), \( \frac{7(m_f + \bar{m}_f)}{2} - N_0 - 2N_1 \) D-branes wrapped around \([b_1, a_1]\), \( \frac{7(m_f + \bar{m}_f)}{2} - N_0 - 2N_1 \) D-branes wrapped around \([a_1, a_2]\). In the limit \( b_0 \to b_1 \), the gauge group becomes \( SU(\tilde{N}_c) = SU(\frac{7(m_f + \bar{m}_f)}{2} - N_c) \).

For general value of \( k \), we wrap \( N_0 \) D-branes around the cycle \([a_1, b_0]\) and \( 2N_i \) D-branes around the cycle \([a_1, b_i]\) for each \( i = 1, 2, \ldots, k \) such that \( N_0 + 2N_1 + \cdots + 2N_k = N_c \) and \( \frac{3(m_f + \bar{m}_f)}{2} \) D-branes around the cycle \([b_0, a_2]\) and \( 2(m_f + \bar{m}_f) \) D-branes
around the cycle $[b_i, a_2]$ for each $i$. The final configuration is a configuration of points ordered as $(b_0, b_1, \ldots, b_k, a_1, a_2)$ with $(\frac{3(m_f + \tilde{m}_f)}{2} - N_0)$ D-branes wrapped around $[b_0, b_1]$, $(\frac{7(m_f + \tilde{m}_f)}{2} - N_0 - 2N_1)$ D-branes wrapped around $[b_1, b_2], \ldots, ((2k + \frac{3}{2})(m_f + \tilde{m}_f) - N_0 - 2N_1 - \cdots - 2N_k)$ D-branes wrapped around $[b_k, a_1]$, and $(2k + \frac{3}{2})(m_f + \tilde{m}_f)$ D-branes wrapped around $[a_1, a_2]$. In the limit $b_i (i = 0, 1, \ldots, k - 1) \to b_k$, the gauge group $SU(\bar{N}_c) = SU((2k + \frac{3}{2})(m_f + \tilde{m}_f) - N_c)$ appears.

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