CHARACTERISTIC POLYNOMIALS OF THE WEAK ORDER ON CLASSICAL AND AFFINE COXETER GROUPS

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ABSTRACT. We find a simple product formula for the characteristic polynomial of the permutations with a fixed descent set under the weak order. As a corollary we obtain a simple product formula for the characteristic polynomial of alternating permutations. We generalize these results to Coxeter groups. We also find a formula for the generating function for the characteristic polynomials of classical Coxeter groups, which is then related to affine Coxeter groups.

1. Introduction

The Möbius function of a poset is an important topic of study in many areas of mathematics including number theory, topology, and combinatorics. For example, if the poset is the set of positive integers ordered by the division relation, then its Möbius function is the classical Möbius function in number theory. The Möbius function of a poset expresses the reduced Euler characteristic of the simplicial complex coming from the poset. The principle of inclusion and exclusion is also generalized in terms of Möbius functions. See [9, Chapter 3] for more details on Möbius functions.

The characteristic polynomial of a finite ranked poset is a generating function for the Möbius function on the poset. A useful application of characteristic polynomials is that one can compute the number of regions and bounded regions of a hyperplane arrangement using the characteristic polynomial of its intersection poset, see [10].

There are many interesting posets whose characteristic polynomials factor nicely. For example, the characteristic polynomials of the Boolean poset and the partition poset are given by \((q - 1)^n\) and \((q - 1)(q - 2) \cdots (q - n + 1)\) respectively. See [4, 7, 8] for more examples.

The main objective of this paper is to study the characteristic polynomials of the (left) weak order on classical and affine Coxeter groups. The motivation of this paper was the following observation on the characteristic polynomial of the poset \(\text{Alt}_n\) of alternating permutations ordered by the weak order. Here an alternating permutation is a permutation \(\pi = \pi_1 \pi_2 \pi_3 \ldots \pi_n\) satisfying \(\pi_1 < \pi_2 > \pi_3 < \cdots\).

**Theorem 1.1.** The characteristic polynomial of \(\text{Alt}_n\) is

\[
\chi_{\text{Alt}_n}(q) = q^{\binom{n-1}{2} - \lfloor \frac{n}{2} \rfloor} (q - 1)^{\lfloor \frac{n}{2} \rfloor}.
\]

Let \(\mathfrak{S}_n\) denote the symmetric group of order \(n\), that is, the set of permutations on \([n] := \{1, 2, \ldots, n\}\). Note that an element in \(\mathfrak{S}_n\) is an alternating permutation if and only if its (right) descent set is equal to \(\{2, 4, 6, \ldots\} \cap [n]\). In Theorem [5.1] we show that the characteristic polynomial of the subposet of \(\mathfrak{S}_n\) with a fixed descent set has a simple factorization. Theorem [1.1] then follows immediately from Theorem [5.1].

A Coxeter group is a group defined by generators and certain relations. Coxeter groups are also studied in many different areas of mathematics, and their classification is well known. In particular, the symmetric group \(\mathfrak{S}_n\) is the Coxeter group of type \(A_{n-1}\), see [2, Proposition 1.5.4]. There are two important orders on Coxeter groups, the Bruhat order and the (left) weak order. In this paper we will only consider the weak order. See [1, 2, 6, 11] and references therein for combinatorial properties of the weak order.

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In search of a generalization of Theorem 1.1, we study characteristic polynomials of Coxeter groups under the weak order. By slightly modifying the definition of the characteristic polynomial so that it can be defined on an infinite poset, we also compute (modified) characteristic polynomials of affine Coxeter groups.

The rest of this paper is organized as follows.

In Section 2, we give some definitions and known results on posets and Coxeter groups. In Section 3, we show that the characteristic polynomial of an interval of a Coxeter group with the weak order is decomposed into the product of characteristic polynomials of its subgroups. In Section 4, we show that the descent class of a Coxeter group is an interval. In Section 5, we give weak order is decomposed into the product of characteristic polynomials of its subgroups. In Section 6, we slightly modify the characteristic polynomial so that it is defined for affine Coxeter groups. We then compute the generating functions for the modified characteristic polynomials of the classical Coxeter groups $A_n, B_n,$ and $D_n$. Finally, in Section 7, we express the modified characteristic polynomials for affine Coxeter groups $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n,$ and $\tilde{D}_n$ in terms of the corresponding finite Coxeter groups.

2. Preliminaries

In this section, we give basic definitions and results on posets and Coxeter groups.

2.1. Posets. Let $(P, \leq)$ be a poset. For $x, y \in P$ we write $x < y$ to mean $x \leq y$ and $x \neq y$. For $x, y \in P$, if $x < y$ and there is no element $t \in P$ such that $x < t < y$, then we say that $y$ covers $x$ and denote by $x \lessdot y$. The (closed) interval $[x, y]$ is the subset $\{t \in P : x \leq t \leq y\}$ of $P$. If there is an element $x \in P$ such that $x \leq y$ (resp. $x \geq y$) for all $y \in P$, then $x$ is called the bottom (resp. the top) of $P$ and is denoted by $\hat{0}$ (resp. 1).

Let $C = \{t_1, t_2, \ldots, t_k\}$ be a subset of $P$. If any two elements of $C$ are comparable, then $C$ is called a chain of $P$. The length of a chain $C$ is defined to be $|C|$. A chain of $P$ is called a maximal chain if it is not contained in any other chain of $P$. If every maximal chain of $P$ has the same length, say $m$, then $P$ is called a ranked poset (or a graded poset). In this case, there is a unique rank function $r_P : P \to \{0, 1, 2, \ldots, m\}$ such that $r_P(t) = 0$ for every minimal element $t \in P$ and $r_P(y) = r_P(x) + 1$ if $x < y$. We say that $r_P(x)$ is the rank of $x \in P$. The length $m$ of a maximal chain of $P$ is called the rank of $P$ and denoted by $r_P(P)$.

For two elements $x, y$ of $P$, a lower bound of $x$ and $y$ is an element $u$ such that $u \leq x$ and $u \leq y$. If $u$ is a lower bound of $x$ and $y$ satisfying $u \leq v$ for any lower bound $v$ of $x$ and $y$, then $u$ is called the meet of $x$ and $y$ and denoted by $x \wedge y$. Similarly, an upper bound of $x$ and $y$ is an element $u$ such that $x \leq u$ and $y \leq u$. If $u$ is an upper bound of $x$ and $y$ satisfying $u \leq v$ for any upper bound $v$ of $x$ and $y$, then $u$ is called the join of $x$ and $y$ and denoted by $x \vee y$. For a subset $A = \{a_1, a_2, \ldots, a_i\}$ of $P$, we denote by $\bigvee A$ the join of all elements in $A$, i.e., $\bigvee A = a_1 \vee a_2 \vee \cdots \vee a_i$.

The Möbius function $\mu_P$ of $P$ is the unique function defined on the pairs $(x, y)$ of elements $x, y \in P$ with $x \leq y$ satisfying the following condition:

$$\sum_{x \leq t \leq y} \mu_P(x, t) = \delta_{x,y} \quad \text{for all } x \leq y \text{ in } P,$$

where $\delta_{x,y}$ is 1 if $x = y$ and 0 otherwise.

**Definition 2.1.** Let $P$ be a finite ranked poset with $\hat{0}$. The characteristic polynomial $\chi_P(q)$ of $P$ is defined by

$$\chi_P(q) = \sum_{t \in P} \mu_P(\hat{0}, t)q^{r_P(P) - r_P(t)}.$$
say that \( W \) is a **Coxeter group**, \( S \) is a set of **Coxeter generators**, and the pair \((W, S)\) is a **Coxeter system**. Note that, for any Coxeter generators \( s, s' \) with \( s \neq s' \), we have

\[
s^2 = e \quad \text{and} \quad m(s, s') s s' s s' \cdots = s' s s' s s' \cdots. \tag{1}
\]

In particular, distinct Coxeter generators \( s \) and \( s' \) commute if and only if \( m(s, s') = 2 \).

A **Coxeter graph** (or a **Coxeter diagram**) is a graph such that the nodes are the Coxeter generators, nodes \( s \) and \( s' \) are connected by an edge if \( m(s, s') \geq 3 \), and the edge has the label \( m(s, s') \) if \( m(s, s') \geq 4 \). A Coxeter system \((W, S)\) is **irreducible** if the Coxeter graph of \((W, S)\) is connected. The irreducible Coxeter groups have been classified and every reducible Coxeter group decomposes uniquely into a product of irreducible Coxeter groups [23 p.4].

From now on, we assume that \((W, S)\) is a Coxeter system.

Since \( S \) is a set of generators of \( W \), each element \( w \in W \) can be represented as a product of generators, say \( w = s_1 s_2 \cdots s_k \), where \( s_i \in S \). Among all such expressions for \( w \), the smallest \( k \) is called the **length** of \( w \) and denoted by \( \ell(w) \). If \( k = \ell(w) \), then the expression \( s_1 s_2 \cdots s_k \) is called a **reduced word** for \( w \).

Let \( \alpha_{s, s'} = \overbrace{s s' s s' \cdots}^{m(s, s')} \). Then the second equation in (1) is rewritten by \( \alpha_{s, s'} = \alpha_{s', s} \). The replacement of \( \alpha_{s, s'} \) by \( \alpha_{s', s} \) in a word is called a **braid-move**. It is known that every two reduced words of \( w \in W \) is connected via a sequence of braid-moves, see [12].

**Definition 2.2.** The Coxeter generators \( s \in S \) are called **simple reflections**. For \( w \in W \), we define the following sets:

\[
D_L(w) = \{ s \in S : \ell(sw) < \ell(w) \}, \\
D_R(w) = \{ s \in S : \ell(ws) < \ell(w) \}.
\]

The set \( D_L(w) \) (resp. \( D_R(w) \)) is called the **left** (resp. **right**) **descent set** of \( w \) and its elements are called **left** (resp. **right**) **descents** of \( w \). Note that \( D_L(w) = D_R(w^{-1}) \).

**Definition 2.3.** For \( I \subseteq J \subseteq S \), the set \( D_I^J = \{ w \in W : I \subseteq D_R(w) \subseteq J \} \) is called a **(right) descent class**. We also denote \( D_I = D_I^S = \{ w \in W : D_R(w) = I \} \) and \( W_J = D_0^S \backslash J = \{ w \in W : D_R(w) \subseteq S \backslash J \} \). For \( J \subseteq S \), the subgroup of \( W \) generated by \( J \) is called a **parabolic subgroup** and denoted by \( W_J \).

Note that \((W_J, J)\) is also a Coxeter system [2] Proposition 2.4.1).

**Definition 2.4.** For \( u, w \in W \), we denote by \( u \leq w \) if \( w = s_k \cdots s_2 s_1 u \) for some \( s_i \in S \) with \( \ell(s_i \cdots s_2 s_1 u) = \ell(u) + i \) for \( 0 \leq i \leq k \). This partial order \( \leq \) is called the (**left**) **weak order**.

Throughout this paper we consider a Coxeter group \( W \) as a poset using the weak order.

If \( W \) is finite then \( W \) is a lattice \([1]\). In this case there exists the top element of \( W \), which we denote by \( w_0 \). For some \( J \subseteq S \), if \( W_J \) is finite, then we denote the top element of \( W_J \) by \( w_0(J) \). Since the weak order on \( W \) is graded, the rank function is defined on \( W \). By the definition of the weak order, the rank and the length of each element are the same.

Let \( \mu_W \) denote the Möbius function of \( W \). For \( u, w \in W \), it is known that \( \mu_W \) only takes the values 0, 1, and \(-1\).

**Proposition 2.5.** [2] Corollary 3.2.8) **The Möbius function** \( \mu_W(u, w) \) of \( W \) is given by

\[
\mu_W(u, w) = \begin{cases} (-1)^{|J|} & \text{if } w = w_0(J)u \text{ for some } J \subseteq S, \\ 0 & \text{otherwise}. \end{cases}
\]

If \( u = e \) in Proposition 2.5 then \( \mu_W(e, w) \) is nonzero if and only if \( w = w_0(J) \) for some \( J \subseteq S \). Hence the characteristic polynomial \( \chi_W(q) \) of \( W \) can be rewritten as

\[
\chi_W(q) = \sum_{w \in W} \mu_W(e, w) q^{\ell(w)} = \sum_{J \subseteq S} (-1)^{|J|} q^{\ell(w_0(J))}. \tag{2}
\]
We note that the Möbius function \( \mu(u, w) \) for the Bruhat order, which is another important partial order on a Coxeter group, is given by
\[
\mu(u, w) = (-1)^{\ell(u) + \ell(w)},
\]
see \[13\] Theorem.

The following proposition is useful in this paper.

**Proposition 2.6.** \[2\] Proposition 3.1.6] If \( u \leq w \), then \([u, w] \cong [e, wu^{-1}]\).

3. THE CHARACTERISTIC POLYNOMIAL OF AN INTERVAL OF A COXETER GROUP

In this section we show that the characteristic polynomial for an interval of a Coxeter group \( W \) is decomposed into the product of the characteristic polynomials of some Coxeter subgroups of \( W \).

Recall that since the weak order on a finite Coxeter group is a lattice, we can define its characteristic polynomial. Even if a Coxeter group is infinite, its interval can be thought of as a finite graded poset with the bottom and top elements so the characteristic polynomial of the interval is also defined.

**Lemma 3.1.** Let \( I, J \subseteq S \) such that \( W_I \) and \( W_J \) are finite subgroups of \( W \). Then we have
\[
w_0(I) \cup w_0(J) = w_0(I \cup J).
\]

**Proof.** Since \( W_I \) and \( W_J \) are finite, we have \( \bigvee I = w_0(I) \) and \( \bigvee J = w_0(J) \), see \[2\] Lemma 3.2.3]. Hence
\[
w_0(I) \cup w_0(J) = \left( \bigvee I \right) \cup \left( \bigvee J \right) = \bigvee (I \cup J) = w_0(I \cup J).
\]

**Proposition 3.2.** For \( w \in W \), let \( \mathcal{M}(w) = \{ u \in W : u \leq w \ \text{and} \ \ u = w_0(J) \ \text{for some} \ J \subseteq S \} \). Then, \( w_0(D_R(w)) \) is the unique maximal element of \( \mathcal{M}(w) \).

**Proof.** Let \( K \) be the union of subsets \( J \subseteq S \) such that \( w_0(J) \leq w \), i.e., \( K = \bigcup_{w_0(J) \leq w} J \). By Lemma 3.1 and the definition of \( \mathcal{M}(w) \), we have
\[
w_0(K) = \bigvee \{ w_0(J) : w_0(J) \leq w, J \subseteq S \} = \bigvee \mathcal{M}(w),
\]
which means that \( w_0(J) \leq w_0(K) \) for all \( J \) with \( w_0(J) \leq w \). By the definition of join, we have \( w_0(K) = \bigvee_{w_0(J) \leq w} w_0(J) \leq w \), which implies \( w_0(K) \in \mathcal{M}(w) \). By \(3\), this implies that \( w_0(K) \) is the unique maximal element of \( \mathcal{M}(w) \).

Now it remains to show that \( K = D_R(w) \). Suppose \( s \in K \subseteq S \). Since \( s \leq w_0(K) \leq w \) and by the definition of the weak order, there is a reduced expression of \( w \) ending with \( s \). This means that \( s \in D_R(w) \). Conversely, suppose \( s \in D_R(w) \). Then, since \( s \leq w \) and \( s = w_0(\{s\}) \in \mathcal{M}(w) \), we have \( s \in K \). Hence we obtain \( K = D_R(w) \), which completes the proof.

To decompose the characteristic polynomial of an interval of \( W \), we first decompose the interval itself into a product of Coxeter subgroups of \( W \) as follows.

**Lemma 3.3.** Let \( K \) be a subset of \( S \) such that \( W_K \) is finite. Let \( K_1, K_2, \ldots, K_r \) be the subsets of \( K \) such that each \( K_i \) is a connected component of the Coxeter graph for \((W_K, K)\) and \( \bigcup_{i=1}^r K_i = K \). Then the interval \([e, w_0(K)]\) of \( W \) is isomorphic to \( W_{K_1} \times W_{K_2} \times \cdots \times W_{K_r} \) as posets.

**Proof.** It is well known \[5\] Proposition 6.1] that \( W_K \) is isomorphic to \( W_{K_1} \times W_{K_2} \times \cdots \times W_{K_r} \) as Coxeter groups, hence as posets. Thus it suffices to show that \([e, w_0(K)] = W_K \). For \( u \in W \), let \( u \in [e, w_0(K)] \). Since \( u \leq w_0(K) \), there is a reduced expression \( s_{i_k} \cdots s_{i_1} \) of \( w_0(K) \) such that \( s_{i_k} \cdots s_{i_1} = u \), where \( j \leq k \). Since every reduced word of \( w_0(K) \in W_K \) consists of elements in \( K \), we have \( u \in W_K \). Conversely, if \( u \in W_K \) then \( u \leq w_0(K) \) since \( w_0(K) \) is the unique maximal element in \( W_K \). Thus \( u \in [e, w_0(K)] \), which completes the proof.

The following lemma shows that the characteristic polynomial of the direct product of posets is the product of the characteristic polynomials of the posets.
Lemma 3.4. Let \( P \) and \( Q \) be finite graded posets with the bottom and top elements. Then
\[
\chi_{P \times Q}(q) = \chi_P(q) \chi_Q(q).
\]

Proof. It is well known \cite[Proposition 3.8.2]{9} that \( \mu_{P \times Q}((s, t), (s', t')) = \mu_P(s, s') \mu_Q(t, t') \) for all \( s, s' \in P \) and \( t, t' \in Q \). It is easy to see that \( r_{P \times Q}((s, t)) = r_P(s) + r_Q(t) \). Hence,
\[
\begin{align*}
\chi_{P \times Q}(q) &= \sum_{(s, t) \in P \times Q} \mu_{P \times Q}(\hat{0}_{P \times Q}, (s, t)) q^{r_{P \times Q}(P \times Q) - r_{P \times Q}((s, t))} \\
&= \sum_{s \in P, t \in Q} \mu_P(\hat{0}_P, s) \mu_Q(\hat{0}_Q, t) q^{r_P(P) + r_Q(Q) - (r_P(s) + r_Q(t))} \\
&= \chi_P(q) \chi_Q(q). \quad \blacksquare
\end{align*}
\]

Let \( u, w \in W \) with \( u \leq w \). Then the interval \([u, w]\) is a graded poset with the bottom element \( u \) and the top element \( w \). To compute the characteristic polynomial of \([u, w]\), we need the rank function of the poset \([u, w]\). Since the rank of \([u, w]\) is the length of maximal chains of \([u, w]\), it is the same as the difference of the length of \( w \) and the length of \( u \), i.e., \( \ell(w) - \ell(u) \). Similarly, the rank of an element \( t \in [u, w] \) is \( \ell(t) - \ell(u) \).

For simplicity, we use the notation \( \chi_K \) instead of \( \chi_{W_K} \) for the characteristic polynomial of \( W_K \) for \( K \subseteq S \).

Theorem 3.5. Let \( u, w \in W \) with \( u \leq w \) and let \( K = D_R(wu^{-1}) \). Suppose that \( K_1, K_2, \ldots, K_r \) are the subsets of \( K \) such that each \( K_i \) is a connected component of the Coxeter graph for \((W_K, K)\) and \( \bigcup_{i=1}^{r} K_i = K \). Then, the characteristic polynomial of \([u, w]\) is given by
\[
\chi_{[u, w]}(q) = q^{\ell(wu^{-1}) - \ell(w_0(K))} \chi_{K_1}(q) \chi_{K_2}(q) \cdots \chi_{K_r}(q).
\]

Proof. By Proposition 2.6 we have
\[
\chi_{[u, w]}(q) = \chi_{[e, wu^{-1}]}(q) = \sum_{t \in [e, wu^{-1}]} \mu_{[e, wu^{-1}]}(e, t) q^{\ell(wu^{-1}) - \ell(t)}. \tag{4}
\]

By Proposition 2.5 we have \( \mu_{[e, wu^{-1}]}(e, t) \neq 0 \) if and only if \( t = w_0(J) \) for some \( J \subseteq S \). Therefore we can rewrite (4) as
\[
\chi_{[u, w]}(q) = \sum_{t \in [e, wu^{-1}]} \mu_{[e, wu^{-1}]}(e, t) q^{\ell(wu^{-1}) - \ell(t)}, \tag{5}
\]

where \( M(wu^{-1}) = \{ v \in W : v \leq wu^{-1} \} \) and \( v = w_0(J) \) for some \( J \subseteq S \).

Let \( K = D_R(wu^{-1}) \). By Proposition 3.2 we have \( M(wu^{-1}) \subseteq [e, w_0(K)] \) and \( w_0(K) \leq wu^{-1} \).

By Proposition 2.5 again, if \( t \in [e, w_0(K)] \setminus M(wu^{-1}) \), then \( \mu_{[e, wu^{-1}]}(e, t) = 0 \). Therefore we can rewrite (4) as
\[
\chi_{[u, w]}(q) = \sum_{t \in [e, w_0(K)]} \mu_{[e, w_0(K)]}(e, t) q^{\ell(wu^{-1}) - \ell(t)}. \tag{6}
\]

Since \( w_0(K) \leq wu^{-1} \), we have \( \mu_{[e, w_0(K)]}(e, t) = \mu_{[e, w_0(K)]}(e, t) = \mu_{[e, w_0(K)]}(e, t) = 0 \) for all \( t \in [e, w_0(K)] \). Therefore the right-hand side of (6) is equal to
\[
q^{\ell(wu^{-1}) - \ell(w_0(K))} \sum_{t \in [e, w_0(K)]} \mu_{[e, w_0(K)]}(e, t) q^{\ell(w_0(K)) - \ell(t)} = q^{\ell(wu^{-1}) - \ell(w_0(K))} \chi_{[e, w_0(K)]}(q),
\]

Finally, by Lemmas 3.3 and 3.4 we have
\[
\chi_{[e, w_0(K)]}(q) = \chi_{K_1}(q) \chi_{K_2}(q) \cdots \chi_{K_r}(q),
\]
which completes the proof. \( \blacksquare \)
4. PROPERTIES OF DESCENT CLASSES

In this section we show that the descent class $D_I^J$ of a Coxeter group $W$ is an interval of $W$ and obtain some properties of the descent classes.

Throughout this section we assume that $(W, S)$ is a finite Coxeter system. Then, for any $J \subseteq S$, the subgroup $W_J$ is also finite and $w_0(J)$ exists. Now we show that $D_I^J$ is an interval of $W$.

**Proposition 4.1.** For $I \subseteq J \subseteq S$, each descent class $D_I^J$ is equal to the interval $[w_0(I), w_0w_0(J^c)]$ of $W$, where $J^c = S \setminus J$.

**Proof.** Björner and Wachs [3] showed that $D_I^J = [w_0(I), w_0w_0(J^c)]$, where $w_0(J^c)$ is the top element of $W^{J^c}$. By the relation $w_0 = w_0d_0(J)$ in [2] p.44, we are done. □

Note that Theorem 3.5 implies that the characteristic polynomial of an interval $[u, w]$ is determined by $D_R(\{u\})$. Therefore the characteristic polynomial of the descent class $D_I^J = [w_0(I), w_0w_0(J^c)]$ is determined by $D_R(w_0w_0(J^c)w_0(I))$. (Note that $(w_0(I))^{-1} = w_0(I)$.) We will find a simple description for $D_R(w_0w_0(J^c)w_0(I))$; To do this, we need the following lemma.

**Lemma 4.2.** Let $u = u_e \ldots u_1$ and $v = v_r \ldots v_1$ be reduced expressions of $u, v \in W$ such that $u_i \neq v_j$ for all $1 \leq i \leq t$ and $1 \leq j \leq r$. Then, for each $v_k \in D_R(v)$, there exists $u_i$ such that $v_ku_i \neq u_iv_k$ if and only if $v_k \notin D_R(vu)$.

**Proof.** Let $v_k \in D_R(v)$. Then there is a reduced expression $v'_r \ldots v'_1$ of $v$ ending with $v'_1 = v_k$.

For the “if” part, suppose that $v_k$ commutes with every $u_i$. Then $v'_r \ldots v'_2u_1 \ldots v'_1$ is a reduced expression of $vu$ ending with $v_k$, which implies $v_k \in D_R(vu)$.

For the “only if” part, suppose that there exists $u_i$ that do not commute with $v_k$. For a contradiction suppose that $v_k \in D_R(vu)$. Then there is a reduced expression $w_{r+1} \ldots w_1$ of $vu$ ending with $w_1 = v_k$. Since any two reduced expressions can be obtained from each other via a sequence of braid-moves, we can find a sequence $R_1, \ldots, R_p$ of reduced expressions of $vu$ such that $R_1 = v_r \ldots v_1u_t \ldots u_1, R_p = w_{r+1} \ldots w_1$, and each $R_{j+1}$ is obtained from $R_j$ by applying a single braid-move.

Let $s = v_k$ and $s' = u_i$. Observe that every simple reflection $s$ is to the left of every simple reflection $s'$ in $R_1$. Therefore we can find the smallest integer $d$ such that $s$ is to the left (resp. right) of $s'$ in $R_d$ (resp. $R_{d+1}$). This can only happen if the braid-move $ss's's \ldots = ss'ss \ldots$, for some integer $m \geq 3$, is used when we obtain $R_{d+1}$ from $R_d$. However, since $m \geq 3$, we can apply this braid-move only if there is already at least one $s$ to the right of $s'$, which is a contradiction because every $s$ is to the left of each $s'$ in $R_d$. Therefore every $u_i$ must commute with $v_k$, which completes the proof. □

Now we give a simple expression for the set $D_R(w_0w_0(J^c)w_0(I))$.

**Lemma 4.3.** Let $I \subseteq J \subseteq S$ and $I^+ = \{v \in S : iv \neq vi \text{ for some } i \in I\}$. Then we have $D_R(w_0w_0(J^c)w_0(I)) = (J \cup I^+) \setminus I$.

**Proof.** The set $D_R(w_0w_0(J^c)w_0(I))$ is a subset of $[e, w_0w_0(J^c)w_0(I)]$. By Proposition 2.6, the interval $[e, w_0w_0(J^c)w_0(I)]$ is isomorphic to $D_I^J = [w_0(I), w_0w_0(J^c)]$ via $x \mapsto xw_0(I)$ for $x \in [e, w_0w_0(J^c)w_0(I)]$. Since $D_R(w_0w_0(J^c)w_0(I)) = \{v \in S : e \leq v \leq w_0w_0(J^c)w_0(I)\}$, by the isomorphism $D_R(w_0w_0(J^c)w_0(I))$ can also be written as $\{v \in S : w_0(I) \leq w_0v \leq w_0w_0(J^c)w_0(I)\}$, say $K$. We claim that $K = (J \cup I^+) \setminus I$.

Let $v \in K$. Since $w_0(I) \leq w_0v$, we have $v \notin I$. Since $w_0v \in D_I^J$, we have $I \subseteq D_R(w_0v) \subseteq J$. Note that $D_R(w_0v) = I$ by the maximality of $w_0(I)$. So $D_R(w_0v)$ is $I$ or $I \cup \{v\}$. In the case of $D_R(w_0v) = I$, in other words $v \notin D_R(w_0v)$, there exists $i \in I$ such that $iv \neq vi$ by Lemma 4.2. Hence, in this case, we have $v \notin I^+$. For the other case $D_R(w_0v) = I \cup \{v\}$, we have $v \in J$ since $I \cup \{v\} = D_R(w_0v) \subseteq J$. So we have $v \in (J \cup I^+) \setminus I$.

Conversely, let $t \in (J \cup I^+) \setminus I$ and consider $tw_0(I)$. Since $t \notin I$, we have $w_0(I) \leq tw_0(I)$. If $t \in J$, then $D_R(tw_0(I))$ is $I$ or $I \cup \{t\}$ so $I \subseteq D_R(tw_0(I)) \subseteq J$. If $t \in I^+$, then $t \notin D_R(tw_0(I))$ by Lemma 4.2. In other words $D_R(tw_0(I)) = I$. Hence we have $tw_0(I) \in D_I^J$, which means that $t \in K$. □
Lemma 4.3 will be used in later to compute the characteristic polynomial of $D_I$ for the Coxeter group $A_n$.

5. Permutations with a fixed descent set

In this section we give an explicit formula for the characteristic polynomial of the set of permutations with a fixed descent set ordered by the weak order. As a corollary we obtain a simple formula for the characteristic polynomial of the poset of alternating permutations.

An important example of Coxeter groups is the finite irreducible Coxeter group $A_{n-1}$, which can be identified with the symmetric group as follows. For a positive integer $n$, let $[n] := \{1, 2, \ldots, n\}$. The symmetric group $S_n$ is the set of all bijections from $[n]$ to $[n]$. Each element $\pi \in S_n$ is called a permutation and we write $\pi = \pi_1 \pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$. The simple transposition $s_i$ is the permutation that exchanges the integers $i$ and $i + 1$ and fixes all the other integers. Let $S$ be the set of simple transpositions. Then $(S_n, S)$ is the Coxeter group $A_{n-1}$, see [2] Proposition 1.5.4.

From now on we identify each Coxeter generator $s_i$ and its index $i$.

The Coxeter graph of $A_n$ is shown in Figure 1. Note that a subgraph of the Coxeter graph of $A_n$ is connected if and only if the vertices of the subgraph are consecutive integers. Let $S$ denote the set of alternating permutations in $S_n$. Then $|S|$ is 1 and the rank of $A_n$ is 2, every element of $M_i$ is contained in $I^+ \setminus I$, and if $|M_i| \geq 3$, only the smallest and largest elements of $M_i$ are contained in $I^+ \setminus I$. Hence every maximal subset of consecutive integers in $I^+ \setminus I$ has one or two elements and the number of subsets of size 1 (resp. 2) is $\alpha + 2\gamma$ (resp. $\beta$). Therefore, by Lemma 3.3 we have

$$\chi_{D_I}(q) = q^{d - \alpha - 2\gamma - 3\beta}(q - 1)^{\alpha + 2\gamma + \beta}(q^2 - q - 1)\beta.$$  

**Proof.** By Proposition 4.1 we have $D_I = [w_0(I), w_0w_0(I')]$. In order to use Theorem 3.5 we investigate the structure of $D_R(w_0w_0(I')w_0(I))$. By Lemma 4.3 we have

$$D_R(w_0w_0(I')w_0(I)) = (I \cup I^+) \setminus I = I^+ \setminus I,$$

where $I^+ = \{j \in S : i j \neq j i \text{ for some } i \in I\}$. Since $I^+ = \{j \in S : |i - j| = 1 \text{ for some } i \in I\}$, we have

$$I^+ \setminus I = \{j \in S \setminus I : |i - j| = 1 \text{ for some } i \in I\}. \tag{7}$$

Let $S \setminus I = M_1 \cup M_2 \cup \cdots \cup M_k$ as in the statement. By (7), if $|M_i| = 1$ or $|M_i| = 2$, every element of $M_i$ is contained in $I^+ \setminus I$, and if $|M_i| \geq 3$, only the smallest and largest elements of $M_i$ are contained in $I^+ \setminus I$. Hence every maximal subset of consecutive integers in $I^+ \setminus I$ has one or two elements and the number of subsets of size 1 (resp. 2) is $\alpha + 2\gamma$ (resp. $\beta$). Therefore, by Lemma 3.3 we have

$$[e, w_0(I^+ \setminus I)] \cong \langle \alpha + 2\gamma \rangle A_1 \times \beta A_2. \tag{8}$$

Since the rank of $A_1$ is 1 and the rank of $A_2$ is 3, the rank of $[e, w_0(I^+ \setminus I)]$ is $\alpha + 2\gamma + 3\beta$. Thus, by Theorem 3.5 and (8),

$$\chi_{D_I}(q) = q^{d - (\alpha + 2\gamma + 3\beta)}\chi_{A_1}(q)^{\alpha + 2\gamma}\chi_{A_2}(q)\beta.$$  

Since $\chi_{A_1}(q) = q - 1$ and $\chi_{A_2}(q) = (q - 1)(q^2 - q - 1)$, we obtain the theorem. \hfill $\square$

A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ satisfying $\pi_1 < \pi_2 > \pi_3 < \cdots$ is called an alternating permutation. Let $Alt_n$ denote the set of alternating permutations in $S_n$. Then one can easily check that $Alt_n = D_I$ in the Coxeter system $(A_{n-1}, S)$, where $I = \{2, 4, 6, \ldots\} \cap [n - 1]$. Observe that $S \setminus I = \{1, 3, 5, \ldots\} \cap [n - 1]$ in which every maximal subset of consecutive integers has size
1. Since \(|S \setminus I| = \lfloor \frac{n}{2} \rfloor\) and the rank of \(\text{Alt}_n\) is \(\binom{n-1}{2}\), we obtain the characteristic polynomial of \(\text{Alt}_n\) as a corollary of Theorem 5.1.

**Corollary 5.2.** The characteristic polynomial of \(\text{Alt}_n\) is
\[
\chi_{\text{Alt}_n}(q) = q^{\left(\binom{n-1}{2} - \lfloor \frac{n}{2} \rfloor \right)}(q - 1)^{\lfloor \frac{n}{2} \rfloor}.
\]

Stanley [8] showed that the characteristic polynomial of a supersolvable lattice is decomposed into linear factors, see also [7, Theorem 6.2]. Although the characteristic polynomial of \(\text{Alt}_n\) has only linear factors, one can check that \(\text{Alt}_n\) is not supersolvable.

### 6. Modified Characteristic Polynomials for Classical Coxeter Groups

In this section we find generating functions for modified characteristic polynomials for the classical Coxeter groups \(A_n, B_n\) and \(D_n\).

**Definition 6.1.** Let \(W\) be a ranked poset with the bottom element \(\hat{0}\). The modified characteristic polynomial \(\hat{\chi}_W(q)\) of \(W\) is defined by
\[
\hat{\chi}_W(q) = \sum_{w \in W} \mu_W(\hat{0}, w)q^{r_W(w)},
\]
where \(r_W(w)\) is the rank of \(w \in W\).

If \(W\) is a finite Coxeter group, there is a simple relation between the modified characteristic polynomial \(\hat{\chi}_W(q)\) and the characteristic polynomial \(\chi_W(q)\) as follows.

**Proposition 6.2.** Let \((W, S)\) be a finite Coxeter group and \(w_0\) the unique maximal element of \(W\). Then we have
\[
\hat{\chi}_W(q) = q^{\ell(w_0)}\chi_W(q^{-1}).
\]

**Proof.** This follows immediately from the definitions of \(\hat{\chi}_W(q)\) and \(\chi_W(q)\). \(\square\)

In what follows we find an expression for the generating functions for \(\hat{\chi}_{A_n}(q)\), \(\hat{\chi}_{B_n}(q)\), and \(\hat{\chi}_{D_n}(q)\). First we consider refinements of these generating functions.

**Definition 6.3.** For the Coxeter system \((A_n, S)\) (resp. \((B_n, S)\), \((D_n, S)\)), let \(A_{n,k,l}\) (resp. \(B_{n,k,l}\), \(D_{n,k,l}\)) be the number of subsets \(J \subseteq S\) such that \(|J| = k\) and \(\ell(w_0(J)) = l\). We define
\[
T_A(x, y, z) = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{l \geq 0} A_{n,k,l} x^n y^k z^l,
\]
\[
T_B(x, y, z) = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{l \geq 0} B_{n,k,l} x^n y^k z^l,
\]
\[
T_D(x, y, z) = \sum_{n \geq 2} \sum_{k \geq 0} \sum_{l \geq 0} D_{n,k,l} x^n y^k z^l,
\]
where \(A_{0,k,l} = B_{0,k,l} = 1\) if \(k = l = 0\) and \(A_{0,k,l} = B_{0,k,l} = 0\) otherwise.

In the definition of \(T_D(x, y, z)\) the sum is over \(n \geq 2\) for computational convenience. Note that \(B_1 \cong A_1, D_2 \cong A_1 \times A_1,\) and \(D_3 \cong A_3\).

Similar to \([2]\), the polynomial \(\hat{\chi}_W(q)\) can be written as
\[
\hat{\chi}_W(q) = \sum_{J \subseteq S} (-1)^{|J|}q^{\ell(w_0(J))}.
\]

If \(W = A_n\), we have
\[
\hat{\chi}_{A_n}(q) = \sum_{k \geq 0} \sum_{l \geq 0} A_{n,k,l} (-1)^k q^l,
\]
which implies that
\[
T_A(x, -1, q) = \sum_{n \geq 0} \left( \sum_{k \geq 0} \sum_{l \geq 0} A_{n,k,l} (-1)^k q^l \right) x^n = \sum_{n \geq 0} \hat{\chi}_{A_n}(q)x^n. \tag{10}
\]
By the same arguments we have

\[
T_B(x, -1, q) = \sum_{n \geq 0} \hat{\chi}_{B_n}(q)x^n, \quad (11)
\]

\[
T_D(x, -1, q) = \sum_{n \geq 2} \hat{\chi}_{D_n}(q)x^n. \quad (12)
\]

We use the fact that \(\ell(w_0(A_n)) = (n+1)/2\), \(\ell(w_0(B_n)) = n^2\), and \(\ell(w_0(D_n)) = n(n-1)\), see \(\text{[5, p.16, p.80 Table 2]}\). Here the notation \(w_0(A_n)\) (resp. \(w_0(B_n), w_0(D_n)\)) means that the unique maximal element of \(A_n\) (resp. \(B_n, D_n\)).

Now we give explicit formulas for \(T_A(x, y, z)\), \(T_B(x, y, z)\), and \(T_D(x, y, z)\).

**Proposition 6.4.** We have

\[
T_A(x, y, z) = \frac{P}{1-xP}, \quad (13)
\]

\[
T_B(x, y, z) = \frac{Q}{1-xP}, \quad (14)
\]

\[
T_D(x, y, z) = \frac{x^2P + 2x(P-1) + R}{1-xP}, \quad (15)
\]

where

\[
P = \sum_{n \geq 0} x^n y^n z^{(n+1)/2}, \quad Q = \sum_{n \geq 0} x^n y^n z^{n^2}, \quad \text{and} \quad R = \sum_{n \geq 2} x^n y^n z^{n^2 - n}.
\]

**Proof.** Recall that \(A_{k,l}\) is the number of subsets \(J\) of \(S = [n]\) satisfying \(|J| = k\) and \(\ell(w_0(J)) = l\) in the Coxeter group \(W = A_n\). For such a subset \(J\), let \(J = J_1 \cup \cdots \cup J_m\), where each \(J_i\) is a connected component of the Coxeter graph of \((W_j, J)\). Then \(\ell(w_0(J)) = \sum_{i=1}^m \ell(w_0(J_i)) = \sum_{i=1}^m (|J_i|+1)/2\) and the contribution of \(J \subseteq [n]\) to \(T_A(x, y, z)\) is

\[
wt(n, J) := x^ny^{j_1} \cdots y^{j_m}z^{(|j_1|+1)/2} \cdots z^{(|j_m|+1)/2}.
\]

Therefore

\[
T_A(x, y, z) = \sum_{(n,J) \in X} wt(n, J),
\]

where \(X\) is the set of all pairs \((n, J)\) of a nonnegative integer \(n\) and a set \(J \subseteq [n]\).

We will divide the sum in (10) into two cases \(J = [n]\) and \(J \neq [n]\). First, note that

\[
\sum_{(n,[n]) \in X} wt(n, [n]) = \sum_{n \geq 0} x^n y^n z^{(n+1)/2} = P.
\]

Now consider \((n, J) \in X\) with \(J \neq [n]\). Let \(t\) be the smallest positive integer not contained in \(J\). Then we have \(J = J_1 \cup \cdots \cup J_m\) with \(J_1 = [t-1]\) and \(J \setminus J_1 \subseteq \{t+1, t+2, \ldots, n\}\). Let \(J' = \{j-t : j \in J \setminus J_1\}\). Then \((n-t, J') \in X\) and \(wt(n, J) = x^ty^{t-1}z^{(t)}wt(n-t, J')\). Since every \((n, J) \in X\) with \(J \neq [n]\) is obtained from \((n-t, J') \in X\) for some \(t \geq 1\) and \(J'\) in this way, we have

\[
\sum_{(n,J) \in X, J \neq [n]} wt(n, J) = \sum_{t \geq 1} x^ty^{t-1}z^{(t)} \sum_{(n-t, J') \in X} wt(n-t, J') = xP \cdot T_A(x, y, z).
\]

By (16), (17), and (18), we have \(T_A(x, y, z) = P + xP \cdot T_A(x, y, z)\), which implies the first identity (13).

The second identity (14) can be proved similarly if we consider the Coxeter system \((B_n, S)\) with \(S = [n]\) and the Coxeter graph as shown in Figure 2. The only difference is that if \(J = [n]\), then \(wt(n, J) = x^n y^n z^{n^2}\) since \(\ell(w_0(B_n)) = n^2\).

For the third identity (15), we consider Coxeter system \((D_n, S)\) with \(S = [n]\) and the Coxeter graph as shown in Figure 2. The proof is similar to the case of the first identity except that \(\ell(w_0(D_n)) = n^2 - n\) and we need to consider the cases that the smallest positive integer \(t\) not contained in \(J\) is \(n-1\) or \(n\) separately. It is not hard to check that \(T_D(x, y, z)\) satisfies \(T_D(x, y, z) = R + xPT_D(x, y, z) + 2x(P-1) + x^2P\), which implies the third identity. \(\square\)
Theorem 7.1. We have

\[ \hat{\chi}_{W_n}(q) = \sum_{n \geq 0} \hat{\chi}_{W_n}(q) x^n = \sum_{n \geq 0} \frac{(-1)^n q^{\frac{n+1}{2}}}{\sum_{n \geq 0} (-1)^n q^{\frac{n}{2}}} x^n, \]

\[ \hat{\chi}_{B_n}(q) = \sum_{n \geq 0} \frac{(-1)^n q^{2n}}{\sum_{n \geq 0} (-1)^n q^{2n}} x^n, \]

\[ \hat{\chi}_{D_n}(q) = \sum_{n \geq 2} \frac{(-1)^n (q^{n-1} - 2q^{n/2} + q^{n/2})}{\sum_{n \geq 0} (-1)^n q^{n/2}} x^n. \]

7. Modified characteristic polynomials for affine Coxeter groups

In this section we express the modified characteristic polynomial \( \hat{\chi}_W(q) \) of an affine Coxeter group \( W \) using those of finite Coxeter groups.

Recall that the modified characteristic polynomial \( \hat{\chi}_W(q) \) is defined as a series if \( W \) is an infinite poset. However, if \( W \) is an affine Coxeter group with finite generators, \( \hat{\chi}_W(q) \) is a polynomial. Note also that, for an affine Coxeter group, we only need to consider the elements \( w_0(J) \) for \( J \subseteq S \) since \( w_0(S) \) does not exist, i.e.,

\[ \hat{\chi}_W(q) = \sum_{J \subseteq S} (-1)^{|J|} q^{(w_0(J))}. \] (19)

The Coxeter graph of an irreducible affine Coxeter group contains the Coxeter graph of a finite Coxeter group as a subgraph. Using this observation we can express the characteristic polynomial of an affine Coxeter group in terms of those of finite Coxeter groups.

Theorem 7.1. We have

\[ \hat{\chi}_{A_n}(q) = \hat{\chi}_{B_n}(q) + \sum_{k=1}^{n} k(-1)^k q^{k+1} \hat{\chi}_{A_{n-k-1}}(q), \]

\[ \hat{\chi}_{B_n}(q) = \hat{\chi}_{B_n}(q) + \sum_{k=0}^{n} (-1)^k q^{k+1} \hat{\chi}_{B_{n-k}}(q), \]

\[ \hat{\chi}_{C_n}(q) = \sum_{k=0}^{n} (-1)^k q^k \hat{\chi}_{D_{n-k}}(q), \]

\[ \hat{\chi}_{D_n}(q) = \hat{\chi}_{D_n}(q) + (-1)^n q^{n-1} + \sum_{k=0}^{n} (-1)^k q^{k+1} \hat{\chi}_{D_{n-k}}(q), \]

where we assume \( n \geq 2, n \geq 3, n \geq 2, \) and \( n \geq 4 \) for \( A_n, B_n, C_n, \) and \( D_n, \) respectively. We define \( \hat{\chi}_{A_i}(q) = \hat{\chi}_{B_i}(q) = 1 \) for \( i \leq 0 \) and \( \hat{\chi}_{D_i}(q) = 1 \) for \( i \leq 1. \)
CHARACTERISTIC POLYNOMIALS OF THE WEAK ORDER ON CLASSICAL AND AFFINE COXETER GROUPS

Figure 3. The Coxeter graph of $\tilde{A}_n$ containing the Coxeter graph of $A_n$.

Figure 4. The Coxeter graph of $\tilde{A}_n$ and the subset $Q$.

Proof. Type $\tilde{A}_n$: The Coxeter graph of $W = \tilde{A}_n$ contains that of $A_n$ as shown in Figure 3. Let $S = \{s_1, s_2, \ldots, s_{n+1}\}$ be the set of generators. By (19), we have

$$\hat{\chi}_{\tilde{A}_n}(q) = \sum_{J \subseteq S, s_{n+1} \notin J} (-1)^{|J|} q^{\ell(w_0(J))} + \sum_{J \subseteq S, s_{n+1} \in J} (-1)^{|J|} q^{\ell(w_0(J))}. \quad (20)$$

Note that the first sum of the right-hand side of (20) is given by

$$\sum_{J \subseteq S, s_{n+1} \notin J} (-1)^{|J|} q^{\ell(w_0(J))} = \hat{\chi}_{A_n}(q). \quad (21)$$

Thus we only need to compute the second sum.

Suppose $J \subseteq S$ with $s_{n+1} \in J$. Let $Q \subseteq J$ be the set of generators in the connected component containing $s_{n+1}$ and let $|Q| = k$. We also define $J' = J \setminus Q$ so that $J = Q \cup J'$. If $k \geq n-1$, then $J' = \emptyset$. Suppose $k < n-1$ and let $s_a, s_b$ be the generators that are adjacent to an end vertex of $Q$ in the Coxeter graph of $\tilde{A}_n$, see Figure 4. Then $J'$ is a subset of $S' = S \setminus (Q \cup \{s_a, s_b\})$. Note that $W_{S'} \cong A_{n-k-1}$ since the graph of $S'$ is a connected path with $|S'| = n - k - 1$. Then

$$(-1)^{|J|} q^{\ell(w_0(J))} = (-1)^{|Q|+|J'|} q^{\ell(w_0(Q))} + \ell(w_0(J')) = (-1)^{k+|J'|} q^{\frac{k+1}{2}} + \ell(w_0(J')).$$

This implies that

$$\sum_{J \subseteq S, s_{n+1} \in J} (-1)^{|J|} q^{\ell(w_0(J))} = \sum_{k=1}^{n} k(-1)^k q^{\frac{k+1}{2}} \hat{\chi}_{A_{n-k-1}}(q). \quad (22)$$

By (20), (21), and (22), we obtain the formula for $\hat{\chi}_{\tilde{A}_n}(q)$.

For the rest of the proof we denote by $S = \{s_0, s_1, \ldots, s_n\}$ the set of generators of $\tilde{B}_n$, $\tilde{C}_n$, or $\tilde{D}_n$. 

\[1\] Coxeter graph of $A_n$

\[2\] Coxeter graph of $\tilde{A}_n$ containing the Coxeter graph of $A_n$. 

\[3\] Coxeter graph of $\tilde{A}_n$ and the subset $Q$.
(19), we have
\[ W \equiv \tilde{W} \]
This can be proved by similar arguments as in the proof of \( \tilde{W} \).

Therefore
\[ Q \subseteq \{ s_0, s_1 \} \not\subseteq J. \]
Suppose now that \( \{ s_0, s_1 \} \subseteq J. \) Let \( k \) be the smallest integer such that \( s_k \not\in J \) and let
\[ Q = \{ s_0, \ldots, s_{k-1} \} \text{ and } J' = J \setminus Q. \]
Then we have \( 2 \leq k \leq n, W_Q \cong D_k, \) and \( J' \subseteq \{ s_{k+1}, \ldots, s_n \} \).
Since \( \ell(w_0(Q)) = k(k-1) \), we have
\[ (-1)^{|J|} q^{\ell(w_0(J))} = (-1)^Q |+J' \rangle q^{\ell(w_0(Q))} + \ell(w_0(J')) = (-1)^k q^{k(k-1)}(-1)^{|J'|} q^{\ell(w_0(J'))}. \]
Therefore
\[ \sum_{J \subseteq S, \{ s_0, s_1 \} \subseteq J} (-1)^{|J|} q^{\ell(w_0(J))} = \sum_{k=2}^n (-1)^k q^{k(k-1)} \sum_{J' \subseteq \{ s_{k+1}, \ldots, s_n \}} (-1)^{|J'|} q^{\ell(w_0(J'))} \]
\[ = \sum_{k=2}^n (-1)^k q^{k(k-1)} \tilde{\chi}_{B_{n-k}}(q). \]  
Combining (23), (24), and (25) gives the desired formula for \( \tilde{\chi}_{B_n}(q) \).

**Type \( \tilde{C}_n \):** The Coxeter graph of \( W = \tilde{C}_n \) includes the graph of \( B_n \) as shown in Figure 6. By (19), we have
\[ \tilde{\chi}_{\tilde{C}_n}(q) = \sum_{J \subseteq S, s_0 \not\in J} (-1)^{|J|} q^{\ell(w_0(J))} + \sum_{J \subseteq S, s_0 \in J} (-1)^{|J|} q^{\ell(w_0(J))}. \]  
This can be proved by similar arguments as in the proof of \( \tilde{B}_n \), where in this case, for \( 1 \leq k \leq n \), we have \( W_Q \cong B_k \) and \( \ell(w_0(Q)) = k^2 \).

**Type \( \tilde{D}_n \):** The Coxeter graph of \( W = \tilde{D}_n \) includes the graph of \( D_n \) as shown in Figure 7. By (19), we have
\[ \tilde{\chi}_{\tilde{D}_n}(q) = \sum_{J \subseteq S, \{ s_0, s_1 \} \not\subseteq J} (-1)^{|J|} q^{\ell(w_0(J))} + \sum_{J \subseteq S, \{ s_0, s_1 \} \subseteq J} (-1)^{|J|} q^{\ell(w_0(J))}. \]  
We can obtain the formula by similar arguments as in the proof of \( \tilde{B}_n \). If \( \{ s_0, s_1 \} \not\subseteq J \), we have
\[ \sum_{J \subseteq S, \{ s_0, s_1 \} \not\subseteq J} (-1)^{|J|} q^{\ell(w_0(J))} = 2 \tilde{\chi}_{\tilde{D}_n}(q) - \tilde{\chi}_{\tilde{D}_{n-1}}(q). \]
The Coxeter graph of $\tilde{D}_n$ containing the Coxeter graph of $D_n$.

Suppose that $\{s_0, s_1\} \subseteq J$. Let $k$ be the smallest integer such that $s_k \not\in J$ and let $Q = \{s_0, \ldots, s_{k-1}\}$ and $J' = J \setminus Q$. Then, for $2 \leq k \leq n-2$ or $k = n$, we have $W_Q \cong D_k$ and $J' \subseteq \{s_{k+1}, \ldots, s_n\}$. For $k = n-1$, there are two cases such that $Q$ and $Q \cup \{s_n\}$. Then we have $W_Q \cong D_{n-1}$ and $W_{Q \cup \{s_n\}} \cong D_n$. In both cases we have $J' = \emptyset$. Therefore

$$
\sum_{J \subseteq S, \{s_0, s_1\} \subseteq J} (-1)^{|J|} q^{|w_J(J)|} = \sum_{k=2}^{n-2} (-1)^k q^{k(k-1)} \sum_{J' \subseteq \{s_{k+1}, \ldots, s_n\}} (-1)^{|J'|} q^{|w_{J'}(J')|} + (-1)^{n-1} q^{(n-1)(n-2)} + 2(-1)^n q^{n(n-1)} = \sum_{k=2}^{n-2} (-1)^k q^{k(k-1)} \chi_{D_{n-k}}(q) + (-1)^{n-1} q^{(n-1)(n-2)} + 2(-1)^n q^{n(n-1)}.
$$

(29)

Combining (27), (28), and (29) gives the desired formula for $\hat{\chi}_{D_n}(q)$. 

Note that using Proposition 6.2, the polynomials in Theorem 7.1 can also be written as sums of characteristic polynomials of finite Coxeter groups.

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