2-Period Balanced Travelling Salesman Problem: a polynomially solvable case and heuristics

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Abstract

We consider the NP-hard 2-period balanced travelling salesman problem. In this problem the salesman needs to visit a set of customers in two time periods. A given subset of the customers has to be visited in both periods while the rest of the customers need to be visited only once, in any of the two periods. Moreover, it is required that the number of customers visited in the first period does not differ from the respective number in the second period by more than \( p \), where \( p \) is a given small constant. The objective is to find tours for visiting the customers in the two periods with minimal total length. We show that in the case when the underlying distance matrix is a Kalmanson matrix, the problem can be solved in polynomial time. For the general case, we propose two new heuristics. A combination of these two heuristics has shown very favourable results in computational experiments on benchmark instances from the literature.

Keywords. Combinatorial optimization; Kalmanson matrix; 2-period balanced travelling salesman problem; vehicle routing problem; polynomially solvable case.

1 Introduction

1.1 Definitions and related work

In the well-known travelling salesman problem (TSP), an \( n \times n \) distance matrix \((c_{ij})\) among \( n \) locations is given: location 1 is where the salesman starts and ends the journey, and the other \( n-1 \) locations are for the customers to be visited. The objective of the salesman is to find a shortest route for visiting all customers. The reader is referred to the books \cite{1,19,29} for the wealth of knowledge on this problem.

A feasible solution to the TSP can be represented as a tour \( \tau = \langle 1 = \tau_1, \tau_2, \ldots, \tau_n, 1 = \tau_{n+1} \rangle \) on the set \( \{1,2,\ldots,n\} \). We will refer to the items in tours interchangeably as points or nodes. The salesman starts from node 1, visits node \( \tau_2 \), and so on, and eventually returns to the initial node 1. Given a distance matrix \((c_{ij})\), the length of the tour \( \tau \) is calculated as \( c(\tau) = \sum_{i=1}^{n} c_{\tau_i,\tau_{i+1}} \). We assume that all distance matrices considered in this paper are symmetric.

In the 2-period travelling salesman problem (2-TSP) the salesman has to visit a set of customers in two periods. A given subset of the customers has to be visited twice – once in each of the two periods; while the rest of the customers have to be visited only once, in any of the two periods. The objective is to find tours for the two periods with the minimal total length.

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We assume that the salesman always starts and ends the journeys in location 1. For the 2-TSP this assumption means that location 1 is visited twice, i.e. 1 belongs to both tours in any feasible solution.

The 2-TSP can be viewed as a simplification of the m-period TSP (see e.g. [4, 9, 30, 31]) and also as a model for some practical applications of the vehicle routing problem (VRP). In particular, Butler, Williams, and Yarrow [6] considered the 2-TSP as a model for a milk collection problem in Ireland. Doerner et al. [14] have described a similar model for the two-day blood delivery service of the Austrian red cross. Hamzadayi, Topaloglu, and Kose [20] considered m-period TSP as a model for delivery goods by a soft drinks company in Turkey.

In this paper we focus on the balanced 2-TSP, where an additional constraint requests that the number of customers visited in each period differs by no more than \( p \), where \( p \) is a given small constant. Different variants of balanced TSPs can also be found in [18, 24, 28].

Bassetto and Mason [5] considered a special case of the balanced 2-TSP where \( p = 1 \) and where the points are located in the Euclidean plane. Fig. 1 illustrates a solution to an instance of the balanced 2-TSP with 10 points in the Euclidean plane. Four out of ten points in this instance are visited in both periods. The set of points shown in Fig. 1 happens to lead to a distance matrix which is a so-called Kalmanson matrix, cf. [12].

A symmetric \( n \times n \) matrix \( C \) is called a Kalmanson matrix if it fulfils the Kalmanson conditions:

\[
c_{ij} + c_{\ell m} \leq c_{i\ell} + c_{jm}, \quad c_{im} + c_{j\ell} \leq c_{i\ell} + c_{jm}, \quad \text{for all } 1 \leq i < j < \ell < m \leq n. \tag{1}
\]

Kalmanson [23] noticed that if \( n \) points in the Euclidean plane are located on the boundary of their convex hull and numbered along the convex hull, then the distance matrix for these points satisfies conditions (1). He proved that the TSP with a Kalmanson distance matrix is solved to optimality by the tour \( \pi = (1, 2, 3, \ldots, n-2, n-1, n, 1) \). Notice that if a distance matrix satisfies conditions (1), it does not necessarily mean that the points are on the boundary of a convex hull (or in the Euclidean plane). For example, the points in the example in Fig. 1 are not on the boundary of the convex hull, however the distance matrix for these points is a Kalmanson matrix (for details and the coordinates of the points see Fig. 1 in [12]).

It can be easily shown that cyclic renumbering of rows and columns does not destroy property (1): if matrix \( C = (c_{ij}) \) is a Kalmanson matrix, then the matrix \( C_\sigma = (c_{\sigma(i)\sigma(j)}) \) with rows and columns permuted with the permutation \( \sigma = (k, k+1, \ldots, n, 1, 2, \ldots, k-1) \), is also a Kalmanson matrix. Permutation \( \sigma \) is called a cyclic shift of the identity permutation. Adding constants (positive or negative) to rows and columns in the distance matrix does not destroy property (1) either.

Kalmanson matrices possess nice structural properties that have been used to identify special cases in a wide range of combinatorial optimization problems (see [8, 10, 13, 25, 32]). In particular, Kalmanson matrices play a key role in the so-called master tour problem [13].

An optimal tour for the TSP is called a master tour if after deleting any subset of points from the tour the so obtained sub-tour is still optimal for the remaining set of points. Given a distance matrix, the master tour problem is to find out whether it is possible to construct a master tour.

A master tour exists if and only if the underlying distance matrix is a Kalmanson matrix (Theorem 5.1 in [13]). In this case the master tour is either the tour \( \pi \) or the inverse of it, tour \( \pi^{-1} = (1, n, n-1, \ldots, 2, 1) \).

The 2-TSP is related to the more general vehicle routing problem (VRP). In the VRP a set of given customers are to be visited by a given fleet of vehicles which are located in one or
The task is to find an optimal set of routes for the vehicles fulfilling the requests of the customers while respecting the given constraints which may vary vastly depending on which VRP variant is considered. A standard VRP objective is to minimize the total travelled distance. For further details the reader is referred to [7, 15, 17, 27, 33, 34, 35].

Note that the distance matrices which result from points located on a line, a cycle, or a tree are all Kalmanson matrices. Hence the papers [3, 21, 26, 36] can be viewed as investigations of the VRP for special classes of Kalmanson matrices.

The 2-TSP can be modelled within the VRP framework as a VRP with only 2 vehicles (2-VRP) as follows. One of the \( m = |S| \) customers that have to be visited in both periods is chosen as a depot. For each of the other \( m - 1 \) customers in \( S \) an identical copy (i.e. a new customer in the same location) is created. Every two identical customers are allocated to two different vehicles. The allocation of these customers to the vehicles is fixed: each vehicle has to serve \( m - 1 \) fixed customers, while \( k = n - |S| \) customers are to be served by only one of the vehicles. In Section 4 we use a VRP related terminology and the 2-VRP model as a useful tool for representing our algorithms for the 2-TSP. Recall that “2” in the abbreviation 2-VRP refers to the number of vehicles and not to the capacity of the vehicle as for instance in the abbreviation 2SDVRP for the Split Delivery Vehicle Routing Problem [2] (see also [21]).

### 1.2 Our contribution and structure of the paper

In Section 2 we focus on the balanced 2-TSP with a Kalmanson distance matrix. We prove that this special case can be solved in polynomial time by dynamic programming (DP). We then use the DP recursions in a heuristic for the 2-TSP on arbitrary distance matrices. Computational experiments on the benchmark problems from the literature demonstrate that our heuristic which is motivated by a polynomially solvable case is competitive with previously published algorithms. Thus we demonstrate that the approach of using efficiently solvable special cases to devise heuristics for general hard problems could be a fruitful one also in other settings. This is a major contribution of this paper.

While designing an algorithm for finding an optimal solution for the 2-TSP with a Kalmanson distance matrix, we use a simple methodological approach - “one sequence - two tours”: an optimal solution is found among sequences which combine two tours. In this way the problem...
of allocating nodes to tours and the problem of optimising the tours are solved at one go. In Section 4, we generalise this approach and adapt the well known Held and Karp \cite{22} DP recursions for finding an optimal sequence containing two tours for the 2-TSP with arbitrary distance matrices. The 2-VRP, which is a generalisation of the 2-TSP is then considered. To deal with large size instances of the 2-VRP, a disassemble-aggregate heuristic is suggested. The heuristic transforms a 2-VRP instance into a collection of small size instances and solves them by using the DP recursions. Due to the flexibility of the DP technique, various additional constraints can be included in the consideration. The new heuristic for the 2-VRP (balanced 2-TSP) is another contribution of this paper.

In Section 5 the results of computational experiments are reported. In the experiments, two types of initial solutions have been used: solutions obtained by a heuristic based on the special solvable case and solutions obtained with a random allocation of nodes to tours. The test results have shown that the approach “one sequence - two tours” incorporated in the suggested heuristic is very robust to changes in the quality of the initial solutions. The set of 60 benchmark instances introduced in \cite{5} have been used in the experiments. The authors in \cite{5} have presented two types of results: (a) results obtained by the algorithms proposed, and (b) results obtained by a hybrid approach where a computer algorithm was run first and the obtained results were then improved by human intervention on the basis of a graphical visualisation of the results. In our experiments, 58 results of type (a), and 48 results of type (b) have been improved.

Section 6 provides summary and conclusions. Technical details and supportive material are placed in the appendices.

2 Polynomially solvable case of the balanced 2-TSP

In this section we consider the balanced 2-TSP with an $n \times n$ Kalmanson distance matrix $C$. Let $S \subset \{1, 2, \ldots, n\}$ denote the subset of customers to be visited in both periods. We refer to these customers as fixed nodes. Given an integer parameter $p$, the objective in the balanced 2-TSP is to find two tours with the minimal total length such that every fixed node appears in both tours, every node which is not in $S$ appears only in one of the tours, and the number of nodes in the tours differs by no more than $p$. For example, if $p = 1$, then for even $n$ the tours in the balanced 2-TSP have to contain the same number of nodes. For arbitrary $p$, each tour has to contain at least $\lceil (n + |S| - p)/2 \rceil$ nodes or equivalently no more than $\lfloor (n + |S| + p)/2 \rfloor$ nodes. Without loss of generality we assume that $1 \in S$ (note that the Kalmanson property is conserved by a cyclic shift, hence any node can be chosen to be the start node). Let $s^*$ denote the maximal item in $S$.

The tours $\langle 1, 2, \ldots, n - 1, n, 1 \rangle$ and $\langle 1, n, n - 1, \ldots, 2, 1 \rangle$ are the master tours for the TSP with a Kalmanson matrix. If a set of indices is removed from the set of rows and the set of columns in a Kalmanson matrix, the so obtained submatrix is still a Kalmanson matrix. It means that if an optimal allocation of items to two tours in the 2-TSP is found, an optimal ordering in each tour can be found by sorting the items either in increasing or decreasing order. We will search for an optimal solution of the balanced 2-TSP among the solutions where the items in the first tour are sorted in increasing order, and in the second tour in decreasing order.

Two feasible tours can be combined into a single feasible sequence $Q$ as follows. We start with a partially created sequence $Q = \langle 1, \ldots, 1 \rangle$ where only the placement of the start and end node is known. There are the following options for placing node 2 in the sequence.

(a) If node 2 is a fixed node, then we place one copy of 2 after the start node 1, and another
copy of 2 before the end node 1 which leads to \( Q = (1, 2, \ldots, 2, 1) \).

(b) If 2 is not a fixed node, we have two choices for the placement of 2: either next to the start which leads to \( Q = (1, 2, \ldots, 1) \), or next to the end which leads to \( Q' = (1, \ldots, 2, 1) \).

After having placed item 2, we decide on the location of 3, and so on. At each step of the construction, the feasible sequence describes two partially constructed feasible tours for the 2-TSP. The node placed in construction, the feasible sequence describes two partially constructed feasible tours for the 2-TSP. The node placed in

...en the final step is node 1 which separates the nodes in the two tours and serves as the last node in the first tour and as the first node in the second tour.

According to this description, the tours depicted in Fig. 1 can be represented as the sequence \( \langle 1, 2, 3, 5, 8, 9, 10, 1, 8, 7, 6, 5, 4, 3, 1 \rangle \), with the set of fixed nodes \( S = \{1, 3, 5, 8\} \) and \( s^* = 8 \). The reader who is familiar with the notion of pyramidal tours (see e.g. Chapter 7 in [16]) can recognise a pyramidal-like structure in the sequence \( Q \). A pyramidal tour consists of two parts: in the first part of the tour the nodes are sorted in increasing order, in the second part of the tour the nodes are sorted in decreasing order.

To find an optimal sequence among all feasible sequences with the structure described above, we use a DP approach which is similar to the approach used for finding an optimal pyramidal tour. Assume that the nodes \( \{1, 2, \ldots, i, \ldots, j\} \), \( i < j \), have already got placed into the sequence \( Q \). Furthermore, assume that node \( i \) is currently the last node in the first partially constructed tour, and \( j \) is currently the first node in the second partially constructed tour. Note that in this case the nodes \( i + 1, \ldots, j - 1 \) have already been placed in the second tour.

Let \( V(i,j,m) \) denote the minimal length of a feasible subsequence that starts at node \( i \), goes through the nodes \( \{j + 1, \ldots, n\} \cup \{1\} \) and stops at node \( j \) where \( m \) denotes the number of nodes in the second partially constructed tour in the sequence, i.e. the sub-path \( \langle j, j - 1, \ldots, i + 1, \ldots, 1 \rangle \). For the placement of \( j + 1 \) we have to distinguish two cases. If \( j + 1 \) is a fixed node, it has to be placed in both tours, after \( i \) and before \( j \). If \( j + 1 \) is not a fixed node, one has to decide whether to place it after \( i \) or before \( j \) depending on which option leads to a shorter total length.

If there exists an item \( s \) in the set of fixed items \( S \) such that \( i < s < j + 1 \), it is not possible to find a feasible subsequence from \( i \) to \( j \). In this case we define \( V(i,j,m) := \infty \).

Values \( V(j, i, m) \) are defined similarly for the subsequences that start at \( j \) and end at \( i \). For the case when \( j \) is a fixed node, we also need to define \( V(j,j,m) \) as the length of the shortest feasible sequence from \( j \) to \( j \).

It follows from the definitions above that the length of an optimal sequence \( Q \), i.e. the total length of an optimal pair of tours, can be calculated as

\[
c(Q) = \begin{cases} 
\min\{c_{12} + V(2, 1, 1), c_{21} + V(1, 2, 2)\}, & \text{if } 2 \notin S; \\
c_{12} + c_{21} + V(2, 2, 2), & \text{if } 2 \in S.
\end{cases}
\]  

The \( V \) values which involve node \( n \) can be defined as shown below.

\[
V(i,n,m) = c_{11} + c_{1n}, \quad \text{if } \lfloor (n + |S| - p)/2 \rfloor \leq m \leq \lfloor (n + |S| + p)/2 \rfloor, \\
V(n,i,m) = c_{n1} + c_{11}, \quad \text{if } \lfloor (n + |S| - p)/2 \rfloor \leq m \leq \lfloor (n + |S| + p)/2 \rfloor,
\]

\[\text{if } i = s^*, s^* + 1, \ldots, n.\]  

The above definitions ensure that balanced sequences that include all fixed nodes are considered. It is easy to check that the remaining values \( V(i,j,m) \) and \( V(j,i,m) \) satisfy the
following DP recursions where we assume that undefined values \( V \) are set to infinity.

\[
V(i, j, m) = \begin{cases} 
  c_{i,j+1} + V(j + 1, j, m) & \text{if } j + 1 \notin S; \\
  c_{j+1,j} + V(i, j + 1, m + 1) & \text{if } j + 1 \in S, \\
  c_{i,j+1} + c_{j+1,j} + V(j + 1, j + 1, m + 1) & \forall i, j : \# s \in S \text{ with } i < s < j + 1.
\end{cases}
\]

Summarizing this proves the main result of the section.

**Proposition 2.1** The balanced 2-TSP on a Kalmanson matrix can be solved in \( O(n^3) \) time.

Note that the recursions (2)–(4) could be further simplified by taking into account the symmetricity of the distance matrix \( C \). We believe though that formulating the recursions as done above provides better insight into the underlying structure.

3 Heuristic motivated by the Kalmanson solvable case

Given an instance of the 2-TSP with a symmetric \( n \times n \) distance matrix \( C \) and a subset \( S \) of customers to be visited twice, one can find a feasible solution to the problem by using recursions (2)–(4). The so obtained solution is the best among exponentially many feasible solutions and is optimal if \( C \) is a Kalmanson matrix. Obviously, for an arbitrary matrix \( C \) the solution found is not necessarily optimal. Clearly, permuting rows and columns in matrix \( C \), or in other words renumbering the cities in the problem, yields a new exponential neighbourhood searched in (2)–(4), and hence a potentially new feasible solution. We describe here a heuristic that uses this idea for finding a solution for the 2-TSP.

One of the intuitive approaches to number the cities is to choose for this a heuristic solution to the TSP. To find a good TSP tour, we use a combination of the simplest and the best know TSP heuristics, nearest neighbour heuristic and the 2-opt heuristic (see e.g. Chapters 5 and 7 in [16]). The tour found is used for renumbering the rows and columns in the distance matrix, and recursions (2)–(4) are then used to find a solution to the 2-TSP.

This heuristic is formally presented in the pseudo-code below. We call it Kalmanson Sequence (KS-)Heuristic. The heuristic takes as input an \( n \times n \) matrix \( C \), subset \( S \), node \( \text{start} \), and returns a sequence \( Q(S) \) which is a heuristic solution to the 2-TSP with the distance matrix \( C \). The points from set \( S \) are visited in each of the two tours in \( Q \).

**KS-Heuristic for the balanced 2-TSP(\( n, C, S, \text{start}, Q \))**

\[
\begin{align*}
&\text{KS-Heuristic for the balanced 2-TSP(} n, C, S, \text{start}, Q \text{)} \\
&\{ \text{Starting with node } \text{start} \text{ find the nearest neighbour tour on the set } \{1, \ldots, n\}; \\
&\quad \text{Apply 2-opt to improve the tour; use obtained tour } \tau \text{ for renumbering the cities;} \\
&\quad \text{Construct sequence } Q(S) \text{ with two 2-TSP tours by applying recursions (2)–(4) to the permuted matrix } C_\tau; \\
&\quad \text{Apply 2-opt to each of the tours in } Q(S); \text{ save the record;} \\
&\quad \text{return the record, i.e. the best } Q(S) \text{ found; } \\
\}
\end{align*}
\]
We run computational experiments to test the performance of the KS-Heuristic. In the experiments we recorded the best solutions found by KS in \( n \) runs (on the number of possible start points in the nearest neighbour heuristic).

The only published benchmark instances for the balanced 2-TSP are the instances from Bassetto & Mason [5]. They considered a Euclidean version of the balanced 2-TSP. A short summary of their approach is as follows. First a TSP tour on the set of all customers, called a general tour (GT), is constructed. The GT is used to obtain a partition of the set of customers into two sub-tours. The initial partition of GT into two sub-tours is improved by applying decision rules motivated by geometry (e.g. solutions get improved by removing crossing edges). For each of the sub-tours, an optimal TSP tour is constructed by applying an exact TSP algorithm (see [1], Chapter 16, and [11]). The authors also use visualisation and human intervention for improvements of the solutions found by the algorithms.

The set of benchmark instances from [5] contains 60 randomly generated instances with 48 customers each. The set of these instances is divided into three subsets with a different number of customers to be visited in both periods: 8, 12, and 24 customers. We use these instances to test our KS-Heuristic. The summary of the computational experiments is presented in Table 1. We compare our results with both types of solutions presented in [5]: with the best solutions found by a computer, which we refer to as “PC” solutions, and with the solutions obtained after visualisation and additional human intervention referred to as “PC/h”. For each set of 20 instances, Table 1 shows the mean of the percentage above the best solution, the percentages for the “best” and “worst” instance, and the number of instances where we improved the previously best solution, found solutions with the same value, or failed to improve the best known solution (cf. the rows \#(<, =, >) in the tables). A negative percentage means that the best known solution was improved. Results for individual test instances can be found in Appendix A.

For our experiments we used a laptop with an Intel(R) i5-8350U 1.70 GHz CPU and 16 GB of RAM. The algorithms were implemented in C++ and compiled with MinGW-w64. Computational times for the instances were less than 30 milliseconds ([5] mentioned times of a “few seconds” in their experiments).

As Table 1 shows, the suggested heuristic improves solutions for 22 out of 60 instances tested in [5], with the quality of other solutions not far away from 1% accuracy. For the case with 8 fixed points our results are on overall better than the results in [5]. This backs up our belief that polynomially solvable cases of hard optimization problems can provide the basis for the construction of heuristics with a performance which is competitive with other approaches.

For the solutions obtained by visualisation of the computer solutions and follow-up human intervention our heuristic improved 6 out of 60 results. Still the average quality of our solutions which are obtained in milliseconds is within 2% from the previously found solutions. In the next section we describe a new heuristic which is able to further improve many of the published solutions.

The DP recursions (2)–(4) make use of the structure “one sequence - two tours” and simultaneously optimise an allocation of nodes to tours and the sequence of the nodes in the tours. We generalise this approach and develop a heuristic which is based on the same methodology and uses the well-known Held and Karp [22] DP recursions for the TSP with arbitrary distance matrices.
4 “One sequence - two tours” heuristic for the 2-TSP

4.1 Formulation of the 2-TSP as a vehicle routing problem

In this section we model the 2-TSP within the framework of a vehicle routing problem (VRP) with two vehicles (2-VRP). We believe that the 2-VRP model is better suited for presenting our ideas.

Consider the 2-TSP with \( n \) nodes with \(|S|\) of them to be visited twice. Let one of the nodes from set \( S \) be a depot and renumber the rest of the nodes as 1,2,\ldots,\( n-1 \). Let \( F_1 \subset \{1,\ldots,n-1\} \) be the set of nodes to be visited twice. Define \( F_2 := F_1 \) and then renumber the nodes in \( F_2 \) as \( \{n,n+1,\ldots,N := n + |S| - 2\} \). In the context of the 2-VRP, set \( F_1 \) is the set of fixed customers to be visited by vehicle one, \( F_2 \) is the set of fixed customers to be visited by vehicle two.

Let \( W_1 \) and \( W_2 \) be the capacities of vehicles one and two. If each customer \( i \) has a unit demand \( w(i) = 1 \), then the balancing constraint requiring the same number of customers visited by each vehicle can be met by setting the capacities as \( W_1 = W_2 = \lfloor (n - 1 + |S| + p)/2 \rfloor \), with \( p = 1 \).

In this manner we end up with a 2-VRP instance with one depot and the set of customers \( \{1,2,\ldots,N\} \). Set \( F_1 \) is the set of customers to be always allocated to the vehicle one, and \( F_2 \) is the set of customers to be always allocated to the vehicle two \( (F_1 \cap F_2 = \emptyset) \). Notice, in real life applications one often needs to allocate customers to particular vehicles, e.g. due to a restricted site access or drivers’ preferences.

4.2 Vehicle routing problem with customers as subsets

Our approach to solving the 2-VRP uses the well-known Held & Karp [22] DP algorithm for the TSP. This is an exact algorithm and naturally can solve only small size instances. To deal with large size instances, we will repeatedly combine several customers into a single customer. With a new customer we will associate a new demand which is the total demand of the customers combined. We also need to consider an extra travel distance while visiting a new customer. It may be helpful to use the following interpretation.

Table 1: Summary of computational experiments for the KS-Heuristic.

|                      | PC     | PC/h   |
|----------------------|--------|--------|
| **8 nodes visited twice** |        |        |
| Mean %               | -0.56  | 1.18   |
| Best %               | -6.11  | -1.18  |
| Worst %              | 3.7    | 4.92   |
| # (<,=,>)            | (15,0,5) | (3,0,17) |
| **16 nodes visited twice** |        |        |
| Mean %               | 0.72   | 1.78   |
| Best %               | -1.74  | -1.54  |
| Worst %              | 4.21   | 6.31   |
| # (<,=,>)            | (6,0,14) | (3,0,17) |
| **24 nodes visited twice** |        |        |
| Mean %               | 1.25   | 1.88   |
| Best %               | -0.04  | 0.32   |
| Worst %              | 3.16   | 4.65   |
| # (<,=,>)            | (1,0,19) | (0,0,20) |
| **Total**            | (22,0,38) | (6,0,54) |
We assume that each customer $i$ is located in an estate with only two entry points. To
distinguish between two entry points to the estate, we refer to one of the entry points as the
left end and denote it by $L(i)$. The other of the entries is referred to as the right end, and
is denoted by $R(i)$. Representing the locations in the described way makes the task of data
aggregation more intuitive. Assume that there is a collection of customers which are located
in the same estate. If it is possible for these customers to be serviced by one vehicle, then
all these customers can be substituted by one new customer whose demand equals the total
demand of the customers. The distance from $L(i)$ to $R(i)$ (and from $R(i)$ to $L(i)$, since the
distances are symmetric), denoted as $l(i)$, can be calculated as the length of the shortest path
visiting all customers in the estate.

Summarising, each customer in our model will have four attributes: the left end $L(i)$, the
right end $R(i)$, the distance $l(i)$ of travelling between the left and the right ends, and the
demand $w(i)$.

It will be convenient to assume that one vehicle travels from depot $d^1$ to depot 0, and
another vehicle travels from depot 0 to depot $d^2$. Depot 0 will be considered as a special cus-
tomer with the set of attributes $\{ L(0), R(0), l(0) := 0, w(0) := 0 \}$. The reason for introducing
the customer 0 is to use it for separating customers and allocating them to the two different
vehicles. This restricts positions for placing customer 0: the total demand of customers visited
by each vehicle should not exceed the vehicle capacities. This constraint can be easily dealt
with in the DP recursions.

The main idea of our approach is to view two routes for the two vehicles as a single two-
vehicle route ("one sequence - two routes"), and then to apply an algorithm for the TSP to
find an optimal route. In the literature on the VRP the term route is used widely, therefore
from now on we use both terms, route and tour.

4.3 Dynamic programming recursions
In this section we adapt the Held & Karp [22] DP recursions for finding the shortest path from
d$^1$ to $d^2$ through the set of customers $U = \{ 0 \} \cup \{ 1, 2, \ldots, N \}$. In this path the customers from
set $F_1$ are placed before customer 0, the customers from set $F_2$ are placed after customer 0.
Moreover, the total demand of customers placed before customer 0 does not exceed capacity
$W_1$, the total demand of customers placed after customer 0 does not exceed capacity $W_2$.

Let $J$ be a subset of customers not containing $i$, so $J \subseteq U \setminus \{ i \}$. Let $VL[i, J]$ be the minimum length of an optimal 2-vehicle route among all routes that start visiting customer $i$
from the left end, then visiting all the customers in set $J$, and stopping in depot $d^2$. Similarly,
define $VR[i, J]$ to be the length of the optimal route that starts visiting customer $i$ from the
right end. Although we are dealing with symmetric distance matrices here, it is important
while sequencing the customers to distinguish whether we pass a customer from the left or
from the right end. The optimal length of the 2-vehicle route can be calculated as
\[
V = \min_{i \in U \setminus \{ 0 \} \cup F_2} \{ c_{d^1, L(i)} + VL[i, U \setminus \{ i \}], c_{d^1, R(i)} + VR[i, U \setminus \{ i \}] \}.
\]  

In the formula above, $i$ is the first customer in the path. We assume that the total demand
of customers is greater than the capacity of each of the vehicles, so item 0 cannot be the first
customer. Items from set $F_2$ are also excluded from the consideration in this step.

Values $VL[i, J]$ and $VR[i, J]$ for all items $i$ and subsets $J$, $J \subseteq \{ 0 \} \cup \{ 1, 2, \ldots, N \}$, are
calculated as shown in the recursions below. In the preprocessing step we calculate the total
demand \( w(J) \) for subsets \( J \) and define initially all values \( \text{VL}[i, J] \) and \( \text{VR}[i, J] \) to be infinity.

\[
\text{VL}[i, J]_{i \neq 0} = \begin{cases} 
\min_{j \in J \setminus F_2} \left\{ l(i) + c_{R(i),L(j)} + \text{VL}[j, J \setminus \{j\}] \right\} \quad & \text{if } 0 \in J \\
\min_{j \in J \setminus F_1} \left\{ l(i) + c_{R(i),R(j)} + \text{VL}[j, J \setminus \{j\}] \right\} \quad & \text{if } 0 \notin J, w(\{i\} \cup J) \leq W_2
\end{cases}
\] (6)

\[
\text{VR}[i, J]_{i \neq 0} = \begin{cases} 
\min_{j \in J \setminus F_2} \left\{ l(i) + c_{R(i),L(j)} + \text{VL}[j, J \setminus \{j\}] \right\} \quad & \text{if } 0 \in J, \\
\min_{j \in J \setminus F_1} \left\{ l(i) + c_{R(i),R(j)} + \text{VL}[j, J \setminus \{j\}] \right\} \quad & \text{if } 0 \notin J, w(\{i\} \cup J) \leq W_2
\end{cases}
\] (7)

\[
\text{VL}[0, J] = \min_{j \in J \setminus F_1} \left\{ c_{R(0),L(j)} + \text{VL}[j, J \setminus \{j\}] \right\} \quad \text{if } w(U \setminus J) \leq W_1, w(J) \leq W_2.
\] (8)

The boundary conditions are:

\[
\text{VL}[i, \emptyset]_{i \notin \{0\} \cup F_1} = l(i) + c_{R(i),d^2}, \quad \text{VR}[i, \emptyset]_{i \notin \{0\} \cup F_1} = l(i) + c_{L(i),d^2}.
\] (9)

In the recursions above, if subset \( J \) does not contain item 0, we eliminate from the calculations the items which are in \( F_1 \). If subset \( J \) contains item 0, we eliminate the items which are in \( F_2 \). It ensures that the fixed items are properly positioned. Since there are only two vehicles, the capacity constraint is also easily checked without any additional calculations.

For instances with small sizes (\( N \leq 20 \) in our experiments) the system \([5, 6] \) was solved within seconds. However when the number of customers increases, the computations turn out to be too time consuming which is hardly surprising. To make the DP approach practical for larger instances as well, we suggest the heuristic approach described below.

### 4.4 Aggregation strategy and local search

Consider the 2-VRP with \( N \) customers, where \( N \) is big enough to make the DP recursions computationally intractable. We start with a simple heuristic to find a feasible solution. We then “disassemble” this solution into a small number of sub-paths, and represent each sub-path as a customer. Recalculation of the attributes associated with the new customer \( i \) is straightforward. The left end \( L(i) \) for the new customer is the first node in the sub-path; the right end \( R(i) \) of the new customer is the last node in the sub-path; the demand \( w(i) \) is the total demand of customers in the sub-path; value \( l(i) \) is the length of the sub-path.

We then apply recursions \([3, 4] \) and find an optimal solution to the small-size 2-VRP with the new set of customers. The optimal solution found can now be transformed into a solution for the initial 2-VRP by replacing the aggregated customers with the original sub-paths. Obviously, the so obtained solution to the initial 2-VRP instance will in general not be optimal. Note however that the obtained solution can be “disassembled” again to get a new small-size instance. This process is repeated until a certain stopping criterion is satisfied.
We suggest the following approach for disassembling, which we call the sliding subset method. Assume that we have an initial solution to the 2-VRP which we present as a two vehicle route: $Q = \langle d_1, t_{11}^1, t_{12}^1, \ldots, t_{1k_1}^1, 0, t_{21}^2, t_{22}^2, \ldots, t_{2k_2}^2, d_2 \rangle$. Here the route of vehicle 1 is $\langle d_1, t_{11}^1, t_{12}^1, \ldots, t_{1k_1}^1, 0 \rangle$ and the route of vehicle 2 is $\langle 0, t_{21}^2, t_{22}^2, \ldots, t_{2k_2}^2, d_2 \rangle$. Fig. 2 illustrates the concept of sliding subset for an instance with $k_1 = 6$ and $k_2 = 5$.

We disassemble the initial solution into a new set of customers as follows. First, we choose customer 0 as a customer in the new 2-VRP instance, and delete it from $Q$. Let $s$ be a small integer constant, a parameter of the algorithm ($s = 2$ in Fig. 2). Choose two subsets of customers $S_1$ and $S_2$ containing $s$ items each. The customers in each subset are at consecutive positions in $Q$. Subset $S_1$ will always contain at least one customer from the route of vehicle 1, and $S_2$ contains at least one customer from the route of vehicle 2. By defining subsets in this way we try to always have a possibility to move the customers between the vehicles (unless the customers are fixed).

In the first disassembling phase define $S_1 = \{t_{11}^1, \ldots, t_{1s}^1\}$, and $S_2 = \{t_{1k_1-s+1}^1, \ldots, t_{1k_1}^1, t_{21}^2\}$. Delete the nodes in $S_1$ and $S_2$ from $Q$ and add them to the set of customers in the new 2-VRP instance. Depot $d_2$ stays as depot in the new problem; delete it from $Q$. The sub-paths which are left in $Q$ after the deletions will be replaced by aggregated customers and added to the new 2-VRP instance. These customers are: depot $d_1$ (as a sub-path with a single node), the sub-path $\langle t_{s+1}^1, \ldots, t_{k_1-s+1}^1 \rangle$ and the sub-path $\langle t_{2s}^2, \ldots, t_{2k_2}^2 \rangle$.

We redefine the subset $S_2$ by deleting $l$ first elements from $S_2$ and adding $l$ new elements; $l$ is a parameter, to which we refer as step. We repeat the process until we reach the end of the route. When subsets $S_2$ of the described type have been enumerated, we change set $S_1$ (with the step $l$) and redefine set $S_2$ to follow set $S_1$ in a manner as described above.

If the solution to the initial 2-VRP instance is improved, it is saved as the record and the process of disassembling is applied to the new solution. The procedure stops when all feasible subsets $S_1$ and $S_2$ got enumerated and no improvements were found.

To keep the size of all small 2-VRP instances the same, we slightly modified the disassembling step used in our implementation (for further details see Appendix B).
5 Computational experiments

For the sliding subset heuristic with the size of sliding subsets $s$ and step $l$, we use the notation $H_{s,l}$. In computational experiments reported in this section we test the influence of parameters $s$ and $l$ on the quality of solutions obtained as well as the importance of initial solutions. The algorithm used in the experiments is formally described below.

One Sequence - Two Tours Heuristic for the 2-VRP($n$, $C$, $s$, $l$)

\{
  \text{ until (stop criteria is satisfied) do }
  \{
    \text{ Generate next feasible sequence } Q \text{ containing two tours of the 2-TSP; }
    \text{ while improvement found do }
    \{
      \text{ Apply recursions (6)–(9) implemented in } H_{s,l} \text{ to improve the tours in } Q; \\
      \text{ Apply 2-opt to each of the two tours; save the record; }
      \text{ Swap tours in } Q \text{ to apply (6)–(9) to the new ordering in the next iteration; }
    \}
    \}
  \}
  \text{ return record, i.e. the best } Q \text{ found; }
\}

The stop criteria in the algorithm above will be defined as the number of iterations or as the total time limit for all iterations.

On each step in the improvement loop of the algorithm, recursions (6)–(9) are used repeatedly in the disassemble-assemble steps in heuristic $H_{s,l}$ as described in Section 4.4. The outcomes of these steps can be different, if the order of tours in sequence $Q$ is changed. Hence the swap step in the algorithm.

Two different approaches are used to generate feasible sequences $Q$. In the first set of experiments, the KS-heuristic (see Section 3) is used to generate sequences $Q$. The number of start points in the KS-Heuristic is the stop criteria in this case. The number of start points is bounded by the number of nodes in the problem. Moreover, for some of the start points the combination of the nearest neighbour and 2-opt heuristics may produce the same solutions. On the other hand, there is an exponentially many feasible solutions partitioning the initial set of points into two routes in the sequence $Q$. Motivated by these observations we also considered a random partitioning of nodes in two routes as an alternative procedure for generating sequences $Q$. In this case the number of possible iterations is practically unrestricted. It can be restricted by a chosen constant or by the bound on the total time the algorithm is allowed to run.

Several sliding subset heuristics have been chosen for the experiments. To manage the computational time, different numbers of initial solutions have been chosen for different heuristics. The heuristics are: $H_{4,2}$ (run with 48 initial solutions), $H_{5,3}$ (36 initial solutions), $H_{6,4}$ (24 initial solutions), and $H_{7,5}$ (12 initial solutions). The set of 60 benchmark problems from [5] have been used.

Table 2 summarises results of the first set of experiments where the initial solutions have been obtained by the KS-heuristic. Notation $[KS + H_{4,2}] \times 48$ means that the corresponding columns contain the summary of results where the KS heuristic was followed by $H_{4,2}$; 48 is the number of generated initial solutions. The other notations in the table are the same as the notations used in Table 1 in Section 3.

In the second set of computational experiments we tested the importance of good initial solutions. Instead of KS-Heuristic for generating initial solutions the following Random Partition (RP) heuristic is used.
RP-Heuristic to get a random feasible 2-TSP solution\( (n, C, S, Q) \)\
\[
\{ \\
\quad \text{Create a copy of set } S \setminus \{1\}; \text{ add the copies to the set } \{1, \ldots, n\}; \\
\quad \text{Use C++ function to \textit{random\_shuffle} the set of indices } \{2, \ldots, n + |S| - 1\}; \\
\quad \text{Transform the obtained permutation into sequence } Q \text{ with two tours: } \\
\quad \quad \text{points from } S \text{ are left in the first tour, and their copies are in the second; } \\
\quad \text{Apply 2-opt to each of the tours in } Q; \\
\}
\]

Used the info from Table 2 we aimed to reach the same computational time as was spent on instances in the first set of experiments. Therefore the stop criteria in this case was chosen to be the computational time spent in the first set of experiments. The mean time achieved in experiments could still be slightly different from the target, so we keep this info in the table. The results of the experiments are summarised in Table 3.

Analysing the results of computational experiments and the information presented in Tables 2 and 3 (see also the table in Appendix A) we can draw the following conclusions.

- By changing parameters \( s \) and \( l \) in the heuristics \( H_{s,l} \) it is possible to trade off between the computational time and accuracy of algorithms. Increasing the number of iterations is another possibility to influence the quality of the solutions obtained;

- Surprisingly, algorithms with random initial partitions performed better than the algorithms with the initial solutions obtained with the KS-heuristic (better results in the tables are highlighted with the bold font). It means that the 2-VRP heuristic is robust to changes in the quality in initial solutions used;

- Use of both types of initial solutions would have increased the number of improved solutions. In the table shown in Appendix A the results which are obtained in the two sets of experiments are compared. For 20 benchmark instance with 24 fixed points, the suggested heuristics improved all results in [5] obtained with the algorithms. For the results obtained there with a human intervention, 17 results are improved, two results are the same, and for only one “difficult” instance (instance 52) the obtained result was 0.03\% worse than the result in [5]. By using random initial solutions with heuristic \( H_{6,4} \) we managed to improve the instance 52 solution (in 37 iterations, 170 seconds), which is now 0.24\% better than the solution in [5]. The solution found is shown in Figure 3.

6 Summary

In this paper we have considered the balanced two period TSP. For this \( \mathcal{NP} \)-hard problem, we have described a new polynomially solvable case which results from the restriction to Kalmanson matrices. Motivated by this solvable case we suggested a simple heuristic. The heuristic managed to improve more than one third of the solutions for the published instances.

In our approach to the special case and in the simple heuristic we made use of a methodological approach “one sequence - two tours”. We formulated then a new heuristic based on the same methodological principle. The core “engine” in the new heuristic is though different. Instead of DP recursions designed for a special solvable case, we used the DP recursions developed by Held and Karp for the general case of the TSP. To address the computations time and the curse of dimensionality we suggested an “assemble-disassemble” heuristic.
Table 2: Test results with initial solutions obtained by Kalmanon Sequence heuristic

|          | [KS+H_{4,2}] × 48 | [KS+H_{5,3}] × 36 | [KS+H_{6,4}] × 24 | [KS+H_{7,5}] × 12 |
|----------|--------------------|--------------------|--------------------|--------------------|
|          | PC | PC/h | PC | PC/h | PC | PC/h | PC | PC/h |
| 8 nodes  |     |      |    |      |    |      |    |      |
| visited  |     |      |    |      |    |      |    |      |
| twice    |     |      |    |      |    |      |    |      |
| Mean %   | -2.37 | -0.66 | -2.41 | -0.70 | -2.44 | -0.73 | -2.59 | -0.79 |
| Best %   | -7.31 | -1.89 | -7.31 | -2.65 | -7.31 | -1.84 | -7.31 | -2.03 |
| Worst %  | -0.36 | +0.51 | -0.36 | +0.87 | -0.24 | +0.94 | -0.39 | +0.11 |
| #(,<,>)   | (20.0,0) | (14.2,4) | (20.0,0) | (15.3,2) | (20.0,0) | (16.2,2) | (20.0,0) | (18.0,2) |
| t_{m} ≈ 29s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 70s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 166s |      |      |    |      |    |      |    |      |
| 16 nodes |     |      |    |      |    |      |    |      |
| visited  |     |      |    |      |    |      |    |      |
| twice    |     |      |    |      |    |      |    |      |
| Mean %   | -1.56 | -0.52 | -1.80 | -0.77 | -1.83 | -0.80 | -1.91 | -0.88 |
| Best %   | -3.77 | -1.84 | -3.93 | -2.59 | -3.89 | -2.41 | -3.89 | -2.24 |
| Worst %  | +0.22 | +1.00 | -0.10 | +1.23 | -0.30 | +0.94 | -0.49 | +0.70 |
| #(,<,>)   | (19.0,1) | (15.0,5) | (20.0,0) | (18.0,2) | (20.0,0) | (18.0,2) | (20.0,0) | (17.0,3) |
| t_{m} ≈ 30s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 83s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 206s |      |      |    |      |    |      |    |      |
| 24 nodes |     |      |    |      |    |      |    |      |
| visited  |     |      |    |      |    |      |    |      |
| twice    |     |      |    |      |    |      |    |      |
| Mean %   | -0.44 | +0.18 | -0.97 | -0.35 | -0.98 | -0.36 | -0.97 | -0.35 |
| Best %   | -2.39 | -0.64 | -3.80 | -1.41 | -4.44 | -1.49 | -3.70 | -1.43 |
| Worst %  | +0.49 | +0.88 | +0.32 | +0.80 | +0.62 | +1.30 | +0.37 | +0.72 |
| #(,<,>)   | (10.0,4) | (10.0,10) | (19.0,1) | (14.1,5) | (18.0,2) | (14.1,5) | (18.0,2) | (12.2,6) |
| t_{m} ≈ 32s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 91s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 230s |      |      |    |      |    |      |    |      |
| Total    | #(,<,>) | (55.0,5) | (39.2,19) | (59.0,1) | (47.4,17) | (58.0,2) | (48.3,9) | (58.0,2) | (47.2,11) |

Table 3: Test results with initial solutions obtained by Random Partition heuristic

|          | [RP+H_{4,2}] | [RP+H_{5,3}] | [RP+H_{6,4}] | [RP+H_{7,5}] |
|----------|--------------|--------------|--------------|--------------|
|          | PC | PC/h | PC | PC/h | PC | PC/h | PC | PC/h |
| 8 nodes  |     |      |    |      |    |      |    |      |
| visited  |     |      |    |      |    |      |    |      |
| twice    |     |      |    |      |    |      |    |      |
| Mean %   | -2.20 | -0.49 | -2.45 | -0.74 | -2.56 | -0.82 | -2.52 | -0.81 |
| Best %   | -6.90 | -2.51 | -7.19 | -2.65 | -6.85 | -2.44 | -7.31 | -2.77 |
| Worst %  | +1.71 | +1.71 | +1.60 | +1.60 | +0.28 | +0.39 | +0.28 | +0.75 |
| #(,<,>)   | (19.0,1) | (11.1,8) | (19.0,1) | (14.2,4) | (19.0,1) | (15.1,4) | (19.0,1) | (13.0,7) |
| t_{m} ≈ 29s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 83s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 160s |      |      |    |      |    |      |    |      |
| 16 nodes |     |      |    |      |    |      |    |      |
| visited  |     |      |    |      |    |      |    |      |
| twice    |     |      |    |      |    |      |    |      |
| Mean %   | -1.75 | -0.71 | -1.88 | -0.85 | -2.09 | -1.07 | -1.90 | -0.87 |
| Best %   | -3.75 | -2.41 | -3.75 | -2.41 | -3.75 | -2.41 | -3.86 | -2.52 |
| Worst %  | -0.31 | +0.41 | -0.07 | +0.72 | -0.52 | +0.19 | -0.07 | +2.09 |
| #(,<,>)   | (20.0,0) | (16.0,4) | (20.0,0) | (17.0,3) | (20.0,0) | (17.0,3) | (20.0,0) | (16.1,3) |
| t_{m} ≈ 29s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 109s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 205s |      |      |    |      |    |      |    |      |
| 24 nodes |     |      |    |      |    |      |    |      |
| visited  |     |      |    |      |    |      |    |      |
| twice    |     |      |    |      |    |      |    |      |
| Mean %   | -0.93 | -0.31 | -1.08 | -0.46 | -1.02 | -0.40 | -1.03 | -0.41 |
| Best %   | -3.34 | -1.50 | -3.67 | -1.11 | -3.48 | -1.52 | -3.98 | -1.52 |
| Worst %  | +0.64 | +0.94 | +0.76 | +0.76 | +0.76 | +0.76 | +0.74 | +0.76 |
| #(,<,>)   | (16.0,4) | (12.1,7) | (18.0,2) | (15.1,4) | (19.0,1) | (15.0,5) | (19.0,1) | (13.2,5) |
| t_{m} ≈ 31s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 91s |      |      |    |      |    |      |    |      |
| t_{m} ≈ 229s |      |      |    |      |    |      |    |      |
| Total    | #(,<,>) | (55.0,5) | (39.2,19) | (57.0,3) | (46.3,11) | (58.0,2) | (47.1,2) | (58.0,2) | (46.2,12) |

14
Computational experiments on test instances have shown good performance: solutions for 58 out of 60 published benchmark instances have been improved (we refer here to the benchmark results obtained by algorithms in [5]).

Our new 2-TSP heuristic has been described by using the terminology for the VRP with two vehicles. We believe that the underlying idea can be used to deal with the case of a larger number of vehicles. We have successfully used this approach in real life applications (to be reported in a forthcoming work) and also successfully took part in international contests in logistic optimizations (third prize at the 2015 VEROLOG competition). We believe that our approach can be used in a variety of VRP settings. This will be the subject of future work.

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A Appendix: Results of computational experiments

Results for the instances with 24 nodes visited twice (48 initial solutions)

| Instance | Solutions from [5] | KS × 48 | [KS + H5,3] × 36 | [RP + H5,3] |
|----------|-------------------|---------|-----------------|-------------|
| PC       | PC/h              | length  | time (s)        | length      |
| I_{41}   | 30253             | 30147   | 30666           | 71          | 30349 | 91 | 30062 |
| I_{42}   | 33008             | 31920   | 31754           | 93          | 31233 | 90 | 31194 |
| I_{43}   | 31500             | 30351   | 29882           | 86          | 27843 | 90 | 27780 |
| I_{44}   | 30313             | 30170   | 30317           | 71          | 31233 | 90 | 30913 |
| I_{45}   | 27986             | 27974   | 28982           | 64          | 27843 | 90 | 27780 |
| I_{46}   | 30073             | 30170   | 30017           | 88          | 30017 | 90 | 30013 |
| I_{47}   | 31004             | 30942   | 30507           | 96          | 31163 | 90 | 30977 |
| I_{48}   | 33663             | 33185   | 34728           | 159         | 33181 | 91 | 33088 |
| I_{49}   | 31266             | 31266   | 31236           | 97          | 31236 | 90 | 31504 |
| I_{50}   | 33722             | 33627   | 33412           | 98          | 33412 | 90 | 33384 |
| I_{51}   | 32353             | 32280   | 32323           | 99          | 32323 | 90 | 32290 |
| I_{52}   | 33287             | 33200   | 33086           | 104         | 33086 | 91 | 32883 |
| I_{53}   | 31973             | 31600   | 31370           | 96          | 31370 | 91 | 31368 |
| I_{54}   | 33837             | 33507   | 33507           | 63          | 33507 | 90 | 33560 |
| I_{55}   | 32696             | 32246   | 32526           | 104         | 32526 | 90 | 31432 |
| I_{56}   | 31549             | 31549   | 31487           | 96          | 31487 | 91 | 30183 |
| I_{57}   | 33287             | 33200   | 33086           | 104         | 33086 | 91 | 32883 |
| I_{58}   | 31973             | 31600   | 31370           | 96          | 31370 | 91 | 31368 |

B Appendix: On implementation of the sliding subset approach

Figure 4 illustrates the various steps (and possibilities) of the disassembling process for the case $s = 2$ and $l = 2$.

Fig. 4(a) illustrates the first step of the disassembling process as was described above. Sets $S_1$ and $S_2$ are separated by one sub-path in this case. The size of the small 2-VRP is $2s + 5$.

Fig. 4(b) illustrates the outcome of disassembling the route on the next iteration. Notice that depot 0 is always treated as a separate customer, therefore setting $l = 2$ yields the position of $S_2$ as shown in the figure. There are two sub-paths between sets $S_1$ and $S_2$, and the size of the small 2-VRP is $2s + 6$.

Consider the step when $S_1$ and $S_2$ are chosen as shown in Fig. 4(c). If the first sub-path did not contain the depot, the size of the problem would have been $2s + 7$. It means that we can end up with instances with $2s + 5$, $2s + 6$, and $2s + 7$ customers.

For the implementation, it was convenient to keep the size of the instances fixed at $2s + 6$. Therefore we decided (1) to “glue” the first sub-path with the depot and define it as the depot in the new problem, as shown in Fig. 4(c); (2) in case when subsets $S_1$ and $S_2$ are separated by a single path (for example, in the first step of disassembling), the last node in the sub-path is considered as a sub-path with one node: in this case the instance with $2s + 5$ customers becomes an instance with $2s + 6$ customers (compare Fig. 4(a) and Fig. 4(d)).
Figure 4: Illustrations of disassembling: (a) first step: $S_1$ and $S_2$ are separated by one sub-path only; (b) an example when $S_1$ and $S_2$ are separated by two sub-paths and a depot; (c) depot and the first sub-path are considered as one customer; (d) modified first step: a sub-path between $S_1$ and $S_2$ (case (a)) is partitioned into a sub-path and a single customer to keep the size of the new 2-VRP fixed.

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