Decentralized non-convex optimization via bi-level SQP and ADMM

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Abstract—Decentralized non-convex optimization is important in many problems of practical relevance. Existing decentralized methods, however, typically either lack convergence guarantees for general non-convex problems, or they suffer from a high subproblem complexity. We present a novel bi-level SQP method, where the inner quadratic problems are solved via ADMM. A decentralized stopping criterion from inexact Newton methods allows the early termination of ADMM as an inner algorithm to improve computational efficiency. The method has local convergence guarantees for non-convex programs. Moreover, it only solves sequences of Quadratic Programs, whereas many existing algorithms solve sequences of Nonlinear Programs. The method shows competitive numerical performance for an optimal power flow problem.

I. INTRODUCTION

Decentralized optimization methods, i.e. methods which solve optimization problems purely based on neighbor-to-neighbor communication, are of interest in many applications such as Optimal Power Flow (OPF) and distributed Model Predictive Control (MPC) [1, 2]. In many cases, these applications require to solve optimization problems with non-convex objectives and non-convex constraints. At large, existing algorithms can be categorized as follows: a) they are decentralized, but lack guarantees for problems with non-convex constraints; b) they are not decentralized, i.e. they require central communication/coordination; or c) they solve non-convex Nonlinear Programs (NLP)s at a subsystem level, which increases complexity and impedes convergence guarantees.

The Alternating Direction Method of Multipliers (ADMM) is a decentralized method, which shows promising performance for a large variety of problems. ADMM is guaranteed to converge only for special classes of non-convex problems [3]. Decentralized schemes with global convergence guarantees based on augmented Lagrangian methods are presented in [4, 5]. Whereas [4] assumes polynomial objectives and equality constraints, [5] allows for more general constraints. A distributed method with local convergence guarantees for non-convex problems is the Augmented Lagrangian Alternating Direction Inexact Newton (ALADIN) method [6]. Bi-level ALADIN variants employ inner algorithms to decompose the coordination step of ALADIN [7]. All the above methods require solving constrained non-convex NLPs to optimality, which can often not be guaranteed by numerical solvers for an arbitrary initialization [8].

An essentially decentralized interior point method with local convergence guarantees for general non-convex problems is proposed in [9]. Therein, the solution of NLPs by subsystems is avoided. However, the step-size selection requires scalar global communication. Tailored algorithms for distributed NMPC can be found e.g. in [2, 10]. These algorithms also suffer from either of the above drawbacks.

For centralized non-convex optimization, Sequential Quadratic Programming (SQP) has local or global convergence guarantees [8]. Bi-level algorithms with SQP on the outer level and ADMM as an inner algorithm are presented in [11, 12]. The scheme of [11] has global convergence guarantees for problems with non-convex objective and linearly coupled subproblems. However, it does not allow for local constraints or inequality constraints. In [12], non-convex objectives and non-convex local constraints are considered, but no convergence guarantees are established. Both approaches assume that ADMM solves the subproblem Quadratic Programs (QP)s to optimality in each SQP step, which is unrealistic in many applications. A further bi-level approach with ADMM as an inner method is presented in [13] for a Sequential Convex Programming-based distributed MPC scheme.

The focus of this work is a bi-level decentralized SQP method (d-SQP) with ADMM as inner algorithm. As the proposed scheme combines SQP with inner decentralized ADMM we obtain a fully decentralized algorithm. This paper presents two contributions. First, in contrast to the bi-level SQP method in [11], we establish convergence for nonlinear programs with non-convex objective and non-convex constraints. Second, we derive a novel stopping criterion for the inner ADMM iterations that guarantees local convergence of d-SQP despite inexact QP solutions. This is crucial as it may not be computationally feasible to obtain an exact subproblem solution via ADMM in each SQP step.

The paper is organized as follows: Section II introduces the problem formulation, SQP schemes, and ADMM. Section III derives d-SQP and establishes local convergence. Section IV analyses numerical results from OPF.

Notation: Given a matrix $A$ and an integer $j$, $[A]_j$ denotes the $j$th row of $A$. For an index set $A$, $[A]_A$ denotes the matrix consisting of rows $[A]_j$ for all $j \in A$. Likewise, $[a]_j$ is the $j$th component of vector $a$ and $a_A$ is the vector of components $[a]_j$ for all $j \in A$. The concatenation of vectors $x$ and $y$ into a column vector is $\text{col}(x, y)$. The ε-neighborhood around a point $x$ is denoted by $B_\varepsilon(x)$, i.e., $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\}$, where $\| \cdot \|$ is some norm on $\mathbb{R}^n$. The condition number of a matrix $A \in \mathbb{R}^{n \times n}$ is defined as $\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$. $I$ is the identity matrix.
II. PROBLEM STATEMENT AND PRELIMINARIES

We consider Nonlinear Programs of the form
\[
\min_{x_1, \ldots, x_S} \sum_{i \in S} f_i(x_i)
\]
subject to
\[
\begin{align*}
  g_i(x_i) &= 0 \quad \forall i \in S, \\
  h_i(x_i) &\leq 0 \quad \forall i \in S,
\end{align*}
\]
\[
\sum_{i \in S} E_i x_i = c \quad \lambda,
\]
where \( S = \{1, \ldots, S\} \) is a set of subsystems, each of which is equipped with \( x_i \in \mathbb{R}^{n_i} \) and twice continuously differentiable functions \( f_i : \mathbb{R}^{n_i} \to \mathbb{R}, g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{gi}}, \) and \( h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{hi}}. \) The matrices \( E_i \in \mathbb{R}^{n_e \times n_i} \) and the vector \( c \in \mathbb{R}^{n_e} \) couple the subsystems. The notation in (1) highlights that \( v_i \in \mathbb{R}^{n_{vi}}, \mu_i \in \mathbb{R}^{n_{hi}}, \) and \( \lambda \in \mathbb{R}^{n_{\lambda}} \) are Lagrange multipliers associated with the respective constraints. The centralized variables are \( x = \text{col}(x_i)_{i \in S}, \nu = \text{col}(\nu_i)_{i \in S}, \) and \( \mu = \text{col}(\mu_i)_{i \in S}. \) Define the Lagrangian of (1),
\[
L(x, \nu, \mu, \lambda) = \left( \sum_{i \in S} L_i(x_i, \nu_i, \mu_i, \lambda) \right) - \lambda^T c,
\]
where \( L_i(\cdot) = f_i(x_i) + \nu_i^T g_i(x_i) + \mu_i^T h_i(x_i) + \lambda^T E_i x_i. \) Then, the Karush-Kuhn-Tucker (KKT) conditions of (1) read
\[
\begin{align*}
  \nabla f_i(x_i) + \nabla g_i(x_i) \nu_i + \nabla h_i(x_i) \mu_i + \lambda^T E_i x_i &= 0, \quad i \in S, \\
  g_i(x_i) &= 0, \quad i \in S, \\
  h_i(x_i) &\leq 0, \quad \mu_i \geq 0, \quad \lambda^T E_i x_i &= 0, \quad i \in S,
\end{align*}
\]
\[
\sum_{i \in S} E_i x_i = c.
\]

A. Sequential quadratic programming

SQP methods repeatedly solve quadratic approximations of (1), cf. [8, Ch. 18]. A quadratic approximation of (1) at a primal-dual iterate \((x^k, \nu^k, \mu^k, \lambda^k)\) is
\[
\min_{s_1, \ldots, s_S} \left\{ \frac{1}{2} \sum_{i \in S} \nabla^2 f_i(x^k) s_i \right\}
\]
subject to
\[
\begin{align*}
  g_i(x^k) + \nabla g_i(x^k) \nu_i + \nabla h_i(x^k) \mu_i + \lambda^T E_i x^k &= 0, \quad i \in S, \\
  h_i(x^k) &\leq 0, \quad i \in S,
\end{align*}
\]
\[
\sum_{i \in S} E_i x^k + s_i = c + \lambda^T E_i x^k.
\]
where \( H_i = \nabla^2 f_i(x^k), \nabla^2 g_i(x^k), \nabla^2 h_i(x^k), \lambda. \) Observe that the KKT system (2) can be written as \( F(x, \nu, \mu, \lambda) \) where
\[
F(x, \nu, \mu, \lambda) =
\begin{bmatrix}
\nabla x_i L_i(x_i, \nu_i, \mu_i, \lambda) \\
\nabla x_i g_i(x_i) \\
\nabla x_i h_i(x_i) \\
\n\sum_{i \in S} E_i x_i - c
\end{bmatrix}.
\]
The block rows \( \min(-h_i(x), \mu_i) \) represent (2c). Algorithm 1 summarizes an SQP method for solving (1).

Algorithm 1 Inequality-constrained SQP for solving (1)

1: Initialization: \( x_i^0, \nu_i^0, \mu_i^0, \lambda^0 \) for all \( i \in S, \lambda^0, \epsilon \)
2: while \( \|F^k\| \leq \epsilon \) do
3: \( \) compute \( \nabla f_i(x_i^k), \nabla g_i(x_i^k), \nabla h_i(x_i^k), H_i^k, \) \( i \in S \)
4: solve QP (3)
5: \( x_i^{k+1} = x_i^k + s_i, \nu_i^{k+1} = \nu_i^P, \mu_i^{k+1} = \mu_i^P, \lambda = \lambda^P \)
6: end while
7: return \( x^* \)

Define \( p = \text{col}(x, \nu, \mu, \lambda), \) let \( p^* = \text{col}(x^*, \nu^*, \mu^*, \lambda^*) \) be a KKT point of (1), and let \( A_i \) and \( I_i \) denote the sets of active and inactive inequality constraints at \( x_i^* \) respectively, i.e.,
\[
A_i = \{ j \in \{1, \ldots, n_{hi}\} | h_i(x_i^*)_j = 0 \}, \\
I_i = \{ j \in \{1, \ldots, n_{hi}\} | h_i(x_i^*)_j < 0 \}.
\]

Assumption 1: The point \( p^* \) is a KKT point of (1) which, for all \( i \in S, \) satisfies
\[
\begin{align*}
  h_i(x_i^*) + \mu_i^* &\not= 0 \quad \text{(strict complementarity)}, \\
  s_i \nabla^2 h_i(x_i^*) \lambda_i &> 0 \quad \text{for all } s_i &\not= 0 \quad \text{with } \nabla g_i(x_i^*)^T s_i = 0.2
\end{align*}
\]
Furthermore, the matrix
\[
\begin{bmatrix}
  \nabla x_i g_i(x_i^*)^T \\
  \nabla x_i (h_i(x_i^*))^A_i \\
  \nabla x_i (h_i(x_i^*))^A_S \\
  E_i \\
  E_S
\end{bmatrix}
\]
has full row rank, i.e., it satisfies the Linear Independence Constraint Qualification (LICQ).

Definition 1 (Convergence rates): We say that the sequence \( \{p^k\} \subset \mathbb{R}^{n_p} \) converges to \( p^* \in \mathbb{R}^{n_p} \)
\[
i \) q-linearly, if \( \|p^{k+1} - p^*\| \leq c \|p^k - p^*\| \quad \forall k \geq k_0 \quad \text{for some } c < 1 \quad \text{and } k_0 \geq 0.
\]

ii) q-superlinearly, quadratically, if \( \|p^k - p^*\| \leq c \|p^k - p^*\| \quad \forall k \geq k_0 \quad \text{for some } c < 1 \quad \text{and } k_0 \geq 0.
\]

Theorem 1 (Local convergence of SQP, [14, Thm. 5.31]): Let Assumption 1 hold. Then, there exists \( \epsilon_k \) such that for all \( p^k \in B_{\epsilon_k}(p^*) \) the following holds.

i) The sequence \( \{p^k\} \) generated by Algorithm 1 converges q-superlinearly to \( p^* \).

ii) If \( \nabla^2 f_i(x_i^k), \nabla^2 g_i(x_i^k), j \in \{1, \ldots, n_{gi}\}, \) \( \nabla^2 h_i(x_i^k), j \in \{1, \ldots, n_{hi}\}, \) \( i \in S, \) \( x_i^k, \) \( \lambda_i \) \{ locally Lipschitz continuous for all \( i \in S, \) then the convergence rate is q-quadratic.

The proof of Theorem 1 in [14] shows that inside \( B_{\epsilon_k}(p^*) \), the iterates generated by Algorithm 1 correspond to the iterates generated by Newton’s method applied to the nonlinear system of equations \( F(p) = 0. \) In particular, for all \( p^k \in B_{\epsilon_k}(p^*) \), \( F^k = \text{col}(F^k) \) is continuously differentiable and \( \nabla F^k = \nabla F(p^k) \) is regular. Hence the Newton iteration \( p^{k+1} = p^k + d \) with \( F^k + \nabla F^k d = 0, \) where
\[
d = \frac{(\text{col}(\nu_i^k)_{i \in S}, \Delta \lambda, \Delta \nu_i^k, \Delta \mu_i^k, \Delta x_i^k, \Delta s_i^k)}{\lambda^k},
\]
\[
d_i \equiv \text{col}(s_i, \Delta \nu_i, \Delta \mu_i, \Delta x_i^k, \Delta s_i^k) = \nu_i^P - \nu_i^k, \Delta \mu_i^k = \mu_i^P - \mu_i^k
\]
and \( \Delta \lambda = \lambda^P - \lambda^k \) is well-defined inside \( B_{\epsilon_k}, \)

2This is a slightly stronger assumption than the Second-Order Sufficient Condition as we exclude the conditions \( \nabla h_i(x_i^*)^T s_i = 0 \) and \( E_i s_i = 0. \)
B. Alternating direction method of multipliers

We next recall how to solve (3) via ADMM. First, we reformulate (3) as

\[
\begin{align*}
\min_{s_i \in \mathbb{R}^{n_i}} & \quad \sum_{i \in S} \phi_i(s_i) \quad \text{(6a)} \\
\text{subject to} & \quad s_i - \bar{s}_i = 0 \quad \forall i \in S, \quad \text{(6b)} \\
\sum_{i \in S} E_i(x^k_i + \bar{s}_i) = c \quad \lambda^{QP}, \quad \text{(6c)}
\end{align*}
\]

where \(\bar{s}_i \in \mathbb{R}^{n_i}\) is an auxiliary decision variable for each subsystem \(i \in S\), \(\phi_i(s_i) \equiv 1 / 2 s_i^T H_i s_i + \nabla f_i(x^k_i)^T s_i\) and

\[
\bar{s}_i = \begin{cases}
\begin{array}{l}
\bar{s}_i \in \mathbb{R}^{n_i} \quad g_i(x^k_i) + \nabla g_i(x^k_i)^T s_i = 0 \\
\quad h_i(x^k_i) + \nabla h_i(x^k_i)^T s_i \leq 0
\end{array}
\end{cases}.
\]

Let \(\bar{s} \equiv \text{col}(\bar{s}_i)_{i \in S}, s \equiv \text{col}(s_i)_{i \in S}\), and \(\gamma \equiv \text{col}(\gamma_i)_{i \in S}\). The augmented Lagrangian of (6) w.r.t. (6b) reads

\[
L^k_{\rho}(s, \bar{s}, \gamma) = \sum_{i \in S} L^k_{\rho_i}(s_i, \bar{s}_i, \gamma_i) = \sum_{i \in S} \phi_i(s_i) + \gamma_i(\bar{s}_i - s_i) + \frac{\rho}{2} \|s_i - \bar{s}_i\|^2.
\]

Denote the feasible set of the coupling constraints by \(E = \{\bar{s} \in \mathbb{R}^n | \sum_{i \in S} E_i(x^k_i + \bar{s}_i) = c\}\). Algorithm 2 summarizes ADMM, where the dual variables \(\mu_i^{QP}\) and \(\mu_i^{QP}\) are obtained in Step 3. Step 3 can be executed by each subsystem in parallel by solving the local QP

\[
\begin{align*}
\min_{s_i} & \quad \frac{1}{2} s_i^T (H_i + \rho I) s_i + (\nabla f_i(x^k_i) + \gamma_i - \rho \bar{s}_i)^T s_i \quad \text{(7a)} \\
\text{subject to} & \quad g_i(x^k_i) + \nabla g_i(x^k_i)^T s_i = 0 \quad \mu_i^{QP}, \quad \text{(7b)} \\
& \quad h_i(x^k_i) + \nabla h_i(x^k_i)^T s_i \leq 0 \quad \mu_i^{QP}, \quad \text{(7c)}
\end{align*}
\]

where the iterates \(\hat{s}_i^k\) and \(\gamma_i^k\) serve as parameters for the local QP of subsystem \(i \in S\).

Remark 1 (Decentralized ADMM): Step 4 in Algorithm 2 solves a QP subject to the coupling constraints and in general requires centralized computation. However, if (1) is given in so-called consensus form, and if \(\gamma_i^0\) is chosen appropriately, then Step 4 is equivalent to a decentralized averaging step, i.e., it only requires neighbor-to-neighbor communication and local computation [15, Ch. 7]. A common setup where Problem (1) is given in consensus form occurs if the constraints (1d) couple original and copied variables between neighboring subsystems, cf. [16]. Then, if the SQP initialization satisfies \(\sum_{i \in S} E_i v_i^0 = 0\), we may replace Step 4 by a decentralized averaging step.

III. DECENTRALIZED SEQUENTIAL QUADRATIC PROGRAMMING

The key idea of our approach is to solve QP (3) with ADMM. It may not be tractable to solve (3) to high precision in every SQP step. Therefore we next derive convergence results of the inexact outer SQP steps and then provide technical results for the subproblems based on well-known ADMM convergence properties.

A. Outer convergence

Theorem 1 establishes convergence if (3) is solved exactly in Step 4 of Algorithm 1. Instead, we allow inexact solutions of (3) and rely on the inexact Newton stopping criterion [17]

\[
\|F^k + \nabla F^k d\| \leq \eta_k \|F^k\| \quad \text{(8)}
\]

with \(0 < \eta_k < 1\).

Lemma 1 (Local convergence with inexact SQP steps):

Let Assumption 1 hold. Solve (3) inexactly in Step 4 of Algorithm 1, form d according to (5), and let d satisfy (8). Then there exist \(\varepsilon_2 > 0\) and \(\eta > 0\) such that for all \(p^0 \in B_{\varepsilon_2}(p^*)\) the following holds.

i) If \(\eta_k \leq \eta\), then the sequence \(\{p^k\}\) generated by Algorithm 1 converges q-linearly to \(p^*\).

ii) If \(\eta_k \to 0\), then the convergence rate is q-superlinear.

iii) If \(\eta_k = O(\|F^k\|)\) and if \(\nabla^2_{x_{i_j}} F, \nabla^2_{x_{j_i}} g_i, j \in \{1, \ldots, n_i\}\) are Lipschitz continuous inside \(B_{\varepsilon_2}\), then the convergence rate is q-quadratic.

Proof: We choose \(\varepsilon_2 \leq \varepsilon_1 \) from Theorem 1. Then, \(\nabla F\) is regular inside \(B_{\varepsilon_2}(p^*)\). To obtain statement i), we modify Theorem 2.3 from [17], which proves linear convergence in the sense that \(\|\nabla F(p^k + p^{k+1} - p^*)\| \leq t \|\nabla F(p^k)(p^k - p^*)\|\) with some \(t\) such that \(\eta < t < 1\). From the second to last inequality follows \(\|p^{k+1} - p^*\| \leq \text{cond}(\nabla F(p^k)) t \|p^k - p^*\|\) [18, Eq. 7]. Since \(\nabla F(p^*)\) is regular, \((\nabla \nabla F(p^*))\) is finite and choosing \(t\) sufficiently small yields q-linear convergence with \(c = \text{cond}(\nabla F(p^*)) t < 1\). Statements ii) and iii) follow directly from [17, Thm. 3.3].

The stopping criterion (8) guarantees local convergence of Algorithm 1 for iterates starting in \(B_{\varepsilon_2}\). However, in practice it is often not known whether \(p^0\) lies inside \(B_{\varepsilon_2}\). For an initialization outside \(B_{\varepsilon_2}\), it may not be possible to evaluate (8) as the derivatives of the block rows min(\(-h_i(x^k_i), \mu_i^{QP}\)), \(i \in S\) in (4) are ill-defined. Hence, we develop a modified stopping criterion that is equivalent to (8) inside \(B_{\varepsilon_2}\), but which can also be evaluated outside \(B_{\varepsilon_2}\). We propose the modified stopping criterion

\[
\|\tilde{F} + \nabla \tilde{F} d\| \leq \eta_k \|\tilde{F}^k\|, \quad \text{(9)}
\]

with

\[
\begin{bmatrix}
\nabla_{x_1} L_1(x_1, \nu_1, \mu_1, \lambda) \\
\vdots \\
\nabla_{x_S} L_S(x_S, \nu_S, \mu_S, \lambda) \\
\sum_{i \in S} E_i v_i^0 - c
\end{bmatrix} = \begin{bmatrix}
\nabla F_1 & \cdots & 0 & E_1^T \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \nabla F_S E_S^T \\
E_1 & \cdots & E_S & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
\nabla_{x_1} L_1(x_1, \nu_1, \mu_1, \lambda) \\
\vdots \\
\nabla_{x_S} L_S(x_S, \nu_S, \mu_S, \lambda) \\
\sum_{i \in S} E_i v_i^0 - c
\end{bmatrix} = \begin{bmatrix}
\nabla \tilde{F}_1 & \cdots & 0 & \nabla \tilde{F}_1^T \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \nabla \tilde{F}_S E_S^T \\
\tilde{E}_1 & \cdots & \tilde{E}_S & 0
\end{bmatrix}.
\]
with \( \tilde{E}_i = [E_i \ 0 \ 0] \) and \( \hat{E}_i = [E_i \ 0] \). Observe that \( \tilde{F} \) does not include the block rows \( \min(-h_i(x_k^i), \mu_k^i) \), \( i \in S \) to avoid differentiability issues outside \( B_{\epsilon} \). The subsystems can evaluate (9) locally and only communicate convergence flags, if \( \sum_i E_i(s_k^i + s_k^i) = c \) and if \( \| \cdot \|_\infty \) is chosen.

**Lemma 2 (Modified stopping criterion):** Let Assumption 1 hold, let \( p^k \in B_{\epsilon, 1}(p^*) \) with \( \epsilon \) from Theorem 1 and let \( (s_i^{QP}, \mu_i^{QP}, \lambda_i^{QP}) \) be an inexact solution to (3) which has the same active set as \( p^* \), i.e.

\[
[h_i(x_k^i)]_{ij} + [\nabla h_i(x_k^i)^T s_i^k]_{ij} = 0, \ \forall j \in A_i, \forall i \in S \quad (10a) \\
[\mu_i^{QP}]_{ij} = 0, \ \forall j \in I_i, \forall i \in S. \quad (10b)
\]

Then, \( \| F^k + \nabla F^k d \|_2 \leq \eta \| F^k \|_2 \implies \| F^k + \nabla F^k d \|_2 \leq \eta \| F^k \|_2. \)  

**Proof:** From (10) follows \( \| F^k + \nabla F^k d \|_2 \leq \| F^k + \nabla F^k d \|_2. \) Since \( \| F^k \|_2 \), we obtain the implication.

**B. Inner convergence**

To apply the stopping criterion (9), we show that ADMM stops at the correct active set in a neighborhood of \( p^* \). We first show that the optimal active sets of (1), (6), and (7) are equivalent in a neighborhood of \( p^* \), if ADMM is initialized appropriately. Let \( q = \cos(\bar{s}, \gamma) \) and \( q_0 = \cos(\bar{s}, \gamma_0). \)

**Lemma 3 (Local active set equivalence of (1), (6), (7)):** Let Assumption 1 hold, let \( p^k \in B_{\epsilon, 1}(p^*) \), and let \((s_k^{QP}, \mu_k^{QP}, \lambda_k^{QP}) \) be a KKT point of (6) formulated at \( p^k \). Then there exists \( \epsilon_3 > 0 \) such that for all \( i \in S \) the following holds. If \( q = \cos(\bar{s}, \gamma) \), then the solution \((s_i^{QP}, \mu_i^{QP})\) to (7) with parameters \( q_0 \) has the same active set as \( p^* \), i.e.

\[
[h_i(x_k^i)]_{ij} + [\nabla h_i(x_k^i)^T s_i^k]_{ij} = 0, \ \forall j \in A_i, \forall i \in S, \quad (10a) \\
[\mu_i^{QP}]_{ij} = 0, \ \forall j \in I_i, \forall i \in S. \quad (10b)
\]

**Proof:** We first show that, (a), for \( p^k \in B_{\epsilon, 1}(p^*) \) problems (1), (6), and (7) have the same active set at their respective solution, if (7) is parameterized with \( q^{k*} \). We then show that, (b), this also holds for small variations in \( q \).

(a) Assumption 1 implies that QPs (3) and (7) each have a unique KKT point. Furthermore, the KKT point of (3) has the same active set as \( p^* \) and satisfies strict complementarity for all \( p^k \in B_{\epsilon, 1}(p^*) \) [14, Thm. 5.31]. Comparing the KKT conditions of (3) and (6), we see that if \((s_k^{QP}, \mu_k^{QP}, \lambda_k^{QP}) \) is the KKT point of (3), then \((s_k^{QP}, \mu_k^{QP}, \lambda_k^{QP}) \) with \( s_k^{QP} = s_k^{QP} \) and \( \gamma^{QP} = \gamma^{QP} \) in the KKT point of (6). Comparing the KKT conditions of (6) and (7), we see that if \((s_k^{QP}, \mu_k^{QP}, \lambda_k^{QP}) \) is the KKT point of (6), then \((s_k^{QP}, \mu_k^{QP}, \lambda_k^{QP}) \) is a KKT point of (7) of subsystem \( i \) with parameters \( q^{k*} \).

(b) Since \((s_k^{QP}, \mu_k^{QP}, \lambda_k^{QP}) \) is the only KKT point of (7), it satisfies strict complementarity for all \( i \in S \):

\[
-[h_i(x_k^i)]_{ij} - [\nabla h_i(x_k^i)^T s_i^k]_{ij} < [\mu_i^{QP}]_{ij}, \ \forall j \in A_i, \\
-[h_i(x_k^i)]_{ij} - [\nabla h_i(x_k^i)^T s_i^k]_{ij} > [\mu_i^{QP}]_{ij}, \ \forall j \in I_i.
\]

Therefore, there exists \( \epsilon_4 > 0 \) such that the active set of (7) is constant for all \( \text{col}(\bar{s}_i, \mu_i^{QP}) \in B_{\epsilon, 1}(s_i^{QP}, \mu_i^{QP}) \).

**C. Local convergence of decentralized SQP**

Algorithm 3 summarizes the decentralized SQP method (d-SQP). The algorithm follows a bi-level structure and we denote the SQP (outer) iterations by \( k \) and the ADMM (inner) iterations by \( l \). We are now ready to state our main result.

**Theorem 2 (Local convergence of d-SQP):** Let Assumption 1 hold and initialize ADMM in each SQP step with \( \bar{s}_i^0 = 0 \) and \( \gamma_i^0 = E_i \lambda_k^0 \) for all \( i \in S \). Then, there exist \( \epsilon > 0 \) and \( \eta > 0 \) such that for all \( p^k \in B_{\epsilon, 1}(p^*) \) the following holds.

(i) If \( \eta^k \leq \eta \), then the sequence \( \{p^k\} \) generated by Algorithm 3 converges to \( p^* \) and the convergence rate is \( \eta \)-linear in the outer iterations.
Algorithm 3 d-SQP for solving (1)

1: SQP initialization: $k = 0, x_0^i, ν_0^i, µ_0^i$ for all $i \in S$, $λ_0, η_0, ϵ$
2: while $\|F^k\| \leq ϵ$ do
3: compute $∇f_i(x^k_i)$, $∇g_i(x^k_i)$, $∇h_i(x^k_i)$, $H^k_i$ locally
4: compute $F^k, ∇F^k$ locally
5: ADMM initialization: $l = 0, s_0^i, γ_0^i$ for all $i \in S$
6: while $\|F^k + ∇F^k d\| \leq η_0\|F^k\|$ do
7: $s_i^{l+1} = \text{argmin}_{s_i} \{L^p_i(s_i, s_i^l, γ_i^l)\}$ for all $i \in S$
8: $s_i^{l+1} = \text{argmin}_{s_i} \sum_{j \in S} L^p_i(s_i^{l+1}, s_i^l, γ_i^l)$
9: $γ_i^{l+1} = s_i^l + ρ(s_i^{l+1} - s_i^l)$ for all $i \in S$
10: $l = l + 1$
11: end while
12: $x_i^{k+1} = x_i^k + s_i^l, ν_i^{k+1} = ν_i^{QP}, µ_i^{k+1} = µ_i^{QP}, γ_i^{k+1} = γ_i^l$
13: update $η_i^{k+1}$
14: $k = k + 1$
15: end while
16: return $x^*$

ii) If $η_i \to 0$, then the convergence rate is q-superlinear in the outer iterations.
iii) If $η_i = O(\|F^k\|)$ and if $∇^2_{x,y} f_j, ∇^2_{x,y} g_i, j \in \{1, \ldots, n_{ij}\}$, and $∇^2_{x,y} h_j, j \in \{1, \ldots, n_{ij}\}$ are Lipschitz continuous inside $B_ε$, then the convergence rate is q-quadratic in the outer iterations.

Proof: We first show that, (a), the ADMM initialization $\text{col} (0, E^T λ_k)$ lies within a neighborhood of the subproblem solution $q^{k*}$ for all SQP iterations. We then show that, (b), this neighborhood can be chosen such that the ADMM iterations are at the correct active set at all iterations. Finally, (c), we invoke Lemmas 1–2 to prove convergence.

(a) We first choose $ε ≤ ε_2$. The convergence of the Newton and inexact Newton methods implies that if $p^k \in B_ε(p^*)$, then $p^{k+1} \in B_ε(p^*)$ holds for the Newton method as well as for the inexact Newton method [17, Thm. 2.3]. Therefore $\|p^{k+1} - p^k\| ≤ 2ε$. Recall that $\|\text{col}(x,y)\| ≥ |x||y|$ for any vectors $x$ and $y$. Hence, $\|\text{col}(x^{k+1} - x^k, λ^{k+1} - λ_k)\| ≤ 2ε$. Recall that $γ = E^T λ$. Hence, $\|\text{col}(x^{k+1} - x^k, γ^{k+1} - γ_k)\| ≤ 2ε$ with $ε_7 = \max(\|E^T\|, 1)$. That is, col$(0, γ_k) \in B_{2ε_7}(0, E^T λ_k)$, the iterations of the Newton method read $x^{k+1} = x^k + s^{k*}$ and $λ^{k+1} = λ_k^{QP}$ and we hence get col$(0, E^T λ_k) \in B_{2ε_7}(0, γ^{k*})$.

(b) Lemma 4 shows that the ADMM iterates $(s^i, µ^i, L^i, Q^P)$ are at the correct active set, if $q^0 \in B_{ε_2}(q^{k*})$. We therefore choose $ε ≤ \min(ε_2, ε_5/(2\max(1, \|E^T\|)))$. The ADMM initialization then satisfies $q^0 = \text{col}(0, E^T λ) \in B_{ε_2}(q^{k*})$. By Lemma 4, the iterates $(s^i, µ^i, L^i, Q^P)$ therefore are at the correct active set for all ADMM iterations $l ≥ 0$ and for all SQP iterations $k ≥ 0$.

(c) The ADMM convergence invoked in the proof of Lemma 4 ensures that ADMM can satisfy the stopping criterion (9). By Lemma 2, satisfaction of (9) together with the correct active set implies satisfaction of the inexact Newton stopping criterion (8). Lemma 1 then implies local q-convergence in the outer iterations.

IV. Numerical Results

We compare the performance of d-SQP to four other methods for an OPF problem for the IEEE 118-bus system. The first method for comparison is stand-alone ADMM, which solves (1) directly and which we denote as ADMM in the following. The second and third methods are bi-level ALADIN variants, where the ALADIN coordination QP is solved via essentially decentralized Conjugate Gradients (d-CG) or via an ADMM variant (d-ADMM). The fourth method is an essentially decentralized Interior Point method (d-IP) [9].

The problem consists of four subsystems with a total of $n = 576$ variables, $n_g = 470$ non-linear equality constraints, $n_h = 792$ linear inequality constraints, and $n_c = 52$ coupling constraints. We initialize voltage magnitudes as 1 and remaining variables as 0.

For ADMM, we tune $ρ$ for fastest convergence in the set $\{10^2, 10^3, 10^4\}$ and obtain $ρ = 10^3$. The respective subproblems in ADMM, and ALADIN are solved with IPOPT [21] and the QPs in d-SQP are solved with qpOases [22]. For d-SQP, we choose $η_i = 0.8$, $η_k^i = 0.9ρ^k$, use $ρ = 10^3$, and initialize the inner iterations with $(s_i^0, γ_i^1) = (0, E^T λ_k)$.

D. Communication requirements and discussion

Four steps of Algorithm 3 require communication between subsystems: Step 1 to initialize $λ_0$, Steps 2 and 6 to evaluate the stopping criteria, and Step 8 for the $s$ update in ADMM.

It is well-known that Step 8 may be computed efficiently as a decentralized averaging step, if (1) is a consensus-type problem, cf. [15, 16]. That is, the update of $s$ in Step 8 then requires only the communication of vectors between neighboring subsystems if we choose $γ_i^0 = E^T s_i^*$.

If $\|\cdot\|_∞$ is chosen, then the stopping criteria can be evaluated locally and in Steps 2 and 6 only convergence flags must be communicated, because the update $x^{k+1} = x^k + s^k$ ensures $\sum_{i \in S} E_i x_i^k = c$ for all $k ≥ 1$.

Remark 2 (Dual iterates of the coupling constraints): Theorem 2 proves convergence with the dual variable $λ_k$. However, instead of computing $λ_k$ explicitly, Algorithm 3 exploits that $E_i^T λ = γ_i^l$ to evaluate Steps 2, 4, and 6.

Remark 3 (Hessian regularization): ADMM is guaranteed to converge if the cost functions $q^k_i(s_i)$ are convex on the constraints $S^k_i$. That is, we do not require $H^k_i$ to be positive definite, but only positive definite on the equality constraint null space. However, this may not be the case if d-SQP is initialized far away from $p^*$. To this end, we regularize the reduced Hessian via [8, Eq. 3.43] with $δ = 10^{-4}$, where $δ$ is in the notation of [8].

Remark 4 (Inner method and application to QPs): Here, we present d-SQP with ADMM as an inner method for solving (3). Instead of ADMM, other decentralized methods with guaranteed convergence to a KKT point for convex QPs can also be used. If (1) itself is a QP, then d-SQP is equivalent to applying the inner method to a possibly regularized version of (1).
the parameters from [9]. We use CasADi [23] to compute sensitivities via algorithmic differentiation in all methods.

The top part of Figure 1 shows the convergence to the minimizer found by IPOPT.\(^3\) For d-SQP, bi-level ALADIN, and d-IP we count the inner iterations. Figure 1 shows that d-SQP requires more iterations than ADMM to achieve an equivalent accuracy in \(\|x - x^*\|_\infty<10^{-6}\). However, at the level of each subsystem, d-SQP only solves QPs whereas ADMM solves NLPs. As a result, in our prototypical implementation ADMM takes 67 s to achieve \(\|x - x^*\|_\infty<10^{-6}\), whereas d-SQP only takes 30 s. Moreover, d-IP is guaranteed to converge locally for this problem, whereas ADMM might diverge [1]. The bottom plot of Figure 1 shows the number of inequality constraints which toggle between being active and inactive. The active set stays constant after 127 iterations, which indicates that the area of local convergence is reached.

All methods require local neighbor-to-neighbor communication of the same complexity: \(2n_{c}\) floats for all subsystems combined per inner iteration. ALADIN/d-CG and d-IP further communicate two floats globally per d-CG iteration. The faster convergence of d-IP in Figure 1 therefore comes at the cost of global communication with low complexity. Hence, the bi-level SQP scheme shows competitive decentralized performance and it admits local convergence guarantees.

V. CONCLUSION AND OUTLOOK

This paper has established new convergence guarantees for decentralized SQP schemes under non-convex constraints. The proposed method solves quadratic subproblems with ADMM. In contrast to existing decentralized SQP methods, we allow for inexact ADMM solutions via an appropriate decentralized stopping criterion for the inner iterations. Numerical results show competitive performance to further optimization schemes for an example from power systems. Future work will consider the globalization of the method and its application to distributed model predictive control.

\(^3\)The convergence results for ADMM shown in Figure 1 differ from the results shown in [9]. Here we use a different formulation of the augmented Lagrangian to facilitate the averaging step, cf. [16].

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