A Fluctuation-Dissipation Process without Time Scale

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We study the influence of a dissipation process on diffusion dynamics triggered by fluctuations with long-range correlations. We make the assumption that the perturbation process involved is of the same kind as those recently studied numerically and theoretically, with a good agreement between theory and numerical treatment. As a result of this assumption the equilibrium distribution departs from the ordinary canonical distribution. The distribution tails are truncated, the distribution border is signalled by sharp peaks and, in the weak dissipation limit, the central distribution body becomes identical to a truncated Lévy distribution.

I. INTRODUCTION

The derivation of thermodynamics from dynamics is still an open field of investigation \cite{1,2,3}. Hereby we focus on a related but seemingly less ambitious purpose, the derivation of fluctuation-dissipation processes from deterministic dynamics \cite{4}. It has been recently pointed out \cite{5} that a genuinely dynamic derivation of Brownian motion would be essentially equivalent to a mechanical foundation of thermodynamics, thereby implying that also this avenue might be fraught by strong conceptual difficulties. It has been remarked \cite{6} that the dynamic foundation of Brownian motion, as described by an ordinary Fokker-Planck equation, imply fluctuations with a finite correlation time $\tau$, namely, rests on the existence of a finite microscopic time $\tau$, or, equivalently, on the microscopical foundation of the linear response theory \cite{7}. However, the resulting transport equation can be identified with a \textit{bona fide} Fokker-Planck equation only if \cite{8} the corresponding relaxation process is exactly, not approximately, exponential: a property in harsh conflict with both quantum \cite{9} and classical \cite{10} dynamics. This is the main reason why the problem of the dynamic foundation of the ordinary Fokker-Planck equation is not yet settled and further efforts must be made not excluding the possibility of either non-Newtonian effects \cite{11} or spontaneous fluctuations \cite{12}, both implying a kind of generalization of ordinary classical and quantum mechanics. Here we reverse the perspective and rather than imposing the Markovian approximation, incompatible with the deterministic nature of the system under study, we discuss the consequence of explicitly rejecting the requirement of a finite microscopic time scale. To conduct this discussion, we adopt Occam’s principle, namely we study the simplest dynamical system with the essential features necessary to produce a fluctuation-dissipation process without using the requirement of a finite microscopic time scale.

Let us consider the Liouville-like equation

\begin{equation}
\frac{\partial}{\partial t} \rho_T(x, \xi, w, t) = \hat{L}\rho_T(x, \xi, w, t) + \hat{\Gamma}(\Delta^2 x) \rho_T(x, \xi, w, t) \quad \text{for} \quad (-\xi \frac{\partial}{\partial x} + \hat{\Gamma}(\Delta^2 x)) \rho_T(x, \xi, w, t). \quad (1)
\end{equation}

This means that we study for simplicity a one-dimensional case. The one-dimensional variable of interest $x$ undergoes the influence of a “fluctuation”, called $\xi$. The dynamics of $\xi$ is driven by the operator $\hat{\Gamma}$ which expresses concisely the action that a set of variables $w$ can exert on $\xi$ so as to render disordered its time evolution. Thus, in principle, the “stochastic” dynamics of $x$ can be either provoked by nonlinearity or by the large number of degrees of freedom. The unperturbed fluctuations $\xi$ are the source of diffusion of the variable $x$. To undergo also dissipation, the second key ingredient of a fluctuation-dissipation process, the variable $x$ must also exert a feedback on the dynamics of $\xi$. This important property is expressed by the dependence of $\hat{\Gamma}$ on $\Delta^2$, left unspecified. As we shall see, we shall assume a linear departure from the unperturbed condition $\hat{\Gamma}_0$ given by $\hat{\Gamma}(\Delta^2 x) = \hat{\Gamma}_0 - \Delta^2 x \hat{\Gamma}_1$. The operator $\hat{\Gamma}_1$ drives the bath response to an external perturbation $\hat{\Gamma}_0$. The parameter $-\Delta^2 x$ denotes the strength of the feedback and the minus sign alludes to the reaction nature of the effect. Pursuing our program inspired to Occam’s criterion, we are forced to assume the variable $\xi$ to be dichotomous. It would be surprising if $\xi$, and so the microscopic statistics, were Gaussian. In a sense, there would be no problem to solve at all. Therefore we must adopt a non Gaussian statistics. Thus, we fix the statistics of $\xi$ to be dichotomous, since dichotomous statistics seem to be the simplest example of non-Gaussian statistics.

We plan to prove that a bath, described by the unperturbed operator $\hat{\Gamma}_0$ with a diverging correlation time, $\tau = \infty$, yields a form of equilibrium strongly departing from the ordinary canonical prescription. The proof is organized as follows. In Section II we derive a generalized master equation. We make the basic assumption that the memory kernel of this master equation only depends on the unperturbed bath dynamics. Using the additional
assumption that the bath response is of the same kind as that studied in earlier publications, we derive the central theoretical result of this paper. In Section III we discuss the error associated with the basic assumption of Section II. In Section IV we use the central theoretical result of this paper to predict the resulting, non-canonical, equilibrium. In Section V we check numerically the effect of the error discussed in Section II. Finally, in Section VI we make a balance of the results obtained in this paper.

II. THE PROJECTION METHOD

In the case of no feedback, namely when $\Delta = 0$, it is shown \cite{10} that an immediate benefit of the dichotomous choice is that the projection operator method \cite{11} allows us to express the dynamics of the variable of interest, $x$, in terms of an exact and simple diffusion-like equation of motion. When the feedback is included, unfortunately, the projection method does not produce a simple equation of motion, and delicate assumptions must be made if we want to keep the elegance and simplicity of the earlier treatment. The main purpose of this section is to discuss these delicate assumptions. The adoption of the projection method \cite{11} yields for the part of interest $\hat{P}\rho_T$ of the total distribution the following equation:

$$\frac{\partial}{\partial t} \hat{P} \rho_T(x, \xi, w, t) = \int_0^t \left\{ \hat{P} \left[ -\xi \frac{\partial}{\partial x} \right] \exp[\hat{Q} \hat{L} \hat{Q}(t-t')] \right\} \hat{Q} \left[ -\xi \frac{\partial}{\partial x} + \Gamma (-\Delta^2 x) \right] \hat{P} \rho_T(x, \xi, w, t') \right\} dt'. \quad (2)$$

As usual, we assume the bath to have an equilibrium distribution satisfying the condition:

$$\hat{\Gamma}_0 \rho_{eq}(\xi, w) = 0. \quad (3)$$

This dictates the choice of the projection operator $\hat{P}$:

$$\hat{P} \rho_T(x, \xi, w, t) = \sigma(x, t) \rho_{eq}(\xi, w), \quad (4)$$

where $\sigma(x, t)$ is obtained tracing the total distribution $\rho_T(x, \xi, w, t)$ over the irrelevant degrees of freedom $\xi$ and $w$. Note that Eq. (4) is an exact equation provided that the initial condition is given by:

$$\rho_T(x, \xi, w; 0) = \sigma(x; 0) \rho_{eq}(\xi, w). \quad (5)$$

For simplicity we assume that:

$$\hat{\Gamma}(K) = \hat{\Gamma}_0 + K \hat{\Gamma}_1. \quad (6)$$

We carry out our calculations setting the condition: $K = -\Delta^2 x$. This is equivalent to assuming a form of linear response to external perturbation in agreement with Refs. \cite{12, 13}. We make now the assumption that, in spite of $\Delta \neq 0$, the exponential operator $\exp(\hat{Q} \hat{L} \hat{Q}(t-t'))$ appearing in Eq. (4) only depends on the unperturbed operator $\hat{\Gamma}_0$, a property that, as earlier pointed out, is valid only in the free diffusion case. We also use Eq. (3). Under all these approximations, we are allowed to rewrite Eq. (4) as

$$\frac{\partial}{\partial t} \hat{P} \rho_T(x, \xi, w, t) = A(\rho_T(x, \xi, w, t)) + B(\rho_T(x, \xi, w, t)), \quad (7)$$

where

$$A(\rho_T(x, \xi, w, t)) \equiv \frac{\partial^2}{\partial x^2} \int_0^t dt' \left\{ \hat{P} \left[ \xi \left[ \exp[\hat{\Gamma}_0(t-t')] \right] \hat{Q}[x] \hat{P} \rho_T(x, \xi, w, t') \right] \right\} \quad (8)$$

and

$$B(\rho_T(x, \xi, w, t)) \equiv \Delta^2 \frac{\partial}{\partial x} \int_0^t dt' \left\{ \hat{P} \left[ \xi \left[ \exp[\hat{\Gamma}_0(t-t')] \right] \hat{Q}[\hat{\Gamma}_1] \hat{P} \rho_T(x, \xi, w, t') \right] \right\}. \quad (9)$$

In conclusion to derive this result we have used a major assumption that will be referred to as assumption (i). This assumption can be expressed as follows:

assumption (i). We assume that the exponential operator appearing on the r.h.s of Eq. (4) only depends on the unperturbed bath dynamics.

In the case where no feedback process is considered \cite{11} there is no error associated with this assumption, since this is shown to be an exact consequence of the dichotomous nature of the variable $\xi$. It is not so in the more general case of this paper. We shall devote Section III and the numerical treatment of Section V to assessing the consequences of the error associated with this basic assumption. Adopting the formalism of the response theory \cite{14} we rewrite Eq. (3) in the form:

$$\frac{\partial}{\partial t} \sigma(x, t) = <\xi^2 >_{eq} \int_0^t dt' \Phi_\xi(t-t') \frac{\partial^2}{\partial x^2} \sigma(x, t') + \Delta^2 <\xi \Gamma_1 >_{eq} \int_0^t dt' C(t-t') \frac{\partial}{\partial x} \sigma(x, t'), \quad (10)$$

where

$$\Phi_\xi(t) \equiv \frac{<\xi \exp[\hat{\Gamma}_0(t)] >_{eq}}{<\xi^2 >_{eq}} \quad (11)$$

and

$$C(t) \equiv \frac{<\xi \exp[\hat{\Gamma}_0(t)] >_{eq}}{<\xi \Gamma_1 >_{eq}}. \quad (12)$$

This result has been obtained by evaluating the diffusion term at the zero-th order in the feedback interaction, and considering, in agreement with the linear response criterion \cite{11}, the first non-vanishing contribution, proportional to the friction. It is worth remarking that the
correlation function of Eq. (11) affords the most convenient way of defining the microscopic time $\tau$ mentioned in Section 1. This is given by:

$$\tau \equiv \int_0^\infty \Phi_\xi(t)dt. \quad (13)$$

We now have recourse to the second approximation on which our crucial theoretical results rests. This approximation will be referred to as assumption (ii) and can be expressed as follows:

assumption (ii). We assume that the function $C(t)$ has a finite time scale.

This assumption is dictated by the theoretical and numerical conclusion of the earlier work of Refs. [12–14]. This assumption cannot be mislead as a property of ordinary statistical mechanics. Actually, this assumption means a deviation from ordinary statistical mechanics, which, as shown in Ref. [3], would imply

$$C(t) = \Phi_\xi(t). \quad (14)$$

In the case where the correlation function $\Phi_\xi(t)$ is not integrable and the correlation time of Eq. (13) diverges, the condition of Eq. (14) would imply a field of finite intensity to produce a current of infinite intensity [12].

The numerical calculations show that this striking physical condition is not realized [12], thereby implying a violation of Eq. (14). This violation, in turn, is due to the fact that the function $C(t)$ has a finite time scale even when the function $\Phi_\xi(t)$ does not. Note that under the assumption that the function $C(t)$ has a finite time scale it is possible to define

$$\gamma \equiv \Delta^2 < \xi \Gamma_1 >_{eq} \int_0^\infty dt' C(t'). \quad (15)$$

In conclusion, we obtain the following important equation:

$$\frac{\partial}{\partial t} \sigma(x,t) = <\xi^2 >_{eq} \int_0^t dt' \Phi_\xi(t - t') \frac{\partial^2}{\partial x^2} \sigma(x,t')$$

$$+ \gamma \frac{\partial}{\partial x} x\sigma(x,t). \quad (16)$$

This result rests on both assumption (i) and (ii). However, it is evident that special attention must be devoted to the first assumption. In a sense the validity of assumption (ii) has already been assessed by the theoretical and numerical work of Refs. [12–14]. The validity of assumption (i), on the contrary, requires further discussion. This will be done in Sections III and V.

Before ending this Section, we want to remark that in the special case where the condition of Eq. (14) applies, the important result of Eq. (14) becomes very similar to the Fokker-Planck type of equation recently found by the authors of Ref. [13]. These authors pointed out that an equation of this kind shows that anomalous diffusion can be compatible with Boltzmann statistics.

We note that this conclusion does not apply to the case of super-diffusion under study in this paper, because, as we have seen, Eq. (14) does not apply. In the subdiffusional case studied in Ref. [13], however, there are no compelling reasons leading to the breakdown of Eq. (14) thereby leaving open the possibility that the subdiffusional condition is compatible with Boltzmann statistics.

### III. NO INTERFERENCE BETWEEN FREE FLUCTUATION AND DISSIPATION: TIME EVOLUTION

To a first sight, one might be led to think that Eq. (16) is equivalent to the Langevin-like equation:

$$\dot{x}(t) = -\gamma x(t) + \xi(t), \quad (17)$$

supplemented, of course, by the set of equations necessary to determine the time evolution of the dichotomous variable $\xi(t)$. In this Section we show that Eq. (16) is not identical to the equation of motion for $\sigma(x,t)$ generated by Eq. (17). This will help us to estimate the error affecting the main prediction of this paper about the condition of equilibrium established by the feedback on the generator of fluctuation without time scale.

#### A. Second moment time evolution

In Section IV we shall point out that Eq. (17) implies that throughout system’s time evolution the trajectory $x(t)$ departing from the initial condition $x(0) = 0$ never leaves the interval $[−W/\gamma, W/\gamma]$. This property means that the second moment of the distribution is kept finite at all time and can never exceed the maximum value $(W/\gamma)^2$. Here we show that, on the contrary, the second moment of the distribution driven by Eq. (16) diverges for $t \to \infty$.

Using Eq. (17) we get:

$$\frac{\partial}{\partial t} x(t) = -\gamma x(t) + \xi(t)$$

$$+ \frac{\partial}{\partial x} x\sigma(x,t). \quad (18)$$

Using the method of integration by parts it is shown that Eq. (18) yields:

$$\frac{\partial}{\partial t} x(t) = -2\gamma x(t) + 2\xi^2 > \int_0^t \Phi(t')dt'.$$
that the distribution $\sigma(x, t)$ cannot be a Lévy process at any finite time $t > 0$. We know that at $\gamma = 0$ the diffusing distribution is in fact a Lévy process with ballistic peaks signalling the presence of a propagation front, thereby ensuring the validity of the method of integration by parts. It is plausible to assume that the action of a dissipation process makes the distribution spreading less intense, so favoring rather than opposing the method of integration by parts.

The solution of Eq. (13) is given by:

$$<x^2(t)> = \frac{<\xi^2>}{\gamma} \int_0^t \Phi_\xi(t - t')[1 - \exp(-2\gamma t')]dt'.$$

Let us adopt for the correlation function $\Phi_\xi(t)$ the choice:

$$\Phi_\xi(t) = \frac{(\beta T)^\beta}{(\beta T + t)^{\beta + 1}}.$$

where $T$ is the mean waiting time in a state of the velocity. In fact, as a consequence of the one-dimensional assumption we are allowed to use the relation [10]:

$$\psi(t) = T \int_0^t \frac{d^2}{dt^2} \Phi_\xi(t) = \frac{(\beta T)^{\beta + 1}(\beta + 1)}{(\beta T + t)^{\beta + 2}},$$

where $\psi(t)$ is the distribution density of sojourn times. Plugging the analytical form of Eq. (21) into the r.h.s. of Eq. (20) and making a time asymptotic analysis, we get:

$$\lim_{t \to \infty} <x^2(t)>> t^{1-\beta}, \quad \gamma > 0$$

and

$$\lim_{\gamma \to 0} <x^2(t)>> t^{2-\beta}, \quad t >> 1.$$

### B. Exact equation of motion for $\sigma(x, t)$

We note that the use of the same projection method as that applied in Section II to the dynamic system described by Eq. (7) yields

$$\frac{\partial}{\partial t} \sigma(x, t) = \gamma \frac{\partial}{\partial x} \sigma(x, t) + \frac{\partial}{\partial x} \int_0^t dt' \left\{ \Phi_\xi(t - t') \cdot \exp[\gamma \frac{\partial}{\partial x} x(t - t')] \frac{\partial}{\partial x} \sigma(x, t') \right\}.$$

We immediately see that the same approximation as that applied to Eq. (2), namely the approximation of neglecting the influence of the feedback on the memory kernel, makes Eq. (25) identical to Eq. (18). Consequently, the numerical treatment of Eq. (17) is expected to depart from the prediction of Eq. (16) and the amount of this departure can be used as a way to establish the error caused by assumption (i) in the derivation of Eq. (16), which is the central result of this paper.

Eq. (25) can be used to derive an analytical expression for the second moment time evolution. The Taylor series expansion of the exponential operator on the r.h.s. of Eq. (25) and the use of integration by parts yield:

$$\frac{\partial}{\partial t} <x^2(t)> + 2\gamma <x^2(t)> = 2 <\xi^2> \cdot \int_0^t \Phi_\xi(t') \exp(-\gamma t')dt',$$

which, in turn, yields the following time evolution:

$$<x^2(t)> = 2 <\xi^2> \exp(-2\gamma t) \int_0^t \exp(2\gamma t') \cdot \int_0^{t'} \Phi_\xi(\tau) \exp(-\gamma \tau) d\tau d\tau'.$$

It is worth remarking that the general expression for the asymptotic value of the second moment is:

$$<x^2(\infty)> = \frac{<\xi^2>}{\gamma} \int_0^\infty \Phi_\xi(t') \exp(-\gamma t')dt'.$$

We see that the asymptotic value for the second moment, as it must be, is finite and in the special case of Eq. (21) the analytical expression for the second moment at $t = \infty$ is:

$$<x^2(\infty)> = \frac{<\xi^2>(\beta T)^\beta \exp(\gamma \beta T) \frac{\Gamma(1 - \beta, \gamma \beta T)}{\gamma^{1-\beta}}}{\gamma^{2-\beta}}.$$

In conclusion, we see that assumption (i) produces the seemingly unacceptable effect of making the asymptotic second moment diverge, whereas the exact equation of motion yields a second moment which is always finite and gets at equilibrium the finite value predicted by Eq. (29). The discussion of Sections IV and V will explain in which sense the error associated with assumption (i) does not invalidate our main conclusion that the final equilibrium distribution is of Lévy kind.

### IV. EQUILIBRIUM PROPERTIES

The fact that the second moment does not converge to a finite value is a consequence of the central approximation yielding Eq. (16). This does not conflict with the possibility that for $t \to \infty$ the distribution approaches asymptotically a time independent shape. Using the recent results of Refs [10] and [14] it is shown that the Fourier transform of Eq. (16) obey the following time evolution equation:

$$\frac{\partial}{\partial t} \hat{\sigma}(k, t) = -b|k|^\alpha \hat{\sigma}(k, t) - \gamma k \frac{\partial}{\partial k} \hat{\sigma}(k, t).$$


the following equilibrium distribution:

\[ \tilde{\sigma}(k, \infty) = \exp \left( -\frac{b}{\alpha \gamma} |k|^\alpha \right), \quad \text{(31)} \]

which, in turns, according to [13], coincides with the equilibrium distribution corresponding to the equation of motion:

\[ \frac{d}{dt} x(t) = -\gamma x(t) + \eta(t), \quad \text{(32)} \]

where \( \eta(t) \) is an uncorrelated noise, with probability distribution \( p(\eta) \), obeying Lévy statistics, and thus defined in the Fourier space by:

\[ p(k) = \int d\eta \exp(-ik\eta)p(\eta) = \exp(-b |k|^{\alpha}), \quad \text{(33)} \]

where \( 0 < \mu < 2 \).

It must be pointed out that Eq. (32) does not coincide with Eq. (17). In the case of very weak friction they do in the sense that will be illustrated in Section V.

V. NUMERICAL RESULTS

The numerical results of this Section are based on the numerical treatment of Eq. (17), and consequently on the numerical implementation of

\[ x(t) = \int_0^t \exp[-\gamma(t - t')]\xi(t') \, dt' + x(0) \exp(-\gamma t). \quad \text{(34)} \]

The fact that the variable \( \xi \) is dichotomous with the correlation function of Eq. (2) naturally leads us to adopt the same numerical approach as that used in Refs. [10, 19]. This means that two random number generators are used. The first results in random number homogeneously distributed in the interval \([0, 1]\). With a proper non-linear deformation this is made equivalent to a random generation of waiting times with the distribution of Eq. (21). This is the way we adopt to build numerically the time evolution of \( x(t) \). We also set an initial condition fitting the crucial condition of Eq. (7) and we make the trajectory run for times larger than \( 20/\gamma \). We evaluate the same way \( 10^4 \) trajectories, then these trajectories are recorded in a bin.

In Fig. 1 we see a sample of the resulting equilibrium distribution with \( \beta = 0.6, T = 50, W = 1, \gamma = 10^{-4}, \) spanning from \( x = -W/\gamma \) to \( x = W/\gamma \). Fig. 1 is a crystal clear illustration of what we mean by statistics of Lévy kind. We see that the equilibrium distribution is truncated and that at the borders two sharp peaks emerge. These sharp peaks are an equilibrium reflection of the peaks already revealed by the numerical treatment of free diffusion [10, 20]. However, the distribution enclosed by these peaks is shown to fit very well the Lévy distribution predicted by the theory of Section IV.

Fig. 2 is devoted to the comparison between the theoretical prediction of Eq. (29) and the result of our numerical treatment. The agreement between theory and
VI. CONCLUDING REMARKS

Which is then the interest of our results? We think that the interest of them lies on this: This paper forces us to change the conventional perspective concerning the microscopic foundation of the canonical statistical behavior. Some years ago, the findings of Zhu and Robinson [21] have been criticized by Keirstad and Wilson [22] with arguments which are a nice example of the conventional wisdom. Let us see why. Zhu and Robinson [21] had detected significant deviations from the canonical Maxwell velocity distribution, in a physical condition characterized by a system of interest very fast compared to its thermal bath. This condition seems to be related to that considered in this paper where the dynamical system playing the role of bath is in fact so slow as to break the condition itself of time scale separation. The reaction of the scientific community, of which the authors of [22] are a significant example, has been that the non-canonical behavior detected numerically by Zhu and Robinson [21] is an artefact of numerical inaccuracy and limited computation time. This paper shows, on the contrary, that the opposite condition might apply, namely, that ordinary rather that anomalous statistics might be the result of numerical inaccuracy. We know that the round-off errors are equivalent to the influence of fluctuations of a given intensity $\epsilon$. The larger the computer accuracy, the smaller is the intensity of the equivalent fluctuations. On the other hand, we know [23] that the effect of these fluctuations is that of changing the correlation function of Eq.(11) into the correlation function $\Phi^*_{\xi}(t)$ related to the original by

$$\Phi^*_{\xi}(t) = \Phi_{\xi}(t) \exp(-t/t_C),$$

with $t_C$ proportional to $e^\delta$ and $\delta$ being a positive coefficient, of the order of unity, determined by the microscopic dynamics under study [23]. It is evident that at times $t > t_C$ the Markov approximation is valid, and as an effect of it the non standard equation of [11] becomes identical to a conventional Fokker-Planck equation. The non conventional equilibrium of [23] is a time asymptotic property, and at any given time $t \gg 1/\gamma$ we can produce a transition from the regime of non-ordinary statistics to a regime of canonical Gaussian equilibrium by increasing the intensity of the parameter $\epsilon$, so as to realize the condition $1/\gamma > \tau_C$.

Finally, we want to stress a problem worth of future investigation. This has to do with the increasing attention devoted to the non-extensive thermodynamics of Tsallis [24,25]. Non-extensive thermodynamics means that the deviation from the canonical equilibrium distribution is not more perceived as a violation of statistical mechanics. This is a very valuable aspect of this research work [24,25]. In fact, as a result of the interest that Tsallis’s non-extensive statistical mechanics is raising, a deviation from the ordinary prescription, of the kind earlier mentioned, would be judged these days as a possible manifestation of non-extensive thermodynamics triggered by the long-range correlations of the dynamical system under study, rather than a consequence of numerical inaccuracy.

However, the arguments of this paper show that under the specific form here adopted to establish a fluctuation-dissipation process in a case of dynamics without time scale, the basin of attraction for equilibrium distribution is given by Lévy statistics. It is interesting to point out that Lévy statistics share with Tsallis statistics the power law behavior of the distribution tails. However, the central part of the Lévy distribution significantly depart from the generalized canonical distribution of Tsallis. In an earlier paper [27] it has been shown that the adoption of Tsallis’ non-extensive thermodynamics naturally leads, via entropy maximisation under a proper constraint, to a transition probability with an inverse power law decay at large distances. By repeated application of this kind of transition, as a consequence of the Lévy-Gnedenko theorem [28], the diffusion process is attracted by the basin of Lévy statistics. In the case of extremely weak friction, equilibrium is reached as a result of a very large number of elementary transitions, and this is probably the main reason why eventually the resulting statistics is of Lévy rather than Tsallis kind.

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