Astrophysical configurations with background cosmology: probing dark energy at astrophysical scales

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ABSTRACT

We explore the effects of a positive cosmological constant on astrophysical and cosmological configurations described by a polytropic equation of state. We derive the conditions for equilibrium and stability of such configurations and consider some astrophysical examples where our analysis may be relevant. We show that in the presence of the cosmological constant the isothermal sphere is not a viable astrophysical model since the density in this model does not go asymptotically to zero. The cosmological constant implies that, for a polytropic index smaller than 5, the central density has to exceed a certain minimal value in terms of the vacuum density in order to guarantee the existence of a finite-size object. We examine such configurations together with effects of Λ in other exotic possibilities, such as neutrino and boson stars, and we compare our results to N-body simulations. The astrophysical properties and configurations found in this paper are specific features resulting from the existence of a dark energy component. Hence, if found in nature they would be an independent probe of a cosmological constant, complementary to other observations.

Key words: cosmology: theory – dark matter.

1 INTRODUCTION

Some of the most relevant properties of the Universe have been established through astronomical data associated with light curves of distant Type Ia supernovae (Riess et al. 2004), the temperature anisotropies in the cosmic microwave background radiation (Spergel et al. 2007) and the matter power spectrum of large-scale structures (Tegmark et al. 2004). Such observations give strong evidence that the geometry of the Universe is flat and that our Universe is undertaking an accelerated expansion at the present epoch. This acceleration is attributed to a dominant dark energy component, whose most popular candidate is the cosmological constant, Λ.

The present-day dominance of dark energy makes us wonder if this component may affect the formation and stability of large astrophysical structures, whose physics is basically Newtonian. This is in fact an old question put forward already by Einstein (Einstein & Straus 1945) and pursued by many other authors (Noordlinger & Petrosian 1971; Baryshev et al. 2001; Chernin Nagirner & Starikova 2003; Jetzer & Serena 2006; Kagramanova et al. 2006; Manera & Mota 2005; Nunes & Mota 2006). In general, the problem is rooted in the question whether the expansion of the Universe, which in the Newtonian sense could be understood as a repulsive force, affects local astrophysical properties of large objects. The answer is certainly affirmative if part of the terms responsible for the expansion of the Universe survives the Newtonian limit of the Einstein equations. This is indeed the case of Λ which is part of the Einstein tensor. In fact, as explained in the main text below, all the effects of the Universe expansion can be taken into account, regardless of the model, by generalizing the Newtonian limit. This approach allows us to calculate the impact of a given cosmological model on astrophysical structures.

Although there are several candidates to dark energy which have their own cosmological signatures (see e.g. Daly & Djorgovski 2003; Seo & Eisenstein 2003; Daly & Djorgovski 2004; Wang & Mukherjee 2004; Brookfield et al. 2006; Koivisto & Mota 2006, 2007a; Shaw & Mota 2006; Koivisto & Mota 2007b; Shaw & Mota 2007b), in this paper we will investigate the Λ cold dark matter (ΛCDM) model only. Such consideration is in fact not restrictive and our results will be common to most dark energy models. At astrophysical scales and within the Newtonian limit, one does not expect to find important differences among the different dark energy models. This, however, should not be interpreted as if Λ has no effect at smaller astrophysical scales. In fact, the effects of a cosmological constant on the equilibrium and stability...
of astrophysical structures are not negligible, and can be of relevance to describe features of astrophysical systems such as globular clusters, galaxy clusters or even galaxies (Nowakowski, Sanabria & García 2002; Iorio 2005; Balaguera-Antolínez, Böhmer & Nowakowski 2006a; Cardoso & Gualtieri 2006; Chernin et al. 2007). Motivated by this, we investigate the effects of a dark energy component on the Newtonian limit of Einstein gravity and its consequences at astrophysical scales.

In this paper, we investigate how the cosmological constant changes certain aspects of astrophysical hydrostatic equilibrium. In particular, we search for specific imprints which are unique to the existence of a dark energy fluid: for instance, the instability of previously viable astrophysical models when \( \Lambda \) is included. We explore such possibilities using spherical configurations described by a polytropic equation of state (EOS) \( p \sim \rho^n \). The polytropic EOS derives its importance from its success and consistency, and it is widely used in determining the properties of gravitational structures ranging from stars (Chandrasekhar 1967) to galaxies (Binney & Tremaine 1987). It leads us to an acceptable description of the behaviour of astrophysical objects in accordance with observations and numerical simulations (Kanididakis, Lavagno & Quarati 1996; Ruffet, Rampp & Hanka 1996; Kennedy & Bludman 1999; Gruzinov 2000; Pinzon & Calvo-Mozo 2001; Sadeth & Rephaeli 2004). The description of such configurations can be verified in the general relativistic framework (Herrera & Barreto 2003) and applications of these models to the dark energy problem have in fact been explored (Mukhopadhyay & Ray 2005).

The effect of a positive cosmological constant can be best visualized as a repulsive non-local force acting on the matter distribution. It is clear that this extra force will result into a minimum density (either central or average) which is possible for the distribution to be in equilibrium. This minimum density is a crucial crossing point: below this value no matter can be in equilibrium, and above this value low-density objects exist (Lahav et al. 1991). Both effects are novel features due to \( \Lambda \). We will demonstrate such inequalities, which are generalizations of corresponding inequalities found in Nowakowski et al. (2002) and Balaguera-Antolínez, Böhmer & Nowakowski (2005), for every polytropic index \( n \). However, the most drastic effect can be found in the limiting case of the polytropic EOS, that is, the isothermal sphere where the polytropic index \( n \) goes to infinity. This case captures, as far as the effects of \( \Lambda \) are concerned, many features also for higher, but finite \( n \). The model of the isothermal sphere is often used to model galaxies and galactic clusters (Natarajan & Lynden-Bell 1997) and used in describing effects of gravitational lensing (Kawano et al. 2004; Maccio 2004; Sereno 2005). Herein lies the importance of the model. Regarding the isothermal sphere, we will show that \( \Lambda \) renders the model unacceptable on general grounds. This essentially means that the model does not even have an appealing asymptotic behaviour for large radii and any attempt to define a physically acceptable radius has its severe drawbacks.

The positive cosmological constant offers, however, yet another unique opportunity, namely the possible existence of young low-density virialized objects, understood as configurations that have reached virial equilibrium just at the vacuum-dominated epoch (in contrast to the structures forming during the matter-dominated era, where the criterion for virialization is roughly \( \tilde{\rho} \sim 200 \rho_{\text{crit}} \)). These low-density hydrostatic/virialized objects can be explained again due to \( \Lambda \) which now partly plays the role of the outward pressure.

The applicability of fluid models, virial theorem and hydrostatic equilibrium to large astrophysical bodies has been discussed many times in the literature. For a small survey on this topic, we refer the reader to Balaguera-Antolínez, Mota & Nowakowski (2006b) where one can also find the relevant references. It is interesting to note that dark matter haloes (DMHs) represent a constant-density background such that, in the Newtonian limit, objects embedded in them feel the analogue to a negative cosmological constant. The equilibrium analysis for such configurations has been performed in Umemura & Ikeuchi (1986) and Horedt (2000). A negative \( \Lambda \) will just enhance the attractive gravity effect, whereas a positive one opposes this attraction. As a result the case \( \Lambda > 0 \) reveals different physical concepts as discussed in this paper.

This paper is organized as follows. In the next section, we introduce the equations relevant to astrophysical systems as a result of the weak field limit and the non-relativistic limit of Einstein field equations taking into account a cosmological constant. There we derive such a limit taking into account the background expansion independently of the dark energy model. In Section 3, we derive the equations governing polytropic configurations, the equilibrium conditions and stability criteria. In Section 4, we describe the isothermal sphere and investigate its applicability in the presence of \( \Lambda \). In Section 5, we explore some examples of astrophysical configurations where the cosmological constant may play a relevant role. In particular, we probe into low-density objects, fermion (neutrino) stars and boson stars. Finally, we perform an important comparison between polytropic configurations with \( \Lambda \) and parametrized density of DMHs. We end with conclusions. We use units \( G_N = c = 1 \) except in Section 5.4 where we restore \( G_N \) and use natural units \( \hat{a} = \hat{c} = 1 \).

## 2 Local Dynamics in the Cosmology Background

The dynamics of the isotropic and homogeneous cosmological background is determined by the evolution of the (dimensionless) scalefactor given through the Friedman–Robertson–Walker line element as a solution of Einstein field equations,

\[
\frac{\dot{a}(t)}{a(t)} = -\frac{4}{3} \pi \left[ \rho(t) + 3 p(t) \right] \quad \text{and} \quad \left[ \frac{\dot{a}(t)}{a(t)} \right]^2 = H(t)^2 = \frac{8}{3} \pi \rho(t) - \frac{k}{a^2(t)},
\]

corresponding to the Raychaudhury equation and Friedman equation, respectively. The total energy density \( \rho \) is a contribution from a matter component – baryonic plus dark matter \(- (\rho_{\text{mat}} \sim a^{-3})\), radiation \( (\rho_{\text{rad}} \sim a^{-4})\) and a dark energy component \( (\rho_{\Lambda} \sim a^{-3(\omega_{\Lambda})}) \) with \( p = \omega_{\Lambda} \rho \). The function \( f(a) \) is given as

\[
f(a) = \frac{3}{\ln a} \int_1^a \frac{\alpha(a')}{a'} da',
\]
where the term \( \omega(a) \) represents the EOS for the dark energy component. The case \( \omega = -1 \) corresponds to the cosmological constant \( \rho_{M} = \rho_{\Lambda} = \Lambda/8\pi. \) The effects of the background on virialized structures can be explored through the Newtonian limit of field equations from which one can derive a modified Poisson equation (see e.g. Noerdlinger & Petrosian 1971; Nowakowski 2001). Recalling that pressure is also a source of gravity, the gravitational potential produced by an overdensity is given by

\[
\nabla^2 \Phi = 4\pi(\rho_\ell + 3P_\ell),
\]

where \( \rho_\ell = \delta \rho + \rho \) and \( P_\ell = \delta P + P, \) and \( \delta \rho \) is the local overdensity with respect to the background density \( \rho. \) Note that equation (3) reduces to the usual Poisson equation, \( \nabla^2 \Phi = 4\pi\delta \rho, \) when non-relativistic matter dominates the Universe, and \( \delta \rho \gg \rho. \) However, at present times, when dark energy dominates, the pressure is non-negligible and \( \delta P \) might even be non-zero, such as in the case of quintessence models (Wang & Steinhardt 1998; Mota & van de Bruck 2004; Maor & Lahav 2005). In this work, however, we will focus in the case of an homogeneous dark energy component where \( \delta \rho = \delta P = 0. \) With this in mind, one can then write the modified Poisson equation as

\[
\nabla^2 \Phi = 4\pi\delta \rho - 3\ddot{a}(t)/a(t),
\]

Note that this equation allows one to probe local effects of different dark energy models through the term \( \ddot{a}/a \) given in equation (1).

Since we will be investigating the configuration and stability of astrophysical objects nowadays, when dark energy dominates, it is more instructive to write the above equations in terms of an effective vacuum density, that is,

\[
\nabla^2 \Phi = 4\pi\delta \rho - 8\pi\rho_{\text{vac}}^\text{eff}(a),
\]

where by using (1) \( \rho_{\text{vac}}^\text{eff}(a) \) has been defined as

\[
\rho_{\text{vac}}^\text{eff}(a) \equiv -\frac{1}{2} \left[ \frac{\Omega_{\text{dm}}}{\Omega_{\text{vac}}} a^{-3} + (1 + 3\omega(a)) a^{-f(a)} \right] \rho_{\text{vac}},
\]

which reduces to \( \rho_{\text{vac}} \) for \( \omega(a) = -1 \) and negligible contribution from the CDM component with respect to the overdensity \( \delta \rho. \) With \( \Phi_{\text{grav}} \) being the solution associated with the pure gravitational interaction, the full solution for the potential can be simply written as

\[
\Phi(r, a) = \Phi_{\text{grav}}(r) - \frac{4}{3} \pi\rho_{\text{vac}}^\text{eff}(a)r^2,
\]

which defines the Newton–Hooke space–time for a scalefactor close to the present time (vacuum dominated epoch), \( \omega(a) = -1 \) and \( \Omega_{\text{dm}} \ll \Omega_{\text{vac}} \) (Aldrovandi et al. 1998; Gibbons & Patricot 2003). For a \( \Lambda \)CDM universe with \( \Omega_{\text{dm}} = 0.27 \) and \( \Omega_{\text{vac}} = 0.73, \) we get \( \rho_{\text{vac}}^\text{eff}(\text{today}) = 0.81\rho_{\text{vac}} \) that is, the positive density of matter which has an attractive effect opposing the repulsive one of \( \Lambda \) reduces effectively the strength of the ‘external force’ in equation (5). Note that, although in the text we will use the notation \( \rho_{\text{vac}} \) which would be valid in the case of a Newton–Hooke space–time, it must be understood that we can replace \( \rho_{\text{vac}} \) by \( \rho_{\text{vac}}^\text{eff}(\text{today}) = 0.81\rho_{\text{vac}} \) for a \( \Lambda \)CDM background.

Given the potential \( \Phi(r, a) \), we can write Euler’s equation for a self-gravitating configuration as

\[
\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} + \partial_i P + \partial_i \Phi = 0
\]

where \( \langle v_i \rangle \) is the (statistical) mean velocity and \( \rho \) is the total energy density in the system. We can go beyond Euler’s equation and write down the tensor virial equation which reduces to its scalar version for spherical configurations. The (scalar) virial equation with the background contribution reads as (Balaguera-Antolínez & Nowakowski 2005; Caimmi 2007):

\[
\frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2} = 2T + \mathcal{W}^{\text{grav}} + 3\Pi + \frac{8}{3} \pi\rho_{\text{vac}}^\text{eff}(a)\mathcal{I} - \int_{\mathcal{V}} p(r, \hat{n}) \ dA,
\]

where \( \mathcal{W}^{\text{grav}} \) is the gravitational potential energy defined by

\[
\mathcal{W}^{\text{grav}} = \frac{1}{2} \int_{\mathcal{V}} \rho(r) \Phi_{\text{grav}}(r) \ d^3r,
\]

and \( T = \frac{1}{2} \int_{\mathcal{V}} \rho(v^2) \ d^3r \) is the contribution of ordered motions to the kinetic energy. Also, \( \mathcal{I} \equiv \int_{\mathcal{V}} \rho r^2 \ d^3r \) is the moment of inertia about the centre of the configuration and \( \Pi \equiv \int_{\mathcal{V}} \rho r \ d^3r \) is the trace of the dispersion tensor. The full description of a self-gravitating configuration is completed with an equation for mass conservation, energy conservation and an EOS \( p = p(\delta \rho, s). \) If we assume equilibrium via \( \mathcal{I} \approx 0, \) we obtain the known virial theorem (Balaguera-Antolínez & Nowakowski 2005):

\[
|\mathcal{W}^{\text{grav}}| = 2T + 3\Pi + \frac{8}{3} \pi\rho_{\text{vac}}^\text{eff}(a)\mathcal{I},
\]

where we have neglected the surface term in equation (8), which is valid in the case when we define the boundary of the configuration where \( p = 0. \)

With \( \rho_{\text{vac}}^\text{eff} \) given in equation (6), one must be aware that an equilibrium configuration is at the most a dynamical one. This is to say that the ‘external repulsive force’ in equation (6) is time-dependent through the inclusion of the background expansion and so are the terms in equation (8). This leads to a violation of energy conservation, which also occurs in the virialization process (Mota & van de Bruck 2004; Maor & Lahav 2005; Wang 2006; Caimmi 2007; Shaw & Mota 2007a). Traced over cosmological times, this implies that if we insist on the second derivative of the inertial tensor being zero, then the internal properties of the object like angular velocity or the internal mean velocity of the components will change with time. Even in this case, one can assume that the object’s shape and density remain constant. Hence, equilibrium
here can be thought of as represented by long-time averages in which case the second derivative also vanishes, not because of constant volume and density, but because of stability (Balaguera-Antolínez & Nowakowski 2006; Balaguera-Antolínez et al. 2006b).

The expressions derived in the last section, especially equations (5) and (10), can be used for testing dark energy models on configurations in a dynamical state of equilibrium. However, in these cases, one should point out that in this approach there is no energy conservation within the overdensity: dark energy flows in and out of the overdensity. Such a feature is a consequence of the assumption that dark energy does not cluster at small scales (homogeneity of dark energy). This is in fact the most common assumption in the literature (Wang & Steinhardt 1998; Chen & Ratra 2004; Horellou & Berge 2005), with a few exceptions investigated in Wang (2006), Maor & Lahav (2005), Caimmi (2007) and Mota & van de Bruck (2004).

In this paper, we will concentrate on the possible effects of a background dominated by a dark energy component represented by the cosmological constant at late times ($z\ll 1$). It implies that the total density involved in the definitions of the integral quantities appearing in the virial equation can be approximated to $\rho \sim \delta \rho$. In that case, the Poisson equation reduces to the form $\nabla^2 \Phi = 4\pi \delta \rho - 8\pi \rho_{\text{vac}}$. As mentioned before, the symbol $\rho_{\text{vac}}$ has no multiplicative factors in the case of a Newton–Hooke space–time, while for a $\Lambda$CDM model it must be understood as $\rho_{\text{vac}} \sim 0.8 \rho_{\text{vac}}$. As the reader will see, the most relevant quantities derived here come in the form of the ratio of a characterizing density and $\rho_{\text{vac}}$, and hence the extra factor appearing in the $\Lambda$CDM can be re-introduced in the characterizing density. For general consideration of equilibrium in the spherical case, see Böhm (2004) and Böhm & Harko (2005), while the quasi-spherical collapse with a cosmological constant has been discussed in Deb Nath et al. (2006).

3 POLYTROPIC CONFIGURATIONS AND THE A LANE–EMDEN EQUATION

We can determine the relevant features of astrophysical systems by solving the dynamical equations describing a self-gravitating configuration (Euler equation, Poisson equation, continuity equations). In order to achieve this goal, we must first know the potential $\Phi$ to be able to calculate the gravitational potential energy. To obtain $\Phi$, one must supply the density profile and solve the Poisson equation. In certain cases, the potential is given and we therefore can solve for the density profile in a simple way. Here, we face the situation where no information on the potential (aside from its boundary conditions) is available and we also do not have a priori information about the density profile (see for instance Binney & Tremaine 1987 for related examples). In order to determine both the potential and the density profile, a complete description of astrophysical systems is required. This means we need to know an EOS $p = p(\rho)$ (here we change notation and we call $\rho$ the proper density of the system). The EOS can take several forms and the most widely used one is the so-called polytropic EOS, expressed as

$$p = \kappa \rho^n, \quad n \equiv 1 + \frac{1}{\gamma},$$

where $\gamma$ is the polytropic index and $\kappa$ is a parameter that depends on the polytropic index, the central density, the mass and the radius of the system. The exponent $\gamma$ is defined as $\gamma = (c_s - c)/(c_s - c)$ and is associated with processes with constant (non-zero) specific heat $c$. It reduces to the adiabatic exponent if $c = 0$. The polytropic EOS was introduced to model fully convective configurations. From a statistical point of view, equation (11) represents a collisionless system whose distribution function can be written in the form $f = f(\dot{E}) \sim \dot{E}^{-3/2}$, with $\dot{E} \equiv \dot{\Phi} - (1/2)\rho v^2$ being the relative energy and $\dot{\Phi}(\rho) \equiv \Phi_0 - \Phi(\rho)$ being the relative potential [where $\Phi_0$ is a constant chosen such that $\dot{\Phi}(\rho = R) = 0$ (Binney & Tremaine 1987)]. In astrophysical contexts, the polytropic EOS is widely used to describe astrophysical systems such as the Sun, compact objects, galaxies and galaxy clusters (Chandrasekhar 1967; Binney & Tremaine 1987; Kennedy & Blandin 1999).

We now derive the well-known Lane–Emden (LE) equation. We start from the Poisson equation and Euler equation for spherically symmetric configurations, written as

$$\frac{d^2 \Phi}{dr^2} + \frac{2}{r} \frac{d \Phi}{dr} = 4\pi \rho - 8\pi \rho_{\text{vac}}, \quad \frac{dp}{dr} = -\frac{d \Phi}{dr}. \quad \text{(12)}$$

This set of equations together with equation (11) can be integrated in order to solve for the density in terms of the potential as

$$\rho(r) = \rho_c \left[ 1 - \left( \frac{\gamma - 1}{\kappa \gamma} \right) \rho_{\text{vac}}^{\gamma-\gamma} (\Phi(r) - \Phi(0)) \right]^{-\frac{1}{\gamma}}, \quad \text{(13)}$$

where $\rho_c$ is the central density. In view of equation (13) in conjunction with equation (7), it is clear that $\rho_{\text{vac}}$ will have the effect to increase the value of $\rho(r)$. Therefore, the boundary of the configuration will be located at a greater $R$ as compared to the case $\rho_{\text{vac}} = 0$. In order to determine the behaviour of the density profile, we again combine equations (11) and (12) in order to eliminate the potential $\Phi(r)$. We obtain

$$\frac{1}{n} \left( \frac{\nabla \rho}{\rho} \right)^2 + \nabla^2 \ln \rho = -\frac{4\pi \rho_{\text{vac}}^{1-\frac{1}{n}}}{\kappa(n+1)} (1 - \zeta), \quad \text{(14)}$$

where we defined the function

$$\zeta = \zeta(r) = 2 \left[ \frac{\rho_{\text{vac}}}{\rho(r)} \right]. \quad \text{(15)}$$

We can rewrite equation (14) by introducing the variable $\psi$ defined by $\rho = \rho_c \psi^n$, where $\rho_c$ is the central density. We also introduce the variable $\xi = r/\alpha$, where

$$\alpha \equiv \sqrt{\frac{\kappa(n+1)}{4\pi \rho_{\text{vac}}^{1-\frac{1}{n}}}} \quad \text{(16)}$$

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Figure 1. Solutions of the ALE equation for different \( \zeta_c \) and different values of the index \( n \) ranging from \( n = 3 \) (red short-dashed line), \( n = 4 \) (blue long-dashed line) and \( n = 5 \) (green dot–short-dashed line).

is a length-scale. Equation (14) is finally written as (Balaguera-Antolínez et al. 2005; Chandrasekhar 1967):

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \frac{\xi^2 d\psi}{d\xi} \right) = \zeta_c - \psi^n, \quad \zeta_c \equiv 2 \left( \frac{\rho_{\text{vac}}}{\rho_c} \right). \tag{17}
\]

This is the \( \Lambda \) Lane–Emden (ALE) equation. Note that for a constant density, we recover \( \rho = 2\rho_{\text{vac}} \) as the first non-trivial solution of the ALE equation. This is consistent with the results from the virial theorem for constant-density spherical objects which tell us that \( \rho \geq 2\rho_{\text{vac}} \) (Nowakowski et al. 2002). Note that using equation (13) we can write the solution \( \psi(\xi) \) with the explicit contribution of \( \rho_{\text{vac}} \) as

\[
\psi(\frac{r}{a}) = 1 - \left( 4\pi a^2 \rho_c \right)^{-1} \left[ \Phi_{\text{grav}}(r) - \Phi_{\text{grav}}(0) \right] + 6\zeta_c\xi^2, \tag{18}
\]

so that for a given \( r \) smaller than the radius we will obtain

\[
\psi(\frac{r}{a}) > \psi(\frac{r}{a})_{\Lambda=0}, \tag{19}
\]
as already pointed out before. The differential equation (17) must then be solved with the initial conditions \( \psi(0) = 1, \; \psi'(0) = 0 \), satisfied by equation (18). Numerical solutions were obtained for the first time in Balaguera-Antolínez & Nowakowski (2005). The solutions presented in Fig. 1 are given in terms of the ratio \( \rho/\rho_{\text{vac}} \) for \( n = 3, 4 \) and 5. This choice of variables is useful also since \( \rho_{\text{vac}} \) sets a fundamental scale of density (the choice \( \rho_0 = \rho_{\text{vac}} \) will be explored for the isothermal sphere, where Fig. 1 will be helpful for discussions). The radius of a polytropic configuration is determined as the value of \( r \) when the density of matter with the EOS (11) vanishes. This happens at a radius located at

\[
R = a\xi_1 \quad \text{such that} \quad \psi(\xi_1) = 0. \tag{20}
\]

Note that equation (17) yields a transcendental equation to determine \( \xi_1 \). Also one notes from Fig. 1 that not all values of \( \zeta_c \) yield allowed configurations in the sense that we cannot find a value of \( \xi_1 \) such that \( \psi(\xi_1) = 0 \). This might not be surprising since for \( n = 5 \) and \( \Lambda = 0 \) we find the situation where the asymptotic behaviour is \( \rho \to 0 \) as \( \xi \to \infty \) (we consider this still as an acceptable behaviour). There is, however, one crucial difference when we switch on a non-zero \( \Lambda \). For \( \Lambda \neq 0 \) not only can we not reach a definite radius, but also the derivative of the density changes sign and hence becomes non-physical. The situation for the cases \( n \geq 5 \) is somewhat similar to the extreme case of \( n \to \infty \) (isothermal sphere). Clearly, these features are responsible for the last term in equation (18), which for high values of \( \zeta \) may become dominant over the remaining (gravitational) terms. We will discuss this case in Section 4 where we will attempt another definition of a finite radius with the constraint \( \psi(\xi) < 0 \). For now it is sufficient to mention that, as expected, the radius of the allowed configurations is larger than the corresponding radius when \( \Lambda = 0 \).
3.1 Equilibrium and stability for polytropes

In this section, we will derive the equilibrium conditions for polytropic configurations in the presence of a positive cosmological constant. We will use the results of the last section in order to write down the virial theorem. The total mass of the configuration can be determined as usual with \( M = \int \rho \, d\mathbf{r} \) together with equation (17). One then has a relation between the mass, the radius and the central density:

\[
R = M^{1/3} \rho_0^{-1/3} f_0(\xi; n) = \left( M R_\Lambda^{1/3} \right)^{1/3} \left( 4\pi \frac{1}{3} \xi_0^{1/3} f_0(\xi; n) \right),
\]

where

\[
f_0(\xi; n) = \left( \frac{\xi^3}{4\pi} \right)^{1/3} \left( \int_0^1 \xi^2 \psi(\xi) d\xi \right)^{-1/3}.
\]

Note that we have introduced the cosmological constant in the equation for the radius, leading to the appearance of the astrophysical length-scale \( \left( M R_\Lambda^{1/3} \right)^{1/3} \) with \( r_\Lambda = \Lambda^{-1/2} = (8\pi \rho_\Lambda)^{-1/2} = 2.4 \times 10^7 \left( \Omega_\Lambda h^3 \right)^{-1/2} \) Mpc \( \approx 4.14 \times 10^7 \) Mpc for the concordance values \( \Omega_\Lambda = 0.7 \) and \( h = 0.7 \). This scale has been already found in the context of the Schwarzschild–de Sitter metric where it is the maximum allowed radius for bound orbits. At the same time, it is the scale of the maximum radius for a self-gravitating spherical and homogeneous configuration in the presence of a positive \( \Lambda \) (Balaguera-Antolínez & Nowakowski 2005). This also lets us relate the mean density of the configuration with its central density and/or cosmological parameters as \( \bar{\rho} = (3/4\pi f_0^3) \rho_0 = (3/2\pi \xi_0^3) \rho_\Lambda = (3 \Omega_\Lambda / 2\pi \xi_0^3) \rho_\Lambda \).

Similarly, we can determine the other relevant quantities appearing in the equations for the energy and the scalar virial theorem (10). For the traces of the moment of inertia tensor and the dispersion tensor \( \Pi \), we can write

\[
I = MR^2 f_1, \quad \Pi = \kappa \rho_0^{4/3} M f_2,
\]

where the functions \( f_{1,2} \) have been defined as

\[
f_1(\xi; n) = \int_0^1 \xi^4 \psi(\xi) d\xi, \quad f_2(\xi; n) = \int_0^1 \xi^2 \psi(\xi) d\xi
\]

using equation (21). These functions are numerically determined in the sequence \( \psi(\xi; c) \rightarrow \xi_1(\xi; c) \rightarrow f(\xi; c) \), such that for a given mass we obtain the radius as \( R = a(1 - \xi_1)^{\xi_1} \).

Let us consider the virial theorem (10) for a polytropic configuration. The gravitational potential energy \( W^{Grav} \) can be obtained following the same arguments as shown in Chandrasekhar (1967). The method consists of integrating the Euler equation and solving for \( \Phi^{Grav} \), and then using equation (9) one obtains \( W^{Grav} \). The final result is written as

\[
W^{Grav} = \frac{M^2}{2R} - \frac{1}{2}(n+1) \Pi + \frac{2}{3} \pi \rho_\Lambda (I - M R^2).
\]

To show the behaviour of \( W^{Grav} \) with respect to the index \( n \), we can solve the virial theorem (10) for \( \Pi \) and replace it in equation (25). We obtain

\[
W^{Grav} = -\frac{3}{5 - n} \left\{ 1 - \rho_\Lambda \frac{\rho}{\bar{\rho}} \left[ \frac{1}{3} (5 + 2n) f_1 - 1 \right] \right\} \frac{M^2}{R}.
\]

This expression shows the typical behaviour of a \( n = 5 \) polytrope (even if \( \rho_\Lambda \neq 0 \)); the configuration has an infinite potential energy, due to the fact that the matter is distributed in an infinite volume. The energy of the configuration in terms of the polytropic index can be easily obtained by using equations (10), (25) and (26):

\[
E = W^{Grav} + 8 \left\{ \frac{\pi \rho_\Lambda}{3} I + n \Pi \right\} = -\frac{3}{5 - n} \left( \frac{3 - n}{5 - n} \right) \frac{M^2}{R} \left\{ 1 - \rho_\Lambda \frac{\rho}{\bar{\rho}} \left[ 5 f_1(\xi) - 1 \right] \right\}.
\]

One is tempted to use \( E < 0 \) as the condition to be fulfilled for a gravitationally bound system. For \( \rho_\Lambda = 0 \), we recover the condition \( \gamma > 4/3 \) \((n < 3)\) for gravitationally bound configurations in equilibrium. On the other hand, for \( \rho_\Lambda \neq 0 \) this condition might not be completely true due to the following reasoning. The two-body effective potential in the presence of a positive cosmological constant does not go asymptotically to zero for large distances, which is to say that \( E < 0 \) is not stringent enough to guarantee a bound system. Therefore, we rather rely on the numerical solutions from which, for every \( n \), we infer the value of \( A_\Lambda \) such that

\[
\rho_c = A_\Lambda \rho_\Lambda.
\]

This gives us the lowest possible central density in terms of \( \rho_\Lambda \). The behaviour of \( \xi_{crit} \), the functions \( f_i \), the solution \( \xi_1(\xi; \xi_{crit}, n) \) and the values of \( A_\Lambda \) are shown in Fig. 2. Note that this inequality can be understood as a generalization of the equilibrium condition \( \rho > A \rho_\Lambda \) which, when applied for a spherical homogeneous configuration, yields \( A = 2 \) with \( \rho = \rho = \text{constant} \) (Balaguera-Antolínez & Nowakowski 2005; Balaguera-Antolínez et al. 2005).

Note that at \( n = 5 \) the radius of the configuration becomes undefined, as well as the energy. An \( n = 5 \) polytrope is highly concentrated at the centre (Chandrasekhar 1967). No criteria can be written since even for \( \rho_\Lambda = 0 \) there is not a finite radius. However, it is this high concentration at the centre and a smooth asymptotic behaviour which make this case still a viable phenomenological model if \( \Lambda \) is zero. In contrast, for non-zero, positive \( \Lambda \) the solutions start oscillating around \( 2 \rho_\Lambda \) which makes the definition of the radius more problematic. For \( n \rightarrow \infty \) the polytropic EOS describes an ideal gas (isothermal sphere). Since in this limit the expressions derived before are not well defined, this case will be explored in more detail in the next section. In spite of the mathematical differences, the isothermal sphere bears many similarities to the cases \( n \geq 5 \) and our conclusions regarding the definition of a radius in the \( n \rightarrow \infty \) case equally apply to finite \( n \) bigger than 5.
3.2 Effects with generalized dark energy EOS

In the last section, we have explored the effects of a dark energy dominated background with the EOS $\omega = -1$. Other dark energy models are often used with $\omega = -1/3$ and $\omega = -2/3$ or even $\omega < -1$, in the so-called phantom regime, or even a time-dependent dark energy model (quintessence). A simple generalization to such models can be easily done by making the following replacement in our equations:

$$\zeta \rightarrow \zeta(a)_{\text{eff}} = -\frac{1}{2} \xi \eta(a) a^{-f(a)},$$

where $\eta(a) = 1 + 3\omega_k(a)$, and where the function $f(a)$ is defined in equation (2). Note that for this generalization to be consistent with the derivation of the ALE equation in equation (17), one needs to consider that the EOS is close to $-1$, so that there is almost no time dependence, and the energy density for dark energy is almost constant. This is indeed the case for most popular candidates of dark energy, especially at low redshifts $z < 1$. Also, note once again that we are still assuming a homogeneous dark energy component which flows freely to and from the overdensity, hence violating energy conservation inside it. Clearly, other models of dark energy will possess dynamical properties that the cosmological constant does not have. For instance, we could allow some fraction of dark energy to take part in the collapse and virialization (Maor & Lahav 2005; Caimmi 2007; Mota & van de Bruck 2004), which would lead to the presence of self- and cross-interaction terms for dark energy and the (polytropic-like) matter in the Euler equation, which at the end modifies the LE equation. With this simplistic approach, we see that the effects with a general EOS are smaller than those associated with the cosmological constant. In particular, the EOS $\omega_k = -1/3$ displays a null effect since it implies $\eta = 0$ (note that this EOS can also resemble the curvature term in the evolution equation for the background). On the other hand, phantom models of dark energy, which are associated with EOSs $\omega_k < -1$ (Caldwell 2002; Nojiri 2005), have quite a strong effect. In Fig. 3, we show numerical solutions of LE equations for a background dominated with dark energy with $\omega_k = -2/3$ and a phantom dark energy with $\omega_k = -2$, with $\zeta = 10^{-3}$. These curves are to be compared with those in Fig. 1. Clearly, EOSs with $\omega_k < -1$ will generate larger radius than the case described in the main text. Furthermore, the asymptotic behaviour of the ratio between the density and $\rho_{\text{vac}}$ is $\rho/\rho_{\text{vac}} \rightarrow |\eta|$.

3.3 Stability criteria with cosmological constant

Stability criteria for polytropic configurations can be derived from the virial equation. Using equations (23)–(25), we can write the virial theorem (10) in terms of the radius $R$ and the mass $M$:

$$-\frac{M^2}{2R} + \frac{1}{2}(5-n)\kappa\rho_c^{1/n} M f_2 + \frac{2}{3} \pi \rho_{\text{vac}} M R^2 (f_1 - 1) = 0,$$

where we have assumed that the only contribution to the kinetic energy comes from the pressure in the form of $K = \frac{1}{2} \Pi$. Note that for $\rho_{\text{vac}} = 0$ and a finite mass, one obtains $R \propto (5-n)^{-1}$ while for $\rho_{\text{vac}} \neq 0$ we would obtain a cubic equation for the radius. Instead of solving for the virial radius, we solve for the mass as a function of central density with the help of equation (21). We have

$$M = \mathcal{G} \rho_c^\frac{1}{3}, \quad \mathcal{G} = \mathcal{G}(\zeta; n) \equiv \left( \frac{\kappa f_1 f_2 (5-n)}{1 - \frac{2\zeta}{3c} f_2 (5f_1 - 1)} \right)^{3/2}.$$

The explicit dependence of the mass with respect to the central density splits into two parts: on the one hand, it has the same form as the usual case with $\Lambda = 0$, that is, $\rho_c^{3-\omega} 2^\omega$, while, on the other hand, the function $\mathcal{G}$ has a complicated dependence on the central density because of
the term $\zeta_c$. With the help of equations (21) and (31) we can write a mass–radius relation and the radius–central density relation:

$$M = \left[ G \frac{1}{\gamma (\pi + 1)} f_0^{\gamma + 1} \right] R^{\frac{1}{\gamma - 1}}, \quad R = \left( \frac{G + f_0}{f_0} \right)^{\frac{1}{\gamma - 1}} \rho_c^{\frac{1}{\gamma - 1}}. \tag{32}$$

Following the stability theorem (see for instance in Weinberg 1972), the stability criteria can be determined from the variations in the mass in equilibrium with respect to the central density. We derive from equation (32)

$$\frac{\partial M}{\partial \rho_c} = \left[ \frac{3}{2} \left( \gamma - \frac{4}{3} \right) \rho_c^{-1} G + \frac{\partial G}{\partial \rho_c} \right] \rho_c^{\frac{4}{\gamma - 1}}. \tag{33}$$

Stability (instability) stands for $\partial M/\partial \rho_c > 0$ ($\partial M/\partial \rho_c < 0$). This yields a critical value of the polytropic exponent $\gamma_{\text{crit}}$ when $\partial M/\partial \rho_c = 0$ given by

$$\gamma_{\text{crit}} = \gamma_{\text{crit}}(\zeta_c) = \frac{4}{3} + 2 \frac{\partial \ln G}{\partial \ln \rho_c}, \tag{34}$$

in the sense that polytropic configurations are stable under small radial perturbations if $\gamma > \gamma_{\text{crit}}$. It is clear that the second term in equation (34) also depends on the polytropic index and therefore this equation is essentially a transcendental expression for $\gamma_{\text{crit}}$.

It is worth mentioning that by including the corrections due to general relativity, the critical value for $\gamma_{\text{crit}}$ is also modified as $\gamma_{\text{crit}} = (4/3) + R_0/R$ (Shapiro & Teukolsky 1983) and hence for compact objects the correction to the critical polytropic index is stronger from the effects of general relativity than from the effects of the background. This is as we would expect it. Stability of relativistic configurations with non-zero cosmological constant has been explored in Böhmer & Harko (2005) and Böhmer (2004).

Going back to equation (31), we can write the mass of the configuration as $M = \alpha_M M(0)$, where $M(0)$ is the mass when $\Lambda = 0$ and $\alpha_M = \alpha_M(\zeta_c, n)$ is the enhancement factor. Both quantities can be calculated to give

$$M(0) = \left[ \kappa (S - n) f_0^{\alpha_0} f_2^{\alpha_2} \right]^{3/2} \rho_c^{\frac{1}{2}}, \quad \alpha_M = \left\{ \frac{f_0 f_2}{f_0^{\alpha_0} f_2^{\alpha_2} \left[ \frac{1}{2} \pi \xi c f_0^2 (5 f_1 - 1) \right]} \right\}^{3/2}, \tag{35}$$

where $f_i^{\alpha_i} \equiv f_i(\xi_c = 0, n)$ are numerical factors (tabulated in Table 1) that can be determined in a straightforward way. Similarly, by using equation (21), the radius can be written as $R = \alpha_R R(0)$, where

$$R(0) = \left[ \kappa (S - n) f_0^{\alpha_0} f_2^{\alpha_2} \right]^{1/2} \rho_c^{\frac{1}{2}}, \quad \alpha_R = \left\{ \frac{f_0 f_2}{f_0^{\alpha_0} f_2^{\alpha_2} \left[ \frac{1}{2} \pi \xi c f_0^2 (5 f_1 - 1) \right]} \right\}^{1/2}. \tag{36}$$
As was done in equation (13), we can integrate the equilibrium equations (Euler and Poisson equations) and obtain an explicit solution for the density with a cosmological constant. Because of the limiting case provided that \( \zeta \) is the critical ratio \( \zeta_{\text{crit}} \) which separates the configurations with a definite ratio such that a zero \( \xi_1 \) exists, provided that \( \zeta < \zeta_{\text{crit}} \). We will show some examples where the enhancement factors may be relevant in Section 5.

| \( n \) | \( \zeta_c = 0.1 \) | \( \zeta_c = 0.05 \) | \( \zeta_c = 0.001 \) |
|---|---|---|---|
| 1 | (1.12, 1.29) | (1.05, 1.12) | (1.001, 1.002) |
| 1.5 | (–, –) | (1.11, 1.17) | (1.002, 1.003) |
| 3 | (–, –) | (–, –) | (1.022, 1.01) |

In Table 1, we show the values of the enhancement factors \( \alpha_M \) and \( \alpha_R \) for different values of \( \zeta_c \) and different values of the polytropic index \( n \). We also show the values of the critical ratio \( \zeta_{\text{crit}} \) which separates the configurations with a definite ratio such that a zero \( \xi_1 \) exists, provided that \( \zeta < \zeta_{\text{crit}} \). We will show some examples where the enhancement factors may be relevant in Section 5.

### 4 THE ISOTHERMAL SPHERE

The isothermal sphere is a popular model in astrophysics, to model large astrophysical and cosmological objects (galaxies, galaxy clusters) (Lynden-Bell & Wood 1968; Yabushita 1968; Penston 1969; Sommer-Larsen, Vedel & Hellsten 1996; Chavanis 2001), to examine the so-called gravothermal catastrophe (Binney & Tremaine 1987; Natarajan & Lynden-Bell 1997; Lombardi & Berti 2001) and finally to compare observations with model predictions (Rines et al. 2002). In the limit \( n \rightarrow \infty \) in the polytropic EOS, one obtains the description for an isothermal sphere (ideal gas configuration) with

\[
p = \sigma^2 \rho,
\]

where \( \sigma \) is the velocity dispersion \( (\sigma^2 \propto T) \). The pattern we found in Section 3 for \( n \geq 5 \) gets confirmed here: no finite radius of the configuration is found with \( \Lambda \), the asymptotic behaviour is not \( \rho \rightarrow 0 \) as \( r \rightarrow \infty \) (but rather \( \rho \rightarrow 2 \rho_{\text{vac}} \)) and, as we will show below, other attempts to define a proper finite radius are not satisfactory.

The results for a finite value of the index \( n \) are defined in the limit \( n \rightarrow \infty \) only asymptotically in the case \( \Lambda = 0 \). We consider this as an acceptable behaviour of the density. Because of the limiting case \( n \rightarrow \infty \), the analysis for the isothermal sphere must be done in a slightly different way. As was done in equation (13), we can integrate the equilibrium equations (Euler and Poisson equations) and obtain an explicit dependence of the density with \( \rho_{\text{vac}} \) as

\[
\rho(r) = \rho_c \exp \left\{ -\frac{1}{\sigma^2} \left[ \Phi_{\text{grav}}(r) - \Phi_{\text{grav}}(0) \right] \right\} \exp \left( \frac{8}{3\sigma^2} \pi \rho_{\text{vac}} r^2 \right).
\]

The resulting differential equation for the density with a cosmological constant can be written as

\[
\frac{\sigma^2}{r^2} \frac{d}{dr} \left( r^2 d \ln \rho \right) = -4\pi \rho + 8\pi \rho_{\text{vac}}.
\]

This differential equation could be treated in the same way as we did for the polytropic EOS, that is, by defining a new function \( \psi \sim \rho/\rho_c \), but here we can already use the fact that the cosmological constant introduces scales of density, length and time (Balaguera-Antolín & Nowakowski 2005a). Let us then define the function \( \psi(\xi) = \ln(\rho_c / \rho_{\text{vac}}) \) and \( r = r_0\xi \), with \( r_0 \) the associated length-scale. Since we are now scaling the density with \( \rho_{\text{vac}} \), the associated length-scale \( r_0 \) should also be scaled by the length-scale imposed by \( \Lambda \):

\[
r_0 = \sigma r_N = 13.34 \left( \frac{\sigma}{10^7 \text{km s}^{-1}} \right) \text{Mpc}.
\]

For a hydrogen cloud with \( \sigma \sim 4 \text{ km s}^{-1} \), we have \( r_0 \approx 40 \text{ kpc} \) which is approximately the radius of an elliptical (E0) galaxy. In terms of the function \( \psi(\xi) \), the differential equation governing the density profile is then written as

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = 1 - \frac{1}{2}\xi \psi,
\]

so that according to equation (38) we may write

\[
\psi \left( \frac{r}{r_0} \right) = \ln \left( \frac{\rho_c}{\rho_{\text{vac}}} \right) - \sigma^2 \left[ \Phi_{\text{grav}}(r) - \frac{8}{3} \pi \rho_{\text{vac}} r^2 \right].
\]

From this we can derive different solutions \( \psi \) depending on the initial condition \( \psi(\xi = 0) = \ln(\rho_c / \rho_{\text{vac}}) \) and \( d\psi(\xi)/d\xi = 0 \) at \( \xi = 0 \). In Fig. 4, we show numerical results for the solutions of equation (41) using different values of \( \rho_c/\rho_{\text{vac}} \). As is the case for \( n > 5 \), the radius cannot be defined by searching the first zero of the density, that is, the value \( \xi_1 \) such that \( e^{\psi(\xi)} = 0 \) (including \( \psi \rightarrow -\infty \)). In this case, the behaviour of the derivative of the density profile changes as compared with the \( \Lambda = 0 \) case since with increasing \( \xi \) the density starts oscillating around the

\[1\] Compare with equation (374) of Chandrasekhar (1967) or equation (1) of Natarajan & Lynden-Bell (1997) where the density is scaled by the central density. The factor 1 on the right-hand side of equation (41) is due to \( \rho_{\text{vac}} \).
value \( \rho = 2\rho_{\text{vac}} \) such that for \( \xi \to \infty \) one has a solution \( \rho \to 2\rho_{\text{vac}} \). This can be checked from equation (41) which corresponds to the first non-trivial solution for \( \rho \). This behaviour implies that there exists a value of \( \xi = \xi_1 \) where the derivative changes sign and hence the validity of the physical condition required for any realistic model i.e. \( d\rho/dr < 0 \) should be given up, unless we define the size of the configuration as the radius at the value of the first local minimum. We will come back to this option below to show that it is not acceptable. A second option would be to set the radius at the position where the density acquires for the first time its asymptotic value \( 2\rho_{\text{vac}} \). We could motivate such a definition by demanding that the density at the boundary goes smoothly to the background density. This is for two reasons, however, not justified. First, we recall that a positive cosmological constant leads to a repulsive ‘force’ as it accelerates the expansion. A negative cosmological constant could be modelled in a Newtonian sense by a constant positive density which, however, strictly speaking, is still not a background density. Secondly, if we include the background density \( \rho_b \) we would have started with \( \rho + \rho_b \) (with a dynamical equation for \( \rho \) being \( \rho_b = \) constant) in which case the boundary condition would again be \( \rho(R) = 0 \) to define the extension of the body (or at least, \( \rho \to \infty \) as \( r \to \infty \)). Hence, this second option can be excluded on general grounds. In any case as can be seen from Fig. 4, both definitions would yield two different values of the radius. Since the first candidate to define a radius is based on a physical condition of the configuration, we could expect this definition to be the more suitable one. However, such a definition must be in agreement with the observed values for masses and radii of specific configurations and the validity of this definition can be put to the test by the total mass of the configuration, given as

\[
M = 1.55 \times 10^{15} \left( \frac{\sigma}{10^3 \text{ km s}^{-1}} \right)^3 f(\xi_1) M_\odot .
\]

Combining equations (40) and (43), we can write

\[
M = 653.55 \left( \frac{R}{\text{kpc}} \right)^5 \xi_1^{-3} f(\xi_1) M_\odot .
\]

If we define the radius at the first minimum (see Fig. 4), we find \( M \approx 2 \times 10^6 (R/\text{kpc})^3 M_\odot \). Although this might set the right order of magnitude for the mass of an E0 galaxy if we insist on realistic values for the respective radius, say \( R \approx 10 \) kpc, the picture changes again as the radius is fixed by equation (40) which gives \( R \approx 5.3 \times 10^6 (\sigma/10^3 \text{ km s}^{-1}) \) kpc. In order to get a radius of the order of kpc with masses of the order of \( 10^{10} M_\odot \), we would require \( \sigma \sim 10^{-3} \text{ km s}^{-1} \). This differs by almost eight orders of magnitude from the measured values for the velocity dispersion \( \sigma \) in elliptical galaxies (\( \sigma \sim 300 \text{ km s}^{-1} \)) or from the Faber–Jackson law for velocity dispersion (Padmanabhan 1993). We conclude that defining the radius by the position of the first minimum is not a realistic solution. As the last option to get realistic values for the parameters of the configuration, we consider the brute force method to simply fix the value of \( \xi_1 \). It is understood, however, that this method is not acceptable if we insist that the model under consideration has some appealing features (without such features almost any model would be phenomenologically viable). Therefore, we discuss this option only for completeness. For a configuration with \( \rho_c = 10^5 \rho_{\text{vac}} \), say an elliptical galaxy, we fix the radius at \( R \approx 50 \) kpc in equation (40) and using a typical value for the velocity dispersion \( \sigma \sim 300 \text{ km s}^{-1} \) we get \( \xi_1 \sim 0.012 \) which implies \( f(\xi_1) \sim 0.036 \). The mass in equation (43) is then given as \( M \sim 1.5 \times 10^5 M_\odot \) while the density at the boundary is \( \rho_b \sim 36000 \rho_{\text{vac}} \), that is, \( \rho_c \sim 2.35 \rho_b \). In Table 2, we perform the same exercise for other radii. The resulting mean density is in accordance with the observed values of the mean density of astrophysical objects ranging from a small elliptical galaxy to a galaxy cluster. However, as mentioned above, the model introduces an arbitrary cut-off and cannot be considered as a consistent model of hydrostatic equilibrium.
Table 2. Values for $\xi_1$, $\rho_c/\rho(R)$ and the masses for different values of radius and for $\rho_c = 10^7 \rho_{\text{vac}}$ with $\sigma = 300 \text{ km s}^{-1}$.

| $R$ (kpc$^{-1}$) | $\xi_1$ | $\rho_c/\rho(R)$ | $M$ (M$\odot$) | $\bar{\rho}$ (g cm$^{-3}$) |
|-----------------|---------|-----------------|----------------|------------------|
| 10              | 0.00249 | 1.0005          | $2.17 \times 10^8$ | $3.4 \times 10^{-25}$ |
| 50              | 0.01248 | 1.0129          | $2.7 \times 10^9$  | $1.94 \times 10^{-25}$ |
| 100             | 0.0249  | 1.052           | $2.11 \times 10^{11}$ | $7.2 \times 10^{-26}$ |

Figure 5. The same as Fig. 2, but now for the functions $f_i$ being evaluated at the value $\xi_{\text{vir}}$ solution of equation (46).

In summary, the attempts to define a finite radius for the isothermal sphere fail in the presence of a cosmological constant either because such a model fails to reproduce certain phenomenological values (if the definition of the radius is fixed by the first minimum) or because the definition is, technically speaking, quite artificial to the extent of introducing arbitrary cut-offs. Note that this conclusion is valid almost for any object as the density of the isothermal sphere with $\Lambda$ has a minimum whatever the central density we choose.

5 EXOTIC ASTROPHYSICAL CONFIGURATIONS

In this section, we will probe into the possibilities of exotic, low-density configurations. The global interest in such structures is twofold. First, the cosmological constant will affect the properties of low-density objects. Secondly, $\Lambda$ plays effectively the role of an external, repulsive force. Hence, a relevant issue that arises in this context is to see whether the vacuum energy density can partly replace the pressure which essentially is encoded in the parameter $\kappa (p = \kappa \rho \gamma)$. By the word replace we mean that we want to explore the possibility of a finite radius as long as the pressure effects are small in the presence of $\rho_{\text{vac}}$.

5.1 Minimal density configurations

As mentioned above, the effect of a positive cosmological constant on matter is best understood as an external repulsive force. In previous sections, we have probed into one extreme which describes the situation where a relatively low-density object is pulled apart by this force (to an extent that we concluded that the isothermal sphere is not a viable model in the presence of $\Lambda$). Limiting conditions when this happens were derived. On the other hand, approaching with our parameters these limiting conditions, but remaining still on the side of equilibrium, means that relatively low-density objects can be still in equilibrium, thanks to the positive cosmological constant. The best way to investigate low-density structures is to use the lowest possible central density. As explained in Section 3, for every $n$ there exists an $A_n$ such that $\rho_c \geq A_n \rho_{\text{vac}}$ which defines the lowest central density. Certainly, a question of interest is to see what such objects would look like. We start with the parameters of the configuration. The radius at the critical value $\xi_{\text{crit}}$ is given by equation (21) after taking the limit $\rho_c \to A_n \rho_{\text{vac}}$ (see equation 28). It is given by

$$R_{\text{crit}} = 2.175 \left( \frac{M}{10^{12} \text{ M}_\odot} \right)^{1/3} f_0(\xi_{\text{crit}}; n) \frac{\xi_{\text{crit}}^{1/3}}{r_{\text{crit}}} \text{ Mpc}. \quad (45)$$

From Fig. 5, we can see that the product $f_0(\xi_{\text{crit}}; n)\xi_{\text{crit}}^{1/3}$ changes a little round the value $\sim 0.2$ as we change the index $n$, so that these polytropic configurations will have roughly the same radius (for a given mass). This implies that such configurations have approximately the same average density $\bar{\rho}(\xi_{\text{crit}}) = 1.64 \rho_{\text{crit}} \approx 2.34 \rho_{\text{vac}}$, that is, such configurations have a mean density of the order of the critical density of the.$\odot$2007 The Authors. Journal compilation $\odot$ 2007 RAS, MNRAS 382, 621–640
Universe. Given such a density, we would, at first glance, suspect that the object described by this density cannot be in equilibrium. However, our result follows strictly from hydrostatic equilibrium and therefore there is no doubt that such an object can theoretically exist. Furthermore, \( \bar{\rho} \) satisfies the inequalities derived in Nowakowski et al. (2002) and Balaguera-Antolínez et al. (2005) from virial equations and Buchdahl inequalities which guarantee that the object is in equilibrium (\( \bar{\rho} > 2 \rho_{\text{vac}} \)). Interestingly, the central density for such objects has to be much higher than \( \bar{\rho} \) as, for example, for \( n = 3 \) we have \( A_3 \approx 300 \) and therefore \( \rho_c > 300 \rho_{\text{vac}} \). Note that these values have been given from the solution of the LE equation, which is a consequence of dynamical equations reduced to describe our system in a steady state. However, we have not tried to solve explicitly quantities from the virial theorem. This makes sense as for DMHs the parametrized density profiles go often only asymptotically to zero and the radius of DMHs is defined as a virial radius where the density is approximately 200 times over the critical one. Therefore, an analysis using virial equations seems to be adequate here. For constant density equation (30) can be expressed as a cubic equation (Balaguera-Antolínez et al. 2006a) for the radius at which the virial theorem is satisfied (let us ignore for this analysis any surface terms coming from the tensor virial equation). However, if the density is not constant, this expression becomes a transcendental equation for the dimensionless radius \( \xi_{\text{vir}} = R_{\text{vir}} / a \). This equation is

\[
\xi_{\text{vir}}^2 = \frac{6(n - 1) f_0}{n + 1} \left[ 3 - 2\xi f_0 (5f_0 - 1) f_0 \right],
\]

understanding the functions \( f_i \) now as integrals up to the value \( \xi_{\text{vir}} \). Once we fix \( \xi_{\text{crit}} \) for a given index \( n \), we use as a first guess for the iteration process the value \( \xi_{\text{vir}}(\xi_{\text{crit}}) \). In Fig. 5, we show the behaviour of the functions \( f_i \) and the solutions of equation (46). For these values, equation (45) gives for \( n = 3 \) a radius

\[
R_{\text{vir}} = 25.8 \left( \frac{M}{M_{\odot}} \right)^{1/3} \text{pc},
\]

which can be compared with the radius–mass relation derived in the top-hat spherical collapse (Padmanabhan 1993):

\[
R_{\text{vir}} = 21.5 h^{-2/3} (1 + z_{\text{vir}})^{-1} \left( \frac{M}{M_{\odot}} \right)^{1/3} \text{pc},
\]

where \( h \) is the dimensionless Hubble parameter and \( z_{\text{vir}} \) is the redshift of virialization. The resulting average density is then of the order of the value predicted by the top-hat spherical model:

\[
\bar{\rho} = \left( \frac{3 \Omega_{\text{vac}}}{2 \pi c^2 f_0} \right) \rho_{\text{crit}} \approx 200 \rho_{\text{crit}}.
\]

Note that with the help of equations (16) and (45) the mass can be written as proportional to the parameter \( \kappa^{3/2} \) (introduced in the polytropic EOS equation 11). Therefore, \( \kappa \to 0 \) is equivalent to choosing a small pressure and, at the same time, a small mass which, in case of a relatively small radius, amounts to a diluted configuration with small density and pressure. Without \( \Lambda \) such configurations would be hardly in equilibrium. Hence, for the configuration which has an extent of parsecs, equation (47), with one solar mass we conclude that the equilibrium is not fully due to the pressure, but partially maintained also by \( \Lambda \). This is possible, as \( \Lambda \) exerts an outwardly directed non-local force on the body. Other mean densities, also independent of \( M \), are for \( n = 1.5 \) and 4, respectively,

\[
\bar{\rho} = 15.3 \rho_{\text{crit}} \quad \text{and} \quad \bar{\rho} = 2.6 \times 10^9 \rho_{\text{crit}}.
\]

The first value is close to \( \rho_{\text{crit}} \) and therefore also to \( \rho_{\text{vac}} \). Certainly, if in this example we choose a small mass, equivalent to choosing a negligible pressure, a part of the equilibrium is maintained by the repulsive force of \( \Lambda \). In Balaguera-Antolínez et al. (2005), we found a simple solution of the hydrostatic equation which has a constant density of the order of \( \rho_{\text{vac}} \). The above is a non-constant and non-trivial generalization of this solution.

5.2 Cold white dwarfs

The neutrino stars which we will discuss in the subsequent section are modelled in close analogy to white dwarfs. Therefore, it makes sense to recall some part of the physics of white dwarfs. In addition, we can contrast the example of white dwarfs to the low-density cases affected by \( \Lambda \).

In the limit where the thermal energy \( k_B T \) of a (Newtonian) white dwarf is much smaller than the energy at rest of the electrons (\( \rho_{\text{e}} \)), these configurations can be treated as polytropic configuration with \( n = 3 \). This is the ultrarelativistic limit where the mass of electrons is much smaller than Fermi’s momentum \( p_{\text{F}} \). In the opposite case, we obtain a polytrope or configuration with negligible pressure, a part of the equilibrium is maintained by the repulsive force of \( \Lambda \). In Balaguera-Antolínez & Teukolsky 1983). In both cases, the parameter \( \kappa_\gamma \) from the polytropic EOS is given as

\[
\kappa_3 = \frac{1}{12\pi^2} \left( \frac{3 \pi^2}{m_\text{e} \mu} \right)^{4/3}, \quad \kappa_{3/2} = \frac{1}{15 m_\text{e} \pi^2} \left( \frac{3 \pi^2}{m_\text{e} \mu} \right)^{5/3},
\]

where \( m_\text{e} \) is the electron mass, \( m_\mu \) is the electron mass and \( \mu \) is the number of nucleons per electron. Using the Newtonian limit with cosmological constant, we can derive the mass and radius of these configurations in equilibrium. In the first case, for \( n = 3 \) the mass is written using equation (31) as \( M_\odot = G(n = 3) \) which corresponds approximately to the Chandrasekhar limit (strictly speaking a configuration would have the critical mass, that is, the Chandrasekhar limit, if its polytropic index \( \gamma \) is such that \( \gamma = \gamma_{\text{crit}} \)). For this situation, one has \( M_\odot(n = 3) = 5.87 \times 10^5 M_\odot \).
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Figure 6. Mass and radius for different central densities, for indices $n = 3$ and $3/2$. The masses range from $m_f = 1$ up to $5 \text{ eV}$. and $R_0(n = 3) = 6.8(\bar{\rho}_c/\rho_\odot)^{1/3} \mu^{-5/2} \text{ R}_\odot$. On the other hand, for $n = 3/2$ one obtains $M_0(n = 3/2) = 3.3 \times 10^{-3}(\bar{\rho}_c/\rho_\odot)^{-1/2} \mu^{-5/2} \text{ M}_\odot$ and the radius is given by $R_0(n = 3/2) = 0.27(\bar{\rho}_c/\rho_\odot)^{1/6} \mu^{-5/2} \text{ R}_\odot$, where $\bar{\rho}_c$ is the mean density of the Sun. Since for these configurations the ratio $\zeta_c$ is much smaller than $10^{-4}$, we see from Fig. 2 that the effects of $\Lambda$ are almost negligible. The critical value of the ratio $\zeta_c$ gives for $n = 3$ the inequality $\rho_c > 307.69 \rho_{\text{vac}}$ and for $n = 3/2$ the same limit reads $\rho_c > 24.24 \rho_{\text{vac}}$. Central densities of white dwarfs are of the order of $10^5 \text{ g cm}^{-3}$ which corresponds to a deviation of nearly 30 orders of magnitude of $\rho_{\text{vac}}$.

5.3 Neutrino stars

An interesting possibility is to determine the effects of $\rho_{\text{vac}}$ on configurations formed by light fermions. Such configurations can be used, for instance, to model galactichaloes (Jetzer 1996; Dolgov & Hansen 2002; Lattanzi et al. 2003; Börner 2004). While discussing the phenomenological interest of fermion stars below, we intend to describe such a halo. Clearly, these kinds of systems will maintain equilibrium by counterbalancing gravity with the degeneracy pressure as in a white dwarf. For stable configurations, that is, $n = 3/2$, one must replace the mass of the electron and nucleon by the mass of the considered fermion and set $\mu = 1$ in equations (31) and (51). We then get for the mass and the radius

$$M_0 = 3.28 \times 10^{28} \left( \frac{\rho_c}{\bar{\rho}_c} \right)^{1/8} \left( \frac{\text{eV}}{m_f} \right)^4 \text{ M}_\odot = 3.64 \times 10^{44} \zeta^{-1/2} \left( \frac{\text{eV}}{m_f} \right)^4 \text{ M}_\odot$$

and

$$R_0 = 1.31 \times 10^{-4} \left( \frac{\rho_c}{\bar{\rho}_c} \right)^{-1/8} \left( \frac{\text{eV}}{m_f} \right)^{1/4} \text{ Mpc} = 6.16 \zeta^{1/8} \left( \frac{\text{eV}}{m_f} \right)^{1/4} \text{ Mpc},$$

respectively. On the other hand, for $n = 3$ one has

$$M_0 = 5.16 \times 10^{18} \left( \frac{\text{eV}}{m_f} \right)^2 \text{ M}_\odot$$

and

$$R_0 = 0.14 \left( \frac{\text{eV}}{m_f} \right)^{1/8} \left( \frac{\rho_c}{\bar{\rho}_c} \right)^{-1/8} \text{ pc} = 3.1 \times 10^2 \zeta^{1/3} \left( \frac{\text{eV}}{m_f} \right)^{2/3} \text{ pc}.$$
values for the mass and radius are sensitive to the choice of $m_f$. Indeed, the dependence on the fermion mass is much stronger than on the central density. It is justified to speculate that a relatively low-density object, affected by $\Lambda$, might exist. If then, as in an example, we choose $m_f \sim 5\,\text{eV}$ and $\rho_{\text{vac}} \sim 40\rho_{\text{vac}}$, then the mass comes out as $10^{12}$ solar masses with a radius of the order of magnitude of half a Mpc which might indeed be the DMH of a galaxy (or at least part of the halo). As pointed out before, such configurations must have a central density greater than $24.4\rho_{\text{vac}}$, in order to be in equilibrium. From Table 1, we see that the effect on a configuration with $\xi_c \sim 0.05$ is represented in an increase in the mass by 17 per cent with respect to $M_0$, and an increment of 11 per cent in the radius. The conclusion would then be that $\Lambda$ affects such a DMH. This is to be taken with some caution as the fermions in such a configuration would be essentially non-relativistic. Note also that it is not clear if neutrinos make up a large fraction of the halo; however, we can also speculate about a low-density clustering around luminous matter. Of course, allowing larger central density might change the picture. However, the emerging scenario would not necessarily be a viable phenomenological model. For instance, changing the value of $m_f$ from eV to keV (MeV) would reduce the mass by 12 (24) orders of magnitude which is definitely too small to be of interest. We could counterbalance this by increasing the central density by 24 (48!) orders of magnitude. Such a ‘countermeasure’ would, however, result in a reduction in the radius by eight (16!) orders of magnitude, again too small a length-scale to be of importance for DMHs. In other words, the example with a fermion mass of the order of 1 eV and low central density is certainly of some phenomenological interest.

The case $n = 3$ is similarly stringent. A neutrino mass of 1–5 eV gives a mass for the entire object of the order of 10–18 solar masses which is too large. A fermion mass of several keV would be suitable for a galaxy halo (a mass of the order of MeV and higher would give too small a total mass). With a relatively low central density as before (see Fig. 6), we then obtain the right order of magnitude for the halo. However, then we will have to live with the fact that on the average the radius of such a halo reaches the next large galaxy. Briefly, we touch upon the other possible applications of fermion stars which have been discussed as candidates for the central object in our Galaxy. If we allow the extent of this object to be 120 au and the mass roughly 2.6 million solar masses, then the fermion mass would come out as $10^4\,\text{eV}$ for $n = 1.5\,(10^6\,\text{eV}\) for $n = 3$) and the central density as $10^{22}\rho_{\text{vac}}\,\text{for} \,n = 3)$.

5.4 Boson stars

We end the section by putting forward a speculative question in connection with boson stars. The latter are general relativistic geons and can be treated exactly only in a general relativistic framework, that is, these kinds of configurations are based on the interaction of a massive scalar field with gravitation which leads to gravitationally bound systems. These objects have been also widely discussed as candidates for dark matter (Ruffini & Bonazzola 1969; Lai 2004). On the other hand, variational methods in the connection with the Thomas–Fermi equations give relatively good results even without invoking the whole general relativistic formalism. By including $\Lambda$, we essentially introduce into the theory a new scale, say in this case a length-scale $r_{\Lambda} = 1/\sqrt{\Lambda}$. The basic parameters of a dimension length in a theory with a boson mass $m_B$ and a total mass $M = N_B m_B$, where $N_B$ is the number of bosons, are (for a better distinction of the different length-scales, we restore in this section the value of $G_N$):

$$L_1 = r_s = G_N M, \quad L_2 = r_B = \frac{1}{m_B}, \quad L_3 = r_{\Lambda} = \frac{1}{\sqrt{\Lambda}}. \quad (54)$$

The resulting radius of the object’s extent can be a combination of these scales, that is,

$$R_{(1)} \propto L_1, \quad R_{(2)} \propto N_B^2 L_2, \quad R_{(3)} \propto (L^2_1 L_2)^{1/3}, \quad R \propto N_B^2 R_{(1)} \quad (55)$$

and a similar combination of higher order. Which one of these combinations gets chosen depends on the details of the model. In a close analogy to Spruch (1991) and Eckehard & Schunck (1998), we can examine this taking into account the presence of a positive cosmological constant by considering the energy of such a configuration as a two-variable function of the mass and the radius, $E = E(R, M)$:

$$E \sim \frac{N_B}{R} G_N n_B^2 \frac{N_B}{R} + \frac{8}{3} G_N \rho_{\text{vac}} m_B N_B R^2. \quad (56)$$

The first term corresponds to the total kinetic energy written as $K = N_B p = N_B/\lambda$ and taking $\lambda \sim R$. The second term is the gravitational potential energy and the third term corresponds to the contribution of the background (see equation 8). By treating mass and radius as independent variables (we think this is the right procedure since the radius will depend on the ‘external force’ due to $\rho_{\text{vac}}$), we extremize the energy leading to the following values of mass and radius:

$$M \sim \frac{1}{G_N m_B} \sim 10^{-10} \left(\frac{\text{eV}}{m_B}\right) M_\odot, \quad R \sim \left(\frac{1}{8\pi G_N m_B \rho_{\text{vac}}}\right)^{1/3} \sim 10^3 \left(\frac{\text{eV}}{m_B}\right)^{1/3} R_\odot. \quad (57)$$

Such values would lead to a mean density of the order of $\bar{\rho} \sim 6\rho_{\text{vac}}$, that is, an extremely low-density configuration. The mass given in the last expression is the so-called Kaup limit (Spruch 1991). Of course, this relatively simple treatment does not guarantee that the full, general relativistic treatment will give the same results. Therefore, we consider it as a conjecture. However, it is also obvious from the discussion above that low-density boson stars are a real possibility worth pursuing with more rigour (we intend to do so in the near future).
5.5 A comparison between ALE profiles and DMH profiles

It is of some importance to see whether our results from the examination of polytropic hydrostatic equilibrium or from the virial equations can be applied to dark matter configurations. N-body simulations based on a ΛCDM model of the Universe show that the density profile of virialized DMHs can be described by a profile of the form (Navarro et al. 1996, hereafter NFW):

$$\rho(r) = (2)^{1-m} \rho_s \left( \frac{r}{r_s} \right)^{-m} \left( 1 + \frac{r}{r_s} \right)^{-3},$$

(58)

where $r_s$ is the characteristic radius [the logarithmic slope is $\ln \rho / \ln r_c = -m + \frac{1}{2}(m - 3)$], $\rho_s = \rho(r_c)$ and the index $m$ characterizes the slope of the profile in the central regions of the halo. The mass of the configuration enclosed in the virial radius $r_{\text{vir}}$ is given as $M_{\text{vir}} = 4\pi(2)^{3-m}\rho_s r_s^3 F(c)$ such that one can write $\rho_s = (1/3)(2)^{m-1}\Delta_{\text{vir}} \rho_{\text{crit}} c^3 F^{-1}(c)$, where $c = r_{\text{vir}}/r_s$ is the concentration parameter, $\Delta_{\text{vir}}$ is the ratio between the mean density at the time of virialization and the critical density of the universe [$\Delta_{\text{vir}} \approx 18\pi^2$ in the top-hat model for a flat Einstein–de Sitter universe, while $\Delta_{\text{vir}} \approx 104$ in the ΛCDM cosmological model (Diemand, Kuhlen & Madau 2007)] and $F(c) = \int_0^c x^{2-m}(1+x)^{-m-1}dx$. This universal profile has been widely used in modelling DMHs in galaxy clusters, and the a comparison of these models with a stellar polytropic-like profile – without the explicit contribution from the cosmological constant in the LE equation – can be found in Cabral-Rosetti et al. (2004) and Arieli (2003).

Some differences can be described between the ALE and the NFW profiles. First, on fundamental grounds, it is clear that the NFW profile does not satisfy the LE equation; a basic reason for this is that dark matter is assumed to be collisionless and is only affected by gravity. However, the fact that the real nature of dark matter is still an unsolved issue leaves an open door through which one can introduce interaction between dark matter particles leading to a different EOS (see for instance Ren et al. 2006). On functional forms, one sees that at the central region the difference is abrupt, since the LE equation has a flat density profile at $r = 0$, while the NFW profile has a cuspy profile of the form $\rho \sim r^{-m}$. It has been widely discussed how such a cuspy profile is inconsistent with data showing central regions of clusters with homogeneous cores (see discussion at the end of this section). Such a small slope in the inner regions can be reproduced by the universal profile for the case $m = 0$, and for the profile derived from the ALE equation (which is just a consequence of initial conditions, and hence it is independent of $\Lambda$). In this case, the effects of the cosmological constant can be reduced to explore the outer regions of the haloes and compare, for instance, the slope of the profiles and the virial radius predicted by each one.

The density profiles predicted by the ALE and the NFW profile (for $m = 1$) are presented in Fig. 7 for the virial radius given by $r_{\text{vir}}$ and in Fig. 8 for the radius given by $r_s$, both with $\xi_c = \xi_{\text{crit}}$. Also, the behaviours of the radius for different values of masses and polytropic...
indices are given in Fig. 9 [with $r_s \approx 25.3(M_{\text{vir}}/10^{12} \, M_\odot)^{0.26} \, \text{kpc}$ and $r_{\text{vir}} = 1.498 \Delta^{-1/3} (M_{\text{vir}}/10^{12} \, M_\odot)^{1/3} \, \text{Mpc} \approx 255(M_{\text{vir}}/10^{12} \, M_\odot)^{1/3} \, \text{kpc}$ (Gentile, Tonini & Salucci 2007)].

We see that the virial radius given by the ALE equation is surprisingly close to the virial radius given by the NFW profile for $n = 3$ in the range of masses shown in Fig. 9, and as pointed out in equation (49), this value yields for this model $\Delta_{\text{A-L}} \approx 200$. However, as can be seen from the plots, the ALE density profile is almost flat until the virial radius. On the other hand, the $m = 0$ profile allows us to parametrize it as $\rho(r)/\rho_{\text{vac}} = 2c^{-1}(1 + r/r_s)^{-3}$, where the concentration parameter $c$ can be written as $c + 1 \approx 1.111 \Delta^{1/3} \Delta_{\text{A-L}}^{-1/3}$.

In Figs 10 and 11, we have compared both profiles with the corresponding $\xi_{\text{crit}}$ for the polytropic indices $n = 3$ and 4.9. For the first case, we see that the virial radii are of the same order of magnitude as the one predicted by the NFW profile. For $n = 4.9$, these quantities differ by one order of magnitude. In all cases, the slope of the NFW profiles changes faster than the ALE profile, which implies that the polytropic configurations enclosed by $r_{\text{vir}}$ display almost a constant density.

The comparison we made above between the polytropic configuration and the NFW profiles shows that up to the cuspy behaviour the polytropic results agree with NFW for the polytropic index $n = 3$. It is worth pointing out here that it is exactly this cuspy behaviour which seems to be at odds with observational facts (Arieli 2003; Hoeft & Mueckert & Gottlöber 2004; Matos, Nunez & Sussman 2005). Our result for $n = 3$ is then a good candidate to describe DMHs. Indeed, the undesired feature of the cuspy behaviour led at least some groups to model the DMHs as a polytropic configuration (Dehnen & Rose 1993; Debattista & Sellwood 1998; Hogan & Dalcanton 2000; McKee 2001; Arieli 2003; Gonzalez-Casado et al. 2004; Henriksen 2004; Yepes et al. 2004; Matos et al. 2005). Sometimes it is claimed that a polytropic model is favoured over the results from $N$-body simulations (Zavala et al. 2006). However, the oscillatory behaviour of the $n = 3$ solutions of the ALE equation represents a disadvantage when compared with the NFW profiles, although the region of physical interest (below the virial radius) is well represented by the solutions of the ALE equation.

6 CONCLUSIONS

In this paper, we have explored the effects of a positive cosmological constant on the equilibrium and stability of astrophysical configurations with a polytropic EOS. We have found that the radius of these kinds of configurations is affected in the sense that not all polytropic indices yield configurations with a definite radius even in the asymptotic sense. Among other things, the widely used isothermal sphere model becomes a non-viable model in the presence of $\Lambda$, unless we are ready to introduce an arbitrary cut-off which renders the model unappealing. Indeed, in this particular case we have tried different definitions of a finite radius with the result that none of them seems to be justified, either from
the phenomenological or from the theoretical point of view. This is then an interesting global result: \( \Lambda \) not only affects quantitatively certain properties of large, low-density astrophysical structures, but also excludes certain commonly used models regardless of what density we use.

For polytropic indices \( n < 5 \) and for certain values of the central density, we cannot find a definite radius. These certain values are encoded in a generalization of the equilibrium condition found for spherical configurations (i.e., \( \rho > 2\rho_{\text{vac}} \)) written now as \( \rho > A_n \rho_{\text{vac}} \). We obtain \( A_1 = 10.8, A_{3/2} = 24.2, A_3 = 307.7, A_4 \approx 4000 \). These values set a minimal central density for a given polytropic index \( n \). We have discussed such minimal density configurations and determined their average density which strongly depends on \( n \). Interestingly, the radius of such configurations has a connection to the length-scale which appears in the Schwarzschild–de Sitter metric as the maximum possible radius for bound orbits (Balaguera-Antolínez et al. 2006a). In this framework, we found also a solution of very low, non-constant density. Indeed, the limiting value of the central density in the above equations is a crucial point. Below this value no matter can be in equilibrium. However, above this value low-density objects can still exist. Both effects are due to \( \Lambda \): in the first case the external repulsive force is too strong for the matter to be in equilibrium, and in the second case this force can counterbalance the attractive Newtonian gravity effects (even if the pressure is small).

Other examples of low-density configurations which we examined in some detail are neutrino stars with masses of the order of 1 eV and 1 keV. In there we found that nowadays a dominating cosmological constant affects both the mass and the radius of such exotic objects. Such effects could change those physical quantities by several orders of magnitude. The magnitude of such effects, however, depends on the fermionic masses and on the assumption that the fermions in such a configuration would be essentially non-relativistic.

Finally, we made a conjecture regarding boson stars, and used variational methods in connection with the Thomas–Fermi equations which could give relatively good results even without invoking the whole general relativistic formalism. We then found extremely low-density configurations for such astrophysical objects. Note, however, that this conjecture relied purely on arguments based on scales and in fact needs a full general relativistic investigation to confirm the results obtained here.

We have compared polytropic configurations with dark matter density profiles from N-body simulations. Surprisingly, we find a reasonable agreement between both approaches for the polytropic index \( n = 3 \) and restricting ourselves to the virial radius. Our model does not have the undesired features of the cuspy behaviour of the NFW profiles.

The importance of the astrophysical properties and configurations found in this paper is that they are specific features to the existence of a dark energy component. Hence, such configurations (e.g. low-density configurations) or properties, if ever found in nature, would imply strong evidence for the presence of a dark energy component. Such observations would be completely independent of, and so complementary to, other cosmological probes of dark energy such as Type Ia supernovae or the cosmic microwave background radiation.

Figure 9. Mass–radius relation for the NFW (red dotted line) profile compared with the solutions of equation (46) (blue solid line) for four different values of \( n \).
Figure 10. Generalized NFW profile (black solid line) for $m = 0$ compared with the $n = 3$ ALE profile, with its corresponding $\zeta_{\text{crit}}$, for different masses.

Figure 11. The same as Fig. 10 but for $n = 4.9$. 
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