Quantum theory of feedback cooling of an anelastic macro-mechanical oscillator

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Conventional techniques for laser cooling, by coherent scattering off of internal states or through an optical cavity mode, have so far proved inefficient on mechanical oscillators heavier than a few nanograms. That is because larger oscillators vibrate at frequencies much too small compared to the scattering rates achievable by their coupling to auxiliary modes. Decoherence mechanisms typically observed in heavy low frequency elastically suspended oscillators also differ markedly from what is assumed in conventional treatments of laser cooling. We show that for a low-frequency anelastic oscillator forming the mechanically compliant end-mirror of a cavity, detuned optical readout, together with measurement-based feedback to stiffen and dampen it, can harness ponderomotively generated quantum correlations, to realize efficient cooling to the motional ground state. This will pave the way for experiments that call for milligram-scale mechanical oscillators prepared in pure motional states, for example, for tests of gravity’s effect on massive quantum systems.

I. INTRODUCTION

The purity with which quantum states of tangible massive objects can be prepared remains an open experimental challenge [1–3]. Although workers in the fields of atomic physics [4–9], and more recently cavity optomechanics [10–18], have succeeded in addressing this challenge at sub-nanogram mass scales, objects with a significantly larger mass feature a qualitatively different behavior. The central pathology remains the same, namely, decoherence, but the precise symptom is unique at large masses.

Small-mass objects, elastically or electromagnetically bound, can be taken to be a mechanical oscillator that is subject to a viscous damping force proportional to its velocity (called velocity damping). For trapped atoms, this is due to the fact that there is little internal dissipation, and any external dissipation arises predominately from background gas collisions, which are naturally described through impulsive momentum kicks; fluctuation-dissipation theorem then assigns a velocity-damped model for motional decoherence. Levitated nano-mechanical systems, recently prepared in their motional ground state [15, 16], appear to be immune to internal dissipation, despite large internal temperatures [19], apparently due to negligible coupling between internal modes and center-of-mass motion. Nano-mechanical objects are elastically bound so as to realize radiofrequency mechanical oscillators; the effects of internal dissipation are largely masked at such high frequencies [20]. The upshot is that all existing theoretical consideration of laser cooling of mechanical oscillators implicitly assumes a velocity damped oscillator [21–24].

Large-mass objects have been isolated so as to be largely immune to external damping. To wit, gas damping (in the high Knudsen number — “low pressure” — regime) scales inversely with the mass [25–27], while suspension techniques have been developed (such as that employed in LIGO, or proposed schemes for levitation) that are not limited by external influence. Internal damping therefore dominates their decoherence. A most ubiquitous form of internal damping in elastic oscillators is so called anelasticity [20, 28, 29], for which the damping is not velocity-proportional, but is described by a frequency dependent “structural damping” rate, \( \Gamma_\text{th}[\Omega] = (\Omega_0/Q)(\Omega_0/\Omega) \), where \( \Omega_0 \) is the resonance frequency, and \( Q \) is the (frequency-independent) quality factor.

The decoherence rate of a structurally damped oscillator, when exposed to a thermal bath of mean occupation \( n_\text{th}[\Omega] \approx k_B T/\hbar \Omega \),

\[
\Gamma_\text{th}[\Omega] \approx n_\text{th}[\Omega]\Gamma_0[\Omega] \approx k_B T \left( \frac{\Omega_0}{\Omega} \right)^2 ,
\]
decreases quadratically with frequency, in marked contrast to a velocity damped oscillator (for which the scaling is linear). This can be harnessed by stiffening the oscillator — for example by radiation pressure forces from a cavity field [30–32] — so as to establish an oscillator mode at the frequency \( \Omega_{\text{eff}} \gg \Omega_0 \), whose thermal decoherence rate, \( \Gamma_\text{th}[\Omega_{\text{eff}}] = \Gamma_\text{th}[\Omega_0](\Omega_0/\Omega_{\text{eff}})^2 \), can be significantly lower than that of the intrinsic mode (at frequency \( \Omega_0 \)). However, this will be counteracted by additional decoherence from quantum fluctuations of the optical field used to produce the optical spring. The interplay of these two effects, given the scaling of the decoherence rate for a structurally damped oscillator, call for a re-examination of the conventional theory of laser cooling [21–24] as applied to macroscopic mechanical oscillators. As we will show, this naturally brings us the opportunity to consider improvement of the cooling performance through
back-action evasion.

In the following we study laser cooling of structurally damped and optically stiffened mechanical oscillators via their coupling to an optical cavity field. Because typical macroscopic mechanical oscillators coupled to optical cavities tend to be in the broadband cavity regime (i.e. mechanical frequency much lower than the cavity decay rate), cooling from cavity dynamical back-action is not practical to realize pure quantum states, so we consider active feedback based on cavity-enhanced measurement of the oscillator position as the cooling mechanism [14, 21, 24, 33]. In fact, significant optical stiffening, by blue-detuning the cavity mode it is coupled to, requires external feedback to stabilize the oscillator against parametric instabilities [34]. The natural rotation of the field quadratures due to cavity detuning, possibly enhanced by choice of homodyne measurement angle to derive the error signal for feedback, gives rise to the possibility of enhancing the performance of feedback cooling using quantum correlations developed intrinsically in the radiation-pressure interaction [35].

II. FEEDBACK COOLING WITH ACTIVE AND DETUNED OPTICAL SPRING

We consider here the following scenario (depicted in fig. 1a): a mechanical oscillator, with displacement fluctuations $\delta x$, forms the end-mirror of an optical cavity, whose motion modulates the cavity frequency as $G \cdot \delta x$; the cavity is probed by an ideal coherent field detuned from resonance by $\Delta$, and is otherwise lossless; the reflected light is subjected to homodyne detection with a local oscillator whose phase differs from that of the cavity input by $\theta$; the resulting photocurrent fluctuations are passed through a causal filter to synthesize a force — the feedback force — that impinges upon the oscillator. Despite the complexity of the scenario, the motion of the oscillator can be described by a simple linear equation (in the frequency domain),

$$\chi^0[\Omega] \delta x[\Omega] = \delta F_{\text{th}}[\Omega] + F_{\text{rp}}[\Omega] + F_{\text{fb}}[\Omega].$$

(1)

It describes the intrinsic response,

$$\chi^0[\Omega] = [m(-\Omega^2 + \Omega_0^2 + i\Gamma_0[\Omega])]^{-1},$$

(2)

of the oscillator — with mass $m$, intrinsic resonance frequency $\Omega_0$, and damping $\Gamma_0[\Omega]$ — to three forces.

The thermal force $\delta F_{\text{th}}$ is characterized by its (symmetrized double-sided) spectral density,

$$S^\text{th}_{\text{rp}}[\Omega] = 2\hbar n_{\text{th}}[\Omega](1 + \frac{1}{2})m\Omega^2[\Omega],$$

(3)

where, $n_{\text{th}}[\Omega] \approx kT/(\hbar \Omega)$, is the average thermal phonon occupation. Note that structural thermal force decreases with frequency (i.e., $S^\text{th}_{\text{rp}}[\Omega] \approx 2\hbar m\Omega^2 n_{\text{th}}[\Omega] \propto 1/\Omega$, see fig. 1b).

The radiation pressure force $F_{\text{rp}}[\Omega]$ arises from an interaction between the oscillator displacement and intracavity field $(a)$ described by the interaction hamiltonian [36], $H_{\text{rp}} = -\hbar G a^\dagger n$, where $n = a^\dagger a$ is the intracavity photon number. In a linearized description, the radiation pressure force can be expressed as the sum of two components [37],

$$F_{\text{rp}}[\Omega] = -\chi^{-1}_{\text{rp}}[\Omega] \delta x[\Omega] + \delta F_{\text{rp}}[\Omega].$$

(4)

The first is a detuning-dependent force that is proportional to the oscillator position, and leads to optical damping/anti-damping and spring shift, while the second is a quantum radiation pressure force fluctuation due to intracavity photon number fluctuations. In the broadband cavity regime (i.e. where the cavity decay rate ($\kappa$) is much larger than the mechanical frequency) the former is described by a susceptibility of the form,

$$\chi_{\text{rp}}^{-1} = m\Omega^2 + i\Omega \Gamma_{\text{rp}},$$

(5)

where,

$$\Omega_{\text{rp}}^2 \approx \Omega_{\text{SQL}}^2 \frac{\delta}{2(1 + \delta^2)}, \quad \Gamma_{\text{rp}} \approx \frac{\Omega_{\text{SQL}}^2 \delta}{\kappa(1 + \delta^2)^2},$$

(6)
are the shifts in the oscillator frequency and damping rate due to the radiation pressure interaction. Here, \( \delta = \Delta / (\kappa / 2) \) is the detuning normalized to the cavity’s FWHM, \( n_c \) is the mean intracavity photon number, and we have defined,

\[
\Omega_{\text{SQL}} = \sqrt{\frac{8\hbar G^2 n_c}{mk}},
\]

the frequency at which the free-mass standard quantum limit (SQL) is attained.

Note that in the theory of cavity optomechanics applied to high-frequency oscillators, the radiation-pressure-induced change in the oscillator frequency is typically small compared to the intrinsic resonance frequency (i.e. \( \Omega_{\text{rp}} \ll \Omega_0 \)). In that case, the characteristic interaction frequency is the vacuum optomechanical coupling rate, \( G\sqrt{n_c}\sqrt{\hbar/(2m\Omega_0)} \), defined with respect to the zero-point motion of the intrinsic oscillator at frequency \( \Omega_0 \). In contrast, here we consider the scenario where the optical spring frequency can be much larger than the intrinsic frequency (i.e. \( \Omega_{\text{rp}} \gg \Omega_0 \)); and we are interested in the properties of the oscillator mode established at the shifted frequency. Even when the shifted mechanical frequency is much different from its intrinsic frequency, the tradeoff between measurement sensitivity and back-action force remains constrained by fundamental constants (explicated below).

Thus, in the terminology of imprecision and back-action noises (see below), the SQL frequency, implicitly defined by, \( \langle m\Omega_{\text{SQL}}^2 \rangle_{\text{imp}}[\Omega_{\text{SQL}}] = \bar{S}_{\text{FF}}[\Omega_{\text{SQL}}] \), is a more convenient measure of the interaction strength that is independent of the mechanical resonance frequency. Note that this implicit definition clarifies the interpretation that it is the frequency at which the SQL is achieved. When both the imprecision and back-action noises are white, for example for displacement readout using a broadband cavity, explicit expressions for these noises give the form in eq. (7). The implicit definition is however valid more generally.

The effect of the position-dependent term in the radiation pressure force (first term in eq. (4)) is to change the effective response of the oscillator. Indeed, inserting the form of the radiation-pressure-modified response [eq. (5)] in eq. (1) and re-arranging terms shows that the radiation-pressure-modified response,

\[
(\delta_0 + 1)^{-1} \approx [m(-\Omega^2 + (\Omega_0^2 + \Omega_{\text{rp}}^2) + i\Omega(\Gamma_0 + \Gamma_{\text{rp}}))]^{-1},
\]

features a mechanical oscillator at a higher frequency \( (\Omega_0^2 + \Omega_{\text{rp}}^2)^{1/2} \gg \Omega_0 \) for blue-detuned (i.e. \( \delta > 0 \)) operation. Since the thermal force decreases with frequency, the displacement fluctuations due to thermal noise at \( \Omega_{\text{rp}} \) can be lower than that at the oscillator’s intrinsic resonance frequency \( \Omega_0 \).

Two effects however affect this conclusion. Firstly, quantum fluctuations in the intracavity photon number, due to the blue-detuned light used to stiffen the oscillator, creates an additional radiation pressure force fluctuation [see appendix A],

\[
\bar{S}_{\text{FF}} = (hG)^2 \bar{S}_{nn} \approx 4h(G^2 n_c / \kappa)/(1 + \delta^2),
\]

or equivalently,

\[
\bar{S}_{\text{FF}}[\omega] = \hbar \frac{m\Omega_{\text{SQL}}^2}{2(1 + \delta^2)} = \hbar \frac{m\Omega_{\text{rp}}^2}{\delta} \mid_{\delta > 0}.
\]

Note that this quantum back-action noise increases quadratically with the stiffened oscillator frequency (for fixed detuning and increasing laser power). The second problem is that as the blue-detuned optical power is increased to realize a stiffer oscillator, the total damping rate can become negative (i.e. \( \Gamma_0 + \Gamma_{\text{rp}} < 0 \)) and render the oscillator unstable — an example of radiation-pressure-induced parametric instability.

Both these problems — increased quantum back-action with oscillator frequency, and parametric instability — can be controlled by applying a feedback force on the oscillator based on an estimate of its position.

The optical field used to pump the cavity — the same one that when detuned produces the optical spring — is modulated by the motion of the mechanical oscillator. Measuring the quadratures of the field leaking out of the cavity, for example by homodyne detection, realizes a linear measurement of the mechanical oscillator’s displacement. The homodyne photocurrent, appropriately normalized, produces a linear estimate of the position,

\[
\delta x_{\text{obs}}[\Omega] = \delta x[\Omega] + \delta x_{\text{imp}}[\Omega],
\]

contaminated by the displacement-equivalent imprecision noise \( \delta x_{\text{imp}} \), due to shot noise fluctuations of the field quadrature that is detected. Since the back-action force \( \delta F_{\text{rp}} \) also arises from the vacuum fluctuations of the same field, the imprecision and back-action satisfy two constraints (see appendix A),

\[
\bar{S}_{\text{FF}}[\Omega] \bar{S}_{\text{imp}}[\Omega] = \frac{\hbar^2}{4\eta} \csc^2 \theta_{\text{eff}}
\]

\[
\bar{S}_{\text{imp}}[\Omega] = -\frac{\hbar}{2\sqrt{\eta}} \cot \theta_{\text{eff}},
\]

where

\[
\theta_{\text{eff}} = \theta - \tan^{-1} \delta
\]

is the effective quadrature angle of the reflected light that is measured (for example using a homodyne detector), and \( \eta \) is the detection efficiency. The first expresses the essence of the uncertainty principle: the measurement imprecision and back-action force are a mutual trade-off. The conventional measurement strategy — for phase quadrature homodyne readout of the reflection (\( \theta_{\text{eff}} = \pi / 2 \)) — can realize a quantum-ideal measurement (i.e. \( \bar{S}_{\text{FF}} \bar{S}_{\text{imp}} = \hbar^2 / 4 \)) if the detection efficiency is unity. The second expression relays the fact that the back-action force and imprecision noise can be correlated — but only for finite detuning and/or homodyne readout of non-phase quadratures — due to the fact that traces of the same optical field fluctuations that produce the back-action force manifest also in the imprecision noise. When
these are anti-correlated (i.e. $S_{FB}^{\text{imp}} < 0$), the detected field quadrature can be squeezed, and (some of) the back-action of the measurement avoided. Equations (10) and (11) exhaustively characterize the constraints on the measurement due to quantum mechanics at the level of spectral densities; in fact they verify the generalized uncertainty principle, $S_{FB}^{\text{imp}}|S_{FB}^{\text{imp}}|^\dagger = \hbar^2/4$ \cite{38,39}.

Finally, a feedback force can be applied on the mechanical oscillator, based on such a measurement, i.e.,

$$F_{fb}[\Omega] = -\chi_{fb}[\Omega]\delta x_{\text{obs}}[\Omega],$$  \hspace{1cm} (13)

where $\chi_{fb}[\Omega]$ is a causal function chosen to produce the desired modification of the oscillator's effective response. (In principle there could be an additional force noise associated with the feedback force — for example from the actuator in the feedback path, or technical noises in the photocurrent inside the passband of $\chi_{fb}[\Omega]$.) Inserting eqs. (4) and (13) in eq. (1) produces the equation of motion modified by radiation pressure and feedback:

$$\chi_{fb}^{-1}[\Omega]\delta x[\Omega] = \delta F_{\text{th}}[\Omega] + \delta F_{\text{rp}}[\Omega] - \chi_{fb}^{-1}\delta x_{\text{imp}}[\Omega],$$  \hspace{1cm} (14)

where $\chi_{fb}[\Omega]$ is the effective response given by,

$$\chi_{fb}^{-1} = \chi_0^{-1} + \chi_{\text{rp}}^{-1} + \chi_{fb}^{-1}. $$  \hspace{1cm} (15)

In order to affect active spring stiffening and cooling, the feedback susceptibility needs to approximate the form,

$$\chi_{fb}^{-1} = m(\Omega^2 + i\Omega \Gamma_{fb}),$$

around the oscillator’s stiffened frequency; here $\Omega_{fb}, \Gamma_{fb} > 0$. This form is comparable to the radiation-pressure-induced susceptibility in eq. (5). The effective susceptibility then takes the form,

$$\chi_{fb}^{-1} = m(-\Omega^2 + \Omega^2 \epsilon + i\Omega \Gamma_{\text{eff}}),$$  \hspace{1cm} (17)

which is the response of an oscillator at the shifted frequency, $\Omega_{\text{eff}} = \Omega_0 + \Omega_{\text{rp}} + \Omega_{fb}$, with a modified damping rate, $\Gamma_{\text{eff}}[\Omega] = \Gamma_{\text{rp}}[\Omega] + \Gamma_{\text{rp}} + \Gamma_{fb}$.

The displacement spectrum of the oscillator so realized takes the form,

$$S_{xx}[\Omega] = |\chi_{\text{eff}}[\Omega]|^2 \left( S_{FB}^{\text{imp}}[\Omega] + S_{FP}^{\text{imp}}[\Omega] + |\chi_{fb}[\Omega]|^{-2} S_{xx}^{\text{imp}}[\Omega] \right)^2 + 2\text{Re} \left( \chi_{fb}^{-1}[\Omega] S_{FB}^{\text{imp}}[\Omega] \right).$$

Here the first line represents the physical motion of the oscillator due to the thermal, radiation pressure back-action, and ‘feedback back-action’ forces; the latter is due to imprecision noise feedback as a force through the filter $\chi_{fb}^{-1}$. The second term is due to imprecision-back-action correlations arising from detuning of the cavity from resonance, or detuning of the homodyne detector from phase quadrature.

When the objective is to cool the mechanical oscillator, a convenient figure of merit is the average phonon number, $n_{\text{eff}}$, defined through the average energy,

$$\langle \delta p^2 \rangle + m\Omega_{\text{eff}}^2 \langle \delta x^2 \rangle = \hbar \Omega_{\text{eff}} \left( n_{\text{eff}} + \frac{1}{2} \right);$$

where $\delta p$ is the fluctuation in the momentum of the oscillator, which is unobserved. However, it can be estimated from the observed displacement as, $\delta p[\Omega] = -im\Omega \delta x[\Omega]$, so that the required variances $\langle \delta x^2 \rangle$, $\langle \delta p^2 \rangle$ can be inferred from the spectral density $S_{xx}$ alone as,

$$\langle \delta x^2 \rangle = \int_{-\infty}^{\infty} d\Omega \frac{1}{2\pi} S_{xx}[\Omega], \quad \langle \delta p^2 \rangle = \int_{-\infty}^{\infty} d\Omega \frac{1}{2\pi} (m\Omega^2 S_{xx}[\Omega]).$$

(19)

Mathematically carrying out this program to estimate the phonon number for a structurally damped oscillator that is controlled with the feedback filter in eq. (16) turns out to be impossible. That is for two reasons:

1. At low frequencies, even without feedback, the variance in the displacement of a structurally damped oscillator is formally infinite \cite{28}. The physical reason is that anelastic damping, just like any physical process dominated by $1/f$ noise, is due to non-equilibrium processes at slower and slower time-scales \cite{40–42} which precludes thermal equilibrium.

2. At high frequencies, feedback of imprecision noise as a force noise leads to a formally infinite momentum \cite{43}. This can be seen as follows: when eq. (18) is used to estimate the momentum variance as the integral of $(m\Omega)^2 S_{xx}$, the term in the integrand proportional to the imprecision noise, $\Omega^2 |\chi_{\text{eff}}|^{-2} S_{xx}^{\text{imp}}$, is a constant at high frequencies, since $|\chi_{\text{eff}}[\Omega] \gg \Omega_{\text{eff}}|^{-2} \sim \Omega^{-4}$, while, $|\chi_{fb}[\Omega] \gg \Omega_{\text{eff}}|^{-2} \sim \Omega^2$, and (at best) $S_{xx}^{\text{imp}}$ is frequency independent.

In other words, a structurally damped oscillator does not strictly satisfy the equipartition principle; naive feedback compounds the problem.

In practice, all experiments have a finite bandwidth and observation time which regulates the singularities at high and low frequencies respectively. In particular, for a large spring ($\Omega_{\text{eff}} \gg \Omega_0$) the effect of structural damping can be well approximated by taking the damping rate to be constant around resonance, i.e. $\Gamma_0[\Omega] \approx \Gamma_0[\Omega_{\text{eff}}] = \Omega_0^2/(Q\Omega_{\text{eff}})$. To regulate the problem with the momentum variance, we modify the feedback filter to the form,

$$\chi_{fb}^{-1}[\Omega] = m\Omega_0^{-1} + \frac{i\Omega_0}{\Omega_{\text{eff}}} \Omega_{\text{eff}},$$

(20)
where \( \Omega_H, \Omega_L \) are high- and low-pass frequencies between which feedback is active (\( \Omega_L > \Omega_H \)), and \( g_{\text{fb}} > 0 \) is the dimensionless gain. In this case, unlike the naive filter in eq. (16), we have that \( \chi_{\text{fb}}[\Omega] \gg \Omega_{\text{eff}} \) \( \approx m\Omega_H^2 g_{\text{fb}} \), so that, \( \Omega^2 \chi_{\text{eff}}[\Omega] \chi_{\text{fb}} \approx \Omega^{-2} \), which regulates the high-frequency divergence of the momentum integral. However, in order to realize a spring and damping, the filter in eq. (20) must conform to the form in eq. (16) at some frequencies; indeed we have,

\[
\chi_{\text{fb}}^{-1}[\Omega \ll \Omega_L] \approx m\Omega_0^2 \left( 1 + i \frac{\Omega}{\Omega_H} \right) g_{\text{fb}}.
\]

Comparing this with eq. (16) implies that the feedback damping is \( \Gamma_{\text{fb}} = g_{\text{fb}}\Omega_0^2/\Omega_H \), and the spring shift is, \( \Omega_{\text{fb}} = \sqrt{g_{\text{fb}}\Omega_0^2} = \sqrt{\Omega_H\Gamma_{\text{fb}}} \).

An additional complication of this choice of the feedback filter is that it need not render the system unconditionally stable in the presence of radiation pressure back-action. A simple Routh-Hurwitz analysis of the effective susceptibility \( \chi_{\text{eff}} \) shows that the system is stable if, \( g_{\text{fb}} > -\Gamma_{\text{rp}}\Omega_L\Omega_H/[\Omega_0^2(\Omega_L - \Omega_H)] \). We assume that sufficient feedback damping can be realized to satisfy this condition.

With these issues addressed, the oscillator’s mean phonon number can be computed from the displacement spectral density. The result can be expressed in closed form (see appendix B):

\[
2n_{\text{eff}} + 1 = \left[ 2\tilde{\Omega}_{\text{eff}}^2 + (\Omega_L - \Omega_H)\Gamma_{\text{eff}} + 2\tilde{\Omega}_{\text{eff}}^2 \left( n_{\text{th,eff}} + n_{\text{ba}} + \frac{1}{2} \right) \frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}} \right] + \left( \frac{2\tilde{\Omega}_{\text{eff}}^2 + (\Omega_L - \Omega_H)\Gamma_{\text{eff}} + 2\tilde{\Omega}_{\text{eff}}^2}{\Omega_L\Omega_{\text{eff}}} \right) n_{\text{imp}} \frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}} - 2\tilde{\Omega}_{\text{eff}}^2 \right] \frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}} \left( 1 - \frac{\Omega_H}{\Omega_L} \right)^{-1}.
\]

Here, \( n_{\text{th,eff}} = n_{\text{th}}[\Omega_{\text{eff}}] = n_{\text{th}}[\Omega_0](\Omega_0/\Omega_{\text{eff}}) \), is the average phonon occupation of the stiffened oscillator, \( n_{\text{ba}} = n_{\text{th,eff}} \cdot \frac{S_{\text{FP}}^c}{S_{\text{FP}}^c[\Omega_{\text{eff}}]} \) is the average phonon occupation due to quantum back-action, \( n_{\text{imp}} \) is the phonon-equivalent imprecision noise defined through the uncertainty relation [eq. (10)], \( n_{\text{imp}}n_{\text{ba}} = (16\pi)^{-1} \csc^2 \theta_{\text{eff}} \), and \( n_{\text{cor}} = (2\sqrt{\pi})^{-1} \cot \theta_{\text{eff}} \) is the phonon-equivalent correlation between imprecision and back-action.

### III. DISCUSSION

The first and second terms in eq. (21) denote the feedback-suppression of the total energy of the stiffened oscillator (\( \propto n_{\text{th,eff}} + n_{\text{ba}} \)) and the heating due to feedback-injection of imprecision noise, respectively. The third term, negative in contribution, is the effect of back-action cancellation originating from imprecision-back-action correlations developed through the radiation pressure interaction [44–46]. Such quantum correlations can be harnessed when feedback is predicated on readout of the outgoing field’s quadrature that is away from phase quadrature (as shown in Ref. [35] for feedback damping with resonant cavity readout for a velocity-damped oscillator).

#### A. Conventional case: feedback with resonant phase-quadrature readout

Before delving into further discussion, note first that the practice of estimating the phonon occupation by assuming the equipartition principle, i.e. taking \( 2n_{\text{eff}} + 1 = (2m\tilde{\Omega}_{\text{eff}}/\hbar) \langle \delta x^2 \rangle \), is equivalent to taking the low-pass cutoff to be \( \Omega_L \to \infty \); in this case, eq. (21) reduces to,

\[
2n_{\text{eff}} + 1 \approx \left( n_{\text{th,eff}} + n_{\text{ba}} + \frac{1}{2} \right) \frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}} + \left( \frac{\Omega_L}{\Omega_{\text{eff}}} \right) n_{\text{imp}} \frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}} \Omega_H n_{\text{cor}}.
\]

For phase measurement at zero-detuning (i.e. \( \theta_{\text{eff}} = \pi/2, \delta = 0 \)), the effective resonance frequency is \( \Omega_{\text{eff}} \approx \Omega_L \Gamma_{\text{eff}} \), so that the above expression can be cast as,

\[
2n_{\text{eff}} + 1 \approx \left[ n_{\text{th,eff}} + n_{\text{ba}} + \frac{1}{2} + \left( \frac{\Omega_{\text{eff}}}{\Gamma_0[\Omega_{\text{eff}}]} \right)^2 \frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}} \right] n_{\text{imp}}
\]

consistent with the experiments on feedback cooling of a structurally damped actively stiffened oscillator near its ground state [1].

The case of no active spring corresponds to setting \( \Omega_{\text{eff}} = 0 \) (because we have assumed that \( \Omega_{\text{eff}} \gg \Omega_L \) in the above equation). With resonant readout, there is
no additional source of spring stiffening either. In this case, the above equation reduces to, $n_{\text{eff}} + \frac{1}{2} \Gamma_0 \approx (n_{\text{th}} + n_{\text{ba}} + \frac{1}{2}) \Gamma_0 + n_{\text{imp}} \Gamma_{\text{eff}}$, which can be interpreted as a detailed balance relation describing a velocity-damped oscillator simultaneously coupled to its thermal and back-action baths at rate $\Gamma_0$, and via feedback to the bath due to measurement imprecision at rate $\Gamma_{\text{eff}}$. Optimizing over the damping rate shows that,

$$n_{\text{eff}} + \frac{1}{2} \geq 2\sqrt{(n_{\text{th}} + n_{\text{ba}} + \frac{1}{2}) n_{\text{imp}}};$$

using the uncertainty relation, $n_{\text{imp}} n_{\text{ba}} \geq \frac{1}{16}$ further gives, $n_{\text{eff}} + \frac{1}{2} \geq 2\sqrt{(n_{\text{th}} + \frac{1}{2}) n_{\text{imp}} + \frac{1}{16}}$. Thus, to realize $n_{\text{eff}} < 1$ requires that $n_{\text{imp}} < 1/(2n_{\text{th}} + 1)$, which is the well-understood requirement on the measurement sensitivity to feedback cool a velocity-damped oscillator to its motional ground state [14, 33].

In marked contrast, for a structurally damped oscillator that is actively stiffened, the apparent initial occupation (i.e. before feedback damping has commenced) in eq. (22), $n_{\text{eff}} + \frac{1}{2} \geq 2\sqrt{(n_{\text{th}} + n_{\text{ba}} + \frac{1}{2}) n_{\text{imp}}}$, has a thermal component (first term), $n_{\text{th}, \text{eff}} = n_{\text{th}}[\Omega_0 / \Omega_{\text{eff}}]$, that decreases with increasing spring frequency — a form of thermal noise dilution [31, 32], and an additional term (third term), $n_{\text{imp}}[\Omega_{\text{eff}} / \Gamma_0[\Omega_{\text{eff}}]]^2$, that increases with the spring frequency — a form of feedback-back-action arising from imprecision noise feedback as a force noise in realizing the active spring. The opposing scaling of these two effects with the spring frequency, with the former scaling as $\Omega_{\text{eff}}^{-2}$ and the latter as $\Omega_{\text{eff}}$, implies an optimal value of the spring frequency beyond which the dilution of thermal noise is nullified by increase in feedback-back-action from the spring. For a given measurement imprecision, which is independent from the effective frequency for the structurally damped oscillator

$$n_{\text{imp}} = \frac{1}{4nQ_0} \left( \frac{\Omega_0}{\Omega_{\text{SQL}}} \right)^2,$$  (23)

that optimal spring frequency is given by (for the relevant case, $n_{\text{imp}} \ll 1$),

$$\Omega_{\text{eff}, \text{opt}} \approx \Omega_0 \left( \frac{n_{\text{th}}[\Omega_0]}{n_{\text{imp}}[4Q_0]} \right)^{1/5} \left( \frac{\Omega_{\text{SQL}}}{\Omega_0} \right)^{2/5},$$  (24)

where $Q_0 \equiv \Omega_0 / \Gamma_0[\Omega_0]$ is the intrinsic quality factor of the oscillator. Inserting this back into eq. (22) gives,

$$n_{\text{eff}} + \frac{1}{2} \approx \left( \frac{5}{2n_{\text{imp}}[\Omega_0 Q_0]^{3/5}} n_{\text{imp}}^{1/5} + \frac{1}{2} \mathcal{G}_0[\Omega_{\text{eff}}] \right) \Gamma_{\text{eff}} (\frac{\Gamma_0[\Omega_{\text{eff}}]}{\Gamma_{\text{eff}}}),$$

$$\geq 2\sqrt{\left( \frac{5}{2n_{\text{imp}}[\Omega_0 Q_0]^{3/5}} n_{\text{imp}}^{1/5} + \frac{1}{2} \right) n_{\text{imp}}},$$

here the second line is the result of optimizing over the feedback damping rate $\Gamma_{\text{eff}}$, while the last line is from the uncertainty principle (and we have omitted a small $O(n_{\text{imp}})$ term). In order that $n_{\text{eff}} < 1$, the last equation implies the requirement,

$$n_{\text{imp}} \leq \sqrt{\frac{2}{5/8 n_{\text{th}} Q_0^{-1/3}}},$$  (26)

on the measurement sensitivity. For the experimentally relevant regime where the oscillator begins in a large thermal state, i.e. $n_{\text{th}} \gg 1$, the requirement on the measurement sensitivity is slightly weaker for the case where the oscillator is structurally damped and actively stiffened (scaling as $n_{\text{th}}^{-2/3}$) compared to the case of a velocity damped oscillator (scaling as $n_{\text{th}}^{-1}$). The condition for $n_{\text{imp}}$ in eq. (26) can be rewritten as that for the mechanical Q-value,

$$Q_0 > \left( \frac{5}{8} \right)^{5/4} n_{\text{th}} \left( \frac{\Omega_0}{\Omega_{\text{SQL}}} \right)^3,$$  (27)

or in terms of the oft-quoted “Qf product”, $Q_0 f_0 > (k_B T / h) \times (5/8)^{5/4} (\Omega_0 / \Omega_{\text{SQL}})^3$. Note that the necessary condition on the mechanical quality factor is relaxed for a low frequency oscillator strongly coupled to a quantum-noise-limited optical field (i.e. $\Omega_{\text{SQL}} \gg \Omega_0$). The unique $\Omega_{\text{SQL}}^{-3}$ scaling on the Q-factor requirement is consistent with the idea that as $\Omega_{\text{SQL}}$ increases, larger active spring frequencies can be realized; for a structurally damped oscillator, its thermal occupation reduces as $\Omega_{\text{eff}}^{-1}$, while the penalty from feedback-back-action in realizing the spring worsens as $n_{\text{imp}} \Omega_{\text{eff}}^4 \propto (\Omega_{\text{eff}}^2 / \Omega_{\text{SQL}})^2$; their ratio is upper-bounded by a factor that scales as $\Omega_{\text{SQL}}^{-3}$.

B. General case: detuned readout with finite-bandwidth feedback

The more general case harnesses the freedom to both detune the readout field from the optical cavity resonance — which produces an optical spring and rotates the quadrature of the the outgoing field with respect to the input field — and a variable-quadrature homodyne detection of the outgoing field — which can be sensitive to the quantum correlations developed via the radiation
pressure interaction. In this case, for a fixed optomechanical system, an experimenter has control over six parameters: the gain of the feedback filter $g_{th}$ — which effectively sets the feedback damping rate $\Gamma_{th}$; the cutoff frequencies $\Omega_L$ and $\Omega_H$ — which together with the feedback gain determines the feedback spring frequency $\Omega_{th}$; the detuning, which contributes to the radiation pressure induced spring and damping; and the effective readout phase $\theta_{eff}$.

In the following we will interchangeably use the phonon number and the purity as figures of merit to assess the quality of the quantum state that is realized. The purity satisfies $0 \leq \mu \leq 1$, where the upper (lower) bound corresponds to a maximally pure (mixed) state. In the scenario we consider, where the initial state of the oscillator is Gaussian (specifically, assumed to be thermal), and measurement and feedback are linear in the oscillator’s position, the state realized by feedback is also Gaussian. For Gaussian states, the purity is related to the average quantum number of its thermal component as, $\mu^{-1} = 2n_{\text{eff}} + 1$. Thus eq. (21) directly gives the inverse of the purity. Note however than the conventionally employed criteria for having realized the ground state of motion, $n_{\text{eff}} < 1$, corresponds to a purity of, $\mu > 1/3$. In the following we will employ purity as a figure of merit.

For fixed detuning, the dependence of the readout angle is through the imprecision and the imprecision-backaction correlations,

$$n_{\text{imp}} \equiv n_{\text{imp}}^{\eta_{\text{eff}}} = n_{\text{imp}}^{\eta_{\text{eff}}} \left( 1 + \cot^2 \theta_{\text{eff}} \right)$$

$$n_{\text{cor}} \equiv n_{\text{cor}}^{\eta_{\text{eff}}} = n_{\text{cor}}^{\eta_{\text{eff}}} \cot \theta_{\text{eff}},$$

where, $n_{\text{imp}}^{\eta_{\text{eff}}} = 1/(16\eta_{\text{ba}})$ is the imprecision for conventional phase quadrature readout, and $n_{\text{cor}}^{\eta_{\text{eff}}} = 1/(2\sqrt{\eta})$. Clearly, phase quadrature readout ($\theta_{\text{eff}} = \pi/2$) minimizes imprecision without harnessing any quantum correlations ($n_{\text{cor}}^{\eta_{\text{eff}}} = 0$), while amplitude quadrature readout contains no information about the motion (i.e. $n_{\text{imp}}^{0} \to \infty$). In the context of displacement measurement, the trade-off between these two scenarios is the principle of so-called “variational measurement” that can realize displacement sensitivity better than that by phase quadrature readout [44-48]. Improved displacement sensitivity, in the context of feedback control, produces less feedback back-action; thus, optimizing the readout angle to harness quantum correlations can lead to better state purity (with other parameters fixed).

Inserting eq. (28) in eq. (21), the latter can be put into the form,

$$R\mu^{-1} = \frac{C_{\text{tot}}}{g} \left( n_{\text{th},\text{eff}} + n_{\text{ba}} + \frac{1}{2} \right) + gC_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}} \left( 1 + \cot^2 \theta \right) - C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}} \cot \theta,$$

where, $g \equiv \Gamma_0/\Gamma_{\text{eff}}$ is the factor by which the damping rate has increased due to feedback, $R \equiv 1 - \Omega_H/\Omega_L$, and $C_{\text{tot,imp,cor}}$ are the dimensionless pre-factors for each of the three terms in eq. (21):

$$C_{\text{tot}} = \frac{20^2 \text{eff} + (\Omega_L - \Omega_H) \Gamma_{\text{eff}} + 2\Omega_L^2}{\Omega_L^2}$$

$$C_{\text{imp}} = \frac{20^2 \text{eff} + (\Omega_L - \Omega_H) \Gamma_{\text{eff}} + 2\Omega_L^2}{\Omega_L^2}$$

$$C_{\text{cor}} = \frac{20^2 \text{eff} + (\Omega_L - \Omega_H) \Gamma_{\text{eff}} + 2\Omega_L \Omega_H}{\Omega_L \Omega_H},$$

which are themselves functions of $g$, $\Omega_{H,L}$. In this sense, the final occupation depends on five parameters: effective readout angle (which includes the detuning), the increase in damping due to feedback, quantified by $g$, and the filter cutoff frequencies $\Omega_{H,L}$; the filter DC gain ($g_{th}$) and spring frequency ($\Omega_{th}$) can be determined in terms of these.

The optimal readout angle is defined to be the one that maximizes the final state purity $\mu$. Completing the square in the angle-dependendent terms of eq. (30):

$$R\mu^{-1} = \frac{C_{\text{tot}}}{g} \left( n_{\text{th},\text{eff}} + n_{\text{ba}} + \frac{1}{2} \right) + gC_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}} \left( \cot \theta - \frac{C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}}}{2gC_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}}} \right)^2$$

$$\geq \frac{C_{\text{tot}}}{g} \left( n_{\text{th},\text{eff}} + n_{\text{ba}} + \frac{1}{2} \right) \frac{(C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}})^2}{4C_{\text{tot}} C_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}}}$$

$$\geq n_{\text{ba}} \left( 1 - \frac{(C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}})^2}{4C_{\text{tot}} C_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}}} \right),$$

where the inequality is true for the choice, $\cot \theta = C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}}/(2gC_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}})$, which minimizes the expression in the first line, and dictates the optimal readout angle.

The negative term in the first parenthesis of the last inequality above represents the decrease in back-action due to back-action cancellation in variable-quadrature readout. Indeed re-writing the back-action related part inside that parenthesis in the form,

$$n_{\text{ba}} - \frac{(C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}})^2}{4C_{\text{tot}} C_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}}} = n_{\text{ba}} \left( 1 - \frac{(C_{\text{cor}} n_{\text{cor}}^{\eta_{\text{eff}}})^2}{4C_{\text{tot}} C_{\text{imp}} n_{\text{imp}}^{\eta_{\text{eff}}}} \right),$$

shows the ideal efficacy of back-action evasion with variable-quadrature readout. (Here the inequality is a result of the statements of the uncertainty principle, $n_{\text{ba}}^{\eta_{\text{imp}}} \geq 1/16$, and $n_{\text{cor}}^{\eta_{\text{imp}}} \leq 1/2$.) Ideally, all back-action is cancelled, corresponding to the condition, $C_{\text{cor}} = C_{\text{tot}} C_{\text{imp}}$; as it turns out, this happens
when \([49] \ C_{tot} = 2\). From eq. (31), and the fact that \(\Omega_H < \Omega_{eff} < \Omega_L\), it follows that \(\Omega_L > \sqrt{\Omega_H \Gamma_{eff}}/2\). This further implies that, \(\ C_{imp} \approx 2\).

Thus, in the ideal case where these conditions can be met, all back-action can be suppressed, and so,

\[
\mu^{-1} \geq \frac{2}{g} \left( n_{th,eff} + \frac{1}{2} \right) + 2gn_{imp}^{\pi/2} \geq 4\sqrt{\left( n_{th,eff} + \frac{1}{2} \right) n_{imp}^{\pi/2}},
\]

indicating that the ground-state can be realized if, \(n_{imp}^{\pi/2} \approx (3/4)^2 n_{th,eff} = (3/4)^2 n_{th} (\Omega_0/\Omega_{eff})^2\), for a readout angle \(\theta \approx \pi/2\).

The practical benefit of variable-quadrature readout is that for a given measurement imprecision, it can materialize moderate back-action cancellation so that the occupation achieved by feedback damping is lower than if phase readout were employed. To illustrate this practical scenario, we numerically optimize the purity as a function of the five parameters: the two cutoff frequencies \(\Omega_L\) and \(\Omega_H\), the feedback gain \(\Gamma_{th} \approx \Gamma_{eff}\), the normalized detuning \(\delta\), and the readout angle \(\theta_{eff}\). The purity is optimized calculated for varying quantum cooperativities, \(C_Q \equiv n_{th}/n_{th}\). In order to emulate conditions of fixed input power, the cutoff frequencies are normalized by the SQL frequency with zero detuning,

\[
\Omega_{SQL,0} = \sqrt{1 + \delta^2} \Omega_{SQL}.
\]

For the same reason, we define \(C_{Q,SQL}\) to be the quantum cooperativity at the SQL frequency with zero detuning.

Figure 2 shows the result of numerically optimizing the achievable purity as a function of quantum cooperativity \(C_{Q,SQL}\). Green lines show the performance of phase quadrature readout \(\theta = \pi/2\), while blue shows the case where the readout laser is blue-detuned, and the cavity output is subjected to variable-quadrature homodyne measurement. Variable-quadrature readout performs better in terms of the achievable state purity at all values of the cooperativity (a result also known in the context of velocity-damped oscillators [35]). Ground state cooling, where \(\mu > 1/3\), can be achieved at \(C_{Q,SQL} \gtrsim 1\), for a readout angle \(\theta \approx \pi/3\). The ultimate purity that can be achieved remains asymptotically bounded by \(\mu < \sqrt{n}\).

At small (large) cooperativity, the optimal filter cutoff frequencies are relatively high (small), while the optimal detuning is small (high). This implies that at small (large) cooperativity, the feedback (optical) spring must be dominant. The reason that is that in the small cooperativity regime, the optomechanical coupling is not strong enough to realize radiation pressure springs large enough to take advantage of the unique scaling of structural thermal noise. Whereas in the high cooperativity regime, the feedback spring introduces additional decoherence from feedback back-action, so that optical spring is ideal in this regime. In either case, the optimal spring frequency is around the SQL frequency.

IV. CONCLUSION

We have investigated the implications of structural damping on feedback-based motional ground-state preparation of elastically bound macroscopic mechani-
cal oscillators. We find that the requirement to realize the ground state is less stringent compared to the oft-studied case of a velocity-damped oscillator. That is because structural thermal noise reduces with increasing frequency much faster than velocity-proportional thermal noise. Hence actively stiffening the oscillator mode to take advantage of this decrease can be fruitful. However, that decrease comes at the expense of increasing back-action force fluctuations from the agency that realizes the stiffened spring. The tradeoff between these competing sources of decoherence is optimized when the mechanical oscillator is subjected to ideal homodyne detection with a local oscillator phase shifted by \( \theta \); the resulting photocurrent fluctuations are proportional to fluctuations of the oscillator phase shifted by \( \theta \). 

The imprecision-backaction product for the displacement \( \delta x \) is given by

\[
\delta a_{\text{ref}} = \delta a_{\text{in}} - \sqrt{\kappa} \delta a,
\]

(A5)

which outperforms feedback based on phase-quadrature measurements at all values of the radiation-pressure cooperativity; this is due to back-action cancellation intrinsic to the variable-quadrature measurement scheme.

All of the above conclusions crucially rely on the implicit assumption that the favourable frequency-scaling of structural thermal noise continues well beyond the SQL frequency of the mechanical mode of interest. This necessitates careful suspension design to eliminate other mechanical modes in that vicinity.

These observations are directly relevant to experiments that hope to realize pure quantum states of macroscopic mechanical oscillators to explore the interface between quantum physics and gravity.

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Appendix A: Imprecision-backaction product for arbitrary detuning

In this section, we present the general form of the imprecision-backaction product for the displacement measurement of a mechanical oscillator at arbitrary detuning and homodyne angle.

Let us consider a mechanical oscillator embedded as the end mirror of a single-sided optical cavity, pumped by an ideal coherent state at the effective detuning \( \Delta \). The intracavity optical fluctuations (\( \delta a \)) and mechanical displacement fluctuations (\( \delta x \)) are described by the quantum Langevin equations [36]:

\[
\dot{\delta a} = (i\Delta - \frac{\kappa}{2}) \delta a + \sqrt{\kappa} \delta a_{\text{in}} + iG \sqrt{\kappa} \delta x,
\]

(A1)

\[
\dot{\delta x} + \Gamma_m \delta x + \Omega_m^2 \delta x = \frac{1}{m} (\delta F_{\text{th}} - hG \sqrt{\kappa} (\delta a + \delta a^\dagger)).
\]

(A2)

Note that Equation (A1) implies that the entry port is the dominant source of intracavity field losses. We may rewrite these equations in the Fourier domain as

\[
\delta a[\Omega] = \sqrt{\kappa} \delta a_{\text{in}} + iG \sqrt{\kappa} \delta x - i(\Delta + \Omega) \delta a + \kappa/2.
\]

(A3)

\[
\delta x[\Omega] = \chi_m (\delta F_{\text{th}} + \delta F_{\text{opt}}).
\]

(A4)

where \( \chi_m = (m(\Omega_m^2 - \Omega^2 - i\Omega \Gamma_m))^{-1} \) is the intrinsic susceptibility of the mechanical oscillator, and \( \delta F_{\text{opt}} = -hG \sqrt{\kappa} (\delta a + \delta a^\dagger) \) is the total backaction force exerted on the mechanical oscillator due to the radiation pressure interaction. The reflected field, given by,

\[
\delta a_{\text{ref}} = \delta a_{\text{in}} - \sqrt{\kappa} \delta a,
\]

is subjected to ideal homodyne detection with a local oscillator phase shifted by \( \theta \); the resulting photocurrent fluctuations are proportional to fluctuations of the quadrature,

\[
\delta q^\theta_{\text{ref}} = \frac{1}{\sqrt{2}} \left( \delta a_{\text{ref}} e^{-i\theta} + \delta a_{\text{ref}}^\dagger e^{i\theta} \right).
\]

(A6)

We may compute the spectral density of the homodyne quadrature as

\[
\tilde{S}^\theta_{\delta a_{\text{ref}}[\Omega]} 2\pi \delta [0] = \langle \delta q^\theta_{\text{ref}}[\Omega] \delta q^\theta_{\text{ref}}[-\Omega] \rangle,
\]

(A7)

which may be written as

\[
\tilde{S}^\theta_{\delta a_{\text{ref}}[\Omega]} \propto \tilde{S}^\theta_{\delta a_{\text{imp}}[\Omega]} + \left| \chi_m \right|^2 \left( \tilde{S}^{\text{th}}_{\delta x[\Omega]} + \tilde{S}^{\text{rp}}_{\delta F_{\text{F}}[\Omega]} \right) + 2 \text{Re} \left( \chi_m \sqrt{\kappa} \tilde{F}_{\text{ref}} \right). \quad \text{(A8)}
\]

Computing this spectrum from eq. (A6) following eq. (A7) and noting the only non-zero correlator for the input vacuum fluctuations, \( \langle \delta a_{\text{in}}[\Omega] \delta a^\dagger_{\text{in}}[-\Omega] \rangle = 2\pi \delta [0] \), we can identify the imprecision noise spectral density, \( \tilde{S}^\theta_{\delta a_{\text{imp}}[\Omega]} \), the backaction force spectral density, \( \tilde{S}^{\text{rp}}_{\delta F_{\text{F}}[\Omega]} \), as well as the correlation term, \( \text{Re} \left( \chi_m \sqrt{\kappa} \tilde{F}_{\text{ref}} \right) \) as follows.

Defining two frequency scale factors \( \delta \equiv \frac{\Delta}{\kappa} \) and \( \omega \equiv \frac{2 \pi}{\kappa} \), the spectral density of the imprecision noise in eq. (A8) is given by

\[
\tilde{S}^\theta_{\delta x[\Omega]} = A^\text{imp} \tilde{S}^\theta_{\delta x[\Omega]} \tilde{S}^{\text{imp}}_{\delta x[\Omega]}[\omega],
\]

(A9)

where

\[
A^\text{imp} = (1 + \omega^2)^{-1} \frac{1 + (\delta + \omega)^2}{(\sin \theta - \delta \cos \theta)^2 + 2\omega^2 \sin^2 \theta}.
\]

(A10)

\[
\tilde{S}^\theta_{\delta x[\Omega]} \tilde{S}^{\text{imp}}_{\delta x[\Omega]}[\omega] = \left( \frac{\kappa}{16 G^2 \bar{n}} \right) (1 + \omega^2) .
\]

(A11)

Next, the optical back-action force \( \delta F_{\text{opt}} \) is the part of \( \delta F_{\text{opt}} \) that only depends on the incoming vacuum field.
fluctuations, \( \delta a_n \) and \( \delta a_n^\dagger \). Its symmetrized spectral density as identified in eq. (A8) is expressed as
\[
S_{FF}^{\delta \text{rp}}[\omega] = A^\text{rp} S_{FF}^{0 \text{rp}}[\omega],
\] (A12)
where
\[
A^\text{rp} = (1 + \omega^2)^2 \left( \frac{1 + \delta^2 + \omega^2}{(1 + (\delta + \omega)^2)(1 + (\delta - \omega)^2)} \right),
\] (A13)
\[
S_{FF}^{0 \text{rp}}[\omega] = \frac{\hbar^2}{4} \frac{1 + \delta^2 + \omega^2}{(\sin \theta - \delta \cos \theta)^2 + 2\omega^2 \sin^2(\theta)},
\] (A14)
Hence, the imprecision-backaction product in the general case of an arbitrary effective detuning and homodyne measurement angle is given by;
\[
S_{xx}^{\theta, \text{imp}}[\omega] S_{FF}^{\delta \text{rp}}[\omega] = \frac{\hbar^2}{4} \frac{1 + \delta^2 + \omega^2}{\sin^2(\theta - \arctan \delta) + O(\omega^2)},
\] (A15)
Note that the minimum value of the product is precisely \( \hbar^2/4 \), for \( \delta = 0 \) and \( \theta = \pi/2 \); i.e. resonant readout of the phase quadrature is ideal.

In the broadband cavity regime (i.e. \( \Omega \ll \kappa \) or \( \omega \ll 1 \)), we have that,
\[
S_{xx}^{\theta, \text{imp}}[\omega] \approx \frac{\hbar^2}{4} \frac{1 + \delta^2 + \omega^2}{\sin^2(\theta - \arctan \delta) + O(\omega^2)},
\] (A16)
\[
S_{FF}^{\delta \text{rp}}[\omega] \approx \frac{\hbar^2}{4} \frac{1 + \delta^2 + \omega^2}{(1 + \delta^2 + \omega^2) + O(\omega^2)}.
\] (A17)
and therefore,
\[
S_{xx}^{\theta, \text{imp}}[\omega] S_{FF}^{\delta \text{rp}}[\omega] \approx \frac{\hbar^2}{4} \csc^2(\theta - \arctan \delta),
\] (A18)
i.e. the effect of detuning, on the imprecision-backaction product in the broadband cavity regime, is equivalent to a quadrature rotation \( \text{arctan}(2\Delta/\kappa) \) by the cavity. Furthermore, in the limit of \( \delta, \omega \ll 1 \), measuring the phase quadrature \( \theta = \pi/2 \) is indeed the optimal strategy.

In the case of lossy homodyne detection, quantified by a non-unit detection efficiency \( \eta \leq 1 \), the imprecision-backaction product is modified as: \( [S_{xx}^{\text{imp}} S_{FF}^{\text{rp}}]_{\eta} = [S_{xx}^{\text{imp}} S_{FF}^{\text{rp}}]_{\eta = 1/\eta} \). Thus, the effect of detuning and general readout quadrature in Equation (A18) can be interpreted as an effective loss, \( \sin^2(\theta - \arctan \delta) \).

Lastly, eq. (A8) allows us to identify the real part of the cross-correlation spectral density between the backaction force and the imprecision noise. In the broadband cavity regime \( (\Omega \ll \kappa \) or \( \omega \ll 1 \), it is given by
\[
\text{Re} \left( S_{FF}^{\theta, \text{imp}}[\omega] \right) = -\frac{\hbar}{2} \left( \cot(\theta - \arctan \delta) + O(\omega^2) \right).
\] (A19)
Note that in the presence of lossy detection, the correlation term in eq. (A19) is modified as \( [S_{FF}^{\text{imp}}]_{\eta} = [S_{FF}^{\text{imp}}]_{\eta = 1/\sqrt{\eta}} \).

**Appendix B: Calculation of the mean phonon number**

Here we describe the details of the integration of the power spectra \( S_{xx} \) of the oscillator under the combined action of feedback and detuned optical spring. Using the feedback filter in eq. (20), eq. (18) takes the form,

\[
S_{xx}[\Omega] = |\chi_{\text{eff}}[\Omega]|^2 \left[ \left( 1 + \frac{1}{C_Q[\Omega_{\text{eff}}]} \right) S_{FF}^{\text{rp}} + \frac{m_0^2 \Omega_L^2 \Omega_{\text{eff}}^2}{\Omega_H^2 + \Omega_{\text{eff}}^2} g_{\text{h}}^2 S_{xx}^{\text{imp}} + \frac{2 m \Omega_0^2 \Omega_L^2 \Omega_{\text{eff}}^2}{\Omega_H^2 + \Omega_{\text{eff}}^2} g_{\text{h}}^2 S_{xx}^{\text{imp}} \right],
\] (B1)
where, \( C_Q[\Omega_{\text{eff}}] = S_{FF}^{\text{rp}}/S_{xx}[\Omega_{\text{eff}}] \) is the quantum cooperativity. Considering that the typical feedback damping is dominant in the total effective damping \( (\Gamma_{\text{eff}} \approx \Gamma_H) \), this can be recast as,

\[
(\Omega^2 + \Omega_{\text{eff}}^2) |\chi_{\text{eff}}[\Omega]|^2 S_{xx}[\Omega] = \Lambda_L \Omega^2 + \Lambda_H \Omega_{\text{eff}}^2,
\] (B2)
where,
\[
\Lambda_L = \left( 1 + \frac{1}{C_Q[\Omega_{\text{eff}}]} \right) S_{FF}^{\text{rp}} + m_0^2 \Omega_L^2 \Omega_{\text{eff}}^2 g_{\text{h}}^2 S_{xx}^{\text{imp}} + 2 m \Omega_0 \Gamma_{\text{eff}} S_{xx}^{\text{imp}}, \quad (B3)
\]
\[
\Lambda_H = \left( 1 + \frac{1}{C_Q[\Omega_{\text{eff}}]} \right) S_{FF}^{\text{rp}} + m_0^2 \Omega_H^2 \Omega_{\text{eff}}^2 g_{\text{h}}^2 S_{xx}^{\text{imp}} + 2 m \Omega_0 \Gamma_{\text{eff}} S_{xx}^{\text{imp}}. \quad (B4)
\]
The inverse of the effective susceptibility is represented as

\[
\frac{\Omega_L + i \Omega_{\text{eff}}^{-1}}{m} \chi_{\text{eff}}^\dagger[\Omega] = -i \Omega^3 - s_1 \Omega^2 + i s_2 \Omega + \Omega_L \Omega_{\text{eff}}^2, \quad (B5)
\]
where
\[
s_1 = \Omega_L + \Gamma_0 + \Gamma_{\text{rp}}, \quad (B6)
\]
\[
s_2 = \Omega_{\text{eff}}^2 + (\Omega_L - \Omega_H) \Gamma_{\text{eff}}, \quad (B7)
\]
and \( \Omega_{\text{eff}}^2 = \Omega_H^2 + \Omega_{\text{eff}}^2 + \Omega_H \Gamma_{\text{eff}} \). In order to calculate the integration in eq. (19), we use the following identity [50, 3.112],

\[
\int_{-\infty}^{\infty} dx \frac{g_n(x)}{h_n(x) h_n(-x)} = (-1)^{n+1} \frac{\pi i}{a_0} M_n \Delta_n, \quad (B8)
\]
where

\[ g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots + b_{n-1}, \quad \text{(B9)} \]
\[ h_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad \text{(B10)} \]
\[ \Delta_n = \begin{bmatrix} a_1 & a_3 & \cdots & a_n \\ a_0 & a_2 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & a_n \end{bmatrix}, \quad \text{(B11)} \]
\[ M_n = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ a_0 & a_2 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & a_n \end{bmatrix}. \quad \text{(B12)} \]

Working to first order in \( \Omega/\kappa \) (since we assume the system is in the broadband cavity regime), this identity can be applied with \( n = 3 \). Here,

\[ \Delta_3 = a_3(a_1a_2 - a_0a_3), \quad \text{(B13)} \]
\[ M_3 = b_0a_2a_3 - b_1a_0a_3 + b_2a_0a_1. \quad \text{(B14)} \]

We use

\[ \begin{cases} a_0 = 1 \\ a_1 = s_1 \\ a_2 = s_2 \\ a_3 = \Omega_L\Omega_{\text{eff}}^2 \end{cases}, \quad \begin{cases} b_0 = 0 \\ b_1 = \Lambda_L \end{cases}, \quad \begin{cases} b_2 = -\Lambda_H\Omega_L^2. \end{cases} \quad \text{(B15)} \]

to calculate \( \langle \delta x^2 \rangle \), and

\[ \begin{cases} a_0 = 1 \\ a_1 = s_1 \\ a_2 = s_2 \\ a_3 = \Omega_L\Omega_{\text{eff}}^2, \end{cases} \quad \begin{cases} b_0 = -\Lambda_L \\ b_1 = \Lambda_H\Omega_L^2 \\ b_2 = 0. \end{cases} \quad \text{(B16)} \]

to calculate \( \langle \delta p^2 \rangle \), respectively. The integrals are performed as

\[ \langle \delta x^2 \rangle = \frac{1}{2m^2\Omega_{\text{eff}}^2} \frac{\Lambda_L\Omega_{\text{eff}}^2 + \Lambda_H\Omega_Ls_1}{s_1s_2 - \Omega_L\Omega_{\text{eff}}^2}, \quad \text{(B17)} \]
\[ \langle \delta p^2 \rangle = \frac{1}{2} \frac{\Lambda_Ls_2 + \Lambda_H\Omega_L^2}{s_1s_2 - \Omega_L\Omega_{\text{eff}}^2}. \quad \text{(B18)} \]

Typically we choose the low pass cutoff frequency of the filter which is much larger than the optical dissipation, so \( s_1 \approx \Omega_L \). Thus, the inverse of the purity is given by

\[ \mu^{-1} = \frac{\Lambda_L(\Omega_{\text{eff}}^2 + s_2) + 2\Lambda_H\Omega_L^2}{2\hbar m\Omega_{\text{eff}}\Omega_L(s_2 - \Omega_L^2)}, \quad \text{(B19)} \]

which is the result in eq. (21).
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[49] In order to see how this works out, it is essential to observe that $C_{tot,imp,cor}$ are constrained: $C_{cor}^2/(2C_{imp}) = \frac{1+(C_{tot}-2)\Omega_L/\Omega_H)^2}{1+1/2(C_{tot}-2)\Omega_L/\Omega_H}$. Then, $1-C_{cor}^2/(C_{tot}C_{imp}) \propto C_{tot} - 2$.
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