LINEAR COMBINATIONS OF HARMONIC MAPPINGS

SUBZAR BEIG AND V. RAVICHANDRAN

ABSTRACT. For the harmonic mappings \( f_k = h_k + \overline{g_k} \) \((k = 1, 2)\), define the combination \( f = h + \overline{g} := \eta h_1 + (1 - \eta) h_2 + \eta g_1 + (1 - \eta) g_2 \) \((\eta \in \mathbb{C})\). Some sufficient conditions are found for \( f \) to be (i) univalent and convex in a particular direction, and (ii) convex. When \( \eta \) is taken as real, these results extend some of the earlier results obtained in this direction. Furthermore, we find sufficient conditions for the general linear combination \( \eta f_1 + (1 - \eta) f_2 \) \((\eta \in \mathbb{C})\) to be univalent and convex in a particular direction. In particular, it is shown to be univalent and convex in the direction \(-\mu\) for \( |\eta| \leq 1\), if \( h_k(z) + e^{-2i\mu} g_k(z) = \int_0^z \psi_{\mu,\nu}(\xi) d\xi \), where \( \psi_{\mu,\nu}(z) = (1 - 2\nu e^{i\mu} \cos \nu + z^2 e^{2i\mu})^{-1} \), \( \mu, \nu \in \mathbb{R} \), and \( \omega_k \), the dilatation of \( f_k \), satisfies \(|\omega_k(z)| < \alpha_k\) with \( \alpha_1 \leq 1/5 \) and \( \alpha_2 \leq 1/7\).

1. Introduction

A complex-valued function \( f \) defined on a domain \( \Omega \subseteq \mathbb{C} \) is known as a harmonic mapping if, and only if both real as well as imaginary parts of \( f \) are real harmonic. If \( \Omega \) is simply connected domain, then \( f \) can be written as \( f = h + \overline{g} \), where both \( h \) and \( g \) are analytic and are, respectively, known as analytic and co-analytic parts of \( f \). By a result of Lewy, a mapping \( f = h + \overline{g} \) is locally univalent and sense-preserving on \( \Omega \) if, and only if its Jacobian \( |h'(z)|^2 - |g'(z)|^2 > 0 \), or equivalently its dilatation \( \omega \), defined by \( \omega = g'/h' \), satisfies \(|\omega(z)| < 1\), for \( z \in \Omega \). Let \( \mathcal{H} \) denotes the class of all locally univalent and sense-preserving harmonic mappings \( f = h + \overline{g} \) defined on the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( h(0) = h'(0) = 1 = 0 \). Also, let \( \mathcal{S}_H \) be the sub-class of \( \mathcal{H} \) consisting of all univalent harmonic mappings, and let \( \mathcal{S}_H^0 \) be sub-class of all mappings \( f = h + \overline{g} \) in \( \mathcal{S}_H \) with \( g'(0) = 0 \). Furthermore, let \( \mathcal{K}_H \) (resp. \( \mathcal{K}_H^0 \)) be sub-class of \( \mathcal{S}_H \) (resp. \( \mathcal{S}_H^0 \)) consisting of all mappings which maps \( D \) onto convex domains. A domain \( \Omega \) is said to be convex in the direction \( \gamma \) \((0 \leq \gamma < \pi)\), if every line parallel to the line joining the origin to the point \( e^{i\gamma} \) has connected intersection with \( \Omega \). If \( \gamma = 0 \) (or \( \pi/2 \)), such a domain is said to be convex in the real (or imaginary) direction. A domain convex in some direction is close-to-convex. A mapping \( f \) is said to be convex in the direction \( \gamma \), if \( f(D) \) is convex in the direction \( \gamma \), and a mapping convex in every direction is a convex mapping.

Let \( f_k = h_k + \overline{g_k} \) \((k = 1, 2)\) be two harmonic mappings defined on \( D \). Their convex combination \( f \) is defined as

\[
(1.1) \quad f = tf_1 + (1 - t)f_2 = th_1 + (1 - t)h_2 + tg_1 + (1 - t)g_2, \quad 0 \leq t \leq 1.
\]

Generally, the convex combination of two convex harmonic mappings need not be convex harmonic, and indeed it need not be univalent. However, using the Clunie and Sheil-Small’s \[2\] method of “shear construction”, several authors \[4,7,11,13,14\] have studied the univalency and convexity in a particular direction of the convex combination defined

\[2010 \text{ Mathematics Subject Classification.} \ 31A05; 30C45.\]

\[\text{Key words and phrases.} \ \text{harmonic mappings; convex mappings; directional convex mappings; convex combination; linear combination; dilatation.}\]

The first author is supported by a Senior Research Fellowship from University Grants Commission, New Delhi, India.
by (1.1) of functions belonging to some sub-classes of harmonic mappings. See [15, 16, 8, 12] and the references therein for the other related work on the directional convexity of harmonic mappings and some of their combinations. In particular, Dorff and Rolf [4] proved that the convex combination of two locally univalent and sense-preserving harmonic mappings convex in the imaginary direction with the same dilatations is univalent and convex in the imaginary direction. Also, Wang et al. [14] proved that the convex combination of two locally univalent and sense-preserving harmonic mappings and some of their combinations. In particular, Dorff and Rolf [4] proved that the convex combination of two locally univalent and sense-preserving harmonic mappings convex in the imaginary direction with the same dilatations is univalent and convex in the imaginary direction.

Theorem 2.2. Let \( f_k = h_k + \overline{g_k} \) \( k = 1, 2 \) be two harmonic mappings in \( \mathbb{D} \). Define a combination \( f \) of \( f_1 \) and \( f_2 \) as

\[
 f = h + \overline{g} := \eta h_1 + (1 - \eta)h_2 + \eta g_1 + (1 - \eta)g_2, \quad \eta \in \mathbb{C}.
\]

Both the analytic and co-analytic parts \( h \) and \( g \) of \( f \) in the above combination are the general linear combination of the analytic and co-analytic parts \( h_k \) and \( g_k \) of \( f_k \), respectively, with the same parameter \( \eta \). For \( \eta \) real, \( f \) is the general real linear combination of \( f_1 \) and \( f_2 \), and is same as the combination given by (1.1) when \( 0 \leq \eta \leq 1 \).

We study the combination defined by (1.3) of harmonic mappings \( f_k = h_k + \overline{g_k} \) \( k = 1, 2 \) obtained by shearing of analytic mapping \( \psi = h_k - e^{2i\phi}g_k \), which is convex in the direction \( \phi \), and find sufficient conditions for this combination to be univalent and convex in the direction of \( \phi \). Also, we find sufficient conditions for the above combination of harmonic mappings, obtained by shearing the analytic mapping \( \phi(z) = \int_0^z \psi_{\mu,\nu}(\xi)\,d\xi \), where \( \psi_{\mu,\nu}(z) = (1 - 2ze^{\mu}\cos\nu + z^2e^{2\mu})^{-1} \), \( \mu, \nu \in \mathbb{R} \), to be convex. Moreover, we study the general linear combination \( \eta f_1 + (1 - \eta)f_2 \) \( \eta \in \mathbb{C} \) of \( f_1 \) and \( f_2 \), and find sufficient conditions for this combination to be univalent and convex in the direction \( -\mu \), when \( f_k \) is obtained by shearing the analytic mapping \( \phi(z) = \int_0^z \psi_{\mu,\nu}(\xi)\,d\xi \). In particular, it is shown that the function \( \eta f_1 + (1 - \eta)f_2 \) is univalent and convex in the direction \( -\mu \) for \( \eta \in \mathbb{D} \) if \( \omega_k \), the dilatation of \( f_k \), satisfies \( |\omega_k(z)| < \alpha_k \) with \( \alpha_1 \leq 1/5 \) and \( \alpha_2 \leq 1/7 \). For \( \eta \) real in the above defined combinations, the results we obtain generalize some of the results already established in [7, 11, 14] to a larger classes of mappings.

2. Main Results

The following result due to Chui and Sheil-Small [2], known as the method of shear construction, is very useful in checking the univalency and convexity in a particular direction/convexity of harmonic mappings.

Lemma 2.1. [2] A locally univalent and sense-preserving harmonic mapping \( f = h + \overline{g} \) on \( \mathbb{D} \) is univalent and maps \( \mathbb{D} \) onto a domain convex in the direction \( \phi \) if and only if the analytic mapping \( h - e^{2i\phi}g \) is univalent and maps \( \mathbb{D} \) onto a domain convex in the direction \( \phi \).

The following theorem gives sufficient conditions for the combination \( f \) of the mappings \( f_1 \) and \( f_2 \) given in (1.3) to be univalent and convex in a particular direction.

Theorem 2.2. For \( k = 1, 2 \), let the locally univalent and sense-preserving harmonic mapping \( f_k = h_k + \overline{g_k} \) satisfy

\[
 \lambda(h_1 - e^{2i\phi}g_1) = h_2 - e^{2i\phi}g_2 = \lambda\psi, \quad \lambda, \varphi \in \mathbb{R},
\]

where \( \psi \) is convex in the direction of \( \varphi \). Then for the mapping \( f \) given by (1.3), we have the following:
1. If $\eta$ is real and $\lambda > 0$, then the mapping $f$ is univalent and convex in the direction $\varphi$ for all $\eta$ with $0 \leq \eta \leq 1$.
2. If $\eta$ is real and $\lambda < 0$, then the mapping $f$ is univalent and convex in the direction $\varphi$ for all $\eta$ with $\eta \leq 0$.
3. If $\lambda = 1$ and $\omega_k$, the dilatation of $f_k$, satisfies $|\omega_k(z)| < \alpha_k$, $z \in \mathbb{D}$, then the mapping $f$ is univalent and convex in the direction $\varphi$ for all $\eta$ with $|\eta| \leq (1 - \alpha_1)/(2(\alpha_1 + \alpha_2))$.

Proof. Since

$$f = h + \overline{g} := \eta h_1 + (1 - \eta)h_2 + \eta g_1 + (1 - \eta)g_2,$$

by using (2.1), we have

$$h - e^{2\imath \varphi}g = \eta \left(h_1 - e^{2\imath \varphi}g_1 - h_2 + e^{2\imath \varphi}g_2\right) + h_2 - e^{2\imath \varphi}g_2$$

$$= \eta(\psi - \lambda \psi) + \lambda \psi$$

$$= (\eta + \lambda (1 - \eta)) \psi.$$ 

Therefore, in view of the assumptions on $\psi$ and $\lambda$, the mapping $h - e^{2\imath \varphi}g$ is convex in the direction $\varphi$. Thus the result follows from Lemma 2.1 whence we prove that $f$ is locally univalent and sense-preserving. Let $\omega_k$ be the dilatation of $f_k$. Therefore $g'_k = \omega_k h'_k$.

Hence the dilatation $\omega$ of $f$ is given by

$$\omega = \frac{g'}{h'} = \frac{\eta g'_1 + (1 - \eta)g'_2}{\eta h'_1 + (1 - \eta)h'_2} = \frac{\eta \omega_1 h'_1 + (1 - \eta)\omega_2 h'_2}{\eta h'_1 + (1 - \eta)h'_2}.$$ 

Solving $g'_k = \omega_k h'_k$ along with the equations obtained after differentiation of (2.1) for $h'_1$ and $h'_2$, we get

$$h'_1 = \frac{\psi'}{1 - e^{2\imath \varphi} \omega_1} \quad \text{and} \quad h'_2 = \frac{\lambda \psi'}{1 - e^{2\imath \varphi} \omega_2}.$$ 

On substituting the values of $h'_1$ and $h'_2$ from (2.3) in (2.2), the dilatation of $f$ becomes

$$\omega = \frac{\eta \omega_1 (1 - e^{2\imath \varphi} \omega_2) + \lambda (1 - \eta) \omega_2 (1 - e^{2\imath \varphi} \omega_1)}{\eta (1 - e^{2\imath \varphi} \omega_2) + \lambda (1 - \eta) (1 - e^{2\imath \varphi} \omega_1)}.$$ 

With $\omega_k$ replaced by $e^{-2\imath \varphi} \omega_k$, the above equation gives

$$e^{2\imath \varphi} \omega = \frac{\eta \omega_1 (1 - \omega_2) + \lambda (1 - \eta) \omega_2 (1 - \omega_1)}{\eta (1 - \omega_2) + \lambda (1 - \eta) (1 - \omega_1)}.$$ 

Therefore, by using (2.5), we see that

$$\operatorname{Re} \left( \frac{1 + e^{2\imath \varphi} \omega}{1 - e^{2\imath \varphi} \omega} \right) = \operatorname{Re} \left( \frac{\eta (1 + \omega_1)(1 - \omega_2) + \lambda (1 - \eta)(1 + \omega_2)(1 - \omega_1)}{(\eta + \lambda(1 - \eta))(1 - \omega_2)(1 - \omega_1)} \right)$$

$$= \operatorname{Re} \left( \frac{\eta}{\eta + \lambda(1 - \eta)(1 - \omega_1)} \right) + \operatorname{Re} \left( \frac{\lambda(1 - \eta)}{\eta + \lambda(1 - \eta)(1 - \omega_2)} \right).$$ 

If either $\eta$ is real with $0 \leq \eta \leq 1$ and $\lambda > 0$, or $\eta$ is real with $\eta \leq 0$ and $\lambda < 0$, then both

$$\frac{\eta}{\eta + \lambda(1 - \eta)} \quad \text{and} \quad \frac{\lambda(1 - \eta)}{\eta + \lambda(1 - \eta)}$$
are non-negative, and at least one of them is positive. Since $|\omega_k(z)| = |e^{2i\eta} \omega_k(z)| < 1$, we have $\text{Re}((1 + \omega_k(z))/(1 + \omega_k(z))) > 0$ for $z \in \mathbb{D}$. Therefore, equation (2.6) gives that $\text{Re}((1 + e^{2i\eta} \omega(z))/(1 + e^{2i\eta} \omega(z))) > 0$ for $z \in \mathbb{D}$. Hence $|\omega(z)| = |e^{2i\eta} \omega(z)| < 1$ for $z \in \mathbb{D}$, which implies that $f$ is locally univalent and sense-preserving. For $\lambda = 1$, we see from (2.5) that

$$e^{2i\eta} \omega = \frac{\eta \omega_1(1 - \omega_2) + (1 - \eta) \omega_2(1 - \omega_1)}{\eta(1 - \omega_2) + (1 - \eta)(1 - \omega_1)}.$$ 

Above equation shows that $|\omega(z)| < 1$, $z \in \mathbb{D}$, if, and only if,

$$|\eta \omega_1(z)(1 - \omega_2(z)) + (1 - \eta) \omega_2(z)(1 - \omega_1(z))|^2 < |\eta(1 - \omega_2(z)) + (1 - \eta)(1 - \omega_1(z))|^2;$$

or equivalently if, and only if

$$(2.7) \quad |1 - \omega_1(z)|^2(1 - |\omega_2(z)|^2) + 2 \text{Re} \left( \eta(\omega_1(z) - \omega_2(z))(1 - \omega_1(z))(1 - \omega_2(z)) \right) > 0.$$ 

Therefore, $|\omega(z)| < 1$ for $z \in \mathbb{D}$ if

$$|\eta| < \frac{|1 - \omega_1(z)|(1 - |\omega_2(z)|^2)}{2(|\omega_1(z) - \omega_2(z))(1 - \omega_2(z))|}.$$

But, for $k = 1, 2$ and $z \in \mathbb{D}$, $|\omega_k(z)| < \alpha_k$ implies that

$$\frac{|1 - \omega_1(z)|(1 - |\omega_2(z)|^2)}{2(|\omega_1(z) - \omega_2(z))(1 - \omega_2(z))|} > \frac{(1 - \alpha_1)(1 - \alpha_2)}{2(\alpha_1 + \alpha_2)}.$$ 

Hence, $|\omega(z)| < 1$ for $z \in \mathbb{D}$, for all $\eta$ with $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2)).$

Remark 2.3. The result in part 3 is sharp in the sense that it doesn’t holds good for all the values of $\eta$ in any disk of radius greater than $(1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$. Put $\omega_1(z) = \alpha z$ and $\omega_2(z) = -\alpha z$ and take $\eta$ to be negative real number in the left-hand side of inequality (2.7), and then by letting $z \to 1$ through the real values, we see that the inequality (2.7) doesn’t hold good for $\eta < -(1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2)).$

Since the mapping $\phi(z) := \int_0^z \psi_{\mu,\nu}(\xi)d\xi$ is convex (convexity of $\phi$ is easily seen by observing that $\text{Re}(1 + z\phi''(z)/\phi'(z)) > 0$ for $z \in \mathbb{D}$), Theorem 2.2 gives the following result.

Corollary 2.4. For $k = 1, 2$ and $\mu, \nu, \varphi \in \mathbb{R}$, let the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ satisfy

$$h_k(z) - e^{2i\varphi}g_k(z) = \phi(z), \quad z \in \mathbb{D}.$$ 

Then for the mapping $f$ given by (1.3), we have the following:

1. If $\eta$ is real, then the mapping $f \in S^0_H$ and convex in the direction $\varphi$ for all $\eta$ with $0 \leq \eta \leq 1$.
2. If $\omega_k$, the dilatation of the mapping $f_k$ satisfies $|\omega_k(z)| < \alpha_k$, $z \in \mathbb{D}$, then the mapping $f \in S^0_H$ and convex in the direction $\varphi$ for all $\eta$ with $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2)).$

However, if we take $\varphi = \pi/2 + \mu$ in the above corollary, we get $f \in \mathcal{K}_H^0$. In fact, in such a case, we have a more generalized result, see Theorem 2.6. For any non-negative integer $n$, define the differential operator $\mathcal{D}^n : A \to A$ on the family $A$ of all analytic mappings $f$ as: $\mathcal{D}^0 f(z) = f(z)$ and $\mathcal{D}^n f(z) = z(\mathcal{D}^{n-1} f)'(z)$ for $n \geq 1$. For the harmonic mapping $f = h + \overline{g}$, define $\mathcal{D}^n f := \mathcal{D}^n h + \overline{\mathcal{D}^n g}$. In order to prove our next result, we use the following straight forward generalization of Sheil-Small’s [10] result on the relation between the starlike and convex harmonic mappings.
Theorem 2.5. If $f = h + \overline{g}$ is a starlike harmonic mapping, and $H$ and $G$ are the analytic mappings defined by

$$D^nH = h, \quad D^nG = (-1)^n g, \quad H(0) = H'(0) - 1 = G(0) = 0,$$

then the mapping $F = H + \overline{G} \in \mathcal{K}_H$.

Theorem 2.6. For $k = 1, 2$ and $\mu, \nu \in \mathbb{R}$, let the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ be such that, the mapping $D^{n-1}f_k$ is locally univalent, sense-preserving, and

$$D^{n-1}h_k(z) + e^{2\mu}(-1)^{n-1}D^{n-1}g_k(z) = \frac{1}{z} \int_0^{z_n=z} \left( \ldots \frac{1}{z_1} \int_0^{z_1} \psi_{\mu,\nu}(\xi) d\xi \ldots \right) dz_{n-1}. \quad (2.8)$$

Then for the mapping $f$ given by (1.3), we have the following:

1. If $\eta$ is real, then the mapping $f \in \mathcal{K}_H^0$ for all $\eta$ with $0 \leq \eta \leq 1$.
2. If $\omega_k$, the dilatation of $D^{n-1}f_k$, satisfies $|\omega_k(z)| < \alpha_k$ for $z \in \mathbb{D}$, then the mapping $f \in \mathcal{K}_H^0$ for all $\eta$ with $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$.

Proof. Since

$$f = h + \overline{g} := \eta h_1 + (1 - \eta)h_2 + \eta g_1 + (1 - \eta)g_2, \quad (2.9)$$

we have

$$h(z) + e^{2\mu}g(z) = \eta \left( h_1(z) + e^{2\mu}g_1(z) - h_2(z) - e^{2\mu}g_2(z) \right) + h_2(z) + e^{2\mu}g_2(z)
= h_2(z) + e^{2\mu}g_2(z).$$

Therefore, in view of (2.8), we see that

$$D^{n-1}h(z) + e^{2\mu}(-1)^{n-1}D^{n-1}g(z) = \int_0^z \psi_{\mu,\nu}(\xi) d\xi,$$

or equivalently

$$H(z) + e^{2\mu}G(z) = \int_0^z \psi(\xi) d\xi, \quad (2.10)$$

where $H(z) := D^{n-1}h(z)$ and $G(z) := (-1)^{n-1}D^{n-1}g(z)$. In view of the assumptions on $D^{n-1}f_k$, Theorem 2.5 shows that the mapping $F := H + \overline{G}$ is locally univalent and sense-preserving. We will show it is convex. To prove this, in view of Lemma 2.1, it suffices to show that the mapping $e^{-\mu}(H - e^{2\theta}G)$ is convex in the direction $\theta$ for all $\theta$ ranging in an interval of length $\pi$. In other words, it is sufficient to show that the mapping $e^{-i(\mu+\theta)}(H - e^{2i\theta}G)$ is convex in the direction $-\mu$ for all $\theta$ such that $0 \leq \mu + \theta < \pi$.

Consider the case $0 \leq \mu + \theta < \pi/2$. Since $f$ is locally univalent and sense-preserving, $|G'(z)/H'(z)| < 1$, $z \in \mathbb{D}$, and hence

$$\text{Re} \left( \frac{H'(z) - e^{-2\mu}G'(z)}{H'(z) + e^{-2\mu}G'(z)} \right) > 0, \quad z \in \mathbb{D}. \quad (2.11)$$

Using above inequality, we have

$$\text{Re} \left( \frac{e^{-i(\mu+\theta)}(H - e^{2i\theta}G)'(z)}{H'(z) + e^{-2\mu}G'(z)} \right) = \text{Re} \left( \frac{(e^{-i(\mu+\theta)}H'(z) - e^{-2\mu}e^{i(\mu+\theta)}G'(z))}{H'(z) + e^{-2\mu}G'(z)} \right)
= \text{Re} \left( \frac{H'(z) - e^{-2\mu}G'(z)}{H'(z) + e^{-2\mu}G'(z)} \cos(\mu + \theta) - i\sin(\mu + \theta) \right)
= \cos(\mu + \theta) \text{Re} \left( \frac{H'(z) - e^{-2\mu}G'(z)}{H'(z) + e^{-2\mu}G'(z)} \right) > 0, \quad z \in \mathbb{D}. \quad (2.11)$$
Corollary 2.7. For $k = 1, 2, \ldots, n$ and $\mu, \nu \in \mathbb{R}$, let the normalized harmonic mapping $f_k = h_k + \overline{g_k}$ satisfy equation (2.8), and let the mapping $D^{n-1} f_k$ be locally univalent and sense-preserving. Then we have the following:

1. If $\sum_{k=1}^{n} t_k = 1$, $0 \leq t_k \leq 1$, then the mapping $f = \sum_{k=1}^{n} t_k f_k \in \mathcal{K}_H^n$.

2. Let $\omega_k$, the dilatation of the mapping $D^{n-1} f_k$, satisfy $|\omega_k(z)| < \alpha_1$ for $k = 1, 2, \ldots, n-1$ and $|\omega_n(z)| < \alpha_2$, $z \in \mathbb{D}$. Also, let $t_1, t_2, \ldots, 1-t_n$ are of same sign and $\sum_{k=1}^{n} t_k = 1$. Then the mapping $f = \sum_{k=1}^{n} t_k f_k \in \mathcal{K}_H^n$ for all $t_n$ with $0 \leq t_n \leq (1-\alpha_1)(1-\alpha_2)/(2(\alpha_1 + \alpha_2))$.

Proof. Part 1 simply follows by the repeated application of part 1 in Theorem 2.6. However, for part 2, we see that

$$f = t_1 f_1 + t_2 f_2 + \cdots + t_n f_n$$

$$= (1-t_n) \left( \frac{t_1}{1-t_n} f_1 + \cdots + \frac{t_{n-1}}{1-t_n} f_{n-1} \right) + t_n f_n$$

$$= (1-t_n) F + t_n f_n.$$ 

By the assumptions on $t_k$, we see that $t_1/(1-t_n), t_2/(1-t_n), \ldots, t_{n-1}/(1-t_n)$ are all positive and $\sum_{k=1}^{n-1} t_k/(1-t_n) = 1$. Also, $|\omega_k(z)| < \alpha_1$, $z \in \mathbb{D}$, for $k = 1, 2, \ldots, n-1$. Therefore by [13, Lemma 6], the dilatation $\omega$ of the mapping $D^{n-1} F = \frac{t_1}{1-t_n} D^{n-1} f_1 + \cdots + \frac{t_{n-1}}{1-t_n} D^{n-1} f_{n-1}$ satisfies $|\omega(z)| < \alpha_1$, $z \in \mathbb{D}$. Since $\sum_{k=1}^{n-1} t_k/(1-t_n) = 1$, the mapping $F$ satisfies equation (2.8), and the normalization of the mappings $f_k$ implies the normalization of the mapping $F$. Therefore the result follows from part 2 of Theorem 2.6.

Observe that

$$\int_0^z \psi_{0,0}(\xi) d\xi = \int_0^z \frac{d\xi}{1 - 2\xi + \xi^2} = \frac{z}{1-z}$$

and

$$\int_0^z \psi_{\pi/2,\pi/2}(\xi) d\xi = \int_0^z \frac{d\xi}{1 - \xi^2} = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right).$$

Therefore, Theorem 2.6 gives the following results of Sun et al..

Corollary 2.8. [13, Theorems 4, 5] For $k = 1, 2$, let the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ satisfy

$$h_k(z) + g_k(z) = \frac{z}{1-z} \quad \text{or} \quad h_k(z) - g_k(z) = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right), \quad z \in \mathbb{D}.$$ 

Then, for $0 \leq t \leq 1$, the mapping $f = tf_1 + (1-t)f_2 \in \mathcal{K}_H^n$.
The following result of Royster and Zeigler [9] can be used to check the directional convexity of the harmonic mappings (arising from shearing of analytic mappings) through the method of shear construction.

**Theorem 2.9.** [9] Let φ be a non-constant analytic mapping in $\mathbb{D}$. Then φ maps $\mathbb{D}$ onto a domain convex in the direction $\gamma$ if, and only if, there are real numbers $\mu$ ($0 \leq \mu < 2\pi$) and $\nu$ ($0 \leq \nu < \pi$), such that

$$
(2.12) \quad \Re \left\{ e^{i(\mu-\gamma)}(1 - 2z e^{-i\mu} \cos \nu + z^2 e^{-2i\mu}) \phi'(z) \right\} \geq 0, \quad z \in \mathbb{D}.
$$

Put $\gamma = \mu$ and then replace it by $-\mu$ in the above theorem, we see the non-constant analytic mapping $\phi$ is convex in the direction $-\mu$, if for some $\nu \in \mathbb{R}$, $\Re (\phi'(z)/\psi_{\mu,\nu}(z)) > 0$. Using this argument, Theorem 2.9 gives the following result.

**Theorem 2.10.** For $\mu, \nu \in \mathbb{R}$, let the locally univalent and sense-preserving mapping $f_k = h_k + \frac{g_k}{\psi_k}$ ($k = 1, 2$) satisfy

$$
(2.13) \quad h_k(z) - e^{-2i\mu} g_k(z) = \int_0^z \psi_{\mu,\nu}(\xi)p(\xi)d\xi, \quad z \in \mathbb{D},
$$

where $p$ is an analytic mapping with $\Re p(z) > 0$ for $z \in \mathbb{D}$. Then for the mapping $f$ given by (1.3), we have the following:

1. If $\eta$ is real, then the mapping $f$ is univalent and convex in the direction $-\mu$ for all $\eta$ with $0 \leq \eta \leq 1$.
2. If $\lambda = 1$ and $\omega_k$, the dilatation of $f_k$, satisfies $|\omega_k(z)| < \alpha_k$, $z \in \mathbb{D}$, then the mapping $f$ is univalent and convex in the direction $-\mu$ for all $\eta$ with $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$.

**Proof.** Since $\Re p(z) > 0$, $z \in \mathbb{D}$, we have

$$
\Re \left( \frac{1}{\psi_{\mu,\nu}} \left( \int_0^z \psi_{\mu,\nu}(\xi)p(\xi)d\xi \right)' \right) = \Re \left( \frac{1}{\psi_{\mu,\nu}(z)} \psi_{\mu,\nu}(z)p(z) \right) = \Re p(z) > 0, \quad z \in \mathbb{D}.
$$

Hence, by Theorem 2.9 the mapping $\int_0^z \psi_{\mu,\nu}(\xi)p(\xi)d\xi$ is convex in the direction $-\mu$. Therefore, in view of equation (2.13), Theorem 2.2 follows the result.

On varying the mapping $p$ in the above theorem, we get different results which in a special cases not only generalizes the already established results on convex combination of harmonic mappings (e.g see [7,13]), but on taking smaller values of $\alpha_1$ and $\alpha_2$, part (2) shows that such results are extended to a wider range of the values of $\eta$ as well.

**Corollary 2.11.** For $k = 1, 2$, and $\mu, \nu_1, \nu_2 \in \mathbb{R}$, let the locally univalent and sense-preserving mapping $f_k = h_k + \frac{g_k}{\psi_k}$ ($k = 1, 2$) satisfy

$$
(2.14) \quad h_k(z) + e^{-2i\mu} g_k(z) = A \frac{z(1 - ze^{i\mu} \cos \nu_1)}{1 - ze^{2i\mu}} + B \int_0^z \psi_{\mu,\nu_2}(\xi)d\xi, \quad z \in \mathbb{D},
$$

where $A, B \geq 0$ with $A + B > 0$. Then for the mapping $f$ given by (1.3), we have the following:

1. If $\eta$ is real, then the mapping $f$ is univalent and convex in the direction $-\left(\mu + \pi/2\right)$ for all $\eta$ with $0 \leq \eta \leq 1$.
2. If $\lambda = 1$ and $\omega_k$, the dilatation of $f_k$, satisfies $|\omega_k(z)| < \alpha_k$, $z \in \mathbb{D}$, then the mapping $f$ is univalent and convex in the direction $-\left(\mu + \pi/2\right)$ for all $\eta$ with $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$. 

Proof. Let

\[ p(z) = A \frac{1 - 2ze^{i\mu} \cos \nu_1 + z^2 e^{2i\mu}}{1 - z^2 e^{2i\mu}} + B \frac{1 - z^2 e^{2i\mu}}{1 - 2ze^{i\mu} \cos \nu_2 + z^2 e^{2i\mu}} \]

Since, for any \( \gamma \) real and \( z \in \mathbb{D} \),

\[ \text{Re} \left( \frac{1 - z^2 e^{2i\mu}}{1 - 2ze^{i\mu} \cos \gamma + z^2 e^{2i\mu}} \right) = \frac{1 - |z|^4 - 2 \cos \gamma (1 - |z|^2) \text{Re}(e^{i\mu}z)}{|1 - 2ze^{i\mu} \cos \gamma + z^2 e^{2i\mu}|^2} \geq (1 - |z|^2)(1 + |z|^2 - 2|\cos \gamma| \text{Re}(e^{i\mu}z)) > 0, \]

we get \( \text{Re} p(z) > 0 \) for \( z \in \mathbb{D} \). Also, in view of equation (2.14), we see that

\[ \int_0^z \psi_{\mu+i/2,0}(\xi)p(\xi)d\xi = \int_0^z \frac{p(\xi)d\xi}{1 - \xi^2 e^{2i\mu}} = \int_0^z \left( A \frac{1 - 2\xi e^{i\mu} \cos \nu_1 + \xi^2 e^{2i\mu}}{(1 - \xi^2 e^{2i\mu})^2} + B \psi_{\mu,\nu_2} \right) d\xi = A \frac{z(1 - ze^{i\mu} \cos \nu_1)}{1 - z^2 e^{2i\mu}} + B \int_0^z \psi_{\mu,\nu_2} d\xi = h_k(z) + e^{-2\mu}g_k(z). \]

Therefore, the result follows by Theorem 2.10.

Remark 2.12. By taking \( A = 1, B = 0, \mu = \pi \) and \( \gamma_1 = 0 \), and \( A = 1, B = 0, \mu = \pi \) in the above corollary, we get Theorem 3 of Wang et al. [14] and Theorem 2.1 of Kumar et al. [7], respectively. Also, observe that

\[ \int_0^z \psi_{\mu,\nu_2}(\xi)d\xi = -\frac{e^{-i\mu}}{2i \sin \nu_2} \log \left( \frac{1 - ze^{i(\mu+\nu_2)}}{1 - ze^{i(\mu-\nu_2)}} \right). \]

for \( \nu_2 \neq n\pi, n \in \mathbb{N} \). Therefore by taking \( \mu = \pi \) in the above corollary, we get a result of Sun et al. [13] p.371.

Taking

\[ p(z) = A \frac{1 - 2ze^{i\mu} \cos \gamma_1 + z^2 e^{2i\mu}}{1 - z^2 e^{2i\mu}} + B \frac{1 - z^2 e^{2i\mu}}{1 - 2ze^{i\mu} \cos \gamma_1 + z^2 e^{2i\mu}} \]

in Theorem 2.10 and proceeding as in Corollary 2.11 we get the following result.

Corollary 2.13. For \( k = 1, 2 \), and \( \mu, \nu \in \mathbb{R} \), let the locally univalent and sense-preserving harmonic mapping \( f_k = h_k + \bar{g}_k \) (\( k = 1, 2 \)) satisfy

\[ h_k(z) - e^{-2\mu}g_k(z) = A \frac{1 + ze^{i\mu}}{2e^{i\mu}} \log \left( \frac{1 + ze^{i\mu}}{1 - ze^{i\mu}} \right) + B \bar{z} \psi_{\mu,\nu}(z), \quad z \in \mathbb{D}, \]

where \( A, B \geq 0 \) with \( A + B > 0 \). Then for the mapping \( f \) given by (1.3), we have the following:

1. If \( \eta \) is real, then the mapping \( f \) is univalent and convex in the direction \(- (\mu + \pi/2)\) for all \( \eta \) with \( 0 \leq \eta \leq 1 \).
2. If \( \lambda = 1 \) and \( \omega_k \), the dilatation of \( f_k \), satisfies \( |\omega_k(z)| < \alpha_k, z \in \mathbb{D} \), then the mapping \( f \) is univalent and convex in the direction \(- (\mu + \pi/2)\) for all \( \eta \) with \( |\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2)) \).
Lastly, we consider the general linear combination $\mathcal{F}$ of the mappings $f_1$ and $f_2$, and is defined as

$$
\mathcal{F} = \eta f_1 + (1 - \eta) f_2 = \eta h_1 + (1 - \eta) h_2 + \eta \overline{g_1} + (1 - \eta) \overline{g_2} =: h + \overline{g}, \quad \eta \in \mathbb{C}.
$$

Like the $f$ combination defined, above combination $\mathcal{F}$ too have both the analytic and co-analytic parts as general linear combination of the corresponding analytic and co-analytic parts of $f_1$ and $f_2$, respectively, except that the parameters of the combination are not same, but complex conjugates. If $\eta$ is real, the above combination is simply a special case of the combination $f$ of the mappings $f_1$ and $f_2$ defined by (1.3). We find sufficient conditions for this combination of the locally univalent and sense-preserving harmonic mappings to be univalent and convex in a particular direction. First, we prove the following lemma.

**Lemma 2.14.** For $k = 1, 2$, let the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ satisfy

$$
(2.15) \quad h_k(z) + e^{-2\mu} g_k(z) = \int_0^z \psi_{\mu,\nu}(\xi) d\xi
$$

for some $\mu, \nu \in \mathbb{R}$. Then the mapping $\mathcal{F} = \eta f_1 + (1 - \eta) f_2$ is univalent and convex in the direction $-\mu$ if it is locally univalent and sense-preserving for all $\eta \in \mathbb{C}$ satisfying anyone of the following:

(i) $0 \leq \text{Re} \eta \leq 1$.

(ii) $-1 \leq \text{Re} \eta \leq 1$ provided $\omega_k$, the dilatation of $f_k$, satisfies $|\omega_k(z)| < \alpha_k$ with $\alpha_1 \leq 1/5$ and $\alpha_2 \leq 1/7$.

**Proof.** Let $\eta = |\eta| e^{i\theta}$. Since

$$
\mathcal{F} = \eta f_1 + (1 - \eta) f_2 = \eta h_1 + (1 - \eta) h_2 + \eta \overline{g_1} + (1 - \eta) \overline{g_2} =: h + \overline{g},
$$

we have

$$
h - e^{-2\mu} g = |\eta| \left( e^{i\theta} (h_1 - h_2) - e^{-2\mu} e^{-i\theta} (g_1 - g_2) \right) + h_2 - e^{-2\mu} g_2
$$

$$
= |\eta| (h_1 - h_2 - e^{-2\mu} (g_1 - g_2)) \cos \theta - i \sin \theta (h_1 - h_2 + e^{-2\mu} (g_1 - g_2))
$$

$$
+ h_2 - e^{-2\mu} g_2
$$

$$
= |\eta| \cos \theta (h_1 - e^{-2\mu} g_1) + (1 - |\eta| \cos \theta) (h_2 - e^{-2\mu} g_2).
$$

Therefore, in view of (2.15), we see that

$$
(2.16) \quad \frac{h' - e^{-2\mu} g'}{\psi_{\mu,\nu}} = \text{Re}(\eta) \frac{h'_1 - e^{-2\mu} g'_1}{h'_1 + e^{-2\mu} g'_1} + (1 - \text{Re} \eta) \frac{h'_2 - e^{-2\mu} g'_2}{h'_2 + e^{-2\mu} g'_2}
$$

Since the mapping $f_k = h_k + \overline{g_k}$ ($k = 1, 2$) is locally univalent and sense-preserving, $|g_k'(z)/h_k'(z)| < 1$ for $z \in \mathbb{D}$, or equivalently

$$
\text{Re} \left( \frac{h'_k(z) - e^{-2\mu} g'_k(z)}{h'_k(z) + e^{-2\mu} g'_k(z)} \right) > 0, \quad z \in \mathbb{D}.
$$

Hence, above equation along with (2.16) shows that

$$
\text{Re} \left( \frac{h(z) - e^{-2\mu} g(z)}{\psi_{\mu,\nu}(z)} \right) > 0, \quad z \in \mathbb{D},
$$

as required.
if $0 \leq \Re \eta \leq 1$. Therefore by Theorem 2.9 the function $h - e^{-2\mu}g$ is convex in the direction $-\mu$. Thus, for all $\eta$ with $0 \leq \Re \eta \leq 1$, lemma 2.14 shows that the mapping $\mathcal{F}$ is univalent and convex in the direction of $-\mu$ provided it is locally univalent and sense-preserving. This follows the result in case (ii). Let $\omega_k$ be the dilatation of the mapping $f_k$. Therefore equation (2.16) is equivalent to

$$
\frac{h' - e^{-2\mu}g'}{\psi_{\mu,\nu}} = \Re(\eta)\frac{1 - e^{-2i\mu}\omega_1}{1 + e^{-2i\mu}\omega_1} + (1 - \Re \eta)\frac{1 - e^{-2i\mu}\omega_2}{1 + e^{-2i\mu}\omega_2}.
$$

Note that

$$
\Re \frac{1 - z/5}{1 + z/5} < \frac{3}{2} \text{ and } \Re \frac{1 - z/7}{1 + z/7} > \frac{3}{4}, \quad z \in \mathbb{D}.
$$

Hence, if $|\omega_1(z)| < 1/5$ and $|\omega_1(z)| < 1/7$, then for $-1 \leq \Re \eta < 0$, equation (2.17) shows that

$$
\Re \frac{h'(z) - e^{-2\mu}g'(z)}{\psi_{\mu,\nu}(z)} > \frac{3}{2} \Re \eta + \frac{3}{4}(1 - \Re \eta) = \frac{3}{4}(1 + \Re \eta) \geq 0, \quad z \in \mathbb{D}.
$$

Above equation along with Theorem 2.9 and Lemma 2.14 shows that, for all $\eta$ with $-1 \leq \Re \eta < 0$, the mapping $\mathcal{F}$ is univalent and convex in the direction of $-\mu$ provided it is locally univalent and sense-preserving. This along with the result in case (i) follows the result in case (ii).

The problem in the above lemma is now to find out the values of $\eta$ for which $\mathcal{F}$ is locally univalent and sense-preserving. In the next theorem, we prove this for the values of $\eta$ lying in the disk with origin as center and radius given in terms of the bounds of the dilatations of $f_k$.

**Theorem 2.15.** For $k = 1, 2$, let the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ satisfy

$$
h_k(z) + e^{-2\mu}g_k(z) = \int_0^z \psi_{\mu,\nu}(\xi)d\xi
$$

for some $\mu \in \mathbb{R}$. Let the dilatation $\omega_k$ of $f_k$ satisfies $|\omega_k| < \alpha_k$, $\alpha_k \in \mathbb{R}$. Then the mapping $\mathcal{F} = \eta f_1 + (1 - \eta)f_2$ $(\eta \in \mathbb{C})$ is locally univalent and sense-preserving for $|\eta| \leq (1 - \alpha_1)/(2(\alpha_1 + \alpha_2))$.

**Proof.** Since $\omega_k$ is the dilatation of the mapping $f_k$, following similarly as in Theorem 2.2, the dilatation $\omega$ of $\mathcal{F}$ is given by

$$
\omega = \eta \omega_1(1 - e^{-2i\mu}\omega_2) + (1 - te^{i\theta}\omega_2(1 - \eta\omega_1) \overline{\eta}(1 - e^{-2i\mu}\omega_2) + (1 - \overline{\eta})(1 - e^{-2i\mu}\omega_1).
$$

With $\omega_k$ replaced by $e^{2i\mu}\omega_k$, the above equation gives

$$
e^{-2i\theta}\omega = \eta \omega_1(1 - \omega_2) + (1 - \eta)\omega_2(1 - \omega_1) \overline{\eta}(1 - \omega_2) + (1 - \overline{\eta})(1 - \omega_1).
$$

Thus $|\omega(z)| < 1$ for $z \in \mathbb{D}$ if, and only if

$$
|\eta \omega_1(1 - \omega_2(z)) + (1 - \eta)\omega_2(z)(1 - \omega_1(z))|^2 < |\overline{\eta}(1 - \omega_2(z)) + (1 - \overline{\eta})(1 - \omega_1(z))|^2,
$$

or equivalently, if and only if

$$
|1 - \omega_1(z)|^2 (1 - |\omega_2(z)|^2) + 2 \Re \left(\eta \omega_1(z) - \omega_2(z))(1 - \overline{\omega_1(z)})(e^{-2i\theta} - \overline{\omega_2(z)})\right) > 0,
$$

for all $z \in \mathbb{D}$.
where $\theta$ is the argument of $\eta$. Therefore $|\omega(z)| < 1$, $z \in \mathbb{D}$, if
\[
|\eta| < \frac{|1 - \omega_1(z)| (1 - |\omega_2(z)|^2)}{2(|\omega_1(z) - \omega_2(z)| (e^{-2\theta} - \omega_2(z))|).
\]

But, for $k = 1, 2$ and $z \in \mathbb{D}$, $|\omega_k(z)| < \alpha_k$ implies that
\[
\frac{|1 - \omega_1(z)| (1 - |\omega_2(z)|^2)}{2(|\omega_1(z) - \omega_2(z)| (e^{-2\theta} - \omega_2(z))|) > \frac{(1 - \alpha_1)(1 - \alpha_2)}{2(\alpha_1 + \alpha_2)}.
\]

Hence $|\omega(z)| < 1$, $z \in \mathbb{D}$, for all $\eta$ with $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$.

**Remark 2.16.** Clearly Remark 2.3 shows that the result in the above theorem is sharp in the sense that it doesn’t hold for all the values of $\eta$ in any disk of radius greater than $(1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$.

Theorem 2.15 along with Lemma 2.14 gives the following result.

**Theorem 2.17.** For $k = 1, 2$, let the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ satisfy
\[
(2.18) \quad h_k(z) + e^{-2\mu}g_k(z) = \int_0^z \psi_{\mu, \nu}(\xi) d\xi
\]
for some $\mu, \nu \in \mathbb{R}$. Let, $\omega_k$, the dilatation of $f_k$ satisfies $|\omega_k| < \alpha_k$, $\alpha_k \in \mathbb{R}$. Then the mapping $F = \eta f_1 + (1 - \eta)f_2$ ($\eta \in \mathbb{C}$) is univalent and convex in the direction $-\mu$ for $|\eta| \leq (1 - \alpha_1)(1 - \alpha_2)/(2(\alpha_1 + \alpha_2))$ and satisfying anyone of the following:

(i) $0 \leq \text{Re} \eta \leq 1$.

(ii) $-1 \leq \text{Re} \eta \leq 1$ provided $\omega_k$, the dilatation of $f_k$, satisfies $|\omega_k(z)| < \alpha_k$ with $\alpha_1 \leq 1/5$ and $\alpha_2 \leq 1/7$.

**Remark 2.18.** For $k = 1, 2$, if the mapping $f_k = h_k + \overline{g_k} \in \mathcal{H}$ satisfy (2.18) and its dilatation $\omega_k$ satisfies $|\omega_k(z)| < \alpha_k$, with $\alpha_1 \leq 1/5$ and $\alpha_2 \leq 1/7$, then the mapping $F = \eta f_1 + (1 - \eta)f_2$ is univalent and convex in the direction $-\mu$ for all $\eta \in \mathbb{D}$.

**References**

[1] Z. Boyd, M. Dorff, M. Nowak, M. Romney and M. Wołoszkiewicz, Univalency of convolutions of harmonic mappings. Appl. Math. Comput. 234 (2014), 326–332.

[2] J. Clunie, and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 3–25.

[3] Cohn, A. ber die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise. (German) Math. Z. 14 (1922), no. 1, 110–148.

[4] M. Dorff, and S. Rolf, Anamorphosis, mapping problems, and harmonic univalent functions. Explo- rations in complex analysis, 197–269, Classr. Res. Mater. Ser., Math. Assoc. America, Washington, DC, 2012.

[5] M. Dorff, M. Nowak, and M. Wołoszkiewicz, Convolutions of harmonic convex mappings, Complex Var. Elliptic Equ. 57 (2012), no. 5, 489–503.

[6] M. Dorff; M. Nowak and M. Wołoszkiewicz, Harmonic mappings onto parallel slit domains. Ann. Polon. Math. 101 (2011), no. 2, 149–162.

[7] R. Kumar, S. Gupta and S. Singh, Linear combinations of univalent harmonic mappings convex in the direction of the imaginary axis. Bull. Malays. Math. Sci. Soc. 39 (2016), no. 2, 751–763.

[8] S. Ponnusamy and A. Rasila, Harmonic shears of slit and polygonal mappings. Appl. Math. Comput. 233 (2014), 588–598.

[9] W. C. Royster and M. Ziegler, Univalent functions convex in one direction. Publ. Math. Debrecen 23 (1976), no. 3-4, 339–345.

[10] T. Sheil-Small, Constants for planar harmonic mappings. J. London Math. Soc. 42 (1990), 237-248.
[11] L. Shi, Z.-G. Wang, A. Rasila and Y. Sun, Convex combinations of harmonic shears of slit mappings Bull. Iranian. Math. Sci. Soc. 43 (2017), 1495–1510.
[12] B. Subzar and V. Ravichandran, Directional convexity of harmonic mappings. Bull. Malays. Math. Sci. Soc. 41 (2018), 1045–1060.
[13] Y. Sun, A. Rasila and Y.-P. Jiang, Linear combinations of harmonic quasiconformal mappings convex in one direction. Kodai Math. J. 39 (2016), no. 2, 366–377.
[14] Z.-G Wang, Z.-H Liu and Y.-C Li, On the linear combinations of harmonic univalent mappings. J. Math. Anal. Appl. 400 (2013), no. 2, 452–459.

Department of Mathematics, University of Delhi, Delhi–110 007, India
E-mail address: beighsubzar@gmail.com

Department of Mathematics, University of Delhi, Delhi–110 007, India
E-mail address: vravi68@gmail.com