A Parallel Algorithm for \((3 + \varepsilon)\)-Approximate Correlation Clustering

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Abstract

Grouping together similar elements in datasets is a common task in data mining and machine learning. In this paper, we study parallel algorithms for correlation clustering, where each pair of items is labeled either similar or dissimilar. The task is to partition the elements and the objective is to minimize disagreements, that is, the number of dissimilar elements grouped together and similar elements grouped separately.

Our main contribution is a parallel algorithm that achieves a \((3 + \varepsilon)\)-approximation to the minimum number of disagreements. Our algorithm builds on the analysis of the PIVOT algorithm by Ailon, Charikar, and Newman [JACM’08] that obtains a 3-approximation in the centralized setting. Our design allows us to sparsify the input graph by ignoring a large portion of the nodes and edges without a large extra cost as compared to the analysis of PIVOT. This sparsification makes our technique applicable on several models of massive graph processing, such as Massively Parallel Computing (MPC) and graph streaming, where sparse graphs can typically be handled much more efficiently.

For linear memory models, such as the linear memory MPC and streaming, our approach yields \(O(1)\) time algorithms, where the runtime is independent of \(\varepsilon\), which only appears in the memory demand.

1 Introduction

In this paper, we consider the correlation clustering problem introduced by [5], where the goal is to group together similar elements and separate dissimilar elements. We model the similarity as a complete signed graph \(G = (V, E^+ \cup E^-)\), where a positive edge \(\{u, v\} = e \in E^+\) indicates that \(u\) and \(v\) are similar. In case \(e \in E^-\), the edge is negative and the nodes are dissimilar. The goal is to minimize the disagreements, where a disagreement is induced by grouping together dissimilar nodes or separating similar ones. As pointed out by [11], it is typically the case that the set of negative edges is much larger than the set of positive edges. Hence, in this paper, we identify the input graph with the set of positive edges, i.e., \(G = (V, E^+)\) and the negative edges are defined implicitly. Correlation clustering is a natural abstraction for central problems in data mining and machine learning such as community and duplicate detection [3, 10], link prediction [20], and image segmentation [17]. A key feature of correlation clustering, as opposed to, for example, the standard \(k\)-means clustering, is that the number of clusters is not predetermined.

As the volume of data sets is growing fast, there is an increasing demand for parallel solutions to clustering problems. In this paper, we study the parallel complexity of correlation clustering. Our main contribution is a parallel \((3 + \varepsilon)\)-approximation algorithm for correlation clustering that can be adapted to various modern models of computation tailored for processing massive data sets.

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**Theorem** (Informal Main Result). There is a parallel algorithm to the correlation clustering problem that returns a $(3 + \varepsilon)$-approximation in expectation and with high probability\(^1\). It can be implemented in

- $O(1)$ rounds in the MPC model with $\tilde{O}(n)$ local memory and $\tilde{O}(m)$ total memory\(^2\). The runtime is independent of $\varepsilon$.
- $O(1)$ passes in the semi-streaming model with $\tilde{O}(n)$ space. The runtime is independent of $\varepsilon$.

### 1.1 Previous Works on Correlation Clustering

In the centralized setting, finding an optimal clustering that minimizes disagreements is known to be NP-hard [5], which motivates the study of approximation algorithms. We note that maximizing agreements, where an agreement corresponds to clustering together positive edges and separating negative edges, is also NP-hard since the optimum solutions are the same for the maximization and the minimization problems. However, the approximation problems are very different. For maximum agreements, a trivial algorithm yields a $1/2$-approximation. Furthermore, 0.7664-approximation and 0.7666 algorithms are known, even for weighted graphs [19, 8].

In this work, we focus on the minimization problem. The current state-of-the-art approximation ratio is 2.06 due to [9]. This result is obtained through a careful LP-rounding scheme and almost matches the integrality gap 2 for the problem [8]. The downside of LP-based algorithms is that it is not known how to efficiently solve LPs in the parallel setting. The best-known algorithm not based on solving an LP, called the PIVOT algorithm, yields a 3-approximation [2]. The PIVOT algorithm works as follows:

- In each sequential step, pick a node $u$ uniformly at random.
- Create a cluster $C$ that contains $u$ and all of its neighbors in the current graph.
- Remove $C$ from the graph and recurse on the remaining graph.

An equivalent formulation is through a (randomized) greedy maximal independent set, or greedy MIS for short, where we pick a random permutation and pick the pivot nodes according to the order of the permutation.

**Parallel Pivot.** While the original formulation of the PIVOT algorithm is inherently sequential, an early work towards a parallel PIVOT showed how to pick several pivot nodes in parallel, in each parallel step. The authors obtained a $(3 + \varepsilon)$-approximation in $O(\log^2 n/\varepsilon)$ rounds. The formulation through the greedy MIS gave a stronger handle towards an efficient parallel algorithm, resulting in an $O(\log^2 n)$-round 3-approximation algorithm with high probability [7]. For graphs with the maximum degree $\Delta \ll n$, the runtime was improved to $O(\log \Delta \cdot \log n/\varepsilon)$ yielding a $(3 + \varepsilon)$-approximation. In a more recent work, the analysis of the greedy MIS was improved to $O(\log n)$ rounds, yielding a 3-approximation in $O(\log n)$ parallel rounds with high probability [15]. It was also shown that the analysis cannot be improved and novel ideas are required for faster algorithms.

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\(^1\)Following the standard, we say that an event holds with high probability if it holds with probability $1 - 1/n^c$ for a constant $c$ we can choose.

\(^2\)\(\tilde{O}(\cdot)\) notation hides logarithmic factors
Sublogarithmic Parallel Algorithms. The algorithms based on a parallel implementation we mentioned so far, are well-suited for distributed and parallel models of computation, such as LOCAL, where a communication network is modeled as a graph and the nodes can communicate through messages over the edges of the graph. In models of modern parallel computation, where the communication is not restricted to local message passing, we typically aim for much faster runtimes. For example, the logarithmic “barrier” for obtaining good approximations can be overcome in the semi-streaming model, where the greedy MIS can be computed in $O(\log \log n)$ passes [1] yielding a 3-approximation. We refer to Section 3 for more details on these models and keep the discussion here independent of model details.

However, all the above algorithms have some dependency on the size of the input graph. In recent works, the sparse-dense decomposition [12, 4] was utilized to obtain a constant approximation to the correlation clustering problem in constant many parallel rounds. Furthermore, this approach is quite model independent. On the downside, the approximation ratios, while being constant, are very high. In the case of [12], they obtain an approximation ratio over 700 and the ratio of [4] is over 6400. In a very recent manuscript [6] (to appear in FOCS'22), a new method was introduced that significantly improved the approximation ratio to $(3 + \varepsilon)$. The new algorithm runs in $O(1/\varepsilon)$-round in various models of parallel and distributed computing, such as LOCAL and CONGEST models of message passing, semi-streaming and sublinear memory MPC.

We underline that our work matches this approximation ratio and results in algorithms whose runtime is constant independently of the desired approximation parameter $\varepsilon$. However, our algorithm runs only in semi-streaming and linear memory MPC.

1.2 Our Contributions and a High Level Technical Overview

Our main contribution is a novel graph sparsification technique for correlation clustering that allows us to turn a potentially dense graph into a much sparser representation. Then, we show that running the PIVOT algorithm (or equivalently finding a greedy MIS) on the sparse graph gives a clustering that is almost as good as running PIVOT in the original graph.

The intuition for our sparsification technique is as follows. Consider some node $u$ and a sequential execution of the PIVOT algorithm for $t := n \log n / (\varepsilon \deg(u))$ iterations and suppose that $u$ is not clustered yet in iteration $t$. Then, it is likely that (roughly) a $(1 - \varepsilon)$-fraction of the neighbors of $u$ are already clustered by iteration $t$. Now, we can cluster node $u$ into a singleton cluster and charge the disagreements caused by the remaining edges by spreading a charge of $\varepsilon$ on the edges clustered earlier. Since PIVOT gives a 3-approximation, these singleton clusters turn the approximation ratio into $3 + \varepsilon$. We start the technical part of the paper with the approximation analysis in Section 2, which can be done without going into the details of the specific model of computation.

For a parallel implementation, we show that excluding these nodes can be done a priori. For this, it is convenient to think of the PIVOT algorithm through the greedy MIS on a random permutation. Similarly to the sequential setting, we expect that a node with a rank higher than $n \log n / (\varepsilon \deg(u))$ will have lost most of its neighbors before it is processed by the greedy MIS. We can check whether the rank is “high” immediately after picking the random permutation and we are sure that such a node will never be a node in the greedy MIS. However, such a node might still be part of a pivot cluster. Fortunately, we can check this after finding the greedy MIS, which allows us to effectively exclude the high-rank nodes when computing the greedy MIS.

This exclusion roughly corresponds to sampling a node $u$ with probability $\log n / (\varepsilon \deg(u))$ into the sparsified graph. This sparsified graph is then small enough to be processed locally on a single machine (Lemma 6).

A major technical challenge is to show that our charging argument also works for nodes whose
Algorithm 1 Sequential Truncated-Pivot

Input: Graph $G = (V, E^+)$, each node is active in the beginning. Let $\text{deg}(u) = |N(u)|$ be the initial degree of node $u$.

1: Pick a random permutation $\pi$ over the nodes.
2: for iteration $i = 1, 2, \ldots$ do
   $\triangleright$ Iterate over $\pi$
3: \hspace{1em} Let $\ell := c \cdot \frac{n \log n}{i}$ \hspace{1em} $\triangleright c$ is a well-chosen constant.
4: \hspace{1em} Let $u \in V$ be the $i$th node in $\pi$.
5: \hspace{2em} Each active node $v$ with $\text{deg}(v) \geq \ell$ becomes inactive and creates a singleton cluster.
6: \hspace{2em} If $u$ is active, create a pivot cluster $C$ consisting of $u$ and its active neighbors.
7: \hspace{2em} Each node in $C$ becomes inactive.
8: end for

degree drops due to non-sampled neighbors. We effectively show that only a few nodes can lose many neighbors due to the sampling and hence, the charge can be spread to the nodes with many neighbors clustered by the greedy MIS (Lemma 4). We give the details of the parallel implementation and the structural properties of the sparsiﬁed graph in Section 2.4 and show how to implement the algorithm in various models of modern parallel computation in Section 3.

2 The Truncated-Pivot Algorithm

The goal of this section is to prove the approximation guarantee of the Truncated-Pivot algorithm. For a more comfortable analysis, the algorithm is designed for the sequential setting. Later, in Lemma 5, we give a parallel version of the algorithm that returns the same clustering as obtained from the sequential process.

Theorem 1 (Main Theorem). For any $\varepsilon > 0$, Truncated-Pivot (Algorithm 1) is a $(3 + \varepsilon)$-approximation algorithm to the correlation clustering problem. The approximation guarantee is in expectation.

A Sequential Process. The high-level idea of our algorithm is to compute a greedy MIS with a small twist. Informally, we exclude nodes whose degree is likely to drop signiﬁcantly before they are processed in the greedy MIS algorithm, where the MIS nodes will correspond to the PIVOT nodes, or simply pivots. This will then allow us to effectively ignore a large fraction of the nodes that will never be chosen as pivots. Consider the following (sequential) algorithm and refer to Algorithm 1 for a pseudo-code representation. Initially, each node is considered active. For each node $u$, we store the degree $\text{deg}(u)$ of $u$ in the input graph. We pick a random permutation $\pi$ on the nodes and in each iteration, we pick a node following the permutation. If this node is still active, it is chosen as a pivot and we create a pivot cluster consisting of the pivot node and its active neighbors (Line 6 of Algorithm 1). The clustered nodes then become inactive and will not be chosen as pivots later.

Additionally, in iteration $i$, we check whether each active node $v$ has a degree significantly larger than $(n \log n)/i$. If so, we expect that the previous pivot choices have removed a large fraction of the neighbors of $v$ from the graph. In this case, $v$ becomes a singleton cluster (Line 5 in Algorithm 1) and we charge the remaining edges of $v$ to the edges incident on neighbors that joined some pivot clusters in previous iterations. Notice that the edges of $v$ that got removed before iteration $i$ can be due to a neighbor joining a pivot cluster or due to creating a singleton cluster. As a technical challenge, we must show that most of the neighbors joined pivot clusters.
2.1 Analyzing the Pivot Clusters

As the first step of our approximation analysis, we bound the number of disagreements caused by the pivot nodes and their respective clusters. The analysis is an adaptation of the approach by [2], where we only focus on a subset of the nodes.

2.2 Bad Triangles

Recall the PIVOT algorithm [2] that computes a greedy MIS. Initially, each node is considered active. The PIVOT algorithm picks a random permutation of the nodes and iteratively considers each node in the permutation. For each active node \( u \) (iterating over the permutation), PIVOT forms a cluster with the active neighbors of \( u \). The cluster is then deleted from the graph by marking the nodes in the new cluster inactive. This is repeated until the graph is empty, i.e., all the nodes are clustered. The PIVOT algorithm gives a solution with the expected cost being a 3-approximation of the optimum solution.

The 3-approximation given by the PIVOT algorithm is due to the nature of the mistakes that can be made through the clustering process. Consider \( u, v, w \in V \): if \( e_1 := \{u, v\} \) and \( e_2 := \{v, w\} \) are in \( E^+ \) but \( e_3 := \{w, u\} \in E^- \), then clustering those nodes has to produce at least one mistake. The triplet \( (e_1, e_2, e_3) \) is called a bad triangle. Because a bad triangle induces at least one mistake in any optimum clustering, the number of disjoint bad triangles gives a lower bound on the disagreement produced by an optimum clustering. In the case of the pivot algorithm, since only direct neighbors of a pivot are added to a cluster, then the following mistakes can happen. Either two neighbors are included in the same cluster being dissimilar, which includes a negative edge in the cluster (the pivot was the endpoint of two positive edges in a bad triangle), or the pivot was an endpoint of the negative edge in a bad triangle which implies that only one positive edge of this bad triangle is included in the cluster and the second positive edge is cut. The authors of [2] show that the expected number of mistakes produced by the PIVOT algorithm is the sum of the probability that we make a mistake on every single bad triangle (not necessarily disjoint) in the graph. The 3-approximation is obtained by comparing this expected cost to the cost of a packing LP which is a lower bound on the cost of an optimum clustering. Our analysis for the mistakes caused by the pivot clusters (Lemmas 1 and 2) is almost the same as in the previous work [2]. Our analysis of the singleton clusters requires us to have an explicit handle on the positive disagreements between the pivot clusters and the singleton clusters provided by the analysis of the pivot clusters.

The Cost of Pivot Clusters in Truncated-Pivot. Let us phrase the expected cost of pivot clusters of Truncated-Pivot (Line 6 of Algorithm 1). Recall that a bad triangle refers to a 3-cycle with two positive and one negative edge.

Definition 1. Consider the set of all bad triangles \( T \) and let \( t \in T \) be a bad triangle on nodes \( u, v \) and \( w \). Define \( A_t \) to be the event that, in some iteration, all three nodes are active and one of \( \{u, v, w\} \) is chosen as a pivot (Line 6 in Algorithm 1). Let \( p_t = \Pr[A_t] \).

Definition 2. Let \( C_{TP} \) be the cost, i.e., the disagreements induced by the pivot clusters (Line 6 in Algorithm 1). This includes the negative edges inside the pivot clusters and, if a pivot cluster \( C \) is created in iteration \( i \), the positive edges to nodes that are active in iteration \( i \) and adjacent to \( C \) but not contained in \( C \). The edges that correspond to positive disagreements caused by the pivot clusters are said to be cut by the pivot clusters.

Lemma 1. Let \( T \) be the set of bad triangles in the input graph. Then, \( \mathbb{E}[C_{TP}] \leq \sum_{t \in T} p_t \).
Proof. Consider a bad triangle $t \in T$ and suppose that in some iteration $i$ all nodes in $t$ are active and one of them is chosen as a pivot node (Line 6 in Algorithm 1), i.e. the event $A_t$ happens at iteration $i$. Then, our algorithm creates one disagreement on this triangle on one of its edges $e \in t$. We charge this disagreement on edge $e$.

We observe that each triangle $t$ can be charged at most once: An edge $e \in t$ is charged only if it is not incident on the pivot node and hence, cannot be charged twice in the same iteration. Hence, at most one edge of $t$ can be charged in one iteration. Furthermore, if $e \in t$ gets charged in iteration $i$, its endpoints will not be both active in any later iteration $j > i$. Which implies that $t$ cannot charged again in another iteration.

Also, creating clusters with neighbors can only create disagreements on bad triangles. Since dropping certain nodes of the graph cannot create bad triangles, the number of disagreements created on a subgraph by this process cannot be higher than the number of disagreements created on the whole graph. Therefore, $\mathbb{E}[C^{TP}] \leq \sum_{t \in T} p_t$.

Bounding $OPT$. In order to give an approximation guarantee to the clustered nodes, we first define the following fractional LP. It was argued by [2] that the cost of the optimal solution $LP_{OPT}$ to this LP is a lower bound for the cost $OPT$ of the optimal solution for correlation clustering.

$$\begin{align*}
\min & \sum_{e \in E^- \cup E^+} x_e, \quad \text{s.t. } \sum_{e \in t} x_e \geq 1, \forall t \in T \\
\end{align*}$$

where $T$ is the set of all bad triangles (non-necessarily disjoint) of the graph. The dual of this LP is the following:

$$\begin{align*}
\max & \sum_{t \in T} y_t, \quad \text{s.t. } \sum_{t \in T} y_t \leq 1, \forall e \in E^- \cup E^+.
\end{align*}$$

By weak duality we have, $\sum_{t \in T} y_t \leq LP_{OPT} \leq OPT$ for all dual feasible solutions $\{y_t\}_{t \in T}$. Therefore, in order to get an approximation guarantee, it suffices to compare the cost $C^{TP}$ with a carefully constructed dual feasible solution.

Lemma 2. Let $C^{TP}$ be the number of disagreements incurred by the pivot clusters (Definition 2). We have that $\mathbb{E}[C^{TP}] \leq 3 \cdot OPT$.

Proof. Let $T$ be the set of bad triangles. Recall the event $A_t$ that all nodes in $t \in T$ are active and one of the nodes in $t$ is chosen as a pivot (Line 6 of Algorithm 1) and let $Pr[A_t] = p_t$. Our goal is to use the probabilities $p_t$ to find a feasible packing of the packing LP defined above.

Let $D_e$ be the event that Algorithm 1 creates a disagreement on $e$ and notice that $D_e \land A_t$ denotes the event that the disagreement caused by $A_t$ was charged on $e$. By the definition of $A_t$, this disagreement cannot be due to creating singleton clusters in Line 5 of Algorithm 1. Consider now the event $A_t$ and observe that as we are iterating over a random permutation of the nodes, each node in $t$ has the same probability to be chosen as the pivot. Furthermore, exactly one choice of pivot can cause $D_e$ for each $e \in t$. Hence, we have that $Pr[D_e \mid A_t] = 1/3$ and therefore, $Pr[D_e \land A_t] = Pr[D_e \mid A_t] \cdot Pr[A_t] = p_t/3$.

Consider the assignment $y_t = p_t/3$. We now show that this is a feasible solution for the dual LP in equation 2. This is because for all edges $e \in E^- \cup E^+$ the events $\{D_e \land A_t\}_{t \in T}$ are disjoint from each other, and hence we have $\sum_{t \in T} y_t = \sum_{t \in T} p_t/3 = \sum_{t \in T} Pr[D_e \land A_t] = Pr[\cup_{t \in T} D_e \land A_t] \leq 1$. As this is a feasible packing, we have that $\sum_{t \in T} p_t/3 \leq OPT$. Finally, by Lemma 1

$$\mathbb{E}[C^{TP}] \leq \sum_{t} p_t = 3 \cdot \sum_{t \in T} \frac{p_t}{3} \leq 3 \cdot OPT.$$

\[\]
2.3 Analyzing the Singleton Clusters

The goal of this section is to bound the number of disagreements caused by the singleton clusters created in Line 5 of Algorithm 1. The high-level idea is that we show that for a node \( u \) of degree \( \deg(u) \), either \( u \) is clustered by some pivot node after \( O(n \log n / \deg(u)) \) iterations or most of its edges have been cut by pivot clusters. In the latter case, we relate the cost of the remaining edges of \( u \) to the ones cut by the pivot clusters and show that the remaining edges do not incur a large additional cost. We also need to account for singleton clusters, where most of the edges are incident on other singleton clusters. For this, we will do a counting argument that shows that there cannot be many singleton clusters that have many edges to other singleton clusters.

Charging the Edges Incident on the Singleton Clusters. Now, our goal is to bound the number of edges cut by the singleton clusters created in Line 5 of Algorithm 1. For intuition, consider a node \( u \) and its neighbors with a smaller degree and suppose that \( u \) will not be included in a pivot cluster. Furthermore, suppose that roughly half of its neighbors have a smaller degree. If any smaller degree neighbor \( v \) is chosen according to the random permutation in the first (roughly) \( n \log n / (\varepsilon \cdot \deg(u)) \) iterations, then \( v \) will be chosen as a pivot. We will show that this implies (w.h.p.) that almost all, roughly a \((1 - \varepsilon)\)-fraction of, the smaller degree neighbors either join a pivot cluster or at least one of them will be chosen as a pivot which would include \( u \) in a pivot cluster (Lemma 3). Once we have this, we can spread the disagreements on the remaining \( \varepsilon \)-fraction of the edges to smaller degree nodes to the edges cut by pivot clusters. As a technical challenge, we also need to account for nodes who have a small number of smaller degree neighbors to begin with. We use a counting argument (Lemma 4) to show that a large fraction of nodes must have many neighbors in pivot clusters which allows us to also spread the cost of the nodes with few smaller degree neighbors.

Consider a node \( u \) and let \( N_i(u) \) be the set of nodes in iteration \( i \) such that for each \( v \in N_i(u) \), we have that \( \deg(v) \leq \deg(u) \) and \( v \) is not in a pivot cluster. Let \( \deg_i(u) = |N_i(u)| \).

Lemma 3. For each active node \( u \), in the beginning of iteration \( i = \frac{\varepsilon}{\varepsilon} \cdot \frac{n \log n}{\deg(u)} \), we have that \( \deg_i(u) \leq \varepsilon \cdot \deg(u) \) with high probability.

Proof. Consider a node \( v \in N_i(u) \). Notice that since \( v \in N_i(u) \), it must be the case that \( \deg(v) \leq \deg(u) \) and hence, if \( v \) is considered in the random permutation before iteration \( i \), node \( v \) is picked as a pivot.

Suppose that throughout iterations \( k = 1, \ldots, i - 1 \), we have that \( \deg_k(u) \geq c \cdot n \log n / i = d \). Then, in each iteration, the probability of choosing one of the \( N_k(u) \) nodes as a pivot node (Line 6 in Algorithm 1) is at least \( d/n \). The probability that none of these nodes is chosen as a pivot through iterations \( 1, \ldots, i - 1 \) is at most

\[
\left( 1 - \frac{d}{n} \right)^{i-1} \leq \left( 1 - \frac{d}{n} \right)^{\frac{i}{2}} \leq e^{-\frac{d}{2n} i} = e^{-\frac{c \cdot n \log n}{2n} i} \leq n^{-c/5},
\]

where \( c/5 \) accounts for the base change of the logarithm. The lemma follows by union bounding over all nodes. \( \square \)

Good Edges and Counting. Consider a positive edge incident on a singleton cluster that contains node \( u \). Suppose that the singleton cluster was created in iteration \( i \). We define an edge \( e = \{u, v\} \) to be good if the other endpoint, node \( v \), was included in a pivot cluster (Line 6 of Algorithm 1) in some iteration \( j < i \). Otherwise, edge \( e \) is in bad. Intuitively, an edge is good
since we can charge this edge to the set $C_{TP}$ which we know how to bound through Lemma 2. Furthermore, if we can show that most edges incident on singleton clusters are good, we can bound the cost of the bad edges.

**Lemma 4.** Let $E^{\text{sin}}$ be the set of positive edges incident on the singleton clusters (Line 5 of Algorithm 1). Then, at most $2\varepsilon \cdot |E^{\text{sin}}|$ are bad with high probability.

**Proof.** For the analysis, let us consider the following orientation on the bad edges. Consider an iteration $i$, where a node $u$ is put into a singleton cluster by Line 5 of Algorithm 1. Notice that this implies that $i \geq \frac{\varepsilon \cdot n \log n}{\deg(u)}$. Then, we orient each unoriented edge from $u$ to $v$ for each neighbor $v$ such that $\deg(v) \leq \deg(u)$ and $v$ is not in a pivot cluster. Recall that $\deg(u)$ is the degree of $u$ in the input graph and $\deg_i(u) = |N_i(u)|$ where $N_i(u)$ is the set of neighbors of $u$ such that for $v \in N_i(u)$, $\deg(v) \leq \deg(u)$ and $v$ is not in a pivot cluster. By Lemma 3, with high probability, we have that $\deg_i(u) \leq \varepsilon \cdot \deg(u)$. Hence, with high probability, the out-degree of each singleton node $u$ is at most $\varepsilon \cdot \deg(u)$.

Denote the out-degree of a node $u$ by $\deg_{\text{out}}(u)$ and $V^{\text{sin}}$ the set of nodes in singleton clusters. Suppose that this high probability event holds for all nodes $u \in V^{\text{sin}}$. By summing over the oriented edges and the handshake lemma, we have that

$$\sum_{u \in V^{\text{sin}}} \deg_{\text{out}}(u) \leq \sum_{u \in V^{\text{sin}}} \varepsilon \cdot \deg(u) \leq 2\varepsilon \cdot |E^{\text{sin}}| .$$

**Theorem 1** (Main Theorem). For any $\varepsilon > 0$, Truncated-Pivot (Algorithm 1) is a $(3 + \varepsilon)$-approximation algorithm to the correlation clustering problem. The approximation guarantee is in expectation.

**Proof.** Recall the following definitions.

- We denote the cost of the pivot clusters by $C_{TP}$ (see Definition 2). This cost also covers the cost of the positive edges between pivot clusters and singleton clusters that were cut by the pivot clusters. These edges are called good and the set of those edges is denoted by $E^{\text{good}}$.
- Bad edges are the positive edges incident on singleton clusters that were not cut by the pivot cluster. Either they are between singletons or the singleton was created before the pivot cluster. Denote those edges by $E^{\text{bad}}$.
- $E^{\text{sin}} = E^{\text{good}} \cup E^{\text{bad}}$

We can split the cost of Algorithm 1 into two parts. By Lemma 2, we have that $\mathbb{E}[C_{TP}] \leq 3 \cdot \text{OPT}$. Then, by Lemma 4, we have that, $|E^{\text{bad}}| \leq 2\varepsilon \cdot |E^{\text{sin}}| \leq \frac{2\varepsilon}{1 - 2\varepsilon} \cdot |E^{\text{good}}| \leq 4\varepsilon \cdot C_{TP}$. We note that the bound on the bad edges is only with high probability. However, in the worst case, the cost of any clustering is at most $n^2$. Therefore, if we assume that the optimum cost is at least one, the cost of the low-probability event becomes negligible.\footnote{The only case where the optimum does not do any mistakes is when the input graph is a set of cliques. This case can be handled separately.}

By combining the above observations, we have that the expected cost of Algorithm 1 is at most

$$\mathbb{E} \left[ C_{TP} + |E^{\text{bad}}| \right] \leq \mathbb{E} \left[ C_{TP} + 4\varepsilon \cdot C_{TP} \right] \leq (1 + 4\varepsilon) \cdot \mathbb{E} \left[ C_{TP} \right] \leq (3 + 12\varepsilon) \cdot \text{OPT} .$$

We can substitute $\varepsilon' := 12\varepsilon$, (where $\varepsilon$ can be arbitrarily small) to obtain a $(3 + \varepsilon')$-approximation and obtain Theorem 1.\qed
Algorithm 2 Parallel Truncated-Pivot

**Input:** Graph $G = (V, E)$, each node $v \in V$ knows its degree $\deg(v)$ in $G$.

1: Fix a random permutation $\pi$ over the nodes.
2: Initially all nodes are unclustered and interesting.
3: A node $v$ marks itself uninteresting if $\pi_v \geq \tau_v$, where $\tau_v = \frac{c}{\epsilon} \cdot n \log n / \deg(v)$
4: Let $G_{store}$ be the graph induced by the interesting nodes.
5: Let $I$ be the output of running greedy MIS on $G_{store}$ with ordering $\pi$.
6: Nodes in $I$ become cluster centers (pivots).
7: Each node in $v \in V \setminus I$ joins the cluster of the smallest rank pivot neighbor $u$, if $\pi_u < \tau_v$.
8: Each unclustered node forms a singleton cluster.

2.4 Parallel Implementation

In this section, we give a parallel version, Parallel Truncated-Pivot, of Truncated-Pivot that is a better fit for parallel implementations. The main difference is that in the beginning of the execution, Line 3 of Algorithm 2, we declare a node $v$ interesting, if its rank is at most $(c/\epsilon) \cdot (n \log n / \deg(v))$. This corresponds to the case where $v$ is excluded and put into a singleton cluster in Line 5 in Algorithm 1. First, we show that this algorithm returns the same clustering as the sequential one and then, in Section 3, we show how a greedy MIS can be computed efficiently in the sparse graph induced by the interesting nodes.

**Lemma 5.** Fix a (random) permutation $\pi$ over the nodes of $G = (V, E)$. Running the sequential Truncated-Pivot (Algorithm 1) with $\pi$ outputs the same clustering as the parallel Truncated-Pivot (Algorithm 2) with $\pi$.

**Proof.** Our goal is to show that both algorithms output the same clustering. First, we show that in both cases the singleton clusters are the same. Then, we show that in both cases the greedy MIS runs on the same subgraph, hence outputting the same pivot clusters.

Consider a node $u$ that is active in the beginning of iteration $i$ ($i \leq \pi_u$), and becomes a singleton cluster due to Line 5 of Algorithm 1. By definition, $i$ is the smallest integer such that $\deg(u) \geq \frac{c}{\epsilon} \cdot \frac{n \log n}{\pi_u}$ and therefore, $i = \lceil \tau_u \rceil$. Since $i \leq \pi_u$, we have $\deg(u) \geq \frac{c}{\epsilon} \cdot \frac{n \log n}{\pi_u}$, which corresponds to being uninteresting in Algorithm 2. Since $u$ is in a singleton cluster after Algorithm 1, it did not join any pivot cluster, implying that no neighbor of $u$ was picked as a pivot before $u$ became a singleton cluster (i.e. $\forall v \in N(u), \pi_v > i$ or $v$ was clustered before iteration $\pi_v$). Hence, no neighbor $v$ of $u$ s.t. $\pi_v < \tau_u$ becomes a pivot. Since $\pi_v$ is an integer, this is equivalent to saying no neighbor $v$ of $u$ s.t. $\pi_v < \tau_u$ becomes a pivot, so by Line 7 of Algorithm 2, $u$ creates a singleton cluster in Algorithm 2 as well.

Now consider a node $u$ that creates a singleton cluster in Algorithm 2. Node $u$ must have been labeled uninteresting (implying $\pi_u \geq \tau_u$), and $u$ can neither be a pivot nor have a neighboring pivot $v$ satisfying $\pi_v < \tau_u$. By definition of $\tau_u$, iteration $\lceil \tau_u \rceil$ is the smallest iteration such that $\deg(u) \geq \frac{c}{\epsilon} \cdot \frac{n \log n}{\lceil \tau_u \rceil}$. This implies that $u$ must be active at the beginning of iteration $\lceil \tau_u \rceil$ in Algorithm 1, and forms a singleton cluster in that iteration.

Since the nodes forming singleton clusters in both algorithms are the same, the subgraph induced by nodes not forming singleton clusters $G[V \setminus V^{\text{sin}}]$ is the same in both cases. Both algorithms find a greedy MIS on $G[V \setminus V^{\text{sin}}]$, which implies that the pivot nodes will be the same in both cases. Finally, we observe that in both algorithms, a non-pivot node $u$ joins the cluster of the first neighbor $v$ s.t. $\pi_v < \tau_u$. Hence, the pivot clusters are the same for both Algorithm 1 and Algorithm 2.
Corollary 1. The Parallel Truncated-Pivot algorithm (Algorithm 2) is a $(3 + \varepsilon)$-approximation algorithm to the Correlation Clustering problem. The approximation guarantee is in expectation. By running $O(\log n)$ independent instances of the algorithm, the approximation guarantee also holds with high probability.

Proof. The corollary follows from Theorem 1 and Lemma 5. The high probability statement follows by Markov’s inequality and standard probability boosting arguments.

2.5 Structural Properties of the Sparsified Graph

Now, we show structural properties of the sparsified graph $G_{\text{store}}$ that will be crucial for the implementations in the models of processing massive data. First, we show that the number of edges in $G_{\text{store}}$ is small and then, we show how to reduce the maximum degree of $G_{\text{store}}$ to logarithmic without loss in the approximation. We note that the bound on the maximum degree would suffice for the implementations, but we believe that Lemma 6 is simpler and can be of independent interest.

Lemma 6. The number of edges in $G_{\text{store}} = (V, E)$ is $O\left(\frac{n \log n}{\varepsilon}\right)$.

Proof. Our goal is to show that each node $v$ is interesting with probability roughly $\log n / \deg(v)$. Then, we bound the number of edges as a sum over the degrees of interesting nodes.

For each node $v$, let $X_v$ be an independent random variable taking values from the real interval $[0, 1]$. The ranking obtained from these values is equivalent (with probability 1) to a random permutation over the nodes. Let $Y_{2r}$ be the number of nodes such that $X_v \leq \frac{2r}{n}$, for some (integer) $r \in [0, n/2]$. Then, $\mathbb{E}[Y_{2r}] = 2r$. By a standard Chernoff bound, we get that $Y_{2r} \leq r$ with low probability when $r \geq c \log n$ for some sufficiently large constant $c$. Let $R_v = \#\{u \in V \text{ s.t. } X_u < X_v\}$ be the rank of $v$. Because $Y_{2r} \leq r \Leftrightarrow \#\{u \in V \text{ s.t. } X_u \leq \frac{2r}{n}\} \leq r$ with low probability, we have that $X_v > \frac{2r}{n} \Rightarrow R_v > r$ with high probability. Hence, $\Pr(R_v \leq r) \leq \Pr(X_v \leq \frac{2r}{n}) = \frac{2r}{n}$ with high probability.

Now, recall that a node is interesting if $\pi_v \leq (c/\varepsilon) \cdot (n \log n) / \deg(v)$ for the random permutation $\pi$ of Algorithm 2. Let us consider $r = \frac{\varepsilon}{c} \cdot \frac{n \log n}{\deg(v)}$. Then,

$$\Pr(v \text{ is interesting}) = \Pr(R_v < r) \leq \frac{2r}{n} = \frac{2c}{\varepsilon} \cdot \frac{\log n}{\deg(v)}.$$  

Hence, we get that the expected number of edges in $G_{\text{store}}$ is at most

$$\sum_{v \in V} \Pr(v \text{ is interesting}) \cdot \deg(v) \leq n \cdot \frac{2c}{\varepsilon} \cdot \frac{\log n}{\deg(v)} \cdot \deg(v) = O\left(\frac{n \log n}{\varepsilon}\right) \text{ whp},$$

since $r = \frac{c}{\varepsilon} \cdot \frac{n \log n}{\deg(v)} \geq c \cdot \log n$.

Notice that $X_v$ is independent of $X_u$, for all $u \neq v$. We can then apply a Chernoff bound to get that the number of edges in $G_{\text{store}}$ is $O(n \log n)/\varepsilon$ with high probability. \qed
3 Implementation in Massive Graph Processing Models

In this section, we show how to implement our algorithm, Parallel Truncated-Pivot, in several models of parallel computation. While the base of the algorithm is the same for each model, the model choice slightly influences the way we construct the graph \( G_{\text{store}} \) where the greedy MIS algorithm is executed. In each case of the implementations, we only need to bound the runtime and memory usage of the algorithm in the corresponding model. Since we execute the Parallel Truncated-Pivot algorithm, the approximation guarantee follows by Corollary 1.

3.1 Linear-space MPC

In this section, we consider the Massively Parallel Computation (MPC) model introduced by [16]. In the MPC model we have \( M \) machines with \( S \) words of memory each, where each word corresponds to \( O(\log n) \) bits. Notice that an identifier of an edge or a node requires one word to store. The machines communicate in an all-to-all fashion. The input graph is divided among the machines and for simplicity and without loss of generality, we assume that the edges of each node are placed on the same machine. In the linear-space MPC model, we set \( S = \tilde{\Theta}(n) \), where the \( \tilde{\Theta}(n) \) hides terms polylogarithmic in \( n \). Furthermore, the total space is defined as \( M \cdot S \) and in our case, we have \( M \cdot S = \tilde{\Theta}(m) \). Notice that \( M \cdot S = \Omega(m) \), for the number of edges \( m \) in the input graph, simply to store the input.

**Implementation** By the definition of Parallel Truncated-Pivot, we create the graph \( G_{\text{store}} \) and we need to find a greedy MIS on that graph. Nodes can decide whether they are interesting, i.e., belong to \( G_{\text{store}} \) once the permutation is fixed in the beginning of Parallel Truncated-Pivot. Hence, we simply need to show that \( G_{\text{store}} \) fits onto a single machine and we can find the greedy MIS locally on a single machine. By Lemma 6, we have that the number of edges in \( G_{\text{store}} \) is \( O(n \log n)/\varepsilon \). Since the local memory \( S = \tilde{\Theta}(n) \), we can indeed send the whole graph to a single machine in \( O(1) \) rounds and use this machine to find a greedy MIS. We obtain Theorem 2.

**Theorem 2.** There is an MPC algorithm with \( \tilde{O}(n)/\varepsilon \) local and \( \tilde{O}(m)/\varepsilon \) total memory to find a \((3 + \varepsilon)\)-approximation for correlation clustering. The algorithm runs in \( O(1) \) rounds.

3.2 Semi-streaming

Typically in the distributed and parallel settings, the input graph is too large to fit a single computer. Hence, it is divided among several computers (in one way or another) and the computers need to communicate with each other to solve a problem. Another angle at tackling large datasets and graphs is through the graph streaming models [13, 14, 18]. In these models, the graph is not stored centrally but an algorithm has access to the edges one by one in an input stream, chosen randomly or by an adversary (the choice sometimes makes a difference). We assume that each edge is processed before the next pass starts. In the semi-streaming setting, the algorithm has \( \tilde{O}(n) \) working space, that it can use to store its state. The goal is to make as few passes over the edge-stream as possible, ideally just a small constant amount. Notice that in the case of many problems, such as matching approximation or correlation clustering, simply storing the output might demand \( \Omega(n) \) words.

**Implementation.** The implementation of Truncated-Pivot is straightforward in the semi-streaming model and it follows along the lines of the implementation in linear-space MPC. In \( O(1) \) passes, compute the degree of each node. Storing the degrees takes at \( O(n) \) words of \( O(\log n) \) bits. With
this information, we can compute $G_{\text{store}}$ and, by Lemma 6, $G_{\text{store}}$ requires $O(n \log n)$ words to store. Hence, using $O(n \log n)$ words of space, we can gather all the information required to simulate Truncated-Pivot and we obtain an $O(1)$-pass semi-streaming algorithm as posed in Theorem 3.

**Theorem 3.** There is an $O(1)$-pass semi-streaming algorithm with $O(n \log n) / \varepsilon$ words of memory to find a $(3 + \varepsilon)$-approximation for correlation clustering.

### 4 Open Questions

While our work improves on the current approximation ratios of the most efficient algorithms, we still do not reach the 3-approximation guaranteed by the classic PIVOT algorithm [2]. As intriguing open questions, we ask:

- Can we find a (random) greedy MIS in $O(1)$ time linear memory models such as linear-space MPC, Congested Clique, or semi-streaming?
- Can we find a 3-approximation to the correlation clustering problem in $O(\log \log n)$ rounds in the low-space MPC model?

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