Dirac Monopoles in Rogue Waves

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We for the first time demonstrate that the widely existed nonlinear waves such as rogue waves, contain Dirac monopoles. We find that the density zeros of these nonlinear waves on an extended complex plane can constitute the Dirac virtual magnetic monopole fields with a quantized flux of elementary π. We then can explain the exotic property of “appearing from nowhere and disappearing without a trace” of rogue waves by means of a Dirac monopole collision mechanism. The maximum amplification ratio and multiple phase steps of high-order rogue waves are found to be closely related to the number of their contained monopoles. Important implications of the intrinsic virtual magnetic monopole fields are discussed.

Introduction—Magnetic monopole is a hypothetical elementary particle representing an isolated source of magnetic field with only one magnetic pole (N without S, or vice versa) and a quantized magnetic charge [1, 2]. Dirac first demonstrated that a virtual monopole is associated with nodal singularity of wave function and can be described well by topological vector potential with a line singularity [3]. The concept of Dirac monopoles provide a fertile playground for theoretical ideas [4] and are also relevant to physical considerations as they are necessarily present in all grand unified models [5, 6]. Although experimental searches for magnetic monopole particles have so far been unsuccessful, virtual magnetic monopoles are found to emerge in momentum space of topological materials [7–10], parameter space of Berry phase theory [11], and in the synthetic magnetic field produced by a spinor Bose-Einstein condensate [12, 13]. Recently, a classical analog to these elusive particles has emerged as topological excitations within pyrochlore spin ice systems [14, 15]. The investigation of these virtual magnetic monopoles not only provides conclusive evidence of the existence of Dirac magnetic monopoles but also is very crucial to understand the topological excitation properties in various physical systems.

In this Letter, we for the first time report that the rogue waves (RWs) contain Dirac monopoles. RWs are a kind of usual nonlinear excitations that have been observed experimentally in optical fibers [16], water wave tanks [17, 18], and plasma systems [19]. Because of its exotic characteristic of “appearing from nowhere and disappearing without a trace” [20, 21], RW is believed to be the cause of many ocean disasters and therefore has attracted much attention [22, 23]. By investigating density zeros for RWs in an extended complex coordinate plane, we find that an nth-order RW contains n(n + 1) pairs of monopoles with opposite charges, and that the collision of these monopoles and the reconnection of the corresponding vector field will lead to energy conversion from interaction energy to kinetic energy and are responsible for the exotic property of RWs. Important implications of our findings on other nonlinear waves such as breathers or even some non-integrable excitations are also discussed.

Dirac virtual magnetic monopole and topological vector potential—We choose one of the simplest models, i.e., the scalar nonlinear Schrödinger equation (NLSE)

\[ \frac{i}{2} \frac{\partial \psi}{\partial t} + \frac{1}{2} \nabla^2 \psi + g|\psi|^2 \psi, \]

to demonstrate our theory. NLSE can be used to describe dynamics of nonlinear waves in atomic gases [24], plasma [25], water waves [17], and ferromagnetic materials [26] and plays important roles in both integrability theory [27] and optical communications [28]. It has a nonlinear wave solution ψ(x, t), which can be a dark soliton (for g > 0), a bright soliton, a RW, a breather (for g < 0), or some superposition thereof. Nevertheless, the topological properties associated with these one-dimensional (1D) nonlinear waves are seldom discussed [29].

In fact, almost a century ago, P. A. M. Dirac has discussed the phase change and its underlying topological property for a general wave function ψ(r, t) = |ψ|eig [3]. He introduced a vector potential field by the phase gradient, i.e., \( \mathbf{A} = (A_x, A_y, A_z) = \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z \), and found that the noncommutative property of \( \frac{\partial A_{\mu}}{\partial r^\nu} \neq \frac{\partial A_{\nu}}{\partial r^\mu} \) (ν, μ = x, y, z and ν ≠ μ) would give rise to a non-integrable phase factor. This striking property is caused by the emergence of the nodal line, where the wave function vanishes and its phase does not have a meaning. More importantly, Dirac claimed that for calculating the change in phase round a closed curve, one has to take into account the influence of the end point of the nodal line, which acts as a virtual magnetic monopole of charge ±1/2 with a vector potential taking the form of

\[ \mathbf{A}_{3D} = \frac{\mathbf{r} \times \mathbf{e}_3}{2r^2} \] with \( r = \sqrt{x^2 + y^2 + z^2} \) [3].

However, when directly applying the above Dirac’s theory to the nonlinear waves such as RWs, we find that the non-integrable phase factor cannot appear because the spatial coordinate is 1D so that the noncommutative property has no meaning. It is noted that some new phe-

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nomena would remain hidden if one were to restrict one’s attention to real physical parameters [30]. Two famous examples are the Lee-Yang zeros [31] and Fisher zeros [32] reported for imaginary magnetic fields and imaginary temperatures, respectively. Very recently, the complex extension is also found to be helpful to study the fluid of Korteweg-de Vries equation [33]. Here, we attempt to extend the real coordinate variable $x$ to a complex variable $Z = x + iy$, and introduce a complex vector potential also by the phase gradient, i.e.,

$$
\alpha_c = \frac{\partial \phi(Z)}{\partial x} e_x + \frac{\partial \phi(Z)}{\partial y} e_y = \frac{\partial \phi(Z)}{\partial Z} (e_x + ie_y).
$$

(1)

The vector potential is now defined on 2D plane of the variables of $x$ and $y$, and we focus on the real part of the above vector potential, i.e., $A = \text{Re} [\alpha_c]$. Following Dirac’s spirit, we then pay attention to the singularity property of the vector potential field of $A = (A_x, A_y)$ and its noncommutative property of $\frac{\partial A_x}{\partial y} \neq \frac{\partial A_y}{\partial x}$. In the 2D situation, the nodal line will reduce to some scattered points corresponding to density zeros where the wave function vanishes and its phase does not have a meaning. More interestingly, we claim that for calculating the change in phase along $x$ axis, the density zeros (i.e., denoted by $r_N = x_N e_x + y_N e_y$) of these nonlinear waves on an extended complex plane can constitute virtual magnetic monopole fields with a quantized flux of elementary $\Omega = \pi$, i.e., the charge of $\mu = 1/2$. The corresponding vector potential takes following explicit form:

$$
A = \text{Re} [\alpha_c] = \sum_N \pm \mu \left[\frac{(x-x_N)e_x - (y-y_N)e_y}{(x-x_N)^2 + (y-y_N)^2}\right],
$$

(2)

where $\pm$ is the sign of monopole. Near each singular point, the above 2D expression (2) of vector potential field is exactly the Dirac’s monopole field of $A_{3D}$ with setting $z = 0$ therein. The corresponding magnetic field will be zero everywhere except at those singular points, that is, $B = \nabla \times A = e_z \sum_N \pm \Omega \delta (x-x_N, y-y_N)$.

The phase variations of such a nonlinear wave between $x_1$ and $x_2$ can then be expressed by the line integral of the vector potential, namely,

$$
\Delta \phi = \phi(x_2) - \phi(x_1) = \int_{x_1}^{x_2} A(x, y = 0) \cdot e_x dx
= \int_{(x_2, 0)}^{(x_1, 0)} A(x, y) \cdot d\vec{l} - \sum n \pi,
$$

(3)

where $\vec{l}$ is an arbitrary curve that connects $(x_1, 0)$ and $(x_2, 0)$, and $\sum n \pi$ denotes the total magnetic flux enclosed by the curve $\vec{l}$ and the line connecting $x_1$ and $x_2$ in $(x, y)$ plane.

It is interesting to compare the above results with the Aharonov-Bohm effect [34], which predicts a topological phase when an electron moves on a close path around a solenoid. A 1D nonlinear wave moving on the real axis cannot see the magnetic fields scattered on the complex plane: however, it will acquire a phase due to the presence of the vector potential. The evolution of such a nonlinear wave can be understood from the transformed equation based on the topological vector potential, $i \partial_t \psi_1 = \left(\frac{\partial_x + i A(x, y = 0) e_y}{2}\right)^2 + (g|\psi_1|^2 + \frac{\partial}{\partial t} A(x, y = 0) e_y dx) \psi_1$ with a transformation $\psi_1 = \psi \exp \left[-i \int A(x, y = 0) \cdot e_x dx\right]$. The effective magnetic field and electric field can be derived as $B = \nabla \times A$ and $E = \frac{\partial A}{\partial t} - \nabla \frac{\partial}{\partial t} A(x, y = 0) e_y dx$, respectively [35]. In this sense, the phase variations of these nonlinear waves can be viewed as a 1D counterpart to the Aharonov-Bohm phase. Usually, the topological vector potential exhibits time dependence, which provides an alternative way to understand the dynamics of such nonlinear waves.

**Topological vector potentials for rogue waves**—The above NLSE with $g = -1$ has a Peregrine breather which is also called as the first-order RW (FRW) solution on a uniform background [20, 36, 37], $\psi = \left[1 - \frac{4(1+2it)}{4t^2 + 4t^2 + 1}\right] e^{it}$. 

![FIG: (a) Amplitude distribution, (b) phase distribution, and (c-f) evolution of the topological vector potential for a Peregrine breather (the first-order RW). The Dirac monopoles with positive and negative charges (i.e., $\pm \frac{1}{2}$) are indicated by $\bigcirc$ and $\bigotimes$, respectively.](image319x425 to 560x740)
The temporal evolution of the RW amplitude depicted in Fig. 1 (a) shows that the wave density remains almost constant until \( t = -5 \), after which sudden growth occurs. At \( t = 0 \), the amplitude amplification ratio (defined as the peak amplitude divided by the background amplitude) reaches its maximum value of 3. Subsequently, the RW quickly decays, and the wave density recovers to be nearly constant. The amplitude peak is located at \( x = 0 \), and on either side of the peak, there are two valleys at \( x = \pm a_0 (a_0 = \frac{\sqrt{2}}{\pi}) \). Interestingly, phase jumps accompany the rise in amplitude. In Fig. 1 (b), we see that there is a \( \pi \) phase jump corresponding to each amplitude valley [38, 39], and the direction of the phase jump at \( x = -a_0 \) suddenly inverts to \(-\pi\) slightly after the moment when the maximum amplitude peak emerges.

Linear modulational instability (MI) analysis describes well the growth of perturbation [21–23, 40, 41]. Nonlinear MI theory was further developed to address nonlinear behavior of MI beyond its growth stage [42, 43]. A truncated three-wave model on frequency was suggested to describe well MI dynamics [44–46]. Those could be used to understand the RW dynamics qualitatively. Especially, analytical studies have indicated that the amplitude amplification ratios for RWs of different orders are subject to certain limits [20, 47, 48], but no physical mechanism for these ceiling values has been discovered. The above \( \pi \) phase jumps and abrupt inversion are also not fully understood [38, 39]. Here, we attempt to elucidate these issues with the help of our developed topological vector potential theory.

It has four density zeros, i.e., \( Z_{1,3} = \pm (a + ib) \) and \( Z_{2,4} = \pm (-a + ib) \), where \( a = \frac{\sqrt{4t^2 + \sqrt{16t^4 + 40t^2 + 9} - 3}}{2\sqrt{3}} \) (the absolute operation in \( |t| \) is introduced for marking monopole conveniently) and \( b = \frac{\sqrt{4t^2 + \sqrt{16t^4 + 40t^2 + 9} - 3}}{2\sqrt{3}} \).

According to Eq. (2), the vector potential \( \mathbf{A} \) underlying the FRW takes an explicit form of \( \mathbf{A}(x, y) = \sum_{N=1, \ldots, 4} \frac{(-1)^N \mu(x-x_N)e(-y-y_N)\delta}{(x-x_N)^2+(y-y_N)^2} \). We can see that the vector potential \( \mathbf{A} \) is composed of two pairs of Dirac monopoles, and in each pair, the two monopoles have charges of 1/2 with opposite signs. The temporal evolution of the potential is shown in Fig. 1 (c-f). The monopole with opposite charges in each pair approach each other (see Fig. 1 (c-d)), collide elastically in the vertical direction at time \( t = 0 \) with speeds of \( db/dt = \frac{\sqrt{3}}{\sqrt{2}} \) and \( da/dt = 0 \), and then bounce back after exchanging their charges (see Fig. 1 (e-f) and a video in [49]). As \( t \to \pm 0 \), \( a \to a_0 \) and \( b \to 0 \). The vector potential on the real axis then takes the following form:

\[
\lim_{t \to \pm 0} \mathbf{A} = \lim_{t \to 0} \left\{ \frac{-\mu[(x \mp a_0)e_y + b_1e_x]}{(x \pm a_0)^2 + b^2} + \frac{\mu[(x \pm a_0)e_y - b_1e_x]}{(x \pm a_0)^2 + b^2} \right\} \\
+ \frac{\mu[(x \mp a_0)e_y + b_2e_x]}{(x \pm a_0)^2 + b^2} + \frac{-\mu[(x \mp a_0)e_y - b_2e_x]}{(x \pm a_0)^2 + b^2} \\
= [\mp \pi \delta(x \mp a_0) + \pi \delta(x \mp a_0)] \mathbf{e}_x. \tag{4}
\]

The line integral of the above vector potential can explain the \( \pi \) phase jumps shown in Fig. 1 (b). The phase gradient determines the density flow, and the change in the phase distribution can provide an understanding of the growth and decay of RWs [38]. The collision of the monopoles leads to a sudden rise in the wave amplitude. The exchange of the monopole charges after collision can well explain the striking phase reversal that induces the RW’s rapid decay. This provides another possible way to understand RW’s dynamical properties, as a good supplement for the previously known mechanisms [21, 42, 44].

Higher-order RWs admit higher amplitude peaks, more density valleys (or humps) and multiple phase steps, as shown in Fig. 2 [20, 37, 38, 47]. For a second-order RW, the maximum amplitude amplification ratio is 3. There are five phase steps distributed symmetrically with respect to the \( x = 0 \) axis, each of which is associated with a phase jump of \( \pm \pi \) (see Fig. 2 (a)). The vector field is composed of six pairs of Dirac monopoles, as shown in Fig. 2 (b). They can be divided into two classes: the four pairs that are closer to the \( x \)-axis collide with each other on the \( x \)-axis, leading to four \( \pi \) phase jumps, whereas the other two pairs (upper and lower pairs) collide on the imaginary axis, mainly contributing to a sudden rise in the wave amplitude. For a third-order RW, the maximum amplitude amplification ratio is 5. There are seven phase steps distributed symmetrically with respect to the \( x = 0 \) axis (see Fig. 2 (c)). The vector field is composed of twelve pairs of monopoles, as shown in Fig. 2 (d). The paired monopoles collide and merge when the RW reaches its highest peak. Among them, six pairs collide...
and found that they are closely related. According to our peak amplitude of a RW and the number of monopoles [50].

As the time integral of the kinetic energy (\(\int_{-\infty}^{+\infty}E_kdt\)) where the first term is the kinetic energy \(E_k\) and the second term is the interaction energy \(E_{int}\). For an FRW, the time-dependent kinetic energy is \(E_k = \frac{\pi^2}{(4n+1)^2}\). Because the total energy is conserved, the amplitude amplification of the RW corresponds to the energy transfer process from interaction energy to kinetic energy, as indicated in Fig. 3 (a).

Quantitatively, the total energy transfer can be evaluated as the time integral of the kinetic energy (\(\int_{-\infty}^{+\infty}E_kdt\)). We have numerically calculated the integrals for RWs whose orders are up to 10 and have found that they are equal to the sum of the absolute magnetic flux of the monopoles (see Fig. 3 (b)). From the vector field perspective, we know that monopole collisions induce the conversion of interaction energy into kinetic energy. This process is analogous to the magnetic field reconnection process identified in astrophysics, in which magnetic field energy is transformed into the kinetic energy of a plasma [50].

We have also investigated the relation between the peak amplitude of a RW and the number of monopoles and found that they are closely related. According to our discussion above, a FRW admits 4 Dirac monopoles. Because the \(n\)th-order RW solution is a nonlinear superposition of \(\frac{n(2n+1)}{2}\) FRWs [47], it will contain \(N = 2n(n+1)\) monopoles on the complex plane. Among them, there are \(4n\) monopoles colliding on the real axis that are responsible for the multiple phase steps (\(n\)th-order RW with the highest peak), whereas the other \(2n(n-1)\) monopoles will collide in other locations on the complex plane. On the other hand, the square of the maximum amplitude amplification ratio \(P\) for a \(n\)th-order RW can be calculated to be \((2n + 1)^2\) [20, 48]. We thus obtain the following explicit relation: \(P = 2N + 1\). Based on this observation, we can predict that the amplitude amplification ratio of a high-order RW will be subject to a certain limit because a RW contains only a limited number of Dirac monopoles due to the finite number of valleys in its geometric configuration [37, 47].

**Conclusion and discussion**—With extending to complex coordinate plane and following Dirac’s theoretical formulation of virtual magnetic monopole, we reveal a monopole topological potential hidden in the phase variations of 1D nonlinear waves. The monopole collision process can be used to understand the striking phase jump of RWs. Moreover, we find that the ceiling values of the amplitude amplification ratios for RWs of different orders are determined by the total magnetic flux. The unintelligible \(\pi\) phase jump is found to be the manifestation of the quantized magnetic flux of a magnetic monopole. Our finding provides a distinct explanation for the exotic property of “appearing from nowhere and disappearing without a trace” of RWs. This could be a good supplement for the previously known mechanisms [22, 23].

Our theory is also applicable to other nonlinear waves [51–54] (see details in [49]) and nonlinear models [55–62]. It can be further extended to study the semiclassical dynamics of BdG excitations in Bose-Einstein condensation under the action of intrinsic topological potentials [63], and the interaction between nonlinear waves even in non-integrable systems [64]. In the latter case, the density zeros in the complex plane that constitute the magnetic monopole fields could be traced with the help of numerical algorithm.

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