Supersymmetric Warped Conformal Field Theory

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Abstract

In this work, we study the supersymmetric warped conformal field theory in two dimensions. We show that the Hofman-Strominger theorem on symmetry enhancement could be generalized to the supersymmetric case. More precisely, we find that within a chiral superspace \((x^+, \theta)\), a two-dimensional field theory with two translational invariance and a chiral scaling symmetry can have enhanced local symmetry, under the assumption that the dilatation spectrum is discrete and non-negative. Similar to the pure bosonic case, there are two kinds of minimal models, one being \(N = (1,0)\) supersymmetric conformal field theories, while the other being \(N = 1\) supersymmetric warped conformal field theories (SWCFT). We study the properties of SWCFT, including the representations of the algebra, the space of states and the correlation functions of the superprimaries.
1 Introduction

Symmetry plays an essential role in quantum field theories. The theories with more symmetries could be better constrained such that their dynamics might be investigated even non-perturbatively. For example, the supersymmetric field theories have better UV behaviors, and the conformal invariant theories are expected to be solvable in the framework of conformal bootstrap.

In two dimensions (2D), the global symmetries in a quantum field theory with scaling symmetry could be enhanced. As shown by J. Polchinski in late 1980s [1], a 2D Poincaré invariant QFT with scale invariance could become conformal invariant, provided that the theory is unitary and the dilation spectrum is discrete and non-negative. In 2011, D. Hofman and A. Strominger [2] relaxed the requirement of Lorentz invariance and studied the enhanced symmetries of the theory with chiral scaling. They obtained two kinds of minimal theories, one being the two-dimensional conformal field theory (CFT$_2$) [3] and the other being the so-called the warped conformal field theory (WCFT$_2$) [4]. In a warped CFT$_2$, the global symmetry group is $SL(2, R) \times U(1)$, and it is enhanced to an infinite-dimensional group generated by an Virasoro-Kac-Moody algebra. Very recently, the symmetry enhancement in 2D QFT was generalized to the cases with global translations and anisotropic scaling symmetries [5]. In such 2D Galilean field theories with anisotropic scaling, the enhanced local symmetries are generated by the infinite dimensional spin-$\ell$ Galilean algebra with possible central extensions, under the
assumption that the dilation operator is diagonalizable and has a discrete and non-negative spectrum.

WCFT\(_2\) has rich structures similar to CFT\(_2\). Though they are not Lorentzian invariant, WCFT\(_2\) shares the modular covariance like CFT\(_2\). For finite temperature WCFT\(_2\) defined on a torus, the modular property can be used to evaluate the density of states at high temperature, which gives a Cardy-like formula for the thermal entropy of WCFT\(_2\) [4]. Due to the infinite symmetries, WCFT\(_2\) is highly constrained. The form of the two- and three-point functions are determined by the global warped conformal symmetry while the four-point functions can be determined up to an arbitrary function of the cross ratio [5]. Specific models of WCFT\(_2\) include chiral Liouville gravity [4], free Weyl fermion [8, 9], free scalars [10] and also the Sachdev-Ye-Kitaev models with complex fermions [11, 12]. For the study on other aspects of WCFT\(_2\), see [11, 13–18].

On the other hand, WCFT\(_2\) plays an important role in the study of holography beyond the usual AdS/CFT correspondence. In [19], it has been shown that under the Compère-Song-Strominger (CSS) boundary conditions, the asymptotic symmetry group of the AdS\(_3\) gravity is generated by an Virasoro-Kac-Moody algebra. This leads to the conjecture that under the CSS boundary conditions, the AdS\(_3\) gravity could be dual to a holographic warped conformal field theory. This AdS\(_3\)/WCFT correspondence has been studied in [14, 16, 20–24]. Moreover WCFT\(_2\) could also appear in the WAdS\(_3\)/WCFT\(_2\) correspondence [25–28], in which the bulk gravity is a three-dimensional topological massive gravity.

In this paper, we would like to generalize the study on WCFT\(_2\) to the supersymmetric case. We first study the supersymmetries on the warped flat geometry [9] in two dimensions, which is essentially equivalent to a Newton-Cartan geometry [29–34] with an additional scaling structure. The supersymmetrization could be done by including Grassmannian coordinates into the bosonic directions to make warped "superspace". However, it turns out that the minimal supersymmetry could be realized in a chiral N = (1, 0) superspace. We then study the enhanced local symmetries, following the approach developed in [1] and [2]. Just as in bosonic case, we find two classes of minimal enhanced algebra. One generates the local symmetries of N = (1, 0) SCFT\(_2\), while the other one generates the symmetries of the supersymmetric warped conformal field theory (SWCFT\(_2\)). Furthermore, we discuss the radial quantization and the state-operator correspondence in SWCFT\(_2\), analogous to the usual WCFT\(_2\) case. We study the correlation functions of superprimaries in SWCFT\(_2\) as well. We notice that the correlation functions share the similar structure as the ones in the holomorphic sector of N = (1, 0) SCFT\(_2\), with additional modifications from U(1) symmetry.

The remaining parts are organized as follows. In Section 2, we discuss the supersymmetries
on the warped geometry and set our notations. In Section 3, we generalize the Hofman-Strominger theorem to the supersymmetric case and show that the global symmetries are enhanced to the local ones. In Section 4, we consider the Hilbert space and the representation of the NS sector of the SWCFT$_2$. After establishing the state-operator correspondence, we discuss the transformations of the super-primaries. Then we calculate the two-point functions and three-point functions of the superprimary operators in the NS sector of the SWCFT, and discuss the higher-point functions. We conclude and give some discussions in Section 5. In Appendix, we discuss the conserved currents in the superspace and show that we can consistently work in the $N = (1, 0)$ superspace.

## 2 Supersymmetries on Warped Geometry

Let us start from a two-dimensional unitary local field theory with translational invariance and a chiral scaling symmetry. The transformation of coordinates under these symmetries are

$$x^a \rightarrow x^a + \delta^a, \quad x^a \rightarrow \lambda^a_{\ b} x^b, \quad (2.1)$$

where $\lambda^a_{\ b}$ is a scaling matrix:

$$\lambda^a_{\ b} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \quad (2.2)$$

As shown in [2], the theory would have enhanced local symmetries. There are two kinds of minimal theories. One kind is the two-dimensional conformal field theory (CFT$_2$), while the other kind is the two-dimensional warped conformal field theory (WCFT$_2$). For WCFT$_2$, in addition to the symmetries (2.1), there is a generalized boost symmetry

$$x^a \rightarrow \Lambda^a_{\ b} x^b \quad (2.3)$$

where $\Lambda^a_{\ b}$ is the boost matrix

$$\lambda^a_{\ b} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}. \quad (2.4)$$

The WCFT$_2$ can be defined consistently in a warped geometry, which is a variant of the Newton-Cartan geometry with an additional scaling structure [3]. In the warped geometry, there are one vector and one one-form

$$q^a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q_a = (1, 0), \quad (2.5)$$

which are invariant under the boost. And there is an antisymmetric tensor $h_{ab}$ which is also invariant under the boost,

$$h_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h^{ab} = -h_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.6)$$
$h_{ab}$ can be used to lower the indices, but one should keep in mind that $h_{ab}$ is not the metric of the warped geometry.

In the warped geometry, one may define the fermionic representations. The first step is to consider the gamma matrix algebra. The gamma matrix algebra is given by the warped Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2q^a q^b,$$

(2.7)

where the gamma matrices are:

$$\Gamma^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.8)

The lower-index gamma matrices are defined by

$$\Gamma_a = h_{ab} \Gamma^b.$$  

(2.9)

This definition proves quite useful as it allows us to define the boost generator as

$$\overline{B} = \frac{1}{8} h_{ab} [\Gamma^a, \Gamma^b].$$

(2.10)

One can check that it acts on the gamma matrices as they are in a vector representation

$$[\overline{B}, \Gamma^c] = q_a \Gamma^a q^c, \quad \text{or} \quad [\overline{B}, \Gamma] = -q^a \Gamma_a q^c.$$  

(2.11)

The operators generating the translations will be donated by $H_a = (H_0, H_1) = (H, \overline{P})$, and they are of course in a vector representation of the boost generator

$$[\overline{B}, H^c] = q_a H^a q^c, \quad \text{or} \quad [\overline{B}, H_c] = -q^a H_a q_c.$$  

(2.12)

The two-dimensional spinor space is spanned by $\Psi_0, \Psi_1$ as follows

$$\Psi_A = \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix}.$$  

(2.13)

From this it is easy to see that

$$\overline{B} \Psi_0 = 0, \quad \overline{B} \Psi_1 = \frac{1}{2} \Psi_0.$$  

(2.14)

The definition for the dual representation is

$$\overline{\Psi}^A = \epsilon^{AB} \Psi_B = (\Psi^0, \Psi^1) = (\Psi_1, -\Psi_0).$$

(2.15)

where the $\epsilon^{AB}$ is given by

$$\epsilon^{AB} = -\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(2.16)
One can easily show that the quantity $\overline{\Psi}\Psi$ is a scalar under the boost.

Now, let us introduce the supercharge operator $Q = (Q_0, Q_1)^T$. The commutators of supercharges are

$$i\{Q_A, \overline{Q}^B\} = 2(\Gamma^a H_a)_A^B.$$  

(2.17)

They can be written in terms of the component operators

$$i\{Q_1, Q_1\} = 2H, \quad i\{Q_0, Q_1\} = 2\mathcal{T}, \quad i\{Q_0, Q_0\} = 0.$$  

(2.18)

From these commutators, one can easily find that $H$ and $Q_1$ are superpartners under the action of $Q_1$, so are $P$ and $Q_0$.

For simplicity, we will denote $Q_1$ by $Q_+$ and $Q_0$ by $Q_-$ in the following discussion. Moreover, we will denote $x^a = (x^0, x^1)$ by $x^a = (x^+, x^-)$, and the Grassmannian coordinates by

$$\theta^A = (\theta^+, \theta^-).$$  

(2.19)

A general superfield is defined on the superspace, and can be expanded as a power series in $\theta^+$ and $\theta^-$

$$\Phi(z) \equiv \Phi(x^+, x^-, \theta^+, \theta^-) = A_1(x^+, x^-) + \theta^+ A_2(x^+, x^-) + \theta^- A_3(x^+, x^-) + \theta^+ \theta^- A_4(x^+, x^-).$$  

(2.20)

The transformation of any field $\Phi$ under the generator $G$ is given by

$$\delta \Phi = i[\epsilon G, \Phi].$$  

(2.21)

In the appendix, we discuss the conserved charges of the theory in the superspace and their corresponding supercurrents. We find that there exists a minimal superspace in which the right-moving supersymmetry can be turned off consistently. As we show in the next section, even only with the supersymmetry in the left-moving sector, the right-moving global symmetry gets enhanced and supersymmetrized as well.

3 Enhanced Symmetries

In this section, we study the enhanced symmetry of two dimensional quantum field theory, whose global symmetry is generated by the left-moving translation $H$, the dilation $D$, the right-moving translation $\mathcal{T}$ and the supersymmetries $Q_+$. We will work in the chiral superspace $(x^+, x^-, \theta^+)$. For simplicity, we denote $\theta^+ = \theta$. Now that for the global charges their related supercurrents depend only on one Grassmannian coordinate $\theta$, we will discuss the enhanced symmetry in this $N = 1$ chiral superspace. As in [2], we assume that the eigenvalue spectrum
of $D$ is discrete and non-negative and there exists a complete basis of $N = 1$ local superfields $\Phi_i$.

A general superfield can be expanded as

$$\Phi(x^+, x^-, \theta) = \varphi(x^+, x^-) + \theta \psi(x^+, x^-).$$

(3.1)

It satisfy:

$$i[H, \Phi_i] = \partial_+ \Phi_i, \quad i[\overline{P}, \Phi_i] = \partial_- \Phi_i,$$

(3.2)

$$i[D, \Phi_i] = (x^+ \partial_+ + \theta \partial_0 + \lambda_i)\Phi_i,$$

(3.3)

$$i[\epsilon Q_+, \Phi_i] = \epsilon(\partial_0 - \theta \partial_+)\Phi_i,$$

(3.4)

$$i[\epsilon Q_-, \Phi_i] = \epsilon(-2 \theta \partial_-)\Phi_i,$$

(3.5)

where $\lambda_i$ is the superweight of $\Phi_i$ and $\int_C d\Phi_i = 0$ for any closed contour $C$. The translational plus the dilational invariance and the supersymmetry restrict the form of the vacuum two-point functions

$$\langle \Phi_i(x^+, x^-, \theta)\Phi_j(x^+, x^-, \theta') \rangle = \frac{f_{ij}(x^--x'^-)}{(x^+ - x'^+ - \theta \theta')}^{\lambda_i + \lambda_j},$$

(3.6)

where $f_{ij}$ are some unknown functions.

### 3.1 From left global symmetries to local symmetries

The global charges $H, D, \overline{P}$ are associated to the supercurrents $H, D$ and $\overline{P}$, respectively. All of these supercurrents have shift freedom [2], which can be used to “gauge” the currents to satisfy the canonical commutation relations

$$i[H, H_{\pm}] = \partial_+ H_{\pm}, \quad i[H, \overline{P}_{\pm}] = \partial_+ \overline{P}_{\pm},$$

(3.7)

$$i[\overline{P}, H_{\pm}] = \partial_- H_{\pm}, \quad i[\overline{P}, \overline{P}_{\pm}] = \partial_- \overline{P}_{\pm},$$

(3.8)

This implies that $H_{\pm}, \overline{P}_{\pm}$ are local operators, but $D_{\pm}$ must have explicit dependence on the $x^+$ coordinate. The weights of the global charges [2, 3, 4] imply

$$i[D, H_{\pm}] = x^+ \partial_+ H_{\pm} + \frac{\theta}{2} \partial_0 H_{\pm} + \frac{3}{2} H_{\pm},$$

$$i[D, H_{\pm}] = x^+ \partial_+ H_{\pm} + \frac{\theta}{2} \partial_0 H_{\pm} + \frac{1}{2} H_{\pm},$$

$$i[D, \overline{P}_{\pm}] = x^+ \partial_+ \overline{P}_{\pm} + \frac{\theta}{2} \partial_0 \overline{P}_{\pm} + \frac{3}{2} \overline{P}_{\pm},$$

$$i[D, \overline{P}_{\pm}] = x^+ \partial_+ \overline{P}_{\pm} + \frac{\theta}{2} \partial_0 \overline{P}_{\pm} - \frac{1}{2} \overline{P}_{\pm},$$

$$i[D, D_{\pm}] = x^+ \partial_+ D_{\pm} + \frac{\theta}{2} \partial_0 D_{\pm} + \frac{1}{2} D_{\pm},$$

$$i[D, D_{\pm}] = x^+ \partial_+ D_{\pm} + \frac{\theta}{2} \partial_0 D_{\pm} - \frac{1}{2} D_{\pm}.$$
The $D_\pm$’s have explicit coordinate dependence. Let us write the current in terms of local operators as in \[2\]. Defining $S_\pm$ by
\[
D_\pm = x^+H_\pm + \theta S_\pm.
\] (3.10)

One can easily find that
\[
i[H, S_\pm] = \partial_+ S_\pm, \quad i[\mathcal{P}, S_\pm] = \partial_- S_\pm,
\] (3.11)

and
\[
i[D, S_+] = x^+ \partial_+ S_+ + S_+, \quad i[D, S_-] = x^+ \partial_+ S_-.
\] (3.12)

So we conclude $(S_+, S_-)$ are local operators of weight $(1, 0)$.

The conservations of the dilation current and left-translation current yield
\[
\partial_+ D_- + \partial_- D_+ = x^+(\partial_+ H_- + \partial_- H_+) + \theta (\partial_+ S_- + \partial_- S_+) + H_- = 0,
\] (3.13)

which leads to
\[
H_- = -\theta (\partial_- S_+ + \partial_+ S_-).
\] (3.14)

Then we use the shift freedom in the currents to shift away $S_+$
\[
H_\pm \rightarrow H_\pm \pm \theta \partial_\pm S_+, \quad D_\pm \rightarrow D_\pm \pm \theta \partial_\pm (x^+ S_+).
\] (3.15)

One can check that the commutators and the conservations of the currents remain consistent.

Now, the equation (3.10) becomes
\[
D_+ = x^+ H_+, \quad D_- = x^+ H_- + \theta S_-,
\] (3.16)

and
\[
H_- = -\theta \partial_+ S_-.
\] (3.17)

Because of $S_-$ is a local operator of weight zero, then from the general form of the two-point function, we have
\[
\langle S_- S_- \rangle = f_{S_-}(x^-),
\] (3.18)

which implies
\[
H_- = 0.
\] (3.19)

Assuming
\[
H_+(x^+, x^-, \theta) = h_0(x^+, x^-) + 2\theta h_1(x^+, x^-),
\] (3.20)
the conservation law and the equation (3.19) yield
\[ H_+(x^+, x^-, \theta) = h_0(x^+) + 2\theta h_1(x^+). \] (3.21)

This fact immediately leads to the existence of two sets of conserved charges. Defining
\[ T_\xi = -\frac{1}{2\pi} \int dx^+ d\theta \xi(x^+, \theta) H_+, \] (3.22)
where \( \xi = \frac{1}{2} \alpha(x^+) + \theta a(x^+) \) with \( \alpha \) and \( a \) being the function of \( x^+ \), we have
\[ T_{0\alpha} = -\frac{1}{2\pi} \int dx^+ a(x^+) h_0, \] (3.23)
\[ T_{1\alpha} = -\frac{1}{2\pi} \int dx^+ \alpha(x^+) h_1. \] (3.24)

There is another set of conserved charges
\[ J_\chi = \frac{1}{2\pi} \int dx^+ \chi(x^+) S_-, \] (3.25)
which may lead to other symmetries [2, 9]. As we are going to discuss the minimal algebra, we choose \( S_- = 0 \).

The algebra spanned by the conserved bosonic charges \( T_{1\alpha} \) has been done in [2]:
\[ i[T_{1\alpha}, T_{1\beta}] = T_{1(\alpha'\beta' - \alpha\beta')} + \frac{c_0}{48\pi} \int dx^+ (\alpha''\beta' - \alpha'\beta''), \] (3.26)
where the prime denotes the derivative with respect to \( x^+ \), \( \alpha' \equiv \partial_+ \alpha \). This is the same as the algebra of the left-moving conformal generators on the Minkowski plane with the central charge \( c_0 \).

Let us now work out the algebra spanned by adding the fermionic charges \( T_{0\alpha} \). The global charges are
\[ H = -\frac{1}{4\pi} \int dx^+ d\theta 1 \cdot H_+ = -\frac{1}{2\pi} \int dx^+ h_1, \] (3.27)
\[ D = -\frac{1}{4\pi} \int dx^+ d\theta \theta x^+ \cdot H_+ = -\frac{1}{2\pi} \int dx^+ x^+ h_1, \] (3.28)
\[ Q_+ = -\frac{1}{2\pi} \int dx^+ d\theta \theta H_+ = -\frac{1}{2\pi} \int dx^+ h_0. \] (3.29)

The actions of \( H \) and \( D \) on \( H_+ \) imply
\[ i[H, T_{0\alpha}] = -T_{0\alpha'}, \] (3.30)
\[ i[D, T_{0\alpha}] = -T_0(\frac{a}{2} - a'x^+). \] (3.31)

This in turn implies that the action of \( T_{0\alpha} \) on \( h_1 \) is
\[ i[h_1, T_{0\alpha}] = -\frac{3}{2} a' h_0 - \frac{1}{2} a \partial_+ h_0 + \partial_+^2 O_\alpha. \] (3.32)
Furthermore we have

\[ i[T_{1\alpha}, T_{0\alpha}] = T_0(\alpha' \alpha - \alpha'' \alpha') - \frac{1}{2\pi} \int dx^+ \alpha \partial_x^2 O_a. \]  

(3.33)

The scaling symmetry plus the locality imply that \( O_a \) must be of the form \( O_a = c_1 a \) with \( c_1 \) being a local operator of weight \( \frac{1}{2} \). But the Jacobi identity with the third operator \( T_{1\beta} \) implies that \( c_1 = 0 \). So we arrive at

\[ i[T_{1\alpha}, T_{0\alpha}] = T_0(\alpha' \alpha - \alpha'' \alpha'). \]  

(3.34)

Next, the action of \( Q_+ = T_{01} \) on \( h_0 \) is \( i\{T_{01}, h_0\} = 2h_1 \). This implies \( i\{T_{01}, T_{0\alpha}\} = 2T_{1\alpha} \) and hence

\[ i\{T_{0\alpha}, h_0\} = 2ah_1 + \partial_+ g_a, \]  

(3.35)

where \( g_a \) is to be determined. Integrating both sides with \(-\frac{1}{2\pi}dx^+b(x^+)\) gives

\[ i\{T_{0\alpha}, T_{0\beta}\} = 2T_{1\alpha} - \frac{1}{2\pi} \int dx^+ b\partial_+ g_a. \]  

(3.36)

The scaling symmetry and the exchange symmetry under \( a \leftrightarrow b \) imply \( g_a = c_2 a' \), where \( c_2 \) is a constant number. The Jacobi identity with the third operator \( T_{1\alpha} \) implies that \( c_2 = \frac{c_0}{3} \). Then

\[ i\{T_{0\alpha}, T_{0\beta}\} = 2T_{1\alpha} + \frac{c_0}{6\pi} \int dx^+ a'b'. \]  

(3.37)

We recognize the equations (3.26), (3.34), (3.37) as the superconformal algebra on the Minkowski plane with the central charge \( c_0 \).

### 3.2 From right global symmetries to local symmetries

In general, \( \mathbb{P}_\pm \) can be written in form of

\[ \mathbb{P}_+ = p_0(x^+, x^-) + 2\theta p_1(x^+, x^-), \]  

(3.38)

\[ \mathbb{P}_- = p_3(x^+, x^-) + 2\theta p_2(x^+, x^-). \]  

(3.39)

The fact that \( \mathbb{P}_- \) is a local superfield of weight \(-\frac{1}{2}\) implies that \( p_2 \) is a weight-zero local field. From the two-point function of \( p_2 \), we get \( \partial_+ p_2 = 0 \). The current conservation then implies \( \partial_- p_1 = 0 \). It follows that

\[ p_1 = p_1(x^+), \quad p_2 = p_2(x^-). \]  

(3.40)

The supersymmetry requires \( p_0 = p_0(x^+) \) and \( p_3 = 0 \), hence \( p_- \) is a singlet under the supersymmetry. Now we have

\[ \mathbb{P}_+ = p_0(x^+) + 2\theta p_1(x^+), \]  

(3.41)

\[ \mathbb{P}_- = p_2(x^-). \]  

(3.42)
In the case $P_+ = 0$, we have infinitely many charges given by

$$T_{1\alpha} = \frac{1}{2\pi} \int dx^- \alpha(x^-) p_2. \quad (3.43)$$

The algebra spanned by $T_{1\alpha}$ gives the right-moving Virasoro algebra on Minkowski plane $[2]$. In this case, the enhanced local symmetry is generated by the left-moving super-Virasoro algebra and the right-moving Virasoro algebra. It gives the local symmetry of $N = (1, 0)$ SCFT$_2$.

In the case $P_- = 0$, we have infinitely many left-moving charges

$$J_{\eta} = -\frac{1}{2\pi} \int dx^+ d\theta \eta(x^+, \theta) P_+, \quad (3.44)$$

where $\eta = \frac{1}{2} \eta(x^+) + \theta c(x^+)$. Then

$$J_{0\epsilon} = -\frac{1}{2\pi} \int dx^+ d\theta \epsilon P_+, \quad (3.45)$$

$$J_{1\eta} = -\frac{1}{2\pi} \int dx^+ d\theta \cdot \eta P_+ = -\int dx^+ \eta p_1. \quad (3.46)$$

The bosonic sector of the algebra are simply $[2]

$$i[J_{1\eta}, J_{1\chi}] = \frac{k}{8\pi} \int dx^+ (\chi' \eta - \chi \eta'), \quad (3.47)$$

$$i[T_{1\alpha}, J_{1\eta}] = -J_{1\alpha \eta'}. \quad (3.48)$$

The equation (3.47) is a $U(1)$ Kac-Moody current algebra and the constant $k$ parameterizes the central element.

To find the fermionic sector of the enhanced symmetry, we need to consider other commutators. Firstly we study the commutator $[T_{0\alpha}, J_{1\eta}]$. Note that the action $Q_+ = T_{01}$ on $P_+$ implies

$$i[T_{01}, J_{1\eta}] = \frac{1}{2} J_{0\eta'}. \quad (3.49)$$

This in turn implies that the action of $J_{1\eta}$ on $h_0$ is

$$[h_0, J_{1\eta}] = \frac{1}{2} \eta' p_0 + \partial_+ O_{0\eta}. \quad (3.50)$$

The scaling symmetry plus the locality imply that $O_{0\eta}$ must be of the form $O_{0\eta} = c_3 \eta$, where $c_3$ is a local operator of weight $\frac{1}{2}$. Consider the zero mode of $J_{1\eta}(\eta = 1) \equiv J_{11}$, which act as $\partial_-$, we have $i[J_{11}, h_0] = 0 = \partial_+ c_3$. This leads to the fact that $c_3$ must be independent of $x^+$. On the other hand, $c_3$ is an operator of weight $\frac{1}{2}$ under the chiral scaling, we conclude that $c_3$ must be zero. Now integrating both sides of the equation (3.50) with $-\frac{1}{2\pi} dx^+ a$ gives

$$i[T_{0\alpha}, J_{1\eta}] = \frac{1}{2} J_{0(\alpha \eta')}. \quad (3.51)$$
Let us now work out the algebra spanned by \( J_{0c} \). Due to the fact that the zero mode \( J_{11} \) acts as \( \partial_+ \), we have

\[
i[J_{11}, p_0] = 0.
\]

This implies

\[
i[J_{11}, J_{0c}] = 0 \quad \text{and hence} \quad i[J_{0c}, p_1] = \partial_+ X_c. \tag{3.52}
\]

Again, the scaling symmetry plus the locality imply \( X_c = 0 \), then

\[
i[J_{0c}, J_{1\eta}] = 0. \tag{3.53}
\]

We also need the commutator \([T_{1\alpha}, J_{0c}]\). The action of \( H \) on \( p_0 \) implies

\[
i[H, J_{01}] = 0,
\]

which in turn implies

\[
i[h_1, J_{01}] = \partial_+(p_0 + Y) \quad \text{with } Y \text{ a local operator of weight } \frac{1}{2}.
\]

Integrating both sides with \(-\frac{1}{2\pi}dx^\alpha \) gives

\[
i[T_{1\alpha}, J_{01}] = \frac{1}{2\pi} \int dx^\alpha (p_0 + Y).
\]

This gives

\[
i[T_{1\alpha}, p_0] = -\alpha'(p_0 + Y) + \partial_+ Z_\alpha. \tag{3.54}
\]

The scaling symmetry plus the locality implies \( Z_\alpha \) must be of the form

\[
Z_\alpha = c_4 \alpha \quad \text{with } c_4 \text{ being a local operator of weight } \frac{1}{2}.
\]

The action of \( D_1 \) on \( p_0 \) implies that \( Y = -\frac{p_0}{2} \) and \( c_4 = p_0 \), hence

\[
i[T_{1\alpha}, p_0] = \alpha p'_0 + \frac{\alpha' p_0}{2}. \tag{3.55}
\]

Then we have

\[
i[T_{1\alpha}, J_{0c}] = -J_{0(\alpha c' + c_6)}. \tag{3.56}
\]

Next we turn to the anti-commutator \( \{T_{0a}, J_{0c}\} \). The fact \( i\{Q_+, p_0\} = 2p_1 \) implies \( i\{Q_+, J_{0c}\} = 2J_{1c} \). This in turn implies

\[
i\{h_0, J_{0c}\} = 2cp_1 + \partial_+ O_c. \tag{3.57}
\]

The scaling symmetry plus the locality imply \( O_c \) must be of form

\[
O_c = c_5 c \quad \text{with } c_5 \text{ being a local operator of weight zero. After integrating both sides with } -\frac{1}{2\pi}dx^c a(x^+) \text{, we get}
\]

\[
i\{T_{0a}, J_{0c}\} = 2J_{1(ac)} - \frac{c_5}{2\pi} \int dx^c a c'. \tag{3.58}
\]

The Jacobi identity with the third operator \( T_{1\alpha} \) implies \( c_5 = 0 \). Finally we have

\[
i\{T_{0a}, J_{0c}\} = 2J_{1(ac)}. \tag{3.59}
\]

Finally we consider the anti-commutator \( \{J_{0c}, J_{0d}\} \). The scaling symmetry implies that

\[
\{J_{01}, p_0\} = c_6, \quad \text{where } c_6 \text{ must be a weight-zero constant number. This implies } \{J_{01}, J_{0c}\} = -c_6 \int dx^c, \text{ which in turn gives}
\]

\[
i\{p_0, J_{0c}\} = c_6 c + \partial_+ W_c. \tag{3.60}
\]
Again, the scaling symmetry plus the locality imply that $W_c = 0$. At last, we get

$$i\{J_{0c}, J_{0d}\} = -c_6 \int dx^+ cd.$$  \hfill(3.61)

The appropriate normalizations of $J_{0c}, J_{1\eta}$ can always help us to set $c_6 = -\frac{k}{4\pi}$, then we recognize (3.47), (3.53), (3.61) as the $U(1)$ super-Kac-Moody algebra (SKMA) on the Minkowski plane with the central charge $k$.

### 3.3 Mode expansion

The supersymmetric Virasoro-Kac-Moody algebra consists of a super-Virasoro algebra

\begin{align*}
i[T_{1a}, T_{1\beta}] &= T_{1(\alpha'\beta' - \alpha\beta')} + \frac{c_0}{48\pi} \int dx^+ (\alpha''\beta' - \alpha'\beta''), \hfill (3.62) \\
i[T_{1a}, T_{0a}] &= T_{0(\alpha'\eta - \alpha\eta')}, \hfill (3.63) \\
i\{T_{0a}, T_{0b}\} &= 2T_{1ab} + \frac{c_0}{6\pi} \int dx^+ a'b', \hfill (3.64)
\end{align*}

a super-Kac-Moody algebra

\begin{align*}
i[J_{1\eta}, J_{1\chi}] &= -\frac{k}{8\pi} \int dx^+ (\chi'\eta - \chi\eta'), \hfill (3.65) \\
i[J_{0c}, J_{1\eta}] &= 0, \hfill (3.66) \\
i\{J_{0c}, J_{0d}\} &= \frac{k}{4\pi} \int dx^+ cd, \hfill (3.67)
\end{align*}

and the semidirect product of the super-Virasoro and super-Kac-Moody algebras

\begin{align*}
i[T_{1a}, J_{1\eta}] &= -J_{1\alpha\eta'}, \hfill (3.68) \\
i[T_{0a}, J_{1\eta}] &= \frac{J_{0(\alpha\eta')}}{2}, \hfill (3.69) \\
i[T_{1a}, J_{0c}] &= -J_{0(\alpha\eta' + \eta'\alpha')} , \hfill (3.70) \\
i[T_{0a}, J_{0c}] &= 2J_{1(\alpha\eta')} . \hfill (3.71)
\end{align*}

Let us put the theory on a cylinder and find the mode expansion of the above algebra. The coordinate transformation is

$$x^+ = e^{i\phi}.$$  \hfill(3.72)

Using the new coordinate $\phi$, we choose test functions $\alpha_n = (x^+)^{n+1} = e^{i(n+1)\phi}$, $a_r = e^{i(r+\frac{1}{2})\phi}$, $\eta_n = e^{in\phi}$ and $c_r = e^{i(r-\frac{1}{2})\phi}$, where $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$ for the Neveu-Schwarz (NS) sector or $r \in \mathbb{Z}$ for the Ramond (R) sector. Letting $L_n = iT_{1a_n}, G_r = iT_{0a_r}, P_n = J_{1\eta_n}$ and $S_r = J_{0c_r}$, then the commutation relations in terms of the charges $\{L_n, P_m, G_r, S_s\}$ are as follows. The super-Virasoro algebra is generated by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n},$$  \hfill(3.73)

\hspace{1cm} \text{We have set } c_0 = c \text{ and } \delta_{n+m} = \delta_{n+m,0}.$
\[ [L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \quad (3.74) \]
\[ \{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s}. \quad (3.75) \]

The super-Kac-Moody algebra is generated by
\[ [P_m, P_n] = \frac{k}{2} m\delta_{m+n}, \quad [P_m, S_r] = 0, \quad \{S_r, S_s\} = \frac{k}{2}\delta_{r+s}. \quad (3.76) \]

The semi-direct product part of the super-Virasoro and super-Kac-Moody algebras is generated by
\[ [L_m, P_n] = -nP_{m+n}, \quad [G_r, P_m] = -\frac{m}{2} S_{m+r}, \quad (3.77) \]
\[ [L_m, S_r] = -(\frac{m}{2} + r)S_{m+r}, \quad \{G_r, S_s\} = 2P_{r+s}. \quad (3.78) \]

These algebras are the same as those appearing in the supersymmetric Wess-Zumino-Witten (SWZW) model \cite{35-38}. We would like to stress that the super-Kac-Moody algebra in the SWZW model is generated by internal symmetries, while here it is generated by the symmetric transformations in the superspace in SWCFT\(_2\). The algebra of the SWZW model consists of two copies (the left-moving and the right-moving) of algebras, while there is only one copy (left-moving) of the algebra in SWCFT\(_2\).

Another remarkable point is on the supersymmetry in the right-moving sector. Our construction starts from the left-moving superspace, but the right-moving sector gets supersymmetrized as well. This could be understood from the diffeomorphism of WCFT\(_2\). Recall that the diffeomorphism of WCFT\(_2\) is generated by
\[ x^+ \to f(x^+), \quad x^- \to x^-, \quad (3.79) \]
and
\[ x^+ \to x^+, \quad x^- \to x^- + g(x^+) \quad (3.80) \]

The diffeomorphism could be generalize to chiral superspace such that the supersymmetry in the left-moving sector is transferred to the right-moving sector. Actually, the superconformal transformation in the left-moving sector is \cite{39}
\[ x^+ \mapsto f(x^+) + \theta F(x^+), \quad (3.81) \]
\[ \theta \mapsto \phi(x^+) + \theta p(x^+). \quad (3.82) \]

Here \(f(x^+), p(x^+)\) are holomorphic functions and \(F(x^+), \phi(x^+)\) are anticommuting holomorphic functions, satisfying the following relations
\[ F(x^+) = \phi(x^+)p(x^+), \quad p(x^+)^2 = \partial_+ f(x^+) + \phi(x^+)^2 \phi(x^+). \quad (3.83) \]

\footnote{we have substitute \(-P\) for \(P\).}
The transformation in the right-moving sector is
\[ x^+ = x^- + g(x^+ ) + \theta G(x^+), \]
where \( G(x^+) \) is an anticommuting holomorphic function. Considering the infinitesimal version of the above transformations, we find that the generators of the super-Virasoro-Kac-Moody algebra could be realized by

\[ L_n = (x^+)^{n+1} \partial_+ + \frac{1}{2}(n+1)(x^+) \theta \partial_\theta, \]
\[ P_n = (x^+) n \partial_-, \]
\[ G_r = (x^+)^{r+1/2} (\partial_\theta - \theta \partial_+), \]
\[ S_s = -2(x^+)^s \theta \partial_-. \]

They satisfy the above commutation relations without central extensions. Then from the Jacobi identity, the central extensions could be recovered. This fact shows that the chiral superspace \((x^+, \theta)\) is enough for our study.

4 Properties of SWCFT

Now we have found two kinds of minimal theories in \( N = (1,0) \) superspace, starting from a 2D QFT with chiral scaling and translation symmetry. One is the \( N = (1,0) \) supersymmetric conformal field theory, whose local symmetries consist of a left-moving super-Virasoro algebra (SVA) and a right-moving Virasoro algebra. The other is the supersymmetric warped conformal field theory, whose local symmetries are generated by supersymmetric Virasoro-Kac-Moody algebra (SVCMA). In this section, we discuss the representations of this algebra, the state-operator correspondence and then the correlation functions in SWCFT.

4.1 Primary states and descendants

In all our subsequent discussions, we consider the NS sector of SWCFT and hence \( r,s \in \mathbb{Z} + \frac{1}{2} \). We want to define the states in this theory at \( t = 0 \) by doing radial quantization. For this purpose, we consider the following complex coordinates

\[ x^+ = e^{-i(t-\phi)} = e^{i\phi + tE}, \quad x^- = t + 2\gamma(\phi - t). \]

where \( t \) is interpreted as the Lorentzian time, and \( tE = -it \) as the Euclidean time. Having an initial state at very early Euclidean time corresponds to insert an operator at \( x^+ = 0 \). Using translational symmetry, we can further put the operator at \( x^- = 0 \). A primary operator \( \Phi \) of weight \( \Delta \) and charge \( Q \) at \( x^+ = 0 \) corresponds to a state

\[ \Phi(0,0,0) \sim |\Delta, Q>. \]
In particular, because of the global sub-algebra of SVKMA is $osp(1|2) \times u(1)$, the identity operator corresponds to the $OSP(1|2) \times U(1)$ invariant vacuum. The vacuum state $|0\rangle$ is defined as

$$L_n|0\rangle = 0, \quad n \geq -1,$$
$$P_m|0\rangle = 0, \quad m \geq 0,$$
$$G_r|0\rangle = 0, \quad r \geq -\frac{1}{2},$$
$$S_s|0\rangle = 0, \quad s \geq \frac{1}{2}.$$  \hfill (4.3)

We now construct the representations by considering the states having definite scaling dimensions and $U(1)$ charges. The state $|\Delta, Q\rangle$ is of scaling dimension $\Delta$ and charge $Q$

$$L_0|\Delta, Q\rangle = |\Delta, Q\rangle,$$
$$P_0|\Delta, Q\rangle = Q|\Delta, Q\rangle.$$  \hfill (4.4)

Using the algebra obtained previously, we have

$$L_0L_n|\Delta, Q\rangle = (\Delta - n)L_n|\Delta, Q\rangle,$$
$$L_0P_m|\Delta, Q\rangle = (\Delta - m)P_m|\Delta, Q\rangle,$$
$$L_0G_r|\Delta, Q\rangle = (\Delta - r)G_r|\Delta, Q\rangle,$$
$$L_0S_s|\Delta, Q\rangle = (\Delta - s)S_s|\Delta, Q\rangle.$$  \hfill (4.5)

We can see that the positive modes $L_n, P_m, G_r, S_s$ lower the value of the scaling dimension while the negative modes $L_{-n}, P_{-m}, G_{-r}, S_{-s}$ raise the value of the scaling dimension. The super-primary states in the theory are defined to have the following properties

$$L_n|\Delta, Q\rangle = 0, \quad n > 0,$$
$$P_n|\Delta, Q\rangle = 0, \quad n > 0,$$
$$G_r|\Delta, Q\rangle = 0, \quad r > 0,$$
$$S_s|\Delta, Q\rangle = 0, \quad s > 0.$$  \hfill (4.6)

The modules (analogue to the Verma modules in CFT) in SWCFT$_2$ are then defined by acting the raising operators $L_{-n}, P_{-m}, G_{-r}, S_{-s}$ on the primary states. The descendant states at level $N$ is

$$|\Delta, Q, \{N\}\rangle = L_{-n_1} \cdots L_{-n_k}P_{-m_1} \cdots P_{-m_l}G_{-r_1} \cdots G_{-r_i}S_{-s_1} \cdots S_{-s_j}|\Delta, Q\rangle,$$  \hfill (4.7)

where $\{N\}$ denotes four sets of $\{n\}, \{m\}, \{r\}$ and $\{s\}$ and the total level $N$ is the sum of all elements in the sets. A primary module consists of a primary state and all its descendant states.
In our chiral superspace, the superfield has two component fields which are related to each other by supersymmetric transformation. The states corresponding to the component fields can be obtained from the highest weight state

$$|\varphi\rangle = |\Delta, Q\rangle,$$
$$|\psi\rangle = G_{-\frac{1}{2}}|\Delta, Q\rangle.$$  \tag{4.8}

They share the same $P_0$ charge

$$P_0|\varphi\rangle = Q|\varphi\rangle,$$
$$P_0|\psi\rangle = Q|\psi\rangle.$$  \tag{4.9}

The matrix of inner products of the states including the descendants defines the SWCFT analogue of the Kac matrix in CFT. We will denote it by $\mathcal{M}_{N}$ and its matrix elements are

$$\mathcal{M}_{\{N\},\{N'\}} = \langle \Delta, Q, \{N\}|\Delta, Q, \{N'\}\rangle.$$  \tag{4.10}

At level $\frac{1}{2}$, we have

$$\mathcal{M}_{\frac{1}{2}} = \begin{bmatrix} \langle \Delta, Q, |G_{\frac{1}{2}} \rangle & \langle \Delta, Q, |S_{\frac{1}{2}} G_{\frac{1}{2}} \rangle \end{bmatrix} \begin{bmatrix} G_{-\frac{1}{2}}|\Delta, Q\rangle, S_{-\frac{1}{2}}|\Delta, Q\rangle \end{bmatrix} = \begin{bmatrix} 2\Delta & 2Q \\ 2Q & \frac{k}{2} \end{bmatrix}. \tag{4.11}
$$

At level 1, we have

$$\mathcal{M}_{1} = \begin{bmatrix} \langle \Delta, Q, |L_{1} \rangle & \langle \Delta, Q, |S_{1} G_{\frac{1}{2}} \rangle \end{bmatrix} \begin{bmatrix} L_{-1}|\Delta, Q\rangle, G_{-\frac{1}{2}} S_{-\frac{1}{2}}|\Delta, Q\rangle, P_{-1}|\Delta, Q\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 2\Delta & 2Q \\ 2Q & \frac{(2\Delta+1)k - 8Q^2}{k} & \frac{kQ}{k} \\ Q & \frac{k}{2} & \frac{kQ}{2} \end{bmatrix}. \tag{4.12}
$$

At level $\frac{3}{2}$, we have

$$\mathcal{M}_{\frac{3}{2}} = \begin{bmatrix} 2\Delta(2\Delta + 1) & 4Q & (2\Delta + 1)2Q & 2\Delta Q & 2Q & 2Q^2 \\ 4Q & 2\Delta + \frac{2}{k} & 4Q & \frac{(2\Delta+1)k}{2} & 2Q^2 & \frac{k}{2} \frac{kQ}{2} \\ 2\Delta Q & 4Q & \frac{(2\Delta+1)k}{2} & 2Q^2 & k\Delta & 0 \frac{kQ}{2} \frac{kQ}{2} \\ 2Q & 2Q & \frac{k}{2} & 0 & \frac{k}{2} & 0 \\ 2Q^2 & \frac{k}{2} & \frac{kQ}{2} & k\Delta & 0 & \frac{k^2}{4} \end{bmatrix}, \tag{4.13}
$$

which is in the base

$$\{L_{-1}G_{-\frac{1}{2}}|\Delta, Q\rangle, G_{-\frac{1}{2}}|\Delta, Q\rangle, L_{-1}S_{-\frac{1}{2}}|\Delta, Q\rangle, P_{-1}G_{-\frac{1}{2}}|\Delta, Q\rangle, S_{-\frac{1}{2}}|\Delta, Q\rangle, P_{-1}S_{-\frac{1}{2}}|\Delta, Q\rangle\}.\{\}$$
Using the Baker-Campbell-Hausdorff (BCH) formula, we get
\begin{align*}
\|L_n|\Delta, Q\| \geq 0 \implies \Delta \geq 0, \quad c \geq 0, \\
\|P_n|\Delta, Q\| \geq 0 \implies Q \in \mathbb{R}, \quad k \geq 0.
\end{align*}
Furthermore, the matrix \( \mathcal{M} \) gives
\begin{equation}
k\Delta - 4Q^2 \geq 0 \implies k \geq \frac{4Q^2}{\Delta}.
\end{equation}

### 4.2 Transformation laws of superprimary fields

We now consider the transformation laws of the primary superfields. The local operator at position \((x^+, x^-, \theta)\) is related to the one at the origin by the transformation
\begin{equation}
\Phi(z) \equiv \Phi(x^+, x^-, \theta) = U\Phi(0)U^{-1}, \quad \text{with} \quad U = e^{x^+L_{-1} + \theta G_{-\frac{1}{2}} + x^- P_0}.
\end{equation}
Next we would like to find the explicit form of the commutator \([L_n, \Phi(z)](n \geq 0)\) for a primary field \(\Phi(z)\). First, we have
\begin{equation}
[L_n, \Phi(z)] = U[U^{-1}L_nU, \Phi(0)]U^{-1}.
\end{equation}
Using the Baker-Campbell-Hausdorff (BCH) formula, we get
\begin{align*}
U^{-1}L_nU &= \sum_{k=0}^{n+1} \frac{(n+1)!}{k!(n+1-k)!}[(x^+)^k L_{n-k} + \frac{k}{2}(x^+)^{k-1}\theta G_{n+\frac{1}{2}-k}], \\
U^{-1}P_mU &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}[(x^+)^k P_{m-k} + (x^+)^{k-1}\theta S_{m+\frac{1}{2}-k}], \\
U^{-1}G_rU &= \sum_{k=0}^{r+\frac{1}{2}} \frac{(r+\frac{1}{2})!}{k!(r+\frac{1}{2}-k)!}[(x^+)^k G_{r-k} - \sum_{k=0}^{r+\frac{1}{2}} \frac{(r+\frac{1}{2})!}{k!(r+\frac{1}{2}-k)!}2k(x^+)^{k-1}\theta L_{r+\frac{1}{2}-k}], \\
U^{-1}S_sU &= \sum_{k=0}^{s-\frac{1}{2}} \frac{(s-\frac{1}{2})!}{k!(s-\frac{1}{2}-k)!}[(x^+)^k S_{s-k} - 2\sum_{k=0}^{s+\frac{1}{2}} \frac{(s-\frac{1}{2})!}{k!(s+\frac{1}{2}-k)!}\theta(x^+)^{k-1}P_{s+\frac{1}{2}-k}].
\end{align*}
Then we obtain
\begin{align*}
[L_n, \Phi(z)] &= U[(x^+)^{n+1}L_{-1} + \frac{n+1}{2}(x^+)n\theta G_{-\frac{1}{2}} + (n+1)(x^+)nL_0, \Phi(0)]U^{-1}, \quad n \geq -1, \\
[P_m, \Phi(z)] &= U[(x^+)^m P_0, \Phi(0)]U^{-1}, \quad m \geq 0, \\
[G_r, \Phi(z)] &= U[(x^+)^{r+\frac{1}{2}}(G_{-\frac{1}{2}} - 2\theta L_{-1}) - 2(r+\frac{1}{2})(x^+)^{r-\frac{1}{2}}\theta L_0, \Phi(0)]U^{-1}, \quad r \geq -\frac{1}{2}, \\
[S_s, \Phi(z)] &= (x^+)^{s-\frac{1}{2}}U[-2\theta P_0, \Phi(0)]U^{-1}, \quad s > 0.
\end{align*}
In particular, we have

\[ U L_{-1} U^{-1} = L_{-1}, \quad U R_0 U^{-1} = R_0, \]
\[ U G_{-\frac{1}{2}} U^{-1} = G_{-\frac{1}{2}} + 2\theta L_{-1}, \quad U S_{\frac{1}{2}} U^{-1} = S_{\frac{1}{2}} + 2\theta P_0. \]  \hspace{1cm} (4.21)

Using the above relations, we finally obtain

\[ [L_n, \Phi(z)] = [(x^+)^{n+1} \partial_+ + \frac{n}{2} (x^+) n \theta \partial_0 + (n+1)(x^+) n \Delta] \Phi(z), \quad n \geq -1, \]  \hspace{1cm} (4.22)

\[ [P_m, \Phi(z)] = (x^+)^m \partial_- \Phi(z), \quad m \geq 0, \]  \hspace{1cm} (4.23)

\[ [G_r, \Phi(z)] = [(x^+)^{r+\frac{1}{2}} (\partial_0 - \theta \partial_+) - 2(r + \frac{1}{2})(x^+)^{r-\frac{1}{2}} \theta \Delta] \Phi(z), \quad r \geq -\frac{1}{2}, \]  \hspace{1cm} (4.24)

\[ [S_s, \Phi(z)] = -2(x^+) s - \frac{1}{2} \theta \partial_+ \Phi(z), \quad s > 0. \]  \hspace{1cm} (4.25)

### 4.3 Ward identities and correlation functions

The vacuum of the NS sector in SWCFT$_2$ is invariant under the global group $OSP(1|2) \times U(1)$, which is generated by \{\(L_0, L_{\pm 1}, P_0, G_{\pm \frac{1}{2}}\}\). Thus the correlation functions obey the Ward identities coming from the generators of $OSP(1|2) \times U(1)$. One can solve the differential equations from the Ward identities to find the correlation functions directly using (4.22)-(4.25).

Consider an \(n\)-point function of super-primary fields

\[ G^{(n)}(\{z_i\}) \equiv G^{(n)}(\{x_{i+}, x_{i-}, \theta_i\}) \]
\[ = \langle 0| T[\Phi_1(x_{1+}, x_{1-}, \theta_1) \Phi_2(x_{2+}, x_{2-}, \theta_2) \ldots \Phi_n(x_{n+}, x_{n-}, \theta_n)] |0\rangle, \]  \hspace{1cm} (4.26)

where \(T\) stands for time ordering, and \(z_i \equiv \{x_{i+}, x_{i-}, \theta_i\}\). Throughout this paper, we will always assume \(x_{i+} > x_{j+}\) for \(i < j\). Since the vacuum state \(|0\rangle\) is $OSP(1|2) \times U(1)$ invariant, the \(n\)-point function is invariant under the action of \(L_0, L_{\pm 1}, G_{\pm \frac{1}{2}}\). This leads to the following differential equations corresponding to the generators \(L_{-1}, P_0, L_0, G_{-1/2}, G_{1/2}\) respectively

\[ 0 = \left( \sum_{i=1}^{n} \partial_{i+} \right) G^{(n)}, \]
\[ 0 = (\partial_{i-} - Q_i) G^{(n)}, \quad \text{with} \quad \sum_{i=1}^{n} Q_i = 0, \]
\[ 0 = \left( \sum_{i=1}^{n} (x_{i+}^{\partial} \partial_{i+} + \frac{1}{2} \theta_i \partial_{\theta_i} + \Delta_i) \right) G^{(n)}, \]
\[ 0 = \left( \sum_{i=1}^{n} ((x_{i+}^{\partial})^2 \partial_{i+} + x_{i+}^\theta \partial_{\theta_i} + 2x_{i+}^\Delta \Delta_i) \right) G^{(n)}, \]
\[ 0 = \left( \sum_{i=1}^{n} \partial_{\theta_i} - \theta_i \partial_{i+} \right) G^{(n)}, \]
\[ 0 = \left( \sum_{i=1}^{n} x_{i+}^{\partial} (\partial_{\theta_i} - \theta_i \partial_{i+} - 2\theta_i \Delta) \right) G^{(n)}. \]  \hspace{1cm} (4.27)
The first equation implies that $G^{(n)}$ should be a function of $x^+_{ij} = x^+_i - x^+_j$. While the fourth equation implies that $G^{(n)}$ should be a function of

$$s_{ij} \equiv x^+_i - x^+_j - \theta_i \theta_j,$$

with

$$\theta_{ij} \equiv \theta_i - \theta_j,$$

For the $x^-$ part, the second equation implies $G^{(n)}$ should be a function of

$$r_{ij} \equiv x^-_i Q_i + x^-_j Q_j.$$

Consequently, the correlation function is of form

$$G^{(n)}(s_{ij}, r_{ij}, \theta^+_{ij}).$$

We stress that because of the $OSP(1|2) \times U(1)$ symmetry of the vacuum, the correlation functions must have the $OSP(1|2) \times U(1)$ structure.

Let us first consider the two-point function $G^{(2)}$ which must be of form

$$G^{(2)}(z_1, z_2) = \frac{f_1(r_{12})}{(s_{12})^{\kappa_1}} + \frac{f_2(r_{12}) \theta_{12}}{(s_{12})^{\kappa_2}},$$

where $\{f_i\}$ are the functions to be determined. The equation from the invariance under $P_0$ gives

$$\frac{\partial f_i}{\partial r_{12}} = f_i, \quad Q_1 + Q_2 = 0,$$

which has the solution

$$f_i = C_i e^{r_{12}}, \quad i = 1, 2$$

where $\{C_i\}$ are constants.

Next consider the differential equation arising from dilatons $L_0$, we find the conditions

$$-\kappa_1 + \Delta_1 + \Delta_2 = 0,$$

$$-\kappa_2 + \frac{1}{2} + \Delta_1 + \Delta_2 = 0.$$

Moreover, the special transformation $L_{+1}$ leads to the condition

$$0 = C_1(s_{12})^\frac{1}{2} \left[ (\Delta_1 x^+_{12}) + \frac{1}{2} x^+_{12} \theta_1 - (\Delta_2 - \frac{1}{2}) x^+_{12} \theta_2 \right].$$
which gives
\[
\Delta_{12} = 0, \quad C_2 = 0. \tag{4.37}
\]
As a result, we have the two-point function
\[
G^{(2)}(z_1, z_2) = \delta_{\Delta_1, \Delta_2} \delta Q_1, -Q_2 \frac{1}{s_{12}} e^{x_{\bar{1}2} Q_1}, \tag{4.38}
\]
where we have set the normalization to unit. In the component fields, the nonvanishing two-point functions are
\[
\langle \varphi_1 \varphi_2 \rangle = \delta_{\Delta_1, \Delta_2} \delta Q_1, -Q_2 \frac{1}{s_{12}} e^{x_{\bar{1}2} Q_1},
\]
\[
\langle \psi_1 \psi_2 \rangle = \delta_{\Delta_1, \Delta_2} \delta Q_1, -Q_2 \frac{1}{s_{12}} e^{x_{\bar{1}2} Q_1}, \tag{4.39}
\]
where \(\varphi_i\) and \(\psi_i\) are the component fields of the superfield \(\Phi_i\).

It is clear that the two-point functions respect the \(OSP(1|2) \times U(1)\) symmetry. The \(OSP(1|2)\) part is determined by the modes \(\{L_0, L_{\pm 1}, G_{\pm \frac{1}{2}}\}\), just as the usual \(N = (1,0)\) SCFT\(_2\), while the \(U(1)\) part is totally determined by the zero mode \(P_0\). Moreover, we note that the two-point-functions of the superprimaries in SWCFT\(_2\) could be obtained straightforwardly from the bosonic one [6] by replacing the difference of the two bosonic coordinates \(x_{\bar{1}2}\) with its supersymmetric generalization \(s_{12}\).

From the structures of two-point function, we know that the higher-point correlation functions of SWCFT\(_2\) must also include two parts, one being determined by the modes \(\{L_0, L_{\pm 1}, G_{\pm \frac{1}{2}}\}\), the other being determined by the zero mode \(P_0\). We can use the well known results of \(N = 1\) super conformal theory to determine the \(OSP(1|2)\) structures of the correlation functions of SWCFT\(_2\). For example, the holomorphic three-point function in the NS sector of \(N = 1\) super-conformal theory is given in [40], and the \(U(1)\) part is given by [6], thus the three-point function in the NS sector of the SWCFT primaries is
\[
G^{(3)}(z_1, z_2, z_3) = \delta_{Q_1, Q_2, Q_3} \delta_{s_{12} + s_{23} + s_{13} - s_{12} - s_{23} - s_{13}} e^{\frac{1}{2} Q_1 x_{\bar{1}2} + \frac{1}{2} Q_2 x_{\bar{1}3} + \frac{1}{2} Q_3 x_{\bar{2}3}}, \tag{4.40}
\]
where we have defined \(x_{ij} \equiv x_{\bar{i}j} - x_{\bar{j}i}, Q_{ij} \equiv Q_i - Q_j\) and \(\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k\). The \(C_{123}\) and \(\tilde{C}_{123}\) are two structure constants of the three-point function. The quantity \(\Xi_{ijk}\) is given by
\[
\Xi_{ijk} \equiv \frac{s_{ik} \theta_k + s_{jk} \theta_i + s_{ki} \theta_j}{\sqrt{s_{ij} s_{jk} s_{ki}}}. \tag{4.41}
\]

For the \(n\)-point function \((n > 3)\) there are \(3n\) coordinates \(\{x_i^+ + x_i^-, \theta_i\}\), and \(5\) constraints from \(OSP(1|2)\) invariance and one constraint from \(U(1)\) invariance. The \(x_i^-\) dependence can
be totally determined by the $U(1)$ invariance, thus the $n$-point function is essentially a function of $(2n - 5)$ $OSP(1|2)$ invariants, which are given by

$$\Xi_{ijk}, \quad \Theta_{ijkl} \equiv \frac{s_{ij} s_{kl}}{s_{li} s_{jk}}. \quad (4.42)$$

The general form of $n$-point function can be written as

$$G^{(n)}(\{z_i\}) = \delta_{\sum_{i=1}^n Q_i,0} \left( \prod_{i<j=1}^n e^{r_{ij}} \right) \left( \prod_{i<j=1}^n s^{\Delta_{ij}} \right) F(\Xi_{ijk}, \Theta_{ijkl}). \quad (4.43)$$

Here $F(\Xi_{ijk}, \Theta_{ijkl})$ is an undetermined function, and $\Delta_{ij}$ are real constants which satisfy

$$\sum_{i \neq j} \Delta_{ij} = 2\Delta_j, \quad \Delta_{ij} = \Delta_{ji}. \quad (4.44)$$

### 5 Conclusion and Discussion

In the present work we studied supersymmetric extension of the warped conformal field theory. Under the assumption that the dilation operator is diagonalizable, and has a discrete, non-negative spectrum, we generalized the Hofman-Strominger theorem to the supersymmetric case. Specifically, we showed that a two-dimensional quantum field theory with two translational symmetries, a chiral scaling symmetry and a chiral supersymmetry may have enhanced local symmetry. The global symmetry could be enhanced to two kinds of minimal algebra. One consists of one copy of the Virasoro algebra and one copy of the super-Virasoro algebra, which leads to the $N = (1,0)$ SCFT_{2}. The other consists of one copy of the super-Virasoro algebra plus a $U(1)$ super-Kac-Moody algebra, which leads to the $N = 1$ supersymmetric warped conformal field theory.

We discussed some properties of the SWCFT_{2}, including the representations of the algebra, the space of the states and the transformations of the superfields. We furthermore calculated the two-point and three-point correlation functions of the SWCFT_{2} with the help of chiral superspace. The form of the correlation functions can be fixed without involving a specific model. Particularly, the vacuum of NS sector in SWCFT_{2} is $OSP(1|2) \times U(1)$ invariant such that the correlation functions must have the $OSP(1|2) \times U(1)$ structure, in which the $U(1)$ symmetry determined the dependence on $x^-$ completely.

One possible future direction is to generalize the minimal supersymmetry to the extended one. Our construction is based on the chiral superspace $(x^+, \theta)$. It is worthy of generalizing the study to the full superspace, including the Grassmannian partner of the $x^-$ coordinate. In the minimal CFT_{2} case, this may lead to the $N = (1,1)$ SCFT_{2}. But it is not clear of its consequence in the WCFT_{2} case. The study can be pushed to the case of $N \geq 2$ extended...
supersymmetry as well. Besides, it is interesting to study the supersymmetrization of the other 2D models with scaling symmetry. The supersymmetric GCA has been studied in [41], but for more general AGFT [5] its supersymmetric version has not been worked out.

It would be interesting to study the other properties of SWCFT$_2$: the modular properties of the torus partition function, the warped conformal bootstrap [6, 16], the entanglement entropy, etc.. It is also interesting to construct explicitly simple examples realizing the SWCFT$_2$. This may help us to understand the theory better.

It could be expected that for the holographic SWCFT$_2$, it is dual to a supersymmetric AdS$_3$ gravity under appropriate asymptotic boundary conditions. It would be nice to find the explicit boundary conditions and see how they break half of the supersymmetries.

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Appendix: conserved charges in the superspace

We start from the global symmetries of the theory. It is generated by the left-moving translation $H$, the dilation $D$, the right-moving translation $\overline{P}$ and the supersymmetries $Q_+$ and $Q_-$. By assumption these charges annihilate the vacuum. Their non-vanishing (anti)commutation relations are

$$i\{Q_+, Q_+\} = 2H, \quad i\{Q_+, Q_-\} = 2\overline{P}, \quad i\{Q_-, Q_-\} = 0,$$

$$i[D, H] = H, \quad i[D, P] = 0,$$

$$i[D, Q_+] = \frac{1}{2}Q_+, \quad i[D, Q_-] = -\frac{1}{2}Q_-.$$  (5.1)

The superspace is a coset space $G/I$, where $G$ is the whole symmetry group and $I$ is the dilation symmetry. A group element in $G$ may be written in the form

$$g_0 = e^{i(\delta H + \epsilon^+ Q_+ + \overline{\theta} Q + \epsilon^- Q_-)} e^{i\lambda D},$$  (5.4)

where $\delta, \overline{\delta}, \epsilon^+, \epsilon^-$ and $\lambda$ are some infinitesimal constants. The coset element can be written as

$$g_1 = e^{i(x^+ H + \theta^+ Q_+ + x^- \overline{P} + \theta^ - Q_-)}.$$  (5.5)

The transformations on the superspace are the natural action of the group $G$ on the coset space

$$g_0 g_1 = e^{i(x^+ H + \theta^+ Q_+ + x^- \overline{P} + \theta^ - Q_-)} e^{i\lambda D},$$  (5.6)

from which we read the induced transformations in the superspace

$$x^+ \to x^+ + \delta + \lambda x^+ - \epsilon^+ \theta^+,$$

$$\theta^+ \to \theta^+ + \epsilon^+ + \frac{\lambda}{2} \theta^+,$$

$$\theta^- \to \theta^- + \epsilon^- - \frac{\lambda}{2} \theta^-,$$

$$x^- \to x^- + \overline{\delta} - \epsilon^- \theta^+ - \epsilon^+ \theta^-.$$  (5.7)

Then we can obtain the differential representations of the global charges

$$H = -i\partial_+, \quad \overline{P} = -i\partial_-,$$

$$Q_+ = -i(\partial_{\theta^+} - \theta^+ \partial_+ - \theta^- \partial_-),$$

$$Q_- = -i(\partial_{\theta^-} - \theta^+ \partial_+ - \theta^- \partial_-),$$

$$D = -i(x^+ \partial_+ + \frac{\theta^+}{2} \partial_{\theta^+} - \frac{\theta^-}{2} \partial_{\theta^-}).$$  (5.11)
For each of the charges $H$, $D$, $\overline{P}$, $Q^+$ and $Q^-$, there is a conserved Noether current. In particular, with the supersymmetries, there exist corresponding supercurrents. In general the supercurrents may have the form

$$
\begin{align*}
\mathbb{O}_+(x^+, x^-, \theta^+, \theta^-) &= a_1 o_{+1} + a_2 \theta^+ o_{+2} + a_3 \theta^- o_{+3} + a_4 \theta^+ \theta^- o_{+4}, \\
\mathbb{O}_-(x^+, x^-, \theta^+, \theta^-) &= a_1 o_{-1} + a_2 \theta^+ o_{-2} + a_3 \theta^- o_{-3} + a_4 \theta^+ \theta^- o_{-4}, 
\end{align*}
$$

(5.12)

where $a_i$ ($i = 1, 2, 3, 4$) are constant numbers. The charges associated to the components of supercurrents can be read by

$$
O_i = -\frac{1}{2\pi} \int dx^+ o_{+i} + \frac{1}{2\pi} \int dx^- o_{-i}.
$$

(5.13)

The supersymmetric transformations of the supercurrents are

$$
\begin{align*}
i [\epsilon_1 Q_+, \mathbb{O}_+(x^+, x^-, \theta^+, \theta^-)] &= \epsilon_1 (\theta_+ - \theta^+ \theta_-) (a_1 o_{+1} + a_2 \theta^+ o_{+2} + a_3 \theta^- o_{+3} + a_4 \theta^+ \theta^- o_{+4}), \\
&= \epsilon_1 (a_2 o_{+2} - \theta^+ a_1 \theta_+ o_{+1} + \theta^- (a_4 o_{+4} - a_1 \theta_+ o_{+1}) + \theta^+ \theta^- (a_2 \theta_+ o_{+2} - a_3 \theta_+ o_{+3})), \\
&= \epsilon_1 (a_2 o_{+2} - a_1 \theta_+ o_{+1} + \theta^- a_4 o_{+4} + a_1 \theta_+ o_{+1} - a_3 \theta^+ \theta^- o_{+3}), \\
&= \epsilon_1 (a_2 o_{+2} - a_1 \theta_+ o_{+1} + \theta^- a_4 o_{+4} + a_1 \theta_+ o_{+1} - a_3 \theta^+ \theta^- o_{+3}).
\end{align*}
$$

(5.14)

$$
\begin{align*}
i [\epsilon_2 Q_-, \mathbb{O}_+(x^+, x^-, \theta^+, \theta^-)] &= \epsilon_2 (\theta_+ - \theta^+ \theta_-) (a_1 o_{-1} + a_2 \theta^+ o_{-2} + a_3 \theta^- o_{-3} + a_4 \theta^+ \theta^- o_{-4}), \\
&= \epsilon_2 (a_3 o_{-3} - \theta^+ a_1 \theta_- o_{-1} + \theta^- (a_4 o_{-4} - a_1 \theta_- o_{-1}) + \theta^+ \theta^- (a_2 \theta_- o_{-2} - a_3 \theta_- o_{-3})), \\
&= \epsilon_2 (a_3 o_{-3} - a_1 \theta_- o_{-1} + \theta^- a_4 o_{-4} + a_1 \theta_- o_{-1} - a_3 \theta^+ \theta^- o_{-3}),
\end{align*}
$$

(5.15)

where $\epsilon_i$ ($i = 1, 2$) are the Grassmannian constants. Then we have

$$
\begin{align*}
i [\epsilon_1 Q_+, a_1 o_{+1}] &= \epsilon_1 a_1 o_{+2}, \\
i [\epsilon_1 Q_+, a_2 o_{+2}] &= -\epsilon_1 a_1 \theta_+ o_{+1}, \\
i [\epsilon_1 Q_+, a_3 o_{+3}] &= \epsilon_1 (a_4 o_{+4} - a_1 \theta_+ o_{+1}), \\
i [\epsilon_1 Q_+, a_4 o_{+4}] &= \epsilon_1 (a_2 \theta_- o_{+2} - a_3 \theta_- o_{+3}),
\end{align*}
$$

(5.16)

and

$$
\begin{align*}
i [\epsilon_2 Q_-, a_1 o_{-1}] &= \epsilon_2 a_3 o_{-3}, \\
i [\epsilon_2 Q_-, a_2 o_{-2}] &= -\epsilon_2 (a_4 o_{-4} + a_1 \theta_- o_{-1}), \\
i [\epsilon_2 Q_-, a_3 o_{-3}] &= 0, \\
i [\epsilon_2 Q_-, a_4 o_{-4}] &= -\epsilon_2 a_3 \theta_+ o_{-3},
\end{align*}
$$

(5.17)

and similarly for $\mathbb{O}_-(x^+, x^-, \theta^+, \theta^-)$. After the integration, we get the transformations of the
charges associated to the components of supercurrents

\[\begin{align*}
  i[\epsilon_1 Q_+, a_1 O_1] &= \epsilon_1 a_2 O_2, \\
  i[\epsilon_1 Q_+, a_3 O_3] &= \epsilon_1 a_4 O_4, \\
  i[\epsilon_2 Q_-, a_1 O_1] &= \epsilon_2 a_3 O_3, \\
  i[\epsilon_2 Q_-, a_3 O_3] &= 0,
\end{align*}\]

(5.18)

\[\begin{align*}
  i[\epsilon_1 Q_+, a_2 O_2] &= 0, \\
  i[\epsilon_1 Q_+, a_4 O_4] &= 0, \\
  i[\epsilon_2 Q_-, a_2 O_2] &= -\epsilon_2 a_4 O_4, \\
  i[\epsilon_2 Q_-, a_4 O_4] &= 0.
\end{align*}\]

(5.19)

In the following, we will donate the currents associated to the charges \(H, P, Q_+\) and \(Q_-\) by \(h_\pm, p_\pm, q_\pm\) and \(q_\mp\), respectively. As it is not clear at this moment how these currents are related to each other by the supersymmetries, we first assume each of them belongs to a supercurrent donated by \(H_\pm, P_\pm, Q_\pm_+\) and \(Q_\pm_-\), then we will find out the relationship between the currents by their transformations under the supersymmetries. For \(Q_\pm_+\), in order to be consistent with (5.18) and (5.19) it must satisfy

\[\begin{align*}
  i[\epsilon_1 Q_+, a_1 Q_+ + 1] &= \epsilon_1 a_2 Q_2, \\
  i[\epsilon_1 Q_+, a_3 Q_+ + 2] &= 0, \\
  i[\epsilon_1 Q_+, a_4 Q_+ + 4] &= 0, \\
  i[\epsilon_2 Q_-, a_1 Q_+ + 1] &= \epsilon_2 a_3 Q_3, \\
  i[\epsilon_2 Q_-, a_3 Q_+ + 3] &= 0, \\
  i[\epsilon_2 Q_-, a_4 Q_+ + 4] &= 0.
\end{align*}\]

(5.20)

We have similar relations for \(H_\pm, P_\pm, Q_\pm_-\). From these relations, we find that the form of \(Q_+\) can only be

\[Q_{\pm} = a_1 q_{\pm} + a_2 \theta^\pm h_\pm + a_3 \theta^- p_\pm,\]

(5.22)

and the form of \(Q_-\) must be

\[Q_{\mp} = a_1 q_{\mp} + a_2 \theta^+ p_\pm,\]

(5.23)

with all none-zero coefficients \(a_i\), for \(i = 1, 2, 3\). We see that the \(P\) belongs to two different supermultiplets. On the other hand, the fact that the operator \(Q_-\) is nilpotent indicates that we may consider a smaller superspace. In fact we can regard \(Q_-\) as the superpartner of \(P\), and consider only one global supercharge \(Q_+\) in the theory. It turns out that the smaller superspace \(\{x^+, x^-, \theta^+\}\) is enough to describe our theories consistently.

The superspace \(\{x^+, x^-, \theta^+\}\) is the coset space \(G/\tilde{I}\), where \(G\) is the whole symmetry group and \(\tilde{I}\) consists of the dilation symmetry and \(Q_-\). In this smaller superspace, we have

\[a_3 = a_4 = 0, \quad a_2 = 2a_1,\]

(5.24)

26
and
\[ H_\pm = Q_{\pm}, \quad \mathcal{F}_\pm = Q_{-\pm}. \] (5.25)

We may choose \( a_1 = 1 \), and find that
\[ H_\pm = Q_{\pm} = h_{0\pm}(x^+, x^-) + 2\theta^+ h_{1\pm}(x^+, x^-), \] (5.26)
\[ \mathcal{F}_\pm = Q_{-\pm} = p_{0\pm}(x^+, x^-) + 2\theta^+ p_{1\pm}(x^+, x^-). \] (5.27)

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