PAIRS OF COMMUTING ISOMETRIES - I

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Abstract. We present a sharper version of Berger, Coburn and Lebow’s classification result for pure pairs of commuting isometries in the sense of an explicit recipe for constructing pairs of commuting isometric multipliers with precise coefficients. One of our main results states that: Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(H\), and let \(V = V_1 V_2\) be a shift (that is, a pure isometry). Let \(W_1, W_2\) and \(W\) be the wandering subspaces for \(V_1, V_2\), and \(V\), respectively, and let \(H^2_W(D)\) denote the \(W\)-valued Hardy space over the unit disc \(D\). Then \((V_1, V_2, V)\) on \(H\) and \((M_{\Phi_1}, M_{\Phi_2}, M_z)\) on \(H^2_W(D)\) are jointly unitarily equivalent, where

\[
\Phi_1(z) = V_1|_{W_2} + V_2^*|_{V_2 W_1} z = U^*(P_{W_2} + z(I_{W} - P_{W_2})),
\]

\[
\Phi_2(z) = V_2|_{W_1} + V_1^*|_{V_1 W_2} z = (I_{W} - P_{W_2} + zP_{W_2})U,
\]

for all \(z \in D\), and

\[
U = \begin{bmatrix} V_2|_{W_1} & 0 \\ 0 & V_1^*|_{V_1 W_2} \end{bmatrix} : \begin{bmatrix} W_1 \\ V_1 W_2 \end{bmatrix} \to \begin{bmatrix} V_2 W_1 \\ W_2 \end{bmatrix},
\]

is a unitary operator on \(W = W_1 \oplus V_1 W_2 = V_2 W_1 \oplus W_2\). As a consequence it follows that the pair \((V_1|_{W_2}, V_2^*|_{V_2 W_1})\) is a complete set of (joint) unitary invariants for pure pairs of commuting isometries. We also compare the above representation with other natural analytic representations of \((V_1, V_2)\). Finally, we study the defect operators of (not necessarily pure) pairs of commuting isometries. We prove that the defect operator of a pair of commuting isometries is negative if and only if the defect operator is the zero operator.

1. Introduction

A very general and fundamental question in the theory of bounded linear operators on Hilbert spaces, analytic function theory and operator algebras is the classification and representations of \(n\)-tuples, \(n \geq 1\), of commuting isometries on Hilbert spaces.

In the case of single isometries this question has a complete and explicit answer: If \(V\) is an isometry on a Hilbert space \(\mathcal{H}\) then there exist a Hilbert spaces \(\mathcal{H}_u\) and a unitary operator \(U\) on \(\mathcal{H}_u\) such that \(V\) on \(\mathcal{H}\) and \(\begin{bmatrix} S \otimes I_{\mathcal{W}} & 0 \\ 0 & U \end{bmatrix}\) on \((l^2(N) \otimes \mathcal{W}) \oplus \mathcal{H}_u\) are unitarily equivalent, where \(\mathcal{W} = \ker V^*\) is the wandering subspace for \(V\) and \(S\) is the forward shift operator on \(l^2(N)\) \([20]\). This fundamental result is due to J. von Neumann \([30]\) and H. Wold \([31]\) (see Theorem 2.1 for more details).

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The case of pairs (and $n$-tuples) of commuting isometries is more subtle, and is directly related to the commutant lifting theorem \[16\] (in terms of an explicit, and then unique solution), invariant subspace problem \[22\] and representations of contractions on Hilbert spaces in function Hilbert spaces \[23\]. For instance:

(a) Let $S$ be a closed joint $\langle M_{z_1}, M_{z_2} \rangle$-invariant subspace of the Hardy space $H^2(\mathbb{D}^2)$. Then $(M_{z_1}|_S, M_{z_2}|_S)$ on $S$ is a pure (see Section 3) pair of commuting isometries. Classification of such pairs of isometries is largely unknown (see Rudin \[27\]).

(b) Let $T$ be a contraction on a Hilbert space $\mathcal{H}$. Then there exists a pair of commuting isometries $(V_1, V_2)$ on a Hilbert space $\mathcal{K}$ such that $T$ and $P_{\ker V_2}V_1|_{\ker V_2}$ are unitarily equivalent (see Bercovici, Douglas and Foias \[9\]).

(c) The celebrated Ando dilation theorem (see Ando \[5\]) states that a commuting pair of contractions dilates to a commuting pair of isometries. Again, the structure of Ando’s pairs of commuting isometries is largely unknown (however, see Agler and McCarthy \[3\], and also \[14\]).

The main purpose of this paper is to explore and relate various natural representations of a large class of pairs of commuting isometries on Hilbert spaces. The geometry of Hilbert spaces, the classical Wold-von Neumann decomposition for isometries, and the analytic structure of the commutator of the unilateral shift are the main guiding principles for our study.

One of our main results, Theorem 3.4 (also see Theorem 3.5), is related to the earlier work of Berger, Coburn and Lebow \[10\] (and also Bercovici, Douglas and Foias \[9\]). The Berger, Coburn and Lebow theorem states that: Let $(V_1, V_2)$ be a pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let $V = V_1V_2$ be a shift (or, a pure isometry - see Section 2). Then there exist a Hilbert space $\mathcal{W}$, an orthogonal projection $P$ and a unitary operator $U$ on $\mathcal{W}$ such that

\[
\Phi_1(z) = U^* (P + zP^\perp), \quad \Phi_2(z) = (P^\perp + zP)U \quad (z \in \mathbb{D}),
\]

are commuting isometric multipliers in $H^\infty_{B(W)}(\mathbb{D})$, and $(V_1, V_2, V)$ on $\mathcal{H}$ and $(M_{\Phi_1}, M_{\Phi_2}, M_z)$ on $H^2_W(\mathbb{D})$ are unitarily equivalent. See Bercovici, Douglas and Foias \[9\] for an elegant proof.

An interesting observation is that the inner multipliers $\Phi_1$ and $\Phi_2$ are commuting factors of the (pure) inner multiplier $zI_{\mathcal{W}} \in H^\infty_{B(W)}(\mathbb{D})$ (cf. \[9\]):

\[
\Phi_1(z)\Phi_2(z) = \Phi_2(z)\Phi_1(z) = zI_{\mathcal{W}} \quad (z \in \mathbb{D}).
\]

Here and further on, given a Hilbert space $\mathcal{H}$ and a closed subspace $S$ of $\mathcal{H}$, $P_S$ denotes the orthogonal projection of $\mathcal{H}$ onto $S$. We also set

\[
P_S^\perp = I_\mathcal{H} - P_S.
\]

In this paper we give a new and more concrete treatment of this topic which includes explicit representations and analytic descriptions of pure pairs of isometries. More specifically, we provide an explicit recipe for constructing the isometric multipliers $(\Phi_1(z), \Phi_2(z))$, and the operators $U$ and $P$ involved in the coefficients of $\Phi_1$ and $\Phi_2$ (see Theorems 3.4 and 3.5). Then we compare the Berger, Coburn and Lebow representations with other possible analytic representations of pairs of commuting isometries.

In Section 6, which is independent of the remaining part of the paper, we analyze defect operators for (not necessarily pure) pairs of commuting isometries. Here our main motivation
is to study the class of pairs of commuting isometries with negative defect operators. This class was analyzed and characterized by He, Qin and Yang (see Theorem 3.5 in [21]) in the setting of pure pairs of commuting isometries. Here, on the contrary, in a more general setting than [21], by removing the assumption of pureness, in Theorem 6.6 we prove that the defect operator of a pair of commuting isometries is negative if and only if the defect operator is the zero operator. Therefore, the class of pairs of commuting isometries with negative defect operators, in a sense, is trivial. Along the way, we provide a list of characterizations of pairs of commuting isometries with positive defect operators (see Theorem 6.2). Our characterization results holds in a more general setting and somewhat simpler than the one by He, Qin and Yang [21].

The paper is organized as follows. In Section 2 we review the classical Wold-von Neumann theorem for isometries and then prove a representation theorem for commutators of shifts. In Section 3 we discuss some basic relationships between wandering subspaces for commuting isometries, followed by a new and explicit proof of the Berger, Coburn and Lebow characterizations of pure pairs of commuting isometries. Section 4 is devoted to a short discussion about joint unitary invariants of pure pairs of commuting isometries. Section 5 ties together the explicit Berger, Coburn and Lebow representation and other possible analytic representations of a pair of commuting isometries. Then, in Section 6, we present a general theory for pairs of commuting isometries and analyze the class of pairs with negative defect operators. Concluding remarks, future directions and a close connection of our consideration with the Sz.-Nagy and Foias characteristic functions for contractions are discussed in Section 7.

2. WOLD-VON NEUMANN DECOMPOSITION AND COMMUTATORS

We begin this section by briefly recalling the construction of the classical Wold-von Neumann decomposition of isometric operators on Hilbert spaces. Here our presentation is more algebraic and geared towards the main theme of the paper.

Recall that an isometry $V$ on a Hilbert space $\mathcal{H}$ is said to be pure, or a shift, if it has no unitary direct summand, or equivalently, if

$$\lim_{m \to \infty} V^{*m} = 0,$$

in the strong operator topology (see Halmos [20]).

Let $V$ be an isometry on a Hilbert space $\mathcal{H}$, and let $W(V)$ be the wandering subspace [20] for $V$, that is,

$$W(V) = \mathcal{H} \ominus V\mathcal{H}.$$ 

Set

$$\mathcal{H}_s(V) = \bigoplus_{m=0}^{\infty} V^m W(V).$$

An easy computation yields that $\mathcal{H}_s(V)$ is a $V$-reducing subspace of $\mathcal{H}$. Then if

$$\mathcal{H}_u(V) = \mathcal{H} \ominus \mathcal{H}_s(V),$$

it follows that (cf. [20], [23])

$$\mathcal{H}_u(V) = \bigcap_{m=0}^{\infty} V^m \mathcal{H}.$$
Moreover, with respect to the orthogonal decomposition
\[ \mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V), \]
it follows that
\[ V_s = V|_{\mathcal{H}_s(V)}, \]
is a shift operator and
\[ V_u = V|_{\mathcal{H}_u(V)}, \]
is a unitary operator.

We summarize the above as:

**Theorem 2.1. (Wold-von Neumann decomposition)** Let \( V \) be an isometry on a Hilbert space \( \mathcal{H} \). Then \( \mathcal{H} \) decomposes as a direct sum of \( V \)-reducing subspaces \( \mathcal{H}_s(V) \) and \( \mathcal{H}_u(V) \) and
\[ V = \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} \in \mathcal{B}(\mathcal{H}_s(V) \oplus \mathcal{H}_u(V)), \]
where \( V_s \) is a shift operator and \( V_u \) is a unitary operator.

We will refer to this decomposition as the **Wold-von Neumann orthogonal decomposition** of \( V \).

Recall that the **Hardy space** \( H^2(D) \) is the Hilbert space of all square summable holomorphic functions on the unit disc \( D \) (cf. [20], [26]). The Hardy space is also a reproducing kernel Hilbert space corresponding to the Szegö kernel
\[ S(z, w) = (1 - z\bar{w})^{-1} \quad (z, w \in D). \]

For any Hilbert space \( \mathcal{E} \), the \( \mathcal{E} \)-valued Hardy space with reproducing kernel
\[ \mathcal{D} \times \mathcal{D} \to \mathcal{B}(\mathcal{E}), \quad (z, w) \mapsto S(z, w)I_{\mathcal{E}}, \]
can canonically be identified with the tensor product Hilbert space \( H^2(D) \otimes \mathcal{E} \). To simplify the notation, we often identify \( H^2(D) \otimes \mathcal{E} \) with the \( \mathcal{E} \)-valued Hardy space \( H^2_{\mathcal{E}}(D) \). The space of \( \mathcal{B}(\mathcal{E}) \)-valued bounded holomorphic functions on \( D \) will be denoted by \( H^\infty_{\mathcal{B}(\mathcal{E})}(D) \).

Let \( M^\mathcal{E}_z \) denote the multiplication operator by the coordinate function \( z \) on \( H^2_{\mathcal{E}}(D) \), that is
\[ (M^\mathcal{E}_z f)(w) = wf(w) \quad (f \in H^2_{\mathcal{E}}(D), w \in \mathcal{D}). \]

Then \( M^\mathcal{E}_z \) is a shift operator and
\[ \mathcal{W}(M^\mathcal{E}_z) = \mathcal{E}. \]

Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two Hilbert spaces. Then it follows that \( M^\mathcal{E}_1 \) and \( M^\mathcal{E}_2 \) are unitarily equivalent if and only if the dimensions of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are the same, or equivalently, if there exists a unitary operator \( U : \mathcal{E}_1 \to \mathcal{E}_2 \). Subsequently we say that the dimension of \( \mathcal{E} \) is the **multiplicity** of \( M^\mathcal{E}_z \) [26].

To simplify the notation we often omit the superscript and denote \( M^\mathcal{E}_z \) by \( M_z \), if \( \mathcal{E} \) is clear from the context.

We now proceed to give an analytic description of the Wold-von Neumann construction.
Let $V$ be an isometry on $H$, and let $\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V)$ be the Wold-von Neumann orthogonal decomposition of $V$. Define
\[ \Pi_V : \mathcal{H}_s(V) \oplus \mathcal{H}_u(V) \to H^2_{\mathcal{W}(V)}(\mathbb{D}) \oplus \mathcal{H}_u \]
by
\[ \Pi_V(V^m \eta \oplus f) = z^m \eta \oplus f \quad (m \geq 0, f \in \mathcal{H}_u). \]
Then $\Pi_V$ is a unitary and $\Pi_V[V^s_0] = [M^W_{\mathcal{V}} z_0] \Pi_V$, that is, $V$ on $\mathcal{H}$ and $[M^W_{\mathcal{V}} z_0]$ on $H^2_{\mathcal{W}(V)}(\mathbb{D}) \oplus \mathcal{H}_u$ are unitarily equivalent. In particular, if $V$ is a shift, then $\mathcal{H}_u = \{0\}$ and hence
\[ \Pi_V V = M^W_{\mathcal{V}} \Pi_V. \]
Therefore, an isometry $V$ on $\mathcal{H}$ is a shift operator if and only if $V$ is unitarily equivalent to $M^E_z$ on $H^2_\mathcal{E}(\mathbb{D})$, where $\dim \mathcal{E} = \dim \mathcal{W}(V)$.

In the sequel we denote by $(\Pi_V, M^W_{\mathcal{V}})$, or simply by $(\Pi_V, M_z)$, the Wold-von Neumann decomposition of the pure isometry $V$ in the above sense.

Let $\mathcal{E}$ be a Hilbert space, and let $C$ be a bounded linear operator on $H^2_\mathcal{E}(\mathbb{D})$. Then $C \in \{ M_z \}'$, that is, $CM_z = M_z C$, if and only if (cf. [23])
\[ C = M_\Theta, \]
for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$, where $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ denotes the Banach space of bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions on $\mathbb{D}$, and
\[ (M_\Theta f)(w) = \Theta(w)f(w) \quad (f \in H^2_\mathcal{E}(\mathbb{D}), w \in \mathbb{D}). \]

Now let $V$ be a pure isometry, and let $C \in \{ V \}'$. Let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$, and let $\mathcal{W} = \mathcal{W}(V)$. Then $\Pi_V C \Pi^*_V$ on $H^2_{\mathcal{W}(V)}(\mathbb{D})$ is the representation of $C$ on $\mathcal{H}$ and
\[ (\Pi_V C \Pi^*_V) M_z = M_z (\Pi_V C \Pi^*_V). \]
Then
\[ \Pi_V C \Pi^*_V = M_\Theta, \]
for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. The main result of this section is the following explicit representation of $\Theta$.

**Theorem 2.1.** Let $V$ be a pure isometry on $\mathcal{H}$, and let $C$ be a bounded operator on $\mathcal{H}$. Let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$. Set $\mathcal{W} = \mathcal{W}(V)$ and assume that $M = \Pi_V C \Pi^*_V$. Then
\[ CV = VC, \]
if and only if
\[ M = M_\Theta, \]
where
\[ \Theta(z) = P_W(I_H - zV^*)^{-1}C \mid_W \quad (z \in \mathbb{D}). \]

**Proof.** Let \( h \in \mathcal{H} \). Since \( V \) is a pure isometry, \( \mathcal{H} = \bigoplus_{m=0}^{\infty} V^m \mathcal{W} \), and hence one can express \( h \) as
\[ h = \sum_{m=0}^{\infty} V^m \eta_m, \]
for some \( \eta_m \in \mathcal{W}, m \geq 0 \). Applying \( P_W V^*l \) to both sides and using the fact that \( W = W(V) = \ker V^* \), we obtain
\[ \eta_l = P_W V^*l \quad (l \geq 0). \]
This implies, for any \( h \in \mathcal{H} \),
\begin{equation} \label{2.1}
(2.1) \quad h = \sum_{m=0}^{\infty} V^m P_W V^*m h.
\end{equation}

Now let \( CV = VC \). Then there exists a bounded analytic function \( \Theta \in H_B(\mathcal{W})(\mathbb{D}) \) such that \( \Pi_V C \Pi_V^t = M_\Theta \). For each \( w \in \mathbb{D} \) and \( \eta \in \mathcal{W} \) we have
\[ \Theta(w)\eta = (M_\Theta \eta)(w) = (\Pi_V C \Pi_V^t \eta)(w) = (\Pi_V \eta)(w), \]
as \( \Pi_V^t \eta = \eta \). Since in view of (2.1)
\[ C\eta = \sum_{m=0}^{\infty} V^m P_W V^*m C\eta, \]
it follows that
\[ \Theta(w)\eta = (\Pi_V \sum_{m=0}^{\infty} V^m P_W V^*m C\eta)(w) \]
\[ = (\sum_{m=0}^{\infty} M_z^m (P_W V^*m C\eta))(w). \]
But
\[ P_W V^*m C\eta \in \mathcal{W}, \]
for all \( m \geq 0 \). Hence
\[ \Theta(w)\eta = \sum_{m=0}^{\infty} w^m (P_W V^*m C\eta) \]
\[ = P_W(I_H - wV^*)^{-1}C\eta. \]
Therefore
\[ \Theta(z) = P_W(I - zV^*)^{-1}C|_W \quad (z \in \mathbb{D}), \]
as required. Finally, since the sufficient part is trivial, the proof is complete.

Note that in the above proof we have used the standard projection formula (see, for example, Rosenblum and Rovnyak [26])

\[ I_H = \text{SOT} - \sum_{m=0}^{\infty} V^m P_W V^{*m}. \]

3. Berger, Coburn and Lebow representations

This section is devoted to a detailed study of Berger, Coburn and Lebow’s representation of pure pairs of commuting isometries. Our approach is different and yields sharper results, along with new proofs, in terms of explicit coefficients of the one variable polynomials associated with the class of pure pairs of commuting isometries. Before dealing more specifically with pure pairs of commuting isometries we begin with some general observations about pairs of commuting isometries.

Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(H\). In the sequel, we will adopt the following notations:

\[ W = W(V) = W(V_1V_2) = H \oplus V_1V_2H, \]

and

\[ W_j = W(V_j) = H \oplus V_jH \quad (j = 1, 2). \]

Also we will denote

\[ V = V_1V_2. \]

A pair of commuting isometries \((V_1, V_2)\) on \(H\) is said to be pure if \(V\) is a pure isometry.

The following useful lemma on wandering subspaces for commuting isometries is simple.

**Lemma 3.1.** Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(H\). Then

\[ W = W_1 \oplus V_1W_2 = V_2W_1 \oplus W_2, \]

and the operator \(U\) on \(W\) defined by

\[ U(\eta_1 \oplus V_1\eta_2) = V_2\eta_1 \oplus \eta_2, \]

for \(\eta_1 \in W_1\) and \(\eta_2 \in W_2\), is a unitary operator. Moreover,

\[ P_WV_i = V_iP_W \quad (i \neq j). \]

**Proof.** The first equality follows from

\[ I - VV^* = (I - V_1V_1^*) \oplus V_1(I - V_2V_2^*)V_1^* = V_2(I - V_1V_1^*)V_2^* \oplus (I - V_2V_2^*). \]

The second part directly follows from the first part, and the last claim follows from

\[ (I - VV^*)V_i = V_i(I - V_jV_j^*) \quad (i \neq j). \]

This concludes the proof of the lemma.

This immediately implies:
Corollary 3.2. Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(\mathcal{H}\). Then

\[ V_i^* W = W_j \quad (i \neq j). \]

The next simple result will turn out to be useful in what follows.

Corollary 3.3. Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(\mathcal{H}\). Then

\[ V_1^*|_W = V_1^*|_{V_1 W_2} = (V_1|_{V_2})^*, \]

and

\[ V_2^*|_W = V_2^*|_{V_2 W_1} = (V_2|_{V_1})^*. \]

Moreover, \(V_i^*|_W \in \mathcal{B}(W, W_j)\) for \(i \neq j\).

Proof. The result follows from Lemma 3.1, the fact that

\[ V_j^* W_j = \{0\} \quad (j = 1, 2), \]

and Corollary 3.2. \(\blacksquare\)

Let \((V_1, V_2)\) be a pure pair of commuting isometries on a Hilbert space \(\mathcal{H}\), and let \((\Pi_V, M_z)\) be the Wold-von Neumann decomposition of \(V\). Since

\[ VV_i = V_i V \quad (i = 1, 2), \]

there exist isometric multipliers (that is, inner functions \([23]\)) \(\Phi_1\) and \(\Phi_2\) in \(H^\infty_{\mathcal{B}(W)}(\mathbb{D})\) such that

\[ \Pi_V V_i = M_{\Phi_i} \Pi_V \quad (i = 1, 2). \]

In other words, \((M_{\Phi_1}, M_{\Phi_2})\) on \(\mathcal{H}_W^2(\mathbb{D})\) is the representation of \((V_1, V_2)\) on \(\mathcal{H}\). Following Berger, Coburn and Lebow \([10]\), we say that \((M_{\Phi_1}, M_{\Phi_2})\) is the BCL representation of \((V_1, V_2)\), or simply the BCL pair corresponding to \((V_1, V_2)\).

We now present an explicit description of the BCL pair \((M_{\Phi_1}, M_{\Phi_2})\).

Theorem 3.4. Let \((V_1, V_2)\) be a pure pair of commuting isometries on a Hilbert space \(\mathcal{H}\), and let \((M_{\Phi_1}, M_{\Phi_2})\) be the BCL representation of \((V_1, V_2)\). Then

\[ \Phi_1(z) = V_1|_{W_2} + V_2^*|_{V_2 W_1} z, \]

\[ \Phi_2(z) = V_2|_{W_1} + V_1^*|_{V_1 W_2} z, \]

for all \(z \in \mathbb{D}\), and

\[ (M_{\Phi_1}, M_{\Phi_2}) = (V_1|_{W_2} + V_2^*|_{V_2 W_1} M_z, V_2|_{W_1} + V_1^*|_{V_1 W_2} M_z). \]

Proof. Let \(\eta\) in \(W = V_2 W_1 \oplus W_2\), and let \(w \in \mathbb{D}\). Then there exist \(\eta_1 \in W_1\) and \(\eta_2 \in W_2\) such that

\[ \eta = V_2 \eta_1 \oplus \eta_2. \]

Hence

\[ V_1 \eta = V \eta_1 + V_1 \eta_2. \]
This, along with the fact that $V_1\eta_2 \in W$ (see Lemma 3.1), shows that
\[
\Phi_1(w)\eta = (M_{\Phi_1}\eta)(w)
= (\Pi_V V_1 \Pi^*_V \eta)(w)
= (\Pi_V V_1 \eta)(w)
= (\Pi_V V_1 \eta_1 + \Pi_V V_1 \eta_2)(w)
= (M_2 \Pi_V \eta_1 + V_1 \eta_2)(w)
= w\eta_1 + V_1 \eta_2,
\]
as $\Pi_V V = M_2 \Pi_V$ and $\Pi_V \eta_1 = \eta_1$. Moreover, as
\[
\eta_1 = V_2^*|_W \eta,
\]
and so
\[
V_1 \eta_2 = V_1 \eta - V \eta_1
= V_1 (I - V V_2^*)|_W \eta
= V_1|_W \eta,
\]
it follows that
\[
\Phi_1(z) = z V_2^*|_W + V_1|_W z \quad (z \in \mathbb{D}).
\]
Now Corollary 3.3 gives
\[
\Phi_1(z) = V_1|_W z + V_2^*|_W z \quad (z \in \mathbb{D}),
\]
and so
\[
M_{\Phi_1} = V_1|_W + V_2^*|_W M_z,
\]
as desired. Finally, using the equality
\[
M_{\Phi_2} = M_{\Phi_1}^* M_z,
\]
we obtain
\[
M_{\Phi_2} = (V_1|_W + V_2^*|_W M_z)^* M_z
= ((V_1|_W)^* + (V_2^*|_W M_z)^* M_z) M_z
= (V_2^*|_W)^* + (V_1|_W)^* M_z,
\]
and hence once again by Corollary 3.3
\[
M_{\Phi_2} = V_2|_W + V_1^*|_W M_z.
\]
Therefore
\[
\Phi_2(z) = V_2|_W z + V_1^*|_W M_z \quad (z \in \mathbb{D}).
\]
This completes the proof of the theorem.
In the following, we present Berger, Coburn and Lebow’s version of representations of pure pairs of commuting isometries. Along with a new proof, our approach yields an explicit representations of the auxiliary operators $U$ and $P$ (see Section 1).

**Theorem 3.5.** Let $(V_1, V_2)$ be a pure pair of commuting isometries on $\mathcal{H}$. Then the BCL pair $(M_{\Phi_1}, M_{\Phi_2})$ corresponding to $(V_1, V_2)$ is given by

$$
\Phi_1(z) = U^*(P_{W_2} + zP_{W_2}^\perp),
$$

and

$$
\Phi_2(z) = (P_{W_2}^\perp + zP_{W_2})U,
$$

where

$$
U = \begin{bmatrix} V_2|_{W_1} & 0 \\ 0 & V_1^*|_{V_1W_2} \end{bmatrix} : W_1 \oplus V_2W_1 \to V_1W_2 \oplus W_2,
$$

is a unitary operator on $W$.

**Proof.** According to Lemma 3.1, the linear map $U : W_1 \oplus V_1W_2 \to V_2W_1 \oplus W_2$ defined by

$$
U(\eta_1 \oplus V_1\eta_2) = V_2\eta_1 \oplus \eta_2 \quad (\eta_1 \in W_1, \eta_2 \in W_2),
$$

is a unitary operator. It follows that the block matrix representation of $U$ is given by

$$
U = \begin{bmatrix} V_2|_{W_1} & 0 \\ 0 & V_1^*|_{V_1W_2} \end{bmatrix} : W_1 \oplus V_2W_1 \to V_1W_2 \oplus W_2,
$$

and so by Corollary 3.3, we have

$$
U^* = \begin{bmatrix} V_2^*|_{V_2W_1} & 0 \\ 0 & V_1^*|_{V_1W_2} \end{bmatrix} : W_2 \oplus V_1W_2 \to W_1 \oplus V_1W_2.
$$

Let $(M_{\Phi_1}, M_{\Phi_2})$ be the BCL pair corresponding to $(V_1, V_2)$ as in Theorem 3.4, that is

$$
\Phi_1(z) = V_1|_{W_2} + V_2^*|_{V_2W_1}z, \quad \text{and} \quad \Phi_2(z) = V_2|_{W_1} + V_1^*|_{V_1W_2}z,
$$

for $z \in \mathbb{D}$. Then

$$
\Phi_1(z) = U^* \begin{bmatrix} zI_{V_2W_1} & 0 \\ 0 & I_{W_2} \end{bmatrix} : V_2W_1 \oplus W_2 \to V_1W_2 \oplus W_1,
$$

and

$$
\Phi_2(z) = \begin{bmatrix} I_{V_2W_1} & 0 \\ 0 & zI_{W_2} \end{bmatrix} U : W_2 \oplus V_1W_2 \to W_1 \oplus V_2W_1.
$$

Hence

$$
\Phi_1(z) = U^*(P_{W_2} + zP_{W_2}^\perp),
$$

and

$$
\Phi_2(z) = (P_{W_2}^\perp + zP_{W_2})U,
$$

for all $z \in \mathbb{D}$. This completes the proof. \(\blacksquare\)
Therefore, \((V_1, V_2, V_1V_2)\) on \(\mathcal{H}\) and \((M_{\Phi_1}, M_{\Phi_2}, M_z^W)\) on \(H^2_{\mathcal{W}}(\mathbb{D})\) are unitarily equivalent, where \(\mathcal{W}\) is the wandering subspace for \(V = V_1V_2\).

4. Unitary invariants

In this short section we present a complete set of unitary invariants for pure pairs of commuting isometries.

Recall that two commuting pairs \((T_1, T_2)\) and \((\tilde{T}_1, \tilde{T}_2)\) on \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\), respectively, are said to be (jointly) unitarily equivalent if there exists a unitary operator \(U : \mathcal{H} \to \tilde{\mathcal{H}}\) such that

\[
UT_j = \tilde{T}_jU \quad (j = 1, 2).
\]

First we note that, by virtue of Theorem 2.9 of [9], the orthogonal projection \(P_{\mathcal{W}_2}\) and the unitary operator \(U\) on \(\mathcal{W}\), as in Theorem 3.5 is a complete set of (joint) unitary invariants of pure pairs of commuting isometries. More specifically: Let \((\tilde{V}_1, \tilde{V}_2)\) on \(\mathcal{H}\), respectively, be two pure pairs of commuting isometries on \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\), respectively. Let \(\mathcal{W}_j\) be the wandering subspace for \(\tilde{V}_j\), \(j = 1, 2\). Then \((V_1, V_2)\) and \((\tilde{V}_1, \tilde{V}_2)\) are unitarily equivalent if and only if

\[
\begin{bmatrix}
V_2|_{\mathcal{W}_1} & 0 \\
0 & V_1^*|_{\mathcal{W}_2}
\end{bmatrix}, P_{\mathcal{W}_2}\quad \text{and} \quad \begin{bmatrix}
\tilde{V}_2|_{\tilde{\mathcal{W}}_1} & 0 \\
0 & \tilde{V}_1^*|_{\tilde{\mathcal{W}}_2}
\end{bmatrix}, P_{\tilde{\mathcal{W}}_2}
\]

are unitarily equivalent.

In addition to the above, the following unitary invariants are also explicit. The proof is an easy consequence of Theorem 3.4. Here we will make use of the identifications of \(\tilde{\mathcal{H}}\) with \(H^2_{\mathcal{W}}(\mathbb{D})\) and \(AM_z\) on \(H^2_{\mathcal{W}}(\mathbb{D})\) with \(I_{H^2(\mathbb{D})} \otimes A\) on \(H^2(\mathbb{D}) \otimes \mathcal{W}\) and \(M_z \otimes A\) on \(H^2(\mathbb{D}) \otimes \mathcal{W}\), respectively, where \(A \in \mathcal{B}(\mathcal{W})\) (see Section 2).

**Theorem 4.1.** Let \((V_1, V_2)\) and \((\tilde{V}_1, \tilde{V}_2)\) be two pure pairs of commuting isometries on \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\), respectively. Then \((V_1, V_2)\) and \((\tilde{V}_1, \tilde{V}_2)\) are unitarily equivalent if and only if \((V_1|_{\mathcal{W}_2}, V_2^*|_{V_2\mathcal{W}_1})\) and \((\tilde{V}_1|_{\tilde{\mathcal{W}}_2}, \tilde{V}_2^*|_{\tilde{\mathcal{W}}_2\tilde{\mathcal{V}}_1})\) are unitarily equivalent.

**Proof.** Let \((M_{\Phi_1}, M_{\Phi_2})\) and \((M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})\) be the BCL pairs corresponding to \((V_1, V_2)\) and \((\tilde{V}_1, \tilde{V}_2)\), respectively, as in Theorem 3.4. Let \(C_1 = V_1|_{\mathcal{W}_2}\) and \(C_2 = V_2^*|_{V_2\mathcal{W}_1}\) be the coefficients of \(\Phi_1\). Similarly, let \(\tilde{C}_1\) and \(\tilde{C}_2\) be the coefficients of \(\tilde{\Phi}_1\).

Now let \(Z : \mathcal{W} \to \tilde{\mathcal{W}}\) be a unitary such that \(ZC_j = \tilde{C}_jZ, j = 1, 2\). Then

\[
M_{\Phi_1} = I_{H^2(\mathbb{D})} \otimes C_1 + M_z \otimes C_2 \\
= I_{H^2(\mathbb{D})} \otimes Z^*\tilde{C}_1Z + M_z \otimes Z^*\tilde{C}_2Z \\
= (I_{H^2(\mathbb{D})} \otimes Z^*)(I_{H^2(\mathbb{D})} \otimes \tilde{C}_1 + M_z \otimes \tilde{C}_2)(I_{H^2(\mathbb{D})} \otimes Z) \\
= (I_{H^2(\mathbb{D})} \otimes Z^*)M_{\Phi_1}(I_{H^2(\mathbb{D})} \otimes Z).
\]

Because \(M_{\Phi_2} = M_zM_{\Phi_1}^*\) and \(M_{\tilde{\Phi}_2} = M_zM_{\tilde{\Phi}_1}^*\), it follows that \((M_{\Phi_1}, M_{\Phi_2})\) and \((M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})\) are unitarily equivalent, that is, \((V_1, V_2)\) and \((\tilde{V}_1, \tilde{V}_2)\) are unitarily equivalent.

To prove the necessary part, let \((M_{\Phi_1}, M_{\Phi_2})\) and \((M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})\) are unitarily equivalent. Then there exists a unitary operator \(X : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\tilde{\mathcal{W}}}(\mathbb{D})\) such that

\[
XM_{\Phi_j} = M_{\tilde{\Phi}_j}X \quad (j = 1, 2).
\]
Since
\[ XM^W_2 = XM_{\Phi_1} M_{\Phi_2} \]
\[ = M_{\Phi_1} XX^* M_{\Phi_2} X \]
\[ = M_{\Phi_1} M_{\Phi_2} X \]
\[ = M^W_2 X, \]
there exists a unitary operator \( Z : W \to \tilde{W} \) such that
\[ X = I_{H^2(D)} \otimes Z. \]
This and \( XM_{\Phi_1} = M_{\Phi_1} X \) implies that
\[ (I_{H^2(D)} \otimes Z)(I_{H^2(D)} \otimes C_1 + M_2 \otimes C_2) = (I_{H^2(D)} \otimes \tilde{C}_1 + M_2 \otimes \tilde{C}_2)(I_{H^2(D)} \otimes Z). \]
Hence \((C_1, C_2)\) and \((\tilde{C}_1, \tilde{C}_2)\) are unitarily equivalent. This completes the proof of the theorem.

Observe that the set of joint unitary invariants \( \{V_1|W_1, V_2^*|W_2\} \), as above, is associated with the coefficients of \( \Phi_1 \) of the BCL pair \((M_{\Phi_1}, M_{\Phi_2})\) corresponding to \((V_1, V_2)\). Clearly, by duality, a similar statement holds for the coefficients of \( \Phi_2 \) as well: \( \{V_2|W_1, V_1^*|W_2\} \) is a complete set of joint unitary invariants for pure pairs of commuting isometries.

5. Pure isometries

In this section we will analyze pairs of commuting isometries \((V_1, V_2)\) such that either \(V_1\) or \(V_2\) is a pure isometry, or both \(V_1\) and \(V_2\) are pure isometries. We begin with a concrete example which illustrate this particular class and also exhibit its complex structure.

Recall that the Hardy space \( H^2(D^2) \) over the bidisc \( D^2 \) is the Hilbert space of all square summable holomorphic functions on \( D^2 \) (see Rudin [27]). Let \( M_{z_j} \) on \( H^2(D^2) \) be the multiplication operator by the coordinate function \( z_j, \ j = 1, 2 \). Note that \((M_{z_1}, M_{z_2})\) on \( H^2(D^2) \) can be identified with \((M_2 \otimes I_{H^2(D)}, I_{H^2(D)} \otimes M_2)\) on \( H^2(D) \otimes H^2(D) \), and consequently, \((M_{z_1}, M_{z_2})\) on \( H^2(D^2) \) is a pair of doubly commuting (that is, \( M_{z_1}^* M_{z_2} = M_{z_2} M_{z_1}^* \)) pure isometries.

Now let \( S \) be a joint \((M_{z_1}, M_{z_2})\)-invariant closed subspace of \( H^2(D^2) \), that is
\[ M_{z_j} S \subseteq S. \]
Set
\[ V_j = M_{z_j}|_S \quad (j = 1, 2). \]
It follows immediately that \( V_j \) is a pure isometry and
\[ V_1 V_2 = V_2 V_1, \]
and hence \((V_1, V_2)\) is a pair of commuting pure isometries on \( S \).

If we assume, in addition, that \((V_1, V_2)\) is doubly commuting (that is, \( V_1^* V_2 = V_2 V_1^* \)), then it follows that \((V_1, V_2)\) on \( S \) and \((M_{z_1}, M_{z_2})\) on \( H^2(D^2) \) are unitarily equivalent. See Slocinski [28] for more details. In general, however, the classification of pairs of commuting isometries, up to unitary equivalence, is complicated and very little seems to be known. For instance, see Rudin [27] for a list of pathological examples (also see Qin and Yang [23]).
We now turn our attention to the general problem. Let \((V_1, V_2)\) be a pair of commuting isometries on \(H\), and let \(V_1\) be a pure isometry. Then, in particular, \(V = V_1V_2\) is a pure isometry, and hence \((V_1, V_2)\) is a pure pair of commuting isometries. Since 

\[ V_1V_2 = V_2V_1, \]

by Theorem 2.1 it follows that

\[ \Pi_{V_1}V_2 = M_{\Theta V_2} \Pi_{V_1}, \]

where \(\Theta V_2 \in H^\infty_{E(W_1)}(\mathbb{D})\) is an inner multiplier and

\[ \Theta V_2(z) = P_{W_1}(I_H - zV_1^*)^{-1}V_2|_{W_1} \quad (z \in \mathbb{D}). \]

Let \((M_{\Phi_1}, M_{\Phi_2})\) be the BCL pair (see Theorem 3.5) corresponding to \((V_1, V_2)\), that is

\[ \Pi_{V_i}V_i = M_{\Phi_i} \Pi_V \quad (i = 1, 2). \]

Set

\[ \tilde{\Pi}_1 = \Pi_{V_1} \Pi_V^*. \]

Then \(\tilde{\Pi}_1 : H^2_{W_1}(\mathbb{D}) \to H^2_{W_1}(\mathbb{D})\) is a unitary operator. Moreover, we have

\[ \tilde{\Pi}_1 M_{\Phi_1} = \Pi_{V_1} V_1 \Pi_V^* = M_{z}^{W_1} \Pi_{V_1} \Pi_V^* = M_{z}^{W_1} \tilde{\Pi}_1, \]

and by \((5.1)\)

\[ \tilde{\Pi}_1 M_{\Phi_2} = \Pi_{V_1} V_2 \Pi_V^* = M_{\Theta V_2} \Pi_{V_1} \Pi_V^* = M_{\Theta V_2} \tilde{\Pi}_1. \]

Therefore, we have the following commutative diagram:

\[
\begin{array}{ccc}
H & \xrightarrow{\Pi_V} & H^2_{W_1}(\mathbb{D}) \\
\downarrow{\Pi_{V_1}} & & \downarrow{\tilde{\Pi}_1} \\
H^2_{W_1}(\mathbb{D}) & & 
\end{array}
\]

where \((M_{\Phi_1}, M_{\Phi_2})\) on \(H^2_{W_1}(\mathbb{D})\) and \((M_{z}^{W_1}, M_{\Theta V_2})\) on \(H^2_{W_1}(\mathbb{D})\) are the representations of \((V_1, V_2)\) on \(H\).

We now proceed to settle the non-trivial part of this consideration: An analytic description of the unitary map \(\tilde{\Pi}_1\). To this end, observe first that since \(\Pi_{V_1} V_1 = M_{z}^{W_1} \Pi_{V_1}\), \((5.1)\) gives

\[ \Pi_{V_1} V = M_{z}^{W_1} M_{\Theta V_2} \Pi_{V_1}. \]

Then

\[ \tilde{\Pi}_1 M_{z}^{W_1} = \Pi_{V_1} V \Pi_V^* = M_{z}^{W_1} M_{\Theta V_2} \Pi_{V_1} \Pi_V^*. \]
that is,
\[ (5.3) \quad \bar{\Pi}_1 M_z^W = (M_z^{W_1} M_{\Theta V_2}) \bar{\Pi}_1. \]

Now we let \( \eta_1 \in W_1 \) and \( \eta_2 \in W_2 \), and set
\[ \eta = \eta_1 \oplus V_1 \eta_2 \in W_1 \oplus V_1 W_2 = W. \]

Since \( V_2 \) is a pure isometry, by Equation (2.1) it follows that
\[ \eta_2 = \sum_{m=0}^{\infty} V_1^m P_{W_1} V_1^{*m} \eta_2. \]

Then \( P_{W_1} V_1^{*m} \eta_2 \in W_1, \ m \geq 0, \) and
\[ \Pi_{V_1} \zeta_1 = \zeta_1 \quad (\zeta_1 \in W_1), \]
implies that
\[ (\Pi_{V_1} \eta_2)(w) = \left( \sum_{m=0}^{\infty} \Pi_{V_1} V_1^m P_{W_1} V_1^{*m} \eta_2 \right)(w) \]
\[ = \left( \sum_{m=0}^{\infty} M_z^{m} P_{W_1} V_1^{*m} \eta_2 \right)(w) \]
\[ = \sum_{m=0}^{\infty} w^m (P_{W_1} V_1^{*m} \eta_2), \]
which yields
\[ (5.4) \quad \Pi_{V_1} \eta_2 = P_{W_1} (I - zV_1^*)^{-1} \eta_2. \]

Then
\[ \bar{\Pi}_1 \eta = \Pi_{V_1} \Pi_{V_1}^{*} \eta \]
\[ = \Pi_{V_1} \eta \]
\[ = \Pi_{V_1} \eta_1 + \Pi_{V_1} \eta_2 \]
\[ = \eta_1 + M_z^{V_1} \Pi_{V_1} \eta_2 \]
\[ = \eta_1 + z P_{W_1} (I - zV_1^*)^{-1} \eta_2. \]

Since
\[ \eta_2 = V_1^{*} \eta, \]
and
\[ \eta_1 = (I - V_1 V_1^{*}) \eta \]
\[ = P_{W_1} \eta, \]
we have
\[ \bar{\Pi}_1 \eta = P_{W_1} [I + z(I - zV_1^*)^{-1} V_1^{*}] \eta, \]
for all \( \eta \in W \). It now follows from \( (5.3) \) that
\[ \bar{\Pi}_1 (z^m \eta) = (z \Theta V_2(z))^m P_{W_1} [I + z(I - zV_1^*)^{-1} V_1^{*}] \eta, \]
for all $m \geq 0$, and so, by
\[ S(\cdot, w)\eta = \sum_{m=0}^{\infty} z^m \bar{w}^m \eta, \]
it follows that
\[ \Pi_1(S(\cdot, w)\eta) = \Pi_1 \left( \sum_{m=0}^{\infty} z^m \bar{w}^m \eta \right) \]
\[ = (I_{W_1} - \bar{w}z\Theta_{V_2}(z))^{-1} P_{W_1} [I_{H} + z(I_{H} - zV_1^*)^{-1}V_2]\eta, \]
for all $w \in \mathbb{D}$ and $\eta \in W$. Finally, from
\[ \Pi_1^* M_z^{W_1} = M_{\Phi_1} \Pi_1^*, \]
and
\[ \Pi_1^* \eta_1 = \eta_1 \quad (\eta_1 \in W), \]
it follows that
\[ \Pi_1^* (z^m \eta_1) = M_{\Phi_1}^m \eta_1 \quad (m \in \mathbb{N}), \]
and hence
\[ \Pi_1^*(S(\cdot, w)\eta_1) = (I_{W} - \Phi_1(z)\bar{w})^{-1} \eta_1, \]
for all $w \in \mathbb{D}$ and $\eta_1 \in W_1$.

We summarize the above observations in the following theorem.

**Theorem 5.1.** Let $(V_1, V_2)$ be a pair of commuting isometries on $H$.

(i) If $V_1$ is a pure isometry, then
\[ \Pi_1 = \Pi_{V_1} \Pi_{V}^* \in \mathcal{B}(H_{W_1}^2(\mathbb{D}), H_{W_1}^2(\mathbb{D})), \]
is a unitary operator,
\[ \Pi_1 M_z^W = M_z \Theta_{V_2} \Pi_1, \]
\[ \Pi_1^* M_z^{W_1} = M_{\Phi_1} \Pi_1^*, \]
and
\[ \Pi_1(S(\cdot, w)\eta) = (I_{W_1} - \bar{w}z\Theta_{V_2}(z))^{-1} P_{W_1} [I_{H} + z(I_{H} - zV_1^*)^{-1}V_2]\eta, \]
for all $w \in \mathbb{D}$ and $\eta \in W$, where
\[ \Theta_{V_2}(z) = P_{W_1} (I_{H} - zV_1^*)^{-1}V_2|_{W_1} \]
for all $z \in \mathbb{D}$. Moreover
\[ \Pi_1^*(S(\cdot, w)\eta_1) = (I_{W} - \Phi_1(z)\bar{w})^{-1} \eta_1 \quad (w \in \mathbb{D}, \eta_1 \in W_1). \]

(ii) If $V_2$ is a pure isometry, then
\[ \Pi_2 = \Pi_{V_2} \Pi_{V}^* \in \mathcal{B}(H_{W_1}^2(\mathbb{D}), H_{W_2}^2(\mathbb{D})), \]
is a unitary operator,
\[ \Pi_2 M_z^W = M_z \Theta_{V_2} \Pi_2, \]
\[ \Pi_2^* M_z^{W_2} = M_{\Phi_2} \Pi_2^*, \]
and
\[ \tilde{\Pi}_2(S(\cdot, w)\eta) = (I_{W_2} - \bar{w}\Theta_{V_1}(z))^{-1}P_{W_2}[I + z(I_{H} - zV_2^*)^{-1}V_2^*]\eta, \]
for all \( w \in \mathbb{D} \) and \( \eta \in \mathcal{W} \), where
\[ \Theta_{V_1}(z) = P_{W_2}(I - zV_1^*)^{-1}V_1|_{W_2}, \]
for all \( z \in \mathbb{D} \). Moreover
\[ \tilde{\Pi}_2^*(S(\cdot, w)\eta_2) = (I_W - \Phi_2(z)\bar{w})^{-1}\eta_2 \quad (w \in \mathbb{D}, \eta_2 \in \mathcal{W}_2). \]

Note that the inner multipliers \( \Theta_{V_1} \in H^\infty_{B(W_2)}(\mathbb{D}) \) and \( \Theta_{V_2} \in H^\infty_{B(W_1)}(\mathbb{D}) \) above satisfies the following equalities:
\[ \Pi_{V_2}V_1 = M_{\Theta_{V_1}}\Pi_{V_2}, \]
and
\[ \Pi_{V_1}V_2 = M_{\Theta_{V_2}}\Pi_{V_1}. \]

Now let \((V_1, V_2)\) be a pair of commuting isometries such that both \( V_1 \) and \( V_2 \) are pure isometries. The above result leads to an analytic representation of such pair.

**Corollary 5.2.** Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \( H \), and let both \( V_1 \) and \( V_2 \) are pure isometries. If \((M_{\Phi_1}, M_{\Phi_2})\) is the BCL representation corresponding to \((V_1, V_2)\), then \( M_{\Phi_1} \) and \( M_{\Phi_2} \) are pure isometries,
\[ \tilde{\Pi}_1M_{\Phi_2} = M_{\Theta_{V_2}}\tilde{\Pi}_1, \]
\[ \tilde{\Pi}_2M_{\Phi_1} = M_{\Theta_{V_1}}\tilde{\Pi}_2, \]
\[ \tilde{\Pi} = \tilde{\Pi}_2\tilde{\Pi}_1^*: H^2_{W_1}(\mathbb{D}) \rightarrow H^2_{W_2}(\mathbb{D}) \text{ is a unitary operator}, \]
\[ \tilde{\Pi}M_{W_1}^{z_{W_1}} = M_{\Theta_{V_1}}\tilde{\Pi}, \]
and
\[ \tilde{\Pi}M_{W_2}^{z_{W_2}} = M_{\Theta_{V_2}}\tilde{\Pi}. \]
Moreover, for each \( w \in \mathbb{D} \) and \( \eta_j \in \mathcal{W}_j \), \( j = 1, 2, \)
\[ \tilde{\Pi}(S(\cdot, w)\eta_j) = (I_{W_2} - \bar{w}\Theta_{V_1}(z))^{-1}P_{W_2}[I + z(I_{H} - zV_2^*)^{-1}V_2^*]\eta_j, \]
and
\[ \tilde{\Pi}^*(S(\cdot, w)\eta_2) = (I_{W_1} - \bar{w}\Theta_{V_2}(z))^{-1}P_{W_1}[I + z(I_{H} - zV_1^*)^{-1}V_1^*]\eta_2. \]

**Proof.** A repeated application of Theorem 5.1 yields
\[ \tilde{\Pi}_1M_{\Phi_2} = \tilde{\Pi}_1M_{\Phi_2}^*(M_{\Phi_1}M_{\Phi_2}) \]
\[ = \tilde{\Pi}_1M_{\Phi_1}M^{z_{W_2}} \]
\[ = (M^{z_{W_1}})\tilde{\Pi}_1M^{z_{W_2}} \]
\[ = (M^{z_{W_1}})^{*}M_{\Theta_{V_2}}\tilde{\Pi}_1, \]
that is
\[ \tilde{\Pi}_1M_{\Phi_2} = M_{\Theta_{V_2}}\tilde{\Pi}_1, \]
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and similarly
\[ \tilde{\Pi}_2 M \Phi_1 = M_{\Theta_{V_1}} \tilde{\Pi}_2. \]

For \( \eta_1 \in \mathcal{W}_1 \), following (5.4), we have
\[ \Pi_{V_2} \eta_1 = P_{W_2} (I - z V_2^*)^{-1} \eta_1. \]

Since \( \tilde{\Pi}_1^* \eta_1 = \eta_1 \) and \( \Pi_{V_1}^* \eta_1 = \eta_1 \), it follows that
\[ \tilde{\Pi} \eta_1 = \tilde{\Pi}_2 \eta_1 = \Pi_{V_2} \Pi_{V_1}^* \eta_1 = \Pi_{V_2} \eta_1, \]
that is
\[ \tilde{\Pi} \eta_1 = P_{W_2} (I - z V_2^*)^{-1} \eta_1. \]

Now using the identity
\[ \tilde{\Pi} (z \eta_1) = M_{\Theta_{V_1}} \tilde{\Pi} \eta_1, \]
we have
\[ \tilde{\Pi} (z^m \eta_1) = \Theta_{V_1} (z)^m P_{W_2} (I - z V_2^*)^{-1} \eta_1, \]
for all \( m \geq 0 \) and \( \eta_1 \in \mathcal{W}_1 \). Finally
\[ S(\cdot, w) \eta_1 = \sum_{m=0}^{\infty} \bar{w}^m z^m \eta_1, \]
gives
\[ \tilde{\Pi} (S(\cdot, w) \eta_1) = (I_{W_2} - \bar{w} \Theta_{V_1} (z))^{-1} P_{W_2} (I - z V_2^*)^{-1} \eta_1. \]

The final equality of the corollary holds because
\[ \tilde{\Pi} \eta_2 = P_{W_1} (I - z V_1^*)^{-1} \eta_2, \]
by (5.4), and so
\[ \tilde{\Pi}^* (z^m \eta_2) = \Theta_{V_2} (z)^m (\tilde{\Pi}^* \eta_2) = \Theta_{V_2} (z)^m P_{W_1} (I - z V_1^*)^{-1} \eta_2, \]
for all \( m \geq 0 \) and \( \eta_2 \in \mathcal{W}_2 \). This concludes the proof.

In the final section, we will connect the analytic descriptions of \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) as in Theorem 5.1 with the classical notion of the Sz.-Nagy and Foias characteristic functions of contractions on Hilbert spaces [23].
6. Defect Operators

Throughout this section, we will work on general (not necessarily pure) pairs of commuting isometries. Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(\mathcal{H}\). The defect operator \(C(V_1, V_2)\) of \((V_1, V_2)\) (see Guo and Yang [19]) is defined by
\[
C(V_1, V_2) = I - V_1V_1^* - V_2V_2^* + V_1V_2V_1^*V_2^*.
\]
It is easy to see that \(C(V_1, V_2)\) is a self adjoint operator.

In this section we prove that, in contrast to the characterization result of negative defect operators for pure pairs of commuting isometries (see Theorem 3.5 in [21]), the defect operator of a (not necessarily pure) pair of commuting isometries is negative if and only if the defect operator is the zero operator. As a consequence, it follows that the defect operator of a pair of commuting isometries is negative if and only if the pair is doubly commuting, and the pair has no non-trivial joint shift-reducing subspaces.

Recall from Section 3 that given a pair of commuting isometries \((V_1, V_2)\), we write \(V = V_1V_2\),
and denote by
\[
\mathcal{W}_j = \mathcal{W}(V_j) = \ker V_j^* = \mathcal{H} \ominus V_j\mathcal{H},
\]
the wandering subspace for \(V_j\), \(j = 1, 2\). The wandering subspace for \(V\) is denoted by \(\mathcal{W}\). Finally, we recall that (see Lemma 3.1)
\[
\mathcal{W} = \mathcal{W}_1 \oplus V_1\mathcal{W}_2 = V_2\mathcal{W}_1 \oplus \mathcal{W}_2.
\]
This readily implies
\[
R_V = P_{\mathcal{W}_1} \oplus P_{V_1}\mathcal{W}_2 = P_{V_2}\mathcal{W}_1 \oplus P_{\mathcal{W}_2}.
\]

The following lemma is well known to the experts, but for the sake of completeness we provide a proof of the statement.

**Lemma 6.1.** Let \((V_1, V_2)\) be a commuting pair of isometries on \(\mathcal{H}\). Then \(\mathcal{H}_s(V)\) and \(\mathcal{H}_u(V)\) are \(V_j\)-reducing subspaces,
\[
\mathcal{H}_s(V_j) \subseteq \mathcal{H}_s(V),
\]
and
\[
\mathcal{H}_u(V_j) \supseteq \mathcal{H}_u(V),
\]
for all \(j = 1, 2\).

**Proof.** For the first part we only need to prove that \(\mathcal{H}_s(V)\) is a \(V_1\)-reducing subspace. Note that since (see Lemma 3.1)
\[
V_1\mathcal{W} \subseteq \mathcal{W} \oplus V\mathcal{W},
\]
it follows that
\[
V_1V^m\mathcal{W} \subseteq V^m(\mathcal{W} \oplus V\mathcal{W}) \subseteq \mathcal{H}_s(V),
\]
for all \(m \geq 0\). This clearly implies that \(V_1\mathcal{H}_s(V) \subseteq \mathcal{H}_s(V)\). On the other hand, since
\[
V_1^*\mathcal{W} = \mathcal{W}_2 \subseteq \mathcal{W},
\]
and

\[ V_1^*V^m \mathcal{W} = V^{m-1}(V_2 \mathcal{W}) \subseteq V^{m-1}(\mathcal{W} \oplus V \mathcal{W}), \]

it follows that \( V_1^* \mathcal{H}_s(V) \subseteq \mathcal{H}_s(V) \). To prove the second part of the statement, it is enough to observe that

\[ V^m \mathcal{H} = V_1^m(V_2^m \mathcal{H}) = V_2^m(V_1^m \mathcal{H}) \subseteq V_1^m \mathcal{H}, V_2^m \mathcal{H}, \]

for all \( m \geq 0 \), and as \( n \to \infty \)

\[ V_1^*h \to 0, \text{ or } V_2^*h \to 0 \Rightarrow V^n h \to 0, \]

for any \( h \in \mathcal{H} \). This concludes the proof of the lemma.

The following characterizations of doubly commuting isometries will prove important in the sequel.

**Lemma 6.2.** Let \( (V_1, V_2) \) be a pair of commuting isometries on a Hilbert space \( \mathcal{H} \). Then the following are equivalent:

(i) \( (V_1, V_2) \) is doubly commuting.

(ii) \( V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1 \).

(iii) \( V_1 \mathcal{W}_2 \subseteq \mathcal{W}_2 \).

**Proof.** Since (i) implies (ii) and (iii) follows from

\[ V_i(I - V_j V_j^*)V_i^* = (I - V_j V_j^*)V_i V_i^* \quad (i \neq j), \]

we only have to show that (ii) implies (i). Let \( V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1 \). Consider the Wold-von Neumann orthogonal decomposition of \( V \) (see Theorem 2.1):

\[ \mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V). \]

Then \( \mathcal{H}_s(V) \) and \( \mathcal{H}_u(V) \) are joint \( (V_1, V_2) \)-reducing subspaces, and the pair \( (V_1|_{\mathcal{H}_s(V)}, V_2|_{\mathcal{H}_u(V)}) \)
on \( \mathcal{H}_u \) is doubly commuting, because \( V_j|_{\mathcal{H}_u(V)}, j = 1, 2, \) are unitary operators, by Lemma 6.1.

Now it only remains to prove that \( V_1 V_2 = V_2 V_1^* \) on \( \mathcal{H}_s(V) \). Since

\[ (V_1^* V_2 - V_2 V_1^*)V^m = V_1^* V^m V_2 - V_2 V_1^* V^m = V^m - V_2 V_1^* V^m = 0 \]

it follows that \( V_1^* V_2 - V_2 V_1^* = 0 \) on \( V^m \mathcal{W} \) for all \( m \geq 1 \). In order to complete the proof we must show that

\[ V_1^* V_2 = V_2 V_1^*, \]

on \( \mathcal{W} \). To this end, let \( \eta = \eta_1 \oplus V_1 \eta_2 \in \mathcal{W} \) for some \( \eta_1 \in \mathcal{W}_1 \) and \( \eta_2 \in \mathcal{W}_2 \). Then

\[ V_1^* V_2(\eta_1 \oplus V_1 \eta_2) = V_1^* V_2 \eta_1 + V_1^* V_2 V_1 \eta_2 = V_2 \eta_2, \]

as \( V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1 \), and on the other hand

\[ V_2 V_1^*(\eta_1 \oplus V_1 \eta_2) = V_2 V_1^* \eta_1 + V_2 V_1^* V_1 \eta_2 = V_2 \eta_2. \]

This completes the proof.
The key of our geometric approach is the following simple representations of defect operators.

**Lemma 6.3.**

\[
C(V_1, V_2) = P_{W_1} - P_{V_2W_1} = P_{W_2} - P_{V_1W_2}.
\]

**Proof.** The result readily follows from (6.1) and

\[
C(V_1, V_2) = (I - V_1V_1^*) + (I - V_2V_2^*) - (I - V_1^*V_1)
\]

\[= P_{W_1} + P_{W_2} - P_{W}.
\]

\[\blacksquare\]

The final ingredient to our analysis is the fringe operator \(F_2\). The notion of fringe operators, introduced by R. Yang [32], plays a significant role in the study of joint shift-invariant closed subspaces of the Hardy space over \(\mathbb{D}^2\) [19] (see the discussion at the beginning of Section 5).

Given a pair of commuting isometries \((V_1, V_2)\) on \(\mathcal{H}\), the fringe operators \(F_1 \in \mathcal{B}(W_2)\) and \(F_2 \in \mathcal{B}(W_1)\) are defined by

\[
F_j = P_{W_i}V_j|_{W_i} \quad (i \neq j).
\]

Of particular interest to us is the isometric fringe operators. Note that

\[
F^*_2F_2 = P_{W_1}V_2^*P_{W_1}V_2|_{W_1}.
\]

**Lemma 6.4.** The fringe operator \(F_2\) on \(W_1\) is an isometry if and only if \(V_2W_1 \subseteq W_1\).

**Proof.** As

\[
I_{W_1} - F^*_2F_2 = I_{W_1} - P_{W_1}V_2^*P_{W_1}V_2|_{W_1},
\]

(6.1) implies that

\[
I_{W_1} - F^*_2F_2 = P_{W_1}V_2^*P_{W_1}V_2|_{W_1}.
\]

Then \(F^*_2F_2 = I_{W_1}\) if and only if \(P_{V_1W_2}V_2|_{W_1} = 0\), or, equivalently, if and only if

\[V_2W_1 \perp V_1W_2 = W_1^\perp,
\]

by Lemma 3.1. This completes the proof. \[\blacksquare\]

Therefore, the fringe operator \(F_2\) is an isometry if and only if the pair \((V_1, V_2)\) is doubly commuting.

We are now ready to formulate a generalization of Theorem 3.4 in [21] by He, Qin and Yang. Here we do not assume that \((V_1, V_2)\) is pure.

**Theorem 6.5.** Let \((V_1, V_2)\) be a pair of commuting isometries on \(\mathcal{H}\). Then the following are equivalent:

(a) \(C(V_1, V_2) \geq 0\).
(b) \(V_2W_1 \subseteq W_1\).
(c) \((V_1, V_2)\) is doubly commuting.
(d) \(C(V_1, V_2)\) is a projection.
(e) The fringe operator \(F_2\) is an isometry.
Proof. The equivalences of (a) and (b), (b) and (c), and (b) and (e) are given in Lemma 6.3, Lemma 6.2 and Lemma 6.4, respectively. The implication (c) implies (d) follows from
\[ C(V_1, V_2) = P_{W_1} P_{W_2} = P_{W_2} P_{W_1}. \]
Clearly (d) implies (a). This completes the proof.

We now prove that the class of pairs of commuting isometries with negative defect operators is trivial.

**Theorem 6.6.** If \((V_1, V_2)\) is a pair of commuting isometries on \(\mathcal{H}\), then \(C(V_1, V_2) \leq 0\) if and only if \(C(V_1, V_2) = 0\).

**Proof.** If \(C(V_1, V_2) \leq 0\), then by Lemma 6.3, we have
\[ P_{W_1} \leq P_{V_2 W_1}, \]
or, equivalently
\[ \mathcal{W}_1 \subseteq V_2 \mathcal{W}_1, \]
and hence
\[ \mathcal{W}_1 \subseteq V_2^n \mathcal{W}_1, \]
for all \(m \geq 0\). Therefore
\[ \mathcal{W}_1 = \bigcap_{m=0}^{\infty} V_2^m \mathcal{W}_1 \subseteq \bigcap_{m=0}^{\infty} V_2^m \mathcal{H} = \mathcal{H}_u(V_2). \]
This and \(V_1 \mathcal{H}_u(V_2) \subseteq \mathcal{H}_u(V_2)\) implies that
\[ \mathcal{H}_s(V_1) \subseteq \mathcal{H}_u(V_2). \]
Using this and the Wold-von Neumann orthogonal decomposition of \(V_2\) (see Theorem 2.1) we have
\[ \mathcal{H} = \mathcal{H}_s(V_2) \oplus \mathcal{H}_u(V_2) \]
\[ = \mathcal{H}_s(V_2) \oplus \mathcal{H}_s(V_1) \oplus (\mathcal{H}_u(V_2) \ominus \mathcal{H}_s(V_1)). \]
But
\[ \mathcal{H}_u(V_2) \ominus \mathcal{H}_s(V_1) = \mathcal{H}_u(V_2) \cap \mathcal{H}_u(V_1) = \mathcal{H}_u(V), \]
and so
\[ \mathcal{H} = \mathcal{H}_s(V_1) \oplus \mathcal{H}_s(V_2) \oplus \mathcal{H}_u(V). \]
Applying Lemma 6.1, it follows that \(\mathcal{H}_s(V_2), \mathcal{H}_s(V_1)\) and \(\mathcal{H}_u(V)\) are joint \((V_1, V_2)\)-reducing subspaces of \(\mathcal{H}\). Then with respect to the above orthogonal decomposition of \(\mathcal{H}\), \(V_1\) and \(V_2\) admits the following representations:
\[ V_1 = \begin{bmatrix} V_{1s} & 0 \\ 0 & V_{1u} \end{bmatrix}, \]
\[ V_2 = \begin{bmatrix} V_{2s} & 0 \\ 0 & V_{2u} \end{bmatrix}. \]
and

\[
V_2 = \begin{bmatrix}
\tilde{V}_{2u} & 0 \\
0 & V_{2s} \\
0 & 0 & V_{2u}
\end{bmatrix}
\]

where

\[
V_{js} = V_j|_{\mathcal{H}_s(V)} \quad \text{and} \quad V_{ju} = V_j|_{\mathcal{H}_u(V)}
\]

are the shift and the unitary parts of \(V_j\) on \(\mathcal{H}_s(V_j)\) and \(\mathcal{H}_u(V)\), respectively, and

\[
\tilde{V}_{1u} = V_1|_{\mathcal{H}_u(V_2)} \quad \text{and} \quad \tilde{V}_{2u} = V_2|_{\mathcal{H}_u(V_1)}.
\]

Since here

\[
\mathcal{H}_u(V_1) = \mathcal{H}_s(V_2) \oplus \mathcal{H}_u(V),
\]

and

\[
\mathcal{H}_u(V_2) = \mathcal{H}_s(V_1) \oplus \mathcal{H}_u(V),
\]

it follows that \(\tilde{V}_{1u}\) and \(\tilde{V}_{2u}\) are unitary operators. Furthermore, since

\[
V_{1s}\tilde{V}_{2u} = \tilde{V}_{2s}V_{1s},
\]

and

\[
\tilde{V}_{1u}V_{2s} = V_{2s}\tilde{V}_{1u},
\]

it follows that \((V_1, V_2)\) is a pair of doubly commuting isometries. Hence Theorem 6.5 implies that \(C(V_1, V_2) \geq 0\), and so \(C(V_1, V_2) = 0\). This completes the proof.

In particular, in the above we have proved that if \(C(V_1, V_2) = 0\), then \((V_1, V_2)\) has no joint shift-reducing subspaces. The converse is true, and follows easily [28], for pairs of doubly commuting isometries.

7. Concluding Remarks

As pointed out in the introduction, a general theory for pairs of commuting isometries is mostly unknown and unexplored (however, see Popovici [24]). In comparison, we would like to add that a great deal is known about the structure of pairs (and even of \(n\)-tuples) of commuting isometries with finite rank defect operators (see [1], [11], [12], [13]). A complete classification result is also known for \(n\)-tuples of doubly commuting isometries (cf. [18], [28], [29]). See also [2] for relevant results. It is now natural to ask whether the present results for pure pairs of commuting isometries can be extended to arbitrary pairs of commuting isometries (see [1], [7], [8], [15], [17] and [18] for closely related results). Another relevant question is to analyze the joint shift invariant subspaces of the Hardy space over the unit bidisc [11] from our analytic and geometric point of views. More detailed discussion on these issues will be given in forthcoming papers.

Also we point out that some of the results of this paper can be extended to \(n\)-tuples of commuting isometries and will be discussed in a future paper.

We conclude this paper by inspecting a connection between the Sz.-Nagy and Foias characteristic functions of contractions on Hilbert spaces [23] and the analytic representations of \(\tilde{\Pi}_1\) and \(\tilde{\Pi}_2\) as described in Theorem 5.1.
Let $T$ be a contraction on a Hilbert space $\mathcal{H}$. The defect operators of $T$, denoted by $D_{T^*}$ and $D_T$, are defined by

$$D_{T^*} = (I - TT^*)^{1/2}, \quad D_T = (I - T^*T)^{1/2}. $$

The defect spaces, denoted by $\mathcal{D}_{T^*}$ and $\mathcal{D}_T$, are the closure of the ranges of $D_{T^*}$ and $D_T$, respectively. The characteristic function of the contraction $T$ is defined by

$$\theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \quad (z \in \mathbb{D}).$$

It follows that $\theta_T \in H^\infty_{\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})}(\mathbb{D})$. The characteristic function is a complete unitary invariant for the class of completely non-unitary contractions. This function is also closely related to the Beurling-Lax-Halmos inner functions for shift invariant subspaces of vector-valued Hardy spaces. For a more detailed discussion of the theory and applications of characteristic functions we refer to the monographs by Bercovici [6] and Sz.-Nagy and Foias [23].

Now let us return to the study of pairs of commuting isometries. Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. We compute

$$P_{W_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_W = [P_{W_1} + zP_{W_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_W$$

$$= [I_{\mathcal{H}} - V_1V_1^* + zP_{W_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_W$$

$$= I_W + [-V_1 + zP_{W_1}(I_{\mathcal{H}} - zV_1^*)^{-1}]V_1^*|_W.$$ 

Since $V_1^*W = W_2$ (see Corollary 3.2), it follows that

$$[-V_1 + zP_{W_1}(I_{\mathcal{H}} - zV_1^*)^{-1}]V_1^*|_W = [-V_1 + zD_{V_1^*}(I_{\mathcal{H}} - zV_1^*)^{-1}D_{V_1^*}]|_{\mathcal{D}_{V_1^*}}(V_1^*|_W).$$

Therefore, setting

$$\theta_{V_1, V_2}(z) = [-V_1 + zD_{V_1^*}(I_{\mathcal{H}} - zV_1^*)^{-1}D_{V_1^*}]|_{\mathcal{D}_{V_1^*}},$$

for $z \in \mathbb{D}$, we have

$$P_{W_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_W = I_W + \theta_{V_1, V_2}(z)V_1^*|_W,$$

for all $z \in \mathbb{D}$. Therefore, if $V_1$ is a pure isometry, then the formula for $\mathbb{P}_1$ in Theorem 5.1 (ii) can be expressed as

$$\mathbb{P}_1(S^*(\cdot, w)\eta) = (I_{W_1} - w\Theta_{V_2}(z))^{-1}P_{W_1}[I_W + \theta_{V_1, V_2}(z)V_1^*]|_W\eta.$$ 

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Similarly, if $V_2$ is a pure isometry, then the formula for $\mathbb{P}_2$ in Theorem 5.1 (ii) can be expressed as

$$\mathbb{P}_2(S^*(\cdot, w)\eta) = (I_{W_2} - w\Theta_{V_1}(z))^{-1}P_{W_2}[I_W + \theta_{V_2, V_1}(z)V_2^*]|_W\eta,$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$, where

$$\theta_{V_2, V_1}(z) = [-V_2 + zD_{V_2^*}(I_{\mathcal{H}} - zV_2^*)^{-1}D_{V_2^*}]|_{\mathcal{D}_{V_2^*}},$$

for all $z \in \mathbb{D}$.

It is easy to see that $\theta_{V_i, V_j}(z) \in \mathcal{B}(\mathcal{W}_j, \mathcal{W})$ for all $z \in \mathbb{D}$ and $i \neq j$.

Note that since the defect operator $D_{V_j} = 0$, the characteristic function $\theta_{V_j}$ of $V_j$, $j = 1, 2$, is the zero function. From this point of view, it is expected that the pair of analytic invariants $\{\theta_{V_i, V_j} : i \neq j\}$ will provide more information about pairs of commuting isometries.
Subsequent theory for pairs of commuting contractions and a more detailed connection between pairs of commuting pure isometries \((V_1, V_2)\) and the analytic invariants \(\{\theta_{V_i V_j} : i \neq j\}\) as defined in (7.1) and (7.2) will be exhibited in more details in future occasion.

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