HIGHER-ORDER ANALOGUES OF THE SLICE GENUS OF A KNOT

PETER D. HORN

Abstract. For certain classes of knots we define geometric invariants called higher-order genera. Each of these invariants is a refinement of the slice genus of a knot. We find lower bounds for the higher-order genera in terms of certain von Neumann \( \rho \)-invariants, which we call higher-order signatures. The higher-order genera offer a refinement of the Grope filtration of the knot concordance group.

1. Introduction

A knot is an embedding of the circle into the three-sphere. All embeddings are required to be topologically flat or smooth. Two knots \( K_0, K_1 \) are concordant if there is an annulus \( A \) embedded in \( S^3 \times [0,1] \) in such a way that \( A \cap (S^3 \times \{i\}) = K_i \) for \( i = 0, 1 \). If a knot \( K \) is concordant to the unknot, we call \( K \) a slice knot. Given two knots, one can “add” them via the connected sum operation \( \# \), defined in [Rol76]. Equipped with the connected sum operation, the set of knots modulo (topological or smooth) concordance forms the (topological or smooth) knot concordance group \( \mathcal{C} \). The class of slice knots serves as the identity element of this group.

Cochran, Orr and Teichner have introduced two filtrations of the topological knot concordance group \( \mathcal{C} \) [COT03]. The \((n)\)-solvable filtration

\[ \cdots \subset F_{n,5} \subset F_n \subset \cdots \subset F_{1,5} \subset F_1 \subset F_{0,5} \subset F_0 \subset \mathcal{C} \]

is defined in terms of algebraic properties on the second homology of certain 4-manifolds, each of whose boundary is 0-surgery on a knot. The Grope filtration

\[ \cdots \subset G_{n+2,5} \subset G_{n+2} \subset \cdots \subset G_3 \subset G_{2,5} \subset G_2 \subset \mathcal{C} \]

is defined much more geometrically. Rigorous definitions of these filtrations will be provided below. These filtrations are related to one another in the sense that \( G_{n+2} \subset F_n \) for all \( n \in \frac{1}{2} \mathbb{N} \) [COT03, Theorem 8.11]. Recently, Cochran, Harvey and Leidy proved that \( F_n/F_{n,5} \) has infinite rank for all \( n \) [CHL07]. Subsequently, the author proved the analogous result for the Grope filtration [Hor]. These results were proven using signatures of certain 4-manifolds. While algebraic techniques are appropriate when working with the \((n)\)-solvable filtration, they do not reflect the geometric nature of the Grope filtration. The main focus of this paper is to define a geometric invariant that will distinguish knots in \( G_{n+2} \).

Given a knot \( K \), the slice genus of \( K \) is the minimal genus of surfaces embedded in \( D^4 \) with boundary equal to \( K \subset S^3 = \partial D^4 \). The slice genus is a concordance invariant of \( K \). In the spirit of the Cochran-Orr-Teichner filtrations of \( \mathcal{C} \), we introduce a series of refinements of the slice genus. For knots in \( G_{n+2} \), we will define a concordance invariant called the \( n \)-th-order genus. Our main result is that the \( n \)-th-order genus distinguishes knots in \( G_{n+2} \) that are not distinguished by the slice genus. That is, each of our higher-order genera is a refinement of the notion of slice genus.

**Theorem 4.5.** For any \( n \geq 1 \), there is a fixed \( g \) and a knot in \( G_{n+2} \) with slice genus bounded above by \( g \) and arbitrarily high \( n \)-th-order genus. Furthermore, this knot has infinite order in \( G_{n+2}/F_{n,5} \).

**Corollary 4.6.** For any \( n \geq 1 \), there are infinitely many knots that lie in \( G_{n+2} \) whose slice genera are equal but whose \( n \)-th-order genera are distinct.

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Murasugi proved [Mur65] Theorem 9.1 that the ordinary signature of a knot is a lower bound for the slice genus of that knot (henceforth “Murasugi’s inequality”). Gilmer later proved [Gil82] Theorem 1 that the sum of certain Casson-Gordon invariants and the ordinary signature bounds the slice genus from below (henceforth “Gilmer’s inequality”). Cochran, Orr and Teichner first used $L^2$-signatures to study knots. First, we define higher-order analogues of slice genus, and to any $(n)$-solvable knot we assign a set of real numbers, called the $n^{th}$-order signatures. This begs the question of whether there is a higher-order analogue of Murasugi’s inequality. Our primary tool is the desired higher-order analogue.

**Theorem 4.2** If $K \in \mathcal{G}_{n+2}$, there is an $n^{th}$-order signature of $K$ that gives a lower bound for the $n^{th}$-order genus of $K$.

We are not the first to utilize $L^2$-signatures in the study of genus-like invariants. Cha used metabelian $L^2$-signatures to obtain new lower bounds on the minimal genus of embedded surfaces representing a given 2-dimensional homology class in certain 4-manifolds [Cha88]. An application of Cha’s methods was to find bounds for the slice genus of knots [Cha88 Proposition 5.1]. Our Theorem 4.2 uses the $L^2$-signatures to obtain lower bounds for the higher-order genera. While Cha obtained obstructions to slice genus, we obtain higher-order obstructions to the higher-order genera. It seems that the only (classical) sliceness obstruction our higher-order genera give is that if one of the higher-order genera of a knot is positive, then that knot cannot be slice. However, a knot having large higher-order genera does not in general obstruct the knot from having a small (but positive) slice genus.

We should note that our higher-order signatures give a lower bound on the topological higher-order genera and often fail to be accurate in the smooth category. Consequently, we choose to work in the topological category, except for Section 3 which contains examples in the smooth category.

2. Definitions

We start with the geometric definitions.

**Definition 2.1.** [FT95] A grope is a special pair (2-complex, base circle). A grope has a **height** $n \in \frac{1}{2} \mathbb{N}$. A grope of height 1 is precisely a compact, oriented surface $\Sigma$ with a single boundary component (the base circle). For $n \in \mathbb{N}$, a grope of height $n+1$ is defined recursively as follows: let $\{\alpha_i, \beta_i : i = 1, \ldots, g\}$ be a symplectic basis of curves for $\Sigma$, the first stage of the grope. Then a grope of height $n+1$ is formed by attaching gropes of height $n$ to each $\alpha_i$ and $\beta_i$ along the base circles.

A grope of height 1.5 is formed by attaching gropes of height 1 (i.e. surfaces) to a Lagrangian of a symplectic basis of curves for $\Sigma$. That is, a grope of height 1.5 is a surface with surfaces glued to “half” of the basis curves. In general, a grope of height $n + 1.5$ is obtained by attaching gropes of height $n$ to the $\alpha_i$ and gropes of height $n + 1$ to the $\beta_i$.

Given a 4-manifold $W$ with boundary $M$ and a framed circle $\gamma \subset M$, we say that $\gamma$ bounds a **grope** in $W$ if $\gamma$ extends to an embedding of a grope with its untwisted framing. That is, a Grope has a trivial normal bundle, so parallel push-offs can be taken. Knots in $S^3$ are always equipped with the zero framing.

The set of all concordance classes of knots that bound Gropes of height $n$ in $D^4$ is denoted $\mathcal{G}_n$, which is a subgroup of $C$. We may choose to forget the top stages of a Grope. Thus, if $K$ bounds a Grope of height $n+1$ in $D^4$, $K$ also bounds a Grope of height $n$ in $D^4$. We see that $\mathcal{G}_{n+1} \subset \mathcal{G}_n$ as subgroups of $C$, and this series of subgroups is the **grope filtration of the knot concordance group**. By ‘$K \in \mathcal{G}_n$,’ we mean a knot $K$ whose concordance class lies in $\mathcal{G}_n$, or equivalently, a knot that bounds a Grope of height $n$ in $D^4$.

**Definition 2.2.** For $K \in \mathcal{G}_{n+2}$, define the **$n^{th}$-order genus** of $K$ to be the minimum of the genera of the first stage surfaces of Gropes of height $n+2$ in $D^4$ bounded by $K$. Denote the $n^{th}$-order genus of $K$ by $g_n(K)$. With this numbering scheme, the slice genus of $K$ is the $-1^{st}$-order genus of $K$.

It is immediately clear that for $K \in \mathcal{G}_{n+2}$, $0 \leq g_{-1}(K) \leq g_0(K) \leq \cdots \leq g_n(K)$, and that $g_n(K) = g_n(J)$ if $K$ and $J$ are concordant. Also, $K$ is slice if and only if $g_n(K) = 0$ for some $n \geq -1$. 

Now we turn to the algebraic definitions. If $G$ is a group, the \textit{derived series} of $G$ is defined recursively by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. The \textit{rational derived series} of $G$ is defined recursively by setting $G_r^{(0)} = G$ and $G_r^{(i+1)} = \left\{ g \in G : g^k \in [G_r^{(i)}, G_r^{(i)}], \text{ for some } k > 0 \right\}$.

\textbf{Definition 2.3.} \cite{COT03} Let $M$ be closed, orientable 3-manifold. A spin 4-manifold $W$ with $\partial W = M$ is an \textit{(n)-solution} for $M$ if the inclusion-induced map $i_* : H_1(M) \to H_1(W)$ is an isomorphism and if there are embedded surfaces $L_i$ and $D_i$ (with product neighborhoods) for $i = 1, \ldots, m$ that satisfy the following conditions:

1. the homology classes $\{[L_1], [D_1], \ldots, [L_m], [D_m]\}$ form an ordered basis for $H_2(W)$,
2. the intersection form $(H_2(W), \cdot)$ with respect to this ordered basis is a direct sum of hyperbolics,
3. $L_i \cap D_j$ is empty if $i \neq j$,
4. for each $i$, $L_i$ and $D_i$ intersect transversely at a point, and
5. each $L_i$ and $D_i$ are \textit{(n)-surfaces}, i.e. $\pi_1(L_i) \subset \pi_1(W)^{(n)}$ and $\pi_1(D_i) \subset \pi_1(W)^{(n)}$.

If, in addition, $\pi_1(L_i) \subset \pi_1(W)^{(n+1)}$ for each $i$, we say $W$ is an \textit{(n,5)-solution} for $M$. Since $H_1(W)$ has no 2-torsion and the intersection form of $W$ is even, $W$ is necessarily spin.

If a closed, orientable 3-manifold has an \textit{(n)-solution}, we say $M$ is \textit{(n)-solvable}. A knot $K$ in $S^3$ is an \textit{(n)-solvable knot} if the zero surgery on $K$ is \textit{(n)-solvable}.

As in \cite{COT03}, the set of all \textit{(n)-solvable knots} is denoted $\mathcal{F}_n$, and Cochran-Orr-Teichner showed that the $\mathcal{F}_n$ form a nested series of subgroups of $\mathcal{C}$. This series of subgroups is the \textit{(n)-solvable filtration} of the knot concordance group.

Given a closed 3-manifold and a homomorphism $\phi : \pi_1(M) \to \Gamma$ where $\Gamma$ is any group, one can define the \textit{von Neumann} $\rho$-invariant $\rho(M, \phi) \in \mathbb{R}$ \cite[Section 4]{CG85}. See \cite{COT07} for an analytical interpretation of these von Neumann $\rho$-invariants.

\textbf{Definition 2.4.} For $K \in \mathcal{F}_n$, we define the \textit{$n$th-order signatures} of $K$ to be the elements of the set $\mathcal{S}^n(K) = \left\{ \rho(M_K, \phi) \in \mathbb{R} \mid \phi : \pi_1(M_K) \xrightarrow{i_*} \pi_1(W) \right\}$ where $\pi_1(W)$, $W$ is an \textit{(n)-solution} for $M_K$, $i : M_K \to W$ is the inclusion map, and $\rho(M_K, \phi)$ is the associated von Neumann $\rho$-invariant. While this set of signatures is an isotopy invariant of $K$, it is not a concordance invariant \cite[Example 3.2]{Hor09}.

Recall the Cheeger-Gromov estimate for the von Neumann $\rho$-invariants of a given closed, orientable 3-manifold \cite{CG85}. That is, given a closed, orientable 3-manifold $M$, there is a constant $C_M$ such that

$$|\rho(M, \phi)| \leq C_M$$

for all homomorphisms $\phi : \pi_1(M) \to \Gamma$ for any group $\Gamma$. Thus for a fixed knot $K$ and a fixed $n$, the set $\mathcal{S}^n(K)$ is a bounded set of real numbers.

3. \textbf{Concrete examples in the smooth category}

In this section we work in the smooth category. The purpose of this section is to construct non-slice knots that bound gropes of a fixed height. We compute the higher-order genera in these examples and conclude that for any positive integers $n$ and $m$, there is a knot whose smooth $n$th-order genus is equal to $m$. The computations do not make use of our $n$th-order signatures.

Let $K$ denote any knot with non-negative maximal Thurston-Bennequin number. For example, if $K$ is the right-handed trefoil, then $TB(K) = 1$. Let $D(K)$ denote the positively-clasped, untwisted Whitehead double of $K$ as depicted in Figure 1. For $i \geq 1$, let $D^i(K) = D(D^{i-1}(K))$ denote the $i$th iterated Whitehead double of $K$. By Livingston \cite{Liv94}, we know that $TB(K) \geq 0$ implies that the Ozsváth-Szabó $\tau$-invariant of $D^i(K)$ is nontrivial, i.e. $\tau(D^i(K)) = 1$. It follows from \cite[Corollary 1.3]{OS03} that $D^i(K)$ is not smoothly slice for all $i \geq 1$. It should be noted that earlier work of Lee Rudolph implies that $D^i(K)$ is not slice for all $i \geq 1$ if $K$ is the right-handed trefoil \cite{Rud93}.
We describe a Grope of height 2 in $S^3 \times I$ bounded by $D(K)$. The standard Seifert surface for $D(K)$ has a symplectic basis of curves, each of which inherits the zero framing from this surface. This basis is pictured in Figure 2. Let $\alpha$ denote the basis curve that “goes over the bridge” of this Seifert surface, and let $\beta$ denote the other curve. Pull $\alpha$ slightly out of the page so that the intersection point with $\beta$ is removed. Observe that the link $\alpha^+ \cup \beta$ is two parallel copies of $K$. Now push these two curves down in the $I$ direction and glue parallel Seifert surfaces for $K$. The Seifert surface for $D(K)$ together with the pushing annuli and Seifert surfaces for $K$ comprise a height 2 Grope for $D(K)$ in $S^3 \times I$. The genus of the first stage of this Grope is 1. By [OS03, Corollary 1.3] $1 = \tau(D(K)) \leq g-1(D(K)) \leq g_0(D(K)) \leq g_0(\#mD^n(K)) \leq \tau(\#_mD^n(K)) = m \cdot \tau(D^n(K)) = m$ and $g_0(D(K)) \leq 1$ by construction, we have $g_0(D(K)) = 1$.

We can iterate this procedure to build a Grope of height $n+1$ in $S^3 \times I$ bounded by $D^n(K)$, and the first stage of this Grope has genus 1. As before, we have $1 \leq \tau(D^n(K)) \leq g-1(D^n(K)) \leq g_0(D^n(K)) \leq \cdots \leq g_{n-1}(D^n(K)) \leq 1$, whence $g_{n-1}(D^n(K)) = 1$.

Since $\tau : C \to \mathbb{Z}$ is a homomorphism [OS03 Theorem 1.2], we conclude that $g_{n-1}(\#_mD^n(K)) \geq \tau(\#_mD^n(K)) = m \cdot \tau(D^n(K)) = m$ and $g_{n-1}(\#_mD^n(K)) \leq m$ by construction. To summarize, we have the following theorem.

**Theorem 3.1.** For any $n \geq 0$ and $m \geq 1$, there is a knot $K \in \mathcal{G}^\text{smooth}_{n+2}$ of infinite order, and $g_n(K) = m$.

**Remark.** Since the Alexander polynomial of $D(K)$ is trivial, it can be shown that $D(K)$ is smoothly $(n)$-solvable for all $n$. However, whether $D(K) \in \mathcal{G}^\text{smooth}_{n+2}$ for all $n$ is still an open question.
4. Lower bounds on higher-order genera

We now turn to our higher-order signatures as tools for estimating the higher-order genera. While the higher-order signatures are not explicitly computable, we demonstrate how to ensure that all higher-order signatures are large enough to guarantee that the higher-order genera are large.

Lemma 4.1. Let \( K \in \mathcal{F}_n \) and \( W \) be an \((n)-solution \) for \( M_K \). Then the \( n \)-th signature of \( K \) associated to \( W \) satisfies \(|\rho(M_K, \phi)| \leq \beta_2(W)\).

Proof. Let \( \phi : \pi_1(M_K) \to \pi_1(W) \) be a homomorphism that factors through \( \pi_1(W) \). By the definition of an \((n)-solution \), the ordinary intersection form of \( W \) is a direct sum of hyperbolics, implying that the ordinary signature of \( W \) is zero. Since \( \phi \) factors through \( \pi_1(W) \), we have that \[ \rho(M_K, \phi) = \sigma^{(2)}(W, \pi_1(W)/\pi_1(W)^{(n+1)}) - \sigma(W) = \sigma^{(2)}(W, \pi_1(W)/\pi_1(W)^{(n+1)}) \]

Here \( \sigma^{(2)}(W, \pi_1(W)/\pi_1(W)^{(n+1)}) \) refers to the \( L^2 \)-signature of \( W \) associated to the quotient \( \pi_1(W)/\pi_1(W)^{(n+1)} \). We refer the reader to Section 5 of [COT03] for a thorough explanation of \( L^2 \)-signatures. Cha has shown that \[ \left| \sigma^{(2)}(W, \pi_1(W)/\pi_1(W)^{(n+1)}) \right| \leq \beta_2(W) \] [Cha08] Lemma 2.7].

That the homomorphism \( \phi : \pi_1(M_K) \to \pi_1(W)/\pi_1(W)^{(n+1)} \) factors through \( \pi_1(W) \) of bounding 4-manifold \( W \) is crucial. Our philosophy differs from Cha’s [Cha08] in that we assume our homomorphisms factor through bounding 4-manifolds (cf. Definition 2.4), whereas Cha takes a homomorphism \( \pi_1(M_K) \to \Gamma \) and tries to extend it to a bounding 4-manifold. In particular, Cha finds a homomorphism \( \phi_\sigma \) : \( \pi_1(M_K) \to \mathbb{Z} \) that factors through a certain bounding 4-manifold, and the von Neumann \( \rho \)-invariant associated to this homomorphism satisfies \(|\rho(M_K, \phi_\sigma)| \leq 4 g_{-1}(K)\), where \( g_{-1}(K) \) is the slice genus of \( K \) [Cha08] Theorem 1.1 and Proposition 1.2]. We, however, consider many homomorphisms that we assume extend to bounding 4-manifolds, and we show that (at least) one of the associated \( \rho \)-invariants satisfies \(|\rho| \leq 4 g_n(K)\), where \( g_n(K) \) is the \( n \)-th order genus of \( K \).

Theorem 4.2. If \( K \in \mathcal{G}_{n+2} \), one of the \( n \)-th order signatures \( \rho \in \mathfrak{S}^n(K) \) satisfies \(|\rho| \leq 4 g_n(K)\).

Proof. Let \( \Sigma \) be the first stage of a Grope of height \( n+2 \) that realizes \( g_n(K) \), i.e. \( g(\Sigma) = g_n(K) \). Cochran-Orr-Teichner construct an \((n)-solution \) \( W \) by surgering \( \Sigma \), and \( \beta_2(W) = 4 g(\Sigma) = 4 g_n(K) \) [COT03] Theorem 8.11]. The conclusion follows from Lemma 4.1.

Remark. Theorem 4.2 may be thought of as a higher-order analogue of Murasugi’s inequality [Mur65] Theorem 9.1]. Unlike the subsequent inequalities of Gilmour [Gil82] Theorem 1] and Cha [Cha08] Proposition 5.1], our result gives higher-order obstructions to the higher-order genera.

Corollary 4.3. If \( K \) is a slice knot, then for any \( n \), one of the \( n \)-th order signatures of \( K \) vanishes.

Proposition 4.4. Suppose \( K \) is \((n)-solvable \). If \( K \) is \((n.5)-solvable \), then one of the \( n \)-th order signatures of \( K \) vanishes.

Proof. Let \( W \) be an \((n.5)-solution \) for \( K \). It follows from [COT03] Theorem 4.2] that the \( n \)-th order signature of \( K \) associated to \( W \) vanishes.

Remark. The conclusion holds even if \( K \) is assumed to be merely rationally \((n.5)-solvable \) [COT03] Definition 4.1].

If the Alexander polynomial of a knot is trivial, then the knot is topologically slice [FQ90]. In particular, Alexander polynomial one knots are \((n)-solvable \) for all \( n \). Consequently, the \( n \)-th order signatures of an Alexander polynomial one knot are all equal to the classical signature, namely zero. As the \( n \)-th order signatures are topological invariants, they will not give accurate bounds for the smooth higher-order genera. For example, the knots constructed in Section 3 had trivial Alexander polynomial but large smooth \( n \)-th order genera.
Theorem 4.5. For any \( n \geq 1 \), there is a fixed \( g \) and a knot in \( G_{n+2} \) with slice genus bounded above \( g \) and arbitrarily high \( n \)-th order genus. Furthermore, this knot has infinite order in \( G_{n+2}/F_{n,5} \).

Remark. The statement of Theorem 4.5 seems to be false for \( n = 0 \). For example, if \( K \in G_2 \), one can construct a Grope of height 2 bounded by \( K \) whose first stage has genus equal to the Seifert genus of \( K \). See [COT03, Remark 8.14] for a discussion.

Proof. We construct knots according to Cochran-Orr-Teichner [COT03] and Cochran-Teichner [CT07]. We borrow the knot \( J \) from [CT07, Figure 3.6]. Let \( J_m = \#_m J \); then \( J_m \) bounds a Grope of height 2 (and is (0)-solvable), and \( \rho_0(J_m) = \frac{4m^3}{3} \) [CT07, Lemma 4.5]. Let \( R \) denote the knot pictured in Figure 3 (ignore the curve \( \eta \) for now). \( R \) is a fibered, genus 2, ribbon knot [COT03, p. 447].

![Figure 3. The ribbon knot R and a curve \( \eta \in \pi_1(S^3 - R)^{(2)} \).]

By [CT07, Theorem 4.3], there is a collection of unknotted curves \( \eta_i, 1 \leq i \leq j \), in \( S^3 - R \) with \( [\eta_i] \in \pi_1(M_R)^{(n)} \) and for any \((n)\)-solution \( V \) of \( M_R \), some \( i_*(\eta_i) \notin \pi_1(V)^{(n+1)} \). For example, Figure 3 shows an unknotted curve \( \eta \) whose homotopy class lies in \( \pi_1(S^3 - R)^{(2)} \) \( \cong \pi_1(M_R)^{(2)} \), and this curve never maps into \( \pi_1(V)^{(3)} \) for any (2)-solution \( V \) for \( M_R \) [COT03, Theorem 4.2]. Let \( K = K_m \) denote the knot obtained by infecting \( R \) by \( J_m \) along \( \eta_i \) (for each \( i \)).

Infecting \( R \) by \( J_m \) along \( \eta_i \) means to grab the strands of \( R \) passing through the unknotted curve \( \eta_i \) and tie them collectively into the knot \( J_m \). Below is a schematic diagram of the infection operation.

![Figure 4. Infecting R by J_m along \( \eta_i \).]

We claim that by choosing \( m \) sufficiently large, we can guarantee all \( \rho \in \mathfrak{S}^n(K) \) are arbitrarily large. It will follow from Theorem 4.2 that \( K \) will have arbitrarily large \( n \)-th order genus, modulo verifying \( K \) bounds a Grope of height \( n + 2 \), which Cochran and Teichner proved in [CT07, Theorem 3.8].
Since the $\eta_i$ have linking number zero with $R$, we can take a Seifert surface for $R$ and tube around the $\eta_i$ so that the tubes are disjoint. We are left with a Seifert surface for $R$ which the $\eta_i$ do not intersect. The knot $K$ will have genus bounded above by the genus of our tubed surface for $R$. We now explain how to increase the $n^{th}$-order genus of $K$ without increasing the genus.

Since our $J_m$ are (0)-solvable, let $W_m$ denote a (0)-solution for $J_m$. We form a 4-manifold $E$ from

$$M_R \times [0, 1] \bigcup_{i=1}^{j} -M_{J_m} \times [0, 1]$$

by identifying, for each $i$, the copy of $\eta_i \times D^2$ in $M_R \times \{1\}$ with the tubular neighborhood of $J_m$ in $M_{J_m} \times \{0\}$ as in Figure 5. The dashed arcs represent the solid tori $\eta_i \times D^2$. As indicated in Figure 3, $\partial E = M_R \sqcup -M_K \sqcup M_{J_m} \sqcup \cdots \sqcup M_{J_m}$. We form another 4-manifold $C$ from $E$ by gluing a copy of $W_m$ to each $M_{J_m} \subset \partial E$.

Now let $W$ be any $(n)$-solution for $M_K$. Let $V = C \cup -M_K -W$ so that $\partial V = M_R$. Then $V$ is an $(n)$-solution for $M_R$ [CT07, Proof of Theorem 4.2]. From our previous discussion, there is a $\eta_k$ with $i_\ast ([\eta_k]) \notin \pi_1(V)^{(n+1)}$. Since $\eta_k$ lives in $M_K$, we may include $\eta_k$ into $W$. Since $W \subset V$, $i_\ast ([\eta_k]) \notin \pi_1(W)^{(n+1)}$.

![Figure 5. The 4-manifold $E$.](image)

Consider the homomorphism $\phi : \pi_1(M_K) \xrightarrow{i_\ast} \pi_1(W) \xrightarrow{\pi_1} \pi_1(W)/\pi_1(W)^{(n+1)}$. Let $\Gamma = \pi_1(W)/\pi_1(W)^{(n+1)}$. Now $M_R - (\sqcup \eta_i) \subset M_K$, so $\phi$ induces a homomorphism $\phi' : \pi_1(M_R - (\sqcup \eta_i)) \to \Gamma$. Since $M_R$ is obtained by $M_R - (\sqcup \eta_i)$ by adding $j$ 2-cells along the meridians of the $\eta_i$ and then by adding $j$ 3-cells, this $\phi'$ will extend to a homomorphism $\phi_R : \pi_1(M_R) \to \Gamma$ if the meridians of the $\eta_i$ die under $\phi$. Now $\eta_i \in \pi_1(M_R)^{(n)}$ and $\Gamma^{(n+1)} = 1$, so [CT07, Theorem 8.1] implies that $\eta_i \in \pi_1(M_K)^{(n)}$. Since the meridian $\mu_i$ of each $J_m$ is identified with the longitude of $\eta_i$, $\mu_i \in \pi_1(M_K)^{(n)}$. Thus $\phi(\mu_i) \in \Gamma^{(n)}$. Since $\mu_i$ generates $\pi_1(S^3 - J_m)/\pi_1(S^3 - J_m)^{(1)}$, we see $\phi(\pi_1(S^3 - J_m)^{(1)}) \subset \Gamma^{(n+1)} = 1$. In particular the meridian of each $\eta_i$ dies under $\phi$, and hence $\phi'$ extends to a map $\phi_R : \pi_1(M_R) \to \Gamma$.

By [CT07, Proposition 4.4], the $\rho$-invariants of $M_K$ and $M_R$ are related by

$$\rho(M_K, \phi) - \rho(M_R, \phi_R) = \sum_{i=1}^{j} \epsilon_i \rho_0(J_m)$$

where $\epsilon_i = 0$ or 1 according to whether $\phi_R([\eta_i]) = 1$ or not. We argued that previously that $i_\ast ([\eta_k]) \notin \pi_1(W)^{(n+1)}$, so $\phi_R([\eta_k]) \neq 1$. Recall that the set of $\rho$-invariants of $M_R$ are bounded above by the Cheeger-Gromov constant $C_{M_R}$ (cf. equation 1). Thus, by choosing $n$ sufficiently large, we will obtain a knot $K$ with $|\rho(M_K, \phi)| > B$ for some large constant $B$. Since $W$ was an arbitrary $(n)$-solution for
$K$, we have proved that every $n^{th}$-order signature for $K$ is larger than $B$. Appealing to Theorem 4.2 we see that $g_{n}(K)$ is arbitrarily large. We should note here that since $0 \notin \mathcal{S}^{n}(K)$, $K \notin \mathcal{F}_{n,5}$ (by Proposition 4.4). [CT07, Theorem 4.2] establishes that $K$ has infinite order in $\mathcal{G}_{n+2}/\mathcal{F}_{n,5}$. □

**Corollary 4.6.** Given any $n \geq 1$, there exist infinitely many knots in $\mathcal{G}_{n+2}$ whose slice genus agree but whose $n^{th}$-order genera are distinct.

*Proof.* By Theorem 4.5 there is a positive integer $g$ and a sequence $\{K_{i}\}_{i=1}^{\infty}$ of knots in $\mathcal{G}_{n+2}$ with $g_{-1}(K_{i}) \leq g$ and $g_{n}(K_{i}) < g_{n}(K_{i+1})$ for all $i \geq 1$. Since the set $\{g_{-1}(K_{i})\}$ is a finite set, we can pass to a subsequence of knots with the same slice genera but different $n^{th}$-order genera. □

**Remark.** We can improve the statement of Corollary 4.6 to say that for each $n \geq 2$, there are infinitely many knots in $\mathcal{G}_{n+2}$ with identical $n^{th}$-order genera for $i \leq n - 1$ and distinct $n^{th}$-order genera. However, the proof is too lengthy to include in this paper. We refer the reader to the author’s thesis for a proof [Hor09, Theorem 5.4]. This result implies that the lower-order genera of knots are inadequate measures of the complexity of $\mathcal{G}_{n+2}$ and that the higher-order genera capture some of the missed information. Examples of this phenomenon can be constructed by infection on the 9$_{46}$ knot as in [Hor].

**Example 4.7.** We provide a concrete family of examples of knots $\{L_{m}\}_{m=1}^{\infty}$ in $\mathcal{G}_{3}$ with slice genus bounded above by 3 and for any $C \in \mathbb{N}$ there is a positive integer $N$ such that for all $n \geq N$, $g_{1}(L_{n}) > C$. Our family is inspired by Cochran-Harvey-Leidy’s family $J_{n}$ (cf. [CHL07]).

Cochran-Harvey-Leidy defined their knots by infecting along the curves $\alpha$ and $\beta$ in Figure 6. We cannot use these curves for the purpose of constructing knots bounding Gropes because the two punctured tori bounded by $\alpha$ and $\beta$ intersect. As per [CHL07, Lemma 3.9], we find curves $\alpha'$ and $\beta'$ that are homotopic to $\alpha$ and $\beta$, respectively, and that bound disjoint height 1 Gropes in $S^{3} - R$. Since these curves are homotopic, the $n^{th}$-order signatures will not distinguish our examples from the examples of [CHL07]. However, our examples are probably not concordant to theirs.

![Figure 6. The infection curves α and β, and homotopic infection curves α' and β'.](image)

Now, let $J$ be the knot from [CT07] and let $J_{m} = \#_{m}J$. $J_{m}$ no longer refers to the knots from [CHL07]. Let $L_{m}$ be infection on $R = 9_{46}$ along $\alpha'$ and $\beta'$ by $J_{m}$. We chose $\alpha'$ and $\beta'$ so that they bound disjointly embedded punctured tori in the complement of $R$, so by [CT07] Proof of Theorem 3.7 the knots $L_{m}$ will bound Gropes of height 3 in $D^{4}$. Since $\alpha'$ and $\beta'$ lie off of a genus 3 Seifert surface for $R$, $L_{m}$ will have slice genus less than or equal to three.

Let $V$ be a (1)-solution for $M = M_{L_{m}}$. Let $\pi = \pi_{1}(V)$. Since $H_{1}(V) \cong \mathbb{Z}$ is torsion-free, we conclude $H_{1}(V) \cong \pi/\pi^{1} \cong \pi/\pi^{1}_1 \cong \mathbb{Z}$. Let $\phi : \pi_{1}(M) \xrightarrow{i} \pi \twoheadrightarrow \pi/\pi^{1}_1$. Since $i_{*} : H_{1}(M) \xrightarrow{\cong} H_{1}(V) \cong \pi/\pi^{1}_1$, we see that $\phi : \pi_{1}(M) \rightarrow H_{1}(M) \xrightarrow{i_{*}} H_{1}(V)$. For emphasis, let $H_{1}(M; \mathbb{Q}[s, s^{-1}])$ denote the first homology
of the infinite cyclic cover of $M$ as a $\mathbb{Q}[s,s^{-1}]$-module, where $H_1(M) = \langle s \rangle$, and let $H_1(M;\mathbb{Q}[t,t^{-1}])$ denote the first homology induced by the coefficient system $\phi : \pi_1(M) \to \pi_1$. The curves $\alpha$ and $\beta$ generate $H_1(M;\mathbb{Q}[s,s^{-1}])$, and since $\alpha'$ and $\beta'$ are homotopic to these generators, $\alpha'$ and $\beta'$ also generate $H_1(M;\mathbb{Q}[s,s^{-1}])$. Since the coefficient system $\phi$ is $\pi_1(M) \to H_1(M)$ followed by an isomorphism, $\alpha'$ and $\beta'$ generate $H_1(M;\mathbb{Q}[t,t^{-1}])$.

Cochran-Orr-Teichner proved that the coefficient system $\phi$ induces a hyperbolic bilinear form $Bl(\cdot,\cdot)$ defined on $H_1(M;\mathbb{Q}[t,t^{-1}])$ [COT03, Theorem 2.13] and that
\[ \mathfrak{t} := \ker \{ i_* : H_1(M;\mathbb{Q}[t,t^{-1}]) \to H_1(V;\mathbb{Q}[t,t^{-1}]) \} \]
satisfies $\mathfrak{t} = \mathfrak{t}^\perp$ with respect to this form [COT03 Theorem 4.4]. Since this form is hyperbolic and $\alpha'$ and $\beta'$ generate $H_1(M;\mathbb{Q}[t,t^{-1}])$, $Bl(\alpha',\beta')$ is nonzero, and hence one of $\alpha'$ and $\beta'$ is not in $\mathfrak{t}$. By the bilinearity of $Bl$, all integer multiples of $\alpha'$ or $\beta'$ are not in $\mathfrak{t}$. Recall that $H_1(V;\mathbb{Q}[t,t^{-1}])$ is the first homology of the infinite-cyclic cover $\tilde{V}$ of $V$, viewed as a $\mathbb{Q}[t,t^{-1}]$-module, and $\pi_1(\tilde{V}) = \pi_1(V)$. If $\alpha'$ were to map to zero in $H_1(V;\mathbb{Q}[t,t^{-1}])$, then $\alpha'$ would map into $\pi_1(V)^{(2)}$. Since no multiple of $\alpha'$ (or of $\beta'$) lie in $\mathfrak{t}$, we conclude that $\alpha'$ or $\beta'$ does not map into $\pi_1(V)^{(2)}$. As in Theorem 4.5, we have the following relationship between the $\rho$-invariants:
\[ \rho(M,\phi) - \rho(M_R,\phi_R) = \epsilon_{\alpha'}\rho_0(J_m) + \epsilon_{\beta'}\rho_0(J_m) \]

Since one of $\alpha'$ and $\beta'$ does not map into $\pi_1(V)^{(2)}$, one of $\epsilon_{\alpha'}$ or $\epsilon_{\beta'}$ is one, as discussed in the proof of Theorem 4.5. By choosing $m$ sufficiently large, the number $|\rho(M,\phi)|$ can be made arbitrarily large. Since $V$ was an arbitrary (1)-solution, we have that $g_1(L_m)$ is arbitrarily large by Theorem 4.2.

5. Applications to a Geometric Structure on the Grope Filtration

Let $B'_n$ denote the subset of all $K$ in $\mathcal{G}_{n+2}$ such that $g_n(K) \leq r$. Since $g_{n-1} \leq g_0 \leq \cdots \leq g_n$, we see that $B'_{n-1} \supseteq B'_0 \supseteq \cdots \supseteq B'_n$. Our main result (Theorem 4.5) is that the higher-order genera are finer measures than the slice genus. Furthermore, by the remark after Corollary 4.6 the $n$th-order genus is a finer measure than the lower-order genera, up to order at least $n-2$. That is, some (depending on $n$ and $r$) of these subset containments are proper. Consequently, these higher-order genera provide a further refinement of the Grope filtration of the knot concordance group. That is, after determining how deep a knot lies in the Grope filtration (say in $\mathcal{G}_{n+2}$), one might try to determine the knot’s $n$th-order genus.

We attempt to complement these comments with the diagram in Figure 7. The ambient three-dimensional space represents $\mathcal{G}_{n+2}$, the plane represents $\mathcal{G}_{n+3}$, the line represents $\mathcal{G}_{n+4}$, and the origin represents $\bigcap_{n \geq 0} \mathcal{G}_n$. The corresponding balls have been drawn. The diagram suggests the existence of knots in $B'_n - B'_{n+1}$, which was proven in Theorem 4.5 and Corollary 4.6 for certain $n$ and $r$. 
Figure 7. The refinement of the Grope filtration by the higher-order genera.

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