Nonzero-sum stochastic games with impulse controls

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Abstract
We consider a general class of nonzero-sum $N$-player stochastic games with impulse controls, where players control the underlying dynamics with discrete interventions. We adopt a verification approach and provide sufficient conditions for the Nash equilibria (NEs) of the game. We then consider the limit situation of $N \to \infty$, that is, a suitable mean-field game (MFG) with impulse controls. We show that under appropriate technical conditions, the MFG is an $\epsilon$-NE approximation to the $N$-player game, with $\epsilon = O\left(\frac{1}{\sqrt{N}}\right)$. As an example, we analyze in details a class of stochastic games which extends the classical cash management problem to the game setting. In particular, we characterize the NEs for its two-player case and compare the results to the single-player case, showing the impact of competition on the player’s optimal strategy, with sensitivity analysis of the model parameters.

Keywords: stochastic games, impulse controls, quasi-variational inequalities, mean-field games, cash management.

1 Introduction
Recent development in the theory of Mean-Field Games (MFGs) has seen an exponential growth in studies of nonzero-sum stochastic games, especially in analyzing $N$-player games and their MFGs counterpart. This paper focuses on nonzero-sum stochastic $N$-player games and the corresponding MFGs in an impulse control setting.

A motivating game problem. The impulse control game is motivated by the classical cash management problem in the seminal work of [12]. Instead of a single-player game in [12], now consider $N$ players, each managing a flow of cash balance. For player $i \in \{1, \ldots, N\}$, the uncontrolled cash balance is driven by

$$dX^i_t = b_i(X^i_{t-})dt + \sigma_i(X^i_{t-})dW^i_t, \quad X^i_0 = x_i,$$

where $W^i$ are independent real Brownian motions. Each player, say player $i$, chooses a sequence of random (stopping) times $(\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,k}, \ldots)$ to intervene and excise her control. At each
$\tau_{i,k}$, the time of this player’s $k$-th intervention, her control is denoted as $\tilde{\xi}_{i,k}$. Given the sequence $\{ (\tau_{i,k}, \tilde{\xi}_{i,k}) \}_{k \geq 1}$ for player $i$, the dynamic of $X^i$ becomes

$$dX^i_t = b_i(X^i_{t-})dt + \sigma_i(X^i_{t-})dW^i_t + \sum_{\tau_{i,k} \leq t} \delta(t - \tau_{i,k})\tilde{\xi}_{i,k}, \quad X^i_0 = x_i,$$

with $\delta(\cdot)$ the Dirac function. The payoff for player $i$ is

$$\mathbb{E}_x \left[ \int_0^\infty e^{-rt} f_i(X_t)dt + \sum_{k=1}^\infty e^{-r\tau_{i,k}}\phi_i(\tilde{\xi}_{i,k}) + \sum_{j \neq i} \sum_{k=1}^\infty e^{-r\tau_{j,k}}\psi_{i,j}(\tilde{\xi}_{j,k}) \right].$$

Here $X_t = (X^1_t, \cdots, X^N_t)$ with $x = (x_1, \cdots, x_N)$ the starting state, $r > 0$ the discount rate, $f_i$ the running cost, $\phi_i$ the cost of control for player $i$, and $\psi_{i,j}$ the cost for player $i$ incurred from player $j$’s control, subject to appropriate conditions to be specified in Section 2.3. The goal of each player $i$ is to find the best policy to minimize her cost among a set of admissible game strategies, to be defined in Section 2.

Our work. Motivated by the above game example, this paper analyzes a class of stochastic games with impulse controls, for both $N$ players and its corresponding MFGs. For the $N$-player game, it establishes a general form of Quasi-Variational Inequalities (QVIs) and provides the sufficient conditions for the Nash equilibria (NEs) of the game, via the verification theorem approach. For the corresponding MFG, it presents sufficient conditions for the existence of NEs and shows that the solution of the MFG is an $\epsilon$-NE approximation to the $N$-player game, with $\epsilon = \frac{1}{\sqrt{N}}$. Through sensitivity analysis and comparisons among $N = 1, 2$ and $N = \infty$ (i.e., MFG), it analyzes the cash management game problem and the effect of competition in games and the collapse of MFG to the single-player game. In particular, it shows that in a game setting players have to take the opponents’ strategies into consideration due to competition. Consequently, it is optimal (in the NE sense) that players choose to intervene less frequently; but once set to intervene, players will exert larger amount of controls. In some sense, competition induces more efficient control strategies from players.

Despite the rapid growth in recent literature on stochastic games and MFGs, stochastic games involving impulse controls are almost non-existent, except for the work in [1]. Compared to games with regular controls and singular controls, impulse control is a more natural mathematical framework for applied problems allowing for discontinuous state space. See the examples of cash management [12], inventory controls [17, 18, 31], transaction cost in portfolio analysis [13, 27, 21, 22, 7, 30], insurance model [20, 8], liquidity risk [25], exchange rates [28, 6], and real options [32, 26, 3]. The presence of discontinuity makes the analysis of control problems hard and even harder for stochastic games. From a PDEs perspective, the corresponding Hamilton-Jacobi-Bellman equation system is coupled with an additional non-local operator for which most PDEs techniques are not applicable. Indeed, there had been little progress in the theory of impulse controls after Bensoussan and Lions’ classical work [4], until the work of [15] where the non-local operator was found to be connected with the infinitesimal differential operator in the nonlinear PDEs via the value functions in the action region and the waiting region. (See also [2].)

Related work on impulse control games. [1] studied a class of two-player impulse control games. In the example proposed, players control the same one-dimensional diffusion process. Interestingly, their value of the game could be derived without having to deal directly with the non-local
operator. In comparison, this paper considers a general class of multi-dimensional \( N \)-player impulse control games and their MFGs. The interaction among players in a discontinuous fashion requires a general mathematical framework, starting from the very definition of game formulation, to admissible game strategies, to the appropriate choices of filtrations, as detailed in Section 2. The structure of the game value also changes when two players do not necessarily control the same dynamics even for the special \( N = 2 \) case: First, one needs to introduce the notion of common waiting region for analyzing general \( N \)-player games; secondly, one can no longer avoid dealing with the tricky non-local operator in the QVIs for impulse controls. Moreover, \cite{1} did not consider the MFG of impulse controls and the relation between the MFG and its \( N \)-player counterpart, nor did they study the impact of competition as their single-player game degenerates.

Related work comparing N-player game and MFG. For an introduction to MFG, we refer to the seminal works \cite{19, 24} and to the recent books \cite{9, 10}. In the regular control setting where the controls are absolutely continuous with respect to the Lebesgue measure, \cite{11} solved the \( N \)-player game of systemic risk with no common noise and studied the MFG counterpart with common noise; \cite{5} focused on a class of linear-quadratic problems; \cite{23} analyzed both the \( N \)-player and MFG of optimal portfolio problem under the competition criterion, as well as the formulations of both types of games with relative criterion. In the singular control setting where controls are càdlàg type, \cite{16} explicitly solved the \( N \)-player game and MFG of a classical fuel followers problem. The \( \epsilon \)-NE approximation of MFGs was first established for regular controls in \cite{9, 10} with \( \epsilon = \frac{1}{N} \) and then for singular controls in \cite{16} and \cite{14} with \( \epsilon = \frac{1}{\sqrt{N}} \). Our result with \( \epsilon = \frac{1}{\sqrt{N}} \) for impulse controls is consistent with those for singular controls as both allow discontinuity in the state space.

2 \ N-player stochastic games with impulse controls

2.1 Problem formulation

In this section, we provide the mathematical definition for the \( N \)-player stochastic games with impulse controls. The idea is clear and intuitive: \( N \) players intervening on a stochastic process by discrete-time intervention. However, the precise mathematical definition presents some non-trivial technicalities with the presence of discontinuous multi-dimensional controlled process.

Domain and underlying process. Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a filtered probability space and let \( \{W_t\}_{t \geq 0} \) be an \( M \)-dimensional \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted Brownian motion. Let \( S \) be a fixed non-empty subset of \( \mathbb{R}^d \), representing the set where the game takes place, in the sense that the game ends when the controlled process exits from \( S \). For example, in portfolio optimization problems the game ends in case of bankruptcy, which may be modelled by choosing \( S = (0, \infty) \).

For \( t \geq 0 \) and \( \zeta \in L^2(\mathcal{F}_t) \), we denote by \( Y^{t, \zeta} = \{Y_s^{t, \zeta}\}_{s \geq t} \) a solution to the stochastic differential equation

\[
\begin{aligned}
    dY_s^{t, \zeta} &= b(Y_s^{t, \zeta})ds + \sigma(Y_s^{t, \zeta})dW_s, \quad s \geq t, \\
    Y_t^{t, \zeta} &= \zeta.
\end{aligned}
\]

(2.1)

Here, \( b : S \to \mathbb{R}^d \) and \( \sigma : S \to \mathbb{R}^{d \times M} \) are given Lipschitz-continuous functions, i.e., there exists a constant \( K > 0 \) such that for all \( y_1, y_2 \in S \),

\[ |b(y_1) - b(y_2)| + |\sigma(y_1) - \sigma(y_2)| \leq K |y_1 - y_2|. \]

The equation in (2.1) models the underlying process when none of the players intervenes.
Interventions of the players and impulse controls.} $N$ players, indexed by $i \in \{1, \ldots, N\}$, can intervene on the process in (2.1) by means of discrete-time interventions. Namely, if player $i$ intervenes with impulse shifts with a simple translation, i.e., $\Gamma_i x_{\tau}$, Remark 2.4 below for details), with $\tau \in \mathbb{N}$ denote by $\Phi_i$ the set of strategies for player $i$.

The interventions of player $i$ are described by the sequence $\{(\tau_{i,k}, \xi_{i,k})\}_{k \geq 1}$ (impulse control), where $\{\tau_{i,k}\}_{k \geq 1}$ represent the intervention times and $\{\xi_{i,k}\}_{k \geq 1}$ the corresponding amount of adjustment. Mathematically, $\tau_{i,k}$ is a stopping time with respect to a suitable filtration $\{\mathcal{F}_t\}_{t \geq 0}$ (see Remark 2.4 below for details), with $\tau_{i,k+1} \geq \tau_{i,k}$, and $\xi_{i,k}$ is a $\mathcal{F}_{\tau_{i,k}}$-measurable variable, for each $k \geq 1$ and $i \in \{1, \ldots, N\}$.

Intervening has a cost or a gain, both for the acting player and for all her opponents. Namely, if $x$ is the current state and player $i$ intervenes with an impulse $\delta$, her cost is $\phi_i(x, \delta)$, whereas the cost for player $j \neq i$ is $\psi_{j,i}(x, \delta)$, for given functions $\phi, \psi: S \times Z_i \to \mathbb{R}$. For the game to be well defined, it is necessary to have $\phi_i > 0$. That is, intervening corresponds to a cost, otherwise the game degenerates and the players could improve their payoff by continuously intervening.

**Action regions, impulse functions, strategies.** As seen, players’ interventions on the underlying process are modelled by impulse controls. In the model we propose here, impulse controls originate from a precise strategy that each player preliminarily fixes.

**Definition 2.1.** A strategy for player $i \in \{1, \ldots, N\}$ is a pair $\varphi_i = (A_i, \xi_i)$, where $A_i$ is a fixed subset of $\mathbb{R}^d$ (action region) and $\xi_i : S \to Z_i$ is a continuous function (impulse function). We denote by $\Phi_i$ the set of strategies for player $i$.

Strategies determine the behaviour of the players, as follows. Fix a starting point $x \in S$ and an $N$-tuple of strategies $\varphi = (\varphi_1, \ldots, \varphi_N)$, where $\varphi_i = (A_i, \xi_i) \in \Phi_i$ is the strategy of player $i$ and the sets $A_i$ are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ for $i \neq j$. Then, $N$ impulse controls $\{(\tau_{i,k}^{\varphi_i}, \xi_{i,k}^{\varphi_i})\}_{k \geq 1}$ (the players’ interventions), a right-continuous process $X^{\varphi}$ (the controlled process), a stopping time $\tau_{S}^{\varphi}$ (the end of the game) are uniquely defined by the following two rules.

1. Player $i$ intervenes if and only if the process enters the set $A_i$, in which case the impulse is given by $\xi_i(y)$, where $y$ is the current state. Recall that choosing $\xi_i(y)$ as the intervention impulse means that player $i$ shifts the process from state $y$ to state $\Gamma_i(y, \xi_i(y))$, as introduced earlier.

2. The game ends when the process exits from $S$.

More precisely, $\{(\tau_{i,k}^{\varphi_i}, \xi_{i,k}^{\varphi_i})\}_{k \geq 1}$, $X^{\varphi}$, $\tau_{S}^{\varphi}$ are defined in the following Definition 2.2, where we use the conventions $\inf \emptyset = \infty$ and $[\infty, \infty) = \emptyset$.

**Definition 2.2.** Let $x \in S$ and $\varphi = (\varphi_1, \ldots, \varphi_N)$, where $\varphi_i = (A_i, \xi_i) \in \Phi_i$ is a strategy for player $i \in \{1, \ldots, N\}$. Assume that $A_i \cap A_j = \emptyset$, for $i \neq j$. For $k \in \{0, \ldots, k\}$, where $k = \sup\{k \in \mathbb{N} \cup \{0\} : \tau_{k} < \alpha^{S}_{k}\}$, we define by induction $\tau_{k} = 0$, $x_0 = x$, $X_0 = Y^{\tau_0}, x_0, \alpha_0^S = \infty$, and

\[
\begin{align*}
\alpha_0^O & = \inf\{s > \tau_{k-1} : X^{k-1}_s \not\in O\}, \\
\tau_{k} & = \min\{\alpha_{k}^{A_{1}}, \ldots, \alpha_{k}^{A_{N}}\}, \\
m_k & = 1_{\{\tau_{k} = \alpha_{k}^{A_{1}}\}} + \cdots + N 1_{\{\tau_{k} = \alpha_{k}^{A_{N}}\}}, \\
\bar{\xi}_k & = \xi_{m_k}(X^{k-1}_{\tau_{k}}),
\end{align*}
\]

[exit time from $O \subseteq S$]

[intervention time]

[index of the player interv. at $\bar{\tau}_k$]

[impulse]
Let \( \bar{k}_i \) be the number of interventions by player \( i \in \{1, \ldots, N\} \) before the end of the game, and, in the case where \( \bar{k}_i \neq 0 \), let \( \eta(i, k) \) be the index of her \( k \)-th intervention \( (1 \leq k \leq \bar{k}_i) \):

\[
\bar{k}_i = \sum_{1 \leq h \leq k} \{m_h = i\}, \quad \eta(i, k) = \min \left\{ l \in \mathbb{N} : \sum_{1 \leq h \leq l} \{m_h = i\} = k \right\}.
\]

Assume now that the times \( \{\bar{\tau}_k\}_{0 \leq k \leq \bar{k}} \) never accumulate strictly before \( \alpha^S_k \). That is, we assume that

\[
\lim_{k \to 1} \bar{\tau}_k = \alpha^S_k \quad \text{in the event} \quad [k = +\infty], \quad \text{with the convention} \quad \alpha^S_\infty = \sup_k \alpha^S_k.
\]

The controlled process \( X^{x,\varphi} \) and the exit time \( \tau^{x,\varphi}_S \) are defined by

\[
X^{x,\varphi} = \tilde{X}^\bar{k}, \quad \tau^{x,\varphi}_S = \alpha^S_k = \inf \{ s \geq 0 : X^{x,\varphi} \notin S \},
\]

with the convention \( \tilde{X}^\infty = \lim_{k \to +\infty} \tilde{X}^\bar{k} \). Finally, the impulse controls \( \{ (\tau^{x,\varphi}_{i,k}, \xi^{x,\varphi}_{i,k}) \}_{k \geq 1} \), with \( i \in \{1, \ldots, N\} \), are defined by

\[
\tau^{x,\varphi}_{i,k} := \begin{cases} 
\bar{\tau}_{\eta(i,k),} & k \leq \bar{k}_i, \\
\tau^{x,\varphi}_S, & k > \bar{k}_i,
\end{cases} \quad \xi^{x,\varphi}_{i,k} := \begin{cases} 
\bar{\xi}_{\eta(i,k),} & k \leq \bar{k}_i, \\
0, & k > \bar{k}_i.
\end{cases}
\]

Notice that, if player \( i \) intervenes a finite number of times, i.e., \( \bar{k}_i = \bar{k}_i(\omega) \) is finite, then the tail of the control is conventionally set to \( (\tau^{x,\varphi}_{i,k}, \xi^{x,\varphi}_{i,k}) = (\tau_S, 0) \) for \( k > \bar{k}_i \). The following lemma characterizes precisely the controlled process \( X^{x,\varphi} \).

**Lemma 2.3.** Let \( x \in S \) and \( \varphi = (\varphi_1, \ldots, \varphi_N) \), where \( \varphi_i = (A_i, \xi_i) \in \Phi_i \) is a strategy for player \( i \in \{1, \ldots, N\} \). Let \( X = X^{x,\varphi} \), \( \tau_S = \tau^{x,\varphi}_S \), \( \tau_{i,k} = \tau^{x,\varphi}_{i,k} \), \( \xi_{i,k} = \xi^{x,\varphi}_{i,k} \) be as in Definition 2.2, for \( i \in \{1, \ldots, N\} \) and \( k \geq 1 \). Then,

- \( X \) admits the following representation, with \( \bar{\tau}_k, x_k \) as in Definition 2.2 and \( Y \) as in (2.1):

\[
X_s = \sum_{k=0}^{\bar{k}_i-1} Y_{\bar{\tau}_k}^{x_k,\bar{\tau}_{k+1}} \mathbb{1}_{[\bar{\tau}_k, \bar{\tau}_{k+1}](s)} + Y_{\bar{\tau}_\bar{k}_i}^{x_{\bar{k}_i},x} \mathbb{1}_{[\bar{\tau}_{\bar{k}_i}, \infty)(s)}.
\]

- \( X \) is right-continuous. More precisely, \( X \) is continuous in \( [0, \infty) \setminus \{ \tau_{i,k} : \tau_{i,k} < \tau_S \} \) and discontinuous in \( \{ \tau_{i,k} : \tau_{i,k} < \tau_S \} \), where

\[
X_{\tau_{i,k}} = \Gamma_i(X_{(\tau_{i,k})^-}, \xi_{i,k}), \quad \xi_{i,k} = \xi_i(X_{(\tau_{i,k})^-}), \quad X_{(\tau_{i,k})^-} \in \partial A_i.
\]

- \( X \) never exits from the set \( (A_1 \cup \cdots \cup A_N)^c \).

**Proof.** We just prove the first property in (2.4), the other ones being immediate. Let \( i \in \{1, \ldots, N\} \), \( k \geq 1 \) with \( \tau_{i,k} < \tau_S \) and set \( \sigma = \eta(i, k) \), with \( \eta \) as in Definition 2.2. By (2.2), (2.3) and Definition 2.2, we have

\[
X_{\tau_{i,k}} = X_{\bar{\tau}_\sigma} = Y_{\bar{\tau}_\sigma}^{\bar{\tau}_\sigma, x}= x_\sigma = \Gamma_i(X_{\bar{\tau}_\sigma}^{\bar{\tau}_\sigma -1}, \bar{\xi}_\sigma) = \Gamma_i(X_{(\bar{\tau}_\sigma)^-}, \bar{\xi}_\sigma) = \Gamma_i(X_{(\tau_{i,k})^-}, \xi_{i,k}),
\]

where the fifth equality is by the continuity of the process \( \bar{X}^{\sigma -1} \) in \( [\bar{\tau}_\sigma, \infty) \) and the next-to-last equality follows from \( \bar{X}^{\sigma -1} = X \) in \( [0, \bar{\tau}_\sigma) \).
Remark 2.4. For $x \in S$ and $\varphi \in \Phi_x$, let $\{F^X_t\}_{t \geq 0}$ denote the natural filtration of the process $X = X^{x,\varphi}$. Then by construction, $\tau_{i,k}$ is a stopping time with respect to the filtration $\{F^X_t\}_{t \geq 0}$ and $\xi_{i,k}$ is a $F^X_{\tau_{i,k}}$-measurable random variable, for $i \in \{1, \ldots, N\}$ and $k \in \mathbb{N}$.

Remark 2.5. In single-player impulse control problems (e.g., [30]), the optimal intervention times are recursively defined by

$$\tau_{k+1} = \inf \{s \geq \tau_k : X^k \in A\},$$

for a suitable set $A$, where $X^k$ represents the controlled process after the $k$-th intervention. Notice that this procedure cannot be directly extended to $N$-player impulse control: In a game setting, the intervention times of player $i$ also depend on her opponents’ past interventions, so that (2.5) would not be well defined in this case. To overcome this technical difficulty and provide a rigorous framework, here we introduce the definition of strategy.

**Objective functions.** Each player aims at minimizing her objective function, made up of four terms: a continuous-time running cost in $[0, \tau_S]$, the discrete-time costs associated to her own interventions, the discrete-time costs associated to her opponents’ interventions, a terminal cost if the game ends.

More precisely, let $f_i, h_i : S \to \mathbb{R}^d$ be given functions, and let $\rho_i > 0$ be strictly positive constants, for $i \in \{1, \ldots, N\}$. For more technical details on the existence and uniqueness of the solution to impulse control problems, see [15]. The functional that player $i$ aims at minimizing is defined as follows.

**Definition 2.6.** Let $x \in S$ and $\varphi = (\varphi_1, \ldots, \varphi_N)$ be a $N$-tuple of strategies. For $i \in \{1, \ldots, N\}$, provided that the right-hand side exists and is finite, we set

$$J^i(x; \varphi) := \mathbb{E}_x \left[ \int_0^{\tau_S} e^{-\rho_i s} f_i(X_s) \, ds + \sum_{k \in \mathbb{N}} \sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} \phi_i(X_{\tau_{i,k}}^-), \xi_{i,k} \right]$$

$$+ \sum_{1 \leq j \leq N} \sum_{\tau_{j,k} \leq \tau_S} e^{-\rho_j \tau_{j,k}} \psi_{i,j}(X_{\tau_{j,k}}, \xi_{j,k}^-) + e^{-\rho_i \tau_S} h_i(X_{\tau_S}) 1_{\{\tau_S < \infty\}} \right],$$

with $X = X^{x,\varphi}$, $\tau_S = \tau_S^{x,\varphi}$, $\{(\tau_{i,k}, \xi_{i,k})\}_{k \geq 1} = \{(\tau_{i,k}^{x,\varphi}, \xi_{i,k}^{x,\varphi})\}_{k \geq 1}$ as in Definition 2.2.

The subscript in the expectation denotes, as in control theory, conditioning with respect to starting point $X_0^{x,\varphi} = x$. To shorten the notations, we will often omit the initial state and write $\mathbb{E}$. Also, notice that in the summations we only consider times strictly smaller than $\tau_S$: indeed, since the game ends in $(\tau_S)^-$, interventions in the form $\tau_{i,k} = \tau_S$ are meaningless for the game.

**Admissible strategies and Nash equilibria.** Before defining a Nash equilibrium (NE) for the game, we define, for each starting point $x \in S$, the set $\Phi_x$ of admissible strategies, i.e., strategies as in Definition 2.1 with additional properties assuring that the game is well defined.

**Definition 2.7.** Let $x \in S$ and $\varphi_i = (A_i, \xi_i)$ be a strategy for player $i \in \{1, \ldots, N\}$. We say that the $N$-tuple $\varphi = (\varphi_1, \ldots, \varphi_N)$ is $x$-admissible, written as $\varphi \in \Phi_x$, if:

1. the sets $A_1, \ldots, A_N$ are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ for $i \neq j$;
2. for $i, j \in \{1, \ldots, N\}$ with $i \neq j$, the following random variables are in $L^1(\Omega)$:

$$
\int_0^{\tau_S} e^{-\rho_i s} |f_i| (X_s) ds, \quad e^{-\rho_i \tau_S} |h_i| (X_{\tau_S}),
$$

$$
\sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} |\phi_i| (X_{(\tau_{i,k})}, \xi_{i,k}), \quad \sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} |\psi_{i,j}| (X_{(\tau_{j,k})}, \xi_{j,k});
$$

(2.7)

3. for each $i \in \{1, \ldots, N\}$ and $p \in \mathbb{N}$, the random variable $\|X\|_{\infty} = \sup_{t \geq 0} |X_t|$ is in $L^p(\Omega)$:

$$
\mathbb{E}[\|X\|_{\infty}^p] < \infty;
$$

(2.8)

4. for $i \in \{1, \ldots, N\}$, we have

$$
\lim_{k \to +\infty} \tau_{i,k} = \tau_S.
$$

(2.9)

The first condition in Definition 2.7 ensures that there are no simultaneous intervening from multiple players. This is essential for the feasibility of controls in stochastic games where the controlled dynamics could be discontinuous. The second condition assures that the functionals $J^i(x; \varphi)$ in (2.6) are well-defined, for each $i \in \{1, \ldots, N\}$. The third condition will be used in the proof of the verification theorem where sufficient conditions for the NEs are specified. Finally, the fourth condition prevents the players from accumulating the interventions before the end of the game.

We now provide the definition of NE and value functions. Given a tuple of strategies $\varphi = (\varphi_1, \ldots, \varphi_N)$, an index $i \in \{1, \ldots, N\}$ and a strategy $\bar{\varphi} \in \Phi_i$, we denote by $s_i, \bar{\varphi} (\varphi)$ the $N$-tuple we get when substituting the $i$-th component of $\varphi$ by $\bar{\varphi}$, that is

$$
s_i, \bar{\varphi} (\varphi) := (\varphi_1, \ldots, \varphi_{i-1}, \bar{\varphi}, \varphi_{i+1}, \ldots, \varphi_N).
$$

Definition 2.8. Given $x \in S$, we say that the admissible $N$-tuple of strategies $\varphi^* \in \Phi_x$ is a NE of the game if

$$
J^i(x; \varphi^*) \leq J^i(x; s_{i, \bar{\varphi}} (\varphi^*)),
$$

for each $i \in \{1, \ldots, N\}$ and each $\varphi_i \in \Phi_i$ such that $s_{i, \bar{\varphi}} (\varphi^*) \in \Phi_x$. Finally, the valued functions of the game are defined as follows: if $x \in S$ and $\varphi^* \in \Phi_x$ is a NE, the value of the game for player $i \in \{1, \ldots, N\}$ is

$$
V_i(x) := J^i(x; \varphi^*).
$$

2.2 Verification theorem

In this section we establish a verification theorem for the games defined in Section 2, providing sufficient conditions to determine the value functions and an NE. This verification theorem links the impulse games with a suitable system of quasi-variational inequalities (QVI). Note that a special case of this verification theorem for $N = 2$ was presented in [1].

In Section 2.2.1 we heuristically introduce the system of QVIs, providing the intuition behind each equation involved. These arguments are made rigorous in Section 2.2.2, with the precise statement and proof of the verification theorem.
2.2.1 The quasi-variational inequalities

We start by heuristically guessing an expression for a NE $\varphi^* = (\varphi^*_1, \ldots, \varphi^*_N)$ and for the corresponding value functions $V_i$ of the game.

Consider a game as in Section 2. Assume for a moment that the value functions $V_i$, $i \in \{1, \ldots, N\}$ are known. Moreover, assume that for every $i$ there exists a (unique) function $\xi_i : S \to Z_i$ such that

$$\{\xi_i(x)\} = \arg \min_{\delta \in Z_i} \{V_i(\Gamma^i(x, \delta)) + \phi_i(x, \delta)\}, \quad (2.10)$$

for each $x \in S$. We define the intervention operators by

$$M_i V_i(x) = V_i(\Gamma^i(x, \xi_i(x))) + \phi_i(x, \xi_i(x)),$$
$$H_{i,j} V_i(x) = V_i(\Gamma^j(x, \xi_j(x))) + \psi_{i,j}(x, \xi_j(x)), \quad (2.11)$$

for $x \in S$ and $i, j \in \{1, \ldots, N\}$, with $i \neq j$.

The functions in (2.10) and (2.11) are intuitive. If $x$ is the current state of the process, and player $i$ (resp. player $j$) intervenes with impulse $\delta$, the value of the game for player $i$ can be represented as $V_i(\Gamma^i(x, \delta)) + \phi_i(x, \delta)$ (resp. $V_i(\Gamma^j(x, \delta)) + \psi_{i,j}(x, \delta)$), that is, as the sum of the intervention cost and the value in the new state. As a consequence, $\xi_i(x)$ in (2.10) is the impulse that player $i$ would use in case she decides to intervene. Similarly, $M_i V_i(x)$ (resp. $H_{i,j} V_i(x)$) represents the payoff for player $i$ when player $i$ (resp. player $j \neq i$) takes the best immediate action and behaves optimally afterwards.

Notice that it is not always optimal to intervene, so $M_i V_i(x) \geq V_i(x)$, for each $x \in S$, and that player $i$ should intervene (with impulse $\xi_i(x)$) only if $M_i V_i(x) = V_i(x)$. As a consequence, provided that an explicit expression for $V_i$ is available, an NE is heuristically given by $\varphi^* = (\varphi^*_1, \ldots, \varphi^*_N)$, where $\varphi^*_i = (A^*_i, \xi^*_i)$ is given, for each $i \in \{1, \ldots, N\}$, by

$$A^*_i = \{M_i V_i - V_i = 0\}, \quad \xi^*_i = \xi_i.$$

Practically, this means that player $i$ intervenes when the process enters the region $\{M_i V_i - V_i = 0\}$, i.e., when $M_i V_i(x) = V_i(x)$. When this happens, her impulse is $\xi_i(x)$, where $x$ is the current state. The verification theorem in the next section will give a rigorous proof to this heuristic argument.

To complete the argument, we need to determine the value functions $V_i$. Assume that $V_i$ are smooth enough so that we can define

$$L V_i = b \cdot \nabla V_i + \frac{1}{2} \text{tr} (\sigma^t D^2 V_i), \quad (2.12)$$

where $b, \sigma$ are as in (2.1), $\sigma^t$ denotes the transpose of $\sigma$ and $\nabla V_i, D^2 V_i$ are the gradient and the Hessian matrix of $V_i$, respectively. Then $V_i$ should satisfy the following quasi-variational inequalities (QVIs), where $i, j \in \{1, \ldots, N\}$:

$$\begin{cases}
V_i = h_i, & \text{in } \partial S, \\
M_j V_j - V_j \geq 0, & \text{in } S, \\
H_{i,j} V_i - V_i = 0, & \text{in } \bigcup_{j \neq i} \{M_j V_j - V_j = 0\}, \\
\min \{L V_i - \rho_i V_i + f_i, M_i V_i - V_i\} = 0, & \text{in } \bigcap_{j \neq i} \{M_j V_j - V_j > 0\}. 
\end{cases} \quad (2.13)$$

Notice that there is a small abuse of notation in (2.13a), as $V_i$ is not defined in $\partial S$, so that (2.13a) means $\lim_{y \to x} V_i(y) = h_i(x)$, for each $x \in \partial S$. 

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Intuition behind each in (2.13): the terminal condition (2.13a) is obvious, and (2.13b), already stated above, is a standard condition in impulse control theory. As for (2.13c), if player $j$ intervenes (i.e., $M_jV_j - V_j = 0$), by the definition on NE, we expect no losses for player $i \neq j$, that is $H_{i,j}V_i - V_i = 0$. Meanwhile, if all the players except $i$ are not intervening (hence, $M_jV_j - V_j > 0$ for all $j \neq i$), then player $i$ faces a classical one-player impulse control problem, so that $V_i$ satisfies the corresponding QVI of $\min \{ LV_i - \rho_iV_i + f_i, M_iV_i - V_i \} = 0$, which is (2.13d). In short, the latter condition says that $LV_i - \rho_iV_i + f_i = 0$ when she does not intervene, whereas $LV_i - \rho_iV_i + f_i \geq 0$ when she intervenes.

Remark 2.9. For any player $i$, the region where she chooses not to intervene, as in (2.13d) when $\min \{ LV_i - \rho_iV_i + f_i, M_iV_i - V_i \} = 0$, is decided by not just player $i$ but all $N$ players; it is indeed the common non-action region $C$. On $C$, it is necessary to have $LV_i - \rho_iV_i + f_i = 0$ for all $i \in \{1, \ldots, N\}$. The condition that $M_iV_i - V_i \geq 0$, however, needs an extra verifying step: it is not entirely player $i$’s decision to wait, yet this choice has to be the best one she can make at a NE. This marks the subtlety of the NE and one crucial difference between the single-player control problem and the multi-player game.

### 2.2.2 Statement and proof

We now provide a rigorous proof of the results heuristically introduced in the previous section. Notations and assumptions from Section 2 are adopted from now on.

**Theorem 2.10 (Verification Theorem).** Let $V_1, \ldots, V_N$ be functions from $S$ to $\mathbb{R}$, assume that

$$\{ \xi_i(x) \} = \arg \min_{\delta \in \mathcal{D}_i} \left\{ V_i(\Gamma^i(x, \delta)) + \phi_i(x, \delta) \right\}$$

holds and set $D_i := \{ M_iV_i - V_i > 0 \}$. Moreover, for $i \in \{1, \ldots, N\}$ assume that:

(i) $V_i$ is a solution to (2.13a)-(2.13d);

(ii) $V_i \in C^2(\bigcap_{j \neq i} D_j \setminus \partial D_i) \cap C^1(\bigcap_{j \neq i} D_j) \cap C(\bigcap_{j \neq i} D_j)$ and it has polynomial growth;

(iii) $\partial D_i$ is a Lipschitz surface, and $V_i$ has locally bounded derivatives up to the second order in some neighbourhood of $\partial D_i$.

Finally, let $x \in S$ and define $\varphi^* = (\varphi^*_1, \ldots, \varphi^*_N)$, with

$$\varphi^*_i := (A^*_i; \xi^*_i), \quad A^*_i := \{ M_iV_i - V_i = 0 \}, \quad \xi^*_i := \xi_i,$$

where $i \in \{1, \ldots, N\}$ and the function $\xi_i$ is as in (2.10). Then, provided that $\varphi^* \in \Phi_x$,

$\varphi^*$ is an NE and $V_i(x) = J^i(x; \varphi^*)$ for $i \in \{1, \ldots, N\}$.

**Proof.** Let $x \in S$, $i \in \{1, \ldots, N\}$ and $\varphi_i \in \Phi_i$ such that $s_i, \varphi_i(\varphi^*) \in \Phi_x$. Notice that $s_i, \varphi_i(\varphi^*)$ corresponds to the case where all the players except player $i$ behave optimally. By Definition 2.8, we have to prove that

$$V_i(x) = J^i(x; \varphi^*), \quad V_i(x) \leq J^i(x; s_i, \varphi_i(\varphi^*)).$$

**Step 1:** $V_i(x) \leq J^i(x; s_i, \varphi_i(\varphi^*))$. To simplify the notations, we omit the dependence on $i, x, \varphi$ and write

$$X = X_{x, s_i, \varphi_i}(\varphi^*), \quad \tau_{j,k} = \tau^x_{j,k}(\varphi^*), \quad \xi_{j,k} = \xi^x_{j,k}(\varphi^*). \quad (2.14)$$
The properties in Lemma 2.3 imply that, for \( j \neq i \), \( s \geq 0 \), \( \tau_{j,k} < \infty \),

\[
(M_j V_j - V_j)(X_s) > 0,  \quad (M_j V_j - V_j)(X_{(\tau_{j,k})^-}) = 0,  \quad \xi_{j,k} = \xi_j(X_{(\tau_{j,k})^-}).
\]

(2.15a)  \hspace{1cm} (2.15b)  \hspace{1cm} (2.15c)

We first approximate \( V_i \) with regular functions. Since (ii) and (iii) hold, by [29, proof of Thm. 10.4.1 and App. D] there exists a sequence of functions \( \{V_{i,m}\}_{m \in \mathbb{N}} \) such that:

(a) \( V_{i,m} \in C^2(\cap_{j \neq i} D_j) \cap C^0(\cap_{j \neq i} \bar{D}_j) \) for each \( m \in \mathbb{N} \) (in particular, the function \( \mathcal{L} V_{i,m} \) is well-defined in \( \cap_{j \neq i} D_j \));

(b) \( V_{i,m} \to V_i \) as \( m \to \infty \), uniformly on the compact subsets of \( \cap_{j \neq i} D_j \);

(c) \( \{\mathcal{L} V_{i,m}\}_{m \in \mathbb{N}} \) is locally bounded in \( \cap_{j \neq i} D_j \) and \( \mathcal{L} V_{i,m} \to \mathcal{L} V_i \) as \( m \to \infty \), uniformly on the compact subsets of \( \cap_{j \neq i} D_j \setminus \partial D_i \).

For each \( r > 0 \) and \( \ell \in \mathbb{N} \), we set

\[
\tau_{r,\ell} = \tau_r \wedge \tau_{1,\ell} \wedge \cdots \wedge \tau_{N,\ell},
\]

(2.16)

where \( \tau_r = \inf \{ s > 0 : X_s \notin B(0,r) \} \) is the exit time from the ball with radius \( r \). By (2.15a) we have that \( X_s \in \cap_{j \neq i} D_j \) for each \( s > 0 \). Since \( V_{i,m} \in C^2(\cap_{j \neq i} D_j) \) by (a), for each \( m \in \mathbb{N} \) we can apply Itô’s formula to the process \( e^{-\rho_t V_i} V_{i,m}(X_t) \) over the interval \([0,\tau_{r,\ell})\). Taking the conditional expectations, we get

\[
V_{i,m}(x) = \mathbb{E}_x \left[ -\int_0^{\tau_{r,\ell}} e^{-\rho_t (\mathcal{L} V_{i,m} - \rho_t V_i)(X_s)} ds - \sum_{\tau_{i,k} < \tau_{r,\ell}} e^{-\rho_{\tau_{i,k}} (V_{i,m}(X_{(\tau_{i,k})^-}) - V_{i,m}(X_{(\tau_{i,k})^-}))} \\
- \sum_{j \neq i} \sum_{\tau_{j,k} < \tau_{r,\ell}} e^{-\rho_{\tau_{j,k}} (V_{i,m}(X_{(\tau_{j,k})^-}) - V_{i,m}(X_{(\tau_{j,k})^-})) + e^{-\rho_{\tau_{r,\ell}} V_{i,m}(X_{(\tau_{r,\ell})^-})} \right].
\]

(2.17)

Notice that (2.17) is well-defined: since \( \tau_{r,\ell} \leq \tau_r \), \( X \) belongs to the compact set \( \overline{B(0,r)} \), where the continuous function \( V_{i,m} \) is bounded; moreover, the two summations consist in a finite number of terms since \( \tau_{r,\ell} \leq \tau_{i,\ell} \) for each \( i \in \{1, \ldots, n\} \). Also, notice that in (2.17) we need to write \( V_{i,m}(X_{(\tau_{i,\ell})^-}) \), as we have a jump at time \( \tau_{r,\ell} \). We now pass to the limit in (2.17) as \( m \to \infty \): since \( X \) belongs to the compact set \( \overline{B(0,r)} \), by the uniform convergence in (b) and (c) we get

\[
V_i(x) = \mathbb{E}_x \left[ -\int_0^{\tau_{r,\ell}} e^{-\rho_t (\mathcal{L} V_i - \rho_t V_i)(X_s)} ds - \sum_{\tau_{i,k} < \tau_{r,\ell}} e^{-\rho_{\tau_{i,k}} (V_i(X_{(\tau_{i,k})^-}) - V_i(X_{(\tau_{i,k})^-}))} \\
- \sum_{j \neq i} \sum_{\tau_{j,k} < \tau_{r,\ell}} e^{-\rho_{\tau_{j,k}} (V_i(X_{(\tau_{j,k})^-}) - V_i(X_{(\tau_{j,k})^-})) + e^{-\rho_{\tau_{r,\ell}} V_i(X_{(\tau_{r,\ell})^-})} \right].
\]

(2.18)

We now estimate each term in the right-hand side of (2.18). As for the first term, since \( (M_j V_j - V_j)(X_s) > 0 \) for each \( j \neq i \) by (2.15a), from (2.13d) it follows that

\[
(\mathcal{L} V_i - \rho_t V_i)(X_s) \geq -f_i(X_s),
\]

(2.19)

for all \( s \in [0, \tau_S] \). Let us now consider the second term: by (2.13b) and the definition of \( M_i V_i \) in (2.11), for every stopping time \( \tau_{i,k} < \tau_S \) we have

\[
V_i(X_{(\tau_{i,k})^-}) \leq M_i V_i(X_{(\tau_{i,k})^-})
\]
As for the third term, let us consider any stopping time $\tau_{j,k} < \tau_S$, with $j \neq i$. By (2.15a) we have $(\mathcal{M}_j V_j - V_j)(X_{(\tau_{j,k})^-}) = 0$; hence, the condition in (2.13c), the definition of $\mathcal{H}_{i,j} V_i$ in (2.11) and the expression of $\xi_{j,k}$ in (2.15c) imply that

$$V_i(X_{(\tau_{j,k})^-}) = \mathcal{H}_{i,j} V_i(X_{(\tau_{j,k})^-}) = V_i(\Gamma^j(X_{(\tau_{j,k})^-}; \xi_j(X_{(\tau_{j,k})^-}))) + \psi_{i,j}(X_{(\tau_{j,k})^-}; \xi_j(X_{(\tau_{j,k})^-})) = V_i(\Gamma^j(X_{(\tau_{j,k})^-}; \xi_j(X_{(\tau_{j,k})^-}))) + \psi_{i,j}(X_{(\tau_{j,k})^-}; \xi_j(X_{(\tau_{j,k})^-})).$$

(2.20)

By (2.18) and the estimates in (2.19)-(2.21) it follows that

$$V_i(x) \leq \mathbb{E}_x \left[ \int_0^{\tau_{r,\ell}} e^{-\rho s} f_i(X_s) ds + \sum_{\tau_{i,k} < \tau_{r,\ell}} e^{-\rho \tau_{i,k}} \phi_i(X_{(\tau_{i,k})^-}; \xi_{i,k}) \right.$$ 

$$+ \sum_{j \neq i} \sum_{\tau_{j,k} < \tau_{r,\ell}} e^{-\rho \tau_{j,k}} \psi_{i,j}(X_{(\tau_{j,k})^-}; \xi_{j,k}) + e^{-\rho \tau_{r,\ell}} V_i(X_{(\tau_{r,\ell})^-}) \right].$$

Thanks to the conditions in (2.7) and (2.8) together with the polynomial growth of $V_i$ in (ii), we now use the dominated convergence theorem and pass to the limit, first as $r \to \infty$ and then as $\ell \to \infty$, so that the stopping times $\tau_{r,\ell}$ converge to $\tau_S$ by (2.9). In particular, for the fourth term we notice that by (ii) and (2.8) we have

$$V_i(X_{(\tau_{r,\ell})^-}) \leq C(1 + |X_{(\tau_{r,\ell})^-}|^p) \leq C(1 + \|X\|^p) \in L^1(\Omega),$$

(2.22)

for suitable constants $C > 0$ and $p \in \mathbb{N}$. Therefore, the corresponding limit for the fourth term immediately follows by the continuity of $V_i$ in the case $\tau_S < \infty$ and by (2.22) itself in the case $\tau_S = \infty$ (as a direct consequence of (2.8), we have $\|X\|^p \in L^1(\Omega)$ a.s.). Hence,

$$V_i(x) \leq \mathbb{E}_x \left[ \int_0^{\tau_S} e^{-\rho s} f_i(X_s) ds + \sum_{\tau_{i,k} < \tau_S} e^{-\rho \tau_{i,k}} \phi_i(X_{(\tau_{i,k})^-}; \xi_{i,k}) \right.$$ 

$$+ \sum_{j \neq i} \sum_{\tau_{j,k} < \tau_S} e^{-\rho \tau_{j,k}} \psi_{i,j}(X_{(\tau_{j,k})^-}; \xi_{j,k}) + e^{-\rho \tau_S} h_i(X_{(\tau_S)^-}) 1_{\{\tau_S < +\infty\}} = J^i(x; s_j \varphi_j(\varphi^*)).$$

Step 2: $V_i(x) = J^i(x; \varphi^*)$. Similar as in Step 1, except that all the inequalities are equalities by the properties of $\varphi^*$. 

2.3 Example: two-player cash management problem

Now let us revisit the cash management game in Section 1, using the notations introduced in Section 2. We here consider the two-player game: $N = 2$, $b_i = 0$, $\sigma_i = \sigma > 0$. The uncontrolled cash level of the two players $X_t = (X_t^1, X_t^2)$ is

$$dX_t = \sigma dW_t, \quad X_{0-} = x.$$
where \( W \) is a two-dimensional standard Brownian motion and \( x \in \mathbb{R}^2 \). Let

\[
\varphi = (\varphi_1, \varphi_2), \quad \varphi_1 = (A_1, \xi_1), \quad \varphi_2 = (A_2, \xi_2),
\]
denote the strategies of the players for this game, as in Definition 2.1. Since player \( i \in \{1, 2\} \) intervenes by shifting her own component \( X_i \) of the cash level, we have

\[
\Gamma_i(x, \delta) = x + \delta, \quad Z_1 = \{ (\delta_1, 0) : \delta_1 \in \mathbb{R} \}, \quad Z_2 = \{ (0, \delta_2) : \delta_2 \in \mathbb{R} \}.
\]

This means that player 1 (resp. player 2) intervenes by moving the process from state \((x_1, x_2)\) to state

\[
\xi_1(x_1, x_2) = (x_1 + \xi_1(x_1, x_2), x_2) \quad \text{(resp. } \xi_2(x_1, x_2) = (x_1, x_2 + \xi_2(x_1, x_2)) \text{)},
\]

for suitable functions \( \xi_i \). Notice that, as a consequence, the controlled process \( X_i = (X^1_i, X^2_i) \) satisfies

\[
dX^i_\tau = \sigma dW^i_\tau + \sum_{\tau_i, k \leq \tau} \delta(t - \tau_i, k) \tilde{\xi}_{i,k}, \quad X^{i}_0 = x_i,
\]

where

\[
\tilde{\xi}_{i,k} = \hat{\xi}_i\left( X^{i}_{(\tau_i,k)^-}, X^{2}_{(\tau_i,k)^-} \right).
\]

Let now \( i, j \in \{1, 2\} \) with \( j \neq i \). The cost function for player \( i \) under the control policy \( \varphi = (\varphi_1, \varphi_2) \) is given by

\[
J^i(x, \varphi) = \mathbb{E}_x \left[ \int_0^\infty e^{-rt} f_i(X_t) dt + \sum_{k \geq 1} e^{-r\tau_i, k} \phi_i(\xi_{i,k}) + \sum_{k \geq 1} e^{-r\tau_{j,k}} \psi_{i,j}(\xi_{j,k}) \right],
\]

where

\[
\begin{align*}
&f_i(x) = h \left| x_i - \frac{1}{N} \sum_{j=1}^N x_j \right|, \quad x \in \mathbb{R}^2, N = 2, \\
&\phi_i(\xi) = K + k|\xi|, \quad \xi \in \mathbb{R}, \\
&\psi_{i,j}(\xi) = c, \quad \xi \in \mathbb{R},
\end{align*}
\]

for positive constants \( h, K, k, c \). The goal of player \( i \) is to minimize the cost \( J^i \): we are interested in finding \( \varphi^* = (\varphi^*_1, \varphi^*_2) \) such that Definition 2.8 holds.

By the symmetry of the problem structure, we seek for an NE where the action regions take the form of

\[
A_1 = \{ x : x_1 - x_2 \geq u \}, \quad A_2 = \{ x : x_2 - x_1 \geq u \}
\]

for some \( u > 0 \), with appropriate impulse functions such that

\[
\xi_1(x) = (U - x_1 + x_2, 0), \quad \xi_2(x) = (0, U - x_2 + x_1)
\]

for some \( U < u \). Recall that this means that player 1 (resp. player 2) intervenes when \( X^1_1 - X^2_1 \geq u \) (resp. \( X^2_2 - X^1_1 \geq u \)) and shifts her component so as to have \( X^1_1 - X^2_1 = U \) (resp. \( X^2_2 - X^1_1 = U \)). Note that \( A_1 \cap A_2 = \emptyset \) and that \( C = \{ x : -u < x_1 - x_2 < u \} \) is the common waiting region, i.e., where no player intervenes.

By the same symmetry argument, we look for value functions in the form of

\[
V_i(x_1, x_2) = w_i(x_i - x_j), \quad i, j \in \{1, 2\}, \quad i \neq j,
\]

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for some functions \(w_i\). In this case, player 1 and player 2 are indistinguishable, therefore it suffices to study the value function of player 1. Now the waiting region for player 1 is \(D_1 = \{x: x_1 - x_2 < u\}\), and define \(D_{-1} = \{x: x_2 - x_1 < u\} = \{x: x_1 - x_2 > -u\}\). By the corresponding QVI and the regularity requirement in the Verification Theorem 2.10, the function \(w_1 : \mathbb{R} \rightarrow \mathbb{R}\) need to satisfy the following system of equations and inequalities:

\[
\begin{align*}
\frac{d}{ds} w_1(s) &= \begin{cases} 
w_1(u) + k(s - u), & s \geq u; \\
\frac{h_2}{2} s + c_1 e^{\lambda_2 s} + c_2 e^{-\lambda_2 s}, & 0 \leq s < u; \\
-\frac{h_2}{2} s + \left( c_1 + \frac{h_2}{r\lambda_2} \right) e^{\lambda_2 s} + \left( c_2 - \frac{h_2}{r\lambda_2} \right) e^{-\lambda_2 s}, & -u \leq s \leq 0; \\
w_1(-u), & s \leq -u;
\end{cases} \\
\lambda w_1(s) - w_1(s) &\geq 0, \quad \forall s \in D_{-1};
\end{align*}
\]

where \(h_2 = \frac{h}{2}, \sigma_2 = \sqrt{2}\sigma, \lambda_2 = \frac{\sqrt{2}}{\sigma_2}\) and \((c_1, c_2, u, U)\) remain to be determined. Accordingly, \(w_2(s) = w_1(-s)\) for any \(s \in \mathbb{R}\).

Now, similar argument as in [12] shows that when \(h_2 - rk > 0, c > 0\), there exists a solution \(w_1\) to Equations (2.23a) to (2.23d) satisfying \(c_1 < 0, c_2 > 0, 0 < U < u\). Moreover, if such solution \(w_1\) as above satisfies (2.23e), then an NE to the cash management problem \(\varphi^* = (\varphi_1^*, \varphi_2^*)\) is characterized by

\[
\begin{align*}
A_1^* &= \{x: x_1 - x_2 \geq u\}, \quad A_2^* = \{x: x_2 - x_1 \geq u\}, \\
\xi_1^*(x) &= (U - x_1 + x_2, 0), \quad \xi_2^*(x) = (0, U - x_2 + x_1).
\end{align*}
\]

(NE-1)

The corresponding value functions are given by

\[
V_1(x_1, x_2) = w_1(x_1 - x_2), \quad V_2(x_1, x_2) = w_1(x_2 - x_1).
\]

The optimality of (NE-1) can be easily verified by checking the conditions in the Verification Theorem 2.10.

Figure 1a shows the value functions for both players if they adopt the control policy specified in (NE-1). Figure 1b illustrates the control policy, with \(h = 2, K = 3, k = 1, r = 0.5, \sigma = \frac{\sqrt{2}}{2}\) and \(c = 1\), where the thresholds can be solved as \(U = 0.686\) and \(u = 5.658\), with \(c_1 = -0.003\) and \(c_2 = 1.972\).
Multiple NEs. In general, NE for nonzero-sum games may not be unique. In this example, an alternative NE can be derived by switching action regions between the two players. For instance, if player 1 is to dictate the game whereas player 2 is a complete follower, the action region for player 1 can be characterized by
\[ A_1 = \{ x \in \mathbb{R}^2 : |x_1 - x_2| > u \}, \text{ and } A_2 = \emptyset. \]
That is, let \( s = x_1 - x_2 \) then
\[
V_1(x) = w_1(x_1 - x_2) \text{ and } V_2(x) = w_2(x_1 - x_2),
\]
where
\[
w_1(s) = \begin{cases} 
  w_1(u) + k(s - u), & s \geq u; \\
  w_1(-s), & 0 \leq s \leq u;
\end{cases}
\]
\[
w_1(u) = \dot{w}_1(U) = k; \quad w_1(U) = w_1(u) + K + k(u - U); \quad (2.24a)
\]
\[
Mw_1(s) - w_1(s) \geq 0, \quad s \in \mathbb{R}; \quad (2.24b)
\]
\[
w_2(s) = \begin{cases} 
  w_2(u), & s \geq u; \\
  w_2(-s), & 0 \leq s \leq u;
\end{cases}
\]
\[
w_2(u) = w_2(U) + c; \quad (2.24c)
\]
\[
Mw_2(s) - w_2(s) \geq 0, \quad -u \leq s \leq u. \quad (2.25b)
\]

Now, assume that \( h_2 - rk > 0, c > 0 \), then again one can show that there exists a solution \( w_1 \) satisfying Equations (2.24a) to (2.24d) with \( c_1 \in (-\frac{h_2}{r\lambda_2}, 0) \) as well as \( 0 < U < u \), and \( w_2 \) satisfying Equations (2.25a) to (2.25c). Moreover, if such solution \( w_2 \) satisfies (2.25c), then an NE to the cash management problem in Section 2.3 \( \varphi^* = (\varphi^*_1, \varphi^*_2) \) is characterized by
\[
\begin{align*}
A_1^* &= \{ x : |x_1 - x_2| \geq u \}, \quad A_2^* = \emptyset; \\
\xi_1^*(x) &= \begin{cases} 
  (U - x_1 + x_2, 0), & \text{if } x_1 - x_2 \geq u, \\
  (-U - x_1 + x_2, 0), & \text{if } x_1 - x_2 \leq -u.
\end{cases}
\end{align*}
\]

Notice that we do not need to define \( \xi_2^* \), as player 2 never intervenes.

Figure 2a shows the value functions and Figure 2b demonstrates the NE, under the same values of \( h, K, k, r \) and \( \sigma \), with thresholds \( U = 0.993 \) and \( u = 1.999 \), and \( c_1 = -0.101 \) and \( c_2 = -0.133 \).
3 Mean-field game with impulse controls

As $N$ increases, it becomes more difficult and infeasible to solve the $N$-player game analytically. We will now introduce an MFG framework here for the impulse control game and show that this MFG provides a reasonable approximation to the $N$-player game.

3.1 Mean-field games (MFGs)

Consider an infinite number of rational and indistinguishable players who interact through the cost structure consisting of a running cost $f$ and the cost of control $\phi$. Each player seeks for the optimal impulse control policy $\varphi^*$ among the set of admissible to minimize the total discounted cost. That is,

$$V(x) = \inf_{\varphi=(\tau_n,\xi_n)_{n \in \mathbb{N}}} J_\infty(x, \varphi),$$

$$J_\infty(x, \varphi) = \mathbb{E}_x \left[ \int_0^\infty e^{-rt} f(X_t, m) dt + \sum_{n=1}^\infty e^{-r\tau_n} \phi(\xi_n) \right],$$

$$dX_t = b(X_{t-}) dt + \sigma(X_{t-}) dW_t + \sum_{\tau_n \leq t} \delta(t - \tau_n) \xi_n, \quad X_0^- = x, \quad m = \lim_{t \to \infty} \mathbb{E}_x[X_t].$$

Here $m = \lim_{t \to \infty} \mathbb{E}_x[X_t]$. Note that any individual player now loses sight of individual opponents, we can assume $\psi_{i,j} = 0$.

Assumptions. To ensure that the MFG is well-posed, we have the following assumptions as in [15] the classical impulse control setting.

(A1) The drift $b$ and the volatility $\sigma$ are all Lipschitz, that is, $\exists C_b, C_\sigma$, both strictly positive, such that

$$\begin{aligned}
|b(x) - b(y)| &< C_b |x - y|, \\
|\sigma(x) - \sigma(y)| &< C_\sigma |x - y|. 
\end{aligned} \quad (3.1)$$

(A2) The running cost $f(x, m) \geq 0$ is continuous in $m$, and for any fixed $m$, $\exists C_f = C_f(m) > 0$ such that $|f(x, m) - f(y, m)| < C_f |x - y|$.
(A3) The cost of control satisfies

\[
\begin{cases}
K := \inf_{\xi \in \mathbb{R}} \phi(\xi) > 0, \\
\phi \in \mathcal{C}(\mathbb{R} \setminus \{0\}), \\
\lim_{|\xi| \to \infty} \phi(\xi) = +\infty, \\
(\phi(\xi_1) + \phi(\xi_2) - \phi(\xi_1 + \xi_2)) + K, \quad \forall \xi_1, \xi_2 \in \mathbb{R}.
\end{cases}
\]  

(A4) \( r > 2C_b + C^2_\sigma \).

**Definition 3.1.** A pair of control policy and mean information \((\varphi^*, (\tau^*_n, \xi^*_n))_{n \in \mathbb{N}}, m^* \) is said to be a solution to the (MFG) if

- \( v(x) = J_\infty(x, \varphi^*) \),
- \( m^* = \lim_{t \to \infty} \mathbb{E}_x [X^*_t] \) where

\[
dX^*_t = b(X^*_t)dt + \sigma(X^*_t)dW_t + \sum_{t \leq \tau^*_n} \delta(t - \tau^*_n)\xi^*_n.
\]

Note from Definition 3.1 we are looking for a stationary solution to the MFG. To solve the MFG, we propose the following three-step approach. This is parallel to the solution under regular control setting.

**Step 1. Solve the optimal impulse control problem under a fixed mean information \( m \).**

Fixed \( m \), the quasi-variational inequality (QVI) associated with (MFG) is given by

\[
\min \{ \mathcal{L}V(\cdot) - rV(\cdot) + f(\cdot, m), \mathcal{M}V(\cdot) - V(\cdot) \} = 0. 
\]  

Here \( \mathcal{L} \) is given in Equation 2.12.

By [15, Theorem 4.2], \( v \) is the unique \( W^{2,p}_{\text{loc}} \) solution to (3.3) in the viscosity sense; in particular, the value function is of \( C^1 \).

Now, define the following regions.

- Waiting region: \( \mathcal{D}(m) = \{ x \in \mathbb{R} : \mathcal{L}V(x) - rV(x) + f(x, m) = 0 \} \),
- Action region: \( \mathcal{A}(m) = \{ x \in \mathbb{R} : V(x) = \mathcal{M}V(x) \} \).

Then, the optimal control \( \varphi^*(m) = \{(\tau^*_n(m), \xi^*_n(m))\}_{n \in \mathbb{N}} \) is

\[
\tau^*_n(m) = \inf \{ t \geq 0 : X_{t-} \in \mathcal{A}(m) \}, \\
\tau^*_n(m) = \inf \{ t > \tau^*_{n-1}(m) : X_{t-} \in \mathcal{A}(m) \}, \forall n \geq 1; \\
\xi^*_n(m) = \inf \{ \xi \in \mathbb{R} : \xi^* = \arg\min_{\xi \in \mathbb{R}} v \left( X_{\tau^*_n(m)-} + \xi \right) + \phi(\xi) \}, \forall n \geq 1. 
\]

We can establish a proper mapping from any given fixed mean information \( m \) to the corresponding optimal impulse control \( \varphi^*(m) \), denoted by \( \Gamma_1 : m \mapsto \varphi^*(m) \).

For the well-definedness of the problem, here we make an additional assumption.

(A5) For any \( m \in \mathbb{R} \), there exists \(-\infty < b_l(m) < b_u(m) < \infty \) such that the above waiting region \( \mathcal{D}(m) \subset [b_l(m), b_u(m)] \).
Step 2. Update the mean information. Under the optimal control $\varphi^*$ given in Step 1, the dynamic of $X_t$ becomes

$$dX_t^* = b(X_t^*)dt + \sigma(X_t^*)dW_t + \sum_{\tau^*_n \leq t} \delta(t - \tau^*_n)\xi_n^*.$$  
Assumption (A3) guarantees that $X_t \in \mathcal{D}(m)$ almost surely for all $t \geq 0$. If Assumption (A5) holds, then by Bounded Convergence Theorem, we can define the second mapping $\Gamma_2 : \varphi^* \mapsto \bar{m} = \lim_{t \to \infty} \mathbb{E}_x[X_t^*] \in [b_l(m), b_u(m)]$.

Step 3. Locate the fixed point. Denote $\Gamma = \Gamma_2 \circ \Gamma_1$. With appropriate fixed-point argument as in the proof of Theorem 2.3 in [24], we can establish the following result.

**Theorem 3.2.** Given Assumptions (A1)-(A5) and assuming that $\Gamma$ is a compact mapping on $\mathbb{R}$, (MFG) admits a solution in the sense of Definition 3.1.

3.2 Example

Recall that in the two-player game in Section 2.3, for tractability we take the running cost as the penalty on the deviation from mean information. In the MFG framework, we first analyze a more general version of the game considered in Section 2.3, with the cost function $f(x,m) = C(x - \alpha(m))$. Here the target level $\alpha$ can be dependent on the mean information $m$. For instance, in inventory controls, this $\alpha$ can be a forecast demand given by Cobb-Douglas function; in interest rates problems, $\alpha$ can be a target interest rate level, see [6]. Take

$$C(x) = \max\{hx, -px\},$$

where $h, p > 0$, and

$$\phi(\xi) = \begin{cases} K^+ + k^+\xi, & \xi \geq 0, \\ K^- - k^-\xi, & \xi < 0, \end{cases} \quad (3.5)$$

where $K^\pm, k^\pm > 0$ and

$$h - k^+r > 0, \quad p - k^-r > 0. \quad (3.6)$$

For simplicity, we fix $b(x) \equiv 0$ and $\sigma(x) \equiv \sigma > 0$ in (MFG).

Let us fix $m \in \mathbb{R}$. Then the corresponding QVI for the control problem would be

$$\min \{LV - rV + C(x - \alpha(m)), MV - V\} = 0. \quad (3.7)$$

Similar as in [12], here we focus on finding an optimal simple policy characterized by the vector $(d, D, U, u)$ with $d < D < 0 < U < u$. This is accomplished by solving (3.7) for a classical solution
with enough regularity, namely, by smooth-fitting principle. The value function satisfies

\[
V(x) = \begin{cases}
V(u + \alpha(m)) - k^{-}(u - x + \alpha(m)), & x - \alpha(m) \geq u, \\
h \frac{r}{\lambda} (x - \alpha(m)) + c_1 \exp \{\lambda(x - \alpha(m))\} + c_2 \exp \{-\lambda(x - \alpha(m))\}, & 0 \leq x - \alpha(m) \leq u, \\
-\frac{p}{\lambda} (x - \alpha(m)) + \left(c_1 + \frac{h + p}{2r\lambda}\right) \exp \{\lambda(x - \alpha(m))\} + \left(c_2 - \frac{h + p}{2r\lambda}\right) \exp \{-\lambda(x - \alpha(m))\}, & d \leq x - \alpha(m) \leq 0; \\
V(d + \alpha(m)) + k^{+}(d - x + \alpha(m)), & x - \alpha(m) \leq d;
\end{cases}
\tag{3.8}
\]

\[
\dot{V}(U + \alpha(m)) = \dot{V}(u + \alpha(m)) = k^{-}, \quad \dot{V}(D + \alpha(m)) = \dot{V}(d + \alpha(m)) = -k^{-};
\tag{3.9}
\]

\[
V(u + \alpha(m)) = K^{-} + k^{-}(u - U) + V(U + \alpha(m)),
\]

\[
V(d + \alpha(m)) = K^{+} + k^{+}(D - d) + V(D + \alpha(m)).
\tag{3.10}
\]

Here \( \lambda = \frac{\sqrt{2\sigma^2}}{\rho} \). By [12], we have the following proposition.

**Theorem 3.3.** Given \( K^{\pm}, k^{\pm} > 0 \) and \( h - \rho k^{-}, p - \rho k^{+} > 0 \), there exists a six-tuple \((c_1, c_2, d, D, U, u)\) satisfying (3.8), (3.9) and (3.10) such that \( d < D < 0 < U < u \) and

\[
\begin{align*}
&c_1 = \frac{h + p}{r\lambda} \frac{(e^{-\lambda u} - e^{-\lambda U})(\cosh(\lambda d) - \cosh(\lambda D))}{e^{\lambda u} - e^{-\lambda U}} - \frac{(h + p)}{2r\lambda} = 0, \\
&c_2 = \frac{h + p}{2r\lambda} \exp \{\lambda(D - d)\} - \frac{(h + p)}{2r\lambda} \exp \{-\lambda(D - d)\} = 0, \\
&K^{-} - \left(\frac{h}{r} - k^{-}\right)(u - U) - 2c_1(e^{\lambda u} - e^{\lambda U}) = 0, \\
&\lambda \left[c_1 e^{\lambda u} - c_2 e^{-\lambda u}\right] + \left(\frac{h}{r} - k^{-}\right) = 0, \\
&K^{+} - \left(\frac{p}{r} - k^{+}\right)(D - d) - 2\left(c_1 + \frac{h + p}{2r\lambda}\right)(e^{\lambda u} - e^{\lambda U}) = 0, \\
&\lambda \left[c_1 + \frac{h + p}{2r\lambda} e^{\lambda u} - \left(c_2 - \frac{h + p}{2r\lambda}\right) e^{-\lambda u}\right] - \left(\frac{p}{r} - k^{+}\right) = 0,
\end{align*}
\tag{3.11}
\]

where the values of the thresholds only depend on \( K^{\pm}, k^{\pm}, h, p, r, \sigma \). The optimal simple control policy \( \varphi^* = \{(\tau_{n}^*, \xi_{n}^*)\}_{n \geq 1} \) is given by

\[
\begin{align*}
&\tau_{1}^* = \inf \{t \geq 0 : |X_{t} - \alpha(m)| \not\in (\alpha(m) + d, \alpha(m) + u)\}, \\
&\tau_{n}^* = \inf \{t > \tau_{n-1}^* : |X_{t} - \alpha(m)| \not\in (\alpha(m) + d, \alpha(m) + u)\}, \quad n \geq 2, \\
&\xi_{n}^* = \begin{cases}
U - X_{\tau_{n}^*} + \alpha(m), & \text{if } X_{\tau_{n}^*} - \alpha(m) \geq \alpha(m) + u, \\
D - X_{\tau_{n}^*} + \alpha(m), & \text{if } X_{\tau_{n}^*} - \alpha(m) \leq \alpha(m) + d.
\end{cases}
\tag{3.17}
\end{align*}
\]

Assume the initial position \( X_{0-} \) follows any given distribution \( \mu_{0-} \). Recall from (MFG) that

\[
V(x) = \inf_{\varphi} \mathbb{E}_x \left[ \int_{0}^{\infty} e^{-rt} f(X_{t}, m) dt + \sum_{n \geq 1} e^{-r\tau_{n}\phi(\xi_{n})} \right]
\]

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Hence, \( \lim \phi \) denotes the fixed point of \( \Gamma \). Denote the fixed point of \( \Gamma \) as \( \bar{\mu}_m \). Therefore, \( \Gamma : m \rightarrow \bar{\mu}_m \) is a contraction mapping. Denote the updated mean information as \( \bar{m} = \lim_{t \rightarrow \infty} E_x[X_t] \). We will show that this \( \bar{m} \) is well-defined and indifferent with respect to \( \mu_{0-} \).

**Claim.** There exists a solution to (MFG), assuming \( \alpha \) is a contraction mapping on \( \mathbb{R} \), or the cost structure is symmetric in the sense that \( h = p, K^+ = K^- \) and \( k^+ = k^- \).

To see this, notice that \( \bar{m} = E_x[X_\infty] = \lim_{n \rightarrow \infty} E_x[X_{\tau_n}] \) by symmetry and a Fubini argument. Note that \( E_x[X_{\tau_n}] = \alpha(m) + U \mathbb{P}\{X_{\tau_n} = \alpha(m) + U\} + D \left[ 1 - \mathbb{P}\{X_{\tau_n} = \alpha(m) + U\} \right] \). For simplification, denote \( \mathbb{P}\{X_{\tau_n} = \alpha(m) + U\} \) as \( p_n(x) \). Then, by the strong Markovian property of \( X_t \)

\[
q_1 \equiv \mathbb{P}\{X_{\tau_{n+1}} = \alpha(m) + U|X_{\tau_n} = \alpha(m) + U\} = \frac{U - d}{u - d}, \forall n \in \mathbb{N},
\]

\[
q_2 \equiv \mathbb{P}\{X_{\tau_{n+1}} = \alpha(m) + U|X_{\tau_n} = \alpha(m) + D\} = \frac{D - d}{u - d}, \forall n \in \mathbb{N},
\]

\[
p_{n+1}(x) = p_n(x)q_1 + [1 - p_n(x)]q_2.
\]

Therefore, we have

\[
p_{n+1}(x) = q_2 + (q_1 - q_2)p_n(x) \Rightarrow p_n(x) - \frac{q_2}{1 - q_1 + q_2} = (q_1 - q_2)^{n-1} \left[ p_1(x) - \frac{q_2}{1 - q_1 + q_2} \right].
\]

Hence, \( \lim_{n \rightarrow \infty} p_n(x) = \frac{q_2}{1 - q_1 + q_2} \) and this is independent of the initial position \( x \). We then have \( \bar{m} = \alpha(m) + \frac{uD + dt}{u - D - d} \). Therefore, \( \Gamma : m \rightarrow \bar{m} \) is given by \( \Gamma(m) = \alpha(m) + \frac{uD + dt}{u - D - d} \). If \( \alpha \) is a contraction mapping, so is \( \Gamma \). Denote the fixed point of \( \Gamma \) as \( m^* \) and let \( \phi^* = \varphi(m^*) \) be in the form of Theorem 3.3. Then \( (\phi^*, m^*) \) is a solution to the (MFG) in the sense of Definition 3.1. If the cost structure is symmetric as given in Section 2.3, then the symmetry can guarantee the existence of a fixed point without the assumption of \( \alpha \) being a contraction mapping.

**Mean-Field Game vs Monopoly.** To compare (MFG) directly with the example in Section 2.3, let us consider a symmetric cost structure with \( \alpha(m) = m, h = p, K^\pm = K, k^\pm = k \). Then due to the symmetric nature of the cost functions, \( d = -u \) and \( D = -U \). Therefore \( \frac{uD + dt}{u - D - d} = 0 \). Moreover, \( \Gamma(m) = \alpha(m) = m \) for any \( m \in \mathbb{R} \) is the solution. In particular, \( m^* = E_{\mu_{0-}} E[X_{0-}] \) is a solution. In other words, due to the aggregation effect and the symmetric structure, the MFG collapses to a single-player game, and its game strategy fails to reveal the interaction among players or bear any game nature.

**MFG as Approximation of N-Player Game.** We will demonstrate that the solution to the (MFG) is an approximation to the N-Player game in the following sense.
Definition 3.4. A strategy $\varphi^* = (\varphi_1^*, \ldots, \varphi_N^*)$ is called an $\epsilon$-Nash equilibrium to the $N$-player game introduced in Section 2 if

$$E_\mu \left[ J^i(X_{0-}, \varphi^*) \right] \leq E_\mu \left[ J^i(X_{0-}, s_i, \varphi_i(\varphi^*)) \right] + \epsilon, \quad \forall \varphi_i \in \Phi_i(X_{0-}) \text{ s.t. } s_i(\varphi^*_i) \in \Phi(X_{0-}),$$

where $X_{0-} \sim \mu$, $i = 1, \ldots, N$.

Let $(\tilde{\varphi}^*, m^*)$ be a solution to the (MFG) where $\tilde{\varphi}^*$ is characterized by $(-u^*, -U^*, U^*, u^*)$ with $0 < U^* < u^*$. Define the following priority sets

$$P_i = \{ x \in \mathbb{R}^N : |x_i - m^*| > |x_j - m^*| \forall j > i, |x_i - m^*| \geq |x_k - m^*| \forall k > i \}, \forall i \in \{1, \ldots, N\}.$$  

Then for $i = 1, \ldots, N$, let $A_i^* = \{ x \in \mathbb{R}^N : |x_i - m^*| > u^* \} \cap P_i$ and

$$\xi_i^* = \begin{cases} m^* + U^* - x_i, & x_i - m^* > u^*, \\ m^* - U^* - x_i, & x_i - m^* < -u^*. \end{cases}$$

Denote $\varphi_i^* = (A_i^*, \xi_i^*)$ and $\varphi^* = (\varphi_1^*, \ldots, \varphi_N^*)$.

Theorem 3.5. $\varphi^*$ is an $\epsilon$-NE for the $N$-player cash management game introduced in Section 2.3 for generic $N$, with $\epsilon = O\left(\frac{1}{\sqrt{N}}\right)$, provided that $\mu$ is symmetric around $m^*$.

Proof. Fix $i \in \{1, \ldots, N\}$. Consider $\bar{\varphi} = s_i, \varphi_i(\varphi^*)$ such that $\varphi_i \in \Phi_i(x)$ and $\bar{\varphi} \in \Phi(x)$. For $j \neq i$, $\varphi_j = \varphi_j^*$ whose action region independent from the strategy of player $i$.

We first look at the running cost.

$$h \left| X_i^i - \frac{1}{N} \sum_{j=1}^{N} X_i^j \right| = h \left| \frac{N-1}{N} (X_i^i - m^*) - \frac{\sum_{j \neq i}(X_i^j - m^*)}{N} \right|$$

so that

$$\frac{N-1}{N} h |X_i^i - m^*| - \frac{N-1}{N} h \left| \frac{\sum_{j \neq i}(X_i^j - m^*)}{N-1} \right| \leq h \left| \frac{1}{N} \sum_{j=1}^{N} X_i^j \right| \leq \frac{N-1}{N} h |X_i^i - m^*| + \frac{N-1}{N} h \left| \frac{\sum_{j \neq i}(X_i^j - m^*)}{N-1} \right|.$$  

Note that

$$\left| \frac{\sum_{j \neq i}(X_i^j - m^*)}{N-1} \right| \leq u^*$$

and by the i.i.d. assumption,

$$E_\mu \left| \frac{\sum_{j \neq i}(X_i^j - m^*)}{N-1} \right| \leq \left( E_\mu \left| \frac{\sum_{j \neq i}(X_i^j - m^*)}{N-1} \right|^2 \right)^{1/2} \Rightarrow E_\mu \left| \frac{\sum_{j \neq i}(X_i^j - m^*)}{N-1} \right| = O\left(\frac{1}{\sqrt{N}}\right).$$

Without loss of generality, let us consider $\varphi_i$ such that

$$E_\mu \left[ \int_0^\infty e^{-rt} h |X_i^i - m^*| dt \right] < M.$$
for some sufficiently large $M > 0$. Then

$$
\mathbb{E}_\mu \left[ \int_0^\infty e^{-rt} h \left| X_t - \frac{\sum_{j=1}^N X_j}{N} \right| dt \right] - \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} \left| X_t^i - m^* \right| dt \right] \leq \frac{1}{N} \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} h \left| X_t^i - m^* \right| dt \right]
$$

$$
+ \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} \left| \frac{\sum_{j \neq i} (X_j^i - m^*)}{N-1} \right| dt \right]
$$

\begin{equation}
= \left( \text{Fubini's} \right) \frac{1}{N} \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} h \left| X_t^i - m^* \right| dt \right] + \int_0^\infty e^{-r \tau} h \mathbb{E}_\mu \left[ \left| \frac{\sum_{j \neq i} (X_j^i - m^*)}{N-1} \right| \right] dt
\end{equation}

$$
= O \left( \frac{1}{N} \right) + O \left( \frac{1}{\sqrt{N}} \right)
$$

$$
\Rightarrow \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} h \left| X_t^i - \frac{\sum_{j=1}^N X_j}{N} \right| dt \right] = \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} h \left| X_t^i - m^* \right| dt \right] + O \left( \frac{1}{\sqrt{N}} \right),
$$

and therefore,

$$
\mathbb{E}_\mu \left[ J^i (X_{0-}, \varphi) \right] = \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} h \left| X_t^i - \frac{\sum_{j=1}^N X_j}{N} \right| dt + \sum_{n \geq 1} e^{-r \tau_i,n} \phi (\xi_{i,n}) + \sum_{j \neq i} \sum_{n \geq 1} e^{-r \tau_j,n} \psi_{j,i} (\xi_{j,n}) \right]
$$

$$
= \mathbb{E}_\mu \left[ \int_0^\infty e^{-r \tau} h \left| X_t^i - m^* \right| dt + \sum_{n \geq 1} e^{-r \tau_i,n} \phi (\xi_{i,n}) \right] + \mathbb{E}_\mu \left[ \sum_{j \neq i} \sum_{n \geq 1} e^{-r \tau_j,n} \psi_{j,i} (\xi_{j,n}) \right] + O \left( \frac{1}{\sqrt{N}} \right)
$$

$$
\geq V(\mu) + \mathbb{E}_\mu \left[ \sum_{j \neq i} \sum_{n \geq 1} e^{-r \tau_j,n} \psi_{j,i} (\xi_{j,n}) \right] + O \left( \frac{1}{\sqrt{N}} \right)
$$

$$
= \mathbb{E}_\mu \left[ J^i (X_{0-}, \varphi^*) + O \left( \frac{1}{\sqrt{N}} \right) \right].
$$

\[\square\]

4 Sensitivity Analyses

To compare the cases of monopoly $N = 1$ and duopoly $N = 2$, and to see how parameters $h$, $K$, $k$, $r$ and $\sigma$ influence the control policies and the thresholds $d$, $D$, $U$, $u$, we conduct a series of sensitivity analyses. We start with $h = 2$, $K = 3$, $k = 1$, $r = 0.5$, $\sigma = \frac{\sqrt{2}}{2}$ and $c = 1$.

We shall see similar behaviors for both the monopoly and the duopoly cases in terms of the thresholds and policy changes with respect to the underlying parameter changes. One distinction is that the thresholds and policy changes are more sensitive to parameter changes in the duopoly case due to competition.

**Duopoly vs Monopoly.** Putting the thresholds for the duopoly and those of the monopoly together in Figures 3, one can see that due to competition in a game setting, players take the opponents’ strategies into consideration. Consequently, it is optimal in the NE sense that players choose to intervene less frequently; but once set to intervene, players will exert larger amount of controls. In some sense, competition induces more efficient control strategies from players.
Running Cost $h$. When the running cost $h$ increases, players have the incentive to intervene more frequently to prevent controlled process from deviating too far away from the target level. See Figure 4a. On the other hand, the presence of the cost of control makes players more cautious when exercising controls. Thus, an increased running cost encourages the players to intervene more frequently but with smaller amount of adjustment. See Figure 4b.

Cost of Control $K$ and $k$. The parameter $K$ is the fixed cost when players choose to intervene. High fixed cost $K$ discourages the player from intervening too frequently. Therefore players have the incentive to tolerate a larger deviation from the target; and once a player chooses to intervene, the size of control needs to be bigger to compensate for less frequent controls. Meanwhile, a decreasing frequency of intervention leads to an increasing action boundary $u$. See Figure 5a and Figure 5b. For the per unit control cost $k$, similar results are shown in Figure 6.
**Discount Rate** $r$. When the discount rate $r$ increases, players are more tolerant with a larger deviation from the target level as the penalty is discounted by a larger factor. That is, a higher discount rate $r$ effectively reduces both the running and control cost, hence resulting in a decreased intervention frequency with an increased size of controls, as shown in Figure 7.

**Volatility** $\sigma$. When the volatility $\sigma$ is bigger, players tend to intervene less as the controlled process is more likely to move closer to the target level with a higher volatility. Therefore, a higher
volatility allows players to intervene less frequently with a larger amount of adjustment, as shown in Figure 8.

![Figure 8: Sensitivity w.r.t. σ](image)

(a) Action Boundary of Monopoly and Duopoly  
(b) Amount of Adjustment of Monopoly and Duopoly

References

[1] R. Aid, M. Basei, G. Callegaro, L. Campi, and T. Vargiolu. Nonzero-sum stochastic differential games with impulse controls: a verification theorem with applications. *Math. Oper. Res.*, to appear, 2018.

[2] E. Bayraktar, T. Emmerling, and J.-L. Menaldi. On the impulse control of jump diffusions. *SIAM J. Control Optim.*, 51(3):2612–2637.

[3] A. Bensoussan and B. Chevalier-Roignant. Sequential capacity expansion options. *Oper. Res.*, to appear, 2018.

[4] A. Bensoussan and J.-L. Lions. *Impulse Control and Quasi-Variational Inequalities*. Bordas, Paris, 1982.

[5] A. Bensoussan, K. C. J. Sung, S. C. P. Yam, and S. P. Yung. Linear-quadratic mean field games. *J. Optim. Theory Appl.*, 169(2):496–529, 2016.

[6] G. Bertola, W. J. Runggaldier, and K. Yasuda. On classical and restricted impulse stochastic control for the exchange rate. *Appl. Math. Optim.*, 74(2):423–545, 2016.

[7] T. R. Bielecki and S. R. Pliska. Risk sensitive asset management with transaction costs. *Finance Stoch.*, 4(1):1–33, 2000.

[8] A. Candenillas, T. Choulli, M. Taksar, and L. Zhang. Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm. *Math. Finance*, 16(1):181–202, 2006.

[9] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications I*, volume 83. Springer.

[10] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications II*, volume 84. Springer.
[11] R. Carmona, J. P. Fouque, and L. H. Sun. Mean field games and systemic risk. *Commun. Math. Sci.*, 13(4):911–933, 2015.

[12] G. M. Constantinides and S. F. Richard. Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time. *Oper. Res.*, 26(4):620–636, 1978.

[13] J. F. Eastham and K. J. Hastings. Optimal impulse control of portfolios. *Math. Oper. Res.*, 13(4):588–605, 1988.

[14] X. Guo and J. S. Lee. Mean field games with singular controls of bounded velocity. *arXiv:1703.04437*, 2018.

[15] X. Guo and G. Wu. Smooth fit principle for impulse control of multi-dimensional diffusion processes. *SIAM J. Control Optim.*, 48(2):594–617, 2010.

[16] X. Guo and R. Xu. Stochastic games for fuel followers problem: N vs MFG. *SIAM J. Control Optim.*, to appear, 2018.

[17] J. M. Harrison, T. M. Sellke, and A. J. Taylor. Impulse control of Brownian Motion. *Math. Oper. Res.*, 8(3):454–466, 1983.

[18] J. M. Harrison and M. I. Taksar. Instantaneous control of Brownian Motion. *Math. Oper. Res.*, 8(3):439–453, 1983.

[19] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Comm. Inform. Systems*, 6(3):221–252, 2006.

[20] M. Jeanblanc-Picqué and A. N. Shiryaev. Optimization of the flow of dividends. *Russian Math. Surveys*, 50(2):257–277, 1995.

[21] R. Korn. Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance Stoch.*, 2(2):85–114, 1998.

[22] R. Korn. Some applications of impulse control in mathematical finance. *Math. Meth. Oper. Res.*, 50(3):493–518, 1999.

[23] D. Lacker and T. Zariphopoulou. Mean field and N-agent games for optimal investment under relative performance criteria. 2017. *arXiv:1703.07685*.

[24] J.-M. Lasry and P.-L. Lions. Mean field games. *Japanese J. Math.*, 2(1):229–260, 2007.

[25] V. Ly Vath, M. Mnif, and H. Pham. A model of optimal portfolio selection under liquidity risk and price impact. *Finance Stoch.*, 11(1):51–90, 2007.

[26] D. C. Mauer and A. J. Triantis. Interactions of corporate financing and investment decisions: a dynamic framework. *J. Finance*, 49(4):1253–1277, 1994.

[27] A. J. Morton and S. R. Pliska. Optimal portfolio management with fixed transaction costs. *Math. Finance*, 5(4):337–356, 1995.

[28] G. Mundaca and B. Øksendal. Optimal stochastic intervention control with application to the exchange rate. *Economics*, 29(2):225–243, 1998.
[29] B. Øksendal. *Stochastic Differential Equations*. Springer-Verlag, 2003.

[30] B. Øksendal and A. Sulem. Optimal consumption and portfolio with both fixed and proportional transaction costs. *SIAM J. Control Optim.*, 40(6):1765–1790, 2002.

[31] A. Sulem. A solvable one-dimensional model of a diffusion inventory system. *Math. Oper. Res.*, 11(1):125–133, 1986.

[32] A. J. Triantis and J. E. Hodder. Valuing flexibility as a complex option. *J. Finance*, 45(2):549–565, 1990.