Note on ultraviolet renormalization and ground state energy of the Nelson model

Fumio Hiroshima
Faculty of Mathematics, Kyushu University,
Fukuoka, 819-0395, Japan

July 21, 2015

Abstract
Ultraviolet (UV) renormalization of the Nelson model $H_\epsilon$ in quantum field theory is considered. E. Nelson proved that $\lim_{\epsilon \to 0} e^{-T(H_\epsilon - E_{\text{ren}}^\epsilon)}$ converges to $e^{-TH_{\text{ren}}}$ in [Nel64a]. A relationship between a ultraviolet renormalization term $E_{\text{ren}}^\epsilon$ and the ground state energy $E_\epsilon(g^2)$ of the Hamiltonian with total momentum zero $H_\epsilon(0)$ is studied by functional integrations. Here $g$ denotes a coupling constant involved in $H_\epsilon(0)$. It can be derived from the formula

$$E_\epsilon(g^2) = -\lim_{T \to \infty} \frac{1}{2T} \log(1, e^{-2TH_\epsilon(0)})$$

that $E_{\text{ren}}^\epsilon$ coincides with the coefficient of $g^2$ in the expansion of $E_\epsilon(g^2)$ in $g^2$, i.e., $E_{\text{ren}}^\epsilon = \lim_{g \to 0} E_\epsilon(g^2)/g^2$, and $E_\epsilon(g^2) - g^2 E_{\text{ren}}^\epsilon$ converges as ultraviolet cutoff is removed.

1 The Nelson model

In this paper we consider a relationship between a ultraviolet (UV) renormalization and the ground state energy of the Nelson model in quantum field theory by functional integrations. The Nelson model describes an interaction system between a scalar bose field and particles governed by a Schrödinger operator with an external potential. We prepare tools used in this paper. The boson Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3)$ is defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} [\otimes_n L^2(\mathbb{R}^3)].$$

(1.1)

Here $\otimes_n L^2(\mathbb{R}^3)$ describes $n$ fold symmetric tensor product of $L^2(\mathbb{R}^3)$ with $\otimes_0 L^2(\mathbb{R}^3) = \mathbb{C}$. Let $a^*(f)$ and $a(f), f \in L^2(\mathbb{R}^3)$, be the creation operator and the annihilation operator, respectively, in $\mathcal{F}$, which satisfy $(a^*(f))^* = a(f)$ and canonical commutation
relations:

\[ [a(f), a^*(g)] = (\bar{f}, g)\mathbb{I}, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)]. \]

Note that \((f, g)\) denotes the scalar product on \(L^2(\mathbb{R}^3)\) and it is linear in \(g\) and anti-linear in \(f\). We also note that \(f \mapsto a^*(f)\) and \(f \mapsto a(f)\) are linear. Denote the dispersion relation by \(\omega(k) = |k|\). Then the free field Hamiltonian \(H_f\) of \(\mathcal{F}\) is then defined by the second quantization of \(\omega\), i.e.,

\[ H_f = \mathcal{D}(\omega) = \int \omega(k) a^*(k)a(k) dk. \]

It satisfies that

\[ e^{-itH_f} a^*(f) e^{-itH_f} = a^*(e^{-it\omega}f), \quad e^{-itH_f} a(f) e^{-itH_f} = a(e^{it\omega}f). \]

Hence it follows that

\[ [H_f, a(f)] = -a(\omega f), \quad [H_f, a^*(f)] = -a^*(\omega f). \]

Furthermore for the Fock vacuum \(1_{\mathcal{F}} = 1 \oplus 0 \oplus 0 \cdots \in \mathcal{F}\), it follows that \(H_f 1_{\mathcal{F}} = 0\).

**Definition 1.1** The Nelson Hamiltonian \(H\) is a self-adjoint operator acting in the Hilbert space \(L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong L^2(\mathbb{R}^3, \mathcal{F})\), which is given by

\[ H = (-\frac{1}{2}\Delta + V) \otimes \mathbb{I} + \mathbb{I} \otimes H_f + g\phi, \]

where \(g \in \mathbb{R}\) is a coupling constant, \(V : \mathbb{R}^3 \to \mathbb{R}\) an external potential, the interaction is defined by \((\phi \Phi)(x) = \phi(x)\Phi(x)\) for a.e. \(x \in \mathbb{R}^3\) and the field operator \(\phi(x)\) by

\[ \phi(x) = \frac{1}{\sqrt{2}} \left( a^*(\bar{\phi}(\sqrt{\omega}e^{i\cdot x})) + a(\tilde{\phi}(\sqrt{\omega}e^{-i\cdot x})) \right) \]

with \(\tilde{\phi}(k) = \bar{\phi}(-k)\). Let \(H_0\) be the operator defined by \(H\) with coupling constant \(g\) replaced by 0. We have to mention the self-adjointness of \(H\). Suppose that

\[ \tilde{\phi}/\sqrt{\omega}, \phi/\omega \in L^2(\mathbb{R}^3), \quad \phi(-k) = \bar{\phi}(k). \]

Then the interaction \(H_1\) is well defined, symmetric and infinitesimally \(H_0\)-bounded, i.e., for arbitrary \(\varepsilon > 0\), there exists a \(b_\varepsilon > 0\) such that

\[ \|H_1\Phi\| \leq \varepsilon\|H_0\Phi\| + b_\varepsilon\|\Phi\| \]

for all \(\Phi \in D(H_0)\). Thus \(H\) is self-adjoint on \(D(H_0)\) by the Kato-Rellich theorem. Throughout this paper we assume condition (1.6).
2 UV renormalization and ground state energy

A point charge limit of $H$, $\tilde{\phi}(k) \to 1$, is studied in [Nel64a, Nel64b] and recently in [GHPS12, GHL14, Hir15]. Let $\lambda > 0$ be a strictly positive infrared cutoff parameter and we fix it throughout. This assumption is used in the proof of Lemma 3.7. Consider the cutoff function

$$\hat{\phi}_\varepsilon(k) = e^{-\varepsilon |k|^2/2} 1_{|k| \geq \lambda}, \quad \varepsilon > 0,$$

and define the regularized Hamiltonian by

$$H_\varepsilon = (-\frac{1}{2} \Delta + V) \otimes 1 + 1 \otimes H_{f} + g \phi_\varepsilon, \quad \varepsilon > 0,$$

where $\hat{\phi}_\varepsilon$ is defined by $\phi$ with $\hat{\phi}$ replaced by $\hat{\phi}_\varepsilon$. Here $\varepsilon > 0$ is regarded as the UV cutoff parameter. Let

$$E_\varepsilon^{\text{ren}} = -g^2 \int_{|k| > \lambda} e^{-\varepsilon |k|^2/2} 2\omega(k) \beta(k) dk,$$

where $\beta$ is given by

$$\beta(k) = \frac{1}{\omega(k) + |k|^2/2}. \quad (2.4)$$

Notice that $E_\varepsilon^{\text{ren}} \to -\infty$ as $\varepsilon \downarrow 0$. E. Nelson proved the proposition below in [Nel64a].

**Proposition 2.1** There exists a constant $C$ such that $H_\varepsilon - E_\varepsilon^{\text{ren}} > -C$ uniformly in $\varepsilon$ and there exists a self-adjoint operator $H_\varepsilon^{\text{ren}}$ such that

$$s - \lim_{\varepsilon \downarrow 0} e^{-T(H_\varepsilon - g^2E_\varepsilon^{\text{ren}})} = e^{-TH_\varepsilon^{\text{ren}}}. \quad (2.5)$$

**Proof.** Refer to see [Nel64a]. \hfill \Box

Let $V = 0$. Then $H_\varepsilon$ is translation invariant, i.e.,

$$[H_\varepsilon, P_{\text{tot}, \mu}] = 0, \quad \mu = 1, 2, 3,$$

where $P_{\text{tot}}$ is the total momentum defined by $P_{\text{tot}} = -i \nabla \otimes 1 + 1 \otimes P_f$. Here $P_f$ denotes the field momentum operator given by $P_f = d\Gamma(k) = \int k a^\ast(k) a(k) dk$. Thus $H_\varepsilon$ can be decomposed as $H_\varepsilon = \int_{\mathbb{R}^3} H_\varepsilon(P) dP$, where

$$H_\varepsilon(P) = \frac{1}{2} (P - P_f)^2 + H_f + g \phi_\varepsilon(0) \quad (2.6)$$

is a self-adjoint operator in $\mathcal{F}$ for each $P \in \mathbb{R}^3$. Let $E_\varepsilon(g^2) = \inf \sigma(H_\varepsilon(0))$ be the bottom of the spectrum of the Nelson model with zero-total momentum, $\mu = 0 \in \mathbb{R}^3$. 

3
Suppose that formally $E_\varepsilon(g^2)$ can be expanded in $g^2$ as $E_\varepsilon(g^2) = E_\varepsilon(0) + a_2g^2 + a_4g^4 + \cdots$, and the ground state energy as $\varphi_g = 1 + g\phi_1 + g^2\phi_2 + \cdots$. Note that $E_\varepsilon(0) = 0$. Then from equation $H_\varepsilon(0)\varphi_g = E_\varepsilon(g^2)\varphi_g$, we can derive the identity

$$a_2 = -\left(\frac{1}{2}P_t^2 + H_t\right)^{-1}\phi_1(0)\mathbb{1}_\mathcal{F}$$

Hence $a_2 = E_\varepsilon^{\text{ren}}$ is derived. Furthermore we expect that $a_n, n \geq 4$, converges as $\varepsilon \downarrow 0$, and hence

$$\lim_{\varepsilon \downarrow 0} |E_\varepsilon(g^2) - g^2E_\varepsilon^{\text{ren}}| < \infty. \tag{2.7}$$

All the statements mentioned above are however informal. In this paper we are concerned with these facts by functional integrations in non-perturbative way. We can show (2.7) for arbitrary values of $g$, and $\lim_{g \to 0} E_\varepsilon(g^2)/g^2 = E_\varepsilon^{\text{ren}}$ in Theorem 3.4.

**Remark 2.2** In Proposition 2.1, (2.7) is proven by an operator theory, we however prove this by applying functional integrations.

## 3 Functional integrations

Let $(B_t)_{t \in \mathbb{R}}$ denote the 3-dimensional Brownian motion on $C(\mathbb{R}, \mathbb{R}^3)$ with the Wiener measure $W$. $\mathbb{E}[\cdots]$ denotes the expectation with respect to $W$ describing the Wiener measure starting from $0 \in \mathbb{R}^3$.

**Lemma 3.1** For $P \in \mathbb{R}^3$, it follows that

$$\left(\mathbb{1}_\mathcal{F}, e^{-2TH_\varepsilon(P)}\mathbb{1}_\mathcal{F}\right) = \mathbb{E}\left[e^{iP \cdot (B_T - B_{-T})} e^{g^2S_\varepsilon}\right], \tag{3.1}$$

where

$$S_\varepsilon = \int_{-T}^{T} ds \int_{-T}^{T} dt W_\varepsilon(B_t - B_s, t - s) \tag{3.2}$$

and $W_\varepsilon: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is given by

$$W_\varepsilon(x,t) = \int_{|k| \geq \lambda} \frac{e^{-\varepsilon|k|^2}e^{-ik \cdot x}e^{-\omega(k)|t|}}{2\omega(k)} dk. \tag{3.3}$$

**Proof.** Refer to see [Hir15, Lemma 2.2]. \hfill $\square$

Putting $P = 0$ in Lemma 3.1, we have

$$\left(\mathbb{1}_\mathcal{F}, e^{-2TH_\varepsilon(0)}\mathbb{1}_\mathcal{F}\right) = \mathbb{E}\left[e^{g^2S_\varepsilon}\right]. \tag{3.4}$$
Lemma 3.2 Let $\lambda > 0$. Then
\[ E_\varepsilon(g^2) = -\lim_{T \to \infty} \frac{1}{2T} \log(\mathbb{1}, e^{-2TH_\varepsilon(0)} \mathbb{1}). \] (3.5)

In particular
\[ E_\varepsilon(g^2) = -\lim_{T \to \infty} \frac{1}{2T} \log \mathbb{E}\left[e^{\frac{\varepsilon^2}{T} S_\varepsilon}\right]. \] (3.6)

Proof. Since $\lambda > 0$, it is shown that $H_\varepsilon(0)$ has the unique ground state $\varphi_\varepsilon$ and it is strictly positive. See also Appendix. Hence $(\mathbb{1}, \varphi_\varepsilon) > 0$. In particular $(\mathbb{1}, \varphi_\varepsilon) \neq 0$. Thus (3.5) follows. \[ \square \]

It can be seen that the pair potential $W_\varepsilon(B_t - B_s, t - s)$ is singular at the diagonal part $t = s$. We shall remove the diagonal part by using the Itô formula, which is done in [GHL14]. We introduce the function
\[ \varrho_\varepsilon(x, t) = \int_{|k| \geq \lambda} \frac{e^{-|k|^2} e^{-ik \cdot x - \omega(k) |t|}}{2\omega(k)} \beta(k) dk, \quad \varepsilon \geq 0. \] (3.7)

Lemma 3.3 It follows that
\[ S_\varepsilon = S_{\varepsilon \text{ren}} + 4T \varrho_\varepsilon(0, 0), \quad \varepsilon > 0, \]
where
\[ S_{\varepsilon \text{ren}} = S_{\varepsilon \text{OD}} + 2 \int_{-T}^{T} \left( \int_{s}^{[s + \tau]} \nabla \varrho_\varepsilon(B_t - B_s, t - s) \cdot dB_t \right) ds \]
\[ - 2 \int_{-T}^{T} \varrho_\varepsilon(B_{[s + \tau]} - B_s, [s + \tau] - s) ds. \] (3.8)

Here $0 < \tau < T$ is an arbitrary number, and $[t] = -T \lor t \land T$, $S_{\varepsilon \text{OD}}$ denotes the off-diagonal part given by
\[ S_{\varepsilon \text{OD}} = 2 \int_{-T}^{T} ds \int_{[s + \tau]}^{T} W_\varepsilon(B_t - B_s, t - s) dt \]
and the integrand is
\[ \nabla \mu \varrho_\varepsilon(X, t) = \int_{|k| \geq \lambda} \frac{-ik \mu e^{-ik \mu} e^{-|t| \omega(k)} e^{-\varepsilon |k|^2}}{2\omega(k)} \beta(k) dk. \]

Proof. It is shown by the Itô formula that
\[ \int_{s}^{S} W_\varepsilon(B_t - B_s, t - s) dt = \varrho_\varepsilon(0, 0) - \varrho_\varepsilon(B_S - B_s, S - s) + \int_{s}^{S} \nabla \varrho_\varepsilon(B_t - B_s, t - s) \cdot dB_t. \] (3.9)

Then the lemma follows directly. \[ \square \]

$-\varrho_\varepsilon(0, 0)$ can be regarded as the diagonal part of $W_\varepsilon$ and turns to be a renormalization term. I.e., we have the lemma below.
Lemma 3.4 Let $\varepsilon > 0$. Then $E_{\varepsilon}^{\text{ren}} = -\varrho_{\varepsilon}(0,0)$.

Lemma 3.5 There exist constants $b > 0$ and $c > 0$ independent of $g$ such that for all $\varepsilon > 0$,

$$E\left[e^{\frac{\alpha^2}{T}S_{\varepsilon}^{\text{ren}}}ight] \leq e^{b(c + g^2T + g^2\log T) + c(\tau)(g^2/2)T},$$  \hfill (3.10)

where

$$c(\tau) = 8\pi \int_{\lambda}^{\infty} e^{-\varepsilon r^2} e^{-\tau r} dr.$$  \hfill (3.11)

Proof. Let $S_{\varepsilon}^{\text{ren}} = S_{\varepsilon}^{\text{OD}} + Y + Z$, where

$$Y = 2 \int_{-T}^{T} \left( \int_{s}^{[s+\tau]} \nabla \varrho_{\varepsilon}(B_t - B_s, t - s) dB_t \right) ds,$$

$$Z = -2 \int_{-T}^{T} \varrho_{\varepsilon}(B_{[s+\tau]} - B_s, [s+\tau] - s) ds.$$

It is established in [GHL14, Lemma 2.10] that

$$E[e^{\alpha Y}] \leq e^{\alpha^2 T b_1}$$  \hfill (3.12)

with some constant $b_1$. We estimate $E[e^{\alpha Z}]$. Straightforwardly there exists a constant $M > 0$ such that $|\varrho_{\varepsilon}(B_T - B_s, T - s)| \leq |\varrho_{\varepsilon}(0, T - s)| < M$ for all $T$, and

$$\varrho_{\varepsilon}(0, T - s) \leq \frac{1}{2} e^{-\lambda |T-s|}.$$

Then we have

$$|Z| \leq 2 \int_{0}^{2T} du \varrho_{\varepsilon}(0, u) = 2 \left( \int_{0}^{1} + \int_{1}^{2T} \right) du \varrho_{\varepsilon}(0, u) \leq 2M + \int_{1}^{2T} du \frac{1}{u} = 2M + \log(2T) - 1.$$  \hfill (3.13)

Finally we can compute $S_{\varepsilon}^{\text{OD}}$. We have

$$|S_{\varepsilon}^{\text{OD}}| \leq 2 \int_{-T}^{T-\tau} ds \int_{s+\tau}^{T} dt \int_{|k| \geq \lambda} \frac{1}{2\omega(k)} e^{-\varepsilon |k|^2} e^{-\omega(k)|t-s|} dk \leq 4\pi \int_{\lambda}^{\infty} e^{-\varepsilon r^2} e^{-\tau r} \left( e^{-(2T-\tau)r} - 1 + (2T-\tau)r \right) dr \leq c(\tau)T.$$  \hfill (3.14)

Then bound (3.10) follows from (3.12), (3.13), (3.14) and the Schwarz inequality

$$E[e^{g^2/2(S_{\varepsilon}^{\text{OD}} + Y + Z)}] \leq E[e^{g^2 Y}]^{1/2} E[e^{g^2(S_{\varepsilon}^{\text{OD}} + Z)}/2].$$
Remark 3.6 Constants $b$ and $c$ given in Lemma 3.5 also depend on $\tau$. See [GHL14, Lemma 2.8, Lemma 2.10, (2.36)].

Now we state the key lemma.

**Lemma 3.7** Let $b > 0$ and $c(\tau)$ be those in Lemma 3.5. Then
\[
\left| \frac{E_{\varepsilon}(g^2)}{g^2} + \varrho_{\varepsilon}(0,0) \right| \leq \frac{1}{2} (g^2 b + \frac{1}{2} c(\tau)). \tag{3.15}
\]

**Proof.** By Lemmas 3.1 and 3.2 we have
\[
E_{\varepsilon}(g^2) = -\lim_{T \to \infty} \frac{1}{2T} \log \mathbb{E} \left[ e^{\frac{g^2}{2} (S_{\varepsilon}^{\text{ren}} + 4T \varrho_{\varepsilon}(0,0))} \right]. \tag{3.16}
\]
We then have
\[
E_{\varepsilon}(g^2) = -g^2 \varrho_{\varepsilon}(0,0) - \lim_{T \to \infty} \frac{1}{2T} \log \mathbb{E} \left[ e^{\frac{g^2}{2} S_{\varepsilon}^{\text{ren}}} \right]
\]
Hence
\[
\left| E_{\varepsilon}(g^2) + g^2 \varrho_{\varepsilon}(0,0) \right| \leq \lim_{T \to \infty} \frac{1}{2T} \log \mathbb{E} \left[ e^{\frac{g^2}{2} S_{\varepsilon}^{\text{ren}}} \right].
\]
By Lemma 3.5 we can obtain (3.15).\hfill \Box

We now state the main theorem in this paper.

**Theorem 3.8** It follows that
\[
\lim_{g \to 0} \frac{E_{\varepsilon}(g^2)}{g^2} = E_{\varepsilon}^{\text{ren}} \tag{3.17}
\]
and
\[
\lim_{\varepsilon \downarrow 0} \left| E_{\varepsilon}(g^2) - g^2 E_{\varepsilon}^{\text{ren}} \right| < \infty. \tag{3.18}
\]

**Proof.** By Lemmas 3.4 and 3.7 we see that
\[
\left| \frac{E_{\varepsilon}(g^2)}{g^2} - E_{\varepsilon}^{\text{ren}} \right| \leq \frac{1}{2} (g^2 b + \frac{1}{2} c(\tau)). \tag{3.19}
\]
Take $g \to 0$. We have
\[
\lim_{g \to 0} \left| \frac{E_{\varepsilon}(g^2)}{g^2} - E_{\varepsilon}^{\text{ren}} \right| \leq \frac{1}{4} c(\tau) \tag{3.20}
\]
holds for arbitrary $\tau > 0$. $\lim_{\tau \to \infty} c(\tau) = 0$ implies (3.17). Furthermore (3.18) can be derived from (3.19) and the fact $\lim_{\varepsilon \downarrow 0} c(\tau) < \infty$.\hfill \Box

7
A Existence of the ground state

For the self-consistency of the paper we show the uniqueness and the existence of ground state of $H_\varepsilon(0)$. The proof mentioned below is taken from [Hir15, Lemma 2.9]. Let $\varphi^T_g = e^{-TH_\varepsilon(0)} \mathbb{1}_\mathcal{F} / \|e^{-TH_\varepsilon(0)} \mathbb{1}_\mathcal{F}\|$ and $\gamma(T) = (\mathbb{1}_\mathcal{F}, \varphi^T_g)^2$, i.e.,

$$
\gamma(T) = \frac{(\mathbb{1}_\mathcal{F}, e^{-TH_\varepsilon(0)} \mathbb{1}_\mathcal{F})^2}{(\mathbb{1}_\mathcal{F}, e^{-2TH_\varepsilon(0)} \mathbb{1}_\mathcal{F})}.
$$

(A.1)

**Proposition A.1** For all $\varepsilon > 0$ and $\lambda > 0$, $H_\varepsilon(0)$ has the ground state and it is unique.

**Proof.** The uniqueness follows from the fact that $e^{-tH_\varepsilon(0)}$ is positivity improving. It remains to show the existence of ground state. The useful criteria is as follows. There exists a ground state of $H_\varepsilon(0)$ if and only if $\lim_{T \to \infty} \gamma(T) > 0$ [LMS02]. Thus it is enough to show $\lim_{T \to \infty} \gamma(T) > 0$. By Lemma 3.1 we have

$$
\gamma(T) = \frac{\mathbb{E}[e^{\frac{u^2}{2} \int_0^T dt \int_0^T ds W_\varepsilon}]^2}{\mathbb{E}[e^{\frac{u^2}{2} \int_0^T dt \int_0^T ds W_\varepsilon}]}.
$$

Here $W_\varepsilon = W_\varepsilon(B_t - B_s, t - s)$. By the reflection symmetry and the Markov property of the Brownian motion we have

$$
\gamma(T) = \frac{\mathbb{E}[e^{\frac{u^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_\varepsilon - g^2 \int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}]}{\mathbb{E}[e^{\frac{u^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}]}.
$$

By estimating $\int_{-T}^T dt \int_0^T W_\varepsilon$ straightforwardly, we have

$$
\gamma(T) \geq \exp \left( -g^2 \int_{\mathbb{R}^2} \mathbb{1}_{|k| \geq \lambda} \frac{e^{-c|k|^2}}{\omega(k)^3} dk \right) > 0 \quad (A.2)
$$

for all $T > 0$. Note that $\lambda > 0$. Then the proposition follows. \qed

**Acknowledgments:** The author acknowledges support of Challenging Exploratory Research 15K13445 from JSPS.

**References**

[GHPS12] C. Gérard, F. Hiroshima, A. Panati and A. Suzuki, Removal of the UV cutoff for the Nelson model with variable coefficients, *Lett Math Phys*, **101** (2012), 305–322.
[GHL14] M. Gubinelli, F. Hiroshima and J. Lorinczi, Ultraviolet renormalization of the Nelson Hamiltonian through functional integration, *J. Funct. Anal.* **267** (2014), 3125–3153.

[Hir15] F. Hiroshima, Translation invariant models in QFT without ultraviolet cut-offs, arXiv:1506.07514, preprint 2015.

[LMS02] J. Lörinczi, R.A.Minlos and H. Spohn, The infrared behavior in Nelson’s model of quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3** (2002), 1–28.

[Nel64a] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[Nel64b] E. Nelson, Schrödinger particles interacting with a quantized scalar field, in: *Proc. Conference on Analysis in Function Space*, W. T. Martin and I. Segal (eds.), p. 87, MIT Press, 1964.