Mean Row Values in $(u, v)$-Calkin-Wilf Trees

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Abstract

We fix integers $u, v \geq 1$, and consider an infinite binary tree $T^{(u,v)}(z)$ with a root node whose value is a positive rational number $z$. For every vertex $a/b$, we label the left child as $a/(ua+b)$ and right child as $(a+vb)/b$. The resulting tree is known as the $(u, v)$-Calkin-Wilf tree. As $z$ runs over $[1/u,v] \cap \mathbb{Q}$, the vertex sets of $T^{(u,v)}(z)$ form a partition of $\mathbb{Q}^+$. When $u = v = 1$, the mean row value converges to $3/2$ as the row depth increases. Our goal is to extend this result for any $u, v \geq 1$. We show that, when $z \in [1/u,v] \cap \mathbb{Q}$, the mean row value in $T^{(u,v)}(z)$ converges to a value close to $v + \log 2/u$ uniformly on $z$.

1 Introduction

In [8], Nathanson defines an infinite binary tree generated by the following rules:
1. fix two positive integers $u$ and $v$, 
2. label the root of the tree by a rational $z$, and 
3. for any vertex labeled $a/b$, label its left and right children by $a/(ua+b)$ and $(a+vb)/b$, respectively.

In the case where $u, v$, and $z$ are equal to 1, the tree generated is the well-known Calkin-Wilf tree [3] (see Figure 1). Since Nathanson’s definition represents a generalization of the Calkin-Wilf tree, we refer to trees defined in the above manner as $(u, v)$-Calkin-Wilf trees, and we denote them by $T^{(u,v)}(z)$ (see Figure 2). The set of depth $n$ vertices of $T^{(u,v)}(z)$ is denoted by $T^{(u,v)}(z; n)$. For example, we see from Figure 1 that $T^{(1,1)}(1; 1) = \{1/2, 2\}$.

![Figure 1: The first four rows of the Calkin-Wilf tree.](image-url)
Figure 2: The first three rows of $T^{(u,v)}(z)$.

The vertices of $T^{(1,1)}(1)$ are all positive rational numbers without any repetition [3]. More generally, the trees $T^{(u,v)}(z)$ form a partition of $\mathbb{Q}^+$ as $z$ runs over $[1/u, v] \cap \mathbb{Q}$; see [8]. The Calkin-Wilf tree has many other interesting properties [3, 5, 6, 8, 9], one of which is the fact that the mean value of vertices of depth $n$ converges to $3/2$ as $n \to \infty$ [1, 10]. Our main result generalizes this property for all $(u,v)$-Calkin-Wilf trees.

The proof that the mean value of vertices of depth $n$ converges to $3/2$ is not difficult and only makes use of one property of the Calkin-Wilf tree; namely, both $a/b$ and $b/a$ appear (in symmetric positions) on every row; see Figure 1.

**Proposition 1.** If $a/b \in T^{(1,1)}(1; n)$, then $b/a \in T^{(1,1)}(1; n)$.

The proof of Proposition 1 follows quickly from induction on the depth $n$. We omit the details.

**Theorem 1.** For $n \geq 0$, let $A(n) = \frac{1}{2n} \sum_{y \in T^{(1,1)}(1; n)} y$. Then $\lim_{n \to \infty} A(n) = \frac{3}{2}$.

**Proof.** Let $S(n) = \sum_{y \in T^{(1,1)}(1; n)} y$. Rewriting $y$ as $a/b$ and using both the definition of the Calkin-Wilf tree and Proposition 1, we see that, for $n \geq 1$,

$$
2S(n) = \sum_{y \in T^{(1,1)}(1; n-1)} \left( \frac{a}{a+b} + \frac{a}{b} + 1 + \frac{b}{b+a} + \frac{b}{a} + 1 \right)
= \sum_{y \in T^{(1,1)}(1; n-1)} \left( \frac{a}{b} + \frac{b}{a} + 3 \right)
= 2S(n-1) + 3 \cdot 2^{n-1}.
$$

This gives the recurrence relation $S(0) = 1$ and $S(n) = S(n-1) + 3 \cdot 2^{n-2}$ for $n \geq 1$. Solving the recurrence relation gives that $S(n) = \frac{3}{4} \cdot 2^n - \frac{1}{2}$ for $n \geq 0$. The desired result follows immediately since $A(n) = S(n)/2^n$.

Let $S^{(u,v)}(z; n) = \sum_{y \in T^{(u,v)}(z; n)} y$ and $A^{(u,v)}(z; n) = S^{(u,v)}(z; n)/2^n$. Suppose $uv > 1$. As a consequence of Lemma 5 and Theorem 2, we show that if $z \in [1/u, v] \cap \mathbb{Q}$, then $\lim_{n \to \infty} A^{(u,v)}(z; n)$
exist\textsuperscript{2}, that the limit is independent of the value of \( z \), and that the limit has a value close to \( v + \log 2/u \). Unfortunately, Proposition \textsuperscript{[1]} does not generalize to other \((u,v)\)-Calkin-Wilf trees by Lemma \textsuperscript{[3]} so a different approach is needed in this broader setting.

At first the value \( v + \log 2/u \) may seem surprising, but a simple heuristic argument quickly leads to this quantity. Note that if \( a/b \) is a vertex in a \((u,v)\)-Calkin-Wilf tree, then its children are given by
\[
\frac{a}{ua + b} = \frac{1}{u + \frac{b}{a}} < \frac{1}{u} \quad \text{and} \quad \frac{a + vb}{b} = \frac{a}{b} + v > v.
\]

Following this pattern from depth \( n \) to depth \( n + 1 \) suggests that a quarter of all elements of a fixed (large) depth have integer part of roughly size \( v \), an eighth have integer part of roughly size \( 2v \), etc. Similarly, half of all elements have a fractional part of roughly size \( 1/u \), a quarter have a fractional part of roughly size \( 1/(2u) \), etc. So we expect that
\[
A^{(u,v)}(z; n) \approx \frac{1}{2^n} \left( \frac{2^n}{4} \left( v + \frac{2}{u} \right) + \frac{2^n}{8} \left( 2v + \frac{2}{u} \right) + \frac{2^n}{16} \left( 3v + \frac{2}{u} \right) + \cdots \right)
\]
\[
= v + \frac{\log 2}{u},
\]
where the last equality follows from the Taylor series expansions for \( 1/(1 - x)^2 \) and \( \log(1 - x) \).

This heuristic throws away a lot of information from the denominator in the fractional part of each element. We would therefore expect the true value of \( A^{(u,v)}(z; n) \) to be smaller than \( v + \log 2/u \).

As for the independence of the limit of \( A^{(u,v)}(z; n) \) from \( z \in [1/u, v] \) \( \cap \mathbb{Q} \), we note that if \( a/b \) is a vertex in a \((u,v)\)-Calkin-Wilf tree with continued fraction representation \( a/b = [q_0, q_1, \ldots, q_r] \), then the children of \( a/b \) have easily computable continued fractions, as the next result shows.

**Lemma 1.** (\textsuperscript{[2]} Lemma 5) Let \( a/b \) be a positive rational number with continued fraction representation \( a/b = [q_0, q_1, \ldots, q_r] \). It follows that
\begin{enumerate}[(a)]  
  \item if \( q_0 = 0 \), then \( a/(ua + b) = [0, u + q_1, \ldots, q_r] \);  
  \item if \( q_0 \neq 0 \), then \( a/(ua + b) = [0, u, q_0, q_1, \ldots, q_r] \);  
  \item \( (a + vb)/b = [v, q_0, q_1, \ldots, q_r] \).  
\end{enumerate}

It follows from the result above that, for large \( n \), most vertices of depth \( n \) will have approximately \( n/2 \) coefficients in their continued fraction expansions. This lowers the influence of the root on the value of \( A^{(u,v)}(z; n) \) as it is quickly buried by the above process. We will make this notion precise in Lemma \textsuperscript{[7]}.

## 2 Main Result

We show that for \( z \in [1/u, v] \) \( \cap \mathbb{Q} \), the limit of \( A^{(u,v)}(z; n) \) exists as \( n \to \infty \) in two main steps:

(A) First we show that, for \( z = 1/u \) or \( z = v \), the mean \( A^{(u,v)}(z; n) \) is monotonic increasing and bounded above as \( n \to \infty \).

(B) Second we show that \( A^{(u,v)}(z_1; n) - A^{(u,v)}(z_2; n) \to 0 \) as \( n \to \infty \) for any \( z_1, z_2 \in [1/u, v] \) \( \cap \mathbb{Q} \).

\textsuperscript{2}The reason for limiting our choice of roots to \([1/u, v] \cap \mathbb{Q}\) is that these rationals are the “orphan” roots in the sense that they are not the children of any rational in \( \textup{any} (u,v)\)-Calkin-Wilf tree \textsuperscript{[8]}. 

\textsuperscript{3}Note that \( A^{(u,v)}(z; n) \) is non-negative for all \( z \) and \( n \).

\textsuperscript{4}The mean \( A^{(u,v)}(z; n) \) is the average of the \( (2^n - 1) \) elements in the \((u,v)\)-Calkin-Wilf tree that have \( z \) as their root.

\textsuperscript{5}When \( a/b \) is the root of a \((u,v)\)-Calkin-Wilf tree, the \((u,v)\)-Calkin-Wilf tree is said to be \( \text{balanced} \) if \( b \leq 2a \) for \( b \geq 1 \).

\textsuperscript{6}The mean \( A^{(u,v)}(z; n) \) is the average of the \( (2^n - 1) \) elements in the \((u,v)\)-Calkin-Wilf tree that have \( z \) as their root.

\textsuperscript{7}The mean \( A^{(u,v)}(z; n) \) is the average of the \( (2^n - 1) \) elements in the \((u,v)\)-Calkin-Wilf tree that have \( z \) as their root.
We begin with a useful lemma for comparing rational numbers based on their continued fraction coefficients.

**Lemma 2.** (p. 101) Suppose that $\alpha, \beta \in \mathbb{Q}$ are distinct with $\alpha = [p_0, p_1, \ldots, p_s]$ and $\beta = [q_0, q_1, \ldots, q_t]$. Let $k$ be the smallest index such that $p_k \neq q_k$. Then $\alpha < \beta$ if and only if $p_k < q_k$ when $k$ is even and $p_k > q_k$ when $k$ is odd. If no such $k$ exists and $n < m$, then $\alpha < \beta$ if and only if $n$ is even.

We note here two useful results from [5] that will be used to obtain our main result. Lemma 3 and Corollary 4 show two things: that there is a very close relationship between two vertices if and only if $n$ is even.

**Lemma 3.** (Theorem 3) Suppose that $z$ and $z'$ are positive rational numbers with continued fraction representations $z = [q_0, q_1, \ldots, q_r]$ and $z' = [p_0, p_1, \ldots, p_s]$. Then $z'$ is a descendant of $z$ in the $(u, v)$-Calkin-Wilf tree with root $z$ if and only if the following conditions all hold:

(a) $s \geq r$ and $2 \mid (s - r)$;
(b) for $0 \leq j \leq s - r - 1$, $v \mid p_j$ when $j$ is even and $u \mid p_j$ when $j$ is odd;
(c) for $2 \leq i \leq r$, $p_{s-r+i} = q_i$;
(d) and
   (i) if $q_0 \neq 0$, then $p_{s-r} \geq q_0$, $v \mid (p_{s-r} - q_0)$ and $p_{s-r+1} = q_1$;
   (ii) otherwise, if $q_0 = 0$, then $v \mid p_{s-r}$, $p_{s-r+1} \geq q_1$, and $u \mid (p_{s-r+1} - q_1)$.

**Lemma 4.** (Corollary 3) Using the same hypothesis as Lemma 3 if $n$ is the depth of $z'$, then

$$n = \frac{1}{v} \left( \sum_{0 \leq j \leq s-r-1 \atop j \text{ even}} p_j + \sum_{0 \leq i \leq r \atop i \text{ even}} (p_{s-r+i} - q_i) \right)$$

$$+ \frac{1}{u} \left( \sum_{0 \leq j \leq s-r-1 \atop j \text{ odd}} p_j + \sum_{0 \leq i \leq r \atop i \text{ odd}} (p_{s-r+i} - q_i) \right).$$

The following lemma gives us the desired monotonicity for $A(u, v)(z; n)$ when $z = 1/u$ or $z = v$.

**Lemma 5.** For any $n \geq 0$, if $z = 1/u$ or $z = v$, then $S^{(u, v)}(z; n + 1) > 2S^{(u, v)}(z; n)$.

**Proof.** Let $n \geq 0$ be given. Enumerate the elements in $T^{(u, v)}(z; n)$ and $T^{(u, v)}(z; n + 1)$ as they appear from left to right in the $(u, v)$-Calkin-Wilf tree by $s_0, s_1, \ldots, s_{2^n-1}$ and $t_0, t_1, \ldots, t_{2^{n+1}-1}$, respectively. Clearly, for $0 \leq i \leq 2^n - 1$, $t_{2i}$ and $t_{2i+1}$ are the left and right children of $s_i$. Our goal is therefore to show that

$$2 \sum_{i=0}^{2^n-1} s_i < \sum_{i=0}^{2^{n+1}-1} t_i.$$

This desired inequality can be reduced further by noting that $t_{2i+1} = s_i + v$. In other words, we obtain the desired result if we can show that

$$\sum_{i=0}^{2^n-1} s_i < 2^nv + \sum_{i=0}^{2^n-1} t_{2i}.$$
Let $I_n = \sum_{i=0}^{2^n-1} \{s_i\}$. That is, $I_n$ is the sum of the integer parts of all of the depth $n$ elements of the $(u, v)$-Calkin-Wilf tree.

Claim: $I_n = (2^n - 1)v + \lfloor w \rfloor$ for $n \geq 0$.

We prove the above claim by induction. Clearly $I_0 = \lfloor w \rfloor$. Suppose that the claim holds for some $k \geq 1$. Since the left child of any number appearing in the $(u, v)$-Calkin-Wilf tree is smaller than $1/u$ and the right child of any element is always the original element plus $v$, it follows that $I_{k+1} = I_k + 2^k v$. By assumption, $I_k = (2^k - 1)v + \lfloor w \rfloor$, from which the desired result immediately follows.

Our previous claim shows that we obtain the desired result if we can show that

$$\lfloor w \rfloor + \sum_{i=0}^{2^n-1} \{s_i\} < v + \sum_{i=0}^{2^n-1} t_{2i}.$$  

If we take $w = 1/u$, then $\lfloor w \rfloor = 0$ and, by Lemma 1 the short continued fraction representation of $\{s_i\}$ must be of the form $[0, \alpha_1 u, \alpha_2 v, \ldots, \alpha_k u]$ with $m := m(s_i) = n + 2 - \sum_{i=1}^{k} \alpha_i > 0$. Since $\{s_{2^n-1}\} = [0, u]$ and $t_0 = [0, (n + 2)u]$, we see that, in this case, (1) reduces further to the inequality

$$\sum_{i=0}^{2^n-2} \{s_i\} < \sum_{i=1}^{2^n-1} t_{2i}.$$  

If $\alpha_k = 1$, then there is an $1 \leq i^* \leq 2^n - 1$ such that

$$t_{2i^*} = [0, \alpha_1 u, \alpha_2 v, \ldots, (\alpha_{k-1} + 1)v, mu].$$  

If $\alpha_k > 1$, then there is an $1 \leq i^* \leq 2^n - 1$ such that

$$t_{2i^*} = [0, \alpha_1 u, \alpha_2 v, \ldots, (\alpha_k - 1)u, v, mu].$$

In either case, it follows that $\{s_i\} < t_{2i^*}$ by Lemma 2. Note that the above association between $\{\{s_i\}_{i=0}^{2^n-2}$ and $\{t_{2i}\}_{i=1}^{2^n-1}$ is bijective, from which (1) follows in this case.

If we take $w = v$, then $\lfloor w \rfloor = v$ and, by Lemma 1 the short continued fraction representation of $\{s_i\}$ must be of the form $[0, \alpha_1 u, \alpha_2 v, \ldots, \alpha_k v]$ with $m$ defined as in the previous case. Since $\{s_{2^n-1}\} = 0$ and $t_0 = [0, (n + 1)u, v]$, we see that, in this case, (1) also reduces to (2). If $m = 1$, then there is an $1 \leq i^* \leq 2^n - 1$ such that

$$t_{2i^*} = [0, \alpha_1 u, \alpha_2 v, \ldots, (\alpha_k + 1)v].$$

If $m > 1$, then there is an $1 \leq i^* \leq 2^n - 1$ such that

$$t_{2i^*} = [0, \alpha_1 u, \alpha_2 v, \ldots, \alpha_k v, (m - 1)u, v].$$

As in the previous case, (2) follows, completing the proof of the lemma.

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The following theorem establishes $v + \log 2/u$ as an upper bound of $A^{(u,v)}(z; n)$. Note that by $f(x) = O(g(x))$ we mean that $|f(x)| \leq C|g(x)|$ for some constant $C$ (which may differ depending on context) and all sufficiently large $x$.

**Theorem 2.** If $u$ and $v$ are positive integers with $uv > 1$ and $z \in \mathbb{Q}$, then $A^{(u,v)}(z; n)$ is bounded above for all $n \geq 0$. In particular,

$$v + \frac{\log 2}{u} = \lim_{n \to \infty} A^{(u,v)}(z; n) = O\left(\frac{1}{u^2 v}\right).$$
Proof. For brevity, we let $S(n) := S(n, v)(z; n)$, $A(n) := A(n, v)(z; n)$, and $T(n) := T(n, v)(z; n)$.

For $n \geq 1$, every rational number in the set $T(n)$ is either the left-child or right-child of a rational number in the set $T(n - 1)$. In particular, for every $y \in T(n - 1)$, there is a unique $x \in T(n)$ that is the right-child of $y$. By definition, $x = y + v$. Likewise, there is a unique $z \in T(n)$ that is the left-child of $y$, making $z = \frac{1}{u} y$.

It follows that

$$S(n) = S(n - 1) + 2^{n-1} v + \sum_{y \in T(n-1)} \frac{1}{u + \frac{1}{y}}. \quad (3)$$

By dividing both sides of (3) by $2^n$, we immediately obtain the equality

$$A(n) = \frac{1}{2} A(n - 1) + \frac{v}{2^n} \sum_{y \in T(n-1)} \frac{1}{u + \frac{1}{y}}. \quad (4)$$

By induction on (4), we can express $A(n)$ as

$$A(n) = \frac{1}{2^n} A(0) + v \sum_{k=1}^{n} \frac{1}{2^{k}} + \frac{1}{2^n} \sum_{k=1}^{n} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}}$$

$$= \frac{z}{2^n} + v \left(1 - \frac{1}{2^n}\right) + \frac{1}{2^n} \sum_{k=1}^{n} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} \quad (5)$$

Taking the limit as $n \to \infty$ of both sides of (5) shows that, to complete the proof, it is enough to prove that

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{n} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \frac{\log 2}{u} + O\left(\frac{1}{u^2 v}\right). \quad (6)$$

Let $m = \lfloor n/2 \rfloor$. We split the double sum in (6) into two parts,

$$\sum_{k=1}^{m} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} + \sum_{k=m+1}^{n} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} \quad (7)$$

For $m < k \leq n$, we apply the following simple upper bound in (7),

$$\sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} \leq \frac{2^{n-k}}{u}. \quad (8)$$

It follows that

$$\sum_{k=m+1}^{n} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} \leq \frac{2^{n-k}}{u} \sum_{k=m+1}^{n} \frac{1}{u}$$

$$= \frac{1}{u} \sum_{i=0}^{n-(m+1)} \frac{2^i}{u}$$

$$= \frac{2^{n-m} - 1}{u}. \quad (8)$$

Since $m \to \infty$ as $n \to \infty$, if we apply (8) to (7), then, by (6), we have reduced the problem to showing that

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{m} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \frac{\log 2}{u} + O\left(\frac{1}{u^2 v}\right). \quad (9)$$
Using the same reasoning on the sum \( \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} \) that led to (3), we see that, for \( n-k > 2 \),

\[
\sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \sum_{y \in T(n-(k+1))} \frac{1}{2u + \frac{1}{y}} + \sum_{y \in T(n-(k+1))} \frac{1}{u + \frac{1}{y}}. \tag{10}
\]

We convert the rightmost sum on the right-hand side of (10) into a sum of geometric series,

\[
\sum_{y \in T(n-(k+1))} \frac{1}{u + \frac{1}{y}} = \frac{1}{u} \sum_{y \in T(n-(k+1))} \left( \frac{1}{u(v+y)} \right)^j. \tag{11}
\]

The justification for (11) follows from the fact that \( 0 < \frac{1}{u(v+y)} \leq \frac{1}{uv} \leq \frac{1}{2} \) for any positive rational \( y \). So

\[
\sum_{y \in T(n-(k+1))} \frac{1}{u + \frac{1}{y}} = \frac{2^{n-(k+2)}}{u} \left( 1 + O\left( \frac{1}{uv} \right) \right). \tag{12}
\]

Combining (12) with (10), we see that

\[
\sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \sum_{y \in T(n-(k+1))} \frac{1}{2u + \frac{1}{y}} + \frac{2^{n-(k+2)}}{u} \left( 1 + O\left( \frac{1}{uv} \right) \right). \tag{13}
\]

We can now repeat all of the above steps starting from (10) with the sum

\[
\sum_{y \in T(n-k)} \frac{1}{2u + \frac{1}{y}}.
\]

Inductively, for any positive integer \( j < n-k \), it follows that

\[
\sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \sum_{y \in T(n-(k+j))} \frac{1}{(j+1)u + \frac{1}{y}} + \sum_{i=1}^{j} \frac{2^{n-(k+i+1)}}{iu} \left( 1 + O\left( \frac{1}{uv} \right) \right) \tag{13}
\]

where the constant associated with the big-oh term is uniform for all of the sums.

Let \( m' = \lfloor n/4 \rfloor \). Then, from (13), for \( 1 \leq k \leq m \),

\[
\sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \sum_{y \in T(n-(k+m'))} \frac{1}{(m'+1)u + \frac{1}{y}} + \sum_{i=1}^{m'} \frac{2^{n-(k+i+1)}}{iu} \left( 1 + O\left( \frac{1}{uv} \right) \right) = O\left( \frac{2^{n-(k+m'+1)}}{(m'+1)u} \right) + \sum_{i=1}^{m'} \frac{2^{n-(k+i+1)}}{iu} \left( 1 + O\left( \frac{1}{uv} \right) \right). \tag{14}
\]

(Note that for \( n \) sufficiently large, since \( k \leq m \), then \( k + m' \leq 3n/4 \), so \( n - (k + m') \geq 1 \). In particular, we can apply (13) with \( j = m' \).)
Using the Taylor series expansion of \( \log(1 - x) \) for \(|x| < 1\), we see that
\[
\sum_{i=1}^{m'} \frac{2n-(k+i+1)}{iu} = 2^{n-(k+1)} \sum_{i=1}^{m'} \frac{1}{i2^i} = \frac{2^{n-(k+1)}}{u} \left( \log 2 - \sum_{i>m'} \frac{1}{i2^i} \right).
\]  
(15)

Combining (14) and (15) with the double sum from (9), it follows that
\[
\frac{1}{2n-1} \sum_{k=1}^{m} \sum_{y \in T(n-k)} \frac{1}{u + \frac{1}{y}} = \frac{1}{u} \sum_{k=1}^{m} \frac{1}{2^k} \left( \log 2 - \sum_{i>m'} \frac{1}{i2^i} \right) \left( 1 + O\left( \frac{1}{uv} \right) \right) + O\left( \frac{1}{(m'+1)u} \right)
\]  
(16)

The result (9) now follows from taking the limit of (16) as \( n \to \infty \).

Lemma 5 and Theorem 2 immediately give (A). To show (B), we give a crude estimate of the difference between two rational numbers based on their short continued fraction representations.

**Lemma 6.** Suppose that \( \alpha, \beta \in \mathbb{Q} \) are distinct with \( \alpha = [p_0, p_1, \ldots, p_s] \) and \( \beta = [q_0, q_1, \ldots, q_r] \). Let \( k \) be the largest index such that \( p_k = q_k \). Then
\[
|\alpha - \beta| \leq \prod_{j=1}^{k} \frac{1}{p_j^2}.
\]

**Proof.** We rewrite the continued fraction representations of \( \alpha \) and \( \beta \) as
\[
\alpha = [p_0, p_1, \ldots, p_k, p_{k+1}, \ldots, p_s] \quad \text{and} \quad \beta = [q_0, p_1, \ldots, q_k, q_{k+1}, \ldots, q_r].
\]
(Note that we cannot have \( k = r = s \) and that if \( k = r \) or \( k = s \), the estimates below still apply.)

Now, for \( A_i = [p_i, \ldots, p_s] \) and \( B_i = [q_i, \ldots, q_r] \) with \( 1 \leq i \leq k + 1 \),
\[
|\alpha - \beta| = \left| \frac{1}{p_1 + A_1} - \frac{1}{p_1 + B_1} \right| \\
\leq \left| \frac{1}{p_1 + A_1} - \frac{1}{p_1 + B_1} \right| \cdot \frac{1}{p_1^2} \\
\vdots \\
\leq \left| \frac{1}{p_{k+1} + A_{k+1}} - \frac{1}{p_{k+1} + B_{k+1}} \right| \cdot \prod_{j=1}^{k} \frac{1}{p_j^2} \\
\leq \prod_{j=1}^{k} \frac{1}{p_j^2}.
\]

In the case where the rationals from Lemma 6 are vertices of possibly two different \((u,v)\)-Calkin-Wilf trees, we get the following corollary.
Corollary 1. With $\alpha$ and $\beta$ as in Lemma 2 and, additionally, suppose that $\alpha$ and $\beta$ are vertices of possibly two different $(u,v)$-Calkin-Wilf trees, then

$$\alpha - \beta = O\left(\frac{\max\{u,v\}}{2^k}\right).$$

Proof. The corollary follows from the fact that if the two rationals $\alpha$ and $\beta$ are vertices on $(u,v)$-Calkin-Wilf trees, then $p_i$ is divisible by $v$ for even $i$ and divisible by $u$ for odd $i$ by Lemma 3.

Before we begin our proof of (B), we need one additional lemma.

Lemma 7. Let $y = [q_0, q_1, \ldots, q_r]$ with $q_r \neq 1$ when $y \neq 1$ and $r = 0$ when $y = 1$ and define $\ell(y) = r$. Let $f_z(n, m) = #\{y \in T^{(u,v)}(z; n) : \ell(y) = m + \ell(z)\}$, then for $m \geq 0$,

$$f_z(n, m) = \begin{cases} \binom{n + 1}{m} & \text{if } 2 \nmid m \text{ and } z > 1 \\ \binom{n + 1}{m} & \text{if } 2 \nmid m \text{ and } z < 1 \\ \binom{n}{m} & \text{if } z = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The desired result can be shown to be true for $n < 2$ by inspection.

Assume that the statement is true for all $0 \leq j \leq k$ for some $k \geq 2$ and let $y \in T^{(u,v)}(z; k+1)$ be such that $\ell(y) = m + \ell(z)$. That is, we assume $y$ is a rational number counted by $f_z(k+1, m) = 1$. There is a sequence of rational numbers $z_0 = z, z_1, \ldots, z_{k+1} = y$ such that $z_{i+1}$ is a descendant of $z_i$ for $0 \leq i < k + 1$. By Lemma 3 we see that $\ell(z_{i+1}) - \ell(z_i) \in \{0, 1, 2\}$. In fact, for $i \geq 1$, $\ell(z_{i+1}) - \ell(z_i) = 2$ if and only if $z_{i+1}$ is a left child of $z_i$ and $z_i$ is a right child of $z_{i-1}$. $\ell(z_1) - \ell(z_0) = 2$ if and only if $z_1$ is a left child of $z_0$ with $z_0 > 1$, and $\ell(z_1) - \ell(z_0) = 1$ if and only if $z_1$ is a left child of $z_0$ with $z_0 = 1$.

We now consider the following three cases:

Case 1: $z_2$ is a right child of $z_1$ and $z_1$ is a right child of $z_0$.
In this case we have that $y \in T^{(u,v)}(z_2; k - 1)$ with $\ell(y) = m + \ell(z_2)$.

Case 2: $z_2$ is a left child of $z_1$ and $z_1$ is a right child of $z_0$.
In this case we have that $y \in T^{(u,v)}(z_2; k - 1)$ with $\ell(y) = m + 2 + \ell(z_2)$.

Case 3: $z_1$ is a left child of $z_0$.
In this case we have that $y \in T^{(u,v)}(z_1; k)$ with

$$\ell(y) = \begin{cases} m - 2 + \ell(z_1) & \text{if } z_0 > 1 \\ m + \ell(z_1) & \text{if } z_0 < 1 \\ m - 1 + \ell(z_1) & \text{if } z_0 = 1. \end{cases}$$

It follows from the three cases above that,

$$f_z(k + 1, m) = \begin{cases} f_z'(k - 1, m) + f_z''(k - 1, m - 2) + f_z''(k, m - 2) & \text{if } z_0 > 1 \\ f_z'(k - 1, m) + f_z''(k - 1, m - 2) + f_z''(k, m) & \text{if } z_0 < 1 \\ f_z'(k - 1, m) + f_z''(k - 1, m - 2) + f_z''(k, m - 1) & \text{if } z_0 = 1. \end{cases}$$

where $z' = z_0 + 2v > 1$, $z'' = \frac{1}{u + \frac{1}{z_0}} < 1$, and $z''' = \frac{1}{u + \frac{1}{z_0}} < 1$.

We will now make heavy use of the well-known binomial coefficient identity $\binom{n}{m} = \binom{n}{m-1} + \binom{n-1}{m-1}$ to complete the proof.
For $z_0 > 1$, the desired result is trivially true when $2 \mid m$, so we assume otherwise. Therefore, by assumption

$$f_z(k+1, m) = \left(\frac{k}{m}\right) + \left(\frac{k}{m-1}\right) + \left(\frac{k+1}{m-1}\right)$$

$$= \left(\frac{k+1}{m}\right) + \left(\frac{k+1}{m-1}\right)$$

$$= \left(\frac{k+2}{m}\right).$$

Similarly, for $z_0 < 1$, the desired result is also trivially true when $2 \mid m$, so we assume otherwise. Therefore, by assumption

$$f_z(k+1, m) = \left(\frac{k}{m}\right) + \left(\frac{k}{m-1}\right) + \left(\frac{k+1}{m+1}\right)$$

$$= \left(\frac{k+1}{m}\right) + \left(\frac{k+1}{m+1}\right)$$

$$= \left(\frac{k+2}{m+1}\right).$$

Finally, for $z_0 = 1$, by assumption, when $m$ is odd,

$$f_z(k+1, m) = \left(\frac{k}{m}\right) + \left(\frac{k}{m-1}\right) + 0$$

$$= \left(\frac{k}{m}\right) + \left(\frac{k}{m-1}\right)$$

$$= \left(\frac{k+1}{m}\right)$$

and when $m$ is even,

$$f_z(k+1, m) = 0 + 0 + \left(\frac{k+1}{m}\right)$$

$$= \left(\frac{k+1}{m}\right).$$

Having exhausted all possibilities, we complete the proof by induction.

An application of the de Moivre-Laplace limit theorem [4, p. 186] shows that the number of continued fraction coefficients in depth $n$ elements is normally distributed with mean approximately $n/2$.

Corollary 11 and Lemma 14 can now be used to compare the difference between rationals in different $(u, v)$-Calkin-Wilf trees that are in the same position relative to the root, showing that the mean values of the rows for different trees are asymptotically the same.

**Proposition 2.** For any $z_1, z_2 \in [1/u, v] \cap \mathbb{Q}$, we have that

$$A_{(u,v)}^{(u,v)}(z_1; n) - A_{(u,v)}^{(u,v)}(z_2; n) \to 0$$

as $n \to \infty$. 

10
Proof. We begin by considering the case where \( z_1 = 1/u \) and \( z_2 = v \). Let \( y \in \mathcal{T}^{(u,v)}(v;n) \). Then by Lemma 3 and Lemma 4, \( y \) has a continued fraction representation of the form \( y = [\alpha_0 v, \alpha_1 u, \ldots, \alpha_k v] \) with \( \sum_{i=0}^{k} \alpha_i = n + 1 \). Consider the map \( f : \mathcal{T}^{(u,v)}(v;n) \rightarrow \mathcal{T}^{(u,v)}(1/u;n) \) given by

\[
f(y) = \begin{cases} 
[\alpha_0 v, \alpha_1 u, \ldots, (\alpha_{k-1} + 1)u] & \text{if } \alpha_k = 1 \\
[\alpha_0 v, \alpha_1 u, \ldots, (\alpha_k - 1)v, u] & \text{otherwise}.
\end{cases}
\]

It is clear that \( f \) represents a well-defined bijection. In particular, by Corollary 1 and Lemma 7,

\[
A^{(u,v)}\left(\frac{1}{u};n\right) - A^{(u,v)}(v;n) = \frac{1}{2^n} \sum_{y \in \mathcal{T}^{(u,v)}(v;n)} f(y) - y
= O\left(\frac{\max\{u,v\}}{2^n} \left( \sum_{y \in \mathcal{T}^{(u,v)}(v;n), \alpha_k = 1} \frac{1}{2^{k-1}} + \sum_{y \in \mathcal{T}^{(u,v)}(v;n), \alpha_k > 1} \frac{1}{2^k} \right) \right)
= O\left(\frac{\max\{u,v\}}{2^n} \sum_{y \in \mathcal{T}^{(u,v)}(v;n)} \frac{1}{2^k} \right)
= O\left(\frac{\max\{u,v\}}{2^n} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{2^k} \right)
= O\left(\frac{\max\{u,v\}}{2^n} \cdot \left(\frac{3}{4}\right)^n \right),
\]

which goes to 0 as \( n \to \infty \).

The cases \( z_1 = 1/u \) and \( z_2 \in (1/u,1] \cap \mathbb{Q} \) and \( z_1 = v \) and \( z_2 \in [1,v) \cap \mathbb{Q} \) can be handled in a similar way. These three cases complete the proof of the proposition.

Proposition 2 completes the proof of (B), giving the desired result.

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