First exit time from a bounded interval for pseudo-processes driven by the equation
\[
\partial / \partial t = (-1)^{N-1}\partial^{2N}/\partial x^{2N}
\]

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Abstract

Let \( N \) be an integer greater than 1. We consider the pseudo-process \( X = (X_t)_{t \geq 0} \) driven by the high-order heat-type equation \( \partial / \partial t = (-1)^{N-1}\partial^{2N}/\partial x^{2N} \). Let us introduce the first exit time \( \tau_{ab} \) from a bounded interval \((a, b)\) by \( X (a, b \in \mathbb{R}) \) together with the related location, namely \( X_{\tau_{ab}} \).

In this paper, we provide a representation of the joint pseudo-distribution of the vector \((\tau_{ab}, X_{\tau_{ab}})\) by means of some determinants. The method we use is based on a Feynman-Kac-like functional related to the pseudo-process \( X \) which leads to a boundary value problem. In particular, the pseudo-distribution of \( X_{\tau_{ab}} \) admits a fine expression involving famous Hermite interpolating polynomials.

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1 Introduction

Let \( N \) be an integer greater than 1 and set \( \kappa_N = (-1)^{N-1} \). We consider the pseudo-process \((X_t)_{t \geq 0}\) driven by the high-order heat-type equation \( \partial / \partial t = \kappa_N \partial^{2N}/\partial x^{2N} \), the so-called pseudo-Brownian motion. This is the pseudo-Markov process with independent and stationary increments, associated with the signed heat-type kernel \( p(t; x) \) which is the elementary solution of the foregoing equation. The kernel \( p(t; x) \) is characterized by its Fourier transform:

\[
\int_{-\infty}^{+\infty} e^{iux} p(t; x) \, dx = e^{-tu^{2N}}.
\]

We define the related transition kernel as \( p(t; x, y) = p(t; x - y) \) for any time \( t > 0 \) and any real numbers \( x, y \), which represents the pseudo-probability that the pseudo-process started at \( x \) is in state \( y \) at time \( t \). In symbols,

\[
P_X\{X_t \in dy\} = p(t; x, y) \, dy.
\]

The \( \mathbb{P}_x, x \in \mathbb{R} \), define a family of signed measures whose total mass equals one:

\[
\mathbb{P}_x\{X_t \in \mathbb{R}\} = \int_{-\infty}^{+\infty} p(t; x, y) \, dy = 1.
\]

The transition kernel \( p(t; x, y) \) satisfies the backward and forward Kolmogorov equations

\[
\frac{\partial p}{\partial t}(t; x, y) = \kappa_N \frac{\partial^{2N} p}{\partial y^{2N}}(t; x, y) = \kappa_N \frac{\partial^{2N} p}{\partial x^{2N}}(t; x, y).
\]

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To be more precise, let us recall that the pseudo-Markov process \((X_t)_{t \geq 0}\) is defined according to the usual chain rule: for any positive integer \(n\), for any times \(t_1, \ldots, t_n\) such that \(0 < t_1 < \cdots < t_n\) and any real numbers \(x_1, \ldots, x_n\), and setting \(t_0 = 0, x_0 = x\),

\[
\mathbb{P}_x \{X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n\} = \left( \prod_{k=1}^n p(t_k - t_{k-1}; x_{k-1}, x_k) \right) dx_1 \cdots dx_n. \tag{1.1}
\]

In particular, by setting \(T_t \phi(x) = \mathbb{E}_x[\phi(X_t)]\) for any time \(t\), any real number \(x\) and any bounded \(C^2\)-function \(\phi\), the family \((T_t)_{t \geq 0}\) is a semi-group of operators whose infinitesimal generator \(\mathcal{G}\) is given by

\[
\mathcal{G} \phi(x) = \lim_{h \to 0^+} \frac{1}{h} \left( \mathbb{E}_x[\phi(X_h)] - \phi(x) \right) = \kappa_x \phi^{(2N)}(x). \tag{1.2}
\]

Above and throughout the paper, for any non-negative integer \(\ell\), \(\phi^{(\ell)}\) stands for the derivative of \(\phi\) of order \(\ell\).

The very notion of pseudo-process in a general framework goes back to Daletskii and Fomin in 1965 ([7]). The reader can find an extensive literature on the particular case of pseudo-Brownian motion. For instance, let us quote the works of Beghin, Cammarota, Hochberg, Krylov, Lachal, Nakajima, Nikitin, Nishioka, Orsingher, Ragozina ([2] to [6], [8], [9], [11] to [23]) and the references therein. These papers deal with several functionals related to pseudo-Brownian motion: sojourn time in a bounded or not interval, first overshooting time of a single level, maximum or minimum time in a bounded or not interval, first overshooting time of a double level, maximum or minimum up to a fixed time... Let us mention also other interesting works: one dealing with high-order Schrödinger-type equation \(\partial \phi / \partial t = i \theta^{2N} / \partial x^{2N}\) which is related to the so-called Feynman-Kac measure [1], as well as [10] in which the authors develop an alternative and more probabilistic approach to pseudo-processes.

In [13, 14], we obtained the pseudo-distribution of the first overshooting time of a single threshold, together with the corresponding location at this time. In symbols, if \(\tau_a\) denotes the first overshooting time of a fixed level \(a\) (upwards or downwards), we derived the joint pseudo-distribution of the couple \((\tau_a, X_{\tau_a})\). Therein, we used an extension of famous Spitzer’s identity. In [13, 14] and, in the particular case \(N = 2\), in [20, 21], the authors observed a curious fact concerning the pseudo-distribution of \(X_{\tau_a}\): it is a linear combination of the Dirac distribution and its successive derivatives (in the sense of Schwartz distributions):

\[
\mathbb{P}_x \{X_{\tau_a} \in dz\} / dz = \sum_{k=0}^{N-1} \frac{(a-x)^k}{k!} \delta_a^{(k)}(z). \tag{1.3}
\]

The quantity \(\delta_a^{(k)}\) is to be understood as the functional acting on test functions \(\phi\) according as \(\langle \delta_a^{(k)}, \phi \rangle = (-1)^k \phi^{(k)}(a)\). Formula (1.3) says that the overshoot through level \(a\) should be actually concentrated at \(a\). The appearance of the Schwartz-Dirac distribution \(\delta_a\) together with its successive derivatives can be interpreted by means of “multipoles” in reference to electric dipoles as in [13, 14] and, for \(N = 2\), in [20, 21]. In particular, therein, \(\delta_a\) and \(\delta_b\) are respectively named “monopole” and “dipole”. We refer the reader to [22] for a detailed account on monopoles and dipoles. An explanation of this curious fact should be found in considering a linear pseudo-random walk with \(2N\) consecutive neighbours around each sites. Indeed, after suitably normalizing such a walk, the neighbours cluster into a single site and form a multipole; see the draft [17].

Till now, the first exit time from a bounded interval, or, equivalently, the first overshooting time of a double threshold has not yet been considered. This is the purpose of this work.

Let us introduce the first exit time from \((a,b)\) for \((X_t)_{t \geq 0}\):

\[
\tau_{ab} = \inf \{t \geq 0 : X_t \notin (a,b)\}
\]

with the usual convention that \(\inf \emptyset = +\infty\). In this paper, we tackle the problem of finding the pseudo-distribution related to the double threshold: we provide a representation for the joint pseudo-distribution of the couple \((\tau_{ab}, X_{\tau_{ab}})\). This representation involves some determinants; this is the object of Theorems 1 and 2. For the location \(X_{\tau_{ab}}\), we have the following counterpart to (1.3); see Theorem 4:

\[
\mathbb{P}_x \{X_{\tau_{ab}} \in dz\} / dz = \sum_{k=0}^{N-1} (-1)^k H_k^{-}(x) \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k H_k^{+}(x) \delta_b^{(k)}(z). \tag{1.4}
\]
where the functions $H_k^\pm$, $0 \leq k \leq N - 1$, are the classical Hermite interpolating polynomials of degree $(2N - 1)$ related to points $a$ and $b$ satisfying
\[
(H_k^-)^{(\ell)}(a) = (H_k^+)^{(\ell)}(b) = \delta_{k\ell}, \quad (H_k^+)^{(\ell)}(a) = (H_k^-)^{(\ell)}(b) = 0, \quad 0 \leq \ell \leq N - 1.
\]
Above, the quantity $\delta_{k\ell}$ denotes the usual Kronecker symbol: if $k = \ell$, $\delta_{k\ell} = 1$, else $\delta_{k\ell} = 0$. They explicitly write as
\[
H_k^-(x) = \left(\frac{x - a}{b - a}\right)^N \frac{(x - a)^k}{k!} \sum_{\ell=0}^{N-k-1} \binom{\ell + N - 1}{\ell} \left(\frac{x - a}{b - a}\right)^\ell,
\]
\[
H_k^+(x) = \left(\frac{x - a}{b - a}\right)^N \frac{(x - b)^k}{k!} \sum_{\ell=0}^{N-k-1} \binom{\ell + N - 1}{\ell} \left(\frac{b - x}{b - a}\right)^\ell.
\]
In particular, we can deduce from (1.4) the “ruin pseudo-probabilities”, that is, the pseudo-probabilities of overshooting one level ($a$ or $b$) before the other one; see Corollary 3.

These results have been announced without any proof in a survey on pseudo-Brownian motion, [16], after a conference held in Madrid (IWAP 2010).

Throughout the paper, the function $\varphi$ denotes any $(N - 1)$ times differentiable function.

## 2 Feynman-Kac functional

We start from the following fact: in [13, 14], we first obtained the pseudo-distribution of the couple $(\sup_{0 \leq s \leq t} X_a, X_t)$ by making use of an extension of Spitzer’s identity. From this, we deduced that of the couple $(\tau_a, X_{\tau_a})$ and we made the observation that, for any $\lambda \geq 0$ and any $(N - 1)$ times differentiable bounded function $\varphi$, the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x \left( e^{-\lambda \tau_a} \varphi(X_{\tau_a}) \mathbb{1}_{\{\tau_a < +\infty\}} \right)$ solves the boundary value problem
\[
\begin{align*}
\{ & x, \Phi^{(2N)}(x) = \lambda \Phi(x), \quad x \in (-\infty, a) \text{ (or } x \in (a, +\infty)), \\
& \Phi^{(k)}(a) = \varphi^{(k)}(a) \text{ for } k \in \{0, 1, \ldots, N - 1\}.
\end{align*}
\]

So, we state the heuristic that an analogous boundary value problem should hold for the Feynman-Kac functional related to $\tau_{ab}$. The results obtained here through this approach coincide with limiting results deduced from a suitable pseudo-random walk studied in [17]. Moreover, when taking the limit as $a$ goes to $-\infty$ or $b$ goes to $+\infty$ in the present results, we retrieve the pseudo-distribution of $(\tau_a, X_{\tau_a})$ obtained in [14]. So, these observations comfort us in our heuristic. Actually, our purpose in this work is essentially concentrated in calculating the pseudo-distribution of $(\tau_{ab}, X_{\tau_{ab}})$.

As pointed out in several works on pseudo-processes, pseudo-Brownian motion is properly defined only on the set of dyadic times and ad-hoc definitions should be taken for computing certain functionals of this pseudo-process depending on a continuous set of times; see, e.g., [14] and, in the particular case $N = 2$, [21]. Roughly speaking, the dense subset of dyadic times is appropriate because of the usual property that for any $n \in \mathbb{N}$, $\{k/2^n, k \in \mathbb{N}\} \subset \{k/2^{n+1}, k \in \mathbb{N}\}$. Indeed, this latter permits to view the pseudo-process $(X_t)_{t \geq 0}$ as an informal limit of the family of step-processes $(X_{n,t})_{t \geq 0}$ defined according to the following sampling procedure:
\[
X_{n,t} = \sum_{k=0}^{n} \mathbb{1}_{\{k/2^n, (k+1)/2^n\}}(t)X_{k/2^n}.
\]

For each fixed $n \in \mathbb{N}$, the sequence $(X_{k/2^n})_{k \in \mathbb{N}}$ can be correctly defined thanks to (1.1). But the fact that $\int_{-\infty}^{+\infty} |p(t; x)| \, dx > 1$ prevent us from applying the classical extension theorem of Kolmogorov for finding a priori a $\sigma$-additive measure on the usual space of right-continuous functions on $[0, +\infty)$ which have left-hand limits, measure whose finite projections would yield the finite-dimensional pseudo-distributions of the sequence $(X_{k/2^n})_{k \in \mathbb{N}}$.

For our concern, we set
\[
\tau_{ab,n} = \frac{1}{2n} \min\{k \in \mathbb{N} : X_{k/2^n} \notin (a, b)\}
\]
and, for \( x \in (a, b) \),
\[
\Phi_n(x) = \mathbb{E}_x \left( e^{-\lambda \tau_{ab, n}} \varphi(X_{\tau_{ab, n}}) \mathbbm{1}_{\{\tau_{ab, n} < +\infty\}} \right).
\]

Then, we define the Feynman-Kac functional \( \Phi(x) = \mathbb{E}_x \left( e^{-\lambda \tau_{ab}} \varphi(X_{\tau_{ab}}) \mathbbm{1}_{\{\tau_{ab} < +\infty\}} \right) \) as the limit
\[
\Phi(x) \overset{\text{def}}{=} \lim_{n \to +\infty} \Phi_n(x)
\]
and we state below the analogue to (2.1).

**Heuristic.** For any \( \lambda \geq 0 \) and any \((N - 1)\) times differentiable bounded function \( \varphi \), the Feynman-Kac functional \( \Phi(x) = \mathbb{E}_x \left( e^{-\lambda \tau_{ab}} \varphi(X_{\tau_{ab}}) \mathbbm{1}_{\{\tau_{ab} < +\infty\}} \right) \) solves the boundary value problem
\[
\begin{aligned}
\kappa_n \Phi^{(2N)}(x) &= \lambda \Phi(x), \quad x \in (a, b), \\
\Phi^{(k)}(a) &= \varphi^{(k)}(a) \quad \text{and} \quad \Phi^{(k)}(b) = \varphi^{(k)}(b) \quad \text{for} \quad k \in \{0, 1, \ldots, N - 1\}.
\end{aligned}
\tag{2.2}
\]

### 3 Joint pseudo-distribution of \((\tau_{ab}, X_{\tau_{ab}})\)

In this section, we solve boundary value problem (2.2) in order to derive the joint pseudo-probability of \((\tau_{ab}, X_{\tau_{ab}})\). In this way, if we choose \( \varphi(x) = e^{\mu x}, \mu \in \mathbb{R} \), we first obtain its Laplace-Fourier transform. Actually, the results we derived hold true for any \((N - 1)\) times differentiable function \( \varphi \).

Let us introduce the \((2N)\)th roots of \( \kappa_n \): \( \theta_{\ell} = e^{\frac{2\pi \ell}{2N} - \frac{1}{2} \pi}, 1 \leq \ell \leq 2N \). We have \( \theta_{\ell}^{2N} = \kappa_n \). For any complex number \( z \), we set \( \theta_{\ell, a} = e^{\lambda(1/2N)z} \).

**Theorem 1.** The Feynman-Kac functional related to \((\tau_{ab}, X_{\tau_{ab}})\) admits the following representation:
\[
\mathbb{E}_x \left( e^{-\lambda \tau_{ab}} \varphi(X_{\tau_{ab}}) \mathbbm{1}_{\{\tau_{ab} < +\infty\}} \right) = \sum_{k=0}^{N-1} \lambda^{-\frac{k}{N}} \Delta_k^+(\lambda; x) \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-\frac{k}{N}} \Delta_k^-(\lambda; x) \varphi^{(k)}(b) \tag{3.1}
\]
where the quantities \( \Delta(\lambda) \) and \( \Delta_k^\pm(\lambda; x) \) are the determinants below:
\[
\Delta(\lambda) = \begin{vmatrix}
\theta_{1, a} e_{\lambda}^a & \ldots & \theta_{2N-1, a} e_{\lambda}^{2N-1a} \\
\theta_{1} e_{\lambda}^a & \ldots & \theta_{2N} e_{\lambda}^{2Na} \\
\vdots & & \vdots \\
\theta_{1} e_{\lambda}^{N-1a} & \ldots & \theta_{2N} e_{\lambda}^{2N(N-1)a} \\
\epsilon_{\lambda}^b & \ldots & \theta_{2N} e_{\lambda}^{2N(N-1)b} \\
\theta_{1} e_{\lambda}^{N-1b} & \ldots & \theta_{2N} e_{\lambda}^{2N(N-1)b} \\
\vdots & & \vdots \\
\theta_{1} e_{\lambda}^{(N-1)b} & \ldots & \theta_{2N} e_{\lambda}^{2N(N-1)b}
\end{vmatrix}
\]
and

\[
\Delta^{-}_k (\lambda; x) = \begin{vmatrix}
\lambda e^{\theta_{1}^b} & \cdots & \lambda e^{\theta_{2N}^b} \\
\vdots & \ddots & \vdots \\
\lambda e^{\theta_{N}^b} & \cdots & \lambda e^{\theta_{2N}^b}
\end{vmatrix}, \quad \Delta^{+}_k (\lambda; x) = \begin{vmatrix}
\lambda e^{\theta_{1}^a} & \cdots & \lambda e^{\theta_{2N}^a} \\
\vdots & \ddots & \vdots \\
\lambda e^{\theta_{N}^a} & \cdots & \lambda e^{\theta_{2N}^a}
\end{vmatrix}.
\]

The functions \( x \mapsto \Delta^{\pm}_k (\lambda; x), 0 \leq k \leq N - 1, \) are the solutions of the boundary value problems

\[
\begin{align*}
(\Delta^{\pm}_k)^{(2N)} (\lambda; x) & = \kappa_k \lambda \Delta^{\mp}_k (\lambda; x), \\
(\Delta^{\pm}_k)^{(l)} (\lambda; a) & = \delta_{k l} \lambda^{l/(2N)} \Delta (\lambda), \quad (\Delta^{\mp}_k)^{(l)} (\lambda; b) = 0 \quad \text{for } l \in \{0, \ldots, N - 1\}, \\
(\Delta^{\mp}_k)^{(2N)} (\lambda; x) & = \kappa_k \lambda \Delta^{\pm}_k (\lambda; x), \\
(\Delta^{\pm}_k)^{(l)} (\lambda; a) & = 0, \quad (\Delta^{\pm}_k)^{(l)} (\lambda; b) = \delta_{k l} \lambda^{l/(2N)} \Delta (\lambda) \quad \text{for } l \in \{0, \ldots, N - 1\}.
\end{align*}
\]

Proof. The solution of linear boundary value problem (2.2) has the form \( \Phi (x) = \sum_{\ell=1}^{2N} a_{\ell} e^{\theta^\alpha_{\ell x}} \) where the coefficients \( a_{\ell}, 1 \leq \ell \leq 2N, \) satisfy the linear system below:

\[
\begin{align*}
\sum_{\ell=1}^{2N} \lambda^{-\frac{1}{2N}} \Phi^{(k)} (a) e^{\theta^\alpha_{\ell} x} a_{\ell} = \lambda^{-\frac{1}{2N}} \Phi^{(k)} (a), & \quad 0 \leq k \leq N - 1, \\
\sum_{\ell=1}^{2N} \lambda^{-\frac{1}{2N}} \Phi^{(k)} (b) e^{\theta^\alpha_{\ell} x} a_{\ell} = \lambda^{-\frac{1}{2N}} \Phi^{(k)} (b), & \quad 0 \leq k \leq N - 1.
\end{align*}
\]

This system can be solved by using Cramer’s formulæ:

\[
a_{\ell} = \frac{\Delta_\ell (\lambda, \varphi)}{\Delta (\lambda)}, \quad 1 \leq \ell \leq 2N,
\]

where \( \Delta (\lambda) \) is the determinant displayed in Theorem 1 and \( \Delta_\ell (\lambda, \varphi) \) is the determinant deduced from \( \Delta (\lambda) \) by replacing its \( \ell \)th column by the right-hand side of (3.2), that is

\[
\Delta_\ell (\lambda, \varphi) = \begin{vmatrix}
e^{\theta_{1}^a} & \cdots & \varphi (a) & e^{\theta_{l+1}^a} & \cdots & e^{\theta_{2N}^a} \\
\theta_1 e^{\theta_{1}^a} & \cdots & \theta_{l-1} e^{\theta_{l-1}^a} & \lambda^{-\frac{1}{2N}} \varphi' (a) & \theta_{l+1} e^{\theta_{l+1}^a} & \cdots & \theta_{2N} e^{\theta_{2N}^a} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\theta_{N-1} e^{\theta_{N-1}^a} & \cdots & \theta_{2N} e^{\theta_{2N}^a} & \lambda^{-\frac{1}{2N}} \varphi^{(N-1)} (a) & \theta_{N-1} e^{\theta_{N-1}^a} & \cdots & \theta_{2N} e^{\theta_{2N}^a} \\
\varphi (b) & \cdots & \varphi (b) & e^{\theta_{l+1}^b} & \varphi (b) & \cdots & e^{\theta_{2N}^b} \\
\theta_1 e^{\theta_{1}^b} & \cdots & \theta_{l-1} e^{\theta_{l-1}^b} & \lambda^{-\frac{1}{2N}} \varphi' (b) & \theta_{l+1} e^{\theta_{l+1}^b} & \cdots & \theta_{2N} e^{\theta_{2N}^b} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\varphi (b) & \cdots & \varphi (b) & e^{\theta_{l+1}^b} & \varphi (b) & \cdots & e^{\theta_{2N}^b}
\end{vmatrix}.
\]

The determinant \( \Delta_\ell (\lambda, \varphi) \) can be expanded with respect to its \( \ell \)th column:

\[
\Delta_\ell (\lambda, \varphi) = \sum_{k=0}^{N-1} \lambda^{-\frac{1}{2N}} \Delta^{-}_k (\lambda) \varphi^{(k)} (a) + \sum_{k=0}^{N-1} \lambda^{-\frac{1}{2N}} \Delta^{+}_k (\lambda) \varphi^{(k)} (b)
\]

\[
\Delta (\lambda) = \begin{vmatrix}
e^{\theta_{1}^a} & \cdots & e^{\theta_{2N}^a} \\
\theta_1 e^{\theta_{1}^a} & \cdots & \theta_{2N} e^{\theta_{2N}^a} \\
\vdots & \ddots & \vdots \\
\theta_{N-1} e^{\theta_{N-1}^a} & \cdots & \theta_{2N} e^{\theta_{2N}^a} \\
e^{\theta_{1}^b} & \cdots & e^{\theta_{2N}^b} \\
\theta_1 e^{\theta_{1}^b} & \cdots & \theta_{2N} e^{\theta_{2N}^b} \\
\vdots & \ddots & \vdots \\
\theta_{N-1} e^{\theta_{N-1}^b} & \cdots & \theta_{2N} e^{\theta_{2N}^b}
\end{vmatrix}.
\]
with

\[\Delta_{\mathcal{K}}(\lambda) = \begin{vmatrix}
\varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \varepsilon_{\lambda}^{\theta_{2N}a} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{2N}a} \\
\theta_{1}^{k} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{k} \varepsilon_{\lambda}^{\theta_{j-1}a} & 1 & \theta_{1}^{k} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{k} \varepsilon_{\lambda}^{\theta_{2N}a} \\
\theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{2N}a} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{2N}a} \\
\varepsilon_{\lambda}^{\theta_{1}b} & \cdots & \varepsilon_{\lambda}^{\theta_{j-1}b} & 0 & \varepsilon_{\lambda}^{\theta_{j+1}b} & \cdots & \varepsilon_{\lambda}^{\theta_{2N}b} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{1}b} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j-1}b} & 0 & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j+1}b} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{2N}b} \\
\end{vmatrix}
\]

\[= \begin{vmatrix}
\varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \varepsilon_{\lambda}^{\theta_{2N}a} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{k-1} \varepsilon_{\lambda}^{\theta_{2N}a} \\
0 & \cdots & 0 & 1 & 0 & \cdots & \vdots \\
\theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{k+1} \varepsilon_{\lambda}^{\theta_{2N}a} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{1}a} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j-1}a} & 0 & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j+1}a} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{2N}a} \\
\varepsilon_{\lambda}^{\theta_{1}b} & \cdots & \varepsilon_{\lambda}^{\theta_{j-1}b} & 0 & \varepsilon_{\lambda}^{\theta_{j+1}b} & \cdots & \varepsilon_{\lambda}^{\theta_{2N}b} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{1}b} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j-1}b} & 0 & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{j+1}b} & \cdots & \theta_{1}^{N-1} \varepsilon_{\lambda}^{\theta_{2N}b} \\
\end{vmatrix}
\]
and, in the same manner,

\[
\begin{align*}
\Delta_{k\ell}^{+}(\lambda) &= \\
&= \begin{vmatrix}
\theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} & \theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} & \theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} \\
\theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} & \theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} \\
\theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} & \theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} \\
\theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} & \theta_{\ell+1}^{N-1} e_{\lambda}^{1b} & \ldots & \theta_{\ell+1}^{N-1} e_{\lambda}^{b} \\
\end{vmatrix}
\end{align*}
\]

With these settings at hand, we can write the solution of (2.2):

\[
\Phi(x) = \sum_{\ell=1}^{2N} \alpha_{\ell} e_{\lambda}^{\ell x} = \sum_{\ell=1}^{2N} \frac{\Delta_{\ell}(\lambda, \varphi)}{\Delta(\lambda)} e_{\lambda}^{\ell x}
\]

\[
= \frac{1}{\Delta(\lambda)} \left[ \sum_{k=0}^{N-1} \lambda^{-k} \left( \sum_{\ell=1}^{2N} \Delta_{\ell}^{-}(\lambda) e_{\lambda}^{\ell x} \right) \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-k} \left( \sum_{\ell=1}^{2N} \Delta_{\ell}^{+}(\lambda) e_{\lambda}^{\ell x} \right) \varphi^{(k)}(b) \right]
\]

\[
= \sum_{k=0}^{N-1} \lambda^{-k} \frac{\Delta_{\lambda}^{-}(\lambda; x)}{\Delta(\lambda)} \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-k} \frac{\Delta_{\lambda}^{+}(\lambda; x)}{\Delta(\lambda)} \varphi^{(k)}(b)
\]

with

\[
\Delta_{\lambda}^{\pm}(\lambda; x) = \sum_{\ell=1}^{2N} \Delta_{\ell}^{\pm}(\lambda) e_{\lambda}^{\ell x}, \quad \Delta_{\lambda}^{\pm}(\lambda; x) = \sum_{\ell=1}^{2N} \Delta_{\ell}^{\pm}(\lambda) e_{\lambda}^{\ell x}.
\]  \hfill (3.3)

We immediately see that equalities (3.3) are the expansions of the determinants displayed in Theorem 1 with respect to their \((k-1)\)th raw and \((k+N-1)\)th raw respectively. Formula (3.1) is proved.

Finally, it is easy to check the boundary value problems satisfied by the functions \(x \mapsto \Delta_{\lambda}^{\pm}(\lambda; x)\) by using elementary rules on differentiating a determinant. In particular for, e.g., \(\Delta_{\lambda}^{\pm},\) the determinants defining \((\Delta_{\lambda}^{\pm})^{(\ell)}(\lambda; a), \ell \in \{0, \ldots, N-1\} \setminus \{k\},\) and \((\Delta_{\lambda}^{\pm})^{(\ell)}(\lambda; b), \ell \in \{0, \ldots, N-1\},\) have two identical rows, thus they vanish. The determinant \((\Delta_{\lambda}^{\pm})^{(k)}(\lambda; a)\) has the same rows as \(\Delta(\lambda)\) up to the multiplicative factor \(\lambda^{k/(2N)}\) for its \(k\)th row, then it coincides with \(\lambda^{k/(2N)} \Delta(\lambda).\) The proof of Theorem 1 is finished.  \(\square\)

Now, by eliminating the function \(\varphi\) in (3.1), we get the following result which should be understood in the sense of Schwartz distributions:

\[
\mathbb{E}_{X} \left( e^{-\lambda \tau_{ab}} \mathbb{1}_{(\tau_{ab} < +\infty)} ; X_{\tau_{ab}} \in dz \right) / dz
\]

\[
= \sum_{k=0}^{N-1} (-1)^k \lambda^{-k} \frac{\Delta_{\lambda}^{-}(\lambda; x)}{\Delta(\lambda)} \delta^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k \lambda^{-k} \frac{\Delta_{\lambda}^{+}(\lambda; x)}{\Delta(\lambda)} \delta^{(k)}(z)
\]  \hfill (3.4)

from which we derive the following representation for the pseudo-distribution of \((\tau_{ab}, X_{\tau_{ab}}).\)

**Theorem 2.** The joint pseudo-distribution of \((\tau_{ab}, X_{\tau_{ab}})\) admits the following representation:

\[
\mathbb{P}_{X} \{ \tau_{ab} \in dt, X_{\tau_{ab}} \in dz \} / dt \, dz = \sum_{k=0}^{N-1} (-1)^k I_{k}^{-}(t; x) \, \delta^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k I_{k}^{+}(t; x) \, \delta^{(k)}(z)
\]  \hfill (3.5)

where the functions \(I_{k}^{\pm}(t; x), 0 \leq k \leq N-1,\) are characterized by their Laplace transforms:

\[
\int_{0}^{\infty} I_{k}^{\pm}(t; x) e^{-\lambda t} \, dt = \lambda^{-k} \frac{\Delta_{\lambda}^{\pm}(\lambda; x)}{\Delta(\lambda)}.
\]
They are also characterized by the boundary value problems

\[
\begin{align*}
\frac{\partial I_k^-}{\partial t}(t; x) &= \kappa_N \frac{\partial^2 N I_k^-}{\partial^2 X}(t; x) \\
\frac{\partial^j}{\partial x^j} (t; a) &= \frac{\partial^j}{\partial x^j} (t; b) = 0 \quad \text{for } \ell \in \{0, 1, \ldots, N - 1\}, \\
\frac{\partial I_k^+}{\partial t}(t; x) &= \kappa_N \frac{\partial^2 N I_k^+}{\partial^2 X}(t; x) \\
\frac{\partial^j}{\partial x^j} (t; a) &= 0, \quad \frac{\partial^j}{\partial x^j} (t; b) = \delta_{k\ell} \quad \text{for } \ell \in \{0, 1, \ldots, N - 1\}.
\end{align*}
\]

The boundary value problems satisfied by the functions \(I_k^\pm, 0 \leq k \leq N - 1\), come from those satisfied by the functions \(\Delta_k^\pm\) displayed in Theorem 1. The only details we have to check are that \(I_k^\pm(t; x)\) goes to 0 as \(t\) tends to 0\(^+\) and that \(I_k^\pm(t; x)\) is bounded as \(t\) tends to \(+\infty\) (in order to have \(\int_0^\infty (\partial/\partial t) I_k^\pm(t; x) e^{-\lambda t} dt = \lambda \int_0^\infty I_k^\pm(t; x) e^{-\lambda t} dt\) which can be deduced from the fact that their Laplace transforms go to 0 exponentially quickly as \(\lambda\) goes to \(+\infty\) and are bounded as \(\lambda\) goes to 0\(^+\). These facts are proved in Appendix A; see (A.2) and (A.5).

**Remark 1.** The functions \(I_k^\pm, 0 \leq k \leq N - 1\), are real-valued. Indeed, observing that the complex numbers \(\theta_i, 1 \leq \ell \leq 2N\), are conjugate two by two, it is easily seen that the determinants contain conjugate columns two by two, so they are real numbers. More precisely, conjugating \(\theta_1, \ldots, \theta_N, \theta_{N+1}, \ldots, \theta_{2N}\) respectively yields \(\theta_N, \ldots, \theta_1, \theta_{2N}, \ldots, \theta_{N+1}\). Therefore, conjugating the determinants \(\Delta\) and \(\Delta_k^\pm\) boils down to interchanging their 1st and \(N\)th columns, their 2nd and \((N - 1)\)th columns, ..., their \((N + 1)\)th and \((2N)\)th columns, their \((N + 2)\)th and \((2N - 1)\)th columns, and so on. In this way, we perform an even number of transpositions and we retrieve the original determinants: \(\overline{\Delta} = \Delta\) and \(\overline{\Delta_k^\pm} = \Delta_k^\pm\), proving that they are real numbers.

Moreover, the functions \(I_k^+\) and \(I_k^-\) are related according to the identity \(I_k^+ (t; x + b - x)\) as it can be seen by proving the same identity concerning their Laplace transforms; see (A.1) in Appendix A.

**Remark 2.** Let us compute the limit of (3.4) as \(b\) tends towards \(+\infty\). To this aim, we find that

\[
\frac{\Delta_k^-(\lambda; x)}{\Delta(\lambda)} \xrightarrow{b \to +\infty} \sum_{\ell=1}^N a_{k\ell} e^{\theta_\ell (x - a)} \quad \text{and} \quad \frac{\Delta_k^+(\lambda; x)}{\Delta(\lambda)} \xrightarrow{b \to +\infty} 0
\]

with

\[
a_{k\ell} = \frac{1}{\det(V)} \begin{bmatrix}
1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\
\theta_1 & \ldots & \theta_{\ell-1} & 0 & \theta_{\ell+1} & \ldots & \theta_N \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\theta_{k-1}^{\ell-1} & \ldots & \theta_{\ell-1}^{k-1} & 0 & \theta_{\ell+1}^{k-1} & \ldots & \theta_N^{k-1} \\
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
\theta_{k+1}^{\ell-1} & \ldots & \theta_{\ell-1}^{k+1} & 0 & \theta_{\ell+1}^{k+1} & \ldots & \theta_N^{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\theta_{N-1}^{\ell-1} & \ldots & \theta_{\ell-1}^{N-1} & 0 & \theta_{\ell+1}^{N-1} & \ldots & \theta_N^{N-1}
\end{bmatrix}
\]

The coefficients \(a_{k\ell}\) are characterized by the identity

\[
\sum_{k=0}^{N-1} a_{k\ell} x^k = \frac{1}{\det(V)} \begin{bmatrix}
1 & \ldots & 1 & 1 & \ldots & 1 \\
\theta_1 & \ldots & \theta_{\ell-1} & x & \theta_{\ell+1} & \ldots & \theta_N \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\theta_1^{N-1} & \ldots & \theta_{\ell-1}^{N-1} & x^{N-1} & \theta_{\ell+1}^{N-1} & \ldots & \theta_N^{N-1}
\end{bmatrix} = \prod_{1 \leq i < j \leq N} \left( \frac{x - \theta_j}{\theta_i - \theta_k} \right)
\]

as it is easily seen by appealing to the well-known Vandermonde determinant \(\det(V) = \prod_{1 \leq i < j \leq N} (\theta_j - \theta_i)\). Notice that polynomial (3.7) is nothing but an elementary Lagrange interpolating polynomial related to the numbers \(\theta_i, 1 \leq i \leq N\). The details of these limiting results being cumbersome, we postpone them to Appendix A.
In regards to (3.4), (3.5) and (3.6), we conclude that, for \( x > a \),
\[
\lim_{b \to +\infty} \mathbb{P}_x \{ \tau_{ab} \in dt, X_{\tau_{ab}} \in dz \} / dt dz = \sum_{k=0}^{N-1} (-1)^k K_k(t; x) \delta_a^{(k)}(z)
\]
where \( K_k \) is the function whose Laplace transform is given by
\[
\int_0^\infty K_k(t; x) e^{-\lambda t} dt = \lambda^{-\frac{k}{N}} \sum_{k=1}^{N} \alpha_{nk} e^{\theta_k 2\sqrt{x}(x-a)}.
\]
We retrieve at the limit the pseudo-distribution of \((\tau_{ab}, X_{\tau_{ab}})\) related to the first overshooting time of level \( a \) displayed in [14], formula (5.15).

4 Pseudo-distribution of \( \tau_{ab} \)

By applying the Schwartz distribution (3.5) to the test function \( 1 \), we immediately extract the pseudo-distribution of \( \tau_{ab} \): \( \mathbb{P}_x \{ \tau_{ab} \in dt \} / dt = I_0^0 (t; x) + I_0^1 (t; x) \) that we state as follows.

**Theorem 3.** The pseudo-distribution of \( \tau_{ab} \) is given either by one of both formulae below:
\[
\mathbb{P}_x \{ \tau_{ab} \in dt \} / dt = I(t; x), \quad \mathbb{P}_x \{ \tau_{ab} \leq t \} = J(t; x)
\]
with
\[
\int_0^\infty I(t; x) e^{-\lambda t} dt = \frac{\Delta_0^+ (\lambda; x) + \Delta_0^- (\lambda; x)}{\Delta(\lambda)}, \quad \int_0^\infty J(t; x) e^{-\lambda t} dt = \frac{1}{\lambda} \frac{\Delta_0^+ (\lambda; x) + \Delta_0^- (\lambda; x)}{\Delta(\lambda)}.
\]
Let us introduce the up-to-date minimum and maximum functionals of \( X \):
\[
m_t = \min_{s \in [0, t]} X_s, \quad M_t = \max_{s \in [0, t]} X_s.
\]
It is plain that the functionals \( m_t, M_t \) and time \( \tau_{ab} \) are related according as \( a < m_t \leq M_t < b \iff \tau_{ab} > t \). Then \( \mathbb{P}_x \{ a < m_t \leq M_t < b \} = 1 - \mathbb{P}_x \{ \tau_{ab} \leq t \} \).

**Corollary 1.** The joint pseudo-distribution of \((m_t, M_t)\) is given by
\[
\mathbb{P}_x \{ a < m_t \leq M_t < b \} = 1 - J(t; x)
\]
and its Laplace transform with respect to \( t \) writes
\[
\int_0^\infty \mathbb{P}_x \{ a < m_t \leq M_t < b \} e^{-\lambda t} dt = \frac{1}{\lambda} \Delta(\lambda) - \Delta_0^+ (\lambda; x) - \Delta_0^- (\lambda; x).
\]

5 Pseudo-distribution of \( X_{\tau_{ab}} \)

In this part, we focus on the exit location of \( X \) at time \( \tau_{ab} \) whose pseudo-distribution admits a remarkable expression by means of Hermite interpolating polynomials whose expressions are displayed in the introduction.

**Theorem 4.** The pseudo-distribution of the exit location \( X_{\tau_{ab}} \mathbb{1}_{\{\tau_{ab} < +\infty\}} \) is given, in the sense of Schwartz distributions, by
\[
\mathbb{P}_x \{ X_{\tau_{ab}} \in dz, \tau_{ab} < +\infty \} / dz = \sum_{k=0}^{N-1} (-1)^k H_k(x) \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k H_k(x) \delta_b^{(k)}(z).
\]
\[
(5.1)
\]
**Proof.** We directly solve boundary value problem (2.2) in the case where \( \lambda = 0 \) therein. Namely, by setting \( \Psi(x) = \mathbb{E}_x \left( \varphi(X_{\tau_{ab}}) \mathbb{1}_{\{\tau_{ab} < +\infty\}} \right) \),
\[
\begin{align*}
\Psi(0)(x) &= 0, \quad x \in (a, b), \\
\Psi^{(2N)}(a) &= 0, \quad \Psi^{(2N)}(b) = 0
\end{align*}
\]
for \( k \in \{0, 1, \ldots, N-1\} \).
It is clear that \( \Psi \) is the polynomial of degree not greater than \((2N - 1) \) whose derivatives at \( a \) and \( b \) up to order \((N - 1) \) are the given numbers \( q^{(k)}(a) \) and \( q^{(k)}(b) \), \( 0 \leq k \leq N - 1 \). It can be written as a linear combination of the Hermite interpolating fundamental polynomials \( H_k^\pm \), \( 0 \leq k \leq N - 1 \), displayed in Theorem 4 as follows: for any test functions \( \varphi \),

\[
\mathbb{E}_x\left( \varphi(X_{\tau_{ab}}) \mathbb{I}_{\{\tau_{ab} < +\infty\}} \right) = \sum_{k=0}^{N-1} H_k^-(x) \varphi^{(k)}(a) + \sum_{k=0}^{N-1} H_k^+(x) \varphi^{(k)}(b). \tag{5.2}
\]

Formula (5.1) is nothing but (5.2) rephrased by means of Schwartz distributions. \( \square \)

**Remark 3.** Formula (5.2) yields for \( \varphi = H_k^+ \), \( 0 \leq k \leq N - 1 \), that

\[
\mathbb{E}_x\left( H_k^+ (X_{\tau_{ab}}) \mathbb{I}_{\{\tau_{ab} < +\infty\}} \right) = H_k^+(x).
\]

**Remark 4.** By letting \( b \) tend to \(+\infty\), we see that \( H_k^+(x) \) tends to 0 while \( H_k^-(x) \) tends to \((x - a)^k / (k!) \).

Hence, we find that

\[
\lim_{b \to +\infty} \mathbb{P}_x\{ X_{\tau_{ab}} \in d\tau, \tau_{ab} < +\infty \} / d\tau = \sum_{k=0}^{N-1} \frac{(a - x)^k}{k!} \delta_a^{(k)}(\tau).
\]

We retrieve at the limit the pseudo-distribution (1.3) of the location \( X_{\tau_a} \) of \( X \) at the first overshooting time of level \( a \), which is displayed in [14], formula (5.18).

**Corollary 2.** Time \( \tau_{ab} \) is \( \mathbb{P}_x \)-almost surely finite in the sense that

\[
\mathbb{P}_x\{ \tau_{ab} < +\infty \} = 1.
\]

Because of this, in the sequel of the paper, we shall omit the condition \( \tau_{ab} < +\infty \) when considering the pseudo-random variable \( X_{\tau_{ab}} \). Actually, let us recall that, in the framework of signed measures, if \( A \) is a set of \( \mathbb{P}_x \)-measure 1, it does not entail that for any set \( B \) that \( \mathbb{P}_x(A \cap B) = \mathbb{P}_x(B) \) contrary to the case of ordinary probability.

**Proof.** The pseudo-probability \( \mathbb{P}\{ \tau_{ab} < +\infty \} \) can be deduced from (5.1) by choosing \( \varphi = 1 \). Indeed, we have that

\[
H_0^-(x) = \left( \frac{b - x}{b - a} \right)^N \sum_{\ell=0}^{N-1} \binom{N - 1}{\ell} \left( \frac{x - a}{b - a} \right)^\ell = \left( \frac{b - x}{b - a} \right)^N \sum_{\ell=0}^{N-1} \binom{N - 1}{\ell} (x - a)\ell (b - a)^{N-1-\ell}.
\]

By writing the term \( (b - a)^{N-1-\ell} \) as

\[
(b - a)^{N-1-\ell} = [(x - a) + (b - x)]^{N-1-\ell} = \sum_{k=0}^{N-1-\ell} \binom{N - 1 - \ell}{k} (x - a)k (b - x)^{N-1-k-\ell},
\]

it follows that

\[
H_0^-(x) = \frac{1}{(b - a)^{2N-1}} \sum_{\ell=0}^{N-1} \sum_{0 \leq k \leq N-1-\ell} \binom{N - 1 - \ell}{k} \binom{\ell + N - 1}{\ell} (x - a)k (b - x)^{2N-1-k-\ell}
\]

\[
= \frac{1}{(b - a)^{2N-1}} \sum_{m=0}^{N-1} \sum_{\ell=0}^{m} \binom{\ell + N - 1}{\ell} \binom{m - \ell}{m - \ell} (x - a)^m (b - x)^{2N-1-m}.
\]

By using the elementary identity \( \sum_{\ell=0}^{n} \binom{\ell + p}{\ell} \binom{n + q - \ell}{n - \ell} = \binom{n + p + q + 1}{n} \) which comes from the equality \( (1 + u)^{-p}(1 + u)^{-q} = (1 + u)^{-p+q} \), together with the expansion, e.g., for \( p \), \( (1 + u)^{-p} = \sum_{\ell=0}^{\infty}(-1)^\ell\binom{\ell + p - 1}{\ell}u^\ell \), we get that

\[
\sum_{\ell=0}^{m} \binom{\ell + N - 1}{\ell} \binom{m - \ell}{m - \ell} = \binom{2N - 1}{m}.
\]
As a byproduct,
\[ H_0^{-}(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m}. \]

Similarly,
\[ H_0^{+}(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{2N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m} \]
and we immediately deduce that
\[ \mathbb{P}\{\tau_{ab} < +\infty\} = H_0^{-}(x) + H_0^{+}(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{2N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m} = 1. \]

Let us introduce the first down- and up-overshooting times of the single thresholds \(a\) and \(b\) for \((X_t)_{t \geq 0}\):
\[ \tau_a^- = \inf\{t \geq 0 : X_t < a\}, \quad \tau_b^+ = \inf\{t \geq 0 : X_t > b\}. \]

The famous problem of the ruin of the gambler in the context of pseudo-Brownian motion consists in computing the pseudo-probability of overshooting one level (\(a\) or \(b\)) before the other one. For instance, we have that
\[ \mathbb{P}_x\{\tau_a^- < \tau_b^+\} = \mathbb{P}_x\{X_{\tau_{ab}} \leq a\}. \]

Hence, in view of formula (5.1), we obtain the following result.

**Corollary 3.** The “ruin” pseudo-probabilities related to pseudo-Brownian motion are given by
\[ \mathbb{P}_x\{\tau_a^- < \tau_b^+\} = H_0^{-}(x), \quad \mathbb{P}_x\{\tau_b^+ < \tau_a^-\} = H_0^{+}(x). \]

In the corollary below, we provide a way for computing the pseudo-moments of \(X_{\tau_{ab}}\).

**Corollary 4.** Let \(P\) be a polynomial and \(R\) the remainder of the Euclidean division of \(P(x)\) by \((x-a)^N(x-b)^N\). We have that
\[ \mathbb{E}_x[P(X_{\tau_{ab}})] = R(x). \]

In particular, the pseudo-moments of \(X_{\tau_{ab}}\) are given, for any \(p \in \{0, 1, \ldots, 2N-1\}\), by
\[ \mathbb{E}_x[(X_{\tau_{ab}})^p] = x^p \]
and for any positive integer \(p\), by setting \(c_n = \sum_{k=0}^{n} \binom{N+k-1}{k} \binom{N+n-1-k}{n-k} a^k b^{n-k}\), by
\[ \mathbb{E}_x[(X_{\tau_{ab}})^{2N+p}] = x^{2N+p} - \left( \sum_{n=0}^{p} c_{p-n} x^n \right) (x-a)^N (x-b)^N \]
\[ = x^{2N+p} - \left[ x^p + N(a+b) x^{p-1} \right. \]
\[ \left. + \left( \frac{1}{2} N(N+1)(a^2 + b^2) + N^2 ab \right) x^{p-2} + \cdots \right] (x-a)^N (x-b)^N. \]

For instance,
\[ \mathbb{E}_x[(X_{\tau_{ab}})^{2N}] = x^{2N} - (x-a)^N (x-b)^N, \]
\[ \mathbb{E}_x[(X_{\tau_{ab}})^{2N+1}] = x^{2N+1} - [x + N(a+b)] (x-a)^N (x-b)^N, \]
\[ \mathbb{E}_x[(X_{\tau_{ab}})^{2N+2}] = x^{2N+2} - \left[ x^2 + N(a+b) x + \left( \frac{1}{2} N(N+1)(a^2 + b^2) + N^2 ab \right) \right] (x-a)^N (x-b)^N. \]

**Proof.** Let us introduce the quotient \(Q\) of the Euclidean division of \(P(x)\) by \((x-a)^N(x-b)^N\); we have \(P(x) = Q(x)(x-a)^N(x-b)^N + R(x)\). The polynomial \(R\) is of degree not greater than \((2N-1)\). Since \(a\) and \(b\) are roots of the polynomial \(P(x) - R(x) = Q(x)(x-a)^N(x-b)^N\) with a
Next, we compute the quotient \( Q(x) \). Then, the quotient of \( \theta \) is \( \frac{\sqrt{2}}{2} \), the pseudo-Brownian motion is the so-called biharmonic-pseudo-process. In this case, the multiplicity not less than \( N \), the successive derivatives of \( P - R \) up to order \((N - 1)\) vanish at \( a \) and \( b \). Therefore, by (5.2), we deduce that

\[
\mathbb{E}_x \left[ Q(X_{a,b}) (X_{a,b} - a)^N (X_{a,b} - b)^N \right] = 0
\]

and then

\[
\mathbb{E}_x [P(X_{a,b})] = \mathbb{E}_x [R(X_{a,b})].
\]

Since the polynomial \( R \) is of degree not greater than \((2N - 1)\), we can write the decomposition

\[
R(x) = \sum_{k=0}^{N-1} R^{(k)}(a) H^k_{-}(x) + \sum_{k=0}^{N-1} R^{(k)}(b) H^k_{+}(x).
\]

Therefore, appealing to Remark 3, we obtain that

\[
\mathbb{E}_x [P(X_{a,b})] = \mathbb{E}_x [R(X_{a,b})] = R(x) = P(x) - Q(x)(x-a)^N(x-b)^N.
\]

Next, we compute the quotient \( Q \) when \( P(x) = x^{2N+p} \):

\[
\frac{x^{2N+p}}{(x-a)^N(x-b)^N} = x^p \left( 1 - \frac{a}{x} \right)^{-N} \left( 1 - \frac{b}{x} \right)^{-N} = x^p \left( \sum_{k=0}^{\infty} \binom{N+k-1}{k} a^k \right)^{N} \left( \sum_{\ell=0}^{\infty} \binom{N+\ell-1}{\ell} b^\ell \right)^{N} = \sum_{n=0}^{\infty} c_n x^p - n = \sum_{n=0}^{p} c_p x^n + \sum_{n=0}^{\infty} c_{n+p} x^n
\]

where

\[
c_n = \sum_{k \ell = n}^{N+k-1} \binom{N+k-1}{k} \binom{N+\ell-1}{\ell} a^k b^\ell = \sum_{k=0}^{n} \binom{N+k-1}{k} \binom{N+n-1-k}{n-k} a^k b^{n-k}.
\]

Then, the quotient of \( x^{2N+p} \) by \( (x-a)^N(x-b)^N \) is equal to \( \sum_{n=0}^{\infty} c_n x^p - n \). In particular,

\[
c_0 = 1, \quad c_1 = N(a+b), \quad c_2 = \frac{1}{2} N(N+1)(a^2 + b^2) + N^2 ab.
\]

\[\Box\]

**Remark 5.** By (5.2), we easily get that

\[
\mathbb{E}_x \left[ (X_{a,b} - b)^p 1_{\{\tau^*_a < \tau^*_b\}} \right] = \begin{cases} p! H^p_{+}(x) & \text{if } p \leq N, \\ 0 & \text{if } p \geq N + 1. \end{cases}
\]

This formula suggests the following interpretation of Hermite polynomials in terms of pseudo-Brownian motion: for \( p \in \{0, \ldots, N - 1\} \),

\[
H^p_{+}(x) = \frac{1}{p!} \mathbb{E}_x \left[ (X_{a,b} - b)^p 1_{\{\tau^*_a < \tau^*_b\}} \right].
\]

### 6. The case \( N = 2 \)

For \( N = 2 \), pseudo-Brownian motion is the so-called biharmonic-pseudo-process. In this case, the settings write \( \theta_1 = e^{i3\pi/4}, \theta_2 = e^{i5\pi/4}, \theta_3 = e^{i7\pi/4}, \theta_4 = e^{i\pi/4} \). and, by setting \( \nu = \lambda/4 \),

\[
\Delta(\lambda) = \begin{bmatrix} e^{i\lambda} & e^{i\lambda} & e^{i\lambda} & e^{i\lambda} \\
\theta_1 e^{i\lambda} & \theta_2 e^{i\lambda} & \theta_3 e^{i\lambda} & \theta_4 e^{i\lambda} \\
e^{i\lambda} & e^{i\lambda} & e^{i\lambda} & e^{i\lambda} \\
\theta_1 e^{i\lambda} & \theta_2 e^{i\lambda} & \theta_3 e^{i\lambda} & \theta_4 e^{i\lambda} \\
e^{i\lambda} & e^{i\lambda} & e^{i\lambda} & e^{i\lambda} \\
\theta_1 e^{i\lambda} & \theta_2 e^{i\lambda} & \theta_3 e^{i\lambda} & \theta_4 e^{i\lambda} \\
e^{i\lambda} & e^{i\lambda} & e^{i\lambda} & e^{i\lambda} \\
\theta_1 e^{i\lambda} & \theta_2 e^{i\lambda} & \theta_3 e^{i\lambda} & \theta_4 e^{i\lambda} \\
e^{i\lambda} & e^{i\lambda} & e^{i\lambda} & e^{i\lambda} \\
\theta_1 e^{i\lambda} & \theta_2 e^{i\lambda} & \theta_3 e^{i\lambda} & \theta_4 e^{i\lambda} \\
e^{i\lambda} & e^{i\lambda} & e^{i\lambda} & e^{i\lambda} \\
\theta_1 e^{i\lambda} & \theta_2 e^{i\lambda} & \theta_3 e^{i\lambda} & \theta_4 e^{i\lambda} \end{bmatrix}
\]
Quite similar computations yield that

$$\Delta_0^{-}(\lambda; x) = \begin{vmatrix} e^\lambda_{1x} & e^\lambda_{2x} & e^\lambda_{3x} & e^\lambda_{4x} \\ e^\lambda_{1b} & e^\lambda_{2b} & e^\lambda_{3b} & e^\lambda_{4b} \\ e^\lambda_{1a} & e^\lambda_{2a} & e^\lambda_{3a} & e^\lambda_{4a} \\ e^\lambda_{1c} & e^\lambda_{2c} & e^\lambda_{3c} & e^\lambda_{4c} \end{vmatrix}, \quad \Delta_0^{+}(\lambda; x) = \begin{vmatrix} e^\lambda_{1a} & e^\lambda_{2a} & e^\lambda_{3a} & e^\lambda_{4a} \\ e^\lambda_{1b} & e^\lambda_{2b} & e^\lambda_{3b} & e^\lambda_{4b} \\ e^\lambda_{1c} & e^\lambda_{2c} & e^\lambda_{3c} & e^\lambda_{4c} \\ e^\lambda_{1d} & e^\lambda_{2d} & e^\lambda_{3d} & e^\lambda_{4d} \end{vmatrix},$$

Elementary computations yield that

$$\Delta(\lambda) = 4 \left[ \cosh(2\sqrt{\lambda}(b-a)) + \cos(2\sqrt{\lambda}(b-a)) - 2 \right].$$

Let us expand, e.g., $\Delta_0^{-}(\lambda; x)$ with respect to its first row:

$$\Delta_0^{-}(\lambda; x) = c_1 e^\lambda_{1x} + c_2 e^\lambda_{2x} + c_3 e^\lambda_{3x} + c_4 e^\lambda_{4x}$$

where $c_1, c_2, c_3, c_4$ are the cofactors of $\Delta_0^{-}(\lambda; x)$ related to the first row. Straightforward (but cumbersome) computations yield that $c_1 = \frac{1}{2}$ and $c_4 = \frac{1}{2}$ and

$$c_1 = (1 - i) e^{\frac{\lambda}{2\sqrt{\lambda}}((2b-a)-ia)} + (1 + i) e^{\frac{\lambda}{2\sqrt{\lambda}}((a-(2b-a))} - 2 e^{(1-i)\frac{\lambda}{2\sqrt{\lambda}}a},$$

$$c_3 = (1 - i) e^{-\frac{\lambda}{2\sqrt{\lambda}}((2b-a)-ia)} + (1 + i) e^{-\frac{\lambda}{2\sqrt{\lambda}}((a-(2b-a))} - 2 e^{-(1-i)\frac{\lambda}{2\sqrt{\lambda}}a}.$$ 

Therefore, we have that

$$\Delta_0^{-}(\lambda; x) = 2 Re \left( c_1 e^\lambda_{1x} + c_3 e^\lambda_{3x} \right)$$

$$= 2 \left[ e^{\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \cos \left( \sqrt{\lambda} (x + a - 2b) \right) + e^{-\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \cos \left( \sqrt{\lambda} (x + a - 2b) \right) 
+ e^{\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \sin \left( \sqrt{\lambda} (x + a - 2b) \right) - e^{-\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \sin \left( \sqrt{\lambda} (x + a - 2b) \right) 
- 2 e^{\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \cos \left( \sqrt{\lambda} (x - a) \right) - 2 e^{-\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \cos \left( \sqrt{\lambda} (x - a) \right) 
+ e^{\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \cos \left( \sqrt{\lambda} (x - a) \right) + e^{-\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \cos \left( \sqrt{\lambda} (x - a) \right) 
- e^{\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \sin \left( \sqrt{\lambda} (x - a) \right) + e^{-\frac{\lambda}{2\sqrt{\lambda}}((x-a))} \sin \left( \sqrt{\lambda} (x - a) \right) \right]$$

which simplifies by means of hyperbolic functions into

$$\Delta_0^{-}(\lambda; x) = 4 \left[ \cosh \left( \sqrt{\lambda} (x - a) \right) \cos \left( \sqrt{\lambda} (x + a - 2b) \right) + \sinh \left( \sqrt{\lambda} (x - a) \right) \sin \left( \sqrt{\lambda} (x + a - 2b) \right) \right. 
+ \cosh \left( \sqrt{\lambda} (x + a - 2b) \right) \cos \left( \sqrt{\lambda} (x - a) \right) - \sinh \left( \sqrt{\lambda} (x + a - 2b) \right) \sin \left( \sqrt{\lambda} (x - a) \right) 
- 2 \cosh \left( \sqrt{\lambda} (x - a) \right) \cos \left( \sqrt{\lambda} (x - a) \right) \right].$$

Quite similar computations yield that

$$\Delta_0^{+}(\lambda; x) = 4 \left[ \cosh \left( \sqrt{\lambda} (x + a - 2b) \right) \sin \left( \sqrt{\lambda} (x - a) \right) + \sinh \left( \sqrt{\lambda} (x + a - 2b) \right) \cos \left( \sqrt{\lambda} (x - a) \right) \right. 
- \cosh \left( \sqrt{\lambda} (x - a) \right) \sin \left( \sqrt{\lambda} (x - a) \right) - \sinh \left( \sqrt{\lambda} (x - a) \right) \cos \left( \sqrt{\lambda} (x - a) \right) \right].$$

The determinants $\Delta_0^{+}$ and $\Delta_0^{-}$ can be immediately deduced from $\Delta_0^{-}$ and $\Delta_0^{+}$ by interchanging the roles of $a$ and $b$ as it can be seen upon interchanging certain rows therein. We obtain that

$$\Delta_0^{+}(\lambda; x) = 4 \left[ \cosh \left( \sqrt{\lambda} (x - b) \right) \cos \left( \sqrt{\lambda} (x + b - 2a) \right) + \sinh \left( \sqrt{\lambda} (x - b) \right) \sin \left( \sqrt{\lambda} (x + b - 2a) \right) \right. 
+ \cosh \left( \sqrt{\lambda} (x + b - 2a) \right) \cos \left( \sqrt{\lambda} (x - b) \right) - \sinh \left( \sqrt{\lambda} (x + b - 2a) \right) \sin \left( \sqrt{\lambda} (x - b) \right) 
- 2 \cosh \left( \sqrt{\lambda} (x - b) \right) \cos \left( \sqrt{\lambda} (x - b) \right) \right],$$

$$\Delta_0^{+}(\lambda; x) = 4 \left[ \cosh \left( \sqrt{\lambda} (x + b - 2a) \right) \sin \left( \sqrt{\lambda} (x - b) \right) + \sinh \left( \sqrt{\lambda} (x + b - 2a) \right) \cos \left( \sqrt{\lambda} (x - b) \right) 
- \cosh \left( \sqrt{\lambda} (x - b) \right) \sin \left( \sqrt{\lambda} (x - b) \right) - \sinh \left( \sqrt{\lambda} (x - b) \right) \cos \left( \sqrt{\lambda} (x - b) \right) \right].$$
Now, formula (3.5) reads
\[ P_x \left\{ \tau_{ab} \in dt, X_{\tau_{ab}} \in dz \right\} = I_0^+ \left( t; \frac{a+b}{2} \right) \delta_a(z) + I_1^+ \left( t; \frac{a+b}{2} \right) \delta_b(z) \]
where the functions \( I_0^+ \) and \( I_1^+ \) are characterized by
\[ \int_0^\infty I_0^+(t;x) e^{-\lambda t} \, dt = \frac{\Delta_0^+(\lambda;x)}{\Delta(\lambda)}, \quad \int_0^\infty I_1^+(t;x) e^{-\lambda t} \, dt = \frac{1}{\sqrt{\lambda}} \frac{\Delta_1^+(\lambda;x)}{\Delta(\lambda)}. \]

Concerning the pseudo-distribution of the exit location \( X_{\tau_{ab}} \), it is given by
\[ P_x \left\{ X_{\tau_{ab}} \in dz \right\} / dz = H_0^{-}(x) \delta_a(z) - H_1^{-}(x) \delta_b(z) + H_0^+(x) \delta_a(z) - H_1^+(x) \delta_b(z) \]
with
\[ H_0^{-}(x) = \frac{(x-b)^2(2x-3a+b)}{(b-a)^3}, \quad H_1^{-}(x) = \frac{(x-a)(x-b)^2}{(b-a)^2}, \]
\[ H_0^+(x) = -\frac{(x-a)^2(2x+a-3b)}{(b-a)^3}, \quad H_1^+(x) = \frac{(x-b)^2x-b}{(b-a)^2}. \]

When the pseudo-process starts at the middle of the interval \( [a, b] \), we obtain the following expressions for the determinants of interest: by setting \( L = (b-a)/2 \),
\[ \Delta(\lambda) = 32 \left[ \cosh(\sqrt{\nu} L) \sinh(\sqrt{\nu} L) - \cos^2(\sqrt{\nu} L) \sin^2(\sqrt{\nu} L) \right], \]
\[ \Delta_0^+ \left( \lambda; \frac{a+b}{2} \right) = \Delta_0^- \left( \lambda; \frac{a+b}{2} \right) = 4 \left[ \cosh(\sqrt{\nu} L) \cos(\sqrt{\nu} L) \left( \cosh^2(\sqrt{\nu} L) + \cos^2(\sqrt{\nu} L) - 2 \right) \right. \]
\[ + \sinh(\sqrt{\nu} L) \sin(\sqrt{\nu} L) \left( \cosh^2(\sqrt{\nu} L) - \cos^2(\sqrt{\nu} L) \right) \], \]
\[ \Delta_1^+ \left( \lambda; \frac{a+b}{2} \right) = -\Delta_1^- \left( \lambda; \frac{a+b}{2} \right) = 4 \left[ \cosh(\sqrt{\nu} L) \sin(\sqrt{\nu} L) \sin^2(\sqrt{\nu} L) \right. \]
\[ - \sin(\sqrt{\nu} L) \cos(\sqrt{\nu} L) \sin^2(\sqrt{\nu} L) \].

Hence, in this case, we have the following symmetric expression:
\[ P_{a+b} \left\{ \tau_{ab} \in dt, X_{\tau_{ab}} \in dz \right\} / dt \, dz = I_0^+ \left( t; \frac{a+b}{2} \right) \left( \delta_a(z) + \delta_b(z) \right) + I_1^+ \left( t; \frac{a+b}{2} \right) \left( \delta_b(z) - \delta_a(z) \right). \]

Moreover,
\[ H_0^+ \left( \lambda; \frac{a+b}{2} \right) = H_0^+ \left( \lambda; \frac{a+b}{2} \right) = \frac{1}{2}, \quad H_1^- \left( \lambda; \frac{a+b}{2} \right) = -H_1^+ \left( \lambda; \frac{a+b}{2} \right) = \frac{L}{4}. \]

Then,
\[ P_{a+b} \left\{ X_{\tau_{ab}} \in dz \right\} / dz = \frac{1}{2} (\delta_a(z) + \delta_b(z)) + \frac{b-a}{8} (\delta_b(z) - \delta_a(z)). \]
A Appendix

A.1 Asymptotics of $\Delta(\lambda)$ and $\Delta_k^\pm(\lambda; x)$ as $b$ tends to $+\infty$

In this appendix, we check limits (3.6). By factorizing the $\ell$th column of the determinant $\Delta(\lambda)$ by $e^{\theta_{\ell}a}$ for each $\ell \in \{1, \ldots, 2N\}$ and observing that $\sum_{\ell=1}^{2N} \theta_{\ell} = 0$, we find that

$$\Delta(\lambda) = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_{2N-1} & \cdots & \theta_{2N} \end{vmatrix} \times \begin{vmatrix} e^{\theta_1(b-a)} & \cdots & e^{\theta_{2N}(b-a)} \\ \theta_1 e^{\theta_1(b-a)} & \cdots & \theta_{2N} e^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_{2N-1} e^{\theta_{2N-1}(b-a)} & \cdots & \theta_{2N} e^{\theta_{2N}(b-a)} \end{vmatrix}.$$ 

We separate $\Delta(\lambda)$ into four squared blocks as follows:

$$\Delta(\lambda) = \begin{vmatrix} V & \tilde{V} \\ W(\lambda) & \tilde{W}(\lambda) \end{vmatrix},$$

with

$$V = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_N \\ \vdots & & \vdots \\ \theta_{2N-1} & \cdots & \theta_{2N} \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} \theta_{N+1} & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_{2N-1} & \cdots & \theta_{2N} \end{pmatrix},$$

$$W(\lambda) = \begin{pmatrix} e^{\theta_1(b-a)} & \cdots & e^{\theta_{2N}(b-a)} \\ \theta_1 e^{\theta_1(b-a)} & \cdots & \theta_{2N} e^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_{2N-1} e^{\theta_{2N-1}(b-a)} & \cdots & \theta_{2N} e^{\theta_{2N}(b-a)} \end{pmatrix},$$

$$\tilde{W}(\lambda) = \begin{pmatrix} e^{\theta_{N+1}(b-a)} & \cdots & e^{\theta_{2N}(b-a)} \\ \theta_{N+1} e^{\theta_{N+1}(b-a)} & \cdots & \theta_{2N} e^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_{2N-1} e^{\theta_{2N-1}(b-a)} & \cdots & \theta_{2N} e^{\theta_{2N}(b-a)} \end{pmatrix}.$$ 

Due to the fact that $\Re(\theta_{\ell}) < 0$ for $\ell \in \{1, \ldots, N\}$ and $\Re(\theta_{\ell}) > 0$ for $\ell \in \{N+1, \ldots, 2N\}$, it may be easily seen by using an expansion by blocks of type $N \times N$ that the leading terms of $\Delta(\lambda)$ are obtained by performing the product of the determinants of both diagonal blocks $V$ and $\tilde{W}(\lambda)$, namely:

$$\Delta(\lambda) \sim_{b \to +\infty} \det(V) \times \det(\tilde{W}(\lambda)).$$

Similarly, we decompose $\Delta_k^-(\lambda; x)$ into

$$\Delta_k^-(\lambda; x) = \begin{vmatrix} V_k(\lambda; x) & \tilde{V}_k(\lambda; x) \\ W(\lambda) & \tilde{W}(\lambda) \end{vmatrix}.$$
with
\[
V_k(\lambda; x) = \begin{pmatrix}
1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_N \\
\vdots & \ddots & \vdots \\
\theta_1^{k-1} & \cdots & \theta_N^{k-1} \\
e^{\theta_1(x-a)}_\lambda & \cdots & e^{\theta_N(x-a)}_\lambda \\
\theta_1^{k+1} & \cdots & \theta_N^{k+1} \\
\vdots & \ddots & \vdots \\
\theta_1^{N-1} & \cdots & \theta_N^{N-1}
\end{pmatrix}, \quad \bar{V}_k(\lambda; x) = \begin{pmatrix}
1 & \cdots & 1 \\
\theta_{N+1} & \cdots & \theta_{2N} \\
\vdots & \ddots & \vdots \\
\theta_{N+1}^{k-1} & \cdots & \theta_{2N}^{k-1} \\
e^{\theta_{N+1}(x-a)}_\lambda & \cdots & e^{\theta_{2N}(x-a)}_\lambda \\
\theta_{N+1}^{k+1} & \cdots & \theta_{2N}^{k+1} \\
\vdots & \ddots & \vdots \\
\theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1}
\end{pmatrix}.
\]

We can easily see that, for \( x \in (a, b) \),
\[
\Delta_k^- (\lambda; x) \sim \frac{\det(V_k(\lambda; x)) \times \det(\bar{W}(\lambda))}{\det(\bar{V})}.
\]

As a byproduct, we get the first limit of (3.6):
\[
\lim_{b \to +\infty} \frac{\Delta_k^- (\lambda; x)}{\Delta(\lambda)} = \frac{\det(V_k(\lambda; x))}{\det(\bar{V})}.
\]

By expanding the determinant of \( V_k(\lambda; x) \) with respect to its \( k \)th row, we obtain that
\[
\frac{\det(V_k(\lambda; x))}{\det(\bar{V})} = \sum_{\ell=1}^{N} a_{k\ell} e^{\theta_\ell(x-a)}_\lambda
\]
where the coefficients \( a_{k\ell} \) are explicitly written in Remark 2.

Next, concerning the determinant \( \Delta_k^+ (\lambda; x) \), by factorizing the \( \ell \)th column by \( e^{\theta_\ell b}_\lambda \) for each \( \ell \in \{1, \ldots, 2N\} \), using the identity \( \sum_{\ell=1}^{2N} \theta_\ell = 0 \) and permuting the \( k \)th and \((N + k)\)th rows for each \( k \in \{1, \ldots, N\} \), we get that
\[
\Delta_k^+ (\lambda; x) = (-1)^N \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\
e^{\theta_1(x-b)}_\lambda & \cdots & e^{\theta_{2N}(x-b)}_\lambda \\
\theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\
\vdots & \ddots & \vdots \\
\theta_1^{N-1} & \cdots & \theta_{2N}^{N-1}
\end{pmatrix}.
\]

As previously, we decompose \( \Delta_k^+ (\lambda; x) \) into
\[
\Delta_k^+(\lambda; x) = (-1)^N \begin{vmatrix}
\gamma_k(\lambda; x) & \tilde{\gamma}_k(\lambda; x) \\
\vdots & \vdots \\
\gamma_k(\lambda; x) & \tilde{\gamma}_k(\lambda; x) \\
Z(\lambda) & \tilde{Z}(\lambda)
\end{vmatrix} = \begin{vmatrix}
\gamma_k(\lambda; x) & \gamma_k(\lambda; x) \\
\tilde{Z}(\lambda) & Z(\lambda)
\end{vmatrix}.
\]
From this, we deduce the second limit of (3.6):

\[ \lambda \]

where

\[ I \]

Finally, by remarking that \( \theta_{N+\ell} = -\theta_{\ell} \) for any \( \ell \in \{1, \ldots, N\} \), we derive that

\[ \Delta_k^+(\lambda; x) = (-1)^k \left| \begin{array}{c|c} U_k(\lambda; x) & \bar{U}_k(\lambda; x) \\ \hline W(\lambda) & \bar{W}(\lambda) \end{array} \right| \]

where the matrices \( U_k(\lambda; x) \) and \( \bar{U}_k(\lambda; x) \) are deduced from \( V_k(\lambda; x) \) and \( \bar{V}_k(\lambda; x) \) by changing \( (x - a) \) into \( (b - x) \), that is, \( U_k(\lambda; x) = V_k(\lambda; a + b - x) \) and \( \bar{U}_k(\lambda; x) = \bar{V}_k(\lambda; a + b - x) \). As a byproduct, we derive the identity

\[ \Delta_k^+(\lambda; x) = (-1)^k \Delta_k^-(\lambda; a + b - x) \]

which is evoked in Remark 1. Thanks to an expansion by blocks, we can see that, for \( x \in (a, b) \),

\[ \Delta_k^+(\lambda; x) \sim (-1)^k \det(V_k(\lambda; a + b - x)) \times \det(\bar{W}(\lambda)) = o(\det(\bar{W}(\lambda))). \]

From this, we deduce the second limit of (3.6):

\[ \frac{\Delta_k^+(\lambda; x)}{\Delta(\lambda)} \xrightarrow{b \to +\infty} 0. \]

### A.2 Asymptotics of \( \Delta(\lambda) \) and \( \Delta_k^+(\lambda; x) \) as \( \lambda \) tends to 0\(^+\) or \(+\infty\)

The procedure depicted in the previous subparagraph can be carried out mutatis mutandis in the case where \( \lambda \) tends to \(+\infty\). This yields the following limiting result:

\[ \frac{\Delta_k^+(\lambda; x)}{\Delta(\lambda)} \xrightarrow{\lambda \to +\infty} 0, \]

the rate of convergence being exponential. Then \( I_k^+(t; x) \xrightarrow{t \to 0^+} 0 \). Below, we examine the case where \( \lambda \) tends to 0\(^+\).
A.2.1 Asymptotics of $\Delta(\lambda)$ as $\lambda$ tends to $0^+$

Set $c = \lambda^{1/(2N)}(b - a)$. The number $c$ tends to $0$. We expand the exponentials lying in $\Delta(\lambda)$ into power series: for $k \in \{0, \ldots, N - 1\},$

$$\theta_k^j e^{\theta_k(b-a)} = \sum_{i=0}^{\infty} \theta_k^j c_i^i \frac{c_i^i}{i!} = \sum_{i=k}^{\infty} \theta_i^j \frac{c_i^j}{(i-k)!}. $$

Then,

$$\Delta(\lambda) = \begin{vmatrix} 1 & \ldots & 1 \\ \theta_1 & \ldots & \theta_{2N} \\ \vdots & \ddots & \vdots \\ \theta_1^{N-1} & \ldots & \theta_{2N}^{N-1} \end{vmatrix} + \begin{vmatrix} \sum_{i=0}^{\infty} \theta_1^{iN+1} & \ldots & \sum_{i=0}^{\infty} \theta_{2N}^{iN+1} \\ \sum_{i=1}^{\infty} \theta_1^{c_i^1} \frac{c_i^1}{(i-1)!} & \ldots & \sum_{i=1}^{\infty} \theta_{2N}^{c_i^1} \frac{c_i^1}{(i-1)!} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{\infty} \theta_1^{c_i^{i-N+1}} \frac{c_i^{i-N+1}}{(i-N+1)!} & \ldots & \sum_{i=1}^{\infty} \theta_{2N}^{c_i^{i-N+1}} \frac{c_i^{i-N+1}}{(i-N+1)!} \end{vmatrix}. $$

By multilinearity, we see that the terms including a power of $\theta_i$ less than $N$ can be discarded (for these terms, the corresponding determinant has two or more identical rows, thus it vanishes). Hence, the determinant $\Delta(\lambda)$ does not change if we only keep the sums $\sum_{i=N}^{\infty} \theta_i^j c_i^{j-k}/(i-k)!$:

$$\Delta(\lambda) = \begin{vmatrix} 1 & \ldots & 1 \\ \theta_1 & \ldots & \theta_{2N} \\ \vdots & \ddots & \vdots \\ \theta_1^{N-1} & \ldots & \theta_{2N}^{N-1} \end{vmatrix} + \begin{vmatrix} \sum_{i=N}^{\infty} \theta_1^{iN+1} & \ldots & \sum_{i=N}^{\infty} \theta_{2N}^{iN+1} \\ \sum_{i=N}^{\infty} \theta_1^{c_i^1} \frac{c_i^1}{(i-1)!} & \ldots & \sum_{i=N}^{\infty} \theta_{2N}^{c_i^1} \frac{c_i^1}{(i-1)!} \\ \vdots & \ddots & \vdots \\ \sum_{i=N}^{\infty} \theta_1^{c_i^{i-N+1}} \frac{c_i^{i-N+1}}{(i-N+1)!} & \ldots & \sum_{i=N}^{\infty} \theta_{2N}^{c_i^{i-N+1}} \frac{c_i^{i-N+1}}{(i-N+1)!} \end{vmatrix}. $$

By multilinearity, we can rewrite $\Delta(\lambda)$ as

$$\Delta(\lambda) = \sum_{i_1, \ldots, i_N \text{ all distinct}} \frac{c^{i_1+(i_2-1)+\cdots+(i_N-N+1)}}{i_1!(i_2-1)! \cdots (i_N-N+1)!} \begin{vmatrix} 1 & \ldots & 1 \\ \theta_1 & \ldots & \theta_{2N} \\ \vdots & \ddots & \vdots \\ \theta_1^{N-1} & \ldots & \theta_{2N}^{N-1} \end{vmatrix}. $$

Because of the conditions on the indices $i_1, \ldots, i_N$, the least power of $c$ is not less than $N^2$: indeed, the indices being distinct and not less than $N$, we have $i_1 + i_2 + \cdots + i_N \geq N + (N+1) + \cdots + (2N-1)$ or, equivalently, $i_1 + (i_2-1) + \cdots + (i_N-N+1) \geq N^2$. Moreover, if an index is greater than $(2N-1)$, say $i_N \geq 2N$, then $i_1 + i_2 + \cdots + i_N-1 + i_{N} \geq N + (N+1) + \cdots + (2N-2) + (2N)$, that is, $i_1 + (i_2-1) + \cdots + (i_N-N+1) \geq N^2 + 1$. In words, the term $c^{N^2}$ is obtained at
most for the indices not greater than \((2N -1)\). Consequently, we see that the terms of the sums corresponding to \(i\) greater than \((2N -1)\) can be neglected when \(c\) tends to 0, namely:

\[
\Delta(\lambda) = \begin{pmatrix}
1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_{2N} \\
\vdots & & \vdots \\
\theta_{1}^{N-1} & \cdots & \theta_{2N}^{N-1}
\end{pmatrix} + o(c^{N^2})
\]

We observe that the matrix lying in the foregoing determinant can be factorized into the product of the two following matrices:

\[
A_1 = \begin{pmatrix} I : O \\ \vdots \\ O : B \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_{2N} \\
\vdots & & \vdots \\
\theta_{1}^{N-1} & \cdots & \theta_{2N}^{N-1}
\end{pmatrix}
\]

where \(I\) and \(O\) are respectively the unit and zero matrices of type \(N \times N\), and

\[
B = \begin{pmatrix}
\frac{c^N}{N!} & \frac{c^{N+1}}{(N+1)!} & \cdots & \frac{c^{N-1}}{(2N-1)!} \\
\frac{c^N}{(N-1)!} & \frac{c^{N-1}}{N!} & \cdots & \frac{c^{2N-2}}{(2N-2)!}
\end{pmatrix}.
\]

We can decompose \(B\) into \(C_1 B C_2\) where \(C_1\) and \(C_2\) are the diagonal matrices with \(c^N, c^{N-1}, \ldots, c\) and 1, \(c, \ldots, c^{N-1}\) as diagonal terms respectively, and

\[
\tilde{B} = \begin{pmatrix}
\frac{1}{N!} & \frac{1}{(N+1)!} & \cdots & \frac{1}{(2N-1)!} \\
\frac{1}{(N-1)!} & \frac{1}{N!} & \cdots & \frac{1}{(2N-2)!}
\end{pmatrix}.
\]

Hence, all this discussion plainly entails that

\[
\Delta(\lambda) \sim \det(A_1) \times \det(A_2) = \det(A_2) \times \det(\tilde{B}) \times \det(C_1) \times \det(C_2) = constant \times c^{N^2}
\]

where the constant does not vanish, or, by means of the variable \(\lambda\),

\[
\Delta(\lambda) \sim constant \times \lambda^{N/2}. \tag{A.3}
\]

### A.2.2 Asymptotics of \(\Delta_{x}^{\pm}(\lambda; x)\) as \(\lambda\) tends to 0+

A similar analysis can be carried out in the case of the determinant \(\Delta_{x}^{\pm}(\lambda; x)\). Recall that \(c = \lambda^{1/(2N)}(b - a)\) and set \(\gamma = \lambda^{1/(2N)}(x - a)\). The numbers \(c\) and \(\gamma\) tend to 0 as \(\lambda\) tends to 0+. E.g.,
for $\Delta_k^-(\lambda; x)$, we have that

$$
\Delta_k^-(\lambda; x) = \begin{vmatrix}
1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_{2N} \\
\vdots & & \vdots \\
\theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\
\sum_{i=0}^{\infty} \theta_1^i N_i & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i N_i \\
\theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\
\vdots & & \vdots \\
\theta_{1}^{N-1} & \cdots & \theta_{2N}^{N-1} \\
\sum_{i=0}^{\infty} \theta_1^i N_i & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i N_i \\
\sum_{i=1}^{\infty} \theta_1^i \frac{\gamma_i}{(i-1)!} & \cdots & \sum_{i=1}^{\infty} \theta_{2N}^i \frac{\gamma_i}{(i-1)!} \\
\vdots & & \vdots \\
\sum_{i=N-1}^{\infty} \theta_1^i \frac{\gamma_i}{(N-i+1)!} & \cdots & \sum_{i=N-1}^{\infty} \theta_{2N}^i \frac{\gamma_i}{(N-i+1)!}
\end{vmatrix}.
$$

As previously, this determinant remains unchanged by removing the terms related to the indices $0, 1, \ldots, k-1, k+1, \ldots, N-1$ in each sum. Moreover, for obtaining an asymptotics when $c, \gamma$ tend to $0$ (actually $c$ and $\gamma$ have the same order of growth when $\lambda$ tends to $0$), it is enough to keep the terms related to the indices not greater than $(2N-1)$. Then, by setting $I_k = \{k\} \cup \{N, N+1, \ldots, 2N-1\}$,

$$
\Delta_k^-(\lambda; x) = \begin{vmatrix}
1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_{2N} \\
\vdots & & \vdots \\
\theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\
\sum_{i=0}^{\infty} \theta_1^i N_i & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i N_i \\
\theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\
\vdots & & \vdots \\
\theta_{1}^{N-1} & \cdots & \theta_{2N}^{N-1} \\
\sum_{i=0}^{\infty} \theta_1^i N_i & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i N_i \\
\sum_{i=0}^{\infty} \theta_1^i \frac{\gamma_i}{(i-1)!} & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i \frac{\gamma_i}{(i-1)!} \\
\vdots & & \vdots \\
\sum_{i=0}^{\infty} \theta_1^i \frac{\gamma_i}{(N-i+1)!} & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i \frac{\gamma_i}{(N-i+1)!}
\end{vmatrix} + o\left(\gamma^k c N^2\right).
$$

We observe that the matrix lying in the above determinant is the product of $\bar{A}_1$ by $A_2$ where

- $A_1 = \left(\begin{array}{cc} I_k & \vdots \\ \vdots & O_{2,k} \\ O_{1,k} & \vdots \\ \vdots & B \end{array}\right)$;
- $I_k$ is the diagonal matrix of type $N \times N$ with diagonal terms equal to $1$ except for the $(k+1)$th which is $\gamma^k / k!$;
- $O_{1,k}$ is the matrix of type $N \times N$ with all terms equal to $0$ except for the $(k+1)$th column which is made of $\gamma^k / k!, \gamma^k / (k-1)!, \ldots, c, 1, 0, \ldots, 0$;
- $O_{2,k}$ is the matrix of type $N \times N$ with all terms equal to $0$ except for the $(k+1)$th row which is made of $\gamma^N / N!, \gamma^{N+1} / (N+1)!, \ldots, \gamma^{2N-1} / (2N-1)!$. 

The determinant of $\tilde{A}_1$ remains unchanged by interchanging its $(k+1)$th and $N$th columns and its $(k+1)$th and $(k+1)$th rows. This yields that

$$\det(\tilde{A}_1) = \begin{vmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & c & \cdots & 1 \end{vmatrix}$$

By expanding this last determinant with respect to its first row, it is not difficult to see that $\det(\tilde{A}_1) = O\left(\gamma^k c N^2\right)$ (recall that $c$ and $\gamma$ have the same order of growth when $\lambda$ tends to 0). Therefore, in terms of the variable $\lambda$,

$$\Delta^-_k (\lambda; x) = O(\lambda^{k/(2N)+N/2}) \quad (A.4)$$

and the same holds for $\Delta^+_k (\lambda; x)$. Finally, by (A.3) and (A.4), we derive that

$$\frac{\Delta^-_k (\lambda; x)}{\Delta(\lambda)} \xrightarrow{\lambda \to 0^+} O(\lambda^{k/(2N)}) \quad (A.5)$$

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