Non-Gaussianity and intermittency in an ensemble of Gaussian fields

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Keywords: non-equilibrium statistical mechanics, turbulence, intermittency, non-Gaussianity

Abstract
Motivated by the need to capture statistical properties of turbulent systems in simple, analytically tractable models, an ensemble of Gaussian sub-ensembles with varying properties of the correlation function such as variance and length scale is investigated. The ensemble statistics naturally exhibit non-Gaussianity and intermittency. Due to the simplicity of Gaussian random fields, many explicit results can be obtained analytically, revealing the origin of non-Gaussianity in this framework. Potential applications of the proposed model ensemble for the description of non-equilibrium statistical mechanics of complex turbulent systems are briefly discussed.

1. Introduction
The frequent occurrence of extreme fluctuations is a hallmark for complex systems far from equilibrium. Such systems therefore generically display non-Gaussian statistics, which is a central topic of non-equilibrium statistical mechanics. It is a common feature of spatio-temporally extended systems that the statistics of fluctuations change with scale. As a prominent example, probability density functions (PDFs) of velocity fluctuations across a given scale in hydrodynamic turbulence transition from a highly non-Gaussian shape at small scales to a close-to-Gaussian shape at large scales [1]. Understanding and characterizing this phenomenon, known as intermittency, is one of the key steps in formulating a statistical theory of turbulent flows.

Intermittency and non-Gaussianity have been reported in hydrodynamic (see, e.g., [1–6]) and magnetohydrodynamic [7–10] turbulence as well as in fusion plasmas [11–14] and in solar wind turbulence [15–22]. Non-Gaussian fluctuations also play a role in many other branches of sciences including the economy of financial markets [23–25], demonstrating that non-Gaussian and scale-dependent statistics are ubiquitous in non-equilibrium systems.

For field-theoretic problems, including the closure problem in exact statistical approaches to the turbulence problem [26], knowledge of the complete statistics of entire random fields is desirable. A general theory of non-Gaussian random fields encompassing these phenomena, however, is currently lacking. Therefore, there is a fundamental interest in the construction and investigation of ensembles of random fields that are analytically tractable and at the same time capable of capturing the non-Gaussian and intermittent features of the aforementioned systems.

Various approaches exist to tackle this problem, which are briefly summarized in the following. For example, random fields considered within the multifractal framework are associated to fractal sets of Hölder exponents [27]. As a result, statistical properties obtained for such fields are determined by a superposition of various Hölder exponents weighted by their statistical measure. The multifractal framework has been extremely successful in capturing both Eulerian [28–31] and Lagrangian [32–34] statistical features of turbulence.

With respect to hydrodynamic turbulence, Kolmogorov [35] and Obukhov [36] modeled intermittency by taking into account spatial fluctuations of the energy dissipation. Similar ideas have been used to model velocity increment PDFs. Motivated by the Kolmogorov–Obukhov picture of turbulence, Castaing and co-workers introduced a model in which Gaussian PDFs with varying standard deviations are superposed according to a
scale-dependent distribution of those standard deviations \cite{3}. The same approach recently has found application in solar wind turbulence \cite{37}.

In the framework of superstatistics \cite{38–40}, a superposition of realizations of a complex system, each in local thermodynamic equilibrium, is considered. On a macroscopic level, fluctuations of parameters such as the inverse temperature lead to non-Gaussian statistics capable of describing non-equilibrium systems including Eulerian \cite{41} and Lagrangian \cite{42–44} hydrodynamic turbulence.

Furthermore, various nonlinear mapping techniques \cite{45–47} as well as field-type stochastic processes \cite{48, 49} have been proposed to capture statistical features of turbulence.

As discussed, the non-Gaussianity of systems far from equilibrium rules out simple Gaussian fields as candidates for a statistical description. To overcome this limitation, here an ensemble of Gaussian random fields is introduced and explored, in which each of the sub-ensembles differs with respect to certain parameters encoded in the correlation structure. Gaussianity of the individual sub-ensembles allows for an analytical treatment while the superposition introduces non-Gaussian features. This approach shares similarities with some of the above-mentioned ones in the sense that a statistical description of non-equilibrium systems is attempted by superposing simpler (in this case Gaussian) statistics. The profoundly new aspect is the fact that the proposed ensemble consists of entire fields, i.e. it contains arbitrary multi-point statistics and therefore offers a comprehensive statistical characterization.

The paper is organized as follows. In section 2 the ensemble will be introduced along with the necessary mathematical background. In section 3 its features are investigated. There, focus is put on the impact of variances or length scales fluctuating across the ensemble on non-Gaussianity and intermittency. The implications of the current findings on the modeling of non-equilibrium systems are discussed in the conclusions.

2. Ensemble of Gaussian fields

In the following a field \(a(x)\) is considered, which in principle could be an arbitrary scalar, vectorial or tensorial field. Here, \(x\) denotes a suitable set of space-time coordinates. In favor of a simple presentation, a scalar field on a line is discussed in the following; generalizations are straightforward. A comprehensive statistical description of an ensemble of such fields can be achieved in terms of the characteristic functional, which is defined as

\[
\phi [\alpha] = \left\{ \exp \left[ i \int_{-\infty}^{\infty} dx \, \alpha(x) \, a(x) \right] \right\}.
\] (1)

Here, the angular brackets denote a suitably defined ensemble average. Multi-point correlation functions, for example, can be obtained from the characteristic functional by taking functional derivatives

\[
\langle a(x_1) \, a(x_2) \ldots a(x_n) \rangle = \left[ \frac{\delta^n}{\delta \alpha(x_1) \delta \alpha(x_2) \ldots \delta \alpha(x_n)} \, \phi [\alpha] \right]_{\alpha = 0}.
\] (2)

Single- and two-point characteristic functions, respectively, can be obtained by

\[
\phi (\alpha_1) = \phi [\alpha = \alpha_1 \delta (x - x_1)],
\] (3)

\[
\phi (\alpha_1, \alpha_2) = \phi [\alpha = \alpha_1 \delta (x - x_1) + \alpha_2 \delta (x - x_2)].
\] (4)

Fourier transforming these expressions leads to the corresponding PDFs. As the complete statistical information of the ensemble of fields is contained in the characteristic functional, it also contains the statistics of derived quantities, such as the coarse-grained fields or the gradient fields. To obtain the characteristic functional \(\psi [\beta]\) of the gradient field, for example, one needs to evaluate

\[
\psi [\beta] = \phi \left[ \alpha = - \frac{\partial \beta}{\partial x} \right]
\] (5)

as can be shown by integration by parts. The single-point characteristic function of the gradient is then again obtained simply by projection:

\[
\psi (\beta_1) = \psi [\beta = \beta_1 \delta (x - x_1)].
\] (6)

For coarse-grained fields

\[
\bar{a}(x) = \int_{-\infty}^{\infty} dy \, G(x - y) \, a(y),
\] (7)

where \(G\) is a suitably defined symmetric filter kernel, a similar relation can be shown: the characteristic functional \(\bar{\phi} [\alpha]\) of the coarse-grained field can be obtained from the characteristic functional of the unfiltered field by
\[ \tilde{\phi}[\alpha] = \phi[\tilde{\alpha}] . \] (8)

Multi-scale features can be investigated, for example, by studying the statistics of increments. The corresponding characteristic function can be obtained from the full functional by

\[ \psi(\Delta; r) = \phi[\alpha = \Delta(\delta(x - r) - \delta(x))] . \] (9)

These examples serve to illustrate the comprehensive statistical information contained in the characteristic functional.

In general, the ensemble average (1) cannot be evaluated explicitly for arbitrary random fields. However, for Gaussian fields an analytical expression is readily available \[50]\:

\[ \phi^{G}[\alpha] = \exp \left[ i \int_{-\infty}^{\infty} dx \mu(x) \alpha(x) - \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' C(x, x') \alpha(x') \right] . \] (10)

Here, \( \mu(x) = \langle a(x) \rangle \) denotes the mean whereas \( C(x, x') = \langle [a(x) - \mu(x)][a(x') - \mu(x')] \rangle \) denotes the covariance. Due to the closed-form expression (10), arbitrary statistical information can be obtained analytically and expressed in terms of the mean and covariance.

In the following, the discussion will be restricted to random fields with zero mean, \( \mu(x) = 0 \). To construct an ensemble which displays non-Gaussian features, an ensemble of Gaussian sub-ensembles is considered. Each of the Gaussian sub-ensembles differs with respect to a set of parameters \( \lambda \) characterizing its statistical properties. These parameters, for instance, could be the variance, characteristic scales etc. Some examples are presented in the next section. The dependence on the set of parameters enters in the specification of the covariance, \( \langle a(x) a(x') \rangle = C_{\lambda}(x, x') \), which uniquely characterizes the individual Gaussian sub-ensembles. As a result, the characteristic functional of the Gaussian sub-ensembles takes the form

\[ \phi_{\lambda}^{G}[\alpha] = \exp \left[ - \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \alpha(x) C_{\lambda}(x, x') \alpha(x') \right] . \] (11)

The full ensemble is then constructed as

\[ \phi[\alpha] = \int d\lambda P(\lambda) \phi^{G}_{\lambda}[\alpha] , \] (12)

where \( P(\lambda) \) is non-negative and normalized,

\[ \int d\lambda P(\lambda) = 1 , \] (13)

i.e. it defines a PDF of the set of parameters. The ‘convex combination’ of characteristic functionals (12) maintains the analytical accessibility of Gaussian fields, yet allowing for non-Gaussian features as will be demonstrated in the next section. It is worth noting that if the distribution of parameters is sharp,

\[ P(\lambda) = \delta(\lambda - \lambda_0) , \] (14)

the ensemble naturally reduces to one of the Gaussian sub-ensembles, i.e. \( \phi[\alpha] = \phi^{G}_{\lambda_0}[\alpha] \).

Furthermore, it has to be pointed out that a superposition of characteristic functionals does not correspond to a superposition of fields. If statistically independent fields were superposed, the resulting characteristic functional would be composed of the product of the individual characteristic functionals. If the individual fields are Gaussian, so is the superposed field. And even for the superposition of a large number of non-Gaussian fields, a generalized central limit theorem should imply asymptotically Gaussian statistics.

The approach presented here shares similarities with superstatistics known from statistical mechanics as well as with so-called Gaussian mixture models \[51\] and in particular Gaussian scale mixtures \[52\]. Scale mixture there refers to the superposition of Gaussian distributions of varying widths. Here we explore a more general situation, namely the statistical superposition of Gaussian fields, which allows for (spatially or temporally) scale-dependent non-Gaussianity controlled by the parameter PDF \( P(\lambda) \) and the covariance \( C_{\lambda}(x, x') \).

### 3. Features of the ensemble

The purpose of this section is to present a number of analytically tractable examples to explicitly highlight the origin of non-Gaussianity and intermittency in the ensemble. In particular, the impact of various fluctuating quantities on non-Gaussianity and intermittency in the ensemble is studied.

In the following, statistically homogeneous fields with correlation functions of the form \( C_{\lambda}(x, x') = \sigma^2 f_{\eta}(r) \) where \( r = |x - x'| \) are considered. Here, \( \sigma \) denotes the standard deviation of the sub-ensemble of Gaussian fields, whereas \( \eta \) denotes a length scale. The normalized autocorrelation function \( f_{\eta} \) is assumed to be a
monotonically decaying function with \( f_{\eta}(0) = 1 \) and \( \lim_{r \to \infty} f_{\eta}(r) = 0 \). It is furthermore assumed that \( f_{\eta} \) has a Taylor expansion of the form

\[
f_{\eta}(r) = 1 - \frac{r^2}{2\eta^2} + \mathcal{O}(r^4),
\]

i.e. \( \eta \) is a length scale characterizing the decorrelation.

The set of parameters that potentially vary across the ensemble is \( \lambda = \{ \sigma, \eta \} \). In principle, these parameters could be correlated as reflected in their joint PDF \( P(\lambda) \). In the following examples, only one of these parameters fluctuates across the ensemble, whereas the other one is fixed. In these cases \( \sigma \) or \( \eta \) will be explicitly used as a subscript to the characteristic functional as opposed to \( \lambda \).

### 3.1. Fluctuating standard deviation

As a first example, it is assumed that the length scale \( \eta \) is fixed and that only the standard deviation \( \sigma \) fluctuates across the ensemble,

\[
\phi[\alpha] = \int_0^\infty d\sigma \ P(\sigma) \ \phi_{\sigma}^\alpha(\alpha).
\]

To obtain explicit results, one needs to choose a distribution function \( P(\sigma) \). For the sake of analytical tractability, it is assumed that \( \sigma \) has a Rayleigh distribution

\[
P(\sigma) = \frac{\sigma}{\gamma_{\sigma}} \exp\left(-\frac{\sigma^2}{2\gamma_{\sigma}}\right) \quad (17)
\]

The parameter \( \gamma_{\sigma} \) determines the variance of the distribution of standard deviations \( \sigma \). By insertion into (16), projecting to the single-point level according to (3), and evaluating the integral one obtains

\[
\phi(\alpha_1) = \frac{1}{1 + \frac{\gamma_{\sigma}^2}{2\gamma_{\sigma}} \alpha_1^2}.
\]

A Fourier transform leads to the Laplace distribution

\[
P(a_1) = \frac{1}{2\gamma_{\sigma}} \exp\left(-\frac{|a_1|}{\gamma_{\sigma}}\right),
\]

which exhibits exponential tails. This example illustrates how the ensemble generates non-Gaussian statistics related to the frequent occurrence of extreme fluctuations. The non-Gaussian shape of the PDF can be intuitively understood with the help of figure 1, which shows the Rayleigh distribution (17) in the left panel. For three values of standard deviations indicated in that plot, the corresponding Gaussian distributions are shown in the middle panel. The same panel shows their superposition weighted with the probability density of the example standard deviations, which demonstrates the mechanism of generating non-Gaussian PDFs. The exponential PDF (19), which results from a continuous superposition of Gaussians with Rayleigh-distributed standard deviations, is shown in the right panel. In essence, this mechanism of generating non-Gaussian PDFs from a superposition of Gaussian PDFs underlies also the superstatistical and multifractal approaches as well as the intermittency model for turbulence first introduced by Castaing and co-workers [3].

Because the statistics of entire fields are available, one can proceed to calculate additional statistical quantities such as gradient and increment PDFs. The gradient PDF can be computed by evaluating (3) and subsequently (6), leading to
Therefore, also the gradient exhibits a Laplace distribution

\[ P(h_i) = \frac{\eta}{2\gamma_\eta} \exp \left( -\frac{\eta |h_i|}{\gamma_\eta} \right). \]  

As a consequence, the single-point distributions of the field and its gradient have identical shapes, however, their standard deviations differ. The same applies to the increment distributions. They can be obtained by first projecting the characteristic functional according to (9) which leads to

\[ \psi(\Delta; r) = \int_0^\infty d\sigma P(\sigma) \exp \left( -\frac{S_\sigma(r) \Delta^2}{2} \right). \]  

Here, \( S_\sigma(r) \) is the second-order structure function

\[ S_\sigma(r) = 2\sigma^2 [1 - f_\sigma(r)]. \]  

The subscript \( \sigma \) indicates that the second-order structure function depends on the standard deviation which varies across the ensemble while the length scale \( \eta \) is kept fixed. Evaluating this integral for the Rayleigh-distributed standard deviation (17) yields

\[ \psi(\Delta; r) = \frac{1}{1 + 2\gamma_\sigma^2 [1 - f_\sigma(r)] \Delta^2}, \]  

which again implies a Laplace distribution for the increments:

\[ P(\delta a; r) = \frac{1}{2\gamma_\sigma \sqrt{2[1 - f_\sigma(r)]}} \exp \left( -\frac{\delta a}{\gamma_\sigma \sqrt{2[1 - f_\sigma(r)]}} \right). \]  

This confirms that the ensemble exhibits statistical self-similarity: the single-point PDFs of the field and its gradients as well as the increment PDF display the same shape and differ only by their variance. The interesting finding therefore is that an ensemble, in which only the standard deviation fluctuates, displays non-Gaussian, but non-intermittent PDFs.

The observation of statistical self-similarity in fact does depend neither on the choice of the distribution of the variance \( P(\sigma) \) nor on the choice of the normalized autocorrelation function \( f_\sigma(r) \). This is evident from the fact that the corresponding characteristic functions, obtained by suitable projection of (16), can be mapped onto each other by simple rescaling. As a consequence, the resulting PDFs are self-similar, i.e. non-intermittent.

### 3.2. Fluctuating length scale

For the next example, it is assumed that all ensemble members share the same standard deviation, but differ with respect to their length scale \( \eta \). Therefore the characteristic functional of the ensemble takes the form

\[ \phi[\alpha] = \int_0^\infty d\eta P(\eta) \phi_\eta^2[\alpha]. \]  

For the length scale a Maxwell distribution,

\[ P(\eta) = \sqrt{\frac{2}{\pi \gamma_\eta^3}} \frac{\eta^2}{2\gamma_\eta^2} \exp \left( -\frac{\eta^2}{2\gamma_\eta^2} \right), \]  

is chosen for the sake of analytical tractability. Furthermore an explicit choice for the autocorrelation function is made,

\[ C_\eta(r) = \sigma^2 \exp \left( -\frac{r^2}{2\gamma_\eta^2} \right), \]  

which complies with the Taylor expansion (15). In this example \( \eta \) corresponds (up to a constant of proportionality) to the integral length scale. While the correlation functions of the individual sub-ensembles show a Gaussian decay, it is interesting to note that the ensemble-averaged correlation function takes the form

\[ C(r) = \int_0^\infty d\eta P(\eta) C_\eta(r) = \sigma^2 \left( 1 + \frac{r}{\gamma_\eta} \right) \exp \left( -\frac{r}{\gamma_\eta} \right). \]  

This demonstrates that (a) the functional form of the correlation changes under averaging and that (b) the integral length scale of the ensemble is determined by the standard deviation controlled by \( \gamma_\eta \).
Proceeding to PDFs, it is straightforward to show that the single-point statistics are insensitive to the fluctuating length scale, which can be seen from
\[ \phi(\alpha_l) = \phi(\alpha + \alpha_l \delta(x - x_l)) \]
\[ = \int_0^\infty d\eta \, P(\eta) \exp\left[-\frac{1}{2} \sigma^2 \alpha^2_l \right] \]
\[ = \exp\left[-\frac{1}{2} \sigma^2 \alpha^2_l \right] \int_0^\infty d\eta \, P(\eta) \]
\[ = 1 \]
(30)
Therefore a Gaussian single-point PDF for the field is obtained:
\[ P(\alpha_l) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\alpha^2_l}{2\sigma^2}\right] \]
(33)
The gradient PDF, however, turns out to be pronouncedly non-Gaussian due to the fluctuating length scale. To show this, first (3) is evaluated and then projected to the single-point level which yields
\[ \psi(\beta_l) = \int_0^\infty d\eta \, P(\eta) \exp\left[-\frac{\sigma^2 \beta^2_l}{2\eta^2}\right] \]
(34)
Comparing this to (31) explicitly reveals the dependence of the Gaussian characteristic function for the gradient on the fluctuating length scale. To obtain an expression for the PDF \( P(b_l) \) of the gradient, the Fourier transform of (34) is taken and the integration under the example assumption of a Maxwell-distributed length scale according to (27) is evaluated. The result is
\[ P(b_l) = \frac{2\gamma_0}{\pi\sigma [b_l^2 \eta_0^2/\sigma^2 + 1]^2} \]
(35)
a heavy-tailed distribution with algebraic tails. As a consequence of the algebraic decay of the PDF, even moments of orders four and higher do not exist. It is interesting to note that this result depends only on the curvature of the correlation function in the origin, i.e. it is valid for all correlation functions with a Taylor expansion according to (15).
Having established the single-point PDF of the field as Gaussian and the single-point gradient field as non-Gaussian, it is instructive to consider the statistics of increments. To study the behavior of increments across scale \( r \), the full correlation function (28) comes into play. To obtain the increment PDF, the characteristic functional is projected to the increment variable according to (9) which leads to
\[ \psi(\Delta; r) = \int_0^\infty d\eta \, P(\eta) \exp\left[-\frac{S_\eta(r) \Delta^2}{2}\right] \]
(36)
Here \( S_\eta(r) \) is the second-order structure function
\[ S_\eta(r) = 2\sigma^2 [1 - f_\eta(r)], \]
(37)
where the subscript \( \eta \) now indicates that the second-order structure function depends on the length scale, which varies across the ensemble while the standard deviation \( \sigma \) is kept fixed. A Fourier transform leads to the corresponding expression for the increment PDF which takes the form
\[ P(\delta a; r) = \int_0^\infty d\eta \, P(\eta) \frac{\exp\left[-\frac{\delta a^2}{2S_\eta(r)}\right]}{\sqrt{2\pi S_\eta(r)}} \]
(38)
This integration is carried out numerically with SciPy [53] and the result is shown in the left panel of figure 2. The increment PDFs show a continuous shape deformation from the non-Gaussian gradient PDF on small scales to a Gaussian PDF on large scales, i.e. they display intermittency. The two limiting cases of very small and very large scales are explained by the above analytical calculations. The continuous transition from non-Gaussian to Gaussian PDFs can be understood from (38), which shows that the distribution of length-scales is probed by Gaussian increment PDFs with an \( \eta \)-dependent variance varying with scale. This demonstrates that a Gaussian ensemble with varying length scales is capable of displaying intermittency.
This can be further illustrated by computing structure functions of the ensemble. Because zero-mean Gaussian fields are considered, all odd-order structure functions are zero. For the even orders one obtains
\[ S_n(r) = \langle (\delta a)^n \rangle (r) = 2^{n/2}(n - 1)!! \sigma^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left( \frac{n/2}{k} \right) \left( 1 + \sqrt{\frac{k \tau}{\gamma_\eta}} \right) \exp\left[-\frac{k \tau}{\gamma_\eta}\right] \]
(39)
A framework to model non-Gaussian statistics and intermittency based on an ensemble of Gaussian fields has been introduced. Non-Gaussian effects thereby arise from the fact that the individual Gaussian sub-ensembles differ in their correlation structure. The proposed framework shares conceptual aspects with superstatistics, multifractality and Gaussian mixture models. With respect to the superstatistical approach, the main novelty is that the correlation structure differs in their correlation structure. The proposed framework shares conceptual aspects with superstatistics, been introduced. Non-Gaussian effects thereby arise from the fact that the individual Gaussian sub-ensembles diverge whereas they drop to the Gaussian value for large $r$.

On a very general level the results of this section are qualitatively in line with the results of the Kolmogorov-Obukhov phenomenology known from hydrodynamic turbulence [35]. In the framework of the refined similarity hypothesis, intermittency is related to a spatially fluctuating dissipation rate. The dissipation rate determines the length scale of the small-scale structures in turbulence; so assuming a fluctuating dissipation rate implies fluctuating length scales. Instead of assuming a spatial variation of parameters, we here let them vary across ensemble members which allows for a mathematically clear-cut treatment. With the focus lying on analytically tractable results, the current choice of correlation function and parameter PDF is not expected to quantitatively capture turbulent statistics. For instance, the correlation function of the current ensemble is characterized by a single length scale only. For modeling turbulence, a correlation function with two length scales, characterizing the smallest and largest scales of the flow, would be more adequate. It will be an important next step to apply the presented framework to turbulence in a quantitative manner.

4. Discussion and conclusions

A framework to model non-Gaussian statistics and intermittency based on an ensemble of Gaussian fields has been introduced. Non-Gaussian effects thereby arise from the fact that the individual Gaussian sub-ensembles differ in their correlation structure. The proposed framework shares conceptual aspects with superstatistics, multifractality and Gaussian mixture models. With respect to the superstatistical approach, the main novelty is that the extension to an ensemble of fields as compared to the superposition of random variables. In comparison to multifractal approaches, in which fields with a multifractal spectrum of Hölder exponents are considered, the current approach sources its simplicity from the simple analytical structure of Gaussian random fields. Two explicit analytical examples showed that (a) a fluctuating variance leads to non-Gaussian but non-intermittent statistics, whereas (b) a fluctuating length scale produces both, non-Gaussianity and intermittency. The current examples focused on intermittency and non-Gaussianity. Skewed statistics, occurring in many turbulent systems, have not been addressed so far. Skewness can be incorporated into the modeling approach by considering Gaussian fields with non-vanishing mean and including a suitable distribution of these mean values.

The main purpose of this paper is to demonstrate the potential of the approach with a number of explicit, analytically tractable examples. An application to hydrodynamic turbulence, specifically Lagrangian intermittency, will be the topic of a forthcoming publication [54]. As it stands, the framework is rather general and not limited to the application to hydrodynamic turbulence. It may be worthwhile evaluating applications...
related to plasma turbulence such as the description of intermittency in magnetohydrodynamic flows and solar wind turbulence.

Another interesting question for future studies is under which conditions non-Gaussian data can be described with this Gaussian ensemble as a surrogate. To this end, the inverse problem of (a) identifying the correct set of fluctuating parameters and (b) determining their distribution from the experimental data needs to be addressed.

The current approach may furthermore turn out to be useful to address the closure problem arising in the formulation of moment or PDF equations for turbulent systems. The PDF of parameters as well as the correlation function, which control the non-Gaussian features of the ensemble, may be good starting points for phenomenological modeling. Simultaneously the comprehensive statistical information contained in the characteristic functional allows the calculation of the statistical quantities necessary to obtain closure. Therefore the current approach potentially establishes a connection between phenomenological modeling approaches on one side, and exact, but generically unclosed statistical approaches on the other.

Acknowledgments

I would like to acknowledge many useful and interesting discussions with Laura J Lukassen, Dimitar G Vlaykov and Cristian C. Laclescu. This work was supported by the Max Planck Society.

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