On Delaunay Ends in the DPW Method

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Abstract

We consider constant mean curvature 1 surfaces in $\mathbb{R}^3$ arising via the DPW method from a holomorphic perturbation of the standard Delaunay potential on the punctured disk. Kilian, Rossman and Schmitt have proven that such a surface is asymptotic to a Delaunay surface. We consider families of such potentials parametrised by the necksize of the model Delaunay surface and prove the existence of a uniform disk on which the surfaces are close to the model Delaunay surface and are embedded in the unduloid case.

Introduction

Beside the sphere, the simplest non-zero constant mean curvature (CMC) surface is the cylinder, which belongs to a one-parameter family of surfaces generated by the revolution of an elliptic function: the Delaunay surfaces, first described in [1]. Like the cylinder, Delaunay surfaces have two annular type ends, and Delaunay ends are the only possible embedded annular ends for a non-zero CMC surface. Indeed, as proven in [11] by Korevaar, Kusner and Solomon, if $\mathcal{M} \subset \mathbb{R}^3$ is a proper, embedded, non-zero CMC surface of finite topological type, then every annular end of $\mathcal{M}$ is asymptotic to a Delaunay surface and if $\mathcal{M}$ has exactly two ends which are of annular type, then $\mathcal{M}$ is a Delaunay surface. Thus, the status of Delaunay surfaces for non-zero CMC surfaces is very much alike the catenoid position in the study of minimal surfaces (see the result of Schoen in [15]), and one has to understand the behaviour of Delaunay ends in order to construct examples of non-compact CMC surfaces with annular ends, as Kapouleas did in 1990 [6].

For an immersion, having a constant mean curvature and having a harmonic Gauss map are equivalent. This is why the Weierstrass type representation of Dorfmeister, Pedit
and Wu [2] has been used since the publication of their article to construct CMC surfaces. The DPW method can construct any conformal non-zero CMC immersion in $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$ with three ingredients: a holomorphic potential which takes its values in a loop group, a loop group factorisation, and a Sym-Bobenko formula. Several examples of CMC surfaces with annular ends, like $n$-noids and bubbletons, have been made by Dorfmeister, Wu, Killian, Kobayashi, McIntosh, Rossmann, Schmitt and Sterling [3, 14, 8, 9, 10, 13]. These constructions often rely on a holomorphic perturbation of the holomorphic potential giving rise to a Delaunay surface via the DPW method, and Kilian, Rossmann and Schmitt [7] have proven that such perturbations always induce asymptotically a Delaunay end.

More precisely, any Delaunay embedding can be obtained with a holomorphic potential of the form $\xi = Az^{-1}dz$ where

$$A = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}.$$  

The main result of [7] states that any immersion obtained from a perturbed potential of the form $\tilde{\xi} = Az^{-1}dz + O(z^0)$ is asymptotic to an embedded half-Delaunay surface in a neighbourhood of $z = 0$, provided that the monodromy problem is solved. In this paper, we allow the perturbed potential to move in the family of Delaunay potentials by introducing a real parameter $t$, proportional to the weight of the model Delaunay surface, and consider $\xi_t = A_tz^{-1}dz + O_t(z)$ where $\xi_t$ gives rise to a chain of spheres if $t \to 0$. The main theorem of [7] states that for every $t$, there exists a small neighbourhood of $z = 0$ on which the surface produced by the potential $\xi_t$ is embedded and asymptotic to a half-Delaunay surface. Unfortunately, as $t$ tends to zero, the neighbourhood gets smaller and smaller. Using similar tools as in [7] and adding a few assumptions, we prove here that one can find a uniform neighbourhood of $z = 0$ on which the surfaces induced by $\xi_t$ are all embedded and asymptotic to a half-Delaunay surface. Consequently, the main benefit of our theorem is that it can be used to show that an entire CMC surface built with DPW is embedded, as Martin Traizet did in [18] to construct embedded $n$-noids without symmetries and in [20] where he uses an opening nodes method in the DPW framework.

In Section 1, we present the DPW method for constructing CMC surfaces in $\mathbb{R}^3$ and introduce the perturbed potential $\xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz + O(t, z^0)dz$, where $A_t$ is a Delaunay residue for all $t$. We also present the three assumptions we make on the holomorphic frame $\Phi_t(z, \lambda)$ to prove our theorem:

- We suppose that the monodromy is unitary;
- We suppose that $\Phi_t(z, \cdot)$ is defined for $\lambda$ (the loop parameter) in a neighbourhood of the unit circle;
• We suppose that the initial condition at \( t = 0 \) is the identity matrix.

The last assumption is very restrictive and nearly never satisfied when one wants to build examples (see [18] or [19]). Fortunately, it is not essential to our theorem and we give in Section 2 a step-by-step method to apply the theorem in the case of a non-identititary initial condition at \( t = 0 \).

In Section 3 we use some basic facts about Fuchsian systems in order to find a gauge and a change of coordinates that increase the convergence of \( \Phi_t \) as \( z \) tends to zero. Indeed, after these changes, our potential will be of the form 

\[
\xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz + O(t, z)dz
\]

and the holomorphic frame will be of the form 

\[
\Phi_t(z, \lambda) = M_t(\lambda)z^{A_t(\lambda)}(I_2 + O(t, z^2)).
\]

In Section 4 we deduce, from the convergence of the holomorphic frame, and using a Cauchy formula, the convergence of the induced immersion \( f_t \) to a Delaunay immersion. Sections 3 and 4 are inspired by [7] where the same techniques are used for a fixed value of the parameter \( t \).

Finally, in Section 5 we show that the family \( f_t \) is embedded on a uniform neighbourhood of \( z = 0 \), provided that it comes from the perturbation of an embedded family of Delaunay potentials.

These different steps lead us to the following theorem (definitions and notations are clarified in Section 1):

**Theorem 1.** Let \( \Phi_t \) be a holomorphic frame arising from a perturbed Delaunay potential \( \xi_t \) defined on a punctured neighbourhood of \( z = 0 \). Suppose that \( \Phi_0(1, \lambda) = I_2 \) and that the monodromy of \( \Phi_t \) is unitary. Then, if \( f_t \) denotes the immersion obtained via the DPW method,

- There exists a family \( f^D_t \) of Delaunay immersions such that for all \( \alpha < 1 \) and \( t \) small enough,
  \[
  \|f_t(z) - f^D_t(z)\|_{\mathbb{R}^3} \leq C_\alpha t|z|^\alpha
  \]
  on a uniform neighbourhood of \( z = 0 \).

- If \( \xi_t \) is a perturbation of an unduloid potential, then, for \( t \) small enough, \( f_t \) is an embedding of a uniform neighbourhood of \( z = 0 \).

- The limit axis of \( f^D_t \) as \( t \) tends to 0 can be made explicit.
1 The DPW method

1.1 Loop groups

Let \( R \geq 1 \). Our maps will often depend on a spectral parameter \( \lambda \) that can be in one of the following subsets of \( C \):

- \( D_R = \{ \lambda \in \mathbb{C}, \ |\lambda| < R \} \)
- \( A_{R_1,R_2} = \{ \lambda \in \mathbb{C}, \ R_1 < |\lambda| < R_2 \} \)
- \( A_R = A_{\frac{1}{R},R} \)
- \( A_1 = \{ \lambda \in \mathbb{C}, \ |\lambda| = 1 \} \).

We define the \( \Lambda \)-adjoint operator as follow: given a holomorphic map \( f : A_R \rightarrow \mathcal{M}_n(\mathbb{C}) \ (n = 1, 2) \), the \( \Lambda \)-adjoint of \( f \) is the holomorphic map defined by

\[
\begin{align*}
f^* : A_R &\rightarrow \mathcal{M}_n(\mathbb{C}) \\
\lambda &\mapsto f\left(\frac{1}{\lambda}\right).
\end{align*}
\]

Let us define the following (untwisted) loop groups and algebras:

- \( \Lambda SL_2 \mathbb{C} \) is the set of smooth maps \( \Phi : A_1 \rightarrow SL_2 \mathbb{C} \).
- \( \Lambda SU_2 \subset \Lambda SL_2 \mathbb{C} \) is the set of smooth maps \( F \in \Lambda SL_2 \mathbb{C} \) that satisfy the reality condition: \( F^* = F^{-1} \).
- \( \Lambda_+ SL_2 \mathbb{C} \subset \Lambda SL_2 \mathbb{C} \) is the set of smooth maps \( G \in \Lambda SL_2 \mathbb{C} \) that can be holomorphically extended to \( D_1 \) and such that \( G(0) \) is upper triangular.
- \( \Lambda_{\pm} SL_2 \mathbb{C} \subset \Lambda_+ SL_2 \mathbb{C} \) is the set of maps \( B \in \Lambda_+ SL_2 \mathbb{C} \) such that \( B(0) \) has real elements on the diagonal.
- \( \Lambda sl_2 \mathbb{C} \) is the set of smooth maps \( A : A_1 \rightarrow sl_2 \mathbb{C} \).
- \( \Lambda su_2 \) is the set of smooth maps \( m \in \Lambda sl_2 \mathbb{C} \) such that \( m^* = -m \).
- \( \Lambda_+ sl_2 \mathbb{C} \subset \Lambda sl_2 \mathbb{C} \) is the set of smooth maps \( g \in \Lambda sl_2 \mathbb{C} \) that can be holomorphically extended to \( D_1 \) and such that \( g(0) \) is upper triangular.
- \( \Lambda_{\pm} sl_2 \mathbb{C} \subset \Lambda_+ sl_2 \mathbb{C} \) is the set of maps \( b \in \Lambda_+ sl_2 \mathbb{C} \) such that \( b(0) \) has real elements on the diagonal.
We will use the following norms:

- For \( v = (v_1, v_2) \in \mathbb{C}^2, |v| = (|v_1|^2 + |v_2|^2)^{\frac{1}{2}} \).
- For \( M \in \mathcal{M}_2(\mathbb{C}), \|M\| = \sup_{|v|=1} |Mv| \).
- For \( \Psi : \mathcal{E} \rightarrow \mathcal{M}_2(\mathbb{C}), \|\Phi\|_\mathcal{E} = \sup_{\lambda \in \mathcal{E}} \|\Psi(\lambda)\| \).

We also use the following notation:

\[
\mathcal{O}(t^\alpha, z^\beta, \lambda^\gamma) = t^\alpha z^\beta \lambda^\gamma f(t, z, \lambda)
\]

where \( f \), on its domain of definition, is holomorphic with respect to \( z, \lambda \) and continuous with respect to \( t \). If one variable is not specified, its exponent is assumed to be 0.

One step of the DPW method relies on the decomposition theorem:

**Theorem 2** (Iwasawa decomposition). The multiplication map

\[
\Lambda_{SU_2} \times \Lambda^R_{+SL_2\mathbb{C}} \rightarrow \Lambda_{SL_2\mathbb{C}}
\]

is a \( C^1 \)-diffeomorphism (for the intersection of the \( C^k \)-topologies) called “Iwasawa decomposition”. The Iwasawa decomposition of \( \Phi \in \Lambda_{SL_2\mathbb{C}} \) will be written:

\[
\Phi = \text{Uni} (\Phi) \times \text{Pos} (\Phi),
\]

where \( \text{Uni} (\Phi) \in \Lambda_{SU_2} \) is called “the unitary factor” of \( \Phi \) and \( \text{Pos}(\Phi) \in \Lambda^R_{+SL_2\mathbb{C}} \) is “the positive factor” of \( \Phi \).

Iwasawa decomposition only deals with maps in \( \Lambda_{SL_2\mathbb{C}} \), yet most of our maps are defined on \( \mathcal{A}_R \). The following corollary ensures that Iwasawa splittings can be extended to \( \mathcal{A}_R \).

**Corollary 1.** Let \( R > 1 \) and \( \Phi : \mathcal{A}_R \rightarrow SL_2\mathbb{C} \) holomorphic. There exists a unique pair \( (F, B) \) such that \( F \) and \( B \) are holomorphic on \( \mathcal{A}_R \) with \( B \) holomorphically extendable to \( \mathcal{D}_R \) and satisfy

\[
\begin{cases}
\Phi = FB, \\
F|_{\mathcal{A}_1} \in \Lambda_{SU_2}, \\
B|_{\mathcal{A}_1} \in \Lambda^R_{+SL_2\mathbb{C}}.
\end{cases}
\]
Proof. We first show existence. Let $\widetilde{\Phi} = \Phi_{|A_1} \in \Lambda_{SL_2^C}$. Iwasawa decompose $\widetilde{\Phi}$ into $\widetilde{F}\widetilde{B}$. Then $\widehat{F} = \Phi\widehat{B}^{-1}$ is holomorphic on $A_{1,R}$, continuous on $A_1$ and can be extended to $A_R$ via

$$F = \begin{cases} 
\widehat{F} & \text{on } A_{1,R}, \\
\widehat{F}^* & \text{on } A_{1,R},
\end{cases}$$

so that $F$ satisfies the reality condition on $A_R$. In order to show that this extension is holomorphic on $A_R$, we use the Schwarz reflection principle: writing

$$F = \begin{pmatrix} a & b \\
-\bar{c} & \bar{b}
\end{pmatrix},$$

the following equations hold on $A_1$:

$$\begin{aligned}
a &= d^* \\
-\bar{b} &= c^*.
\end{aligned}$$

Thus, writing

$$f_1 = a + d, \\
f_2 = i(a - d), \\
f_3 = b - c, \\
f_4 = i(b + c),$$

for all $i = 1, \ldots, 4$, one has $f_i^* = f_i$. Hence, by the Schwarz reflection principle, the functions $f_i$ are holomorphic on $A_R$ and so are the entries of $F$. The map $F$ is then holomorphic on $A_R$. Defining $B = F^{-1}\Phi$ on $A_R$ ends the proof of existence because $B_{|A_1} = \widetilde{B} \in \Lambda_{R,SL_2^C}$.

To prove unicity, write $\Phi = FB = F'B'$ and denote the restrictions of these maps to $A_1$ with a tilde. Because Iwasawa decompostion is unique, $\widetilde{F} = \widetilde{F}'$ and $\widetilde{B} = \widetilde{B}'$. By unicity of the analytic continuation, the holomorphic extension of $\widetilde{B}$ to $D_1$ is the same than the holomorphic extension of $\widetilde{B}'$ to $D_1$. Thus, $F_{|A_{1,R}} = F'_{|A_{1,R}}$, so $F = F'$ and $B = B'$.

Note that we often cannot give an explicit form for the Iwasawa decomposition of a given $\Phi \in \Lambda_{SL_2^C}$.
1.2 The $\mathfrak{su}_2$ model of $\mathbb{R}^3$

In the DPW method, immersions are given in a matrix model. The euclidean space $\mathbb{R}^3$ is thus identified with the Lie algebra $\mathfrak{su}_2$ by

$$x = (x_1, x_2, x_3) \simeq X = \frac{-i}{2} \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix}.$$ 

The canonical basis of $\mathbb{R}^3$ identified as $\mathfrak{su}_2$ is denoted $(e_1, e_2, e_3)$. In this model, the euclidean scalar product and norm are given by

$$\langle x, y \rangle = -2\text{tr}(XY),$$

$$\|x\|^2 = 4 \text{det}(X).$$

Linear isometries are represented by the conjugacy action of $\text{SU}_2$ on $\mathfrak{su}_2$:

$$H \cdot X = HXH^{-1}.$$ 

The kernel of this action is $\{\pm I_2\}$ and $\text{SO}_3 \simeq \text{SU}_2/\{\pm I_2\}$. We recall how to represent affine isometries in the same fashion in section 1.4.

1.3 The recipe

The DPW method takes for input data:

- A Riemann surface $\Sigma$;
- A $\Lambda_{\mathfrak{sl}_2\mathbb{C}}$-valued holomorphic 1-form $\xi = \xi(z, \lambda)$ on $\Sigma$ called “the DPW potential” which extends meromorphically to $\mathcal{D}_1$ with a pole only at $\lambda = 0$, and which must be of the form

$$\xi(z, \lambda) = \sum_{j=-1}^{\infty} \xi_j(z)\lambda^j$$

where each matrix $\xi_j(z)$ depends holomorphically on $z$ and all the entries of $\xi_{-1}(z)$ are zero except for the upper right entry which must never vanish;
- A base point $z_0 \in \Sigma$;
- An initial condition $\Phi_{z_0} \in \Lambda_{\mathfrak{sl}_2\mathbb{C}}$.

Given such data, here are the three steps of the DPW method for constructing CMC-1 surfaces in $\mathbb{R}^3$ (in the untwisted setting):
1. Solve for $\Phi$ the Cauchy problem with parameter $\lambda \in \mathcal{A}_1$:

$$\begin{align*}
\frac{d}{dz} \Phi(z, \lambda) &= \Phi(z, \lambda) \xi(z, \lambda), \\
\Phi(z_0, \lambda) &= \Phi_{z_0}(\lambda).
\end{align*}$$

The solution $\Phi(z, \cdot) \in \Lambda_{SL_2} \subset \mathbb{C}$ is called the “holomorphic frame” of the surface. In general, $\Phi(z, \cdot)$ is only defined on the universal cover $\tilde{\Sigma}$ of $\Sigma$ (see Section 1.6). Note that if $\xi(z, \cdot)$ can be holomorphically extended to $\mathcal{A}_R (R > 1)$, then $\Phi(z, \cdot)$ can also be holomorphically extended to $\mathcal{A}_R$ provided that $\Phi_{z_0}$ is holomorphic on $\mathcal{A}_R$.

2. For all $z \in \tilde{\Sigma}$, Iwasawa decompose $\Phi(z, \lambda) = F(z, \lambda)B(z, \lambda)$. The decomposition is done pointwise in $z$, but $F(z, \lambda)$ and $B(z, \lambda)$ depend real-analytic on $z$. The map $F$ is called the “unitary frame” of the surface.

3. Define $f : \tilde{\Sigma} \to \mathfrak{su}_2$ by the Sym-Bobenko formula:

$$f(z) = \text{Sym}(F) = i \frac{\partial F}{\partial \lambda}(z, 1) F(z, 1)^{-1}.$$ 

The map $f$ is then a conformal CMC-1 immersion whose normal map is given by

$$\mathcal{N}(z) = \frac{-i}{2} F(z, 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F(z, 1)^{-1}.$$ 

(1)

Its metric and Hopf differential are

$$ds = 2 \rho^2 |\xi_{12}||dz|,$$

$$Q = -2 \xi_{12} \xi_{21} dz^2$$

where $\xi_{kl}$ is the $(k,l)$-entry of the matrix $\xi_j(z)$ and $\rho$ is the upper-left entry of $B(z, 0)$.

The theory states that every conformal CMC-1 immersion can be obtained this way but recall that in general, step 2 (Iwasawa decomposition) is not explicit.

As an example, let us look at the Delaunay data

$$\Sigma = \mathbb{C}^*, \quad \xi(z, \lambda) = A(\lambda) z^{-1} dz, \quad z_0 = 1, \quad \Phi_{z_0} = I_2,$$

where

$$A(\lambda) = \begin{pmatrix} 0 & r \lambda^{-1} + s \\ r \lambda + s & 0 \end{pmatrix}, \quad r, s \in \mathbb{R}^*, \quad r + s = \frac{1}{2}.$$
With such data, the axis of the surface is given by \( \{(x, 0, -2r), \ x \in \mathbb{R} \} \) and its weight (as defined in [5]) is \( 8\pi rs \). Thus, the Delaunay surface is an unduloid if \( rs > 0 \) and it is a nodoid if \( rs < 0 \). If \( s = 0 \), then the induced immersion is a round sphere centered at \((0, 0, -1)\). If \( r = 0 \), the immersion degenerates into a point, but has the normal map of a catenoid (see [19]). In this paper, we will introduce a perturbation in the potential \( \xi \) and prove that on a uniform neighbourhood of \( z = 0 \), the induced surface is still embedded if \( rs > 0 \).

1.4 Dressing and isometries

Let \( \xi \) be a DPW potential and \( \Phi \in \Lambda SL_2 \mathbb{C} \) a solution of \( d \Phi = \Phi \xi \). Take a loop \( H \in \Lambda SL_2 \mathbb{C} \) that does not depend on \( z \). Then \( \tilde{\Phi} = H \Phi \) also satisfies \( d \tilde{\Phi} = \tilde{\Phi} \xi \). This procedure is called “dressing \( \tilde{\Phi} \)” and is equivalent to a change of initial condition \( \Phi_{z_0} \) in the DPW data \((\Sigma, \xi, z_0, \Phi_{z_0})\). In general, the effect of dressing on the induced immersion \( f \) is unknown because Iwasawa decomposition is rarely explicit, and that is why the choice of the initial condition \( \Phi_{z_0} \) in the DPW data plays a major role in the construction of surfaces.

However, note that if \( H \in \Lambda SU_2 \), then dressing \( \Phi \) with \( H \) corresponds to applying an affine isometry on \( f \). Indeed, by uniqueness of Iwasawa decomposition,

\[
\tilde{F} = \text{Uni}(\tilde{\Phi}) = HF
\]

and the induced immersion is given by

\[
\tilde{f}(z) = \text{Sym}(HF) = H(1)f(z)H(1)^{-1} + i \frac{\partial H}{\partial \lambda}(1)H(1)^{-1}
\]

This enjoins us to extend the action of section 1.2 to affine isometries by

\[
H(\lambda) \cdot X = H(1)XH(1)^{-1} + i \frac{\partial H}{\partial \lambda}(1)H(1)^{-1}.
\]

Note that \( \Lambda SU_2 \) also acts on the tangent bundle of \( \mathbb{R}^3 \) via:

\[
H \cdot (p, \vec{v}) = (H \cdot p, H(1) \cdot \vec{v}). \tag{2}
\]

This action will be useful to follow the axis of our surfaces: oriented affine lines are generated by pairs \((p, \vec{v})\) and the action of \( \Lambda SU_2 \) on a given oriented affine line corresponds to the action (2) on its generators.
1.5 Gauging

Let \((\Sigma, \xi, z_0, \Phi_{z_0})\) be a set of DPW data with \(d\Phi = \Phi \xi\). Let \(G(z, \lambda)\) be a holomorphic map with respect to \(z \in \Sigma\) such that \(G(z, \cdot) \in \Lambda_{+} SL_2 \mathbb{C}\) (such a map is called a “gauge”). If we define \(\tilde{\Phi} = \Phi G\), then \(\Phi\) and \(\tilde{\Phi}\) give rise to the same immersion \(f\). This operation is called “gauging” and one can retrieve \(\tilde{\Phi}\) by applying the DPW method to the data \((\Sigma, \xi \cdot G, z_0, \Phi_{z_0} G(z_0, \cdot))\) where

\[
\xi \cdot G = G^{-1} \xi G + G^{-1} dG
\]

is the action of gauges on potentials. Gauging often allows us to simplify a DPW potential and an initial condition without changing the immersion.

1.6 The monodromy problem

Since \(\Phi\) is defined as the solution of a Cauchy problem on \(\Sigma\), it is only defined on the universal cover \(\tilde{\Sigma}\) of \(\Sigma\). For any deck transformation \(\tau\) of \(\tilde{\Sigma}\), we define the monodromy \(M_{\tau}(\Phi) \in \Lambda SL_2 \mathbb{C}\) as follow:

\[
\Phi(\tau(z), \lambda) = M_{\tau}(\Phi)(\lambda) \Phi(z, \lambda).
\]

Note that \(M_{\tau}(\Phi)\) does not depend on \(z\). The standard sufficient condition for the immersion \(f\) to be be well-defined on \(\Sigma\) is the following set of equations, called the monodromy problem in \(\mathbb{R}^3\):

\[
\begin{align*}
\mathcal{M}_\tau(\Phi) &\in \Lambda SU_2, \quad (i) \\
\mathcal{M}_\tau(\Phi)(1) &= \pm I_2, \quad (ii) \\
\frac{\partial}{\partial \lambda} \mathcal{M}_\tau(\Phi)(1) &= 0. \quad (iii)
\end{align*}
\]

Indeed, condition \((i)\) implies that \(\mathcal{M}_\tau(\Phi)\) acts as an affine isometry on \(f\), and the other two conditions imply that this isometry is the identity. However, note that condition \((i)\) is not necessary.

As an example, let us compute the monodromy around \(z = 0\) of \(\Phi\) coming from the Delaunay data of section 1.3 \((\mathbb{C}^*, A(\lambda) z^{-1} dz, 1, I_2)\). The solution of the Cauchy problem is \(\Phi(z, \lambda) = z^{A(\lambda)}\) and

\[
\mathcal{M}_\tau(\Phi)(\lambda) = \exp(2i\pi A(\lambda))
\]

\[
= \begin{pmatrix}
\cos(2\pi \mu(\lambda)) & \frac{i \sin(2\pi \mu(\lambda))}{\mu(\lambda)} (r\lambda^{-1} + s) \\
\frac{i \sin(2\pi \mu(\lambda))}{\mu(\lambda)} (r\lambda + s) & \cos(2\pi \mu(\lambda))
\end{pmatrix}
\]
where
\[ \mu(\lambda)^2 = -\det A(\lambda) = (r + s)^2 + rs\lambda^{-1}(\lambda - 1)^2. \]

This is why we suppose that \( r + s = \frac{1}{2} \) and that \( r, s \in \mathbb{R} \): with these conditions, the monodromy problem for \( \Phi \) is solved.

**Remark 1.** In this paper, the Riemann surface \( \Sigma \) will always be a punctured neighbourhood of \( z = 0 \). Thus, all the deck transformations \( \tau \) will be associated to a closed loop around \( z = 0 \) and we will write \( \mathcal{M}(\Phi) \) instead of \( \mathcal{M}_\tau(\Phi) \).

**Remark 2.** Let \( \Phi : \mathbb{C}^* \to \Lambda_{SL_2}\mathbb{C} \) such that \( \mathcal{M}(\Phi) \in \Lambda_{SU_2} \). Let \( \tilde{\Phi} = H(h^*\Phi) \cdot G \) where \( H \in \Lambda_{SL_2}\mathbb{C}, G \) is holomorphic at \( z = 0 \) and \( h \) is a Möbius transformation that leaves \( z = 0 \) invariant. Then
\[ \mathcal{M}(\tilde{\Phi}) = H \mathcal{M}(\Phi) H^{-1}. \]

Thus, if the monodromy problem for \( \Phi \) is solved, a sufficient condition for the monodromy problem for \( \tilde{\Phi} \) to be solved is that \( H \in \Lambda_{SU_2} \).

### 1.7 Perturbed Delaunay DPW data

We take a Delaunay potential as in section 1.3 and we perturb it with a small parameter \( t \geq 0 \) for \( z \) in a small uniform neighbourhood of 0:

**Definition 1** (Perturbed Delaunay potential). Let \( U \subset \mathbb{C} \) be a neighbourhood of 0 and \( R > 1 \). A perturbed Delaunay potential of speed \( \omega \in \mathbb{R}^* \) is a one-parameter family \( \xi_t \) of DPW potentials defined for \( t \in [0, T] \), holomorphic on \( U^* \times A_R \) and of the form
\[ \xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz + tC_t(\lambda)dz + \mathcal{O}(t, z)dz \]
where \( A_t \) is a Delaunay residue of weight \( 8\pi\omega t \) for all \( t \), i.e.
\[ A_t(\lambda) = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix} \]  \hspace{1cm} (3)

where
\[ r + s = \frac{1}{2}, \quad rs = \omega t \] \hspace{1cm} (4)

and
\[ C_t(\lambda) = \begin{pmatrix} c_{11}(t, \lambda) & \lambda^{-1}c_{12}(t, \lambda) \\ c_{21}(t, \lambda) & -c_{11}(t, \lambda) \end{pmatrix} \] \hspace{1cm} (5)

with \( c_{ij}(t, \lambda) = \mathcal{O}(t^0, \lambda^0) \) holomorphic for \( \lambda \in \mathcal{D}_R \) and \( C^1 \) for \( t \) in a neighbourhood of 0. One may call “nodoidal” the cases where \( \omega < 0 \) and “unduloidal” the cases where \( \omega > 0 \).
Note that when $t$ tends to 0, \((4)\) splits into two cases:

- if \((r, s) \xrightarrow{t \to 0} (\frac{1}{2}, 0)\), $\xi_0$ is a spherical potential,
- if \((r, s) \xrightarrow{t \to 0} (0, \frac{1}{2})\), $\xi_0$ is a catenoidal point potential, as explained in \[19\].

**Remark 3.** We will sometimes write

$$A_t(\lambda) = \begin{pmatrix} 0 & \alpha_t(\lambda) \\ \alpha_t^*(\lambda) & 0 \end{pmatrix}$$

where

$$\alpha_t(\lambda) = r\lambda^{-1} + s$$

and diagonalise $A_t = H_tD_tD_t^{-1}$ where

$$H_t(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \pm \alpha_t(\lambda) \\ \mu_t(\lambda) & 1 \end{pmatrix} \in \Lambda SU_2,$$

$$D_t(\lambda) = \begin{pmatrix} \mu_t(\lambda) & 0 \\ 0 & -\mu_t(\lambda) \end{pmatrix}$$

with

$$\mu_t(\lambda)^2 = -\det A_t(\lambda) = (r + s)^2 + rs\lambda^{-1}(\lambda - 1)^2.$$  \((7)\)

**Definition 2** (Holomorphic frame). Let $\xi_t$ be a perturbed Delaunay potential defined on $\hat{U}^* \times \mathcal{A}_R$. A holomorphic frame associated to $\xi_t$ is a map $\Phi_t : \hat{U}^* \times \mathcal{A}_R \to SL_2\mathbb{C}$ such that

\[
\begin{aligned}
&d\Phi_t = \Phi_t \xi_t, \\
&\Phi_t(z, \lambda) \text{ is holomorphic with respect to } \lambda \in \mathcal{A}_R, \\
&\Phi_t \text{ is continuous with respect to } t \geq 0.
\end{aligned}
\]

The theorem we prove in this paper is the following:

**Theorem 3.** Let $\xi_t$ be a perturbed Delaunay potential of speed $\omega$ and $\Phi_t$ a holomorphic frame associated to $\xi_t$ such that $\mathcal{M}(\Phi_t) \in \Lambda SU_2$ and $\Phi_0(1, \lambda) = I_2$. Let $f_t = \text{Sym} (\text{Uni}(\Phi_t))$. Then,

- For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $t < T$,

$$\|f_t(z) - f_t^D(z)\|_{\mathbb{R}^3} \leq Ct|z|^\alpha$$

where $f_t^D$ is a Delaunay immersion of weight $8\pi\omega t$.  

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• If the residue of $\xi_t$ is unduloidal, then there exist $T' > 0$ and $\varepsilon' > 0$ such that for all $t < T'$, $f_t$ is an embedding of $\{0 < |z| < \varepsilon'\}$.

• If $(r, s) \xrightarrow{t \to 0} \left(\frac{1}{2}, 0\right)$, the limit axis as $t$ tends to 0 of $f_t^D$ is the oriented line generated by $(-e_3, -e_1)$.

  If $(r, s) \xrightarrow{t \to 0} (0, \frac{1}{2})$, the limit axis as $t$ tends to 0 of $f_t^D$ is the oriented line generated by $(0, -e_1)$.

Remark 4. We do not have to assume that $1 \in U$ for $\Phi_0$ to be defined at $z = 1$. This only comes from the fact that $\xi_0$ is defined on $\mathbb{C}^*$, which implies that $\Phi_0$ is defined on the universal cover $\tilde{\mathbb{C}}^*$.

2 An application

In practice (as in [18] and [19]), rare are the cases when $\Phi_0(1, \lambda) = I_2$. We give in this section a method to apply Theorem 3 when this condition is not satisfied. In all the section, $\xi_t$ is a perturbed Delaunay potential of speed $\omega$ with

$$(r, s) \xrightarrow{t \to 0} \left(\frac{1}{2}, 0\right)$$

and $\Phi_t$ a holomorphic frame associated to $\xi_t$ such that $\mathcal{M}(\Phi_t) \in \Lambda SU_2$ and $\Phi_0(1, \lambda) = M(\lambda)$ where

$$M(\lambda) = \begin{pmatrix} a & b\lambda^{-1} \\ c\lambda & d \end{pmatrix} \quad (a, b, c, d \in \mathbb{C}). \quad (8)$$

After some simplifications, one can apply Theorem 3 and prove:

Corollary 2. Let $f_t = \text{Sym}(\text{Uni}(\Phi_t))$, then,

• For all $\alpha < 1$ there exist constants $\varepsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \varepsilon$ and $t < T$,

  $$\|f_t(z) - f_t^D(z)\|_{\mathbb{R}^3} \leq Ct|z|^\alpha$$

  where $f_t^D$ is a Delaunay immersion of weight $8\pi \omega t$.

• If the residue of $\xi_t$ is unduloidal, then there exist $T' > 0$ and $\varepsilon' > 0$ such that for all $t < T'$, $f_t$ is an embedding of $\{0 < |z| < \varepsilon'\}$. 

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• The limit axis of $f_t^D$ as $t$ tends to 0 is the oriented line generated by $Q \cdot (0, e_3)$ where

$$Q = \text{Uni} \left[ MH_0 \right]$$

(9)

and $H_t$ is defined in (6).

The method involves gauging, changing coordinates and applying an isometry, and relies on the fact that one can explicitly compute the Iwasawa decomposition of this kind of matrix:

$$
\begin{pmatrix}
a & b \lambda^{-1} \\
c \lambda & d
\end{pmatrix} = \frac{1}{\sqrt{|b|^2 + |d|^2}} \left( \frac{\overline{d}}{-b \lambda} \begin{pmatrix} b \lambda^{-1} \\ -b \lambda \end{pmatrix} \right) \times \frac{1}{\sqrt{|b|^2 + |d|^2}} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)
\begin{pmatrix} |b|^2 + |d|^2 \\ 0 \\
|b|^2 + |d|^2
\end{pmatrix}.
$$

(10)

We first find appropriate gauge and change of coordinates, and we then prove that Theorem 3 implies Corollary 2.

### 2.1 Gauging and changing coordinates

We want to make $\Phi_t$ satisfy the hypotheses of Theorem 3.

**Proposition 1.** There exist a gauge $G$, a change of coordinate $h$ and an isometry $J \in \Lambda SU_2$ such that, denoting

$$\tilde{\Phi}_t = J (h^* \Phi_t) G$$

and

$$\tilde{\xi}_t = (h^* \xi) \cdot G,$$

$\tilde{\xi}_t$ is a perturbed Delaunay potential of speed $\omega$ defined on $V^* \subset U^*$ and $\tilde{\Phi}_t$ is a holomorphic frame associated to $\tilde{\xi}_t$ such that $\mathcal{M}(\tilde{\Phi}_t) \in \Lambda SU_2$ and $\tilde{\Phi}_0(1, \lambda) = I_2$.

In order to prove Proposition 1, we will need the following lemma:

**Lemma 1.** Let $\xi_t$ be a perturbed Delaunay potential of speed $\omega$ defined on $\{z \in U^*\}$ with

$$(r, s) \rightarrow (\frac{1}{2}, 0).$$

(11)

Let $p \in \mathbb{C}$, $q \in \mathbb{C}^*$,

$$h(z) = \frac{z}{pz + q},$$

(12)

$$G(z, \lambda) = \begin{pmatrix}
\sqrt{\frac{|pz + q|}{q(pz + q)}} & 0 \\
\lambda pz \sqrt{\frac{|pz + q|}{q(pz + q)}} & \sqrt{\frac{|pz + q|}{q}}
\end{pmatrix}$$

(13)
\[ \tilde{\xi}_t = (h^*\xi_t) \cdot G. \]  

(14)

Then \( \tilde{\xi}_t \) is still a perturbed Delaunay potential of speed \( \omega \) defined on \( \{ z \in V^* \} \) where \( V \subset U \) is a uniform neighbourhood of \( z = 0 \).

Proof. First, the gauge \( G \) may not be defined on \( U^* \), so we restrict ourselves to \( V^* \) where

\[ V = \left\{ |z| < \frac{p}{q} \right\} \cap U. \]

Then, note that \( G(0, \lambda) = I_2 \), so \( G^{-1}dG \) is holomorphic at \( z = 0 \), and \( h^{-1}dh = z^{-1}dz + O(z^0)dz \). Thus,

\[ \text{Res}_0 \tilde{\xi}_t = A_t. \]  

(15)

Equation (11) implies that

\[ A_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \]

and \( G \) has been chosen to satisfy

\[ \tilde{\xi}_0 = G^{-1}A_0Gh^{-1}dh + G^{-1}dG = A_0z^{-1}dz = \xi_0. \]  

(16)

Finally, we make sure that

\[ \tilde{C}_t = \frac{1}{t} \left[ \tilde{\xi}_t - A_t z^{-1} \right] \]

satisfies Equation (5) by computing

\[ \frac{h^*\xi_t}{dz} = A_t \left( z^{-1} + \frac{p}{q} + O(z) \right) + tC_t \left( \frac{1}{q} + O(z) \right) + O(z), \]

\[ \frac{\tilde{\xi}_t}{dz} = A_t z^{-1} + \left[ A_t, \frac{dG(0)}{dz} \right] - \frac{p}{q} A_t + \frac{t}{q} C_t + \frac{dG(0)}{dz} + O(z), \]

and

\[ \frac{s}{t} \left( \lambda - 1 \right) \frac{p}{q} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} + \frac{1}{q} C_t. \]  

(17)

Thus, \( \tilde{C}_t \) satisfies Equation (5) and its entries are \( C^1 \) for \( t \) in a neighbourhood of 0.

Hence, Equations (15), (16) and (17) show that

\[ \tilde{\xi}_t(z, \lambda) = A_t z^{-1}dz + t\tilde{C}_t dz + O(t, z)dz \]

is a perturbed Delaunay potential of speed \( \omega \).
We can now prove Proposition 1: let \( h \) and \( G \) as in Lemma 1. We look forward to finding \( p \in \mathbb{C} \) and \( q \in \mathbb{C}^* \) such that
\[
\Phi_0 (h(1), \lambda) G (1, \lambda) \in \Lambda SU_2. \tag{18}
\]
We note that \( \Phi_0 = M z^{A_0} \) and use Remark 3 to compute
\[
\Phi_0 (h(1), \lambda) G (1, \lambda) = M(\lambda) H_0(\lambda) \left( \frac{\lambda p}{\sqrt{q}} \quad 0 \right) H_0(\lambda)^{-1}.
\]
Equation (10) allows us to write
\[
\text{Pos} (MH_0) = \begin{pmatrix} \rho & 0 \\ \lambda \mu & \rho^{-1} \end{pmatrix}
\]
where \( \rho \in \mathbb{R}_+^* \) and \( \mu \in \mathbb{C} \). With this notation we set
\[
\begin{cases}
  p = -\rho \mu, \\
  q = \rho^2,
\end{cases}
\]
so that equation (18) is satisfied with \( \Phi_0 (h(1), \lambda) G (1, \lambda) = Q(\lambda) H_0(\lambda)^{-1} \in \Lambda SU_2 \) (see Equation (9)). Setting
\[
J = H_0 Q^{-1} \in \Lambda SU_2, \tag{19}
\]
the gauged frame \( \tilde{\Phi}_t \) is defined on \( \tilde{V}^* \) and satisfies the hypotheses of Theorem 3 (see Remark 4 for the monodromy):
\[
\begin{cases}
  d\tilde{\Phi}_t = \tilde{\Phi}_t \xi_t, \\
  \mathcal{M}(\tilde{\Phi}_t) \in \Lambda SU_2, \\
  \tilde{\Phi}_0(1, \lambda) = I_2.
\end{cases}
\]

2.2 Theorem 3 implies Corollary 2

Using Proposition 1, one can construct a holomorphic frame \( \tilde{\Phi}_t \) satisfying the hypotheses of Theorem 3. Let \( \tilde{f}_t = \text{Sym}(\text{Uni}(\tilde{\Phi}_t)) \) and \( f_t = \text{Sym}(\text{Uni}(\Phi_t)) \). Then
\[
f_t(z) = J^{-1} \cdot \tilde{f}_t (h^{-1}(z)) \tag{20}
\]
Suppose that Theorem 3 is true and apply it to \( \tilde{\Phi}_t \). Take \( \alpha < 1 \) and write
\[
||\tilde{f}_t(z) - \tilde{f}_t^{\text{mp}}(z)||_{\mathbb{R}^3} \leq C t |z|^\alpha.
\]
Then,
• Let
\[ f_t^D = J^{-1} \cdot \tilde{f}_t^D \left( h^{-1}(z) \right). \]
This map is a Delaunay immersion of weight $8\pi \omega t$ because so is $\tilde{f}_t^D$. Moreover, for all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $t < T$,
\[ \| f_t(z) - f_t^D(z) \|_{\mathbb{R}^3} = \left\| J^{-1} \cdot \tilde{f}_t \left( h^{-1}(z) \right) - J^{-1} \cdot \tilde{f}_t^D \left( h^{-1}(z) \right) \right\|_{\mathbb{R}^3} \]
\[ \leq \tilde{C} t |h^{-1}(z)|^\alpha \]
\[ \leq C t |z|^\alpha \]

• If the residue of $\xi_t$ is unduloidal, then so is the residue of $\tilde{\xi}_t = \tilde{\Phi}_t^{-1} d\tilde{\Phi}_t$. According to Theorem 3, $\tilde{f}_t$ is an embedding and by Equation (20) so is $f_t$.

• The limit axis as $t$ tends to 0 of $\tilde{f}_t^D$ is generated by $(-e_3, -e_1^*)$ and
\[ J^{-1} \cdot (-e_3, -e_1^*) = Q \cdot (H_0^{-1} \cdot (-e_3, -e_1^*)) \]
\[ = Q \cdot (-e_3, e_3^*) \]
\[ \simeq Q \cdot (0, e_3^*). \]
and the limit axis as $t$ tends to 0 of $f_t^D$ is then generated by $Q \cdot (0, e_3^*)$.

### 3 The $z^AP$ form of $\Phi_t$

Let us start the proof of Theorem 3: let $\xi_t$ be a perturbed Delaunay potential of speed $\omega$ and $\Phi_t$ a holomorphic frame associated to $\xi_t$ such that $\mathcal{M}(\Phi_t) \in \Lambda SU_2$ and $\Phi_0(1, \lambda) = I_2$.

In this section, we want to apply the Fröbenius method and write $\Phi_t$ in a $z^AP$ form. Unfortunately, the underlying Fuchsian system seems to admit resonance points. Our goal is to avoid them and to gain an order of convergence in the matrix $P$ of the $z^AP$ form. We will obtain the following result:

**Proposition 2.** There exist a change of coordinate $h_t$ and a gauge $G_t$ such that, denoting
\[ \tilde{\Phi}_t = h_t^* (\Phi_t G_t) \]
and
\[ \tilde{\xi}_t = h_t^* (\xi_t \cdot G_t), \]
\( \tilde{\xi}_t \) is a perturbed Delaunay potential of speed \( \omega \) and \( \tilde{\Phi}_t \) is a holomorphic frame associated to \( \tilde{\xi}_t \) such that \( \mathcal{M}(\tilde{\Phi}_t) \in \Lambda SU_2 \) and \( \tilde{\Phi}_0(1, \lambda) = I_2 \). Moreover,

\[
\tilde{\Phi}_t(z, \lambda) = \tilde{M}_t(\lambda) z^{A_t(\lambda)} \tilde{P}_t(z, \lambda)
\]

where \( \tilde{M}_t, \tilde{P}_t \in \Lambda SL_2 \mathbb{C} \) are holomorphic on \( \mathcal{A}_R \) for all \( t \geq 0 \) and \( \tilde{P}_t : V \rightarrow \Lambda SL_2 \mathbb{C} \) is holomorphic with respect to \( z \in V \subset U \) and satisfies \( \tilde{P}_t(z, \lambda) = I_2 + \mathcal{O}(t, z^2) \).

### 3.1 Extending to the resonance points

In this section, we use the Fröbenius method to write \( \Phi_t \) in a \( z^A P \) form, and extend this form to the resonance points. We will thus prove:

**Proposition 3.** There exist \( M_t, P_t \in \Lambda SL_2 \mathbb{C} \) holomorphic on \( \mathcal{A}_R \) for all \( t \in [0, T] \) such that \( P_t : U \rightarrow \Lambda SL_2 \mathbb{C} \) is holomorphic with respect to \( z \in U \) and satisfies \( P_t(z, \lambda) = I_2 + \mathcal{O}(t^0, z) \), and

\[
\Phi_t(z, \lambda) = M_t(\lambda) z^{A_t(\lambda)} P_t(z, \lambda).
\]

Let us first recall the Fröbenius method:

**Proposition 4 (Fröbenius method).** Let

\[
\xi(z) = A z^{-1} dz + \sum_{k=0}^{+\infty} C_k z^k dz.
\]

Looking forward to solve the equation \( d\Phi = \Phi \xi \) on a neighbourhood of \( z = 0 \), one can use the Ansatz

\[
\begin{cases}
\Phi(z) = M z^A P(z), \\
P(z) = \sum_{k=0}^{\infty} P_k z^k,
\end{cases}
\]

where \( P \) is holomorphic and satisfies \( P(0) = I_2 \). Then, such a \( \Phi \) is solution if \( P \) satisfies the recurrence equation

\[
\begin{cases}
P_0 = I_2, \\
\mathcal{L}_{k+1}(P_{k+1}) = \sum_{i+j=k} P_i C_j, \quad k \in \mathbb{N},
\end{cases}
\]

where

\[
\mathcal{L}_n(X) = [A, X] + nX
\]

is a linear map for all \( n \in \mathbb{N}^* \). The system is called “resonnant” whenever some \( \mathcal{L}_n \) is not invertible.
One might ask what are the resonance points in our case. The answer is given by the following lemma:

**Lemma 2.** Let $\xi_t$ be a perturbed Delaunay potential of speed $\omega$ and $\Phi_t$ a holomorphic frame associated to $\xi_t$. Let $\{L_{t,n}\}_{n \in \mathbb{N}^*}$ be the linear maps associated to $(\xi_t, \Phi_t)$ via the Fröbenius method, as defined in (23). There exist $T > 0$ and $R > 1$ such that:

- For all $n \geq 2$, $L_{t,n}$ is invertible on $(t, \lambda) \in [0, T] \times \mathcal{D}_R^*$. 
- For $n = 1$, $L_{t,1}$ is invertible on $(t, \lambda) \in (0, T] \times \mathcal{D}_R^* \setminus \{1\}$.

In every case, its inverse is given by

$$
L_{t,n}^{-1}(X) = -n \left( \frac{-n}{n^2(n^2 - 4\mu_t^2)} \left( (2\mu_t^2 - n^2)X + n[A_t, X] + 2A_tA_t \right) \right)
$$

(24)

where $\mu_t$ is defined by (7).

**Proof.** As in Remark 3, we write

$$A_t = \begin{pmatrix} 0 & \alpha_t \\ \alpha_t^* & 0 \end{pmatrix}$$

and work in the canonical basis $(E_{11}, E_{12}, E_{21}, E_{22})$ of $\mathcal{M}_2(\mathbb{C})$. In this basis, the map $L_{t,n}$ is represented by the matrix

$$\text{Mat}(L_{t,n}) = \begin{pmatrix} n & -\alpha_t^* & \alpha_t & 0 \\ -\alpha_t & n & 0 & \alpha_t \\ \alpha_t^* & 0 & n & -\alpha_t^* \\ 0 & \alpha_t^* & -\alpha_t & n \end{pmatrix}$$

whose determinant is

$$\det(\text{Mat}(L_{t,n})) = n^2 \left( n^2 - 4\mu_t(\lambda)^2 \right).$$

(25)

Recalling Equation (4), we have

$$\mu_t(\lambda)^2 = \frac{1}{4} + \omega t \lambda^{-1} (\lambda - 1)^2$$

and with $t > 0$ and $R - 1 > 0$ small enough, $L_{t,1}$ is invertible on $(t, \lambda) \in (0, T] \times \mathcal{D}_R^* \setminus \{1\}$ and $L_{t,n}$ is invertible on $(t, \lambda) \in [0, T] \times \mathcal{A}_R$ for all $n \geq 2$. The inverse matrix is

$$\text{Mat}(L_{t,n})^{-1} = \frac{-1}{n(n^2 - 4\mu_t^2)} \begin{pmatrix} 2\mu_t^2 - n^2 & -\alpha_t^*n & \alpha_tn & 2\mu_t^2 \\ -\alpha_tn & 2\mu_t^2 - n^2 & 2\alpha_t^2 & \alpha_tn \\ \alpha_t^*n & 2\alpha_t & 2\mu_t^2 - n^2 & -\alpha_t^*n \\ 2\mu_t^2 & \alpha_t^*n & -\alpha_tn & 2\mu_t^2 - n^2 \end{pmatrix}$$

which is the matrix of the map given in (24).
From now on, we assume that $T$ and $R$ are chosen for $L_{t,n}$ to be invertible on $(t, \lambda) \in (0, T] \times A_R \backslash \{1\}$ for all $n \in \mathbb{N}^*$.

**Remark 5.** If we use the Ansatz given by the Fröbenius method and write

$$\Phi_t(z, \lambda) = M_t(\lambda) z^{A_t(\lambda)} P_t(z, \lambda)$$

where

$$P_t(z, \lambda) = \sum_{k=0}^{\infty} P_{t,k}(\lambda) z^k,$$

note that the resonance points only occur in the computation of $P_{t,1}(\lambda)$ because $L_{t,n}$ is invertible on $(t, \lambda) \in [0, T] \times A_R$ for all $n \geq 2$. Thus, we only need to extend $P_{t,1}(\lambda)$ at $t = 0$ and $\lambda = 1$ to extend the $z^A P$ form of $\Phi_t$. According to (22),

$$P_{t,1}(\lambda) = L_{t,1}^{-1}(tC_t(\lambda))$$

and the form of $\det L_{t,1}$ shows that $P_{t,1}$ has at most a pole of order 2 at $\lambda = 1$. Moreover, $\det L_{t,1} = O(t)$ and $tC_t = O(t)$, so we already know that $P_t$ (and as a consequence, $M_t$) extends to $t = 0$.

It remains to extend the $z^A P$ form (26) to $\lambda = 1$. To do this, we adapt the techniques used in Lemma 2.5 of [13] to prove the following unitary $\times$ commutator lemma:

**Lemma 3.** Let $M : A_R \backslash \{1\} \to SL_2 \mathbb{C}$ holomorphic on $A_R \backslash \{1\}$ with at most a pole at $\lambda = 1$. Let $t > 0$, $Q = \exp (2i\pi A_t) \in \Lambda SU_2$ and suppose that for all $\lambda \in A_1 \backslash \{1\}$,

$$M Q M^{-1} \in SU_2.$$  \hfill (29)

Then there exist $U \in \Lambda SU_2$, $R' \in (1, R]$ and $K : A_{R'} \backslash \{1\} \to SL_2 \mathbb{C}$ holomorphic on $A_{R'} \backslash \{1\}$ such that

$$\left\{ \begin{array}{l} M = UK \\ [A_t, K] = 0. \end{array} \right.$$  

**Proof.** We first define

$$V = M + (M^{-1})^*$$

holomorphic on $A_R \backslash \{1\}$ with at most a pole at $\lambda = 1$. Then

$$\det V = 2 + \sum_{i,j=1}^{2} M_{ij} M_{ij}^*$$

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because \( \det M = 1 \), and \( \det V > 0 \) on \( \mathcal{A}_1 \setminus \{1\} \). Let \( \mathcal{A} \subset \mathcal{A}_R \) be a small neighbourhood of \( \mathcal{A}_1 \) and define on \( \mathcal{A} \setminus \{1\} \):

\[
U = (\det V)^{-1/2} V
\]

so that \( UU^* = I_2 \) on \( \mathcal{A}_1 \setminus \{1\} \). The map \( U \) is then bounded on \( \mathcal{A}_1 \setminus \{1\} \) and holomorphic on \( \mathcal{A} \setminus \{1\} \) with at most a pole at \( \lambda = 1 \). We can thus extend \( U \) to \( \mathcal{A} \) and \( U \in \Lambda SU_2 \).

Let \( K = U^{-1}M \) defined on \( \mathcal{A} \setminus \{1\} \). On \( \mathcal{A} \setminus \{1\} \),

\[
QM^*M = (MQ^{-1})^* M
= (MQ^{-1}M^{-1}M)^* M
= M^* (MQ^{-1}M^{-1})^{-1} M
= M^* M Q,
\]

the first equality coming from the fact that \( Q \in \Lambda SU_2 \) and (29) giving the third. Moreover,

\[
[Q, M^* M] = 0 \implies [Q, (M^{-1} + M^*) M] = 0
\implies [Q, V^* M] = 0
\implies [Q, K] = 0.
\]

Hence the map \( \lambda \mapsto [Q(\lambda), K(\lambda)] \) is holomorphic on \( \mathcal{A} \setminus \{1\} \) and vanishes on \( \mathcal{A}_1 \setminus \{1\} \). Thus, for all \( \lambda \in \mathcal{A} \setminus \{1\} \),

\[
[Q(\lambda), K(\lambda)] = 0. \tag{31}
\]

Writing

\[
Q = \cos(2\pi \mu_1) I_2 + \frac{\sin(2\pi \mu_1)}{\mu_1} A_t
\]

where \( \mu_t \) is defined by (1), Equation (31) implies that \( [A_t, K] = 0 \) wherever \( \mu_t(\lambda)^2 \neq \frac{1}{4} \). Using (4), \( [A_t(\lambda), K(\lambda)] = 0 \) for all \((t, \lambda) \in (0, T] \times \mathcal{A} \setminus \{1\} \). Choose \( R' \in (1, R] \) such that \( \mathcal{A}_{R'} \subset \mathcal{A} \) to end the proof.

Without loss of generality, we suppose from now on that \( R \) is small enough for \( R' \) to equal \( R \). We can now extend the \( z^AP \) form of \( \Phi_t \) to \( \lambda = 1 \). For \( t > 0 \) and \( \lambda \in \mathcal{A}_1 \setminus \{1\} \), use Lemma 3 to write

\[
\Phi_t(z, \lambda) = U_t(\lambda)z^{A_t(\lambda)}K_t(\lambda)P_t(z, \lambda).
\]

Let \( \epsilon > 0 \), \( \mathcal{S}_\epsilon = \{ z \in U : |z| = \epsilon \} \) and \( \mathbb{D}_\epsilon = \{ z \in U : |z| \leq \epsilon \} \). On \( \mathcal{S}_\epsilon \times \mathcal{A}_1 \setminus \{1\} \), \( \Phi_t \) and \( z^{A_t} \) are bounded. Then the map \((z, \lambda) \mapsto K_tP_t \) is bounded on \( \mathcal{S}_\epsilon \times \mathcal{A}_1 \setminus \{1\} \) and holomorphic on \( \mathbb{D}_\epsilon \times \mathcal{A}_1 \setminus \{1\} \), so it is bounded on \( \mathbb{D}_\epsilon \times \mathcal{A}_1 \setminus \{1\} \). But \( P_t(0, \lambda) = I_2 \), so \( K_t \) is bounded on \( \mathcal{A}_1 \setminus \{1\} \). Thus, \( P_t \) is bounded on \( \mathbb{D}_\epsilon \times \mathcal{A}_1 \setminus \{1\} \). But \( P_t \) is holomorphic on \( \mathbb{D}_\epsilon \times \mathcal{A}_{R'} \setminus \{1\} \) with at most a pole at \( \lambda = 1 \), so \( P_t \) is holomorphic on \( \mathbb{D}_\epsilon \times \mathcal{A}_R \) and \( M_t \) is holomorphic on \( \mathcal{A}_R \). This ends the proof of Proposition 3.
3.2 A property of $\xi_t$

The fact that there exists a holomorphic frame $\Phi_t$ associated to $\xi_t$ such that $M(\Phi_t) \in \Lambda SU_2$ and $\Phi_0(1, \lambda) = I_2$ gives us an information on the potential $\xi_t$. Namely, 

$$p_t = \frac{t}{2} \left( \frac{c_{12}(t, 0)}{r} + \frac{c_{21}(t, 0)}{s} \right) = O(t),$$

where $c_{ij}$ are defined in Equation (5). We will need the following lemma:

**Lemma 4.** Let $\xi_t(z, \lambda)$ be a perturbed Delaunay potential and $\Phi_t$ a holomorphic frame associated to $\xi_t$ such that $M(\Phi_t) \in \Lambda SU_2$ and $\Phi_0(1, \lambda) = I_2$. Then

$$i \text{Res}_0 (z^{A_0} C_0 z^{-A_0}) \in \Lambda su_2.$$ 

**Proof.** Note that $\Phi_0(1, \lambda) = I_2$ implies that $\Phi_0(z, \lambda) = z^{A_0(\lambda)}$, and thus $M(\Phi_0) = -I_2$. Let $\gamma \subset U^*$ be a closed loop around 0. Apply Proposition 7 of Appendix A to get ($X'$ denotes the derivative of $X$ at $t = 0$)

$$M(\Phi_t)' = -\int_{\gamma} z^{A_0} \xi' z^{-A_0},$$

$$= -\int_{\gamma} z^{A_0} (A' z^{-1}) z^{-A_0} \text{d}z - \int_{\gamma} z^{A_0} (C_0 + O(z)) z^{-A_0} \text{d}z,$$

$$= M(z^{A_1})' - \int_{\gamma} z^{A_0} (C_0 + O(z)) z^{-A_0} \text{d}z,$$

$$= M(z^{A_1})' - 2i\pi \text{Res}_0 (z^{A_0} C_0 z^{-A_0}).$$

Finally, use the fact that $M(\Phi_t), M(z^{A_1}) \in \Lambda SU_2$ and that $M(\Phi_0) = M(z^{A_0}) = -I_2$ to ensure that $M(\Phi_t)', M(z^{A_1})' \in \Lambda su_2$, so that $2i\pi \text{Res}_0 (z^{A_0} C_0 z^{-A_0}) \in \Lambda su_2$. 

In order to show that $p_t = O(t)$, recall Remark 3 and compute

$$H_0^{-1} \text{Res}_0 (z^{A_0} C_0 z^{-A_0}) H_0 = \text{Res}_0 (z^{D_0} H_0^{-1} C_0 H_0 z^{-D_0}) = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},$$

where

$$c(\lambda) = -4\alpha_0^*(\lambda) c_{11}(0, \lambda) - 4\lambda^{-1} (\alpha_0^*(\lambda))^2 c_{12}(0, \lambda) + c_{21}(0, \lambda).$$

Then, according to Lemma 4 for all $\lambda \in D_R^*$,

$$c(\lambda) = 0$$

(33)
because $H_0 \in \Lambda SU_2$.

If $(r, s) \xrightarrow{t \to 0} (\frac{1}{2}, 0)$, then $\alpha_0^*(\lambda) = \lambda/2$ and

$$c(\lambda) = -2\lambda c_{11}(0, \lambda) - \lambda c_{12}(0, \lambda) + c_{21}(0, \lambda)$$

so $c_{21}(0, 0) = 0$, $c_{21}(t, 0) = \mathcal{O}(t)$ and $p_t = \mathcal{O}(t)$.

If $(r, s) \xrightarrow{t \to 0} (0, \frac{1}{2})$, then $\alpha_0^*(\lambda) = 1/2$ and

$$c(\lambda) = -2\lambda c_{11}(0, \lambda) - \lambda^{-1} c_{12}(0, \lambda) + c_{21}(0, \lambda)$$

so $c_{12}(0, 0) = 0$, $c_{12}(t, 0) = \mathcal{O}(t)$ and $p_t = \mathcal{O}(t)$.

### 3.3 Gaining an order of convergence

We can now prove Proposition 2 by following the method used in Section 2.2 of [7]:

gauging the potential. The gauge we will use is of the following form:

$$G_t(z, \lambda) = \exp \left(g_t(\lambda)z\right)$$

which is an admissible gauge provided that $g_t \in \Lambda^+ \mathfrak{sl}_2 \mathbb{C}$. This is why we need the following lemma:

**Lemma 5.** Let

$$g_t(\lambda) = p_t A_t(\lambda) - P_{t,1}(\lambda)$$

where $P_{t,1}$ is defined in Equation (28). Then $g_t \in \Lambda^+ \mathfrak{sl}_2 \mathbb{C}$ is holomorphic on $\mathcal{D}_R$ and $g_t = \mathcal{O}(t)$.

**Proof.** Use Equation (28) and Lemma 2 to compute

$$P_{t,1}(\lambda) = \mathcal{L}^{-1}_{t,1}(t C_t(\lambda))$$

$$= \lambda^{-1} \frac{t}{2} \begin{pmatrix} 0 & c_{12}(t, 0) + \frac{r}{s} c_{21}(t, 0) \\ 0 & 0 \end{pmatrix} + \lambda^0 \frac{t}{2} \begin{pmatrix} c_{21}(t, 0) & * \\ * & c_{12}(t, 0) \end{pmatrix} + \mathcal{O}(\lambda).$$

Thus, for all $\lambda \in \mathcal{D}_R$,

$$g_t(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & rp_t - \frac{t}{2} \left(c_{12}(t, 0) + \frac{r}{s} c_{21}(t, 0)\right) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} sp_t - \frac{t}{2} \left(c_{21}(t, 0) + \frac{r}{s} c_{12}(t, 0)\right) & * \\ * & \end{pmatrix} + \mathcal{O}(\lambda).$$

By definition of $p_t$, $g_t \in \Lambda^+ \mathfrak{sl}_2 \mathbb{C}$ is holomorphic in $\mathcal{D}_R$.

A straightforward calculation using Equation (28) and Lemma 2 shows that

$$g_0(\lambda) = p_0 A_0(\lambda) - P_{0,1}(\lambda) = \frac{\lambda}{4 \omega (\lambda - 1)^2} H_0(\lambda) \begin{pmatrix} 0 & 0 \\ c(\lambda) & 0 \end{pmatrix} H_0(\lambda)^{-1} = 0$$

because of Equation (33). Thus, $g_t = \mathcal{O}(t)$.  

\[ \square \]
Let $G_t$ be the gauge defined by (34). Then the gauged potential has the form
\[
\xi_t \cdot G_t(z, \lambda) = A_t(\lambda)z^{-1}dz + ([A_t(\lambda), g_t(\lambda)] + g_t(\lambda) + tC_t(\lambda)) dz + O(t, z)dz + O(g_t^2 z)dz
\]
\[
= A_t(\lambda)z^{-1}dz + (L_{t,1}(g_t(\lambda)) + tC_t(\lambda)) dz + O(t, z)dz
\]
\[
= A_t(\lambda)z^{-1}dz + p_t A_t(\lambda) dz + O(t, z)dz,
\]
because of Equation (28). This gauge has been chosen to fit with the following change of coordinate:

\[
h_t(z) = \frac{z}{1 + p_t z}.
\]

(35)

The resulting potential (defined in Proposition 2) is then

\[
\tilde{\xi}_t(z, \lambda) = A_t(\lambda)z^{-1}dz + O(t, z)dz
\]

(36)

because $p_t = O(t)$. Apply the Fröbenius method to $\tilde{\xi}_t$ to obtain (21) and choose

\[
V = U \cap \bigcap_{t \in [0, T]} \{ z \in \mathbb{C} : |z| < |p_t|^{-1} \}
\]

to end the proof of Proposition 2.

Use the same techniques as in Section 2.2 to show that one can suppose that $\Phi_t$ is of the form given by Proposition 2 in order to prove Theorem 3.

4 Convergence of immersions

In this section, we prove the first and third points of our theorem:

Proposition 5.

• For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $t < T$,

\[
\|f_t(z) - f_t^D(z)\|_{\mathbb{R}^3} \leq Ct|z|^\alpha
\]

(37)

where $f_t^D$ is a Delaunay immersion of weight $8\pi \omega t$.

• If $(r, s) \to (\frac{1}{2}, 0)$, the limit axis as $t$ tends to 0 of $f_t^D$ is generated by $(-e_3, -\vec{e}_1)$.

If $(r, s) \to (0, \frac{1}{2})$, the limit axis as $t$ tends to 0 of $f_t^D$ is generated by $(0, -\vec{e}_1)$.
In the end, we want to compare \( \Phi_t(z, \lambda) = M_t(\lambda)z^{A_t(\lambda)}(I_2 + O(t, z^2)) \) to
\[
\Phi^D_t(z, \lambda) = M_t(\lambda)z^{A_t(\lambda)}(I_2 + O(t, z^2)).
\] (38)

We will denote
\[
F^D_t = \text{Uni}(\Phi^D_t)
\] (39)
and
\[
f^D_t = \text{Sym}(F^D_t).
\] (40)

We first want to make sure that \( \Phi^D_t \) induces a Delaunay surface for all \( t > 0 \). It is the purpose of the following lemma.

**Lemma 6.** Let \( A_t \) be a Delaunay residue of weight \( 8\pi\omega t \), \( t > 0 \) and \( M \in \Lambda_{\text{SL}_2 \mathbb{C}} \) holomorphic on \( A_R \). Let \( \Phi^D = Mz^{A_t} \).

If \( M(\Phi^D) \in \Lambda_{\text{SU}_2}, \) then \( \Phi^D \) induces a Delaunay surface of weight \( 8\pi\omega t \).

**Proof.** Using lemma [3] we write
\[
\Phi^D = Uz^{A_t}K
\]
with \( U \in \Lambda_{\text{SU}_2} \) and \( K \in \Lambda_{\text{SL}_2 \mathbb{C}} \) holomorphic on \( A_R \). Let \( \theta \in [0, 2\pi] \) and
\[
\tilde{R}_\theta = (e^{i\theta}z)^{A_t}z^{-A_t} = e^{i\theta A_t}.
\] (41)

A study of the Delaunay holomorphic frame \( \tilde{\Phi} = z^{A_t} \) shows that \( \tilde{R}_\theta \in \Lambda_{\text{SU}_2} \) represents a rotation of angle \( \theta \) around the axis \( \tilde{L} = (-2re_3, e_1^*) \). Let us define
\[
R_\theta(\lambda) = \Phi^D(e^{i\theta}z, \lambda)\Phi^D(z, \lambda)^{-1} = U(\lambda)\tilde{R}_\theta(\lambda)U(\lambda)^{-1}.
\]

Then \( R_\theta \in \Lambda_{\text{SU}_2} \) represents a rotation of angle \( \theta \) around the axis \( L = U \cdot \tilde{L} \). Thus, the holomorphic frame \( \Phi^D \) induces an immersion \( f^D : \mathbb{C}^* \rightarrow \mathbb{R}^3 \) such that \( f(\mathbb{C}^*) \) is a non-zero CMC surface of revolution, i.e. a Delaunay surface.

Denoting by \( \tilde{f} \) the Delaunay immersion induced by \( \tilde{\Phi} \), we note that \( (\Phi^D)^{-1}d\Phi^D = \tilde{\Phi}^{-1}d\tilde{\Phi} \), so \( f^D \) and \( \tilde{f} \) have the same Hopf differential. As a consequence, \( f^D(\mathbb{C}^*) \) and \( \tilde{f}(\mathbb{C}^*) \) are Delaunay surfaces of the same weight \( 8\pi\omega t \).

**Remark 6.** The proof of Lemma 6 implies that there exists a rigid motion \( \phi \) of \( \mathbb{R}^3 \) such that \( \phi \circ f^D \) has the following parametrisation:
\[
\phi \circ f^D : \Sigma \ni z = re^{i\theta} \quad \mapsto \quad \mathbb{R}^3 \ni (r, \sigma(r)\cos \theta, \sigma(r)\sin \theta)
\]
where \( \sigma \) is the profile curve of the surface.
Let us compare the asymptotic behaviours of the unitary parts of $\Phi_t$ and $\Phi_t^D$ for $\lambda \in A_1$ using, as in [7], a Cauchy formula:

**Lemma 7.** For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $t < T$,

$$\left\| (F_t^D)^{-1} F_t - I_2 \right\|_{A_1} \leq C t |z|^\alpha$$  \hspace{1cm} (42)

and

$$\left\| \frac{\partial}{\partial \lambda} \right[ (F_t^D)^{-1} F_t \right]\right\|_{A_1} \leq C t |z|^\alpha.$$  \hspace{1cm} (43)

**Proof.** The first step is to compute the asymptotic behaviour of the positive part $B_t^D$ of $\Phi_t^D$. We first compute the asymptotic behaviour of $\Phi_t^D$ for $|z| < 1$: recalling Remark 3

$$\left\| \Phi_t^D(z, \lambda) \right\|_{A_R} = \left\| M_t(\lambda) z A_t(\lambda) \right\|_{A_R}$$

$$\leq \left\| M_t(\lambda) \right\|_{A_R} \left\| H_t(\lambda) \right\|_{A_R} |z|^{-\left\| D_t(\lambda) \right\|_{A_R}}$$

$$\leq \left( \left\| M_0(\lambda) \right\|_{A_R} + o(t) \right) (1 + o(|R - 1|)) \frac{1}{2} |z|^{-\frac{1}{2} + o(t)}.$$

Thus, for all $\alpha < 1$, there exists $T > 0$ and $C_1 = C_1(T) > 1$ such that for all $t < T$,

$$\left\| \Phi_t^D(z, \lambda) \right\|_{A_R} \leq C_1 |z|^{-\frac{1}{2} - \frac{1-\alpha}{4}}.$$  \hspace{1cm} (44)

We then compute the asymptotic behaviour of $F_t^D$: using Appendix B

$$\left\| F_t^D(z, \lambda) \right\|_{A_R} \leq \left\| F_t^D(1, \lambda) \right\|_{A_R} \times \exp \left( \frac{1}{2} \int_1^1 |ds_t| \right)$$

where $ds_t$ is the metric of $f_t^D$. Recalling Remark 4 and the fact that $f_t^D$ is a conformal immersion, we get

$$ds_t^2 = 2 \frac{\sigma_t^2}{|z|^2} |dz|^2$$

where $\sigma_t$ is the profile curve of $f_t^D$. This curve is uniformly bounded because it is the profile curve of a Delaunay surface, so

$$\int_1^1 |ds_t| \leq -C_2 \log |z|.$$

Thus, for $R > 1$ small enough, there exists $C_3 \geq 1$ such that

$$\left\| F_t^D(z, \lambda) \right\|_{A_R} \leq C_3 |z|^{-\frac{1-\alpha}{4}}.$$  \hspace{1cm} (45)
We can now compute the asymptotic behaviour of $B^D_t$: for all $\alpha < 1$ there exist $T > 0$, $R > 1$ and $C_4 \geq 1$ such that

$$\| B^D_t (z, \lambda) \|_{A_R} \leq C_4 |z|^{\alpha - 1}. $$

We then write

$$\begin{align*}
\tilde{\Phi}_t &= \left( (F^D_t)^{-1} F_t \right) \times \left( B_t (B^D_t)^{-1} \right) = B^D_t (\Phi^D_t)^{-1} \Phi_t (B^D_t)^{-1} \\
&= \tilde{F}_t \times \tilde{B}_t
\end{align*}$$

with $\tilde{F}_t \in \Lambda SU_2$ and $\tilde{B}_t \in \Lambda^R SL_2 \mathbb{C}$ and thus have

$$\begin{align*}
\| \tilde{\Phi}_t (z, \lambda) - I_2 \|_{A_R} &= \left\| B^D_t (z, \lambda) \left( P_t (z, \lambda) - I_2 \right) \left( B^D_t (z, \lambda) \right)^{-1} \right\|_{A_R} \\
&\leq \| B^D_t (z, \lambda) \|^2_{A_R} O(t, |z|^2) \\
&\leq Ct |z|^\alpha.
\end{align*}$$

Let $n_k$ denotes the seminorms

$$n_k(X) = \sum_{j=0}^k \left\| \frac{\partial^k X}{\partial \lambda^k} \right\|_{A_1}. \quad (45)$$

Apply Cauchy formula with $\lambda \in \partial A_R$ to get

$$n_k \left( \tilde{\Phi}_t - I_2 \right) \leq c_k \sum_{j=0}^k t |z|^\alpha, \; \forall k \in \mathbb{N}$$

where $c_k > 0$ are uniform constants. But $\text{Uni}(\tilde{\Phi}_t) = \tilde{F}_t = (F^D_t)^{-1} F_t$ and Iwasawa decomposition is a $C^1$-diffeomorphism, so $n_0 \left( \tilde{F}_t - I_2 \right) \leq Ct |z|^\alpha$ and $n_1 \left( \tilde{F}_t - I_2 \right) \leq Ct |z|^\alpha$. We then have (42) and (43).

The asymptotic behaviour of $\frac{\partial \tilde{\Phi}}{\partial \lambda}$ allows us to prove the convergence of immersions as stated in Proposition 5. We need the following lemma.

**Lemma 8.** Let $F_t, F^D_t \in \Lambda SU_2$ satisfy (43). Then (37) is satisfied.
Proof. First, note that the Sym-Bobenko formula for $\mathbb{R}^3$ implies that (we omit the index $t$)

$$iF(z, 1)\frac{\partial(F^{-1}F^D)}{\partial\lambda}(z, 1)F^D(z, 1)^{-1} = i\frac{\partial F^D}{\partial\lambda}(z, 1)F^D(z, 1)^{-1} - i\frac{\partial F}{\partial\lambda}(z, 1)F(z, 1)^{-1}$$

$$= f^D(z) - f(z).$$

We can then compute

$$\|f_t(z) - f_t^D(z)\|_{\mathbb{R}^3}^2 = 4 \det \left( f_t(z) - f_t^D(z) \right)$$

$$= -4 \det \frac{\partial(F^{-1}F^D)}{\partial\lambda}(z, 1)$$

$$\leq C^2 t^2 |z|^{2\alpha}.$$

This proves the first point of Proposition 5. To prove the second one, use (38) and note that $M_0 = I_2$. So the axis of $f_t^D$ as $t \to 0$ is the same that the axis of the unperturbed Delaunay surface induced by $z^{A_t}$.

It remains to show that the surface is embedded if $\omega > 0$.

5 Embeddedness

We suppose in this section that $\omega > 0$. The asymptotic behaviour of $f_t$ and the fact that $f_t^D$ is an embedding for all $t$ allow us to show that $f_t$ is an embedding of a sufficiently small uniform neighbourhood of $z = 0$ for $t$ small enough. We begin by recalling some basic results of differential geometry, we then give a general result of embeddedness that can be applied for Delaunay asymptotics. We finally show the embeddedness of our surfaces by applying this result to perturbed Delaunay immersions.

Lemma 9. Let $\mathcal{M}$ be a complete surface in $\mathbb{R}^3$. Let $\eta : \mathcal{M} \to \mathbb{S}^2$ denote the Gauss map of $\mathcal{M}$. Let $r > 0$ such that the tubular neighbourhood $\text{Tub}_r \mathcal{M}$ is embedded. Then $\|d\eta\| \leq \frac{1}{r}$.

Moreover, denoting $\pi$ the projection from $\text{Tub}_{r'} \mathcal{M}$ with $0 < r' < r$, $\pi$ satisfies $\|d\pi\| \leq \frac{r}{r-r'}$.

Proof. By definition of the embedded tubular neighbourhood of $\mathcal{M}$, the map

$$\psi : \mathcal{M} \times (-r, r) \to \mathbb{R}^3$$

$$(x, t) \mapsto x + t\eta(x)$$
is a diffeomorphism. So
\[ d\psi_{(x,t)} \cdot (X, T) = (\text{id} + t d\eta_x) \cdot X + T \eta(x) \]
is an isomorphism. Using \( \langle X, \eta \rangle = 0 \), \( I_2 + t d\eta_x \) is an isomorphism for all \( t \in (-r, r) \), so the eigenvalues of \( d\eta_x \) are in \( (-\frac{1}{r}, \frac{1}{r}) \) and \( \|d\eta\| < \frac{1}{r} \).
Restricting \( \psi \) to \( \mathcal{M} \times (-r', r') \), we deduce
\[ \|d\psi_{(x,t)} \cdot (X, T)\| \geq \left(1 - \frac{r'}{r}\right)\|X\|. \tag{46} \]
Let \( p = \psi(x, t) \in \text{Tub}_r \mathcal{M} \) and \( Y \in \mathbb{R}^3 \). There exist \( X \) and \( T \) such that \( Y = d\psi_{(x,t)} \cdot (X, T) \). Noting that \( \pi \circ \psi(x, t) = x \), we have
\[ X = d(\pi \circ \psi)_{(x,t)} \cdot (X, T) = d\pi_p \cdot Y. \]
Using Equation (46),
\[ \|Y\| \geq \frac{r - r'}{r} \|d\pi_p \cdot Y\| \]
and
\[ \|d\pi\| \leq \frac{r}{r - r'}. \]

**Proposition 6.** Let \( f_n^R \colon \mathbb{C}^* \rightarrow \mathcal{M}_n^R = f_n^R(\mathbb{C}^*) \subset \mathbb{R}^3 \) be a sequence of complete immersions of normal maps \( \mathcal{N}_n^R \) with an end at \( z = 0 \). Suppose that for all \( n \) there exists \( r_n > 0 \) such that the tubular neighbourhood \( \text{Tub}_{r_n} \mathcal{M}_n^R \) of \( \mathcal{M}_n^R \) is embedded. Suppose that for all \( \rho > 0 \) there exists \( 0 < \rho' < \rho \) such that for all \( n \in \mathbb{N} \), \( x \in \partial \mathbb{D}_\rho \) and \( y \in \mathbb{D}_{\rho'}^* \),
\[ \|f_n^R(x) - f_n^R(y)\|_{\mathbb{R}^3} > 2r_n. \tag{47} \]
Let \( U^* \subset \mathbb{C}^* \) be a neighbourhood of \( z = 0 \) and \( f_n : U^* \rightarrow \mathbb{R}^3 \) a sequence of immersions of normal maps \( \mathcal{N}_n \) satisfying
\[ \sup_{n \in \mathbb{N}} \|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} \rightarrow 0 \quad \text{as} \quad z \rightarrow 0 \quad \tag{48} \]
and
\[ \sup_{z \in U^*} \|\mathcal{N}_n(z) - \mathcal{N}_n^R(z)\|_{\mathbb{R}^3} \rightarrow 0. \tag{49} \]
Then there exist \( \rho' > 0 \) and \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( f_n \) is an embedding of \( \mathbb{D}_{\rho'}^* \).
Proof. Let us split the proof in several steps.

- **Claim 1**: there exists $\rho > 0$ such that the map

$$\varphi_n : \mathbb{D}_\rho \setminus \{0\} \to M^R_n \quad z \mapsto \pi_n \circ f_n(z)$$

(where $\pi_n$ is the projection from $\text{Tub}_{r_n} M^R_n$ onto $M^R_n$) is well-defined and satisfies

$$\|\varphi_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < r_n$$

for all $z \in \mathbb{D}_\rho \setminus \{0\}$. We then fix such a $\rho$ and fix $\rho'$ so that Equation (47) is satisfied.

- **Claim 2**: there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\varphi_n$ is a local diffeomorphism on $\mathbb{D}^*_\rho$. We then fix such $N$ and $n$.

- **Claim 3**: the restriction

$$\tilde{\varphi} : \varphi_n^{-1} \left( \varphi_n(\mathbb{D}^*_\rho) \right) \cap \mathbb{D}^*_\rho \to \varphi_n(\mathbb{D}^*_\rho) \quad z \mapsto \varphi_n(z)$$

is a covering map.

- **Claim 4**: this covering map is one-sheeted.

- **Conclusion**: the map $\tilde{\varphi}$ is an injective diffeomorphism, so $f_n(\mathbb{D}^*_\rho)$ is a graph over $M^R_n$ contained in its embedded tubular neighbourhood and $f_n(\mathbb{D}^*_\rho)$ is thus embedded.

To prove the first claim, use Hypothesis (48): there exists $\rho > 0$ such that for all $n \in \mathbb{N}$ and $z \in \mathbb{D}_\rho \setminus \{0\}$,

$$\|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < \frac{r_n}{2}.$$  

(51)

So $f_n(\mathbb{D}^*_\rho) \subset \text{Tub}_{r_n/2} M^R_n$ and $\varphi_n$ is well-defined. Moreover, recalling Lemma 9 and using (51), for all $z \in \mathbb{D}_\rho \setminus \{0\}$

$$\|\varphi_n(z) - f_n^R(z)\|_{\mathbb{R}^3} \leq \sup_{p \in \text{Tub}_{r_n/2} M^R_n} \|d\pi_n(p)\| \times \|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} \leq \frac{1}{1 - \frac{1}{2}} \times \frac{r_n}{2} \leq r_n$$

30
and Equation (50) holds.

Let $z \in \mathbb{D}_\rho^*$. In order to show that $\varphi_n$ is a local diffeomorphism, we show that

$$\langle \mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n(z) \rangle > 0 \quad (52)$$

where $\mathcal{N}_{\varphi_n}$ is defined by

$$\mathcal{N}_{\varphi_n} : \mathbb{D}_\rho^* \rightarrow S^2 \subset \mathbb{R}^3 \quad z \mapsto \eta^R_n(\varphi_n(z))$$

and $\eta^R_n$ is the Gauss map of $\mathcal{M}_n^R$. Using the triangle inequality,

$$\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n(z)\|_{\mathbb{R}^3} \leq \|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\|_{\mathbb{R}^3} + \|\mathcal{N}_n^R(z) - \mathcal{N}_n(z)\|_{\mathbb{R}^3}.$$

The first term of the right-hand side satisfies

$$\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\|_{\mathbb{R}^3} \leq \sup_{p \in \mathcal{M}_n^R} \|d\eta^R_n(p)\| \times |\gamma|$$

where $\gamma \subset \mathcal{M}_n^R$ is a path joining $\varphi_n(z)$ to $f_n^R(z)$. Recalling Lemma 9 and the fact that Tub$_{r_n}$ is embedded, we get

$$\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\|_{\mathbb{R}^3} \leq \frac{1}{r_n} \times |\gamma|.$$

Let $\sigma(t) = (1 - t)f_n(z) + tf_n^R(z)$, $t \in [0, 1]$. Then,

$$\|\sigma(t) - f_n^R(z)\|_{\mathbb{R}^3} \leq (1 - t)\|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < \frac{r_n}{2}$$

because of Equation (51). Then, $\sigma \subset$ Tub$_{r_n/2}\mathcal{M}_n^R$ and setting $\gamma = \pi_n \circ \sigma$,

$$|\gamma| \leq \sup_{p \in \text{Tub}_{r_n/2}\mathcal{M}_n^R} \|d\pi_n(p)\| \times \|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < r_n$$

because of Lemma 9 and Equation (51). Thus,

$$\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\| < \frac{1}{r_n} \times r_n = 1.$$
Recalling Hypothesis (49), we can choose a uniform \( N \in \mathbb{N} \) such that for all \( n < N \),
\[
\| \mathcal{N}_{\rho_n}(z) - \mathcal{N}_n(z) \| < \sqrt{2},
\]
which proves Equation (52) and the second claim.

Recall that \( 0 < \rho' < \rho \) and \( n \geq N \) are now fixed. To prove the third claim we show that \( \tilde{\varphi} \) is a proper map. Let \( (x_i)_{i \in \mathbb{N}} \subset \varphi_n^{-1} (\varphi_n (\mathbb{D}^*_\rho)) \cap \mathbb{D}^*_\rho \) such that \( (\tilde{\varphi}(x_i))_{i \in \mathbb{N}} \) converges to \( p \in \varphi_n (\mathbb{D}^*_\rho) \). Then \( (x_i)_i \) converges to \( x \in \mathbb{D}_\rho \). Using Equation (50) and the fact that \( f_n^R \) has an end at \( 0, x \neq 0 \). If \( x \in \partial \mathbb{D}_\rho \), denoting \( \tilde{x} \in \mathbb{D}^*_\rho \) such that \( \tilde{\varphi}(\tilde{x}) = p \), one has
\[
\| f_n^R(x) - f_n^R(\tilde{x}) \|_{\mathbb{R}^3} < \| f_n^R(x) - \tilde{\varphi}(x) \|_{\mathbb{R}^3} + \| f_n^R(\tilde{x}) - \tilde{\varphi}(\tilde{x}) \|_{\mathbb{R}^3} < 2r_n
\]
which contradicts the definition of \( \rho' \). Thus, \( \tilde{\varphi} \) is a proper local diffeomorphism between locally compact spaces, i.e. a covering map.

To compute the number of sheets, let \( \gamma : [0, 1] \rightarrow \mathbb{D}^*_\rho' \) be a loop of winding number 1 around 0, \( \Gamma = f_n^R(\gamma) \) and \( \widetilde{\Gamma} = \tilde{\varphi}(\gamma) \subset \mathcal{M}_n^R \) and let us construct a homotopy between \( \Gamma \) and \( \widetilde{\Gamma} \). Let
\[
\sigma_t : [0, 1] \rightarrow \mathbb{R}^3
\]
\[
s \mapsto (1 - s)\widetilde{\Gamma}(t) + s\Gamma(t).
\]
For all \( t, s \in [0, 1] \),
\[
\inf \{ \| \sigma_t(s) - p \|_{\mathbb{R}^3}, p \in \mathcal{M}_n^R \} \leq \| \sigma_t(s) - \Gamma(t) \|_{\mathbb{R}^3} < r_n
\]
which implies that \( \sigma_t(s) \in \text{Tub}_{\pi_n, \mathcal{M}_n^R} \) because \( \mathcal{M}_n^R \) is complete. One can thus define the following homotopy between \( \Gamma \) and \( \widetilde{\Gamma} \)
\[
H : [0, 1]^2 \rightarrow \mathcal{M}_n^R
\]
\[
(s, t) \mapsto \pi_n \circ \sigma_t(s)
\]
where \( \pi_n \) is the projection from \( \text{Tub}_{\pi_n, \mathcal{M}_n^R} \) to \( \mathcal{M}_n^R \). Using the fact that \( f_n^R \) is an embedding, the degree of \( \Gamma \) is one, and the degree of \( \widetilde{\Gamma} \) is also one. Hence, \( \tilde{\varphi} \) is one-sheeted.

We can now apply Proposition 3 to each case:

- If \( (r, s) \rightarrow (1/2, 0) \), we set \( f_n^R = f_t^D \) and \( f_n = f_t \) with \( t = 1/n \). The tubular radius \( r_n \) is of the order of \( 4\omega t \) and Hypothesis (17) is satisfied because \( f_0^D \) is the immersion of a sphere. Lemmas 7 and 8 ensure that Hypotheses (48) and (49) hold.

- If \( (r, s) \rightarrow (0, 1/2) \), we set \( f_n^R = \frac{1}{t} f_t^D \) and \( f_n = \frac{1}{t} f_t \) with \( t = 1/n \). The tubular radius \( r_n \) is of the order of \( 4\omega \) and Hypothesis (17) is satisfied because \( f_0^D \) is the immersion of a catenoid. Lemmas 7 and 8 ensure that Hypotheses (48) and (49) hold.

The second point of our theorem is then proved.
A Derivative of the monodromy

The following proposition, used in Section 3, is derived from Proposition 5 in [18].

**Proposition 7.** Let \( \xi_t : \Omega \rightarrow \mathcal{M}_n(\mathbb{C}) \) be a family of matrix-valued 1-forms on a domain \( \Omega \subset \mathbb{C} \) depending holomorphically on the complex parameter \( t \) in a neighbourhood of \( t_0 \). Let \( \Phi_t(z) \) be the solution of \( d\Phi_t = \Phi_t \xi_t \) with initial value \( \Phi_t(z_0) \in \text{GL}_2\mathbb{C} \) and continuous at \( t_0 \). Let \( \gamma \in \pi_1(\Omega, z_0) \) and let \( \mathcal{M}(t) \) be the monodromy of \( \Phi_t \) along \( \gamma \). If for all \( t \) in a neighbourhood of \( t_0 \),

\[
[\mathcal{M}(t_0), \Phi_{t_0}(z_0)\Phi_t(z_0)^{-1}] = 0,
\]

then \( \mathcal{M} \) is differentiable at \( t_0 \) and

\[
\mathcal{M}'(t_0) = \left( \int_\gamma \Phi_{t_0}(z) \frac{\partial \xi_t(z)}{\partial t} \bigg|_{t=t_0} \Phi_{t_0}(z)^{-1} \right) \times \mathcal{M}(t_0).
\]

In particular, if \( \mathcal{M}(t_0) = \pm I_2 \) or if \( \Phi_t(z_0) \) is constant, then (53) is satisfied.

**Proof.** Let \( \tilde{\Phi}_t(z) = \Phi_t(z_0)^{-1} \Phi_t(z) \), so that \( d\tilde{\Phi}_t = \tilde{\Phi}_t \xi_t \) and \( \tilde{\Phi}_t(z_0) = I_n \). Let \( \tilde{\mathcal{M}}(t) \) be the monodromy of \( \tilde{\Phi}_t \) along \( \gamma \). Then Proposition 5 of [18] applies and

\[
\tilde{\mathcal{M}}'(t_0) = \left( \int_\gamma \tilde{\Phi}_{t_0}(z) \frac{\partial \xi_t(z)}{\partial t} \bigg|_{t=t_0} \tilde{\Phi}_{t_0}(z)^{-1} \right) \times \tilde{\mathcal{M}}(t_0).
\]

On the other hand,

\[
\mathcal{M}(t) = \Phi_t(z_0)\tilde{\mathcal{M}}(t)\Phi_t(z_0)^{-1}
\]

and because of Equation (53),

\[
\mathcal{M}(t_0) = \Phi_t(z_0)\tilde{\mathcal{M}}(t_0)\Phi_t(z_0)^{-1}.
\]

Thus, \( \mathcal{M} \) is differentiable at \( t_0 \) and

\[
\mathcal{M}'(t_0) = \Phi_{t_0}(z_0)\tilde{\mathcal{M}}'(t_0)\Phi_{t_0}(z_0)^{-1}
\]

which proves the proposition.

\[\Box\]

B A control formula on the unitary frame

The following proposition is used in Section 4.

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Proposition 8. Let \((\Sigma, \xi, z_0, \Phi_{z_0})\) be a set of untwisted DPW data, holomorphic for \(\lambda \in \mathcal{A}_R\) with \(R \geq 1\). Then for all \(z_1, z_2 \in \Sigma\),

\[
\|F(z_1, \lambda)\|_{\mathcal{A}_R} \leq \|F(z_2, \lambda)\|_{\mathcal{A}_R} \times \exp \left( (R - 1) \int_{z_1}^{z_2} \rho^2(w) |a_{-1}(w)| \, dw \right)
\]

where \(a_{-1}(z)dz\) is the upper-right residue of \(\xi\) at \(\lambda = 0\) and \(\rho(z)\) is the upper-left entry of Pos(\(\Phi\))(\(z, 0\)).

Proof. Write

\[
\xi(z, \lambda) = \lambda^{-1} \begin{pmatrix} 0 & a_{-1}(z) \\ 0 & 0 \end{pmatrix} dz + \lambda^{\alpha} \begin{pmatrix} c_0(z) & a_0(z) \\ b_0(z) & -c_0(z) \end{pmatrix} dz + \mathcal{O}(\lambda).
\]

Let \(\Phi = FB\) be the Iwasawa decomposition of \(\Phi\). By computations similar to Section 4.3 of [5], \(dF = FL\) where

\[
L(z, \lambda) = \begin{pmatrix} \rho^{-1}_z & \lambda^{-1} \rho^2 a_{-1} \\ b_0 \rho^{-2} & -\rho^{-1}_z \end{pmatrix} dz + \begin{pmatrix} -\rho^{-1}_z & -b_0 \rho^{-2} \\ -\lambda \rho^2 a_{-1} & \rho^{-1}_z \end{pmatrix} d\bar{z}.
\]

Let

\[
\tilde{F}(z, \lambda) = F \left( z, \frac{\lambda}{|\lambda|} \right)
\]

so that \(\tilde{F}(z, \lambda) \in SU_2\) for all \(\lambda \in \mathcal{A}_R\). Then \(d\tilde{F} = \tilde{F} \tilde{L}\) where

\[
\tilde{L}(z, \lambda) = L \left( z, \frac{\lambda}{|\lambda|} \right).
\]

Using the variation of constants method, for all \(z_1, z_2 \in \Sigma\) (we omit the variable \(\lambda\)),

\[
F(z_1) = F(z_2) \tilde{F}(z_2)^{-1} \tilde{F}(z_1) + \left( \int_{z_2}^{z_1} F(w) \left( L(w) - \tilde{L}(w) \right) \tilde{F}(w)^{-1} \right) \tilde{F}(z_1).
\]

But

\[
L(w, \lambda) - \tilde{L}(w, \lambda) = \rho^2(w) \begin{pmatrix} 0 & a_{-1}(w) \lambda^{-1} (1 - |\lambda|) \, dw \\ -\bar{a}_{-1}(w) \lambda (1 - |\lambda|^{-1}) \, d\bar{w} & 0 \end{pmatrix}
\]

so

\[
\|L(w, \lambda) - \tilde{L}(w, \lambda)\|_{\mathcal{A}_R} \leq (R - 1) \rho^2(w) |a_{-1}(w)| \|dw\|
\]

and the result follows from Gronwall’s inequality (Lemma 2.7 in [16]).
As an application, recall that in the untwisted $\mathbb{R}^3$ setting, if $f = \text{Sym}(F)$, then $f$ is a CMC 1 conformal immersion whose metric is given by

$$ds = 2\rho^2|a_{-1}||dz|.$$ 

So let $z_1, z_2 \in \Sigma$ and $\gamma \subset \Sigma$ be a path joining $f(z_1)$ to $f(z_2)$. Then,

$$\|F(z_1, \lambda)\|_{A_R} \leq \|F(z_2, \lambda)\|_{A_R} \exp\left(\frac{(R - 1)}{2} |\gamma|\right).$$
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