Integral Transforms and $\mathcal{PT}$-symmetric Hamiltonians

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Abstract

Integral transforms can be used as a tool to simplify the computations of differential equations. In this work, we systematically study integral transforms in the context of $\mathcal{PT}$-symmetric Hamiltonians. First, we obtained a closed analytical formula for the exponential Fourier transformed $\mathcal{PT}$-symmetric Hamiltonians. Using Segal-Bargmann transform, we investigate the effect of the Fourier transform on the eigenfunctions of the original Hamiltonian. Moreover, we comment on the holomorphic representation of non-Hermitian spin chains in which the Hamiltonian operator is written in terms of analytical phase-space coordinates and their partial derivatives in the Bargmann space rather than matrices in the complex Hilbert space. Specifying to non-Hermitian $XX$ spin chain, we prove by numerically solving the quantum master equation its ability to flip from dynamical to static system by running the coupling constant from weak to strong. Finally, we solve the Swanson Hamiltonian and discuss its behavior under integral transforms.
In [1], Bender and Boettcher proved the reality of energy eigenvalues for a large class of non-Hermitian Hamiltonians numerically. The common property between these Hamiltonians is their invariance under the action of both time-reversal \( T \) and parity \( P \) operations i.e. \( [PT, H] = 0 \). Since then, many studies has been done in this area with potential applications in optics and other areas of physics such as open quantum systems, supersymmetric quantum mechanics, topological matter, cold atoms and magnonics waveguides [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

In quantum mechanics, wave functions can be written both in position (\( x \)-space) and momentum (\( p \)-space) spaces using exponential Fourier transform [24, 25, 26],

\[
\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) dx, \quad (1.1)
\]

and its inverse transform

\[
\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \phi(p) dp. \quad (1.2)
\]

The Schrödinger equation in the \( p \)-space is

\[
(E - \frac{p^2}{2m})\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{V}(p - p')\phi(p') dp', \quad (1.3)
\]

which is an integral equation. In some problems it is more convenient to work in the \( p \)-space such as delta-potential i.e. \( V = A \delta(x) \)[25, 26, 27, 28]. Furthermore, in one version of non-commutative quantum mechanics we have the commutation relations defined as \([p_i, p_j] = 0, [x_i, x_j] = i\theta_{ij}, [x_i, p_j] = i\hbar\delta_{ij}\) where \( \theta_{ij} \) is antisymmetric tensor with dimension (length)\(^2\)[29]. In this case, the ordinary product of position-dependent functions is promoted to the Moyal star-product \( f \star g(x) = e^{i\theta_{ij}\partial^{(3)}\partial^{(2)}} f(x_1)g(x_2)|_{x_1=x_2=x} \)[30, 31, 32]. Consequently, working in the momentum space is more efficient since the momentum-dependent functions are multiplied in the usual way. Noncommutative quantum mechanics appears naturally in the study of charged particle motion under the effect of magnetic field and thus it has deep connection with Landau levels, anomalous Hall effect in ferromagnetic metals and in the context of magnetic skyrmions [33, 34, 35, 36, 37, 38, 39, 40, 41].

In this work, we give a detailed study of \( PT \)-symmetric Hamiltonians under integral transforms such as the Fourier transform. Moreover, we use the Segal-Bargmann transform to
explore the effect of the Fourier transform on the eigenfunctions. Furthermore, the Bargmann representation of spin operators can be used in the study of non-Hermitian spin chains analytically. In this case, spin operators are expressed in terms of orthonormal phase-space coordinates obtained from Segal-Bargmann transform of the coordinate-dependent wavefunctions. Finally, we consider the non-Hermitian spin chains and Swanson Hamiltonian as examples of applying integral transforms within the context of non-Hermitian Hamiltonians.

2 Integral Transforms of $\mathcal{PT}$-symmetric Hamiltonians

The integral equation problem is to find the solution to:

$$ h(x)f(x) = g(x) + \lambda \int_a^b K(x,y)f(y)dx, \quad (2.1) $$

We are given $h(x), g(x), K(x,y)$ and want to determine $f(x)$. The quantity $\lambda$ is a parameter which may be complex in general and $K(x,y)$ is called the kernel of the integral equation. If $h(x) = 0$, we may take $\lambda = -1$ without loss of generality and we end up with Fredholm equation of the first kind or an integral transform.

**Definition 2.1.** The integral transform of a function $f(x)$ in the interval $a \leq x \leq b$ denoted by $\mathcal{I}\{f(x)\} = F(k)$ is

$$ \mathcal{I}\{f(x)\} = F(k) = \int_a^b K(x,k)f(x)dx, \quad (2.2) $$

where $K(x,k)$ is called the kernel of the transform and $\mathcal{I}$ is the transform operator.

The integral transform for a function $f(x)$ with several variables is

$$ \mathcal{I}\{f(x)\} = F(k) = \int_D K(x,k)f(x)dx, \quad (2.3) $$

where $x = (x_1, x_2, \ldots, x_n)$, $k = (k_1, k_2, \ldots, k_n)$ and $D \in \mathbb{R}^n$. Apparently $\mathcal{I}$ is a linear operator since

$$ \mathcal{I}\{\alpha f(x) + \beta g(x)\} = \int_a^b \{\alpha f(x) + \beta g(x)\}K(x,k)dx $$

$$ = \alpha \mathcal{I}\{f(x)\} + \beta \mathcal{I}\{g(x)\}. \quad (2.4) $$

Let $f \in L^1(\mathbb{R})$, the Fourier transform of $f(x)$ in $L^1(\mathbb{R})$ is defined by the integral

$$ \mathcal{F}_x(f(x)) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx. \quad (2.6) $$
where the kernel of the Fourier transform (in our convention) is $e^{-ikx}$. Clearly, $|\int_{-\infty}^{\infty} e^{-ikx} f(x) dx| \leq \int_{-\infty}^{\infty} |f(x)| dx$. This implies that the integral $2.6$ exist for all $k \in \mathbb{R}$. One interesting feature of the Fourier transform is described by the generalized Parseval relation. Let $f, g \in L^2(\mathbb{R})$, then

$$\langle f|g \rangle = \int_{-\infty}^{\infty} f(x)g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) \hat{g}(k) dk = \frac{1}{2\pi} \langle \hat{f}|\hat{g} \rangle$$

(2.7)

It is not difficult to realize that Fourier transform can be viewed as a special case of the bilateral Laplace transform with imaginary argument,

$$\hat{f}(k) = \mathcal{F}_x \{f(x)\} = \mathcal{L}\{f(x)\}|_{s=ik}$$

(2.8)

where $\mathcal{L}\{f(x)\} = \int_{-\infty}^{\infty} e^{-sx} f(x) dx$. The definition $2.2$ of integral transforms holds for quantum mechanical systems where the arguments of these transforms are in principle functions of canonical variables $F(\hat{q}_i, \hat{p}_i)$ satisfying $[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$ (in our case, we simply choose these canonical variables to be the position and momentum operators $x$ and $p$ defined in the continuous Hilbert space).

We derive the expression for the Fourier transform of a general $\mathcal{PT}$-symmetric Hamiltonians. The exponential Fourier transform $\mathcal{F}$ of the operator $\hat{f}(x)$ is $43, 44$

$$\mathcal{F}_x(\hat{f}(x)) = \hat{F}(x) = \int_{-\infty}^{\infty} \hat{f}(x)e^{-ikx} dx$$

(2.9)

and the inverse Fourier transform is defined by

$$\mathcal{F}^{-1}_k(\hat{F}(k)) = \hat{f}(x) = 2\pi \int_{-\infty}^{\infty} \hat{F}(k)e^{ikx} dk$$

(2.10)

where $x, k$ are operators in the continuous Hilbert space. Under parity (spatial reflection) $p \rightarrow -p, x \rightarrow -x$ and under time-reversal $p \rightarrow -p, x \rightarrow x$ and $i \rightarrow -i$, the general $\mathcal{PT}$-symmetric Hamiltonian (in units $\hbar = 2m = 1$ for simplicity) is

$$H = p^2 + V_{Re}(x) + iV_{Im}(x)$$

(2.11)

where the real part of the potential $V_{Re}$ is even and the imaginary part $V_{Im}$ is odd under parity transformation. Note that both $V_{Re}, V_{Im} \in \mathbb{R}$. We define their exponential Fourier transforms as

$$\tilde{V}_{Re}(p) = 2 \int_0^{\infty} V_{Re}(x) \cos(p \cdot x) \, dq,$$

(2.12)

$$\tilde{V}_{Im}(p) = -2i \int_0^{\infty} V_{Im}(x) \sin(p \cdot x) \, dq.$$
Now if we plug 3.1, 3.2 in the Schrödinger equation, we find

\[ \tilde{H}\phi(p) = p^2\phi(p) + 2\pi \int_{-\infty}^{\infty} \tilde{V}(p - p')\phi(p')dp', \]  

(2.14)

where

\[ \tilde{V}(p) = 2\int_{0}^{\infty} (V_{\text{Re}}(x)\cos(x \cdot p) + V_{\text{Im}}(x)\sin(x \cdot p)) \, dx, \]  

(2.15)

Interestingly, performing the exponential Fourier transform of the Hamiltonian 2.11 removes the imaginary unit \( i \) in front of \( V_{\text{Im}} \). In the above we have considered Hamiltonians with one position coordinate \( x \). However, the generalization to higher-dimensional spaces is straightforward.

Next, we need to investigate the effect of the exponential Fourier transform on the orthonormal basis attached to a quantum system. Consider, for example, Hermite polynomials \( H_n(x) = (-)^ne^{x^2}(\frac{d}{dx})^n(e^{-x^2}) \) which appears in many quantum mechanical problems. The Fourier transform of Hermite polynomials is \[ \mathcal{F}\{e^{-x^2/2}H_n(x)\} = (-i)^n e^{-p^2/2}H_n(p) \]  

(2.16)

It is worthy to mention that the imaginary coefficient which appears in the front of the second hand-side \( (-i)^n \) has no effect on the measured energy eigenvalues. To prove this we compute the Segal-Bargmann transform of \( H_n(p) \). Let \( f(p) \) be a function in the momentum Hilbert space \( L^2(\mathbb{R}^d) \), we define the Segal-Bargmann transform with respect to \( p \) as \[ \mathcal{B}(f(z)) = \int_{\mathbb{R}^d} e^{-(z - 2\sqrt{z}\cdot p - p \cdot p)/2} f(p) dp \]  

[46, 47, 48, 49, 50]. The Segal-Bargmann transform of Hermite polynomials \( H_n \) are simply complex monomials of the form \( \{\frac{z^n}{\sqrt{n!}}\} \). These monomials form an orthonormal basis since

\[ \int \frac{dz \, d\bar{z}}{\pi} \exp[-z\bar{z}] \, z^n z^m = n! \delta_{mn}, \]  

(2.17)

where \( z \) is arbitrary complex coordinate (physically it is the phase-space coordinate \( z = x + ip \)). As we emphasized in [51], both \( \frac{z^n}{\sqrt{n!}} \) and \( \frac{(-iz)^n}{\sqrt{n!}} \) correspond to the same energy eigenvalue for each level characterized by the same quantum number \( n \) and the effect of \( (-i)^n \) in front of Hermite polynomials have no effect on the measured energy eigenvalues.

**Definition 2.2.** The Bargmann space also known as the Fock-Bargmann or Segal-Bargmann space, denoted by \( \mathcal{HL}^2(\mathbb{C}^n, \mu) \), is the space of holomorphic functions with Gaussian integral measure \( \mu(z) = (\pi)^{-n}e^{-|z|^2} \) and \( |z|^2 = |z_1|^2 + \cdots + |z_n|^2 \).
Any entire analytic function \( f(z) \) in this space satisfy a square-integrability condition of the type \[ ||f||^2 := \langle f|f \rangle_\mu = (\pi)^{-n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dz < \infty, \] \[ (2.18) \]

where \( dz \) is the \( 2n \)-dimensional Lebesgue measure on \( \mathbb{C}^n = \mathbb{R}^{2n} \).

**Remark :** The inner-product between analytic functions \( f(z) \) and \( g(z) \) in the Bargmann space \( \mathcal{H}L^2(\mathbb{C}^n, \mu) \) is \[ \langle f|g \rangle_\mu = (\pi)^{-n} \int_{\mathbb{C}^n} \overline{f}(z) g(z) e^{-|z|^2} dz. \] \[ (2.19) \]

**Definition 2.3.** The Segal-Bargmann transform is an integral transformation from \( L^2(\mathbb{R}^n) \) to \( \mathcal{H}L^2(\mathbb{C}^n, \mu) \) defined as \[ \mathcal{B}(f(z)) = \int_{\mathbb{R}^n} K_n(z, x)f(x)d^n x \text{ for } z \in \mathbb{C}^n \text{ and } f(x) \in \mathbb{R}^n \] \[ (2.20) \]

with \( K_n(z, x) = e^{(z \cdot z - 2\sqrt{2} z \cdot x - x \cdot x)/2} \) the kernel integral.

**Theorem 2.1** (Unitarity of the Segal-Bargmann transform). The Segal-Bargmann transform defined by \[ (2.20) \] is a unitary mapping between the Hilbert space \( L^2(\mathbb{R}^n) \) and \( \mathcal{H}L^2(\mathbb{C}^n, \mu) \).

In Bargmann space \( \mathcal{H}L^2(\mathbb{C}, \mu) \), the creation and annihilation operators of the harmonic oscillator are \( z \) and \( \frac{\partial}{\partial z} \) since \[ \left[ \frac{\partial}{\partial z}, z \right] f(z) = \frac{\partial}{\partial z}(zf(z)) - z \frac{\partial f(z)}{\partial z} = 1 \] \[ (2.21) \]

similar to the canonical commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). In this representation, the harmonic oscillator Hamiltonian operator is \( H = \hbar \omega \left( z \frac{d}{dz} + \frac{1}{2} \right) \). Applying to energy eigenstates (Fock states), we find \[ H|n\rangle = \hbar \omega \left( z \frac{d}{dz} + \frac{1}{2} \right) \frac{z^n}{\sqrt{n!}} = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle. \] \[ (2.22) \]

### 3 Model Examples

#### 3.1 Non-Hermitian Spin Chains

For a given entire analytic function \( f_{\alpha,\beta}(z, w) = \frac{z^\alpha \omega^\beta}{\sqrt{\alpha! \beta!}} \), we define the spin operators in the two-dimensional Bargmann space \( \mathcal{H}L^2(\mathbb{C}^2, \mu) \) using the holomorphic representation of Jordan-
Schwinger map as [52, 53]

\[ S_x = \frac{\hbar}{2} \left( z \frac{\partial}{\partial w} + w \frac{\partial}{\partial z} \right), \quad (3.1) \]
\[ S_y = \frac{\hbar}{2i} \left( z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z} \right), \quad (3.2) \]
\[ S_z = \frac{\hbar}{2} \left( z \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} \right). \quad (3.3) \]

These operators belong to the $SU(2)$ Lie algebra and obey the commutation relations $[S_i, S_j] = i \hbar \varepsilon_{ijk} S_k$ since the only non-trivial commutators between $z$, $w$ and their partial derivatives are $[\frac{\partial}{\partial z}, z] = [\frac{\partial}{\partial w}, w] = 1$. The total number operator is

\[ N = n_z + n_w, \quad (3.4) \]

where $n_z = z \frac{\partial}{\partial z}$ and $n_w = w \frac{\partial}{\partial w}$. The functions $\{f_{\alpha, \beta}\}$ form an orthonormal basis in the two-dimensional Bargmann space since

\[ \langle f_{\alpha', \beta'}(z, w) | f_{\alpha, \beta}(z, w) \rangle_{\mu} = \int dz \, dw \, \exp[-\bar{z}z - \bar{w}w] f_{\alpha', \beta'}(z, w) f_{\alpha, \beta}(z, w) = \pi^{2} \alpha! \beta! \delta_{\alpha', \alpha} \delta_{\beta', \beta} \]

where $\alpha, \beta \in \mathbb{N}$ is the set of all natural numbers including zero i.e. $\{0, 1, 2, \ldots \}$.

One advantage of the holomorphic representation of spin operators appears clearly in the study of spin chains. Consider the non-Hermitian $XX$ spin chain Hamiltonian [54]

\[ H = \frac{1}{2} \sum_{j=1}^{J-1} \left[ S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + ig \left( S_j^z - S_{j+1}^z \right) \right], \quad (3.6) \]

where for small values of the coupling constant $g$, it becomes spectral equivalence to Hermitian Hamiltonian [54]. In the holomorphic representation, [3.6] becomes

\[ H = \frac{\hbar^2}{4} \sum_{j=1}^{J-1} \left( z_{j+1} w_j \frac{\partial^2}{\partial z_j \partial w_{j+1}} + z_j w_{j+1} \frac{\partial^2}{\partial w_j \partial z_{j+1}} \right) \]
\[ + \frac{ig}{4} \sum_{j=1}^{J-1} \left( z_j \frac{\partial}{\partial z_j} - z_{j+1} \frac{\partial}{\partial z_{j+1}} + w_{j+1} \frac{\partial}{\partial w_{j+1}} - w_j \frac{\partial}{\partial w_j} \right). \]

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Interestingly, the non-Hermitian $XX$ spin chain described in 3.6 has unique properties as it becomes overall static system in the strong coupling regime and spectral equivalent to the Hermitian $XX$ spin chain in the weak coupling regime. We prove these assumptions numerically by solving the quantum master equation. Generally, the dynamics of spin chains in contact to bath(s) are well described by a quantum master equation of the form

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho,$$

where $\rho$ is the density operator and $\mathcal{L}$ is the Liouville operator. In the Markov-Born approximation Liouville operator reads

$$\mathcal{L}\rho = \frac{1}{i\hbar}[H_s, \rho] + \sum_k \frac{\gamma_k}{2} \mathcal{D}[X_k]\rho,$$

where $\gamma_k > 0$ are real damping constants and $H_s$ is the spin chain Hamiltonian (system) the dissipator can be written in the Lindblad form as

$$\mathcal{D}[X]\rho = 2X\rho X^\dagger - X^\dagger X\rho - \rho X^\dagger X.$$

Note that the spin chain Hamiltonian in the quantum master equation can be written both in usual operator form or in holomorphic analytical form as the both approaches are equivalent physically [50]. By numerically solving the quantum master equation for the $XX$ spin chain, we plot the dynamics of a system with $N = 10$ spins in figures 1,2,3,4. By looking at figure 1 and 2 we realize that both cases are almost identical and thus for weak coupling regime, the dynamics of the non-Hermitian $XX$ spin chain is spectral equivalent to its Hermitian counterpart. Figures 3,4 describe the strong coupling regime of the non-Hermitian $XX$ spin chains and its Hermitian counterpart. It is important to mention that the unitarity is broken in the strong coupling regime as shown in figure 3 for non-Hermitian $XX$ spin chains. From figure 3, We realize that the dynamics of spins oscillates near zero where half of spins evolve in opposite to the second half of spins leaving us with static system globally. This unique property provides new arena for investigation and enrich our space of possibilities in controlling spin systems. This would be of great interest in the design of novel spintronic devices.

### 3.2 The Swanson Hamiltonian

The Swanson Hamiltonian is [56, 57, 58, 59],

$$H(\omega, \alpha, \beta) = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar \alpha \hat{a}^2 + \hbar \beta \hat{a}^\dagger 2,$$

where

$$\hat{a}^\dagger \hat{a} = \frac{1}{2}, \quad \hbar = \frac{\pi}{\tau}, \quad \tau = \beta^{-1}.$$
Figure 1: The dynamics of a non-Hermitian $XX$ spin chain for $N = 10$ spins and small coupling constant $g = 0.1$ with dephasing rate $\gamma = 0.01$ in arbitrary units.

Figure 2: The dynamics of a Hermitian $XX$ spin chain for $N = 10$ spins and small coupling constant $g = 0.1$ with dephasing rate $\gamma = 0.01$ in arbitrary units.
Figure 3: The dynamics of a non-Hermitian $XX$ spin chain for $N = 10$ spins and large coupling constant $g = 1$ with dephasing rate $\gamma = 0.01$ in arbitrary units.

Figure 4: The dynamics of a Hermitian $XX$ spin chain for $N = 10$ spins and large coupling constant $g = 1$ with dephasing rate $\gamma = 0.01$ in arbitrary units.
where $\alpha$ and $\beta$ are real unequal constants in general. When $\alpha = \beta$, Eq.\ref{eq:3.11} reduces to the squeezed harmonic oscillator and the energy spectrum becomes real.

Eq.\ref{eq:3.11} can be written in terms of position and momentum operators by plugging the exact form of creation and annihilation operators,

$$\hat{a} = \frac{1}{\sqrt{2}} \left( \frac{x}{\xi_0} + i \frac{\xi_0}{\hbar} p \right), \quad (3.12)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( \frac{x}{\xi_0} - i \frac{\xi_0}{\hbar} p \right), \quad (3.13)$$

with $\xi_0$ being the characteristic length scale of the system. We obtain

$$H(\omega, \alpha, \beta) = \frac{1}{2} \hbar (\omega + \alpha + \beta) \left( \frac{x}{\xi_0} \right)^2 + \frac{1}{2} \hbar (\omega - \alpha - \beta) \left( \frac{\xi_0 p}{\hbar} \right)^2 + \frac{1}{2} \hbar (\alpha - \beta) \left( \frac{2i}{\hbar} xp + 1 \right) \quad (3.14)$$

It is convenient in our analysis to make the Swanson-harmonic Hamiltonian mapping. This can be done by proper definitions of the position and momentum operators. Following \cite{58}, we have

$$P = \left( p + i \hbar \frac{\alpha - \beta}{(\omega - \alpha - \beta)\xi_0^2} x \right), \quad (3.15)$$

$$X = x. \quad (3.16)$$

It is straightforward to verify the commutation relation $[X, P] = i\hbar$.

In terms of $X$ and $P$, equation \ref{eq:3.14} becomes

$$H = \frac{1}{2m} P^2 + \frac{k}{2} X^2 \quad (3.17)$$

where

$$m = \frac{\hbar}{(\omega - \alpha - \beta)\xi_0^2} \in \mathbb{R}, \quad (3.18)$$

$$k = m\Omega^2 = m\sqrt{\omega^2 - 4\alpha\beta} \in \mathbb{C}. \quad (3.19)$$

As shown by Swanson in \cite{56}, Eq.\ref{eq:3.11} possesses real and positive energy eigenvalues whenever the condition $\omega^2 - 4\alpha\beta \geq 0$ ($\Omega^2 \geq 0$) is satisfied. Indeed, the limiting case when $\Omega = 0$ and $\omega - \alpha - \beta \neq 0$ corresponds to free particle\cite{58}, the creation and annihilation operators are

$$\hat{A} = \sqrt{\frac{m\Omega}{2\hbar}} \left( X + \frac{i}{m\Omega} P \right), \quad (3.20)$$

$$\hat{A}^\dagger = \sqrt{\frac{m\Omega}{2\hbar}} \left( X - \frac{i}{m\Omega} P \right) \quad (3.21)$$
Since $\Omega$ is complex in general, it can be written in the form $\Omega = |\Omega|e^{i\theta}$. Thus, (3.20) and (3.21) becomes

$$\hat{A} = \sqrt{\frac{m|\Omega|}{2\hbar}} \left( Xe^{i\theta/2} + \frac{i}{m|\Omega|} Pe^{-i\theta/2} \right), \quad (3.22)$$

$$\hat{A}^\dagger = \sqrt{\frac{m|\Omega|}{2\hbar}} \left( Xe^{-i\theta/2} - \frac{i}{m|\Omega|} Pe^{i\theta/2} \right), \quad (3.23)$$

Obviously $[\hat{A}, \hat{A}^\dagger] = 1$. By virtue of Euler’s formula i.e. $e^{i\theta} = \cos \theta + i \sin \theta$, (3.22) and (3.23) becomes in matrix form

$$\begin{pmatrix} \hat{A} \\ \hat{A}^\dagger \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix} = \mathbf{M} \cdot \mathbf{b} \quad (3.24)$$

where $\det(\mathbf{M}) = \cos \theta$. When $\theta = 2\pi N$ where $N$ is integer, $\det(\mathbf{M}) = 1$ and this corresponds to unitary transformation and the energy spectrum in this case is real. The operators $\hat{b}$ and $\hat{b}^\dagger$ are the usual creation and annihilation operators for harmonic oscillator with frequency $|\Omega|$ and generalized momentum $P$. Next, consider $|\tilde{n}\rangle$ to be the energy eigenvectors for (3.17) and $|n\rangle$ to be the energy eigenvectors for harmonic oscillator with frequency $|\Omega|$, then we have

$$\hat{A}|\tilde{n}\rangle = \sqrt{n}|\tilde{n} - 1\rangle = \cos \frac{\theta}{2} \hat{b}|n\rangle + i \sin \frac{\theta}{2} \hat{b}^\dagger|n\rangle = \cos \frac{\theta}{2} \sqrt{n}|n - 1\rangle + i \sin \frac{\theta}{2} \sqrt{n + 1}|n + 1\rangle$$

$$\hat{A}^\dagger|\tilde{n}\rangle = \sqrt{n + 1}|\tilde{n} + 1\rangle = \cos \frac{\theta}{2} \hat{b}^\dagger|n\rangle - i \sin \frac{\theta}{2} \hat{b}|n\rangle = \cos \frac{\theta}{2} \sqrt{n + 1}|n + 1\rangle - i \sin \frac{\theta}{2} \sqrt{n}|n - 1\rangle$$

(3.25) (3.26)

From these results we notice that when $\sin \frac{\theta}{2} = 0$, the Hamiltonian (3.17) assumes real energy eigenvalues. The potential term in (3.17) can acquire imaginary value and thus breaks the $\mathcal{PT}$-symmetry. Its exponential Fourier transform has the form

$$\tilde{V}(P) = \frac{m|\Omega|^2}{2} e^{2i\theta} \int_{-\infty}^{\infty} X^2 e^{-iP \cdot X} dX = -\pi m|\Omega|^2 e^{2i\theta} \delta''(P) \quad (3.27)$$

where $\delta''$ is the second-derivative of the Dirac delta function. The non-zero phase $\theta$ breaks the $\mathcal{PT}$-symmetry in the Swanson model. When $\theta = 0$, the energy eigenvalues of the Swanson Hamiltonian are real and the $\mathcal{PT}$-symmetry is exact.
The creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ can be written in terms of the operators $\hat{A}$ and $\hat{A}^\dagger$ as

$$\hat{A} = \frac{1}{2} \sqrt{\frac{m|\Omega|}{\hbar}} \left( s\xi_0 + \frac{\hbar e^{-i\theta/2}}{m|\Omega|\xi_0} \right) \hat{a} + \left( s\xi_0 - \frac{\hbar e^{-i\theta/2}}{m|\Omega|\xi_0} \right) \hat{a}^\dagger, \quad (3.28)$$

$$\hat{A}^\dagger = \frac{1}{2} \sqrt{\frac{m|\Omega|}{\hbar}} \left( s\xi_0 + \frac{\hbar e^{i\theta/2}}{m|\Omega|\xi_0} \right) \hat{a}^\dagger + \left( s\xi_0 - \frac{\hbar e^{i\theta/2}}{m|\Omega|\xi_0} \right) \hat{a} \quad (3.29)$$

where $s = \left( e^{i\theta/2} - \frac{\hbar(\alpha - \beta) e^{-i\theta/2}}{m|\Omega|(|\omega\alpha - \alpha - \beta|\xi_0)} \right)$. The operators $\hat{A}$ and $\hat{A}^\dagger$ belong to pseudo-boson annihilation and creation operators introduced in [60]. Using Bargmann representation, we may write $\hat{A}$ and $\hat{A}^\dagger$ respectively as

$$\frac{\partial}{\partial w} = \frac{1}{2} \sqrt{\frac{m|\Omega|}{\hbar}} \left( s\xi_0 + \frac{\hbar e^{-i\theta/2}}{m|\Omega|\xi_0} \right) \frac{\partial}{\partial z} + \left( s\xi_0 - \frac{\hbar e^{-i\theta/2}}{m|\Omega|\xi_0} \right) z \quad (3.30)$$

$$w = \frac{1}{2} \sqrt{\frac{m|\Omega|}{\hbar}} \left( s\xi_0 + \frac{\hbar e^{i\theta/2}}{m|\Omega|\xi_0} \right) z + \left( s\xi_0 - \frac{\hbar e^{i\theta/2}}{m|\Omega|\xi_0} \right) \frac{\partial}{\partial z}, \quad (3.31)$$

with $[\frac{\partial}{\partial w}, w] = 1$. The eigenfunctions of the Swanson model is monomials of the phase-coordinate $w$ i.e. $\frac{w^n}{\sqrt{n!}}$. While the eigenfunctions associated with operators $\hat{a}$ and $\hat{a}^\dagger$ are $\frac{z^n}{\sqrt{n!}}$ which by construction the ordinary harmonic oscillator energy states.

4 Conclusion

We have investigated the behavior of $\mathcal{PT}$-symmetric Hamiltonians under integral transforms. First, we started with the exponential Fourier transform to write the Hamiltonian in the momentum space and investigated the resulting eigenfunctions in the momentum space. More interestingly, we applied another type of integral transform called the Segal-Bargmann transform to write the Hamiltonian in terms of the phase-space coordinates starting from the Hamiltonians in the position space. This was applied to the case of non-Hermitian $XX$ spin chains. By solving the quantum master equation for this specific spin chain, we prove numerically its spectral equivalent with its Hermitian counterpart in the weak coupling regime. Furthermore, we prove that the dynamics of this system becomes globally static in the strong coupling regime. We argue that such behavior can be used in building novel spintronic devices. One possible application of the Fourier transforms in the context of $\mathcal{PT}$-symmetric Hamiltonians can be found in the study of double well delta function potentials [61].

$$V(x) = (1 + i\Gamma) \delta(x + a) + (1 - i\Gamma) \delta(x - a) \quad (4.1)$$
which has applications in the study of $\mathcal{PT}$-symmetric Bose-Einstein condensates.

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