CONCENTRATION AND CONVERGENCE RATES
FOR SPECTRAL MEASURES OF RANDOM MATRICES

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Abstract. The topic of this paper is the typical behavior of the spectral measures of large random matrices drawn from several ensembles of interest, including in particular matrices drawn from Haar measure on the classical Lie groups, random compressions of random Hermitian matrices, and the so-called random sum of two independent random matrices. In each case, we estimate the expected Wasserstein distance from the empirical spectral measure to a deterministic reference measure, and prove a concentration result for that distance. As a consequence we obtain almost sure convergence of the empirical spectral measures in all cases.

1. Introduction

The topic of this paper is the typical behavior of the spectral measures of large random matrices drawn from several ensembles of interest. Specifically, we consider random matrices drawn from Haar measure on the classical Lie groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, and $Sp(2n)$; Dyson’s circular ensembles; random compressions of random Hermitian matrices satisfying a concentration hypothesis (including random Wigner matrices as a special case); and a random matrix model considered in free probability described by the sum of two random Hermitian matrices, one of which has been subjected to a random basis change. In each case, we estimate the expected Wasserstein distance from the empirical spectral measure to a deterministic reference measure, and prove a concentration result for that distance. Our bounds are sufficient to obtain almost sure convergence of the empirical spectral measures (with rates in the Wasserstein distance) in all cases.

The proofs follow the same approach as the recent work of E. Meckes [19] on random projections of high-dimensional probability measures. The central idea is to view the Wasserstein distance $d_1(\mu_M, \mu)$ from the empirical spectral measure of a random matrix $M$ to a deterministic reference measure $\mu$ as the supremum of a stochastic process indexed by the unit ball of the (infinite-dimensional) space $\text{Lip}(C)$ of real-valued Lipschitz functions on $C$. Concentration properties of the random matrices considered imply that the stochastic process in question satisfies a subgaussian increment condition; Dudley’s entropy bound together with approximation arguments are then used to bound the expected supremum of the process. In the case of the classical Lie groups, earlier work by Diaconis and Mallows [6], Diaconis and Shahshahani [7], and Rains [25] is used to show that the deterministic reference measure can be taken to be the uniform measure on the circle, and the classical measure concentration results of Gromov and Milman [12] are used to obtain the needed concentration properties. For the Hermitian models, the deterministic reference measure used is simply the average of the empirical spectral measure and the matrices are assumed at the outset to satisfy a concentration hypothesis.

Further history and motivation are discussed in sections [2] and [3] below; the remainder of this section is devoted to notation and conventions.
For a subset \( A \subseteq \mathbb{C} \), the space of Lipschitz functions \( f : A \to \mathbb{R} \) is denoted by Lip\((A)\), and is equipped with the Lipschitz seminorm \( |\cdot|_{\text{Lip}} \). Denote by \( \mathcal{P}(A) \) the space of all probability measures supported in \( A \), and by \( \mathcal{P}_p(A) \) be the space of probability measures in \( \mathcal{P} \) with finite \( p \)th moment, equipped with the \( L_p \) Wasserstein distance \( d_p \) defined by

\[
(1.1) \quad d_p(\mu, \nu) := \inf_{\pi} \left( \int |x - y|^p \, d\pi(x,y) \right)^{1/p}.
\]

The infimum above is over probability measures \( \pi \) on \( A \times A \) with marginals \( \mu \) and \( \nu \). Note that \( d_p \leq d_q \) when \( p \leq q \). The \( L_1 \) Wasserstein distance can be equivalently defined (see, e.g., [9]) by

\[
(1.2) \quad d_1(\mu, \nu) := \sup_f \int [f(x) - f(y)] \, d\mu(x)d\nu(y),
\]

where the supremum is over \( f \) in the unit ball \( B(\text{Lip}(A)) \) of Lip\((A)\). In what follows, “Wasserstein distance” with \( p \) unspecified refers to \( d_1 \).

Denote by \( M_n^{sa} \) the space of \( n \times n \) Hermitian matrices, by \( N_n \) the space of \( n \times n \) normal matrices. Denote by \( U(n) \) the group of \( n \times n \) unitary matrices, by \( O(n) \) the group of \( n \times n \) real orthogonal matrices, by \( SU(n) \) and \( SO(n) \) respectively the special unitary and orthogonal groups, and by \( Sp(2n) \subseteq U(2n) \) the compact symplectic group. In all results below these are understood to be equipped with the Hilbert–Schmidt norm \( \|\cdot\|_{\text{HS}} \). For any \( A \in N_n \), let \( \mu_A \) denote the spectral distribution of \( A \); that is, if \( \{\lambda_i\}_{i=1}^n \) are the eigenvalues of \( A \), then

\[
\mu_A := \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}.
\]

For \( A \in M_n^{sa} \), denote by \( \delta(A) := \lambda_{\max}(A) - \lambda_{\min}(A) \) the spectral diameter of \( A \). Note in particular that

\[
\delta(A) = 2 \inf_{\lambda \in \mathbb{R}} \|A - \lambda I\|_{\text{op}},
\]

where \( \|\cdot\|_{\text{op}} \) denotes the operator norm.

Throughout Sections 2 and 3, \( c \), \( C \), and similar symbols denote absolute positive constants, whose exact values may vary from one instance to another.

2. Random matrices in classical Lie groups

This section is concerned primarily with a random matrix \( U \) drawn according to Haar measure from one of the classical compact Lie groups \( O(n) \), \( SO(n) \), \( U(n) \), \( SU(n) \), and \( Sp(2n) \). It will be shown (see Corollary 2.7 below) that for fixed \( n \), the empirical spectral measure \( \mu_U \) is tightly concentrated near the uniform measure \( \nu \) on \( S^1 = \{z \in \mathbb{C} : |z| = 1\} \), with mean Wasserstein distance of order at most \( n^{-2/3} \) and subgaussian tail bounds. As a consequence, it is shown (see Corollary 2.8) that the Wasserstein distance between \( \mu_U \) and \( \nu \) is almost surely of order at most \( n^{-2/3} \). We do not claim that these results are sharp; in fact, there is reason to suspect that \( n^{-2/3} \) could be replaced by \( n^{-1} \), up to logarithmic factors. However, to the best of our knowledge these are the first results which achieve any bounds for these quantities.

Random matrices from these groups have been extensively studied, and much is already known. In particular, we use results from [6], [7], and [25] below in order to show that the uniform distribution on the circle is the correct reference measure for these ensembles. In the case of the unitary and special unitary groups \( U(n) \) and \( SU(n) \), large deviations principles for the empirical spectral measures have been proved by Hiai and Petz [14] and Hiai, Petz, and Ueda [15], respectively. The rates in those LDPs are consistent with the level
of concentration we obtain for the distance, and both results imply in particular the almost sure convergence of the spectral measures, although the LDPs do not give information about the rates of convergence. It should be noted that almost sure convergence for random unitary matrices was proved prior to the results of Hiai and Petz in Voiculescu’s paper [30]. As far as we know, almost sure convergence for the spectral distributions of matrices from the other groups above was not previously known.

The approach taken in this section has three main steps:

1. The mean ESD $\mu = \mathbb{E} \mu_U$ approximates $\nu$ in Wasserstein distance (Theorem 2.1). This is shown using known moments of $\mu$ and classical results on approximating Lipschitz functions on $\mathbb{S}^1$ by polynomials.

2. The mean Wasserstein distance $\mathbb{E} d_1(\mu_U, \mu)$ is small (Theorem 2.6). Using definition (1.2), the Wasserstein distance is interpreted as the supremum of a stochastic process indexed by test functions. Concentration of measure on the classical Lie groups implies that this process has subgaussian increments, allowing the expected supremum to be estimated via entropy methods.

3. The Wasserstein distance $d_1(\mu_U, \nu)$ is tightly concentrated near its mean (Theorem 2.7), and almost sure convergence of $\mu_U$ — with the indicated rate in Wasserstein distance — follows from the Borel–Cantelli lemma (Corollary 2.8). This concentration is again shown using concentration of measure on the classical Lie groups.

In contrast to the proofs of the LDPs in [14, 15], the proofs here make no use of the joint densities of eigenvalues in the classical Lie groups.

There is an important technical caveat to the strategy outlined above, which is that the general concentration of measure results known for $\mathbb{SO}(n)$, $\mathbb{SU}(n)$, and $\mathbb{Sp}(2n)$ do not extend to $\mathbb{O}(n)$ and $\mathbb{U}(n)$. The latter two cases will instead be handled basically by reducing to the corresponding special groups. For this purpose it will be useful also to consider Haar measure on the coset $\mathbb{SO}^{-}(n) = \{U \in \mathbb{O}(n) : \det U = -1\}$. (In this case Haar measure refers to invariance under the action of $\mathbb{SO}(n)$.)

The same strategy can also be carried out for random matrices from Dyson’s Circular Ensembles, as indicated in Theorem 2.9.

The first step of the plan of this section is achieved in the following theorem. Here and in the following, $U \in G$ means that $U$ is distributed according to Haar measure on the group (or coset) $G$. Recall that $\nu$ denotes the uniform probability measure on $\mathbb{S}^1$, and that $\mu = \mathbb{E} \mu_U$.

**Theorem 2.1.**

1. If $U \in \mathbb{U}(n)$ then $\mu = \nu$.
2. If $U \in \mathbb{SU}(n)$ then $d_1(\mu, \nu) \leq \frac{C}{n}$.
3. If $U \in \mathbb{SO}(n)$, $\mathbb{O}(n)$, $\mathbb{SO}^{-}(n)$, or $\mathbb{Sp}(2n)$, then $d_1(\mu, \nu) \leq C \frac{\log n}{n}$.

**Proof.**

1. For any fixed $\omega \in \mathbb{S}^1$, $\omega U$ is also Haar-distributed in $\mathbb{U}(n)$. Therefore $\mu$ is a rotation-invariant probability measure on $\mathbb{S}^1$, hence equal to $\nu$.
2. Observe first that $e^{2\pi i/n} I_n \in \mathbb{SU}(n)$, and so $e^{2\pi i/n} U$ is Haar-distributed in $\mathbb{SU}(n)$. Thus for any integer $k$,

$$
\mathbb{E} \text{tr} U^k = \mathbb{E} \text{tr}(e^{2\pi i/n} U)^k = e^{2\pi ik/n} \mathbb{E} \text{tr} U^k.
$$
Therefore $E \text{tr} U^k = 0$ for $1 \leq |k| < n$. If $g(z) = \sum_{|k|<n} a_k z^k$ is a trigonometric polynomial on $S^1$, it follows that

$$\int g \, d\mu = E \int g \, d\mu_U = \frac{1}{n} \sum_{|k|<n} a_k E \text{tr} U^k = a_0 = \int g \, d\nu.$$

Now given $f : S^1 \to \mathbb{R}$ which is 1-Lipschitz, Jackson’s theorem (see, e.g. [26, Theorem 1.4]) implies that there is such a polynomial $g$ such that $\|f - g\|_{\infty} \leq \frac{C}{n}$. Thus

$$\left| \int f \, d\mu - \int f \, d\nu \right| \leq \frac{2 \|f - g\|_{\infty}}{n} \leq \frac{2C}{n}.$$ 

(3) By results of Diaconis and Mallows (see [3]), Diaconis and Shahshahani [7], and Rains [25], in each of these cases $E \text{tr} U^k \leq 1$ for $1 \leq |k| < n$.

Given $f : S^1 \to \mathbb{R}$ which is 1-Lipschitz, it is easy to check that $|\hat{f}(k)| \leq \frac{C}{k}$ for $|k| \geq 1$ (see, e.g., Theorem 4.6 of [17]). If

$$S_n(z) = \sum_{k=-n}^{n-1} \hat{f}(k) z^k,$$

then

$$\left| \int S_n \, d\mu - \int S_n \, d\nu \right| = \frac{1}{n} \sum_{1 \leq |k| \leq n-1} \left| \hat{f}(k) E \text{tr} U^k \right| \leq \frac{C}{n} \sum_{1 \leq k \leq n-1} \frac{1}{k} \leq C \frac{\log n}{n}.$$ 

A theorem of Lebesgue (see, e.g., [26, Theorem 2.2]) implies that

$$\|f - S_n\|_{\infty} \leq C(\log n) \inf_g \|f - g\|_{\infty},$$

where the infimum is over all trigonometric polynomials $g(z) = \sum_{|k|<n} a_k z^k$. Combined with Jackson’s theorem [26, Theorem 1.4] this implies that $\|f - S_n\|_{\infty} \leq C \frac{\log n}{n}$, and thus

$$\left| \int f \, d\mu - \int f \, d\nu \right| \leq \left| \int f \, d\mu - \int S_n \, d\mu \right| + \left| \int S_n \, d\mu - \int S_n \, d\nu \right| + \left| \int S_n \, d\nu - \int f \, d\nu \right| \leq C \frac{\log n}{n}.$$ 

The second and third steps of the plan of this section rely on the following concentration of measure property. This essentially follows from a general isoperimetric inequality for Riemannian manifolds due to Gromov and Milman [12] and calculations of the Ricci curvature of the classical Lie groups (for which see [11, Appendix F]). In the precise form stated it follows from a result of Bakry and Émery [2] which shows that the same Ricci curvature bounds imply a logarithmic Sobolev inequality, which in turn implies such a concentration inequality (cf. [18, Chapter 5]).
Proposition 2.2 (See [1, Theorem 4.4.27]). Let $G$ be one of $\text{SO}(n)$, $\text{SO}^-(n)$, $\text{SU}(n)$, or $\text{Sp}(2n)$. Let $F : G \to \mathbb{R}$ be 1-Lipschitz with respect to the geodesic metric (induced by the standard embedding in matrix space with the Hilbert–Schmidt norm). If $U \in G$, then
\[
\mathbb{P}\left[ F(U) - \mathbb{E} F(U) \geq t \right] \leq e^{-cnt^2}
\]
for every $t > 0$.

The geodesic metric on $G$ dominates the Hilbert–Schmidt metric on matrix space, so the conclusion of Proposition 2.2 applies in particular to $F$ which is 1-Lipschitz with respect to the Hilbert–Schmidt metric.

The following lemma provides the necessary Lipschitz estimates for the functions to which the concentration property will be applied in this and the subsequent section.

Lemma 2.3. The map $A \mapsto \mu_A$ from $\mathbb{N}_n$ to $\mathcal{P}_1(\mathbb{C})$ taking a normal matrix to its spectral measure is $n^{-1/2}$-Lipschitz. Furthermore, if $\rho \in \mathcal{P}_1(\mathbb{C})$ is any fixed probability measure, the following statements hold.

1. For any 1-Lipschitz function $f : \mathbb{C} \to \mathbb{R}$, the function $A \mapsto \int f \, d\mu_A - \int f \, d\rho$ is $n^{-1/2}$-Lipschitz.
2. The map $A \mapsto d_1(\mu_A, \rho)$ is $n^{1/2}$-Lipschitz.

Proof. If $A$ and $B$ are $n \times n$ normal matrices, then the Hoffman–Wielandt inequality [3, Theorem VI.4.1] implies that
\[
\min_{\sigma \in S_n} \sum_{j=1}^n |\lambda_j(A) - \lambda_{\sigma(j)}(B)|^2 \leq \|A - B\|_{\text{HS}}^2,
\]
where $\lambda_1(A), \ldots, \lambda_n(A)$ and $\lambda_1(B), \ldots, \lambda_n(B)$ are the eigenvalues (with multiplicity, in any order) of $A$ and $B$ respectively. Defining couplings of $\mu_A$ and $\mu_B$ given by
\[
\pi_{\sigma} = \frac{1}{n} \sum_{j=1}^n \delta(\lambda_j(A), \lambda_{\sigma(j)}(B))
\]
for $\sigma \in S_n$, it follows from (1.1) and (2.1) that
\[
d_1(\mu_A, \mu_B)^2 \leq d_2(\mu_A, \mu_B)^2 \leq \min_{\sigma \in S_n} \int |w - z|^2 \, d\pi_{\sigma}(w, z)
\]
\[
= \min_{\sigma \in S_n} \frac{1}{n} \sum_{j=1}^n |\lambda_j(A) - \lambda_{\sigma(j)}(B)|^2 \leq \frac{1}{n} \|A - B\|_{\text{HS}}^2,
\]
proving the first statement of the lemma. The final claim that $A \mapsto d_1(\mu_A, \rho)$ is $n^{-1/2}$-Lipschitz is now immediate.

By the definition in (1.2) of $d_1$, given a 1-Lipschitz $f : \mathbb{C} \to \mathbb{R}$, the mapping $\mathcal{P}_1(\mathbb{C}) \to \mathbb{R}$, $\mu \mapsto \int f \, d\mu$ is 1-Lipschitz. Combined with the above argument, this implies that the function
\[
A \mapsto \int f \, d\mu_A - \int f \, d\rho
\]
is $n^{-1/2}$-Lipschitz on $\mathbb{N}_n$. □
Corollary 2.4. Let \( G \) be one of \( SO(n) \), \( SO^{-}(n) \), \( SU(n) \), or \( S\mathbb{P}(2n) \), and let \( U \in G \).

1. For any fixed probability measure \( \rho \in \mathcal{P}(S^1) \) and 1-Lipschitz \( f : S^1 \to \mathbb{R} \), define the random variable

\[
X_f = \int f \, d\mu_U - \int f \, d\rho.
\]

Then

\[
\mathbb{P}[|X_f - \mathbb{E}X_f| \geq t] \leq 2e^{-ct^2}
\]

for every \( t > 0 \).

2. For any fixed probability measure \( \rho \in \mathcal{P}(S^1) \),

\[
\mathbb{P}[d_1(\mu_U, \rho) - \mathbb{E}d_1(\mu_U, \rho) \geq t] \leq e^{-ct^2}
\]

for every \( t > 0 \).

Proof. The first part of the corollary follows from Proposition 2.2 and part (1) of Lemma 2.3 (applied to both \( X_f \) and \( -X_f = X_{-f} \)).

The second part of the corollary follows from Proposition 2.2 and part (2) of Lemma 2.3. \( \square \)

As noted earlier, the strategy outlined above does not apply directly to the full unitary and orthogonal groups, due to the lack of the concentration property of Proposition 2.2. The results of Gromov–Milman and Bakry–Émery fail to apply to \( O(n) \) because it is not connected, and to \( U(n) \) because its Ricci tensor is degenerate. Nevertheless, the main results of this section can be extended to \( U(n) \) and \( O(n) \). In the orthogonal case this will be done by conditioning on \( \det U \), which is why it is convenient to consider also the case of random matrices in \( SO^{-}(n) \). The unitary case could be handled in a similar way, but can also be deduced immediately from the special unitary case via the following lemma.

Lemma 2.5. If \( U \in U(n) \) and \( V \in SU(n) \), then \( d_1(\mu_U, \nu) \) and \( d_1(\mu_V, \nu) \) are identically distributed.

Proof. Define a coupling of \( U \) and \( V \) as follows. Let \( V \in SU(n) \) be Haar-distributed, and let \( \omega \in S^1 \) be uniformly distributed independently of \( V \). Define \( U = \omega V \).

Now given any fixed \( W \in U(n) \), \( W = \xi Y \) for some \( \xi \in S^1 \) and \( Y \in SU(n) \), and thus

\[
UW = (\omega \xi)(VY)
\]

and

\[
WU = (\xi \omega)(YV)
\]

both have the same distribution as \( \omega V = U \). Therefore \( U \in U(n) \) is Haar-distributed.

It follows that

\[
d_1(\mu_U, \nu) = d_1(\mu_{\omega V}, \nu) = d_1(\mu_V, \nu)
\]

since \( \mu_{\omega V} \) is a translation (in \( S^1 \)) of \( \mu_V \) and \( \nu \) is translation-invariant. \( \square \)

An analogous statement to Lemma 2.5 holds for \( O(n) \) and \( SO(n) \) when \( n \) is odd; in that case \( \omega, \xi \in \{-1, 1\} \) in the proof above. When \( n \) is even, \( -I_n \in SO(n) \) and so the argument breaks down, requiring a different approach to deducing the main results for \( O(n) \).

The next result carries out the second step in the plan of this section.

Theorem 2.6. Let \( G \) be one of \( O(n) \), \( SO(n) \), \( SO^{-}(n) \), \( U(n) \), \( SU(n) \), or \( S\mathbb{P}(2n) \), and let \( U \in G \). Then

\[
(2.2) \quad \mathbb{E}d_1(\mu_U, \nu) \leq Cn^{-2/3}.
\]
Proof. Assume for now that $G$ is one of $\mathrm{SO}(n)$, $\mathrm{SO}^-(n)$, $\mathrm{SU}(n)$, or $\mathrm{Sp}(2n)$.

Let $\operatorname{Lip}_0(\mathbb{S}^1) = \{ f \in \operatorname{Lip}(\mathbb{S}^1) : f(1) = 0 \}$, and observe that the Lipschitz seminorm $|\cdot|_{\operatorname{Lip}}$ is a norm on this space; denote by $B(\operatorname{Lip}_0(\mathbb{S}^1))$ its unit ball. For $f : \mathbb{S}^1 \to \mathbb{R}$, define the random variable

$$X_f = \int f \, d\mu_U - \int f \, d\mu.$$ 

Note that $\mathbb{E}X_f = 0$ for every $f$. Since the value of $X_f$ is unchanged by adding a constant to $f$, by [12],

$$d_1(\mu_U, \mu) = \sup \{ X_f : f \in B(\operatorname{Lip}_0(\mathbb{S}^1)) \}.$$ 

Fix $m \in \mathbb{N}$, to be determined later, and let $\operatorname{Lip}_0^m(\mathbb{S}^1)$ be the $(m-1)$-dimensional subspace of $\operatorname{Lip}_0(\mathbb{S}^1)$ consisting of functions which, when interpreted instead as $2\pi$-periodic functions on $\mathbb{R}$, are affine on each subinterval $[\frac{2(k-1)}{m}, \frac{2k}{m}]$ for $k \in \mathbb{Z}$. Given $f \in B(\operatorname{Lip}_0(\mathbb{S}^1))$, there is a unique $g \in B(\operatorname{Lip}_0^m(\mathbb{S}^1))$ such that $g(\exp(i\frac{2k\pi}{m})) = f(\exp(i\frac{2k\pi}{m}))$ for every $k$. Then $\|f - g\|_\infty \leq \frac{2\pi}{m}$, so that

$$|X_f - X_g| \leq \frac{2\pi}{m}$$

almost surely. It follows that

$$d_1(\mu_U, \mu) \leq \sup \{ X_g : g \in B(\operatorname{Lip}_0^m(\mathbb{S}^1)) \} + \frac{2\pi}{m}.$$ 

By Corollary [2.1] for $g, h \in B(\operatorname{Lip}_0^m(\mathbb{S}^1))$,

$$\mathbb{P}[|X_g - X_h| \geq t] = \mathbb{P}[|X_{g-h}| \geq t] \leq 2e^{-ct^2/|g-h|_{\operatorname{Lip}}^2}$$

for every $t > 0$. Thus by Dudley’s entropy bound [3],

$$\mathbb{E} \sup \{ X_g : g \in B(\operatorname{Lip}_0^m(\mathbb{S}^1)) \} \leq \frac{C}{n} \int_0^\infty \sqrt{\log N(B(\operatorname{Lip}_0^m(\mathbb{S}^1)), |\cdot|_{\operatorname{Lip}}, \epsilon)} \, d\epsilon,$$

where $N(B(\operatorname{Lip}_0^m(\mathbb{S}^1)), |\cdot|_{\operatorname{Lip}}, \epsilon)$ denotes the minimum number of $\epsilon$-balls with respect to $|\cdot|_{\operatorname{Lip}}$ needed to cover $B(\operatorname{Lip}_0^m(\mathbb{S}^1))$. (For a very neat exposition of Dudley’s bound, see Section 1.2 of [20].) Since $B(\operatorname{Lip}_0^m(\mathbb{S}^1))$ is itself a ball with respect to the norm $|\cdot|_{\operatorname{Lip}}$, there is the standard volumetric estimate [21] Lemma 2.6]

$$N(B(\operatorname{Lip}_0^m(\mathbb{S}^1)), |\cdot|_{\operatorname{Lip}}, \epsilon) \leq \left( \frac{3}{\epsilon} \right)^{m-1}.$$ 

Inserting this into (2.4) and then inserting the resulting estimate into (2.3) yields

$$\mathbb{E}d_1(\mu_U, \mu) \leq C\sqrt{\frac{m}{n}} + \frac{2\pi}{m}.$$ 

Picking $m$ of the order $n^{2/3}$ yields that

$$\mathbb{E}d_1(\mu_U, \mu) \leq \frac{C}{n^{2/3}},$$

and so the theorem (except for the cases of $\mathrm{O}(n)$ and $\mathrm{U}(n)$) follows by Theorem 2.1 and the triangle inequality for $d_1$.

If $G = \mathrm{U}(n)$, then the theorem follows from Lemma 2.5 and the case of $\mathrm{SU}(n)$.

If $G = \mathrm{O}(n)$, then conditionally on $\det U$, $U$ is Haar-distributed in either $\mathrm{SO}(n)$ or $\mathrm{SO}^-(n)$. Since

$$\mathbb{E}d_1(\mu_U, \nu) = \mathbb{E}\left[ d_1(\mu_U, \nu) \mid \det U \right],$$

...
the theorem follows from the cases of $\mathbb{O}(n)$ and $\mathbb{SO}^-(n)$. □

A direct union bound argument can also be used in place of Dudley’s theorem in the proof of Theorem 2.6, but the argument given above is considerably more elegant and concise.

The next two results complete the plan of this section.

**Corollary 2.7.** Let $G$ be one of $\mathbb{O}(n)$, $\mathbb{SO}(n)$, $\mathbb{SO}^-(n)$, $\mathbb{U}(n)$, $\mathbb{SU}(n)$, or $\mathbb{Sp}(2n)$, and let $U \in G$. Then
\[
\Pr \left[ d_1(\mu_U, \nu) \geq C n^{-2/3} + t \right] \leq e^{-cn^2 t^2}
\]
for every $t > 0$.

**Proof.** This follows immediately from Proposition 2.2 and Theorem 2.6, except in the cases of $\mathbb{O}(n)$ and $\mathbb{U}(n)$. If $G = \mathbb{U}(n)$, the corollary follows from Lemma 2.5 and the case of $\mathbb{SU}(n)$. If $G = \mathbb{O}(n)$, then
\[
\Pr \left[ d_1(\mu_U, \nu) \geq C n^{-2/3} + t \right] \leq E \left( \Pr \left[ d_1(\mu_U, \nu) \geq C n^{-2/3} + t \left| \det U \right| \right] \right)
\]
and the corollary follows from the cases of $\mathbb{SO}(n)$ and $\mathbb{SO}^-(n)$. □

**Corollary 2.8.** For each $n$ let $G_n$ be one of $\mathbb{O}(n)$, $\mathbb{SO}(n)$, $\mathbb{SO}^-(n)$, $\mathbb{U}(n)$, $\mathbb{SU}(n)$, or $\mathbb{Sp}(2n)$, and let $U_n \in G_n$. Then with probability 1,
\[
d_1(\mu_{U_n}, \nu) \leq C n^{-2/3}
\]
for all sufficiently large $n$.

**Proof.** Let $t = n^{-2/3}$ in Corollary 2.7 and apply the Borel–Cantelli lemma. □

The main results of this section can all be extended to Dyson’s circular ensembles (for extensive discussion, see [23]), by a slight variation of the same methods. The Circular Unitary Ensemble $\text{CUE}(n)$ is the same as the Haar distribution on $\mathbb{U}(n)$. The Circular Orthogonal Ensemble $\text{COE}(n)$ is distributed as $V^T V$, where $V$ is Haar-distributed in $\mathbb{U}(n)$. The Circular Symplectic Ensemble $\text{CSE}(2n)$ is distributed as $J V^T J^T V$, where $V$ is Haar-distributed in $\mathbb{U}(2n)$ and
\[
J = \begin{bmatrix}
0 & -1 & & & \\
1 & 0 & & & \\
0 & 0 & -1 & & \\
1 & 0 & 0 & & \\
& & & & \\
& & & & \\
0 & -1 & & & \\
1 & 0 & & &
\end{bmatrix}.
\]

**Theorem 2.9.** Let $U$ be drawn from COE$(n)$ or CSE$(2n)$. Then
\[
\mathbb{E} \mu_U = \nu,
\]
\[
\mathbb{E} d_1(\mu_U, \nu) \leq \frac{C}{n^{2/3}},
\]
and
\[
\Pr \left[ d_1(\mu_U, \nu) \geq C n^{-2/3} + t \right] \leq e^{-cn^2 t^2}
\]
for every $t > 0$. 

If, for each $n$, $U_n$ is drawn from COE($n$) or CSE($2n$), then with probability 1,
\[ d_1(\mu_{U_n}, \nu) \leq Cn^{-2/3} \]
for all sufficiently large $n$.

Proof. For brevity the proof is given only in the case of the COE, the argument for the CSE being nearly identical.

Let $U = V^T V$, where $V \in U(n)$ is Haar-distributed, and fix $e^{i\theta} \in \mathbb{S}^1$. Then $e^{i\theta/2} V$ is also Haar-distributed in $U(n)$, so $U$ has the same distribution as $(e^{i\theta/2} V)^T (e^{i\theta/2} V) = e^{i\theta} U$. Therefore $E_{\mu_U}$ is a rotation-invariant probability measure on $\mathbb{S}^1$, and is hence equal to $\nu$.

Next, arguing as in the proof of Lemma 2.5, it follows that $d_1(\mu_U, \nu)$ has the same distribution as $d_1(\mu_{W^T W}, \nu)$.

Now given $W_1, W_2 \in SU(n)$,
\[ \|W_1^T W_1 - W_2^T W_2\|_{HS} \leq \|W_1^T (W_1 - W_2)\|_{HS} + \|(W_1 - W_2) W_2\|_{HS} \]
\[ = \|W_1 - W_2\|_{HS} + \|W_1^T - W_2^T\|_{HS} \leq 2 \|W_1 - W_2\|_{HS}. \]

Thus the map $SU(n) \to SU(n)$ given by $W \mapsto W^T W$ is 2-Lipschitz, and so by Proposition 2.2
\[ \mathbb{P}[F(W^T W) - EF(W^T W) \geq t] \leq e^{-ct^2} \]
for every $t > 0$ and every 1-Lipschitz function $F : SU(n) \to \mathbb{R}$.

The remainder of the proof is the same as the proofs of Theorem 2.6, Corollary 2.7, and Corollary 2.8.

3. Some random Hermitian matrices

In this section, we prove results comparable to Theorem 2.6 and Corollaries 2.7 and 2.8 for two models of Hermitian random matrices. An essential condition on some of the random matrices used in the constructions below is the following.

Let $A$ be a random $n \times n$ Hermitian matrix. Suppose that for some $C, c > 0$,
\[ \mathbb{P}[|F(A) - EF(A)| \geq t] \leq C \exp[-ct^2] \]
for every $t > 0$ and $F : M_n^a \to \mathbb{R}$ which is 1-Lipschitz with respect to the Hilbert–Schmidt norm. Examples in which this condition is satisfied include:

1. The diagonal and upper-diagonal entries of $M$ are independent and each satisfy a quadratic transportation cost inequality with constant $c/\sqrt{n}$. This is slightly more general than assuming a log-Sobolev inequality (see [18, Section 6.2]), and is essentially the most general condition with independent entries (see [11]). It holds, e.g., for Gaussian entries and, more generally, for entries with densities of the form $e^{-n u_{ij}(x)}$ where $u_{ij}(x) \geq c > 0$.

2. The distribution of $M$ itself has a density proportional to $e^{-n \operatorname{tr} u(M)}$ with $u : \mathbb{R} \to \mathbb{R}$ such that $u''(x) \geq c > 0$. This is a subclass of the so-called unitarily invariant ensembles, studied extensively in mathematical physics (see [5]). The hypothesis on $u$, via the Bakry–Émery theorem, guarantees that $M$ satisfies a log-Sobolev inequality; cf. [11] Proposition 4.4.26.
One could also consider the situation in which (3.1) is only assumed to hold for convex Lipschitz functions $F$. By Talagrand’s theorem (see e.g. [18, Section 4.2]), this is the case if the diagonal and upper-diagonal entries of $M$ are independent and supported in sets of diameter at most $c/\sqrt{n}$. Under this weaker condition, the arguments below can be applied to prove results analogous to Theorem 2.6 and Corollaries 2.7 and 2.8, not for $d_1(\mu_M, E\mu_M)$ but for a “convex-Wasserstein distance” defined by

$$d_{1,c}(\mu, \nu) := \sup_{f \in B(\text{Lip}(\mathbb{R})), f \text{ convex}} \left| \int f d\mu - \int f d\nu \right|.$$  

This distance is also a metric for weak convergence of laws (see, e.g., the proof of [21, Theorem 2]).

The first model of random Hermitian matrix considered in this section is the following. Let $U \in U(n)$ distributed according to Haar measure, independent of $A$, and let $P_k$ denote the projection of $\mathbb{R}^n$ onto the span of the first $k$ basis elements. Define a random matrix $M$ by

$$M := P_k U A U^* P_k^*.$$  

Then $M$ is a compression of $A$ (as an operator on $\mathbb{R}^n$) to a random $k$-dimensional subspace chosen independently of $A$. In the case that $\{A_n\}_{n \in \mathbb{N}}$ is a deterministic sequence of matrices with a limiting spectral distribution and $\frac{\mu_A}{n} \to \alpha$, the limiting spectral distribution of $M$ can be determined using techniques of free probability (see [28]); the limit is given by a free-convolution power related to the limiting spectral distribution of $A_n$ and the value $\alpha$. The concentration properties of the spectral distribution of $M$ for $A$ deterministic were treated in [20], and the results below improve on those appearing in that paper.

In the case that $k = n$, the empirical spectral measure $\mu_M$ of $M$ is the same as $\mu_A$; in particular, if $A$ satisfies a log-Sobolev inequality and $k = n$, then the results below on the concentration of $\mu_M$ about its mean improve on the comparable results of Guionnet and Zeitouni from [13], both in terms of the specific bounds and in the metric used. (The metric used in [13], although referred to there as Wasserstein, is more commonly referred to as the bounded-Lipschitz distance and metrizes a slightly weaker topology than the metric used here.) We show below that the expected Wasserstein distance of $\mu_M$ to $E\mu_M$ is of order $n^{-2/3}$, whereas what follows from the results of [13] is that the expected bounded-Lipschitz distance of $\mu_M$ to $E\mu_M$ is of order $n^{-2/5}$.

In the further special case that the entries on and above the diagonal are assumed to be independent, the results below have been surpassed (in Kolmogorov distance) in the very recent work of Götze and Tikhomirov [10], who proved for such matrices that the Kolmogorov distance between the empirical spectral distribution and the semicircular law is almost surely of order $O(n^{-1} \log^b n)$ with some positive constant $b > 0$, under mild conditions on the distributions of the entries.

The proofs below follow the same approach as described in the final two steps of the outline given in Section 2. Namely, measure concentration, both on $U(n)$ and from the hypothesis of (3.1), is used together with entropy methods to show that $E d_1(\mu_M, \mu)$ is small, and moreover that $d_1(\mu_M, \mu)$ is strongly concentrated near its mean. Here, $\mu_M$ is again the empirical spectral measure of $M$ and $\mu = E\mu_M$; in this section, $\mu$ is always used as a reference measure. An additional truncation argument will be necessary, since the support of $\mu_M$ is not necessarily uniformly bounded in this context.
The following lemma is proved using a standard discretization argument.

**Lemma 3.1** (cf. [22, Proof of Proposition 4]). Suppose that \( \|E\|_{op} \leq C \) and \( A \) satisfies (5.1) for every convex 1-Lipschitz function \( F : \mathcal{M}^sa_n \to \mathbb{R} \). Then there is a constant \( K \) depending only on \( C, c, C' \) such that

\[
\mathbb{E} \|A\|_{op} \leq K.
\]

Observe that it follows from Lemma 3.1 that \( \mathbb{E} \|M\|_{op} \leq K \) for \( M \) defined in (3.2).

The next preliminary lemma and corollary are needed to obtain concentration properties for \( M \) from those of \( A \) and \( U \).

**Lemma 3.2.** Let \( A \in \mathcal{M}^sa_n \) be fixed. The map \( \mathbb{U}(n) \to \mathcal{M}^sa_n \), \( U \mapsto UA^* \) is \( \delta(A) \)-Lipschitz.

**Proof.** For \( \lambda \in \mathbb{R} \), let \( A_\lambda = A - \lambda I \). For any \( U, V \in \mathbb{U}(n) \),

\[
\|UA^* - VAV^*\|_{HS} = \|UA_\lambda U^* - V A_\lambda V^*\|_{HS}
\]

\[
= \|UA_\lambda(U - V)^* + (U - V) A_\lambda V^*\|_{HS}
\]

\[
\leq \|UA_\lambda(U - V)^*\|_{HS} + \|(U - V) A_\lambda V^*\|_{HS}
\]

\[
\leq \|UA_\lambda\|_{op} \|(U - V)^*\|_{HS} + \|U - V\|_{HS} \|A_\lambda V^*\|_{op}
\]

\[
= 2 \|A_\lambda\|_{op} \|U - V\|_{HS}.
\]

Here we have used the facts that

(1) \( \|U\|_{op} = 1 \) for \( U \in \mathbb{U}(n) \),

(2) \( \|AB\|_{op} \leq \|A\|_{op} \|B\|_{op} \) for \( A, B \in \mathcal{M}_n \), and

(3) \( \|AB\|_{HS} \leq \|A\|_{op} \|B\|_{HS} \) for \( A, B \in \mathcal{M}_n \).

Recalling that \( \delta(A) = \inf_{\lambda} \|A_\lambda\|_{op} \), optimizing over \( \lambda \) proves the lemma.

In [20] a weaker result is proved, essentially using instead of the third fact above the weaker estimate \( \|AB\|_{HS} \leq \|A\|_{HS} \|B\|_{HS} \).

**Corollary 3.3.** Let \( A \in \mathcal{M}^sa_n \) be fixed and let \( 1 \leq k \leq n \). Then the map \( \mathbb{U}(n) \to \mathcal{M}^sa_k \) given by \( U \mapsto P_kUA^*P_k^* \) is \( \delta(A) \)-Lipschitz.

**Proof.** Combine the Lemma 3.2 with the obvious fact that \( A \mapsto P_kAP_k^* \) is 1-Lipschitz \( \mathcal{M}^sa_n \to \mathcal{M}^sa_k \) (since \( P_kAP_k^* \) is just a submatrix of \( A \)).

**Theorem 3.4.** Suppose that \( A \) satisfies (5.1) for every 1-Lipschitz function \( F : \mathcal{M}^sa_n \to \mathbb{R} \).

(1) If \( F : \mathcal{M}^sa_n \to \mathbb{R} \) is 1-Lipschitz, then for \( M = P_kUA^*P_k^* \),

\[
\mathbb{P} \left[ \|F(M) - \mathbb{E}F(M)\|_{op} \geq t \right] \leq C \exp \left[ -cnt^2 \right]
\]

for every \( t > 0 \).

(2) In particular,

\[
\mathbb{P} \left[ \|M\|_{op} - \mathbb{E} \|M\|_{op} \| \geq t \right] \leq C \exp \left[ -cnt^2 \right]
\]

for every \( t > 0 \).

(3) For any fixed probability measure \( \mu \in \mathcal{P}_2(\mathbb{C}) \) and 1-Lipschitz \( f : \mathbb{R} \to \mathbb{R} \), if

\[
X_f = \int f \, d\mu - \int f \, d\mu_M,
\]

then

\[
\mathbb{P} \left[ \|X_f - \mathbb{E}X_f\|_{op} \geq t \right] \leq C \exp \left[ -cnt^2 \right]
\]

for every \( t > 0 \).
(4) For any fixed probability measure $\mu \in P_2(C)$ and $1 \leq p \leq 2$,
\[
\mathbb{P}\left[|d_p(\mu, \mu) - \mathbb{E}d_p(\mu, \mu)| \geq t\right] \leq Ce^{-cknt^2}
\]
for every $t > 0$.

Proof. For the first part, observe that
\[
\mathbb{P}\left[|F(M) - \mathbb{E}F(M)| \geq t\right] \leq \mathbb{E}\left(\mathbb{P}\left[|F(M) - \mathbb{E}[F(M)|U]| \geq \frac{t}{2}\right|U\right) + \mathbb{P}\left[|\mathbb{E}[F(M)|U] - \mathbb{E}F(M)| \geq \frac{t}{2}\right].
\]
Conditional on $U$, $F(M)$ is a 1-Lipschitz function of $A$, and by taking expectation over $U$ in Corollary 3.3, it follows that $\mathbb{E}[F(M)|U]$ is an $\mathbb{E}[\delta(A)]$-Lipschitz function of $U$. The first part thus follows from the hypothesis on $A$ and Lemma 3.1. Part (2) follows from part (1) and the fact that the operator norm is a 1-Lipschitz function with respect to the Hilbert–Schmidt norm on $\mathcal{M}_n^{sa}$. The remaining parts follow from Lemma 2.3 and part (1). □

To estimate $\mathbb{E}d_1(\mu_M, \mu)$ (where, as before, $\mu = \mathbb{E}\mu_M$) the arguments in the previous section can be supplemented with a truncation argument using the lemma above to obtain the following.

Theorem 3.5. Suppose that $A$ satisfies (3.1) for every 1-Lipschitz function $F: \mathcal{M}_n^{sa} \rightarrow \mathbb{R}$. Let $M = P_kUAU^*P_k$, and let $\mu_M$ denote the empirical spectral distribution of $M$ with $\mu = \mathbb{E}\mu_M$. Then
\[
\mathbb{E}d_1(\mu_M, \mu) \leq \frac{C''(\|M\|_{op})^{1/3}}{(kn)^{1/3}} \leq \frac{C''}{(kn)^{1/3}},
\]
and so
\[
\mathbb{P}\left[d_1(\mu_M, \mu) > \frac{C''}{(kn)^{1/3}} + t\right] \leq Ce^{-cknt^2}
\]
for each $t > 0$.

Proof. Denote by $\text{Lip}_0(\mathbb{R}) = \{f \in \text{Lip}(\mathbb{R}) : f(0) = 0\}$, and observe that
\[
\mathbb{E}d_1(\mu_M, \mu) = \mathbb{E}\sup \{X_f : f \in B(\text{Lip}_0(R))\},
\]
where
\[
X_f := \int f \, d\mu_M - \int f \, d\mu
\]
as before. The indexing space can be reduced to compactly supported functions via a truncation argument, as follows. Fix $R > 0$, and let
\[
f_R(x) = \begin{cases} 
 f(x) & \text{if } |x| \leq R; \\
 f(R) + \left[\text{sgn}(f(R))\right](R - x) & \text{if } R < x < R + |f(R)| \\
 f(-R) + \left[\text{sgn}(f(-R))\right](x - R) & \text{if } -|f(-R)| - R < x < -R; \\
 0 & \text{if } x \leq -R - |f(-R)| \text{ or } x \geq R + |f(R)|;
\end{cases}
\]
that is, \( f_R = f \) for \( |x| \leq R \) and then drops off linearly to zero, so that \( f_R \) is 1-Lipschitz, \( f(x) = 0 \) for \( |x| > 2R \), and \( |f(x) - f_R(x)| \leq |x| \) for all \( x \in \mathbb{R} \). Then by Fubini’s theorem,

\[
\left| \int f \, d\mu_M - \int f_R \, d\mu_M \right| \leq \int_{|x|>2R} |x| \, d\mu_M(x)
\]

\[
\leq 2R \int_{|x|>2R} d\mu_M(x) + \int_{2R}^{\infty} \mu_M((t, \infty)) \, dt + \int_{-\infty}^{-2R} \mu_M((\infty, t)) \, dt.
\]

Taking the supremum over \( f \) followed by expectation over \( M \), and making use of part (2) of Theorem 3.4 together with the trivial bound \( \mathbb{E} \mu_M((-\infty, t) \cup (t, \infty)) \leq n \mathbb{P}([\|M\|_{op} \geq t]) \)

\[
\mathbb{E} \sup \left\{ \int (f - f_R) \, d\mu_M : f \in B(\text{Lip}_0(\mathbb{R})) \right\} \leq CRn \exp \left[ -cn(2R - \mathbb{E} \|M\|_{op})^2 \right],
\]

and the same holds if \( \mu_M \) is replaced by \( \mu \). Taking, for example, \( 2R = \mathbb{E} \|M\|_{op} + 1 \) gives that

\[
\mathbb{E} \sup \{ |X_f - X_{f_R}| : f \in B(\text{Lip}_0(\mathbb{R})) \} \leq Cn \mathbb{E} \|M\|_{op} e^{-cn}.
\]

Consider therefore the process \( X_f \) indexed by \( \text{Lip}_{1/2}^{(\mathbb{E} \|M\|_{op} + 1)} \) (with norm \( \|\cdot\|_{\text{Lip}} \)), where

\[
\text{Lip}_{a,b} := \left\{ f : \mathbb{R} \to \mathbb{R} : |f|_{\text{Lip}} \leq a; f(x) = 0 \text{ if } |x| > b \right\}.
\]

The above argument shows that

\[
\mathbb{E} \left[ d_1(\mu_M, \mu) \right] \leq \mathbb{E} \left[ \sup \left\{ X_f : f \in \text{Lip}_{1/2}^{(\mathbb{E} \|M\|_{op} + 1)} \right\} \right] + Cn (\mathbb{E} \|M\|_{op}) e^{-cn}.
\]

Now that the indexing space of the process has been reduced to compactly supported functions, the proof can be completed exactly as in the case of Theorem 2.6; the additional error incurred by the truncation above is negligible compared to the errors produced by the earlier argument. The factor \( (\mathbb{E} \|M\|_{op})^{1/3} \) in the final bound is due to the size of the truncation parameter \( R \) (in the proof of Theorem 2.6, the corresponding quantity was simply \( 2\pi \) and therefore disappeared into the constants in the statement).

**Corollary 3.6.** For each \( n \), let \( A_n \in \mathcal{M}_n^{sa} \) be fixed with spectrum bounded independently of \( n \). Let \( U_n \in \mathcal{U}(n) \) be Haar-distributed and fix \( k \). Let \( M_n = P_k U A_n U^* P_k^* \) and let \( \mu_n = \mathbb{E} \mu_{M_n} \). Then with probability 1,

\[
d_1(\mu_{M_n}, \mu_n) \leq C n^{-1/3},
\]

where \( C \) depends only on \( k \) and the bounds on the sizes of the spectra of \( A_n \).

**Proof.** This follows from Theorem 3.5 using \( t = n^{-1/3} \) and the Borel–Cantelli lemma.

The second model of random matrix considered in this section is defined as follows. Let \( A,B \in \mathcal{M}_n^{sa} \) satisfy condition (3.1) let \( U \in \mathcal{U}(n) \) be Haar distributed, with \( A,B,U \) independent. Define

\[
M = U A U^* + B,
\]

the “randomized sum” of \( A \) and \( B \). In the case of deterministic sequences \( \{A_n\} \) and \( \{B_n\} \), this model has been studied at some length. The limiting spectral measure was studied first by Voiculescu [30] and Speicher [27], who showed that if \( \{A_n\} \) and \( \{B_n\} \) have limiting eigenvalue distributions \( \mu_A \) and \( \mu_B \) respectively, and if \( M_n := U A_n U^* + B_n \), then the limiting spectral distribution of \( M_n \) is given by the free convolution \( \mu_A \boxplus \mu_B \). More recently,
Chatterjee [4] showed subexponential concentration (up to a logarithmic factor) of $\mu_{M_n}$ about its mean; Kargin [16] improved this to subgaussian concentration (again up to a logarithmic factor), and was furthermore able to consider the distance to $\mu_{A_n} \boxplus \mu_{B_n}$ itself, rather than $E\mu_{M_n}$. Theorem 3.8 below gives a similar level of concentration to Kargin’s result. The main differences are that here the reference measure is $E\mu_{M_n}$ rather than a free convolution; the matrices $A_n$ and $B_n$ may be random here, whereas Kargin’s result requires $A_n$ and $B_n$ to be deterministic; and Kargin’s result is in terms of Kolmogorov distance, rather than Wasserstein distance.

The proofs below once again follow the same approach as described in the final two steps of the outline given in Section 2.

Note that by Weyl’s inequalities [3, Theorem III.2.1], the spectrum of $M$ always lies in the interval $[\lambda_{\min}(A) + \lambda_{\min}(B), \lambda_{\max}(A) + \lambda_{\max}(B)]$, of length $\delta(A) + \delta(B)$, and so by Lemma 3.1, $E\|M\|_{op}$ is bounded in terms of the constants in (3.1) for $A$ and $B$. We also have the following analog of Theorem 3.4.

**Theorem 3.7** (cf. [1, Corollary 4.4.30]). Let $A, B \in M_n^{sa}$ satisfying (3.1) and let $U \in U(n)$ be Haar-distributed with $A, B, U$ independent. Define $M = UAU^* + B$.

1. There exist $C, c$ depending only on the constants in (3.1) for $A$ and $B$, such that if $F : M_k^{sa} \rightarrow \mathbb{R}$ is 1-Lipschitz, then
   $$P[|F(M) - E F(M)| \geq t] \leq C \exp \left[-cn^2 t^2\right]$$
   for every $t > 0$.
2. In particular,
   $$P[\|M\|_{op} - E \|M\|_{op} \geq t] \leq C \exp \left[-cn^2 t^2\right]$$
   for every $t > 0$.
3. For any fixed probability measure $\rho \in \mathcal{P}_2(\mathbb{R})$ and 1-Lipschitz $f : \mathbb{R} \rightarrow \mathbb{R}$, let
   $$X_f = \int f \, d\mu_M - \int f \, d\rho.$$
   Then
   $$P[|X_f - E X_f| \geq t] \leq C \exp \left[-cn^2 t^2\right]$$
   for every $t > 0$.
4. For any fixed probability measure $\rho \in \mathcal{P}_2(\mathbb{R})$ and $1 \leq p \leq 2$,
   $$P[|d_p(\mu_M, \mu) - E d_p(\mu_M, \mu)| \geq t] \leq C \exp \left[-cn^2 t^2\right]$$
   for every $t > 0$.

**Proof.** (1) By the coupling described in the proof of Lemma 2.5, we may equivalently define
   $$M = (\omega V)A(\omega V)^* + B = VAV^* + B$$
for \( \omega \) and \( V \) independent with \( \omega \) uniformly distributed in \( S^1 \) and \( V \) Haar-distributed in \( SU(n) \). Now,

\[
P[|F(M) - \mathbb{E}[F(M)]| \geq t] \leq \mathbb{E}\left( P\left( |F(M) - \mathbb{E}[F(M)|A,V]| \geq \frac{t}{3} |A,V| \right) \right) + \mathbb{E}\left( P\left( |\mathbb{E}[F(M)|A,V] - \mathbb{E}[F(M)|V]| \geq \frac{t}{3} |V| \right) \right) + P\left( |\mathbb{E}[F(M)|V] - \mathbb{E}[F(M)]| \geq \frac{t}{3} \right).
\]

Conditional on \( A \) and \( V \), \( F(M) \) is a 1-Lipschitz function of \( B \), and by independence, the distribution of \( B \) is unchanged by conditioning on \( A \) and \( V \). The conditional distribution of \( B \) therefore still satisfies the concentration hypothesis and so the first summand above is bounded as desired. Similarly, conditional on \( V \), \( \mathbb{E}[F(M)|A,V] \) is a 1-Lipschitz function of \( A \), and the bound on the second summand follows from independence and the concentration hypothesis for \( A \). By Corollary 3.3, \( M \) is \( \delta(A) \)-Lipschitz as a function of \( V \); it follows that \( \mathbb{E}[F(M)|V] \) is an \( \mathbb{E}[\delta(A)] \)-Lipschitz function of \( V \), and the claim then follows from Lemma 3.1 and Proposition 2.2.

(2) This follows from the previous part and the fact that the operator norm is a 1-Lipschitz function with respect to the Hilbert-Schmidt norm on \( M_n^{sa} \).

(3) As a function of \( \mu_M \in P_1(\mathbb{R}) \), \( X_f \) is 1-Lipschitz by the duality between \( d_1 \) and 1-Lipschitz functions on \( \mathbb{R} \). By Lemma 2.3, \( \mu_M \) is \( n^{-1/2} \)-Lipschitz as a function of \( M \), and so the claim follows from the first part.

(4) This also follows from the first part and Lemma 2.3.

**Theorem 3.8.** In the setting of Theorem 3.7, there are constants \( c, C, C', C'' \) depending only on the concentration hypotheses for \( A \) and \( B \), such that

\[
\mathbb{E}d_1(\mu_M, \mu) \leq \frac{C(\mathbb{E}\|M\|_{op})^{1/3}}{n^{2/3}} \leq \frac{C'}{n^{2/3}},
\]

and so

\[
P\left( d_1(\mu_M, \mu) \geq \frac{C'}{n^{2/3}} + t \right) \leq C'' e^{-Ct^2}
\]

for \( t > 0 \).

The proof is exactly the same as the proof of Theorem 3.5.

**Corollary 3.9.** For each \( n \), let \( A_n, B_n \in M_n^{sa} \) be fixed matrices with spectra bounded independently of \( n \). Let \( U_n \in U(n) \) be Haar-distributed. Let \( M_n = U A_n U^* + B_n \) and let \( \mu_n = \mathbb{E}[\mu_{M_n}] \). Then with probability 1,

\[
d_1(\mu_{M_n}, \mu_n) \leq Cn^{-2/3}
\]

for all sufficiently large \( n \), where \( C \) depends only on the bounds on the sizes of the spectra of \( A_n \) and \( B_n \).

**Proof.** This follows from Theorem 3.8 using \( t = n^{-2/3} \) and the Borel–Cantelli lemma. \( \square \)
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