Bounded Verification of Doubly-Unbounded Distributed Agreement-Based Systems

Christopher Wagner, Nouraldin Jaber, and Roopsha Samanta

Purdue University, West Lafayette, USA
{wagne279,njaber,roopsha}@purdue.edu

Abstract. The ubiquity of distributed agreement protocols, such as consensus, has galvanized interest in verification of such protocols as well as applications built on top of them. The complexity and unboundedness of such systems, however, makes their verification onerous in general, and, particularly prohibitive for full automation. An exciting, recent breakthrough reveals that, through careful modeling, it becomes possible for verification of interesting distributed agreement-based (DAB) systems, that are unbounded in the number of processes, to be reduced to model checking of small, finite-state systems.

It is an open question if such reductions are also possible for DAB systems that are doubly-unbounded, in particular, DAB systems that additionally have unbounded data domains. We answer this question in the affirmative in this work for models of DAB systems, thereby broadening the class of DAB systems which can be automatically verified. We present a new symmetry-based reduction and develop a tool, Venus, that can efficiently verify sophisticated DAB system models.

1 Introduction

A recent breakthrough in formal reasoning about distributed systems builds on the modularity inherent in their design to enable modularity in their verification [26,22,36]. The central approach incorporates abstractions of common core protocols, such as distributed consensus, into verification of applications built on top of such core protocols. This approach inspires an interesting epiphany: modular models of distributed systems based on protocol abstractions may permit fully-automated verification à la model checking, even when their monolithic and intricate counterparts do not [20,24]. Amenability to full automation is significant because the problem of algorithmically verifying correctness of systems with an unbounded number of processes, popularly known as the parameterized model checking problem (PMCP), is a well-known undecidable problem [37]. For instance, recent work [20] identifies a class of modular models of distributed services, based on distributed agreement protocols, for which PMCP can be reduced to model checking of small, finite-state systems. However, a limitation of this approach (as well as most decision procedures for PMCP [19,20,24]) is that it requires each process in the distributed system model to be finite-state. This stipulation is not surprising—the distributed system becomes doubly-unbounded if, in addition to an unbounded number of processes, it has infinite-state processes.
In this paper, we seek to substantially expand the class of unbounded distributed agreement-based (DAB) systems which can be automatically and scalably verified through modular and bounded verification. Towards this goal, we characterize modular models of DAB systems, unbounded in the number of processes as well as variable domains whose verification can be reduced to that of small, bounded systems.

We tackle PMCP for systems that are unbounded along two dimensions, one dimension at a time. We first focus on the verification of DAB systems with unbounded variable domains and some fixed number \( n \) of processes. We identify conditions under which their verification can be reduced to model checking of \( n \)-process systems with small, bounded variable domains. Intuitively, our approach has the following features.

**Value Symmetry.** We first analyze if a system exhibits symmetry in its use and access of variables with unbounded domains, specifically, if the system’s correctness is preserved under permutations of values from these domains. As is standard, verification of such symmetric systems can exploit the induced redundancies, for instance, through the use of a smaller quotient structure.

**Data Saturation.** A quotient structure is not guaranteed to be finite-state in general. Hence, we seek to check if our value-symmetric system also exhibits data saturation; that is, if the size of its symmetry-reduced quotient structure has a finite upper bound, even when the systems’ data domains are unbounded.

**Domain Cutoffs.** Finally, to enhance the applicability of our domain reduction, we automatically infer bounds on the data domains of the system’s processes. We refer to these bounds as domain cutoffs. We argue that domain cutoffs are more flexible than an upper bound for a symmetry-reduced quotient structure.

With the data domains reduced to a fixed, finite space, it now becomes possible to leverage existing results \cite{26} for PMCP of DAB systems with an unbounded number of finite-state processes. Specifically, PMCP can now use process cutoffs to reduce verification to that of systems with a fixed, finite number of processes. We emphasize that our domain reduction and cutoffs are not restricted to be used only with \cite{26}; instead, our inference of domain cutoffs can facilitate flexible combination with varied techniques to obtain process cutoffs.

To summarize, we make the following contributions for DAB system models:

1. **Symmetry-based Domain Reduction** (Sec. 4): We characterize systems with unbounded data domains that exhibit symmetry and saturation in their data usage, thereby enabling a reduction of their verification to that of systems with finite, bounded domains.
2. **Domain Cutoffs** (Sec. 5): We present a sound procedure to infer bounds on the data domains of systems that permit our symmetry reduction.
3. **VENUS** (Sec. 7): We develop a tool, VENUS, for parameterized verification of doubly-unbounded systems that can efficiently compute domain/process cutoffs and verify sophisticated DAB systems.
We motivate and illustrate our contributions with an example distributed system, **Consortium**, that uses different types of distributed agreement to achieve trust-based consensus.

### 2.1 Motivation: Distributed Agreement-Based Systems

**Consortium.** The Consortium distributed system involves a set of actors who want to mutually make a decision based on information they gather individually. In order to do this efficiently, a subset of the actors is elected and trusted with making the decision and announcing it to the rest of the actors. This resembles scenarios where a trade-off between trust and performance is needed (e.g., a consortium blockchain [6,23]).

We model each actor/process of Consortium in **Mercury** [26], a modeling language with inbuilt primitives for distributed agreement. As can be seen in Fig. 1, **Mercury** facilitates a clean, *modular* design of DAB systems such as Consortium with its encapsulation of the intricacies of agreement protocols into agreement primitives.

#### Safety Property: Actors always have equal values of the variable `decision` in locations `ReplicaDone` and `LeaderDone`.

```mercury
1 process Consortium 24 location Decided
2 variables 25 on partition<share>(elect.winS,1)
3 int[1,3] motion 26 win: goto Announce
4 int[1,3] decision 27 lose: goto LeaderDone
5 events 28 location Announce
6 br inform : int[1,3] 29 on _ do
7 env rz influence : int[1,3] 30 broadcast(inform[decision])
8 env br reset : unit 31 goto LeaderDone
9 initial location Election 32 goto Election
10 on recv(influence) do 33 on recv(reset) do
11 motion := influence.payld 34 decision := default(decision)
12 on partition<elect>(All,2) 35 on recv(influence) do
13 win: goto Deliberate 36 decision := inform.payld 37 goto ReplicaDone
14 on recv(influence) do 38 goto Election
15 motion := influence.payld 39 location Wait
16 on recv(reset) do 40 location Announce
17 location Deliberate 41 on recv(inform) do
18 on recv(influence) do 42 decision := inform.payld 43 goto ReplicaDone
19 decision := vc.decVar[1] 44 goto Decided
20 on consensus<vc>(elect.winS,1,motion) do 45 location ReplicaDone
21 decision := vc.decVar[1] 46 on recv(reset) do
22 goto Decided 47 decision := default(decision)
23 goto Election 48 goto Election
```

Fig. 1: **Mercury** model of a Consortium process.
An actor initially starts in the **Election** location and coordinates with all other actors (Line 13) to elect at most two actors. Notice that this election is performed using a “**partition**” agreement primitive with identifier **elect** (Line 13); this instance of **partition** expresses that 2 actors are elected from among all actors. The elected actors move to the **Deliberate** location where they are trusted to make a decision for everyone. The remaining actors move to the **Wait** location (Line 15), where they wait to be informed once the elected actors agree on a decision (Line 41). The elected actors in **Deliberate** may be influenced (by the environment) to update their proposal, stored in variable **motion** (Line 18).

Further, the elected actors make a decision using a “**consensus**” agreement primitive with identifier **vc** (Line 21); this instance of **consensus** models agreement on 1 value proposed in the **motion** variable of processes elected in **Election**. After storing the decided value in the **decision** variable, the elected actors move to **Decided** where they elect one actor (Line 25) to announce the agreed-upon value to all actors (Line 31). Next, the elected actors move to **LeaderDone** and all other actors move to **ReplicaDone**. All actors then go back to the initial location, reinitializing their variables, to start further rounds (Lines 35, 46).

**Correctness Specification and Parameterized Verification.** The correctness specification of interest for a Consortium distributed system \( M(n) \) with \( n \) instantiations of the above actor is a safety property: all \( n \) actors in locations **ReplicaDone** and **LeaderDone** agree on the value of the variable **decision**.

We wish to ensure that a Consortium distributed system with an arbitrary, unbounded number of actors is correct. In particular, we are interested in fully-automated parameterized verification of Consortium that seeks to algorithmically check if \( M(n) \) satisfies its correctness specification for all values of \( n \). Furthermore, we are interested in modular verification that effectively exploits the modularity of MERCURY’s Consortium model.

**Prior Work: Unbounded Number of Processes and Bounded Domains.** While there exist multiple algorithms and tools for parameterized verification, we are aware of only two fully-automated, modular approaches \([25,26]\) that can tackle DAB systems, of which only one (QUICKSILVER \([26]\)) can tackle systems modeled in MERCURY. In fact, QUICKSILVER can perform parameterized verification of Consortium, as modeled in Fig. 1, efficiently, in less than a second.

Sadly, QUICKSILVER is limited to MERCURY systems composed of finite-state processes with a relatively modest number of states. Thus, even though QUICKSILVER can do verification of systems parameterized with an unbounded number of processes, it cannot handle systems with an unbounded number of processes and variable domains. Notice that a Consortium actor is finite-state with integer datatypes over small subranges. If the integer subrange datatypes of variables **motion** and **decision** in Fig. 1 are replaced with true integers (or even standard 32-bit integers), QUICKSILVER is unable to perform parameterized verification of the new system.

**This work: Unbounded Number of Processes and Unbounded Domains.** The inspiration for this work is two fold: (a) the effectiveness of MERCURY in enabling modular design and verification of DAB systems and (b) the confirm-
tion provided by QuickSilver that fully-automated parameterized verification of such systems is possible! We ask a natural question to help address QuickSilver’s limitations:

Is fully-automated, modular, and scalable parameterized verification possible for DAB systems with large, unbounded, or infinite-state processes?

In particular, can we algorithmically and efficiently perform parameterized verification of a doubly-unbounded version of Consortium where the the integer subrange datatypes of variables and events are replaced with (unbounded) integers? In what follows, we continue to denote the Consortium distributed system with $n$ instantiations of the finite-state actor in Fig. 1 as $M(n)$. Further, we denote Consortium with $n$ instantiations of the corresponding infinite-state actor (with unbounded integer datatypes) as $M_\Delta(n)$.

2.2 Contributions

Domain Reduction. Our key contribution employs notions of value symmetry and data saturation, to enable a reachability-preserving transformation of Mercury systems with unbounded variable domains to Mercury systems with finite domains. We illustrate these concepts and our domain reduction on our target (doubly-unbounded) Consortium system.

Notice that an actor of $M_\Delta(n)$ uses and accesses integer datatypes in a restricted way, in particular, as a scalarset datatype [24]—expressions over this datatype are restricted to (dis)equality checks and variables. Notice also that the safety specification accesses integer-valued variables as scalarsets. This “value symmetry” in an actor of $M_\Delta(n)$ as well as the safety specification implies preservation of correctness under permutations of scalarsets—an execution $\eta$ of $M_\Delta(n)$ violates the safety specification iff any execution corresponding to a permutation of the scalarset values in $\eta$ violates the safety specification.

This preservation of reachability under scalarset permutations suggests that many system executions are redundant for verification, and not all of them need to be explored. Quotient structures are used commonly to exploit such redundancies to verify properties of symmetric systems more efficiently. For value-symmetric systems, if the number of distinct values appearing in any reachable state has a finite upper bound, the size of the quotient structure can have a finite upper bound, even when the systems’ scalarset domains become unbounded or infinite; this property is called data saturation [24]. In particular, if such an upper bound can be established on the number of distinct scalarset values appearing in the reachable states of $M_\Delta(n)$, then safety verification of $M_\Delta(n)$ can be reduced to safety verification of a system with finite integer domains!

We now argue that a bound of three distinct values can be established for $M_\Delta(n)$, thereby reducing safety verification of $M_\Delta(n)$ to safety verification of $M(n)$ with integer subrange $[1,3]$. Any execution of $M_\Delta(n)$ is equivalent (w.r.t. the safety property) to one in which only three distinct integer values appear. This stems from the fact that (a) only one value at a time is needed to represent the most recent result of consensus, and (b) each actor has only two local
variables: motion and decision. While, in general, different actors may have distinct values in their local variables, because these values are never compared or communicated between actors (except during consensus), any execution in which actors have more than two distinct values in their local variables can be shown to be equivalent to an execution in which the actors have the same two values in their local variables. Ergo, to verify the unbounded-domain system $M_\Delta(n)$, it suffices to verify the $[1,3]$-bounded-domain system $M(n)$.

In this work, we identify conditions under which value symmetry and data saturation enable domain reduction for safety verification of Mercury systems with unbounded or infinite-state processes. In particular, if (a) all values from an unbounded scalarset domain held by a process in the system can be partitioned into two regions at any point in a system execution—a region of globally known values of which all processes are aware and a region of locally known values which may be known only to a single process—and (b) the maximum sizes of these two regions can be statically bounded, then the “large” unbounded scalarset domain can be replaced by a “small” finite one with size equal to the sum of the bounds of these two regions. We then show that if an unsafe execution exists in the “large” system, there must exist a related unsafe execution in the “small” system.

**Domain Cutoffs.** To support practical application of domain reduction, we present a procedure which analyzes Mercury processes to identify two regions as described above and their associated bounds. In doing so, we provide a concrete path to verification of Mercury models of DAB systems. As explained above, the two bounded regions yield Mercury systems with finite, bounded variable domains, which can then be verified using existing parameterized verification engines such as QuickSilver.

**Venus and Evaluation.** We have implemented a tool, Venus, that combines our domain reduction and cutoffs with QuickSilver and can perform fully-automated, modular, and scalable parameterized verification of Mercury system models with large, unbounded, or infinite-state processes. In particular, Venus is able to verify, in under a second, that $M_\Delta(n)$ satisfies its correctness specification for all values of $n$.

## 3 Problem Definition

Mercury is a modeling language for DAB systems consisting of an arbitrary number of identical processes and built on top of verified agreement protocols. As a native feature, Mercury includes two special event handlers, partition and consensus, which capture the general behavior of common agreement protocols (e.g., leader election and consensus). In what follows, we first review the syntax and semantics of Mercury systems, and then define the parameterized verification problem for such systems.

### 3.1 Mercury Systems

A Mercury process definition $P$ has three main components: (i) a set of typed process-local variable declarations, (ii) a set of typed event signatures defining
the set of coordination events which may occur during program execution, and (iii) a set of action handlers describing the behavior of a process when an event is initiated or received.

**Variable Declarations.** Each variable declaration consists of a variable name \( v \) and an associated variable domain which may be either a bounded integer range or the powerset of the set of process identifiers (i.e. \( v \) may store a set of process identifiers). We denote the set of variables as \( V \).

**Event Declarations and Actions.** Event declarations correspond to four different types of process coordination: (i) pairwise communication, denoted \( \text{pw} \), where one process sends a message, and one other process receives it, (ii) broadcast communication, denoted \( \text{bc} \), where a process sends a message, and all other processes receive it, (iii) partition coordination, denoted \( \text{pc} \), where a set of participating processes designates a finite subset of themselves as “winners”, and all other participating processes as “losers”, or (iv) consensus coordination, denoted \( \text{vc} \), where each process in a set of participating processes proposes a value, and all participating processes agree on a finite subset of the proposed values.

Each event declaration consists of an event type (i.e., one of \( \text{pw}, \text{bc}, \text{pc}, \text{vc} \)), event name \( \text{eID} \), a payload data type (either a bounded integer range or \( \text{unit} \)), and optionally a keyword \( \text{env} \) if the event is an interaction with the environment. Each event declaration describes a set of events and a set of actions. An event \( e \) corresponds to a particular payload value \( val \) of the associated payload data type.

An action is a polarized event where the polarity indicates if the event is acting (denoted \( e! \)) or reacting (denoted \( e? \)) to the event. In particular, acting (resp. reacting) events are sends (resp. receives) of broadcasts and pairwise communication, and correspond to “winners” (resp. “losers”) of agreement coordination.

We denote the sets of events, acting events, reacting events, and actions as \( E \), \( E! \), \( E? \), and \( E\overset{\triangleright}{} \).

**Example 1.** Consider the \( \text{inform} \) event declaration defined on Line 8 in Fig. 1. The event has type \( \text{bc} \), name \( \text{inform} \), and payload type \( \text{int} \). The set \( E \) of events associated with this event declaration (and induced by the values of the payload) is \{ \( \text{inform}[0] \), \( \text{inform}[1] \), \ldots \}. For each event \( e \in E \), there is an acting and a reacting action. For instance, the acting and reacting actions for event \( \text{inform}[0] \) are \( \text{inform}[0]! \) and \( \text{inform}[0]? \), respectively.

**Action Handlers.** Each location in the process definition is associated with a set of action handlers. An action handler comprises an action name, a guard, and a set of updates. A guard is a Boolean predicate over variables in \( V \) and the set of updates is essentially a parallel assignment of expressions to each variable.

**Mercury Semantics.** We refer to the semantics of a Mercury process as local and the semantics of a Mercury system composed of multiple identical Mercury processes as global.

The local semantics of a Mercury process, \( P \) is given by a labeled state-transition system \( M_P = (S, S_0, T) \) with a state space \( S \), a set \( S_0 \) of initial states,
and a set $T \subseteq S \times \mathcal{E} \times S$ of transitions induced by the set of action handlers. The state space $S$ corresponds to all possible valuations of variables in $\mathcal{V} \cup \{v_{\text{loc}}\}$, where $v_{\text{loc}}$ is a special variable defined to store the location. We denote the value of a variable $v$ in a local state $s \in S$ as $s(v)$. Action handlers for an event $e$ induce acting and reacting transitions in $T$ labeled with $e!$ and $e?$, respectively. In particular, $T$ contains a transition $(s, e!, s')$ for $e! \in \{e!, e?\}$ iff there exists an action handler for $e!$ such that the handler’s guard is true in $s$ and $s'$ is obtained by applying the handler’s updates to $s$.

The global semantics of a MERCURY system consisting of $n$ identical processes $P_1, \ldots, P_n$ and an environment process $E$ is given by a labeled transition system $\mathcal{M}(n) = (Q, Q_0, R)$ describing their parallel composition. Here, $Q = S^n \times \mathcal{E}_E$ is the set of global states, $Q_0 = S_0^n \times S_0, E$ is the set of initial global states, and $R \subseteq Q \times \mathcal{E} \times Q$ is the global transition relation capturing the process coordination necessary for different event types and payloads. For instance,

1. $R$ contains a “broadcast transition” $(q, e, q')$ for broadcast event $e$ iff (1) one process $P_i$ has a local “broadcast send” transition $(q[i], e!, q'[i])$ and (2) all other processes $P_j$ have corresponding local “broadcast receive” transitions $(q[j], e?, q'[j])$ with $q'[j]$ updated with the payload value.

2. $R$ contains a “Consensus transition” $(q, e_{\text{ID}}[V'], q')$ for Consensus event $e = e_{\text{ID}}[V']$, participant set $Prt$, and set of winning values $V'$ iff (1) each participating process in $Prt$ has a consistent view of the other participants, (2) every value $v$ in $V'$ is proposed by some process in $Prt$ with a local transition for $e!$, (3) each process in $Prt$ has a corresponding local $e?$ local transition, and (4) the local states of all other processes remain unchanged.

**Example 2.** Consider a Consortium system with two processes $P_1$ and $P_2$. Fig. 2 shows a broadcast transition on event $e = \text{inform}[99]$. Since $P_1$ has local transition $(q[1], e!, r[1]) \in T$ and $P_2$ has local transition $(q[2], e?, r[2]) \in T$, the global transition $(q, e, r)$ is in the global transition relation $R$.

We refer the reader to App. A for the complete local and global MERCURY semantics. An execution of a global transition system $\mathcal{M}(n)$ is defined in a standard way. A global execution is a (possibly infinite) sequence of global states, $q_0, q_1, \ldots$ in $Q$ such that for each $j \geq 0$, $(q_j, e, q_{j+1}) \in R$ for some event $e$. Global state $q$ is reachable if there exists a finite execution of $\mathcal{M}(n)$ that ends in $q$.

### 3.2 The Parameterized Verification Problem for MERCURY Systems

For a system $\mathcal{M}(n)$ with some number $n \in \mathbb{N}$ of finite-state processes $P$ and a correctness specification $\Phi$, we use $\mathcal{M}(n) \models \Phi$ to denote that the system
\( \mathcal{M}(n) \) satisfies \( \Phi \). The parameterized model checking problem (PMCP) targets the verification of a family \( \mathcal{M}(N) \) of systems \{\( \mathcal{M}(0), \mathcal{M}(1), \ldots \)\} w.r.t. correctness specification \( \Phi \). In particular, PMCP seeks to check if \( \forall n. \mathcal{M}(n) \models \Phi \) [9]. Note that this standard formulation of PMCP assumes that each process \( P \) has a finite-state space. In order to enable reasoning about MERCURY processes with unbounded and possibly infinite state spaces, we introduce new notation and a new formulation for the parameterized verification problem.

We denote a MERCURY process with a set \( \Delta \) of (possibly infinite) data domains as \( P_\Delta \) and a process with a set \( \overline{\Delta} \) of finite data domains as \( P_{\overline{\Delta}} \). We denote a MERCURY system with \( n \) instances of processes \( P_\Delta \) (resp. \( P_{\overline{\Delta}} \)) as \( \mathcal{M}_\Delta(n) \) (resp. \( \mathcal{M}_{\overline{\Delta}}(n) \)). In this paper, we target the parameterized verification problem over a family \( \mathcal{M}_\Delta(N) \) of MERCURY systems \{\( \mathcal{M}_\Delta(0), \mathcal{M}_\Delta(1), \ldots \)\}, defined as:

\[
\forall n. \mathcal{M}_\Delta(n) \models \Phi.
\]

The above problem generalizes PMCP for MERCURY systems from the verification of an infinite family \( \mathcal{M}_{\overline{\Delta}}(N) \) of finite-state systems \{\( \mathcal{M}_{\overline{\Delta}}(0), \mathcal{M}_{\overline{\Delta}}(1), \ldots \)\} to an infinite family \( \mathcal{M}_\Delta(N) \) of infinite-state systems \{\( \mathcal{M}_\Delta(0), \mathcal{M}_\Delta(1), \ldots \)\}.

## 4 Domain Cutoffs for MERCURY Systems

To enable parameterized verification over a family \( \mathcal{M}_\Delta(N) \) of MERCURY systems, we utilize value symmetry and data saturation to present a reduction of verification of the infinite-state MERCURY system \( \mathcal{M}_\Delta(n) = (Q, Q_0, R) \) to verification of a finite-domain MERCURY system. In particular, we characterize verification problems, denoted \( \langle \mathcal{M}_\Delta(n), \Phi \rangle \), that are domain-reducible and hence permit a reachability-preserving transformation that replaces the data domains of \( \mathcal{M}_\Delta(n) \) with finite, bounded ones. Let \( \mathcal{M}_{\overline{\Delta}}(n) = (\overline{Q}, \overline{Q_0}, \overline{R}) \) denote the MERCURY system with \( n \) finite-state processes \( P_{\overline{\Delta}} \), where \( P_{\overline{\Delta}} \) is obtained from \( P_\Delta \) by replacing the latter’s data domains \( \Delta \) with finite data domains \( \overline{\Delta} \). We show the following.

**Theorem 1.** For domain-reducible \( \langle \mathcal{M}_\Delta(n), \Phi \rangle \), \( \mathcal{M}_\Delta(n) \models \Phi \iff \mathcal{M}_{\overline{\Delta}}(n) \models \Phi \)

In what follows, we present a proof of this claim. Our proof relies on establishing a backward simulation relation \( \approx \) over pairs of global states in \( \mathcal{M}_\Delta(n) \) and \( \mathcal{M}_{\overline{\Delta}}(n) \): For all \( r \in Q, \tau \in \overline{Q} \), if \( r \approx \tau \) and \( (q, e, r) \in R \), then there is \( (\overline{q}, \overline{e}, \tau) \in \overline{R} \) such that \( q \approx \overline{q} \). This relation enables a proof of Theorem 1 by induction on an arbitrary error path in \( \mathcal{M}_\Delta(n) \) to show that a related error path exists in \( \mathcal{M}_{\overline{\Delta}}(n) \).

To simplify presentation, we assume, initially, that \( \Delta \) contains exactly one unbounded, scalarset domain \( \delta \). Further, we elide the environment process, which is not impacted by our reachability-preserving transformation.

### 4.1 Permutations and Scalarsets

We first review standard notions of permutations and scalarsets. Then, we introduce a notion of component-wise permutations. All notions of permutations are w.r.t. the unbounded, symmetric domain \( \delta \in \Delta \).
**Permutation.** A δ-permutation, \( \pi : \delta \mapsto \delta \), is a bijection mapping the set \( \delta \) onto itself. With some abuse of notation, we set \( \pi(\text{val}) = \text{val} \) for any value \( \text{val} \) with domain in \( \Delta \setminus \delta \). We further define liftings of δ-permutations to local states and events. An application \( \pi(s) \) of a δ-permutation \( \pi \) to a local state \( s \) is defined as:
\[
\forall v \in V : \pi(s)(v) = \pi(s(v)).
\]
An application \( \pi(e) \) of a δ-permutation \( \pi \) to an event \( e = \text{eID}[\text{val}] \) is \( \text{eID}[\pi(\text{val})] \), the permutation \( \pi(e^\gamma) \) of an action of event \( e \) is \( \pi(e)^\gamma \), and the permutation of a local transition \( (s, e^\gamma, s') \in T \) is \( (\pi(s), \pi(e^\gamma), \pi(s')) \).

**Example 3.** We illustrate permutations on our running example, Consortium. The figure to the right shows an example of applying a δ-permutation \( \pi \) to the local state of process \( P_1 \) from Fig. [2]. Let \( \delta \) denote the type \( \text{int} \). Recall that the local variables decision and motion (denoted \( D \) and \( M \) in the figure) are of type \( \text{int} \). The permutation \( \pi \) maps values 7 to 1, 99 to 2, and all other values appropriately so that \( \pi \) is a valid permutation (i.e., a bijection over the \( \text{int} \) domain). Notice that the value of the variable \( \nu_{\text{loc}} \) (denoted \( L \) in the figure) is not of type \( \text{int} \) and, hence, it is not changed in the permuted state (i.e. \( \pi(LD) = LD \)).

**Scalarsets.** A scalarset domain [24] is a set of distinct elements with restricted operations. Specifically, (i) all valid scalarset terms are variable references; there are no scalarset constants, (ii) scalarset terms may only be compared using (dis)equality and only with terms of the same scalarset type, and (iii) scalarset variables may only be assigned values of exactly the same scalarset type. These restrictions ensure that the local transition relation is invariant over permutations of a scalarset:

**Lemma 1.** \( \forall \pi \in G, (s, \alpha, s') \in T : \pi((s, \alpha, s')) \in T \)

For the proof, see App. [3.1]. We note that the bounded integer domains in MERCURY can be treated as scalarsets if used according to the restrictions above. For instance, the decision variable in Fig. [3] is of type \( \text{int} \) but can be treated as a scalarset variable because it conforms to the constraints (i) through (iii).

**Component-wise Permutation.** Next, we define a new type of transformation on global system states, called a **component-wise permutation** (CWP). CWP consists of a series of separate δ-permutations which are applied, component-wise, to each local state in a global state. For a global state \( q = (s_1, s_2, \ldots, s_n) \) let \( \gamma = (\pi_1, \pi_2, \ldots, \pi_n) \) be a CWP over the values of \( \delta \) and \( \gamma(q) \) be the component-wise application of permutations in \( \gamma \) to local states in \( q \). That is, \( \gamma(q) = (\pi_1(s_1), \pi_2(s_2), \ldots, \pi_n(s_n)) \). Informally, one can think of a CWP as a tool permute a global state \( q \) in \( M_\Delta(n) \) with an unbounded data domain to a global state \( \gamma(q) \) in the target system, \( M_\Delta(n) \), with a bounded data domain, effectively

---

*Fig. 3: A CWP.*
"collapsing" it. Such a reduction from unbounded to bounded domains is possible because a CWP is allowed to permute different values in different components to the same value, as shown in the following example.

Example 4. An example CWP $\gamma = (\pi_1, \pi_2)$ applied to a global state $r$ of Consortium, yielding $r'$, is shown in Fig. 3. The state of process $P_1$ (resp. $P_2$) is permuted according to $\pi_1$ (resp. $\pi_2$). Observe that the larger values in $r$ are permuted to smaller values in $r'$. Further, notice that while there are three distinct values in $r$, there are only two in $r'$.

4.2 Scalarset Domain Reduction

While, in general, a CWP $\gamma$ can permute the same value in different components to different permuted values, at certain global states in an execution, equality between some values in different processes is significant (e.g., when the results of consensus are determined). Hence, component permutations of $\gamma$ must permute such values consistently to preserve their equality (i.e., if such values are equal before applying $\gamma$, they should be after $\gamma$ is applied). Next, we characterize these regions in the execution where values must be permuted consistently.

Value-Stable and Bounded Regions. A region $\Psi \subseteq S \times V$ is a set of pairs of process-local states and variables. Given a global state $q$, we denote the set of $\delta$-values appearing in $q$ as $\Pi_\delta(q)$, appearing in the local state $q[i]$ of the $i^{th}$ process as $\Pi_\delta(q[i])$. Given a global state $q$ and a region $\Psi$, we denote the set of $\delta$-values appearing in corresponding elements of $\Psi$ as $\Pi_{\Psi}(q)$, i.e., $\Pi_{\Psi}(q) = \{val = s(v) \mid (s, v) \in \Psi \text{ and } \exists i : q[i] = s\}$.

We say a region $\Psi$ is value-stable if there exists a finite upper bound $\rho$ on the number of distinct values in $\Pi_{\Psi}(q)$ over all reachable global states $q$. Additionally, a value-stable region $\Psi$ is bounded if there exists a finite upper bound, $\lambda$, over all reachable global states $q$, on the number of distinct $\delta$-values that are held by any single process and are not in $\Pi_{\Psi}(q)$. More precisely, for a bounded, value-stable region, there exists $\lambda$ such that for all reachable global states $q$: $\forall i : |\Pi_\delta(q[i]) \setminus \Pi_{\Psi}(q)| \leq \lambda$.

Consistent and Minimal CWPs. For a given region $\Psi$, we say that a CWP $\gamma$ is consistent w.r.t. a global state $q$, denoted $\text{cons}(\gamma, q, \Psi)$, iff all component permutations of $\gamma$ map each value in $\Pi_{\Psi}(q)$ to the same permuted value. Formally, $\text{cons}(\gamma, q, \Psi)$ iff $\forall i, j, \forall val \in \Pi_{\Psi}(q) : \gamma[i](val) = \gamma[j](val)$. Additionally, we say that a CWP $\gamma$ is minimal with respect to a global state $q$, denoted $\text{min}(\gamma, q, \Psi)$, iff each component permutation $\pi_i$ of $\gamma$ permutes all values in $\Pi_\delta(q[i]) \setminus \Pi_{\Psi}(q)$ to the smallest available values i.e., to the smallest values, according to an arbitrary total order over $\delta$, that are not in the image of any component permutation.

Example 5. For Fig. 3, consider a bounded region $\Psi$ that contains the state-variable pair $(r[2], \text{decision})$. A CWP $\gamma$ is consistent w.r.t $r$ iff all the component permutations of $\gamma$ permute the value 99 (in the decision variable in $r[2]$) consistently to the same permuted value (e.g., to 2 as done by $\gamma = (\pi_1, \pi_2)$). On the other hand, if $\pi_1$ is changed to permute 99 to, say 3 instead of 2, then
the resulting $\gamma$ is not consistent. Further, assume that $\Psi$ does not contain the state-variable pairs $(r[1], \text{motion})$ and $(r[2], \text{motion})$. The CWP $\gamma$ in Fig. 3 is minimal since it permutes the values outside of $\Psi$ (7 and 1 in $(r[1], \text{mention})$ and $(r[2], \text{mention}$, respectively) to the smallest available value, 1. If $\pi_1$ is changed to permute 7 to, say 3 instead of 1, then the resulting $\gamma$ would not be minimal, because $\pi_1$ could have permuted 7 to a smaller value, 1.

**Consistency.** We denote by $\Gamma_{s}$ the set of all CWPs over the domain $\delta$. For a given region $\Psi$, we define the corresponding consistency set $\Gamma^\Psi_{s}(q)$ w.r.t. a global state $q$ to be the set of CWPs which are consistent and minimal w.r.t $q$. That is, $\Gamma^\Psi_{s}(q) = \{q \in \Gamma_{s} \mid \text{cons}(\gamma, q, \Psi) \land \text{min}(\gamma, q, \Psi)\}$.

For a domain $\delta$ and bounded region $\Psi$, we define an equivalence relation $\approx_{\Psi}$, over pairs of global states in $\mathcal{M}(n)$ and $\mathcal{M}(\Sigma(n))$, derived from the consistency sets of global states in $\mathcal{M}(n)$: $\approx_{\Psi} = \{(q, \bar{q}) \in Q \times \bar{Q} \mid \exists \gamma \in \Gamma^\Psi_{s}(q) : \gamma(q) = \bar{q}\}$.

**Scalarset Domain Reduction.** For the system $\mathcal{M}(n)$ composed of $n$ instances of $P_\delta$, we say that a region $\Psi$ is encompassing if it is value-stable and bounded, and it satisfies a set of conditions (detailed in section 5.1) which ensure its consistency set preserves the global semantics of $\mathcal{M}(n)$. In other words, for any $q \in Q$ and CWP $\gamma \in \Gamma^\Psi_{s}(q)$, $\gamma$ consistently permutes any values which may have been received in $q$, and if $q$ is an initial (resp. error) state, $\gamma(q)$ is also an initial (resp. error) state.

Then, let us assume the existence of an encompassing region $\Psi$, with bounds $\rho$ and $\lambda$ as defined earlier. When such a bounded region $\Psi$ exists for $\mathcal{M}(n, \Phi)$, we say that $(\mathcal{M}(n, \Phi))$ is domain-reducible. Let $\delta$ be a scalarset domain of size $\rho + \lambda$ and. We denote by $P_{\delta}$, the process obtained from $P_\delta$ by replacing all variable and event declarations of type $\delta$ with declarations of type $\tilde{\delta}$. Let $\mathcal{M}(\Sigma(n))$ be the system composed of $n$ instances of $P_\delta$.

**Lemma 2.** $\forall r \in Q, \tau \in \bar{Q}, (q, e, r) \in R : r \approx_{\Psi} \tau \implies (\exists(q, \bar{q}, \bar{r}) \in \bar{R} : q \approx_{\Psi} \bar{q})$.

**Proof sketch.** Since $r \approx_{\Psi} \tau$, we know there must exist a CWP $\gamma$ such that $\gamma(r) = \tau$. Then, we must identify an appropriate global state $\bar{q}$ and $\bar{r}$ such that $(\bar{q}, \bar{r}, \bar{\tau}) \in \bar{R}$ and $q \approx_{\Psi} \bar{q}$. Since $\bar{q}$ must be related to $q$ by $\approx_{\Psi}$, we need to show that there exists another CWP $\gamma'(q)$ such that $\gamma'(q) = \bar{q}$ and $\gamma'(r)$ such that $\gamma'(r), \bar{r}, \bar{\tau} \in \bar{R}$. The lemma is proven by letting $\bar{q} = \gamma'(q)$. We carefully define $\gamma'$ to (i) agree with $\gamma$ on values preserved in the transition from $q$ to $r$ so that $\gamma'(q)$ may transition to $\tau$, (ii) permute values in $\Pi^\Psi_{\delta}(q)$ consistently and values outside of $\Pi^\Psi_{\delta}(q)$ to the smallest possible values so that $\gamma' \in \Gamma^\Psi_{\delta}(q)$. With $\gamma'$ in hand, it remains to show that there exists an event $\bar{e}$ such that the transition $(\gamma'(q), \bar{e}, \bar{\tau} \in \bar{R}$, which can be achieved by permuting each transmitted value to match the permutation (in $\gamma$) of a process that receives it. We refer the reader to App. B.22 for the full proof.

**Reducing Multiple Domains.** We focused on the single reduction $\mathcal{M}(n, \Phi) \xrightarrow{\delta} \mathcal{M}(n)$ with respect to unbounded scalarset domain $\delta \in \Delta$. Our results extend immediately to the case with multiple unbounded data domains via a sequence of reductions w.r.t. each $\delta \in \Delta$: $\mathcal{M}(n) \xrightarrow{\delta_1} \mathcal{M}(n) \xrightarrow{\delta_2} \ldots \xrightarrow{\delta_n} \mathcal{M}(\Sigma(n))$. 


5 Determining Bounded Regions

Recall that if there exists a bounded region $\Psi$ with (i) an upper bound $\rho$, over all reachable global states $q$, on the number of unique $\delta$-values $\Pi^\Psi_\delta(q)$ stored in variables of processes in $\Psi$, and (ii) an upper bound $\lambda$, over every local state $s$ in any reachable global state $q$, on the number of unique $\delta$-values not in $\Pi^\Psi_\delta(q)$, then we can reduce the domain-reducible verification problem $\langle M^{\Delta(n)}, \Phi \rangle$ to the verification for the bounded-domain system $M^{\Delta}(n)$, w.r.t. $\Phi$. In this section, we discuss how such a bounded region $\Psi$ and bounds $\rho$, $\lambda$ may be determined.

**Location Transition System (LTS).** To determine $\Psi$ with bound $\rho$, we inspect the movement of values in the MERCURY system. In particular, we wish to examine values that may flow into a set of local states, without constructing the unbounded local semantics of $P^{\Delta}$. So, we begin by defining a finite-state transition system $A$ capturing the control-flow of a MERCURY process: $A = \langle L, E \rangle$, where each node in $L$ corresponds to a location in $P^{\Delta}$ and there exists an edge $(a, b)$ in $E$ iff there exists a transition in the local semantics of $P^{\delta}$ between states in locations $a$ and $b$. Let the location of node $x \in L$ be denoted $loc(x)$ and the set of local states with $v_{loc} = loc(x)$ be denoted $[x]$. To track relevant data flow, we extend $A$ with some bookkeeping information for each edge, in particular, the set of action handlers yielding the edge. Let the set of edges derived from a handler $h$ be $edges(h)$. Notice that $A$ constitutes an over-approximation of the local semantics of $P^{\Delta}$.

**Example 6.** The Partition handler (Fig. 1 Line 25) yields an edge $e = (n_1, n_2) \in E$, where $loc(n_1) = Decided$ and $loc(n_2) = Announce$.

Finally, we extend our notion of a region from Sec. 4.2 to LTSs. Let an abstract region $r \subseteq L \times \cal{V}$ be a set of node-variable pairs such that each node-variable pair $(x, v)$ represents the set $\{(s, v) \mid s \in [x]\}$ of state-variable pairs. Let $[r]$ be the concrete region $\{(s, v) \mid (s, v) \in [(x, v)] \land (x, v) \in r\}$ that the abstract region $r$ represents. We say an abstract region $r$ is value-stable (resp. bounded) iff the corresponding concrete region $[r]$ is value-stable (resp. bounded). The bounds $\rho(r)$ and $\lambda(r)$ of an abstract region $r$ are equal to the bounds $\rho$ and $\lambda$ of $[r]$.

5.1 Domain-Cutoff Analysis

We now present a procedure for determining a suitable abstract bounded region (ABRs), defined in Algo. 1.

**Initial Regions.** We identify a set of (initial) abstract regions where for each region $r$ in that set, it is clear that any number of processes in $r$ have a finite number of distinct values of type $\delta$ among the associated variables of $r$ (i.e., $\rho(r)$ is finite). To that end, we identify three types of such initial ABRs as defined

---

2 The LTS constructed in our implementation is refined further based on the local guards in the process definition. We omit detailing this refinement to simplify the presentation.
Algorithm 1: Determining an abstract bounded region \( r \)

1. procedure getAbstractBoundedRegion(\( A, P_3 \))
   
   Inputs: A process \( P_3 \), and the LTS \( A \)
   
   Output: \( r \), an ABR
   
   2. regions = getInitialRegions(\( P_3, A \))
   
   3. regions = expandRegions(regions, \( A \))
   
   4. regions = mergeRegions(regions, \( A \))
   
   5. \( r = \text{getMinimalRegion}(\text{regions}) \)
   
   6. return \( r \);

in Algo. 2. First, for each consensus action \( vc \), create an ABR composed of all
the nodes corresponding to locations where \( vc \) terminates along with the variable
holding the decided values (Line 3). The bound \( \rho \) of this ABR matches the
cardinality of the consensus action \( vc \). Intuitively, by the nature of consensus,
we know that these variables hold the agreed-upon values, which are consistent
across participating processes in these locations. Second, if the locations corre-
spanding to a set of nodes can only be occupied by a single process at a time
(e.g. a server), we create an ABR for each of these nodes with every variable of
type \( \delta \) in the system (Line 8). The bound \( \rho \) of each of these ABRs is the number
of variables of type \( \delta \), as one process can only hold that many unique \( \delta \)-values.
Finally, we create a single ABR (with \( \rho = 1 \)) encoding variables of type \( \delta \) in the
initial location, where all processes hold some initial default value (Line 12).

Expanding Regions. Initial regions are expanded to maintain the bound \( \rho \).
Intuitively, we follow the flow of values between variables so that any node-
variable pair that only gets values from some ABR region is added to that region
(Line 6). We denote by \( \text{gets}(x, v, x', v') \) that for some \( s' \in [x'] \) the value \( s'(v') \)
goes to variable \( v \) in some state \( s \in [x] \) via a direct assignment or a transmission.

Merging Regions. If two ABRs \( r_1 \) and \( r_2 \) are mutually exclusive (i.e., processes
are only in one of them at a time), we create a new ABR \( r = r_1 \cup r_2 \) with
\( \rho(r) = \max(\rho(r_1), \rho(r_2)) \). Since processes may not be in \( r_1 \) and \( r_2 \) simultaneously,
their bounds are independent and the larger applies to both regions.

A Minimal Region. After constructing a set of ABRs, we select those which meet
a few conditions to support the domain-reduction technique from Sec. 3. In
particular, for some ABR \( r \), we require that the corresponding bounded region \([r]\)
must include (i) all initial local states, (ii) all destination local states of consensus
actions, (iii) all source local states of broadcast actions with payloads, and (iv)
all state-variable pairs referred to by the global specification \( \Phi \). We select the
final ABR \( r \) that satisfies these criteria and has the minimal bound \( \rho \) of all such
candidate ABRs. The bounded region \( \Psi \) is then \([r]\) with bound \( \rho = \rho(r) \). We
determine the local bound \( \lambda \) by simply counting the number of variables of type
\( \delta \) in the program \( P_3 \) that appear in state-variable pairs outside of \( \Psi \).

Example 7. Consider the Consortium system (Fig. 1). According to Algo. 2 the
initial regions include \( r_1 = \{(\text{Decided}, \text{decision})\} \) with \( \rho(r_1) = 1 \) which captures
the results of the \( vc \) consensus event, and \( r_2 = \{(\text{Election}, \text{decision)},
(\text{Election}, \text{motion})\} \) with \( \rho(r_2) = 1 \) which captures the initial default values.
Algorithm 2: Constructing initial regions

```plaintext
procedure getInitialRegions(\(P_\delta, A\))
Inputs : A MERCURY program \(P_\delta\) and an LTS \(A\)
Output: a set of ABRs
1 regions = {}  
2 foreach a \(\in\) consActions(\(P_\delta\)) do
3 \(H = \text{handlersOf}(a)\)
4 \(r = \{(\text{dst}(e), a.decVar) | h \in H \land e \in \text{edges}(h)\}\)
5 \(\rho(r) = \text{cardOf}(h)\)
6 regions = regions \(\cup\) r
7 foreach s \(\in\) serverRegion(\(A\)) do
8 \(r = \{(s) \times V\}
9 \(\rho(r) = |V|\)
10 regions = regions \(\cup\) r
11 r = \{(x,v) | \text{\(\not\exists\)} (x',v') : \text{gets}(x,v,x',v')\}
12 \(\rho(r) = 1\)
13 regions = regions \(\cup\) r
14 return regions
```

Algorithm 3: Expanding regions

```plaintext
procedure expandRegions(regions, \(A\))
Inputs : The LTS \(A\) and an initial set regions of regions
Output: a set of ABRs
1 foreach r \(\in\) regions do
2 do
3 changeMade = false
4 foreach (x,v) \(\not\in\) r do
5 if \((x',v') : \text{gets}(x,v,x',v')\) \(\subseteq\) r then
6 \(r = r \cup \{(x,v)\}\)
7 changeMade = true
8 while changeMade;
9 return regions
```

Algo. 3 expands the initial regions. One such expansion step of region \(r_1\) through edge \(e\) from Example 6 adds the node-variable pair (Announce, decision) to \(r_1\) since the value in the decision variable is preserved along edge \(e\). A minimal valid ABR that meets conditions (i) through (iv) is \(r = \{(\text{Election, motion}), (\text{Election, decision}), (\text{Decided, decision}), (\text{Announce, decision}), (\text{LeaderDone, decision}), (\text{ReplicaDone, decision})\}\). This \(r\) is the result of merging the expansions of \(r_1\) and \(r_2\) and hence has \(\rho(r) = \max(\rho(r_1), \rho(r_2)) = 1\). Finally, \(\lambda\) is 2 since both the decision and motion variables appear in node-variable pairs outside of \(r\) (e.g., in the (Wait, decision) and (Wait, motion) node-variable pairs).
6 Solving PMCP for Effectively Bounded $\langle M_\Delta(n), \Phi \rangle$

Let $\langle M_\Delta(n), \Phi \rangle$ be a domain-reducible verification problem with domain cut-off $\Delta$. Let $c$ be a process cutoff computable by some cutoff-based procedure (e.g., [26]) for parameterized verification of the reduced system $M_{\overline{\Delta}}(n)$ w.r.t. safety specification $\Phi$. We then refer to the tuple $\langle M_\Delta(n), \Phi \rangle$ as effectively bounded. The procedure for parameterized verification of effectively bounded $\langle M_\Delta(n), \Phi \rangle$ is summarized by the following result.

**Theorem 2.** $\forall n : M_\Delta(n) \models \Phi$ iff $\forall n : M_{\overline{\Delta}}(n) \models \Phi$ iff $M_{\Delta}(c) \models \Phi$.

7 Evaluation

In this section, we present the implementation of our technique and evaluate it on a set of doubly-unbounded DAB systems.

**VENUS.** We build a tool, VENUS, for verification of doubly-unbounded DAB systems that combines our domain reduction with a recent tool, QUICKSILVER [26], for parameterized verification of DAB systems with finite-state processes. As detailed in Sec. 5, VENUS yields a program with finite data domains, paving the way for QUICKSILVER to perform its reasoning.

While each expansion step in Algo. 3 can only expand a region with a single node-variable pair, in VENUS, we generalize an expansion step to expand a region with a strongly-connected set of node-variable pairs, even when its pairs cannot be used individually. Further, we emphasize that VENUS automatically identifies data domains which can be treated as scalar sets.

**Case Studies.** We now demonstrate the efficacy of our technique on two case studies, Consortium and Distributed Register, that are representative of commonly used DAB systems.

Consortium was introduced in Sec. 2 along with its safety property. We now introduce three variants of this system. The first variant, Consortium-Three, elects a consortium of size three instead of two. The second variant, Consortium-BCast, does not elect a particular trusted actor to announce the decision, but rather allows either elected actor to perform a broadcast of the decided value to the rest of the system. The third variant, Consortium-Check, forces both of the trusted actors to share the decided value, and allows the rest of the actors to check if the shared values are identical, and if not, move to an error state.

Distributed Register is based on Atomix’s AtomicValue [7] which gives a consistent view of some stored value under concurrent updates. In Distributed Register, clients in the environment submit requests to read from and update a register. Processes service read requests from their local copy of the register. For update requests, processes use consensus to determine a consistent value to be stored in the shared register. The safety property is that any two processes that are in a location where they serve client read requests always have equal values in their local copy of the register. We define a variant of this system, called DistReg-Two, that allows two registers to be manipulated simultaneously.
For each case study, we also define versions, *Consortium-32Bit* and *DistReg-32Bit*, respectively, with 32-bit integer domains.

**Evaluation.** We now discuss the result of evaluating our case studies in *Venus*. All experiments are performed on a MacBook Pro with Intel Core i5 CPU and 16 GB of RAM.

| Benchmark          | LoC | Domain Size | Domain Cutoff | Process Cutoff | Time(s)    |
|--------------------|-----|-------------|---------------|----------------|------------|
| Consortium         | 62  | ∞           | 3             | 3              | 0.464 ± 0.012 |
| Consortium-Three   | 62  | ∞           | 3             | 4              | 1.340 ± 0.007 |
| Consortium-BCast   | 58  | ∞           | 3             | 3              | 0.231 ± 0.003 |
| Consortium-Check   | 68  | ∞           | 3             | 5              | 3.209 ± 0.013 |
| Consortium-32Bit   | 62  | 2^{32}      | 3             | 3              | 0.450 ± 0.002 |
| DistReg            | 34  | ∞           | 2             | 2              | 0.118 ± 0.005 |
| DistReg-Two        | 79  | ∞           | 2             | 2              | 2.135 ± 0.024 |
| DistReg-32Bit      | 34  | 2^{32}      | 2             | 2              | 0.116 ± 0.003 |

The performance of *Venus* is shown in Table 1. For each benchmark, we provide the number of lines of code needed to model the benchmark in *Mercury*, the initial size of the data domain (marked ∞ if unbounded), the domain cutoff computed by *Venus*, the process cutoff used for verification, and the mean run time for 10 verification runs as well as the 95% confidence intervals.

*Venus* successfully reduces both large and unbounded data domains to relatively small, bounded domains (2 and 3). Notably, for the DistReg-Two benchmark, *Venus* identifies two unbounded data domains (corresponding to the two registers) and computes domain cutoffs for each independently. Finally, *Venus* is able to verify each of our benchmarks rather efficiently in under 4 seconds.

### 8 Related Work

The formal methods community has developed a variety of techniques for verification of infinite-state concurrent/distributed systems. At the coarsest level, these approaches can be divided into semi-automated [21, 29, 31, 34, 38] and fully-automated [12, 13, 14, 15, 16, 17, 18, 20, 22, 25, 27, 28, 30, 32, 35] techniques. Semi-automated verification typically involves manually discovering inductive invariants [3, 39]. Fully automated verification (the focus of this paper) is able to compute a domain cutoff for DistReg-Two, *QuickSilver* is unable to compute a process cutoff. Because *Venus* can potentially be combined with tools beyond *QuickSilver* for computing process cutoffs, we make an exception for this case by manually computing a process cutoff of 2.
Symbolic representations and reduction strategies are typically employed to scale or enable automated verification for systems with large or infinite state spaces. For instance, symmetry reduction \cite{12,13,14,15,24} is used to explore the state space of a system via a reduced quotient structure. Automated verification of distributed systems is often complicated further by an unbounded number of processes, infinite data domains, or both. A popular approach to handling systems with an unbounded number of processes is to use (process) cutoffs \cite{2,5,8,9,13,16,17,25,26,27,28,32}. Another approach for dealing with an unbounded number of processes and/or unbounded variable domains is to impose a well-order \cite{20,35} over the state space of the system.

In what follows, we briefly explore prior work on verification of distributed systems with unbounded processes or infinite process state spaces.

**Bounded Number of Processes, Unbounded Process States.** One inspiration to this paper is the notion of data saturation, first broached by \cite{30} and explored further by \cite{24}. Infinite state spaces are also handled in well-ordering based frameworks \cite{1} or using “temporal case splitting” wherein each case identifies a particular value and all other values are represented with a symbolic constant \cite{11}. None of these approaches target doubly-unbounded systems.

**Unbounded Number of Processes, Bounded Process States.** The two common approaches for parameterized verification of finite-state processes are based on well-ordering \cite{19,20,35} and cutoff results computed either statically \cite{5,8,13,16,17,25,27,32} or dynamically \cite{2,28}. Notably, many of these cutoff results can potentially be combined with our domain reduction approach to enable verification of doubly-unbounded systems.

**Doubly-Unbounded Systems.** A related effort \cite{3} tackles parameterized verification of infinite-state systems using a well-ordering over global states and performing a backward-reachability analysis. Our approach differs from this effort in the target distributed systems and their models, as well as the technical approach. The effort in \cite{3} targets generic distributed systems that are modeled using the more-traditional system model where processes are defined as extended finite-state automata with local variables and global conditions on transitions. On the other hand, we target modularly-designed DAB systems built with abstractions of agreement protocols and, in particular, modeled in an easy-to-use modeling language Mercury. Further, the reduction in \cite{3} does not require systems to be value-symmetric and tackles the unboundedness of both the number of processes and the process state space in one consolidated step. In contrast, while the reduction in our approach requires the system to be value-symmetric, it is separable, and, arguably, more flexible. Specifically, our approach relies on two separate reductions, one for the process state spaces and one for the number of processes, each of which can potentially be replaced with other reductions.

The authors of \cite{13} combine predicate and counter abstractions into an “environment abstraction” to verify doubly-unbounded systems. The environment abstraction designates one process as the “reference” process and models all other...
processes in relation to it. Our work differs from their approach in two ways: (i) they do not verify systems which incorporate abstractions of agreement protocols, and (ii) they capture the relationships between values in the system using predicates describing the environment of the reference process, while we capture such relationships during our bounded region analysis.

The authors of [32] reduce parameterized verification of consensus algorithms over inputs from infinite domains to parameterized verification of these algorithms over binary inputs. They provide a process cutoff to reduce the problem to finite verification of consensus algorithms over binary inputs. Aside from focusing on consensus algorithms (as opposed to systems built upon them), their work also differs from ours in that their reduction of infinite domains to finite ones relies on the zero-one principle for sorting networks rather than a static analysis of the system being verified, like our bounded region analysis.
References

1. Abdulla, P.A., Cerans, K., Jonsson, B., Tsay, Y.K.: General decidability theorems for infinite-state systems. In: Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science. p. 313. LICS ’96. IEEE Computer Society, USA (1996)

2. Abdulla, P., Haziza, F., Holik, L.: Parameterized Verification Through View Abstraction. International Journal on Software Tools for Technology Transfer 18(5), 495–516 (2016)

3. Abdulla, P.A., Delzanno, G., Rezine, A.: Parameterized verification of infinite-state processes with global conditions. In: Proceedings of the 19th International Conference on Computer Aided Verification. p. 145–157. CAV’07, Springer-Verlag, Berlin, Heidelberg (2007)

4. Alur, R., Raghothaman, M., Stergiou, C., Tripakis, S., Udupa, A.: Automatic completion of distributed protocols with symmetry. In: Kroening, D., Păsăreanu, C.S. (eds.) Computer Aided Verification. pp. 395–412. Springer International Publishing, Cham (2015)

5. Aminof, B., Kotek, T., Rubin, S., Spegni, F., Veith, H.: Parameterized model checking of rendezvous systems. Distributed Computing 31(3), 187–222 (2018). https://doi.org/10.1007/s00446-017-0302-6

6. Amsden, Z., Arora, R., Bano, S., Baudet, M., Blackshear, S., Bothra, A., and Christian Catalini, G.C., Chalkias, K., Cheng, E., Ching, A., Chursin, A., and Gerardo Di Giacomo, G.D., Dill, D.L., Ding, H., Doudchenko, N., Gao, V., Gao, Z., Garillot, F., Gorven, M., Hayes, P., Hou, J.M., Hu, Y., Hurley, K., Lewi, K., Li, C., Li, Z., and Sonia Margulis, D.M., Maurer, B., Mohassel, P., de Naurois, L., Nikolaenko, V., Nowacki, T., and Dmitri Perelman, O.O., Pott, A., Proctor, B., Qadeer, S., Rain, Russi, D., Schwab, B., Sezer, S., Sonnino, A., Venter, H., Wei, L., Wernerfelt, N., Williams, B., Wu, Q., Yan, X., Zakian, T., Zhou, R.: The Libra Blockchain. Tech. rep. (2020), https://developers.libra.org/docs/assets/papers/the-libra-blockchain/2020-05-26.pdf

7. Atomix: Atomix (2021), https://atomix.io/docs/latest/user-manual/primitives/AtomicValue/

8. Außerlechner, S., Jacobs, S., Khalimov, A.: Tight Cutoffs for Guarded Protocols with Fairness. In: Jobstmann, B., Leino, K.R.M. (eds.) Verification, Model Checking, and Abstract Interpretation - 17th International Conference, VMCAI 2016, St. Petersburg, FL, USA, January 17-19, 2016. Proceedings. Lecture Notes in Computer Science, vol. 9583, pp. 476–494. Springer (2016). https://doi.org/10.1007/978-3-662-49122-5_23

9. Bloem, R., Jacobs, S., Khalimov, A., Konnov, I., Rubin, S., Veith, H., Widder, J.: Decidability of Parameterized Verification. Synthesis Lectures on Distributed Computing Theory, Morgan & Claypool Publishers (2015)

10. Burch, J.R., Clarke, E.M., McMillan, K.L., Dill, D.L., Hwang, L.J.: Symbolic model checking: 10<sup>20</sup> states and beyond. Inf. Comput. 98(2), 142–170 (jun 1992). https://doi.org/10.1016/0890-5401(92)90017-A. https://doi.org/10.1016/0890-5401(92)90017-A

11. Clarke, E., Long, D., McMillan, K.: Compositional model checking. In: Proceedings of the Fourth Annual Symposium on Logic in Computer Science. p. 353–362. IEEE Press (1989)

12. Clarke, E.M., Emerson, E.A., Jha, S., Sistla, A.P.: Symmetry reductions in model checking. In: Proceedings of the 10th International Conference on Computer Aided Verification. p. 147–158. CAV ’98, Springer-Verlag, Berlin, Heidelberg (1998)
13. Clarke, E.M., Talupur, M., Veith, H.: Environment abstraction for parameterized verification. In: VMCAI. Lecture Notes in Computer Science, vol. 3855, pp. 126–141. Springer (2006)
14. Emerson, E.A., Sistla, A.P.: Utilizing symmetry when model-checking under fairness assumptions: An automata-theoretic approach. ACM Trans. Program. Lang. Syst. 19(4), 617–638 (jul 1997). [https://doi.org/10.1145/262004.262008]
15. Emerson, E.A., Havlicek, J.W., Trefler, R.J.: Virtual symmetry reduction. In: Proceedings of the 15th Annual IEEE Symposium on Logic in Computer Science. p. 121. LICS ’00. IEEE Computer Society, USA (2000)
16. Emerson, E.A., Kuhlau, V.: Reducing Model Checking of the Many to the Few. In: McAllester, D.A. (ed.) CADE. Lecture Notes in Computer Science, vol. 1831, pp. 236–254. Springer (2000)
17. Emerson, E.A., Kuhlau, V.: Exact and Efficient Verification of Parameterized Cache Coherence Protocols. In: CHARME. Lecture Notes in Computer Science, vol. 2860, pp. 247–262. Springer (2003)
18. Emerson, E.A., Wahl, T.: On Combining Symmetry Reduction and Symbolic Representation for Efficient Model Checking. In: Advanced Research Working Conference on Correct Hardware Design and Verification Methods. pp. 216–230. Springer (2003)
19. Esparza, J., Finkel, A., Mayr, R.: On the Verification of Broadcast Protocols. In: 14th Annual IEEE Symposium on Logic in Computer Science, Trento, Italy, July 2-5, 1999. pp. 352–359. IEEE Computer Society (1999). [https://doi.org/10.1109/LICS.1999.782630]
20. Finkel, A., Schnoebelen, P.: Well-structured Transition Systems Everywhere! Theor. Comput. Sci. 256(1-2), 63–92 (2001)
21. v. Gleissenthall, K., Kıcı, R.G., Bakst, A., Stefan, D., Jhala, R.: Pretend synchrony: Synchronous verification of asynchronous distributed programs. Proc. ACM Program. Lang. 3(POPL) (Jan 2019). [https://doi.org/10.1145/3290372]
22. Griffin, J., Lesani, M., Shadab, N., Yin, X.: Tcl: Temporal logic of distributed components. Proc. ACM Program. Lang. 4(ICFP) (Aug 2020). [https://doi.org/10.1145/3409005]
23. Hyperledger: The Hyperledger Project (2021), [https://www.hyperledger.org/]
24. Ip, C.N., Dill, D.L.: Better Verification Through Symmetry. Formal methods in system design 9(1-2), 41–75 (1996)
25. Jaber, N., Jacobs, S., Wagner, C., Kulkarni, M., Samanta, R.: Parameterized verification of systems with global synchronization and guards. In: Lahiri, S.K., Wang, C. (eds.) Computer Aided Verification. pp. 299–323. Springer International Publishing, Cham (2020)
26. Jaber, N., Wagner, C., Jacobs, S., Kulkarni, M., Samanta, R.: Quicksilver: Modeling and parameterized verification for distributed agreement-based systems. Proc. ACM Program. Lang. 5(OOPSLA) (oct 2021). [https://doi.org/10.1145/3485534]
27. Jacobs, S., Sakr, M.: Analyzing guarded protocols: Better cutoffs, more systems, more expressivity. In: Dillig, I., Palsberg, J. (eds.) Verification, Model Checking, and Abstract Interpretation - 19th International Conference, VMCAI 2018, Los Angeles, CA, USA, January 7-9, 2018, Proceedings. Lecture Notes in Computer Science, vol. 10747, pp. 247–268. Springer (2018). [https://doi.org/10.1007/978-3-319-73721-8_12]
28. Kaiser, A., Kroening, D., Wahl, T.: Dynamic Cutoff Detection in Parameterized Concurrent Programs. In: Touili, T., Cook, B., Jackson, P.B. (eds.) Computer Aided Verification, 22nd International Conference, CAV 2010, Edinburgh, UK, July 15-19, 2010. Proceedings. Lecture Notes in Computer Science, vol. 6174, pp. 645–659. Springer (2010)

29. Krogh-Jespersen, M., Timany, A., Ohlenbusch, M.E., Gregersen, S.O., Birkedal, L.: Aneris: A mechanised logic for modular reasoning about distributed systems. In: Müller, P. (ed.) Programming Languages and Systems. pp. 336–365. Springer International Publishing, Cham (2020)

30. Lubachevsky, B.D.: An approach to automating the verification of compact parallel coordination programs. i. Acta Inf. 21(2), 125–169 (aug 1984). https://doi.org/10.1007/BF00289237

31. Ma, H., Goel, A., Jeannin, J.B., Kapritsos, M., Kasikci, B., Sabelfah, K.A.: I4: Incremental inference of inductive invariants for verification of distributed protocols. In: Proceedings of the 27th ACM Symposium on Operating Systems Principles. p. 370–384. SOSP ’19, Association for Computing Machinery, New York, NY, USA (2019). https://doi.org/10.1145/3341301.3359651

32. Marić, O., Sprenger, C., Basin, D.: Cutoff Bounds for Consensus Algorithms. In: International Conference on Computer Aided Verification. pp. 217–237. Springer (2017)

33. Padon, O., Losa, G., Sagiv, M., Shoham, S.: Paxos made epr: Decidable reasoning about distributed protocols. Proc. ACM Program. Lang. 1(OOPSLA) (Oct 2017). https://doi.org/10.1145/3140568

34. Padon, O., McMillan, K.L., Panda, A., Sagiv, M., Shoham, S.: Ivy: Safety verification by interactive generalization. In: Proceedings of the 37th ACM SIGPLAN Conference on Programming Language Design and Implementation. p. 614–630. PLDI ’16, Association for Computing Machinery, New York, NY, USA (2016). https://doi.org/10.1145/2908080.2908118

35. Schmitz, S., Schnoebeelen, P.: The power of well-structured systems. In: D’Argenio, P.R., Melgratti, H.C. (eds.) CONCUR 2013. Lecture Notes in Computer Science, vol. 8052, pp. 5–24. Springer (2013). https://doi.org/10.1007/978-3-642-40184-8_2

36. Sergey, I., Wilcox, J.R., Tatlock, Z.: Programming and proving with distributed protocols. Proc. ACM Program. Lang. 2(POPL) (Dec 2017). https://doi.org/10.1145/3158116

37. Suzuki, I.: Proving properties of a ring of finite-state machines. Inf. Process. Lett. 28(4), 213–214 (Jul 1988). https://doi.org/10.1016/0020-0190(88)90211-6

38. Taube, M., Losa, G., McMillan, K.L., Padon, O., Sagiv, M., Shoham, S., Wilcox, J.R., Woos, D.: Modularity for decidability of deductive verification with applications to distributed systems. In: Proceedings of the 39th ACM SIGPLAN Conference on Programming Language Design and Implementation. p. 662–677. PLDI 2018, Association for Computing Machinery, New York, NY, USA (2018). https://doi.org/10.1145/3192366.3192414

39. Wilcox, J.R., Woos, D., Panchehka, P., Tatlock, Z., Wang, X., Ernst, M.D., Anderson, T.: Verdi: A framework for implementing and formally verifying distributed systems. In: Proceedings of the 36th ACM SIGPLAN Conference on Programming Language Design and Implementation. p. 357–368.
A Formal Semantics

In this section, we present formal definitions of the local and global semantics of Mercury program definitions, as presented in figures Fig. 4 and Fig. 5 respectively.

![Local operational semantics of Mercury programs.](image)

In addition to the broadcast and consensus transitions defined in Sec. 3, we formally present the remaining transition types below:

1. A rendezvous transition for event eID can occur when one process has a local internal transition and all other processes remain in the same local states,
2. a rendezvous transition for event eID with coordination type pw can occur when one process has a local transition for eID, one other process has corresponding eID? local transition, and all other processes remain in the same local states,
3. A “Partition transition” (q, e, q’) for partition event e with participant set Prt and winning set W iff (1) each participating process in Prt has a consistent view of the other participants by holding Prt in a local participant set variable, (2) each process P_i in W has a local “partition win” transition (q[i], e!, q’[i]), (3) each process P_j in Prt \ W has a corresponding local “partition lose” transition (q[j], e?, q’[j]), and (4) the local states of all other processes remain unchanged.
Lemma 3. B.1 Local Transitions under Permutations

\[ \exists i : (q[i], \text{in}, r[i]) \in T \quad \forall j \neq i : r[j] = q[j] \]
\[ (q, \text{in}, r) \in R \]

B Proofs

\[ \exists i : (q[i], \text{el}, r[i]) \in T \quad \forall j \neq i : (q[j], e?, r[j]) \in T \]
\[ (q, e, r) \in R \]

RENDEZVOUS

\[ e = (a, \text{val, bc, f}) \]
\[ \exists i : (q[i], \text{el}, r[i]) \in T \quad \exists j : (q[j], e?, r[j]) \in T \quad \forall k : k \notin \{i, j\} \implies r[k] = q[k] \]
\[ (q, e, r) \in R \]

PARTITION

\[ e = (a, W, \text{pc, f}) \]
\[ \forall i \in W : (q[i], \text{el}, r[i]) \in T \quad \forall j \in S \setminus W : (q[j], e?, r[j]) \in T \quad \forall k \in I_n \setminus S : r[k] = q[k] \]
\[ (q, e, r) \in R \]

CONSENSUS

\[ e = (a, \text{val, wc, f}) \in [e] \]
\[ \exists S \subseteq I_n : \forall i \in S : q[i](\text{pcct},) = S \quad \forall w \in \text{val} : \exists i \in S : q[i](\text{var},) = w \]
\[ \forall j \in S : (q[j], e?, r[j]) \in T \quad \forall j \in I_n \setminus S : r[j] = q[j] \]
\[ (q, e, r) \in R \]

Fig. 5: Global operational semantics of MERCURY programs.

B Proofs

B.1 Local Transitions under Permutations

Lemma 3. \( \forall \pi : \delta \mapsto \delta, (s, \alpha, s') \in T : \pi((s, \alpha, s')) \in T \)

Proof. We examine an arbitrary permutation \( \pi \) over \( \delta \) and an arbitrary local transition \( (s, \alpha, s') \in T \). By the definition of scalar sets, values in \( \delta \) may only be compared with (dis)equality, so the satisfaction of any local guard \( \phi \) associated with a handler is invariant to permutation; i.e. \( s \models \phi \implies \pi(s) \models \phi \).

Additionally, since all updates \( U \) involving values from \( \delta \) must be direct assignments, it follows immediately that \( U(s) = s' \implies U(\pi(s)) = \pi(s') \).

Finally, since \( \pi \) maps values in \( \delta \) to values in \( \delta \), it is trivial that for any action \( \alpha = (a, \text{val, } \tau, f) \in [\varepsilon] \), the permuted action \( \pi(\alpha) = (a, \pi(\text{val}), \tau, f) \) is also in \( [\varepsilon] \). So, by the appropriate local operational semantics rule in Fig. 4, \( \pi((s, \alpha, s')) = (\pi(s), \pi(\alpha), \pi(s')) \in T \).

B.2 Domain Reduction Congruence

Lemma 4. \( \forall r \in Q, r' \in \overline{Q}, (q, e, r) \in R : r \approx_{\psi} r' \implies (\exists(q, e, r) \in \overline{R} : q \approx_{\psi} q') \).

Since \( r \approx_{\psi} r' \), we know there must exist a CWP \( \gamma' \) such that \( \gamma'(r) = r' \). To prove the lemma, we must identify an appropriate global state \( \overline{\gamma} \) and \( \overline{\tau} \) such that \( (\overline{\gamma}, \overline{\tau}, r') \in \overline{R} \) and \( q \approx_{\psi} \overline{\gamma} \). Since \( \overline{\gamma} \) must be related to \( q \) by \( \approx_{\psi} \), we need to show that there exists another CWP \( \gamma \in I_{\psi}'(q) \) such that \( \text{min}(\gamma, q) \) and \( \gamma(q) = \overline{\gamma} \). By identifying a \( \gamma \) and \( \overline{\tau} \) such that \( (\gamma(q), \overline{\tau}, r) \in \overline{R} \), the lemma is proven by letting \( \overline{\gamma} = \gamma(q) \). We carefully define \( \gamma \) to (i) agree with \( \gamma' \) on values preserved in the
transition from $q$ to $r$ so that $\gamma(q)$ may transition to $\tau$, (ii) permute values in $\Pi^\delta_\gamma(q)$ consistently and values outside of $\Pi^\delta_\gamma(q)$ to the smallest available values so that $\gamma \in \Pi^\delta_\gamma(q)$. More precisely, we detail a procedure $mk\gamma$ below which, given $\gamma'$, $q$, and $r$, constructs the CWP $\gamma = mk\gamma(\gamma', q, r)$ as follows:

1. For any two process indices $i, j \in [1, n]$ and value $val \in \delta$, if $val$ is in both $\Pi^\delta_\gamma(q[i])$ and $\Pi^\delta_\gamma(r[j])$, then for each such $i$, let $\gamma[i]$ permute $val$ to the same value as the permuted value of $val$ in $\gamma'[j]$. That is,

$$\forall i, j \in [1, n], val \in \delta : val \in \Pi^\delta_\gamma(q[i]) \land val \in \Pi^\delta_\gamma(r[j]) \implies \gamma[i](val) = \gamma'[j](val).$$

2. For each process indices $i, j \in [1, n]$ and variables $v, v'$, if $gets(r[j], v', q[i], v)$ then let $\gamma[i]$ permute $q[i](v)$ to the same value as the permuted value of $r[j](v')$ in $\gamma'[j]$. That is,

$$\forall i, j \in [1, n], v, v' \in \mathcal{V} : \text{gets}(r[j], v', q[i], v) \land r[j](v') = q[i](v) \implies \gamma[i](q[i](v)) = \gamma'[j](r[j](v')).$$

3. For any value $val \in \delta$ for which there exists some process index $i \in [1, n]$ such that $val \in \Pi^\delta_\gamma(q[i])$ but there does not exist a process index $j \in [1, n]$ such that $val \in \Pi^\delta_\gamma(r[j])$, then for all $i \in [1, n]$ let $\gamma[i]$ permute $val$ to the smallest value $val'$ (according to $\gamma_3$) that was not used in any of the previous steps (for any process).

4. For each process $i \in [1, n]$ and variable $v \in \mathcal{V}$ of type $\delta$, if there does not exist another process $j \in [1, n]$ and variable $v'$ such that $gets(r[j], v', q[i], v)$ let $\gamma[i]$ permute $q[i](v)$ to the smallest value $val'$ (according to $\gamma_3$) such that no value from steps 1 or 3 was permuted to $val'$ in $q[i]$.

5. For each $i \in [1, n]$, permute any remaining values to the next value $val'$ (according to $\gamma_3$) that was not used in any of the previous steps (for any process).

Properties of $mk\gamma(\gamma', q, r)$. The above construction gives the CWP $\gamma = mk\gamma(\gamma', q, r)$ several interesting properties. The first of which is that a value which stays in the system during the transition (by being retained in a variable) is permuted consistently in $\gamma$ and $\gamma'$:

$$\forall i \in [1, n], v \in \mathcal{V} : \exists j \in [1, n], v' \in \mathcal{V} : \text{gets}(r[j], v', q[i], v) \implies mk\gamma(\gamma', q, r)[i](q[i](v)) = \gamma'[j](r[j](v')) \quad (\gamma\text{-kept})$$

Another property of the CWP $\gamma$ is that any value in $\Pi^\delta_\gamma(q)$ is permuted consistently by all components of $\gamma$:

$$\forall i, j \in [1, n], val \in \Pi^\delta_\gamma(q) : \text{mk}\gamma(\gamma', q, r)[i](val) = \text{mk}\gamma(\gamma', q, r)[j](val) \quad (\gamma\text{-stable})$$
The construction also ensures that $\gamma$ is a bijection:

$$\forall i \in [1, n], val, val' \in \delta : val = val' \iff mk\gamma(\gamma', q, r)[i](val) = mk\gamma(\gamma', q, r)[i](val') \quad (\gamma\text{-bijection})$$

Ultimately, we are able to conclude that the permutation $\gamma$ maps the values of $q$ into the bounded range of $\delta$.

$$\gamma'(r) \in Q \implies \forall i \in [1, n], val \in \delta : mk\gamma(\gamma', q, r)[i](val) \in T_{\delta}[1, y] \quad (\gamma\text{-bounded})$$

where $y = \rho + \lambda$ is the size of the reduced scalarset domain $\bar{\delta}$, and $T_{\delta}[1, y]$ is the set of the $y$ most minimal values according to $T_{\delta}$.

This property comes from the assumption that there are no more than $\rho$ values $val$ in $\Pi_{\delta}^y(r[i])$ for any process $i$, and any of them which are also in $\Pi_{\delta}^y(q)$ will be permuted to the same values as $\gamma'(r)$ (all of which are in $T_{\delta}[1, y]$, since $\gamma'(r) = \tau \in Q$).

Then since each $r[j]$ may only hold up to $\lambda$ values which are not stable according to any process (i.e. $val \notin \Pi_{\delta}^y(q)$), these values are all permuted in $\tau$ to a particular set of $\lambda$ values in $T_{\delta}[1, y]$ (by our minimality constraint), and our assumptions prevent more than 1 process $j$ from getting a value $val \notin \Pi_{\delta}^y(q)$ from $q[i]$, so for some variable(s) $v$ and $v'$, if $q[i](v) = val$ and $\text{gets}(r[j], v', q[i], v)$, $\gamma[i]$ permutes $val$ to the same single value as $\gamma'[j]$ permutes it, which is in the set of $\lambda$ values in $\tau$ to which unstable values in $\tau$ are permuted.

Then, if there are only $\rho' < \rho$ values $val$ in $q[i]$ such that $val \in \Pi_{\delta}^y(q)$, there must be $\rho - \rho'$ values in $T_{\delta}[1, y]$ such that no value has been permuted to them in the first two steps, any additional values $val \in \Pi_{\delta}^y(q)$, of which there may be at most $\rho - \rho'$, are permuted to the next available value according to $T_{\delta}$ (which must necessarily be in $T_{\delta}[1, y]$, otherwise there could not have been only $\rho'$ values $val \in \Pi_{\delta}^y(q)$).

Lastly, if there are only $\lambda' < \lambda$ values $val \in \Pi_{\delta}(q[i]) \setminus \Pi_{\delta}^y(q)$ for $i \in [1, n]$, there must be at most $\lambda - \lambda'$ values in $T_{\delta}[1, y]$ such that no value has been permuted to them in $\gamma[i]$ by the first three steps, any additional values $val \in \Pi_{\delta}(q[i]) \setminus \Pi_{\delta}^y(q)$, of which may be at most $\lambda - \lambda'$, are permuted to the next available value according to $T_{\delta}$ which no value in $q[i]$ has been permuted yet (and which must necessarily be in $T_{\delta}[1, y]$, otherwise there could not have been only $\lambda'$ values in $\Pi_{\delta}(q[i]) \setminus \Pi_{\delta}^y(q)$).

Thus, $\forall i \in [1, n], val \in \delta : mk\gamma(\gamma', q, r)[i](val) \in T_{\delta}[1, y]$, which leads us to the following conclusion:

$$\gamma'(r) \in Q \implies mk\gamma(\gamma', q, r)(q) \in \overline{Q} \quad (\gamma\text{-bounded-upshot})$$

**Transition Case Analysis** With such $\gamma$ in hand, it remains to show that there exists an event $\tau$ such that the transition $(\gamma(q), \tau, \tau)$ is a valid transition in $\overline{R}$.

In what follows, we perform a case analysis over all types of events $e$ and show, for each type of global transition $(q, e, r) \in R$, that such an event $\tau$ exists.

**Lemma 5.** $\exists \gamma, \tau : (\gamma(q), \tau, \tau) \in \overline{R}$
Proof.

In the case that $e$ is an internal coordination event, we prove the lemma by contradiction, as follows.

1. $(q, e, r) \in Q$ (assumption)
2. $\gamma'(r) = \tau \in \overline{Q}$ (assumption)
3. $r \approx_{\psi} \tau$ (assumption)
4. $\neg \exists \gamma, \pi : (\gamma(q), \pi, \tau) \in \overline{R}$ (assumption)
5. $e = \text{in}$ (assumption)
6. $\exists \gamma' : (q[i'], \text{in}, r[i']) \in T \land \forall j' \neq i : r[j'] = q[j']'$ (1, 5)
7. $(q[i], \text{in}, r[i]) \in T \land \forall j' \neq i : r[j'] = q[j']'$ (6, $\exists \gamma' = i$ (arbitrary))
8. $q[i], \text{in}, r[i] \in T$ (7, $\land$)
9. $\forall j' \neq i : r[j'] = q[j']'$ (7, $\land$
10. $\forall \gamma, \pi : (\gamma(q), \pi, \tau) \notin \overline{R}$ (4, $\neg \exists$
11. $\forall \gamma : (\gamma(q), \pi, \tau) \notin \overline{R}$ (10, $\pi = \text{in}$)
12. $\gamma = mk\gamma(q', q, r) \land (\gamma(q), \text{in}, \tau) \notin \overline{R}$ (11, $\forall \gamma = mk\gamma(q', q, r)$
13. $\gamma = mk\gamma(q', q, r)$ (12, $\land$
14. $(\gamma(q), \text{in}, \tau) \notin \overline{R}$ (12, $\land$
15. $\forall v' \in V : \exists v \in V : \text{gets}(r[i], v', q[i], v)$ (8, kept)
16. $\exists \gamma' : (\gamma(q)[i'], \text{in}, \tau[i']) \in T \land \forall j' \neq i' : \tau[j']' = \gamma(q)[j'] \land \forall \gamma(q) \in \overline{Q} \land \tau \in \overline{Q}$ (14, internal-global)
17. $\forall \gamma' : (\gamma(q)[i'], \text{in}, \tau[i']) \in T \land \forall j' \neq i' : \tau[j']' \neq \gamma(q)[j'] \lor \forall \gamma(q) \in \overline{Q} \land \tau \in \overline{Q}$ (16, DeMorgan’s)
18. a) $\tau \notin \overline{Q}$ (17, $\lor$
19. a) $\bot$ (2, 18a)
20. b) $\gamma(q) \notin \overline{Q}$ (17, $\lor$
21. b) $\gamma(q) \in \overline{Q}$ (2, bounded-upshot)
22. b) $\bot$ (20b, 21)
23. $\forall \gamma' : (\gamma(q)[i'], \text{in}, \tau[i']) \in T \lor \exists \gamma' : \tau[i']' \neq \gamma(q)[j]'$ (17, 18-19a, 20-22b)
24. $(\gamma(q)[i], \text{in}, \tau[i]) \in T \land \exists j' \neq i : \tau[j'] = \gamma(q)[j']'$ (23, $\exists \gamma' = i$ (from 7))
25. a) $\exists j' \neq i : \tau[j'] = \gamma(q)[j']'$ (24, $\lor$
26. a) $j \neq i \land \forall \gamma(q) \in \overline{Q}$ (25, $\exists j' = j$ (arbitrary))
27. a) $j \neq i$ (26, $\land$
28. a) $\tau[j] \neq \gamma(q)[j]$ (26, $\land$
29. a) $\gamma'(r)[j] \neq \gamma(q)[j]$ (28, 2
30. a) $\forall v \in V : j' \neq i : \text{gets}(r[j'][v], v, q[j'][v], v)$ (9, internal-global)
31. a) $\forall v' \in V : \text{gets}(r[j][v'], q[j'][v])$ (30, $\forall v' = j$ (from 26))
32. a) $r[j] = q[j]$ (9, 27)
33. a) $\forall v \in V : \gamma(q)[j][v] = \gamma'(r)[j][v]$ (31, $\gamma$-kept)
34. a) $\forall v' \in V : \gamma(q)[j][v'] = \gamma'(r)[j][v]$ (33, $\gamma(q)[i] = \gamma(q)[i][v][v]$)
35. a) $\gamma(q)[j] = \gamma'(r)[j]$ (34, all vars equal)
36. a) $\gamma'(r)[j] \neq \gamma(q)[j]$ (28, 2
37. a) $\bot$ (35a, 36a)
38. b) $(\gamma(q)[i], \text{in}, \tau[i]) \in T$ (24, $\lor$
39. b) $(\gamma(q)[i], \text{in}, \gamma'(r)[i]) \in T$ (38, 2)
40. b) \((γ[i](q[i]), \text{ in, } γ'[i](r[i])) ∈ T\)  
41. b) \((γ[i](q[i]), \text{ in, } γ'[i](r[i])) ∈ T\)  
42. b) \(∀v ∈ V : ∃ j' \in [1, n], v' ∈ V : \)
\(\text{gets}(r[j], v', q[i], v)\)  
43. b) \(∀v' ∈ V : γ[i](q[i](v)) = γ'[i](r[i](v))\)  
44. b) \(γ(q[i]) = γ'(r[i])\)  
45. b) \(γ[i](q[i]) = γ'[i](r[i])\)  
46. b) \(∀v ∈ V : ∃ v' ∈ V : \text{gets}(r[i], v', q[i], v)\)  
47. b) \(∀v ∈ V : γ[i](r[i](v)) = γ'[i](r[i](v))\)  
48. b) \(γ[i](r[i]) = γ'[i](r[i])\)  
49. b) \((γ[i](q[i]), \text{ in, } γ'[i](r[i])) ∈ T\)  
50. b) \(\bot\)  

All branches closed.

The proof is almost identical when \(e\) is a partition event, or any interaction with the environment, as in these transitions do not transmit values between system processes. As such, we elide the proof details for these transitions here.

For the remaining cases, we make two observations which will aid in our proof effort. The first is that when two local permutations treat the right-hand-side values of all assignments (i.e., those being assigned from) in a set of updates identically, the local state resulting from that set of updates is identical, regardless of which permutation is applied to the assigned values.

\[
∀\pi, π', U, s, s' : U(s) = s' ∧ (∀v, v' ∈ U : π(s(v)) = π'(s'(v))) \implies U(π(s)) = π'(s') \quad \text{(update-equiv)}
\]

The second is that, during a local transition, when a value moves from one variable \(v\) in some state \(s\) to another variable \(v'\) in another state \(s'\), if two permutations \(π\) and \(π'\) agree on the permutation of the value of \(s(v)\), and the transition is preserved when both \(s\) and \(s'\) are permuted by \(π\), then the transition is preserved when the source state \(s\) is permuted by \(π'\) instead of \(π\).

\[
∀\pi, π' : (∀s, v, s', v' : \text{getsFrom}(s', v', s, v) \implies π(s(v)) = π'(s'(v))) \implies ((π(s), e, π(s')) ∈ T \iff (π'(s), e, π(s')) ∈ T) \quad \text{(src-equiv)}
\]

Next, we consider the case in which \(e\) is a pairwise transmission between two non-environment processes. Again, we prove the lemma by contradiction, as follows.

1. \(\langle q, e, r \rangle ∈ R\) (assumption)
2. \(γ'(r) = \pi ∈ \mathcal{R}\) (assumption)
3. \(r ≈_φ \pi\) (assumption)
4. \(¬∃γ, π : (γ(q), π, r) ∈ \mathcal{R}\) (assumption)
5. \(e = (a, \text{ val}, \text{ pw}, \bot)\) (assumption)
6. \(∃ j' : (q[j'], e, r[j']) ∈ T \land \exists j'' : (q[j''], e', r[j'']) ∈ T \land \forall k' \notin \{i', j'\} \neq i' : r[k'] = q[k']\) (1, pw-global)
∀γ (q, q'[i]) ∈ T ∧
∀γ (q[j], q'[j]) ∈ T ∧
∀γ (q[i], q'[i]) ∈ T ∧
∀γ (q[j], q'[j]) ∈ T ∧
∀γ (q[i], q'[i]) ∈ T ∧
∀γ (q[j], q'[j]) ∈ T ∧ (6, ∃i' = i, j' = j)
(7, ∧)
(7, ∧)
(4, ¬∃)
∀γ (γ(q), γ'[j]e, τ) /∈ R
(10, ∀σ = γ'[j]e)
(12, ∀γ = mkγ(γ', q, r))
(12, ∧)
(15, pw-global)
γ = mkγ(γ', q, r) (12, bounded-upshot)
(16, ¬∃, ¬v)
¬∃i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
¬∃j': (γ(q)[j'], γ'[j](e, τ[j'])) /∈ T ∨
¬∀k’ /∈ {i', j'} : τ[k'] = γ(q)[k'] ∨
(17, 18-19a, 20-22b)
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
(17, 18-19a, 20-22b)
∀j': (γ(q)[j'], γ'[j](e, τ[j'])) /∈ T ∨
∀j': (γ(q)[j'], γ'[j](e, τ[j'])) /∈ T ∨
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
(23, ∀i' = i, j' = j (from 7))
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
k /∈ {i, j} ∧ τ[k] /∈ γ(q)[k] (24, ∃k' = k (arbitrary))
(25, ∨)
(26, ∧)
(26, ∧)
(28, 2)
0. a) ∀σ /∈ R
(17, 17)
11. a) ⊥ (2, 18a)
20. b) γ(q) /∈ R
(17, 17)
(20b, 21)
21. b) γ(q) ∈ R
(17, 17)
22. b) ⊥
(20b, 21)
23. ∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
∀j': (γ(q)[j'], γ'[j](e, τ[j'])) /∈ T ∨
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
(25, ∨)
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
∀j': (γ(q)[j'], γ'[j](e, τ[j'])) /∈ T ∨
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
(23, ∀i' = i, j' = j (from 7))
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
∀j': (γ(q)[j'], γ'[j](e, τ[j'])) /∈ T ∨
∀i': (γ(q)[i'], γ'[j](e, τ[i'])) /∈ T ∨
k /∈ {i, j} ∧ τ[k] /∈ γ(q)[k] (24, ∃k' = k (arbitrary))
(25, ∨)
(26, ∧)
(26, ∧)
(28, 2)
30. a) ∀σ ∈ V : K' /∈ {i, j} :
gets(r[k], v, q[k], v) (10, pw-global)
(30, ∀k = k (from 25))
31. a) ∀σ ∈ V : gets(r[k], v, q[k], v) (31, γ-kept)
(32, γ(q)[i] = γ[i](q[i]))
(33, all vars equal)
34. a) γ(q)[j] = γ'(r)[j] (33, all vars equal)
35. a) ⊥ (29a, 34a)
36. b) γ(q)[i], γ'[j](e, τ[i]), τ[i] /∈ T
(25, ∨)
37. b) γ(q)[i], γ'[j](e, τ[i]), τ[i] /∈ T
(36b, 2)
38. b) γ(q)[i], γ'[j](e, τ[i]), τ[i](q[i]) /∈ T
(37b, γ(q)[i] = γ[i](q[i]))
39. b) γ(q)[i], γ'[j](e, τ[i]), τ[i](q[i]) /∈ T
(8, π preserves τ)
40. b) gets(q'[j], varγ, q[i], varΩ) (7, gets, pw-global)
between system processes as well so, again, we elide the proof details for those

We have

\begin{align*}
  (q, e, r) &\in R \\
  \gamma'(r) &\equiv \Phi \in \mathcal{Q} \\
  r &\equiv \psi \Phi \\
  \neg \exists \gamma, e' : (\gamma(q), e', \Phi) &\in \overline{R} \\
  e &\equiv (a, \text{\textit{val}}^*, \text{\textit{vc}}, \perp) \\
  \exists S \subseteq I_n : \forall i \in S : q[i]\text{\textit{pctct}}_e = S \\
  &\text{(assumption, participant set)} \\
  \forall w' \in \text{\textit{val}}^* : \exists i' \in S : q[i'][\text{\textit{var}}_e] = w' \\
  &\land \forall j' \in S : (q[j'], e', r[j']) \in T^\land \\
  &\forall k' \in I_n \setminus S : r[k'] = q[k'] \\
  &\text{(1, vc-global)} \\
  \forall w' \in \text{\textit{val}}^* : \exists i' \in S : q[i'][\text{\textit{var}}_e] = w' \\
  &\land \forall j' \in S : (q[j'], e', r[j']) \in T^\land \\
  &\forall \gamma, e' : (\gamma(q), e', \Phi) \not\in \overline{R} \\
  &\text{(4, \neg \exists)} \\
  \gamma &\equiv mk\gamma(\gamma', q, r) \\
  &\land \forall e' : (\gamma(q), e', \Phi) \not\in \overline{R} \\
  &\text{(9, \forall \gamma = mk\gamma(\gamma', q, r))} \\
  \gamma &\equiv mk\gamma(\gamma', q, r) \\
  &\land \forall e' : (\gamma(q), e', \Phi) \not\in \overline{R} \\
  &\text{(10, \land)} \\
  \forall e' : (\gamma(q), e', \Phi) \not\in \overline{R} \\
  &\text{(10, \land)} \\
  \forall e' \in [1, n] : q[i'][\text{\textit{var}}_e] = r[i'][\text{\textit{var}}_e] \\
  &\text{(1, vc-local)} \\
  \forall w' \in \text{\textit{val}}^*, i' \in S : \pi(w') = \gamma[i'](w') \\
  &\land (\gamma(q), \pi(e), \Phi) \not\in \overline{R} \\
  &\text{(12, \forall e' = \pi(e), \text{\textit{val}}^* \text{\textit{stable}})}
\end{align*}
15. \( \forall w' \in \text{val}' , i' \in S : \pi(w') = \gamma'[i'][w'] \) (14, \( \land \))
16. \( (\gamma(q), \pi(e), \overline{r}) \notin R \) (14, \( \land \))
17. \( \exists w' \in \pi(\text{val}'') : \forall i' \in S : \gamma(q)[i'][\text{var}_\text{st}] \neq w' \lor \exists w' \in \pi(\text{val}'') : \forall i' \in S : \gamma(q)[i'][\text{var}_\text{st}] \neq w' \lor \forall j' \in S : (\gamma(q)[j'], \pi(e), \overline{r}[j']) \notin T \lor \exists k' \in I_n \setminus S : \pi(k') \neq \gamma(q)[k'] \lor \gamma(q) \notin \overline{Q} \lor \forall \gamma \notin \overline{Q} \) (16, vc-global)

20. a) \( \overline{r} \notin \overline{Q} \) (17, \( \lor \))
21. a) \( \perp \) (2, 18a)
22. b) \( \gamma(q) \notin \overline{Q} \) (17, \( \lor \))
23. c) \( \exists k' \in I_n \setminus S : \pi[k'] \neq \gamma(q)[k'] \) (17, \( \lor \))
24. c) \( k \in I_n \setminus S \land \pi[k] \neq \gamma(q)[k] \) (23, \( \exists k' = k(\text{arbitrary}) \))
25. c) \( k \in I_n \setminus S \) (24, \( \land \))
26. c) \( \pi[k] \neq \gamma(q)[k] \) (24, \( \land \))
27. c) \( \gamma'[r][k] \neq \gamma(q)[k] \) (28, 2)
28. c) \( \forall k' \in I_n \setminus S : r[k'] = q[k'] \) (7, \( \land \))
29. c) \( \forall v \in V, k' \in I_n \setminus S : \) 
30. c) \( \forall v \in V : \) 
31. c) \( \forall w' \in V : \gamma[k](\pi[k](v)) = \gamma'[k](\pi[k](v)) \) (29c, \( \forall k' = k(\text{from 24}) \))
32. c) \( \forall w' \in V : \gamma(q)[j](\pi(e)[j]) = \gamma'[r](\pi[k](v)) \) (30c, \( \gamma\)-gets)
33. c) \( \gamma(q)[k] = \gamma'[r][k] \) (31c, \( \gamma(q)[i] = \gamma'[i][q[i]] \))
34. c) \( \perp \) (32c, all vars equal)
35. d) \( \exists j' \in S : (\gamma(q)[j'][, \pi(e)[, \overline{r}[j']]) \notin T \) (27c, 33c)
36. d) \( j \in S \land (\gamma(q)[j], \pi(e)[, \overline{r}[j]) \notin T \) (35d, \( \exists j' = j(\text{arbitrary}) \))
37. d) \( j \in S \) (36d, \( \land \))
38. d) \( (\gamma(q)[j], \pi(e)[, \overline{r}[j]) \notin T \) (36d, \( \land \))
39. d) \( (\gamma(q)[j], \pi(e)[, \gamma'[r][j]) \notin T \) (38d, 2)
40. d) \( (\gamma[q[j]], \pi(e)[, \overline{r}[j]) \notin T \) (39d, \( \gamma(q)[i] = \gamma'[i][q[i]] \))
41. d) \( \forall j' \in S : (\gamma[j][j,e], \overline{r}[j]) \in T \) (7, \( \land \))
42. d) \( (\gamma'[q[j], j, \overline{r}[j]) \in T \) (41d, 37d, \( \forall j' = j(\text{from 36d}) \))
43. d) \( (\gamma'[q[j], j, \overline{r}[j]) \in T \) (42d, \( \pi \) preserves \( T \))
44. d) \( \forall v, w \in V : \) 
45. d) \( \gamma[j](\pi(e)[v]) = \gamma'[j](\pi[e][v]) \) (11, \( \gamma\)-gets)
46. d) \( \forall w' \in \text{val}' : \pi(w') = \gamma'[j](w') \) (43d, 44d, src-equiv)
47. d) \( \pi(e) = \gamma'[j](e) \) (45d, 47d, src-equiv)
48. d) \( \gamma[q[j], j, \overline{r}[j]) \in T \) (46d, all vars equal)
49. d) \( \perp \) (40d, 48d)
50. e) \( \exists w' \in \pi(\text{val}'') : \forall i' \in S : \gamma(q)[i'][\text{var}_\text{st}] \neq w' \) (17, \( \lor \))
51. e) \( w \in \pi(\text{val}'') \land \forall i' \in S : \gamma[q[i'][\text{var}_\text{st}] \neq w \) (50e, \( \exists w' = w(\text{arbitrary}) \))
52. e) \( w \in \pi(\text{val}'') \) (51e, \( \land \))
53. e) \( \forall i' \in S : \gamma(q)[i'][\text{var}_\text{st}] \neq w \) (51e, \( \land \))
54. e) \( \exists w' \in \pi(\text{val}'') : \forall i' \in S : \gamma[q[i'][\text{var}_\text{st}] = w' \) (7, \( \land \))
55. e) \( \exists i' \in S : q[i'][\text{var}_\text{st}] = \pi^{-1}(w) \) (52e, 54e)
56. e) $i \in S \land q[i](\text{var}_{e_1}) = \pi^{-1}(w)$ (55e, $\exists i' = i(\text{arbitrary})$
57. e) $i \in S$ (56e, $\land$
58. e) $q[i](\text{var}_{e_1}) = \pi^{-1}(w)$ (56e, $\land$
59. e) $\gamma(q)[i](\text{var}_{e_1}) \neq w$ (53e, 57e, $\forall i' = i (\text{from } 36d)$
60. e) $\gamma'[i](q[i](\text{var}_{e_1})) = w$ (57e, 58e, 15)
61. e) $\text{gets}(r[i], \text{var}_{e_1}, q[i], \text{var}_{e_1})$ (13, vc-local)
62. e) $\gamma[i](q[i](\text{var}_{e_1})) = \gamma'[i](q[i](\text{var}_{e_1}))$ (61e, 37d, $\gamma$-gets)
63. e) $\gamma[i](q[i](\text{var}_{e_1})) = w$ (62e, 60e)
64. e) $\gamma[i](q[i](\text{var}_{e_1})) \neq w$ (59e, $\gamma(q)[i] = \gamma[i](q[i])$)
65. e) $\perp$ (63e, 64e)

All branches closed.

Having exhausted all types of events that $e$ can be, we can finally conclude that it is indeed the case that, using our construction of $\gamma = mk\gamma(\gamma', q, r)$, it is always possible to identify an event $\bar{e}$ such that $\exists(q, \bar{e}, \tau) \in \mathcal{R} : q \approx_{\bar{e}} \bar{q}$.