Formulas for Partial Entanglement Entropy

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Abstract: Partial entanglement entropy (PEE) $s_{\mathcal{A}}(\mathcal{A}_i)$ characterizes how much the subsets $\mathcal{A}_i$ of $\mathcal{A}$ contribute to the entanglement entropy $S_{\mathcal{A}}$. We find one additional physical requirement for $s_{\mathcal{A}}(\mathcal{A}_i)$, which is the invariance under a permutation between $\mathcal{A}_i$ and $\bar{\mathcal{A}}$, the complementary region of $\mathcal{A}$. Together with the other known requirements we find that, in quantum field theories with Poincaré symmetry PEE should satisfy a general formula thus can be uniquely defined. This formula shows a decoupling between the information of the subjective choice of $\mathcal{A}$, $\mathcal{A}_i$ and the objective entanglement structure of the whole system. Based on this formula we can in general write $s_{\mathcal{A}}(\mathcal{A}_i)$ as a linear combination of subset entanglement entropies for a generic choice of $\mathcal{A}$ and $\mathcal{A}_i$, thus prove the partial entanglement entropy proposal and generalize it to a generic set up.
1 Introduction

The entanglement structure of a many-body quantum system is the correlation structure between any parts of the system. Entanglement entropy $S_A$ which characterizes the correlation between a region $A$ and its complement $\bar{A}$ is the most important quantity that we have used to explore the entanglement structure. Nevertheless, entanglement entropy in quantum field theory is an ambiguous quantity. Because of the short distance correlation, entanglement entropy in quantum field theory is infinite thus needs to be regularized. The regularization is to ignore certain types of correlations, which can be done by introducing certain types of cutoffs. Nevertheless, different cutoffs mean different ways to count entanglement thus lead to different values for entanglement entropy. Even with a chosen cutoff, the typical size fluctuations of the region still make the sub-leading contributions to the entanglement entropy ambiguous [1, 2]. Due to these ambiguities, people turn to the mutual information which is cutoff independent but still capture the information of entanglement. For any two non-intersecting regions $A$ and $B$, the mutual information is defined as

$$I(A, B) = S_A + S_B - S_{A\cup B}. \quad (1.1)$$

Recently, several papers [3–17] arise to study the so-called entanglement contour [5], which is a function that characterizes how much each degrees of freedom in a region $A$ contributes to the entanglement entropy $S_A$. In other words, consider a quantum field theory in $d$ dimensions, the entanglement contour is a density function of entanglement entropy that depends on $A$ and satisfy

$$S_A = \int_A f_A(x) d\sigma_x. \quad (1.2)$$
where \( x \) denotes a point on \( \mathcal{A} \) and \( \sigma_x \) denotes the infinitesimal subset of \( \mathcal{A} \) at \( x \). It is also convenient to define the partial entanglement entropy (PEE) \( s_{\mathcal{A}}(\mathcal{A}_i) \) for any subset \( \mathcal{A}_i \) of \( \mathcal{A} \) in the following way

\[
s_{\mathcal{A}}(\mathcal{A}_i) = \int_{\mathcal{A}_i} f_{\mathcal{A}}(x) d\sigma_x. \tag{1.3}
\]

Hence \( s_{\mathcal{A}}(\mathcal{A}_i) \) gives the contribution from \( \mathcal{A}_i \) to the entanglement entropy \( S_{\mathcal{A}} \). Like the mutual information, PEE is finite and cutoff independent when the boundaries of \( \mathcal{A} \) and \( \mathcal{A}_i \) do not overlap. PEE is particularly useful to explore the local and dynamical properties of entanglement. Physically PEE should satisfy the following requirements [5]:

1. **Additivity**: by definition we should have
   \[
s_{\mathcal{A}}(\mathcal{A}_i) = s_{\mathcal{A}}(\mathcal{A}_i^a) + s_{\mathcal{A}}(\mathcal{A}_i^b), \quad \mathcal{A}_i = \mathcal{A}_i^a \cup \mathcal{A}_i^b. \tag{1.4}
\]

2. **Invariance under local unitary transformations**: \( s_{\mathcal{A}}(\mathcal{A}_i) \) is invariant under any local unitary transformations act only inside \( \mathcal{A}_i \) and \( \bar{\mathcal{A}} \).

3. **Symmetry**: For any symmetry transformation \( \mathcal{T} \) under which \( \mathcal{T}\mathcal{A} = \mathcal{A}' \) and \( \mathcal{T}\mathcal{A}_i = \mathcal{A}_i' \), we have
   \[
s_{\mathcal{A}}(\mathcal{A}_i) = s_{\mathcal{A}}(\mathcal{A}_i'). \tag{1.5}
\]

4. **Normalization**: \( S_{\mathcal{A}} = s_{\mathcal{A}}(\mathcal{A}_i)|_{\mathcal{A}_i \rightarrow \mathcal{A}} \).

5. **Positivity**: \( s_{\mathcal{A}}(\mathcal{A}_i) \geq 0 \).

6. **Upper bound**: \( s_{\mathcal{A}}(\mathcal{A}_i) \leq S_{\mathcal{A}_i} \).

However, the above requirements are not enough to uniquely determine the entanglement contour function. So far, there are three proposals to construct entanglement contour and each of them are restricted to some special cases. The first one is the Gaussian formula [4–9, 14, 17] which only applies to Gaussian states in free theories. The second proposal is a geometric construction [10, 11, 15] based on the boundary and bulk modular flows in holography and applies to static spherical regions (or intervals) for holographic field theories. The third one is the partial entanglement entropy proposal [10, 13] (see also Ref.[12] for its reformulation using conditional entropy, Ref.[17] for its extension to define the contour of entanglement negativity, and Ref.[16] for its extension to explore the contour of holographic complexity) that claims PEE is given by a linear combination of subset entanglement entropies which is additive.

Is there a unique way to determine PEE (or entanglement contour) that satisfies all the physical requirements? In this paper, we answer this fundational question. We point out that PEE should satisfy another requirement, which is a symmetry under permutation. Based on this permutation symmetry and requirements-1, 2, 3, we follow the discussions[18] by Casini and Huerta, and show that in generic quantum field theories with Poincaré symmetry, PEE should satisfy a general formula. The requirement of normalization can
also be satisfied after we properly choose the coefficient for the formula, thus PEE can be uniquely defined. The requirement of positivity leads to a c-function. Then we prove the PEE proposal, and remarkably we extend this proposal to a generic set up with arbitrary choices of \( \mathcal{A} \) and \( \mathcal{A}_i \).

### 2 The general formula for partial entanglement entropy

Since \( s_{\mathcal{A}}(\mathcal{A}_i) \) captures the contribution from \( \mathcal{A}_i \) to \( S_{\mathcal{A}} \), in some sense \( s_{\mathcal{A}}(\mathcal{A}_i) \) captures the correlation between \( \mathcal{A}_i \) and \( \bar{\mathcal{A}} \). Therefore it is natural to require \( s_{\mathcal{A}}(\mathcal{A}_i) \) to be invariant under the following permutation,

\[
s_{\mathcal{A}}(A_i) = I(\bar{A}, \mathcal{A}_i) = I(\mathcal{A}_i, \bar{A}) = s_{\bar{\mathcal{A}}}(\bar{A}),
\]

where \( I \) is a symmetric function with respect to \( \mathcal{A}_i \) and \( \bar{\mathcal{A}} \). This together with the requirement of additivity indicate that, \( s_{\mathcal{A}}(\mathcal{A}_i) \) can be written as a double integration over \( \bar{\mathcal{A}} \) and \( \mathcal{A}_i \),

\[
I(\bar{A}, \mathcal{A}_i) = \int_{\bar{A}} d\sigma_x \int_{\mathcal{A}_i} d\sigma_y \ j(x, y, \eta_x, \eta_y),
\]

where \( x \) \((y)\) represents points in \( \bar{A} \) \((\mathcal{A}_i)\), \( \sigma_x \) \((\sigma_y)\) represents the infinitesimal subset of \( \bar{A} \) \((\mathcal{A}_i)\) at \( x \) \((y)\), and \( \eta_x \) \((\eta_y)\) represents the outward-pointing unit vector normal to \( \sigma_x \) \((\sigma_y)\). It will be enough to only consider \( I(\bar{A}, \mathcal{A}_i) \) between connected regions without loss of generality.

Since any fluctuations of a region with its causal development fixed can be given by a local unitary transformation confined in this region, the requirement-2 is equivalent to the requirement of causality. This means \( I(\bar{A}, \mathcal{A}_i) \) is invariant under any fluctuations of \( \bar{A} \) and \( \mathcal{A}_i \) with their causal development fixed. The requirement-3 means that PEE respect Poincaré symmetry in a Poincaré invariant theory. Under these requirements \( j(x, y, \eta_x, \eta_y) \) should be written as a conserved current in a formula that is strongly constrained by Poincaré symmetry. Finally we get the following conclusion[18] (see Appendix A).

- **For Poincaré invariant theories, the PEE can in general be written as**

\[
s_{\mathcal{A}}(\mathcal{A}_i) = \int_{\partial \bar{A}} d\sigma_x \int_{\partial \mathcal{A}_i} d\sigma_y \ H(|x - y|)(\eta_x \cdot \eta_y).
\]

Note that, in the above equation, \( x, y, \sigma_x \) and \( \sigma_y \) represent points or infinitesimal subsets on the boundaries \( \partial \bar{A} \) and \( \partial \mathcal{A}_i \). The function \( H(l) \), which is defined by \( C(l) = (d - 1)l^{2d-3}H'(l) \), depends on the other details of the theory. The function \( C(l) \) is extract from \( j(x, y, \eta_x, \eta_y) \), and the requirement of positivity shows \([18]\) \( C'(l) \leq 0 \) which implies that \( C(l) \) is a c-function \([19, 20]\) characterizing the renormalization group flow. For CFTs, \( C(l) = 2C_d(d - 1)(d - 2) \) is a constant, hence the \( H(l) \) has the particular formula \([18]\)

\[
H(l) = -\frac{C_d}{l^{2d-4}}.
\]
The constant $C_d$ can be determined by the requirement of normalization.

Since PEE reduces to an integration on relevant boundaries, we can write it as a functional of the boundaries with directions,

$$s_A(A_i) = \tilde{I}(\partial A_i, \partial A),$$  \hspace{1cm} (2.5)

where $\partial A_i$ is defined as the boundary $\partial A_i$ with an outward-pointing direction. Under this notation, we should have properties like $\partial A = -\partial \bar{A}$ and $\tilde{I}(\partial A_i, -\partial \bar{A}) = -\tilde{I}(\partial A_i, \partial A)$.

For self-contained, we summarize the detailed derivation [18] of (2.3) in Appendix A.

Before going on we would like to comment on the physical interpretation of (2.3). The authors of Ref.[18] interpreted the formula Eq.(2.3) as an additive (extensive) mutual information, which is true for special theories (if they exist) with additive mutual information. However, in general the mutual information is not additive, so their results seem to be much less generic. Actually they did not use the definition (1.1) for mutual information in the derivation, so any quantity that satisfies the requirements-1,2,3 and Eq.(2.1), should be given by Eq.(2.3). The more natural interpretation for the formula (2.3) can come from PEE. Later we will write Eq.(2.3) as a linear combination of entanglement entropies, which is usually not mutual information.

So far, there are some known results for PEE. For example the entanglement contour for static spherical regions (or intervals) in holographic CFTs were carried out in Ref.[10, 15] (see also Ref.[12]) using a geometric construction in the bulk. The strategy is to consider the bulk extension of the boundary modular flow lines, which are two dimensional surfaces that form a natural slicing of the bulk entanglement wedge. This slicing relates the points on the boundary region $A$ to the points on the RT surface $E_A$ by static spacelike geodesics normal to $E_A$. Based on this fine relation we read the contour functions for static spheres [12, 15],

$$f_A(r) = \frac{c_d}{6} \left( \frac{2R}{R^2 - r^2} \right)^{d-1},$$  \hspace{1cm} (2.6)

where $R$ is the radius of the spherical region $A$, $d$ is the spacetime dimension, and $c_d = a_d^* \frac{2\Gamma(d/2)}{\pi^{d/2-1}}$ (see Ref.[21, 22] for the definition of $a_d^*$) is a constant related to the central charge.

Following Eq.(1.3) it is easy to calculate the PEE [15],

$$\tilde{s}_A(A_2) = \frac{c_d}{6} \int_0^{R_0} \left( \frac{2R}{R^2 - r^2} \right)^{d-1} \Omega_{d-2} r^{d-2} \, dr$$

$$= \frac{c_d}{6} \frac{(4\pi z^2)^{d-1}}{\Gamma(2)} 2\tilde{F}_1 \left( \frac{d-1}{2}, d-1; \frac{d+1}{2}; z^2 \right),$$  \hspace{1cm} (2.7)

where $A_2$ is a cocentric sphere with radius $R_0 < R$, $2\tilde{F}_1(a, b, c, x) = \frac{2F_1(a, b; c; x)}{\Gamma(c)}$ is the regularized hypergeometric function, $\Omega_{d-2}$ is the volume of the unit $(d-2)$-sphere $S^{d-2}$ and $z$ is the ratio $z = R_0/R$. 

- 4 -
On the other hand, we can also calculate PEE using the generic formula (2.3). Plugging (2.4) into (2.3), we find

\[ s_A(A_2) = C_d \int_{\partial A} d\sigma_x \int_{\partial A_2} d\sigma_y \frac{(RR_0)^{d-2} \vec{n}_x \cdot \vec{n}_y}{|\vec{n}_x R - \vec{n}_y R_0|^{2d-4}} \]

\[ = C_d \Omega_d^{-2} \Omega_d^{-3} \int_0^\pi d\theta \frac{\frac{2z^{d-2}}{(1 + z^2 - 2z \cos \theta)} - \frac{4}{(1 - z^2)^3}}. \]  

(2.8)

Though it is not easy to write the above integration in a compact form as in Eq.(2.7), one can check that the integration (2.8) coincide with (2.7) up to a coefficient that depend on \( d \). For example when \( d = \{3, 4, 5\} \) we have

\[ \tilde{s}_A(A_2) \sim s_A(A_2) \sim \left\{ \frac{z^2}{1 - z^2}, \frac{z^3 + z}{(z^2 - 1)^2} = \frac{1}{2} \tanh^{-1} \left( \frac{2z}{z^2 + 1} \right), \frac{z^4 (z^2 - 3)}{(z^2 - 1)^3} \right\}. \]  

(2.9)

### 3 Partial entanglement entropy as a linear combination of entanglement entropies

![Figure 1](image1.png)

Figure 1. These figures show examples for the partitions of an interval, a strip and an annulus. The subsets \( A_1 \) and \( A_3 \) share boundary with \( A \) while \( A_2 \) lies between them.

In Ref.[10, 13] the author proposed that for two dimensional QFTs, PEE is given by a linear combination of subset entanglement entropies. More explicitly, given a connected region \( A \), for any connected subset \( A_2 \), which in general partition the region \( A \) into three subsets \( \{A_1, A_2, A_3\} \), \( s_A(A_2) \) is given by

\[ s_A(A_2) = \frac{1}{2} (S_{A_1 \cup A_2} + S_{A_2 \cup A_3} - S_{A_1} - S_{A_3}) . \]  

(3.1)

This proposal can be extended to higher dimensional configurations with rotation symmetry or translation symmetry (see Fig. 1), which we call “quasi one-dimensional” configurations. In such cases the requirement of additivity is still satisfied without imposing extra constrains on entanglement entropy. In Ref.[12, 13], it was shown that the proposed \( s_A(A_2) \) (3.1) satisfies the requirements 1-6. Assuming \( \tilde{A} \) is the system that purifies the region \( A \), then

\[ \mathcal{I}(\tilde{A}, A_2) = s_{\tilde{A}}(A_2) = \frac{1}{2} (S_{\tilde{A}_1 \cup A_2} + S_{\tilde{A}_2 \cup A_3} - S_{A_1} - S_{A_3}) \]

\[ = s_{\tilde{A}_1}(A) = \mathcal{I}(A_2, \tilde{A}), \]  

(3.2)
thus, the requirement of invariance under a permutation (2.1) is also satisfied. With all the requirements satisfied, our previous discussion implies that, in theories with Poincaré symmetry PEE (3.1) should be given by the generic formula (2.3).

Figure 2. The colored region is $\mathcal{A}$, and the subset $\mathcal{A}_2$ divide $\mathcal{A}$ into three regions, the directions of the boundaries $L_1, l_1, l_2$ are shown by the arrows.

The requirement of normalization means entanglement entropy should be recovered from PEE under the limit $\mathcal{A}_2 \to \mathcal{A}$ and properly choosing the coefficient,

$$S_{\mathcal{A}} = \tilde{I}(\partial \mathcal{A}, \partial \mathcal{A}).$$

(3.3)

The integration $\tilde{I}(\partial \mathcal{A}, \partial \mathcal{A})$ is divergent as $l$ can be vanished when $x$ and $y$ overlap. Certain prescriptions are needed to prevent divergence, thus we can test Eq.(3.3). We give an example in Appendix B. In CFTs Eq.(3.3) has passed several non-trivial tests in generic dimensional extensive mutual information (EMI) models [2, 23–26]. Our discussion implies that the results in Ref.[2, 23–26] apply to generic CFTs.

Consider the case in Fig. 2, where $\mathcal{A}$ is partitioned into $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$. We denote the boundary of $\mathcal{A}$ as $L_1$ while the two boundaries of $\mathcal{A}_2$ as $l_1$ and $l_2$. Since we need to specify the direction of the boundaries when calculating PEE, it is convenient to define that $L_1 (l_i)$ points outward from $\mathcal{A}$ ($\mathcal{A}_2$), while $-L_1 (-l_i)$ points inward, hence $\pm L_1$ are just $\partial \mathcal{A}$ and $\partial \mathcal{A}$. Following the above notations we have

$$\partial \mathcal{A}_1 : \{L_1, -l_2\}, \quad \partial \mathcal{A}_2 : \{l_2, l_1\}, \quad \partial (\mathcal{A}_1 \cup \mathcal{A}_2) : \{L_1, l_1\},$$

$$\partial \mathcal{A}_3 : \{-l_1\}, \quad \partial (\mathcal{A}_2 \cup \mathcal{A}_3) : \{l_2\}, \quad \partial \mathcal{A} : \{L_1\}. \quad (3.4)$$

Since $\partial \mathcal{A} : \{-L_1\}$, the left hand side of Eq.(3.1) should be given by

$$s_{\mathcal{A}}(\mathcal{A}_2) = \tilde{I}(\partial \mathcal{A}_2, \partial \mathcal{A}) = \tilde{I}(\{l_2, l_1\}, \{-L_1\})$$

$$= \tilde{I}(l_1, -L_1) + \tilde{I}(l_2, -L_1). \quad (3.5)$$

Then we calculate the entanglement entropies on the right hand side of Eq.(3.1) using Eq.(3.3). Since $\partial \mathcal{A}_1 : \{-L_1, l_2\}$, the entanglement entropy $S_{\mathcal{A}_1}$ is given by

$$S_{\mathcal{A}_1} = \tilde{I}(\{L_1, -l_2\}, \{-L_1, l_2\})$$

$$= \tilde{I}(L_1, -L_1) + 2\tilde{I}(L_1, l_2) + \tilde{I}(-l_2, l_2). \quad (3.6)$$
Similarly, we have

\[ S_{A_1 \cup A_2} = \tilde{I}(l_1, -l_1) - 2\tilde{I}(l_1, L_1) + \tilde{I}(L_1, -L_1), \]
\[ S_{A_3} = \tilde{I}(l_1, -l_1), \quad S_{A_2 \cup A_3} = \tilde{I}(-l_2, l_2). \]  

(3.7)

We then plug the above entanglement entropies into Eq. (3.1) to find

\[ s_A(A_2) = \tilde{I}(l_1, -L_1) - \tilde{I}(L_1, l_2), \]  

(3.8)

which exactly matches with (3.5). Hence the proposal (3.1) is justified.

In Ref. [13] the author addressed that, when \( d > 2 \) the construction (3.1) is additive only for “quasi one-dimensional” configurations. For more generic cases we need to impose further constraints on entanglement entropy. Nevertheless, the above discussion does not assume the configuration to be “quasi one-dimensional”, which implies that the proposal (3.1) indeed hold for a generic set up. This is because the entanglement entropy calculated by Eq. (3.3) is already highly constraint and satisfies all those further constraints needed to keep additivity for (3.1).

Now we are ready to generalize the PEE proposal (3.1) to a generic partition. We consider a generic region \( \mathcal{A} \) with \( m \) outward-pointing boundaries \( L_j (1 \leq j \leq m) \), and a subset \( \mathcal{A}_1 \) with \( n \) outward-pointing boundaries \( l_i (1 \leq i \leq n) \) (see, for example, Fig. 3). Then the PEE \( s_A(A_1) = \tilde{I}(\mathcal{A}_1, \bar{\mathcal{A}}) \) is given by

\[ s_A(A_1) = \tilde{I}([l_1, \ldots, l_n], \{-L_1, \ldots, -L_m\}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{I}(l_i, -L_j). \]  

(3.9)

We try to write the summation as a linear combination of entanglement entropies. We denote the region enclosed by the two boundaries \( l_i \) and \( L_j \) as \( \mathcal{A}_{ij} \). Since \( \mathcal{A}_{ij} \) is always inside \( \mathcal{A} \), so \( L_j \) is also the outward-pointing boundary of \( \mathcal{A}_{ij} \). However \( l_i \) could be either the outward or inward-pointing boundary of \( \mathcal{A}_{ij} \). For example, in the right figure of Fig. 3, \( l_1 \) is the inward-pointing boundary of \( \mathcal{A}_{1,j} \), while \( l_2 \) (\( l_3 \)) is the outward-pointing boundary of \( \mathcal{A}_{2,j} \) (\( \mathcal{A}_{3,j} \)). When \( l_i \) is the outward-pointing boundary of \( \mathcal{A}_{ij} \) (which we denote as \( l_i//L_j \)), according to (3.3) we have

\[ S_{\mathcal{A}_{ij}} = \tilde{I}(l_i, -l_i) + \tilde{I}(L_j, -L_j) + 2\tilde{I}(l_i, -L_j), \]  

(3.10)

\[ \tilde{I}(l_i, -L_j) = \frac{1}{2} (S_{\mathcal{A}_{ij}} - S_{l_i} - S_{L_j}) , \]  

(3.11)

where \( S_{l_i} \) (\( S_{L_j} \)) is the entanglement entropy of the region enclosed by \( l_i \) (\( L_j \)). When \( l_i \) is the inward-pointing boundary of \( \mathcal{A}_{ij} \) (or \(-i//j\)), we should have

\[ S_{\mathcal{A}_{ij}} = \tilde{I}(l_i, -l_i) + \tilde{I}(L_j, -L_j) + 2\tilde{I}(l_i, L_j), \]  

(3.12)

\[ \tilde{I}(l_i, -L_j) = \frac{1}{2} (S_{l_i} + S_{L_j} - S_{\mathcal{A}_{ij}}) . \]  

(3.13)
Then we conclude that, given a generic $A$ with outward-pointing boundaries $L_j$ and a subset $A_1$ with outward-pointing boundaries $l_i$, the PEE $s_A(A_1)$ should be given by

$$s_A(A_1) = \sum_{l_i//L_j} \frac{1}{2} (S_{A_i} - S_{l_i} - S_{L_j}) - \sum_{-l_i//L_j} \frac{1}{2} (S_{A_i} - S_{l_i} - S_{L_j}) . \tag{3.14}$$

In the following we applies Eq.(3.14) to three cases.

- **Case 1**: In Fig.2 we calculate $s_A(A_2)$. So we have $S_{A_{11}} = S_{A_1}, S_{L_1} = S_{A, S_{l_1}} = S_{A_2//A_3}, S_{L_2} = S_{A_1, -l_1//L_1, l_2//L_1}$. Then according to (3.14) we find

  $$2s_A(A_2) = S_{A_1\cup A_2} + S_{A_2\cup A_3} - S_{A_1} - S_{A_3}, \tag{3.15}$$

  which is absolutely not the mutual information $I(\bar{A}, A_2)$.

- **Case 2**: In the left figure of Fig.3, the subset $A_1$ partition the $A$ into two parts and $l_1//L_1$, thus

  $$s_A(A_1) = \frac{1}{2} (S_A + S_{A_1} - S_A) = \frac{I(\bar{A}, A_1)}{2}. \tag{3.16}$$

In this case $2s_A(A_1)$ is a mutual information. If we further divide $A_1$ into subsets $A_1^l$ with the same topology of a sphere, then $2s_A(A_1)$ is also the mutual information $I(\bar{A}, A_1^l)$. Because PEE is additive, thus $I(\bar{A}, A_1) = \sum_I(\bar{A}, A_1^l)$ looks additive. However, when any of the $A_1^l$ has other topologies (like an annulus as in the previous case), this additivity for mutual information breaks down.

- **Case 3**: In the case of the right figure of Fig.3, we find

  $$2s_A(A_1) = S_{A_1\cup A_2\cup A_3\cup A_4} + S_{A_1\cup A_2\cup A_4\cup A_2} + 2S_{A_1\cup A_2\cup A_3} + S_{A_1\cup A_2\cup A_3\cup A_4\cup A_1}$$

  $$+ S_{A_1\cup A_2\cup A_4\cup A_1} - S_{A_4\cup A_2} - 2S_{A_3} - S_{A_4\cup A_1} - S_{A_2} - S_{A_2}. \tag{3.17}$$

Here $\bar{A}$ is disconnected, one can check that

$$I(A_1, \bar{A}_1) = s_A(A_1)\big|_{A_2 \to 0}$$

$$= \frac{1}{2} (S_{A_1\cup A_4\cup A_4} + S_{A_1\cup A_2\cup A_4} + S_{A_1\cup A_2\cup A_3} - S_{A_4} - S_{A_2} - S_{A_3} - S_A), \tag{3.18}$$

and $s_A(A_1) = I(A_1, \bar{A}_1) + I(A_1, \bar{A}_2)$. So it is unnecessary to consider the partial entanglement entropies between regions that are disconnected.

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**Figure 3.** The colored region is $A$. We calculate $s_A(A_1)$, so $l_i (L_j)$ points outward $A_1 (A)$. Note that $A_i$ is not the complement of $A_i$. In the following we applies Eq.(3.14) to three cases.
4 Discussion

The motivation of Ref.[18] came from the entanglement entropy for multi-intervals in 2-dimension al free massless fermions [27] (also see Ref.[28–30] for similar results), which indicates that the mutual information is additive. Eq.(2.3) was thought to be restrictive to theories with EMI (or EMI models). Nevertheless, no specific theory with additive mutual information were known except the free massless fermions [27]. Here we introduced an additional key requirement of symmetry under permutation (2.1) for PEE, and clarified that the general formula (2.3) indeed gives PEE which can be defined in general Poincaré invariant theories.

We want to stress that a choice for the region $A$ (plus a cutoff for quantum field theories) is subjective thus not a physical operation. It tells us what kind of entanglement we should “count” to refer to the (regularized) entanglement entropy $S_A$. While entanglement structure of the whole system is objective thus should not be affected by a choice of $A$ (and the cutoff). One would expect the formula that characterizes entanglement between sub-regions should show a decoupling between the information of the subjective partition and the objective entanglement structure of the whole system. The general formula (2.3) for PEE is just what we expect. All the information of the subjective partition and cutoffs are contained in the domain of the double integration. While the entanglement structure of the whole system is characterized by the bi-local function $j(x, y, \eta_x, \eta_y)$ in Eq.(2.2) or $H(l)$ in Eq.(2.3). We call these functions the entanglement structure functions. More interestingly, we can extract a $c$-function from them. Thanks to this decoupling, the computation of entanglement entropy for regions with arbitrary shape of boundaries (for example singular regions) reduces to a simple geometric computation [2, 23–26].

We showed that in general PEE (2.3) can be expressed as a unique linear combination of subset entanglement entropy (3.14) when entanglement entropy are calculated as a limit (3.3) of PEE. Note that the topology of the regions play an important role in Eq.(3.14). This proves the PEE proposal [10, 13] and generalize it to a generic set up. One can directly take (2.3) or (3.14) as the definition for PEE. The definition (3.14) can even be applied to few-body systems or lattice models (see the comparison with the Gaussian formula [12] in free theories) as long as the entanglement entropy can be defined. Also, quantities like entanglement entropy and mutual information can be expressed as a linear combination of partial entanglement entropies under some certain limits. For theories that are not conformal invariant, the entanglement structure functions are usually unknown, this can be explored by matching the partial entanglement entropies to explicit results of entanglement entropy.

Acknowledgments

We would like to thank Chong-Sun Chu, Jonah Kudler-Flam, Rong-xin Miao, Tatsuma Nishioka, Zhuo-Yu Xian, and Gang Yang for helpful discussions. Especially we would like to thank Chen-Te Ma for a careful reading of the manuscript and Horacio Casini for pointing out their work on mutual information and $c$-functions. We thank the Yukawa Institute for
A Derivation of the general formula for partial entanglement entropy

Consider a quantum field theory defined on a \((d-1)\)-dimensional Cauchy slice where \(A\) and \(B\) are any two non-intersecting sub-regions. Assuming there exist an information theoretic quantity \(\mathcal{I}(A, B)\) defined on \(A\) and \(B\), which is invariant under permutation and additive,

\[
\mathcal{I}(A, B) = \mathcal{I}(B, A) , \quad \mathcal{I}(A, B \cup C) = \mathcal{I}(A, B) + \mathcal{I}(A, C).
\]

Then \(\mathcal{I}(A, B)\) can be written as a double integration of the bi-local function

\[
\mathcal{I}(A, B) = \int_{A} d\sigma_{x} \int_{B} d\sigma_{y} j(x, y, \eta_{x}, \eta_{y}) ,
\]

where \(x (y)\) represents points in \(\bar{A} (A_{i})\), \(\sigma_{x} (\sigma_{y})\) is any infinitesimal subset of \(\bar{A} (A_{i})\) at \(x (y)\), and \(\eta_{x} (\eta_{y})\) represents the outward-pointing unit vector normal to \(\sigma_{x} (\sigma_{y})\). The causality requires \(\mathcal{I}(A, B)\) to be invariant under any fluctuations of the sub-regions \(A\) and \(B\) with their causal development fixed. This constrains the formula (2.2) further to be

\[
\mathcal{I}(A, B) = \int_{A} d\sigma_{x} \int_{B} d\sigma_{y} \eta_{y}^{\nu} \eta_{x}^{\mu} J_{\mu\nu}(x, y) ,
\]

with \(J_{\mu\nu}(x, y)\) being a conserved current \(\partial_{\mu} J_{\mu\nu}(x, y) = 0\). This can be understood by the fact that the flux of a conserved current that passes through a region is invariant under any fluctuation of the region with its boundary fixed. The Poincaré invariance indicates

\[
J_{\mu\nu}(x, y) = \frac{(x - y)^{\mu}(x - y)^{\nu}}{(x - y)^{2d}} G(l) - \frac{g_{\mu\nu}}{(x - y)^{2(d-1)}} F(l) ,
\]

where \(l\) is the distance between \(x\) and \(y\), \(F\) and \(G\) are two dimensionless functions. The conservation of \(J_{\mu\nu}\) furthermore gives

\[
(G(l) - F(l))' = -(d - 1) \frac{2F(l) - G(l)}{l} .
\]

The requirement of positivity implies that for any time like vectors \(\eta_{\nu}^{\mu}\) and \(\eta_{x}^{\mu}\), we should have

\[
\eta_{\nu}^{\mu} \eta_{x}^{\mu} J_{\mu\nu}(x, y) \geq 0 .
\]

This furthermore implies that,

\[
2F(l) \geq G(l) \geq 0 .
\]
Define \( C(l) = G(l) - F(l) \), then according to (A.5) we have

\[
C'(l) \leq 0,
\]

which implies \( C(l) \) is always decreasing under the RG flow, hence can be considered as a \( c \)-function. For theories with an infrared fixed point, we have

\[
C(l) \geq 0,
\]

for any \( l \). From (A.5), we also have

\[
F(l) = -\frac{lC'(l)}{d-1} + C(l), \quad G(l) = -\frac{lC'(l)}{d-1} + C(l).
\]

Then it is convenient to define another function \( H(l) \) by

\[
C(l) = (d-1)l^{2d-3}H'(l).
\]

Thus

\[
J_{\mu\nu}(l) = -\partial_{\mu}\partial_{\nu}H(l) + g_{\mu\nu}\partial_\alpha\partial^\alpha H(l)
\]

and

\[
\mathcal{I}(A, B) = \int_{\partial A} d\sigma_x \int_{\partial B} d\sigma_y H(|x - y|)(\vec{\eta}_x \cdot \vec{\eta}_y).
\]

Since \( C(l) \) is a \( c \)-function, for CFTs \( C(l) = 2C_d(d-1)(d-2) \) is a constant, so we have

\[
H(l) = -\frac{C_d}{l^{2d-4}},
\]

where \( C_d \) is a constant that depend on \( d \).

**B Entanglement entropy as a limit of partial entanglement entropy**

Here we give an example about how entanglement entropy can be approached as a limit of (2.3). The requirement of normalization indicates that, the entanglement entropy should be recovered in some way under the limit \( A_2 \to A \). In other words, when \( \partial A_2 \) approaches \( \partial A \) the integration (2.3) should give \( S_A \) in some way. Let us consider the case of holographic CFT\(_3\), where two concentric circles with radius \( R_+ \) (\( R_+ > R_- \)) partition the systems into three regions: a disk \( A \) with radius \( R_- \), the region \( B \) outside the circle with radius \( R_+ \) and an annulus \( C \) between them. When \( R_- \) approaches \( R_+ \), we may expect that the PEE \( s_{AUC}(A) = s_{BUC}(B) \) will approach the entanglement entropy of a disk with radius \( R \), which is given by

\[
R = \frac{R_+ + R_-}{2} + \alpha \left( R_+ - R_- \right), \quad -1 \leq \alpha \leq 1.
\]

---
According to (2.7), for holographic CFT$_3$ we have $s_{A∪C}(A) = \frac{2\pi c R^2}{3(R_+^2 - R_-^2)}$. Then we take the limit $R_+ - R_- = \epsilon \to 0$, and find

$$s_{A∪C}(A) = \frac{c}{6} \frac{2\pi R}{\epsilon} + \frac{c}{6} \pi (\alpha - 2) + \mathcal{O}(\epsilon). \quad (B.2)$$

On the other hand, in this case the entanglement entropy for a disk with radius $R$ can also be calculated by the Ryu-Takayanagi (RT) formula [31, 32],

$$S_{EE} = \frac{c}{6} \frac{2\pi R}{\delta} - \frac{c}{3} \pi + \mathcal{O}(\delta), \quad (B.3)$$

where $\delta$ is the UV cutoff on the RT surface.

If we match Eq.(B.2) to the holographic entanglement entropy Eq.(B.3), then the first term is the standard area term. Nevertheless, the universal term is ambiguous due to the undertermined parameter $\alpha$. In Ref.[2] it was argued that in order to protect the universal term from the UV physics, we should choose $\alpha = 0$. It is also obvious that when $\alpha = 0$ Eq.(B.2) exactly matches with Eq.(B.3) after we replace $\epsilon$ with $\delta$. This argument can be extend to boundaries with any shape [2]. Then we consider a general connected region $A$ with boundary $\partial A$. In order to calculate its entanglement entropy using PEE, we should extend the boundary $\partial A$ into an infinitely narrow strip (or shell when $d > 3$) with the original boundary $\partial A$ in the middle. We may denote the inner side of the strip as $\partial A$, while denote the outer side as $\partial \bar{A}$. Then the entanglement entropy of $A$ can be approximated by

$$S_A = \tilde{I}(\partial A, \partial \bar{A})|_{\epsilon \to 0}, \quad (B.4)$$

where $\epsilon$ is the width of the strip (or shell) and $\partial \bar{A}$ (or $\partial A$) points outward $A$ ($\bar{A}$).

For theories beyond CFTs, if $C(l) \geq 0$, then when $l$ decreases $|H(l)|$ will monotonically increase (to infinity). Eq.(2.3) indicates the requirement-6 of positivity is always satisfied.

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