Basics of Autonomous Nonlinear Oscillators: 
Limit Cycle, Orbital Stability, and Synchronization 

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Abstract: Periodic motions are often desired in engineering applications. The traditional method of trajectory planning followed by feedback tracking may work well for maintaining the planned motion. However, it is desired for certain applications to achieve the motion as autonomous rather than forced oscillations so as to adapt the motion profile in response to disturbance and changing environment. This motivates us to consider feedback control design to embed a limit cycle oscillation in the closed-loop dynamics with the notion of orbital stability. Here we provide a tutorial review of basic concepts and analysis tools for limit cycles arising from autonomous nonlinear dynamics. The Floquet theory and contraction analysis are reviewed, simple planar oscillators are illustrated, and synchronization and coordination of coupled oscillators are discussed.

Key Words: oscillation, limit cycle, orbital stability, synchronization, coordination.

1. Introduction

While traditional theories of feedback control have focused on the regulation around an equilibrium point, much less attention has been paid to control specifications involving periodic motion. Capability of generating coordinated autonomous oscillations can be useful and even crucial in many engineering applications. A standard approach to realize periodic motion has been trajectory planning followed by feedback regulation. A limitation of this approach is the difficulty of incorporating adaptivity to changing environment, for a trajectory is calculated off-line and implemented as a fixed reference. Another approach would be to design a feedback controller so that the closed-loop system has the desired periodic motion as (part of) a stable limit cycle oscillations. In this case, the motion is achieved not as a forced response but as a free response of an autonomous system. This approach can be advantageous in certain scenarios.

A scenario is where we aim to achieve a periodic trajectory \( \chi(t) \) for the state \( x(t) \), but when perturbed by a disturbance, the state \( x(t) \) does not have to come back to \( \chi(t) \) with originally set timing. That is, convergence of \( x(t) \) to \( \chi(t) \) is not required, and it suffices if \( x(t) \) converges to \( \chi(t + \tau) \) with some arbitrary time shift \( \tau \). For example, a walking human would have a nominal movement \( \chi(t) \) specifying periodic oscillations of state variables \( x(t) \) (e.g., knee and hip angles). Reference tracking \( x(t) = \chi(t) \) would surely achieve walking, but so does \( x(t) = \chi(t + \tau) \) since the timing is unimportant. Thus, if we trip on a stone after making steps at \( t = 1, 2, 3 \), motor control would not aim for stepping at \( t = 7, 8, 9 \) after the recovery, but rather choose the timing flexibly (e.g., \( t = 7.2, 8.2, 9.2 \)). Such recovery is possible if \( \chi(t) \) is embedded in the closed-loop state space as a limit cycle with orbital stability. This type of property, convergence to an orbit rather than a timed trajectory, would be useful for robotic locomotion systems and has been pursued in the literature (e.g. [1]).

Another scenario is the pattern formation in complex dynamical systems, especially in the context of synchronization and coordination of coupled oscillators. An interconnection of multiple elements with simple individual dynamics can exhibit a complex collective behavior perceived as a pattern. Scientific understanding of the mechanisms underlying such emergent behaviors in an uncertain varying environment could provide a central idea for innovative design of engineered systems with new functionalities. Along this line, coupled oscillators have been studied in various contexts including neuronal circuits called the central pattern generators (CPGs) [2]–[4], chaos and theoretical physics [5]–[7], and electrical power networks [8]. Again, the notion of orbitally stable limit cycles characterizes the dynamical property of interest.

Various theoretical analysis tools for nonlinear oscillators have been developed in the literature, including the harmonic balance [9],[10], the Poincaré-Bendixson theorem [11],[12], the Hopf bifurcation theorem [13],[14], and perturbation theory and averaging [15],[16]. Especially well developed is the body of work for synchronization of coupled oscillators, including the Malkin theorem for phase coordinate transformation under weak coupling [2],[3],[17],[18], synchrony conditions for Kuramoto-type phase oscillators [5],[19], contraction analysis for global convergence with strong coupling [20]–[22], and master stability equation based on eigenvalues of the coupling matrix [6],[23].

In this article, we will provide a tutorial on some basic theories that characterize orbital stability of limit cycles and their applications to synchronization and coordination problems. In particular, we will first present the classical Floquet theory for stability analysis of linear periodic systems. The result will be directly useful for local orbital stability analysis of limit cycles through linearization around the orbit. The result will be applied to coordination of weakly coupled oscillators where we see an averaging technique simplifies the orbital stability
Let $t_0$ be a positive integer and $X(t)$ be a fundamental matrix of the linear system, and every solution $x(t)$ can be given as a linear combination of the columns of $X(t)$, that is, there exists a constant vector $v$ such that $x(t) = X(t)v$. Clearly, it is important for $X(t)$ to have linearly independent columns. It turns out that the linear independence is ensured for all $t$ if the initial condition is chosen such that $\det(X_0) \neq 0$ due to the following result.

**Lemma 1** Let $k$ be a positive integer and $X(t) \in \mathbb{R}^{n \times n}$ be a matrix function satisfying

$$\dot{X}(t) = A(t)X(t), \quad \text{rank}(X(0)) = k.$$ 

Then the rank of $X(t)$ is equal to $k$ for all $t \in \mathbb{R}$.

**Proof.** Since the rank of $X(t)$ is less than $k+1$ at time $t = 0$, there exists a full rank matrix $V \in \mathbb{R}^{n \times (n-k)}$ such that $X(0)V = 0$.

Define $W(t) := X(t)V$. Then $W(0) = 0$ and $\dot{W} = AW$. Hence $W(t) \equiv 0$, implying that $X(t)$ has rank less than $k + 1$ for all $t$. If the rank of $X(t)$ were less than $k$ at some time $t = t_0$, following the same argument, the rank of $X(t)$ would have to be less than $k$ for all $t$ including $t = 0$, which cannot be the case. Therefore, the rank of $X(t)$ is equal to $k$ for all $t \in \mathbb{R}$. ■

We will use a fundamental matrix $X(t)$ to define a proper Lyapunov transformation. Let

$$Q(t) := X(t)^{-1}X(t + T).$$

Then $Q(t)$ is constant because

$$X(t)Q(t) = X(t + T)$$

$$\Rightarrow \dot{X}(t)Q(t) + X(t)\dot{Q}(t) = A(t)X(t)$$

$$\Rightarrow A(t)X(t)Q(t) + X(t)\dot{Q}(t) = A(t)X(t)Q(t)$$

$$\Rightarrow \dot{Q}(t) = 0.$$

Let $B \in \mathbb{C}^{n \times n}$ be a matrix such that $Q = e^{BT}$. Such $B$ exists since $Q$ is nonsingular, but $B$ may be a complex matrix in general even though $Q$ is real. Define $P(t)$ by

$$P(t) := X(t)e^{-BT}, \quad Q = e^{BT}.$$ 

Then $P(t)$ is $T$-periodic because

$$P(t + T) = X(t + T)e^{-BT} = X(t)e^{-BT} = P(t)$$

and $P(t)$ satisfies

$$\dot{P} = AP - PB.$$

By the coordinate transformation $x = Pz$, we have $\dot{z} = Bz$, noting that

$$\dot{z} = P\dot{z} + P\dot{z} = (AP - PB)z + P\dot{z}.$$ 

Thus, stability of the periodic system $\dot{x} = A(t)x$ is equivalent to stability of the linear time-invariant system $\dot{z} = Bz$. Summarizing the result, we have the following.

**Theorem 1** Consider the linear periodic system in (1). Let $X(t)$ be a square matrix function such that

$$X(t) = A(t)X(t), \quad \det(X(t)) \neq 0.$$ 

Then $X(t)$ is invertible for all $t \in \mathbb{R}$ and

$$Q(t) := X(t)^{-1}X(t + T)$$

is constant over time. Let $B, P(t) \in \mathbb{C}^{n \times n}$ be defined by

$$P(t) := X(t)e^{-BT}, \quad Q = e^{BT}.$$ 

Then $P(t)$ is $T$-periodic. Moreover, with transformation $x = Pz$, we have the linear time-invariant system

$$\dot{z}(t) = Bz(t).$$

Thus the original linear periodic system is stable if and only if $B$ is Hurwitz.
In the above result, \( B \) is a complex matrix in general and the Lyapunov transformation \( P(t) \) is also complex. This is sometimes inconvenient. The result can be restated using real matrices; \( B \) is defined as a matrix such that \( Q^2 = e^{2BT} \) and \( P(t) \) becomes 2\( T \)-periodic, but otherwise the statement remains the same. In this case, \( B \) and \( P(t) \) can always be chosen to be real. To see this, recall that, given a real square matrix \( M \), there exists a real square matrix \( X \) satisfying \( M = e^{iX} \) if and only if \( M \) is nonsingular and each Jordan block of \( M \) belonging to a negative eigenvalue occurs an even number of times [25]. Reference [26] then argues that this is the case for \( Q \) nonsingular, reasoning that any negative real eigenvalue of \( M \) comes from an eigenvalue of \( Q \) on the imaginary axis, its complex conjugate is also an eigenvalue since \( Q \) is real, and it makes the Jordan block of the negative eigenvalue repeated.

Theorem 1 shows that the linear periodic system (1) is stable if and only if all the eigenvalues of \( B \) have negative real parts. By definition, \( B \) is related to \( Q \) by \( Q = e^{BT} \), which implies \( \mu = e^{\lambda T} \) where \( \mu \) and \( \lambda \) are the eigenvalues of \( B \) and \( Q \), respectively. Hence, \( B \) is Hurwitz if and only if all the eigenvalues of \( Q \) have magnitude less than 1. Now, the value of \( Q \) depends on the initial value chosen for \( X(0) \), but the eigenvalues of \( Q \) are independent of \( X(0) \). To see this, let \( \Phi(t) \) be the solution to \( \dot{\Phi} = A(\Phi) \) with the initial condition \( \Phi(t) \). Then it is easily verified that \( X(t) = \Phi(t)X(0) \) holds, and hence \( Q = X(0)^{-1}X(t) \) satisfies \( \dot{X}(0)\Phi = \Phi(T)X(0) \), indicating that the eigenvalues of \( \Phi(T) \) coincide with those of \( Q \) generated by an arbitrary (non-singular) initial value \( X(0) \). Thus we have one of the fundamental results in the stability analysis of linear periodic systems as follows.

**Corollary 1** ([27]) Consider the linear periodic system in (1) and the fundamental matrix \( \Phi(t) \) defined by

\[
\Phi(t) = A(t)\Phi(t), \quad \Phi(0) = I.
\]

Then the system is stable if and only if all the eigenvalues of \( \Phi(T) \) are strictly inside the unit circle.

The eigenvalues of \( \Phi(T) \) are called the Floquet multipliers of system (1), which coincide with the Floquet multipliers of the transformed system \( \tilde{x} = B\tilde{x} \) in Theorem 1, i.e., the eigenvalues of \( e^{2BT} \). In fact, the Floquet multipliers are invariant under arbitrary periodic Lyapunov transformation. That is, the systems (1) and (2) share the same set of Floquet multipliers [27].

Finally, the following lemmas turn out to be instrumental for stability analysis of periodic solutions to general nonlinear systems discussed in later sections.

**Lemma 2** ([28], p.285) The linear periodic system in (1) has a nontrivial \( T \)-periodic solution if and only if it has a Floquet multiplier at 1.

**Lemma 3** ([29], p.22) The fundamental matrix \( \Phi(t) \) solving

\[
\dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0) = I
\]

is given by the Peano-Baker series

\[
\Phi(t) = I + \int_0^t A(t_1)dt_1 + \int_0^t A(t_1)\int_0^{t_1} A(t_2)dt_2dt_1 + \cdots
\]

Moreover, under the commuting property

\[
A(t)B(\tau) = B(\tau)A(t), \quad B(\tau) := \int_0^\tau A(\sigma)d\sigma,
\]

the series converges to

\[
\Phi(t) = e^{Bt}.
\]

**Lemma 4** ([30]) Consider the linear periodic system

\[
\dot{x}(t) = eA(t)x(t), \quad A(t + T) = A(t), \quad x(t) \in \mathbb{R}^n.
\]

Define the average dynamics as

\[
\bar{A} := \int_0^T A(t)dt.
\]

Then there exists \( \epsilon > 0 \) such that the system is exponentially stable for all \( 0 < \epsilon < \epsilon_0 \) if and only if \( \bar{A} \) is Hurwitz.

**Proof.** An outline of the proof is given here using an argument similar to the proof of Theorem 3 on page 206 of [29]. Using Lemma 3, the fundamental matrix is given by

\[
\Phi(T) = I + e\bar{A} + O(e^2).
\]

Let \( \lambda \) be an eigenvalue of \( \Phi(T) \). Then

\[
\lambda = 1 + e(\sigma + j\omega) + O(e^2)
\]

for an eigenvalue \( \sigma + j\omega \) of \( \bar{A} \). Note that

\[
|\lambda|^2 = (1 + e\sigma)^2 + (e\omega)^2 + O(e^2) = 1 + 2e\sigma + O(e^2).
\]

Thus, \( |\lambda| < 1 \) for sufficiently small \( e > 0 \) if \( \sigma < 0 \), and the result then follows from Corollary 1.

**3. Stability Analysis of Periodic Orbits**

Consider the nonlinear system

\[
\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n,
\]

where \( f \) is continuously differentiable and the Jacobian matrix \( \frac{\partial f}{\partial x} \) is bounded and Lipschitz. Let \( x = \chi \) be a solution. The purpose of this section is to provide a set of tools for analyzing stability of the solution \( x = \chi \). We briefly consider arbitrary solutions and then focus on periodic solutions. We will characterize convergence properties in the neighborhood of \( x = \chi \), based on analyses of the linearized system around \( x = \chi \) given by

\[
\dot{x}(t) = A(t)x(t), \quad x := x - \chi, \quad A(t) := \frac{\partial f}{\partial x}(\chi(t)).
\]

To this end, let us first introduce a notion of stability.

**Definition 2** Consider the system (5) and let \( x = \chi \) be a (not necessarily periodic) solution. The solution is said to be stable if there exist positive scalars \( \epsilon, k, \) and \( \gamma \), independent of initial time \( t_0 \), such that

\[
||x(t) - \chi(t)|| < \epsilon
\]

\[
||x(t) - \chi(t)|| \leq ||x(t_0) - \chi(t_0)||ke^{-\gamma t},
\]

i.e., all trajectories \( x(t) \) in the neighborhood of \( \chi(t) \) locally exponentially converge to \( \chi(t) \). We express the convergence property as \( x(t) \to \chi(t) \).
For autonomous nonlinear systems, it is well known as the Hartman-Grobman theorem that trajectories in the neighborhood of a hyperbolic equilibrium point are qualitatively similar (or topologically equivalent) to trajectories in the neighborhood of the origin of the linearized system around the equilibrium. Consequently, stability of an equilibrium point of an autonomous nonlinear system is equivalent to stability of the linearized (time-invariant) system. A similar statement can be made for time-varying nonlinear systems ([12], p.152), which leads to the following result by a simple coordinate change.

**Lemma 5** Consider the nonlinear system (5) and let \( x = \chi \) be a solution. The solution \( \chi \) is stable if and only if the linearized system (6) is stable.

Let us now consider the case where the solution \( \chi \) is periodic, where the linearized system (6) is periodic, and hence its stability can be checked by the Floquet multipliers as stated in Corollary 1. It turns out, however, that there is no point in checking the linear stability since no (nontrivial) periodic solution can be checked by the Floquet multipliers as stated in Corollary 1. The initial states \( \chi(t_0) \) and \( \chi(t_0 + \tau) \) can be arbitrarily close to each other when \( \tau > 0 \) is small, but \( \chi(t) \) will never converge to \( \chi(t) \) since the minimum of \( \| \chi(t + \tau) - \chi(t) \| \) over \( t \) is bounded away from zero. Consistently with this observation, the linearized system (6) is not stable. Note that \( x = \dot{\chi} \) is a solution of (6) since

\[
\dot{x} = f(x) \quad \Rightarrow \quad \dot{\chi} = \frac{\partial f}{\partial x}(\chi, \tau)\dot{\chi},
\]

where we took the time derivative on both sides. Thus the linearized system has a nontrivial periodic solution \( x = \chi \) not converging to the origin, and hence cannot be stable.

While a periodic solution cannot be stable, there are cases where the orbit of a periodic solution attracts nearby trajectories. Let us make it precise.

**Definition 3** Consider the system (5) and let \( x = \chi \) be a \( T \)-periodic solution. The orbit of the solution is defined as the closed curve in the state space traced by the solution:

\[
\mathcal{O} := \{ \chi(t) : t \in \mathbb{R} \}.
\]

Define the distance between a point \( x \in \mathbb{R}^n \) and the orbit:

\[
\text{dist}(x, \mathcal{O}) := \min_{\chi \in \mathcal{O}} \| x - \chi \|.
\]

The solution is said to be orbitally stable if there exist positive scalars \( \epsilon, k \), and \( g \) such that

\[
\text{dist}(x(0), \mathcal{O}) < \epsilon \quad \Rightarrow \quad \text{dist}(x(t), \mathcal{O}) \leq \text{dist}(x(0), \mathcal{O}) e^{-\gamma t},
\]

i.e., all trajectories \( x(t) \) in the neighborhood of \( \mathcal{O} \) locally exponentially converge to \( \mathcal{O} \). We express the convergence property as \( x(t) \rightarrow \mathcal{O} \).

The notion of orbital stability is weaker than that of stability. In fact, \( \chi \) is orbitally stable when \( x(t) \rightarrow \chi(t + \tau) \) holds for some \( \tau \) (depending on \( x(0) \)) in the neighborhood of \( \mathcal{O} \). Thus, orbital stability does not require convergence to \( \chi(t) \), but allows for an arbitrary time shift \( \chi(t + \tau) \). With this relaxation, it is possible for a time-invariant system to have an orbitally stable periodic solution. The following result gives a condition for orbital stability.

**Theorem 2** Consider the nonlinear system in (5) and let \( x = \chi \) be a \( T \)-periodic solution. The linearized system (6) has a Floquet multiplier at 1. The solution \( \chi \) is orbitally stable if and only if the other \( n-1 \) Floquet multipliers of (6) lie strictly inside the unit circle on the complex plane.

**Proof.** One of the Floquet multipliers is at 1 due to Lemma 2 since (6) has a periodic solution \( x = \dot{\chi} \). See Theorem 1.1 of [31] for the stability statement.

This result is nontrivial because the linearization (6) is supposed to be valid only in the neighborhood of the solution \( \chi \), yet Theorem 2 shows that the linearization is useful for determining a convergence property of a trajectory \( x \) that may have an initial value \( x(0) \) far away from \( \chi(0) \). The notion of orbital stability is defined for a specific periodic solution, but in fact it is a property of its orbit. There are infinitely many periodic solutions \( \chi(t) = \chi(t + \tau) \) parametrized by \( \tau \in \mathbb{R} \) that share the same orbit \( \mathcal{O} \). If one of these solutions is orbitally stable, then so are the rest. Indeed, all the linearized systems around these solutions share the same set of Floquet multipliers ([28], p.323) and thus the Floquet multipliers are defined for the orbit rather than for each solution.

Based on Theorem 2, orbital stability of a periodic solution can be analyzed by the \( n-1 \) Floquet multipliers of the linearized system other than 1. The Floquet multiplier at 1 represents the periodic mode corresponding to the solution \( x = \dot{\chi} \) to (6), inherent to linearization around a periodic solution \( x = \chi \). Direct calculations of the Floquet multipliers are sufficient for stability analysis. However, for the purpose of designing nonlinear oscillators with orbitally stable periodic solution, it is convenient to isolate the periodic mode from the dynamics of the other \( n-1 \) Floquet multipliers. Such isolation may become possible through a state coordinate transformation before linearizing the system. This process is justified by the fact that, under a mild condition, the linear system obtained by a Lyapunov transformation followed by linearization of a nonlinear system shares the same Floquet multipliers as those for the linear system obtained by linearization followed by the corresponding coordinate transformation.

### 4. Convergence to Invariant Subspace

This section reviews the contraction theory [20],[22] that is useful later for analyzing synchronization property of coupled oscillators. Let us first define the contraction property for nonlinear systems.

**Definition 4** A matrix-valued function \( F(x,t) \in \mathbb{R}^{n \times n} \) of \((x,t) \in \mathbb{R}^n \times \mathbb{R} \) is said to be uniformly negative definite if there exists \( \varepsilon > 0 \), independent of \((x,t)\), such that

\[
F(x,t) + F(x,t)^T \leq -\varepsilon I, \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}.
\]

A nonlinear time-varying system \( \dot{x} = f(x,t) \) with \( x(t) \in \mathbb{R}^n \) is said to be contracting if its Jacobian matrix \( \frac{\partial f}{\partial x}(x,t) \) is uniformly negative definite.
The following result states a global convergence property of contracting systems.

**Lemma 6 (201)** Consider a nonlinear time-varying system

\[ \dot{x} = f(x, t), \quad x(t) \in \mathbb{R}^n. \]  

(7)

Suppose the system is contracting. Then every solution \( x(t) \) is stable, and all the solutions globally converge to a single trajectory.

**Proof.** Let \( x = \chi \) be an arbitrary solution. Then the linearization around the solution is given by

\[ \dot{x}(t) = A(t)x(t), \quad x := x - \chi, \quad A(t) := \frac{\partial f}{\partial x}(x(t), t). \]

The linear system is stable since \( V(x) := ||x||^2 \) is a Lyapunov function satisfying

\[ V(x) := \dot{x}^T(A(t) + A(t)^T)x \leq -\varepsilon ||x||^2 \]

for a constant \( \varepsilon > 0 \). Hence, by a Hartman-Grobman theorem for time-varying systems ([12], p.152), the solution \( \chi \) is (locally exponentially) stable. By the same argument, every solution is stable and attracts nearby trajectories. Thus, all the solutions converge to a single trajectory.

In the literature, global convergence to a single trajectory is guaranteed by the contracting property as in Lemma 6. In fact, the global convergence is equivalent to asymptotic stability of every solution as shown below, but the stronger contracting property is easier to analyze and thus has been found useful.

**Lemma 7** Consider the system (7). Suppose every solution is locally asymptotically stable. Then all the trajectories globally converge to a single trajectory.

**Proof.** Let \( x_0 \) and \( x_1 \) be any solutions of (7) defined for \( t \geq 0 \). Fix \( t_0 \geq 0 \) and let

\[ \eta_t := \lambda x_1(t_0) + (1 - \lambda)x_0(t_0) \]

for \( \lambda \in \mathbb{R} \). Denote by \( x_1 \) the trajectory starting from \( \eta_t \) at \( t = t_0 \).

Define

\[ \Lambda := \{ \lambda > 0 : x_1 \text{ converges to } x_0 \text{ for all } \lambda \in (0, \lambda] \}. \]

By local asymptotic stability of \( x_0 \), the set \( \Lambda \) is nonempty. Suppose \( 1 \notin \Lambda \). Then the set \( \Lambda \) is a convex interval contained in \((0, 1)\). Let \( \Lambda_0 \) be the supremum of \( \lambda \in \Lambda \). There are two cases:

Case \( \Lambda_0 \in \Lambda \): The trajectory \( x_{\Lambda_0} \) converges to \( x_0 \) but the trajectory \( x_{\lambda} \) does not converge to \( x_0 \) for \( \lambda := \lambda_0 + \varepsilon \) with sufficiently small \( \varepsilon > 0 \). However, due to local asymptotic stability of \( x_{\Lambda_0} \), the trajectory \( x_{\lambda} \) must converge to \( x_{\Lambda_0} \) and therefore \( x_{\lambda} \) must converge to \( x_0 \). By contradiction, this is not the case.

Case \( \Lambda_0 \notin \Lambda \): The trajectory \( x_{\Lambda_0} \) does not converge to \( x_0 \) but the trajectory \( x_{\lambda} \) converges to \( x_0 \) for \( \lambda := \lambda_0 - \varepsilon \) with sufficiently small \( \varepsilon > 0 \). However, due to local asymptotic stability of \( x_{\lambda} \), the trajectory \( x_{\lambda} \) must converge to \( x_{\Lambda_0} \) and therefore \( x_{\lambda} \) must converge to \( x_0 \). By contradiction, this is not the case.

Thus, neither case is a possibility, and we conclude that \( 1 \in \Lambda \) and hence \( x_0 \) and \( x_1 \) converge to each other.

An important implication of Lemma 6 is that, when \( x(t) \equiv 0 \) is a solution of (7), the contracting property implies global convergence to the origin since the “single trajectory” in the statement can be chosen as the null solution. The contracting property is a very strong requirement that is hard to be satisfied. One of the strong factors is that the distance to the trajectory, \( ||x(t)|| \), is monotonically decreasing. A relaxation is available [20] to add a state scaling and consider a Lyapunov function of the form \( V(x, t) = ||\Theta(t)x||^2 \). The strong condition turns out to be useful for some cases, especially when the contraction property is required only in a subspace of the state space, as discussed below.

Let us first consider the simple linear time-invariant case, where the system is given by

\[ \dot{x}(t) = Ax(t), \quad AU = UA, \quad x(t) \in \mathbb{R}^n, \]

where the eigenvalues of \( A \in \mathbb{R}^{n \times n} \) coincide with some of those of \( \Lambda \), and the columns of \( U \in \mathbb{R}^{n \times r} \) form an orthonormal basis for the eigenspace spanned by the associated eigenvectors. We assume \( r < n \). The linear subspace \( U \subset \mathbb{R}^n \) spanned by the columns of \( U \) is an invariant subspace of the state space in the sense that every solution \( x(t) \) starting with initial value \( x(0) \in U \) remains in \( U \). This is easy to see by introducing a coordinate transformation

\[ \begin{bmatrix} y \\ z \end{bmatrix} := \begin{bmatrix} V^T \\ U^T \end{bmatrix} x \quad \text{or} \quad x = Vy + Uz. \]  

(8)

where \( V \in \mathbb{R}^{n \times (n-r)} \) is a matrix such that \([ V \ U \]) is an orthogonal matrix. The system can then be described as

\[ \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \Omega & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, \quad \Omega := V^TAV. \]

Clearly, if \( \Omega \) is Hurwitz, then \( y(t) \) converges to zero and \( z(t) \) satisfies \( z = Az \) in the steady state. Thus, stability of the \( y \) dynamics implies convergence to the invariant subspace. In particular, if \( \Lambda \) is a \( 2 \times 2 \) matrix with eigenvalues at \( \pm \mu \omega \), then \( z(t) \) will oscillate in the steady state at frequency \( \omega \). This type of analysis has a natural extension to the nonlinear case using the concept of contraction.

Let us consider the situation where we would like to determine whether every trajectory of (7) converges to a subspace \( U \) of the state space \( \mathbb{R}^n \). First of all, it makes sense to examine such convergence property when \( U \) is flow-invariant, i.e., every trajectory starting in \( U \) stays in \( U \) for all time. More formally, a linear subspace \( U \subset \mathbb{R}^n \) is said to be a flow-invariant subspace of \( f \) if \( f(\mathcal{U}, t) \subset U \) for all \( t \). Let \( U \subset \mathbb{R}^n \) be a flow-invariant subspace with an orthonormal basis given by the columns of \( U \in \mathbb{R}^{n \times r} \) where \( r \) is the dimension of \( U \). Consider the coordinate transformation in (8). Note that \( x \in U \) if and only if \( y = 0 \). The dynamics for \( y \) is given by

\[ \dot{y} = V^Tf(Vy + Uz, t). \]

Clearly, \( y(t) \equiv 0 \) is a solution because \( f(Uz, t) \in U \) holds for any \( z \) and \( t \) due to the flow invariance of \( U \), which implies that \( V^Tf(Uz, t) = 0 \) since \( V \) is orthogonal to \( U \). Hence, every solution \( y(t) \) converges to the origin, and therefore every solution \( x(t) \) of the original system converges to \( U \). This idea leads to the following result.

**Theorem 3 (22)** Consider the system (7) and let \( x = \chi \) be an arbitrary solution. Let \( \mathcal{U} \subset \mathbb{R}^n \) be a flow-invariant subspace with a basis given by the columns of \( U \in \mathbb{R}^{n \times r} \) where \( r \) is the dimension of \( \mathcal{U} \). Let \( V \in \mathbb{R}^{n \times (n-r)} \) be such that \([ V \ U \]) is an orthogonal matrix. Suppose
\[ \dot{y} = h(y, r), \quad h(y, r) := V^T f(Vy + U\zeta(t), t), \]
is contracting where \( \zeta := U^T \chi \). Then the solution \( \chi(t) \) approaches the flow-invariant subspace \( \mathbb{U} \). Moreover, if
\[ A(x, t) := V^T \left( \frac{\partial f}{\partial x}(x, t) \right) V \]
is uniformly negative definite, then every trajectory \( x \) of (7) globally exponentially converges to \( \mathbb{U} \).

**Proof.** Due to the flow invariance, \( y(t) = 0 \) is a solution to \( \dot{y} = h(y, r) \) as discussed earlier. On the other hand, \( y = V^T \chi \) is also a solution to \( \dot{y} = h(y, r) \) by definition. Since \( h \) is contracting, \( V^T \chi \) must converge to 0, which implies that \( \chi = VV^T \chi + UU^T \chi \) converges to \( \mathbb{U} \). When \( A(x, t) \) is uniformly negative definite, \( h \) is contracting regardless of \( \zeta(t) \), and hence every solution \( x(t) \) converges to \( \mathbb{U} \).

To visualize the utility of Theorem 3, consider a system with three dimensional \((n = 3)\) state space containing a periodic orbit that lies on a two dimensional \((r = 2)\) plane \( \mathbb{U} \). If the contraction property guarantees convergence to \( \mathbb{U} \), then the steady state behavior is governed by the planar dynamics of the system on \( \mathbb{U} \). In particular, if the planar system is an oscillator in which almost all trajectories converge to the periodic orbit, generic convergence to the orbit would be guaranteed for the original system. This idea extends to the case with arbitrary dimensions \( n \) and \( r \), although the analysis of the steady state behavior in \( \mathbb{R}^r \) would be more involved when \( r \geq 3 \). The result will not work if the system has no flow-invariant subspace \( (e.g. \) when the steady state trajectories evolve on a manifold not contained in any linear subspace \( r < n \)), in which case, the method of partial contraction [21] may be useful.

The following result is a simple application of Theorem 3 and presents a condition for stability of a flow-invariant subspace under linear perturbation. The result will turn out to be useful later for analysis of coupled oscillators.

**Corollary 2** Consider the system
\[ \dot{x} = f(x) - Gx, \quad x(t) \in \mathbb{R}^n, \tag{9} \]
where \( G \) is a constant matrix. Let \( \mathbb{U} \subset \mathbb{R}^n \) be a flow-invariant subspace of \( f \) and consider the coordinate transformation in (8). Suppose \( GU = 0 \) and
\[ \text{He}(V^T G V) > \kappa I, \quad \kappa := \sup_{x \in \mathbb{U}^\circ} \lambda_{\text{max}} \left[ \text{He} V^T \left( \frac{\partial f}{\partial x}(x) \right) V \right], \]
where \( \kappa \) is a finite number, \( \lambda_{\text{max}} \) denotes the largest eigenvalue, and \( \text{He}(M) := M + M^T \). Then every trajectory \( x \) globally exponentially converges to \( \mathbb{U} \).

## 5. Planar Oscillators

This section presents simple oscillators with two scalar variables, which illustrate utility of the basic analysis results in the previous sections, and provide a foundation for coupled oscillator analysis in the next section.

### 5.1 Andronov-Hopf Oscillator

Consider the nonlinear system
\[ \dot{x} = \begin{bmatrix} \sigma(x) & \omega \\ -\omega & \sigma(x) \end{bmatrix} x, \quad \sigma(x) := \mu(a^2 - |x|^2), \tag{10} \]
where \( a, \omega, \mu \in \mathbb{R} \) are scalar constants. This is called the Andronov-Hopf oscillator. When \( \mu = 0 \), the system is a linear harmonic oscillator and it is easy to see that
\[ \chi(t) = \begin{bmatrix} \alpha \sin \omega t \\ \alpha \cos \omega t \end{bmatrix}, \tag{11} \]
is a periodic solution. Noting that \( \sigma(\chi) = 0 \), this remains to be a solution for arbitrary \( \mu \). It turns out that the periodic solution is orbitally stable when \( \alpha \) and \( \omega \) are nonzero and \( \mu \) is positive. We will illustrate the general analysis methods presented earlier using this system as an example.

A condition for orbital stability is given by Theorem 2. Linearizing the system (10) around \( x = \chi \) as in (6),
\[ \dot{x} = Ax, \quad x := x - \chi, \quad A := \begin{bmatrix} -\gamma s^2 & \omega - \gamma xc \\ -\omega - \gamma xc & -\gamma c^2 \end{bmatrix}, \tag{12} \]
\[ s := \sin \omega t, \quad c := \cos \omega t, \quad \gamma := 2\mu a^2. \]
To calculate the Floquet multipliers, we need to solve (3) for the principal fundamental matrix \( \Phi(t) \). This can be done by numerical integration, but it is not obvious how to find an analytical expression for \( \Phi(t) \). We will find \( \Phi(t) \) analytically through a coordinate transformation.

The orbit of the periodic solution \( \chi \) is the circle of radius \( \alpha \) with center at the origin in the two dimensional state space. Based on this observation, consider the transformation to the polar coordinates
\[ x = h(z), \quad z := \begin{bmatrix} \theta \\ r \end{bmatrix}, \quad h(z) := \begin{bmatrix} r \sin \theta \\ r \cos \theta \end{bmatrix}, \tag{13} \]
where
\[ z = \zeta := \begin{bmatrix} \omega \theta \\ \alpha \end{bmatrix}, \]
is the solution corresponding to the periodic solution \( x = \chi \). Taking the time derivative of \( x \) in (13) and substituting into (10), we have
\[ \dot{\theta} = \omega, \quad \dot{r} = \mu(\alpha - r)(\alpha + r). \tag{14} \]
At this point, we see that the solution for the first equation is \( \theta = \omega t + \theta_0 \) where \( \theta_0 \) is the initial value \( \theta(0) \), and the second equation has three equilibrium points at \( r = 0 \) and \( \pm \alpha \), where the former is unstable and the latter are stable due to the sign of \( \dot{r} \). Hence, given an arbitrary nonzero initial state \( x(0) \neq 0 \), the trajectory \( x(t) \) converges to \( \chi(t + \tau) \) with \( \tau := \theta_0/\omega \) (when \( r \rightarrow \alpha \)) or \( \tau := \theta_0/\omega + \pi \) (when \( r \rightarrow -\alpha \)) which is on the circular orbit defined by \( \chi(t) \). Thus \( x = \chi \) is orbitally stable, and in fact, the convergence is guaranteed from all nonzero initial states.

For illustrative purposes, let us continue our Floquet analysis. Linearizing the system (14) around \( z = \zeta \),
\[ \dot{z} = Bz, \quad z := \begin{bmatrix} \theta - \omega t \\ r - \alpha \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}. \tag{15} \]
A fundamental matrix \( \Psi(t) \) is given by
\[ \Psi = B^t \Psi(t) \Rightarrow \Psi(t) = e^{Bt} \Psi(0) \approx \begin{bmatrix} 1 & 0 \\ 0 & e^{-\gamma t} \end{bmatrix} \Psi(0). \]
With \( \Psi(0) = I \) and \( T := 2\pi/\omega \), we see that the Floquet multipliers (i.e. eigenvalues of \( \Psi(T) \)) are 1 and \( e^{-\gamma T} \). Since systems (12) and (15) are related by the Lyapunov transformation.
they share the same Floquet multipliers. When \(\alpha, \omega, \mu\) are positive, we have \(|e^{-\gamma t}| < 1\) and thus conclude that \(x = \chi\) is orbitally stable. Finally, the principal fundamental matrix \(\Phi(t)\) for (12), defined in (3), is given by

\[
\Phi = L(\zeta)\Psi = \begin{bmatrix} \alpha & s \\ -\alpha s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\gamma t} \end{bmatrix} \Psi(0) = \begin{bmatrix} 1/\alpha & 0 \\ 0 & 1 \end{bmatrix},
\]

where \(\Psi(0)\) is chosen so that \(\Phi(0) = I\). Noting that \(\Phi(T) = \text{diag}(1, e^{-\gamma T})\), we confirm that (12) and (15) share the same Floquet multipliers.

5.2 Mechanical Oscillator

Consider the mechanical system

\[
M\ddot{z} + Kz = w, \quad z(t), w(t) \in \mathbb{R}^n,
\]

where \(M\) and \(K\) are symmetric positive definite matrices representing the mass and stiffness, \(z(t)\) are the generalized coordinates, and \(w(t)\) are the generalized forces. A natural mode of oscillation is given by

\[
z(t) = ae \sin \omega t, \quad (\omega^2 M - K)e = 0, \quad e^2 M e = 1,
\]

where \(\omega \in \mathbb{R}\) is the natural frequency, \(e \in \mathbb{R}^n\) is the mode shape, and \(\alpha \in \mathbb{R}\) is the oscillation amplitude. We design a feedback controller so that the natural oscillation is an orbitally stable periodic solution of the closed-loop system.

Let us introduce the modal coordinates \(v\) and transformed force inputs \(p\):

\[
z = Ep, \quad w = MEv, \quad E := \begin{bmatrix} e & * \end{bmatrix}, \quad E^T ME = I,
\]

where * is chosen to satisfy \(E^T ME = I\). Then the equation of motion becomes

\[
p + \Lambda p = v, \quad \Lambda := E^T KE = \text{diag}(\omega^2, *).
\]

Using the control input

\[
v = \begin{bmatrix} u \\ -D\dot{\eta} \end{bmatrix}, \quad p = \begin{bmatrix} q \\ \eta \end{bmatrix}, \quad (q, \eta) \in \mathbb{R}^n, \quad (\eta, \dot{\eta}) \in \mathbb{R}^{n-1},
\]

with positive definite \(D\), the other oscillation modes \(\eta\) are stabilized, and the remaining dynamics are described by

\[
\ddot{\eta} + \omega^2 q = u,
\]

where \(u(t) \in \mathbb{R}\) is the auxiliary input. The system is a linear oscillator with natural frequency \(\omega\). The oscillation can have an arbitrarily large amplitude depending on the initial condition. The task is to design a feedback controller \(u = h(q, \dot{q})\) such that the closed-loop system has a sinusoidal oscillation \(q(t) = a \sin \omega t\) as an orbitally stable periodic solution, where \(\alpha > 0\) is a prescribed amplitude.

Motivated by the Andronov-Hopf oscillator, we consider the nonlinear damping compensation

\[
u = h(q, \dot{q}) = \mu(\alpha^2 - \dot{q}^2 - \dot{q}^2/\omega^2)\dot{q},
\]

which vanishes on the target solution, and provides positive/negative damping when the amplitude of \(q\) is larger/smaller than \(\alpha\). The closed-loop system is

\[
\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & \sigma(x) \end{bmatrix} x, \quad x := \begin{bmatrix} q \\ \dot{q}/\omega \end{bmatrix},
\]

\[
\sigma(x) := \mu(\alpha^2 - ||x||^2).
\]

We see that the system is identical to the Andronov-Hopf oscillator except for the lack of the \(\sigma(x)\) term in the \((1,1)\) entry of the coefficient matrix. Also, it should be noted that the system coincides with the van der Pol oscillator if function \(\sigma(x)\) is modified to \(\sigma(x) = \mu(1 - x_1^2)\). This function does not vanish when \(x_1\) is a sinusoid, and the van der Pol oscillator has a non-sinusoidal periodic solution. It can readily be verified that the system (16) has a sinusoidal periodic solution \(x = \chi\) in (11). We will illustrate how orbital stability of this solution can be established, which is more difficult than the case of the Andronov-Hopf oscillator due to the lack of “symmetry,” and is simpler than the case of the van der Pol oscillator due to the availability of the closed-form expression for the periodic solution.

As before, we consider the polar coordinates \(z\) as in (13) and its target solution \(\zeta\). The transformation gives

\[
\dot{\theta} = \omega - \mu(\alpha^2 - r^2) \sin \theta, \quad s_\theta := \sin \theta,
\]

\[
\dot{r} = \mu(\alpha^2 - r^2) r c_\theta^2 \quad c_\theta := \cos \theta.
\]

Linearizing the system around \(z = \zeta\), we obtain

\[
\dot{z} = Bz, \quad z := \begin{bmatrix} \theta - \omega t \\ r - \alpha \end{bmatrix},
\]

\[
B := \mu a \begin{bmatrix} 0 & \sin(2\omega t) \\ 0 & -\alpha(1 + \cos(2\omega t)) \end{bmatrix}.
\]

The second row of equation \(\dot{z} = Bz\) can be solved analytically as

\[
\rho(t) := r(t) - \alpha = e^{-\mu \omega T} e^{\frac{\sin(2\omega t)}{2\omega}} + \frac{\sin(2\omega t)}{2\omega},
\]

which converges exponentially to zero. Hence, the solution \(\chi\) is orbitally stable whenever \(\mu, \alpha, \omega\) are positive. Indeed, the principal fundamental matrix of the system \(\dot{z} = Bz\) is

\[
\Phi(t) = \begin{bmatrix} 1 & e^{-\mu \omega T} e^{\frac{\sin(2\omega t)}{2\omega}} \\ 0 & e^{-\mu \omega T} e^{\frac{\sin(2\omega t)}{2\omega}} \end{bmatrix},
\]

and the Floquet multipliers are 1 and \(e^{-\mu \omega T}\), the latter of which has a magnitude less than 1. Note that the exponent is half of that for the Andronov-Hopf oscillator, indicating that the rate of convergence is reduced by half.

Finally, if the analytical solution for \(\rho(t)\) were not available, then we may use Lemma 4 to show orbital stability when \(\mu > 0\) is small. In particular, the second row of \(\dot{z} = Bz\) gives

\[
\dot{\rho} = B(t)\rho, \quad B(t) := -\mu \omega^2 (1 + \cos(2\omega t)),
\]

and the average dynamics are described as

\[
\overline{B} := \int_0^T B(t) dt = -\mu \omega^2 T.
\]

Since \(\overline{B}\) is Hurwitz, we conclude that the solution \(\chi\) is orbitally stable for sufficiently small \(\mu > 0\).

5.3 Pendulum with Energy Regulation

Consider a simple pendulum in a gravity field, the dynamics of which are described by

\[
x = L(\zeta)c, \quad L(\zeta) = \frac{\partial h}{\partial \zeta}(\zeta) = \begin{bmatrix} \alpha c & s \\ -\alpha s & c \end{bmatrix},
\]
\[ \ddot{\theta} + \omega^2 \sin \theta = u, \]  

(17)

where \( \theta(t) \) is the pendulum angle, \( u(t) \) is the torque input, and \( \omega \in \mathbb{R} \) is a parameter determined by the gravity constant \( g \) and pendulum length \( l \) as \( \omega := \sqrt{g/l} \). Without input (\( u = 0 \)), the system has infinitely many periodic solutions of natural oscillations parametrized by various amplitudes. The orbits of these solutions can be visualized as ovals in the state space \((\theta, \dot{\theta})\). Each solution is not orbitally stable because the trajectory after a small perturbation will be placed on a nearby orbit and will not return to the original orbit. We consider the design of a feedback controller so that the closed-loop system has a natural oscillation of prescribed amplitude \( \alpha \) as an orbitally stable periodic solution. Let us denote the targeted oscillation by \( \theta_\alpha(t) \) which satisfies (17) with the maximum value \( \alpha \).

A strategy for the control design is energy regulation. Note that the sum of the kinetic and potential energies remains constant on each orbit. In particular, if the amplitude of angle oscillation is \( \alpha \), then

\[
e(\theta, \dot{\theta}) := \frac{1}{2} \dot{\theta}^2 + \omega^2 (1 - \cos \theta)
\]

holds for all \((\theta, \dot{\theta})\) on the orbit. This motivates us to consider the control law

\[
u = \kappa (c - e(\theta, \dot{\theta})) \dot{\theta}, \quad \kappa > 0,
\]

which provides positive/negative damping when the energy level is higher/lower than \( c \).

To analyze the orbital stability property of \((\theta_\alpha, \dot{\theta}_\alpha)\), first note that the total energy can be expressed as

\[
e = \frac{1}{2} (\dot{\theta}^2 + 2 \omega \sin \theta \frac{\theta^2}{2})\]

This expression motivates us to introduce the following transformation \((\theta, \dot{\theta}) \to (r, \varphi)\) into the polar coordinates:

\[
\dot{\theta} = r \sin \varphi, \quad 2 \omega \sin \theta \frac{\theta}{2} = r \cos \varphi.
\]

Note that \( r(t) = \pm r_0 \) with \( r_0 := \sqrt{2c} \) on the targeted periodic orbit. The dynamics can then be described by

\[
\dot{r} = \kappa (c - r^2/2) r \sin^2 \varphi,
\]

\[
\dot{\varphi} = \kappa (c - r^2/2) \sin \varphi \cos \varphi - \omega \sqrt{1 - \left( \frac{\sin \varphi}{\sin \theta} \right)^2}.
\]

The trajectory \(\theta(t)\) would converge to the target \(\theta_\alpha\) if \( r(t) \) converges to \( r_0 \). The equation for \( r \) can be written in terms of \( \rho := r - r_0 \), and if this dynamics are contracting with respect to \( \rho \) for arbitrary \( \varphi \), then global convergence can be established using Theorem 3. However, the dynamics are not contracting since \( \rho = -r_0 \) (or \( r = 0 \)) is an equilibrium point. On the other hand, local convergence can be established by Lemma 4 and Theorem 2. Linearizing the \( r \) dynamics around \( r_0 \), we have

\[
\dot{\rho} = -\kappa (2 \omega \sin \theta (1/2))^2 \rho.
\]

This linear periodic system is stable for sufficiently small \( \kappa > 0 \) if the averaged system is stable (Lemma 4). Since the average of \( \sin^2(\theta_\alpha/2) \) over a cycle is positive, we conclude that the solution \((\theta_\alpha, \dot{\theta}_\alpha)\) is orbitally stable when \( \kappa > 0 \) is sufficiently small.

As an example, the closed-loop is simulated with

\[
\omega = 2\pi, \quad \kappa = 0.1, \quad \alpha = 170^\circ, \quad \theta(0) = 17^\circ, \quad \dot{\theta}(0) = 0,
\]

and the result is shown in Figs. 1 and 2. The trajectory converged to a limit cycle, which is verified to be the natural oscillation by noting that the control input \( u(t) \) goes to zero in the steady state.

### 6. Coupled Oscillators

This section analyzes synchronization and coordination of coupled nonlinear oscillators using the contraction analysis and the Floquet theory with linearization. The former provides a condition for global convergence, while the latter ensures local convergence.

#### 6.1 Global Convergence via Contraction

Consider synchronization of a set of \( n \) identical subsystems interacting with each other via diffusive coupling:

\[
\dot{x}_i = f_0(x_i) + \sum_{j=1}^n \delta_{ij} (x_i - x_j), \quad i \in I_n := \{1, \ldots, n\},
\]

(18)

where function \( f_0 \) describes the dynamics of each subsystem, and \( \delta_{ij} \in \mathbb{R} \) are coupling parameters. In the summation, the parameters \( \delta_{ij} \) are multiplied by zero \( (x_i = x_j) \) and hence their values do not affect the system behavior. For notational convenience, we choose \( \delta_{ij} \) so that the sum of \( \delta_{ij} \) over \( j = 1, \ldots, n \) is equal to zero. The subsystems are said to be synchronized if \( \|x_i(t) - x_j(t)\| \) converges to zero for all pairs \((i, j)\). We seek a condition under which global convergence to the synchronized state is guaranteed.

![Fig. 1 Natural entrainment with energy regulation.](image1)

![Fig. 2 Trajectory on the phase plane.](image2)
The coupled subsystems can be described as (9) with

\[
\begin{align*}
x(t) & := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad f(x) := \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{bmatrix}, \quad G := \Delta \otimes I,
\end{align*}
\]

where \( \Delta \in \mathbb{R}^{n \times n} \) is the matrix whose \((i, j)\) entry is \( \delta_{ij} \), the notation \( \otimes \) means the Kronecker product, and the dimension of the identity matrix \( I \) in the definition of \( G \) is equal to the state dimension of \( x_i \). Here we noted that \( \Delta I = 0 \) due to the choice made for \( \delta_{ij} \), where \( I \in \mathbb{R}^n \) is the vector with all entries equal to 1/\( \sqrt{n} \). The synchronized state, \( x_i = x_j \) for all \((i, j)\), can be seen as a flow-invariant subspace \( U \) spanned by the basis \( U := \otimes I \) because \( G U = (\Delta I) \otimes I = 0 \) due to the zero row-sum condition on \( \Delta \) and \( f(Uz) = U f_i(z) \) due to the homogeneity of the subsystems. Hence, a sufficient condition for synchronization is obtained as a special case of Corollary 2.

**Theorem 4** (\([22]\)) Consider the set of coupled subsystems in (18). Suppose

\[
N^T(\Delta + \Delta^T)N > \kappa I, \quad \Delta I = 0, \quad \kappa := \sup_z \lambda_{\max} \left( \text{He} \left( \frac{\partial f_i}{\partial z} \right)(z) \right),
\]

where \( \Delta \) is the matrix whose \((i, j)\) entry is equal to \( \delta_{ij} \), the supremum is assumed to exist, and \( N \) is a matrix such that \([N \ I]\) is an orthogonal matrix. Then all the trajectories of (18) globally converge to the synchronized state where \( x_i = x_j \) for all \((i, j)\).

**Proof.** The result can be verified using Corollary 2 by noting that

\[
V = N \otimes I, \quad V^T V = I, \quad V^T G V = (N^T \Delta N) \otimes I,
\]

and, for a symmetric matrix \( M, M \otimes I \) is positive definite if and only if \( M \) is positive definite. The parameter \( \kappa \) here is actually an upper bound on the \( \kappa \) in Corollary 2, where we note that the largest eigenvalue of symmetric matrix \( M \) is larger than or equal to the largest eigenvalue of \( V^T M V \) when \( V^T V = I \). \( \square \)

The result indicates that global synchronization of identical subsystems is always possible by strong coupling (i.e., large values of \([\delta_{ij}]\)) if the subsystem dynamics \( f_i \) are bounded in the sense that \( \kappa \) is well defined. An example is coupled Andronov-Hopf oscillators for which

\[
f_i(z) := \begin{bmatrix} \sigma(z) & \omega \\ -\omega & \sigma(z) \end{bmatrix} z, \quad \sigma(z) := \mu(\alpha^2 - \|z\|^2), \quad z(t) \in \mathbb{R}^2.
\]

A straightforward calculation shows that

\[
\text{He} \left( \frac{\partial f_i}{\partial z} \right)(z) = 2\mu(\alpha^2 - \|z\|^2)I - 4\mu z z^T \leq 2\mu \alpha^2 I
\]

where the inequality holds for all \( z \in \mathbb{R}^2 \). Hence \( \kappa \) is well defined as \( \kappa = 2\mu \alpha^2 \), and sufficiently strong coupling will synchronize the oscillators. In particular, global convergence to the synchronized state is guaranteed when the symmetric part of \( N^T \Delta N - \mu \alpha^2 I \) is positive definite. The diffusive coupling (the summation in (18)) vanishes when synchronized, and due to the orbital stability property of the individual Andronov-Hopf oscillator, \( x_i(t) \) for all \( i \) generically converges to a single periodic solution \( \chi \) for some \( \tau \in \mathbb{R} \) where \( \chi \) is defined in (11).

The synchronization result for the Andronov-Hopf oscillators can readily be generalized to deal with the coordination problem for which the objective is to orbitally stabilize the periodic solution \( x = \chi \) with

\[
x_i(t) := \begin{bmatrix} a \sin(\alpha t + \phi_i) \\ a \cos(\alpha t + \phi_i) \end{bmatrix},
\]

where \( \phi_i \in \mathbb{R} \) are prescribed phases. The synchronization is the special case where \( \phi_i = 0 \) for all \( i \in \mathcal{I}_n \). Noting that

\[
\Omega_i \chi_i = \begin{bmatrix} \alpha \sin \phi_i \\ \alpha \cos \phi_i \end{bmatrix}, \quad \Omega_i := \begin{bmatrix} \cos \phi_i & \sin \phi_i \\ -\sin \phi_i & \cos \phi_i \end{bmatrix}
\]

holds for all \( i \in \mathcal{I}_n \), we see that \( \Omega_i \chi_i - \Omega_j \chi_j \) vanishes for all \( i, j \in \mathcal{I}_n \) when the coordination objective is achieved. This motivates us to modify the coupling terms in (18) and consider the following:

\[
\dot{x}_i = f_i(x_i) + \sum_{j=1}^n \delta_{ij}(x_i - \Omega_j x_j),
\]

for \( i \in \mathcal{I}_n \), where \( f_i \) is defined by (19) and \( \Omega_i = \Omega_i \Omega_i^T \).

In this case, the coordinated state, \( \Omega_i \chi_i = \Omega_j \chi_j \) for all \( i, j \), is a flow-invariant subspace \( U \) spanned by the columns of \( \{\Omega_1, \ldots, \Omega_n\} \). The flow-invariance property relies on the special structure of the Andronov-Hopf oscillator for which \( f_i(\Omega_i z) = f_i(z) \) holds for arbitrary \( z \in \mathbb{R}^2 \). In fact, multiplying (21) by \( \Omega_i^T \) from left and redefining \( \Omega_i \chi_i \) as \( x_i \), the equation becomes identical to (18) with the particular \( f_i \) for the Andronov-Hopf oscillator. Hence, convergence to the flow-invariant subspace (or the coordination objective) is achieved when \( \delta_{ij} \) are chosen to satisfy the condition in Theorem 4.

### 6.2 Local Convergence via Linear Stability

The contraction to the flow-invariant subspace, presented in the previous section, is a powerful tool for ensuring global convergence. However, its applicability can be limited due to various factors such as physical constraints. Also, weaker requirements of local convergence may suffice or even be desirable for certain practical purposes. In such cases, the method of perturbation and averaging may be a good alternative to the contraction method. This section will review this alternative method.

Let us first consider the set of coupled Andronov-Hopf oscillators described by (21). As we have seen earlier, the system has a periodic solution \( x_i = \chi_i \) with (20). Here, we derive a condition for (local) orbital stability via linearization. Introducing the polar coordinates

\[
\begin{align*}
x_i &= \begin{bmatrix} r_i \sin \theta_i \\ r_i \cos \theta_i \end{bmatrix},
\end{align*}
\]

the system is transformed into

\[
\dot{r}_i = \omega + \sum_{j=1}^n \delta_{ij}(r_j / r_i) \sin(\phi_i - \phi_j),
\]

\[
\dot{\theta}_i = \mu(\alpha^2 - r_i^2) \omega + \sum_{j=1}^n \delta_{ij} r_j \cos(\phi_i - \phi_j),
\]
invariant since the first term in (24) lacks symmetry due to the absence of the $\sigma(x_i)$ term in the (1, 1) entry. In the special case where $\varphi_i = 0$ for all $i \in I$, we have $Q_i = I$ and the synchronized state $U$ becomes flow invariant. Even for this case, however, the first term in (24) is unbounded and $k$ in Theorem 4 is undefined. Hence, it appears difficult to establish global convergence through the contracting property.

Local convergence can be proven through linearization. Introducing the polar coordinates $(\theta_i, r_i)$ for each $x_i$ in (13), the system (24) is described as

\[
\begin{bmatrix}
    r_i \dot{\theta}_i \\
    \dot{r}_i
\end{bmatrix} = \begin{bmatrix}
    \omega - \mu \frac{r_i^2}{r_i^2} + \frac{\rho_i}{r_i} \\
    \frac{\rho_i}{r_i} \cos \theta_i \cos \theta_i - \cos \theta_i \cos \theta_i
\end{bmatrix} r_i + \epsilon \sum_{j=1}^{n} \delta_{ij} \begin{bmatrix}
    \sin \theta_i \\
    -\cos \theta_i
\end{bmatrix} \cos(\theta_j + \varphi_j) r_j,
\]

where we have chosen $\delta_i$ so that $\Delta I = 0$ as before, and $\varphi_i := \varphi_i - \varphi_j$ are the relative phases. Introducing the perturbation variables around the target oscillation as in (22), linearizing the system, and assembling the scalar equations into a vector form, we have

\[
\begin{bmatrix}
    \alpha \dot{\theta} \\
    \dot{\rho}
\end{bmatrix} = \epsilon \begin{bmatrix}
    -S_2^2 \Delta & S_\theta \varsigma_\Omega (\Phi + \Delta) \\
    S_\theta \varsigma_\Omega (\Phi + \Delta) & -C_\Omega^2 (\Phi + \Delta)
\end{bmatrix} \begin{bmatrix}
    \alpha \dot{\theta} \\
    \dot{\rho}
\end{bmatrix},
\]

where

\[
\Phi := (2\mu a^2/\epsilon I), \quad S_\theta := \text{diag}(\sin(\eta_1, \ldots, \sin(\eta_n)), \text{cos}(\cos(\eta_1, \ldots, \cos(\eta_n)).
\]

The orbital stability of the solution $\chi = \chi$ is guaranteed if this linear periodic system has all its Floquet multipliers inside the unit circle except for one at 1. A simpler stability condition can be obtained for the weak coupling case. In particular, taking the average of the coefficient matrix over a cycle period, the system equation reduces to

\[
\dot{\theta} = -\frac{\epsilon}{2} \Delta \theta, \quad \dot{\rho} = -\frac{\epsilon}{2} (\Phi + \Delta) \rho,
\]

where the dynamics for the amplitude $\rho$ and phase $\theta$ are decoupled. If $\epsilon > 0$ is sufficiently small and $\mu > 0$ is of order $\epsilon$, then the orbital stability is achieved when the averaged system has eigenvalues in the open left half plane except for one at the origin. This condition holds if and only if the eigenvalues of $\Delta$ have negative real parts except for one at the origin as discussed earlier. This property is ensured if $\Delta$ is a Laplacian matrix representing a directed graph with positive weights, having a spanning tree [24]. Thus, coordination of the coupled oscillators can be achieved with a distributed network.

All the results in this section are summarized below.

**Theorem 5** ([24]) Consider the coupled oscillators described by (21) or (24) where $\alpha, \omega, \mu, \epsilon \in \mathbb{R}$ are positive parameters and $\delta_{ij}, \varphi_i \in \mathbb{R}$ are parameters given for $i, j \in I_n$. Then each system has a periodic solution

\[
x_i = \begin{bmatrix}
    \alpha \sin(\omega t + \varphi_i) \\
    \alpha \cos(\omega t + \varphi_i)
\end{bmatrix},
\]

and the following statements are equivalent:

(i) The periodic solution is (locally) orbitally stable.
(ii) The linearized system around the solution, (23) or (25), has all the Floquet multipliers inside the unit circle except for one at 1.

(iii) The matrix $\Delta$ has one eigenvalue at the origin and the rest of the eigenvalues have positive real parts.

Moreover, these conditions hold if all the off-diagonal entries of $\Delta$ are negative or zero, and the nonzero entries form a directed graph with a spanning tree.

6.3 Examples

Consider two planar oscillators coupled to each other:

$$\dot{x}_1 = f_0(x_1) + \varepsilon \delta_{12} B (x_1 - \Omega^t x_2), \quad x_1(t) \in \mathbb{R}^2,$$

$$\dot{x}_2 = f_0(x_2) + \varepsilon \delta_{21} B (x_2 - \Omega x_1), \quad x_2(t) \in \mathbb{R}^2,$$

where $\alpha, \omega, \phi, \varepsilon, \mu, \mu_1, \mu_2, \delta_{12}, \delta_{21} \in \mathbb{R}$ are constant parameters and

$$f_0(z) := \begin{bmatrix} \mu_1 (\alpha^2 - ||z||^2) & \omega \\ -\omega & \mu_2 (\alpha^2 - ||z||^2) \end{bmatrix} z,$$

$$\Omega := \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}, \quad B := \frac{1}{\mu} \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

The system is seen as coupled Andronov-Hopf oscillators when $\mu_1 = \mu_2 = \mu$, and as coupled mechanical oscillators when $\mu_1 = 0$ and $\mu_2 = \mu$. In either case, $x_i = \chi_i$ with $i = 1, 2$ and

$$\chi_1 = \begin{bmatrix} \alpha \sin \omega t \\ \alpha \cos \omega t \end{bmatrix}, \quad \chi_2 = \begin{bmatrix} \alpha \sin (\omega t + \phi) \\ \alpha \cos (\omega t + \phi) \end{bmatrix}$$

is a periodic solution.

For the case of mechanical oscillators, the periodic solution is (locally) orbitally stable when condition (iii) in Theorem 5 is satisfied. In this case, $\Delta$ is defined as

$$\Delta = \begin{bmatrix} -\delta_{12} & \delta_{12} \\ \delta_{21} & -\delta_{21} \end{bmatrix},$$

and its eigenvalues are 0 and $-(\delta_{12} + \delta_{21})$. Hence, the orbital stability condition is that $\varepsilon > 0$ and $\mu > 0$ are sufficiently small and

$$\delta_{12} + \delta_{21} < 0.$$

Figure 3 shows the simulation result for

$$\mu = 5, \quad \alpha = 1, \quad \omega = 2\pi, \quad \phi = -\pi/2,$$

$$\delta_{12} = -12, \quad \delta_{21} = 4, \quad \varepsilon = 1,$$

in which case, the periodic solution is orbitally stable. However, the stability property is local, and the trajectory diverges if the initial state is outside of the domain of attraction for the stable limit cycle. Figure 4 shows an example of such a case.

For the case of the Andronov-Hopf oscillators, global convergence to the coordinated state $x_1 = \Omega^t x_2$ is guaranteed by the condition in Theorem 4. Noting that

$$\Delta = \varepsilon \begin{bmatrix} -\delta_{12} & \delta_{12} \\ \delta_{21} & -\delta_{21} \end{bmatrix}, \quad N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and $\kappa = 2\mu \alpha^2$, the condition is given by

$$\varepsilon (\delta_{12} + \delta_{21}) < -\mu \alpha^2.$$

Figure 5 shows the simulation result for the parameter values in (26) that satisfy the global convergence condition. With the same initial state as the one for Fig. 4, the trajectory converges to the coordinated oscillation with a very short transient. This illustrates the difficulty associated with mechanical systems where coupling of subsystems can occur only through the force actuators and $B$ is rank deficient.

7. Conclusion

We have reviewed basic methods for orbital stability analysis of limit cycle orbits for general nonlinear dynamical systems. In addition to simple planar oscillators, synchronization and coordination of coupled oscillators are considered. Local convergence can be established through the Floquet analysis of the linearized system, while global convergence can be proven using the contraction analysis. The basic methods will be useful for developing a general theory of feedback control design for achieving stable limit cycles in various contexts.

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