PRIMORDIAL NON-GAUSSIANITY AND ANALYTICAL FORMULA FOR MINKOWSKI FUNCTIONALS OF THE COSMIC MICROWAVE BACKGROUND AND LARGE-SCALE STRUCTURE

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ABSTRACT

We derive analytical formulae for the Minkowski functionals of the cosmic microwave background (CMB) and large-scale structure (LSS) from primordial non-Gaussianity. These formulae enable us to estimate a nonlinear coupling parameter, $f_{\text{NL}}$, directly from the CMB and LSS data without relying on numerical simulations of non-Gaussian primordial fluctuations. One can use these formulae to estimate statistical errors on $f_{\text{NL}}$ from Gaussian realizations, which are much faster to generate than non-Gaussian ones, fully taking into account the cosmic/sampling variance, beam smearing, survey mask, etc. We show that the CMB data from the Wilkinson Microwave Anisotropy Probe (WMAP) should be sensitive to $|f_{\text{NL}}| \gtrsim 40$ at the 68% confidence level. The Planck data should be sensitive to $|f_{\text{NL}}| \gtrsim 20$. As for the LSS data, the late-time non-Gaussianity arising from gravitational instability and galaxy biasing makes it more challenging to detect primordial non-Gaussianity at low redshifts. The late-time effects obscure the primordial signals at small spatial scales. High-redshift galaxy surveys at $z > 2$ covering $\sim 10 \text{Gpc}^3$ volume would be required for the LSS data to detect $|f_{\text{NL}}| \gtrsim 100$. Minkowski functionals are nicely complementary to the bispectrum because the Minkowski functionals are defined in real space and the bispectrum is defined in Fourier space. This property makes the Minkowski functionals a useful tool in the presence of real-world issues such as anisotropic noise, foreground, and survey masks. Our formalism can be easily extended to scale-dependent $f_{\text{NL}}$.

Subject headings: cosmic microwave background — large-scale structure of universe — methods: analytical

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1. INTRODUCTION

Recent observations of cosmological fluctuations in the CMB and LSS strongly support basic predictions of inflationary scenarios: primordial fluctuations are nearly scale-invariant (Spergel et al. 2003, 2006; Teegarden et al. 2004; Seljak et al. 2005), adiabatic (Peiris et al. 2003; Bucher et al. 2004; Bean et al. 2006), and Gaussian (Komatsu et al. 2002, 2003; Creminelli et al. 2006; Spergel et al. 2006 and references therein). In order to discriminate between more than 100 candidate inflationary models, however, one needs to look for deviations from scale invariance, and adiabaticity, and Gaussianity, for which different inflationary models make specific predictions. Inflationary models based on a slowly rolling single scalar field predict very small deviations from Gaussianity; however, the postinflationary evolution of nonlinear perturbations inevitably generates ubiquitous non-Gaussian fluctuations. On the other hand, a broad class of inflationary models based on different assumptions about the nature of scalar field(s) can generate significant primordial non-Gaussianity (Lyth et al. 2003; Dvali et al. 2004; Arkani-Hamed et al. 2004; Alishahiha et al. 2004; Bartolo et al. 2004). Therefore, Gaussianity of the primordial fluctuations offers a direct test of inflationary models based on different assumptions about the nature of scalar field(s) can generate significant primordial non-Gaussianity (Komatsu et al. 2003; Spergel et al. 2006). (See Creminelli et al. [2006] for an alternative parameterization of $f_{\text{NL}}$.) As for the LSS, the analytical formula is known only for the three-dimensional perturbations. Note that $\Phi$ is related to the primordial comoving curvature perturbations generated during inflation, $R$, by $\Phi = (3/5)R$. While this quadratic form is motivated by inflationary models based on a single, slowly rolling scalar field, the actual predictions usually include momentum dependence in $f_{\text{NL}}$. (That is to say, $f_{\text{NL}}$ is not a constant.) Therefore, when precision is required, one should use the actual formula given by specific processes, either from primordial non-Gaussianity from inflation or the postinflationary evolution of nonlinear perturbations, to calculate a more accurate form of statistical quantities such as the angular bispectrum of the CMB (Babich et al. 2004; Liguori et al. 2006). Nevertheless, a constant $f_{\text{NL}}$ is still a useful parameterization of non-Gaussianity that enables us to obtain simple analytical formulae for the statistical quantities to compare with observations. The use of a constant $f_{\text{NL}}$ is also justified by the fact that the current observations are not sensitive enough to detect the momentum dependence of $f_{\text{NL}}$. Therefore, we adopt the constant $f_{\text{NL}}$ for our analysis throughout this paper. Note that it is actually straightforward to extend our formalism to a momentum-dependent $f_{\text{NL}}$.

So far, analytical formulae for the statistical quantities of the CMB from primordial non-Gaussianity are known only for the angular bispectrum (Komatsu & Spergel 2001; Babich & Zaldarriaga 2004; Liguori et al. 2006) and trispectrum (Okamoto & Hu 2002; Kogo & Komatsu 2006). The analytical formulae are extremely valuable especially when one tries to measure non-Gaussian signals from the data. Fast, nearly optimal estimators for $f_{\text{NL}}$ have been derived on the basis of these analytical formulae (Komatsu et al. 2005; Creminelli et al. 2006) and have been successfully applied to the CMB data from the Wilkinson Microwave Anisotropy Probe (WMAP); the current constraint on $f_{\text{NL}}$ from the angular bispectrum is $-54$ to $114$ at the 95% confidence level (Komatsu et al. 2003; Spergel et al. 2006). (See Creminelli et al. [2006] for an alternative parameterization of $f_{\text{NL}}$.)
bispectrum (Verde et al. 2000; Scoccimarro et al. 2004). The LSS bispectrum contains not only the primordial non-Gaussianity, but also the late-time non-Gaussianity from gravitational instability and galaxy biasing, which potentially obscure the primordial signatures.

In this paper, we derive analytical formulae for another statistical tool, namely, the Minkowski functionals (MFs), which describe morphological properties of fluctuating fields (Mecke et al. 1994; Schmalzing & Buchert 1997; Schmalzing & Gӧrski 1998; Winizki & Kosowsky 1998). In d-dimensional space \(d = 2\) for the CMB and \(d = 3\) for LSS, \(d + 1\) MFs are defined, as listed in Table 1. The “Euler characteristic” measures topology of the fields and is essentially given by the number of hot spots minus the number of cold spots when \(d = 2\). This quantity is sometimes called the “genus statistics,” which was independently rediscovered by Gott et al. (1986) in search of a topological measure of non-Gaussianity in the cosmic density fields. (The Euler characteristic and genus are different only by a numerical coefficient, \(-1/2\).)

Why study MFs? Since different statistical methods are sensitive to different aspects of non-Gaussianity, one should study as many statistical methods as possible. Most importantly, the MFs and bispectrum are very different in that MFs are defined in real space, whereas the bispectrum is defined in Fourier (or harmonic) space. Therefore, these statistical methods are nicely complementary to each other. Previously there were several attempts to give constraints on the primordial non-Gaussianity using MFs (e.g., Novikov et al. 2000). Although we show in this paper that the MFs do not contain more information than the bispectrum in the limit that non-Gaussianity is weak, the complementarity is still powerful in the presence of complicated real-world issues such as inhomogeneous noise, survey mask, foreground contamination, etc. The MFs have also been used to constrain \(f_{\text{NL}}\) (Komatsu et al. 2003) and Spergel et al. (2006) have used numerical simulations of non-Gaussian CMB sky maps to calculate the predicted form of MFs as a function of \(f_{\text{NL}}\) and compared the predictions with the WMAP data to constrain \(f_{\text{NL}}\), obtaining similar constraints to the bispectrum ones. This method (calculating the form of MFs from non-Gaussian simulations) is, however, a painstaking process: it takes about 3 hr to simulate one non-Gaussian map on one processor of SGI Origin 300. When cosmological parameters are varied, one needs to resimulate a whole batch of non-Gaussian maps from the beginning — this is a highly inefficient approach. Once we have the analytical formula for the MFs as a function of \(f_{\text{NL}}\), however, we no longer need to simulate non-Gaussian maps, greatly speeding up the measurement of \(f_{\text{NL}}\) from the data. We use the perturbative formula for MFs originally derived by Matsubara (1994, 2003): assuming that non-Gaussianity is weak, which has been justified by the current constraints on \(f_{\text{NL}}\), we consider the lowest-order corrections to the MFs using the multidimensional Edgeworth expansion around a Gaussian distribution function.

The organization of this paper is as follows. In § 2 we review the generic perturbative formula for the MFs. In § 3 we derive the analytical formula for MFs of the CMB from primordial non-Gaussian fluctuations parameterized by \(f_{\text{NL}}\). We also estimate projected statistical errors on \(f_{\text{NL}}\) expected from the WMAP data from multiyear observations as well as from the Planck data. In § 4 we derive the analytical formula for MFs of the LSS from primordial non-Gaussianity, nonlinear gravitational evolution, and galaxy biasing in a perturbative manner. Section 5 is devoted to summary and conclusions. In Appendix A we outline our method for computing the MFs from the CMB and LSS data. We also describe our simulations of the CMB and LSS. In Appendix B we derive the analytical formula for the galaxy bispectrum. In Appendix C we compare the analytical MFs of the CMB with non-Gaussian simulations in the Sachs-Wolfe limit. In Appendix D, we extend the corrections of primordial potential to nth order, in order to examine more carefully the validity of our perturbative expansion.

Throughout this paper, we assume a \(\Lambda\)CDM cosmology with the cosmological parameters at the maximum likelihood peak from the WMAP 1 yr data — only fit (Spergel et al. 2003). Specifically, we adopt \(\Omega_b = 0.049\), \(\Omega_{\text{CDM}} = 0.271\), \(\Omega_{\Lambda} = 0.68\), \(H_0 = 68.2\) km s\(^{-1}\) Mpc\(^{-1}\), \(\tau = 0.0987\), and \(n_s = 0.967\). The amplitude of primordial fluctuations has been normalized by the first acoustic peak of the temperature power spectrum, \(l(l + 1)C_l/(2\pi) = (74.7\) \(\mu\)K\(^2\)) at \(l = 220\) (Page et al. 2003b).

### 2. GENERAL PERTURBATIVE FORMULA FOR MFs

Suppose that we have a d-dimensional fluctuating field, \(f\), which has zero mean. Then, one may define the MFs for a given threshold, \(\nu \equiv f/\sigma_0\), where \(\sigma_0 \equiv (\langle f^2 \rangle)^{1/2}\) is the standard deviation of \(f\). Matsubara (2003) has obtained the analytical formulae for the \(k\)th MFs of weakly non-Gaussian fields in \(d\) dimensions, \(V_k(\nu)\), as (eq. [133] of Matsubara 2003)

\[
V_k(\nu) = \frac{1}{(2\pi)^k(k + 1/2)} \omega_d \omega_d \omega_k \left( \frac{\omega_d}{\sqrt{\sigma_0}} \right)^k e^{-\nu^2/2} \left\{ H_{k-1}(\nu) + \left[ \frac{1}{6} S^{(0)} H_{k+1}(\nu) + \frac{k}{3} S^{(1)} H_k(\nu) + \frac{k(k - 1)}{6} S^{(2)} H_{k-2}(\nu) \right] \sigma_0 + O(\sigma_0^2) \right\},
\]

where \(H_k(\nu)\) are the Hermite polynomials and \(\omega_d \equiv \pi^{k/2}/\Gamma(k/2 + 1)\) gives \(\omega_1 = 0, \omega_2 = 2, \omega_3 = \pi,\) and \(\omega_4 = 4\pi/3\). Here, the \(S^{(0)}\) are the “skewness parameters” defined by

\[
S^{(0)} = \frac{\langle f^3 \rangle}{\sigma_0^3},
\]

\[
S^{(1)} = \frac{3}{4} \frac{\langle f^2 (\nabla f^2) \rangle}{\sigma_0^3 \sigma_1^1},
\]

\[
S^{(2)} = -\frac{3d}{2(d-1)} \frac{\langle (\nabla f)(\nabla f)(\nabla^2 f) \rangle}{\sigma_1^3},
\]
which characterize the skewness of fluctuating fields and their derivatives. The quantity $\sigma_1$ characterizes the variance of fluctuating fields and their derivatives, and is given by

$$\sigma_1^2 \equiv \int_0^\infty \frac{k^2 dk}{2\pi^2} k^2 P(k) W^2(k R), \quad (6)$$

for $d = 3$, and

$$\sigma_2^2 \equiv \frac{1}{4\pi} \sum_l (2l+1)(l(l+1)) C_l W_l^2, \quad (7)$$

for $d = 2$. In both cases $W$ represents a smoothing kernel, or a window function, which is given by a product of the experimental beam transfer function, the pixelization window function, and an extra Gaussian smoothing. The power spectra, $P(k)$ for $d = 3$ and $C_l$ for $d = 2$, are defined as

$$\langle \hat{h} \hat{f} \rangle = (2\pi)^3 P(k) \delta_D(k - k'), \quad (8)$$

$$\langle a_{lm} a^{*}_{lm'} \rangle = C_l \delta_{ll'} \delta_{mm'}, \quad (9)$$

where $\delta_D(k)$ is the Dirac delta function and the Fourier and harmonic coefficients are given by

$$f(x) = \int \frac{d^3k}{(2\pi)^3} f(k) e^{ik \cdot x}, \quad (10)$$

$$f(\Omega) = \sum_{lm} a_{lm} Y_{lm}(\Omega), \quad (11)$$

for $d = 3$ and $d = 2$, respectively. Finally, the most relevant Hermite polynomials are given by

$$H_{-1}(\nu) = \sqrt{\frac{\pi}{2}} e^{\nu^2/2} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right), \quad (12)$$

$$H_0 = 1, \quad (13)$$

$$H_1(\nu) = \nu, \quad (14)$$

$$H_2(\nu) = \nu^2 - 1, \quad (15)$$

$$H_3(\nu) = \nu^3 - 3\nu, \quad (16)$$

$$H_4(\nu) = \nu^4 - 6\nu^2 + 3, \quad (17)$$

$$H_5(\nu) = \nu^5 - 10\nu^3 + 15\nu. \quad (18)$$

3. APPLICATION I: CMB

3.1. Analytical Formula for MFs of CMB

For the CMB, we have $d = 2$ and $f = \Delta T/T$. We define the angular bispectrum as

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} \equiv \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle. \quad (19)$$

Then, by expanding skewness parameters into spherical harmonics, we obtain

$$S_0^{(0)} = \frac{1}{4\pi} \sum_{lm} B_{l_1 l_2 l_3}^{m_1 m_2 m_3} G_{l_1 l_2 l_3}^{m_1 m_2 m_3} W_{l_1} W_{l_2} W_{l_3}, \quad (20)$$

$$S_0^{(1)} = \frac{3}{16\pi^2} \sum_{lm} \left\{ l_1(l_1 + 1) + l_2(l_2 + 1) + l_3(l_3 + 1) \right\}$$

$$\quad \times B_{l_1 l_2 l_3}^{m_1 m_2 m_3} G_{l_1 l_2 l_3}^{m_1 m_2 m_3} W_{l_1} W_{l_2} W_{l_3}, \quad (21)$$

$$S_0^{(2)} = \frac{3}{8\pi^2} \sum_{l_m} \left\{ \left| l_1(l_1 + 1) + l_2(l_2 + 1) - l_3(l_3 + 1) \right| \right\}$$

$$\quad \times W_{l_1} W_{l_2} W_{l_3} B_{l_1 l_2 l_3}^{m_1 m_2 m_3} G_{l_1 l_2 l_3}^{m_1 m_2 m_3}, \quad (22)$$

where $(cyc.)$ means the addition of terms with the same cyclic order as the subscripts of the previous term, $W_l$ is a smoothing kernel in $l$-space, and $G_{l_1 l_2 l_3}^{m_1 m_2 m_3}$ is the Gaunt integral,

$$G_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \int d\mathbf{n} Y_{l_1 m_1}(\mathbf{n}) Y_{l_2 m_2}(\mathbf{n}) Y_{l_3 m_3}(\mathbf{n}). \quad (23)$$

Note that we have used the following properties of $Y_{lm}(\hat{n})$:

$$\nabla^2 Y_{lm}(\hat{n}) = -l(l+1)Y_{lm}(\hat{n}), \quad (24)$$

$$\int d\mathbf{n} \left[ \nabla Y_{l_1 m_1}(\mathbf{n}) \right] \left[ \nabla Y_{l_2 m_2}(\mathbf{n}) \right] Y_{l_3 m_3}(\mathbf{n}) = \frac{l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1)}{2} G_{l_1 l_2 l_3}^{m_1 m_2 m_3}. \quad (25)$$

The summation over $m$ can be done by writing

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = G_{l_1 l_2 l_3}^{m_1 m_2 m_3} b_{l_1 l_2 l_3}, \quad (26)$$

where $b_{l_1 l_2 l_3}$ is the reduced bispectrum that depends on specific non-Gaussian models (Komatsu & Spergel 2001). Using the reduced bispectrum, we finally obtain the analytical formula for MFs of the CMB:

$$S_0^{(0)} = \frac{3}{2\pi} \sum_{2 \leq l_l \leq l_3} \frac{l_1^2}{l_l} b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}, \quad (27)$$

$$S_0^{(1)} = \frac{3}{8\pi} \sum_{2 \leq l_l \leq l_3} \frac{l_1^2}{l_l} \left( l_1(l_1 + 1) + l_2(l_2 + 1) + l_3(l_3 + 1) \right)$$

$$\quad \times l_l b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}, \quad (28)$$

$$S_0^{(2)} = \frac{3}{4\pi} \sum_{2 \leq l_l \leq l_3} \left\{ \left[ l_1(l_1 + 1) + l_2(l_2 + 1) - l_3(l_3 + 1) \right] l_l \right\}$$

$$\quad \times \left( cyc. \right) l_l^2 b_{l_1 l_2 l_3} W_{l_1} W_{l_2} W_{l_3}, \quad (29)$$

where

$$l_l b_{l_1 l_2 l_3} \equiv \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \quad (30)$$

and we have used

$$\sum_{m_1 m_2 m_3} \left( G_{l_1 l_2 l_3}^{m_1 m_2 m_3} \right)^2 = l_l^2 l_l b_{l_1 l_2 l_3}, \quad (31)$$

When $f_{NL}$ is a constant, the form of $b_{l_1 l_2 l_3}$ is given by (Komatsu & Spergel 2001)

$$b_{l_1 l_2 l_3} = 2 \int_0^\infty r^2 dr \left[ b_{l_1}^N(r) b_{l_2}^N(r) b_{l_3}^N(r) \right. $$

$$\left. + b_{l_1}^N(r) b_{l_2}^N(r) b_{l_3}^N(r) + b_{l_1}^N(r) b_{l_2}^N(r) b_{l_3}^N(r) \right], \quad (32)$$
where

\[ b^L_l(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 \, dk \, P_{\phi}(k) \, g_{RL}(k) \, j_l(kr), \]  

(33)

and \( P_{\phi}(k) \propto k^{n_{\phi}-4} \) is the primordial power spectrum of \( \phi \). The amplitude of \( P_{\phi}(k) \) is fixed by the first peak amplitude of the temperature power spectrum, \( l(l + 1)C_l/(2\pi) = (74.7 \mu K)^2 \) at \( l = 220 \) (Page et al. 2003a), and the temperature power spectrum is given by

\[ C_l = \frac{2}{\pi} \int_0^\infty k^2 \, dk \, P_{\phi}(k) \, g_{RL}^2(k). \]  

(35)

We calculate the full radiation transfer function, \( g_{RL}(k) \), using the publicly available CMBFAST code (Seljak & Zaldarriaga 1996). Note that our formalism is completely generic. One can easily generalize our results to non-Gaussian models with a momentum-dependent \( f_{NL} \) by using an appropriate form of \( b_{l0lb} \) given in Liguori et al. (2006). Our results suggest that the MFs do not contain information beyond the bispectrum when non-Gaussianity is weak. The MFs of the CMB basically measure the weighted sum of the CMB angular bispectrum.

In Figure 1, we plot the skewness parameters, \( S^{(a)} \) (eqs. [27]–[29]), and \( S^{(a)} \) multiplied by \( \sigma_0 \), for a pure signal of CMB anisotropy (without noise) as a function of a Gaussian smoothing width, \( \theta_0 \), which determines a Gaussian smoothing kernel, \( W_l = \exp[-l(l + 1)\theta_0^2/2] \). The perturbative expansion of MFs works only when \( S^{(a)} \sigma_0 \) is much smaller than unity (see eq. [2]). We find that the perturbative expansion is valid for \( f_{NL} \ll 3000 \) from the results plotted in the right panel of Figure 1 that show \( |S^{(a)}|\sigma_0 \leq 3 \times 10^{-4} f_{NL} \).

This figure also shows how MFs may be as powerful as the angular bispectrum in measuring \( f_{NL} \). Komatsu & Spergel (2001) have shown that sensitivity of the first skewness parameter, \( S^{(0)} \), to \( f_{NL} \) is much worse than that of the angular bispectrum, as acoustic oscillations in \( l \)-space smear out non-Gaussian signals in the skewness, which is the weighted sum of the angular bispectrum over \( l \). (The angular bispectrum is negative in the Sachs-Wolfe regime at low \( l \) and oscillates about zero by changing its sign at higher \( l \).) The MFs are sensitive to not only \( S^{(0)} \), but also to \( S^{(1)} \) and \( S^{(2)} \). The weight of the sum over multipole differs among the three skewness parameters: \( S^{(2)} \) has the largest weight at high \( l \), \( S^{(0)} \) has the largest weight at low \( l \), and \( S^{(1)} \) is somewhere in between (eqs. [27]–[29]). In particular, because \( S^{(2)} \) picks up the highest multipoles efficiently, \( S^{(2)} \) changes its sign depending on \( \theta_0 \). The parameter \( S^{(2)} \) is negative on very large angular scales. As \( \theta_0 \) decreases (as the small-scale information is included), \( S^{(2)} \) increases and eventually changes its sign to positive values near the scale of the first acoustic peak, \( \theta_0 \sim 40' \), while the bispectrum has the largest amplitude, while the other two skewness parameters do not change their signs. Therefore, \( S^{(2)} \) keeps information about the acoustic oscillations. This property is crucial for obtaining a better signal-to-noise ratio for primordial non-Gaussianity in the CMB.

Figure 2 shows the predicted MFs of CMB temperature anisotropy, \( V_0, V_1, \) and \( V_2 \), as a function of \( f_{NL} \). The MFs for \( f_{NL} = 100 \) and 50 are plotted in the solid lines, while the MFs for \( f_{NL} = 0, -50 \), and \( -100 \) are plotted in the dotted lines. The lower panels show the difference between the Gaussian and non-Gaussian MFs divided by the maximum amplitude of the Gaussian MFs. Primordial non-Gaussianity with \( f_{NL} = 100 \) changes \( V_0, V_1, \) and \( V_2 \) by 0.2%, 0.4%, and 1%, respectively, relative to the maximum amplitude of the corresponding Gaussian MFs. While \( V_0 \) (area) has little dependence on \( \theta_0 \), \( V_2 \) (Euler characteristic) strongly depends on \( \theta_0 \), which is mainly due to the sign change of \( S^{(2)} \) at \( \theta_0 \sim 40' \).

In Appendix C we show that these analytical predictions agree very well with non-Gaussian simulations in the Sachs-Wolfe limit. The comparison with the full simulations that include the full radiation transfer function will be reported elsewhere.
3.2. Measuring $f_{\rm NL}$ from MFs of the CMB

The MFs at different threshold values, $\nu$, are correlated, and different MFs are also correlated, i.e.,

$$
\Sigma_{kk'}^{(d)}(\nu, \nu') = \left\langle V_k^{(d)}(\nu)V_{k'}^{(d)}(\nu') \right\rangle - \delta_{kk'} \delta(\nu - \nu'),
$$

where $\delta$ denotes the Kronecker delta. Therefore, it is important to use the full covariance matrix, $\Sigma$, in the data analysis. We obtain the full covariance matrix from Monte Carlo simulations of Gaussian temperature anisotropy. Because non-Gaussianity is weak, the covariance matrix estimated from Gaussian simulations is a good approximation of the exact one. In Appendix A.1 we describe our methods for computing MFs from the pixelized CMB maps. In Appendices A.2 and A.3 we describe our methods for simulating sky maps of CMB temperature anisotropy with noise and instrumental characteristics of the WMAP and Planck experiments, respectively.

We use the Fisher information matrix formalism to estimate the projected errors on $f_{\rm NL}$ from given measurement errors on MFs. The Fisher matrix, $F_{ij}$, is written in terms of the inverse of the covariance matrix, $\Sigma^{-1}$, as

$$
F_{ij}^{(d)} = \sum_{kk'} \int d\nu \int d\nu' \frac{\partial V_k^{(d)}(\nu)}{\partial p_i} \left( \Sigma^{-1} \right)^{kk'}_{kk'}(\nu, \nu') \frac{\partial V_{k'}^{(d)}(\nu')}{\partial p_j},
$$

where $p_i$ is the $i$th parameter. For the CMB, we consider only one parameter, $p_1 = f_{\rm NL}$, whereas for LSS we also include galaxy biasing parameters (see § 4). The projected 1 $\sigma$ error on $f_{\rm NL}$ is given by the square root of $(F^{-1})_{11}$. While equation (37) may be evaluated for a given smoothing scale, $\theta_s$, one will eventually need to combine all combinations of MFs at different $\theta_s$ to obtain the best constraint on $f_{\rm NL}$ from given data. The MFs at different $\theta_s$ are also correlated. We therefore calculate the full covariance matrix of MFs that consists of $N_\nu \times N_{\text{MF}} \times N_s$ elements, where $N_\nu = 25$ is the number of bins for $\nu$ per each MF in the range of $\nu$ from $-3$ to $3$; $N_{\text{MF}} = 3$ and $4$ is the number of MFs for $d = 2$ and $3$, respectively (i.e., $N_{\text{MF}} = 3$ for the CMB and $4$ for LSS); and $N_s$ is the number of
smoothing scales used in the analysis. The Fisher matrix may be written as

$$F_{ij}^{(d)} = \sum_{\alpha} \frac{\partial V_{ij}^{(d)}}{\partial \alpha} (\Sigma^{-1})_{\alpha \beta} \frac{\partial V_{\beta j}^{(d)}}{\partial \alpha},$$  

where $\alpha$ is a single index denoting $k$, $\nu$, and $\theta_s$.

Let us comment on the effect of noise on MFs. The instrumental noise increases $\sigma_3^2$ by adding extra power at small scales. On the other hand, the signal part of the angular bispectrum is unaffected by noise because noise is Gaussian. As a result, the instrumental noise always reduces the skewness parameters, as the skewness parameters contain $\sigma_3$ in their denominator. Therefore, the MFs would approach Gaussian predictions in the noise-dominated limit, as expected. We compute the increase in $\sigma_3$ and $\sigma_1$ due to noise from Monte Carlo simulations and then rescale $S^{(d)}$ and $\partial V_{ij}^{(d)}/\partial \theta_s$ for a given smoothing scale, $\theta_s$. The window function, $W_i$, includes the beam-smearing effect, the pixel window function, and a Gaussian smoothing.

Using the method described above and in Appendix A, we estimate the projected 1σ error on $f_{NL}$ expected from WMAP's 1 and 8yr observations. We also consider an ideal WMAP experiment without noise or sky cut (but the beam smearing is still included). We consider six different smoothing scales, $\theta_s = 5\prime, 10\prime, 100\prime, 20\prime, 40\prime, 70\prime,$ and $100\prime$. The results from various combinations of $\theta_s$ are summarized in Table 2 for WMAP observations.

For the WMAP 1yr data, the error on $f_{NL}$ is the smallest for $\theta_s = 20\prime$. At smaller angular scales, say $\theta_s = 10\prime$, the noise dominates more, and thus the error on $f_{NL}$ increases at $\theta_s \leq 20\prime$. For the WMAP 8yr data, on the other hand, a better signal-to-noise ratio at smaller angular scales enables us to constrain MFs at $\theta_s = 10\prime$. As the beam size of WMAP in W band is about $\theta_s = 10\prime$, one cannot constrain MFs at angular scales smaller than this. When all the smoothing scales are combined, the projected 1σ error on $f_{NL}$ reaches $\sim 40$ for the WMAP data, which is in rough agreement with the result reported in Komatsu et al. (2003) and Spergel et al. (2006). The best constraint that can be obtained from the WMAP data, in the limit of zero noise and full sky coverage, is $f_{NL} \sim 22$.

We also estimate the Planck constraint on $f_{NL}$ listed at Table 3. As Planck's beam and noise are $\approx 4$ and 10 times smaller than WMAP's, respectively, one can constrain MFs even at $\theta_s = 5\prime$. Planck should be sensitive to $|f_{NL}| \sim 20$.

### Table 2

| $\theta_s$ (arcmin) | WMAP 1 yr | WMAP 8 yr | No Noise and No Sky Cut |
|---------------------|-----------|-----------|-------------------------|
|                     | $f_{NL}$  | $f_{NL}$  | $f_{NL}$                |
|                     | (V0, V1, V2) | (V0, V1, V2) | (V0, V1, V2) |
| 100                  | 271       | 269       | 135                     |
| 70                   | 153       | 153       | 94                      |
| 40                   | 86        | 84        | 59                      |
| 20                   | 54        | 51        | 41                      |
| 10                   | 73        | 41        | 35 ($120, 47, 51$)      |
| 5                    | ...       | 75        | 30 ($106, 42, 43$)      |
| 10, 20, 40           | 46        | 36        | 30 ($49, 43, 45$)       |
| 5, 10, and 20        | 36        | 36        | 22 ($46, 32, 37$)       |

**Notes.**—The errors are from each of the MFs of CMB temperature anisotropy (V0: area; V1: total circumference; V2: Euler characteristic) as well as from the combined analysis of all the MFs, for various combinations of smoothing scales, $\theta_s$. We use noise and beam properties of WMAP 1 and 8 yr observations with the Kp0 mask. The last two rows of table show the 1σ errors from the “ultimate WMAP,” which uses the entire sky with zero noise. (The beam smearing is still included.)

### Table 3

| $\theta_s$ (arcmin) | Planck | No Noise and No Sky Cut |
|---------------------|--------|-------------------------|
|                     | (V0, V1, V2) | (V0, V1, V2) |
| 100                  | 271     | 136                     |
| 70                   | 157     | 95                      |
| 40                   | 77      | 54                      |
| 20                   | 46      | 37 ($140, 49, 55$)      |
| 10                   | 32      | 28 ($102, 40, 39$)      |
| 5                    | 24      | 21 ($87, 27, 30$)       |
| 5, 10, and 20        | 19      | 15 ($38, 22, 26$)       |

4. **APPLICATION II: LSS**

#### 4.1. **Analytical Formula for MFs of LSS**

For LSS, we have $d = 3$ and $f = \delta_3(\xi, z)$, where $\delta_3$ is the density contrast of galaxies. Statistical isotropy of the universe gives the following form of the bispectrum:

$$\langle \delta_3(z) (k_1) \delta_3(z) (k_2) \delta_3(z) (k_3) \rangle = \langle \delta_3(z) \rangle (2\pi)^3 \delta_0 (k_1 + k_2 + k_3) B_g(k_1, k_2, k_3).$$  

We obtain the skewness parameters by integrating $B_g(k_1, k_2, k_3)$ over $k_1, k_2,$ and $\mu \equiv \langle k_1 \cdot k_2 \rangle / (k_1 k_2)$ with appropriate weights as

$$S_g^{(0)}(z) = \langle \delta_3(z) \rangle (2\pi)^3 \delta_0 (k_1 + k_2 + k_3) B_g(k_1, k_2, k_3),$$

$$S_g^{(1)}(z) = \langle \delta_3(z) \rangle (2\pi)^3 \delta_0 (k_1 + k_2 + k_3) \langle S_g^{(0)}(z) \rangle,$$

$$S_g^{(2)}(z) = \langle \delta_3(z) \rangle (2\pi)^3 \delta_0 (k_1 + k_2 + k_3) \langle S_g^{(0)}(z) \rangle,$$

$$S_g^{(3)}(z) = \langle \delta_3(z) \rangle (2\pi)^3 \delta_0 (k_1 + k_2 + k_3) \langle S_g^{(0)}(z) \rangle,$$

where $k_1 \equiv |k_1 + k_2| = (k_1^2 + k_2^2 + 2\mu k_1 k_2)^{1/2}$.  

Note that $\theta_s$ is a Gaussian width, which is $1/2.35$ times the FWHM.
Unlike for the CMB, for which we needed to consider only the effect of primordial non-Gaussianity, there are three sources of non-Gaussianity in $B_g(k_1, k_2, k_3, z)$: primordial non-Gaussianity, nonlinearity in the gravitational evolution, and nonlinearity in the galaxy bias. In Appendix B we show that $B_g$ is given by

$$B_g(k_1, k_2, k_3, z) = b_1(z) [B_{\text{pri}}(k_1, k_2, k_3, z) + B_{\text{gr}}(k_1, k_2, k_3, z)] + b_2(z) [P_m(k_1, z)P_m(k_2, z) + (\text{cyclic})],$$

(43)

where $P_m(k, z)$ is the linear matter power spectrum; $b_1(z)$ and $b_2(z)$ are the linear and nonlinear galaxy bias parameters, respectively (see eq. [B1] for the precise definition); and $B_{\text{pri}}$ and $B_{\text{gr}}$ represent the contributions from primordial non-Gaussianity and nonlinearity in gravitational clustering, respectively:

$$B_{\text{pri}}(k_1, k_2, k_3, z) = \frac{2f_{\text{NL}}}{D(z)} \left[ \frac{P_m(k_1, z)P_m(k_2, z)M(k_3)}{M(k_1)M(k_2)} + (\text{cyclic}) \right],$$

(44)

$$B_{\text{gr}}(k_1, k_2, k_3, z) = 2[F_2(k_1, k_2)P_m(k_1, z)P_m(k_2, z) + (\text{cyclic})],$$

(45)

where $D(z)$ is the growth rate of linear density fluctuations normalized such that $D(z) \to 1/(1 + z)$ during the matter era and $M(k)$ and $F_2(k_1, k_2)$ are given by equations (B10) and (B5), respectively. These equations suggest that $f_{\text{NL}}$ and the galaxy bias parameters must be determined simultaneously from the LSS data. Moreover, even if the galaxy bias is perfectly linear, $b_2 \equiv 0$, the primordial signal might be swamped by non-Gaussianity due to nonlinear gravitational clustering, $B_{\text{gr}}$.

In order to investigate how important the effect of $B_{\text{gr}}$ is, let us define the skewness parameters that are contributed solely by $B_{\text{pri}}$ or $B_{\text{gr}}$. Substituting $B_{\text{gr}}$ and $\sigma_{m,j}(z)$ for $B_g$ and $\sigma_{j,g}$, respectively, in equations (40)–(42), we obtain $S_{g}^{(a)} (a = 0, 1, 2)$ given by (Matsubara 2003)

$$S_{g}^{(0)}(z) = \frac{3}{28\pi^4 \sigma^4 m_{gr}(z)} [5I_{220}(z) + 7I_{133}(z) + 2I_{122}(z)],$$

(46)

$$S_{g}^{(1)}(z) = \frac{3}{56\pi^4 \sigma^2 m_{gr}(z) \sigma_{m,1}(z)} \times [10I_{240}(z) + 12I_{233}(z) + 7I_{151}(z) + 11I_{242}(z) + 2I_{333}(z)],$$

(47)

$$S_{g}^{(2)}(z) = \frac{9}{56\pi^4 \sigma^4 m_{13}(z)} \times [5I_{440}(z) + 7I_{251}(z) + 3I_{442}(z) - 7I_{553}(z) - 2I_{444}(z)],$$

(48)

where

$$I_{mn}(z) = \int_{-1}^{1} dk_1 \int_{0}^{\infty} dk_2 \int_{0}^{\infty} \int_{0}^{\infty} d\mu W(k_1 R)W(k_2 R)W(k_3 R)k_1^n k_2^m k_3^r \langle P_m(k_1, z)P_m(k_2, z) \rangle.$$

(49)

Note that $\sigma_{j,g} = b_1 \sigma_{m,j}$. Similarly, we also calculate the primordial skewness parameters, $S_{g}^{(a)}$, by substituting $B_{\text{pri}}$ and $\sigma_{m,j}(z)$ for $B_g$ and $\sigma_{j,g}(z)$, respectively, in equations (40)–(42). These skewness parameters are related to the skewness parameters of the total galaxy bispectrum as

$$S_{g}^{(a)} = S_{g}^{(a) pri} + S_{g}^{(a) gr} + \frac{3b_2}{b_1}.$$  

(50)

In Figure 3 we compare $S_{g}^{(a) pri}$ and $S_{g}^{(a) gr}$ at $z = 0, 2, 5$, as a function of a smoothing length, $R$, which is in units of $h^{-1}$ Mpc. The smoothing kernel $W(kR)$ is set to be a Gaussian filter, $W(kR) = \exp(-k^2 R^2/2)$. Non-Gaussianity from nonlinear gravitational clustering always gives positively skewed density
fluctuations, $S_{\delta}^{(a)} > 0$. Primordial non-Gaussianity with a positive $f_{NL}$ also yields positively skewed density fluctuations; however, primordial non-Gaussianity with a negative $f_{NL}$ yields negatively skewed density fluctuations, which may be more easily distinguished from $S_{\delta}^{(a)}$. As the smoothing scale, $R$, increases (i.e., density fluctuations become more linear), non-Gaussianity from nonlinear clustering, $S_{\delta}^{(a)}\sigma_{m,0}$, becomes weaker, while primordial non-Gaussianity, $S_{\delta_{p}}^{(a)}\sigma_{p,0}$, remains nearly the same. At $z = 0$, the primordial contribution exceeds nonlinear gravity only at very large scales, $R > 200 h^{-1}$ Mpc for $f_{NL} = 100$ and $R > 800 h^{-1}$ Mpc for $f_{NL} = 10$. At higher redshifts, on the other hand, nonlinearity is much weaker, and therefore the primordial contribution dominates at relatively smaller spatial scales, $R > 120$ and $80 h^{-1}$ Mpc at $z = 2$ and 5, respectively, for $f_{NL} = 100$. Unlike for the CMB, all the skewness parameters of galaxies exhibit similar dependence on the smoothing scales. The perturbation formula is valid when the amplitude of the second-order correction of MFs is small, $S^{(a)}\sigma_0 \ll 1$, that is, $f_{NL} \ll 5000$.

4.2. Measuring $f_{NL}$ from MFs of LSS

Figure 4 shows the perturbation predictions for the MFs from primordial non-Gaussianity with $f_{NL} = 100$, 50, 0, $-50$, and $-100$. For comparison, we also show the MFs computed from numerical simulations with $f_{NL} = 100$. In Appendix A.4 we describe our methods for computing MFs from the LSS data. In Appendix A.5 we describe our methods for simulating the LSS data with primordial non-Gaussianity (but without any effects from nonlinear gravitational clustering or galaxy bias). The error bars are estimated from variance among 2000 realizations divided by $200^{1/2}$. The left panels show the MFs, while the right panels show the difference between non-Gaussian and Gaussian MFs, divided by the maximum amplitude of each MF. We find that the analytical perturbation predictions agree with the numerical simulations very well.

Let us comment on some subtlety that exists in the comparison between the perturbation predictions and numerical simulations. The MFs measured from numerical simulations often deviate from the analytical predictions, even for Gaussian fluctuations, due to subtle pixelization effects. The MFs from our Gaussian simulations deviate from the analytical predictions at the level of $10^{-2}$ when normalized to the maximum amplitude of each MF. It is important to remove this bias, as the magnitude of this effect is comparable to or larger than the effect of primordial non-Gaussianity with $f_{NL} = 100$ ($10^{-3}$ for $V_0$ and $10^{-2}$ for $V_1$).
Therefore, it is often necessary to recalibrate the Gaussian predictions for the pixelization effects by running a large number of Gaussian realizations. However, we have found that the difference between the Gaussian and non-Gaussian MFs measured from simulations agrees with the perturbation predictions without any corrections. Therefore, one may use the following procedure for calculating the correct non-Gaussian MFs:

1. Use the analytical formulae (eq. [2]) to calculate the difference between the Gaussian and non-Gaussian MFs,

\[
\Delta V_k^{(d)}(\nu, f_{\text{NL}}) \equiv V_k^{(d)}(\nu, f_{\text{NL}}) - V_k^{(d)}(\nu, f_{\text{NL}} = 0). \tag{51}
\]

2. Run Gaussian simulations. Estimate the average MFs from these Gaussian simulations, \( V_k^{(d)}(\nu, f_{\text{NL}} = 0) \). These would be slightly different from the analytical formula, \( V_k^{(d)}(\nu, f_{\text{NL}} = 0) \), due to the pixelization and boundary effects. Note that the same simulations may be used to obtain the covariance matrix of the MFs.

3. Calculate the final non-Gaussian predictions as

\[
P_k^{(d)}(\nu, f_{\text{NL}}) = P_k^{(d)}(\nu, f_{\text{NL}} = 0) + \Delta V_k^{(d)}(\nu, f_{\text{NL}}). \tag{52}
\]

We estimate the projected errors on \( f_{\text{NL}} \) from LSS using the Fisher information matrix in the same way as the CMB (see § 3.2). We compute the covariance matrix of MFs from 2000 realizations of Gaussian density fluctuations in a 1 Gpc\(^3\) cubic box. For simplicity, we focus on the large scales and ignore non-Gaussianity from nonlinear gravitational evolution. (The minimum smoothing scale is \( R = 60 h^{-1} \) Mpc.) We also ignore shot noise.

Table 4 shows the projected \( \sigma \) errors on \( f_{\text{NL}} \) from a galaxy survey covering a \( 1 h^{-3} \) Gpc\(^3\) volume. This volume would correspond to, e.g., a galaxy survey covering 300 deg\(^2\) on the sky at \( 5.5 < z < 6.5 \). Note that these constraints are independent of \( z \) when nonlinear gravitational evolution is ignored, as \( S_{\text{NL}}^1 \sigma_0 \) is independent of \( z \). We have assumed a linear galaxy bias \( (b_z = 0) \) in column (2), while we have marginalized \( b_2/b_1 \) in column (3). A more realistic prediction would lie between these two cases, as we can use the power spectrum and bispectrum to put some constraints on \( b_2 \). We find better constraints from smaller smoothing scales for a given survey volume, simply because we have more modes on smaller scales. The limits on \( f_{\text{NL}} \) from a \( 1 h^{-3} \) Gpc\(^3\) survey are not very promising, \( f_{\text{NL}} \sim 270 \) at the 68% confidence level; thus, one would need the survey volume to be as large as \( 25 h^{-3} \) Gpc\(^3\) to make the LSS constraints comparable to the WMAP constraints (using the MFs only). This could be done by a survey covering \( \sim 2000 \) deg\(^2\) at \( 3.5 < z < 6.5 \). One would obviously need more volume to make it comparable to the Planck data.

### 5. SUMMARY AND CONCLUSIONS

We have derived analytical formulae of the MFs for the CMB and LSS using a perturbation approach. The analytical formula is useful for studying the behavior of MFs and estimating the observational constraints on \( f_{\text{NL}} \) without relying on non-Gaussian numerical simulations. The perturbation approach works when the skewness parameters multiplied by variance, \( S_{\text{NL}} f_{\text{NL}} \), is much smaller than unity, i.e., \( |f_{\text{NL}}| \ll 3300 \) for the CMB and \( |f_{\text{NL}}| \ll 5000 \) for LSS, both of which are satisfied by the current observational constraints from WMAP (Komatsu et al. 2003; Creminelli et al. 2006; Spergel et al. 2006). We have shown that the perturbation predictions agree with non-Gaussian numerical realizations very well.

We have used the Fisher matrix analysis to estimate the projected constraints on \( f_{\text{NL}} \) expected from the observations of the CMB and LSS. We have found that the projected \( \sigma \) error on \( f_{\text{NL}} \) from the WMAP should reach 50, which is consistent with the MF analysis given in Komatsu et al. (2003) and Spergel et al. (2006) and is comparable to the current constraints from the bispectrum analysis given in Komatsu et al. (2003), Creminelli et al. (2006), and Spergel et al. (2006). The MFs from the WMAP 8 yr and Planck observations should be sensitive to \( |f_{\text{NL}}| \sim 40 \) and 20, respectively, at the 68% confidence level.

As the MFs are solely determined by the weighted sum of the bispectrum for \( |f_{\text{NL}}| \ll 3300 \) for the CMB and \( |f_{\text{NL}}| \ll 5000 \) for LSS, the MFs do not contain information more than the bispectrum. However, this does not imply that the MFs are useless for measuring primordial non-Gaussianity by any means. The important distinction between the MFs and bispectrum is that the MFs are intrinsically defined in real space, while the bispectrum is defined in Fourier space. The systematics in the data are mostly dealt with in real space, and thus the MFs should be quite useful in this regard. Therefore, in the presence of real world issues such as inhomogeneous noise, foreground, masks, etc., these two approaches should be used to check for consistency of the results.

In this paper we have calculated the MFs from primordial non-Gaussianity with a scale-independent \( f_{\text{NL}} \). It is easy to extend our calculations to a scale-dependent \( f_{\text{NL}} \). All one needs to do is to calculate the form of the bispectrum with a scale-dependent \( f_{\text{NL}} \) (e.g., Babich et al. 2004; Liguori et al. 2006) and use it to obtain the skewness parameters, \( S_{\text{NL}} f_{\text{NL}} \) (eqs. [20]–[22] for the CMB and eqs. [40]–[42] for LSS). The MFs are then given by equation (2) in terms of the skewness parameters. Also, we have not included non-Gaussianity from secondary anisotropies such as the Sunyaev-Zel’dovich effect, Rees-Sciama effect, patchy reionization, weak-lensing effect, extragalactic radio sources, etc. It is again straightforward to calculate the MFs from these sources using our formalism, as long as the form of the bispectrum is known (e.g., Spergel & Goldberg 1999; Goldberg & Spergel 1999; Cooray & Hu 2000; Komatsu & Spergel 2001; Verde & Spergel 2002).

For LSS, the late-time non-Gaussianity from gravitational evolution and galaxy biasing are strong, and thus it is necessary to smooth the field at large scales, more than 100 \( h^{-1} \) Mpc, to reduce their uncertainty. Under such large-scale smoothing, MFs are insensitive to the morphology of LSS, for which they have been
designed in the first place. The constraints on primordial non-Gaussianity from the MFs of LSS in a galaxy survey covering 1 \( h^{-3} \) Gpc\(^3\) volume are about 5 times weaker than those from the MFs of the CMB in the \( WMAP \) data. One would therefore need the survey volume as large as 25 \( h^{-3} \) Gpc\(^3\) to make the LSS constraints comparable to the \( WMAP \) constraints (using the MFs only). This could be done by a survey covering \( \sim 2000 \) deg\(^2\) at \( 3.5 < z < 6.5 \). One would obviously need more volume to make it comparable to the \( Planck \) data. The MFs from LSS are less sensitive to primordial non-Gaussianity because non-Gaussianity from the nonlinear evolution of gravitational clustering exceeds the primordial contribution at \( R < 200, 120, \) and 80 \( h^{-1} \) Mpc at \( z = 0, 2, \) and 5, respectively, which severely limits the amount of LSS data available to constrain primordial non-Gaussianity.

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APPENDIX A

MEASURING MFs FROM CMB AND LSS

In this appendix we describe our methods for computing the MFs from the CMB and LSS data. We also describe our simulations of the CMB and LSS.

A1. COMPUTATIONAL METHOD: CMB

We estimate the MFs from pixelized CMB sky maps by integrating a combination of first and second angular derivatives of temperature anisotropy, \( I_k \), over the sky (Schmalzing & Górski 1998),

\[
F_k^{(2)}(\nu) = \frac{\sum_{i=1}^{n_{\text{pix}}} w_i I_k(\Omega_i)}{\sum_{i=1}^{n_{\text{pix}}} w_i},
\]

(A1)

where \( \Omega_i \) is the unit vector pointing toward a given position on the sky. We set the weight of the \( i \)th pixel, \( w_i \), to be 1 when the pixel at \( \Omega_i \) is outside of the survey mask, and 0 otherwise. We calculate \( I_k \) at \( \Omega_i \) from covariant derivatives of temperature anisotropies divided by its standard deviation, \( u(\Omega_i) = (\Delta T/T)/\sigma_0 \).

\[
I_0 = \Theta(u - \nu),
\]

(A2)

\[
I_1 = \frac{1}{4} F(u - \nu) \sqrt{u_{\theta}^2 + u_{\phi}^2},
\]

(A3)

\[
I_2 = \frac{1}{2\pi} F(u - \nu) \frac{2u_{\theta}u_{\phi}u_{\theta\phi} - u_{\theta}^2u_{\phi\phi} - u_{\phi}^2u_{\theta\theta}}{u_{\theta}^2 + u_{\phi}^2}.
\]

(A4)

The function \( F(u - \nu) \) has a value of \( 1/\Delta \nu \) (\( \Delta \nu \) is the binning width of \( \nu \)) when \( u \) is within \([\nu - \Delta \nu/2, \nu + \Delta \nu/2]\), and 0 otherwise. Covariant derivatives are related to the partial derivatives as

\[
u_{\theta} = \frac{1}{\sin \theta} u_{\phi},
\]

(A5)

\[
u_{\phi} = \frac{1}{\sin \theta} u_{\theta},
\]

(A6)

\[
u_{\theta\theta} = \frac{\cos \theta}{\sin \theta} u_{\phi},
\]

(A7)

\[
u_{\theta\phi} = \frac{1}{\sin \theta} u_{\theta\phi} - \frac{\cos \theta}{\sin \theta} u_{\phi},
\]

(A8)

\[
u_{\phi\phi} = \frac{1}{\sin \theta} u_{\phi\phi} + \frac{\cos \theta}{\sin \theta} u_{\theta}.
\]

(A9)

The first derivative of the temperature field \( u,(i = \theta \text{ or } \phi) \) are calculated in Fourier space as

\[
\begin{align*}
I_{lm, i} &= \sum_{l'm} a_{lm} Y_{lm,i},
\end{align*}
\]

(A10)

\[
Y_{lm, \theta} = \begin{cases} 
\frac{l}{\tan \theta} Y_{lm} - \sqrt{\frac{2l+1}{2l-1}} \left( 1^2 - m^2 \right) \frac{1}{\sin \theta} Y_{l-1m}, & |m| < l, \\
\frac{l}{\tan \theta} Y_{lm}, & m = l,
\end{cases}
\]

(A11)

\[
Y_{lm, \phi} = im Y_{lm}.
\]

(A12)
The second derivatives are also computed as

\[ u_{ij} = \sum_{lm} a_{lm} Y_{lm}, \]

\[ Y_{lm,\theta\theta} = \left\{ -\left[ l(l + 1) - \frac{m^2}{\sin^2 \theta} \right] Y_{lm} - \frac{1}{\tan \theta} Y_{lm,\theta} \right\}, \]

\[ Y_{lm,\theta\phi} = im Y_{lm,\theta}, \]

\[ Y_{lm,\phi\phi} = -m^2 Y_{lm}. \]

We calculate the MFs of temperature anisotropy for \( \nu \) from \(-3\) to \(3\) with \( \Delta_\nu = 0.24 \), which yields 25 bins per each MF.

### A2. Simulating WMAP Data

In order to quantify uncertainties in the estimated MFs from cosmic variance and various effects, we simulate Gaussian temperature anisotropy maps with noise characteristics of the WMAP data. We generate 1000 Gaussian realizations of CMB temperature anisotropy using the HEALPix package (Górski et al. 2005). We use \( N_{\text{side}} = 128 \), 256, and 512 for \( \theta_s \geq 40', 40' > \theta_s \geq 10', \) and \( 10' > \theta_s \), respectively. The number of pixels is given by \( N_{\text{pix}} = 12 N_{\text{side}}^2 \). From each sky realization we construct eight simulated maps of WMAP differential assemblies (DAs), Q1, Q2, V1, V2, W1, W2, W3, and W4, by convolving the sky map with the beam transfer function in each DA (Page et al. 2003a) and adding independent Gaussian noise realizations following the noise pattern in each DA. Each pixel is given noise variance of \( \sigma_{0, \text{noise}}^2 / N_{\text{obs}}(\Omega_i) \), where \( N_{\text{obs}}(\Omega_i) \) is the number of observations per pixel, and \( \sigma_{0, \text{noise}} \) is given in Bennett et al. (2003a). We then co-add the eight maps by weighting each map by \( N_{\text{obs}} / \sigma_{0, \text{noise}}^2 \), where \( N_{\text{obs}} \) is the full-sky average of \( N_{\text{obs}}(\Omega_i) \). Finally, we mask the co-added map by the Kp0 galaxy mask (including point-source mask) provided by Bennett et al. (2003b). This mask leaves 76.8\% of the sky available for the subsequent data analysis. In addition to WMAP 1 yr data, we also simulate the future WMAP 8 yr data by simply multiplying \( N_{\text{obs}} \) by a factor of 8.

Before we estimate the MFs from each simulated map, we smooth it using a Gaussian filter with a smoothing scale of \( \theta_s \),

\[ W_T = \exp\left[-\frac{1}{2}(l+1)(\theta_s/\theta)^2\right]. \]

To remove the effect of the survey mask, we calculate the MFs by limiting the pixels where the five measurements, \( \sigma, \sigma_1, \) and \( S^{(k)}(k = 0, 1, \text{and} 2) \), have nearly equal values between the field with the survey mask and that without the survey mask. We use the pixels where the difference is within 5\% of each standard deviation for \( \theta_s = 20' \) and \( 40' \) and 10\% for \( \theta_s = 70' \) and \( 100' \). For \( \theta_s = 5' \) and \( 10' \), we use the pixels as long as they are away from the boundary of the mask by more than 2\( \theta_s \). Table 5 lists the sky fraction used in the analysis for each smoothing scale. The mean density over the pixels that are used for the calculation of MFs is not completely zero, and thus we subtract it from the density field to satisfy zero mean in each realization.

### A3. Simulating Planck Data

For the simulations of the Planck data, we follow the same procedure as the WMAP simulations. Each realization is a co-added map of nine bands with the inverse weight of the noise variance listed in Table 6. We approximate the beam transfer function as a Gaussian function, \( \exp\left[-\theta_{\text{beam}}^2 l(l+1)/2\right] \), where \( \theta_{\text{beam}} = \theta_{\text{FWHM}} [8 \ln(2)]^{1/2} \). We add the homogeneous noise distribution with the noise variance per pixel \( \sigma_{\text{noise}}^2 \) given by

\[ \frac{1}{\sigma_{\text{noise}}^2} = \sum_i \left( \frac{T}{\Delta T} \right)^{\nu^2} \frac{4\pi/n_{\text{pix}}}{\theta_{\text{FWHM}}^4} \right)^2, \]

where \( i \) denotes each band and \( n_{\text{pix}} = 12 N_{\text{side}}^2 \) is the number of pixels in simulated maps. We use the Kp0 mask to define the survey area for the Planck simulations, in exactly the same manner as for the WMAP simulations.
A4. COMPUTATIONAL METHOD: LSS

We use two complementary routines to compute the MFs of a density field on the grids. The first approach is often called Koenderink invariants (Koenderink 1984) in which the surface integrals of the curvature are transformed into the volume integral of invariants formed from the first and second derivatives of the density fluctuations. The second method, which is called Crofton’s formula (Crofton 1868; Schmalzing & Buchert 1997; Koenderink 1984), is based on the integral geometry, and the calculation reduces to simply counting the elementary cells (e.g., cubes, squares, lines, and points for the cubic meshes). The outline of these methods is summarized in Schmalzing & Buchert (1997), and the observational application to Sloan Digital Sky Survey galaxy samples is performed by Hikage et al. (2003).

A5. SIMULATING LSS DATA WITH PRIMORDIAL NON-GAUSSIANITY

We calculate the MFs of density field in a cubic box with a length of 1 h⁻¹ Gpc but ignore the observational effects such as survey geometry, for simplicity. We also ignore nonlinear gravitational clustering or galaxy bias in order to isolate the effect from primordial non-Gaussianity. (The purpose of this simulation is to check the accuracy of our perturbation predictions for the form of MFs from primordial non-Gaussianity.) To simulate the LSS data with primordial non-Gaussianity, we first generate a Gaussian potential field in a cubic box with a length of 1 h⁻¹ Gpc, assuming that the power spectrum of potential is \( P_g(k) \propto k^{n_s - 4} \), where \( n_s = 0.967 \). We inversely Fourier transform it into real space to obtain \( \delta(x) \). (The number of grids is 128³.) We then construct a non-Gaussian potential field, \( \Phi(x) \), using equation (1) for a given \( f_{NL} \). We finally convert it to the matter density field by multiplying \( \Phi \) by \( M(k) \) in Fourier space (see eq. [B10]). We have generated 2000 realizations of the non-Gaussian density field.

APPENDIX B
DERIVATION OF GALAXY BISPECTRUM

In this appendix we derive the perturbative formula for the galaxy bispectrum including primordial non-Gaussianity, nonlinearity in gravitational clustering, and nonlinearity in galaxy biasing, in the weakly nonlinear regime (Verde et al. 2000; Scoccimarro et al. 2004). In the weakly nonlinear regime, it would be reasonable to assume that the galaxy biasing is local and deterministic. We then expand the galaxy density contrast, \( \delta_g \), perturbatively in terms of the underlying matter density contrast, \( \delta_m \), as (Fry & Gaztanaga 1993)

\[
\delta_g(z) = b_0(z) + b_1(z)\delta_m(z) + \frac{b_2(z)}{2} \delta_m^2(z) + O(\delta_m^3),
\]

where \( b_0(z) \) is determined such that \( \langle \delta_g(z) \rangle = 0 \). Here, \( b_1(z) \) and \( b_2(z) \) are the time-dependent galaxy bias parameters. The power spectrum and bispectrum of the galaxy distribution, \( P_g \) and \( B_g \), respectively, are then given by those of the underlying matter distribution as

\[
P_g(k, z) = b_1^2(z) P_m(k, z),
\]

\[
B_g(k_1, k_2, k_3, z) = b_1^3(z) B_m(k_1, k_2, k_3, z) + b_1^2(z) b_2(z) [P_m(k_1, z) P_m(k_2, z) + \text{(cyc.)}] + O(\delta_m^5).
\]

If the underlying mass distribution obeyed Gaussian statistics, its bispectrum would vanish exactly, \( B_m \equiv 0 \); however, the nonlinear evolution of density fluctuations due to gravitational instability makes \( \delta_m \) slightly non-Gaussian in the weakly non-Gaussian regime, yielding a nonzero bispectrum.

The second-order correction to the density fluctuations from nonlinear gravitational clustering gives the following equation,

\[
\delta_m(k) = \delta_{L,k}(z) + \int d^3q F_2(k, k - q) \delta_{L,q}(z) \delta_{L,k - q}(z),
\]

where \( \delta_{L,k}(z) \) is the linear (but non-Gaussian) density fluctuations and

\[
F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{2k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1^2 k_2^2} \right)^2.
\]
is the time-independent kernel describing mode-coupling due to nonlinear clustering of matter density fluctuations in the weakly nonlinear regime. Equation (B5) is exact only in an Einstein–de Sitter universe, but the corrections in other cosmological models are small (e.g., Bernardeau 1994). The power spectrum and bispectrum of the underlying mass density distribution, \( P_m \) and \( B_m \), respectively, are thus given in terms of the linear and nonlinear contributions:

\[
P_m(k, z) = P_L(k, z),
\]

\[
B_m(k_1, k_2, k_3, z) = B_{\text{pr}}(k_1, k_2, k_3, z) + 2[f_{NL}(k_1, k_2)P_L(k_1, z)P_L(k_2, z) + \text{cyc.}].
\]

Note that we have ignored the nonlinear contributions in the power spectrum. That is to say, the power spectrum is still described by linear perturbation theory.

The remaining task is to relate \( \delta_L(z) \) to Bardeen’s curvature perturbations during the matter era, \( \Phi \). One may use Poisson’s equation for doing this:

\[
k^2 \delta \Phi k T(k) = 4\pi G \rho_m(z) \frac{\delta_{\text{m},k}(z)}{(1 + z)^2} = \frac{3}{2} \Omega_m H_0^2 \delta_{\text{m},k}(z)(1 + z),
\]

where \( T(k) \) is the linear transfer function that describes the evolution of density fluctuations during the radiation era and the interactions between photons and baryons (Eisenstein & Hu 1999). Note that \( \delta \Phi \) is independent of time during the matter era. At very early times, say, \( z = z_* \sim 10^3 \), the nonlinear evolution may be safely ignored at the scales of interest, and one obtains

\[
\delta_{L,k}(z) = \frac{M(k)\delta \Phi_k}{1 + z_*},
\]

where

\[
M(k) \equiv \frac{2}{3} k^2 T(k) \Omega_m H_0^2.
\]

Therefore, using the quadratic non-Gaussian model given in equation (1), one obtains the linear power spectrum at \( z = z_* \),

\[
P_L(k, z) = \frac{M^2(k)}{(1 + z_*)^2} P_\phi(k),
\]

and the primordial bispectrum at \( z = z_* \),

\[
B_{\text{pr}}(k_1, k_2, k_3, z) = 2f_{NL} M(k_1)M(k_2)M(k_3)\frac{P_\phi(k_1)P_\phi(k_2)}{(1 + z_*)^2}\left[ P_L(k_1, z)P_L(k_2, z)M(k_3) + \text{cyc.} \right]
\]

\[
= 2f_{NL}(1 + z_*) \left[ \frac{P_L(k_1, z_*)P_L(k_2, z_*)M(k_3)}{M(k_1)M(k_2)} + \text{cyc.} \right],
\]

where \( P_\phi(k) \propto k^{n_s-4} \) is the power spectrum of \( \phi \) and we have ignored the higher order terms. We then use the linear growth rate of density fluctuations, \( \delta_L \propto D(z) \), to evolve the linear bispectrum forward in time:

\[
B_{\text{pr}}(k_1, k_2, k_3, z) = 2f_{NL} \frac{1 + z_*)D(z_*)}{D(z)} \left[ \frac{P_L(k_1, z_*)P_L(k_2, z_*)M(k_3)}{M(k_1)M(k_2)} + \text{cyc.} \right].
\]

One may simplify this expression by normalizing the growth rate such that

\[
D(z_*) = \frac{1}{1 + z_*}.
\]

Note that this condition gives the normalization of \( D(z) \) that is actually independent of the choice of \( z_* \), when \( z_* \) is taken to be during the matter era.

By putting all the terms together, we finally obtain the following form of \( B_\delta(k_1, k_2, k_3, z) \):

\[
B_\delta(k_1, k_2, k_3, z) = 2f_{NL} \frac{b_\delta(z)}{D(z)} \left[ \frac{P_m(k_1, z)P_m(k_2, z)M(k_3)}{M(k_1)M(k_2)} + \text{cyc.} \right] + 2b_\delta(z)f_{NL}[P_L(k_1, z)P_L(k_2, z) + \text{cyc.} + b_\delta(z)b_\delta(z)[P_m(k_1, z)P_m(k_2, z) + \text{cyc.}].
\]

We use this formula to calculate the skewness parameters that are used for the MFs of the galaxy distribution.

While the perturbative formula for the MFs derived by Matsubara (2003) (see § 2) works well for \( \Phi \), it is not immediately clear if it works for \( \delta \) because of the \( k \)-dependent coefficient, \( \delta_{L,k} \propto M(k)\delta \Phi_k \). In Appendix D we show that the perturbative formula still works,
as long as $f_{\text{NL}}$ is not very large. The current observational constraints on $f_{\text{NL}}$ already guarantee that the perturbative formula for the MFs of the galaxy distribution provides an excellent approximation.

APPENDIX C

MFs OF CMB: ANALYTICAL FORMULA VERSUS SIMULATIONS IN THE SACHS-WOLFE LIMIT

We compare the perturbative formula of MFs for the CMB with Monte Carlo realizations of non-Gaussian temperature anisotropy in the Sachs-Wolfe regime, in order to check the accuracy of our formalism. The angular power spectrum is set to be $(l+1)C_{l}^{\text{SW}}/2\pi = 10^{-10}$. In the Sachs-Wolfe limit, $\Delta T^{\text{SW}}/T = -\Phi/3$, the non-Gaussian maps of CMB temperature anisotropy may be constructed from the Gaussian maps, $\Delta T_{G}/T$, by the following simple mapping:

$$\frac{\Delta T^{\text{SW}}}{T} = \frac{\Delta T_{G}}{T} - 3f_{\text{NL}} \left( \frac{\Delta T_{G}}{T} \right)^2 - \left( \left( \frac{\Delta T_{G}}{T} \right)^2 \right)^{1/2}.$$  \hspace{1cm} (C1)

We calculate the MFs from 6000 realizations of the non-Gaussian CMB maps and compare them with the perturbation predictions (eq. [2]). The skewness parameters can be calculated from the reduced bispectrum of the CMB in the Sachs-Wolfe limit,

$$b_{l_1l_2l_3} = -6f_{\text{NL}} \left( C_{l_1}^{l_2} C_{l_2}^{l_3} + C_{l_2}^{l_1} C_{l_3}^{l_1} + C_{l_3}^{l_1} C_{l_1}^{l_2} \right).$$  \hspace{1cm} (C2)

Figure 5 shows that the MFs from Monte Carlo realizations agree with the perturbation predictions very well. The comparison with the full simulations that include the full radiation transfer function will be reported elsewhere. (For subtleties in this comparison arising from pixelization and boundary effects, see §4.2.)

Fig. 5.—Comparison between the MFs calculated from the analytical perturbation predictions (solid lines) and the numerical simulations (symbols) in the Sachs-Wolfe limit. We have used $\theta_{s} = 100$ arcmin and $f_{\text{NL}} = 100$. [See the electronic edition of the Journal for a color version of this figure.]
APPENDIX D

VALIDITY OF PERTURBATIVE FORMULAE FOR nTH-ORDER CORRECTIONS OF PRIMORDIAL NON-GAUSSIANITY

We consider the primordial non-Gaussianity extended to nth-order corrections of a primordial potential field:

\[ \Phi = \epsilon \phi_0 + \frac{\epsilon^2 f_{NL}^{(1)}}{2!} (\phi_0^2 - \langle \phi_0^2 \rangle) + \frac{\epsilon^3 f_{NL}^{(2)}}{3!} \phi_0^3 + \frac{\epsilon^4 f_{NL}^{(3)}}{4!} \left( \phi_0^4 - \langle \phi_0^4 \rangle \right) + \cdots + \frac{\epsilon^n f_{NL}^{(n-1)}}{n!} \left( \phi_0^n - \langle \phi_0^n \rangle \right) + \cdots, \tag{D1} \]

where \( \epsilon \phi_0 \) is an auxiliary random Gaussian field and \( f_{NL}^{(n)} \) represents the coefficients of nth order of \( \epsilon \). For convenience, we separate the random Gaussian field into two parts; the \( \epsilon \) represents the amplitude of the primordial potential power spectrum, which is of order \( 10^{-5} \), and thereby the fluctuation of \( \phi_0 \) is of order unity. The parameter \( f_{NL} \) in the equation (1) corresponds to \( f_{NL}^{(1)/2} \).

The Fourier transform of \( \Phi \) is written by

\[ \hat{\Phi} = \epsilon \hat{\phi}_0 + \frac{\epsilon^2 f_{NL}^{(1)}}{2!} \hat{\phi}_0^{(2)} + \cdots + \frac{\epsilon^n f_{NL}^{(n-1)}}{n!} \hat{\phi}_0^{(n)} + \cdots, \tag{D2} \]

where \( \hat{\phi}_0^{(n)}(k) \) is the Fourier transform of the nth-order term, \( \hat{\phi}_0^n - \langle \hat{\phi}_0^n \rangle \), given by

\[ \hat{\phi}_0^{(n)}(k) = \frac{1}{(2\pi)^{3n-3}} \int d\mathbf{k}_1 \cdots \int d\mathbf{k}_{n-1} \hat{\phi}_0^{(n)}(k_1) \cdots \hat{\phi}_0^{(n)}(k_{n-1}) \hat{\phi}_0(k + k_1 + \cdots + k_{n-1}) - (2\pi)^3 \delta_D(k) \langle \hat{\phi}_0^n \rangle. \tag{D3} \]

The polyspectra of \( \Phi, P_{\Phi}^{(n)} \) \((n \geq 2)\), are defined by

\[ \langle \Phi(k_1) \cdots \Phi(k_n) \rangle_c = (2\pi)^3 \delta_D(k_1 + \cdots + k_n) P_{\Phi}^{(n)}(k_1, \cdots, k_n), \tag{D4} \]

where \( P_{\Phi}^{(2)} \) and \( P_{\Phi}^{(3)} \) correspond to the power spectrum and the bispectrum of \( \Phi \), respectively.

According to the diagrammatic method by Matsubara (1995), the lowest order of \( \epsilon \) for the connected part of the nth-order moment, called cumulants, are the \( n - 1 \) products of the quadratic moment as follows:

\[ P_{\Phi}^{(2)} = \epsilon^2 P_{\phi_0}, \tag{D5} \]

\[ P_{\Phi}^{(3)} = \epsilon^4 f_{NL}^{(1)} [ P_{\phi_0}(k_1) P_{\phi_0}(k_2) + \text{(cycl.)} ], \tag{D6} \]

\[ P_{\Phi}^{(4)} = \epsilon^6 f_{NL}^{(1)} [ P_{\phi_0}(k_1) P_{\phi_0}(k_2) P_{\phi_0}(k_1 + k_2) + \text{(sym.)(12)} ] + \epsilon^2 f_{NL}^{(2)} [ P_{\phi_0}(k_1) P_{\phi_0}(k_2) P_{\phi_0}(k_3) + \text{(cycl.)} ], \tag{D7} \]

\[ P_{\Phi}^{(5)} = \epsilon^4 f_{NL}^{(1)} [ P_{\phi_0}(k_1) P_{\phi_0}(k_2) P_{\phi_0}(k_3) + \text{(sym.)(60)} ], \tag{D8} \]

\[ + \epsilon^8 f_{NL}^{(1)} [ P_{\phi_0}(k_1) P_{\phi_0}(k_2) P_{\phi_0}(k_3) P_{\phi_0}(k_1 + k_2 + k_3) + \text{(sym.)(60)} ] + \epsilon^4 f_{NL}^{(2)} [ P_{\phi_0}(k_1) P_{\phi_0}(k_2) P_{\phi_0}(k_3) P_{\phi_0}(k_4) + \text{(cycl.)} ], \]

\[ \cdots \]

\[ P_{\Phi}^{(n)} = \epsilon^{2n-2} \sum_{d_1, d_2, \ldots, d_n} \prod_{m=1}^{n} (f_{NL}^{(m)})^{d_m} \prod_{k_a(d_1, d_2, \ldots, d_n)} P_{\phi_0}(k_a + k_{a+1}) + \text{(sym.)}, \tag{D9} \]

where (sym.)\((n)\) means the addition of \( n \) terms with the subscripts symmetric to the previous term and the edge \((AB)\) is one of the edges in a tree graph \((d_1, d_2, \ldots, d_n)\) that satisfies the condition that \( d_1 + 2d_2 + \cdots + nd_n = n - 2 \).

We obtain the nth polyspectrum of \( \delta_{L,k}, P_L^{(n)} \), by

\[ P_L^{(n)}(k_1, \cdots, k_{n-1}) = M(k_1) M(k_2) \cdots M(k_n) P_{\Phi}^{(n)}. \tag{D10} \]

The nth-order terms of the perturbative formula (eq. [1]) are represented by

\[ \frac{\langle \delta_L^{(n)} \rangle_c}{\langle \delta_L^{(2)} \rangle^{n/2}} = \frac{\langle \delta_L^{(n-2)} \delta_L \rangle_c}{\langle \delta_L^{(2)} \rangle^{n/2-1} \langle (\nabla \delta_L) (\nabla \delta_L) \rangle}, \quad \frac{\langle \delta_L^{(n-3)} (\nabla \delta_L) (\nabla \delta_L) \rangle_c}{\langle \delta_L^{(2)} \rangle^{n/2-2} \langle (\nabla \delta_L) (\nabla \delta_L) \rangle}, \tag{D11} \]

These terms are nth-order cumulants of the products of \( \delta_L, \nabla \delta_L, \) and \( \nabla^2 \delta_L \) divided by \( n/2 \) times the product of the corresponding combination of the second moments \( \langle \delta_L^{(2)} \rangle \) and \( \langle (\nabla \delta_L) (\nabla \delta_L) \rangle \).

The nth-order cumulants \( \langle \delta_L^{(n)} \rangle_c \) are obtained by the inverse Fourier transform of \( P_L^{(n)} \) as

\[ \langle \delta_L^{(n)} \rangle_c = \frac{1}{(2\pi)^3} \int d\mathbf{k} P_L^{(n)}(k) M(k) W(kR)^2 \tag{D12} \]

\[ \langle \delta_L^{(n)} \rangle_c = \frac{1}{(2\pi)^{3n-3}} \int d\mathbf{k}_1 \cdots \int d\mathbf{k}_{n-1} P_{\Phi}^{(n)} M(k_1) W(k_1R) \cdots M(k_n) W(k_nR). \tag{D13} \]
The $n$th-order term of the perturbative formula (eq. [1]) has the following order:

$$\frac{\langle \delta_L^n \rangle_c}{\langle \delta_L^2 \rangle_c^{n/2}} \sim \epsilon^{n-2} \sum_{d_1, d_2, \ldots, d_n} \left( \prod_{m=1}^{n} \left( f^{(m)}_{NL} \right)^{d_m} \right).$$  \hspace{1cm} (D14)

The other $n$th-order terms in equation (D11) have the same order of $\epsilon$ as $\langle \delta_L^n \rangle_c / \langle \delta_L^2 \rangle_c^{n/2}$.

The above equation is different from the well-known hierarchical condition from the gravitational evolution:

$$\langle f^n \rangle_c \sim \langle f^2 \rangle_c^{n-1}. \hspace{1cm} (D15)$$

Indeed, the skewness parameters due to the primordial non-Gaussianity have a scale dependence of $M(k)$. Nevertheless, the perturbation works well as long as equation (D14) is much smaller than unity, which corresponds to

$$|f^{(n)}_{NL}| \ll \epsilon^{-n} \sim 10^{4n}. \hspace{1cm} (D16)$$

Recent observations presented by WMAP (Komatsu et al. 2003) gave constraints on $|f^{(1)}_{NL}| < O(10^{-2})$. Standard inflation models predict that higher order coefficients are the same order as $f_{NL}$, and thus the perturbation is applicable to the actual primordial non-Gaussianity.

REFERENCES

Alishahiha, M., Silverstein, E., & Tong, D. 2004, Phys. Rev. D, 70, 123505
Arkani-Hamed, N., Creminelli, P., Mukohyama, S., & Zaldarriaga, M. 2004, J. Cosmol. Astropart. Phys., 04, 001
Babich, D., Creminelli, P., & Zaldarriaga, M. 2004, J. Cosmol. Astropart. Phys., 08, 009
Babich, D., & Zaldarriaga, M. 2004, Phys. Rev. D, 70, 083005
Bartolo, N., Komatsu, E., Matarrese, S., & Riotto, A. 2004, Phys. Rep., 402, 103
Bean, R., Dunkley, J., & Pierpaoli, E. 2006, Phys. Rev. D, 74, 063503
Bennett, C. L., et al. 2003a, ApJS, 148, 39
Bennett, C. L., et al. 2003b, ApJS, 148, 119
Bennett, C. L., et al. 2005, ApJ, 622, 759
Bernardeau, F. 1994, ApJ, 433, 1
Buchert, M., Dunkley, J., Ferreira, P. G., Moodley, K., & Skordis, C. 2004, Phys. Rev. Lett., 93, 081301
Cooray, A., & Hu, W. 2000, ApJ, 534, 533
Creminelli, P., Nicollin, A., Senatore, L., Tegmark, M., & Zaldarriaga, M. 2006, J. Cosmol. Astropart. Phys., 05, 004
Crofton, M. W. 1868, Philos. Trans. R. Soc. London A, 158, 181
Dvali, G., Gruzinov, A., & Zaldarriaga, M. 2004, Phys. Rev. D, 69, 083505
Eisenstein, D. J., & Hu, W. 1999, ApJ, 511, 5
Fry, J. N., & Gaztanaga, E. 1993, ApJ, 413, 447
Goldberg, D. M., & Spergel, D. N. 1999, Phys. Rev. D, 59, 103002
Görski, K. M., Hivon, E., Banday, A. J., Wandelt, B. D., Hansen, F. K., Reinecke, M., & Bartelman, M. 2005, ApJ, 622, 759
Gott, J. R., Illlott, A. L., & Dickinson, M. 1986, ApJ, 306, 341
Hikage, C., et al. 2003, PASJ, 55, 911
Koenderink, J. J. 1984, Biol. Cybern., 50, 363
Kogo, N., & Komatsu, E. 2006, Phys. Rev. D, 73, 083007
Komatsu, E., & Spergel, D. N. 2001, Phys. Rev. D, 63, 063002
Komatsu, E., Spergel, D. N., & Wandelt, W. D. 2005, ApJ, 634, 14
Komatsu, E., Wandelt, B. D., Spergel, D. N., Banday, A. J., & Górski, K. M. 2002, ApJ, 566, 19
Komatsu, E., et al. 2003, ApJS, 148, 119
Liguori, M., Hansen, F. K., Komatsu, E., Matarrese, S., & Riotto, A. 2006, Phys. Rev. D, 73, 043505
Lyth, D. H., Ungarelli, C., & Wanders, D. 2003, Phys. Rev. D, 67, 23503
Matsubara, T. 1994, ApJ, 434, L43
———. 1995, ApJS, 101, 1
———. 2003b, ApJS, 148, 1
Mecke, K. R., Buchert, T., & Wagner, H. 1994, A&A, 288, 697
Novikov, D., Schmalzing, J., & Mukhanov, V. F. 2000, A&A, 364, 17
Okamoto, T., & Hu, W. 2002, Phys. Rev. D, 66, 063008
Page, L., et al. 2003a, ApJS, 148, 39
———. 2003b, ApJS, 148, 233
Peiris, H. V., et al. 2003, ApJS, 148, 213
Schmalzing, J., & Buchert, T. 1997, ApJ, 482, L1
Schmalzing, J., & Górski, K. M. 1998, MNRAS, 297, 355
Scoccimarro, R., Sefusatti, E., & Zaldarriaga, M. 2004, Phys. Rev. D, 69, 103513
Seljak, U., & Zaldarriaga, M. 1996, ApJ, 469, 437
Seljak, U., et al. 2005, Phys. Rev. D, 71, 103515
Spergel, D. N., & Goldberg, D. M. 1999, Phys. Rev. D, 59, 103001
Spergel, D. N., et al. 2003, ApJS, 148, 175
———. 2006, ApJ, submitted (astro-ph/0603449)
Tegmark, M., et al. 2004, Phys. Rev. D, 69, 103501
Verde, L., & Spergel, D. N. 2002, Phys. Rev. D, 65, 043007
Verde, L., Wang, L.-M., Heavens, A., & Kamionkowski, M. 2000, MNRAS, 313, 141
Winitzki, S., & Kosowsky, A. 1998, NewA, 3, 75...