Duality and cosmological compactification of superstrings with unbroken supersymmetry

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Abstract

The cosmological compactification of $D = 10$, $N = 1$ supergravity–super–Yang–Mills theory obtained from superstring theory is studied. The constraint of unbroken $N = 1$ supersymmetry is imposed. A duality transformation is performed on the resulting consistency conditions. The original equations as well as the transformed equations are solved numerically to obtain new configurations with a nontrivial scale factor and a dynamical dilaton. It is shown that various classes of solutions are possible, which include cosmological solutions with no initial singularity.

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1 Introduction

String theory \cite{1}, at present, is the most promising candidate for a unified theory of gravity with other forces of nature. Therefore, there is considerable interest in investigating string effects on the evolution of the universe. In principle, string theory is expected to explain the curved nature of spacetime, provide a consistent description of the very early history of the universe and generate corrections to the cosmological evolution equations which are in agreement with current observations.

String theory is defined on a $D$–dimensional manifold ($D=26$ for bosonic strings and $D=10$ for supersymmetric strings). Thus to make a connection with physical four–dimensional – in particular, cosmologically relevant – geometries, the extra $D-4$ dimensions have to be compactified. Specifically, we require the $D$–dimensional manifold to be a direct product of the four–dimensional manifold and the $(D-4)$–dimensional compact internal space whose physical dimensions are of the order of the Planck scale.

Candelas et al. \cite{2} considered the compactification of ten–dimensional supergravity coupled to super–Yang–Mills theory, assuming unbroken supersymmetry and maximal symmetry in four dimensions. It was found that consistent compactification was possible only for Minkowskian geometry. The condition of maximal symmetry in four dimensions is restrictive, and relaxing this condition allows compactification to curved spacetime \cite{3}. Explicit solutions to the consistency conditions for unbroken supersymmetry were obtained \cite{4} within the ansatz of spherical symmetry for three–dimensional space. It was found that the scale factor has a Milne type evolution, with a constant dilaton field in four dimensions. In general, if a dynamical dilaton $\phi$ and a non–vanishing antisymmetric field strength tensor $H_{\mu\nu\rho}$ are considered, nontrivial evolution \cite{5} of the scale factor and the dilaton field is obtained in an open Friedmann–Robertson–Walker (FRW) universe.

Recent work in string cosmology has focussed on solutions of the
equations of motion obtained from the low–energy effective action \cite{1}, i.e., the action describing the dynamics of massless modes (the metric $g_{\mu\nu}$, the dilaton $\phi$ and the antisymmetric field $B_{\mu\nu}$) of the string. It was pointed out by Veneziano \cite{7} that some of the duality symmetries of string theory are manifested as symmetries of the low-energy field equations. In particular, he emphasised the importance of scale factor duality (SFD), which relates physically inequivalent solutions of the string modified Einstein–Friedmann equations. For an expanding universe with increasing scale factor $R(t)$, we have a dual scenario with $R(t) \to R(t)^{-1}$ which describes a contracting universe. Although the “proof” of SFD \cite{7} is valid only for spatially flat geometries, it seems reasonable to assume that it is more generally true (see, e.g., \cite{8}). Several attempts have been made \cite{9, 10, 11, 12, 13} to use SFD to solve the graceful exit problem of inflationary cosmology. This typically requires introducing by hand an \textit{ad-hoc} dilaton potential. At a deeper level, scale factor duality offers a way to solve the initial singularity problem of big-bang cosmologies \cite{12, 14}.

The above considerations, however, may be insufficient for a truly “stringy” description of cosmological evolution. For instance, more theoretical support is required to justify the form of the dilaton potential. For our purpose, we do not assume any explicit form for the effective action. Taking an alternative approach, we work within the framework of supersymmetric compactification. We are encouraged in this by the fact that unbroken supersymmetry in four dimensions, at some energy scale high compared to the electroweak scale, seems to be required in most viable theoretical scenarios \cite{15} for the unification of forces. Assuming that scale factor duality holds for spatially non-flat geometries, we explore the consequences on the consistency conditions for compactification with unbroken supersymmetry. In particular, we find out the four–dimensional geometries consistent with the duality transformed equations. It is shown that different classes of cosmological solutions are allowed, including those with no initial singularity. Since we do not assume any arbitrary potential, these considerations may, in general, be applicable to a wider range of cosmological evolution possibilities.
The paper is organised as follows. In section II we consider the compactification of ten–dimensional supergravity–super–Yang–Mills theory with \( N = 1 \) supersymmetry. It is shown that compactification with unbroken supersymmetry admits different classes of non-trivial cosmological solutions. Section III is devoted to the solutions of equations which are transformed under the duality transformations. We show that an entirely different class of cosmological solutions is obtained. Section IV contains some concluding remarks.

2 Compactification

We consider the compactification of ten–dimensional supergravity coupled to super–Yang–Mills theory \([16]\), i.e., the infinite tension limit of superstring theory. The 10–dimensional manifold is of the form \( M_D = M_4 \otimes K_6 \) where \( M_4 \) is the four–dimensional space-time and \( K_6 \) is the six–dimensional compact internal manifold whose physical dimensions are of the order of the Planck scale. For simplicity, we assume the FRW form for the metric in four–dimensional spacetime.

The fermionic fields in the supermultiplet are assumed to have vanishing background values. The supersymmetric transformations for the fermi fields for some Majorana–Weyl spinor \( \epsilon \) are

\[
\delta \varphi_\mu = \nabla_\mu \epsilon + \frac{1}{48} e^{2\phi} (\gamma_\mu H_{\rho\sigma\gamma}^{\rho\sigma\gamma} - 12 H_{\mu\rho\sigma}^{\rho\mu\sigma} + \gamma_\mu \gamma_5 \otimes H) \epsilon \tag{1}
\]

\[
\delta \lambda = \gamma^\mu (\nabla_\mu \phi) \epsilon + \frac{1}{24} e^{2\phi} H_{\rho\sigma\gamma}^{\rho\mu\sigma} \epsilon + \gamma_5 \otimes [\gamma^m (\nabla_m \phi) + \frac{1}{24} e^{2\phi} H] \epsilon \tag{2}
\]

Here \( \gamma^{\rho\nu} = \frac{1}{2} \gamma^{[\rho} \gamma^{\nu]} \) and \( H \equiv H_{mnp} \gamma^{mnp} \), \( H_{\mu\nu\rho} \) being the field strength tensor for the antisymmetric (torsion) tensor. The Greek indices
represent the four–dimensional spacetime, m, n... represent the six–dimensional compactified manifold and M, N... denote the full ten–dimensional spacetime.

Since we demand spacetime supersymmetry for the ground state, \( \delta \varphi_\mu \) and \( \delta \lambda \) must vanish for any arbitrary \( \epsilon \). The requirement of maximal symmetry in three dimensions leads us to the following ansatz for \( H_{\mu\nu\rho} \) and the dilaton field \( \phi \)

\[
\phi = \phi(r, t) \quad (3)
\]

\[
H_{\mu\nu\rho} = e^{l(r)} \varepsilon_{\mu\nu\rho\sigma} b^\sigma \quad (4)
\]

where \( l(r) \) is any arbitrary function of \( r \) and \( b^\sigma \) is a constant timelike vector.

Taking the dilaton field to be a constant in internal space \[17\], i.e., \( \nabla_m \phi = 0 \), the requirement of unbroken supersymmetry, using equations (1) and (2), yields

\[
[\nabla_\mu, \nabla_\lambda] \epsilon = \frac{1}{2} \left[ \gamma_\lambda (\nabla_\mu \nabla_\sigma \phi) - \gamma_\mu (\nabla_\lambda \nabla_\sigma \phi) \right] \epsilon \\
+ \frac{1}{4} \left[ \gamma_\lambda \gamma^\sigma \gamma_\mu \gamma^\tau - \gamma_\mu \gamma^\sigma \gamma_\lambda \gamma^\tau \right] (\nabla_\sigma \phi)(\nabla_\tau \phi) \epsilon \\
+ \frac{1}{4} \left[ \nabla_\mu (2\phi + l) \varepsilon_{\lambda\rho\sigma} - \nabla_\lambda (2\phi + l) \varepsilon_{\mu\rho\sigma} \right] \gamma^{\rho\nu} b^\sigma b^\tau \epsilon \\
+ \frac{1}{16} e^{4\phi + 2l} \left[ \varepsilon_{\lambda\rho\sigma} \varepsilon_{\mu\delta\eta\tau} - \varepsilon_{\mu\rho\sigma} \varepsilon_{\lambda\delta\eta\tau} \right] \gamma^{\rho\nu} \gamma^{\delta\eta} b^\sigma b^\tau \epsilon
\]

As in Candelas et al. \[2\], we have \( [\nabla_\mu, \nabla_\lambda] \epsilon = -\frac{1}{4} R_{\mu\lambda\rho\nu} \gamma^{\mu\rho} \epsilon \), where \( R_{\mu\lambda\nu\rho} \) is the Riemann curvature tensor. To avoid torsion in four–dimensions, we do not consider the case when both \( \phi \) and \( H_{\mu\nu\rho} \) have nontrivial behaviour, i.e., either we take the dilaton field to be a constant in four dimensions or we take \( H_{\mu\nu\rho} = 0 \). The above integrability condition gives the Riemann tensor in terms of
fields $\phi(t)$ and $H_{\mu\nu\rho}$, which in turn gives rise to a set of coupled equations, which are similar to the equations of motion obtained from the action in scalar–tensor theories of gravity. These equations can be solved to find four-dimensional geometries consistent with the requirement of unbroken supersymmetry.

It was shown \[5\] that, in the case where the dilaton field is a constant, i.e., for $\nabla_\mu \phi = 0$, we have

$$\frac{\dddot{R}}{R} = 0 \quad (5)$$

$$\dddot{R}^2 + c\dddot{R}^2 + k = 0 \quad (6)$$

Here we have imposed $\nabla_\mu l = 0$, and the constant $c$ is defined by $c = e^{4\phi + 2l}$. Clearly, the only solution in this case is the Minkowskian geometry.

If $H_{\mu\nu\rho} = 0$, then for a time dependent dilaton field, the equations take the form \[5\]

$$\frac{\dddot{R}}{R} - \dddot{\phi} = 0 \quad (7)$$

$$\dddot{R}^2 + \dddot{R}^2 \dddot{\phi}^2 + k = 0 \quad (8)$$

If $\dddot{\phi} = 0$, we again get the Minkowskian solution. To get consistent nontrivial solutions, $k$, the FRW spatial curvature factor cannot have any value greater than or equal to zero. Thus, we have $k = -1$, corresponding to an open universe.

Numerical solutions of these equations \[5\] show that the dilaton field approaches a linear growth in time while the scale factor grows subluminally (see Fig.(1) and Fig.(2)). This shows that compactifi-
cation of 10–dimensional supergravity with unbroken supersymmetry is consistent with nontrivial evolution for the scale factor and the dilaton field in four dimensions.

In addition to the solutions presented in [5], there also exists a class of one–parameter solutions with the initial conditions \( R(0) = 1, \dot{R}(0) = \alpha, \dot{\phi}(0) = \beta, \) with \( \alpha^2 + \beta^2 = 1. \) The numerical solutions with these initial conditions are shown in Fig.(3) and Fig.(4). It is clear that the consistency conditions allow non–singular solutions. Note that again the dilaton field approaches a linear growth but the scale factor approaches a constant for large time.

In general, the equations can be reparametrised as

\[
\dot{R}(t) = \cos \theta(t)
\]

\[
R(t)\dot{\phi}(t) = \sin \theta(t)
\]

which give the following solutions

\[
R(t) = Ae^{\theta(t)} \sin \theta(t)
\]

\[
\dot{\phi}(t) = \frac{1}{A} e^{-\theta(t)}
\]

Interestingly, the non–singular solutions obtained above are a special case of this more general parameterization, i.e., that corresponding to

\[
Ae^{\theta(0)} \sin \theta(0) = 1
\]
3 Dual Equations

We now transform the consistency conditions (equations (7) and (8)) under SFD, i.e., the scale factor is replaced by its inverse, namely

\[ R(t) \rightarrow \frac{1}{R(t)} \]  

(9)

It was shown [7] that the string–modified Einstein–Friedmann equations are invariant under the above transformation if the time–dependent dilaton field \( \phi(t) \) transforms nontrivially as

\[ \phi(t) \rightarrow \phi(t) - \ln|g_{ii}| \]  

(10)

where the spatial indices \( i \) are summed over. For an FRW metric with scale factor \( R(t) \), the transformation for the dilaton field translates to

\[ \phi(t) \rightarrow \phi(t) - 6 \ln R(t). \]  

(11)

We make a working hypothesis that the transformation given by eqs. (9) and (11) describes a duality symmetry of the theory even though we cannot establish it in the absence of an action. We adopt the point of view that, just as the consistency conditions (eqs. (7) and (8)) lead to cosmological background configurations for the original theory, the transformed equations describe cosmological configurations for the dual theory. Applying the duality transformations to equations (7) and (8), we get

\[ 5 \frac{\dddot{R}}{R} - 4 \left( \frac{\dot{R}}{R} \right)^2 - \dddot{\phi} = 0 \]  

(12)

\[ 37 \dot{R}^2 + R^2 \dddot{\phi} - 12 R \dot{R} \ddot{\phi} + k R^4 = 0 \]  

(13)
Changing to a convenient variable $u = \frac{1}{t}$, the equations can be rewritten as

$$5u\frac{d^2R}{du^2} + 10\frac{dR}{du} - \frac{4}{R}\left(\frac{dR}{du}\right)^2 - Ru\frac{d^2\phi}{du^2} - 2Ru\frac{d\phi}{du} = 0 \quad (14)$$

$$37u^4\left(\frac{dR}{du}\right)^2 + R^2u^4\left(\frac{d\phi}{du}\right)^2 - 12Ru^4\frac{dR}{du}\frac{d\phi}{du} + kR^4 = 0 \quad (15)$$

Again, for consistent and nontrivial solutions, we can only have open FRW configurations, i.e., $k = -1$.

For small $u$, the scale factor $R$ and $\chi \equiv \frac{d\phi}{du}$ have the following leading order behaviour

$$R \xrightarrow{u \to 0} u$$

$$\chi \xrightarrow{u \to 0} \frac{c}{u}$$

It is convenient to make the transformation of variable given by

$$R = uf_1(u) \quad (16)$$

$$\chi = \frac{c}{u}f_2(u) \quad (17)$$

where $f_1(0) = 1$, $f_2(0) = 1$ and $f_1'(0)$ is a free parameter. It is interesting to note that the value of the constant $c$ is fixed by
the equations themselves; on substituting equations (16) and (17) in equations (14) and (15), it turns out that \( c = 6 \).

In terms of the new variables \( f_1 \) and \( f_2 \), equations (14) and (15) take the form

\[
5u^2 f_1'' + 10(u f_1' + f_1) - \frac{4}{f_1}(u f_1' + f_1)^2 - 6u f_1' f_2' - 6f_1 f_2 = 0 \quad (18)
\]

\[
37(u f_1' + f_1)^2 + 36 f_1^2 f_2^2 - 72 f_1 f_2 (u f_1' + f_1) - f_1^4 = 0 \quad (19)
\]

Numerical solutions of these equations, showing the evolution of \( R \) and \( \chi \) as functions of \( u \), are shown in Fig.(5) and Fig.(6).

The leading order analysis of equations (14) and (15) shows the following asymptotic behaviour for \( R \) and \( \chi \)

\[
R \quad \stackrel{u \to \infty}{\longrightarrow} \quad A + \frac{B}{u} \\
\chi \quad \stackrel{u \to \infty}{\longrightarrow} \quad \frac{C}{u^2}
\]

This can be considered as a consistency check for the numerical solutions of the duality transformed equations.

If, instead of \( u \), we consider the evolution with respect to \( t (= \frac{1}{u}) \) the solutions behave as shown in Fig.(7) and Fig.(8). Clearly, the scale factor decreases linearly with \( t \) while \( \chi \) is proportional to \( t^2 \), as expected from the argument above. Notice that for \( t \to 0 \) the scale factor approaches a finite value; thus these solutions have no initial singularity. From the behaviour of \( \chi \) it is clear that the dilaton field
approaches a constant for large \( u \) (see Fig.(9)). We can see this more explicitly by plotting \( \phi \) against \( t \) (see Fig.(10)), i.e., the dilaton field is constant for small \( t \) while it vanishes as \( t \) becomes large. This shows an entirely different evolution behaviour from the one we had obtained in the case of the original equations.

4 Summary and Conclusions

We have considered the compactification of ten dimensional supergravity coupled to super–Yang–Mills theory and obtained numerical solutions to the consistency conditions for unbroken supersymmetry in four dimensions. It is clear that supersymmetric compactification allows nontrivial geometries which are of cosmological interest. In the original form, the solutions can describe the evolution of an FRW universe at late times. The dual solutions, on the other hand, correspond to interesting nonsingular cosmologies at early times. These features are similar to those found by other authors who solve the low- energy equations of motion. These results are a reflection of the fact that scale factor duality does not simply reparametrise the equations, but relates two different physical domains of the theory.

In the above, “early times” refers to the time scale at which supersymmetry is unbroken. Our classical configurations, however, should not be extrapolated back to times of the order of the Planck scale. The “stringy” dynamics of the universe at or before the Planck epoch is a subject of current interest \[12, 14\]. Work on quantum effects within the overall scheme of supersymmetric compactification is in progress.

One objection to our approach could be that supersymmetry consistency conditions are not, after all, equations of motion. Can we, without invoking the latter, make definite statements about dynamics? Our answer is yes. The uncertainties inherent in our approach are no greater than the uncertainty regarding the form of the dilaton potential in other approaches \[9, 10, 11, 12\]. We can go a step further, and conjecture that the strong “equation of motion” look of
the consistency conditions is not accidental. There is a deep dynamical content in these equations which, we believe, will be discovered at some time in the future.

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Figure Captions

1. Evolution of the scale factor ref. [5].

2. Evolution of the dilaton field ref. [5].

3. The scale factor $R(t)$—new solutions.

4. The dilaton field $\phi(t)$—new solutions.

5. Solutions to the dual equations—$R(u)$ vs $u$.

6. Solutions to the dual equations—$\chi(u)$ vs $u$.

7. Solutions to the dual equations—$R(t)$ vs $t$.

8. Solutions to the dual equations—$\chi(t)$ vs $t^2$.

9. Solutions to the dual equations—$\phi$ vs $u$.

10. Solutions to the dual equations—$\phi(t)$ vs $t$. 
FIG. 1
FIG. 3
FIG. 4
\[f'_1 = -5\]
\[f'_1 = -10\]
\[f'_1 = -15\]
\[f'_1 = -20\]
\[f'_1 = -25\]
\[ f_1' = -5 \]

\[ f_1' = -25 \]

FIG. 6
FIG. 7
\[ f_1 = -5 \quad f_1' = -10 \quad f_1 = -15 \quad f_1' = -20 \quad f_1' = -25 \]

**FIG. 8**
\[
f_1' = -5 \\
f_1' = -25
\]
