SEPARATION OF THE TWO-DIMENSIONAL UNSTEADY PRANDTL BOUNDARY LAYER UNDER AN ADVERSE PRESSURE GRADIENT

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Abstract. In this paper, we study the separation of the two-dimensional unsteady Prandtl boundary layer under an adverse pressure gradient. For monotonic initial data we prove that the first critical point of the tangential velocity profile with respect to the normal variable, if exists, must appear on the boundary if the pressure gradient of the outer flow is adverse. Moreover, we give a condition on the initial tangential velocity and outer flow such that there exists separation of the Prandtl boundary layer under the adverse pressure gradient. Finally, we introduce two examples showing that separation occurs either when the flow distance is long in the streamwise direction for a given initial monotonic tangential velocity field, or when the initial tangential velocity grows slow in a large neighborhood of the boundary for a fixed flow distance in the streamwise direction.

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1. INTRODUCTION

This paper is devoted to the study of separation of the two-dimensional unsteady boundary layer under an adverse pressure gradient. Consider the following problem for the Prandtl boundary layer equation with non-slip boundary condition for an unsteady incompressible flow in a domain $Q_T = \{(t, x, y)|0 \leq t < T, 0 \leq x \leq L, 0 \leq y \leq H\}$:

\begin{align*}
\begin{aligned}
  \partial_t u + u \cdot \nabla u + v \cdot \nabla v &= \frac{1}{\Pr} \Delta v, \\
  \partial_t v + u \cdot \nabla v + v \cdot \nabla u &= -\nabla p + \frac{1}{\Pr} \Delta u, \\
  \nabla \cdot v &= 0, \\
  u \cdot \nabla v &= 0, \\
  v \cdot \nabla u &= 0, \\
  v(t, 0, y) &= v(t, L, y), \\
  u(t, x, 0) &= u(t, x, H), \\
  u(t, x, y) &= u(t, x, y) = 0.
\end{aligned}
\end{align*}

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\[ y < +\infty \}, \]

\[
\begin{align*}
\partial_t u + u\partial_x u + v\partial_y u &= \partial_y^2 u - \partial_x P, \\
\partial_x u + \partial_y v &= 0, \\
u|_{y=0} &= v|_{y=0} = 0, \quad \lim_{y \to +\infty} u(t, x, y) = U(t, x), \\
u|_{t=0} &= u_0(x, y), \quad u|_{x=0} = u_1(t, y),
\end{align*}
\]

where \((u(t, x, y), v(t, x, y))\) is the velocity field in the boundary layer, \(U(t, x)\) and \(P(t, x)\) are traces at the boundary \(\{y = 0\}\) of the tangential velocity and pressure of the Euler outer flow respectively, interrelated through Bernoulli’s law

\[
\partial_t U + U\partial_x U = -\partial_x P.
\]

The equation in (1.1) was first proposed by Prandtl (27) to describe the behavior of boundary layers in the small viscosity limit for the incompressible viscous flow with non-slip boundary condition. So far, there are some interesting results on well-posedness of problems for the Prandtl equation. In [25], Oleinik obtained the well-posedness of the problem (1.1) locally in time, for data satisfying

\[ u_0(x, y) > 0, \quad u_1(t, y) > 0, \quad \forall t \in [0, T], \quad x \in [0, L], \quad y \in [0, \infty) \]

and the monotonicity assumption,

\[ \partial_y u_0(x, y) > 0, \quad \partial_y u_1(t, y) > 0, \quad \forall t \in [0, T], \quad x \in [0, L], \quad y \in [0, \infty) \]

This result was surveyed in the monograph [26]. Recently, the local well-posedness result in the monotonic class was also obtained in the Sobolev spaces in [1] and [22] by using the energy method. When this monotonicity assumption does not hold for the initial data, there are several interesting results on blowup or instability of solutions to the problem (1.1), cf. [6,10,12,13,17] and references therein. The monotonicity assumption of the tangential velocity is believed to be essential for the well-posedness of (1.1) in the two-dimensional problem, except in the spaces of analytic functions, cf. [15,16,21,23,29,36] or Gevrey functions ([11,18,20]).

According to the phenomena observed in fluid mechanics, cf. [27,28], when the pressure gradient is favourable, i.e. the pressure in the outer flow is decreasing in the streamwise direction,

\[ \partial_x P(t, x) \leq 0, \quad \forall t > 0, \quad x \in [0, L] \]

the Prandtl boundary layer is expected to be stable globally in time. Mathematically, Xin and Zhang ([35]) obtained a global existence of a solution to the two-dimensional Prandtl equation for monotonic data providing that the pressure is favourable in the sense of (1.5). On the other hand, adverse pressure gradients may lead to the phenomenon of boundary layer separation, because an adverse pressure gradient will retard the fluid in the boundary layer, then there may exist a separation point, behind which the flow follows the pressure gradient and moves...
in a direction opposite to the outer flow. The point of separation is defined as 
\((t_0, x_0, 0)\) at which

\[
\partial_y u(t_0, x_0, 0) = 0
\]

and

\[
\partial_y u(t, x, 0) > 0, \quad \forall 0 < x < x_0, \ 0 < t < t_0.
\]

From mathematical point of view, the monotonicity condition is violated when the point of separation exists, and thus the problem (1.1) may be ill-posed when \(t\) across \(t_0\).

As pointed out in [8], the importance of boundary layer separation lies in the fact that it is the mechanism through which the vorticity generated at the boundary is ejected into the outer flow, and it is considered as one of the important factors for the transition to turbulence in wall-bounded fluids. The seminal work of van Dommelen and Shen [33, 34] clarified how separation is related to the formulation of singularity for the solution of the Prandtl equation, and raised the question when and how the boundary layer separates. Since then, many works have been done for studying singularities of solutions numerically to the Prandtl equation, e.g. in [7, 9, 32] and references therein. To our knowledge, the first theoretical result on the appearance of separation in the two-dimensional steady Prandtl boundary layer under an adverse pressure gradient was given in [24, 31] (see also [26]). The asymptotic behavior of flow near the separation point was formally investigated by Goldstein [13], and the rigorous analysis of separation for the two-dimensional steady Prandtl equation was first carried out by E and Caffarelli in an unpublished manuscript mentioned in [5], then was developed recently in detail by Dalibard and Masmoudi in (4) when \(\partial_x P \equiv 1\).

The separation in unsteady flow has been studied by many mechanicians experimentally and numerically, cf. in [3, 30], the review article [2] and references therein. However, there is no any mathematical theory on separation of unsteady boundary layers. The goal of this paper is to study rigorously the existence of a separation point for the problem (1.1) of the unsteady Prandtl equation when the data satisfy the monotonicity assumption (1.4), but with an adverse pressure gradient.

The main result of this paper is as follows:

**Theorem 1.1.** (1) Assume that the trace at the boundary of the outer Euler flow satisfies

\[
U \in C^1([0, T) \times [0, L]) \text{ and } U(t, x) > 0, \forall t \in [0, T), \ x \in [0, L],
\]

and the uniformly adverse pressure gradient in the sense that

\[
\partial_x P(t, x) > 0, \quad \forall t \in [0, T), \ x \in [0, L].
\]
Let \((u, v)\) be the local classical solution to the problem \((1.1)\) corresponding to the data \(u_0, u_1\) satisfying \((1.3)\) and the monotonicity condition \((1.4)\). Then, the first zero point of \(\partial_y u(t, x, y)\) should be at the boundary \(\{y = 0\}\), if it exists for some \(t > 0\).

(2) Moreover, when the initial velocity \(u_0(x, y)\) satisfies
\[
\int_0^\infty \int_0^L \frac{\partial_y u_0}{\sqrt{(\partial_y u_0)^2 + u_0^2}} (L - x)^{\frac{3}{2}} dxdy \geq C_*,
\]
for a positive constant \(C_*\) depending only on \(L, T, U\) and \(\partial_x P\), then there is a separation point \((t^*, x^*)\) \(\in (0, T) \times [0, L]\), such that
\[
\begin{cases}
\partial_y u(t^*, x^*, 0) = 0, \\
\partial_y u(t, x, y) > 0, \quad \forall 0 < t < t^*, x \in [0, L], y \geq 0.
\end{cases}
\]
Moreover, we have \(\partial_y^2 u(t^*, x^*, 0) \neq 0\).

To get the conclusion, we shall prove the first result given in Theorem 1.1 by using a contradiction argument and the maximum principle for a scalar degenerate parabolic problem derived from \((1.1)\) by employing the Crocco transformation. Then, the existence of a separation point will be obtained by developing a Lyapunov functional argument.

This paper is organized as follows. In Section 2, we investigate the possible location of the first critical point of the tangential velocity profile in the Prandtl boundary layer. Then, we obtain the existence of a separation point under the assumption \((1.10)\) in Section 3. Finally, in Section 4, we introduce examples to show that the condition \((1.10)\) of separation holds, either when the flow distance \(L\) is large in the streamwise direction for a given initial monotonic velocity \(u_0(x, y)\), or when the initial tangential velocity \(u_0(x, y)\) grows slow with respect to \(y\) in a large neighborhood of the boundary \(\{y = 0\}\) for a fixed flow distance \(L\).

2. Position of the First Critical Point of the Tangential Velocity

In this section, we shall prove that the first possible critical point of the tangential velocity \(u(t, x, y)\) with respect to \(y\) should be on the boundary \(\{y = 0\}\).

Since the initial data given in \((1.1)\) are strictly monotonic as given in \((1.4)\), there is a local classical solution to the problem \((1.1)\) for \(0 \leq t < T\), in the class of \(\partial_y u > 0\) for all \(y \geq 0\), as obtained in \([25, 26]\) and \([1, 22]\). In this monotonic class, as in \([26]\), the following Crocco transformation is invertible,
\[
\tau = t, \quad \xi = x, \quad \eta = \frac{u(t, x, y)}{U(t, x)},
\]
and with \((2.1)\),
\[
w(\tau, \xi, \eta) = \frac{\partial_y u(t, x, y)}{U(t, x)},
\]
satisfies the following initial boundary value problem in $Q_T^* = \{(\tau, \xi, \eta)| 0 \leq \tau < T, \ \xi \in [0, L], \ \eta \in [0, 1]\}$,

\[
\begin{aligned}
\begin{cases}
\partial_\tau w + \eta U \partial_\xi w + A \partial_\eta w + Bw = w^2 \partial_\eta w, \\
w \partial_\eta w|_{\eta=0} = \frac{\partial P}{U}, \ \ w|_{\eta=1} = 0, \\
w|_{\tau=0} = w_0 := \frac{\partial u_0}{U}, \ \ w|_{\xi=0} = w_1 := \frac{\partial v_0}{U},
\end{cases}
\end{aligned}
\]

(2.3)

where $A = (1 - \eta^2) \partial_\xi U + (1 - \eta) \frac{\partial U}{U}$, and $B = \eta \partial_\xi U + \frac{\partial U}{U}$.

In this section, we shall have the following result:

**Proposition 2.1.** Under the same assumption as given in Theorem 1.1(1), let $(u, v)$ be a local classical solution to the problem (1.1), then the first critical point of $u(t, x, y)$ with respect to $y$, if exists, can only be at the boundary $\{y = 0\}$.

To prove this proposition, we first have the following lemma.

**Lemma 2.1.** Let $w$ be a local classical solution to the problem (2.3) for $0 \leq \tau < T$. Under the same assumption as given in Proposition 2.1, there is a constant $C_0$ depending only on $U$ and $T$, such that we have

\[
\sup_{Q_T^*} w^2(\tau, \xi, \eta) \leq C_0 \max \left( \sup_{0 \leq \xi \leq L, 0 \leq \eta \leq 1} w_0^2(\xi, \eta), \sup_{0 \leq \tau < T, 0 \leq \eta \leq 1} w_1^2(\tau, \eta) \right).
\]

(2.4)

Proof. For the solution $w$ of the problem (2.3), set $f(\tau, \xi, \eta) := e^{-N\tau}w^2$ with $N$ satisfying $N + 2B \geq 0$. Then, from (2.3), we know that the function $f(\tau, \xi, \eta)$ satisfies the following problem in $Q_T^*$,

\[
\begin{aligned}
\begin{cases}
\partial_\tau f + \eta U \partial_\xi f + A \partial_\eta f + (N + 2B)f = w^2 \partial_\eta f - 2e^{-N\tau}w^2(\partial_\eta w)^2, \\
\partial_\eta f|_{\eta=0} = \frac{\partial P}{U}e^{-N\tau}, \ \ f|_{\eta=1} = 0, \\
f|_{\tau=0} = w_0^2, \ \ f|_{\xi=0} = w_1^2 e^{-N\tau}.
\end{cases}
\end{aligned}
\]

(2.5)

From the adverse pressure gradient condition (1.9), we know that $\partial_\eta f$ is positive at $\{\eta = 0\}$, thus $f$ can not attain its supremum in $Q_T^*$ at $\{\eta = 0\}$. Applying the maximum principle of degenerately parabolic equations for the problem (2.4), we deduce that $f$ attains its supremum in $Q_T^*$ at $\{\tau = 0\}$ or $\{\xi = 0\}$, from which we get the conclusion (2.4) immediately. \hfill \Box

Proof of Proposition 2.1. We prove this proposition by a contradiction argument. Assume the first critical point of $u$ with respect to $y$ is an inner point $(t_0, x_0, y_0)$ with $0 < t_0 < T, \ 0 \leq x_0 \leq L, \ 0 < y_0 < \infty$. Then for any $0 \leq t < t_0$, the Crocco transformation given in (2.1) is invertible. For a sufficiently small $\epsilon$, by continuity of $u$ in $\tau$, there must exist a point $\frac{\epsilon}{4} < \tau_0 < t_0$ near $t_0$ such that

\[
0 < w(\tau_0, \xi_0, \eta_0) < \epsilon^2,
\]

(2.6)

where $\xi_0 = x_0$ and $\eta_0 = \frac{u(\tau_0, \xi_0, \eta_0)}{U(\tau_0, \xi_0)}$. Moreover, we have

\[
w(\tau, \xi, \eta) > 0 \ \ \text{in} \ \{(\tau, \xi, \eta)| 0 \leq \tau \leq \tau_0, \ 0 \leq \xi \leq \xi_0, \ 0 \leq \eta < 1\}.
\]

(2.7)
(1) Consider a function $F(\tau, \xi, \eta)$ defined in
\[ D_\epsilon = \{ (\tau, \xi, \eta) | 0 \leq \tau \leq \tau_\epsilon, \ 0 \leq \xi \leq \xi_0, \ 0 \leq \eta \leq 1 \} \]
as
\[ F(\tau, \xi, \eta) := w(\tau, \xi, \eta) - \epsilon \phi(\eta) e^{-M\tau}, \]
where parameters $M$ and $\epsilon$ will be determined later, $\phi \in C^2([0, 1])$ satisfies
\[ \phi(\eta) = \begin{cases} \frac{1}{2\eta_1} \eta, & \eta \in [0, \eta_1], \\ 1, & \eta = \eta_0, \\ 1 - \eta, & \eta \in [\eta_2, 1], \end{cases} \]
for some $\eta_1, \eta_2 \in (0, 1)$ satisfying $\eta_1 < \eta_0 < \eta_2$, moreover, $\partial_\eta \phi \geq 0$ for any $\eta \in [0, \eta_0]$, and $\partial_\eta \phi \leq 0$ for any $\eta \in [\eta_0, 1]$.

From (2.3), we know that $F$ satisfies the following problem in $D_\epsilon$,
\[ \left\{ \begin{array}{l} \partial_\tau F + \eta U \partial_\xi F + A \partial_\eta F + BF = w^2 \partial_\eta^2 F + \epsilon F, \\ F|_{\eta=0} = w(\tau, \xi, 0), \ F|_{\eta=1} = 0, \\ F|_{\tau=0} = w_0 - \epsilon \phi(\eta), \ F|_{\xi=0} = w_1 - \epsilon \phi(\eta) e^{-M\tau}, \end{array} \right. \tag{2.8} \]
where
\[ F = M \phi(\eta) e^{-M\tau} - A \partial_\eta \phi(\eta) e^{-M\tau} - B \phi(\eta) e^{-M\tau} + w^2 \partial_\eta^2 \phi(\eta) e^{-M\tau}. \]

(2) We claim that $F \geq 0$ in $D_\epsilon$ when $M$ is properly large. To this end, divide the interval $[0, 1]$ of $\eta$ into three parts, $[0, 1] = [0, \eta_1] \cup [\eta_1, \eta_2] \cup (\eta_2, 1]$ and study $F$ in these subintervals respectively.

i) When $\eta \in [0, \eta_1)$, from the definition of $\phi(\eta)$ we know
\[ -A \partial_\eta \phi(\eta) e^{-M\tau} - B \phi(\eta) e^{-M\tau} = -\frac{1}{2\eta_1} \left[ \partial_\xi U + \frac{\partial_\eta U}{U} \right] e^{-M\tau} \]
\[ = \frac{1}{2\eta_1} \frac{\partial_\xi P}{U} e^{-M\tau} > 0, \]
by using the assumption (1.9).

Noticing that $\partial_\eta^2 \phi(\eta) = 0$ on $[0, \eta_1)$, we have
\[ F(\tau, \xi, \eta) > M \phi(\eta) e^{-M\tau} \geq 0, \ \forall \eta \in [0, \eta_1) \]
for any non-negative $M$.

ii) When $\eta \in [\eta_1, \eta_2]$, by using Lemma 2.1 we know that $A \partial_\eta \phi(\eta) + B \phi(\eta) - w^2 \partial_\eta^2 \phi(\eta)$ is bounded. Since $\phi(\eta) \geq \min\{\frac{1}{2}, 1 - \eta_2\}$ on $[\eta_1, \eta_2]$, we can choose $M$ large enough such that
\[ M \phi(\eta) - A \partial_\eta \phi(\eta) + B \phi(\eta) + w^2 \partial_\eta^2 \phi(\eta) \geq 0, \]
which implies that $F(\tau, \xi, \eta) \geq 0$ in $[\eta_1, \eta_2]$. 

Noticing that $\partial_\xi F \geq \partial_\eta F$ on $[\eta_1, \eta_2]$, we have
\[ \partial_\xi F \geq M \phi(\eta) e^{-M\tau} \geq 0, \ \forall \eta \in [\eta_1, \eta_2] \]
for any non-negative $M$.
iii) Noticing that \( A = (1 - \eta) \left[ (1 + \eta) \partial_\xi U + \frac{\partial_\xi U}{U} \right] \), for \( \eta \in (\eta_2, 1] \) we have
\[
A \partial_\eta \phi = \left[ (1 + \eta) \partial_\xi U + \frac{\partial_\xi U}{U} \right] \phi.
\]
Moreover, \( \partial_\eta^2 \phi(\eta) = 0 \) when \( \eta \in (\eta_2, 1] \). Thus we have
\[
\mathcal{F} = \left[ M + (1 + \eta) \partial_\xi U + \frac{\partial_\xi U}{U} - B \right] \phi(\eta) e^{-M\tau} = (M + \partial_\xi U) \phi(\eta) e^{-M\tau}
\]
which implies \( \mathcal{F} \geq 0 \) on \((\eta_2, 1]\), by choosing \( M \) properly large such that \( M + \partial_\xi U \geq 0 \). From now on, we fix \( M \) large such that \( \mathcal{F}(\tau, \xi, \eta) \geq 0 \) in the whole \( D_\epsilon \).

(3) To apply for the maximum principle for the problem (2.8), choose another constant \( N \) properly large such that \( N + B \geq 0 \). Denote by \( G(\tau, \xi, \eta) := e^{-N\tau} F(\tau, \xi, \eta) \). From (2.8), we know that \( G(\tau, \xi, \eta) \) satisfies the following problem in \( D_\epsilon \),
\[
\begin{aligned}
&\partial_\tau G + \eta U \partial_\xi G + A \partial_\eta G + (N + B)G = w^2 \partial_\eta^2 G + \epsilon \mathcal{F} e^{-N\tau}, \\
&G|_{\eta=0} = w(\tau, \xi, 0) e^{-N\tau}, \quad G|_{\eta=1} = 0, \\
&G|_{\tau=0} = w_0 - \epsilon \phi(\eta), \quad G|_{\xi=0} = \left[ w_1 - \epsilon \phi(\eta) e^{-M\tau} \right] e^{-N\tau},
\end{aligned}
\]
Choose \( \epsilon_1 > 0 \) small enough, such that when \( 0 < \epsilon < \epsilon_1 \),
\[
G(\tau, \xi, \eta_0) < (\epsilon^2 - \epsilon e^{-M\tau}) e^{-N\tau} < 0,
\]
and
\[
1 - e^{-(M+N)\tau_\epsilon} + \epsilon e^{-N\tau_\epsilon} < 1
\]
hold. Set \( \bar{\eta} = \max(\eta_2, 1 - e^{-(M+N)\tau_\epsilon} + \epsilon e^{-N\tau_\epsilon}) \), and let \( 0 < \epsilon_2 \leq \epsilon_1 \) be small such that when \( 0 < \epsilon \leq \epsilon_2 \), we have
\[
\epsilon \geq \min_{\xi \in [0, \xi_0], \eta \in [0, \bar{\eta}]} \frac{w_0(\xi, \eta) - \epsilon^2 e^{-N\tau_\epsilon}}{\phi(\eta)} \quad \text{and} \quad \epsilon \geq \min_{\tau \in [0, \tau_\epsilon], \eta \in (0, \bar{\eta})} \frac{w_1(\tau, \eta) - \epsilon^2 e^{-N\tau_\epsilon}}{\phi(\eta) e^{-M\tau}},
\]
From (2.10) and (2.12), we get immediately that
\[
\min_{\xi \in [0, \xi_0], \eta \in [0, \bar{\eta}]} G(0, \xi, \eta) \geq \epsilon^2 e^{-N\tau_\epsilon} > G(\tau, \xi_0, \eta_0),
\]
and
\[
\min_{\tau \in [0, \tau_\epsilon], \eta \in [0, \bar{\eta}]} G(\tau, 0, \eta) \geq \epsilon^2 e^{-N\tau_\epsilon} > G(\tau, \xi_0, \eta_0).
\]
On the other hand, when \( \eta \in [\bar{\eta}, 1] \), obviously from (2.11), we have
\[
\min_{\tau \in [0, \tau_\epsilon], \eta \in [0, 0]} G(\tau, 0, \eta) \geq \epsilon^2 e^{-N\tau_\epsilon} > G(\tau, \xi_0, \eta_0),
\]
and
\[
\min_{\tau \in [0, \tau_\epsilon], \eta \in [0, 0]} G(\tau, 0, \eta) \geq \epsilon^2 e^{-N\tau_\epsilon} > G(\tau, \xi_0, \eta_0),
\]
when \( 0 < \epsilon \leq \epsilon_2 \).
Combining (2.13), (2.14) with (2.15), (2.16) respectively, it follows that
\[
\min_{\xi \in [0, \xi_0], \eta \in [0, 1]} G(0, \xi, \eta) > G(\tau, \xi_0, \eta_0),
\]
and
\[
\min_{\tau \in [0, \tau_\epsilon], \eta \in [0, 1]} G(\tau, 0, \eta) > G(\tau, \xi_0, \eta_0),
\]
hold.

Applying the maximum principle in the problem (2.9), and by using (2.17)-(2.18) it follows that
\[
\min_{\tau \in [0, \tau_\epsilon], \xi \in [0, \xi_0]} w(\tau, \xi, 0) e^{-\eta T} \leq G(\tau, \xi_0, \eta_0) < 0.
\]
Since \(\min_{\xi \in [0, \xi_0]} w(0, \xi, 0) > 0\), there must exist a point \((\tau^*, \xi^*) \in (0, \tau_\epsilon) \times (0, \xi_0)\), such that \(w(\tau^*, \xi^*, 0) = 0\). This is in contradiction with (2.7). Therefore, we conclude the assertion given in Proposition 2.1.

\[\Box\]

3. Existence of a separation point

In this section, under the assumptions given in Theorem 1.1(2), we shall prove there exists a critical point of the tangential velocity \(u(\cdot, y)\) of (1.1) with respect to \(y\) at the boundary \(\{y = 0\}\). It will be obtained by a contradiction approach. From Proposition 2.1, we know that under the monotonicity condition (1.4) and the adverse pressure gradient assumption (1.9), the first zero point of \(\partial_y u(t, x, y)\), if exists, could not be an interior of \(0 \leq y < +\infty\). From now on, we assume that \(\partial_y u(t, x, y)\) is positive everywhere in the domain
\[Q_T = \{(t, x, y) | 0 \leq t < T, 0 \leq x \leq L, 0 \leq y < +\infty\},\]
for the problem (1.1) under the assumptions (1.4) and (1.9). Then, the Crocco transformation (2.1) is invertible in \(Q_T\), and \(w(\tau, \xi, \eta) = \partial_y u > 0\) in
\[Q_T^* = \{(\tau, \xi, \eta) | 0 \leq \tau < T, 0 \leq \xi \leq L, 0 \leq \eta < 1\}.
\]

Denote by \(W(\tau, \xi, \eta) = W^{-\frac{2}{3}}(\tau, \xi, \eta)\), with \(W(\tau, \xi, \eta) = w^2 + \eta^2\). From (2.3), we know that \(W > 0\) satisfies the following problem in \(Q_T^*:\)
\[
\left\{\begin{array}{l}
\partial_\tau W + \eta U \partial_\xi W + A \partial_\eta W = BW - \frac{\eta}{W^2} \partial_\eta^2 w + \eta \frac{\partial P}{\partial y} W^3, \\
(\partial_\xi W + \frac{\partial P}{\partial y} W^3)|_{\eta=0} = 0, \quad W|_{\eta=1} = 1, \\
W|_{\tau=0} = (w_0^2 + \eta_0^2)^{-\frac{1}{2}}, \quad W|_{\xi=0} = (w_0^2 + \eta_0^2)^{-\frac{1}{2}}.
\end{array}\right.
\]

For the problem (3.1), we have

**Proposition 3.1.** Under the same assumptions as given in Theorem 1.1(2), there exists a point \((\tau^*, \xi^*) \in (0, T) \times [0, L]\), such that
\[
W(\tau^*, \xi^*, 0) = \infty.
\]
Obviously, from (3.2) we have \( w(\tau^*, \xi^*, 0) = 0 \), which is in contradiction with \( w > 0 \) in \( Q^*_T \). We shall prove this proposition by constructing a Lyapunov functional and concluding that this functional blows up within the time interval \((0, T)\), provided that the initial data of \( W \) is suitable large.

**Proof of Proposition 3.1.** Let \( \varphi(\xi) = (L - \xi)^\frac{3}{2} \), denote by

\[
\mathcal{G}(\tau) = \int_{\Omega} W(\tau, \xi, \eta) \varphi(\xi) d\xi d\eta,
\]

where \( \Omega = [0, L]_{\xi} \times [0, 1]_{\eta} \).

From (3.1), we know

\[
\frac{d}{d\tau} \mathcal{G} = \int_{\Omega} \eta \partial_\xi W \varphi d\xi d\eta - \int_{\Omega} (\partial_\eta W - BW) \varphi d\xi d\eta + \int_{\Omega} \eta \frac{\partial_\xi P}{U} W^3 \varphi d\xi d\eta
\]

\[
- \int_{\Omega} \frac{w^3}{W^2} \partial_\eta^2 \varphi d\xi d\eta
\]

\[
= \sum_{i=1}^{4} \mathcal{R}_i,
\]

with obvious notations \( \mathcal{R}_i (1 \leq i \leq 4) \). Now we shall estimate each \( \mathcal{R}_i \) step by step.

i) By using integration by parts, we have

\[
\mathcal{R}_1 = \int_{\Omega} \eta \partial_\xi U W \varphi d\xi d\eta + \int_{\Omega} \eta U W \partial_\xi \varphi d\xi d\eta + C_0(\tau),
\]

where

\[
C_0(\tau) = L^\frac{3}{2} U(\tau, 0) \int_{0}^{1} \frac{\eta}{\sqrt{w_1^2 + \eta^2}} d\eta > 0.
\]

For the second term on the right hand side of (3.4), we use the Young inequality to obtain

\[
\int_{\Omega} \eta U W \partial_\xi \varphi d\xi d\eta \geq - \int_{\Omega} \eta \left[ \frac{U^4}{\partial_\xi P} \right]^\frac{3}{2} d\xi d\eta - \frac{1}{2} \int_{\Omega} \eta \frac{\partial_\xi P}{U} W^3 \varphi d\xi d\eta
\]

\[
= - C_1(\tau) - \frac{1}{2} \int_{\Omega} \eta \frac{\partial_\xi P}{U} W^3 \varphi d\xi d\eta,
\]

where

\[
C_1(\tau) = \frac{1}{2} \int_{0}^{L} \left[ \frac{U^4}{\partial_\xi P} \right]^\frac{3}{2} d\xi > 0.
\]

Thus, from (3.4) we have

\[
\mathcal{R}_1 \geq \int_{\Omega} \eta \partial_\xi U W \varphi d\xi d\eta - \frac{1}{2} \int_{\Omega} \eta \frac{\partial_\xi P}{U} W^3 \varphi d\xi d\eta + C_0(\tau) - C_1(\tau).
\]

ii) Recall that \( A = (1 - \eta^2) \partial_\xi U + (1 - \eta) \frac{\partial_\xi U}{U} \) and \( B = \eta \partial_\xi U + \frac{\partial_\xi U}{U} \). By using integration by parts we have,

\[
\mathcal{R}_2 = \int_{\Omega} \partial_\eta AW \varphi d\xi d\eta - \int_{0}^{L} \frac{\partial_\xi P}{U} W(\tau, \xi, 0) \varphi d\xi + \int_{\Omega} BW \varphi d\xi d\eta
\]
\[=- \int_{\Omega} \eta \partial_\xi W \varphi d\eta - \int_{L}^{L} \frac{\partial_k P}{U} W(\tau, \xi, 0) \varphi d\xi.\]

iii) By using the Hölder inequality,
\[
\int_{\Omega} \eta \partial_\xi P \frac{W^3}{U} \varphi d\eta \geq 2C_2(\tau) \left( \int_{\Omega} W \varphi d\eta \right)^3 = 2C_2(\tau)G^3,
\]
where
\[C_2(\tau) = \frac{1}{2} \left( 2 \int_{0}^{L} \left[ \frac{U}{\partial_k P} \right]^{\frac{1}{2}} \varphi d\xi \right)^2 > 0.\]
Thus we have
\[(3.7) \quad \mathcal{R}_3 \geq \frac{1}{2} \int_{\Omega} \eta \partial_\xi P \frac{W^3}{U} \varphi d\eta + C_2(\tau)G^3,\]
iv) Finally, we use integration by parts and the Young inequality to obtain
\[(3.8) \quad \mathcal{R}_4 = 3 \int_{\Omega} \frac{w^2}{W} (\partial_\xi w)^2 \varphi d\xi d\eta - 3 \int_{\Omega} \frac{w^4}{W^2} (\partial_\xi w)^2 \varphi d\xi d\eta
- 3 \int_{\Omega} \frac{\eta w^3}{W^2} \partial_\xi w \varphi d\xi d\eta + \int_{0}^{L} \frac{\partial_k P}{U} W(\tau, \xi, 0) \varphi d\xi
= 3 \int_{\Omega} \frac{\eta w^2}{W^2} (\partial_\xi w)^2 \varphi d\xi d\eta - 3 \int_{\Omega} \frac{\eta w^3}{W^2} \partial_\xi w \varphi d\xi d\eta + \int_{0}^{L} \frac{\partial_k P}{U} W(\tau, \xi, 0) \varphi d\xi
\geq - \frac{3}{4} \int_{\Omega} \frac{w^4}{W^2} \varphi d\xi d\eta + \int_{0}^{L} \frac{\partial_k P}{U} W(\tau, \xi, 0) \varphi d\xi
\geq - \frac{3}{4} G + \int_{0}^{L} \frac{\partial_k P}{U} W(\tau, \xi, 0) \varphi d\xi.
\]
Combining \((3.5)-(3.8)\) with \((3.3)\) we obtain
\[(3.9) \quad \frac{d}{d\tau} G \geq C_2(\tau)G^3 - \frac{3}{4} G + C_0(\tau) - C_1(\tau).\]

Since for any \((\tau, \xi) \in [0, T) \times [0, L)\), \(\partial_k P\) is bounded below away from zero as well as \(U\), moreover \(U\) is bounded above, there exists positive constants \(\lambda_0, \lambda_1\) and \(\lambda_2\) such that \(C_0(\tau) \geq \lambda_0, C_1(\tau) \leq \lambda_1, C_2(\tau) \geq \lambda_2\). Thus, from \((3.9)\) it follows
\[(3.10) \quad \frac{d}{d\tau} G \geq \lambda_2 G^3 - \frac{3}{4} G + \lambda_0 - \lambda_1.\]

Therefore, if the initial data of \(G\) is properly large, that is when the assumption \((1.10)\) holds, \(G(\tau)\) blows up within the time interval \((0, T)\). That means there must exist a point \((\tau^*, \xi^*, \eta^*) \in (0, T) \times [0, L] \times [0, 1)\), such that \(W(\tau^*, \xi^*, \eta^*) = \infty\). Recall that \(W(\tau, \xi, \eta) = (w^2 + \eta^2)^{-\frac{1}{2}}\), we get \(\eta^* = 0\) and \(w(\tau^*, \xi^*, 0) = 0\). \(\square\)

By combining Proposition 2.1 with Proposition 3.1, we get there is a separation point \((t^*, x^*, 0)\) with \((t^*, x^*) \in (0, T) \times [0, L]\) for the tangential velocity profile \(u(t, x, y)\) to the problem \((1.1)\). Moreover, from the first equation given in \((1.1)\) we
know that \( \partial_y^2 u = \partial_x P > 0 \) at this separation point, thus this critical point of \( u \) in \( y \) is non-degenerate.

4. Examples of separation of boundary layers

In this section, we shall give two examples on the existence of separation points of boundary layers, from which we know that the sufficient condition (1.10) holds when the space interval \([0, L]\) is properly large or when the initial velocity \( u_0(x, y) \) grows slow with respect to \( y \) in a large neighborhood of the boundary \( \{y = 0\} \) for a fixed \( L > 0 \).

**Example 4.1.** Consider the tangential velocity of the outer flow being given

\[
U(x) = L^{-3}(2L - x), \quad 0 \leq x \leq L.
\]

From the Bernoulli law, we have

\[
\partial_x P = L^{-6}(2L - x) > 0, \quad \forall \ 0 \leq x \leq L.
\]

Choose the initial tangential velocity \( u_0(x, y) \) of the problem (1.1) being monotonic in \( y \geq 0 \) such that

\[
\int_0^\infty \frac{\partial_y u_0}{\sqrt{\left(\partial_y u_0\right)^2 + u_0^2}} dy \geq c_0, \quad \forall \ x \in [0, L],
\]

holds for a fixed constant \( c_0 > 0 \).

For the solution of (1.1) with the above data, let \( w(\tau, \xi, \eta) \) be determined by using the Crocco transformation (2.1)-(2.2). From the computation given in Section 3, we know that

\[
\frac{d}{d\tau} \mathcal{G}(\tau) \geq \frac{25}{32} L^{-8} \mathcal{G}^3 - \frac{3}{4} \mathcal{G} - \frac{4\sqrt{2} - 1}{5} L^{-\frac{3}{2}},
\]

which implies that \( \mathcal{G}(\tau) \) will blow up in a finite time when \( \mathcal{G}(0) \geq C_0 L^4 \) for a positive constant \( C_0 \) independent of \( L \).

On the other hand, by using the Crocco transformation and (1.1) we have

\[
\mathcal{G}(0) = \int_0^L \int_0^L \frac{\partial_y u_0}{\sqrt{\left(\partial_y u_0\right)^2 + u_0^2}} \frac{(L-x)^{\frac{3}{2}}}{U(x)} dx dy \\
\geq \frac{4}{5} c_0 L^2 \int_0^L (L-x)^{\frac{3}{2}} dx = \frac{4}{5} c_0 L^{\frac{5}{2}}
\]

which is larger than or equal to \( C_0 L^4 \) when \( L \) is properly large. Thus, by using Theorem 1.1 there exists separation in this case.
Remark 4.1. In contrast to one well-posedness result given in Oleinik and Samokhin [20, Theorem 4.2.3], in which they obtained the global existence of a classical solution to the problem \((1.1)\) when \(L > 0\) is small, the above example shows that for a general monotonic initial datum satisfying \((4.1)\), the Prandtl boundary layer will separate in a finite time when the space interval \([0, L]\) is properly large, and separation occurs earlier as \(L\) larger.

Example 4.2. Fix \(L = 1\), and assume that the tangential velocity of the Euler outer flow on the boundary is the following one,

\[ U(x) = 2 - x, \quad \text{for } x \in [0, 1]. \]

From the Bernoulli law, we have

\[ \partial_x P = 2 - x, \quad \forall \ 0 \leq x \leq 1. \]

Choose the initial data of the problem \((1.1)\) being \(u_0(x, y) = U(x)\phi(y)\), where \(\phi(y)\) satisfies

\[
\begin{cases}
\phi(y) = \alpha y, & \text{for } 0 \leq y \leq M, \\
\lim_{y \to +\infty} \phi(y) = 1, \\
\phi'(y) > 0, & \text{for all } y \in [0, \infty), \\
\int_0^\infty \frac{\phi'(y)}{\phi(y) + \phi'(y)} \, dy < +\infty,
\end{cases}
\]

for some \(M \gg 1\) to be determined later, with \(0 < \alpha < M^{-1}\) being fixed. For the solution of \((1.1)\) with the above data, let \(w(\tau, \xi, \eta)\) be determined by using the Corocco transformation \((2.1)-(2.2)\). From the computation given in §3, we know that

\[ G(\tau) = \int_0^1 \int_0^1 (w^2(\tau, \xi, \eta) + \eta^2)^{-\frac{1}{2}} (1 - \xi)^{\frac{3}{2}} \, d\xi d\eta \]

satisfies the following inequality

\[
\frac{d}{d\tau} G \geq \frac{25}{32} G^3 - \frac{3}{4} G - \frac{4\sqrt{2} - 1}{5},
\]

which implies that \(G(\tau)\) blows up in a finite time when \(G(0) \geq C_0\) for a positive constant \(C_0\). On the other hand, it is easy to have

\[
G(0) = \int_0^\infty \int_0^1 \frac{\partial_y u_0}{\sqrt{(\partial_y u_0)^2 + u_0^2}} \frac{(1 - x)^{\frac{3}{2}}}{U(x)} \, dx dy \geq \frac{1}{5} \int_0^M \frac{1}{\sqrt{1 + y^2}} \, dy
\]

which is larger than or equal to \(C_0\) when \(M\) is properly large. Thus, by using Theorem 1.1, there exists separation in this case.

Remark 4.2. This example shows that for a fixed flow distance \(L\) in the streamwise direction, the boundary layer shall separate when the initial velocity \(u_0(x, y)\) grows slow with respect to \(y\) in a large neighborhood of the boundary \(\{y = 0\}\), and separation occurs earlier as the neighborhood larger.
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