The ordinarity of an isotrivial elliptic fibration

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Abstract: In this paper, we study the ordinarity of an isotrivial elliptic surface defined over a field of positive characteristic. If an isotrivial elliptic fibration \( \pi : X \to C \) is given, \( X \) is ordinary when the common fiber of \( \pi \) is ordinary and a certain finite cover of the base \( C \) is ordinary. By this result, we may obtain the ordinary reduction theorem for some kinds of isotrivial elliptic fibrations.

1 Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and \( A \) be an abelian variety of \( d \)-dimensional over \( k \). The number of \( p \)-torsion points of \( A(k) \) is given by \( p^\gamma (0 \leq \gamma \leq d) \) and \( \gamma \) is called the \( p \)-rank of \( A \). By definition, \( A \) is ordinary if \( \gamma = d \). There are several equivalent conditions for \( A \) to be ordinary. Let \( F_A \) be the absolute Frobenius morphism of \( A \). \( A \) is ordinary if and only if \( F_A^*: H^1(O_A) \to H^1(O_A) \) is bijective. Also, \( A \) is ordinary if and only if the Frobenius morphism on \( H^1(WO_A) \) is bijective, here \( WO_A \) is the sheaf of the Witt vectors of \( O_A \). ([6],p.651) The ordinarity of \( A \) is equivalent to that the Newton polygon of \( A \) and the Hodge polygon of \( A \) coincide at every level. The definition of ordinarity can be generalized to an arbitrary proper smooth variety over \( k \). Let us recall the definition of an ordinary variety. ([7]) Let \( X \) be a proper smooth variety of \( d \)-dimensional over \( k \) and \( F_X \) be the absolute Frobenius morphism on \( X \). Let

\[
0 \to O_X \xrightarrow{d} \Omega^1_{X/k} \to \cdots \xrightarrow{d} \Omega^d_{X/k} \to 0
\]

be the deRham complex of \( X/k \). Consider the push forward of the deRham complex by \( F_X \)

\[
0 \to F_X^*O_X \xrightarrow{d} F_X^*\Omega^1_{X/k} \to \cdots \xrightarrow{d} F_X^*\Omega^d_{X/k} \to 0.
\]

In this complex, the exterior differential \( d \) is \( O_X \)-linear, so the kernel and the image of \( d \) are coherent \( O_X \)-modules. In fact, they are vector bundles. We denote the kernel and image of \( d : F_X^*\Omega^{i-1}_{X/k} \to F_X^*\Omega^i_{X/k} \) by \( Z\Omega^{i-1}_{X/k} \) and \( B\Omega^i_{X/k} \) respectively.

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Definition 1.1. $X$ is (Bloch-Kato-Illusie-Raynaud) ordinary if $H^i(B\Omega^1_{X/k}) = 0$ for all $i$ and $j$.

When the crystalline cohomology of $X/k$, $H^i_{cris}(X/W(k))$ is torsion free for each $i$, $X$ is ordinary if and only if the Hodge polygon and the Newton polygon coincide at every level. ([7] p.209) Since $\Omega^0_{X/k} \subset F_X^*\mathcal{O}_X$ is equal to the image of $F_X^* : \mathcal{O}_X \hookrightarrow F_X^*\mathcal{O}_X$, there is an exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{F_X^*} F_X^*\mathcal{O}_X \rightarrow B\Omega^1 \rightarrow 0.$$ 

If $X$ is ordinary, $H^i(B\Omega^1) = 0$ for each $i$, so the Frobenius morphism on $H^i(\mathcal{O}_X)$ is bijective for each $i$. When $X$ is an abelian variety or a curve, the converse also holds.

Let $F$ be a number field and $E$ be an elliptic curve defined over $F$. The following theorem is well known.

Theorem 1.2 ( Ordinary reduction theorem, [15] ). The set of ordinary reduction places of $F$ for $E$ has a positive density. Moreover for a suitable finite field extension $F'/F$, the set of ordinary reduction places of $F'$ for $E$ has density 1.

The ordinary reduction theorem holds for all abelian surfaces ([11]) and for some kinds of abelian varieties of higher dimension. ([12], [9]) It is conjectured that the ordinary reduction theorem holds for all abelian varieties of all dimensions defined over a number field. We may think of the ordinary reduction problem for an arbitrary proper smooth variety defined over a number field. It is known that for an K3-surface, the ordinary reduction theorem holds. ([9])

In this paper we study the ordinarity of a proper smooth surfaces which admit an isotrivial elliptic fibration to a proper smooth curve. An isotrivial elliptic fibration $\pi : X \rightarrow C$ is given by a desingularization of a quotient of a product of an elliptic curve and another curve $E \times D$ by a finite group $G$ which acts on both curves faithfully. (Section 3) Hence the finite group $G$ should be a semi-product of a finite abelian group of rank at most two and a cyclic rotation group which is $\mathbb{Z}/6, \mathbb{Z}/4, \mathbb{Z}/3, \mathbb{Z}/2$ or the trivial group. If the order of $G$ is relatively prime to the characteristic of the base field, the ordinarity of the surface $X$ can be determined by the ordinarity of the common fiber $E$ and a certain finite cover of $C$. (Theorem 4.1) Using this result we can prove the ordinary reduction theorem holds for some kinds of isotrivial elliptic surfaces. (Corollary 4.10) And also we construct an ordinary surface which admits an isotrivial fibration such that the common fiber is not ordinary. (Example 4.3, 4.4)

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2 Ordinary surfaces

Let $k$ be an algebraically closed field of characteristic $p > 0$. $X$ is a smooth variety of $d$-dimensional over $k$. Let $x$ be a closed point in $X$. Étale locally around $x$, we
may assume $X$ is isomorphic to $\text{Spec } k[t_1, \ldots, t_d]$. The $i$-th cohomology sheaf of the deRham complex of $X$, $\mathcal{H}^i\Omega_{X/k} = Z^i/\mathcal{B}_i^1$ is a vector bundle and the set of elements $t_{j_1}^{p-1} \cdots t_{j_i}^{p-1} dt_{j_1} \cdots dt_{j_i}$ $(1 \leq j_1 < j_2 < \cdots < j_i \leq d)$ forms a local basis of $\mathcal{H}^i\Omega_{X/k}$. The Cartier morphism $C : \mathcal{H}^i\Omega_{X/k} \to \Omega_{X/k}^i$ defined by

$$t_{j_1}^{p-1} \cdots t_{j_i}^{p-1} dt_{j_1} \cdots dt_{j_i} \mapsto dt_{j_1} \cdots dt_{j_i}$$

is globally well defined and is an $\mathcal{O}_X$-linear isomorphism. ([6], Chapter 0)

Assume $X$ is a smooth surface over $k$. Let us think of a pairing

$$\varphi' : F_X^* \mathcal{O}_X \otimes B \Omega^2 \to \Omega_{X/k}^2$$

defined by $\varphi'(\alpha \otimes \omega) = C(\alpha \omega)$. $\mathcal{O}_X \subset F_X^* \mathcal{O}_X$ is the kernel of $\varphi'$, so there is the induced pairing $\varphi : B \Omega^1 \otimes B \Omega^2 \to \Omega_{X/k}^2$ which is perfect. By the Serre duality, we have $H^i(B \Omega^1) = H^{2-i}(B \Omega^2)$.

**Lemma 2.1.** Let $X$ be a proper smooth surface over $k$. $X$ is ordinary if and only if $F_X^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$ is bijective for $i = 1, 2$.

Assume $X$ is a proper smooth variety and $Y \subset X$ is a smooth subvariety of codimension $\geq 2$. Let $\tilde{X} \to X$ be the blow-up along $Y$. Then $\tilde{X}$ is ordinary if and only if both of $X$ and $Y$ are ordinary. ([8], p.118) Hence for surfaces, the ordinarity is a birational invariant.

**Lemma 2.2.** Let $f : X' \to X$ be a birational morphism of proper smooth surfaces. $X$ is ordinary if and only if $X'$ is ordinary.

By lemma 2.1., if $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$, $X$ is ordinary. In particular, a rational surface or a classical Godeaux surface is ordinary. Moreover for a rational surface or a Godeaux surface defined over a number field, the ordinary reduction theorem is trivial. If $H^2(\mathcal{O}_X) = 0$, the image of the Albanese map $\phi : X \to \text{Alb } X$ is a smooth curve $C$. ([2], p.86) $\text{Alb } X = \text{Jac } C$ and $\phi^* : H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_X)$ is isomorphic. Therefore $X$ is ordinary if and only if $C$ is ordinary.

**Proposition 2.3.** Let $F$ be a number field and $X$ be a proper smooth surface defined over $F$.

(a) If the Kodaira dimension of $X$, $\kappa(X) = -\infty$, i.e. $X$ is birationally ruled and $\pi : X \to C$ is the structure morphism, then the ordinary reduction theorem for $X$ is equivalent to the ordinary reduction theorem for $C$.

(b) If $\kappa(X) = 0$, the ordinary reduction theorem holds for $X$.

**Proof.** If $\kappa(X) = -\infty$, $\text{Alb } X = \text{Jac } C$ and $X \to C \to \text{Jac } C$ is the Albanese map of $X$. Hence (a) follows by the above argument. Assume $\kappa(X) = 0$. Because of Lemma 2.2, we may assume $X$ is minimal. Assume $X$ is an abelian surface, a K3 surface, an Enrique surface or a bielliptic surface. If $X$ is an abelian surface or a K3 surface, the ordinary reduction theorem holds. (Section 1.) If $X$ is an Enrique surface, there exists an étale double cover $X' \to X$ where $X'$ is a K3 surface. If $\nu$ is a place of $F$ whose
residue characteristic $\neq 2$ and the reduction $X'_v$ is ordinary, $X_v$ is also ordinary. Hence the ordinary reduction theorem holds for $X$. If $X$ is a bielliptic surface, $X$ admits a smooth elliptic fibration $\pi : X \to C$ to an elliptic curve, $C$. Here all the fibers of $\pi$ are isomorphic to each other and there exists a finite Galois étale cover $D \to C$ such that the fiber product $X \times_C D$ is isomorphic to a trivial fibration $E \times_k D$, here $E$ is the common fiber. $E \times_k D \to X$ is finite Galois étale and $E \times_k D$ is an abelian surface, so the ordinary reduction theorem holds for $E_k D$. Hence by a similar argument to the above, the ordinary reduction theorem holds for $X$.

Now we consider the ordinarity of fibred surfaces. The fiber product of two curves, $X = C \times_k D$ is ordinary if and only if both of $C$ and $D$ are ordinary. ([4], p.91) Indeed, in the case of a product of two curves,

$$H^1(\mathcal{O}_{C \times D}) = H^1(\mathcal{O}_C) \oplus H^1(\mathcal{O}_D) \quad \text{and} \quad H^2(\mathcal{O}_{C \times D}) = H^1(\mathcal{O}_C) \otimes H^1(\mathcal{O}_D).$$

Since the Frobenius morphism acts separately, the claim follows. If we regard the fiber product $X = C \times_k D$ as a fibred surface equipped with a trivial fibration $X \to D$, then $X$ is ordinary if and only if both of the base and the generic fiber are ordinary. However this is not true in general. There is a non-ordinary 3-fold which admits a fibration over $\mathbb{P}^1$ of which every smooth fiber is ordinary. ([8] p.119)

**Example 2.4.** There is also a non-ordinary fibred surface over $\mathbb{P}^1$ of which each smooth fiber is ordinary. Assume the characteristic of $k > 2$. Assume $C$ is an ordinary elliptic curve over $k$. $A = C \times_k D$ is an abelian surface of $p$-rank 1. The height of the formal Brauer group of $A$ is 2. Let $X$ be the Kummer surface of $A$. The height of $X$ is equal to the height of $A$, ([5], p.109) so $X$ is not ordinary. But $X$ admits a Kummer fibration $\pi : X \to \mathbb{P}^1$ given by the quotient of the projection $A \to D$ by the involution. Each smooth fiber of $\pi$ is isomorphic to $C$, so ordinary, and there are 4 singular fibers which are of type $I^*_0$ in Kodaira’s classification.

In the above example the image of singular fibers in $\mathbb{P}^1$ consists of 4 points. The double cover of $\mathbb{P}^1$ ramified over those 4 points is isomorphic to $D$. Hence although the base itself is ordinary, the fibration encodes information about $D$ which is not ordinary. Later we will see this phenomenon in detail.

Let $\pi : X \to C$ be a fibered surface. Since $\pi_* \mathcal{O}_X = \mathcal{O}_C$, considering the Leray spectral sequence, $\pi^*: H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_X)$ is injective. And the diagram

$$\begin{array}{ccc}
X & \xrightarrow{F_X} & X \\
\downarrow \pi & & \downarrow \pi \\
C & \xrightarrow{F_C} & C
\end{array}$$

is commutative, so if

$$F^*_X : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$$
is bijective,
\[ F^*_C : H^1(O_C) \to H^1(O_C) \]
is bijective. Hence if \( X \) is ordinary, \( C \) is also ordinary. Assume \( \pi \) is generically smooth.

**Definition 2.5.** We call \( \pi \) is generically ordinary if almost all fibers are ordinary.

\( \pi \) is generically ordinary if and only if the generic fiber of \( \pi \) is ordinary by the Grothendieck specialization theorem.

**Proposition 2.6.** Let \( \pi : X \to C \) be an elliptic fibration. If \( H^2(O_X) \neq 0 \) and \( X \) is ordinary, then \( \pi \) is generically ordinary.

**Proof.** Let us consider the Frobenius diagram of \( \pi \)

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/C}} & X^{(p)} \\
\downarrow{\pi} & & \downarrow{\pi} \\
C & \xrightarrow{F_C} & C.
\end{array}
\]

Here the right square is cartesian and \( F_X = W \circ F_{X/C} \). There is an exact sequence of \( O_{X^{(p)}} \)-modules

\[ 0 \to O_{X^{(p)}} \xrightarrow{F^*_{X/C}} F_{X/C*}O_X \to B \to 0. \] (2.1)

Note that the restriction of the cokernel \( B \) to a smooth fiber \( X_x \) over \( x \in C \) is just \( B \Omega_{X_x/k}^1 \). \( B \) is not \( O_C \)-flat at a non-reduced fiber in general. Let \( N = R^1 \pi_*O_X \). \( N \) is a line bundle on \( C \). We have the long exact sequence associated to the above exact sequence via \( \pi_* \),

\[ 0 \to O_C \simeq O_C \to \pi_*^{(p)}B \to F^*_C N \xrightarrow{F^*_{X/C}} N \to R^1 \pi_*^{(p)}B \to 0. \] (2.2)

Note that \( H^2(O_X) = H^1(N) \) and \( H^2(O_{X^{(p)}}) = H^1(F^*_C N) \). If \( \pi \) is generically ordinary, in (2.2), \( F^*_X \) is injective, otherwise \( F^*_X = 0 \). Since

\[ F^*_X = F^*_{X/C} \circ W^* : H^2(O_X) \to H^2(O_{X^{(p)}}) \to H^2(O_X), \]

if \( \pi \) is not generically ordinary, \( F^*_X|H^2(O_X) = 0 \), so \( X \) is not ordinary.

**Remark 2.7.** Because the moduli space of elliptic curves is 1-dimensional and the generic elliptic curve is ordinary, if an elliptic fibration is not generically ordinary, it should be isotrivial. (Section 3.) If \( H^2(O_X) = 0 \) or the fiber genus is large, the above proposition may fail. We will see counterexamples later.
3 Isotrivial elliptic fibrations

Let \( k \) be an algebraically closed field and \( \text{char}(k) \neq 2, 3 \). \( \pi : X \to C \) is an elliptic fibration over \( k \).

**Definition 3.1.** We say \( \pi \) is isotrivial if all the smooth fibers of \( \pi \) are isomorphic to each other.

We recall a systematical construction of isotrivial elliptic fibrations. ([2] chapter 6, [14]) Let \( \pi : X \to C \) be an isotrivial elliptic fibration and \( E \) be the common fiber of \( \pi \). Let \( U = C - \{ \text{the image of the singular fibers of } \pi \} \). There exists a finite cover \( U' \) of \( U \) such that

\[
U' \times_U V = U' \times_k E.
\]

Let \( C' \) be the proper smooth model of \( U' \). We may assume \( f : C' \to C \) is Galois. Let \( H \) be the Galois group of \( C'/C \) and \( X' = X \times_C C' \). \( X' \) has an \( H \)-action coming from the \( \pi \)-action of \( C' \). This \( H \)-action on \( X' \) is compatible with the \( \pi \)-action on \( C' \) via the projection \( X' \to C' \). Since \( X' \) is birationally equivalent to \( C' \times_k E \) and \( C' \times_k E \) is minimal, there is an \( H \)-action on \( C' \times_k E \)

\[
\rho : H \times C' \times_k E \to C' \times_k E
\]

which is compatible with the action on \( C' \) via the projection. By [2], p.102 with a slight modification, there exist a finite étale cover \( g : D \to C' \) and a finite group \( G \) which acts on both of \( D \) and \( E \) such that

\[
D/G \simeq C'/H = C \quad \text{and} \quad (D \times_k E)/G \simeq (C' \times_k E)/H.
\]

Here the action of \( G \) on \( D \times_k E \) is the product of the actions on \( D \) and \( E \). We may assume the actions of \( G \) on \( E \) and \( D \) are faithful. \( (D \times_k E)/G \) may have isolated singularities and is birationally equivalent to \( X \). Let \( \text{Iso} \) be the group of \( k \)-variety automorphisms of \( E \) and \( \text{Aut} \) be the group of elliptic curve automorphisms of \( E \). \( \text{Aut} E = \mathbb{Z}/2 \) if \( j(E) \neq 0,1728 \), \( \text{Aut} E = \mathbb{Z}/4 \) if \( j(E) = 1728 \) and \( \text{Aut} E = \mathbb{Z}/6 \) if \( j(E) = 0 \). ([16], p.103) \( \text{Iso} E = E(k) \ltimes \text{Aut} E \). Since \( G \) acts faithfully on \( E \), \( G \subset \text{Iso} E \) and \( G = T \rtimes R \) where \( T \subset E(k) \) and \( R \) is embedded into \( \text{Aut} E \) by the projection \( \text{Iso} E \to \text{Aut} E \), so \( R \) is cyclic. Adjusting the 0-point of \( E \), we may regard \( R \subset \text{Aut} E \). In particular \( G \) is solvable. Let \( Y = (E \times D)/G \). We assume the characteristic of \( k \) does not divides \( |G| \). \( T = \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 \) where \( n_2 \mid n_1 \) and \( R = \mathbb{Z}/6, \mathbb{Z}/4, \mathbb{Z}/3, \mathbb{Z}/2 \), or trivial. Let \( f : \tilde{X} \to Y \) be the standard desingularization which resolves the quotient singularities by Hirzebruch-Jung string. ([14], p.64) \( \tilde{\pi} : X \to C = D/G \) is an isotrivial elliptic fibration. In general \( \tilde{\pi} \) is not relatively minimal. Since \( |G| \) is relatively prime to the characteristic of the base field, each singularity of \( Y \) is a rational singularity, so \( f^* : H^i(\mathcal{O}_Y) \to H^i(\mathcal{O}_{\tilde{X}}) \) is isomorphic for each \( i \). Let us think of the \( G \)-action on \( E \). The following lemma is well-known.

**Lemma 3.2.** A finite abelian group \( G \) acts on a variety \( X \) over \( k \). Assume \( |G| \) is relatively prime to the characteristic of the base field \( k \). Let \( Y \) be the quotient \( X/G \) and \( p : X \to Y \) be the projection. Then \( p_* \mathcal{O}_X = \bigoplus \chi L_\chi \) where \( L_\chi \) is a coherent \( \mathcal{O}_Y \)-module on which \( G \) acts through the character \( \chi : G \to k^* \).
Let a finite group $G$ act on a variety $S$ and $T = S/G$. If $G$ is solvable and $|G|$ is relative prime to the characteristic of the base field, by the above lemma, $H^i(\mathcal{O}_T) = H^i(\mathcal{O}_S)^G$.

Let $E' = E/T$. Since $T$ is a translation group of $E$, $E'$ is an elliptic curve and $E \to E'$ is isogeny. If $R$ is $\mathbb{Z}/3$, $\mathbb{Z}/4$ or $\mathbb{Z}/6$, $E'$ is isomorphic to $E$. Assume $R$ is not trivial, and $g : E' \to E'/R = \mathbb{P}^1$ is the canonical map. The following lemma is trivial.

**Lemma 3.3.** (a) If $R = \mathbb{Z}/2$, $g_*\mathcal{O}_{E'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $1 \in \mathbb{Z}/2$ acts on $\mathcal{O}_{\mathbb{P}^1}(-2)$ by multiplying $-1$. There are 4 ramification points of ramification index 2.

(b) If $R = \mathbb{Z}/3$, $g_*\mathcal{O}_{E'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and there is a primitive 3rd root of unity $\xi_3$ such that $1 \in \mathbb{Z}/3$ acts on $\mathcal{O}_{\mathbb{P}^1}(-2)$ by multiplying $\xi_3$. There are 3 ramification points of ramification index 3.

(c) If $R = \mathbb{Z}/4$, $g_*\mathcal{O}_{E'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and there is a primitive 4th root of unity $\xi_4$ such that $1 \in \mathbb{Z}/4$ acts on $\mathcal{O}_{\mathbb{P}^1}(-2)$ by multiplying $\xi_4$. There are 2 ramification points of ramification index 4 and 2 ramification points of ramification index 2 which are conjugate to each other.

(d) If $R = \mathbb{Z}/6$, $g_*\mathcal{O}_{E'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and there is a primitive 6th root of unity $\xi_6$ such that $1 \in \mathbb{Z}/6$ acts on $\mathcal{O}_{\mathbb{P}^1}(-2)$ by multiplying $\xi_6$. Here $\xi_6^2 = \xi_3$. There are 1 point of ramification index 6, 2 points of ramification index 3 and 3 points of ramification index 2.

Let $\pi : X \to C$ be the minimal relative model of $\tilde{\pi} : \tilde{X} \to C$. $\tilde{X} \to X$ contracts every $-1$-curve in a fiber of $\tilde{\pi}$. Let us recall the figure of a singular fiber of $\pi : X \to C$. ([1], p.157, [10], [14]) Since $|G|$ is relative prime to the characteristic of $k$, for any $x \in D$, the stabilizer of $x$, $\text{Stab}_x(G) \subset G$ is a cyclic group. By an easy calculation, we can check $\text{Stab}_x(G) \subset T$ or $\text{Stab}_x(G) \subset R$. Let $y \in C$ be the image of $x$. $Y_y$ is the fiber of $Y \to C$ over $y$, $\tilde{X}_y$ is the fiber of $\tilde{\pi} : \tilde{X} \to C$ over $y$ and $X_y$ is the fiber of $\pi : X \to C$ over $y$. $E_l(X_y)$ is the $l$-adic Euler characteristic of $X_y$,

$$E_l(X_y) = \sum (-1)^i \dim_{\mathbb{Q}_l} H^i(X_y, \mathbb{Q}_l).$$

Here $l$ is a prime number different from the characteristic of $k$. We have several cases.

(i) Assume $\text{Stab}_x(G) \subset T$. $Y_y$ is $|\text{Stab}_x(G)|(E/\text{Stab}_x(G))$. Note that every point $Y_y$ is non-singular on $Y$. Hence $X_y = \tilde{X}_y = Y_y$ and $E_l(X_y) = 0$.

(ii) Assume $\text{Stab}_x(G) \subset R$ and isomorphic to $\mathbb{Z}/2$. $Y_y$ has 4 singular points of type $A_1$. After the Hirzebruch-Jung resolution, $\tilde{X}$ is a fiber of type $I_0^2$ which is minimal. $X_y = \tilde{X}_y$ and $E_l(X_y) = 6$.

(iii) Assume $\text{Stab}_x(G) \subset R$ and isomorphic to $\mathbb{Z}/3$. Let

$$\hat{O}_{D,x} = \lim_{\leftarrow} \mathcal{O}_{D,x}/m_x^n \simeq k[[l]].$$

Assume $1 \in \mathbb{Z}/3$ acts on $\hat{O}_{D,x}$ by multiplying $\xi_3$. Then $Y_y$ has 3 singular points of type $A_{3,1}$ and $\tilde{X}_y$ is $3L_1 + L_2 + L_3 + L_4$, where each $L_i$ is a non-singular rational curve and
the intersection matrix is

\[
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -3 & 0 & 0 \\
1 & 0 & -3 & 0 \\
1 & 0 & 0 & -3 \\
\end{pmatrix}.
\]

After blowing down \( L_1 \), \( X_y \) is type of IV and \( E_l(X_y) = 4 \). Assume \( 1 \in \mathbb{Z}/3 \) acts on \( \mathcal{O}_{D,x} \) by multiplying \( \xi_3^{-1} \). \( Y_y \) has 3 singular points of type \( A_2 \). \( \bar{X}_y = X_y \) is a fiber of type \( IV^* \). \( E_l(X_y) = 8 \).

(iv) Assume \( Stab_x(G) \subset R \) and isomorphic to \( \mathbb{Z}/4 \). Assume \( 1 \in \mathbb{Z}/4 \) acts on \( \mathcal{O}_{D,x} \) by multiplying \( \xi_4 \). \( Y_y \) has 2 singular points of type \( A_{4,1} \) and 1 singular point of type \( A_1 \). \( \bar{X}_y \) is \( 4L_1 + L_2 + 2L_3 + L_4 \), where each \( L_i \) is a non-singular rational curve and the intersection matrix is

\[
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -4 & 0 & 0 \\
1 & 0 & -2 & 0 \\
1 & 0 & 0 & -4 \\
\end{pmatrix}.
\]

After blowing down twice, the fiber \( X_y \) is of type III. \( E_l(X_y) = 3 \). Assume \( 1 \in \mathbb{Z}/4 \) acts on \( \mathcal{O}_{D,x} \) by multiplying \( \xi_4^{-1} \). \( Y_y \) has 1 singular point of type \( A_1 \) and 2 singular points of type \( A_3 \). \( \bar{X}_y \) is minimal and of type III\(^*\). \( X_y = \bar{X}_y \) and \( E_l(X_y) = 9 \).

(v) Assume \( Stab_x(G) \subset R \) and isomorphic to \( \mathbb{Z}/6 \). Assume \( 1 \in \mathbb{Z}/6 \) acts on \( \mathcal{O}_{D,x} \) by multiplying \( \xi_6 \). \( Y_y \) has 1 singular point of type \( A_{6,1} \), 1 singular point of type \( A_{3,1} \) and 1 singular point of type \( A_1 \). \( \bar{X}_y \) is \( 6L_1 + L_2 + 2L_3 + 3L_4 \), where each \( L_i \) is a non-singular rational curve and the intersection matrix is

\[
\begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -6 & 0 & 0 \\
1 & 0 & -3 & 0 \\
1 & 0 & 0 & -2 \\
\end{pmatrix}.
\]

After blowing down three times, \( X_y \) is of type II and \( E_l(X_y) = 2 \). Assume \( 1 \in \mathbb{Z}/6 \) acts on \( \mathcal{O}_{D,x} \) by multiplying \( \xi_6^{-1} \). \( Y_y \) has 1 singular point of type \( A_5 \), 1 singular point of type \( A_2 \) and 1 singular point of type \( A_1 \). \( \bar{X}_y \) is minimal and of type II\(^*\). \( X_y = \bar{X}_y \) and \( E_l(X_y) = 10 \).

Let \( E_l(X) = \sum (-1)^i \dim_{\mathbb{Q}_l} H^i_{\text{ét}}(X, \mathbb{Q}_l) \). Then

\[
E_l(X) = \sum E_l(X_y),
\]

here \( E_l(X_y) = 0 \) if \( X_y \) is smooth, so the right hand side is a finite sum. Since \( X \) is relatively minimal, \( K_X^2 = 0 \). By the Noether formula, we obtain

\[
12 \chi(\mathcal{O}_X) = E_l(X).
\]
On the other hand, \( H^i(\mathcal{O}_X) = H^i(\mathcal{O}_Y) \) and
\[
H^1(\mathcal{O}_Y) = H^1(\mathcal{O}_C) \oplus H^1(\mathcal{O}_E)^G \quad \text{and} \quad H^2(\mathcal{O}_Y) = (H^1(\mathcal{O}_E) \otimes H^1(\mathcal{O}_D))^G.
\]

Let \( D' = D/T \). \( R \) is the Galois group of \( D'/C \). Let \( \rho_{D'} \) be the \( R \)-action on \( D' \). Let \( g : D' \to C \) be the canonical quotient map. Let \( n = |R| \) and \( \chi : R \to k^* \) be a character satisfying \( \chi(1) = \xi_n \). Then \( g_*\mathcal{O}_{D'} = \oplus_{i=0}^{n-1} L_i \), where \( R \) acts on \( L_i \) through \( \chi^i \). Since \( R \) acts on \( D' \) faithfully, each \( L_i \) is a line bundle on \( C \) and \( L_0 = \mathcal{O}_C \). Note that if \( n > 2 \), there is another \( R \)-action \( \rho_{D'}^\prime \) on \( D' \) such that \( \rho_{D'}^\prime(1) \) acts on \( L_1 \) by multiplying \( \xi_n^{-1} \).

Let \( a_1 \) be the number of branch points of ramification index 2 for \( D'/C \). For \( i = 3, 4, 6 \), let \( a_i^+ \) be the number of branch points of ramification index \( i \) for \( D'/C \) such that at a corresponding ramification point, \( 1 \in \mathbb{Z}/i \subset R \) acts by multiplying \( \chi \). \( a_i^- \) is the number of branch points such that at a corresponding ramification point, \( 1 \) acts by multiplying \( \chi^{-1} \).

**Proposition 3.4.** We assume all the above notations.

(a) Assume \( R \) is trivial. \( Y \) is smooth \( X = Y \). \( H^1(\mathcal{O}_X) = H^1(\mathcal{O}_E) \oplus H^1(\mathcal{O}_C) \), \( H^2(\mathcal{O}_X) = H^1(\mathcal{O}_E) \otimes H^1(\mathcal{O}_C) \). \( R^1\pi_\ast \mathcal{O}_X = \mathcal{O}_C \). \( \chi(\mathcal{O}_X) = E_\ell(X) = 0 \).

(b) Assume \( R = \mathbb{Z}/2 \). \( H^1(\mathcal{O}_X) = H^1(\mathcal{O}_C) \) and \( H^2(\mathcal{O}_X) = H^1(\mathcal{O}_E) \otimes H^1(\mathcal{L}_1) \). \( R^1\pi_\ast \mathcal{O}_X = L_1 \). \( \deg L_1 = -a_2/2 \). \( \chi(\mathcal{O}_X) = a_2/2 \) and \( E_\ell(X) = 6a_2 \).

(c) Assume \( R = \mathbb{Z}/3 \). \( H^1(\mathcal{O}_X) = H^1(\mathcal{O}_C) \) and \( H^2(\mathcal{O}_X) = H^1(\mathcal{O}_E) \otimes H^1(\mathcal{L}_2) \). \( R^1\pi_\ast \mathcal{O}_X = L_2 \). \( \deg L_2 = \chi(\mathcal{O}_X) = \frac{1}{3}(a_3^+ + 2a_3^-) \) and \( \deg L_1 = -\frac{1}{3}(2a_3^+ + a_3^-) \). \( E_\ell(X) = 4a_3^+ + 8a_3^- \).

(d) Assume \( R = \mathbb{Z}/4 \). \( H^1(\mathcal{O}_X) = H^1(\mathcal{O}_C) \) and \( H^2(\mathcal{O}_X) = H^1(\mathcal{O}_E) \otimes H^1(\mathcal{L}_3) \). \( R^1\pi_\ast \mathcal{O}_X = L_3 \). \( \deg L_3 = \chi(\mathcal{O}_X) = \frac{1}{4}(a_4^+ + 3a_4^- + 2a_2^-) \) and \( \deg L_1 = -\frac{1}{4}(3a_4^+ + a_4^- + 2a_2^-) \). \( E_\ell(X) = 3a_4^+ + 9a_4^- + 6a_2^- \).

(e) Assume \( R = \mathbb{Z}/6 \). \( H^1(\mathcal{O}_X) = H^1(\mathcal{O}_C) \) and \( H^2(\mathcal{O}_X) = H^1(\mathcal{O}_E) \otimes H^1(\mathcal{L}_5) \). \( R^1\pi_\ast \mathcal{O}_X = L_5 \). \( \deg L_5 = \chi(\mathcal{O}_X) = \frac{1}{6}(a_6^+ + 5a_6^- + 2a_3^+ + 4a_3^- + 3a_2^-) \). \( \deg L_1 = -\frac{1}{6}(5a_6^+ + a_6^- + 4a_3^+ + 2a_3^- + 3a_2^-) \). \( \deg L_2 = -\frac{1}{2}(a_2^+ + a_6^- + a_3^- + 2a_0^- + 2a_2^-) \). \( \deg L_3 = -\frac{1}{2}(a_2^+ + a_6^- + a_3^- + 2a_0^- + 2a_2^-) \). \( \deg L_4 = -\frac{1}{2}(2a_0^+ + 2a_3^- + a_6^- + a_3^- + 2a_0^- + 2a_2^-) \). \( E_\ell(X) = 2a_0^- + 10a_0^- + 4a_3^- + 8a_3^- + 6a_2^- \).

**Proof.** If \( R \) is trivial, any \( g \in G \) has no fixed point on \( D \times E \), so \( Y \) is smooth and \( X = Y \). Moreover \( G = T \) and \( T \) acts trivially on \( H^1(\mathcal{O}_E) \). Since
\[
\dim H^1(\mathcal{O}_X) = \dim H^0(R^1\pi_\ast \mathcal{O}_X) \quad \text{and} \quad \dim H^2(\mathcal{O}_X) = \dim H^1(R^1\pi_\ast \mathcal{O}_X),
\]
by the Riemann-Roch theorem, \( \deg R^1\pi_\ast \mathcal{O}_X = 0 \) and \( \dim H^0(R^1\pi_\ast \mathcal{O}_X) = 1 \). Hence \( R^1\pi_\ast \mathcal{O}_X = \mathcal{O}_C \). This proves (a). Let \( Z = (E \times D)/T \) and consider the following diagram

\[
\begin{array}{ccc}
E \times D & \xrightarrow{\alpha} & Z \\
| & | & | \\
| & p & | \quad \beta \\
D & \xrightarrow{h} & D' \xrightarrow{g} C.
\end{array}
\]
Because $\alpha, \beta, h, g$ are finite morphisms,

$$R^1r_*((\beta \circ \alpha)_*\mathcal{O}_E) = (g \circ h)_*R^1p_*\mathcal{O}_{E \times D}.$$ 

One hand, we have

$$(R^1r_*((\beta \circ \alpha)_*\mathcal{O}_E))^G = R^1r_*\mathcal{O}_Y = R^1\pi_*\mathcal{O}_X.$$ 

On the other hand,

$$((g \circ h)_*R^1p_*\mathcal{O}_{E \times D})^R = (H^1(\mathcal{O}_E) \otimes \mathcal{O}_D)^G = (H^1(\mathcal{O}_E) \otimes \mathcal{O}_{D'})^R = L_{n-1}.$$ 

When $R = \mathbb{Z}/2$, $\pi$ has $a_2$ fibers of type $I_6^*$. Therefore

$$E_l(X) = 6a_2 = 12\chi(O_X) = -12 \deg R^1\pi_*\mathcal{O}_X.$$ 

This proves (b). Now let $\rho_D$ be the given action of $G$ on $D$. $\lambda : G \to G$ is an automorphism of $G$ given by $(a, b) \in T \times R \mapsto (a, -b)$. Let

$$\rho'_D = \rho_D \circ \lambda : G \to \text{Aut } D$$

and $\pi' : X' \to C$ be the minimal isotrivial elliptic fibration associated to the action $\rho_E \times \rho'_D$ on $E \times D$. Assume $R = \mathbb{Z}/3$. $\pi$ has $a_3^+$ fibers of type $IV$ and $a_3^-$ fibers of type $IV^*$. Thus

$$E_l(X) = 4a_3^+ + 8a_3^-.$$ 

On the other hand, $\pi'$ has $a_3^+$ fibers of type $IV^*$ and $a_3^-$ fibers of type $IV$, so

$$E_l(X') = 8a_3^+ + 4a_3^-$$

and $R^1\pi'_*\mathcal{O}_{X'} = L_1$.

This proves (c). If $R = \mathbb{Z}/4$, let $g''' : D'' \to C$ be the intermediate double covering of $D/C$. Then $g''_*\mathcal{O}_{D''} = \mathcal{O}_C \oplus L_2$. Since the number of branch points of $g''$ is $a_4^+ + a_4^-$,

$$\deg L_2 = -\frac{3}{2}(a_4^+ + a_4^-).$$

The rest of proof of (d) is similar to that of (c). If $R = \mathbb{Z}/6$, let $g''' : D''' \to C$ be the intermediate triple cover of $D/C$. $g'''_*\mathcal{O}_{D'''} = \mathcal{O}_C \oplus L_2 \oplus L_4$. $1 \in \mathbb{Z}/3 = \text{Gal}(D''/C)$ acts on $L_2$ by multiplying $\xi_3^2 = \xi_3$. Let $x \in D'$ be a point of ramification index 3. Assume $2 \in \mathbb{Z}/6 = R$ is the generator of $\mathbb{Z}/3 \subset \mathbb{Z}/6$ acting on $\mathcal{O}_{D',x}$ by multiplying $\xi_3$. $y = g(x)$ and $z$ is the unique point in $D'''$ lying over $x$. Then $1 = 4 \in \text{Gal}(D'''/C)$ acts on $\mathcal{O}_{D'''_x}$ by multiplying $\xi_3^2$. Hence the number of ramification points of $D'''/C$ corresponding to the character $\chi_3$ is $a_6^+ + a_3^-$. In the same way, the number of ramification points corresponding $\chi_3^2$ is $a_6^- + a_3^+$. By part (c),

$$\deg L_2 = -\frac{1}{3}(a_6^+ + a_3^- + 2a_6^- + 2a_3^+) \text{ and } \deg L_4 = -\frac{1}{3}(2a_6^+ + 2a_3^- + a_6^- + a_3^+).$$

This finishes the proof. \(\square\)
4 Frobenius morphism of isotrivial elliptic fibrations

Let $k$ be an algebraically closed field of characteristic $p > 3$. We resume the notations of the last section. Remind that $\pi' : X' \to C$ is the minimal isotrivial fibration associated to the $G$-action $\rho_E \times \rho_D'$ on $E \times D$.

**Theorem 4.1.** (a) If $R$ is trivial, $X$ is ordinary if and only if both of $E$ and $C$ are ordinary.

(b) If $R = \mathbb{Z}/2$ and $X$ is not rational, $X$ is ordinary if and only if both of $E$ and $D'$ are ordinary.

(c) If $R = \mathbb{Z}/3$, unless both of $X$ and $X'$ are rational, both of $X$ and $X'$ are ordinary if and only if both of $D'$ and $E$ are ordinary.

(d) If $R = \mathbb{Z}/4$, unless both of $X$ and $X'$ are rational, both of $X, X'$ are ordinary if and only if both of $E$ and $C$ are ordinary and gen $D' - \text{gen } D'' = p$-rank of $D' - p$-rank of $D''$.

(e) If $R = \mathbb{Z}/6$, unless both of $X$ and $X'$ are rational, all of $X, X', D''$ are ordinary if and only if both of $E$ and $C$ are ordinary and gen $D' - \text{gen } D'' = p$-rank of $D' - p$-rank of $D''$.

**Proof.** When $R$ is trivial,

$$H^1(O_X) = H^1(O_E) \oplus H^1(O_C) \text{ and } H^2(O_X) = H^1(O_E) \otimes H^1(O_C).$$

Since the Frobenius morphism acts separately, $F^*_X$ is bijective on $H^1(O_X)$ and $H^2(O_X)$ if and only if $F^*_E|H^1(O_E)$ and $F^*_C|H^1(O_C)$ are bijective. When $R = \mathbb{Z}/2$,

$$H^1(O_X) = H^1(O_C) \text{ and } H^2(O_X) = H^1(O_E) \otimes H^1(L_1).$$

Under the assumption, $X$ is ordinary if and only if $F^*_E|H^1(O_E), F^*_C|H^1(O_C)$ and $F^*_D'|H^1(L_1)$ are bijective, i.e. $E$ and $D'$ are ordinary. When $R = \mathbb{Z}/3$,

$$H^1(O_X) = H^1(O_{X'}) = H^1(O_C), \quad H^2(O_X) = H^1(O_E) \otimes H^1(L_2)$$

and

$$H^2(O_{X'}) = H^1(O_E) \otimes H^1(L_1).$$

Under the assumption, if $X$ and $X'$ are ordinary, $C$ and $E$ are ordinary and $F^*_D'$ are bijective on $H^1(L_1)$ and $H^1(L_2)$. Hence $F^*_D'|((H^1(O_C) \oplus H^1(L_1) \oplus H^1(L_2))$ is bijective, so $D'$ is ordinary. The converse also holds. Note that when $R = \mathbb{Z}/3, \ j(E) = 0$ and $E$ is ordinary if and only if $p \equiv 1 \pmod{6}$. If so, $F^*_D'$ sends $L_1$ into $L_1$ and $L_2$ into $L_2$ respectively. When $R = \mathbb{Z}/4$,

$$H^1(O_X) = H^1(O_{X'}) = H^1(O_C), \quad H^2(O_X) = H^1(O_E) \otimes H^1(L_3)$$

and

$$H^2(O_{X'}) = H^1(O_E) \otimes H^1(L_1).$$

Under the assumption, if $X$ and $X'$ are ordinary, $C$ and $E$ are ordinary and $F^*_D'$ are bijective on $H^1(L_1)$ and $H^1(L_3)$. Again note that in this situation, $j(E) = 1728$ and
$p \equiv 1 \pmod{4}$. Let $\mathcal{O}_{D'} = \mathcal{O}_{D''} \oplus M$ as an $\mathcal{O}_{D''}$-module. $g''M = L_1 \oplus L_3$. $F_{D'}|H^1(M)$ is bijective if and only if
\[
\text{gen } D' - \text{gen } D'' = p\text{-rank of } D' - p\text{-rank of } D''.
\]
Hence (d) is valid. For (e), note that as an $\mathcal{O}_{D''}$-module, $\mathcal{O}_{D'} = \mathcal{O}_{D''} \oplus M$ where $M = L_1 \oplus L_3 \oplus L_5$ and $\mathcal{O}_{D''} = \mathcal{O}_C \oplus L_3$. The rest of proof is similar to that of (d). \qed

**Corollary 4.2.** Let $\pi : X \to C$ be an isotrivial elliptic fibration and $E$ be the generic common fiber. If $X$ is ordinary and $E$ is supersingular, then $C = \mathbb{P}^1$ and $X$ is rational.

**Proof.** By the proof of the above theorem, $H^1(R^1\pi_*\mathcal{O}_X)$ should be 0. Since $\deg R^1\pi_*\mathcal{O}_X \leq 0$, $C = \mathbb{P}^1$ and $R^1\pi_*\mathcal{O}_X = \mathcal{O}_C$ or $\mathcal{O}(-1)$. If $R^1\pi_*\mathcal{O}_X = \mathcal{O}_C$, $R$ is trivial and $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_E)$, so $X$ is not ordinary. Hence $R^1\pi_*\mathcal{O}_X = \mathcal{O}(-1)$ and $X$ is rational. \qed

**Example 4.3.** There exists an isotrivial elliptic fibration whose generic fiber is supersingular. Let $E$ be a supersingular elliptic curve and $\rho_E$ be the action of $\mathbb{Z}/2$ on $E$ by $(-1)$-involution. $D = \mathbb{P}^1$ and $\rho_D$ is the action of $\mathbb{Z}/2$ given by $x \mapsto -x$ where $x$ is an affine coordinate of $\mathbb{P}^1$. Let $X \to \mathbb{P}^1$ be the isotrivial elliptic fibration associated to the $\mathbb{Z}/2$ action $\rho_E \times \rho_D$ on $E \times D$. Then $E_l(X) = 12$, so $X$ is rational and ordinary, but the generic fiber, $E$, is supersingular.

**Example 4.4.** There exists an ordinary fibred surface $\pi : X \to C$ of fiber genus $\geq 2$ such that the generic fiber is not ordinary. There is a cyclic étale covering of curves $D \to C$ of order $n$ such that $p \nmid n$, $C$ is ordinary and $D$ is not ordinary. ([13], p.76) Let $G = T = \mathbb{Z}/n$. $G$ has the canonical action on $D$. Let $E$ be an ordinary elliptic curve. $P \in E(k)$ is an $n$-torsion point. G acts on $E$ through the translation by $P$. Let $Y = (D \times E)/G$. $Y \to C$ is an isotrivial elliptic fibration with $R = 0$. Since $C$ and $E$ are ordinary, $Y$ is ordinary by Theorem 4.1.(a). But for the fibration $\pi : Y \to E/G$, every fiber of $\pi$ is isomorphic to $D$, so not ordinary.

Now assume the generic fiber $E$ of an isotrivial elliptic fibration $\pi$ is ordinary. Let us recall the Frobenius diagram of $\pi$ (2.1)

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/C}} & X^{(p)} \\
\downarrow \pi & & \downarrow \pi^{(p)} \\
C & \xrightarrow{F_C} & C
\end{array}
\]

and an exact sequence of $\mathcal{O}_{X^{(p)}}$-modules
\[
0 \to \mathcal{O}_{X^{(p)}} \to F_{X/C}\mathcal{O}_X \to B \to 0.
\]
Since $\pi$ is generically ordinary, from (2.2) we obtain
\[
0 \to F_{C^{(p)}}\mathcal{O}_X \xrightarrow{F_{X^{(p)}}} \pi_*\mathcal{O}_X \to R^1\pi_*B \to 0.
\]
Let $H = R^1\pi_*(p)B$. $H$ is a positive divisor concentrated on the image of singular fibers. Assume $d = -\deg R^1\pi_*\mathcal{O}_X$. The degree of $H$ is $d(p - 1)$.

**Definition 4.5.** We say the divisor $H$, the Hasse divisor of $\pi$ and the multiplicity of $H$ at $x \in H$, the Hasse multiplicity of $\pi$ at $x$.

**Proposition 4.6.** Let $\pi : X \to C$ be a minimal generically ordinary isotrivial elliptic fibration. For $x \in C$, $X_x$ is the fiber of $\pi$ over $x$. Then the Hasse multiplicity of $\pi$ at $x$ is

$$\frac{1}{12}(p - 1)$$

where $A$ is the Euler characteristic of $\pi$ and $a_0$ is the degree of $\pi^*$. Assume $A$ is not a multiple fiber. The Hasse multiplicity at $x$ depends only on the type of $X_x$. Indeed, $X \otimes_{\mathcal{O}_C} \mathcal{O}_{C,x}$ is the relatively minimal desingularization of the quotient of $E \times_k \mathcal{O}_{D,y}$ by $\text{Stab}_y(G)$ where $y$ is a ramification point lying over $x$.

It depends only on the type of the fiber $X_x$. Since the relative Frobenius morphism is compatible with the base change, the Hasse multiplicity is also determined by $X \otimes_{\mathcal{O}_C} \mathcal{O}_{C,x}$. Now consider the case $D = \mathbb{P}^1$, $G = R = \mathbb{Z}/2$ and $G$ action on $D$ is given by $x \mapsto -x$, where $x$ is an affine coordinate of $\mathbb{P}^1$. Let $X$ be the minimal isotrivial elliptic fibration associated to $(E \times D)/G$. Then $E_i(X) = 12$ and $G$ has two fibers of type $I_0^0$. The degree of $H$ is $p - 1$, so the Hasse multiplicity at the image of a fiber of type $I_0^0$ is $\frac{1}{12}(p - 1)$ which is $\frac{1}{12}(p - 1)$ times the Euler characteristic of the fiber. Assume $j(E) = 1728$. $G = R = \mathbb{Z}/4$ acts on $E$ via an isomorphism $\epsilon : \mathbb{Z}/4 \simeq \text{Aut} E$. The minimal fibration $\pi : X \to \mathbb{P}^1$ associated to $(E \times E)/G$ has 1 fiber of type $I_0^0$ and 2 fibers of type $III$. $E_i(X) = 12$ and $\deg H = (p - 1)$. Hence the Hasse multiplicity at the image of a fiber of type $III$ is $\frac{1}{12}(p - 1)$. Now assume $E' = E$ and $G = \mathbb{Z}/4$ acts on $E'$ through the other isomorphism $-\epsilon : \mathbb{Z}/4 \to \text{Aut} E$. The fibration $\pi' : X' \to \mathbb{P}^1$ associated to $(E \times E)/G$ has 1 fiber of type $I_0^0$ and 2 fibers of type $III^*$ and $E_i(X) = 24$. Hence $\deg H = 2(p - 1)$ and the Hasse multiplicity at the image of the fiber of type $III^*$ is $\frac{1}{12}(p - 1)$. When $j(E) = 0$, there are two actions of $\mathbb{Z}/3$ on $E$ and two actions of $\mathbb{Z}/6$ on $E$. It follows by a similar arguments to the above, that the Hasse invariants at the image of fiber of type $II$, $II^*$, $IV$ and $IV^*$ are $\frac{1}{6}(p - 1)$, $\frac{1}{6}(p - 1)$, $\frac{1}{3}(p - 1)$ and $\frac{1}{3}(p - 1)$ respectively. \[\square\]

**Remark 4.7.** Let $h$ be the Hasse multiplicity of $\pi$ at $x \in C$. Let $\tilde{X} = X \otimes_{\mathcal{O}_C} \mathcal{O}_{C,x}$ and $\tilde{\pi} : \tilde{X} \to \text{Spec} \mathcal{O}_{C,x}$. Then $R\tilde{\pi}_*(B|\tilde{X})$ is quasi isomorphic to

$$0 \to 0 \to \mathcal{O}_{C,x}/p^h \to 0,$$

here the non-zero term occurs only at the 1st level. Therefore,

$$RH(B|X_x) = R\tilde{\pi}_*(B|\tilde{X}) \otimes (\mathcal{O}_{X,x}/m_x^h)$$

is quasi isomorphic to

$$0 \to \mathcal{O}_{X,x}/m_x^h \xrightarrow{\pi_x} \mathcal{O}_{X,x}/m_x^h \to 0,$$

here $\pi_x$ is a uniformizer of $\mathcal{O}_{X,x}$. Hence $H^0(B|X_x) = H^1(B|X_x) = k$ if $h > 0$ and both are 0 if $h = 0$. Therefore the cohomologies of $B|X_x$ does not contain information about the Hasse multiplicity, but only the “ordinarity” of the fiber. Note that $B|X_x$ is the
cokernel of the Frobenius morphism of the structure sheaf of $X_x$, so it depend only on the fiber $X_x$. The above proposition shows that for an isotrivial elliptic fibration with a ordinary generic fiber, the Hasse multiplicity of a singular fiber is just a multiple of the Euler characteristic of the fiber by a constant determined by the characteristic of the base field. However for a non-isotrivial elliptic fibration, this problem is subtle because of the existence of smooth supersingular fibers.

**Remark 4.8.** The Frobenius morphism on $H^2(O_X)$ is determined by $R^1\pi_*O_X$ and the Hasse divisor. Recall from the Frobenius diagram (2.1), $F_X = W \circ F_{X/C}$ and $H^2(O_X) = H^1(R^1\pi_*O_X)$. $W^* : H^2(O_X) \rightarrow H^2(O_{X(0)})$ is equal to the $H^1$ of the morphism

$$F_C^* : R^1\pi_*O_X \rightarrow F_{C*}O_C \otimes R^1\pi_*O_X$$

which depends only on $R^1\pi_*O_X$. And $F_{X/C}^* : H^2(O_{X(0)}) \rightarrow H^2(O_X)$ is equal to the $H^1$ of

$$F_{X/C}^* : R^1\pi_*O_X \rightarrow R^1\pi_*O_X.$$ 

When $C = \mathbb{P}^1$, $R = \mathbb{Z}/2$ and $D'$ is a hyperelliptic curve, these morphisms can be expressed precisely. Let $[x, y]$ be a projective coordinate of $\mathbb{P}^1$. Assume $R^1\pi_*O_X = O(-d)$.

$$\frac{1}{x^{d-1}y}, \frac{1}{x^{d-2}y^2}, \cdots, \frac{1}{xy^{d-1}}$$

forms a basis of $H^1(O(-d))$. The image of $W^* : H^1(O(-d)) \rightarrow H^1(O(-pd))$ is generated by

$$\frac{1}{x^{p(d-1)}y^p}, \frac{1}{x^{p(d-2)}y^{2p}}, \cdots, \frac{1}{x^{yp(d-1)}}.$$ 

We may regard the Hasse divisor $D$ as a homogeneous polynomial in $x, y$ of degree $(p - 1)d$, say $k[x, y] = \sum_{i=0}^{(p-1)d} a_i x^{i(p-1)d-i}$. The image of $\frac{1}{x^iy^k}$ by $F_{X/C}^* : H^1(O(-pd)) \rightarrow H^1(O(-d))$ is

$$a_{j-1} \frac{1}{xy^{d-1}} + \cdots + a_{j-d+1} \frac{1}{xy^{d-1}}$$

the sum of the $\frac{1}{x^{d-1}y}, \frac{1}{x^{d-2}y^2}, \cdots, \frac{1}{xy^{d-1}}$ terms of $k[x, y] = \sum_{i=0}^{(p-1)d} a_i x^{i(p-1)d-i}$. If we arrange the coefficients of $\frac{1}{x^{d-1}y}, \cdots, \frac{1}{xy^{d-1}}$ of $F_{X/C}^*(\frac{1}{x^iy^k})$ properly, it makes the Hasse matrix of the hyperelliptic curve $D'$. Hence $X$ is ordinary if and only if $D'$ is ordinary. This recovers Theorem 4.1.(b) for this case.

**Theorem 4.9.** Let $\pi : X \rightarrow C$ be an isotrivial elliptic fibration associated to an action of a finite group $G = T \times R$ on $E \times D$ defined over a number field $F$ and $D' = D/T$.

(a) If $R$ is trivial, the ordinary reduction theorem holds for $X$ if and only if it holds for $C$.

(b) If $R$ is non-trivial, the ordinary reduction theorem holds for $X$ if it holds for $D'$. 

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Proof. Since the ordinary reduction theorem holds for $E$, there are a number field $F' \supset F$ and $\mathcal{M}_{F'}$, a set of places of $F'$ of density 1, such that for each $\nu \in \mathcal{M}_{F'}$, the reductions of $E \otimes F'$ is ordinary. If $C$ satisfies the ordinary reduction theorem, there are also a number field $F'' \supset F$ and $\mathcal{M}_{F''}$, a set of places of $F''$ of density 1, such that for each $\nu \in \mathcal{M}_{F''}$, the reduction of $C \otimes F''$ is ordinary. Hence for a number field $F''' \supset F' \cup F''$, the set of places of $F'''$, $\mathcal{M}_{F'} \cap \mathcal{M}_{F''}$ has density 1 and at a place in $\mathcal{M}_{F'} \cap \mathcal{M}_{F''}$, the reductions of both of $C$ and $E$ are ordinary. By theorem 4.1, (a) is valid. The same argument can be applied to (b).

Corollary 4.10. Let $\pi : X \to C$ be an isotrivial elliptic fibration defined over a number field $F$. Assume each non-multiple singular fiber of $\pi$ is of type $I^*_0$.

(a) If $\text{gen } C = 0$ and $\chi(O_X) \leq 3$, then the ordinary reduction theorem holds for $X$.
(b) If $\text{gen } C = 1$ and $\chi(O_X) \leq 1$, then the ordinary reduction theorem holds for $X$.
(c) If $\text{gen } C = 2$ and $R^1\pi_*O_X = O_C$, then the ordinary reduction theorem holds for $X$.

Proof. Note that if $j(E) \neq 0,1728$, the assumption is valid. Since a proper smooth curve of genus $\leq 2$ satisfies the ordinary reduction theorem, by the above theorem, the claim follows.

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