Abstract. — We observe that the notion of a trivial Serre fibration, a Serre fibration, and being contractible, for finite CW complexes, can be defined in terms of the Quillen lifting property with respect to a single map $M \to \Lambda$ of finite topological spaces (preorders) of size 5 and 3. In particular, we observe that the double Quillen orthogonal $\{M \to \Lambda\}^{lr}$ is precisely the class of trivial Serre fibrations if calculated in a certain category of nice topological spaces. This suggests a question whether there is a finitistic/combinatorial definition of a model structure on the category of topological spaces entirely in terms of the single morphism $M \to \Lambda$, apparently related to the Michael continuous selection theory.

1. Introduction

Being contractible, compact (for nice spaces), trivial Serre fibration (for nice spaces, with caveats), connected, dense, extremely disconnected, zero-dimensional, and separation axioms $T_0, T_1, T_4, T_5$, can each be defined in terms of the Quillen lifting property [1] and a single map of topological spaces (preorders), usually with less than 7 points [2]. This suggests a combinatorial, computational notation for these topological properties, which could perhaps be of use in computer algebra and proof verification. This notation shows there is finite combinatorics implicit in the basic definitions of topology—what does it tell us?

In this note we show the finite combinatorics implicit in the basic definitions of contractible, trivial fibrations, and fibrations. We observe that for a certain map $M \to \Lambda$ of finite topological spaces (see Fig. 1), the double Quillen orthogonal (negation) $\{M \to \Lambda\}^{lr}$, defined below, is exactly the class of trivial Serre fibrations when calculated in a certain category of nice spaces. If we calculate the same orthogonal in the category of (all) topological spaces, we only prove that

$$a \text{ finite CW complex } X \text{ is contractible iff } X \to \{o\} \in \{M \to \Lambda\}^{lr}$$
In fact, the precise choice of the map $M \to A$ in the double Quillen orthogonal (negation) $(M \to A)^{(2)}$ is a way to add precise “niceness” assumptions to the “naive” lifting property defining fibrations:

(wf) a map $p : Y \to B$ is a trivial fibration iff the lifting property $A \to X \times Y \rightrightarrows B$
holds whenever $A \subset X$ is a “nice” closed subset of a “nice” space $X$.

(f) a map $p : Y \to B$ is a fibration iff, whenever $A \subset X$ closed and “nice”, for any
lifting problem $A \to X \times Y \rightrightarrows B$, there exists a diagonal lifting defined on some
open neighbourhood of $A^{(2)}$

We use the word “nice”, in this paper, to mean various precise assumptions of the
kind made to avoid spurious difficulties related to wild phenomena such as curves
cheerfully filling cubes, which are irrelevant from the point of view of the topological
intuition of shapes, cf. [4 §5, pp.28/29].

The definition of Serre fibration chooses the nicest possible $A \subset X$ – the inclusions of
a sphere as the boundary of a ball. Michael continuous selection theory [6 Thm.1.2]
chooses leas(?) nice ones: an arbitrary closed subset of a Hausdorff paracompact
space of finite Lebesgue dimension (see §3.1 esp. Thm.3.1.1 for a summary of [6]
Thm.1.2] of Michael continuous selection theory; also see Lemma 2.1.1(4), (2.4(ii),
and Conjecture [2.5.1].

As noted above, the map $M \to A$ does capture the implicit combinatorics of the
definition of a trivial fibration in presence of the right “niceness” assumptions, i.e. if
calculated in a certain subcategory of nice spaces, but it is not clear to us whether
this implicit combinatorics is sufficient if calculated in the category of all topological
spaces. Perhaps the reader would see this right away.

It is easy to see that the map $M \to A$ captures the “combinatorics” implicit in
the definition of normality: a space $X$ is normal ($T_4$ but not necessarily $T_1$) iff $\emptyset \to
X \times M \to \Lambda$; indeed, to give a map $X \to \Lambda$ is to give two disjoint closed subsets of

\[ \{ i : \forall p \in P \ i \prec p \}, \ P^{*}\ = \{ p : \forall i \in P \ i \prec p \}, \ P^{*}_{r} := (P^{*})^{r} \ldots \]

Taking the orthogonal of a class $P$ is a simple way to define a class of morphisms excluding non-
iso morphisms from $P$, in a way which is useful in a diagram chasing computation, and is often used
to define properties of morphisms starting from an explicitly given class of (counter)examples. For
this reason, it is convenient and intuitive to refer to $P^{d}$ and $P^{*}$ as left, resp. right, Quillen negation
of property $P$. See [1] for a quick explanation and some examples.

\footnote{(1)Recall that a morphism $i$ in a category has the left lifting property with respect to a morphism $p$,
and $p$ also has the right lifting property with respect to $i$, denoted $i \prec p$, iff for each $f : A \to X$ and
g : $B \to Y$ such that $p \circ f = g \circ i$ there exists $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.

For a class $P$ of morphisms in a category, its left orthogonal $P^{d}$ is the class of all morphisms which have the left, respectively
right, lifting property with respect to each morphism in the class $P$. In notation,

\[ P^{d} = \{ i : \forall p \in P \ i \prec p \} \]

Taking the orthogonal of a class $P$ is a simple way to define a class of morphisms excluding non-
iso morphisms from $P$, in a way which is useful in a diagram chasing computation, and is often used
to define properties of morphisms starting from an explicitly given class of (counter)examples. For
this reason, it is convenient and intuitive to refer to $P^{d}$ and $P^{*}$ as left, resp. right, Quillen negation
of property $P$. See [1] for a quick explanation and some examples.

\footnote{(2)Formally in notation, for any commutative square

\[ \begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow p \\
X & \xrightarrow{\emptyset} & B
\end{array} \]

there is an open $A \subset U \subset X$ and a map $\tilde{f}_{U} : U \to Y$ such that the diagram

\[ \begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow p \\
U \xrightarrow{\emptyset_{U}} & \xrightarrow{\tilde{f}_{U}} & B
\end{array} \]

commutes. The diagram chasing rendering of this uses the non-Hausdorff mapping cone of $Y \rightrightarrows B$.}
X (the preimages of the two closed points of Λ), and to give a factorisation X → Λ is to give their disjoint neighbourhoods (the preimages of the open subsets of M separating the preimages of the two closed points of Λ). Instead of M → Λ, one may consider the more complicated map implicit in the definition of hereditary normal (separation axiom T5), see the proof of Lemma 2.3.1 for a discussion. Seeing that the map M → Λ captures the “combinatorics” implicit in the proof of Tietze extension theorem and, arguably, the notion of contractability, is slightly less obvious, see the proof of Lemma 2.3.1(2).

Everything in this note is very elementary: a reader is likely to improve upon our claims, and any proofs can be given as exercise to any student familiar with the terminology.

Structure of the paper. — As a warm-up the reader may want to skip, §2.1-2.2 define connected, quotient, and compact in terms of maps of spaces with at most 2 points, as

\[
\{\varnothing \to \{o\}\}_{rll}, \{\varnothing \to \{o\}\}_{lrrrl}, \text{ and } \left(\{\{o\} \to \{o \to c\}\}\right)_{lr}^{r}
\]

In §2.1 we also define a few other notions starting with the simplest possible map, the inclusion of the empty space into a singleton, and in Appendix §3.3 we list a few more. In §2.2 we define the class of proper maps of nice spaces. §2.3 and §2.4 is the main body of the paper. In §2.3 we discuss the definition of trivial fibrations, and in §2.4 we discuss the definitions of fibrations, trivial fibrations, and Michael selection theory. This enables us to conjecture a “finitistic”/computational model structure in §2.5.

In Appendix §3.1 we state [6, Thm.1.2] of continuous selection theory we use, and Appendix §3.2 we state the theorems of [12] we use for compactness.

2. Observations

A number of basic notions in topology can be concisely defined, often starting from simplest examples, by repeatedly taking the orthogonal with respect to the Quillen lifting property in the category of topological spaces [1, 2, 3].

Here is a sample: connected, compact, and contractible; see [2] for a longer list.

2.1. Connected. — Being connected can be defined using the simplest possible map, the embedding of the empty set into a singleton.

**Lemma 2.1.1 (ϕ → \{o\}).** — In the category of (all) topological spaces,

- r: \{ϕ → \{o\}\}\textsuperscript{r} is the class of surjections
- rl: \{ϕ → \{o\}\}\textsuperscript{rl} is the class of maps A → A ∪ D where D is discrete
- rrr: \{ϕ → \{o\}\}\textsuperscript{rrr} is the class of subsets, i.e. the inclusions i: A → B where A is a subset of B, and i(a) = a, a ∈ A.
- lrrrl: \{ϕ → \{o\}\}\textsuperscript{lrrrl} is the class of quotients, i.e. the maps f: A → B such that a subset U ⊆ B is open in B iff its preimage f⁻¹(U) ⊆ A is open in A.
- rll: A map f: A → B of “nice” spaces belongs to \{ϕ → \{o\}\}\textsuperscript{rll} iff the induced map π₀(f): π₀(A) → π₀(B) of connected components is surjective. In particular,
A topological space \( X \) is connected iff for each, equiv. any, map \( \{ o \} \to X \) from a singleton it holds

\[ \{ o \} \to X \in \{ \emptyset \to \{ o \} \}^{rl} \]

Here in (r) and (rl), \( A \cup D \) denotes the disconnected union of \( A \) and \( D \), i.e. both subsets \( A \) and \( D \) are closed and open, and the topology on both \( A \) and \( D \) is induced.

In (rll), by a space being “nice” we mean that it splits into a disconnected union of closed and open connected components.

Proof. — 1. By definition

\[ \{ \emptyset \to \{ o \} \}^{r} := \left\{ X \overset{\varnothing}{\to} Y : \emptyset \to \{ o \} \times X \overset{\varnothing}{\to} Y \right\} \]

is the class of maps which have the right lifting property with respect to the embedding of the empty subset into a singleton. This lifting property says that any point of \( Y \) (the image of \( \{ o \} \) in \( Y \)) has a preimage in \( X \) (the image of \( \{ o \} \) in \( X \)), i.e. is surjective.

2. By definition

\[ \{ \emptyset \to \{ o \} \}^{rr} = \left\{ X \overset{\varnothing}{\to} Y : f \circ g \text{ for any } f \in \{ \emptyset \to \{ o \} \}^{r} \right\} \]

is the class of maps which have the right lifting property with respect to any surjection.

If map \( g : X \to Y \) represents a subset, i.e. \( X \subset Y \), the topology on \( X \) is induced from \( Y \), and if \( X = id \) \( X \), then the image of \( B \to Y \) is contained in \( X \), and, as the topology on \( X \) is induced, the lifting is continuous. In the opposite direction, take \( B \) to be the image of \( g : X \to Y \), and \( A \) to be the preimage of \( g : X \to Y \) with topology induced from \( Y \). Then \( f \circ g \) lifts iff \( g : X \to Y \) represents a subset. Rest is similar.

2.2. Compact. — Perhaps the simplest example of a map which is not closed (and thereby not proper), is the embedding of a point as the open point in the two-point space with one point open and one point closed. We denote this map by \( \{ o \} \to \{ o \to c \} \).

Lemma 2.2.1 (\( \{ o \} \to \{ o \to c \} \)). — In the category of (all) topological spaces, the class \( \{ \{ \{ o \} \to \{ o \to c \} \}^{r} \}_{5}^{r} \) is a class of proper maps, and

- a map of “nice” spaces is proper iff it lies in \( \{ \{ o \} \to \{ o \to c \} \}^{r} \}_{5}^{r} \)

In particular, a Hausdorff space \( K \) is compact iff

\[ K \to \{ o \} \in \{ \{ o \} \to \{ o \to c \} \}^{r} \}_{5}^{r} \]

Here, “nice” may be taken to mean Hausdorff hereditary normal (separation axioms \( T_{1} \) and \( T_{3} \)), and \( \{ o \} \to \{ o \to c \} \) \( r \) \( 5 \) denotes the subclass of \( \{ o \} \to \{ o \to c \} \) \( r \) consisting of maps of spaces with less than 5 points.

Proof. — See [3.2] or [14] §2.2 for a verbose explanation; here we are brief. First check that a map \( f \) of finite spaces is closed, equiv. proper, iff \( \{ o \} \to \{ o \to c \} \times f \). The definition of being proper via ultrafilters (see Bourbaki [12] I§10.2, Th.1(d)], quoted
A SUGGESTION TOWARDS A FINITIST’S REALISATION OF TOPOLOGY

5

in \[3.2\] expresses being proper as a lifting property with respect to a class of maps associated with ultrafilters: \(f\) is proper iff

\[A \to A \cup U \{\infty\} \setminus X \overset{f}{\to} Y\]

where the topology on \(A \cup U \{\infty\}\) is such that \(\infty\) is closed, \(U\) is the neighbourhood filter of \(\infty\), and the topology on \(A\) is induced \[12\] \S 6.5, Def. 5, Example]. These maps belong to \(\{(\{o\} \to \{o \to c\})_{<5}\}^l\), hence any map in \(\{(\{o\} \to \{o \to c\})_{<5}\}^r\) is proper.

Smirnov-Vulikh-Taimanov theorem \[13\] 3.2.1,p.136] gives sufficient conditions to extend a map to a compact Hausdorff space, and can be generalised to give the required lifting property. It says that a map to a compact Hausdorff space can be extended to the whole space \(X\) from a dense subset \(A\) satisfying (in fact the necessary) condition for every pair \(B_1, B_2\) of disjoint closed subsets of \(A\) the inverse images \(f^{-1}(B_1)\) and \(f^{-1}(B_2)\) have disjoint closures in the space \(X\). A verification shows that the following four maps are closed and their left orthogonals define these sufficient conditions on \(A \to X\) \[3\]

\[
\begin{align*}
&\{a \leftarrow u \to b\} \to \{a = u = b\} & (\text{disjoint closures}) & \{a \leftrightarrow b\} \to \{a = b\} & (\text{injective}) & \{o \to c\} \to \{o = c\} & (\text{pullback topology}) & \{c\} \to \{o \to c\} & (\text{dense image}) \\
&\{a \leftarrow u \to b\} & \{a \leftrightarrow b\} & \{o \to c\} & \{o \to c\}
\end{align*}
\]

Hence, the Smirnov-Vulikh-Taimanov theorem \[13\] 3.2.1,p.136] implies that a Hausdorff space \(K\) to \(\{o\}\) is in

\(\{(a \leftarrow u \to b) \to \{a = u = b\}, (a \leftrightarrow b) \to \{a = b\}, (o \to c) \to \{o = c\}, (c) \to \{o \to c\}\}^l\),

and the latter is a subclass of \(\{(\{o\} \to \{o \to c\})_{<5}\}^r\).

Is it useful to say that these four maps of preorders reveal combinatorics implicit in the notion of compactness?

Note that for this statement it is important that the category of topological spaces contains spaces associated with ultrafilters that would usually be considered to belong to wild phenomena such as curves cheerfully filling cubes, which are irrelevant from the point of view of the topological intuition of shapes, cf. \[4\] § 5, pp 28/29.

2.3. Contractible. — To define contractible (among “nice” spaces), it is enough to consider a morphism \(M \to \Lambda\) from a space \(M\) with 5 points (two open and three closed), into a space \(\Lambda\) with 3 points (one open and two closed), see Fig. 1

**Lemma 2.3.1** \((M \to \Lambda)\). — *In the category of (all) topological spaces, \(\{M \to \Lambda\}^l\) is a class of trivial Serre fibrations, and*

\[\text{(3)}\] Our notation represents finite topological space as preorders or finite categories with each diagram commuting, and is hopefully self-explanatory; see \[3\] for details. In short, an arrow \(a \to c\) indicates that \(c \in cl a\), and each point goes to “itself”; the list in \(\{..\}\) after the arrow indicates new relations/morphisms added, thus in \(\{o \to c\} \to \{o = c\}\) the equality indicates that the two points are glued together or that we added an identity morphism between \(o\) and \(c\). The notation in the 3rd line informal (red indicates new/added elements), and in the 4th line reminds of a computer syntax.
1. A “nice” space \( Y \) is contractible iff
   \[ Y \rightarrow \{ o \} \in \{ M \rightarrow \Lambda \}^{lr} \]

2. \( X \) is normal (not necessarily Hausdorff) iff \( \varnothing \rightarrow X \in \{ M \rightarrow \Lambda \} \), i.e.
   \[ \varnothing \rightarrow X \times M \rightarrow \Lambda \]

3. For a map \( A \rightarrow X \) from a Hausdorff space \( A \) to a “nice” (meaning Hausdorff hereditary normal) space \( X \), it represents a closed subset \( A \subset X \) iff
   \[ A/\text{uni}B \rightarrow X \in \{ M \rightarrow \Lambda \} \], i.e.
   \[ A/\text{uni}B \rightarrow X/\text{uni}M \rightarrow \Lambda \]

In (1), “nice” may be taken to mean “being a finite CW complex”\(^{(4)}\). What we need is that \( Y \) is a retract of some Euclidean space \( \mathbb{R}^n \) iff \( Y \) is weakly contractible.

Of course, this Lemma tempts a conjecture

**Conjecture 2.3.2.** — A map of “nice” spaces is a trivial fibration iff it belongs to \( \{ M \rightarrow \Lambda \}^{lr} \).

**Proof.** — Recall that \( \{ M \rightarrow \Lambda \}^{lr} = \left\{ Y \xrightarrow{p} B : A \xrightarrow{i} X \xrightarrow{p} B \text{ whenever } A \xrightarrow{i} X \times M \rightarrow \Lambda \right\} \)

Thus, to see that \( \{ M \rightarrow \Lambda \}^{lr} \) is a class of trivial Serre fibrations it is enough to verify that \( S^n \rightarrow \mathbb{D}^{n+1} \in \{ M \rightarrow \Lambda \}^{lr} \), where \( S^n \rightarrow \mathbb{D}^{n+1} \) denotes the standard embedding of an \( n \)-sphere into the \( n+1 \)-ball as the boundary. We skip this, and only remark that to verify that \( S^n \rightarrow \mathbb{D}^{n+1} \times M \rightarrow \Lambda \) we need to use that \( \mathbb{D}^{n+1} \) is hereditary normal.\(^{(5)}\)

(2). To give a map \( X \rightarrow \Lambda \) is to give two disjoint closed subsets of \( X \); to give a lifting to \( M \) is to find their disjoint neighbourhoods. (1). It is enough to show that for \( Y = [0,1] \): indeed, \( \sim \)-orthogonals are closed under products and retracts, and any contractible finite CW complex is a retract of some \([0,1]^{n+1}, n > 0\] \(^{(10)}\). The proof for \( Y = [0,1] \) we give is the standard proof of the Tietze extension theorem retold in a diagram chasing notation.

Represent the interval \([0,1] \) as a union

\[ [0,1] = \{ 0 \} \cup (0,t_1) \cup \{ t_1 \} \cup (t_1,t_2) \cup \ldots \cup (t_{n-1},1) \cup \{ 1 \} \]

(4) As pointed out by Tyrone at mathoverflow.net “nice” may not taken to mean being a CW complex: Let \( C^\infty \) be the cone over a countably infinite discrete complex (this is a contractible 1-dimensional polyhedron). van Douwen and Pol \(^{(7)}\) have constructed a countable regular \( T_2 \) space \( X \) which is perfectly normal and a function \( A \rightarrow C^\infty \), defined on a certain closed \( A \subset X \), which does not extend over any neighbourhood in \( X \). In particular, the map of countable complexes \( C^\infty \rightarrow (o) \) is both a Hurewicz fibration and a homotopy equivalence, but is not soft wrt all perfectly normal pairs.

(5) Namely, use the following characterisation: a space is hereditary normal iff whenever each of two disjoint subsets can be separated from the other by an open neighbourhood, they have disjoint open neighbourhoods. \(^{(3)}\) represents this as a lifting property.
Contract the open intervals to (open) points, and denote the resulting map by $[0,1] \to \Lambda_n$ where $\Lambda_n = \{ 0 \leftarrow \{ t_0, t_1 \}, \ldots, \leftarrow \{ t_{n-1}, t_n \} \}$. Subdividing the open intervals gives maps $\Lambda_{2n} \to \Lambda_n$. The map $\Lambda_1 = M \to \Lambda = \Lambda_0$ corresponds to subdividing a single open interval into two. Use that $^r$-orthogonals are closed under pullbacks to see that $\Lambda_{2n} \to \Lambda_n \in \{ M \to \Lambda \}^{lr}$, and that $^r$-orthogonals are closed under inverse limits to see that $\Lambda_\omega \to \Lambda \in \{ M \to \Lambda \}^{lr}$ where $\Lambda_\omega := \lim_{\Lambda_{2n} \to \Lambda_n} \Lambda$. Finally, the maps $[0,1] \to \Lambda_n$ induce an embedding $[0,1] \to \Lambda_\omega$ of $[0,1]$ into $\Lambda_\omega$ as a retract, hence, an orthogonals are closed under retract, we get the required result. (3). Pick a map sending $X$ to the open point of $\Lambda$, and the separating neighbourhoods of two distinct points of $A$ to the two open points of $M$. A lifting would provide separating neighbourhoods of their images. Therefore, the map $A \to X$ is injective. To see that it is closed, pick a map sending the whole of $A$ to the closed point in the “middle” of $M$, and an arbitrary point $x$ of $X - A$ into a closed point of $\Lambda$. A lifting would provide neighbourhood of $x$ disjoint from $A$. To see that the topology on $A$ is induced, Pick a map $X \to \Lambda$ sending $X$ to the open point of $\Lambda$, and a map $A \to M$ sending an arbitrary open subset $U$ of $A$ into an open point of $\Lambda$. A lifting would provide an open subset of $X$ whose intersection with $A$ is $U$.  

2.4. The naive defining lifting property of a fibration. — If all spaces were “nice”, we could perhaps define fibrations and trivial fibrations as follows:

(wf) a map $p : Y \to B$ is a trivial fibration iff the lifting property $A \to X \times Y \mathrel{\xrightarrow{p}} B$ holds whenever $A \subset X$ is a closed subset of a space $X$.

(f) a map $p : Y \to B$ is a fibration iff, whenever $A \subset X$ closed, for any lifting problem $A \to X \times Y \mathrel{\xrightarrow{p}} B$, there exists a diagonal lifting defined on some open neighborhood of $A$.\(^{[6]}\)

\(^{[6]}\) Formally in notation, for any commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \phi \\
X & \xrightarrow{p} & B
\end{array}
\]

there is an open $A \subset U \subset X$ and a map $f_U : U \to Y$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \phi \\
U & \xrightarrow{\phi_U} & B
\end{array}
\]

commutes.
In (f), we get the definition of trivial Serre fibration if we restrict \( A \subset X \) to be cellular inclusions of finite CW complexes, or indeed just the inclusions \( S^n \to \mathbb{B}^{n+1} \) of an \( n \)-sphere as the boundary of \( n+1 \)-ball, \( n \geq 0 \).

Michael selection theory (see §3.1) says that we do get the standard notions of a trivial fibration, and of a fibration, if we take \( X \) to vary among paracompact spaces of finite Lebesgue dimension; then it is sufficient for \( p : Y \to B \) to be a map of complete metric spaces with uniformly contractible fibres, i.e. a map of topological spaces admitting complete metrics such that there are \( \delta, \varepsilon > 0 \) such that in any fibre any ball of radius \( \delta \) is contractible within a ball of radius \( \varepsilon \) (in the fibre). These assumptions come from Michael continuous selection theory [6, Thm.1.2], see §3.1.

We rewrite (wf) and (f) in the diagram chasing manner using Lemma 2.4.1 and the notion of non-Hausdorff mapping cone/cylinder:

\[
\text{Lemma 2.4.1. — In a full subcategory of \"nice\" topological spaces,}
\]
\[
\begin{align*}
\text{wf}' & \quad \text{a \"very nice\" map is a trivial fibration iff it belongs to } \{ M \to \Lambda \}^{ir} \\
\text{f}' & \quad \text{a \"very nice\" map is a fibration iff the map from its non-Hausdorff mapping cone to the base belongs to } \{ M \to \Lambda \}^{ir}
\end{align*}
\]

\[
Y^p_{\to B} \to B \in \{ M \to \Lambda \}^{ir}
\]

Here, being \"nice\" means being (possibly non-Hausdorff) paracompact of finite Lebesgue dimension, and \"very nice\" means say a map of finite CW complexes or being smooth in a suitable sense (we need something to ensure that a fibration is necessarily a map of complete metrisable spaces with uniformly locally contractible fibres), and \( Y^p_{\to B} \) denotes the non-Hausdorff mapping cone of \( Y \to B \).

\[\text{Proof. — Recall that}\]
\[
\{ M \to \Lambda \}^{ir} = \left\{ Y^p_{\to B} : A^i \otimes X^p_{\to B} \text{ whenever } A^i \otimes X \to M \to \Lambda \right\}
\]

A map to a Hausdorff spaces necessarily glues together points which cannot be separated by neighbourhoods (for their images can if distinct), hence we may assume that both \( A \) and \( X \) are Hausdorff and by Lemma 2.4.1(3) that \( A^i \otimes X \) is the inclusion of a closed subset. Hence, (f)' states precisely (f) above, i.e. the conclusion of Michael selection theorem Theorem [5.1.1] for trivial fibrations.

Similarly, (wf)' is (wf) using the diagram chasing property of the non-Hausdorff mapping cone:

---

\[\text{Intuitively, this is the usual (Hausdorff) mapping cone } Y \times [0,1]/\{(y,1) = p(y)\} \text{ where we replaced } [0,1] \text{ by the two-point Sierpinski-Kolmogorov space } \{\cdot, \cdot\}. \text{ Formally, the non-Hausdorff mapping cone/cylinder of a map } p : Y \to B, \text{ denoted by } Y^p_{\to B}, \text{ is } Y \times \{\cdot, \cdot\}/\{\cdot, c\} = p(\cdot), \text{ i.e. the disjoint union } Y \cup B \text{ equipped with the following topology: an open subset is either an open subset of } X, \text{ or the union of an open subset of } B \text{ and its preimage.}\]
– is to give a map $X \to Y_p \leftarrow B$ is the same as to give a commutative square

\[
\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\downarrow & & \downarrow p \\
X & \xleftarrow{\phi} & B \\
\end{array}
\]

for some open subset $U$ of $X$.

Indeed, this means that the lifting property $A \xrightarrow{i} X \xleftarrow{\phi} Y_p \leftarrow B$ of item (f)' holds iff for any open subset $U$ of $A$ and a commutative square

\[
\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\downarrow & & \downarrow p \\
X & \xleftarrow{\phi} & B \\
\end{array}
\]

there is an open $U \subset V \subset X$ and a map $\tilde{f}_V : V \to Y$ such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\downarrow & & \downarrow p \\
V & \xleftarrow{\jmath_V} & \tilde{f}_V \\
\end{array}
\]

commutes. This is almost the conclusion of Michael selection theorem Theorem 3.1.1 for fibrations as stated.

Finally, by Lemma 2.4.1(3), $\{M \to \Lambda\}_l$ contains the inclusion $S^n \to \mathbb{B}^{n+1}$ of an $n$-sphere as the boundary of $n + 1$-ball, $n \geq 0$, and thereby $\{M \to \Lambda\}_r$ is a subclass of trivial Serre fibrations. The reader will find it an exercise (check this!) to see that (f)' implies that $Y \xrightarrow{p} B$ is a fibration under suitable assumptions.

2.5. A naive “combinatorial” model structure. — Considerations above suggest the following conjecture. The idea is to use $M \to \Lambda$ to make precise niceness assumptions in the naive lifting property of fibrations.

**Conjecture 2.5.1** ($M \to \Lambda$). — A closed model structure on the category of topological spaces is defined as follows:

- $\{M \to \Lambda\}_l$ is the class of cofibrations.
- $\{M \to \Lambda\}_r$ is the class of trivial fibrations.
- $\{Y \xrightarrow{\jmath} B : |Y| < \infty, |B| < \infty, \text{ and } Y \xrightarrow{\jmath} B \in \{M \to \Lambda\}_r\}_l$ is the class of trivial cofibrations.
- $\{Y \xrightarrow{p} B : |Y| < \infty, |B| < \infty, \text{ and } Y \xrightarrow{p} B \in \{M \to \Lambda\}_r\}_r$ is the class of fibrations.
- a weak equivalence is the composition of a trivial cofibration with a trivial fibration.
The language of this conjecture is purely combinatorial. Can we define a model category of “formal” topological spaces (“formal” as in formal power series), i.e. a model category whose objects and arrows belong to a calculus of diagram chasing computations, so to say? A naive hope is that the size of spaces appearing in the Quillen orthogonals (negations) representing basic notions of topology \[2, 3\] is small enough (< 7) to make feasible the exponential growth in the computer processing of such a calculus.

The following would represent a rule in such a diagram chasing calculus of formal topological spaces.

**Conjecture 2.5.2 (M2).** — For each finite set \( P \) of maps of finite spaces, and each string consisting of letters \( l \) and \( r \), each map in the category of topological spaces decomposes as a map in \((P)^{st}\) followed by a map in \((P)^{sr}\), and as a map in \((P)^{rt}\) followed by a map in \((P)^{rs}\):

Of course, the real temptation is to develop a computer algebra system doing topology using a syntax extending the concise syntax for topology we discuss, and to use it in teaching.

3. **Appendix.**

3.1. **Michael continuous selections.** — We sketch the statement of the Michael continuous selections theorem \[6\] Thm.1.2 we use, see also \[5, 7\].

Let \((F_x)_{x \in X}\) be a family of non-empty subsets of a topological space \( Y \). Michael selection theory thinks of such a family as a multivalued function \( \phi : X \to 2^Y \) and refers to the family as a *carrier*. Michael selection theory gives sufficient conditions for existence of a continuous choice function \( f(x) \in F_x, x \in X \). These conditions are satisfied when the family \((F_x)_{x \in X}\) is the family of fibres of a fibration of "nice" spaces.

\[7\] considers families of convex subsets of a Banach space but we do not discuss it here.

The family \((F_x)_{x \in X}\) is *lower semi-continuous* iff, whenever \( U \subset Y \) is open in \( Y \), the subset \( \{ x \in X : F_x \cap U \neq \emptyset \} \) is open in \( X \). This subset can be thought of as the preimage of \( U \) under the multivalued function \((F_x)_{x \in X}\).

The family \((F_x)_{x \in X}\) is *uniformly locally \( n \)-contractible* iff, for every \( x \in X \) and every \( y \in F_x \), and every neighbourhood \( U \) of \( y \) in \( Y \), there exists a neighbourhood \( V \) of \( y \) in \( Y \) such that, for every \( F_x, x' \in X \), every continuous image of an \( m \)-sphere \((m \leq n)\) in \( F_x \cap V \) is contractible in \( F_x \cap U \). By convention, as there are no \( m \)-spheres for
We repeat the conclusion in notation: for every continuous choice function \( f : A \to Y \) such that \( f(x) \in F_x \) whenever \( x \in A \), there is an open neighbourhood \( U \subset A \) of \( A \)

---

(8) The usual definition is in terms of open coverings. We combine [7] §9 and [8]:

"A open covering of a topological space \( X \) is, in [8], a collection of open subsets of \( X \) whose union is \( X \). Its elements need not be open unless that is specifically assumed. A refinement of a covering \( \mathcal{U} \) is a covering \( \mathcal{V} \) such that every \( V \in \mathcal{V} \) is a subset of some \( U \in \mathcal{U} \). A covering \( \mathcal{U} \) is point-finite if every \( x \in X \) is an element of only finitely many \( U \in \mathcal{U} \), it is locally finite if every \( x \in X \) has a neighbourhood intersecting only finitely many \( U \in \mathcal{U} \).

Call a collection \( \mathcal{U} \) of subsets of a topological space closure-preserving if, for every subcollection \( \mathcal{V} \in \mathcal{U} \), the union of closures is the closure of the union (i.e. every \( x \in X \) such that \( \text{cl}(\cup \mathcal{V}) \ni y \) whenever \( y \in x \)) is in \( \text{cl}(\cup \mathcal{U}) \).

---

(9) These notions can probably be expressed as lifting properties as follows. To give a finite open, resp. closed, covering \( \mathcal{U} \) is to give a map \( X \to \{ V : \emptyset \neq V \subset \mathcal{U} \} \) where the topology is defined by the order \( V_1 \to V_2 \) if \( V_1 \supset V_2 \), resp. \( V_1 \subset V_2 \). To give a finite open covering \( \mathcal{U} \) of order \( \leq n \) is to give a map \( X \to \{ V : \emptyset \neq V \subset \mathcal{U}, |V| \leq n + 1 \} \). A finite open covering \( \mathcal{U} \) of \( X \) has a finite, open refinement \( \mathcal{V} \) of order \( \leq n \) if \( \emptyset \to X \times \{ (W, V) : \emptyset \neq W \subset \mathcal{U}, |V| \leq n + 1 \} \) to \( \{ V : \emptyset \neq V \subset \mathcal{U} \} \) where the topology is generated by the orders \( W_1, V_2 \to (W_2, V_2) \) iff \( W_1 \supset W_2 \) and \( V_1 \supset V_2 \), and \( V_1 \to V_2 \) iff \( V_1 \supset V_2 \).

To give a finite \( \emptyset \)-finite covering-preserving closed covering \( \mathcal{V} \) of order \( \leq n \) is to give a map \( X \to \{ V : \emptyset \neq V \subset \mathcal{U}, |V| < \omega \} \) where the topology is defined by the order \( V_1 \to V_2 \) if \( V_1 \subset V_2 \) (sic!). An open covering \( \mathcal{U} \) has a finite-pointwise closed-preserving refinement \( \mathcal{V} \) iff \( \emptyset \to X \times \{ (W, V) : \emptyset \neq W \subset \mathcal{U}, |V| < \omega \} \) to \( \{ V : \emptyset \neq V \subset \mathcal{U} \} \) where the topology on the domain is defined by order \( (W_1, V_2) \to (W_2, V_2) \) iff \( W_1 \subset W_2 \) (sic!) and \( V_1 \supset V_2 \), on the target by the open subsets \( \{ V \subset \mathcal{U} : \emptyset \notin V \} \), for \( U \in \mathcal{U} \).
and a continuous choice function \( \tilde{f} : U \to Y \) such that \( \tilde{f}(u) \in F_u \) whenever \( u \in U \), and \( f(a) = \tilde{f}(a) \) whenever \( a \in A \).

### 3.2. Extending maps to compact spaces.

We explain in more detail the proof in §2.3 of the characterisation of compactness. The reader may find a verbose exposition focusing on logical ideas in \([14], \S2.2\).

#### 3.2.1. Compactness via ultrafilters by Bourbaki.

Item d) of the following characterisation of proper maps by Bourbaki \([12]\) states almost a lifting property. Arguably, this suggests that the ideas/technique of category theory were present in \([12]\), although not the notation or language of category theory.

**Theorem 1.** Let \( f : X \to Y \) be a continuous mapping. Then the following four statements are equivalent:

a) \( f \) is proper.

b) \( f \) is closed and \( \tilde{f}(y) \) is quasi-compact for each \( y \in Y \).

c) If \( \mathcal{F} \) is a filter on \( X \) and if \( y \in Y \) is a cluster point of \( f(\mathcal{F}) \) then there is a cluster point \( x \) of \( \mathcal{F} \) such that \( f(x) = y \).

d) If \( \mathcal{U} \) is an ultrafilter on \( X \) and if \( y \in Y \) is a limit point of the ultrafilter base \( f(\mathcal{U}) \), then there is a limit point \( x \) of \( \mathcal{U} \) such that \( f(x) = y \).

Item d) expresses the following lifting property (almost): \( |X| \to |X| \cup \{\infty\} \times X \xleftarrow{f} Y \) where \( |X| \) denotes the set of points of \( X \) equipped with discrete topology, and the topology on \( |X| \cup \{\infty\} \) is such that \( \mathcal{U} \) is the neighbourhood filter of \( \infty \), and the induced topology on subset \( |X| \) is discrete \([12] \S6.5, \text{Def.5, Example}\).

#### 3.2.2. Extending maps to compact Hausdorff spaces.

The theorem of Vulikh-Smirnov-Taimanov \([13], 3.2.1, \text{p.136}\) is stated in the language of lifting properties almost explicitly (“compact” below stands for “compact Hausdorff”):

**Theorem.** Let \( A \) be a dense subspace of a topological space \( X \) and \( f \) a continuous mapping of \( A \) to a compact space \( Y \). The mapping \( f \) has a continuous extension over \( X \) if and only if for every pair \( B_1, B_2 \) of disjoint closed subsets of \( Y \) the inverse images \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) have disjoint closures in the space \( X \).

Let us transcribe this to the language/notation of finite topological spaces and lifting properties. We are given a dense subspace \( A \xrightarrow{f} X \) of a topological space \( X \) and a continuous mapping \( A \xrightarrow{f} Y \) of \( A \) to a [Hausdorff] compact space \( Y \). The mapping \( f \) has a continuous extension over \( X \) means that the arrow \( A \xrightarrow{f} Y \) factors via \( A \xrightarrow{i} X \) (cf. Figure 2f). A pair \( B_1, B_2 \) of disjoint closed subsets of \( Y \) is an arrow \( Y \xleftarrow{f} \{B_1 \xleftarrow{i} O \to B_2\} \) where \( \{B_1 \xleftarrow{i} O \to B_2\} \) is the space with one open point denoted by \( O \) and two closed points denoted by \( B_1 \) and \( B_2 \). To say the inverse images \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) have disjoint closures in the space \( X \) is to say that the composition \( A \xrightarrow{f} Y \xrightarrow{\{B_1 \xleftarrow{i} O \to B_2\}} \) factors as \( A \xrightarrow{i} X \xrightarrow{\{B_1 \xleftarrow{i} O \to B_2\}} \) (cf. Figure 2g).

Now we need to define the class of dense subspaces. A dense subspace is an injective map with dense image such that the topology on the domain is induced from the
target. This suggests we try to define this class by taking left Quillen negations (orthogonals) of the simplest archetypal examples of a map whose image is not dense \( \{U \to U'\} \), a non-injective map \( \{x \leftrightarrow y\} \to \{x = y\} \), and a map such that the topology on the domain is not induced from the target \( \{o \to c\} \to \{o = c\} \).

Doing so leads to the following reformulation.

**Theorem 3.2.1.** — Let \( Y \) be Hausdorff compact and let \( A \xrightarrow{i} X \) satisfy (cf. Figure 2(ijk))

(i) (dense) \( A \xrightarrow{i} X \times \{U\} \to \{U \to U'\} \)

(ii) (injective) \( A \xrightarrow{i} X \times \{x \leftrightarrow y\} \to \{x = y\} \)

(iii) (induced topology) \( A \xrightarrow{i} X \times \{o \to c\} \to \{o = c\} \)

Then the properties of \( A \xrightarrow{i} Y \) defined by Figure 2(f) and Figure 2(g) are equivalent.

This implies that, for Hausdorff compact \( Y \), items 3.2.1(i-iii) and \( A \xrightarrow{i} X \times \{B_1 \leftarrow O \to B_2\} \to \{B_1 = O = B_2\} \) imply that \( A \xrightarrow{i} X \) is of form \( A \xrightarrow{i} A \cup \{\infty\} \).

Further, note that if \( X = A \cup \{\infty\} \) is obtained from \( A \) by adjoining a single closed non-open point, then

\[
A \xrightarrow{i} X \times \{B_1 \leftarrow O \to B_2\} \to \{B_1 = O = B_2\}
\]

iff there exists an ultrafilter \( U \) such that \( A \xrightarrow{i} X \) is of form \( A \to A \cup \{\infty\} \).

This implies that maps of form \( A \to A \cup \{\infty\} \) are in \( P^{l} \) where \( P \) consists of

\[
\{B_1 \leftarrow O \to B_2\} \to \{\bullet\} \quad \{U\} \to \{U \cup U'\} \quad \{x \leftrightarrow y\} \to \{x = y\} \quad \{o \cup c\} \to \{o = c\}
\]

3.2.3. A logical point of view: the simplest counterexample negated three times.—
We took a (the?) simplest possible non-proper map, took Quillen negation thrice (although once passing to the subclass of finite spaces), and got (almost?) the definition of a proper map.

Let us explicitly state the conjecture.

**Conjecture** \( (\{\{o\} \to \{o \to c\}\})^{3}_{lr} \). — In the category of topological spaces, the following Quillen ortho-\( g \) (negation) defines the class of proper maps:

\[
(\{\{o\} \to \{o \to c\}\})^{3}_{lr}
\]

3.3. Appendix. Properties of the empty subspace of a singleton. — We give a list of properties of maps one can define starting with the simplest possible map \( \emptyset \to \{o\} \). Note that the notion of connectivity, discreteness, and quotient arises in this way.

\[2\] gives a longer list of notions one can obtain in this way starting from more complicated maps of finite topological spaces, of up to 7 points. Note compactness arises in this way, and also contractible, as we saw above.

**Lemma 3.3.1.** — In the category of (all) topological spaces,
r: \((\varnothing \to \{o\})^r\) is the class of surjections
l: \((\varnothing \to \{o\})^l\) is the class of maps \(A \to B\) where \(A \neq \varnothing\) or \(A = B\)
rr: \((\varnothing \to \{o\})^{rr} = \{(x \leftrightarrow y \leftrightarrow c) \to (x = y = c)\} = \{(x \leftrightarrow y \leftrightarrow c) \to (x = y = c)\}^l\) is the class of subsets, i.e. injective maps \(A \to B\) where the topology on \(A\) is induced from \(B\)
lr: \((\varnothing \to \{o\})^{lr}\) is the class of maps \(\varnothing \to B\), \(B\) arbitrary, and \(A = B\)
lrr: \((\varnothing \to \{o\})^{lrr}\) is the class of maps \(A \to B\) which admit a section
rl: \((\varnothing \to \{o\})^{rl}\) is the class of maps of form \(A \to A \cup D\) where \(D\) is discrete
rll: \((\varnothing \to \{o\})^{rll}\) is the class of maps \(A \to B\) such that each connected subset of \(B\) intersects the image of \(A\); for “nice” spaces it means that the map \(\pi_0(A) \to \pi_0(B)\) is surjective, where “nice” means that connected components are both open and closed.
rllr: \((\varnothing \to \{o\})^{rlr}\) is the class of maps of form \(A \to A \cup B\) where \(A \cup B\) denotes the disconnected union of \(A\) and \(B\).
lrr: \((\varnothing \to \{o\})^{lrr}\) is the class of injective maps, i.e. such that \(f(x) \neq f(y)\) whenever \(x \neq y\)
lrrr: \((\varnothing \to \{o\})^{lrrr}\) is the class of maps such that an arbitrary (not necessarily continuous) section is necessarily continuous
lrrrl: \((\varnothing \to \{o\})^{lrrrl}\) is the class of quotients, i.e. the maps \(f: A \to B\) such that a subset \(U \subseteq B\) is open in \(B\) iff its preimage \(f^{-1}(U) \subseteq A\) is open in \(A\).

Proof. — Each is an easy exercise in diagram chasing and point set topology. 

In lrrrl, we apply Quillen negation 5 times and get a meaningful notion. Can it be more than 5? I.e. can we apply Quillen negation > 5 times to something simple or natural, and still get a meaningful and/or well-known notion?

3.4. Acknowledgements. — Tyrone (Cutler?) at mathoverflow.net brought Michael selection theory to our attention. We thank Martin Bays, Sergei Ivanov, Vladimir Sosnilo, participants of the A.Smirnov seminar, and Nicolas Cianci, for helpful discussions. Readability of 2, 3 is due to Urs Schreiber. We thank Alexandroff St.Petersburg topology seminar for the invitation. It is wishful to think that our expression \((\{(o \to \{o \to c\})^5\})^r\) for compactness would have not have fit the strict finitist criteria of Nikolai Schanin, teacher of Grigori Mints, and not have been considered inherently vague.

References
[1] Wikipedia. Lifting property. https://en.wikipedia.org/wiki/Lifting_property
[2] Ncatlab. Lifting property. https://ncatlab.org/nlab/show/lift
[3] Ncatlab. Separation axioms in terms of lifting properties. https://ncatlab.org/nlab/show/separation+axioms+in+terms+of+lifting+properties
[4] Alexandre Grothendieck. Esquisse D’un programme. https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/EsquisseEng.pdf https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/EsquisseFr.pdf
A SUGGESTION TOWARDS A FINITIST’S REALISATION OF TOPOLOGY

[5] Ernst Michael. Selected Selection Theorems. The American Mathematical Monthly, (1956). 63(4), 233. doi:10.2307/2310346 https://doi.org/10.2307/2310346

[6] Ernst Michael. Continuous Selections II. The Annals of Mathematics, 64(3), 562. doi:10.2307/1969603 https://doi.org/10.2307/1969603

[7] Ernst Michael. Continuous Selections I. The Annals of Mathematics, 63(2), 361. doi:10.2307/1969615 https://doi.org/10.2307/1969615

[8] Ernst Michael. Another note on paracompact spaces. Proceedings of the American Mathematical Society. 8(4). 1957. https://www.ams.org/journals/proc/1957-008-04/S0002-9939-1957-0087079-9/S0002-9939-1957-0087079-9.pdf

[9] Eric Wofsey. On the algebraic topology of finite spaces. 3/9/2008. https://www.docdroid.net/zJ3nw2/finite-spaces-wosey-pdf

[10] Takashi Nishimura, Goo Ishikawa. Smooth retracts of Euclidean space. Kodai Math. J. 18(2): 260-265 (1995). DOI: 10.2996/kmj/1138043422.

[11] Eric K. van Douwen, Roman Pol. Countable spaces without extension properties. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), no. 10, 987-991.

[12] Nicolas Bourbaki. General Topology. §10.2, Thm.1(d), p.101 (p.106 of file) General Topology. §10.2, Thm.1(d), p.101 (p.106 of file)

[13] Ryszard Engelking. General Topology. Thm.3.2.1, p.136.

[14] A. D. Taimanov. On extension of continuous mappings of topological spaces. Mat. Sb. (N.S.), 31(73):2 (1952), 459-463. www.mathnet.ru/eng/sm5540

[15] Point set topology as diagram chasing computations. Lifting properties as instances of negation. The De Morgan Gazette 5 no. 4 (2014), 23–32, ISSN 2053-1451. http://mishap.sdf.org/mints/mints-lifting-property-as-negation-DMG_5_no_4_2014.pdf

● E-mail : mishaM3a.sdf.org kip302002@yahoo.com https://t.me/McVlr Konstantin Pimenov (SbGU) & Masha Gavrilovich (IPRERAN)
Figure 2. These are equivalent reformulations of quasi-compactness of spaces and its generalisation to maps, that of properness of maps. (a) the identity map \( K \xrightarrow{id} K \) factors as \( K \xrightarrow{id} K \cup F \{\infty\} \xrightarrow{\cdot} K \) (b) this is also equivalent to \( K \) being quasi-compact (we no longer require the arrow \( K \xrightarrow{id} K \) to be identity) (c) and in fact quasi-compact spaces are orthogonal to maps associated with ultrafilters (d) \( X \xrightarrow{id} Y \) is proper, i.e. \( d) \) If \( \mathcal{U} \) is an ultrafilter on \( X \) and if \( y \in Y \) is a limit point of the ultrafilter base \( f(\mathcal{U}) \), then there is a limit point \( x \) of \( \mathcal{U} \) such that \( f(x) = y \). [Bourbaki, General Topology, §10.2,Th.1(d)] (e) this is also equivalent to \( X \xrightarrow{id} Y \) is proper, i.e. this holds for each ultrafilter \( \mathcal{U} \) on each space \( A \) (f) The mapping \( f \) has a continuous extension over \( X \) (h) for every pair \( B_1, B_2 \) of disjoint closed subsets of \( Y \) the inverse images \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) have disjoint closures in the space \( X \) (i) the image of \( A \) is dense in \( B \) (j) the map \( A \xrightarrow{id} B \) is injective (k) the topology on \( A \) is induced from \( B \) (l) for \( X \) and \( Y \) finite, this means that the map \( X \xrightarrow{id} Y \) is closed, or, equivalently, proper.