ALGEBRAIC METHODS IN SUM-PRODUCT PHENOMENA

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Abstract. We classify the polynomials \( f(x, y) \in \mathbb{R}[x, y] \) such that given any finite set \( A \subset \mathbb{R} \) if \( |A + A| \) is small, then \( |f(A, A)| \) is large. In particular, the following bound holds: \( |A + A||f(A, A)| \gtrsim |A|^{5/2} \). The Bezout’s theorem and a theorem by Y. Stein play important roles in our proof.

1. INTRODUCTION

The sum-product problems have been intensively studied since the work by Erdős and Szemerédi [6] that there exists \( c > 0 \) such that for any finite set \( A \subset \mathbb{Z} \), one has
\[
\max(|A + A|, |A \cdot A|) \gtrsim |A|^{1+c}.
\]
Later, much work has been done either to give an explicit bound of \( c \) or to give a generalization of the sum-product theorem. One of the important generalizations is the work by Elekes, Nathanson and Ruzsa [5] who showed that given any finite set \( A \subset \mathbb{R} \), let \( f \) be a strictly convex (or concave) function defined on an interval containing \( A \). Then
\[
\max(|A + A|, |f(A) + f(A)|) \gtrsim |A|^{5/4}.
\]
Taking \( f(x) = \log x \) recovers the sum-product theorem mentioned above by Erdős and Szemerédi. An analogous result in finite field \( \mathbb{F}_p \) with \( p \) prime was proven in 2004 by Bourgain, Katz and Tao [3] that if \( p^{\delta} < |A| < p^{1-\delta} \), for some \( \delta > 0 \), then there exists \( \epsilon = \epsilon(\delta) > 0 \) such that
\[
\max(|A + A|, |A \cdot A|) \gtrsim |A|^{1+\epsilon}.
\]
This remarkable result has found many important applications in various areas (see [1], [2] for further discussions). Recently, Solymosi ([8]) applied spectral graph theory to give a similar result mentioned above by Elekes, Nathanson and Ruzsa showing that for a class of functions \( f \), one has the following bound.
\[
\max(|A + B|, |f(A) + C|) \gtrsim \min(|A|^{1/2}q^{1/2}, |A||B|^{1/2}|C|^{1/2}q^{-1/2}),
\]
for any \( A, B, C \subset \mathbb{F}_q \). This was further studied by Hart, Li and the author [7] using Fourier analytic methods showing that for suitable assumptions on the functions \( f \) and \( g \), one has the bound.
\[
\max(|f(A) + B|, |g(A) + C|) \gtrsim \min(|A|^{1/2}q^{1/2}, |A||B|^{1/2}|C|^{1/2}q^{-1/2}).
\]

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using spectral graph theory showing that if \( f \) have
\(|f| \geq 1 \)
then one has the following bound.

One may find the difficulties come from the reducibilities of the polynomials result in the reals. Namely, given non-degenerate polynomials generalization of Szemerédi-Trotter theorem by Székely [9] to establish an analogous above Erdős-Szemerédi’s sum-product theorem. Indeed, in this paper we apply a connection to the problems in incidence geometry. In particular, he applied the so-called Szemerédi-Trotter theorem to show that one can take \( c = 1/4 \) in the above Erdős-Szemerédi’s sum-product theorem. Indeed, in this paper we apply a generalization of Szemerédi-Trotter theorem by Székely [9] to establish an analogous result in the reals. Namely, given non-degenerate polynomial \( f(x,y) \in \mathbb{R}[x,y] \) (see section 2 for the definition), then for any finite set \( A \subset \mathbb{R} \) one has the following bound.

\[
\max(|A + A|, |f(A,A)|) \gtrsim \min(|A|^{2/3}q^{1/3}, |A|^{3/2}q^{-1/4}),
\]

for any \( A \subset \mathbb{F}_q \). This was also the first time using spectral graph theory to study the incidence problems (see [7], in which Fourier analytic methods were given to reprove the results by Vu). However this result is only effective when \(|A| \geq q^{1/2}\). Therefore it turns out that if one wants to extend this result to the real setting, new tools are required. As observed by Elekes [4], the sum-product problems have interesting connections to the problems in incidence geometry. In particular, he applied the so-called Szemerédi-Trotter theorem to show that one can take \( c = 1/4 \) in the above Erdős-Szemerédi’s sum-product theorem. Indeed, in this paper we apply a generalization of Szemerédi-Trotter theorem by Székely [9] to establish an analogous result in the reals. Namely, given non-degenerate polynomial \( f(x,y) \in \mathbb{R}[x,y] \) (see section 2 for the definition), then for any finite set \( A \subset \mathbb{R} \) one has the following bound.

\[
\max(|A + A|, |f(A,A)|) \gtrsim |A|^{5/4}.
\]

One may find the difficulties come from the reducibilities of the polynomials \( f(x,y) \), and this is how the Bezout’s theorem and a theorem by Y. Stein concerning the reducibility of a multi-variables polynomial come into our proof.

## 2. Algebraic Preliminaries

Given quantities \( X \) and \( Y \) we use the notation \( X \gtrsim Y \) to mean \( X \leq CY \), where the constant \( C \) is universal (i.e. independent of \( A \)). The constant \( C \) may vary from line to line but are universal. It is also clear that when one of the quantities \( X \) and \( Y \) has polynomial \( f(x,y) \) involved, the constant \( C \) may also depend on the degree of \( f \). We now state some definitions and give some preliminary lemmas. The first two definitions can be found in [11] and [10] respectively. For the convenience of the reader, we state the definitions here.

**Definition 2.1.** A polynomial \( f(x,y) \in \mathbb{R}[x,y] \) is degenerate if it can be written as \( Q(L(x,y)) \) where \( Q \) is a one-variable polynomial and \( L \) is a linear form in \( x, y \).

**Definition 2.2.** A polynomial \( f(x,y) \in \mathbb{C}[x,y] \) is composite if it can be written as \( Q(g(x,y)) \) for some \( g(x,y) \in \mathbb{C}[x,y] \), and some \( Q(t) \in \mathbb{C}[t] \) of degree \( \geq 2 \).
Definition 2.3. Given a polynomial \( f(x, y) \in \mathbb{R}[x, y] \), we use \( \deg_x(f) \) to denote the degree of \( f \) in \( x \) variable (i.e. consider \( y \) as a constant). Similarly, denote \( \deg_y(f) \) the degree of \( f \) in \( y \) variable.

The following theorem is the celebrated Bezout theorem, and the next one is a theorem by Y. Stein [9].

Theorem 2.4. (Bezout’s theorem) Two algebraic curves of degree \( m \) and \( n \) intersect in at most \( mn \) points unless they have a common factor.

Theorem 2.5. (Y. Stein) Given \( f(x, y) \in \mathbb{C}[x, y] \) of degree \( k \). Let \( \sigma(f) = \{ \lambda : f(x, y) - \lambda \text{ is reducible} \} \). Suppose \( f(x, y) \) is not composite, then \( |\sigma(f)| < k \).

We shall need a theorem by Székely [9], which is a generalization of Szemerédi-Trotter incidence theorem in the plane.

Theorem 2.6. Let \( P \) be a finite collection of points in \( \mathbb{R}^2 \), and \( L \) be a finite collection of curves in \( \mathbb{R}^2 \). Suppose that for any two curves in \( L \) intersect in at most \( \alpha \) points, and any two points in \( P \) are simultaneously incident to at most \( \beta \) curves. Then

\[
I(P, L) = |\{(p, \ell) \in P \times L : p \in \ell\}| \leq (\alpha^{1/2} \beta^{1/3}|P|^{2/3}|L|^{2/3} + |L| + \beta|P|).
\]

3. Main Results

As discussed in section 1, we will be applying the Székely’s theorem. Therefore we need to take the advantage of the non-degeneracy property of the polynomial to construct a bunch of curves which each of them has large intersections with some appropriate points set \( P \). In order to apply the Székely’s theorem efficiently, we need to control the number of the curves. It turns out that we shall need the following theorems.

Theorem 3.1. Given \( f(x, y) \in \mathbb{R}[x, y] \) of degree \( k \geq 2 \), and assume that \( \deg_x(f) \geq \deg_y(f) \). Suppose there exists distinct \( a_1, \ldots, a_{k+1}, \) and \( b_1, \ldots, b_{k+1} \) and a polynomial

\[
Q(t) = q_m t^m + q_{m-1} t^{m-1} + \cdots + q_0
\]

so that

\[
f(x, a_i) = Q(x + bi)
\]

for each \( i \). Then \( f(x, y) = Q(g(x, y)) \) for some \( g(x, y) \), and \( \deg Q \geq 2 \).

Proof. First we write \( f(x, y) = c_k x^k + \cdots + x^m (a'_{k-m} y^{k-m} + a'_{k-m-1} y^{k-m-1} + \cdots + a'_0) + x^{m-1} (a_{k-m+1} y^{k-m+1} + a_{k-m} y^{k-m} + \cdots + a_0) + h(x, y) \), where \( h(x, y) \) is the lower degree terms in \( x \) of \( f(x, y) \). By assumptions, for each \( i \) we have

\[
f(x, a_i) = c_k x^k + \cdots + x^m \left( \sum_{h=0}^{k-m} a'_h a_i^h \right) + x^{m-1} \left( \sum_{h=0}^{k-m+1} a_h a_i^h \right) + h(x, a_i)
\]

which is equal to

\[
Q(x + bi) = q_m (x + bi)^m + q_{m-1} (x + bi)^{m-1} + \cdots.
\]
We compare the coefficients of the term $x^m$. By our assumption on $a_i$, we first conclude that $f(x, y)$ doesn’t have $x^l$ terms for $l > m$, and $a'_h = 0$ for $h = 1 \sim k - m$, and $a'_0 = q_m$. We compare the coefficients of the term $x^{m-1}$ to get

$$q_mmb_i + q_{m-1} = \sum_{h=0}^{k-m+1} a_ha_i^h,$$

which gives $b_i = \sum_{h=0}^{k-m+1} a_ha_i^h - q_{m-1}$, for each $1 \leq i \leq k^2 + 1$. Now given any $x_0 \in \mathbb{R}$, $f(x_0, y) - Q(x_0 + \sum_{h=0}^{k-m+1} a_hy^h - q_{m-1})$ is a polynomial in $y$ of degree $\leq \max\{k, m(k-m+1)\} \leq k^2$, but is zero for distinct $k^2 + 1$ values of $a_i$. Therefore we conclude that $f(x, y) = Q(x + \sum_{h=0}^{k-m+1} a_hy^h - q_{m-1})$. Since we assume deg$(f) \geq 2$, we also conclude that deg$Q \geq 2$, otherwise it will contradict the assumption that deg$\sigma(x) \geq$ deg$y(f)$.

\[\square\]

**Corollary 3.2.** Given $f(x, y) \in \mathbb{R}[x, y]$ of degree $k \geq 2$. Suppose deg$_\sigma(f) \geq$ deg$_y(f)$, and there exists $(k^3 + 1)$ distinct points $S = \{(a_i, b_i)\}_{i=1}^{k^3+1}$ in the plane such that

$$f(x-a_1, b_1) = f(x-a_2, b_2) = \cdots = f(x-a_{k^3+1}, b_{k^3+1}).$$

Then $f$ is composite.

**Proof.** Suppose there are $\geq k^2 + 1$ distinct $b_i$ such that $f(x-a_1, b_1) = f(x-a_2, b_2) = \cdots = f(x-a_i, b_i)$, we then apply theorem 3.1 to get that $f$ is composite. If not, there must exist one $b \in \{b_i : (a_i, b_i) \in S\}$ such that there are $\geq k + 1$ distinct $a_i$ so that $f(x-a_1, b) = f(x-a_2, b) = \cdots = f(x-a_{k+1}, b)$. A direction computation shows that in this case the only possible is that $f$ is a one variable polynomial in $y$ of degree $\geq 2$, which is composite.

**Remark 3.3.** The assumption deg$_\sigma(f) \geq$ deg$_y(f)$ is necessary because we might have the case $f(x, y) = x + y^2$.

**Theorem 3.4.** Given non degenerate polynomial $f(x, y)$ of degree $k$. Then for any finite set $A \subset \mathbb{R}$, one has

$$|A + A||f(A, A)| \geq |A|^{5/2}.$$

Before we proceed to prove our main theorem, we observe that our non-degenerate polynomial $f(x, y)$ could be $Q(g(x, y))$ for some $Q(t) \in \mathbb{R}[t]$ and some non-degenerate polynomial $g(x, y)$. In this case, we will work on $g(x, y)$ instead of $f(x, y)$, since we are concerned the cardinality and we use a fact that $|f(A, A)| \geq \frac{1}{\text{deg}g}|g(A, A)|$, which in turn says that we can assume $f(x, y)$ is not composite. In addition, we can always assume the deg$_\sigma(f) \geq$ deg$_y(f)$, since again we are concerned $|f(A, A)|$ (for example if $f(x, y) = x + y^2$, we write it as $x^2 + y$).

**Proof.** Given $y_0 \in \mathbb{R}$, let $f_{y_0}(x) = f(x, y_0)$. We first remove the elements $b$ in $A$ such that $f(x, b)$ is identically zero. Since $f$ is of degree $k$, there are at most $k$ elements $b$ which make this happen. We now abuse the notation, let $A = A - \{b_1, .., b_k\}$, where
Let us write $T_b(x + a) = \sum_{j=0}^{k} c_j(x + a)^j$ for some $c_j$ and $T_{(a,b)}(x) = \sum_{j=0}^{k} c_jx^j$. We note that for each $(a, b)$, $T_{(a,b)}(x)$ is a polynomial of degree $\leq k$. Therefore for any two pairs $(a, b), (c, d) \in A \times A$, if $T_{(a,b)}(x)$ intersects $T_{(c,d)}(x)$ more than $k + 1$ points, then $T_{(a,b)}(x) = T_{(c,d)}(x)$. Now given $(a, b), (c, d) \in A \times A$, we say $(a, b) \sim (c, d)$ if and only if $T_{(a,b)}(x) = T_{(c,d)}(x)$. This is equivalently saying the Taylor polynomials of $f_b(x)$ and $f_d(x)$ about $a$ and $c$ respectively have the same form, i.e.

\begin{align*}
T_b(x + a) &= c_k(x + a)^k + c_{k-1}(x + a)^{k-1} + \cdots + c_0 \\
T_d(x + c) &= c_k(x + c)^k + c_{k-1}(x + c)^{k-1} + \cdots + c_0.
\end{align*}

First we observe that $T_{(a,b)}(x) = f(x - a, b)$, we now apply Corollary 3.2 to get that each equivalence class has at most $k^3$ elements. Therefore $L' = \{T_{(a,b)}(x) : a, b \in A\} / \sim$ has at least $\frac{|A|^2}{k^3}$ equivalence classes. For each equivalence class we choose one represented curve, and conclude that there are $\geq \frac{|A|^2}{k^3}$ curves, and any two of them intersect at most $k$ points. We now show that for most pairs of points in $P = (A + A) \times f(A, A)$, there are at most $k^2$ curves from $L'$ which are incident to them simultaneously. We note that if the curve $T_{(a,b)}(x)$ incident to some point $p = (x', y') \in P$, we have $f(x' - a, b) = y'$. Given any two points $p_1 = (x_0, y_0), p_2 = (x'_0, y'_0)$ in $P = (A + A) \times f(A, A)$. Consider two algebraic curves $f(x_0 - x, y) - y_0 = 0$ and $f(x'_0 - x, y) - y'_0 = 0$. If there is a curve incident to $p_1$ and $p_2$ simultaneously, then there exists a pair $(a, b)$ such that these two algebraic curves intersect at $(a, b)$. By Bezout’s theorem, there are at most $k^2$ pairs $(a, b)$ so that

\begin{align*}
f(x_0 - a, b) - y_0 &= 0 \\
f(x'_0 - a, b) - y'_0 &= 0,
\end{align*}

unless these two algebraic curves $f(x_0 - x, y) - y_0 = 0$ and $f(x'_0 - x, y) - y'_0 = 0$ have a common factor, which means

\begin{align*}
f(x_0 - x, y) - y_0 &= G(x, y)H(x, y),
\end{align*}

and

\begin{align*}
f(x'_0 - x, y) - y'_0 &= G(x, y)H'(x, y).
\end{align*}

This implies

\begin{align*}
f(x, y) - y_0 &= G(x_0 - x, y)H(x_0 - x, y),
\end{align*}

and

\begin{align*}
f(x, y) - y'_0 &= G(x'_0 - x, y)H'(x'_0 - x, y),
\end{align*}

which in turn shows that the $y$ coordinates of the points $p_1$ and $p_2$ are from $\sigma(f)$. Therefore by Stein’s theorem, we conclude that we can remove at most $|A + A|$ points from $P = (A + A) \times f(A, A)$, and any pair in the rest of points in $P$ has at most $k^2$ curves incident to them simultaneously. Therefore we let $P' = P - \{(A + A)}.
and observe that each curve \( T_{(a,b)} \in L' \) incidents to at least \( |A|/k \) points in \( P' \). We now apply theorem 2.5 on \( P' \) and \( L' \) to get

\[
\frac{|A|^2 |A|}{k^3} \leq \left( \frac{|P'| |A|^2}{k^3} \right)^{2/3},
\]

which implies \( |A + A||f(A, A)| \gtrsim |A|^{5/2} \). \( \square \)

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