SCALING LIMITS OF RANDOM PÓLYA TREES

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Abstract. Pólya trees are rooted unordered trees considered up to symmetry. We establish the convergence of uniform random Pólya trees with arbitrary degree restrictions to Aldous’s Continuum Random Tree with respect to the Gromov-Hausdorff metric. Our proof is short and elementary, and it shows that a random Pólya tree contains in a well-defined sense a large simply generated tree that is decorated with small forests.

1. Introduction and main results

The Continuum Random Tree (CRT), which was developed in the series [3, 4, 5] of seminal papers by Aldous, is known to be the universal limit of various families of random discrete combinatorial structures, see [1, 12, 11, 8, 18, 24, 26]. The most prominent examples are random plane trees given by the family trees of critical Galton-Watson processes, where the offspring distribution has finite nonzero variance.

Plane trees have a distinguished vertex, called the root, and any offspring set is endowed with an ordering. Here we consider classes of trees that are unordered and unlabelled. In particular, Pólya trees are rooted unordered trees considered up to symmetry. They are named after George Pólya who developed a framework based on generating functions in order to study their properties [25]. The main difficulty of analysing these objects in a random setting is that they do not fit into well-studied models of random trees such as simply generated trees, a fact that was widely believed and which has been rigorously established by Drmota and Gittenberger [13, Thm. 1].

Since the construction of the CRT in the early nineties it has been a long-standing conjecture [4, p. 55] by Aldous that this model of random trees also allows the CRT as a scaling limit. The convergence of binary Pólya trees, i.e., where the vertex outdegrees are restricted to the set \( \{0, 2\} \), was shown by Marckert and Miermont [23] using an appropriate trimming procedure on trees. Later, Haas and Miermont [17] proved the conjecture by extending this result. Actually, they proved a far more general result on the scaling limit of random trees satisfying a certain Markov branching property. They established the convergence for Pólya trees without degree restrictions or with vertex outdegrees in a set of the form \( \{0, 1, \ldots, d\} \) or \( \{0, d\} \) for \( d \geq 2 \). However, the question about the convergence of Pólya trees with arbitrary degree restrictions has remained open since.

The contribution of this paper is twofold. First, we establish the scaling limit for uniform random Pólya trees with arbitrary degree restrictions. Second, our proof is very short, almost elementary, and it actually reveals a striking structural property: we show that a random Pólya tree “contains” in a well-defined sense a large simply generated tree that is decorated with small forests. This enables us to construct an appropriate coupling that relates the distances in the Pólya tree with those in the corresponding simply generated tree.

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Theorem 1.1. Let $\Omega$ be an arbitrary set of nonnegative integers containing zero and at least one integer greater than or equal to two. Let $A_n$ denote the uniform random Pólya tree with $n$ vertices and vertex outdegrees in $\Omega$. Then there exists a constant $c_\Omega > 0$ such that the rescaled space $c_\Omega A_n / \sqrt{n}$ converges in the Gromov-Hausdorff sense towards the continuum random tree $T_e$ as $n \equiv 1 \mod \gcd(\Omega)$ tends to infinity.

We obtain explicit expressions for the “scaling constant” $c_\Omega$ in our proof, see (3.8) and the subsequent equations. Our method relies on a (known) way of generating random Pólya trees using the framework of Boltzmann samplers [15, 9]. Our main insight is that this allows us to show that with high probability, i.e., with probability tending to one as $n \to \infty$, the shape of the Pólya tree $A_n$ is given by a subtree $T_n$ with small subtrees that contain $O(\log n)$ vertices attached to each vertex, see also Figure 1 below. As a metric space, we argue that $T_n$ is distributed like a critical Galton-Watson tree, whose offspring distribution has finite exponential moments, conditioned on having a randomly drawn size concentrating around $n$ times a constant. In particular, by using Aldous’s fundamental result [5], see also [21, 22], we obtain that $T_n$ converges to a multiple of the CRT; moreover, the Gromov-Hausdorff distance of $A_n / \sqrt{n}$ and $T_n / \sqrt{n}$ converges in probability to zero, and this yields our result.

2. Preliminaries

2.1. Gromov-Hausdorff Convergence and the CRT. We let $K$ denote the set of isometry classes of pointed compact metric spaces and let $d_{GH}$ denote the (pointed) Gromov-Hausdorff metric on $K$. See [10, 14] for the definition of this metric and basic properties. The Continuum Random Tree (CRT) $T_e$ is a random metric space that is encoded by the Brownian excursion of duration one. We refer to [22, 20] for a detailed introduction and an extensive treatment.

2.2. Cycle Index Sums and Ordinary Generating Series. For any nonnegative integer $k$ we let $S_k$ denote the symmetric group of order $k$. Note that $S_0$ consists of a single element, the identity map of the empty set. Any permutation $\sigma \in S_k$ can be decomposed into a product of disjoint cycles. For any $i$ we let $\sigma_i$ denote the number of $i$-cycles of $\sigma$. Here we count fixpoints as 1-cycles. Let $\Omega \subset \mathbb{N}_0$ denote a subset containing zero and at least one integer $k \geq 2$. We define the cycle index sum $Z_{SET_\Omega}$ as the formal power series

$$Z_{SET_\Omega}(s_1, s_2, \ldots) = \sum_{k \in \Omega} \sum_{\sigma \in S_k} \frac{1}{k!} s_1^{\sigma_1} s_2^{\sigma_2} \cdots s_k^{\sigma_k} \in \mathbb{Q}[[s_1, s_2, \ldots]].$$

Moreover, we define the ordinary generating series $\tilde{A}_\Omega(z)$ to be the power series whose $n$th coefficient is the number of Pólya-trees with vertex outdegrees in $\Omega$. By standard combinatorial theory [7, 19], the two series are related by the equation

$$\tilde{A}_\Omega(z) = z Z_{SET_\Omega}(\tilde{A}_\Omega(z), \tilde{A}_\Omega(z^2), \ldots).$$

(2.1)

2.3. Deviation Inequalities. We will make use of the following moderate deviation inequality for one-dimensional random walks found in most textbooks on the subject.

Lemma 2.1. Let $(X_i)_{i \in \mathbb{N}}$ be family of independent copies of a real-valued random variable $X$ with $\mathbb{E}[X] = 0$. Let $S_n = X_1 + \ldots + X_n$. Suppose that there is a $\delta > 0$ such that $\mathbb{E}[e^{\delta X}] < \infty$.

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1. Arborescence is the French word for rooted tree, hence the notation $A_n$. 
for $|\theta| < \delta$. Then there is a $c > 0$ such that for every $1/2 < p < 1$ there is a number $N$ such that for all $n \geq N$ and $0 < \epsilon < 1$

$$\mathbb{P}(|S_n/n^p| \geq \epsilon) \leq 2\exp(-cn^{2p-1}).$$

3. Proof of the main theorem

In the following $\Omega$ will always denote a set of nonnegative integers containing zero and at least one integer greater than or equal to two. Moreover, $n$ will always denote a natural number that satisfies $n \equiv 1 \mod \gcd(\Omega)$ and is large enough such that rooted trees with $n$ vertices and outdegrees in $\Omega$ exist.

We start with a standard enumerative result required for later arguments.

**Proposition 3.1.** Let $\rho_\Omega$ denote the radius of convergence of the ordinary generating function $\tilde{A}_\Omega(z)$ and let $Z_{\lambda_\Omega}(s_1, s_2, \ldots)$ denote the cycle index sum. Then the following holds.

i) We have that $0 < \rho_\Omega < 1$ and $\tilde{A}_\Omega(\rho_\Omega) < \infty$.

ii) For some $\epsilon > 0$, the function $E(z, w) = zZ_{\lambda_\Omega}(w, \tilde{A}_\Omega(z^2), \tilde{A}_\Omega(z^3), \ldots)$ satisfies

$$E(\tilde{A}_\Omega(\rho_\Omega) + \epsilon, \rho_\Omega + \epsilon) < \infty.$$

iii) For some constant $d_3 > 0$, the number of Pólya trees with $n$ vertices and outdegrees in $\Omega$ is asymptotically given by

$$[z^n]\tilde{A}_\Omega(z) \sim d_3 n^{-3/2}\rho_\Omega^{-n}.$$ 

**Proof.** The proof of these well-known properties is by an application of a general enumeration result given in Bill, Burris and Yeats [6, Thm. 28]. See for example [26, Prop. 3.4 and 3.5] for a detailed argument.

The following procedure provides a way of generating Pólya trees at random. It will play a central role in the proof of our main theorem.

**Lemma 3.2.** The following recursive procedure $\Gamma\tilde{A}_\Omega(x)$ terminates almost surely and draws a random Pólya tree with outdegrees in $\Omega$ according to the Boltzmann distribution with parameter $0 < x \leq \rho_\Omega$, i.e. any tree with $n$ vertices gets drawn with probability $x^n/\tilde{A}_\Omega(x)$.

1. Start with a root vertex $v$.
2. Draw a random permutation $\sigma(v)$ from the union of permutation groups $\bigcup_{k \in \Omega} S_k$ with distribution given by

$$\mathbb{P}(\sigma(v) = \nu) = \frac{x}{\tilde{A}_\Omega(x)\nu!} \tilde{A}_\Omega(x)\nu_1 \tilde{A}_\Omega(x^2)\nu_2 \cdots \tilde{A}_\Omega(x^k)\nu_k$$

for each $k \in \Omega$ and $\nu \in S_k$. Here $\nu_i$ denotes the number of cycles of length $i$ of the permutation $\nu$. In particular, $\nu_1$ is the number of fixpoints of $\nu$.
3. If $\sigma(v) \in S_0$ return the tree consisting of the root only and stop. Otherwise, for each cycle $\tau$ of $\sigma(v)$ let $\ell_\tau \geq 1$ denote its length and draw a Pólya tree $A_\tau$ by an independent call to the sampler $\Gamma\tilde{A}_\Omega(x^{\ell_\tau})$. Make $\ell_\tau$ identical copies of the tree $A_\tau$ and connect their roots to the vertex $v$ by adding edges. Return the resulting tree and stop.

**Proof.** Note that [2.1] ensures that the weights we assign to the permutation add up to 1. The rules for the construction of Boltzmann samplers guarantee that the above procedure terminates almost surely and satisfies the claimed Boltzmann distribution, see Flajolet, Fusy and Pivoteau [15] and Bodirsky, Fusy, Kang and Vigerske [9, Ch. 5].
The Boltzmann distribution is a measure on Pólya trees with an arbitrary number of vertices. However, any tree with \( n \) vertices has the same probability, i.e., the distribution conditioned on the event that the generated tree has \( n \) vertices is uniform. This will allow us to reduce the study of properties of a random Pólya tree with exactly \( n \) vertices to the study of \( \Gamma \hat{A}_\Omega \).

**Proof of Theorem 1.1.** We begin the proof with a couple of auxiliary observations about the sampler \( \Gamma \hat{A}_\Omega(x) \) from Lemma 3.2. Let us fix \( x = \rho_\Omega \) throughout. Suppose that we modify Step 1 to ”Start with a root vertex \( v \). If the argument of the sampler is \( \rho_\Omega \) (as opposed to \( \rho_i^\Omega \) for some \( i \geq 2 \)), then mark this vertex with the color blue.”. Then the resulting tree is still Boltzmann-distributed, but comes with a colored subtree which we denote by \( T \).

Note that \( T \) is distributed like a Galton-Watson tree without the ordering on the offspring sets. By construction, the offspring distribution \( \xi \) of \( T \) is given by the number of vertices \( |F(v)| \) in \( F(v) \). Then the probability generating function of \( \xi \) is

\[
E[z^\xi] = \frac{\rho_\Omega}{\hat{A}_\Omega(\rho_\Omega)} Z_{\text{SET}}(z\hat{A}_\Omega(\rho_\Omega), \hat{A}_\Omega(\rho_\Omega^2), \hat{A}_\Omega(\rho_\Omega^3), \ldots).
\]

Moreover, for any blue vertex \( v \) we may consider the forest \( F(v) \) of the trees dangling from \( v \) that correspond to cycles of the permutation \( \sigma(v) \) with length at least two. Let \( \zeta \) denote a random variable that is distributed like the number of vertices \( |F(v)| \) in \( F(v) \). Then the probability generating function of \( \zeta \) is

\[
E[z^\zeta] = \frac{\rho_\Omega}{\hat{A}_\Omega(\rho_\Omega)} Z_{\text{SET}}(\hat{A}_\Omega(\rho_\Omega), \hat{A}_\Omega((z\rho_\Omega)^2), \hat{A}_\Omega((z\rho_\Omega)^3), \ldots).
\]

By Proposition 3.1 ii) and (3.1) and (3.2) it follows that \( \xi \) and \( \zeta \) have finite exponential moments. In particular, there are constants \( c, c' > 0 \) such that for any \( s \geq 0 \)

\[
\mathbb{P}(\xi \geq s), \mathbb{P}(\zeta \geq s) \leq ce^{-c's}.
\]

Moreover, as we argue below, \( \xi \) has average value

\[
E[\xi] = \left( \frac{\partial}{\partial s_1} Z_{\text{SET}} \right)(\hat{A}_\Omega(\rho_\Omega), \hat{A}_\Omega(\rho_\Omega^2), \ldots) \rho_\Omega = 1.
\]

This can be shown as follows. Recall that the ordinary generating series satisfies the identity \( \hat{A}_\Omega(z) = E(z, \hat{A}_\Omega(z)) \) with the series \( E(z, w) \) given by

\[
E(z, w) = zZ_{\text{SET}}(w, \hat{A}_\Omega(z^2), \hat{A}_\Omega(z^3), \ldots).
\]

In particular, we have that \( F(z, \hat{A}_\Omega(z)) = 0 \) with \( F(z, w) = E(z, w) - w \). Suppose that \( (\frac{\partial}{\partial w} F)(\rho, \hat{A}_\Omega(\rho)) \neq 0 \). Then by the implicit function theorem the function \( \hat{A}_\Omega(z) \) has an analytic continuation in a neighbourhood of \( \rho_\Omega \). But this contradicts Pringsheim’s theorem [16] Thm. IV.6], which states that the series \( \hat{A}_\Omega(z) \) must have a singularity at the point \( \rho_\Omega \) since all its coefficients are nonnegative real numbers. Hence we have \( (\frac{\partial}{\partial w} F)(\rho, \hat{A}_\Omega(\rho)) = 0 \) which is equivalent to \( E[\xi] = 1 \).

With all these facts at hand we proceed with the proof of the theorem. Slightly abusing notation, we let \( A_n \) denote the colored random tree drawn by conditioning the (modified) sampler \( \Gamma \hat{A}_\Omega(\rho_\Omega) \) on having exactly \( n \) vertices. That is, if we ignore the colors, \( A_n \) is drawn uniformly among all Pólya trees of size \( n \) with outdegrees in \( \Omega \). Moreover, let \( T_n \) denote the colored subtree of \( A_n \), and for any vertex \( v \) of \( T_n \) let \( F_n(v) \) denote the corresponding forest.
that consists of non-blue vertices. We will argue that with high probability there is a constant $C > 0$ such that $|F_n(v)| \leq C \log n$ for all $v \in T_n$. Indeed, note that by Proposition 3.1 iii)

$$
\mathbb{P}\left( |\Gamma_\rho| = n \right) = \Theta(n^{-3/2}),
$$

i.e. the probability is (only) polynomially small. Thus, for any $s \geq 0$, if we denote by $\zeta_1, \zeta_2, \ldots$ independent random variables that are distributed like $\zeta$

$$
\mathbb{P}\left( \exists v \in T_n : |F_n(v)| \geq s \right) = \mathbb{P}\left( \exists v \in \mathcal{T} : |F(v)| \geq s \mid |\Gamma_\rho| = n \right)
\leq O(n^{3/2}) \mathbb{P}(\exists 1 \leq i \leq n : \zeta_i \geq s).
$$

Using (3.3) and setting $s = C \log n$ we get that $\mathbb{P}(\zeta_i \geq s) = o(n^{-5/2})$ for an appropriate choice of $C > 0$. Thus, by the union bound

$$
\mathbb{P}\left( \forall v \in T_n : |F_n(v)| \leq C \log n \right) = 1 - o(1).
$$

The typical shape of $A_n$ thus consists of a colored tree with small forests attached to each of its vertices, compare with Figure 1. In particular, we have that the Gromov-Hausdorff distance between the rescaled trees $A_n/\sqrt{n}$ and $T_n/\sqrt{n}$ converges in probability to zero. We are going to show that there is a constant $c_\Omega > 0$ such that $c_\Omega T_n/\sqrt{n}$ converges weakly towards the Brownian continuum random tree $\mathcal{T}_e$. This immediately implies that

$$
c_\Omega A_n/\sqrt{n} \xrightarrow{(d)} \mathcal{T}_e
$$
and we are done.

We are going to argue that the number of vertices in $T_n$ concentrates around a constant multiple of $n$. More precisely, we are going to show that for any exponent $0 < s < 1/2$ we have with high probability that

$$
|T_n| \in (1 \pm n^{-s}) \frac{n}{1 + \mathbb{E}[\zeta]}.
$$

To this end, consider the corresponding complementary event in the unconditioned setting

$$
|\mathcal{T}| \notin (1 \pm n^{-s}) \frac{|\Gamma_\rho|}{1 + \mathbb{E}[\zeta]}.
$$

If this occurs, then we clearly also have that

$$
\sum_{v \in \mathcal{T}} (1 + |F(v)|) = |\Gamma_\rho| \notin (1 \pm \Theta(n^{-s}))(1 + \mathbb{E}[\zeta])|\mathcal{T}|.
$$

Let $\mathcal{E}$ denote the corresponding event. From (3.5) we know that with high probability $|F_n(v)| = O(\log n)$ for all vertices $v$ of $T_n$. Hence, with high probability, say, $|T_n| \geq n/\log^2 n$. Using again (3.4)

$$
\mathbb{P}\left( \mathcal{E} \mid |\Gamma_\rho| = n \right) = O(n^{3/2}) \mathbb{P}\left( \frac{n}{\log^2 n} \leq |\mathcal{T}| \leq n, \mathcal{E} \right) + o(1).
$$
By applying the union bound, the latter probability is at most
\[
\sum_{n/\log^2 n \leq \ell \leq n} \mathbb{P} \left( \sum_{i=1}^{\ell} (1 + \zeta_i) \notin (1 \pm \Theta(n^{-s}))(1 + \mathbb{E} [\zeta])\ell \right).
\]
Since the random variable \( \zeta \) has finite exponential moments and \( 0 < s < 1/2 \), by applying Lemma 2.1 we infer that this expression is in \( n^{-\omega(1)} \). Hence, (3.6) holds with high probability.

We are now going to prove that
\[
\sqrt{1 + \mathbb{E} [\zeta]}\sigma \mathcal{T}_n \overset{(d)}{\to} \mathcal{T}_e
\]
with \( \sigma^2 \) denoting the variance of the random variable \( \xi \). This implies that
\[
c_\Omega \mathcal{A}_n / \sqrt{n} \overset{(d)}{\to} \mathcal{T}_e \quad \text{with} \quad c_\Omega = \frac{\sqrt{1 + \mathbb{E} [\zeta]}\sigma}{2}
\]
and we are done. Note that \( \sigma \) and \( \mathbb{E} [\zeta] \) may be computed explicitly from the expression of the probability generating functions in (3.1) and (3.2), in particular we obtain
\[
\sigma^2 = \left( \tilde{\mathcal{A}}_\Omega(\rho_\Omega) \frac{\partial^2}{\partial s_1^2} Z_{\text{SET}_\Omega} + \frac{\partial}{\partial s_1} Z_{\text{SET}_\Omega} - \rho_\Omega \left( \frac{\partial}{\partial s_1} Z_{\text{SET}_\Omega} \right)^2 \right) (\tilde{\mathcal{A}}_\Omega(\rho_\Omega), \tilde{\mathcal{A}}_\Omega(\rho_\Omega^2), \ldots) \rho_\Omega
\]
and, after a slightly tedious calculation,
\[
\mathbb{E} [\zeta] = \left( \frac{\partial}{\partial z} \mathbb{E} \left[ z^{\zeta} \right] \right) (1) = \frac{1}{\tilde{\mathcal{A}}_\Omega(\rho_\Omega)} \sum_{i \geq 2} \left( \frac{\partial}{\partial s_i} Z_{\text{SET}_\Omega} \right) (\tilde{\mathcal{A}}_\Omega(\rho_\Omega), \tilde{\mathcal{A}}_\Omega(\rho_\Omega^2), \ldots) i \rho_\Omega \tilde{\mathcal{A}}'_\Omega(\rho_\Omega^i),
\]
where \( \tilde{\mathcal{A}}'_\Omega = \frac{\partial}{\partial z} \tilde{\mathcal{A}}_\Omega \). Note that this expression is well-defined, since \( 0 < \rho_\Omega < 1 \).

In order to show (3.8), let \( f : \mathbb{R} \to \mathbb{R} \) denote a bounded, Lipschitz-continuous function defined on the space \( \mathbb{R} \) of isometry classes of compact metric spaces. Note that the tree \( \mathcal{T}_n \) conditioned on having \( \ell \) vertices is distributed like the tree \( \mathcal{T} \) conditioned on having \( \ell \) vertices. In particular, it is identically distributed to a \( \xi \)-Galton-Watson tree \( \mathcal{T}^\xi \) conditioned on having \( \ell \) vertices, which we denote by \( \mathcal{T}^\xi_\ell \). Since (3.6) holds with high probability it follows that
\[
\mathbb{E} \left[ f(c_\Omega \mathcal{T}_n / \sqrt{n}) \right] = o(1) + \sum_{\ell \in (1 + n^{-s})} \mathbb{E} \left[ f(c_\Omega \mathcal{T}^\xi_\ell / \sqrt{n}) \right] \mathbb{P} (|T| = \ell).
\]
Let \( D(T) \) denote the diameter of \( T \), i.e., the number of vertices on a longest path in \( T \). Since \( f \) was assumed to be Lipschitz-continuous it follows that
\[
\left| \mathbb{E} \left[ f(c_\Omega \mathcal{T}^\xi_\ell / \sqrt{n}) \right] - \mathbb{E} \left[ f(\sigma \mathcal{T}^\xi_\ell / 2\sqrt{\ell}) \right] \right| \leq a_{n,\ell} \mathbb{E} \left[ D(\mathcal{T}^\xi_\ell) / \sqrt{\ell} \right]
\]
for a sequence \( a_{n,\ell} \) with \( \sup_{\ell} (a_{n,\ell}) \to 0 \) as \( n \) becomes large. Moreover, the average rescaled diameter \( \mathbb{E}[D(\mathcal{T}^\xi_\ell) / \sqrt{\ell}] \) converges to a multiple of the expected diameter of the CRT \( \mathcal{T}_e \) as \( \ell \) tends to infinity, see e.g. [2]. In particular, it is a bounded sequence. Since \( \mathbb{E}[f(\sigma \mathcal{T}^\xi_\ell / 2\sqrt{\ell})] \to \mathbb{E}[f(\mathcal{T}_e)] \) as \( \ell \to \infty \), it follows that \( \mathbb{E}[f(c_\Omega \mathcal{T}_n / \sqrt{n})] \to \mathbb{E}[f(\mathcal{T}_e)] \) as \( n \) becomes large. This completes the proof. \( \square \)
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