Gröbner-Shirshov Bases and Hilbert Series of Free Dendriform Algebras*

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Abstract: In this paper, we give a Gröbner-Shirshov basis of the free dendriform algebra as a quotient algebra of an $L$-algebra. As applications, we obtain a normal form of the free dendriform algebra. Moreover, Hilbert series and Gelfand-Kirillov dimension of finitely generated free dendriform algebras are obtained.

Key words: Gröbner-Shirshov basis; $L$-algebra; dendriform algebra; Hilbert series; Gelfand-Kirillov dimension.

AMS 2000 Subject Classification: 13P10, 16S15, 17D99, 16P90.

1 Introduction

The theories of Gröbner-Shirshov bases and Gröbner bases were invented independently by A.I. Shirshov ([8], 1962) for (commutative, anti-commutative) non-associative algebras, by H. Hironaka ([4], 1964) for infinite series algebras (both formal and convergent) and by B. Buchberger (first publication in [3], 1965) for polynomial algebras. Gröbner–Shirshov technique is very useful in the study of presentations of many kinds of algebras defined by generators and defining relations.

An $L$-algebra (see [5]) is a vector space over a field $k$ with two operations $\prec$, $\succ$ satisfying one identity: $(x \succ y) \prec z = x \prec (y \prec z)$. A dendriform algebra (see [6, 7]) is an $L$-algebra with two identities: $(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z)$ and $x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z$.

The Composition-Diamond lemma for $L$-algebras is established in a recent paper [1]. In this paper, by using the Composition-Diamond lemma for $L$-algebras in [1], we give a Gröbner-Shirshov basis of the free dendriform algebra as a quotient algebra of a free $L$-algebra and then a normal form of a free dendriform algebra is obtained. As applications, we obtain the Hilbert series and Gelfand-Kirillov dimension of the free dendriform algebra generated by a finite set.

*Supported by the NNSF of China (No.10771077) and the NSF of Guangdong Province (No.06025062).
2 \textit{L-algebras}

We first introduce some concepts and results from the literature which are related to the Gr"{o}bner-Shirshov bases for \textit{L}-algebras. We will use some definitions and notations which are mentioned in [1].

Let $k$ be a field, $X$ a set of variables, $\Omega$ a set of multilinear operations, and

$$\Omega = \cup_{n\geq 1} \Omega_n,$$

where $\Omega_n = \{\delta^{(n)}_i \mid i \in I_n\}$ is the set of $n$-ary operations, $n = 1, 2, \ldots$. Now, we define “\Omega-words”.

Define $(X, \Omega)_0 = X$.

For $m \geq 1$, define $(X, \Omega)_m = X \cup \Omega((X, \Omega)_{m-1})$ where

$$\Omega((X, \Omega)_{m-1}) = \cup_{t=1}^{\infty} \{\delta^{(t)}_i (u_1, u_2, \ldots, u_t) \mid \delta^{(t)}_i \in \Omega_t, u_j \in (X, \Omega)_{m-1}\}.$$

Let

$$(X, \Omega) = \bigcup_{m=0}^{\infty} (X, \Omega)_m.$$ Then each element in $(X, \Omega)$ is called an $\Omega$-word.

\textbf{Definition 2.1} [2] An \textit{L}-algebra is a $k$-vector space $L$ equipped with two bilinear operations $\prec, \succ : L \otimes^2 \rightarrow L$ verifying the so-called entanglement relation:

$$(x \succ y) \prec z = x \prec (y \prec z), \forall x, y, z \in L.$$ Let $\Omega = \{\prec, \succ\}$. In this case, we call an $\Omega$-word as an \textit{L}-word.

\textbf{Definition 2.2} [1] An \textit{L-word} $u$ is a normal \textit{L}-word if $u$ is one of the following:

i) \quad $u = x$, where $x \in X$.

ii) \quad $u = v \succ w$, where $v$ and $w$ are normal \textit{L}-words.

iii) \quad $u = v \prec w$ with $v \neq v_1 \succ v_2$, where $v_1, v_2, v, w$ are normal \textit{L}-words.

We denote $u$ by $[u]$ if $u$ is a normal \textit{L}-word.

We denote the set of all the normal \textit{L}-words by $N$. Then, the free \textit{L}-algebra has an expression $L(X) = kN = \{\sum_{i} \alpha_i u_i \mid \alpha_i \in k, u_i \in N\}$ with $k$-basis $N$ and the operations $\prec, \succ$: for any $u, v \in N$,

$$u \prec v = [u \prec v], \quad u \succ v = [u \succ v].$$
Clearly, \([u > v] = u > v\) and
\[
[u < v] = \begin{cases} 
  u < v & \text{if } u = u_1 < u_2, \text{ or } u \in X, \\
  u_1 > [u_2 < v] & \text{if } u = u_1 > u_2.
\end{cases}
\]

Now, we order \(N\) in the same way as in \([1]\).

Let \(X\) be a well ordered set. We denote \(\succ\) by \(\delta_1\), \(<\) by \(\delta_2\). For any normal \(L\)-word \(u\), define
\[
wt(u) = \begin{cases} 
  (1, x), & \text{if } u = x \in X; \\
  (|u|, \delta_1, u_1, u_2), & \text{if } u = \delta_1(u_1, u_2) \in N,
\end{cases}
\]
where \(|u|\) is the number of \(x \in X\) in \(u\). Then we order \(N\) as follows:
\[
 u > v \iff wt(u) > wt(v) \text{ lexicographically}
\]
by induction on \(|u| + |v|\), where \(\delta_2 > \delta_1\).

Let \(* \notin X\). By a \(*\)-L-word we mean any expression in \((X \cup \{\ast\}, \{<, >\})\) with only one occurrence of \(*\).

Let \(u\) be a \(*\)-L-word and \(s \in L(X)\). Then we call \(u|s = u|\ast \mapsto s\) an \(s\)-word in \(L(X)\).

An \(s\)-word \(u|s\) is called a normal \(s\)-word if \(u|s \in N\).

It is shown in \([1]\) that the above ordering on \(N\) is monomial in the sense that for any \(*\)-L-word \(w\) and any \(u, v \in N\), \(u > v\) implies \(|w|_u > |w|_v\).

Assume that \(L(X)\) is equipped with the monomial ordering \(>\) as above. For any \(L\)-polynomial \(f \in L(X)\), let \(\bar{f}\) be the leading normal \(L\)-word of \(f\). If the coefficient of \(\bar{f}\) is 1, then \(f\) is called monic.

**Definition 2.3** \([1]\) Let \(f, g \in L(X)\) are two monic polynomials.

1) Composition of right multiplication.
   If \(\bar{f} = u_1 > u_2\) for some \(u_1, u_2 \in N\), then for any \(v \in N\), \(f < v\) is called a composition of right multiplication.

2) Composition of inclusion.
   If \(w = \bar{f} = u|\ast\) where \(u|\ast\) is a normal \(g\)-word, then
   \[
   (f, g)_w = f - u|g
   \]
   is called the composition of inclusion and \(w\) is called the ambiguity of the composition \((f, g)_w\).

**Definition 2.4** \([1]\) Let the ordering on \(N\) be as before, \(S \subset L(X)\) a monic set and \(f, g \in S\).

1) The composition of right multiplication \(f < v\) is called trivial modulo \(S\), denoted by \(f < v \equiv 0 \mod(S)\), if
   \[
   f < v = \sum \alpha_i u_i|s_i,
   \]
   where each \(\alpha_i \in k\), \(s_i \in S\), \(u_i|s_i\) normal \(s_i\)-word, and \(u_i|\ast \leq \bar{f} < v\).
2) The composition of inclusion \((f, g)_w\) is called trivial modulo \((S, w)\), denoted by 
\[(f, g)_w \equiv 0 \mod (S, w),\]
where each \(\alpha_i \in k\), \(s_i \in S\), \(u_i|_{s_i}\) normal \(s_i\)-word, and \(u_i|_{s_i} < w\).

\(S\) is called a Gröbner-Shirshov basis in \(L(X)\) if any composition of polynomials in \(S\) is trivial modulo \(S\) (and \(w\)).

**Theorem 2.5** \([1]\) (Composition-Diamond lemma for \(L\)-algebras) Let \(S \subset L(X)\) be a monic set and the ordering on \(N\) as before. Let \(Id(S)\) be the ideal of \(L(X)\) generated by \(S\). Then the following statements are equivalent:

(I) \(S\) is a Gröbner-Shirshov basis in \(L(X)\).

(II) \(f \in Id(S) \Rightarrow \overline{f} = u|_s\) for some \(s \in S\), where \(u|_s\) is a normal \(s\)-word.

(III) The set \(Irr(S) = \{u \in N \mid u \neq v|_s, s \in S, v|_s\) is a normal \(s\)-word\} is a \(k\)-basis of the \(L\)-algebra \(L(X) \mid S) = L(X)/Id(S)\).

3 Gröbner-Shirshov bases for free dendriform algebras

In this section, we give a Gröbner-Shirshov basis of the free dendriform algebra \(DD(X)\) generated by \(X\). As an application, we obtain a normal form of \(DD(X)\).

**Definition 3.1** \([6]\) A dendriform algebra is a \(k\)-vector space \(DD\) with two bilinear operations \(\prec, \succ\) subject to the three axioms below: for any \(x, y, z \in DD\),

1) \((x \succ y) \prec z = x \prec (y \prec z),\)
2) \((x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z),\)
3) \(x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z,\)

Thus, any dendriform algebra is an \(L\)-algebra.

It is clear that the free dendriform algebra generated by \(X\), denoted by \(DD(X)\), has an expression

\[L(X) \mid (x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z),\]
\[x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z, x, y, z \in N).\]

The following theorem gives a Gröbner-Shirshov basis for \(DD(X)\).
Theorem 3.2 Let the ordering on \( N \) be as before. Let
\[
\begin{align*}
  f_1(x, y, z) &= (x < y) < z - x < (y < z) - x < (y > z), \\
  f_2(x, y, z) &= (x < y) > z + (x > y) > z - x > (y > z), \\
  f_3(x, y, z, v) &= ((x > y) > z) > v - (x > y) > (z > v) + (x > (y < z)) > v.
\end{align*}
\]
Then, \( S = \{ f_1(x, y, z), \ f_2(x, y, z), \ f_3(x, y, z, v) | x, y, z, v \in N \} \) is a Gröbner-Shirshov basis in \( L(X) \).

Proof. All the possible compositions of right multiplication in \( S \) are as follows.

1) \( f_2(x, y, z) \prec u, \ u \in N \). We have
\[
\begin{align*}
  f_2(x, y, z) &\prec u = ((x < y) > z) \prec u + ((x > y) > z) \prec u - (x > (y > z)) \prec u \\
  &= (x < y) > (z < u) + (x > y) > (z < u) - x > ((y > z) < u) \\
  &\equiv -(x > y) > (z < u) + x > (y > (z < u)) + (x > y) > (z < u) \\
  &\quad - x > ((y > z) < u) \\
  &\equiv 0 \mod(S).
\end{align*}
\]

2) \( f_3(x, y, z, v) \prec u, \ u \in N \). We have
\[
\begin{align*}
  f_3(x, y, z, v) &\prec u = (((x > y) > z) > v) \prec u - ((x > y) > (z > v)) \prec u \\
  &\quad + ((x > (y < z)) > v) \prec u \\
  &= ((x > y) > z) > (v < u) - (x > y) > (z > (v < u)) \\
  &\quad + (x > (y < z)) > (v < u) \\
  &\equiv (x > y) > (z > (v < u)) - (x > (y < z)) > (v < u) \\
  &\quad - (x > y) > (z > (v < u)) + ((x > (y < z)) > v) \prec u \\
  &\equiv 0 \mod(S).
\end{align*}
\]

We denote by \( f_i \land f_j \) an inclusion composition of the polynomials \( f_i \) and \( f_j \), \( i, j = 1, 2, 3 \). All the possible ambiguities in \( S \) are listed as follows:

3) \( f_1(x, y, z) \land f_1(a, b, c) \):
\[3.1 \ w_{3.1} = (x|_{(a \prec b) \prec c} < y) \prec z, \]
\[3.3 \ w_{3.3} = (x < y) < z|_{(a \prec b) \prec c}, \]
\[3.2 \ w_{3.2} = (x < y|_{(a \prec b) \prec c}) \prec z, \]
\[3.4 \ w_{3.4} = ((a < b) \prec c) \prec z. \]

4) \( f_1(x, y, z) \land f_2(a, b, c) \):
\[4.1 \ w_{4.1} = (x|_{(a \prec b) \prec c} < y) \prec z, \]
\[4.3 \ w_{4.3} = (x < y) < z|_{(a \prec b) \prec c}, \]
\[4.2 \ w_{4.2} = (x < y|_{(a \prec b) \prec c}) \prec z, \]

5) \( f_1(x, y, z) \land f_3(a, b, c, d) \):
\[5.1 \ w_{5.1} = (x|_{((a \prec b) \prec c) \prec d} < y) \prec z, \]
\[5.3 \ w_{5.3} = (x < y) < z|_{((a \prec b) \prec c) \prec d}, \]
\[5.2 \ w_{5.2} = (x < y|_{((a \prec b) \prec c) \prec d}) \prec z, \]

6) \( f_2(a, b, c) \land f_1(x, y, z) \):
6.1 $w_{6.1} = (a|((x < y) < z) < b) > c$,
6.3 $w_{6.3} = (a < b) > c|((x < y) < z)$,

7) $f_2(a, b, c) \land f_2(x, y, z)$:
7.1 $w_{7.1} = (a|((x < y) < z) < b) > c$,
7.3 $w_{7.3} = (a < b) > c|((x < y) < z)$.

8) $f_2(a, b, c) \land f_3(x, y, z, v)$:
8.1 $w_{8.1} = (a|((x < y) < z) < v) < b) > c$,
8.3 $w_{8.1} = (a < b) > c|((x < y) < z) > v)$.

9) $f_3(x, y, z, v) \land f_1(a, b, c)$:
9.1 $w_{9.1} = ((x|((a < b) < c) > y) > z) > v$,
9.3 $w_{9.3} = ((x > y) > z|((a < b) < c) > v$,
9.2 $w_{9.2} = ((x > y|((a < b) < c) > z) > v$,
9.4 $w_{9.4} = ((x > y) > z) > v|((a < b) < c)$.

10) $f_3(x, y, z, v) \land f_2(a, b, c)$:
10.1 $w_{10.1} = ((x|((a < b) < c) > y) > z) > v$,
10.3 $w_{10.3} = ((x > y) > z|((a < b) < c) > v$,
10.5 $w_{10.5} = (((a < b) > c) > z) > v$.

11) $f_3(x, y, z, v) \land f_3(a, b, c, d)$:
11.1 $w_{11.1} = ((x|((a < b) < c) > d) > y) > z) > v$,
11.2 $w_{11.2} = ((x > y|((a < b) < c) > d) > z) > v$,
11.3 $w_{11.3} = ((x > y) > z|((a < b) < c) > d) > v$,
11.4 $w_{11.4} = ((x > y) > z) > v|((a < b) < c) > d$,
11.5 $w_{11.5} = ((((a > b) > c) > d) > z) > v$,
11.6 $w_{11.6} = ((((a > b) > c) > d) > z) > v$.

We will prove that all compositions are trivial mod$(S, w)$. Here, for example, we only check Case 6.4, Case 10.5 and Case 11.6. The others are easy to check.

Case 6.4:

$$(f_2(a, b, c), f_1(x, y, z))_{w_{6.4}}
\equiv (x < (y < z)) > c + (x < (y > z)) > c - (x < y) > (z > c)
+((x < y) > z) > c
\equiv x > ((y < z) > c) - (x > (y < z)) > c + x > ((y > z) > c)
- (x > (y > z)) > c - x > (y > (z > c)) + (x > y) > (z > c)
\equiv x > (y > (z > c)) - (x > (y < z)) > c - x > (y > (z < c))
\equiv 0 \text{ mod}(S, w_{6.4})$$
Case 10.5:

\[(f_3(x, y, z, v), f_2(a, b, c))_{w_{10.5}} \equiv ((a \succ (b \succ c)) \succ z) \succ v - (((a \succ b) \succ c) \succ z) \succ v \]
\[-((a \prec b) \succ (z \succ v) + ((a \prec b) \succ (c \prec z)) \succ v) \]
\[\equiv (a \succ (b \succ c)) \succ (z \succ v) - (a \succ ((b \prec c) \prec z)) \succ v \]
\[-((a \succ b) \succ (z \succ v) + ((a \succ b) \succ (c \prec z)) \succ v) \]
\[-(a \succ (b \succ c)) \succ (z \succ v) + ((a \succ b) \succ (c \prec z)) \succ (z \succ v) \]
\[+ (a \succ (b \succ (c \prec z)) \succ v - ((a \succ b) \succ (c \prec z)) \succ v \]
\[\equiv 0 \text{ mod}(S, w_{10.5}).\]

Case 11.6:

\[(f_3(x, y, z, v), f_3(a, b, c, d))_{w_{11.6}} \equiv (((a \succ b) \succ (c \succ d)) \succ z) \succ v - (((a \succ b) \succ (d \succ c) \succ d) \succ z) \succ v \]
\[-((a \succ b) \succ (z \succ v) + ((a \succ b) \succ (d \prec z)) \succ v) \]
\[\equiv ((a \succ b) \succ (c \succ d)) \succ (z \succ v) - ((a \succ b) \succ (c \succ d) \succ (z \succ v)) \]
\[-((a \succ b) \succ (c \succ d)) \succ (z \succ v) + ((a \succ b) \succ (d \prec c) \succ d) \succ (z \succ v) \]
\[+ ((a \succ b) \succ (c \succ (d \prec z)) \succ v - ((a \succ (b \prec c)) \succ (d \prec z)) \succ v \]
\[\equiv 0 \text{ mod}(S, w_{11.6}).\]

The proof is complete.

\[\square\]

**Definition 3.3** An L-word \(u\) is called a normal DD-word, denoted by \(\lfloor u \rfloor\), if

1) \(u = x, x \in X\),
2) \(u = x \prec \lfloor v \rfloor, x \in X\),
3) \(u = x \succ \lfloor v \rfloor, x \in X\),
4) \(u = (x \succ [u_1]) \succ [u_2], x \in X\).

**Remark** From Definition 2.2 and Definition 3.3, we know that any normal DD-word is a normal L-word.

The following corollary follows from Theorem 2.5 and Theorem 3.2.

**Corollary 3.4** The set \(\operatorname{Irr}(S) = \{u| u \text{ is a normal DD-word }\}\) is a \(k\)-basis of the free dendriform algebra \(DD(X)\).
4 Hilbert series and Gelfand-Kirillov dimension of the free dendriform algebra

In this section, we give Hilbert series of the free dendriform algebra $DD(X)$ where $|X|$ is finite. As an application, we prove that Gelfand-Kirillov dimension of the free dendriform algebra $DD(X)$ is infinite.

We introduce some basic definitions and concepts that we will use throughout this section.

**Definition 4.1** Let $V = (V, \prec, \succ)$ be a dendriform algebra. Then $V$ is called a finitely graded algebra if $V = \bigoplus_{m \geq 1} V_m$ as $k$-vector spaces such that $\dim_k V_m < \infty$ and $\delta(V_i, V_j) \subseteq V_{i+j}$ for all $i, j \geq 1, \delta \in \{\prec, \succ\}$.

**Definition 4.2** Let $V = \bigoplus_{m \geq 1} V_m$ be a finitely graded dendriform algebra and $\dim_k(V_m)$, the dimension of the vector space $V_m$. Then the Hilbert series of $V$ is defined to be

$$H(V, t) = \sum_{m=1}^{\infty} \dim_k(V_m) t^m.$$ 

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $DD_m$ the subspace of $DD(X)$ generated by all normal $DD$-words in $DD(X)$ of degree $m$. Then $DD(X) = \bigoplus_{m \geq 1} DD_m$ is a finitely graded dendriform algebra.

By the definition of normal $DD$-words, one has

$$\dim_k(DD_1) = n, \quad \dim_k(DD_2) = 2n^2.$$ 

Assume that for any $m \geq 1$, $\dim_k(DD_m) = f(m)n^m$. Then $f(1) = 1$, $f(2) = 2$. For convenience, let $f(0) = 1$.

For any $m > 2$, it is clear that $DD_m$ has a $k$-basis

$$\{x \prec [u] | x \in X, |u| > 1, [u] \text{ is a normal } DD\text{-word}\}$$

$$\bigcup \{x \succ [u] | x \in X, |u| > 1[u] \text{ is a normal } DD\text{-word}\}$$

$$\bigcup \{(x \succ [u_1]) \succ [u_2] | x \in X, |u_1|, |u_2| \geq 1, [u_1],[u_2] \text{ are normal } DD\text{-words}\}.$$ 

It follows that

$$f(m) = 2 \times f(m-1) + 1 \times 1 \times f(m-2) + 1 \times f(m-2) \times 1$$

$$+ 1 \times \sum_{i=2}^{m-3} f(i)f(m-3-i)$$

$$= \sum_{i=0}^{m-1} f(i)f(m-1-i).$$ 

Therefore, we prove the following lemma.
Lemma 4.3 Let $X$ be a finite set with $|X| = n$. Then the Hilbert series of the free dendriform algebra $DD(X)$ is
\[
\mathcal{H}(DD(X), t) = \sum_{m \geq 1} f(m)n^m t^m,
\]
where $f(m)$ satisfies the recursive relation ($f(0) = 1$):
\[
f(m) = \sum_{i=0}^{m-1} f(i)f(m-1-i), \quad m \geq 1.
\]

Now, we describe the Hilbert series of $DD(X)$ with another way.

Let $A, B, C$ be the subspaces of $DD(X)$ with $k$-bases
\[
\{ x < [u] \mid x \in X, \ [u] \text{ is a normal } DD\text{-word} \},
\]
\[
\{ x > [u] \mid x \in X, \ [u] \text{ is a normal } DD\text{-word} \},
\]
\[
\{ (x > [u_1]) > [u_2] \mid x \in X, \ [u_1], [u_2] \text{ are normal } DD\text{-words} \},
\]
respectively. Assume that their Hilbert series are $\mathcal{H}(A, t)$, $\mathcal{H}(B, t)$, $\mathcal{H}(C, t)$, respectively. Clearly, we have
\[
\mathcal{H}(B, t) = \mathcal{H}(A, t).
\]

Noting that $A$ has a $k$-basis
\[
\{ x_i < x_j \mid x_i, x_j \in X \} \bigcup \{ x < [u] \mid |u| > 1, \ x \in X, \ [u] \text{ is a normal } DD\text{-word} \},
\]
we have
\[
\mathcal{H}(A, t) = n^2 t^2 + nt \times (\mathcal{H}(A, t) + \mathcal{H}(B, t) + \mathcal{H}(C, t))
\]
\[
= n^2 t^2 + nt \times (2\mathcal{H}(A, t) + \mathcal{H}(C, t)). \tag{1}
\]

Since $C$ has a $k$-basis
\[
\bigcup \{ (x > [u]) > x_j \mid x_i, x_j, x_k \in X \}
\]
\[
\bigcup \{ (x_i > x_j) > [u] \mid x_i, x_j \in X, |u| > 1, \ [u] \text{ is a normal } DD\text{-word} \}
\]
\[
\bigcup \{ (x > [u]) > x_j \mid x_i, x_j \in X, |u| > 1, \ [u] \text{ is a normal } DD\text{-word} \}
\]
\[
\bigcup \{ (x > [u]) > [v] \mid |u|, |v| > 1, \ [u], [v] \text{ are normal } DD\text{-words} \},
\]
we have
\[
\mathcal{H}(C, t) = n^3 t^3 + 2n^2 t^2 \times (\mathcal{H}(A, t) + \mathcal{H}(B, t) + \mathcal{H}(C, t)) + nt \times (\mathcal{H}(A, t) + \mathcal{H}(B, t) + \mathcal{H}(C, t))^2
\]
\[
= nt \times (nt + (2\mathcal{H}(A, t) + \mathcal{H}(C, t)))^2 \tag{2}
\]

From equations (1) and (2), we obtain
\[
\mathcal{H}(A, t) = \frac{1 - 2nt \pm \sqrt{1 - 4nt^2}}{2}.
\]
Since $H(A,0) = 0$, we have

$$H(A,t) = \frac{1 - 2nt - \sqrt{1-4nt}}{2}.$$ 

Therefore,

$$H(C,t) = \frac{1 - (1 - 2nt)\sqrt{1-4nt}}{2nt} - 2 + nt.$$ 

Thus, we have the following theorem.

**Theorem 4.4** Let $X$ be a finite set with $|X| = n$. The Hilbert series of the free dendriform algebra $DD(X)$ is

$$H(DD(X),t) = \frac{1 - 2nt - \sqrt{1-4nt}}{2nt}.$$ 

We now give an exact expression of the function $H(DD(X),t)$.

For $t \leq \frac{1}{4n}$, we have

$$\sqrt{1-4nt} = (1 + (-4nt))^{\frac{1}{2}} = 1 + \sum_{i=1}^{\infty} \frac{\frac{1}{2} \cdot \cdots \cdot \left(\frac{1}{2} - i + 1\right)}{i!} \times \left(-4n^i t^i\right).$$

From this and Lemma 4.3 we get the following theorem.

**Theorem 4.5** Let $X$ be a finite set with $|X| = n$. Then the Hilbert series of the free dendriform algebra $DD(X)$ is

$$H(DD(X),t) = \sum_{m=1}^{\infty} \frac{1 \times 3 \times \cdots \times (2m-1) \times 2^m}{(m+1)!} n^m t^m.$$ 

Therefore, $dim_k(DD_m) = \frac{(2m)! \times n^m}{(m+1)! m!}$, $m \geq 1$.

Now, by using Theorem 4.4, we show that Gelfand-Kirillov dimension of the free dendriform algebra $DD(X)$ is infinite when $|X|$ is finite.

**Definition 4.6** Let $R$ be a finitely presented algebra over a field $k$ and $x_1, x_2, \ldots, x_n$ be its generators. Consider $R = \bigcup_{d \in N} V(d)$, where $V(d)$ is spanned by all the monomials in $x_i$ of length $\leq d$. The quantity

$$GKR = \lim_{d \to \infty} \frac{\log dim_k V(d)}{\log d}$$

is called the Gelfand-Kirillov dimension of $R$. 

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Theorem 4.7 Let $X$ be a finite set with $|X| = n$. Then the Gelfand-Kirillov dimension of free dendriform algebra $DD(X)$ is

$$GKDD(X) = \infty.$$ 

Proof. For a fixed natural $d$, let $DD_{(d)}$ be the subspace spanned by all the monomials in $x_i$ of length $\leq d$. Then

$$\dim DD_{(d)} = \sum_{i=1}^{d} \dim_k(DD_{i}) \geq \dim_k(DD_{d}).$$

Therefore,

$$GKDD(X) \geq \lim_{d \to \infty} \frac{\log \dim_k(DD_{d})}{\log d} = \lim_{d \to \infty} \frac{\ln (2d)! \times n^d}{(d+1)d} \ln d$$

$$= \lim_{d \to \infty} d \ln 2n + \sum_{i=1}^{d} \ln(2i - 1) - \sum_{i=1}^{d+1} \ln i \ln d$$

$$= \lim_{d \to \infty} d \ln 2n + \lim_{d \to \infty} \sum_{i=1}^{d} \ln \frac{2i-1}{2i} - 1$$

$$= \lim_{d \to \infty} (d \ln 2n) + \lim_{d \to \infty} \left( \frac{d}{\ln d} \sum_{i=1}^{d} \frac{\ln (2 - \frac{1}{i})}{d} \right) - 1$$

$$= \infty.$$  

□

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