SHADOWS OF 3-UNIFORM HYPERGRAPHS UNDER A MINIMUM DEGREE CONDITION

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Abstract. We prove a minimum degree version of the Kruskal–Katona theorem for triple systems: given \( d \geq 1/4 \) and a triple system \( F \) on \( n \) vertices with minimum degree \( \delta(F) \geq d \binom{n}{3} \), we obtain asymptotically tight lower bounds for the size of its shadow. Equivalently, for \( t \geq n/2 - 1 \), we asymptotically determine the minimum size of a graph on \( n \) vertices, in which every vertex is contained in at least \( \binom{t}{2} \) triangles. This can be viewed as a variant of the Rademacher–Turán problem.

1. Introduction

Given a set \( X \) and a family \( F \) of \( k \)-subsets of \( X \), the shadow \( \partial F \) of \( F \) is the family of all \((k - 1)\)-subsets of \( X \) contained in some member of \( F \). The Kruskal–Katona theorem \([12, 13]\) is one of the most important results in extremal set theory – it gives a tight lower bound for the size of shadows of all \( k \)-uniform families of a given size. The following is a version due to Lovász \([17]\), where \( \binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!} \) for a real number \( t \). Note that it is tight when \( t \) is an integer by considering the family of all \( k \)-subsets of a set of \( t \) vertices.

**Theorem 1** (Kruskal–Katona theorem). If \( F \) is a family of \( k \)-sets with \( |F| \geq \binom{t}{k} \) for some real number \( t \), then \( |\partial F| \geq \binom{t}{k-1} \).

A family \( F \) of \( k \)-subsets of \( X \) is often regarded as a \( k \)-uniform hypergraph, or \( k \)-graph \((X,F)\) with \( X \) as the vertex set and \( F \) as the edge set. For every \( x \in X \), define \( F_x = \{ F \setminus x : x \in F \text{ and } F \in F \} \). The minimum (vertex) degree of \( F \) is denoted by \( \delta(F) := \min_x |F_x| \). The following minimum degree version of the Kruskal–Katona theorem has not been studied before but emerged naturally when Han, Zang, and Zhao \([9]\) investigated a packing problem for 3-graphs.

**Problem 2.** Given \( k \geq 3 \) and \( 0 < d < 1 \), let \( X \) be a set of \( n \) vertices and \( F \) be a family of \( k \)-subsets of \( X \) with \( \delta(F) \geq d \binom{n}{k-1} \). How small can \( |\partial F| \) be?

Problem 2 belongs to an area of active research on extremal problems under maximum or minimum degree conditions. Two early examples are the work of Bollobás, Daykin, and Erdős \([1]\), who studied the minimum degree version of the Erdős matching conjecture, and of Frankl \([6]\), who studied the Erdős–Ko–Rado theorem under maximum degree conditions. More recent examples include the minimum (co)degree Turán’s problems \([15, 18]\), the minimum degree version of the Erdős–Ko–Rado theorem \([8, 10, 14]\), and the minimum degree version of Hilton–Milner...
Since $\delta(F) \geq d\binom{n}{k-1}$ implies that $|F| \geq d\binom{n}{k}$, we could apply Theorem 1 to $F$ but will not obtain a tight bound for $|\partial F|$. A better approach is applying Theorem 1 to $F_x$ for each vertex $x$.

Since $|F_x| \geq d\binom{n}{k-1} \geq \left(\frac{d^{k-1}}{k-1}\right)\binom{n}{k-2}$, by Theorem 1 we have $|\partial F_x| \geq \left(\frac{d^{k-1}}{k-1}\right)\binom{n}{k-2} + O(n^{k-3})$. Consequently,

$$|\partial F| = \sum_x |\partial F_x| \geq \frac{n}{k-1} d^{k-1} \left(\binom{n}{k-2}\right) + O(n^{k-2}) \geq \left(\frac{n}{k-1} d^{k-1}\right) + O(n^{k-2}).$$

This bound is tight (up to the error term) when the first inequality in (1) is asymptotically an equality, which occurs when $F_x$ is a clique of order $\frac{d^{k-1}}{k-1}n$ for every $x$. Thus, the bound in (1) is asymptotically tight when $F$ consists of $d\frac{n}{k-1}$ vertex-disjoint cliques of order $\frac{d^{k-1}}{k-1}n$, in particular, when $d = \ell^{-k}$ for some $\ell \in \mathbb{N}$.

In this paper we improve (1) and answer Problem 2 asymptotically for $k = 3$ and $d \geq \frac{1}{4}$. Two overlapping cliques of order about $\sqrt{dn} + 1$ is a natural candidate for extremal hypergraphs – the following theorem confirms this for $\frac{1}{4} \leq d < \frac{47-5\sqrt{57}}{24} \approx 0.385$. However, there is a different extremal hypergraph for larger values of $d$.

**Theorem 3.** Let $1/4 \leq d < 1$ and $n \in \mathbb{N}$ be sufficiently large. If $F$ is a triple system on $n$ vertices with $\delta(F) \geq d\binom{n}{2}$, then

$$|\partial F| \geq \begin{cases} 
\left(4\sqrt{d} - 2d - 1\right) \binom{n}{2} & \text{if } \frac{1}{4} \leq d < \frac{47-5\sqrt{57}}{24} \\
\left(\frac{1}{2} + \sqrt{\frac{4d-12}{12}}\right) \binom{n}{2} & \text{if } d \geq \frac{47-5\sqrt{57}}{24}.
\end{cases}$$

These bounds are best possible up to an additive term of $O(n)$.

Although seemingly technical, Theorem 3 has an interesting application on 3-graph packing and covering. Given positive integers $a, b, c$, let $K_{a,b,c}^3$ denote the complete 3-partite 3-graph with parts of size $a, b, c$. Answering a question of Mycroft [9], Han, Zang, and Zhao [8] determined the minimum $\delta(H)$ of a 3-graph $H$ that forces a perfect $K_{a,b,c}^3$-packing in $H$ for any given $a, b, c$. One of the main steps in their proof is determining the smallest $\delta(H)$ of a 3-graph $H$ that guarantees that every vertex of $H$ is contained in a copy of $K_{a,b,c}^3$ (this is necessary for $H$ containing a perfect $K_{a,b,c}^3$-packing).

**Corollary 4.** [8 Lemma 3.7] Let $d_0 = 6 - 4\sqrt{2} \approx 0.343$. For any $\gamma > 0$, there exists $\eta > 0$ such that the following holds for sufficiently large $n$. If $H$ is an $n$-vertex 3-graph with $\delta(H) \geq (d_0 + \gamma)\binom{n}{2}$, then each vertex of $H$ is contained in at least $\eta n^{a+b+c-1}$ copies of $K_{a,b,c}^3$.

It was shown [8 Construction 2.6] that $d_0$ in Corollary 4 is best possible. We give a proof outline of Corollary 4 at the end of Section 2 – a complete proof can be found in [8].

Our approach towards Theorem 3 is viewing it as an extremal problem on graphs. The following is an equivalent form of Problem 2 in which $K_k^t$ denotes the complete $k$-graph on $t$ vertices (and we omit the superscript when $k = 2$).

**Problem 5.** Given a $(k-1)$-graph $G$ on $n$ vertices such that every vertex is contained in at least $d\binom{n}{k-1}$ copies of $K_k^{k-1}$, how many edges must $G$ have?

\footnote{Given hypergraphs $H$ and $F$, a perfect $F$-packing in $H$ is a spanning subgraph of $H$ that consists of vertex-disjoint copies of $F$.}
To see why Problems 2 and 5 are equivalent, let \( m_1 \) be the minimum \( |\partial F| \) for Problem 2 and \( m_2 \) be the minimum \( e(G) \) for Problem 5. To see why \( m_1 \geq m_2 \), consider a \( k \)-uniform family \( \mathcal{F} \) with \( \delta(\mathcal{F}) \geq d(\binom{n}{k-1}) \). Let \( G = (V(\mathcal{F}), \partial \mathcal{F}) \) be the \((k-1)\)-graph of its shadow. Since every member of \( \mathcal{F} \) gives rise to a copy of \( K_{k-1}^k \) in \( G \), \( \delta(\mathcal{F}) \geq d(\binom{n}{k-1}) \) implies that every vertex is contained in at least \( d(\binom{n}{k-1}) \) copies of \( K_{k-1}^k \). Thus \( |\partial \mathcal{F}| = e(G) \geq m_2 \). To see why \( m_2 \geq m_1 \), consider a \((k-1)\)-graph \( G \) such that every vertex is contained in at least \( d(\binom{n}{k-1}) \) copies of \( K_{k-1}^k \). Let \( \mathcal{F} \) be the family of \( k \)-subsets of \( V(G) \) that span copies of \( K_{k-1}^k \) in \( G \). Then \( \partial \mathcal{F} \subseteq G \) and for every \( v \in V(G) \), we have \( |\mathcal{F}_v| \geq d(\binom{n}{k-1}) \). Thus \( e(G) \geq |\partial \mathcal{F}| \geq m_1 \) as desired.

In order to prove Theorem 3, we solve the \( k = 3 \) case of Problem 5 with \( d \geq 1/4 \). For convenience, we assume that every vertex of \( G \) is contained in at least \( \binom{t}{2} \) triangles. There are essentially two extremal graphs: the first one consists of two copies of \( K_{t+1} \) that share \( 2t + 2 - n \) vertices; the second one is obtained from two disjoint copies of \( K_{n/2} \) by adding a regular bipartite graph between them. The size of these two extremal graphs can be conveniently represented by a quadratic function \( f(x) \), which arises naturally from a lower bound for \( e(G) \) in Proposition 7.

**Theorem 6.** Let \( n \in \mathbb{N}, t, r \in \mathbb{R} \) such that \( n/2 \leq t + 1 \leq n, r \geq 0, \) and

\[
(2) \quad \left( \frac{n}{2} - 1 \right) + 3 \left( \frac{n}{2} \right) = \left( \frac{t}{2} \right).
\]

Define a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) as

\[
(3) \quad f(x) = \left( \frac{t}{2} \right) + x(n - x) - \left( \frac{n - x - 1}{2} \right).
\]

If \( G \) is an \( n \)-vertex graph such that each vertex is contained in at least \( \binom{t}{2} \) triangles, then

\[
(4) \quad e(G) \geq \begin{cases} 
 f(t) & \text{if } r + t \leq \frac{5n}{6} \text{ or approximately } t \leq 0.6208n, \\
 f\left( \frac{n}{2} + r - 1 \right) & \text{otherwise.}
\end{cases}
\]

Furthermore, these bounds are tight when \( n/2, t, r \) are integers, and tight up to an additive \( O(n) \) in general.

Theorem 6 can be viewed as a variant of the well-studied Rademacher–Turán problem. Starting with the work of Rademacher (unpublished) and of Erdős [4], the Rademacher–Turán problem studies the minimum number of triangles in a graph with given order and size. Instead of the total number of triangles in a graph, one may ask for the maximum or minimum number of triangles containing a fixed vertex. Given a graph \( G \), we define the triangle-degree of a vertex as the number of triangles that contain this vertex. Let \( \Delta_{K_3}(G) \) and \( \delta_{K_3}(G) \) denote the maximum and minimum triangle-degree in \( G \), respectively. The contrapositive of Theorem 6 states that if \( G \) is a graph on \( n \) vertices that fails [4], then \( \delta_{K_3}(G) < \binom{t}{2} \). Correspondingly, the maximum triangle-degree version of Rademacher–Turán problem was recently studied by Falgas-Ravry, Markström, and Zhao [5]. In addition, Theorem 6 looks similar to the question of Erdős and Rothschild [3] on the book size of graphs: in the complementary form, it asks for the maximum size of a graph on \( n \) vertices, in which every edge is contained in at most \( d \) triangles.

We prove Theorem 6 and Theorem 3 in the next section. When \( t < n/2 - 1 \), it is reasonable to speculate that an extremal graph is a disjoint union of copies of \( K_{t+1} \) and an extremal graph for Theorem 6. Unfortunately, we cannot verify this. We provide some evidence for this speculation in the last section.

**Notation.** Given a family \( \mathcal{F} \) of sets, \( |\mathcal{F}| \) is the size of \( \mathcal{F} \), namely, the number of sets in \( \mathcal{F} \). A \( k \)-uniform hypergraph \( H \), or \( k \)-graph, consists of a vertex set \( V(H) \) and an edge set \( E(H) \),
which is a family of $k$-subsets of $V(H)$. Given a vertex set $S$, denote by $e_H(S)$ the number of edges of $H$ induced on $S$. Suppose $G$ is a graph. For a vertex $v \in V(G)$, let $N_G(v)$ denote the neighborhood of $v$, the set of vertices adjacent to $v$, and let $d_G(v) = |N_G(v)|$ be the degree of $v$. Let $N_G[v] := N_G(v) \cup \{v\}$ denote the closed neighborhood of $v$. When the underlying (hyper)graph is clear from the context, we omit the subscript in these notations.

2. Proofs of Theorem 6 and Theorem 3

Suppose that $G = (V, E)$ is a graph on $n$ vertices such that each vertex is contained in at least $\binom{t}{2}$ triangles, in other words,

$$\forall v \in V, \quad e(N(v)) \geq \binom{t}{2},$$

where $t$ is a positive real number. Trivially $t \leq \delta(G) \leq n - 1$ because $e(N(v)) \leq \binom{d(v)}{2}$ for every vertex $v \in V$. Therefore

$$e(G) \geq \frac{\delta(G)n}{2} \geq \frac{tn}{2}.$$

When $t + 1$ divides $n$, this bound is tight because $G$ can be a disjoint union of $\frac{n}{t+1}$ copies of $K_{t+1}$. Below we often assume that $t \leq n - 2$ because when $t = n - 1$, we must have $G = K_n$.

Let us derive another lower bound for $e(G)$ by using the function $f$ defined in (3).

**Proposition 7.** If $G = (V, E)$ is a graph on $n$ vertices satisfying (5), then $e(G) \geq f(\delta(G))$, and the equality holds if and only if there exists $v_0 \in V$ such that $e(N(v_0)) = \binom{t}{2}$, $d(v) = \delta(G)$ for all $v \notin N(v_0)$, and $V \setminus N[v_0]$ induces a clique.

**Proof.** Suppose $\delta(G) = \delta$ and $v_0 \in V$ satisfies $d(v_0) = \delta$. Since we may partition $E(G)$ into the edges induced on $N(v_0)$ and the edges incident to some vertex $v \notin N(v_0)$, we have

$$e(G) = e(N(v_0)) + \left(\sum_{v \notin N(v_0)} d(v)\right) - e(V \setminus N(v_0)).$$

Because of (5), $d(v) \geq \delta$ for all $v \notin N(v_0)$, and $e(V \setminus N(v_0)) \leq \binom{n-\delta-1}{2}$ (note that $v_0$ has no neighbor outside $N(v_0)$), we derive that $e(G) \geq \binom{t}{2} + \delta(n-\delta) - \binom{n-\delta-1}{2}$. Furthermore, the equality holds exactly when $e(N(v_0)) = \binom{t}{2}$, $d(v) = \delta(G)$ for all $v \notin N(v_0)$, and $V \setminus N[v_0]$ induces a clique. \hfill \Box

Let us construct three graphs satisfying (5). Note that, if $r$ satisfies (2), then $r \leq n/2$ because $\binom{n/2-1}{2} + 3\binom{n/2}{2} = \binom{n-1}{2} \geq \binom{t}{2}$.

**Construction 8.** Suppose $t, r \in \mathbb{R}$ satisfy $\frac{t^2}{2} - 1 \leq t \leq n - 2$, $r \geq 0$, and (2).

1. Let $G_1$ be the union of two copies of $K_{[t]+1}$ sharing $2[t] + 2 - n$ vertices.
2. When $n$ is even, let $G_2$ be the $n$-vertex graph obtained from two disjoint copies of $K_{n/2}$ by adding an $\lfloor r \rfloor$-regular bipartite graph between two cliques.
3. When $n$ is odd, let $r' \in \mathbb{R}^+$ satisfy $\binom{n-3}{2} + 3\binom{r'}{2} = \binom{t}{2}$. Let $G_2'$ be the $n$-vertex graph obtained from two disjoint copies of $K_{(n-1)/2}$ by adding an $\lceil r' \rceil$-regular bipartite graph between them, and a new vertex whose adjacency is the exactly the same as one of the existing vertices.

It is easy to see that $G_1, G_2, G_2'$ all satisfy (5). For example, consider a vertex $x \in V(G_2)$. Let $A$ and $B$ denote the vertex sets of the two copies of $K_{n/2}$ of $G_2$ and assume $x \in A$. Then
\( N(x) \) contains \( \binom{n/2 - 1}{2} \) edges from \( A \), \( \binom{|r|}{2} \) edges from \( B \), and \( |r|(|r| - 1) \) edges between \( A \) and \( B \). Hence \( e(N(x)) = \binom{n/2 - 1}{2} + 3\binom{|r|}{2} \geq \binom{5}{2} \).

The following proposition gives the sizes of \( G_1, G_2 \), and \( G'_2 \).

**Proposition 9.** Suppose \( n \in \mathbb{N} \), \( t, r \geq 0 \) satisfy \( \frac{n}{2} - 1 \leq t \leq n - 1 \) and \( (2) \). If all \( n/2, t, r \) are integers, then \( e(G_1) = f(t) \) and \( e(G_2) = f(n/2 + r - 1) \), otherwise \( e(G_1) \leq f(t) + n \) and \( e(G_2) \leq f(n/2 + r - 1) + n/2 \). Furthermore, \( e(G'_2) = f(n/2 + r - 1) + O(n) \) when \( r', r = \Omega(n) \).

**Proof.** First, by the definition of \( f(x) \), it is easy to see that

\[
(6) \quad f(t) = \binom{n}{2} - (n - 1 - t)^2
\]

(alternatively when \( t \in \mathbb{Z} \), we can apply Proposition (7) by letting \( v_0 \) be any vertex not in the intersection of the two cliques). We know that

\[
e(G_1) = \binom{n}{2} - (n - 1 - \lceil t \rceil)^2 \geq \binom{n}{2} - (n - 1 - t)^2 = f(t)
\]

and equality holds when \( t \in \mathbb{Z} \). In addition, we have \( e(G_1) \leq f(t) + n \) because

\[
(n - 1 - \lceil t \rceil)^2 - (n - 1 - t)^2 = (2(n - 1) - (\lceil t \rceil + t))(\lceil t \rceil - t) \leq n
\]

by using \( t + 1 \geq \lceil t \rceil \geq t \geq n/2 - 1 \).

Second, using the definitions of \( f(x) \) and \( r \), it is not hard to see that

\[
(7) \quad f\left(\frac{n}{2} + r - 1\right) = \frac{n}{2} \left(\frac{n}{2} + r - 1\right).
\]

It follows that

\[
e(G_2) = \frac{n}{2} \left(\frac{n}{2} + \lceil r \rceil - 1\right) \leq f\left(\frac{n}{2} + r - 1\right) + \frac{n}{2}
\]

and equality holds when \( r \in \mathbb{Z} \).

Third, it is easy to see that

\[
e(G'_2) = \frac{n + 1}{2} \left(\frac{n - 1}{2} + \lceil r' \rceil - 1\right).
\]

By the definitions of \( r \) and \( r' \), we have \( \binom{r'}{2} - \binom{|r|}{2} = \frac{2n - \frac{7}{2}r}{24} \). When \( r, r' = \Omega(n) \), we have \( r' - r = O(1) \) and consequently,

\[
e(G'_2) - f\left(\frac{n}{2} + r - 1\right) \leq \frac{n + 1}{2} \left(\frac{n - 1}{2} + r' - 1\right) - \frac{n}{2} \left(\frac{n}{2} + r - 1\right)
\]

\[
= \frac{n}{2}(r' - r) + \frac{r'}{2} - \frac{3}{4} = O(n).
\]

We compare \( f(t) \), the approximate size of \( G_1 \), with \( f(\frac{n}{2} + r - 1) \), the approximate size of \( G_2 \) and \( G'_2 \), in the next proposition.

**Proposition 10.** Suppose \( \frac{n}{2} - 1 \leq t \leq n - 1 \), \( f(x) \) and \( r \) are defined as in \( (3) \) and \( (2) \), respectively. We have \( f(t) \leq f(\frac{n}{2} + r - 1) \) if and only if \( r + t \leq \frac{5n}{6} \), equivalently,

\[
(8) \quad t \leq \frac{5}{4}n - \frac{\sqrt{57n^2 - 72n}}{12} - 1 \approx 0.6208n.
\]

To prove Proposition (10) we need a simple fact on quadratic functions.

**Fact 11.** Suppose \( g(x) \) is a quadratic function with a maximum at \( x = a \) and assume \( x_1 \leq x_2 \). Then \( g(x_1) \leq g(x_2) \) if and only if \( x_1 + x_2 \leq 2a \).  \( \Box \)
Proof of Proposition 11. First note that
\[ f(x) = -\frac{3}{2}x^2 + \frac{4n - 3}{2} - \frac{n^2}{2} + \left(\frac{t}{2}\right) + \frac{3}{2}n - 1 \]
is a quadratic function with a maximum at \( x = \frac{2n}{3} - \frac{1}{2} \). Second, since \( r \leq \frac{n}{2} \), it follows that
\[
\left(\frac{\frac{n}{2} + r - 1}{2}\right) = \left(\frac{\frac{n}{2} - 1}{2}\right) + \left(\frac{r}{2}\right) \geq \left(\frac{\frac{n}{2} - 1}{2}\right) + 3\left(\frac{r}{2}\right) = \left(\frac{t}{2}\right).
\]
Consequently \( \frac{n}{2} + r - 1 \geq t \). By Fact 11, \( f(t) \leq f\left(\frac{n}{2} + r - 1\right) \) if and only if \( t + \frac{n}{2} + r - 1 \leq \frac{4n}{3} - 1 \) or \( r + t \leq \frac{5n}{6} \). By (2), this is equivalent to
\[
\left(\frac{n}{2} - 1\right) + 3\left(\frac{5n}{6} - t\right) \geq \left(\frac{t}{2}\right) \quad \text{or} \quad (t + 1)^2 - \frac{5}{2}(t + 1)n + \frac{7}{6}n^2 + \frac{n}{2} \geq 0,
\]
which holds exactly when \( t + 1 \leq \frac{5}{4}n - \sqrt{\frac{5n^2 - 72n}{12}} \) (because \( t < n \)).

We are ready to prove Theorem 3.

Proof of Theorem 3. Assume that \( \delta = \delta(G) \). We separate two cases.

Case 1: \( r + t \leq \frac{5n}{6} \), equivalently, (3).

First assume that \( \delta \geq \frac{4}{3}n - t - 1 \). Since \( t \leq \frac{2n}{3} - r \), we have \( \delta \geq \frac{n}{2} + r - 1 \) and consequently,
\[ e(G) \geq \frac{n}{2}\left(\frac{n}{2} + r - 1\right) = f\left(\frac{n}{2} + r - 1\right) \geq f(t) \]
by (7) and Proposition 11.

Second assume that \( \delta < \frac{4}{3}n - t - 1 \). By Proposition 7 we have \( e(G) \geq f(\delta) \). Recall that (5) forces \( t \leq \delta \). Since \( t \leq \delta < \frac{4}{3}n - t - 1 \) and \( f(x) \) is a quadratic function maximized at \( \frac{2n}{3} - \frac{1}{2} \), we derive from Fact 11 that \( f(\delta) \geq f(t) \). Hence \( e(G) \geq f(\delta) \geq f(t) \).

Case 2: \( r + t > \frac{5n}{6} \).

If \( \delta \geq \frac{n}{2} + r - 1 \), then \( e(G) \geq \frac{n}{2}\left(\frac{n}{2} + r - 1\right) = f\left(\frac{n}{2} + r - 1\right) \) by (7). Otherwise \( \delta < \frac{n}{2} + r - 1 \). Note that
\[ \delta + \frac{n}{2} + r - 1 \geq t + \frac{n}{2} + r - 1 > \frac{5n}{6} + \frac{n}{2} - 1 = \frac{4n}{3} - 1. \]
Since the quadratic function \( f(x) \) is maximized at \( \frac{2n}{3} - \frac{1}{2} \), we derive from Fact 11 that \( f(\delta) \geq f\left(\frac{n}{2} + r - 1\right) \).

By Proposition 9, when \( n/2, t, r \) are all integers, we have \( e(G_1) = f(t) \) and \( e(G_2) = f\left(\frac{n}{2} + r - 1\right) \). In other cases, we have \( e(G_1) \leq f(t) + n \) and \( e(G_2) \leq f\left(\frac{n}{2} + r - 1\right) + n/2 \). When \( n \) is odd and \( r + t > 5n/6 \), we have \( r' = \Omega(n) \) and thus \( e(G'_2) = f\left(\frac{n}{2} + r - 1\right) + O(n) \).

Remark 12. When \( n/2, t, r \) are all integers, we actually learn the following about extremal graphs from the proof of Theorem 3. Suppose that \( G \) is an extremal graph. We claim that \( G = G_1 \) when \( r + t < 5n/6 \), and \( G \) is \((n/2 + r - 1)\)-regular when \( r + t > 5n/6 \).

Indeed, first assume \( r + t < 5n/6 \). If \( \delta \geq \frac{4}{3}n - t - 1 \), then \( \delta > \frac{n}{2} + r - 1 \) and consequently, \( e(G) > \frac{n}{2}\left(\frac{n}{2} + r - 1\right) = f(t) \), a contradiction. Following the second case of Case 1, we obtain that \( e(G) = f(\delta) = f(t) \) and consequently, \( \delta = t \). Using Proposition 7, we can derive that \( G = G_1 \). When \( r + t > 5n/6 \), the second case of Case 2 shows that \( e(G) \geq f(\delta) > f\left(\frac{n}{2} + r - 1\right) \), a contradiction. Thus \( \delta \geq \frac{n}{2} + r - 1 \) and \( e(G) = \frac{n}{2}\left(\frac{n}{2} + r - 1\right) \), which forces \( G \) to be \((n/2 + r - 1)\)-regular.

We now prove Theorem 3 by applying Theorem 3 and the arguments that show the equivalence of Problems 2 and 5 in Section 1.
Case 1: \(\frac{1}{4} ≤ d < \frac{47−5\sqrt{57}}{24}\).

Thus \(\frac{1}{4} ≤ \sqrt{d} < \frac{47−5\sqrt{57}}{12}\). Since \(n\) is sufficiently large, we have \(\sqrt{dn} ≤ \frac{15−\sqrt{57}}{12}n - 2\). Since \(\sqrt{dn} < t < \sqrt{dn} + 1\), it follows that

\[
\frac{n}{2} < t < \frac{15−\sqrt{57}}{12}n - 1 < \frac{5}{4}n - \frac{\sqrt{57}n^2 - 72n}{12} - 1.
\]

This allows us to apply the first case of Theorem 6 and (6) to derive that

\[
e(G) ≥ f(t) = \binom{n}{2} - (n - 1 - t)^2 ≥ \binom{n}{2} - (n - 1 - \sqrt{dn})^2
\]

\[
= (4\sqrt{d} - 2d - 1)\binom{n}{2} + n - dn - 1
\]

\[
≥ (4\sqrt{d} - 2d - 1)\binom{n}{2}
\]

as \(d < 1\) and \(n\) is sufficiently large.

Case 2: \(d ≥ \frac{47−5\sqrt{57}}{24}\).

Thus \(\sqrt{d} ≥ \frac{15−\sqrt{57}}{12}\). Since \(t > \sqrt{dn}\), it follows that

\[
t + 1 > \frac{15−\sqrt{57}}{12}n + 1 > \frac{5}{4}n - \frac{\sqrt{57}n^2 - 72n}{12}
\]

because \(\sqrt{57}n^2 - 72 > \sqrt{57}n^2 - 6\) for \(n ≥ 2\). Since (8) fails, we will apply the second case of Theorem 6. Since \(\binom{1}{2} = d\binom{n}{2}\) and \(r ≥ 0\), we can obtain from (2) that

\[
r = \frac{1}{6} \left(3 + \sqrt{3(n-1)((4d-1)n + 5)}\right) = \frac{1}{2} + \frac{n}{2} \sqrt{\frac{4d-1}{3}} + h(n),
\]

where

\[h(n) = \frac{1}{2\sqrt{3}} \left(\sqrt{(4d-1)n^2 + (6-4d)n - 5 - \sqrt{4d-1}}\right).
\]

It is easy to see that \(0 ≤ h(n) = O(1)\). Theorem 6 thus gives that

\[
e(G) ≥ f\left(\binom{n}{2} + r - 1\right) = \binom{n}{2} r - 1)
\]

\[
= \binom{n}{2} \left(\frac{1}{2} + \frac{n}{2} \sqrt{\frac{4d-1}{3}} + h(n)\right)
\]

\[
= \binom{n}{2} \left(\frac{1}{2} + \frac{\sqrt{\frac{4d-1}{12}}}{12}\right) + \frac{n}{4} \sqrt{\frac{4d-1}{3}} + \frac{n}{2} h(n)
\]

\[
≥ \binom{n}{2} \left(\frac{1}{2} + \sqrt{\frac{4d-1}{12}}\right).
\]

To see why these bounds are asymptotically tight, for every graph \(G ∈ \{G_1, G_2, G_2'\}\), we construct a triple system \(F_G\) whose members are all triangles of \(G\). Then \(\partial F_G ⊆ E(G)\) and \(δ(F_G) ≥ \binom{1}{2} = d\binom{n}{2}\).
Proposition \ref{prop:delta} gives that $|\partial F_{G_1}| \leq e(G_1) \leq f(t) + n$. By \eqref{eq:delta} and the assumption $t \leq \sqrt{d}n + 1$,

$$|\partial F_{G_1}| \leq f(t) + n \leq \left(\frac{n}{2}\right) - (n - 2 - \sqrt{d}n)^2 + n$$

$$= \left(4\sqrt{d} - 2d - 1\right) \left(\frac{n}{2}\right) + \left(3 - 2\sqrt{d} - d\right)n - 4 + n$$

$$= \left(4\sqrt{d} - 2d - 1\right) \left(\frac{n}{2}\right) + O(n).$$

When $n$ is even, we apply Proposition \ref{prop:delta} and \ref{prop:delta} obtaining that

$$|\partial F_{G_2}| \leq e(G_2) \leq f \left(\frac{n}{2} + r - 1\right) + \frac{n}{2} = \left(\frac{n}{2}\right) \left(\frac{1}{2} + \sqrt{\frac{4d - 1}{12}}\right) + O(n).$$

When $n$ is odd, we assume $r + t > 5n/6$ and thus $r, r' = \Omega(n)$. By Proposition \ref{prop:delta} and \ref{prop:delta}, we conclude that

$$|\partial F_{G_2}| \leq e(G_2) = f \left(\frac{n}{2} + r - 1\right) + O(n) = \left(\frac{n}{2}\right) \left(\frac{1}{2} + \sqrt{\frac{4d - 1}{12}}\right) + O(n). \quad \Box$$

We outline the proof of Corollary \ref{cor:delta} emphasizing how Theorem \ref{thm:delta} is applied. In a 3-graph, the degree of a pair $p$ of vertices is the number of the edges that contains $p$.

Proof Outline of Corollary \ref{cor:delta} Assume $\eta \ll \gamma$ and $\varepsilon = \gamma/12$. Let $H$ be an $n$-vertex 3-graph and $x$ be a vertex of $H$. In order to find $\eta n^{a+b+c-1}$ copies of $K^3_{a,b,c}$, it suffices to find $\frac{\sqrt{n}}{2} \binom{n}{2}$ pairs of vertices of $H$ with degree at least $\varepsilon^2 n$ – this follows from standard counting arguments in extremal (hyper)graph theory, or conveniently \cite{LoMarkstrom} Lemma 4.2 of Lo and Markström.

Suppose $\delta_1(H) \geq (d_0 + \gamma) \binom{n}{3}$ with $d_0 = 6 - 4\sqrt{2} \approx 0.343$. As shown in \cite[Lemma 3.3]{LoMarkstrom}, it is easy to find a set $V_0$ of at most $3\varepsilon n$ vertices and a subgraph $H'$ of $H$ on $V \setminus V_0$ such that $\delta(H') \geq d_0 \binom{n}{3}$, where $n' = |V \setminus V_0|$, and every pair in $\partial H'$ has degree at least $\varepsilon^2 n$ in $H$. Since $\frac{1}{4} < d_0 < \frac{\sqrt{2} - 5\sqrt{2}}{24} \approx 0.385$, by the first case of Theorem \ref{thm:delta} we have

$$|\partial H'| \geq \left(4\sqrt{d_0} - 2d_0 - 1\right) \binom{n'}{2} \geq \left(4\sqrt{d_0} - 2d_0 - 1 - \frac{\gamma}{2}\right) \binom{n}{2}.$$  

For every vertex $x \in V(H)$, since $d(x) \geq (d_0 + \gamma) \binom{n}{2}$ and crucially $4\sqrt{d_0} - 2d_0 - 1 = 1 - d_0$, at least $\frac{\gamma}{2} \binom{n}{2}$ pairs in $H$ are also in $\partial H'$ thus having degree at least $\varepsilon^2 n$, as desired.  

$\Box$

3. Concluding remarks

Let us restate the $k = 3$ case of Problem \ref{problem:triangle}

**Problem 13.** Let $G$ be a graph on $n$ vertices such that each vertex is contained in at least $\binom{n}{2}$ triangles, where $t$ is a positive real number. How many edges must $G$ have?

Our Theorem \ref{thm:delta} (asymptotically) answers Problem \ref{problem:triangle} for $n/2 \leq t + 1 \leq n$. The following proposition shows that for larger $n$, all but $O(t^3)$ vertices of an extremal graph are contained in isolated copies of $K_{t+1}$.

**Proposition 14.** When $n > (t + 1)^2(t + 2)/4$, every extremal graph for Problem \ref{problem:triangle} contains an isolated copy of $K_{t+1}$.
Proof. Let $G = (V, E)$ be an extremal graph with $|V| = n$. Since every vertex lies in at least $\binom{t}{2}$ triangles, it suffices to show that $G$ contains a vertex of degree $t$ and all of its neighbors also have degree $t$ (thus inducing an isolated copy of $K_{t+1}$).

Suppose $n = a(t + 1) + b$, where $0 < b \leq t$. Let $G'$ be the disjoint union of $a - 1$ copies of $K_{t+1}$ together with two copies of $K_{t+1}$ sharing $t + 1 - b$ vertices. Since $G$ is extremal, we have

$$2e(G) \leq 2e(G') = tn + (t + 1 - b)b \leq tn + (t + 1)^2/4.$$

Partition $V(G)$ into $A \cup B$ such that $A$ consists of all vertices of degree greater than $t$ and $B$ consists of all vertices of degree exactly $t$. Then

$$\sum_{v \in A} (d_G(v) - t) = \sum_{v \in V} (d_G(v) - t) = 2e(G) - tn \leq (t + 1)^2/4.$$

This implies that $|A| \leq (t + 1)^2/4$. Let $e(A, B)$ denote the number of edges (of $G$) between $A$ and $B$. It follows that

$$e(A, B) \leq \sum_{v \in A} d(v) \leq \frac{1}{4}(t + 1)^2 + t|A| \leq \frac{1}{4}(t + 1)^3.$$

Let $B'$ consists of the vertices of $B$ that are adjacent to some vertex of $A$. Then $|B'| \leq e(A, B) \leq (t + 1)^3/4$. If $n > (t + 1)^2(t + 2)/4$, then $n > |A| + |B'|$ and consequently, there exists a vertex of $B$ whose $t$ neighbors are all in $B$, as desired. \hfill $\square$

The $t = 2$ case of Problem 13 assumes that every vertex in an $n$-vertex graph is contained in a triangle. Since $\delta(G) \geq 2$, it follows that $e(G) \geq n$, which is best possible when $3$ divides $n$. Recently, Chakraborti and Loh [2] determined the minimum number of edges an $n$-vertex graph in which every vertex is contained in a copy of $K_s$, for arbitrary $s \leq n$. Their extremal graph is the union of copies of $K_s$, all but two of which are isolated.

Finally, using careful case analysis, we can answer Problem 13 exactly when $t$ is very close to $n$. This falls into the $r + t > 5n/6$ case of Theorem 6 but $G_2$ defined in Construction 8 is not necessarily extremal (unless both $r$ and $n/2$ are integers).

- When $n = t + 2$, the (unique) extremal graph is $K_n$, the complete graph on $n$ vertices minus one edge.
- When $n = t + 3$ is even, the (unique) extremal graph is $K_n$ minus a perfect matching (provided $t > 5$). When $n = t + 3$ is odd, $K_n$ minus a matching of size $\frac{n-1}{2}$ is an extremal graph (provided $t > 6$).
- When $n = t + 4$, the complement of any $K_3$-free 2-regular graph on $n$ vertices is an extremal graph. Note that $r = n/2 - 2$ in this case and thus $G_2$ is one of the extremal graphs when $n$ is even.

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