Multiple fibers of holomorphic Lagrangian fibrations
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Abstract

We determine all possible multiplicities of general singular fibers of a holomorphic Lagrangian fibration, under the assumption that all components of the fibers are of Fujiki class. The multiplicities are at most 6 and the possible values are intricately related to the Kodaira type of the characteristic cycle.

1 Introduction

We work in the category of complex analytic spaces. A proper morphism between two complex analytic spaces is a fibration if it has connected fibers. Throughout the paper, we will assume that all components of the fibers of a fibration are of Fujiki class, i.e., bimeromorphic to compact Kähler manifolds.

Let \((M, \sigma_M)\) be a holomorphic symplectic manifold and \(f : M \to B\) be a fibration over a complex manifold \(B\) whose fibers are Lagrangian with respect to \(\sigma_M\). We say that \(f\) is a Lagrangian fibration. A smooth fiber of \(f\) is a complex torus. We do not assume that \(M\) is compact. Our main interest is to understand the structure of a general singular fiber of \(f\). In \([\text{HO}]\), a basic structure theory of a general singular fiber was developed, modulo the multiplicity of the fiber. A crucial ingredient is the concept of characteristic cycles, certain connected 1-cycles in the singular fiber naturally determined by the symplectic form (see Section 2 for the definition). In fact, when the coefficients are divided by the multiplicity of the fiber, the structure of the characteristic cycles is exactly the same as the structure of Kodaira’s elliptic singular fibers, with one exceptional case of infinite chain (cf. Proposition 2.3 and Theorem 2.4). This gives a quite satisfactory geometric description of a general singular fiber, modulo its multiplicity. The central remaining question is the possible values of the multiplicity of a general singular fiber. This question was not touched upon in \([\text{HO}]\). The main goal of the present paper is to give a complete answer to this question.

Here the theory gets somewhat different from Kodaira’s theory of elliptic fibrations, which corresponds to the case of \(\dim M = 2\). When \(\dim M = 2\), except fibers of type \(I_m\), a singular fiber is simply connected and the multiplicity is always 1. So the theory of multiplicity is quite simple and the detailed geometry of singular fibers plays little role in the study of the multiplicity. When \(\dim M \geq 4\), however, a general singular fiber is not simply connected. For many types of the characteristic cycles, the multiplicity can be bigger than 1. The precise values of the possible multiplicities are intricately related to the geometry of the singular fiber, as we will see below. We will determine all the possible values for each type of the characteristic cycle. Our main result is the following.

Theorem 1.1 (Main Theorem) Let \(f : M \to B\) be a Lagrangian fibration, \(D \subset B\) be an irreducible component of the discriminant divisor, and \(b \in D\) be a general point. Assume that

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dim $M = 2d \geq 4$. Then the multiplicity $n$ of $f^{-1}(b)$ satisfies $n \leq 6$. More precisely, let $\Theta$ be the characteristic cycle with coefficients divided by $n$. Set $\zeta_k = e^{2\pi i/k}$ and denote by $E_\tau$ the elliptic curve of period $\tau$. Then we have the following possibilities.

(i) If $n = 6$, then $\Theta$ is isomorphic to the elliptic curve $E_3$;
(ii) If $n = 5$, then $\Theta$ is of Kodaira’s type $II$;
(iii) If $n = 4$, then $\Theta$ is either isomorphic to the elliptic curve $E_4$ or of Kodaira’s type $IV$;
(iv) If $n = 3$, then $\Theta$ is isomorphic to the elliptic curve $E_3$, of Kodaira’s type $III$ or of Kodaira’s type $I_0^*$;
(v) If $n = 2$, then $\Theta$ is of Kodaira’s type $I_{2m}$ ($m \geq 0$), $I_0^*$, $IV$, $IV^*$, or of type $A_\infty (=I_\infty)$.

In particular, if $\Theta$ is of Kodaira’s type $I_{2m-1}$, $I_m^*$ ($m \geq 1$), $III^*$ or $II^*$, then $n = 1$, i.e., $f^{-1}(b)$ is not multiple.

Moreover, all the cases in (i)-(v) are realizable in each dimension $2d \geq 4$.

In the course of proving our main theorem, we obtain a couple of results which are of independent interest. In the classification of characteristic cycles in [HO], a hypothetical case of type $D_\infty$ remained. In Theorem 2.4 below, we remove this case, thus giving a complete picture of the classification of characteristic cycles (modulo multiplicities). Another result of independent interest is the stable reduction theory presented in Section 4. This generalizes Kodaira’s stable reduction of singular fibers of elliptic fibrations to general singular fibers of higher dimensional Lagrangian fibrations. In fact, by exploiting the geometry of the characteristic cycles, the stable reduction can be constructed from Kodaira’s construction for surfaces.

As was the case with [HO], our results have some overlap with Matsushita’s paper [Ma]. He studies the structure of general singular fibers and their multiplicities under the additional assumption that the Lagrangian fibration $f : M \to B$ is a projective morphism. He used the sophisticated theory of the toroidal degeneration of abelian varieties, which requires the assumption that $f$ is projective. Our approach is quite different and elementary. Other than the general tools of complex geometry, all we need is the classical work of Kodaira on elliptic fibrations. We believe that our approach gives a simpler and geometrically clearer explanation for both the structure theory and the classification result, even in the case when $f$ is projective. Also it should be mentioned that, compared with our results, some cases are missing in the classification list of [Ma].

2 Review of [HO] with some complements

In this section, we recall the definition and basic properties of characteristic cycles from [HO] and prove some complementary results.

Let $M = (M, \sigma_M)$ be a holomorphic symplectic manifold of dimension $2d \geq 4$ and $f : M \to B$ be a Lagrangian fibration over a $d$-dimensional complex manifold $B$ as defined in the introduction. Let $b \in D \subset B$ be a general point of the discriminant set $D$. By $f^{-1}(b)$ we denote the scheme theoretic fiber over $b$. We denote by $f^{-1}(b)_{\text{red}}$ the reduction of $f^{-1}(b)$, i.e., the underlying reduced analytic space of $f^{-1}(b)$. Since we are interested in the local geometry of $f$ near $b$, we will consider $B$ as a germ of a neighborhood of $b$. Let us denote by $T$ the complex Lie group $\mathbb{C}^{d-1}$.

Proposition 2.1 There exists an action of $T$ on $M$ which preserves each fiber of $f$ and the symplectic form $\sigma_M$, such that the isotropy subgroup at each point of $M$ is discrete. In particular, the induced action on the dualizing sheaf $\omega_M$ is trivial, and consequently, the induced action on
the dualizing sheaf of each fiber $f^{-1}(b)$ is trivial. The singular locus of the reduction $f^{-1}(b)_{\text{red}}$ consists of finitely many $T$-orbits, each of which is a complex torus.

**Proof.** This is immediate from [HO, Proposition 2.2] except the invariance of the symplectic form. However, the latter is immediate from the way the $T$-action is constructed. In fact, the $T$-action is generated by $d - 1$ commuting Hamiltonian vector fields which clearly leaves the symplectic form invariant. \(\square\)

The following was proved in [HO, Theorem 1.3].

**Proposition 2.2** For each component $V$ of $f^{-1}(b)_{\text{red}}$, the normalization $\tilde{V}$ is a compact complex manifold of Fujiki class and the Albanese map $\text{alb}: \tilde{V} \to \text{Alb}(V)$ is a fiber bundle whose fiber is either $P_1$ or an elliptic curve.

Let us call the image of a fiber of the Albanese map in $f^{-1}(b)_{\text{red}}$ a characteristic leaf. We define an equivalence relation on the set $f^{-1}(b)_{\text{red}}$ by declaring two points $x$ and $x'$ equivalent if there is a finite number of characteristic leaves $C_1, \ldots, C_\ell$ such that $x \in C_i$, $x' \in C_i$ and $C_i \cap C_i \neq \emptyset$ for all $2 \leq i \leq \ell$. Let us call an equivalence class a characteristic curve. Thus a characteristic curve $C$ consists of a countable union of characteristic leaves $\{C_i\}$. For a characteristic curve $C = \cup_i C_i$, we define the characteristic cycle as the analytic cycle $\Sigma_i m_i C_i$ where $m_i$ is the multiplicity of the component of the divisor $f^{-1}(D)$ which contains $C_i$. The greatest common divisor of $\{m_i\}$ is the multiplicity of the fiber $f^{-1}(b)$. In [HO, Theorem 1.4] we have shown the following. We say that the intersection of two smooth curves is quasi-transversal if the tangent spaces of the curves at the intersection point are distinct.

**Proposition 2.3** The $T$-action on $f^{-1}(b)_{\text{red}}$ induces a transitive action on the set of characteristic curves. In particular, all characteristic curves on $f^{-1}(b)_{\text{red}}$ are isomorphic. Moreover, when $n$ is the multiplicity of the fiber $f^{-1}(b)$ and $\Sigma_i m_i C_i$ is a characteristic cycle, the cycle $\Sigma_i \frac{m_i}{n} C_i$, is of the form of

(1) one of the singular fibers of a relatively minimal elliptic fibration listed by Kodaira [Kd, Theorem 6.2];

(2) 1-cycle of Type $A_\infty$, i.e., 1-cycle $\sum_{i \in \mathbb{Z}} C_i$ consisting of infinitely many $P_1$’s such that $C_i \cap C_{i+1} = \{P_i\}$ (the intersections are quasi-transversal and $P_i \neq P_j$ if $i \neq j$), and such that $C_i \cap C_j = \emptyset$ if $|i - j| \geq 2$;

(3) 1-cycle of Type $D_\infty$, i.e., 1-cycle $C_0 + C_1 + \sum_{i \geq 2} 2C_i$ consisting of infinitely many $P_1$’s such that $C_i \cap C_{i+1} = \{P_i\}$ for each $i \geq 1$, $C_0 \cap C_2 = \{P_0\}$ (all the intersections are quasi-transversal and $P_i \neq P_j$ if $i \neq j$) and such that $C_i \cap C_j = \emptyset$ for other pairs $i \neq j$.

Examples of characteristic cycles of the type (1) in Proposition 2.3 are provided by Kodaira’s construction. In [HO, Proposition 4.13], an example belonging to the type (2) was given. At the time when [HO] was completed, the authors were not aware whether examples of the type (3) exist or not. It turns out that the following simple argument excludes the type (3).

**Theorem 2.4** Characteristic cycles belonging to the type (3) in Proposition 2.3 do not exist.

**Proof.** Set $F = f^{-1}(b)$ and let $n$ be its multiplicity. Assume to the contrary that a characteristic cycle $\Theta$ is of Type $D_\infty$. The characteristic cycles are parametrized by the $T$-action;

$$\Theta_t = nC_{0,t} + nC_{1,t} + \sum_{i \geq 2} 2nC_{i,t} \ (t \in T).$$
These cycles $\Theta_t$ cover $F$. Consider the Zariski closures in $F_{\text{red}}$:

$$E_1 := \bigcup_{t \in T} C_{0,t} \cup C_{1,t}, \ E_k := \bigcup_{t \in T} C_{k,t} \ (k \geq 2).$$

Then $E_1$ is a finite union of the irreducible components of $F_{\text{red}}$ and each $E_k \ (k \geq 2)$ is an irreducible component of $F_{\text{red}}$. For the multiplicity reason, $E_k \not\subset E_1$ if $k \geq 2$. On the other hand, $F_{\text{red}}$ has only finitely many irreducible components. Thus, there are $\ell > k \geq 2$ such that $E_{\ell} = E_k$.

Choose $k$ that is the minimum among all such pairs. By definition of the characteristic cycle of Type $D_{\infty}$, we have $E_{k-1} \cap E_k \neq \emptyset$. Then, $E_{k-1} \cap E_k$ is the image of the multi-section of the Albanese map from $E_k$ (the normalization of $E_k$) and $C_{\ell,t} \subset E_k$ is the image of a fiber of the Albanese map, under the normalization map. Thus, $C_{\ell,t}$ meets $E_{k-1} \cap E_k$, whence, meets $C_{k-1,t'}$ for some $t'$. This implies that $E_{\ell-1} = E_{k-1}$. If $k \geq 3$, this contradicts the minimality of $k$. If $k = 2$, then $C_{\ell,t}, \ell \geq 3$ meets a component $C_{1,t'}$ of multiplicity 1, a contradiction to the definition of Type $D_{\infty}$.

Now we describe the local structure of the analytic space (scheme) $f^{-1}(b)$. For an analytic space $F$ and a point $x \in F$, let us denote by $(F, x)$ the germ of $F$ at $x$. The following is a consequence of a combination of some results and arguments in [HO].

**Proposition 2.5** In the setting of Proposition 2.3, let $x \in f^{-1}(b)$ be a point. Then there exist a germ $\mathcal{R}$ of an effective divisor in the germ $(C^2, 0)$ and an isomorphism of germs

$$(f^{-1}(b), x) \cong \mathcal{R} \times (C^{d-1}, 0),$$

such that the factor $(C^{d-1}, 0)$ is tangent to the $T$-orbits near $x$. Moreover, the germ of the characteristic cycle $\Theta \subset f^{-1}(b)$ through $x$ is biholomorphic to the germ of $\mathcal{R}$ as a cycle in $(C^2, 0)$ by the projection $\mathcal{R} \times (C^{d-1}, 0) \to \mathcal{R}$:

$$\begin{array}{ccc}
(f^{-1}(b), x) & \cong & \mathcal{R} \times (C^{d-1}, 0) \\
\cup & & \\
(\Theta, x) & \cong & \mathcal{R}.
\end{array}$$

**Proof.** In [HO, Proposition 4.4 (2)], the isomorphism of germs

$$(f^{-1}(b)_{\text{red}}, x) \cong \mathcal{R}_{\text{red}} \times (C^{d-1}, 0)$$

is given for the reduction $f^{-1}(b)_{\text{red}}$ with some reduced divisor $\mathcal{R}_{\text{red}}$ in $(C^2, 0)$. Since $b$ is a general point of the discriminant set, $(f^{-1}(b), 0)$ has the structure of a divisor in $(C^{d+1}, 0)$. Thus if we define the divisor $\mathcal{R} \subset (C^2, 0)$ whose reduction is $\mathcal{R}_{\text{red}}$, by assigning the multiplicities of its components to match those of $(f^{-1}(b), 0)$, we get the desired isomorphism

$$\begin{array}{ccc}
(f^{-1}(b), x) & \cong & \mathcal{R} \times (C^{d-1}, 0).
\end{array}$$

Now, to establish the asserted isomorphism of cycles, it suffices to prove that $(\Theta_{\text{red}}, x)$ is biholomorphic to $\mathcal{R}_{\text{red}}$ by the projection $\mathcal{R}_{\text{red}} \times (C^{d-1}, 0) \to \mathcal{R}_{\text{red}}$. From [HO, Proposition 4.4 (3)], it is clear that the projection gives an isomorphism $(\Theta_{\text{red}}, x) \cong \mathcal{R}_{\text{red}}$, except possibly when $\Theta_{\text{red}}$ is of Kodaira’s type II or III. When it is of type II, the isomorphism was proved in the proof of [HO, Proposition 4.7 (2) and (3)]. When it is of type III and $x$ is the point where two smooth components of $\Theta_{\text{red}}$ intersect tangentially, by [HO, Proposition 4.4 (3)] again, $(\Theta_{\text{red}}, x) \to \mathcal{R}_{\text{red}}$ is
bijective. It suffices to show that the two components of $R_{\text{red}}$ has intersection number 2 in $(C^2, 0)$. This is immediate from the intersection number consideration in the proof of [HO, Proposition 4.11]. □

Proposition 2.5 provides a structure of analytic space (scheme) on a characteristic cycle. From now on, we will consider characteristic cycles with this scheme structure.

3 Multiplicity in the stable case

Following [BHPV, Section V.8], we will say that a characteristic cycle is stable, if it is of type $I_b$, $0 \leq b \leq \infty$, modulo the multiplicity, where $I_{\infty}$ denotes the case (2) in Proposition 2.3. In this section, we will determine the multiplicity of a general singular fiber in the case where the characteristic cycle is stable (Proposition 3.5).

To start with, we recall some general facts on the multiplicity. Let $\Delta$ be the unit disk in the complex plane with the origin $0 \in \Delta$ and $h : Z \to \Delta$ be a fibration of a complex manifold $Z$. The multiplicity of the fiber $h^{-1}(0)$ is the largest positive integer which divides the multiplicity (as a divisor in $Z$) of each component of $h^{-1}(0)$.

**Proposition 3.1** Given a fibration of a complex manifold $h : Z \to \Delta$, suppose the fiber $h^{-1}(b)$ has multiplicity $n$. Let $\nu : \Delta \to \Delta$ be the cyclic branched cover of degree $n$ and let $\tilde{Z}$ be the normalization of the fiber product of $h$ and $\nu$. Then in the natural commuting diagram

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\pi} & Z \\
\varphi \downarrow & & \downarrow h \\
\Delta & \xrightarrow{\nu} & \Delta,
\end{array}
$$

$\pi$ is an unramified covering of degree $n$ and the fiber $\tilde{h}^{-1}(0)$ has multiplicity 1. The cyclic Galois group of order $n$ acts on the fiber $\tilde{h}^{-1}(0)$ freely. Assume further that the dualizing sheaf $\omega_Z$ is trivial, $\omega_Z \cong \mathcal{O}_Z$. Then the induced action of the cyclic Galois group on the dualizing sheaf $\omega_{\tilde{Z}}$ is trivial, i.e., the action has weight 1.

**Proof.** This is essentially [BHPV, Chapter III, Proposition 9.1] where it is stated and proved when $\dim Z = 2$. The same proof works in any dimension. □

Recall that we have provided a characteristic cycle $\Theta$ with the structure of a complex analytic space (cf. the remark after Proposition 2.5). Denote by $\text{Aut}(\Theta)$ the biholomorphic automorphism group of $\Theta$ and by $\text{Aut}^\omega(\Theta)$ the subgroup consisting of the automorphisms acting trivially on the dualizing sheaf $\omega_\Theta$. Then the quotient group $\text{Aut}(\Theta)/\text{Aut}^\omega(\Theta)$ acts faithfully on $\omega_\Theta$.

**Proposition 3.2** Let $F$ be a general singular fiber of a Lagrangian fibration with multiplicity 1. We are given a $T$-action on $F$ by Proposition 2.1. Let $\text{Aut}_T(F)$ be the group of automorphisms of $F$ commuting with the $T$-action. Let $\text{Aut}_T^F$ be the subgroup of $\text{Aut}_T(F)$ consisting of automorphisms acting trivially on $\omega_F$. Fix a characteristic cycle $\Theta \subset F$. Then there exists a canonical injective homomorphism

$$
\theta : \text{Aut}_T(F)/\text{Aut}_T^F(F) \to \text{Aut}(\Theta)/\text{Aut}^\omega(\Theta)
$$

such that the eigenvalue of an element $g \in \text{Aut}_T(F)$ on $\omega_F$ agrees with that of $\theta(g)$ on $\omega_\Theta$. 

5
Proof. Given $g \in \text{Aut}_T(F)$ and a fixed choice $\Theta \subset F$, there exists $t \in T$ such that $t \cdot \Theta = g \cdot \Theta$. Then $t^{-1} \circ g : \Theta \to \Theta$ is an automorphism of $\Theta$. Define
\[
\theta(g) := t^{-1} \circ g \mod \text{Aut}^\omega(\Theta) \in \text{Aut}(\Theta)/\text{Aut}^\omega(\Theta).
\]
We claim that this definition is independent of the choice of $t$. In fact, if $(t_1 \circ t_2) \Theta = \Theta$, then $t_1 \circ t_2$ regraded as an automorphism of $\Theta$ is in $\text{Aut}^\omega(\Theta)$. This is because $T$-action on $F$ gives a trivial action on $\omega_F$ by Proposition 2.1. Since $\text{Aut}_T(F)$ commutes with the $T$-action, it is immediate that $\theta$ is a group homomorphism. Let us determine the kernel of $\theta$. Suppose that $g$ is in the kernel of $\theta$. Then $t^{-1} \circ g \in \text{Aut}^\omega(\Theta)$. Since $T$ action on $\omega_F$ is trivial, this means that $g$ acts trivially on $\omega_F$, i.e., $g \in \text{Aut}_T^\omega(F)$. It follows that $\theta$ descends to an injection
\[
\theta : \text{Aut}_T(F)/\text{Aut}_T^\omega(F) \to \text{Aut}(\Theta)/\text{Aut}^\omega(\Theta).
\]
From the definition of $\theta$, it is clear that the eigenvalue of an element $g \in \text{Aut}_T(F)$ on $\omega_F$ agrees with that of $\theta(g)$ on $\omega_\Theta$. □

Now let $\Theta$ be a characteristic cycles of type $I_b$, $0 \leq b \leq \infty$. More precisely, we have the following description.

1. When $b = 0$, $\Theta$ is an elliptic curve.
2. When $1 \leq b < \infty$, $\Theta$ is the Kodaira fiber of type $I_b$, consisting of $b$ smooth rational curves.
3. When $b = \infty$, $\Theta$ is the analytic space with infinitely many irreducible components described in (2) of Proposition 2.3.

Proposition 3.3 Let $\Theta$ be of type $I_b$, $0 \leq b \leq \infty$. Then $\text{Aut}(\Theta)/\text{Aut}^\omega(\Theta)$ is a finite cyclic group. Its order is 6 when $\Theta \cong E_3$, 4 when $\Theta \cong E_4$, and 2 otherwise. When $b$ is odd, an element $\tau \in \text{Aut}(\Theta)$ which is not in $\text{Aut}^\omega(\Theta)$ fixes a singular point $P$ of $\Theta$ and exchanges the two irreducible components of the germ of $\Theta$ at $P$.

Proof. The statement for $I_0$ and $I_1$ is well-known fact from the theory of elliptic curves (see for instance [Mc, Proposition 4.2] for the explicit statement with a proof). The statement for $I_b$, $b > 0$ then follows from the fact that $\Theta$ of type $I_b$ is an unramified cyclic Gorenstein covering of $I_1$. □

Proposition 3.4 Let $F$ be a general singular fiber of a Lagrangian fibration with multiplicity 1 whose characteristic cycle is of type $I_b$. Suppose that $\Gamma \subset \text{Aut}_T(F)$ acts faithfully on $\omega_F$. Then the order of $\Gamma$ is 1 or 2 when $b > 0$ and 2,3,4 or 6 when $b = 0$. When it is 3 or 6, the characteristic cycle is isomorphic to $E_3^b$ and when it is 4, the characteristic cycle is isomorphic to $E_4^b$.

Proof. By the assumption, $\Gamma \to \text{Aut}_T(F)/\text{Aut}_T^\omega(F)$ is injective. Thus the result follows from Proposition 3.2 and Proposition 3.3. □

Proposition 3.5 The multiplicity of a general singular fiber of a Lagrangian fibration with stable characteristic cycles of type $I_b$ (modulo multiplicity) can take only the following values: 1 or 2 when $b \geq 1$ and $b$ is even, 1 when $b$ is odd, and 1,2,3,4, or 6 when $b = 0$. When it is 3 or 6, the reduced characteristic cycle is isomorphic to $E_3^b$, and when it is 4, the reduced characteristic cycle is isomorphic to $E_4^b$.

Proof. Let $f : M \to B$ be a holomorphic Lagrangian fibration over a complex manifold $B$. Let $D \subset B$ be a component of the discriminant of $f$ and $b \in D$ be a general point. Choose an arc $\alpha : \Delta \to B$ with $\alpha(0) = b$ which intersects $D$ transversally and denote by $h : Z \to \Delta$ the pull-back
of \( f \) by \( \alpha \). Then \( Z \) is a complex manifold and the multiplicity of the fiber \( h^{-1}(0) \) is the same as the multiplicity of \( f^{-1}(b) \). The \( T \)-action on \( M \) induces a \( T \)-action on \( Z \) preserving the fibers of \( h \). By applying Proposition 3.1, we get an unramified cover \( \tilde{Z} \) of \( Z \) with a fibration \( \tilde{h} : \tilde{Z} \to \Delta \). The \( T \)-action on \( Z \) lifts to a \( T \)-action on \( \tilde{Z} \) and the action of the cyclic Galois group on \( \tilde{Z} \) commutes with this \( T \)-action. Since \( \omega_Z \) is trivial, the Galois group acts on \( \omega_{\tilde{Z}} \) trivially. But the action on \( \omega_{\Delta} \) is faithful. Thus the Galois group acts faithfully on the dualizing sheaf of the fiber \( \tilde{h}^{-1}(0) \).

From the way it is constructed, the fiber \( F := \tilde{h}^{-1}(0) \) can be realized as a general singular fiber of multiplicity 1 in some holomorphic Lagrangian fibration, say, \( \tilde{M} \to \tilde{B} \), where \( \tilde{B} \) is the cyclic covering of order \( n \) branched along \( D \) and \( M \) is the normalization of the fiber product, near over \( b \). Thus we can apply Proposition 3.4 with the cyclic Galois group as the group \( \Gamma \). This proves Proposition 3.5 except that the order of \( \Gamma \) is not 2 when \( b \) is odd.

To complete the proof, assuming that \( b \) is odd and \( \Gamma = \langle \iota \rangle \) is of order 2, we shall derive a contradiction. Put \( F = \tilde{h}^{-1}(0) \) and choose a characteristic cycle \( \Theta \subset F \). \( \Theta \) is either of type \( I_b \) or of Type \( I_{2b} \).

If it is of type \( I_{2b} \), then \( \iota(\Theta) = \Theta \). The action is a cyclic permutation of the components of \( \Theta \). This is because the characteristic cycle of \( h \) is also of type \( I_b \). However, then \( \iota^*\omega_{\Theta} = \omega_{\Theta} \), a contradiction to Proposition 3.2.

Consider the case where \( \Theta \) is of type \( I_b \). By Proposition 3.2 and its proof, there is \( t \in T \) such that \( t^{-1} \circ \iota(\Theta) = \Theta \). Put \( \tau = t^{-1} \circ \iota \). Then \( \tau^*\omega_F = -\omega_F \). This is because \( \iota \) acts on the base as \(-1\) but the action on \( \omega_{\tilde{Z}} \) is trivial. Then, by Proposition 3.2, \( \tau^*\omega_{\Theta} = -\omega_{\Theta} \). As \( b \) is odd, by Proposition 3.3, \( \tau \) fixes a singular point \( P \) of \( \Theta \) and changes the two components, say \( C_1 \) and \( C_2 \), of the germ of \( \Theta \) at \( P \). Now consider the isotropy action of \( \tau \) on the \((d + 1)\)-dimensional tangent space \( T_P(\tilde{Z}) \) of \( \tilde{Z} \) at \( P \). Because \( \iota \) commutes with \( T \), the isotropy action fixes the \((d - 1)\)-dimensional subspace \( T_P(T \cdot P) \subset T_P(\tilde{Z}) \), the tangent to the orbit \( T \cdot P \). Let \( N = T_P(\tilde{Z})/T_P(T \cdot P) \) be the two-dimensional quotient space. The induced action on \( N \) is an order-2 automorphism of \( N \) which changes two distinct subspaces of dimension 1 in \( N \) corresponding to the tangent directions of \( C_1 \) and \( C_2 \) at \( P \). Any automorphism of order 2 on a 2-dimensional vector space with this property must act on \( \wedge^2 N \) by \(-1\). This implies \( \tau^*\omega_{\tilde{Z}} = -\omega_{\tilde{Z}} \), a contradiction. \( \square \)

## 4 Stable reduction

When \( \Theta \) is unstable, the group \( \text{Aut}(\Theta)/\text{Aut}^-(\Theta) \) is no longer as simple as in Proposition 3.3. To bound the multiplicity in this case, we need the stable reduction of the unstable fiber. The goal of this section is to explain the construction of the stable reduction.

We start with recalling Kodaira’s stable reduction for a minimal elliptic fibration with unstable singular fiber ([Kd, Sections 8,9]). Let \( j : S \to \Delta \) be a minimal elliptic fibration with a singular fiber \( j^{-1}(0) \) of unstable type. There are three different cases: (Case 1) Type \( I_7^b ; b \geq 1 \), (Case 2) Type \( I_2^b, II^b, III^b, IV^b \) and (Case 3) Type \( II, III, IV \). Let us recall the construction of the stable reduction in each case. A good reference is [BHPV, Section V.10].

(Case 1) Type \( I_7^b ; b \geq 1 \). There are four \((-2)\)-curves in the central fiber \( j^{-1}(0) \). Contracting these four curves gives a normal surface \( S' \) with a fibration \( j' : S' \to \Delta \) whose central fiber is a string of \( b + 1 \) nonsingular rational curves of multiplicity \( m = 2 \). Take a cyclic cover of order \( m = 2, \nu : \Delta \to \Delta \) and let \( j^2 : S^2 \to \Delta \) be the normalization of the fiber product of \( j \) and \( \nu \). Then \( S^2 \) is non-singular and the central fiber of \( j^2 \) is a reduced fiber of type \( I_{2b} \). This \( j^2 \) is the stable
reduction of $j$. The Galois group $\langle -1 \rangle$ of order 2 acts on $S^2$ with 4 fixed points. The generator $-1$ of the Galois group acts on $\omega_{j^{-1}(0)}$ by $-1$. The quotient surface $S^2/\langle -1 \rangle$ has 4 singular points of type $1/(1,1)$ (under the notation [BPHV, Section V, 10]). From the construction, we can choose a coordinate system $(x,u)$ at a fixed point, say $P$, and a coordinate system $(y,v)$ on an open subset in $S$ where $x$ (resp. $y$) is the pull-back of the coordinate on $\Delta$ by $j^2$ (resp. $j$) such that there exists a meromorphic correspondence

$$y = x^2, \quad v = \frac{ux}{x^2} = \frac{u}{x}.$$  

Here $v$ is an affine coordinate of the exceptional curve arising from the minimal resolution of $S^2/\langle -1 \rangle$ at the image of $P$. The dualizing sheaves of $S^2$ and $S$ are related by

$$dy \wedge dv = 2dx \wedge du.$$

(Case 2) Type $I_0^*, II^*, III^*, IV^*$ There is one irreducible component $C$ of maximal multiplicity $m$ in $j^{-1}(0)$. The values of $m$ are (cf. Line 4 of Table 5 in [BHPV, Section V.10]):

$$m = 2 \text{ for Type } I_0^*, \quad m = 6 \text{ for Type } II^*, \quad m = 4 \text{ for Type } III^*, \quad m = 3 \text{ for Type } IV^*.$$

The connected components of $j^{-1}(0) \setminus mC$ are four $(-2)$-curves for Type $I_0^*$ and three Hirzebruch-Jung strings for Type $II^*, III^*, IV^*$. Contract these Hirzebruch-Jung strings to get a fibration $j_2 : S_2 \to \Delta$ of a normal surface $S_2$ with irreducible central fiber of multiplicity $m$. Then take a cyclic cover $\nu : \Delta \to \Delta$ of order $m$ and let $j^2 : S^2 \to \Delta$ be the normalization of the fiber product of $\nu$ and $j_2$. Then $j^2$ is a non-singular elliptic fibration. This $j^2$ is the stable reduction of $j$. The cyclic Galois group of order $m$ acts on $S^2$ with fixed points. The generator $\zeta_m$ of the Galois group acts on the dualizing sheaf $\omega_{j^2(0)}$ by $\zeta_m^{-1}$ (cf. Line 7 of Table 5 in [BHPV, Section V.10]) while it acts on $\omega_{\Delta}$ by $\zeta_m$. Consequently, it acts on $\omega_{S^2}$ trivially. From the construction, $\zeta_m$ has a fixed point, say $P$, and the quotient surface $S^2/\langle \zeta_m \rangle$ has a singular point of type $1/m(1,-1)$ at the image of $P$. So, we can choose a coordinate system $(x,u)$ at a fixed point of $S^2$ and a coordinate system $(y,v)$ on an open subset in $S$ where $x$ (resp. $y$) is the pull-back of the coordinate on $\Delta$ by $j^2$ (resp. $j$) such that there exists a meromorphic correspondence

$$y = x^m, \quad v = \frac{ux}{x^m} = \frac{u}{x^{m-1}}.$$  

Here $v$ is again an affine coordinate of one of the exceptional curves arising from the minimal resolution $S$ of $S^2/\langle \zeta_m \rangle$ at the image of $P$. The dualizing sheaves of $S^2$ and $S$ are related by

$$dy \wedge dv = mdx \wedge du.$$

(Case 3) Type $II, III, IV$. First apply a finite number of blow-ups of $S$ to get $j_1 : S_1 \to \Delta_1$ such that the central fiber has normal crossing support. The blow-ups needed in this process are listed in Page 209 of [BHPV, Section V.10]. In each step, the blow-up center is the unique singular point of the reduced fiber. There is one component of maximal multiplicity $m$ in $j_1^{-1}(0)$. The values of $m$ are (cf. Line 4 of Table 5 in [BHPV, Section V.10]):

$$m = 6 \text{ for Type } II, \quad m = 4 \text{ for Type } III, \quad m = 3 \text{ for Type } IV.$$

Beside this component with multiplicity $m$, there are three smooth rational curves with negative self-intersection in the singular fiber of $j_1^{-1}(0)$. So they are Hirzebruch-Jung strings of length 1.
Contract these Hirzebruch-Jung strings to get a fibration \( j_2 : S_2 \to \Delta \) of a normal surface \( S_2 \) with irreducible central fiber of multiplicity \( m \). Then take a cyclic cover \( \nu : \Delta \to \Delta \) of order \( m \) and let \( j^2 : S^2 \to \Delta \) be the normalization of the fiber product of \( \nu \) and \( j_2 \). Then \( j^2 \) is a non-singular elliptic fibration. This \( j^2 \) is the stable reduction of \( j \). The cyclic Galois group \( \langle \zeta_m \rangle \) of order \( m \) acts on \( S^2 \) with a fixed point, say \( P \). The generator \( \zeta_m \) of the Galois group acts on the dualizing sheaf \( \omega_{j^2-1(0)} \) by \( \zeta_m \) (cf. Line 7 in Table 5 in [BHPV, Section V.10]) while it acts on \( \omega_{\Delta} \) by \( \zeta_m \). Consequently, it acts on \( \omega_{S^2}/\langle \zeta_m \rangle \) and the quotient surface \( \omega_{S^2}/\langle \zeta_m \rangle \) has a singular point of type \( \frac{1}{m}(1,1) \) at the image of \( P \). From this description, we can choose a coordinate system \((x,u)\) at a fixed point \( P \) of \( \zeta_m \) and a coordinate system \((y,v)\) on an open subset in \( S \) where \( x \) (resp. \( y \)) is the pull-back of the coordinate on \( \Delta \) by \( j^2 \) (resp. \( j \)) such that there exists a meromorphic correspondence

\[
y = x^m, \quad v = \frac{ux^{m-1}}{x} = \frac{u}{x}.
\]

Again, as before, \( v \) is an affine coordinate of the exceptional curve of the minimal resolution \( S_1 \) of \( \omega_{S^2}/\langle \zeta_m \rangle \) at the image of \( P \). Note that this exceptional curve certainly survives under taking the relative minimal model over the base, of the minimal resolution of \( \omega_{S^2}/\langle \zeta_m \rangle \). Thus \((y,v)\) can be regarded as a coordinate system on an open subset in \( S \). The dualizing sheaves of \( S^2 \) and \( S \) are related by

\[
dy \wedge dv = m x^{m-2} dx \wedge du.
\]

To generalize this construction to higher dimensions, it is convenient to introduce the following notion.

**Definition 4.1** Let \( j : S \to \Delta \) be a fibration of a normal surface \( S \) over a disk which is smooth over \( \Delta \setminus \{0\} \). A \((d+1)\)-dimensional normal complex analytic variety \( Z \) and a fibration \( h : Z \to \Delta \) which is smooth over \( \Delta \setminus \{0\} \), is called a fibration modeled on \( j : S \to \Delta \) if the following holds.

(i) The Lie group \( T = \mathbb{C}^{d-1} \) acts on \( Z \) by a holomorphic map \( \gamma : Z \times T \to Z \) which preserves the fibers of \( h \), i.e., the following diagram commutes, where \( \hat{h} \) is the composition of \( h \) and the projection to \( Z \).

\[
\begin{array}{ccc}
Z \times T & \xrightarrow{\gamma} & Z \\
\hat{h} \downarrow & & \downarrow h \\
\Delta & \xrightarrow{=} & \Delta,
\end{array}
\]

The stabilizer of the \( T \)-action at each point of \( Z \) is discrete and the singular loci of \( h^{-1}(0)_{\text{red}} \) consists of finitely many \( T \)-orbits, which are \((d-1)\)-dimensional tori.

(ii) For each point \( x \in h^{-1}(0) \), there exists a 1-dimensional compact connected analytic subscheme \( \Psi_x \subset h^{-1}(0) \) containing \( x \) such that for \( x, y \in h^{-1}(0) \), either \( \Psi_x \cap \Psi_y = \emptyset \) or \( \Psi_x = \Psi_y \), and for \( g \in T \),

\[
g \cdot \Psi_x = \Psi_{g \cdot x}, \quad \text{i.e.,} \quad \gamma(\Psi_x, g) = \Psi_x(g \cdot x).
\]

(iii) For each \( x \in h^{-1}(0) \), there exist a point \( s \in j^{-1}(0) \subset S \) and a morphism

\[
\iota : j^{-1}(0) \to Z, \quad \iota(s) = x,
\]

inducing an isomorphism \( j^{-1}(0) \cong \Psi_x \). Furthermore, there exists isomorphisms of germs \( \rho : (S, s) \times (T, 0) \cong (Z, x) \) and \( \rho|_{j^{-1}(0) \times T} : (j^{-1}(0), s) \times (T, 0) \cong (h^{-1}(0), x) \) with the commuting
(S, s) \times (T, 0) \xrightarrow{\rho} (Z, x) \\
\downarrow (j^{-1}(0), s) \times (T, 0) \xrightarrow{\rho|_{j^{-1}(0) \times T}} (h^{-1}(0), x)

\Delta \quad \Rightarrow \quad \Delta \\
(S, s) \times (T, 0) \quad \xrightarrow{\rho} \quad (Z, x),

compatible with the T-action

\begin{align*}
(j^{-1}(0), s) \times (T, 0) & \xrightarrow{\iota \times \text{id}_T} (Z, x) \times (T, 0) \\
\rho \downarrow & \quad \quad \quad \quad \downarrow \gamma \\
(Z, x) & \xrightarrow{=} (Z, x).
\end{align*}

**Proposition 4.2** In the setting of Definition 4.1, assume that \( h^{-1}(0)_{\text{red}} \) is not smooth. Then there exists a complex torus \( T' \) of dimension \( d - 1 \) with a \( T' \)-action and a \( T' \)-equivariant morphism \( q : h^{-1}(0) \to T' \) such that for each \( t \in T' \), \( q^{-1}(t) \) is \( \Psi_x \) for some \( x \).

**Proof.** For a given \( \Psi_x, x \in h^{-1}(0) \), let \( I_{\Psi_x} \) be the ideal sheaf on \( h^{-1}(0) \) defining \( \Psi_x \) as a subscheme. By Definition 4.1 (iii), a germ of \( h^{-1}(0) \) is the product of a germ of \( \Psi_x \) with a smooth germ of dimension \( (d - 1) = \dim T \). Thus the conormal sheaf \( I_{\Psi_x}/I_{\Psi_x}^2 \) is a locally free sheaf on \( \Psi_x \) of rank \( d - 1 \). Moreover, the vector fields generating \( T' \)-action on \( h^{-1}(0) \) determine a \( d - 1 \)-pointwise independent sections of \( \text{Hom}(I_{\Psi_x}/I_{\Psi_x}^2, O_{h^{-1}(0)}) \). Thus \( I_{\Psi_x}/I_{\Psi_x}^2 \) is free and \( \text{Hom}(I_{\Psi_x}/I_{\Psi_x}^2, O_{h^{-1}(0)}) \), which is the tangent space to the Hilbert scheme (or Douady space) of \( h^{-1}(0) \), has dimension \( d - 1 \). It follows that the Hilbert scheme is smooth and of dimension \( d - 1 \) at the point parametrizing \( \Psi_x \). By Definition 4.1 (ii), all subschemes \( \Psi_x, x \in h^{-1}(0) \) belong to one \( T \)-orbit in the Hilbert scheme. Thus there exists a component \( T' \) of the Hilbert scheme with an open \( T \)-orbit \( T'_0 \subset T' \) such that \( T'_0 \) parametrizes \( \Psi_x \)’s. But when \( h^{-1}(0)_{\text{red}} \) is not smooth, an orbit of \( T \) in \( h^{-1}(0) \) is a torus by Definition 4.1 (i). This implies that \( T'_0 = T' \) is a torus. Let \( \hat{\varphi} : U \to T' \) and \( p : U \to h^{-1}(0) \) be the universal family morphisms associated with the Hilbert scheme \( T' \) such that fibers of \( \hat{\varphi} \) are sent to \( \Psi_x \)’s by \( p \). By Definition 4.1 (ii), \( p \) must be a bijective morphism. By Definition 4.1 (iii), \( p \) is unramified, hence it must be an isomorphism. Then \( \hat{\varphi} \) induces a \( T \)-equivariant morphism \( q : h^{-1}(0) \to T' \) whose fibers are \( \Psi_x \)’s. \( \square \)

Let \( S \) be a normal surface and \( j : S \to \Delta \) be a fibration which is smooth over \( \Delta \setminus \{0\} \). Starting from \( j : S \to \Delta \), we can consider the following three operations to get a new fibration \( j' : S' \to \Delta \).

(a) **Blow-up:** Assume that \( S \) is smooth and \( j^{-1}(0)_{\text{red}} \) has a unique singular point \( s \). Let \( S' \) be the blow-up of \( S \) at \( s \). The morphism \( j' : S' \to \Delta \) is just the composition of the blow-up with \( j \).

(b) **Contraction:** Assume that \( S \) is smooth and there exists a unique irreducible component \( C \) of maximal multiplicity \( m \) in \( j^{-1}(0) \) such that each connected component of \( j^{-1}(0) \setminus mC \) is a Hirzebruch-Jung string (in the sense of [BHPV, Section III.2]). Contract these Hirzebruch-Jung strings to get a normal surface \( S' \) with the induced morphism \( j' : S' \to \Delta \).

(c) **Cyclic cover:** For any positive integer \( m \), we take the cyclic cover \( \nu : \Delta \to \Delta \) of degree \( m \). Let \( j' : S' \to \Delta \) be the normalization of the base change of \( j \) by \( \nu \).

Note that the stable reduction of a given minimal elliptic singular fibration is obtained by a finite number of operations of the above three kinds. To generalize the stable reduction to higher dimensions, we will explain how the above three operations on surfaces can be generalized to higher dimensions.
Let \( j' : S' \to \Delta \) be obtained from \( j : S \to \Delta \) by one of the three operations (a), (b), (c) explained above. Given a fibration \( h : Z \to \Delta \) modeled on \( j \) and a fixed choice of an isomorphism \( \imath : j^{-1}(0) \cong \Psi_x \subset h^{-1}(0) \) of Definition 4.1 (iii), we can construct a fibration \( h' : Z' \to \Delta \) modeled on \( j' \) in a canonical way as follows.

(A) Blow-up: If \( j' \) is obtained by the operation (a), then we blow-up \( Z \) along the compact \( T \)-orbit \( T \cdot (\imath(s)) \) to get \( h' : Z' \to \Delta \), which is clearly a fibration modeled on \( j' \).

(B) Contraction: If \( j' \) is obtained by the operation (b), let \( C_1, \ldots, C_k \) be the Hirzebruch-Jung strings in \( j^{-1}(0) \). Then the union of \( T \cdot (\imath(C_1)), \ldots, T \cdot (\imath(C_k)) \) is a divisor in \( Z \). We assign the multiplicity on the component of this divisor by the multiplicity of \( C_i \) in \( j^{-1}(0) \) and call the resulting divisor \( D \). \( D \) consists of some components of \( h^{-1}(0) \). There exists a fibration \( f : D \to T' \) onto a complex torus \( T' \) induced by the morphism in Proposition 4.2. Let \( L \) be the line bundle on \( Z \) corresponding to the Cartier divisor \( D \) and \( L^* \) be the dual bundle of \( L \). Since \( C_i \)'s are Hirzebruch-Jung strings, we have (i) \( L^* \) restricted to \( D \) is \( f \)-ample and (ii) \( R^1 f_*(L^* \otimes L) = 0 \) for \( \ell > 0 \). Here (i) comes from the negative definiteness of the intersection matrix of a Hirzebruch-Jung string and (ii) comes from the proof of [BHPV, Proposition III (3.1)]. Thus this divisor \( D \) can be contracted to give a normal variety \( Z' \) inducing \( h' : Z' \to \Delta \) by [Fu, Theorem 2]. It is immediate to check that \( h' \) is a fibration modeled on \( j' \).

(C) Cyclic cover: If \( j' \) is obtained by the operation (c), then \( h' : Z' \to \Delta \) is defined to be the normalization of the base change of \( h : Z \to \Delta \) by \( \nu : \Delta \to \Delta \). The \( T \)-action on \( Z \) naturally lifts to a \( T \)-action on \( Z' \) and one can check that \( h' \) is a fibration modeled on \( j' \).

Since all the steps in the construction of the stable reduction \( j^s \) from the minimal elliptic fibration \( j \) with an unstable fiber are coming from the three operations (a), (b) and (c), if we are given a fibration \( h : Z \to \Delta \) modeled on \( j \), we get a fibration \( h^s : Z^s \to \Delta \) modeled on \( j^s \) by applying the three operations (A), (B) and (C). The resulting fibration \( h^s \) is the stable reduction of \( h \). Now using the property (iii) of Definition 4.1 and the local coordinate expression of the meromorphic correspondence for the stable reduction for surfaces, we have the following.

**Proposition 4.3** Let \( h : Z \to \Delta \) be a fibration modeled on a minimal elliptic fibration \( j : S \to \Delta \) with an unstable singular fiber. Let \( h^s : Z^s \to \Delta \) be the stable reduction constructed above. The cyclic Galois group \( \langle \zeta_m \rangle \) of order \( m \) acts on \( Z^s \) with fixed points. We can choose a coordinate system \( (x, u_1, u_2, \ldots, u_d) \) at a fixed point and a coordinate system \( (y, v_1, v_2, \ldots, v_d) \) on an open subset in \( Z \) where \( x \) (resp. \( y \)) is the pull-back of the coordinate on \( \Delta \) by \( h^s \) (resp. \( h \)) with the following meromorphic correspondence, depending on the three cases of \( j \).

(Case 1) \( y = x^2, v_1 = \frac{u_1}{x}, v_i = u_i \) for \( 2 \leq i \leq d \), and

\[
dy \wedge dv_1 \wedge \cdots \wedge dv_d = 2dx \wedge du_1 \wedge \cdots \wedge du_d.
\]

(Case 2) \( y = x^m, v_1 = \frac{u_1}{x^{m-1}}, v_i = u_i \) for \( 2 \leq i \leq d \), and

\[
dy \wedge dv_1 \wedge \cdots \wedge dv_d = m dx \wedge du_1 \wedge \cdots \wedge du_d.
\]

(Case 3) \( y = x^m, v_1 = \frac{u_1}{x^m}, v_i = u_i \) for \( 2 \leq i \leq d \), and

\[
dy \wedge dv_1 \wedge \cdots \wedge dv_d = m x^{m-2} dx \wedge du_1 \wedge \cdots \wedge du_d.
\]
5 Multiplicity in the unstable case

In this section, we will determine the possible values of the multiplicity when the characteristic cycle is unstable. A key observation is the following proposition.

**Proposition 5.1** Let \( j : S \to \Delta \) be a fibration of a smooth surface \( S \). Let \( h : Z \to \Delta \) be a fibration modeled on \( j : S \to \Delta \) in the sense of Definition 4.1 such that \( \omega_Z \) is trivial. Suppose \( \chi : Z \to Z \) is a bimeromorphic map on \( Z \) inducing an isomorphism \( h^{-1}(\Delta \setminus \{0\}) \cong h^{-1}(\Delta \setminus \{0\}) \) commuting with the projection \( h \) and the \( T \)-action. Then \( \chi \) extends to a biholomorphic map \( \tilde{\chi} : Z \to Z \).

**Proof.** Let \( B \subset Z \) (possibly empty) be the indeterminacy locus of the bimeromorphic map \( \chi : Z \to Z \). Let \( \pi : \tilde{Z} \to Z \) be a composition of blow-ups such that the induced map \( \tilde{\chi} : \tilde{Z} \to Z \) is holomorphic. As \( \chi \) commutes with \( T \)-action, it follows that each component of the exceptional divisor \( E = \bigcup_{i=1}^m E_i \) of \( \pi \) is \( T \)-equivariant under \( \pi \). Since \( T = \mathbb{C}^{d-1}, \dim Z = d + 1 \) and \( T \) acts on \( Z \) with discrete isotropy at every point, each \( E_i \) is the blow-up of \( T \cdot P_i \) for some point \( P_i \). On the other hand, as \( \omega_Z \cong \mathcal{O}_Z \) and \( Z \) is smooth, it follows from [Ko, Lemma 4.3] that the bimeromorphic map \( \chi \) is isomorphic in codimension 1. Thus \( \tilde{\chi}(E_i) \) is a subvariety of codimension \( \geq 2 \). Moreover it is \( T \)-stable as \( \tilde{\chi} \) is \( T \)-equivariant. Thus \( \tilde{\chi}(E_i) = T \cdot Q_i \) for some point \( Q_i \). As \( \chi \) is \( T \)-equivariant, this means that any fiber of \( \pi \) is contracted by \( \tilde{\chi} \). This implies that \( \chi \) itself is holomorphic. \( \square \)

Now let \( f : M \to B \) be a holomorphic Lagrangian fibration over a complex manifold \( B \). Let \( D \subset B \) be a component of the discriminant of \( f \) and \( b \in D \) be a general point. Choose an arc \( \alpha : \Delta \to B \) with \( \alpha(0) = b \) which intersects \( D \) transversally and denote by \( h : Z \to \Delta \) the pull-back of \( f \) by \( \alpha \). Then \( Z \) is a complex manifold and the multiplicity of the fiber \( h^{-1}(0) \) is the same as the multiplicity of \( f^{-1}(b) \). The \( T \)-action on \( M \) induces a \( T \)-action on \( Z \) preserving the fibers of \( h \).

**Proposition 5.2** Let \( h : Z \to \Delta \) be the above fibration with a central fiber of multiplicity \( n \geq 1 \). Assume that the characteristic cycle is unstable, i.e., belongs to one of the three cases considered in Section 4. Let

\[
\begin{array}{c}
Z_1 \\
h_1 \downarrow \\
\Delta
\end{array} \quad \xrightarrow{\pi_1} \quad \begin{array}{c}
Z \\
h \downarrow \\
\Delta_1
\end{array}
\]

be the normalization of the fiber product where \( \nu_1 \) is a cyclic covering of degree \( n \), as in Proposition 3.1. By Proposition 2.3 and Proposition 2.5, \( h_1 \) is a fibration modeled on a minimal elliptic fibration with an unstable fiber, in the sense of Definition 4.1. Let \( h_2 : Z_2 \to \Delta \) be the stable reduction of \( h_1 : Z_1 \to \Delta \) constructed in Section 4 with a dominant meromorphic map \( \pi_2 : Z_2 \to Z_1 \) of degree \( m \). Let \( \Gamma_1 \) be the cyclic group of order \( mn \) with a subgroup \( \Gamma_2 \subset \Gamma_1 \) of order \( m \) and the quotient group \( \Gamma_1 = \Gamma_1/\Gamma_2 \). Then there exists an action of \( \Gamma_1 \) on \( Z_2 \) commuting with the \( T \)-action and compatible with the fiberation \( h_2 \), such that the induced action of \( \Gamma_2 \) on \( Z_2 \) agrees with the Galois action of the cyclic group of order \( m \) on the stable reduction. Via \( \pi_2 \), the action of \( \Gamma_1 \) on \( Z_2 \) induces an action of \( \Gamma_1 \) on \( Z_1 \), which agrees with the unramified Galois action of the cyclic group of order \( n \) on \( Z_1 \). In particular, the induced \( \Gamma_1 \)-action on \( \omega_{Z_1} \) is trivial.
Proof. We have a cyclic covering of degree \( m \), \( \nu_2 : \Delta \to \Delta \) and the commuting diagram

\[
\begin{array}{c}
Z_2 \\
\downarrow h_2
\end{array} \quad \begin{array}{c}
\cdots \quad \pi_2 \\
\downarrow h_1
\end{array} \quad \begin{array}{c}
Z_1 \\
\downarrow \pi_1
\end{array} \quad \begin{array}{c}
Z \\
\downarrow h
\end{array}
\end{array}
\]

Let \( \hat{h} : \hat{Z} \to \Delta \) be the normalization of the base change of \( h : Z \to \Delta \) by the cyclic covering \( \hat{\nu} : \hat{Z} \to Z \) of degree \( mn \). By the construction of the stable reduction, there exists a bimeromorphic map \( \varphi : \hat{Z} \cdots \to Z_2 \) commuting with the fibrations \( \hat{h} \) and \( h_2 \), inducing a biholomorphism outside the central fibers. We have the Galois action of the cyclic group \( \Gamma_{12} \) on \( \hat{Z} \) respecting the fibration \( \hat{h} \). Thus we have a bimeromorphic action of \( \Gamma_{12} \) on \( Z_2 \) respecting \( h_2 \) which induces a biholomorphic action on \( Z_2 \setminus h_2^{-1}(0) \). By Proposition 5.1, this extends to a biholomorphic action of \( \Gamma_{12} \) on \( Z_2 \).

By the construction, a cyclic subgroup \( \Gamma_2 \subset \Gamma_{12} \) acts on \( Z_2 \) as the Galois action for the cyclic covering of degree \( m \) in the construction of the stable reduction of \( h : Z \to \Delta \). The action of the quotient group \( \Gamma_1 \) on \( Z_1 \) must be the Galois action on \( Z_1 \) induced by \( \nu_1 \), which preserves \( \omega_{Z_1} \) from Proposition 3.1. \( \Box \)

Now we determine the possible values of \( n \). We will treat the three cases of Section 5 separately.

**Proposition 5.3** In the setting of Proposition 5.2, suppose that the characteristic cycle is of type \( \Gamma_b \), \( b \geq 1 \). Then the multiplicity \( n \) must be 1.

Proof. By Proposition 5.2, we have an action of the cyclic group \( \Gamma_{12} \) generated by the root of unity \( \zeta_{mn} \) on \( Z_2 \) with \( m = 2 \). We claim that \( \zeta_{mn} \) acts on the dualizing sheaf of the fiber \( \omega_{F_2} \) by \(-1\). By the construction of the stable reduction, the generator \( \zeta_m = \zeta_{mn}^a \) of the Galois group \( \Gamma_2 \) of the covering \( \nu_2 \) acts on the dualizing sheaf \( \omega_{Z_2} \) by \(-1\). Thus \( \operatorname{Aut}_T(F_2)/\operatorname{Aut}_T^\nu(F_2) \) is non-trivial and by Proposition 3.2,

\[
\operatorname{Aut}_T(F_2)/\operatorname{Aut}_T^\nu(F_2) \cong \mathbb{Z}/2.
\]

Let \( \bar{\zeta}_{mn} \in \operatorname{Aut}_T(F_2)/\operatorname{Aut}_T^\nu(F_2) \) be the image of \( \zeta_{mn} \in \operatorname{Aut}_T(F_2) \) in the quotient group. Since \( \bar{\zeta}_{mn} = -1 \), \( \bar{\zeta}_{mn} \) must be \(-1\), too. This means that it acts on \( \omega_{F_2} \) by \(-1\). Since its weight on \( \omega_{Z_1} \) is \( \zeta_{mn} \), it acts on \( \omega_{Z_2} \) by \(-\zeta_{mn} \). By Proposition 4.3, the induced action of \( \zeta_{mn} \) on \( \omega_{Z_1} \) is also by \(-\zeta_{mn} \). But by Proposition 5.2, the induced action on \( \omega_{Z_1} \) must be trivial. Hence \( -\zeta_{mn} = 1 \), which implies that \( n = 1 \). \( \Box \)

**Proposition 5.4** In the setting of Proposition 5.2, suppose that the characteristic cycle is of type \( \Gamma_0^*, \Gamma_1^*, \Gamma_2^* \) or \( \Gamma_4^* \). Then the possible values of \( (m, n) \) are \((2, 2), (2, 3), (3, 2)\).

Proof. By Proposition 5.2, we have an action of the cyclic group \( \Gamma_{12} \) generated by the root of unity \( \zeta_{mn} \) on \( Z_2 \). The generator \( \zeta_{mn} \) acts on the dualizing sheaf of the fiber \( \omega_{F_2} \) by some weight \( \zeta_{mn}^a \) where \( 0 \leq a < mn \) is some integer. By the construction of the stable reduction, the generator \( \zeta_m = \zeta_{mn}^a \) of the Galois group \( \Gamma_2 \) of the covering \( \nu_2 \) acts on the dualizing sheaf \( \omega_{Z_2} \) by \( \zeta_{mn}^{a-1} \). It follows that \( a \equiv m - 1 \mod m \).

Since \( \zeta_{mn} \) acts on \( \omega_{Z_2} \) by \( \zeta_{mn} \), the action on \( \omega_{Z_2} \) is by \( \zeta_{mn}^{a+1} \). By Proposition 4.3, the induced action on \( \omega_{Z_1} \) is also by \( \zeta_{mn}^{a+1} \). But by Proposition 5.2, the induced action on \( \omega_{Z_1} \) must be trivial. It follows that \( a + 1 \equiv 0 \mod mn \) and \( a = mn - 1 \). Thus the action on \( \omega_{F_2} \) is by \( \zeta_{mn}^{mn-1} \). By Proposition 3.2, \( \zeta_{mn}^{mn-1} = \zeta_{mn}^{-1} \) must have order \( 2, 3, 4, \) or \( 6 \). The possible values of \( (m, n) \) with \( m, n \geq 2 \) are \((2, 2), (2, 3), (3, 2)\). \( \Box \)

**Proposition 5.5** In the setting of Proposition 5.2, suppose that the characteristic cycle is of type \( \Gamma_2^*, \Gamma_3^*, \Gamma_4^* \) or \( \Gamma_5^* \). Then the possible values of \( (m, n) \) are \((3, 2), (3, 4), (4, 3) \) and \((6, 5)\).
Proof. By Proposition 5.2, we have an action of the cyclic group \( \Gamma_{12} \) generated by the root of unity \( \zeta_{mn} \) on \( \mathbb{Z}_2 \). The generator \( \zeta_{mn} \) acts on the dualizing sheaf of the fiber \( \omega_{F_2} \) by some weight \( \zeta_{mn}^a \) where \( 0 \leq a < mn \) is an integer. By the construction of the stable reduction, the generator \( \zeta_m = \zeta_{mn}^n \) of the Galois group of the covering \( \nu_2 \) acts on the dualizing sheaf \( \omega_{Z_2} \) by \( \zeta_m \). It follows that \( a \equiv 1 \mod m \).

Since \( \zeta_{mn} \) acts on \( \omega_F \) by \( \zeta_{mn}^a \), the action on \( \omega_{Z_2} \) is by \( \zeta_{mn}^{a+1} \). By the construction of the stable reduction, the generator \( \zeta_m = \zeta_{mn}^n \) of the Galois group of the covering \( \nu_2 \) acts on the dualizing sheaf \( \omega_{Z_2} \) by \( \zeta_m \). It follows that \( a + m - 1 \equiv 0 \mod mn \). From \( a \equiv 1 \mod m \), let \( a = mq + 1 \). Then

\[
a = m(nq' + 1) + 1 \equiv 1 - m \mod mn.
\]

It follows that the action on \( \omega_{F_2} \) is by \( \zeta_{mn}^{1-m} \). By Proposition 3.2, \( \zeta_{mn}^{1-m} \) must have order 2, 3, 4, or 6. The possible values of \((m, n)\) with \( m = 3, 4, 6 \) and \( n \geq 2 \) are \((3, 2), (3, 4), (4, 3)\) or \((6, 5)\). \( \square \)

Proposition 5.3, Proposition 5.4, Proposition 5.5 and Proposition 3.5 complete the proof of Theorem 1.1, modulo the realization part, which will be given in the next section.

6 Explicit examples of multiple fibers

In this section, we show the last statement of Theorem 1.1 by giving explicit examples of multiple fibers described in (i) - (v) there. It suffices to construct them in dimension 4.

We shall first give an explicit example of (ii) in Theorem 1.1 with full details. In fact, this is one of the cases missing in [Ma] and the discovery of this case is one of the starting points of our Theorem 1.1.

Example 6.1 (Type II with multiplicity 5)

Let \( \phi : S \rightarrow \Delta_t \) be a relatively minimal elliptic surface over the unit disk \( \Delta_t \) given by the Weierstrass equation

\[
y^2 = x^3 + t.
\]

There is a very convenient algorithm, called the Tate’s algorithm, to determine the type of singular fiber from the Weierstrass equation. This algorithm is given by [Ta, Pages 34-35, Summary 0 with additional definition in Page 36 (3.6)]. Applying to our equation, we readily see that \( \phi \) has a singular fiber of Type II over \( t = 0 \). (This algorithm can be also used to determine the singular fiber in Example 6.2 below.) We also note that the 2-form

\[
\sigma_S := \frac{dx \wedge dt}{y}
\]

gives the generator of \( \omega_S \simeq \mathcal{O}_S \). Note also that \( S \) admits an automorphism given by

\[
\tau^* (x, y, t) = (\zeta_5^2 x, \zeta_5^3 y, \zeta_5 t).
\]

Now consider the product 4-fold

\[
M := S \times E \times \Delta_s
\]

where \( E \) is an elliptic curve and \( \Delta_s \) is a unit disk. \( M \) has a fibration

\[
f : M \rightarrow \Delta_t \times \Delta_s; ((x, y, t), (z, s)) \mapsto (t, s).
\]
The 2-form

$$\sigma_M := \sigma_S + dz \wedge ds$$

is a symplectic form on $M$ and makes $f$ Lagrangian. We define the automorphism $g$ of $M$ by

$$g^* ((x, y, t), z, s) = ((\zeta_5^2 x, \zeta_5^3 y, \zeta_5 t), z + p, s)$$

where $p$ is a 5-torsion point on $E$. Then, $\langle g \rangle \cong \mathbb{Z}/5$ and $\langle g \rangle$ acts on $f : M \to \Delta_t \times \Delta_s$ freely. Moreover $g^* \sigma_M = \sigma_M$ by the explicit form of $g$. The quotient manifold (where $u = t^5$)

$$M/\langle g \rangle \to \Delta_u \times \Delta_s$$

is then a Lagrangian fibration with multiple fibers of multiplicity 5 along $u = 0$ whose characteristic cycles are of Type $II$. Note that the fiber $F = f^{-1}(0, s) \simeq f^{-1}(0) \times E$ is stable under $g$ and satisfies

$$g^* \frac{dx}{y} \wedge dz = \zeta_5^{-1} \frac{dx}{y} \wedge dz,$$

for the generator of $\omega_F$. We also note that the stable reduction of $S$ is $y^2 = x^3 + 1$. Geometrically, the stable reduction is the second projection from the product $E_{\zeta_3} \times \Delta$ of the elliptic curve $E_{\zeta_3}$ and the unit disk.

**Example 6.2 (Examples of (i), (iii), (iv), (v) when characteristic cycle is not of Type $I_{2m}$ ($1 \leq m \leq \infty$))**

The construction for other cases (i), (iii), (iv), (v) in Theorem 1.1 are quite similar if the characteristic cycle is not of Type $I_m$. In fact, in the quotient $(S \times E \times \Delta_s)/\langle g \rangle$ in the example above, we just replace the pair $(S, \tau, g)$ as follows (with the same $E$ and $\Delta_s$ and the same expression for $\sigma_S$ and $\sigma_M$), according to the cases. Here $S$ is given by the Weierstrass equation and $p_n$ is an $n$-torsion point of the elliptic curve $E$:

(i) ($n = 6$ and $E_{\zeta_3}$):

$$S : y^2 = x^3 + 1, \quad \tau^* (x, y, t) = (\zeta_6^2 x, \zeta_6^3 y, \zeta_6 t)$$

$$g^* ((x, y, t), z, s) = ((\zeta_6^2 x, \zeta_6^3 y, \zeta_6 t), z + p_6, s).$$

(iii) ($n = 4$ and $E_{\zeta_4}$):

$$S : y^2 = x^3 + x, \quad \tau^* (x, y, t) = (-x, \zeta_4^3 y, \zeta_4 t)$$

$$g^* ((x, y, t), z, s) = ((-x, \zeta_4^3 y, \zeta_4 t), z + p_4, s).$$

(iii) ($n = 4$ and Type $IV$):

$$S : y^2 = x^3 + t^2, \quad \tau^* (x, y, t) = (-x, \zeta_4^3 y, \zeta_4 t)$$

$$g^* ((x, y, t), z, s) = ((-x, \zeta_4^3 y, \zeta_4 t), z + p_4, s).$$

(iv) ($n = 3$ and $E_{\zeta_3}$):

$$S : y^2 = x^3 + 1, \quad \tau^* (x, y, t) = (\zeta_3^2 x, y, \zeta_3 t)$$

$$g^* ((x, y, t), z, s) = ((\zeta_3^2 x, y, \zeta_3 t), z + p_3, s).$$
(iv) \((n = 3 \text{ and Type III})\):
\[
S : y^2 = x^3 + tx, \quad \tau^*(x, y, t) = (\zeta_3^2 x, y, \zeta_3 t)
\]
\[
g^*(x, y, t, z, s) = ((\zeta_3^2 x, y, \zeta_3 t), z + p_3, s).
\]

(iv) \((n = 3 \text{ and Type } I_{0}^0)\):
\[
S : y^2 = x^3 + t^3, \quad \tau^*(x, y, t) = (\zeta_3^2 x, y, \zeta_3 t)
\]
\[
g^*(x, y, t, z, s) = ((\zeta_3^2 x, y, \zeta_3 t), z + p_3, s).
\]

(v) \((n = 2 \text{ and Type } I_0)\):
\[
S : y^2 = x^3 + ax + b, \quad \tau^*(x, y, t) = (x, -y, -t)
\]
\[
g^*(x, y, t, z, s) = ((x, -y, -t), z + p_2, s).
\]

(v) \((n = 2 \text{ and Type } I_{0}^*)\):
\[
S : y^2 = x^3 + t^2 x, \quad \tau^*(x, y, t) = (x, -y, -t)
\]
\[
g^*(x, y, t, z, s) = ((x, -y, -t), z + p_2, s).
\]

(v) \((n = 2 \text{ and Type } IV)\):
\[
S : y^2 = x^3 + t^2, \quad \tau^*(x, y, t) = (x, -y, -t)
\]
\[
g^*(x, y, t, z, s) = ((x, -y, -t), z + p_2, s).
\]

(v) \((n = 2 \text{ and Type } IV^*)\):
\[
S : y^2 = x^3 + t^4, \quad \tau^*(x, y, t) = (x, -y, -t)
\]
\[
g^*(x, y, t, z, s) = ((x, -y, -t), z + p_2, s).
\]

Example 6.3 (Examples of \(n = 2 \text{ and Type } I_{2m} \quad (1 \leq m \leq \infty)\))

We will use the example in [HO, Proposition 4.13]. Let us recall the setting. Let \(R_k = \text{Specan } \mathbb{C}[[u^{k+1}v^{-1}, u^{-k}v]] \quad (k \in \mathbb{Z})\). There is a natural morphism \(g_k : R_k \longrightarrow \text{Specan } \mathbb{C}[u]\). Let \(E\) be an elliptic curve. Using the morphisms \(g_k\), which are compatible with the natural gluing of the spaces \(R_k\), we obtain a morphism
\[
(\bigcup_{k \in \mathbb{Z}} R_k) \times E \times \text{Specan } \mathbb{C}[y] \longrightarrow \text{Specan } \mathbb{C}[u, y].
\]
Restricting this morphism over a sufficiently small 2-dimensional disk \(\Delta^2\) (centered at \((u, y) = (0, 0)\)), we obtain a fibration
\[
\tilde{f} : \tilde{M} \longrightarrow \Delta^2_{(u, y)}.
\]
The fiber over \(u = 0\) is an infinite chain of \(\mathbb{P}^1 \times E\), while the fiber over \(u \neq 0\) is \(\mathbb{C}^* \times E\). Let \(\alpha\) be a point of \(E\). Then \(\mathbb{Z}\) acts on \(\tilde{f} : \tilde{M} \longrightarrow \Delta^2\) if we define the action of \(m \in \mathbb{Z}\) by
\[
(u^{k+1}v^{-1}, u^{-k}v, x, y) \mapsto (u^{k+1+m}v^{-1}, u^{-k-m}v, x + m\alpha, y).
\]
As explained in [HO], this action is properly discontinuous and free. Let \( p \) be a 2-torsion point of \( E \). Define the automorphism of \( \tilde{M} \) by
\[
g^*(u, v, x, y) = (-u, v^{-1}, x + p, y).
\]
g does not commute with the \( \mathbb{Z} \)-action, but it commutes with the action of the index two subgroup \( 2\mathbb{Z} \). This is why our example below of fibers of multiplicity 2 can be constructed for Type \( I_b \) with even \( b \) even, but not with odd \( b \).

First choose \( \alpha \) to be a nontorsion point of \( E \) and set \( M_2 = \tilde{M}/2\mathbb{Z} \). The symplectic 2-form
\[
du \wedge \frac{dv}{v} + dx \wedge dy
\]
on \( \tilde{M} \) descends to a symplectic 2-form on \( M_2 \). We regard \( M_2 \) as a symplectic manifold with this symplectic from. Then, \( g \) descends to the free symplectic involution of the Lagrangian fibration
\[
f : M_2 \longrightarrow \Delta^2.
\]
Thus, the quotient fibration
\[
\overline{f} : M_2/\langle g \rangle \longrightarrow \Delta^2
\]
gives an example of a Lagrangian fibration such that a general singular fiber is of multiplicity 2 with characteristic cycle of Type \( A_{\infty} = I_{\infty} \).

In the above construction, if we choose \( \alpha \in E \) to be a torsion element of order \( 1 \leq 2\ell < \infty \) and choose the 2-torsion point \( p \) with \( p \notin \langle \alpha \rangle \), we obtain an example of a Lagrangian fibration such that a general singular fiber is of multiplicity 2 with characteristic cycle of Type \( I_{2\ell} \) (\( \ell \geq 1 \)).

(6.1)-(6.3) complete the proof of the realizability.

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