AGGREGATION AND DISAGGREGATION OF ACTIVE PARTICLES ON THE UNIT SPHERE WITH TIME-DEPENDENT FREQUENCIES

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Abstract. We introduce an active swarming model on the sphere which contains additional temporal dynamics for the natural frequency, inspired from the recently introduced modified Kuramoto model, where the natural frequency has its own dynamics. For the attractive interacting particle system, we provide a sufficient framework that leads to the asymptotic aggregation, i.e., all the particles are aggregated to the single point and the natural frequencies also tend to a common value. On the other hand, for the repulsive interacting particle system, we present a sufficient condition for the disaggregation, i.e., the order parameter of the system decays to 0, which implies that the particles are uniformly distributed over the sphere asymptotically. Finally, we also provide several numerical simulation results that support the theoretical results of the paper.

1. Introduction. The cooperative motion of self-propelled particles has received considerable attention from several scientific disciplines: swarming behavior in quantitative biology [3, 15, 16, 24, 42], control theory in engineering [28, 40] and even for nonconvex optimization in machine learning [6, 7, 25]. It is worthwhile mentioning the pioneering work of Winfree [44] and Kuramoto [31] in which the authors initiated a mathematically rigorous study of collective behavior for a large population. Since then, the study on collective dynamics has been extensively and intensively investigated in the field of applied mathematics, and so far, several mathematical models have been proposed to describe a collective motion, for instance, the Cucker-Smale model [14], the Kuramoto model [31], the Vicsek model [43], etc. On the other hand, there has been substantial literature concerned with the
emergent dynamics on Riemannian manifolds, to name a few, on the unit sphere [38], hyperboloid [41] and matrix Lie groups [17, 32]. Among them, we are here interested in a first-order aggregation model on the unit sphere which has been widely studied in the previous researches.

For $i = 1, 2, \ldots, N$, let $x_i \in \mathbb{R}^d$ be the position of $i$-th particle (or agents). Then, the following swarming model is mainly used to describe the dynamics of position $x_i$ on the unit sphere [38, 39]:

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_k, x_i \rangle x_i), \quad x_i \in \mathbb{R}^d, \quad t > 0, \quad (1)$$

subject to the initial data $x_i(0) = x_i^0 \in \mathbb{S}^{d-1}$. Here, a $d \times d$ skew-symmetric matrix $\Omega_i \in \text{Skew}_d(\mathbb{R})$ is called a natural frequency of the $i$-th particle and $\mathbb{S}^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \}$ is the unit sphere in $\mathbb{R}^d$. The natural frequency for each agent is employed so that under the absence of couplings between agents, all agents rotate the unit sphere with their own frequencies. The constant $\kappa$ denotes the coupling strength. If $\kappa$ is positive, the particle attracts the other particles, while it repulses when $\kappa$ is negative. Finally, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the standard inner product and $\ell^2$-norm in $\mathbb{R}^d$, respectively.

We briefly discuss the swarming dynamics (1). First of all, thanks to the structure of the dynamics, it is easy to observe that the particles stay on the sphere for all time, i.e., $x_i(t) \in \mathbb{S}^{d-1}$ for $t > 0$ (see [11, 13] or the proof of Lemma 2.1 in Section 2). Also, by introducing the mean position $x_c := \frac{1}{N} \sum_{i=1}^{N} x_i$, the dynamics (1) can be rewritten as

$$\dot{x}_i = \Omega_i x_i + \kappa(x_c - \langle x_c, x_i \rangle x_i) = : \Omega_i x_i + \kappa P_{x_c}(x_c - x_i),$$

where the operator $P_{x_c}(y) = y \mapsto y - \langle y, x \rangle x$ is a projection of the vector $y$ onto the tangent space of $\mathbb{S}^{d-1}$ at $x \in \mathbb{S}^{d-1}$. Therefore, the dynamics (1) can be understood as a system that each agent moves toward the projected average position of its neighbors on the $d$-dimensional unit sphere, leading to the aggregation of the particles on the sphere.

We review the swarming model (1) which has been extensively studied in diverse scientific disciplines: applied mathematics [5, 11, 13, 22, 26, 27, 29], applied physics [8, 9, 32, 33, 34, 45, 46] and engineering [35, 36, 37, 38]. Since most of the work considered the homogeneous case $\Omega_i \equiv \Omega$, we restrict ourselves to the homogeneous case at this moment. In [38], equilibria are classified into three classes: a global aggregation, a local aggregation, and a splay state. In addition, a linear stability analysis is performed so that the last two equilibria are unstable under an attractive regime. After then, the authors in [27] presented a more delicate estimate to provide a dichotomy for generic initial data. Precisely, if the initial mean position is positive, then only two possible cases arise: a globally aggregated state and a locally aggregated state with only one antipodal point. On the other hand, global aggregation using gradient flow structure has been studied in different contexts including [27]. For instance, the author in [45] constructed a potential function defined as the total distance and applied LaSalle’s invariance principle to verify that all points converge to the same point. In [35], the author used Łojasiewicz’s inequality and performed stability argument to show that all equilibria except a globally aggregated state are unstable. We also mention the heterogeneous case in which natural frequencies are different in general. The authors in [8] numerically observed the phase transition which occurs non-hysteretically in an odd dimension $d \geq 3$. Recently in [37], the
heterogeneous case is partially considered to show that any dispersed equilibrium is unstable if the Frobenius norms of natural frequencies are sufficiently small. It should be noted that the unit sphere as an underlying manifold is generalized to the compact Riemannian manifold \([1]\) and even noncompact one \([41]\). On the other hand, the kinetic description for the swarming model \((1)\) is also studied in previous works. Asymptotic behavior of a measure-valued solution to the kinetic swarming equation on the sphere was investigated in \([22]\), while the uniform-in-time mean-field limit and the stability of the solution is studied in \([26]\). Other than \((1)\), the kinetic descriptions for the other swarming models have been also investigated in numerous literature. For instance, the existence of the solutions \([18, 23]\) or the continuum limit \([19]\) to the kinetic Vicsek-type models were studied, while the hyperbolic moment system for two-dimensional kinetic Vicsek model was derived in \([20]\) by using projection operator model reduction method \([21]\). For the details, we refer the reader to the articles above and references therein.

Recently, the extended Kuramoto-type model that takes into account the dynamics of the natural frequencies of the particle was introduced in \([10]\). In this “active particle” model, the particles interact with each other so that not only the phase but also the natural frequency of each particle is dynamically affected by neighboring particles. We extend this dynamical natural frequency model to the swarming model on the sphere, which can be understood as the high-dimensional Kuramoto model. Thus, in this paper, we consider the time-dependent natural frequency whose temporal evolution is adjusted by an adaptive law between its interaction. To this end, we introduce the active swarming model on sphere whose positions and natural frequencies of particles are described by the following equation:

\[
\begin{align*}
\dot{x}_i &= \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle) x_i, \quad t > 0, \\
\dot{\Omega}_i &= -\gamma \Omega_i + \frac{\mu}{N} \sum_{k=1}^{N} \Gamma(\Omega_i, \Omega_k), \quad i = 1, \ldots, N,
\end{align*}
\]

(2)

subject to the initial data on \(\mathbb{S}^{d-1} \times \text{Skew}_d(\mathbb{R})\):

\[
(x_i, \Omega_i)(0) = (x_i^0, \Omega_i^0) \in \mathbb{S}^{d-1} \times \text{Skew}_d(\mathbb{R}), \quad i = 1, \ldots, N.
\]

(3)

Here, \(\kappa \in \mathbb{R}\) and \(\mu > 0\) denote the coupling coefficient between position and the learning rate respectively, and \(\gamma > 0\) implies the damping coefficient of the natural frequency. The reason \(\mu\) is called the learning rate is because it represents the degree of impact of \(\Omega_j\) to the update (= learning) of \(\Omega_i\). The reason \(\gamma\) is called the damping rate is because the term \(-\gamma \Omega_i\) drives \(\Omega_i\) toward zero matrix \(O\), which implies the damping effect on it. In particular, when \(\kappa > 0\), the position of the particles tend to aggregate, while they repulse each other for the case of \(\kappa < 0\). Finally, the general function \(\Gamma : \text{Skew}_d(\mathbb{R}) \times \text{Skew}_d(\mathbb{R}) \to \text{Skew}_d(\mathbb{R})\) is called as an adaptive communication law on the natural frequencies between the particles.

The most preceding question on the active particle model \((2)\) is again that whether the particle stays on the unit sphere for \(t > 0\). In order to guarantee the positive invariance of the underlying manifold, it should be verified that \(\Omega_i(t)\) becomes a skew-symmetric matrix for \(t > 0\). Thanks to the skew-symmetry of the adaptive law \(\Gamma\), one can easily check the positive invariance of \(\mathbb{S}^{d-1}\) and \(\text{Skew}_d(\mathbb{R})\) for \(x_i\) and \(\Omega_i\), respectively (see Lemma 2.1). By concerning the global-in-time existence of a solution to \((2)\), it would mainly depend on the choice of the adaptive law, and
the solution would not be uniformly bounded in time for a specific choice of \( \Gamma \) (see Remark 1).

The purpose of this article is to study the time-asymptotic behavior of (2) under several types of adaptive laws and investigate how the emergent dynamics can be affected by the temporal evolution of the natural frequency. Throughout the paper, we employ the following three types of the adaptive law \( \Gamma \):

• (C1) (A Lipschitz adaptive law): There exists a positive constant \( \Gamma_{\text{Lip}} \) such that
  \[
  \| \Gamma(P_1, Q) - \Gamma(P_2, Q) \|_F \leq \Gamma_{\text{Lip}} \| P_1 - P_2 \|_F.
  \]
  Moreover, there exists a modulus function \( \omega_0 : \mathbb{R}_+ \to \mathbb{R}_+ \), i.e., increasing function with \( \omega_0(0) = 0 \), such that
  \[
  \| \Gamma(P, Q) \|_F \leq \omega_0(\| P - Q \|_F).
  \]
  For instance, the simplest example of this type for the adaptive law \( \Gamma \) would be a linear difference \( \Gamma(P, Q) = Q - P \).

• (C2) (A commutator adaptive law): In this case, \( \Gamma \) is given by the (standard) commutator
  \[
  \Gamma(P, Q) = [P, Q].
  \]
  We note that \( \text{Skew}_d(\mathbb{R}) \) is a Lie algebra of the orthogonal group \( \mathbb{O}(d) \) on which, the commutator operator is natural since it can be identified with the Lie bracket.

• (C3) (A perturbation of a constant matrix): For a given scalar-valued function \( f : \text{Skew}_d(\mathbb{R}) \times \text{Skew}_d(\mathbb{R}) \to \mathbb{R} \) and a constant matrix \( \Psi \in \text{Skew}_d(\mathbb{R}) \), we construct \( \Gamma \) as
  \[
  \Gamma(P, Q) = f(P, Q)\Psi,
  \]
  where \( f(P, Q) \) satisfies the following growth condition: for any \( P_1, P_2 \) and \( Q \) in \( \text{Skew}_d(\mathbb{R}) \) with \( \| P_1 \|_F, \| P_2 \|_F, \| Q \|_F < R \), there exists a modulus function \( \omega_1 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that
  \[
  |f(P_1, Q) - f(P_2, Q)| \leq \omega_1(R) \| P_1 - P_2 \|_F.
  \]
  Moreover, there exists another modulus function \( \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that
  \[
  f(P, Q) \leq f(O, O) + \omega_2(\| P - Q \|_F).
  \]

The main results of this work are two-fold: emergence of the complete aggregation for the attractive coupling and the complete disaggregation for the repulsive one. Intuitively, aggregation means that all particles are concentrated at a single point on the sphere, whereas the disaggregation represents that the particles are uniformly distributed over the entire sphere. See Definition 2.2 for the rigorous definition of aggregation and disaggregation. First of all, regardless of the sign of the coupling strength \( \kappa \) and the choice of the type for the adaptive law \( \Gamma \), we show that the maximal difference between \( \Omega_i \) tends to zero exponentially for a large damping \( \gamma \):

\[
\max_{1 \leq i,j \leq N} \| \Omega_i(t) - \Omega_j(t) \|_F \lesssim e^{-\Omega(1)t}, \quad t > 0.
\]

See Proposition 1 for the precise statement. For the attractive regime, i.e., \( \kappa > 0 \), we provide sufficient conditions leading to the complete aggregation in terms of the system parameters and the initial data. Precisely, under the well-prepared initial conditions:

\[
\max_{1 \leq i,j \leq N} |x_i^0 - x_j^0| \ll 1 \quad \text{and} \quad \text{either } \gamma \gg 1 \text{ or } \kappa \gg 1,
\]
the complete aggregation occurs (see Theorem 3.1 for details). On the other hand for the repulsive regime, our goal is to find a sufficient condition which results in the emergence of the complete disaggregation. In this case, if we assume \( \gamma \gg 1 \), then we achieve
\[
\lim_{t \to \infty} \left| \frac{1}{N} \sum_{i=1}^{N} x_i(t) \right| = 0.
\]
We refer the reader to Theorem 4.3 for details.

The rest of this paper is organized as follows. In Section 2, we present several basic estimates on the active particle model (2) and preparatory mathematical lemmas that will be used in the later sections. Section 3 provides the sufficient condition for the complete aggregation when \( \kappa > 0 \) for each case of adaptive laws. In Section 4, we present a sufficient condition for the complete disaggregation for the case of \( \kappa < 0 \). Section 5 illustrates several numerical simulations for the active particle model for each case of parameters and adaptive laws. Finally, Section 6 is devoted to a brief summary of the paper and discussion on future work.

**Notation:** Throughout the paper, we use the notation \( X = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} \) and \( \Omega = (\Omega_1, \ldots, \Omega_N) \in \text{Skew}_d(\mathbb{R})^N \). For a vector \( u = (u^1, \ldots, u^d) \in \mathbb{R}^d \) and \( A = (a_{ij}) \in \mathbb{R}^{d \times d} \), we define the \( \ell^2 \)-norm of \( u \) and the Frobenius norm of \( A \) as
\[
|u| := \left( \sum_{i=1}^{d} (u^i)^2 \right)^{\frac{1}{2}}, \quad \|A\|_F := \left( \sum_{i,j=1}^{d} |a_{ij}|^2 \right)^{\frac{1}{2}} = \left( \text{tr}(A^T A) \right)^{\frac{1}{2}}.
\]

2. **Preliminaries.** In this section, we discuss basic properties of (2) including positive invariance of the governing manifolds. Several *a priori* estimates and elementary lemmas are also presented in this section.

2.1. **Basic properties.** In this subsection, basic properties of (2) are investigated. First of all, we show that \( \mathbb{S}^{d-1} \times \text{Skew}_d(\mathbb{R}) \) is positively invariant under the system (2).

**Lemma 2.1.** Let \( (X, \Omega) \) be a solution to (2)–(3). Suppose the initial data satisfy the following condition:
\[
x_i^0 \in \mathbb{S}^{d-1} \quad \text{and} \quad \Omega_i^0 \in \text{Skew}_d(\mathbb{R}), \quad i = 1, \ldots, N.
\]
Then, one has
\[
x_i(t) \in \mathbb{S}^{d-1} \quad \text{and} \quad \Omega_i(t) \in \text{Skew}_d(\mathbb{R}), \quad t > 0, \quad i = 1, \ldots, N.
\]
Therefore, \( \mathbb{S}^{d-1} \times \text{Skew}_d(\mathbb{R}) \) is an invariant manifold of the dynamics (2).

**Proof.** For the skew-symmetry of \( \Omega_i \), it directly follows from (2)_2 and skew-symmetry of \( \Gamma \) that
\[
\frac{d}{dt} (\Omega_i^\top + \Omega_i) = -\gamma (\Omega_i^\top + \Omega_i) + \frac{\kappa}{N} \sum_{k=1}^{N} \left( \Gamma(\Omega_i, \Omega_k)^\top + \Gamma(\Omega_i, \Omega_k) \right) = -\gamma (\Omega_i^\top + \Omega_i),
\]
which yields
\[
\Omega_i(t)^\top + \Omega_i(t) = ((\Omega_i^0)^\top + \Omega_i^0)e^{-\gamma t} = 0, \quad t > 0.
\]
Hence, $\Omega_i(t) \in \text{Skew}_d(\mathbb{R})$ for all $i = 1, 2, \ldots, N$ and $t > 0$. Next, for the unit modulus property of $x_i$, we use (2) to obtain

$$\frac{d}{dt}|x_i|^2 = 2\langle x_i, \dot{x}_i \rangle = 2(x_i, \Omega_i x_i) + \frac{2\kappa}{N} \sum_{k=1}^{N} (\langle x_i, x_k \rangle - \langle x_i, x_k \rangle) = 2\langle x_i, \Omega_i x_i \rangle = 0,$$

where the last equality comes from the skew-symmetric property of $\Omega_i$. Therefore, we conclude that $\frac{d}{dt}|x_i|^2 = 0$ and this gives the desired invariance property.

As mentioned in the introduction, since $\Omega_i$ is not a constant but a time-dependent function, uniform boundedness of $\Omega_i$ would not be guaranteed in some situations. Below, we show that under a small damping regime with a specific choice of $\Gamma$, a solution to (2) would diverge in finite time.

**Remark 1.** We consider the two-oscillator system, i.e., the case when $N = 2$ and choose an adaptive law

$$\Gamma(P, Q) = \|P - Q\|_F^p P, \quad p > 0.$$

Then, one observes that

$$\frac{d}{dt}(\Omega_1 - \Omega_2) = -\gamma (\Omega_1 - \Omega_2) + \frac{\mu}{2} \|\Omega_1 - \Omega_2\|_F^p (\Omega_1 - \Omega_2),$$

which yields

$$\frac{d}{dt}\|\Omega_1 - \Omega_2\|_F = -\gamma \|\Omega_1 - \Omega_2\|_F + \frac{\mu}{2} \|\Omega_1 - \Omega_2\|_F^{p+1}. \quad (4)$$

Hence, if the initial data satisfy

$$\|\Omega_1^0 - \Omega_2^0\|_F > \frac{2\gamma}{\mu}, \quad \text{or equivalently,} \quad \gamma < \frac{\mu\|\Omega_1^0 - \Omega_2^0\|_F^p}{2},$$

then we always have

$$\|\Omega_1(t) - \Omega_2(t)\|_F \geq \|\Omega_1^0 - \Omega_2^0\|_F > \left(\frac{2\gamma}{\mu}\right)^{1/p}, \quad t > 0.$$ 

Moreover, by introducing a handy notation $D(t) = \|\Omega_1(t) - \Omega_2(t)\|_F$ and $D_0 := D(0)$, we have

$$-\gamma D(t) + \frac{\mu}{2} D(t)^{p+1} > \left(\frac{\mu}{4} - \frac{\gamma}{2} D_0^{-p}\right) D(t)^{p+1}, \quad t > 0.$$

Thus, it follows from (4) that

$$\frac{dD(t)}{dt} > \left(\frac{\mu}{4} - \frac{\gamma}{2} D_0^{-p}\right) D(t)^{p+1}, \quad \text{i.e.,} \quad D(t)^p > \frac{1}{D_0^{-p} - \left(\frac{\mu}{4} - \frac{\gamma}{2} D_0^{-p}\right) t}.$$ 

Therefore, if we define a positive critical time $T_* := \frac{D_0^{-p}}{\mu \left(\frac{\mu}{4} - \frac{\gamma}{2} D_0^{-p}\right)}$, then we have the finite time blow-up phenomenon:

$$\lim_{t \to T_*} \|\Omega_1(t) - \Omega_2(t)\|_F = +\infty.$$
2.2. A priori estimates. In this subsection, we provide a priori estimates on several physical quantities which play key roles for the emergent behavior estimates. To this end, we introduce the following notation for the inner product $\langle x_i, x_j \rangle$ between two particles and the average position $\frac{1}{N} \sum_{i=1}^{N} x_i$:

$$h_{ij} := \langle x_i, x_j \rangle, \quad x_c := \frac{1}{N} \sum_{i=1}^{N} x_i.$$ 

Finally, we denote the magnitude of the average position of the particles, i.e., $\rho := |x_c|$. The quantity $\rho$ is usually called an “order parameter”, denoting the degree of aggregation (synchronization) or disaggregation (asynchronization). Let us explain briefly why the order parameter can measure the degree of aggregation for the readers who are not familiar with it. Since the particles are on the unit sphere, it is straightforward to see that $0 \leq \rho \leq 1$. When $x_i$ are aggregated on a single point, then the average position $x_c$ should also be the same point, and therefore, we have $\rho = 1$. On the other hand, if the position is totally disaggregated on the unit sphere, i.e., if the positions of particles are uniformly distributed on the unit sphere, then the average position $x_c$ should be close to 0, implying that the order parameter $\rho$ is also close to 0. In this sense, the magnitude of the order parameter $\rho$ can represent the degree of aggregation. The argument above is summarized in the following definition.

**Definition 2.2.** Let $(X, \Omega)$ be a global solution to (2)–(3). 
(i) We say that the system (2) exhibits the complete aggregation if and only if the following relation holds:

$$\lim_{t \to \infty} \rho(t) = 1.$$ 

(ii) We say that the system (2) exhibits the complete disaggregation if and only if the following relation holds:

$$\lim_{t \to \infty} \rho(t) = 0.$$ 

In the following lemma, we present the estimate on $h_{ij}, \rho$ and $\Omega_i - \Omega_j$, which are the key estimates throughout the paper.

**Lemma 2.3.** Let $(X, \Omega)$ be a global solution to (2)–(3). Then, the following relations hold:

(i) $\frac{d}{dt} (1 - h_{ij}) = -2\kappa (1 - h_{ij}) \quad \frac{\kappa}{N} \sum_{k=1}^{N} (1 - h_{ik} + 1 - h_{kj}) (1 - h_{ij}) + \langle x_i, (\Omega_j - \Omega_i)x_j \rangle.$

(ii) $\frac{d\rho^2}{dt} = 2\kappa \left( \rho^2 - \frac{1}{N} \sum_{i=1}^{N} \langle x_i, x_c \rangle^2 \right) + \frac{2}{N} \sum_{i=1}^{N} \langle \Omega_i x_i, x_c \rangle.$

(iii) $\frac{d}{dt} (\Omega_i - \Omega_j) = -\gamma (\Omega_i - \Omega_j) + \frac{\mu}{N} \sum_{k=1}^{N} \left( \Gamma(\Omega_i, \Omega_k) - \Gamma(\Omega_j, \Omega_k) \right).$

**Proof.** (i) For the first assertion, it follows from the definition of $h_{ij}$ and skew-symmetry of $\Omega_i$ that

$$\frac{d}{dt} (1 - h_{ij})$$
Then, we have
\[
(iii) \text{ The last assertion is directly obtained by substituting the equation (2) for } \langle x_i, x_j \rangle.
\]

Moreover, the convergence rate is integrable.

\[
\begin{align*}
(i) & \quad \text{We first note that} \\
(ii) & \quad \text{Preparatory lemmas. In this subsection, we present elementary lemmas to be used later.}
\end{align*}
\]

\[\text{Lemma 2.4. Suppose that } f \in (L^1 \cap L^\infty)(0, \infty) \text{ is an integrable and bounded function, and } a \text{ is a positive constant. Let } y = y(t) \text{ be a nonnegative } C^1 \text{-function satisfying}
\]

\[\dot{y} \leq -ay + f, \quad t > 0, \quad y(0) = y_0.\]

Then, we have
\[\lim_{t \to \infty} y(t) = 0.\]

Moreover, the convergence rate is integrable.

\[\text{Proof. For the proof, we refer the reader to Lemma A.1 in [12].}\]
Next, we provide a (small) perturbation of the Riccati differential inequality which will be crucially used for our argument.

**Lemma 2.5.** Suppose that \( f \in (L^1 \cap L^\infty)(0, \infty) \) is an integrable and bounded function and \( a, b \) are positive constants. Let \( y = y(t) \) be a nonnegative \( C^1 \)-function satisfying
\[
\dot{y} \leq -ay + by^2 + f(t), \quad t > 0, \quad y(0) = y_0.
\]
(i) If the system parameters and initial datum satisfy
\[
\|f\|_{L^1} < \frac{a}{b}, \quad y_0 < \frac{a}{b} - \|f\|_{L^1},
\]
then we have
\[
\lim_{t \to \infty} y(t) = 0.
\]
(ii) Instead, if the system parameters and initial datum satisfy
\[
\|f\|_{\infty} < \frac{a^2}{4b}, \quad y_0 < \frac{a + \sqrt{a^2 - 4b\|f\|_{\infty}}}{2b},
\]
then we also have
\[
\lim_{t \to \infty} y(t) = 0.
\]
(iii) In particular, suppose \( f \) exponentially decays to zero, that is, there exist constants \( C_0 > 0 \) and \( \lambda > 0 \) such that \( f(t) \leq C_0 e^{-\lambda t} \) for \( t > 0 \). Suppose that one of the following conditions holds:
\[
(1) \quad \frac{C_0}{\lambda} < \frac{a}{b}, \quad y_0 < \frac{a}{b} - \frac{C_0}{\lambda},
\]
\[
(2) \quad a^2 - 4bC_0 > 0, \quad y_0 < \frac{a + \sqrt{a^2 - 4bC_0}}{2b}.
\]
Then, there exist positive constants \( C, \eta \) and \( \tilde{T} \) such that
\[
y(t) \leq Ce^{-\eta t}, \quad t > \tilde{T}.
\]

**Proof.** (i) First, we assume that the condition (6) holds. Our claim is to show that
\[
y(t) < y_* := \frac{1}{2} \left( y_0 + \|f\|_{L^1} + \frac{a}{b} \right) < \frac{a}{b}, \quad t > 0.
\]
The second inequality in (9) trivially holds due to (6). To show the first inequality, we define the critical time \( T_* \) such that
\[
T_* = \inf \left\{ t > 0 : \sup_{0 \leq s \leq t} y(s) < y_* \right\}.
\]
Since \( y(0) = y_0 < y_0 + \|f\|_{L^1} < y_* \), we have \( T_* > 0 \). Now, suppose to the contrary that \( T_* < +\infty \). Then, it follows from the continuity for \( y \) that \( y(T_*) = y_* \). Thus, we obtain
\[
y_* = y(T_*) = y_0 + \int_0^{T_*} (-ay(t) + by(t)^2 + f(t)) \, dt \leq y_0 + \int_0^{T_*} |f(t)| \, dt
\]
\[
\leq y_0 + \|f\|_{L^1} < y_*,
\]
which yields contradiction. Here, we use \( -ay(t) + by(t)^2 < 0 \) for \( 0 \leq t \leq T_* \).
Therefore, \( T_* = +\infty \) and (9) holds. Since \( y(t) < y_* \) for \( t > 0 \), one has
\[
-ay + by^2 \leq \frac{-ay_* + by_*^2}{y_*} y = -(a - by_*) y.
\]
Therefore, the Riccati equation (5) implies
\[ \dot{y} \leq -ay + by^2 + f(t) \leq -(a - by_*) y + f(t), \quad t > 0. \]

Finally, it follows from Lemma 2.4 that
\[ \lim_{t \to \infty} y(t) = 0. \]

(ii) For the second assertion, since (7) holds, the quadratic equation \( bz^2 - az + \|f\|_\infty = 0 \) has two distinct positive roots, say \( 0 < \alpha_1 < \alpha_2 \). Moreover, since the initial datum satisfies (7), i.e., \( y_0 < \alpha_2 \), there exists a finite entrance time \( T_1 > 0 \) such that
\[ y(t) < \frac{\alpha_1 + \alpha_2}{2} = \frac{a}{2b}, \quad t > T_1. \]

We substitute (10) into (5) to find
\[ \dot{y} \leq -\frac{a}{2} y + f(t), \quad t > T_1. \]

Finally, we again use Lemma 2.4 for \( t > T_1 \) to obtain the desired zero convergence of \( y \).

(iii) For the last assertion on the exponential convergence of \( y \), we note that each condition (1) or (2) in (8) corresponds to the condition (6) in (i) or (7) in (ii), respectively. Hence, \( y(t) \) converges to 0 as \( t \to \infty \), and thus, for any \( 0 < \delta < \frac{a}{2b} \), there exists \( T_\delta \) such that
\[ \sup_{t \geq T_\delta} y(t) < \delta. \]

Then, for \( t \geq T_\delta \), one finds
\[ \dot{y} \leq -ay + by^2 + f(t) \leq -(a - b\delta) y + C_0 e^{-\lambda t}. \]

Thus, we estimate \( y \) as follows:
\[ y(t) \leq e^{-(a-b\delta)(t-T_\delta)} y(T_\delta) + \frac{C_0}{\lambda} e^{-(a-b\delta)\lambda t} \left( e^{-\lambda T_\delta} - e^{-\lambda t} \right) \]
\[ < e^{-(a-b\delta)T_\delta + C_0 e^{-\lambda T_\delta}} e^{-(a-b\delta)t} = \left( e^{(a-b\delta)T_\delta + C_0 e^{-\lambda T_\delta}} \right) e^{-(a-b\delta)t}. \]

Finally, for any choice of \( 0 < \delta < \frac{a}{2b} \), we may choose
\[ C = e^{(a-b\delta)T_\delta + C_0 e^{-\lambda T_\delta}}, \quad \eta = a - b\delta \]
to obtain a desired estimate. \( \square \)

**Remark 2.** Lemma 2.5 (iii) states that if the function \( f \) has an exponential decay rate, then \( y \) satisfying (5) also decays to zero exponentially.

Finally, we end this section with classical Barbalat’s lemma.

**Lemma 2.6.** [2] Let \( f : [0, \infty) \to \mathbb{R} \) be a real-valued function. If \( f \) is uniformly continuous and
\[ \lim_{t \to \infty} \int_0^t f(s) ds \]
exists and is finite,
then \( f \) tends to zero as \( t \to \infty \):
\[ \lim_{t \to \infty} f(t) = 0. \]
3. **Emergence of the complete aggregation.** In this section, we study the asymptotic behavior of (2) with attractive couplings $\kappa > 0$. Our goal is to find a sufficient condition leading to the globally aggregated state. To this end, we define the maximal diameters for $h_{ij}$ and $\Omega_i$ as follows:

$$D(H(t)) := \max_{1 \leq i,j \leq N} |1 - h_{ij}(t)|, \quad D(\Omega(t)) := \max_{1 \leq i,j \leq N} \|\Omega_i(t) - \Omega_j(t)\|_F.$$

**Remark 3.** For the functionals $D(H)$, if one observes the following relation:

$$1 - \rho^2 = \frac{1}{2N^2} \sum_{i,j=1}^{N} |x_i - x_j|^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} (1 - h_{ij}),$$

then, together with Definition 2.2, we notice that the complete aggregation occurs if and only if $D(H)$ tends to 0.

We first show that if the adaptive law satisfies one of the conditions (C1)–(C3), then the maximal diameter $D(\Omega)$ converges to zero with an exponential rate for sufficiently large damping coefficient $\gamma$.

**Proposition 1.** Let $\{\Omega_i\}_{i=1}^{N}$ be a solution to (2) with the initial data $\{\Omega_i^0\}_{i=1}^{N} \in \text{Skew}_d(\mathbb{R})^N$. Suppose one of the conditions (C1)–(C3) for the adaptive law $\Gamma$ holds. Then, there exist constants $C > 0$ and $\gamma_0 \geq 0$ such that if $\gamma > \gamma_0$,

$$D(\Omega(t)) \leq Ce^{-(\gamma - \gamma_0)t}, \quad t > 0.$$

**Proof.** We prove the assertion for the three cases (C1), (C2), and (C3) separately.

- **Case (C1):** We first present the estimate for the case of Lipschitz continuous adaptive law:

$$\|\Gamma(P_1, Q) - \Gamma(P_2, Q)\|_F \leq \Gamma_{\text{Lip}}\|P_1 - P_2\|_F.$$

Since $\Omega_i \in \text{Skew}_d(\mathbb{R})$, we have

$$\frac{d}{dt}\|\Omega_i - \Omega_j\|_F^2 = \frac{d}{dt}\text{tr}((\Omega_i - \Omega_j)^\top (\Omega_i - \Omega_j)) = -\text{tr}\left(\frac{d}{dt}(\Omega_i - \Omega_j)^2\right)$$

$$= -2\text{tr}\left((\Omega_i - \Omega_j)\frac{d}{dt}(\Omega_i - \Omega_j)\right) = 2\text{tr}\left((\Omega_i - \Omega_j)^\top \frac{d}{dt}(\Omega_i - \Omega_j)\right)$$

$$= -2\gamma \text{tr}\left((\Omega_i - \Omega_j)^\top (\Omega_i - \Omega_j)\right) + 2\text{tr}\left((\Omega_i - \Omega_j)^\top \left(\frac{\mu}{N} \sum_{k=1}^{N} (\Gamma(\Omega_i, \Omega_k) - \Gamma(\Omega_j, \Omega_k))\right)\right)$$

$$\leq -2\gamma\|\Omega_i - \Omega_j\|_F^2 + \frac{2\mu}{N}\|\Omega_i - \Omega_j\|_F \sum_{k=1}^{N} \|\Gamma(\Omega_i, \Omega_k) - \Gamma(\Omega_j, \Omega_k)\|_F$$

$$\leq -2\gamma\|\Omega_i - \Omega_j\|_F^2 + 2\mu\Gamma_{\text{Lip}}\|\Omega_i - \Omega_j\|_F^2 = -2(\gamma - \mu\Gamma_{\text{Lip}})\|\Omega_i - \Omega_j\|_F^2.$$

Therefore, in this case, we set $\gamma_0 = \mu\Gamma_{\text{Lip}}$ and for $\gamma > \gamma_0$, we have

$$\frac{d}{dt}\|\Omega_i - \Omega_j\|_F \leq -(\gamma - \gamma_0)\|\Omega_i - \Omega_j\|_F,$$

which yields the desired convergence: for $t > 0$,

$$\|\Omega_i(t) - \Omega_j(t)\|_F \leq \|\Omega_i^0 - \Omega_j^0\|_F e^{-(\gamma - \gamma_0)t}.$$
We now choose $\gamma$ which, together with the Grönwall inequality, implies
\[
\dot{\Omega}_i = -\gamma \Omega_i + \frac{\mu}{N} \sum_{k=1}^{N} (\Omega_i^t \Omega_k - \Omega_k^t \Omega_i) = -\gamma \Omega_i + \mu (\Omega_i \Omega_c - \Omega_c \Omega_i),
\]
and therefore by taking the sum of the above estimate for $i = 1, \cdots, N$, we have
\[
\dot{\Omega}_c = -\gamma \Omega_c.
\]
Thus, we have an exponential decay of $\Omega_c$, i.e., $\|\Omega_c\|_F \leq e^{-\gamma t} \|\Omega_c^0\|_F$. We now use the same estimate as in Case A to derive
\[
\frac{d}{dt} \|\Omega_i - \Omega_j\|_F^2 = -2\gamma \text{tr} ((\Omega_i - \Omega_j)^\top (\Omega_i - \Omega_j)) + 2\text{tr} \left( (\Omega_i - \Omega_j)^\top \left( \frac{\mu}{N} \sum_{k=1}^{N} (\Gamma(\Omega_i, \Omega_k) - \Gamma(\Omega_j, \Omega_k)) \right) \right) = -2\gamma \|\Omega_i - \Omega_j\|_F^2 + 2\text{tr} ((\Omega_i - \Omega_j)^\top ((\Omega_i - \Omega_j)^\top (\Omega_i - \Omega_j)))
\]
\[
\leq -2\gamma \|\Omega_i - \Omega_j\|_F^2 + 4\mu \|\Omega_i - \Omega_j\|_F^2 \|\Omega_c\|_F \leq -2(\gamma - 4\mu \|\Omega_c^0\|_F e^{-\gamma t}) \|\Omega_i - \Omega_j\|_F^2.
\]
Therefore, we have
\[
\frac{d}{dt} \|\Omega_i - \Omega_j\|_F \leq - (\gamma - 2\mu \|\Omega_c^0\|_F e^{-\gamma t}) \|\Omega_i - \Omega_j\|_F,
\]
which, together with the Grönwall inequality, implies
\[
\|\Omega_i - \Omega_j\|_F \leq \|\Omega_i^0 - \Omega_j^0\|_F \exp \left( - \int_0^t (\gamma - 2\mu \|\Omega_c^0\|_F e^{-\gamma s}) ds \right) = \|\Omega_i^0 - \Omega_j^0\|_F \exp \left( -\gamma t + 2\mu \|\Omega_c^0\|_F \frac{1 - e^{-\gamma t}}{\gamma} \right) \leq \|\Omega_i^0 - \Omega_j^0\|_F \exp \left( \frac{2\mu \|\Omega_c^0\|_F}{\gamma} \right) e^{-\gamma t}.
\]
Hence, the desired estimate for $D(\Omega(t))$ holds with $\gamma_0 = 0$ and $C = \|\Omega_i^0 - \Omega_j^0\|_F \exp \left( \frac{2\mu \|\Omega_c^0\|_F}{\gamma} \right)$.

**Case (C3):** Finally, we consider the case when the adaptive law is given by $\Gamma(P, Q) = f(P, Q)\Psi$. We define $R$ as
\[
R := \max_{1 \leq i \leq N} \|\Omega_i^0\|_F.
\]
We now choose $\gamma_0$ as
\[
\gamma_0 := \max \left\{ \frac{\mu \|\Psi\|_F (f(O, O) + \omega_2(D(\Omega^0)))}{\min_{1 \leq i \leq N} \|\Omega_i^0\|_F}, 2\mu \omega_1(R) \|\Psi\|_F \right\}.
\]
Our goal is to show that for $\gamma > \gamma_0$,
\[
\sup_{t \geq 0} \max_{1 \leq i \leq N} \|\Omega_i(t)\|_F \leq R.
To this end, we fix an arbitrary \( \varepsilon > 0 \) and we define the critical time \( T_\ast \) as
\[
T_\ast := \sup \left\{ t > 0 : \sup_{0 \leq s \leq t} \max_{1 \leq i \leq N} \| \Omega_i(s) \|_F \leq R + \varepsilon \right\}.
\]
Then, it follows from the choice of \( R \) that \( T_\ast > 0 \). We claim that \( T_\ast = +\infty \). To show this, we use a method by contradiction. Suppose to the contrary that \( T_\ast \) is finite. Then, it follows from the continuity of \( \Omega_t \) that
\[
\max_{1 \leq i \leq N} \| \Omega_i(T_\ast) \|_F = R + \varepsilon.
\]
We use (2) again to observe that, until time \( T_\ast \),
\[
\frac{d}{dt} \| \Omega_i - \Omega_j \|_F^2
= -2\gamma \text{tr} \left( (\Omega_i - \Omega_j)^\top (\Omega_i - \Omega_j) \right)
+ 2\text{tr} \left( \Omega_i - \Omega_j \right)^\top \left( \frac{\mu}{N} \sum_{k=1}^N (\Gamma(\Omega_i, \Omega_k) - \Gamma(\Omega_j, \Omega_k)) \right)
= -2\gamma \| \Omega_i - \Omega_j \|_F^2 + 2\text{tr} \left( \Omega_i - \Omega_j \right)^\top \left( \frac{\mu}{N} \sum_{k=1}^N (f(\Omega_i, \Omega_k) - f(\Omega_j, \Omega_k)) \Psi \right)
\leq -2\gamma \| \Omega_i - \Omega_j \|_F^2 + 2\mu \| \Psi \|_F \| \Omega_i - \Omega_j \|_F \sum_{k=1}^N |f(\Omega_i, \Omega_k) - f(\Omega_j, \Omega_k)|
\leq -2\gamma \| \Omega_i - \Omega_j \|_F^2 + 2\omega_1(R) \mu \| \Psi \|_F \| \Omega_i - \Omega_j \|_F^2.
\]
Therefore, it follows from the choice of \( \gamma_0 \) that for any \( \gamma > \gamma_0 \), we have \( \gamma > 2\mu \omega_1(R) \| \Psi \|_F \) and
\[
\| \Omega_i(t) - \Omega_j(t) \|_F \leq \| \Omega_i^0 - \Omega_j^0 \|_F e^{-(\gamma - 2\mu \omega_1(R) \| \Psi \|_F) t}, \quad 0 \leq t \leq T_\ast.
\] (12)
On the other hands, for any \( i = 1, 2, \ldots, N \),
\[
\frac{d}{dt} \| \Omega_i \|_F
= -\gamma \| \Omega_i \|_F + \frac{\mu}{N} \| \Psi \|_F \sum_{k=1}^N |f(\Omega_i, \Omega_k)|
\leq -\gamma \| \Omega_i \|_F + \mu \| \Psi \|_F (f(O, O) + \omega_2(\| \Omega_i - \Omega_k \|_F)),
\]
and therefore,
\[
\frac{d}{dt} \| \Omega_i \|_F \leq -\gamma \| \Omega_i \|_F + \mu \| \Psi \|_F (f(O, O) + \omega_2(\| \Omega_i - \Omega_k \|_F)).
\]
Thus,
\[
\| \Omega_i \|_F \leq \frac{\mu \| \Psi \|_F (f(O, O) + \omega_2(\| \Omega^0 \|_F))}{\gamma} \left( 1 - e^{-\gamma t} \right) + \| \Omega_i^0 \|_F e^{-\gamma t}.
\]
Again, it follows from the choice of \( \gamma_0 \) that
\[
\| \Omega_i^0 \|_F > \frac{\mu \| \Psi \|_F (f(O, O) + \omega_2(\| \Omega^0 \|_F))}{\gamma},
\]
and thus,
\[
\| \Omega_i(t) \|_F \leq \| \Omega_i^0 \|_F \leq R, \quad 0 \leq t \leq T_\ast.
\]
which contradicts the choice of $T_\ast$. Thus, we have $T_\ast = +\infty$ and consequently,

$$\sup_{t \geq 0} \max_{1 \leq i \leq N} \|\Omega_i(t)\|_F \leq R + \varepsilon.$$ 

Since $\varepsilon$ was arbitrary, we have

$$\sup_{t \geq 0} \max_{1 \leq i \leq N} \|\Omega_i(t)\|_F \leq R$$

and the estimate (12) holds for whole time interval. Then, (12) can be estimated as

$$\|\Omega_i(t) - \Omega_j(t)\|_F \leq \|\Omega_i^0 - \Omega_j^0\|_F e^{- (\gamma_0 - 2\mu\omega_1(R)) \|\Psi\|_F} e^{-(\gamma - \gamma_0) t} \leq D(\Omega(0)) e^{-(\gamma - \gamma_0) t}.$$ 

Therefore, we choose $C = D(\Omega(0))$, and $\gamma_0 = \max \left\{ \frac{\mu \|\Psi\|_F (f(O,O) + \omega_2(D(\Omega^0)))}{\min_{1 \leq i \leq N} \|\Omega_i^0\|_F}, 2\mu\omega_1(R) \|\Psi\|_F \right\}$ to obtain desired estimate.

Below, we summarize the choices for $C$ and $\gamma_0$ for each case corresponding to $(C1)$–$(C3)$.

**Table 1.** Choices for $(C, \gamma_0)$ for each case

|     | $C$                        | $\gamma_0$            |
|-----|---------------------------|-----------------------|
| $(C_1)$ | $D(\Omega^0)$            | $\mu \Gamma_{\text{Lip}}$ |
| $(C_2)$ | $D(\Omega^0) \exp \left( \frac{2\mu \|\Omega_0\|_F}{\gamma} \right)$ | 0 |
| $(C_3)$ | $D(\Omega^0)$            | $\max \left\{ \frac{\mu \|\Psi\|_F (f(O,O) + \omega_2(D(\Omega^0)))}{\min_{1 \leq i \leq N} \|\Omega_i^0\|_F}, 2\mu\omega_1(R) \|\Psi\|_F \right\}$ |

We are now ready to provide the first main result concerning the emergence of global aggregation. To obtain the global aggregation, we introduce the following alternative assumptions on the system parameters and the initial data.

- **(A1):** The damping coefficient $\gamma$ is sufficiently large so that it is larger than $\gamma_0$, specified in Proposition 1 for each case.

  We also assume one of the following sets of assumptions holds where the constant $C$ is defined in Proposition 1.

  - **(A2):**
    - (i) The damping coefficient is further large so that
      $$\frac{C}{\gamma - \gamma_0} < 1,$$
      i.e., $\gamma > \gamma_0 + C$.
    - (ii) The initial data $\{x_i^0\}_{i=1}^N$ is well-prepared in the sense that
      $$D(H^0) < 1 - \frac{C}{\gamma - \gamma_0}.$$

  or

  - **(A3):**
    - (i) The coupling strength $\kappa$ is sufficiently large so that
      $$\kappa > C.$$
    - (ii) The initial data $\{x_i^0\}_{i=1}^N$ is well-prepared in the sense that
      $$D(H^0) < \frac{\kappa + \sqrt{\kappa^2 - \kappa C}}{2\kappa} = 1 - \frac{C}{2\kappa + 2\sqrt{\kappa^2 - \kappa C}}.$$
In the following theorem, our goal is to show that \( D(H(t)) \) decays to 0 under the assumptions above.

**Theorem 3.1.** Suppose that one of the assumptions \{\( A1 \), \( A2 \)\} or \{\( A1 \), \( A3 \)\} holds. Let \((X, \Omega)\) be a global solution to (2) with the initial data \((X^0, \Omega^0)\). Suppose further that one of the conditions (C1)–(C3) for the adaptive law \( \Gamma \) holds. Then, there exist positive constants \( C_1 \) and \( \eta_1 \) such that

\[
D(H(t)) \leq C_1 e^{-\eta_1 t}, \quad D(\Omega(t)) \leq C e^{-(\gamma-\gamma_0)t}, \quad t > 0.
\]

Moreover, the convergence rate is at most exponential.

**Proof.** We choose the maximal indices \( i, j \) so that \( 1 - h_{ij} = D(H(t)) \). Then, it follows from Lemma 2.3 (i) that the maximal diameter satisfies

\[
\frac{d}{dt}D(H) = -2\kappa D(H) + \kappa \sum_{k=1}^{N} (1 - h_{ik} + 1 - h_{kj})D(H) + \langle x_i, (\Omega_j - \Omega_i)x_j \rangle
\]

\[
\leq -2\kappa D(H) + 2\kappa D(H)^2 + D(\Omega).
\]

Since we assume \( A1 \), Proposition 1 implies that there exist positive constants \( C \) and \( \gamma_0 \) such that

\[
D(\Omega(t)) \leq C e^{-(\gamma-\gamma_0)t}, \quad t > 0.
\]

Hence, under the assumption \( A1 \), \( D(H) \) satisfies the Riccati inequality (5) with \( a = b = 2\kappa \), \( f(t) = C e^{-A t} \) and \( \lambda = \gamma - \gamma_0 \). Therefore, the each assumption \( A2 \) and \( A3 \) corresponds to the condition (1) and (2) of Lemma 2.5 (iii), respectively. Thus, we conclude that there exist positive constants \( C_1 \) and \( \eta_1 \) satisfying the desired exponential estimate:

\[
D(H(t)) \leq C_1 e^{-\eta_1 t}, \quad t > 0.
\]

\( \square \)

**Remark 4.** (i) The initial data for \( \{x^0\} \) in \( A2 \) and \( A3 \) can be visualized as a subset of a quadrant of the unit sphere. Indeed, the condition for initial data in \( A2 \) implies that the initial data \( \{x^0\} \) satisfies

\[
1 - h_{ij}^0 = 1 - \langle x_i^0, x_j^0 \rangle < 1 - \frac{C}{\gamma - \gamma_0}, \quad \text{i.e.,} \quad \langle x_i^0, x_j^0 \rangle > \frac{C}{\gamma - \gamma_0}, \quad i, j = 1, 2, \ldots, N.
\]

Similarly, the condition for initial data in \( A3 \) is

\[
\langle x_i^0, x_j^0 \rangle > \frac{C}{2\kappa + 2\sqrt{\kappa^2 - \kappa C}}, \quad i, j = 1, 2, \ldots, N.
\]

In particular, the inner product between two initial data should be positive, and therefore, the angle between two initial data should be less than \( \frac{\pi}{2} \). For instance, under the condition \( A2 \), the angle \( \theta \) between two initial data \( x_i^0 \) and \( x_j^0 \) should be less than \( \arccos \left( \frac{C}{\gamma - \gamma_0} \right) < \frac{\pi}{2} \) (See Figure 1). We also note that the condition for initial data is alleviated as the parameters \( \gamma \) or \( \kappa \) getting larger. Compared to [13] in which the natural frequency is given to be a common constant, the initial data leading to the complete aggregation are exact quadrant of the unit sphere.

(ii) Concerning the system parameters \( \gamma \) and \( \kappa \), in \( A2 \), we do not need additional assumption on \( \kappa > 0 \) for \( A2 \), whereas a large coupling strength is required in \( A3 \).
4. Emergence of the complete disaggregation. In this section, we provide a sufficient condition for the complete disaggregation, i.e., \( \lim_{t \to \infty} \rho(t) = 0 \) with the repulsive couplings \( \kappa < 0 \). For convenience, we introduce a positive constant \( \lambda := -\kappa > 0 \). Then, obviously, (2) can be rewritten as

\[
\begin{cases}
\dot{x}_i = \Omega_i x_i - \frac{\lambda}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle) x_i, & t > 0, \\
\dot{\Omega}_i = -\gamma \Omega_i + \frac{\mu}{N} \sum_{k=1}^{N} \Gamma(\Omega_i, \Omega_k), & i = 1, \ldots, N.
\end{cases}
\]

(13)

We assume that the damping coefficient \( \gamma \) is sufficiently large so that \( \gamma > \gamma_0 \) as in Proposition 1. Then, we have an exponential decay of \( D(\Omega(t)) \), i.e., \( D(\Omega(t)) \leq Ce^{-(\gamma - \gamma_0)t} \). In order to achieve the complete disaggregation, we recall the functional \( \mathcal{H} \) in [13]:

\[ \mathcal{H}(t) := \prod_{i \neq j} (1 - h_{ij}(t)) \]

which also measures the entire degree of aggregation. In the following lemma, we present the lower bound estimate on the term \( 1 - h_{ij}(t) \) under the assumptions on parameters and initial data.

**Lemma 4.1.** Let \((X, \Omega)\) be a global solution to (13) and \( \alpha > 2\lambda \) be a fixed positive constant. Suppose one of the conditions (C1)–(C3) for the adaptive law \( \Gamma \) holds and the initial data and system parameters satisfy

\[ x_i^0 \neq x_j^0, \quad i \neq j, \quad \gamma > \gamma_0 + \alpha + \frac{2C}{\delta \log 2}, \quad \delta := \min_{1 \leq i \neq j \leq N} \left( 1 - h_{ij}^0 \right). \]

(14)

Then, we have

\[ 1 - h_{ij}(t) \geq \frac{1}{2} \delta e^{-\alpha t}, \quad t > 0, \quad i \neq j. \]

(15)

**Proof.** We use the estimate in the proof of Lemma 2.3 (i) to obtain

\[ \frac{d}{dt} h_{ij} = -\frac{\lambda}{N} \sum_{k=1}^{N} (h_{ik} + h_{kj})(1 - h_{ij}) + \langle x_i, (\Omega_j - \Omega_i) x_j \rangle, \]

which yields

\[ \frac{d}{dt} \log(1 - h_{ij}) = -\frac{1}{1 - h_{ij}} \frac{d}{dt} h_{ij} = \frac{\lambda}{N} \sum_{k=1}^{N} (h_{ik} + h_{kj}) - \frac{\langle x_i, (\Omega_j - \Omega_i) x_j \rangle}{1 - h_{ij}} \]
We now define the critical time $T_*$. This in turn implies the desired estimate (15).

Thus, we find an explicit relation for $\mathcal{H}$:

$$
\mathcal{H}(t) = \mathcal{H}^0 \exp \left( 2\lambda(N^2 - N) \int_0^t \rho^2(s) \, ds + \int_0^t F(s) \, ds \right).
$$

We now define the critical time $T_*$ as

$$
T_* := \sup \left\{ t \in (0, \infty) : 1 - h_{ij}(s) \geq \frac{1}{2} \delta e^{-\alpha s}, \quad s \in [0, t) \quad \forall i \neq j \right\}.
$$

We claim that $T_* = +\infty$. To prove the claim, suppose to the contrary that $T_* < \infty$. Then, by the continuity of $h_{ij}$, there exists a pair of indices $(i,j)$ such that we have

$$
1 - h_{ij}(t) \geq \frac{1}{2} \delta e^{-\alpha t}, \quad 0 \leq t \leq T_*, \quad \text{and} \quad 1 - h_{ij}(T_*) = \frac{1}{2} \delta e^{-\alpha T_*}.
$$

On the other hand, since $\gamma > \gamma_0$ by (14), we note that

$$
\left| \int_0^t \frac{\langle x_i, (\Omega_i - \Omega_j)x_j \rangle}{1 - h_{ij}} \, ds \right| \leq \frac{2}{\delta} \int_0^t e^{\alpha s} D(\Omega(s)) \, ds
$$

$$
\leq \frac{2C}{\delta} \int_0^t e^{-(\gamma - \gamma_0 - \alpha)s} \, ds \leq \frac{2C}{\delta(\gamma - \gamma_0 - \alpha)}.
$$

Since $\langle x_i, x_i + x_j \rangle \leq |x_i|(|x_i| + |x_j|) \leq 2$, it follows from (16) that

$$
1 - h_{ij}(t) = (1 - h^0_{ij}) \exp \left( \int_0^t \lambda(x_c, x_i + x_j) \, ds + \int_0^t \frac{\langle x_i, (\Omega_i - \Omega_j)x_j \rangle}{1 - h_{ij}} \, ds \right)
$$

$$
\geq \min_{1 \leq i \neq j \leq N} (1 - h^0_{ij}) e^{-\frac{2C}{\gamma - \gamma_0 - \alpha}} e^{-2\lambda t} = \delta e^{-\frac{2C}{\gamma - \gamma_0 - \alpha}} e^{-2\lambda t}.
$$

Since we assume (14) which can be rewritten as

$$
\delta e^{\frac{2C}{\gamma - \gamma_0 - \alpha}} > \frac{\delta}{2},
$$

the relation (20) yields

$$
1 - h_{ij}(t) > \frac{\delta}{2} e^{-2\lambda t}, \quad \text{for} \quad 0 \leq t \leq T_*, \quad \text{in particular,} \quad 1 - h_{ij}(T_*) > \frac{\delta}{2} e^{-2\lambda T_*}.
$$

However, since $\alpha > 2\lambda$, (21) contradicts (18). Hence, we conclude that $T_* = +\infty$.

Next, we present the uniform boundedness of the natural frequency $\Omega_i$. Unlike the case of complete aggregation result in the previous section, the boundedness of the natural frequency is required to show the complete disaggregation.

**Lemma 4.2.** Let $\{\Omega_i\}_{i=1}^N$ be a solution to (13) with the initial data $\{\Omega^0_i\}_{i=1}^N \in \text{Skew}_d(\mathbb{R})^N$. Suppose one of the conditions (C1)–(C3) for the adaptive law $\Gamma$ holds.
Moreover, suppose that the damping coefficient $\gamma$ is larger than $\gamma_0$ in Proposition 1. Then, there exists a uniform constant $R > 0$ such that
\[
\max_{1 \leq i \leq N} \|\Omega_i(t)\|_F \leq R, \quad t \geq 0.
\]

Proof. Since the Case of condition (C3) is already proved in the proof of Proposition 1, we only consider the cases (C1) and (C2).

- Case (C1): For the case of Lipschitz continuous adaptive law, it follows from Proposition 1 that we have
\[
D(\Omega(t)) \leq Ce^{-(\gamma - \gamma_0)t}, \quad t > 0.
\]
On the other hand, We use (2) to derive
\[
\frac{1}{2} \frac{d}{dt} \|\Omega_i\|_F^2 \leq -\gamma \|\Omega_i\|_F^2 + \frac{\mu}{N} \sum_{k=1}^{N} \|\Gamma(\Omega_i, \Omega_k)\|_F \|\Omega_i\|_F
\]
\[
\leq -\gamma \|\Omega_i\|_F^2 + \frac{\mu}{N} \sum_{k=1}^{N} \omega_0(\|\Omega_i - \Omega_k\|_F) \|\Omega_i\|_F
\]
\[
\leq -\gamma \|\Omega_i\|_F^2 + \frac{\mu}{N} \sum_{k=1}^{N} \omega_0(Ce^{-(\gamma - \gamma_0)t}) \|\Omega_i\|_F
\]
\[
= -\gamma \|\Omega_i\|_F^2 + \mu \|\Omega_i\|_F \omega_0(Ce^{-(\gamma - \gamma_0)t}).
\]

Therefore, we have
\[
\frac{d}{dt} \|\Omega_i\|_F \leq -\gamma \|\Omega_i\|_F + \mu \omega_0(C),
\]
which implies $\|\Omega_i(t)\|_F \leq \max \left\{ \|\Omega_i^0\|_F, \frac{\mu \omega_0(C)}{\gamma} \right\}$. Thus, in this case, we choose $R$ as
\[
R = \max \left\{ \max_{1 \leq i \leq N} \|\Omega_i^0\|_F, \frac{\mu \omega_0(C)}{\gamma} \right\}
\]
to obtain the desired estimate.

- Case (C2): When $\Gamma(P, Q) = [P, Q] = PQ - QP$, Proposition 1 again yields
\[
D(\Omega(t)) \leq Ce^{-\gamma t}, \quad t > 0.
\]
Here, $\gamma_0$ can be chosen as 0 (see Table 1). Recall the dynamics can be shortened in this case as
\[
\dot{\Omega}_i = -\gamma \Omega_i + \mu (\Omega_i \Omega_c - \Omega_c \Omega_i).
\]
Therefore, we have
\[
\frac{1}{2} \frac{d}{dt} \|\Omega_i\|_F^2 = -\gamma \|\Omega_i\|_F^2 + 2\mu \|\Omega_i\|_F \|\Omega_c\|_F \leq -\gamma \|\Omega_i\|_F^2 + 2\mu e^{-\gamma t} \|\Omega_i\|_F \|\Omega_c^0\|_F,
\]
which implies
\[
\frac{d}{dt} \|\Omega_i\|_F \leq -\gamma \|\Omega_i\|_F + 2\mu \|\Omega_c^0\|_F e^{-\gamma t} \|\Omega_i\|_F = -\gamma \|\Omega_i\|_F - 2\mu \|\Omega_c^0\|_F e^{-\gamma t} \|\Omega_i\|_F
\]
Note that this is exactly the same type of differential inequality as (11). Therefore, we conclude that
\[
\|\Omega_i\|_F \leq \|\Omega_i^0\|_F \exp \left( \frac{2\mu \|\Omega_c^0\|_F}{\gamma} \right) e^{-\gamma t}.
\]
Hence, we may choose $R$ as

$$R = \max_{1 \leq i \leq N} \|\Omega_i^0\|_F \exp \left( \frac{2\mu \|\Omega_i^0\|_F}{\gamma} \right)$$

to obtain the desired estimate. In fact, in this case, we have a stronger result that $\|\Omega_i\|_F$ converges to 0.

We are now ready to show the complete disaggregation of the system (13).

**Theorem 4.3.** Let $(X, \Omega)$ be a global solution to (13) and $\alpha > 2 \lambda$ be a fixed positive constant. Suppose one of the conditions (C1)–(C3) for the adaptive law $\Gamma$ holds and the initial data and system parameters satisfy (14). Then, the order parameter converges to zero:

$$\lim_{t \to \infty} \rho(t) = 0.$$

**Proof.** We first recall that (19) holds for all $t > 0$ and $i \neq j$. Therefore, we estimate

$$\int_0^t F(s) \, ds \leq (N^2 - N) \frac{2C}{\delta(\gamma - \gamma_0 - \alpha)} =: M, \quad t > 0. \quad (22)$$

In (17), we use (22) to find

$$2^{N^2 - N} \geq \mathcal{H}(t) \geq \mathcal{H}^0 \exp \left( \lambda(2N^2 - N) \int_0^t \rho^2(s) \, ds \right) e^{-M},$$

which yields

$$\mathcal{H}^0 \exp \left( \lambda(2N^2 - N) \int_0^t \rho^2(s) \, ds \right) \leq 2^{N^2 - N} e^M, \quad t > 0.$$ 

Hence, $\rho^2 \in L^1(\mathbb{R}_+)$. On the other hand, it follows from Lemma 2.3 (ii) that

$$\frac{d\rho^2}{dt} = -2\lambda \left( \rho^2 - \frac{1}{N} \sum_{i=1}^N (x_i, x_c)^2 \right) + \frac{2}{N} \sum_{i=1}^N \langle \Omega_i x_i, x_c \rangle \leq 2 \sup_{t \geq 0} \max_{1 \leq i \leq N} \|\Omega_i\|_F$$

and

$$\frac{d\rho^2}{dt} \geq -2\lambda - 2 \sup_{t \geq 0} \max_{1 \leq i \leq N} \|\Omega_i\|_F.$$

Therefore, thanks to Lemma 4.2, we have

$$\left| \frac{d\rho^2}{dt} \right| \leq 2\lambda + 2 \sup_{t \geq 0} \max_{1 \leq i \leq N} \|\Omega_i\|_F \leq 2\lambda + 2R,$$

which implies that $\rho^2$ is absolutely continuous. Then, by Barbalat’s lemma in Lemma 2.6, we obtain the desired zero convergence of $\rho$. \qed

**Remark 5.** For the emergence of the complete disaggregation in Theorem 4.3, we do not impose initial assumption on the position $\{x_i^0\}$ except for being distinct as in [13]. As a compensate, a large damping is required due to the additional dynamics for the natural frequency: if the minimal initial relative distance $\delta$ is given, then the damping strength $\gamma$ should be larger than $\frac{1}{\delta}$.
5. Numerical simulations. In this section, we provide several numerical simulations for the active swarm sphere model (2) for various cases. Precisely, we present the simulations for the attractive case ($\kappa > 0$) and repulsive case ($\kappa < 0$) separately in the following subsections. Before we illustrate the figure, we explain the setup of the numerical simulations.

- (Choice of adaptive laws): In what follows, we consider the following specific examples for each case of the adaptive law $\Gamma$:
  - (Case 1): $\Gamma(P, Q) = \frac{1}{1 + \|P - Q\|_F^2}(P - Q)$.
  - (Case 2): $\Gamma(P, Q) = [P, Q] = PQ -QP$.
  - (Case 3): $\Gamma(P, Q) = (1 + \|P - Q\|_F^2)\Psi$.

Here, we choose a constant skew-symmetric matrix $\Psi$ as

$$
\Psi_{ij} = \begin{cases} 
1, & \text{if } i > j \\
-1, & \text{if } i < j \\
0, & \text{if } i = j
\end{cases}
$$

for $1 \leq i, j \leq d$.

The example for Case 1 would be one of the simplest, but non-linear Lipschitz coupling term between $P$ and $Q$. The example for Case 3 can also be regarded as one of the simplest example for the perturbation of a constant skew-symmetric matrix.

- (Choice of parameters): We choose the number of particles as $N = 20$ and the dimension $d = 3$. One may also conduct numerical simulations with a larger number of particles. However, we simulate with $N = 20$ particles to provide the a clear figure of trajectory. When the number of particles $N$ is too large, it is not effective, or even impossible to numerically solve the entire large system of ordinary differential equations. In this case, one may adopt a single kinetic description of the ODE model, or its moment equations as introduced in [4, 30], to effectively simulate the large system. Moreover, we choose $d = 3$ to illustrate the numerical simulation over the standard 2-dimensional sphere. Finally, we fix the parameters $\kappa$ as 1 for the attractive case and $-1$ for the repulsive case. We choose these specific $\kappa$, since $\kappa$ mainly contributes to the convergence rate of aggregation or disaggregation, not the overall asymptotic dynamics. Moreover, it follows from the equation for $\Omega_i$ that the ratio between $\gamma$ and $\mu$ is important, not the individual values. Therefore, we fix the parameter $\mu = 1$ so that we concentrate on the effect of $\gamma$ for the asymptotic behaviors. To sum up, we use the following set of parameters:

$$N = 20, \quad d = 3, \quad \kappa = \pm 1, \quad \mu = 1. $$

- (Choice of initial data): The initial data $x_i^0$ for the particle position are generated from the uniform distribution over the sphere $S^2$, by normalizing the Gaussian random vector:

$$x_i^0 = \frac{Z_i}{|Z_i|}, \quad Z_i \sim \mathcal{N}(0, 1)^3, \quad \text{for } i = 1, 2, \ldots, N.$$

On the other hand, the initial data $\Omega_i^0$ for natural frequencies are generated by skew-symmetrizing the $3 \times 3$ uniform random matrices:

$$\Omega_i^0 = \frac{U_i - U_i^\top}{2}, \quad U_i \sim [-1, 1]^{3 \times 3}.$$

- (Simulation details): All of the following simulations were done by using MATLAB software. We use the fourth-order Runge-Kutta (RK4) method for solving ODEs (2) with prediction-correction type update, with a time step size $h = 0.05$. Precisely,
the numerical solution is obtained by using the RK4 method, and then normalized after one discrete time step, so that the solution stays on the sphere. We choose the simulation time $T = 50$ which is a sufficiently long time to present asymptotic behaviors.

5.1. **Attractive case.** First, we present the numerical simulation for the attractive case, i.e. when the coupling strength is $\kappa = 1$. Figure 2 shows the trajectory of the particle for each case, with different $\gamma$, while the dynamics of the diameter of $\Omega$ and the dynamics of the order parameter $\rho$ are presented in Figure 3.

![Figure 2. Particle trajectories for the attractive case ($\kappa = 1$): Simulation for Case 1 (Left), Case 2 (Middle) and Case 3 (Right). The blue marks on the sphere denote the initial positions of particles, while the red marks illustrate the positions of particles at the terminal time $t = 50$. The trajectory for Case 3 with $\gamma = 1$ blows up in finite time.](image)

The left, middle and right columns in Figure 2 correspond to the simulation results for Case 1, Case 2, and Case 3, respectively. We use $\gamma = 1, 2, 3$ for the simulation of Case 1 and Case 2, and use $\gamma = 1, 4, 5$ for the simulation of Case 3. We choose the specific choices of parameter $\gamma$ to show that the different asymptotic behaviors occur depending on the choice of $\gamma$. The reason why we choose different values of $\gamma$ for Case 3 is that $\gamma = 2, 3$ shows the same result as the case of $\gamma = 1$. In fact, the values of $\gamma_0$ presented in Table 1 imply that $\gamma$ should be chosen sufficiently large to present the
desired asymptotic behavior. It should be larger than $\gamma_0$, which is represented by the other parameters and initial data, to attain the aggregation/disaggregation results. Below, we explain the different asymptotic behaviors for each case, depending on the value of $\gamma$.

○ (Case 1): When $\gamma = 1$, the particles continuously swirl and converge to a single stationary point, while the particles lump to a single point and then move together for the case of $\gamma = 2$. When $\gamma$ becomes larger, i.e., $\gamma = 3$, the particles converge to a single stationary point. Each case of $\gamma$ represents the three different regimes. Precisely, when $\gamma = 1$, the diameter of $\Omega$ does not converge to 0, which is also described in Figure 3. Surprisingly, although the diameter of $\Omega$ does not vanish, the particles converge to a single point. Indeed, we numerically found that $\Omega_t^\infty := \lim_{t \to \infty} \Omega_t(t)$ are constant multiplication of a common skew-symmetric matrix, and the common stationary point $x^\infty := \lim_{t \to \infty} x_t(t)$ is contained in the kernel of them, i.e., $\Omega_t^\infty x^\infty = 0$. Moreover, as it is shown in the graph for the order parameter, the order parameter does not monotonically increase to 1 when $\gamma = 1$. This implies that the particles almost aggregate when $t \approx 5$, and are scattered for a while ($t \approx 10$) and then re-aggregate to the single point ($t \geq 20$). When $\gamma = 2$, now the diameter of $\Omega$ converges to 0, while the limit value of $\Omega$ is not zero. This results in the common movement of the aggregated point. Finally, when $\gamma$ is sufficiently large ($\gamma = 3$), not only the diameter of $\Omega$ converges to 0, but also $\Omega_t$ itself converges to 0, which leads to the convergence of the aggregated point toward the stationary point.

○ (Case 2): In Case 2, we first note that $\gamma_0$ was 0. Therefore, regardless of the choice of $\gamma$, we expect that the solutions show the complete aggregation, i.e., the diameter of $\Omega$ decays to 0 and the particles aggregate to a single point. The numerical
simulation results in Figure 2 and Figure 3 show exactly what we expected from analytical results.

- (Case 3): Since we choose a non-Lipschitz adaptive law in Case 3, the existence of the solution is not guaranteed for a small value of $\gamma$. When $\gamma = 1$, the solution only exists local-in-time and then blows up. On the other hand, for sufficiently large $\gamma$, there exist global-in-time solutions and they show the aggregation phenomena. However, different from Case 1 or Case 2, the particles continuously move over the sphere, in particular, following the stationary orbit. The reason why the particles continuously move is that the asymptotic values of $\Omega_i$ are not 0. It follows from (2) that the common asymptotic value of $\Omega^\infty := \lim_{t \to \infty} \Omega_i(t)$ should satisfy

$$\gamma \Omega^\infty = \mu \Gamma(\Omega^\infty, \Omega^\infty) = \mu \Psi.$$ 

Therefore, after the particles aggregate to the single point, they follow the common dynamics $\dot{x}_i = \Omega^\infty x_i = \frac{\gamma}{\mu} \Psi x_i$.

![Figure 4. Trajectory of the $x_1(t)$ for the attractive case ($\kappa = 1$). Trajectory of the Case 1 (Left), trajectory for Case 2 (Middle) and trajectory for Case 3 (Right).](image)

To provide the clear differences in the asymptotic behavior for each case, we present the particle trajectories of a single particle $x_1(t) = (x_1(t), x_2(t), x_3(t))$ in Figure 4. We plot the case of $\gamma = 2$ for Case 1 and 2 and $\gamma = 4$ for Case 3, since they can show the difference between each case well. These figures show different asymptotic behaviors such as converging to a stationary point or stationary orbit, for each case.

5.2. Repulsive case. We now present the numerical simulation for the repulsive case, when the coupling strength is $\kappa = -1$. Figure 5 and Figure 6 illustrate the dynamics of the particles over the sphere, the diameter of $\Omega$, and the order parameter.

Again, we provide the particle trajectories for Case 1, Case 2, and Case 3 at the left, middle, and right columns of Figure 5, respectively. We again use the same values of $\gamma$ as in the attractive case in the previous subsection. We also explain the asymptotic behaviors, case by case.

- (Case 1): The overall dynamics show similar behaviors with the attractive case. When $\gamma$ is too small, i.e., $\gamma = 1$, then the diameter of $\Omega$ does not decay to 0 as illustrated in Figure 6, and continuous swirling over the sphere with different natural frequency $\Omega_i$ occurs. Moreover, the order parameter also does not converge to 0. However, as $\gamma$ getting larger, the diameter of $\Omega$ decays to 0. As a consequence, the particle moves with the same natural frequency when $\gamma = 2$, and then, the particles asymptotically converge to their own stationary points when $\gamma = 3$. 

Figure 5. Numerical simulations for the repulsive case ($\kappa = -1$): Simulation for Case 1 (Left), Case 2 (Middle) and Case 3 (Right). The blue marks on sphere denote the initial positions of particles, while the red marks illustrate the positions of particles at the terminal time $t = 50$.

• (Case 2): The asymptotic behavior of Case 2 is similar to the Case 2 of attractive cases, except that the particles are now disaggregated and the order parameter converges to 0. Since $\gamma_0 = 0$, the asymptotic behaviors are the same, regardless of the choice of $\gamma$.

• (Case 3): Finally, the dynamics for Case 3 is also similar to the attractive case. Specifically, when $\gamma$ is too small, the solution does not globally exist and blows up in finite time. However for a sufficiently large $\gamma$, then the diameter of $\Omega$ decays exponentially to 0, and the solution globally exists. However, different from the attractive case, the particles continue swirling over the sphere with their own stationary orbits, but with the same natural frequencies.

Finally, we plot the trajectory of the single particle $x_1$ in Figure 7 with $\gamma = 2$ for Case 1 and Case 2 and $\gamma = 4$ for Case 3. As in the attractive case, Figure 7 shows the obvious difference between the asymptotic behaviors of each case.

6. Conclusion. In this paper, we introduce the variation of the swarming model on the sphere, whose natural frequency also has an interacting dynamics, inspired by the recent Kuramoto-type model in [10]. In order to model the positions on the
Figure 6. The dynamics of the diameter of $\Omega$ and the order parameters for the repulsive case ($\kappa = -1$) with Case 1 (Left), Case 2 (Middle) and Case 3 (Right). The graphs for Case 3 with $\gamma = 1$ again blows up.

Figure 7. Trajectory of the $x_1(t)$ for the repulsive case ($\kappa = -1$). Trajectory of the Case 1 (Left), trajectory for Case 2 (Middle) and trajectory for Case 3 (Right).

sphere, the well-known swarming model [38, 39] is applied, while three different candidates for the dynamics of a natural frequency are introduced. For the case of an attractive coupling, we provide sufficient conditions for the complete aggregation for both positions and natural frequencies in terms of the system parameters and the initial data. On the other hand, for the case of a repulsive coupling, we provide a sufficient framework for the complete disaggregation, when the order parameter converges to 0. We present several numerical simulations that confirm our theoretical results, and illustrate diverse phenomena depending on the types of an adaptive law for the natural frequency. Still, several interesting questions for the active swarming model (2) remains. First of all, as in [10], we may introduce additional stochastic terms in (2), such as a Brownian motion on the sphere. On the other hand, the mean-field limit and the existence theory for the kinetic counterpart of the active swarming model (2) are also interesting subjects. For example, since the swarming
model without the dynamics of natural frequency already studied in recent paper [26], it would be interesting to reveal the relation between the PDE model in [26] and that of (2). Also, as in the Vicsek-type model, the derivation of a reduced hyperbolic model of the system (2) by using a projection operator introduced in [21] is also one of the possible future perspectives. These will be explored in future work.

REFERENCES

[1] A. Aydoğan, S. T. McQuade and N. P. Duteil, Opinion dynamics on a general compact Riemannian manifold, Netw. Heterog. Media, 12 (2017), 489–523.
[2] I. Barbălat, Systèmes d’équations différentielles d’oscillations non linéaires, Rev. Math. Pures Appl., 4 (1959), 267–270.
[3] J. Buck and E. Buck, Biology of synchronous flashing of fireflies, Nature, 211 (1966), 562–564.
[4] Z. Cai and R. Li, Numerical regularized moment method of arbitrary order for Boltzmann-BGK equation, SIAM J. Sci. Comput., 32 (2010), 2875–2907.
[5] M. Caponigro, A. C. Lai and B. Piccoli, A nonlinear model of opinion formation on the sphere, Discrete Contin. Dyn. Syst., 35 (2015), 4241–4268.
[6] J. A. Carrillo, Y.-P. Choi, C. Totz and O. Tse, An analytical framework for consensus-based global optimization method, Math. Models Methods Appl. Sci., 28 (2018), 1037–1066.
[7] J. A. Carrillo, S. Jin, L. Li and Y. Zhu, A consensus-based global optimization method for high dimensional machine learning problems, ESAIM Control Optim. Calc. Var., 27 (2021), Paper No. S5, 22 pp.
[8] S. Chandra, M. Girvan and E. Ott, Observing microscopic transitions from macroscopic bursts: Instability-mediated resetting in the incoherent regime of the D-dimensional generalized Kuramoto model, Chaos, 29 (2019), 033124.
[9] C. Chen, S. Liu, X.-q. Shi, H. Chaté and Y. Wu, Weak synchronization and large-scale collective oscillation in dense bacterial suspensions, Nature, 542 (2017), 210–214.
[10] D. Chi, S.-H. Choi and S.-Y. Ha, Emergent behaviors of a holonomic particle system on a sphere, J. Math. Phys., 55 (2014), 052703.
[11] J. Cho, S.-Y. Ha, F. Huang, C. Jin and D. Ko, Emergence of bi-cluster flocking for the Cucker-Smale model, Math. Models Methods Appl. Sci., 26 (2016), 1191–1218.
[12] S.-H. Choi and S.-Y. Ha, Complete entrainment of Lohe oscillators under attractive and repulsive couplings, SIAM J. Appl. Dyn. Syst., 13 (2014), 1417–1441.
[13] F. Cucker and S. Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control, 52 (2007), 852–862.
[14] D. Cumin and C. P. Unsworth, Generalising the Kuramoto model for the study of neuronal synchronisation in the brain, Phys. D, 226 (2007), 181–196.
[15] T. Danino, O. Mondragon-Palomino, L. Tsunrming and J. Hasty, A synchronized quorum of genetic clocks, Nature, 463 (2010), 326–330.
[16] P. Degond, A. Frouvelle and S. Merino-Aceituno, A new flocking model through body attitude coordination, Math. Models Methods Appl. Sci., 27 (2017), 1005–1049.
[17] P. Degond, J.-G. Liu, S. Motsch and V. Panferov, Hydrodynamic models of self-organized dynamics: Derivation and existence theory, Methods Appl. Anal., 20 (2013), 89–114.
[18] P. Degond and S. Motsch, Continuum limit of self-driven particles with orientation interaction, Math. Models Methods Appl. Sci., 18 (2008), 1193–1215.
[19] J. Duan, Y. Kuang and H. Tang, Model reduction of a two-dimensional kinetic swarming model by operator projections, East Asian J. Appl. Math., 8 (2018), 151–180.
[20] Y. Fan, J. Koellermeiner, J. Li, R. Li and M. Torrilhon, Model reduction of kinetic equations by operator projection, J. Stat. Phys. 162 (2016), 457–486.
[21] A. Frouvelle and J.-G. Liu, Long-time dynamics for a simple aggregation equation on the sphere, Springer Proc. Math. Stat., 282 Springer, Cham, 2019, 457–479.
[22] I. M. Gamba and M.-J. Kang, Global weak solutions for Kolmogorov-Vicsek type equations with orientational interactions, Arch. Rational Mech. Anal., 222, (2016), 317–342.
[23] T. Gregor, K. Fujimoto, N. Masaki and S. Sawai, The onset of collective behavior in social amoebae, Science, 328 (2010), 1021–1025.
[25] S.-Y. Ha, S. Jin and D. Kim, Convergence of a first-order consensus-based global optimization algorithm, Math. Models Methods Appl. Sci., 30 (2020), 2417–2444.
[26] S.-Y. Ha, D. Kim, J. Lee and S. E. No, Particle and kinetic models for swarming particles on a sphere and stability properties, J. Stat. Phys., 174 (2019), 622–655.
[27] S.-Y. Ha, D. Ko and S. W. Ryoo, Emergent dynamics of a generalized Lohe model on some class of Lie groups, J. Stat. Phys., 168 (2017), 171–207.
[28] S.-M. Hung and S. N. Givigi, A Q-learning approach to flocking with UAVs in a stochastic environment, IEEE Trans. Cybern., 47 (2017), 186–197.
[29] D. Kim and J. Kim, Stochastic Lohe matrix model on the Lie group and mean-field limit, J. Stat. Phys., 178 (2020), 1467–1514.
[30] J. Koellermeier and M. Torrilhon, Numerical study of partially conservative moment equations in kinetic theory, Commun. Comput. Phys., 21 (2017), 981–1011.
[31] Y. Kuramoto, Self-entrainment of a population of coupled non-linear oscillators, International Symposium on Mathematical Problems in Mathematical Physics., Lecture Notes in Theoretical Physics 39, 1975, 420–422.
[32] M. A. Lohe, Non-Abelian Kuramoto model and synchronization, J. Phys. A, 42 (2009), 395101.
[33] M. A. Lohe, Higher-dimensional generalizations of the Watanabe-Strogatz transform for vector models of synchronization, J. Phys. A, 51 (2018), 225101, 24 pp.
[34] M. A. Lohe, On the double sphere model of synchronization, Phys. D, 412 (2020), 132642, 13 pp.
[35] J. Markdahl and J. Gonçalves, Global convergence properties of a consensus protocol on the n-sphere, 2016 55th IEEE Conference on Decision and Control (CDC), (2016), pp. 2487–2492.
[36] J. Markdahl, J. Thunberg and J. Gonçalves, Almost global consensus on the n-sphere, IEEE Trans. Automat. Control 63 (2018), 1664–1675.
[37] J. Markdahl, D. Proverbio and J. Gonçalves, Robust synchronization of heterogeneous robot swarms on the sphere, 2020 59th IEEE Conference on Decision and Control (CDC), (2020), pp. 5798–5803.
[38] R. Olfati-Saber, Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks, 2006 45th IEEE Conference on Decision and Control (CDC), (2006), pp. 5060–5066.
[39] R. Olfati-Saber, Flocking for multi-agent dynamic systems: Algorithms and theory, IEEE Trans. Automat. Control, 51 (2006), 401–420.
[40] L. Perea, G. Gomez and P. Elosegui, Extension of the Cucker-Smale control law to space flight formations, J. Guid. Control, 32 (2009), 527–537.
[41] L. M. Ritchie, M. A. Lohe and A. G. Williams, Synchronization of relativistic particles in the hyperbolic Kuramoto model, Chaos, 28 (2018), 053116.
[42] M. Rubenstein, A. Cornejo and R. Nagapal, Programmable self-assembly in a thousand-robot swarm, Science, 345 (2014), 795–799.
[43] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett., 75 (1995), 1226–1229.
[44] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, J. Theor. Biol., 16 (1967), 15–42.
[45] J. Zhu, Synchronization of Kuramoto model in a high-dimensional linear space, Phys. Lett. A, 377 (2013), 2939–2943.
[46] J. Zhu, J. Zhu and C. Qian, On equilibria and consensus of the Lohe model with identical oscillators, SIAM J. Appl. Dyn. Syst., 17 (2018), 1716–1741.

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