Core phase structure of cosmic strings and monopoles

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Global and local symmetries may or may not be restored inside topological defects depending upon the values of the parameters of the model. A detailed study of this parameter dependence of the core structure of strings and monopoles is presented in the context of simple models.

I. INTRODUCTION

A lot of theoretical and experimental effort has been devoted over the past twenty years to the search for defects predicted in field theory models and to the study of their dynamics and role in particle physics and cosmology [1]. Despite of the fact that to date there is no direct evidence of any such object, their potential importance in cosmology (structure formation, baryogenesis) and the interest in the non-perturbative structure of field theories has provided enough motivation for theorists to search for all possible extended solutions and to analyze their physical properties. The absolutely stable topological solitons were studied first [2], and were followed by the discovery of the metastable semilocal defects [3], the electroweak strings [4], and the ribbons [5] in realistic models of high energy physics, including topologically trivial ones.

The role of solitons is known to depend to some extent on the detailed profile and the unbroken symmetries in their interior. For instance, the cosmological evolution of a superconducting string [6] differs significantly from that of a normal string [7]. The purpose of this note is to study the parameter dependence of the core structure of strings and monopoles. As a first step, we restrict ourselves to models which are simple enough to be analyzed in detail, and which furthermore could easily be embedded into larger realistic theories. This analysis is particularly interesting in the context of Cosmology, where due to the temperature dependence of the parameters of field theories, one may encounter during the cosmological evolution ”phase transitions” inside the cores of topological defects. In fact, laboratory experiments on $^3$He, designed to investigate the physics of phase transitions in the Early Universe, have explicitly provided us with strong experimental evidence for defect-core transitions in the interior of vortices which appear in the superfluid $^3$He – B phase [8].

In section II we study the core structure of cosmic strings. The analysis is done in the context of a U(1) gauge model with two classically relevant parameters and with the original semilocal string as a limiting case. A similar analysis is performed in section III where the ’t Hooft-Polyakov monopole is embedded in an SU(2) gauge model with an extended Higgs sector. The parameter space is divided in regions corresponding to the two possibilities of the monopole-core global symmetry. Possible extensions and applications are commented upon in the final discussion section.

II. EMBEDDED NIELSEN-OLESEN VORTEX

Consider the simple extension of the Abelian Higgs model with two equally charged scalars $\Phi_1$ and $\Phi_2$ described by the lagrangian density

$$
\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} (D_\mu \Phi)^2 D^\mu \Phi - V(\Phi_1, \Phi_2)
$$

(1)

where $\Phi = (\Phi_1, \Phi_2)$, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $D_\mu = \partial_\mu - ig B_\mu$, and with $\tau_3$ the third Pauli matrix

$$
V(\Phi_1, \Phi_2) = -\frac{M^2}{2} \Phi^\dagger \Phi - \frac{m^2}{2} \Phi^\dagger \tau_3 \Phi + \frac{\lambda}{4} (\Phi^\dagger \Phi)^2
$$

(2)

For generic values of the parameters the vacuum has $\Phi_1^* \Phi_1 = (M^2 + m^2)/\lambda$ and $\Phi_2 = 0$. The model is an example of a U(1) gauge theory with the gauge group broken spontaneously to the identity, and as such it supports the existence of topologically stable strings. $\Phi_1$, the field with non-vanishing vacuum expectation value, carries their winding and vanishes at the center. As for $\Phi_2$, it vanishes at infinity, but its profile inside the string has no geometric or topological constraints and is determined dynamically by the field equations. In this section we will study in detail the structure of the string core.
and how it varies with the parameters of the model. One might imagine (1), (2) embedded in a more realistic theory with $\Phi_2$ coupled to electromagnetism. A non-zero core value of $\Phi_2$ in this case would render the string superconducting with well known observable effects [3].

Of the four parameters of the model, one sets the scale, another may be pulled outside of the action to play the role of the semiclassical parameter $\hbar$ and there remain two classically relevant ones. Specifically, by rescaling fields and coordinates according to

$$\Phi_{1(2)} \rightarrow M\Phi_{1(2)}/\sqrt{\lambda}, \quad B_\mu \rightarrow MB_\mu/g, \quad x_i \rightarrow x_i/M$$

one is left with the parameters

$$\alpha \equiv m/M \quad \text{and} \quad \beta \equiv g/\sqrt{\lambda} \quad (4)$$

Various limiting cases of (1), (2) have been studied before. First, for both $\alpha$ and $\beta$ equal to zero a simple scaling argument shows that the model does not possess any kind of stable defect solution. Second, for $\alpha = 0$ one obtains an SU(2)$_{\text{global}} \times$ U(1)$_{\text{local}}$ symmetric model, with the U(1) gauge field distinguishing an S$^1$ fiber out of the S$^3$ vacuum. The "strength" of this fibration is proportional to $\beta$. For $\beta > \beta_0 \equiv \sqrt{2}$ the embedded Nielsen-Olesen configuration i.e. with $\Phi_2 = 0$ is a classically stable solution of the model [4], and is the first example of a semilocal vortex studied before [5]. For smaller values of $\beta$ an instability arises due to the development of a non-vanishing $\Phi_2$ inside the core and the string blows up to the vacuum. Finally, the global model with $\beta = 0$ was studied recently in [6]. In analogy to the U(1) gauging of the previous limiting case, one may think of the term $|\Phi_1|^2 - |\Phi_2|^2$ in (3) as defining an S$^1$ "scalar" fibration with "strength" $\alpha$ on the S$^3$ vacuum manifold of the $\alpha = 0$ model. Topologically stable global strings (with logarithmically infinite energy per unit length) exist in this case for all positive values of $\alpha$. Furthermore, for $\alpha > \alpha_0 \simeq 0.4$ they have $\Phi_2 = 0$, while $\Phi_2 \neq 0$ when $\alpha < \alpha_0$.

The simplest guess for the parameter dependence of the core structure of the strings in (1), (2) is then the following: In the $(\alpha, \beta)$ plane there is a curve connecting the critical points $(\alpha_0, 0)$ and $(0, \beta_0)$ on the two axes, outside of which the embedded Nielsen-Olesen strings are stable, and inside of which the stable strings are characterized by non-vanishing $\Phi_2$. This picture will indeed be confirmed below and the curve $\alpha_{\text{crit}}(\beta)$ will be determined numerically.

The ansatz for the string solutions with winding number $n$ has the axially symmetric form

$$\Phi_1 = f(\rho)e^{in\theta}, \quad \Phi_2 = G(\rho), \quad B = \hat{\epsilon}_\theta \frac{B(\rho)}{\rho} \quad (5)$$

and the corresponding energy functional is in units of $M^4/2\lambda$

$$E = \int d^2x \left[ f'^2 + G'^2 + \frac{1}{\beta^2} \frac{B^2}{\rho^2} + \frac{(n - B)^2}{\rho^2} \right]$$

$$+ \frac{B^2}{\rho^2} G^2 + \frac{1}{2} \left( f^2 + G^2 \right)^2 - \left( 1 + \alpha^2 \right) f^2 - \left( 1 - \alpha^2 \right) G^2 \right]$$

$$+ \frac{B^2}{\rho^2} G^2 + \frac{1}{2} \left( f^2 + G^2 \right)^2 - \left( 1 + \alpha^2 \right) f^2 - \left( 1 - \alpha^2 \right) G^2 \right]$$

$$+ \frac{B^2}{\rho^2} G^2 + \frac{1}{2} \left( f^2 + G^2 \right)^2 - \left( 1 + \alpha^2 \right) f^2 - \left( 1 - \alpha^2 \right) G^2 \right]$$

$$= \int d^2x \left[ f'^2 + G'^2 + \frac{1}{\beta^2} \frac{B^2}{\rho^2} + \frac{(n - B)^2}{\rho^2} \right]$$

$$+ \frac{B^2}{\rho^2} G^2 + \frac{1}{2} \left( f^2 + G^2 \right)^2 - \left( 1 + \alpha^2 \right) f^2 - \left( 1 - \alpha^2 \right) G^2 \right]$$

It leads to the following field equations:

$$f'' + f' - \frac{(n - B)^2}{\rho^2} f + (1 + \alpha^2 - f^2 - G^2) f = 0$$

$$B'' - \frac{B'}{\rho} + \beta^2 f^2 (n - B) - \beta^2 G^2 B = 0 \quad (7)$$

$$G'' + \frac{G'}{\rho} - \frac{B^2}{\rho^2} G + (1 - \alpha^2 - f^2 - G^2) G = 0$$

We restrict our analysis to the minimal $n = 1$ case. In the end we comment briefly on the results for higher values of $n$. Finiteness of the energy and the field equations at the origin imply the boundary conditions

$$f(\infty) = \sqrt{\alpha^2 + 1}, \quad G(\infty) = 0, \quad B(\infty) = n$$

$$f(0) = 0, \quad G'(0) = 0, \quad B(0) = 0 \quad (8)$$

For all values of the parameters $\alpha$ and $\beta$, equations (6) admit the embedded Nielsen-Olesen solution with $G(\rho) = 0$. As explained above, this is known to be classically stable on the $\alpha$-axis for $\alpha > \alpha_0$ and on the $\beta$-axis for $\beta > \beta_0$. For arbitrary $(\alpha, \beta)$, the region of stability of the corresponding solution is determined by the requirement that in the expansion of the energy functional around it the first non-trivial term in $\delta E$ is strictly positive for all field variations. To quadratic order in field changes $\delta E = \delta E_{NO} + \delta E_G$, where $\delta E_{NO}$ is independent of $G(\rho)$ and identical in form with the perturbation obtained for the topologically stable Nielsen-Olesen vortex. Thus $\delta E_{NO} > 0$ for all $\delta f$ and $\delta A$ and may be ignored. $\delta E_G$ on the other hand represents the quadratic correction to the energy of the Nielsen-Olesen string due to the excitation of $G(\rho)$. It may be written in the form:

$$\delta E_G = \int d^2x \ G \hat{O} G \quad (9)$$

with

$$\hat{O} = -\frac{1}{\rho} \frac{d}{d\rho} \frac{d}{d\rho} - \frac{B^2}{\rho^2} + f^2 + \alpha^2 - 1 \quad (10)$$

where $f, B$ are the Nielsen-Olesen fields obtained by solving the first two equations of system (6) with $G = 0$. For stability of the embedded Nielsen-Olesen vortex we require that $\delta E_G > 0$ for arbitrary small perturbation $G$. 


This is equivalent to demanding that the operator $\hat{O}$ has no negative eigenvalues. The region of stability of the embedded Nielsen-Olesen vortex was determined numerically. The method is straightforward. We pick a point in the $(\alpha, \beta)$ plane, set $G = 0$ and solve the remaining equations (7) to find the corresponding Nielsen-Olesen solution. We then construct the operator $\hat{O}$ and consider the eigenvalue problem

$$\hat{O} G = \omega^2 G$$

(11)

In Figure 2 we show a stable embedded Nielsen-Olesen configuration and in Figure 3 an example of a vortex solution for parameter values in the superconducting region. The latter has a non-zero order parameter in the core. For $\alpha \neq 0$ this relaxed configuration is stable due to a non-trivial topology. For $\alpha = 0$ there is no topological stability and the Nielsen-Olesen configuration spreads its flux to infinity [11].

Figure 1: The string-core structure for all values of parameters $\alpha, \beta$. Inside the region bounded by the solid line $\Phi_2 \neq 0$, while outside it $\Phi_2$ vanishes.

We use a fourth order Runge-Kutta routine with initial conditions $G(0) = 1, G'(0) = 0$ to investigate if there is a bound state in equation (11). If there is a $\rho_0$ such that for $\rho > \rho_0$ we have $G(\rho) < 0$ with the above initial condition, then there is clearly a bound state for the considered values of $\alpha, \beta$. This implies an instability of the embedded Nielsen-Olesen vortex towards another stable configuration with non-zero order parameter in the core. In Figure 1 we plot the parameter space $(\alpha, \beta)$ and display the regions of stability and instability of the embedded Nielsen-Olesen vortex. We have repeated the above analysis to the $|n| = 2$ case. As shown in Figure 1 increasing $|n|$ leads to an expansion of the superconducting region. This is related to the fact that the behaviour of $f$ near zero is $f \sim r^{|n|}$. Thus increasing $|n|$ leads to a broadening of the potential of the Schroedinger operator $\hat{O}$ and favours the existence of negative eigenvalues.

Figures 2 and 3 contain the results of the numerical integration via a relaxation method of the system (7).

Figure 2: For $\alpha = 0.1$ and $\beta = 1.3$ the stable string configuration shown here is the embedded Nielsen-Olesen solution.

Figure 3: The profile of the stable string for $\alpha = 0.1$ and $\beta = 0.6$. It has $\Phi_2 \neq 0$ inside the core.
To leading order in the temperature $T$ the parameter $\alpha$ becomes $T$-dependent $\alpha^2(T) = -m^2/[\pi^2 M^4 + (2\lambda + g^2)T^2/4]$, while $\beta$ is constant. Strings form at $T$ slightly below the critical temperature $T_C$. $\alpha^2(T \sim T_C)$ is very large and the strings start off normal. When $T$ is such that $\alpha(T) = \alpha_{\text{crit}}(\beta)$ for the given $\beta$ the string becomes superconducting.

III. STABILITY OF THE EMBEDDED MONOPOLE

It is straightforward to apply the above method to the study of the fate of global symmetries inside the monopole core. We choose to work in the context of the simple model

$$L = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \frac{1}{2} D_{\mu} \Phi^{a} D^{\mu} \Phi^{a} + \frac{1}{2} (\partial_{\mu} \Phi^{4})^2 - V$$  (12)

describing the dynamics of an $O(3)$ gauge field $A_{\mu}^{a}$ coupled to the scalar triplet $\Phi^{a}$ $a=1,2,3$, which in addition interacts with the gauge singlet $\Phi^{4}$ through the potential

$$V = \frac{\lambda}{4} (\Phi^{a} \Phi^{a} + \Phi^{4} \Phi^{4} - \nu^2)^2 - \frac{m^2}{2} (\Phi^{a} \Phi^{a} - \Phi^{4} \Phi^{4})$$  (13)

The field strength and the covariant derivative are given by $F_{\mu \nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g e^{abc} A_{\mu}^b A_{\nu}^c$ and $D_{\mu} \Phi^{a} = \partial_{\mu} \Phi^{a} + g e^{abc} A_{\mu}^b \Phi^{c}$, respectively.

The above is a simple extension of the Georgi-Glashow $O(3)$ model with the two classically relevant parameters

$$\tilde{\alpha} = \frac{m}{\sqrt{\lambda}v} \quad \text{and} \quad \tilde{\beta} = \frac{g}{\sqrt{\lambda}}$$  (14)

as revealed after the rescaling $x^{i} \rightarrow x^{i}/(v \sqrt{\lambda})$, $\Phi^{a} \rightarrow v \Phi^{a}$, $\Phi^{4} \rightarrow v \Phi^{4}$, and $A_{\mu}^{a} \rightarrow v A_{\mu}^{a}$.

For $\tilde{\alpha} = \tilde{\beta} = 0$ $\{12\}$ possesses an $O(4)$ global symmetry. A non-zero $m$ breaks this symmetry explicitly to the gauged $O(3)$ subgroup, and this in turn is spontaneously broken to $O(2)$ by the vacuum of the model. According to the standard lore the model admits for generic values of $\tilde{\alpha}$ and $\tilde{\beta}$ a whole tower of topologically stable magnetic monopoles. The minimal one in its "ground state" has the spherically symmetric form

$$\Phi^{a} = \delta_{ia} \frac{x^{i}}{r} f(r), \quad A_{i}^{a} = \epsilon_{aij} \frac{x^{j}}{r} W(r)$$  (15)

$$\Phi^{4} = G(r)$$

with $f(r)$ and $W(r)$ necessarily vanishing at the origin $r = 0$. Depending on the profile of $G(r)$, which as in the string case can only be decided by solving the field equations, the symmetry inside the monopole core will be either $O(3)$ ($G(r) \neq 0$) or the full $O(4)$ ($G(r) = 0$). Which one of the two is realized depends on the values of the parameters of the model. Using the same numerical method as in the string case we give below a complete map of the parameter space based on the core symmetry of the corresponding magnetic monopole.

It is convenient to define $K(r) \equiv 1 - \beta r W(r)$, in which case the field equations for the three unknown functions of the ansatz take the form:

$$f'' + \frac{2f''}{r} - \frac{2f'}{r^2} K^2 + (1 + \tilde{\alpha}^2 - f^2 - G^2) f = 0$$
$$K'' - \frac{K(K^2 - 1)}{r^2} - \tilde{\beta}^2 f^2 K = 0$$  (16)
$$G'' + \frac{2G'}{r} + (1 - \tilde{\alpha}^2 - f^2 - G^2) G = 0$$

while the corresponding boundary conditions, dictated by the finiteness of the energy and the field equations at the origin, are

$$f(\infty) = \sqrt{1 + \tilde{\alpha}^2}, \quad G(\infty) = 0, \quad K(\infty) = 0$$
$$f(0) = 0, \quad G'(0) = 0, \quad K(0) = 1$$  (17)

Figure 4: The dependence on the parameters $\tilde{\alpha}$ and $\tilde{\beta}$ of the monopole-core global symmetry.

We now look for instability modes of the embedded 't Hooft-Polyakov monopole solution (i.e. with $G(r) = 0$ at all $r$) of equations (16). Using the same approach as in the case of the Nielsen-Olesen vortex we obtain the
linearized eigenvalue problem corresponding to the last equation in (16)

\[-G'' - \frac{2G'}{r} + (f^2 - 1 + \tilde{\alpha}^2)G = \omega^2 G\]  

(18)

where \( f \) is obtained from the 't Hooft-Polyakov monopole solution i.e. solving the system of the first two equations in (16) with \( G = 0 \). Notice that contrary to the string case there is no coupling of the gauge field to \( G \). In models where such coupling exists and the gauge symmetry is \( O(4) \) the embedded 't Hooft-Polyakov monopole is always unstable due to the Brandt-Neri-Coleman instability \([13]\). This instability is realized as a screening of the long range magnetic field of the monopole by the other massless gauge fields of the theory \([14]\). No such instability exists in the model under discussion because the gauge symmetry is \( O(3) \) and not \( O(4) \).

We have solved the eigenvalue problem (18) for several parameter pairs \((\tilde{\alpha}, \tilde{\beta})\). The parameter space shown in Figure 4 is divided into two regions according to the symmetry properties of the core of the stable monopole solution. For strong scalar and gauge fibrations i.e. for \( \tilde{\alpha}, \tilde{\beta} \) large, the core symmetry of the stable monopole solution is \( O(4) \). Negative modes (instabilities) towards an \( O(3) \) symmetric core develop for weak fibrations as in the case of the embedded Nielsen-Olesen vortex. The embedded 't Hooft-Polyakov monopole in the semilocal limit was first discussed in \([5]\). Our results confirm the qualitative analysis presented there.

We solved numerically equations (16) and (17) using a relaxation method for a variety of parameter values. In Figure 5 an example of a stable embedded 't Hooft-Polyakov monopole solution is shown, while in Figure 6 we plot the profile of a stable monopole solution with \( O(3) \) symmetric core.

\[f(r)\]
\[K(r)\]
\[G(r)\]

Figure 5: For \( \tilde{\alpha} = 0.2 \) and \( \tilde{\beta} = 0.6 \) the stable monopole is the embedded 't Hooft-Polyakov solution with \( G(r)=0 \).

\[f(r)\]
\[K(r)\]
\[G(r)\]

Figure 6: The stable monopole shown here for \( \tilde{\alpha} = 0.2 \) and \( \tilde{\beta} = 0.1 \) has \( \Phi^4 \) excited inside the core.

**IV. CONCLUSION**

The dependence of the core phase structure of flux vortices and magnetic monopoles was studied explicitly in the context of simple gauge models carrying these topological defects. The combination of "scalar" and "gauge" fibrations on the vacuum manifold leads to instabilities of the core of the embedded Nielsen-Olesen vortices and 't Hooft-Polyakov monopoles, which result in a non-trivial interior in the stable solution. The models discussed here are very simple and allow for only two phases in the corresponding defect interiors. A larger variety of core phases is of course expected in realistic models with more Higgs multiplets and richer symmetry pattern. The cosmological implications of the transitions between these different core possibilities deserve further study.

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