COMPLEX LAGRANGIAN EMBEDDINGS OF
MODULI SPACES OF VECTOR BUNDLES

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Abstract. By means of a Fourier-Mukai transform we embed moduli spaces $\mathcal{M}_C(r,d)$ of stable bundles on an algebraic curve $C$ of genus $g(C) \geq 2$ as isotropic subvarieties of moduli spaces of $\mu$-stable bundles on the Jacobian variety $J(C)$. When $g(C) = 2$ this provides new examples of special Lagrangian submanifolds.

1. Introduction

Throughout this paper we shall fix $\mathbb{C}$ as the ground field. Let $C$ be a smooth algebraic curve of genus $g > 1$, and denote by $J(C)$ its Jacobian variety and by $\Theta \in H^2(J(C), \mathbb{Z})$ the cohomology class corresponding to the theta divisor. Fix coprime positive integers $r, d$ such that $d > 2rg$, and let $\mathcal{M}_C(r,d)$ be the moduli space of stable vector bundles on $C$ of Chern character $(r,d)$. We show that $\mathcal{M}_C(r,d)$ can be embedded as an isotropic holomorphic submanifold of the complex symplectic variety $\mathcal{M}_J^\mu(r,d) = \mathcal{M}_{J(C)}^\mu(d + r(1 - g), -r\Theta, 0, \ldots, 0)$ — the moduli space of $\mu$-stable vector bundles on $J(C)$ with Chern character $(d + r(1 - g), -r\Theta, 0, \ldots, 0)$ (cf. Theorem 2.1 for a precise statement). When $g(C) = 2$ one has $\dim \mathcal{M}_J^\mu(r,d) = 2 \dim \mathcal{M}_C(r,d)$, and by using the hyper-Kähler structure of $\mathcal{M}_J^\mu(r,d)$, one can choose on this space a complex structure such that $\mathcal{M}_C(r,d)$ embeds as a special Lagrangian submanifold, thus providing new examples of such objects.
We recall a few facts about the Fourier-Mukai transform in the context of Abelian varieties \[8\]. Let \(X\) be an Abelian variety and \(\hat{X} = \text{Pic}^0(X)\) its dual variety. Let \(\mathcal{P}\) be the normalized Poincaré bundle on \(X \times \hat{X}\). The Mukai functor is defined as

\[
\mathbb{R}S: D(X) \to D(\hat{X})
\]

\[
\mathbb{R}S(-) = \mathbb{R}\pi_{\hat{X}}^* (\pi_X^*(-) \otimes \mathcal{P})
\]

where \(D(X)\) and \(D(\hat{X})\) are the bounded derived categories of coherent sheaves on \(X\) and \(\hat{X}\), respectively. Mukai has shown that the functor \(\mathbb{R}S\) is invertible and preserves families of sheaves (cf. \([8, 10]\)). If \(E\) is a \(\text{WIT}_i\) sheaf on \(X\), that is, a sheaf whose transform is concentrated in degree \(i\), then the functor \(\mathbb{R}S\) preserves the Ext groups:

\[
\text{Ext}^j_X(E, E) \cong \text{Ext}^j_{\hat{X}}(\hat{E}, \hat{E}) \quad \text{for every } j,
\]

where \(\hat{E}\) indicates the transform of \(E\).

Let \(C\) be a smooth projective curve of genus \(g > 1\) and \(J(C)\) the Jacobian of \(C\). If we fix a base point \(x_0\) on \(C\), and let \(\alpha_{x_0}: C \to J(C)\) be the Abel-Jacobi embedding given by \(\alpha_{x_0}(x) = \mathcal{O}_C(x - x_0)\), the normalized Poincaré bundle \(\mathcal{P}_C\) on \(C \times J(C)\) is the pullback of the Poincaré bundle on \(J(C) \times J(C)\), where we identify \(J(C)\) with \(\hat{J}(C)\) via the isomorphism \(-\phi_{\Theta}: J(C) \to \hat{J}(C)\). The Poincaré bundle on \(C \times J(C)\) gives rise to a derived functor (which is not invertible):

\[
\mathbb{R}\Phi_C: D(C) \to D(J(C))
\]

\[
\mathbb{R}\Phi_C(-) = \mathbb{R}\pi_{J(C)}^* (\pi_C^*(-) \otimes \mathcal{P}_C).
\]

Since \(\alpha_{x_0}\) is a closed immersion we have a natural isomorphism of functors

\[
\mathbb{R}\Phi_C \cong \mathbb{R}S \circ \alpha_{x_0,*}.
\]

Thus the study of the transforms of bundles \(F\) on \(C\) with respect to \(\mathbb{R}\Phi_C\) is equivalent to studying the transforms of sheaves of pure dimension 1 of the form \(\alpha_{x_0,*}(F)\) with respect to \(\mathbb{R}S\). We recall the following fact which is proven in \([1]\).

**Proposition 1.1.** If \(E\) is a stable bundle on \(C\) of rank \(r\) and degree \(d\) such that \(d > 2rg\), then \(E\) is \(\text{WIT}_0\), and the transformed sheaf \(\hat{E} = \)
\( R^0 \Phi C(E) \) is locally free and \( \mu \)-stable with respect to the theta divisor on \( J(C) \).

2. Complex Lagrangian embeddings

If we consider the moduli space \( M_C(r, d) \) of stable bundles of rank \( r \) and degree \( d \) on a projective smooth curve of genus \( g > 1 \) such that \( d > 2rg \) and \( r, d \) are coprime, the functor \( R\Phi C \) gives rise to an injective morphism

\[
j: M_C(r, d) \rightarrow M^\mu_{J(C)}(r, d) = M^\mu_{J(C)}(d + r(1 - g), -r, 0, \ldots, 0)
\]

where the sheaves in \( M^\mu_{J(C)}(r, d) \) are stable with respect to the polarization \( \Theta \).

Before studying the morphism \( j \) we need to recall some elementary facts about the Yoneda product of Ext groups. Let \( A \) be an abelian category with enough injectives. The elements of \( \text{Ext}^1_A(E, E) \) are identified with equivalence classes of exact sequences

\[
0 \rightarrow E \rightarrow F \rightarrow E \rightarrow 0
\]

with respect to the usual relation. This can be generalized to the groups \( \text{Ext}^2_A(E, E) \) as follows. We refer to [2] for proofs and details.

Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
E & : & 0 & \rightarrow & B & \rightarrow & G_1 & \rightarrow & G_2 & \rightarrow & A & \rightarrow & 0 \\
E' & : & 0 & \rightarrow & B & \rightarrow & G_1' & \rightarrow & G_2' & \rightarrow & A & \rightarrow & 0.
\end{array}
\]

We write \( E \rightarrow E' \) when such a diagram holds. The relation \( \rightarrow \) is not symmetric, but it generates the following equivalence relation: \( E \sim E' \) if and only if there exists a chain of sequences \( E_0, E_1, \ldots, E_k \) such that

\[
E = E_0 \rightarrow E_1 \leftarrow E_2 \rightarrow \cdots \leftarrow E_k = E'.
\]

Let \( \text{Yext}^2_A(\_ , \_ ) \) the set of such equivalence classes.

There is an isomorphism

\[
\text{Yext}^2_A(\_ , \_ ) \cong \text{Ext}^2_A(\_ , \_ ).
\]
From now on we shall identify the above groups. Observe that the identity of \( \text{Ext}_A^2(A, B) \) is given by the class of the sequence

\[
0 \rightarrow B \xrightarrow{\text{Id}_B} B \xrightarrow{0} A \xrightarrow{\text{Id}_A} A \rightarrow 0.
\]

Moreover the Yoneda product

\[
\text{Ext}_A^1(B, A) \times \text{Ext}_A^1(A, C) \rightarrow \text{Ext}_A^2(B, C)
\]

is obtained in the following way: let \( E \) and \( E' \) be two elements of \( \text{Ext}_A^1(B, A) \) and \( \text{Ext}_A^1(A, C) \) represented respectively by the sequences

\[
E : \quad 0 \rightarrow A \xrightarrow{\nu} F \xrightarrow{p} B \rightarrow 0
\]

\[
E' : \quad 0 \rightarrow C \xrightarrow{i} G \xrightarrow{\lambda} A \rightarrow 0.
\]

Then the class of the exact sequence

\[
0 \rightarrow C \xrightarrow{i} G \xrightarrow{\nu \circ \lambda} F \xrightarrow{p} B \rightarrow 0
\]

in \( \text{Ext}_A^2(B, C) \) is the image of \( E, E' \) with respect to the Yoneda product.

We shall also need to introduce a moduli space of stable sheaves in Simpson's sense \[13\]. For simplicity we denote the Abel-Jacobi map as \( j : C \rightarrow J(C) \). Observe that if \( E \) is a stable bundle on \( C \) then \( j_*(E) \) is a stable sheaf of pure dimension 1 on \( J(C) \) with respect to the polarization \( \Theta \). Let \( \mathcal{M}^{\text{pure}}_{J(C)}(r, d) \) be the moduli space of all stable pure sheaves on \( J(C) \) with Chern character \((0, \ldots, 0, r\Theta, d + r(1 - g))\). If \( E \) is a flat family of vector bundles on \( C \) parametrized by a Noetherian scheme \( S \), then \( j_{S, *}(E) \) is a flat family of sheaves on \( J(C) \times S \) over \( S \), where \( j_S : C \times S \rightarrow J(C) \times S \) is the embedding \( j \times \text{Id}_S \). Therefore one has a morphism of moduli spaces

\[
(3) \quad j_* : \mathcal{M}(r, d) \rightarrow \mathcal{M}^{\text{pure}}_{J(C)}(r, d).
\]

**Lemma 2.1.** The morphism \( \tilde{j} : \mathcal{M}_C(r, d) \rightarrow \mathcal{M}^{\mu}_{J(C)}(r, d) \) is an immersion (i.e., its tangent map is injective).

**Proof.** From the isomorphism given by Eq. (1) and recalling that the transform \( RS \) preserves the Ext groups of WIT sheaves, it is enough to show that the same claim holds for the morphism (3). By the very
definition of the Kodaira-Spencer map, the tangent map to \( j_* \) may be identified with the map
\[
\Ext^1_C(E, E) \xrightarrow{\phi} \Ext^1_{j_*(C)}(j_*(E), j_*(E))
\]
obtained in the following way. Let
\[
A : 0 \to E \to F \to E \to 0
\]
be a sequence representing an element of \( \Ext^1_C(E, E) \). If we apply the functor \( j_* \) to the above sequence we obtain the exact sequence
\[
B : 0 \to j_*(E) \to j_*(F) \to j_*(E) \to 0.
\]
One checks immediately that the map \( \phi([A]) = [B] \) is well defined. If \( \phi([A]) = 0 \) then \( \phi([A]) \) is represented by the extension
\[
0 \to j_*(E) \to j_*(E) \oplus j_*(E) \to j_*(E) \to 0.
\]
Now applying the functor \( j^* \) to the above sequence and noting that \( j^*(j_*(H)) \cong H \) for every vector bundle \( H \) on \( C \) we obtain the split exact sequence
\[
0 \to E \to E \oplus E \to E \to 0.
\]
Therefore \( \phi([A]) = 0 \) implies \([A] = 0\) and \( \phi \) is injective.

Mukai proved that the moduli space of simple sheaves on an abelian surface \( X \) is symplectic; more precisely, the Yoneda pairing
\[
v : \Ext^1_X(E, E) \times \Ext^1_X(E, E) \to \Ext^2_X(E, E) \cong \mathbb{C}
\]
defines a holomorphic symplectic form on the moduli of simple sheaves on \( X \) (cf. [3, 11]). When \( \dim X = 2n > 2 \) to define a symplectic form on the smooth locus of the moduli space one needs to choose a symplectic form \( \omega \) on \( X \). The symplectic form is then defined by the compositions (cf. [3])
\[
\Ext^1_X(E, E) \times \Ext^1_X(E, E) \to \Ext^2_X(E, E) \xrightarrow{\text{tr}} H^2(X, \mathcal{O}_X) \xrightarrow{\lambda} H^{0,2}(X, \mathbb{C}) \cong \mathbb{C}
\]
where \( \text{tr} \) is the trace morphisms and the map \( \lambda \) is obtained by wedging by \( \omega^n \wedge \bar{\omega}^{n-1} \).
Theorem 2.1. If $g(C)$ is even, and the map $\bar{j}$ embeds $\mathcal{M}(r, d)$ into the smooth locus $\mathcal{M}^0_{J(C)}(r, d)$ of $\mathcal{M}^\mu_{J(C)}(r, d)$, then the subvarieties $\mathcal{M}_C(r, d)$ are isotropic with respect to any of the symplectic forms defined by equation (8). In particular, when $g(C) = 2$ the subvarieties $\mathcal{M}_C(r, d)$ are Lagrangian with respect to the Mukai form of $\mathcal{M}^\mu_{J(C)}(r, d)$.

Proof. Since $\mathcal{M}^0_{J(C)}(r, d)$ is smooth, and $\bar{j} : \mathcal{M}(r, d) \to \mathcal{M}^0_{J(C)}(r, d)$ is injective and is an immersion, it is also an embedding. Now, let $E \in \mathcal{M}_C(r, d)$. It is enough to show that the Yoneda product

$$\text{Ext}^1_{J(C)}(j_*(E), j_*(E)) \times \text{Ext}^1_{J(C)}(j_*(E), j_*(E)) \to \text{Ext}^2_{J(C)}(j_*(E), j_*(E))$$

vanishes when applied to pairs $([A], [B])$ of elements in $\text{Ext}^1_{J(C)}(j_*(E), j_*(E))$ where $[A]$ and $[B]$ are represented, respectively, by the sequences

$$A : \quad 0 \to j_*(E) \overset{\nu}{\to} j_*(F) \overset{p}{\to} j_*(E) \to 0$$

$$B : \quad 0 \to j_*(E) \overset{i}{\to} j_*(G) \overset{\lambda}{\to} j_*(E) \to 0$$

with $F, G \in \mathcal{M}_C(r, d)$. It is enough to remark that the product of the classes of the sequences of sheaves on $C$

$$0 \to E \to F \to E \to 0, \quad 0 \to E \to G \to E \to 0$$

is zero for dimensional reasons, and apply the functor $j_*$.

In the case $g(C) = 2$ the moduli space is smooth by the results in [9]; moreover,

$$\dim \mathcal{M}^\mu_{J(C)}(r, d) = 2(r^2 + 1) = 2 \dim \mathcal{M}_C(r, d).$$

Remark 3. If we consider the moduli space $\mathcal{M}_C(r, \xi)$ of stable bundles on $C$ of rank $r$ and fixed determinant isomorphic to $\xi$, then the result is trivial: the variety $\mathcal{M}_C(r, \xi)$ is Fano, so that it carries no holomorphic forms. ▲
4. The case $g(C) = 2$

In this section we elaborate on the case $g(C) = 2$. One can characterize situations where the moduli space $\mathcal{M}^μ_{J(C)}(r, d)$ is compact. This happens for instance in the following case.

**Proposition 4.1.** Assume $g(C) = 2$, $d > 4r$ and that $\rho = d - r$ is a prime number. Then every Gieseker-semistable sheaf on $J(C)$ with Chern character $(d - r, -r\Theta, 0)$ is $\mu$-stable. Moreover, if $d > r^2 + r$, every such sheaf is locally free (this always happens when $r = 1, 2, 3$).

**Proof.** Since $d - r$ is prime, every sheaf in $\mathcal{M}_{J(C)}(r, d)$ is properly stable. Let $[F] \in \mathcal{M}_{J(C)}(r, d)$ and assume that the subsheaf $G$ destabilizes $F$. Let $\text{ch}(G) = (\sigma, \xi, s)$. Standard computations show that if $F$ is not $\mu$-stable then

$$\frac{\xi \cdot \Theta}{\sigma} = -\frac{2r}{\rho} \quad \text{and} \quad s < 0.$$  

Setting $n = \xi \cdot \Theta$ we have $|n| = 2r\sigma/\rho$, with $\sigma < \rho$ and $\rho > 3r$. This is impossible whenever $\rho$ is prime.

The statement about local freeness follows from the Bogomolov inequality.

In the case $g(C) = 2$ the complex Lagrangian embedding $\tilde{j}: \mathcal{M}_C(r, d) \to \mathcal{M}^μ_{J(C)}(r, d)$ provides new examples of special Lagrangian submanifolds. We refer to [1, 7] for the definition and the main properties of these objects. Now, if $X$ is a hyper-Kähler manifold of complex dimension $2n$, let $I, J, K$ be three complex structures compatible with the hyper-Kähler metric, and such that $IJ = K$. Let $\omega_I, \omega_J, \omega_K$ be the corresponding Kähler forms. Then the 2-form $\Omega = \omega_I + i\omega_J$ is a holomorphic symplectic form in the complex structure $K$. It is easy to check that a $K$-complex $n$-dimensional submanifold which is Lagrangian with respect to $\Omega$ is special Lagrangian in the structure $J$ [3].

One should notice that via the Hitchin-Kobayashi correspondence (which identifies $\mu$-stable bundles on a Kähler manifold with irreducible Einstein-Hermite bundles, cf. [4]), the space $\mathcal{M}^μ_{J(C)}(r, d)$ acquires a hyper-Kähler structure, compatible with a Kähler form provided by
the Weil-Petersson metric, and with a holomorphic symplectic form which may be identified with the Mukai form \[4\].

Therefore we obtain the following result.

**Proposition 4.2.** The space \(\mathcal{M}_{\mu J}(C)(r, d)\) has a complex structure such that \(\tilde{j}: \mathcal{M}_C(r, d) \to \mathcal{M}_{\mu J}(C)(r, d)\) is a special Lagrangian submanifold.

The elements of the Jacobian variety \(J(C)\) act on the embedding \(j: C \to J(C)\) by translation, so that for every \(x \in J(C)\) we have a special Lagrangian submanifold \(\tilde{j}_x: \mathcal{M}_C(r, d) \to \mathcal{M}_{\mu J}(C)(r, d)\). This provides a family of deformations of \(\tilde{j}(\mathcal{M}_C(r, d))\) through special Lagrangian submanifolds. As one easily shows, this embeds \(J(C)\) into the moduli space \(\mathcal{M}_{SL}\) of special Lagrangian deformations of \(\tilde{j}(\mathcal{M}_C(r, d))\) (notice that \(\dim_\mathbb{R} \mathcal{M}_{SL} = b_1(\mathcal{M}_C(r, d)) = 4 = \dim_\mathbb{R}(J(C))\)) \[12\]. The case \(r = 1\) is somehow trivial because \(\mathcal{M}_{\mu J}(C)(1, d) \simeq J(C) \times J(C)\) by a result of Mukai \[8\].

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