First-order phase transition in 1d Potts model with long-range interactions

K Uzelac and Z Glumac

Institute of Physics, Bijenička 46, POB 304, 10000 Zagreb, Croatia

Abstract

The first-order phase transition in the one-dimensional $q$-state Potts model with long-range interactions decaying with distance as $1/r^{1+\sigma}$ has been studied by Monte Carlo numerical simulations for $0 < \sigma \leq 1$ and integer values of $q > 2$. On the basis of finite-size scaling analysis of interface free energy $\Delta F_L$, specific heat and Binder’s fourth order cumulant, we obtain the first-order transition which occurs for $\sigma$ below a threshold value $\sigma_c(q)$.

Physics Abstracts classification number: 05.50.+q, 64.60.Cn
The subject of our study is the one-dimensional (1d) Potts model with ferromagnetic long-range (LR) interactions decaying with distance as $1/r^{1+\sigma}$, defined by the Hamiltonian

$$H = -\sum_{i<j} \frac{J}{|i-j|^{1+\sigma}} \delta(s_i, s_j),$$

where $J > 0$, $s_i$ denotes the $q$-state Potts variable at site $i$, $\delta$ is Kronecker symbol, and summation is taken over all pairs in the system. The phase transition at nonzero temperature, shown rigorously \cite{1} for the Ising ($q = 2$) case with $\sigma \leq 1$ and by renormalization group for the continuous $n$-component models with $\sigma < 1$ \cite{2}, exists also in model (1) for $\sigma \leq 1$ and $q > 0$ \cite{3,4} and goes through a variety of universality classes by variation of $q$ and $\sigma$. Model (1) has been used as a relevant model for describing a number of phenomena involving LR interactions, from spin glasses to neural networks, but may also be of interest for possible analogies \cite{5}, in some cases very direct \cite{6}, with short-range (SR) models in higher dimensions.

An important feature of Potts models with SR interactions is the onset of the first-order phase transition for $q$ above some threshold value $q_c(d)$, which depends on dimensionality \cite{7}. For example, in $d = 2$ \cite{8} and $d = 4$ \cite{9} analytical results yield $q_c$ equal to 4 and 2 respectively, while for $d = 3$, the approximate methods give a non-integer value for $q_c$, slightly lower than 3 \cite{10}, so that the 3d 3-state Potts model has an extremely weak first-order transition, very difficult to detect \cite{11}. Such a distinct situations created by variation of $d$ and $q$ make this model a canonical example for study of various aspects of temperature-driven first-order phase transitions \cite{12}.

One can expect similar behaviour to occur for the Potts model with LR interactions, although certain related quantities (e.g. the interface energy) may require different interpretation. Generally, this model has been much less explored than its SR version, due to non-locality of interactions, which makes difficult the application of standard renormalization group techniques in direct space, otherwise appropriate for discrete models. A few studies that have been done concern mostly the mean-field region and its vicinity by using an $\epsilon$-expansion within Ginzburg-Landau continuous formalism \cite{13}, and the special case of $\sigma = 1$ \cite{3}. We have recently proposed
the finite-range scaling (FRS) approach [14] suitable for study of the Hamiltonian (1), with both $\sigma$ and $q$ arbitrary and continuous [4]. This approach is, however, inherently insensitive to discern the first-order transitions and remained inconclusive with this respect. The problem was not resolved in other recent works [15, 16] on the LR Potts model either.

For this reason we present here the results based on simple numerical simulations performed with intention to examine the existence of a first-order transition in this model and get a qualitative estimate of its dependence on $q$ and $\sigma$.

It has been recently pointed out [17], that the temperature-driven first-order transitions can be identified from finite-size scaling (FSS) analysis of maxima of the energy probability distribution

$$P_L(E) = \frac{1}{Z_L(K)} N_L(E) e^{-KE},$$

where $K = J/k_B T$, $Z_L(K)$ is the partition function, $N_L(E)$ is the number of configurations with energy $E$, and index $L$ denotes the system size. Due to coexistence of phases, at the temperature of a first-order phase transition $P_L(E)$ has two maxima, corresponding to the two wells in the free energy. The barrier separating them, which represents the interface free energy is defined by

$$\Delta F_L = \ln \left. \frac{P_L(E_{\min})}{P_L(E_{\max})} \right|_{K_L},$$

where $E_{\min}$ and $E_{\max}$ denote the energies corresponding to the minimum and one of the two maxima respectively, the finite-size temperature $K_L$ being adjusted so as to make the two maxima equal. For a first-order phase transition $\Delta F_L$ should diverge with $L$. In systems with SR interactions it scales like a surface, i.e. $\sim L^{d-1}$, while in the present case it is expected to scale rather like a volume, i.e. $\sim L$.

The calculations were performed on chains of size $100 \leq L \leq 400$ with periodic boundary conditions. We have used the simple Metropolis single-spin-flip algorithm with $1 \times 10^6 - 3 \times 10^6$ Monte Carlo (MC) sweeps per spin. The number of necessary runs for the precise localization of each of the size-dependent critical temperatures $K_L$ has been reduced by applying the Ferrenberg and Swendsen [18] histogram method.  

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We have considered integer values of \( q \geq 2 \) in the interval \( 0 < \sigma \leq 1 \). While for \( q = 2 \) (Ising case), the simulations at \( T_c \) show only a single maximum in \( P_L(E) \) in the entire range of \( \sigma \), for higher values of \( q \), the two peaks emerge for \( \sigma \) sufficiently low. They become more pronounced with increasing \( q \) or decreasing \( \sigma \). For illustration, in Figure 1 are shown the maxima corresponding to different values of \( \sigma \), taken from three typical sets of simulations with fixed \( q = 3 \) and \( L = 400 \).

We report here the systematic results for two chosen values, \( q = 3 \) and \( q = 5 \), in the whole interval \( 0 < \sigma \leq 1 \) taken with increment 0.1.

In Figures 2a and b are summarized the results for the free energy barrier plotted as a function of chain size \( L \) for \( q = 3 \) and \( q = 5 \) respectively. The corresponding critical temperatures extrapolated to \( L \to \infty \), given in table 1, are found to be in good agreement with our earlier FRS results [4], as well as with other known approximate results [16].

| \( \sigma \) | \( q \) | \( K_{e}^{MC} \) | \( K_{e}^{FRS} \) | \( K_{e}^{MC} \) | \( K_{e}^{FRS} \) |
|------------|------|-------------|-------------|-------------|-------------|
| 0.1        | 3    | 0.190       | 0.136       | 5           | 0.262       | 0.28        |
| 0.2        |      | 0.279       | 0.270       | 0.333       | 0.45        |
| 0.3        |      | 0.380       | 0.386       | 0.492       | 0.576       |
| 0.4        |      | 0.489       | 0.494       | 0.637       | 0.690       |
| 0.5        |      |             |             | 0.771       | 0.803       |
| 0.6        |      |             |             | 0.901       | 0.920       |
| 0.7        |      |             |             | 1.019       | 1.046       |

Table 1: Inverse critical temperatures (\( K_{e}^{MC} \)), obtained by extrapolation of \( K_{e} \) compared to FRS extrapolated values (\( K_{e}^{FRS} \)) [4].

For both considered values of \( q \), there is a wide range of \( \sigma \), where \( \Delta F_L \) increases with size, indicating the first-order transition. As expected for the LR interactions, \( \Delta F_L \) is proportional to volume rather than surface and depends linearly on \( L \). The slope is larger by an order of magnitude for \( q = 5 \) in comparison to \( q = 3 \), show-
ing that the first-order character becomes stronger with increasing $q$, like for the SR interactions. In both cases the slope decreases with increasing $\sigma$ to the point where $\Delta F_L$ becomes of the order of error bars. Beyond this value, at least for sizes considered here, the $P_L(E)$ exhibits a single maximum indicating the onset of the second-order phase transition. In present calculation, taken with a rough increment of 0.1 in $\sigma$, this change is observed around $\sigma = 0.5$ and $\sigma = 0.8$ for $q = 3$ and $q = 5$, respectively. These values should be taken with caution and only as a lower limit for the threshold value $\sigma_c$ between the first- and the second-order transition. Namely, the present model allows the continuous approach to the threshold value $\sigma_c$, whereby the first-order transition becomes arbitrarily weak and very difficult to detect, comparable to the situation with the 3d 3-state Potts model with SR interactions. The above results, however, strongly suggest that $\sigma_c$ is considerably larger for $q = 5$ than for $q = 3$, and that dependence $\sigma_c(q)$ should be expected, analogous to threshold dependence $d_c(q)$ in the SR model.

Two other energy-related quantities are more conventionally [13] used for determination of the first-order transition in context of FSS analysis of MC simulation results: specific heat and Binder’s fourth order cumulant [20], which both can be derived from $P_L(E)$, and expressed in terms of higher energy momenta $\langle E^n \rangle_L = \sum_{E^n} E^n P_L(E)$. Specific heat is given by

$$C_L = \frac{K^2}{L^d} \left( \langle E^2 \rangle_L - \langle E \rangle_L^2 \right).$$  

According to the FSS theory, for second-order transitions its maximum scales as $C_L^{\text{max}} \sim L^{\alpha/\nu}$, where $\alpha$ and $\nu$ are the critical exponents of the specific heat and the correlation length respectively. When the transition is of the first order, it scales as a volume, i.e. $C_L^{\text{max}} \sim L^d$. Instead of the Binder fourth cumulant $V_L^{(4)} = 1 - U_L^{(4)}/3$, we consider here the ratio

$$U_L^{(4)} = \frac{\langle E^4 \rangle_L}{\langle E^2 \rangle_L^2}. $$

For the first-order transitions, $\lim_{L \to \infty} U_L^{(4)} = 1$ when $T \neq T_c$, while at $T = T_c$ $\lim_{L \to \infty} U_L^{(4)} = \text{const} > 1$. For second-order transitions it always tends to one.
We present the behaviour of those two quantities on two examples: $q = 5$, $\sigma = 0.2$ and $q = 3$, $\sigma = 0.8$, representative of the first- and second-order regimes, respectively.

In Figure 3a and b one can observe two kinds of behaviour of $C_{L_{\text{max}}}^{\text{max}}$ in the two cases: linear and power law. The fit to the form $C_{L_{\text{max}}}^{\text{max}} \sim L^x$ in the latter case gives the value $x = 0.24$ for $q = 3, \sigma = 0.8$. The bare extrapolation error bars for $x$ are estimated to be of order of 10%. The hyper-scaling relation with substitution of FRS result $\nu = 1.74$ gives $\alpha/\nu = 0.15$. The difference can be attributed to the general difficulty in extracting the critical exponents from the specific heat, and to the fact that the calculated exponent is small and additional correction terms due to finite-size gain importance.

The convergence of the maxima of $U_L^{(4)}$ with size is presented in figure 2((c) and (d)). The points for the Ising model with $\sigma = 0.5$ are added as a reference for second-order transition behaviour. The case $q = 3, \sigma = 0.8$ shows, within numerical error bars, clear convergence towards 1. The points for $q = 5, \sigma = 0.2$ converge towards a much larger value, which is approximately 2.4 when we take into account larger values of $L$ and use the linear extrapolation.

Thus, the two quantities confirm earlier conclusions based upon the behaviour of $\Delta F_L$. However, at present stage, we could not extract from these quantities any better precision in determination of $\sigma_c(q)$, so we do not reproduce any systematic study for them.

In summary, by simple numerical calculations, in combination with FSS arguments, we have shown that the 1$d$ LR Potts model for integer $q > 2$ exhibits the first-order phase transition for $\sigma$ below some threshold value $\sigma_c$ generally depending on $q$. First-order character becomes weaker with the increase of $\sigma$, which represents a continuous parameter leading from first- to second-order phase transition regime. More intensive numerical approach should be needed in future in order to determine the threshold value $\sigma_c(q)$. 
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Figure captions:

Figure 1:

Maxima of $\ln P_L(E')$ for $q = 3, L = 400$ and $\sigma = 0.3, 0.4$ and 0.5, taken at respective values of $K_L$. $E' = E/|E_0(\sigma)|$, where the $E_0(\sigma)$ stay for the zero-temperature energies.

Figure 2:

MC results of the free energy barrier $\Delta F_L$ for sizes $L = 100$ to 400 for: (a) $q = 3, \sigma = 0.1$ to 0.4, (b) $q = 5, \sigma = 0.1$ to 0.7. Notice that the slope is by an order of magnitude larger in the case (b). The size of the numerical error bars is comparable to or smaller than the size of the points.

Figure 3:

MC data for sizes $L = 100$ to 400: specific heat maxima are plotted for (a) $q = 3, \sigma = 0.8$, (b) $q = 5, \sigma = 0.2$; ratio $U_L^{(4)}$ maxima are plotted in function of inverse size for: (c) $q = 3, \sigma = 0.8$, (d) $q = 5, \sigma = 0.2$ (full circles). The diamonds correspond to $q = 2, \sigma = 0.5$ taken as a reference. Lines show the linear extrapolations. Notice the common scale on x-axis, but different scales on y-axis. The size of the numerical error bars is smaller than the size of the points.
Fazni prijelaz prvog reda u 1d Pottsovom modelu s dugodosežnim medjudjelovanjem

K Uzelac i Z Glumac

_Institut za Fiziku, Bijenička 46, POB 304, 10000 Zagreb, Hrvatska_

Sažetak

U jednodimenzionalnom Pottsovom modelu s q stanja i s dugodosežnim medjudjelovanjima koja opadaju s udaljenosću kao 1/r^{1+\sigma}, Monte Carlo simulacijama je promatran fazni prijelaz prvog reda za 0 < \sigma \leq 1 i cjelobrojne vrijednosti q > 2. Na temelju scaling analize slobodne energije medjuplohe, specifične topline i Binderovog kumulanta četvrtog reda, dobivamo prijelaz prvoga reda za \sigma manji od granične vrijednosti \sigma_c(q).
\[
\ln P_L(E') = \left\{ \begin{array}{ll}
\sigma = 0.3 \\
\sigma = 0.4 \\
\sigma = 0.5 
\end{array} \right.
\]
(a) $q = 3$

$\Delta F_L$ vs. $L$ for different values of $\sigma$:
- $\sigma = 0.1$
- $\sigma = 0.2$
- $\sigma = 0.3$
- $\sigma = 0.4$

(b) $q = 5$

$\Delta F_L$ vs. $L$ for different values of $\sigma$:
- $\sigma = 0.1$
- $\sigma = 0.2$
- $\sigma = 0.3$
- $\sigma = 0.4$
- $\sigma = 0.5$
- $\sigma = 0.6$
- $\sigma = 0.7$
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(1)

where $J > 0$, $s_i$ denotes the $q$-state Potts variable at site $i$, $\delta$ is Kronecker symbol, and summation is taken over all pairs in the system. The phase transition at nonzero temperature, shown rigorously [1] for the Ising ($q = 2$) case with $\sigma \leq 1$ and by renormalization group for the continuous $n$-component models with $\sigma < 1$ [2], exists also in model (1) for $\sigma \leq 1$ and $q > 0$ [3, 4] and goes through a variety of universality classes by variation of $q$ and $\sigma$. Model (1) has been used as a relevant model for describing a number of phenomena involving LR interactions, from spin glasses to neural networks, but may also be of interest for possible analogies [5], in some cases very direct [6], with short-range (SR) models in higher dimensions.

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For this reason we present here the results based on simple numerical simulations performed with intention to examine the existence of a first-order transition in this model and get a qualitative estimate of its dependence on $q$ and $\sigma$.

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where $K = J/k_B T$, $Z_L(K)$ is the partition function, $\mathcal{N}_L(E)$ is the number of configurations with energy $E$, and index $L$ denotes the system size. Due to coexistence of phases, at the temperature of a first-order phase transition $P_L(E)$ has two maxima, corresponding to the two wells in the free energy. The barrier separating them, which represents the interface free energy is defined by

\[ \Delta F_L = \ln \left. \frac{P_L(E_{\text{min}})}{P_L(E_{\text{max}})} \right|_{K_L}, \]

where $E_{\text{min}}$ and $E_{\text{max}}$ denote the energies corresponding to the minimum and one of the two maxima respectively, the finite-size temperature $K_L$ being adjusted so as to make the two maxima equal. For a first-order phase transition $\Delta F_L$ should diverge with $L$. In systems with SR interactions it scales like a surface, i.e. $\sim L^{d-1}$, while in the present case it is expected to scale rather like a volume, i.e. $\sim L$.

The calculations were performed on chains of size $100 \leq L \leq 400$ with periodic boundary conditions. We have used the simple Metropolis single-spin-flip algorithm with $1 \times 10^6 - 3 \times 10^6$ Monte Carlo (MC) sweeps per spin. The number of necessary runs for the precise localization of each of the size-dependent critical temperatures $K_L$ has been reduced by applying the Ferrenberg and Swendsen [18] histogram method.
We have considered integer values of $q \geq 2$ in the interval $0 < \sigma \leq 1$. While for $q = 2$ (Ising case), the simulations at $T_c$ show only a single maximum in $P_L(E)$ in the entire range of $\sigma$, for higher values of $q$, the two peaks emerge for $\sigma$ sufficiently low. They become more pronounced with increasing $q$ or decreasing $\sigma$. For illustration, in Figure 1 are shown the maxima corresponding to different values of $\sigma$, taken from three typical sets of simulations with fixed $q = 3$ and $L = 400$.

We report here the systematic results for two chosen values, $q = 3$ and $q = 5$, in the whole interval $0 < \sigma \leq 1$ taken with increment 0.1.

In Figures 2a and b are summarized the results for the free energy barrier plotted as a function of chain size $L$ for $q = 3$ and $q = 5$ respectively. The corresponding critical temperatures extrapolated to $L \to \infty$, given in table 1, are found to be in good agreement with our earlier FRS results [4], as well as with other known approximate results [16].

| $\sigma$ | $q$ | $K_e^{MC}$ | $K_e^{FRS}$ | $q$ | $K_e^{MC}$ | $K_e^{FRS}$ |
|----------|-----|------------|------------|-----|------------|------------|
| 0.1      | 3   | 0.190      | 0.136      | 5   | 0.262      | 0.28       |
| 0.2      |     | 0.279      | 0.270      |      | 0.333      | 0.45       |
| 0.3      |     | 0.380      | 0.386      |      | 0.492      | 0.576      |
| 0.4      |     | 0.489      | 0.494      |      | 0.637      | 0.690      |
| 0.5      |     |            | 0.771      |      | 0.803      |            |
| 0.6      |     |            | 0.901      |      | 0.920      |            |
| 0.7      |     |            | 1.019      |      | 1.046      |            |

Table 1: Inverse critical temperatures ($K_e^{MC}$), obtained by extrapolation of $K_L$ compared to FRS extrapolated values ($K_e^{FRS}$) [4].

For both considered values of $q$, there is a wide range of $\sigma$, where $\Delta F_L$ increases with size, indicating the first-order transition. As expected for the LR interactions, $\Delta F_L$ is proportional to volume rather than surface and depends linearly on $L$. The slope is larger by an order of magnitude for $q = 5$ in comparison to $q = 3$, show-
ing that the first-order character becomes stronger with increasing \( q \), like for the SR interactions. In both cases the slope decreases with increasing \( \sigma \) to the point where \( \Delta F_L \) becomes of the order of error bars. Beyond this value, at least for sizes considered here, the \( P_L(E) \) exhibits a single maximum indicating the onset of the second-order phase transition. In present calculation, taken with a rough increment of 0.1 in \( \sigma \), this change is observed around \( \sigma = 0.5 \) and \( \sigma = 0.8 \) for \( q = 3 \) and \( q = 5 \), respectively. These values should be taken with caution and only as a lower limit for the threshold value \( \sigma_c \) between the first- and the second-order transition. Namely, the present model allows the continuous approach to the threshold value \( \sigma_c \), whereby the first-order transition becomes arbitrarily weak and very difficult to detect, comparable to the situation with the 3d 3-state Potts model with SR interactions. The above results, however, strongly suggest that \( \sigma_c \) is considerably larger for \( q = 5 \) than for \( q = 3 \), and that dependence \( \sigma_c(q) \) should be expected, analogous to threshold dependence \( d_c(q) \) in the SR model.

Two other energy-related quantities are more conventionally [19] used for determination of the first-order transition in context of FSS analysis of MC simulation results: specific heat and Binder’s fourth order cumulant [20], which both can be derived from \( P_L(E) \), and expressed in terms of higher energy momenta \( \langle E^n \rangle_L = \sum_E E^n P_L(E) \). Specific heat is given by

\[
C_L = \frac{K^2}{L^d} \left( \langle E^2 \rangle_L - \langle E \rangle_L^2 \right). \tag{4}
\]

According to the FSS theory, for second-order transitions its maximum scales as \( C_L^{max} \sim L^{\alpha/\nu} \), where \( \alpha \) and \( \nu \) are the critical exponents of the specific heat and the correlation length respectively. When the transition is of the first order, it scales as a volume, i.e. \( C_L^{max} \sim L^d \). Instead of the Binder fourth cumulant \( V_L^{(4)} = 1 - U_L^{(4)}/3 \), we consider here the ratio

\[
U_L^{(4)} = \frac{\langle E^4 \rangle_L}{\langle E^2 \rangle_L^2}. \tag{5}
\]

For the first-order transitions, \( \lim_{L \to \infty} U_L^{(4)} = 1 \) when \( T \neq T_c \), while at \( T = T_c \) \( \lim_{L \to \infty} U_L^{(4)} = const > 1 \). For second-order transitions it always tends to one.
We present the behaviour of those two quantities on two examples: \( q = 5, \sigma = 0.2 \) and \( q = 3, \sigma = 0.8 \), representative of the first- and second-order regimes, respectively.

In Figure 3a and b one can observe two kinds of behaviour of \( C_L^{\text{max}} \) in the two cases: linear and power law. The fit to the form \( C_L^{\text{max}} \sim L^x \) in the latter case gives the value \( x = 0.24 \) for \( q = 3, \sigma = 0.8 \). The bare extrapolation error bars for \( x \) are estimated to be of order of 10%. The hyper-scaling relation with substitution of FRS result [4] \( \nu = 1.74 \) gives \( \alpha/\nu = 0.15 \). The difference can be attributed to the general difficulty in extracting the critical exponents from the specific heat, and to the fact that the calculated exponent is small and additional correction terms due to finite-size gain importance.

The convergence of the maxima of \( U_L^{(4)} \) with size is presented in figure 2((c) and (d)). The points for the Ising model with \( \sigma = 0.5 \) are added as a reference for second-order transition behaviour. The case \( q = 3, \sigma = 0.8 \) shows, within numerical error bars, clear convergence towards 1. The points for \( q = 5, \sigma = 0.2 \) converge towards a much larger value, which is approximately 2.4 when we take into account larger values of \( L \) and use the linear extrapolation.

Thus, the two quantities confirm earlier conclusions based upon the behaviour of \( \Delta F_L \). However, at present stage, we could not extract from these quantities any better precision in determination of \( \sigma_c(q) \), so we do not reproduce any systematic study for them.

In summary, by simple numerical calculations, in combination with FSS arguments, we have shown that the 1d LR Potts model for integer \( q > 2 \) exhibits the first-order phase transition for \( \sigma \) below some threshold value \( \sigma_c \) generally depending on \( q \). First-order character becomes weaker with the increase of \( \sigma \), which represents a continuous parameter leading from first- to second-order phase transition regime. More intensive numerical approach [21] should be needed in future in order to determine the threshold value \( \sigma_c(q) \).
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Figure captions:

Figure 1:
Maxima of ln $P_L(E')$ for $q = 3$, $L = 400$ and $\sigma = 0.3, 0.4$ and $0.5$, taken at respective values of $K_L$. $E' = E / |E_0(\sigma)|$, where the $E_0(\sigma)$ stay for the zero-temperature energies.

Figure 2:
MC results of the free energy barrier $\Delta F_L$ for sizes $L = 100$ to $400$ for: (a) $q = 3, \sigma = 0.1$ to $0.4$, (b) $q = 5, \sigma = 0.1$ to $0.7$. Notice that the slope is by an order of magnitude larger in the case (b). The size of the numerical error bars is comparable to or smaller than the size of the points.

Figure 3:
MC data for sizes $L = 100$ to $400$: specific heat maxima are plotted for (a) $q = 3, \sigma = 0.8$, (b) $q = 5, \sigma = 0.2$; ratio $U_L^{(4)}$ maxima are plotted in function of inverse size for: (c) $q = 3, \sigma = 0.8$, (d) $q = 5, \sigma = 0.2$ (full circles). The diamonds correspond to $q = 2, \sigma = 0.5$ taken as a reference. Lines show the linear extrapolations. Notice the common scale on x-axis, but different scales on y-axis. The size of the numerical error bars is smaller than the size of the points.
Fazni prijelaz prvog reda u 1d Pottsovom modelu s dugodosežnim međudjelovanjem

K Uzelac i Z Glumac

Institut za Fiziku, Bijenička 46, POB 304, 10000 Zagreb, Hrvatska

Sažetak

U jednodimenzionalnom Pottsovom modelu s $q$ stanja i s dugodosežnim međudjelovanjima koja opadaju s udaljenošću kao $1/r^{1+\sigma}$, Monte Carlo simulacijama je promatran fazni prijelaz prvog reda za $0 < \sigma \leq 1$ i cjelobrojne vrijednosti $q > 2$. Na temelju scaling analize slobodne energije međuplohe, specifične topline i Binderovog kumulanta četvrtog reda, dobivamo prijelaz prvoga reda za $\sigma$ manji od granične vrijednosti $\sigma_c(q)$. 