We study the dynamical stripe instabilities on the geometries with hyperscaling violation in the IR, which asymptotically approach AdS$_4$ in the UV. The instabilities break the translational invariance spontaneously and are induced by the axion term $\sim a F \wedge F$ in the bulk action. We first study the perturbation equations in the probe limit, and find that there is a strong correlation between the stripe instabilities caused by the axion term and parameters of the theories which determine the IR hyperscaling violation. Contrary to the IR AdS$_2$ case, the effect of the axion term for the stripe instabilities can be enhanced/suppressed at low temperature depending on the parameters. For a certain one-parameter family of the hyperscaling violation, we find the onset of the stripe instability analytically in the axion coupling tuned model. For more generic parameter range of hyperscaling violation, we study the instability onset by searching for the zero mode numerically on the full geometries. We also argue that quite analogous results hold, after taking into account the graviton fluctuation, i.e., beyond the probe limit.

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Acknowledgments

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since they are, if realized in the IR, dual to the field theories which break the Lorentz-invariance but respect spatial rotation and translational invariance in the IR. IR Lifshitz and hyperscaling violating geometries can be realized as the near horizon geometries of some black brane solutions, and they can be obtained, for example, in theories where dilaton has run-away behavior governed by the exponential potentials [28,30].

Another interesting nature of these geometries is that these geometries do not admit large entropy unlike Reissner-Nordstrom black branes [35]. On the boundary field theory side, this means that it has vanishing entropy density at the zero temperature limit, which is more natural from thermodynamical view point. In addition, rich behavior of the hyperscaling violating geometries allows more exotic behavior for the fermion correlators. For examples, it gives various $\omega$-dependence for the non-Fermi liquid decay-rates, as shown in [30].

In the meantime, recently a lot of recent progress was made in understanding the translational symmetry breaking in the holographic setting [10,47]. The symmetry breaking can be induced by the axion term $a F \wedge F$ in 4d gravity (where $a$ is neutral pseudo-scalar field, and we call it axion in this paper), and can occur both spontaneously and by the source term. In many of the situations, the analysis is mainly done for the Reissner-Nordstrom black brane which admits AdS$_2$ geometries in the IR. Given this, it is very natural to study if above IR Lifshitz or geometries with hyperscaling violation can survive after we take into account such axion effect and see if it induces the translational symmetry breaking. Furthermore, since this instability changes the IR nature of the geometries, studying this instability on the hyperscaling violating geometries is by itself interesting questions as general relativity problem. In this paper, we study the instability on the IR hyperscaling violating geometries, focusing on the onset of the stripe instability.

The organization of this paper is as follows: We first review the geometries with hyperscaling violation in the 4d Einstein-Maxwell-dilaton-axion system in §II. The geometries with hyperscaling violation emerge in the IR, i.e., in the near horizon limit of the full solutions which approach AdS$_4$ in the UV asymptotically. In §III, we study the perturbation on these geometries in the probe limit. We first analyze the perturbation focusing on the IR geometries at small nonzero temperature in §III.B and at zero temperature limit in §III.C.1. These studies allow us to find the dependence of the stripe instabilities on the parameters of the theories which determine the IR hyperscaling violation. Given this, we identify the instability onset on a certain one-parameter family of the hyperscaling violating geometries, in §III.C.2. In §IIID, we search for the zero mode on the full geometries for more generic parameters numerically. In §IV, we study the perturbation equations without taking the probe limit, and see that essentially the same results hold compared to the probe limit case. We end with a summary and discussion in §V.

II. EINSTEIN-MAXWELL-DILATON-AXION SYSTEM

A. The set-up

The action we consider is Einstein-Maxwell theory coupled to a dilaton-axion, given by

$$S = \int d^{4}x \sqrt{-g} \left( R - 2(\nabla \phi)^2 - 2e^{2\xi\phi}(\nabla a)^2 - V(\phi) - f(\phi)F_{\mu\nu}F^{\mu\nu} - \theta(a)F_{\mu\nu}\tilde{F}^{\mu\nu} \right).$$

Here, $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\kappa} F_{\rho\kappa}$, and $\epsilon^{\mu\nu\rho\kappa}$ has a factor of $1/\sqrt{-g}$ in its definition such that axion term $\theta(a)F_{\mu\nu}\tilde{F}^{\mu\nu}$ is independent of the metric. We take the convention that $\epsilon_{\tau\tau\gamma} > 0$.

As an explicit example, in this paper we consider

$$f(\phi) = e^{2\alpha\phi}, \quad V(\phi) = 2V_0 \cosh 2\phi, \quad \theta(a) = c_1 a,$$

for the explicit stability/symmetry breaking by taking various real parameters $\alpha, \delta, c_1,$ and $V_0$, but we take $V_0 < 0$. Note that $V(\phi) \to V_0 e^{2\delta\phi}$ in the $\delta \phi \to \infty$ limit.

The Einstein equation in trace reversed form becomes

$$R_{\mu\nu} - 2\partial_\mu \phi \partial_\nu \phi - 2e^{2\xi\phi} \partial_\mu a \partial_\nu a = f(\phi) \left( 2F_{\mu\lambda} F^\lambda_{\nu} - \frac{1}{2} g_{\mu\nu} F^2 \right) + \frac{1}{2} g_{\mu\nu} V(\phi, a),$$

which is irrelevant to the axionic $\theta(a) F \wedge F$ term. The equations of motion for dilaton and axion are

$$4 \sqrt{-g} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = \partial_\nu V(\phi, a) + (\partial_\nu f(\phi)) F^2 + 4\xi e^{2\xi\phi}(\nabla a)^2,$$

$$4 \sqrt{-g} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu a) = (\partial_\nu \theta(a)) F \tilde{F},$$

and for the gauge field, we have

$$\partial_\mu \left( \sqrt{-g} \left( f(\phi) F^{\mu\nu} + \theta(a) \tilde{F}^{\mu\nu} \right) \right) = 0,$$

with the Bianchi identity

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0.$$

B. Background solutions

1. IR Hyperscaling violating geometries

First we review the background hyperscaling violating geometries [29,30], on which we later add the small fluctuation to study its stability. We consider the background geometries where all quantities are functions of
only the radial direction \( r \). By using the \( r \) coordinate re-definition, we make the metric in the form
\[
ds^2 = -\tilde{a}(r)^2dt^2 + \frac{dr^2}{\tilde{a}(r)^2} + b(r)^2(dx^2 + dy^2)
\] by the coordinate transformation \( \tilde{r} \equiv r^{1-\gamma-\beta} \), where \( d = 4 \).

These are the so-called “geometries with hyperscaling violation”. The parameter \( z \) is called dynamical critical exponent and \( \theta \) is called hyperscaling violation parameter. For generic values of \( \alpha \) and \( \delta \), we have \( \gamma \neq 1 \) and \( \theta \neq 0 \). If \( \gamma = 1 \), and equivalently, \( \theta = 0 \), this IR metric has additional scale invariance symmetry, therefore \( \theta \) characterizes the “deviation” from the scale invariant limit. Similarly, if \( z = 1 \), then time and spatial coordinate transform equivalently, therefore \( z \) characterizes the “deviation” from the relativistic limit.

Before we continue, let us pose to check the number of the parameters of our system. Our model, given by eqs. \([1] \) and \([2] \), has 5 real parameters, \( \alpha, \delta, V_0, \xi, c_1 \). However \( V_0 \) is a parameter to set the scale (of asymptotic \( \text{AdS}_4 \)), so we can set \( V_0 = -1 \), without loss of generality. Furthermore, since our background has zero VEV for the axion field, \( a = 0 \), parameters \( \xi \) and \( c_1 \) are irrelevant for the background. In this way, the background is characterized by two real parameters \( \alpha \) and \( \delta \) only. Especially in the IR, these two parameters determine the hyperscaling violation \( \theta \) and \( z \), or equivalently \( \beta \) and \( \gamma \).

We will now review the full solutions which interpolate these \( IR \beta, \gamma \) geometries with hyperscaling violation to UV \( \text{AdS}_4 \) asymptotically.

2. Full solutions interpolating IR hyperscaling violating geometries to UV \( \text{AdS}_4 \)

The solution we reviewed is the IR limit of the full solution which asymptotically approaches \( \text{AdS}_4 \) in UV. In order to obtain the full solution which connects the IR to UV, we perturb the IR solution by \( O(\epsilon) \) and numerically follow the evolution of it. To obtain asymptotic \( \text{AdS}_4 \) with appropriate boundary conditions, we tune \( \epsilon \), where \( \epsilon \) is a small parameter. In the zero temperature case, we set \([30] \)
\[
\tilde{a}(r) = C_a r^{\gamma} (1 + \epsilon d_1 r^\nu), \quad b(r) = r^{\beta} (1 + \epsilon d_2 r^\nu), \quad \phi(r) = k \log r + \epsilon d_3 r^\nu,
\]
and the perturbation equations of motion are all satisfied up to the \( O(\epsilon) \) by setting
\[
\nu_1 = \nu_2 = \nu, \quad d_2 = \frac{B_1}{B_2} d_1, \quad d_3 = \frac{4(-1 + \nu) + (\alpha + \delta)^2(1 + \nu)}{4(\alpha + \delta)} d_2, \quad \nu = \frac{2(\delta(\alpha + \delta) + 2) - 3}{2} \left( \frac{A}{(\alpha + \delta)^2 + 4} \right)^{1/2},
\]
\[
A = \left( \left( 4 + (3\alpha - \delta)(\alpha + \delta) \right) [(\alpha + \delta)^2 + 4] \right)^{1/2} \times \left( 36 - (\alpha + \delta)(\alpha(8\delta(\alpha + \delta) - 19) + 17\delta) \right),
\]
\[
\nu = \frac{2(\delta(\alpha + \delta) + 2) - 3}{2} \left( \frac{A}{(\alpha + \delta)^2 + 4} \right)^{1/2},
\]
\[
A = \left( \left( 4 + (3\alpha - \delta)(\alpha + \delta) \right) [(\alpha + \delta)^2 + 4] \right)^{1/2} \times \left( 36 - (\alpha + \delta)(\alpha(8\delta(\alpha + \delta) - 19) + 17\delta) \right),
\]
\[
\nu = \frac{2(\delta(\alpha + \delta) + 2) - 3}{2} \left( \frac{A}{(\alpha + \delta)^2 + 4} \right)^{1/2},
\]
\[
A = \left( \left( 4 + (3\alpha - \delta)(\alpha + \delta) \right) [(\alpha + \delta)^2 + 4] \right)^{1/2} \times \left( 36 - (\alpha + \delta)(\alpha(8\delta(\alpha + \delta) - 19) + 17\delta) \right),
\]
\[
\nu = \frac{2(\delta(\alpha + \delta) + 2) - 3}{2} \left( \frac{A}{(\alpha + \delta)^2 + 4} \right)^{1/2},
\]
We solve numerically by setting \( V \) equations from \( \epsilon \). We can obtain the appropriate boundary condition in the \( \epsilon \) AdS geometries to the UV AdS which smoothly connect the IR hyperscaling violating geometries. To connect our interpolating solutions, we now add the \( x \)-dependent perturbation on this background to study the stripe stability on the background geometry.

\[
B_1 = (\alpha + \delta) \left( 5\alpha^3 + 3\alpha^2\delta - 9\alpha\delta^2 + 32\alpha - 7\delta^3 - 16\delta \right) \\
+ 48 + \left( [4 + (3\alpha - \delta)(\alpha + \delta)] [(\alpha + \delta)^2 + 4] \right)^2 \\
\times (36 - (\alpha + \delta)(\alpha(8\delta(\alpha + \delta) - 19) + 17\delta))^{1/2},
\]

\[
B_2 = 2 \left( 2(\alpha^2 + 2) \delta^2 + \alpha (3\alpha^2 + 4) \delta - 4 (\alpha^2 + 2) \\
- \alpha \delta^3 \right) \times ((\alpha + \delta)^2 + 4).
\]

With this data, we can obtain the interpolating solutions which smoothly connect the IR hyperscaling violating geometries to the UV AdS\(_4\) geometries. To connect our analytic IR hyperscaling violating solutions to an asymptotically AdS\(_4\) metric, we numerically solve the Einstein equations from \( r = r_0 \) to \( r = \infty \). Here \( r_0 \) is some small radius where the solutions are given by eq. (19) and (20). We solve numerically by setting \( V_0 = -1 \). At \( r = r_0 \), the condition \( |\epsilon d_i r_0^i| \ll 1 \) \( (i = 1, 2) \) needs to be satisfied [31]. We can obtain the appropriate boundary condition in the UV by fine-tuning \( \epsilon \).

The results of interpolating functions \( \tilde{a}(r), b(r), \phi(r) \), for the parameter choice \( \alpha = -1/3, \delta = 0.55, d_1 = 0.01 \) are shown in Fig. 1 - 3. For this parameter choice, we obtain \( \gamma \approx 1 - 0.06, \beta \approx 0.01, k \approx -0.1, C_a \approx 1, Q_a^2 \approx 0.5 \), and we have positive \( \nu, \nu \approx 0.9 > 0 \), and real \( d_2 \) and \( d_3 \), \( d_2 \approx -1.4 d_3 \), \( d_3 \approx 0.5 d_1 \). The results shown in Fig. 1 - 3 are essentially the same as given in the appendix F of [39].

Similarly in the finite temperature case, we can obtain the interpolating solutions. We now add the \( x \)-dependent perturbation on this background to study the stripe stability on the background geometry.

III. PERTURBATION ANALYSIS IN THE PROBE LIMIT

A. Perturbation equations in the probe limit

Given the full solutions which interpolate between the IR hyperscaling violating geometries and UV AdS\(_4\), we add the \( x \)-dependent perturbation and study the onset of the stripe instability. First, for the sake of simplicity, we neglect the graviton fluctuation, namely, we consider the stability in the “probe limit,” where matter fluctuation will not back-react to the graviton fluctuation. In this limit, the translational symmetry breaking can occur and there can exist unstable mode, but the background geometries are fixed. We will see even in this case, we have interesting phenomena for a dynamical instability of the translationally invariant vacuum.

On the background metric and flux given by eq. [8]
and eq. 9, we add the following small fluctuation,
\[ \delta A_y, \quad \delta a, \quad \delta \phi, \quad (28) \]
and we choose these fluctuations are x-coordinate dependent. If any of these modes show nonzero vacuum expectation value with the boundary condition that their non-normalizable modes disappear at the boundary, we have gravity dual of the translational symmetry breaking.

From the dilaton fluctuation equation eq. (4), we have
\[ \frac{4}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\delta \phi) = \partial_\phi^2 V(\phi, a) \delta \phi \\
+ (\partial_\phi f(\phi))F^2 \delta \phi + 2(\partial f(\phi))F^{\mu\nu} \delta F_{\mu\nu}. \quad (29) \]

Note that the third term on the RHS vanish, since \( F^{\mu\nu} \neq 0 \) only for \( F^{11} \) but we excite only \( \delta A_y \). Therefore, dilaton equation of motion can be trivially satisfied by setting
\[ \delta \phi = 0, \quad (30) \]
so we can set the dilaton fluctuation to be zero. In this paper, we will set the fluctuation of \( \phi \) to be zero, \( \delta \phi = 0 \). In such case, the operators which are dual to \( A_y, a \) can condensate.

The axion fluctuation equation of motion eq. (5) gives
\[ \frac{4}{\sqrt{-g}} \partial_{\mu}(e^{2\phi} \sqrt{-g}g^{\mu\nu}\partial_{\nu}\delta a) \\
= (\partial_\phi \theta(a))e^{\mu\nu\lambda\eta}F_{\mu\nu} \delta F_{\lambda\eta}. \quad (31) \]
Here we have used the fact that background flux is purely electrical, \( \mathcal{F} \mathcal{F} = 0 \).

The gauge field equation of motion eq. (6) is
\[ \partial_{\mu} \left( \sqrt{-g} (f(\phi) \delta F^{\mu\nu} + \theta(a) \delta \tilde{F}^{\mu\nu}) + (\partial_\phi \theta(a)) \tilde{F}^{\mu\nu} \delta a \right) = 0, \quad (32) \]
here we used \( \delta \phi = 0 \). We need to study the coupled equation for \( \delta A_y \) and \( \delta a \) given by eq. (31) and eq. (32).

We follow the analysis of [41] closely. We set,\[ \delta A_y = \delta A_y(r) \sin qx e^{-i \omega t}, \quad (33) \]
\[ \delta a = \delta a(r) \cos qx e^{-i \omega t}. \quad (34) \]

By using the background metric and flux given previously, after a bit algebra, the \( \nu = y \) component of eq. (32) gives,
\[ \sqrt{-g}g^{\mu\nu}f(\phi) (-g^{xx}q^2 - g^{tt} \omega^2) \delta A_y(r) \\
+ \partial_r \left( \sqrt{-g}g^{\mu\nu}f(\phi)g^{rr} \partial_r \delta A_y(r) \right) \\
- q\sqrt{-g} \partial_r \theta(a) \tilde{F}^{xy} \delta a(r) = 0. \quad (35) \]
With the metric ansatz eq. (8), and by using \( \tilde{F}^{\mu\nu} = 0 \), and
\[ \sqrt{-g} \tilde{F}^{xy} = \sqrt{-g}e^{xytr}F_{tr} = -\frac{Q_e}{b^2 f(\phi)} \],
\[ \delta \phi = 0, \quad (30) \]
this equation is re-written as
\[ f(\phi) (-b^{-2}q^2 + \tilde{a}^{-2} \omega^2) \delta A_y(r) \\
+ \partial_r \left( f(\phi) \tilde{a}^2 \partial_r \delta A_y(r) \right) \\
+ (\partial_\phi \theta(a)) \frac{Q_e}{b^2 f(\phi)} \delta a(r) = 0. \quad (37) \]

Note that both a term proportional to \( q^2 \), and a term induced by axion term, have explicit common factor \( b^{-2} \).

Using \( e^{4\pi y} = -e^{4\pi r} = 1/\sqrt{-g} \) and the fact that the background axion takes the zero VEV, it is straightforward to check that \( \nu \neq y \) component of eq. (32) is automatically satisfied.

Similarly, the axion fluctuation equation of motion is written as
\[ e^{2\xi b^2} (-b^{-2}q^2 + \tilde{a}^{-2} \omega^2) \delta a(r) \\
+ \partial_r \left( e^{2\xi b^2} \tilde{a}^2 \partial_r \delta a(r) \right) \\
+ (\partial_\phi \theta(a)) \frac{Q_e}{b^2 f(\phi)} \delta A_y(r) = 0. \quad (38) \]

We will solve the coupled equations eq. (37) and eq. (38) to study the translational symmetry breaking and stability of the translationally invariant vacuum. However since eq. (37) and (38) are coupled, generically it is quite difficult to solve these for the stability. However since there are analytical expressions for the metrics in the IR, which are the hyperscaling violating geometries given by eq. (10) and (11), we first restrict our attention to the near horizon IR geometries.

**B. Negative momentum square mode on the hyperscaling violating geometry: finite temperature analysis**

First, let’s consider the finite temperature case, where the near horizon metric is given by eq. (10) and (11). In the near horizon region, where \( r \to r_h \), we can approximate the metric as
\[ \tilde{a}^2 = (2\beta + 2\gamma - 1)C^2 \alpha^2 (r - r_h), \quad (39) \]
\[ b^2 \to r_h^{2/3}, \quad f(\phi) \to r_h^{2\alpha k}, \quad (40) \]
With this approximation, the eq. (37) and eq. (38) reduce to
\[ r_h^{2\alpha k} \left( -r_h^{-2\beta}q^2 + \tilde{a}^{-2} \omega^2 \right) \delta A_y(r) \quad (41) \]
\[ + \partial_r \left( r_h^{2\alpha k} \tilde{a}^2 \partial_r \delta A_y(r) \right) + c_1 \frac{Q_e}{r_h^{2\alpha k + 2\beta}} \delta a(r) = 0. \quad (42) \]
\[ r_h^{2\alpha k + 2\beta} \left( -r_h^{-2\beta}q^2 + \tilde{a}^{-2} \omega^2 \right) \delta a(r) \quad (42) \]
\[ + \partial_r \left( r_h^{2\alpha k + 2\beta} \tilde{a}^2 \partial_r \delta a(r) \right) + c_1 \frac{Q_e}{r_h^{2\alpha k + 2\beta}} \delta A_y(r) = 0. \]
It is more convenient to re-write these as
\[ \left(-r_h^{-2\beta} q^2 + \tilde{a}^{-2}\omega^2\right) \delta A_y + \partial_r \left(\tilde{a}^2 \partial_r \delta A_y\right) + c_1 \frac{qQ_e}{r_h^{2\beta+4\alpha k}} \delta a = 0, \] (43)
\[ \left(-r_h^{-2\beta} q^2 + \tilde{a}^{-2}\omega^2\right) \delta a + \partial_r \left(\tilde{a}^2 \partial_r \delta a\right) + c_1 \frac{qQ_e}{r_h^{4\beta+2\alpha k+2\delta k}} \delta A_y = 0. \] (44)

By multiplying \(c_2 = r_h^{\beta+\xi k-\alpha k}\) to eq. (44), we obtain
\[ \left(-r_h^{-2\beta} q^2 + \tilde{a}^{-2}\omega^2\right) \psi_1 + \partial_r \left(\tilde{a}^2 \partial_r \psi_1\right) + c_1 \frac{qQ_e}{c_2 r_h^{2\beta+4\alpha k}} \psi_2 = 0, \] (45)
\[ \left(-r_h^{-2\beta} q^2 + \tilde{a}^{-2}\omega^2\right) \psi_2 + \partial_r \left(\tilde{a}^2 \partial_r \psi_2\right) + c_1 \frac{qQ_e}{c_2 r_h^{2\beta+4\alpha k}} \psi_1 = 0. \] (46)

where
\[ \delta A_y \equiv \psi_1, \quad c_2 \delta a \equiv \psi_2 \] (47)

By choosing \(\psi_\pm \equiv \psi_1 \pm \psi_2\), we have two independent equations
\[ \left(-r_h^{-2\beta} q^2 + \tilde{a}^{-2}\omega^2\right) \psi_\pm + \partial_r \left(\tilde{a}^2 \partial_r \psi_\pm\right) \pm c_1 \frac{qQ_e}{r_h^{3\beta+3\alpha k+\xi k}} \psi_\pm = 0. \] (48)

These equations have very similar structure to the equations of motion for massive scalars in the finite temperature black brane. However due to the axion term proportional to \(c_1\), it has an “effective momentum square” defined as
\[ q_{eff,\pm}^2 \equiv q^2 + c_1 \frac{qQ_e}{r_h^{3\beta+3\alpha k+\xi k}}. \] (49)

We take \(c_1 qQ_e > 0\) without loss of generality. Note that this guarantees that at the horizon, one of the “effective momentum square”, \(q_{eff,\pm}^2\) above for the mode \(\psi_+(r)\), becomes always negative. The minimum value is given at critical wave number \(q_c\),
\[ q_c = \frac{c_1 Q_e}{2r_h^{3\beta+3\alpha k+\xi k}}, \] (50)
with minimum momentum square
\[ q_{eff,\pm}\big|_{q=q_c} = -\frac{c_1^2 Q_e^2}{4r_h^{2(3\beta+3\alpha k+\xi k)}} < 0, \] (51)
which is always negative.

This is very similar to the situations where the striped phase instability is studied by [40, 41]. The axion term \(c_1 \neq 0\) makes the system have lower energy by having the nonzero momentum “stripe” type of the spatially modulated mode.

However contrary to the IR AdS case, there are differences on the study of the instability of the IR hyperscaling violating geometries with finite but small temperature; Due to the additional warping factor in hyperscaling violating geometries, there is extra temperature \((r_h)\) dependence for this effective momentum square, and \(c_1 Q_e\) appears only in the combinations of \(c_1 Q_e/r_h^{\beta+3\alpha k+\xi k}\). In the AdS case, where \(\alpha = \delta = 0\) for \(\beta = k = 0\) and \(\gamma = 1\) in eq. (12) and (13), \(c_1 Q_e\) does not involve any temperature dependence, since \(r_h^{\beta+3\alpha k+\xi k}\) becomes constant.

If \(c_1 Q_e/r_h^{\beta+3\alpha k+\xi k}\) are very large, for \(q \neq 0\), \(\psi_+\) mode can have lower energy with nonzero momentum \(q\) mode, and the existence of such negative momentum square mode gives the possibility that such mode induces the instability of the translationally invariant vacuum.

The fact that \(c_1 Q_e\) appears in the combinations of \(c_1 Q_e/r_h^{\beta+3\alpha k+\xi k}\) implies that the situation is very different, depending on \(\beta + 3\alpha k + \xi k\) is positive or negative. If
\[ \beta + 3\alpha k + \xi k > 0, \] (52)
as we lower the temperature \((i.e., as we lower r_h)\), the instability are more likely to occur even for the small values of \(c_1 Q_e\), due to the enhancement by the factor \(1/r_h^{\beta+3\alpha k+\xi k}\). In this case, \(c_1 Q_e/r_h^{\beta+3\alpha k+\xi k}\) diverges positively at \(r_h \to 0\) and \(q_{eff,\pm}^2\) goes to negative infinity, and these suggest that the instabilities are expected to occur. On the other hand, if \(\beta + 3\alpha k + \xi k\) is negative, as we lower the temperature, we need to increase \(c_1 Q_e\) in order to keep the same value for \(q_{eff,\pm}^2\), due to the suppression by the factor \(1/r_h^{\beta+3\alpha k+\xi k}\).

However, whether the existence of the negative effective momentum square mode, \(q_{eff,\pm}^2 < 0\), which is evaluated at the horizon, is enough or not for instability is another question. This is because in general hyperscaling violating geometries, we do not know the critical values of the instability. This is in sharp contrast to the AdS case. In the AdS case, partially due to the scale invariance of the geometry, we have a sharp bound for the instability, \(i.e., Breitenlohner - Freedman (BF)\) bound. However for geometries with hyperscaling violation, we generically do not expect sharp bound for generic radius and therefore in order to show that there is an unstable mode, local argument is not enough in general and we need to study the full bulk system to search for the unstable mode. Note that since hyperscaling violating geometries are IR limit and they approach AdS in the UV, this means that we need to study the perturbation equations on the whole geometries, with the normalizable boundary condition in the UV AdS region.

However, before we study the full bulk system for the unstable mode numerically, one might wonder if something special occurs for the special parameter range
where the equality of eq. \((52)\) holds. To understand this marginal parameter range in more detail, we continue the study of perturbation equations on the IR hyperscaling violating geometries by taking the zero temperature limit.

C. Zero temperature analysis and instability criteria

1. Zero temperature perturbation

We analyze the instabilities at zero temperature limit, where the metric is
\[
\delta^2 = C_a^2 r^{2\gamma}, \quad b^2 = r^{2\beta}.
\] (53)

In this case, by redefining the field as
\[
\delta A_y = r^{1-\gamma-\alpha k}\delta \hat{A}_y, \quad \delta a = r^{1-\beta-\gamma-\xi k}\delta \hat{a},
\] (54)
the gauge field and axion equations of motion are written as
\[
\nabla^2 \delta \hat{X} = M^2 \delta \hat{X}
\] (55)
where
\[
\nabla^2 \equiv C_a^2 \partial_r r^2 \partial_r, \quad \delta \hat{X} = \left( \delta \hat{A}_y \right), \quad M^2 = \left( \begin{array}{cc} A(r) & B(r) \\ B(r) & C(r) \end{array} \right),
\] (56)
\[
A(r) = C_a^2 (-1 + \gamma + \alpha k) (r^{\gamma + \alpha k}) + q^2 r^{2-2\beta-2\gamma},
\] (57)
\[
B(r) = -c_1 q Q \epsilon r^{-3\beta-2\gamma-k(3\alpha+\xi)},
\] (58)
\[
C(r) = C_a^2 (-1 + \beta + \gamma + k \xi) (r^{\beta + \gamma + k \xi}) + q^2 r^{2-2\beta-2\gamma},
\] (59)
where we have set \(\omega = 0\), since it allows us to see the instability onset.

Note that if both
\[
2 - 3\beta - 2\gamma - k(3\alpha + \xi) < 0,
\] (60)
and
\[
2 - 3\beta - 2\gamma - k(3\alpha + \xi) < 2 - 2\beta - 2\gamma,
\] (61)
are satisfied, the off-diagonal terms, which are proportional to \(c_1\) and which come from the axion term, dominate at the extremal horizon, \(r \to 0\). In such case, we are forced to have the situation where the matrix \(M^2\) has at least one negative eigenvalue, which goes to negative infinity at \(r \to 0\). Therefore in this parameter range, it is expected that there is an unstable mode for the stripe instability.

On the other hand, if the parameters of hyperscaling violation do not satisfy above inequality (61) and (62), then the matrix \(M^2\) does not always have negative eigenvalue at \(r \to 0\). However even in this case, if we take large \(c_1\), then at finite \(r\), \(M^2\) has one eigenvalue which becomes negative at some intermediate radius \(r\). From these, we expect that if inequality (61) and (62) hold, the instability is more likely to occur compared to the case where inequality (61) and (62) do not hold. We will see this actually later in the numerical analysis.

Note that the second condition eq. (62) is equivalent to
\[
\beta + 3\alpha k + \xi k > 0,
\] (63)
which is the same as eq. (52), i.e., the second term of effective negative momentum square in eq. (19) dominates at the extremal limit \(r_h \to 0\). The first condition eq. (61) is to guarantee that the off-diagonal term will not vanish at the horizon.

2. Analytical criteria for instability onset for special type of hyperscaling violation

There is a special parameter range in hyperscaling violating geometries, which occurs when the equality of both eq. (61) and (62) are satisfied. This is when \(\xi\), the parameter for the axion kinetic term in eq. (1), is tuned as
\[
\xi = -3\alpha - \frac{3\beta + 2\gamma - 2}{k},
\] (64)
and \(\beta\) and \(\gamma\) in the IR hyperscaling violating geometries satisfy the relation
\[
\beta + \gamma = 1.
\] (65)

For this special type of hyperscaling violating geometries, we can identify the critical mass just as Breitenlohner - Freedman (BF) bound, since the matrix \(M^2\), given in eq. (57), becomes independent on the radius \(r\), and therefore constant matrix. From the solution eq. (12) and (13), we can see that if we do not have parameter \(\xi\), the only way to satisfy eq. (64) and (65) are AdS\(_2\) case, i.e., \(\beta = 0\) and \(\gamma = 1\) case. However for the theories with tuned axion kinetic terms \(\xi\) as eq. (64), \(M^2\) becomes constant matrix for one real parameter family of hyperscaling violating geometries with eq. (65), and AdS\(_2\) is a special parameter point in this parameter range.

For such IR geometries with hyperscaling violation satisfying (65), \(M^2\) has two eigenvalues, \(m_{\pm}^2\),
\[
m_{\pm}^2(q) = \frac{1}{2} \left( A_0 + C_0 \pm \sqrt{(A_0 - C_0)^2 + 4B_0^2} \right),
\] (66)
where
\[
A_0 = C_a^2 (-1 + \gamma + \alpha k) (\gamma + \alpha k) + q^2,
\] (67)
\[
B_0 = -c_1 q Q \epsilon,
\] (68)
\[
C_0 = C_a^2 (-1 + \beta + \gamma + k \xi) (\beta + \gamma + k \xi) + q^2.
\] (69)

\(\delta \hat{X}\) admits the power law ansatz
\[
\delta \hat{X} \sim r^\Delta,
\] (70)
where $\Delta$ satisfies

$$ C_a^2 \Delta(\Delta + 1) = m_{\pm}^2(q) , $$

(71)

and critical values for the instability onset are given when $\Delta$ becomes complex \[69\]. By minimizing $m_{\pm}^2(q)$ with respect to $q$ at $q = q_{\text{min}}$, it becomes

$$ C_a^2 \Delta(\Delta + 1) = m_{\pm}^2(q_{\text{min}}) , $$

(72)

where $q_{\text{min}}$ satisfies $\partial_q m_{\pm}^2(q)|_{q=q_{\text{min}}} = 0$. $\Delta$ becomes complex when

$$ m_{\pm}^2(q_{\text{min}}) \leq -\frac{1}{4} C_a^2 $$

(73)

holds. In the large $c_1 Q_e$ and $q$ case,

$$ m_{\pm}^2(q) \approx q^2 - |c_1 Q_e| q , $$

(74)

which allows $q_{\text{min}} \approx |c_1 Q_e|/2$, and $m_{\pm}^2(q_{\text{min}}) \approx -|c_1 Q_e|^2/4$. Therefore in this case, eq. (73) is satisfied and we have stripe instability.

The special relation eq. (65) is the point where both $z$ and $\theta$ diverge ($z$ and $\theta$ are given by eq. (17)). More precisely, it is the special point of hyperscaling violation where either $z \to -\infty$ and $\theta \to +\infty$, or $z \to +\infty$ and $\theta \to -\infty$ holds.

It is interesting to see how above results are different from the AdS$_2$ case. Let’s first consider the AdS$_2$ case, which is the special limit of above and we take $\alpha = \delta = 0$ so that we have $\beta = 0$, $\gamma = 1$, $k = 0$ in eq. (12), and eq. (13). In such case, $A_0$, $B_0$, $C_0$ in eq. (67) - (69) simplifies and we obtain

$$ m_{\pm}^2_{\text{AdS}2}(q) = q^2 \pm c_1 q Q_e . $$

(75)

This $m_{\pm}^2_{\text{AdS}2}$ is the same effective momentum square $q^2_{\text{eff} \pm}$ given in eq. (49). Therefore in AdS$_2$ case, the onset value of the instability, which we call $c_{\text{min AdS}2}$, is given by the equality of eq. (73) as

$$ c_{\text{min AdS}2} = \frac{C_a}{Q_e}^{\alpha=\delta=0} = \sqrt{2} . $$

(76)

Here we have used eq. (13) and (14). The stripe instability occurs at $c_1 \geq c_{\text{min AdS}2}$.

Next, we consider non-AdS$_2$ case. Note that eq. (65) is satisfied when $\alpha = \pm \delta$ from eq. (12) and (13). However since $\alpha = -\delta$ gives $\beta = 0$, $\gamma = 1$, which is AdS$_2$, we consider $\alpha = \delta$ case. From eq. (12) - (14), in this case the hyperscaling violating geometries are parametrized by only one real parameter $\alpha$, with

$$ \beta = \frac{\alpha^2}{1 + \alpha^2} , \quad \gamma = \frac{1}{1 + \alpha^2} , \quad k = -\frac{\alpha}{1 + \alpha^2} , $$

(77)

$$ Q_e^2 = -V_0 \frac{1 - \alpha^2}{2(1 + \alpha^2)} , \quad C_a^2 = -V_0 , $$

(78)

with $V_0 < 0$, and eq. (64) gives

$$ \xi = -2\alpha . $$

(79)

Then, the eigenvalue $m_{\pm}^2(q)$ of $M^2$ given by eq. (66) becomes

$$ m_{\pm}^2(q) = \frac{1}{2(\alpha^2 + 1)^2} \left( 2(\alpha^2 + 1)^4 q^2 - 8 (\alpha^2 + 1)^2 \alpha^4 V_0 \right. $$

$$ -\sqrt{2} (\alpha^2 + 1)^6 V_0 \left( (\alpha^4 - 1) c_1^2 q^2 + 8 \alpha^4 V_0 \right) \right) . $$

(80)

Since this has minima at

$$ q = q_{\text{min}} = \frac{-\sqrt{2} V_0 \sqrt{(1 - \alpha^2)^2 c_1^2 - \frac{64 \alpha^4}{c_1^4}}}{2 \sqrt{2(1 - \alpha^2)}} , $$

(81)

the instability condition eq. (73) becomes

$$ -2 \sqrt{(\alpha^2 - 1)^2 (\alpha^2 + 1)^6 c_1^4 V_0^2} $$

$$ + \frac{(\alpha^2 + 1)^2 V_0}{8 c_1^2 (\alpha^2 - 1)(\alpha^2 + 1)} \times \left( (\alpha^2 - 1)^2 (\alpha^2 + 1) c_1^4 $$

$$ - 64 (\alpha^6 + \alpha^4) - 32 (\alpha^2 - 1) \alpha^4 c_1^4 \right) \leq \frac{V_0}{4} . $$

(82)

Therefore, the onset $c_1$ value of the instability, which we call $c_{\text{min}}^0$, is given by the equality of this and it is

$$ c_{\text{min}}^0 = \sqrt{2} \left( 1 + \frac{\alpha^2}{1 - \alpha^2} \right)^{1/2} . $$

(83)

The instability occurs when $c_1 \geq c_{\text{min}}^0$, which is larger than the critical value $c_{\text{min AdS}2}$ for AdS$_2$ case, $\sqrt{2}$. Therefore it is more stable than the AdS$_2$ case, regarding the stripe instability.

This expression loses its meaning when $\alpha^2 \geq 1$, corresponding to $\gamma \leq 1/2$. But $\gamma$ cannot be less than $1/2$, since then the entropy density of these black brane are proportional to the negative power of the temperature, which, thermodynamically, does not make sense \[60\]. Another reason for $\gamma > 1/2$ is that if $\gamma \leq 1/2$, then the null rays from the zero temperature horizon $r = 0$ can reach nonzero $r$ at finite time, which contradicts with the “horizon” property at $r = 0$ \[61\].

3. $\xi = 0$ case

In the next section, we conduct numerical analysis of generic parameter points where such an equality eq. (65) does not hold. For that purpose, let’s investigate the $\xi = 0$ case a bit more, since this is the case where axion has canonical kinetic term.

The parameter range which satisfies eq. (62) is given by

$$ (\alpha + \delta)(5\alpha - \delta) < 0 , $$

(84)

and for eq. (61), it is given by

$$ (\alpha + \delta)(3\alpha + \delta) < 0 . $$

(85)
This gives
\[-3\alpha < \delta < -\alpha \quad (\text{for } \alpha > 0) , \quad (86)\]
\[-3\alpha > \delta > -\alpha \quad (\text{for } \alpha < 0) . \quad (87)\]

This range is written in terms of \(\beta\) and \(\gamma\), or \(\theta\) and \(z\). From eq. (12), we have
\[\alpha = \pm \frac{1 - 2\beta - \gamma}{\sqrt{\beta(1 - \beta)}}, \quad \delta = \pm \frac{\gamma - 1}{\sqrt{\beta(1 - \beta)}}. \quad (88)\]

Remember that we need
\[\frac{1}{2} < \gamma , \quad 0 < \beta < 1 . \quad (89)\]

Therefore, the parameter range, eq. (86) or (87), is equivalent to
\[0 < \beta < \frac{1}{3}(1 - \gamma) , \quad \frac{1}{2} < \gamma < 1 . \quad (90)\]

In terms of \(\theta\) and \(z\), using
\[\beta = \frac{\theta - 4}{2(\theta - 2z)}, \quad \gamma = \frac{\theta - 4z}{2(\theta - 2z)}, \quad (91)\]

from eq. (17), the parameter range eq. (90) is written as
\[4 < \theta < 6, \quad z < 0 . \quad (92)\]

So far we consider the parameter range satisfying eq. (61) and (62). In addition to these, we have more restriction on the parameters. First, we need \(\delta k < 0\) which restricts \(\delta(\alpha + \delta) > 0\). Second, we want to study the stripe instability on the background which is stable at \(q = 0\). Since the IR hyperscaling violating geometries asymptotically approach AdS4 in the UV, the mass of the dilaton must be above the AdS4 BF bound for the stability at zero momentum \(q = 0\). From our Lagrangian eq. (1), we can read off the mass of the dilaton at \(\phi = 0\) and this gives additional constraint
\[-\sqrt{\frac{3}{8}} < \delta < \sqrt{\frac{3}{8}} . \quad (93)\]

Combined these additional conditions with eq. (86) and (87), finally they are summarized as
\[0 > -\frac{1}{3} \delta > \alpha > -\delta > -\sqrt{\frac{3}{8}} . \quad (94)\]

or,
\[0 < -\frac{1}{3} \delta < \alpha < -\delta < \sqrt{\frac{3}{8}} . \quad (95)\]

It is clear that there are parameter ranges satisfying above in the “physical parameter ranges” \(62\) parametrized by \(\alpha\) and \(\delta\), as is shown in Figure 1 of \(50\). In this parameter range, the effective momentum square in eq. (51) becomes large negative value at low temperature, and also the off-diagonal components of matrix \(M^2\) in eq. (54) dominates. Therefore it is expected that the instability is more likely to occur at small value of \(c_1\) in this case.

We next conduct numerical analysis for the parameter choice where eq. (94) is satisfied and also for another parameter choice where it is not satisfied. We call the case where eq. (94) is satisfied as case I, and the case where eq. (94) (or eq. (63) in more generic case where \(\xi \neq 0\)) is not satisfied as case II.

The limit \(\alpha \rightarrow 0\) and \(\delta \rightarrow 0\) corresponds to the AdS2 and its stripe instability is well studied in \(41, 42\). We will now study numerically the whole bulk system in these parameter range for the stripe instability next.

D. Numerical analysis for the bulk zero mode on the full geometries

Now we conduct numerical investigation of the coupled equations eq. (37) and eq. (38) to study the dynamical translational symmetry breaking. So far we have been concentrating on the perturbation equation analysis restricting our attention on the IR hyperscaling violating geometries. However, now we will conduct the numerical analysis on the full geometries, which approach geometries with hyperscaling violation in the IR and AdS4 in the UV.

In order to find the onset of the instability, we will look for the zero mode, namely the solution of eq. (37) and eq. (38) with \(\omega = 0\). If there is such a zero frequency mode for some value of \(c_1\), then it is the critical mode. By increasing \(c_1\) above the critical value, the instability occurs. This is because if there is an unstable mode, \(\Im(\omega) > 0\) which grows as time evolves, then there should also be zero mode solution \(\omega = 0\) at the instability onset point.

Let us emphasize again why searching the zero mode on the full background geometries is important: In the case of IR AdS2, we have a sharp local criteria for the onset of the stripe instabilities, which is given by the condition \(c_1 \geq c_{\text{min AdS2}}\) with eq. (70). This is in contrast to our generic hyperscaling violating geometries in the IR; in two-parameter (\(\beta\) and \(\gamma\), or \(\theta\) and \(z\)) hyperscaling violating IR geometries, we do not have sharp criteria for the stripe instability from the IR geometries.

However even in the IR AdS2 case, it is important to find the zero mode on the full geometries which approach AdS4 in the UV. This is because generically the stability is not determined by the IR region only, but rather it is determined by the full geometries. It is especially so if two modes are coupled.

To see this, let us consider the IR AdS2 geometries which approach AdS4 in the UV. For the IR AdS2 region, the matrix \(M^2\) in eq. (54) becomes constant matrix and we can obtain two eigenvector modes (let us call these as mode A and B) made by some specific linear combination
of mode $\delta A_y$ and $\delta a$. Suppose that the mode $A$ gives the lower eigenvalue of the matrix $M^2$ than the mode $B$, and furthermore that the eigenvalue of the mode $A$ is lower than the AdS$_2$ BF bound and the eigenvalue of the mode $B$ is higher than the AdS$_2$ BF bound. It is true that if we excite only the mode $A$, then we can lower the energy by the mode $A$ perturbation and this indicates instability. However due to the UV boundary condition and the mixing of the two modes, generically we cannot excite only the mode $A$, but rather we need to excite the mode $B$ too in general [63]. At what ratio, we excite the mode $A$ and $B$, is determined by the full (whole) geometries. Therefore, depending on the ratio of the mode $A$ and $B$ excitations, we can see if the system is unstable or not by the perturbation. This is because the mode $A$ lowers the energy but the mode $B$ increases the energy in the IR AdS$_2$. In this way, it is generically determined not by the local IR geometry only, but rather by the full geometries. Therefore it is important to find the zero mode on the whole geometries including IR and UV.

Furthermore, in the case of IR hyperscaling violating geometries, which is parametrized by two parameters, we do not generically have sharp BF-like bound. Therefore it is more important to search for the zero mode on the full geometries which include both IR and UV region.

We now seek for the zero mode on the full geometries. For numerics, we set $V_0 = -1$. Introducing new variables $\psi_{\pm} = \delta A_y \pm c_2 \psi_2$ as eq. (17), we can obtain the boundary condition at the horizon $r = r_h$ by imposing regularity as

$$\partial_r \psi_{\pm} |_{r=r_h} = \frac{r_h^{1-2\beta-2\gamma} q_{eff\pm} \psi_{\pm} |_{r=r_h}}{(2\beta + 2\gamma - 1)C_a}, \quad (96)$$

where $q_{eff\pm}$ is the effective momentum square defined in eq. (19).

In order to find the onset of the spontaneous translational symmetry breaking in the holographic dual setting, we impose Dirichlet boundary condition for the variables $\delta A_y$ and $\delta a$, at the AdS$_4$ boundary,

$$\delta A_y = \delta a = 0 \quad \text{for} \quad r = \infty. \quad (97)$$

This corresponds to the requirement that there is no non-normalizable mode for $\delta A_y$ and $\delta a$ in the UV AdS$_4$ boundary. Non-normalizable mode for $\delta A_y$ and $\delta a$ approach constant in the UV AdS$_4$.

It is useful to consider the parameter counting. For given fixed parameters $\alpha, \delta, \xi, r_h$, and $c_1$, there are two free parameters $\psi_-|_{r=r_h}$ and $q$. Note that $\psi_+|_{r=r_h}$ is not a free parameter, since we can fix $\psi_+|_{r=r_h}$ to unity without loss of generality in linearized perturbations. We tune $\psi_-|_{r=r_h}$ and $q$ such that the two boundary conditions eqs. (97) are satisfied. After this tuning, we have no parameter left, and as a results, we have nonzero normalizable mode for both $\psi_+$ and $\psi_-$ at $r \to \infty$. This implies that there are spatially modulated VEV for the scalar and vector current $< j_\mu >$, which are dual to axion and and gauge boson $A_y$, and this implies that dual theories at IR show the “current density wave” phase. Note also that for a given temperature $T(r_h)$, we expect normalizable zero modes to appear at specific values of momentum $q$.

As we have discussed, we expect that the effect of the axion term and the negative “effective momentum square” is enhanced or suppressed depending on eq. (94) is satisfied or not. In order to see the difference, we investigate two typical cases; Case I corresponding to $\alpha = -0.33$, $\delta = 0.55$, $\xi = 0$, which gives $\beta = 0.01$, $\gamma = 0.94$, $k = -0.10$, and it satisfies $\beta + 3\alpha \gamma + \xi k > 0$. Case II corresponding to $\alpha = 0.2$, $\delta = 0.55$, $\xi = 0.4$, which gives $\beta = 0.12$, $\gamma = 0.82$, $k = -0.33$, and it satisfies $\beta + 3\alpha \gamma + \xi k < 0$.

It is easily checked that for the case I, it satisfies eq. (94), and therefore, (91) - (63). So case I corresponds to the case where axion term is expected to be enhanced at low temperature $r_h \to 0$ due to $c_1 Q_y/r_h^{\beta+3\alpha \gamma+\xi k} \to \infty$, and we expect that axion term induces instability even at very small values of $c_1$ at low temperature.

On the other hand, for the case II, it violates eq. (94) (more precisely it violates eq. (63) since $\xi \neq 0$), but does not violate the UV AdS$_4$ BF bound (93). Therefore case II corresponds to the case where axion term is expected to be rather suppressed at low temperature $r_h \to 0$ due to $c_1 Q_y/r_h^{\beta+3\alpha \gamma+\xi k} \to 0$, and we expect that we need large values of $c_1$ in order to induce instability at low temperature.

For both cases, we first numerically construct the background solutions interpolating the IR analytical hyperscaling violating geometries and UV AdS$_4$ as we reviewed in [11B2]. Then we solve the coupled eqs. (37) and eq. (38), and find numerically the solutions $\psi_{\pm}$ satisfying the boundary conditions eqs. (97) for a suitable $c_1$, $q$ and $\psi_-$ for each temperature $T$ ($r_h$), where eq. (15) gives the relationship between $r_h$ and $T$.

For each temperature we find the zero mode solution by tuning $q$ for $c_1$, as far as $c_1 > c_{min}$. There is a minimum value $c_{min}$ for $c_1$, namely, we could not find any zero mode solution for any $q$ and $\psi_-$ when $c_1 < c_{min}$.

We plot $c_{min}$ for each temperature $T$ in Fig. 4 (case I) and Fig. 5 (case II). As expected, for the case I, $c_{min}$ decreases significantly as $T$ decreases and seems to approach zero. On the other hand, for the case II, $c_{min}$ does not change drastically as $T$ decreases. This suggests that spontaneous translational symmetry breaking easily occurs for much lower temperature when eq. (94) is satisfied, although we cannot further evaluate the minimum value $c_{min}$ for much lower temperature. This is because highly numerical accuracy is required. Similarly we face another difficulties at low $T$.

This is because in such temperature, $r_h$ is not so small and as a result, $\phi|_{r=r_h}$ is not large enough. Then, our perturbative method in [11B2] to construct full interpolating solutions breaks down. Therefore we conduct numerical analysis in rather restricted low temperature range.
Note that the value range of $c_{\text{min}}$ for case II in Fig. 5 is higher than the critical value for the AdS$_2$ case, suggesting that these are instabilities triggered by the suppressed axion term effect due to the radius dependent factor, $c_1 Q_h/r_h^{3+3\alpha k+\xi k} \to \infty$ at $r_h \to 0$. One might expect that in case I, at the zero temperature limit where $r_h \to 0$, the critical value $c_{\text{min}}$ approaches zero. Fig. 4 is consistent with this expectation. However due to the numerical difficulties, we could not confirm this at the very low temperature $T \lesssim 6 \times 10^{-7}$.

On the other hand, in case II, we find that the critical value $c_{\text{min}}$ is almost constant as we lower the temperature $r_h$. Note that the value range of $c_{\text{min}}$ for case II in Fig. 5 is higher than the critical value for the AdS$_2$ case, suggesting that these are instabilities triggered by the suppressed axion term effect due to the radius dependent factor in case II. However we do not have clear physical interpretation of the result in the case II. From the IR analysis, it might suggest that the critical value $c_{\text{min}}$ increases as we lower the temperature, but the result of Fig. 5 is not so. One possible reason for this behavior is that the negative momentum square is more dominating away from the horizon, $r \gg r_h$. As we have discussed in III C 1 one can see that even in the case II with zero temperature limit, one of the eigenvalue of $M^2$ can become negative at some finite radius $r$. In other words, one of the eigenvalue of $M^2$ can become negative at some finite radius $r$, but as $r \to 0$, that value approaches zero. Because of this, even though one of the eigenvalue of $M^2$ becomes zero at the horizon in the case II, it can become some negative value at some finite radius $r$ and therefore, there could exist a zero mode in the whole bulk for the stripe instability. In such case, if the bulk region, where $M^2$ eigenvalue becomes negative, is away from the horizon $r_h$, then it is possible that lowering the temperature does not influence these bulk region much. As a result, in such a case, changing the $r_h$ does not change the $c_{\text{min}}$. However in order to obtain clear physical understanding of these results, we need more detail analysis.

It is also useful to draw the figure for the zero mode in $(q, T)$ plot with fixed $c_1$ value. We plot the critical temperatures $T$ versus $q$ for the normalizable zero mode in Fig. 6 (case I parameter choice with $c_1 = 1.3$) and Fig. 7 (case II parameter choice with $c_1 = 3.3$).

In the case I, for given temperature, we generically have two $q$’s allowing the normalizable zero modes. Un-
stable modes should exist for momentum $q$ between $q_{\text{min}}(T)$ and $q_{\text{max}}(T)$, and at $T \approx 2.05 \times 10^{-4}$, $q_{\text{min}}$ and $q_{\text{max}}$ coincide at $q \approx 0.83$. As we lower the temperature, $q_{\text{min}}(T)$ decreases and $q_{\text{max}}(T)$ increases. In the case I, because $c_1 Q_e/r_h^{3+3\kappa+k+\epsilon}$ goes to infinity as $r_h \to 0$, we expect to have very large effect of the axion term, and therefore we expect that $q_{\text{max}}$ becomes very large, at zero temperature limit. It would be really nice to confirm this numerically, however we could not conduct numerics at this very low temperature, $T \lesssim 4.2 \times 10^{-7}$ due to the difficulties of numerical analysis.

Again in the case II, we do not have clear physical interpretation of the results.

### IV. PERTURBATION ANALYSIS BEYOND THE PROBE LIMIT

We have analyzed so far the system without the graviton fluctuation, namely in the probe limit. However, it is pretty straightforward to conduct the similar analysis with the graviton fluctuation, and we will see that the analysis with graviton fluctuation shows essentially the same results, compared with the analysis without graviton. We will see here that there is a negative momentum squared mode in the hyperscaling violating geometry even after we take into account the coupling to the graviton.

We consider again the action (I). The fluctuations we consider are the following components, $\delta g_{ty}$, $\delta A_y$, $\delta a$, $\delta \phi$, and we take the following mode dependence,

$$
\begin{align*}
\delta g_{ty} &= \delta g_{ty}(r) \sin qx, \\
\delta A_y &= \delta A_y(r) \sin qx, \\
\delta a &= \delta a(r) \cos qx, \\
\delta \phi &= 0.
\end{align*}
$$

which has additional graviton mode. Here we have set $\omega = 0$, i.e., no time-dependence from the beginning compared with eq. (33) and (34), in order to discuss the onset of the instability.

Quite analogously to the previous analysis in [III A] in the probe limit, given the background geometry eq. (3), we have the equations of motion for the gauge field $\delta A_y$,

$$
\begin{align*}
-f(\phi)b^{-2}q^2 \delta A_y(r)
+\partial_r \left( f(\phi)a^2 \partial_r \delta A_y(r) + f(\phi)(\partial_r A_t)\delta g_{ty}(r) \right)
+(\partial_a \theta(a)) \frac{qQ_x}{b^2 f(\phi)} \delta a(r) &= 0, \\
(102)
\end{align*}
$$

and for the axion,

$$
\begin{align*}
\partial_r (e^{2\xi (\phi)} \sqrt{-g} g^{rr} \partial_r \delta a(r)) &= q^2 e^{2\xi (\phi)} \sqrt{-g} g^{xx} \delta a(r)
+q(\partial_a \theta(a)) F_{tr} \delta A_y(r) = 0.
(103)
\end{align*}
$$

This equation is unmodified by $\delta g_{ty} \neq 0$.

In addition, from the $(t, r)$ component of the trace reversed Einstein equations, we have fluctuation equation for the graviton,

$$
\begin{align*}
\delta h_{tr} &= \left( (\partial_t^2 \delta g_{ty}(r)) - 4f(\phi) F_{tr} (\partial_r \delta A_y(r)) \right)
+\left( 4ab(\partial_r \hat{a})(\partial_r b) + b^2 (2f(\phi)(F_{tr})^2 + V(\phi) \right)
-q^2 \right) \delta g_{ty}(r) = 0,
(104)
\end{align*}
$$

and all the other components of Einstein equations and equations of motion are automatically satisfied.

Let’s investigate the near horizon limit in similar way to the analysis of [III B] and [III C]. From graviton fluctuation eq. (104), by imposing the regularity condition of the solution at the horizon, we can see that we need the boundary condition $\delta g_{ty}(r = r_h) = 0$ at the horizon. Then, it is more convenient to set the new variable $\delta h_{ty} = \partial_t \delta g_{ty}$. And quite analogously to the case where we neglect graviton fluctuation in [III C], the three equations eq. (102) - (104) are approximated and written in the near horizon as

$$
\begin{align*}
\hat{\nabla}^2 \delta \tilde{Y} |_{r=r_h} &\approx \check{M}^2 |_{r=r_h} \delta \tilde{Y} |_{r=r_h}, \\
\text{where}
\end{align*}
$$

$$
\begin{align*}
\hat{\nabla}^2 &= \partial_t \tilde{a}^2 \partial_t, \\
\delta \tilde{Y} &= \left( \begin{array}{c}
\delta A_y \\
\delta a \\
\partial_r (\delta g_{ty})
\end{array} \right),
(106)
\end{align*}
$$

and

$$
\check{M}^2 = \left( \begin{array}{ccc}
q^2/b^2 & (M_{12}(r))^2 F_{tr} & 0 \\
(M_{21}(r))^2 F_{tr} & q^2/b^2 & 0 \\
4f(\phi) F_{tr} q^2/b^2 & (M_{23}(r))^2 (M_{33}(r))^2
\end{array} \right),
(107)
$$

$$
\begin{align*}
(M_{12}(r))^2 &= -(\partial_a \theta(a)) q F_{tr}/f(\phi), \\
(M_{21}(r))^2 &= -(\partial_\alpha \theta(a)) q F_{tr}/e^{2\xi (\phi)} b^2, \\
(M_{32}(r))^2 &= -4q(\partial_a \theta(a)) q (F_{tr})^2, \\
(M_{33}(r))^2 &= \frac{q^2}{b^2} + 2f(\phi)(F_{tr})^2 - \frac{4ab(\partial_r \hat{a})(\partial_r b)}{b^2} - V(\phi),
(111)
\end{align*}
$$
Therefore even with the graviton fluctuation, \( \delta A \) is not satisfied, so we expect that the effect of axion term are suppressed as we lower the temperature and intrigues more stripe instability, and the behavior of minimum \( c_1 \) for stripe instability is expected to show very similar behavior to the Fig. 4. On the other hand, if \( \delta A \) is satisfied, we expect that the effect of axion term are enhanced as we lower the temperature at the horizon and this intrigues less instability, and the behavior of minimum \( c_1 \) for stripe instability is expected to show very similar behavior to the Fig. 5. It would be best if we can confirm this by solving the eq. (102) - (104) numerically with the normalizable boundary condition at the UV AdS boundary as we have done in the probe limit in III. However in this case, the parameter range we seek for the normalizable boundary conditions becomes 3-dimensional, instead of 2-dimensional for the probe limit case, and this turns out quite hard task. Therefore we leave this as future work on this paper.

Clearly at the large \( c_1 \) limit, we can have a mode which has large negative eigenvalue at some radius, and this indicates the striped phase instability can occur. One difference, compared to the probe limit, is that the analysis for the analytic expression for the onset of the instability in III.C.2 does not work. This is because even if both eq. [64] and [65] hold, the determinant of the matrix \( M^2 \) is proportional to \( r^{-6\beta} \). So we need \( \beta = 0 \) and \( \gamma = 1 \) for the matrix \( M^2 \) to become constant matrix, which is AdS2 case. This implies that we need to introduce one more parameter in the Lagrangian to be tuned, so that we can have constant matrix \( M^2 \) in the presence of graviton.

\[ \text{V. SUMMARY AND DISCUSSION} \]

In this paper, we studied the stripe instabilities (spatially modulated instabilities) of the geometries with hyperscaling violation in the IR, which approach AdS4 metric in the UV asymptotically. The instabilities are induced by the axion term \( \delta S = \int d^4 x c_1 a F \wedge F \) in the bulk 4d action. We first study the perturbation equations in the probe limit, and saw that there is a strong correlation between the stripe instabilities caused by the axion term and parameters of the theories which determine the hyperscaling violation. Contrary to the IR AdS2 case, we found that, due to the lack of scale invariance and the nontrivial radial dependence of the IR hyperscaling violating geometries, the effect of axion term can be either enhanced or suppressed depending on the parameters. In the parameter range where the effect of the axion term is expected to be enhanced, the stripe instability occurs and \( c_{\min} \) decreases as we lower the temperature, where \( c_{\min} \) is the critical value for the instability and instability occurs only at \( c_1 \geq c_{\min} \). On the other hand, in the parameter range where the effect of axion term is expected to be suppressed, we find that \( c_{\min} \) does not change much as we lower the temperature. We have explicitly obtained the zero mode solutions for the coupled fluctuations of gauge boson \( \delta A_y \) and axion \( \delta a \) numerically in the probe limit, with the boundary condition that there are no non-normalizable modes. This implies that in the dual field theories, the scalar and vector current \( \langle j_y \rangle \), which are dual to axion \( a \) and gauge boson \( A_y \) in the bulk, acquire the spatially modulated VEV spontaneously, and that dual theories at IR show the “current density wave” phase. We identify the instability onset on a certain one-
parameter family of the hyperscaling violating geometries analytically, where the relation eq. (65) holds. We also argue that quite analogous results are expected to hold beyond the probe limit.

There are several open issues which should be understood in better way. We have done our search of the zero mode on rather limited temperature range in §III D. This is due to the numerical difficulties, and it comes from the fact that our background solutions, which are hyperscaling violating geometries in IR and approach AdS4 in UV, are constructed only numerically. If we could construct an analytical solution, we can search numerically for the zero mode more accurately. So it is interesting and important to look for analytical background solutions which interpolate between UV AdS4 and IR hyperscaling violating geometries. By conducting the numerical analysis in better way, we can check if the $c_{\text{min}}$ goes to zero or not in the zero temperature limit in Fig. 4. Similarly, we can check how the $T - q$ curve behaves in the zero temperature limit in Fig. 6. It is interesting to check these.

We argue in §IV that the results of instability analyses are essentially the same by taking into account the graviton effects. Of course, it is better if we could confirm this by searching for the zero mode explicitly on the full geometries, as we have done in §III D in the probe limit.

There are other issues which we would like to understand in better way. In this paper, we did not argue the validity of the action eq. (1) for the background solutions. However, it can happens that the starting action is not valid for describing the solutions, depending on the behavior of the solutions. For example, our background dilaton has run-away behavior, and if the background dilaton runs to the strong coupling direction in IR, then we have to worry about the possibility that our starting action is highly corrected due to the strong coupling effects. Such a possibility exists if the starting action is derived under the weak coupling approximation. For example, from our action eq. (1), the effective coupling of the gauge field $g_U(\phi)$ is given by $g_U(\phi) \equiv f(\phi) = e^{2\phi\phi}$. In our background solution this behaves as $g_U(\phi) \rightarrow r^{-\alpha k}$ in IR. If we assume that our theory eq. (1) is derived under the weak coupling condition, this with eq. (13) forces us $-\alpha k \propto \alpha(\alpha + \delta) > 0$ for consistency. But this implies that our condition eq. (84) and (85), for the case $\xi = 0$, cannot be satisfied. And we have only the parameter range where the axion term is expected to be suppressed, which corresponds to case II in the analysis of §III D. These issues should be understood more from the string theory embedding view point. However rather in this paper, we study the stability analysis with the assumption that the action eq. (1) is valid for any solutions, we do not “derive” our action eq. (1) from string theory. It would be nice to study these consistency points in more detail.

It could be that if the background dilaton blows up in the IR hyperscaling violating geometries, then we need to take into account the higher loop corrections and this might make the geometries into the AdS2 metric in further IR, as studied, for examples, in [51–53]. This viewpoint resolves the problematic singularities of the zero temperature limit of the hyperscaling violating geometries at $r \rightarrow 0$ [66]. However, it is not clear if this is always the case. For examples, once higher loop corrections (corresponds to higher string coupling $g_s$ corrections) enter the game, we always need to worry about full loop correction effects. Namely, once we face the situations where higher order $g_s$ effects are as important as leading order in $g_s$ expansion, this implies that $g_s$ expansion is no more valid. But in general, we do not have such a fully non-perturbative effective action, and the validity of the loop corrected effective action in order to derive the deep IR AdS2 metric is unclear.

Another interesting question is the end point of the stripe instability. In this paper, we studied the onset of the stripe instabilities. However to see what is the end point of these instabilities, perturbation analysis is not enough. We need to study the equations of motion where $A_\phi$ and $a$ are coupled in the probe limit, and the full back reacted Einstein equations to go beyond the probe limit. For successful examples of the end point of stripe instabilities, see [40, 42, 46].

In the probe limit, we identify the onset of the stripe instability when the relation eq. (65) holds. It is interesting to note that this relation also holds at the transition point between quasi-particle picture holds/breaks down from the study of the fermion Green’s function on these background [30]. It is interesting to investigate to see if there are any deep reason for this coincident.

There are many open questions to be understood better. However one thing which is very clear is that these geometries with hyperscaling violation and stripe instabilities are rich subject and it is worth understanding more in great detail. We hope to return these questions in future.

**Note added:** When we have almost finished preparing for the draft, a paper appeared [54], where they also studied the stripe instability on the hyperscaling violating geometries. However our set-up and analysis is different from the one in [54]. The authors of [54] studied the geometries whose IR $(r \rightarrow 0)$ is AdS2, and in large $r$, they approach the hyperscaling violating geometries. They identified the onset of stripe instability in this IR AdS2 region by using the AdS2 BF bound, i.e., local AdS2 argument. Then they interpret that onset in terms of the large $r$ hyperscaling violating parameters. On the other hand, in this paper we studied the geometries whose IR are hyperscaling violating geometries, which interpolate to the AdS4 in the UV.

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[63] One can also construct black brane solutions which do not have large entropy by admitting spatial anisotropy, see for example, [59].

[64] This solution is obtained under the assumption $Q_c \neq 0$. For the case $Q_c = 0$, the derivation of this solution in [30] breaks down since we do not have to require the condition eq. (2.19) in [30]. Actually AdS$_3$ is such a limit, where we have $\beta = 1, \gamma = 1, k = 0, \delta = 0, C_2^a = -\nu_0/6$ and $Q_c = 0$ with arbitrary choice of $\alpha$.

[65] In the asymptotic AdS$_3$ region, $\phi$ behaves as $\phi \approx D_+ r^{\nu_0 + \delta} - D_+ r^{-\nu_0} - k$ as $r \to \infty$, where $\lambda_+ = -3/2 \pm \sqrt{9/2 - 24\nu_0^2}/2$. As shown in [15], the theory is generally unstable for the perturbation with respect to $\phi$ when both modes exist. However for the perturbation analysis we conduct in this paper, the dilaton $\phi$ does not fluctuate. Therefore we consider that the instability of [15] is not related to the stripe instability we investigate in this paper. In addition, for the numerical analysis in [15], we choose $D_+ \propto \nu_0$ as small as possible in the range $|\nu_0| < 1$ (i.e., $1 - 2\nu_0^2 \approx 0$).

[66] $Q_c$ is actually not a parameter, since it takes definite value as eq. (13).

[67] This simple method gives the generalized Breitenlohner - Freedman (BF) bound for the stability in non-AdS case, for examples, [79, 50].

[68] See eq. (15), if $\gamma < 1/2$, then $c_{\nu_0} \to 0$ with $T \to \infty$.

[69] We thank Noriaki Ogawa for pointing this out to us.

[70] “Physical” in the sense that $Q^2 > 0, C_a^2 > 0, \gamma > 0, \gamma - \beta > 0, \beta > 0, \text{and} \delta k < 0$ are satisfied.

[71] To see this, note that in the UV, we can similarly construct the matrix $M^2$ from the equations of motion for $\delta A_\mu$ and $\delta \sigma$. The eigenmode in the UV (let us call these as $A'$ and $B'$) are different from the eigenmode model $A$ and $B$ (defined in IR) generically since the metric is different in UV and IR. The mode $A'$ and $B'$ are specific linear combinations of the mode $A$ and $B$. Now we need to impose boundary condition both at the IR horizon and UV boundary. At the horizon, we have ingoing boundary conditions or regularity condition for both $A$ and $B$, and at the UV boundary we impose that non-normalizable mode vanishes for both $A'$ and $B'$. Since for the second order differential equations, we have imposed two boundary conditions (one at the horizon and one at the boundary), we do not have parameter left and this generically implies that we are forced to have excitation both mode $A$ and $B$ in the IR near horizon region generically.

[72] We thank Aristos Donos and Jerome Gauntlett for raising this question to us.

[73] In order to derive this result, we have used two conditions: 1) terms proportional to $\delta g_{yy}$, are neglected
since $\delta g_{tr}|_{r=r_h} \to 0$, and 2) using the regularity of the flux $F_{tr}$ and dilaton $f(\phi)$ at the finite temperature horizon $r = r_h$, we can neglect term proportional to $\partial_r \delta A_y(r)$ since $|\left[ (\partial_r (f(\phi) F_{tr})) \tilde{a}^2 \partial_r \delta A_y(r) \right]|_{r=r_h} \ll |\left[ f(\phi) F_{tr} (\partial_\tilde{r} \tilde{a}^2) \partial_r \delta A_y(r) \right]|_{r=r_h}$ holds.

[66] This singularities can be avoided by introducing small but non-zero temperature. And our studies of the stripe instability in small but non-zero temperature case are not affected by this singularities.