THE $[n_1,n_2,\ldots,n_s]$–TH REDUCED KP HIERARCHY AND $W_{1+\infty}$ CONSTRAINTS

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ABSTRACT. To every partition $n = n_1 + n_2 + \cdots + n_s$ one can associate a vertex operator realization of the Lie algebras $a_\infty$ and $\hat{gl}_n$. Using this construction we obtain reductions of the $s$–component KP hierarchy, reductions which are related to these partitions. In this way we obtain matrix KdV type equations. We show that the following two constraints on a KP $\tau$–function are equivalent (1) $\tau$ is a $\tau$–function of the $[n_1,n_2,\ldots,n_s]$–th reduced KP hierarchy which satisfies string equation, $L^{-1}\tau = 0$, (2) $\tau$ satisfies the vacuum constraints of the $W_{1+\infty}$ algebra.

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§1. $a_\infty$ and the KP hierarchy in the fermionic picture [KV].

Consider the infinite dimensional complex Lie algebra $a_\infty := \overline{gl}_\infty \oplus \mathbb{C} c$, where

$$\overline{gl}_\infty = \{ a = (a_{ij})_{i,j \in \mathbb{Z}^+} | a_{ij} = 0 \text{ if } |i - j| >> 0 \},$$

with Lie bracket defined by

(1.1) \[ [a + \alpha c, b + \beta c] = ab - ba + \mu(a,b)c, \]

for $a, b \in \overline{gl}_\infty$ and $\alpha, \beta \in \mathbb{C}$. Here $\mu$ is the following 2–cocycle:

(1.2) \[ \mu(E_{ij}, E_{kl}) = \delta_{il} \delta_{jk} (\theta(i) - \theta(j)), \]

where $E_{ij}$ is the matrix with a 1 on the $(i,j)$-th entry and zeros elsewhere and $\theta : \mathbb{R} \to \mathbb{C}$ is the step–function defined by

(1.3) \[ \theta(i) := \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \leq 0. \end{cases} \]

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Let $\mathbb{C}^\infty = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} v_j$ be the infinite dimensional complex vector space with fixed basis \( \{ v_j \}_{j \in \mathbb{Z} + \frac{1}{2}}. \) The Lie algebra $a_\infty$ acts linearly on $\mathbb{C}^\infty$ via the usual formula: 

\[
E_{ij}(v_k) = \delta_{jk} v_i.
\]

Let $\mathcal{C}_\ell$ be the Clifford algebra with generators $\psi^{\lambda}_i, i \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -$, satisfying the following relations:

(1.4) 

\[
\psi^{\lambda}_i \psi^{\mu}_j + \psi^{\mu}_j \psi^{\lambda}_i = \delta_{\lambda, -\mu} \delta_{i, -j}.
\]

We define an irreducible $\mathcal{C}_\ell$–module $F$ by introducing a vacuum vector $|0\rangle$ such that

(1.5) 

\[
\psi^\pm_j |0\rangle = 0 \text{ for } j > 0.
\]

Define a representation $\hat{r}$ of $a_\infty$ on $F$ by

\[
\hat{r}(E_{ij}) =: \psi^-_i \psi^+_j :, \quad \hat{r}(c) = I,
\]

where $: :$ stands for the normal ordered product defined in the usual way ($\lambda, \mu = +$ or $-$):

(1.6) 

\[
: \psi^{\lambda(i)}_k \psi^{\mu(j)}_\ell : = \begin{cases} 
\psi^{\lambda(i)}_k \psi^{\mu(j)}_\ell & \text{if } \ell \geq k \\
-\psi^{\mu(j)}_\ell \psi^{\lambda(i)}_k & \text{if } \ell < k.
\end{cases}
\]

Define the charge decomposition

(1.7) 

\[
F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}
\]

by letting

(1.8) 

\[
\text{charge}(|0\rangle) = 0 \text{ and } \text{charge}(\psi^\pm_j) = \pm 1.
\]

It is easy to see that each $F^{(m)}$ is irreducible with respect to $a_\infty$.

We are now able to define the KP hierarchy in the fermionic picture, it is the equation

(1.9) 

\[
\sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^+_k \tau \otimes \psi^-_{-k} \tau = 0,
\]

for $\tau \in F^{(0)}$. One can prove (see e. g. [KP2] or [KR]) that this equation characterizes the group orbit of the vacuum vector $|0\rangle$ for the the group $GL_\infty$. Since the group does not play an important role in this paper, we will not introduce it here.
§2. Vertex operator constructions.

We will now sketch how one can construct vertex realizations of the affine Lie algebra \( \hat{\mathfrak{gl}}_n \), following [TV] (see also [KP1] and [L]). From now on let \( n = n_1 + n_2 + \cdots + n_s \) be a partition of \( n \) into \( s \) parts, and denote by \( N_a = n_1 + n_2 + \cdots + n_{a-1} \). We begin by relabeling the basis vectors \( v_j \) and with them the corresponding fermionic operators:

\[
\begin{align*}
\psi^{(a)}_{n_a j - p + \frac{1}{2}} &= v_{n_j - N_a - p + \frac{1}{2}} \\
\psi^{\pm(a)}_{n_a j + p + \frac{1}{2}} &= \psi^{\pm}_{n_a j + N_a + p + \frac{1}{2}}
\end{align*}
\]  

(2.1)

Notice that with this relabeling we have: \( \psi^{\pm(a)}_k |0\rangle = 0 \) for \( k > 0 \). We also rewrite the \( E_{ij} \)'s:

\[
E^{(ab)}_{n_a j - p + \frac{1}{2}, n_b k - q + \frac{1}{2}} = E_{n_j - N_a - p + \frac{1}{2}, n_k - N_b - q + \frac{1}{2}};
\]

then \( \hat{r}(E^{(ab)}_{jk}) =: \psi^{-(a)}_j \psi^{+(a)}_k \).

Introduce the fermionic fields \((z \in \mathbb{C}^\times)\):

\[
\psi^{\pm(a)}(z) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{(a)}_k z^{-k - \frac{1}{2}}.
\]

(2.3)

Let \( N \) be the least common multiple of \( n_1, n_2, \ldots, n_s \). It was shown in [TV] that the modes of the fields

\[
: \psi^{+(a)}(\omega^p_a z^{N_a}) \psi^{-(b)}(\omega^q_b z^{N_b}) :,
\]

(2.4)

for \( 1 \leq a, b \leq s, 1 \leq p \leq n_a, 1 \leq q \leq n_b \), where \( \omega_a = e^{2\pi i / n_a} \), together with the identity, generate a representation of \( \hat{\mathfrak{gl}}_n \) with center \( K = 1 \). It is easy to see that restricted to \( \hat{\mathfrak{gl}}_n \), \( F^{(a)} \) is its basic highest weight representation (see [K, Chapter 12]). From (2.4) it is obvious, that if we obtain vertex operators for the fermionic fields \( \psi^{\pm(a)}(z) \), we also have a vertex operator realization of \( \hat{\mathfrak{gl}}_n \).

Next we introduce special bosonic fields \((1 \leq a \leq s)\):

\[
\alpha^{(a)}(z) \equiv \sum_{k \in \mathbb{Z}} \alpha^{(a)}_k z^{-k - 1} \overset{\text{def}}{=} : \psi^{+(a)}(z) \psi^{-(a)}(z) :.
\]

(2.5)

The operators \( \alpha^{(a)}_k \) satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by \( \mathfrak{a} \):

\[
[\alpha^{(i)}_k, \alpha^{(j)}_\ell] = k \delta_{ij} \delta_{k,-\ell},
\]

(2.6)
and one has
\[ \alpha^{(i)}_k |m \rangle = 0 \text{ for } k > 0. \]

This realization of \( \hat{gl}_n \), has a natural Virasoro algebra. In [TV], it was shown that the following two sets of operators have the same action on \( F \).

\[
L_k = \sum_{i=1}^{s} \sum_{j \in \mathbb{Z}} \frac{1}{2n_i} \cdot \alpha^{(i)}_{-j} \alpha^{(i)}_{j+n_i k} : + \delta_{k0} \frac{n_i^2 - 1}{24n_i},
\]

\[
H_k = \sum_{i=1}^{s} \sum_{j \in \mathbb{Z} + \frac{1}{2}} \left( \frac{j}{n_i} + \frac{k}{2} \right) : \psi^{+(i)}_{-j} \psi^{-(i)}_{j+n_i k} : + \delta_{k0} \frac{n_i^2 - 1}{24n_i}.
\]

So \( L_k = H_k \),

\[
[L_k, \psi^{\pm(i)}_j] = -(\frac{j}{n_i} + \frac{k}{2}) \psi^{\pm(i)}_{j+n_i k}
\]

and

\[
[L_k, L_\ell] = (k - \ell)L_{k+\ell} + \delta_{k,-\ell} \frac{k^3 - k}{12} n.
\]

We will now describe the \( s \)-component boson-fermion correspondence (see [KV]). Let \( \mathbb{C}[x] \) be the space of polynomials in indeterminates \( x = \{x_{(i)}_k\}, k = 1, 2, \ldots, i = 1, 2, \ldots, s \). Let \( L \) be a lattice with a basis \( \delta_1, \ldots, \delta_s \) over \( \mathbb{Z} \) and the symmetric bilinear form \( (\delta_i | \delta_j) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. Let

\[
\varepsilon_{ij} = \begin{cases} 
-1 & \text{if } i > j \\
1 & \text{if } i \leq j.
\end{cases}
\]

Define a bimultiplicative function \( \varepsilon : \ L \times L \rightarrow \{\pm1\} \) by letting

\[
\varepsilon(\delta_i, \delta_j) = \varepsilon_{ij}.
\]

Let \( \delta = \delta_1 + \ldots + \delta_s, \ Q = \{ \gamma \in L | (\delta | \gamma) = 0 \}, \Delta = \{ \alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, s, i \neq j \}. \) Of course \( Q \) is the root lattice of \( \text{sl}_s(\mathbb{C}) \), the set \( \Delta \) being the root system.

Consider the vector space \( \mathbb{C}[L] \) with basis \( e^\gamma, \gamma \in L \), and the following twisted group algebra product:

\[
e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}.
\]

Let \( B = \mathbb{C}[x] \otimes_\mathbb{C} \mathbb{C}[L] \) be the tensor product of algebras. Then the \( s \)-component boson-fermion correspondence is the vector space isomorphism

\[
\sigma : F \xrightarrow{\sim} B,
\]
given by $\sigma(|0\rangle) = 1$ and

\begin{equation}
\sigma \psi^{\pm(a)}(z)\sigma^{-1} = e^{\pm \delta a} z^{\pm \delta a} \exp(\pm \sum_{k=1}^{\infty} x_k^{(a)} z_k) \exp(\pm \sum_{k=1}^{\infty} \frac{\partial}{\partial x_k^{(a)}} \frac{z^{-k}}{k}),
\end{equation}

where

\begin{equation}
\delta_a(p(x) \otimes e^\gamma) = (\delta_a |\gamma\rangle)p(x) \otimes e^\gamma.
\end{equation}

The transported charge then is as follows:

\[ \text{charge}(p(x) \otimes e^\gamma) = (\delta |\gamma\rangle). \]

We denote the transported charge decomposition by

\[ B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}. \]

The transported action of the operators $\alpha_m^{(i)}$ is given by

\begin{equation}
\begin{cases}
\sigma \alpha_{-m}^{(j)} \sigma^{-1}(p(x) \otimes e^\gamma) = mx_m^{(j)} p(x) \otimes e^\gamma, & \text{if } m > 0, \\
\sigma \alpha_m^{(j)} \sigma^{-1}(p(x) \otimes e^\gamma) = \frac{\partial p(x)}{\partial x_m} \otimes e^\gamma, & \text{if } m > 0, \\
\sigma \alpha_0^{(j)} \sigma^{-1}(p(x) \otimes e^\gamma) = (\delta_j |\gamma\rangle)p(x) \otimes e^\gamma.
\end{cases}
\end{equation}

If one substitutes (2.13) into (2.4), one obtains the vertex operator realization of $\hat{gl}_n$ which is related to the partition $n = n_1 + n_2 + \cdots + n_s$ (see [TV] for more details).

Using the isomorphism $\sigma$ we can reformulate the KP hierarchy (1.9) in the bosonic picture.

We start by observing that (1.9) can be rewritten as follows:

\begin{equation}
\text{Res}_{z=0} \frac{dz}{z} \sum_{j=1}^{s} \psi^{(j)}(z) \tau \otimes \psi^{-(j)}(z) \tau = 0, \; \tau \in F^{(0)}.
\end{equation}

Notice that for $\tau \in F^{(0)}$, $\sigma(\tau) = \sum_{\gamma \in Q} \tau_\gamma(x^\gamma) e^\gamma$. Here and further we write $\tau_\gamma(x^\gamma) e^\gamma$ for $\tau_\gamma \otimes e^\gamma$. Using (2.13), equation (2.16) turns under $\sigma \otimes \sigma : F \otimes F \sim \mathbb{C}[x', x''] \otimes (\mathbb{C}[L'] \otimes \mathbb{C}[L''])$ into the following set of equations; for all $\alpha, \beta \in L$ such that $\langle \alpha |\delta\rangle = -\langle \beta |\delta\rangle = 1$ we have:

\begin{equation}
\begin{aligned}
\text{Res}_{z=0} &\left( \frac{dz}{z} \sum_{j=1}^{s} \mathcal{E}(\delta_j, \alpha - \beta) z^{(\delta_j |\alpha - \beta - 2\delta_j)} \right) \\
&\times \exp(\sum_{k=1}^{\infty} (x_k^{(j)' - x_k^{(j)''})} z^k) \exp(-\sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_k^{(j)'}} - \frac{\partial}{\partial x_k^{(j)''}} \right) \frac{z^{-k}}{k}) \\
&\tau_{\alpha - \delta_j}(x') (e_\alpha)' \tau_{\beta + \delta_j}(x'') (e_\beta)' = 0.
\end{aligned}
\end{equation}
§3. The algebra of formal pseudo-differential operators and the s-component KP hierarchy as a dynamical system [KV].

We proceed now to rewrite the formulation (2.17) of the s-component KP hierarchy in terms of formal pseudo-differential operators, generalizing the results of [DJKM] and [JM]. For each \( \alpha \in \text{supp } \tau := \{ \alpha \in Q | \tau = \sum_{\alpha \in Q} \tau_{\alpha} e^{\alpha}, \tau_{\alpha} \neq 0 \} \) we define the (matrix valued) functions

\[
V^\pm(\alpha, x, z) = (V^\pm_{ij}(\alpha, x, z))^{s}_{i,j=1}
\]

as follows:

\[
V^\pm_{ij}(\alpha, x, z) \stackrel{\text{def}}{=} \varepsilon(\delta_j, \alpha + \delta_i)z^{(\delta_j, \pm \alpha + \delta_i - \delta_j)} \\
\times \exp(\pm \sum_{k=1}^{\infty} x^{(j)}_k z^k) \exp(\pm \sum_{k=1}^{\infty} \frac{\partial}{\partial x^{(j)}_k} z^{-k}) \tau_{\alpha \pm (\delta_i - \delta_j)}(x)/\tau_{\alpha}(x).
\]

It is easy to see that equation (2.17) is equivalent to the following bilinear identity:

\[
\text{Res}_{z=0} V^+(\alpha, x, z)^t V^-(\beta, x', z) dz = 0 \text{ for all } \alpha, \beta \in Q.
\]

Define \( s \times s \) matrices \( W_{\pm}^{(m)}(\alpha, x) \) by the following generating series (cf. (3.2)):

\[
\sum_{m=0}^{\infty} W_{ij}^{\pm(m)}(\alpha, x) (\pm z)^{-m} = \varepsilon_{ji} z^{\delta_{ij} - 1} (\exp(\pm \sum_{k=1}^{\infty} \frac{\partial}{\partial x^{(j)}_k} z^{-k}) \tau_{\alpha \pm (\delta_i - \delta_j)}(x))/\tau_{\alpha}(x).
\]

We see from (3.2) that \( V^\pm(\alpha, x, z) \) can be written in the following form:

\[
V^\pm(\alpha, x, z) = (\sum_{m=0}^{\infty} W_{ij}^{\pm(m)}(\alpha, x) R_{\pm}^{\pm}(\alpha, \pm z)(\pm z)^{-m})e^{\pm z \cdot x},
\]

where

\[
z \cdot x^{(j)} = \sum_{k=1}^{\infty} x^{(j)}_k z^k, \quad e^{z \cdot x} = \text{diag}(e^{z \cdot x^{(1)}}, \ldots, e^{z \cdot x^{(s)}})
\]

and

\[
R_{\pm}^{\pm}(\alpha, z) = \sum_{i=1}^{s} \varepsilon(\delta_i, \alpha) E_{ii}(\pm z)^{\pm (\delta_i | \alpha)}.
\]

Here and further \( E_{ij} \) stands for the \( s \times s \) matrix whose \( (i, j) \) entry is 1 and all other entries are zero. Let

\[
\partial = \frac{\partial}{\partial x^{(1)}_1} + \ldots + \frac{\partial}{\partial x^{(s)}_1}.
\]
we can now rewrite \( V^\pm(\alpha, x, z) \) in terms of formal pseudo-differential operators

\[
V^\pm(\alpha, x, z) = P^\pm(\alpha, x, \partial) = I_s + \sum_{m=1}^{\infty} W^\pm(m)(\alpha, x)\partial^{-m} \quad \text{and} \quad R^\pm(\alpha) = R^\pm(\alpha, \partial)
\]

as follows:

\[
V^\pm(\alpha, x, z) = P^\pm(\alpha)R^\pm(\alpha)e^{\pm z: x}.
\]

As usual one denotes the differential part of \( P(x, \partial) \) by \( P_+(x, \partial) = \sum_{j=0}^{N} P_j(x)\partial^j \), and writes \( P_- = P - P_+ \). The linear anti-involution \( \ast \) is defined by the following formula:

\[
(\sum_j P_j\partial^j)^\ast = \sum_j (-\partial)^j \circ t P_j.
\]

Here and further \( tP \) stands for the transpose of the matrix \( P \). Then one has the following fundamental lemma (see [KV]):

**Lemma 3.1.** If \( P, Q \in \Psi \) are such that

\[
\text{Res}_{z=0}(P(x, \partial)e^{z: x})^t(Q(x', \partial')e^{-z: x'})dz = 0,
\]

then \( (P \circ Q^\ast)_- = 0 \).

Using this Lemma, Victor Kac and the author showed in [KV] that given \( \beta \in \text{supp} \, \tau \), all the pseudo-differential operators \( P^\pm(\alpha), \alpha \in \text{supp} \, \tau \), are completely determined by \( P^+(\beta) \) from the following equations

\[
R^-(\alpha, \partial)^{-1} = R^+(\alpha, \partial)^\ast, \quad P^-(\alpha) = (P^+(\alpha)^\ast)^{-1}, \quad (P^+(\alpha)R^+(\alpha - \beta)P^+(\beta)^{-1})_- = 0 \text{ for all } \alpha, \beta \in \text{supp} \, \tau.
\]

They also showed the following

**Proposition 3.2.** Consider \( V^+(\alpha, x, z) \) and \( V^-(\alpha, x, z) \), \( \alpha \in Q \), of the form (3.8), where \( R^\pm(\alpha, z) \) are given by (3.6). Then the bilinear identity (3.3) for all \( \alpha, \beta \in \text{supp} \, \tau \) is equivalent to the Sato equation:

\[
\frac{\partial P}{\partial x^{(j)}} = -(PE_{jj} \circ \partial^k \circ P^{-1})_- \circ P.
\]

for each \( P = P^+(\alpha) \) and the matching conditions (3.10-12) for all \( \alpha, \beta \in \text{supp} \, \tau \).

Fix \( \alpha \in Q \), introduce the following formal pseudo-differential operators \( L(\alpha), C^{(j)}(\alpha), \) and differential operators \( B_{m}^{(j)}(\alpha) : \)

\[
L \equiv L(\alpha) = P^+(\alpha) \circ \partial \circ P^+(\alpha)^{-1},
\]

\[
C^{(j)} \equiv C^{(j)}(\alpha) = P^+(\alpha)E_{jj}P^+(\alpha)^{-1},
\]

\[
B_{m}^{(j)} \equiv B_{m}^{(j)}(\alpha) = (P^+(\alpha)E_{jj} \circ \partial^m \circ P^+(\alpha)^{-1})_+.
\]

Then

\[
\sum_{i=1}^{s} C^{(i)} = I_s, \quad C^{(i)}L = LC^{(i)}, \quad C^{(i)}C^{(j)} = \delta_{ij}C^{(i)}.
\]
Proposition 3.3. If for every $\alpha \in Q$ the formal pseudo-differential operators $L \equiv L(\alpha)$ and $C^{(j)}(\alpha)$ of the form (3.14) satisfy conditions (3.15) and if the equations

\[
\begin{cases}
LP = P\partial \\
C^{(i)}P = PE_{ii} \\
\frac{\partial P}{\partial x_k^{(i)}} = -(L^{(i)})^{-1}P, \text{ where } L^{(i)} = C^{(i)}L.
\end{cases}
\]

have a solution $P \equiv P^+(\alpha)$ of the form (3.7), then the differential operators $B_k^{(j)} \equiv B_k^{(j)}(\alpha)$ satisfies the following conditions:

\[
\begin{cases}
\frac{\partial L}{\partial x_k^{(j)}} = [B_k^{(j)}, L], \\
\frac{\partial C^{(i)}}{\partial x_k^{(j)}} = [B_k^{(j)}, C^{(i)}],
\end{cases}
\]

Finally, we introduce one more pseudo–differential operator

\[
M(\alpha) := P(\alpha)R(\alpha) \sum_{a=1}^{\infty} \sum_{k=1}^{\infty} k x_k^{(a)} \partial^{k-1} E_{aa} R(\alpha)^{-1} P(\alpha)^{-1}.
\]

Then

\[
M(\alpha)V^+(\alpha, x, z) = \frac{\partial V^+(\alpha, x, z)}{\partial z}
\]

and one easily checks that $[L(\alpha), M(\alpha)] = 1$

§4. $[n_1, n_2, \ldots, n_s]$-reductions of the s-component KP hierarchy.

If one restricts the, to the partition $n = n_1 + n_2 + \cdots + n_s$ related, vertex operator construction of $\hat{gl}_n$ in the vector space $B^{(0)}$ to $\hat{sl}_n$, the representation is not irreducible anymore. In order to obtain irreducible representations of $\hat{sl}_n$, one has to ‘remove’ all elements

\[
\sum_{i=1}^{s} \alpha_{kn_i}^{(i)}, \quad k \in \mathbb{Z}.
\]

Hence, a KP $\tau$–function is an $\hat{sl}_n \tau$–function if

\[
\sum_{j=1}^{s} \frac{\partial \tau}{\partial x_k^{(j)}_{kn_j}} = 0, \quad \text{for all } k \in \mathbb{N}.
\]
We will call this the \([n_1, n_2, \ldots, n_s]\)-th reduced KP hierarchy. Using the Sato equation (3.13), this implies the following two equivalent conditions:

\[
\sum_{j=1}^{s} \frac{\partial V^{+}(\alpha, x, z)}{\partial x_{kn_j}^{(j)}} = V^{+}(\alpha, x, z) \sum_{j=1}^{s} z^{kn_j} E_{jj},
\]

(4.3) \[Q(\alpha) = 0,\]

where

\[
Q(\alpha) = \sum_{j=1}^{s} L(\alpha)^{kn_j} C(j).
\]

§5. The string equation and \(W_{1+\infty}\) constraints.

From now on we assume that \(\tau\) is any solution of the KP hierarchy. In particular, we no longer assume that \(\tau_{\alpha}\) is a polynomial. Of course this means that the corresponding wave functions \(V^{\pm}(\alpha, x, z)\) will be of a more general nature than before.

Let \(L_{-1}\) be given by (2.7a), the string equation is the following constraint on \(\tau \in F^{(0)}\):

\[
L_{-1} \tau = 0.
\]

(5.1)

Using (2.15) we rewrite \(L_{-1}\) in terms of operators on \(B^{(0)}\):

\[
L_{-1} = \sum_{a=1}^{s} \{ \delta_a x_{n_a}^{(a)} + \frac{1}{2n_a} \sum_{p=1}^{n_a-1} p(n_a - p) x_{p}^{(a)} x_{n_a-p}^{(a)} + \frac{1}{n_a} \sum_{k=1}^{\infty} (k + n_a) x_{k+n_a}^{(a)} \frac{\partial}{\partial x_{k}^{(a)}} \}.
\]

Since \(\tau = \sum_{\alpha \in Q} \tau_{\alpha} e^{\alpha}\), we find that \(L_{-1} \tau_{\alpha} = 0\) for all \(\alpha \in Q\). Using a calculation similar to the one in [D], one deduces from (5.1) that

\[
N(\alpha) = 0,
\]

(5.2)

where

\[
N(\alpha) := \sum_{a=1}^{s} \left\{ \frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} C^{(a)}(\alpha) - \frac{n_a-1}{2n_a} L(\alpha)^{-n_a} C^{(a)}(\alpha) \right\}.
\]

(5.3)

Hence \(N(\alpha)\) is a differential operator that satisfies

\[
[Q(\alpha), N(\alpha)] = 1.
\]

From now on we assume that \(\tau\) is a \(\tau\)-function of the \([n_1, n_2, \ldots, n_s]\)-th reduced KP hierarchy, which satisfies the string equation. So, we assume that (4.2) and (5.1) holds. Hence, for all \(\alpha \in \text{supp} \ \tau\) both \(Q(\alpha)\) and \(N(\alpha)\) are differential operators. Thus, also \(N(\alpha)^p Q(\alpha)^q\) is a differential operator, i.e.,

\[
(\sum_{a=1}^{s} \left\{ \frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} - \frac{n_a-1}{2n_a} L(\alpha)^{-n_a} \right\}^p L(\alpha)^{qn_a} C^{(a)}(\alpha) \} = 0 \quad \text{for} \ p, q \in \mathbb{Z}_+.
\]

(5.4)

This leads to
Lemma 5.1. For all $p, q \in \mathbb{Z}_+$ one has

$$
\text{Res}_{z=0} dz \sum_{a=1}^{s} z^{q_a} \left( \frac{1}{n_a} \frac{1-n_a}{2} \frac{\partial}{\partial z} \frac{1-n_a}{2} \right)^p \psi^+(a) (z) \tau_{\alpha+\delta_i-\delta_j} = 0.
$$

(5.5) Taking the $(i, j)$-th coefficient of (5.5) one obtains

Corollary 5.2. For all $1 \leq i, j \leq s$ and $p, q \in \mathbb{Z}_+$ one has

$$
\text{Res}_{z=0} dz \sum_{a=1}^{s} z^{q_a} \left( \frac{1}{n_a} \frac{1-n_a}{2} \frac{\partial}{\partial z} \frac{1-n_a}{2} \right)^p \psi^+(a) (z) \tau_{\alpha+\delta_i-\delta_j} = 0.
$$

(5.6) Let

$$
W^{(p+1)}_{q-p} = \text{Res}_{z=0} dz \sum_{a=1}^{s} z^{q_a} \left( \frac{1}{n_a} \frac{1-n_a}{2} \frac{\partial}{\partial z} \frac{1-n_a}{2} \right)^p \psi^+(a) (z) \tau_{\alpha+\delta_i-\delta_j}.
$$

(5.7) Using a generalization of an identity of Date, Jimbo, Kashiwara and Miwa [DJKM3] (see also [G], [V]), (5.6) is equivalent to

$$
- t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^\ell E_{ii} = \sum_{m \in \mathbb{Z}} -m(m-1) \cdots (m-\ell+1) E_{ii} \left( -m-k-\frac{1}{2}, -m-\frac{1}{2} \right).
$$

(5.8) If we ignore the cocycle term for a moment, then it is obvious from (5.7), that the elements $W^{(p+1)}_k$ are the generators of the $W$-algebra $W_{1+\infty}$ [Ra], [KRa] (the cocycle term, however, will be slightly different). Upto some modification of the elements $W^{(p+1)}_0$, one gets the standard commutation relations of $W_{1+\infty}$, where $c = nI$.

As the next step, we take in (5.9) $x^{(i)}_k = x^{(i)}_{k'}$, for all $k \in \mathbb{N}$, $1 \leq i \leq s$, we then obtain

$$
\left\{ \begin{array}{ll}
\frac{\partial}{\partial x^{(i)}_1} \left( \frac{W^{(p+1)}_{q-p}}{\tau_{\alpha}} \right) = 0 & \text{if } i = j, \\
\tau_{\alpha+\delta_i-\delta_j} W^{(p+1)}_{q-p} \tau_{\alpha} &= \tau_{\alpha} W^{(p+1)}_{q-p} \tau_{\alpha+\delta_i-\delta_j} & \text{if } i \neq j.
\end{array} \right.
$$

(5.10)
The last equation means that for all $\alpha, \beta \in \text{supp } \tau$ one has

$$
\frac{W_{q-p}^{(p+1)} \tau_\alpha}{\tau_\alpha} = \frac{W_{q-p}^{(p+1)} \tau_\beta}{\tau_\beta}.
$$

Next we divide (5.9) by $\tau_\alpha(x) \tau_\alpha(x')$, of course only for $\alpha \in \text{supp } \tau$, and use (5.11). Then for all $\alpha, \beta \in \text{supp } \tau$ and $p, q \in \mathbb{Z}_+$ one has

$$
\text{Res}_{z=0} dz \sum_{a=1}^s \left\{ \exp \left( - \sum_{k=1}^\infty \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)} \frac{\psi^+(a)(z) \tau_\alpha + \delta_a - \delta_a(x)}{\tau_\alpha(x)} e^{\alpha + \delta_a - \delta_a} \times
$$

$$
\frac{\psi^-(a)(z) \tau_\alpha + \delta_a - \delta_a(x')}{\tau_\alpha(x')} (e^{\alpha + \delta_a - \delta_a})' = 0.
$$

Since one also has the bilinear identity (3.3), we can subtract that part and thus obtain the following

**Lemma 5.3.** For all $\alpha, \beta \in \text{supp } \tau$ and $p, q \in \mathbb{Z}_+$ one has

$$
\text{Res}_{z=0} dz \sum_{a=1}^s \left\{ \exp \left( - \sum_{k=1}^\infty \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)} \times
$$

$$
\frac{\psi^+(a)(z) \tau_\alpha + \delta_a - \delta_a(x)}{\tau_\alpha(x)} e^{\alpha + \delta_a - \delta_a} \frac{\psi^-(a)(z) \tau_\alpha + \delta_a - \delta_a(x')}{\tau_\alpha(x')} (e^{\alpha + \delta_a - \delta_a})' = 0.
$$

Define

$$
S(\beta, p, q, x, z) := \sum_{a=1}^s \left\{ \exp \left( - \sum_{k=1}^\infty \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)} E_{aa}.
$$

Notice that the first equation of (5.10) implies that $\partial \circ S(\beta, p, q, x, \partial) = S(\beta, p, q, x, \partial) \circ \partial$. Then viewing (5.12) as the $(i, j)$-th entry of a matrix, (5.12) is equivalent to

$$
\text{Res}_{z=0} dz P^+(\alpha) R^+(\alpha) S(\beta, p, q, x, \partial) e^{x \cdot z} t(P^-(\alpha)' R^-(\alpha)' e^{-x \cdot z}) = 0.
$$

Now using Lemma 3.1, one deduces

$$
(P^+(\alpha) R^+(\alpha) S(\beta, p, q, x, \partial) R^+(\alpha)^{-1} P^+(\alpha)^{-1})_+ = 0,
$$

hence

$$
P^+(\alpha) S(\beta, p, q, x, \partial) P^+(\alpha)^{-1} = (P^+(\alpha) S(\beta, p, q, x, \partial) P^+(\alpha)^{-1})_+ = 0.
$$

So $S(\beta, p, q, x, \partial) = 0$ and therefore

$$
\left\{ \exp \left( - \sum_{k=1}^\infty \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)} = 0.
$$
From which we conclude that

\[(5.15)\quad W_{q-p}^{(p+1)} \tau_\beta = \text{constant} \quad \tau_\beta \quad \text{for all } p, q \geq 0.\]

In order to determine the constants on the right–hand–side of \((5.15)\) we calculate the Lie brackets

\[(5.16)\quad [W_1^{(2)}, \frac{-1}{q+1} W_{q-p}^{(p+1)}] \tau_\beta = 0.\]

Notice that the right–hand–side of \((5.16)\) is equal to

\[(W_{p-q}^{(p+1)} + \mu(W_1^{(2)}, \frac{-1}{q+1} W_{q-p}^{(p+1)})) \tau_\beta.\]

Now using \((1.2-3), (2.2)\) and \((5.8)\) we thus obtain the main result, generalizing some of the results of \([FKN]\), see also \([AV]\):

**Theorem 5.4.** The following two conditions for \(\tau \in F^{(0)}\) are equivalent:

1. \(\tau\) is a \(\tau\)–function of the \([n_1, n_2, \ldots, n_s]\)–th reduced \(s\)–component KP hierarchy which satisfies the string equation \((5.1)\).
2. For all \(p, q \geq 0\):

\[(5.17)\quad (W_{q-p}^{(p+1)} + \delta_{pq} c_{p+1}) \tau = 0,\]

where

\[(5.18)\quad c_{p+1} = \frac{1}{p+1} \sum_{b=1}^{s} \sum_{j=1}^{n_b} \left(\frac{n_b - 2j + 1}{2n_b}\right)p+1,\]

and \((k)_\ell = k(k-1)(k-2) \cdots (k-\ell+1)\)

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