Wave-kinetic description of nonlinear photons

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Abstract. The nonlinear interaction, due to quantum electrodynamical (QED) effects, between photons is investigated using a wave-kinetic description. Starting from a coherent wave description, we use the Wigner transform technique to obtain a set of wave-kinetic equations, the so called Wigner–Moyal equations. These equations are coupled to a background radiation fluid, whose dynamics is determined by an acoustic wave equation. In the slowly varying acoustic limit, we analyse the resulting system of kinetic equations, and show that they describe instabilities, as well as Landau-like damping. The instabilities may lead to break-up and focusing of ultra-high intensity multi-beam systems, which in conjunction with the damping may result in stationary strong field structures. The results could be of relevance for the next generation of laser-plasma systems.

1. Introduction

Currently, the development of laser technology and laser-plasma accelerators is pushing the limits of the achievable field strengths in laboratories to levels unprecedented in human history [1–5]. The successes in laser-plasma based acceleration may even hold the promise of reaching the critical Schwinger limit, when the vacuum becomes fully nonlinear [3]. Thus, it is clear that the nonlinear quantum electrodynamical (QED) vacuum effects will become important. Another possibility that has been pointed out is the formation of plasma channels, evacuated plasma cavities which could support immense field strengths. It has been suggested that elastic photon–photon scattering could be detected within these systems, using the next generation of laser-plasma facilities [6,7]. Moreover, a large number of astrophysical systems, such as magnetars [8], gives rise to more extreme conditions than one could ever produce in earth-based laboratories. As an example of nonlinear QED effects, the possibility of photon–photon scattering is perhaps the most prominent [9–11]. There has been much interest in this particular effect, both from an experimental and an astrophysical point of view (see, e.g., [12–29] and references therein, and [30])

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for an up-to-date discussion and [31] for an overview). The formulation of this effect in terms of the Heisenberg–Euler Lagrangian has been used to investigate such diverse topics as photon splitting and magnetic lensing [24–27], direct detection via second harmonic generation [18], self-focusing [19], nonlinear wave mixing in cavities [21, 22] and waveguide propagation [20]. The approach presented here will be of importance for both the experimental and theoretical questions that may be posed concerning photon–photon scattering.

2. The governing equations

Let us here start by giving a short review of the necessary equations. This will serve as a guide for the steps to follow. We thus set up a coupled system of nonlinear Schrödinger equations (NLSE) for two electromagnetic pulses, and the corresponding acoustic wave equation for the fluid background. We then apply the Wigner transformation to the NLSE, and obtain a coupled set of Vlasov-like equations, the Wigner–Moyal equations. In previous work [32, 33], it has been found that the effect of photon–photon scattering is to introduce modulational and filamentational instabilities in coherent photon systems. Here we will show that the Wigner–Moyal system also leads to modulational instabilities, and that these can be constrained by Landau-like damping. The results and applications thereof are then discussed.

The nonlinear self-interaction of photons can be expressed in terms of the Heisenberg–Euler effective Lagrangian [9–11]

\[ L = \varepsilon_0 F + \kappa \varepsilon_0^2 \left[ 4 F^2 + 7 G^2 \right] , \]  

(2.1)

where \( F = (E^2 - c^2 B^2)/2 \) and \( G = c E \cdot B \). The parameter \( \kappa = 2 \alpha^2 \hbar^3 / 45 m_e^4 c^5 \approx 1.63 \times 10^{-30} \text{ms}^2 / \text{kg} \) represents the inverse of a critical energy density. Here \( \alpha \) is the fine-structure constant, \( \hbar \) is the Planck constant, \( m_e \) the electron mass, and \( c \) the velocity of light in vacuum. The Lagrangian (2.1) is valid when there is no electron–positron pair creation and the field strength is smaller than the critical field, i.e.,

\[ \omega \ll m_e c^2 / \hbar \quad \text{and} \quad |E| \ll E_{\text{crit}} \equiv m_e c^2 / e \lambda_c \]  

(2.2)

respectively. Here \( e \) is the elementary charge, \( \lambda_c \) is the Compton wave length, and \( E_{\text{crit}} \approx 10^{18} \text{V/m} \).

In Ref. [33], the coupled equations

\[ i \left( \frac{\partial}{\partial t} + c \hat{k}_{01} \cdot \nabla \right) E_1 + \frac{c}{2 k_{01}} \left[ \nabla^2 - (\hat{k}_{01} \cdot \nabla)^2 \right] E_1 + \frac{\lambda c k_{01}}{2} \left( \frac{4}{3} \varepsilon_g + \alpha_1 \varepsilon_2 \right) E_1 = 0, \]  

(2.3a)

and

\[ i \left( \frac{\partial}{\partial t} + c \hat{k}_{02} \cdot \nabla \right) E_2 + \frac{c}{2 k_{02}} \left[ \nabla^2 - (\hat{k}_{02} \cdot \nabla)^2 \right] E_2 + \frac{\lambda c k_{02}}{2} \left( \frac{4}{3} \varepsilon_g + \alpha_2 \varepsilon_1 \right) E_2 = 0, \]  

(2.3b)

were derived. They describe the propagation of two electromagnetic pulses \( E_1 \) and \( E_2 \) on an incoherent radiation background. Here \( k_{0j} \ (j = 1, 2) \) is the unperturbed vacuum wave vector (with a hat denoting the corresponding unit vector), \( \lambda = 14 \kappa \) or \( 8 \kappa \) depending on the photon polarisation state, while

\[ \alpha_{1,2} = 2 - 2 k_{01} \cdot \hat{k}_{02} - (k_{01,2} \cdot \hat{\varepsilon}_{02,1})^2 - [k_{01,2} \cdot (\hat{k}_{02,1} \times \hat{\varepsilon}_{02,1})]^2 , \]  

(2.4)
depends on the relative polarisation and propagation directions of the two pulses in vacuum. Moreover, \( \mathcal{E}_g \) and \( \mathcal{E}_i = \varepsilon_0 \langle |E_i|^2 \rangle \) is the energy density of the radiation gas and the pulse \( i \), respectively. Here the angular brackets denote the ensemble average.

We note that in the co-linearly propagating stationary case, Eq. (3.5) below yields \( \mathcal{E}_g = 2\lambda \mathcal{E}_0 (\mathcal{E}_1 + \mathcal{E}_2) \), while Eqs. (2.3) may be written as

\[
\begin{align*}
\frac{i}{\hbar} \frac{\partial E_1}{\partial x} + \frac{1}{2k_{01}} \nabla_1^2 E_1 + \lambda k_{01} \left[ \frac{4}{3} \lambda \mathcal{E}_0 (\mathcal{E}_1 + \mathcal{E}_2) + \frac{1}{2} \alpha_1 \mathcal{E}_2 \right] E_1 &= 0, \\
\frac{i}{\hbar} \frac{\partial E_2}{\partial x} + \frac{1}{2k_{02}} \nabla_2^2 E_2 + \lambda k_{02} \left[ \frac{4}{3} \lambda \mathcal{E}_0 (\mathcal{E}_1 + \mathcal{E}_2) + \frac{1}{2} \alpha_2 \mathcal{E}_1 \right] E_2 &= 0,
\end{align*}
\]

(2.5a)

where \( \epsilon = k_{01} \cdot k_{02} \), and we have chosen the direction of propagation along the \( x \)-axis. Note that \( \epsilon = \pm 1 \) depending on whether the pulses are parallel or anti-parallel. For parallel propagating beams, \( \alpha_{1,2} = 0 \), and the direct coupling between the pulses vanishes. Still, because of the response of the radiation background, the pulses are coupled, and Eqs. (2.5) exhibit spatial self-focusing [34]. Thus, we here generalise the two-dimensional self-focusing results found perturbatively in Ref. [30] and numerically in Ref. [32]. The results of the two-dimensional self-focusing due to photon–photon scattering can also be understood in the context of the modulational instability exhibited by the NLSE [32].

The equations (2.3) are coupled to the acoustic wave equation [33]

\[
\begin{align*}
\frac{\partial^2 \mathcal{E}_g}{\partial t^2} - \frac{c^2}{3} \nabla^2 \mathcal{E}_g &= - \frac{2}{3} \lambda \mathcal{E}_0 \left\{ \left( 1 + \frac{\beta}{2} \sqrt{\frac{\mathcal{E}_2}{\mathcal{E}_1}} \right) \left( \frac{\partial^2}{\partial t^2} + c^2 \nabla^2 \right) \mathcal{E}_1 \right. \\
&\quad + \left( 1 + \frac{\beta}{2} \sqrt{\frac{\mathcal{E}_1}{\mathcal{E}_2}} \right) \left( \frac{\partial^2}{\partial t^2} + c^2 \nabla^2 \right) \mathcal{E}_2 \right.
\end{align*}
\]

(2.6)

\[-\frac{\beta}{4} \sqrt{\frac{\mathcal{E}_2}{\mathcal{E}_1}} \left[ \left( \frac{\partial \mathcal{E}_1}{\partial t} \right)^2 + c^2 |\nabla \mathcal{E}_1|^2 \right] - \frac{\beta}{4} \sqrt{\frac{\mathcal{E}_1}{\mathcal{E}_2}} \left[ \left( \frac{\partial \mathcal{E}_2}{\partial t} \right)^2 + c^2 |\nabla \mathcal{E}_2|^2 \right]
\]

\[+ \frac{\beta}{2 \sqrt{\mathcal{E}_1 \mathcal{E}_2}} \left[ \frac{\partial \mathcal{E}_1}{\partial t} \frac{\partial \mathcal{E}_2}{\partial t} + c^2 (\nabla \mathcal{E}_1) \cdot (\nabla \mathcal{E}_2) \right],\]

for the radiation gas energy density \( \mathcal{E}_g \),† where \( \beta = \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_2 + (\mathbf{k}_1 \cdot \mathbf{k}_2) \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_2 - (\mathbf{k}_1 \cdot \mathbf{\hat{e}}_2)(\mathbf{k}_2 \cdot \mathbf{\hat{e}}_1). \)

3. Kinetic description

We assume that the polarisation of the pulses remains constant, and define the Wigner functions \( \mathcal{G}_j \) as the Fourier transform of the spatial coherence function of \( E_j \), \( j = 1, 2 \) [35–37]

\[
\mathcal{G}_j(t, \mathbf{r}, \mathbf{\kappa}) = \frac{1}{(2\pi)^d} \int d\mathbf{y} \exp(i\mathbf{\kappa} \cdot \mathbf{y})(E_j^*(t, \mathbf{r} + \mathbf{y}/2)E_j(t, \mathbf{r} - \mathbf{y}/2)),
\]

(3.1)

† Note that we have made the split \( \mathcal{E}_g \rightarrow \mathcal{E}_0 + \mathcal{E}_g \), with \( \mathcal{E}_g \ll \mathcal{E}_0 \), in accordance with Ref. [33], and transformed away the resulting phase shift term in Eqs. (2.3).
where $\hbar \kappa$ can be viewed as representing the momentum of the individual photons. The Wigner function $\varrho_j$ has the property

$$\langle |E_j|^2 \rangle = \int d\kappa \varrho_j(t, r, \kappa). \quad (3.2)$$

The transform (3.1) in conjunction with Eqs. (2.3) leads to the Wigner–Moyal equation

$$\frac{\partial \varrho_j}{\partial t} + [c \hat{k}_{0j} + \frac{c}{k_{0j}} \kappa - \frac{c}{k_{0j}} (\hat{k}_{0j} \cdot \kappa) \hat{k}_{0j}] \cdot \nabla \varrho_j + 2i \varrho_0 \sin \left(\frac{1}{2} \nabla \cdot \nabla \kappa \right) \varrho_j = 0. \quad (3.3)$$

where the potentials are defined according to

$$\varrho_1 = \lambda c \kappa \left(\frac{2}{3} \varrho_0 + \frac{1}{2} \alpha_1 \varrho_0 \right) \quad \text{and} \quad \varrho_2 = \lambda c \kappa \left(\frac{2}{3} \varrho_0 + \frac{1}{2} \alpha_2 \varrho_0 \right) \quad (3.4)$$

and the intensities $\langle |E_j|^2 \rangle$ are given by Eq. (3.2). Moreover, the sin-operator in Eq. (3.3) is defined in terms of its Taylor expansion, the arrows indicate the direction of operation, and $\nabla \kappa$ denotes the derivative with respect to $\kappa$.

We will now assume that the fields have perpendicular polarisations, i.e. $\beta = 0$, and $\alpha_j = 0$ or 4 if the beams are co- or counter-propagating, respectively. Equation (2.6) then simplifies considerably, and takes the form

$$\frac{\partial^2 \varrho_0}{\partial t^2} - \frac{2}{3} \lambda \kappa \left(\frac{\partial^2 \varrho_0}{\partial t^2} + c^2 \nabla^2 \right) (\varrho_1 + \varrho_2), \quad (3.5)$$

4. Stability analysis

In order to analyse the stability of the system (3.3) and (3.5), we linearise according the following scheme. Let $\varrho_j = \varrho_{0j}(\kappa) + \tilde{\varrho}_j(\kappa) \exp(i \mathbf{K} \cdot r - i \Omega t)$ (and the corresponding expression for $\varrho_j$), where $\tilde{\varrho}_j \ll \varrho_{0j}$, and $\varrho_0 = \tilde{\varrho}_0 \exp(i \mathbf{K} \cdot r - i \Omega t)$. For co-propagating waves, Eqs. (3.3) (in the Vlasov limit) and (3.5) give the dispersion relation

$$1 = W_1 I_1^+ + W_2 I_2^+, \quad (4.1)$$

while in the case of counter-propagating pulses, we obtain

$$\left(\frac{4}{9} \lambda \kappa \Delta \right)^2 (1 - W_1 I_1^+ - W_2 I_2^-) = (1 + \frac{4}{9} \lambda \kappa \Delta) W_1 W_2 I_1^+ I_2^- \quad (4.2)$$

where $W_j \equiv (4/9) \lambda^2 c k_{0j} \Delta$, $\Delta \equiv (\Omega^2 + c^2 K^2) / (\Omega^2 - c^2 K^2 / 3)$, and

$$I_j^\pm \equiv \int d\kappa \frac{\mathbf{K} \cdot \nabla_k \tilde{\varrho}_{0j}}{\Omega + c K x (1 - \kappa_x / k_{0j}) - (c / k_{0j}) \mathbf{K} \cdot \mathbf{k}}, \quad (4.3)$$

where $K_x = \mathbf{k}_0 \cdot \mathbf{K}$. For $\varrho_2 = 0$, we obtain a result similar to that of Ref. [44] concerning Landau damping.

For mono-energetic beams, i.e., $\varrho_{0j}(\kappa) = \langle |E_{0j}|^2 \rangle \delta(\kappa - k_{0j})$, the integral (4.3) can be reduced, and we obtain the well-known beam modulational instability. In the case of parallel propagating beams, with $E_{01} = E_{02} \equiv E_0$, $k_{01} = k_{02} \equiv k_0$, and $\kappa_{01} = \kappa_{02} \equiv k_0 \hat{x}$, we obtain the growth rate [33]

$$\Gamma \approx \frac{1}{2} \left( c K_x \right) \sqrt{\frac{16}{3} \eta_0 \eta_p \frac{2 K_x^2 + K_y^2}{-2 K_x^2 + K_y^2} \frac{K_z^2}{K_z^2}}, \quad (4.4)$$
where the dimensionless parameters are defined according to $\eta_0 = \lambda\epsilon_0$ and $\eta_p = \lambda\epsilon_p$. Furthermore, $\Omega = cK_x + i\Gamma$ and $K_\perp = K - K_\parallel$.

In general, however, the background distribution is not mono-energetic, and due to the spectral width of $\rho_0$ we will have Landau-like damping manifested by the poles of the integral (4.3). In the one-dimensional case, we can take the spectral width $\kappa_W$ into account by considering two equal incoherent backgrounds with $\rho_0(\kappa) = \langle |E_0|^2 \rangle / (\sqrt{2}\pi\kappa W) \exp[-\kappa^2 / 2\kappa_W^2]$, so that $I^+ = 0$, and

$$I^- = \frac{\langle |E_0|^2 \rangle k_0}{2c\kappa_W} \left\{ 1 + \frac{i\sqrt{\pi} k_0(\Omega + cK)}{2\sqrt{2}\kappa_W K} \text{erf} \left[ \frac{k_0(\Omega + cK)}{2\sqrt{2}\kappa_W K} \right] \exp \left[ -\frac{k_0^2(\Omega + cK)^2}{8\kappa^2_W K^2} \right] \right\}$$

$$+ \frac{i\sqrt{\pi} \langle |E_0|^2 \rangle k_0^2(\Omega + cK)}{4c^2\kappa_W^2} \frac{1}{\sqrt{2}K} \exp \left[ -\frac{k_0^2(\Omega + cK)^2}{8\kappa^2_W K^2} \right].$$

(4.5)

Here we clearly see that the effect of the non-zero spectral width is to introduce a damping.

5. Discussion and conclusion

The existence of a modulational instability for mono-energetic beams strongly suggests that the full nonlinear effect due to photon–photon scattering of incoherent waves in a radiation background should be taken into account, since any small perturbation can grow to form a large amplitude structure, given sufficient time. On the other hand, as the example with a one-dimensional Gaussian distribution shows, we must also expect the evolution to be damped by the resonant interaction between incoherent modes and fluid modes. Thus, it is not unlikely that the full system can account for very interesting structure formation, where the initial growth is governed by a modulational type of instability, and that this growth is stabilised by the Landau-like damping at a later stage of the dynamical phase of the photon system. Thus, one can therefore conjecture about the existence of three-dimensional stable photon structures generated by the vacuum nonlinearities. In order to investigate this conjecture, a numerical analysis of the full system would be necessary.

Situations where the effects presented in this paper may occur range from earth-based laboratory systems, such as ultra-high intensity lasers [1, 2] and plasma accelerators [3, 7], to astrophysical scenarios, such as the early Universe [30], gamma ray bursts [45] and magnetars [8]. If the presents effects do occur, they could also show up on small angular scales within high precision cosmology measurements, such as the ones presented by WMAP [46–48]. A common feature for all these systems is that they at some stage of their evolution lead to extreme radiation energy densities, which is a key feature for probing photon–photon scattering. Thus, there are good possibilities that in the near future one can put QED through new tests, both in laboratory and astrophysical environments, where the latter is perhaps the most fascinating, since it will connect very small scales to very large scales as a fundamental theory test-bed.

In the present paper we have used a system of nonlinear Wigner–Moyal equations. These equations are coupled through a radiation fluid background, where the dynamics is determined by an acoustic wave equation driven by the incoherent photons. Using these equations, we have shown that incoherent electromagnetic pulses can transfer energy between each other by means of the radiation fluid background. Consistent with previous work, we have moreover shown that the resulting system of
equations is subject to a modulational instability, and that a nonzero spectral width of the background incoherent photons gives rise to a damping, much like the well-known Landau-damping of electrostatic waves in plasmas. The implications of the results have been discussed, and it was conjectured that stable three-dimensional electromagnetic structures may form as a result of photon–photon scattering.

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