QUANTUM HEISENBERG GROUP AND ALGEBRA:
CONTRACTION, LEFT AND RIGHT REGULAR REPRESENTATIONS

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ABSTRACT

We show that the quantum Heisenberg group $H_q(1)$ can be obtained by means of contraction from quantum $SU_q(2)$ group. Its dual Hopf algebra is the quantum Heisenberg algebra $U_q(h(1))$. We derive left and right regular representations for $U_q(h(1))$ as acting on its dual $H_q(1)$. Imposing conditions on the right representation the left representation is reduced to an irreducible holomorphic representation with an associated quantum coherent state. By duality, left and right regular representations for quantum Heisenberg group with the quantum Heisenberg algebra as representation module are also constructed. As before reduction of left representations leads to finite dimensional irreducible ones for which the intertwining operator is also investigated.

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1. Following the advent of the so called q-deformed oscillator [1], [2], [3], some alternatives were put forward for deformation of the oscillator group or Heisenberg algebra [4], [5]. The basic idea of that scheme was a generalized version of the standard Wigner-Inönü contraction technique of Lie algebras, which starting from the quantum $U_q(su(2))$ algebra could lead to one-dimensional Heisenberg algebra $U_q(h(1))$. The corresponding contraction of the universal $R$-matrix and its resulting fundamental matrix representation was further employed to introduce the Heisenberg quantum matrix pseudogroup $H_q(1)$ which also was possessing a non trivial *-Hopf algebra structure.

One the other hand, recent developments in the representation theory of quantum groups have provided explicit construction for the left and right regular representations of some classes of quantum algebras e.g. $su_q(1,1)$ [6], quantum Lorentz algebra [7], $sl_q(2)$ and $e_q(2)$ [8], $sl_q(3)$ [9] and $sl_q(n)$ [10]. Interesting contributions by these works include, the extension to the case of quantum algebras of the notion of differential operators intertwining representations [12] and some reduction schemes of the module of representation of the corresponding algebras to holomorphic functions which could turn out to be useful in the elucidation of the relation between representation theory and geometry of quantum groups.

In the present work we shall be concerned with the quantum Heisenberg group and its algebra. This is an example of fundamental importance both for potential applications of quantum group theory and from its pedagogical character. Here a new quantization procedure of the Heisenberg group $H_q(1)$, will be introduced by means of a contraction scheme operating on the quantum group $SU_q(2)$, and its *-Hopf algebra. Then the quantum Heisenberg algebra $U_q(h(1))$ is the dual Hopf algebra of $H_q(1)$ . Then we shall construct the left and right regular representations of the $U_q(h(1))$ algebra generators acting on the algebra of functions $H_q(1)$, taken as the representation module. Next a reduction to an irreducible representation submodule will be made, which will provide a holomorphic representation for the $U_q(h(1))$ generators, to be regarded as quantum analogue of the Bargmann representation. The eigenfunction of the left annihilation operator will also be obtained as a quantum analogue of the canonical coherent states. Moreover, on duality grounds, the left and right regular representations of the quantum Heisenberg group acting on its dual quantum Heisenberg algebra can also be determined by employing techniques analogous to those applied in the case of the algebra representations. In this case also reduction of the regular group representations to finite dimensional ones and their intertwiner operators will be found. Conclusions and some of the perspectives can be found at the end of the paper.
2. The quantum Heisenberg group $H_q(1)$ can be obtained by means of contraction method from the quantum $SU_q(2)$ group. The latter [13], [14], [15], is defined by the relations

$$RT_1T_2 = T_2T_1R$$

(1)

where

$$R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q - q^{-1} & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}$$

(2)

is the R-matrix, while the co-multiplication $\Delta : SU_q(2) \to SU_q(2)^{\otimes 2}$ is defined by

$$\Delta T_{in} = \sum_k T_{ik} \otimes T_{kn}$$

(3)

and the co-unit

$$\varepsilon(T_{ij}) = \delta_{ij}$$

(4)

and antipode

$$ST_{ij} = T_{ij}^{-1}$$

(5)

together with the *-conjugation operation $T^{*}_{ij} = ST_{ji}$, $q^{*} = q$, determine the *-Hopf algebra structure of $SU_q(2)$ group, for which in addition the reality conditions,

$$d = a^{*} \quad c = -q^{-1}b^{*}$$

(6)

and the determinant condition,

$$Det_q \equiv ad - qbc = aa^{*} + bb^{*} = 1,$$

(7)

are valid.

Let us assume that element $d$ is invertible [16]. Then $a$ can be expressed in terms of $b, c, d$. The substitutions $d = e^{l\beta}$, $b = -l^{1/2}\alpha$, $c = l^{1/2}\delta$ and $q = e^{\omega l}$ leads in the limit $l \to 0$, to the following commutation relations for elements $\alpha, \beta$ and $\delta$

$$[\alpha, \delta] = 0, \quad [\delta, \beta] = \omega \delta, \quad [\alpha, \beta] = \omega \alpha$$

(8)

and co-multiplication $\Delta : H_q(1) \to H_q(1)^{\otimes 2}$,

$$\Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta - \delta \otimes \alpha$$

(9)
\[
\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta \quad \Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha
\]  
(10)

with antipode,

\[
S(\delta) = -\delta, \quad S(\alpha) = -\alpha, \quad S(\beta) = -\beta - \alpha \delta
\]  
(11)

and co-unit,

\[
\varepsilon(\alpha) = \varepsilon(\beta) = \varepsilon(\delta) = 0.
\]  
(12)

These are precisely the relations obtained in [6] up to a redefinition of the algebra generators, viz. \(\omega \rightarrow \omega/2\), \(\alpha \rightarrow -\delta\), \(\delta \rightarrow \alpha\). Moreover the *-conjugation can also be obtained by means of this contraction and it reads,

\[
\delta^* = \alpha, \quad \beta^* = -\beta - \alpha \delta.
\]  
(13)

which enjoys the property

\[
(\ast \otimes \ast) \circ \Delta = \Delta \circ \ast
\]  
(14)

Altogether we obtain the *-Hopf algebra \(H_q(1)\) which is dual to the corresponding quantum Heisenberg algebra \(U_q(h(1))\) [6]. The latter is defined by the commutators,

\[
[H, A] = 0, \quad [H, A^+] = 0, \quad [A, A^+] = \frac{\sinh(\omega H)}{\omega},
\]  
(15)

cooplication \(\Delta : H_q(1) \rightarrow H_q(1)^{\otimes 2}\),

\[
\Delta(H) = H \otimes 1 + 1 \otimes H
\]

\[
\Delta(A) = A \otimes e^{\omega H/2} + e^{-\omega H/2} \otimes A, \quad \Delta(A^+) = A^+ \otimes e^{\omega H/2} + e^{-\omega H/2} \otimes A^+
\]  
(16)

and antipode and co-unit given by

\[
S(X) = -X, \quad \text{and} \quad \varepsilon(X) = 0, \quad X \in \{A, A^+, H\}.
\]  
(17)

It possesses a R-matrix obtained by contraction of the \(U_q(su(2))\) [6].

3. Left and right actions of the quantum enveloping algebra \(A \equiv U_q(h(1))\) on its dual quantum group \(A^* \equiv H_q(1)\) are defined respectively by \(L : A \times A^* \rightarrow A^*, (a, \phi) \rightarrow L(a)\phi,\)

\[
(L(a)\phi)(b) \equiv <b, L(a)\phi> = <S(a)b, \phi> \equiv \phi(S(a)b)
\]  
(18)

and \(R : A \times A^* \rightarrow A^*, (a, \phi) \rightarrow R(a)\phi,\)

\[
(R(a)\phi)(b) \equiv <b, R(a)\phi> = <ba, \phi> \equiv \phi(ba)
\]  
(19)
where \(a, b \in A\) and \(\phi, \psi \in A^*\). Using the properties of the duality pair (we use the notation \(\Delta \phi = \sum_{(\phi)} \phi^{(1)} \otimes \phi^{(2)}\))

\[
< b, L(a) \phi >= < S(a)b, \phi > = \sum_{(\phi)} < S(a), \phi^{(1)}> < b, \phi^{(2)} >
\]  

(20)

and

\[
< b, R(a) \phi >= < ba, \phi > = \sum_{(\phi)} < b, \phi^{(1)}> < a, \phi^{(2)} > ,
\]  

(21)

the twisted derivation rule for the left action

\[
L(a)\phi\psi = \sum_{(a)} L(a^{(2)})\phi L(a^{(1)})\psi ,
\]  

(22)

the derivation rule for the right action

\[
R(a)\phi\psi = \sum_{(a)} R(a^{(1)})\phi R(a^{(2)})\psi ,
\]  

(23)

and by virtue of the pairing,

\[
< A^n A^{+l} H^m, \beta >= \delta_{k,0}\delta_{1,0}\delta_{m,1},
\]

\[
< A^n A^{+l} H^m, \delta >= \delta_{k,0}\delta_{1,1}\delta_{n,0},
\]

\[
< A^n A^{+l} H^m, \alpha >= \delta_{k,1}\delta_{1,0}\delta_{m,0}.
\]  

(24)

and of the properties[17]

\[
< XY, \phi >= < X \otimes Y, \Delta(\phi) > ,
\]

\[
< X, \phi\psi >= < \Delta(X), \phi \otimes \psi > ,
\]  

(25)

\(X, Y \in A\) and \(\phi, \psi \in A^*\), valid for dual pairs of Hopf algebras, we arrive at following left and right regular action of the algebra:

\[
L(A)\beta^n \alpha^m \delta^l = -m(\beta - \frac{\omega}{2})^n \alpha^{m-1} \delta^l ,
\]

\[
L(A^+)\beta^n \alpha^m \delta^l = -l(\beta - \frac{\omega}{2})^n \alpha^m \delta^{l-1} + P_n(\beta)\alpha^{m+1} \delta^l ,
\]

\[
L(H)\beta^n \alpha^m \delta^l = -n\beta^{n-1} \alpha^m \delta^l
\]  

(26)
and

\[ R(A^+) \beta^n \alpha^m \delta^l = l(\beta - \frac{\omega}{2})^n \alpha^m \delta^{l-1}, \]
\[ R(A) \beta^n \alpha^m \delta^l = m(\beta - \frac{\omega}{2})^n \alpha^{m-1} \delta^l - P_{(n)}(\beta) \alpha^m \delta^{l+1}, \]
\[ R(H) \beta^n \alpha^m \delta^l = n \beta^{n-1} \alpha^m \delta^l. \]  

(27)

These expressions could also have been obtained from \( su_q(2) \) by a contraction as in the case of \( e_q(2) \) \[9\].
In the above formulae

\[ P_{(n)}(\beta) = \sum_{j=0}^{n-1} (\beta - \frac{\omega}{2})^j (\beta + \frac{3\omega}{2})^{n-1-j} . \]  

(28)

Due to the property

\[ \frac{dP_{(n)}(\beta)}{d\beta} = nP_{(n-1)}(\beta), \]  

(29)

valid for \( n \geq 2 \) for \( P_{(n)} \) taken as a function of \( \beta \), it turns out to be convenient to work in the following basis of \( H_q(1) \):

\[ x(r,m,l) \equiv e^{r\beta} \alpha^m \delta^l \]  

where \( r \in \mathbb{Z}, m, l \in \mathbb{N} \). We observe that

\[ \sum_{n=1}^{\infty} \frac{r^n}{n!} P_{(n)}(\beta) = C_r e^{r\beta}, \]  

(30)

where

\[ C_r \equiv \frac{1}{2}\omega (e^{3\omega r} - e^{-3\omega r}). \]  

(31)

In this new basis the left and right regular actions are taken respectively the following forms,

\[ L(A) x(r,m,l) = -me^{-\frac{\omega}{2}} x(r,m-1,l), \]
\[ L(A^+) x(r,m,l) = -le^{-\frac{\omega}{2}} x(r,m,l-1) + C_r x(r,m+1,l), \]
\[ L(H) x(r,m,l) = -rx(r,m,l) \]  

(32)

and

\[ R(A^+) x(r,m,l) = le^{-\frac{\omega}{2}} x(r,m,l-1), \]
\[ R(A) x(r,m,l) = me^{-\frac{\omega}{2}} x(r,m-1,l) - C_r x(r,m,l+1), \]
\[ R(H) x(r,m,l) = rx(r,m,l). \]  

(33)

One can easily verify that \( L \) and \( R \) are in fact representations of the \( U_q(h(1)) \) and that \([R(X), L(Y)] = 0 \) for \( X, Y \in \{A, A^+, H\} \).
We turn now to the reduction of the above representations. First we observe that $L(A)$ and $R(A^+)$ act like "annihilation" operators and their action is bounded from below. The "vacuum" condition $R(A^+) = 0$ implies $l = 0$ and analogously $L(A) = 0$ implies $m = 0$. As we want to obtain an irreducible left representation (right representation reduction goes along same lines), we choose to impose on the representation space the condition $l = 0$ as well as the condition that the action of $R(H)$ is just a multiplication by a chosen integer, say $s$. This condition fixes $r$ to be $r = s$. Under this conditions the form of the left regular representation becomes

\begin{align}
L(A)x(s, m, 0) &= -me^{-\frac{s\omega}{2}}x(s, m - 1, 0), \\
L(A^+)x(s, m, 0) &= Csx(s, m + 1, 0), \\
L(H)x(s, m, 0) &= -sx(s, m, 0). \tag{34}
\end{align}

We see that the label $s$ (the term $e^{s\beta}$) remains unchanged under the left action of our algebra. It is therefore natural to pass to the induced representation on monomials (and then by linearity to holomorphic functions) $\alpha^m, m \in \mathbb{N}$. Allowing the formal derivation with respect to $\alpha$ (strictly speaking it is an operator) we arrive at a holomorphic realization of the quantum Heisenberg algebra generators:

\begin{align}
L(A) &= -e^{-\frac{s\omega}{2}} \frac{\partial}{\partial \alpha}, \\
L(A^+) &= Cs \alpha, \\
L(H) &= -s. \tag{35}
\end{align}

If we introduce the following elements of the representation space $|s; m \rangle \equiv \frac{\alpha^m}{\sqrt{m!}}$ we get

\begin{align}
L(A)|s; m \rangle &= -\sqrt{m}e^{-\frac{2s}{2}}|s; m - 1 \rangle, \\
L(A^+)|s; m \rangle &= \sqrt{m + 1}Cs|s; m + 1 \rangle, \\
L(H)|s; m \rangle &= -s|s; m \rangle. \tag{36}
\end{align}

Finally we can redefine elements $A, A^+, H$ in the following way (here we have to assume that $C_s \neq 0$)

\[ \tilde{H} = -H, \]
\[ \tilde{A} = -e^{\frac{\omega}{2}} \sqrt{\frac{\sinh(\omega s)}{\omega}} A, \]

\[ \tilde{A}^+ = \frac{1}{C_s} \sqrt{\frac{\sinh(\omega s)}{\omega}} A^+. \]  

(37)

This transformation does not change the quantum Heisenberg algebra relations (15). Moreover we get the following expressions for the left representation

\[ L(\tilde{A})|s; m> = \sqrt{m} \frac{\sinh(\omega s)}{\omega} |s; m - 1>, \]

\[ L(\tilde{A}^+)|s; m> = \sqrt{(m + 1)} \frac{\sinh(\omega s)}{\omega} |s; m + 1>, \]

\[ L(\tilde{H})|s; m> = s|s; m>, \]  

(38)

which are exactly the same as those obtained in [5]. Let us also mention for completeness that the operator which in the case of semisimple algebras plays a role of interwinner takes here the form

\[ R(A)x(r, m, l) = me^{-\frac{\omega}{2}} x(r, m - 1, l) - C_r x(r, m, l + 1). \]  

(39)

This introduces an unwanted operator \( \delta \), thus leading us outside the subspace of elements defined by the condition \( l = 0 \). Finally we observe that the analytic vectors defined as \( (\mu^2 \equiv \frac{\sinh(\omega s)}{\omega}) \),

\[ |\alpha > \equiv e^{\frac{\omega}{2}\mu^2}, \]

(40)

for each irreducible left regular representation labelled by \( s \), determine eigenfunctions of the annihilation operator i.e.

\[ L(\tilde{A})|\alpha > = \alpha|\alpha > \]  

(41)

and define a formal deformed canonical unnormalized coherent state for the quantum Heisenberg algebra [15], [19].

4. As mentioned in the introduction proceeding in analogous way as in the case of the quantum algebra we can find a representation of the quantum Heisenberg group acting on its dual quantum Heisenberg algebra. We obtain the following expressions for the left regular representation of \( H_q(1) \),

\[ L(\alpha)H^k(A^+)m A^n = -nH^k e^{\frac{H\omega}{2}} (A^+)m A^{n-1}, \]
\[
L(\delta)H^k(A^+)mA^n = -mH^k e^{\frac{\omega}{2}H} (A^+)^{m-1}A^n,
\]
\[
L(\beta)H^k(A^+)mA^n = -kH^{-1}(A^+)mA^n + \frac{\omega(m+n)}{2} H^k(A^+)mA^n,
\]
while its right regular representation reads,
\[
R(\alpha)H^k(A^+)mA^n = nH^k e^{-\frac{\omega}{2}H} (A^+)^{m-1}A^n,
\]
\[
R(\delta)H^k(A^+)mA^n = mH^k e^{-\frac{\omega}{2}H} (A^+)^{m-1}A^n,
\]
\[
R(\beta)H^k(A^+)mA^n = kH^{-1}(A^+)mA^n + \frac{\omega(m+n)}{2} H^k(A^+)mA^n - mnH^k e^{\omega} (A^+)^{m-1}A^{n-1}.
\]
(43)

Of course in the limit as \( \omega \to 0 \) \( \alpha,\beta,\delta \) become commuting among themselves. Applying the procedure of reducing the \( L \) representation by imposing some conditions on \( R \), we can arrive at finite dimensional irreducible representations of the Heisenberg group, which we recall that as an algebra it is a Lie algebra but has non trivial bialgebra structure. To this end we first introduce a more convenient basis in the quantum Heisenberg algebra consisting of elements \( y(k,m,n) \equiv e^{\frac{\omega}{2}H} (A^+)mA^n \). In this basis the left and right action become correspondingly,
\[
L(\alpha)y(k,m,n) = -ny(k+1,m,n-1),
\]
\[
L(\delta)y(k,m,n) = -my(k+1,m-1,n),
\]
\[
L(\beta)y(k,m,n) = \frac{\omega}{2} (m+n-k)y(k,m,n)
\]
(44)

and
\[
R(\alpha)y(k,m,n) = ny(k-1,m,n-1),
\]
\[
R(\delta)y(k,m,n) = my(k-1,m-1,n),
\]
\[
R(\beta)y(k,m,n) = \frac{\omega}{2} (m+n+k)y(k,m,n) - mny(k-2,m-1,n-1).
\]
(45)

We first impose the condition \( R(\alpha) = 0 \), which fixes \( n = 0 \). Then we demand that all the vectors of the representation space are \( R(\beta) \) eigenstates to the eigenvalue \( \frac{\omega}{2}p \) for some integer \( p \). This imposes a constraint on possible values of \( k \) and \( m \) as \( k+m = p \). The action of elements of left representation and of \( R(\delta) \) becomes (we use a notation \( |p;m> \equiv y(p-m,m,0) \))
\[
L(\alpha)|p;m> = 0,
\]
\[ L(\delta)|p; m > = -m|p; m - 1 >, \]
\[ L(\beta)|p; m > = \frac{\omega}{2} (2m - p)|p; m > \quad (46) \]

and
\[ R(\delta)|p; m > = |p - 2; m - 1 >. \quad (47) \]

If we do not impose any restrictions on \( m \) it is clear that (46) are infinite dimensional irreducible representations of the quantum Heisenberg group and representations are labelled by an integer \( p \). Then \( R(\delta) \) can be viewed as an intertwiner as it is a map between representations labelled by \( p \) and \( p - 2 \). We can write it as
\[ L_{p-2}(X) R_p(\delta) = R_p(\delta) L_p(X), \quad (48) \]

where we have added subscripts \( p \) and \( p - 2 \) to operators \( L \) and \( R \) (\( X \) can be any generator of the Heisenberg group) in order to make clear on which representation space do they act. Finally we observe that for given \( p \) the set of vectors in \( \ker(R(\delta)^M) \) form an invariant subspace under the action of \( L \)'s and thus provide a \( M \)-dimensional irreducible representation of the quantum Heisenberg algebra.

5. In conclusion, we have presented a construction of the Hopf algebra structure of the quantum Heisenberg group by a the contraction method; this group is dual as a Hopf algebra to the quantum Heisenberg algebra. Also regular representations of the quantum Heisenberg group and algebra where obtained together with intertwinners. Reduction schemes of regular representations to holomorphic realizations for quantum Heisenberg algebra where also provided and the associated quantum analogues of canonical coherent states where pointed out. There are a number of interesting topics to pursue further: given the R-matrix of the quantum Heisenberg group, a quantum differential calculus can be developed and studied; additionally employing the contraction method, the algebraic integral calculus available for \( SU_q(2) \) could be induced into the Heisenberg group and used further to establish hermiticity of the regular representations and to develop a notion of square integrability in the modules of these representations; to these and related problems we aim to return elsewhere.

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