Proof of Consistency of Nonlinear Massive Gravity in the St"uckelberg Formulation

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ABSTRACT: We address some recent concerns about the absence of the Boulware-Deser ghost in the St"uckelberg formulation of nonlinear massive gravity. First we provide general arguments for why any ghost analysis in the St"uckelberg formulation has to agree with existing consistency proofs that have been carried out without using St"uckelberg fields. We then demonstrate the absence of the ghost at the completely nonlinear level in the St"uckelberg formulation of the minimal massive gravity action. The constraint that removes the ghost field and the associated secondary constraint that eliminates its conjugate momentum are computed explicitly, confirming the consistency of the theory in the St"uckelberg formulation.

KEYWORDS: Massive Gravity

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1 Introduction and statement of the problem

General relativity describes a massless spin-2 field $g_{\mu\nu}$. Constructing a mass term for $g_{\mu\nu}$ which is reparametrization invariant requires introducing another “metric”, say $f_{\mu\nu}$. Then a potential can be constructed as a function of $g^{\mu\lambda}f_{\lambda\nu}$, and one can write a massive gravity action in the form,

$$S = M_p^2 \int d^4x \sqrt{-g} \left[ R(g) - 2m^2 V(g^{-1}f) \right].$$  \hspace{1cm} (1.1)

Alternatively, this describes a massive spin-2 field in fixed background metric $f_{\mu\nu}$.

It has long been known that for a generic $V$ such a theory is inconsistent, containing the Boulware-Deser ghost [1, 2]. Only recently potentially consistent massive gravity theories were proposed mostly based on a perturbative analysis in the St"uckelberg formulation of the theory [3, 4]. These were then generalized and shown to be ghost free in a completely nonlinear analysis without using the St"uckelberg formalism [5–8]. In spite of these proofs, concerns about the consistency of massive gravity in the St"uckelberg formulation has arisen, both at the perturbative [9, 10] as well as nonlinear levels [11, 12]. At the perturbative level the problem has been addressed in [13–15]. Here these issues will be addressed and resolved at the completely nonlinear level.

We start with stating the problem. The simplest choice for $f_{\mu\nu}$ in (1.1) is the flat metric,

$$f_{\mu\nu} = \frac{\partial \phi^A}{\partial x^\mu} \eta_{AB} \frac{\partial \phi^B}{\partial x^\nu}. \hspace{1cm} (1.2)$$

The four scalar fields $\phi^A$ ensure reparametrization invariance of $V(g^{-1}f)$ with flat $f$ and can be gauged away by a reparametrization $\tilde{x}^A = \phi^A(x)$ to set $\tilde{f}_{\mu\nu} = \eta_{\mu\nu}$. These are the St"uckelberg fields that inherit their dynamics from the form of $V$. It was pointed out in [16, 17] that the consistency of the $\phi^A$ theory is correlated to the existence of the Boulware-Deser ghost. This allowed for a simpler analysis of the ghost which was otherwise an involved problem. In particular, one could work in a “decoupling limit” that ignores the nonlinear dynamics of $g_{\mu\nu}$ and retains only the $\phi^A$. By the argument of [16], this retains also a part of the ghost information that survives the decoupling limit. The potentially ghost-free potential $V(g^{-1}f)$ based on (1.2) was first constructed [3, 4] using a perturbative analysis of the St"uckelberg fields in the decoupling limit. It was established to be ghost free to linear order in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. A fourth order analysis was also carried out [4].

The subsequent development in the field did not rely on the St"uckelberg formulation. The nonlinear demonstration of the absence of ghost in the flat $f$ theory of [4] was carried out in the unitary gauge $f = \eta$ [6]. Subsequently, these actions were extended, first, to any arbitrary non-dynamical $f_{\mu\nu}$ [7, 8], and then to a dynamical $f_{\mu\nu}$ [8, 18] and shown to be ghost free at the complete nonlinear level, without a need for gauge fixing. This established that such theories contained the right constraints to eliminate a propagating ghost.

On the other hand, while the St"uckelberg analysis, based on studying the dynamics of the $\phi^A$ (1.2), proved very powerful in the decoupling limit, extending it beyond this limit led to
some speculation that the ghost may return at higher orders in $h_{\mu\nu}$ [9, 10]. This concern was addressed in [13] where it was asserted that the constraints that eliminated the ghost [6–8, 18] must also arise in the Stückelberg formulation of the same theory. This was demonstrated perturbatively at lowest orders in $h_{\mu\nu}$ (as well as exactly in 2 dimensions) and argued to be extendable to higher orders. Recently, a nonlinear analysis of constraints in the Stückelberg setup has been performed [11, 12] and seems to indicate that the required constraints may not exist in the Stückelberg formulation. If true, this would be in contradiction with the proofs of [6–8, 18] as well as the arguments in [13].

Due to these lingering doubts about the consistency of massive gravity in the Stückelberg formulations and in order to conclusively resolve this confusion, we reconsider this set up. Our main results are summarized below.

• With simple arguments based on general covariance, we show that the Stückelberg formulation of massive gravity must agree with the existing consistency proofs already obtained in the standard formulation.

• We perform a Hamiltonian (ADM) analysis of the minimal model of massive gravity in the Stückelberg formulation and explicitly obtain 2 constraints that remove both the ghost and its canonical momentum. The analysis is performed at the fully nonlinear level and for arbitrary metric $f_{\mu\nu}$. Our results extend those of [13] which, for a flat $f_{\mu\nu}$, found the corresponding constraints exactly in 2 dimensions and perturbatively in 4 dimensions. This explicit nonlinear proof should settle the issue of absence of ghost in the Stückelberg formulation of massive gravity.

In section 2, we discuss the consistency between the Stückelberg and standard non-Stückelberg formulations of massive gravity. In section 3, we perform the consistency analysis directly in the Stückelberg formalism and prove the absence of ghost at the nonlinear level.

2 Consistency of the Stückelberg and non-Stückelberg formulations

The absence of ghost in massive gravity has been proven in the standard, non-Stückelberg, formulation of the theory [6–8]. It is easy to show, on general grounds, that this also implies the absence of ghost in the Stückelberg formulation.

In the standard formulation, the $f_{\mu\nu}$ in (1.1) is a non-dynamical tensor that ensures general covariance, while the equations of motion are obtained by varying the action with respect to $g_{\mu\nu}$ alone,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 2m^2 V_{\mu\nu} = 0,$$

where, $V_{\mu\nu} = \frac{\delta V}{\delta g_{\mu\nu}} - \frac{1}{2}g_{\mu\nu}V$. The Bianchi identity for the Einstein tensor then implies,

$$\nabla^{\mu}V_{\mu\nu} = 0.$$  

1This is different from the objection raised in [19] and addressed in [8].
On the other hand, in the Stückelberg formulation of (1.1), one writes [16],

$$f_{\mu\nu} = \frac{\partial \phi^A}{\partial x^\mu} \tilde{f}_{AB}(\phi) \frac{\partial \phi^B}{\partial x^\nu}.$$  \hspace{1cm} (2.3)

$\tilde{f}_{AB}$ are given scalar functions of $x^\mu$, but the Stückelberg fields $\phi^A$ are now treated as dynamical. Hence along with the $g_{\mu\nu}$ equation above, one also obtains the $\phi^A$ equations of motion,

$$\frac{\delta V}{\delta \phi^A} = 0.$$ \hspace{1cm} (2.4)

However, general covariance implies that (2.4) is already contained in the $g_{\mu\nu}$ equations (2.2). To see this, consider infinitesimal reparametrizations $\delta x^\mu = \xi^\mu$ under which, $\delta g_{\mu\nu} = -2 \nabla^{(\mu} \xi^{\nu)}$ and $\delta \phi^A = -\xi^\mu \partial_\mu \phi^A$. The invariance of the action implies,

$$\delta S = -\frac{M_p^2}{2} \int d^4 x \sqrt{-g} \left[ (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 2m^2 V_{\mu\nu}) \delta g^{\mu\nu} + 2m^2 \frac{\delta V}{\delta \phi^A} \delta \phi^A \right] = 0.$$ \hspace{1cm} (2.5)

For the above variations, integrating by parts and using the Bianchi identity then gives,

$$\nabla^\mu V_{\mu\nu} = \frac{\delta V}{\delta \phi^A} \partial_\nu \phi^A.$$ \hspace{1cm} (2.6)

Since $\phi^A$ are non-singular coordinate transformations of $x^\mu$, $\partial_\mu \phi^A$ is an invertible matrix. Then (2.6) implies that (2.4) is equivalent to (2.2), as can be explicitly checked [5]. This is a consequence of general covariance and shows that the Stückelberg fields do not lead to extra equations beyond the metric equations of motion. Adding matter does not change this argument.

Now, the proof of absence of ghost in massive gravity in [6–8] only involves the $g_{\mu\nu}$ equations of motion. In particular, it does not involve the $f_{\mu\nu}$ equations nor gauge fixing (except for the unitary gauge analysis in [6]). Therefore, in the final action for the five physical components of the massive spin-2 field, obtained on imposing the constraints and eliminating the ghost, one may express $f$ as in (2.3) and obtain the $\phi$ equations. But as discussed above, these are already contained in the $g$ equations, showing that the Stückelberg formalism cannot be inconsistent with the ghost analysis of [6–8].

After this general argument, let us briefly review the ghost analysis for the simplest of the massive gravity actions, the minimal model of [5], following [7, 8],

$$S = \frac{M_p^2}{2} \int d^4 x \sqrt{-g} \left[ R(g) - 2m^2 \text{Tr}(\sqrt{g} f) \right].$$ \hspace{1cm} (2.7)

Here, $\sqrt{g^{-1} f}$ is a square-root matrix defined such that $\sqrt{E} \sqrt{E} = E$. To obtain the physical degrees of freedom one uses the ADM parametrization,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i \gamma_{ij} N_j & N_i \\ N_j & \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \frac{1}{N^2} \begin{pmatrix} -1 & N^i \\ N^j & N^2 \gamma_{ij} - N^i N^j \end{pmatrix}.$$ \hspace{1cm} (2.8)
and writes the action in the Hamiltonian formulation,

\[ S = M_p^2 \int d^4x \left[ \pi^{ij} \dot{\gamma}_{ij} + NR_0 + N^iR_i - 2m^2 N \sqrt{\text{det} \gamma} V_{\text{min}}(N, N^i, \gamma, f) \right]. \quad (2.9) \]

where, \( V_{\text{min}} \) stands for \( \text{Tr}(\sqrt{g^{-1}f}) \) expressed in the ADM parameterization and,

\[ R_0 = \sqrt{\gamma} \left[ \frac{3}{2} R + \left( \frac{1}{2} \pi^2 - \pi^{ij} \pi_{ij} \right) / \sqrt{\gamma} \right], \quad R_i = 2\sqrt{\gamma} \nabla_j \left( \pi^{ij} / \sqrt{\gamma} \right). \quad (2.10) \]

In the Hamiltonian formulation, the action contains 6 potentially propagating modes \( \gamma_{ij} \) and their canonically conjugate momenta \( \pi^{ij} = \delta S / \delta \dot{\gamma}_{ij} \). Of these, 5 conjugate pairs describe the massive spin-2 graviton, while the sixth one is the Boulware-Deser ghost. The \( N \) and \( N_i \) have no canonical momenta and are non-propagating. All these fields give rise to equations of motion. The crucial point is to show that these equations contain the two constraints that can eliminate the ghost and its conjugate momentum. There are no equations of motion for the spectator fields \( f_{\mu\nu} \).

Of the four \( N \) and \( N^i \) equations of motion, three combinations determine the \( N^i \) in terms of \( N \), \( \gamma_{ij} \) and \( \pi^{ij} \) (the explicit solutions are obtained for three functions \( n^i(N, N^i) \) in terms of \( \gamma_{ij} \) and \( \pi^{ij} \)). The fourth combination becomes the Hamiltonian constraint on \( \gamma_{ij} \) and \( \pi^{ij} \),

\[ C(\gamma_{ij}, \pi^{ij}) = 0, \quad (2.11) \]

where \( C \) does not involve time-derivatives of its arguments. The preservation of this constraint by the time evolution of the system requires \( dC / dt = 0 \). Eliminating all time derivatives of fields using their dynamical equations leads to a second constraint,

\[ C_2(\gamma_{ij}, \pi^{ij}) = 0. \quad (2.12) \]

These two constraints eliminate the ghost field and its conjugate momentum. Finally \( N \) is eliminated by the equation following from the preservation of \( C_2 \) in time. All these equations are part of the \( g_{\mu\nu} \) equations of motion and leave behind a theory for the 5 physical modes of a massive spin-2 field.

The final theory also contains \( f_{\mu\nu} \) which was treated as a spectator all along. If this is now expressed in terms of the Stückelberg fields \( \phi^A(2.3) \), we know that their equations of motion are already part of the \( g_{\mu\nu} \) equations and there is no inconsistency. In addition, general covariance should be used to eliminate four gauge modes. Below, we perform a Hamiltonian analysis directly in the Stückelberg formulation of the theory.

### 3 ADM analysis in the Stückelberg formulation

In this section we will perform a Hamiltonian analysis of massive gravity in the Stückelberg formulation and derive the constraints needed to eliminate the ghost. This explicitly answers the concerns raised in [9–12], about the consistency of massive gravity in the Stückelberg formulation and extends the perturbative arguments of [13] to the completely nonlinear theory.
3.1 Hamiltonian form of the action with St"uckelberg fields

Here we explicitly focus on the minimal massive gravity action (2.7) in the St"uckelberg formulation with $f_{\mu \nu}$ given by (2.3) and reconsider the analysis of this model in [11, 12]. To avoid working with the square-root matrix, we follow [20, 21] to recast the action as,

$$S = M_P^2 \int d^4x \sqrt{g} \left[ R(g) - m^2 \left[ \Phi^A_A + (\Phi^{-1})^A_B \Phi^B_A \right] \right],$$

(3.1)

where,

$$A^B_A = \partial_\mu \phi^B g^{\mu \nu} \partial_\nu \phi^C \bar{f}_{CA}. \quad (3.2)$$

Note that since $A^B_A \equiv A^{AB} \bar{f}_{AB}$ is symmetric, only the symmetric piece of $\Phi^{AB} \equiv \bar{f}_{AC} \Phi^{CB}$ appears in the action. We therefore treat $\Phi^{AB}$ as symmetric in the following. Solving the $\Phi^{AB}$ equation of motion and plugging the solution back into the action gives back (2.7).

To rewrite the action (3.1) in the Hamiltonian formulation, we summarize the analysis of [11, 12] until our conclusions diverge, and mainly adapt their notation to facilitate comparison. In the ADM parametrization (2.8), the matrix $A^B_A$ of (3.2) can be written as,

$$A^B_A = -\nabla_n \phi^B \nabla_n \phi^C \bar{f}_{CA} + V^B_A, \quad (3.3)$$

where,

$$\nabla_n \phi^A \equiv \frac{1}{N} \left( \partial_0 \phi^A - N^i \partial_i \phi^A \right), \quad V^B_A \equiv \gamma^{ij} \partial_i \phi^B \partial_j \phi^C \bar{f}_{CA}. \quad (3.4)$$

In this notation, the canonical momentum conjugate to $\phi^B$ is,

$$p_B = 2m^2 M_P^2 \sqrt{\gamma} (\Phi^{-1})_{AB} \nabla_n \phi^A. \quad (3.5)$$

Hence, the action in the Hamiltonian formulation becomes,

$$S = M_P^2 \int d^4x \left[ \pi^{ij} \dot{\gamma}_{ij} + p_A \dot{\phi}^A - \mathcal{H} \right], \quad (3.6)$$

with the Hamiltonian density given by,

$$\mathcal{H} = -NR_0 - N^i R_i + N \mathcal{H}_T^{sc} + N^i \mathcal{H}_i^{sc}. \quad (3.7)$$

The first two terms are familiar from general relativity, whereas the rest read as,

$$\mathcal{H}_T^{sc} = \sqrt{\gamma} M_P^2 m^2 \left( \Phi^A_A + (\Phi^{-1})^A_B \Phi^B_A \right) + \frac{1}{4m^2 M_P^2 \sqrt{\gamma}} \Phi^{AB} p_A p_B,$$

$$\mathcal{H}_i^{sc} = p_A \partial_i \phi^A. \quad (3.8)$$

Before we set out to determine the physical content of the theory, let us begin by listing its field content in the Hamiltonian description:

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\[ \text{To avoid lengthy expressions, we mostly work with the action principle in the Hamiltonian formulation. Only in section 3 we work with Poisson brackets in a minimal way.} \]
Thus there are 10 potentially propagating modes given by $\gamma_{ij}$ and the Stückelberg fields $\phi^A$, while 14 further fields are non-propagating. There are also 4 gauge invariances.

A naive counting of the field components and non-dynamical equations of motion (ones without time derivatives) derived from (3.6) may give the impression that the theory contains the Boulware-Deser ghost: At first sight, the 10 equations of motion for the $\Phi_{AB}$ depend on $\Phi_{AB}$ and will therefore determine this matrix rather than serve as constraints on other fields. Since the action is linear in $N$ and $N^i$, their equations of motion turn into 4 non-dynamical equations for the remaining variables. But unlike general relativity, now these also contain the $p_A$ and $\phi^A$ and can be solved, for example, for the $p_A$ (after gauge fixing the $\phi^A$) rather than impose a constraint on the Boulware-Deser ghost contained in $\gamma_{ij}$. Hence, it seems that the theory may not have the required constraints to reduce the number of propagating modes of $\gamma_{ij}$ below 6, as concluded in [11, 12].

However, as we will demonstrate in the following, the 10 equations of motion for $\Phi_{AB}$ depend only on 9 independent combinations of the matrix elements $\Phi_{AB}$. One combination remains undetermined and the corresponding equation of motion will give an additional constraint on the remaining variables instead, which will remove the ghost. Below we obtain this and the associated secondary constraint.

### 3.2 The first constraint

Varying the action (3.6) with respect to $\Phi^{AB}$ gives the equations of motion [12],

$$
\Psi_{AB} \equiv \sqrt{\gamma} m^2 \left( \bar{f}_{AB} - (\Phi^{-1})_{AC} V^{CD} (\Phi^{-1})_{DB} \right) + \frac{1}{4 m^2 M_p^2 \sqrt{\gamma}} p_A p_B = 0.
$$

To arrive at these equations we have divided by $N$ which is non-zero since $g_{\mu\nu}$ is invertible. Since $\Psi_{AB}$ is symmetric, the naive expectation would be that (3.9) provides 10 conditions that determine all 10 components of the symmetric $(\Phi^{-1})_{AB}$.

However, the crucial observation is that the $4 \times 4$ matrix $V^{AB} \equiv \gamma^{ij} \partial_i \phi^A \partial_j \phi^B$ always has rank 3 since it is composed of $3 \times 4$ and $3 \times 3$ matrices each of maximum rank 3. This also implies that $W \equiv \Phi^{-1} V \Phi^{-1}$ has rank 3 and hence cannot depend on more than 9 independent combination of the 10 $\Phi_{AB}$. To see this, note that, being a symmetric rank-3 matrix, $W$ can be diagonalized by an appropriate orthogonal transformation $O$ to,

$$
W_D = O^T W O = \text{diag}\{w_1, w_2, w_3, 0\},
$$

Table:

| fields   | components | canonical momenta |
|----------|------------|-------------------|
| $N$      | 1          | -                 |
| $N_i$    | 3          | -                 |
| $\gamma_{ij}$ | 6          | $\pi^{ij}$       |
| $\phi^A$ | 4          | $p_A$             |
| $\Phi^{AB}$ | 10        | -                 |
where $W_D$ always has one zero eigenvalue. Since the orthogonal matrix $O$ has six independent parameters, on inverting the transformation we see that $W$ depends on at most 9 independent parameters. This shows that the 10 equations (3.9) depend only on the 9 independent combinations of $\Phi^{AB}$ that appear in $\Phi^{-1} V \Phi^{-1}$ and can only determine these combinations. One combination of the equations therefore cannot fix $\Phi^{AB}$, rather it constrains $\gamma_{ij}$ and $p_A$.

The constraint hidden in (3.9) is extracted by multiplying on both sides with $\Phi$ to get,

$$
\Phi^{CA} \left( \tilde{f}_{AB} + \frac{1}{4(m^2 M^2_\text{P} \sqrt{\gamma})^2} p_A p_B \right) \Phi^{BD} = V^{CD}.
$$

(3.11)

As noted above, $V^{CD}$ is a rank-3 matrix and so the left-hand side also has to have rank 3. By definition, $\Phi$ is invertible and thus has rank 4. Therefore the matrix

$$
C_{AB} \equiv \tilde{f}_{AB} + \frac{1}{4(m^2 M^2_\text{P} \sqrt{\gamma})^2} p_A p_B
$$

(3.12)

is constrained to have rank 3 by equation (3.11). This implies the constraint $C \equiv \det C = 0$. Since $p_A p_B$ is a rank-1 matrix, $\det C$ is particularly simple and the constraint becomes,

$$
C \equiv \tilde{f} \left( \frac{p_A p_B}{\alpha^2} + 1 \right) = 0,
$$

(3.13)

where $\tilde{f} \equiv \det(\tilde{f}_{AB})$ and $\alpha = 2m^2 M^2_\text{P}$. This is a single constraint on $\gamma_{ij}$ and $p_A$ and can be used to determine $\gamma = \det(\gamma_{ij})$ in terms of the $p_A$. Together with the equations obtained from varying the action with respect to $N$ and $N^i$ (which can be solved for the $p_A$ on gauge fixing the $\phi^A$ fields), this gives a single constraint on the $\gamma_{ij}$ and the $\pi^{ij}$ that can remove the ghost field. The associated secondary constraint will be obtained in the next subsection.

To compare with existing results, in [13] the analogue of (3.13) was obtained in 2 dimensions and also perturbatively in 4 dimensions to linear order in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ (in this case, for a different choice of the potential). Here we have derived the constraint in the minimal model of massive gravity in St"uckelberg formulation at the fully nonlinear level. The proof can be generalized to any dimension and, unlike previous studies, is valid for general $f_{AB}$.

We now obtain a result that will be used in the next subsection. Let us write (3.9) in the form $\Phi^{CA} \Psi_{AB} p^B = 0$. On imposing $C = 0$ and using $V$ given in (3.4) this becomes,

$$
V^{CD}(\Phi^{-1})_D A p_A = \partial_i \phi^C \gamma^{ij} \partial_j \phi^D (\Phi^{-1})_D A p_A \approx 0.
$$

(3.14)

In words, $(\Phi^{-1})_A^C p_C$ is the eigenvector with zero eigenvalue of the rank-3 matrix $V_B^A$. These equations imply a result that will be used later,

$$
\gamma^{kj} \partial_j \phi^D (\Phi^{-1})_D A p_A \approx 0.
$$

(3.15)

The last step can be justified explicitly. The map $\partial_i \phi^C$ can produce any 3-vector $B_i$ as a map from some 4-vector $b_C$. In particular, one can always find three 4-vectors $b^a_C$ ($a = 1, 2, 3$) that map to 3 linearly independent 3-vectors $B_i^a$,

$$
B_i^a \equiv \partial_i \phi^C b^a_C, \quad a = 1, 2, 3.
$$

(3.16)
Then, $B_i^a$, regarded as a $3 \times 3$ matrix is invertible (easiest to verify in a unitary gauge $\delta^C = x^C$). Now multiplying (3.14) with $(B^{-1})^k_a (b^a)_C$ gives (3.15).

3.3 The secondary constraint

In order to eliminate both the ghost and its conjugate momentum, a second constraint is needed. This should arise as a secondary constraint that preserves (3.13) under time evolution, i.e., $C_2(\gamma, \pi) \propto dC/dt \approx 0$. It is most efficient to compute $dC/dt$ using Poisson brackets, \(^3\)

$$
\frac{dC}{dt} = \{C, H_{\text{tot}}\}. \tag{3.17}
$$

Below, we compute this and verify that it indeed leads to a constraint.

An economical way of going over from the action (3.6) to the Poisson bracket formulation, is to regard only $(\gamma_{ij}, \pi^{ij})$ and $(\phi^A, p_A)$ as canonical pairs and obtain all non-dynamical equations in terms of Lagrange multipliers. This requires introducing the Hamiltonian,

$$
H_{\text{tot}} = \int d^3 x \left[ N (H^\text{sc}_T - R_0 + \Gamma^{AB} \Psi_{BA}) + N^i (H^\text{sc}_i - R_i) \right]. \tag{3.18}
$$

Note that as compared to (3.7) this contains the extra Lagrange multipliers $\Gamma^{AB}$ to obtain the $\Phi^{AB}$ equations of motion $N \Psi_{BA}$ (3.9). Of course, the theory can also be extended by introducing extra momenta and Lagrange multipliers as in [12]. But that will not affect the computation here. The $\Gamma^{AB}$ are determined by the new $\Phi^{AB}$ equations of motion.

To compute the Poisson brackets we need the following identities,

$$
\{p_A(y), \partial_x \phi^B(x)\} = -\delta^B_A \partial_y \delta(x - y), \quad \{p_A(y), p_B(x)\} = 0,
$$

$$
\{f_{BC}(y), p_A(x)\} = \frac{\delta f_{BC}(y)}{\delta \phi^A(x)} , \quad \{\gamma^{-1}(y), \gamma_{ij}(x)\} = 0, \tag{3.19}
$$

together with (where $\nabla$ is the $\gamma_{ij}$ compatible covariant derivative),

$$
\frac{\delta R_i(x)}{\delta \pi^{jk}(z)} = -\gamma_{ik}(x) \nabla_{x^j} + \gamma_{ij}(x) \nabla_{x^k} \delta(x - z),
$$

$$
\frac{\delta R_0(x)}{\delta \pi^{jk}(z)} = \frac{1}{\sqrt{\gamma(x)}} \left( \gamma_{jk}(x) \pi(x) - 2 \pi_{jk}(x) \right) \delta(x - z). \tag{3.20}
$$

For $\{C, H_{\text{tot}}\}$ to give a constraint on $\gamma$ and $\pi$, it should not depend on $N, N^i$ and the single component of $\Phi^{AB}$ not determined by (3.9). Using the above identities we find,

$$
\{C(y), H^\text{sc}_T(x)\} \approx -2 \bar{f}(y) \nabla y \delta(x - y) \approx \{C(y), R_i(x)\}. \tag{3.21}
$$

Hence the term proportional to the $N^i$ in $H_{\text{tot}}$ does not contribute to $C_2$ on the constraint surface. The remaining terms are,

$$
\{C(y), H_{\text{tot}}\} = \int d^3 x \ N(x) \{C(y), H^\text{sc}_T(x) + \Gamma^{AB} \Psi_{BA}(x) - R_0(x) \}. \tag{3.22}
$$

\(^3\)The Poisson bracket is defined as $\{f(x), g(y)\} \equiv \int d^3 z \left( \frac{\delta f(x)}{\delta q^a(x)} \frac{\delta g(y)}{\delta p_a(z)} - \frac{\delta f(x)}{\delta p_a(z)} \frac{\delta g(y)}{\delta q^a(x)} \right)$, where $q^a$ and $p_a$ stand for the variables $\phi^A$, $\gamma_{ij}$ and their canonical momenta $p_A$, $\pi^{ij}$, respectively.
Each bracket can be evaluated on the constraint surface using (3.19) along with,
\[
\{p_A(y), V^C_D(x)\} = -\gamma^j(x) \partial_{x^j} \phi^B(x) \left[ (\delta^C_B \bar{f}_{BD} + \delta^C_B \bar{f}_{AD}) \partial_{y^l} + \partial_{x^j} \phi^C \frac{\delta \bar{f}_{BD}}{\delta \phi^A} \right] \delta(x - y),
\]
\[
\{C(y), R_0(x)\} \approx \bar{f} \pi \gamma^{-1/2} \delta(x - y).
\] (3.23)

The coefficient of the term proportional to \(\partial_j N\) vanishes by virtue of (3.15). It is then straightforward to see that the result is proportional to \(N(y)\) which therefore appears as an overall factor in \(\{C, H_{tot}\}\) and can be divided out. Then the secondary constraint reads,
\[
C_2 \approx N^{-1} \{C(y), H_{tot}\} \approx -\gamma^{-1/2} \bar{f} \pi - 4 m^2 M^2 \gamma^{-1/2} \bar{f} (\Phi^{-1})_{BD} \partial_j \phi^D \gamma^{jk} \nabla^{(f)}_k p^B.
\] (3.24)

Here, \(\nabla^{(f)}_k p^B = \partial_k p^B + \Gamma^B_{CD} \partial_k \phi^C p^D\), with \(\Gamma^B_{CD}\) being the Levi-Civita connection of the metric \(\bar{f}_{AB}\). To get (3.24), we have eliminated \(\Gamma^B_{AB}\) using the \(\Phi^{-1}\) equation,
\[
\Gamma^B_{AB} (\Phi^{-1})^{BC} V_{CD} \approx 0,
\] (3.25)
and in deriving the second term we have made use of (3.15). Note that (3.24) still contains the \(\Phi^{AB}\), but it is easy to see that it depends only on the 9 combinations of these that are determined in terms of other variables by \(\Psi_{AB} = 0\): Multiplying (3.9) by \((B^{-1})^k_a (b^a)_C \Phi^{CA}\), where \(B^a_k\) is defined in (3.16), we obtain,
\[
(B^{-1})^k_a (b^a)_C \Phi^{CA} \left( \bar{f}_{AB} + \frac{p_{APB}}{\alpha^2} \right) = (\Phi^{-1})_{BD} \partial_j \phi^D \gamma^{jk}.
\] (3.26)

Hence the equation \(\Psi_{AB} = 0\) that determines only 9 components of \(\Phi^{AB}\), contains this matrix in the same combination in which it appears in (3.24). Thus on gauge fixing the \(\phi^A\) by using coordinate transformations, solving for the \(p^A\) using the \(N\) and \(N^i\) equations and eliminating the 9 components of \(\Phi\) using \(\Psi_{AB} = 0\), we obtain a constraint \(C_2(\gamma, \pi) \approx 0\). This has the desired form to eliminate the momentum canonically conjugate to the ghost field. Finally note that (3.25) along with (3.14) implies that \(\Gamma_{AB} = \lambda p_{APB}\).

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