Operational causality in spacetime

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We consider the general evolution of binary statistics in a, possibly curved, spacetime with the help of the optimal transport theory. It covers a wide range of models including classical statistics, quantum wave-packets and general, possibly non-linear, post-quantum theories. We postulate that any such evolution must satisfy a rigorous constraint encoding the intuition that (infinitesimal) probability parcels ought to travel along classical future-directed causal curves. We show that, surprisingly, its violation always has operational consequences leading to some form of superluminal signalling. In consequence, it establishes concrete limits on the validity of various models, in particular, the ones based on the Schrödinger equation. Finally, we explain how the presented formalism establishes an unheralded link between the general theory of no-signalling boxes and relativistic spacetime physics.

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Introduction

The problem of causality in quantum theory, ignited by the famous Einstein–Bohr debate [6, 16], has long been a controversial topic (cf., for instance, [8, 20]). It took a few decades to realise that although quantum correlations are stronger than the classically available ones, they do not allow for a superluminal transfer of any information [12, 39]. The latter demand is usually phrased in the domain of quantum information as the “no-signalling principle” and adopted as the “micro-causality axiom” in the axiomatic approaches to quantum field theory [18, 35]. Nevertheless, the actual relationship between the no-signalling principle and the structure of relativistic spacetime remains a subject of controversies and intense studies [7, 10, 21, 22, 30, 33]. The problem becomes even more acute if one takes into account the spacetime curvature, coerced by the laws of general relativity [9, 19].

Recently, a rigorous mathematical formalism for the study of causality of ‘nonlocal phenomena’ evolving on a fixed spacetime has been proposed [15] and developed [23, 36]. It is based on a combination of optimal transport techniques and the framework of Lorentzian geometry. The formalism was based on classical intuitions of causal flow of spread-out objects, such as dust or charge/energy distributions. Surprisingly, however, it was found out [14] that such an approach induces a constraint on the evolution law, which is satisfied not only by statistics of the position of a classical relativistic particle, but also by the one generated by relativistic quantum wave dynamics in the absence of the measurement. But the wave dynamics has an inherently potential character, so it is unclear why it obeys a classical-like constraint. Indeed, any signalling must eventually involve well-defined objective information.

Here we consider the general question: When does the evolution of the measurement statistics on detectors in a spacetime disallow superluminal signalling? We show that the violation of the constraint formulated in [14], when complemented by an elementary axiom about the measurement, leads to operational faster-than-light signalling.

Our basic assumption is that there exists a globally hyperbolic objective spacetime [25], in which the evolution of physical phenomena is modelled and measurements are performed. This is a sufficient condition for the existence of an objective causal structure, which gives a precise sense to the notions of “space-like separation”, “causal precedence” and “time-evolution”.

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The asset of our approach is its universality. The structural hypothesis about the existence of spacetime is the only vital one. Indeed, we consequently adopt here a ‘box-like perspective’ (cf. [23]) asking only about the evolution of potential statistics of events on the detectors. Hence, firstly, we do not make any assumptions about the dynamics of the underlying physical systems subject to measurements — the evolution of resulting probabilities could be classical, quantum, non-linear quantum, discontinuous, or whatever. Secondly, the underlying spacetime is a priori arbitrary — in particular, it need not be a solution to Einstein equations. Finally, the spacetime need not be a background — we could allow for a dynamical coupling of it with the studied phenomenon (as, e.g., in the Einstein–Maxwell system), as long as the spacetime solving the dynamical equations is globally hyperbolic.

Preliminaries.—

To make the article self-contained and fix the notions we recall some basic notions from spacetime geometry and measure theory [20].

Firstly, every spacetime $M$ is naturally endowed with the causal and chronological relations. Concretely, for any two events $p, q \in M$ one says that $p$ causally (chronologically) precedes $q$ if there exists a future-directed causal (timelike) curve from $p$ to $q$ [24], in which case one writes $p \preceq q$ ($p \ll q$). For any $X \subset M$, one defines the set $J^+(X)$ ($I^+(X)$) called the causal (chronological) future of $X$, as the set of all events in $M$ that can reach by means of future-directed causal (timelike) curves starting within $X$. The set $J^-(X)$ ($I^-(X)$), called the causal (chronological) past of $X$, is defined analogously.

A globally hyperbolic spacetime $M$ can always be regarded as $M = \mathbb{R} \times S$ with $\{t\} \times S =: \mathcal{S}_t$, $t \in \mathbb{R}$ being a Cauchy hypersurface [3, 17]. Global hyperbolicity is a necessary condition for the well-posedness of the Cauchy problem, i.e. a sensible formulation of dynamical evolution of given initial data [25].

Let now $\mathcal{P}(M)$ denote the space of all (Borel) probability measures on $M$ and let $I \subset \mathbb{R}$ be an interval. In [14, 23] the notion of a causal time-evolution of probability measures was introduced and studied. It is defined as a map $I \ni t \mapsto \mu_t \in \mathcal{P}(M)$ such that $\text{supp} \mu_t \subset \mathcal{S}_t$ and the following causality condition holds:

$$\forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t. \quad (1)$$

Here the symbol $\preceq$ denotes the natural extension of the causal precedence relation onto $\mathcal{P}(M)$, introduced in [17]. It is rooted in the optimal transport theory and encodes the following intuition: “Each infinitesimal portion of probability should travel along a future-directed causal curve.” The relation $\preceq$ can be characterised in various ways, in particular [14] Theorem 1:

$$\mu_s \preceq \mu_t \Leftrightarrow \mu_s(K) \leq \mu_t(J^+(K)) \quad (2)$$

for all compact $K \subset \text{supp} \mu_s$.

Since $\mu_t$ is supported on $\mathcal{S}_t$, one can replace the set $J^+(K)$ on the right-hand side of (2) with the set $J^+(K) \cap \mathcal{S}_t$, which is compact [24 Corollary A.5.4].

Actually, in our result only the initial and final measures — $\mu_s$ and $\mu_t$, respectively — are relevant, and therefore we shall simplify the notation and set $\mu := \mu_s$ and $\nu := \mu_t$.

Finally, let us note that the splitting $M = \mathbb{R} \times S$ is non-unique — various observers might employ different splittings of the spacetime into time and space. Consequently, those observers would employ different evolutions of measures to describe the same dynamical phenomenon. Nevertheless, the notion of a causal evolution is observer-independent [29].

Main result.—

A measure $\mu \in \mathcal{P}(M)$ supported on a Cauchy hypersurface is a natural mathematical object tailored to model the measurement statistics on detectors. Indeed, if the detector is located in a compact region of space $K$, $\mu$ yields a concrete number $\mu(K) \in [0, 1]$. The compactness of $K$ reflects the demand of the locality of the detector [29].

We adopt here the minimalistic definition of what it means that the signal is detected in the region $K$. Namely, it only means that the signal was not detected outside of $K$ with certainty. If the signal was carried by a classical-like particle the above would mean that the detection was an actual event located somewhere in $K$. Would that be a quantum particle instead, this conclusion would not be justified.

As the fundamental principle of the evolution of potential statistics we consider the one following directly from (1) in the formulation based upon (2):

Causal Evolution (CE) condition. The following inequality

$$\mu(K) \leq \nu(J^+(K)) \quad (3)$$

must hold for all compact $K \subset \text{supp} \mu$.

In order to unravel the “true” operational reason behind the formal property CE, let us first carefully explain the basic relationship between an active measurement and the measures studied.

Consider a measurement checking whether the signal at the initial time $s$ is within the compact set $K \subset \text{supp} \mu$. The impact of the measurement on the final measure at some later time $t$ requires introducing the conditional measures

$$\nu(\cdot | m_K), \quad m_K \in \{0, 1\},$$

where $m_K = 1$ ($m_K = 0$) corresponds to the situation where the measurement has been (has not been) performed at time $s$. The statistics of the measurement $\{P(r|m_K)\}$ with the possible results $r \in \{+, -, 0\}$ (corresponding to “signal detected”, “signal not detected”,

$$\nu(\cdot | m_K) \text{ must be such that}\nu(\cdot | m_K) \leq \nu(J^+(K)) \quad (3)$$

for all compact $K \subset \text{supp} \mu$.
“not applicable”, respectively) satisfies the following consistency conditions:

\[ P(+|1) = \mu(K), \quad P(-|1) = 1 - \mu(K), \quad P(0|1) = 0, \quad P(0|0) = 1. \]  

(4)

Another consistency condition:

\[ \nu(., |0) = \nu \]  

(5)

is natural, because the absence of the measurement clearly should not disturb the assumed dynamics. We shall also consider the extension of \( \nu \) to the joint probability measure \( \tilde{\nu} \in \mathcal{P}(\mathcal{M} \times \{+, -, 0\}) \) satisfying, for any measurable \( \mathcal{X} \subset \mathcal{M} \),

\[ \tilde{\nu}(\mathcal{X}, r|m_K) = \nu(\mathcal{X}|r)m_K P(r|m_K), \]  

(6)

with the marginals \( \sum_r \tilde{\nu}(\mathcal{X}, r|m_K) = \nu(\mathcal{X}|m_K) \).

We now present the incarnation of the no-signalling principle in the measure-theoretic framework:

**No-signalling condition (NS).** For any compact set \( \mathcal{K} \subset \mathcal{S}_t \), the statistics of any later potential measurement outside of the causal future of \( \mathcal{K} \) must not depend on \( m_K \). In the language of measures:

\[ \nu(\mathcal{C} | 1) = \nu(\mathcal{C} | 0), \]  

(7)

for any compact \( \mathcal{C} \subset \mathcal{S}_t \setminus j^+(\mathcal{K}) \).

The operationality of the thus formulated no-signalling principle will be discussed in the next section.

Having established the language, we can now formulate a natural and fairly uncontroversial axiom about the active measurement.

**Axiom 1 (A1).** If the signal has been detected \( (r = +) \) at time \( s \) in the region \( \mathcal{K} \) \( (m_K = 1) \), then the signal must be present with certainty in that region’s future \( j^+(\mathcal{K}) \) for any later time \( t \):

\[ \nu(j^+(\mathcal{K})|+, 1) = 1. \]  

(8)

Now, we are ready to formulate the main result of the article:

**Theorem 1.** Under the assumption of A1, the violation of CE entails the violation of NS.

Below we present the proof of the Theorem. Then, in the next section we demonstrate, with the help of an explicit protocol, that the condition NS is operational. Hence, Theorem 1 shows that the violation of the formal constraint CE leads, under an uncontroversial assumption on the measurement process, to the operational possibility of superluminal signalling.

**Proof.** Suppose there exists a compact set \( \mathcal{K} \subset \text{supp} \mu \) such that

\[ \mu(\mathcal{K}) > \nu(j^+(\mathcal{K})). \]  

(9)

Our aim is to find a compact \( \mathcal{C} \subset \mathcal{S}_t \setminus j^+(\mathcal{K}) \) violating \( \mathcal{C} \). To this end, observe first that

\[ \nu(j^+(\mathcal{K})|1) = \sum_{r \in \{+, -, 0\}} \tilde{\nu}(j^+(\mathcal{K}), r|1) \]

\[ = \sum_{r \in \{+, -, 0\}} \nu(j^+(\mathcal{K})|r, 1) P(r|1) \]

\[ = 1 \cdot \mu(\mathcal{K}) + \nu(j^+(\mathcal{K})|-, 1)(1 - \mu(\mathcal{K})) \]

\[ \geq \mu(\mathcal{K}) > \nu(j^+(\mathcal{K})) = \nu(j^+(\mathcal{K})|0), \]

where we have employed: (6), (4), (8), (9) and, finally, (5). Passing to the complement, we thus have

\[ \nu(\mathcal{S}_t \setminus j^+(\mathcal{K})|1) < \nu(\mathcal{S}_t \setminus j^+(\mathcal{K})|0). \]

Invoking the tightness argument (cf. Supplementary Materials) one can now easily infer the existence of a compact subset \( \mathcal{C} \subset \mathcal{S}_t \setminus j^+(\mathcal{K}) \) for which the above inequality remains valid, that is

\[ \nu(\mathcal{C}|1) < \nu(\mathcal{C}|0). \]  

(10)

**Operational signalling via the violation of NS.**

In order to uncover the operational meaning of the NS condition, we shall need the following spin-off of Theorem 1.

**Theorem 2.** If NS is violated, the set \( \mathcal{C} \) for which \( \mathcal{C} \) does not hold can always be chosen so that there exist spacetime points \( q, p_1, \ldots, p_k \) (for some \( k \in \mathbb{N} \)) such that

\[ \mathcal{K} \subset \bigcup_{i=1}^{k} I^+(p_i), \quad \mathcal{C} \subset I^-(q), \quad \text{and} \quad p_i \not\equiv q, \quad i = 1, \ldots, k. \]

The somewhat technical proof of this fact can be found in the Supplementary Materials.

The essence of Theorem 2 is illustrated in Fig. 1 (a). In order to send a superluminal signal we shall first fill both regions \( \mathcal{K} \) and \( \mathcal{C} \) with detectors, as in Fig. 1 (b). The violation of NS impels that the measurement effectuated by devices in \( \mathcal{K} \) changes the detection probability in \( \mathcal{C} \). However, to actually execute the (statistical) superluminal signalling we would firstly need to orchestrate the measurement in \( \mathcal{K} \) and, secondly, gather the statistical information from \( \mathcal{C} \). This amounts to the existence of sending event(s) \( p_1, \ldots, p_k \) and a readout event \( q \), such that \( p_i \not\equiv q \) for all \( i \).

If \( k = 1 \) (set \( K_2 = \emptyset \) in Fig. 1), what happens in particular when \( \mathcal{K} \) is a convex set, then the statistical signalling from \( p \) to the spacelike \( q \) via the sharp inequality \( \mathcal{C} \) is straightforward. Theorem 2 says that the set \( \mathcal{C} \) can always be chosen in such a way that the readout is performed at a single event \( q \). On the other hand, the set \( \mathcal{K} \) might in general be more complicated — for instance, disconnected —, in which case \( k > 1 \) sending events are needed.
able to communicate bits outside of their future light-cones.

Conclusion.- A violation of NS enables a protocol suitable for operational superluminal communication.

Complementary axiom and the triad of interrelated conditions.—

We now briefly discuss yet another condition that one might associate with the no-signalling principle. Just as Axiom 1 puts constraints on the possible evolution of the measure \( \mu(.|+,1) \), i.e. the measure conditioned upon the positive result of the detection measurement in \( K \), the following condition deals with the measure \( \mu(.|-1) \), i.e., conditioned upon the negative result.

Axiom 2 (A2) - If the particle has not been detected at time \( s \) within \( K \), then outside of \( J^+(K) \) the evolution of \( \mu(.|-1) \) proceeds with no modification other than re-normalisation:

\[
\nu(C|-1) = \frac{\nu(C)}{1 - \mu(K)},
\]

for any compact \( C \subset S_2 \setminus j^+(K) \).

The axiom can be intuitively justified as follows: Immediately after the measurement, the result of which was positive, we must have \( \mu(K|+,1) = 1 \) and thus \( \mu(.|+,1) \) is zero outside of \( K \). Moreover, irrespectively of the measurement’s result, the statistics of the potential measurements outside of \( K \) must remain unchanged so as not to allow for an instantaneous signalling, i.e., \( \mu(K'|0) = \mu(K'|0) \) for every \( K' \subset S_2 \setminus K \). Altogether, one gets

\[
\mu(K') = \mu(K'|0) = \mu(K'|1) \\
= \tilde{\mu}(K',+1) + \tilde{\mu}(K',-1) \\
= \mu(K') P(+1) + \mu(K') P(-1) \\
= \mu(K'|-1)(1 - \mu(K)).
\]

We see that \( \mu(K'|-1) = \mu(K')/(1 - \mu(K)) \) for any \( K' \) disjoint with \( K \). Since the evolution outside of \( J^+(K) \) should not be altered, this formula “time-evolves” into \( (11) \).

The three conditions A1, A2, NS, together with the overarching constraint CE, turn out to enjoy an intimate logical interplay captured by the following result:

Theorem 3. If any two conditions from the set \{ NS, A1, A2 \} hold true, then the third one and CE hold true as well.

In fact, Theorem 3 exhausts the logical dependencies between the four conditions considered. More rigorously speaking: Any combination of the true/false values assigned to the conditions NS, A1, A2, CE which is not deemed impossible by Theorem 3, can be realised with suitably defined measures \( \mu, \nu \) and \( \nu(.|\pm,1) \). For more details as well as for the proof of Theorem 3, consult the Supplementary Materials.
The unravelled connection between spacetime structure and the concept of no-signalling boxes opens up new interesting research directions. The first natural step \cite{10} is to extend the formalism to the “multi-signal” setting, which would allow to take into account the evolution of correlations between multiple measurements. Secondly, one should incorporate the possible internal degrees of freedom. This would enable, in particular, to take into account the case when the internal degrees of freedom are the actual source of the statistics, while the space coordinates are merely parameters \cite{21}. Finally, the presented measure-theoretic approach allows for linking the spacetime structure with general probabilistic theories (GPT) \cite{10} and/or category theory, in a way inspired by \cite{11,22}.

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\section*{Discussion}

We have provided a minimalistic set of necessary conditions for causality of an evolution of binary statistics in spacetime. The asset of the approach is its generality: The potential statistics can equally well be modelled by a classical probability density, the modulus square of a quantum wave function or any other post-quantum object. Moreover, the formalism works equally well in any globally hyperbolic spacetime, giving justice to the demand of general covariance.

The presented result has immediate consequences for modelling of statistics via quantum wave packets: It shows that the standard Schrödinger equation implies operational faster-than-light signalling. Even if the employed Hamiltonian is obtained from a Lorentz-invariant dynamics the causality of the evolution of probability density is not guaranteed. In particular, the Salpeter Hamiltonian $\hat{H} = \sqrt{\hat{p}^2 + m^2}$ also does lead to superluminal signalling \cite{14,37}. On the other hand, the Dirac and photon wave function \cite{4,5} evolutions are strictly no-signalling \cite{14}. Moreover, the measure-theoretic approach allows for a straightforward quantification of the violation of no-signalling involving the time-scale $t - s$, the size $\text{vol} K$ and the “capacity of the superluminal channel” $\nu(S \setminus j^+(K)|0) - \nu(S \setminus j^+(K)|1)$ — see \cite{14} for concrete examples.

The adopted approach applies beyond the wave packet formalism. In fact, it can be related to the concept of no-signalling boxes \cite{33,34}, which was initially designed as a description of discrete degrees of freedom. While keeping the output a binary, we promote the input to run over all $K$’s being compact subsets of space at a given (observer’s) time, which results in a family of boxes $p_i(a|K)$. The gain is an explicit prescription of causality for single signal’s evolution, which was not available in the standard binary box approach.

The unravelling connection between spacetime structure and the concept of no-signalling boxes opens up new interesting research directions. The first natural step \cite{10} is to extend the formalism to the “multi-signal” setting, which would allow to take into account the evolution of correlations between multiple measurements. Secondly, one should incorporate the possible internal degrees of freedom. This would enable, in particular, to take into account the case when the internal degrees of freedom are the actual source of the statistics, while the space coordinates are merely parameters \cite{21}. Finally, the presented measure-theoretic approach allows for linking the spacetime structure with general probabilistic theories (GPT) \cite{10} and/or category theory, in a way inspired by \cite{11,22}.

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[25] We adopt the standard notions [35]: A spacetime is a 4-dimensional time-oriented smooth Lorentzian manifold. It is globally hyperbolic, if it admits a Cauchy hypersurface. However, using the techniques developed in [35], our results could be generalised to the setting of n-dimensional Lorentz–Finsler geometry. Also, the smoothness assumption, could possibly be relaxed [24].

[26] For the full story, the Reader is referred to [15] and references therein.

[27] A piecewise smooth curve γ : [0,1] → M is future-directed causal if its tangent vectors, wherever they exist, are future-directed causal (i.e. timelike or lightlike).

[28] Actually, we need not assume the global hyperbolicity of the “whole universe”. Strictly speaking, we only require the communication protocol to lie in a simple convex neighbourhood, which exist in any spacetime [31].

[29] This assumption is based on the standard requirements of relativistic quantum mechanics [18] [22]. Actually, the adopted mathematical framework is flexible enough to accommodate for quasi-local detectors, in which case a compact subset of space should be replaced by a suitable test function.

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Supplementary Materials

**Tightness of measures.**—

Any Borel probability measure $\mu$ living on a Polish space $\mathcal{M}$ (that is a separable completely metrizable topological space) is **tight**, by which we mean that

$$
\mu(B) := \sup \{ \mu(C) \mid C \subset B, C \text{ is compact} \}. 
$$ (12)

for any measurable $B \subset \mathcal{M}$ (for details, see [1 Chapter 12]). Let us explain how this property has been used in the proof of Theorem 1. to obtain (10). First, observe that both $\nu(.) \mid 1$ and $\nu(.) \mid 0$ are Borel probability measures and are hence tight. The inequality

$$
\nu(S_t \setminus j^+(K)) \mid 1 < \nu(S_t \setminus j^+(K)) \mid 0
$$

can thus be rewritten as the strict inequality of the respective suprema over all compact subsets of $S_t \setminus j^+(K)$. Therefore, there exists a compact set $C \subset S_t \setminus j^+(K)$ realising this strict inequality. Notice that $C$ must be nonempty in this case.

**Proof of Theorem 2.**—

Let $C \subset S_t \setminus j^+(K)$ be the (nonempty) set violating (7). In the first three steps of the proof we construct a compact subset $C' \subset C$ still violating (7), together with $q \in \mathcal{M} \setminus j^+(K)$ such that $C' \subset I^-(q)$. Then, in the last step, we show how to find $p_1, \ldots, p_k \in \mathcal{M} \setminus J^-(q)$ (for some $k \in \mathbb{N}$) such that $C' \subset \bigcup_{i=1}^k I^+(p_i)$.

**Step 1.** Consider the family $\{I^-(q)\}_{q \in \mathcal{M} \setminus J^+(K)}$ of open subsets of $\mathcal{M}$. We claim that it covers $C$, i.e. that

$$
\forall p \in C \exists q \in \mathcal{M} \setminus I^+(K) \quad p \in I^-(q).
$$

Indeed, assuming the contrary, we would have that

$$
\exists p \in C \forall q \in \mathcal{M} \setminus I^+(K) \quad p \notin I^-(q),
$$

which in fact can be equivalently rewritten as

$$
\exists p \in C \quad I^+(p) \subset J^+(K).
$$

But since $p$ lies in the closure of $I^+(p)$ and the set $J^+(K)$ is closed [2], we obtain that $C \cap J^+(K) \neq \emptyset$, in contradiction with the inclusion $C \subset S_t \setminus j^+(K) = S_t \setminus J^+(K)$.

**Step 2.** Since $C$ is compact, there exists a finite subcover $\{I^-(q_i)\}_{i \in F}$, where $F$ is a finite set of indices. However, for a technical reason to become clear soon, we shall need a pairwise disjoint refinement of this subcover. To this end, one might construct the family $U := \{U_S\}$, where the index $S$ runs over all nonempty subsets of $F$, by defining

$$
U_S := \bigcap_{i \in S} I^-(q_i) \setminus \bigcup_{j \in F \setminus S} I^-(q_j) = \{p \in \mathcal{M} \mid \forall i \in \{1, \ldots, l\} \quad p \notin I^-(q_i) \iff i \in S\}.
$$

Observe that every $U_S$ is measurable. Clearly, thus defined $U$ is a pairwise disjoint family of sets which covers $C$. That it also a refinement of the cover $\{I^-(q_i)\}_{i \in F}$ stems from the fact that every $S$ is nonempty, and hence every $U_S$ is contained in at least one of the $I^-(q_i)$’s. We now claim that

$$
\exists \emptyset \neq S^* \subset F \quad \nu(C \cap U_{S^*} \mid 1) < \nu(C \cap U_{S^*} \mid 0).
$$ (13)

Indeed, assuming the contrary, i.e. that $\nu(C \cap U_{S^*} \mid 1) \geq \nu(C \cap U_{S^*} \mid 0)$ for all nonempty $S \subset F$, one would get

$$
\nu(C) = \sum_{\emptyset \neq S \subset F} \nu(C \cap U_S \mid 1) \geq \sum_{\emptyset \neq S \subset F} \nu(C \cap U_S \mid 0) = \nu(C) \mid 0,
$$

in contradiction with (10). It is at this step that we needed the cover to be pairwise disjoint – otherwise we would not be able to use the measures’ additivity property.

**Step 3.** Invoking tightness, we obtain the existence of a compact $C' \subset C \cap U_{S^*}$ such that

$$
\nu(C') \mid 1 < \nu(C') \mid 0.
$$ (14)

Moreover, picking any $i \in S^*$, we obtain that $C' \subset U_{S^*} \subset I^-(q_i)$ with $q_i \in \mathcal{M} \setminus J^+(K)$, because only such events were involved in the construction of the original open cover in Step 2. Of course, we define now $q := q_i$.

**Step 4.** Consider now the family $\{I^+(q_i)\}_{q_i \in \mathcal{M} \setminus J^+(K)}$. We claim it is an open cover of $\mathcal{K}$. Indeed, assuming the contrary and proceeding analogously as in Step 1., one would obtain that $K \cap J^-(q_i) \neq \emptyset$, in contradiction with how $q$ was defined. By the compactness of $\mathcal{K}$, one can take now the finite subcover $\{I^+(p_1), \ldots, I^+(p_k)\}$. Observe that the spacetime points $p_i$ satisfy $p_i \notin q$, $i = 1, \ldots, k$, as desired.

**Proof of Theorem 3.**—

In what follows, $\mathcal{C}$ is always understood as bound by the quantifier “for any compact $C \subset S_t \setminus j^+(K)$”.

On the strength of [5], the NS condition can be written as

$$
\check{\nu}(C \mid 1) = \nu(C).
$$ (15)

Let us also reexpress Axiom 1. in the following form

$$
\nu(C \mid +, 1) = 0
$$

or, in the language of the joint probability measure $\check{\nu}$, as

$$
\check{\nu}(C, + \mid 1) = 0.
$$ (16)

Also Axiom 2. can be expressed in terms of $\check{\nu}$, namely

$$
\check{\nu}(C, - \mid 1) = \nu(C).
$$ (17)

Observe now that, by the obvious identity $\nu(C \mid 1) = \check{\nu}(C, + \mid 1) + \check{\nu}(C, - \mid 1)$, any two equalities from (15, 16, 17) imply the third one, what completes the first part of the proof.
It now suffices to show, e.g., that A2 implies CE. Indeed, plugging \( C := S_t \setminus j^+(K) \) into (11) we obtain that

\[
\frac{\nu(S_t \setminus j^+(K))}{1 - \mu(K)} = \nu(S_t \setminus j^+(K) | -, 1) \leq 1
\]

and hence

\[
1 - \nu(j^+(K)) \leq 1 - \mu(K),
\]

what is equivalent with (3). This completes the proof of Theorem 3.

Finally, let us demonstrate that Theorem 3 completely describes the logical relations between the four considered conditions. Namely, we shall find concrete realizations of all the logical situations not excluded by Theorem 3.

To this end, let us fix \( p, q \in S_s \) and \( p', q' \in S_t \) such that \( p \preceq p', q \preceq p', q \preceq q' \) and \( p \not\preceq q' \). Let us consider the following family of discrete (one- or two-point) measures

\[
\mu := \frac{1}{2} \delta_p + \frac{1}{2} \delta_q,
\]

\[
\nu := A \delta_{p'} + (1 - A) \delta_{q'},
\]

\[
\nu(\cdot | +, 1) := B \delta_{p'} + (1 - B) \delta_{q'},
\]

\[
\nu(\cdot | -, 1) := C \delta_{p'} + (1 - C) \delta_{q'},
\]

where \( A, B, C \in [0, 1] \) are parameters. Without much effort one can convince oneself that, for such defined measures, the four conditions considered here amount to

- **NS:** \( 2A = B + C, \)
- **A1:** \( B = 1, \)
- **A2:** \( 2A = 1 + C, \)
- **CE:** \( 2A \geq 1. \)

It is now not difficult to find sample values of the parameter triple \((A, B, C)\) providing a realization of each combination of truth values assigned to the four conditions that is not excluded by Theorem 3. The following table sums that up, with T, F denoting “true” and “false”, respectively.

| NS | A1 | A2 | CE | sample \((A, B, C)\) |
|----|----|----|----|-------------------|
| T  | T  | T  | T  | \((1, 1, 1)\)     |
| F  | F  | T  | T  | \((1, 0, 1)\)     |
| F  | T  | F  | T  | \((1, 1, 0)\)     |
| T  | F  | F  | T  | \((\frac{5}{6}, \frac{1}{6}, 0)\) |
| F  | F  | F  | T  | \((1, 0, 0)\)     |
| F  | T  | F  | F  | \((0, 1, 0), (0, 1, 1)\) |
| T  | F  | F  | F  | \((0, 0, 0)\)     |
| F  | F  | F  | F  | \((0, 0, 1)\)     |