Quantum behavior of a many photons cavity field revealed by quantum discord

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Abstract

We investigate the quantum-to-classical crossover of a dissipative optical cavity mode based on measurement of the correlations between two atoms which do not interact with each other, but interact with the cavity mode. Firstly, we note that there is a time window where the mode has a classical behavior, which depends on the cavity decay rate, the atom-field coupling strength and the number of atoms. Then, considering only two atoms inside the cavity and working in the steady state of the system, we note that the entanglement between the atoms disappears with the increasing the mean number of photons of the cavity field \((\pi)\). However, the quantum discord reaches an asymptotic non-zero value, even in the limit of \(\pi \rightarrow \infty\). This happens either by increasing \(\pi\) coherently (applying a coherent driving field) or incoherently (raising the temperature of the reservoir coupled to the cavity mode). Therefore, the cavity mode, which is quantum by construction, always preserves its quantum behavior in the asymptotic limit and this is revealed only by the quantum discord.

Although the quantum theory predicts many non-classical and intriguing phenomena as quantum superposition of states and quantum nonlocality \([1, 2]\), such phenomena can hardly be observed in the macroscopic world, being the classical physics recovered from the quantum theory for large excitation numbers and many particle systems \([3]\). The emergence of classical physics from quantum mechanics is therefore actively studied and suggestions to explain it include decoherence due to the interaction with the environment \([4]\), impossibility of macroscopic superposition of distinct states \([5]\), and restrictions due to imprecise measurements \([6]\). However, to verify the quantum or classical behavior of a given system we need to introduce the meter to observe its properties, which in quantum theory is not a simple task. The simple interaction between the system and the meter modifies its dynamics in a way that the quantum-to-classical crossover depends on the meter involved. For cavity fields, e.g., in microwave \([7]\) or circuit \([8]\) systems, usually one employs a single atom as the meter for the cavity field properties, as for example in Refs. \([9, 10]\). In Ref. \([10]\) the quantum-to-classical crossover was investigated by raising gradually the effective temperature of a coplanar transmission line resonator strongly coupled to a superconducting artificial atom, i.e., a circuit QED system. At low temperatures they could observe vacuum Rabi oscillations and mode splitting, revealing the quantum nature of the light field. However, these effects disappear when one raises the effective temperature, i.e., increasing the mean number of photons of the cavity mode. Naturally, the conditions needed for a bosonic mode to have a classical behavior depends on the system parameters and even a cavity mode with a very small mean number of photons may behave classically \([11]\). But, usually, the increasing of the mean number of photons of the cavity field kills its quantum properties, as shown in Refs. \([9, 11]\). However, in these works this was observed using only one atom interacting with the cavity mode so that the question about what happens when we use a different meter is still open. Here we try to answer this question, investigating the behavior of a dissipative cavity field interacting with \(N\) atoms instead of just one. As in \([11]\) here we also assume that the cavity mode is pumped by a classical (external) field which controls the mean number of photons inside the cavity. It can be easily shown that a purely classical field is not able to generate any kind of quantum correlation between the atoms. Then, when the cavity mode behaves classically, all the quantum correlations must be null. Here these quantum correlations are quantified by quantum discord \([12]\) and entanglement of formation \([13]\). On the other hand, the quantum correlations can be non-zero only when the cavity mode has a quantum behavior, owing of the indistinguishability of the photons inside the cavity. But, once some quantum states of the cavity mode are not able to generate quantum correlations between the atoms, the absence of quantum correlations does not give us information about the character of the cavity field. In this way, the presence of quantum correlations between the atoms works out as a signature of the non-classical behavior of the cavity field.

Considering a cavity mode interacting with \(N\) identical two-level atoms \((|g\rangle = \text{ground state}, |e\rangle = \text{excited state})\) and simultaneously driven by a classical field, the total Hamiltonian which describes such a system is \((\hbar = 1)\)

\[
H = \frac{\omega_0}{2} S_z + \omega a^\dagger a + H_P + H_I, \tag{1}
\]

where \(S_z = \sum_{j=1}^{N} \sigma_z^j\) being \(\omega_0\) and \(\sigma_z^j = (|e\rangle_j \langle e| - |g\rangle_j \langle g|)\) the atomic transition frequency and the \(z\)-Pauli matrix of the atom \(j\), respectively; \(\omega\) is the cavity mode fre-
frequency and a (a†) its annihilation (creation) operator; H_P = ε (a^ε a† + h.c.) describes the pumping field on the cavity mode, ε and ω_L being the strength and frequency of the driving field, respectively. Here h.c. stands for hermitean conjugate. H_I = gνN (aS_+ + h.c.) is the interaction Hamiltonian between the cavity mode and atoms, with S_+ = S_− 1/2 = ∑_{j=1}^{N} 1/2 σ_j^+ - σ_j^- = |ε⟩⟨g|, and g the atom-field coupling. Writing the Hamiltonian in a rotating frame with the laser frequency through the unitary transformation T = exp [−iω_L t (a† a + S_+ / 2)] we have
\[ V_L (t) = \delta a^\dagger a + \frac{1}{2} \Delta S_+ + (g\nu N)S_+ a + \varepsilon a + h.c. , \] (2)
being Δ = ω_0 - ω_L and δ = ω - ω_L. Assuming a leaking cavity, the dynamics of the system is governed by the master equation (T = 0K)
\[ \dot{\rho} = -i [V_L (t), \rho] + \kappa L [a] \rho , \] (3)
where \( \kappa \) is the dissipation rate of the cavity mode and \( L [a] \rho = 2A^\dagger a^\dagger A - A^\dagger A \rho - \rho A^\dagger A \). As in [10], we neglect the atomic decay as the atoms work out as a meter to monitor the behavior of the cavity mode. To observe the action of the driven cavity field on the atoms, firstly we apply a time-independent unitary transformation which consists of a displacement operation \( D (\alpha) = e^{a\alpha^\dagger - a^\alpha} \), i.e., \( \hat{\rho} = D^\dagger (\alpha) \rho D (\alpha) \). Imposing \( \alpha = -i\varepsilon / (\kappa + i\delta) \), we finally obtain
\[ \frac{d\hat{\rho}}{dt} = -i [H_{JC} + H_{SC}, \hat{\rho}] + \kappa L [a] \hat{\rho} , \] (4)
with \( H_{JC} = \delta a^\dagger a + g\nu N (aS_+ + h.c.) \), \( H_{SC} = \frac{1}{2} \Delta S_+ + (\Omega S_+ + h.c.) \) and \( \Omega = g\nu N \alpha \). We must observe that the chosen value for \( \alpha \) is exactly the amplitude of the asymptotic coherent field of the cavity mode for \( \varepsilon \gg g\nu \sqrt{N} \). Under this condition we verify, in all numerical calculations we have done below, the cavity field presents the statistical properties of a coherent field (i.e., correlation function \( g^{(2)} (0) = 1 \), Mandel factor \( Q = 0 \), and mean number of photons \( \bar{N} = |\alpha|^2 \)). Looking at the atoms, it is clear from Eq. (4) that, in this displaced picture, we have two kinds of dynamics: one governed by a classical field and another one governed by a quantum field. Now we proceed to investigate what happens to the dynamics of the N two-level atoms when the cavity mode dissipates strongly.

Firstly we consider \( \delta = \Delta_0 = 0 \) and assume the weak coupling limit such that \( \kappa \gg g_{eff} \sqrt{\bar{n}_D} + 1 \), with \( g_{eff} = g\nu \sqrt{N} \) and \( \bar{n}_D \) the mean number of photons in the cavity mode in the displaced representation. For \( t \gg 1/\kappa \) we can do an adiabatic elimination of the field variables [11], resulting in a reduced master equation for the atoms, which in the interaction picture is given by
\[ \dot{\rho}_a = -i [H_{SC}, \rho_a] + \Gamma_{eff} L [S_-] \rho_a , \] (5)
with \( H_{SC} = (\Omega_{eff} S_+ + h.c.) \), \( \Omega_{eff} = g\nu \sqrt{N} \), and \( \Gamma_{eff} = g^2 N / \kappa \).

We have to notice that [12] describes a set of N atoms driven by a classical field with effective Rabi frequency \( \Omega_{eff} \) and interacting with a common effective reservoir with an effective decay rate \( \Gamma_{eff} \). For \( \Omega_{eff} \gg \Gamma_{eff} \), i.e., for \( \varepsilon \gg g\nu \sqrt{N} \), and for interaction time \( t \ll 1/\Gamma_{eff} \), according to Eq. (5) we can see that the dynamics of the system will be governed mainly by a free evolution. Then, as the effective master equation above was derived for \( t \gg 1/\kappa \), we can see that, for interaction times limited to the time window
\[ 1 \ll kt \ll \frac{\kappa}{\kappa_{eff}} = \left( \frac{\kappa}{g\nu \sqrt{N}} \right)^2 , \] (6)
the effective master equation can be approximated by \( \rho_a \simeq -i [H_{SC}, \rho_a] \), which represents an atomic system interacting with a classical electromagnetic field. Starting in a separable atomic state, the atomic system remains separable, i.e., the interaction of the atoms with a common classical field is not able to generate any kind of correlations between the atoms. Moreover, if an initial atomic state is pure, it will remain pure for any time. In this way, the atomic purity is a good parameter to visualize the validity this semi-classical approximation, once out of the time window (5) the dynamics of the system is governed by Eq. (6) where the presence of the effective dissipative term introduces decoherence in the atomic system.

Looking at the time window we see that, for a fixed g, the interaction time where the semi-classical regime is still valid decreases with 1/N, i.e., as bigger the number of atoms inside the cavity, the smaller the time window in which the semi-classical regime is valid. In Fig. 1(a) we show the atomic purity for 1, 2 and 3 atoms inside the cavity, assuming \( g = 0.01\kappa, \varepsilon = 1.0\kappa \) (which results in a maximum mean number of photons \( \bar{N}_{max} = |\varepsilon / \kappa|^2 = 1 \), and considering the atoms prepared initially in the excited state \( |e⟩ \)). For atoms initially prepared in the ground state \( |g⟩ \) the graphic is qualitatively the same. We can see in this figure that the purity of the system decreases quickly when we go out of the time window which defines the semi-classical regime.

The next step consists in determining the quantum correlations between two atoms (here A and B). The measure of total quantum correlations used is the quantum discord (QD) [12]. Nonzero QD in a bipartite system implies that it is impossible to extract all information about one subsystem without perturbing its complement. In every cases studies here the reduced density matrix for the atomic system \( \rho_{AB} \) has the X structure defined by its elements \( \rho_{12} = \rho_{13} = \rho_{24} = \rho_{34} = 0 \), with real coherences and \( \rho_{22} = \rho_{33} \). For this class of density matrix the QD can be analytically calculated [10]: \( QD (\rho_{AB}) = S (\rho_A) - S (\rho_{AB}) - \max \{ D_1, D_2 \} \), where \( D_1 = \)
\[ \sum_{i=1,3} \rho_{ii} \log_2 \left( \frac{\rho_{ii}}{\rho_{ii} + \rho_{i+1,i+1}} \right) + \sum_{i=2,4} \rho_{ii} \log_2 \left( \frac{\rho_{ii}}{\rho_{ii} + \rho_{i-1,i-1}} \right) \]

and

\[ D_2 = \sum_{i=0,1} \frac{(1+i-1)^2}{2} \log_2 \left( \frac{1+(1+i-1)^2}{2} \right), \]

where

\[ \beta^2 = (\rho_{11} - \rho_{44})^2 + 4(\rho_{23} + |\rho_{14}|)^2. \]

Here \( S(\cdot) \) denotes the von Neumann entropy \[17\] and \( \rho_A = Tr_{B \rho_{AB}} \). The entanglement, another kind of quantum correlation, is computed through entanglement of formation (EOF) \[13\]. For X form density matrix the EOF is \[14, 16\]:

\[ EOF(\rho_{AB}) = -\eta \log_2 \eta - (1-\eta) \log_2 (1-\eta), \]

where \( \eta = \frac{1}{2} \left( 1 + \sqrt{1-C^2} \right) \) with \( C = 2 \max \{ 0, |\rho_{14}| - \sqrt{|\rho_{22}\rho_{33}|}, |\rho_{23}| - \sqrt{|\rho_{11}\rho_{44}|} \} \) being the concurrence. Although for pure states the QD is equal to EOF, this is not the case for mixed states, existing quantum correlated states with null entanglement.

Now, for two atoms initially prepared in the state \( |e, e\rangle \), we see that the EOF is zero all the time (for initial state \( |e, g\rangle \) or \( |g, g\rangle \) the EOF is null for long interaction times and almost zero for very short interaction times), as we see in Fig. 1(b). In this way, the EOF is not useful to distinguish the quantum and classical character of a cavity field in the steady state of the system. However, the QD has very small values within the time window \( [0, 1] \), so that the correlations generated by the cavity field in the atoms are negligible, which confirms the classical character of the field within this window. Moreover, the QD grows continuously until it reaches a stationary value asymptotically. This considerable value \( (\simeq 0.33) \) for the QD for initial state \( |e, e\rangle \) (or \( |g, g\rangle \)) shows that the quantum correlations between atoms, generated via interaction of the atoms with the cavity mode, is significant for long interaction times and reveals the quantum nature of this cavity field. Thus, to determine the classical or quantum behavior of the field, we must calculate the correlations between atoms in the steady state.

**Entanglement and Quantum Discord in the stationary regime:** here we analyze the stationary behavior of the QD and EOF as function of the ratios \( g/\kappa \) and \( \varepsilon/\kappa \) through the numerical solution of the Eq. 3 without any approximation. In Figs. 2(a) and (b), for initial atomic state \( |e, g\rangle \) (or \( |g, e\rangle \)), we see that EOF always goes to zero for \( \varepsilon \gg g \), being different of zero only for \( \varepsilon \ll g \), i.e., for small mean number of photons. These results are in accordance with the equivalence principle since for very high mean number of photons we expect an agreement between quantum and semi-classical description, which means no quantum correlations between the atoms. But, surprisingly, the QD is always different of zero, reaching a significative value in the limit of \( \varepsilon \gg g \) (\( QD_{ss} \simeq 0.12 \) for initial state \( |e, g\rangle \) and \( QD_{ss} \simeq 0.33 \) for initial state \( |g, g\rangle \)). This means that the cavity mode is able to generate quantum correlations between the atoms for any finite mean number of photons inside the cavity, even for extremely intense fields. In this way the cavity mode, which is quantum by construction, has its quantum character revealed only through the QD between the atoms.

The origin of the quantum character is the indistinguishability of paths in the exchange of photons between atoms and field. As the photon is indivisible, when a photon is absorbed from the field it generates a superposition of possibilities: either the photon is absorbed by the first atom or is absorbed by the second one, but these two possibilities happen simultaneously. One could argue that the origin of this quantum character could be in the coherence of the driving field, which generates a coherent
field inside the cavity as in [18]. However, being this the case, an incoherent pumping of photons into the cavity would not generate correlations between the atoms in the stationary regime. To analyze this point more carefully, we have neglected the driving field and assumed that the cavity mode is at a finite temperature $T$, which implies in an incoherent injection of photons into the cavity. In this case we have to consider $\varepsilon = \delta = 0$ and replace the master equation (3) by

$$\dot{\rho} = -i [V_L (t), \rho] + \kappa (n_{th} + 1) \mathcal{L} [a] \rho + \kappa n_{th} \mathcal{L} [a^\dagger] \rho,$$

with $n_{th}$ being the mean number of thermal photons. For one atom interacting with a cavity mode and in the limit of large photon numbers, the classical limit derived in [10] requires $n_{th} > (g/\kappa)^2$. In Fig. 3 we have assumed $g = 0.1\kappa$ which implies that $n_{th} = 1$ is already much bigger than $(g/\kappa)^2$. We see that the EoF goes to zero in the steady state as increase the temperature. However, this does not happen to the QD, so that the cavity mode is still able to generate quantum correlations between the atoms even for a cavity mode interacting with a thermal reservoir, as we can see in Figs. 3(a) and (b). Taking in to account the initial atomic state $|e, g \rangle$ we note that the QD and the EoF decay when we increase $n_{th}$. The EoF goes to zero while the QD reaches the asymptotic value $\simeq 0.12$ as in the case we have a coherent injection of photons. For the initial atomic state $|g, g \rangle$ we see that the EoF is always zero and for low temperatures the QD is negligible. But, increasing $n_{th}$, the QD increases, reaching the asymptotic value $QD_{as} \simeq 0.33$, as when we have a coherent driving field. In this case exactly the same behavior of quantum correlations is obtained for any value of atom-field coupling. This happens due the thermalization of the system, i.e., the cavity mode thermalizes with the reservoir and the atoms effectively thermalize with the cavity mode. The atom-field coupling just determines the interaction time required for the thermalization of the atoms with the cavity field. Therefore, the stronger this coupling, the shorter the interaction time required to reach the stationary state of the system. Thus we see that the cavity field is always able of generating quantum correlations, no matter the temperature of the reservoir, revealing the quantum character of the field for any temperature.

As a conclusion, we showed that a non-zero QD between atoms works out as a signature of the non-classical behavior of the cavity mode. It is important to notice that, for high mean number of photons inside the cavity, other quantum correlations such as entanglement is not present in the atomic system such that the quantum character of the cavity mode can be revealed only by QD. Also, the QD is different of zero for any value of atom-field coupling and even in the limit of very strong driving field or high temperatures, both implying in many photons inside the cavity, revealing us the quantum nature of the cavity mode even in these limits. (The atom-field coupling is important to determine the interaction time required to reach the stationary state of the system.) Then, we can conclude that the QD between the atoms show us that classical behavior of the cavity field is not recovered in the limit of many photons, contrary to the correspondence principle. Moreover, there is a time window where the mode has an effective classical behavior, which depends on the cavity decay rate, the strength of the atom-field coupling and the number of atoms.

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