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ON SOME ISOMORPHISM OF COMPACTIFICATIONS OF MODULI SCHEME OF VECTOR BUNDLES

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A morphism of the reduced Gieseker – Maruyama moduli functor (of semistable coherent torsion-free sheaves) to the reduced moduli functor of admissible semistable pairs with the same Hilbert polynomial, is constructed. It is shown that main components of reduced moduli scheme for semistable admissible pairs \((\tilde{S}, \tilde{L}), \tilde{E}\) are isomorphic to main components of reduced Gieseker – Maruyama moduli scheme.

Keywords: semistable admissible pairs, moduli functor, vector bundles, algebraic surface.

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Introduction

In the present article \(S\) is a smooth irreducible projective algebraic surface over an algebraically closed field \(k\) of characteristic 0, \(O_S\) is its structure sheaf, \(E\) coherent torsion-free \(O_S\)-module, \(E' := \text{Hom}_{O_S}(E, O_S)\) dual \(O_S\)-module. In this case \(E'\) is reflexive and, consequently, locally free. In the sequel we make no difference between locally free sheaf and the corresponding vector bundle, and both terms are used as synonyms. Let \(L\) be very ample invertible sheaf on \(S\); it is fixed and will be referred to as the polarization. The symbol \(\chi(\cdot)\) denotes the Euler characteristic, \(c_i(\cdot)\) \(i\)-th Chern class. Also if \(Y \subset X\) be the locally closed subscheme of the scheme \(X\), then \(\overline{Y}\) be its scheme-theoretic closure in \(X\).

Definition 0.1. [6, 7] A polarized algebraic scheme \((\tilde{S}, \tilde{L})\) is called admissible if the scheme \((\tilde{S}, \tilde{L})\) satisfies one of following two conditions

\[ i) (\tilde{S}, \tilde{L}) \cong (S, L), \]
\[ ii) \tilde{S} \cong \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t)')^s/(t^{s+1}), \]
where \(I = \text{Fitt}^0 \text{Ext}^2(\kappa, O_S)\) for Artinian quotient sheaf \(q: \bigoplus \tilde{O}_S \twoheadrightarrow \kappa\) of length \(l(\kappa) \leq c_2\), and \(\tilde{L} = L \otimes (\sigma^{-1} I \cdot O_S)\) is very ample invertible sheaf on the scheme \(\tilde{S}\). This polarization \(\tilde{L}\) will be referred to as the distinguished polarization.

Remark 0.2. In the further considerations, if necessary, we replace \(L\) by its big enough tensor power. As shown in [7], this power can be chosen constant and fixed. All Hilbert polynomials are compute with respect to these new \(L\) and \(\tilde{L}\) correspondingly.

Definition 0.3. [7, 8] \(S\)-(semi)stable pair \((\tilde{S}, \tilde{L}), \tilde{E}\) is the following data:

\[ \bullet \tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \ \text{admissible scheme}, \sigma: \tilde{S} \to S \ \text{morphism which is called canonical}, \sigma_i : \tilde{S}_i \to S \ \text{its restrictions onto components} \tilde{S}_i, i \geq 0; \]
\[ \bullet \tilde{E} \ \text{vector bundle on the scheme} \tilde{S}; \]
\[ \bullet \tilde{L} \in \text{Pic} \tilde{S} \ \text{distinguished polarization}; \]
such that

- \( \chi(\tilde{E} \otimes \tilde{L}^m) = rp(m) \), the polynomial \( p(m) \) and the rank \( r \) of the sheaf \( \tilde{E} \) are fixed;
- the sheaf \( \tilde{E} \) on the scheme \( \tilde{S} \) is stable (respectively, semistable) in the sense of Gieseker, i.e.
  for any proper subsheaf \( \tilde{F} \subset \tilde{E} \) for \( m \gg 0 \)
  \[
  \frac{h^0(\tilde{F} \otimes \tilde{L}^m)}{\text{rank } \tilde{F}} < \frac{h^0(\tilde{E} \otimes \tilde{L}^m)}{\text{rank } \tilde{E}},
  \]
  \[
  (\text{respectively, } \frac{h^0(\tilde{F} \otimes \tilde{L}^m)}{\text{rank } \tilde{F}} \leq \frac{h^0(\tilde{E} \otimes \tilde{L}^m)}{\text{rank } \tilde{E}}); \]
- for each additional component \( \tilde{S}_i, i > 0 \), the sheaf \( \tilde{E}_i := \tilde{E}|_{\tilde{S}_i} \) is quasi-ideal, namely it has a description
  \[
  \tilde{E}_i = \sigma_i^* \text{ker } q_0/tors. \tag{0.1}
  \]

In the series of papers of the author \[3\] — \[8\] the projective algebraic scheme \( \tilde{M} \) is constructed as reduced moduli scheme for \( S \)-semistable pairs.

The scheme \( \tilde{M} \) contains an open subscheme \( \tilde{M}_0 \) which is isomorphic to the subscheme \( M_0 \) of Gieseker-semistable vector bundles, in Gieseker – Maruyama moduli scheme \( \overline{M} \) for semistable torsion-free sheaves with Hilbert polynomial equal to \( \chi(\tilde{E} \otimes \tilde{L}^m) = rp(m) \). We make use of the Gieseker’s definition of semistability.

**Definition 0.4.** Coherent \( O_S \)-sheaf \( E \) is stable (respectively, semistable) if for any proper subsheaf \( F \subset E \) of rank \( r' = \text{rank } F \) for \( m \gg 0 \) the following holds:

\[
\frac{\chi(E \otimes \tilde{L}^m)}{r} > \frac{\chi(F \otimes \tilde{L}^m)}{r'}, \quad \left( \text{respectively, } \frac{\chi(E \otimes \tilde{L}^m)}{r} \geq \frac{\chi(F \otimes \tilde{L}^m)}{r'} \right).
\]

Let \( E \) be a semistably locally free coherent sheaf. Then, obviously, the sheaf \( I = \text{Fitt}^0 \text{Ext}^1(E, O_S) \) is trivial and \( \tilde{S} \cong S \). Consequently, \( ((\tilde{S}, \tilde{L}), \tilde{E}) \cong ((S, L), E) \) and there is a bijection \( \tilde{M}_0 \cong M_0 \).

Let \( E \) be a semistable nonlocally free sheaf, then the scheme \( \tilde{S} \) contains reduced irreducible component \( S_0 \) such that \( \sigma_0 := |S_0 : \tilde{S}_0 \to \tilde{S} \) is a morphism of blowing up of the scheme \( S \) in the sheaf of ideals \( I = \text{Fitt}^0 \text{Ext}^1(E, O_S) \). Formation of the sheaf \( I \) is a way to characterize singularities of the sheaf \( E \), i.e. its difference from local freeness. Indeed, the quotient sheaf \( \kappa := \frac{E \cap E}{E} \) \( \tilde{E} \) is an Artinian sheaf and its length is not greater than \( c_2(E) \), and \( E \cap (E, O_S) \cong E \cap (\kappa, O_S) \). Then \( \text{Fitt}^0 \text{Ext}^1(\kappa, O_S) \) is a sheaf of ideals of (in general case nonreduced) subscheme \( Z \) of bounded length \( Z \) supported in a finite number of points on the surface \( S \). Hence, as it is shown in \[6\], the rest irreducible components \( \tilde{S}_i, i > 0 \) of the scheme \( \tilde{S} \) in general case can carry nonreduced scheme structure.

We assign to each semistable torsion-free coherent sheaf \( E \) a pair \( ((\tilde{S}, \tilde{L}), \tilde{E}) \) with \( (\tilde{S}, \tilde{L}) \) defined as described before.

Let \( U \) be Zariski-open subset in one of components \( \tilde{S}_i, i \geq 0 \), and \( \sigma^* E|_{\tilde{S}_i}(U) \) be the corresponding group of sections which is a \( O_{\tilde{S}_i}(U) \)-module. Sections \( s \in \sigma^* E|_{\tilde{S}_i}(U) \) which are annihilated by prime ideals of positive codimension in \( O_{\tilde{S}_i}(U) \), form a submodule in \( \sigma^* E|_{\tilde{S}_i}(U) \). This submodule will be denoted as \( \text{tors}_i(U) \). The correspondence \( U \to \text{tors}_i(U) \) defines a subsheaf \( \text{tors}_i \subset \sigma^* E|_{\tilde{S}_i} \). Note that associated primes of positive codimension which annihilate sections \( s \in \sigma^* E|_{\tilde{S}_i}(U) \), correspond to subschemes supported in the preimage \( \sigma^* \text{Supp } \kappa = \bigcup_{i \geq 0} \tilde{S}_i \). Since by the construction the scheme \( \tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \) is connected, then subsheaves \( \text{tors}_i, i \geq 0 \), allow to form a subsheaf \( \text{tors} \subset \sigma^* E \). This subsheaf is defined in the following way. A section \( s \in \sigma^* E|_{\tilde{S}_i}(U) \) satisfies the condition \( s \in \text{tors}_i|_{\tilde{S}_i}(U) \) if and only if

- there exists a section \( y \in O_{\tilde{S}_i}(U) \) such that \( ys = 0 \).
• at least one of following two conditions is satisfied: either \( y \in \mathfrak{p} \), where \( \mathfrak{p} \) is prime ideal of positive codimension; or there exist a Zariski-open subset \( V \subset \tilde{S} \) and a section \( s' \in \sigma^*E(V) \) such that \( V \supset U \), \( s'|_U = s \), and \( s'|_{V \cap \tilde{S}_0} \in \text{tors}(\sigma^*E|_{\tilde{S}_0})(V \cap \tilde{S}_0) \). The torsion subsheaf \( \text{tors}(\sigma^*E|_{\tilde{S}_0}) \) in the recent expression is understood in the usual sense.

In our construction the subsheaf \( \text{tors} \subset \sigma^*E \) plays the role which is completely analogous to the role of the subsheaf of torsion for the case of reduced and irreducible base scheme. Since no ambiguity occur, the symbol \( \text{tors} \) anywhere in this article is understood as mentioned above. The subsheaf \( \text{tors} \) is called a torsion subsheaf.

In \([7]\) it is proven that sheaves \( \sigma^*E/\text{tors} \) are locally free. The sheaf \( \tilde{E} \) in the pair \((\tilde{S}, \tilde{L}), \tilde{E} \) is defined by the formula \( \tilde{E} = \sigma^*E/\text{tors} \). In this case there is an isomorphism \( H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}) \cong H^0(S, E \otimes L) \).

In \([5]\) it is shown that the restriction of the sheaf \( \tilde{E} \) on each component \( \tilde{S}_i \), \( i > 0 \), is given by the condition of quasi-ideality \( q_0 : \mathcal{O}_{\tilde{S}}^\otimes \to \mathcal{X} \) is an epimorphism defined by the exact triple \( 0 \to E \to E^\vee \otimes \to \mathcal{X} \rightarrow 0 \) and by local freeness of the sheaf \( E^\vee \).

Resolution of singularities of a semistable sheaf \( E \) can be globalized in a flat family by the procedure developed in articles \([3, 4, 5, 7]\) in different settings. Let \( T \) be reduced irreducible quasi-projective scheme, \( \mathcal{E} \) be a sheaf of \( \mathcal{O}_{T \times S} \)-modules, \( L \) invertible \( \mathcal{O}_{T \times S} \)-sheaf very ample relatively to \( T \) and such that \( L|_{t \times S} = L \) and \( \chi(\mathcal{E} \otimes L^m|_{t \times S}) = rp(m) \) for all closed points \( t \in T \). Also suppose that \( T \) contains nonempty open subset \( T_0 \) such that \( \mathcal{E}|_{T_0 \times S} \) is locally free \( \mathcal{O}_{T_0 \times S} \)-module. Then the following data is defined:

- \( \tilde{T} \) integral normal scheme obtained by some blowing up \( \phi : \tilde{T} \to T \) of the scheme \( T \),
- \( \pi : \tilde{\Sigma} \to \tilde{T} \) flat family of admissible schemes with invertible \( \mathcal{O}_{\tilde{\Sigma}} \)-module \( \tilde{L} \) such that \( \tilde{L}|_{t \times S} \) is the distinguished polarization of the scheme \( \pi^{-1}(t) \),
- \( \tilde{E} \) locally free \( \mathcal{O}_{\tilde{\Sigma}} \)-module and \( ((\pi^{-1}(t), \tilde{L}|_{\pi^{-1}(t)}), \tilde{E}|_{\pi^{-1}(t)}) \) is \( S \)-semistable admissible pair.

In this case there is a blowup morphism \( \Phi : \tilde{\Sigma} \to \tilde{T} \times S \), and \( (\Phi_*\tilde{E})^\vee = \phi^*E \). This follows from the coincidence of sheaves from the left hand side and from the right hand side, on the open subset out of a subscheme of codimension 3. Both sheaves are reflexive. The scheme \( \tilde{T} \times S \) is integral and normal.

The mechanism described is called in \([7]\) as \textit{standard resolution}.

In section 1 we recall definitions of reduced functors \((f^{GM}/\sim)\) of moduli of coherent semistable torsion-free sheaves ("the Gieseker – Maruyama moduli functor") and \((f/\sim)\) of moduli of admissible semistable pairs. The rank \( r \) and the polynomial \( p(m) \) are fixed and equal for both moduli functors.

In the present article we prove following results.

\textbf{Theorem 0.5.} There is a morphism of reduced moduli \( \mathfrak{t} : (f^{GM}/\sim) \to (f/\sim) \), defined by the procedure of standard resolution.

\textbf{Theorem 0.6.} Main components of the reduced scheme \( \tilde{M} \) are isomorphic to main components of the reduced Gieseker – Maruyama scheme.

These theorems are proven in sections 1 and 2 respectively.

1 \textbf{Morphism of moduli functors: proof of theorem 0.5}

Following \([2]\) ch. 2, sect. 2.2 we recall some definitions. Let \( \mathcal{C} \) be a category, \( \mathcal{C}^\circ \) its dual, \( \mathcal{C}' = \text{Funct}(\mathcal{C}^\circ, \text{Sets}) \) the category of functors to the category of sets. By Yoneda’s lemma the functor \( \mathcal{C} \to \mathcal{C}' : F \mapsto (\mathcal{F} : X \mapsto \text{Hom}_C(X, F)) \) includes \( \mathcal{C} \) in \( \mathcal{C}' \) as a full subcategory.
Definition 1.1. [2, ch. 2, definition 2.2.1] The functor \( f \in \text{Ob}\mathcal{C}' \) is corepresented by the object \( F \in \text{Ob}\mathcal{C} \) if there is a \( \mathcal{C}' \)-morphism \( \psi : f \to F \) such that \( \psi' : f \to F' \) factors through the unique morphism \( \omega : F \to F' \).

Definition 1.2. The scheme \( \widetilde{M} \) is coarse moduli space for the functor \( f \) if \( f \) is corepresented by the scheme \( M \).

Let \( T \) be a scheme over the field \( k \). Consider families of semistable pairs

\[
\mathfrak{T}_T = \left\{ \begin{array}{l}
\bar{\pi} : \Sigma_T \to T, \quad \bar{L}_T \in \text{Pic} \Sigma_T, \forall t \in T \quad \bar{L}_t = \bar{L}_T|_{\bar{\pi}^{-1}(t)} \quad \text{very ample}; \\
(\bar{\pi}^{-1}(t), \bar{L}_t) \text{ admissible scheme with distinguished polarization;}
\bar{E}_T \text{ locally free } \mathcal{O}_{\Sigma_T} \text{ sheaf;}
\chi(\bar{E}_T \otimes \bar{L}_T^n)|_{\bar{\pi}^{-1}(t)} = rp(m);
(\bar{\pi}^{-1}(t), \bar{L}_t, \bar{E}_T|_{\bar{\pi}^{-1}(t)}) \text{ (semi)stable admissible pair }
\end{array} \right\}
\]

and a functor \( f : (\text{Schemes}_k)^o \to (\text{Sets}) \) from the category of \( k \)-schemes to the category of sets, assigning to a scheme \( T \) the set \( \{\mathfrak{T}_T\} \). The moduli functor \( (f/\sim) \) assigns to a scheme \( T \) the set of equivalence classes \( (\{\mathfrak{T}_T\})/\sim \).

The equivalence relation \( \sim \) is defined as follows. Families \( ((\bar{\pi} : \Sigma_T \to T, \bar{L}_T), \bar{E}_T) \) and \( ((\bar{\pi}' : \Sigma'_T \to T, L'_T), E'_T) \) from the set \( \{\mathfrak{T}_T\} \) are said to be equivalent (notation: \( ((\bar{\pi} : \Sigma_T \to T, \bar{L}_T), \bar{E}_T) \sim ((\bar{\pi}' : \Sigma'_T \to T, L'_T), E'_T)) \) if

1) there is an isomorphism \( \Sigma_T \sim \Sigma'_T \) such that the diagram

\[
\begin{array}{ccc}
\Sigma_T & \sim & \Sigma'_T \\
\downarrow{\pi} & & \downarrow{\pi'} \\
T & & T
\end{array}
\]

commutes;

2) there exist linear bundles \( L', L'' \) on \( T \) such that \( \bar{E}_T = \bar{E}_T \otimes \bar{\pi}^*L' \) and \( \bar{L}_T = \bar{L}_T \otimes \bar{\pi}^*L'' \).

Convention 1.3. In this paper we restrict ourselves by the consideration of the full subcategory \( \text{RSch}_k \) of reduced schemes and of the reduced moduli functor \( (f_{\text{red}}/\sim) = (f|_{\text{RSch}_k})/\sim \) [9]. Since no ambiguity occur, we use the symbol \( (f/\sim) \) for the reduced moduli functor.

In the general case results of the paper [7] provide existence of a coarse moduli space for any maximal under inclusion irreducible substack in \( \prod_{T \in \text{Ob}(\text{RSch}_k)} \{\mathfrak{T}_T\}/\sim \) if this substack contains such pairs \( ((\bar{\pi}^{-1}(t), \bar{L}_t), \bar{E}_T|_{\bar{\pi}^{-1}(t)}) \) that \( (\pi^{-1}(t), \bar{L}_t) \cong (S, L) \). Such pairs will be referred to as S-pairs. We mention under \( \widetilde{M} \) namely the moduli space of a substack containing semistable S-pairs and emphasize this speaking about main components of the moduli scheme.

Analogously, we mention the Gieseker – Maruyama scheme \( \overline{M} \) as union of those components of reduced moduli scheme of semistable torsion-free sheaves, that contain locally free sheaves.

The Gieseker – Maruyama functor \( f^{GM} : (\text{Schemes}_k)^o \to \text{Sets} \) is defined as follows: \( T \to \{\mathfrak{T}_T^{GM}\} \), where

\[
\mathfrak{T}_T^{GM} = \left\{ \begin{array}{l}
\mathcal{E}_T \text{is a sheaf of } \mathcal{O}_{T \times S} \text{ – modules, flat over } T; \\
\mathcal{L}_T \text{ is invertible sheaf of } \mathcal{O}_{T \times S} \text{ – modules,}
L_i := \mathcal{L}_T|_{T \times S} \text{ is very ample};
E_i := \mathcal{E}_T|_{T \times S} \text{ is torsion-free and Gieseker-semistable with respect to } L_i;
\chi(E_i \otimes L_i^n) = rp(m).
\end{array} \right\}
\]

Families \( (L_T, E_T) \) and \( (L'_T, E'_T) \) are said to be equivalent if there are invertible \( \mathcal{O}_T \)-sheaves \( L' \) and \( L'' \) such that for the projection \( p : T \times S \to T \) one has

\[
\begin{array}{l}
E'_T \cong E_T \otimes p^*L', \\
L'_T \cong L_T \otimes p^*L''.
\end{array}
\]
For this functor we use the convention which is analogous to the convention \[1.3\].

The functor morphism \( t : (\mathcal{F}^{	ext{GM}} / \sim) \to (\mathcal{F} / \sim) \) is defined by commutative diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
T & \xrightarrow{\mathcal{F}} & \{\mathcal{F}^{	ext{GM}}\} / \sim \\
\downarrow{t_F} & & \downarrow{t_T} \\
\{\mathcal{F}_T\} / \sim & \quad & \{\mathcal{F}_T\} / \sim
\end{array}
\end{array}
\]

(1.1)

where \( T \in \text{Ob} \text{Sch}_k \), \( t_T : (\{\mathcal{F}^{	ext{GM}}_T\} / \sim) \to (\{\mathcal{F}_T\} / \sim) \) is a morphism in the category of sets (the map of sets).

Remark 1.4. We consider subfunctors in \( \mathcal{F}^{	ext{GM}} \) and in \( \mathcal{F} \) which correspond to maximal under inclusion irreducible substacks containing locally free sheaves and \( S \)-pairs respectively. Hence any family \( \mathcal{F}^{	ext{GM}}_\Sigma \) (respectively, \( \mathcal{F}_\Sigma \)) can be include in a family \( \mathcal{F}^{	ext{GM}}_{\Sigma'} \) (respectively, \( \mathcal{F}_{\Sigma'} \)) with some connected base \( T' \) and containing locally free sheaves (respectively, \( S \)-pairs) according to the fibres diagram

\[
\begin{array}{c}
\begin{array}{ccc}
T & \xleftarrow{\mathcal{F}_{\Sigma'}} & \{\mathcal{F}^{	ext{GM}}_{\Sigma'}\} \\
\downarrow{t'} & & \downarrow{t} \\
\{\mathcal{F}_{\Sigma'}\} & \quad & \{\mathcal{F}_\Sigma\}
\end{array}
\end{array}
\]

Namely, \( \mathcal{E}_T = i^*\mathcal{E}_{T'} \) (respectively, \( \overline{\Sigma} = \overline{\Sigma}_{T'} \times T, \overline{\Sigma} : \overline{\Sigma}_T \hookrightarrow \overline{\Sigma}_{T'} \) is the induced morphism of immersion, \( \overline{\mathcal{E}}_T = i^*\overline{\mathcal{E}}_{T'}, \overline{L}_T = \overline{i}^*\overline{L}_{T'} \)). In particular, this restriction excludes embedded components of moduli scheme from our consideration whenever these components do not contain locally free sheaves (respectively, \( S \)-pairs). Then it is enough to construct diagrams \[1.1\] only for families which contain locally free sheaves (respectively, \( S \)-pairs), where \( T \) is reduced scheme.

Let \( p : \Sigma_T \to T \) be a flat family of schemes such that its fibre is isomorphic to \( S \), \( L_T \) be a family of very ample invertible sheaves on fibres of the family \( p \), \( \mathcal{E}_T \) be a flat family of coherent torsion-free sheaves on fibres of \( p \). The sheaves are mentioned to have rank \( r \) and Hilbert polynomial \( rp(m) \) and to be semistable with respect to polarizations induced by the family \( L_T \). The application of the standard resolution leads to the collection of data \( (\overline{\phi}, \overline{\mathcal{E}}_T, \overline{L}_T, \overline{\mathcal{E}}_T) \). Let \( \Sigma_T := \Sigma_T \times_T \overline{T} \) where \( \phi : \overline{T} \to T \) is birational morphism also provided by the standard resolution, and \( (\phi, \text{id}_S) : \Sigma_T \to \Sigma_T \) is the induced morphism.

Further, due to the considerations of \[4, 5, 6\] there is a partial morphism of functors \( t^{-1} : (\mathcal{F} / \sim) \to (\mathcal{F}^{	ext{GM}} / \sim) \) defined by the morphism \( \sigma : \Sigma_T \to \Sigma_{\overline{T}} \) and the operation \( (\sigma_\ast -) \ast \) on those families which are obtained by standard resolution from families of coherent semistable torsion-free sheaves. Then \( t^{-1} \circ t = \text{id}_{\text{Sets}} \). Since \( t^{-1} \) is defined only partially, it is impossible to claim that \( t \) is an isomorphism.

Remark 1.5. Also, as it is shown in \[7\], there is a birational morphism of moduli schemes \( \kappa : \overline{\Sigma} \to \overline{\mathcal{M}} \) which are mentioned to be reduced \[8\]. The scheme \( \overline{\mathcal{M}} \) can be not normal. Hence, although \( \kappa \) is a bijective morphism and becomes a morphism of integral schemes if restricted on each of irreducible components, this does not imply that \( \kappa \) is an isomorphism.

In the further text we will show that there is a morphism of reduced Gieseker – Maruyama moduli functor to the reduced moduli functor of admissible semistable pairs. Namely, for any reduced scheme \( T \) the correspondence \( \mathcal{E}_T \rightarrow (\Sigma_T, \overline{L}_T, \overline{\mathcal{E}}_T) \) will be constructed. This correspondence defines the map of sets \( \{\mathcal{E}_T\} / \sim \rightarrow \{(\Sigma_T, \overline{L}_T, \overline{\mathcal{E}}_T)\} / \sim \). This all means that for any family of semistable coherent sheaves \( \mathcal{E}_T \) which is flat over its base \( T \) one can build up a family \( (\Sigma_T, \overline{L}_T, \overline{\mathcal{E}}_T) \) of admissible semistable pairs with the same base \( T \).

The procedure of standard resolution yields in the family of admissible schemes \( \pi : \Sigma_{\overline{T}} \to \overline{T} \) which is flat over \( \overline{T} \), in the locally free \( \mathcal{O}_{\Sigma_{\overline{T}}} \)-sheaf \( \overline{\mathcal{E}}_T \) and in the invertible \( \mathcal{O}_{\Sigma_{\overline{T}}} \)-sheaf \( \overline{L}_T \), which is very ample with respect to the morphism \( \pi \).
Proposition 1.6. There exist

- \( \widetilde{\Sigma}_T \) scheme,
- \( \pi : \widetilde{\Sigma}_T \to T \) flat morphism,
- \( \overline{\phi} : \overline{\Sigma}_T \to \widetilde{\Sigma}_T \) birational morphism,
- \( \overline{E}_T \) locally free \( \mathcal{O}_{\Sigma_T} \)-sheaf,
- \( \overline{L}_T \) invertible \( \mathcal{O}_{\Sigma_T} \)-sheaf,

such that

- the square

\[
\begin{array}{ccc}
\widetilde{\Sigma}_T & \xrightarrow{\overline{\phi}} & \overline{\Sigma}_T \\
\downarrow{\overline{\pi}} & & \downarrow{\pi} \\
\overline{T} & \xrightarrow{\phi} & T
\end{array}
\]

is Cartesian,

- \( \overline{E}_T \otimes \mathcal{L}' = \overline{\pi}^* \overline{E}_T \) for some \( \mathcal{L}' \in \text{Pic} \overline{T} \),
- \( \overline{L}_T \otimes \mathcal{L}'' = \overline{\phi}^* \overline{L}_T \) for some \( \mathcal{L}'' \in \text{Pic} \overline{T} \).

The proposition formulated implies the functor morphism of interest \( t : (\mathcal{I}^{GM}/\sim) \to (f/\sim) \). It is defined for any reduced scheme \( T \in \text{Ob} \text{RSch}_k \) by the commutative diagram (1.1)

\[
\begin{array}{ccc}
T & \xrightarrow{((\mathcal{E}_T)/\sim)} & \{(\mathcal{E}_T)/\sim\} \\
\downarrow & & \downarrow \\
& \{(\{\Sigma_T, \mathcal{L}_T\}, \mathcal{E}_T)/\sim\}
\end{array}
\]

where the right vertical arrow is a morphism (mapping) in the category of sets. This mapping is defined by the proposition 1.6. The horizontal and the skew arrows are defined by functorial correspondences \((\mathcal{I}^{GM}/\sim)\) and \((f/\sim)\) respectively.

Proof of proposition 1.6. For the construction of the scheme \( \widetilde{\Sigma}_T \) we assume that \( m \gg 0 \) is as big as the sheaf of \( \mathcal{O}_T \)-modules \( \overline{\pi}_*(\overline{E}_T \otimes \overline{L}_T^m) \) is locally free, the canonically defined morphism \( \overline{\pi}_* \overline{\pi}_*(\overline{E}_T \otimes \overline{L}_T^m) \to \overline{E}_T \otimes \overline{L}_T^m \) is surjective, and there is in induced closed immersion \( \overline{\Sigma}_T \hookrightarrow G(\overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m), r) \) into Grassmannian bundle \( G(\overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m), r) \). Also there is a (relative) Plücker immersion of the Grassmannian bundle into the (relative) projective space

\[
G(\overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m), r) \hookrightarrow P(\bigwedge^r \overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m)).
\]

Besides consider the isomorphism of \( \mathcal{O}_T \)-sheaves \( p_*(\phi, id_S)^* (\mathcal{E}_T \otimes \mathcal{L}_T^m) = \phi^* p_*(\mathcal{E}_T \otimes \mathcal{L}_T^m) \) and the sheaf \( \overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m) \). These sheaves are locally free and coincide on open subset of the scheme \( \overline{T} \). Then, if \( \mathcal{L}' \) denotes the invertible sheaf of the form \( \mathcal{L}' := \text{det} p_*(\phi, id_S)^* (\mathcal{E}_T \otimes \mathcal{L}_T^m) \otimes (\text{det} \overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m))^{\vee} \), then there are two locally free \( \mathcal{O}_T \)-sheaves \( \overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m) \otimes \mathcal{L}' \) and \( p_*(\phi, id_S)^* (\mathcal{E}_T \otimes \mathcal{L}_T^m) \). They coincide along the open subset of the normal integral scheme \( \overline{T} \). This subset is obtained by excision of some subscheme of codimension \( \geq 2 \). This implies that these sheaves coincide, namely,

\[
\overline{\pi}_* (\overline{E}_T \otimes \overline{L}_T^m) \otimes \mathcal{L}' = p_*(\phi, id_S)^* (\mathcal{E}_T \otimes \mathcal{L}_T^m) = \phi^* p_*(\mathcal{E}_T \otimes \mathcal{L}_T^m).
\]
Consequently, formation of exterior powers and passing to projectivizations and to Grassmannian bundles induces the fibred diagram

\[
P(\Lambda^r(\pi_* (E_T \otimes L^n_T) \otimes L')) \rightarrow P(\Lambda^r p_*(E_T \otimes L^m_T))
\]

It is compatible with Plücker embeddings in the sense that the square

\[
P(\Lambda^r(\pi_* (E_T \otimes L^n_T) \otimes L')) \rightarrow P(\Lambda^r p_*(E_T \otimes L^m_T))
\]

\[G(\pi_*(E_T \otimes L^n_T) \otimes L'), r) \rightarrow G(p_*(E_T \otimes L^m_T), r)
\]

is also Cartesian. Denote by \(\tilde{\Sigma}_T\) the (scheme-theoretic) image of the composite map \(\tilde{\Sigma}_T = G(\pi_*(E_T \otimes L^n_T) \otimes L'), r) \rightarrow G(p_*(E_T \otimes L^m_T), r)\). The immersion here is induced by the sheaf \(\tilde{\pi}_*(E_T \otimes L^n_T) \otimes L'\); the morphism \(g\) comes from the previous diagram. To convince that the scheme \(\tilde{\Sigma}_T\) is flat over \(T\), it is enough to verify the coincidence of images for fibres of the scheme \(\tilde{\Sigma}_T\) over closed points of the fibre \(\phi^{-1}(t)\) for each closed point \(t \in T\). But by the construction, images of all fibres of the scheme \(\tilde{\Sigma}_T\) over points of the subscheme \(\phi^{-1}(t)\), have equal collections of local equations in fibres of the projective bundle \(P(\Lambda^r p_*(E_T \otimes L^m_T))\). Consequently, they are also given by equal local equations in fibres of the Grassmannian bundle \(G(p_*(E_T \otimes L^m_T), r)\). This completes the proof of flatness of \(\tilde{\Sigma}_T\) over \(T\).

Also it is clear that the morphism \(\tilde{\phi} : \tilde{\Sigma}_T \rightarrow \tilde{\Sigma}_T\) of the scheme \(\tilde{\Sigma}_T\) onto its image becomes an isomorphism when restricted to the open subset \(\pi^{-1}(T_0)\). \(T_0\) is open subscheme of the scheme \(T\) where \(\phi\) is an isomorphism. This implies that the morphism \(\tilde{\phi}\) is birational.

Now turn to closed immersions \(j_T : \Sigma_T \rightarrow G(\pi_*(E_T \otimes L^n_T) \otimes L'), r)\) and \(j : \Sigma_T \rightarrow G(p_*(E_T \otimes L^m_T), r)\). We denote by the symbol \(S_T\) the universal (locally free) quotient sheaf of rank \(r\) on \(G(p_*(E_T \otimes L^m_T), r)\). Also the symbol \(S_T\) denotes the universal (locally free) quotient sheaf of rank \(r\) on \(G(\pi_*(E_T \otimes L^n_T) \otimes L'), r)\). It is clear that by the construction \((1.2, 1.3)\) of Grassmannian bundles of our interest we can write \(S_T = g^* S_T\). Then \(j_T^* S_T = E_T \otimes L^n_T \otimes \pi^* L'\), and \(j_T^* S_T\) is a locally free sheaf on the scheme \(\tilde{\Sigma}_T\).

Consider the invertible \(O_{E_T}\)-sheaf \(L_T\) providing the distinguished polarization on fibres of the morphism \(\pi : \Sigma_T \rightarrow \tilde{T}\), and its direct image \(\pi_* L_T\). The recent sheaf is locally free by the choice of the sheaf \(L_T\). Also take (any) invertible \(O_{\Sigma_T}\)-sheaf \(L_T\) which is very ample relatively to the projection \(p : \Sigma_T \rightarrow T\) and such that sheaves \(L_T\) and \(L_T\) induce equal polarizations on fibres over points of open subscheme \(T_0\). The sheaf \(p_* L_T\) is also locally free. Since the Hilbert polynomial on fibres of morphisms \(\pi\) and \(p\) is constant over the base and the morphism \(\Sigma_T \rightarrow \Sigma_T\) is birational, ranks of locally free sheaves \(\pi_* L_T\) and \(p_* L_T\) are equal. Moreover, their restrictions on the open subscheme \(T_0 \subset T\) are isomorphic. Then there exists an invertible \(O_{\tilde{T}}\)-sheaf \(L''\) such that \(\pi_* L_T \otimes L'' = \phi^* p_* L_T = p_*(\phi, 0_{\Sigma_T})^* L_T\).

Now consider the relative projective space \(P(\pi_* L_T \otimes L'')\) together with closed immersion \(i : \tilde{\Sigma}_T \rightarrow P(\pi_* L_T \otimes L'')\). Also take the relative projective space of same dimension \(P(p_* L_T)\). Then
the square

\[
P(\pi_*L_T \otimes L'') \xrightarrow{p'} P(p_*L_T) \xrightarrow{p''} P(p_*L_T)
\]

is Cartesian. Denote by \(\tilde{\Sigma}_T\) the image of the composite map \(\tilde{\Sigma}_T \hookrightarrow P(\pi_*L_T \otimes L'') \to P(p_*L_T)\).

By the construction the scheme \(\Sigma_T'\) is flat over \(T\). Denote by \(j : \Sigma_T' \to P(p_*L_T)\) the corresponding closed immersion of this subscheme; let \(\mathcal{O}_T(1)\) be the canonical invertible sheaf on the projective bundle \(P(p_*L_T)\). Then define \(\mathbb{L}_T' := j^*\mathcal{O}_T(1)\).

Now form the fibred product \(G(p_*(E_T \otimes L_T^m), r) \otimes_T P(p_*L_T)\) and consider the mapping \(\tilde{\Sigma}_T \to G(p_*(E_T \otimes L_T^m), r) \times_T P(p_*L_T)\). This map is induced by mappings into each factor constructed earlier. Let \(\Sigma_T\) be the scheme image of this map. Then there is the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{\Sigma}_T' & \xrightarrow{p'} & \tilde{\Sigma}_T' \\
\downarrow & & \downarrow \\
G(p_*(E_T \otimes L_T^m), r) & \xleftarrow{\quad p''} & G(p_*(E_T \otimes L_T^m), r) \times_T P(p_*L_T) & \rightarrow & P(p_*L_T)
\end{array}
\]

Birational morphisms \(p', p''\) are defined by projections of the product \(G(p_*(E_T \otimes L_T^m), r) \times_T P(p_*L_T)\) to each factor.

It rests to define sheaves \(\mathbb{L}_T\) and \(\mathbb{E}_T\) on the scheme \(\Sigma_T\) by formulas:

\[
\begin{align*}
\mathbb{L}_T & := p''\mathbb{L}_T', \\
\mathbb{E}_T & := p'(j_*\mathbb{T}_T) \otimes \mathbb{L}_T^m
\end{align*}
\]

**Remark 1.7.** If \(T\) is an integral and normal scheme then the functorial correspondence which has been constructed is invertible. Then the class of schemes where the partial functor \(t^{-1}\) is defined, contains all integral and normal schemes.

## 2 Isomorphism of moduli schemes: proof of theorem 0.6

First recall some objects and constructions which are used in classical built-up of Gieseker – Maruyama scheme. By choice of the polarization \(L\) we assume that \(H^0(S, E \otimes L_m^\nu) \otimes L^\nu \to E\) is an epimorphism. Let \(\text{Quot}^{r_p(m)}(V \otimes L^{-m})\) be the Grothendieck’s \(\text{Quot}\) scheme. It parameterizes quotient sheaves \(V \otimes L^{-m} \to E\) for \(V \cong H^0(S, E \otimes L^m)\), with Hilbert polynomial \(r_p(m)\) with respect to \(L\). The scheme \(\text{Quot}\) carries universal family of quotient sheaves

\[
V \otimes L^{-m} \boxtimes \mathcal{O}_{\text{Quot}}^{r_p(m)}(V \otimes L^{-m}) \to E_{\text{Quot}}.
\]

Let \(Q'\) be the quasiprojective subscheme in \(\text{Quot}^{r_p(m)}(V \otimes L^{-m})\), consisting of points corresponding to semistable sheaves, \(\xi : Q \to Q'\) be any its smooth resolution, \(E_Q := \xi^*E_{\text{Quot}} |_{Q'}\) family of coherent sheaves which is flat over \(Q\). It comes from \(E_{\text{Quot}}\). Classical way to construct Gieseker – Maruyama scheme is to form a GIT-quotient \(Q'_{ss}/PGL(V)\) of the set \(Q'_{ss}\) of semistable points with respect to the action of the group \(PGL(V)\) upon the scheme \(Q'\). The action is induced by linear transformations of vector space \(V\).

Application of standard resolution to the data \(Q \times S, L = \mathcal{O}_Q \boxtimes L, E_Q\) leads to the collection of data \(\mathcal{Q}, \mathcal{L}, \mathcal{E}_Q\).

Note that the sheaf \(\mathcal{E} \otimes L^m\) defines a closed immersion \(j : \mathcal{S} \hookrightarrow G(V, r)\) where \(G(V, r)\) is Grassmann variety parameterizing quotient spaces of dimension \(r\) of the vector space \(V \cong H^0(S, \mathcal{E} \otimes L^m)\).
The immersion $j$ is defined non-uniquely but up to isomorphy class $H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \to V$ modulo multiplication by nonzero scalars $\vartheta \in k^*$. Let $P(m) = \chi(j^*\mathcal{O}_{G(V,r)}(m))$ be the Hilbter polynomial of the subscheme $j(\tilde{S}) \subset G(V, r)$. Hence the point corresponding to the subscheme $j(\tilde{S}) \subset G(V, r)$, is defined in the Hilbert scheme $\text{Hilb} P(m) G(V, r)$ up to the action of the group $PGL(V)$.

In [7] it is shown that the data $\tilde{Q}, \tilde{L}, \tilde{E}, \tilde{Q}$ defines the morphism $\mu : \tilde{Q} \to \text{Hilb} P(m) G(V, r)$. Also it is proven there that there is a morphism of projective schemes $\kappa : \overline{\mathcal{M}} \to \overline{M}$.

For the further consideration we also need the notion of $M$-equivalence for semistable admissible pairs. This notion is introduced and investigated in [7].

For any two schemes of the form $\tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0} (I_1[t] + (t))^{s}/(t)^{s+1} \quad \tilde{S}_2 = \text{Proj} \bigoplus_{s \geq 0} (I_2[t] + (t))^{s}/(t)^{s+1}$ define a scheme

$$\tilde{S}_{12} = \text{Proj} \bigoplus_{s \geq 0} (I_1'[t] + (t))^{s}/(t)^{s+1} = \text{Proj} \bigoplus_{s \geq 0} (I_2'[t] + (t))^{s}/(t)^{s+1}$$

together with morphisms $\tilde{S}_1 \xrightarrow{\sigma_1} \tilde{S}_{12} \xrightarrow{\sigma_2'} \tilde{S}_2$, such that the diagram

$$\begin{array}{ccc}
\tilde{S}_2 & \xrightarrow{\sigma_2} & \tilde{S}_1 \\
\sigma_1' \downarrow & & \downarrow \sigma_1 \\
\tilde{S}_{12} & \xrightarrow{\sigma_2'} & \tilde{S}_1
\end{array}$$

commutes. The operation $(\tilde{S}_1, \tilde{S}_2) \mapsto \tilde{S}_1 \circ \tilde{S}_2 = \tilde{S}_{12}$ defined as before, is associative. Moreover, since for any admissible morphism $\sigma : \tilde{S} \to S$ the relation $\tilde{S} \circ S = S \circ \tilde{S} = \tilde{S}$ holds, admissible morphisms for each class $[E]$ of $E$-equivalent semistable coherent sheaves, generate a commutative monoid $\diamond [E]$ with binary operation $\circ$ and neutral element $\text{id}_S : S \to S$.

**Definition 2.1.** [7] Semistable pairs $(\tilde{S}, \tilde{E})$ and $(\tilde{S}', \tilde{E}')$ are $M$-equivalent (monoidally equivalent) if for morphisms of $\circ$-product $\tilde{S} \circ \tilde{S}'$ to both factors $\pi' : \tilde{S} \circ \tilde{S}' \to \tilde{S}$ and $\pi : \tilde{S} \circ \tilde{S}' \to \tilde{S}'$ and for associated polystable sheaves $\bigoplus_i \text{gr}_i(\tilde{E})$ and $\bigoplus_i \text{gr}_i(\tilde{E}')$ there are isomorphims

$$\pi'^* \bigoplus_i \text{gr}_i(\tilde{E})/\text{tors} \cong \pi^* \bigoplus_i \text{gr}_i(\tilde{E}')/\text{tors}.$$
morphism becomes an immersion $j_b : \pi^{-1}(b) \hookrightarrow G(H^0(\pi^{-1}(b), \tilde{E} \otimes \tilde{L}^m|_{\pi^{-1}(b)}), r)$ when restricted to fibres of the morphism $\pi$.

The sheaf $\pi_*(\tilde{E} \otimes \tilde{L}^m)$ is locally free and then the Grassmannian bundle $G(\pi_*(\tilde{E} \otimes \tilde{L}^m), r)$ is locally trivial over $B$. Let $\bigcup_i B_i = B$ be Zariski-open trivializing cover. Subfamilies $\Sigma_i$ are given by fibred squares

\[
\begin{array}{ccc}
\bar{\Sigma} & \xleftarrow{\Sigma_i} & \Sigma_i \\
\xrightarrow{\pi} & & \xrightarrow{\pi_i} \\
B & \xleftarrow{B_i} & B_i
\end{array}
\]

where horizontal arrows are open immersions. Fix isomorphisms of trivialization $\tau_i : G(\pi_*(\tilde{E} \otimes \tilde{L}^m), r)|_{B_i} \to G(V, r) \times B_i$. The composite map $\bar{\Sigma_i} \xrightarrow{\rho_i} G(\pi_*(\tilde{E} \otimes \tilde{L}^m), r)|_{B_i} \xrightarrow{\tau_i} G(V, r) \times B_i \xrightarrow{pr_2} G(V, r)$ defines a morphism of the base into Hilbert scheme $\mu_i : B_i \to \text{Hilb}_{^{(m)}}G(V, r)$. The morphism $\mu_i$ factors through the subscheme $\mu(Q)$. For universal scheme $\text{Univ}^{(m)}G(V, r)$ over Hilbert scheme $\text{Hilb}^{(m)}G(V, r)$ we have the fibre diagram

\[
\begin{array}{ccc}
\Sigma_i & \xrightarrow{\text{Univ}^{(m)}G(V, r)} & \text{Univ}^{(m)}G(V, r) \\
\xrightarrow{\text{Univ}^{(m)}G(V, r)|_{\mu(Q)}} & & \xrightarrow{\text{Univ}^{(m)}G(V, r)|_{\mu(Q)}} \\
B_i & \xrightarrow{\text{Hilb}^{(m)}G(V, r)} & \text{Hilb}^{(m)}G(V, r)
\end{array}
\]

Define schemes $B_i Q$ and $\bar{\Sigma}_{i Q}$ as fibred products $B_i Q := B_i \times_{\mu(Q)} \bar{Q}$ and $\bar{\Sigma}_{i Q} := \bar{\Sigma}_i \times \bar{Q} B_i Q$. Let $\bar{\beta} : \bar{\Sigma}_{i Q} \to \bar{\Sigma}_i$ be the projection onto the factor. Define $\bar{E}_{i Q} := \bar{\beta}^* \bar{E}_i$. Then there is a morphism $\Phi_i : \bar{\Sigma}_{i Q} \to B_i Q \times S$ obtained by the fibred product from the morphism $\Phi : \bar{\Sigma}_i \to \bar{Q} \times S$. By Serre’s theorem and by choice of the involutive sheaf $\tilde{L}$ one has an epimorphism $\pi^* \pi_*(\tilde{E}_i Q \otimes \tilde{L}^m) \otimes \tilde{L}^{-m} \to \bar{E}_{i Q}$. Let $B_0$ be the open subset of the scheme $B$, where $\bar{\Sigma}$ is locally trivial and a fibre is isomorphic to $S$. Refining the cover $\{B_i\}$ if necessary, we come to epimorphisms $V \otimes L^{-m} \otimes \mathcal{O}_{B_i Q} \to \bar{E}_{i Q}|_{B_i Q}$ on overlaps $B_i Q = B_0 \cap B_i$. Then there is a morphism $q_{i 0} : B_{i 0} \to \text{Quot}_{^{(m)}}(V \otimes \tilde{L}^{-m})$. Since $\bar{E}_{i Q}|_{B_{i 0}} = \bar{E}_{i Q}|_{B_{i 0}}$ are families of semistable locally free sheaves, the morphism $q_{i 0}$ factors through the subscheme $Q'$.

Form a closure $q_{i 0} B_{i 0} \supset q_{i 0} B_{i 0}$ in the scheme $Q'$, and a product $B_i \times q_{i 0} B_{i 0}$. Let $\tilde{B}_i$ be a closure of the subset of points of the view $(b, q_{i 0}(b)), b \in B_{i 0}$ in the product $B_i \times q_{i 0} B_{i 0}$. The projection $\rho : \tilde{B}_i \to B_i$ onto the first factor is a birational morphism of integral schemes. Introduce the notation for the composite map $q_i : \tilde{B}_i \hookrightarrow B_i \times q_{i 0} B_{i 0} \xrightarrow{pr_1} q_{i 0} B_{i 0} \subset Q'$. Other necessary notations are fixed by the following fibre diagram

\[
\begin{array}{ccc}
\Sigma_{i Q} & \xrightarrow{\bar{\rho}} & \bar{\Sigma}_{i Q} \\
\xrightarrow{\Phi_i} & & \xrightarrow{\Phi} \\
\tilde{B}_i \times S & \xrightarrow{\rho \times id_S} & B_i Q \times S \xrightarrow{\Phi} \bar{Q} \times S
\end{array}
\]

Consider sheaves $E_{i Q} := (\Phi_{i Q}^* \bar{E}_{i Q})^{vv}$ and $(q_i, id_S)^* E_Q$. Both sheaves are reflexive on integral and normal scheme $\tilde{B}_i \times S$ and coincide on open subset out of a subscheme of codimension 3 where
they are nonlocally free. Then $\mathbb{E}_{iQ} = (q_i, id_S)^*\mathbb{E}_Q$, what, in particular, proves that the sheaf $\mathbb{E}_{iQ}$ is flat over $\overline{B}_i$.

Composite maps $\overline{B}_i \to B_{iQ} \to \overline{Q} \to Q'$ factor through morphisms $B_i \to \overline{M}$ under formation of $PGL(V)$-quotient, since isomorphic semistable admissible pairs correspond to $PGL(V)$-equivalent points. Morphisms $B_i \to \overline{M}$ are glued together in the morphism $B \to \overline{M}$. The dual morphism in the dual category $(\text{Schemes}_k)^o$ leads to the natural transformation $\omega : \overline{M} \to \overline{L}'$.

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