BETTI TABLES FOR INDECOMPOSABLE MATRIX FACTORIZATIONS OF $XY(X - Y)(X - \lambda Y)$

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ABSTRACT. We classify the Betti tables of indecomposable graded matrix factorizations over the simple elliptic singularity $f_\lambda = XY(X - Y)(X - \lambda Y)$ by making use of an associated weighted projective line of genus one.

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1. INTRODUCTION

Let $\Lambda$ be a Cohen-Macaulay algebra of infinite CM-type, occurring as the local algebra of a reduced curve singularity over an algebraically closed field $k$. Drozd and Greuel [5] proved (in char $k \neq 2$) that $\Lambda$ is of tame CM-type if and only if it birationally dominates a curve singularity of type $T_{pq}$ given by the equation $X^p + \alpha X^2 Y^2 + Y^q = 0$ ($\alpha \neq 0, 1$). Furthermore, $\Lambda$ is tame of finite growth if and only if it dominates a curve of type $T_{44}$ or $T_{36}$. Tameness of the $T_{pq}$ singularities can be shown indirectly via deformation theory, with abstract classifications known from [6]. Tameness of $T_{44}$ in particular was established by Dieterich [4] who related it to a particular tubular quiver path algebra. In [7], Drozd and Tovpyha raised the question of finding explicit presentations of indecomposable maximal Cohen-Macaulay modules over the completion of $T_{44} \sim XY(X - Y)(X - \lambda Y)$ ($\lambda \neq 0, 1$), or equivalently to produce the indecomposable matrix factorizations of $f_\lambda$. They reduced this problem to a “matrix problem”, namely representations of bunches of chains, and used this to produce some of the indecomposables.

In this paper we investigate the indecomposable graded matrix factorizations of $T_{44}$ using triangulated category methods, and produce a classification closer in spirit to the work of Dieterich. One can deal with $T_{36}$ with similar methods but for simplicity we will restrict ourselves to the case $T_{44}$. We deduce a complete classification of Betti tables of indecomposable graded MCM modules over $k[X,Y]/f_\lambda$.

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We will see that all MCM modules over the algebra \( k[[X,Y]]/f_\lambda \) are gradable and so this subsumes the above problem. The main result of this paper is (see 5.1, 5.2):

**Theorem.** There is a classification of Betti tables of indecomposable graded matrix factorizations of \( f_\lambda \). Up to internal degree shifts, these fall into 5 general and 4 exceptional shapes (respectively 3 and 2 up to shifts and syzygies).

We also describe the set of indecomposables with fixed Betti table, characterize exceptional objects by a numerical criterion, study natural functionals on \( K_0 \) and give a classification of the indecomposable Ulrich modules over \( k[ X,Y ]/f_\lambda \). Our work is close in nature to that of A. Pavlov who classified the Betti tables of matrix factorizations of a Hesse plane cubic in [15], although the presence of exceptional objects in our setting adds a layer of complexity.

Order the linear factors as \( XY(X - Y)(X - \lambda Y) = l_1 l_2 l_3 l_4 \). We will study the classification of indecomposable graded matrix factorizations of \( f_\lambda \) by means of the finite dimensional “Squid algebra” \( Sq(2,2,2,2;\lambda) \), which is the path algebra of the quiver

```
      1          2
     / \          / \
    5   X -- Y   6
     \ /          \ /
       3          4
     / \          / \   p_3
    p_1       p_2  p_4
```

subject to the relations \( p_i l_i(X,Y) = 0 \). The algebra \( Sq(2,2,2,2;\lambda) \) arises as the endomorphism algebra of a tilting sheaf on the weighted projective line \( X = P^1(2,2,2,2;\lambda) \) introduced by Geigle and Lenzing, and this will reduce most of our work to sheaf cohomology calculations on the latter. More precisely we will use

**Theorem** (Buchweitz-Iyama-Yamaura). There is a tilting MCM module \( T \) with \( \text{End}_{gr R}(T) \cong Sq(2,2,2,2;\lambda) \), and so we have induced equivalences of triangulated categories

\[
\text{MCM}(gr S/f_\lambda) \cong D^b(Sq(2,2,2,2;\lambda)) \cong D^b(X).
\]

This result is a special case of a recent result of Buchweitz-Iyama-Yamaura [2], and the author has learned it from Buchweitz. That matrix factorizations of \( f_\lambda \) be related to \( X = P^1(2,2,2,2;\lambda) \) should also be well-known to experts. Indeed one can, at least in \( \text{char } k \neq 2 \), compare \( \mathbb{Z}\)-graded matrix factorizations of \( f_\lambda \) to \( \mathbb{Z}\oplus \mathbb{Z}_2 \)-graded matrix factorizations of \( f_\lambda + z^2 \) by results of Kn"orrer. The elliptic curve \( E_\lambda = \{ f_\lambda + z^2 = 0 \} \) in \( \mathbb{P}(1,1,2) \) is a branched double cover \( \pi : E_\lambda \to \mathbb{P}^1 \) ramified over \( \{ f_\lambda = 0 \} \), with hyperelliptic involution \( \sigma \). One should then obtain from Kn"orrer’s and Orlov’s theorem an equivalence

\[
\text{HMF}^\mathbb{Z}(S,f_\lambda) \cong \text{HMF}^\mathbb{Z}\mathbb{Z}_2(S[z],f_\lambda + z^2) \cong D^b(\text{coh}_\sigma(E_\lambda))
\]

with the derived category of \( \sigma \)-equivariant coherent sheaves on \( E_\lambda \). Coherent sheaves on the weighted projective line \( \mathbb{P}^1(2,2,2,2;\lambda) \) are equivalent to the latter in \( \text{char } k \neq 2 \), and so we will prefer the above characteristic-free setup via tilting theory, where calculations are more tractable in any case.
2. Background and setup

Throughout this paper, modules will refer to finitely generated right graded modules. We denote by $M \mapsto M(i)$ the grade shift autoequivalence with $M(i)_n = M_{n+i}$. Fix a field $k$ and let $f \in S = k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d$. The triangulated homotopy category of graded matrix factorizations $HMF^\perp(S, f)$ has for objects pairs of morphisms $(\varphi, \psi)$ of $S$-free modules

$$F \xrightarrow{\varphi} G \xleftarrow{\psi} F(-d)$$

with compositions $\varphi \psi = f \cdot \text{Id}$ and $\psi \varphi = f \cdot \text{Id}$, and for morphisms the natural notion of homotopy classes of chain-maps. Matrix factorizations give $S$-free presentations of graded maximal Cohen-Macaulay (MCM) modules over $R = S/f$, with the short resolution

$$0 \leftarrow M \leftarrow F \xrightarrow{\varphi} G \leftarrow 0$$
of $M = \text{coker}(\varphi)$ descending to a 2-periodic $R$-free resolution

$$P_* : 0 \leftarrow M \leftarrow F \xrightarrow{\varphi} G \xleftarrow{\psi} F(-d) \xrightarrow{\varphi} G(-d) \leftarrow \cdots$$

This induces an equivalence of triangulated categories $HMF^\perp(S, f) \cong \text{MCM}^{\perp}(gr R)$ with the projectively stable category of graded MCM modules. The latter inherits a triangulated structure from Buchweitz’s equivalence \cite{Buchweitz}

$$D_{\text{sg}}(gr R) = D^b(gr R)/D^{\text{perf}}(gr R) \cong \text{MCM}^{\perp}(gr R)$$
or equivalently from an equivalence with the homotopy category of complete resolutions, obtained by extending the above resolution of $M$ by periodicity to the acyclic complex of graded free modules

$$C_* : \cdots \leftarrow F(d) \xrightarrow{\varphi} G(d) \xleftarrow{\psi} F \xrightarrow{\varphi} G \xleftarrow{\psi} F(-d) \xrightarrow{\varphi} G(-d) \leftarrow \cdots$$

We have for suspension $\Sigma M = M[1] = \text{cosyz}_R(M)$ with inverse $\Omega M = M[-1] = \text{syz}_R(M)$. The category $\text{MCM}(gr R)$ also has the grade shift exact autoequivalence $M \mapsto M(1)$. In particular note the 2-periodicity natural isomorphism $(d) = [2]$.

Define the Tate cohomology groups $\text{Ext}^n_{gr R}(M, N)$ for $M$ an MCM module and $N$ any module by $H^*\text{Hom}_{gr R}(C_*, N)$ for any $n \in \mathbb{Z}$. The module $N$ admits an MCM approximation $N^{st}$ fitting in a short exact sequence

$$0 \rightarrow Q \rightarrow N^{st} \rightarrow N \rightarrow 0$$

with $Q$ perfect, so that $N$ is sent to $N^{st}$ under Buchweitz’s equivalence. Tate cohomology vanishes against perfect complexes, and we have $\text{Ext}^n_{gr R}(M, N^{st}) = \text{Hom}_{gr R}(M, N^{st}[n])$. We define the (complete) graded Betti numbers $\beta_{i,j}$ of $M$ by

$$\beta_{i,j} = \dim_k \text{Ext}^i_{gr R}(M, k(-j)) = \dim_k \text{Ext}^i_{gr R}(M, k^{st}(-j)).$$

When the resolution $P_*$ is minimal, it follows that $F = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\oplus \beta_0,j}$ and $G = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\oplus \beta_1,j}$, with the other Betti numbers obtained by periodicity $\beta_{i,j} = \beta_{i+2,j+4}$. Hence calculating Betti tables reduces to calculating dimensions of morphism spaces into the various MCM modules $k^{st}(-j)$. We will need the following:
Proposition 2.1. Assume that $R$ has isolated singularities. Then

$$S_R(-) = - \otimes_R \omega_R[\dim R - 1] = (a)[\dim R - 1]$$

is a Serre functor for $\text{MCM}(\text{gr} R)$, where $a = d - n$.

Since $R$ is graded connected, the category $\text{MCM}(\text{gr} R)$ is Ext-finite and Krull-Schmidt, and in particular idempotent complete.

Weighted projective lines. We refer to [9, 12, 14] for basic definitions and results, see also [3]. In particular we will only use the weighted projective line of genus one $X = \mathbb{P}^1(\mathbb{p}, \lambda) = \mathbb{P}^1(2, 2, 2, \lambda)$ with $\lambda = (0, 1, \infty, \lambda)$. The derived categories of weighted projective lines of genus one were thoroughly investigated by Lenzing and Meltzer in [12, 14].

We fix the notation for the rest of the paper. Recall that to a general set of weights $\mathbb{p} = (p_1, \ldots, p_n)$ one associates a rank one abelian group $L = L(\mathbb{p})$ with

$$L(\mathbb{p}) = \langle \vec{x}_1, \ldots, \vec{x}_n, \vec{c} | p_1 \vec{x}_1 = \cdots = p_n \vec{x}_n = \vec{c} \rangle.$$ Setting $p = \text{lcm}(p_1, \ldots, p_n)$, there is a group homomorphism $\delta: L \to \mathbb{Z}$ given by $\delta(\vec{x}_i) = \frac{p}{p_i}$, with finite kernel. We denote by $\bar{\mathbb{p}} = (n - 2)\vec{c} - \sum_{i=1}^n \vec{x}_i$ the canonical element in $L$. We have an isomorphism of abelian groups $\mathbb{L} \cong \text{Pic}(X)$ sending $\vec{x} \mapsto \mathcal{O}(\vec{x})$, and we denote by $\omega_X = \mathcal{O}(\bar{\mathbb{p}})$ the canonical line bundle of $X$.

Every coherent sheaf on $X$ is the direct sum of its torsion subsheaf and quotient torsion-free sheaf, which is then a vector bundle. Vector bundles admit finite filtrations by line bundles. The indecomposable torsion sheaves are supported over a single point $x \in \mathbb{P}^1(k)$. We say that $x$ is ordinary if it lies outside of the set $\lambda$, and exceptional otherwise. Torsion sheaves supported over $x$ form a serial abelian subcategory, with unique simple sheaf over $x$ ordinary and $p_i$-many simple sheaves $\{S_{i,j}\}_{j \in \mathbb{Z}/p_i\mathbb{Z}}$ over $x = \lambda_i$ exceptional. These have presentations

$$0 \to \mathcal{O}((j - 1)\vec{x}_i) \to \mathcal{O}(j\vec{x}_i) \to S_{i,j} \to 0.$$ In particular we single out $S_{i,0}$ as the unique simple sheaf with a non-zero section and we have $\text{Hom}(\mathcal{O}, S_{i,0}) = k$ and $S_{i,j} \otimes \omega_X = S_{i,j+1}$. There is a family of indecomposable “ordinary” torsion sheaves $S_x$ for any $x$, with presentation

$$0 \to \mathcal{O}(-\vec{c}) \to \mathcal{O} \to S_x \to 0.$$ The sheaf $S_x$ has length one when $x$ is ordinary and length $p_i$ over $x = \lambda_i$, with the $S_{i,j}$ as simple composition factors. Lastly, there are additive rank and degree functionals on $K_0(X) := K_0(\text{coh}(X))$, uniquely determined by their value on line bundles as

$$rk(\mathcal{O}(\vec{x})) = 1$$
$$\deg(\mathcal{O}(\vec{x})) = \delta(\vec{x}).$$ In particular $\deg(S_x) = \deg(\mathcal{O}(\vec{c})) = p$ and $\deg(S_{i,j}) = 1$.

\footnote{This agrees with the notation in [9] but disagrees with [14].}
Mutations of exceptional pairs. Given objects $A, B, C$ in an Ext-finite $k$-linear triangulated category $T$, denote by $\mathbb{R}\text{Hom}(A, B) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_T(A, B[n])[-n]$ the object in $D^b(k)$, and define

$$
\mathbb{R}\text{Hom}(A, B) \otimes_k C = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_T(A, B[n]) \otimes_k C[-n].
$$

Interpret the dual $\mathbb{R}\text{Hom}(A, B)^*$ in $D^b(k)$ accordingly. Following Gorodentsev, we have canonical distinguished triangles in $T$

$$
L_A(B)[-1] \rightarrow \mathbb{R}\text{Hom}(A, B) \otimes_k A \xrightarrow{CV} B \rightarrow L_A(B)
$$

$$
R_A(B) \rightarrow A \xrightarrow{\text{cex}} \mathbb{R}\text{Hom}(A, B)^* \otimes B \rightarrow R_A(B)[1]
$$

which can be taken to define $L_A(B), R_A(B)$. We have the standard result.

**Proposition 2.2** (Gorodentsev, [16]). Let $(E, F)$ be an exceptional pair in $T$. The operations $L, R$ descend to an action on the set of exceptional pairs

$$
R : (E, F) \mapsto (F, R_E(F))
$$

$$
L : (E, F) \mapsto (L_E(F), E).
$$

Furthermore, $R, L$ are inverses in that $L \circ R(E, F) \simeq (E, F)$ and $R \circ L(E, F) \simeq (E, F)$.

**The equivalence.** Now fix $k$ algebraically closed and $f_\lambda = XY(X - Y)(X - \lambda Y)$ in $S = k[X, Y]$ with $\lambda \neq 0, 1$ in $k$. Let $R = S/f_\lambda$, with $m = R_{\geq 1}$. We have $a = d - 2 = 2$. Writing $f_\lambda = l_1l_2l_3l_4$, we have natural matrix factorizations $(l_i, f_\lambda/l_i)$ which present the MCM module $L_i = R/l_i \simeq k[z_i]$. Next, since $R$ is Cohen-Macaulay, $m$ contains a non-zero divisor and $m^n$ has depth 1 for all $n \geq 1$, and so are MCM modules. Now, the modules $L_i$ have particular simple complete resolutions

$$
\cdots \leftarrow R(3) \xrightarrow{f_\lambda/l_i} R \xleftarrow{l_i} R(-1) \xleftarrow{f_\lambda/l_i} R(-4) \leftarrow \cdots
$$

Since $m = \Omega^1_{k^e}(k) = k^{et}[-1]$, using Serre duality we note that the Tate cohomology groups between $m(r)$ and $L_i(s)$ are easily computed for any $r, s \in \mathbb{Z}$. We will need:

**Lemma 2.3.** We have

$$
\dim_k \mathbb{E}xt^n_{gr R}(m(1), L_i) = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0
\end{cases}
$$

$$
\dim_k \mathbb{E}xt^n_{gr R}(m(2), L_i) = \begin{cases} 
1 & n = 1 \\
0 & n \neq 1
\end{cases}
$$

We will give a proof of the following theorem, which in this case follows standard lines.

**Theorem 2.4** (Buchweitz-Iyama-Yamaura). The collection $(m(1), m^2(2), L_1, L_2, L_3, L_4)$ forms a full strong exceptional collection in $\text{MCM}(gr R)$.

**Proof.** We use Orlov’s theorem with cutoff $i = -a = -2$. Hence there is a fully faithful embedding

$$
\Phi_{-2} : \text{MCM}(gr R) \rightarrow D^b(gr_{\geq -2}R)
$$
which is right-inverse to Buchweitz’s stabilization functor $\text{st} : D^b(grR) \to D_{sg}(grR) \cong \text{MCM}(grR)$, and a semiorthogonal decomposition involving $Z = V(f_\lambda) \subset \mathbb{P}^1$ given by

$$\Phi_2(\text{MCM}(grR)) = \langle k(2), k(1), R\Gamma_{\geq 0}(D^b(Z)) \rangle.$$  

The category $D^b(Z)$ is semisimple generated by the skyscraper sheaves of 4 points, and we have $R\Gamma_{\geq 0}(D^b(Z)) = \langle L_1, \ldots L_4 \rangle$. This gives a full exceptional sequence $\langle k(2), k(1), L_1, \ldots, L_4 \rangle$ which is however not strong, and so we mutate it. Now, the extension

$$\xi : m/m^2 \to R/m^2 \to k \to m/m^2[1]$$

agrees with the universal extension

$$\text{Ext}^1(k, k(-1))^* \otimes_k k(-1) \to R_k(k(-1)) \to k \xrightarrow{\text{coev}} \text{Ext}^1(k, k(-1))^* \otimes_k k(-1)[1].$$

One has $R\Hom(k, k(-1)) = \text{Ext}^1_{grR}(k, k(-1))[-1]$, and so we have the right mutation $R_k(k(-1)) = R/m^2$, and similarly $R_k(k(1)) = (R/m^2)(2)$. After mutating and desuspending the first two terms, we obtain the resulting exceptional collection

$$\langle k(1)[-1], (R/m^2)(2)[-1], L_1, \ldots, L_4 \rangle$$

of $\Phi_2(\text{MCM}(grR))$, which descends to the full exceptional collection

$$\langle m(1), m^2(2), L_1, \ldots, L_4 \rangle$$

in $\text{MCM}(grR)$. Direct calculations with lemma [23] and the extension $\xi$ shows that this collection is strong. □

Setting $U = m(1) \oplus m^2(2) \oplus \left( \bigoplus_{i=1}^4 L_i \right)$, we obtain that $U$ is a tilting object for $\text{MCM}(grR)$. We can calculate the stable endomorphism ring.

**Proposition 2.5** (Buchweitz-Iyama-Yamaura). We have algebra isomorphisms $\text{End}_{grR}(U) = \text{End}_{grR}(U) \cong S_q(2, 2, 2; \lambda)$.

**Proof.** Consider the following morphisms

$$\xymatrix{ m(1) \ar[r]^X \ar[rr]^Y & m^2(2) \ar[rr]^2 & & L_1 \ar[rr]^1 & & L_2 \ar[rr]^q_1 & & L_3 \ar[rr]^q_3 & & L_4 \ar[rr]^q_4 & & L_2 \ar[rr]^1 & & L_1 }$$

with $q_i : m^2(2) \to L_i$ induced by $X, Y \mapsto \overline{X}, \overline{Y} \in R/l_i = L_i$. These are easily seen to generate the endomorphism algebra and satisfy the squid relations, thus showing $\text{End}_{grR}(U) \cong S_q(2, 2, 2; \lambda)$. Lastly, a simple dimension count shows that $\text{End}_{grR}(U) = \text{End}_{grR}(U)$. □

Thinking ahead, we will normalize our tilting object by using $T = U(-3)[1]$ instead of $U$, which has the same endomorphism algebra. Using $T$, and making use of [13] example 4.4, we obtain:

**Corollary 2.6.** For $\mathbb{X} = \mathbb{P}^1(2, 2, 2; \lambda)$, we have equivalences of triangulated categories

$$\text{MCM}(grR) \cong D^b(\Lambda) \cong D^b(\mathbb{X}).$$
The composed equivalence sends the full strong exceptional collection
\[(k^s(-2), (R/m^2)^s(-1), L_1(-3)[1], \ldots, L_4(-3)[1])\]
to
\[(\mathcal{O}, \mathcal{O}(\bar{c}), S_{1,0}, \ldots, S_{4,0}).\]

As alluded to in the introduction, we also have:

**Corollary 2.7.** Let \(\hat{R}\) be the completion of \(R\) at \(m\). Every MCM \(\hat{R}\)-module \(M\) is the completion of a graded MCM \(R\)-module.

**Proof.** By [11] proposition 1.5, the completion functor \((\_): \text{MCM}(grR) \to \text{MCM}(\hat{R})\) identifies with the universal morphism to the triangulated hull of the orbit category \(\text{MCM}(grR)/1\). Since \((4) = [2]\), the functor \((1)\) moves away from the hereditary \(\mathcal{O}\), and so the completion functor is essentially surjective by results of Keller [10]. \(\square\)

### 3. Betti tables and cohomology tables

We can calculate graded Betti numbers as \(\beta_{i,j}(M) = \dim_k \Ext^{i-j}_R(M, k^s(-j))\), hence as dimension of corresponding Ext-spaces in \(D^b(\mathcal{X})\). Due to the periodicity \(k^s(-j-4) = k^s(-j)[-2]\), it suffices to compute the images of
\[k^s, k^s(-1), k^s(-2), k^s(-3).\]

The Serre functor \(S_R(M) = M(2)\) is sent to the Serre functor \(S_\mathcal{X}(\mathcal{F}) = \mathcal{F} \otimes \omega_\mathcal{X}[1]\). The periodicity identity \((4) = [2]\) corresponds to the fact that \(\omega_\mathcal{X}\) is 2-torsion. Keeping this in mind, we will prove the following:

**Theorem 3.1.** Under the above equivalence, we have
\[
\begin{align*}
k^s & \mapsto \omega_\mathcal{X}[1] \\
k^s(-1) & \mapsto \mathcal{O}(-\bar{c})[1] \\
k^s(-2) & \mapsto \mathcal{O} \\
k^s(-3) & \mapsto \mathcal{O}(-\bar{c}) \otimes \omega_\mathcal{X}.
\end{align*}
\]

**Proof.** By corollary [22] we already have \((k^s(-2)) \mapsto \mathcal{O}\), and so \(k^s = S_R(k^s(-2)) \mapsto S_\mathcal{X}(\mathcal{O}) = \omega_\mathcal{X}[1]\). Let \(F_{k^s(-1)}\) correspond to \(k^s(-1)\). We can obtain the exceptional pair \((k^s(-2), (R/m^2)^s(-1))\) as the right mutation of
\[R: (k^s(-1), k^s(-2)) \mapsto (k^s(-2), R_{k^s(-1)}(k^s(-2)))\]
and so we can obtain \((\mathcal{O}, \mathcal{O}(\bar{c}))\) as the right mutation of
\[R: (F_{k^s(-1)}, \mathcal{O}) \mapsto (\mathcal{O}, R_{F_{k^s(-1)}}(\mathcal{O}))\]

By [22] we can recover \(F_{k^s(-1)}\) as the left mutation \(F_{k^s(-1)} \cong L_\mathcal{O} \left(R_{F_{k^s(-1)}}(\mathcal{O})\right) \cong L_\mathcal{O}(\mathcal{O}(\bar{c}))\), calculated by the distinguished triangle
\[
L_\mathcal{O}(\mathcal{O}(\bar{c}))[1] \to R\Hom_\mathcal{X}(\mathcal{O}, \mathcal{O}(\bar{c})) \otimes_k \mathcal{O} \xrightarrow{\text{ev}} \mathcal{O}(\bar{c}) \to L_\mathcal{O}(\mathcal{O}(\bar{c})).
\]

Finally we have a fully faithful embedding \(D^b(\mathbb{P}^1) \hookrightarrow D^b(\mathcal{X})\) sending \(\mathcal{O}(n)\) to \(\mathcal{O}(n\bar{c})\). Calculating in the former one sees that left mutation gives \(F_{k^s(-1)} = L_\mathcal{O}(\mathcal{O}(\bar{c})) = \mathcal{O}(-\bar{c})[1]\), and so \(k^s(-1) \mapsto \mathcal{O}(-\bar{c})[1]\) and \(k^s(-3) \mapsto \mathcal{O}(-\bar{c}) \otimes \omega_\mathcal{X}. \square\
Given an MCM module $M$, write $\mathcal{F}_M$ for the corresponding complex of coherent sheaves on $\mathcal{X}$.

**Corollary 3.2.** We can calculate Betti numbers $\beta_{i,j} = \beta_{i,j}(M)$ as follows:

\[
\begin{align*}
\beta_{0,0} &= \dim_k \text{Ext}^i(\mathcal{F}_M, \omega_{\mathcal{X}}[1]) = h^{-i}(\mathcal{F}_M) \\
\beta_{1,1} &= \dim_k \text{Ext}^i(\mathcal{F}_M, \mathcal{O}(-\mathcal{C})[1]) = h^{-i}(\mathcal{F}_M(\mathcal{C}) \otimes \omega_{\mathcal{X}}) \\
\beta_{1,2} &= \dim_k \text{Ext}^i(\mathcal{F}_M, \mathcal{O}) = h^{1-i}(\mathcal{F}_M \otimes \omega_{\mathcal{X}}) \\
\beta_{1,3} &= \dim_k \text{Ext}^i(\mathcal{F}_M, \mathcal{O}(-\mathcal{C}) \otimes \omega_{\mathcal{X}}) = h^{1-i}(\mathcal{F}_M(\mathcal{C})).
\end{align*}
\]

When $\mathcal{F}_M$ is a coherent sheaf, collecting terms via the periodicity $\beta_{i,j} = \beta_{i+2,j+4}$, the only possible non-trivial Betti numbers for $\mathcal{F}$ of the form $\beta_{0,\ast}, \beta_{1,\ast}$ are

$$\beta_{0,0}, \beta_{0,1}, \beta_{0,2}, \beta_{0,3}, \beta_{1,2}, \beta_{1,3}, \beta_{1,4}, \beta_{1,5}.$$

Since $\text{coh}(\mathcal{X})$ is hereditary, indecomposables in $D^b(\mathcal{X})$ are of the form $\mathcal{F}[-n]$ for $\mathcal{F}$ an indecomposable coherent sheaf and $n \in \mathbb{Z}$, and it suffices to work out Betti tables corresponding to coherent sheaves. In this case, the data is best expressed in the following table:

$$
\beta(M) = \begin{pmatrix}
\beta_{0,0} & \beta_{1,2} \\
\beta_{0,1} & \beta_{1,3} \\
\beta_{0,2} & \beta_{1,4} \\
\beta_{0,3} & \beta_{1,5}
\end{pmatrix} = \begin{pmatrix}
h^0(\mathcal{F}) & h^0(\mathcal{F} \otimes \omega_{\mathcal{X}}) \\
h^0(\mathcal{F}(\mathcal{C}) \otimes \omega_{\mathcal{X}}) & h^0(\mathcal{F}(\mathcal{C})) \\
h^1(\mathcal{F} \otimes \omega_{\mathcal{X}}) & h^1(\mathcal{F}) \\
h^1(\mathcal{F}(\mathcal{C})) & h^1(\mathcal{F}(\mathcal{C}) \otimes \omega_{\mathcal{X}})
\end{pmatrix}
$$

where $\mathcal{F} = \mathcal{F}_M$. We will refer to the latter table as the cohomology table $\beta(\mathcal{F})$.

**Weighted projective lines of genus one.** Let us recall the classification scheme for indecomposable coherent sheaves over a weighted projective line of genus one due to Lenzing and Meltzer [12], which closely mirrors Atiyah’s classification of sheaves on an elliptic curve. Let $rk, \deg : K_0(\mathcal{X}) \to \mathbb{Z}$ be the rank and degree functionals, and let $\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{rk(\mathcal{F})}$ be the slope of $\mathcal{F}$. For $q \in \mathbb{Q} \cup \{\infty\}$, let $C_q$ be the category of semistable sheaves of slope $q$ (see [12] 2.5). One can show that indecomposables are semistable, and thus part of $C_q$ for some $q$. As $\deg(\omega_{\mathcal{X}}) = 0$ for $\mathcal{X}$ of genus one, we see that each $C_q$ is closed under $- \otimes \omega_{\mathcal{X}}$. In [12], [14] chapter 5], Lenzing and Meltzer construct two autoequivalences

$$R, S : D^b(\mathcal{X}) \to D^b(\mathcal{X})$$

given by tubular mutations (or spherical twists), which fit into canonical distinguished triangles

$$R\mathcal{F} \to \mathcal{F} \xrightarrow{\text{cuv}} \bigoplus_{j \in \mathbb{Z}_2} \text{RHom}(\omega_{\mathcal{X}}^j, \mathcal{F})^* \otimes_k \omega_{\mathcal{X}}^j \to R\mathcal{F}[1]$$

and (for some fixed choice of $i$)

$$\bigoplus_{j \in \mathbb{Z}_2} \text{RHom}(S_{i,j}, \mathcal{F}) \otimes_k S_{i,j} \xrightarrow{\text{cuv}} \mathcal{F} \to S\mathcal{F} \to \bigoplus_{j \in \mathbb{Z}_2} \text{RHom}(S_{i,j}, \mathcal{F})[1]$$

These act on rank and degree by

$$
\begin{pmatrix}
\text{rk}(R\mathcal{F}) \\
\text{deg}(R\mathcal{F})
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\text{rk}(\mathcal{F}) \\
\text{deg}(\mathcal{F})
\end{pmatrix}
$$

$$
\begin{pmatrix}
\text{rk}(S\mathcal{F}) \\
\text{deg}(S\mathcal{F})
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
\text{rk}(\mathcal{F}) \\
\text{deg}(\mathcal{F})
\end{pmatrix}
$$
and restrict to equivalences

\[ R : C_q \xrightarrow{\cong} C_{q+1} \]
\[ S : C_q \xrightarrow{\cong} C_{q+1} \]

for \( q \neq \infty \), and

\[ R : C_\infty \xrightarrow{\cong} C_1. \]

By a known result, the transformations \( R : q \mapsto \frac{q}{q+1} \) and \( S : q \mapsto q + 1 \) defines an action of the free semigroup on two words \( F\{R, S\} \) on the positive rationals \( \mathbb{Q}_+ \), which is free and transitive with single generator 1. Writing \( q \in \mathbb{Q}_+ \) as \( w_q(R, S) \cdot 1 \) for some unique word \( w_q(R, S) \), one deduces the existence of an autoequivalence

\[ w_q(R, S) \circ R : D^b(X) \xrightarrow{\cong} D^b(X) \]

restricting to \( C_\infty \xrightarrow{\cong} C_q \) for any \( q > 0 \), then extended to any \( q \leq 0 \) by composing with sufficient powers of \( S^{-1} \). Since the category \( C_\infty \) consists of all skyscraper sheaves, the category \( C_q \) is serial for any \( q \), with simples given by the stable sheaves. Since the type of \( X \) is \((2,2,2,2)\), the Auslander-Reiten quiver of \( C_q \) breaks down into components which are rank one tubes indexed by the ordinary points of \( X \), as well as 4 tubes of rank two indexed by the exceptional points. These correspond to indecomposables for which \( F \otimes_{\omega_X} \cong F \) and \( F \otimes_{\omega_X} \not\cong F \), respectively.

For computing morphism spaces, we have the following well-known results.

**Lemma 3.3 (lemma 4.1, [12]).** Let \( F, G \) be semistable sheaves of slopes \( q, q' \).

1. If \( q > q' \), then \( \text{Hom}(F, G) = 0 \).
2. If \( q < q' \), then \( \text{Ext}^1(F, G) = 0 \).

**Proposition 3.4 (Weighted Riemann-Roch, [12]).** We have

\[ \chi(F, G) + \chi(F, G \otimes \omega_X) = \begin{vmatrix} rk(F) & rk(G) \\ deg(F) & deg(G) \end{vmatrix}. \]

In particular, we get

\[ \chi(F) + \chi(F \otimes \omega_X) = deg(F). \]

**The shift autoequivalence.** The autoequivalence \( M \mapsto M(1) \) on \( \text{MCM}(grR) \) induces an autoequivalence \( F_M \mapsto F_M \{1\} \) on \( D^b(X) \). Since shifting the grading acts by translation on Betti tables, it suffices to compute one Betti table in the orbit \( \{M(n)\}_{n \in \mathbb{Z}} \), or equivalently one cohomology table in the orbit \( \{F_M\{n\}\}_{n \in \mathbb{Z}} \). First, we calculate the effect of (1) on rank and degree. Observe that these pull back to functionals \( rk, deg : K_0(\text{MCM}(grR)) \to \mathbb{Z} \).

**Lemma 3.5.** For any graded MCM module \( M \), we have

\[ deg(M) = \chi(M, k^st) - \chi(M, k^st(-2)) \]
\[ = \sum_{i \in \mathbb{Z}} (-1)^i \beta_{i,0} - \sum_{i \in \mathbb{Z}} (-1)^i \beta_{i,2}, \]

\[ rk(M) = \frac{1}{2} \chi(M, k^st(-1) \oplus k^st(-2)) - \frac{1}{2} \chi(M, k^st \oplus k^st(-3)) \]
\[ = \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^i (\beta_{i,1} + \beta_{i,3}) - \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^i (\beta_{i,0} + \beta_{i,3}). \]
When $M$ corresponds to a coherent sheaf, this simplifies to
\[
\deg(M) = (\beta_{0,0} + \beta_{1,2}) - (\beta_{0,2} + \beta_{1,4}),
\]
\[
\rk(M) = \frac{1}{2}(\beta_{0,1} + \beta_{0,2} + \beta_{1,3} + \beta_{1,4}) - \frac{1}{2}(\beta_{0,0} + \beta_{0,3} + \beta_{1,2} + \beta_{1,5}).
\]

Proof. The formula for $\deg(M)$ falls out of the weighted Riemann-Roch theorem via Theorem 3.1. To deduce the formula for $\rk(M)$, we use $\deg(F(\C)) = \deg(F) + 2\rk(F)$, so that $\rk(F) = \frac{1}{2}(\deg(F(\C)) - \deg(F))$, then collect terms via 3.1 \hfill \Box

**Proposition 3.6.** For any graded MCM module $M$, we have
\[
\begin{pmatrix}
\rk(M(1)) \\
\deg(M(1))
\end{pmatrix} =
\begin{pmatrix}
-1 & -1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
\rk(M) \\
\deg(M)
\end{pmatrix}.
\]

Proof. Writing $Z(M)$ for $(\rk(M) \ \deg(M))^T$, we need to verify that
\[
\begin{array}{ccc}
K_0 & \rightarrow & Z \\
\downarrow & & \downarrow \\
(1) & \rightarrow & (-1, -1) \\
\downarrow & & \downarrow \\
K_0 & \rightarrow & Z^2
\end{array}
\]
commutes. The full exceptional collection $(k^{st}(-1), k^{st}(-2), L_1(-3)[1], \ldots, L_4(-3)[1])$ descends to a $\mathbb{Z}$-basis of $K_0$, on which this can be checked. We have
\[
\begin{align*}
Z(k^{st}(-2)) &= Z(O) = (1 \ 0)^T \\
Z(k^{st}(-1)) &= Z(O(-\C)[1]) = (-1 \ 2)^T \\
Z(k^{st}) &= Z(\omega_{\C}[1]) = (-1 \ 0)^T \\
Z(L_i(-3)[1]) &= Z(S_{i,0}) = (0 \ 1)^T \\
Z(L_i(-2)[1]) &= (-1 \ 1)^T.
\end{align*}
\]

The last line is calculated by Lemma 3.5 and so the above diagram commutes. \hfill \Box

**Remark 3.7.** The above matrix has order 4 in $\text{SL}(2, \mathbb{Z})$, corresponding to the identity $(4) = [2]$ on $\text{MCM}(\text{gr}R)$.

Writing $r, d$ for rank and degree, it is not hard to find a fundamental domain for the action of $(1)$ on the $(r, d)$ lattice. We will take our fundamental domain as the union of 3 regions, as shown in figure 1. Note that since $r \geq 0$ throughout, each pair $(r, d)$ is realized by a coherent sheaf and we need not consider complexes. Another reason for this choice is to maximize vanishing patterns in the cohomology table
\[
\beta(F) = \begin{pmatrix}
h^0(F) & h^0(F \otimes \omega_{\mathbb{C}}) \\
h^0(F(\C) \otimes \omega_{\mathbb{C}}) & h^0(F(\C)) \\
h^1(F) & h^1(F(\C)) \\
h^1(F(\C) \otimes \omega_{\mathbb{C}})
\end{pmatrix}
\]

**Lemma 3.8.** Let $F$ be an indecomposable coherent sheaf.

(1) For $F$ in region $(1)$, $h^1(F) = h^1(F \otimes \omega_{\mathbb{C}}) = h^1(F(\C)) = h^1(F(\C) \otimes \omega_{\mathbb{C}}) = 0$.

(2) For $F$ in region $(3)$, $h^0(F) = h^0(F \otimes \omega_{\mathbb{C}}) = h^0(F(\C)) = h^0(F(\C) \otimes \omega_{\mathbb{C}}) = 0$.

\footnote{We will ignore $(0, 0)$ since any indecomposable with $rk(M) = 0 = \deg(M)$ must be zero.}
BETTI TABLES FOR MATRIX FACTORIZATIONS OF $X(Y - Y)(X - \lambda Y)$

\[ (X - Y)(X - \lambda Y) \]

\[ r \geq 0, \ d > 0 \]  \hspace{2cm} (1)

\[ r > 0, \ d = 0 \]  \hspace{2cm} (2)

\[ r > 0, \ d < -2r. \]  \hspace{2cm} (3)

**Figure 1.** Fundamental domain for the action of (1) on $(r, d)$.

**Proof.** These follow from Lemma 3.3 by slope arguments, using the formula

\[ \mu(F \otimes L) = \mu(F) + \deg(L) \]

for a line bundle $L$.

This lemma reduces calculations in regions (1), (3) to computing Euler characteristics, and region (2) can be dealt with by hand.

4. COHOMOLOGY TABLES OF INDECOMPOSABLE COHERENT SHEAVES

**Cohomology tables for rank one tubes.** We are now in a position to compute the cohomology tables of indecomposable sheaves. We will list the corresponding possible Betti tables of matrix factorizations in a later section, under a different normalization. We begin with indecomposables living in rank one tubes, or equivalently which satisfy $F \otimes \omega_X \sim F$.  

**Theorem 4.1.** Let $(r, d)$ be in the fundamental domain with $q = \frac{d}{r}$. Consider $F$ with $F \otimes \omega_X \sim F$ and $(rk(F), \deg(F)) = (r, d)$. An indecomposable such $F$ exists if and only if $\gcd(r, d)$ is even, in which case $\frac{1}{2} \gcd(r, d)$ gives the length of $F$ in $\mathcal{C}_q$. The cohomology table $\beta(F)$ is then given as:

\[
\begin{pmatrix}
\frac{d}{2} & \frac{d}{2} & r \\
0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
r & r \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
-d - \frac{d}{2} & -d - \frac{d}{2} & -d - \frac{d}{2} - r
\end{pmatrix}
\]

**Proof.** Let $\Phi_q, \infty : \mathcal{C}_q \rightarrow \mathcal{C}_\infty$ be the Lenzing-Meltzer autoequivalence, which is a composite of $R^\pm, S^\pm$ and thus acts on $(r, d)$ by $SL(2, \mathbb{Z})$ transformations. The torsion sheaf $\Phi_q, \infty(F)$ has type $(r', d') = (0, d')$ with $d' = |\gcd(r', d')| = |\gcd(r, d)|$. Since $F \otimes \omega_X \sim F$, $\Phi_q, \infty(F)$ is supported on an ordinary point and $d'$ must be even, and is simple in $\mathcal{C}_\infty$ precisely when $d' = 2$. This proves the claim except for the shape of $\beta(F)$.  

Now, $F \mapsto F \otimes \omega_X$ acts by column change on cohomology tables, and so $\beta(F)$ is symmetrical. By Riemann-Roch we have $2 \cdot \chi(F) = d$ and $2 \cdot \chi(F(\vec{c})) = d + 2r$. Combining this with Lemma 5.8 determines tables in region $(1), (3)$. Next, let $F$ be in region $(2)$. Then both $O, F$ are in $C_0$, and $O$ lives in a rank two tube. Applying $\Phi_{q,\infty}$ to both sends them to skyscraper sheaves with disjoint support, and thus $\text{Ext}^*(O, F) = 0$. An application of Lemma 3.8 and Riemann-Roch as above determines $\beta(F)$ in $(2)$. □

Cohomology tables for rank two tubes. We now study indecomposable sheaves $F$ with $F \otimes \omega_X \not\cong F$. We begin with some generalities.

**Proposition 4.2.** Let $(r, d)$ be in the fundamental domain and $q = \frac{d}{r}$. The following hold:

1. For any $(r, d)$, there is an indecomposable $F$ with $(\text{rk}(F), \deg(F)) = (r, d)$ and $F \otimes \omega_X \not\cong F$.
2. Any such indecomposable $F$ has length $|\gcd(r, d)|$ in $C_q$.
3. There are finitely many indecomposable sheaves of type $(r, d)$ if and only if $\gcd(r, d)$ is odd, in which case there are exactly eight.
4. There is an exceptional sheaf of type $(r, d)$ if and only if $|\gcd(r, d)| = 1$, in which case all such indecomposables are exceptional.
5. When $\gcd(r, d)$ is even and $d \neq 0$, $\beta(F) = \beta(\tilde{F})$ where $\tilde{F}$ is indecomposable of same rank and degree, and $\tilde{F} \otimes \omega_X \cong \tilde{F}$.

**Proof.** The first four points follow from the autoequivalence $\Phi_{q,\infty}$ as in the proof of theorem 4.1. For (5), similarly reduce to skyscraper sheaves. Let $S^{[2n]}$ be an indecomposable torsion sheaf supported over the exceptional point $x_i$ of degree $2n$. Then $[S^{[2n]}]$ has height $2n$ in its tube, and computing Grothendieck classes gives $[S^{[2n]}] = n[S_{i,0}] + n[S_{i,1}]$. In particular the “ordinary” torsion sheaf $S_x$ for $x = x_i$ has degree 2, and we have $[S_x] = [S_{i,0}] + [S_{i,1}]$. From the presentation $0 \to O(-\vec{c}) \to O \to S_x \to 0$ we see that $[S_x] = [S_{x'}]$ for any ordinary point $x'$, and so $[S^{[2n]}] = n[S_x] = n[S_{x'}] = [S_{x'}^{[n]}]$ where $S_{x'}^{[n]}$ is a length $n$ indecomposable sheaf supported at $x'$.

Pulling back through $\Phi_{q,\infty}$, we deduce that for any indecomposable $F$ with $\gcd(r, d)$ even, there is another indecomposable $\tilde{F}$ of type $(r, d)$ with $[\tilde{F}] = [F]$ and $\tilde{F} \otimes \omega_X \cong \tilde{F}$. Since $[\tilde{F}] = [F]$, we have $\chi(\tilde{F} \otimes L) = \chi(F \otimes L)$ for any line bundle $L$, and outside of the case $d = 0$, those values determine the cohomology table, hence $\beta(F) = \beta(\tilde{F})$. □

The remainder of the section will be aimed at proving the next theorem.
Theorem 4.3. Let \((r, d)\) be in the fundamental domain. The \(\beta(F)\) of indecomposables of type \((r, d)\) satisfying \(F \otimes \omega_\mathcal{X} \not\cong F\) are listed as follows:

| \((r, d)\) | \(r \geq 0, d > 0\) | \(r > 0, d < -2r\) |
|---|---|---|
| \(d \text{ odd}\) | \[
\begin{pmatrix}
\frac{d+1}{2} + r & \frac{d+1}{2} + r \\
0 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\frac{d+1}{2} + r & \frac{d+1}{2} + r \\
\end{pmatrix}
\] |
| \(\text{(odd, even)}\) | \[
\begin{pmatrix}
\frac{d \pm 1}{2} + r & \frac{d \pm 1}{2} + r \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\frac{d \pm 1}{2} + r & \frac{d \pm 1}{2} + r \\
\end{pmatrix}
\] |
| \(\text{(even, even)}\) | \[
\begin{pmatrix}
\frac{d}{2} + r & \frac{d}{2} + r \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\frac{d}{2} - r & \frac{d}{2} - r \\
\end{pmatrix}
\] |

For \(r > 0, d = 0\), the tube contains \(O_\mathcal{X}\) if

| \((r, d)\) | \(r odd\) | \(r even\) |
|---|---|---|
| | \[
\begin{pmatrix}
1 & 0 & r - 1 & r + 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 \\
r & r \\
0 & 0 \\
\end{pmatrix}
\] |
| | \[
\begin{pmatrix}
0 & 1 & r + 1 & r - 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
r & r \\
0 & 0 \\
\end{pmatrix}
\] |
| | \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\] |

The subscript counts the number of indecomposables satisfying \(F \otimes \omega_\mathcal{X} \not\cong F\) with given cohomology table.

The proof strategy. We will make use of Crawley-Boevey’s generalization of Kac’s Theorem for weighted projective lines. By the previous proposition, indecomposables with \(\gcd(r, d)\) odd correspond to real roots of the associated root system, which are enumerated in a standard basis for \(K_0\). Going through the list, one tabulates all triples \((rk(F), \deg(F), \chi(F))\) coming from real roots \([F]\), and this triple completely determines \(\beta(F)\) in regions (1), (3). The region (2) is then dealt with by hand. We first recall the needed notions, following [3][17].
**Kac’s theorem, after Schiffmann-Crawley-Boevey.** Let $X = \mathbb{P}^1(p, \lambda)$ be a general weighted projective line for now, and let $T_{\text{can}} = \bigoplus_{0 \leq x \leq e} \mathcal{O}(x)$ be the canonical tilting object with endomorphism algebra $\Lambda_{\text{can}} = kQ/I$ with quiver $Q$:

\[ x_1' \rightarrow 2x_1' \rightarrow \ldots \rightarrow (p_1 - 1)x_1' \]

\[ 0 \rightarrow 2x_2' \rightarrow \ldots \rightarrow (p_2 - 1)x_2' \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[ x_n' \rightarrow 2x_n' \rightarrow \ldots \rightarrow (p_n - 1)x_n' \]

Let $Q'$ be the tree subquiver corresponding to $T' = \bigoplus_{0 \leq x < e} \mathcal{O}(x)$, which we label differently as

\[ 1,1 \rightarrow 1,2 \rightarrow \ldots \rightarrow 1,p_1 - 1 \]

\[ 2,1 \rightarrow 2,2 \rightarrow \ldots \rightarrow 2,p_2 - 1 \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[ n,1 \rightarrow n,2 \rightarrow \ldots \rightarrow n,p_n - 1 \]

Let $g$ be its associated Kac-Moody algebra with root system $\Gamma$, with simple roots $\epsilon_0, \epsilon_{i,j}$, and let $L_q = g[t, t^{-1}]$ its loop algebra with root system $\hat{\Gamma} = \mathbb{Z}\delta \oplus \Gamma$, with $(\delta, -) = 0$. The derived equivalence

\[ D^b(X) \cong D^b(\Lambda_{\text{can}}) \]

sends $\{\mathcal{O}, S(i,j)\}_{j \neq 0}$ to the simple modules $S(0), S(i, j)$ supported over $Q'$. This identifies the summand $\mathbb{Z}[\mathcal{O}] \oplus \bigoplus_{i,j} \mathbb{Z}[S(i,j)]$ of $K_0(X)$ with $\Gamma$, sending the symmetrized Euler form with the Weyl-invariant symmetric bilinear form on $\Gamma$. As Schiffmann then shows [17], this extends to a full isomorphism $K_0(X) \xrightarrow{\cong} \hat{\Gamma}$ sending $[S_x]$ to $\delta$. The induced positive cone given by classes of coherent sheaves on $\hat{\Gamma}$ is given by nonnegative combinations of

\[ \epsilon_0, \epsilon_0 + n\delta, \epsilon_{i,j}, \delta - \sum_{j \neq 0} \epsilon_{i,j}, \quad n \in \mathbb{Z} \]

with $[\mathcal{O}(n\delta)] \mapsto \epsilon_0 + n\delta$ and $[S_{i,0}] \mapsto \delta - \sum_{j \neq 0} \epsilon_{i,j}$. A version of Kac’s Theorem then holds for coherent sheaves on $X$, which we only state in a weak form:

**Proposition 4.4** (Crawley-Boevey, [3]). The isomorphism $K_0(X) \xrightarrow{\cong} \hat{\Gamma}$ induces a bijection between Grothendieck classes of indecomposable coherent sheaves and the positive roots of $L_q$.

1. When $\beta$ is a positive real root, then there is a unique indecomposable $F$ such that $[F] \mapsto \beta$.

2. When $\beta$ is a positive imaginary root, then there are infinitely many indecomposables $F$ for which $[F] \mapsto \beta$.

This extends naturally to indecomposables in $D^b(X)$ by considering all roots and complexes up to $F \mapsto F[2]$. Letting $\Delta$ be the set of roots of $g$, the roots of $L_q$ are given by $\{\alpha + n\delta \mid \alpha \in \Delta, \ n \in \mathbb{Z}\}$, and the real roots are those of the form $\alpha + n\delta$ with $\alpha \in \Delta^{re}$. 
Coming back to $X = \mathbb{P}^1(2, 2, 2; \lambda)$, $g$ is of affine type $\tilde{D}_4$ and we write $\epsilon_i$ for $\epsilon_{i,1}$. The positive real roots of $\tilde{D}_4$ are given by solutions $\alpha = \sum_{i=0}^4 \alpha_i \epsilon_i$ to

$$q(\alpha) = (\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_4^2) - (\alpha_0 \alpha_1 + \cdots + \alpha_0 \alpha_4) = 1$$

with $\alpha_i \in \mathbb{Z}_{\geq 0}$. Writing $q(\alpha) = \sum_{i=1}^4 (\alpha_i - \frac{1}{2} \alpha_0)^2 \geq 0$, one sees that the solutions are as listed in figure 2. The last three columns record $(rk(F), deg(F), \chi(F))$ where $F$ is any indecomposable corresponding to $\beta = \alpha + n \delta$. To see this, note that we must have $|F| = \alpha_0 |O| + \sum_{i=1}^4 \alpha_i |S_{i,1}| + n |S_x|$ and that $\chi(S_{i,1}) = 0$. A general real root then has the form $\beta = \pm \alpha + n \delta$, with $\alpha$ in the above table and $n \in \mathbb{Z}$. Finally, let us record a lemma before moving on to the proof of theorem 4.3.

| $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $r$ | $d$ | $\chi$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 2m        | m+1       | m         | m         | m         | 2m       | 4m+1+2n   | 2m+n     |
| 2m        | m         | m+1       | m         | m         | 2m       | 4m+1+2n   | 2m+n     |
| 2m        | m         | m         | m+1       | m         | 2m       | 4m+1+2n   | 2m+n     |
| 2m        | m         | m         | m         | m+1       | 2m       | 4m+1+2n   | 2m+n     |
| 2m+1      | m         | m         | m         | m         | 2m+1     | 4m+2n     | 2m+1+n   |
| 2m+1      | m+1       | m         | m         | m         | 2m+1     | 4m+2n     | 2m+1+n   |
| 2m+1      | m         | m+1       | m         | m         | 2m+1     | 4m+2n     | 2m+1+n   |
| 2m+1      | m         | m         | m         | m+1       | 2m+1     | 4m+2n     | 2m+1+n   |
| 2m+1      | m+1       | m+1       | m         | m         | 2m+1     | 4m+2n     | 2m+1+n   |
| 2m+1      | m         | m         | m         | m+1       | 2m+1     | 4m+2n     | 2m+1+n   |

**Figure 2.** Positive real roots of affine $D_4$ and triples $(r, d, \chi)$, where $m \geq 0$, $n \in \mathbb{Z}$.
**Lemma 4.5.** Let $\mathcal{F}$ be an indecomposable with $[\mathcal{F}] \mapsto \beta$ real, with $r \geq 0$, $d \in \mathbb{Z}$. Then the possible values of $\chi(\mathcal{F})$ only depend on $d$ and are listed below:

| $d$   | $\chi$                                      |
|-------|---------------------------------------------|
| $d$ odd | $(\frac{d-1}{2})_4$, $(\frac{d+1}{2})_4$   |
| $d$ even | $(\frac{d}{2} - 1)_1$, $(\frac{d}{2})_6$, $(\frac{d}{2} + 1)_1$ |

The subscript indicates how many times $\chi$ appears for fixed $(r,d)$.

**Proof.** This follows by inspection of figure 2, where the case $r > 0$ corresponds to $\beta = \alpha + n\delta$ for $\alpha$ in the table, and $r = 0$ uses $\pm \alpha$ for $\alpha$ in the first four rows. □

We can now compute cohomology tables of indecomposables in rank two tubes

$$
\beta(\mathcal{F}) = \begin{pmatrix}
    h^0(\mathcal{F}) & h^0(\mathcal{F} \otimes \omega_X) \\
    h^0(\mathcal{F}(\bar{c}) \otimes \omega_X) & h^0(\mathcal{F}(\bar{c})) \\
    h^1(\mathcal{F} \otimes \omega_X) & h^1(\mathcal{F}) \\
    h^1(\mathcal{F}(\bar{c}) \otimes \omega_X) & h^1(\mathcal{F}(\bar{c}))
\end{pmatrix}
$$

**Proof.** Let $\mathcal{F}$ be indecomposable with $\mathcal{F} \otimes \omega_X \not\cong \mathcal{F}$, of type $(r,d)$. First assume $(r,d)$ in region (1), the case (3) being similar. The bottom half of $\beta(\mathcal{F})$ vanishes for slope reasons (lemma 3.8). When $\gcd(r,d)$ is even then by proposition 4.2 the table $\beta(\mathcal{F})$ is as in theorem 4.1. When $\gcd(r,d)$ is odd, then by 4.2 and Kac’s Theorem the class $[\mathcal{F}]$ must correspond to a real root, and so the possible values of $\chi$ are listed in lemma 4.5, which determines the possible values of $h^0(\mathcal{F})$. By Riemann-Roch we have

$$
h^0(\mathcal{F}) + h^0(\mathcal{F} \otimes \omega_X) = d$$

$$
h^0(\mathcal{F}(\bar{c})) + h^0(\mathcal{F}(\bar{c}) \otimes \omega_X) = d + 2r$$

Now, keeping in mind that $\beta(\mathcal{F}) = \beta(M)$ where $M$ is presented by a matrix factorization, the sum of each column must be equal. From this one sees that the tables must be of the form

| $(r,d)$ | $r \geq 0$, $d > 0$ |
|--------|---------------------|
| $d$ odd | \[
\begin{pmatrix}
    \frac{d+1}{2} + r & \frac{d+1}{2} + r \\
    0 & 0 \\
    0 & 0
\end{pmatrix}_4
\]
| $d$ even | \[
\begin{pmatrix}
    \frac{d}{2} \pm 1 & \frac{d}{2} \pm 1 \\
    \frac{d}{2} \pm 1 + r & \frac{d}{2} \pm 1 + r \\
    0 & 0 \\
    0 & 0
\end{pmatrix}_6
\]

The same argument determines tables in region (3), so we are left with region (2) where $r > 0$, $d = 0$. We first note that the base of rank two tubes in $C_0$ consists
of line bundles of degree 0, with one distinguished tube having \( \{ \mathcal{O}, \omega_X \} \) as base. For the other tubes, an appeal to the autoequivalence \( \Phi_{0, \infty} : \mathcal{C}_0 \xrightarrow{\cong} \mathcal{C}_\infty \) shows that \( \text{Ext}^*(\mathcal{O}, \mathcal{F}) = 0 = \text{Ext}^*(\mathcal{O}, \mathcal{F} \otimes \omega_X) \). The first row and third row of \( \beta(\mathcal{F}) \) vanishes, and slope considerations show vanishing of the fourth row. The table must then be

\[
\begin{pmatrix}
0 & 0 \\
r & r \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

We are down to \( \mathcal{F} \) in the Auslander-Reiten component of \( \{ \mathcal{O}, \omega_X \} \). Now, \( \mathcal{F} \) is uniserial with socle either \( \mathcal{O} \) or \( \omega_X \). Assume the first. From the structure of a rank two tube, the simple top of \( \mathcal{F} \) is \( \omega_X \) when \( r \) is even, and \( \mathcal{O} \) for \( r \) odd. This determines the dimensions of \( \text{Hom}(\mathcal{O}, \mathcal{F}) \), \( \text{Hom}(\omega_X, \mathcal{F}) \), \( \text{Hom}(\mathcal{F}, \mathcal{O}) \), \( \text{Hom}(\mathcal{F}, \omega_X) \) as

\[
\begin{align*}
\dim_k \text{Hom}(\mathcal{O}, \mathcal{F}) &= 1 \\
\dim_k \text{Hom}(\omega_X, \mathcal{F}) &= 0 \\
\dim_k \text{Hom}(\mathcal{F}, \mathcal{O}) &= \begin{cases} 1 & r \text{ odd} \\ 0 & r \text{ even} \end{cases} \\
\dim_k \text{Hom}(\mathcal{F}, \omega_X) &= \begin{cases} 0 & r \text{ odd} \\ 1 & r \text{ even} \end{cases}
\end{align*}
\]

and from Serre duality one deduces the shape of the first and third rows. This is enough to determine the tables as

\[
\begin{pmatrix}
r \text{ odd} & r \text{ even} \\
1 & 0 \\
r - 1 & r + 1 \\
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
r & r \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]

The case of \( \mathcal{F} \) with socle \( \omega_X \) is then given by the mirrored table.

\[\square\]

5. **Betti tables of indecomposable matrix factorizations**

The previous fundamental domain for \((r, d)\) was a natural choice when thinking about coherent sheaves on \( X \), but it is clear that a simpler normalization exists for Betti tables. Translating results of the previous sections, we will normalize and display them in the standard format

\[
\begin{array}{c|cc}
\beta_{i,j} & 0 & 1 \\
\hline
0 & \beta_{0,0} & \beta_{1,1} \\
1 & \beta_{0,1} & \beta_{1,2} \\
2 & \beta_{0,2} & \beta_{1,3} \\
3 & \beta_{0,3} & \beta_{1,4} \\
4 & \beta_{0,4} & \beta_{1,5} \\
5 & \vdots & \vdots \\
\end{array}
\]
Call an indecomposable \( M \) of the first kind if it belongs to the same Auslander-Reiten component as some \( \Sigma^n k^\text{st}(m) \), and of the second kind otherwise.

**Corollary 5.1.** The indecomposables of the first kind are uniquely determined by their Betti table. Up to translation, these are all tables of the form

\[
\begin{array}{ccc}
0 & 1 & 0 \\
1 & r-1 & 0 \\
2 & 1 & r+1 \\
3 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & r-1 \\
2 & 0 & 1 \\
3 & 0 & 0 \\
\end{array}
\]

for \( r > 0 \) odd,

\[
\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & r \\
2 & 0 & 1 \\
3 & 0 & r \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 0 \\
1 & 1 & 0 \\
2 & 0 & 1 \\
3 & 0 & r \\
\end{array}
\]

for \( r > 0 \) even.

These have degree 0 and rank \( r \).

**Proof.** Assuming the hypothesis, \( \Sigma^{-n} k^{\text{st}}(-m-2) \) is in the same Auslander-Reiten component as \( k^{\text{st}}(-2) \) which corresponds to \( O_X \), then apply theorem 4.3 and translate the resulting tables in the above form. \( \square \)

Most indecomposables are of the second kind.

**Corollary 5.2.** Up to translation, the Betti tables of indecomposables of the second kind are all tables of type \( I - V \):

\[
\begin{array}{ccc}
I & 0 & 1 \\
0 & a & 0 \\
1 & b & a \\
2 & 2 & b \\
3 & 3 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
II & 0 & 1 \\
0 & a+1 & 0 \\
1 & b & a \\
2 & 2 & b+1 \\
3 & 3 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
III & 0 & 1 \\
0 & a & 0 \\
1 & b+1 & a+1 \\
2 & 2 & b \\
3 & 3 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
IV & 0 & 1 \\
0 & a+2 & 0 \\
1 & b & a \\
2 & 2 & b+2 \\
3 & 3 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
V & 0 & 1 \\
0 & a & 0 \\
1 & b+2 & a+2 \\
2 & 2 & b \\
3 & 3 & 0 \\
\end{array}
\]

with \( a, b \geq 0 \), where we have \( b \neq 0 \) for tables of type I and \( b - a \) odd for tables of type IV - V. Here the degree is given by \( d = \beta_{0,0} + \beta_{1,2} = 2a, 2a+1, 2a+2 \) and the rank by \( r = b - a \).

**Proof.** Let \( M \) be indecomposable of the second kind. We claim that, up to translation, \( \beta(M) \) can be put in the form

\[
\beta(M) = \begin{pmatrix}
\beta_{0,0} & \beta_{1,2} \\
\beta_{0,1} & \beta_{1,3} \\
\beta_{0,2} & \beta_{1,4} \\
\beta_{0,3} & \beta_{1,5}
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\beta + r & \alpha + r \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

for some \( \alpha, \beta \) and \( r \in \mathbb{Z} \). Running over possibilities in theorem 4.1, 4.2, 4.3 this is already the case in regions (1), (2), where \( d, r \geq 0 \). For \( (rk(M), \deg(M)) \) belonging to region (3), applying \( M \mapsto M(2) \) will put \( \beta(M) \) in the above form. Note that this sends \( (r, d) \mapsto (-r, -d) \), and so the above tables coming from region (3) will
have \( r < 0 \). This will change our fundamental domain to \( r > -\frac{d}{2} \), \( d \geq 0 \):

Now, the case \( \alpha = \beta = 0 \) corresponds to \( d = 0 \), or tables in region (2). The other two regions run over the same pairs \((\alpha, \beta)\), with only difference whether \( r \geq 0 \) or \( r < 0 \). Next, set \( a = \alpha \), \( b = \alpha + r \). Running over the possible \( \alpha \) in theorem 4.1, 4.3 shows that tables must have shapes I – V. In particular in type I, note that \( b = 0 \) implies \( r = -\frac{d}{2} \) which falls outside of our domain. Lastly, fixing the type I – V of a table, note that \((a, b)\) and \((r, d)\) uniquely determine each other via \( r = b - a \) and \( d = 2a \), \( 2a + 1 \), \( 2a + 2 \), and so the classification is complete.

We can also describe indecomposables of the second kind with fixed Betti table. We will say that a family \( \mathcal{M} \) of indecomposables is parameterized by \( \mathbb{X} \) (resp. \( U \subseteq \mathbb{X} \)) of level \( n \) if there is a fully faithful functor \( \Phi_{\mathcal{M}} : C_{\infty} \hookrightarrow \text{MCM}(grR) \) which sends the skyscraper sheaves of degree \( 2n \) to \( \mathcal{M} \) (resp. skyscraper sheaves supported over \( U \)). As corollaries of theorem 4.1, 4.3, we have

**Corollary 5.3.** There are four indecomposables for each table of type II – III, and a unique indecomposable for each table of type IV – V. Type I breaks down as follows for fixed \((a, b)\):

i. For \( r = b - a \) odd, there are six indecomposables per table.

ii. For \( d = 2a \neq 0 \) and \( r = b - a \) even, indecomposables are parameterized by \( \mathbb{X} \) of level \( n = \frac{gcd(r,d)}{2} \).

iii. For \( d = 2a = 0 \) and \( r = b \) even, indecomposables are parameterized by \( \mathbb{X} \setminus \{\infty\} \) of level \( n = \frac{r}{2} \).

**Ulrich modules.** Let \( M \) be a (graded) MCM \( R \)-module. Let \( \mu = \mu(M) \) denote the minimal number of generators of a module \( M \), and \( e = e(M) \) its multiplicity. The latter can be calculated from the Hilbert series of \( M \), given by

\[
H_M(t) = \frac{P_M(t)}{1 - t}
\]

for some Laurent polynomial \( P_M(t) \) with \( P_M(1) \neq 0 \). We then have \( e(M) = P_M(1) \). Note that \( e(M(1)) = e(M) \). There is a general bound \( \mu(M) \leq e(M) \), and Ulrich modules are defined as MCM modules meeting this bound. We can calculate the
Hilbert series $H_M(t)$ from the Betti table $\beta(M)$ as

$$H_M(t) = \frac{H_R(t)}{1 - t^4} \left( \sum_{j \in \mathbb{Z}} \beta_{0,j} t^j - \sum_{j \in \mathbb{Z}} \beta_{1,j} t^j \right)$$

and so we will deduce a classification of Ulrich modules. We first note that $R$ possesses some Ulrich modules: the module $L_i = R/I_i$ is cyclic with Hilbert series $1 - t^4$, and so $\mu = e = 1$.

**Theorem 5.4.** Let $M$ be an indecomposable graded Ulrich module. Then up to degree shift we have $M \cong L_i$ for some $i = 1, 2, 3, 4$.

**Proof.** First assume that $M$ is of second kind, so that up to grade shift $M$ has Betti table

$$\begin{array}{c|cc}
0 & 0 & 1 \\
1 & \beta_{0,1} & \beta_{1,2} \\
2 & \beta_{0,2} & \beta_{1,3} \\
3 & \beta_{0,3} & \beta_{1,4}
\end{array} = \begin{array}{c|cc}
0 & \alpha & 0 \\
1 & \beta + r & \beta \\
2 & 0 & \alpha + r \\
3 & 0 & 0
\end{array}$$

for some $\alpha, \beta \geq 0$ and $r \in \mathbb{Z}$. Then

$$H_M(t) = \frac{1}{1 - t^4} H_R(t) \left( \alpha t^0 + (\beta + r)t^1 - \beta t^2 - (\alpha + r)t^3 \right)$$

$$= \frac{1}{1 - t^4} \frac{1 - t^4}{(1 - t)^2} \left( \alpha t^0 + (\beta + r)t^1 - \beta t^2 - (\alpha + r)t^3 \right)$$

$$= \frac{1}{(1 - t)^2} \left( \alpha(1 - t^2) + \beta(t - t^3) + (\alpha - \beta)(t^2 - t^3) + r(t - t^3) \right)$$

$$= \frac{1}{(1 - t)^2} \left( 1 - t \right) \left( \alpha(1 + t) + \beta t(1 + t) + (\alpha - \beta)t^2 + rt(1 + t) \right)$$

and so $e = 3\alpha + \beta + 2r \geq \alpha + \beta + r = \mu$, with equality when $\alpha = r = 0$. The only possibility is given by the type III table

$$\begin{array}{c|cc}
0 & 1 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
2 & 0 & 0 \\
3 & 0 & 0
\end{array}$$

There are four indecomposables with this table, given by $L_i(-1), i = 1, 2, 3, 4$. One easily verifies that $e > \mu$ for indecomposables of the first kind, and so the result holds. \hfill \Box

6. Matrix factorizations corresponding to simple torsion sheaves

We will produce the indecomposable matrix factorizations corresponding to the simple objects in $\mathbb{C}_\infty$. For simplicity we assume $\text{char } k \neq 2$ in this section. The pair $(k^{st}(-1)[-1], k^{st}(-2))$ corresponds to $(\mathcal{O}(\mathcal{C}), \mathcal{O}),$ with 2-dimensional morphism space. Taking a basis $\phi_0, \phi_\infty$, for any $p = [p_0 : p_1] \in \mathbb{P}^1(k)$ we define the MCM module $M_p$ as the cone of $\phi_p = p_1 \phi_0 + p_0 \phi_\infty$

$$k^{st}(-1)[1] \xrightarrow{\phi_p} k^{st}(-2) \rightarrow M_p \rightarrow k^{st}(-1)[2]$$
induces natural isomorphisms on Tate cohomology following:

\[ \mathcal{O}(-\partial) \xrightarrow{s} \mathcal{O} \to \text{coker}(s_p) \to \mathcal{O}(-\partial)[1]. \]

for the corresponding cosection \( s_p \), with cokernel an ordinary skyscraper sheaf. We can produce the associated matrix factorizations. We have \( f_\lambda = xy(x-y)(x-\lambda y) = x^3y - (1 + \lambda)x^2y^2 + \lambda xy^3 \). Write \( f_\lambda = xf_z + yf_y \) for \( f_x = \frac{\partial f_\lambda}{\partial x} \) and \( f_y = \frac{\partial f_\lambda}{\partial y} \), and note that \( x|f_y \) and \( y|f_x \).

**Proposition 6.1.** We have the following explicit presentations.

1. \( k^{st} \) corresponds to the matrix factorization

\[
\begin{array}{ccc}
S \oplus S(2) & \xrightarrow{A} & S(-1) \oplus S(-1) \\
& & \xleftarrow{B} S(-4) \oplus S(-2)
\end{array}
\]

with

\[
A = \begin{pmatrix}
x & y \\
-f_y & f_x
\end{pmatrix} \\
B = \begin{pmatrix}
f_x & -y \\
f_y & x
\end{pmatrix}.
\]

2. The morphisms \( \phi_0, \phi_\infty : k^{st}(-1)[-1] \to k^{st}(-2) \) can be realized as

\[
\begin{array}{ccc}
S(-2) \oplus S(-2) & \xrightarrow{-B} & S(-5) \oplus S(-3) \\
& \xleftarrow{A} & S(-6) \oplus S(-6)
\end{array}
\]

with matrices given by

\[
\varphi_0 = \begin{pmatrix}
0 & 1 \\
0 & -f_x
\end{pmatrix} \\
\psi_0 = \begin{pmatrix}
-f_x & -1 \\
0 & 0
\end{pmatrix}
\]

\[
\varphi_\infty = \begin{pmatrix}
1_x & 0 \\
f_y & 0
\end{pmatrix} \\
\psi_\infty = \begin{pmatrix}
0 & 0 \\
-f_y & 1
\end{pmatrix}.
\]

**Proof.** Part (1) follows from the Tate resolution [8]. For part (2), note that the MCM approximation \( k^{st}(-2) \to k(-2) \) corresponds to the natural projection and induces natural isomorphisms on Tate cohomology

\[
\text{Ext}^0_{grR}(k^{st}(-1)[-1], k^{st}(-2)) \xrightarrow{s} \text{Ext}^0_{grR}(k^{st}(-1)[-1], k(-2)).
\]

The morphisms \( \phi_0, \phi_\infty \) descend to the natural basis on the latter. \( \square \)

Now let \( \phi_p = p_1\phi_0 + p_0\phi_\infty \). Taking \( M_p = \text{Cone}(\phi_p) \) yields a \( 4 \times 4 \) matrix factorization

\[
\begin{array}{ccc}
S(-1) & \xrightarrow{A} & S(-2) \\
S(1) & \xleftarrow{-\varphi_p} & S(-2) \\
S(-2) & \xrightarrow{-\psi_p} & S(-3) \\
S & & S(-3)
\end{array}
\]

Comparing with the classification of Betti tables, this matrix factorization must be stably equivalent to a \( 2 \times 2 \) matrix factorization. Direct calculations show the following:
Proposition 6.2. The module $M_p$ is given by the reduced matrix factorization

$$S(-1) \oplus S \xrightarrow{A_p} S(-2) \oplus S(-3) \xrightarrow{B_p} S(-5) \oplus S(-4)$$

where $(A_p, B_p)$ for $p_1 \neq 0$ are given by

$$A_p = \begin{pmatrix} x - \frac{p_0}{p_1} y & \frac{1}{p_1} y^2 \\ -p_0 \frac{x}{y} & \frac{1}{x} \end{pmatrix} \quad B_p = \begin{pmatrix} \frac{p_0}{x} & -\frac{1}{p_1} y^2 \\ p_0 \frac{x}{y} & x - \frac{p_0}{p_1} y \end{pmatrix}$$

and for $p_1 = 0, p_0 \neq 0$ by

$$A_p = \begin{pmatrix} y & x^2 \\ 0 & \frac{1}{y} \end{pmatrix} \quad B_p = \begin{pmatrix} \frac{1}{y} & -x^2 \\ 0 & y \end{pmatrix}$$

Recall that the modules $L_i(-3)[1] = R/(f_{l_i})$ correspond to the simple sheaves $S_{i,0}$, where $L_i = R/l_i$. From the above presentation, one verifies

Lemma 6.3. For each $p_i = V(l_i) \in \mathbb{P}^1(k)$, there are short exact sequences of MCM modules

$$0 \to R/l_i(-1) \to M_{p_i} \to R/(f_{l_i}) \to 0.$$

Hence the $M_{p_i}$ corresponds to $S_{p_i}$. Summarising, we have shown

Proposition 6.4. The indecomposable MCM modules

$$\{ M_p \}_{p \neq 0, 1, \infty, \lambda} \cup \{ R/(f_{l_i}), R/l_i(-1) \}_{i=1,2,3,4}$$

correspond under the equivalence of corollary 2.6 to the simple torsion sheaves in $C_\infty$.

The remaining indecomposables are constructed from these by taking extensions and applying tubular mutations.

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