GLOBAL LIPSCHITZ STABILITY FOR INVERSE PROBLEMS FOR RADIATIVE TRANSPORT EQUATIONS

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Abstract. We consider inverse problems of determining coefficients or time independent factors of source terms in radiative transport equations by means of Carleman estimate. We establish global Lipschitz stability results with an additional condition which requires some strict positivity for initial value or given factor of source, but we need not any extra conditions on domains of velocities, which is the main achievement of this article compared with the existing work by Machida and Yamamoto (Inverse Problems 30 035010, 2014). The proof relies on a Carleman estimate with a piecewise linear weight function according to the partition of the velocity domain.

1. Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n, \ n \geq 2 \) with \( C^1 \)-boundary \( \partial \Omega \). Let \( V \) be a domain in \( \mathbb{R}^n \) with \( 0 \notin \overline{V} \), where \( \overline{V} \) is the closure of \( V \). We use symbols \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \). Moreover, \( v \cdot v' \) denotes the scalar product of vectors \( v, v' \). Let \( u(x, v, t) \) be the solution to the following radiative transport equation.

\[
P_0 u + \sigma(x, v)u - \int_V k(x, v, v')u(x, v', t) dv' = F(x, v, t), \quad x \in \Omega, \ v \in V, \ 0 < t < T,
\]

(1.1)

\[
u(x, v, 0) = a(x, v), \quad x \in \Omega, \ v \in V; \quad (1.2)
\]

\[
u(x, v, t) = g(x, v, t) \quad \text{on} \ \Gamma_- \times (0, T), \quad (1.3)
\]

where \( F \) is a source term and

\[
P_0 u(x, v, t) = \partial_t u(x, v, t) + v \cdot \nabla u(x, v, t).
\]

Let \( \nu(x) \) be the outward unit vector normal to \( \partial \Omega \) at \( x \in \partial \Omega \). We define \( \Gamma_+ \) and \( \Gamma_- \) by

\[
\Gamma_\pm = \{(x, v) \in \partial \Omega \times V; \ (\pm \nu(x) \cdot v) > 0\}.
\]

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The coefficients are assumed as follows.

\[ \sigma \in L^\infty(\Omega \times V), \quad k \in L^\infty(\Omega \times V \times V). \]

In this article, we will consider the inverse problem of determining \( \sigma \) and a time independent factor of the source term \( F(x, v, t) \) by \( u \) on \( \Gamma_+ \), \( 0 < t < T \), assuming that \( k \) is known.

For an arbitrary fixed constant \( M > 0 \), we set

\[ U = \{ u \in X; \| u \|_X + \| \nabla u \|_{H^1(0, T; L^2(\Omega \times V))} \leq M \}, \]

where

\[ X = H^1(0, T; L^\infty(\Omega \times V) \cap H^2(0, T; L^2(\Omega \times V))). \]

Here, \( H^1, H^2 \) denote usual Sobolev spaces over the specified domains. A solution which satisfies (1.1) - (1.3) can be obtained in \( U \) due to the regularity and compatibility conditions of the initial value \( a \) and the boundary value \( g \). As for the direct problems, see Bardos [7], Douglis [16], Prilepko and Ivankov [30], and Ukai [32].

The study of inverse transport problems has started in radiative transfer by Bellman, Kagiwada, Kalaba and Ueno [11] and neutron transport by Case [14]. As for application aspects of related inverse problems, see Arridge [1] and Arridge and Schotland [2].

For the mathematical analysis for inverse problems for radiative transport equations, we can have two main methodologies by (i) albedo operators and (ii) Carleman estimate. Limited to the non-stationary case and not aiming at any comprehensive literature, we refer to works below.

First, the albedo operator can be interpreted as a mapping from boundary input on some subboundary to boundary data of the solution on other part of the boundary. As for the approach by the albedo operator, we first refer to a review article by Bal [3]. Moreover, the uniqueness was studied by Choulli and Stefanov [15] and Stefanov [31]. In general, one can prove the stability of Hölder type. See Bal and Jollivet [4, 5, 6]. This approach does not require strong assumptions such as nonzero initial values, but measurements have to be performed infinitely many times for obtaining the uniqueness and the stability.

Second, as for the approach by Carleman estimate, we refer to Bukhgeim and Klibanov [10] and Klibanov [22, 23] as pioneering works, which apply Carleman estimates to inverse problems for second-order partial differential equations such as hyperbolic equations. Such an approach yields the uniqueness and the stability for inverse problems for partial differential equations with a single measurement. Moreover, as for the inverse problems by Carleman estimates, one can consult Belina and Klibanov [8], Bellassoued and Yamamoto [9], Imanuvilov and Yamamoto [19, 20], Isakov [21], Klibanov [24], Klibanov and Timonov [27], and Yamamoto [33].

With the Carleman-estimate technique for the radiative transport equations, Klibanov and Pamyatnykh [26] proved the Lipschitz stability in determining \( \sigma \) provided that \( a(x, v) := u(x, v, 0) \neq 0 \) for all \( (x, v) \in \Omega \times V \) and

\[ (a(x, v) \sigma(x, v))^2 = (a(x, -v) \sigma(x, -v))^2 \quad \text{for} \quad x \in \Omega \quad \text{and} \quad v \in V \quad (1.4) \]

in the case of \( V = \{ v; \ |v| = 1 \} \). The extra condition (1.4) is required because in [26], the extension of \( u \) to the time interval \( (-T, T) \) is necessary for the Carleman estimate, and (1.4) is essential for the regularity of the extension. The condition (1.4) is concerned also with unknown \( \sigma \), and so restrictive. See also Klibanov and
Pamyatnykh [25] and Klibanov and Yamamoto [28] as for related problems on a radiative transport equation.

After [26], we refer to Machida and Yamamoto [29]: it established a Carleman estimate with a linear weight function which is different from [26] and the global Lipschitz stability for the inverse problems without any extension of the solution \( u \) to \((-T, T)\). In particular, any extra conditions for \( \sigma(x, v) \) such as (1.4) are not required. However, it must be assumed in [29] that \( V \) is a sectional domain, which means that \( v \) is confined in narrow directions. More precisely, \( v \in V \) must satisfy

\[
(\gamma \cdot v) > 0 \quad \text{for an arbitrary fixed } \gamma \in \mathbb{R}^n. \tag{1.5}
\]

The main purpose of this article is to remove (1.5) and improve [29]. More precisely, we prove the global Lipschitz stability results for the inverse problems with any bounded domain \( V \) with \( 0 \not\in V \), not necessarily satisfying (1.5).

In this article, the weight function for the Carleman estimate is linear in \( x, t \), similar to [29], but the main difference is that we make choices of the weight according to suitably partitioned subdomains of \( V \) for deleting (1.5), so that the weight function can be understood as piecewise linear function in \( v \).

As for similar inverse problems for transport equations with \( k \equiv 0 \) in (1.1), we refer to Gaitan-Ouzzane [17]. Moreover one can consult Cannarsa, Floridia, Gögeberlyen and Yamamoto [12], Cannarsa, Floridia and Yamamoto [13] and Gögeberlyen and Yamamoto [18], where a linear weight function is used.

The remainder of the paper is organized as follows. We state our main results in Section 2. The inverse coefficient problem reduces to an inverse source problem. Section 3 is devoted to the introduction of coupled radiative transport equations. In Section 4, we prove our key Carleman estimate. The energy estimate for the coupled radiative transport equations is established in Section 5. The proof for the main theorem is given in Section 6. Finally, we give concluding remarks in Section 7.

2. Main results

Let us choose \( \widetilde{V} = \{v_0 < |v| < v_1\} \) with sufficiently small \( v_0 \) and large \( v_1 \) such that \( V \subset \widetilde{V} \). Then we take the zero extension for \( \sigma(x, v) \) in \( v \) and \( k(x, v, v') \) in \( (v, v') \), i.e., \( \sigma = 0 \) for \( v \in \widetilde{V} \setminus \overline{V} \), and \( k = 0 \) if \( v \in \widetilde{V} \setminus \overline{V} \) or \( v' \in \widetilde{V} \setminus \overline{V} \). Moreover we set \( a = g = 0 \) for \( v \in \widetilde{V} \setminus \overline{V} \). Then we can replace \( V \) in (1.1), (1.2), and (1.3) with \( \widetilde{V} \). The integral term can be further expressed as

\[
\int_{V} k(x, v, v')u(x, v', t) \, dv' = \int_{\widetilde{V}} k(x, v, v')u(x, v', t) \, dv' = \sum_{j=0}^{m-1} \int_{V_j} k(x, v, v')u(x, v', t) \, dv',
\]

where \( m \) subdomains \( V_j \) \((j = 0, \ldots, m - 1)\) are set as follows. We note that in spherical coordinates \( v \in \widetilde{V} \) is specified by \((r, \theta, \phi_1, \ldots, \phi_{n-2})\), where \( v_0 < r < v_1, \ 0 \leq \theta < 2\pi, \ 0 \leq \phi_i \leq \pi \ (i = 1, \ldots, n-2) \ ((r, \theta) \text{ in the case of } n = 2) \). We define subdomains \( V_j \) \((j = 0, \ldots, m - 1)\) as

\[
V_j = \left\{ v \in \widetilde{V}; v_0 < r < v_1, \ \frac{2\pi l_0 - 1}{L_0} \leq \theta < \frac{2\pi l_0}{L_0} \right\},
\]
\[
\frac{\pi(l_i - 1)}{L_i} \leq \phi_i < \frac{\pi l_i}{L_i}, \quad i = 1, \ldots, n - 2, \quad (2.1)
\]

where \( l_0 = 1, \ldots, L_0, \ l_i = 1, \ldots, L_i \ (i = 1, \ldots, n - 2), \ m = L_0L_1 \ldots L_{n-2} \) and
\[
j = l_0 + (l_1 - 1)L_0 + (l_2 - 1)L_0L_1 + \cdots + (l_{n-2} - 1)L_0 \ldots L_{n-3}.
\]

Let \( \gamma_j \in \mathcal{V}_j \) be arbitrarily chosen vectors \((j = 0, 1, \ldots, m - 1)\). We take sufficiently large \( m \) such that if \( v \in \mathcal{V}_j \), then \( \gamma_j \cdot v \geq \kappa \) for an arbitrary fixed constant \( \kappa > 0 \). That is, we have
\[
\min_{v \in \mathcal{V}_j} (\gamma_j \cdot v) > 0, \quad j = 0, \ldots, m - 1.
\]

Our main results are stated as follows.

**Theorem 2.1.** Let \( u^i(x, v, t) \ (i = 1, 2) \) be solutions to the radiative transport equation for \( \sigma^i, a^i \), i.e.,
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
(\partial_t + v \cdot \nabla + \sigma^i(x, v)) u(x, v, t) - \int_{\mathcal{V}} k(x, v, v') u(x, v', t) dv' = 0, \\
x \in \Omega, \ v \in \mathcal{V}, \ 0 < t < T,
\end{array} \right.
\end{aligned}
\]

\[
u(x, v, 0) = a^i(x, v), \quad x \in \Omega, \ v \in \mathcal{V};
\]

\[
u(x, v, t) = g(x, v, t), \quad (x, v) \in \Gamma_-, \ 0 < t < T.
\]

Let \( u^i \in \mathcal{U} \) and assume \( ||\sigma^i||_{L^\infty(\Omega \times \mathcal{V})}, ||k||_{L^\infty(\Omega \times \mathcal{V} \times \mathcal{V})} \leq M \). Suppose that \( T \) is large enough to satisfy
\[
T > \frac{\max_{0 \leq j \leq m-1} \max_{x \in \Omega} (\gamma_j \cdot x) - \min_{0 \leq j \leq m-1} \min_{v \in \mathcal{V}_j} (\gamma_j \cdot v)}{\min_{0 \leq j \leq m-1} \min_{v \in \mathcal{V}_j} (\gamma_j \cdot v)}.
\]

We assume that there exists a constant \( a_0 > 0 \) such that
\[
a^1(x, v) \geq a_0 \quad \text{or} \quad a^2(x, v) \geq a_0, \quad \text{a.e. in } (x, v) \in \Omega \times \mathcal{V}. \quad (2.3)
\]

Then there exists a constant \( C = C(M, a_0) > 0 \) such that
\[
||\sigma^1 - \sigma^2||_{L^2(\Omega \times \mathcal{V})} \leq C \left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t (u^1 - u^2)(x, v, t)|^2 dSdvdt \right)^{1/2}
\]
\[
+ C \left( \|a^1 - a^2\|_{L^2(\Omega \times \mathcal{V})} + \|\nabla a^1 - \nabla a^2\|_{L^2(\Omega \times \mathcal{V})} \right)
\]

and
\[
\left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v)|\partial_t (u^1 - u^2)(x, v, t)|^2 dSdvdt \right)^{1/2}
\]
\[
\leq ||\sigma^1 - \sigma^2||_{L^2(\Omega \times \mathcal{V})} + \|a^1 - a^2\|_{L^2(\Omega \times \mathcal{V})} + \|\nabla a^1 - \nabla a^2\|_{L^2(\Omega \times \mathcal{V})}.
\]

Here we have \( C(M, a_0) \to \infty \) as \( M \to \infty \) or \( a_0 \to 0 \).

Thus we have removed extra conditions \((1.4)\) and \((1.5)\) to prove the global Lipschitz stability for the inverse coefficient problem. Condition \((2.2)\) means that the critical length \( T \) of the time interval depends on the partition of \( \mathcal{V} \).
Corollary 2.2. If we assume $a^1 = a^2$ in $\Omega \times V$, then we have the following both-sided estimate.

\[
\begin{align*}
\left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t (u^1 - u^2)(x,v,t)|^2 \, dSdvdt \right)^{1/2} &\leq \|\sigma^1 - \sigma^2\|_{L^2(\Omega \times V)} \\
\leq C \left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t (u^1 - u^2)(x,v,t)|^2 \, dSdvdt \right)^{1/2}.
\end{align*}
\]

This means that the choice of the norm for the boundary data $\Gamma_+ \times (0,T)$ is the best possible for our inverse problem.

Remark 2.3. Positive initial values in (2.3) can be set up by a combination of some control procedures. More precisely, let us assume that $\Omega$ is strictly convex. Suppose $\sigma^2$ is known and we consider the radiative transport equation for $\sigma^2$ during the time interval $(-T_0, T)$ with some $T_0 > 0$. We extend $g$ such that $g$ belongs to some weighted $L^2$-space in $\Gamma_- \times (-T_0, 0)$. The value $u^2(x,v,-T_0)$ may be either zero or nonzero. By the exact controllability result [28], we can have $u^2(x,v,0) = a^2(x,v) \geq a_0$, a.e. in $(x,v) \in \Omega \times V$, by adjusting the boundary value $g$ for sufficiently large $T_0 > 0$.

In the case of optical tomography [1, 2], the boundary value $g$ is the incident laser beam of near-infrared light. We can prepare positive initial values by turning on the laser at $t = -T_0$ before starting to detect the out-going light on the surface of biological tissue at $t = 0$.

Remark 2.4. It is also possible to determine the scattering coefficient $\sigma_s$ if we write $k$ as $k(x,v,v') = \sigma_s(x,v)p(x,v,v')$. The proof is similar to that of Theorem 2.1.

We refer the reader to [29].

Next we state the second main result for an inverse source problem. We consider the following radiative transport equation with an internal source term $F(x,v,t)$.

\[
\left\{ \begin{array}{l}
(\partial_t + v \cdot \nabla + \sigma(x,v))u(x,v,t) - \int_V k(x,v,v')u(x,v',t) \, dv' = F(x,v,t), \\
x \in \Omega, \ v \in V, \ 0 < t < T, \\
u(x,v,0) = a(x,v), \ x \in \Omega, \ v \in V, \\
u(x,v,t) = g(x,v,t) \text{ on } \Gamma_- \times (0,T).
\end{array} \right.
\tag{2.4}
\]

Let us assume that $F(x,v,t)$ has the following form.

\[F(x,v,t) = f(x,v)R(x,v,t),\]

By subtraction for the equations in Theorem 2.1, we obtain (2.4) with

\[
\begin{align*}
\frac{d}{dt} u(x,v,t) - \nabla \cdot [\sigma(x,v) u(x,v,t)] &= f(x,v), \\
\sigma(x,v) &= \sigma^1(x,v), \quad \sigma^2(x,v),
\end{align*}
\tag{2.5}
\]

\[
\begin{align*}
a(x,v) &= a^1(x,v) - a^2(x,v), \\
g(x,v,t) &= 0, \\
f(x,v) &= \sigma^1(x,v) - \sigma^2(x,v), \quad R(x,v,t) = -u^2(x,v,t).
\end{align*}
\tag{2.7}
\]

The following global Lipschitz stability is obtained for the inverse source problem for (2.4).
Theorem 2.5. Let $u(x,v,t)$ be the solution to (2.4). Suppose $u \in U$. We assume that $\|\sigma\|_{L^\infty(\Omega \times V)}$, $\|k\|_{L^\infty(\Omega \times V \times V)} \leq M$. Suppose that $T$ satisfies (2.2). We assume $R, \partial_t R \in L^2(0,T;L^\infty(\Omega \times V))$. For an arbitrary fixed constant $a_0 > 0$, we further assume that $R(x,v,0) > a_0$ almost all $(x,v) \in \Omega \times V$. Then there exists a constant $C = C(M, a_0) > 0$ such that

$$
\|f\|_{L^2(\Omega \times V)} \leq C \left( \int_0^T \int_{\partial\Omega} (\nu \cdot v) |\partial_t u|^2 dSdvdt \right)^{1/2} + C \left( \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)} \right)
$$

and

$$
\left( \int_0^T \int_{\partial\Omega} (\nu \cdot v) |\partial_t u|^2 dSdvdt \right)^{1/2} \leq \|f\|_{L^2(\Omega \times V)} + \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)}
$$

for any $f \in L^2(\Omega \times V)$. If $g = 0$, we have

$$
\|f\|_{L^2(\Omega \times V)} \leq C \left( \int_0^T \int_{\Gamma_+} (\nu \cdot v) |\partial_t u|^2 dSdvdt \right)^{1/2} + C \left( \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)} \right)
$$

and

$$
\left( \int_0^T \int_{\Gamma_+} (\nu \cdot v) |\partial_t u|^2 dSdvdt \right)^{1/2} \leq \|f\|_{L^2(\Omega \times V)} + \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)}
$$

for any $f \in L^2(\Omega \times V)$.

The proof of Theorem 2.1 is reduced to Theorem 2.5 by substituting (2.5) - (2.7) in (2.8) and (2.9). Thus it suffices to prove Theorem 2.5. The proof of Theorem 2.5 relies on Lemma 4.1 which is a Carleman estimate proved in Section 4.

3. Coupled radiative transport equations

We recall (2.1). Then we can construct mappings $R_j : V_0 \to V_j$ ($j = 1, \ldots, m-1$) as

$$
R_jv = R_j(r, \theta, \phi_1, \ldots, \phi_{n-2}) = \left( r, \theta + \frac{2\pi(l_0-1)}{L_0}, \phi_1 + \frac{\pi(l_1-1)}{L_1}, \ldots, \phi_{n-2} + \frac{\pi(l_{n-2}-1)}{L_{n-2}} \right).
$$

We define $R_0 = 1$. Then for $v \in V_0$,

$$
R_1v \in V_1, \quad R_2v \in V_2, \ldots, \quad R_{m-1}v \in V_{m-1}, \quad R_mv \in V_0.
$$

Let us define

$$
u_i(x,v,t) = u(x, R_iv, t), \quad v \in V_0, \quad i = 0, \ldots, m-1. \quad (3.1)
$$

We introduce

$$
P_i u_i(x,v,t) = \partial_t u_i(x,v,t) + w_i \cdot \nabla u_i(x,v,t),
$$

where $P_i$ are the radiative transport operators.
where
\[ w_i = \mathcal{R}_i v. \]

Let us define
\[ \Gamma_\pm = \{(x, v) \in \partial \Omega \times V_0; (\pm v(x) \cdot \mathcal{R}_i v) > 0\}. \]

Furthermore we define \( a_i(x, v) = a(x, \mathcal{R}_i v), g_i(x, v, t) = g(x, \mathcal{R}_i v, t), \) and
\[ \sigma_i(x, v) = \sigma(x, \mathcal{R}_i v), \quad k_{ij}(x, v, v') = k(x, \mathcal{R}_i v, \mathcal{R}_j v), \quad x \in \Omega, \quad v, v' \in V_0. \]

Thus (2.4) can be rewritten as

\[
\begin{cases}
P_i u_i(x, v, t) + \sigma_i(x, v) u_i(x, v, t) - \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(x, v, v') u_j(x, v', t) \, dv' = F_i(x, v, t), \\
x \in \Omega, \ v \in V_0, \quad 0 < t < T, \\
u_i(x, v, 0) = a_i(x, v), \quad x \in \Omega, \ v \in V_0, \\
u_i(x, v, t) = g_i(x, v, t) \quad \text{on} \ \Gamma_\pm \times (0, T)
\end{cases}
\]  

(3.2)

for \( i = 0, 1, \ldots, m - 1. \) Here we defined
\[ F_i(x, v, t) = f_i(x, v) R_i(x, v, t), \quad f_i(x, v) = f(x, \mathcal{R}_i v), \quad R_i(x, v, t) = R(x, \mathcal{R}_i v, t). \]

4. Key Carleman estimate for coupled equations

Let us introduce weight functions as
\[ \varphi_j(x, t) = (\gamma_j \cdot x) - \beta t, \quad j = 0, 1, \ldots, m - 1, \]
where
\[ 0 < \beta < \min_{0 \leq j \leq m-1} \min_{v \in V_0} (\gamma_j \cdot v). \]  

(4.1)

We set
\[ \mathcal{P}_i u_i(x, v, t) = P_i u_i(x, v, t) + \sigma_i(x, v) u_i(x, v, t) - \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(x, v, v') u_j(x, v', t) \, dv' \]
for \( i = 0, 1, \ldots, m - 1. \) Then the following inequality holds.

**Proposition 4.1.** We assume that \( \sigma_{ij} \in L^\infty(\Omega \times V_0), k_{ij} \in L^\infty(\Omega \times V_0 \times V_0) \) \((i, j = 0, \ldots, m - 1). \) Furthermore we assume \( u_i \in H^1(0, T; L^2(\Omega \times V_0)), \nabla u_i \in L^2(\Omega \times V_0 \times (0, T)), i = 0, \ldots, m - 1. \) Suppose \( u_i(\cdot, \cdot, T) = 0 \) in \( \Omega \times V_0. \) Then there exist constants \( s_0 > 0 \) and \( C > 0 \) such that

\[
\begin{align*}
s \int_{V_0} \int_{\Omega} \sum_{i=0}^{m-1} |u_i(x, v, 0)|^2 e^{2s\varphi_i(x, 0)} \, dx \, dv + s^2 \int_{0}^{T} \int_{V_0} \int_{\Omega} \sum_{i=0}^{m-1} |u_i|^2 e^{2s\varphi_i} \, dx \, dv \, dt \\
& \leq C \int_{0}^{T} \int_{V_0} \int_{\Omega} \sum_{i=0}^{m-1} |\mathcal{P}_i u_i|^2 e^{2s\varphi_i} \, dx \, dv \, dt + s \int_{0}^{T} \int_{\Gamma_\pm} (w_i \cdot \nu)|u_i|^2 e^{2s\varphi_i} \, dS \, dv \, dt
\end{align*}
\]

(4.2)

for \( s \geq s_0. \)
Proposition 4.1 is proved using Lemma 4.2 below. For a fixed \( j \in \{0, \ldots, m-1\} \), we define

\[
\tilde{\mathcal{P}}_j = P_0 u + \sigma u - \int_{V_j} k(x,v,v') u(x,v,t) \, dv', \quad x \in \Omega, \ v \in V_j, \ t \in (0,T),
\]

where \( \sigma \in L^\infty(\Omega \times V_j) \) and \( k \in L^\infty(\Omega \times V_j \times V_j) \). Furthermore we set

\[
Q = \Omega \times (0,T).
\]

The following Carleman estimate is obtained in \cite{29}.

**Lemma 4.2.** For a fixed \( j \in \{0, \ldots, m-1\} \), there exist constants \( s_0 > 0 \) and \( C > 0 \) such that

\[
s \int_{V_j} \Omega |u(x,v,0)|^2 e^{2s\varphi_j(x,0)} \, dv + 2 \int_{Q} \int_{V_j} |u(x,v,t)|^2 e^{2s\varphi_j(x,v)} \, dx \, dv \, dt \leq C \int_{Q} \int_{V_j} |\tilde{\mathcal{P}}_j u|^2 e^{2s\varphi_j(x,v)} \, dx \, dv \, dt + s \int_0^T \int_{\partial \Omega \setminus \{|v-v'|>0\}} (v \cdot \nu)|u|^2 e^{2s\varphi_j(x,v)} \, dS \, dt
\]

for all \( s \geq s_0 \) and \( u \in H^1(0,T;L^2(\Omega \times V)) \) satisfying \( \nabla u \in L^2(\Omega \times V \times (0,T)) \) and \( u(\cdot, \cdot, T) = 0 \) in \( \Omega \times V_j \).

**Proof.** In the proof below we write \( \varphi(x,t) = \varphi_j(x,t) \). See also \cite{29}.

Let us set \( z(x,v,t) = e^{s\varphi_j(x,t)} u(x,v,t) \) and \( Lz(x,v,t) = e^{s\varphi_j(x,t)} P_0 (e^{-s\varphi_j(x,t)} z(x,v,t)) \).

That is, we have

\[
Lz(x,v,t) = P_0 z(x,v,t) - sBz(x,v,t),
\]

where

\[
B = (\partial_t + v \nabla) \varphi(x,t) = -\beta + v \cdot \gamma > 0.
\]

We note that

\[
\int_Q |P_0 u(x,v,t)|^2 e^{2s\varphi_j(x,t)} \, dx = \int_Q |Lz(x,v,t)|^2 \, dxdt, \quad v \in V_j.
\]

The following calculation holds for almost all \( v \in V_j \).

\[
\int_Q |P_0 u|^2 e^{2s\varphi} \, dxdt
\]

\[
= \int_Q |\partial_t z + v \cdot \nabla z|^2 \, dxdt + \int_Q |sB|^2 z^2 \, dxdt - 2s \int_Q Bz(\partial_t z + v \cdot \nabla z) \, dxdt
\]

\[
\geq -2s \int_Q Bz(\partial_t z + v \cdot \nabla z) \, dxdt + s^2 \int_Q B^2 z^2 \, dxdt
\]

\[
= -sB \int_Q (\partial_t (z^2) + v \cdot \nabla (z^2)) \, dxdt + s^2 B^2 \int_Q z^2 \, dxdt
\]

\[
= sB \int_Q |z(x,v,0)|^2 \, dx - sB \int_0^T \int_{\partial \Omega} (v \cdot \nu) z^2 \, dS \, dt + s^2 B^2 \int_Q z^2 \, dxdt
\]

\[
\geq sB \int_Q |z(x,v,0)|^2 \, dx - sB \int_0^T \int_{\partial \Omega \cap \{|v-v'|>0\}} (v \cdot \nu) z^2 \, dS \, dt + s^2 B^2 \int_Q z^2 \, dxdt.
\]
By substituting \( z = e^{s\varphi} u \) and integrating over \( v \) in the above inequality, we obtain
\[
C \int_Q \int_{V_j} |P_0 u|^2 e^{2s\varphi} \, dv \, dx dt \geq s \int_\Omega \int_{V_j} |u(x, v, 0)|^2 e^{2s\varphi(x, 0)} \, dx
\]
\[
- s \int_0^T \int_{\partial \Omega \cap \{(v, \nu(x)) > 0\}} \int_{V_j} (v \cdot \nu) |u|^2 e^{2s\varphi} \, dv dt + s^2 \int_Q \int_{V_j} |u|^2 e^{2s\varphi} \, dv dt.
\]
It is straightforward to replace \( |P_0 u|^2 \) in the above inequality with \( |\tilde{P}_j u|^2 \) and the proof is complete. \( \square \)

We can rewrite the inequality in Lemma 4.2 as
\[
s \int_{V_0} \int_\Omega |u_j(x, v, 0)|^2 e^{2s\varphi_j(x, 0)} \, dx dv + s^2 \int_Q \int_{V_0} |u_j(x, v, t)|^2 e^{2s\varphi_j(x, v)} \, dx dv dt
\]
\[
\leq C \int_Q \int_{V_0} |\tilde{P}_j u_j|^2 e^{2s\varphi_j(x, v)} \, dx dv dt + s \int_0^T \int_{\Gamma_j^+} (v \cdot \nu) |u_j|^2 e^{2s\varphi_j(x, v)} \, dS dv dt.
\]
By summing up the inequalities from \( j = 0 \) through \( j = m - 1 \), we obtain the inequality 4.2. Thus Proposition 4.1 is proved.

5. Energy estimates

Henceforth in this section, \( C > 0 \) denotes generic constants which are independent of \( f_i \).

Lemma 5.1. The following inequalities hold for the solutions \( u_i(x, v, t), i = 0, \ldots, m - 1 \), which satisfy (4.2).

\[
\int_\Omega \int_{V_0} \sum_{i=0}^{m-1} |\partial_t u_i(x, v, t)|^2 \, dx dv
\]
\[
\leq C \sum_{i=0}^{m-1} \left( \|f_i\|^2_{L^2(\Omega \times V_0)} + \|a_i\|^2_{L^2(\Omega \times V_0)} + \|\nabla a_i\|^2_{L^2(\Omega \times V_0)} \right)
\]
\[
+ C \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i^-} |(w_i - \nu)| |\partial_t u_i|^2 \, dS dv dt
\]
(5.1)

and

\[
\int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS dv dt \leq \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i^-} |(w_i - \nu)| |\partial_t u_i|^2 \, dS dv dt
\]
\[
+ \sum_{i=0}^{m-1} \left( \|f_i\|^2_{L^2(\Omega \times V_0)} + \|a_i\|^2_{L^2(\Omega \times V_0)} + \|\nabla a_i\|^2_{L^2(\Omega \times V_0)} \right).
\]
(5.2)

Proof. We differentiate the coupled transport equation in (3.2) with respect to \( t \) and obtain
\[
\partial_t (\partial_t u_i) + w_i \cdot \nabla (\partial_t u_i) + \sigma_i (\partial_t u_i) - \int_{V_0} \sum_{j=0}^{m-1} k_{ij} (\partial_t u_j) \, dv' = f_i \partial_t R_i.
\]
By multiplying $2\partial_t u_i$, we have

$$\partial_t (\partial_t u_i)^2 + w_i \cdot \nabla (\partial_t u_i)^2 + 2\sigma_i (\partial_t u_i)^2 - 2(\partial_t u_i) \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(\partial_t u_j(x,v',t)) dv' = 2(\partial_t u_i) f_i \partial_t R_i.$$ 

By integrating over $\Omega \times V_0$, we obtain

$$\partial_t \int_{V_0} \left| \partial_t u_i \right|^2 dv + \int_{V_0} w_i \cdot \nabla \left| \partial_t u_i \right|^2 dv + 2 \int_{V_0} \sigma_i \left| \partial_t u_i \right|^2 dv
- 2 \int_{V_0} \left( \partial_t u_i(x,v,t) \right) \left( \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(x,v') \partial_t u_j(x,v',t) dv' \right) dv
= 2 \int_{V_0} \left( \partial_t u_i \right) f_i \left( \partial_t R_i \right) dv.$$ 

Setting $E(t) = \int_{V_0} \sum_{i=0}^{m-1} |\partial_t u_i(x,v,t)|^2 dv dx$ and integrating the second term on the left-hand side, we obtain

$$\partial_t E(t) = -\int_{V_0} \sum_{i=0}^{m-1} (w_i \cdot \nabla) |\partial_t u_i|^2 dv S - 2 \int_{V_0} \sum_{i=0}^{m-1} \sigma_i |\partial_t u_i|^2 dv dx
+ 2 \int_{V_0} \sum_{i=0}^{m-1} (\partial_t u_i(x,v,t)) \left( \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(\partial_t u_j(x,v',t)) dv' \right) dv dx
+ 2 \int_{V_0} \sum_{i=0}^{m-1} (\partial_t u_i) f_i (\partial_t R_i) dv dx.$$ 

Then by integrating over $t$, we have

$$E(t) - E(0)
= -\int_0^t \sum_{i=0}^{m-1} \left( \int_{\Gamma^+} + \int_{\Gamma^-} \right) (w_i \cdot \nu) |\partial_t u_i|^2 dS dt - 2 \int_0^t \int_{V_0} \sum_{i=0}^{m-1} \sigma_i |\partial_t u_i|^2 dv dx dt
+ 2 \int_0^t \int_{V_0} \sum_{i=0}^{m-1} (\partial_t u_i) \left( \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(\partial_t u_j(x,v',t)) dv' \right) dv dx dt
+ 2 \int_0^t \int_{V_0} \sum_{i=0}^{m-1} (\partial_t u_i) f_i (\partial_t R_i) dv dx dt.$$
We note that by the Cauchy-Schwarz inequality,
\[
\left| \int_{V_0} \sum_{i=0}^{m-1} (\partial_t u_i(x,v,t)) \left( \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(x,v,v') (\partial_t u_j(x,v',t))\right) dv \right|
\]
\[
\leq C \int_{V_0} \sum_{i=0}^{m-1} |\partial_t u_i(x,v,t)| \left( \int_{V_0} \sum_{j=0}^{m-1} |\partial_t u_j(x,v',t)| dv \right) dv
\]
\[
\leq C \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \left( \int_{V_0} |\partial_t u_i(x,v,t)|^2 dv \right)^{1/2} \left( \int_{V_0} |\partial_t u_j(x,v',t)|^2 dv \right)^{1/2} |V_0|^{1/2}
\]
\[
\leq C |V_0| \int_{V_0} \sum_{i=0}^{m-1} |\partial_t u_i(x,v,t)|^2 dv,
\]
where $|V_0| = \int_{V_0} dv$, and
\[
2 \int_{\Omega} \int_{V_0} |(\partial_t u_i) f_i(\partial_t R_i)| dv dx \leq \int_{\Omega} \int_{V_0} |f_i(\partial_t R_i)|^2 dv dx + \int_{\Omega} \int_{V_0} |\partial_t u_i|^2 dv dx
\]
\[
\leq C \int_{\Omega} \int_{V_0} |f_i|^2 dv dx + \int_{\Omega} \int_{V_0} |\partial_t u_i|^2 dv dx.
\]
Hence we obtain for $0 \leq t \leq T$,
\[
E(t) - E(0) \leq - \int_{t_0}^t \sum_{i=0}^{m-1} \left( \int_{\Gamma^+_k} + \int_{\Gamma^-_k} \right) (w_i \cdot \nu)|\partial_t u_i|^2 dS dt + C \int_{t_0}^t E(\eta) d\eta
\]
\[
+ C \sum_{i=0}^{m-1} ||f_i||^2_{L^2(\Omega)}.
\]  \hfill (5.3)

We note that from (5.2),
\[
\partial_t u_i(x,v,0) + w_i \cdot \nabla a_i(x,v) + \sigma_i a_i(x,v) - \int_{V_0} \sum_{j=0}^{m-1} k_{ij} a_j(x,v') dv' = f_i R_i(x,v,0),
\]
and hence
\[
E(0) \leq C \sum_{i=0}^{m-1} \left( ||f_i||^2_{L^2(\Omega \times V_0)} + ||a_i||^2_{L^2(\Omega \times V_0)} + ||\nabla a_i||^2_{L^2(\Omega \times V_0)} \right).
\]

Using the Gronwall inequality, we arrive at
\[
E(t) \leq C \sum_{i=0}^{m-1} \left( \int_{t_0}^t \left( \int_{\Gamma^+_k} + \int_{\Gamma^-_k} \right) (w_i \cdot \nu)|\partial_t u_i|^2 dS dt + ||f_i||^2_{L^2(\Omega \times V_0)} + ||a_i||^2_{L^2(\Omega \times V_0)} + ||\nabla a_i||^2_{L^2(\Omega \times V_0)} \right). \hfill (5.4)
\]

Noting that $\int_{t_0}^t (w_i \cdot \nu)|\partial_t u_i|^2 dS dt \geq 0$, the first inequality (5.1) in Lemma 5.1 is proved from (5.4).
Equation (5.4) yields
\[
\int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i^-} (w_i \cdot \nu) |\partial_t u_i|^2 \, dSdvdt 
\leq - \frac{E(T)}{C} - \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dSdvdt 
+ \sum_{i=0}^{m-1} \left( \|f_i\|_{L^2(\Omega \times V_0)}^2 + \|a_i\|_{L^2(\Omega \times V_0)}^2 + \|
abla a_i\|_{L^2(\Omega \times V_0)}^2 \right)
\leq - \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i^-} (w_i \cdot \nu) |\partial_t u_i|^2 \, dSdvdt 
+ \sum_{i=0}^{m-1} \left( \|f_i\|_{L^2(\Omega \times V_0)}^2 + \|a_i\|_{L^2(\Omega \times V_0)}^2 + \|
abla a_i\|_{L^2(\Omega \times V_0)}^2 \right).
\]
Thus the second inequality (5.2) in Lemma 5.1 is proved.

6. PROOF OF THEOREM 2.5

Hereafter \( C > 0 \) denotes generic constants which are independent of \( s > 0 \).
We set
\[
r_{\text{max}} = \max_{0 \leq j \leq m-1} \max_{x \in \Omega} \gamma_j \cdot x, \quad r_{\text{min}} = \min_{0 \leq j \leq m-1} \min_{x \in \Omega} \gamma_j \cdot x.
\]
Since \( T \) satisfies
\[
T > \frac{r_{\text{max}} - r_{\text{min}}}{\min_{0 \leq j \leq m-1} \min_{v \in \Omega} \gamma_j \cdot v},
\]
we can choose \( \beta \) such that (4.1) and
\[
r_{\text{max}} - \beta T < r_{\text{min}}.
\]
Then we have
\[
\varphi_i(x, T) \leq r_{\text{max}} - \beta T < r_{\text{min}} \leq \varphi_i(x, 0), \quad i = 0, \ldots, m-1, \ x \in \Omega.
\]
Therefore there exist \( \delta > 0 \) and \( r_0, r_1 \) such that \( r_{\text{max}} - \beta T < r_0 < r_1 < r_{\text{min}} \),
\[
\varphi_i(x, t) > r_1, \quad x \in \Omega, \ 0 \leq t \leq \delta,
\]
and
\[
\varphi_i(x, t) < r_0, \quad x \in \Omega, \ T - 2\delta \leq t \leq T
\]
for \( i = 0, 1, \ldots, m-1 \). Let us introduce a cut-off function \( \chi \in C_0^\infty(\mathbb{R}) \) such that \( 0 \leq \chi \leq 1 \) and
\[
\chi(t) = \begin{cases} 
1, & 0 \leq t \leq T - 2\delta, \\
0, & T - \delta \leq t \leq T.
\end{cases}
\]
Let us set
\[
z_i(x, v, t) = (\partial_t u_i(x, v, t))\chi(t), \quad i = 0, \ldots, m-1.
\]
Then we have $z_i(x, v, T) = 0$. By differentiating the equation in (6.2), we obtain

$$
\tilde{P}_i z_i(x, v, t) - \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(x, v, v') z_j(x, v', t) \, dv' = \chi f_i(\partial_t R_i) + (\partial_t \chi) \partial_t u_i \quad (6.1)
$$

for $(x, t) \in Q, \; v \in V_0$, where

$$
\tilde{P}_i u_i(x, v, t) = P_i u_i(x, v, t) + \sigma_i u_i(x, v, t), \quad (x, t) \in Q, \; v \in V_0.
$$

From the coupled radiative transport equation in (5.2), we have at $s = 0$,

$$
z_i(x, v, 0) = f_i(x, v) R_i(x, v, 0) - w_i \cdot \nabla a_i(x, v) - \sigma_i(x, v) a_i(x, v) + \int_{V_0} \sum_{j=0}^{m-1} k_{ij}(x, v, v') a_j(x, v') \, dv', \quad x \in \Omega, \; v \in V_0. \quad (6.2)
$$

Using the Carleman estimate in Theorem 4.1, we obtain for $z_i$,

$$
s \int_{V_0} \int_{\Omega} \sum_{i=0}^{m-1} |z_i(x, v, 0)|^2 e^{2s \varphi_i(x, 0)} \, dx dv \leq C \int_Q \int_{V_0} \sum_{i=0}^{m-1} |\chi f_i(\partial_t R_i)|^2 e^{2s \varphi_i(x, t)} \, dv dx dt + C \int_Q \int_{V_0} \sum_{i=0}^{m-1} |(\partial_t \chi) \partial_t u_i|^2 e^{2s \varphi_i(x, t)} \, dv dx dt + C s \int_Q \sum_{i=0}^{m-1} \int_{\Gamma_+^i} (w_i \cdot \nu) |z_i|^2 e^{2s \varphi_i(x, t)} \, dS dv dt. \quad (6.3)
$$

Let us set

$$
d_0 = \left( \int_0^T \int_{\Omega} \sum_{i=0}^{m-1} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS dv dt \right)^{1/2}.
$$

The last term in (6.3) is estimated as

$$
C s \int_0^T \int_{\Omega} \sum_{i=0}^{m-1} \int_{\Gamma_+^i} (w_i \cdot \nu) |z_i|^2 e^{2s \varphi_i(x, t)} \, dS dv dt \leq Ce^{C_1 s} d_0^2,
$$

where we used $s e^{Cs} \leq e^{(C + 1)s}$ for $s > 0$ and set $C_1 = C + 1$. Since $\partial_t \chi = 0$ for $0 \leq t \leq T - 2\delta$ or $T - \delta \leq t \leq T$, we have

$$
\int_Q \int_{V_0} \sum_{i=0}^{m-1} |(\partial_t \chi) \partial_t u_i|^2 e^{2s \varphi_i(x, t)} \, dv dx dt = \int_{T-\delta}^{T-\delta} \int_{\Omega} \int_{V_0} \sum_{i=0}^{m-1} |(\partial_t \chi) \partial_t u_i|^2 e^{2s \varphi_i(x, t)} \, dv dx dt \\
\leq C e^{2s \delta_0} \int_{T-2\delta}^{T-\delta} \int_{\Omega} \int_{V_0} \sum_{i=0}^{m-1} |\partial_t u_i|^2 \, dv dx dt. \quad (6.4)
$$
Thus from (6.4) with the help of (5.1) in Lemma 5.1,

$$\int \int_{V_0} \sum_{i=0}^{m-1} |(\partial_t \chi) \partial_t u_i|^2 e^{2s\varphi} \, dv \, dx dt$$

$$\leq C e^{2s\varphi_0} \sum_{i=0}^{m-1} \left( \|f_i\|_{L^2(\Omega \times V_0)}^2 + \|a_i\|_{L^2(\Omega \times V_0)}^2 + \|\nabla a_i\|_{L^2(\Omega \times V_0)}^2 \right)$$

$$- C e^{2s\varphi_0} \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS \, dv \, dt.$$

Note that (6.2) holds for $x \in \Omega$, $v \in V_0$, and recall the fact that $R_i(\cdot, \cdot, 0) \neq 0$ in $\Omega \times V_0$. We obtain

$$\int \int_{V_0} \sum_{i=0}^{m-1} |z_i(x, v, 0)|^2 e^{2s\varphi_i(x, 0)} \, dv \, dx + C e^{Cs} \sum_{i=0}^{m-1} \left( \|a_i\|_{L^2(\Omega \times V_0)}^2 + \|\nabla a_i\|_{L^2(\Omega \times V_0)}^2 \right)$$

$$\geq C \int \int_{V_0} \sum_{i=0}^{m-1} |f_i(x, v)|^2 e^{2s\varphi_i(x, 0)} \, dv \, dx.$$

Therefore (6.3) yields

$$s \int \int_{V_0} \sum_{i=0}^{m-1} |f_i(x, v)|^2 e^{2s\varphi_i(x, 0)} \, dv \, dx \leq C \int \int_{V_0} \sum_{i=0}^{m-1} |f_i(x, v)|^2 e^{2s\varphi_i(x, t)} \, dv \, dx \, dt$$

$$+ C e^{2s\varphi_0} \sum_{i=0}^{m-1} \|f_i\|_{L^2(\Omega \times V_0)}^2 - C e^{2s\varphi_0} \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS \, dv \, dt + C e^{Cs} d^2,$$

where we defined

$$d = \left( \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS \, dv \, dt \right)^{1/2} + \sum_{i=0}^{m-1} \left( \|a_i\|_{L^2(\Omega \times V_0)} + \|\nabla a_i\|_{L^2(\Omega \times V_0)} \right).$$

Since $\varphi_i(x, t) \leq \varphi_i(x, 0)$ for $(x, t) \in Q$,

$$(s - CT) \int \int_{V_0} \sum_{i=0}^{m-1} |f_i(x, v)|^2 e^{2s\varphi_i(x, 0)} \, dv \, dx \leq C e^{2s\varphi_0} \sum_{i=0}^{m-1} \|f_i\|_{L^2(\Omega \times V_0)}^2$$

$$- C e^{2s\varphi_0} \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS \, dv \, dt + C e^{Cs} d^2.$$

Noting that $\varphi_i(x, 0) > r_1$, we have

$$s e^{2s_1} \int \int_{V_0} \sum_{i=0}^{m-1} |f_i|^2 \, dv \, dx \leq C e^{2s\varphi_0} \sum_{i=0}^{m-1} \|f_i\|_{L^2(\Omega \times V_0)}^2$$

$$- C e^{Cs} \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma^+} (w_i \cdot \nu) |\partial_t u_i|^2 \, dS \, dv \, dt + C e^{Cs} d^2.$$
Hence, for sufficiently large $s$,
\[
\sum_{i=0}^{m-1} \|f_i\|^2_{L^2(\Omega \times V_0)} \leq C e^{-2s(r_1 - r_0)} \sum_{i=0}^{m-1} \|f_i\|^2_{L^2(\Omega \times V_0)} - C e^{Cs} \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i} (w_i \cdot \nu) |\partial_t u_i|^2 dS dvt \\
+ C e^{Cs} \int_0^T \sum_{i=0}^{m-1} \int_{\Gamma_i} (w_i \cdot \nu) |\partial_t u_i|^2 dS dvt \\
+ C e^{Cs} \sum_{i=0}^{m-1} \left( \|a_i\|^2_{L^2(\Omega \times V_0)} + \|\nabla a_i\|^2_{L^2(\Omega \times V_0)} \right). \quad (6.5)
\]

The first term on the right-hand side of (6.5) will vanish as $s$ becomes large. We can rewrite (6.5) as
\[
\|f\|^2_{L^2(\Omega \times V)} \leq -C e^{Cs} \int_0^T \int_{\partial \Omega} \int_V |(v \cdot \nu)||\partial_t u|^2 dvdS dt \\
+ C e^{Cs} \left( \|a\|^2_{L^2(\Omega \times V)} + \|\nabla a\|^2_{L^2(\Omega \times V)} \right)
\]
for sufficiently large $s$. Hence the first inequality in Theorem 2.5 is proved.

The second inequality in Theorem 2.5 is immediately obtained from (5.2) in Lemma 5.1. Thus the proof is complete. □

7. Concluding remarks

In [29], the velocity $v$ must satisfy $(v \cdot \gamma) > 0$ with some fixed vector $\gamma \in \mathbb{R}^n$, for which there are limited applications in transport phenomena.

In the present article, we deleted such an extra assumption and our global stability results Theorems 2.1 and 2.5 require only the positivity (2.3) of initial values and $R(x, v, 0) > 0$ on $(x, v) \in \Omega \times V$ respectively.

Moreover, it is understood that such positivity is essential for the methodology by Carleman estimate, in general.

With the partition by choosing multiple fixed vectors $\gamma_j$, $j = 0, \ldots, m-1$, which are dependent on $v$, we construct the weight functions in the form $\varphi_i(x, t) := (\gamma_j \cdot x) - \beta t$ to derive the key Carleman estimate Proposition 4.1. Such dependence of the weight on $v$ still admits to prove the relevant Carleman estimate for $u$.

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