Algebraically Special Class of Space-Times
and
(1+1)-Dimensional Field Theories

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We present the (1+1)-dimensional description of the algebraically special class of space-times of 4-dimensions. It is described by the (1+1)-dimensional Yang-Mills action interacting with matter fields, with diffeomorphisms of 2-surface as the gauge symmetry. Parts of the constraints are identified as the gauge fixing condition. We also show that the representations of $w_{\infty}$-gravity appear naturally as special cases of this description, and discuss the geometry of $w_{\infty}$-gravity in term of the fibre bundle.
Lower dimensional field theories have received considerable attention in connection with self-dual spaces of 4-dimensions [1] and the physics of black-holes [2, 3]. Recently it was realized that general relativity itself can be also viewed as a (1+1)-dimensional field theory, where the other two space-like dimensions are regarded as the fibre 1, and the action principle from the (1+1)-dimensional perspective was obtained [5]. In contrast to the cases of the self-dual spaces and black-hole space-times, however, the (1+1)-dimensional action principle for general space-times appears to be rather formal and consequently, of little practical use. In this letter we therefore draw attention following the Petrov classification [4] to a specific class of space-times, namely, the algebraically special class, and interpret the entire class from the (1+1)-dimensional point of view. Namely, we shall show that space-times of this class can be regarded as (1+1)-dimensional gauge theories, where the local gauge group is the diffeomorphism group of the 2-dimensional fibre 2.

As a bonus of this (1+1)-dimensional analysis of space-times, we find the fibre bundle as the natural framework for the geometric description of the so-called \( w_\infty \)-gravity, which was lacking so far [7, 8]. In this picture the local gauge fields for \( w_\infty \)-gravity are the connections valued in the Lie algebra of the area-preserving diffeomorphisms of the 2-dimensional fibre. Due to this picture of \( w_\infty \)-geometry, we are able to construct field theoretic realizations of \( w_\infty \)-gravity in a straightforward way, as we shall show later.

Let us consider a class of space-times that contain a twist-free null vector field \( k^A \) \((A, B, \cdots = 0, 1, 2, 3)\). These space-times belong to the algebraically special class

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1Here we are regarding space-times as a fibre bundle, treating the (1+1)-dimensional section as the base manifold and the remaining two space-like dimensions as the fibre. For the algebraically special class of space-times we consider here, the 2-dimensional fibre may be interpreted as the transverse wave-surface [4].

2It is worth mentioning that Einstein's field equations can be 'derived' from the source-free Yang-Mills equations [5]. Here, however, we are essentially making the (2+2)-decomposition of vacuum GR.
of space-times, according to the Petrov classification. This class of space-times is rather broad, since most of the known exact solutions of Einstein’s equations are algebraically special. Being twist-free, the null vector field may be chosen to be a gradient field, so that $k_A = \partial_A u$ for some function $u$. The null hypersurface $N_2$ defined by $u = \text{constant}$ spans the 2-dimensional subspace for which we introduce two space-like coordinates $y^a\ (a, b, \cdots = 2, 3)$. The general line element for this class has the form \[ ds^2 = \phi_{ab} dy^a dy^b - 2du (dv + m_a dy^a + H du), \] (1)

where $v$ is the affine parameter, and $\phi_{ab}$, $m_a$ and $H$ are functions of all of the four coordinates $(u, v, y^a)$, as we assume no Killing vector fields here (and afterwards).

For the class of space-times (1), we shall find the (1+1)-dimensional action principle defined on the $(u, v)$-surface. Namely, we wish to show that space-times of the above type can be viewed as (1+1)-dimensional gauge theories on the $(u, v)$-surface, where the gauge fields are valued in the infinite dimensional Lie algebra associated with the diffeomorphism group $\text{diff}N_2$ of the 2-dimensional surface $N_2$. To show how this works, let us first recall that the general line element of space-time, as viewed as a local product of two 2-dimensional submanifolds $M_{1+1} \times N_2$, can be written as, at least locally, \[ ds^2 = \phi_{ab} dy^a dy^b + (\gamma_{\mu\nu} + \phi_{ab} A^a_\mu A^b_\nu) dx^\mu dx^\nu + 2\phi_{ab} A^b_\mu dx^\mu dy^a, \] (2)

where $\gamma_{\mu\nu}$ $(\mu, \nu = 0, 1)$ resp. $\phi_{ab}$ $(a, b = 2, 3)$ is the metric on the (1+1)-dimensional surface $M_{1+1}$ resp. the 2-dimensional surface $N_2$ spanned by $\partial/\partial x^\mu$ resp. $\partial/\partial y^a$. In this (2+2)-decomposition, the Einstein-Hilbert action for the line element (2) can be
written as, \[3\]

\[
\mathcal{L} = -\sqrt{-\gamma} \sqrt{\phi} \left[ \gamma_{\mu\nu} R_{\mu\nu} + \phi^{ab} R_{ab} + \frac{1}{4} \phi_{ab} F_{\mu\nu}^a F^{\mu\nu b} \right. \\
+ \frac{1}{4} \gamma_{\mu\nu} \phi^{cd} \left\{ (D_{\mu} \phi_{ac})(D_{\nu} \phi_{bd}) - (D_{\mu} \phi_{ab})(D_{\nu} \phi_{cd}) \right\} \\
+ \frac{1}{4} \phi^{ab} \gamma_{\mu\nu} \gamma_{\alpha\beta} \left\{ (\partial_{\alpha} \gamma_{\mu\nu})(\partial_{\beta} \gamma_{\nu\alpha}) - (\partial_{\alpha} \gamma_{\mu\nu})(\partial_{\beta} \gamma_{\alpha\nu}) \right\} \right],
\]

up to the surface terms. Here is the summary of our notations (for details, see \[3\]):

1. The covariant derivative $D_{\mu} \phi_{ab}$ is defined by

\[
D_{\mu} \phi_{ab} = \partial_{\mu} \phi_{ab} - [A_{\mu}, \phi]_{ab}
= \partial_{\mu} \phi_{ab} - \left\{ A_{\mu}^c \partial_c \phi_{ab} + (\partial_a A_{\mu}^c) \phi_{cb} + (\partial_b A_{\mu}^c) \phi_{ac} \right\},
\]

where the bracket means the Lie derivative here and afterwards, an infinite dimensional generalization of the finite dimensional matrix commutators.

2. The field strength $F_{\mu\nu}^a$ is defined as usual,

\[
F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a - [A_{\mu}, A_{\nu}]^a.
\]

3. The Levi-Civita connections $\Gamma$'s, and $R_{\mu\nu}$ and $R_{ac}$ are defined as

\[
\Gamma_{\mu\nu}^a = \frac{1}{2} \gamma^{\alpha\beta} \left( D_{\mu} \gamma_{\nu\beta} + D_{\nu} \gamma_{\mu\beta} - D_{\beta} \gamma_{\mu\nu} \right),
\]

\[
\Gamma_{ab}^c = \frac{1}{2} \phi^{cd} \left( \partial_a \phi_{bd} + \partial_b \phi_{ad} - \partial_{[a} \phi_{b]} \right),
\]

\[
R_{\mu\nu} = D_{\mu} \Gamma_{\nu\alpha}^a - D_{\nu} \Gamma_{\mu\alpha}^a + \Gamma_{\mu\beta}^a \Gamma_{\alpha\nu}^\beta - \Gamma_{\beta\alpha}^\beta \Gamma_{\mu\nu}^a,
\]

\[
R_{ab} = \partial_a \Gamma_{cb}^c - \partial_c \Gamma_{ab}^c + \Gamma_{ad}^c \Gamma_{cb}^d - \Gamma_{dc}^d \Gamma_{ab}^c.
\]

Notice that here $R_{\mu\nu}$ is covariantized, which we might call the ‘gauged’ Ricci tensor \([3, 10]\).
Although the action (3) brings general relativity (up to a surface term) into a form of (1+1)-dimensional field theories for the general line element (2), it appears rather formal. For the algebraically special class of space-times that we wish to consider in this letter, however, the action reduces to a remarkably simple form. In order to show this, let us first introduce the ‘light-cone’ coordinates \((u, v)\) such that
\[
u = \frac{1}{\sqrt{2}}(x^0 + x^1), \quad v = \frac{1}{\sqrt{2}}(x^0 - x^1),
\]
and define \(A^a_u\) and \(A^a_v\)
\[
A^a_u = \frac{1}{\sqrt{2}}(A^a_0 + A^a_1), \quad A^a_v = \frac{1}{\sqrt{2}}(A^a_0 - A^a_1).
\]
For \(\gamma_{\mu\nu}\), we assume the Polyakov ansatz \([11]\)
\[
\gamma_{\mu\nu} = \begin{pmatrix} -2h & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 2h \end{pmatrix}, \quad (\det\gamma_{\mu\nu} = -1),
\]
in the \((u, v)\)-coordinates. Then the line element (2) becomes
\[
ds^2 = \phi_{ab}dy^a dy^b - 2dudv - 2h(du)^2 + \phi_{ab}(A^a_u du + A^a_v dv)(A^b_u du + A^b_v dv)
+ 2\phi_{ab}(A^a_u du + A^a_v dv)dy^b.
\]
If we choose, at least locally, the ‘light-cone’ gauge \[3A^a_v = 0\], then this becomes
\[
ds^2 = \phi_{ab}dy^a dy^b - 2 du\left[ dv - \phi_{ab}A^b_u dy^a + \left( h - \frac{1}{2}\phi_{ab}A^a_u A^b_u \right) du \right].
\]
A quick glance at (3) and (11) tells us that if the following identifications
\[
m_a = -\phi_{ab}A^b_u, \quad H = h - \frac{1}{2}\phi_{ab}A^a_u A^b_u
\]
\[3\text{We are viewing the action (3) as a (1+1)-dimensional gauge theory, as it should be clear now. Here we are referring to the disposable gauge degrees of freedom in the action. There could be topological obstruction against globalizing this choice, as the general coordinate transformation of } N_2 \text{ corresponds to the gauge transformation.}\]
are made, then the two line elements are the same. Thus the Polyakov ansatz amounts to the restriction (modulo the gauge choice $A_v^a = 0$) to the algebraically special class of space-times that contain a twist-free null vector field.

Let us now examine the transformation properties of $h$, $A^{\, a}_u$, $A^{\, a}_v$, and $\phi_{ab}$ under the arbitrary diffeomorphic changes of the coordinates $y^a$ on $N_2$,

$$y'{}^a = y^{{}' a}(y^b, u, v), \quad u' = u, \quad v' = v. \quad (13)$$

Under these transformations, we find that

$$h'(y', u, v) = h(y, u, v), \quad \phi'_{ab}(y', u, v) = \frac{\partial y^c}{\partial y^{{}' a}} \frac{\partial y^d}{\partial y^{{}' b}} \phi_{cd}(y, u, v),$$

$$A^{\, a}_u(y', u, v) = \frac{\partial y^a}{\partial y^c} A^c_u(y, u, v) - \partial_a y^{{}' a}, \quad A^{\, a}_v(y', u, v) = -\partial_v y^{{}' a}, \quad (14)$$

which become, under the infinitesimal variations, $\delta y^{{}' a} = \xi^a(y, u, v)$,

$$\delta h = -[\xi, h] = -\xi^a \partial_a h, \quad (15a)$$

$$\delta \phi_{ab} = -[\xi, \phi]_{ab} = -\xi^c \partial_c \phi_{ab} - (\partial_a \xi^c) \phi_{cb} - (\partial_b \xi^c) \phi_{ac}, \quad (15b)$$

$$\delta A^{\, a}_u = -D_u \xi^a = -\partial_u \xi^a + [A_u, \xi]^a, \quad (15c)$$

$$\delta A^{\, a}_v = -\partial_v \xi^a. \quad (15d)$$

This shows that $h$ is a scalar field, and $A^{\, a}_u$ and $A^{\, a}_v$ are the gauge fields valued in the infinite dimensional Lie algebra associated with the group of diffeomorphisms of $N_2$. That $A^{\, a}_v$ is a pure gauge is clear, as it depends on the gauge function $\xi^a$ only. Therefore it can be always set to zero, at least locally, by a suitable coordinate transformation $\ref{13}$. To maintain the explicit gauge invariance, however, we shall work with the line element $\ref{10}$ in the following, with the understanding that $A^{\, a}_v$ is a pure gauge.
Let us now proceed to write down the action principle for (10) in terms of the fields \( h, A^a_u, A^a_v, \) and \( \phi_{ab} \). For this purpose, it is convenient to decompose the 2-dimensional metric \( \phi_{ab} \) into the conformal classes

\[
\phi_{ab} = \Omega \rho_{ab}, \quad (\Omega > 0 \text{ and } \det \rho_{ab} = 1).
\]

The kinetic term \( K \) of \( \phi_{ab} \) in (3) then becomes

\[
K = \frac{1}{4} \sqrt{-\gamma} \sqrt{\phi} \gamma_{\mu\nu} \phi^{ab} \phi^{cd} \left\{ (D_\mu \phi_{ac}) (D_\nu \phi_{bd}) - (D_\mu \phi_{ab}) (D_\nu \phi_{cd}) \right\}
\]

\[
= -\frac{1}{2} e^\sigma (D_\mu \sigma)^2 + \frac{1}{4} e^\sigma \gamma_{\mu\nu} \rho^{ab} \rho^{cd} (D_\mu \rho_{ac}) (D_\nu \rho_{bd}),
\]

(17)

where we defined \( \sigma \) by \( \sigma = \ln \Omega \), and the covariant derivatives \( D_\mu \Omega, D_\mu \rho_{ab}, \) and \( D_\mu \sigma \) are

\[
D_\mu \Omega = \partial_\mu \Omega - A^a_\mu \partial_a \Omega - (\partial_a A^a_\mu) \Omega, \quad (18a)
\]

\[
D_\mu \rho_{ab} = \partial_\mu \rho_{ab} - [A_\mu, \rho]_{ab} + (\partial_c A^c_\mu) \rho_{ab}, \quad (18b)
\]

\[
D_\mu \sigma = \partial_\mu \sigma - A^a_\mu \partial_a \sigma - \partial_a A^a_\mu, \quad (18c)
\]

respectively, where \([A_\mu, \rho]_{ab}\) is given by

\[
[A_\mu, \rho]_{ab} = \partial_\mu \rho_{ab} - \left\{ A^c_\mu \partial_c \rho_{ab} + (\partial_a A^c_\mu) \rho_{cb} + (\partial_b A^c_\mu) \rho_{ac} \right\}. \quad (19)
\]

The inclusion of the divergence term \( \partial_\alpha A^a_\mu \) in (18) is necessary to ensure (18) transform covariantly (as the tensor fields) under \( \text{diff}N_2 \), \( \Omega \) and \( \rho_{ab} \) being densities of weight \(-1\) and \(+1\), respectively. Using the ansatz (9), the kinetic term (17) becomes

\[
K = e^\sigma (D_+ \sigma) (D_- \sigma) - \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac}) (D_- \rho_{bd})
\]

\[
- h e^\sigma \left\{ (D_- \sigma)^2 - \frac{1}{2} \rho^{ab} \rho^{cd} (D_- \rho_{ac}) (D_- \rho_{bd}) \right\},
\]

(20)

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where \(+(-)\) stands for \(u(v)\). The Polyakov ansatz (9) simplifies enormously the remaining terms in the action (3), as we now show. Let us first notice that \(\det \gamma_{\mu\nu} = -1\). Therefore the term \(\sqrt{-\gamma} \sqrt{\phi} \phi^{ab} R_{ab}\) can be removed from the action being a surface term. Moreover, we have that \(\gamma^{\mu\nu} \partial_\alpha \gamma_{\mu\nu} = 2(-\gamma)^{-1/2} \partial_\alpha (-\gamma)^{1/2} = 0\). Furthermore, one can easily verify that \(\phi^{ab} \gamma^{\mu\nu} \gamma^{\alpha\beta} (\partial_\alpha \gamma_{\mu\nu})(\partial_\beta \gamma_{\nu\beta})\) vanishes identically. The only remaining terms that contribute to the action (3) are thus the \((1+1)\)-dimensional Yang-Mills action and the ‘gauged’ gravity action. The Yang-Mills action becomes

\[
\frac{1}{4} \phi_{ab} F_\mu^a F^{\mu b} = -\frac{1}{2} e^\sigma \rho_{ab} F_+^a F_-^b. \tag{21}
\]

To express the ‘gauged’ Ricci scalar \(\gamma^{\mu\nu} R_{\mu\nu}\) in terms of \(h\) and \(A_\sigma^a\), etc., we have to compute the Levi-Civita connections first. They are given by

\[
\begin{align*}
\Gamma_{++} &= -D_- h, & \Gamma_{+-} &= D_+ h + 2h D_- h, \\
\Gamma_{+-} &= \Gamma_{--} = D_- h, \tag{22}
\end{align*}
\]

and vanishing otherwise. Thus the ‘gauged’ Ricci tensor becomes

\[
R_{+-} = R_{-+} = -D_-^2 h, \quad R_{--} = 0. \tag{23}
\]

From (9) and (23), the ‘gauged’ Ricci scalar \(\gamma^{\mu\nu} R_{\mu\nu}\) is given by

\[
\gamma^{\mu\nu} R_{\mu\nu} = 2\gamma^{--} R_{--} = 2D_-^2 h, \tag{24}
\]

since \(\gamma^{++} = R_{--} = 0\). Putting together (20), (21), and (24) into (3), the action becomes

\[
\mathcal{L}_2 = -\frac{1}{2} e^{2\sigma} \rho_{ab} F_+^a F_-^b + e^\sigma (D_+ \sigma)(D_- \sigma) - \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) \\
+ h e^\sigma \left\{ \frac{1}{2} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) - (D_- \sigma)^2 \right\} + 2 e^\sigma D_-^2 h. \tag{25}
\]

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The last term in (25) can be expressed as
\[ e^{\sigma} D^2 h = e^{\sigma} \left( \partial_- - A^b \partial_b \right) \left( \partial_- h - A^a \partial_a h \right) \]
\[ = e^{\sigma} \left\{ \partial_-^2 h - \partial_- \left( A^a \partial_a h \right) - A^a \partial_a (D_- h) \right\} \]
\[ = -e^{\sigma} (D_- \sigma)(D_- h) + \partial_- \left( e^{\sigma} D_- h \right) - \partial_a \left( e^{\sigma} A^a D_- h \right), \quad (26) \]
using the Stoke’s theorem, where the last two terms in (26) are the surface terms which we may drop. The first term in (26) can be written as
\[ e^{\sigma} (D_- \sigma)(D_- h) = -he^{\sigma} (D_- \sigma)^2 - he^{\sigma} D^2 \sigma + \partial_- \left\{ e^{\sigma} h D_- \sigma \right\} - \partial_a \left\{ e^{\sigma} h A^a D_- \sigma \right\}, \quad (27) \]
where the last two terms in (27) are also the surface terms. From (26) and (27), we therefore have
\[ e^{\sigma} D_-^2 h \simeq he^{\sigma} \left\{ (D_- \sigma)^2 + D^2 \sigma \right\}, \quad (28) \]
neglecting the surface terms. The resulting (1+1)-dimensional action principle therefore becomes
\[ \mathcal{L}_2 = -\frac{1}{2} e^{2\sigma} \rho_{ab} F^+_{a} F^+_{b} + e^{\sigma} (D_+ \sigma)(D_- \sigma) - \frac{1}{2} e^{\sigma} \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) \]
\[ + he^{\sigma} \left\{ 2D_-^2 \sigma - (D_- \sigma)^2 + \frac{1}{2} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right\}, \quad (29) \]
up to the surface terms. Notice that \( h \) is a Lagrange multiplier, whose variation yields the constraint
\[ H_0 = D_-^2 \sigma - \frac{1}{2} (D_- \sigma)^2 + \frac{1}{4} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \approx 0. \quad (30) \]
From this (1+1)-dimensional point of view, \( h \) is the lapse function (or a pure gauge) that prescribes how to ‘move forward in the \( u \)-time’, carrying the surface \( N_2 \) at
each point of the section \( u = \text{constant} \). The (Hamiltonian) constraint, \( H_0 \approx 0 \), is polynomial in \( \sigma \) and \( A_\alpha \), and contains a non-polynomial term of the non-linear sigma model type but in (1+1)-dimensions. The remaining (momentum) constraints associated with diffeomorphisms of \( N_2 \) are replaced by the gauge condition \( A_\alpha = 0 \), which allows us to view the problem of the constraints of general relativity [12] from a new perspective.

We now have the (1+1)-dimensional action principle for the algebraically special class of space-times that contain a twist-free null vector field. It is described by the Yang-Mills action, interacting with the scalar fields \( \sigma \) and \( \rho_{ab} \) on the flat (1+1)-dimensional surface, which however must satisfy the (Hamiltonian) constraint \( H_0 \approx 0 \). (The flatness of the (1+1)-dimensional surface can be seen from the fact that the lapse function, \( h \), can be chosen as zero, provided that \( H_0 \approx 0 \) holds.) The infinite dimensional group of diffeomorphisms of the surface \( N_2 \) is built-in as the local gauge symmetry, via the minimal couplings to the gauge fields.

Having formulated the algebraically special class of space-times as a (1+1)-dimensional field theory, we may wish to apply varieties of field theoretic methods developed in (1+1)-dimensions. For instance, the action (29) can be viewed as the bosonized form [13] of some version of the (1+1)-dimensional QCD in the infinite dimensional limit of the gauge group [14]. For small fluctuations of \( \sigma \), the action (29) becomes

\[
\mathcal{L}_2 = -\frac{1}{2} \rho_{ab} F_{a+}^b + (D_+ \sigma)(D_- \sigma) - \frac{1}{2} \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}),
\]

modulo the constraint \( H_0 \approx 0 \). It is beyond the scope of this letter to investigate these theories as (1+1)-dimensional quantum field theories. However, this formulation raises many intriguing questions such as: would there be any phase transition
in quantum gravity as viewed as the (1+1)-dimensional quantum field theories? If it does, then what does that mean in quantum geometrical terms? Thus, general relativity, as viewed from the (1+1)-dimensional perspective, renders itself to be studied as a gauge theory in full sense \[15\], at least for the class of space-times discussed in this letter.

So far we derived the action principle on the flat (1+1)-dimensions as the vantage point of studying general relativity for this algebraically special class of space-times. We now ask different but related questions: what kinds of other (1+1)-dimensional field theories related to this problem can we study? For these, let us consider the case where the local gauge symmetry is replaced by the area-preserving diffeomorphisms of \(N_2\). (For these varieties of field theories, we shall drop the constraint \(30\) for the moment. It is at this point that we are departing from general relativity.) This class of field theories naturally realizes the so-called \(w_\infty\)-gravity \[7, 8\] in a linear as well as geometric way, as we now describe.

The area-preserving diffeomorphisms are generated by the vector fields \(\xi^a\), tangent to the surface \(N_2\) and divergence-free,

\[
\partial_a \xi^a = 0. \tag{32}
\]

Let us find the gauge fields \(A_\pm^a\) compatible with the divergence-free condition \(32\). Taking the divergence of both sides of \(15c\), we have

\[
\partial_a \delta A_\pm^a = -\partial_\pm (\partial_a \xi^a) + \partial_a [A_\pm, \xi]^a. \tag{33}
\]

This shows that the condition \(\partial_a A_\pm^a = 0\) is invariant under the area-preserving diffeomorphisms, and characterizes a special subclass of the gauge fields, compatible with
the condition (32). Moreover, when $\partial_a A^+_a = 0$, the fields $\rho_{ab}$ and $\sigma$ behave under the area-preserving diffeomorphisms as a tensor and a scalar field, respectively, as (18b) and (18d) suggest. Indeed, the Jacobian for the area-preserving diffeomorphisms is just 1, disregarding the distinction between the tensor fields and the tensor densities. The (1+1)-dimensional action principle now becomes

$$L'_2 = -\frac{1}{2}e^{2\sigma} \rho_{ab} F^+_a F^-_b + e^{\sigma} (D_+ \sigma)(D_- \sigma) - \frac{1}{2}e^{\sigma} \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}),$$

(34)

where $D_\mu \sigma$, $D_\mu \rho_{ab}$, and $F^+_a$ are

$$D_{\pm} \sigma = \partial_{\pm} \sigma - A_{\pm}^a \partial_a \sigma,$$

(35a)

$$D_{\pm} \rho_{ab} = \partial_{\pm} \rho_{ab} - [A_{\pm}, \rho]_{ab},$$

(35b)

$$F^+_a = \partial_+ A_+^a - \partial_- A_-^a - [A_+, A_-]^a.$$  

(35c)

Under the infinitesimal variations

$$\delta y^a = \xi^a(y, u, v), \quad (\partial_a \xi^a = 0),$$

(36)

the fields transform as

$$\delta \sigma = -[\xi, \sigma] = -\xi^a \partial_a \sigma,$$

(37a)

$$\delta \rho_{ab} = -[\xi, \rho]_{ab} = -\xi^c \partial_c \rho_{ab} - (\partial_a \xi^c) \rho_{cb} - (\partial_b \xi^c) \rho_{ac},$$

(37b)

$$\delta A_+^a = -D_+ \xi^a = -\partial_+ \xi^a + [A_+, \xi]^a,$$

(37c)

$$\delta A_-^a = -\partial_- \xi^a,$$

(37d)

which shows that it is a linear realization of the area-preserving diffeomorphisms. The geometric picture of the action principle (34) is now clear: it is equipped with
the natural bundle structure, where the gauge fields are the connections valued in the Lie algebra associated with the area-preserving diffeomorphisms of the fibre $N_2$. Thus the action principle (34) provides a field theoretical realization of $w_\infty$-gravity \cite{7,8} in a linear as well as geometric way, with the built-in area-preserving diffeomorphisms as the local gauge symmetry.

With this picture of $w_\infty$-geometry at hands, we may construct as many different realizations of $w_\infty$-gravity as one wishes. The simplest example would be a single real scalar field representation, which we may write

$$L''_2 = -\frac{1}{2} F^a_+ F^a_- + (D_+ \sigma)(D_- \sigma), \quad (38)$$

where we used $\delta_{ab}$ in the summation, and $D_\pm \sigma$ and $F^a_\pm$ are as given in (35a) and (35c). By choosing the gauge $A^a_- = 0$ and eliminating the auxiliary field $A^a_+$ in terms of $\sigma$ using the equations of motion of $A^a_+$, we recognize (38) as a single real scalar field realization of $w_\infty$-gravity. In presence of the auxiliary field $A^a_+$, (38) provides an example of the linearized realization of $w_\infty$-gravity for a single real scalar field. It would be interesting to see if the representation (38) is related to the ones constructed in the literatures \cite{7,8}.

Now we summarize our discussions. In this letter, we described the algebraically special class of space-times as the (1+1)-dimensional field theories that possess as the local gauge symmetry the diffeomorphisms of the 2-dimensional null hypersurface. Parts of the constraints are identified as the gauge fixing condition, and the Hamiltonian constraint appears in a manageable form (for the class of space-times investigated here). As a related problem, a special subclass of (1+1)-dimensional field theories associated with the area-preserving diffeomorphisms was also discussed.
in connection with the geometrical formulation of $w_\infty$-gravity. There seem to be many interesting questions to be asked about general relativity from this lower dimensional perspective, which we might be able to address somewhere else.

Acknowledgements

The author thanks Q-Han Park and Soonkeon Nam for many enlightening discussions. This work is supported in part by the Ministry of Education and by the Korea Science and Engineering Foundation.

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