Complete and Consistent Non-Minimal String Corrections to Supergravity

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Abstract

We give a complete and consistent solution to string corrected (deformed), $D = 10$, $N = 1$ supergravity as the non-minimal low energy limit of string theory. We solve the Bianchi identities with suitable constraints to second order in the string slope parameter. In so doing we pave the way for continuing the study of the many applications of these results. We also modify, reaffirm and correct a previously given incomplete solution, and we introduce an important adjustment to the known first order results.
1 Introduction

This work is inspired by the celebration of the activity of Prof. Buchbinder. One of us (S.B.) has been a steady collaborator of the Tomsk group in the past decade, with 16 original papers published in collaboration. Most of them involved Prof. Anton Galajinsky and two were directly related to the work of Prof. Buchbinder. The latter ones dealt with noncommutative field theory, however many discussions and interactions with Prof. Buchbinder over the years, concerned issues related to supersymmetry and supergravity theories, an area where Prof. Buchbinder obtained some of his numerous prestigious achievements. Hence it is quite suitable to present in this dedicated volume an investigation that connects to the stimulating and seemingly almost everlasting issue of string-corrected ten-dimensional supergravity theories.

The route to finding a manifestly supersymmetric theory of D= 10, N=1 supergravity at second order in the string slope parameter has encountered many difficulties over the years. Some years ago a solution to D=10, N=1 Supergravity as the low energy limit of String Theory was given at first order in the string slope parameter and was recently re-calculated [1]. In a sense this was a minimal solution. This approach was founded on what is nowadays referred to, as the scenario of Gates and collaborators; (see [1], [2], and references therein). Other varied approaches are also pursued, however the power of this older approach is now being vindicated. A partial second order solution was recently given in [3] and [4]. It was incomplete and therefore in doubt due do an unsatisfactory assumption in the curvature sector, as well as a computational error. Here we reaffirm that that solution is correct up to a curvature term, and in particular that the proposed X tensor is valid. We then show that the results obtained can be used to solve the curvature Bianchi identity, equation (3). We achieve this by introducing $R^{(1)}_{ab\gamma\delta}$, and then imposing a condition on it which also modifies the old first order results. The difficulties that prevented the complete closing of the Bianchi identities at second order are overcome. We present the full set of equations that consistently satisfy all required Bianchi identities. As the work in itself is lengthy we leave finding the equations of motion and other applications for another letter. We also do not list results explicitly solved by Bianchi identities. For this approach it is required that we solve the Bianchi identities for D=10 N=1 supergravity in superspace at second order in the string slope parameter, and in the presence of the Lorentz Chern-Simons Form, using the so called Beta Function Favored Constraints [5]. This approach has been detailed to first order in [1], and to second order in [3], so we will not recount it here. We show that all results fall neatly into place in a very elegant way, therefore further vindicating the whole original scenario. We note here that it appears also to work consistently at third order, as we have proceeded to that order, and that is for yet another work.

\footnote{In earlier works, this made the determination of a D=10 globally supersymmetric and Lorentz covariant higher derivative Yang-Mills action possible, to order $\gamma^3$ (see e.g. [6]), an important result for topologically nontrivial gauge vector field configurations, as in the case of compactified string theories on manifolds with topologically nontrivial properties.}
2 Review of Solution and Notation

The Bianchi identities in Superspace are as follows:

\[ [[\nabla_{[A}, \nabla_{B}], \nabla_{C}]] = 0 \] (1)

Here we have switched off Yang-Mills fields and the commutator is given by

\[ [\nabla_{A}, \nabla_{B}] = T_{AB}^{C} \nabla_{C} + \frac{1}{2} R_{ABd} e M_{d} \] (2)

A solution must be found in such a way that all if the identities are simultaneously satisfied. A small alteration in one sector will change the whole picture. Most of the resulting identities are listed in [1] and [4], so we will not list them here. The second order solution given in part in [3] and [4] to some extent was based upon an Ansatz for the so called X tensor, as well as extensive algebraic manipulations. The necessity for introducing the X tensor was predicted by Gates et. al., [1]. In [3], and [4], the following Bianchi identity was not properly solved:

\[ T_{(\alpha\beta|}^{\lambda} R_{|\gamma)}{\lambda}de - T_{(\alpha\beta|}^{g} R_{|\gamma)}{g}de - \nabla_{(\alpha} R_{\beta\gamma)}{d}e = 0 \] (3)

It is crucial to show that all of the second order torsions and curvatures satisfy this identity. Also \( R^{(2)}_{\gamma}de \) is required, in order to complete the set. Various ideas, such as finding a new X tensor, imposing constraints on the spinor derivative \( \nabla_{\alpha} \chi_{\beta} \) at second order or adjusting the super current \( A_{abc} \) at second order were considered. We have found that including these adjustments and constraints is unnecessary, and might in fact be wrong.

In this paper we find a complete and consistent solution. We also point out that equation (58) in reference [3] (or equation (115) in reference [4]) is wrong.

In order to avoid a proliferation of terms we maintain the same notation and conventions as in [1], but to avoid recasting the first order results, we denote all quantities by the order in the slope parameter

\[ R_{ABde} = R^{(0)}_{ABde} + R^{(1)}_{ABde} + R^{(2)}_{ABde} + ... \]

\[ T_{AD}^{G} = T^{(0)}_{AD} G + T^{(1)}_{AD} G + T^{(2)}_{AD} G + ... \]

In this work we make some improvements to the notation of references [3]. For example an apparently fundamental object is the following:
\[ \Omega^{(1) \, gef} = L^{(1) \, gef} - \frac{1}{4} A^{(1) \, gef} \] (4)

and its spinor derivative which we denote simply as

\[ \Omega^{(1) \, agef} = \nabla_\gamma \{ L^{(1) \, gef} - \frac{1}{4} A^{(1) \, gef} \} \] (5)

We leave it like this for brevity of notation. The numerical superscript refers to the order of the quantity. A crucial input at first order is that for the super-current \( A^{(1) \, gef} \).

The choice made for on-shell conditions in [1] and hence also [3], is as follows:

\[ A^{(1) \, gef} = i \gamma \sigma_{gefe} T^{mn \tau} T_{mn} \tau \] (6)

In [3], we proposed the form of the X tensor to read

\[ T^{(2) \, \alpha \beta \, d} = \sigma^{pqref}_{\alpha \beta} X_{pqref} d = - \frac{i \gamma}{6} \sigma^{pqref}_{\alpha \beta} H^{(0) \, ef} A^{(1) \, pqr}. \] (7)

A fundamental result which was used in every Bianchi identity and which is very lengthy to derive is the following:

\[ T^{(0) \, \lambda}_{\, (\alpha \beta) \, \sigma^{pqref} \, (\gamma) \, pqr} H^{(0) \, def} - \sigma^{pqref}_{\alpha \beta} H^{(0) \, def} \nabla_{(\gamma)} A^{(1) \, pqr} = -24 \sigma^{pqref}_{\alpha \beta} H^{(0) \, def} [\Omega^{(1) \, \gamma gef}] \] (8)

We note however in this paper that this result can be achieved indirectly by using the first order results found in [1], in conjunction with the Bianchi identity (3), listed in this paper. We found that the following dimension one half torsion is given uniquely by:

\[ T^{(2) \, \alpha \beta \, \lambda} = - \frac{i \gamma}{12} \sigma^{pqref}_{\alpha \beta} A^{(1) \, pqr} T_{ef} \lambda. \] (9)

It was then shown that together with the proposed X tensor Ansatz as well as equation (8) and other observations and results, the H sector Bianchi identities as listed in [1], [2] could be solved simultaneously with the torsions (10) and (11) as listed below

\[ T_{(\alpha \beta) \, \lambda T_{\gamma} \lambda} d - T_{(\alpha \beta) \, g T_{\gamma} g} d - \nabla_{(\alpha} T_{\beta \gamma) \, d} = 0 \] (10)

and

\[ T_{(\alpha \beta) \, \lambda T_{\gamma} \delta} - T_{(\alpha \beta) \, g T_{\gamma} g} \delta - \nabla_{(\alpha} T_{\beta \gamma) \delta} - \frac{1}{4} R_{(\alpha \beta) \, de \sigma^{de \, \gamma}} \delta = 0. \] (11)
We find the second order solutions to (10) to be given by (7) and the following

\[ \sigma^g_{\alpha\beta}[T^{(2)}_{\gamma\delta}]_{g\delta} = 4\gamma\sigma^g_{\alpha\beta}[\Omega^{(1)}_{\gamma\delta}g\gamma]_{d\gamma}f H^{(0)}_{d\gamma}f - \frac{i\gamma}{6}\sigma^g_{\alpha\beta}(\sigma^{pqre}_{g\gamma})_f A^{(1)}_{pqr}T^{(0)}_{d\gamma}f, \]  

(12)

\[ T^{(2)}_{\gamma\delta} = +2\gamma[\Omega^{(1)}_{\gamma\delta}]_{g\gamma}f H^{(0)}_{g\gamma}f + \sigma_{\alpha\beta} \gamma = \frac{2\gamma}{3}\phi_H^{(0)} g\gamma f H^{(0)}_{g\gamma}f \]  

(13)

In equation (11), we notice the occurrence of the term

\[ -\nabla(\alpha\beta)_{\gamma\delta}^{\delta(\text{Order}2)} = [2\delta_{\alpha}\delta_{\beta}\lambda + \sigma^g_{\alpha\beta}] \nabla_{\gamma\delta}^{(2)}. \]  

(14)

This was not properly considered in references [3]. In this work we find that there is no need to modify the spinor derivative of \( \chi_\alpha \) at second order so that an additional constraint on this derivative is unnecessary. For the solution of (11) we extract after some algebra, and neat cancelations, the candidates

\[ T^{(2)}_{\gamma\delta} = 2\gamma T^{(0)}_{\gamma\delta}f + 2\gamma f \Omega^{(1)}_{\gamma\delta} = A^{(1)}_{pqr}T^{(0)}_{d\gamma}f. \]  

(15)

\[ R^{(2)}_{\alpha\beta de} = -\frac{i\gamma}{12}\sigma^{pqre}_{\alpha\beta} A^{(1)}_{pqr}R^{(0)}_{d\gamma}f. \]  

(16)

We now must show that all of the above found results satisfy (3).

### 3  New Solution for \( R^{(2)}_{\lambda gde} \)

We must show that we can close equation (3) using the results (7), (9), (15), and (16). As mentioned, various approaches such as implementing the previously suggested constraints did not work, nor was there any way to manipulate the terms using the sigma matrix algebra. Eventually the following procedure provides a solution. At second order the Bianchi identity (3) becomes
\[
T^{(0)}_{(\alpha\beta)} R^{(2)}_{(\gamma)} \lambda de + T^{(2)}_{(\alpha\beta)} R^{(0)}_{(\gamma)} \lambda de - T^{(0)}_{(\alpha\beta)} R^{(2)}_{(\gamma)} g de - T^{(2)}_{(\alpha\beta)} R^{(0)}_{(\gamma)} g de - \nabla_{(\alpha)} \left[ R^{0}_{(\beta\gamma)} \right]_{de} \text{Order}(2) + R^{(1)}_{(\beta\gamma)} \text{Order}(2) + R^{(2)}_{(\beta\gamma)} \text{Order}(2) = 0.
\]  

(17)

Using the results listed above we arrive at

\[
- i \sigma^{g}_{(\alpha\beta)} R^{(2)}_{(\gamma)} g de + T^{(0)}_{(\alpha\beta)} \lambda \left[ - \frac{i \gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} R^{(0)}_{abde} \right] - \frac{i \gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} T_{ab} \lambda R^{(0)}_{(\gamma)} \lambda de + \frac{i \gamma}{6} \sigma^{pqrab}_{(\alpha\beta)} H^{(0)}_{g ab} A^{(1)}_{pqr} R^{(0)}_{(\gamma)} g de - \nabla_{(\gamma)} \left\{ -2 i \sigma^{g}_{(\alpha\beta)} \Pi^{(0)+(1)}_{\gamma g de} + \frac{i}{24} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} \right\} = 0.
\]  

(18)

Here we encounter second order contributions from zero order terms but in solvable form. (That is where we can extract a quantity symmetrized with a sigma matrix) We define

\[
\Pi^{\text{ef}} g = L^{\text{ef}} g - \frac{1}{8} A^{\text{ef}} g.
\]  

(19)

Now again using out key relation (8) we obtain

\[
- i \sigma^{g}_{(\alpha\beta)} R^{(2)}_{(\gamma)} g de + 2 i \gamma \sigma^{g}_{(\alpha\beta)} R^{(0)}_{abde} \left[ \Omega^{(1)}_{(\gamma)gab} \right] - \nabla_{(\gamma)} \left\{ -2 i \sigma^{g}_{(\alpha\beta)} \Pi^{(0)+(1)}_{\gamma g de} \right\} - \frac{i \gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} T_{ab} \lambda R^{(0)}_{(\gamma)} \lambda de + \frac{i \gamma}{6} \sigma^{pqrab}_{(\alpha\beta)} H^{(0)}_{g ab} A^{(1)}_{pqr} R^{(0)}_{(\gamma)} g de + \frac{i \gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} \left[ \nabla_{(\gamma)} R^{(0)}_{abde} \right] - \frac{i}{24} \sigma^{pqrab}_{(\alpha\beta)} \left[ \nabla_{(\gamma)} A^{(1)} \text{Order}(2) \right] = 0
\]  

(20)

Of particular concern and interest is the last term in (20). It was thought that a possible modification of \( A^{(1)}_{pqr} \), or a contribution from \( A^{(2)}_{pqr} \) would be necessary. Here we may avoid such a modification. In advance we anticipate that the solution will be as follows:

\[
+ i \sigma^{g}_{(\alpha\beta)} R^{(2)}_{(\gamma)g de} = + 2 i \gamma \sigma^{g}_{(\alpha\beta)} R^{(0)}_{abde} \left[ \Omega^{(1)}_{(\gamma)g ab} \right] + \nabla_{(\gamma)} \left\{ 2 i \sigma^{g}_{(\alpha\beta)} \Pi^{(0)+(1)}_{\gamma g de} \right\} \text{Order}(2)
\]  

(21)

And
\[
- \frac{i\gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} T_{ab} \lambda R^{(0)}_{(\gamma)\lambda de} + \frac{i\gamma}{6} \sigma^{pqrab}_{(\alpha\beta)} H^{(0)}_{ab} A^{(1)}_{pqr} R^{(0)}_{(\gamma) gde}
+ \frac{i\gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} A^{(1)}_{pqr} [\nabla_{(\gamma)} R^{(0)}_{abde}] - \frac{i}{24} \sigma^{pqr}_{de(\alpha\beta)} [\nabla_{(\gamma)} A^{(1)}_{pqr}(\text{Order}(2))] = 0 \quad (22)
\]

We need to show that (22) does in fact vanish. We must begin with the Bianchi identity that gives the spinor derivative of \( T_{kl}^\tau \).

\[
\nabla_\gamma T_{kl}^\tau = T_{[k}^{\lambda} T_{l] \lambda}^\tau + T_{[k}^{\gamma} T_{g[l]}^\tau + T_{kl}^{\lambda} T_{\lambda \gamma}^\tau + T_{kl}^{\gamma} T_{q \gamma}^\tau - \nabla_{[k} T_{l] \gamma}^\tau - R_{kl}^\gamma \tau \quad (23)
\]

At first order this simplifies to

\[
\nabla_\gamma T_{kl}^\tau \text{Order}(1) = -R_{kl}^\gamma \tau \quad (24)
\]

We now write the last term in (22), using the ten dimensional metric so that the unsolved part becomes

\[
- \frac{i}{12} \sigma^{pqrab}_{(\alpha\beta)} \{ \gamma A^{(1)}_{pqr} T_{ab} \lambda R^{(0)}_{(\gamma)\lambda de} + T^{(0)}_{ab} R^{(0)}_{(\gamma) gde} - \nabla_{(\gamma)} R^{(0)}_{abde} \}
+ \frac{1}{2} \eta_{ad} \eta_{be} \nabla_{(\gamma)} A^{(1)}_{pqr}(\text{Order}(2)) = 0 \quad (25)
\]

Using the definition of \( A^{(1)}_{pqr} \) (6), yields

\[
+ \frac{\gamma}{12} \sigma^{pqrab}_{(\alpha\beta)} \sigma_{pqrst} T^{kle} \{ \gamma T_{kl}^\tau [T_{ab} \lambda R^{(0)}_{(\gamma)\lambda de} + T^{(0)}_{ab} R^{(0)}_{(\gamma) gde} - \nabla_{(\gamma)} R^{(0)}_{abde}] \\
+ \eta_{ad} \eta_{be} \nabla_{(\gamma)} T_{kl}^\tau \} = 0 \quad (26)
\]

We now use equation (24) and the properties of the sigma matrices. After some algebra we choose to impose a condition on \( R^{(1)}_{kl\gamma} \). We require

\[
R^{(1)}_{kl\gamma} = \frac{\gamma}{100} T_{kl}^\tau [T_{mn} \lambda R^{(0)}_{(\gamma)\lambda mn} + T^{(0)}_{mn} g R^{(0)}_{(\gamma) mn} - \nabla_{(\gamma)} R^{(0)}_{mnmn}] \\
- \frac{1}{48} [2H^{(0)}_{klg} \sigma^{g \lambda \sigma rst \lambda r} A^{(1)}_{rst} - 2 \sigma[k_{\gamma} \lambda \sigma rst \lambda r] \nabla_{(\gamma)} A^{(1)}_{rst}] \quad (27)
\]

This can now be added to the list of first order results quoted in [1]. \( R^{(1)}_{kl\gamma} \) was not defined in [1]. We obtain as we required,

\[
R^{(2)}_{\gamma gde} = 2\gamma R^{(0)}_{abde} [\Omega^{(1)}_{\gamma g} ab] + 2 \nabla_{\gamma} [\Pi^{(0)}_{(0)+(1)} gde] \text{Order}(2) \quad (28)
\]
As a check we can also examine another Bianchi identity. The following Bianchi identity also includes $R^{(2)}_{abcd}$:

$$
\frac{1}{4} R_{\alpha|amn} \sigma^{mn}_{|\beta} \gamma + T_{\alpha \beta} g T_{ga} \gamma + T_{\alpha \beta} \lambda T_{\lambda a} \gamma + T_{a(\alpha} \lambda T_{\beta)\lambda} \gamma - T_{a(\alpha} g T_{|\beta)g} \gamma \\
- \nabla_{(\alpha} T_{|\beta)a} \gamma - \nabla_a T_{\alpha \beta} \gamma = 0 \quad (29)
$$

This Bianchi identity after some cancelations results in the following expression:

$$
\frac{1}{4} R^{(2)}_{(\alpha|amn} \sigma^{mn}_{|\beta)\gamma} + 2 \gamma \{ \nabla_{(\alpha} \Omega^{(1)}_{ae f}} \} \left[ -\frac{1}{4} R^{(0)}_{ef} \sigma^{mn}_{|\beta) \gamma} \right] \\
+ i \sigma^g_{\alpha \beta} T^{(2)}_{ga} \gamma - \frac{i \gamma}{6} \sigma^{pqref}_{\alpha \beta} A^{(1)}_{pqr} H^{(0)}_{ef} T^{(0)}_{ga} \gamma + T^{(0)}_{\alpha \beta} \lambda T^{(2)}_{\lambda a} \gamma \\
- 2 \gamma \tau^{(0)}_{ef} \{ \nabla_{(\alpha} \nabla_{|\beta)} \{ \Omega^{(1)}_{ae f}} \} \\
+ i \gamma \sigma^{pqref}_{\alpha \beta} \{ \nabla_a A^{(1)}_{pqr} \} T^{(0)}_{ef} \gamma + \frac{i \gamma}{12} \sigma^{pqref}_{\alpha \beta} A^{(1)}_{pqr} \{ \nabla_a T^{(0)}_{ef} \gamma \} \\
+ \left[ \delta_{(\alpha}^{\lambda} \delta_{|\beta)}^{\phi} + \sigma^g_{\alpha \beta} g^{\lambda \phi} \right] \nabla_a \chi^{\phi \text{Order2}} \\
+ \frac{1}{4} \sigma^{mn}_{(\alpha} \gamma \nabla_{|\beta)} \Pi_{amn} \text{Order(2)} = 0 
$$

This identity also predicts the same form for $R^{(2)}_{aamn}$. However it also includes a great deal of other information which we plan to include in another letter.

4 Conclusions

We have found a consistent and manifestly supersymmetric solution to the Bianchi identities for D=10, N=1 supergravity, with string corrections to second order in the slope parameter. We have reaffirmed the results and the proposed X tensor of [3] and [4], and we have solved the remaining previously intractable curvature. We have used the first

$$
T_{ab}^{\delta} = -\frac{1}{48} \sigma_{ba\lambda} \sigma^{pqr} \lambda^{\delta} A_{pqr} \quad (31)
$$

at second order. This is a conventional constraint and so could have been imposed to all orders. However we dropped it in favor of requiring an adjustment to $T_{ab}^{\delta(2)}$ as given by equation (15).
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6 Appendix I

Here for convenience we list the torsions curvatures and H sector results to second order, simply by including the results found at first order in [1].

\[ H_{\alpha\beta\gamma} = 0 + \text{Order}(\gamma^3) \] (32)

\[ H_{\alpha\beta\delta} = \frac{i}{2} \sigma_{\delta\alpha\beta} + 4i\gamma \sigma^g_{\alpha\beta} H_{\gamma}{}^{ef} H_{d}{}^{ef} \]

\[ \sigma_{\alpha\beta}{}^g \left[ 8i\gamma \left( H^{(0)}_d{}^{ef} L^{(1)}_g + i\gamma \left( H^{(0)}_d{}^{ef} A^{(1)}_g \right) \right) \right] \]

\[ + \sigma^{pqr} \alpha\beta \left[ \frac{i\gamma}{12} \left( H^{(0)}_d{}^{ef} A^{(1)}_pqr \right) + \text{Order}(\gamma^3) \right] \] (33)

\[ H_{\alpha ab} = +2i\gamma \left[ -\sigma_{[a]}{}^{\alpha\beta} T_{ef}{}^{\beta} G_{[b]}{}^{ef} - 2\sigma_{\epsilon\alpha\beta} T_{f}{}^{[\alpha} G_{\beta]}{}^{ef} \right] \]

\[ + 2\gamma \left[ \nabla_{[a} \left( H^{(0)}_d{}^{ef} H^{(0)}_{[b]} \right) - \sigma_{\epsilon a b} \phi \nabla_{[a b} H^{(0)}_{d} H^{(0)}_{g f} \right] \]

\[ + 2i\gamma \sigma_{[a]}{}^{\alpha\beta} T_{ef}{}^{\beta} \phi(1)_{[b]}{}^{ef} - 2i\gamma \sigma_{\epsilon a b} \lambda \sigma_{[a}{}^{\beta} T_{ef}{}^{\epsilon} \phi(1)_{b]}{}^{ef} \]

\[ - \frac{\gamma}{6} \sigma_{[a}{}^{\alpha} \phi \sigma_{[b]}{}^{\beta} T_{ef}{}^{\lambda} \phi(1)_{g]{}^{ef} - \frac{\gamma}{6} \sigma_{[a}{}^{\alpha} \sigma_{b]{}^{\phi} T_{ef}{}^{\epsilon} \phi(1)_{g]}{}^{ef} \]

\[ + 4\gamma R^{(1)}_{[a}{}^{ef} H^{(0)}_{g]ef} + T^{(2)}_{\alpha ab} + \text{Order}(\gamma^3) \] (34)

\[ T_{\alpha\beta}{}^g = i\sigma_{\alpha\beta}{}^g - \frac{\gamma}{6} \sigma^{pqr} \alpha\beta H^{(0)}_{d}{}^{ef} A^{(1)}_{pqr} + \text{Order}(\gamma^3) \] (35)

\[ T_{abc} = -2L_{abc} \] (36)

\[ T_{\alpha\beta}{}^\gamma = -[\delta_{(\alpha}]{}^{\gamma} \delta_{[\beta]}{}^\delta + \sigma^g_{\alpha\beta} \sigma_{g}{}^{\gamma\delta}] \chi_{\delta} - \frac{i\gamma}{12} \sigma^{pqr} \alpha\beta A^{(1)}_{pqr} T_{ef}{}^{\gamma} + \text{Order}(\gamma^3) \] (37)
\[ T_{\alpha g}^\delta = -\frac{1}{48}\sigma_{g\alpha\sigma}^{\delta \phi\rho\phi}A_{\rho\phi}^{(1)} + 2\gamma T(0)_{\sigma\phi\delta}^\gamma \Omega_{\alpha\sigma\phi}^{(1)} + \text{Order}(\gamma^3) \] (38)

\[ \sigma_g^{(\alpha\beta)}T^{(2)}_{\gamma gd} = 4\gamma^\sigma\alpha\gamma\Omega_{\gamma gd}^f H(0)_{d ef}^f - \frac{i\gamma}{6}\sigma_g^{(\alpha\beta)}\sigma^{pqre}_{g\gamma}A_{pq}^{(1)} T(0)_{de}^0 \] (39)

Or symmetrized,

\[ T_{\gamma ab} = +2\gamma[\Omega_{(1)}^{\gamma[a\phi]f} H(0)_{b\phi}^f + \sigma_{ab\gamma^\phi}^\delta \frac{\gamma}{3}\Omega_{\gamma df}^f] \]

\[ -\frac{\gamma}{6}\sigma_{[a\gamma}^{\phi} \phi 하 \Omega_{(1)\phi b\phi d}^f H(0)_{\phi d}^f + \Omega_{\gamma df}^f \phi 하 \Omega_{\gamma df}^f] \]

\[ -\frac{i\gamma}{12}\sigma_{pqg}^{\phi\lambda} A^{(1)}_{pqrs} \phi \lambda T(0)_{de}^\phi \]

\[ \frac{i\gamma}{144}\gamma_{pqg}^{\phi\lambda} T(0)_{de}^\phi + \sigma_{pqg}^{\phi\lambda} T(0)_{\phi\lambda} + \text{Order}(\gamma^3) \] (40)

\[ R_{\alpha\beta\gamma\delta} = -2i\sigma_{\alpha\beta}^g \Pi_{def}^{(1)} + \frac{i}{24}\sigma_{def}^{pq\alpha\beta} A_{pq}^{(1)} \]

\[ -\frac{i\gamma}{12}\sigma_{pqg}^{\phi\lambda} A^{(1)}_{pqrs} \phi \lambda T(0)_{de}^\phi + \text{Order}(\gamma^3) \] (41)

Where

\[ \Pi_{def}^{(1)} = L_{def}^{(1)} \]

\[ R_{detd} = -i\sigma_{d\alpha\beta}^g L_{\alpha\betade}^{(1)} + \frac{i}{24}\sigma_{def}^{pq\alpha\beta} A_{pq}^{(1)} \]

\[ -\frac{i\gamma}{12}\sigma_{pqg}^{\phi\lambda} A^{(1)}_{pqrs} \phi \lambda T(0)_{de}^\phi + \text{Order}(\gamma^3) \] (42)

\[ R_{agde} = -i\sigma_{d\alpha\beta}^g T_{\alpha\betade}^{(1)} + i\gamma\sigma_{d\alpha\beta}^g T_{kl}^\phi R_{kl}^{(1)} \]

\[ +2\gamma R(abde)_{0}^{(1)} + 2\nabla_{\alpha} \{ \Pi_{(0)+1}^{(1)} gde \}^{(2)} + \text{Order}(\gamma^3) \] (43)

\[ A_{abc} = i\gamma\sigma_{g\phi\lambda}^m T^{mn\gamma} T_{mn}^\lambda \] (44)

\[ R_{lt\phi}^{(1)} = +\frac{\gamma}{100} T_{lt}^\phi T^{\phi\lambda} R(0)_{\gamma\lambda}^{m\phi} + T^{(0)}_{\gamma\lambda} R(0)_{\gamma\lambda}^{m\phi} - \nabla_{\gamma} R(0)_{\gamma\lambda}^{m\phi} \]

\[ -\frac{1}{144}[2H(0)_{kl\phi}^{\gamma\lambda\\\gamma\lambda} T^{(1)}_{rst} + 2\sigma_{k\gamma\lambda}^{\gamma\lambda} T^{(1)}_{rst} - 2\sigma_{k\gamma\lambda}^{\gamma\lambda} T^{(1)}_{rst} \nabla_{\phi} A(1)_{rst}] \] (45)

The spinor derivative of $L_{abc}$ is solved and available from a Bianchi identity. We will list it in a later paper. $R_{lt\phi}^{(2)}$ if it exits will likely show up from third order calculations of (3).
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