New Results on Equilibria in Strategic Candidacy

Jérôme Lang\textsuperscript{1}, Nicolas Maudet\textsuperscript{2}, and Maria Polukarov\textsuperscript{3}

\textsuperscript{1} LAMSADE, Université Paris-Dauphine, Paris, France
lang@lamsade.dauphine.fr
\textsuperscript{2} LIP6, Université Pierre et Marie Curie, Paris, France
nicolas.maudet@lip6.fr
\textsuperscript{3} University of Southampton, United Kingdom
mp3@ecs.soton.ac.uk

Abstract. We consider a voting setting where candidates have preferences about the outcome of the election and are free to join or leave the election. The corresponding candidacy game, where candidates choose strategically to participate or not, has been studied in very few papers, mainly by Dutta et al. \cite{dutta2007candidate, dutta2009candidate}, who showed that no non-dictatorial voting procedure satisfying unanimity is candidacy-strategyproof, or equivalently, is such that the joint action where all candidates enter the election is always a pure strategy Nash equilibrium. They also showed that for voting trees, there are candidacy games with no pure strategy equilibria. However, no results were known about other voting rules. Here we prove several such results. Some are positive (a pure strategy Nash equilibrium is guaranteed for Copeland and the uncovered set, whichever the number of candidates, and for all Condorcet-consistent rules, for 4 candidates). Some are negative, namely for plurality and maximin. We also study the existence of strong equilibria and relate it to a stronger variant of candidate control.

1 Introduction

Two main issues for the evaluation of voting rules is their ability to resist various sorts of strategic behaviour and to adapt to changes in the environment. Many (if not most) papers in computational social choice deal with (at least) one of both issues. Strategic behaviour can come from the voters reporting insincere votes (manipulation); from a third party, typically the chair, acting on the set of voters or candidates (control), or on the votes (bribery and lobbying), or on the voting rule (e.g., agenda control\footnote{There are also some forms of strategic behaviour that are specific to multiwinner elections, such as gerrymandering (by the chair) or vote pairing (by the voters).}; finally, it may come from the candidates themselves, who may have preferences as well about the outcome of the election. However, strategic behaviour by the candidates has received less attention in (computational) social choice than strategic behaviour by the voters and (to a lesser extent) by the chair. One form thereof involves choosing optimal political platforms. But probably the simplest form comes from the ability of candidates to decide whether to run for the election or not, which is the issue we address here. The following table summarizes this rough classification of strategic behavior in voting, according to the identity of the strategizing agent(s) and also to another relevant dimension, namely...
what the strategic actions bear on—voters, votes or candidates (we omit the agenda to keep the table small).

| actions → agents | voters | votes | candidates |
|------------------|--------|-------|------------|
| voters           | strategic participation | manipulation | -          |
| third party / chair | voter control | bribery, lobbying | candidate control, cloning |
| candidates       | -      | -     | strategic candidacy |

Strategic candidacy does happen frequently in real-life elections, both in large-scale political elections and in small-scale, low-stake elections (e.g., electing a chair in a research group, or—moving a little bit away from elections—reputation systems). Throughout the paper we consider a finite set of potential candidates, which we simply call candidates when this is not ambiguous, and we make the following assumptions:

1. each candidate may choose to run or not for the election;
2. each candidate has a preference ranking over candidates;
3. each candidate ranks himself on top of his ranking;
4. the candidates’ preferences are common knowledge among them;
5. the outcome of the election as a function of the set of candidates who choose to run is common knowledge among the candidates.

With the exception of 3, these assumptions were also made in the original model of Dutta et al. [4] which we discuss below. Assumption 2 amounts to saying that a candidate is interested only in the winner of the election[3] and has no indifferences or incomparabilities. Assumption 3 (considered as optional in [4]) is a natural domain restriction in most contexts. Assumptions 4 and 5 are common game-theoretic assumptions; note that we do not have to assume that the candidates know precisely how voters will vote, nor even the number of voters; they just have to know the choice function mapping every subset of candidates to a winner.

Existing work on strategic candidacy is rather scarce. It starts with [4] and [5], that formulate the strategic candidacy game and prove the following results (among others): (i) no non-dictatorial voting procedure satisfying unanimity is candidacy-strategyproof—or equivalently, is such that the joint action where all candidates enter the election is always a pure strategy Nash equilibrium; (ii) for the specific case of voting trees, there are candidacy games with no pure strategy Nash equilibria. These results are discussed further (together with simpler proofs) [6], and extended to voting correspondences [8], [14] and to probabilistic voting rules [13].

Many questions remain unsolved. In particular, studying the solution concepts (such as Nash equilibria or strong equilibria) of a candidacy game would help predicting the set of actual candidates and therefore the outcome of the vote, and therefore help to better design elections. However, little is known about this: we only know that for

---

5 In some contexts, candidates may have more refined preferences that bear for instance on the number of votes they get, how their score compares to that of other candidates, etc. We do not consider these here.
any reasonable voting rule, there are some candidacy games for which the set of all candidates is not a Nash equilibrium, and that for voting trees, there exist a candidacy game with no pure strategy Nash equilibrium.

In this paper, we go further in this direction and prove some positive as well as some negative results. We first consider the case of 4 candidates and show that a pure strategy Nash equilibrium always exists for Condorcet-consistent rules. Then we show that for Copeland and uncovered set there is always an equilibrium in pure strategies, whichever the number of candidates (although strong equilibria are not guaranteed to exist). On the negative side, we show that for plurality, for at least 4 candidates, and for maximin for at least 5 candidates, there are candidacy games without Nash equilibria.

Although it seems that strategic candidacy has not been considered yet in computational social choice, it is related to some questions that have received some attention in this community. First, the existence of strong equilibria is highly related to a stronger variant of candidate control, termed consenting control, where candidates have their word to say about being deleted or added. Other somewhat less related works that also consider a dynamic set of candidates are candidate cloning [7], possible winners with new candidates [3], and the unavailable candidate model [11].

The paper unfolds as follows. In Section 2 we define the strategic candidacy games and give a few preliminary results. In Section 3 we focus on the case of 4 candidates, whereas the case of 5 candidates or more is considered in Section 4. Section 5 relates the candidacy game to candidate control. Finally, in Section 6 we discuss further issues.

2 Model and preliminaries

In this section, we formally define the model of strategic candidacy and show that it induces a normal form game. We then present two simple results on the existence of Nash equilibria and strong equilibria in this setting.

2.1 Voting rules

For completeness, we first define the common voting rules that we study in this paper.

There is a set of $n$ voters electing from a set of $m$ candidates. A single vote is a strict ordering of the candidates. A voting rule takes all the votes as input, and produces an outcome—a candidate, called the winner of the election. Although voting rules are usually defined for a fixed number of candidates, here we naturally extend the definition to an arbitrary number of candidates. All voting rules we consider in this work are deterministic: we first define their nondeterministic version and assume that ties are broken up according to a fixed priority relation over candidates. Because voting rules are applied to varying sets of candidates, we assume that tie-breaking rule is defined for the whole set of potential candidates, and projected to smaller sets of candidates; in other terms, if $x$ has priority over $y$ when all potential candidates run, this will still be the case for any set of candidates that contains $x$ and $y$.

The plurality winner is the candidate ranked first by the largest number of voters. The Borda winner is the candidate who gets the highest Borda score: for each voter, a candidate $c$ receives $m - 1$ points if it is ranked first by that voter, $m - 2$ if it is ranked
second, and so on; the Borda score $B(c)$ of $c$ is the total number of points he receives from all the voters.

Let $N(c, x)$ be the number of votes that rank $c$ higher than $x$. The majority graph associated with a set of votes is the graph whose vertices are the candidates and containing an edge from $x$ to $y$ whenever $N(x, y) > \frac{n}{2}$ (when this holds we say that $x$ beats $y$). A candidate $c$ is a Condorcet winner if $x$ beats $y$ for all $y \neq x$. A voting rule is Condorcet-consistent if it always elects a Condorcet winner when one exists.

The maximin rule chooses the candidate $c$ for whom $\min_{x \in X \setminus \{c\}} N(c, x)$ is maximal. The Copeland$^0$ (resp., Copeland$^1$) rule elects the candidate $c$ maximizing the number of candidates $x$ such that $N(c, x) > \frac{n}{2}$ (resp., $N(c, x) \geq \frac{n}{2}$). The uncovered set (UC) rule selects the winner from the uncovered set of candidates: a candidate $c$ belongs to the uncovered set if and only if, for any other candidate $x$, if $x$ beats $c$ then $c$ beats some $y$ that beats $x$.

### 2.2 Strategic candidacy

There is a set $X = \{x_1, x_2, \ldots x_m\}$ of $m$ potential candidates, and a set $V = \{1, 2, \ldots n\}$ of $n$ voters. We assume that voters and candidates are disjoint. As is classical in social choice theory, each voter $i \in V$ has a preference, $P_i$, over the different candidates—i.e., a strict order ranking the candidates. The combination $P = (P_1, P_2, \ldots, P_n)$ of all the voters’ preferences defines their preference profile.

Furthermore, each candidate has also a strict preference ordering over the candidates. We naturally assume that the candidates’ preferences are self-supported—that is, the candidates rank themselves at the top of their ordering. Let $P^X = (P^X_c)_{c \in X}$ denote the candidates’ preference profile. Following $P^X$, the potential candidates may decide to enter an election or withdraw their candidacy. Thus, the voters will only express their preferences over a subset $Y \subseteq X$ of the candidates that will have chosen to participate in the election, and we denote by $P^{1Y}$ the restriction of $P$ to $Y$. We assume that the voters are sincere.

Given a profile $P$ of the voters’ preferences, a voting rule $r$ defines a (single) winner among the actual candidates—i.e., given a subset $Y \subseteq X$ of candidates, it assigns to a (restricted) profile $P^{1Y}$ a member of $Y$. Each such voting rule $r$ induces a natural game form, where the set of players is given by the set of potential candidates $X$, and the strategy set available to each player is $\{0, 1\}$ with 1 corresponding to entering the election and 0 standing for withdrawal of candidacy. A state $s$ of the game is a vector of strategies $(s_c)_{c \in X}$, where $s_c \in \{0, 1\}$. For convenience, we use $s_{-z}$ to denote $(s_c)_{c \in X \setminus \{z\}}$—i.e., $s$ reduced by the single entry of player $z$. Similarly, for a state $s$ we use $s_Z$ to denote the strategy choices of a coalition $Z \subseteq X$ and $s_{-Z}$ for the complement, and we write $s = (s_z, s_{-z})$.

The outcome of a state $s$ is $r\left(P^{1Y}\right)$ where $c \in Y$ if and only if $s_c = 1$\(^6\). Coupled with a profile $P^X$ of the candidates’ preferences, this defines a normal form game $\Gamma = \langle X, P, r, P^X \rangle$ with $m$ players. Here, player $c$ prefers outcome $\Gamma(s)$ over outcome $\Gamma(s')$ if ordering $P^X$ ranks $\Gamma(s)$ higher than $\Gamma(s')$.

---

\(^6\) When clear from the context, we use vector $s$ to also denote the set of candidates $Y$ that corresponds to state $s$; e.g., if $X = \{x_1, x_2, x_3\}$, we note $\{x_1, x_3\}$ and $(1,0,1)$ interchangeably.
2.3 Game-theoretic concepts

Having defined a normal form game, we can now apply standard game-theoretic solution concepts. Let \( \Gamma = (X, P, r, P^X) \) be a candidacy game, and let \( s \) be a state in \( \Gamma \). We say that a coalition \( Z \subseteq X \) has an improving move in \( s \) if there is \( s'_Z \) such that \( \Gamma(s-Z, s'_Z) \) is preferable over \( \Gamma(s) \) by every player \( z \in Z \). In particular, the improving move is unilateral if \(|Z| = 1\). A (pure strategy) Nash equilibrium (NE) \([12]\) is a state that has no unilateral improving moves. More generally, a state is a \( k \)-NE if no coalition with \(|Z| \leq k\) has an improving move. A strong equilibrium (SE) \([1]\) is a state that has no improving moves.

Example 1. Consider the game \( \langle \{a, b, c, d\}, P, r, P^X \rangle \), where \( r \) is the Borda rule, and \( P \) and \( P^X \) are as follows:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
b & c & c & d & b & a & b & e \\
d & d & c & e & b & d & a & a \\
a & a & b & c & d & c & d & c \\
e & b & a & a & d & c & e & d \\
\end{array}
\]

The state \((1,1,1,1)\) is not an NE: \(abcd \mapsto c\), but \(abc \mapsto a\), and \(d\) prefers \(a\) to \(c\), so for \(d\), leaving is an improving move. Now, \((1,1,1,0)\) is an NE, as no one has an improving move neither by joining (\(d\) prefers \(a\) over \(c\)), or by leaving (obviously not \(a\); if \(b\) or \(c\) leaves then the winner is still \(a\)). It can be checked that this is also an SE.

2.4 Preliminary results

Regardless of the number of voters and the voting rule used, a straightforward observation is that a candidacy game with three candidates is guaranteed to possess a NE. Note that this does not hold for SE\(^8\).

The following result applies to any number of candidates, but is limited to Condorcet-consistent rules.

Proposition 1. Let \( \Gamma = (X, P, r, P^X) \) be a candidacy game where \( r \) is Condorcet-consistent. If \( P \) has a Condorcet winner \( c \) then for any \( Y \subseteq X \),

\[
Y \text{ is a SE } \iff Y \text{ is an NE } \iff c \in Y.
\]

The very easy proof can be found in Appendix. If \( P \) has no Condorcet winner, the analysis becomes more complicated. We provide results for this more general case in

---

7 In our examples, we assume a lexicographic tie-breaking. We also use the simplified notation \( Y \mapsto x \) to denote that rule \( r \) applied to the subset of candidates \( Y \subseteq X \) is \( x \), and we omit curly brackets. The first row in \( P \) indicates the number of voters casting the different ballots.

8 Here is a counterexample (for which we thank an anonymous reviewer of a previous version of the paper). The selection rule is \( abc \mapsto b; ab \mapsto a; ac \mapsto c; bc \mapsto c \); it can be easily implemented by the scoring rule with scoring vector \((5, 4, 0)\) with 5 voters. Preferences of candidates are: \(a : a \succ b \succ c; b : b \succ c \succ a; c : c \succ a \succ b\). The group deviations are: in \(\{a, b, c\}\), \(c\) leaves; in \(\{a, b\}\), \(b\) leaves and \(c\) joins; in \(\{a, c\}\), \(c\) joins; in \(\{b\}\), \(c\) joins; in \(\{c\}\), \(a\) and \(b\) join.
the following sections. Interestingly, as we demonstrate, some Condorcet-consistent rules (e.g., Copeland and UC) do always possess a Nash equilibrium in this case, while some other (e.g., maximin) do not.

3 Four candidates

With only 4 potential candidates, we exhibit a sharp contrast between Condorcet consistent rules, which all possess an NE, and scoring rules.

3.1 Scoring rules

To study scoring rules, we make use of a very powerful result by Saari [15]. It states that for almost all scoring rules, any conceivable choice function can result from a voting profile. This means that our question boils down to check whether a choice function, together with some coherent candidates’ preferences, can be found such that no NE exists with 4 candidates. We solved this question by encoding the problem as an Integer Linear Program (ILP), the details of which can be found in Appendix. It turns out that such choice functions do exist: it then follows from Saari’s result that counter-examples can be obtained for “most” scoring rules. We show this for plurality.

Proposition 2. For plurality and \( m = 4 \), there may be no NE.

Proof. We exhibit a counter-example with 13 voters, whose preferences are contained in the left part of the table below. The top line indicates the number of voters with each particular profile. The right part of the table represents the preferences of the candidates.

| a b c d | a b c d | a b c d | a b c d |
|--------|--------|--------|--------|
| d d d a a b b c | c b a b c d c a b | b a d a |
| a c b c b c d c | c c a b |
| b a c d d c a d | d b c |

Similar constructions of profiles can thus be obtained for other scoring rules. However, Borda comes out as a very peculiar case [15] among scoring rules. This is also verified for strategic candidacy.

Proposition 3. For Borda and \( m = 4 \), there is always an NE.

We could check this by relying on the fact that Borda rule is represented by a weighted majority graph, and by adding the corresponding constraints into the ILP. The infeasibility of the resulting set of constraints shows that no instances without NE can be constructed. However, it takes only coalitions of pairs of agents to ruin this stability.

\footnote{For a more detailed statement of this result, we point the reader to the work of Saari, in particular [16]. In a nutshell, for the case of 4 candidates, families of scoring rules such that, when the scoring vector for 3 candidates is of the form \( \langle w_1, w_2, 0 \rangle \), the vector for 4 candidates is of the form \( \langle 3w_1, w_1 + 2w_2, 2w_2, 0 \rangle \) are an exception. For instance, the scoring rule \( \langle \langle 3, 1, 0, 0 \rangle, \langle 1, 0, 0 \rangle, \langle 1, 0 \rangle \rangle \) satisfies this.}
Remark 1. For Borda and $m = 4$, there may be no 2-NE.

Proof. Consider the following game:

\[\begin{array}{cccc}
\text{abcd} & \text{bcd} & \text{d} & \text{a} \\
\text{bcd} & \text{c} & \text{d} & \text{a} \\
\text{d} & \text{a} & \text{c} & \text{a} \\
\text{a} & \text{b} & \text{d} & \text{d} \\
\end{array}\]

Here, only $s_1 = (0, 1, 1, 1)$ and $s_2 = (1, 1, 0, 1)$ are NE, with $bcd \mapsto b$, and $abd \mapsto d$. But from $s_1$ the coalition $\{a, c\}$ has an improving move to $s_2$ as they both prefer $d$ over $b$. Now take $s_2$: if $b$ leaves and $c$ joins, they reach $(1, 0, 1, 1)$, with $acd \mapsto c$ and both prefer $c$ over $d$. □

3.2 Condorcet-consistent rules

We now turn our attention to Condorcet-consistent rules. It turns out that for all of them, the existence of a NE can be guaranteed.

Proposition 4. For $m = 4$, if $r$ is Condorcet-consistent there always exists a NE.

Proof. We start with a remark: although we do not assume that $r$ is based on the majority graph, we nevertheless prove our result by considering all possible cases for the majority graphs (we get back to this point at the end of the proof). There are four graphs to consider (all others are obtained from these ones by symmetry).

\[\begin{array}{cccc}
G_1 & G_2 & G_3 & G_4 \\
\begin{array}{c}
a \mapsto b \\
c \mapsto d
\end{array} & \begin{array}{c}
a \mapsto b \\
c \mapsto d
\end{array} & \begin{array}{c}
a \mapsto b \\
c \mapsto d
\end{array} & \begin{array}{c}
a \mapsto b \\
c \mapsto d
\end{array}
\end{array}\]

For $G_1$ and $G_2$, any subset of $X$ containing the Condorcet winner is an NE (see Proposition [9]). For $G_3$, we note that $a$ is a Condorcet loser. That is, $N(a, x) < N(x, a)$ for all $x \in \{b, c, d\}$. Note that in this case, there is no Condorcet winner in the reduced profile $P_{\{b, c, d\}}^1$ as this would imply the existence of a Condorcet winner in $P$ (case $G_1$ or $G_2$). W.l.o.g., assume that $b$ beats $c$, $c$ beats $d$, and $d$ beats $b$. W.l.o.g. again, assume that $bcd \mapsto b$. Then, $\{b, c\}$ is an NE. Indeed, in any set of just two candidates, none has an incentive to leave. Now, $a$ or $d$ have no incentive to join as this would not change the winner; in the former case, observe that $b$ is the (unique) Condorcet winner in $P_{\{a, b, c\}}$, and the latter follows by our assumption. There is always an NE for $G_3$.

The proof for $G_4$ is more complex and proceeds case by case. Since $r$ is Condorcet-consistent, we have $acd \mapsto a$, $bcd \mapsto c$, $ab \mapsto b$, $ac \mapsto a$, $ad \mapsto a$, $bc \mapsto c$, and $bd \mapsto d$. The sets of candidates for which $r$ is undetermined are $abcd$, $abc$ and $abd$.

We have the following easy facts: (i) if $abcd \mapsto a$ then $abd$ is an NE, (ii) if $abcd \mapsto c$ then $bcd$ is an NE, (iii) if $abc \mapsto a$ then $ac$ is an NE, (iv) if $abcd \mapsto a$ then $ad$ is an NE, (v) if $abc \mapsto c$ then $bc$ is an NE. The only remaining cases are:
1. $abcd \leftrightarrow b, abc \leftrightarrow b, abd \leftrightarrow b$.
2. $abcd \leftrightarrow b, abc \leftrightarrow b, abd \leftrightarrow d$.
3. $abcd \leftrightarrow d, abc \leftrightarrow b, abd \leftrightarrow b$.
4. $abcd \leftrightarrow d, abc \leftrightarrow b, abd \leftrightarrow d$.

In cases 1 and 3, $ab$ is an NE. In case 2, if $a$ prefers $b$ to $c$ then $abc$ is an NE, and if $a$ prefers $c$ to $b$, then $bcd$ is an NE. In case 4, if $a$ prefers $c$ to $d$, then $bcd$ is an NE; if $b$ prefers $a$ to $d$, then $ad$ is an NE; finally, if $a$ prefers $d$ to $c$ and $b$ prefers $d$ to $a$, then $abcd$ is an NE. To conclude, observe that the proof never uses the fact that two profiles having the same majority graph have the same winner. □

The picture for 4 candidates shows a sharp contrast. On the one hand, the existence of choice functions shows that “almost all scoring rules” [15] may fail to have an NE. On the other hand, Condorcet-consistency alone suffices to guarantee the existence of a NE. (However this criteria is not sufficient to guarantee stronger notion of stability: e.g., for Copeland, we could exhibit examples without any 2-NE).

4 More candidates

The first question which comes to mind is whether examples showing the absence of NE transfer to larger set of candidates. They indeed do, under an extremely mild assumption. We say that a voting rule is insensitive to bottom-ranked candidates (IBC) if given any profile $P$ over $X = \{x_1, \ldots, x_m\}$, if $P'$ is the profile over $X \cup \{x_{m+1}\}$ obtained by adding $x_{m+1}$ at the bottom of every vote of $P$, then $r(P') = r(P)$. This property is extremely weak (much weaker than Pareto) and is satisfied by almost all voting rules studied in the literature (a noticeable exception being the veto rule).

**Lemma 1.** For any voting rule $r$ satisfying IBC, if there exists $\Gamma = \langle X, P, r, P_X \rangle$ with no NE, then there exists $\Gamma' = \langle X', P', r, P_Y \rangle$ with no NE, where $|X'| = |X| + 1$.

**Proof.** Take $\Gamma$ with no NE, with $X = \{x_1, \ldots, x_m\}$. Let $X' = X \cup \{x_{m+1}\}$; $P'$ the profile obtained from $P$ by adding $x_{m+1}$ at the bottom of every vote, and $P_Y$ be the candidate profile obtained by adding $x_{m+1}$ at the bottom of every ranking of a candidate $x_i$, $i < m$, and whatever ranking for $x_{m+1}$. Let $Y \subseteq X$. Because $Y$ is not an NE for $\Gamma$, some candidate $x_i \in X$ has an interest to leave or to join, therefore $Y$ is not an NE either for $\Gamma'$. Now, consider $Y' = Y \cup \{x_{m+1}\}$. If $x_i \in X$ has an interest to leave (resp., join) $Y$, then because $r$ satisfies IBC, the winner in $Y' \setminus \{x_i\}$ (resp., $Y' \cup \{x_i\}$) is the same as in $Y \setminus \{x_i\}$ (resp., $Y \cup \{x_i\}$), therefore $x_i \in X$ has an interest to leave (resp., join) $Y'$, therefore $Y'$ is not an NE. □

**Corollary 1.** For plurality and $m \geq 4$, there may be no NE.

We now turn our attention to Condorcet-consistent rules, which all admit NE with 4 candidates. However, 5 candidates suffice to show that NE are not guaranteed any longer.

---

10 For instance, we may have two profiles $P, P'$ both corresponding to $G_4$, such that $r(P) = a$ and $r(P') = b$; the proof perfectly works in such a case.
Proposition 5. For maximin with $m = 5$ there may be no NE.

Proof. The counterexample is the following weighted majority graph along with the candidates’ preference profile.

| a | b | c | d | e |
|---|---|---|---|---|
| $a$ | 1 | 4 | 2 | 3 |
| $b$ | 4 | $-$1 | 4 | 3 |
| $c$ | 1 | 4 | $-$2 | 2 |
| $d$ | 3 | 1 | 3 | 0 |
| $e$ | 2 | 2 | 3 | 5 |

We have $abde \rightarrow c$, the maximin winner. Furthermore, $abcd \rightarrow a, abde \rightarrow b, abce \rightarrow e, acde \rightarrow a, bede \rightarrow c, ...$  □

Corollary 2. For maximin and $m \geq 5$, there may be no NE.

This negative result does not extend to all Condorcet-consistent rules. In particular, next we show the existence of NE for Copeland and the uncovered set (UC) rules, under deterministic tie-breaking, for any number of candidates.

Proposition 6. For Copeland$^0$ and Copeland$^1$, with any number of candidates, there is always an NE.

Proof. We only give the proof for Copeland$^0$. The proof for Copeland$^1$ is the same, replacing majority by weak majority.

Let $P$ be a profile and $\rightarrow_P$ its associated majority graph. Let $C(x, P)$ be the number of candidates $y \neq x$ such that $x \rightarrow_P y$. Let $\text{COP}^0(P)$ be the set of the Copeland$^0$ cowinners for $P$, i.e., the set of candidates maximizing $C(\cdot, P)$, and $\text{Cop}^0(P) = c$ the Copeland$^0$ winner, i.e., the most prioritary candidate in $\text{COP}^0(P)$. Consider $\text{Dom}(c) = \{c\} \cup \{y | c \rightarrow_P y\}$. Note that $C(c, \text{P}_1\text{Dom}(c)) = |\text{Dom}(c)| - 1 = q = C(c, P)$. Also, since any $y \in \text{Dom}(c)$ is beaten by $c$, we have $C(y, \text{P}_1\text{Dom}(c)) \leq q - 1$.

We claim that $\text{Dom}(c)$ is an NE. Note that $c$ is a Condorcet winner in the restriction of $P$ to $\text{Dom}(c)$, and a fortiori, in the restriction of $P$ to any subset of $\text{Dom}(c)$. Hence, $c$ is the Copeland$^0$ winner in $\text{Dom}(c)$ and any of its subsets, and no candidate in $\text{Dom}(c)$ has an incentive to leave.

Now, assume there is a candidate $z \in X \setminus \text{Dom}(c)$ such that $\text{Cop}^0\left(\text{P}_1\text{Dom}(c) \cup \{z\}\right) \neq c$. Note that $c \not\rightarrow_P z$ as $z$ does not belong to $\text{Dom}(c)$; so, $C(c, \text{P}_1\text{Dom}(c) \cup \{z\}) = q$. For any $y \in \text{Dom}(c)$ we have $C(y, \text{P}_1\text{Dom}(c) \cup \{z\}) \leq (q-1) + 1 = q = C(c, \text{P}_1\text{Dom}(c) \cup \{z\})$. If $C(y, \text{P}_1\text{Dom}(c) \cup \{z\}) \leq C(c, \text{P}_1\text{Dom}(c) \cup \{z\})$, then $y$ is not the Copeland$^0$ winner in $\text{P}_1\text{Dom}(c) \cup \{z\}$. If $C(y, \text{P}_1\text{Dom}(c) \cup \{z\}) = C(c, \text{P}_1\text{Dom}(c) \cup \{z\})$, then $C(y, P) \geq C(c, P)$. That is, either $c \notin \text{COP}^0(P)$, a contradiction, or both $y, c$ are in $\text{COP}^0(P)$.

In that case, the tie-breaking priority ensures that $\text{Cop}^0\left(\text{P}_1\text{Dom}(c) \cup \{z\}\right) \neq y$.

Hence, $\text{Cop}^0\left(\text{P}_1\text{Dom}(c) \cup \{z\}\right) = z$. By $\text{Cop}^0(P) = c$ we have $C(z, \text{P}_1\text{Dom}(c) \cup \{z\}) \leq C(z, P) \leq q$; therefore, $C(z, \text{P}_1\text{Dom}(c) \cup \{z\}) = q$ and the tie-breaking priority favors $z$ over $c$. But then, $C(z, P) = C(c, P)$, i.e., both $c$ and $z$ are in $\text{COP}^0(P)$, and the tie-breaking priority ensures that $\text{Cop}^0\left(\text{P}_1\text{Dom}(c) \cup \{z\}\right) \neq z$, a contradiction. Therefore, the Copeland$^0$ winner in $\text{P}_1\text{Dom}(c) \cup \{z\}$ must be $c$, which implies that $z$ has no incentive to join $\text{Dom}(c)$.
Note that if the number of voters is odd, we do not have to care about head-to-head ties. In this case, all Copeland rules, where agents in a head-to-head election each get $0 \geq \alpha \geq 1$ points in the case of a tie (Copeland and Copeland$^1$ are special cases), are equivalent, and the result above holds. However, if the number of voters is even, this is not necessarily the case. Thus, in particular, for Copeland$^{0.5}$ (more often referred to as Copeland), $Dom(c)$ is generally no more a NE.

**Proposition 7.** For UC, with any number of candidates, there is always an NE.

**Proof.** Let $c$ be the UC winner in $P$, i.e., the most prioritary candidate in $UC(P)$. Consider (again) $Dom(c) = \{c\} \cup \{y|c \rightarrow_P y\}$. We claim that $Dom(c)$ is an NE.

Since $c$ is a Condorcet winner in the restriction of $P$ to $Dom(c)$, and a fortiori, in the restriction of $P$ to any subset of $Dom(c)$, it is the UC winner in $Dom(c)$ and in any of its subsets, and no candidate in $Dom(c)$ wants to leave.

Now, let $z \in X \setminus Dom(c)$. Since $z \notin Dom(c)$, we have $c \not\rightarrow_P z$. If $z \rightarrow_P c$, then since $x \in UC(P)$, there must be $y \in Dom(c)$ such that $y \rightarrow_P z$. This implies that $x \in UC(p^1|Dom(c)\cup\{z\})$, which, due to tie-breaking priority, yields that $c$ is the UC winner in $p|Dom(c)\cup\{z\}$. Thus, $z$ has no incentive to join $Dom(c)$. \qed

# 5 Strong Equilibria and Consenting Control

Now we briefly discuss the connection to control by deleting or adding candidates. Bartholdi et al. [2] define constructive control by deleting candidates (CCDC) and constructive control by adding candidates (CCAC): an instance of CCDC consists of a profile $P$ over set of candidates $C$, a distinguished candidate $c$, an integer $k$, and we ask whether there is a subset $C'$ of $C$ with $|C \setminus C'| \leq k$ such that $c$ is the unique winner in $C'$. An instance of CCAC consists of a profile $P$ over set of candidates $C_1 \cup C_2$, a distinguished candidate $c$, and we ask whether there is a subset $C'$ of $C_2$ such that the unique winner in $C_1 \cup C'$ is $c$. Destructive versions of control are defined by Hemaspaandra [10]: destructive control by deleting (DCDC) is similar to CCDC, except that we ask whether there is a subset $C'$ of $C \setminus \{c\}$ with $|C \setminus C'| \leq k$ such that $c$ is not the unique winner in $C \setminus C'$; and destructive control by adding candidates (DCAC) is similar to CCAC, except that $c$ should not be the unique winner in $C'$. There are also multimode versions of control [9]: e.g., CC(DC+AC) allows the chair to delete some candidates and to add some others (subject to some cardinality constraints).

Nash equilibria and strong equilibria in strategic candidacy relate to a slightly more demanding notion of control, which we can call consenting control, and that we find an interesting notion per se. In traditional control, candidates have no preferences and no choice—the chair may add or delete them as he likes. An instance of consenting CCDC consists of an instance of CCDC plus, for each candidate in $C$, a preference ranking over $C$, and we ask whether there is a subset $C'$ of $C$ with $|C \setminus C'| \leq k$ such that $c$ is the unique winner in $C'$, and every candidate in $C \setminus C'$ prefers $c$ to the candidate which would win if all candidates in $C$ were running. An instance of consenting CCAC consists of an instance of CCAC plus, for each candidate in $C_2$, a ranking over $C_1 \cup C_2$, and we ask whether there is a subset $C'$ of $C_2$ such that $c$ is the unique winner in $C_1 \cup C'$ and every candidate in $C'$ prefers $c$ to the candidate which would win if only the
candidates in $C_1$ were running. Consenting versions of destructive control are defined similarly: here the goal is to have a different candidate from the current winner elected.

Clearly, for profile $P$, $(1, \ldots, 1)$ is an SE iff there is no consenting destructive control by removing candidates against the current winner $r(X)$, with the value of $k$ being fixed to $m$ (the chair has no limit on the number of candidates to be deleted; the limits come here from the fact that the candidates must consent), and $(1, \ldots, 1)$ is an NE iff there is no consenting destructive control by removing candidates against the current winner $r(X)$, with the upper bound of $k = 1$ on the number of candidates to be deleted.

For candidate sets that are different from the set $X$ of all candidates (as some may leave and some other may join), we have to resort to consenting destructive control by removing+adding candidates, as in [9]. Let $s$ be a state and $X_s$ the set of running candidate in $s$: $s$ is an SE if there is no consenting destructive control by removing+adding candidates against the current winner $r(X_s)$, without any constraint on the number of candidates to be removed or added. For an NE, this is similar, but with the bound $k = 1$ on the number of candidates to be deleted or added.

For some versions of control, it is not difficult to show that consenting control is, computationally, at least as hard as standard control (the proof is in Appendix).

**Proposition 8.** For any (deterministic) voting rule $r$, consenting DCDC (respectively, DCAC, CCAC) is at least as hard as DCDC (respectively, DCAC, CCAC).

Because of the restriction to $k = m$, we can unfortunately not derive a hardness result about deciding whether a state is an SE from hardness of control. We can only say that deciding whether $X$ is a $k$-NE is NP-hard for a given voting rule $r$ if DCDC is NP-hard for $r$. We would have to inspect the proofs of these results about control to see whether the reductions can be easily adapted so that $k = m - 1$.

6 Conclusions

We have explored further the landscape of strategic candidacy in elections by obtaining several positive results (for Condorcet-consistent rules with 4 candidates; for two versions of Copeland, as well as for uncovered set, with any number of candidates) and several negative results (for plurality and maximin); we have also given a strong connection to a new variant of candidate control, which we believe to have its own interest.

Many open problems remain, especially Borda with more than 4 candidates. Another issue for further research is the study of the set of states that can be reached by some (e.g. best response) dynamics starting from the set or all potential candidates. In some cases, even when the existence of NE is guaranteed (e.g. for Copeland), we could already come up with examples such that none is reachable by a sequence of best responses. But other types of dynamics may be studied. Another issue for further research is the computational complexity of deciding whether there is an NE or SNE.

Obtaining a similar result for consenting CCDC is not easy, and in fact we don’t know whether it is possible; the difficult point is that if the current winner is $a$ while the candidate the chair wants to make win is $c$, a successful control may involve the deletion of $a$, who, clearly, does not consent to leave.
Acknowledgements

We would like to thank Michel Le Breton and Vincent Merlin for helpful discussions.

References

1. R. Aumann. Acceptable points in general cooperative n-person games. In Contributions to the Theory of Games IV, volume 40 of Annals of Mathematics Study, pages 287–324, 1959.
2. J. Bartholdi, C. Tovey, and M. Trick. How hard is it to control an election? Social Choice and Welfare, 16(8-9):27–40, 1992.
3. Y. Chevaleyre, J. Lang, N. Maudet, J. Monnot, and L. Xia. New candidates welcome! possible winners with respect to the addition of new candidates. Mathematical Social Sciences, 64(1):74–88, 2012.
4. B. Dutta, M. L. Breton, and M. O. Jackson. Strategic candidacy and voting procedures. Econometrica, 69:1013–1037, 2001.
5. B. Dutta, M. L. Breton, and M. O. Jackson. Voting by successive elimination and strategic candidacy in committees. Journal of Economic Theory, 103:190–218, 2002.
6. L. Ehlers and J. A. Weymark. Candidate stability and nonbinary social choice. Economic Theory, 22(2):233–243, 2003.
7. E. Elkind, P. Faliszewski, and A. M. Slinko. Cloning in elections: Finding the possible winners. J. Artif. Intell. Res. (JAIR), 42:529–573, 2011.
8. H. Eraslan and A. McLennan. Strategic candidacy for multivalued voting procedures. Journal of Economic Theory, 117(1):29–54, 2004.
9. P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. Multimode control attacks on elections. JAIR, 40:305–351, 2011.
10. E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Anyone but him: The complexity of precluding an alternative. Artificial Intelligence, 171(5-6):255–285, 2007.
11. T. Lu and C. Boutilier. The unavailable candidate model: a decision-theoretic view of social choice. In ACM Conference on Electronic Commerce 2010, pages 263–274, 2010.
12. J. Nash. Non-cooperative games. Annals of Mathematics, 54(2):286–295, 1951.
13. C. Rodriguez-Alvarez. Candidate stability and probabilistic voting procedures. Economic Theory, 27(3):657–677, 2006.
14. C. Rodriguez-Alvarez. Candidate stability and voting correspondences. Social Choice and Welfare, 27(3):545–570, 2006.
15. D. Saari. A dictionary of voting paradoxes. Journal of Economic Theory, 48, 1989.
16. D. Saari. Election results and a partial ordering for positional ordering. In N. J. Schofield, editor, Collective Decision-Making: Social Choice and Political Economy, pages 93–110. Kluwer, 1996.
Appendix A: Proofs

**Proposition 9.** Let \( Γ = (X, P, r, P^X) \) be a candidacy game where \( r \) is Condorcet-consistent. If \( P \) has a Condorcet winner \( c \) then for any \( Y \subseteq X \),
\[ Y \text{ is a SE} \iff Y \text{ is an NE} \iff c \in Y. \]

**Proof.** Assume \( c \) is a Condorcet winner for \( P \) and let \( Y \subseteq X \) such that \( c \in Y \). Because \( r \) is Condorcet-consistent, and because \( c \) is a Condorcet winner for \( P|_Y \), we have \( r(P|_Y) = c \). Assume \( Z = Z^+ \cup Z^- \) is a deviating coalition from \( Y \), with \( Z^+ \) the candidates who join and \( Z^- \) the candidates who leave the election. Clearly, \( c \notin Z \), as \( c \in Y \) and \( c \) has no interest to leave. Therefore, \( c \) is still a Condorcet winner in \( P|_{Y \setminus Z^-} \cup Z^+ \), which by the Condorcet-consistency of \( r \) implies that \( r(P|_{Y \setminus Z^-} \cup Z^+) = c \), which contradicts the assumption that \( Z \) wants to deviate. We thus conclude that \( Y \) is an SE, and a fortiori an NE. Finally, let \( Y \subseteq X \) such that \( c \notin Y \). Then, \( Y \) is not an NE (and a fortiori not an SE), because \( c \) has an interest to join the election. \( \Box \)

**Proposition 8** For any (deterministic) voting rule \( r \): consenting DCDC (respectively, DCAC, CCAC) is at least as hard as DCDC (respectively, DCAC, CCAC)

**Proof.** For the first point, we give a polynomial reduction from DCDC to consenting DCDC. Construct the instance \( F(I) \) of consenting DCDC by adding to \( I \) the candidates’ preferences, defined as follows: \( c \) prefers himself to all others (in an arbitrary order); any \( x \neq c \) prefers himself to all others except \( c \) (in an arbitrary order), \( c \) being her worst choice. If there is a DCDC against \( c \), then there exists a subset \( C' \subseteq C \setminus \{c\} \) such that \(|C'| \leq k \) and the winner from \( C' \) is not \( c \). All candidates in \( C \setminus C' \) are happy with the new outcome and consent to leave, therefore \( F(I) \) is a positive instance of consenting DCDC. The converse implication is straightforward (a consenting DCDC is a DCDC).

The proofs for the other two points are similar, starting respectively from DCAC and CCAC. For DCAC, every candidate \( x \) in \( C_2 \) prefers himself to all others except \( c \) (in an arbitrary order), \( c \) being her worst choice. For CCAC, every candidate \( x \) in \( C_2 \) prefers himself to \( c \), \( c \) to all others (in an arbitrary order) except the current winner \( a \) in \( C_1 \), \( a \) being her worst choice. \( \Box \)

Appendix B: ILP formulation

Let \( S \) be the set of \((2^{|X|})\) states, and \( A(s) \) be the set of agents candidating in state \( s \).

**Choice functions without any NE.** We introduce a binary variable \( w_{s,i} \), meaning that agent \( i \) wins in state \( s \). We add constraints enforcing that there is a single winner in each state \( s \):
\[
\forall i \in X, \forall s \in S : \ w_{s,i} \in \{0, 1\} \quad \quad (1) \\
\forall s \in S : \sum_{i \in X} w_{s,i} = 1 \quad \quad (2) \\
\forall s \in S, \forall i \in X \notin A(s) : \ w_{s,i} = 0 \quad \quad (3)
\]
Now, we shall introduce constraints related to deviations. We denote by \( D(s) \) the set of possible deviations from state \( s \) (state where a single agent’s candidacy differs from \( s \)). We also denote by \( a(s,t) \) an agent potentially deviating from \( s \) to \( t \).

We now define binary variables \( d_{s,t} \) indicating a deviation from a state \( s \) to a state \( t \). In each state, there must be at least one deviation otherwise this state must be a NE.

\[
\forall s \in S, \forall t \in S : \quad d_{s,t} \in \{0, 1\} \quad (4)
\]

\[
\forall s \in S : \sum_{t \in D(s)} d_{s,t} \geq 1 \quad (5)
\]

Now we introduce constraints related to the preferences of the candidates. For this purpose, we introduce a binary variable \( p_{i,j,k} \), meaning that agent \( i \) prefers candidate \( j \) over candidate \( k \). If there is indeed a deviation from \( s \) to \( t \), the deviating agent must prefer the winner of the state new state compared to the winner of the previous state:

\[
\forall s \in S, \forall t \in D(s), \forall i \in X, \forall j \in X : w_{s,i} + w_{t,j} + d_{s,t} - p_{a(s,t),j,i} \leq 2 \quad (6)
\]

Finally we ensure that the preferences are irreflexive and transitive, and respect the constraint of being self-supported.

\[
\forall i \in X, \forall j \in X : \quad p_{i,j,j} = 0 \quad (7)
\]

\[
\forall a \in X, \forall i \in X \forall j \in X, \forall k \in X : p_{a,i,j} + p_{a,j,k} - p_{a,i,k} \leq 1 \quad (8)
\]

\[
\forall i \in X, \forall j \in X : \quad p_{i,i,j} = 1 \quad (9)
\]

**Constraints for Borda.** We introduce a new integer variable \( N_{i,j} \) to represent the number of voters preferring \( i \) over \( j \) in the weighted tournament. We first make sure that the values of \( N_{i,j} \) are coherent throughout the weighted tournament.

\[
\forall i \in X, \forall j \in X, \forall k \in X, \forall l \in X : N_{i,j} + N_{j,i} = N_{k,l} + N_{l,k} \quad (10)
\]

In each state, when agent \( i \) wins, we must make sure that her total amount of points is the highest among all agents in this state (note that \( i \) can simply tie with agents it has priority over in the tie-breaking, we omit this for the sake of readability):

\[
\forall s \in S, \forall i \in A(s), \forall j \in A(s) \setminus \{i\} : \quad (1 - w_{s,i}) \times M + \sum_{j \in A(s) \setminus \{i\}} N_{i,j} > \sum_{j \in A(s) \setminus \{i\}} N_{j,i} \quad (11)
\]

Here \( M \) is an arbitrary large value, used to relax the constraint when \( w_{s,i} \) is 0.