Telegraph-type versus diffusion-type models of turbulent relative dispersion

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Properties of two equations describing the evolution of the probability density function (PDF) of the relative dispersion in turbulent flow are compared by investigating their solutions: the Richardson diffusion equation with the drift term and the self-similar telegraph equation derived by Ogasawara and Toh [J. Phys. Soc. Jpn. 75, 083401 (2006)]. The solution of the self-similar telegraph equation vanishes at a finite point, which represents persistent separation of a particle pair, while that of the Richardson equation extends infinitely just after the initial time. Each equation has a similarity solution, which is found to be an asymptotic solution of the initial value problem. The time lag has a dominant effect on the relaxation process into the similarity solution. The approaching time to the similarity solution can be reduced by advancing the time of the similarity solution appropriately. Batchelor scaling, a scaling law relevant to initial separation, is observed only for the telegraph case. For both models, we estimate the Richardson constant, based on their similarity solutions.

I. INTRODUCTION

The relative dispersion of particle pairs is fundamental to the turbulent research and has many practical applications to environmental and industrial problems, such as the transport of pollutants in the atmosphere and the fuel mixing in engines. The study of turbulent relative dispersion has a long history since the pioneering work by Richardson, who observed the anomalous dispersion or superdiffusion of particle pairs and proposed the diffusion equation describing the evolution of the probability density function (PDF) for the pair separation. Several attempts to modify the Richardson model have been presented. Recent experiments and direct numerical simulations (DNSs) showed that the Richardson model can well represent the separation PDF in the inertial subrange. Nevertheless, the best model of the relative dispersion has not yet been determined decisively. This is because the inertial ranges achieved both in experiments and DNSs up to now are not wide enough to observe the full superdiffusive behavior.

The relative dispersion in turbulent flows has often been modeled with the Reynolds-number dependence and the contribution from the three ranges of turbulence, namely the energy-dissipative, the inertial and the energy-containing ranges. Indeed, the unified description of the relative dispersion is important especially for comparison of data among models, experiments, and DNSs at moderate Reynolds number. However, we limit our attention to the relative dispersion in the infinitely-extended inertial range, namely in the case that \( \text{Re} \rightarrow \infty \), where the superdiffusive behavior can be observed, and discuss the governing equation of the separation PDF there. As is already mentioned, the Richardson model is supported by several experiments and DNSs. However, there are a few shortcomings in this model. One of those of the Richardson model is the absence of the long-time correlation of the Lagrangian relative velocity of a particle pair, which exists in real turbulent flows. This correlation gives rise to persistent separation of pairs, which should obey the self-similarity as shown in the recent DNS.

In order to overcome this shortcoming, Ogasawara and Toh devised another model of turbulent relative dispersion, the self-similar telegraph model. This model is derived taking into account the persistency of pair separation on the basis of Sokolov’s picture, where the relative separation consists of persistent expansion and compression. As a result, the governing equation of the PDF of relative separation in this model has a second-order time derivative term. This is the reason why this model is called ‘telegraph,’ while in the other previous theories such as the Richardson model the governing equation of PDF is diffusion-type. The term ‘self-similar’ comes from scale-dependent coefficients reflecting the self-similarity of the dispersion process in the inertial range. Owing to this inclusion of the self-similarity, solutions of the self-similar telegraph equation do not approach those of its corresponding diffusion equation, whereas those of the usual telegraph equation appearing in the problems of molecular diffusion or Brownian motion quickly relax into those of the diffusion equation. This indicates that the self-similar telegraph model has essentially distinct properties from those of its diffusion-type counterpart.

In this paper, we investigate time-integrated solutions of the self-similar telegraph equation as well as those of its diffusion-type counterpart. In the literature, the realizability of the similarity solution of the Richardson diffusion equation has implicitly been assumed, and thereby the agreements between the similarity solution and the separation PDF obtained by experiments or DNSs have been discussed. However, the similarity solution becomes a delta function at the origin in the limit \( t \to 0 \). This requires that a particle pair be located at the same place at \( t = 0 \). This condition is difficult to set in experiments and DNSs. In addition, the viscous effects in the dissipation range may also contaminate the pure inertial-range behavior of the dispersion process. For this reason, the initial separation of a pair is finite in experiments and DNS, and hence the initial condition with a nonzero separation is desired.
Furthermore, the recent experiments\textsuperscript{17} suggested that the separation PDF obtained by their experiment agreed well with the similarity solution of Batchelor’s diffusion equation rather than Richardson’s mentioned above. However, their claim is contradictory in that the similarity solution of the Batchelor diffusion equation implies the usual Richardson scaling law for the mean-square separation \( \langle r^2 \rangle \propto t^3 \), whereas they found the (modified) Batchelor scaling law instead of it, as was also reported in Ref. \textsuperscript{18}. These results stimulate us to investigate the short time behavior of the solution of the Richardson diffusion equation with finite initial separation. To our knowledge, this issue has never been treated in the literature.

Motivated by the above, we numerically solve the governing equations of the PDF under appropriate initial conditions with finite separations, and observe the behaviors of the time-integrated solutions. Comparisons of the solutions between the telegraph-type and the diffusion-type model are also made to reveal the characteristics of the models.

The remainder of the paper is organized as follows. In Sec. \textbf{II} we introduce the self-similar telegraph equation together with its diffusion-type counterpart. Section \textbf{III} explains the settings of the simulation. The results of the simulation is presented in Sec. \textbf{IV} where the Richardson constants are also obtained and compared for the two models. In Appendix \textbf{A} the decrement of the total probability is examined for the self-similar telegraph model, because the conservation of it is not guaranteed for this model.

\section{II. SELF-SIMILAR TELEGRAPH AND PALM EQUATIONS}

In this section, we introduce the self-similar telegraph model of turbulent relative dispersion. In this model, the evolution of the spherically symmetric PDF of pair separation \( P(r, t) \) in \( d \)-dimensional isotropic turbulence is described by the following self-similar telegraph equation\textsuperscript{12}

\[ \frac{T_c(r)}{\lambda} \frac{\partial P}{\partial t} + \frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left[ D(r)(r^{d-1} \frac{\partial}{\partial r} \left( \frac{P}{r^{d-1}} \right)) \right] + \sigma \frac{\partial}{\partial r} [v(r)P], \tag{1} \]

where \( T_c(r) \), \( D(r) \) and \( v(r) \) are a characteristic time scale, a diffusion coefficient and the Lagrangian relative velocity, respectively, of a particle pair of separation \( r \). The parameters \( \lambda \) and \( \sigma \) characterize the turbulent field considered. The coefficients \( T_c(r) \), \( D(r) \) and \( v(r) \) are assumed to obey the following scaling laws:

\begin{align*}
T_c(r) &= \tilde{A}^{-1}r^s, \tag{2a} \\
D(r) &= \tilde{A}^{-1}r^{2-s}, \tag{2b} \\
v(r) &= \tilde{A}r^{1-s}. \tag{2c}
\end{align*}

Here, \( \tilde{A} \) is a dimensional constant and \( s \) a scaling exponent: \( s = 2/3 \) for Kolmogorov scaling and \( s = 2/5 \) for Bolgiano-Obukhov scaling. The last term on the r.h.s. of Eq. \textbf{(1)} is a drift term, with the drift velocity being \( -\sigma v(r) \). \( \lambda^{-1} \) represents the persistency of the separation, and corresponds to the persistent parameter introduced by Sokolov\textsuperscript{13}.

In the derivation of Eq. \textbf{(1)}\textsuperscript{12}, a parameter \( \delta \) was also used, and has a relation with \( \lambda \) and \( \sigma \) via \( \lambda \sigma = d - 2s + \delta \). The physical meaning of \( \delta \) is the difference between the two transition rates of the direction of the separation, from expansion to compression and from compression to expansion. Hereafter, we use \( \delta \) as a control parameter instead of \( \sigma \), following the previous papers\textsuperscript{15,19}.

If the effects of the finite separation and the finite correlation of the relative velocity are not considered, the first term on the l.h.s. of Eq. \textbf{(1)} is omitted, and then Eq. \textbf{(1)} can be reduced to the following diffusion equation:

\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left[ D(r)(r^{d-1} \frac{\partial}{\partial r} \left( \frac{P}{r^{d-1}} \right)) \right] + \sigma \frac{\partial}{\partial r} [v(r)P]. \tag{3} \]

This equation has the same form as the Richardson diffusion equation with the drift term. If \( \sigma = 0 \), i.e. \( \delta = 2s - d \), Eq. \textbf{(3)} is the Richardson diffusion equation itself. The addition of the drift term to the Richardson equation was discussed by Palm in 1957\textsuperscript{20}. Recently, Goto and Vassilicos\textsuperscript{21} derived the same equation as Eq. \textbf{(3)}. We refer to Eq. \textbf{(3)} as Palm equation.

Before we perform the time integration of Eqs. \textbf{(1)} and \textbf{(3)}, the nondimensionalization is made for convenience. We normalize space by the nonzero initial relative separation \( R (\neq 0) \) of particle pairs and time by the corresponding timescale \( T_c(R) \). Consequently, new dimensionless space and time variables are defined as \( \tilde{r} = r/R \) and \( \tilde{t} = t/T_c(R) = \tilde{A}t/R^s \) respectively. We also introduce the normalized separation PDF \( \tilde{P}(\tilde{r}, \tilde{t}) = RP(r, t) \). Substituting Eqs. \textbf{(2)} into Eqs. \textbf{(1)} and \textbf{(3)}, eliminating the dimensional quantities and dropping the superposed tildes from the dimensionless quantities give the following nondimensionalized versions of the equations:

\[ \tilde{r}^s \frac{\partial^2 P}{\partial \tilde{t}^2} + \frac{\partial P}{\partial \tilde{t}} = \frac{\partial}{\partial \tilde{r}} \left[ \tilde{r}^{1-s+d} \frac{\partial}{\partial \tilde{r}} \left( \frac{P}{\tilde{r}^{d-1}} \right) \right] + (d - 2s + \delta) \frac{\partial}{\partial \tilde{r}} (\tilde{r}^{1-s} P), \tag{4} \]
To mimic this initial condition, we employed the following finite-width PDF as an alternative initial condition:

\[ P(r,0) = \begin{cases} 
\frac{1}{2w} \left( 1 + \cos \left( \pi \frac{r-1}{w} \right) \right) & |r-1| < w \\
0 & |r-1| > w,
\end{cases} \]  

(6)

where \( w \) represents the width of the initial PDF. As is mentioned in the last part of Sec. II, the spatial scale is normalized by the initial separation, so that the initial PDF is distributed around 1. Hereafter, we regard 1 as the typical initial relative separation. Note that this value cannot be strictly referred to as the mean value since the normalized by the initial separation, so that the initial PDF is distributed around 1. Hereafter, we regard 1 as the typical initial relative separation. Note that this value cannot be strictly referred to as the mean value since the PDF should be spherically integrated to obtain the mean value of relative separation in two or three dimension. In the following, we consider the two cases, \( w = 1 \) and 0.1, to see the effects of the width of the initial PDF. Narrower widths than 0.1 cause the numerical instability of the scheme for the present interval of grids and steeper ridges at the bounds of the time-integrated solution of the self-similar telegraph equation (see Subsec. IV A and Fig. II).

For the self-similar telegraph equation (4), another initial condition \( \partial P(r,t)/\partial t|_{t=0} = 0 \) is required. This condition implies the symmetry between the extending and compressing pairs at the initial time, which reflects the situation where the initial placement of pairs is uncorrelated with the velocity field of turbulent flow. Finally, we set the boundary conditions to \( P(0,t) = P(\infty,t) = 0 \).

C. Numerical method

Time integration of Eqs. (4) and (5) is performed with the Crank-Nicolson scheme, which is second order accurate in time, after the transformation of the spatial variable. We define a new spatial coordinate \( r' \) as \( r'/s \) and \( r/(r+L) \) for Eqs. (4) and (5), respectively. Here \( L \) is an arbitrary parameter, which determines "local density" of grids in \( r \) space. The former transformation leads to the constant Courant number \( c = r^{1-s}(\Delta t/\Delta r) = \Delta t/\Delta r' \) with uniform grids in \( r' \) space, while the latter transforms \( r' \) to the finite computational plane \( r' \in [0,1] \).

For the self-similar telegraph model, grids are added above the maximum grid of the computational plane and remove the grids of the same number as the added ones when the maximum separation of the PDF nearly reaches the maximum grid. In this way, the number of the grids is always conserved during the computation (although the interval between the grids are increased at every change in the grids). The grids of 82,000 and 100,000 were used to compute the solutions of Eqs. (4) and (5), respectively. For the self-similar telegraph model, we set the initial grid interval \( \Delta r'_{init} \) to 10^{-4}, while for the Palm model, we set \( L \) to 0.2 or 10 for the short-time and 100 or 1000 for the long-time behavior.
IV. RESULTS AND DISCUSSION

A. Short-time behavior

The time-integrated solutions of Eqs. (4) and (5) are shown in Fig. 1 at different times. While the solution of Eq. (5) extends infinitely at finite time, that of Eq. (4) bounds at a finite point. This point corresponds to the maximum separation of pairs, which is exactly obtained by direct integration of the equation \(dr/dt = v(r)\) with the scaling law of the velocity difference (2c) to give in the nondimensionalized form

\[r_{\text{max}}(t) = (r_{\text{max}}(0) + st)^{1/s}.\]  

The existence of the maximum point in the solution is a manifestation of the finite separation of a pair, which is included in the self-similar telegraph model.

For the case of \(w = 0.1\) (Figs. 1(a) and 1(c)) of the self-similar telegraph model, we can observe ridges at the edges of the PDF. The similar behavior has been found in the model of Ref. 24, based on a Lévy-walk stochastic approach. The PDF predicted by their model has peaks at the side edges. Since our approach is similar to theirs, the reason for the presence of the ridges should be consistent: the distribution of the relative velocity is not taken into account. Particle pairs having never been compressed or expanded from the initial time should be accumulated at these advancing or receding ridges, respectively. To see this, let \(r_{\text{max}}(0)\) in Eq. (7) be equal to 1, the center of the initial PDF. Then, \(r_{\text{max}}(1) \approx 2.32\), consistent with the position of the advancing ridge in Fig. 1(a). However, these ridges get smaller as time elapses owing to the effect of diffusion, i.e. the random changes of the direction of relative velocity. In Fig. 1(c), at \(t = 4\), the receding ridge disappears and only the remnant of the reduced advancing ridge can be observed in the case of \(w = 0.1\).
FIG. 2: Similarity solutions of the self-similar telegraph equation (4) (solid line) and the Palm equation (5) (dashed line) in the similarity form $F(\eta) = (t/\lambda)^{1/s}P_s(r,t)$. The similarity variable $\eta$ is normalized by $\eta_c \equiv s\lambda$, corresponding to the maximum separation for the self-similar telegraph equation (4). $s = 2/5$, $d = 2$, $\lambda = 5.2$ and $\delta = -0.77$. The inset is a linear plot of the same figure.

B. Long-time behavior

Equation (4) can be reduced to a second-order ordinary differential equation with the similarity variable 

\[ \eta = \lambda r^{2/5}/t. \]

Therefore it has a similarity solution, and so for Eq. (5). Although the similarity solution of Eq. (4) has no analytical form, that of Eq. (5) does and reads

\[ P_s(r,t) = C t^{1/2} \left( \frac{\lambda r^{4}}{s^2 t} \right)^{(2s-\delta-1)/s} \exp \left( -\frac{\lambda r^{4}}{s^2 t} \right), \]

where $C$ is the dimensionless normalization factor

\[ C = s \left( \frac{\lambda}{s^2} \right)^{1/2} \Gamma \left( \frac{2s-\delta}{s} \right), \]

and $\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt$ is the gamma function. The subscript $s$ of $P_s$ denotes the similarity solution. In Fig. 2 we show the similarity solutions of Eqs. (4) and (5) in the similarity form $F(\eta) = (t/\lambda)^{1/s}P_s(r,t)$, with the values of the parameters specified in Subsec. III A.

In contrast to the solution of the Palm equation, that of the self-similar telegraph equation has an upper bound in space, and vanishes for $\eta > \eta_c \equiv s\lambda$, which is attributed to the singular point of the reduced ordinary differential equation. This yields the maximum separation $r_{\text{max}}(t) = (st)^{1/s}$, which is also obtained by neglecting $r_{\text{max}}(0)$ in Eq. (7). The similarity solutions tend to a delta function at the origin in the limit $t \to 0$, i.e. $P_s(r,0) = \delta(r)$, as is expected by Eq. (5) and the fact that $r_{\text{max}}(0) = 0$ for the similarity solution of the self-similar telegraph equation. This initial condition is different from that adopted in the present paper. However, we inferred that the effect of the initial separation on the PDF is negligible after a long time, and thus the solutions of Eqs. (4) and (5) starting from our initial condition approach the corresponding similarity solutions. In order to confirm this, we further advanced the time-integrated solutions. The resulting PDFs are transformed into the similarity form $F(\eta)$ and are plotted in Fig. 3. From this figure, we found that the solutions with the initial condition approach the similarity ones as time elapses, and finally almost reach them at $t = 4092$, so that we cannot distinguish them. This suggests that the similarity solutions are asymptotic solutions of the initial value problem. The same behaviors as that shown in Fig. 3 can be obtained for $w = 0.1$.

In order to analyze this behavior quantitatively, we introduce the root-squared difference between the time-integrated and the similarity PDF as

\[ \|\delta P\| = \sqrt{\int_{0}^{\infty} \left| P(r,t) - P_s(r,t) \right|^2 dr}. \] (10)

This quantity vanishes if the separation PDF has the same form as that of the similarity solution. Thus, we can quantify the degree of the proximity of the two solutions by calculating this quantity.
FIG. 3: Time-integrated solutions of (a) the self-similar telegraph equation \( F(\eta) = (t/\lambda)^{1/s} P(r, t) \), under the initial condition \( \text{[6]} \) with \( w = 1 \), at time \( t = 12, 60, 252, 1020 \) and 4092 (from bottom to top at the maximum point), along with each similarity solution (solid line). The insets are linear plots of the same figures.

FIG. 4: Temporal evolution of the root-squared differences between the time-integrated and the similarity solutions, \( \|\delta P\| \) or \( \|\delta P\|_{lt} \), defined in (10) or (15), for both equations and widths. The upper curves denote \( \|\delta P\| \), while the lower \( \|\delta P\|_{lt} \): advancing times \( t_{st} \) in Eq. (15) are set to 8.5, 8.3, 8.3 and 8.2 for the telegraph with \( w = 0.1 \), the telegraph with \( w = 1 \), the Palm with \( w = 0.1 \) and the Palm with \( w = 1 \), respectively.

In Fig. 4 we plotted the temporal evolution of \( \|\delta P\| \) for both models and widths. From this figure, the scaling laws \( \|\delta P\| \propto t^{-\beta} \) can be found in both short-time and long-time regimes with \( \beta \approx 1.1 \) and 2.2, respectively. The transition between the two regimes occurs around \( t = 10 \). No qualitative difference between the two models is found. The difference between the two values of \( w \) in the initial condition \( \text{[6]} \) can scarcely be seen throughout the whole time scales shown in the figures. In this case, the relaxation processes into the similarity solutions appear to be rather universal, independent of both models and widths. However, as is illustrated below, these apparent universal behaviors are mainly due to the effect of time lag.

As an explanation of the origin of the above scaling behaviors, we consider the effect of time lag, because the similarity solutions are localized at the origin in the limit \( t \to 0 \), whereas the initial condition for the time-integrated solutions \( \text{[6]} \) implies nonzero initial separations. We assume that the initial condition \( \text{[6]} \) can be approximately described by the similarity solution at a certain time \( t_0 \). Then, its subsequent time evolution is the same as the original similarity solution after \( t_0 \) only for the Palm case, since for the telegraph case it depends on the additional condition \( \partial P(r, t)/\partial t|_{t=t_0} \). We can now replace \( P(r, t) \) by \( P_s(r, t+t_0) \) in the right hand side of Eq. (10), and substitute Eq. (8) to obtain

\[
\int_0^\infty |P_s(r, t + t_0) - P_s(r, t)|^2 dr = \frac{C^2}{2\gamma s} \left( \frac{s^2}{\lambda t} \right)^{1/s} \Gamma(\gamma) \left\{ 1 + \left( \frac{t}{t + t_0} \right)^{1/s} - 2 \left( \frac{2t}{2t + t_0} \right) \gamma \left( \frac{t + t_0}{t} \right)^{(2s-\delta-1)/s} \right\}^2.
\]
In Fig. 5, we show the approach of the time-integrated solutions to the corresponding similarity solutions advanced with $t_0 = 8.2$ for the square root of Eq. (11). Here, the time lag has the dominant effect on the decay of $\|\delta P\|$ in time.

By removing the effect of the time lag, more rapid approach of the time-integrated solution to the similarity solution can be realized. In Fig. 4 we also plotted the following quantity:

$$\|\delta P\|_{ul} = \sqrt{\int_0^\infty |P_s(r, t + t_{at}) - P(r, t)|^2 dr}.$$  

where

$$\gamma = \frac{4s - 2\delta - 1}{s}.$$  

We examine the two limiting cases for Eq. (11), namely the long-time and the short-time limits. The calculation for the former case is straightforward and gives

$$\int_0^\infty |P_s(r, t + t_0) - P_s(r, t)|^2 dr \simeq \frac{C^2}{2s^{\frac{3}{\gamma}}(\lambda t)^{\gamma s}} \Gamma(\gamma)(\gamma s^2 + 1) \left(\frac{t_0}{t}\right)^2 \quad (t_0 \ll t).$$  

For the latter case, it is obvious that

$$\int_0^\infty |P_s(r, t + t_0) - P_s(r, t)|^2 dr \simeq \int_0^\infty P^2_s(r, t)dr = \frac{C^2}{2s^{\frac{3}{\gamma}}(\lambda t)^{\gamma s}} \Gamma(\gamma) \quad (t_0 \gg t),$$  

since only the $P^2_s(r, t)$ term in the integrand of Eq. (11) diverges in the limit $t \to 0$. Equations (13) and (14) lead to the scaling laws $t^{-1-1/(2s)} = t^{-2.25}$ and $t^{-1/(2s)} = t^{-1.25}$, respectively, for the root-squared difference. This explains the above scaling behaviors of $\|\delta P\|$ in Fig. 4.

Figure 5 displays the result of the fitting of $\|\delta P\|$ in the case of the Palm model with $w = 1$, shown in Fig. 4, with the square root of Eq. (11). Here, the time lag $t_0$ of value about 8.2 for the square root of Eq. (11) gives the best fit of $\|\delta P\|$. The agreement is excellent. This result ensures our expectation that the time lag has the dominant effect on the similarity solution.

By removing the effect of the time lag, more rapid approach of the time-integrated solution to the similarity solution can be realized. In Fig. 4 we also plotted the following quantity:

$$\|\delta P\|_{ul} = \sqrt{\int_0^\infty |P_s(r, t + t_{at}) - P(r, t)|^2 dr}.$$  

Here, $t_{at}$ is an advancing time of the similarity solution, which is chosen such that $\|\delta P\|_{ul}$ has the minimum value around $t = 1000$. Specifically, $t_{at} = 8.5$, 8.3, 8.3 and 8.2 for the telegraph model with $w = 0.1$, the telegraph model with $w = 1$, the Palm model with $w = 0.1$ and the Palm model with $w = 1$, respectively. These values are consistent with the above optimal value of $t_0$ in Eq. (11), resulting from the fitting (Fig. 5). As is obvious from the figure, $\|\delta P\|_{ul}$ is reduced by two or three orders of magnitude compared with $\|\delta P\|$. Moreover, the algebraic decrease is no longer seen for both short and long times.

In Fig. 5 we show the approach of the time-integrated solutions to the corresponding similarity solutions advanced by $t_{at} = 8.3$ and 8.2 for the self-similar telegraph and the Palm model, respectively. From the comparison between Figs. 5 and 6, it is obvious that the time-integrated solutions approach the similarity solutions with the time lag much faster than those without the time lag, as is expected from Fig. 4. At $t = 252$, already, the time-integrated solutions almost accord with the similarity solutions. Hence, the approaching time to the similarity solution becomes shorter than that without the time lag as in Fig. 5 by one order of magnitude.
C. Batchelor scaling

For times shorter than the characteristic time of an eddy of size of the initial separation, the difference between the mean-square separation and its initial value is expected to grow in time as \( t^2 \), reflecting the initial ballistic or persistent motion of particle. This scaling law was first derived by Batchelor, and we refer to it as Batchelor scaling.

To see this Batchelor scaling for the present time-integrated solutions, we plot in Fig. 7 the mean-square separation \( \langle r^2(t) \rangle - \langle r^2(0) \rangle \) versus time for the above solutions. For the self-similar telegraph model this scaling law is clear. This result implies that the time derivative of the mean-square separation vanishes at the initial time for the self-similar telegraph model, which reminds us of the initial condition \( \partial P(r,t)/\partial t|_{t=0} = 0 \). However, the scaling law proportional to \( t \) is observed for the Palm model. This arises from its incapability to represent the condition \( d\langle r^2(t)\rangle/dt|_{t=0} = 0 \), or equivalently \( \partial P(r,t)/\partial t|_{t=0} = 0 \). Therefore, the Palm model cannot satisfy the initial symmetry condition of pair separation mentioned in Subsec. III.B. Notice that from Fig. 7, the behaviors for the two different initial widths are almost the same, and hence it is concluded that the presence of the ridges only in the case of \( w = 0.1 \) for the telegraph model at short times, shown in Figs. II(a) and II(c), does not affect this scaling law.

In the previous subsection, we showed that there exists the optimal time of the similarity solution for minimizing the relaxation time. This may lead to another scaling law of the mean-square separation. If we assume that the root mean-square separation obeys the same temporal evolution as that of the maximum separation except the coefficient of the time, we may write the mean-square separation as

\[
\langle r^2(t) \rangle = (\langle r^2(0) \rangle/s^{2/3} + ct)^{2/s},
\]

where \( c \) is a constant. This expression was also proposed by Goto and Vassilicos using the concept of the doubling
time, and was adopted in the recent paper. Equation (16) can be interpreted as the Richardson law with the advancing time, \( \langle r^2(t) \rangle \propto (t_{at} + t)^{2/s} \), if we regard \( \langle r^2(0) \rangle^{s/2} \) as \( ct_{at} \). This means that the scaling law (16) exactly holds for the similarity solution advanced by \( t_{at} \). In fact, Eq. (16) was employed to determine the Richardson constant in DNS with finite initial separations in Ref. 21.

In Fig. 8(a), this scaling law is observed for the Palm model. Note that this scaling law tells nothing about the Batchelor scaling. In fact, as is seen in Fig. 8(a), Eq. (16) is not satisfied by the self-similar telegraph model, which satisfies the Batchelor scaling.

Figure 8(b) shows the time dependence of the coefficient \( c \) in the right hand side of Eq. (16) for the Palm model with \( w = 1 \) and 0.1. The values of \( c \) slightly vary in the region \( 1 < t < 10 \), which means that Eq. (16) cannot exactly capture the behavior of the mean-square separations of the time-integrated solutions in the whole time regime. Moreover, the basic assumption \( \langle r^2(0) \rangle^{s/2} \simeq ct_{at} \) and the estimated value of the advancing time \( t_{at} \) in the previous subsection lead to the value of about 0.12 for \( c \) for each width of the initial distribution of pair separation. This value conflicts with that obtained in Fig. 8(b). Therefore, the advancing time \( t_{at} \) of the similarity solution, introduced in the previous subsection, does not have a significant meaning for the short time behavior. The variation of \( c \) is smaller for \( w = 1 \), implying that Eq. (16) is more appropriate for the smoother initial PDF, and becomes less appropriate as the initial PDF gets sharper and closer to the shape of the delta function. In the long-time limit this value is related to the Richardson constant, which is the subject of the following subsections.

D. Richardson scaling

For times much longer than the timescale of the initial separation and much shorter than the integral timescale, the well-known Richardson \( t^3 \) law of the mean-square separation is expected to hold in 3DNS or 2DIC turbulence governed by the Kolmogorov scaling. In the Bolgiano-Obukhov scaling, however, the mean-square separation should grow as \( t^5 \). This scaling law is clearly seen for the two models in Fig. 8 for the long time.

However, now that we previously showed the agreement of the time-integrated solutions with the similarity solutions in the long-time limit, we can use the similarity solutions to see that scaling law. From the similarity form of the PDF \( F(\eta) = (t/\lambda)^{1/s} P_s(r, t) \), it is easy to derive

\[
\langle r^2 \rangle = \left( \frac{t}{\lambda} \right)^{2/s} \frac{1}{s} \int \eta^{3/s-1} F(\eta) d\eta.
\]

Here, the integration on the right hand side is regarded as a constant for the similarity solutions, so that we can rewrite Eq. (17) as

\[
\langle r^2 \rangle = G \left( \frac{t}{\lambda} \right)^{2/s}.
\]

Thus we can find the \( t^5 \) dependence of the mean-square separation in the Bolgiano-Obukhov scaling \( s = 2/5 \).
FIG. 9: Long-time behaviors of the mean-square separations. The solid line refers to the self-similar telegraph model and the dashed one the Palm model. \( w = 1 \). The dotted line is proportional to \( t^5 \).

FIG. 10: The coefficient of the Richardson \( t^{2/s} \) law for the similarity solutions, \( G \), as a function of the parameters \( \lambda \) and \( \delta \). The upper dashed lines refer to the Palm model, Eq. (19), and the lower solid lines the self-similar telegraph model. (a): \( s = 2/5 \) for the Bolgiano-Obukhov scaling, (b): \( s = 2/3 \) for the Kolmogorov scaling.

E. Richardson constant

One of the main interests in turbulent relative dispersion is the determination of the universal coefficient of the Richardson law, namely the Richardson constant. From Eq. (18), \( G(C_A/\lambda)^{2/s} \) is regarded as the Richardson constant, where \( C_A \) is the nondimensional part of the dimensional constant \( \hat{A} \) (see Appendix B).

First, we determine the value of \( G \). To do this, we have only to calculate the integration in Eq. (17) for the similarity solution. For the Palm model, this can be easily carried out using the similarity solution, and in terms of the gamma function we write it as

\[
G = s^{4/s} \frac{\Gamma \left( \frac{2s - \delta + 2}{s} \right)}{\Gamma \left( \frac{2s - \delta}{s} \right)}.
\]  

Note that this value is independent of the value of \( \lambda \). This is because the Palm equation (18) becomes independent of \( \lambda \) if we use rescaled time \( \hat{t} = t/\lambda \).

For the self-similar telegraph case, however, the dependence of \( \lambda \) cannot be removed by this rescaled time, and hence \( G \) depends on \( \lambda \). Although we do not have the explicit expression of \( G \) for this case, we can evaluate it numerically, and the evaluated value are shown in Fig. 10 for various realistic values of \( \lambda \) and \( \delta \) with those of the Palm model.

For \( s = 2/5, \lambda = 5.2 \) and \( \delta = -0.77 \), we have \( G \cong 0.34 \) and 0.66 for the self-similar telegraph and the Palm model, respectively.

The value of \( G \) for the self-similar telegraph case accords with that for the Palm case in the limit of \( \lambda \to \infty \), since Eq. (1) with the rescaled time \( \hat{t} \) tends to the Palm equation for this limit. As can be seen from Fig. 10, \( G \) of the Palm model is always larger than that of the self-similar telegraph. This means that the inclusion of the effect of persistent
separation for the model suppresses the relative dispersion.

Now, we calculate the Richardson constant for the two models. The remaining component of the Richardson constant, $C_A/\lambda$, must be determined from the characteristics of turbulent flows. This factor is estimated in the way summarized in the Appendix B. Note that the value of $\lambda$ is also required for the self-similar telegraph model, since $G$ depends on $\lambda$ in this case. For our previous DNS of 2DFC turbulence, we have estimated the Richardson constant as 0.030 and 0.058 for the self-similar telegraph and the Palm model, respectively. These values are smaller than those estimated of the Richardson constant for the 2DIC case, also shown in Table I by two orders of magnitude. This might be one of the peculiarities of relative dispersion in 2DFC turbulence. From the upper two cases in Table I, the Richardson constants estimated using the Palm model appear closer to the values obtained by their DNSs (the rightmost column). However, in their original papers, additional assumptions were imposed to obtain their values, shown in the rightmost column of Table I: the satisfaction of the Richardson diffusion equation for Ref. 14 and the scaling law of Eq. (16) for Ref. 21, both of which do not hold for the telegraph model. Therefore, their values cannot be used to determine which model yields a more correct value of the Richardson constant.

| Paper                        | Paper                        | $s$ | $C_A/\lambda$ | $\lambda$ | $\delta$ | Richardson constant from model (1) | Richardson constant from model (3) | Richardson constant from paper |
|------------------------------|------------------------------|-----|---------------|-----------|----------|------------------------------------|------------------------------------|-----------------------------|
| Boffetta and Sokolov         | 2/3                          | 0.72 | 14            | -1.48     | 1.2      | 4.4                                | 3.8                                | 3.8                         |
| Goto and Vassilicos          | 2/3                          | 0.84 | -             | -0.87     | -        | 3.9                                | 6.9                                | 6.9                         |
| Ogasawara and Toh            | 2/5                          | 0.62 | 5.2           | -0.77     | 0.030    | 0.058                              | -                                  | 0.058                       |

V. CONCLUDING REMARKS

We have numerically solved the two different equations describing the temporal evolution of the PDF of the separation of a particle pair in the inertial range of homogeneous and isotropic turbulence. In the simulations, 2DFC turbulence case is dealt with, because we have the values of the control parameters for that case. The time-integrated solutions of both equations have been compared to characterize the models. There, the two initial conditions of the different widths of the initial PDF are imposed. However, the difference between the two widths only affects the short-time behavior of the PDF for the self-similar telegraph model.

The self-similar telegraph model represents the finiteness of the separation and its persistency from the bounds of PDFs or the maximum relative separations, appeared in Figs. 1 and 2 whose behavior is not seen in the diffusion-type counterpart (the Palm model). The inclusion of the effect of persistent separation in the self-similar telegraph model would have an advantage in describing the initial persistent separation, which was recently found to strongly affect the relative dispersion for a relatively long time in experiments. The unique character of the relative velocity is responsible for this, as is also shown in the model of Ref. 24 based on the similar approach to ours. However, in the real turbulent dispersion, the relative velocity has a distribution, and therefore these ridges cannot be observed in DNS and experiments. We will incorporate the distribution of the relative velocity into our model in future work.

The long-time behaviors of the time-integrated solutions are characterized by a relaxation process into the corresponding similarity solutions. It is found from the numerical simulations that the similarity solutions of both equations, (11) and (13), are asymptotic solutions of the time-integrated solutions. Thus, the realizability of the similarity solutions was corroborated, aside from the very long approaching time. We also investigated the decay rate of the root-squared difference between the time-integrated and the similarity solutions, $||\delta P||$. The dominant contribution to $||\delta P||$ is the time lag between the two solutions. This contribution seems to be independent of both models and widths of the initial PDF.

If we compare the time-integrated solution with the appropriately-advanced similarity solution, more rapid approach to it can be found. This is because the similarity solution is located at the origin at the initial time, while our initial condition allows for nonzero initial separations. This result may give suggestions to the observation of the actual relaxation process of separation PDF in experiments or DNSs.
FIG. 11: Temporal evolution of the total probability, $S$, for various values of the parameters $\lambda$ and $\delta$. (a): $\delta = -0.77$ and $\lambda = 3.5$, 5.2, 6.8 and 8.3 from bottom to top. (b): $\lambda = 5.2$ and $\delta = -1.2$, $-0.77$, $-0.4$ and 0 from bottom to top. $w = 1$ for both. The values remain constant after $t = 2$.

The Batchelor scaling law for the mean-square separation holds only for the self-similar telegraph model, while both models satisfy the Richardson scaling law. This difference for the short-time behavior between the two models is due to their capacities to satisfy the condition $\left[\frac{d\langle r^2(t)\rangle}{dt}\right]_{t=0} = 0$, or $\left[\frac{\partial P(r,t)}{\partial t}\right]_{t=0} = 0$.

On the other hand, the behavior of the mean-square separation for the Palm model can be well described by the Richardson scaling law with the time lag, Eq. (16), to some extent. However, as is suggested by the fact that the Batchelor scaling cannot be derived from Eq. (16), this scaling law neglects the effect of initial persistent separation of a particle pair.

We have also estimated the Richardson constant for both models making use of the data of DNSs to determine the values of the parameters of the models. The Richardson constant for the self-similar telegraph model is generally smaller than that for the Palm model. Furthermore, both models predict smaller values for the 2DFC case than those predicted for the 2DIC case, by a factor of $10^{-2}$. This result should be confirmed in future DNS of 2DFC turbulence.

Although we have dealt with the 2DFC turbulence, we have not yet observed the clear Richardson $t^5$ law in that case. We will attempt to achieve the Richardson $t^5$ law as well as the separation PDF in future work to determine which model is better for describing the relative dispersion.

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APPENDIX A: LOSS OF THE TOTAL PROBABILITY FOR SELF-SIMILAR TELEGRAPH MODEL

Although for the Palm model the total probability $S = \int P(r,t)dr$ is conserved, the self-similar telegraph model does not assure the conservation of $S$. Thus we calculated $S$ using the time-integrated solution and showed its temporal evolution in Fig. 11 varying the values of the parameters $\lambda$ and $\delta$. We can easily see from this figure that $S$ monotonically decreases for $t < 2$ and is conserved afterward. The variation of the value of $w$ does not change these graphs, even quantitatively. Therefore, we may regard these results as universal, independent of the initial condition. Note that the loss of $S$ is not the result of errors in accuracy of the numerical scheme used, since finer grids and time steps make the same figures. As is mentioned in our previous paper, the similarity solution fulfills the probability conservation. However, we cannot conclude from this that the time-integrated solution already has the similarity form for $t > 2$.

We also plotted the dependence on the parameters $\lambda$ and $\delta$ of the loss of the total probability, $dS = 1 - S$, at $t = 2$ in Fig. 12. There the scaling laws $0.44\lambda^{-1.8}$ and $-0.013\delta + 0.011$ are seen. Its decrease with increasing $\lambda$ is
obvious from the fact that the self-similar telegraph equation (4) tends to the Palm equation (5) as $\lambda \to \infty$, keeping the rescaled time $\tilde{t}$ appeared in Subsec. IV E unchanged.

It is not difficult to make the self-similar telegraph model to conserve the total probability by reformulating it in a conserved form. The results will be reported in future papers.

APPENDIX B: ESTIMATE OF $C_A/\lambda$

The dimensional constant $\tilde{A}$ can be decomposed into the nondimensional and dimensional parts as $C_A \tilde{A}^{1/3}$ and $C_A \tilde{A}^{\varepsilon_{\theta}/3} (\alpha g)^{2/5}$ for the Kolmogorov and Bolgiano-Obukhov scaling, respectively. Here, the nondimensional part is denoted by $C_A$. The dimensional constants $\varepsilon$, $\varepsilon_{\theta}$, $\alpha$ and $g$ are the energy dissipation rate, the entropy dissipation rate, the thermal expansion coefficient and the gravitational acceleration, respectively.

In order to determine the value of $C_A/\lambda$, we use the exit time statistics. The exit time, $T_E(r; \rho)$, is the time it takes for the separation of the particle pair to reach the threshold of $\rho r$ from that of $r$. Assuming that the PDF of the pair separation obeys Eq. (3), the following expression of the mean exit time,

$$\langle T_E(r; \rho) \rangle = \lambda \tilde{A} \frac{1}{s(2s - \delta)} (\rho^s - 1)^{r^s},$$

is derived in the same manner as Ref. [14]. The proportionality of the mean exit time to $\tilde{A}^{-1} (\rho^s - 1)^{r^s}$ is also understood from the scaling law for the characteristic time, Eq. (2a). If we set the values of $s$, $\delta$ and $\rho$ and estimate that of the proportionality coefficient of $r^s$ in the right hand side of Eq. (B1) from the data of DNSs or experiments, then the value of $\tilde{A}/\lambda$ can be calculated. Thus, we obtain the value of $C_A/\lambda$ dividing $\tilde{A}/\lambda$ by the dimensional part of $\tilde{A}$, corresponding to the scaling of the DNS or experiment.

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