On the First Cohomology of Local Units

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Abstract

The groups of units $U_i^L$ of a local field $L$ play an important role in algebraic number theory, especially in class field theoretic topics. Therefore, it is interesting to study these groups from a cohomological point of view. In this article, we study and compute the first cohomology of $U_1^L$, $U_2^L$ and $U_3^L$ under certain mild hypotheses, and discuss some results about general $U_i^L$'s.

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1 Notations and basic results

1.1 Local fields

In this section, we summarize the notations and results that will be used in this article. Serre’s book [6] is our main source, where all the results in this section can be found.

Throughout this article, \( K \) denotes a complete field with a discrete valuation \( \nu_K \). \( K \) is endowed with the topology defined by \( \nu_K \). The corresponding valuation ring is denoted by \( \mathcal{O}_K := \{ x \in K | \nu_K(x) \geq 0 \} \), and its maximal ideal by \( \mathfrak{m}_K \). Its residue field is \( \kappa = \mathcal{O}_K / \mathfrak{m}_K \), and \( U_K = \mathcal{O}_K - \mathfrak{m}_K \) is the group of units. We assume that the residue field is finite with positive characteristic \( p > 0 \).

Let \( L \) be a finite Galois extension of \( K \) with \( \text{Gal}(L/K) = G \), and let \( \mathcal{O}_L \) be the integral closure over \( \mathcal{O}_K \) in \( L \). We know that \( \mathcal{O}_L \) is also a complete discrete valuation ring, with a valuation \( \nu_L \), its maximal ideal \( \mathfrak{m}_L \), the residue field \( \lambda \) and the group of units \( U_L \). With the assumption above, the residue field extension is an extension of finite fields. The ramification index is denoted by \( e \) and the degree of the residue field extension is denoted by \( f \). We have \( ef = [L : K] \). Here, we will always make the convention that \( \nu_L(\mathcal{O}_L^\times) = \mathbb{Z} \) and \( \nu_L(K^\times) = e\mathbb{Z} \). We fix a choice of uniformizer of \( L \) and denote it by \( \pi_L \), so \( \nu_L(\pi_L) = 1 \).

1.2 Unit groups and ramification groups

For \( i \geq 0 \), we denote by \( U_L^i \) the elements \( x \in \mathcal{O}_L \) such that \( x \equiv 1 \mod \pi_L^i \). In other words, \( U_L^i = 1 + \mathfrak{m}_L^i \). By convention, we set \( U_L^0 = U_L \). These groups \( U_L^i \) give a decreasing sequence of closed subgroups of \( U_L \), and they form a neighborhood base of 1 for the topology induced on \( U_L \) by \( L^\times \). Further, we have the following result:

\[
U_L \cong \lim_{\leftarrow i \geq 0} U_L/U_L^i.
\]

Remark 1.1. In fact we have for any \( k \),

\[
U_L^k \cong \lim_{\leftarrow j \geq k} U_L^k/U_L^j
\]
There is a filtration for the Galois group $G$. Let $i \geq -1$ be an integer. Define

$$G_i := \{ \sigma \in G \mid \nu_L(\sigma(a) - a) \geq i + 1 \text{ for any } a \in \mathcal{O}_L \}.$$ 

These groups are called ramification groups and $G_i$ is called the $i$-th ramification group of $G$. We have the following well-known result:

**Proposition 1.2.** The $G_i$ form a decreasing sequence of normal subgroups of $G$. $G_{-1}$ is the full Galois group $G$, and $G_0$ is the inertia subgroup of $G$. Moreover, $G/G_0$ is canonically isomorphic to $\text{Gal}(\lambda/\kappa)$.

Next, we consider the relationship between unit groups and ramification groups. We first recall that there is a canonical isomorphism

$$\rho_0 : U_L/U_L^1 \xrightarrow{\sim} \lambda^\times,$$

given by

$$\rho_0(u) = \hat{u},$$

where $\hat{u}$ is the image of $u$ in the residue field $\lambda$. Also, for $i \geq 1$, there are non-canonical isomorphisms

$$\rho_i : U_L^i/U_L^{i+1} \xrightarrow{\sim} \lambda$$

such that if $x = 1 + u \cdot \pi_L^i$, then

$$\rho_i(x \cdot U_L^{i+1}) = \hat{u}.$$ 

When $i \geq 1$, these $\rho_i$’s are not canonical because their definition involves a choice of the uniformizer $\pi_L$. What is worse, $\rho_i$ is only an isomorphism of abelian groups, not an isomorphism of $G$-modules. We will investigate this issue more closely in Section 4.

Next, we recall the following two propositions cited from Serre’s book [6].

**Proposition 1.3.** Let $i$ be a non-negative integer and $\sigma$ be an element in $G_0$. In order for $\sigma$ to belong to $G_i$, it is sufficient and necessary that

$$\frac{\sigma(\pi_L)}{\pi_L} \equiv 1 \mod m_L^i.$$

In other words, $\sigma \in G_i$ if and only if $\frac{\sigma(\pi_L)}{\pi_L} \in U_L^i$. 

3
Since for any $\sigma \in G$, the fraction $\frac{\sigma(\pi L)}{\pi L} \in U_L$ is a unit, and we may consider the map $f_\pi : G \to U_L$ defined by

$$f_\pi(\sigma) = \frac{\sigma(\pi L)}{\pi L}.$$ 

We know from Proposition 1.3 that $f_\pi(G_i) \subseteq U_i^i$. For each index $i$, we may consider the restriction of $f_\pi$ on $G_i$, and compose it with the quotient map

$$G_i \xrightarrow{f_\pi} U_L \to U_i^i/U_i^{i+1}.$$ 

We shall denote by $\theta_i$ this composed map

$$\theta_i : G_i \to U_i^i/U_i^{i+1}.$$ 

Due to the above proposition, $\theta_i$ factors through $G_{i+1}$. By abuse of notations, we will also denote by $\theta_i$ the induced map:

$$\theta_i : G_i/G_{i+1} \to U_i^i/U_i^{i+1}.$$ 

We rephrase

**Proposition 1.4.** Let $i$ be a non-negative integer. The map which assigns $\frac{\sigma(\pi)}{\pi}$ to $\sigma \in G_i$, induces by passage to the quotient an isomorphism $\theta_i$ of the quotient group $G_i/G_{i+1}$ onto a subgroup of the group $U_i^i/U_i^{i+1}$. The isomorphism is independent of the choice of the uniformizer. In particular, the map $\theta_i : G_i/G_{i+1} \to U_i^i/U_i^{i+1}$ is injective.

### 1.3 Invariants of the unit groups

In this subsection, we consider the $G$-invariants $(U_L)^G$ of the groups of units $U_L^i$.

We first show that $(U_L)^G = U_K$. Let $s \in U_L$ be such that $\sigma(s) = s$ for any $\sigma \in G$. This implies that $s \in K$. From the equality $s \cdot t = 1$, we draw $\sigma(t) = t$, for any $\sigma$. Therefore the inverse $t$ is also in $K$. Thus $s \in U_K$, and hence $(U_L)^G \subseteq U_K$. The other inclusion is obvious, hence we conclude that $(U_L)^G = U_K$.
Next, we show \((U_L^1)^G = U_K^1\). Let \(s \in U_L^1\) be such, that \(\sigma(s) = s\) for any \(\sigma \in G\). As in the previous paragraph, this implies \(s \in U_K^1\). Since \(\nu_L(s - 1) > 0\), it follows that \(\nu_K(s - 1) > 0\). Therefore, \(s \in U_K^1\). So, we have \((U_L^1)^G \subseteq U_K^1\). On the other hand, one has \(U_K^1 \subseteq (U_L^1)^G\). Thus, we draw the equality \((U_L^1)^G = U_K^1\).

However, it is not true in general that \((U_L^i)^G = U_K^i\), as the following argument shows. In fact, we have the following:

**Proposition 1.5.** For any \(i \geq 0\), one has:

\[
(U_L^i)^G = U_K^j,
\]

where \(j = \left\lceil \frac{i}{e} \right\rceil\). In particular, if \(L/K\) is unramified, then \((U_L^i)^G = U_K^i\) for any \(i \geq 0\).

**Proof.** Set \(i' = j \cdot e\). Then, one has the obvious inequality \(i' \geq i\). Hence, the inclusion \(U_L^{i'} \subseteq U_L^i\) yields the inclusion of invariants

\[
(U_L^{i'})^G \subseteq (U_L^i)^G.
\]

On the other hand, assume \(s \in (U_L^i)^G\), then, as before, we have \(s \in U_K^1\). Consider \(v = \nu_L(s - 1)\) and \(v' = \nu_K(s - 1)\), \(v, v' \in \mathbb{Z}\). Since \(s \in U_L^i\), one has \(v \geq i\).

Note that one has the identity

\[
\nu_K(x) = \nu_L(x)/e,
\]

for any \(x \in K\). This shows that \(v' = v/e\). Since \(v' \in \mathbb{Z}\), it follows that \(e|v\). Combining this with the inequality, one has

\[
\frac{v}{e} \geq \frac{i}{e},
\]

and since \(\frac{v}{e}\) is an integer, one has

\[
\frac{v}{e} \geq \left\lfloor \frac{i}{e} \right\rfloor = j.
\]
Namely, \( v \geq i' = ej \), and \( s \in U'_L \). Therefore, \( s \in (U'_L)^G \). This shows the other inclusion and we draw:

\[
(U'_L)^G = (U''_L)^G.
\]

It remains to prove that \((U''_L)^G = U'_K\). Indeed, suppose \( s \in U'_K \). Then we have \( \nu_K(s - 1) \geq j \), hence \( \nu_L(s - 1) \geq j \cdot e = i \), meaning \( s \in U'_L \). This shows \( U'_K \subseteq (U''_L)^G \). On the other hand, suppose \( s \in (U'_L)^G \). Then \( \nu_L(s - 1) \geq i' = ej \), hence \( \nu_K(s - 1) \geq j \), meaning \( s \in U'_K \). Thus, \((U''_L)^G \subseteq U'_K\). This settles the proof.

As a final remark, we cite the following result from [5].

**Proposition 1.6.** If the local field extension \( L/K \) is unramified, then the group of units \( U_L \) and the group of principal units \( U^1_L \) are cohomologically trivial.

In fact, the proof therein implies that all the higher unit groups \( U^i_L \) are cohomologically trivial if \( L/K \) is unramified. In this paper, we will study the behavior of the cohomology groups assuming that the local field extension \( L/K \) is ramified, and it turns out that the sizes of the cohomology groups are related to the ramification indices.

## 2 First cohomology of the group of units \( U_L \)

It is known in [1] and [3] that the first cohomology group \( H^1(G, U_L) \) of \( U_L \) is cyclic of order \( e \). We present the proof, with an extra emphasis on the explicit 1-cocycle \( f_\pi \), which generates the cohomology group and plays a fundamental role in later parts of this article.

### 2.1 The cohomology class \( f_\pi \)

Let \( f_\pi : G \to U_L \) be the map defined in the previous section. One sees easily that the map \( f_\pi \) defines a 1-cocycle relation so it represents a cohomology class in \( H^1(G, U_L) \). Note that it is not a coboundary because \( \pi \) itself is not in \( U_L \), so the cohomology class is not trivial. Although the
definition of $f_{\pi}$ as a 1-cocycle depends on the choice of the uniformizer $\pi$, the cohomology class does not. Indeed, let $\pi'$ be another uniformizer of $L$, then $\pi$ and $\pi'$ differ by a unit, i.e., $\pi' = u \cdot \pi$ for some $u \in U_L$. We look at the cocycle defined by $\pi'$:

$$f_{\pi'}(\sigma) = \frac{\sigma(\pi')}{\pi'} = \frac{\sigma(u \cdot \pi)}{u \cdot \pi} = \frac{\sigma(u)}{u} \cdot \frac{\sigma(\pi)}{\pi} = f_u(\sigma) \cdot f_\pi(\sigma).$$

Here, the map $f_u$ defined by $f_u(\sigma) = \frac{\sigma(u)}{u}$ is a 1-coboundary on $U_L$, which gives the trivial cohomology class. Thus, the cohomology classes defined by $f_{\pi'}$ and $f_{\pi}$ are the same. Namely, the cohomology class does not depend on the choice of the uniformizer $\pi_L$. By abuse of notation, we shall denote the corresponding cohomology class by $f_{\pi}$ as well, and show that it is a generator of $H^1(G, U_L)$.

### 2.2 The cohomology group $H^1(G, U_L)$

**Proposition 2.1.** The cohomology group $H^1(G, U_L)$ is a cyclic group whose order is equal to the ramification index $e$ and is generated by the cohomology class $f_{\pi}$.

**Proof.** Consider the exact sequence of $G$-modules:

$$1 \to U_L \to L^\times \xrightarrow{\nu_L} \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ is endowed with the trivial $G$-action. This sequence induces a long exact sequence of cohomology:

$$1 \to (U_L)^G \to (L^\times)^G \xrightarrow{\nu_L} (\mathbb{Z})^G \xrightarrow{\delta} H^1(G, U_L) \to H^1(G, L^\times) \to \cdots.$$ 

By Hilbert 90, we have $H^1(G, L^\times) = 0$. Hence, the first several terms give the following:

$$1 \to U_K \to K^\times \xrightarrow{\nu_K} \mathbb{Z} \xrightarrow{\delta} H^1(G, U_L) \to 0.$$ 


Thus

\[ H^1(G, U_L) \cong \mathbb{Z}/\nu_L(K^\times) \cong \mathbb{Z}/e\mathbb{Z}. \]

This shows that \( H^1(G, U_L) \) is a cyclic group of order 3.

Next, we look closely at the map \( \delta \). It is clear that \( \delta(1) \) is the generator of \( H^1(G, U_L) \). By definition, to obtain \( \delta(1) \), one chooses an element \( x \in L^\times \) such that \( \nu_L(x) = 1 \). The canonical choice is \( \pi_L \), then the fraction \( \sigma(\pi_L)/\pi_L \) defines the 1-cocycle \( \delta(1) \). This is exactly the map \( f_\pi \) we defined in the previous section. As a result, we have the following:

**Fact.** The cohomology class \( \delta(1) \) is represented by the 1-cocycle \( f_\pi \), which is defined by

\[ f_\pi(\sigma) = \frac{\sigma(\pi_L)}{\pi_L} \]

for any \( \sigma \in G \).

Therefore, we conclude that \( H^1(G, U_L) \) is a cyclic group of order \( e \), with \( f_\pi \) as a generator.

\[ \square \]

### 3 First cohomology of principal units \( U^1_L \)

Our goal in this section is to show that \( H^1(G, U^1_L) \) is a cyclic group of order \( w \), with \( f_\pi^t = (f_\pi)^t \) as a generator. Here, \( w \) and \( t \) stand for the wild ramification index and tame ramification index respectively. In fact, our claim can be seen from the factorization

\[ U_L \cong U^1_L \times \lambda^\times, \]

and henceforth

\[ H^1(G, U_L) \cong H^1(G, U^1_L) \times H^1(G, \lambda^\times). \]

It follows that \( H^1(G, U^1_L) \) is the \( p \)-component of the group \( H^1(G, U_L) \) and \( H^1(G, \lambda^\times) \) is the prime-to-\( p \) component. However, we also present the following longer argument, for the method implemented there will play an important role in the next section.
3.1 The preparatory step

We make the following

Claim.

\[ H^1(G, U_L^1) = \ker(H^1(G, U_L) \to H^1(G, \lambda^x)). \]

Proof. The exact sequence

\[ 1 \to U_L^1 \to U_L \to \lambda^x \to 1. \]

induces a long exact sequence of cohomology groups:

\[ 1 \to (U_L^1)^G \to (U_L)^G \to (\lambda^x)^G \xrightarrow{\delta} H^1(G, U_L^1) \to H^1(G, U_L) \to H^1(G, \lambda^x) \to \cdots. \]

Rewrite it as:

\[ 1 \to U_K^1 \to U_K \to \kappa^x \xrightarrow{\delta} H^1(G, U_L^1) \to H^1(G, U_L) \to H^1(G, \lambda^x) \to \cdots. \]

The map \( U_K \to \kappa^x \) is the canonical projection map, and we know the map is surjective, with kernel \( U_K^1 \). Thus, the above long exact sequence slits into two exact sequences:

\[ 1 \to U_K^1 \to U_K \to \kappa^x \to 1, \]

and

\[ 0 \to H^1(G, U_L^1) \to H^1(G, U_L) \to H^1(G, \lambda^x) \to \cdots. \]

In particular, the second sequence implies that the map \( H^1(G, U_L^1) \to H^1(G, U_L) \) induced by the inclusion is injective. As a result, \( H^1(G, U_L^1) \) can be canonically identified as a subgroup of \( H^1(G, U_L) \). More precisely, it is the kernel of the map \( H^1(G, U_L) \to H^1(G, \lambda^x) \). Note here that this map is induced by the canonical projection map \( U_L \to \lambda^x \). This settles our first step. \( \square \)

3.2 Relating with \( T = \text{Hom}(G_0/G_1, \lambda^x) \)

In this subsection, we embed \( H^1(G, \lambda^x) \) into some group \( T \) such that the composition \( H^1(G, U_L) \to T \) is easy to understand. Since \( H^1(G, \lambda^x) \to T \) is injective, the composed map

\[ H^1(G, U_L) \to H^1(G, \lambda^x) \to T \]
has the same kernel as $H^1(G, U_L) \to H^1(G, \lambda^x)$, which is $H^1(G, U_L^1)$. We will compute $H^1(G, U^1_L)$ using the composition in the next subsection.

In order to figure out the group $T$, we make the following considerations. For $G = \text{Gal}(L/K)$, the normal subgroup $G_0 \subseteq G$ and the $G$-module $\lambda^x$, the inflation-restriction exact sequence for this data gives:

$$0 \to H^1(G/G_0, (\lambda^x)^{G_0}) \xrightarrow{\text{inf}} H^1(G, \lambda^x) \xrightarrow{\text{res}} H^1(G_0, \lambda^x).$$

Notice that $G_0$ acts trivially on $\lambda^x$, hence we have $(\lambda^x)^{G_0} = \lambda^x$, and $H^1(G_0, \lambda^x) = \text{Hom}(G_0, \lambda^x)$. As a result, the inflation-restriction sequence reads:

$$0 \to H^1(G/G_0, \lambda^x) \xrightarrow{\text{inf}} H^1(G, \lambda^x) \xrightarrow{\text{res}} \text{Hom}(G_0, \lambda^x).$$

We also notice that $G/G_0 \cong \text{Gal}(\lambda/\kappa)$. Hence, by Hilbert 90, the first term $H^1(G/G_0, \lambda^x)$ is 0. For the third term, note that $\lambda^x$ has cardinality coprime to $p$, while $G_1$ is a $p$-group, so any homomorphism from $G_0$ to $\lambda^x$ factors through $G_0/G_1$. Namely, there is a natural isomorphism

$$\text{Hom}(G_0, \lambda^x) \cong \text{Hom}(G_0/G_1, \lambda^x).$$

As a result, the above sequence becomes:

$$0 \to H^1(G, \lambda^x) \xrightarrow{\text{res}} \text{Hom}(G_0/G_1, \lambda^x).$$

This means that the restriction map is injective. We set

$$T := \text{Hom}(G_0/G_1, \lambda^x).$$

This settles our second step.

### 3.3 Computing the kernel of $H^1(G, U_L) \to T$

To compute the kernel, we take the generator $f_{\pi}$ of the group $H^1(G, U_L)$, and consider its image in $T = \text{Hom}(G_0/G_1, \lambda^x)$. We check that the image of $f_{\pi}$ is $\theta_0$. Indeed, the first map $H^1(G, U_L) \to H^1(G, \lambda^x)$ is induced from the natural projection $U_L \to \lambda^x$, so the image of $f_{\pi}$ in $H^1(G, U_L)$ is the cohomology class represented by the assignment

$$\sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L} \in \lambda^x.$$
Then, the restriction map $H^1(G, U_L) \xrightarrow{\text{res}} \text{Hom}(G_0, \lambda^\times)$ merely narrows the domain of $f_\pi$ to the subgroup $G_0$, we see this assignment coincides with the map $\theta_0$ defined in the first section, under the canonical identification $U_L/U_L^1 = \lambda^\times$.

Notice that $\theta_0$ is an injective homomorphism from the cyclic group $G_0/G_1$ to $\lambda^\times$, so the order of $\theta_0$, as an element in the group $T = \text{Hom}(G_0/G_1, \lambda^\times)$, must $|G_0/G_1| = t$, the tame ramification index. Hence the image of $f_\pi$ in the target $T$ is of order $t$. By an easy group theory exercise, we conclude that the kernel of $H^1(G, U_L) \to T$, which is $H^1(G, U_L^1)$, is generated by $f_\pi^t$. In particular, its order is $e/t = w$, the wild ramification index. This finishes our calculation.

We conclude that:

**Proposition 3.1.** The cohomology group $H^1(G, U_L^1)$ is naturally isomorphic to the subgroup of $H^1(G, U_L)$ generated by $(f_\pi)^t$, where $t = \frac{|G_0|}{|G_1|}$ is the tame ramification index. In particular, the order of $H^1(G, U_L^1)$ is equal to $w = |G_1|$, the wild ramification index.

### 4 The Galois module $\lambda_i$

#### 4.1 Group action twisted by 1-cocycles

Let $G$ be a finite group and $k$ be a field. Suppose that there is an action of $G$ on $k$ denoted by $\mu : G \times k \to k$, such that $\mu$ is induce from a group homomorphism $G \to \text{Aut}(k)$. We denote this action by $g \ast x$ for $g \in G$ and $x \in k$. This action gives rise to an action of $G$ on $k^\times$, and by abuse of notation we shall also denote it by $\ast$.

Regarding the action of $G$ on $k^\times$, let $f : G \to k^\times$ be a 1-cocycle. Then, we make the following:

**Definition 4.1.** For the data above, the twisted action $\mu_f : G \times k \to k$ of $\mu$ by $f$ is given by the following formula:

$$\mu_f(g, a) := f(g) \cdot \mu(g, a),$$

or

$$g \ast_f a := f(g) \cdot (g \ast a),$$
where $\cdot$ denotes the multiplication of the field $k$.

**Proposition 4.2.** $\mu_f : G \times k \to k$ defined above is an action of $G$ on $k$.

**Proof.** To show that $\mu_f$ is indeed a group action, we need to prove that

- $e \ast_f a = a$;
- $(gh) \ast_f a = g \ast_f (h \ast_f a)$.

To prove the first identity, we notice that if $f : G \to k^\times$ is a 1-cocycle, then the equality $f(e) = 1$ holds. Indeed, for any $g \in G$, one has

$$f(g) = f(eg) = f(e) \cdot (e \ast f(g)) = f(e) \cdot f(g).$$

Since $k$ is a field and $f(g) \in k^\times$, we can cancel $f(g)$ from the equality $f(g) = f(e) \cdot f(g)$ to get $1 = f(e)$, as desired.

Then, we have

$$e \ast_f a = f(e) \cdot (e \ast a) = 1 \cdot (e \ast a) = 1 \cdot a = a.$$ 

This settles the first identity.

To prove the second identity

$$(gh) \ast_f a = g \ast_f (h \ast_f a),$$

we first compute the left-hand side. By definition,

$$(gh) \ast_f a = f(gh) \cdot (gh \ast a)$$

On the other hand, the right-hand side is

$$g \ast_f (h \ast_f a) = f(g) \cdot \left( g \ast (h \ast_f a) \right) = f(g) \cdot \left( g \ast (f(h) \cdot (h \ast a)) \right) = f(g) \cdot \left( g \ast f(h) \right) \cdot \left( g \ast (h \ast a) \right).$$
Since \( f \) is a 1-cocycle, we have
\[
f(g) \cdot (g \cdot f(h)) = f(gh).
\]
Therefore, the right-hand side becomes
\[
g \cdot f(h \cdot f(a)) = f(gh) \cdot (gh \cdot f(a)).
\]
This settles the second equality.

We remark that the group elements \( g \in G \) can no longer be regarded as field automorphisms via the new action \( \cdot f \), because the action is not multiplicative in general. Namely, we do not have \( g \cdot f(a \cdot b) = (g \cdot f a) \cdot (g \cdot f b) \).

### 4.2 The Galois module \( U_i^L / U_{i+1}^L \)

In this subsection, we consider the quotient \( U_i^L / U_{i+1}^L \). We know that in the category of abelian groups, \( U_i^L / U_{i+1}^L \) is isomorphic non-canonically to \( \lambda \) when \( i \geq 1 \). In this subsection, we discuss the Galois module structures rather than abelian groups.

We resume the previous notations. Let \( G = \text{Gal}(L/K) \) and \( i \geq 1 \). Let \( s = 1 + u \cdot \pi_L^i \) be an element in \( U_i^L \) but not in \( U_{i+1}^L \). Then, for any \( \sigma \in G \), we have
\[
\sigma(s) = 1 + \sigma(u) \cdot \sigma(\pi_L)^i = 1 + \sigma(u) \cdot \frac{\sigma(\pi_L)^i}{\pi_L^i} \cdot \pi_L^i.
\]
Recall that the group isomorphism \( U_i^L / U_{i+1}^L \to \lambda \) is given by the assignment \( s \mapsto \hat{u} \), so we have
\[
\sigma(s) \mapsto \sigma(u) \cdot \frac{\sigma(\pi_L)^i}{\pi_L^i} = \sigma(u) \cdot f_{\pi}(\sigma),
\]
where \( f_{\pi} \) is the 1-cocycle considered in the previous sections.

From the above discussions, we see that in order for the map \( U_i^L / U_{i+1}^L \to \lambda \) to be \( G \)-equivariant, it is necessary to give a new action of \( G \) to \( \lambda \) such that
\[
\sigma \ast y = \sigma(y) \cdot f_{\pi}(\sigma),
\]
for any \( y \in \lambda \).

We see that the new group action on \( \lambda \) is the Galois action twisted by \( \widehat{f^t} \), in the sense of the previous subsection, where \( G \to G/G_0 = \text{Gal}(\lambda/\kappa) \) is the natural quotient map. We denote this Galois module by \( \lambda_i \). We also remark that when \( L/K \) is unramified, then \( \pi_L = \pi_K \), and \( \sigma(\pi_L)/\pi_L \) is just 1. Hence there is no need to consider twisting in the unramified case. We summarize our discussion as the following:

**Definition 4.3.** Let \( \lambda \) and \( G = \text{Gal}(L/K) \) be as above. We define the \( G \)-module \( \lambda_i \), such that \( \lambda_i = \lambda \) as abelian groups, and the action of \( G \) on \( \lambda_i \) is given by the formula

\[
\sigma *_i a = \sigma(a) \cdot \left( \widehat{f^t(\sigma)} \right)^i = \sigma(a) \cdot \left( \sigma(\pi_L)/\pi_L \right)^i,
\]

for any \( a \in \lambda \) and any \( \sigma \in G \). The \( *_i \) is indeed a \( G \)-action because of Proposition 4.1.

We close this subsection by computing the group of invariants \( (\lambda_i)^G \). Let \( a \in \lambda_i \) be an element which is invariant under the twisted action \( *_i \) of \( G \). Then, for any \( \sigma \in G \), we have

\[
a = \sigma *_i a = \sigma(a) \cdot \widehat{f^t(\sigma)}^i.
\]

Since 0 is trivially an invariant element, we may assume that \( a \neq 0 \), hence \( \sigma(a) \neq 0 \). Thus, the above equality can be written as

\[
\frac{a}{\sigma(a)} = \widehat{f^t(\sigma)}^i.
\]

This implies that \( f^t = \widehat{f^t} \) is a 1-coboundary on \( \lambda^\times \). More precisely, this means the image of \( (f^t)^i \) is trivial via the map \( H^1(G, U_L) \to H^1(G, \lambda^\times) \). By the previous section, we know that the kernel of this map is exactly \( H^1(G, U_L) \) and is generated by \( f^t_\pi \). Thus, we must have \( t|i \), where \( t \) is the tame ramification index.

Hence, we draw the following fact:

\[
(\lambda_i)^G \neq 0 \implies t|i.
\]
Therefore, when $t$ does not divide $i$, we have $(\lambda_i)^G = 0$. Next, we consider the case where $t|i$. By the previous section, $f'$ is a 1-coboundary on $\lambda^\times$. Therefore, there exists $b \in \lambda^\times$ such that the following holds:

$$f'(\sigma) = \frac{\sigma(b)}{b}.$$ 

Hence, for any $a \in \lambda^\times$, we have

$$\sigma *_i a = \sigma(a) \cdot f'(\sigma) = \sigma(a) \cdot \frac{\sigma(b)}{b}.$$ 

If $a$ is invariant under this action, then we have

$$a = \frac{\sigma(a) \sigma(b)}{b},$$ 

hence

$$ab = \sigma(ab).$$

This implies $ab \in (\lambda)^G = \kappa$, so $a \in b^{-1} \cdot \kappa$, which is an additive subgroup of $\lambda$. We remark that this $b$ depends on the index $i$.

### 4.3 The first cohomology of $\lambda_i$

In this subsection, we consider the cohomology group $H^1(G, \lambda_i)$. The inflation-restriction sequence for $\lambda_i$ gives:

$$0 \rightarrow H^1(G/G_0, (\lambda_i)^{G_0}) \rightarrow H^1(G, \lambda_i) \rightarrow H^1(G_0, (\lambda_i)^{G/G_0}).$$

To study these groups, we first look at the invariants $(\lambda_i)^{G_0}$. For $\sigma \in G_0$, we use the notations from the previous subsection and notice that

$$\sigma *_i a = \sigma *_{f'} a = f'(\sigma) \cdot \sigma(a).$$

We note that with the original Galois action, the inertia group $G_0$ acts trivially on the residue field $\lambda$, so $\sigma(a) = a$ for any $\sigma \in G_0$ and any $a \in \lambda$. Hence, the above equality becomes

$$\sigma *_{f'} a = f'(\sigma) \cdot a.$$
Hence, if \( a \in (\lambda_i)^{G_0} \), then we must have \( f'(\sigma) \cdot a = a \). Thus, the equality

\[
f'(\sigma) = \widehat{f}^i_\sigma(\sigma) = 1,
\]

holds in \( \lambda^\times \) for any \( \sigma \in G_0 \). By Proposition 1.4, this yields \( t \mid i \). When this happens, we must have \( (\lambda_i)^{G_0} = \lambda_i \).

Therefore, we conclude:

\[
(\lambda_i)^{G_0} = \begin{cases} 
0 & \text{if } t \nmid i, \\
\lambda_i & \text{if } t \mid i.
\end{cases}
\]

From this, it follows that

\[
H^1(G/G_0, (\lambda_i)^{G_0}) = \begin{cases} 
0 & \text{if } t \nmid i, \\
H^1(G/G_0, \lambda_i) & \text{if } t \mid i.
\end{cases}
\]

Our goal is to show that in the second case where \( t \mid i \), the group \( H^1(G/G_0, \lambda_i) \) is 0. Since \( \lambda_i \) is finite, its Herbrand quotient is 1. Thus, it suffices to show that the 0-th Tate cohomology group \( \widehat{H}^0(G/G_0, \lambda_i) \) is trivial. By definition, we have

\[
\widehat{H}^0(G/G_0, \lambda_i) = \frac{(\lambda_i)^{G/G_0}}{N_{G/G_0}(\lambda_i)}.
\]

By the previous subsection, we know that

\[
(\lambda_i)^{G/G_0} = (\lambda_i)^G = b^{-1} \cdot \kappa,
\]

where \( b \in \lambda^\times \) is the element such that

\[
f'(\sigma) := \widehat{f}_\sigma^i(\sigma) = \frac{\sigma(b)}{b}.
\]

Therefore, it suffices to show that

\[
N_{G/G_0}(\lambda_i) = b^{-1} \cdot \kappa.
\]

For simplicity, we denote by \( g = G/G_0 \). Note that \( g = \text{Gal}(\lambda/\kappa) \), which is a cyclic group. Suppose \( \tau \) is a generator of \( g \), then we can compute \( N_g(\lambda_i) \) as follows:
For any element $a \in \lambda$, we have

$$N_{\mathfrak{g}} \ast_i a = \sum_{k=0}^{\lfloor g \rfloor - 1} \tau^k \ast_i a$$

$$= \sum_{k=0}^{\lfloor g \rfloor - 1} f'(\tau^k) \cdot \tau^k(a)$$

$$= \sum_{k=0}^{\lfloor g \rfloor - 1} \frac{\tau^k(b)}{b} \cdot \tau^k(a)$$

$$= \sum_{k=0}^{\lfloor g \rfloor - 1} \frac{\tau^k(ba)}{b}$$

$$= \sum_{k=0}^{\lfloor g \rfloor - 1} \tau^k(ba)$$

In the above fraction, we notice that the numerator $\sum_{k=0}^{\lfloor g \rfloor} \tau^k(ba)$ is the usual norm $N_{\mathfrak{g}}(ba)$ with respect to the usual Galois action. When $a$ ranges through elements in $\lambda$, so does the product $ba$, since $b \neq 0$ by definition. Thus, the numerator ranges through elements of $N_{\mathfrak{g}}(\lambda)$, with respect to the usual Galois action. By the normal basis theorem, the Galois module $\lambda$ is cohomologically trivial. Thus, $H^1(\mathfrak{g}, \lambda) = 0$, and it follows that $\widehat{H}^0(\mathfrak{g}, \lambda) = 0$ by the Herbrand quotient argument. This means

$$N_{\mathfrak{g}}(\lambda) = (\lambda)^g = \kappa.$$

Therefore, we conclude that

$$N_{\mathfrak{g}}(\lambda_i) = b^{-1} \kappa = (\lambda_i)^g,$$

and this means $\widehat{H}^0(G/G_0, \lambda_i) = 0$, as desired. As a result, $H^1(\mathfrak{g}, (\lambda_i)^{G_0})$ is always 0, settling our claim.

Once we know that the group $H^1(G/G_0, (\lambda_i)^{G_0})$ is identically 0, the inflation-restriction sequence

$$0 \to H^1(G/G_0, (\lambda_i)^{G_0}) \to H^1(G, \lambda_i) \to H^1(G_0, (\lambda_i)^{G/G_0}) \to \cdots$$

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now reads
\[ 0 \to H^1(G, \lambda_i) \to H^1(G_0, \lambda_i)^{G/G_0} \to \cdots \]
In particular, the restriction map \( H^1(G, \lambda_i) \xrightarrow{\text{res}} H^1(G_0, \lambda_i) \) is injective.

We make some further considerations. For the quotient \( G_0/G_1 \) and the module \( \lambda_i \), we have the inflation-restriction sequence
\[ 0 \to H^1(G_0/G_1, (\lambda_i)^{G_1}) \to H^1(G_0, \lambda_i) \to H^1(G_1, \lambda_i)^{G/G_0} \to \cdots \]
We note that the group \( G_0/G_1 \) has order \( t \), which is the tame ramification index and is coprime to \( p \). Thus, the abelian group \( H^1(G_0/G_1, (\lambda_i)^{G_1}) \) is eliminated by both \( t \) and \( p \), so it must be trivial. Therefore, the above sequence reads
\[ 0 \to H^1(G_0, \lambda_i) \to H^1(G_1, \lambda_i)^{G/G_0} \to \cdots \]
In particular, the restriction \( H^1(G_0, \lambda_i) \xrightarrow{\text{res}} H^1(G_1, \lambda_i) \) is injective.

Therefore, the restriction map \( H^1(G_0, \lambda_i) \xrightarrow{\text{res}} H^1(G_1, \lambda_i) \) is injective as well, because it is equal to the composition of \( H^1(G, \lambda_i) \xrightarrow{\text{res}} H^1(G_0, \lambda_i) \) and \( H^1(G_0, \lambda_i) \xrightarrow{\text{res}} H^1(G_1, \lambda_i) \).

Next, we remark that the group \( G_1 \) acts trivially on \( \lambda_i \), because for any \( \sigma \in G_1, f_\pi(\sigma) = \frac{\sigma(\pi_L)}{\pi_L} \equiv 1 \mod \pi_L \). Therefore,
\[ \sigma \ast_i a = f_\pi(\sigma) \cdot \sigma(a) = 1 \cdot \sigma(a) = \sigma(a) = a. \]
The last equal sign is due to the fact that the subgroup \( G_1 \subseteq G_0 \) has trivial Galois action on \( \lambda \). Hence, \( G_1 \) indeed acts trivially on \( \lambda_i \). As a result, \( H^1(G_1, \lambda_i) = \text{Hom}(G_1, \lambda) \). Namely, \( H^1(G_1, \lambda_i) \) is the collection of group homomorphisms from \( G_1 \) to \( \lambda = \lambda_i \) (as abelian groups). We summarize our discussion as the following:

**Proposition 4.4.** The restriction map \( H^1(G, \lambda_i) \xrightarrow{\text{res}} H^1(G_1, \lambda_i) \) is injective, and we have \( H^1(G_1, \lambda_i) = \text{Hom}(G_1, \lambda) \).

### 5 Cohomology of higher unit groups

For simplicity, in this section we shall assume that the local fields \( L/K \) are both local number fields, i.e., they are finite extensions of the field of \( p \)-adic numbers \( \mathbb{Q}_p \). In particular, we have \( \text{char}(L) = 0 \).
5.1 The assumption $t > i$

From Section 4, we have the following short exact sequence of $G$-modules:

\[ 1 \to U_{i+1}^i \to U_i^i \to \lambda_i \to 0. \]

This exact sequence induces a long exact sequence

\[ 1 \to (U_{i+1}^i)^G \to (U_i^i)^G \to (\lambda_i)^G \to H^1(G, U_{i+1}^i) \to H^1(G, U_i^i) \to H^1(G, \lambda_i) \to \cdots \]

When $t$ does not divide $i$, we have $(\lambda_i)^G = 0$. In particular, this is true when $t > i$. Hence, in this case, the above long exact sequence yields:

\[ 0 \to H^1(G, U_{i+1}^i) \to H^1(G, U_i^i) \to H^1(G, \lambda_i) \to \cdots \]

From this, we draw:

\[ H^1(G, U_{i+1}^i) = \ker(H^1(G, U_i^i) \to H^1(G, \lambda_i)). \]

At the end of the previous section, we showed that the restriction map $H^1(G, \lambda_i) \xrightarrow{\text{res}} H^1(G_1, \lambda_i)$ is injective. Therefore, we may consider the composition

\[ H^1(G, U_i^i) \to H^1(G, \lambda_i) \xrightarrow{\text{res}} H^1(G_1, \lambda_i), \]

without changing the kernel:

\[ H^1(G, U_{i+1}^i) = \ker\left( H^1(G, U_i^i) \to H^1(G, \lambda_i) \right) = \ker\left( H^1(G, U_i^i) \to H^1(G_1, \lambda_i) \right). \]

In particular, we have

**Proposition 5.1.** Suppose $t$ does not divide $i$, where $t$ is the tame ramification index. Then, $H^1(G, U_{i+1}^i)$ is a subgroup of $H^1(G, U_i^i)$.

When $i = t, 2t, 3t, \cdots$. It is not easy to determine the structure of $H^1(G, U_{i+1}^i)$ and $H^1(G, U_i^i)$. Therefore, it is relatively easy to assume that $i$ is small. In the next two subsections, we will study the cases where $i = 1, 2$ to determine $U_2^2$ and $U_3^2$, under the assumption $t > i$. 

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5.2 First cohomology of $U_2^L$

For $i = 1$ and $t > i = 1$, we draw the following from the previous discussions:

$$H^1(G, U_2^L) = \text{Ker} \left( H^1(G, U_1^L) \to \text{Hom}(G_1, \lambda_1) \right).$$

In Section 3, we showed that the group $H^1(G, U_1^L)$ is a cyclic group of order $w$, with a generator being $(f_\pi)^t$. We denote by $\beta$ the image of $(f_\pi)^t$ in Hom$(G_1, \lambda_1)$. So $\beta$ is the assignment

$$\beta : \sigma \mapsto \overline{u_{\sigma,t}},$$

where $\sigma \in G_1$, $\overline{u_{\sigma,t}}$ is the image of $u_{\sigma,t}$ in the residue field $\lambda$, and $u_{\sigma,t}$ is the element such that

$$\frac{\sigma(\pi_t^L)}{\pi_t^L} = 1 + u_{\sigma,t} \cdot \pi_L.$$

We first remark that $\beta$ factors through the quotient group $G_1/G_2$, because for elements $\sigma \in G_2$, we have $\frac{\sigma(\pi_L)}{\pi_L} \in U_2^L$. As a result,

$$\frac{\sigma(\pi_t^L)}{\pi_t^L} = \left( \frac{\sigma(\pi_L)}{\pi_L} \right)^t \in U_2^L.$$

Namely, $\beta(\sigma) = 0$ for $\sigma \in G_2$. Hence, we may regard $\beta$ as an element in the subgroup $\text{Hom}(G_1/G_2, \lambda_1) \subseteq \text{Hom}(G_1, \lambda_1)$. We also notice that $\beta$ is the image of the generator of $H^1(G, U_1^L)$, so the entire image of the map $H^1(G, U_1^L) \to \text{Hom}(G_1, \lambda_1)$ is contained in the subset $\text{Hom}(G_1/G_2, \lambda_1)$, i.e.,

$$\text{Im} \left( H^1(G, U_1^L) \to \text{Hom}(G_1, \lambda_1) \right) \subseteq \text{Hom}(G_1/G_2, \lambda_1).$$

We first consider the trivial case where $G_1/G_2 = 1$. Then, the group $\text{Hom}(G_1/G_2, \lambda_1)$ is trivial. Thus, the above inclusion implies:

$$\text{Im} \left( H^1(G, U_1^L) \to \text{Hom}(G_1, \lambda_1) \right) = 0,$$

and hence

$$\text{Ker} \left( H^1(G, U_1^L) \to \text{Hom}(G_1, \lambda_1) \right) = H^1(G, U_1^L).$$
Therefore, we draw $H^1(G, U_L^2) = H^1(G, U_L^1)$.

Next, we assume that $G_1/G_2 \neq 1$.

Suppose, for $\sigma \in G_1$,

$$\frac{\sigma(\pi_L)}{\pi_L} = 1 + u_\sigma \cdot \pi_L,$$

(5.1)

then we see that

$$\frac{\sigma^t(\pi_L)}{\pi_L^t} = \left(\frac{\sigma(\pi_L)}{\pi_L}\right)^t = (1 + u_\sigma \cdot \pi_L)^t = 1 + t \cdot u_\sigma \cdot \pi_L + \cdots,$$

where the omitted terms are all divisible by $(\pi_L)^2$. As a result, we have

$$u_{\sigma,t} \equiv t \cdot u_\sigma \mod \pi_L,$$

i.e., $\widehat{u_{\sigma,t}} = t \cdot \widehat{u_\sigma}$. We remark that $t$ is coprime to $p = \text{char}(\lambda)$, so $t$ is invertible in $\lambda$.

A similar calculation shows that

$$\beta^p(\sigma) = p \cdot \widehat{u_{\sigma,t}} = 0,$$

since $p = \text{char}(\lambda)$. This implies that $(f_\pi)^{pt}$ is in the kernel of the map:

$$(f_\pi)^{pt} \in H^1(G, U_L^2) = \text{Ker} \left( H^1(G, U_L^1) \to \text{Hom}(G_1, \lambda_1) \right).$$

Lastly, we show that $f_\pi^{pt}$ generates the kernel $H^1(G, U_L^2)$. Notice that $H^1(G, U_L^1)$ is a cyclic group, and $\langle f_\pi^{pt} \rangle \subseteq H^1(G, U_L^1)$ is the unique subgroup of index $p$. Thus, to show that the kernel is $\langle f_\pi^{pt} \rangle$, it suffices to show that the generator $f_\pi^t$ is not in the kernel.

Recall that $\beta \in \text{Hom}(G_1/G_2, \lambda_1) \subseteq \text{Hom}(G_1, \lambda_1)$. Consider $\gamma \in \text{Hom}(G_1/G_2, \lambda_1)$, which is the composition

$$\gamma : G_1/G_2 \xrightarrow{\theta_1} U_L^1/U_L^2 \xrightarrow{\rho_1} \lambda = \lambda_1.$$

$\gamma$ is a non-trivial homomorphism because both $\theta_1$ and $\rho_1$ are injective maps, hence $\gamma$ is injective. We see that $\gamma$ is the assignment

$$\gamma : \sigma \mapsto \widehat{u_\sigma},$$

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where $u_\sigma$ is the element in the formula (5.1). Therefore, we find the following relation:

$$\beta(\sigma) = t \cdot \gamma(\sigma),$$

for any $\sigma \in G_1$. Since $t$ in invertible in $\lambda$, $\gamma$ being injective implies that $\beta$ is also injective. Thus $\beta \neq 0$, as desired.

Let us summarize the discussions in this subsection:

**Proposition 5.2.** Suppose that $t > 1$, where $t$ the tame ramification index. Then, the group $H^1(G, U^2_L)$ is a subgroup of $H^1(G, U^1_L)$. Moreover, the following hold:

- If $G_1/G_2 = 1$, then $H^1(G, U^2_L) = H^1(G, U^1_L)$.
- If $G_1/G_2 \neq 1$, then $H^1(G, U^2_L)$ is generated by $(f_\pi)^{pt}$. In particular, it is the unique subgroup of $H^1(G, U^1_L)$ with index $p$ and its order is $w/p$.

Next, we remove the working assumption $t > 1$, so $t = 1$.

### 5.3 First cohomology of $U^3_L$

We now consider $i = 2$ and $t > i = 2$. Again, we consider the sequence of $G$-modules

$$1 \rightarrow U^3_L \rightarrow U^2_L \rightarrow \lambda_2 \rightarrow 0,$$

which yields

$$H^1(G, U^3_L) = \text{Ker} \left( H^1(G, U^2_L) \rightarrow \text{Hom}(G_1, \lambda_2) \right).$$

Based on Proposition 5.2, we have to discuss whether $G_1/G_2 = 1$ or not.

#### 5.3.1 $G_1/G_2 = 1$

If $G_1/G_2 = 1$, then we have $G_1 = G_2$. As a result, we have

$$H^1(G_1, \lambda_2) = H^1(G_2, \lambda_2) = \text{Hom}(G_2, \lambda_2),$$

and we shall study the map

$$H^1(G, U^2_L) \rightarrow \text{Hom}(G_2, \lambda_2)$$
By Proposition 5.2, we have $H^1(G, U_2^2) = H^1(G, U_1^2)$, in particular, $H^1(G, U_2^2)$ is generated by $f^t_\pi$. As before, we denote by $\beta$ the image of $f^t_\pi$ in the group $\text{Hom}(G_2, \lambda_2)$. Thus, $\beta$ is the assignment

$$\beta : \sigma \mapsto s_{\sigma,t},$$

where $\sigma \in G_2$ and $s_{\sigma,t}$ is the element such that

$$\frac{\sigma(\pi_1^t)}{\pi_1^t} = 1 + s_{\sigma,t} \cdot (\pi_L)^2.$$

Like before, this map $\beta$ factors through the quotient $G_2/G_3$. Hence,

$$\text{Im} \left( H^1(G, U_2^2) \to \text{Hom}(G_2, \lambda_2) \right) \subseteq \text{Hom}(G_2/G_3, \lambda_2) \subseteq \text{Hom}(G_2, \lambda_2).$$

Again, if $G_2/G_3 = 1$, then the group $\text{Hom}(G_2/G_3, \lambda_2)$ is trivial. We draw:

$$\text{Ker} \left( H^1(G, U_2^2) \to \text{Hom}(G_2, \lambda_2) \right) = H^1(G, U_2^2).$$

This means $H^1(G, U_3^2) = H^1(G, U_2^2) = H^1(G, U_1^2)$. Moreover, we can prove the following by induction:

**Proposition 5.3.** Suppose $G_1 = G_2 = \cdots = G_j$ for some $j \geq 2$ and $t > j$. Then the following equality holds:

$$H^1(G, U_1^2) = H^1(G, U_2^2) = \cdots H^1(G, U_j^2).$$

Next, we consider the non-trivial situation where $G_2/G_3 \neq 1$. As before, we consider the composition

$$\gamma : G_2/G_3 \xrightarrow{\theta_2} U_2^2/U_1^2 \xrightarrow{\rho_2} \lambda = \lambda_2.$$

This $\gamma$ is an injective map, hence is non-zero. We compare $\gamma$ with $\beta$ in the same way as the previous subsection, and the same relation $\beta(\sigma) = t \cdot \gamma(\sigma)$ holds. The same argument then yields

$$H^1(G, U_3^2) = \langle f^t_\pi \rangle.$$
5.3.2 $G_1/G_2 \neq 1$

When $G_1/G_2$ is non-trivial, by Proposition 5.2 we know that $H^1(G, U^2_L)$ is generated by $f^pt$. Let $\beta$ be the image of $f^pt$ in $\text{Hom}(G_1, \lambda_2)$. For $\sigma \in G_1$, we have

$$f^pt(\sigma) = \left(\frac{\sigma(\pi_L)}{\pi_L}\right)^pt.$$  

Write as before $\frac{\sigma(\pi_L)}{\pi_L} = 1 + u_\sigma \cdot \pi_L$, then

$$\left(\frac{\sigma(\pi_L)}{\pi_L}\right)^pt = (1 + u_\sigma \cdot \pi_L)^pt$$

$$= 1 + pt \cdot u_\sigma \cdot \pi_L + \left(\frac{pt}{2}\right) \cdot (u_\sigma)^2 \cdot (\pi_L)^2 + \cdots,$$  

(5.2)

where the omitted terms are divisible by $(\pi_L)^3$. We denote by $e_L$ the ramification index of $L/\mathbb{Q}_p$ and by $e_K$ the ramification index of $K/\mathbb{Q}_p$, so $e_L = e \cdot e_K$. Then we have

$$p = s \cdot (\pi_L)^{e_L}$$

for some unit $s$ in $L$. Note that $G_1/G_2 \neq 1$ implies $G_1 \neq 1$, hence we have

$$|G_1| = w \geq p.$$  

In particular, $e_L \geq p \geq 2$, so $e_L + 1 \geq 3$. Thus, formula (5.2) becomes

$$\left(\frac{\sigma(\pi_L)}{\pi_L}\right)^pt = 1 + pt \cdot u_\sigma \cdot \pi_L + \left(\frac{pt}{2}\right) \cdot (u_\sigma)^2 \cdot (\pi_L)^2 + \cdots$$

$$= 1 + ts \cdot u_\sigma \cdot (\pi_L)^{e_L+1} + \left(\frac{pt}{2}\right) \cdot (u_\sigma)^2 \cdot (\pi_L)^2 + \cdots$$

$$= 1 + \left(\frac{pt}{2}\right) \cdot (u_\sigma)^2 \cdot (\pi_L)^2 + \cdots$$

We notice that $p|\left(\frac{pt}{2}\right)$, hence the above considerations apply to the remaining term $\left(\frac{pt}{2}\right) \cdot (u_\sigma)^2 \cdot (\pi_L)^2$. It follows that $(\pi_L)^3$ divides $\left(\frac{pt}{2}\right) \cdot (u_\sigma)^2 \cdot (\pi_L)^2$. Thus, we draw $\left(\frac{\sigma(\pi_L)}{\pi_L}\right)^pt \in U^3_L$. As a result,

$$\left(\frac{\sigma(\pi_L)}{\pi_L}\right)^pt \mapsto 0 \in \lambda_2 \cong U^2_L/U^3_L.$$  

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This means $\beta \in \text{Hom}(G_1, \lambda_2)$ is the trivial homomorphism, and therefore $H^1(G, U^2_L) \to \text{Hom}(G_1, \lambda_2)$ is trivial. Hence,

$$H^1(G, U^3_L) = \text{Ker} \left( H^1(G, U^2_L) \to \text{Hom}(G_1, \lambda_2) \right) = H^1(G, U^2_L).$$

We conclude:

**Proposition 5.4.** Suppose $t > 2$ and $G_1/G_2 \not= 1$. Then, we have

$$H^1(G, U^3_L) = H^1(G, U^2_L) = \langle f_{pt}^t \rangle.$$

If we combine the above argument with induction, then the following holds:

**Proposition 5.5.** Suppose $G_1/G_2 \not= 1$. Then, for any integer $j$ such that $j \leq t$ and $2 \leq j \leq e_L + 1$, we have:

$$H^1(G, U^j_L) = H^1(G, U^2_L) = \langle f_{pt}^t \rangle.$$

### 5.4 Discussions on general situations

In this subsection, we discuss the general situation for $U^{i+1}_L \subseteq U^i_L$, where the assumption $t > i$ is removed. In Section 5.1, we already showed that $H^1(G, U^{i+1}_L)$ is a subgroup of $H^1(G, U^i_L)$ if $t$ does not divide $i$. In what follows, we assume $t|i$.

At the beginning we had the sequence

$$1 \to U^{i+1}_L \to U^i_L \to \lambda_i \to 0,$$

which yields a long exact sequence:

$$1 \to (U^{i+1}_L)^G \to (U^i_L)^G \to (\lambda_i)^G \to H^1(G, U^{i+1}_L) \to H^1(G, U^i_L) \to H^1(G, \lambda_i) \to \cdots \quad (5.3)$$

If $t$ divides $i$, then $(\lambda_i)^G \not= 0$, and we have found that $(\lambda_i)^G = b^{-1} \cdot \kappa$, with $b$ depending on $i$. Thus $(\lambda_i)^G$ is isomorphic to $\kappa$ as abelian groups.

**Proposition 5.6.** If the ramification index $e$ divides $i$, then $H^1(G, U^{i+1}_L)$ is a subgroup of $H^1(G, U^i_L)$.
Proof. We have figured out that

\[(U^i_L)^G = U^{i+1}_K.\]

When \(e|i\), we have \(\left\lceil \frac{i+1}{e} \right\rceil = \frac{i}{e} + 1 = \left\lceil \frac{i}{e} \right\rceil + 1\). If we denote the quotient \(\frac{i}{e}\) by \(j\), then we have

\[(U^i_L)^G = U^j_K, \quad \text{and} \quad (U^{i+1}_L)^G = U^{j+1}_K.\]

As a result, the long exact sequence (5.3) becomes:

\[
1 \to U^j_K \to U^j_G \to (\lambda_i)^G \to H^1(G, U^{i+1}_L) \to H^1(G, U^i_L) \to H^1(G, \lambda_i) \to \cdots.
\]

The image of the map \(U^j_K \to (\lambda_i)^G\) is isomorphic to \(U^j_K / U^{j+1}_K \cong \kappa\). Therefore, this image has the same cardinality with \((\lambda_i)^G\). As a result, the map in the above sequence \(U^j_K \to (\lambda_i)^G\) is surjective. The surjectivity of \(U^j_K \to (\lambda_i)^G\) breaks the long sequence into two parts, and the second part reads:

\[
0 \to H^1(G, U^{i+1}_L) \to H^1(G, U^i_L) \to H^1(G, \lambda_i) \to \cdots.
\]

This settles the proof. \(\square\)

Next, we assume that \(e\) does not divide \(i\). We remark that this (along with \(t|\lambda_i\)) implies \(w \neq 1\), which is our assumption from the beginning. Then, we have

\[
\left\lceil \frac{i+1}{e} \right\rceil = \left\lceil \frac{i}{e} \right\rceil.
\]

With the same notation as above, we then have

\[(U^i_L)^G = (U^{i+1}_L)^G = U^j_K.\]

As a result, the sequence (5.3) now becomes:

\[
1 \to U^j_K \to U^j_G \to (\lambda_i)^G \to H^1(G, U^{i+1}_L) \to H^1(G, U^i_L) \to H^1(G, \lambda_i) \to \cdots.
\]

The map \(U^j_K \to U^j_G\) is induced from the inclusion map \(U^{i+1}_L \to U^i_L\) via taking invariants, hence \(U^j_K \to U^j_G\) is the identity map in the above long exact sequence. Therefore, we draw:

\[
0 \to (\lambda_i)^G \to H^1(G, U^{i+1}_L) \to H^1(G, U^i_L) \to H^1(G, \lambda_i) \to \cdots.
\]

In particular, if we denote by \(K_i\) the kernel of the map \(H^1(G, U^i_L) \to H^1(G, \lambda_i)\), then the following holds:

**Proposition 5.7.** Suppose \(t\) divides \(i\) but \(e\) does not divide \(i\), then there is a short exact sequence of abelian groups:

\[
0 \to (\lambda_i)^G \to H^1(G, U^{i+1}_L) \to K_i \to 0.
\]
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