POINCARE ALGEBRA AND SPACE-TIME CRITICAL DIMENSIONS FOR PARABOSONIC STRINGS*

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Abstract

We construct the parabosonic string formalism based on the paraquantization of both the center of mass variables and the excitation modes of the string. A critical study of the different commutators of the Poincaré algebra based on the redefinition of its generators and the direct treatment using trilinear relations is done. Space-time critical dimensions $D$ as functions of the paraquantization order $Q$ are obtained.

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1 Introduction

As far back as 1950, Wigner [1] demonstrated that, for satisfying the wave particle duality, which is a direct consequence of the Heisenberg equations of motion, the set of canonical commutation relations $[q_i, p_j] = i\hbar \delta_{ij}$, $[q_i, q_j] = [p_i, p_j] = 0$ (which correspond to the usual procedure of canonical quantization) is a possible solution, but is by no means the only one. Paraquantization, as a generalization of the quantization, was first introduced by Green [2]. Indeed, one basing itself on trilinear commutation relations, paraquantization consists of a generalization of creation-annihilation operators algebra for bosons and fermions. We note also that the paraquantization is characterized by a parameter $Q$, the order of paraquantization, such that $Q = 1$ corresponds to the ordinary quantization. The details of these questions can be found in [3]. This work consists in doing a critical study of Poincaré algebra in the case of parabosonic strings. To set the notations, we begin with a brief summary of some familiar results in bosonic string theory.

The action is postulated as [4], [5].

$$S = -\int_{\tau_0}^{\tau_1} d\tau \int_0^\pi d\sigma L$$

(1)

with the lagrangian

$$L = \frac{1}{2\pi\alpha'} \sqrt{\left(\dot{X}X'\right)^2 - \dot{X}^2X'^2}$$

(2)

where

$$\dot{X}^\mu = \frac{\partial X^\mu (\sigma, \tau)}{\partial \tau}$$

(3)

$$X'^\mu = \frac{\partial X^\mu (\sigma, \tau)}{\partial \sigma}$$

(4)

$\tau$ is a time like evolution parameter, while the parameter $\sigma$ labels points on the string. Notice that the conjugate momentum to $X^\mu (\sigma, \tau)$ is $P^\mu (\sigma, \tau) = -\frac{\partial L}{\partial \dot{X}_\mu}$

In the orthonormal gauge, the equations of motion become linear. The solutions are:

$$X^\mu (\sigma, \tau) = x^\mu + p^\mu \tau + \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu (0) \exp (-in\tau) \cos n\sigma$$

(5)

where $x^\mu$ and $p^\mu$ are respectively the "center of mass" coordinates and the total energy momentum of the string. The total angular momentum of the string is given by

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_n^\mu \alpha_n^\nu - \alpha_n^\nu \alpha_n^\mu\right)$$

(6)

A first study of the paraquantum Poincaré algebra was done by F. Ardalan and F. Mansouri [6]. This study is based on the particular manner in which the center of mass variables of
the string are to be handled. Indeed, these authors impose on the center of mass coordinates and the total energy momentum operators of the string to satisfy ordinary commutation relations. This is done by the choice of a specific direction in the paraspace of the Green components (see (19)). This requires relative paracommutation relations between the center of mass coordinates and the excitation modes of the string. In this hypothesis, they find that the resulting theory is Poincaré invariant if the dimension $D$ of the space-time and the order $Q$ of the paraquantization are related by the expression $D = 2 + \frac{24}{Q}$.

Notice that, with the same hypothesis, this relation has been found by other methods [7], [8].

This work consists in paraquantizing the theory by reinterpreting the classical dynamical variables $X^\mu (\sigma, \tau)$ and $P^\mu (\sigma, \tau)$ as operators satisfying the so-called trilinear commutation relations (subsection 2.1). This is done by requiring that both the center of mass variables and the excitation modes of the string verify paraquantum relations (11-18) (subsection 2.1). The first commutator of the Poincaré algebra (\([p^\mu, p^\nu] = 0\)) means that the operators $p^\mu$ obey to bilinear commutation relations which is against the paracommutation relations based on trilinear relations as for example $[p^\mu, [p^\nu, p^\rho]] = 0$. Nevertheless, with the only use of trilinear relations we prove that the two other commutators of the algebra are satisfied and that the relation between the space-time dimension $D$ and the order of the paraquantization $Q$ is still $D = 2 + \frac{24}{Q}$.

2 Paraquantum formalism of bosonic strings

2.1 Covariant gauge

The paraquantization of the theory is carried out by reinterpreting the classical dynamical variables $X^\mu (\sigma, \tau)$ and $P^\mu (\sigma, \tau)$ as operators satisfying the so-called trilinear commutation relations

\[
[X^\mu (\sigma, \tau), [P^\nu (\sigma', \tau), P^\rho (\sigma'', \tau)]]_+ = 2i g^{\mu\nu} P^\rho \delta (\sigma - \sigma') + 2i g^{\mu\rho} P^\nu \delta (\sigma - \sigma'')
\]

(7)

\[
[P^\mu (\sigma, \tau), [X^\nu (\sigma', \tau), X^\rho (\sigma'', \tau)]]_+ = -2i g^{\mu\nu} X^\rho \delta (\sigma - \sigma') - 2i g^{\mu\rho} X^\nu \delta (\sigma - \sigma'')
\]

(8)

\[
[X^\mu (\sigma, \tau), [X^\nu (\sigma', \tau), P^\rho (\sigma'', \tau)]]_+ = 2i g^{\mu\rho} X^\nu \delta (\sigma - \sigma'')
\]

(9)

\[
[P^\mu (\sigma, \tau), [X^\nu (\sigma', \tau), P^\rho (\sigma'', \tau)]]_+ = 2i g^{\mu\nu} P^\rho \delta (\sigma - \sigma')
\]

(10)

Rewritten in terms of $x^\mu$, $p^\mu$ and $\alpha^\mu_n$ defined by (5), equations (7-10) are equivalent to:

\[
[x^\mu, [x^\nu, p^\rho]]_+ = 2i g^{\alpha\mu} x^\nu
\]

(11)

\[
[x^\mu, [p^\nu, p^\rho]]_+ = 2i (g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu)
\]

(12)

\[
[x^\mu, [p^\nu, \alpha^\nu_n]]_+ = 2i g^{\nu\mu} \alpha^\rho_n
\]

(13)

\[
[p^\mu, [x^\nu, p^\rho]]_+ = -2i g^{\mu\rho} p^\nu
\]

(14)

\[
[p^\mu, [x^\nu, x^\rho]]_+ = -2i (g^{\mu\nu} x^\rho + g^{\mu\rho} x^\nu)
\]

(15)

\[
[\alpha^\nu_n, [\alpha^\nu_n, \alpha^\nu_l]]_+ = 2 (g^{\mu\nu} n \delta_{n+m,0} \alpha^\nu_l + g^{\mu\rho} n \delta_{n+l,0} \alpha^\nu_m)
\]

(16)

\[
[\alpha^\mu_n, [p^\nu, \alpha^\rho_m]]_+ = 2ng^{\mu\rho} \delta_{n+m,0} p^\nu
\]

(17)

\[
[\alpha^\mu_n, [x^\nu, \alpha^\rho_m]]_+ = 2ng^{\mu\rho} \delta_{n+m,0} x^\nu
\]

(18)
and all the other commutators are null.

Next we introduce the Green decomposition of the operators $x^\mu$, $p^\mu$ and $\alpha_n^\mu$ defined by:

$$x^\mu = \sum_{\alpha=1}^{Q} x^{\mu(\alpha)}; \quad p^\mu = \sum_{\alpha=1}^{Q} p^{\mu(\alpha)}; \quad \alpha_n^\mu = \sum_{\beta=1}^{Q} \alpha_n^{\mu(\beta)}$$

where Q is the order of paraquantization, such that the trilinear commutation relations (7-10) transform to bilinear commutation relations of an anomalous case

$$[X^{\mu(\alpha)}(\sigma, \tau), P^{\nu(\alpha)}(\sigma', \tau)] = ig^{\mu\nu}\delta(\sigma - \sigma')$$

$$[X^{\mu(\alpha)}(\sigma, \tau), P^{\nu(\beta)}(\sigma', \tau)]_+ = 0 \quad \text{for} \quad \alpha \neq \beta$$

$$[X^{\mu(\alpha)}(\sigma, \tau), X^{\nu(\alpha)}(\sigma', \tau)] = [P^{\mu(\alpha)}(\sigma, \tau), P^{\nu(\alpha)}(\sigma', \tau)] = 0$$

$$[X^{\mu(\alpha)}(\sigma, \tau), X^{\nu(\beta)}(\sigma', \tau)]_+ = [P^{\mu(\alpha)}(\sigma, \tau), P^{\nu(\beta)}(\sigma', \tau)]_+ = 0 \quad \text{for} \quad \alpha \neq \beta$$

In the same way for the relations (11-18)

$$[x^{\mu(\sigma)}, p^{\nu(\sigma)}] = ig^{\mu\nu}$$

$$[x^{\mu(\sigma_1)}, p^{\nu(\sigma_2)}]_+ = 0 \quad \sigma_1 \neq \sigma_2$$

$$[p^{\mu(\sigma)}, p^{\nu(\sigma)}]_+ = [X^{\mu(\sigma)}, X^{\nu(\sigma)}]_+ = 0$$

$$[p^{\mu(\sigma_1)}, p^{\nu(\sigma_2)}]_+ = [X^{\mu(\sigma_1)}, X^{\nu(\sigma_2)}]_+ = 0 \quad \sigma_1 \neq \sigma_2$$

$$[\alpha_n^{\mu(\sigma)}, \alpha_n^{\nu(\sigma)}]_+ = n(g^{\mu\nu}\delta_{n+m,0})$$

$$[\alpha_n^{\mu(\sigma_1)}, \alpha_n^{\nu(\sigma_2)}]_+ = 0 \quad \sigma_1 \neq \sigma_2$$

$$[X^{\mu(\sigma)}_n, \alpha_n^{\nu(\sigma)}] = [P^{\mu(\sigma)}_n, \alpha_n^{\nu(\sigma)}] = 0$$

In the same way for the relations (11-18)

$$[x^{\mu(\sigma)}, p^{\nu(\sigma)}] = ig^{\mu\nu}$$

Notice here that in the Ardalan and Mansouri hypothesis [6], $[X^{\mu(\sigma)}(\sigma, \tau), P^{\nu(\alpha)}(\sigma', \tau)] = ig^{\mu\nu} [\delta(\sigma - \sigma') - (1 - \delta_{\alpha 1})]$ which is not compatible with the relations (7-10).

### 2.2 Transverse gauge

In the same way as before, paraquantizing the theory in this gauge comes down to reinterpret the independant classical dynamical variables $x^{-}, p^{+}, x^{i}, p^{i}$ and $\alpha_n^{i}$ as operators satisfying the paracommutation relations:

$$[x^i, [p^j, p^k]]_+ = 2i (g^{ij}p^k + g^{jk}p^i)$$

$$[\alpha_n^i, [\alpha_n^j, \alpha_n^k]]_+ = 2 (g^{ij}n\delta_{n+m,0}\alpha_n^k + g^{jk}n\delta_{n+l,0}\alpha_n^j)$$

$$[x^i, [p^j, A]]_+ = 2i\delta_{ij}A$$

$$[x^-, [p^+, B]]_+ = 2iB$$

$$[\alpha_n^i, [\alpha_n^j, C]]_+ = 2n\delta_{n+m,0}\delta_{ij}C$$

4
where $A, B,$ and $C$ are given by:

$A = x^{-}, p^{+}, x^{k},$ or $\alpha_{n}^{k}.$

$B = x^{+}, x^{k}, p^{k}$ or $\alpha_{n}^{k}.$

$C = x^{-}, p^{+}, x^{k}$ or $p^{k}.$

Similarly, applying the Green decomposition

$$x^{i} = \sum_{\alpha=1}^{Q} x^{i(\alpha)}; \quad p^{i} = \sum_{\alpha=1}^{Q} p^{i(\alpha)}; \quad \alpha^{i}_{n} = \sum_{\beta=1}^{Q} \alpha^{i(\beta)}_{n}$$

$$x^{-} = \sum_{\alpha=1}^{Q} x^{-}(\alpha); \quad p^{+} = \sum_{\alpha=1}^{Q} p^{+}(\alpha)$$

the set of equations (22) is equivalent to the bilinear relations

$$[x^{i(\alpha)}, p^{j(\beta)}] = i \delta^{ij}; \quad [x^{i(\alpha)}, p^{j(\beta)}^+] = 0 \quad \alpha \neq \beta$$

$$[x^{-}(\alpha), p^{+}(\beta)] = i \delta^{ij}; \quad [x^{-}(\alpha), p^{+}(\beta)]^+ = 0 \quad \alpha \neq \beta$$

$$[\alpha^{i(\alpha)}_{n}, \alpha^{j(\beta)}_{m}] = i n \delta^{ij} \delta_{n+m,0}; \quad [\alpha^{i(\alpha)}_{n}, \alpha^{j(\beta)}_{m}]^+ = 0 \quad \alpha \neq \beta$$

and all the other commutators (and anticommutators) of the type $[A^{(\alpha)}, B^{(\beta)}] = 0$ (and $[A^{(\alpha)}, B^{(\beta)}]^+ = 0,$ for $\alpha \neq \beta$).

3 Paraquantum Poincaré algebra

3.1 Introduction

Just as an example, we begin by showing that, if we take the classical form of the Poincaré algebra generators in terms of operators (like in the ordinary case), the direct use of the trilinear paracommutation relations is not possible and, of course, the algebra is completely violated. These generators are then written in the form

$$M^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}.$$ 

where

$$l^{\mu\nu} = \sum_{\alpha,\beta=1}^{Q} [x^{\mu(\alpha)} p^{\nu(\beta)} - x^{\nu(\alpha)} p^{\mu(\beta)}]$$

and

$$E^{\mu\nu} = \sum_{\alpha,\beta=1}^{Q} \left[ -i \sum_{n=1}^{+\infty} \frac{1}{n} (\alpha^{\mu(\alpha)}_{-n} \alpha^{\nu(\beta)}_{n} - \alpha^{\nu(\alpha)}_{-n} \alpha^{\mu(\beta)}_{n}) \right]$$
Let us consider the first commutator \([p^\mu, p^\nu]\). It is an easy matter to perform this commutator which gives

\[
[p^\mu, p^\nu] = 2 \sum_{\alpha \neq \beta} p^{\mu(\alpha)} p^{\nu(\beta)} \neq 0!
\]

(27)

We now perform the second commutator of the algebra

\[
[p^\mu, M^{\alpha\beta}] = [p^\mu, l^{\alpha\beta}] + [p^\mu, E^{\alpha\beta}]
\]

(28)

Taking the Green decomposition with the use of (21), the first term gives

\[
[p^\mu, l^{\alpha\beta}] = -i Q \left( g^{\mu\beta} p^\alpha - g^{\mu\alpha} p^\beta \right) + 2 \sum_{\sigma_1 \neq \sigma_2}^Q \left( p^{\mu(\sigma_1)} x^{\alpha(\sigma_2)} p^\beta - p^{\mu(\sigma_1)} x^{\beta(\sigma_2)} p^\alpha + x^{\alpha} p^{\mu(\sigma_1)} p^{\beta(\sigma_2)} - x^{\beta} p^{\mu(\sigma_1)} p^{\alpha(\sigma_2)} \right)
\]

(29)

Similarly for the second term, we obtain

\[
[p^\mu, E^{\alpha\beta}] = -2i \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{\sigma_1 \neq \sigma_2} \left( p^{\mu(\sigma_1)} \alpha^{\alpha(\sigma_2)} \alpha^{\beta(\sigma_2)} - p^{\mu(\sigma_1)} \alpha^{\beta(\sigma_2)} \alpha^{\alpha(\sigma_2)} + \alpha^{\alpha} p^{\mu(\sigma_1)} \alpha^{\beta(\sigma_2)} \alpha^{\alpha(\sigma_2)} - \alpha^{\beta} p^{\mu(\sigma_1)} \alpha^{\alpha(\sigma_2)} \alpha^{\alpha(\sigma_2)} \right)
\]

(30)

When combined with (29) and (30), (28) gives

\[
[p^\mu, M^{\alpha\beta}] = -i Q \left( g^{\mu\beta} p^\alpha - g^{\mu\alpha} p^\beta \right) + 2 \sum_{\sigma_1 \neq \sigma_2} \left\{ p^{\mu(\sigma_1)} x^{\alpha(\sigma_2)} p^\beta - p^{\mu(\sigma_1)} x^{\beta(\sigma_2)} p^\alpha + x^{\alpha} p^{\mu(\sigma_1)} p^{\beta(\sigma_2)} - x^{\beta} p^{\mu(\sigma_1)} p^{\alpha(\sigma_2)} \right\} - i \sum_{n=1}^{+\infty} \frac{1}{n} \left( p^{\mu(\sigma_1)} \alpha^{\alpha(\sigma_2)} \alpha^{\beta(\sigma_2)} - p^{\mu(\sigma_1)} \alpha^{\beta(\sigma_2)} \alpha^{\alpha(\sigma_2)} + \alpha^{\alpha} p^{\mu(\sigma_1)} \alpha^{\beta(\sigma_2)} \alpha^{\alpha(\sigma_2)} - \alpha^{\beta} p^{\mu(\sigma_1)} \alpha^{\alpha(\sigma_2)} \alpha^{\alpha(\sigma_2)} \right)
\]

(31)

The third commutator of the algebra can be performed as follows:

\[
[M^{\mu\nu}, M^{\alpha\beta}] = [\mu^{\mu\nu}, l^{\alpha\beta}] + ([\mu^{\mu\nu}, E^{\alpha\beta}] - (\mu \leftrightarrow \alpha, \nu \leftrightarrow \beta)) + [E^{\mu\nu}, E^{\alpha\beta}]
\]

(32)

The first term is

\[
[\mu^{\mu\nu}, l^{\alpha\beta}] = [(x^\mu p^\nu - x^\nu p^\mu), l^{\alpha\beta}]
\]

(33)

The relation (29) immediately leads us to

\[
[x^\mu, l^{\alpha\beta}] = i Q \left( g^{\mu\alpha} x^{\beta} - g^{\mu\beta} x^{\alpha} \right) + 2 \sum_{\sigma_1 \neq \sigma_2} \left( x^{\mu(\sigma_1)} x^{\alpha(\sigma_2)} p^\beta - x^{\mu(\sigma_1)} x^{\beta(\sigma_2)} p^\alpha + x^{\alpha} x^{\mu(\sigma_1)} p^{\beta(\sigma_2)} - x^{\beta} x^{\mu(\sigma_1)} p^{\alpha(\sigma_2)} \right)
\]

(34)
The relation (33), together with (29) and (34) lead us to

$$[l^{\mu\nu}, l^{\alpha\beta}] = \imath Q \left( g^{\mu\beta} l^{\nu\alpha} - g^{\mu\alpha} l^{\nu\beta} + g^{\nu\alpha} l^{\beta\mu} - g^{\nu\beta} l^{\alpha\mu} \right) + 2 \sum_{\sigma_1 \neq \sigma_2} \sum_{\sigma_3, \sigma_4} \mathcal{E}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta}$$

(35)

with

$$\mathcal{E}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} = \left\{ x^{\mu(\sigma_1)} x^{\nu(\sigma_2)} + x^{\mu(\sigma_3)} x^{\nu(\sigma_4)} + x^{\mu(\sigma_1)} x^{\nu(\sigma_2)} + x^{\mu(\sigma_3)} x^{\nu(\sigma_4)} + x^{\mu(\sigma_1)} x^{\nu(\sigma_2)} \right\}$$

(36)

Similarly, with the use of (21) one can perform the other commutators of (32) and find:

$$[l^{\mu\nu}, E^{\alpha\beta}] = -2\imath \sum_{\sigma_1 \neq \sigma_2} \sum_{\sigma_3, \sigma_4} \sum_{n=1}^{+\infty} \frac{1}{n} \Theta_{n,\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta}$$

(37)

with

$$\Theta_{n,\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} = \mathcal{E}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} \left( x^{\alpha} \rightarrow \alpha^\alpha_n, p^{\alpha} \rightarrow \alpha^\beta_n \right)$$

(38)

and

$$[E^{\mu\nu}, E^{\alpha\beta}] = +\imath Q \left( g^{\mu\beta} E^{\nu\alpha} - g^{\mu\alpha} E^{\nu\beta} + g^{\nu\alpha} E^{\beta\mu} - g^{\nu\beta} E^{\alpha\mu} \right)$$

$$- 2 \sum_{\sigma_1 \neq \sigma_2} \sum_{\sigma_3, \sigma_4} \sum_{n,m=1}^{+\infty} \frac{1}{nm} K_{n,m,\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta}$$

(39)

where

$$K_{n,m,\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} = \mathcal{E}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} \left( x^{\mu} \rightarrow \alpha^\mu_n, p^{\mu} \rightarrow \alpha^\nu_n, x^{\nu} \rightarrow \alpha^\alpha_n, p^{\nu} \rightarrow \alpha^\beta_n, x^{\alpha} \rightarrow \alpha^\alpha_m, p^{\alpha} \rightarrow \alpha^\beta_m \right)$$

(40)

The substitution of (36), (37) and (39) in (32) gives

$$[M^{\mu\nu}, M^{\alpha\beta}] = \imath Q \left( g^{\mu\beta} M^{\nu\alpha} - g^{\mu\alpha} M^{\nu\beta} + g^{\nu\alpha} M^{\beta\mu} - g^{\nu\beta} M^{\alpha\mu} \right)$$

$$+ 2 \left\{ \sum_{\sigma_1 \neq \sigma_2} \sum_{\sigma_3, \sigma_4} \mathcal{E}_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} + \sum_{n=1}^{+\infty} \frac{1}{n} \left( \Theta_{n,\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\alpha\beta} - \Theta_{n,\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{\mu\nu\alpha\beta} \right) \right\}$$

(41)

Clearly, such unfamiliar results in (27), (31) and (41) are attributable to the fact that if the Green indices are different, the bilinear relations between the Green components become anomalous. We thus conclude that the Poincaré algebra is violated. It should be noted here that, on one hand, if we set \( Q = 1 \) (which correspond to the ordinary case), the Poincaré algebra become satisfied (indeed, \( Q = 1 \) means that all the terms which contain a summation of the type \( \sum_{\sigma_1 \neq \sigma_2} \) will be eliminated), and on the other hand, to perform these commutators we must use the Green decomposition.
3.2 Possible calculation methods

We begin by remarking that, if in Quantum Mechanics, the correspondance principle.

\[ QM : (x^\mu p^\nu - p^\nu x^\mu) \rightarrow (x^\mu_{op} p^\nu_{op} - x^\nu_{op} p^\mu_{op}) \]

doesn’t cause any order ambiguity problem, it is clearly not the case for the Paraquantum Mechanics. This leads us to generalize it by the correspondance

\[ PQM : (x^\mu p^\nu - p^\nu x^\mu) \rightarrow \frac{1}{2} \left\{ [x^\mu_{op}, p^\nu_{op}]_+ - [x^\nu_{op}, p^\mu_{op}]_+ \right\} \]

We then rewrite the generators \( M^{\mu\nu} \) basing on an adequate symetrization which takes the form \( M^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} \) with:

\[ l^{\mu\nu} = \frac{1}{2} [x^\mu, p^\nu]_+ - [x^\nu, p^\mu]_+ \quad (42) \]

and

\[ E^{\mu\nu} = -\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha^\mu_{-n}, \alpha^\nu_n]_+ - [\alpha^\nu_{-n}, \alpha^\mu_n]_+ \right) \quad (43) \]

Now, if this writing allows the elimination of the order ambiguities, it also allows the paraquantum treatment of the problem solely with trilinear relations (11-18) without having recourse to the Green representation (21). One can nevertheless find again the same results with the use of the Green decomposition.

3.2.1 Direct application of the trilinear relations:

Let us perform the second commutator of the Poincaré algebra

\[ [p^\mu, M^{\nu\rho}] = \left[ p^\mu, \left\{ \frac{1}{2} [x^\nu, p^\rho]_+ - [x^\rho, p^\nu]_+ \right\} \right] \]

\[ = -i \sum_{n=1}^{\infty} \frac{1}{n} \left[ p^\mu, \left\{ [\alpha^\nu_{-n}, \alpha^\rho_n]_+ - [\alpha^\rho_{-n}, \alpha^\nu_n]_+ \right\} \right] \quad (44) \]

With the use of (14), (44) gives

\[ [p^\mu, M^{\nu\rho}] = -ig^{\mu\nu} p^\rho + ig^{\mu\rho} p^\nu \quad (45) \]

a result which satisfy Poincaré algebra. Similarly, for the third commutator of the algebra, one can write:

\[ [M^{\mu\nu}, M^{\rho\sigma}] = [l^{\mu\nu}, l^{\rho\sigma}] + [E^{\mu\nu}, E^{\rho\sigma}] + [l^{\mu\nu}, E^{\rho\sigma}] + [E^{\mu\nu}, l^{\rho\sigma}] \quad (46) \]

With the direct use of the trilinear relations (11), (14), one can perform the first term and find (Appendix A)

\[ [l^{\mu\nu}, l^{\rho\sigma}] = ig^{\nu\rho} l^{\mu\sigma} - ig^{\mu\sigma} l^{\nu\rho} - ig^{\nu\sigma} l^{\mu\rho} - ig^{\mu\rho} l^{\nu\sigma} \quad (47) \]
In the same way, one can perform the second term and obtain (Appendix B)

$$[E^\mu{}^\nu, E^\rho{}^\sigma] = i (g^{\nu\rho} E^\sigma{}^\mu + g^{\rho\mu} E^\nu{}^\sigma + g^{\nu\sigma} E^\mu{}^\rho + g^{\rho\sigma} E^\mu{}^\nu)$$

(48)

Now the relation $[x^\mu{}^\rho, [\alpha_{-n}^\rho, \alpha_{n}^\sigma]] = 0$, leads to the result $[l^\mu{}^\nu, E^\rho{}^\sigma] = [E^\mu{}^\nu, l^\rho{}^\sigma] = 0$

When combined with (47) and (48), (46) gives

$$[M^\mu{}^\nu, M^\rho{}^\sigma] = i g^{\nu\rho} M^\sigma{}^\mu - i g^{\rho\sigma} M^\mu{}^\nu - i g^{\nu\sigma} M^\mu{}^\rho + i g^{\rho\mu} M^\nu{}^\sigma$$

(49)

3.2.2 Green decomposition

In terms of Green components, the generators $M^\mu{}^\nu$ are given by:

$$M^\mu{}^\nu = \frac{1}{2} \sum_{\sigma_1}^{Q} \sum_{\sigma_2}^{Q} \left\{ [x^{\mu(\sigma_1)}, p^{\nu(\sigma_2)}]_+ - [x^{\nu(\sigma_1)}, p^{\mu(\sigma_2)}]_+ ight\}$$

$$+ i \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha_{-n}^{\mu(\sigma_1)}, \alpha_{n}^{\nu(\sigma_2)}]_+ - [\alpha_{-n}^{\nu(\sigma_1)}, \alpha_{n}^{\mu(\sigma_2)}]_+ \right)$$

(50)

From the relations (21), (50) can be rewritten as:

$$M^\mu{}^\nu = \sum_{\sigma_1}^{Q} \left\{ x^{\mu(\sigma_1)} p^{\nu(\sigma_2)} - x^{\nu(\sigma_1)} p^{\mu(\sigma_2)} - i \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_{-n}^{\mu(\sigma_1)} \alpha_{n}^{\nu(\sigma_2)} - \alpha_{-n}^{\nu(\sigma_1)} \alpha_{n}^{\mu(\sigma_2)} \right) \right\}$$

(51)

Then, the symetrization of $M^\mu{}^\nu$ allows us to obtain the result:

$$M^\mu{}^\nu = \sum_{\sigma_1, \sigma_2}^{Q} M^{\mu\nu(\sigma_1, \sigma_2)} = \sum_{\sigma_1}^{Q} M^{\mu\nu(\sigma)}$$

where $M^{\mu\nu(\sigma)}$ has the ordinary form.

One can then perform the two commutators $[p^\mu, M^{\nu\rho}]$ and $[M^\mu{}^\nu, M^{\rho\sigma}]$ as follows:

$$[p^\mu, M^{\nu\rho}] = \sum_{\sigma_1, \sigma_2}^{Q} \left[ p^{\mu(\sigma_1)}, M^{\nu\rho(\sigma_2)} \right]$$

$$= \sum_{\sigma_1}^{Q} \left[ p^{\mu(\sigma)}, M^{\nu\rho(\sigma)} \right] + \sum_{\sigma_1 \neq \sigma_2} \left[ p^{\mu(\sigma_1)}, M^{\nu\rho(\sigma_2)} \right]$$

(52)

By the use of

$$[A^{(\sigma_1)}, B^{(\sigma_2)}, C^{(\sigma_2)}] = 0 \quad \forall A, B, C \quad \text{for} \quad \sigma_1 \neq \sigma_2$$

(53)

It is clear that

$$[p^{\mu(\sigma_1)}, M^{\nu\rho(\sigma_2)}] = 0 \quad \sigma_1 \neq \sigma_2$$

(54)
and
\[
[p^\mu, \, M^{\nu\rho}] = \sum_{\sigma=1}^{Q} [p^{\mu(\sigma)}, \, M^{\nu\rho(\sigma)}]
\]
\[
= \sum_{\sigma=1}^{Q} (\imath g^{\mu\rho} p^{\nu(\sigma)} - \imath g^{\mu\nu} p^{\rho(\sigma)})
\]
(55)

Then
\[
[p^\mu, \, M^{\nu\rho}] = \imath g^{\mu\rho} p^\nu - \imath g^{\mu\nu} p^\rho
\]
(56)

In the same way
\[
[M^{\mu\nu}, \, M^{\rho\sigma}] = \sum_{\alpha, \beta=1}^{Q} [M^{\mu\nu(\alpha)}, \, M^{\rho\sigma(\beta)}]
\]
(57)

But
\[
[A^{(\alpha)} B^{(\alpha)}, \, C^{(\beta)} D^{(\beta)}] = 0 \quad \alpha \neq \beta
\]
(58)

Then
\[
[M^{\mu\nu}, \, M^{\rho\sigma}] = \sum_{\alpha=1}^{Q} [M^{\mu\nu(\alpha)}, \, M^{\rho\sigma(\alpha)}]
\]
\[
= \sum_{\alpha=1}^{Q} (\imath g^{\nu\rho} M^{\sigma\mu(\alpha)} - \imath g^{\mu\sigma} M^{\nu\rho(\alpha)} - \imath g^{\nu\sigma} M^{\rho\mu(\alpha)} + \imath g^{\mu\sigma} M^{\rho\nu(\alpha)})
\]
(59)

Lastly
\[
[M^{\mu\nu}, \, M^{\rho\sigma}] = \imath g^{\nu\rho} M^{\sigma\mu} - \imath g^{\mu\sigma} M^{\nu\rho} - \imath g^{\nu\sigma} M^{\rho\mu} + \imath g^{\mu\sigma} M^{\rho\nu}
\]
(60)

4 Space-time critical dimensions

Let us introduce, in the transverse gauge, the generators $M^{\mu\nu}$ in the form :
\[
M^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}
\]
(61)

where
\[
l^{\mu\nu} = \frac{1}{2} \left[ x^i, \, \frac{1}{p^+} \right] + \alpha^-_0 - \frac{1}{2} \left[ x^-, \, p^i \right]_+
\]
(62)

and
\[
E^{\mu\nu} = -\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha^-_n, \, \frac{1}{p^+} \right)_+ - \alpha^-_{-n} \left( \alpha^+_n, \, \frac{1}{p^+} \right)_+
\]
(63)
One can write

\[ [M^i, M^j] = [l^i, l^j] + [l^i, E^j] + [E^i, l^j] + [E^i, E^j] \]  \tag{64} \]

It is easy to verify that:

\[ [\alpha^i, \alpha_0^j] = -ip^j \]  \tag{65} \]

and

\[ [x^i, \alpha_n^j] = -i\alpha_n^i \]  \tag{66} \]

By the use of the trilinear relations (22) and the Heisenberg equations, one can verify that:

\[ [l^i, l^j] = \frac{i}{4} \left( \left[ x^i, \frac{1}{p^+} \right] + \left[ p^j, \frac{1}{p^+} \right] - \left[ x^j, \frac{1}{p^+} \right] + \left[ p^i, \frac{1}{p^+} \right] \right) \alpha_0^- \]

\[ - \frac{i}{2} \left( \frac{1}{(p^+)^2} \left[ x^i, p^j \right] + \frac{1}{(p^+)^2} \left[ x^j, p^i \right] \right) \alpha_0^- \]  \tag{67} \]

and

\[ [l^i, E^j] = \frac{i}{4} \sum_n \frac{1}{n} \left\{ \left( \left[ \alpha^i, \frac{1}{p^+} \right] + \left[ \alpha^j, \frac{1}{p^+} \right] - \left[ \alpha'^i, \frac{1}{p^+} \right] - \left[ \alpha'^j, \frac{1}{p^+} \right] \right) \alpha_0^- \right. \]

\[ + \left[ \frac{2}{(p^+)^2} \left[ p^i, \alpha_{-n}^j \right] + \alpha_0^i - \frac{2}{(p^+)^2} \alpha_{-n}^{-} \left[ p^i, \alpha_{-n}^j \right] \right] \} \]  \tag{68} \]

If we take the mean values on the physical states \((\alpha_{-m}^k | 0)\) and by the use of the trilinear relations (22), one can prove that:

\[ \langle 0 | \alpha_m^i [l^i, l^j] \alpha_{-m}^k | 0 \rangle = 0 \]  \tag{69} \]

\[ \langle 0 | \alpha_m^i [l^i, E^j] \alpha_{-m}^k | 0 \rangle = \frac{m^2}{(p^+)^2} (\delta^{li} \delta^{kj} - \delta^{lj} \delta^{ki}) + \frac{m}{(p^+)^2} (\delta^{lj} p^k p^j - \delta^{jk} p^l p^i) \]  \tag{70} \]

Finally

\[ \langle 0 | \alpha_m^i ([l^i, E^j] + [E^i, l^j]) \alpha_{-m}^k | 0 \rangle = 2m^2 \left( \frac{1}{p^+} \right)^2 (\delta^{li} \delta^{kj} - \delta^{lj} \delta^{ki}) \]

\[ + \frac{m}{(p^+)^2} (\delta^{lj} p^k p^j - \delta^{li} p^l p^j - \delta^{jk} p^i p^j + \delta^{ik} p^i p^j) \]  \tag{71} \]

On the other hand, by the use of the trilinear relations (22), one can write:

\[ \langle 0 | \alpha_m^i E^i E^j \alpha_{-m}^k | 0 \rangle = \sum_{i=1}^{4} C_i \]  \tag{72} \]
Where

\[ C_1 = -\frac{1}{4} \sum_{n,n'} \langle 0 | \alpha_m^l \left[ \frac{\alpha^i_n}{n} : \frac{1}{p^+} \right] + \alpha_n^i \left[ \frac{\alpha^j_{n'}}{n'} : \frac{1}{p^+} \right] \alpha_{-n}^j \alpha_{-m}^k | 0 \rangle \]

\[ = - \frac{m}{p^+} \delta^{ik} \left( \frac{1}{2} m (m - 1) \delta_{jk} + \frac{1}{2} \left[ p^j, \frac{1}{p^+} \right] p^k \right) \quad (73) \]

\[ C_2 = \frac{1}{4} \sum_{n,n'} \langle 0 | \alpha_m^l \alpha_n^i \left[ \frac{\alpha^j_n}{n} : \frac{1}{p^+} \right] + \frac{\alpha^j_{n'}}{n'} : \frac{1}{p^+} \right] \alpha_{-m}^k | 0 \rangle \]

\[ = - \frac{m^2}{(p^+)^2} (m - 1) \delta^{ij} \delta^{jk} \quad (74) \]

\[ C_3 = \frac{1}{4} \sum_{n,n'} \langle 0 | \alpha_m^l \alpha_n^i \left[ \frac{\alpha^j_n}{n} : \frac{1}{p^+} \right] + \alpha_{-n}^i \alpha_{-n'}^j \left[ \frac{1}{n'} : \frac{1}{p^+} \right] \alpha_{-m}^k | 0 \rangle \]

\[ = \frac{m}{p^+} \delta^{ij} \left( m \frac{\delta^{ik}}{2p^+} (m - 1) + p^i p^j \right) \quad (75) \]

\[ C_4 = \frac{1}{4} \sum_{n,n'} \langle 0 | \alpha_m^l \left[ \frac{\alpha^j_n}{n} : \frac{1}{p^+} \right] + \alpha_n^i \alpha_n^j \left[ \frac{1}{n'} : \frac{1}{p^+} \right] \alpha_{-m}^k | 0 \rangle \]

\[ = \frac{1}{(p^+)^2} \delta^{ij} \delta^{jk} \left( \frac{Q}{12} \frac{D - 2}{m} (m^2 - 1) + 2m \alpha (0) \right) \quad (76) \]

Lastly

\[ \langle 0 | \alpha_m^l \left[ E^{i-} E^{j-} \alpha^k_{-m} \right] | 0 \rangle = - \left( \frac{1}{p^+} \right)^2 m^2 (m - 1) \left( \delta^{ij} \delta^{jk} - \delta^{ij} \delta^{ik} \right) \]

\[ + \left( \frac{1}{p^+} \right)^2 \delta^{ij} \delta^{jk} \left( \frac{Q}{12} \frac{D - 2}{m} (m^2 - 1) + 2m \alpha (0) \right) + \frac{m}{(p^+)^2} \left( \delta^{jk} p^j p^k - \delta^{ij} p^i p^k \right) \quad (77) \]

Then

\[ \langle 0 | \alpha_m^l \left[ E^{i-}, E^{j-} \right] \alpha_{-m}^k | 0 \rangle = \]

\[ = - \left( \frac{1}{p^+} \right)^2 \left( \delta^{ij} \delta^{jk} - \delta^{ij} \delta^{ik} \right) \left( -2m^2 (m - 1) + Q \frac{D - 2}{12} m (m^2 - 1) + 2m \alpha (0) \right) \]

\[ - \frac{m}{(p^+)^2} \left( \delta^{ij} p^j p^k - \delta^{ij} p^i p^k - \delta^{jk} p^j p^i + \delta^{ik} p^i p^j \right) \quad (78) \]

Collecting these results, one obtain :

\[ \langle 0 | \alpha_m^l \left[ M^{i-}, M^{j-} \right] \alpha^k_{-m} | 0 \rangle = \]

\[ = - \left( \frac{1}{p^+} \right)^2 \left( \delta^{ij} \delta^{jk} - \delta^{ij} \delta^{ik} \right) \left( -2m^3 + Q \frac{D - 2}{12} m (m^2 - 1) + 2m \alpha (0) \right) \quad (79) \]
In terms of operators, this equation is equivalent to:

\[
[M^{i-}, M^{j-}] = 
- \frac{1}{2(p^+)^2} \sum_{n=1}^{\infty} \left( [\alpha^{-}_n, \alpha^j_n]_+ - [\alpha^-_n, \alpha^i_n]_+ \right) \left( -2n + Q \frac{D-2}{12} \left( n - \frac{1}{n} \right) + 2\alpha(0) \frac{1}{n} \right)
\]

(80)

In conclusion, to have \([[M^{i-}, M^{j-}] = 0]\), equation (80) gives

\[
\begin{cases}
D = 2 + \frac{24}{Q} \\
\alpha(0) = 1
\end{cases}
\]

(81)

5 Conclusion

In view of the importance of the Poincaré algebra, the aim of this work is to study the consequences which result from the paraquantization of this algebra in string theory. Unlike Ardalan and Mansouri work [6], in this study, the paraquantization of the theory requires the same treatment for the center of mass variables and the excitation modes of the string. The first consequence is the appearance of noncommuting momentum coordinates expressed by the relation (27) which is equivalent to \([[p^\mu, [p^\nu, p^\sigma]]_+ = 0]\. For the two other commutators of the algebra, the generalization of the correspondance principle permits to redefine the generators \(M^{\mu\nu}\) in both Lorentz and transverse gauges. As a consequence, one can find again the results:

\[
[p^\mu, M^{\nu\rho}] = -i g^{\mu\rho} p^\nu + i g^{\mu\nu} p^\rho
\]

\[
[M^{\mu\nu}, M^{\rho\sigma}] = i g^{\nu\rho} M^{\sigma\mu} - i g^{\mu\sigma} M^{\nu\rho} - i g^{\nu\sigma} M^{\rho\mu} + i g^{\mu\rho} M^{\nu\sigma}
\]

with the use of the trilinear relations or the Green decomposition. In the transverse gauge, to re-establish the commutator \([[M^{i-}, M^{j-}] = 0]\), the space-time critical dimension \(D\) is calculated with the only use of the trilinear relations. Like in Ardalan and Mansouri work [6], \(D\) is again given as a function of the paraquantization order \(Q\) through the relation \(D = 2 + \frac{24}{Q}\). This result opens other existence possibilities of bosonic strings at space-time dimensions \(D = 26, 14, 10, 8, 6, 5, 4\) respectively in orders \(Q = 1, 2, 3, 4, 6, 8, 12\). Notice here that in terms of Green components one gets \([x^\mu, x^\nu] = 2 \sum_{\alpha \neq \beta} x^{(\alpha)} x^{(\beta)} \neq 0\), that is noncommuting coordinates. One may then wonder if this is a kind of noncommuting space-time, since we have in addition noncommuting momentum coordinates.
Appendices

A

One can write

\[ [l^\mu, l^\rho] = \frac{1}{4} \{ [x^\mu, p^\nu], [x^\rho, p^\sigma] \} = \frac{1}{4} (A - B - C + D) \] (82)

Where

\[ A = x^\mu [p^\nu, [x^\rho, p^\sigma]_+] + [x^\mu, [x^\rho, p^\sigma]_+] p^\nu + p^\nu [x^\mu, [x^\rho, p^\sigma]_+] + [p^\nu, [x^\rho, p^\sigma]_+ x^\mu \] (83)

by the use of (11), (14), (83) gives

\[ A = -2i g^{\nu \rho} [x^\mu, p^\sigma]_+ + 2i g^{\mu \sigma} [x^\rho, p^\nu]_+ \] (84)

In the same way, one can compute

\[ B = A (\mu \leftrightarrow \nu) = -2i g^{\mu \rho} [x^\nu, p^\rho]_+ + 2i g^{\rho \sigma} [x^\mu, p^\nu]_+ \] (85)

\[ C = A (\rho \leftrightarrow \sigma) = -2i g^{\rho \sigma} [x^\mu, p^\nu]_+ + 2i g^{\mu \rho} [x^\nu, p^\rho]_+ \] (86)

\[ D = A (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) = -2i g^{\mu \sigma} [x^\nu, p^\rho]_+ + 2i g^{\nu \rho} [x^\sigma, p^\mu]_+ \] (87)

By substitution in (82), one obtain :

\[ [l^\mu, l^\rho] = \frac{i}{2} g^{\mu \rho} ( [x^\sigma, p^\nu]_+ - [x^\mu, p^\sigma]_+ ) + \frac{i}{2} g^{\mu \sigma} ( [x^\rho, p^\nu]_+ - [x^\nu, p^\rho]_+ ) \]

\[ -\frac{i}{2} g^{\nu \rho} ( [x^\mu, p^\sigma]_+ - [x^\nu, p^\rho]_+ ) - \frac{i}{2} g^{\nu \sigma} ( [x^\rho, p^\mu]_+ - [x^\mu, p^\rho]_+ ) \] (88)

Then

\[ [l^\mu, l^\rho] = ig^{\nu \rho} l^\sigma - ig^{\mu \sigma} l^\nu - ig^{\nu \sigma} l^\mu - ig^{\mu \rho} l^\nu \] (89)

B

One can write

\[ [E^\mu, E^\rho] = -\frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \{ [\alpha_n^\mu, \alpha^\nu_m]_+, [\alpha_{-m}^{\rho}, \alpha^\sigma_m]_+ \} - [\nu, \mu \} , (\rho, \sigma)] \]

\[ - [\nu, \mu \} , (\sigma, \rho)] + [(\mu, \nu \}, (\sigma, \rho)] \} \]

\[ = -\frac{1}{4} (A' - B' - C' + D') \] (90)
where

\[ A' = \sum_{n,m=1}^{\infty} \frac{1}{nm} \left\{ \alpha_{-n}^\mu [\alpha_n^\nu, [\alpha_{-m}^\rho, \alpha_{m}^\sigma]] + [\alpha_{-n}^\nu, [\alpha_n^\rho, \alpha_{m}^\sigma]] + [\alpha_n^\mu, [\alpha_{-m}^\rho, \alpha_{m}^\sigma]] + [\alpha_n^\nu, [\alpha_{-m}^\rho, \alpha_{m}^\sigma]] \right\} \]

(91)

By the use of (16), one can write

\[ A' = 2 \sum_{n,m=1}^{\infty} \frac{1}{nm} \left\{ \alpha_{-n}^\mu (g^{\nu\rho} n \delta_{n-m,0} \alpha_m^\sigma + g^{\nu\sigma} n \delta_{n+m,0} \alpha_{-m}^\rho) \\
+ (g^{\mu\rho} (-n) \delta_{n-m,0} \alpha_m^\sigma + g^{\mu\sigma} (-n) \delta_{n+m,0} \alpha_{-m}^\rho) \alpha_n^\nu \\
+ \alpha_{-n}^\nu (g^{\mu\rho} (-n) \delta_{n-m,0} \alpha_m^\sigma + g^{\mu\sigma} (-n) \delta_{n+m,0} \alpha_{-m}^\rho) \\
+ (g^{\nu\rho} n \delta_{n-m,0} \alpha_m^\sigma + g^{\nu\sigma} n \delta_{n+m,0} \alpha_{-m}^\rho) \alpha_{-n}^\mu \right\} \]

(92)

Then

\[ A' = 2 \sum_{n=1}^{\infty} \frac{1}{n} \left\{ g^{\nu\rho} [\alpha_{-n}^\rho, \alpha_n^\sigma] - g^{\nu\sigma} [\alpha_{-n}^\rho, \alpha_n^\nu] \right\} \]

(93)

We can then deduce

\[ -B' = 2 \sum_{n=1}^{\infty} \frac{1}{n} \left\{ -g^{\mu\rho} [\alpha_n^\nu, \alpha_n^\sigma] + g^{\nu\sigma} [\alpha_{-n}^\rho, \alpha_n^\mu] \right\} \]

(94)

\[ -C' = 2 \sum_{n=1}^{\infty} \frac{1}{n} \left\{ -g^{\nu\sigma} [\alpha_{-n}^\mu, \alpha_n^\rho] + g^{\mu\rho} [\alpha_{-n}^\sigma, \alpha_n^\nu] \right\} \]

(95)

\[ D' = 2 \sum_{n=1}^{\infty} \frac{1}{n} \left\{ g^{\mu\rho} [\alpha_n^\nu, \alpha_n^\rho] - g^{\nu\rho} [\alpha_{-n}^\sigma, \alpha_n^\mu] \right\} \]

(96)

And finally

\[ [E^{\mu\nu}, E^{\rho\sigma}] = \mathcal{I} \left\{ g^{\nu\rho} \left( \frac{-i}{2} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha_{-n}^\sigma, \alpha_n^\mu] + [\alpha_{-n}^\mu, \alpha_n^\sigma] \right) \right. \]

\[ + g^{\nu\sigma} \left( \frac{-i}{2} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha_{-n}^\rho, \alpha_n^\mu] + [\alpha_{-n}^\mu, \alpha_n^\rho] \right) \right. \]

\[ + g^{\mu\rho} \left( \frac{-i}{2} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha_n^\nu, \alpha_n^\sigma] + [\alpha_{-n}^\sigma, \alpha_{n}^\nu] \right) \]

\[ + g^{\mu\sigma} \left( \frac{-i}{2} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left( [\alpha_n^\rho, \alpha_n^\mu] + [\alpha_{-n}^\mu, \alpha_n^\rho] \right) \right\} \]

(97)

Which gives

\[ [E^{\mu\nu}, E^{\rho\sigma}] = \mathcal{I} (g^{\nu\rho} E^{\sigma\mu} + g^{\mu\rho} E^{\nu\sigma} + g^{\nu\sigma} E^{\mu\rho} + g^{\mu\sigma} E^{\nu\mu}) \]

(98)
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