SHARP HARDY INEQUALITIES IN THE HALF SPACE
WITH TRACE REMAINDER TERM

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Abstract. In this paper we deal with a class of inequalities which interpolate the Kato’s inequality and the Hardy’s inequality in the half space. Starting from the classical Hardy’s inequality in the half space \( \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty) \), we show that, if we replace the optimal constant \( \left( \frac{n-2}{2} \right)^2 \) with a smaller one \( \left( \frac{\beta-2}{4} \right)^2 \), \( 2 \leq \beta < n \), then we can add an extra trace-term equals to that one that appears in the Kato’s inequality. The constant in the trace remainder term is optimal and it tends to zero when \( \beta \) goes to \( n \), while it is equal to the optimal constant in the Kato’s inequality when \( \beta = 2 \).

1. Introduction

Sobolev spaces play a fundamental role in the study of differential and integral operators especially for their imbedding characteristics. Most of the embedding results assert that, if \( \Omega \) is an open set of \( \mathbb{R}^n \) with smooth boundary, then \( W^{1,p}(\Omega) \) is imbedded into some Lebesgue spaces \( L^q(\Omega) \) or \( L^q(\partial \Omega) \), with \( q > p \). Here we want to consider the particular case \( p = 2 \) and \( \Omega = \mathbb{R}^n_+ = \{ (x, t) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, t > 0 \} \), \( n \geq 3 \), that is the upper half \( n \)-dimensional euclidean space.

The standard trace embedding theorem asserts that if \( u \) is any real valued function on \( \mathbb{R}^n_+ \), sufficiently smooth up to the boundary and decaying fast enough at infinity, then the trace of \( u \) belongs to the Lebesgue space \( L^{2^*}(\partial \mathbb{R}^n_+) \), with \( 2^* = \frac{2(n-1)}{n-2} \) (cf., e.g., [2]). More precisely, the following inequality holds:

\[
(1) \quad \left( \frac{n-2}{2} \right)^{1/2} \omega_{n-1}^{1/2(n-1)} \| u \|_{L^{2^*}(\partial \mathbb{R}^n_+)} \leq \| \nabla u \|_{L^2(\mathbb{R}^n_+)} \quad \forall u \in W^{1,2}(\mathbb{R}^n_+)
\]

where \( \omega_m \) is, now and in the following, the Lebesgue measure of the unit ball in \( \mathbb{R}^m \). The constant that appears in (1) together with the extremal functions

\[
(2) \quad u_a(x, t) = \left( (a + t)^2 + |x|^2 \right)^{-\frac{n}{4} + 1} \quad a \in (0, \infty).
\]

were found in 1988 by Escobar ([23], see [35] for the case \( p \neq 2 \)).

Although optimal in the framework of Lebesgue spaces the trace imbedding (1) admits an improvement in terms of Lorentz spaces. Indeed the following Kato’s inequality holds

\[
(3) \quad 2 \frac{\Gamma^2 \left( \frac{n}{4} \right)}{\Gamma^2 \left( \frac{n-2}{4} \right)} \int_{\partial \mathbb{R}^n_+} \frac{u^2(x, 0)}{|x|} dx \leq \int_{\mathbb{R}^n_+} |\nabla u|^2(x, t) dt \quad u \in W^{1,2}(\mathbb{R}^n_+)
\]

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where, here and in the following, $\Gamma$ is the usual Gamma function defined as $\Gamma(s) = \int_0^\infty t^{s-1} \exp(-t)dt$. The constant given in (3) is optimal but, unlike what happens for the standard trace inequality (1), it is never attained ([32], [20]). Indeed the functions that are candidates to be extremals, are proportional to the solution of the problem

$$\begin{cases}
\Delta \varphi = 0 & \text{in } \mathbb{R}^n_+,

\varphi = |x|^{-\frac{n}{2}+1} & \text{on } \partial \mathbb{R}^n_+.
\end{cases}$$

(then do not belong to $W^{1,2}(\mathbb{R}^n_+)$) and can be expressed in terms of Legendre functions (see [20]). Inequality (3) is an improvement of (1), in the following sense: from inequality (1) we deduce that if $u \in W^{1,2}(\mathbb{R}^n_+)$ then the trace of $u$ belongs to the Sobolev space $L^2^*(\partial \mathbb{R}^n_+)$. In fact, inequality (3) tells us that the trace of $u$ has a higher degree of summability, since one can deduce that it belongs to the Lorentz space $L^{2^*,2}(\partial \mathbb{R}^n_+)$ which is a proper subspace of $L^2^*(\partial \mathbb{R}^n_+)$.

The relation between (1) and (3) is exactly the same that there exists between the classical Sobolev inequality

$$\sqrt{n(n-2)} \left( \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right)^{1/n} \|u\|_{L^\frac{2n}{n-2}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}$$

(5)

and the Hardy inequality

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2(y)}{|y|^2} \, dy \leq \int_{\mathbb{R}^n} |\nabla u|^2(y) \, dy.$$  

(6)

Indeed, the constants that appear in (5) and (6) are optimal ([10], [19], [39], see also [7], [18], [36]), but only that one in (5) is attained. Moreover, inequality (6) strengthens the standard Sobolev embedding of $W^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$, with $2^* = \frac{2n}{n-2}$ in as much as it states that $u \in L^{2^*,2}(\mathbb{R}^n)$ and $L^{2^*,2}(\mathbb{R}^n) \subseteq L^{2^*}(\mathbb{R}^n)$.

Both the theories of boundary trace in Sobolev spaces and of Hardy inequalities have a large number of applications, especially to boundary value problems for partial differential equations and nonlinear analysis. They have been developed, via different methods and in different settings, by various authors (see, for example, [4], [24], [25], [28] or the references on this topic in the monographs [3], [34], [36], [42]). In particular, the lack of extremals in (3) and in (6) has inspired many mathematicians to consider possible extra terms on the left hand side. As regards inequality (6), it has been proven that no extra terms can be added on the left hand side ([16], [17]), while, if $\mathbb{R}^n$ is replaced by a bounded open subset containing the origin, then different type of remainder terms can be considered (see, for example, [5, 6, 9, 11, 12, 13, 14, 15, 21, 20, 22, 26, 27, 29, 30, 31, 33, 40, 41]). A similar result for inequality (3) has been proven in [20] where a Kato’s inequality with a remainder term is proven in the intersection of $\mathbb{R}^n_+$ with a ball centered at the origin.

In this paper we deal with a class of inequalities which interpolate inequalities (3) and (6). Starting from the classical Hardy’s inequality in the half-space, we show that, if we replace the optimal constant with a smaller one, then we can add an extra term equal to that one that appears on the left hand side of (3). Indeed, our aim is to prove the following Theorem.

**Theorem 1.1.** Let $n \geq 3$ and let $u$ be a real function on $\mathbb{R}^n_+$ vanishing at infinity, such that $|\nabla u| \in L^2(\mathbb{R}^n_+)$. Then, for any $2 \leq \beta < n$, there exists a positive constant $H(n, \beta)$ such that

$$H(n, \beta) \int_{\mathbb{R}^n_+} \frac{u^2}{|x|} \, dx + \frac{(\beta - 2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2 + t^2} \, dx \, dt \leq \int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx \, dt.$$  

(7)
The best value of the constant $H(n, \beta)$ is given by

$$H(n, \beta) = 2 \frac{\Gamma \left( \frac{n + \beta}{2} - \frac{1}{2} \right) \Gamma \left( \frac{n - \beta}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{n + \beta}{2} - 1 \right) \Gamma \left( \frac{n - \beta}{2} \right)}.$$  

(8)

It can be easily checked that when $\beta = 2$, then inequality (7) reduces to (3), while when $\beta$ goes to $n$, then inequality (7) reduces to (6). The optimal constant $H(n, \beta)$, as expected, is never attained since the candidates to be extremal functions are proportional to the solution of the problems

$$\begin{align*}
\Delta \varphi + \frac{(\beta-2)^2}{4} \frac{\varphi}{|x|^2 + t^2} &= 0 \quad \text{in } \mathbb{R}^n_+,
\varphi &= |x|^{-\frac{\beta}{2}+1} \quad \text{on } \partial \mathbb{R}^n_+.
\end{align*}$$

These solutions are explicitly given in Section 2 and they are expressed in terms of the hyper-geometric series

$$F(a, b, c; z) = 1 + \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \sum_{k=1}^{\infty} \frac{\Gamma(a+k) \Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}.$$  

(10)

The following Proposition holds.

**Proposition 1.1.** Let $2 \leq \beta < n$ and let $H(n, \beta)$ be the constant defined in (8). Then the functions

$$\varphi(x, t) = \frac{F \left( \frac{n + \beta}{2} - 1, \frac{n - \beta}{2}, 1; \frac{t^2}{|x|^2 + t^2} \right)}{|x|^2 + t^2} - t \frac{H(n, \beta)}{4} F \left( \frac{n + \beta}{4} - \frac{1}{2}, \frac{n - \beta}{4} + \frac{3}{2}; \frac{t^2}{|x|^2 + t^2} \right)$$

are regular solutions of problems (9).

Incidentally, when $\beta = 2$ a much more handle expression of the solution of (9) can be considered. Indeed such solution is proportional to the harmonic function

$$\phi(x, t) = \int_0^\infty \frac{a^{\frac{\beta-2}{2}}}{(a+t)^2 + |x|^2} \frac{da}{a^{\frac{\beta-2}{2}} - 2}.$$  

(12)

obtained by integrating the harmonic functions given in (2) with respect to a suitable weight of the parameter. The choice of the weight is influenced by the required summability on the boundary of $\mathbb{R}^n_+$.

To conclude, let us describe our approach, based on a very classical method of Calculus of Variation (see [38], p. 167) that recently have been adopted in ([8]) to find an improvement of classical Sobolev inequality. This method can be used to prove many different integral inequalities and differs from that one used by Herbst, based on dilation analytic techniques, and from that one used in [20], based on a suitable change of variables (see also [13, 15, 29, 34]). Our approach, instead, is based on the careful construction of a suitable divergence free vector field $\mathbf{F}$ related to the family $\mathcal{G} = \{k \varphi \}_{k \in (0, \infty)}$, where $\varphi$ is defined in (11). The functions of $\mathcal{S}$ are solutions of the Euler equation of the functional

$$J(u) = \int_{\mathbb{R}^n_+} |\nabla u|^2 dx dt - \frac{(\beta-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2 + t^2} dx dt$$

but, unfortunately, they do not belong to $W^{1,2}(\mathbb{R}^n_+)$, since they do not have the right summability neither at the origin nor at infinity. To overlap this inconvenient, we have to evaluate $J(u)$ on an approximating sequence of bounded sets that do not contain the origin. The very definition of $\mathbf{F}$ allows us to estimate $J(u)$ from below with the flow of $\mathbf{F}$ across the graph of $u$. Since $\mathbf{F}$ is divergence free, when $u$ is sufficiently smooth, we can use divergence theorem to prove that the inequality (7) holds. The optimality of the constant is derived by repeating the previous arguments, replacing the function $u$ by the functions of $\mathcal{G}$ and observing that, in such case, all the inequalities hold as equalities.
2. Proof of Proposition 1.1

Writing

\[ \rho = \sqrt{|x|^2 + t^2}, \quad \theta = \arctan \frac{t}{|x|}, \quad \varphi(x, t) = \rho^{-\frac{\theta}{2} + 1} f(\theta) \]

problem (9) is equivalent to the following limit problem

\[
\begin{cases}
  f''(\theta) - (n - 2) \tan \theta f'(\theta) - \left( \frac{(a-2)^2}{4} - \frac{(b-2)^2}{4} \right) f(\theta) = 0 & \theta \in (0, \frac{\pi}{2}) \\
  f(0) = 1 \lim_{\theta \to \frac{\pi}{2}} f(\theta) \in \mathbb{R}.
\end{cases}
\]

Equation (15) is explicitly solved in [37] (p. 271 eq. 131). Indeed \( f(\theta) = w(\sin^2 \theta) \), where \( w \) is the solution of following limit problem for the hypergeometric equation (see, for instance, [1], [37] for the general theory)

\[
\begin{cases}
  z(z - 1)w''(z) + \left( \frac{\alpha}{2} z - \frac{\beta}{2} \right) w'(z) + \left[ \frac{(a-2)^2}{16} - \frac{(b-2)^2}{16} \right] w(z) = 0 & z \in (0, 1) \\
  w(0) = 1 \lim_{z \to 1} w(z) \in \mathbb{R}
\end{cases}
\]

The general solution of (16) satisfying \( w(0) = 1 \) has the form

\[
w(z) = F \left( \frac{n + \beta}{4} - 1, \frac{n - \beta}{4}, \frac{1}{2}, z \right) + \alpha \sqrt{z} F \left( \frac{n + \beta}{4} - \frac{1}{2}, \frac{n - \beta}{4} + \frac{1}{2}, \frac{3}{2}, z \right)
\]

where \( F(a, b; c) \) is the hypergeometric series given in (10) which, \textit{a fortiori}, is convergent for \( 0 \leq z < 1 \). Since we are looking for bounded solution of (15), we have to analyze the behavior of a hypergeometric function near the point \( z = 1 \). To this aim consider that (see [1] p. 559)

\[
F(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad \text{if} \quad c - a - b > 0;
\]

\[
\lim_{z \to 1} F(a, b, c; z) = -\frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \quad \text{if} \quad c - a - b = 0;
\]

\[
\lim_{z \to 1} \frac{F(a, b, c; z)}{(1 - z)^{c - a - b}} = \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} \quad \text{if} \quad c - a - b < 0.
\]

An easy calculation shows that for both the hypergeometric functions appearing in (17) \( c - a - b = \frac{3 - n}{2} \leq 0 \). Let us, now, first examine the case \( n = 3 \). Since

\[
\lim_{z \to 1} w(z) = \lim_{z \to 1} \ln(1 - z) \left[ \frac{F \left( \frac{3 - \beta}{4}, \frac{3 - \beta}{4}, 1, z \right)}{\ln(1 - z)} + \alpha \sqrt{z} F \left( \frac{1 + \beta}{4}, \frac{5 - \beta}{4}, \frac{3}{2}, z \right) \right],
\]

using (19) and formula 5.3.10 p. 559 of [1], we get that the limit is finite if and only if

\[
\alpha = -\frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{3 - \beta}{4} \right) \Gamma \left( \frac{5 - \beta}{4} \right)}{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{\beta + 1}{4} \right) \Gamma \left( \frac{\beta + 3}{4} \right)} = -H(3, \beta)
\]

where in the last equality we use the fact that \( \Gamma \left( \frac{3}{2} \right) = 2 \Gamma \left( \frac{3}{4} \right) \) and the very definition of \( H(n, \beta) \) given in (8). In a similar way, if \( n > 3 \) consider

\[
\lim_{z \to 1} w(z) = \lim_{z \to 1} \left( 1 - z \right)^{\frac{3n}{2 - n}} \left[ \frac{F \left( \frac{3n + \beta - 1}{4}, \frac{3n - \beta - 1}{4}, 1, z \right)}{(1 - z)^{\frac{3n}{2 - n}}} + \alpha \sqrt{z} F \left( \frac{3n + \beta - 1}{4}, \frac{3n - \beta - 1}{4}, \frac{3}{2}, z \right) \right].
\]
Using (20) instead of (19), de l’Hopital theorem and differentiation formulas for hypergeometric functions we get that the limit is finite if and only if \( \alpha = -H(n, \beta) \).

Finally, if we choose \( \alpha = -H(n, \beta) \) in (17) and we take into account (14), we deduce the thesis since

\[
\varphi(x, t) = \frac{1}{(|x|^2 + t^2)^{n/2}} w \left( \frac{t^2}{|x|^2 + t^2} \right).
\]

3. Proof of Theorem 1.1

Let \( \varphi \) be the function defined in (23): since \( \varphi \) is a solution of problem (9), it is a solution of the Euler Lagrange equation associated to the functional (13). Consider the one-parameter family of \( H^n \)-surfaces \( \mathcal{G} = \{ G_k \}_{k \geq 0} \) given by the graphs of the functions \( \varphi_k = k \varphi \). For each point \((x, t, v) \in \mathbb{R}^n_+ \times \mathbb{R} \) there exist a unique \( G_k \) containing it, that one corresponding to \( k = \frac{v}{\varphi(x,t)} \). Therefore, to each point \((x, t, v) \in \mathbb{R}^n_+ \times \mathbb{R} \) we can associate the Mayer field

\[
(1, \mathbf{p}(x, t, v)) \equiv \left(1, \frac{v}{\varphi(x,t)} \nabla \varphi(x,t)\right) .
\]

Since \( \varphi \) is a solution of problem (9), then it is easy to check that the vector field

\[
\mathbf{F}(x, t, v) \equiv \left(\frac{v}{\varphi(x,t)} \nabla \varphi(x,t), \frac{v^2}{\varphi(x,t)} |\nabla \varphi|^2(x,t) + \frac{(\beta - 2)^2}{4} \frac{v^2}{|x|^2 + t^2}\right)
\]

is divergence free. Let us now take a nonnegative function \( \mathbf{F} \) given by (24), and consider the integral

\[
\int_{\mathbb{R}^n_+ \setminus B_r(0)} \int_B [2 < \mathbf{p}, \nabla u > - |\mathbf{p}|^2 + |\nabla u - \mathbf{p}|^2] - \frac{(\beta - 2)^2}{4} \int_{\mathbb{R}^n_+ \setminus B_r(0)} \frac{u^2(x,t)}{|x|^2 + t^2} \, dx \, dt
\]

where \( \mathbf{p} \equiv \mathbf{p}(x, t, u(x,t)) \) is defined in (24) and \( \cdot > \cdot \) stands for the standard inner products in \( \mathbb{R}^m \). The last integral in (26) is the integral of \( \mathbf{F} \) across the graph of \( u \) on \( \mathbb{R}^n_+ \setminus B_r(0) \). Since \( \mathbf{F} \) is divergence free and \( \mathbf{F}(x, t, 0) \equiv 0 \), by divergence theorem it follows that this flow equals the sum of the outward flows across the two manifolds

\[
\Sigma_1 = \{(x, t, v) \in \mathbb{R}^n_+ \times \mathbb{R} : |x| > r, \quad t = 0, \quad 0 \leq v \leq u(x,0)\}
\]

\[
\Sigma_2 = \{(x, t, v) \in \mathbb{R}^n_+ \times \mathbb{R} : |x|^2 + t^2 = r^2, \quad 0 \leq v \leq u(x, t)\}.
\]

Let us begin evaluating the flow of \( \mathbf{F} \) across \( \Sigma_1 \): since the only non zero component of the unit outward normal \( \nu \) to \( \Sigma_1 \) is the \( n \)-th \( \nu_n \) and \( \nu_n = -1 \), then

\[
\int_{\Sigma_1} < \mathbf{F}, \nu > \, dH^n = -2 \int_{|x| > r} dx \int_0^{u(x,0)} v \frac{\varphi_t(x,0)}{\varphi(x,0)} \, dv
\]

\[
= - \int_{|x| > r} u^2(x,0) \frac{\varphi_t(x,0)}{\varphi(x,0)} \, dx = H(n, \beta) \int_{|x| > r} \frac{u^2(x,0)}{|x|} \, dx.
\]

Indeed the first equality follows from the very definition of \( \mathbf{F} \) given by (25). The last one relies on the fact that, by (9), \( \varphi(x,0) = |x|^{-\frac{n}{2} + 1} \) and on an easy computation that shows \( \varphi_t(x,0) = -H(n, \beta)|x|^{-\frac{n}{2}} \).
As regards the flow of $F$ across $\Sigma_2$, since its outer unit normal is $\nu \equiv \left(-\frac{x}{\sqrt{|x|^2+r^2}}, -\frac{t}{\sqrt{|x|^2+r^2}}, 0\right)$, then
\begin{equation}
\int_{\Sigma_2} \langle F, \nu \rangle_{n+1} dH^n \leq \int_{\Sigma_2} v \left| \frac{\partial \phi}{\partial x}(x, t) \right| dH^n = \frac{n-2}{2r} \int_{\Sigma_2} v dH^n \leq \frac{n-2}{8r} \sup_{\partial B^+_R(0)} u^2(x, t)
\end{equation}
where $\phi_x$ denotes the derivative of $\phi$ in the radial direction. The equality is a trivial consequence of the definitions of $F$ and $\phi$, given in (25) and (14) respectively, and the last inequality follows from the facts that $v \leq u(x, t)$ on $\Sigma_2$ and $u$ is bounded on the boundary of $B^+_R(0) = \mathbb{R}_+^n \cap B_r(0)$. Collecting (26), (29) and (30) we deduce that for some positive constant $C$, depending only on $n$, the following inequality holds
\[\int_{\mathbb{R}_+^n \setminus B_r(0)} |\nabla u|^2(x, t) dx dt - \frac{(\beta - 2)^2}{4} \int_{\mathbb{R}_+^n \setminus B_r(0)} \left| \phi_\beta(x, t) \right|^2 \frac{u^2(x, t)}{|x|^2 + t^2} dx dt \geq H(n, \beta) \int_{|x| = R} \frac{u^2}{|x|} dx + Cr^{n-2}\]
from which (7) follows directly on letting $r$ go to zero.

It remains to show that the constant that appears in (8) is optimal. To do this, let $0 < r < R$ and apply the previous arguments replacing $u$ by $\phi$. Starting from (26) we get
\begin{equation}
\int_{B^+_R(0) \setminus B_r(0)} |\nabla \phi|^2(x, t) dx dt - \frac{(\beta - 2)^2}{4} \int_{B^+_R(0) \setminus B_r(0)} \frac{\phi^2(x, t)}{|x|^2 + t^2} dx dt = H(n, \beta) \int_{r < |x| < R} \frac{\phi^2(x, 0)}{|x|} dx + \frac{n-2}{2r} \int_{S_1} v dH^n - \frac{n-2}{2R} \int_{S_2} v dH^n
\end{equation}
where, as before, $B^+_R(0) = \mathbb{R}_+^n \cap B_R(0)$ and
\begin{align*}
S_1 &= \{(x, t, v) \in \mathbb{R}_+^n \times \mathbb{R} : |x|^2 + t^2 = r^2, \quad 0 \leq v \leq \phi(x, t)\} \\
S_2 &= \{(x, t, v) \in \mathbb{R}_+^n \times \mathbb{R} : |x|^2 + t^2 = R^2, \quad 0 \leq v \leq \phi(x, t)\}
\end{align*}
It is easy to check that the last two integrals in (31) are equal. Indeed, by spherical coordinates and (23)
\begin{equation}
\frac{n-2}{2r} \int_{S_1} v dH^n = \frac{n-2}{2r} \int_{\partial B^+_R(0)} r^{n-1} \left( \int_{0}^{\phi(x', r t')} v dv \right) d\mathcal{H}^{n-1}
= \frac{n-2}{4} r^{n-2} \int_{\partial B^+_R(0)} \phi^2(x', t') d\mathcal{H}^{n-1} = \frac{n-2}{4} \int_{\partial B^+_R(0)} w^2(t^2) d\mathcal{H}^{n-1} = \frac{n-2}{2R} \int_{S_2} v dH^n.
\end{equation}
Collecting (31) and (32) we deduce
\[\lim_{R \to \infty} \lim_{r \to 0} \frac{\int_{B^+_R(0) \setminus B_r(0)} |\nabla \phi|^2(x, t) dx dt}{\int_{r < |x| < R} \frac{\phi^2(x, 0)}{|x|} dx} = H(n, \beta)\]
that shows the optimality of the constant.

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