On reduction and separation of projective sets in Tychonoff spaces

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In what follows $\mathcal{F}$, $\mathcal{G}$, $\mathcal{K}$, and $\mathcal{L}$ denote the classes of closed, open, compact, and zero sets (preimages of $0 \in [0,1] \subseteq \mathbb{R}$ under continuous maps), and $\mathcal{S}$ denotes an unspecified class. The classes are treated as operators: $\mathcal{F}(X)$ consists of all closed sets in $X$, and so on; $\mathcal{S}(X) = \mathcal{F} \cap \mathcal{P}(X)$. Let $\mathcal{S}(Y) \upharpoonright X = \{S \cap X : S \in \mathcal{S}(Y)\}$. For an $F: X \rightarrow Y$, let $FA$ and $F^{-1}A$ denote the image and preimage of $A$. We say that $\Phi$ is an $\omega$-ary Hausdorff (or $\delta s$-) operation if there is an $S \subseteq \omega^\omega$ (the base of $\Phi$) such that $\Phi(A_s)_{s \in \omega^{<\omega}} = \bigcup_{f \in S} \bigcap_{n \in \omega} A_{f|n}$ for all $s \in \omega^{<\omega}$ ([6], [5], [1]; for the operation with base $\kappa^\omega$, see [9] and [8]). For example, if $S = \omega^\omega$, then $\Phi$ is the $A$-operation. A $\Phi$-set is a set obtained by $\Phi$. Let $\Phi(\mathcal{S}, X)$ be the class of $\Phi$-sets generated by $\mathcal{S}(X)$ and let $\Phi(\mathcal{S})$ denote the union of the $\Phi(\mathcal{S}, X)$ for all $X$.

The Borel hierarchy generated by $\mathcal{S}(X)$ is defined by alternating countable unions and complements, and $\Sigma^0_n(\mathcal{S}, X)$ and $\Pi^0_n(\mathcal{S}, X)$ are its $n$th additive and multiplicative classes. For example, $\Sigma^0_2(\mathcal{F}, X)$ is $\mathcal{F}_\sigma(X)$ and $\Pi^0_0(\mathcal{F}, X)$ is $\mathcal{G}_0(X)$.

By induction on $\alpha$, each Borel class is of the form $\Phi(\mathcal{S}, X)$ for some $\Phi$. The projective hierarchy generated by $\mathcal{S}(X)$ for Polish spaces $X$ is defined by alternating the projections of subsets of $X \times \omega^\omega$ onto $X$ and complements, and $\Sigma^1_n(\mathcal{S}, X)$ and $\Pi^1_n(\mathcal{S}, X)$ are its $n$th additive and multiplicative classes. For example, $\Sigma^1_1(\mathcal{F}, \mathbb{R})$ and $\Pi^1_1(\mathcal{F}, \mathbb{R})$ consist of the $A$-sets and CA-sets of reals. By the Fundamental Theorem on Projections ([6], I, p. 264), if $X$ is Polish, then the class of projections of sets in $\Phi(\mathcal{F}, X \times \omega^\omega)$ onto $X$ is of the form $\Psi(\mathcal{F}, X)$ for $\Psi$ with a base in $\Phi(\mathcal{F}_\sigma, \omega^\omega)$.

So, by induction on $n$, each projective class is of the form $\Phi(\mathcal{F}, X)$ for some $\Phi$.

For an arbitrary $X$, we define projective classes as $\Phi(\mathcal{S}, X)$ for $\Phi$ such that the corresponding projective class in $\mathbb{R}$ is $\Phi(\mathcal{S}, \mathbb{R})$. One can define $\sigma$-projective classes in a similar manner (see [2] and [7]).

We say that $\mathcal{S}(X)$ has the reduction property if for any $A, B \in \mathcal{S}(X)$, there are $C, D \in \mathcal{S}(X)$ such that $C \subseteq A$, $D \subseteq B$, $C \cap D = \varnothing$, and $C \cup D = A \cup B$. It has the separation property if for any $A, B \in \mathcal{S}(X)$ such that $A \cap B = \varnothing$, there is a $C \in \mathcal{S}(X)$ such that $C \subseteq A$ and $B \subseteq C$. If $\mathcal{S}(X)$ has the reduction property, then the dual class $\{X \setminus S : S \in \mathcal{S}(X)\}$ has the separation property. The classes $\Sigma^0_\alpha(\mathcal{F}, \mathbb{R})$, $\alpha > 1$, $\Pi^1_1(\mathcal{F}, \mathbb{R})$, $\Sigma^1_2(\mathcal{F}, \mathbb{R})$ have the pre-well-ordering property (we do not formulate it here), which is stronger than reduction. Furthermore, $V = L$ implies reduction in $\Sigma^1_n(\mathcal{F}, \mathbb{R})$ for all $n \geq 2$, and under PD (the projective determinacy axiom), $\Sigma^1_2(\mathcal{F}, \mathbb{R})$ and $\Pi^1_2(\mathcal{F}, \mathbb{R})$ have the pre-well-ordering property (a fact known as the First Periodicity Theorem), and so the reduction property [7], [9], [4], [8]. If $\mathcal{S}(Y)$ has the reduction (separation)
property, then so does \( \mathcal{J}(Y) \upharpoonright X \). We have \( \Phi(\mathcal{J}(Y) \upharpoonright X) = \Phi(\mathcal{J}, Y) \upharpoonright X \) for all \( \Phi \), whence we obtain the following result.

**Lemma 1.** Let \( X \subseteq Y \) and \( \mathcal{J}(X) = \mathcal{J}(Y) \upharpoonright X \). Then \( \Phi(\mathcal{J}, X) = \Phi(\mathcal{J}, Y) \upharpoonright X \), and if \( \Phi(\mathcal{J}, Y) \) has the reduction (separation) property, then so does \( \Phi(\mathcal{J}, X) \).

For example, \( \mathcal{J}(X) = \mathcal{J}(Y) \upharpoonright X \) holds if \( \mathcal{J} \) is \( \mathcal{F} \) or \( \mathcal{G} \), and also if \( \mathcal{J} = \mathcal{L} \) for Tychonoff \( X \) and \( Y \) (Lemma 4). Given \( F : X \to Y \), \( F \) preserves \( \mathcal{J} \) if \( A \in \mathcal{J}(X) \) implies that \( FA \in \mathcal{J}(Y) \), and \( F^{-1} \) preserves \( \mathcal{J} \) if \( B \in \mathcal{J}(Y) \) implies that \( F^{-1}B \in \mathcal{J}(X) \). For example, \( F \) is closed if and only if \( F \) preserves \( \mathcal{F} \), continuous if \( F^{-1} \) preserves \( \mathcal{F} \) (or \( \mathcal{G} \)), compact if and only if \( F^{-1} \) preserves \( \mathcal{H} \), and perfect if and only if it is closed, continuous, and compact. Since \( F^{-1}\Phi(A_s)_{s \in \omega < \omega} = \Phi(F^{-1}A_s)_{s \in \omega < \omega} \) for all \( \Phi, F \), and \( (A_s)_{s \in \omega < \omega} \), we have the following assertion.

**Lemma 2.** If \( F^{-1} \) preserves \( \mathcal{J} \), then \( F^{-1} \) preserves \( \Phi(\mathcal{J}) \).

For example, if \( F \) is continuous, then \( F^{-1} \) preserves \( \Phi(\mathcal{F}), \Phi(\mathcal{G}) \), and \( \Phi(\mathcal{L}) \), and if \( F \) is compact, then \( F^{-1} \) preserves \( \Phi(\mathcal{H}) \). Given \( F : X \to Y \), define its kernel by \( \ker F = \{ F^{-1}\{y\} : y \in Y \} \) and the algebra of preimages by \( \text{alg} F = \{ F^{-1}B : B \subseteq Y \} \). Clearly, \( \text{alg} F = \{ A \subseteq X : F^{-1}FA = A \} \). It is the complete subalgebra of \( \mathcal{P}(X) \) generated by \( \ker F \), so it is closed under all \( \Phi \). By using the diagonal product of maps which show that the \( A_s \) are zero sets, we obtain the following proposition.

**Proposition 1.** If \( (A_s)_{s \in \omega < \omega} \) is in \( \mathcal{L}(X) \), then there is a continuous \( F : X \to [0, 1]^\omega \) such that \( A_s \in \text{alg} F \) for all \( s \in \omega < \omega \), and so \( \Phi(A_s)_{s \in \omega < \omega} \in \text{alg} F \) for all \( \Phi \).

Given \( (I, \leq) \), a family \( (A_i)_{i \in I} \) is decreasing if \( A_i \supseteq A_j \) for all \( i \leq j \). A map \( F : X \to Y \) is closed-to-one if \( \ker F \subseteq \mathcal{F}(X) \). It can be shown that for such an \( F \), \( F \bigcap_{i \in I} A_i = \bigcap_{i \in I} FA_i \) for all directed \( (I, \leq) \) and decreasing \( (A_i)_{i \in I} \) in \( (\mathcal{F} \cap \mathcal{H})(X) \), and so \( F \Phi(A_s)_{s \in \omega < \omega} = \Phi(FA_s)_{s \in \omega < \omega} \) for all decreasing \( (A_s)_{s \in \omega < \omega} \) in \( (\mathcal{F} \cap \mathcal{H})(X) \) and all \( \Phi \), whence we obtain the following lemma.

**Lemma 3.** If \( \mathcal{J}(X) \subseteq (\mathcal{F} \cap \mathcal{H})(X) \) is closed under finite intersections and \( F : X \to Y \) is closed-to-one and preserves \( \mathcal{J} \), then \( F \) preserves \( \Phi(\mathcal{J}) \).

For example, for Hausdorff \( X \) and \( Y \) and a continuous \( F : X \to Y \), if \( X \) is compact, then \( F \) preserves \( \Phi(\mathcal{F}) \). If, moreover, \( Y \) is perfectly normal, then \( F \) also preserves \( \Phi(\mathcal{L}) \). Lemmas 2 and 3 allow one to transfer reduction (separation) to the preimage side.

**Proposition 2.** Let \( \mathcal{J}(X) \subseteq (\mathcal{F} \cap \mathcal{H})(X) \) be closed under finite intersections, and for any \( (A_s)_{s \in \omega < \omega} \) in \( \mathcal{J}(X) \) let there be a \( Y \) and a closed-to-one \( F : X \to Y \) such that \( F \) and \( F^{-1} \) preserve \( \mathcal{J} \), \( (A_s)_{s \in \omega < \omega} \) is in \( \text{alg} F \), and \( \Phi(\mathcal{J}, Y) \) has the reduction (separation) property. Then \( \Phi(\mathcal{J}, X) \) has the same property.

**Lemma 4.** If \( X \subseteq Y \) are Tychonoff, then \( \Phi(\mathcal{Z}, X) = \Phi(\mathcal{Z}, Y) \upharpoonright X \), and whenever \( \Phi(\mathcal{Z}, Y) \) has the reduction (separation) property, then so does \( \Phi(\mathcal{Z}, X) \).

For \( \mathcal{Z}(X) \subseteq \mathcal{Z}(Y) \upharpoonright X \), note that all \( F : X \to [0, 1] \) extend continuously to \( \beta X \), the Čech–Stone compactification of \( X \), and then to \( [0, 1]^\kappa \) for a suitable \( \kappa \) (see [3]). The main result of this note is as follows.
Theorem 1. Let $X$ be a Tychonoff space and let $\Phi$ be a Hausdorff operation. If $\Phi(\mathcal{F}, \mathbb{R})$ has the reduction (separation) property, then so does $\Phi(\mathcal{F}, X)$.

By Lemma 4, it suffices to consider $X = [0, 1]^\kappa$. Using Proposition 1, we verify the assumptions of Proposition 2 for $\mathcal{F} = \mathcal{F}$ and $Y = [0, 1]^\omega$, the same for all $(A_s)_{s \in \omega^\omega}$ in $\mathcal{F}([0, 1]^\kappa)$.

Corollary 1. If $X$ is Tychonoff, then $\Sigma^0_\alpha(\mathcal{F}, X)$, $\Pi^1_1(\mathcal{F}, X)$, and $\Sigma^1_2(\mathcal{F}, X)$ have the reduction property for all $\alpha < \omega_1$ such that $\alpha > 1$. Under PD, $\Sigma^1_{2n}(\mathcal{F}, X)$ and $\Pi^1_{2n+1}(\mathcal{F}, X)$ have the reduction property for all $n < \omega$ such that $n > 0$.

Under $\sigma$-PD, Corollary 1 extends to the $\sigma$-projective classes generated by $\mathcal{F}(X)$.

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