VERIFICATION OF THE JONES UNKNOT CONJECTURE
UP TO 24 CROSSINGS

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Abstract. Extending upon our previous work, we verify the Jones Unknot Conjecture for all knots up to 24 crossings. We describe the method of our approach and analyze the growth of the computational complexity of its different components.

1. Introduction

The Jones conjecture states that the Jones polynomial distinguishes all non-trivial knots from the trivial one. It is a striking statement implying that the Jones polynomial which is a quantum-physics inspired and still mysterious invariant of knots contains a very strong topological information. Jones proposed it as one of the challenges for mathematics in the 21st century in [Jo].

The main result of this paper is:

Theorem 1. The Jones polynomial distinguishes all knots with diagrams up to 24 crossings from the unknot.

This is an extension of our earlier work [TS]. That paper includes also references to other people’s work on this conjecture.

2. Description of the method

By a 2-tangle is a 1-manifold $T$ embedded into a 3-ball $B^3$ with endpoints at distinguished points $b_1, ..., b_4 \in \partial B^3$. Tangles are considered up to isotopy in $B^3$ fixing $\partial B^3$.

Tangles obtained from the zero tangle by a sequence of operations below are called algebraic, [Co].

![Figure 1. Tangle addition, multiplication, and twists](image)

A closure of an algebraic tangle is an algebraic or arborescent link.

A Conway polyhedron is an edge-connected 4-valent simple planar graph with no regions with just two vertices. (Such polyhedra are called simple
polyhedra in [Co].) An important observation of Conway is that each knot is either algebraic or has a diagram obtained by filling the 4-valent vertices of a Conway polyhedron with algebraic tangles. Roughly speaking we use that approach to generate all necessary knot diagrams.

The notion of the Kauffman bracket can be extended to framed tangles, since each of them can be expressed as

\[ T = p \cdot \langle + q \cdot \rangle, \]

where \( p, q \in \mathbb{Z}[A^{\pm 1}] \) are uniquely defined. We call \( [T] = (p, q) \) the Kauffman bracket of \( T \).

A tangle \( T \) is algebraically trivializable if its Kauffman bracket \( [T] = (p(A), q(A)) \) is such that \( p(A)r(A) + q(A)s(A) = 1 \) for some \( r(A), s(A) \in \mathbb{Z}[A^{\pm 1}] \). For example, \( T \) with \( [T] = (3A, 2) \) is algebraically trivializable since \( 3A \cdot A^{-1} - 2 \cdot 1 = 1 \), but \( T \) with \( [T] = (4A, 2) \) is not.

The importance of this property stems from the fact that any potential counterexample to Jones conjecture is either a closure of an algebraically trivializable algebraic tangle or can be obtained by filling vertices of a Conway polyhedron with algebraically trivializable algebraic tangles. Additionally, we can assume that these tangles have no internal loops, like for example the (2,2)-pretzel tangle, since such tangles cannot be completed to a knot.

The method used in this work was that of [TS] with further optimizations implemented. It involved the following steps:

1. Generation of knot diagrams of 24 crossings, by (a) considering all possible insertions of algebraically-trivializable algebraic tangles into Conway polyhedra resulting in 24 crossing diagrams and by (b) considering closures of all 24-crossing algebraic tangles.
2. Elimination of knot diagrams allowing a pass move which reduces either the number of crossings or the number of vertices in the corresponding Conway polyhedron.
3. Computation of the determinants of the remaining knot diagrams by a divide-and-conquer method.
4. Computation of the Kauffman bracket polynomials of determinant 1 knot diagrams by a similar divide-and-conquer method. The diagrams with monomial Kauffman brackets are called candidates.
5. Computation of the knot group presentations for the candidates \( K \) using the computer program SnapPy [Sn]. This program is highly efficient in finding finding the cyclic \( (a) \) presentation of \( \pi_1(S^3 - K) \), does confirming the triviality of \( K \). (This step is repeated if necessary, as SnapPy algorithm is non-deterministic.)
6. Using other methods for knot diagrams for which SnapPy did not find the cyclic presentation.

Among the optimizations introduced for the 23 and 24 crossing knot testing was the analysis of mutations among Conway polyhedra and elimination
of those related by mutation to other polyhedra already on the list and related to 4-valent graphs which contain a bigon.

3. Computational Complexity

Computational complexity depends in part on the growth of number of Conway polyhedra with number of vertices (roughly between $3^v$ and $4^v$, as indicated in Table 1), and the number of trivializable algebraic tangles with number of crossings (roughly about $3^n$, as indicated in Table 3). It should be noted that algebraic tangles with internal loops, such as the $(2,2)$-pretzel tangle, are not allowed since they would result in multi-component links.

Table 1. Numbers of Conway polyhedra with $v$ vertices.

| $v$ | Total | $v$ | Total |
|-----|-------|-----|-------|
| 6   | 1     | 16  | 499   |
| 8   | 1     | 17  | 1473  |
| 9   | 1     | 18  | 4974  |
| 10  | 3     | 19  | 16296 |
| 11  | 3     | 20  | 56102 |
| 12  | 12    | 21  | 192899|
| 13  | 19    | 22  | 674678|
| 14  | 63    | 23  | 2381395|
| 15  | 153   | 24  | 8468424|

Table 2. Numbers of algebraic tangles (total and trivializable) with $c$ crossings (and no internal loops).

| $c$ | Total | Triv | $c$ | Total | Triv |
|-----|-------|------|-----|-------|------|
| 1   | 1     | 1    | 10  | 4334  | 2589 |
| 2   | 1     | 1    | 11  | 15076 | 7754 |
| 3   | 2     | 2    | 12  | 53648 | 23572|
| 4   | 4     | 4    | 13  | 193029| 71124|
| 5   | 12    | 12   | 14  | 698590| 211562|
| 6   | 36    | 30   | 15  | 2560119| 633059|
| 7   | 113   | 94   | 16  | 9422500| 1866458|
| 8   | 374   | 288  | 17  | 34935283| 5478404|
| 9   | 1242  | 836  | 18  | 130250565| 15674910|

Although the number of $v$ vertex Conway polyhedra grows with $v$, Conway polyhedra with the larger number of vertices admit fewer knot diagrams with a given crossing number. The overall effect of these two patterns on determinant computation times is shown in Figure 2. The sudden jump in the computation times between 21 and 22 crossings appears to be due to the relative speed of two different CPU types (i7-4790 for calculations for 21 or fewer crossings, and Intel Xeon L5520 for 22 or more crossings). The
overall computation time for a \( c \) crossing calculation is exponential in \( c \), somewhere between \( 5^c \) and \( 7^c \). Similar trends occur for number of diagrams tested (Figure 3) and number of diagrams with monomial Kauffman brackets (Figure 4).

Figure 2. CPU times for non-algebraic knot diagrams generation and determinant testing.
FIGURE 3. Number of non-algebraic knot diagrams tested for determinant. “nc” is the total crossing number.

FIGURE 4. Number of diagrams with monomial Kauffman bracket tested for unknottedness.
4. Summary

The verification for 24 crossings required testing 59,361,435,729,041 non-algebraic knot diagrams and 185,317,928,640 algebraic knot diagrams, for a total of 59,546,753,657,681 knot diagrams — approximately 6 times the number of knot diagrams for 23 crossings.

Computations for 24 crossings were performed on 8-core Intel Xeon L5520 processors operated by the Center for Computational Research at the University at Buffalo for a total of 41.8 core-years of wall-clock time.

References

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