Covariant EBK quantization of the electromagnetic two-body problem

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Abstract

We discuss a method to transform the covariant Fokker action into an implicit two-degree-of-freedom Hamiltonian for the electromagnetic two-body problem with arbitrary masses. This dynamical system appeared 100 years ago and it was popularized in the 1940’s by the still incomplete Wheeler and Feynman program to quantize it as a means to overcome the divergencies of perturbative QED. Our finite-dimensional implicit Hamiltonian is closed and involves no series expansions. The Hamiltonian formalism is then used to motivate an EBK quantization based on the classical trajectories with a non-perturbative formula that predicts energies free of infinities.

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I. INTRODUCTION

The usual Hamiltonian description of two-body dynamics is surprisingly restrictive within relativity physics: If Lorentz transformations are to be represented by canonical transformations, only non-interacting two-body motion can be described. This is the content of the no-interaction theorem of 1964, which later in 1984 was proved for the usual Lagrangian description as well. A covariant version of Hamiltonian dynamics, constraint dynamics, was invented to overcome this group-theoretical obstacle, but this too has limited applicability. In this work we start from a relativistic physical theory: the time-symmetric relativistic action-at-a-distance electrodynamics, for which no constraint description exists at present. We discuss the passage from the covariant Fokker Lagrangian to an implicit Hamiltonian formalism defined in closed form without power expansions. We derive a two-degree of freedom Hamiltonian for the electromagnetic two-body problem with arbitrary masses and with either repulsive or attractive interaction. In our description a Lorentz transformation is represented by a canonical transformation followed by a rescaling of the evolution parameter. We use this covariant Hamiltonian formalism to motivate an EBK quantization of the electromagnetic two-body problem that uses the classical trajectories to approximate the quantum energies with a formula that is non-perturbative and free of infinite quantities.

In 1903, Schwarzchild proposed a relativistic interaction between charges that was time reversible precisely because it involved retarded and advanced interactions symmetrically. The same model reappeared in the 1920s in the work of Tetrode and Fokker and it finally became electromagnetic theory after Wheeler and Feynman showed that this direct-interaction theory can describe all the classical electromagnetic phenomena (i.e. the classical laws of Coulomb, Faraday, Ampère, and Biot-Savart). In particular, Wheeler and Feynman showed in 1945 that in the limit where the electron interacts with a completely absorbing universe, the response of this universe to the electron's field is equivalent to the local Lorentz-Dirac self-interaction theory without the need of mass renormalization. Complete absorption is added to the theory as an approximation to uncouple away from the detailed neutral-delay dynamics of the other charges of the universe. For other approximations see Ref. The Wheeler and Feynman program to quantize the action-at-distance electrodynamics and overcome the infinities of QED is still not implemented because of the lack of a Hamiltonian description, which was a motivation for the present
work.

The relativistic action-at-a-distance electrodynamics is better regarded as a many-body electromagnetic theory, once it is based on a parametrization-independent action involving two-body interactions only, without the mediation by fields [6, 7]. The isolated electromagnetic two-body problem, away from the other charges of the universe, is a time-reversible dynamical system defined by the action

$$S_F = -\int m_1 ds_1 - \int m_2 ds_2 - e_1 e_2 \int \delta(||x_1 - x_2||^2) \dot{x}_1 \cdot \dot{x}_2 ds_1 ds_2,$$

(1)

where $x_i(s_i)$ represents the four-position of particle $i = 1, 2$ parametrized by its arc-length $s_i$, double bars stand for the four-vector modulus $||x_1 - x_2||^2 \equiv (x_1 - x_2) \cdot (x_1 - x_2)$, and the dot indicates the Minkowski scalar product of four-vectors with the metric tensor $g_{\mu\nu}$ ($g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$) (the speed of light is $c = 1$). The integration in Eq. (1) cannot be restricted to a segment of trajectory going from an initial to a final time, because Lagrangian (1) at the end-points involves the future and past light-cones, which are out of the segment. This difference from our Poincaré-invariant Lagrangian to the usual Hamilton’s principle of Galilei-invariant mechanics demands that action (1) be defined by the whole orbit. A sensible mathematical variation of Eq.(1) is along trajectory variations $\delta x_i(t)$ that vanish at both $t = \pm \infty$, such that $\delta S_F$ is a sensible finite quantity to be minimized. This difference is discussed in [11], and we shall henceforth ignore it and derive our equations of motion formally by extremizing action (1). The attractive two-body problem is defined by Eq.(1) with $e_1 = -e_2 \equiv e$ (Hydrogen atom), while the repulsive problem is defined by Eq. (1) with $e_1 = e_2 \equiv e$. For the electromagnetic two-body problem with arbitrary masses (Eq. (1)), the only known analytical solution is the circular orbit for the attractive problem [12, 13]. A comprehensive discussion of literature on the Fokker Lagrangian is given in Ref. [3]. Some existence and uniqueness results for this two-body system are reviewed in Appendix I.

This paper is divided as follows: In Section II we construct a Hamiltonian that is general enough to describe the non-runaway solutions of the Fokker Lagrangian for the two-body problem with arbitrary masses and with either attractive or repulsive interaction. In Section III we use this Hamiltonian in an EBK quantization procedure that is non-perturbative and can provide a set of energies free of infinite quantities. In Section IV we put the discussions and conclusion. In Appendix I we construct a Lorentz frame where a generic non-runaway
orbit of the Fokker Lagrangian is time-reversible (the CMF frame).

II. OUTLINE OF THE METHOD

The method to describe twice-monotonic orbits of the Fokker Lagrangian by an implicit Hamiltonian formalism was discussed in Ref. [3] for the special case of equal masses with repulsive interaction. In the following we generalize this result for the case of arbitrary masses with either repulsive or attractive interaction.

The first step of the method consists of a transformation to new variables [11]

\[ \begin{align*}
\xi_1 &\equiv t_1 - x_1, \quad \zeta_1 \equiv t_1 + x_1, \\
\xi_2 &\equiv t_2 - x_2, \quad \zeta_2 \equiv t_2 + x_2.
\end{align*} \]

(2)

This transformation splits the action integral (1) into the sum of two separate actions

\[ S_F = \frac{1}{2} (S_a + S_b), \]

(3)

with

\[ S_a = -\int m_1(d\xi_1 d\zeta_1)^{1/2} - \int m_2(d\xi_2 d\zeta_2)^{1/2} \]

\[ -e_1 e_2 \int \int \frac{\delta(\zeta_1 - \zeta_2)}{|\xi_1 - \xi_2|} (d\xi_1 d\zeta_2 + d\xi_2 d\zeta_1), \]

(4)

and

\[ S_b = -\int m_1(d\xi_1 d\zeta_1)^{1/2} - \int m_2(d\xi_2 d\zeta_2)^{1/2} \]

\[ -e_1 e_2 \int \int \frac{\delta(\zeta_1 - \zeta_2)}{|\xi_1 - \xi_2|} (d\xi_1 d\zeta_2 + d\xi_2 d\zeta_1). \]

(5)

Integration of the \( \delta \) function in Eq. (4) gives a nonzero contribution only where \( \zeta_1 \) and \( \zeta_2 \) take equal values \( \zeta_1 = \zeta_2 = \zeta \), and this \( \zeta \) is the natural evolution parameter of action \( S_a \) (henceforth called type a foliation). Analogously, for Eq. (5), integration over the \( \delta \) function produces a nonzero contribution only where \( \xi_1 = \xi_2 = \xi \), and this \( \xi \) is the natural time parameter of action \( S_b \) (henceforth called type b foliation). Notice that with type a foliation particles are in the light-cone condition \( (x_1 - x_2)^2 - (t_1 - t_2)^2 = 0 \) and particle 2 is ahead of particle 1 in time by \( r_a \). The light-cone distance \( r_a \) in the type a foliation is

\[ r_a = -\frac{1}{2} (\xi_1 - \xi_2). \]

(6)
With type $b$ foliation particles are also in the light-cone condition, and particle 2 is behind in time by $r_b$, and the light-cone distance $r_b$ is

$$r_b = \frac{1}{2}(\zeta_1 - \zeta_2).$$  \hspace{1cm} (7)

After expressing action (4) in terms of the time-like parameter $\zeta$, the associated Euler-Lagrange equation is a simple ordinary differential equation. The Euler-Lagrange problem for action (5) is analogous, with $\zeta$ replaced by $\xi$. A Lagrangian (such as Eqs. (4) and (5)), whose Euler-Lagrange equation is an ordinary differential equation is henceforth called to be in the local form. Notice that Lagrangians (4) and (5) are also separately covariant.

The fact that the Fokker Lagrangian is not in the local form is a motivation to search for an equivalent covariant and local Lagrangian, and the natural candidates would be $S_a$ and $S_b$ of Eqs. (4) and (5). With that in mind, one could speculate if an orbit that extremizes Eq. (1) could also extremize actions $S_a$ (Eq. (4)) ($\delta S_a = 0$) and $S_b$ (Eq. (5)) ($\delta S_b = 0$) separately. Each separate minimization, in general, yields a different trajectory, and as discussed in [3], these trajectories are never equal. In Ref. [3] we found that the trajectories of the following alternative pair of Lagrangians

$$\delta S_a = \delta G,$$  \hspace{1cm} (8)

and

$$\delta S_b = -\delta G,$$  \hspace{1cm} (9)

coincide for a choice of the Lagrangian $G$. A trajectory that satisfies Eqs. (8) and (9) will also extremize the Fokker action (1), a simple consequence of Eqs. (3), (8) and (9):

$$\delta S_F = \frac{1}{2}\delta S_a + \frac{1}{2}\delta S_b = \frac{1}{2}\delta G - \frac{1}{2}\delta G = 0.$$  \hspace{1cm} (10)

Our problem is then to fix the Lagrangian $G$ such that Eqs. (8) and (9) yield the same trajectory and such that Eqs. (8) and (9) be ordinary differential equations (rather than delay equations). Such $G$ is henceforth called a bilocal ghost Lagrangian.

In Appendix I we prove that any non-runaway orbit of (1) has a Lorentz frame in which it is time-reversible (henceforth called the CMF frame). In the construction that follows we operate on this CMF Lorentz frame. Since the orbit is time reversible, we can restrict the minimization of Eqs. (8) and (9) to the set of time-reversible orbits ($[x_1(-t) =$...
We also add the physical property that both advanced and retarded distances decrease monotonically to a point of minimum and then start increasing monotonically again. We henceforth call this set of orbits the CMF set. The piecewise monotonic property is a consequence of the velocity being a monotonic function of time, which was proved in Appendix 1 of Ref. [3] for the repulsive case. The generalization to the attractive case along symmetric orbits is trivial. We henceforth refer to a CMF orbit simply as a twice-monotonic orbit. Assuming that a twice-monotonic and time-reversible orbit exists in our given Lorentz frame, as this orbit extremizes Eq. (1) over all orbits, it is necessarily an extremum inside the CMF set.

In the following we generalize some integral identities first used in Ref. [3] for CMF orbits. The time-reversal operation maps type a parametrization onto type b and acts on CMF orbits by the following map: \( \zeta_{1,2} \rightarrow -\xi_{1,2} , \xi_{1,2} \rightarrow -\zeta_{1,2} , r_a \rightarrow r_b \). The simplest type of integral identity, valid for an arbitrary integrable function \( \phi(x) \) of the real variable, is

\[
\int_a \phi(r_a) d\zeta = \int_b \phi(r_b) d\xi. \tag{11}
\]

The lower index of the integral denotes the parametrization type, and Eq. (11) is a consequence of the coordinate transformation induced by the time-reversal symmetry of the CMF set \( \zeta \rightarrow -\xi \) (the Jacobian of this transformation is one). For the attractive problem we can restrict the integration to twice the finite interval between two consecutive collisions (the period), while for the repulsive problem we must define the above integrals over the infinite interval. In this work we ignore questions of convergence of the integrals, an ambiguity inherited from the Wheeler-Feynman theory and discussed in Ref. [11]. The same time-reversal action \( (\zeta_{1,2} \rightarrow -\xi_{1,2} , \xi_{1,2} \rightarrow -\zeta_{1,2} ) \) along CMF orbits generates the following identities for arbitrary functions \( V_1(r) , V_2(r) , \alpha_1(r) \) and \( \alpha_2(r) \) of the real variable

\[
\int_a V_{1,2}(r_a)(\frac{d\xi_1}{d\zeta} + \frac{d\xi_2}{d\zeta})d\zeta = \int_b V_{1,2}(r_b)(\frac{d\zeta_1}{d\xi} + \frac{d\zeta_2}{d\xi})d\xi, \tag{12}
\]

\[
\int_a \alpha_{1,2}(r_a)(d\xi_1 d\zeta_1)^{1/2} = \int_b \alpha_{1,2}(r_b)(d\xi_2 d\zeta_2)^{1/2}. \tag{13}
\]

The above identities suggest that we use a ghost Lagrangian \( G \) of type

\[
G = \int_a [\phi(r_a) + \frac{1}{2} V_1(r_a) \dot{\xi}_1 + \frac{1}{2} V_2(r_a) \dot{\xi}_2 + \alpha_1(r_a) \sqrt{\dot{\xi}_1} + \alpha_2(r_a) \sqrt{\dot{\xi}_2}]d\zeta. \tag{14}
\]

The dot over \( \xi_{1,2} \) in Eq. (14) indicates derivative respect to \( \zeta \) (the time-parameter of case a). This \( G \) is in the local form when added to \( S_a \), where \( \zeta \) plays the role of the time parameter.
and the coordinates are $\xi_1$, $\xi_2$. When this same $G$ is subtracted from action $S_b$, the integral identities allow us to express $G$ as

$$G = \int [\dot{\phi}(r_b) + \frac{1}{2}V_1(r_b)\dot{\xi}_1 + \frac{1}{2}V_2(r_b)\dot{\xi}_2 + \alpha_1(r_b)\sqrt{\xi_1} + \alpha_2(r_b)\sqrt{\xi_2}]d\xi, \quad (15)$$

which is also in the local form for action $S_b$, with $\xi$ being the time parameter and the coordinates being $\xi_1$ and $\xi_2$. Notice that the dot over $\xi_{1,2}$ in equation (15) indicates derivative respect to $\xi$ (the time-parameter of case $b$). Lagrangian (14) is the most general ghost Lagrangian whose associated Hamiltonian involves quadratic rational functions of the momenta, and we shall see that it suffices to describe any non-runaway orbit with two monotonic branches.

The condition $\dot{r} = 0$ divides the phase space of a twice-monotonic orbit in two disjoint regions according to whether $\dot{r} > 0$ or $\dot{r} < 0$. In each disjoint region the Lagrangian has a branch that can be uniquely inverted to a Hamiltonian formalism. To avoid the overloaded notation of Ref. [3], we henceforth drop all indications about branches. Notice that after inclusion of the ghost Lagrangian, the problem $\delta S_b = \delta G$ of Eq. (8) implies the Euler-Lagrange equations for $L_a = S_a - G$

$$L_a = -\int [M_{1a}\sqrt{\dot{\xi}_1} + M_{2a}\sqrt{\dot{\xi}_2} + \frac{e_1e_2}{|\xi_1 - \xi_2|} + \frac{1}{2}V_1(r_a)\dot{\xi}_1 + \frac{1}{2}V_2(r_a)\dot{\xi}_2 + \phi(r_a)]d\xi, \quad (16)$$

where $M_{1a} \equiv m_1 + \alpha_1(r_a)$ and $M_{2a} \equiv m_2 + \alpha_2(r_a)$. The time-reversed problem $\delta S_b = -\delta G$ is described by $L_b = S_b + G$,

$$L_b = -\int [M_{1b}\sqrt{\dot{\xi}_1} + M_{2b}\sqrt{\dot{\xi}_2} + \frac{e_1e_2}{|\xi_1 - \xi_2|} - \frac{1}{2}V_1(r_b)\dot{\xi}_1 - \frac{1}{2}V_2(r_b)\dot{\xi}_2 - \phi(r_b)]d\xi, \quad (17)$$

with $M_{1b} \equiv m_1 - \alpha_1(r_b)$ and $M_{2b} \equiv m_2 - \alpha_2(r_b)$. The Hamiltonian in each case and branch is given by

$$H_a = -\frac{1}{4} \left\{ \frac{M_{1a}^2}{(p_1 - \frac{1}{2}V_1 + \frac{e_1e_2}{|\xi_1 - \xi_2|})} + \frac{M_{2a}^2}{(p_2 - \frac{1}{2}V_2 + \frac{e_1e_2}{|\xi_1 - \xi_2|})} \right\} - \phi(r_a), \quad (18)$$

and

$$H_b = -\frac{1}{4} \left\{ \frac{M_{1b}^2}{(p_1 - \frac{1}{2}V_1 + \frac{e_1e_2}{|\xi_1 - \xi_2|})} + \frac{M_{2b}^2}{(p_2 - \frac{1}{2}V_2 + \frac{e_1e_2}{|\xi_1 - \xi_2|})} \right\} + \phi(r_b). \quad (19)$$

Notice that the Hamiltonian $H_a$ depends only on $r_a = -\frac{1}{2}(\xi_1 - \xi_2)$, such that $P_a = p_1 + p_2$ is a constant of motion. Analogously for type $b$ parametrization, Hamiltonian $H_b$ depends
only on \( r_b = \frac{1}{2}(\zeta_1 - \zeta_2) \), such that \( P_b = p_1 + p_2 \) is a constant of motion. The constant \( P_a = p_1 + p_2 \) of case \( a \) suggests the following canonical change of variables

\[
X \equiv \frac{1}{2}(\xi_1 + \xi_2), \quad P \equiv p_1 + p_2
\]

\[
x \equiv \frac{1}{2}(\xi_1 - \xi_2), \quad p = p_1 - p_2.
\]

For type \( b \) we use the analogous transformation with \( \xi \) replaced by \( \zeta \) in Eq. (20). We use Eq. (20) to express \( p_1 \) and \( p_2 \) of Eq. (18) in terms of \( P \) and \( p \) and substitute into the condition \( H_a(r_a, p_a, E, P) = E_a \), yielding a quadratic equation for \( p \), with solutions

\[
p_a(r_a, E_a, P_a) = \frac{V_2 - V_1}{2} + \frac{\Delta_a}{(E_a + \phi)} \pm \sqrt{(P_a + \frac{Q_a}{(E_a + \phi)}) + \frac{1}{2} V_1 + \frac{1}{2} V_2 + \frac{e_1 e_2}{r_a} + \left( \frac{\Delta_a^2 - Q_a^2}{(E_a + \phi)^2} \right)},
\]

where \( Q_a \equiv \frac{1}{4}(M_{1a}^2 + M_{2a}^2) \), \( \Delta_a \equiv \frac{1}{4}(M_{2a}^2 - M_{1a}^2) \), \( r_a = |x| \) and the plus or minus describes the two branches. The separation for case \( b \) is analogous. Notice that in Ref. \( 3 \) we used only one function \( V = \frac{1}{2}(V_1 + V_2) \), a trivial gauge transformation if \( V_1 \) and \( V_2 \) are functions only of \( r_a \), but nontrivial if \( V_1 \) and \( V_2 \) depend on \( E \) and \( P \), as in the generalization that follows. This generalization is necessary for the different mass case.

As we show below, it is always possible to adjust the potentials such that Hamiltonians (18) and (19) have a common trajectory. The drawback is that these so adjusted potentials depend on \( E \) and \( P \) as well. This dependence suggests that we generalize the potentials of Eqs. (15) to functions of \( E \) and \( P \) \[3\]. For example, the potential \( \phi(r_a) \) should be generalized to \( \phi = \phi(r_a, E_a, P_a) \) in case \( a \) and to \( \phi = \phi(r_b, E_b, P_b) \) in case \( b \) (an analogous generalization goes for \( V_{1,2} \) and \( \alpha_{1,2} \)). The generalized Hamiltonian is still a function of phase space, defined implicitly by Eqs. (18) and (19), because \( E \) and \( P \) are functions of the phase space. This generalization must be done carefully though, because in general the equations of motion of Hamiltonians (18) and (19) with implicitly defined potentials do not correspond to the Lagrangian equations of (16) and (17) anymore, but rather to some highly involved Lagrangian with a complex dependence on the velocities. On the other hand, simultaneous minimization of a pair of Lagrangians with the simple form of (16) and (17) is very desirable, because the Wheeler-Feynman equation of motion (\( \delta S_F = 0 \)) is then a consequence of Eq. (10). In the following we construct a pair of auxiliary Lagrangians with the simple form (16) and (17) for any given orbit. These auxiliary Lagrangians carry different potential functions for each orbit. All we need to know about these auxiliary Lagrangians is their existence,
as it guarantees that our orbit satisfies the Wheeler-Feynman equation of motion by means of Eq. (10).

The auxiliary Lagrangian is constructed from the potential functions of \((r, P, E)\) as follows: For an orbit of energy \(E_o\) and momentum \(P_o\), we define auxiliary fixed form potentials:
\[
\phi^∗(r) = \phi(r, E_o, P_o); \quad V_{1,2}^∗(r) = V_{1,2}(r, E_o, P_o); \quad \alpha_{1,2}^∗(r) = \alpha_{1,2}(r, E_o, P_o),
\]
which in turn define auxiliary Lagrangians by Eqs. (16) and (17). The orbit of the auxiliary Lagrangian (16) satisfies the auxiliary Hamiltonian equation of motion defined by Eq. (18) with the fixed form potentials. This equation is in general different from the Hamiltonian equation of Eq. (18) with the implicit potentials. To fix this, we construct the potentials such that both equations are equal in each case. For this it is sufficient to supplement the following conditions in case \(a\)
\[
\begin{align*}
\frac{\partial H_a(p, P_a, r, E_a)}{\partial E_a} &= 0, \quad \text{(22)} \\
\frac{\partial H_a(p, P_a, r, E_a)}{\partial P_a} &= 0, \quad \text{(23)}
\end{align*}
\]
and analogously in case \(b\)
\[
\begin{align*}
\frac{\partial H_b(p, P_b, r, E_b)}{\partial E_b} &= 0, \quad \text{(24)} \\
\frac{\partial H_b(p, P_b, r, E_a)}{\partial P} &= 0. \quad \text{(25)}
\end{align*}
\]
These are four conditions imposed on the five potentials \(\phi, V_{1,2}\) and \(\alpha_{1,2}\). Before we proceed to construct potentials satisfying the above conditions, it is convenient to find the explicit solution of Hamiltonian (18), which is accomplished by use of a canonical transformation with a generating function \(S\) given by
\[
S = PX + W(x, P, E) - E_\zeta, \quad \text{(26)}
\]
where \(W(x, P, E)\) is defined to satisfy the condition \(p = \partial W/\partial x\), with \(p\) given by Eq. (21). The new momentum associated with the old variable \(X\) is the same old constant \(P = \partial S/\partial X\) and the other new momentum is the energy \(E\) (with this last definition we exploit the fact that \(E\) is already one argument of the implicit potentials). We choose \(S\) in the manner of Hamilton-Jacobi such that the new Hamiltonian vanishes: \(K = H + \frac{\partial S}{\partial \zeta} = 0\). As the Hamiltonian is zero, the new coordinates are defined simply by two constants \(X_0\) and \(C_0\).
\[ X_0 = \partial S/\partial P = X + \partial W/\partial P \]  
\[ C_0 = -\partial S/\partial E = \zeta - \partial W/\partial E \]  

Eq. (27) defines \( \zeta \) and \( X \) as functions of \( r_a \equiv |x| \) for case \( a \) (and \( \xi \) and \( X \) as a function of \( r_b = |x| \) for case \( b \)). We shall also need the differential form of Eq. (27) relative to \( x \),

\[ dX = -(\partial^2 W/\partial x \partial P)dx = -(\partial p/\partial P)dx, \]
\[ d\zeta = (\partial W/\partial x \partial E)dx = (\partial p/\partial E)dx, \]

where we have used \( p = \partial W/\partial x \) (definition of the Hamilton-Jacobi transformation) and exchanged the partial derivatives. The differentials \( dx_i \) and \( dt_i \) along the trajectory are obtained using Eq. (2) to relate particle coordinates to \( X \) and \( \zeta \) followed by use of Eqs. (28) and (29) to relate \( dX \) and \( d\zeta \) to \( dr_a \).

The explicit solution is

\[ dt_{1a,b} = \pm \frac{1}{2}(\partial p_{a,b}/\partial P - \partial p_{a,b}/\partial E - 1)dr_{a,b}, \]
\[ dx_{1a,b} = \frac{1}{2}(1 - \partial p_{a,b}/\partial P - \partial p_{a,b}/\partial E)dr_{a,b}, \]
\[ dx_{2a,b} = -\frac{1}{2}(\partial p_{a,b}/\partial P + \partial p_{a,b}/\partial E + 1)dr_{a,b}, \]

where the plus sign holds for case \( a \) and the minus holds for case \( b \) in the time variables. Eq. (30) gives the explicit solution in terms of \( p_{a,b}(r, E, P) \), as given by Eq. (21).

As time reversal maps case \( a \) into case \( b \), a time-reversible orbit of both (18) and (19) must satisfy \( v_{1a}(r) = -v_{1b}(r) \) and \( v_{2a}(r) = -v_{2b}(r) \) for \( r \in [r_a, \infty) \). An economical way to satisfy this condition is to express the functions \( \partial p/\partial P \) and \( \partial p/\partial E \) as

\[ \frac{\partial p_a}{\partial P} = \frac{\partial p_b}{\partial P} \equiv -\frac{\cosh[s(r, E, P)]}{\sinh[s(r, E)]}, \]
\[ \frac{\partial p_a}{\partial E} = \frac{\partial p_b}{\partial E} \equiv \frac{F(r, E, P)}{\sinh[s(r, E, P)]}, \]

such that \( v_{1a}(r) = -v_{1b}(r) \) and \( v_{2a}(r) = -v_{2b}(r) \) are automatically satisfied by use of Eqs. (30). We are now ready to obtain an analytical solution of Eqs. (22), (24), (31) and (32) in terms of the undetermined functions \( s(r, E, P) \) and \( F(s, E, P) \) defined by Eqs. (31) and (32).
Our solution is defined such that \( E_a = E_b \equiv E \) and \( P_a = P_b \equiv P \). For case \( a \) the equations of motion for \( \xi_1 \) and \( \xi_2 \) derived from Hamiltonian (18) with use of conditions (22) are

\[
\frac{d\xi_1}{d\zeta} = \frac{\exp(s)}{F(r)} = \frac{(m_1 + \alpha_1)^2}{4(p_1 + \frac{1}{2}V_1 + \frac{e_1 e_2}{2\rho_a})^2},
\]

and

\[
\frac{d\xi_2}{d\zeta} = \frac{\exp(-s)}{F(r)} = \frac{(m_2 + \alpha_2)^2}{4(p_2 + \frac{1}{2}V_2 + \frac{e_1 e_2}{2\rho_a})^2}.
\]

As discussed in Ref. [3], there are four possible square roots of Eqs. (33) and (34), but only one choice does not pose a constraint involving \( s(r, E, P) \) and \( F(r, E, P) \), leaving both functions arbitrary. This physical choice of signs is defined by

\[
P + p_a + V_1 + \frac{e_1 e_2}{r} = -\sqrt{F(m_1 + \alpha_1)} \exp(-s/2),
\]

\[
P - p_a + V_2 + \frac{e_1 e_2}{r} = \sqrt{F(m_2 + \alpha_2)} \exp(s/2).
\]

where we have used \( p_1 = \frac{1}{2}(P_a + p_a) \) and \( p_2 = \frac{1}{2}(P_a - p_a) \) with \( P_a = P \). Case \( b \) is analogous, and the motion of \( \zeta_1 \) and \( \zeta_2 \) derived from Hamiltonian (19) with condition Eq. (24) defines the physical choice of the square roots by

\[
P + p_b - V_1 + \frac{e_1 e_2}{r} = \sqrt{F(m_1 - \alpha_1)} \exp(-s/2),
\]

\[
P - p_b - V_2 + \frac{e_1 e_2}{r} = -\sqrt{F(m_2 - \alpha_2)} \exp(s/2).
\]

We can use Eqs. (35) and (36) to eliminate \( p_1 \) and \( p_2 \) from Eq. (18) with \( E_a = E \), yielding

\[
E + \phi = \frac{1}{2\sqrt{F}}[(m_1 + \alpha_1) \exp(s/2) - (m_2 + \alpha_2) \exp(-s/2)],
\]

and an analogous use of Eqs. (37), (38) and (19) with \( E_b = E \) yields

\[
E - \phi = -\frac{1}{2\sqrt{F}}[(m_1 - \alpha_1) \exp(s/2) - (m_2 - \alpha_2) \exp(-s/2)].
\]

Notice that Eqs. (31) and (32) are immediately satisfied if we assume

\[
p_a = p_b = p(E, P, r).
\]

Eqs. (31) and (32) actually allow \( p_a \) and \( p_b \) to differ by a trivial Gauge function \( g(r) \) entering in Eq. (11), but that can be absorbed in the definition of \( V_1 \) and \( V_2 \). We can satisfy the seven
conditions of Eqs. (35)-(41) with completely arbitrary functions $F(E, P, r)$ and $s(E, P, r)$ in the following solution

$$\alpha_1 = \frac{E F \exp(s/2) + (P + \frac{e_1 e_2}{r}) \exp(-s/2)}{\sqrt{F} \sinh(s)},$$

$$\alpha_2 = \frac{E F \exp(-s/2) + (P + \frac{e_1 e_2}{r}) \exp(s/2)}{\sqrt{F} \sinh(s)},$$

$$V_1 = -m_1 \sqrt{F} \exp(-s/2),$$

$$V_2 = m_2 \sqrt{F} \exp(s/2),$$

$$\phi = \frac{m_1 \exp(s/2) - m_2 \exp(-s/2)}{2 \sqrt{F}}.$$

and last with $p$ given by

$$p = -\frac{[E F + (P + \frac{e_1 e_2}{r}) \cosh(s)]}{\sinh(s)}.$$  \hspace{1cm} (43)

Now that we have solved simultaneously all conditions (22), (24), (31), (32) and (35)-(40), some remarks are in order:

(i) To see the generality of our method, we can start from a numerical solution of the delay equation of motion, with energy $E$, and simply adjust the two arbitrary functions $s(E, P, r)$ and $F(E, P, r)$ such that Hamiltonian (18) has this orbit as solution. This can be done with Eqs. (33) and (34) and takes exactly two functions, to describe the two particle positions as a function of the light-cone separation. The method has a branch point at $s = 0$, such that for each separation there should be two values for the position, one in the inbound flight and another in the outbound flight, such that we can describe twice-monotonic orbits only. A general proof of the twice-monotonic property is given in Appendix A of Ref. [3]. At the moment the only available method to solve the delay equations of motion is by numerical calculation. An automatic consequence of (37) and (38) is that the orbit also satisfies Hamiltonian (19).

(ii) Condition (22) was used as an ingredient in obtaining Eqs. (33) and (34) of case $a$, but we never forced it directly. An instructive direct check is to take the partial derivative respect to $E$ of the implicit energy equation ($E = H_a(r, p, E, P)$) defined by Eq. (18), together with substitution of the potentials of Eqs. (42) and use of Eq. (32). The result is \(\partial H_a/\partial E = 0\), and analogously for case $b$ we obtain $\partial H_a/\partial E = 0$, which are the upper lines of (22) and (24). The lower lines of (22) and (24) are obtained in the same way, by taking the derivatives of the implicit energy equation respect to $P$ in each case.
(iii) After we determine \( s(E, P, r) \) and \( F(E, P, r) \), substitution of \( E = E_o \) and \( P = P_o \) into the potentials produces auxiliary Lagrangians that have an orbit in common, precisely because of (ii). And of course, if we did not know that yet, we would prove that this orbit satisfies the Wheeler-Feynman equation of motion (\( \delta S_F = 0 \)), as a consequence of Eq. (11).

Last, as discussed in Ref. [3], at the branch point \( s = 0 \) and \( r = r_o \) the continuity condition for the functions of Eqs. (42) and (43) calculates \( P \) as

\[
P = -EF(r_o) - \frac{e_1 e_2}{r_o},
\]

which will be used in the next section.

III. INFINITE-FREE EBK QUANTIZATION AND COVARIANCE

Under a Lorentz transformation with boost parameter \( w \) the light-cone distance of case \( a \) transforms like

\[
\bar{r}_a = \sqrt{(1 + w)/(1 - w)}r_a,
\]

while in case \( b \) the same formula holds with the replacement \( w \rightarrow -w \). Equation (45) shows that the twice-monotonic property is a Lorentz-invariant property, suggesting that there is no obstacle to describe a CMF orbit in another Lorentz frame.

The Lorentz transformation acts on the coordinates of Hamiltonian (18) (the \( \xi_1 \) and \( \xi_2 \) defined in Eq. (2) ) as a simple rescaling

\[
\bar{\xi}_1 = \frac{1}{\lambda_a}\xi_1,
\]

\[
\bar{\xi}_2 = \frac{1}{\lambda_a}\xi_2
\]

and also rescales the evolution parameter \( \zeta \) of case \( a \) to

\[
\bar{\zeta} = \lambda_a \zeta,
\]

with \( \lambda_a \) given by

\[
\lambda_a = \sqrt{1 + \frac{w}{1 - w}}.
\]
Substituting the above Lorentz-transformed quantities into (16) we find another Lagrangian with the same functional form, which proves that a CMF orbit is described in any Lorentz frame by a Lagrangian $\tilde{L}_a$ of the form (16). To see how the potentials are transformed in the new frame, we make use of the explicit solution (30) in both frames, together with the condition that the new coordinates $\tilde{x}_{1a}, \tilde{t}_{1a}, \tilde{x}_{1b}$ and $\tilde{t}_{1b}$ must be the Lorentz transformed of the old coordinates $x_{1a}, t_{1a}, x_{1b}$ and $t_{1b}$ with the boost parameter $w$, yielding

$$\frac{\partial \tilde{p}_a}{\partial E} = \frac{(1 - w) \partial p_a}{(1 + w) \partial E},$$

$$\frac{\partial \tilde{p}_a}{\partial P} = \frac{\partial p_a}{\partial P},$$

where we used the differential version of Eq. (45) ($d\tilde{r}_a = \sqrt{(1 + w)/(1 - w)} dr_a$). Equation (49) shows that $s_a(r)$ as defined by Eqs. (31) and (32) is a Lorentz scalar:

$$\tilde{s}_a(\tilde{r}_a, \tilde{E}, \tilde{P}) = s_a(\lambda_a \tilde{r}_a, E, P),$$

and that $F_a(r_a)$ (as defined by Eqs. (31) and (32)) transforms like

$$F_a(\tilde{r}_a, E, P) = \lambda_a^2 F(\lambda_a \tilde{r}_a, E, P).$$

Case $b$ transforms in the same way with $\lambda_b = 1/\lambda_a$, which is Eq. (48) with the substitution $w \rightarrow -w$. This last equation allows us to express the Hamiltonian in any frame by use of the CMF form of $F(r)$ and a rescaling depending on the boost parameter.

It can be seen that the Lorentz transformation of Eqs. (16) is a canonical transformation that scales the momenta with the inverse factor, $\tilde{p}_1 = \lambda_a p_1$ and $\tilde{p}_2 = \lambda_a p_2$, such that the canonically transformed Hamiltonian is

$$\tilde{H}_a = \frac{-\lambda_a}{4} \left\{ \frac{\tilde{M}^2_{1a}(\tilde{r}_a)}{(\tilde{p}_1 + \frac{1}{2} \tilde{V}_1 + \frac{e^2}{\tilde{\zeta}_1 - \tilde{\xi}_1})} + \frac{\tilde{M}^2_{2a}(\tilde{r}_a)}{(\tilde{p}_2 + \frac{1}{2} \tilde{V}_2 + \frac{e^2}{\tilde{\zeta}_2 - \tilde{\xi}_2})} \right\} - \lambda_a \tilde{\phi}(\tilde{r}_a),$$

with $\tilde{\phi}(\tilde{r}_a) \equiv \frac{1}{\lambda_a} \phi(r_a)$, $\tilde{V}(\tilde{r}_a) \equiv \lambda_a V(r_a)$, $\tilde{M}^2_{1a}(\tilde{r}_a) \equiv M^2_{1a}(r_a)$, and $\tilde{M}^2_{2a}(\tilde{r}_a) \equiv M^2_{2a}(r_a)$. Notice that the new Hamiltonian $\tilde{H}_a$ picked a multiplicative factor of $\lambda_a$, and if we perform a change to the natural evolution parameter $\tilde{\zeta} = \lambda_a \zeta$ of the new Lorentz frame, it compensates exactly for that factor, going back to the form (18), such that the Hamiltonian has the same form in all Lorentz frames (only the value of the energy is changed to $\tilde{E} = E/\lambda_a$).

As the Lorentz transformation is canonical for our Hamiltonian, the action $I \equiv \oint p dr$ is a Lorentz-invariant, such that EBK quantization [23, 24] is a sensible covariant quantization
procedure. The need to follow the canonical transformation by a rescaling can be avoided by use of a Lorentz-invariant evolution parameter in the Lagrangian of Eq. (1). For example a Lorentz-invariant parameter can be constructed with the help of Eqs. (51) and (47) as

$$du_a \equiv \frac{d\zeta}{\sqrt{F_a(r_a)}} = \frac{d\tilde{\zeta}}{\sqrt{\bar{F}_a(\bar{r}_a)}}.$$  (53)

As the momentum $p \equiv \partial L/\partial \dot{\xi}$ is independent of parametrization, the action integral $\oint pdr$ marks the same orbits with or without rescaling.

The action $I \equiv \oint pdr$ for the attractive problem ($e_1e_2 = -1$) can be expressed, by use of Eqs. (43), (29) and (32), as

$$I = -\oint \left[ E + \frac{1}{F(r)}(P - \frac{1}{r}) \right] d\zeta,$$  (54)

and substitution of $P$ as given by Eq. (44) into Eq. (54) yields

$$I = \oint \left[ \frac{1}{rF(r)} - E(1 - \frac{F(r_o)}{F(r)}) \right] d\zeta.$$  (55)

The function $F(r_o)$ can be calculated by elimination of $s(r)$ from Eqs. (35) and (36) and use of (2) as

$$F^2(r_o) = \frac{(1 + v_1)(1 + v_{2a})}{(1 - v_1)(1 - v_{2a})},$$  (56)

where $v_1$ is the velocity of particle 1 and $v_{2a}$ is the velocity of particle 2 in the advanced light-cone parametrized by $r_a$. For a shallow energy orbit, the particles have a low velocity in a large portion of the orbit, such that Eq. (56) predicts $F(r) \simeq 1$. The main difference from Coulombian orbits to our covariant orbits at shallow energies is at the collision, where Coulombian orbits reach an infinite velocity, while the relativistic motion saturates at the speed of light. Disregarding the collision region, substitution of $F(r) = 1$ together with the low-velocity approximation $d\zeta \simeq dt$ (valid away from the collision), into Eq. (55), yields exactly the one-dimensional Coulombian WKB integral

$$I = \oint \frac{dt}{r}.$$  (57)

Equation (57) is easily derived using the Coulombian energy and the Levi-Civita regularization, see for example the Appendix on Levi-Civita regularization of Ref. [19]. At this point the reader can see that our strange looking Hamiltonian formalism actually reduces nicely to the Coulomb approximation for shallow orbits. The use of $I = (n + \frac{1}{2})\hbar$ together
with the approximate formula (57) predicts the well known Balmer energy terms for the one-dimensional Coulombian two-body system. The corrections can be evaluated using the exact covariant formula (55) along the numerical orbits, as obtained in Ref. [19] for the equal-mass case. As discussed in Ref. [19] for the equal-mass problem, at the outbound collision $F(r_a)$ is singular like $1/\sqrt{r}$, and $d\zeta = (1 + v_1)dt \simeq \sqrt{2}r_a dr_a$, which makes the integral of (55) finite (actually, for the equal-mass case the collision does not even contribute to the integral). The merit of this procedure is that it calculates a Lamb shift without infinities, which might prove useful for understanding the renormalization of QED. For example, on the equal-mass two-body problem, the orbits can be calculated numerically using the regular delay equation developed in Ref. [19]. Along those regular collision orbits, there is nothing infinite involved in the evaluation of the action integral of Eq. (55) and the above covariant EBK quantization procedure yields an electrodynamics free of infinities for the positronium atom (the quantum energies are all finite). It would be highly desirable to generalize this procedure to the 3-dimensional motion of the electromagnetic two-body problem, but at present we do not know how to do that. The EBK quantization of Wheeler-Feynman theory is discussed in [23, 26] along with the only physical result so far (i.e. simple EBK quantization of circular orbits). This quantization predicts energies with finite small shifts in agreement with the Dirac equation to order $\alpha^4$ [26]. This is exactly what one expects physically, as those are circular orbits with a finite angular momentum. The infinite Lamb shift plagues only zero angular-momentum states, which corresponds to the orbits studied in the present work. The possibility of using the regularization of Ref. [19] to produce an electrodynamics free of infinities for the positronium atom suggests that further efforts to understand the regularization of the different-mass electromagnetic two-body problem could shine light on the renormalization of QED.

IV. CONCLUSIONS AND DISCUSSION

The idea to remove the field degrees of freedom from Maxwell’s electrodynamics goes back to Dirac [8] and later Wheeler and Feynman planned to quantize WF2B as a means to avoid the divergencies of QED, as in the action-at-a-distance theory the infinite number of field degrees of freedom are absent. The difficulties in converting the Fokker Lagrangian [11]
to Hamiltonian form have caused the famous seminar that never came from Wheeler (see
[10], page 97). In chapter 5, page 97 of reference [10], Feynman says that ‘I didn’t solve it either—a quantum theory of half-advanced half-retarded potentials—and I worked on it for years...’. This is still an outstanding problem today and the difficulties in casting relativistic Lagrangian interactions into Hamiltonian form are explained in references [28, 29].

Our description does not violate the no-interaction theorem [1], and there are two places where we avoid it: (i) the no-interaction theorem is an obstacle to covariant Hamiltonian description of two interacting particles only in fully 3-dimensional motion. We are restricted to 1-dimensional motion; (ii) Because the evolution parameter in our Hamiltonians is not time but rather \( \zeta \) for case \( a \) (and \( \xi \) for case \( b \)), the no-interaction theorem does not apply. In principle, because time is not the evolution parameter, even in three dimensions the no-interaction theorem would not be an obstacle to a Hamiltonian description, and that is an open problem. The merit of our constructive approach is that it starts from the physical theory of action-at-a-distance and does not involve expansions. This procedure should be compared to the very interesting covariant extrapolation of Wheeler-Feynman theory within constraint dynamics, which is a covariant way to introduce the correction of order \((v/c)^2\) to the Coulomb problem [27].

The energy equation for our Hamiltonian (18) is the following quadratic equation for \( p \):

\[
p^2 + A(E, r)p + U(E, r) = 0,
\]

with

\[
A(E, r) = \left[ \frac{m_{2a}^2 - m_{1a}^2}{2(E + \phi)} + (V_2 - V_1) \right],
\]

and

\[
U(E, r) = \left( \frac{m_{1a}^2 + m_{2a}^2}{2(E + \phi)} \right) (P + \frac{e_1 e_2}{r}) + \left( \frac{m_{1a}^2 V_2 + m_{2a}^2 V_1}{2(E + \phi)} \right).
\]

where \( m_{1a} \equiv m_1 + \alpha_1(r) \), \( m_{2a} \equiv m_2 + \alpha_2(r) \), \( V_1(r) \) and \( V_2(r) \) are determined from \( s(r, P, E) \) and \( F(r, P, E) \) by Eq. (42). The procedure outlined above determines the potentials only along the orbit, but it would be desirable to extend these potentials to the whole phase with some analytical continuation and/or use of symmetry. If that is accomplished, after a symmetrization, Eq. (58) can be used in a quantum canonical formalism with \( p \) and \( P \) being the quantum momentum operators. The trivial separation of the center-of-mass coordinate \( X \) produces a familiar looking Schroedinger equation. This equation should be compared
to the one-dimensional Klein-Gordon equation, that has been studied in connection with
the one-dimensional hydrogen atom \[25\]. The quantization of the one-dimensional hydrogen
atom interacting with the electromagnetic fields is an interesting testbed problem, as the
minimal coupling to the electromagnetic field produces divergencies in perturbation theory
that are exactly like in the 3-dimensional problem (and which are absent in the quantization
outlined above). In that respect, Eq. \[58\] is a properly renormalized version of the Coulomb
Hamiltonian operator.

Last, our procedure could be applied to the problem of calculating level shifts for an atom
in the presence of a strong gravitational field, by incorporating the metric tensor \(g_{\mu\nu}\) in the
definition of the variables of Eq. \[2\], which should be modified to
\[\xi_1 \equiv \sqrt{g_{00}t_1} - \sqrt{|g_{11}|}x_1\]
and
\[\zeta_1 \equiv \sqrt{g_{00}t_1} + \sqrt{|g_{11}|}x_1.\]

V. APPENDIX I: CONSTRUCTION OF THE CMF FRAME

For the repulsive two-electron problem \((m_1 = m_2 \equiv m\text{ and } e_1 = e_2 = e)\) and along
symmetric orbits \([-x_2(t) = x_1(t) \equiv x(t)]\), the minimization of action \[11\] yields the following
equation of motion
\[
m\frac{d}{dt} \left( \frac{v}{\sqrt{1 - v^2}} \right) = \frac{e^2}{2r^2} \left( \frac{1 - v(t - r)}{1 + v(t - r)} \right) + \frac{e^2}{2q^2} \left( \frac{1 + v(t + q)}{1 - v(t + q)} \right),
\]
where \(v(t) \equiv dx/dt\) is the velocity of the first electron, of mass \(m\) and charge \(e\), and the
functions \(r(t)\) and \(q(t)\) are the time-dependent delay and advance, defined implicitly by the
light-cone conditions
\[
r(t) = x(t) + x(t - r),
\]
\[q(t) = x(t) + x(t + q).\]

In general, a neutral-delay equation such as Eq. \[61\] requires a pair of world-line segments
as the initial condition (one world-line segment for each particle). As discussed in Ref.
\[14\], the initial world-line segments can be provided in such a way that Eq. \[61\] is well-
possed by using "maximal independent segments". A pair of world-line segments is called
independent if the end points of each segment lie on the forward and backward light-cones
of a single point interior to the other segment. A surprising existence theorem was proved
in Ref. [15] for the problem of Eq. (61) along symmetric orbits $-x_2(t) = x_1(t) \equiv x(t)$ using a Banach to Banach contraction mapping; This theorem states that for sufficiently low energies, Newtonian initial conditions $[x(0) = x_o$ and $v(0) = v_o]$ determine the unique symmetric orbit that is globally defined (i.e., that does not run away at some point) [15].

Later in 1981 it was proved for this same equal-mass problem that the delay Eq. (61) is equivalent to a local Newtonian equation [16].

The physical meaning of the above order-reduction of the delay equation is the following: of the infinite many initial conditions necessary to solve the delay equation of motion, only the Newtonian type ”position and velocity values at $t = 0$ ” suffice to determine the non-runaway orbit. The remaining conditions either take values determined by the initial position and velocity or the orbit will be a run-away. The above existence/uniqueness results are certainly true in the low-velocity Coulombian limit, as the Coulomb ODE’s have unique solutions determined by initial position and velocity. Therefore the existence/uniqueness result is either true in general or at least up to some high energy where strange new solutions bifurcate out. In the following we use this Newtonian order-reduction to define the CMF frame.

We consider here the two-body problem of Eq. (1) with attractive interaction and arbitrary masses. The construction of the Hamiltonian given in Section II for time-reversible orbits can be generalized to arbitrary orbits if we are able to find a Lorentz frame where this generic non-runaway orbit is time-reversible. This frame is henceforth called the CMF frame for the orbit. To accomplish this we start below by proving the following Lemma:

Lemma 1: Given any non-runaway orbit of Eq. (1), defined in a given Lorentz frame, we can construct another Lorentz frame where the two particles come to rest at the same time ($t = 0$) along this orbit.

The proof will be given for the attractive case and we define the measure time such that at $t = 0$ the particles are at the origin and moving in opposite directions with the speed of light: $x_1(0) = x_2(0) = 0$, $\dot{x}_1(0) = 1$ and $\dot{x}_2(0) = -1$ (the outbound collision). As the force is always attractive, the absolute velocity of each particle will decrease monotonically until it comes to rest at some turning time, and then it accelerates again to the speed of light until the next collision, that happens at a later time $t = T$ (the inbound collision). At this second collision the particles bounce at each other and start again with outbound velocities. The
proof can be made as well for the case where particles pass through each other at the inbound collision, such that the orbit is twice as long. For simplicity we assume that they bounce, such that the unit cell of our periodic orbit is the interval \((0, T)\), and particle one is always on the right-hand side. The turning times of the particles are in general two different values in the interval \((0, T)\). To fix that, we use a Lorentz transformation with boost parameter \(w\), defined such that the transformed of events \((t_1, x_1(t_1))\) and \((t_2, x_2(t_2))\) be simultaneous in the new frame. This imposes the condition

\[
t_1 - wx_1(t_1) = t_2 - wx_2(t_2).
\] (63)

As we also want that both particles be at rest simultaneously in the new frame, we must choose \(w\) such that

\[
\dot{x}_1(t_1) = \dot{x}_2(t_2) = w.
\] (64)

Because the velocities are monotonic in the interval \((0, T)\), condition (64) defines \(t_2(t_1)\) as a one-to-one function from the interval \((0, T)\) onto itself. We can then rewrite condition (63) as the zero of a function of the variable \(t_1\) of the interval \((0, T)\)

\[
F(t_1) = t_1 - t_2(t_1) - \dot{x}_1(t_1)(x_1(t_1) - x_2(t_2)) = 0.
\] (65)

Using the facts that \(t_2(t_1 = 0) = T, \dot{x}_1(0) = 1\) and \(t_2(t_1 = T) = 0, \dot{x}_1(T) = -1\), and also that \(x_1(t_1) - x_2(t_2) > 0\) for all times, we find that the continuous function \(F(t_1)\) is negative at \(t_1 = 0\) and positive at \(t_1 = T\), such that it must have a root in the interval \((0, T)\). This value of \(t_1\) calculates the boost \(w\) by Eq. (64) and defines a Lorentz frame where both particles are at rest simultaneously (the CMF), as we wanted to demonstrate.

We have then constructed the CMF frame with the property that both particles have zero velocity at the same time, say \(t = 0\), which we can now use as "Newtonian initial conditions". As discussed above, position and velocity initial conditions determine a unique non-runaway orbit, and since an initial condition with both particles simultaneously at rest is invariant by time reversal, the unique orbit determined by it must be time-reversible. We actually need the generalization of the Banach to Banach proof of Refs. ([15]) for the
attractive case with different masses to complete our proof.

[1] Currie DG, Jordan TF and Sudarshan ECG 1963 *Rev. Mod. Phys.* **35** 350, Marmo G, Mukunda N and Sudarshan ECG 1984 *Phys. Rev. D* **30** 2110

[2] Komar A 1978 *Phys. Rev. D* **18** 1887, Mukunda N and Goldberg JN 1981 *Phys. Rev. D* **23** 2218

[3] Hollander EB and De Luca J 2003 *Phys. Rev. E* **67** 026219

[4] Schwarzschild K 1903 *Gottinger Nachrichten* **128**, 132

[5] Tetrode H 1922 *Zeits. f. Physik* **10** 137, Fokker AD 1929 *Zeits. f. Physik* **58** 386

[6] Wheeler JA and Feynman RP 1945 *Rev.Mod. Phys.* **17** 157; Wheeler JA and Feynman RP 1949 **21** 425

[7] Leiter D 1970 *Am. J. Phys* **38** 207

[8] Dirac PAM 1938 Proceedings of the Royal Society of London, ser. A **167**,148.

[9] Hoyle F and Narlikar JV 1995 *Rev. Mod. Phys.* **67**,113; Hoyle F and Narlikar JV 1996 *Cosmology and Action at a Distance Electrodynamics*, World Scientific Publishing, Singapore

[10] Mehra J 1994 *The Beat of a Different Drum: The Life and Science of Richard Feynman*, Oxford University Press, Oxford, (chapter 5).

[11] Staruszkiewicz A 1970 *Annalen der Physik* **25** 362

[12] Schonberg M 1946 *Phys. Rev.* **69** 211

[13] Schild A 1963 *Phys. Rev.* **131** 2762; Schild A 1962 *Science* **138** 994

[14] Murdock JA 1974 *Annals of Physics* **84** 432

[15] Driver RD 1979 *Phys. Rev. D* **19** 1098; J. Hoag J and Driver RD *Nonlinear Analysis, Theory, Methods & Applications* **15165**

[16] Zhdanov VI 1981 *Physics Letters* **85A** 138

[17] Andersen CM and von Baeyer HC 1972 *Phys. Rev. D* **5** 2470

[18] Klimenko SV, Nikitin IN and Urazmetov WF 1999 *International Journal of Modern Physics C* **10** 1

[19] Hollander EB and De Luca J 2004 *Chaos* to be published arXiv.org /abs/math-ph/0888

[20] Anderson JL 1967 *Principles of Relativity Physics*, Academic press, New York, page 225.

[21] "The theory of Action-at-a-distance in Relativistic Particle Dynamics", Edited by Kerner EH
1972, Gordon and Breach Publishers, New York

[22] Nikitin IN 1995, *Il Nuovo Cimento* **110** 771; Klimenko SV, I.N. Nikitin and Urazmetov WF 1998 *Il Nuovo Cimento* **111** 1281

[23] von Bayer HC 1975 *Phys. Rev. D* **12** 3086

[24] Percival IC 1977 *Adv. Chem. Phys.* **36** 1; Leopold JG and Percival IC 1980 *J. Phys. B: Atom,Molec.Phys* **13** 1037

[25] Spector HN and Lee J 1985 *Am. J.Phys.* **53** 248

[26] Andersen CM and von Baeyer HC 1970 *Ann. of Phys.* **60** 67

[27] Crater H and Yang D 1991 *J. Math. Phys.* **32** 2374

[28] Tretyak V and Shpytko V 2000 *J. Phys. A* **33** 5719; Crater H and Lusanna L 2001 *Ann. Phys.-New York* **289** 87

[29] Nazarenko A 2001 *Int. J. Mod. Phys. A* **16** 4865