On the Abel-Jacobi map of an elliptic surface
and
the topology of cubic-line arrangements

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Abstract

Let $\varphi : S \to C$ be an elliptic surface over a smooth curve $C$ with a section $O$. We denote its generic fiber by $E_S$. For a divisor $D$ on $S$, we canonically associate a $\mathbb{C}(C)$-rational point $P_D$. In this note, we give a description of $P_D$ of $E_S$, when the rank of the group of $\mathbb{C}(C)$-rational points is one. We apply our description to refine our result on a Zariski pair for a cubic-line arrangement.

Introduction

Let $E$ be an elliptic curve defined over a field $K$ isomorphic to either $\mathbb{C}(t)$ or $\mathbb{C}(t_1, t_2)$ (the rational function field of one variable or two). We denote their group of $K$-rational points by $E(K)$. Since $E$ can be considered the generic fiber of an elliptic surface or 3-fold, they both have arithmetic and geometric aspects. In [18], the second author considered the case when $E$ is the generic fiber of a certain elliptic K3 surface, and made use of 3-torsions of $E(\mathbb{C}(t))$ in order to construct Zariski pairs for irreducible sextic curves. Also, in our previous works [3, 4, 21], we investigated the case when $E$ is the generic fiber of certain rational elliptic surfaces, and constructed Zariski pairs ($N$-plets) for reducible curves by using non-torsion elements in $E(\mathbb{C}(t))$. For the case of $\mathbb{C}(t_1, t_2)$, Cogolludo Agustín, Kloosterman and Libgober have recently investigated $E(\mathbb{C}(t_1, t_2))$ in order to study toric decomposition of plane curves, which is related to the embedded topology of plane curves ([6, 7, 9, 10]). These results shows that the study of arithmetic aspects of elliptic curves over the rational function fields is one of the important tools to study the topology of plane curves. In this article, we continue to study topology of plane curves along this line with more emphasis on the arithmetic aspects, especially the Abel-Jacobi map on an elliptic surface, which we explain below.

Let $\varphi : S \to C$ be an elliptic surface over a smooth projective curve. Throughout this paper, we assume that (i) $\varphi$ is relatively minimal, (ii) there exists a section $O : C \to S$, and (iii) there exists at least one degenerate fiber. As for a section $s : C \to S$, we identify $s$ and its image, i.e., an irreducible curve meeting any fiber at one point. We denote the set of sections of $\varphi : S \to C$ by $\text{MW}(S)$. Note that $\text{MW}(S) \neq \emptyset$ as $O \in \text{MW}(S)$. Let $E_S$, $\mathbb{C}(C)$ and $E_S(\mathbb{C}(C))$ denote the generic fiber of $S$, the rational function field of $C$ and the set of $\mathbb{C}(C)$-rational points of
\(E_S\), respectively. Under our assumption, by restricting a section to the generic fiber, \(\text{MW}(S)\) can be canonically identified with \(E_S(\mathbb{C}(C))\). We identify \(O\) with the corresponding rational point. Thus \((E_S, O)\) is an elliptic curve defined over \(\mathbb{C}(C)\).

Let \(D\) be a divisor on \(S\). By restricting \(D\) to \(E_S\), we have a divisor \(\mathfrak{d}\) on \(E_S\) defined over \(\mathbb{C}(C)\). By applying Abel’s theorem to \(\mathfrak{d}\), we have \(P_D \in E_S(\mathbb{C}(C))\) and thus \(s(D) \in \text{MW}(S)\) (see \([16, \text{Lemma 5.1, \S 5}]\) for the explicit description of \(s(D)\)).

In our previous articles \([3, 4, 19, 21]\), we studied the properties of \(P_D\) in \(E_S(\mathbb{C}(C))\) such as \(p\)-divisibility, for odd primes \(p\), in order to study the topology of reducible plane curves with irreducible components of low degrees. In this article, we consider \(n\)-divisibility of \(P_D\) in the case when \(\text{rank} E_S(\mathbb{C}(C)) = 1\), i.e., \(E_S(\mathbb{C}(C)) = \mathbb{Z}P_o \oplus E_S(\mathbb{C}(C))_{\text{tor}}\) for some \(P_o \in E_S(\mathbb{C}(C))\).

**Proposition 0.1** Let \(\phi_o : \text{NS}(S) \to \text{NS}(S) \otimes \mathbb{Q}\) and \(\phi : E_S(\mathbb{C}(C)) \to \text{NS}(S) \otimes \mathbb{Q}\) be the homomorphism defined in \(\S 2\). Suppose that \(\text{rank} E_S(\mathbb{C}(C)) = 1\). Let \(n\) be an integer such that \(P_D = nP_o + P_\tau\), \(P_\tau \in E_S(\mathbb{C}(C))_{\text{tor}}\). Then we have

\[
n^2 = -\frac{\phi_o(D) \cdot \phi_o(D)}{\langle P_o, P_o \rangle}, \quad n = -\frac{\phi_o(D) \cdot \phi(P_o)}{\langle P_o, P_o \rangle}
\]

where \(\cdot\) and \(\langle , \rangle\) mean the intersection and height pairing, respectively.

**Remark 0.1** As we see in Lemma \([2.1, \S 2]\), \(\phi_o(D) = \phi(P_D)\). Hence \(\phi_o(D)\) can be considered as (almost) a ‘class’ in the Picard group of the generic fiber.

An explicit form for the right hand side in the above formula are given in \(\S 2\). In \(\S 3\) we develop a method to determine the contribution from the torsion part from \(\phi_o(D)\). Hence for the rank one case, it is possible to describe \(P_D\) completely.

In the remaining part of this article, we consider an application of Proposition 0.1 in the investigation of the embedded topology of reducible plane curves. As we have seen in our previous papers, properties of \(P_D\) in \(E_S(\mathbb{C})\) played important role in order to study the topology of plane curves which arise from \(D\). In this article, we compute \(n\)-divisibility of \(P_D\) for a certain trisection and apply it to refine our result for a Zariski pair given in \([5]\) as follows:

Let \((B^1, B^2)\) be the Zariski pair for a nodal cubic \(E\) and four lines considered in \([5]\), i.e., the one with Combinatorics 1-(b). Namely, it is as follows:

**Combinatorics 1-(b).** Let \(E\), and \(L_i\) \((i = 0, 1, 2, 3)\) be as below and we put \(B = E + \sum_{i=0}^3 L_i\):

(i) \(E\): a nodal cubic curve.
(ii) $L_0$: a transversal line to $E$ and we put $E \cap L_0 = \{p_1, p_2, p_3\}$.

(iii) $L_i$: a line through $p_i$ and tangent to $E$ at a point $q_i$ distinct from $p_i$ ($i = 1, 2, 3$).

(iv) $L_1, L_2$ and $L_3$ are not concurrent.

By taking the group structure of $E \setminus \{the \ node\}$ into account, we infer that $q_i$ ($i = 1, 2, 3$) are either collinear or not. For $B$ with Combinatorics 1-(b), we call it Type I (resp. Type II) if $q_1, q_2$ and $q_3$ are collinear (resp. not collinear).

Then with terminologies and notation for $D_{2n}$-covers given in [1, 3], we have

**Theorem 0.1** Let $n$ be an integer $\geq 3$.

(i) If $B$ is of Type I, there exists a $D_{2n}$-cover $\pi : X_n \to \mathbb{P}^2$ branched at $2(\sum_{i=0}^{3} L_i) + nE$ for any $n$.

(ii) If $B$ is of Type II, there exists a $D_{2n}$-cover $\pi : X_n \to \mathbb{P}^2$ branched at $2(\sum_{i=0}^{3} L_i) + nE$ for $n = 4$ only.

**Corollary 0.1** Let $(B^1, B^2)$ be a pair of plane curves with Combinatorics 1-(b) such that their Types are distinct. Then both of the fundamental groups $\pi_1(\mathbb{P}^2 \setminus B^j, \ast)$ ($j = 1, 2$) are non-abelian and there exist no homeomorphisms between $(\mathbb{P}^2, B^1)$ and $(\mathbb{P}^2, B^2)$.

**Remark 0.2** For $B$ of Type I, we denote the line through $q_i$ ($i = 1, 2, 3$) by $T_i$. Then we infer that $\sum_{i=1}^{3} L_i$ is a member of the pencil generated by $E$ and $L_0 + 2T_i$. This shows that there exists a $D_{2n}$-cover of $\mathbb{P}^2$ branched at $2(E + \sum_{i=0}^{3} L_i) + n$ for any $n \geq 3$. If $p_i$ is not an inflection point of $E$, there exist just two lines through $p_i$ that are tangent to $E$ at different points from $p_i$. Hence we infer that for the pair $(B^1, B^2)$ for Combinatorics 1-(b) given in [5], $B^1$ (resp. $B^2$) is Type II (resp. Type I).

**Remark 0.3** For the explicit example for $(B^1, B^2)$ given in [5], the non-abelianness for $\pi_1(\mathbb{P}^2 \setminus B^j, \ast)$ ($j = 1, 2$) was first pointed out by E. Artal Bartolo. In this note, we prove that the same is true for any curve with the same combinatorics.

## 1 Preliminaries on elliptic surfaces

We refer to [11], [13], [14] and [16] for details. In this article, an elliptic surface always means the one introduced in the Introduction. We denote a subset of $C$ over which $\varphi$ has degenerate fibers by $\text{Sing}(\varphi)$. $\text{Red}(\varphi)$ means a subset of $\text{Sing}(\varphi)$.
consisting of a point $v \in \text{Sing}(\varphi)$ such that $\varphi^{-1}(v)$ is reducible. For $v \in \text{Sing}(\varphi)$, we denote the corresponding fiber by $F_v = \varphi^{-1}(v)$. The irreducible decomposition of $F_v$ is denoted by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i} \Theta_{v,i}$$

where $m_v$ is the number of irreducible components of $F_v$ and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0} \cdot O = 1$. We call $\Theta_{v,0}$ the identity component. In order to describe types of singular fibers, we use Kodaira’s symbol. We label irreducible components of singular fibers as in [20, p.81-82].

Let $MW(S)$ be the set of sections of $\varphi: S \rightarrow C$. By our assumption, $MW(S) \neq \emptyset$, as $O \in MW(S)$. By regarding $O$ as the zero element, $MW(S)$ is equipped with the structure of an abelian group through fiberwise addition.

Let $E_S$ be the generic fiber of $\varphi: S \rightarrow C$. We can regard $E_S$ as a curve of genus 1 over $C(C)$, the rational function field of $C$. Under our assumption on $S$, $NS(S)$ is torsion free by [16, Theorem 1.2]. Let $T_\varphi$ be the subgroup of $NS(S)$ generated by $O$, a fiber $F$ of $\varphi$ and $\Theta_{v,i}$ ($v \in \text{Red}(\varphi), 1 \leq i \leq m_v - 1$). In [16, Theorem 1.3], by taking Abel’s theorem on $E_S$ into account, an isomorphism $\psi: NS(S)/T_\varphi \rightarrow E_S(C(C))$ of abelian groups is given as follows:

We first define a homomorphism $\psi$ from the group of divisors $\text{Div}(S)$ to $\text{Pic}^0_{C(C)}(E_S) \cong E_S(C(C))$ by

$$\psi: \text{Div}(S) \ni D \mapsto \alpha(D|_{E_S} - (DF)|_{E_S}) \sim_{E_S} P_D - O,$$

where $\alpha$ is the Abel-Jacobi map on $E_S$ and $\sim_{E_S}$ denotes the linear equivalence on $E_S$. By [16, Lemma 5.2], $\psi$ induces a group isomorphism ([16, Theorem 1.3])

$$\overline{\psi}: NS(S)/T_\varphi \rightarrow E_S(C(C)).$$

We denote the section corresponding to $P_D$ by $s(D)$. By [16, Lemma 5.1], we have a relation in $NS(S)$:

$$D \approx s(D) + (d-1)O + nF + \sum_{v \in \text{Red}(\varphi)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i},$$

where $\approx$ denotes the algebraic equivalence between divisors, and $d, n$ and $b_{v,i}$ are integers defined as follows:

$$d = D \cdot F \quad n = (d-1)\chi(O_S) + O \cdot D - s(D) \cdot O,$$
and
\[
\begin{bmatrix}
  b_{v,1} \\
  \vdots \\
  b_{v,m_v-1}
\end{bmatrix} = A_v^{-1} (\mathbf{c}(v, D) - \mathbf{c}(v, s(D))),
\]
where, for $D \in \text{Div}(S)$, we put
\[
\mathbf{c}(v, D) := \begin{bmatrix}
  D \cdot \Theta_{v,1} \\
  \vdots \\
  D \cdot \Theta_{v,m_v-1}
\end{bmatrix},
\]
and $A_v$ is the intersection matrix $(\Theta_{v,i} \Theta_{v,j})_{1 \leq i,j \leq m_v - 1}$.

**Remark 1.1**

(i) If $D$ is a section (i.e, $D \in \text{MW}(S)$), the above relation (*) becomes trivial.

(ii) Entries of $A_v^{-1}$ are not necessarily integers. On the other hand, the relation (*) is a relation between two divisors of $\mathbb{Z}$-coefficients. This impose some restriction on $D$ and $s(D)$ at which irreducible components of $F_v, D$ and $s(D)$ meet. One of useful facts is a lemma below.

**Lemma 1.1** If $A_v^{-1} \mathbf{c}(v, D) \in \mathbb{Z}^{\oplus m_v - 1}$, then $\mathbf{c}(v, s(D)) = 0$.

**Proof.** Since $s(D) \cdot F = 1$, $\tilde{\mathbf{c}}(v, s(D))$ has a unique entry with 1 and other entries are 0. Also, $s(D)$ meets the component $\Theta_{v,i}$ with $a_{v,i} = 1$ or $\Theta_{v,0}$. Hence if $\mathbf{c}(v, s(D)) \neq 0$, $A_v^{-1} (\mathbf{c}(v, D) - \mathbf{c}(v, s(D))) \notin \mathbb{Z}^{\oplus m_v - 1}$.

\[\square\]

## 2 Proof of Proposition 0.1

Put $\text{NS}_\mathbb{Q} := \text{NS}(S) \otimes \mathbb{Q}$ and $T_{\varphi, \mathbb{Q}} := T_{\varphi} \otimes \mathbb{Q}$. As $\text{NS}(S)$ is torsion free by [16, Theorem 1.2], there is no big difficulty in considering $\text{NS}_\mathbb{Q}$. By using the intersection pairing, we have an orthogonal decomposition
\[
\text{NS}(S)_\mathbb{Q} = T_{\varphi, \mathbb{Q}} \oplus T_{\varphi, \mathbb{Q}}^\perp.
\]

Let $\phi$ denote the homomorphism from $E_S(C(C))$ to $\text{NS}(S)_\mathbb{Q}$ given in [16, Lemma 8.2]. Also we define a homomorphism $\phi_o$ from $\text{Div}(S)$ to $T_{\varphi, \mathbb{Q}}^\perp (\subset \text{NS}(S)_\mathbb{Q})$ by the composition:
\[
\phi_o : \text{Div}(S) \to \text{NS}(S) \to T_{\varphi, \mathbb{Q}}^\perp \subset \text{NS}(S)_\mathbb{Q},
\]
the last morphism is the projection. Explicitly, for $D \in \text{Div}(S)$, $\phi_o$ is given by
\[\text{(**) } \phi_o(D) = D - dO - (dC + (O \cdot D))F - \sum_{v \in \text{Red}(\varphi)} \mathbb{F}_v A_v^{-1} \mathbf{c}(v, D),\]
where \( d = D \cdot F, \chi = \chi(\mathcal{O}_S) \) and \( F_v = [\Theta_{v,1}, \ldots, \Theta_{v,m_v-1}] \). Here we have the following lemma on \( \phi \) and \( \phi_o \):

**Lemma 2.1**

(i) For \( P \in E_S(\mathbb{C}(C)) \) and its corresponding section \( s_P \), we have 
\[ \phi(P) = \phi_o(s_P). \]

(ii) For \( D \in \text{Div}(S) \) and its corresponding point \( P_D \in E_S(\mathbb{C}(C)) \), we have 
\[ \phi(P_D) = \phi_o(D). \]

**Proof.** The statement (i) follows from definition. For (ii), our statement follows from the relation \((*)\) in the previous subsection. \( \square \)

In [16], a \( \mathbb{Q} \)-valued bilinear form \( \langle \cdot, \cdot \rangle \) called the height paring on \( E_S(\mathbb{C}(C)) \) is defined by \( \langle P_1, P_2 \rangle := -\phi(P_1) \cdot \phi(P_2) \) (see [16] for details). By Lemma 2.1 (ii), we have 
\[ \langle P_D, P_D \rangle = -\phi_o(D) \cdot \phi_o(D), \quad \langle P_D, P_o \rangle = -\phi_o(D) \cdot \phi(P_o). \]

As \( \langle P_D, P_D \rangle = n^2 \langle P_o, P_o \rangle \) and \( \langle P_D, P_o \rangle = n \langle P_o, P_o \rangle \), we have our statement in Proposition 0.1. Also by computing the intersection pairing explicitly, we have
\[
\phi_o(D) \cdot \phi_o(D) = D^2 - 2dD \cdot O - d^2 \chi - \sum_{v \in \text{Red}(\varphi)} t c(v, D) A_v^{-1} c(v, D)
\]
\[
\phi_o(D) \cdot \phi(P_o) = (D - dO) \cdot s_{P_o} - d\chi - O \cdot D - \sum_{v \in \text{Red}(\varphi)} t c(v, s_{P_o}) A_v^{-1} c(v, D)
\]
\[
\langle P_o, P_o \rangle = 2\chi + 2s_{P_o} \cdot O + \sum_{v \in \text{Red}(\varphi)} t c(v, s_{P_o}) A_v^{-1} c(v, s_{P_o}).
\]

Note that we do not need any data of \( P_D \) in order to compute \( \phi_o(D) \cdot \phi_o(D) \).

3 The torsion part of \( P_D \)

3.1 The homomorphism \( \gamma_{NS} \)

For a reducible singular fiber \( F_v = \sum_i a_{v,i} \Theta_{v,i} \) \( (v \in \text{Red}(\varphi)) \), we denote a subgroup generated by \( \Theta_{v,1}, \ldots, \Theta_{v,m_v-1} \) by \( R_v \). Let \( R_v^\vee \) be the dual of \( R_v \), which can be embedded into \( R_v \otimes \mathbb{Q} \) by the intersection pairing. Under this circumstance, \( R_v^\vee \) can be regarded as a subgroup generated by the columns of \( A_v^{-1} \).

**Definition 3.1** We define a map \( \gamma_{NS} \) from \( \text{NS}(S) \) to \( \bigoplus_{v \in \text{Red}(\varphi)} R_v^\vee \) by
\[
\gamma_{NS} : \text{NS}(S) \ni D \mapsto (-A_v^{-1} c(v, D))_{v \in \text{Red}(\varphi)} \in \bigoplus_{v \in \text{Red}(\varphi)} R_v^\vee,
\]
where \( c(v, D) \) as in §[1]. We denote the induced map from \( \text{NS}(S) \) to \( \oplus_{v \in \text{Red}(\varphi)} R_v^\varphi / R_v \) by

\[
\tau_{\text{NS}} : \text{NS}(S) \rightarrow \oplus_{v \in \text{Red}(\varphi)} R_v^\varphi / R_v.
\]

**Lemma 3.1** Both \( \gamma_{\text{NS}} \) and \( \tau_{\text{NS}} \) are group homomorphisms.

**Proof.** Since \( \Theta_{v,i}(aD_1 + bD_2) = a\Theta_{v,i} \cdot D_1 + b\Theta_{v,i} \cdot D_2 \) \( (D_1, D_2 \in \text{NS}(S), a, b \in \mathbb{Z}) \), our statement is immediate. \( \square \)

**Lemma 3.2** Let \( \psi : \text{Div}(S) \rightarrow E_S(\mathbb{C}(C)) \) be the homomorphism in the previous section. For \( D_1, D_2 \in \text{Div}(S) \), if \( \psi(D_1) = \psi(D_2) \), then \( \tau_{\text{NS}}(D_1) = \tau_{\text{NS}}(D_2) \), where we identify \( D_i \) with its algebraic equivalence class.

**Proof.** Put \( s(D_i) \) be the corresponding sections to \( \psi(D_i) \) \( (i = 1, 2) \). Then by \((*)\) in §[1] we have

\[
D_i \approx s(D_i) + (d_i - 1)O + n_i F + \sum_{v \in \text{Red}(\varphi)} F_v(-A_v)^{-1}(c(v, D_i) - c(v, s(D_i)));
\]

where \( d_i = D_i F, n_i = (d_i - 1)\chi(O_X) + O \cdot D_i - O \cdot s(D_i) \). Since \( s(D_1) = s(D_2) \), we have

\[
D_1 - D_2 \approx (d_1 - d_2)O + (n_1 - n_2)F + \sum_{v \in \text{Red}(\varphi)} F_v(-A_v)^{-1}(c(v, D_1) - c(v, s(D_2))).
\]

In the above equivalence, all coefficients of \( O, F, \) and \( \Theta_{v,j} \)'s are integers. Hence all the entries of \( (-A_v)^{-1}(c(v, D_1) - c(v, D_2)) \) \( (v \in \text{Red}(\varphi)) \) are integers. Since \( R_v^\varphi \) can be regarded as a \( \mathbb{Z} \)-module obtained by adding column vectors of \( (-A_v)^{-1} \), we infer that \( (-A_v)^{-1}(c(v, D_1) - c(v, D_2)) \in R_v \) for \( \forall v \in \text{Red}(\varphi) \). Hence we have

\[
\tau_{\text{NS}}(D_1) = \tau_{\text{NS}}(D_2).
\]

\( \square \)

**Remark 3.1** For a singular fiber \( F_v = \sum_i a_{v,i} \Theta_{v,i} \), we put

\[
F_v^\sharp = \cup_{a_{v,i}=1} \Theta_{v,i}^\sharp;
\]

where \( \Theta_{v,i}^\sharp := \Theta_{v,i} \setminus (\text{singular points of } (F_v)_{\text{red}}) \). By [11, §9], \( F_v^\sharp \) has a structure of an abelian group, and we define an finite abelian group \( G_{F_v^\sharp} \) as in [20, p.81-82]. Roughly speaking, \( G_{F_v^\sharp} \) is a group given by the indices of the irreducible
components of \( F_v \). Put \( G_{\text{Sing}}(\varphi) := \sum_{v \in \text{Sing}} G_{F_v}^\varphi \) and we define a homomorphism 
\( \gamma : \text{MW}(S) \to G_{\text{Sing}}(\varphi) \), which describes at which irreducible component each
section meets. The homomorphism \( \gamma_{\text{NS}} : \text{NS}(S) \to \bigoplus_{v \in \text{Red}(c)} R_v^\varphi / R_v \) can
be considered as a generalization of \( \gamma \). In fact, \( R_v^\varphi / R_v \) is canonically isomorphic
to \( G_{F_v}^\varphi \) and \( \gamma_{\text{NS}} \) and \( \gamma \) coincide for sections.

**Example 3.1** To illustrate the morphism \( \gamma_{\text{NS}} \) and the isomorphism between
\( R_v^\varphi / R_v \) and \( G_{F_v}^\varphi \) in more detail, let us look in to the case where \( S \) has a unique
reducible singular fiber \( F_v \) of type \( I_0^* \). We will relabel the fiber components of \( I_0^* \) as

\[ \Theta_{v,0}, \Theta_{v,1} = \Theta_{v,01}, \Theta_{v,2} = \Theta_{v,10}, \Theta_{v,3} = \Theta_{v,11}, \Theta_{v,4}. \]

In this case, we have

\[
A_v = (\Theta_{v,i}, \Theta_{v,j}) = \begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}, \quad A_v^{-1} = \begin{bmatrix} -1 & -1/2 & -1/2 & -1 \\ -1/2 & -1 & -1/2 & -1 \\ -1/2 & -1/2 & -1 & -1 \\ -1 & -1 & -1 & -2 \end{bmatrix}.
\]

Let \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \) and suppose that

\[-A_v^{-1}x = \begin{bmatrix} -1 & -1/2 & -1/2 & -1 \\ -1/2 & -1 & -1/2 & -1 \\ -1/2 & -1/2 & -1 & -1 \\ -1 & -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in R_v \]

for some integers \( a, b, c, d \in \mathbb{Z} \). Then, we have \( x_1 = 2a - d, x_2 = 2b - d, x_3 = 2c - d, x_4 = -a - b - c + 2d \), which implies that \( x_1, x_2, x_3 \) must have the same
parity. Conversely, if \( x_1, x_2, x_3 \) have the same parity, \(-A_v^{-1}x \in R_v \). From these
facts, we see that in \( R_v^\varphi / R_v \), any \(-A_v^{-1}x \) is equivalent to one of \( 0, -A_v^{-1}e_1, -A_v^{-1}e_2, -A_v^{-1}e_3 \), where \( e_1, e_2, e_3 \) are the first three of the standard basis vectors.

Also we have \((-A_v)^{-1}e_i + (-A_v)^{-1}e_j = (-A_v)^{-1}e_k\), \((\{i, j, k\} = \{1, 2, 3\})\) and
\(2(-A_v)^{-1}e_i = 0, (i = 1, 2, 3)\). Therefore, we have \( R_v^\varphi / R_v \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \cong G_{F_v}^\varphi \).

### 3.2 The determination of the torsion part

As noted above, Proposition [0.1] allows us to compute the coefficient of the \( P_v \) part
of \( P_D \). In this subsection we study the homomorphism \( \gamma_{\text{NS}} \) in order to determine
the torsion part \( P_\tau \) of \( P_D \).

Let \( P_\tau, P_{\tau'} \in E_S(\mathbb{C}(C)) \) be torsion points and let \( s_\tau, s_{\tau'} \) be the corresponding
sections. Then we have:
Lemma 3.3 $\tilde{\gamma}_{NS}(s_\tau) = \tilde{\gamma}_{NS}(s_\tau') \Leftrightarrow P_\tau = P_{\tau'}$

Proof. We first recall that $\gamma$ and $\tilde{\gamma}_{NS}$ coincide for sections. Suppose $\tilde{\gamma}_{NS}(s_\tau) = \tilde{\gamma}_{NS}(s_\tau')$. Then $s_\tau \approx s_{\tau'}$ by $(*), \text{ which implies that } P_\tau \sim_{E_S} P_{\tau'} \text{ as divisors on } E_S$. Hence we have $P_\tau = P_{\tau'}$. The converse is obvious from the definition of $\gamma_{NS}$. □

From this Lemma 3.3, it is enough to determine $\tilde{\gamma}_{NS}(s_\tau)$ to determine $P_\tau$, but since $\gamma$ is a homeomorphism, we have

$$\tilde{\gamma}_{NS}(s_\tau) = \tilde{\gamma}_{NS}(P_D) - \tilde{\gamma}_{NS}(nP_o)$$

which enables us to compute $\tilde{\gamma}_{NS}(s_\tau)$.

4 A rational elliptic surface attached to general 4 lines

In this section, we apply the above discussions to compute $P_D$ for certain $D$ in the case of a rational elliptic surfaces that is associated to an arrangement of four non-concurrent lines $L_i, i = 0, 1, 2, 3$. The computations will be used to prove Theorem 0.1 in the next section.

Let $z_o$ be a general point of $L_0$, $Q = L_0 + L_1 + L_2 + L_3$ and consider the rational elliptic surface $S_{Q,z_o}$ associated to $Q$ and $z_o$. (For the details of the construction of $S_{Q,z_o}$, see [3, II].) Let $L_0 \cap L_i = p_i (i = 1, 2, 3)$ and $L_i \cap L_j = p_{ij} (i \neq j)$. By the construction, $S_{Q,z_o}$ is a rational elliptic surface whose set of reducible singular fibers is of type $I^{*}_{0}, 3I_{2}$. The $I^{*}_{0}$ fiber arises from the preimage of the line $L_0$ and the $I_{2}$ fibers arise from the preimages of the lines through $z_o$ and $q_i$. We denote the $I^{*}_{0}$ fiber by $F_{v_0}$ and its components by

$$\Theta_{\infty,0}, \Theta_{\infty,1} = \Theta_{\infty,01}, \Theta_{\infty,2} = \Theta_{\infty,10}, \Theta_{\infty,3} = \Theta_{\infty,11}, \Theta_{\infty,4}$$

as in Section 3.2 and for $i = 1, 2, 3$, we label the $I_{2}$ fiber corresponding to the line $\overline{z_oq_i}$ by $F_i$ and its components by $\Theta_{i,0}, \Theta_{i,1}$. Also, by [15], $\text{MW}(S_{Q,z_o}) \cong A^*_1 \oplus (\mathbb{Z}/2\mathbb{Z})^\oplus 2$.

Next we consider configurations consisting of an irreducible cubic $E$ and $Q = L_0 + L_1 + L_2 + L_3$ satisfying the following:

- $E$ passes through the three points $p_1, p_2, p_3$.
- For $i = 1, 2, 3$, $E$ is tangent to $L_i$ at a point $q_i$ distinct from $p_i, p_{ij}$.

Note that if $E$ is a nodal cubic, $E + \sum_{i=0}^{3} L_i$ has Combinatorics 1-$(b)$. Suppose $E$ is a splitting curve with respect to $Q$ and let $E^\pm$ be the irreducible components
of the strict transform of $E$ in $S_{Q,z_0}$. Our goal is to compute $P_{E^\pm}$. We first recall the diagram

\[
\begin{array}{ccc}
S'_Q & \xrightarrow{\mu} & S_Q \\
\downarrow f'_Q & & \downarrow f_Q \\
\mathbb{P}^2 & \xrightarrow{q} & \mathbb{P}^2 \\
\downarrow f_Q & & \downarrow f_{Q,z_0} \\
S_Q & \xleftarrow{\nu_{z_0}} & S_{Q,z_0}
\end{array}
\]

that appears in the construction of $S_{Q,z_0}$ given in [4, Introduction]. Note that the covering transformation of $f_{Q,z_0}$ induces the inversion morphism on the generic fiber $E_{S_{Q,z_0}}$ and we have $P_{E^\pm} = -P_{E^\pm}$. Let $P_{E^+} = nP_o + P_\tau$, where $P_o$ is a generator of the $A_1^+$ part of $ES(\mathbb{C}(t))$. We may assume $n \geq 0$ after relabeling $E_{\pm}$, suitably. First, from the data of the intersection of $E$ and $L_i$ ($i = 0, 1, 2, 3$) we have

\[
c(v_0, E^+) = t \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}, 
\]

where $v_0$ is the $I_0$ fiber and $v_i$ are the $I_2$ fibers. Now, from Proposition 0.1 we have

\[
n^2 = -\frac{\phi_o(E^+) \cdot \phi_o(E^+)}{P_o P_o} = -2((E^+)^2 - 3).
\]

Since $E$ is a cubic and passes through $p_1, p_2, p_3$, the strict transform of $E$ in $(\mathbb{P}^2)_{z_0}$ has self-intersection number 6. Hence, we have

\[
(E^+ + E^-)^2 = 2 \cdot 6 = 12
\]

by which we obtain

\[
(E^+)^2 = (E^-)^2 = 6 - E^+ \cdot E^-.
\]

Now, if $E$ is smooth, then $E^+ \cdot E^- = 3$ and we have $(E^+)^2 = 3$, which implies $n = 0$.

Next, if $E$ is a nodal cubic, there are two possibilities for $E^+ \cdot E^-$, namely $E^+ \cdot E^- = 3$ or 5, depending on the data of the preimage of the node. If the preimage of the node becomes nodes of $E^+, E^-$, then $E^+ \cdot E^- = 3$, and if the preimage of the node becomes intersection points of $E^+$ and $E^-$ then $E^+ \cdot E^- = 5$. Therefore we have $n = 0$ (resp. 2) when $E^+ \cdot E^- = 3$ (resp. $E^+ \cdot E^- = 5$). Finally, by Remark 1.1 (ii) and Lemma 1.1 or Example 3.1 we have

\[
c(v_0, s(E^+)) = t \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, c(v_i, s(E^+)) = 0 (i = 1, 2, 3),
\]

which implies that $P_\tau = 0$ by the arguments in Section 3.2.

So far, we have two possibilities

\[
P_{E^+} = \begin{cases} O \\ 2P_o \end{cases}
\]
We can determine which case occurs as follows:
Suppose $s(E^+) = O$. This implies that for every smooth singular fiber $F$, the sum of the three intersection points of $E^+$ and $F$ under the group operation of $F$ must equal $O \cap F$. Let $L_{12}$ be the line through $q_1, q_2$ and choose $z_0$ to be the intersection of $L_{12}$ and $L_0$. Then the intersection points of $E^+$ and the smooth fiber corresponding to $L_{12}$, lying above $q_1, q_2$ become 2-torsion points on $F$. This and the fact that the sum of these two points with the third intersection point must equal $O \cap F$ implies that the third intersection point must also be a 2-torsion point on $F$, hence must lie over $q_3$. Hence $q_3 \in L_{12}$ and $q_1, q_2, q_3$ become collinear.

On the other hand, if $s(E^+) = s(2P_o)$, we have $s(E^+) \cdot O = 0$. This implies that for every smooth fiber $F$, the sum of the intersection points of $E^+$ and $F$ cannot equal $O$ under the group operation of $F$. Hence in this case, $q_1, q_2, q_3$ cannot be collinear.

Summing up, we have the following proposition.

**Proposition 4.1** Let $E$ be a smooth or nodal cubic having the combinatorics given above. Then, the following statements hold:

1. If $E$ is a smooth cubic and is splitting then $s(E^+) = O$ and $q_1, q_2, q_3$ are collinear.

2. If $E$ is a nodal cubic, then $E$ always splits and
   
   (a) $s(E^+) = O$ if and only if $q_1, q_2, q_3$ are collinear, and
   
   (b) $s(E^+) = s(2P_o)$ otherwise.

**Proof.** All the statements follow from the above discussions. We note that if $E$ is a nodal cubic, $E$ is a splitting curve as it is rational.

## 5 Dihedral covers and proof of Theorem 0.1

### 5.1 Dihedral covers

Let $D_{2n}$ be the dihedral group of order $2n$. In order to prove Theorem 0.1 we consider the existence/non-existence for Galois covers of $\mathbb{P}^2$ whose Galois group is isomorphic to $D_{2n}$ ($D_{2n}$-covers, for short). In this section, we summarize some facts on $D_{2n}$-covers which we need for our proof. As for general terminologies on Galois covers, we refer to [1, Section 3], or [21]. As for $D_{2n}$-covers, we refer [2, 3, 4, 17, 21]. In our previous papers such as [3, 4, 21], we considered the cases when $n$ is odd. In this article, however, we also consider the case where $n$ is even. We here introduce some results on the even case based on [17, Proposition 0.6 and Remark 3.1] as follows:
Proposition 5.1 Let $n$ be an even integer $\geq 4$. Let $f: S \rightarrow \Sigma$ be a smooth finite double cover of a simply connected smooth projective surface $\Sigma$. Let $\sigma_f$ be the involution on $S$ determined by the covering transformation for $f$. Let $C, D$ and $D_o$ be divisors on $S$ satisfying the following properties:

(i) If $C$ is irreducible such that $\sigma_f^* C \neq C$

(ii) $D$ is a reduced divisor or $D = \emptyset$. If $D \neq \emptyset$ or each irreducible component $D_j$, there exists an irreducible divisor $B_j$ on $\Sigma$ such that $D_j = f^* B_j$.

(iii) $C + n/2 D - \sigma_f^* C \sim n D_o$.

Then there exists a $D_{2n}$-cover $\pi: X \rightarrow \Sigma$ such that (a) $D(X/\Sigma) = S$ and (b) $\Delta_\pi = \Delta_f \cup f(\text{Supp}(C + D))$.

Proof. We rewrite $D_o$ as a difference of effective divisors: $D_o = D_1 + D_2 - D_3 - D_4$. Now put $D_1 = C$, $D_2 = D$, $D_3 = D^\sigma$ and $D_4 = D^\sigma$. By [17, Remark 3.1], the condition (e) in [17, Propositions 0.6] is satisfied. Hence, by Proposition 0.6, we have a $D_{2n}$-cover as desired. □

As for the “converse” for Proposition 5.1, it can be stated as follows:

Proposition 5.2 Let $f: S \rightarrow \Sigma$, $\sigma_f$ and $D$ be as in Proposition 5.1. If there exists a $D_{2n}$-cover ($n$: even $\geq 4$) $\pi: X \rightarrow \Sigma$ such that (i) $D(X/\Sigma) = S$ and (ii) $\pi$ is branched at $2(\Delta_f + f(D)) + nC$.

Then:

(a) $f^* C$ is of the form $C + \sigma_f^* C$, $C \neq \sigma_f^* C$.

(b) There exists divisor $D_o$ on $S$ such that $C + n/2 D - \sigma_f^* C \sim n D_o$.

Proof. Let $D_1, D_2, D_3$ and $D_4$ be the divisors in [17] Proposition 0.7. Then the conditions (i) and (v) in [17] Proposition 0.7 and (ii) as above, we may assume:

$$D_1 = aC, 0 < a < (n - 1)/2, D_2 = D, D'_o = D_4 - D_3,$$

and we have $aC - n/2 D - a\sigma_f^* C \sim n D_o$. Take $u \in \mathbb{C}(S)$ such that $(u) = aC - n/2 D - a\sigma_f^* C - n D_o$. By the proof of [17] Proposition 0.7, $X$ is the $\mathbb{C}(S)(\sqrt[n]{u})$-normalization of $S$. This shows that the ramification index along $C$ is given by $n/ \gcd(a,n)$, i.e., $\gcd(a,n) = 1$ by the condition (ii). Let $k$ be an integer such that $ak \equiv 1 \mod n$. Then

$$ak C + \frac{kn}{2} D - a k \sigma_f^* C$$

$$= C + \frac{n}{2} D - \sigma_f^* C + n \left\{ \frac{nk - 1}{n} (C - \sigma_f^* C) + \frac{k - 1}{2} D \right\}$$

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Hence we have
\[
C + \frac{n}{2}D - \sigma^*_f C \sim n \left\{ D'_o - \frac{nk-1}{n}(C - \sigma^*_f C) - \frac{k-1}{2}D \right\}.
\]

\[\square\]

5.2 Proof of Theorem 0.1

We first remark that, as \( S_{Q,z_0} \) is a rational elliptic surface, there is no difference between algebraic and linear equivalence. For the covering transformation of \( f_{Q,z_0} \), we denote it by \( \sigma_{Q,z_0} \) for simplicity.

**Proof for (i).** In this case, \( q_1, q_2 \) and \( q_3 \) are collinear. By Proposition 1.1, we have \( s(E^+) = O \), i.e.,
\[
E^+ \sim 3O + 3F - 2\Theta_{\infty,1} - 2\Theta_{\infty,2} - 2\Theta_{\infty,3} - 3\Theta_{\infty,4}.
\]

\( O, F \) and \( \Theta_{\infty,i} \) are invariant under \( \sigma_{Q,z_0} \), \( E^+ \sim E^- \). Hence, by [2 Proposition 1.1, Corollary 1.2], there exists an \( n \)-cyclic cover \( g_n : \hat{X}_n \to S_{Q,z_0} \) such that (i) \( \Delta_{g_n} = E^+ \cup E^- \) and (ii) \( f_{Q,z_0} \circ g_n : \hat{X}_n \to (\mathbb{P}^2)_{z_0} \) is a \( D_{2n} \)-cover. The stein factorization \( X_n \) of \( q \circ q_{z_0} \circ f_{Q,z_0} \circ g_n : \hat{X}_n \to \mathbb{P}^2 \) is a \( D_{2n} \)-cover of \( \mathbb{P}^n \) as desired.

**Proof for (ii).** We prove the existence of a \( D_{2n} \)-cover. Let us recall the notation: \( L_o \cap L_i = \{ p_i \} \) and \( L_i \cap L_j = \{ p_{ij} \} \). Let \( \overline{p_ip_{jk}} \) \( (\{ i, j, k \} = \{ 1, 2, 3 \}) \) be the lines connecting \( p_i \) and \( p_{jk} \). Each \( \overline{p_ip_{jk}} \) gives rise to sections \( s^+_j \) such that \( \langle P^+_j, P^+_j \rangle = 1/2 \). We choose one of them as a generator \( s_o \) of the free part \( A^+_1 \) (note that the differences between \( s^+_i \) and \( s^+_j \) are a translation-by-2-torsions). Choose \( P^+_1 \) as \( P_o \). Note that we change \( \pm \) so that \( s(E^+) = s(2P_o) \) if necessary. Then as
\[
2s_o \sim s(2P_o) + O + F - 2\Theta_{\infty,1} - \Theta_{\infty,2} - \Theta_{\infty,3} - 2\Theta_{\infty,4} - \Theta_{1,1},
\]
we have
\[
E^+ \sim s(2P_o) + 2O + 2F - 2\Theta_{\infty,1} - 2\Theta_{\infty,2} - 2\Theta_{\infty,3} - 3\Theta_{\infty,4} \\
\sim 2s_o + O + F - \Theta_{\infty,2} - \Theta_{\infty,3} - \Theta_{\infty,4} + \Theta_{1,1}.
\]

Since \( \sigma_{Q,z_0} = [-1]_{\varphi_{Q,z_0}} \), we have \( s(E^-) = s(-2P_o) \). Also as
\[
-2s_o \sim s(-2P_o) - 3O - 3F + 2\Theta_{\infty,1} + \Theta_{\infty,2} + \Theta_{\infty,3} + 2\Theta_{\infty,4} + \Theta_{1,1},
\]
we have
\[
E^- \sim -2s_o + 5(O + F) - 4\Theta_{\infty,1} - 3\Theta_{\infty,2} - 3\Theta_{\infty,3} - 5\Theta_{\infty,4} - \Theta_{1,1}.
\]
Thus we have
\[ E^++2(\Theta_{\infty,2}+\Theta_{\infty,3}+\Theta_{1,1})-E^- \sim 4(s_o-O-F+\Theta_{\infty,1}+\Theta_{\infty,2}+\Theta_{\infty,3}+\Theta_{\infty,4}+\Theta_{1,1}). \]

Now by Proposition 5.1 we have a \( D_8 \)-cover \( \hat{\pi}_4 : \hat{X}_4 \rightarrow (\mathbb{P}^2)_{z_0} \) branched at
\[ 2\Delta_{f_{Q_{z_0}}(E^+} \cup \Theta_{\infty,2} \cup \Theta_{\infty,3} \cup \Theta_{1,1}) \]
and the Stein factorization of \( q \circ q_{z_0} \circ \hat{\pi}_4 \) gives the desired \( D_8 \)-cover \( \pi_4 : X_4 \rightarrow \mathbb{P}^2 \) branched at \( 2Q+4E \).

We now go on to prove non-existence of \( D_{2n} \)-covers as in Theorem 0.1 for \( n \geq 3 \) except \( n = 4 \). Suppose that there exists a \( D_{2n} \)-cover \( \pi_n : X_n \rightarrow \mathbb{P}^2 \) of \( \mathbb{P}^2 \) as in Theorem 0.1 Then we have \( D(X_n/\mathbb{P}^2) = S^\prime_Q \) and \( f_Q^\prime = \beta_1(\pi_n) \) Let \( \hat{X}_n \) be the \( \mathbb{C}(X_n) \)-normalization of \((\mathbb{P}^2)_{z_0}\). Then we have the following:

- The induced morphism \( \hat{\pi}_n : \hat{X}_n \rightarrow (\mathbb{P}^2)_{z_0} \) is a \( D_{2n} \)-cover of \((\mathbb{P}^2)_{z_0}\).
- \( D(\hat{X}_n/(\mathbb{P}^2)_{z_0}) = S^\prime_Q \) and \( S^\prime_Q \) is the \( \mathbb{C}(S^\prime_Q) \) normalization of \((\mathbb{P}^2)_{z_0}\).
- The image of the branch locus \( \Delta_{\beta_2(\hat{\pi}_n)} \) of the form \( f_{Q_{z_0}}(E^+) + \Xi \), where \( \Xi \) is contained in the exceptional set of \( q \circ q_{z_0} \).

Lemma 5.1  
(i) If \( n \) is odd, \( \Xi = 0 \).
(ii) If \( \Xi \neq 0 \), then \( n \) is even and the ramification index along \( \Xi \) is 2.

Proof. Let \( E_o \) be an arbitrary irreducible component of \( \Xi \). In our case, \( f_{Q_{z_0}}^\prime(E_o) \) is irreducible. Now (i) follows from [21] Corollary 2.4, and (ii) from [17] Proposition 0.7. \( \square \)

\( n = \text{odd} \): Choose any odd prime \( p \) dividing \( n \). Since we have a surjective morphism \( D_{2n} \rightarrow D_{2p} \), we have a \( D_{2p} \)-cover \( \hat{\pi}_p : \hat{X}_p \rightarrow (\mathbb{P}^2)_{z_0} \) such that \( D(\hat{X}_p/(\mathbb{P}^2)_{z_0}) = S^\prime_{Q_{z_0}} \) and \( \beta_2(\hat{\pi}_p) \) is branched at \( p(E^+ + E^-) \) by Lemma 5.1 By [21] Theorem 3.2, \( s(E^+) \) is \( p \)-divisible in \( \text{MW}(S^\prime_{Q_{z_0}}) \), but this contradicts to Proposition 4.1.

\( n = \text{even} \): By Proposition 5.2, there exist divisors \( D \) and \( D_o \) on \( S^\prime_{Q_{z_0}} \) such that
\[ E^++\frac{n}{2} D - E^- \sim nD_o, \]
where \( \text{Supp}(D) \) is contained in the exceptional set of \( q \circ q_o \) by Lemma 5.1. By [16] Theorem 1.3, this implies \( P_{E^+} - P_{E^-} \) is \( n \)-divisible in \( \text{MW}(S^\prime_{Q_{z_0}}) \). On the other hand, by Proposition 4.1, we have
\[ P_{E^+} - P_{E^-} = 2P_o - (-2P_o) = 4P_o. \]
This proves the non-existence for \( n \geq 6 \).

Remark 5.1 As we noticed before, there are 4 different choices for \( s_o \). If we choose generator different from \( s_o \), the divisor \( \Theta_{\infty,2} + \Theta_{\infty,3} + \Theta_{1,1} \) changes. This means that we have 4 different \( D_8 \)-covers in the case of curves of Type II.
References

[1] E. Artal Bartolo, J.-I. Codgolludo and H. Tokunaga: A survey on Zariski pairs, Adv.Stud.Pure Math., 50(2008), 1-100.

[2] E. Artal Bartolo and H. Tokunaga: Zariski k-plets of rational curve arrangements and dihedral covers, Topology Appl. 142 (2004), 227-233.

[3] S. Bannai and H. Tokunaga: Geometry of bisections of elliptic surfaces and Zariski N-plets for conic arrangements, Geom. Dedicata 178(2015), 219 - 237.

[4] S. Bannai and H. Tokunaga: Geometry of bisections of elliptic surfaces and Zariski N-plets II, Topology and its Applications, 231(2017), 10 - 25

[5] S. Bannai, H. Tokunaga and M. Yamamoto: Rational points of elliptic surfaces and Zariski N-plets for cubic-line, cubic-conic-line arrangements, arXiv:1710.02691

[6] J.-I. Cogolludo-Agusttin and R. Kloosterman: Mordell-Weil groups and Zariski triples. Geometry and arithmetic, 75-89, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012.

[7] J. -I. Cogolludo-Agustin and A. Libgober: Mordell-Weil groups of elliptic threefolds and the Alexander module of plane curves, J. Reine Angew. Math. 697(2014), 15-55.

[8] E. Horikawa: On deformation of quintic surfaces, Invent. Math. 31 (1975), 43 – 85.

[9] R. Kloosterman: Cuspidal plane curves, syzygies and a bound on the MW-rank, J. Algebra 375 (2013), 216-234

[10] R. Kloosterman: Mordell-Weil lattices and toric decompositions of plane curves, Math. Ann. 367 (2017), 755-783.

[11] K. Kodaira: On compact analytic surfaces II-III, Ann. of Math. 77 (1963), 563-626, 78(1963), 1-40.

[12] A. Libgober: On Mordell-Weil groups of isotrivial abelian varieties over function fields, Math. Ann. 357 (2013), 605-629.

[13] R. Miranda: Basic theory of elliptic surfaces, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989.

[14] R. Miranda and U. Persson: On extremal rational elliptic surfaces, Math. Z. 193(1986), 537-558.

[15] K. Oguiso and T. Shioda: The Mordell-Weil lattice of Rational Elliptic surface, Comment. Math. Univ. St. Pauli 40(1991), 83-99.
[16] T. Shioda: *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Pauli **39** (1990), 211-240.

[17] H. Tokunaga: *On dihedral Galois coverings*, Canadian J. of Math. **46** (1994), 1299 - 1317.

[18] H. Tokunaga: *Some examples of Zariski pairs arising from certain elliptic K3 surfaces*, Math. Z. **227** (1998), 465-477, **II**, Math.Z. **230** (1999), 389-400

[19] H. Tokunaga: *Dihedral covers and an elementary arithmetic on elliptic surfaces*, J. Math. Kyoto Univ. **44**(2004), 55-270.

[20] H. Tokunaga: *Some sections on rational elliptic surfaces and certain special conic-quartic configurations*, Kodai Math. J. **35**(2012), 78-104.

[21] H. Tokunaga: *Sections of elliptic surfaces and Zariski pairs for conic-line arrangements via dihedral covers*, J. Math. Soc. Japan **66**(2014) 613-640.

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