Stochastic local operations and classical communication properties of the n-qubit symmetric Dicke states

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Abstract

Recently, several schemes for the experimental creation of Dicke states were described. In this paper, we show that all the n-qubit symmetric Dicke states with \( l \) \((2 \leq l \leq (n-2))\) excitations are inequivalent to the \(|\text{GHZ}\rangle\) state or the \(|\text{W}\rangle\) state under SLOCC, that the even n-qubit symmetric Dicke state with \( n/2 \) excitations is inequivalent to any even n-qubit symmetric Dicke state with \( l \neq n/2 \) excitations under SLOCC, and that all the n-qubit symmetric Dicke states with \( l \) \((2 \leq l \leq (n-2))\) excitations satisfy Coffman, Kundu and Wootters’ generalized monogamy inequality \( C_{12}^2 + \ldots + C_{1n}^2 < C_{1(2\ldots n)}^2 < 1.\)

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1 Introduction

Stockton et al. pointed out that the symmetric systems are experimentally interesting because it is easier to nonselectively address an entire ensemble of particles rather than individually address each member \[1\]. Due to the symmetry under permutations of the qubits, the symmetric Dicke states become important. Dicke states are considered to be the simultaneous eigenstates of both the square of the total spin operator \( \hat{S}^2 \) and

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its $z$-component $\hat{S}_z$ [2]. In [1], the $n$-qubit symmetric Dicke states with $l$ excitations, where $1 \leq l \leq (n-1)$, were defined as follows.

$$|l, n\rangle = \sum_i P_i |1_1 1_2 \ldots 1_l 0_{l+1} \ldots 0_n\rangle,$$ (1.1)

where $\{P_i\}$ is the set of all the distinct permutations of the qubits. But, from the definition in Eq. (1.1), it is not obvious what position each “1” or “0” occurs in $P_i |1_1 1_2 \ldots 1_l 0_{l+1} \ldots 0_n\rangle$. For our further consideration it is desirable to rewrite Eq. (1.1) by using a binary basis in the following form:

$$|l, n\rangle = \frac{1}{\sqrt{\binom{n}{l}}} \sum_{0 \leq i_1 < i_2 < \ldots < i_l \leq n-1} |2^{i_1} + 2^{i_2} + \ldots + 2^{i_l}\rangle, 1 \leq l \leq n-1.$$ (1.2)

It is easy to see that just $l$ ones occur in each basis term of the state $|l, n\rangle$ and $|l, n\rangle$ is symmetric under permutations of the qubits. $|1, n\rangle$ is just the $|W\rangle$ state, i.e., $|1, n\rangle = (|100\ldots0\rangle + |010\ldots0\rangle + \ldots + |00\ldots1\rangle)/\sqrt{n}$.

Several schemes for the experimental creation of Dicke states were described in [2, 3, 4, 5, 6, 7, 8, 9]. For example, the authors in [10, 11, 12] realized the $|W\rangle$ state of three qubits in a photonic system, while the $|W\rangle$ state of eight qubits was created with trapped ions in [13]. The authors in [14] generated a four-qubit symmetric Dicke state with two excitations and discussed its application in quantum communication. The authors in [2, 7] reported the realistic proposals to generate Dicke states in specific physical systems. The cavity QED schemes for generating symmetric Dicke states were discussed in [15]. The methods for detecting entanglement around symmetric Dicke states were presented in [16].

Quantum entanglement is a quantum mechanical resource and plays a key role in quantum computation and quantum information. Recently, many authors have exploited SLOCC (stochastic local operations and classical communication) entanglement classification [17, 18, 19, 20, 21, 22, 23, 25]. If two states can be obtained from each other by means of local operations and classical communication (LOCC) with nonzero probability, we say that two states have the same kind of entanglement [17]. In [18], Dür et al. showed that for pure states of three qubits there are six inequivalent SLOCC entanglement classes, of which two are true entanglement classes: $|GHZ\rangle$ and $|W\rangle$. Verstraete et al. [19] discussed the entanglement classes of four qubits under SLOCC and pointed out there exist nine families of states corresponding to nine different ways
of entangling four qubits.

As indicated in [18], if two states are SLOCC equivalent, then they are suited to do the same tasks of QIT. In this paper, by means of the SLOCC invariant for \( n \) qubits [23], we investigate the SLOCC properties of the \( n \)-qubit symmetric Dicke states with \( l \) excitations. The different SLOCC invariants were proposed in [20] [24]. For the readability, we list the SLOCC invariant for \( n \) qubits [23] as follows.

Let \( |\psi\rangle \) and \( |\psi'\rangle \) be any states of \( n \) qubits. Then we can write
\[
|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle \quad \text{and} \quad |\psi'\rangle = \sum_{i=0}^{2^n-1} a'_i |i\rangle.
\]
It is well known from [18] that
\[
|\psi\rangle \text{ is equivalent to } |\psi'\rangle \text{ under SLOCC if and only if } |\psi\rangle = \alpha \otimes \beta \otimes \gamma \cdots |\psi'\rangle,
\]
(1.3)
where \( \alpha, \beta, \gamma, \ldots \) are invertible local operators.

For even \( n \) qubits, if \( |\psi\rangle \) is equivalent to \( |\psi'\rangle \) under SLOCC then \( |\psi\rangle \) and \( |\psi'\rangle \) satisfy the following equation [23]:
\[
\tau(\psi) = \tau(\psi') |\det(\alpha) \det(\beta) \det(\gamma)\cdots|_n,
\]
(1.4)
where
\[
\tau(\psi) = 2|\mathcal{I}^*(a, n)|,
\]
\[
\mathcal{I}^*(a, n) = \sum_{i=0}^{2^{n-2}-1} sgn^*(n, i)(a_{2i}a_{(2^n-1)-2i} - a_{2i+1}a_{(2^n-2)-2i}),
\]
(1.5)
in which
\[
sgn^*(n, i) = \begin{cases} 
(-1)^{N(i)} & \text{for } 0 \leq i \leq 2^{n-3} - 1, \\
(-1)^{n+N(i)} & \text{for } 2^{n-3} \leq i \leq 2^{n-2} - 1. 
\end{cases}
\]
The \( N(i) \) above is defined as follows. Let \( i_{n-1} \cdots i_1 i_0 \) be an \( n \)-bit binary representation of \( i \). That is, \( i = i_{n-1}2^{n-1} + \cdots + i_1 2^1 + i_0 2^0 \). Then, let \( N(i) \) be the number of the occurrences of “1” in \( i_{n-1} \cdots i_1 i_0 \). Note that when \( n \) is even, \( (-1)^{n+N(i)} = (-1)^{N(i)} \), and \( sgn^*(n, i) = (-1)^{N(i)} \) for \( 0 \leq i \leq 2^{n-2} - 1 \). Thus, Eq. (1.5) can be simplified as Eq. (3.1) in this paper.
For even \( n \) qubits, we can define \( \tau(\psi') \) in Eq. (1.4) from the definition of \( \tau(\psi) \) in Eq. (1.5) by replacing the amplitudes \( a_k \) of \( |\psi\rangle \) with the amplitudes \( a'_k \) of \( |\psi'\rangle \).

For odd \( n \) qubits, if \( |\psi\rangle \) is equivalent to \( |\psi'\rangle \) under SLOCC then \( |\psi\rangle \) and \( |\psi'\rangle \) satisfy the following equation [23]:

\[
\tau(\psi) = \tau(\psi') \frac{2}{\det(\alpha)} \frac{2}{\det(\beta)} \frac{2}{\det(\gamma)} \cdot \ldots, \tag{1.6}
\]

where

\[
\tau(\psi) = 4|\mathcal{I}(a, n)|^2 - 4\mathcal{I}^*(a, n - 1)\mathcal{I}^*_{+2n-1}(a, n - 1)|, \tag{1.7}
\]

in which

\[
\mathcal{I}(a, n) = \sum_{i=0}^{2^{n-3} - 1} (-1)^{N(i)}(a_{2i} a_{(2n-1) - 2i} - a_{2i+1} a_{(2n-2) - 2i})

- (a_{(2n-1)-2i} a_{(2n-1)+2i} - a_{(2n-1)-2i} a_{2n-1+2i}), \tag{1.8}
\]

and

\[
\mathcal{I}^*_{+2n-1}(a, n - 1) = \sum_{i=0}^{2^{n-3} - 1} \text{sgn}^*(n - 1, i) \times

(a_{2n-1+2i} a_{(2n-1)-2i} - a_{2n-1+2i} a_{(2n-2)-2i}), \tag{1.9}
\]

For odd \( n \) qubits, we can also define \( \tau(\psi') \) in Eq. (1.6) from the definition of \( \tau(\psi) \) in Eqs. (1.7), (1.8) and (1.9) by replacing the amplitudes \( a_k \) of \( |\psi\rangle \) with the amplitudes \( a'_k \) of \( |\psi'\rangle \).

By Eqs. (1.4) and (1.6), we obtain the following corollary 1.

Corollary 1. For any \( n \) qubits, if \( \tau(\psi) = 0 \) but \( \tau(\psi') \neq 0 \) or \( \tau(\psi) \neq 0 \) but \( \tau(\psi') = 0 \), then \( |\psi\rangle \) and \( |\psi'\rangle \) are inequivalent under SLOCC.
A simple calculation shows that for any $n$-qubit $|GHZ⟩ = (|0⟩^{⊗n} + |1⟩^{⊗n})/\sqrt{2}$, $τ(GHZ) = 1$, and for any $n$-qubit $|W⟩$, $τ(W) = 0$. By Corollary 1, the states $|GHZ⟩$ and $|W⟩$ of any $n$ qubits are inequivalent under SLOCC.

This paper is organized as follows. In Sec. 2, we show that all the Dicke states with $l$ excitations are true entangled. In Sec. 3, we discuss the SLOCC properties of the even $n$-qubit symmetric Dicke states. In Sec. 4, we analyze the SLOCC properties of the odd $n$-qubit symmetric Dicke states. In Sec. 5, we discuss the monogamy inequality for Dicke states.

2 All the Dicke states with $l$ excitations are true entangled

When $l = 1$ and $n = 2$, it reduces to Bell state. When $l = 1$ and $n \geq 3$, $|l,n⟩$ is the $|W⟩$ state. Let us consider $l > 1$ and $n \geq 3$. Assume that $|l,n⟩$ is a product state. Then, we can write $|l,n⟩ = |φ⟩ ⊗ |ω⟩$, where $|φ⟩$ is a state of $k$ qubits and $|ω⟩$ is a state of $(n − k)$ qubits.

Case 1. If $l$ ones occur in some basis term of $|φ⟩$, then only zeros occur in each basis term of $|ω⟩$. It is impossible.

Case 2. If only zeros occur in some basis term of $|φ⟩$, then $l$ ones must occur in each basis term of $|ω⟩$. It is also impossible.

Case 3. If $t$ ones, where $1 \leq t \leq l − 1$, occur in some basis term of $|φ⟩$, then each basis term of $|ω⟩$ must contain $(l − t)$ ones. It implies that each basis term of $|φ⟩$ must contain $t$ ones. Thus, each basis term of $|φ⟩$ contains $t$ ones and $|φ⟩$ has $\binom{k}{t}$ terms, while each basis term of $|ω⟩$ contains $(l − t)$ ones and $|ω⟩$ has $\binom{n − k}{l − t}$ terms. Thus, $|φ⟩ ⊗ |ω⟩$ has $\binom{k}{t} \binom{n − k}{l − t}$ terms. However, $|l,n⟩$ has $\binom{n}{l}$ terms. Hence, this case is impossible.

From the above cases, clearly $|l,n⟩$ $(1 \leq l \leq (n − 1))$ is a true entangled state, i.e., not a product state.

For example, for $n$ qubits, $|2,n⟩ = (1/\sqrt{n(n−1)/2})\sum_{0\leq i<j\leq n−1} |2^i + 2^j⟩$. It means that $(n−2)$ zeros and two ones occur in each term of the state $|2,n⟩$. So, $|2,n⟩$ has $(n−1)/2$ terms. For example, $|2,4⟩ = (|0011⟩ + |0101⟩ + |0110⟩ + |1001⟩ + |1010⟩ + |1100⟩)/\sqrt{6}$. It was proven that $|2,4⟩$ is a true entangled state in [25].
3 Even $n$-qubit symmetric Dicke states

In this section, we show that the even $n$-qubit symmetric Dicke state with $n/2$ excitations is inequivalent to any even $n$-qubit symmetric Dicke state with $l \neq n/2$ excitations under SLOCC, and that the even $n$-qubit symmetric Dicke states with $l$ excitations, where $2 \leq l \leq (n-2)$, are different from the even $n$-qubit $|\text{GHZ}\rangle$ and $|\text{W}\rangle$ states under SLOCC, respectively.

Note that the states $|l,n\rangle$ and $|(n-l),n\rangle$ are equivalent under SLOCC by Lemma 1 in Appendix A. Hence, it is enough to consider $2 \leq l \leq n/2$.

3.1 Dicke state with $n/2$ excitations is inequivalent to Dicke state with $l \neq n/2$ excitations under SLOCC.

3.1.1 $\tau(|n/2,n\rangle) = 1$ for Dicke state with $n/2$ excitations

By the definition in Eq. (1.2), for the state $|l,n\rangle$, the amplitudes $a_{2i_1+2i_2+...+2i_l}$, where $0 \leq i_1 < i_2 < ... < i_l \leq n-1$ and $0 \leq j_1 < j_2 < ... < j_{l(n/2)} \leq n-1$, and $2^{i_1} + 2^{i_2} + ... + 2^{i_l} + 2^{j_1} + 2^{j_2} + ... + 2^{j_{l(n/2)}} = 2^n - 1$. Clearly, $a_{2i_1+2i_2+...+2i_l}a_{2j_1+2j_2+...+2j_{l(n/2)}} = 1/n_{l(n/2)}$. Note that $(-1)^{N(2i_1+2i_2+...+2i_l)} = (-1)^{n/2}$. Clearly, there are $n_{l(n/2)}/2$ terms being of the form $a_{2i_1+2i_2+...+2i_l}a_{2j_1+2j_2+...+2j_{l(n/2)}}$, and each term has the same sign $(-1)^{n/2}$. Hence, by Eq. (3.1), $\tau(|n/2,n\rangle) = 1$. For example, $\tau(|2,4\rangle) = 1$ and $\tau(|3,6\rangle) = 1$. 

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3.1.2 \( \tau(|l,n\rangle) = 0 \) for Dicke state with \( 1 \leq l \leq (n-1) \) but \( l \neq n/2 \)

Let us prove that each term \( a_k a_{(2n-1)-k} \) in Eq. (3.1) vanishes when \( l \neq n/2 \). By the definition in Eq. (1.2), if \( N(k) \neq l \), then \( a_k = 0 \). Otherwise, \( N(k) = l \) and \( a_k = 1/\sqrt{{n \choose l}} \). However, when \( N(k) = l \), \( N((2^n - 1) - k) = n - l \neq l \). Thus, \( a_{(2n-1)-k} = 0 \). Therefore, \( \tau(|l,n\rangle) = 0 \). Especially, \( \tau(W) = 0 \).

From the above discussion, \( \tau(|n/2,n\rangle) = 1 \) while \( \tau(|l,n\rangle) = 0 \), where \( 1 \leq l \leq (n-1) \) but \( l \neq n/2 \). By Corollary 1, the state \( |n/2,n\rangle \) is different from the states \( |l,n\rangle \) \((1 \leq l \leq (n-1) \) but \( l \neq n/2) \) under SLOCC.

3.2 All the \( n \)-qubit symmetric Dicke states are inequivalent to the \( n \)-qubit \(|GHZ\rangle \) state under SLOCC.

In order to show that the states \( |l,n\rangle \) \((2 \leq l \leq (n-2)) \) are different from the \(|GHZ\rangle \) and \(|W\rangle \) states under SLOCC respectively, let us consider the following quantity for any state \(|\Omega\rangle = \sum_{k=0}^{2^n-1} b_k|k\rangle\),

\[
D^{(l)}(\Omega) = (b_{1+\Delta}b_{4+\Delta} - b_{0+\Delta}b_{5+\Delta})(b_{11+\Delta}b_{14+\Delta} - b_{10+\Delta}b_{15+\Delta}) \\
-(b_{1+\Delta}b_{6+\Delta} - b_{2+\Delta}b_{7+\Delta})(b_{9+\Delta}b_{12+\Delta} - b_{8+\Delta}b_{13+\Delta}), \quad (3.2)
\]

where

\[
\Delta = \begin{cases} 
0 & : l = 2, \\
2^4 + 2^5 + \ldots + 2^{l+1} & : l \geq 3.
\end{cases}
\]

Lemma 2 in Appendix A says that for \( n \) qubits, if \(|\psi\rangle\) is equivalent to \(|GHZ\rangle \) under SLOCC then \( D^{(l)}(\psi) = 0 \), where \( 2 \leq l \leq (n-2) \). However, Lemma 4 in Appendix A says that for states \(|l,n\rangle \) \( D^{(l)}(|l,n\rangle) \neq 0 \), where \( 2 \leq l \leq (n-2) \). Therefore, the states \(|l,n\rangle \) \((2 \leq l \leq (n-2)) \) are different from the \(|GHZ\rangle \) state under SLOCC. By means of Corollary 1, we can also verify that the states \(|l,n\rangle \) \((2 \leq l \leq (n-1) \) but \( l \neq n/2) \) are different from the \(|GHZ\rangle \) state under SLOCC. This is because that \( \tau(|l,n\rangle) = 0 \) while \( \tau(GHZ) = 1 \) for even \( n \) qubits. See Sec. 3.1.

Remark 1. \( \tau(\psi) \) in Eq. (1.5) is considered as the residual entanglement for even \( n \) qubits in [23] and [26]. From the above discussion, we can say that for even \( n \) qubits, the states \(|n/2,n\rangle \) and \(|GHZ\rangle \) possess the
maximal residual entanglement $\tau = 1$ while the residual entanglement $\tau$ for the Dicke states $|l, n\rangle$ ($l \neq n/2$) vanishes.

### 3.3 All the $n$-qubit symmetric Dicke states with $2 \leq l \leq (n - 2)$ excitations are different from the $n$-qubit $|W\rangle$ state under SLOCC.

Lemma 3 in Appendix A says that for $n$ qubits, if $|\psi\rangle$ is equivalent to $|W\rangle$ under SLOCC then $D^{(l)}(\psi) = 0$, where $2 \leq l \leq (n - 2)$. However, Lemma 4 in Appendix A says for states $|l, n\rangle$, $D^{(l)}(|l, n\rangle) \neq 0$, where $2 \leq l \leq (n - 2)$. Therefore, the states $|l, n\rangle$ ($2 \leq l \leq (n - 2)$) are different from the $|W\rangle$ state under SLOCC. By means of Corollary 1, we can also verify that the state $|n/2, n\rangle$ is different from the $|W\rangle$ state under SLOCC because $\tau(|n/2, n\rangle) = 1$ and $\tau(W) = 0$.

Conjecture. For even $n$ qubits, perhaps $|2, n\rangle$, $|3, n\rangle$, ... , and $|(n/2 - 1), n\rangle$ are different from each other under SLOCC because $D^{(k)}(|l, n\rangle) = 0$ whenever $k \neq l$ while $D^{(l)}(|l, n\rangle) \neq 0$.

### 4 Odd $n$-qubit symmetric Dicke states

In this section, we demonstrate that the odd $n$-qubit Dicke states with $l$ excitations, where $2 \leq l \leq (n - 2)$, are different from the odd $n$-qubit $|GHZ\rangle$ and $|W\rangle$ states under SLOCC, respectively.

Note that the states $|l, n\rangle$ and $|(n - l), n\rangle$ are equivalent under SLOCC by Lemma 1 in Appendix A. Hence, it is enough to consider that $2 \leq l \leq (n - 1)/2$ in this section.

By means of Corollary 1, we demonstrate that for odd $n$ qubits, the states $|l, n\rangle$ ($2 \leq l < (n - 1)/2$) are inequivalent to the $|GHZ\rangle$ state under SLOCC below.

#### 4.1 All the $n$-qubit symmetric Dicke states are inequivalent to the $n$-qubit $|GHZ\rangle$ state under SLOCC.

First let us prove that $\tau(|l, n\rangle) = 0$ when $1 \leq l \leq (n - 1)/2$ as follows.

Note that each term in Eq. (1.9) is of the form $a_{2n-1+k}a_{(2n-1)-k}$. For the state $|l, n\rangle$, we want to show
\( a_{2^n-1+k}a_{(2^n-1)-k} = 0 \). Note that \( 2^{n-1} + k + (2^n - 1) - k = 2^n - 1 + 2^{n-1} \), whose binary number is \( 1011...1_{n-1} \).

That is, \( N(2^n - 1 + 2^{n-1}) = n \). Since \( 2l \leq (n - 1) \), it is impossible that \( N(2^n - 1 + k) = N((2^n - 1) - k) = l \). It says that \( a_{2^n-1+k}a_{(2^n-1)-k} = 0 \). Thus, \( I^*_{2^n-1}(a, n - 1) = 0 \).

Note that each term of \( I(a, n) \) in Eq. (1.8) is of the form \( a_k a_{(2^n-1)-k} \). It is trivial that \( k + (2^n - 1) - k = (2^n - 1) \) and \( N(2^n - 1) = n \). As discussed above, it is impossible that \( N(k) = N((2^n - 1) - k) = l \) because \( 2l \leq (n - 1) \). Hence, \( a_k a_{(2^n-1)-k} = 0 \). Also, \( I(a, n) = 0 \).

From the discussion above, \( \tau(l, n) = 0 \) when \( 1 \leq l \leq (n - 1)/2 \). For example, by calculating it is easy to see that \( \tau(2, 5) = 0 \).

Since \( \tau(GHZ) = 1 \) while \( \tau(l, n) = 0 \) \( (1 \leq l \leq (n - 1)/2) \), by Corollary 1 the states \( |l, nrangle \) \( (1 \leq l \leq (n - 1)/2) \) are different from the \( |GHZ \rangle \) state under SLOCC. Lemmas 2 and 4 in Appendix A also verify this fact. Lemma 2 says that \( D^{(l)}(|\psi \rangle) = 0 \) for any state \( |\psi \rangle \) in the \( |GHZ \rangle \) class while Lemma 4 says that \( D^{(l)}(|l, n\rangle) \neq 0 \) for the state \( |l, n\rangle \), where \( l \geq 2 \). It says that the states \( |l, n\rangle \) \( (l \geq 2) \) are not in the \( |GHZ \rangle \) class.

Remark 2. \( \tau(\psi) \) in Eq. (1.7) is considered as the residual entanglement for odd \( n \) qubits in [23] and [26]. From the above discussion, we can say that for odd \( n \) qubits, \( |GHZ \rangle \) possess the maximal residual entanglement \( \tau = 1 \) while the residual entanglement \( \tau \) for the Dicke states \( |l, n\rangle \) vanishes.

### 4.2 All the \( n \)-qubit symmetric Dicke states with \( 2 \leq l \leq (n - 2) \) excitations are different from the \( n \)-qubit \( |W\rangle \) state under SLOCC.

Lemma 3 in Appendix A says that \( D^{(l)}(|\psi \rangle) = 0 \) for any state \( |\psi \rangle \) in the \( |W\rangle \) class. By Lemma 4 in Appendix A, \( D^{(l)}(|l, n\rangle) \neq 0 \) for states \( |l, n\rangle \), where \( 2 \leq l \leq (n - 1)/2 \). Hence, states \( |l, n\rangle \) \( (2 \leq l \leq (n - 1)/2) \) are not in the \( |W\rangle \) class.

Conjecture. For odd \( n \) qubits, \( |2, n\rangle, |3, n\rangle, \ldots, \) and \(|(n - 1)/2, n\rangle \) are different from each other under SLOCC because \( D^{(k)}(|l, n\rangle) = 0 \) whenever \( k \neq l \) while \( D^{(l)}(|l, n\rangle) \neq 0 \).
5 Monogamy inequality for the $n$-qubit symmetric Dicke states

Osborne and Verstraete in [27] obtained the general Coffman, Kundu and Wootters monogamy inequality [29]. The general inequality is $C_2^1 + \ldots + C_2^n \leq C_2^{1(2\ldots n)}$. Let $\chi_{12\ldots n} = C_2^{1(2\ldots n)} - (C_2^1 + \ldots + C_2^n)$. For the $n$-qubit $|GHZ\rangle$ state, $\chi_{12\ldots n} = 1$ [28]. For the $n$-qubit $|W\rangle$ state, $\chi_{12\ldots n} = 0$ [29]. We show that all the $n$-qubit symmetric Dicke states but the $n$-qubit $|W\rangle$ state satisfy $0 < \chi_{12\ldots n} < 1$ below. For this purpose, first we need to compute the concurrence between any two qubits and the one between qubit 1 and other qubits as follows.

5.1 Concurrence between any two qubits

Let $\rho_{12\ldots n} = |l, n\rangle \langle l, n|$. When any $(n-2)$ qubits are traced out, let $\rho_{ij}$ be the density matrices of the remaining two qubits. For example,

$$
\rho_{12} = tr_{3\ldots n} \rho_{12\ldots n} = \frac{1}{n(n-1)} \left( |00\rangle \langle 00| + \frac{l(l-1)}{n(n-1)} |11\rangle \langle 11| + \frac{2(l-1)}{n(n-1)} |\psi^+\rangle \langle \psi^+| \right),
$$

where $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Note that $|\psi^+\rangle$ is a maximally entangled state of two qubits. By the symmetry of the $n$-qubit symmetric Dicke states, all the reduced density operators $\rho_{ij} = \rho_{12}$. By Coffman, Kundu and Wootters’ definition [29], $\rho_{12} = (\sigma_y \otimes \sigma_y) \rho_{12} (\sigma_y \otimes \sigma_y)$. Then $\rho_{12} \rho_{12}^\dagger$ has the following eigenvalues: $4t^2 + s^2 + 4t^2 n^2, 0, 1, \frac{1}{n^2 + 2n^2 + n^2} (t^4 - 2t^3 n + t^2 n^2 + l^2 n - l^2 - ln^2 + ln)$ (double roots). Thus, the concurrence of the density matrix $\rho_{12}$ is

$$
C_{12} = 2\frac{\sqrt{l(l-1)}}{n(n-1)} \left( \sqrt{\sqrt{|n-1|} - \sqrt{(l-1)(n-l-1)}} \right).
$$

For the definition of the two qubit concurrence, see [30]. By symmetry of the Dicke states, when any $(n-2)$ qubits are traced out the concurrence between the remaining two qubits is $C_{12}$.

The concurrence shows that the $n$-qubit symmetric Dicke states retain bipartite entanglement, even distillable. It is not hard to derive that the concurrence decreases as the number of excitations increases. Therefore, among the $n$-qubit symmetric Dicke states, the $n$-qubit $|W\rangle$ state has the maximal concurrence: $2/n$. It has also been proven in [31] that the $|W\rangle$ state optimizes the concurrence if all but two qubits are traced out. In [31], the concurrence of the $n$-qubit $|W\rangle$ state was also determined to be $2/n$. Among
the even \( n \)-qubit symmetric Dicke states, the state \(|n/2, n\rangle\) possesses minimal concurrence: \(1/(n - 1)\). In [1], Stockton et al. also derived the result. Among the odd \( n \)-qubit symmetric Dicke states, the state with \((n - 1)/2\) excitations possesses the minimal concurrence: \(\frac{(n+1) - \sqrt{(n+1)(n-3)}}{2n}\). While for the state \(|GHZ\rangle\) of any \( n \) qubits, when any \((n - 2)\) qubits are traced out the concurrence between the remaining two qubits vanishes, i.e., the remaining state is not entangled.

5.2 Concurrence between one qubit and other \((n - 1)\) qubits

By calculating,

\[
\rho_1 = tr_{2...n} \rho_{12...n} = \left[ \binom{n-1}{l} |0\rangle\langle 0| + \binom{n-1}{l-1} |1\rangle\langle 1| \right].
\]

By the definition in [29], the concurrence between qubit 1 and other qubits is

\[
C^2_{1(2...n)} = 4 \det(\rho_1) = 4 \frac{l(l - 1)}{n^2}.
\]

This concurrence demonstrates that the \( n \)-qubit symmetric Dicke states retain the entanglement between one qubit and other \((n - 1)\) qubits. It is easy to see that the concurrence \(C^2_{1(2...n)}\) increases as the number of excitations does. Hence, the \( n \)-qubit \(|W\rangle\) state possesses the minimal concurrence between one qubit and other \( n - 1 \) qubits, i.e., \(C^2_{1(2...n)} = 4 \frac{n - 1}{n^2}\). The even \( n \)-qubit symmetric Dicke state \(|n/2, n\rangle\) possesses the maximal concurrence between one qubit and other \((n - 1)\) qubits, i.e., \(C^2_{1(2...n)} = 1\). Among the odd \( n \)-qubit symmetric Dicke states, the state with \((n - 1)/2\) excitations possesses the maximal concurrence between one qubit and other \((n - 1)\) qubits, i.e., \(C^2_{1(2...n)} = 1\).

5.3 Monogamy inequality for Dicke states

By the symmetry of the \( n \)-qubit symmetric Dicke states and from Eq. \([5.1]\), \(C^2_{12} + \ldots + C^2_{1n} = 4\frac{l(l - 1)(\sqrt{l(l - 1)} - \sqrt{(l-1)(n-l-1)})^2}{n^2(n-1)}\).

Then from Eq. \([5.2]\), \(\chi_{12...n} = 8l(l - 1)\frac{\sqrt{n + l(n - l - 1)} + \sqrt{n + l + (n - l - 1)}}{n(n - 1)}\). Clearly, \(0 \leq \chi_{12...n} < 1\). Especially, for the \(|W\rangle\) state \(\chi_{12...n} = 0\). Thus, it verifies Eq. (27) in [29]. When \(l < n/2\), \(\chi_{12...n}\) increases as \(l\) does. For even \( n \) qubits, when \(l = n/2\), \(\chi_{12...n}\) gets the maximum \(\frac{n-2}{n-1}\). For odd \( n \) qubits, when \(l = (n-1)/2\),
\(\chi_{12...n}\) gets the maximum \(\frac{(n+1)(n-1)\sqrt{n-3}(\sqrt{n+1}\sqrt{n-3})}{2n^2}\) < 1. When \(l > n/2\), \(\chi_{12...n}\) decreases as \(l\) increases. Therefore, when \(2 \leq l \leq (n-2)\), \(0 < \chi_{12...n} < 1\). Clearly, \(\chi_{12...n}\) is almost 1 for large \(n\) when \(l\) is about \(n/2\).

For example, when \(n = 5\) and \(l = 2\), \(\chi_{12345} = 0.70277\). When \(n = 6\) and \(l = 2\), \(\chi_{123456} = 0.67519\). When \(n = 4\) and \(l = 2\), \(\chi_{1234} = 2/3\). When \(n = 6\) and \(l = 3\), \(\chi_{123456} = 4/5\).

6 Summary

In this paper, we show that the \(n\)-qubit symmetric Dicke states with \(l\) (\(2 \leq l \leq (n-2)\)) excitations are different from the states \(|GHZ\rangle\) and \(|W\rangle\) of \(n\) qubits under SLOCC, respectively. We also argue that the even \(n\)-qubit symmetric Dicke state with \(n/2\) excitations is different from the even \(n\)-qubit symmetric Dicke states with \(l \neq n/2\) excitations under SLOCC. And we demonstrate that all the \(n\)-qubit symmetric Dicke states with \(l\) (\(2 \leq l \leq (n-2)\)) excitations satisfy Coffman, Kundu and Wootters’ generalized monogamy inequality \(C_{l2}^2 + ... + C_{1n}^2 < C_{l(2...n)}^2 < 1\). We indicate that among the \(n\)-qubit symmetric Dicke states, the \(n\)-qubit \(|W\rangle\) state maximally retains the concurrence between the remaining two qubits when \((n-2)\) qubits are traced out but possesses the minimal concurrence between one qubit and other \(n-1\) qubits, while among the even \(n\)-qubit symmetric Dicke states, the state \(|n/2,n\rangle\) minimally retains the concurrence between the remaining two qubits when \((n-2)\) qubits are traced out but possesses the maximal concurrence between one qubit and other \((n-1)\) qubits, i.e., \(C_{l(2...n)}^2 = 1\).

Appendix A: The properties of \(D^{(l)}(\psi)\)

Lemma 1. The complementary states are SLOCC equivalent

Let the set of the basis states of \(n\) qubits be \(B = \{|0\rangle, |1\rangle, ..., |2^n-1\rangle\}\). Let \(\bar{1}\) (\(\bar{0}\)) be the complement of a bit 1 (0). Then \(\bar{0} = 1\) and \(\bar{1} = 0\). Let \(\bar{z} = \bar{z}_1 \bar{z}_2 ... \bar{z}_n\) denote the complement of a binary string \(z = z_1 z_2 ... z_n\). Also, the set of the basis states \(\bar{B} = \{|\bar{0}\rangle, |\bar{1}\rangle, ..., |2^n-1\rangle\}\). Let \(|\varphi\rangle\) be any state of \(n\) qubits. Then we can write \(|\varphi\rangle = c_0|0\rangle + c_1|1\rangle + ... + c_{2^n-1}|(2^n-1)\rangle\). Let \(|\overline{\varphi}\rangle = c_0|\bar{0}\rangle + c_1|\bar{1}\rangle + ... + c_{2^n-1}|(2^n-1)\rangle\). We call \(|\overline{\varphi}\rangle\)
the complement of $|\phi\rangle$.

Let $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\sigma_x \otimes \ldots \otimes \sigma_x |\phi\rangle = \sum_{i=0}^{2^n-1} c_i (\sigma_x \otimes \ldots \otimes \sigma_x |i\rangle) = \sum_{i=0}^{2^n-1} c_i |i\rangle = |\overline{\phi}\rangle$.

Consequently, if two states of $n$ qubits are complementary then they are SLOCC equivalent.

Lemma 2.

Let $|\psi\rangle$ be any pure state of $n$ qubits. Then, we can write

$$|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle.$$  \hspace{1cm} (A1)

Then, if $|\psi\rangle$ is equivalent to $|GHZ\rangle$ under SLOCC then $D^{(l)}(\psi) = 0$. Especially, $D^{(l)}(GHZ) = 0$.

Proof. It is known that $|\psi\rangle$ is equivalent to $|GHZ\rangle$ under SLOCC if and only if there exist invertible local operators $F^{(1)}, F^{(2)}, \ldots, F^{(n)}$, where

$$F^{(i)} = \begin{pmatrix} f^{(i)}_1 & f^{(i)}_2 \\ f^{(i)}_3 & f^{(i)}_4 \end{pmatrix},$$

such that

$$|\psi\rangle = \bigotimes_{i=0}^{n} F^{(i)} |GHZ\rangle.$$ \hspace{1cm} (A3)

Let $i_n i_{n-1} \ldots i_2 i_1$ be the $n$-bit binary number of $i$, where $i_j \in \{0, 1\}$. By solving Eq. (A1),

$$a_i = \left( f^{(n)}_{2(i_n+1)} \ldots f^{(m)}_{2(m+1)} \ldots f^{(1)}_{2(i_1+1)} \right) + \left( f^{(n)}_{2(i_n+2)} \ldots f^{(m)}_{2(m+2)} \ldots f^{(1)}_{2(i_1+2)} \right) / \sqrt{2},$$ \hspace{1cm} (A2)

$i = 0, 1, \ldots, 2^n - 1$. By substituting $a_i$ in Eq. (A2) into $D^{(l)}(\psi)$ in Eq. (3.2), one can verify that $D^{(l)}(\psi) = 0$, where $l \geq 2$. It implies that $D^{(l)}(\psi) = 0$ for any state $|\psi\rangle$ in $|GHZ\rangle$ class.

Lemma 3. Let $|\psi\rangle$ be any pure state of $n$ qubits. Then, if $|\psi\rangle$ is equivalent to $|W\rangle$ under SLOCC then $D^{(l)}(\psi) = 0$. Especially, $D^{(l)}(W) = 0$.

Proof. It is known that $|\psi\rangle$ is equivalent to $|W\rangle$ under SLOCC if and only if there exist invertible local operators $F^{(1)}, F^{(2)}, \ldots, F^{(n)}$, such that

$$|\psi\rangle = \bigotimes_{i=0}^{n} F^{(i)} |W\rangle.$$ \hspace{1cm} (A3)

Let

$$\delta(i,j) = \begin{cases} 1 & \text{if } i = j, \\
0 & \text{otherwise.} \end{cases}$$
And let $i_n i_{n-1} ... i_2 i_1$ be the $n$-bit binary number of $i$, where $i_j \in \{0, 1\}$. By solving Eq. (A3),

$$a_i = (\sum_{j=0}^{n} \prod_{k=1}^{j} f_{2i_k+2i_{j+1}}^{(k)})/\sqrt{n},$$

(A4)

$i = 0, 1, ..., 2^n - 1$. By substituting $a_i$ in Eq. (A4) into $D^{(l)}(\psi)$ in Eq. (3.2), one can verify that $D^{(l)}(\psi) = 0$, where $l \geq 2$. It implies that $D^{(l)}(\psi) = 0$ for any state $|\psi\rangle$ in the $|W\rangle$ class. Especially, $D^{(l)}(W) = 0$.

Lemma 4. $D^{(l)}(|l, n\rangle) \neq 0$ for the state $|l, n\rangle$, where $2 \leq l \leq (n-2)$.

Proof. For the state $|l, n\rangle$, $a_{3+\Delta}a_{6+\Delta} = a_{9+\Delta}a_{12+\Delta} = 1/\binom{n}{l}$ because $N(3+\Delta) = N(6+\Delta) = N(9+\Delta) = N(12+\Delta) = l$, and $a_{1+\Delta}a_{4+\Delta} = a_{0+\Delta}a_{5+\Delta} = a_{11+\Delta}a_{14+\Delta} = a_{10+\Delta}a_{15+\Delta} = a_{2+\Delta}a_{7+\Delta} = a_{8+\Delta}a_{13+\Delta} = 0$ by the amplitudes of the state $|l, n\rangle$. A simple calculation shows $D^{(l)}(|l, n\rangle) \neq 0$. Especially when $n$ is even and $l = n/2$, $D^{(n/2)}(|n/2, n\rangle) \neq 0$.

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