STABILITY DATA, IRREGULAR CONNECTIONS
AND TROPICAL CURVES

SARA A. FILIPPINI, MARIO GARCIA-FERNANDEZ, AND JACOPO STOPPA

Abstract. We construct isomonodromic families of irregular meromorphic connections $\nabla(Z)$ on $\mathbb{P}^1$, with values in the derivations of a class of infinite dimensional Poisson algebras. Our main results concern the limits of the families $\nabla(Z)$ as we vary a scaling parameter $R$. In the $R \to 0$ “conformal limit” we recover a semi-classical version of the connections introduced by Bridgeland and Toledano Laredo (and so the Joyce holomorphic generating functions). In a different $R \to \infty$ “large complex structure” limit the $\nabla(Z)$ approach a very simple family of connections, while their flat sections display tropical behaviour, and also encode certain tropical/relative Gromov-Witten invariants. The connections $\nabla(Z)$ are a rough but rigorous approximation to the (mostly conjectural) four-dimensional $tt^*$-connections introduced by Gaiotto-Moore-Neitzke. A precise comparison with these is established in a basic example.

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1. Introduction

1.1. Over the last few years a number of different results have appeared that link stability data on a class of graded Lie algebras [KS], Stokes factors for irregular meromorphic connections [JS, BTT, GMN] and tropical/Gromov-Witten invariants [GPS]. The purpose of this paper is to make some progress towards unifying some of these results. Starting from a continuous family of stability data...
on a certain infinite-dimensional Poisson Lie algebra, we will construct a family of irregular meromorphic connections on $\mathbb{P}^1$

$$\nabla(Z) = d - \left( \frac{1}{z^2} A^{(-1)} + \frac{1}{z} A^{(0)} + A^{(1)} \right) dz$$ (1.1)

(parametrised by some linear maps $Z$) which gives a rough mathematical counterpart to the physical construction in [GMNI]. Here, the $A^{(i)}$ are complicated functions of $Z$, but are constant in $z$, and take values in a space of derivations of the algebra. The $\nabla(Z)$ are formally very similar to the irregular connections introduced by Dubrovin [Du1, eq. 2.18] in his study of the $tt^*$-equations [CV1], defining the Zamolodchikov Hermitian metric on a Frobenius manifold.

Our main results concern the limits of the families $\nabla(Z)$ as we vary a scaling parameter $R$. We will show that the limit $\nabla(RZ)$ as $R \to 0$, taken up to scaling $z = Rt$ and gauge transformations (a “conformal limit”), coincides with the semi-classical limit of a motivic version of the family introduced by Bridgeland and Toledano Laredo [BT1]. As we approach an opposite $R \to \infty$ limit (“large complex structure limit”), flat sections of $\nabla(RZ)$ display tropical behaviour and encode a class of tropical/relative Gromov-Witten invariants described in [GPS]. The latter statement uses the Gross-Pandharipande-Siebert wall-crossing theory and is in fact equivalent to part of it.

The graded components of the residue of the Bridgeland-Toledano Laredo connections at 0 are Joyce's holomorphic generating functions for invariants counting stable objects in certain Abelian categories [J5]. Our results give a precise meaning to the intuition that Joyce’s generating functions and the Gross-Pandharipande-Siebert wall-crossing theory, involving tropical counts and Gromov-Witten theory, appear when one expands a single geometric object (in our case, the connections $\nabla$) at different points in parameter space. The limits $R \to 0$, $R \to \infty$ are somewhat reminiscent of the limits $iu \to 0$, $iu \to -\infty$ in the Gromov-Witten/Donaldson-Thomas partition function $Z_{DT}(q,v) = Z_{GW}(u,v)$, $q = -e^{iu}$, and to follow the analogy one would set $R = i/u$.

Very recently, as we had just completed the present work, an interesting physics paper by D. Gaiotto appeared [Ga], proposing a new perspective on the $R \to 0$ scaling limit (called there the “conformal limit”) in the case of differential-geometric $tt^*$-type connections. While the main focus and methods are very different, some of our results in section 6 seem to be related to those in [Ga] section 2. It would be interesting to investigate these connections further.

1.2. The paper is planned as follows. In Section 2 we give a brief introduction to the circle of ideas on which this paper is based, and present our main results in an informal way. In Section 3 we review some examples of continuous stability data following [KS]. The construction of our family $\nabla(Z)$ is carried out in Section 4. It is a very rough approximation to the (mostly conjectural) four-dimensional $tt^*$-connections $\nabla^{GMN}(Z)$ of Gaiotto, Moore and Neitzke [CMN]. The construction of Bridgeland-Toledano Laredo connections $\nabla^{BTL}(Z)$ for our special
choice of stability data, as well as a precise relation with [BT1], are discussed in Section 5. Following [GMN1], a particular scaling limit of $\nabla(RZ)$ as $R \to 0$ should be related with $\nabla^{BTL}(Z)$, and in Section 6 we confirm this expectation, proving that they coincide up to a complex gauge transformation. In Section 7 we will show in a precise sense that flat sections of $\nabla(RZ)$ display tropical behaviour in the limit $R \to \infty$, and relate this behaviour to the tropical invariants which play an important role in [GPS]. Part of the material in Sections 4 and 7 originally appeared (in a different form) in the preprint [FS] (the present paper supersedes a large part of that work). Sections 8 and 9 have a more differential-geometric flavour, and are devoted to a comparison with $tt^*$-type connections. In Section 8 we discuss the main example where the $tt^*$-type connections have been constructed rigorously, relating (1.1) to $\nabla^{GMN}(Z)$ in that case. The subtleties in taking the $R \to 0$ scaling limit in the differential-geometric situation are explained in Section 9.

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2. Background and main results

In this section we give a brief outline of the ideas on which this paper builds, starting with the notion of stability data on a graded Lie algebra, and sketching how it is related to meromorphic connections and tropical invariants. At the same time we will also present our main results in an informal way.

2.1. Fix a rank $n$ lattice $\Gamma$, and let $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ denote a $\Gamma$-graded Lie algebra over $\mathbb{Q}$. The space of stability data $\text{Stab}(\mathfrak{g})$ on $\mathfrak{g}$ is a complex $n$-dimensional manifold introduced by M. Kontsevich and Y. Soibelman in [KS]. Its points are given by pairs $(Z, a)$ where $Z: \Gamma \to \mathbb{C}$ is a group homomorphism and $a: \Gamma \setminus \{0\} \to \mathfrak{g}$ is a map of sets which preserves the grading (that is, such that $a(\gamma) \in \mathfrak{g}_\gamma$). Additionally one requires that $(Z, a)$ satisfy the support property

$$||\gamma|| \leq C|Z(\gamma)|$$

(2.1)

whenever $\gamma \in \text{Supp} a$, that is, $a(\gamma) \neq 0$, for some arbitrary norm on $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ and some constant $C > 0$. In particular the set $\{Z(\gamma) : \gamma \in \text{Supp} a\} \subset \mathbb{C}$ is discrete. In algebro-geometric applications, $Z$ is the central charge for a stability condition on a category $\mathcal{C}$ with Grothendieck group $\Gamma$, and the element $a(\gamma) \in \mathfrak{g}_\gamma$ corresponds to a count of $Z$-semistable objects of class $\gamma$. The Lie algebra $\mathfrak{g}$ is typically infinite-dimensional and the support property is a quantitative analogue of the fact that the central charge of a semistable object should not vanish. In general $\text{Stab}(\mathfrak{g})$ has an uncountable number of paracompact connected components.
2.2. While the definition of \( \text{Stab}(g) \) as a set might look a bit arbitrary at first sight, what really matters is the topology with which it was endowed in [KS, Section 2.3], that is, what it means to have a \textit{continuous} family of stability data on \( g \). This topology is essentially characterised by the following two properties. The first is simply that the projection \( (Z,a) \mapsto Z \) on \( \text{Hom}(\Gamma, \mathbb{C}) \) is a local homeomorphism. The second property is formulated in terms of a (pro-nilpotent) Lie group \( G \) with Lie algebra \( \hat{g} \); the completion of \( g \) with respect to the grading (see Section 3). Given stability data \( (Z,a) \) and a ray \( \ell \subset \mathbb{C}^* \) we define a group element

\[
S_\ell(Z) = \exp \left( \sum_{\gamma \in \Gamma, Z(\gamma) \in -\ell} a(\gamma) \right).
\]  

(2.2)

If a family \( (Z_t, a_t) \) of stability data on \( g \) parametrised by \([0,1]\) is continuous at \( t_0 \), then the inequality (2.1) holds uniformly in a neighborhood of \( t_0 \) and for any strictly convex sector \( V \subset \mathbb{C}^* \) such that \( Z(\text{Supp} a_{t_0}) \cap \partial V = \emptyset \) the following group element is constant in this neighborhood

\[
S_V = \prod_{\ell \subset V} S_\ell(Z_t).
\]

(2.3)

This last property is equivalent to Kontsevich-Soibelman’s \textit{wall-crossing formula} and plays a central role in the theory of Donaldson-Thomas invariants.

2.3. The graded Lie algebra which is most relevant for the present paper is an infinite-dimensional Poisson Lie algebra \( g \) introduced in [KS, Section 2.5], which may be thought of as the algebra of functions on an algebraic Poisson torus. It is canonically attached to the lattice \( \Gamma \), upon the choice of an integral skew pairing. Details will be given in Section 3.

Continuous families of stability data on \( g \) arise naturally in algebraic geometry and mathematical physics. For example, let \( \mathcal{C} \) denote the category of finite-dimensional modules over a finite-dimensional \( \mathbb{C} \)-algebra \( A \) (e.g. the path algebra of a quiver without loops). The Grothendieck group \( K(\mathcal{C}) \) is lattice of finite rank \( n \), and we can choose \( \Gamma = K(\mathcal{C}) \), endowed with the skew-symmetrised Euler form. According to [BT1, J5], this setup leads to a continuous family of stability data on the Lie algebra of derivations of \( g \) parametrised by the set of stability conditions on the Abelian category \( \mathcal{C} \) (which can be identified with \( \mathbb{H}^n, \mathbb{H} \subset \mathbb{C} \) denoting the upper half-plane). In many examples (e.g. for Dynkin quivers, see section 5) this family is in fact induced by a continuous family of stability data on \( g \) via the adjoint representation. In this case we have the following \textit{positivity property}: there is a positive cone \( K_{>0}(\mathcal{C}) \subset \Gamma \) (given by the effective classes in \( K(\mathcal{C}) \)), such that \( a(\gamma) \) vanishes in the complement of \( K_{>0}(\mathcal{C}) \). In other words \( \Gamma \) admits a natural positive basis, such that if \( a(\gamma) \neq 0 \) then \( \gamma \) is a non-negative linear combination of the basis vectors. Furthermore, the image \( Z(\gamma) \) for all \( \gamma \in \Gamma \) with \( a(\gamma) \neq 0 \) is contained in \( \mathbb{H} \); for a stability condition on \( \mathcal{C} \) we have \( Z(K_{>0}(\mathcal{C})) \subset \mathbb{H} \).
Similar constructions appear in the physics literature, e.g. in the context of compactifications of certain $\mathcal{N} = 2$ supersymmetric quantum field theories in four dimensions [GMN1, GMN2], as well as in the theory of complex integrable systems [KS, KS2]. An important feature of the stability data which appear in physics is that the images $Z(\gamma)$ for $a(\gamma) \neq 0$ are never contained in a single half-space, since one requires $a(\gamma) = a(-\gamma)$ (“every BPS particle has a CPT conjugate antiparticle”). In the sense of [KS, Definition 2], this corresponds to symmetric stability data. As we will see, we find here one of the crucial differences between the algebro-geometric setup of [BT1] and the one which is more relevant for applications to differential geometry.

2.4. A key idea in the subject (with many closely related variants) is that continuous families of stability data are related with systems of differential equations. In the case of interest for the present work, stability data parameterise meromorphic connections on a homomorphically trivial principal bundle on the unit disc $\Delta$, with a single order 2 pole at the origin. The collection of group elements $S_\ell$, indexed by rays $\ell \subset \mathbb{C}^*$, define the generalized monodromy (i.e. Stokes factors [B2]) for these connections. The relevant structure group is $\text{Aut}^*(\hat{g})$, the group of automorphisms of $\hat{g}$ as a commutative associative algebra, and each $S_\ell \in G$ is regarded as an element in this group using the adjoint representation. In this setting, the topology of $\text{Stab}(g)$ has a very natural interpretation. Let $(Z_t, a_t)$ be a continuous family of stability data on $g$, and suppose that we can indeed find a suitable family of irregular meromorphic connections $\nabla(Z_t, a_t)$ as above, whose generalized monodromy is given by the collection $S_\ell(Z_t)$, with Stokes rays $\ell_\gamma(Z_t) = -\Re_{>0} Z(\gamma)$ (for $\gamma \in \Gamma$). Then the continuity condition (2.3) precisely says that the family $\nabla(Z_t, a_t)$ is isomonodromic: i.e. their generalized monodromy at 0 is constant.

A solution to the problem of constructing isomonodromic families of connections parameterised by continuous stability data was found by Bridgeland and Toledano Laredo in [BT1], working in the setup of Abelian categories of modules described above. The main difficulty in solving this problem is that it involves calculating and inverting the monodromy map, which is a highly transcendental object. To explain the result in our setup, we assume that we have a continuous family of stability data $(Z', a)$ parametrised by some open set $U \subset \text{Hom}(\Gamma, \mathbb{C})$, which satisfy the positivity property: $\Gamma$ admits a basis such that $a(\gamma) \neq 0$ implies that the coefficients of $\gamma$ are non-negative. Without loss of generality, we assume that image of the elements of the basis by any $Z' \in U$ lies in the lower-half plane $-\mathbb{H}$. Setting $Z = -Z'$, we will denote by $\nabla^{\text{BTL}}(Z)$ the associated Bridgeland-Toledano Laredo isomonodromic family. This is in fact a family of meromorphic connections on $\mathbb{P}^1$, with a pole of order two at 0 and a simple pole at $\infty$, of the form

$$\nabla^{\text{BTL}}(Z) = d - \left( \frac{Z}{t^2} + \frac{f(Z)}{t} \right) dt. \quad (2.4)$$

Here $Z$ and $f \in g$ are regarded as derivations of $g$ (as a commutative associative algebra), acting respectively by $Z(e_\alpha) = Z(\alpha)e_\alpha$ and $f(e_\alpha) = [f, e_\alpha]$, for $e_\alpha \in g_\alpha$. 
According to the main result of [BT1, p. 6], the residue $f$ only has positive graded components, given explicitly by

$$f_{\alpha}(Z) = \sum_{n \geq 1} \prod_{\alpha_1 + \cdots + \alpha_n = \alpha} J_n(Z(\alpha_1), \cdots, Z(\alpha_n))a(\alpha_1) \otimes \cdots \otimes a(\alpha_n).$$

Here $\otimes$ denotes the product in the universal enveloping algebra $U\mathfrak{g}$, and the $J_n : (\mathbb{C}^*)^n \to \mathbb{C}$ are certain special holomorphic functions with branchcuts. It turns out that the graded component $f_{\alpha}(Z)$ is a holomorphic function of $Z$, and in fact it can be identified with the holomorphic generating function for $Z$-semistables in class $\alpha$ introduced by Joyce in [J5]. By the positivity property, the above sum is finite. The main tools in constructing $\nabla^\text{BTL}(Z)$ are the Fourier-Laplace transform, which takes (2.4) to a Fuchsian connection and provides a formula for the monodromy map, and a complicated inversion formula that leads to (2.5). The function $J_n$ is expressed as a sum of multilogarithms indexed by finite trees.

In the original approach [BT1], the connection $\nabla^\text{BTL}(Z)$ takes values in the Hall algebra $\text{CF}(\mathcal{C})$ of constructible functions of an abelian category $\mathcal{C}$ [J2]. Ideally, in algebro-geometric examples (e.g. for Dynkin quivers) one would like to induce (2.4) from the $\text{CF}(\mathcal{C})$-valued connection via a Lie algebra morphism. However this is not so straightforward: the desired morphism $\Phi$ is defined on $\text{SF}(\mathcal{C})$, a Hall algebra of stacky functions, rather than $\text{CF}(\mathcal{C})$.

In section 5, after briefly showing how to carry out the Bridgeland-Toledano Laredo construction starting from stability data on $\mathfrak{g}$, we address the question above for the category $\mathcal{C}$ of representations of a quiver without oriented cycles. We describe a connection $\nabla^\mathcal{C}$ which specialises to the $\text{CF}(\mathcal{C})$-valued Bridgeland-Toledano Laredo connection and also admits a “semi-classical limit” with values in the derivations of $\hat{\mathfrak{g}}$. For Dynkin quivers, this is induced from $\hat{\mathfrak{g}}$ via the adjoint.

2.5. The result of Bridgeland and Toledano Laredo is remarkable, especially since it still holds in the much more interesting Ringel-Hall algebra $\text{CF}(\mathcal{C})$ (rather than just an algebra $\mathfrak{g}$ modelled on $K(\mathcal{C})$). There is a sense, however, in which the methods of [BT1] are not completely satisfactory. The Fourier-Laplace transform relates (2.4) to Fuchsian systems because of the very particular type of this connection, with a second order pole at zero and a logarithmic pole at infinity. In addition, the inversion formula for the monodromy map in [BT1] does not arise from first principles and, to the knowledge of the authors, has no conceptual explanation.

An interesting set of connections for which these methods do not apply goes back to the fundamental works of Cecotti-Vafa [CV1] and Dubrovin [Du1]. This is an important ingredient in the study of mathematical aspects of $\mathcal{N} = 2$ supersymmetric quantum field theories in two dimensions. To regard the $tt^*$-equations of Cecotti-Vafa as equations for isomonodromic deformations, Dubrovin defines in [Du1] eq. 2.18 (see also [CV2, eq. 4.11]) a family of irregular connections,
known as $tt^*$-connections, which take the schematic form

$$\nabla^{tt^*} = d - \left( \frac{1}{z^2} W + \frac{1}{z} A + W^* \right) dz,$$

(2.6)

for certain matrices $A, W$. The point is that $\nabla^{tt^*}$ has a singularity of order two at both 0 and $\infty$, and this is not just a formality but an important feature of the theory. Interestingly, the $tt^*$-equations are a key ingredient to define special hermitian metrics on Frobenius manifolds, known as Zamolodchikov metrics [Du2, p. 90].

According to the important physics paper [GMN1], there should exist close analogues of Dubrovin’s $tt^*$-connections also in the infinite-dimensional case. These connections should be related to $\mathcal{N} = 2$ supersymmetric quantum field theories in four dimensions, with interesting mathematical applications to the construction of hyperKähler metrics on the total space of certain complex integrable systems. The relevant connections in this case are supposed to take the form

$$\nabla^{GMN}(Z) = d - \left( \frac{1}{z^2} A^{(-1)}(Z) + \frac{1}{z} A^{(0)}(Z) + A^{(1)}(Z) \right) dz,$$

(2.7)

where the $A^{(i)}$ take values in the Lie algebra $\mathfrak{X}$ of complex-valued smooth vector fields on the torus $\Gamma^\vee \otimes U(1)$ (thought of as the generic fibre of the integrable system), and $A^{(-1)} = A^{(1)}$. Like in the finite-dimensional case, the symmetry between the singularity order at 0 and $\infty$ plays a crucial role (now reflecting Hitchin’s theorem on twistor families [H2]). There are by now a number of mathematical papers investigating various aspects of these proposals (for example [BS], [C], [E], [KNPS], [Su]) so it seems useful to find at least a rough mathematical incarnation of $\nabla^{GMN}(Z)$. Because of their specific form, we will often refer to $\nabla^{GMN}(Z)$ as $tt^*$-type connections.

2.6. In the rest of the paper, we will assume that we have a continuous family of stability data $(Z, a)$ on $\mathfrak{g}$, parameterised by an open subset $U \subset \text{Hom}(\Gamma, \mathbb{C})$, which satisfies the positivity property described above. From this, we construct an isomonodromic family of irregular meromorphic connections $\nabla(Z)$ on $\mathbb{P}^1$, with $\nabla(RZ)$ degenerating to $\nabla^{BTL}(Z)$ in a suitable $R \to 0$ conformal limit. Our connections take values in the Lie algebra $D^*(\hat{\mathfrak{g}})$ of derivations of $\hat{\mathfrak{g}}$ as a commutative associative algebra. This construction is a very close analogue of the physical $\nabla^{GMN}(Z)$ mentioned above.

The connections $\nabla(Z)$ are defined by the following properties. They are of $tt^*$-type, that is, they have the form (1.1) where the $A^{(i)}$ are (complicated) functions of $Z$, but are constant in $z$. Similarly to $W$ and $W^*$ in (1.1), the functions $A^{(-1)}$ and $A^{(1)}$ are also related by a suitable symmetry. The formal type of $\nabla(Z)$ is

$$d + \frac{Z}{z^2} dz,$$

with Stokes rays $\ell_\gamma(Z) = -\mathbb{R}_{>0} Z(\gamma)$, for $\gamma \in \Gamma$. Finally, the Stokes data for $\nabla(Z)$ are given by the collection $\text{Ad} S_\ell$ defined by (2.2) (hence matching those of
∇^{BTL}). We should notice that the connections \( \nabla(Z) \) are naturally framed, but we ignore this technicality at this point.

Picking a positive basis \( \gamma_i \) for \( \Gamma \), the \( \text{Aut}^*(\hat{g}) \)-connection \( \nabla(Z) \) is uniquely determined by a system of local flat sections \( X_{\gamma_i}(z) = X(z, Z)(\gamma_i) \) given by

\[
X(z; Z)e_\alpha = e_\alpha \exp_s \left( z^{-1}Z(\alpha) + z\tilde{Z}(\alpha) - \langle \alpha, \sum_T W_T(Z)G_T(z; Z) \rangle \right), \tag{2.8}
\]

where \( \exp_s \) is defined via the standard series for the exponential map and the commutative product on \( g \). The \( G_T(z; Z) \) are a collection of special \( z \)-holomorphic functions with branchcuts, with values in \( \hat{g} \), indexed by a set of finite trees \( T \) (the analogues of the multilogarithms appearing in [BT1]), and the \( W_T(Z) \in \Gamma \otimes \mathbb{C} \) are combinatorial weights. We will give explicit formulae for \( W_T \) and \( G_T \) is Section 4.

Following [Du1, CV2, GMN1], the main tool in constructing the \( X_{\gamma_i}(z) \) (and so \( \nabla(Z, R) \)) is a suitable \( \text{Aut}^*(\hat{g}) \)-valued singular integral equation; this replaces the Fourier-Laplace transform and the inversion formula of [BT1], all at once.

In the sequel, we will always write \( \nabla(Z, R) \) for the rescaling \( \nabla(RZ) \), the extra parameter being a real \( R > 0 \); similarly, we will write \( \nabla^{GMN}(Z, R) \) for \( \nabla^{GMN}(RZ) \).

2.7. As pointed out in [GMN1, p. 6], there is a candidate scaling limit that seems to take \( \nabla(Z, R) \) to a connection of the type constructed by Bridgeland and Toledano-Laredo. Namely, one lets \( z = Rt \), getting

\[
\nabla_t(Z, R) = d - \left( \frac{1}{t^2} R^{-1}A^{(-1)}(RZ) + \frac{1}{t} A^{(0)}(RZ) + RA^{(1)}(RZ) \right) dt,
\]

and then takes the limit \( R \to 0 \). As we will explain, however, things are not as simple as that, and one needs to be a little careful when taking the limit. It turns out that if we want to stay in the fixed gauge chosen above, then as \( R \to 0 \) there are divergencies in the coefficients of \( \nabla_t \). To get rid of these divergencies we gauge them away with a sequence of gauge transformations \( g(R) \), i.e. holomorphic maps \( \mathbb{C}^* \to \text{Aut}^*(\hat{g}) \). Remarkably, it is possible to choose the \( g(R) \) to be \( z \)-constant, so their action on the \( A^{(i)} \) is simply

\[
g(R) \cdot A^{(i)} = g^{-1}(R)A^{(i)}g(R)
\]

(the usual composition of endomorphisms). We will give explicit formulae for the transformations \( g(R) \), and prove the following result.

**Theorem 2.1.** The limit \( g(R) \cdot \nabla_t(Z, R) \) as \( R \to 0 \) exists and has the form

\[
\hat{\nabla}_t = d - \left( -\frac{Z}{t^2} + \frac{f}{t} \right) dt,
\]

with the same Stokes data as \( \nabla^{BTL}(Z) \). It follows that \( \hat{\nabla}_t \) actually coincides \( \nabla^{BTL}(-Z) \).
As we will explain, the last implication follows easily from the uniqueness result proved in [BT2]. The opposite sign is due to different conventions; if $\nabla^{BTL}(Z)$ is defined for $Z \in \mathbb{H}^n$ then $\nabla(Z, R)$ is defined on $(-\mathbb{H})^n$.

2.8. Our construction confirms an expected relation between the irregular connections of [BT1] and [GMN1], at least when the latter have been rigorously constructed. The common ground for this relation is a diagram of partially defined morphisms

\[
\begin{array}{ccc}
\text{SF}(\mathcal{C}) & \xrightarrow{I} & \mathfrak{X} \\
\pi & \downarrow & \downarrow \Psi \\
\text{CF}(\mathcal{C}) & \xrightarrow{} & D^*(\mathfrak{g})
\end{array}
\]

where $\text{CF}(\mathcal{C})$ is the Hall algebra of constructible functions of the category $\mathcal{C}$ and $\text{SF}(\mathcal{C})$ is its stacky version [J2], and $\mathfrak{X}$ is the Lie algebra of complex-valued smooth vector fields on the torus $\Gamma^\vee \otimes U(1)$. The map $\pi$ is a surjective morphism of associative algebras and $I$ denotes the semi-classical limit of an integration morphism to a non-commutative $q$-deformation of $\mathfrak{g}$ (composed with the adjoint representation) (see Section 5). The map $\Psi$ corresponds essentially to a Fourier expansion, and will be explained in the main known example when $\nabla^{GMN}(Z, R)$ is well defined, the so called Ooguri-Vafa case, in Section 8. Notice that (2.7) will not quite map to (1.1), because of the issue with Stokes factors for “negative rays” mentioned above: the latter includes only a half of the nontrivial Stokes factors for the former. We will be able to give a version of (1.1) including the single negative Stokes factor in the special Ooguri-Vafa example. In general it is possible to give a version of (1.1) which also includes negative Stokes rays with a nontrivial Stokes factor by working over a different Poisson algebra, such as $\mathfrak{g}[[t]]$.

The $R \to 0$ scaling limit may also be analysed directly for $\nabla^{GMN}(Z, R)$, and we will do this in Section 9 for the Ooguri-Vafa case. This differential-geometric example displays some interesting features which are hidden for $D^*(\mathfrak{g})$-connections. In particular, one can get rid of divergencies in the $R \to 0$ limit in a different way, by a redefinition of the “energy scale”. This limit is different from the one obtained by gauge transformations. In both cases, the limiting connections are not smooth.

2.9. Theorem 2.1 implies that we can recover $\nabla^{BTL}$ from $\nabla$. This alternative construction has some advantages, and carries some more information.

Firstly, although the construction of $\nabla(Z, R)$ a priori only works for $\hat{\mathfrak{g}}$ (rather than some more interesting Ringel-Hall algebra mapping to $\hat{\mathfrak{g}}$), it is arguably more elementary, relying only the basic technique of singular integral equations. On the other hand, as we will see, the construction is entirely based on flat sections $X(z)$ for $\nabla(Z, R)$. Our method of constructing $\hat{\nabla}(Z)$ is to show that the limits

$$\hat{X}(t) = \lim_{R \to 0} q^{-1}(R)X(Rt)$$

exist and solve a suitable Riemann-Hilbert factorization problem. Thus we produce a natural system of fundamental solutions $\hat{X}(t)$ for $\hat{\nabla}(Z)$, for which we give
explicit formulae. Moreover the equality between $\hat{\nabla}(Z)$ and $\nabla^{BTL}(-Z)$ can be seen as a weak version of Conjecture 1 in [St].

2.10. More importantly, the $tt^*$-type connections $\nabla(Z,R)$ seem to provide a “geometric enhancement” of $\nabla^{BTL}(-Z)$. To see this, we draw a parallel with the finite-dimensional case [Du1, CV2, Du2]. In that picture, the physics provides a canonical moduli space $M$ of 2-dimensional topological field theories, endowed with the structure of a Frobenius manifold. The main ingredient in this structure is a solution of the WDVV equations, which (according to Dubrovin [Du2]) can be recast as the isomonodromic deformation equations of a connection of the form (cf. (2.4))

$$\nabla = d - \left( \frac{U}{t^2} + \frac{V}{t} \right) dt,$$

for $U, V$ suitable matrices. The Frobenius manifold $\mathcal{M}$ corresponds to a particular choice of Stokes data for $\nabla$. The role played by the $tt^*$-connections (2.6) is now to fix a special metric on $\mathcal{M}$: as mentioned earlier, Dubrovin characterizes in [Du1] the $tt^*$-equations of Cecotti-Vafa as isomonodromic deformation equations for (2.6), and the former determine the Zamolodchikov hermitian metric. Informally, the agreement of the Stokes data tells the $tt^*$-connection on which manifold it has to construct the metric.

The infinite-dimensional case is, of course, more difficult. Physical considerations in four-dimensional gauge theory led in [GMN1] to a moduli space $\mathcal{M}$, endowed with a structure of complex integrable system over a base $B$. In [KS] it is conjectured that, for suitable $\mathcal{M}$, a generic choice of base point $a \in B$ determines a continuous family of stability data on the poisson Lie algebra $\mathfrak{g}$ attached to the lattice $\Gamma = H_1(\mathcal{M}_a, \mathbb{Z})$. This family is actually parameterised by an open complex submanifold $B^* \subset B$, which according to [BT1] determines an isomonodromic family of connections $\nabla^{BTL}(b)$, for $b \in B^*$. In known examples [KS, BS, Su], the continuous family $B^* \subset \text{Stab}(\mathfrak{g})$ is a half-dimensional complex (Lagrangian) submanifold of the space of stability data. The connection $\nabla^{BTL}$ is, therefore, very much related to the structure of complex integrable system on $\mathcal{M}$. On the other hand, isomonodromic deformations of the $tt^*$-type connection $\nabla^{GMN}$ are conjecturally linked with a complete hyperKähler metric on this space [GMN], arising from physical considerations. The idea is that flat sections for $\nabla^{GMN}$ should give holomorphic Darboux coordinates for the twistor family of holomorphic symplectic forms on $\mathcal{M}$ (with twistor parameter $z$). Again, the agreement of the Stokes data seems to tell the $tt^*$-connection on which integrable system it has to construct the metric.

2.11. The geometric origin of $\nabla(Z,R)$ appears in the opposite, $R \to \infty$ limit, and turns out to be related to rational tropical curves immersed in $\mathbb{R}^2$. This is not unexpected, by the following very informal argument. Suppose, for simplicity, that $\Gamma$ has rank 2 and the skew pairing $\langle -, - \rangle$ is nondegenerate. According to [GMN], $R^{-1}$ should be thought of as proportional to the volume of the fibres of $\mathcal{M} \to B$, with respect to the hyperKähler metric $g$ on the complex surface $\mathcal{M}$. 
The $R \to \infty$ limit should correspond to the limit of very small fibres, which is mirror to a large complex structure limit \((GW)\). Near the large complex structure limit, the complex structure is obtained from a degenerate limit via “instanton corrections”, which encode interesting tropical invariants (see e.g. \(Gr\)).

We will show that a toy model of this behaviour holds for \(\nabla(Z, R)\). This applies to the special local flat sections \(X(z; Z, R)\) for \(\nabla(Z, R)\). For simplicity, we will only examine the model case when \(\Gamma \cong \mathbb{Z}^2\), generated by \(\gamma, \eta\) with \(\langle \gamma, \eta \rangle = \kappa > 0\), with the simplest nontrivial stability data in a chamber \(U^+\) in the parameter space \(U\). For example when \(\kappa = 1\) \((2.3)\) reduces to a simple “pentagon identity”, but nevertheless the connections \(\nabla(Z, R)\) are already far from trivial. Recall that the \(X(z; Z, R)\) are constructed using the special functions \(G_T(z; Z, R)\) (the analogues of multilogarithms in the work of Bridgeland-Toledano Laredo). At a generic point \(z^* \in \mathbb{C}^*\) the \(G_T(z^*; Z, R)\) have discontinuous jumping behaviour when \(Z\) crosses a certain locus in \(U\), while \(X(z; Z; R)\) is continuous. This will enable us to compute how the expansion \((2.8)\) changes across the critical locus, and to related this behaviour to tropical curves and invariants. Fully precise statements of the following results, together with the few basic notions from tropical geometry we need, will be given in Section 7.

**Theorem 2.2.** As \(Z\) crosses the boundary of \(U^+ \subset U\) from the interior, a special function \(G_T(z^*; Z, R)\) appearing in the expansion \((2.8)\) for the flat section \(X(z^*; Z, R)\) is replaced by a linear combination of the form

\[
\sum_{T'} \pm G_{T'}(z^*; Z, R),
\]

where we sum over a finite set of trees (not necessarily distinct). The terms corresponding to a single-vertex tree in the sum above are uniquely characterised by their asymptotic behaviour as \(R \to \infty\). These leading order terms are in bijection with a finite set of weighted graphs \(C_i\), which have a natural interpretation as combinatorial types of rational tropical curves immersed in \(\mathbb{R}^2\).

Note that each (type of a) curve \(C_i\) appears with a sign \(\varepsilon(C_i)\); it turns out that this sign is determined by the residue theorem. Each tree \(T\) appearing in the expansion for \(X(z^*; Z, R)\) in the chamber \(U^+\) defines a pair of unordered partitions \(\text{deg}(T)\), whose parts are positive integral multiples of the generators \(\gamma, \eta\). In the light of Theorem 2.2 it is natural to identify \(\text{deg}(T)\) with a tropical degree \(w\).

**Theorem 2.3.** The sum of contributions \(\varepsilon(C_i(T)) = \pm 1\) over tropical types \(C_i\), weighted by the coefficients \(W_T\) in the expansion \((2.8)\) for flat sections in \(U^+\),

\[
\sum_{\text{deg}(T) = w} W_T \sum_i \varepsilon(C_i(T))
\]

equals a tropical invariant \(N_{\text{trop}}(w)\) enumerating plane rational tropical curves, times a simple combinatorial factor in \(\Gamma \otimes \mathbb{Q}\).

The tropical invariants \(N_{\text{trop}}(w)\) equal in fact certain relative Gromov-Witten invariants of weighted projective planes, and play a crucial role in \([GPS]\).
When taking the large $R$ limit we use the specific form of $\nabla(Z, R)$ (with double poles at 0 and $\infty$), and we have not been able to find a similar tropical structure underlying the special functions used in [BT1].

3. Continuous families of stability data

We review in this section some examples of continuous families of stability data in a graded Lie algebra $\mathfrak{g}$, and recall the definition of Kontsevich-Soibelman’s Poisson Lie algebra [KS] relevant for this work.

3.1. Algebraic groups. The basic finite-dimensional example is given by the Lie algebra $\mathfrak{g}$ of a complex algebraic group $G$ endowed with the choice of an algebraic torus $H \subset G$. The torus $H$ acts on $\mathfrak{g}$ by the adjoint action, and $\mathfrak{g}$ splits into weight spaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

for some finite set $\Phi$ of nonzero elements of the dual of the Lie algebra $\mathfrak{h}$ of $H$ (the roots of $G$ relative to $H$), such that $H$ acts on $\mathfrak{g}_\alpha$ via the torus character $e^n \alpha$. We assume that $G$ is actually defined over $\mathbb{Q}$, so that it makes sense to talk of rational points in $\mathfrak{g}$, and that the above splitting is also defined over $\mathbb{Q}$. Then one can choose $\Gamma$ to be the lattice spanned by $\Phi$ in $\mathfrak{h}^*$. To construct stability data, we consider $a$ such that the elements $a(\gamma), \gamma \in \Gamma$ are zero except possibly when $\gamma \in \Phi$ is one of the roots $\alpha$, in which case $a(\alpha)$ is a rational point in $\mathfrak{g}_\alpha$.

Let $\text{Hom}^o(\Gamma, \mathbb{C})$ denote the locus of $Z \in \text{Hom}(\Gamma, \mathbb{C})$ such that $Z(a(\alpha)) \neq 0$ for all roots $\alpha$. Since there are only finitely many roots, if $Z \in \text{Hom}^o(\Gamma, \mathbb{C})$ then $(Z, a)$ satisfies the support property and gives a point in $\text{Stab}(\mathfrak{g})$.

The special case when $G = GL(n, \mathbb{C})$ with its standard maximal torus $H$ of diagonal matrices is discussed in [KS Section 2.9]. Identifying $\mathfrak{h}^* \cong \mathbb{C}^n$ with canonical basis $e_i$, the lattice $\Gamma$ is the sublattice of $\mathbb{Z}^n$ spanned by the roots $\gamma_{ij} = e_i - e_j$ for $i \neq j$,

$$\Gamma = \{(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n : \sum_{i=1}^n k_i = 0\}.$$ 

Letting $E_{ij}$ denote the usual basis of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{Q})$ given by elementary matrices, we have $\mathfrak{g}_{\gamma_{ij}} = \langle E_{ij} \rangle$. Since $\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{i,j} \mathfrak{g}_{\gamma_{ij}}$, a rational degree-preserving map $a : \Gamma \setminus \{0\} \to \mathfrak{g}$ is the same as a matrix $a_{ij}$ with rational entries such that $a(\gamma_{ij}) = a_{ij}E_{ij}$. We can also identify $\text{Hom}(\Gamma, \mathbb{C})$ with $\mathbb{C}^n/(\langle 1, \ldots, 1 \rangle)$, and so identify $\text{Hom}^o(\Gamma, \mathbb{C})$ with the set of elements represented by

$$[(z_1, \ldots, z_n)] \in \mathbb{C}^n/(\langle 1, \ldots, 1 \rangle)$$

such that $z_i \neq z_j$ for $i \neq j$.

Then a point of $\text{Stab}(\mathfrak{gl}(n, \mathbb{Q}))$ is given by a pair $(Z, a)$ with $Z \in \text{Hom}^o(\Gamma, \mathbb{C})$ and a matrix $a_{ij} \in \mathfrak{gl}(n, \mathbb{Q})$. We have $Z(\gamma_{ij}) = z_i - z_j$. Symmetric stability data in the sense of [KS Definition 2] are such that $a$ is preserved by the Cartan involution acting on $\Gamma$. In the present case this happens precisely when the matrix $a_{ij}$ is skew-symmetric.
3.2. Wall-crossing for algebraic groups. We examine now formula (2.3) and the topology on \(\text{Stab}(\mathfrak{gl}(n, \mathbb{Q}))\). For fixed stability data \((Z, a) = ([z_k], a_{ij})\) we have

\[
\ell_{\gamma_{ij}, Z} = -\mathbb{R}_{>0}(z_i - z_j) \subset \mathbb{C}^*.
\]

The set of \(Z\) for which all rays are distinct is given by

\[
\text{Hom}^{oo}(\Gamma, \mathbb{C}) \subset \text{Hom}(\Gamma, \mathbb{C}) \cong \mathbb{C}^n/\mathbb{C}((1, \cdots, 1))
\]
such that for all distinct triples \((i, j, k)\) the complex numbers \(z_i, z_j, z_k\) do not lie on the same real line. Formula (2.3) forces the matrix \(a_{ij}\) to jump along the locus (the walls)

\[
\text{Hom}^{o}(\Gamma, \mathbb{C}) \setminus \text{Hom}^{oo}(\Gamma, \mathbb{C}).
\]

To illustrate this consider a continuous family of stability data \((Z_t, a_t)\) on \(\mathfrak{gl}(n, \mathbb{Q})\) parameterised by \([0, 1]\) such that only the component \(z_j(t)\) of \(Z_t\) is nonconstant. We assume that \(z_j(t_0)\) belongs to the line through \(z_i\) and \(z_k\) for some distinct triple \((i, j, k)\). Then the rays \(\pm \ell_{\gamma_{ij}}(Z_{t_0}), \pm \ell_{\gamma_{ik}}(Z_{t_0})\) and \(\pm \ell_{\gamma_{jk}}(Z_{t_0})\) coincide, while we assume that all other rays \(\ell_{\gamma_{pq}}(Z_t)\) are distinct for all times. We can pick indices so that \(\ell_{\gamma_{ij}}(Z_t), \ell_{\gamma_{ik}}(Z_t),\) and \(\ell_{\gamma_{jk}}(Z_t)\) all lie in a strictly convex sector; in fact for \(t\) sufficiently close to \(t_0\) we can pick a very narrow sector which contains no other rays. Then the matrices \(a_{pq}\) and \(a'_{pq}\) on the two sides of the wall are related by

\[
\exp(a_{ij}E_{ij}) \exp(a_{ik}E_{ik}) \exp(a_{jk}E_{jk}) = \exp(a'_{ij}E_{ij}) \exp(a'_{ik}E_{ik}) \exp(a'_{jk}E_{jk}).
\]

Since \([E_{ij}, E_{jk}] = E_{ik}\), while all other commutators vanish, by the Baker-Campbell-Hausdorff formula for \(\mathfrak{gl}(n, \mathbb{C})\) we have

\[
\exp(a_{ij}E_{ij}) \exp(a_{ik}E_{ik}) \exp(a_{jk}E_{jk}) = \exp(a_{ij}E_{ij} + a_{ik}E_{ik} + a_{jk}E_{jk} + \frac{1}{2}a_{ij}a_{jk}E_{ik}),
\]

\[
\exp(a'_{ij}E_{ij}) \exp(a'_{ik}E_{ik}) \exp(a'_{jk}E_{jk}) = \exp(a'_{ij}E_{ij} + a'_{ik}E_{ik} + a'_{jk}E_{jk} - \frac{1}{2}a'_{ij}a'_{jk}E_{ik}).
\]

Since \(E_{pq}\) gives a basis for \(\mathfrak{gl}(n, \mathbb{C})\), we must have

\[
a'_{ik} = a_{ik} + a_{ij}a_{jk}. \tag{3.1}
\]

This is the only jump in the matrix \(a_{ij}\).

3.3. Kontsevich-Soibelman’s Poisson Lie algebra. The graded Lie algebra which is most relevant for the present paper is not \(\mathfrak{gl}(n, \mathbb{Q})\), but an infinite-dimensional one introduced by Kontsevich and Soibelman in [KS, Section 2.5]. It may be thought of as the Poisson algebra of functions on an complex affine algebraic torus.

Let \(\Gamma\) denote a lattice of finite rank \(n\) endowed with an integral, skew-symmetric bilinear form \(\langle - , - \rangle\). In the rest of the paper we write \(\mathfrak{g}\) for the infinite-dimensional complex Lie algebra generated by symbols \(e_\gamma\) for \(\gamma \in \Gamma\), with bracket

\[
[e_\gamma, e_\eta] = \langle \gamma, \eta \rangle e_{\gamma + \eta}.
\]

We can also define a commutative product \(\ast\) on \(\mathfrak{g}\) simply by

\[
e_\gamma \ast e_\eta = e_{\gamma + \eta}. \tag{3.2}
\]
The product $\ast$ turns $\mathfrak{g}$ into a Poisson algebra, i.e. the Lie bracket acts as a derivation.

**Remark 3.1.** A different version of $\mathfrak{g}$ which is often used has bracket

$$[e_\gamma, e_\eta] = (-1)^{\langle \gamma, \eta \rangle} \langle \gamma, \eta \rangle e_{\gamma + \eta}.$$  

These two versions are isomorphic (non-canonically). Also notice that $\mathfrak{g}$ is really defined over $\mathbb{Z}$, so in particular it makes sense to speak of integral and rational elements of $\mathfrak{g}$.

To make sense of various objects (such as the group elements $S_\ell$ in (2.2)), we need to work with an amenable subalgebra $\mathfrak{g}_{\geq 0}$ of $\mathfrak{g}$. The algebra $\mathfrak{g}_{\geq 0}$ is the analogue in our setup of the bialgebra in [BT1, 4.3]. We briefly recall now its construction following [BT1, 4.3]. Once and for all, we fix a strict convex cone with vertex at the origin

$$\Gamma_{\geq 0} \subset \Gamma,$$

that is, a non-empty subset closed under addition and multiplication by positive integer numbers which does not contain a straight line. Let $\mathfrak{g}_{\geq 0} \subset \mathfrak{g}$ be the Poisson Lie subalgebra generated by the elements:

$$\mathfrak{g}_{\geq 0} = \langle e_\gamma : \gamma \in \Gamma_{\geq 0} \rangle \subset \mathfrak{g}$$

and note that $\mathfrak{g}_{\geq 0}$ is graded by the semi-group $\Gamma_{\geq 0}$ and has finite-dimensional homogeneous components $\mathfrak{g}_k$.

### 3.4. The completed algebra $\widehat{\mathfrak{g}}$.

We now construct the completion of $\mathfrak{g}_{\geq 0}$. For each $k \geq 1$, we denote by $\Gamma_{>k} \subset \Gamma_{\geq 0}$ the cone generated by elements $\gamma_1 + \ldots + \gamma_m$ for $m > k$ and $\gamma_j \neq 0$ for all $j = 1, \ldots, m$. The subspace $\mathfrak{g}_{>k} \subset \mathfrak{g}_{\geq 0}$ induced by $\Gamma_{>k}$ is an ideal. Consider the finite-dimensional nilpotent Lie algebra

$$\mathfrak{g}_{\leq k} = \mathfrak{g}_{\geq 0} / \mathfrak{g}_{>k}.$$  

and the corresponding inverse system $\ldots \to \mathfrak{g}_{\leq k} \to \mathfrak{g}_{\leq k-1} \to \mathfrak{g}_{\leq 0} = \mathbb{C}$. By definition, the pro-nilpotent graded Lie algebra given by the completion of $\mathfrak{g}_{\geq 0}$ is the limit

$$\widehat{\mathfrak{g}}_{\geq 0} = \lim_{\leftarrow} \mathfrak{g}_{\leq k} = \prod_{k \geq 0} \mathfrak{g}_k.$$  

To simplify the notation we will often write $\widehat{\mathfrak{g}}$ in place of $\widehat{\mathfrak{g}}_{\geq 0}$. Similarly, we define a pro-nilpotent Lie group $\widehat{G}$ with Lie algebra $\widehat{\mathfrak{g}}$ by

$$\widehat{G} = \lim_{\leftarrow} G_{\leq k}$$

where $G_{\leq k} = \exp(\mathfrak{g}_{\leq k})$ is the nilpotent Lie group with Lie algebra $\mathfrak{g}_{\leq k}$.

Note that $\widehat{\mathfrak{g}}$ inherits a natural structure of Poisson Lie algebra. Setting

$$\mathfrak{g}_+ = \{ f = \sum_{\gamma \in \Gamma_{>0}} f_\gamma \in \widehat{\mathfrak{g}} : f_0 = 0 \}$$

$$\mathfrak{g}_\times = \{ f = \sum_{\gamma \in \Gamma_{>0}} f_\gamma \in \widehat{\mathfrak{g}} : f_0 = 1 \}$$
the standard power series for the logarithm and the exponential functions combined with the commutative product * yield well-defined maps \( \exp_*: \mathfrak{g}^+ \to \mathfrak{g}^\times \), \( \log_*: \mathfrak{g}^\times \to \mathfrak{g}^+ \) given by

\[
\exp_*(x) = \sum_{k \geq 0} \frac{1}{n!} x \ast \ldots \ast k \ast x \quad \text{and} \quad \log_*(x) = \sum_{k \geq 1} \frac{(-1)^{n-1}}{n} x \ast \ldots \ast k \ast x
\]

which are each other’s inverse.

Throughout the paper we will write \( \text{Aut}^*(\widehat{\mathfrak{g}}) \) for the group of automorphisms of \( \widehat{\mathfrak{g}} \) as a commutative algebra (we do not require that these preserve the Lie bracket). We will often forget the notation * in (3.2) and simply write \( e_\gamma e_\eta \) for the product, but at some points it will be important to have a special symbol for it to avoid confusion (especially to distinguish between the commutative exponential \( \exp_* \) and the Lie algebra exponential \( \exp \)).

When considering \( \widehat{\mathfrak{g}} \)-valued holomorphic functions, meromorphic connections, Stokes data, etc. we need to be careful, since \( \widehat{\mathfrak{g}} \) is infinite-dimensional. For example, by a \( \widehat{\mathfrak{g}} \)-valued connection \( \nabla \) on \( \mathbb{P}^1 \) we mean an inverse system of connections \( \nabla_k \) on \( \mathfrak{g} \leq k \), and a flat section of \( \nabla \) is an inverse system of \( \nabla_k \)-flat sections. This is the notion used to define the Stokes data (2.2). See [BT2] for a general treatment.

3.5. **Positive stability data and Stokes factors.** For most of the time, we will be interested in stability data \((Z,a)\) on \( \mathfrak{g} \) (in particular, the \( a(\gamma) \) are rational points). A natural compatibility condition of the stability data with the cone \( \Gamma \geq 0 \subset \Gamma \) is given by the following definition, that plays a central role in this paper.

**Definition 3.2.** We say that \((Z,a) \in \text{Stab}(\mathfrak{g})\) is positive if \( a(\gamma) \neq 0 \) implies \( \gamma \in \Gamma \geq 0 \).

Note that for positive stability data all the rays \( \ell_{\gamma,Z} = -\mathbb{R}_{>0}Z(\gamma) \) for \( a(\gamma) \neq 0 \) are contained in a half-space \( \mathbb{H}' \subset \mathbb{C} \).

In the Lie algebra \( \mathfrak{g} \) it is standard to rewrite (2.3) in a different way using the Poisson structure. Firstly notice that for any \( \gamma \in \Gamma \) we may rewrite by Möbius inversion

\[
a(\gamma) = - \sum_{n \geq 1, n|\gamma} \frac{\Omega(\gamma/n)}{n^2} e_\gamma.
\]

So for \( \gamma \in \Gamma^{\text{prim}} \), that is, for primitive \( \gamma \)

\[
a(k\gamma) = - \sum_{p,n \geq 1, pn = k} \frac{\Omega(p\gamma)}{n^2} e_{pn\gamma}.
\]

Summing over all \( k \geq 1 \) and using standard dilogarithm notation we find

\[
\sum_{k \geq 1} a(k\gamma) = - \sum_{p \geq 1} \Omega(p\gamma) \sum_{n \geq 1} \frac{e_{pn\gamma}}{n^2} = - \sum_{p \geq 1} \Omega(p\gamma) \sum_{n \geq 1} \text{Li}_2(e_{p\gamma}).
\]

Since \( \mathfrak{g} \) is Poisson, for all \( \gamma \in \Gamma \geq 0 \) (not necessarily primitive) \([\text{Li}_2(e_\alpha), -]\) acts as a commutative algebra derivation on the completion \( \widehat{\mathfrak{g}} \). Therefore \( \exp \left(-[\text{Li}_2(e_\gamma), -] \right) \)
(the exponential of a derivation) acts as an algebra automorphism \( T_\gamma \) of \( \hat{\mathfrak{g}} \), preserving the Lie bracket (a Poisson automorphism). It turns out that this action is especially nice:

\[
T_\gamma(e_\eta) = e_\eta(1 - e_\gamma)^{\langle \gamma, \eta \rangle}
\]

(see [FS] for an explicit computation).

Recall that we denote by \( \ell_\gamma(Z) \) the ray \(-\mathbb{R}_{>0}Z(\gamma) \subset \mathbb{C}^*\), for any given \( \gamma \in \Gamma \) and \( Z \in \text{Hom}(\Gamma, \mathbb{C})\).

**Definition 3.3.** We say that \((Z, a) \in \text{Stab}(\mathfrak{g})\) is *generic* if when \( a(\gamma), a(\eta) \neq 0 \) and the rays \( \ell_\gamma(Z), \ell_\eta(Z) \) coincide, then \( \langle \gamma, \eta \rangle = 0 \). We say that \((Z, a)\) is *strongly generic* if \( \ell_\gamma(Z) = \ell_\eta(Z) \) with \( a(\gamma) \neq 0 \neq a(\eta) \) imply that \( \gamma \) and \( \eta \) are linearly dependent.

Note that these conditions define open dense subsets of \( \text{Stab}(\mathfrak{g}) \). The locus where the strongly generic condition does not hold corresponds to the so called *walls of marginal stability.*

We show now how to represent the Stokes factors using the first condition. Given generic stability data \((Z, a)\) and a ray \( \ell \subset \mathbb{C}^* \), the image of the group element \( S_\ell \) in (2.2) via the adjoint representation admits the following expression

\[
\text{Ad} S_\ell = \prod_{\gamma \in \Gamma^{\text{prim}}, \ell_\gamma(Z) \in -\ell} \prod_{p \geq 1} T_{p\gamma}^{\Omega(p\gamma, Z)}.
\]  

(3.3)

Here of course we write \( T_{p\gamma}^{\Omega} \) for the automorphism \( \exp(-\Omega[\text{Li}_2(e_{\gamma}), -]) \). Note that we do not need to specify an order for the previous product as the genericity condition implies that all the \( T_{p\gamma} \) with \( Z(\gamma) \in -\ell \) commute. Furthermore, if in addition \((Z, a)\) is positive, then \( S_\ell = 1 \) unless \(-\ell \subset Z(\Gamma_{\geq 0})\).

Finally, we can write (2.3) as an equivalent identity of Poisson automorphisms of \( \hat{\mathfrak{g}}\),

\[
\prod_{\gamma \in \Gamma^{\text{prim}}, \ell_\gamma(Z_{t_0-\varepsilon}) \in V} \prod_{p \geq 1} T_{p\gamma}^{\Omega(p\gamma, Z_{t_0-\varepsilon})} = \prod_{\gamma \in \Gamma^{\text{prim}}, \ell_\gamma(Z_{t_0+\varepsilon}) \in V} \prod_{p \geq 1} T_{p\gamma}^{\Omega(p\gamma, Z_{t_0+\varepsilon})}.
\]  

(3.4)

For this formula we assume that there is a single \( t_0 \in [0, 1] \) for which the stability data is non-generic.

**Example.** A special case of (3.4) which is similar to (3.1) appears in the case of the sublattice \( \Gamma_0 \) generated by elements \( \gamma, \eta \) with \( \langle \gamma, \eta \rangle = 1 \). Then, (3.4) is equivalent to the “pentagon identity”

\[
T_\gamma T_\eta = T_{\eta + \gamma} T_\gamma.
\]

In the rest of the paper, we will assume that we have a continuous family of stability data \((Z, a(Z))\) on \( \mathfrak{g}\), parametrised by some open set \( U \subset \text{Hom}(\Gamma, \mathbb{C})\), which satisfy the positivity property. For a generic point \( Z \in U \), all the rays \( \ell_\gamma(Z) \) with \( a(\gamma) \neq 0 \) are distinct, so in particular \((Z, a(Z))\) is strongly generic in the sense of Definition 3.3.
Remark 3.4. In the mathematical physics literature, it is standard to require that stability data is symmetric in the sense of [KS, Definition 2]. For $\frak{g}$, this is given by the condition $a(\gamma) = a(-\gamma)$. We will need to come back to this at several points in our discussion.

4. The connections $\nabla(Z)$ from stability data

In this section we show that any positive stability data on $\frak{g}$ defines Stokes factors for an irregular (framed) connection on $\mathbb{P}^1$. Given a continuous family of stability data parameterised by an open set $U \subset \text{Hom}(\Gamma, \mathbb{C})$, the connection varies isomonodromically with $Z \in U$.

4.1. Irregular connections on $\mathbb{P}^1$. Let $\mathbb{P}$ be the holomorphically trivial, principal $\text{Aut}^*\widehat{\frak{g}}$-bundle on $\mathbb{P}^1$. By this we mean the inverse limit of the system of holomorphically trivial principal bundles corresponding to the groups $\text{Aut}^*(\frak{g}_{\leq k})$.

By a $D^*\widehat{\frak{g}}$-valued meromorphic function $A$ in $\mathbb{C}$, we mean an inverse system of meromorphic functions $A_{\leq k}: \mathbb{C} \to D^*\frak{g}_{\leq k}$.

For a choice of local coordinate $z$ in $\mathbb{P}^1$, we will consider meromorphic connections on $\mathbb{P}$ of the form

$$\nabla = d - A dz,$$

given by the inverse limit of a system of meromorphic connections

$$\nabla_{\leq k} = d - A_{\leq k} dz. \quad (4.1)$$

Given a local gauge transformation $Y: U \to \text{Aut}^*(\frak{g})$, that is, an inverse system of holomorphic maps $Y_{\leq k}: U \to \text{Aut}^*(\frak{g}_{\leq k})$ on an open $U \subset \mathbb{C}$, we use the standard notation

$$Y \cdot A = (\partial_z Y)Y^{-1} + Y A Y^{-1}.$$

In the rest of the paper, we focus on connections $\nabla$ with a second order pole at $z = 0$ and simple dependence on $z$, of the form

$$\nabla = d - \left( \frac{1}{z^2} A^{-1} + \frac{1}{z} A^0 + A^1 \right) dz, \quad (4.2)$$

where $A^j \in D^*\frak{g}$ are constant in $z$. We choose the formal type at the origin to be

$$d + \frac{Z}{z^2} dz, \quad (4.3)$$

with $Z \in \text{Hom}(\Gamma, \mathbb{C})$ (regarded as a derivation). It will be convenient to work with the following notion (see e.g. [B1]).

**Definition 4.1.** A (compatibly) framed connection is a pair $(\nabla, g)$ given by a connection $\nabla$ as above and an element $g \in \text{Aut}^*\widehat{\frak{g}}$ such that $g^{-1} \cdot A^{-1} = -Z$.

We introduce now Stokes data for a framed connection $(\nabla, g)$ following [B1]. With our choice of formal type (4.3), the Stokes rays for $\nabla_{\leq k}$ are of the form $-R > 0 Z(\alpha)$ for $\alpha$ a root of $g_{\leq k}$ relative to $Z$. We define a Stokes ray of $\nabla$ to be of the form $\ell_\gamma = -R > 0 Z(\gamma)$ for $\gamma \in \Gamma \setminus \{0\}$, and say that a ray is admissible.
if is not a Stokes ray. Note that the set of Stokes rays need not be finite. By definition, an admissible ray for $\nabla$ is admissible for each $\nabla_{\leq k}$.

By a fundamental solution $X: U \to \text{Aut}^* (\hat{g})$ for $\nabla$ on an open $U \subset \mathbb{C}$, we mean an inverse system of holomorphic flat sections $X_{\leq k}: U \to \text{Aut}^* (\hat{g}_{\leq k})$ for $\nabla_{\leq k}$, that is, satisfying

$$\partial_z X_{\leq k} = A_{\leq k} X_{\leq k}.$$  

The previous formula should be understood as acting on an arbitrary element of $\hat{g}_{\leq k}$, where we use standard notation for the composition of maps. Note that a fundamental solution provides a local description of the connection $\nabla = (\partial_z X)$. (4.4)

Given an admissible ray $\ell \subset \mathbb{C}^*$ for $\nabla$, define $H_\ell \subset \mathbb{C}$ to be the open half-plane containing $\ell$ and with boundary perpendicular to $\ell$. Then, there exists a unique fundamental solution $X_\ell: H_\ell \to \text{Aut}^* (\hat{g})$ with prescribed asymptotics $X_\ell e^{\frac{-Z}{z}} \to g$ as $z \to 0$ in $H_\ell$.[BT1, Th. 6.2].

This follows from an analogue result for the corresponding system of connections $\nabla_{\leq k}$ (see [B1, Th. 3.1 & Lem. 3.3]). The solution $X_\ell$ is called the canonical fundamental solution of $(\nabla, g)$ corresponding to the admissible ray $\ell$.

Definition 4.2. $(\nabla, g)$ admits a Stokes factor $S_\ell \in \text{Aut}^* (\hat{g})$ along the Stokes ray $\ell$ if the elements $S_{\ell_1, \ell_2}$ tend to $S_\ell$ as the admissible rays $\ell_1, \ell_2$ tend to $\ell$ in such a way that $\ell$ is always contained in the corresponding closed sector.

Remark 4.3. Similarly as in [BT1, Prop. 6.3], one can verify that $\nabla$ admits a Stokes factor along any Stokes ray.

Our goal in this section is to construct an isomonodromic family of connections as above with prescribed Stokes factors. We start with a continuous family of positive elements $(Z, a(Z))$ of $\text{Stab}(g)$ parametrised by an open set $U \subset \text{Hom}(\Gamma, \mathbb{C})$.

In particular all the rays $\ell_{\gamma, Z} = -R_{>0} Z(\gamma)$ for $a(\gamma) \neq 0$ are contained in a half-space $\mathbb{H}'$. We will prove the following result.

Proposition 4.4. Let $Z \in U$ correspond to generic stability data. Then, there exists a meromorphic framed connection $(\nabla(Z), g(Z))$ on $\mathbb{P}^1$, of the form

$$\nabla(Z) = d - \left( \frac{1}{z^2} A^{-1}(Z) + \frac{1}{z} A^{(0)}(Z) + A^{(1)}(Z) \right) dz,$$

such that $(\nabla(Z), g(Z))$ has Stokes data given by the rays $\ell_{\gamma, Z}$ and factors (3.3). It extends to an isomonodromic family of framed connections on all of $U$. 


The proof consists of several steps and will be given in sections 4.2 - 4.9 below. We will also show that \(A^{(-1)}\) and \(A^{(1)}\) are related by a suitable symmetry, so that \(\nabla(Z)\) only depends on a pair of \(D^*(\hat{g})\)-valued functions. The result can be extended to the case symmetric stability data working with the Lie algebra \(\mathfrak{g}[t]\). In this case, \(A^{(-1)}\) and \(A^{(1)}\) are related by an involution.

4.2. Riemann-Hilbert factorisation problem. Following ideas of [GMN1], the construction of \((\nabla(Z), g(Z))\) from generic \((Z, a(Z))\) can be turned into a Riemann-Hilbert factorisation problem (RH), that is, the construction of a sectionally holomorphic function \(X(Z) = X(z; Z): \mathbb{C}^\ast \to \text{Aut}^*(\hat{g})\), with jumps (3.3) across \(\ell_{\gamma, Z}\), \(\gamma \in \Gamma\). More precisely, we seek a family \(X(Z)\) with the following properties.

1. For all \(\gamma \in \Gamma\), \(X(Z)\) is a \(\text{Aut}^*(\hat{g})\)-valued holomorphic function in the complement of the rays \(\ell_{\alpha, Z}\) with \(a(\gamma) \neq 0\).
2. \(X(Z)(e_\alpha)\) extends to a holomorphic function in a neighborhood of \(\ell_{\alpha, Z}\).
3. Fix a ray \(\ell \subset \mathbb{H}'\). For every \(z_0 \in \ell\) and \(\alpha \in \Gamma\), denote by \(X(z_0^\alpha)(e_\alpha)\) the limit of \(X(z; Z)(e_\alpha)\) as \(z \to z_0\) in the counterclockwise direction. Similarly let \(X(z_0^-\alpha)(e_\alpha)\) denote the limit in the clockwise direction. Both limits exist, and they are related by \(X(z_0^+\alpha) = X(z_0^-\alpha) \text{Ad} S_\ell\).
4. There exists \(g(Z) \in \text{Aut}^*(\hat{g})\) such that \(\lim_{z \to 0} X(\ell e^{-Z/z}) = g(Z)\) along directions non tangential to Stokes rays.

Remark. Condition (1) means that each element of the inverse system \(X_{\in \mathbb{k}}(Z)\) should be holomorphic in the complement of the finite set of rays \(\ell_{\gamma, Z}\) for which the class of \(a(\gamma)\) in \(\mathfrak{g}_{\in \mathbb{k}}\) is nonzero. Similar clarifications apply to conditions (2) and (3).

Given a solution of RH, we can construct the framed connection in the obvious way: \(\nabla(Z)\) is given by formula (4.4) and \(g(Z)\) by condition (4). Note that the jump (4.3) across a Stokes ray \(\ell_{\gamma, Z}\) is independent of \(z\), thus the local expression (4.4) patches over a collection of sectors between Stokes rays to all \(\mathbb{C}^\ast\). Therefore, it defines a meromorphic connection on \(\mathbb{P}^1\) with (possibly) poles at \(z = 0, \infty\). Provided that the restriction of \(X(Z)\) to sectors between Stokes rays admits a suitable analytic continuation, continuity of the family of stability data will assure that \((\nabla(Z), g(Z))\) is isomonodromic. In what follows, we will construct explicit solutions \(X(Z)\) of RH and prove that the corresponding framed connections fulfill the requirements of Proposition (4.4).

4.3. Integral operator. The basic technique to solve Riemann-Hilbert factorization problems consists of finding fixed points for singular integral operators, involving integration along the jump contour (see e.g. [FIKN]). The general form of the integral operators which are relevant for RH formulated above is that of
a \( \mathcal{Z} = \mathcal{Z}(Z) \) acting on suitable \( \text{End}(\widehat{\mathfrak{g}}) \)-valued holomorphic functions \( Y \) by (cf. [GMN1, Eq. 5.11])

\[
\mathcal{Z}[Y](e_\alpha) = e_\alpha \exp \left( L(\alpha) + \sum_\ell \int_{\ell} \frac{dz'}{z'} \rho(z, z') \log \star (Y(e_\alpha)^{-1} \ast (Y \text{ Ad} S_\ell(e_\alpha))) \right),
\]

summing over all rays \( \ell = \ell_{\gamma,Z} \) with \( \gamma \in \Gamma_{>0} \). Here \( L \) is a holomorphic function with values in \( \text{Hom}(\Gamma, \mathbb{C}) \) and \( \rho(z, z') \) is a suitable Cauchy-type integration kernel. We take (4.5) as a formal expression for a moment.

Recall that the stability data \((Z, a(Z))\) is equivalent to a family \((Z, \{\Omega(\gamma, Z)\})\) and note that

\[
Y \text{ Ad} S_\ell(e_\alpha) = Y(e_\alpha) \ast \prod_{Z(\gamma) \in -\ell} (1 - Y(e_\gamma))^{\Omega(\gamma)(\gamma, \alpha)}.
\]

This leads to the equivalent expression

\[
\mathcal{Z}[Y](e_\alpha) = e_\alpha \exp \left( L(\alpha) + \sum_\gamma \Omega(\gamma) \langle \gamma, \alpha \rangle \int_{\ell_{\gamma,Z}} \frac{dz'}{z'} \rho(z, z') \log \star (1 - Y(z')(e_\gamma)) \right),
\]

summing over all \( \gamma \in \Gamma \) with \( \Omega(\gamma) \neq 0 \). Notice that we have formally

\[
\mathcal{Z}[Y](e_\alpha \ast e_\beta) = \mathcal{Z}[Y](e_\alpha) \ast \mathcal{Z}[Y](e_\beta),
\]

so, when it is well defined, \( \mathcal{Z}[Y] \) automatically preserves the commutative product. Following [GMN1], we will actually choose

\[
L(z) = \frac{Z}{z} + z \bar{Z},
\]

\[
\rho(z, z') = \frac{1}{4\pi i} \frac{z' + z}{z' - z}.
\]

**Remark 4.5.** This choice is crucial for the solution of RH to provide a connection of the form (4.2).

To find fixed points for the operator \( \mathcal{Z} \), we wish to iterate in the integral equation (4.6). Initial points for this iteration are provided by the following definition.

**Definition 4.6.** A holomorphic function \( Y : U \to \text{Aut}^\ast(\widehat{\mathfrak{g}}) \) on an open set \( U \subset \mathbb{C}^\ast \) is **admissible** if the following conditions hold.

1. The limit \( \lim_{z \to 0, \infty} Ye^{-L}(z)(e_\alpha) \) exists in \( \widehat{\mathfrak{g}} \) for all \( \alpha \in \Gamma_{>0} \), in any direction non-tangential to \( \ell_{\gamma,Z} \) with \( a(\gamma) \neq 0 \).
2. If \( \ell_{\gamma,Z} \) belongs to the boundary of \( U \), then \( Y(e_\gamma) \) extends to a \( \widehat{\mathfrak{g}} \)-valued holomorphic function in a neighborhood of \( \ell_{\gamma,Z} \) with the same property.

Trivially, the holomorphic function \( X^0 = e^L : \mathbb{C} \to \text{Aut}^\ast(\widehat{\mathfrak{g}}) \) is admissible. This will be our choice of initial point for the iteration. To run the iterative process we need the following.
Lemma 4.7. Assume that $(Z,a(Z))$ is strongly generic. If $Y: U \to \text{Aut}^*(\hat{g})$ is admissible then so is $Z[Y]$.

4.4. Basic estimates. The proof follows from elementary estimates that we now establish. For later applications, we introduce a parameter $R > 0$. The estimates needed for the proof of Lemma 4.7 are obtained setting $R = 1$. We need to study the integral

$$
\int_{\mathbb{R} < 0 e^{i\psi} c} \frac{dz'}{z' - z} \exp(R(z'^{-1}c + z' \bar{c}))
$$

(4.7)

where $c \in \mathbb{C}^*$, $\psi$ is a sufficiently small angle, and $z \notin \mathbb{R} < 0 e^{i\psi} c$. We also look at the integral along an arc,

$$
\int_{|\psi| < \epsilon} \frac{dz'}{z' - z} \exp(R(z'^{-1}c + z' \bar{c}))
$$

(4.8)

where $z' \in \mathbb{R} < 0 e^{i\psi} c$, $|z'|$ is fixed, $z \notin \mathbb{R} < 0 c$ and $\epsilon$ is small enough. We claim that (4.7) converges for sufficiently small $|\psi|$, and in fact its modulus is bounded above by

$$
\frac{C}{2R|c|} \exp(-2R|c|)
$$

for $R$ large enough (for a constant $C$ depending on the angular distance of $z$ from $\mathbb{R} < 0 e^{i\psi} c$). This follows from the change of variable $z' = -e^{s+i\psi} c$ for real $s$ and $\psi$, reducing (4.7) to

$$
\int_{-\infty}^{+\infty} ds \frac{-e^{s+i\psi} c + z}{-e^{s+i\psi} c - z} \exp(-R|c|(e^{-s-log|c|} - i\psi + e^{s+log|c|} + i\psi))
$$

The modulus of this is bounded above by

$$
C \int_{-\infty}^{+\infty} ds \exp(-2R|c| \cosh(s + log |c| + i\psi))
$$

where $C$ is a constant depending on the angular distance of $z$ from $\mathbb{R} < 0 e^{i\psi} c$. For $\psi$ sufficiently small this is in turn bounded by

$$
C \int_{-\infty}^{+\infty} ds' \exp(-2R|c| \cosh(s'))
$$

where $C$ is a new (possibly larger) constant, depending also on $\psi$. We recognize the integral as a Bessel function, and we find that (4.7) converges for $|\psi| \ll 1$, and moreover it is actually bounded above by $\frac{C}{2R|c|} \exp(-2R|c|)$ for $R$ large enough, as required.

Similarly with the same change of variable (4.8) becomes

$$
i \int_{-\epsilon}^{+\epsilon} d\psi \frac{-e^{s+i\psi} c + z}{-e^{s+i\psi} c - z} \exp(-R|c| e^{-s-log|c|} - i\psi) \exp(-R|c| e^{s+log|c|} + i\psi).
$$

One can check that this vanishes for $s \to \pm \infty$ (for fixed, sufficiently small $\epsilon$).
Proof of Lemma 4.7. First we show that for admissible \( Y, \alpha \in \Gamma_{>0}, z \in U \) the right hand side of (4.6) is a well defined element of \( \widehat{\mathfrak{g}} \). Since \( z \) lies away from the rays of integration, the function \((z')^{-1} \rho(z, z')\) is holomorphic on each \( \ell_{\gamma} \). For fixed \( k > 0 \), the projection of \( Z[Y](e_\alpha) \) on \( \mathfrak{g}_{<k} \) involves finitely many \( \gamma \). We claim that all the integrals appearing are convergent. To see this, we first expand \( \log(1 - Y(e_{\gamma})) \) and use the boundary conditions in Definition 4.6 to see that each integral is dominated by the sum of a fixed finite number of integrals of the form

\[
C \int_{\ell_{\gamma, z}} \frac{dz' z' + z}{z' - z} \exp(z'^{-1}Z(\gamma) + z'Z(\gamma))
\]

for some constant \( C \). By the basic estimates for (4.7), these are all convergent. Then it follows from standard theory that \( Z[Y](e_\alpha) \) is a holomorphic function of \( z \in U \) (since \((z')^{-1} \rho(z, z')\) is, and by the above convergence). On the other hand it is also holomorphic in a neighborhood of \( \ell_{\alpha, z} \) since by the genericity property for \( Z \) the integral along \( \ell_{\gamma, z} = \ell_{\alpha, z} \) appears with a factor of \((\gamma, \alpha)\) and therefore vanishes. Finally we can use the basic estimate for (4.7) to take the limit

\[
\lim_{z \to 0} Z[Y]e^{-L}(z)(e_\alpha)
\]

in a direction non-tangential to \( \ell_{\gamma, z} \) with \( a(\gamma) \neq 0 \): by the definition of \( L \) this is the constant element of \( \widehat{\mathfrak{g}} \) given by

\[
e_\alpha \exp_s \left( \frac{1}{4\pi i} \sum_{\gamma} \Omega(\gamma)\langle \gamma, \alpha \rangle \int_{\ell_{\gamma, z}} \frac{dz'}{z'} \log_s (1 - Y(z')(e_{\gamma})) \right)
\]

The same argument applies to the \( z \to \infty \) limit, which completes the proof. \( \square \)

4.5. Application of Plemelj’s theorem. The link between the integral operator \( Z \) and the Riemann-Hilbert problem follows from standard theory, and relies on a result from elementary complex analysis (Plemelj’s theorem).

Suppose \( Y(z) \) is an admissible function and \((Z, a(Z))\) is generic. Fix a ray \( \ell \) in our half-space and a point \( z_0 \in \ell \), and denote by \( Z[Y](z^+)(e_\alpha) \) the limit of \( Z[Y](z)(e_\alpha) \) as \( z \to z_0 \) in the counterclockwise direction. Similarly let \( Z[Y](z^-)(e_\alpha) \) denote the limit in the clockwise direction.

Lemma 4.8. Both limits exist, and they are related by

\[
Z[Y](z^+)(e_\alpha) = Z[Y](z^-)(e_\alpha) \prod_{\ell(\gamma) \in -\ell} (1 - Y(z_0)(e_\gamma))^{\Omega(\gamma)(\gamma, \alpha)}.
\] (4.9)

Proof. Note first that \( Y(z_0)(e_\gamma) \) in (4.9) is well-defined because \( Y \) is admissible. For \( \gamma \in \Gamma_{>0} \) we have

\[
\int_{\ell_{\gamma}} \frac{dz'}{z'} \rho(z, z') \log(1 - Y(z')(e_\gamma)) = \sum_{k \geq 1} \frac{1}{4\pi i} \int_{\ell_{\gamma}} \frac{dz'}{z'} \frac{1}{z' - z} \left( 1 + \frac{z}{z'} \right) \frac{1}{k} (Y[z'](e_\gamma))^k
\]
We may apply Plemelj’s theorem to find
\[
\lim_{z \to z^\pm} \frac{1}{2\pi i} \int_{\ell_\gamma} \frac{dz'}{z' - z} \left(1 + \frac{z}{z'}\right) (Y(z')(e_\gamma))^k = \pm(Y(z_0)(e_\gamma))^k
\]
\[+ \text{pv} \frac{1}{2\pi i} \int_{\ell_\gamma} \frac{dz'}{z' - z_0} (Y(z')(e_\gamma))^k,
\]
where pv denotes a (well defined, convergent) principal value integral. Therefore
\[
\lim_{z \to z^\pm} \frac{1}{4\pi i} \int_{\ell_\gamma} \frac{dz'}{z' - z} \log(1 - Y'(z')(e_\gamma)) = \pm \frac{1}{2} \log(1 - Y(z_0)(e_\gamma))
\]
\[+ \text{pv} \frac{1}{4\pi i} \int_{\ell_\gamma} \frac{dz'}{z' - z_0} \log(1 - Y'(z')(e_\gamma)),
\]
where the last principal value integral is convergent. Equation (4.9) follows. □

4.6. Fixed point. Following [GMN1, App. C], we construct now a solution of the singular integral equation
\[
X = Z[X],
\]
by iteration from \(X^0 = e^L\), and show that it solves the Riemann-Hilbert factorization problem. Recall that \(L(z) = z^{-1}Z + z\overline{Z}\) and hence the solution will depend (in a complicated way) on the parameter \(Z\).

Set \(Z^{(i)} = Z \circ \cdots \circ Z\) (\(i\) times) and consider the sequence
\[
X^{(i)} = Z^{(i)}[X^0]
\]
for \(i \geq 0\). We claim that \(X^{(i)}(z)\) converges as \(i \to \infty\). This follows from an explicit calculation. To calculate \(X^{(i)}\) we start by rewriting, for all admissible \(Y\),
\[
\sum_{\gamma} \Omega(\gamma) \log(1 - Y'(z')(e_\gamma)) = \sum_{\gamma} \Omega(\gamma) \sum_{k>0} \frac{1}{k^2} Y(z')(e_{k_\gamma}) k\gamma
\]
\[= \sum_{\gamma} Y(z')(e_\gamma) \text{DT}(\gamma) \gamma,
\]
using that \(Y(z)\) is an algebra homomorphism, where
\[\text{DT}(\gamma) \gamma = -a(\gamma),\]
that is,
\[\text{DT}(\gamma') = \sum_{n>0, n|\gamma'} \frac{\Omega(n^{-1}\gamma')}{n^2}.
\]

Remark. The notation \(\text{DT}\) reflects the way in which the “BPS state counts” \(\Omega\) are related to Donaldson-Thomas invariants; in the present case it is of course purely formal.

So we can rewrite the action of \(Z\) as
\[
Z[Y](z)(e_\alpha) = X^0(z)(e_\alpha) \exp \left(\sum_{\gamma} \langle \text{DT}(\gamma) \gamma, \alpha \rangle \int_{\ell_\gamma} \frac{dz'}{z'} \rho(z, z') Y(z')(e_\gamma)\right).
\]
It follows that

$$X^{(i)}(z)(e_\alpha) = X^0(z)(e_\alpha) \sum_k \frac{1}{k!} \prod_j \left( \langle DT(\gamma_j)\gamma_j, \alpha \rangle \int_{\ell_{\gamma_j}} \frac{dz'}{z'} \rho(z, z') X^{(i-1)}(z')(e_{\gamma_j}) \right)^{k_j},$$

(4.11)

where we sum over ordered partitions $k$, and we take the product over all unordered collections $\{\gamma_1, \ldots, \gamma_l\} \subset \Gamma$ for $l$ the length of $k$. Let us denote by $T$ a connected rooted tree, decorated by elements $\gamma \in \Gamma$ (i.e. there is a map from the vertex set $T^0$ to $\Gamma$). Similarly, we write $T^1$ for the set of edges of $T$, and we denote by $\alpha(v)$ the decoration at $v \in T^0$. Introduce a factor

$$W_T = (-1)^{|T^1|} \frac{DT(\gamma_T)\gamma_T}{|\text{Aut}(T)|} \prod_{v \rightarrow w} \langle \alpha(v), DT(\alpha(w))\alpha(w) \rangle,$$

where $\gamma_T$ denotes the label of the root of $T$, and $\text{Aut}(T)$ is the automorphism group of $T$ as a decorated, rooted tree. To each $T$ we also attach a “propagator” $G_T$ which is a $g$-valued holomorphic function in the complement of $\ell_\gamma$ rays with $a(\gamma) \neq 0$, defined inductively by

$$G_T(z; Z) = \int_{\ell_{\gamma_T}} \frac{dz'}{z'} \rho(z, z') X^0(z'; Z)(e_{\gamma_T}) \prod_{T'} G_{T'}(z'; Z),$$

(4.12)

where $\{T'\}$ denotes the set of (connected, rooted, decorated) trees obtained by removing the root of $T$ (setting $G_0(z) = 1$). By applying (4.11) $i - 1$ times we obtain

$$X^{(i)}(e_\alpha) = X^0(e_\alpha) \sum_j \prod \langle \alpha, W_{T_j} G_{T_j} \rangle,$$

where we sum over all collections of trees $\{T_1, \ldots, T_i\}$ as above, with depth at most $i$. (Notice that $W_{T_j} G_{T_j}(z) \in \Gamma \otimes g$, and we have extended $\langle -, - \rangle$ by $g$-linearity). By the definition of $\text{Aut}^+(g)$-valued holomorphic functions, we see that the sequence $X^{(i)}$ converges for $i \rightarrow \infty$ to a limit $X = X(Z)$, which is a solution of (4.10), given explicitly by

$$X(e_\alpha) = X^0(e_\alpha) \exp(\alpha, - \sum T W_T G_T),$$

(4.13)

where we sum over arbitrary (decorated, rooted) trees. This is the analogue of \cite{GMN1} equation (C.26) (see also \cite{N} equation (4.12)).

Our discussion so far can be summarised in the following result.

**Lemma 4.9.** The fixed point $X(Z)$ defined by (4.13) solves RH. The automorphism $g(Z) = \lim_{z \rightarrow 0} X e^{-z/z}$ of condition (4) is given by

$$g(Z)(e_\alpha) = e_\alpha \ast \exp(\alpha, - \sum T W_T G^0_T(Z)),$$

(4.14)

where $G^0_T$ defined by

$$G^0_T(Z) = \frac{1}{4\pi i} \int_{\ell_{\gamma_T}} \frac{dz'}{z'} X^0(z'; Z)(e_{\gamma_T}) \prod_{T'} G_{T'}(z'; Z),$$

(4.15)
Picking \( k \geq 0 \) and projecting into \( \text{Aut}^\ast(g_{\leq k}) \), the proof follows from the previous lemmas and the basic estimates for (4.7). The existence of the limit (4.14) follows from our choice of integration kernel:

\[
\lim_{z \to 0} \rho(z, z') = \frac{1}{4\pi i}.
\]

**Remark 4.10.** We observe that it is possible to allow symmetric stability data (i.e. to allow \( \Omega(\gamma) = \Omega(-\gamma) \)) in the construction above by working over the Poisson \( \mathbb{C}[[t]] \)-algebra \( g[[t]] \). The relevant integral operator is simply

\[
Z'[Y](z)(e_\alpha) = e_\alpha \exp \left( L(z)(\alpha) + \sum_\gamma \Omega(\gamma)(\gamma, \alpha) \int_{\ell_\gamma(Z)} \frac{dz'}{z'} \rho(z, z') \log (1 - tY(z')(e_\gamma)) \right),
\]

and we still find a fixed point given by

\[
X'(e_\alpha) = X^0(e_\alpha) \exp(\alpha, -\sum_T t^{[T^0]} W_T G_T).
\]

This solves a Riemann-Hilbert problem which is formally identical to the one we described, but with monodromy (Poisson) automorphisms which are compositions of operators \( T'_\gamma \) acting on \( g[[t]] \) by

\[
T'_\gamma(e_\alpha) = e_\alpha(1 - te_\gamma)^{(\gamma, \alpha)}.
\]

In fact we can perform an identical construction over a local complete or Artin ring, for arbitrary \( \Omega \) (symmetric or not).

### 4.7. Definition of the framed connection.

We define \( (\nabla(Z), g(Z)) \) setting \( g(Z) \) as in (4.14) and

\[
\nabla(Z) = d - (\partial_z X) X^{-1} dz,
\]

where \( X = X(Z) \) is an in Lemma [4.9]. As mentioned earlier, this defines a meromorphic connection on \( \mathbb{P}^1 \) with (possibly) poles at \( z = 0, \infty \). We show now that it fulfills the requirements of Proposition (4.4).

We first prove that \( \nabla(Z) \) has a double pole at zero and infinity, and therefore is of the form (4.2). Consider the \( \text{Aut}^\ast(\hat{g}) \)-valued map

\[
Y = X(X^0)^{-1},
\]

given explicitly by

\[
Y(z, Z)(e_\alpha) = e_\alpha \exp(\alpha, -\sum_T W_T G_T(z, Z)).
\]

In each finite-dimensional quotient \( g_{\leq k} \), it makes sense to consider a sector \( \Sigma \) between consecutive rays \( \ell_\gamma \) with \( a(\gamma) \neq 0 \). So in a fixed \( g_{\leq k} \), and inside \( \Sigma \), we find that \( Y_{\leq k} \) is holomorphic, with well-defined limits as \( z \to 0, \infty \). Taking now \( k \to \infty \) we obtain

\[
\lim_{z \to 0} Y(z) = g(Z), \quad \lim_{z \to \infty} Y(z) = Y_{\infty}
\]

(4.17)
which are elements of $\Aut^*(\hat{\mathfrak{g}})$. The existence of these limits follows from the basic estimates for (4.17) and our choice of integration kernel:

$$\lim_{z \to 0} \rho(z, z') = \frac{1}{4\pi i}, \quad \lim_{z \to \infty} \rho(z, z') = -\frac{1}{4\pi i}.$$

Consider the meromorphic connection $\nabla^0(Z)$ on $\mathbb{P}^1$ given by

$$\nabla^0 = d - \left( -\frac{Z}{z^2} + \bar{Z} \right) dz.$$

and note that $X^0$ provides a fundamental solution. Thus, we find an alternative description of $\nabla(Z)$ as a gauge transformation of $\nabla^0$ inside the sector $\Sigma$, that is,

$$\nabla(Z) = Y \cdot \nabla^0 = d - \left( \frac{1}{z^2} A^{(-1)} + \frac{1}{z} A^{(0)} + A^{(1)} \right) dz$$

for some $A^{(i)} \in D^*(\hat{\mathfrak{g}})$ and also

$$g(Z)^{-1} \cdot A^{(-1)} = -Z$$

i.e. $(\nabla(Z), g(Z))$ is a compatibly framed connection with order two poles at 0 and $\infty$, and no other singularities.

**Remark 4.11.** The automorphism $g(Z)$ lies in the subset of $\Aut^*(\hat{\mathfrak{g}})$ given by elements of the form $e_\alpha \mapsto e_\alpha \exp_x(\langle \alpha, x \rangle)$ for some $x \in \Gamma \otimes \hat{\mathfrak{g}}$. There is an obvious involution $(-)^*$ on this set, induced simply by $x \mapsto -x$. Notice that we have

$$g(Z) = (Y_\infty)^*,$$

that jointly with $Y_\infty \cdot A^{(1)} = \bar{Z}$ shows that $A^{(-1)}$ uniquely determines $A^{(1)}$. In the case of symmetric stability data one can see that the corresponding derivations are related by an involution.

**4.8. Formal type and Stokes data.** If we expand the kernel $\rho(z, z')$ as a formal power series in $z$ around $z = 0$, we can regard $Y$ as a formal gauge transformation (an element of $\Aut^*(\hat{\mathfrak{g}}[[z]])$), taking the germ of $\nabla(Z)$ at 0 to the germ of $\nabla^0$. In other words, the formal type of $\nabla(Z)$ at 0 is the type of $\nabla^0$. The gauge transformation $h(z)$ acting by

$$h(z)(e_\alpha) = e_\alpha \exp_x(-z\bar{Z}(\alpha))$$

is well defined near $z = 0$ (it has an essential singularity at $\infty$), and it takes the formal type of $\nabla^0$ at 0 to

$$d + \frac{Z}{z^2} dz.$$

This proves that the $\nabla(Z)$ has the desired formal type, with Stokes rays given by $\ell_\gamma(Z) = -\Re_{>0} Z(\gamma)$ for $\gamma \in \Gamma$. 
For each finite dimensional quotient \( g_{<k} \), the restriction \( X_{<k} \) to a sector \( \Sigma \) as above is a fundamental solution of \( \nabla(Z)_{<k} \) with the right asymptotics as \( z \to 0 \). These solutions differ by the action of (3.3) along a Stokes ray \( \ell \). To prove that these automorphisms are actually the Stokes factors of \( \nabla(Z) \) it is enough to show that a solution given by \( X_{<k}|_{\Sigma} \) induces, by analytic continuation, a fundamental solution on a supersector \( \hat{\Sigma} \) preserving the asymptotics. We will perform a formally identical check later in Section 6.7, so we do not reproduce the argument here.

Similarly, we can calculate the Stokes data at \( \infty \). Setting \( w = \frac{1}{z} \) and arguing as before, we see that the formal type of \( \nabla(z) \) at \( \infty \) is

\[
d + \frac{\overline{Z}}{w^2} dw,
\]

with Stokes rays given by \( \ell_{\gamma,Z} = -\mathbb{R}_{>0} \overline{Z}(\gamma) \) for \( \gamma \in \Gamma \). We claim that the attached Stokes factors are given again by (3.3). This follows again from the argument in 4.8 applied to \( \check{X}(w; Z) = X(w^{-1}; Z) \). We have

\[
\check{X}(w,Z)(e_\alpha) = X^0(w^{-1})(e_\alpha) \exp(\alpha, - \sum_T W_T G_T(w^{-1})),
\]

with

\[
G_T(w^{-1}; Z) = \int_{\mathbb{R}_{>0} Z(\gamma_T)} \frac{dz' z' + w^{-1}}{z' - w^{-1}} X^0(z'; Z)(e_{\gamma_T}) \prod_{T'} G_{T'}(z'; Z).
\]

Making the change of variable \( z' = \frac{1}{w} \) we can rewrite

\[
G_T(w^{-1}; Z) = \int_{\mathbb{R}_{>0} Z(\gamma_T)} \frac{dw' w'}{w' - w} X^0(w'^{-1}; Z)(e_{\gamma_T}) \prod_{T'} G_{T'}(w'^{-1}; Z).
\]

By induction on \( |T^0| \), this proves that the jump of \( \check{X}(w,Z) \) across \( \ell_{\gamma,Z} \) equals the jump of \( X(z; Z) \) across \( \ell_{\gamma,Z} \).

4.9. Extension and isomonodromy of \( \nabla(Z) \). So far we have constructed \( \nabla(Z) \) under the assumption that \( (Z, a(Z)) \) is generic. We wish to allow for a pair of Stokes rays \( \ell_{\gamma,Z} \) and \( \ell_{\eta,Z} \) to come together without the assumption \( \langle \gamma, \eta \rangle = 0 \).

Fix a strictly convex sector \( V \subset \mathbb{H}' \). Suppose that \( (Z(t), a(t)) \) is a continuous family of stability data on \( g \) parametrised by \([0, 1]\), such that \( V \) contains only two Stokes rays \( \ell, \ell' \), whose \( Z_t \) counterclockwise order is \( \ell, \ell' \) for \( t < t_0 \), respectively \( \ell', \ell \) for \( t > t_0 \). Thus the rays \( \ell, \ell' \) coincide only when \( t = t_0 \). We assume that all other Stokes rays are constant. Let us also write \( \Omega^\pm \) for the obvious limits.

From its construction, we see that \( X(t) = X(Z(t)) \) has finite limits when \( t \to t_0^\pm \). These limits \( X(t_0^\pm) \) are sectionally holomorphic with values in \( \text{Aut}^*(\mathfrak{g}) \). Their jumps across all rays \( \ell_{\gamma,Z}(t_0) \) distinct from \( \ell, \ell' \) are the same. The jump of \( X(t_0^-) \) across \( \ell = \ell' \) is given by

\[
\prod_{\gamma' \in \ell'} T_{\gamma'} \circ \prod_{\gamma \in \ell} T_{\gamma}.
\]
Similarly the jump of \( X(t^+_0) \) across the same rays is given by
\[
\prod_{\gamma \in \ell} T^{\Omega_+} \circ \prod_{\gamma' \in \ell'} T^{\Omega_+} \circ \prod_{\gamma \in \ell} T^{-\Omega} \circ \prod_{\gamma' \in \ell'} T^{-\Omega}.
\]
As the family \((Z(t), a(t))\) is continuous by assumption, we have
\[
\prod_{\gamma' \in \ell'} T^{\Omega} \circ \prod_{\gamma \in \ell} T^{\Omega} = \prod_{\gamma \in \ell} T^{\Omega} \circ \prod_{\gamma' \in \ell'} T^{\Omega}.
\]
Therefore the function \( X^{-1}(t^+_0)X(t^+_0) \) is holomorphic on \( \mathbb{C}^* \). Recall that we have
\[
X(Z) = Y(Z)X^0(Z)
\]
and that \( Y(Z) \) has \( z \to 0, z \to \infty \) limits \( g(Z), Y_\infty \) which do not depend on \( Z \). It follows that
\[
\lim_{z \to 0} X^{-1}(t^+_0)X(t^+_0) = \lim_{z \to \infty} X^{-1}(t^+_0)X(t^+_0) = 1,
\]
and hence \( X^{-1}(t^+_0) = X(t^+_0) \) (this is clear working in any finite dimensional quotient \( \mathfrak{g} \)). From their construction using \( X(Z) \), we conclude that the connections \( \nabla(Z(t)) \) are meromorphic and isomonodromic for \( t \in [0, 1] \). By applying this argument repeatedly, we can extend \( \nabla(Z) \) isomonodromically to all of \( U \).

4.10. **Explicit formula for** \( \nabla(Z) \). We provide now a more explicit formula for the \( D^*(\hat{\mathfrak{g}}) \)-valued function
\[
\mathcal{A} = \frac{1}{z^2} A^{-1}(Z) + \frac{1}{z} A^0(Z) + A^{(1)}(Z)
\]
that defines the connection \( \nabla(Z) \).

Given an automorphism \( T \in \text{Aut}^*(\hat{\mathfrak{g}}) \), it acts on \( D^*(\hat{\mathfrak{g}}) \) via the adjoint representation, inducing a map
\[
\partial T : D^*(\hat{\mathfrak{g}}) \to D^*(\hat{\mathfrak{g}}) : D \mapsto T^{-1}DT.
\]
Unlike the \( \hat{\mathfrak{g}} \)-module structure on \( D^*(\hat{\mathfrak{g}}) \), we stress that this map is not \( \mathfrak{g} \)-linear (it is a formal analogue of pull-back of vector fields by diffeomorphisms on the formal torus \( \text{Spec} \hat{\mathfrak{g}} \)). For \( T = X^{-1} \), we can now write
\[
\mathcal{A} = \partial X^{-1}(X^{-1} \partial X)
\]
where \( X^{-1} \partial X \) is an element in \( D^*(\hat{\mathfrak{g}}) \) defined in the obvious way.

We pick a basis \( \{ \gamma_j \} \) for the lattice \( \Gamma \), with dual basis \( \partial_j \). Then, we can express
\[
X^{-1} \partial X = \sum_j X^{-1}(\partial X(e_{\gamma_j})) \partial_j
\]
in this basis and construct a matrix of “partial derivatives” of \( X^{-1} \)
\[
[\partial X^{-1}]_{ij} = (\partial X^{-1}(\partial_i))(e_{\gamma_j})
\]
(in general \([\partial X^{-1}] \) is not the inverse of \([\partial X]\) with respect to the commutative product, precisely due to the failure of \(\widehat{\mathfrak{g}}\)-linearity of \(\partial X\)). Using this, a direct calculation shows that

\[
\mathcal{A}_z = \sum_{i,j} \partial_z X(e_{\gamma_i})[\partial X^{-1}]_{ij} \partial_j.
\]

This formula provides a rigorous analogue of [GMN1, eq. (5.18)].

In [St, Section 2.8], (4.18) is combined with the general asymptotic expansion for \(X\) in order to write down an expansion for the \(\mathcal{A}\) connection. For this, we recall that \(X = YX^0\) and use this to express

\[
[\partial X^{-1}] = \text{Id} + YB
\]

where \(YB\) denotes the \(\widehat{\mathfrak{g}}\)-valued matrix given by the action of \(Y\) on the components of \(B\), defined by

\[
B_{ij} = \langle \gamma_j, \sum_T W_T \partial_i G_T \rangle.
\]

Finally, denoting \(\mathcal{A}_j = \sum_i \partial_z X(e_{\gamma_i})[\partial X^{-1}]_{ij}\), we obtain the formula

\[
\mathcal{A}_j = \sum_{i} \left( \frac{1}{z^2} Z(\gamma_i) - \bar{Z}(\gamma_i) - \langle \gamma_i, \sum_T W_T \partial_z G_T \rangle \right) \left( \delta_{ij} + \langle \gamma_j, \sum_T W_T Y(\partial_i G_T) \rangle \right).
\]

4.11. Single-ray solution. The following basic example will make contact with differential geometry through the \(tt^t\)-type connections \(\nabla^{GMN} (Z)\) of [GMN1]. This is the case when \(\Gamma \cong \mathbb{Z}^2\) generated by \(\gamma, \eta\) with \(\langle \gamma, \eta \rangle = 1\), and the stability data are such that \(\Omega(\gamma) = 1\) with all other \(\Omega\) vanishing. This computation will play an important role in Sections 6 and 8.

Using (4.19), one can derive explicit expressions for the connections \(\nabla(Z)\) as follows

\[
\mathcal{A}^{(-1)} = Z - Z(\gamma) \sum_{k>0} \frac{1}{4k\pi i} \int_{\ell_\gamma} \frac{dz'}{z'} \exp(L(z')(k\gamma)) \text{ad}(e_{k\gamma}),
\]

\[
\mathcal{A}^{(0)} = -(Z(\gamma) + \bar{Z}(\gamma)) \sum_{k>0} \frac{1}{2k\pi i} \int_{\ell_\gamma} dz' \exp(L(z')(k\gamma)) \text{ad}(e_{k\gamma}),
\]

\[
\mathcal{A}^{(1)} = -\bar{Z} + \bar{Z}(\gamma) \sum_{k>0} \frac{1}{4k\pi i} \int_{\ell_\gamma} \frac{dz'}{z'} \exp(L(z')(k\gamma)) \text{ad}(e_{k\gamma}).
\]
We provide a different argument using the solution of the integral equation. Indeed, it is straightforward to see that $\mathcal{Z}[X] = X$ reduces to

$$X(z)(e_\gamma) = X^0(z)(e_\gamma) = e_\gamma \exp(L(z)(\gamma)),$$

$$X(z)(e_\eta) = X^0(z)(e_\eta) \exp \left( \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') \log(1 - X^0(z')(e_\gamma)) \right)$$

$$= e_\eta \exp(L(z)(\eta)) \exp \left( \sum_{k>0} \frac{1}{k} \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') X^0(z')(e_{k\gamma}) \right).$$

(which is just (4.13) in this case). Denoting $L' = \partial_z L = -z^{-2} Z + \tilde{Z}$, we have

$$\partial_z X(e_{\gamma e}) = \partial_z \exp(L(z)(\gamma)) e_{\gamma e}$$

$$= L'(z) X(e_\gamma).$$

On the other hand

$$L'(z) X(e_\eta) = (L'(z) X^0(e_\eta)) \ast \exp \left( \sum_{k>0} \frac{1}{k} e_{k\gamma} \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') \exp(L(z')(k\gamma)) \right)$$

$$+ X^0(e_\eta) \ast \left( \sum_{k>0} \frac{1}{k} L'(z)(e_{k\gamma}) \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') \exp(L(z')(k\gamma)) \right),$$

which we rewrite as

$$L'(z) X(e_\eta) = \partial_z X(e_\eta) - X^0(e_\eta) \ast \partial_z \left( \sum_{k>0} \frac{1}{k} e_{k\gamma} \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') \exp(L(z')(k\gamma)) \right)$$

$$+ X^0(e_\eta) \ast \left( \sum_{k>0} \frac{1}{k} e_{k\gamma} \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') L'(z)(k\gamma) \exp(L(z')(k\gamma)) \right).$$

Integrating by parts, using the skew-symmetry of the kernel $\rho(z, z')$, we find

$$\partial_z \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') \exp(L(z)(k\gamma)) = \frac{1}{z} \int_{\ell, \gamma} dz' \rho(z, z') L'(z')(k\gamma) \exp(L(z')(k\gamma)),$$

and so

$$\partial_z X(e_\eta) = L'(z) X(e_\eta) + X^0(e_\eta) \ast \left( \sum_{k>0} \frac{1}{k} e_{k\gamma} \int_{\ell, \gamma} \frac{dz'}{z'} \rho(z, z') \left( \frac{z'}{z} L'(z') - L'(z) \right) \exp(L(z')(k\gamma)) \right)$$

$$= L'(z) X(e_\eta) + X^0(e_\eta) \ast \frac{Z(\gamma)}{z} \left( \sum_{k>0} e_{k\gamma} \int_{\ell, \gamma} \frac{dz'}{z} \rho(z, z') \left( \frac{z}{z'} - 1 \right) \exp(L(z')(k\gamma)) \right)$$

$$+ X^0(e_\eta) \ast \frac{\tilde{Z}(\gamma)}{z} \left( \sum_{k>0} e_{k\gamma} \int_{\ell, \gamma} \frac{dz'}{z} \rho(z, z')(z' - z) \exp(L(z')(k\gamma)) \right).$$
Expanding in $z$, we get
\[
\partial_z X(e_\eta) = L'(z)X(e_\eta) + \frac{Z(\gamma)}{z^2}X^0(e_\eta) \ast \left( \sum_{k>0} \frac{e_k \gamma}{4 \pi i} \int_{\ell_k} \frac{dz'}{z'} \exp \left( L(z')(k\gamma) \right) \right) \\
+ \frac{Z(\gamma) + \bar{Z}(\gamma)}{z}X^0(e_\eta) \ast \left( \sum_{k>0} \frac{e_k \gamma}{4 \pi i} \int_{\ell_k} \frac{dz'}{z'} \left( z' + \frac{1}{z'} \right) \exp \left( L(z')(k\gamma) \right) \right) \\
- \bar{Z}(\gamma)X^0(e_\eta) \ast \left( \sum_{k>0} \frac{e_k \gamma}{4 \pi i} \int_{\ell_k} \frac{dz'}{z'} \exp \left( L(z')(k\gamma) \right) \right),
\]
from which the claimed formula for $\nabla(Z)$ follows readily.

**Remark 4.12.** Notice that in this special case it is not harder to allow symmetric stability data with $\Omega(\pm \gamma) = 1$ and all other $\Omega$ vanishing, without the need to pass to $g[[t]]$. The results are formally the same, with the only difference of summing over all nonzero $k$ in the formulae for $A^{(i)}$. We will denote the resulting connections by $\nabla^{\text{sym}}(Z)$.

4.12. **Relation to heat kernel.** We close this Section taking a brief look at the iterative solution of (4.10) from a slightly different point of view (this subsection can be safely skipped). We introduce the scaling $Z \rightarrow RZ$, which will play a crucial role in the rest of the paper. Denote by $K_{m,l}(|x-y|)$ the usual Euclidean heat kernel in dimension 2, with mass $m$,
\[
K_{m,l}(|x-y|) = \frac{1}{4\pi l} \exp \left( -|x-y|^2l^{-1} - |m|^2l \right).
\]
The reason for writing $|m|$ is that we want to allow the “mass” to be a complex number. Let us also introduce the rational function
\[
\rho_{m,l}(z) = \frac{ml - z}{ml + z},
\]
which will serve as an integration kernel. Then the simplest integrals that appear in the solution $X(z)$ (attached to graphs which consist of a single vertex) are very similar to the Euclidean propagator in two dimensions in its so-called parametric representation, but with an extra factor $-i\rho$ inserted,
\[
-i \int_0^\infty ds \rho_{Z_{\beta},s}(Rz) K_{|Z_{\beta}|,s}(R).
\]
(The Euclidean propagator in 2 dimensions is $C(x,y) = \int ds K_{m,s}(|x-y|/2)$; its Fourier transform is $\hat{C}(p) = (p^2 + m^2)^{-1}$.)

To describe further contributions to $X(z)$ one considers graphs $T$ which are rooted trees decorated by $\alpha_i \in \Gamma$, as in the example of Figure 1. To each internal edge $\alpha_i \rightarrow \alpha_j$ is attached the usual Euclidean propagator, deformed by an extra factor $\rho_{Z_{\alpha_j},s_j}(s_i)$,
\[
-i \int_0^\infty ds_j \rho_{Z_{\alpha_j},s_j}(s_i) K_{|Z_{\alpha_j}|,s_j}(R).
\]
The root of $T$ is responsible for the $z$-dependence, by contributing a factor
\[-i \int_0^\infty ds \rho_{Z_\beta,s_1}(Rz)K_{Z_\beta}|s_1}(R).

Thus the element of $\mathfrak{g}$ attached to a diagram $T$ with $n$ vertices is
\[G_T(z) = (-i)^n \prod_k e_{\alpha_k} \int_k ds_k \rho_{Z_\beta,s_1}(Rz)K_{Z_\alpha_1}|s_1}(R) \prod_{i \to j} \rho_{Z_{\alpha_j},s_j}(s_i)K_{Z_{\alpha_j},s_j}(R).\]

To make some contact with more familiar notions, notice that if we suppose all the “masses" $m = |Z_{\alpha_i}|$ coincide and we forget all the $\rho$ insertions, then setting $R = |x - y|/2$ we recover the integrand for a Feynman diagram in 2-dimensional $\phi^n$ theory with $n$ edges, displayed in Figure 2. As usual, each diagram $T$ also comes with a corresponding rational weight $W_T$, which in the present case takes values in $\Gamma \otimes \mathbb{Q}$ and is given by
\[W_T = W_T(\Omega) = (-1)^{n-1} \frac{1}{|\text{Aut}(T)|} \alpha_0 \prod_{i \to j} \langle \alpha_i, \alpha_j \rangle \left( \prod_{1 \to m > 0, m}|\alpha_j| m^2 \right).\]

Notice that this is the only place where the locally constant function $\Omega$ actually shows up in the solution.

5. The connections $\nabla^{BTL}(Z)$ from stability data

In this section we recall the construction of the Bridgeland-Toledano-Laredo (BTL) connections, for our choice of stability data in the Poisson Lie algebra $\mathfrak{g}$,
and study the relation with [BT1]. Then, for the category of representations of a quiver without oriented cycles, we construct a “motivic” isomonodromic family of irregular connections, which recovers the connections in [BT1] using natural Lie algebra morphisms [J2]. For the case of Dynkin quivers, we are also able to recover (2.4) following ideas in [R1]. We use the foundational work of Joyce [J2, J3, J4].

5.1. Irregular $\hat{B}$-connections on $\mathbb{P}^1$. Let $\text{Hom}(\Gamma, \mathbb{C}^*)$ the group of characters of $\Gamma$, acting on $\hat{g}$ by

$$\sum_{\gamma \in \Gamma_{\geq 0}} f_{\gamma} \mapsto \sum_{\gamma \in \Gamma_{\geq 0}} \psi(\gamma) f_{\gamma},$$

for any $\psi \in \text{Hom}(\Gamma, \mathbb{C}^*)$. This action preserves the Poisson Lie algebra structure and induces an action on the pro-nilpotent Lie group $\hat{G}$. We define the pro-solvable, pro-algebraic group $\hat{B}$ with maximal torus $\text{Hom}(\Gamma, \mathbb{C}^*)$ as the semi-direct product

$$\hat{B} = \text{Hom}(\Gamma, \mathbb{C}^*) \ltimes \hat{G} = \lim_{\leftarrow} \text{Hom}(\Gamma, \mathbb{C}^*) \ltimes G_{\leq k}.$$}

The Lie algebra of $\hat{B}$ is given by the extension

$$\hat{b} = \text{Hom}(\Gamma, \mathbb{C}) \ltimes \hat{g}$$

of $\hat{g}$ by the abelian Lie algebra $\text{Hom}(\Gamma, \mathbb{C})$, with bracket $[Z, e_\gamma] = Z(\gamma)e_\gamma$.

Let $P_{\hat{B}}$ be the holomorphically trivial, principal $\hat{B}$-bundle on $\mathbb{P}^1$. By this we mean the inverse limit of the system of holomorphically trivial principal bundles corresponding to the groups $\text{Hom}(\Gamma, \mathbb{C}^*) \ltimes G_{\leq k}$. For $Z \in \text{Hom}(\Gamma, \mathbb{C})$ and $f \in \hat{g}$, consider connections of the form

$$d - \left( \frac{Z}{t^2} + \frac{f}{t} \right) dt$$

(5.1)

with a second order pole at $t = 0$ and a logarithmic pole at $t = \infty$. Any such connection is the inverse limit of a system of connections

$$d - \left( \frac{Z}{t^2} + \frac{f_{\leq k}}{t} \right) dt,$$

(5.2)

where $f_{\leq k}$ denotes the projection of $f$ in $g_{\leq k}$.

We note that any connection of the form (5.1) induces a $D^*(\hat{g})$-valued connections in $\mathbb{P}^1$ using the adjoint representation. We will go back to this when we relate the BTL isomonodromic family to the original construction in [BT1], and also to our family of connections $\nabla(Z)$ in Section 6.

5.2. The Bridgeland-Toledano-Laredo Theorem. The methods of [BT1] apply in our context leading to an existence result for connections of the form (5.1) with prescribed Stokes factors.

Let $(Z', a(Z'))$ be a continuous family in $\text{Stab}(g)$ parametrised by an open set $U \subset \text{Hom}(\Gamma, \mathbb{C})$. We denote by $U_{\text{reg}} \subset U$ the open subset given by elements $Z'$ corresponding to strongly generic stability data (see Definition 3.3). Without
loss of generality, we assume that the image of $\Gamma_{\geq 0}$ by any $Z' \in U$ lies in the lower-half plane $-\mathbb{H}$.

**Theorem 5.1 (BT1).** For any $Z' \in U_{\text{reg}}$, setting $Z = -Z'$, there exists a unique connection

$$\nabla^{\text{BTL}}(Z) = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t}\right) dt$$

(5.3)

with Stokes factors $S_\ell$, given by (2.2), along the set of Stokes rays $\ell_\gamma(Z') = \mathbb{R}_{>0}Z(\gamma)$, with $\gamma \in \Gamma$. The residue $f(Z)$ has only positively graded components $f(Z) = \sum_{\gamma \in \Gamma_{\geq 0}} f_\gamma$, explicitly given by

$$f_\alpha(Z) = \sum_{n \geq 1} \sum_{\alpha_j \in \Gamma_{\geq 0}} J_n(Z(\alpha_1), \ldots, Z(\alpha_n)) a(\alpha_1) \otimes \cdots \otimes a(\alpha_n),$$

(5.4)

where $J_n : (\mathbb{C}^*)^n \to \mathbb{C}$ are suitable holomorphic functions with branchcuts and $\otimes$ denotes the product in the universal enveloping algebra $U\mathfrak{g}$. Furthermore, as $Z'$ varies in $U$, the connections $\nabla^{\text{BTL}}(Z)$ extend to an isomodromic family of connections with holomorphic dependence on $Z$.

Following ideas of Bulsar-Jurkat-Lutz [BJL], the proof follows from the application of the Fourier-Laplace transform to a connection of the form (2.2), and the study of the analytic continuation of solutions of the corresponding Fuchsian connection. Using the well-known fact that the monodromy of such connections can be expressed in terms of multilogarithms, Tannaka duality leads to a formula for the Stokes map [BT2 Thm. 4.7]. Recall that this map sends the irregular connection (5.1), determined by $Z \in \text{Hom}(\Gamma, \mathbb{C})$ and $f \in \mathfrak{g}$, to its Stokes data. For fixed $Z$, it can be seen as a map

$$S(Z) : \mathfrak{g} \to \hat{\mathfrak{g}} : f \mapsto \epsilon,$$

where the Stokes factors are $S_\ell = \exp(\sum_{\alpha \in \Gamma_{\geq 0}} \epsilon_{\alpha})$ for $\epsilon = \sum_{\alpha \in \Gamma_{\geq 0}} \epsilon_{\alpha}$. Explicitly, the Stokes map is given (in a suitable open subset of $U_{\text{reg}}$) by

$$\epsilon_{\alpha} = \sum_{n \geq 0} \sum_{\alpha_j \in \Gamma_{\geq 0}} M_n(Z(\alpha_1), \ldots, Z(\alpha_n)) f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_n},$$

(5.5)

where $f = \sum_{\alpha \in \Gamma_{\geq 0}} f_\alpha$. The multilogarithm functions $M_n : (\mathbb{C}^*)^n \to \mathbb{C}$ are holomorphic and given by iterated integrals (see [BT1] Definition 4.4). Formula (5.4) follows from a universal formula for the Taylor series of $S(Z)^{-1}$ around $f = 0$ [BT2 Th. 4.8], that involves sums of multilogarithms indexed by finite trees. Crucially, by the positivity property of the stability data the sum in the right hand side of (5.4) is finite, and hence the Taylor series of $S(Z)^{-1}$ yields a global inverse in this case.

The functions $J_n$ in (5.4) are holomorphic with branchcuts, making the expression $J_n(Z(\alpha_1), \ldots, Z(\alpha_n))$ a well-defined holomorphic function on the complement of a divisor in $U_{\text{reg}}$ (see [BT2 Thm. 4.9]). The remarkable point is that the discontinuities of $J_n$ precisely balance the specific jumping behaviour of $a(Z)$ in the continuous family of stability data, thus resulting in a continuous,
holomorphic function $f_\alpha(Z)$ in $\mathcal{U}$. By results of Jimbo-Miwa-Ueno \cite{JMU} and Boalch \cite{BT1}, the isomonodromy condition for the holomorphic family $\nabla^{BTL}$ can be recast in terms of Joyce’s differential equation \cite{J5}:

$$df_\alpha = \sum_{\gamma, \beta \in \Gamma \geq 0} [f_\beta, f_\gamma] d\log \gamma.$$ 

**Remark 5.2.** We should stress that formula (5.5) for the Stokes map would not be valid if the residue $f$ has non-zero component in the centralizer of $Z$, that is, the condition $f \in [Z, \hat{b}] = \hat{a}$ (which ensures nilpotency of the residues of the Laplace transformed connection \cite[Sec. 8.2]{BT1}) is essential to prove (5.5) and hence also (5.4) and the uniqueness part of Theorem 5.1.

In the original approach \cite{BT1}, the connections $\nabla^{BTL}(Z)$ take values in the Hall algebra $\text{CF}(\mathcal{C})$ of constructible functions of an abelian category $\mathcal{C}$ with Grothendieck group $\Gamma$. In this context, the right hand side of (5.4) can be identified with the holomorphic generating function for $Z$-semistables in class $\alpha$ introduced by Joyce \cite{J5}. The next four sections are devoted to explain the relation between the connections $\nabla^{BTL}$ in Theorem 5.1 and the Hall-algebra valued connections of \cite{BT1}.

### 5.3. Motivic Hall algebras

We shall restrict ourselves to the case where $\mathcal{C}$ is the abelian category of finite-dimensional representations of a quiver $Q$ without oriented cycles. We work over the field of complex numbers $\mathbb{C}$. It should be possible to generalize our construction to more general abelian categories, but for the sake of simplicity we focus on this well-behaved case. Since the category $\mathcal{C}$ has finite length and finitely many simple modules $S_1, \ldots, S_n$ up to isomorphism (corresponding to the vertices of $Q$), the Grothendieck group $\Gamma = K(\mathcal{C})$ is a rank $n$ lattice, with non-negative cone $\Gamma_{\geq 0}$. We denote by $\Gamma_{>0} = \Gamma_{\geq 0} \setminus \{0\}$.

Given an integer $d \geq 0$, there is an affine variety $\text{Rep}_d$ parameterising $A$-module structures on the vector space $\mathbb{C}^d$. The moduli stack $\mathcal{M}_d$ of $A$-modules of dimension $d$ is the quotient

$$\mathcal{M}_d = \text{Rep}_d / \text{GL}_d(\mathbb{C}).$$

More generally, we can consider the moduli stack of objects in $\mathcal{C}$

$$\mathcal{M} = \bigsqcup_{\gamma \in \Gamma_{\geq 0}} \mathcal{M}_\gamma$$

where $\mathcal{M}_\gamma$ denotes the moduli stack of objects of class $\gamma \in \Gamma$.

Let $\text{SF}(\mathcal{C})$ be the motivic Hall algebra of the category $\mathcal{C}$, as defined by Joyce \cite{J2}. Here we follow closely \cite{Br3}. This is an associative, unital algebra over $\mathbb{C}$ with underlying vector space the relative Grothendieck group $K(\text{St} / \mathcal{M})$. As a vector space, it is generated by classes of morphisms

$$[f : X \to \mathcal{M}],$$

where $X$ is an algebraic stack of finite type over $\mathbb{C}$ with affine stabilizers. The associative product $*$ on $K(\text{St} / \mathcal{M})$ constructed in \cite{J2} endowes $\text{SF}(\mathcal{C})$ with a
structure of graded algebra for the semi-group \( \Gamma \geq 0 \)

\[
\text{SF}(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma \geq 0} \text{SF}_\gamma(\mathcal{C}),
\]

where \( \text{SF}_\gamma(\mathcal{C}) \) is the vector space generated by classes \([f : X \to \mathcal{M}_\gamma]\). We will need later that \( \text{SF}(\mathcal{C}) \) is an algebra over \( K(\text{St}/\mathcal{C}) \) and the explicit description (see [Br3 Lem. 3.9])

\[
K(\text{St}/\mathcal{C}) = K(\text{Var}/\mathcal{C})[[\text{GL}_d]^{-1} : d \geq 1].
\]

There is a canonical graded Lie subalgebra with respect to the commutator bracket \([J4, \text{Cor. 5.6}]\)

\[
\mathfrak{p}(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma > 0} \mathfrak{p}_\gamma(\mathcal{C}) \subset \text{SF}(\mathcal{C}),
\]

generated as a Lie algebra by sets of special elements (see Remark 5.3)

\[
\mathfrak{p}(\mathcal{C}) = \langle \tau_\alpha : \alpha \in \Gamma > 0 \rangle \subset \text{SF}(\mathcal{C}).
\]

By definition, \( \mathfrak{p}(\mathcal{C}) \) is a quotient of the \( \Gamma > 0 \)-graded free Lie algebra with symbols \( e_\alpha, \alpha \in \Gamma > 0 \) and hence it is a pro-nilpotent Lie algebra with finite-dimensional homogeneous components (cf. Section 3.4). Let \( U\mathfrak{p}(\mathcal{C}) \) be the universal enveloping algebra of \( \mathfrak{p}(\mathcal{C}) \). Let \( \hat{\mathfrak{p}}(\mathcal{C}) \) denotes the completion of \( \mathfrak{p}(\mathcal{C}) \) with respect to the grading (cf. Section 3.4) and \( U\hat{\mathfrak{p}}(\mathcal{C}) \) the corresponding universal enveloping algebra. Exponentiation in \( U\hat{\mathfrak{p}}(\mathcal{C}) \) leads to a pro-unipotent Lie group with Lie algebra \( \hat{\mathfrak{n}}(\mathcal{C}) \)

\[
\mathcal{N}' \subset U\hat{\mathfrak{n}}(\mathcal{C}).
\]

To define the structure group of the (trivial) principal bundle we are interested in, we note that there is a unique surjective algebra morphism (see [J3 p. 66])

\[
U\mathfrak{p} \to \mathcal{C}(\mathcal{C}) := \langle 1, \tau_\alpha : \alpha \in \Gamma > 0 \rangle \subset \text{SF}(\mathcal{C}),
\]

which extends the identity on \( \mathfrak{p}(\mathcal{C}) \). We define the pro-unipotent Lie group \( \hat{\mathcal{N}}(\mathcal{C}) \) with Lie algebra \( \hat{\mathfrak{n}}(\mathcal{C}) \), to be the exponentiation of \( \mathfrak{n}(\mathcal{C}) \) in (the completion of) the algebra \( \mathcal{C}(\mathcal{C}) \). As in Section 5.1 one can use the grading of \( \mathfrak{n}(\mathcal{C}) \) to form a larger pro-solvable Lie group

\[
\hat{B}(\mathcal{C}) = \text{Hom}(\Gamma, \mathbb{C}^*) \ltimes \hat{\mathcal{N}}(\mathcal{C})
\]

with pro-solvable Lie algebra \( \hat{\mathfrak{b}}(\mathcal{C}) = \text{Hom}(\Gamma, \mathbb{C}) \ltimes \hat{\mathfrak{n}}(\mathcal{C}) \).

Remark 5.3. In the notation of [J3 Def. 8.9], the algebras \( \mathfrak{p}(\mathcal{C}) \) and \( \mathcal{C}(\mathcal{C}) \) correspond respectively to \( \mathcal{L}_r \) and \( \mathcal{H}_r \).

5.4. Constructible functions and quantum groups. To establish the relation with [BT1], consider the Ringel-Hall algebra of \( \mathcal{C} \)-valued constructible functions on the moduli stack \( \mathcal{M} \)

\[
\text{CF}(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma > 0} \mathcal{H}_\gamma(\mathcal{C}),
\]
where $\mathcal{H}_\gamma(C)$ is the subspace of functions supported on modules of class $\gamma$. As a vector space, $\text{CF}(C)$ is generated by $\text{GL}_d(C)$-invariant constructible functions on $\text{Rep}_d$ for $d \geq 0$. There is a surjective morphism $[J2, \text{Def. } 2.7 \& \text{Thm. } 5.2]$

$$\pi : \text{SF}(C) \to \text{CF}(C) \quad (5.7)$$

and also a natural injection

$$\iota : \text{CF}(C) \to \text{SF}(C) \quad (5.8)$$

satisfying $\pi \circ \iota = \text{Id}$, which in general does not preserve the product (see $[J2, \text{p. } 32]$). The morphism $\pi$ induces surjective algebra morphism from $\mathfrak{n}(C)$ and $C(C)$ to, respectively, the Lie algebra of constructible functions supported on indecomposables $\mathfrak{n}(C)$ considered in $[BT1, \text{Sec. } 4.4]$ and its universal enveloping algebra $C(C)$ $[BT1, \text{Prop. } 4.5]$ (see $[J4, \text{p. } 21]$).

**Remark 5.4.** An important difference between $\mathfrak{n}(C)$ and $C(C)$ is that the universal enveloping algebra $U\mathfrak{n}(C)$ is not embedded in $\text{SF}(C)$, unlike $C(C) \subset \text{CF}(C)$. Instead, one has the surjective morphism $(5.6)$.

Following $[J2, \text{Ex. } 4.25 \& 5.20]$, we can make most of the previous construction very explicit for our choice of abelian category $C$. As for the construction of the motivic connections, the rest of this section can be safely skipped. Let

$$\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

be the Kac-Moody Lie algebra corresponding to the undirected graph underlying the quiver $Q$. Then, $\mathfrak{n}(C)$ is isomorphic to the positive part $\mathfrak{n}_+$ and $C(C)$ is isomorphic to its universal enveloping algebra $U\mathfrak{n}_+$.

The algebras $\mathfrak{n}(C)$ and $C(C)$ provide a “quantized” version of the algebras $\mathfrak{n}_+$ and $U\mathfrak{n}_+$. To see this, we assume that $Q$ is a Dynkin quiver, so that the Lie algebra $\mathfrak{n}_+$ is finite dimensional. We relate a natural quotient of $C(C)$ with Drinfeld’s quantum group $[Dr, \text{Ex. } 6.2]$

$$U_q\mathfrak{n}_+.$$  

Recall that $U_q\mathfrak{n}_+$ is $q$-deformation of the universal enveloping algebra $U\mathfrak{n}_+$, and that $U\mathfrak{n}_+$ is recovered in the “semi-classical” limit $q \to 1$. Let $\mathbb{C}(q^{1/2})$ be the algebra of rational functions in $q^{1/2}$ with coefficients in $\mathbb{C}$ and

$$P : K(\text{St}_/C) \to \mathbb{C}(q^{1/2})$$

be the (unique extension of the) virtual Poincaré polynomial $[JT, \text{Ex. } 4.3]$ (see also $[J2, \text{Th. } 2.21]$). Consider the $\mathbb{C}(q^{1/2})$-module (see $[Br3, \text{Rem. } 3.11]$)

$$\text{SF}(C, P, \mathbb{C}(q^{1/2})) = \text{SF}(C) \otimes_{K(\text{St}_/C)} \mathbb{C}(q^{1/2}).$$

(5.10)

According to $[J2, \text{Th. } 5.2 \& \text{Rem. } 6.5]$, this space can be endowed with an associative product which naturally identifies it with a quotient of $\text{SF}(C)$. We denote by

$$\overline{C}(C, P, \mathbb{C}(q^{1/2})) \subset \text{SF}(C, P, \mathbb{C}(q^{1/2}))$$

the subalgebra induced by $\overline{C}(C)$. Here we note that the algebra $\overline{C}(C)$ coincides with the composition algebra, that is, the subalgebra of $\text{SF}(C)$ generated by the
characteristic functions of simple modules $\iota(\kappa[S_i])$ (see Remark 5.10). Then, there is an isomorphism $U_qn_+ \cong \overline{\mathcal{C}}(C, P, \mathcal{C}(q^{1/2}))$ and we obtain that there is a surjective algebra morphism

$$\overline{\mathcal{C}}(C) \to U_qn_+.$$  

5.5. Motivic extension of Theorem 5.1. Following [BT1], we define a stability condition in $\mathcal{C}$ as a group homomorphism $\mathcal{Z}: \Gamma \to \mathbb{C}$ such that $\mathcal{Z}(\Gamma_{>0}) \subset \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane. Let $\text{Stab}(\mathcal{C})$ denote the set of all stability conditions on $\mathcal{C}$. Since the positive cone $\Gamma_{>0}$ is generated by the classes of the simple modules $S_1, \ldots, S_n$ there is a bijection

$$\text{Stab}(\mathcal{C}) \cong \mathbb{H}^n$$  

and we may therefore regard $\text{Stab}(\mathcal{C})$ as a complex manifold.

Given $Z \in \text{Stab}(\mathcal{C})$ Joyce constructs in [J3, Definition 8.1] positive stability data $(Z, \tau(Z))$ in (the $\mathbb{Q}$-points of) the graded Lie algebra $\mathfrak{n}(\mathcal{C})$ as follows. For any $\gamma \in \Gamma_{>0}$ define $\delta_\gamma \in \text{CF}_\gamma(\mathcal{C})$ to be the characteristic function of $Z$-semistables modules of class $\gamma$. Recall that a module $M$ is $Z$-semistable if it is nonzero and

$$\arg Z([N]) \leq \arg Z([M])$$  

for any submodule $0 \neq N \subset M$. Note that the function $\delta_\gamma$ depends on $Z$ in a discontinuous way because of wall-crossing behaviour. We set

$$\overline{\delta}_\gamma = \iota(\delta_\gamma) \in \text{SF}_\gamma(\mathcal{C}),$$  

using the injection (5.8). Given $\alpha \in \Gamma_{>0}$, define $\tau_\alpha = \tau(Z)(\alpha) \in \text{SF}_\alpha(\mathcal{C})$ by the finite sum

$$\tau_\alpha = \sum_{n>0} \sum_{\gamma_1 + \ldots + \gamma_n = \alpha} (\frac{-1}{n})^{n-1} \overline{\delta}_{\gamma_1} \ast \ldots \ast \overline{\delta}_{\gamma_n}. \quad (5.11)$$  

Formula (5.11) defines a rational element in $\mathfrak{n}(\mathcal{C})$ and, as $Z$ varies in $\text{Stab}(\mathcal{C})$, the set

$$\{(Z, \tau(Z)) : Z \in \text{Stab}(\mathcal{C})\} \subset \text{Stab}(\mathfrak{n}(\mathcal{C}))$$  

(5.12)

defines a continuous family of stability data for the pro-nilpotent group $\hat{\mathcal{N}}(\mathcal{C})$. Continuity of (5.12) follows from the Harder-Narasimhan property for elements in $\text{Stab}(\mathcal{C})$ and the definition of the product in $\text{SF}(\mathcal{C})$, as in the proof of [BT1 Th. 6.6] (cf. [J4 Cor. 5.6]).

Remark 5.5. We note that $\mathfrak{n}(\mathcal{C})$ is defined in [J3 Def. 8.9] as the Lie subalgebra of $\text{SF}(\mathcal{C})$ generated by the elements $\tau_\alpha$, with $\alpha \in \Gamma_{>0}$. In [J4 Cor. 5.6] it is proved that it is independent of $Z \in \text{Stab}(\mathcal{C})$ (under suitable assumptions that hold in our case).

Using the methods of [BT1], the next result constructs the motivic Bridgeland-Toledano-Laredo isomonodromic family for the category $\mathcal{C}$. 
Theorem 5.6. The analogue of Theorem 5.1 holds. It leads to a unique holomorphic, isomonodromic family $\nabla^C$ of irregular connections on the holomorphically trivial principal $\hat{B}(C)$-bundle on $\mathbb{P}^1$ with Stokes data determined by (5.12). The family of connections in [BT1] is induced from $\nabla^C$ using the morphism (5.7).

Proof. The proof is in two steps. First, given a ray $\ell \subset \mathbb{C}^*$, define an element $SS'_\ell(Z)$ in the group $N'$ by

$$SS'_\ell(Z) = \exp \left( \sum_{\alpha \in \Gamma \geq 0, Z(\alpha) \in \ell} \epsilon_\alpha \right) \in N',$$

where the exponential is taken in the universal enveloping algebra $U(\mathfrak{g}(\mathbb{C}))$. The methods of [BT1] (summarised in our sketchy proof of Theorem 5.1) imply now the existence of a unique holomorphic family of irregular connections $\nabla'(Z) = d - \left( \frac{Z}{t^2} + \frac{f'(Z)}{t} \right) dt$ with stokes factors $SS'_\ell$ and residue $f'(Z) \in \hat{n}(\mathbb{C})$ given by the analogue of (5.4). Note that $\nabla'$ is not necessarily isomonodromic, as the condition that (2.3) is constant may not hold in the group $N'$.

The second step deals with the lack of isomodromy of $\nabla'$. We use the surjective morphism (5.6) to induce from $\nabla'(Z)$ a family of $\hat{B}(C)$-connections $\nabla^C(Z) = d - \left( \frac{Z}{t^2} + \frac{f^C(Z)}{t} \right) dt$ (5.13) with residue $f^C(Z) \in \hat{n}(\mathbb{C})$. The fundamental solutions of (5.13) are $\hat{B}(C)$-valued, induced from the $\text{Hom}(\Gamma, \mathbb{C}^*) \rtimes N'$-valued fundamental solutions of $\nabla'(Z)$ using (5.6). This implies that the stokes factors of $\nabla^C(Z)$ along the set of Stokes rays $\ell_\gamma(Z)$, with $\gamma \in \Gamma$, are now given by

$$SS_\ell(Z) = \exp \left( \sum_{\alpha \in \Gamma \geq 0, Z(\alpha) \in \ell} \epsilon_\alpha \right) = 1 + \sum_{\gamma \in \Gamma \geq 0, Z(\gamma) \in \ell} \delta_\gamma \in \hat{N}(\mathbb{C}),$$

where the exponential is taken in $\mathfrak{g}(\mathbb{C})$, and hence the family (5.13) is isomonodromic. The claimed relation between $\nabla^C$ and the isomonodromic family of connections constructed in [BT1] is straightforward by construction.

Remark 5.7. Note that $\nabla^C(Z)$ and $\nabla'(Z)$ are equal as $\hat{n}(\mathbb{C})$-valued 1-forms on $\mathbb{P}^1$, that is, $f^C(Z) = f'(Z)$ since (5.6) is the identity restricted to $\hat{n}(\mathbb{C})$. The effect of applying this morphism if simply to express $f'(Z)$ in terms of the product $\ast$ in $\mathfrak{g}(\mathbb{C})$ rather than $\otimes$. Although this may seem confusing at first, the point is that they are different as connections on $\mathbb{P}^1$, with fundamental solutions taking values in different groups.
5.6. The semi-classical limit. We use now an integration morphism due to Joyce [J2] combined with arguments of Reineke [R2] to induce the connections (5.3) from the universal family $\nabla_C$. For this, we regard (5.3) as $D^*\hat{g}$-valued connections using the adjoint representation.

Consider the Grothendick group $\Gamma = K(C)$ endowed with the skew-pairing $\langle \cdot, \cdot \rangle$, given by skew-symmetrisation of the euler form

$$\chi(N,M) = \sum_j \dim \text{Ext}^j(N,M).$$

Let $g(q)$ be the non-commutative, associative, $\Gamma$-graded algebra over $\mathbb{C}[q^{1/2}]$ generated by symbols $\hat{e}_\gamma$ for $\gamma \in \Gamma$, with product

$$\hat{e}_\gamma \ast \hat{e}_\eta = q^{\frac{1}{2}\langle \gamma, \eta \rangle} \hat{e}_{\gamma + \eta}.$$

Using the virtual Poincare polynomial (5.9), in [J2, Th. 6.4] (cf. [J2, Rem. 6.5]) Joyce defines a morphism of algebras

$$\Phi: SF(C) \to g(q) \quad (5.14)$$

(factorizing through (5.10)). Given a constructible set $C \subset \mathcal{M}_\gamma$, the morphism is such that

$$\Phi \circ \iota(\kappa_C) = P([C])\hat{e}_\gamma,$$

where $\kappa_C \in \text{CF}_\gamma(C)$ is the characteristic function of $C$ and $[C] \in K(\text{St}/\mathbb{C})$.

Let $g$ be the Poisson Lie algebra corresponding to $(\Gamma, \langle \cdot, \cdot \rangle)$, constructed as in Section 3.3. Let $g_q \subset g(q)$ be the $\mathbb{C}[q^{1/2}]$-module generated by the $\hat{e}_\gamma$ for $\gamma \in \Gamma$. Then, $g_q$ is a subalgebra of $g(q)$ endowed with a Poisson Lie bracket (cf. [Br3, Sec. 5.1])

$$\{f,g\} = \frac{f \ast g - g \ast f}{q - 1}$$

and there is an isomorphism of Poisson Lie algebras

$$\text{sc}: g \cong g_q/(q - 1)g_q. \quad (5.15)$$

Informally, we think of $g$ as a degeneration of $g_q$ in the limit $q \to 1$

$$\lim_{q \to 1} (q - 1)^{-1}[\hat{e}_\gamma, \hat{e}_\eta] = [e_\gamma, e_\eta] = \langle \gamma, \eta \rangle e_{\gamma + \eta}.$$

We wish to combine now the morphism (5.14) with the isomorphism (5.15), to induce the connections (5.3) from (5.13). The following statement, which is a consequence of deep results of Joyce, shows that this can be done upon composition of $\Phi$ with the adjoint representation. Let $Z \in \text{Stab}(C)$ be a stability condition and consider $\tau_\alpha \in \pi_\alpha(C)$ as in (5.11).

**Lemma 5.8.** The adjoint action of $\Phi(\tau_\alpha)$ preserves $g_q$.

The proof follows from [J2 Th. 8.7], that implies $(q - 1)\Phi(\tau_\alpha) \in g_q$, combined with (5.15), which gives that $g_q$ is abelian modulo $(q - 1)g_q$. To get a feeling, suppose that $Z \in \text{Stab}(C)$ such that the semi-stable objects of class $\gamma$ are stable.
Then, the moduli space $\mathcal{M}_{\gamma}^{ss}$ of semi-stable representations of class $\gamma$ is a smooth projective variety and (see [R1])

$$\Phi(\delta_{\gamma}) = P(\mathcal{M}_{\gamma}^{ss}) \frac{P}{q-1} \hat{e}_\gamma,$$

where now $P(\mathcal{M}_{\gamma}^{ss})$ is simply the Poincare polynomial in singular cohomology. Explicit combinatorial formulae for this quantity can be found in [R1, J2]. Choosing $\gamma = [S_i]$, the class of a simple element, we have $\tau_\gamma = \delta_{S_i}$ which clearly satisfies the statement.

Consider the push-forward of derivations in the associative algebra $(\mathfrak{g}, *)$ induced by (5.15)

$$sc: D^*(\mathfrak{g}_q) \rightarrow D^*(\mathfrak{g})$$

and the partially defined Lie algebra morphism

$$I = sc \circ \text{ad} \circ \Phi: \mathfrak{m}(\mathcal{C}) \rightarrow D^*(\mathfrak{g}).$$

By Lemma 5.8, (5.16) is well-defined on elements of the continuous family (5.12) and hence yields a continuous family of stability data on $D^*(\mathfrak{g})$. Furthermore, as the formula for the residue of the connection (5.13) is a Lie series in the elements $\tau_\alpha$ (see [BT1, Th. 3.7]), the morphism (5.16) is well-defined on $\mathfrak{f}_C(Z)$. This lead us to the following result.

**Corollary 5.9.** The universal connection $\nabla^C$ induces a holomorphic, isomonodromic family of $D^*(\mathfrak{g})$-valued connections, with prescribed Stokes factors

$$\text{Ad} S_\ell(Z) = \exp \left( \sum_{\alpha \in \Gamma_{>0}} \sum_{\ell(\alpha) \in \ell} I(\tau_\alpha) \right) \in \text{Aut}^*(\hat{\mathfrak{g}}).$$

(5.17)

An explicit, combinatorial formula for (5.17) was provided by Reineke [R1, R2].

We note that it is not obvious that the methods of [BT1] apply directly to the induced continuous family on $\text{Stab}(D^*(\mathfrak{g}))$. The problem may come from elements in the stability data that commute with $Z$ (cf. Remark 5.2). To analyze a simple case, assume that $Q$ is a Dynkin quiver. Then, we can choose a strongly generic $Z \in \text{Stab}(\mathcal{C})$ such that the only stable representations are the simples $S_i$, corresponding to the vertices of $Q$, and semi-stables correspond to direct sums of a unique stable object $[K]$. In this case, the non-trivial Stokes factors correspond to $\ell_i = \mathbb{R}_{>0}Z([S_i])$ and (5.17) reduces to

$$\text{Ad} S_{\ell_i}(Z) = T_{\gamma_i}.$$ 

This shows that, at least for Dynkin quivers, the continuous family in $\text{Stab}(D^*(\mathfrak{g}))$ is induced from a continuous family in $\text{Stab}(\mathfrak{g})$ via the adjoint representation, and hence the connections in Corollary 5.9 are induced from (5.3).

**Remark 5.10.** The previous choice of stability data shows that the algebra $\mathcal{O}(\mathcal{C})$ is generated by characteristic functions of simple modules.
Remain 5.11. Motivation for considering $\text{SF}(\mathcal{C})$ instead of $\text{CF}(\mathcal{C})$ in our discussion comes from the fact that the inclusion (5.8) is not a morphism (see [J2, p. 32]). Thus, the integration morphism $\Phi$ cannot be applied directly to the $\text{CF}(\mathcal{C})$-valued connection in [BT1] to induce (5.3).

6. $\nabla^{BTL}(Z)$ as a scaling limit of $\nabla(Z, R)$

6.1. In this Section (6.2 - 6.7) we prove Theorem 2.1. Consider the rescaled connections

$$\nabla_t(Z, R) = d - \left( \frac{1}{t^2} R^{-1} A^{(-1)}(RZ) + \frac{1}{t} A^{(0)}(RZ) + RA^{(1)}(RZ) \right) dt.$$  

The $R \to 0$ limit of $\nabla_t(Z, R)$ does not exist in this fixed gauge. This is already apparent in the single-ray solution discussed in 4.11: we have

$$R^{-1} A^{(-1)} = Z - Z(\gamma) \sum_{k>0} \frac{1}{4k\pi i} \int_{\ell_{\gamma}^{}} \frac{dz'}{z'^2} \exp(L(z')(k\gamma)) \text{ad}(e_{k\gamma}).$$

The integrals $\int_{\ell_{\gamma}^{}} \frac{dz'}{z'^2} \exp(L(z')(k\gamma))$ are real, positive and diverge logarithmically as $R \to 0$. Another example of $R \to 0$ singularity with multiple rays will be discussed in 6.8 at the end of this section.

In 6.2 - 6.6 below, we construct a sequence of constant gauge transformations $g(R)$ such that $\lim_{R \to 0} g(R) \cdot \nabla_t(Z, R)$ exists and has the form

$$\hat{\nabla}_t = d - \left( -\frac{Z}{t^2} + \frac{\hat{I}}{t} \right) dt.$$  

In 6.7 we will compute the Stokes data for $\hat{\nabla}_t$ and deduce its actual equality with $\nabla^{BTL}(-Z)$.

6.2. An auxiliary integral operator. The sequence of gauge transformations $g(R)$ is given explicitly by

$$g(R) = \lim_{z \to 0} Y(z; Z, R),$$

that is

$$g(Z, R)(e_\alpha) = e_\alpha \exp(\langle \alpha, -\sum_T W_T \lim_{z \to 0} G_T(z; Z, R) \rangle).$$

Our strategy to show that the limit $\hat{\nabla}_t$ exists is to show first that the local flat sections of $\nabla(Z, R)$ given by the restriction of $X(z; Z, R)$ to a sector $\Sigma$ have a finite $R \to 0$ limit after gauging them by $g^{-1}(R)$. Fixing a sector $\Sigma$ between consecutive Stokes rays of $\nabla(Z, R)$ (in particular, working in a finite-dimensional quotient $g_{\leq k}$), we will show that the limit

$$\hat{X}(t) = \lim_{R \to 0} g^{-1}(R) X(Rt)$$

exists. Notice that

$$\lim_{R \to 0} X^0(Rt)(e_\alpha) = e_\alpha \exp(t^{-1} Z(\alpha)).$$
As \( X(Rt) \) is the composition \( Y(Rt)X^0(Rt) \) in \( \text{Aut}^*(\hat{g}) \), it is enough to consider the limit

\[
\lim_{R \to 0} g^{-1}(R)Y(Rt).
\]

To study this we consider the \( \text{Aut}^*(\hat{g}) \)-valued function \( h \), holomorphic in \( \Sigma \), given by

\[
h(t; Z, R) = g^{-1}(R)Y(Rt).
\]

Let us show that \( h(t; Z, R) \) is a fixed point for an integral operator which is very similar to \( Z \). By definition, for all \( \alpha \in \Gamma \) we have

\[
Y(Rt)(e_{\alpha}) = e_{\alpha} \ast \exp_*(\alpha, \sigma(Rt))
\]

where \( \sigma(Rt) \) is a sum of integrals of the form

\[
\frac{1}{4\pi i} \int_{\ell_{\alpha'}} \frac{dz' \ z'}{z' - z'} I(z', R),
\]

one for each graph \( T \) in a class of decorated, rooted trees. Here \( I(z', R) \) is some other iterated integral, with values in \( \Gamma \otimes \hat{g} \), of the form

\[
\alpha' \ DT(\alpha') \exp(R(Z(\alpha')z'^{-1} + \bar{Z}(\alpha')z'))e_{\alpha'} \ast r(z', R)
\]

for some \( g \)-valued function \( r(z', R) \). By rescaling, we can rewrite each term in (6.1) as

\[
\frac{1}{4\pi i} \int_{\ell_{\alpha'}} \frac{dz' \ z'}{z' - z'} I(Rz', R),
\]

so now \( I(Rz', R) \) has the form

\[
\alpha' \ DT(\alpha') \exp(\pi Z(\alpha')z'^{-1} + \pi R^2 \bar{Z}(\alpha')z')e_{\alpha'} \ast r(z', R).
\]

Next we compute

\[
\partial_t Y(Rt)(e_{\alpha}) = \langle \alpha, \sigma'(Rt) \rangle \ast Y(Rt)(e_{\alpha})
\]

where \( \sigma'(Rt) \) is again a sum of terms labelled by the same diagrams \( T \), of the form

\[
\frac{1}{4\pi i} \int_{\ell_{\alpha'}} \frac{dz' \ 2Rz'}{z'(z' - R)^2} I(z', R) = \frac{1}{2\pi i} \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} I(Rz', R).
\]

Going back to \( h(t) \), we have

\[
\partial_t h(t)(e_{\alpha}) = g^{-1}(R)\partial_t Y(Rt)(e_{\alpha})
= g^{-1}(R)(\langle \alpha, \sigma'(Rt) \rangle \ast Y(Rt)(e_{\alpha}))
= g^{-1}(R)(\langle \alpha, \sigma'(Rt) \rangle \ast h(t)(e_{\alpha})).
\]

In the last equation we used the algebra automorphism property of \( Y_{0}^{-1} \). By the same property, the factor \( g^{-1}(R)(\langle \alpha, \sigma'(Rt) \rangle) \) splits into a sum of terms of the form

\[
g^{-1}(R) \left( \frac{1}{2\pi i} \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} \langle \alpha, I(Rz', R) \rangle \right),
\]
We can turn this into the integral equation rewritten as
\[ \partial \text{domain } \Sigma \text{ starting from a fixed base point} \]
where the outer integral is computed along some path in the simply connected one for each \( T \). In each finite-dimensional quotient \( g_{\leq k} \), the latter term can be rewritten as
\[ \frac{1}{2\pi i} \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} g_0^{-1}(\langle \alpha, I(Rz', R) \rangle). \quad (6.2) \]
Recalling the iterative definition of \( I(Rz', R) \), this equals in turn
\[ \frac{1}{2\pi i} \langle \alpha, \alpha' \rangle \text{DT}(\alpha') \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} \exp(Z(\alpha')z'^{-1} + R^2 \bar{Z}(\alpha')z') g_0^{-1}(e_{\alpha'}\langle \alpha', I'(Rz', R) \rangle) \]
for some “residual” (iterated) integral \( I'(Rz', R) \). Indeed, the above rewriting can be seen as the operation of removing the root \( a \) (labelled by \( \alpha' \)) from the fixed tree \( T \) corresponding to \( (6.2) \), leaving a finite set of disconnected trees \( T \setminus \{a\} \). Notice that we allow the empty tree in this residual set, corresponding to a factor 1. Now we let the original \( T \) behind \( (6.2) \) vary among all trees with root \( a \) labelled by \( \alpha' \), and sum over all the corresponding integrals, getting for each fixed \( \alpha' \),
\[ \frac{1}{2\pi i} \langle \alpha, \alpha' \rangle \text{DT}(\alpha') \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} \exp(Z(\alpha')z'^{-1} + R^2 \bar{Z}(\alpha')z') g_0^{-1}(e_{\alpha'}\langle \alpha', \sum_{\text{disconnected}} I'(Rz', R) \rangle). \]
By a standard combinatorial principle,
\[ \sum_{\text{disconnected}} = \exp \left( \sum_{\text{connected}} \right), \]
so for each \( \alpha' \) the last integral equals
\[ \frac{1}{2\pi i} \langle \alpha, \alpha' \rangle \text{DT}(\alpha') \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} \exp(Z(\alpha')z'^{-1} + R^2 \bar{Z}(\alpha')z') h(z')(e_{\alpha'}). \]
The upshot is that we have found an integro-differential equation for \( h(t) \), namely
\[ \partial_t h(t)(e_\alpha) = \left( \frac{1}{2\pi i} \sum_{\alpha'} \langle \alpha, \alpha' \rangle \text{DT}(\alpha') \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t)^2} \exp(Z(\alpha')z'^{-1} + R^2 \bar{Z}(\alpha')z') f(z')(e_{\alpha'}) \right) * h(t)(e_\alpha). \]
We can turn this into the integral equation
\[ h(t)(e_\alpha) = e_\alpha \]
\[ * \exp \left( \frac{1}{2\pi i} \int_{t_0}^t dt' \sum_{\alpha'} \langle \alpha, \alpha' \rangle \text{DT}(\alpha') \int_{\ell_{\alpha'}} \frac{dz'}{(z' - t')^2} \exp(Z(\alpha')z'^{-1} + R^2 \bar{Z}(\alpha')z') h(z')(e_{\alpha'}) \right), \quad (6.3) \]
where the outer integral is computed along some path in the simply connected domain \( \Sigma \) starting from a fixed base point \( t_0 \).
6.3. **Fixed point.** Of course (6.3) looks quite similar to the integral equation for $X(z)$. The advantage is that now it is easy to compute $R \to 0$ limits. To see this we leave aside $h(t) = g^{-1}(R)Y(Rt)$ for a moment and consider solutions of (6.3) obtained by iteration. Since we have, for all $R > 0$,

$$\lim_{t \to 0} g^{-1}(R)Y(Rt) = I,$$

we look at the solution $\hat{h}(t)$ obtained by iteration starting from $h^0(t) = I$. We will then prove that when we choose $t_0 = 0$ (by a limiting argument) we have in fact $h = \hat{h}$.

**Remark 6.1.** Notice that, starting from $h^0(t) = I$, all the integrals

$$2 \int_{t_0}^{t} dt' \int_{\ell_\alpha} \frac{dz'}{(z' - t')^2} \exp(Z(\alpha')z'^{-1} + R^2 Z(\alpha')z')e_{\alpha'},$$

have a well defined $R \to 0$ limit, namely

$$2 \int_{t_0}^{t} dt' \int_{\ell_\alpha} \frac{dz'}{(z' - t')^2} \exp(Z(\alpha')z'^{-1})e_{\alpha'}.$$

So at least the first iteration from $h^0(t) = I$ has a well defined as $R \to 0$. This does not happen for the first iteration of the integral equation for $X$ starting from $X^0$: that is already divergent.

As in the case of $Z$, we have an expression for the iterative solution of (6.3), namely (with the usual notation)

$$\tilde{h}(t)(e_\alpha) = e_\alpha \exp(\alpha, - \sum_T W_T H_T(t)),$$

where

$$H_T(t) = e_{\gamma_T} \frac{1}{2\pi i} \int_{t_0}^{t} dt' \int_{\ell_{\gamma_T}} \frac{dz}{(z - t')^2} \exp(Z(\gamma_T)z^{-1} + R^2 \tilde{Z}(\gamma_T)z) \prod_j H_{T_j}(z).$$

(6.4)

Let us assume inductively that each $H_{T_j}(z)$ is of order $o(|z|^\varepsilon)$ as $z \to \infty$ for all $\varepsilon > 0$ (i.e. it grows less than any positive power), and that this holds uniformly in $R$ (this is certainly true for the identity). Similarly let us assume inductively that each $H_{T_j}(z)$ is bounded as $z \to 0$. Then one can show that the inner integral is convergent and in fact of order $O(|t'|^{-1}) \cdot o(|t'|^\varepsilon)$ for all $\varepsilon > 0$ as $t' \to \infty$ uniformly as $R \to 0$. At the same time it is uniformly bounded near $t' = 0$ for all $R$. In particular $H_T(t)$ does have a well-defined $R \to 0$ limit, namely just

$$\widehat{H}_T(t) = e_{\gamma_T} \frac{1}{2\pi i} \int_{t_0}^{t} dt' \int_{\ell_{\gamma_T}} \frac{dz}{(z - t')^2} \exp(Z(\gamma_T)z^{-1}) \prod_j \widehat{H}_{T_j}(z).$$

Going back to (6.4), we need to check the inductive hypotheses. But since the inner integral is $O(|t'|^{-1}) \cdot o(|t'|^\varepsilon)$ uniformly, $H_T(t)$ is $O(\log |t|) \cdot o(|t|^\varepsilon)$ for all $\varepsilon > 0$ as $|t| \to \infty$, which is again of order $o(|t|^\varepsilon)$ for all $\varepsilon > 0$. Also, the integrand is uniformly bounded as $t' \to 0$ for all $R$, so the same is true for $H_T(t)$ as $t \to 0$. 
The upshot of this is that the limit of $\tilde{h}(t)$ as $R \to 0$ exists and is given by

$$\lim_{R \to 0} \tilde{h}(t)(e_\alpha) = e_\alpha \exp\{\alpha, -\sum_T W_T \tilde{H}_T(t)\}. \quad (6.5)$$

**Example.** In the single-ray case of Section 4.11 we get

$$\tilde{h}(t)(e_\gamma) = e_\gamma,$$

while

$$\tilde{h}(t)(e_\eta) = e_\eta \ast \exp\left(-\sum_{k>0} \frac{1}{k} H_{k\gamma}(t)\right).$$

Now

$$\tilde{H}_{k\gamma}(t) = e_{k\gamma} * \frac{1}{2\pi i} \int_{t_0}^{t} dt' \int_{\ell_\gamma} \frac{dz}{(z-t')^2} \exp(kZ_{\gamma}z^{-1}),$$

and one can check that we can apply Fubini to get

$$\tilde{H}_{k\gamma}(t) = e_{k\gamma} * \frac{1}{2\pi i} \int_{\ell_\gamma} \frac{dz}{z-t} \exp(kZ_{\gamma}z^{-1}) \left[-\frac{1}{z-t}\right]_{t'=t}^{t'=t_0}.$$ 

By a limiting argument, we can take the base point $t_0 = 0$ and find

$$\tilde{H}_{k\gamma}(t) = e_{k\gamma} * \frac{1}{2\pi i} \int_{\ell_\gamma} \frac{dz}{z-t} \exp(kZ_{\gamma}z^{-1}).$$

**6.4. Application of Fubini’s theorem.** The argument above proves in fact that for all $R \geq 0$

$$\int_{t_0}^{t} \int_{\ell_{\gamma T}} dt' \frac{dz}{(z-t')^2} \exp(Z(\gamma T)z^{-1} + R^2 \bar{Z}(\gamma T)z) \prod_j H_{T_j}(z) \mid < \infty.$$ 

At the same time, by the definition of $\ell_{\gamma T}$, the integral

$$\int_{\ell_{\gamma T}} \int_{t_0}^{t} dt' \frac{dz}{(z-t')^2} \exp(Z_{\gamma T}z^{-1} + \pi R^2 \bar{Z}_{\gamma T}z) \prod_j H_{T_j}(z) \mid$$

equals

$$\int_{\ell_{\gamma T}} dz \exp(\pi Z(\gamma T)z^{-1} + \pi R^2 \bar{Z}(\gamma T)z) \prod_j H_{T_j}(z) \mid \int_{t_0}^{t} \frac{dt'}{(z-t')^2}.$$

Since the integration path from $t_0$ to $t$ is compact for a fixed $t$, the inner integral is $O(|z|^{-2})$ as $z \to \infty$. Therefore the second integral is also finite for all $R \geq 0$. Then for a fixed $t$ we can apply Fubini to rewrite

$$H_T(t) = e_{\gamma T} * \frac{1}{2\pi i} \left(t-t_0\right) \int \frac{dz}{(t-z)(t_0-z)} \exp(\pi Z(\gamma T)z^{-1} + \pi R^2 \bar{Z}(\gamma T)z) \prod_j H_{T_j}(z).$$
By a limiting argument, we can choose $t_0 = 0$ (which strictly speaking is not in $\Sigma$), and find for the corresponding solutions

$$H_T(t) = e_{\gamma_T} \frac{1}{2\pi i} t \int_{\ell_{\gamma_T}} \frac{dz}{z} \frac{1}{z - t} \exp(\pi Z(\gamma_T) z^{-1} + \pi R^2 Z(\gamma_T) z) \prod_j H_{T_j}(z), \quad (6.6)$$

and

$$\tilde{H}_T(t) = e_{\gamma_T} \frac{1}{2\pi i} t \int_{\ell_{\gamma_T}} \frac{dz}{z} \frac{1}{z - t} \exp(\pi Z(\gamma_T) z^{-1}) \prod_j \tilde{H}_{T_j}(z). \quad (6.7)$$

6.5. When $t_0 = 0$ we have $\tilde{h}(t) = h(t) = g^{-1}(R) Y(Rt)$. To see this notice that both $h(t)$ and $\tilde{h}(t)$ are solutions to (6.3) (with $t_0 = 0$). One can check that this implies that $h^{-1}(t) \tilde{h}(t)$ (the composition in Aut($\tilde{\mathfrak{g}}$)) is holomorphic on $\mathbb{C}^*$. On the other hand both $h(t)$ and $\tilde{h}(t)$ are bounded as $t \to 0$ and $t \to \infty$. For $h(t)$ this follows from the same property for $Y(Rt)$, while for $\tilde{h}(t)$ it follows from (6.6) above. So $h^{-1}(t) \tilde{h}(t) \in$ Aut($\tilde{\mathfrak{g}}$) is a constant, and since $\lim_{t \to 0} h^{-1}(t) \tilde{h}(t) = I$ by construction, it must be the identity. It follows in particular that $h(t)$ has a finite limit in Aut($\tilde{\mathfrak{g}}$) as $R \to 0$, which we denote by $\tilde{h}(t)$.

6.6. $\vec{\nabla}_t$ from limit flat sections. We are now in a position to set

$$\vec{X}(t) = \lim_{R \to 0} g^{-1}(R) X(Rt) = \lim_{R \to 0} g^{-1}(R) Y(Rt) X_0(Rt) = \tilde{h}(t) \vec{X}_0(t),$$

where

$$\vec{X}_0(t) = e_{\alpha} \exp(t^{-1} Z(\alpha)).$$

Notice that $\vec{X}_0(t)$ is naturally a flat section for the connection

$$\vec{\nabla}_0 = d - \left( -\frac{Z}{t^2} \right) dt.$$

This implies that, in each sector $\Sigma$, $\vec{X}(t; Z)$ is a flat section of the pullback connection $\vec{\nabla}|_{\Sigma} = \tilde{h}^{-1}(t)|_{\Sigma} \cdot \vec{\nabla}_0$. By precisely the same argument as in section 4, the $\vec{\nabla}|_{\Sigma}$ for various $\Sigma$ glue to a connection on $\mathbb{C}^* \subset \mathbb{P}^1$ in each finite-dimensional quotient $\mathfrak{g}_k$, and taking an inverse limit we find a well-defined $D^*(\tilde{\mathfrak{g}})$-valued connection $\vec{\nabla}$.

Notice that $\vec{\nabla}$ is in fact (a posteriori) the $R \to 0$ limit of the connections $g(R) \cdot \nabla_t$, which have the form

$$g(R) \cdot \nabla_t(Z) = d - \left( -\frac{1}{t^2} R^{-1} \tilde{A}^{-1}(R) \tilde{A}(0)(R) + \frac{1}{t} \tilde{A}(0)(R) + R \tilde{A}(1)(R) \right) dt$$

for some $t$-constant $D^*(\tilde{\mathfrak{g}})$-valued functions $\tilde{A}^{(i)}(R)$, such that the limits

$$\lim_{R \to 0} R^{-1} \tilde{A}^{(i)}(R)$$

exist. In fact, we claim that $\lim_{R \to 0} \tilde{A}^{(1)}(R)$ also exists, from which it follows that

$$\vec{\nabla}(Z) = d - \left( -\frac{Z}{t^2} + \frac{\tilde{f}}{t} \right) dt$$
where $\tilde{f} = \lim_{R \to 0} \tilde{A}^{(0)}(R)$. To prove the claim, we notice that our argument in section 6.2 is symmetric, in that it applies equally well to the different scaling limit $z = R^{-1}t$, $R \to 0$, with the same gauge transformations $g(R)$. The rescaled connections $\nabla$ connections take the form

$$\nabla_{z=t/R} = d - \left( -\frac{1}{t^{2}} R\bar{A}^{(-1)}(R) + \frac{1}{t} \bar{A}^{(0)}(R) + R^{-1} \bar{A}^{(1)}(R) \right) dt$$

and so

$$g(R) \cdot \nabla_{z=t/R} = d - \left( -\frac{1}{t^{2}} R\bar{A}^{(-1)}(R) + \frac{1}{t} \bar{A}^{(0)}(R) + R^{-1} \bar{A}^{(1)}(R) \right) dt$$

with well defined, finite limits $\lim_{R \to 0} R^{i} \tilde{A}^{(i)}(R)$.

Regarding $\tilde{h}(t)$ as a formal gauge transformation (i.e. an element of $\text{Aut}^{*}(\mathfrak{g}[[z]])$), we see that the formal equivalence type of $\nabla(t; Z)$ is $d + \frac{z}{R} dt$. Therefore the Stokes rays of $\nabla(t; Z)$ are $\ell_{\alpha}, \alpha \in \Gamma$. Thus the Stokes rays are independent of $R$. Notice also that since $\tilde{X}(t)$ is the $R \to 0$ limit of $g^{-1}(R)X(Rt)$, we have immediately for a Stokes ray $\ell \subset \mathbb{H}$ and a point $z_{0} \in \ell$

$$\tilde{X}(z_{0}^{+})(e_{\alpha}) = \tilde{X}(z_{0}^{-})(e_{\alpha}) \prod_{Z(\gamma) \in \ell} (1 - \tilde{X}(z_{0}; Z)(e_{\gamma}))^{\Omega(\gamma)(\gamma, \alpha)}$$

(informally, $\tilde{X}(z_{0}^{+}) = \tilde{X}(z_{0}^{-}) \circ \prod_{\gamma \in \ell} T_{\gamma}^{\Omega(\gamma)}$).

6.7. **Stokes factors at 0.** To compute the Stokes factors of $\nabla$ rigorously, we need to understand the analytic continuation of the limit $\tilde{h}(t)$ beyond a sector $\Sigma$. By (6.5) this amounts to understanding the continuation of the integrals $H_{T}(t)$.

We wish to prove the following: each $\tilde{H}_{T}(t)$ extends to an analytic function in the supersector $\tilde{\Sigma}$, which vanishes as $t \to 0$ in $\tilde{\Sigma}$. We argue by induction on the length of $T$. The result certainly holds for $\tilde{H}_{0}(t) = 1$. Next look at

$$\tilde{H}_{T}(t) = e_{\gamma T} * \frac{1}{2\pi i} t \int dz \frac{1}{z - t} \exp(\pi Z(\gamma T)z^{-1}) \prod_{j} \tilde{H}_{T_{j}}(z). \quad (6.8)$$

We assume without loss of generality that $\ell_{\alpha}$ lies in $\tilde{\Sigma}$. By induction, we can assume that we have already extended all the $\tilde{H}_{T_{j}}(z)$ to analytic functions on $\tilde{\Sigma}$, vanishing as $t \to 0$ in $\tilde{\Sigma}$. We write $H(z)$ for the product of these analytic continuations, i.e. on $\Sigma$,

$$H(z) = \prod_{j} \tilde{H}_{T_{j}}(z).$$

Write $Z = Z(\gamma T)$ for simplicity. Then setting $z = -Zs$ we have

$$\tilde{H}_{T}(t) = e_{\gamma T} * \frac{1}{2\pi i} t \int_{0}^{\infty} ds \frac{1}{s - Zs - t} \exp(-\pi s^{-1}) H(-Zs). \quad (6.9)$$

Upon the change of variable $s \to s^{-1}$, we may rewrite this as

$$\tilde{H}_{T}(t) = -e_{\gamma T} * \frac{1}{2\pi i} \int_{0}^{\infty} ds \frac{1}{s^{-1} - Z + s} \exp(-\pi s) H(-Zs^{-1}).$$
Set $\sigma = t^{-1}Z + s$ we get of course
\[
\hat{H}_T(t) = -e^{\gamma_T} \cdot \frac{1}{2\pi i} e^{\pi t^{-1}Z} \int_{t^{-1}Z}^{\infty} \frac{d\sigma}{\sigma} \exp(-\pi\sigma) H(-Z(\sigma - t^{-1}Z)^{-1}),
\]
where the integral is taken along the path $t^{-1}Z + \mathbb{R}_{>0}$. By our inductive assumptions on $H(z)$ (in particular since $H(-Z(\sigma - t^{-1}Z)^{-1}) \to 0$ as $\sigma \to \infty$ along the path), the integral
\[
e^{\pi t^{-1}Z} \int_{t^{-1}Z}^{\infty} \frac{d\sigma}{\sigma} \exp(-\pi\sigma) H(-Z(\sigma - t^{-1}Z)^{-1})
\]
is a holomorphic function of $t \in \hat{\Sigma}$, except for a branch cut discontinuity along the ray $t^{-1}Z \in \mathbb{R}_{<0}$, that is $\ell_\alpha$. However we can extend across this branch cut simply by rotating the integration path around its origin: we integrate along the path $t^{-1}Z + e^{\pm i\phi}\mathbb{R}_{>0}$ for some $\phi \in (-\pi/2, \pi/2)$ such that the ray
\[-Z((t^{-1}Z + e^{\pm i\phi}\mathbb{R}_{>0}) - t^{-1}Z)^{-1} = -Ze^{\mp i\phi}\mathbb{R}_{>0} = e^{\mp i\phi}\ell_\alpha
\]
lies in $\hat{\Sigma}$ (see Figure 3). By induction and since $\phi \in (-\pi/2, \pi/2)$ this analytic continuation has the same asymptotics as $t \to 0$ in $\hat{\Sigma}$. Indeed as $-Z((t^{-1}Z + e^{\pm i\phi}\mathbb{R}_{>0}) - t^{-1}Z)^{-1}$ lies in $\hat{\Sigma}$ we have $H(-Z(\sigma - t^{-1}Z)^{-1}) \to 0$ as $\sigma \to \infty$ along the new integration ray, uniformly as $t \to 0$, so it is enough to check that
\[
e^{\pi t^{-1}Z} \int_{t^{-1}Z + e^{\pm i\phi}\mathbb{R}_{>0}} \frac{d\sigma}{\sigma} \exp(-\pi\sigma)
\]
also vanishes as $t \to 0$. But since $\phi \in (-\pi/2, \pi/2)$, the real part of $\sigma$ is strictly positive as $\sigma \to \infty$ along $t^{-1}Z + e^{\pm i\phi}\mathbb{R}_{>0}$. Therefore the integral along the path $t^{-1}Z + e^{\pm i\phi}\mathbb{R}_{>0}$ behaves like the integral along $t^{-1}Z + \mathbb{R}_{>0}$ for $t \to 0$, and the latter is vanishing: one way to see this is to go back to its expression as
\[
2t \int_{0}^{\infty} \frac{ds}{s} \frac{1}{-Zs - t} \exp(-\pi s^{-1}).
\]

Figure 3. Analytic continuation of $\hat{H}_T(t)$
Example. Look at the case
\[ \hat{H}_\alpha(t) = e_{\alpha} \ast \frac{1}{2\pi i} t \int_{\ell_\alpha} \frac{dz}{z} \left( \frac{1}{z - t} \exp(\pi Z(\alpha)z^{-1}) \right). \]

Arguing as above, we can rewrite this as
\[ \hat{H}_\alpha(t) = -e_{\alpha} \ast \frac{1}{2\pi i} \int_0^\infty \frac{ds}{Z(\alpha)t^{-1} + s} \exp(-\pi s). \]

The classical incomplete Gamma special function is defined by
\[ \Gamma(a, t) = \int_t^\infty z^{a-1} e^{-z} dz. \]

It is well known that \( \Gamma(a, t) \) is an analytic function of \( t \in \mathbb{C}^* \), with a branch cut discontinuity along \( \mathbb{R}_{<0} \). Then we have an equality
\[ \hat{H}_\alpha(t) = 2e^{\pi Z(\alpha)t^{-1}} \Gamma(0, Z(\alpha)t^{-1}). \]

It follows that \( \hat{H}_\alpha(t) \) is analytic for all \( t \in \mathbb{C}^* \), with a single branch cut along \( Z(\alpha)t^{-1} \in \mathbb{R}_{<0} \). In this case it is easy to work out the various branches of the function: from the representation
\[ 2e^{\pi Z(\alpha)t^{-1}} \int_0^\infty \frac{dz}{z^{\pi Zt^{-1}}} e^{-z} \]

we only have to choose the integration path from \( \pi Zt^{-1} \) to \( \infty \) suitably to get a branch of the function which extends across \( \ell_\alpha \) (in fact, up to \( e^{\pm \pi/2 \ell_\alpha} \)). Moreover, by the same representation, one can check that all of these branches are vanishing as \( t \to 0 \).

We have proved that \( \hat{\nabla}(Z) \) has the same formal type and Stokes factors as \( \nabla^{BTL}(-Z) \). It follows from the general theory developed in [B1, BT2] that \( \hat{\nabla}(Z) \) and \( \nabla^{BTL}(-Z) \) are gauge equivalent. Note that a gauge transformation taking \( \nabla^{BTL}(-Z) \) to \( \hat{\nabla}(Z) \) must be \( t \)-constant, and such constant gauge transformations preserves the off-diagonal property of the residue \( \tilde{f} \) described in Remark 5.2. The actual equality
\[ \hat{\nabla}(Z) = \nabla^{BTL}(-Z) \]

follows now from the uniqueness part of the main theorem on [BT2].

6.8. Higher order divergencies. Finally we briefly discuss an example of the more complicated \( R \to 0 \) singularities which appear in the general multiple rays case.

We apply (4.19) to a two-dimensional lattice \( \Gamma \) spanned by \( \gamma, \eta \). The integral
\[ \partial_\gamma \int_{\ell_{\gamma}} \frac{dz_1}{z_1} \rho(z, z_1)X^0(z_1)(e_\gamma) \int_{\ell_{\eta}} \frac{dz_2}{z_2} \rho(z_1, z_2)X^0(z_2)(e_\eta) \]
appears in the component $\mathcal{A}_\eta$, at the level of single-edge trees. Integrating by parts, we rewrite this as
\[
\int \frac{dz_1}{z_1} \rho(z, z_1) R \left( -\frac{1}{z_1} Z(\gamma) + \bar{Z}(\gamma) \right) X^0(z_1)(e_\eta) \int \frac{dz_2}{z_2} \rho(z_1, z_2) X^0(z_2)(e_\eta) + \int \frac{dz_1}{z_1} \rho(z, z_1) X^0(z_1)(e_\eta) z_1 \partial z_1 \int \frac{dz_2}{z_2} \rho(z_1, z_2) X^0(z_2)(e_\eta).\]
The first of these two terms couples with
\[
\int \frac{dz_1}{z_1} \rho(z, z_1) R \left( \frac{1}{z_2} Z(\gamma) - \bar{Z}(\gamma) \right) X^0(z_1)(e_\eta) \int \frac{dz_2}{z_2} \rho(z_1, z_2) X^0(z_2)(e_\eta)
\]
which also appears in $\mathcal{A}_\eta$. Summing up, we find a contribution to $\mathcal{A}_\eta^{(-1)}$ given by
\[
-\frac{Z(\gamma)}{z^2} \int \frac{dz_1}{z_1} X^0(\zeta_1)(e_\eta) \int \frac{dz_2}{z_2} \rho(z_1, z_2) X^0(z_2)(e_\eta).
\]
When $\text{Re}(Z(\gamma)/Z(\eta)) < 0$, this term is $O(\log^2(R))$ as $R \to 0$. The key is showing that
\[
\int \frac{dz_1}{z_1} X^0(\zeta_1)(e_\eta) \int \frac{dz_2}{z_2 - z_1} X^0(z_2)(e_\eta)
\]
is $O(\log^2(R))$. Setting $\omega = Z(\gamma)/Z(\eta)$ and after changing variables, we rewrite this as $e_{\gamma + \eta}$ times the factor
\[
\int_0^{+\infty} \frac{ds}{s} \exp(-\pi s^{-1} - \pi R^2 s |Z(\gamma)|^2) \int_0^{+\infty} \frac{dt}{t - \omega s} \exp(-\pi t^{-1} - \pi R^2 t |Z(\eta)|^2)
\]
(6.10)
It is possible to compute the $R \to 0$ asymptotics of (6.10) by iterated Laplace transform. Recall
\[
\mathcal{L}(\phi)(p) = \int_0^{+\infty} \phi(t) e^{-pt} dt.
\]
Define
\[
g(p, z) = \mathcal{L} \left( \frac{1}{t - z} \exp(-\pi t^{-1}) \right)(p) = \int_0^{+\infty} \frac{dt}{t - z} \exp(-\pi t^{-1}) \exp(-pt).
\]
Then (6.10) equals the iterated Laplace transform
\[
\mathcal{L} \left( \frac{1}{s} \exp(-\pi s^{-1}) g(\pi R^2 |Z(\eta)|^2, \omega s) \right)(\pi R^2 |Z(\gamma)|^2).
\]
By general properties of $\mathcal{L}$, we have an identity
\[
\mathcal{L} \left( \frac{1}{t - z} f(t) \right)(p) = e^{-zp} e^z \mathcal{L} \left( \frac{1}{t - z} f(t) \right)(1) - e^{-zp} \int_1^p \mathcal{L}(f)(q)e^{zq} dq.
\]
from which (in terms of the Bessel function $\tilde{K}_1(x) = 2K_1(2x)$)
\[
g(p, z) = e^{-zp} e^z g(1, z) + e^{-z} \int_0^1 \frac{\sqrt{\pi}}{\sqrt{q}} \tilde{K}_1(\sqrt{\pi q}) e^{zq} dq.
\]
Using this one can show that for \( \text{Re}(\omega) < 0 \) the leading order term as \( R \to 0 \) in \( \text{(6.10)} \) can be written as
\[
\int_{\pi R^2|Z(\eta)|^2}^1 \frac{\sqrt{\pi}}{\sqrt{q}} \tilde{K}_1(\sqrt{\pi q}) dq \int_0^{+\infty} \frac{ds}{s} \exp(-\pi s^{-1}-(\pi R^2(|Z(\gamma)|^2+\omega|Z(\eta)|^2)-\omega q)s).
\]
Standard asymptotics for \( \tilde{K}_1 \) now imply that the leading order term is \( O(\log^2(R)) \).

7. The \( R \to \infty \) limit and tropical geometry

7.1. As we explained in the Introduction \( \text{(2.11)} \), it is expected that the four-dimensional \( tt^* \)-connections of mathematical physics \( \nabla^{GMN}(Z,R) \) display some form of tropical behaviour as we approach the \( R \to \infty \) limit. One may wonder if some shadow of this behaviour may still be present in the toy models \( \nabla(Z,R) \). In this section we show that this is indeed the case, by proving fully precise statements of Theorems 2.2 and 2.3.

The precise version of Theorem 2.2 is given in 7.2 below and applies to the distinguished flat sections given by the restriction of \( U \subset \text{open subset} \). The tropical behaviour we describe concerns the functions \( G_T(z;Z,R) \): these play the same role as multilogarithms in \( \text{[BT1], [BT2]} \). The tropical behaviour we describe concerns the functions \( G_T(z;Z,R) \) in the limit when \( Z \) approaches a degenerate linear map and, at the same time, \( R \to \infty \). The proof of Theorem 2.2 is carried out in several steps in 7.3 - 7.6. We will recall the few (basic) notions from tropical geometry we need in 7.6 - 7.7.

The precise version of Theorem 2.3 is given in 7.8 and the proof is carried out in 7.9 - 7.12.

7.2. General setup. For simplicity, we will only describe the model case when \( \Gamma \) is generated by two elements \( \gamma, \eta \) with \( \langle \gamma, \eta \rangle = \kappa > 0 \).

We will choose for definiteness a family \( Z \in \mathcal{U} \) parametrised by a connected, open subset \( \mathcal{U} \subset \text{Hom}(\Gamma, \mathbb{C}) \) for which \( Z(\gamma), Z(\eta) \) lie in the positive quadrant. We will write \( Z^\pm \) for a point in the open subset \( U^\pm \) of \( \mathcal{U} \) where \( \pm \text{Im}(Z(\gamma))/Z(\eta) > 0 \); we assume that \( U^\pm \) are nonempty. We fix a continuous family of stability data on \( g \) characterised by \( \Omega(\gamma, Z^+) = \Omega(\eta, Z^+) = 1 \), with all other \( \Omega(\alpha, Z^+) \) vanish. The locally constant function \( a : \Gamma \to g \) underlying the stability data in \( U^+ \) is given simply by \( a(k\gamma) = -\frac{1}{k^2} \gamma, a(h\eta) = -\frac{1}{k^2} \eta \) for \( h, k > 0 \), with all other values vanishing. While this family is very simple, the corresponding irregular connections \( \nabla(Z,R) \) are already as complicated as in the most general case.

Choose a fixed \( z^* \in \mathbb{C}^* \) with \( \text{Re}z \text{Im}z < 0 \). We consider trees \( T \) such that \( W_T(Z^+) \neq 0 \), i.e. their vertices are decorated by positive multiples of the basic vectors \( \gamma \) or \( \eta \). For each tree \( T \) with more than a single vertex, the special function \( G_T(z^*,Z;R) \) is sectionally holomorphic in \( Z \in \mathcal{U} \): it is discontinuous along the critical locus where \( \text{Im}(Z(\gamma))/Z(\eta) = 0 \).

The idea we wish to implement is very simple: we will rewrite \( G_T(z^*;Z^+,R) \) as a sum of iterated integrals over rays \( \ell(Z^-) \), of the form \( \pm G_T(z^*;Z^-,R) \) for various \( T' \), with the only difference that the integrands involve \( X^0(z;Z^+;R) \). Since \( X^0 \) is continuous in \( Z \), \( G_T(z^*;Z^+,R) \) will be asymptotically equal to the sum of
these terms $\pm G_T(z^*; Z^-, R)$ as $|Z^+ - Z^-| \to 0$; and this gives an effective way to see which linear combination of the special functions $G_T(z^*; Z^-, R)$ replaces $G_T(z^*; Z^+, R)$ in the expansion of $X(z^*; Z^-, R)$. The theorem below makes this idea precise, and characterises the single-vertex term $G_\bullet(z^*; Z^-, R)$ in this linear combination in terms of certain tropical graphs.

**Theorem 2.2 (precise statement).** There is an expansion

$$G_T(z^*; Z^+, R) = \sum_{T'} \pm G_{T'}(z^*; Z^-, R) + r(|Z^+ - Z^-|)$$

(7.1)

where $r(|Z^+ - Z^-|) \to 0$ as $|Z^+ - Z^-| \to 0$, and we sum over a finite set of rooted trees $T'$, not necessarily distinct, decorated by $\Gamma$. Let $\beta \in \Gamma$ denote the sum $\sum_i \alpha(i)$ of all decorations of $T$.

The terms corresponding to a single-vertex tree in (7.1) are labelled by a finite set of graphs $C_i$ containing $|T|\ell$ external 1-valent vertices and with 3-valent internal vertices. These terms are all equal to $G_\beta(z^*; Z^-, R)$ up to sign, and differ by a well defined factor $\varepsilon(C_i) = \pm 1$ which is uniquely attached to the graph $C_i$. Moreover, the graphs $C_i$ come naturally with an extra combinatorial structure, which says precisely that they are the combinatorial types of a finite set of tropical curves immersed in the plane $\mathbb{R}^2$.

Finally, the single-vertex terms in (7.1) are uniquely characterised by the asymptotic behaviour: they are of order

$$(2|Z^-(\beta)|R)^{-1} \exp(-2|Z^-(\beta)|R)e_\beta,$$

as $R \to \infty$, uniformly as $|Z^+ - Z^-| \to 0$.

For the sake of simplicity we will also assume that the lattice element $\beta$ is primitive in $\Gamma$. Similar results hold in the non-primitive case but require keeping track of disconnected curves.

Before tackling the general case, it is helpful to illustrate the asymptotic expansion for $G_T(R)$ in terms of the graphs $C_i$ starting with the simplest case when $T$ is the decorated tree with a single edge $\gamma \to \eta$. By definition, for $Z^+ \in U^+$ we have

$$G_T(z^*; Z^+, R) = \int_{\ell_{\gamma}(Z^+)} \frac{dz_1}{z_1} \rho(z^*_1, z_1)X^0(z_1; Z^+)(e_\gamma) \int_{\ell_{\eta}(Z^+)} \frac{dz_2}{z_2} \rho(z_2, z_2)X^0(z_2; Z^+)(e_\eta).$$

One way to compute $\delta G_T$ is to first rewrite $G_T(z^*; Z^+, R)$ in terms of integrals over rays $\ell(Z^-)$ for a point $Z^- \in U^-$. Recall that $X^0(s)$ is holomorphic in $\mathbb{C}^*$, and that $\rho(s, t)$ is a meromorphic function of $t$ with a simple pole at $s$, with $\text{Res}_s \rho(s, t) = (2\pi i)^{-1}$. By our choice of $z^*$ and the definitions of $U^\pm$, it follows
that we can rewrite

$$G_T(z^*;Z^+,R) = \int_{\ell_\gamma(Z^-)} \frac{dz_1}{z_1} \rho(z^*, z_1) X^0(z_1)(e_\gamma) \int_{\ell_\eta(Z^+)} \frac{dz_2}{z_2} \rho(z_1, z_2) X^0(z_2)(e_\eta)$$

$$= \int_{\ell_\gamma(Z^-)} \frac{dz_1}{z_1} \rho(z^*, z_1) X^0(z_1)(e_\gamma) \int_{\ell_\eta(Z^-)} \frac{dz_2}{z_2} \rho(z_1, z_2) X^0(z_2)(e_\eta)$$

$$+ \int_{\ell_\gamma(Z^-)} \frac{dz_1}{z_1} \rho(z^*, z_1) X^0(z_1)(e_\gamma) \ast X^0(z_1)(e_\eta).$$

(7.2)

The last term (7.3) comes from the residue theorem when we push $\ell_\eta(Z^+)$ over to $\ell_\eta(Z^-)$, crossing the first integration ray $\ell_\gamma(Z^-)$. Notice that we have the simple but crucial property

$$X^0(z_1)(e_\gamma) \ast X^0(z_1)(e_\eta) = X^0(z_1)(e_{\gamma+\eta}).$$

It is this property that allows to relate $G_T$ to tropical curves in $\mathbb{R}^2$ in the general case.

In the present example, pushing $\ell_\gamma(Z^-)$ in the residue term to $\ell_{\gamma+\eta}(Z^0)$, and recalling that $X^0$ is continuous across the critical locus where $\text{Im} Z(\gamma)/Z(\eta) = 0$, we find

$$G_T(z^*;Z^+,R) = G_{\gamma+\eta}(z^*;Z^-,R) + G_T(z^*;Z^-,R) + r|Z^+ - Z^-|$$

where $r \to 0$ as $|Z^+ - Z^-| \to 0$. The single-vertex term has asymptotics

$$G_{\gamma+\eta}(z^*;Z^-,R) \sim (2|Z^-(\gamma + \eta)|R)^{-1} \exp(-2|Z^-(\gamma + \eta)|R)e_{\gamma+\eta}$$

uniformly for $|Z^+ - Z^-| \to 0$.

There is an obvious graph $C$ which we can attach to the computation above, displayed in Figure 4. There are edges $E_1, E_2$ labelled by the two factors in (7.2), and $E_3$ labelled by the residue term (7.3). These edges meet in a single vertex $V$,

![Figure 4](image)

and come with attached integral vectors $\alpha(E_1) = \gamma$, $\alpha(E_2) = \eta$ and $\alpha(E_3) = \gamma+\eta$.

It is natural to think of $E_1, E_2$ as incoming in $V$, and $E_3$ as outgoing from $V$. Keeping track of this orientation, we have the balancing condition

$$-\alpha(E_1) - \alpha(E_2) + \alpha(E_3) = 0.$$
7.3. Expansion for $G_T(z^*; Z^+, R)$ across critical locus. We will show that the simple analysis above can be carried out in general, for an arbitrary $G_T(z^*; Z, R)$ function, up to leading order terms as $R \to 0$.

Fix a decorated tree $T$ as in Section 4.6 with $W_T(Z^+) \neq 0$. The precise form of the expansion (7.1) depends on a choice of total order for the vertices of $T$. We simply fix one such total order, without assuming that it is compatible with the natural orientation of $T$ as a rooted tree (i.e. flowing away from the root).

It will be useful to introduce the notion of a totally ordered tree $T'$ attached to an iterated integral $I(T')$: we mean by this that there is a bijective correspondence between vertices of $T'$ and factors of the form

$$\int_{\ell_j} \frac{dz_j}{z_j} \rho(z_i, z_j)X^0(z_j; Z^+, R)(e_{\alpha_j})$$

appearing in $I(T')$, such that the factor

$$\int_{\ell_i} \frac{dz_i}{z_i} \rho(z_i, z_k)X^0(z_i; Z^+, R)(e_{\alpha_i}) \int_{\ell_j} \frac{dz_j}{z_j} \rho(z_i, z_j)X^0(z_j; Z^+, R)(e_{\alpha_j})$$

appears if and only if there is an arrow $i \to j$ in $T'$. Notice that in particular $T$ is attached to the iterated integral $G_T(z^*; Z^+, R)$ in this sense.

Remark 7.1. In (7.4) we allow $z_i \in \ell$, i.e. we allow factors of the form

$$\lim_{\ell' \to \ell} \int_{\ell'} \frac{dz_j}{z_j} \rho(z_i, z_j)X^0(z_j, Z^+)(e_{\alpha_j})$$

where $z_j \in \ell$. However in this case (7.4) will be decorated with the direction in which $\ell'$ approaches $\ell$, using $\ell' \to \ell^\pm$ for the clockwise (respectively counterclockwise) direction.

Let $T'$ be a tree which is attached to an iterated integral in the sense above. We will construct from $T'$ a finite set of trees $S(T')$ of the same type, obtained by applying the residue theorem. To save some space, we set

$$X^0_0(z) = X^0(z; Z^+, R)(e_\alpha).$$

In the following, we say that a ray $\ell$ separates $\ell_1, \ell_2$ if $\ell_1, \ell_2$ lie in different connected components of the complement of $\ell$ in the sector between $\ell_\gamma(Z^+), \ell_\eta(Z^+)$. We allow the limiting case in which $\ell_1 \to \ell$ in a component which does not contain $\ell_2$, or possibly $\ell_1 \to \ell$ and $\ell_2 \to \ell$ in different components.

Consider the set of vertices $j \in T'$ for which one of the following occurs:

1. the corresponding factor in $I(T')$ has the form

$$\int_{\ell_{\alpha(j)}(Z^+)} \frac{dz_j}{z_j} \rho(z_i, z_j)X^0_{\alpha(j)}(z_i)$$

where $\alpha(j)$ is a positive multiple of $\gamma$ or $\eta$, or

2. it is of the form

$$\int_{\ell} \frac{dz_j}{z_j} \rho(z_i, z_j)X^0_{\alpha(j)}(z_j)$$

for some ray $\ell \subset \mathbb{C}^*$ which is not one of $\ell_{\alpha(j)}(Z^\pm)$. 
In fact we will see (inductively) that there is at most one vertex \( i \) of \( T' \) for which (2) holds. If the set of \( j \) satisfying (1) or (2) is empty we simply set \( S(T') = \{ T' \} \). Otherwise we choose the first element \( j \) in this set (with respect to the total order of \( T' \)). As \( T \) is rooted, there is at most one arrow \( i \to j \), and possibly several arrows \( j \to k \). Since \( j \) satisfies (1) or (2), the factor of \( I(T') \) corresponding to \( j \) fits into

\[
\int_{\ell_{\alpha(i)}(Z^+)} \frac{dz_i}{z_i} \rho(z_h, z_i) X^0_{\alpha(i)}(z_i) \int_{\ell} \frac{dz_j}{z_j} \rho(z_i, z_j) X^0_{\alpha}(z_j) \prod_k \int_{\ell_{\alpha(k)}(Z^+)} \frac{dz_k}{z_k} \rho(z_j, z_k) X^0_{\alpha(k)}(z_k)
\]

where \( \ell \) is either \( \ell^+_{\alpha} \) or a ray which is distinct from \( \ell^-_{\alpha} \), and \( h \to i \).

If none of the rays \( \ell_{\alpha(i)}(Z^+) \) and \( \ell_{\alpha(k)}(Z^-) \) separate \( \ell \) and \( \ell_{\alpha}(Z^-) \), we set \( S(T') = \{ T'' \} \), with \( T'' = T' \) and \( I(T'') \) obtained from \( I(T') \) by replacing \( \ell \) in the factor above with \( \ell_{\alpha(j)} \).

Otherwise we apply Fubini and rewrite the integral above in the form

\[
\int_{\ell_{\alpha(i)}(Z^+)} \frac{dz_i}{z_i} \rho(z_h, z_i) X^0_{\alpha(i)}(z_i) \left( \prod_k \int_{\ell_{\alpha(k)}(Z^+)} \frac{dz_k}{z_k} X^0_{\alpha(k)}(z_k) \right) \int_{\ell} \frac{dz_j}{z_j} \prod_k \rho(z_j, z_k) \rho(z_i, z_j) X^0_{\alpha}(z_j). \tag{7.5}
\]

The function

\[
\frac{1}{z_j} \prod_k \rho(z_j, z_k) \rho(z_i, z_j) X^0_{\alpha}(z_j)
\]

is holomorphic in the variable \( z_j \in \mathbb{C}^* \setminus \{ z_i, z_k \} \), and has simple poles at \( z_i, z_k \) with residues given respectively by \(- (2\pi i)^{-1} \rho(z_i, z_k) X^0_{\alpha}(z_i) \) and \((2\pi i)^{-1} \rho(z_i, z_k) X^0_{\alpha}(z_k) \).

If we apply the residue theorem (justified by the estimates of integrals along an arc given in section 4.4) we can rewrite (7.5) as

\[
\int_{\ell_{\alpha(i)}(Z^+)} \frac{dz_i}{z_i} \rho(z_h, z_i) X^0_{\alpha(i)}(z_i) \left( \prod_k \int_{\ell_{\alpha(k)}(Z^+)} \frac{dz_k}{z_k} X^0_{\alpha(k)}(z_k) \right) \int_{\ell_{\alpha}(Z^-)} \frac{dz_j}{z_j} \prod_k \rho(z_j, z_k) \rho(z_i, z_j) X^0_{\alpha}(z_j) \tag{7.6}
\]

plus residue terms

\[
\mp \int_{\ell_{\alpha(i)}(Z^-)} \frac{dz_i}{z_i} \rho(z_h, z_i) X^0_{\alpha(i)+\alpha}(z_i) \prod_k \int_{\ell_{\alpha(k)}(Z^+)} \frac{dz_k}{z_k} \rho(z_i, z_k) X^0_{\alpha(k)}(z_k) \tag{7.7}
\]
and
\[ \pm \int_{\ell_{\alpha(i)}(Z^\pm)} \frac{dz_i}{z_i} \rho(z_h, z_i) X_{\alpha(i)}^0(z_i) \int_{\ell_{\alpha(k')} (Z^-)} \frac{dz_{k'}}{z_{k'}} \rho(z_i, z_{k'}) X_{\alpha(k')+\alpha}(z_{k'}) \prod_{k \neq k'} \int_{\ell_{\alpha(k)}(Z^\pm)} \frac{dz_k}{z_k} \rho(z_i, z_k) X_{\alpha(k)}^0(z_k). \]  

(7.8)

It is understood that the term (7.7) is only present if \( \ell_{\alpha(i)}(Z^-) \) and \( \ell_{\alpha(i)}(Z^\pm) \) separates \( \ell \) and \( \ell_{\alpha}(Z^-) \), while a term (7.8) appears for each \( \ell_{\alpha(k')}(Z^-) \) separating \( \ell \), \( \ell_{\alpha}(Z^-) \). The signs in (7.7), (7.8) are determined according to whether \( \ell \) moving to \( \ell_{\alpha}(Z^-) \) crosses \( \ell_{\alpha(i)}(Z^-) \) (respectively \( \ell_{\alpha(k')}(Z^-) \)) in the clockwise, respectively counterclockwise direction.

**Remark 7.2.** Following our convention, if \( \ell_{\alpha}(Z^-) \) coincides with \( \ell_{\alpha(i)}(Z^-) \) or a subset of the \( \ell_{\alpha(k)}(Z^-) \), or both, then the integral over \( \ell_{\alpha}(Z^-) \) in (7.6) is actually a limit of integrals over \( \ell \to \ell_{\alpha(\ell)}(Z^-) \).

We define trees \( T'' \) in \( S(T') \) in bijection with the terms (7.6), (7.7), (7.8). There is an obvious (rooted, decorated, totally ordered) tree \( T'' \) attached to each of these integrals, whose underlying bare tree is given simply by contracting the edge \( i \to j \) in \( T \). By construction, the condition (2) can happen for at most a single vertex of \( T'' \).

Starting from our original pair of \( T \) and \( I(T) = G_T(z^*; Z^+, R) \), by construction the sequence of sets \( S(T), S(S(T)), \ldots \) stabilises after a finite number of steps; we let \( S^{(p)}(T) \) denote the first set for which \( S^{(p)}(T) = S^{(p+1)}(T) \).

This finishes the construction of the expansion (7.1). Indeed \( G_T(z^*; Z^+, R) \) is a sum of terms which are in bijection with elements of \( S^{(p)}(T) \), and these all have the form

\[ \pm G_{T'}(z^*; Z^-, R) + r(|Z^+ - Z^-|) + r_{T'}(|Z^+ - Z^-|) \]

where \( r_{T'}(|Z^+ - Z^-|) \to 0 \) as \( |Z^+ - Z^-| \to 0 \). The single-vertex terms in (7.1) are in bijection with trees \( T_p \in S^{(p)}(T) \) which contain a single vertex.

**7.4. Highest order terms.** We characterise the single-vertex terms in (7.1) by their asymptotic behaviour. Let \( T_p \) denote a tree in \( S^{(p)}(T) \) with a single vertex. Then by construction

\[ I(T_p) = \pm \int_{\ell_{\beta}(Z^-)} \frac{dz'}{z'} \rho(z, z') X_{\beta}^0(z'). \]

for a unique sign attached to \( T^p \) by orientations in the residue theorem, and where \( \beta \in \Gamma \) is given by the sum of all the lattice elements attached to the vertices of \( T \). By the results of section 4.4 we have an expansion as \( R \to 0 \)

\[ I(T_p) \sim (2|Z^-(\beta)|R)^{-1} \exp(-2|Z^-(\beta)|R)e_{\beta}, \]

which holds uniformly as \( |Z^+ - Z^-| \to 0 \).
Suppose now that \( T_2 \in S^{(p)}(T) \) is a tree which contains more than a single vertex. We claim that this is subleading, i.e. there is an expansion of the form
\[
I(T_2) \sim \phi(|Z^+ - Z^-|) f(Z^\pm, R) e^\beta
\]
for some function \( f \) such that
\[
f(Z^\pm, R) = o((2|Z^- (\beta)| R)^{-1} \exp(-2|Z^- (\beta)| R)).
\]
Indeed by construction in this case we have
\[
I(T_2) = \int \prod_{(i \to j) \subseteq \tilde{T}_2} \frac{dz_j}{z_j} \rho(z_i, z_j) X^0_{\alpha(j)}(z_j), \tag{7.9}
\]
where the tree \( \tilde{T}_2 \) contains an extra vertex 0 mapping to the root of \( T_2 \), with \( z_0 = z^* \). Iterating the argument in section 4.4 sufficiently many times, one can then show that
\[
I(T_2) \sim \kappa(|Z^+ - Z^-|) f(R) e^\beta \tag{7.10}
\]
for a function \( f(Z^\pm, R) \) bounded by
\[
\prod_i \frac{1}{2R|Z^- (\alpha(i))|} e^{-2R|Z^- (\alpha(i))|}.
\]
This decays faster than \((2|Z^- (\beta)| R)^{-1} \exp(-2|Z^- (\beta)| R)\), because
\[
\sum_i |Z^- (\alpha(i))| > |Z^- (\beta)|.
\]
This follows immediately from our assumption that \( \beta \) is primitive in \( \Gamma \) (using that \( Z^- \) is nondegenerate).

7.5. Tropical graphs attached to highest order terms. It is now a simple matter to show that a tree \( T_p \in S^{(p)}(T) \) containing a single vertex determines a graph \( C \) containing only 1-valent and 3-valent vertices, and whose edges are decorated by elements of \( \Gamma \). We will show that this extra data satisfy two relations, which imply that \( C_i \) is the combinatorial type of a tropical curve in \( \mathbb{R}^2 \) (this notion will also be recalled).

The tree \( T_p \) determines a unique sequence of trees \( T_r \in S^{(i)}(T) \), \( r = 0, \ldots, p \), its ancestors, with \( T_0 = T \). Moreover there are natural maps between the set of vertices
\[
\varphi_r: T^0_r \to T^0_{r+1},
\]
such that \( \varphi_r \) is either a bijection, or maps two vertices \( i_1, i_2 \) to the same vertex \( i \in T^0_{r+1} \) (and is a bijection on \( T^0_r \setminus \{i_1, i_2\} \)). The set of vertices \( \bigcup_r T^0_r \) and gluing maps \( \{ \varphi_r \} \) define a graph \( \tilde{C} \), whose internal vertices are either 2-valent or 3-valent.

Let \( V \) be a 3-valent vertex of \( \tilde{C} \), corresponding to a vertex of \( T_r \). By construction, this determines a unique factor of the form \( \{7.6\} \) in \( I(T_{r-1}) \), and \( V \) corresponds to a unique nonzero residue term of the form \( \{7.7\} \) or \( \{7.8\} \). This means that there there is a natural choice of incoming edges \( E_1, E_2 \), respectively an outgoing edge \( E_3 \). The edges \( E_i \) come naturally with vectors \( \alpha(E_i) \in \Gamma \): in
the notation of (7.6) - (7.8) these are given by \((\alpha(i), \alpha, \alpha(i) + \alpha)\), respectively \((\alpha(i), \alpha, \alpha(k') + \alpha)\). Thus we always have the balancing condition

\[- \alpha(E_1) - \alpha(E_2) + \alpha(E_3) = 0.\]

(7.11)

Notice that the balancing condition is a direct consequence of the residue theorem and the property

\[X_0(z'; Z, R)(e_{\alpha_1}) * X_0(z'; Z, R)(e_{\alpha_2}) = X_0(z'; Z, R)(e_{\alpha_1 + \alpha_2}).\]

At the same time we see that \(\alpha(E_1), \alpha(E_2) \in \Gamma\) are linearly independent over \(\mathbb{Q}\), otherwise the residue term with lattice element \(\alpha(E_3) = \alpha(E_1) + \alpha(E_2)\) would not appear in (7.7) - (7.8). Also, if \(E\) and \(E'\) are edges of \(\tilde{C}\) which are respectively outgoing and incoming to 3-valent vertices \(V, V'\), we must have \(\alpha(E) = \alpha(E')\) (since no application of the residue theorem separates \(V, V'\)).

Finally we define a graph \(C\) obtained from \(\tilde{C}\) by forgetting all the internal 2-valent vertices.

7.6. Rational tropical curves in \(\mathbb{R}^2\). We have attached to our original \(T, G_T(z^*; Z^+, R)\) a finite collection of graphs \(C_i\) with 3-valent internal vertices, one for each single-vertex tree in \(S^{(p)}(T)\) or, equivalently, one for each leading order term in the expansion

\[G_T(z^*; Z^+, R) = \sum_{T'} \pm G_{T'}(z^*; Z^{-}, R) + r(|Z^+ - Z^-|)\]

Each \(C_i\) comes with the extra data of a decoration of its edges \(E\) by elements \(\alpha(E) \in \Gamma\), satisfying the above conditions of balancing (7.11) and linear independence.

The extra data say precisely that \(C_i\) is the combinatorial type of a rational tropical curve immersed in \(\mathbb{R}^2\). Following [GPS] section 2.1, we define plane rational tropical curves as immersions of certain graphs in \(\mathbb{R}^2\).

Let \(C\) denote a connected graph with only 3-valent internal vertices. We suppose that \(C\) is weighted, i.e. we have the extra data of a positive number \(w(E)\) for every edge \(E\) of \(C\). We write \(C\) as well for the topological model of the graph, and \(C^0\) for the topological space obtained by removing all 1-valent (external) vertices. A parametrised tropical curve in \(\mathbb{R}^2\) is a proper map \(h: C^0 \to \mathbb{R}^2\), such that for all \(E\), the map \(h|E\) is an embedding into an affine line of rational slope, and for which the following balancing condition folds. At each image of a vertex \(h(V)\), we have well defined primitive vectors \(m_i \in \mathbb{Z}^2\) pointing out of \(h(V)\) along the directions of the incident edges \(E_1, E_2, E_3\). Then one requires

\[w(E_1)m_1 + w(E_2)m_2 + w(E_3)m_3 = 0.\]

A rational plane tropical curve is then defined as the equivalence class of maps \(h\) up to isomorphisms of the domain graph. Following [GM] section 2, the combinatorial type of a tropical curve is defined as the data of the underlying graph \(C\), together with the vectors \(m_i\) for each internal vertex \(V\).
It is now clear that each of our graphs $C_i$ is the combinatorial type of a class of tropical curves. As an example, consider the tree

$$T = \{ \gamma \rightarrow \eta \rightarrow \gamma \rightarrow 2\eta \}$$

and fix the unique total order of vertices which is compatible with the orientation. Then the expansion for $G_T$ contains two leading order terms, labelled by the tropical types $C_1, C_2$ of Figures 5, 6.

![Figure 5. The tropical type $C_1$](image)

![Figure 6. The tropical type $C_2$](image)

7.7. Tropical invariants. Just as for plane algebraic curves, there is a natural notion of degree for a plane rational tropical curve $(C^0, h)$ as above, see e.g. [GM] section 2: this is just the unordered collection of vectors $-w(E_i)m_i \in \mathbb{Z}^2$ and $w(E_{out})m_{out}$ attached to all the external edges of $C$. Notice that we allow $w(E_i) > 1$ for some or all the external edges.

The enumerative theory of plane tropical curves of fixed degree through the expected number of general points is well established in all genera (going back to
the foundational work of Mikhalikin [M], see [GM] for a result in the generality we need here).

We will only be concerned with a very special enumerative invariant, which is described in detail in [GPS] section 2.3. Choose $l_1$ general lines $d_{1j}$ with the same (positive, primitive) direction $d_1$, respectively $l_2$ general lines $d_{2j}$ in the direction $d_2$. We attach a positive integral weight $w_{ij}$ to the line $d_{ij}$. Look at the set of parametrised plane rational tropical curves $(C^o, h)$ having a collection of unbounded edges $E_{ij}, E_{out}$, such that $h(E_{ij}) \subset d_{ij}$ and $w(E_{ij}) = w_{ij}$. By the balancing condition, the degree of these curves is determined by a weight vector $w = (w_1, w_2)$, where each $w_i$ is the collection of integers $w_{ij}$ (for $1 \leq i \leq 2$ and $1 \leq j \leq l_i$) such that $1 \leq w_{i1} \leq w_{i2} \leq \cdots \leq w_{il_i}$. By the general theory, for generic $d_{ij}$ the number of isomorphism classes of parametrised curves $(C^o, h)$ as above is finite. Counting these tropical curves with the multiplicity of tropical geometry yields a number $N^{trop}(w) \in \mathbb{N}_{>0}$, which is invariant under deformation of the constraints $d_{ij}$.

Recall that the tropical multiplicity $\mu_V$ at a 3-valent vertex $V \in h(C^o)$ with associated primitive vectors $m_i$ is defined as $|w(E_i)m_i \wedge w(E_j)m_j|$ for $i \neq j$ (this is well defined by the balancing condition). The multiplicity of $(C^o, h)$ is $\prod_V \mu_V$, the product over all 3-valent vertices. As an example $N^{trop}((1,1), (1,2)) = 8$ is computed by Figure 7. Notice that for the choice of constraints $d_{ij}$ displayed in the figure two combinatorial types appear: a curve of type $C_1$ and two curves of type $C_2$.

![Figure 7](image_url)

**Figure 7.** $N^{trop}((1,1), (1,2)) = 8$

Remark 7.3. Although we only defined the tropical invariants $N^{trop}(w)$ for two-components weight vectors $w$, as explained in [GPS] section 2.3, there is an obvious extension to an arbitrary number of components (with corresponding directions for the infinite ends).
7.8. **Highest order terms and tropical invariants.** In the rest of this section we will relate the (combinatorial types of) tropical curves $C_i$ constructed above to actual tropical invariants.

The $C_i$ attached to a single $T$, $G_T(z^*; Z^+, R)$ all have the same tropical degree $w$, which we will sometime denote by $\operatorname{deg}(T)$. The component $w_1$ ($w_2$) can be identified with the set of multiples of $\gamma$ (respectively $\eta$) in the set of all decorations $\alpha(i)$ (in particular, $w$ is independent of the arbitrary choice of a total order of vertices). Recall also that $C_i$ comes with a distinguished sign $\varepsilon(C_i) = \pm 1$ (rather than a multiplicity), uniquely determined by the residue theorem through (7.7) - (7.8). It is natural to consider the set of all trees defining the same degree $w$, and to try and relate the sum $\sum_T \sum_i \varepsilon(C_i(T))$ to $N^{\text{trop}}(w)$. Indeed the following holds.

**Theorem 2.3** (precise statement) The sum over trees $T$ with $W_T(Z^+) \neq 0$ (i.e. decorated by positive multiples of $\gamma$ or $\eta$) 

$$\sum_{\operatorname{deg}(T) = w} W_T \sum_i \varepsilon(C_i(T))$$

equals the tropical invariant $N^{\text{trop}}(w)$, times the combinatorial factor in $\Gamma \otimes \mathbb{Q}$ given by

$$\frac{1}{|\operatorname{Aut}(w)|} \prod_{k,l} \frac{1}{w_{kl}} (|w_1| \gamma + |w_2| \eta)$$

Our proof is not direct, but relies instead on the methods of [GPS] section 2.

7.9. **Tropical types and stability data.** The functions $X(z; Z^\pm, R)$ induce flat sections of $\nabla(Z^\pm, R)$ on a supersector $\Sigma$ for $\nabla(Z^-, R)$, with the same asymptotics as $z \to 0$ (uniformly as $R \to \infty$). Since the connections $\nabla(Z^\pm, R)$ glue, choosing $z^* \in \Sigma$, when $|Z^+ - Z^-| \to 0$ we must have

$$X(z^*; Z^+, R) - X(z^*; Z^-, R) \to 0,$$

uniformly as $R \to \infty$. By ?, the same must be true for the difference

$$\sum_T W_T(Z^+)G_T(z^*; Z^+, R) - \sum_T W_T(Z^-)G_T(z^*; Z^-, R). \quad (7.12)$$

Let $T$ be a tree with $W_T(Z^+) \neq 0$ as usual. We have $W_T(Z^+) = W_T(Z^-)$ in this case. This follows since $\operatorname{DT}(h\gamma, Z^+) = \operatorname{DT}(h\gamma, Z^-)$, and similarly $\operatorname{DT}(k\eta, Z^+) = \operatorname{DT}(k\eta, Z^-)$. To check (for example) the first statement notice that for single-vertex trees we have

$$W_{h\gamma}(Z^\pm)G_{h\gamma}(z^*; Z^\pm, R) = \operatorname{DT}(k\eta, Z^\pm) \int_{\ell_{z^*}(Z^\pm)} \frac{dz}{z} \rho(z^*, z) X^0(z; Z^\pm(h\gamma), R).$$

Therefore $W_{h\gamma}(Z^\pm)G_{h\gamma}(z^*; Z^\pm, R) - W_{h\gamma}(Z^-)G_{h\gamma}(z^*; Z^-, R)$ has the form

$$(\operatorname{DT}(h\gamma, Z^+) - \operatorname{DT}(h\gamma, Z^-))(2R|Z^-(h\gamma)|)^{-1} \exp(-2R|Z^-(h\gamma)|) \varepsilon_{h\gamma}$$

as $|Z^+ - Z^-| \to 0$, uniformly as $R \to \infty$, and by Theorem ? it cannot be cancelled by some other term in (7.12).
Let us go back to the difference (7.12). Pick a primitive \( \beta \in \Gamma \). By the expansion (7.1), the \( e_{\beta} \) component of the first summand contains a distinguished sum of highest order terms

\[
\sum_{W_T(Z^+) \neq 0, \sum_i \alpha(i) = \beta} W_T \sum_i \varepsilon(C_i(T)) \int_{\ell_\beta(Z^-)} \frac{dz}{z} \rho(z^*, z) X^0(z; Z^-(\beta), R),
\]

which is uniquely characterised by its asymptotics as \(|Z^+ - Z^-| \to 0, R \to \infty\). The unique term in the second summand of (7.12) with matching asymptotics is

\[
DT(\beta, Z^-) \beta \int_{\ell_\beta(Z^-)} \frac{dz}{z} \rho(z^*, z) X^0(z; Z^-(\beta), R).
\]

We have proved

\[
DT(\beta, Z^-) \beta = \sum_{W_T(Z^+) \neq 0, \sum_i \alpha(i) = \beta} W_T \sum_i \varepsilon(C_i(T)). \tag{7.13}
\]

10. **Refinement.** Consider the set of all trees with \( W_T(Z^+) \neq 0 \), i.e., decorated with positive multiples of \( \gamma, \eta \), and with total decoration \( \sum_i \alpha(i) = \beta \). In the previous section, we related the sum of the signs \( \varepsilon(C_i(T)) \) attached to tropical types over all such trees to the stability data, that is the quantity \( DT(\beta, Z^-) \).

Fix a weight vector \( w \) such that \( \beta = |w|_1 \gamma + |w|_2 \eta \). We need to prove a more refined result, given a similar link between stability data and the sum

\[
\sum_{W_T(Z^+) \neq 0, \deg(T) = w} W_T \sum_i \varepsilon(C_i(T)).
\]

To achieve this we consider a larger lattice \( \overline{\Gamma} \) mapping to \( \Gamma \). Denoting by \( l_i \) the length of \( w_i \), we take \( \overline{\Gamma} \) to be generated by elements \( \gamma_1, \ldots, \gamma_{l_1} \) and \( \eta_1, \ldots, \eta_{l_2} \) such that

\[
\langle \gamma_i, \gamma_j \rangle = \langle \eta_i, \eta_j \rangle = 0, \langle \gamma_i, \eta_j \rangle = 1.
\]

The map \( \pi: \Gamma \to \mathbb{Z}^2 \) is given by \( \pi(\gamma_i) = \gamma_i, \pi(\eta_j) = \eta_j \). There is of course a pullback family of elements of \( \text{Hom}(\overline{\Gamma}, \mathbb{C}) \) induced by our family \( Z \); we will suppress the pullback in our notation. We look at the unique continuous family of stability data on \( \mathcal{F} \) which correspond to setting \( \Omega(\gamma_i, Z^+) = \Omega(\eta_j, Z^+) = 1 \), with all other \( \Omega(\alpha, Z^+) \) vanishing.

The analogues of the asymptotic expansion (7.1), the construction of tropical types and of the argument leading to (7.13) are straightforward; the only difference is that we consider now trees \( \overline{T} \) which are labelled by positive multiples of \( \gamma_i, \eta_j \). We still write \( \sum_i \alpha(i) \) for the sum of all decorations of \( \overline{T} \). Thus we have

\[
DT(\overline{\beta}, Z^-) \overline{\beta} = \sum_{W_{\overline{T}}(Z^+) \neq 0, \sum_i \alpha(i) = \overline{\beta}} W_{\overline{T}} \sum_i \varepsilon(C_i(T)).
\]

This is still not enough for our purposes. We need to impose the condition that the trees over which we sum have precisely \( l_1 + l_2 \) vertices. This is possible if we consider a formal version of our stability data setting \( \Omega(\gamma_i, Z^+) = \Omega(\eta_j, Z^+) = \epsilon \), with all other \( \Omega(\alpha, Z^+) \) vanishing. We can think of \( \epsilon \) as a formal parameter or
as an arbitrary rational number. The setup is unchanged, except that \( W_\mathbf{T} \) and \( \text{DT}(\bar{\beta}, Z^-) \) will now be polynomials in the variable \( \epsilon \). Therefore

\[
\text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1+l_2}]\epsilon^{l_1+l_2}\bar{\beta} = \sum_{W_\mathbf{T}(Z^+)=\emptyset, \sum_i \alpha(i) = \bar{\beta}, |T^0| = l_1 + l_2} W_\mathbf{T} \sum_i \varepsilon(C_i(T)), \tag{7.14}
\]

where \( \text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1+l_2}] \) denotes the coefficient of the monomial \( \epsilon^{l_1+l_2} \). This is the refinement we need. Given a weight vector \( \mathbf{w} \) as above, we construct an element \( \bar{\beta} \) as

\[
\bar{\beta} = \sum_{i=1}^{l_1} |w_{i1}| \gamma_i + \sum_{j=1}^{l_2} |w_{2j}| \eta_j.
\]

The set of trees \( \mathbf{T} \) such that \( W_\mathbf{T}(Z^+) \neq 0 \), \( \sum_i \bar{\alpha}(i) = \bar{\beta} \) and \( |T^0| = l_1 + l_2 \) is precisely the set \( \mathbf{P} \) of rooted trees with \( l_1 + l_2 \) vertices decorated by

\[
\{w_{i1} \gamma_1, \ldots, w_{i1} \gamma_{l_1}, w_{21} \eta_1, \ldots, w_{2l_2} \eta_{l_2}\}.
\]

There is a forgetful map from \( \mathbf{P} \) to the set \( \mathbf{T} \) of rooted trees \( T \) decorated by elements of \( \Gamma \), with \( W_\mathbf{T}(Z^+) \neq 0 \) and \( \deg(T) = \mathbf{w} \), given by replacing \( \gamma_i \) with \( \gamma \) and \( \eta_j \) with \( \eta \). This is clearly onto. For \( \mathbf{T} \) mapping to \( T \), we have (after \( \mathbb{Q} \)-linear extension of \( \pi \))

\[
\pi(W_\mathbf{T}) = \epsilon^{l_1+l_2} \kappa^{-|T^0|} |\text{Aut}(T)| W_\mathbf{T}.
\]

We also have

\[
\varepsilon(C_i(\mathbf{T})) = \varepsilon(C_i(T)),
\]

where the latter is computed with respect to the total order induced from \( \mathbf{T} \). On the other hand, the fibre of \( \mathbf{P} \to \mathbf{T} \) over \( T \) contains \( (\text{Aut}(T))^{-1} \text{Aut}(\mathbf{w}) \) trees. Applying \( \pi \) to both sides of (7.14) proves

\[
\text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1+l_2}]\beta = |\text{Aut}(\mathbf{w})| \sum_{W_\mathbf{T}(Z^+)=\emptyset, \deg(T)=\mathbf{w}} \kappa^{-|T^0|} W_\mathbf{T} \sum_i \varepsilon(C_i(T)). \tag{7.15}
\]

7.11. **Application of a result of [GPS]**. In the last step of the proof we relate the stability data \( \text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1+l_2}]\epsilon^{l_1+l_2} \) to the tropical count \( N^{\text{trop}}(\mathbf{w}) \). This is where the techniques of [GPS] section 2 are required.

By its very definition, \( \text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1+l_2}] \) admits the following description. Consider the ordered factorisation problem in \( \text{Aut}(\mathbf{w}) \) given by

\[
\prod_j \text{Ad exp}(-\varepsilon \text{Li}_2(e_{\eta_j})) \prod_i \text{Ad exp}(-\varepsilon \text{Li}_2(e_{\gamma_i})) = \prod \text{Ad exp}(-\Omega(\bar{\alpha}; Z^-)(\varepsilon) \text{Li}_2(e_{\bar{\alpha}})) \tag{7.16}
\]

where \( \bar{\alpha} = \sum_{i=1}^{l_1} a_i \gamma_i + \sum_{j=1}^{l_2} b_j \eta_j \) and we are writing the operators from left to right in the clockwise order of \( Z^+(\bar{\alpha}) = Z^+(\alpha) \), for \( \alpha = \pi(\bar{\alpha}) \). It is straightforward to check that, by the definition of \( \mathbf{T} \), operators supported on the same ray commute (even if \( Z^- \) is degenerate) so (7.16) is well posed and admits a
unique solution. To compute this, we compare (7.16) with an ordered factorisation problem for automorphisms of a different algebra. As an intermediate step, let $R$ denote the formal power series ring $R = C[[s_1, \ldots, s_{\ell_1}, t_1, \ldots, t_{\ell_2}]]$. If we notice that (7.16) is equivalent to the factorisation problem over $\text{Aut}(g \otimes R)$

$$\prod_j \text{Ad} \exp(-\epsilon \text{Li}_2(t_j e_\eta)) \prod_i \text{Ad} \exp(-\epsilon \text{Li}_2(s_i e_\gamma)) = \prod_{\alpha \in \ell} \text{Ad} \exp \left( - \sum a_{\alpha} \text{DT}(\alpha, Z^-)(\epsilon) e_{\alpha} \right),$$

(7.17)

in the following sense: $\text{DT}(\alpha; Z^-)(\epsilon)$ will now be a polynomial in the variables $s_i, t_j$ (as well as $\epsilon$), and in fact

$$\text{DT}(\alpha; Z^-)(\epsilon) = \sum_{\pi(\bar{\alpha}) = \alpha} \text{DT}(\bar{\alpha}; Z^-)(\epsilon)(s, t)^{\bar{\alpha}},$$

where we set $(s, t)^{\bar{\alpha}} = \prod_i s_i^{a_{ij}} t_j^{b_{ij}}$. Thus $\text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1+l_2}]$ appears as the coefficient of the monomial $\epsilon^{l_1+l_2}(s, t)^{\bar{\beta}}$ in the polynomial $\text{DT}(\beta, Z^-)(\epsilon)$.

We are now in a position to compare with the results of [GPS]. First, as in ibid. section 1, we identify the operators appearing in (7.17) with (symplectic) automorphisms $\theta_{\alpha', f_{\alpha'}}$ of the power series ring $C[\Gamma][[s, t]]$ over the group algebra $C[\Gamma]$, acting (for primitive $\alpha'$) by

$$\theta_{\alpha', f_{\alpha'}}(e_\gamma) = e_\gamma f_{\alpha'}^{(\alpha', \gamma)}, \quad \theta_{\alpha', f_{\alpha'}}(e_\eta) = e_\eta f_{\alpha'}^{(\alpha', \eta)}.$$

The operators appearing on the left hand side of (7.17) act by

$$\theta_{\eta, f_j}(e_\gamma) = e_\gamma (1 - t_j e_\eta)^\epsilon, \quad \theta_{\eta, f_j}(e_\eta) = e_\eta,$$

respectively

$$\theta_{\gamma, f_i}(e_\gamma) = e_\gamma, \quad \theta_{\gamma, f_i}(e_\eta) = e_\eta (1 - s_i e_\gamma)^\epsilon.$$

Therefore

$$\log f_j = \sum_{p \geq 1} \frac{\epsilon}{p^2} e_{p \eta} t_j^{p}, \quad \log f_i = \sum_{q \geq 1} \frac{\epsilon}{q^2} e_{q \gamma} s_i^{q}.$$

In the notation of [GPS] section 1 (p. 312), we have

$$a_{jpp} = \frac{\epsilon}{p^2}, \quad a_{iqq} = \frac{\epsilon}{q^2},$$

with all other $a_{jkt}, a_{ik't'}$ vanishing. For $\alpha'$ primitive, the automorphism

$$\text{Ad} \exp \left( - \sum_{k \geq 1} \text{DT}(k \alpha', Z^-)(\epsilon) e_{k \alpha'} \right)$$

is the same as $\theta_{\alpha', f_{\alpha'}}$, with

$$\log f_{\alpha'} = \sum_{k \geq 1} \text{DT}(k \alpha', Z^-)(\epsilon) e_{k \alpha'}.$$
Let us go back to our $\bar{\beta} \in \Gamma$ with $\beta = \pi(\bar{\beta})$ primitive. According to [GPS] Theorem 2.8, the coefficient of the monomial $(s, t)^{\bar{\beta}} e^{\beta}$ in $\log f_{\beta}$ admits a tropical description: it equals the sum

$$\sum_{w'} \frac{N_{\text{trop}}(w')}{\text{Aut}(w')} \prod_j \prod_m a_{jw'_m} w'_m \prod_i a_{iw'_i} w'_i$$

summing over weight vectors $w' = (w'_{in}, w'_{jm})$, with $l_1 + l_2$ components, such that

$$\sum_n w'_{in} = w_1, \quad \sum_m w'_{jm} = w_j.$$

The invariant $N_{\text{trop}}(w')$ here is computed for a generic choice of constraints $\partial_{in}$ with the same direction, and similarly $\partial_{jm}$ with the same direction. The components $w'_{in}, w'_{jm}$ can be arbitrary increasing collections, satisfying only the condition above. However, we can refine our calculation further by looking only at the coefficient of the monomial $\epsilon^{l_1 + l_2}(s, t)^{\bar{\beta}} e^{\beta}$ in $\log f_{\beta}$. By the specific form of the coefficients $a_{jpp}, a_{iqq}$, this coefficient is given by the sum over weight vectors $w'$ for which the collections $w'_{in}, w'_{jm}$ contain a single element. There is precisely one such $w'$, given by

$$w' = ((w'_{11}), \ldots, (w'_{1l_1}), (w'_{21}), \ldots, (w'_{2l_2})).$$

Clearly, $|\text{Aut}(w')| = 1$, and as $w'$ is just a subdivision of $w$ (the type of constraints is the same), we have

$$N_{\text{trop}}(w') = N_{\text{trop}}(w).$$

Thus the coefficient of the monomial $\epsilon^{l_1 + l_2}(s, t)^{\bar{\beta}} e^{\beta}$ equals

$$N_{\text{trop}}(w) \prod_{k,l} \frac{1}{w_{kl}^2}.$$ 

On the other hand, we know already that this coefficient is precisely $\text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1 + l_2}]$. We have proved

$$\text{DT}(\bar{\beta}, Z^-)[\epsilon^{l_1 + l_2}] = N_{\text{trop}}(w) \prod_{k,l} \frac{1}{w_{kl}^2}. \quad (7.18)$$

7.12. **Comparison.** Comparing our formulae (7.15), (7.18) gives the promised connection between the tropical types attached to flat sections and actual tropical counts,

$$\sum_{\deg(T) = w} W_T \sum_i \varepsilon(C_i(T)) = \kappa^{l_1 + l_2} \frac{N_{\text{trop}}(w)}{|\text{Aut}(w)|} \prod_{k,l} \frac{1}{w_{kl}^2} (|w|_1 \gamma + |w|_2 \eta), \quad (7.19)$$

where we are summing over trees $T$ with $W_T(Z^+) \neq 0$, i.e. decorated by positive multiples of $\gamma$ or $\eta$. This equality holds in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. We can make the relation a bit more explicit. Indeed one has

$$W_T = \prod_{k,l} \frac{1}{w_{kl}^2} \frac{1}{|\text{Aut}(T)|} \prod_{v \to w} \langle \alpha(v), \alpha(w) \rangle \gamma_T,$$
and so
\[ \sum_{\text{deg}(T) = w} \frac{1}{|\text{Aut}(T)|} \prod_{\{v \to w\} \subset T} \langle \alpha(v), \alpha(w) \rangle \sum_i \varepsilon(C_i(T)) \gamma_T = \kappa^{l_1 + l_2} N^{\text{trop}}(w) |\text{Aut}(w)| (|w|_1 \gamma + |w|_2 \eta). \]

8. A comparison with $tt^*$-type connections

8.1. $tt^*$-like picture. Suppose, for simplicity, that $\Gamma$ has even rank $2r$ and $\langle -, - \rangle$ is nondegenerate. The important physics paper [GMN1] considers families of stability data on $\mathfrak{g}$ parametrised by certain special submanifolds $B \subset \text{Hom}(\Gamma, \mathbb{C})$ of complex dimension $r$ ("Coulomb branches of pure $\mathcal{N} = 2$ field theories on $\mathbb{R}^3 \times S^1_R"). Starting from these data the authors propose a construction of a family of meromorphic connections $\nabla^{GMN}(Z, R)$ on $\mathbb{P}^1$, formally very similar to Dubrovin’s $tt^*$-connections, parametrised by $B$ and $R > 0$. These connections take values in the Lie algebra $\mathfrak{X}$ of complex-valued smooth vector fields on the compact torus $\Gamma^\vee \otimes_{\mathbb{Z}} U(1)$. The $\nabla^{GMN}(Z, R)$ have precisely the same form as $\nabla(Z, R)$, but with $A(\gamma) \in \mathfrak{X}$. The family $\nabla^{GMN}(Z, R)$ should be isomonodromic, with Stokes data given essentially by $\ell_{\gamma, Z}$ ($\gamma \in \Gamma^{\text{prim}}$), $\prod_{p \geq 1} T_{\gamma_p}^{\Omega(p, \gamma, Z)}$. The latter point is a bit tricky to make sense of, and it involves thinking of local flat sections $X_{\gamma}(z)$ of $\nabla^{GMN}(Z, R)$ as maps $\Gamma^\vee \otimes_{\mathbb{Z}} U(1) \to \Gamma^\vee \otimes_{\mathbb{C}^*}$, on which the (birational) torus automorphism inducing $T_{\gamma}^{\Omega(p, \gamma)} \in \text{Aut}^*(\hat{\mathfrak{g}})$ acts by acting on the target (here we regard $\hat{\mathfrak{g}}$ as functions on $\Gamma^\vee \otimes \mathbb{C}^*$). Physical arguments predict that the connections $\nabla^{GMN}(Z, R)$ should admit a distinguished set of local flat sections such that $\log X_{\gamma}(z; Z, R)$ gives local holomorphic Darboux coordinates for a global, complete hyperkähler manifold $(M, g_R)$. In the examples coming from pure $\mathcal{N} = 2$ theories, $(M, g_R)$ should be a Hitchin system on $\mathbb{P}^1$ with suitable irregular singularities. There is a well-known Hitchin torus fibration $M \to B$, and $R^{-\tau}$ should be identified essentially with the volume of the fibres.

8.2. Ooguri-Vafa $tt^*$-type connections. The above $tt^*$-like picture has been established rigorously in a few local (incomplete) cases, the basic example being as above that of $\Gamma \cong \mathbb{Z}^2$ generated by $\gamma, \eta$ with $\langle \gamma, \eta \rangle = 1$, and symmetric stability data given by $\Omega(\pm \gamma) = 1$ with all other $\Omega$ vanishing. There are by now a good number of mathematical references for this construction (see e.g. [C]), so we do not reproduce it in detail, but only give a brief reminder, and also collect some formulae we need later on. We write $\theta_\gamma, \theta_\eta$ for the angular variables on $\Gamma^\vee \otimes_{\mathbb{Z}} U(1)$ dual to $\gamma, \eta$.

The special submanifold $B^o \subset \text{Hom}(\Gamma, \mathbb{C})$ in this case is parametrised by $u, \Lambda \in \mathbb{C}$ with $u/\Lambda \notin \mathbb{R}_{\geq 0}$ and $|u| < |\Lambda|$, \[ Z(\gamma) = u, \ Z(\eta) = \frac{1}{2\pi i} \left( u \log \frac{u}{\Lambda} - u \right). \] Similarly to [4.11] the $tt^*$-type connections are given by [GMN1] \[ \nabla^{GMN}(Z, R) = -d - A_z dz = d - \left( \frac{1}{2 \pi i} A_z^{-1} + \frac{1}{2} A_z^0 + A_z^1 \right) dz, \]
with components expressed in terms of Bessel integrals

\[ K_\alpha(x) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{2} s + s^{-1}} s^\alpha \frac{ds}{s} \]

by

\[ A_z^{(-1)} = i\pi RZ \cdot \partial_\theta - \frac{R^2}{2} \sum_{n \neq 0} e^{in\theta} K_0(2\pi R|nu|) \partial_{\theta_n}, \]

\[ A_z^{(0)} = R|u| \sum_{n \neq 0} (\text{sign } n) e^{in\theta} K_1(2\pi R|nu|) \partial_{\theta_n}, \]

\[ A_z^{(1)} = -i\pi R\bar{Z} \cdot \partial_\theta - \frac{R^2}{2} \sum_{n \neq 0} e^{in\theta} K_0(2\pi R|nu|) \partial_{\theta_n}, \]

Notice that \( A_z^{(1)} = \overline{A_z^{(-1)}} \). The flatness equation

\[ \nabla^{GMN} \chi = 0 \]

is a linear PDE for functions \( \chi(z; u, R) \). Fundamental solutions \( \chi_\gamma, \chi_\eta \) for \( \nabla^{GMN} \) were computed in [GMN1]; they are formally very similar to \( X(z)(e_\gamma), X(z)(e_\eta) \), and are obtained from the reference functions

\[ X^{sf}_\gamma(z; u, R) = \exp(\pi Rz^{-1}Z_\alpha + i\theta_\alpha + \pi Rz\bar{Z}_\alpha). \]

through the monodromy-fixing equations

\[ \chi_\gamma(z) = \chi^{sf}_\gamma(z), \]

\[ \chi_\eta(z) = \chi^{sf}_\eta(z) \exp \left[ -\int_{\ell_\gamma} \frac{dz'}{z} \rho(z, z') \log(1 - \chi_\gamma(z')) \right. \]

\[ + \left. \int_{\ell_\gamma} \frac{dz'}{z} \rho(z, z') \log(1 - \chi_\gamma(z')^{-1}) \right]. \]

8.3. Relation to hyperKähler metric. Remarkably, one can show that the family of complex-valued two-forms

\[ \omega(z) = -\frac{1}{4\pi R} d\log \chi_\gamma(z) \wedge d\log \chi_\eta(z) \]

are holomorphic symplectic forms for a hyperKähler structure on \( \mathcal{M}^o = \Gamma^V \otimes Z \times U(1) \times B^o \). (Here of course we write \( d \) for the differential on \( \mathcal{M}^o \). A crucial ingredient for this is that the family \( \omega(z) \) has simple poles at both 0 and \( \infty \), which can be traced back to the double poles at 0 and \( \infty \) for \( \nabla^{GMN} \) (i.e. its \( tt^* \) form). The underlying hyperKähler metric has a \( U(1) \)-symmetry (shifting \( \theta_\eta \)), and therefore by general theory it can be written in Gibbons-Hawking form, i.e. in terms of a positive harmonic function \( V \) and a connection 1-form \( A \) with \( F(A) = *dV \),

\[ V(x) \left( \frac{d\theta_\eta}{2\pi} + A(x) \right)^2 + dx^2 \]
where
\[ x = \left( \text{Re}(u), \text{Im}(u), \frac{\theta_\gamma}{2\pi R} \right). \]

We will need explicit formulae for the potential:
\[
V = \frac{R}{4\pi} \sum_{n=-\infty}^{\infty} \left( \frac{1}{\sqrt{R^2 |u|^2 + \left( \frac{\theta_\gamma}{2\pi} + n \right)^2}} - \kappa_n \right),
\]
where, according to the footnote in [GMN1] p. 22, the regularization constants are given by
\[
\kappa_n = \frac{1}{\sqrt{\Lambda^2 + n^2}}
\]
for some choice of complex parameter $\hat{\Lambda}$. The parameters $\Lambda$, $R$ and the regularization constants $\hat{\Lambda}$ are constrained by
\[
\Lambda = R^{-1} \hat{\Lambda} \exp \left( -2 \sum_{m=1}^{\infty} K_0(2\pi m |\hat{\Lambda}|) \right).
\]

By construction, $V$ is periodic in $\theta_\gamma$, and the metric is easily seen to extend to $0 < |u| < |\Lambda|$. However, it extends even across $u = 0$: the standard way to see this is to compare it with the Taub-NUT hyperKähler metric near the origin. The resulting incomplete hyperKähler metric over the disc of radius $|\Lambda|$ can be seen as a metric on a neighbourhood of a nodal elliptic curve, and is known as Ooguri-Vafa metric.

The potential $V$ is closely related to the Bessel integrals appearing in $\nabla^{GMN}$: one can show that
\[
V = V + V^{\text{sf}},
\]
with
\[
V^{\text{sf}} = -\frac{R}{4\pi} \left( \log \frac{a}{\Lambda} + \log \frac{\bar{a}}{\Lambda} \right),
\]
\[
V^{\text{inst}} = \frac{R}{2\pi} \sum_{n \neq 0} e^{in\theta_\gamma} K_0(2\pi R |na|).
\]

### 8.4. Comparison morphism

Finally, let us show that in this special example there is a Lie algebra morphism $X \to D^\dagger(\hat{\mathfrak{g}})$ taking $\nabla^{GMN}$ to $\nabla^{\text{sym}}$.

Consider a symplectic real torus $(T, \omega)$ of dimension 2. Let
\[
\mathfrak{a} = (C^\infty(T, \mathbb{C}), \{ -,- \}, -)
\]
be the Poisson algebra of $\mathbb{C}$-valued smooth functions on $T$. We also consider the Poisson algebra $\hat{\mathfrak{g}}$ modeled on the lattice $\Gamma = H_1(T, \mathbb{Z})$, and construct a homomorphism of Poisson Lie algebras
\[
\Phi: \mathfrak{a} \to \hat{\mathfrak{g}}.
\]
To define $\Phi$, we choose affine symplectic coordinates on the torus $\theta_\gamma$ and $\theta_\eta$ (so that $\omega = -d\theta_\gamma \wedge d\theta_\eta$) and use the Fourier expansion of an element in $a$. Setting 

$$\Phi(e^{ik\theta_\gamma + in\theta_\eta}) = e^{k\gamma}e^{in\eta},$$

where $\eta \in \Gamma = H_1(T, \mathbb{Z})$ (resp. $\gamma$) is the dual of $d\theta_\gamma$ (resp. $d\theta_\eta$), we have 

$$\Phi\left(\{e^{ik\theta_\gamma}, e^{in\theta_\eta}\}\right) = \Phi(kne^{ik\theta_\gamma + in\theta_\eta}) = [e^{k\gamma}, e^{n\eta}]$$

and hence $\Phi$ extends to an homomorphism. (Note that $\langle \gamma, \eta \rangle = 1$). 

As usual we write $D^*(\widehat{g})$ for Lie algebra of derivations of the commutative algebra $(\widehat{g}, \ast)$; the derivations of $\widehat{g}$ which are Poisson (i.e. also satisfy the Leibniz rule) are denoted by $D(\widehat{g}) \subset D^*(\widehat{g})$. Notice that there is a distinguished sub-Lie algebra $\text{ad}(\widehat{b}) \subset D(\widehat{g})$ generated by $\text{Hom}(\Gamma, \mathbb{C})$ and $\widehat{g}$ (the latter acting via the adjoint representation). The reason for the notation $\text{ad}(\widehat{b})$ will be clarified soon. 

Similarly, we have the Lie algebra $\mathfrak{x}$ of complex valued vector fields on $T$, containing the sub-Lie algebra $D(a)$ of derivations of the Poisson Lie algebra $a$. 

$D(\widehat{g})$ contains distinguished elements $\partial_\gamma = d\theta_\gamma$, $\partial_\eta = d\theta_\eta$ (so e.g. $\partial_\gamma(\gamma) = 1$ and $\partial_\gamma(\eta) = 0$). For a general element $D = f\partial_\gamma + g\partial_\eta \in \mathfrak{x}$, we define 

$$\Phi D = \Phi(f) \ast i\partial_\gamma(\Phi(-)) + \Phi(g) \ast i\partial_\eta(\Phi(-)),$$

thus extending $\Phi$ to a morphism $\mathfrak{x} \to D^*(\widehat{g})$. 

Notice also that $D(a) \subset \mathfrak{x}$ can be identified with the complexification of the Lie algebra of symplectic vector fields, and so admits a decomposition 

$$D(a) = a \oplus \mathfrak{h}$$

where $\mathfrak{h} = \text{Hom}_\mathbb{C}(\Gamma, \mathbb{C})$. We claim that this implies that for any $D \in D(a)$ there exists (very likely unique) $D' \in \text{ad}(\widehat{b})$ such that 

$$\Phi D = D' \Phi.$$

As $\Phi$ is a Poisson Lie algebra morphism, it is enough to check this for an element $Z \in \mathfrak{h}$. As a derivation, $Z$ is identified with a complex vector field of the form 

$$Z = a\partial_\gamma + b\partial_\eta$$

for $a, b \in \mathbb{C}$. Then, one can easily check that 

$$\Phi Z = i(a\partial_\gamma + b\partial_\eta)(\Phi(-)).$$

We now calculate the connection $D'$ related via $\Phi$ with the $D^*(a)$-valued connection $\nabla^{GMN}$. In fact, a little thought shows that in this special case $\nabla^{GMN}$ is in fact $D(a)$-valued, so $D'$ will be in $D(\widehat{g})$. We will show that actually $D'$ lies in $\text{ad}(\widehat{b})$, and indeed 

$$\Phi \nabla^{GMN}(Z) = \nabla^{\text{sym}}(-\pi Z).$$
This follows immediately from the identity
\[ \Phi \left( \frac{R}{2} \sum_{n \neq 0} e^{in} K_0(2\pi R|nu|) \partial \theta_n \left( e^{ik\theta} + e^{in\theta_n} \right) \right) = i \left( \frac{R}{2} \sum_{n \neq 0} K_0(2\pi R|nu|) e^{in\gamma} \right) \ast n' e^{k'\gamma + n'\eta} \]
\[ = - \left[ R\pi Z(\gamma) \sum_{n \neq 0} \frac{1}{2ni\pi} K_0(2\pi R|nu|) e^{n\gamma}, e^{k'\gamma + n'\eta} \right], \]
and the analogous one for \( A_z^{(0)} \).

9. The Conformal Limit

9.1. For most of this paper we have been concerned with some special limits of the connections \( \nabla_{(Z,R)} \) when \( R \to 0 \) (when the connections take Bridgeland-Toledano Laredo shape) or \( R \to \infty \). It is important to understand these limits for the \( tt^* \)-type connections of the previous section. Here we concentrate on the \( R \to 0 \) limit.

Recall that
\[ \nabla_{GMN}^{(Z,R)} = d - \left( \frac{1}{z^2} A_z^{(-1)} + \frac{1}{z} A_z^{(0)} + A_z^{(1)} \right) dz, \]
where the components \( A_z^{(i)} \) are independent of \( z \). Rescaling by \( z = Rt \) takes \( \nabla_{GMN} \) to
\[ \nabla_{t}^{GMN} = d - \left( \frac{1}{t^2} R^{-1} A_z^{(-1)} + \frac{1}{t} A_z^{(0)} + RA_z^{(1)} \right) dt. \]
So at least formally, when \( R \to 0 \), we seem to end up with a connection which has a pole of order two at 0, and a simple pole at \( \infty \), just as for \( \nabla^{BTL} \), although with values in \( X \) rather than \( D^*(\hat{\mathfrak{g}}) \). So one may hope to map this to a \( D^*(\hat{\mathfrak{g}}) \)-valued connection using \( \Phi \).

However this does not work so simply, and we will show that there are divergencies in \( R^{-1} A_z^{(-1)}(R) \) as \( R \to 0 \). (The component \( A_z^{(0)} \) does have a limit, and \( RA_z^{(1)} \) vanishes as \( R \to 0 \).) In this special example, one can get rid of the divergencies in (at least) two different ways: by adjusting the family of central charges \( Z \) as \( R \to 0 \), tweaking the \( \Lambda \) parameter, or by acting with \( R \)-dependent complex gauge transformations. The first method has a nice interpretation, via the physical meaning of \( |\Lambda| \), as a redefinition of the “energy scale” as \( R \to 0 \). The second method seems to be more general, and it is what we can really get to work in the formal case of \( D^*(\hat{\mathfrak{g}}) \)-valued connections. The limits we get are different (not gauge equivalent). An interesting point is that in both cases, the limiting connections are not smooth. The limit obtained with gauge transformations takes values in \( L^2 \) complex vector fields.
9.2. We first compute the limit
\[
\lim_{R \to 0} A_z^{(0)}(R).
\]
It is useful to introduce a slightly different normalization for \(K\)-Bessel functions, namely
\[
\tilde{K}_\alpha(c) = 2K_\alpha(2c) = \int_0^\infty e^{-c(s+s^{-1})} s^\alpha \frac{ds}{s}.
\]

Then we have
\[
A_z^{(0)} = \frac{|u|}{2} \sum_{n \neq 0} (\text{sign } n) R\tilde{K}_1(\pi R|nu|) e^{in\theta} \partial_{\theta_q}.
\]

The one-variable \(\tilde{K}\)-Bessel function has a natural two-variables version,
\[
\tilde{K}_\alpha(v, w) = \int_0^\infty e^{-(v^2s+w^2s^{-1})} s^{\alpha} \frac{ds}{s}.
\]

This is not really more general, due to the identity
\[
\tilde{K}_\alpha(v, w) = (w/v)^\alpha \tilde{K}_\alpha(vw).
\]

So we can rewrite
\[
R\tilde{K}_1(\pi R|nu|) = \pi|nu| \frac{R}{\pi|nu|} \tilde{K}_1(\pi|nu|) \cdot R
\]
\[
= \pi|nu|\tilde{K}_1(\pi|nu|, R).
\]

By definition, for \(R > 0\)
\[
\tilde{K}_1(\pi|nu|, R) = \int_0^\infty e^{-R^2s^{-1}} e^{-(\pi|nu|)^2s} ds < \int_0^\infty e^{-(\pi|nu|)^2s} ds,
\]
from which we get for \(R > 0\)
\[
0 < R\tilde{K}_1(\pi R|nu|) < \frac{1}{\pi|nu|}, \tag{9.1}
\]

Moreover, we have clearly
\[
\lim_{R \to 0} R\tilde{K}_1(\pi R|nu|) = \frac{1}{\pi|nu|}. \tag{9.2}
\]

The series
\[
\sum_{n \neq 0} \frac{1}{n} e^{in\theta_q}
\]
is the the Fourier series of the purely imaginary, periodic \(L^2(S^1)\) seesaw function
\[-\log(e^{i\theta_q}).
\]

Using (9.1), (9.2), dominated convergence in \(L^2(Z)\), and Plancharel's theorem, we see that
\[
\lim_{R \to 0} A_z^{(0)}(R) = \frac{1}{2\pi} \log(e^{i\theta_q}) \partial_{\theta_q}.
\]
9.3. Divergencies and regularisation. Let us now turn to
\[
\lim_{R \to 0} \frac{A_{z}^{(-1)}(R)}{R}.
\]
The decomposition
\[
R^{-1}A_{z}^{(-1)}(R) = i\pi Z \cdot \partial_{\theta} - \frac{u}{2} \sum_{n \neq 0} e^{in\theta} K_{0}(2\pi R|nu|)\partial_{\theta_{n}},
\]
for fixed Z, is badly behaved as \(R \to 0\). The second piece of this decomposition diverges pointwise as \(R \to 0\). To see this recall that \(A, R\) and the regularisation constants \(\kappa_{n}\) are related by
\[
\kappa_{n} = \frac{1}{\sqrt{|\tilde{\Lambda}|^{2} + n^{2}}}
\]
\[
\Lambda = R^{-1}\tilde{\Lambda} \exp \left( -2 \sum_{m=1}^{\infty} K_{0}(2\pi m|\tilde{\Lambda}|) \right).
\]
On the other hand, we have
\[
Z(\eta) = \frac{1}{2\pi i} \left( u \log \frac{u}{\Lambda} - u \right).
\]
Thus the quantity \(Z \cdot \partial_{\theta}\) in (9.3) is also implicitly a function of \(R\). If we want this to be constant, or more generally to have a well defined limit as \(R \to 0\), we need \(\Lambda(R)\) to converge to a finite limit \(\Lambda_{0} \neq 0\) as \(R \to 0\). By the above relation, this implies that \(\tilde{\Lambda}\) must also be chosen to be a function of \(R\), with \(|\tilde{\Lambda}| \to 0\) as \(R \to 0\). But then \(\kappa_{0} = |\tilde{\Lambda}(R)|^{-1}\) becomes divergent as \(R \to 0\). So \(V\) diverges pointwise when \(R \to 0\), and in fact since
\[
\sum_{n \neq 0} e^{in\theta} K_{0}(2\pi R|na|) = -\frac{1}{2} \left( \log \frac{a}{\Lambda} + \log \frac{\bar{a}}{\Lambda} \right) - V,
\]
the second piece of (9.3) diverges pointwise.

This argument seems to show a way out: we can simply choose \(\tilde{\Lambda}\) to be a nonzero constant. Then as \(R \to 0\) the quantity \(R^{-1}V\) converges pointwise to the function of \(\theta_{\gamma} \in (0, 2\pi)\) given by the sum of series
\[
\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left( \frac{1}{\frac{\bar{a}}{2\pi} + n} - \frac{1}{\sqrt{|\tilde{\Lambda}|^{2} + n^{2}}} \right).
\]
If we choose \(\tilde{\Lambda}\) to be a nonzero constant (independent of \(R\)) as above so that \(R^{-1}V(R)\) has a finite limit as \(R \to 0\), then \(R^{-1}A_{z}^{(-1)}(R)\) does have a finite,
nontrivial $R \to 0$ limit, namely

$$i\pi u \partial_{\theta_\gamma} + \frac{u}{4} \left( \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \frac{1}{n + \sqrt{|\Lambda|^2 + n^2}} \right) - \log \left( \frac{\bar{u}}{u} \frac{\bar{\Lambda}}{\Lambda} \right) - 2 \right) \partial_{\eta}. $$

This follows from rewriting (9.3) in terms of $V$,

$$A_z^{(-1)} = -u \left( -i\pi R \partial_{\theta_\gamma} + \pi \left( V - \pi R \frac{1}{4\pi} \log \left( \frac{\bar{u}}{u} \frac{\bar{\Lambda}}{\Lambda} \right) \partial_{\eta} \right) - \frac{Ru}{2} \partial_{\eta} \right). $$

Of course in this case $Z$ is also a function of $R$, which does not have a limit as $R \to 0$ since $|\Lambda(R)| \to \infty$ as $R \to 0$. In the light of the physical interpretation of $|\Lambda|$, we may say that we got rid of the $R \to 0$ divergency in $R^{-1}A_z^{(-1)}$ by a redefinition of the “energy scale”.

9.4. Distributional point of view. For the sake of completeness we also give a simple distributional argument for the above regularisation result, without appealing to an explicit choice of regularisation constants (for which we do not have a purely mathematical reference). For all $R > 0$, we have an equality of distributions

$$f(\theta_\gamma, R) = \frac{1}{2\pi} \sum_{n\neq 0} e^{in\theta_\gamma} K_0(2\pi R|nu|) $$

$$= -\frac{1}{2\pi} \log(R) \sum_{n\neq 0} e^{in\theta_\gamma} + \frac{1}{2\pi} \sum_{n\neq 0} (K_0(2\pi R|nu|) + \log(R)) e^{in\theta_\gamma}. $$

Recalling the distributional identity

$$\sum_{n=-\infty}^{+\infty} e^{i2\pi x} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(x - n) $$

we find

$$f(\theta_\gamma, R) = -\frac{1}{2\pi} \log(R) \left( -1 + \sum_{n=-\infty}^{+\infty} \delta(\theta_\gamma/2\pi - n) \right)$$

$$+ \frac{1}{2\pi} \sum_{n\neq 0} (K_0(2\pi R|nu|) + \log(R)) e^{in\theta_\gamma}. $$

On the other hand we have

$$\lim_{R \to 0} K_0(2\pi R|nu|) + \log(R) = -\log|u| - \log|n| - \gamma,$$
where $\gamma$ denotes Euler’s constant (hopefully without ambiguity with the element $\gamma \in \Gamma$). It follows that, in the sense of distributions,

$$
\begin{align*}
    f(\theta, R) & \sim \frac{1}{2\pi} \log(R) \left( 1 - \sum_{n=-\infty}^{+\infty} \delta(\theta/2\pi - n) \right) \\
    & \quad + \frac{1}{2\pi} \left( \log |u| + \gamma \right) \left( 1 - \sum_{n=-\infty}^{+\infty} \delta(\theta/2\pi - n) \right) + \frac{1}{2\pi} \sum_{n \neq 0} \log |n| e^{in\theta} + O(R).
\end{align*}
\tag{9.5}
$$

We interpret this as the decomposition of the distributional $R \to 0$ limit of $f(\theta, R)$ into a singular plus a regular part. Notice that away from $2\pi \mathbb{Z}$ we have an equality of smooth functions

$$
    f(\theta, R) \sim \frac{1}{2\pi} \log(R) \\
    + \frac{1}{2\pi} \left( \log |u| + \gamma \right) - \frac{i}{2\pi} \partial_{\theta} \sum_{n \neq 0} \frac{\log |n|}{n} e^{in\theta} + O(R).
$$

Now we can just push the infinities into $\Lambda$. By (9.3) we have

$$
    R^{-1} A_{\gamma}^{(-1)} = i\pi u \partial_{\theta} + \frac{1}{2} \left( u \log \frac{u}{\Lambda_0} - u \right) \partial_{\theta} - \pi a f(\theta, R) \partial_{\theta}.
$$

By (9.4) then

$$
    R^{-1} A_{\gamma}^{(-1)} \sim i\pi u \partial_{\theta} + \frac{1}{2} \left( u \log \frac{u}{\Lambda_0} - u \right) \partial_{\theta} - \frac{u}{2} \log(\Lambda/\Lambda_0) \partial_{\theta} \\
    - \frac{u}{2} \log(R) \left( 1 - \sum_{n=-\infty}^{+\infty} \delta(\theta/2\pi - n) \right) \partial_{\theta} \\
    - \frac{u}{2} \left( \log |a| + \gamma \right) \left( 1 - \sum_{n=-\infty}^{+\infty} \delta(\theta/2\pi - n) \right) \partial_{\theta} \\
    - \frac{u}{2} \sum_{n \neq 0} \log |n| e^{in\theta} \partial_{\theta} + O(R).
$$

Letting $\Lambda/\Lambda_0 \sim R^{-1}$ cancels out the singular $R \to 0$ term and gives a smooth expansion, away from $2\pi \mathbb{Z}$,

$$
    R^{-1} A_{\gamma}^{(-1)} \sim i\pi u \partial_{\theta} + \frac{1}{2} \left( u \log \frac{u}{\Lambda_0} - u \right) \partial_{\theta} \\
    - \frac{u}{2} \log(R) \left( 1 - \sum_{n=-\infty}^{+\infty} \delta(\theta/2\pi - n) \right) \partial_{\theta} \\
    - \frac{u}{2} \left( \log |u| + \gamma \right) \partial_{\theta} + \frac{u}{2} i \partial_{\theta} \sum_{n \neq 0} \frac{\log |n|}{n} e^{in\theta} \partial_{\theta} + O(R).
$$

\subsection*{9.5. Gauge transformations.}

A different approach to extract a meaningful $R \to 0$ limit is to act on $A_{\gamma}(R)$ with a (divergent) sequence of complex gauge transformations $g(R)$ as $R \to 0$. This approach seems to be more general and it is the one we will actually follow for the $\nabla(Z, R)$ connections.
In the present example, for $tt^*$-type connections, we consider the family of differentiable maps $\Upsilon$ from $\Gamma^\vee \otimes_\mathbb{Z} U(1) \cong S^1 \times S^1$ to $\Gamma^\vee \otimes_\mathbb{Z} \mathbb{C}^* \cong \mathbb{C}^* \times \mathbb{C}^*$ given by

$$\Upsilon(e^{i\theta}, e^{i\eta}) = \left( e^{i\theta}, e^{i\eta} \exp \left( \int_{\ell_{\pm\gamma}} \frac{dz'}{z'} \rho(z, z') \log(1 - \mathcal{X}_{\pm\gamma}(z')) \right) \right).$$

For $R > 0$ and $z \in \mathbb{C}^*$ the map $\Upsilon(-; z, R)$ is a diffeomorphism on its image. Let us keep $R > 0$ fixed for a moment. As $z$ varies in $\mathbb{C}^* \setminus \{\ell_{\pm\gamma}\}$, one gets a family of differentiable maps $\Upsilon(z): S^1 \times S^1 \to S^1 \times \mathbb{C}^* \subset (\mathbb{C}^*)^2$ (using our fixed identifications). These maps have a well-defined limit $\Upsilon_0$ as $z \to 0$ (nontangentially to $\ell_{\pm\gamma}$), given by

$$\Upsilon_0(e^{i\theta}, e^{i\eta}) = \left( e^{i\theta}, e^{i\eta} \exp \left( \frac{1}{2\pi i} \sum_{n \neq 0} \frac{1}{n} e^{in\eta} K_0(2\pi R|nu|) \right) \right).$$

For all $R > 0$, the function $F(\theta, R)$ given by the sum of the convergent series

$$\frac{1}{2\pi i} \sum_{n \neq 0} \frac{1}{n} e^{in\theta} K_0(2\pi R|nu|)$$

is real and odd. Indeed, we have

$$F(\theta, R) = \int_0^\theta f(\tau, R) d\tau.$$

**Remark 9.1.** One can check that

$$||F(\theta, R)||_\infty > -C \log(R).$$

and so as $R \to 0$ the diffeomorphisms $\Upsilon_0(R)$ diverge rapidly.

We wish to interpret $\Upsilon_0(R)$ as a sequence of complex gauge transformations, i.e. elements in the complexification of the “gauge group” Diff($S^1 \times S^1$). In general, this is a difficult notion to make sense of; in our case we only need to give a meaning to the action of $\Upsilon_0(R)$ on the $tt^*$-type connection. The natural choice is to define

$$\Upsilon_0^{-1}\nabla^{GMN} \Upsilon_0 = d - (\Upsilon_0)_* A_\zeta \, dz,$$

i.e. to push forward the complex-valued vector field $A_z$ by $(\Upsilon_0)_*$. This means that we need to allow connections which take values in the Lie algebra of complex-valued vector fields on $S^1 \times \mathbb{C}^*$. We will check that $\Upsilon_0$ extends to an element of Diff($S^1 \times \mathbb{C}^*$) and $A_z$ extends to a vector field on $S^1 \times \mathbb{C}^*$.

Indeed recall that the coefficients of the complex vector fields $A^{(0)}_z$, $A^{(-1)}_z$ on $S^1 \times S^1$ do not depend on the variable $\theta$. So writing

$$w = \rho e^{i\theta},$$

for a complex variable $w$ on $\mathbb{C}^*$, we can extend $A^{(0)}_z$, $A^{(-1)}_z$ to smooth complex vector fields on $S^1 \times \mathbb{C}^*$ simply by extending $\partial_{\theta}$ to $i(w \partial_w - \bar{w} \partial_{\bar{w}})$ (we still
write $\partial_{\theta_{\eta}}$ for this extension). Similarly, $\Upsilon_0$ admits a natural extension to a self-diffeomorphism of $S^1 \times \mathbb{C}^*$ by replacing $e^{i\theta_0}$ with $w$.

By (9.3) and the definition of $f(\theta_\gamma, R)$ we have

$$R^{-1}A^{(-1)}_z(R) = i\pi Z(\gamma)\partial_{\theta_{\gamma}} + (i\pi Z(\eta) - \pi uf(\theta_\gamma, R))\partial_{\theta_{\eta}}.$$ 

Consider the change of variables

$$\tilde{\theta}_\gamma = \theta_\gamma,$$
$$\tilde{\theta}_\eta = \theta_{m\eta} + \lambda F(\theta_\gamma, R),$$

(9.6)

where $F(\theta_\gamma, R)$ is the primitive for $f(\theta_\gamma, R)$ introduced above. The differential is given by

$$\partial_{\theta_{\gamma}} = \partial_{\tilde{\theta}_\gamma} + \lambda f(\theta_\gamma, R)\partial_{\tilde{\theta}_\eta},$$
$$\partial_{\theta_{\eta}} = \partial_{\tilde{\theta}_\eta};$$

and since $Z(\gamma) = u$, our change of variables transforms $R^{-1}A^{(-1)}_z(R)$ to

$$i\pi Z(\gamma)\partial_{\tilde{\theta}_\gamma} + (i\pi Z(\eta) + \pi Z(\gamma)(\lambda i - 1) f(\tilde{\theta}_\gamma, R))\partial_{\tilde{\theta}_\eta},$$

So if we choose $\lambda = -i$, the pushforward of $R^{-1}A^{(-1)}_z(R)$ under the transformation (9.6) becomes simply

$$i\pi Z \cdot \partial_{\tilde{\theta}}$$

Of course $A^{(0)}_z$ is unchanged. Choosing $\lambda = -i$ in (9.6) means that we are in fact regarding it as a self-diffeomorphism of $S^1 \times \mathbb{C}^*$ given by

$$(\tilde{\theta}_\gamma, w) \mapsto (\theta_\gamma, e^{F(\theta_\gamma, R)}w).$$

On the other hand the map $\Upsilon_0$ can be rewritten as

$$\Upsilon_0(e^{i\theta_\gamma}, e^{i\theta_\eta}) = (e^{i\tilde{\theta}_\gamma}, e^{i(\theta_\gamma - iF(\theta_\gamma, R))}),$$

hence the transformation (9.6) is the same as $\Upsilon_0$.

To summarise this construction, we have “complexified” the connection $\nabla_{t}^{GMN}(R)$ by extending it to one with values in complex vector fields on $S^1 \times \mathbb{C}^*$, and then acted on it with the complex gauge transformation $\Upsilon_0(R)$ (an element of $\text{Diff}(S^1 \times \mathbb{C}^*)$), getting

$$\Upsilon_0(R) \cdot \nabla_{t}^{GMN}(R) = \frac{1}{t^2}(i\pi Z \cdot \partial_{\tilde{\theta}}) - \frac{1}{t} \left( \frac{1}{2\pi} \log(e^{i\theta_\gamma}) \right) \partial_{\theta_\eta}.$$

If we choose our parameters so that $\Lambda(R)$ has a well-defined, nonzero limit as $R \to 0$, this sequence of connections has a well-defined $R \to 0$ limit. For example we can assume $\Lambda \equiv 1$, so the sequence is actually constant ($A^{(-1)}_z(R)$ diverges, but this is not a problem by acting with gauge transformations).
9.6. The limit connection \( \lim_{R \to 0} \mathcal{Y}_0(R) \cdot \nabla^G_{MN}(R) \) is not smooth, but takes values in \( L^2 \) periodic vector fields. Denoting by \( \widehat{\mathcal{X}} \), we can extend the morphism \( \Phi \) to a map \( \widehat{\mathcal{X}} \to D^*(\widehat{\mathfrak{g}}) \) by using the Fourier expansion of periodic \( L^2 \) vector fields. There is no Poisson bracket on \( \widehat{\mathcal{X}} \) (the product of \( L^2 \) functions is not in general \( L^2 \)), however the extension of \( \Phi \) preserves the bracket whenever it is defined. For \( D \in \widehat{\mathcal{X}} \) we have

\[
\Phi D\phi = \Phi(f \partial_{\theta_e} \phi) + \Phi(g \partial_{\theta_m} \phi)
\]

for any smooth \( \phi \) smooth, as in this case \( f \partial_{\theta_e} \phi \) and \( g \partial_{\theta_m} \phi \) are in \( L^2 \). Recall that the dilogarithm function in \( \widehat{\mathfrak{g}} \) is

\[
\text{Li}_2(e^\alpha) = -\sum_{k \geq 1} \frac{e^{k\alpha}}{k^2}.
\]

Then one can show

\[
\lim_{R \to 0} \mathcal{Y}_0(R) \cdot \nabla^G_{MN}(R) = -\frac{\pi}{t^2} - \frac{1}{t} \text{ad} \frac{\text{Li}_2(e_{\gamma_e}) - \text{Li}_2(e_{-\gamma_e})}{2\pi i}.
\]

This is based on the identity

\[
\Phi((\log(1 - e^{i\theta_e}) - \log(1 - e^{-i\theta_e}))\partial_{\theta_m}) = i \text{ad}(\text{Li}_2(e_{\gamma_e}) - \text{Li}_2(e_{-\gamma_e}))*\Phi
\]

which follows from direct computation.

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190 CH-8057 Zürich, Switzerland

E-mail address: saraangela.filippini@math.uzh.ch

École Polytechnique Fédéral de Lausanne, SB MATHGEOM, MA B3495 (Bâtiment MA) Station 8, CH-1015 Lausanne, Switzerland

E-mail address: mario.garcia@epfl.ch

Università di Pavia, Dipartimento di Matematica “F. Casorati”, Via A. Ferrata 1, 27100 Pavia, Italy

E-mail address: jacopo.stoppa@unipv.it