Two-Fold Circle-Covering of the Plane under Congruent Voronoi Polygon Conditions

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Abstract. The $k$-coverage problem is to find the minimum number of circles such that each point in a finite plane is covered by at least $k$ circles. Under equal circle condition, when $k=1$, this problem has been solved by Kershner in 1939. However, when $k > 1$, it becomes hopelessly difficult. One tried to tackle this problem with different restrictions. In this paper, we restrict ourself to congruent Voronoi polygon, and prove the minimum density of the two-coverage with such a restriction. Our proof is simpler and more rigorous than that given recently by Yun et al. [5,6].

Keywords: two-coverage problem, congruent voronoi polygon

1 Introduction

A set of circles is said to form an $k$-fold covering if every point of the plane belongs to at least $k$ circles. The problem of finding the thinnest (or minimum density) $k$-fold covering has been studied for a long time.

In 1939, Kershner [1] solved the one-coverage problem with equal circles, and proved that the triangle lattice pattern is the best, and its minimum density $\vartheta$ is $\vartheta = \pi/A_6$, where $A_6 = 3\sqrt{3}/2$ is the area of a regular hexagon inscribed in a unit circle. The $k$-coverage problem with $k > 1$ is much more difficult than the one-coverage problem. So far only a few special $k$-coverage problem was solved.

For example, in 1957, Blundon [2] restricted himself to lattice-coverings, i.e. suppose that the centres of the circles form a lattice. Let $\vartheta^{(k)}$ be the minimum density of $k$-fold lattice covering by equal circles. Blundon proved that $\vartheta^{(2)} = 2\vartheta$, $\vartheta^{(3)} = 2.841\vartheta$, $\vartheta^{(4)} = 3.608\vartheta$, ... 

In 1960, instead of Blundon’s single lattice-covering, Danzer [3] considered a multiple lattice-covering, and obtained the minimum density $D_2$ of the two-coverage: $2.094 \ldots \leq D_2 \leq 2.347 \ldots$.

In 1976, Toth [4] used the notion of the $k$-th Dirichlet (Voronoi) cell to estimate the the lower bound of the $k$-coverage. Let $D_k$ be the minimum density of the $k$-coverage. Toth proved that $D_k \geq \frac{\pi}{3} \csc \frac{\pi}{k}$.

Recently Yun et al. [5,6] investigated congruent Voronoi polygon coverings, i.e., Suppose that Voronoi polygons generated by the centres of the equal circles are congruent. Then under such a restriction, how many is the minimum density of the two-coverage? Here, the Voronoi polygon $V(c)$ is defined as the set of all
points that are closer to the center $c$ of a circle than to any other center, i.e., $V(c) = \{x | d(x, c) \leq d(x, a), a$ is the center of any other circle$\}$, where $d(\cdot, \cdot)$ denotes Euclidean distance. Under congruent Voronoi polygon conditions, the covering density is defined as the ratio of the area of a circle to the area of the Voronoi polygon inscribed in the circle. Yun et al. [5] claimed that the minimum density of the two-coverage under congruent Voronoi polygon conditions is $2\vartheta$. However, their proof on it is considered not to be rigorous. For example, Lemma 4.1 in [5] is not straightforward. However, Yun et al. did not prove it. That is, they did not show why there are only 12 types of congruent Voronoi polygonal tessellations. In [7,8], Grübaum et al. listed the 107 polygonal isohedral types of tilings. Yun et al. did not show how to reduce the 107 types to 12 types and exclude non-isohedral tilings (Voronoi is not necessarily isohedral). Also, in the proof of Lemma 4.7 of Ref.[5], the authors proved type (h), using the unproved inequality: $2\sqrt{r^2 - \left(\frac{a^2}{\sqrt{a^2 + 1}}\right)^2} \geq \sqrt{a^2 + 1}$. We cannot conclude that this inequality holds.

This paper addresses the same two-coverage problem as Ref.[5], i.e., two-fold covering with congruent Voronoi polygonal restrictions. However, we adopt a different proving strategy to provide a simpler and more rigorous proof than Yun et al.

2 Minimum Density of the Two-Coverage with Congruent Voronoi Polygon Restrictions

![Fig. 1. Three intersections $A, B, C$ of three equal circles $C_1, C_2, C_3$ lie on the red circle of the same radius.](image)

The main theorem of this paper will be presented after the following two lemmas.

**Lemma 1.** As shown in Figure 1, suppose that three equal circles $C_1, C_2, C_3$ with radius $r$ meets at a point $Q$, and their other three intersections are $A, B$ and $C$. Then the radius of circumcircle of triangle $ABC$ is $r$ also, i.e., these four circles are equal.
Proof. By the triangle formed by three centers of circles $C_1, C_2, C_3$, we obtain that arc $AQB + arc BQC + arc CQA = 2\pi$. For the red circle, we have arc $AB + arc BC + arc CA = 2\pi$. Suppose that the radius of the red circle is not $r$. Then it implies arc $AB + arc BC + arc CA \neq 2\pi$, since the same chord on the circles of different radiiuses subtends the arc with different angles. This is a contradiction. □

Lemma 2. Under congruent Voronoi triangle conditions, the minimum density of the two-coverage with equal circles is $2\vartheta$, where $\vartheta$ is the minimum density of the one-coverage.

Proof. To satisfy the conditions of the lemma, i.e., Voronoi triangles inscribed in each circle are equal and achieve two-coverage, the tessellation patterns must be one shown in in Figure 1. In this case, the density is the ratio of the area of a circle to the area of $\triangle ABC$. It is well known that the area of a inscribed triangle reaches its maximum when it is regular. Therefore, the minimum density is $\frac{4\pi}{\sqrt{27}} = 2\vartheta$. □

Here is the main theorem of this paper, which indicates that the density in the case of polygons is no less than that in the case of equilateral triangle.

Theorem 1. Under congruent Voronoi polygon conditions, the minimum density of the two-coverage with equal circles is $2\vartheta$.

Fig. 2. (a) Only one point is shared by three circles. (b) An area is shared by three circles. (c) The tangent point of two circles is shared by three circles

Proof. Here we may assume that there are no coincident circles forming a two-fold covering, since coincident cases cannot generate congruent Voronoi polygons.

To achieve two-coverage, there is at least an interior point of any circle, which belongs to at least the other three circles, since any two circles cannot cover fully a circle unless that circle and one of them are coincident. A point that is shared by three circles may be classified into the following three cases:
Fig. 3. A possible non-triangle tessellation derived by pattern (a) in Figure 2.

Fig. 4. Two types of two-coverage derived by pattern (b) shown in Figure 2.

Fig. 5. (a) The centers of five circles shown in Figure 2(b). (b) Point $S$ generated by rotating $\triangle PQR$ (rotation symmetry tessellation). (c) Point $S$ generated by mirror-mapping $\triangle PQR$ (mirror symmetry tessellation).
Fig. 6. Two types of two-coverage derived by pattern (c) shown in Figure 2. (a) Two pairs of two circles are tangent. (b) Only two circles are tangent.

1) As shown in Figure 2(a), three circles meet at a point;
2) As shown in Figure 2(b), an area is shared by three circles;
3) As shown in Figure 2(c), two circles are tangent. Another circle passes through that tangent point.

Any tessellation pattern is one of the above three cases, or their combination.

First we consider pattern (a) in Figure 2. To achieve two-coverage, it may result in two cases: (1) A circle is fully covered by the other three circles, i.e., a triangle pattern shown in Figure 1; (2) A circle is not fully covered by the other three circles, i.e., a non-triangle pattern shown in Figure 3. Lemma 1 indicates that there exists indeed such a triangle pattern. By Lemma 2, we obtain that the density of the two-coverage in the case of the triangle pattern is at least $2\vartheta$.

For the non-triangle pattern, there must be an area that shared by three circles, e.g., circles $C_1, C_2$ and red circle shown in Figure 3 constructs such an area. Therefore, the non-triangle pattern is actually pattern (b) in Figure 2.

Now consider pattern (b) shown in Figure 2. This may derive two types of two-coverage: (1) As shown in Figure 4(a), the red circle is fully covered by circles $C_1, C_2, C_3$; (2) As shown in Figure 4(b), the red circle requires at least four circles to be fully covered, i.e., circles $C_1, C_2, C_3$ cannot cover fully it. In Figure 4(a), we assume that the red circle passes through the intersection $A$ of circles $C_2, C_3$. If it does not pass through $A$, the density computation is similar.

Under this assumption, the Voronoi polygon lying in the red circle is a triangle $ABC$, as depicted Figure 4(a). Clearly, the area of $\triangle ABC$ is no greater than that of the inscribed triangle, i.e., $\triangle ADE$. From the fact that the area of a regular triangle is maximum among all inscribed triangles, we can obtain that the density of the tessellation pattern shown in Figure 4(a) is at least $2\vartheta$.

Compared with Figure 4(a), computing the density of Figure 4(b) is more difficult. Given the locations of a few centers, in many cases, the locations of the remaining centers can be computed by using the symmetry property, since under congruent Voronoi polygon conditions, there must be the symmetry property between centers of circles, i.e., each center has the same adjacent structure. In
Figure 4(b), there are five circles, the centers of which are shown in Figure 5(a). Now we compute another center $S$ by using three out of the five centers, i.e., $P, Q, R$. As shown in Figure 5, we restrict ourselves to the case where centers $P, Q, R$ are not collinear because the case where centers $P, Q, R$ are collinear is trivial. Since center $R$ must have same adjacent structure as center $Q$, there must exist a triangle congruent to $\triangle PQR$. We can construct such a congruent triangle by only two operations: rotation operation and mirror-map operation. Figure 5(b) depicts $\triangle QRS$ (dotted line) congruent to $\triangle PQR$, which is generated by a rotation operation. Figure 5(c) depicts $\triangle QRS$ (dotted line) congruent to $\triangle PQR$, which is generated by a mirror-map operation. Notice, with respect to line segment $PR$, we can carry out repeatedly at most $360^\circ$ rotation operations. However, suppose that the plane to be cover may be arbitrarily large or infinite, then the number of rotation or mirror-map operations with respect to line segment $PR$ should be able to be arbitrarily large or infinite. Therefore, with respect to the line segment $PR$'s, there must exist mirror-map operations. It is easy to see that once a mirror-map operation is carried out, there must be another mirror-map to construct parallelogram $QRST$ congruent to parallelogram $PQRS$ shown in Figure 5(c), where centers $P, R$ and $T$ are collinear. The rest may be deduced by analogy. It implies that centers lying on the straight line through $PR$ are evenly spaced, and the distance between any two adjacent centers on this line is $|PR|$. Similarly, this claim for straight line through $QS$ holds also. Now we compute the density. Let the radius be 1, $|PR| = 2x$, $y$ and $z$ be the distance from the centers of circles $C_4$ and $C_1$ to straight line $PR$, respectively. We may assume without loss of generality that $y \geq z$. Clearly, from Figure 4(b), we have $y \leq |CD| + 1$, and $|CD|^2 = 1^2 - |DR|^2 = 1 - x^2$. It implies $y \leq \sqrt{1 - x^2} + 1$. The density of Figure 4(b) is at least $\min(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\pi}{2\sqrt{1 - x^2} + 1}$. Notice, centers based on line $PR$ yields at least the density of $\frac{\pi}{2\sqrt{1 - x^2}}$, the density of centers based on line $QS$ is at least $\frac{\pi}{2\sqrt{1 + 1}}$ also. When $x = \frac{\sqrt{3}}{2}$, the formula $x(\sqrt{1 - x^2} + 1)$ reaches maximum, which is equal to $\frac{\pi}{2\sqrt{3}}$. Therefore, the density in this case is at least $\frac{\pi}{2\sqrt{3}} = 2\theta$. (In fact, we found that this proof approach is also suitable for the proof of Lemma 2, i.e., constructing congruent Voronoi triangles are based two straight lines, and centers on each straight line are evenly spaced.)

Finally we consider pattern (c) shown in Figure 2. This pattern may derive two types of two-coverage shown in Figure 6. In Figure 6(a), the red circle is covered by four circles, the relation of which is that two pairs of tangent circles meets at a point. In Figure 6(b), there is only one pair of tangent circles. Both the cases are an extreme case of Figure 4(b) with $xy \leq 1$. In a way similar to Figure 4(b) mentioned above, we can obtain that in either case the density of pattern (c) is at least $\pi > 2\theta$, since the average distance between two adjacent centers in the horizontal direction is $r$, and the average distance between two adjacent centers in the vertical direction is $r$ in the case of Figure 6(a), and at most $r$ in the case of Figure 6(b), where $r$ is the radius of the circle.
Gathering together the above discussion about three cases shown in Figure 2 completes the proof of the theorem.

In fact, any $k$-fold covering under congruent Voronoi polygon conditions is a double lattice-covering, each of which has the same structure, while Ref. [2] is a single lattice-covering. Therefore, we can prove Theorem 1 also by extending some theorems of Ref. [2].

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