Cohomological Properties of Differential Calculi on Hopf Algebras

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1 Introduction

In the approach to the differential calculus on quantum groups proposed in [1], besides the obvious notion of differential \( d \), the theory is founded on the notion of bicovariant bimodule; the algebraic nature of this construction extends the properties of differential forms to the noncommutative situation. Following this idea a general treatment and a classification of differential calculi have been constructed for quantum groups obtained as deformation of semisimple Lie groups making use of the quasi-triangularity property \([2,3]\).

In this report we give an intrinsic treatment of the results we developed in \([4]\) connecting the differential calculi on Hopf algebras to the Drinfeld double \([5]\). In the first place we recover that bicovariant bimodules are in one to one correspondence with the Drinfeld double representations \([6]\); we then introduce a Hochschild cohomology of the algebra of functions and discuss the main result stating that each differential calculus is associated to a 1-cocycle satisfying an additional invariance condition with respect to a natural action \([4]\). Defining a Hochschild cohomology of the double, the above invariance becomes a condition with respect to the enveloping algebra component of the double that must be added to the 1-cocycle relation.

The general classification of differential calculi is therefore reduced to a cohomological problem, which can be performed with the more usual and efficient tools. Moreover a supply of differential calculi is obtained by observing that the coboundary operator maps invariant 0-cochains into invariant 1-coboundaries.

The construction we present is completely independent of the quasi-triangular property of Hopf algebras and can obviously be applied to classical groups, both Lie and discrete or finite. An interesting feature is that all the known differential calculi on quantum and finite groups correspond to coboundaries, at difference with the usual Lie group case, in which no invariant coboundary exists and the classical differential calculus is determined by a nontrivial 1-cocycle.

2 Differential calculus of the first order

Let \( \mathcal{A} \) be an associative algebra and \( \Gamma \) an \( \mathcal{A} \) - bimodule; with \( a . \gamma \) and \( \gamma . a \) we indicate left and right multiplication of \( a \in \mathcal{A} \) with \( \gamma \in \Gamma \).

By differential calculus of the first order we mean the couple \((\Gamma, d)\) where \( d : \mathcal{A} \rightarrow \Gamma \) is a linear application such that

(i) \( d(ab) = a . (db) + (da) . b \quad a, b \in \mathcal{A} \);

(ii) \( \text{Im } d \) generates \( \Gamma \) (i.e. \( \forall \gamma \in \Gamma \) there exist \( a_k, b_k \in \mathcal{A} \) such that \( \gamma = \sum a_k db_k \)).

Two differential calculi \((\Gamma_1, d_1)\) and \((\Gamma_2, d_2)\) are said to be equivalent if there exists a bimodule isomorphism \( i : \Gamma_1 \rightarrow \Gamma_2 \) such that \( d_2 = i \circ d_1 \).
Let \( \mathcal{A}_2 = \text{Ker} \ m \subset \mathcal{A} \otimes \mathcal{A} \), where \( m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) is the multiplication in \( \mathcal{A} \). Let \( \mathcal{A}_2 \) be given the following structure of \( \mathcal{A} \)-bimodule:

\[
\begin{align*}
  a \cdot \left( \sum_k x_k \otimes y_k \right) & = \sum_k a x_k \otimes y_k, \\
  \left( \sum_k x_k \otimes y_k \right) . a & = \sum_k x_k \otimes y_k a,
\end{align*}
\]

with \( \sum_k x_k y_k = 0 \) and \( a, b \in \mathcal{A} \). It is easy to verify that, with

\[ Da = 1 \otimes a - a \otimes 1. \]

\( (\mathcal{A}_2, D) \) is a first order differential calculus. We refer to it as to the universal differential calculus. The universality property is justified by the following

(1) Proposition. Each differential calculus \( (\Gamma, d) \) is obtained as a quotient from \( (\mathcal{A}_2, D) \), i.e. there exists a sub-bimodule \( \mathcal{N} \subset \mathcal{A}_2 \), such that \( \Gamma = \mathcal{A}_2/\mathcal{N} \) and \( d = \pi \circ D \), where \( \pi : \mathcal{A}_2 \to \Gamma \) is the canonical homomorphism.

In the case of an Hopf algebra \( \mathcal{F} \) the universal calculus can be described in a particularly useful form. Let \( r : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F} \) be defined by

\[ r(a \otimes b) = (a \otimes 1) \Delta b, \]

with inverse

\[ r^{-1}(a \otimes b) = (a \otimes 1)(S \otimes 1) \Delta b. \]

Define on \( \mathcal{F} \otimes \text{Ker} \epsilon \) the structure of \( \mathcal{F} \)-bimodule as follows

\[ a \cdot \left( \sum_k b_k \otimes c_k \right) = \sum_k a b_k \otimes c_k, \]

\[ \left( \sum_k b_k \otimes c_k \right) . a = \sum_k (b_k \otimes c_k) \Delta a, \]

with \( \sum_k b_k \otimes c_k \in \mathcal{F} \otimes \text{Ker} \epsilon \). It’s easy to prove that \( r \) is a bimodule isomorphism between \( \mathcal{F}_2 \) and \( \mathcal{F} \otimes \text{Ker} \epsilon \). Moreover, if \( D' = r \circ D : \mathcal{F} \to \mathcal{F} \otimes \text{Ker} \epsilon \), i.e.

\[ D' a = \Delta a - a \otimes 1, \]

then \( (\mathcal{F}_2, D) \) and \( (\mathcal{F} \otimes \text{Ker} \epsilon, D') \) are isomorphic differential calculi. From now on we refer to \( (\mathcal{F} \otimes \text{Ker} \epsilon, D') \) as to the universal differential calculus.

3 Bicovariant bimodules and Drinfeld quantum double

The notion of bicovariant bimodule is an ad hoc notion introduced in [1] to generalize the definition of translations on the space of forms on Lie groups to the case of Hopf algebras. We show in this section the general result that bicovariant bimodules are completely characterized by the representations of the Drinfeld double of \( \mathcal{F} \).

An \( \mathcal{F} \)-bimodule is said to be a left-covariant bimodule if it is defined a coaction \( \delta_\Gamma : \Gamma \to \mathcal{F} \otimes \Gamma \) such that

\[ \delta_\Gamma(a \gamma) = \Delta(a) \delta_\Gamma(\gamma), \quad \delta_\Gamma(\gamma a) = \delta_\Gamma(\gamma) \Delta(a), \quad (\epsilon \otimes \text{id}) \delta_\Gamma(\gamma) = \gamma, \]

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for each \( a \in \mathcal{F} \in \Gamma \). Two left-covariant \( \mathcal{F} \)-bimoduli \((\Gamma, \delta_{\Gamma})\) and \((\Gamma', \delta_{\Gamma'})\) are equivalent if there exists a bimodule isomorphism \( i : \Gamma \rightarrow \Gamma' \) such that

\[
(id \otimes i)\delta_{\Gamma} = \delta_{\Gamma'}i.
\]

A form \( \omega \in \Gamma \) is left invariant if \( \delta_{\Gamma}(\omega) = 1 \otimes \omega \); we call \( \text{inv} \Gamma \) the space of left invariant forms.

Left covariant bimoduli are completely determined by giving a representation of \( \mathcal{F} \) on \( \text{inv} \Gamma \), according to the following result ([1], but for the intrinsic formulation see [7]).

(2) **Proposition.** If \( \Gamma \) is a left covariant \( \mathcal{F} \)-bimodule, then \( \tau_{\Gamma} : \mathcal{F} \rightarrow \text{End}(\text{inv} \Gamma) \) defined by

\[
\tau_{\Gamma}(a)\gamma = \sum (a) S(a(1)) \cdot \gamma \cdot a(2) \quad \gamma \in \text{inv} \Gamma, \ a \in \mathcal{F}
\]

is a right representation of \( \mathcal{F} \). Moreover \( \Gamma \) is isomorphic to \( \mathcal{F} \otimes \text{inv} \Gamma \), which is a free left module and has a right multiplication given by

\[
(1 \otimes \gamma) a = \sum (a) a(1) \otimes \tau_{\Gamma}(a(2)) \gamma \quad a \in \mathcal{F}, \ \gamma \in \text{inv} \Gamma.
\]

Conversely, if \( \tau_{\Gamma} \) is a right representation of \( \mathcal{F} \) on \( \mathcal{V}_{\tau} \), then (3) defines on the free left module \( \Gamma = \mathcal{F} \otimes \mathcal{V}_{\tau} \) the structure of left covariant \( \mathcal{F} \)-bimodule, with \( \text{inv} \Gamma = \mathcal{V}_{\tau} \).

Two left covariant \( \mathcal{F} \)-bimoduli are equivalent if and only if the corresponding representations of \( \mathcal{F} \) are equivalent.

Analogously an \( \mathcal{F} \)-bimodule is **right-covariant** if it is defined a coaction \( \Gamma \delta : \Gamma \rightarrow \Gamma \otimes \mathcal{F} \) such that

\[
\Gamma \delta(a \gamma) = \Delta(a) \Gamma \delta(\gamma), \quad \Gamma \delta(\gamma a) = \Gamma \delta(\gamma) \Delta(a), \quad (id \otimes \epsilon) \Gamma \delta(\gamma) = \gamma,
\]

for each \( a \in \mathcal{F} \in \Gamma \). We omit the obvious definition of isomorphism of right covariant bimoduli.

We finally arrive to the definition of bicovariant bimodule. An \( \mathcal{F} \)-bimodule is **bicovariant** if it is left and right covariant and if left and right translations \( \delta_{\Gamma} \) and \( \Gamma \delta \) verify the following compatibility relation

\[
(id \otimes \Gamma \delta)\delta_{\Gamma} = (\delta_{\Gamma} \otimes id)\Gamma \delta.
\]

Two bicovariant bimoduli are isomorphic if there exists a bimodule isomorphism which is an isomorphism of left and right covariant bimoduli.

As in the case of left covariant bimoduli, there exists a one-to-one correspondence between bicovariant bimoduli and representations of an Hopf algebra \( \mathcal{D} \), the Drinfeld double of \( \mathcal{F} \).

Let \( \mathcal{U} \) be the Hopf algebra dual to \( \mathcal{F} \). Then the Drinfeld double of \( \mathcal{F} \) is the Hopf algebra \( \mathcal{D} \) defined by the following requirements:

(i) \( \mathcal{D} = \mathcal{F} \otimes \mathcal{U} \) as vector space;

(ii) \( \mathcal{F} \) and \( \mathcal{U}^{op} \) (the opposite Hopf algebra of \( \mathcal{U} \)) are Hopf subalgebras of \( \mathcal{D} \);

(iii) for each \( a \in \mathcal{F} \) and \( X \in \mathcal{U} \) we have

\[
a X = \sum (a) X(2) a(2) < X(1), S^{-1}(a(3)) > < X(3), a(1) >.
\]
In the following we will indicate with \( \overline{S} \) the antipode in \( D \), i.e. \( \overline{S}(a) = S(a) \) if \( a \in F \) and \( \overline{S}(X) = S^{-1}(X) \) if \( X \in U \). We remark that \( D \) is a quasitriangular Hopf algebra even if \( F \) is not. The main result of this section is contained in the following proposition, which we state without proof.

(5) **Proposition.** If \((\Gamma, \gamma, \delta)\) is a bicovariant \( F \)-bimodule then \( \tau_U : U \to \text{End}(\text{inv} \Gamma) \) defined by

\[
\tau_U(X) \gamma = (\text{id} \otimes \overline{S}(X)) \delta(\gamma) \quad X \in U, \gamma \in \text{inv} \Gamma,
\]

is a right representation of \( U \) on \( \text{inv} \Gamma \) and \( \tau_D : D \to \text{End}(\text{inv} \Gamma) \), defined by

\[
\tau_D(Xa) = \tau_F(a) \tau_U(X) \quad a \in F \hookrightarrow D, X \in U \hookrightarrow D,
\]

is a right representation of \( D \).

Conversely, if \( \tau_D \) is a right representation on \( D \) on \( \text{inv} \Gamma \), \( \tau_U = \tau_D \mid_U \) and \( \Gamma = F \otimes V_D \) is the left covariant bimodule associated to \( \tau_F = \tau_D \mid_F \), then the right translation \( \gamma \delta \) defined in (6) gives \( \Gamma \) the structure of a bicovariant bimodule.

Two representations of \( D \) are equivalent if and only if the corresponding bicovariant bimoduli are equivalent.

### 4 Bicovariant differential calculi and invariant 1-cocycles

Let \((\Gamma, \delta, \gamma)\) be a bicovariant \( F \)-bimodule. A differential calculus \((\Gamma, d)\) is said to be bicovariant if

\[
\delta \circ d = (1 \otimes d) \Delta, \quad \gamma \circ d = (d \otimes 1) \Delta.
\]

In the case of classical differential calculi these conditions express the commutativity between differential and translations. The requirement of bicovariance strongly restricts the set of admissible calculi. An intrinsic property that in principle classifies them was already given in [1].

(7) **Proposition.** Bicovariant differential calculi are in one-to-one correspondence with right ideals \( R \subset \text{Ker} \epsilon \) of \( F \) which are \( \text{ad} \)-invariant, i.e. such that \( \text{ad}(R) \subset R \otimes F \).

In this section we propose an alternative approach in terms of cocycles of an Hochschild cohomology, invariant with respect to a certain action of \( U \). Let’s recall the basic definitions of the Hochschild cohomology.

Let \( A \) be a generic associative algebra and \( \mathcal{C}^k(A, M) \) the set of \( k \)-linear applications (\( k \)-cochains) from \( A^k \) to an \( A \)-bimodule \( M \), as usual \( \mathcal{C}^0(A, M) \equiv M \) and we let \( \delta : \mathcal{C}^k \to \mathcal{C}^{k+1} \) be the coboundary operator defined by

\[
\delta \psi(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = \alpha_1 \psi(\alpha_2, \ldots, \alpha_{k+1}) + \sum_{i=1}^{k} (-1)^i \psi(\alpha_1, \ldots, \alpha_i \alpha_{i+1}, \ldots, \alpha_{k+1}) + (-1)^{k+1} \psi(\alpha_1, \ldots, \alpha_k) \cdot \alpha_{k+1}.
\]

\( \mathcal{C}(A, M) \) is a complex and we define \( k \)-cocycles and \( k \)-coboundaries as

\[
Z^k(A, M) = \{ \psi \in \mathcal{C}^k \mid \delta \psi = 0 \}, \quad B^k(A, M) = \{ \psi \in \mathcal{C}^k \mid \exists \gamma \in \mathcal{C}^{k-1} \mid \psi = \delta \gamma \}.
\]
respectively. The $k$-th group of cohomology is
\[ H^k(A, M) = Z^k/B^k. \]

Let $\mathcal{F}$ be an Hopf algebra and $(\Gamma, d)$ a bicovariant differential calculus. Let’s give to $\mathcal{F}$ the structure of a $\mathcal{F}$ bimodule, with right multiplication given by $\tau_\mathcal{F}$ and left multiplication by the counit $\epsilon$. Let’s consider the Hochschild complex $C(\mathcal{F}, \mathcal{F}, \Gamma)$, in particular for $\gamma \in \mathcal{F}, \Gamma$, we have $(\delta \gamma)(a) = \epsilon(a)\gamma - \tau_\mathcal{F}(a)\gamma$.

Let $\bullet : C^k(\mathcal{F}, \mathcal{F}, \Gamma) \otimes \mathcal{U} \rightarrow C^k(\mathcal{F}, \mathcal{F}, \Gamma)$ be the right action of $\mathcal{U}$ on $k$-cochains defined by
\[(\psi \bullet X)(a_1, \ldots, a_k) = \sum_{\langle X \rangle} \tau_\mathcal{F}(X_{k+1})\psi(Ad_{X_{k+1}}a_1, \ldots, Ad_{X_1}a_k).\]

If $\psi \bullet X = \epsilon(X)\psi$, for each $X \in \mathcal{U}$, we say that $\psi$ is invariant.

(9) **LEmma.** A $k$-cochain $\psi \in C^k(\mathcal{F}, \mathcal{F}, \Gamma)$ is invariant if and only if, for each $X \in \mathcal{U}$,
\[ \tau_\mathcal{F}(S(X)))\psi(a_1, \ldots, a_k) = \sum_{\langle X \rangle} \psi(Ad_{X_{k+1}}a_1, \ldots, Ad_{X_1}a_k). \]

We finally give the main result of this paper that completely characterizes bicovariant differential calculi.

(10) **Proposition.** (i) There is one-to-one correspondence between bicovariant differential calculi $(\Gamma, d)$ and invariant 1-cocycles $\psi$ such that
\[ da = \sum_{\langle a \rangle} a_{(1)} \psi(a_{(2)}). \]

(ii) The image of invariant 0-cochain under $\delta$ is just the space of invariant 1-cocycle, so that each invariant 0-cochain defines a coboundary differential calculus.

Using the whole representation of $\mathcal{D}$ we can give $\mathcal{F}, \Gamma$ the structure of a $\mathcal{D}$-bimodule, where $\alpha \cdot \gamma = \epsilon(\alpha)\gamma$ and $\gamma \cdot \alpha = \tau_\mathcal{F}(\alpha)\gamma$, for $\alpha \in \mathcal{D}$ and $\gamma \in \Gamma$. The result in (10) can be equivalently reformulated in terms of Hochschild Cohomology $C(\mathcal{D}, \mathcal{F}, \Gamma)$.

(11) **Corollary.** Bicovariant differential calculi are in one-to-one correspondence with $\phi \in Z^1(\mathcal{D}, \mathcal{F}, \Gamma)$ such that $\phi(\mathcal{U}) = 0$.

(12) **Remark.** (Coboundary calculi) Two calculi that differ by a coboundary are not equivalent. Indeed consider an invariant 0-cochain $\gamma$ and the calculus associated to $\delta \gamma$. Using the property of invariance of $\gamma$, $\tau_\mathcal{F}(X)\gamma = (id \otimes S(X))_\mathcal{D}\delta \gamma = \epsilon(X)\gamma$, we see that $\delta \gamma = \gamma \otimes 1$, so that an invariant zero cochain is a form which is right and left invariant. The associated differential
\[ da = \sum_{\langle a \rangle} a_{(1)} \cdot (\delta \gamma)(a_{(2)}) = a \cdot \gamma - \sum_{\langle a \rangle} a_{(1)} \cdot \tau_\mathcal{F}(a_{(2)})\gamma = a \cdot \gamma - \gamma \cdot a, \]
results in an internal derivation but it is obviously not equivalent to the trivial calculus $d = 0$. In [3] this kind of calculus is called *internal* and it is shown that all bicovariant calculi on quantum groups of the series $ABCD$ are coboundary. The same situation is met with finite groups. On the contrary, the ordinary differential calculus on Lie groups, being the space of forms symmetrical, it can’t be coboundary.
5 Quantum Lie algebras

Let \((\Gamma, d)\) be a bicovariant differential calculus and suppose that \(\dim_{\mathbb{C}} \Gamma = n < \infty\) so that we can introduce a basis \(\{\omega_i\} i = 1, n\). We then have

\[
[\tau_D(a)]_{ij} = \langle f_{ij}, a \rangle, \quad [\tau_D(S(X))]_{ij} = \langle X, R_{ij} \rangle,
\]

with \(\{f_{ij}\}, X \in U\) and \(\{R_{ij}\}, a \in \mathcal{F}\). Obviously the usual properties of representative elements are satisfied:

\[
\Delta(f_{ij}) = \sum_k f_{ik} \otimes f_{kj}, \quad \epsilon(f_{ij}) = \delta_{ij},
\]

\[
\Delta(R_{ij}) = \sum_k R_{ik} \otimes R_{kj}, \quad \epsilon(R_{ij}) = \delta_{ij}.
\]

Using the representation \(\tau_D\) the relation (4) reads

\[
\sum_i (f_{ii} \ast a) R_{si} = \sum_i R_{il} (a \ast f_{is}),
\]

where \(f \ast a = (\text{id} \otimes f) \Delta a\) and \(a \ast f = (f \otimes \text{id}) \Delta a\). This is the bicovariance as in [1].

Again in the basis \(\{\omega_i\}\) we find

\[
\omega_i \ast a = \sum_j (f_{ij} \ast a) \omega_j, \quad \rho(\omega_i) = \sum_j \omega_j \otimes R_{ji}.
\]

And the bicovariance of the bimodule implies that the numerical matrix

\[
\Lambda^{ij}_{k\ell} = \langle f_{j\ell}, R_{ki} \rangle
\]

solves quantum Yang Baxter equation \(\Lambda_{12} \Lambda_{13} \Lambda_{23} = \Lambda_{23} \Lambda_{13} \Lambda_{12}\). We used the convention \((A \otimes B)^{ij}_{k\ell} = A_{ij} B_{k\ell}\). This is an obvious consequence of the quasitriangularity of \(D\). In fact if \(\{e_A\}\) is a linear basis for \(\mathcal{F}\), \(\{e^A\}\) for \(U\), then \(R = \sum_A e_A \otimes e^A \in D \otimes D\) and \(\sigma \circ R^{-1} = \sum_A S(e^A) \otimes e_A\), with \(\sigma(a \otimes b) = b \otimes a\), solve the universal quantum Yang-Baxter equation. Using this basis it is possible to write

\[
R_{ij} = \sum_A e_A [\tau_D(S(e^A))]_{ij}, \quad f_{ij} = \sum_A e^A [\tau_D(e_A)]_{ji},
\]

and then, introducing the left representation \(\rho_D = \tau_D^t\),

\[
\Lambda^{ij}_{k\ell} = \rho_D[\sigma \circ R^{-1}]^{ij}_{k\ell}.
\]

Let \(\psi\) be the invariant 1-cocycle associated to \((\Gamma, d)\) and let \(\chi_i \in U\) be defined by

\[
\langle \chi_i, a \rangle = [\psi(a)]_i.
\]

The cocycle properties of \(\psi\) yield the relations

\[
(i) \quad \Delta(\chi_i) = 1 \otimes \chi_i + \sum_j \chi_j \otimes f_{ji}, \quad (ii) \quad \epsilon(\chi_i) = 0.
\]
The invariance of \( \psi \) gives then

\[
(iii) \quad ad_Y(\chi_i) = \sum_{(Y)} S(Y(1))\chi_i Y(2) = \langle Y, R_{i,k} \rangle \chi_k , \quad Y \in \mathcal{U} .
\]

that constitutes an invariant condition of the "fields" \( \chi_i \) with respect to the adjoint action of \( \mathcal{U} \) on itself. Properties (i-iii) identify what is usually called a quantum Lie algebra (see [8]).

The invariants fields allow to the following expression of the differential:

\[
da = \sum_i (\chi_i \ast a) \omega_i .
\]

6 Differential calculus for the finite groups

The results of the previous sections are well illustrated on finite groups. In fact it is possible to give a complete classification of bicovariant bimoduli by means of representations of the double; it is also possible to study the invariant cocycles of the associated Hochschild cohomology, and then obtain a complete classification of bicovariant differential calculi of finite groups.

Let \( G \) be a finite group of order \( n \) and \( \mathcal{F}(G) \) the \( n \)-dimensional vector space of complex functions on \( G \). The elements \( \{ \phi_g \}_{g \in G} \), where \( \phi_g(h) = \delta_{g,h} \), are a basis for \( \mathcal{F}(G) \). As usual the Hopf algebra structure on \( \mathcal{F}(G) \) is given by

\[
\phi_g \phi_h = \delta_{g,h} \phi_g , \quad \Delta \phi(g,h) = \phi(gh) , \quad S(\phi)(g) = \phi(g^{-1}) , \quad \epsilon(\phi) = \phi(1) ,
\]

where \( g, h \in G \) and \( \epsilon \) is the unity of the group.

We can canonically associate to \( \mathcal{F}(G) \) another Hopf algebra, the group algebra \( CG \). The multiplication is induced by the group composition and the coalgebra structure is determined by

\[
\Delta g = g \otimes g , \quad S(g) = g^{-1} , \quad \epsilon(g) = 1 , \quad g \in G .
\]

\( CG \) and \( \mathcal{F}(G) \) are in duality with the coupling

\[
\langle g, \phi \rangle = \phi(g) \quad \phi \in \mathcal{F}(G) , \quad g \in G .
\]

In particular \( \{ \phi_g \}_{g \in G} \) and \( \{ g \}_{g \in G} \) are dual basis, i.e. \( \langle g, \phi_h \rangle = \delta_{g,h} \). The double \( D(G) \) is isomorphic to \( \mathcal{F}(G) \otimes CG \) as a vector space and contains \( \mathcal{F}(G) \) and \( CG \) as Hopf subalgebras. Its definition is completed by the relations

\[
g \phi = \phi^g g \quad \phi \in \mathcal{F}(G) , \quad g \in CG ,
\]

where \( \phi^g(h) = \phi(g^{-1}hg) \), with \( g, h \in G \).

Let \( C = \{ h \in G \mid h = ghg^{-1} \} \) be a conjugacy class and let \( n_C \) be the number of elements of \( C \). Let \( a \in C \) and \( Z_a = \{ g \in G \mid ag = ga \} \) be its centralizer, the following holds

(13) PROPOSITION. Let \( V_C \) be a \( n_C \)-dimensional vector space and \( \{ v_h \}_{h \in C} \) a basis. For each irreducible representation of \( G \) of dimension \( n_\mu \), \( \rho^\mu : G \to \text{End}(W^\mu) \), an irreducible representation of the double \( \rho^\mu_C : D(G) \to \text{End}(V_C \otimes W^\mu) \) is defined by

\[
\rho^\mu_C(\phi) v_h \otimes w_\alpha = \phi(h) v_h \otimes w_\alpha,
\]

\[
\rho^\mu_C(g) v_h \otimes w_\alpha = v_{ghg^{-1}} \otimes \rho^\mu(g) w_\alpha , \quad \alpha = 1, \ldots, n_\mu .
\]
Two representations $\rho^C_1$ and $\rho^C_2$ are equivalent if and only if $C_1 = C_2$ and the restrictions $\rho^C_1|_{Z_a}$ and $\rho^C_2|_{Z_a}$ are equivalent for an arbitrary $a \in C$.

Using propositions (2) and (5) it is now easy to classify bicovariant bimoduli. Let $\Gamma^\mu_C$ the bicovariant bimodule associated to $\rho^C_\mu$; it is generated as a free left module by left invariant forms $\{\omega[h_\alpha]\}, h \in C$ and $\alpha = 1, \ldots, n_\mu$. Right multiplication and right translation are defined by

\[
\omega[h_\alpha] \cdot \phi = \sum_{\ell \in C, \beta} \left( F^{[\ell\beta]}_{[h_\alpha]} \ast \phi \right) \omega_{[\ell\beta]} \\
r_\delta(\omega[h_\alpha]) = \sum_{\ell \in C, \beta} \omega_{[\ell\beta]} \otimes R^{[\ell\beta]}_{[h_\alpha]},
\]

where

\[
F^{[\ell\beta]}_{[h_\alpha]} = h \delta_{h \ell} \delta_{\alpha, \beta}, \quad R^{[\ell\beta]}_{[h_\alpha]} = \sum_{g \in G} \phi_g \delta_{\ell, ghg^{-1}} [\rho^\mu(g)]^{[\beta]}_{[\alpha]}.
\]

Finally we conclude that the $(n_\mu n_C) \times (n_\mu n_C)$ matrix

\[
\Lambda^{[n_\eta][\ell\beta]}_{[n_\mu][k\alpha]} = \left( F^{[\ell\beta]}_{[h_\alpha]} R^{[n\gamma]}_{[n\eta]} \right) = \delta_{\ell, k} \delta_{\alpha, \beta} \delta_{m, k} [\rho^\mu(k)]^\gamma_{\alpha}
\]

satisfies the quantum Yang-Baxter equation.

The study of the corresponding Hochschild cohomologies permits the complete classification of bicovariant differentials calculi on finite groups. Using proposition (10) in fact we arrive at the following result.

(14) PROPOSITION. Let $C$ a conjugacy class and $\Gamma_C$ the bicovariant bimodule associated to $C$ and to the trivial representation of $G$. Then $(\Gamma_C, d_C)$, with $d_C : \mathcal{F}(G) \rightarrow \Gamma_C$ defined by

\[
d_C \phi = \sum_{g \in C} (\chi_g \ast \phi) \omega_g \quad \phi \in \mathcal{F}(G),
\]

where $\chi_g = e - g$, is a coboundary bicovariant differential calculus, associated to the 0-cochain $\sum_g \omega_g$. All non equivalent bicovariant differential calculi on $\mathcal{F}(G)$ are direct sums of these calculi.

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