Adaptive Personalized Federated Learning

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Abstract

Investigation of the degree of personalization in federated learning algorithms has shown that only maximizing the performance of the global model will confine the capacity of the local models to personalize. In this paper, we advocate an adaptive personalized federated learning (APFL) algorithm, where each client will train their local models while contributing to the global model. Theoretically, we show that the mixture of local and global models can reduce the generalization error, using the multi-domain learning theory. We also propose a communication-reduced bilevel optimization method, which reduces the communication rounds to $O(\sqrt{T})$ and show that under strong convexity and smoothness assumptions, the proposed algorithm can achieve a convergence rate of $O(1/T)$ with some residual error. The residual error is related to the gradient diversity among local models, and the gap between optimal local and global models.

1 Introduction

With an enormously growing amount of decentralized data continually generated on a vast number of devices like smartphones, federated learning offers training a high-quality shared global model with a central server while reducing the systemic privacy risks and communication costs (McMahan et al., 2017). Despite the classical approaches, where large-scale datasets are located on massive and expensive data centers for training (Dean et al., 2012; Li et al., 2014), in federated learning, the data and training both reside on the local nodes. This will ensure the privacy of the local data, while enabling us to learn from massive, not available otherwise, data on those devices. Not to mention the enormous reduction in communication sizes due to local training and data.

In federated learning, the ultimate goal is to learn a global model that achieves uniformly good performance over almost all participating clients. Motivated by this goal, most of the existing methods pursue the following procedure to learn a global model: (i) a subset of clients participating in the training is chosen at each round and receive the current copy of the global model; (ii) each chosen client updates the local version of the global model using its own local data, (iii) the server aggregates over the obtained local models to update the global model, and this process continues until convergence (McMahan et al., 2017; Mohri et al., 2019; Karimireddy et al., 2019; Pillutla et al., 2019). Most notably, FedAvg by McMahan et al. (2017) uses averaging as its aggregation method over the local learned models on clients.

Due to inherent diversity among local data shards and highly non-IID distribution of the data across clients, FedAvg is hugely sensitive to its hyperparameters, and as a result, does not benefit from a favorable convergence guarantee (Haddadpour and Mahdavi, 2019; Li et al., 2020). In Karimireddy et al. (2019), authors argue that if these hyperparameters are not carefully tuned, it will result in the divergence of FedAvg, as local models may drift significantly from each other. Therefore, in the presence of statistical device heterogeneity, the global model might not generalize well on the local data of each client individually (Jiang et al., 2019). This is even more crucial in fairness-critical systems such as medical diagnosis (Li and Wang, 2019), where poor performance on local clients could result in damaging consequences.

This problem is exacerbated even further as the diversity among local data of different clients is growing. This is depicted in Figure 1, where the global model generalization error on local validation data diverges dramatically when the diversity among different clients’ data increases. This observation illustrates that solely optimizing for
the global model’s accuracy leads to a poor generalization of local clients. This challenging issue has motivated researchers to embrace the heterogeneity and pursue a personalized model that performs well on local data in addition to the global model.

In this work, we study a new federated learning framework that explicitly optimizes performance on all clients. We argue that information-theoretically, collaboration reduces the generalization error depending on the distribution characteristics of the local data. Individual learning often suffers from an insufficient amount of data, so the model’s generalization error will be large. Hence, we seek help from other users’ data to reduce the generalization error, even though they may come from different distributions. Motivated by this, we tend to learn a personalized model that is a mixture of global and local models. But the key question is, when will the collaboration really help? Intuitively, if the user’s data distribution does not deviate too much from other users’ data distribution, then collaboration can be beneficial to reduce the local generalization error. On the other hand, when the local distribution is far from being a representative sample of the overall distribution (the data distribution among all clients differ significantly), it is too difficult to find a global model that is good for all clients; thereby independent per-device models are preferable.

In this paper, we propose an adaptive personalized federated learning (APFL) algorithm and theoretically analyze its convergence rate. Moreover, utilizing the multi-domain learning theory (Ben-David et al., 2010), we provide a theoretical analysis of the generalization bound for proposed personalization. Motivated by our generalization theory, the proposed algorithm adaptively learns a personalized model by leveraging the relatedness between its local and global model as learning proceeds. As it is shown in Figure 1, the personalized solution found by the proposed algorithm is more robust against increasing the diversity of data among clients and can generalize well.

2 Related Work

The number of research in federated learning is proliferating during the past few years. In federated learning, the main objective is to learn a global model that is good enough for yet to be seen data and has fast convergence to a local optimum. This indicates that there are several uncanny resemblances between federated learning and meta-learning approaches Finn et al. (2017); Nichol et al. (2018). However, despite this similarity, meta-learning approaches are mainly trying to learn multiple models, personalized for each new task, whereas in most federated learning approaches, the main focus is on the single global model. As discussed by Kairouz et al. (2019), the gap between the performance of global and personalized models shows the crucial importance of personalization in federated learning. Several different approaches are trying to personalize the global model, primarily focusing on
optimization error, while the main challenge with personalization is during the inference time. Some of these works on the personalization of models in a decentralized setting can be found in Vanhaesebrouck et al. (2017); Almeida and Xavier (2018), where in addition to the optimization error, they have network constraints or peer-to-peer communication limitation Bellet et al. (2017); Zantedeschi et al. (2019). In general, as discussed by Kairouz et al. (2019), there are three significant categories of personalization methods in federated learning, namely, local fine-tuning, multi-task learning, and contextualization. Yu et al. (2020) argue that the global model learned by federated learning, especially with having differential privacy and robust learning objectives, can hurt the performance of many clients. They indicate that those clients can obtain a better model by using only their own data. Hence, they empirically show that using these three approaches can boost the performance of those clients. In addition to these three, there is also another category that fits the most to our proposed approach, which is mixing the global and local models.

**Local fine-tuning:** The dominant approach for personalization is local fine-tuning, where each client receives a global model and tune it using its own local data and several gradient descent steps. This approach is predominantly used in meta-learning methods such as MAML by Finn et al. (2017) or domain adaptation and transfer learning (Ben-David et al., 2010; Mansour et al., 2009; Pan and Yang, 2009). Jiang et al. (2019) discuss the similarity between federated learning and meta-learning approaches, notably the Reptile algorithm by Nichol et al. (2018) and FedAvg, and combine them to personalize local models. They observed that federated learning with a single objective of performance of the global model could limit the capacity of the learned model for personalization. In Khodak et al. (2019), authors using online convex optimization to introduce a meta-learning approach that can be used in federated learning for better personalization. Fallah et al. (2020) borrow ideas from MAML to learn personalized models for each client with convergence guarantees. Similar to fine-tuning, they update the local models with several gradient steps, but they use second-order information to update the global model, like MAML. Another approach adopted for deep neural networks is introduced by Arivazhagan et al. (2019), where they freeze the base layers and only change the last “personalized” layer for each client locally. The main drawback of local fine-tuning is that it minimizes the optimization error, whereas the more important part is the generalization performance of the personalized model. In this setting, the personalized model is pruned to overfit.

**Multi-task learning:** Another view of the personalization problem is to see it as a multi-task learning problem similar to Smith et al. (2017). In this setting, optimization on each client can be considered as a new task; hence, the approaches of multi-task learning can be applied. One other approach, discussed as an open problem in Kairouz et al. (2019), is to cluster groups of clients based on some features such as region, as similar tasks, similar to one approach proposed by Mansour et al. (2020).

**Contextualization:** An important application of personalization in federated learning is using the model under different contexts. For instance, in the next character recognition task in Hard et al. (2018), based on the context of the use case, the results should be different. Hence, we need a personalized model on one client under different contexts. This requires access to more features about the context during the training. Evaluation of the personalized model in such a setting has been investigated by Wang et al. (2019), which is in line with our approach in experimental results in Section 6.

**Personalization via mixing models:** Parallel to our work, there are other studies to introduce different personalization approaches for federated learning by mixing the global and local models. Hanzely and Richtárik (2020) try to introduce a general approach for federated learning by combining the optimization of the local and global models. In their effort, they use a mixing parameter, which controls the degree of optimization for both local models and the global model. The FedAvg (McMahan et al., 2017) can be considered a special case of this approach. They show that the learned model is in the convex haul of both local and global models, and at each iteration, depend on the local models’ optimization parameters, the global model is getting closer to the global model learned by FedAvg. Perhaps, the closest approach for personalization to our proposal is introduced by Mansour et al. (2020). In fact, they propose three different approaches for personalization with generalization guarantees, namely, client clustering, data interpolation, and model interpolation. Out of these three, the first two approaches need some meta-features from all clients that makes them not a feasible approach for federated learning, due to privacy concerns. The third schema, which is the most promising one in practice as well, has
a close formulation to ours in the interpolation of the local and global models. However, in their theory, the generalization bound does not demonstrate the advantage of mixing models, but in our analysis, we will show how the model mixing can impact the generalization bound, by presenting its dependency on the mixture parameter, data diversity and optimal models on local and global distributions.

Beyond different techniques for personalization in federated learning, Kairouz et al. (2019) ask an essential question of “when is a global FL-trained model better?”, or as we can ask, when is personalization better? The answer to these questions mostly depends on the distribution of data across clients. As we theoretically prove and empirically verify in this paper, when the data is distributed IID, we cannot benefit from personalization, and it is similar to the local SGD scenario (Stich, 2018; Haddadpour et al., 2019a,b). However, when the data is non-IID across clients, which is mostly the case in federated learning, personalization can help to balance between shared and local knowledge. Then, the question becomes, what degree of personalization is best for each client? While this was an open problem in Mohri et al. (2019) on how to appropriately mix the global and local model, we answer this question by adaptively tuning the degree of personalization for each client, as discussed in Section 5.2, so it can perfectly work agnostic to the data distribution.

3 Personalized Federated Learning

In this section, we propose a personalization approach for federated learning and analyze its statistical properties. Following the statistical learning theory, in a federated learning setting each client has access to its own data distribution $D_i$ on domain $\Xi := \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \in \mathbb{R}^d$ is the input domain and $\mathcal{Y}$ is the label domain. For any hypothesis $h \in H$ the loss function is defined as $\ell : H \times \Xi \rightarrow \mathbb{R}$. The true risk at local distribution is denoted by $\bar{L}_{D_i}(h) = \mathbb{E}_{(x,y) \sim D_i}[\ell(h(x), y)]$. We use $\bar{L}_D(h)$ to denote the empirical risk of $h$ on distribution $D$.

We use $\mathcal{D} = (1/n) \sum_{i=1}^n D_i$ to denote the average distribution over all clients. Intrinsically, as in federated learning, the global model is trained to minimize the empirical (i.e., ERM) loss with respect to distribution $\mathcal{D}$, i.e., $\min_{h \in H} \bar{L}_D(h)$.

3.1 Personalized model

In a standard federated learning scenario, where the goal is to learn a global model for all devices cooperatively, the learned global model obtained by minimizing the joint empirical distribution, either by proper weighting or in an agnostic manner, may not perfectly generalize on local users’ data when the heterogeneity among local data shards is high (i.e., the global and local optimal models might drift significantly). However, by assuming that all users’ data come from the (roughly) similar distribution, it is expected that the global model enjoys a better generalization accuracy on any user distribution over its domain than the user’s own local model. Meanwhile, from the local user perspective, the key incentive to participate in “federated” learning is the desire to seek a reduction in the local generalization error with the help of other users’ data. In this case, the ideal situation would be that the user can utilize the information from the global model to compensate for the small number its local training data while minimizing the harm induced by heterogeneity among each user’s local data and the data shared by other devices. Obviously, when the local distribution is highly correlated with global distribution, the global model is preferable; otherwise, the global model might be ineffective to be employed as the local model. This motivates us to mix the global model and local model with an adaptive weight as a joint prediction model, namely, the personalized model.

In the adaptive personalized federated learning the goal is to find the optimal combination of the global model and the local model, in order to achieve a better client-specific model. In this setting, each user trains a local model while incorporating part of the global model, with some mixing weight $\alpha_i$, i.e.,

$$h_{\alpha_i} = \alpha_i \hat{h}^* + (1 - \alpha_i) \tilde{h}^*,$$

where $\hat{h}^* = \arg\min_{h \in H} \bar{L}_D(h)$ is global empirical risk minimizer and $\tilde{h}^*_i = \arg\min_{h \in H} \bar{L}_{D_i}(\alpha_i h + (1 - \alpha_i) \hat{h}^*)$ is the mixture model that minimizes the empirical loss at $i$th client.
It is worth mentioning that, \( h_{\alpha_i} \) is not necessarily the minimizer of empirical risk \( \hat{L}_{D_i}(\cdot) \), because we partially incorporate the global model. In fact, in most cases, as we will show in the convergence of the proposed algorithm, \( h_{\alpha_i} \) will incur a residual risk if evaluated on \( D_i \).

### 3.2 Generalization guarantees

We present the learning bounds for classification and regression tasks. For classification, we consider a binary classification task, and define the loss function as \( \ell(h(x), y) = |h(x) - y| \). In the regression task, we consider the MSE loss: \( \ell(h(x), y) = (h(x) - y)^2 \). To measure the discrepancy between two hypothesis, we use:

\[
\mathcal{L}_D(h, h') = \mathbb{E}_{x \sim \mathcal{D}}[\ell(h(x), h'(x))].
\]  

\( \text{(2)} \)

**Definition 1 \((\mathcal{H}\Delta\mathcal{H}\text{-distance Ben-David et al. (2010)})\).** The \( \mathcal{H}\Delta\mathcal{H}\text{-distance of } \mathcal{H} \text{ over domain } D \text{ and } D' \text{ is defined as:} 

\[
d_{\mathcal{H}\Delta\mathcal{H}}(D, D') = \sup_{h, h' \in \mathcal{H}} |\mathbb{P}_{x \sim D}(h(x) \neq h'(x)) - \mathbb{P}_{x \sim D'}(h(x) \neq h'(x))|.
\]  

\( \text{(3)} \)

Our analysis of generalization of personalized model defined in (1) is build off of analysis in Ben-David et al. (2010) which is proposed for multiple-source domain adaption (unlike our setting, the data from target domain is not employed in learning a model from source domains). To do so, we state a result that is required for our analysis.

**Lemma 1 \((\text{Ben-David et al. (2010)})\).** Let \( \mathcal{H} \) be a hypothesis class with VC dimension \( d \). Let \( D_S \) denote source domain, \( D_T \) denote target domain respectively, and \( \mathcal{L}(\cdot) \) be the classification risk, then with probability at least \( 1 - \delta \), for every \( h \in \mathcal{H} \):

\[
\mathcal{L}_{D_T}(h) \leq \mathcal{L}_{D_S}(h) + \frac{1}{2}d_{\mathcal{H}\Delta\mathcal{H}}(D_T, D_S) + \lambda,
\]  

\( \text{(4)} \)

where \( \lambda = \inf_{h \in \mathcal{H}} (\mathcal{L}_{D_T}(h) + \mathcal{L}_{D_S}(h)) \).

Equipped with this lemma, we now state the main result on the generalization of proposed personalization schema.

**Theorem 1.** Let \( \mathcal{H} \) be a hypothesis class with VC dimension \( d \). Then, assuming loss function \( \ell(\cdot) \) is bounded between \([0, 1]\), with a probability at least \( 1 - \delta \), the generalization error of \( \alpha_i \)-mixed model, \( h_{\alpha_i} = \alpha_i \hat{h}^*_i + (1 - \alpha_i)\hat{h}^* \), on domain \( D_i \) is bounded.

For a classification task:

\[
\mathcal{L}_{D_i}(h_{\alpha_i}) \leq (1 - \alpha_i) \left( \mathcal{L}_D(\hat{h}^*) + \frac{1}{2}d_{\mathcal{H}\Delta\mathcal{H}}(D, D_i) + \lambda + \sqrt{\frac{\log \frac{2}{\delta}}{2m_i}} \right) + \alpha_i \left( \mathcal{L}_{D_i}(h^*_i) + \sqrt{\frac{2\log(2d/\delta)}{m_i}} \right),
\]  

\( \text{(5)} \)

and, for a regression task:

\[
\mathcal{L}_{D_i}(h_{\alpha_i}) \leq (1 - \alpha_i) \left( 3\mathcal{L}_D(\hat{h}^*) + \frac{3}{4}d_{\mathcal{H}\Delta\mathcal{H}}(D, D_i) + 3\lambda + 3\sqrt{\frac{\log \frac{2}{\delta}}{2m_i}} \right) + \alpha_i \left( \mathcal{L}_{D_i}(h^*_i) + \alpha_i \sqrt{\frac{2\log(2d/\delta)}{m_i}} \right),
\]  

\( \text{(6)} \)

where \( m_i, i = 1, 2, \ldots, n \) is the number of training data at \( i \)-th client, and \( m = m_1 + \ldots, m_n \) is the total number of data in all devices, \( \lambda = \inf_{h \in \mathcal{H}} (\mathcal{L}_D(h) + \mathcal{L}_{D_i}(h)) \), and \( h^*_i = \arg \min_{h \in \mathcal{H}} \mathcal{L}_{D_i}(h) \).

**Proof.** The proof is provided in Appendix A.
Theorem 1 shows that the generalization risk of \( h_{\alpha_i} \) on \( D_i \) depends mainly on two parts, the generalization risk of global model \( \bar{h}^* \) on \( \bar{D} \), and the generalization risk of local empirical risk minimizer \( h_i^* \). Usually, the first part can be smaller since the global model will have much more training data. If a client has a tiny amount of local data, then the second term can be very large; hence it should choose a small \( \alpha_i \) to incorporate more proportion of the global model. However, sometimes the global model may not be beneficial to local models as it might hurt the local generalization, as the first part contains the divergence between the average domain and local domain. Therefore, if a local distribution drifts away too much from other clients', taking a majority of the global model is not a wise choice, and it is preferable to use independent per-device models.

Remark 1. We note that a very analogous parallel work to ours is Mansour et al. (2020), where the authors provide the generalization bound for mixing global and local models (please see Theorem 6 in their paper). However, their bound do not show the dependency on the mixing parameter \( \alpha_i \), and hence we cannot see the advantage of personalizing schema, over merely learning model individually, or fully accepting the global model.

4 Optimization Method

The proposed personalized model is rooted in adequately mixing the optimal global and slightly modified local empirical risk minimizers. Also, as it is revealed by generalization analysis, the per-device mixing parameter \( \alpha_i \) is a key quantity for the generalization ability of the mixed model. In this section, we propose a communication efficient adaptive algorithm to learn the personalized local models and the global model.

To do so, we let every hypothesis \( h \) in the hypothesis space \( \mathcal{H} \) to be parameterized by a vector \( w \in \mathbb{R}^d \) and denote the empirical risk at \( i \)th device by local objective function \( f_i(w) \). Adaptive personalized federated learning can be formulated as a two-phase optimization problem: globally update the shared model, and locally update users’ local models. Similar to FedAvg algorithm, the server will solve the following optimization problem:

\[
\min_{w \in \mathbb{R}^d} \left[ F(w) := \frac{1}{n} \sum_{i=1}^{n} \left\{ f_i(w) := \mathbb{E}_{\xi_i}[f_i(w, \xi_i)] \right\} \right],
\]

(7)

where \( f_i(.) \) is the local objective at \( i \)th client, \( \xi_i \) is a minibatch of data in data shard at \( i \)th client, and \( n \) is the total number of clients.

Motivated by the trade-off between the global model and local model generalization errors in Theorem 1, we need to learn a personalized model as in (1) to optimize the local empirical risk. To this end, each client needs to solve the following optimization problem over its local data:

\[
\min_{v \in \mathbb{R}^d} f_i (\alpha_i v + (1 - \alpha_i) w^*),
\]

(8)

where \( w^* = \arg \min_w F(w) \) is the optimal global model. The balance between these two models is governed by a parameter \( \alpha_i \), which is associated with the diversity of the local model and the global model. In general, when the local and global data distributions are well aligned, one would intuitively expect that the optimal choice for the mixing parameter would be small to gain more from the data of other devices. On the other side, when local and global distributions drift significantly, the mixing parameter needs to be closed to one to reduce the contribution from the data of other devices on the optimal local model. In what follows, we propose a local descent approach to optimize both objectives simultaneously.

4.1 Local Descent APFL

In this subsection we propose our bilevel optimization algorithm, namely Local Descent APFL. At each communication round, server uniformly random selects \( K \) clients as a set \( U_t \). Each selected client will maintain three models at iteration \( t \): local version of the global model \( w^{(t)}_i \), its own local model \( v^{(t)}_i \), and the mixed personalized model \( \bar{v}^{(t)}_i = \alpha_i v^{(t)}_i + (1 - \alpha_i) w^{(t)}_i \). Then, selected clients will perform the following updates locally on their own...
Algorithm 1: Local Descent APFL

input: Mixture weights $\alpha_1, \cdots, \alpha_n$, Synchronization gap $\tau$, A set of randomly selected $K$ clients $U_0$, Local models $v_i^{(0)}$ for $i \in [n]$ and initial local version of global model $w_i^{(0)}$ for $i \in [n]$.

for $t = 0, \cdots, T$ do
  parallel for $i \in U_t$ do
    if $t$ not divides $\tau$ then
      $w_i^{(t)} = w_i^{(t-1)} - \eta_t \nabla f_i \left( w_i^{(t-1)}; \xi_t^i \right)$
      $v_i^{(t)} = v_i^{(t-1)} - \frac{n}{K} \eta_t \nabla v f_i \left( v_i^{(t-1)}; \xi_t^i \right)$
      $\bar{v}_i^{(t)} = \alpha_i v_i^{(t)} + \left( 1 - \alpha_i \right) w_i^{(t)}$
      $U_t \leftarrow U_{t-1}$
    else
      each selected client sends $w_i^{(t)}$ to the server
      $w_i^{(t)} = \frac{1}{|U_t|} \sum_{j \in U_t} w_j^{(t)}$
      server uniformly samples a subset $U_t$ of $K$ clients.
      server broadcast $w_i^{(t)}$ to all chosen clients
    end
  end
  for $i \notin U_t$ do
    $v_i^{(t)} = v_i^{(t-1)}$
  end
  end
  for $i = 1, \ldots, n$ do
    output: Personalized model: $\hat{v}_i = \frac{1}{S_T} \sum_{t=1}^T p_t \left( \alpha_i v_i^{(t)} + \left( 1 - \alpha_i \right) \frac{1}{K} \sum_{j \in U_t} w_j^{(t)} \right)$;
    Global model: $\hat{w} = \frac{1}{K S_T} \sum_{t=1}^T p_t \sum_{j \in U_t} w_j^{(t)}$.
  end

data:

\begin{align}
  w_i^{(t)} &= w_i^{(t-1)} - \eta_t \nabla f_i \left( w_i^{(t-1)}; \xi_t^i \right) \\
  v_i^{(t)} &= v_i^{(t-1)} - \eta_t \nabla v f_i \left( v_i^{(t-1)}; \xi_t^i \right)
\end{align}

where $\nabla f_i (.; \xi)$ denotes the stochastic gradient of $f(.)$ evaluated at mini-batch $\xi$. Then, using the updated version of the global model and the local model, we update the personalized model $v_i^{(t)}$ as well. The clients that are not selected in this round will keep their previous step local model $v_i^{(t)} = v_i^{(t-1)}$. Every $\tau$ steps, selected clients will send their localized version of the global model $w_i^{(t)}$ to the server for aggregation by averaging:

\begin{equation}
  w_i^{(t)} = \frac{1}{|U_t|} \sum_{j \in U_t} w_j^{(t)}.
\end{equation}

Then the server will choose another set of $K$ clients for the next round of training and broadcast this new model to them. This process continues until convergence.

5 Convergence Analysis

We will present the convergence analysis of local descent APFL in this section. The analysis is conducted on general smooth and strongly-convex functions. In order to get a tight analysis, as well as putting the optimization
results in the context of generalization bounds discussed above, we define the following parameterization-invariant quantities:

**Definition 2 (Gradient Diversity).** We define the following quantity to measure the diversity among local gradients with respect to the gradient of the *i*th client:

\[
\Lambda(f_i) = \sup_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{j=1}^{n} \| \nabla f_j(w) - \nabla f_i(w) \|_2^2.
\]  

(12)

**Definition 3 (Local-Global Optimality Gap).** We define the following quantity to measure the gap between optimal local model and optimal global model:

\[
\Delta(f_i, F) = \| v_i^* - w^* \|_2^2.
\]  

(13)

where \( v_i^* = \arg\min_v f_i(v) \) is the optimal local model at *i*th client, and \( w^* = \arg\min_w F(w) \) is the global optimal.

We note that these two quantities only depend on the distributions of local data across clients and the geometry of loss functions.

The convergence analysis of proposed APFL algorithm is based on the following standard assumptions about the analytical properties of the loss functions.

**Assumption 1 (Strong Convexity).** There exists a \( \mu > 0 \) such that \( \forall i \in [n] \),

\[
f_i(x) \geq f_i(y) + \langle \nabla f_i(y), y - x \rangle + \frac{\mu}{2} \| x - y \|^2, \quad \forall x, y \in \mathbb{R}^d.
\]  

(14)

**Assumption 2 (Smoothness).** There exists a \( L > 0 \) such that \( \forall i \in [n] \),

\[
\| \nabla f_i(x) - \nabla f_i(y) \| \leq L \| x - y \|, \quad \forall x, y \in \mathbb{R}^d.
\]  

(15)

**Assumption 3 (Bounded Variance).** The gradient of individual local objectives is bounded and the variance of stochastic gradients computed at each local data shard is bounded, i.e., \( \forall i \in [n] \) :

\[
\| \nabla f_i(x) \|^2 \leq G^2, \quad \mathbb{E}[\| \nabla f_i(x; \xi) - \nabla f_i(x) \|^2] \leq \sigma^2.
\]  

(16)

The following theorem establishes the convergence of global model. We note that we state the convergence rate in terms of key parameters and hide constants in \( O(\cdot) \) notation for ease of discussion and defer the detailed analysis to appendix.

**Theorem 2 (Global model convergence of Local Descent APFL).** If each client’s objective function satisfies Assumptions 1-3, using Algorithm 1, by choosing the mixing weight \( \alpha_i \geq \max\{1 - \frac{4}{\sqrt{6}\mu}, 1 - \frac{1}{4\sqrt{6}\sqrt{\mu}}\} \), learning rate \( \eta_t = \frac{16}{\mu(1+\alpha)} \), where \( a = \max\{128\kappa, \tau\} \), \( \kappa = \frac{L}{\mu} \), and using average scheme \( \hat{w} = \frac{1}{\kappa S_T} \sum_{i=1}^{T} \sum_{j \in U_i} w^{(i)}_j \), where \( p_t = (t+a)^2 \), \( S_T = \sum_{t=1}^{T} p_t \), and letting \( F^* \) to denote the minimum of the \( F \), then the following convergence holds:

\[
\mathbb{E}[F(\hat{w})] - F^* \leq O \left( \frac{\mu \mathbb{E}[\| w^{(1)}(1) - w^* \|^2]}{T^3} \right) + O \left( \frac{\kappa \tau^2 G^2}{T^2} \right) + O \left( \frac{L^2 \tau^2 G^2 \ln T}{T^3} \right) + O \left( \frac{G^2 + \sigma^2}{KT} \right),
\]  

(17)

where \( \tau \) is the number of local updates (i.e., synchronization gap) and \( \kappa = \frac{L}{\mu} \) is the condition number of local objectives.

**Proof.** The proof is provided in Appendix B. \( \square \)

**Remark 2.** It is noticeable that the obtained rate matches the convergence rate of the FedAvg, and if we choose \( \tau = \sqrt{T/K} \), we recover the rate \( O(\sqrt{T/K}) \), which is the convergence rate of well-known local SGD with periodic averaging Stich (2018).
We now turn to stating the convergence of the personalized local model to the optimal local model.

**Theorem 3** (Local model convergence of Local Descent APFL). Assume each client’s objective function satisfies Assumptions 1-3, and let $\kappa = L/\mu$. Using Algorithm 1, by choosing the mixing weight $\alpha_i \geq \max \{1 - \frac{1}{4\sqrt{\kappa \tau}}, 1 - \frac{1}{4\sqrt{\kappa \tau}}\}$, learning rate $\eta = \frac{16}{\Delta(f_i, F)}$, where $a = \max\{128\kappa, \tau\}$, and using average scheme $\hat{v}_t = \frac{1}{S_T} \sum_{i=1}^T p_t (\alpha_i v_i^{(t)} + (1 - \alpha_i) \frac{1}{K} \sum_{j \in U_i} w_j^{(t)})$, where $p_t = (t + a)^2$, $S_T = \sum_{t=1}^T p_t$, and letting $f_i^*$ denote the local minimum of the $i$th client, then the following convergence rate holds for all clients $i \in [n]$:

$$
\mathbb{E}[f_i(\hat{v}_t)] - f_i^* \leq O \left( \frac{\mu}{T^3} \right) + O \left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu T^3} \right) + \alpha_i^2 O \left( \frac{\sigma^2 + G^2}{\mu T} \right) + (1 - \alpha_i)^2 \left( O \left( \frac{\kappa \ln T}{T^3} \right) + O \left( \frac{\sigma^2 + G^2}{\mu KT} \right) + O \left( \frac{\kappa^2 \tau^2 G^2}{\mu^3 T^2} \right) + O \left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu^3 T^3} \right) \right)
$$

(18)

**Proof.** The proof is provided in Appendix B.2.2.

An immediate implication of above theorem is the following.

**Corollary 1.** In Theorem 3, if we choose $\tau = \sqrt{T/K}$, then we recover the convergence rate:

$$
\mathbb{E}[f_i(\hat{v}_t)] - f_i^* \leq O \left( \frac{\kappa^2 G^2 \ln T}{K T^2} \right) + \alpha_i^2 O \left( \frac{\sigma^2 + G^2}{\mu T} \right) + (1 - \alpha_i)^2 \left( O \left( \frac{\kappa \ln T}{T^3} \right) + O \left( \frac{\kappa^2 G^2 \ln T}{\mu^3 T^2} \right) \right) + (1 - \alpha_i)^2 O \left( \frac{\Lambda(f_i)}{\mu} + \kappa L \Delta(f_i, F) \right).
$$

(19)

A few remarks about the convergence of personalized local model are in place:

- If we only focus on the terms with $(1 - \alpha_i)^2$, which is contributed by the global model’s convergence, and omit the residual error, we achieve the convergence rate of $O(1/KT)$ using only $\sqrt{KT}$ communication, which recovers the result of local SGD (Stich, 2018) ($K$ is the number of all clients for local SGD).

- The residual error is related to the gradient diversity and local-global optimality gap, multiplied by a factor $1 - \alpha_i$. It shows that taking any proportion of the global model will result in a sub-optimal ERM model. As we discussed in Section 3.1, $h_{\alpha_i}$ will not be the empirical risk minimizer in most cases.

- If all users’ data are generated IID, then the residual error disappears. At this situation, if we give each client equal weight, which means $\alpha_i = \frac{1}{n}$, then we roughly recover the rate $O \left( \frac{(n-1)^2 + K}{n^2 KT} \right) = O \left( \frac{1}{KT} \right)$, which matches the convergence rate of local SGD (Stich, 2018).

We also remark that the analysis of convergence in Theorem 3 relies on a constraint that $\alpha_i$ needs to be larger than some value in order to get a tight rate. In fact, this condition can be alleviated, but the residual error will not be as tight as stated in Theorem 3. To see this, we present the analysis of this relaxation in the following theorem.

**Theorem 4** (Local convergence of Local Descent APFL without assumption on $\alpha_i$). If each client’s objective function is $\mu$-strongly-convex and $L$-smooth, using Algorithm 1, learning rate $\eta = \frac{8}{\Delta(f_i, F)}$, where $a = \max\{64\kappa, \tau\}$, and using average scheme $\hat{v}_t = \frac{1}{S_T} \sum_{i=1}^T p_t (\alpha_i v_i^{(t)} + (1 - \alpha_i) \frac{1}{K} \sum_{j \in U_i} w_j^{(t)})$, where $p_t = (t + a)^2$, $S_T = \sum_{t=1}^T p_t$, and $f_i^*$ is the local minimum of the $i$th client, then the following convergence holds for all $i \in [n]$:

$$
\mathbb{E}[f_i(\hat{v}_t)] - f_i^* \leq O \left( \frac{\mu}{T^3} \right) + O \left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu T^3} \right) + \alpha_i^2 O \left( \frac{G^2 + \sigma^2}{\mu T} \right) + (1 - \alpha_i)^2 \left( O \left( \frac{G^2 + \sigma^2}{\mu KT} \right) + O \left( \frac{\kappa^2 \tau^2 G^2}{\mu T^2} \right) + O(G^2) \right).
$$

(20)
Proof. The proof is provided in Appendix C.

Remark 3. Here we remove the assumption $\alpha_i \geq \max\{1 - \frac{1}{4\sqrt{6}c}, 1 - \frac{1}{4\sqrt{6}c\sqrt{\mu}}\}$. The key difference is that we can only show the residual error with dependency on $G$, instead of more accurate quantities $\Lambda(f_i)$ and $\Delta(f_i, F)$. Apparently, when the diversity among data shards is small, $\Lambda(f_i)$ and $\Delta(f_i, F)$ terms become small which leads to a tighter convergence rate.

5.1 The choice of $\alpha$

As Theorem 1 suggests the generalization bound is a linear function of $\alpha$, which means its optimal is reached when $\alpha = 0$ or 1. However, the convergence of the empirical risk minimization also depends on $\alpha$. When the local distribution is not far from the averaging distribution, and the amount of local data is much less than all other domains’ data, fully accepting the global seems to be the right choice. However, as long as we take any proportion of the global model, the mixed model cannot reach optimal local empirical risk.

5.2 Adaptive $\alpha$ update

As it was discussed before and can be inferred from the empirical results in Section 6, the optimum value for $\alpha$ depends on how the data are distributed in different clients, and how far is each local model from the global averaged model. When the data across different clients are highly non-IID, and the diversity among the models are high, we need to have a higher $\alpha$ value, so each client can mostly depend on its own data while benefiting from the knowledge shared by others. On the other hand, when the data are distributed uniformly random across clients, the local model of each client should be very close to the global model, and hence, the $\alpha$ value should be very close to zero. Motivating by this idea, we want to update $\alpha$ values during the training and for each client individually so they can find a better value based on the objective of the defined personalized model. Based on the local objective defined in (8), the optimum value of $\alpha$ for each client can be found by solving:

$$
\alpha^*_i = \arg \min_{\alpha_i \in [0, 1]} f_i (\alpha_i \nu + (1 - \alpha_i)w),
$$

where we can use the gradient descent to optimize it at every communication round, using the following step:

$$
\alpha_i^{(t)} = \alpha_i^{(t-1)} - \eta_t \nabla f_i \left(v_i^{(t-1)}; \xi_i^{(t)}\right)
$$

$$
= \alpha_i^{(t-1)} - \eta_t \left(v_i^{(t-1)} - w_i^{(t-1)}, \nabla f_i \left(v_i^{(t-1)}; \xi_i^{(t)}\right)\right),
$$

which interestingly shows that the mixing coefficient $\alpha$ is updated based on the correlation between the difference of the personalized and the localized global models, and the gradient at the in-device personalized model. It indicates that, when the global model is drifting from the personalized model, the value of $\alpha$ changes to adjust the balance between local data and shared knowledge among all devices captured by the global model. Obviously, when personalized and global models are very close to each other (IID data), $\alpha$ value does not change that much.

6 Experiments

In this section, we empirically show the effectiveness of the proposed algorithm in personalized federated learning. To that end, we aim at showing the convergence of both optimization and generalization errors of our proposed algorithms. More importantly, we show that using our algorithm, we can utilize the model for each client, in order to get the best generalization error on their own data. First, we describe the experimental setup we used to simulate as close as possible to a real federated learning setup and then present the results.
6.1 Experimental setup

To mimic the real setting of the federated learning, we run our code on Microsoft Azure systems, using Azure Machine Learning API. We developed our code on PyTorch Paszke et al. (2019) using its “distributed” API. We then deploy this code on Standard F64s family of VMs in Microsoft Azure, where each node has 64 vCPUs that enable us to run 100 threads of the training simultaneously. We use the Message Passing Interface (MPI) to connect each node to the server. To use PyTorch in compliance with MPI, we need to build it against the MPI. Thus, we build our user-managed docker container with the aforementioned settings and use it to run our code in it.

6.2 Experimental results

In this section, we discuss the results of applying APFL in a federated setting. For all the experiments, we have 100 users, each of which has access to its own data only. We use logistic regression as our loss function in the experiments.

Datasets. We use two datasets for our experiments, MNIST\(^1\) and a synthetic dataset.

MNIST: For the MNIST dataset to be similar to the setting in federated learning, we need to manually distribute it in a non-IID way. To this end, we follow the steps used by McMahan et al. (2017), where they partitioned the dataset based on labels and for each client draw samples from some limited number of classes. We use the same way to create 3 datasets, that are, MNIST non-IID with 2 classes per client, MNIST non-IID with 4 classes per client, and MNIST IID, where the data is distributed uniformly random across different clients.

Synthetic: For generating the synthetic dataset, we follow the procedure used by Li et al. (2018), where they use two parameters, say \(\text{synthetic}(\gamma, \beta)\), that control how much the local model and the local dataset of each client differ from that of other clients, respectively. The procedure is that for each client we generate a weight matrix \(W_i \in \mathbb{R}^{m \times c}\) and a bias \(b \in \mathbb{R}^c\), where the output for the \(i\)th client is \(y_i = \arg \max \sigma(W_i^\top x_i + b)\), where \(\sigma(.)\) is the softmax. In this setting, the input data \(x_i \in \mathbb{R}^m\) has \(m\) features and the output \(y\) can have \(c\) different values indicating number of classes. The model is generated based on a Gaussian distribution \(W_i \sim \mathcal{N}(\mu_i, 1)\) and \(b_i \sim \mathcal{N}(\mu_i, 1)\), where \(\mu_i \sim \mathcal{N}(0, \gamma)\). The input is drawn from a Gaussian distribution \(x_i \sim \mathcal{N}(v_i, \Sigma)\), where \(v_i \sim \mathcal{N}(V_i, 1)\) and \(V_i \sim \mathcal{N}(0, \beta)\). Also the variance \(\Sigma\) is a diagonal matrix with value of \(\Sigma_{k,k} = k^{-1.2}\). Using this procedure, we generate three different datasets, namely \(\text{synthetic}(0.0, 0.0)\), \(\text{synthetic}(0.5, 0.5)\), and \(\text{synthetic}(1.0, 1.0)\), where we move from an IID dataset to a highly non-IID data.

Normal training. We start by normal training in a federated setting using the APFL algorithm. Note that the model \(\tilde{w}\) and its localized version \(\tilde{w}_i\) are similar to the global and local model of the FedAvg. We refer to its localized version as the “Local” model. Also, we refer to the \(v_i\) as the “Personalized” model in the algorithm.

We run this part on the MNIST dataset, with different levels of non-IID using the three datasets from MNIST discussed before. For this part, we do not have client sampling for federated learning. To evaluate the models, we generate three datasets. One is the test dataset in the server that evaluates the global model. Also, we divide the training data for each client into 80% for training and 20% for validation. We use the training dataset to evaluate the global and personalized models during the training, aggregate the results across the clients, and report it as the global training loss, which is the optimization error. We use validation dataset to evaluate personalized and global model on local data, where it shows the generalization of both models. The test dataset indicates the generalization of the global model on global test data.

The results of running APFL on 100 clients are depicted in Figure 2, where we move from highly non-IID data (left) to IID data (right). The first row shows the training loss of the local model as well as the personalized model with different rates of personalization as \(\alpha\). The second row shows the generalization performance of the

\(^1\)http://yann.lecun.com/exdb/mnist/
Figure 2. Performance of the proposed APFL algorithm on the MNIST dataset with different levels of non-IID data distribution among different clients using a logistic regression model. In (a), each client has data from only 2 classes in the dataset, which is highly non-IID. In (b), we decrease the degree of non-IID by having data from 4 classes in each client. In (c), the data is distributed in IID manner. We refer to \( \theta_i \) as the “Local” version of the global model and \( \hat{\theta}_i \) as the “Personalized” model. The first row shows the training loss of the local version of the global model and personalized model on their training data, averaged over all clients. The second row shows the generalization of the same models on their validation data, as well as the global model on the test data. Note that as we move from highly non-IID to IID, the optimum value for \( \alpha \) is decreasing from 1.0 to 0.0. In this move, although the gap in training loss between different \( \alpha \) values is decreasing, the generalization accuracy on local data is dropping dramatically.

local model on its validation data, personalized model on its validation data, and global model on test data. As can be seen, by increasing \( \alpha \) where the diversity is high (non-IID with 2 classes per client), the performance of the personalized model both in training and generalization outperforms the global and local models. However, as we move toward IID data, this pattern is getting reversed; that is, for IID data, the personalized model with \( \alpha = 0 \) can achieve the same generalization performance as the local and global models. In training, the personalized model with the highest \( \alpha \) has the best performance, which is evident. Hence, in real federated learning, when we are dealing with highly non-IID data, APFL will outperform the local and global model of FedAvg in both training and generalization.

**Effect of sampling.** To understand how the sampling of different clients will affect the performance of the APFL algorithm, we run the same experiment with different sampling rates for the MNIST dataset. The results of this experiment is depicted in Figure 3, where we run the experiment for different sampling rates of \( K \in \{0.3, 0.5, 0.7\} \). Also, we run it with different values of \( \alpha \in \{0.25, 0.5, 0.75\} \). As it can be inferred, decreasing the sampling ratio has a negative impact on both the training and generalization performance of FedAvg. However, we can see that despite the sampling ratio, APFL is outperforming FedAvg in both training and generalization. Also, from the results of Figure 2, we know that for this dataset that is highly non-IID, larger \( \alpha \) values are preferred. Increasing \( \alpha \) can diminish the effects of sampling on personalized models both in training and generalization.

**Adaptive \( \alpha \) update.** Now, we want to show how adaptively learning the value of \( \alpha \) across different clients, based on (22), will affect the training and generalization performance of APFL models. For this experiment, we
Local Validation Accuracy

Global Training Loss

FedAvg (K = 0.3)
FedAvg (K = 0.5)
FedAvg (K = 0.7)
APFL (K = 0.3)
APFL (K = 0.5)
APFL (K = 0.7)

Figure 3. Evaluating the effect of sampling on APFL and FedAvg algorithm using the MNIST dataset that is non-IID with only 2 classes per client. The first row is training performance on the local model of FedAvg and personalized model of APFL with different sampling rates from \( \{0.3, 0.5, 0.7\} \). The second row is the generalization performance of models on local validation data, aggregated over all clients. It can be inferred that despite the sampling ratio, APFL can superbly outperform FedAvg. By increasing the \( \alpha \) values, the performance of different sampling ratios is getting close to each other, which diminishes the effect of sampling.

will use the three mentioned synthetic datasets we generated. We set the \( \alpha_i^{(0)} = 0.01 \) for every \( i \in [n] \). The results of this experiment are depicted in Figure 4, where the first figure shows the training performance of different datasets. The second figure is comparing the generalization of local and personalized models. As it can be inferred, in training, APFL outperforms FedAvg in the same datasets. More interestingly, in generalization, all datasets achieve almost the same performance as a result of adaptively updating \( \alpha \) values, while the FedAvg algorithm has a huge gap with them. This shows that, when we do not know the degree of diversity among data of different clients, we should adaptively update \( \alpha \) values to guarantee the best generalization performance.

Figure 4. Comparing APFL with adaptive \( \alpha \) and FedAvg on training and generalization performance. The first figure is the training performance, where APFL outperforms FedAvg when comparing the same dataset. The second figure shows the generalization of these methods on local validation data. APFL superbly outperforms FedAvg in generalization performance and adaptively updating \( \alpha \) results in the same performance for datasets with different levels of diversity.
7 Discussion and Extensions

Connection Between Learning Guarantee and Convergence As Theorem 1 suggests, the generalization bound depends on the divergence of the local and global distributions. In the language of optimization, the counter-part of divergence of distribution is the gradient diversity; hence, the gradient diversity appears in our empirical loss convergence rate (Theorem 3). The other interesting discovery is in the generalization bound, we have the term $\lambda_i$ and $L_D(h_i^*)$, which are intrinsic to the distributions and hypothesis class. Meanwhile, in the convergence result, we have the term $\|v_i^* - w^*\|^2$, which also only depends on the data distribution and hypothesis class we choose. In addition, $\|v_i^* - w^*\|^2$ also reveals the divergence between local and global optimal solutions.

Why APFL is “Adaptive” Both information-theoretically (Theorem 1) and computationally (Theorem 3), we prove that when the local distribution drifts far away from the average distribution, we have to tune the mixing parameter $\alpha$ to a larger value. Thus it is necessary to make $\alpha$ updated adaptively during empirical risk minimization. In Section 5.2, (22) shows that the update of $\alpha$ depends on the correlation of local gradient and deviation between local and global models. Experimental results show that our method can adaptively tune $\alpha$, and can outperform the training scheme using fixed $\alpha$.

Personalization for New Participant Nodes Suppose we already have a trained global model $\hat{w}$, and now a new device $k$ joins in the network, which is desired to personalize the global model to adapt its own domain. This can be done by performing a few local stochastic gradient descent updates from the given global model as an initial local model:

$$v_k^{(t+1)} = v_k^{(t)} - \eta_t \nabla v f_k(\alpha_k v_k^{(t)} + (1 - \alpha_k) \hat{w}; x_k^{(t)})$$

(23)

to quickly learn a personalized model for the newly joined device. One thing worthy of investigation is the difference between APFL and meta-learning approaches, such as model-agnostic meta-learning Finn et al. (2017). Our goal is to share the knowledge among the different users, in order to reduce the generalization error; while meta-learning cares more about how to build a meta-learner, to help training models faster and with fewer samples. In this scenario, similar to FedAvg, when a new node joins the network, it gets the global model and takes a few stochastic steps based on its own data to update the global model. In Figure 5, we show the results of applying FedAvg and APFL on synthetic data with two different rates of diversity, synthetic(0.0, 0.0) and synthetic(0.5, 0.5). In this experiment, we keep 3 nodes with their data off in the entire training for 100 rounds of communication between 97 nodes. In each round, each client updates its local and personalized models for one epoch. After the training is done, those 3 clients will join the network and get the latest global model and start training local and personalized models of their own. Figure 5 shows the training loss and validation accuracy of these 3 nodes during the 5 epochs of updates. The local model represents the model that will be trained in FedAvg, while the personalized model is the one resulting from APFL. Although the goal of APFL is to adaptively learn the personalized model during the training, it can be inferred that APFL can learn a better personalized model in a meta-learning scenario as well.

Agnostic Global Model and Non-Strongly-Convex Objective As pointed out by Mohri et al. (2019), the global model can be distributionally robust if we optimize the agnostic loss:

$$\min_{w \in \mathbb{R}^d} \max_{q \in \Delta_n} F(w) := \sum_{i=1}^n q_i f_i(w),$$

(24)

where $\Delta_n = \{q \in \mathbb{R}_+^n \mid \sum q_i = 1\}$ is the $n$-dimensional simplex.

We call this scenario “Adaptive Personalized Agnostic Federated Learning”. In this case, the analysis will be more challenging since the global empirical risk minimization is performed at a totally different domain, so the risk upper bound for $h_{\alpha}$ we derived does not hold anymore. Also, from a computational standpoint, since the resulted problem is a minimax optimization problem, the convergence analysis of Agnostic APFL will be more involved, which we will leave as an interesting future work.
Figure 5. Comparing the effect of fine-tuning with FedAvg and with APFL on the synthetic datasets. The model is trained for 100 rounds of communication with 97 clients, and then 3 clients will join in fine-tuning the global model based on their own data. It can be seen that the model from APFL can better personalize the global model with respect to the FedAvg method both in training loss and validation accuracy. Increasing diversity makes it harder to personalize, however, APFL surpasses FedAvg again.

The other challenging future work will be the convergence analysis of APFL for non-strongly-convex objectives. In our work, APFL converges provably fast under strongly-convex setting, but we leave the analysis for general convex or even non-convex settings as open problems.

8 Conclusion and Future Work

In this paper, we proposed an adaptive federated learning algorithm that learns a mixture of local and global models as the personalized model. Motivated by learning theory in domain adaptation, we provided generalization guarantees for our algorithm that demonstrated the dependence on the diversity between each clients’ data distribution and the representative sample of the overall distribution of data, and the number of per-device samples as key factors in personalization. Moreover, we proved the convergence rate for the proposed algorithm under smooth and strongly convex objective functions. Finally, we empirically backed up our theoretical results by conducting various experiments in a federated setting. We leave the extension of our generalization and convergence analysis to non-convex problems as an interesting and challenging open question.

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A Proof of Generalization

In this section, we present the proof of generalization bound of APFL. As pointed out, related works by Ben-David et al. (2010, 2007; Crammer et al. (2008) did the pioneering analysis of learning a single model from different domains. However, in our problem, we are tackling a further difficult case: learning two models from different domains and combining them as the final solution. This difference casts new challenges on theoretical analysis, which arise from how to bound $\mathcal{L}_D(h_i) + (1 - \alpha_i)\bar{h}^*$ (unlike standard risk where we aim at bounding $\mathcal{L}_D(h)$, in our case global and local models are mixed). Hence, as a high-level idea, we propose to view this as the risk in the transformed label space $\mathcal{L}_D'(\bar{h}^*)$, and then employ known bounds in the agnostic PAC learning Shalev-Shwartz and Ben-David (2014). That being said, we will still need multi-domain learning theory in our proof, to deal with risk term containing $\bar{h}^*$, so let us first introduce a regression version of multi-domain learning bound, before delving into the proof of the main theorem on the generalization of APFL.

**Lemma 2.** Let $\mathcal{H}$ be a hypothesis class with VC dimension $d$. Let $\mathcal{D}_S$ and $\mathcal{D}_T$ denote source domain and target domain distributions, respectively, and $\mathcal{L}_D(h) = E_{(x,y) \sim \mathcal{D}}[(h(x) - y)^2]$ be the regression risk with respect to distribution $\mathcal{D}$, then with probability at least $1 - \delta$, for every $h \in \mathcal{H}$ it holds that

$$\mathcal{L}_D(h) \leq 3\mathcal{L}_{\mathcal{D}_S}(h) + \frac{3}{4}d_{\mathcal{H}\mathcal{D}_H}(\mathcal{D}_T, \mathcal{D}_S) + 3\lambda,$$

(25)

where $\lambda = \inf_{h \in \mathcal{H}} (\mathcal{L}_{\mathcal{D}_T}(h) + \mathcal{L}_{\mathcal{D}_S}(h))$.

**Proof.** We define $h^* = \arg\min_{h \in \mathcal{H}} (\mathcal{L}_{\mathcal{D}_T}(h) + \mathcal{L}_{\mathcal{D}_S}(h))$. Since we consider MSE loss for regression task, then by the arithmetic and geometry inequality $\|a\|^2 = \|a - b\|^2 + \|b\|^2 + 2\|a - b\|\|b\| \leq (1 + \frac{1}{2})\|a - b\|^2 + (1 + q)\|b\|^2$, we have:

$$\mathcal{L}_{\mathcal{D}_T}(h) \leq 3\mathcal{L}_{\mathcal{D}_T}(h^*) + \frac{3}{2}\mathcal{L}_{\mathcal{D}_T}(h^*, h)$$
$$\leq 3\mathcal{L}_{\mathcal{D}_S}(h^*) + \frac{3}{2}\mathcal{L}_{\mathcal{D}_S}(h^*, h) + \frac{3}{2}[\mathcal{L}_{\mathcal{D}_S}(h^*, h) - \mathcal{L}_{\mathcal{D}_T}(h^*, h)]$$
$$\leq 3\mathcal{L}_{\mathcal{D}_S}(h^*) + \frac{3}{2}\mathcal{L}_{\mathcal{D}_S}(h^*, h) + \frac{3}{2}d_{\mathcal{H}\mathcal{D}_H}(\mathcal{D}_T, \mathcal{D}_S)$$
$$\leq 3\mathcal{L}_{\mathcal{D}_S}(h^*) + 3\mathcal{L}_{\mathcal{D}_S}(h) + \frac{3}{2}d_{\mathcal{H}\mathcal{D}_H}(\mathcal{D}_T, \mathcal{D}_S) + 3\lambda,$$

(26)

where the third inequality follows from Lemma 3 in Ben-David et al. (2010) \hfill \Box

A.1 Proof of Theorem 1

**Proof.** We define $\hat{h}_i^* = \arg\min_{h \in \mathcal{H}} \mathcal{L}_{\mathcal{D}_i}(\alpha_i h + (1 - \alpha_i)\bar{h}^*)$. In classification task, the risk can be re-written as

$$\mathcal{L}_{\mathcal{D}_i}(\alpha_i \hat{h} + (1 - \alpha_i)\bar{h}^*) = \alpha_i E_{(x,y) \sim \mathcal{D}_i} \left| h(x) - \left( \frac{y}{\alpha_i} - \frac{1 - \alpha_i}{\alpha_i} \bar{h}^*(x) \right) \right|.$$

(27)

So, the risk can be viewed as the risk minimization on a new label space which we denote its domain as $\mathcal{D}_i'$. Then, we use $\hat{h}_i^* = \arg\min_{h \in \mathcal{H}} \alpha_i \mathcal{L}_{\mathcal{D}_i'}(h)$ to denote the risk minimizer over new domain.

Similarly, for a regression task:

$$\mathcal{L}_{\mathcal{D}_i}(\alpha_i \hat{h} + (1 - \alpha_i)\bar{h}^*) = \alpha_i^2 E_{(x,y) \sim \mathcal{D}_i} \left( h(x) - \left( \frac{y}{\alpha_i} - \frac{1 - \alpha_i}{\alpha_i} \bar{h}^*(x) \right) \right)^2.$$

(28)

Hence, $\hat{h}_i^* = \arg\min_{h \in \mathcal{H}} \alpha_i^2 \mathcal{L}_{\mathcal{D}_i'}(h)$. 18
Now we show the proof for the classification problem:

\[ \mathcal{L}_{D_i}(h_{\alpha_i}) = \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) \]

\[ \leq \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) + |\mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) - \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h})| \]

\[ = \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) + |\alpha_i\mathcal{L}_{D_i}'(\hat{h}_i^*) - \alpha_i\mathcal{L}_{D_i}'(\bar{h}_i^*)|, \quad (29) \]

where the second term can be bounded by standard generalization bounds from agnostic PAC learning (e.g., please see Shalev-Shwartz and Ben-David (2014); Mohri et al. (2018)):

\[ |\alpha_i\mathcal{L}_{D_i}'(\hat{h}_i^*) - \alpha_i\mathcal{L}_{D_i}'(\bar{h}_i^*)| \leq \alpha_i \sqrt{\frac{2 \log(2d/\delta)}{m_i}}, \quad (30) \]

Since we assume loss function is convex so we apply the Jensen’s inequality on the first term in (29). Define \( h_i^* = \arg \min_{h \in H} \mathcal{L}_{D_i}(h) \):

\[ \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) \leq \mathcal{L}_{D_i}(\alpha_i h_i^* + (1 - \alpha_i)\bar{h}) \]

\[ \leq \alpha_i \mathcal{L}_{D_i}(h_i^*) + (1 - \alpha_i) \mathcal{L}_{D_i}(\bar{h}). \quad (31) \]

Then, plugging back Lemma 1 in (31) yields:

\[ \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) \leq \alpha_i \mathcal{L}_{D_i}(h_i^*) + (1 - \alpha_i) \left( \mathcal{L}_{\hat{D}}(\bar{h}) + \frac{1}{2} \Delta H_{D}(D_i) + \lambda \right). \quad (32) \]

Now, a simple application of the Hoeffding’s inequality Boucheron et al. (2013) gives

\[ \mathbb{P}\left( |\mathcal{L}_{\hat{D}}(\bar{h}) - \hat{\mathcal{L}}_{\hat{D}}(\bar{h})| \geq \epsilon \right) \leq 2 \exp\left( \frac{-2m \epsilon^2}{\sum_{i=1}^{m} (b - a)^2} \right), \]

where \([a, b]\) is the range of loss function. Assume, without loss of generality, the loss function is bounded in \([0, 1]\) so \((b - a)^2 \leq 1\), thereby with probability at least \(1 - \delta\), \( \mathcal{L}_{\hat{D}}(\bar{h}) \leq \hat{\mathcal{L}}_{\hat{D}}(\bar{h}) + \sqrt{\frac{\log \frac{2}{\delta}}{2m}} \). Therefore, we have:

\[ \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) \leq \alpha_i \mathcal{L}_{D_i}(h_i^*) + (1 - \alpha_i) \left( \hat{\mathcal{L}}_{\hat{D}}(\bar{h}) + \frac{1}{2} \Delta H_{D}(D_i) + \lambda + \sqrt{\frac{\log \frac{2}{\delta}}{2m}} \right) \quad (33) \]

Putting these pieces together yields the claim for the classification case of the theorem:

\[ \mathcal{L}_{D_i}(h^*_{\alpha_i}) \leq \alpha_i \left( \mathcal{L}_{D_i}(h_i^*) + \sqrt{\frac{2 \log(2d/\delta)}{m_i}} \right) + (1 - \alpha_i) \left( \hat{\mathcal{L}}_{\hat{D}}(\bar{h}) + \frac{1}{2} \Delta H_{D}(D_i) + \lambda + \sqrt{\frac{\log \frac{2}{\delta}}{2m}} \right). \quad (34) \]

For the regression case, the proof strategy is similar. Assuming that the range of loss function is bounded to \([a, b]\), we have

\[ \mathcal{L}_{D_i}(h^*_{\alpha_i}) = \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) \]

\[ \leq \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) + |\mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) - \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h})| \]

\[ = \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) + |\alpha_i^2 \mathcal{L}_{D_i}'(\hat{h}_i^*) - \alpha_i^2 \mathcal{L}_{D_i}'(\bar{h}_i^*)| \]

\[ \leq \mathcal{L}_{D_i}(\alpha_i\hat{h}_i^* + (1 - \alpha_i)\bar{h}) + \alpha_i^2 \sqrt{\frac{2(b - a)^2 \log(2d/\delta)}{m_i}} \]

\[ \leq \alpha_i \mathcal{L}_{D_i}(h_i^*) + (1 - \alpha_i) \mathcal{L}_{D_i}(\bar{h}) + \alpha_i^2 \sqrt{\frac{2(b - a)^2 \log(2d/\delta)}{m_i}}. \quad (35) \]
Then, plugging back (35) in Lemma 2 will conclude the proof:

$$L_{D_i}(h^*_i) \leq \alpha_i \left( L_{D_i}(h^*_i) + \alpha_i \sqrt{\frac{2(b-a)^2 \log(2d/\delta)}{m}} \right) + (1 - \alpha_i) \left( 3L_{\mathcal{B}}(\bar{h}^*) + \frac{3}{4} d_{\Delta H}(D_i, \mathcal{D}) + 3\lambda \right)$$

$$\leq \alpha_i \left( L_{D_i}(h^*_i) + \alpha_i \sqrt{\frac{2(b-a)^2 \log(2d/\delta)}{m}} \right)$$

$$+ (1 - \alpha_i) \left( 3L_{\mathcal{B}}(\bar{h}^*) + \frac{3}{4} d_{\Delta H}(D_i, \mathcal{D}) + 3\lambda + 3\sqrt{\frac{(b-a)^2 \log \frac{2}{\delta}}{2m}} \right).$$

(36)

For the simplicity, we assume that the range of loss function is $[0, 1]$, so that we can present an elegant result in Theorem 1. This simplification will only affect the result by a constant factor so there is no loss of generality.

\[ \square \]

**B Proof of Convergence**

In this section, we present the proof of convergence results. For ease of mathematical derivations, we first consider the case without sampling clients at each communication step and then generalize the proof to the setting where $K$ devices are sampled uniformly at random by the server.

**B.1 Proof without Sampling**

Before giving the convergence analysis of the Algorithm 1 in the main paper, we first discuss a warm-up case: local descent APFL without client sampling. As Algorithm 2 shows, all clients will participate in the averaging stage every $\tau$ iterations. The convergence of global and local models in Algorithm 2 are given in the following theorems. We start by stating the convergence of local model.

**Theorem 5** (Local model convergence of Local Descent APFL without Sampling). If each client’s objective function satisfies Assumptions 1-3, using Algorithm 2, choosing the mixing weight $\alpha_i \geq \max\{1 - \frac{1}{4\sqrt{6} \kappa}, 1 - \frac{1}{4\sqrt{6} \kappa}, 1\}$, learning rate $\eta_i = \frac{16}{\mu(t+a)}$, where $a = \max\{128\kappa, \tau\}$, and using average scheme $\bar{v}_i = \frac{1}{8T} \sum_{t=1}^{T} p_t (\alpha_i v_i(t) + (1 - \alpha_i) \frac{1}{T} \sum_{j=1}^{n} w_j^{(t)})$, where $p_t = (t + a)^2$, $S_T = \sum_{t=1}^{T} p_t$, and $f_i^*$ is the local minimum of the $i$th client, then the following convergence holds for all $i \in [n]$:  

$$E[f_i(\bar{v}_i)] - f_i^* \leq O\left( \frac{\mu^2 \tau^2 G^2 \ln T}{\mu T^3} \right) + \alpha_i^2 O\left( \frac{\sigma^2}{\mu T} \right)$$

$$+ (1 - \alpha_i)^2 \left( O\left( \frac{\kappa L \ln T}{T^3} \right) + O\left( \frac{\sigma^2}{\mu n T} \right) + O\left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu^3 T^3} \right) \right)$$

$$+ (1 - \alpha_i)^2 \left( O\left( \frac{\Lambda(f_i)}{\mu} + \kappa L \Delta(f_i, F) \right) \right).$$

(37)

The following theorem obtains the convergence of global model in Algorithm 2.

**Theorem 6** (Global model convergence of Local Descent APFL without Sampling). If each client’s objective function satisfies Assumptions 1-3, using Algorithm 2, choosing the mixing weight $\alpha_i \geq \max\{1 - \frac{1}{4\sqrt{6} \kappa}, 1 - \frac{1}{4\sqrt{6} \kappa}, 1\}$, learning rate $\eta_i = \frac{16}{\mu(t+a)}$, where $a = \max\{128\kappa, \tau\}$, and using average scheme $\bar{w} = \frac{1}{8S_T} \sum_{t=1}^{T} p_t \sum_{j=1}^{n} w_j^{(t)}$, where $p_t = (t + a)^2$, $S_T = \sum_{t=1}^{T} p_t$, then the following convergence holds:

$$E[F(\bar{w})] - F(w^*) \leq O\left( \frac{\mu E\left[ \frac{\|w^{(1)} - w^*\|^2}{T^3} \right]}{T^3} \right) + O\left( \frac{\kappa L \tau^2 G^2}{T^3} \right) + O\left( \frac{L^2 \tau^2 G^2 \ln T}{T^3} \right) + O\left( \frac{\sigma^2}{n T} \right),$$

(38)

where $w_* = \arg\min_w F(w)$ is the optimal global solution.
Algorithm 2: Local Descent APFL (without sampling)

input: Mixture weights $\alpha_1, \cdots, \alpha_n$, Synchronization gap $\tau$, Local models $v_i^{(0)}$ for $i \in [n]$ and local version of global model $w_i^{(0)}$ for $i \in [n]$.

for $t = 0, \cdots, T$ do
  if $t$ not divides $\tau$ then
    \begin{align*}
    w_i^{(t)} &= w_i^{(t-1)} - \eta_t \nabla f_i \left( w_i^{(t-1)}; \xi_i^t \right) \\
    v_i^{(t)} &= v_i^{(t-1)} - \eta_t \nabla v_i \left( v_i^{(t-1)}; \xi_i^t \right) \\
    \bar{v}_i^{(t)} &= \alpha_i v_i^{(t)} + (1 - \alpha_i) w_i^{(t)}
    \end{align*}
 \end{align*}
  else
    each client sends $w_j^{(t)}$ to the server
    \begin{align*}
    w_i^{(t)} &= \frac{1}{n} \sum_{j=1}^{n} w_j^{(t)}\text{ server broadcast $w_i^{(t)}$ to all clients}
    \end{align*}
end

for $i = 1, \cdots, n$ do
  \begin{align*}
  \text{output:} \quad \hat{v}_i &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} p_t (\alpha_i v_i^{(t)} + (1 - \alpha_i) w_i^{(t)}) \\
  \text{Global model:} \quad \hat{w} &= \frac{1}{n \sqrt{T}} \sum_{t=1}^{T} p_t \sum_{j=1}^{n} w_j^{(t)}.
  \end{align*}
end

B.1.1 Proof of Useful Lemmas

Before giving the proof of Theorem 5, we first prove few useful lemmas. We also define a virtual sequence $\{w_i^{(t)}\}_{t=1}^{T}$ where $w_i^{(t)} = \frac{1}{n} \sum_{j=1}^{n} w_j^{(t)}, \bar{v}_i^{(t)} = \alpha_i v_i^{(t)} + (1 - \alpha_i) w_i^{(t)}$.

We start with the following lemma that bounds the difference between the gradients of local objective and global objective at local and global models.

Lemma 3. At each iteration, the gap between local gradient and global gradient is bounded by

\[
\mathbb{E} \left[ \| \nabla f_i(v_i^{(t)}) - \nabla F(w_i^{(t)}) \|^2 \right] \leq 2L^2 \mathbb{E} \left[ \| \bar{v}_i^{(t)} - v^* \|^2 \right] + 6\Delta(f_i, F) + 6L^2 \mathbb{E} \left[ \| w_i^{(t)} - w^* \|^2 \right] + 6L^2 \Delta(f_i, F) \tag{39}
\]

Proof. From the smoothness assumption and by applying the Jensen’s inequality we have:

\[
\mathbb{E} \left[ \| \nabla f_i(\bar{v}_i^{(t)}) - \nabla F(w_i^{(t)}) \|^2 \right] \leq 2L^2 \mathbb{E} \left[ \| \nabla f_i(v_i^{(t)}) - \nabla F(w_i^{(t)}) \|^2 \right]
\]

\[
\leq 2L^2 \mathbb{E} \left[ \| \bar{v}_i^{(t)} - v^*_t \|^2 \right] + 6L^2 \mathbb{E} \left[ \| \nabla f_i(v_i^{(t)}) - \nabla F(w_i^{(t)}) \|^2 \right] + 6L^2 \mathbb{E} \left[ \| \nabla F(w_i^{(t)}) - \nabla F(w_i^{(t)}) \|^2 \right]
\]

\[
\leq 2L^2 \mathbb{E} \left[ \| \bar{v}_i^{(t)} - v^*_t \|^2 \right] + 6L^2 \mathbb{E} \left[ \| v_i^{(t)} - w^*_t \|^2 \right]
\]

\[
+ 6L^2 \mathbb{E} \left[ \| \nabla f_i(w^*_t) - \nabla f_i(w^*_t) \|^2 \right] + 6L^2 \mathbb{E} \left[ \| w_i^{(t)} - w^*_t \|^2 \right]
\]

\[
\leq 2L^2 \mathbb{E} \left[ \| \bar{v}_i^{(t)} - v^*_t \|^2 \right] + 6L^2 \Delta(f_i, F) + 6L^2 \mathbb{E} \left[ \| w_i^{(t)} - w^*_t \|^2 \right] \tag{40}
\]

\[\square\]
Lemma 4. Under the settings of Theorem 5 and Algorithm 2, at each iteration, the deviation between each local version of the global model $w_i^{(t)}$ and the global model $w^{(t)}$ is bounded by:

$$
\mathbb{E} \left[ \|w^{(t)} - w_i^{(t)}\|_2^2 \right] \leq 4\eta_t^2\tau^2G^2, \forall i \in [n].
$$

Proof. The proof is straightforward and can be found in Stich (2018), Lemma 3.3.

Lemma 5. (Convergence of global model) Let $w^{(t)} = \frac{1}{n} \sum_{i=1}^{n} w_i^{(t)}$. Under the setting of Theorem 5, we have:

$$
\mathbb{E} \left[ \|w^{(t+1)} - w^*\|_2^2 \right] \leq \frac{a^3 \mathbb{E} \left[ \|w^{(1)} - w^*\|_2^2 \right]}{(t+a)^3} + \left( t + 16 \left( \frac{1}{a+1} + \ln(t+a) \right) \right) \frac{32768L^2\tau^2G^2}{\mu^4(t+a)^3} + \frac{128\sigma^2(t+2a)}{n\mu^2(t+a)^3}.
$$

Proof. By the updating rule we have:

$$
w^{(t+1)} - w^* \leq w^{(t)} - w^* - \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t).
$$

Then, taking square of norm and expectation on both sides, as well as applying strong convexity and smoothness assumptions yields:

$$
\begin{align*}
\mathbb{E} \left[ \|w^{(t+1)} - w^*\|_2^2 \right] & \leq \mathbb{E} \left[ \|w^{(t)} - w^*\|_2^2 \right] - 2\eta_t \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}), w^{(t)} - w^* \right\rangle \right] + \eta_t^2 \frac{\sigma^2}{n} + \eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) \right\|_2^2 \right] \\
& \leq \mathbb{E} \left[ \|w^{(t)} - w^*\|_2^2 \right] - 2\eta_t \mathbb{E} \left[ \left\langle \nabla F(w^{(t)}), w^{(t)} - w^* \right\rangle \right] + \eta_t^2 \frac{\sigma^2}{n} + \eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) \right\|_2^2 \right] \\
& \quad - 2\eta_t \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla F(w^{(t)}), w^{(t)} - w^* \right\rangle \right] \\
& \leq (1 - \mu\eta_t) \mathbb{E} \left[ \|w^{(t)} - w^*\|_2^2 \right] - 2\eta_t (\mathbb{E}[F(w^{(t)})] - F(w^*)) + \eta_t^2 \frac{\sigma^2}{n} + T_1 + T_2,
\end{align*}
$$

where at the last step we used the strongly convex property.

Now we are going to bound $T_1$. By the Jensen’s inequality and smoothness, we have:

$$
T_1 \leq 2\eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla F(w^{(t)}) \right\|_2^2 \right] + 2\eta_t^2 \mathbb{E} \left[ \left\| \nabla F(w^{(t)}) \right\|_2^2 \right] \\
\leq 2\eta_t^2 L^2 \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \|w_j^{(t)} - w^{(t)}\|_2^2 \right] + 4\eta_t^2 L \left( \mathbb{E} \left[ F(w^{(t)}) \right] - F(w^*) \right)
$$

Then, we bound $T_2$ as:

$$
T_2 \leq \eta_t \left( \frac{2}{\mu} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla F(w^{(t)}) \right\|_2^2 \right] + \frac{\mu}{2} \mathbb{E} \left[ \|w^{(t)} - w^*\|_2^2 \right] \right) \\
\leq \frac{2\eta_t L^2}{\mu} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \|w_j^{(t)} - w^{(t)}\|_2^2 \right] + \frac{\mu\eta_t}{2} \mathbb{E} \left[ \|w^{(t)} - w^*\|_2^2 \right].
$$
Now, by plugging back $T_1$ and $T_2$ from (45) and (46) in (44), we have:

$$
\mathbb{E} \left[ \| \mathbf{w}^{(t+1)} - \mathbf{w}^* \|^2 \right] \leq \left( 1 - \frac{ \mu \eta }{2} \right) \mathbb{E} \left[ \| \mathbf{w}^{(t)} - \mathbf{w}^* \|^2 \right] - \frac{2 \eta L^2 (4 \eta_n^2 \mathcal{L})}{2} \left( \mathbb{E} \left[ F(\mathbf{w}^{(t)}) \right] - F(\mathbf{w}^*) \right) + \frac{\eta_n^2 \sigma^2}{n} + \frac{2 \eta_n^2 L^2 (\mathcal{L})}{2} \left( 1 - \frac{ \mu \eta }{2} \right) \mathbb{E} \left[ \| \mathbf{w}^{(t)} - \mathbf{w}^* \|^2 \right]
$$

$$
+ \left( \frac{2 \eta_n^2 L^2 (\mathcal{L})}{\mu} + 2 \eta_n^2 L^2 \right) \sum_{j=1}^{n} \mathbb{E} \left[ \| \mathbf{w}^{(t)} - \mathbf{w}^{(j)} \|^2 \right].
$$

(47)

Now, by using Lemma 4 we have:

$$
\mathbb{E} \left[ \| \mathbf{w}^{(t+1)} - \mathbf{w}^* \|^2 \right] \leq \left( 1 - \frac{ \mu \eta }{2} \right) \mathbb{E} \left[ \| \mathbf{w}^{(t)} - \mathbf{w}^* \|^2 \right] + \left( \frac{2 \eta_n^2 L^2 (\mathcal{L})}{\mu} + 2 \eta_n^2 L^2 \right) \frac{4 \eta_n^2 \tau^2 G^2 + \eta_n^2 \sigma^2}{n}.
$$

(48)

Note that \((1 - \frac{ \mu \eta }{2}) \frac{ \mu }{ \eta } = \frac{ \mu (t+1)^3 }{16} \leq \frac{ \mu (t-1)^3 }{16} = \frac{ \mu }{ \eta t - 1} \), so we multiply \(\frac{ \mu }{ \eta t} \) on both sides and do the telescoping sum:

$$
\frac{ \mu }{ \eta t} \mathbb{E} \left[ \| \mathbf{w}^{(t+1)} - \mathbf{w}^* \|^2 \right] \leq \frac{ \mu }{ \eta } \mathbb{E} \left[ \| \mathbf{w}^{(1)} - \mathbf{w}^* \|^2 \right] + \sum_{i=1}^{t} \left( \frac{8 \mu \eta^2 L^2 \tau^2 G^2}{\mu} + \frac{8 \mu \eta^2 L^2 \tau^2 G^2}{\mu} \right) + \sum_{i=1}^{t} \mu \eta^2 \frac{ \sigma^2 }{n}.
$$

(49)

Then, by re-arranging the terms will conclude the proof:

$$
\mathbb{E} \left[ \| \mathbf{w}^{(t+1)} - \mathbf{w}^* \|^2 \right] \leq \frac{a^3}{(T + a)^3} \mathbb{E} \left[ \| \mathbf{w}^{(1)} - \mathbf{w}^* \|^2 \right] \left( T + 16 \left( \frac{1}{a + 1} + \ln(T + a) \right) \right) \frac{32768 L^2 \tau^2 G^2}{\mu^3 (T + a)^3} + \frac{128 \sigma^2 T (T + 2a)}{n \mu^2 (T + a)^3},
$$

(50)

where we use the inequality \( \sum_{i=1}^{T} \frac{1}{i+a} \leq \frac{1}{a+1} + \int_{1}^{T} \frac{1}{i+a} < \frac{1}{a+1} + \ln(T + a) \).

\(\square\)

**B.1.2 Proof of Theorem 6**

**Proof.** According to (47) and (49) in the proof of Lemma 5 we have:

$$
\frac{ \mu }{ \eta } \sum_{t=1}^{T} \mathbb{E} \left[ \| \mathbf{w}^{(t)} - \mathbf{w}^* \|^2 \right] \leq \frac{ \mu }{ \eta } \mathbb{E} \left[ \| \mathbf{w}^{(1)} - \mathbf{w}^* \|^2 \right] - \sum_{t=1}^{T} \mu \eta \mathbb{E} \left[ F(\mathbf{w}^{(t)}) \right] - F(\mathbf{w}^*)
$$

$$
+ \sum_{t=1}^{T} \left( \frac{8 \mu \eta^2 L^2 \tau^2 G^2}{\mu} + \frac{8 \mu \eta^2 L^2 \tau^2 G^2}{\mu} \right) + \sum_{t=1}^{T} \mu \eta^2 \frac{ \sigma^2 }{n},
$$

(51)

re-arranging term and dividing both sides by \( S_T = \sum_{t=1}^{T} \mu \) yields:

$$
\frac{1}{S_T} \sum_{t=1}^{T} \mu \mathbb{E} \left[ F(\mathbf{w}^{(t)}) \right] - F(\mathbf{w}^*) \leq \frac{ \mu }{ S_T \eta } \mathbb{E} \left[ \| \mathbf{w}^{(1)} - \mathbf{w}^* \|^2 \right]
$$

$$
+ \frac{1}{S_T} \sum_{t=1}^{T} \left( \frac{8 \mu \eta^2 L^2 \tau^2 G^2}{\mu} + \frac{8 \mu \eta^2 L^2 \tau^2 G^2}{\mu} \right) + \frac{1}{S_T} \sum_{t=1}^{T} \mu \eta^2 \frac{ \sigma^2 }{n}
$$

$$
\leq O \left( \frac{ \mu \mathbb{E} \left[ \| \mathbf{w}^{(1)} - \mathbf{w}^* \|^2 \right] }{ T^3 } \right) + O \left( \frac{ \kappa L^2 \tau G^2 }{ T^2 } \right) + O \left( \frac{ L^2 \tau^2 G^2 \ln T }{ T^3 } \right) + O \left( \frac{ \sigma^2 }{ n T } \right).
$$

(52)
Recall that \( \hat{w} = \frac{1}{nSf} \sum_{t=1}^{T} \sum_{j=1}^{n} w_j^{(t)} \) and convexity of \( F \), we can conclude that:

\[
\mathbb{E} [ F(\hat{w})] - F(w^*) \leq O \left( \mu \mathbb{E} \left[ \| w^{(1)} - w^* \|^2 \right] \right) + O \left( \frac{\kappa L r^2 G^2}{T^2} \right) + O \left( \frac{L^2 r^2 G^2 \ln T}{T^3} \right) + O \left( \sigma^2 \right) + O \left( \frac{1}{nT} \right).
\]

(B.1.3) Proof of Theorem 5

Proof. Recall that we defined virtual sequences \( \{w^{(t)}\}_{t=1}^{T} \) where \( w^{(t)} = \frac{1}{n} \sum_{i=1}^{n} w_i^{(t)} \) and \( \hat{v}_i^{(t)} = \alpha_i v_i^{(t)} + (1 - \alpha_i) w_i^{(t)} \), then by the updating rule we have:

\[
\mathbb{E} \left[ \| \hat{v}_i^{(t+1)} - v_i^* \|^2 \right] = \mathbb{E} \left[ \| \hat{v}_i^{(t)} - \alpha_i^2 \eta_i \nabla f_i(\hat{v}_i^{(t)}) - (1 - \alpha_i) \eta_i \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - v_i^* \|^2 \right]
\]

\[
+ \mathbb{E} \left[ \left\| \alpha_i^2 \eta_i \nabla f_i(\hat{v}_i^{(t)}) - \nabla f_i(\hat{v}_i^{(t)}; \xi_i) \right\|^2 \right] + \mathbb{E} \left[ \left\| (1 - \alpha_i) \eta_i \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla f_j(w_j^{(t)}; \xi_i) \right\|^2 \right]
\]

\[
\leq \mathbb{E} \left[ \| \hat{v}_i^{(t)} - v_i^* \|^2 \right] - 2\mathbb{E} \left[ \left\langle \alpha_i^2 \eta_i \nabla f_i(\hat{v}_i^{(t)}) + (1 - \alpha_i) \eta_i \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}), \hat{v}_i^{(t)} - v_i^* \right\rangle \right]
\]

\[
+ \eta_i^2 \mathbb{E} \left[ \alpha_i^2 \nabla f_i(\hat{v}_i^{(t)}) + (1 - \alpha_i) \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) \right]^2 + \alpha_i^2 \eta_i^2 \sigma^2 + (1 - \alpha_i)^2 \eta_i^2 \frac{\sigma^2}{n}
\]

\[
= \mathbb{E} \left[ \| \hat{v}_i^{(t)} - v_i^* \|^2 \right] - 2(\alpha_i^2 + 1 - \alpha_i) \eta_i \mathbb{E} \left[ \left\langle \nabla f_i(\hat{v}_i^{(t)}), \hat{v}_i^{(t)} - v_i^* \right\rangle \right]
\]

\[
- 2\eta_i (1 - \alpha_i) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla f_i(\hat{v}_i^{(t)}), \hat{v}_i^{(t)} - v_i^* \right\rangle \right]
\]

\[
+ \eta_i^2 \mathbb{E} \left[ \alpha_i^2 \nabla f_i(\hat{v}_i^{(t)}) + (1 - \alpha_i) \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) \right]^2 + \alpha_i^2 \eta_i^2 \sigma^2 + (1 - \alpha_i)^2 \eta_i^2 \frac{\sigma^2}{n}.
\]
Now, we bound the term $T_1$ as follows:

$$T_1 = -2\eta_t (\alpha_i^2 + 1 - \alpha_i) [\nabla f_i (\tilde{v}_i(t), \tilde{v}_i(t) - v_i^*)]$$

$$- 2\eta_t (\alpha_i^2 + 1 - \alpha_i) \mathbb{E} \left[ \nabla f_i (\tilde{v}_i(t)) - \nabla f_i (\tilde{v}_i(t), \tilde{v}_i(t) - v_i^*) \right]$$

$$\leq -2\eta_t (\alpha_i^2 + 1 - \alpha_i) \left( \mathbb{E} \left[ f_i (\tilde{v}_i(t)) \right] - f_i (v_i^*) + \frac{\mu}{2} \mathbb{E} \left[ \| v_i(t) - v_i^* \|^2 \right] \right)$$

$$+ (\alpha_i^2 + 1 - \alpha_i) \eta_t \left( \frac{8L^2}{\mu (1 - 8(\alpha_i - \alpha_i^2))} \mathbb{E} \left[ \| v_i(t) - \tilde{v}_i(t) \|^2 \right] + \frac{\mu (1 - 8(\alpha_i - \alpha_i^2))}{8} \mathbb{E} \left[ \| v_i(t) - v_i^* \|^2 \right] \right)$$

$$\leq -2\eta_t (\alpha_i^2 + 1 - \alpha_i) \left( \mathbb{E} \left[ f_i (\tilde{v}_i(t)) \right] - f_i (v_i^*) + \frac{\mu}{2} \mathbb{E} \left[ \| v_i(t) - v_i^* \|^2 \right] \right)$$

$$+ \eta_t \left( \frac{8L^2(1 - \alpha_i^2)}{\mu (1 - 8(\alpha_i - \alpha_i^2))} \mathbb{E} \left[ \| w_i(t) - w_i^* \|^2 \right] + \frac{\mu (1 - 8(\alpha_i - \alpha_i^2))}{8} \mathbb{E} \left[ \| v_i(t) - v_i^* \|^2 \right] \right)$$

$$\leq -2\eta_t (\alpha_i^2 + 1 - \alpha_i) \left( \mathbb{E} \left[ f_i (\tilde{v}_i(t)) \right] - f_i (v_i^*) + \frac{7\mu \eta_t}{8} \mathbb{E} \left[ \| v_i(t) - v_i^* \|^2 \right] \right)$$

$$+ \frac{8\eta_t L^2(1 - \alpha_i^2)}{\mu (1 - 8(\alpha_i - \alpha_i^2))} \mathbb{E} \left[ \| w_i(t) - w_i^* \|^2 \right], \quad (55)$$

where we use the fact $(\alpha_i^2 + 1 - \alpha_i) \leq 1$. Note that, because we set $\alpha_i \geq \max \{1 - \frac{1}{4\sqrt{6\kappa}}, 1 - \frac{1}{4\sqrt{6\kappa}^2} \}$, and hence $1 - 8(\alpha_i - \alpha_i^2) \geq 0$, so in the second inequality we can use the arithmetic-geometry inequality.

Next, we turn to bounding the term $T_2$ in (54):

$$T_2 = -2\eta_t (1 - \alpha_i) \left( \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} \nabla f_j (w_j(t)) - \nabla f_i (\tilde{v}_i(t), \tilde{v}_i(t) - v_i^*) \right] \right)$$

$$\leq \eta_t (1 - \alpha_i) \left( \frac{2(1 - \alpha_i)}{\mu} \mathbb{E} \left[ \left\| \nabla f_i (\tilde{v}_i(t)) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j (w_j(t)) \right\|^2 \right] + \frac{\mu}{2(1 - \alpha_i)} \mathbb{E} \left[ \| \tilde{v}_i(t) - v_i^* \|^2 \right] \right)$$

$$\leq \frac{6(1 - \alpha_i)^2 \eta_t}{\mu} \times$$

$$\left( \mathbb{E} \left[ \left\| \nabla f_i (\tilde{v}_i(t)) - \nabla f_i (\tilde{v}_i(t)) \right\|^2 \right] + \mathbb{E} \left[ \left\| \nabla f_i (\tilde{v}_i(t)) - \nabla f_i (\tilde{v}_i(t)) \right\|^2 \right] + \mathbb{E} \left[ \left\| \nabla f_i (\tilde{v}_i(t)) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j (w_j(t)) \right\|^2 \right] \right)$$

$$+ \frac{\eta_t \mu}{2} \mathbb{E} \left[ \| \tilde{v}_i(t) - v_i^* \|^2 \right]$$

$$\leq \frac{6(1 - \alpha_i)^2 \eta_t}{\mu} \left( L^2 \mathbb{E} \left[ \| w_i(t) - w_i(t) \|^2 \right] + \mathbb{E} \left[ \left\| \nabla f_i (\tilde{v}_i(t)) - \nabla f_i (\tilde{v}_i(t)) \right\|^2 \right] + \frac{1}{n} \sum_{j=1}^{n} L^2 \mathbb{E} \left[ \| w_i(t) - w_i(t) \|^2 \right] \right)$$

$$+ \frac{\eta_t \mu}{2} \mathbb{E} \left[ \| \tilde{v}_i(t) - v_i^* \|^2 \right]. \quad (56)$$
And finally, we bound the term $T_3$ in (54) as follows:

$$T_3 = E \left[ \left\| \alpha_i^2 \nabla f_i(\bar{\nu}_i) + (1 - \alpha_i) \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\nu_j(t)) \right\|^2 \right]$$

$$\leq 2(\alpha_i^2 + 1 - \alpha_i) E \left[ \left\| \nabla f_i(\bar{\nu}_i(t)) \right\|^2 \right] + 2E \left[ \left\| (1 - \alpha_i) \left( \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\nu_j(t)) - \nabla f_i(\bar{\nu}_i(t)) \right) \right\|^2 \right]$$

$$\leq 2 \left( 2(\alpha_i^2 + 1 - \alpha_i) E \left[ \left\| \nabla f_i(\bar{\nu}_i(t)) - \nabla f_i^*(t) \right\|^2 \right] + 2(\alpha_i^2 + 1 - \alpha_i) E \left[ \left\| \nabla f_i(\bar{\nu}_i(t)) - \nabla f_i(\bar{\nu}_i(t)) \right\|^2 \right] \right)$$

$$+ 2(1 - \alpha_i)^2 E \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\nu_j(t)) - \nabla f_i(\bar{\nu}_i(t)) \right\|^2 \right]$$

$$\leq 8L(\alpha_i^2 + 1 - \alpha_i) \left( E \left[ f_i(\bar{\nu}_i(t)) \right] - f_i^* \right) + 4L^2 E \left[ \left\| w(t) - w_i(t) \right\|^2 \right]$$

$$+ 6(1 - \alpha_i)^2 \left( L^2 E \left[ \left\| w(t) - w_i(t) \right\|^2 \right] \right) + E \left[ \left\| \nabla f_i(\bar{\nu}_i(t)) - \nabla F(w(t)) \right\|^2 \right] + \frac{1}{n} \sum_{j=1}^{n} L^2 E \left[ \left\| w(t) - w_j(t) \right\|^2 \right].$$

(57)

Now, using Lemma 4 that $E \left[ \left\| w_j(t) - w(t) \right\|^2 \right] \leq 4\eta^2 t^2 G^2$, $(1 - \alpha_i)^2 \leq 1$ and plugging back $T_1$, $T_2$, and $T_3$ from (55), (56), and (57) into (54), yields:

$$E \left[ \left\| \bar{\nu}_i(t+1) - v_i^* \right\|^2 \right] \leq \left( 1 - \frac{3\eta(\bar{\mu} - \eta)}{8} \right) E \left[ \left\| \bar{\nu}_i(t) - v_i^* \right\|^2 \right] - 2(\eta - 4\eta^2 t\eta)(\alpha_i^2 + 1 - \alpha_i) \left( E \left[ f_i(\bar{\nu}_i(t)) \right] - f_i(v_i^*) \right)$$

$$+ \alpha_i^2 \eta^2 \sigma^2 + (1 - \alpha_i)^2 \eta^2 \sigma^2 \frac{\sigma^2}{n}$$

$$+ \left( \frac{8\eta L^2(1 - \alpha_i)^2}{\mu(1 - 8(\alpha_i - \alpha_i^2))} + \frac{12(1 - \alpha_i)^2 \eta L^2}{\mu} + 12(1 - \alpha_i) \eta^2 L^2 + 4\eta^2 L^2 \right) 4\eta^2 t^2 G^2$$

$$+ \left( \frac{6\eta}{\mu} + 6\eta^2 \right) (1 - \alpha_i)^2 E \left[ \left\| \nabla F(w(t)) - \nabla f_i(\bar{\nu}_i(t)) \right\|^2 \right].$$

(58)

where using Lemma 3 we can bound $T_4$ as:

$$T_4 \leq \frac{6\eta}{\mu} (1 - \alpha_i)^2 \left( 2L^2 E \left[ \left\| \bar{\nu}_i(t) - v_i^* \right\|^2 \right] + 6\Lambda(f_i) + 6L^2 E \left[ \left\| w(t) - w_i(t) \right\|^2 \right] + 6L^2 \Delta(f_i, F) \right)$$

$$+ 6\eta^2 (1 - \alpha_i)^2 \left( 2L^2 E \left[ \left\| \bar{\nu}_i(t) - w_i^* \right\|^2 \right] + 6\Lambda(F_i) + 6L^2 E \left[ \left\| w(t) - w_i(t) \right\|^2 \right] + 6L^2 \Delta(f_i, F) \right).$$

(59)

Note that we choose $\alpha_i \geq \max\{1 - \frac{1}{4\sqrt{\lambda}}, 1 - \frac{1}{4\sqrt{\lambda} \sqrt{n}}\}$, hence $\frac{12L^2(1 - \alpha_i)^2}{\mu} \leq \frac{\eta}{8} \frac{\mu}{8}$ and $12L^2(1 - \alpha_i)^2 \leq \frac{\eta}{8}$, thereby we have:

$$T_4 \leq \frac{\mu\eta}{4} \left\| \bar{\nu}_i(t) - v_i^* \right\|^2 + 36\eta \left( \frac{1}{\mu} + \eta \right) (1 - \alpha_i)^2 \left( \Lambda(F_i) + L^2 E \left[ \left\| w(t) - w_i(t) \right\|^2 \right] + L^2 \Delta(f_i, F) \right).$$

(60)
Now, using Lemma 5 we have:

\[
T_4 \leq \frac{\mu \eta_t}{4} \mathbb{E} \left[ \left\| \hat{v}_i^{(t)} - v^* \right\|^2 \right] + 36 \eta_t \left( \frac{1}{\mu} + \eta_t \right) (1 - \alpha_t)^2 \left( \Lambda(f_i) + L^2 \left( \frac{a^3 \mathbb{E} \left[ \left\| w^{(1)} - w^* \right\|^2 \right]}{(t - 1 + a)^3} \right) \right)
\]

+ \left( t + 16 \left( \frac{1}{\alpha + 1} + \ln(t + a) \right) \right) 32768 L^2 \tau^2 G^2 \left( \frac{1}{\mu(t - 1 + a)^3} \right) + 128 \sigma^2 t(t + 2a) \left( \frac{1}{n \mu^2(t - 1 + a)^3} \right) + L^2 \Delta(f, F).
\]

By plugging back $T_4$ from (61) in (58) and using the fact $-(\eta_t - 4\eta_t^2 L) \leq -\frac{1}{4} \eta_t$, and $(\alpha_t^2 + 1 - \alpha_t) \geq \frac{3}{4}$, we have:

\[
\mathbb{E} \left[ \left\| \hat{v}_i^{(t+1)} - v^*_i \right\|^2 \right] \leq \left( 1 - \frac{\mu \eta_t}{8} \right) \mathbb{E} \left[ \left\| \hat{v}_i^{(t)} - v^*_i \right\|^2 \right] - \frac{3 \eta_t}{4} \left( \mathbb{E} \left[ f_i(\hat{v}_i^{(t)}) \right] - f_i(v^*_i) \right) + \alpha_t^2 \sigma^2 + (1 - \alpha_t)^2 \eta_t^2 \frac{\sigma^2}{n}
\]

+ \left( \frac{8 \eta_t L^2 (1 - \alpha_t)^2}{\mu(1 - 8(\alpha_t - \alpha_t^2))} + \frac{12(1 - \alpha_t)^2 \eta_t L^2}{\mu} + 12(1 - \alpha_t)^2 \eta_t^2 L^2 \right) 4 \eta_t^2 \tau^2 G^2 + 16 L^2 \eta_t^4 \tau^2 G^2

+ 36 \eta_t \left( \frac{1}{\mu} + \eta_t \right) (1 - \alpha_t)^2 \left( \Lambda(f_i) + L^2 \left( \frac{a^3 \mathbb{E} \left[ \left\| w^{(1)} - w^* \right\|^2 \right]}{(t - 1 + a)^3} \right) \right)

+ \left( t + 16 \left( \frac{1}{\alpha + 1} + \ln(t + a) \right) \right) 32768 L^2 \tau^2 G^2 \left( \frac{1}{\mu(t - 1 + a)^3} \right) + 128 \sigma^2 t(t + 2a) \left( \frac{1}{n \mu^2(t - 1 + a)^3} \right) + \Delta(f, F).
\]

Note that $(1 - \frac{\mu \eta_t}{8}) \frac{p_t}{\eta_t} \leq \frac{p_{t-1}}{\eta_{t-1}}$ where $p_t = (t + a)^2$, so, we multiply $\frac{p_t}{\eta_t}$ on both sides, and re-arrange the terms:

\[
\frac{3 \eta_t}{4} \left( \mathbb{E} \left[ f_i(\hat{v}_i^{(t)}) \right] - f_i(v^*_i) \right) \leq \frac{p_t - 1}{\eta_{t-1}} \mathbb{E} \left[ \left\| \hat{v}_i^{(t)} - v^*_i \right\|^2 \right] - \frac{p_t}{\eta_t} \mathbb{E} \left[ \left\| \hat{v}_i^{(t+1)} - v^*_i \right\|^2 \right] + p_t \eta_t \left( \alpha_t^2 \sigma^2 + (1 - \alpha_t)^2 \frac{\sigma^2}{n} \right)
\]

+ \left( 1 - \alpha_t \right)^2 L^2 \left( \frac{8 \eta_t}{\mu(1 - 8(\alpha_t - \alpha_t^2))} + \frac{12 \eta_t^2}{\mu} + 12 \eta_t^2 \right) \frac{4 \eta_t \tau^2 G^2 + 16 L^2 \eta_t^4 \tau^2 G^2}{n \mu^2(t - 1 + a)^3}

+ 36 \eta_t \left( \frac{1}{\mu} + \eta_t \right) (1 - \alpha_t)^2 \left( \Lambda(f_i) + L^2 \Delta(f, F) \right) \left( \frac{a^3 \mathbb{E} \left[ \left\| w^{(1)} - w^* \right\|^2 \right]}{(t - 1 + a)^3} \right)

+ \left( t + 16 \left( \frac{1}{\alpha + 1} + \ln(t + a) \right) \right) 32768 L^2 \tau^2 G^2 \left( \frac{1}{\mu(t - 1 + a)^3} \right) + 128 \sigma^2 t(t + 2a) \left( \frac{1}{n \mu^2(t - 1 + a)^3} \right) + \Delta(f, F)
\]

(62)
By applying the telescoping sum and dividing both sides by $S_T = \sum_{t=1}^T p_t \geq T^3$ we have:

$$f_i(\hat{v}_i) - f_i(v_i^*)$$

$$\leq \frac{1}{S_T} \sum_{t=1}^T p_t (f_i(\hat{v}_i) - f_i(v_i^*))$$

$$\leq \frac{4\rho_h\mathbb{E} \left[ \| \hat{v}_i(t) - v_i^* \|^2 \right]}{3\eta_0 S_T} + \frac{4}{S_T} \sum_{t=1}^T p_t \eta_t \left( \alpha^2 \frac{\sigma^2}{n} + (1 - \alpha_i)^2 \frac{\sigma^2}{n} \right) + \frac{64 L^2 \tau^2 G^2}{S_T} \sum_{t=1}^T p_t \eta_t^3$$

$$+ \frac{1}{S_T} \sum_{t=1}^T (1 - \alpha_i)^2 L^2 \left( \frac{8\eta_t}{\mu(1 - 8(\alpha_i - \alpha^2_i))} + \frac{12\eta_t}{\mu} + 12\eta_t^2 \right) \frac{16}{3} \frac{1}{S_T} \eta_t \tau^2 G^2$$

$$+ 48(1 - \alpha_i)^2 \left( \lambda(f_i) + L^2 \Delta(f_i, F) \right) \frac{1}{S_T} \sum_{t=1}^T p_t \left( \frac{1}{\mu} + \eta_t \right)$$

$$\leq \frac{4\rho_h\mathbb{E} \left[ \| \hat{v}_i(t) - v_i^* \|^2 \right]}{3\eta_0 S_T} + \frac{32T(T + a)}{3\mu S_T} \left( \alpha^2 \frac{\sigma^2}{n} + (1 - \alpha_i)^2 \frac{\sigma^2}{n} \right) + \frac{L^2 \tau^2 G^2 \Theta(\ln T)}{\mu^3 S_T}$$

$$+ \frac{16(1 - \alpha_i)^2 L^2 G^2}{a^2 S_T} \left( \frac{2048T}{\mu^3(1 - 8(\alpha_i - \alpha^2_i))} + \frac{3072 \tau^2 G^2}{\mu^3} + \frac{49152 \tau^2 G^2 \Theta(\ln T)}{\mu^3} \right)$$

$$+ 48(1 - \alpha_i)^2 \frac{L^2}{a^2 S_T} \left( \frac{a^2 \Theta(\ln T) \mathbb{E} \left[ \| w^{(1)}(t) - w^* \|^2 \right]}{6\mu} + \left( \frac{T}{a} + \Theta(\ln T) + O(1) \right) \frac{32768 L^2 \tau^2 G^2}{\mu^5} + \frac{64(2a + 1) \sigma^2 T(T + a)}{\eta a \mu^3} \right)$$

$$+ 48(1 - \alpha_i)^2 \left( \lambda(f_i) + L^2 \Delta(f_i, F) \right) \frac{1}{S_T} \left( \mu T^2 + \frac{8T(T + 2a)}{\mu} \right)$$

$$= O \left( \frac{\mu}{T^3} \right) + O \left( \frac{\alpha^2 \sigma^2}{\mu T^2} \right)$$

$$+ (1 - \alpha)^2 \left( O \left( \frac{\kappa \tau^2 G^2 \ln T}{\mu T^2} \right) + O \left( \frac{\sigma^2}{\mu T^2} \right) + O \left( \frac{\kappa \tau^2 G^2 \ln T}{\mu^3 T^2} \right) + O \left( \frac{\kappa \tau^2 G^2 \ln T}{\mu^3 T^2} \right) + O \left( \frac{\lambda(f_i) + \kappa L \Delta(f_i, F)}{\mu} \right) \right).$$

(63)

where we use the convergence of $\sum_{t=1}^\infty \frac{\ln t}{T^2} = O(1)$, and $\sum_{t=1}^\infty \frac{1}{T^2} \rightarrow \frac{\pi^2}{6}$.

\[ \square \]

**B.2 Proof with Sampling**

In this section we will provide the formal proof of the Theorem 3. The proof pipeline is similar to what we did in Appendix B.1.3. The only difference is that we use sampling method here, hence, we will introduce the variance depending on sampling size $K$. Now we first begin with the proof of Lemma 6.

**Lemma 6.** (Convergence of Global Model) Let $w^{(t)} = \frac{1}{K} \sum_{j \in U_t} w^{(t)}$. Assume each client’s objective function satisfies Assumption 1-3, then, using Algorithm 1 by choosing learning rate as $\eta_t = \frac{16}{\mu(t+a)}$ and letting $\kappa = L/\mu$,
we have:

\[
\mathbb{E} \left[ ||w^{(t+1)} - w^*||^2 \right] \leq \frac{a^3 \mathbb{E} \left[ ||w^{(1)} - w^*||^2 \right]}{(t + a)^3} + \left( t + 16 \left( \frac{1}{a + 1} + \ln(t + a) \right) \right) \frac{32768 L^2 \tau^2 G^2}{\mu^4 (t + a)^3} + \frac{256(G^2 + \sigma^2)t(t + 2a)}{K \mu^2 (t + a)^3}.
\]

(64)

**Proof.** First, we note that from the updating rule we have

\[
w^{(t+1)} - w^* = w^{(t)} - w^* - \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j; \xi^{(t)}_j).
\]

(65)

Now, making both sides squared and according to the strong convexity we have:

\[
\mathbb{E} \left[ ||w^{(t+1)} - w^*||^2 \right] \leq \mathbb{E} \left[ ||w^{(t)} - w^*||^2 \right] - 2\eta_t \mathbb{E} \left[ \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j), w^{(t)} - w^* \right] + \eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j) \right\|^2 \right] + \eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j; \xi^{(t)}_j) - \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j) \right\|^2 \right] \leq (1 - \mu \eta_t) \mathbb{E} \left[ ||w^{(t)} - w^*||^2 \right] - 2\eta_t (F(w^{(t)}) - F(w^*)) + \frac{\eta_t 2(G^2 + \sigma^2)}{K} + \eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j) - \nabla f_j(w^{(t)}), w^{(t)} - w^* \right\|^2 \right].
\]

(66)

Then, following the same procedure as in Lemma 5 yields:

\[
\frac{p_T}{\eta_T} \mathbb{E} \left[ ||w^{(T+1)} - w^*||^2 \right] \leq \frac{p_T}{\eta_0} \mathbb{E} \left[ ||w^{(1)} - w^*||^2 \right] - \mathbb{E} [F(w^{(t)}) - F(w^*)] + \sum_{t=1}^{T} \left( \frac{8p_t \eta_t^2 L^2 \tau^2 G^2}{\mu} + 8p_t \eta_t^3 L^2 \tau^2 G^2 \right) + \sum_{t=1}^{T} p_t \eta_t \frac{2(G^2 + \sigma^2)}{K}.
\]

(67)

Then, by re-arranging the terms will conclude the proof as

\[
\mathbb{E} \left[ ||w^{(T+1)} - w^*||^2 \right] \leq \frac{a^3}{(T + a)^3} \mathbb{E} \left[ ||w^{(1)} - w^*||^2 \right] + \left( T + 16 \left( \frac{1}{a + 1} + \ln(T + a) \right) \right) \frac{32768 L^2 \tau^2 G^2}{\mu^4 (T + a)^3} + \frac{256\sigma^2 T(T + 2a)}{K \mu^2 (T + a)^3}.
\]

(68)

**B.2.1 Proof of Theorem 2**

**Proof.** According to (67) we have:

\[
\frac{p_T}{\eta_T} \mathbb{E} \left[ ||w^{(T+1)} - w^*||^2 \right] \leq \frac{p_T}{\eta_0} \mathbb{E} \left[ ||w^{(1)} - w^*||^2 \right] - \sum_{t=1}^{T} p_t \left( \mathbb{E} [F(w^{(t)})] - F(w^*) \right) + \sum_{t=1}^{T} \left( \frac{8p_t \eta_t^2 L^2 \tau^2 G^2}{\mu} + 8p_t \eta_t^3 L^2 \tau^2 G^2 \right) + \sum_{t=1}^{T} p_t \eta_t \frac{2(G^2 + \sigma^2)}{K}.
\]

(69)
By re-arranging the terms and dividing both sides by \( S_T = \sum_{t=1}^{T} p_t > T^3 \) yields:

\[
\frac{1}{S_T} \sum_{t=1}^{T} p_t \left( \mathbb{E} \left[ F(w(t)) \right] - F(w^*) \right) \\
\leq \frac{p_0}{S_T\eta_0} \mathbb{E} \left[ \| w^{(1)} - w^* \|^2 \right] + \frac{1}{S_T} \sum_{t=1}^{T} \left( 8p_t \eta_0^2 L^2\tau^2 G^2 + 8p_t \eta_0^3 L^2 \tau^2 G^2 \right) + \frac{1}{S_T} \sum_{t=1}^{T} p_t \eta_t^2 \frac{(G^2 + \sigma^2)}{K} \\
\leq O \left( \frac{\mu \mathbb{E} \left[ \| w^{(1)} - w^* \|^2 \right]}{T^3} \right) + O \left( \frac{\kappa L^2 G^2}{T^2} \right) + O \left( \frac{L^2 \tau^2 G^2 \ln T}{T^3} \right) + O \left( \frac{G^2 + \sigma^2}{KT} \right). \tag{70}
\]

Recalling that \( \hat{w} = \frac{1}{T n} \sum_{t=1}^{T} \sum_{j=1}^{n} w^{(t)}_j \), from the convexity of \( F(\cdot) \), we can conclude that

\[
\mathbb{E} \left[ F(\hat{w}) \right] - F(w^*) \leq O \left( \frac{\mu \mathbb{E} \left[ \| w^{(1)} - w^* \|^2 \right]}{T^3} \right) + O \left( \frac{\kappa L^2 G^2}{T^2} \right) + O \left( \frac{L^2 \tau^2 G^2 \ln T}{T^3} \right) + O \left( \frac{G^2 + \sigma^2}{KT} \right). \tag{71}
\]

\[\square\]

### B.2.2 Proof of Theorem 3

In this section we provide the proof of Theorem 3. The main difference in this case is that only a subset of local models get updated each period due to partial participation of devices, i.e., \( K \) out of all \( n \) devices that are sampled uniformly at random. To generalize the proof, we will use an indicator function to model this stochastic update, and show that while the stochastic gradient is unbiased, it introduces some extra variance that can be taken into account by properly tuning the hyper-parameters.

**Proof.** Recall that we defined virtual sequences of \( \left\{ w^{(t)} \right\}_{t=1}^{T} \) where \( w^{(t)} = \frac{1}{K} \sum_{j \in U_t} w^{(t)}_j \) and \( \hat{v}^{(t)}_i = \alpha_i v^{(t)}_i + (1 - \alpha_i) w^{(t)}_i \). We also define an indicator variable to denote whether \( i \) th client was selected at iteration \( t \):

\[
I^t_i = \begin{cases} 
1 & \text{if } i \in U_t \\
0 & \text{else}
\end{cases}
\]

obviously, \( \mathbb{E} \left[ \frac{n}{K} I^t_i \right] = 1 \), and \( \mathbb{E} \left[ \left\| \frac{n}{K} I^t_i - 1 \right\|^2 \right] = \frac{n-K}{K} \). We start by writing the updating rule:

\[
\hat{v}^{(t+1)}_i = \hat{v}^{(t)}_i - \alpha_i n \frac{n}{K} \eta_t \nabla f_i(\hat{v}^{(t)}_i : \xi^t_i) - (1 - \alpha_i) \eta_t \frac{1}{K} \sum_{j \in U_t} \nabla f_j(w^{(t)}_j : \xi^t_j). \tag{73}
\]

Now, subtracting \( \nu^*_i \) on both sides, taking the square of norm and expectation, yields:

\[
\mathbb{E} \left[ \| \hat{v}^{(t+1)}_i - \nu^*_i \|^2 \right] = \mathbb{E} \left[ \left\| \hat{v}^{(t)}_i - \alpha_i^2 \eta_t \nabla f_i(\hat{v}^{(t)}_i) - (1 - \alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w^{(t)}_j) - \nu^*_i \right\|^2 \right] \\
+ \mathbb{E} \left[ \left\| \alpha_i^2 \eta_t \left( \nabla f_i(\hat{v}^{(t)}_i) - \frac{n}{K} \frac{n}{n} \nabla f_i(\hat{v}^{(t)}_i : \xi^t_i) \right) - (1 - \alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w^{(t)}_j : \xi^t_j) \right\|^2 \right] \\
= \mathbb{E} \left[ \| \hat{v}^{(t)}_i - \nu^*_i \|^2 \right] - 2 \alpha_i^2 \eta_t \nabla f_i(\hat{v}^{(t)}_i) + (1 - \alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w^{(t)}_j), \hat{v}^{(t)}_i - \nu^*_i \\
+ \eta_t^2 \mathbb{E} \left[ \left\| \alpha_i^2 \nabla f_i(\hat{v}^{(t)}_i) + (1 - \alpha_i) \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w^{(t)}_j) \right\|^2 \right] + \alpha_i^2 \eta_t^2 2(nG^2 + (n-K)G^2) + (1 - \alpha_i)^2 \eta_t^2 2(G^2 + \sigma^2). \tag{74}
\]
Note that we obtain the similar formulation as the without sampling case in (54). The only difference is the variance changed from

$$\alpha_i^2 \eta^2 \sigma^2 + (1 - \alpha_i)^2 \frac{\eta^2 \sigma^2}{n},$$

(75)

to

$$\alpha_i^2 \eta^2 \frac{2(n\sigma^2 + (n - K)G^2)}{K} + (1 - \alpha_i)^2 \eta^2 \frac{2(G^2 + \sigma^2)}{K}.$$  

(76)

Then following the same procedure in Appendix B.1.3, together with the application of Lemma 6 we can conclude that:

$$f_i(\hat{v}_i) - f_i(v_i^*)$$

$$\leq 1 \sum_{t=1}^{T} p_t(f_i(\hat{v}_i^{(t)}) - f_i(v_i^*))$$

$$\leq \frac{4p_0}{3\eta_0 ST} \left[ \|\hat{v}_i^{(1)} - v_i^*\|^2 \right] + \frac{1}{ST} \left( \sum_{t=1}^{T} p_t \eta_t \left( \alpha_i^2 \frac{2(n\sigma^2 + (n - K)G^2)}{K} + (1 - \alpha_i)^2 \frac{2(G^2 + \sigma^2)}{K} + \frac{64}{3} L^2 \tau^2 G^2 \right) \right)$$

$$+ \frac{1}{ST} \sum_{t=1}^{T} (1 - \alpha_i)^2 \tau^2 \left( \frac{8\eta_t}{\mu(1 - 8(\alpha_i - \alpha_i^2))} + \frac{12\eta_t}{\mu} + \frac{12\mu}{\eta_t} \right)$$

$$+ \frac{16}{3} p_t \eta_t \tau G^2$$

$$+ 48(1 - \alpha_i)^2 L^2 \sum_{t=1}^{T} p_t \left( \frac{1}{\mu} + \eta_t \right)$$

$$+ \frac{a^2}{\mu^3} \left[ \Theta(\ln T) \cdot \frac{\left[ \|w^{(1)} - w^*\|^2 \right]}{\mu^3} \right] + \frac{T \cdot \Theta(\ln T)}{\mu^3} + \frac{32768L^2 \tau^2 G^2}{\mu^3} + \frac{256(G^2 + \sigma^2)(t(t + 2a))}{K \mu^2 (t + 1)^3}$$

(77)

$$+ 32T(T + a) \left( \alpha_i^2 \frac{2(n\sigma^2 + (n - K)G^2)}{K} + (1 - \alpha_i)^2 \frac{2(G^2 + \sigma^2)}{K} \right)$$

$$+ \frac{16(1 - \alpha_i)^2 \tau^2 L^2 G^2}{a^2 ST} \left( \mu^3 \left( \frac{2048T}{\mu^3} + \frac{3072T}{\mu^3} + \frac{49152\Theta(\ln T)}{\mu^3} \right) \right)$$

$$+ 48(1 - \alpha_i)^2 \frac{(a + 1)^2}{a^2 ST} \left( \frac{a^2 \Theta(\ln T)E \left[ \|w^{(1)} - w^*\|^2 \right]}{\mu} + \frac{\Theta(\ln T)}{\mu^3} \right)$$

$$+ \frac{32768L^2 \tau^2 G^2}{\mu^3} + \frac{128(2a + 1)(G^2 + \sigma^2)T(T + a)}{K \mu^3}$$

$$+ 48(1 - \alpha_i)^2 \frac{(a + 1)^2}{a^2 ST} \left( \frac{16a^3 \pi^2 E \left[ \|w^{(1)} - w^*\|^2 \right]}{6\mu} + \Theta(\ln T) + O(1) \right)$$

$$+ \frac{32768L^2 \tau^2 G^2}{\mu^3} + \frac{4096(2a + 1)(G^2 + \sigma^2)T}{K \mu^3}$$

$$+ 48(1 - \alpha_i)^2 \left( \frac{\lambda(f_i) + L^2 \Delta(f_i, F)}{\mu^3} \right) \frac{1}{\mu^3} \left( \frac{S_T}{\mu} + \frac{8T(T + 2a)}{\mu} \right)$$

$$+ O \left( \frac{\mu}{T^3} \right) + O \left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu T^3} \right) + O \left( \frac{\sigma^2 + G^2}{\mu T} \right)$$

$$+ O \left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu^3 T^2} \right) + O \left( \frac{\kappa^2 \tau^2 G^2 \ln T}{\mu^3 T^2} \right) + O \left( \frac{\lambda(f_i)}{\mu} + \kappa L \Delta(f_i, F) \right).$$

(78)
C Proof of Convergence without Assumption on $\alpha_i$

In this section, we present the proof of Theorem 4.

Proof. By the updating rule, we have:

$$\hat{v}_i^{(t+1)} = \hat{v}_i^{(t)} - \alpha_i^2 \eta_t \nabla f_i(\hat{v}_i^{(t)}; \xi_i^t) - (1-\alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t)$$

(79)

Now, subtracting $v_i^*$ on both sides, taking the square of norm and expectation, yields:

$$\mathbb{E} \left[ \|\hat{v}_i^{(t+1)} - v_i^*\|^2 \right] = \mathbb{E} \left[ \|\hat{v}_i^{(t)} - \alpha_i^2 \eta_t \nabla f_i(\hat{v}_i^{(t)}; \xi_i^t) - (1-\alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t) - (1-\alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t) - v_i^*\|^2 \right]

+ \mathbb{E} \left[ \alpha_i^2 \eta_t \left( \nabla f_i(\hat{v}_i^{(t)}) - \frac{n}{K} \nabla f_i(\hat{v}_i^{(t)}; \xi_i^t) \right) \right] + (1-\alpha_i) \eta_t \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t) \right)^2

= \mathbb{E} \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right] - 2 \eta_t \left( \alpha_i^2 + 1 - \alpha_i \right) \mathbb{E} \left[ \nabla f_i(\hat{v}_i^{(t)}), \hat{v}_i^{(t)} - v_i^* \right]_{t_1}

- \frac{2 \eta_t (1-\alpha_i) \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t) - \nabla f_i(\hat{v}_i^{(t)}), \hat{v}_i^{(t)} - v_i^* \right]}{t_2}

+ \eta_t^2 \mathbb{E} \left[ \|\alpha_i^2 \nabla f_i(\hat{v}_i^{(t)}) + (1-\alpha_i) \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}; \xi_j^t)\|^2 \right]

+ \frac{\alpha_i^2 \eta_t^2 2(n\sigma^2 + (n-K)G^2)}{K} + (1-\alpha_i)^2 \eta_t^2 \frac{2(G^2 + \sigma^2)}{K}.

(80)

First, we bound the term $T_1$:

$$T_1 = -2 \eta_t \left( \alpha_i^2 + 1 - \alpha_i \right) \mathbb{E} \left[ \nabla f_i(\hat{v}_i^{(t)}), \hat{v}_i^{(t)} - v_i^* \right]_{t_1}

- 2 \eta_t \left( \alpha_i^2 + 1 - \alpha_i \right) \mathbb{E} \left[ \nabla f_i(\hat{v}_i^{(t)}) - \nabla f_i(\hat{v}_i^{(t)}), \hat{v}_i^{(t)} - v_i^* \right]_{t_1}

\leq -2 \eta_t \left( \alpha_i^2 + 1 - \alpha_i \right) \mathbb{E} \left[ f_i(\hat{v}_i^{(t)}) - f_i(v_i^*) + \frac{L}{2} \mathbb{E} \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right] \right]

+ \eta_t \left( \frac{4L^2}{\mu} \mathbb{E} \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right] + \frac{L}{4} \mathbb{E} \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right] \right).$$

(81)
where we drop the term $\alpha_i^2 + 1 - \alpha_i$ since it is less or equal than 1. Also, since $\alpha_i^2 + 1 - \alpha_i \geq \frac{3}{4}$, we have:

$$T_1 \leq -2\eta (\alpha_i^2 + 1 - \alpha_i) \left( E \left[ f_i(\hat{v}_i^{(t)}) - f_i(v_i^t) \right] \right) - \frac{3\mu}{4} E \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right] + \eta \frac{4L^2(1 - \alpha_i)^2}{\mu} E \left[ \|w_i^{(t)} - w^{(t)}\|^2 \right].$$

(82)

Next, we bound the term $T_2$:

$$T_2 = -2\eta (1 - \alpha_i) \left( \frac{4 - \alpha_i}{\mu} \right) E \left[ \left( \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla f_i(\hat{v}_i^{(t)}) \right) \cdot \left( \hat{v}_i^{(t)} - v_i^* \right) \right]$$

$$\leq \eta (1 - \alpha_i) \left( \frac{4 - \alpha_i}{\mu} \right) E \left[ \left\| \nabla f_i(\hat{v}_i^{(t)}) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) \right\|^2 \right] + \frac{\mu}{4(1 - \alpha_i)} E \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right]$$

$$\leq \frac{8\eta (1 - \alpha_i)^2}{\mu} G^2 + \frac{\mu \eta}{4} E \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right].$$

(83)

And finally, we bound the term $T_3$:

$$T_3 = E \left[ \left\| \alpha_i^2 \nabla f_i(\hat{v}_i^{(t)}) + (1 - \alpha_i) \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) \right\|^2 \right]$$

$$\leq 2(\alpha_i^2 + 1 - \alpha_i) E \left[ \left\| \nabla f_i(\hat{v}_i^{(t)}) \right\|^2 \right] + 2E \left[ \left\| (1 - \alpha_i) \left( \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla f_i(\hat{v}_i^{(t)}) \right) \right\|^2 \right]$$

$$\leq 2(\alpha_i^2 + 1 - \alpha_i) (2E \left[ \left\| \nabla f_i(\hat{v}_i^{(t)}) - \nabla f_i^* \right\|^2 \right] + 2E \left[ \left\| \nabla f_i(\hat{v}_i^{(t)}) - \nabla f_i^* \right\|^2 \right]$$

$$+ 2(1 - \alpha_i)^2 E \left[ \left\| \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(w_j^{(t)}) - \nabla f_i(\hat{v}_i^{(t)}) \right\|^2 \right]$$

$$\leq 8(\alpha_i^2 + 1 - \alpha_i) L \left( E \left[ f_i(\hat{v}_i^{(t)}) \right] - f_i(v_i^t) \right) + 4L^2 E \left[ \|w_i^{(t)} - w^{(t)}\|^2 \right] + 4(1 - \alpha_i)^2 G^2.$$  

(84)

Using the Lemma 4 that $E \left[ \|w_i^{(t)} - w^{(t)}\|^2 \right] \leq 4\eta_i^2 \tau^2 G^2$ and plugging back $T_1$, $T_2$, and $T_3$ from (82), (83), and (84) in (80), we have:

$$E \left[ \|\hat{v}_i^{(t+1)} - v_i^*\|^2 \right] \leq \left( 1 - \frac{\mu \eta}{4} \right) E \left[ \|\hat{v}_i^{(t)} - v_i^*\|^2 \right] - 2(\alpha_i^2 + 1 - \alpha_i) (\eta \left( 4\eta_i^2 L^2 \right) \left( f_i(\hat{v}_i^{(t)}) \right) - f_i(v_i^t))$$

$$\leq -\frac{3\eta \eta_i^2}{4}$$

$$+ \frac{\alpha_i^2 \eta_i^2 (n\sigma^2 + (n - K)G^2)}{K} + \frac{(1 - \alpha_i)^2 \eta_i^2 (2G^2 + \sigma^2)}{K}$$

$$+ \frac{4(1 - \alpha_i)^2 \eta_i^2 L^2}{\mu} + 4\eta_i^2 L^2 \frac{8\eta (1 - \alpha_i)^2}{\mu} G^2 + 4(1 - \alpha_i)^2 \eta_i^2 G^2.$$

33
Note that \((1 - \frac{p_i}{n}) \leq \frac{p_i}{n} = (t + a)^2\), so we multiply \(\frac{p_i}{n}\) on both sides and re-arrange the terms to get

\[
\frac{3}{4} p_i \left( \mathbb{E} \left[ f_i(\hat{v}_i^{(t)}) \right] - f_i(v_i^*) \right) \\
\leq \frac{p_{t-1}}{\eta_{t-1}} \mathbb{E} \left[ \| \hat{v}_i^{(t)} - v_i^* \|^2 \right] - \frac{p_t}{\eta_t} \mathbb{E} \left[ \| \hat{v}_i^{(t+1)} - v_i^* \|^2 \right] + p_t \eta_t \left( \alpha_i^2 \frac{2(n\sigma^2 + (n - K)G^2)}{K} + (1 - \alpha_i)^2 \frac{2(G^2 + \sigma^2)}{K} \right) \\
+ \frac{16(1 - \alpha_i)^2 p_t \eta_t^2 L^2 \tau^2 G^2}{\mu} + 16L^2 p_t \eta_t^3 \tau^2 G^2 + \frac{8p_t(1 - \alpha_i)^2}{\mu} G^2 + 4(1 - \alpha_i)^2 p_t \eta_t G^2.
\]

By applying the telescoping sum and dividing both sides by \(S_T = \sum_{t=1}^T p_t \geq T^3\), we obtain:

\[
\mathbb{E} [f_i(\hat{v}_i)] - f_i(v_i^*) \\
\leq \frac{1}{S_T} \sum_{t=1}^T p_t \left( \mathbb{E} \left[ f_i(\hat{v}_i^{(t)}) \right] - f_i(v_i^*) \right) \\
\leq \frac{4p_0}{3\mu S_T} \mathbb{E} \left[ \| \hat{v}_i^{(1)} - v_i^* \|^2 \right] + \frac{4}{3} \frac{1}{S_T} \sum_{t=1}^T p_t \eta_t \left( \alpha_i^2 \frac{2(n\sigma^2 + (n - K)G^2)}{K} + (1 - \alpha_i)^2 \frac{2(G^2 + \sigma^2)}{K} \right) \\
+ \frac{4}{3} \frac{1}{S_T} \sum_{t=1}^T \frac{16(1 - \alpha_i)^2 p_t \eta_t^2 L^2 \tau^2 G^2}{\mu} + \frac{64L^2 \tau^2 G^2}{3 \mu} \frac{1}{S_T} \sum_{t=1}^T p_t \eta_t^3 \\
+ \frac{32(1 - \alpha_i)^2 G^2}{\mu} \frac{1}{S_T} \sum_{t=1}^T p_t + \frac{16}{3} (1 - \alpha_i)^2 G^2 \frac{1}{S_T} \sum_{t=1}^T p_t \eta_t \\
\leq \frac{4p_0}{3\mu S_T} \mathbb{E} \left[ \| \hat{v}_i^{(1)} - v_i^* \|^2 \right] + \frac{16T(T + 2a)}{3\mu S_T} \left( \alpha_i^2 \frac{2(n\sigma^2 + (n - K)G^2)}{K} + (1 - \alpha_i)^2 \frac{2(G^2 + \sigma^2)}{K} \right) + \frac{L^2 \tau^2 G^2 \Theta(\ln T)}{\mu^3 S_T} \\
+ \frac{4096(1 - \alpha_i)^2 L^2 \tau^2 G^2 T}{3\mu^3 S_T} + \frac{32(1 - \alpha_i)^2 G^2}{\mu} + \frac{8(1 - \alpha_i)^2 G^2 T(T + 2a)}{3\mu S_T} \\
= O \left( \frac{\mu}{T^3} \right) + O \left( \frac{\kappa^2 \tau^2 \ln T}{\mu T^3} \right) + \alpha_i^2 O \left( \frac{G^2 + \sigma^2}{\mu T} \right) + (1 - \alpha_i)^2 \left( O \left( \frac{G^2 + \sigma^2}{\mu K T} \right) + O \left( \frac{\kappa^2 \tau^2}{\mu T^2} \right) + O(G^2) \right).
\]

\(\square\)