Entanglement renormalization, scale invariance, and quantum criticality

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The use of entanglement renormalization in the presence of scale invariance is investigated. We explain how to compute an accurate approximation of the critical ground state of a lattice model, and how to evaluate local observables, correlators and critical exponents. Our results unveil a precise connection between the multi-scale entanglement renormalization ansatz (MERA) and conformal field theory (CFT). Given a critical Hamiltonian on the lattice, this connection can be exploited to extract most of the conformal data of the CFT that describes the model in the continuum limit.

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The study of quantum critical phenomena through real-space renormalization group (RG) techniques\textsuperscript{[1, 2]} has traditionally been obstructed by the accumulation, over successive RG transformations, of short-range entanglement across block boundaries. Entanglement renormalization\textsuperscript{[3]} was recently proposed as a technique to address this problem. By removing short-range entanglement at each iteration of the RG transformation, not only can arbitrarily large lattice systems be considered, but the scale invariance characteristic of critical phenomena is also seen to be restored\textsuperscript{[3, 4]}.

In this paper we explain how to use the multi-scale entanglement renormalization ansatz (MERA)\textsuperscript{[5]} to investigate scale invariant systems\textsuperscript{[3, 4, 5, 6, 7]}. It has been shown that the scale invariant MERA can represent the infra-red limit of topologically ordered phases\textsuperscript{[6]}. Here we focus instead on its use at quantum criticality. We start by considering a finite 1D lattice $L \rightarrow L_0 \rightarrow \cdots \rightarrow L_1 \rightarrow \cdots \rightarrow L_T$, which corresponds to the limit of infinitely many layers, $T \rightarrow \infty$, and to choosing the disentanglers and isometries in all layers to be copies of a unique pair $u$ and $w$\textsuperscript{[8, 9]}. In this case we refer to the ascending superoperator $A_{\tau}$, which no longer depends on $\tau$, as the scaling superoperator $S_{\alpha}$ (see Fig. 1), and to its dual $D_{\tau}$ as $S^\ast$. Notice that $S$ is a fixed-point RG map. Then, as customary in RG analysis\textsuperscript{[3, 10]}, the scaling operators $\phi_{\alpha}$ and scaling dimensions $\Delta_{\alpha}$ of the theory,

\begin{equation}
S(\phi_{\alpha}) = \lambda_{\alpha} \phi_{\alpha}, \quad \Delta_{\alpha} = -\log_3 \lambda_{\alpha},
\end{equation}

are obtained by diagonalizing this map,

\begin{equation}
S(\bullet) = \sum_{\alpha} \lambda_{\alpha} \phi_{\alpha} (\hat{\phi}_{\alpha} \bullet), \quad \text{tr}(\hat{\phi}_{\alpha} \phi_{\beta}) = \delta_{\alpha\beta},
\end{equation}

where $\phi_{\alpha}$ are the eigenvectors of the dual $S^\ast$, $S^\ast(\phi_{\alpha}) = \lambda_{\alpha} \hat{\phi}_{\alpha}$. Eq. 8 was first discussed in Ref.\textsuperscript{[8]} by Giovannetti, Montangero and Fazio\textsuperscript{[11]}. It formalizes a previous observation (see Eq. 5 of Ref.\textsuperscript{[3]}) that the scale invariant MERA displays polynomial correlations. By
Lattices and isometries α with ∆, pairs of sites of the MERA, two-site operators supported on three different sites of the average of three contributions, each of which (and thus also their average) is unital and contractive thanks to the isometric character of u and w.[3].

construction, S is unital, S(1) = 1, so that the identity operator 1 in V⊗2 is a scaling operator with eigenvalue λ1 = 1; and contractive, meaning |λa| ≤ 1 [12]. Here we will assume, as it is the case in the examples below, that only the identity operator 1 has eigenvalue λ = 1. Then the operator ̂ρ ≡ 1 is a density matrix that corresponds to the unique fixed point of S*, S*( ̂ρ) = ̂ρ, and since

$$\lim_{T \to \infty} \left( S^* \circ \cdots \circ S^* \right)(\rho_T) = ̂ρ$$

(7)

for any starting ρT, it follows that ̂ρ is the state of any pair of contiguous sites of L. [Consistent with scale invariance, ̂ρ is also the state of any pair of contiguous sites of Lr for any finite r]. The computation of the expected value of the local observable o is then straightforward,

$$\langle o \rangle = \text{tr}( ̂ρ o),$$

(8)

which for the scaling operators reduces to $\langle φ_a \rangle = δ_o a$.

Correlators.— Let us now diagonalize the one-site scaling superoperator S(1) of Fig. 2

$$S^{(1)}(\bullet) = \sum_a λ_{α}^{(1)} φ_α^{(1)} \text{tr}( ̂φ_α^{(1)} \bullet),$$

(9)

where the scaling dimensions $\Delta_α^{(1)} \equiv - \log_3 λ_α^{(1)}$ coincide with $\Delta_α$. [13]. The correlator for two scaling operators

$$\phi_α^{(1)} and φ_β^{(1)} placed on contiguous sites reads

$$C_{αβ} ≡ \langle φ_α^{(1)}(1) φ_β^{(1)}(0) \rangle = \text{tr}( ̂φ_α^{(1)} ⊗ ̂φ_β^{(1)} ̂ρ).$$

(10)

Suppose now that φα(1) and φβ(1) are placed in two special sites x, y as in Fig. 2, where rxy ≡ x - y is such that $|r_{xy}| = 3^q$ for q = 1, 2, · · ·. Then after q = log3 |r_{xy}| iterations of the RG transformation, $φ_α^{(1)}$ and $φ_β^{(1)}$ become first neighbors again. Notice that each iteration contributes a factor $λ_α^{(1)} λ_β^{(1)}$. Using the identity $q_{ab} = 2 q a$ we find

$$\langle λ_α^{(1)} λ_β^{(1)} \rangle^{log_q |r_{xy}|} = |r_{xy}|^{log_3(λ_α^{(1)} λ_β^{(1)})} = |r_{xy}|^{−Δ_α^{(1)}−Δ_β^{(1)}}$$

and obtain a closed expression for two-point correlators,

$$\langle φ_α^{(1)}(x) φ_β^{(1)}(y) \rangle = \frac{C_{αβ}}{|r_{xy}|^{Δ_α^{(1)}+Δ_β^{(1)}}}.$$

(11)

For three-point correlators we define the constants

$$Ω_{αγ} ≡ Δ_α^{(1)} + Δ_γ^{(1)} − Δ_α^{(1)}$$

(12)

$$C_{αβγ} ≡ 2Ω_{αβ} \text{tr}( ̂ϕ_α^{(1)} ⊗ ̂ϕ_β^{(1)} ⊗ ̂ϕ_γ^{(1)} ̂ρ)$$

(13)

where the trace corresponds to the correlator on three consecutive sites and $ρ^{(3)}$ is obtained from ̂ρ. For $|r_{xy}| = |r_{xz}| = |r_{yz}|/2 = 3^q$, analogous manipulations lead to

$$\langle φ_α^{(1)}(x) φ_β^{(1)}(y) φ_γ^{(1)}(z) \rangle = \frac{C_{αβγ}}{|r_{xy}|^{Ω_{αγ}} |r_{yz}|^{Ω_{βγ}} |r_{xz}|^{Ω_{αβ}}},$$

(14)

CFT.— The continuous limit of a quantum critical lattice system (scale invariant case) corresponds to a conformal field theory (CFT)[3,10]. A CFT contains an infinite set of quasi-primary fields $φ_{α}^{CFT}$, with scaling dimensions $Δ_{α}^{CFT}$, The correlators involving two or three quasi-primary fields have expressions analogous to Eqs. (11) and (14) and the (symmetric) coefficients $C_{αβγ}^{CFT}$ for three-point
correlators coincide with those in the so-called operator product expansion (OPE). Moreover, quasi-primary fields are organized in conformal towers corresponding to irreducible representations of the Virasoro algebra. Each tower contains one primary field \( \phi^p \) at the top, with conformal dimensions \((t, \bar{t})\) such that its scaling dimension is \( \Delta^p = t + \bar{t} \), and its infinitely many descendants, which are quasi-primary fields with scaling dimension \( \Delta = \Delta^p + n \) for some integer \( n \geq 1 \).

A CFT is completely specified by its symmetries once the following conformal data has been provided: (i) the central charge \( c \), (ii) a complete list of primary fields with their conformal dimensions \((S, 0, (\frac{1}{10}, \frac{1}{10})\) and \((\frac{2}{3}, \frac{2}{3})\), and (iii) the OPE coefficients

\[
C_{\alpha\beta}^{\text{CFT}} = \delta_{\alpha\beta}, \quad C_{\alpha\sigma}^{\text{CFT}} = \frac{1}{2}, \quad C_{\sigma\sigma}^{\text{CFT}} = C_{\epsilon\epsilon}^{\text{CFT}} = C_{\epsilon\sigma}^{\text{CFT}} = 0. \quad (15)
\]

The present analysis readily suggests a correspondence between the scaling operators \( \phi^p \) of the scale invariant MERA, defined on a lattice, and the quasi-primary fields \( \phi^{CFT}_\alpha \) of a CFT, defined in the continuum. Together with the algorithm described below, this correspondence grants us numerical access, given a critical Hamiltonian \( H \) on the lattice, to most of the conformal data of the underlying CFT, namely to scaling dimensions and OPE coefficients. The central charge \( c \) can also be obtained e.g. \([14]\) from the von Neumann entropy \( S(\rho) = -\text{tr}(\rho \log_2 \rho) \), which for a block of \( L \) sites scales, up to some additive constant, as \( S = \frac{3}{2} \log_2 L \) \([13]\). We then have \( S(\hat{\rho}) - S(\hat{\rho}^{(1)}) = \frac{\xi}{3}(\log_2 2 - \log_2 1) = \frac{\xi}{3} \), or simply

\[
c = 3 \left( S(\hat{\rho}) - S(\hat{\rho}^{(1)}) \right). \quad (16)
\]

**Algorithm.**— Given a critical Hamiltonian \( H \) for an infinite lattice, we obtain a scale invariant MERA for its ground state \( |\Psi\rangle \) by adapting the general strategy discussed in Ref. \([5]\). Recall that tensors (disentanglers \( u \) and isometries \( w \)) are optimized so as to minimize the energy \( E = \langle \Psi | H | \Psi \rangle \). After linearization this reads

\[
E = \text{tr}(u \Upsilon_u) + k_1 = \text{tr}(w \Upsilon_w) + k_2,
\]

where \( \Upsilon_u \) and \( \Upsilon_w \) are known as environments and \( k_1, k_2 \) are two irrelevant constants. In the translation invariant case \([3]\) the environment for, say, an isometry \( w \) at layer \( \tau \) of the MERA, \( \Upsilon_w = f(u_\tau, w_\tau, \rho_\tau, h_{\tau-1}) \), is a function of the disentangler \( u_\tau \) and isometry \( w_\tau \) of that layer, a two-site density matrix \( \rho_\tau \) and a two-site Hamiltonian term \( h_{\tau-1} \). In the present case, we replace the above with the unique pair \( (u, w) \), the fixed-point density matrix \( \hat{\rho} \), and an average Hamiltonian \( \bar{h} = \sum_\tau h_\tau / 3^\tau \), where the weights \( 1/3^\tau \) account for the relative number of tensors in different layers of the MERA. Then, starting from some initial pair \( (u, w) \) and the critical Hamiltonian \( H \) made of two-body terms \( \hat{h} \), the following steps are repeated until convergence:

A1. Given the latest \( (u, w) \), compute \( (\hat{\rho}, \bar{h}) \).

A2. Given \( (u, w, \hat{\rho}, \bar{h}) \), update the pair \( (u, w) \).

In step A1, the scaling superoperator \( S \) is built as indicated in Fig. \([1]\). We compute the fixed-point density matrix \( \hat{\rho} \) by sparse diagonalization of \( S \), and the average Hamiltonian \( \bar{h} \) by using \( h_\tau = S(h_{\tau-1}) \). Step A2 is decomposed into a sequence of alternating optimizations for \( u \) and \( w \) as in the generic algorithm of Ref. \([3]\), where each tensor is updated by computing a singular value decomposition of its environment.

**Examples.**— We illustrate the above ideas and the performance of the algorithm by considering the Ising and 3-level Potts quantum critical models in 1D,

\[
H_{\text{Ising}} = \sum_r \left( \lambda \sigma_z^r + \sigma_x^r \sigma_x^{r+1} \right),
\]

\[
H_{\text{Potts}} = \sum_r \left( \lambda \sigma_z^r + M_{x,1}^r M_{x,2}^r + M_{x,3}^r M_{x,1}^{r+1} \right)
\]

where \( \sigma_z \) and \( \sigma_x \) are Pauli matrices, and

\[
M_x = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_{x,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

\( M_{x,2} = (M_{x,1})^2 \). Notice that sites have a vector space of dimension \( d = 2 \) or \( d = 3 \). In order to use a scale invariant MERA with \( \chi > d \), we allow the disentanglers and isometries of the first few (typically one to five) layers to be different from \( u \) and \( w \). We iterate steps A1-A2 about 1000 times. With a cost per iteration that scales as \( \chi^8 \) and using a 3 GHz dual core desktop with 8 Gb of RAM,
TABLE I: Comparison of scaling dimensions of primary fields of the Ising and Potts models calculated using MERA ($\Delta$ (MERA $\chi = 22$)) with exact results known from CFT ($\Delta^{\text{CFT}}$).

|         | $\Delta^{\text{CFT}}$ | $\Delta$ (MERA $\chi = 22$) | rel. error |
|---------|------------------------|-----------------------------|------------|
| Ising   | $\sigma$ 1/8 = 0.125   | 0.124997                    | 0.002%     |
|         | $\epsilon$ 1           | 1.0001                      | 0.01%      |
| Potts   | $\sigma_1$ 2/15 = 0.13  | 0.1339                      | 0.4%       |
|         | $\sigma_2$ 2/15 = 0.13  | 0.1339                      | 0.4%       |
|         | $\epsilon$ 4/5 = 0.8   | 0.8204                      | 2.5%       |
|         | $Z_1$ 4/3 = 1.3         | 1.3346                      | 0.1%       |
|         | $Z_2$ 4/3 = 1.3         | 1.3351                      | 0.1%       |

simulations for $\chi = 4, 8, 16, 22$ take of the order of minutes, hours, days and weeks respectively. The following results correspond to $\chi = 22$.

From Eq. 16 we obtain an estimate for the central charge, namely $c_{\text{Ising}} = 0.5007$ and $c_{\text{Potts}} = 0.806$, to be compared with the exact results 0.5 and 0.8. Fig. 3 shows the smallest scaling dimensions $\Delta_{\alpha}$ of the scaling superoperator $S$. We obtain remarkable agreement with those expected from CFT, as shown in Table I. Recall that all the critical exponents of the model can be obtained from the scaling dimensions of primary fields. For instance, for the Ising model the exponents $\nu$ and $\eta$ are $\nu = 2\Delta_{\sigma}$ and $\eta = \frac{1}{2\Delta_{\sigma}}$, whereas the scaling laws express the critical exponents $\alpha, \beta, \gamma, \delta$ in terms of $\nu$ and $\eta$. Further, the OPE coefficients for primary fields of, say, the critical Ising model are computed as follows. The matrix $C_{\alpha \beta}$ in Eq. 10 is diagonal for the scaling operators corresponding to $I$, $\sigma$ and $\epsilon$, which we normalize so that $C_{\alpha \beta} = \delta_{\alpha \beta}$. With this normalization, we then compute the coefficients $C_{\alpha \beta \gamma}$ using Eq. 10. We reproduce all the values of Eq. 15 with errors bounded by $3 \times 10^{-4}$.

Discussion.— In this paper we have explained how to compute the ground state of a critical Hamiltonian using the scale invariant MERA and how to extract from it the properties that characterize the system at a quantum critical point. Our results, which build upon those of Refs. 3, 6, 8, also unveil a concise connection between the scale invariant MERA and CFT. This correspondence adds significantly to the conceptual foundations of entanglement renormalization. The scale invariant MERA can be regarded as approximately realizing an infinite dimensional representation of the Virasoro algebra. The finite value of $\chi$ effectively implies that only a finite number of the quasi-primary fields of the theory can be included in the description. Fields with small scaling dimension, such as primary fields, are retained foremost. As a result, given a Hamiltonian on an infinite lattice, we can numerically evaluate the scaling dimensions and OPE of the primary fields of the CFT that describes the continuum limit of the model. This approach differs in a fundamental way from, and offer an alternative to, the long-established techniques ofRefs. 17, based instead on finite size scaling.

We conclude by noting that most of our considerations rely on scale invariance alone and can be applied to study also critical ground states in 2D systems 18.

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[13] Our numerics show that the lowest $n_\Delta$ scaling dimensions fulfill $\Delta_n^{(\kappa)} \approx \Delta_\kappa \approx \Delta_{\text{CFT}}$, where $n_\Delta$ grows with $\chi$.
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