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Interplay of superconductivity and spin-density-wave order in doped graphene

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We study the interplay between superconductivity and spin-density-wave order in graphene doped to 3/8 or 5/8 filling (a van Hove doping). At this doping level, the system is known to exhibit weak-coupling instabilities to both chiral $d+i\sigma$ superconductivity and to a uniaxial spin density wave. Right at van Hove doping, the superconducting instability is strongest and emerges at the highest $T_c$, but slightly away from van Hove doping, a spin density wave likely emerges first. We investigate whether at some lower temperature superconductivity and spin density waves coexist. We derive the Landau-Ginzburg functional describing interplay of the two order parameters. Our calculations show that superconductivity and spin-density-wave order do not coexist and are separated by first-order transitions, either as a function of doping or as a function of $T$.

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I. INTRODUCTION

Two-dimensional electron systems provide an ideal environment for exploration of many-body physics. Graphene, as a new two-dimensional electron system, may allow us to access new many-body phases that have not been hitherto observed. Unfortunately, undoped single-layer graphene seems to be well described by a noninteracting model, with the vanishing density of states. One way to do this is by taking a plus sign. The van Hove doping then corresponds to both chiral superconductivity and to a uniaxial spin density wave. Right at van Hove doping, the superconducting instability is strongest and emerges first. We investigate whether at some lower temperature superconductivity and spin density waves coexist. We derive the Landau-Ginzburg functional describing interplay of the two order parameters. Our calculations show that superconductivity and spin-density-wave order do not coexist and are separated by first-order transitions, either as a function of doping or as a function of $T$.

II. THE MODEL

Our point of departure is the tight-binding model with the nearest-neighbor dispersion

$$\varepsilon_k = \pm t \sqrt{1 + 4 \cos \left(\frac{3}{2} k_y \sqrt{3} \right) \cos \left(\frac{3}{2} k_x \sqrt{3} \right) + 4 \cos^2 \left(\frac{k_y \sqrt{3}}{2} \right) - \mu},$$ (1)

where the overall sign is + or − depending on whether we are above or below half-filling. For definiteness, we take a plus sign. The van Hove doping then corresponds to $\mu = t$, at which point the Fermi surface has the form shown in Fig. 1. The Fermi velocity vanishes near the hexagon corners $M_1 = (2\pi/3,0), M_2 = (\pi/3,\pi/\sqrt{3}), M_3 = (-\pi/3,\pi/\sqrt{3})$, which are saddle points of the dispersion:

$$\varepsilon_{k \approx M_1} = \frac{3t}{4} (3k_y^2 - k_x^2), \quad \varepsilon_{k \approx M_2} = \frac{3t}{4} 2k_y(k_y - \sqrt{3}k_x), \quad \varepsilon_{k \approx M_3} = \frac{3t}{4} 2k_y(k_y + \sqrt{3}k_x).$$ (2)

Each time, $k$ is a deviation from a saddle point. Saddle points give rise to a logarithmic singularity in the density of states (DOS) and control physics at weak coupling. There are three inequivalent nesting vectors $Q_{ab}$ connecting inequivalent pairs of saddle points $M_a$ and $M_b$ (see Fig. 1):

$$Q_1 = Q_{23} = (\pi,\pi/\sqrt{3}), \quad Q_2 = Q_{31} = (\pi, -\pi/\sqrt{3}), \quad Q_3 = Q_{12} = (0,2\pi/\sqrt{3}).$$ (3)
Each \( Q_i \) is physically the same as \(-Q_i \) because \( Q_i \) is half of a reciprocal lattice vector.

There are four different interactions between fermions near saddle points \( g_i, i = 1 - 4 \), with momentum transfer near zero and near \( Q_i \).\(^{4,5,12-14} \) For our purposes, relevant interactions are density-density interaction within one patch \( (g_1) \) and between patches \( (g_2) \) and the interaction which describes hopping of a pair of fermions from one patch to the other \( (g_3) \). The fourth interaction \( g_4 \) is the exchange interaction between patches. Interactions \( g_2 \) and \( g_3 \) renormalize particle-hole vertices and control the SDW instability, while interactions \( g_3 \) and \( g_4 \) renormalize particle-particle vertices and control the superconducting instability (note that \( g_3 \) contributes to both instabilities).

The partition function \( Z \) of the model can be written as a functional integral over Grassmann valued (fermionic) fields \( \psi \). We have \( Z = \int \mathcal{D}[\psi, \bar{\psi}] \exp[-S(\bar{\psi}, \psi)] \), where

\[
S = \int_{0}^{1/T} \mathcal{L}(k, \tau) \ (T \text{ is the temperature}) \text{ and}
\]

\[
\mathcal{L} = \sum_{i, a} \left[ \bar{\psi}_{a, \alpha}(\partial_{\tau} + \epsilon_{k} - \mu - g_4 \bar{\psi}_{a, \beta} \psi_{a, \alpha}) \psi_{a, \alpha} 
- \sum_{b, \beta} g_1 \bar{\psi}_{a, \alpha} \bar{\psi}_{a, \beta} \psi_{a, \alpha} \psi_{b, \beta} + g_2 \bar{\psi}_{a, \alpha} \bar{\psi}_{b, \beta} \psi_{b, \beta} \psi_{a, \alpha} 
+ g_3 \bar{\psi}_{a, \alpha} \bar{\psi}_{a, \beta} \psi_{b, \beta} \psi_{b, \alpha} \right].
\]

(4)

Here, \( a, b = 1, 2, 3 \) label which saddle point we are closest to, \( \alpha \) and \( \beta \) are spin labels, and \( \bar{\alpha} \) is the opposite spin state to \( \alpha \). We have retained only those states that are close to the saddle points; this “patch model” is exact in the limit of weak coupling.\(^5\)

### III. THE SUPERCONDUCTING AND SDW ORDERS

This action displays instabilities towards \( d \)-wave superconductivity and SDW. We therefore decouple the interactions in the \( d \)-wave superconducting and SDW channels simultaneously by means of two Hubbard-Stratanovich transformations. We introduce the Hubbard-Stratanovich superconducting fields \( \Delta_{\nu} = (g_3 - g_4)(\psi_{a, \nu} \psi_{a, \nu}) \). Since the superconductivity is known to be \( d + i d^\pm \), we set \( (\Delta_1, \Delta_2, \Delta_3) = \Delta(1, e^{i2\pi/3}, e^{-i2\pi/3}) \) and describe superconducting fields by a single complex order parameter \( \Delta \). We also introduce the three SDW order parameters \( M_{ab} = (g_2 + g_3)(\bar{\psi}_{a, \nu} \psi_{b, \nu}) \). Since the SDW order is known to be uniaxial,\(^{11} \) we can replace the three vector order parameters \( M_{12}, M_{23}, M_{31} \) by a single scalar SDW order parameter \( M \), which represents the magnetic order along the SDW axis.

Since the system has \( O(3) \) spin rotation symmetry, the SDW axis can be chosen to coincide with the \( z \) axis without loss of generality. Finally, we introduce the Nambu spinor \( \chi_a \), a four-component spinor defined according to \( \chi_a = (\psi_{a, \uparrow}, \psi_{a, \downarrow}, \psi_{a, \downarrow}, -\psi_{a, \uparrow}) \). The action after Hubbard-Stratanovich transformation can be written in the Nambu spinor basis as

\[
\mathcal{L} = \frac{M^2}{g_2 + g_3 - g_4} + \frac{|\Delta|^2}{g_3 - g_4} + \sum_{ab} \bar{\chi}_a \mathcal{G}^{-1}_{ab} \chi_b,
\]

(5)

\[
\mathcal{G}_{ab}^{-1} = [\partial_{\tau} + (\epsilon_{k} - \mu) \sigma_3 + \Delta \sigma_+ + \Delta^* \sigma_-] \otimes 1_2,
\]

\[
\mathcal{G}_{a \neq b}^{-1} = M(1_2 \otimes \eta_3).
\]

Here, the \( \sigma_i \) are Pauli matrices acting in the particle-hole space, the \( \eta_i \) are Pauli matrices acting in the spin space, \( 1_2 \) is a two-dimensional identity matrix, and \( \sigma_\pm = \sigma_1 \pm \sigma_2 \). The notation we have used is borrowed from Ref. 21. We emphasize that this expression for the Ginzburg-Landau functional in terms of superconducting (SC) and SDW order parameters is unique because the superconducting and SDW channels are orthogonal.

We can now integrate out the fermions exactly to obtain an action purely in terms of the superconducting and SDW order-parameter fields

\[
\mathcal{L} = \frac{M^2}{g_2 + g_3 - g_4} - \text{Tr} \ln \mathcal{G}^{-1}(\Delta, M),
\]

(7)

where the trace goes over Nambu spinor indices, and also over imaginary time and over momentum. We now define \( G \) to be the “bare” (matrix) Green’s function evaluated at \( \Delta = 0, M = 0 \), and define matrix order parameters \( \Delta \) and \( M \), such that \( \mathcal{G}^{-1} = G^{-1} + \Delta + M \). We can then write

\[
\text{Tr} \ln \mathcal{G}^{-1} = \text{Tr} \ln(G^{-1}[1 + G(\Delta + M)]) = \text{const} + \text{Tr} \ln[1 + G(\Delta + M)].
\]

(8)

It is convenient to explicitly write out the expressions for \( G, \Delta, \) and \( M \). We adopt the shorthand \( F^\pm(\omega_n, k) = 1/[i \omega_n \pm (\epsilon_k - \mu)] \), where \( \omega_n = (2n + 1)\pi T \) are fermionic Matsubara frequencies. Using the shorthand,
we can define the various matrices as

\[
G = \begin{pmatrix}
F^+(\omega_n, k) & 0 & 0 & 0 & 0 \\
0 & F^-(\omega_n, k) & 0 & 0 & 0 \\
0 & 0 & F^+(\omega_n, k + Q_1) & 0 & 0 \\
0 & 0 & 0 & F^-(\omega_n, k + Q_1) & 0 \\
0 & 0 & 0 & 0 & F^+(\omega_n, k + Q_2) \\
0 & 0 & 0 & 0 & F^-(\omega_n, k + Q_2)
\end{pmatrix} \otimes I_2.
\]

\[
\Delta = \begin{pmatrix}
0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Delta e^{-2\pi/3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta^* e^{2\pi/3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta e^{2\pi/3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta^* e^{-2\pi/3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta e^{-2\pi/3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta^* e^{2\pi/3}
\end{pmatrix} \otimes I_2.
\]

\[
M = M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \otimes \sigma_3.
\]

**IV. LANDAU-GINZBURG ANALYSIS**

Thus far, everything we have done has been exact. We now work close to $T_c$ and perform a double expansion of (8) in small $|\Delta|$ and small $M$. We terminate the expansion at quartic order in both fields and drop all terms that are odd in powers of $M$ or $\Delta$ as they vanish upon taking the trace. We then obtain for the $\text{Tr}$ in $\ln$ the expression

\[
\text{Tr} \left[ -\frac{1}{2} (G \Delta A + GMG) - \frac{1}{2} (G \Delta A \Delta A + GMGMGM + 4GMGMG + 2GMGMGMA + 2GMGMGGMG) \right].
\]

We have made use of the fact that the trace of a product of matrices is invariant under a cyclic permutation of the matrices. Evaluating the traces and substituting back into (7) leads to the expression

\[
\mathcal{L} = \alpha_1 (T - T_c)|\Delta|^2 + \alpha_2 (T - T_N) M^2 + K_1 |\Delta|^4 + K_2 M^4 + 2K_3 |\Delta|^2 M^2,
\]

where we have defined the expansion coefficients

\[
K_1 = 3T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \left[ F^+(\omega_n, k) F^-(\omega_n, k) \right]^2,
\]

\[
K_2 = 3T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\omega_n, k)^2 F^+(\omega_n, k + Q_1)^2 + 2 F^+(\omega_n, k)^2 F^+(\omega_n, k + Q_1) F^+(\omega_n, k + Q_2) + (F^+ \rightarrow F^-),
\]

\[
K_3 = 6T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \left[ F^+(\omega_n, k) F^-(\omega_n, k) F^+(\omega_n, k + Q_1) \right] \times F^-((\omega_n, k + Q_1) \cos \frac{2\pi}{3} + F^+(\omega_n, k)^2 F^-(\omega_n, k)) \times F^+(\omega_n, k + Q_1) + (F^+ \leftrightarrow F^-).
\]

Terminating the expansion at quartic order in both order parameters is justified if the quadratic terms for superconducting and SDW order change sign at about the same critical temperature $T_c \approx T_N$.

Renormalization-group analysis shows that the couplings which determine both superconducting and SDW instabilities diverge at the onset of the first instability upon lowering $T_c$.

The critical temperatures, however, are determined by superconducting and SDW susceptibilities, which generally have different exponents (different anomalous dimensions). It has been demonstrated on general grounds that different orders emerge simultaneously when their anomalous dimensions $\eta_i > 1$. In our case, the anomalous exponents for superconducting and SDW susceptibilities have been calculated in Ref. 4. Using results from that work, we find that $\eta_{SC} = 1.48 > 1$, while $\eta_{SDW} = 0.97$ (the results are for perfect nesting). Because $\eta_{SDW} < 1$, $T_N < T_c$. However, since $\eta_{SDW}$ is very close to one, we expect that $T_N$ in (10) is only slightly lower than $T_c$, in which case the expansion up to quartic order in $\Delta$ and $M$ is justified. For dopings slightly away from the van Hove one, we expect SDW to be the leading instability. In this case, $T_N \geq T_c$; again, we assume that the difference between the two critical temperatures is small.

It was shown in the context of pnictides that a free energy of the form (10) leads to coexistence of the two order parameters if $K_1 K_2 - K_3^2 > 0$. We computed the coefficients $K_1$, $K_2$, and $K_3$ in our case by explicitly integrating over fermionic momenta and summing over fermionic frequencies in (12) (for details, see the Appendix). We obtain, with logarithmic accuracy

\[
K_1 = \frac{1.05}{\sqrt{3\pi + T_c^2}} \ln \frac{t - T_c}{T_c} + \text{subleading},
\]

\[
K_2 = \frac{1.05}{\sqrt{3\pi + T_c^2}} \ln \frac{t}{T_c} + \text{subleading} = K_1,
\]

\[
K_3 = \frac{1.05 \left[ \cos \left( \frac{2\pi}{3} \right) + 2 \right]}{\sqrt{3\pi + T_c^2}} \ln \frac{t}{T_c} + \text{subleading}
\]

Note that there are two processes which contribute to the coefficient $K_3$. The processes are represented diagrammatically.
in Fig. 2. The process shown in Fig. 2(a) is sensitive to the chirality of the superconducting order parameter because of the dependence on the phase difference between different saddle points, and gives rise to the \( \cos(\frac{\pi}{3}) \) term. Because \( \cos(2\pi/3) = -1/2 \), this process gives rise to an effective attraction between superconductivity and spin density waves. This effective attraction is, however, outweighed by a larger (chirality-independent) repulsion between the two order parameters, coming from the processes shown in Fig. 2(b). The prefactors in our case are such that \( K_3 = \frac{3}{2} K_1 > 0 \). Comparing \( K_1, K_2 \) and \( K_2 \), we see that in the case of doped graphene \( K_1, K_2 - K_3^2 < 0 \), so that coexistence is disfavored. The system only allows one order parameter to exist, even when \( M \neq 0 \) and \( M \neq 0 \) is expected to be first order (although we can not exclude a non-Landau continuous transition between the two ordered states).

The fact that \( T_c \neq T_N \) makes coexistence even less likely. We therefore conclude that there is no coexistence of superconducting and SDW order in doped graphene.

In pnictides, the structure of \( K_1, K_2, K_3 \) is quite similar (modulo that there is no \( \ln \frac{t}{T_c} \) term), but the argument of \( \cos \) in \( K_3 \) is the phase difference between the gaps on hole and electron FSs. For \( s^+ \) superconductivity, the argument is \( \pi \), in which case \( K_3 = K_1 = K_2 \). Then, \( K_1 K_2 = K_3^2 \), and one has to include subleading terms to verify whether the two orders can coexist. The subleading terms are those which break the nesting between hole and electron pockets, and the analysis shows that \( s^+ \) superconducting and SDW orders coexist in some range of parameters. In graphene, the argument of \( \cos \) is \( 2\pi/3 \), and such coexistence does not occur.

**V. CONCLUSIONS**

To conclude, we have demonstrated that superconductivity and spin-density-wave order are mutually exclusive in graphene doped near the \( M \) point of the Brillouin zone (a van Hove doping). Sufficiently close to the van Hove point, we expect to see pure chiral superconductivity, and somewhat away from the van Hove point, we expect to see pure spin-density-wave order. The results stand in stark contrast to pnictides, where coexistence can be coexistence between spin density waves and superconductivity.

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**APPENDIX**

1. Evaluating \( K_1 \)

We start with the expression

\[
K_1 = 3T_c \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} [F^+(\omega_n, k) F^-(\omega_n, k)]^2
\]

where the sum goes over all Matsubara frequencies, and the integral goes over all wave vectors \( k \) up to a UV cutoff of \( |k| \sim O(1) \), at which point the dispersion relation changes. The UV cutoff must be retained in the integrals because the integrals are log divergent in the UV if we ignore the cutoff (the integrals are convergent in the infrared at nonzero temperature). We now scale out \( \omega_n \), and define the rescaled coordinates \( x = \sqrt{T_c/(4\omega_n)} k_x \), \( y = \sqrt{T_c/(4\omega_n)} k_y \). The expression for \( K_1 \) can then be recast as

\[
K_1 = 3T_c \sum_{\omega_n} \frac{1}{3\sqrt{2}\pi^2 |\omega_n|^3} \int_{-\sqrt{\pi T_c}}^{\sqrt{\pi T_c}} \frac{dx dy}{[1 + (3x^2 - y^2)^2]^{3/2}}.
\]

We now integrate first over \( -\infty < y < \infty \), and then over \( -\sqrt{\pi T_c} < x < \sqrt{\pi T_c} \), and expand the resulting expression to leading order in large \( t/T_c \) (the manipulations are all done on MATHEMATICA). We obtain the result

\[
K_1 = T_c \sum_{\omega_n} \frac{1}{\pi^2 |\omega_n|^3} \left( \frac{\pi}{2\sqrt{3}} \ln \frac{t}{T_c} + \text{subleading} \right).
\]

The same result is obtained, with logarithmic accuracy, if we first integrate over \( -\infty < x < \infty \), and then over \( -\sqrt{\pi T_c} < y < \sqrt{\pi T_c} \). We now recall that \( \omega_n = (2n + 1)\pi T_c \), and that \( \sum_{n} \frac{1}{|n + 1/2|^2} = 14\zeta(3) \approx 16.8 \) (the sum may again be evaluated on MATHEMATICA). Thus, we obtain, with logarithmic accuracy, the result quoted in the main text, i.e.,

\[
K_1 = \frac{14\zeta(3)}{16\sqrt{3}\pi^4 T_c^2} \ln \frac{t}{T_c} \approx 1.05 \frac{1.05}{\sqrt{3}\pi^4 T_c^2} \ln \frac{t}{T_c}.
\]

2. Evaluating \( K_2 \)

The coefficient \( K_2 \) was evaluated already in Ref. 11. We can write \( K_2 = 6Z_1 + 12Z_2 \), where \( Z_1 \) and \( Z_2 \) are coefficients that were defined in Ref. 11 and calculated in the supplement to Ref. 11. (Note that there is an overall factor of 2 relative to Ref. 11, which comes about because we have doubled the number of degrees of freedom in going to the Nambu spinor representation. However, this overall factor of 2 multiplies all terms in our free energy, and thus has no physical significance.)
In Ref. 11, it was shown that the term $Z_1$ was larger than $Z_2$ by a factor of $\ln t/\tau$, which is a large number at weak coupling. Thus, we can neglect $Z_2$ with logarithmic accuracy, and say $K_2 = 6Z_1$. The coefficient $Z_1$ was calculated in Ref. 11, however, the calculation there had a factor of 2 error, which was unimportant for the physics considered in Ref. 11 but is important here. Therefore, we redo the calculation of $Z_1$.

We wish to evaluate

$$Z_1 = T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} F^+(k, \omega_n)^2 F^-(k + Q, \omega_n)^2.$$  \hspace{1cm} (A5)

The integral over the Brillouin zone is dominated by those values of $k$ where both Green’s functions correspond to states near a saddle point. Expanding the energy about the saddle points, we rewrite the integral as

$$Z_1 \approx T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\left[i \omega_n - \frac{3\pi}{4} (3k_x^2 - k_y^2)\right]^2} \left[i \omega_n - \frac{3\pi}{4} 2k_y(k_y - \sqrt{3} k_z)\right]^2.$$  \hspace{1cm} (A6)

where the integral is understood to have a UV cutoff for $k$ of order 1. We now define $a = \sqrt{3}/4(k_y - \sqrt{3} k_z)$ and $b = \sqrt{3}/4(k_y + \sqrt{3} k_z)$, and rewrite the above integral as

$$Z_1 = \frac{2}{3} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \theta}{\sqrt{\pi}} \frac{1}{i \omega_n + ab} \left(i \omega_n - a(a + b)\right)^2.$$  \hspace{1cm} (A7)

We now define $x = ab$ and rewrite the integral as

$$Z_1 = T \sum_{\omega_n} \frac{2}{3} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \theta}{\sqrt{\pi}} \frac{1}{i \omega_n - a} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \xi}{2\pi} \frac{1}{\xi^2}.$$  \hspace{1cm} (A8)

We now assume $\tau_N \ll t_1$ (which should certainly be the case for weak/moderate coupling). In this limit, we can perform the integral over $x$ approximately, using the Cauchy integral formula, to get

$$Z_1 = \frac{2}{3} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \theta}{\sqrt{\pi}} \frac{1}{i \omega_n - a} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \xi}{2\pi} \frac{1}{\xi^2}.$$  \hspace{1cm} (A9)

The imaginary part of the above integral is odd in $\omega$ and hence vanishes upon performing the Matsubara sum to leave an integral that is purely real:

$$Z_1 = \frac{4}{3} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \theta}{\sqrt{\pi}} \frac{1}{i \omega_n - a} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{d \xi}{2\pi} \frac{1}{\xi^2}.$$  \hspace{1cm} (A10)

with logarithmic accuracy. Performing the integral over $\omega$ (again with logarithmic accuracy) gives

$$Z_1 \approx T \sum_{\omega_n} \frac{1}{12\pi \sqrt{3} t_1} \frac{1}{\omega_n^3} \ln \frac{t_1}{\omega_n} = \frac{1}{96\pi^4 \sqrt{3} t_1} \left(16.8 \ln \frac{t_1}{2\pi T} + 10.5\right) \approx 16.8 \ln \frac{t_1}{2\pi T},$$  \hspace{1cm} (A11)

where we take $\omega_n = 2\pi(n + 1/2)T_N$, $T = T_N$, and perform the discrete sum on MATHEMATICA. The error in the supplement to Ref. 11 was in the last line of the calculation.

3. Evaluating $K_3$

There are two distinct contributions to $K_3$, and we evaluate both in turn. We can write $K_3 = K_3^+ + K_3^-$, where

$$K_3^+ = 6T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} F^+(\omega_n, k) F^-(\omega_n, k) F^+(\omega_n, k + Q_1) \times F^-(\omega_n, k + Q_1) \cos(\theta_k - \theta_{k + Q}).$$

$$K_3^- = 6T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} F^+(\omega_n, k) F^-(\omega_n, k) F^+(\omega_n, k + Q_1) + (F^+ \leftrightarrow F^-).$$  \hspace{1cm} (A12)

The first contribution $K_3^+$ comes from processes of the form shown in Fig. 2(a), and is sensitive to the chirality of the superconducting order parameter (it depends on the difference in the phase of the superconducting order parameter at different points on the Fermi surface). This process leads to an attraction between chiral superconductivity and spin density waves. The second contribution $K_3^-$ comes from processes of the form shown in Fig. 2(b), and is insensitive to the chirality of the superconducting order parameter. This process leads to a repulsion between any kind of superconductivity and spin density waves. The second process dominates (because of purely numerical prefactors), so superconductivity and spin density waves do repel, but the repulsion is too weak to prevent coexistence.

Let us first calculate $K_3^+$. For $d + id$ pairing, we have $\delta_k - \theta_{k+Q} = 4\pi/3$. Thus, we have

$$K_3^+ = 6T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} F^+(\omega_n, k) F^-(\omega_n, k) F^+(\omega_n, k + Q_1) \times F^-(\omega_n, k + Q_1) \cos \frac{4\pi}{3}$$

$$= 6T \cos \frac{4\pi}{3} \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\left[i \omega_n^2 + \frac{a^2}{16} (3k_x^2 - k_y^2)^2\right] \left[\omega_n^2 + \frac{a^2}{16} 4k_y^2 (k_y - \sqrt{3} k_z)^2\right]}.$$  \hspace{1cm} (A13)

We scale out $\omega_n$ and define rescaled variables $x = \sqrt{3}/(4a) k_x, y = \sqrt{3}/(4a) k_y$. The expression for $K_3^+$ can
then be recast as

\[ K_3^a = \cos \left( \frac{4\pi}{3} \right) \sum_{a_0} \frac{2T_c}{\sqrt{\pi^2 |\omega_n|^3}} \times \int_{-\sqrt{T_c}}^{\sqrt{T_c}} dx \ dy \]

\[ \times \left[ 1 + \frac{3}{\sqrt{3}x^2} \left( y - \sqrt{3}x \right)^2 \right]. \quad (A14) \]

We define the new coordinates \( a = y - \sqrt{3}x, b = y + \sqrt{3}x, \) and hence reexpress the above integral (with logarithmic accuracy) as

\[ K_3^a = \cos \left( \frac{4\pi}{3} \right) \sum_{a_0} \frac{T_c}{\sqrt{\pi^2 |\omega_n|^3}} \times \int_{-\sqrt{T_c}}^{\sqrt{T_c}} \frac{da \ db}{(1 + a^2 b^2)(1 + a^2(a + b)^2)}. \quad (A15) \]

We integrate over \(-\sqrt{\pi T_c} < b < \sqrt{\pi T_c}\) (on MATHEMATICA), and expand the resulting expression to leading order in large \( t/T_c \). This leads to the expression

\[ K_3^a = \cos \left( \frac{4\pi}{3} \right) \sum_{a_0} \frac{T_c}{\sqrt{\pi^2 |\omega_n|^3}} \left( \int da \ \frac{\pi}{2|a|(1 + a^4/4)} \right). \quad (A16) \]

It should be remembered that the expansion in large \( t/T_c \) is valid only for \( a^2 t/T_c \gg 1 \), thus, the above integral implicitly carries an infrared cutoff on the scale \( a \approx \sqrt{T_c/T_c} \). Performing the integral with this infrared cutoff, we obtain the expression

\[ K_3^a = \cos \left( \frac{4\pi}{3} \right) \sum_{a_0} \frac{T_c}{2\sqrt{\pi^2 |\omega_n|^3}} \ln \frac{t}{T_c}. \quad (A17) \]

Performing the summation over \( n \) on MATHEMATICA, as before, we obtain

\[ K_3^a = \cos \left( \frac{4\pi}{3} \right) \frac{14\zeta^3}{16\sqrt{3} \pi^4 t^2} \ln \frac{t}{T_c} \approx -\frac{1}{2} \frac{16.8}{16\sqrt{3} \pi^4 t^2} \ln \frac{t}{T_c} \]

\[ = -\frac{1.05}{2\sqrt{3} \pi^4 t^2} \ln \frac{t}{T_c} = -\frac{1}{2} K_1. \quad (A18) \]

Note the crucial minus sign that comes from the chirality sensitive cos factor; this particular term represents an attraction between magnetism and chiral superconductivity.

We now turn our attention to the second term \( K_3^b \). We have

\[ K_3^b = 6T_c \sum_{a_0} \int \frac{d^2 k}{(2\pi)^2} F^+(\omega_n, k) F^-(\omega_n, k) F^+(\omega_n, k + Q_1) + (F^+ \leftrightarrow F^-) \]

\[ = 6T_c \sum_{a_0} \int \frac{d^2 k}{(2\pi)^2} \left[ \omega_n^2 + \frac{9}{2} (3k_x^2 - k_y^2)^2 \right] \left[ \omega_n^2 + \frac{9}{16} (3k_x^2 - k_y^2)^2 \right] \left[ \omega_n^2 + \frac{9}{16} (3k_x^2 - k_y^2)^2 \right] \]

\[ = 12T_c \sum_{a_0} \int \frac{d^2 k}{(2\pi)^2} \left[ \omega_n^2 + \frac{9}{16} (3k_x^2 - k_y^2)^2 \right] \left[ \omega_n^2 + \frac{9}{16} (3k_x^2 - k_y^2)^2 \right] \left[ \omega_n^2 + \frac{9}{16} (3k_x^2 - k_y^2)^2 \right] \]

\[ (A19) \]

Again, we scale out \( \omega_n \) and define the rescaled variables \( x = \sqrt{3}t/(4\omega_n)k_x, y = \sqrt{3}t/(4\omega_n)k_y \). The expression for \( K_3^b \) can then be recast as

\[ K_3^b = \sum_{a_0} \frac{4T_c}{\pi^2 |\omega_n|^3} \int_{-\sqrt{T_c}}^{\sqrt{T_c}} dx \ dy \]

\[ \times \left[ 1 - \frac{3x^2 - y^2)2y(y - \sqrt{3}x)}{[1 + (3x^2 - y^2)^2] [1 + 4y^2(y - \sqrt{3}x)^2]} \right]. \quad (A20) \]

We define the new coordinates \( a = y - \sqrt{3}x, b = y + \sqrt{3}x, \) and hence reexpress the above integral (with logarithmic accuracy) as

\[ K_3^b = \sum_{a_0} \frac{2T_c}{\sqrt{\pi^2 |\omega_n|^3}} \int_{-\sqrt{T_c}}^{\sqrt{T_c}} da \ db \]

\[ \times \left[ 1 + \frac{a^2 b^2(a + b)}{(1 + a^2 b^2)(1 + a^2(a + b)^2)} \right]. \quad (A21) \]

We integrate over \(-\sqrt{\pi T_c} < b < \sqrt{\pi T_c}\) (on MATHEMATICA), and expand the resulting expression to leading order in large \( t/T_c \). This leads to the expression

\[ K_3^b = \sum_{a_0} \frac{T_c}{\sqrt{\pi^2 |\omega_n|^3}} \left( \int da \ \frac{\pi}{2|a|(1 + a^4/4)} \right). \quad (A22) \]

It should be remembered that the expansion in large \( t/T_c \) is valid only for \( a^2 t/T_c \gg 1 \), thus, the above integral implicitly carries an infrared cutoff on the scale \( a \approx \sqrt{T_c/T_c} \). Performing the integral with this infrared cutoff, we obtain the expression

\[ K_3^b = \sum_{a_0} \frac{T_c}{\sqrt{\pi^2 |\omega_n|^3}} \ln \frac{t}{T_c} \approx -\frac{2.01}{\sqrt{3} \pi^4 t^2} \ln \frac{t}{T_c} = 2K_1. \quad (A23) \]

Putting things together, we have

\[ K_3 = K_3^a + K_3^b = K_1 \left( \cos \frac{2\pi}{3} + 2 \right) \]

\[ = K_1 \left( \frac{1}{2} + 2 \right) = \frac{3}{2} K_1. \quad (A24) \]

quoted in the main text.
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