Coupled Nonlinear Schrödinger equation  
and  
Toda equation  
(the Root of Integrability)  

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Abstract  
We consider the relation between the discrete coupled nonlinear Schrödinger equation and Toda equation. Introducing complex times we can show the integrability of the discrete coupled nonlinear Schrödinger equation. In the same way we can show the integrability in coupled case of dark and bright equations. Using this method we obtain several integrable equations.  

Key Words: optical fiber, integrable discrete systems, Toda equation.
1 Introduction

The propagation of soliton envelopes in nonlinear optical media has been predicted and demonstrated experimentally. This prediction arises from a reduction the Maxwell equations which govern the electro magnetic fields in the medium to a single, completely integrable, partial differential equation. The well-studied example is the nonlinear Schrödinger (NLS) equation:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + 2|\phi|^2 \phi = 0. \tag{1.1}
\]

This equation describes the wave propagation of picosecond pulse envelopes \( \phi(x, t) \) in a lossless single mode fiber. The NLS equation (1.1) is the one of the completely integrable system. Recently the interactions among several modes are studied. In general the coupled mode approach still permits description of the pulse propagation in a multi-mode waveguide by means of vector version of (1.1). Although these systems of equations are no longer integrable except for the special parameters, one may obtain quantitative information about the pulse propagation restoring to numerical and perturbative methods. Physically interesting situations that can be described by coupled NLS equations include two parallel waveguides coupled through field overlap. The study of the propagation of optical soliton in multi-mode nonlinear couplers is important in the view point of their possible applications in technology. From the detailed theoretical investigations, several integrable coupled NLS equations possessing soliton have been introduced for the special parameters.

In this paper we consider the discrete coupled NLS equation. This equation is integrable and \( N \)-soliton solution is obtained by the Hirota method. In the continuum limit this equation becomes the coupled NLS equation. This equation is embedded in the 2-dimensional Toda equation. There, two times \( t \) and \( \bar{t} \) are complex conjugate. From the relation to the Toda equation we study the integrability of the discrete coupled NLS equation. Using this method we show the integrability of the new equations.
This paper is organized as follows. In §2 we consider the discrete coupled NLS equation. In any coupled case of the bright and dark equations we show the integrability. In §3 we present the integrable new equation using the method in §2. In §4 we discuss this systems from a viewpoint of the conservation laws. The last section is devoted to the concluding remarks.

2 Discrete coupled nonlinear Schrödinger and Toda equation

Let \( \phi_n^{(j)}(t), j = 1, 2, \cdots, l \) denote \( l \) component dynamical variables. We consider the discrete coupled nonlinear Schrödinger (DCNS) equation.[8]

\[
\frac{\partial \phi_n^{(j)}}{\partial t_1} - i(\phi_n^{(j)} - \phi_{n-1}^{(j)})(1 + \kappa \sum_{k=1}^{l} |\phi_n^{(k)}|^2) + 2i\phi_n^{(j)} = 0, \tag{2.1}
\]

where \( t_1 \) is the time and \( \kappa \) is a constant.

We introduce the new time \( t_2 \) and assume that \( \phi_n^{(j)}(t_1, t_2) \) satisfies the following equation:

\[
\frac{\partial \phi_n^{(j)}}{\partial t_2} + (\phi_n^{(j)} - \phi_{n-1}^{(j)})(1 + \kappa \sum_{k=1}^{l} |\phi_n^{(k)}|^2) = 0 \tag{2.2}
\]

This means a restriction that \( \phi_n^{(j)}(t_1, t_2) \) is the solution of (2.1) and (2.2). The relation between (2.1) and (2.2) is “dual”. We call (2.2) the coupled modified Volterra (CMV) equation.

We find it convenient to use complex times \( t \) and \( \bar{t} \) which is related to \( t_1 \) and \( t_2 \).

\[
t = t_2 + it_1, \quad \bar{t} = t_2 - it_1. \tag{2.3}
\]

In terms of the complex times, we rewrite (2.1) and (2.2):

\[
\frac{\partial \phi_n^{(j)}}{\partial t} = \phi_{n+1}^{(j)}(1 + \kappa \sum_{k=1}^{l} |\phi_n^{(k)}|^2) - \phi_n^{(j)}, \tag{2.4a}
\]

\[
\frac{\partial \phi_n^{(j)}}{\partial \bar{t}} = -\phi_{n-1}^{(j)}(1 + \kappa \sum_{k=1}^{l} |\phi_n^{(k)}|^2) + \phi_n^{(j)}. \tag{2.4b}
\]
\[ \frac{\partial \phi_n^{(j)}}{\partial t} = -\phi_{n-1}^{(j)}(1 + \kappa \sum_{k=1}^{l} |\phi_{n}^{(k)}|^2) + \phi_n^{(j)}, \quad (2.4c) \]

\[ \frac{\partial \bar{\phi}_n^{(j)}}{\partial t} = \bar{\phi}_{n+1}^{(j)}(1 + \kappa \sum_{k=1}^{l} |\phi_{n}^{(k)}|^2) - \bar{\phi}_n^{(j)}. \quad (2.4d) \]

Here we introduce new dependent variables:

\[ a_n^{(j)} = \kappa |\phi_n^{(j)}|^2, \quad b_n^{(j)} = \kappa \phi_n^{(j)} \bar{\phi}_{n-1}^{(j)}, \quad \bar{b}_n^{(j)} = \kappa \bar{\phi}_n^{(j)} \phi_{n-1}^{(j)}. \quad (2.5) \]

The physical meanings of \(a_n^{(j)}\) and \(b_n^{(j)}\) are the amplitude and the momentum.

We rewrite (2.4) using these variables

\[ \frac{\partial a_n^{(j)}}{\partial t} = (b_{n+1}^{(j)} - b_n^{(j)})(1 + \kappa \sum_{k=1}^{l} a_{n}^{(k)}), \quad (2.6a) \]

\[ \frac{\partial a_n^{(j)}}{\partial \bar{t}} = (\bar{b}_{n+1}^{(j)} - \bar{b}_n^{(j)})(1 + \kappa \sum_{k=1}^{l} a_{n}^{(k)}), \quad (2.6b) \]

\[ \frac{\partial b_n^{(j)}}{\partial t} = -a_{n-1}^{(j)}(1 + \kappa \sum_{k=1}^{l} a_{n}^{(k)}) + a_n^{(j)}(1 + \kappa \sum_{k=1}^{l} a_{n-1}^{(k)}), \quad (2.6c) \]

\[ \frac{\partial \bar{b}_n^{(j)}}{\partial t} = -a_{n-1}^{(j)}(1 + \kappa \sum_{k=1}^{l} a_{n}^{(k)}) + a_n^{(j)}(1 + \kappa \sum_{k=1}^{l} a_{n-1}^{(k)}), \quad (2.6d) \]

It is remarkable that these equations are the 2-dimensional Toda lattice equation:

\[ \frac{\partial a_n}{\partial t} = (b_{n+1} - b_n)a_n, \quad \frac{\partial a_n}{\partial \bar{t}} = (\bar{b}_{n+1} - \bar{b}_n)a_n, \quad (2.7a) \]

and

\[ \frac{\partial b_n}{\partial t} = \frac{\partial \bar{b}_n}{\partial \bar{t}} = a_n - a_{n-1}, \quad (2.7b) \]

where

\[ a_n = 1 + \sum_{j=1}^{l} a_n^{(j)}, \quad b_n = \sum_{j=1}^{l} b_n^{(j)}, \quad \bar{b}_n = \sum_{j=1}^{l} \bar{b}_n^{(j)} \quad (2.8) \]
We note that $a_n$ and $b_n$ are the sum of the amplitudes and the sum of the momentum over all components. It means that DCNS and CMV equations are the variants of the 2-dimensional Toda lattice equation. However, (2.7a) and (2.7b) are not regular 2-dimensional Toda lattice. This equation contains complex conjugate in $t$ ($\bar{t}$) and $b_n$ ($\bar{b}_n$). The Toda equation has the $N$-soliton solution. Then both DCNS and CMV equations have solutions corresponding to the solutions of the Toda lattice.

Hereafter we call the two equations are “dual” when from these equations we can construct the Toda lattice equation.

Next we consider the coupled case of the bright and dark equations:

$$\begin{align*}
\frac{\partial \phi_n^{(j)}}{\partial t_1} - i(\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{j=1}^{l} |\phi_n^{(j)}|^2 - \kappa \sum_{j=l+1}^{m} |\phi_n^{(j)}|^2) + 2i\phi_n^{(j)} &= 0, \\
j &= 1, 2, \cdots, l \\
\frac{\partial \phi_n^{(j)}}{\partial t_1} + i(\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{j=1}^{l} |\phi_n^{(j)}|^2 - \kappa \sum_{j=l+1}^{m} |\phi_n^{(j)}|^2) + 2i\phi_n^{(j)} &= 0, \\
j &= l + 1, l + 2, \cdots, m
\end{align*}$$

(2.9)

(2.10)

Among the $m$-components equations, $l$ equations are bright-type and $(m-l)$ equations are dark-type. We consider also the “dual” equations:

$$\begin{align*}
\frac{\partial \phi_n^{(j)}}{\partial t_2} + (\phi_{n-1}^{(j)} - \phi_{n+1}^{(j)})(1 + \kappa \sum_{j=1}^{l} |\phi_n^{(j)}|^2 - \kappa \sum_{j=l+1}^{m} |\phi_n^{(j)}|^2) &= 0, \\
j &= 1, 2, \cdots, l \\
\frac{\partial \phi_n^{(j)}}{\partial t_2} - (\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{j=1}^{l} |\phi_n^{(j)}|^2 - \kappa \sum_{j=l+1}^{m} |\phi_n^{(j)}|^2) &= 0, \\
j &= l + 1, l + 2, \cdots, m
\end{align*}$$

(2.11)

(2.12)

In the same way introducing the complex times we can cast these equations as the Toda lattice equation (2.7a) and (2.7b). The only difference is the
definition of $a_n$:

$$a_n = 1 + \sum_{j=1}^{l} a_n^{(j)} - \sum_{j=l+1}^{m} a_n^{(j)}.$$  \hspace{1cm} (2.13)

That is $a_n$ is the sum of the amplitudes but for the dark-type the sign is “-”.

Usual one dark soliton is described as $A - B \text{sech}^2 X$. Then from the viewpoint of the Toda equation the dark and bright soliton are same.

### 3 New Integrable Equations

Here we consider new coupled equations:

$$\frac{\partial \phi_{(j)}^{(i)}}{\partial t_1} - \frac{i}{2} (\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)}) (1 + \kappa \sum_{k,l} g_{kl} \phi_n^{(k)} \phi_n^{(l)}) + 2i \phi_{n}^{(j)} = 0. \hspace{1cm} (3.1)$$

where

$$g_{ij} = g_{ji}, \quad g_{ij} = 0 \text{ or } 1. \hspace{1cm} (3.2)$$

As in the previous section we introduce the “dual” equation

$$\frac{\partial \phi_{(j)}^{(i)}}{\partial t_2} + (\phi_{n-1}^{(j)} - \phi_{n+1}^{(j)}) (1 + \kappa \sum_{k,l} g_{kl} \phi_n^{(k)} \phi_n^{(l)}) = 0. \hspace{1cm} (3.3)$$

Moreover we define new dependent variables:

$$e_n^{(ij)} = \kappa \phi_n^{(i)} \phi_n^{(j)}, \quad b_n^{(ij)} = \kappa \phi_n^{(i)} \phi_n^{(j)}, \quad \bar{b}_n^{(ij)} = \kappa \phi_n^{(i)} \phi_n^{(j)}. \hspace{1cm} (3.4)$$

In the same way introducing the complex times, we can obtain the Toda lattice equation (2.7a) and (2.7b) for the variables:

$$a_n = 1 + \sum_{i,j} g_{ij} a_n^{(ij)}, \quad b_n = \sum_{i,j} g_{ij} b_n^{(ij)}, \quad \bar{b}_n = \sum_{i,j} g_{ij} \bar{b}_n^{(ij)}. \hspace{1cm} (3.5)$$

From these results we can see that (3.1) and (3.3) are integrable.

If we set $g_{ij} = \delta_{ij}$ then (3.1) and (3.3) become DCNS and CMV equations respectively. If we set $g_{ij} = 1$ then (3.1) and (3.3) can be reduced to the (no coupled) discrete nonlinear Schrödinger (DNLS) and the (no coupled) modified Volterra (MV) equations. As examples we obtain the following new integrable equations:
\[
\frac{\partial \phi_n^{(j)}}{\partial t_1} - i(\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{k \neq l} \phi_{n}^{(k)} \phi_{n+1}^{(l)}) + 2i\phi_{n}^{(j)} = 0. \tag{3.6}
\]

ii)

\[
\frac{\partial \phi_n^{(j)}}{\partial t_1} - i(\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{k = 1}^{l} |\phi_{n}^{(k)}|^2) + i|\phi_{n}^{(j)}|^2 \phi_{n+1}^{(j)} + 2i\phi_{n}^{(j)} = 0,
\]

\[
\phi_{n}^{(k+l)} = \phi_{n}^{(k)}, \quad \phi_{n}^{(k+l)} = \phi_{n}^{(k)}. \tag{3.7}
\]

In the case (i) or (ii) the soliton solution in a line must run together with the soliton in the other line. On the other hands in the case DCNS (2.1) equation the soliton solution can run independently. If the difference of the phases between \(j\) and \(k\) lines is \(-\pi/2 < \theta_{ij} < \pi/2\), the amplitude becomes \(a_{n}^{(jk)} < 0\) and the solution is “dark”. (3.1) and (3.3) may be integrable in the continuum limit.

The discrete Hirota equation reads as: \[10\]

\[
\frac{\partial \phi_n}{\partial t_1} - \alpha(\phi_{n-1} + \phi_{n+1})(1 + \kappa |\phi_{n}|^2) + \beta(\phi_{n-1} - \phi_{n+1})(1 + \kappa |\phi_{n}|^2) + 2i\alpha \phi_{n} = 0 \tag{3.8}
\]

This equation is a hybrid of the discrete nonlinear Schrödinger (DNLS) equation and the modified Volterra (MV) equation. In the continuous limit (3.8) becomes

\[
i\frac{\partial \phi}{\partial t} + k_1 \frac{\partial^2 \phi}{\partial x^2} + k_2 |\phi|^2 \phi + i[k_3 \frac{\partial^3 \phi}{\partial x^3} + k_4 |\phi|^2 \frac{\partial \phi}{\partial x}] = 0. \tag{3.9}
\]

where \(k_i\) is the arbitrary parameters. (3.8) contains several generalized (continuous) NLS equations. As pulse lengths become comparable to the wavelength, NLS equation is not adequate, as the additional effects must be considered. For these cases (3.9) is useful.\[11\]

We consider the coupled version of the discrete Hirota (DCH) equation:

\[
\frac{\partial \phi_{n}^{(j)}}{\partial t_1} - \alpha(\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{k = 1}^{l} |\phi_{n}^{(k)}|^2) + \beta(-\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)})(1 + \kappa \sum_{k = 1}^{l} |\phi_{n}^{(k)}|^2)
\]

\[
+ 2i\alpha \phi_{n}^{(j)} = 0. \tag{3.10}
\]
In the same way we introduce the “dual” equation:

\[
\frac{\partial \phi_n^{(j)}}{\partial t_2} - i\beta (\phi_{n-1}^{(j)} + \phi_{n+1}^{(j)}) (1 + \kappa \sum_{k=1}^{l} |\phi_n^{(k)}|^2) + \alpha (\phi_{n-1}^{(j)} - \phi_{n+1}^{(j)}) (1 + \kappa \sum_{k=1}^{l} |\phi_n^{(k)}|^2) + 2i\beta \phi_n^{(j)} = 0.
\]  

(3.11)

Notice that the “dual” equation also becomes the (DCH) equation, with the parameters \(\alpha\) and \(\beta\) being exchanged.

We introduce the following new independent variables:

\[
a_n^{(j)} = \kappa |\phi_n^{(j)}|^2, \quad b_n^{(j)} = \kappa (\alpha + \beta i) \phi_n^{(j)} \bar{\phi}_{n-1}^{(j)}, \quad \bar{b}_n^{(j)} = \kappa (\alpha - \beta i) \bar{\phi}_n^{(j)} \phi_{n-1}^{(j)}. \quad (3.12)
\]

Then we can obtain the Toda lattice equation (2.7a) and (2.7b) for the variables:

\[
a_n = (\alpha^2 + \beta^2) (1 + \sum_{j=1}^{l} a_n^{(j)}), \quad b_n = \sum_{j=1}^{l} b_n^{(j)}, \quad \bar{b}_n = \sum_{j=1}^{l} \bar{b}_n^{(j)}. \quad (3.13)
\]

Form these results (3.10) and (3.11) are seen to be integrable. These equations are also the variants of the Toda lattice equation. In the case that the bright and dark equations are coupled, such equations are also integrable as discussed in the previous section.

Here we change \(\alpha + i\beta\) to \(re^{i\theta}\). If we set \(\tilde{\phi}_n^{(j)} = e^{i\theta n + 2i(\alpha - \beta i)t} \phi_n^{(j)}\), then \(\tilde{\phi}_n^{(j)}\) satisfy the DCNS equation. In this meaning DCH (3.10) equation is the same as the DCNS (2.1) equation. But in the continuum limit there are differences.

In the continuum limit (3.10) becomes generalized coupled NLS equations:

\[
i \frac{\partial \phi^{(j)}}{\partial t} + k_1 \frac{\partial^2 \phi^{(j)}}{\partial x^2} + k_2 \left( \sum_l |\phi^{(l)}|^2 \right) \phi^{(j)} + ik_3 \frac{\partial^3 \phi^{(j)}}{\partial x^3} + ik_4 \left( \sum_l |\phi^{(l)}|^2 \right) \frac{\partial \phi^{(j)}}{\partial x} = 0,
\]  

(3.14)

while (3.14) contains nonlinear derivative term.
4 Conservation Laws

The 2-dimensional Toda lattice system (2.7a) and (2.7b) has infinite number of conserved quantities. The conserved density are obtained from the Lax operator.

\[ D^T_1 = b_n, \quad \bar{D}^T_1 = \bar{b}_n \]
\[ D^T_2 = \frac{1}{2} b_n^2 + \frac{a_n}{1 - a_n} b_n b_{n-1}, \quad \bar{D}^T_2 = \frac{1}{2} \bar{b}_n^2 + \frac{a_n}{1 - a_n} \bar{b}_n \bar{b}_{n-1}, \]
\[ D^T_3 = \frac{1}{3} b_n^3 + \frac{a_n}{1 - a_n} \frac{a_{n-1}}{1 - a_n} b_n b_{n-1} b_{n-2} \]
\[ + \frac{a_n}{1 - a_n} b_n b_{n-1} (b_n + b_{n-1}), \]
\[ \bar{D}^T_3 = \frac{1}{3} \bar{b}_n^3 + \frac{a_n}{1 - a_n} \frac{a_{n-1}}{1 - a_n} \bar{b}_n \bar{b}_{n-1} \bar{b}_{n-2} \]
\[ + \frac{a_n}{1 - a_n} \bar{b}_n \bar{b}_{n-1} (b_n + \bar{b}_{n-1}). \] (4.1)

On the other hand the conserved densities of the DNLS and MV equation are obtained from the Ablowitz and Ladik (AL) system:

\[ D^{AL}_1 = \phi_n \bar{\phi}_{n-1}, \quad \bar{D}^{AL}_1 = \bar{\phi}_n \phi_{n-1} \]
\[ D^{AL}_2 = \frac{1}{2} (\phi_n \bar{\phi}_{n-1})^2 + \phi_{n+1} \bar{\phi}_n (1 + |\phi_n|^2) \]
\[ \bar{D}^{AL}_2 = \frac{1}{2} (\bar{\phi}_{n+1} \phi_{n-1})^2 + \bar{\phi}_{n+1} \phi_n (1 + |\phi_n|^2), \]
\[ D^{AL}_3 = \frac{1}{3} (\phi_n (\bar{\phi}_{n+1} \phi_{n-1}))^3 + \phi_{n+1} \bar{\phi}_{n-2} (1 + |\phi_n|^2) (1 + |\phi_{n-1}|^2) \]
\[ + \phi_{n+1} \bar{\phi}_{n-1} (\phi_{n+1} \bar{\phi}_n + \phi_n \bar{\phi}_{n-1}) (1 + |\phi_n|^2), \]
\[ \bar{D}^{AL}_3 = \frac{1}{3} (\bar{\phi}_n (\bar{\phi}_{n+1} \phi_{n-1}))^3 + \bar{\phi}_{n+1} \phi_{n-2} (1 + |\phi_n|^2) (1 + |\phi_{n-1}|^2) \]
\[ + \bar{\phi}_{n+1} \phi_{n-1} (\bar{\phi}_{n+1} \phi_n + \bar{\phi}_n \phi_{n-1}) (1 + |\phi_n|^2). \] (4.2)

The conserved densities of the Toda equation agree with those of the AL system. For the multi component cases the number of the conserved densities is the same as the one component case. Then, the conserved densities of the Toda equation agree with those the multi component AL systems. From these results we can obtain the conserved density for the DCNS, CMV and new integrable equations.
5 Concluding Remarks

In this paper we have studied the discrete coupled nonlinear Schrödinger (DCNS) and coupled modified Volterra (CMV) equations from the viewpoint of the 2-dimensional Toda equation. Introducing the complex time, coupling the DCNS and CMV equations we have obtained the Toda equation. The DCNS and CMV equation are equivalent to the Toda system. This can be seen from the explicit transformations and conservation laws. From this point of view when the bright and the dark equation are coupled, the DCNS equation is also integrable. In the Toda equation the dark and bright soliton are equivalent. Using this method we can present several new equations. All these equations are embedded in the 2-dimensional Toda equation of which time variable \( t \) and \( \bar{t} \) are complex conjugate. In the continuum limit these new equations may be integrable.

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