Unique Quantum Paths by Continuous Diagonalization of the Density Operator

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In this short note we show that for a Markovian open quantum system it is always possible to construct a unique set of perfectly consistent Schmidt paths, supporting quasi-classicality. Our Schmidt process, elaborated several years ago, is the $\Delta t \to 0$ limit of the Schmidt chain constructed very recently by Paz and Zurek.
I. INTRODUCTION

In a very recent work, Paz and Zurek [1] discuss Markovian open quantum systems and construct Schmidt paths showing exact decoherence. The authors, however, notice their Schmidt paths are quite unstable under, e.g., varying the number of subsequent projections. They think to eliminate the problem by tuning time intervals between subsequent projections larger than the typical decoherence time.

In the present short note I propose the opposite. Let time intervals be infinitely short! It means that the frequency of the subsequent projections is so high that the Schmidt path will be defined for any instant \( t \) in the period considered. This infinite frequency limit exists and provides unique consistent set of Schmidt paths. The issue was discussed [2] and completely solved [3] several years ago. Of course, the above limit is only valid up to Markovian approximation. In fact, the ”infinitesimal” repetition interval is still longer than the response time of the reservoir.

For the sake of better distinction between ordinary [1] and hereinafter advocated [3] Schmidt paths, let us call them Schmidt chain and Schmidt process, respectively.

In the next Sect., Schmidt chains are briefly reviewed. The Sect. III. will recall the earlier results available now for Schmidt processes. Subsequently, in Sect. IV., we propose an application of Schmidt processes to the quantum Brownian motion where classicality might be demonstrated.

II. SCHMIDT PATH—MARKOV CHAIN

Consider the reduced dynamics of a given subsystem:

\[
\rho(t') = J(t' - t)\rho(t), \quad (t' > t)
\]  

where \( \rho \) is the reduced density operator, \( J \) is the Markovian evolution superoperator. For a given sequence \( t_0 < t_1 < \ldots < t_n \) of selection times, let us have the corresponding sequence of pure state (Hermitian) projectors: \( \{P^0, P^1, \ldots, P^n\} \) is a Schmidt chain if
\[ [P^{k+1}, J(t_{k+1} - t_k)P^k] = 0, \quad k = 0, 1, 2, \ldots, n - 1 \]  

(2)

cf. Sect. 4 of Ref. [1]. For fixed initial state \( P^0 = \rho(t_0) \), let the probabilities

\[ p(P^1, P^2, \ldots, P^n) = \text{tr} \left[ P^n J(t_n - t_{n-1}) P^{n-1} \cdots P^1 J(t_1 - t_0) \rho(t_0) \right] \]  

(3)

be assigned to Schmidt chains. Schmidt chains satisfy the following sum rule:

\[
\sum_{\text{Schmidt paths}} p(P^1, P^2, \ldots, P^n) P^1 \otimes P^2 \otimes \cdots \otimes P^n = \rho(t_1) \otimes \rho(t_2) \otimes \cdots \otimes \rho(t_n)
\]  

(4)

assuring the consistency [4] of the probability assignments (3).

Let us construct concrete Schmidt chains. To satisfy the Eq. (2) for \( k = 0 \), let us first diagonalize the positive definite operator \( J(t_1 - t_0) P^0 \):

\[ J(t_1 - t_0) P^0 = \sum_\alpha \alpha P^1_\alpha. \]  

(5)

If our choice is \( P^1 = P^1_{\alpha_1} \), whose probability is \( p_{\alpha_1} \), consider Eq. (2) for \( k = 1 \) and diagonalize \( J(t_2 - t_1) P^1_{\alpha_1} \):

\[ J(t_2 - t_1) P^1_{\alpha_1} = \sum_\alpha \alpha P^2_\alpha. \]  

(6)

Single out \( P^2 = P^2_{\alpha_2} \) at random, with probability \( p^2_{\alpha_2} \), etc.

For the Schmidt chain \( \{P^0, P^1_{\alpha_1}, P^2_{\alpha_2}, \ldots, P^n_{\alpha_n}\} \) one generates from the fixed initial state \( P^0 \), the probability (3) takes the following factorized form:

\[
p(P^1_{\alpha_1}, P^2_{\alpha_2}, \ldots, P^n_{\alpha_n}) \equiv p(\alpha_1, \alpha_2, \ldots, \alpha_n) = p^1_{\alpha_1} p^2_{\alpha_2} \cdots p^n_{\alpha_n}.
\]  

(7)

Schmidt chain is Markov chain. Given \( P^0 = \rho(t_0) \), it will branch at \( t_1 \) into \( P^1_{\alpha_1} \), i.e., into one of the eigenstate projectors of \( J(t_1 - t_0) P^0 \); the branching probability \( p^1_{\alpha_1} \) is the corresponding eigenvalue. In the general case, \( P^k_{\alpha_k} \) will branch into \( P^{k+1}_{\alpha_{k+1}} \), i.e., into a certain eigenstate of \( J(t_{k+1} - t_k) P^k_{\alpha_k} \), with branching probability \( p^{k+1}_{\alpha_{k+1}} \) given by the corresponding eigenvalue.
III. SCHMIDT PATH—MARKOV PROCESS

In this Sect., we consider the limiting case of the Schmidt chains when the separations $t_{k+1} - t_k$ go to zero. The pure state path $\{P(t); t > t_0\}$, starting from the fixed initial state $P(t_0) = \rho(t_0)$, is a Schmidt process if (for $t > t_0$ and $\epsilon \equiv dt > 0$) $P(t + \epsilon)$ branches into an eigenstate projector $P_\alpha(t)$ of $J(\epsilon)P(t)$ while the branching probability is the corresponding eigenvalue $p_\alpha(t)$. Branching rates $w_\alpha(t)$ are worthwhile to introduce by $p_\alpha(t) = \epsilon w_\alpha(t)$.

We follow the general results obtained in Ref. [3]. Let us introduce the Liouville superoperator $L$ generating the Markovian evolution (1):

$$J(\epsilon) = 1 + \epsilon L.$$  

Assume the Lindblad form [5]:

$$L\rho = -i[H, \rho] - \frac{1}{2} \sum_\lambda \left( F_\lambda^\dagger F_\lambda \rho + \rho F_\lambda^\dagger F_\lambda - 2F_\lambda \rho F_\lambda^\dagger \right)$$  

where $H$ is the Hamiltonian and $\{F_\lambda\}$ are the Lindblad generators. Following the method of Ref. [3], introduce the frictional (i.e. nonlinear-nonhermitian) Hamiltonian:

$$H_P = H - \frac{1}{2i} \sum_\lambda \left( < F_\lambda^\dagger > F_\lambda - H.C. \right) - \frac{i}{2} \sum_\lambda \left( F_\lambda^\dagger - < F_\lambda^\dagger > \right) \left( F_\lambda - < F_\lambda > \right) + \frac{i}{2} w$$  

and the nonlinear positive definite transition rate operator:

$$W_P = (F_\lambda - < F_\lambda >) P \left( F_\lambda^\dagger - < F_\lambda^\dagger > \right)$$  

where, e.g., $< F_\lambda > \equiv tr(F_\lambda P)$. We need the unit expansion of the transition rate operator:

$$W_P = \sum_{\alpha=1}^{\infty} w_\alpha P_\alpha.$$  

Observe that, due to the identity $W_P P \equiv 0$, each $P_\alpha$ is orthogonal to $P$. The $w_\alpha$'s are called transition (branching) rates. The total transition (branching) rate then follows from Eqs. (11) and (12):
\[ w \equiv \sum_{\alpha} w_\alpha = < F^\dagger_\lambda F_\lambda > - < F^\dagger_\lambda >= F_\lambda >. \]  

How to generate Schmidt processes? Given the initial pure state \( \rho(t_0) = P(t_0) \), the pure state \( P(t) \) evolves according to the deterministic frictional Schrödinger-von Neumann equation:

\[ \frac{d}{dt} P = -i(H_P P - P H_P^\dagger) \]  except for discrete orthogonal jumps (branches)

\[ P(t + 0) = P_\alpha(t) \]  occurring from time to time at random with \( P(t) \)-dependent partial transition rates \( w_\alpha(t) \).

It is worthwhile to note that neither \( H_P \) nor \( W_P \) depend on the concrete Lindblad representation (9) of \( L \), as it is clear in Ref. [3].

Mathematically, the above Schmidt path is pure-state-valued Markov process of generalized Poissonian type. During a given infinitesimal period \( (t, t + dt) \), the probability of the branch-free (i.e., jump-free, continuous) evolution is \( 1 - w(t) dt \). Consequently, one obtains [6] the a priori probability of continuous evolution for an arbitrarily given period \( (t_1, t_2) \) as

\[ \exp \left( - \int_{t_1}^{t_2} w(t) dt \right). \]  

**IV. CLASSICALITY**

Schmidt processes assure maximum classicality in "measurement situations". It has been shown in Ref. [6] that for large enough \( t \), Schmidt process converges to one of the pointer states while the overall probability of further branches tends to zero. Convergence is then dominated by the asymptotic solutions of the deterministic frictional Schrödinger-von Neumann Eq. (14).

To test classicality of Schmidt processes in less artificial situations, let us start with the (modified [7]) Caldeira-Leggett [8] master equation:
\[
\frac{d\rho}{dt} = L\rho = -i \frac{1}{2M} [p^2, \rho] - i\gamma [q, \{p, \rho\}] - \frac{1}{2} \gamma \lambda_{dB}^2 [q, [q, \rho]] - \frac{1}{2} \kappa \gamma \lambda_{dB}^2 [p, [p, \rho]]
\]

where \(\gamma\) is (two times) the friction constant, \(\lambda_{dB}\) stands for the thermal deBroglie length of the Brownian particle of mass \(M\). In Ref. [7] the value \(\kappa = 4/3\) has been suggested. For simplicity’s, we have omitted the usual renormalized potential term in the Hamiltonian, assuming it is zero or small enough. Hence we can model quantum counterpart of pure frictional motion.

Applying mechanically the Eqs. (10) and (11), we calculate both the frictional Hamiltonian and the transition rate operator:

\[
H_P = \frac{1}{2M} p^2 + \frac{1}{2} \gamma \{q- < q >, p- < p >\}
- \frac{i}{2} \gamma \left(\lambda_{dB}^2 ((q- < q >)^2 - \sigma_{qq}^2) + \kappa \lambda_{dB}^2 ((p- < p >)^2 - \sigma_{pp}^2)\right),
\]

\[
W_P = \gamma \lambda_{dB}^{-2} (q- < q >) P(q- < q >) + \kappa \gamma \lambda_{dB}^2 (p- < p >) P(p- < p >)
- i\gamma ((q- < q >) P(p- < p >) - (p- < p >) P(q- < q >))
\]

where \(\sigma_{qq}^2 = < q^2 > - < q >^2\) and \(\sigma_{pp}^2 = < p^2 > - < p >^2\). The total transition (branching) rate (13) obtains the simple form:

\[
w = \gamma (\lambda_{dB}^{-2} \sigma_{qq}^2 + \kappa \lambda_{dB}^2 \sigma_{pp}^2 - 1)
\]

as can be easily verified by observing \(w = tr W_P\).

For most of the time the Schmidt process is governed by the frictional Hamiltonian (18), via the nonlinear Eq. (14). This equation itself possesses a stationary solution \(P(\infty)\) with simple Gaussian wave function representing a standing particle. Furthermore, one can heuristically guess that the nonhermitian terms establish quasi-classicality for arbitrarily given initial states. Obviously, the random jumps (15) will interrupt the deterministic evolution of the Schmidt process. If, however, in the quasi-classical regime the rate (20) of jumps were much smaller than the speed of relaxation due to the frictional Hamiltonian (18) then the infrequent jumps would only cause slight random walk and breathing to the otherwise quasi-classical wave function.
V. CONCLUSION

Schmidt processes offer a certain solution to the preferred basis problem of quantum mechanics, at least when the subsystem’s reduced dynamics can be considered Markovian. It will be interesting to carry on with analytic calculations for the Schmidt process of the Brownian motion, not at all exhausted in Sec. IV.

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REFERENCES

[1] J. P. Paz and W. Zurek, preprint (1993).

[2] L. Diósi, Phys. Lett. 112A, 288 (1985).

[3] L. Diósi, Phys. Lett. 114A, 451 (1986).

[4] R. B. Griffiths, J. Stat. Phys. 36, 219 (1984).

[5] G. Lindblad, Commun. Math. Phys. 48, 119 (1976).

[6] L. Diósi, J. Phys. A21, 2885 (1988).

[7] L. Diósi, Europhys. Lett. 22, 1 (1993).

[8] A. O. Caldeira and A. J. Leggett, Physica A12, 587 (1983).