Gravitational Lensing by Elliptical Galaxies and the Schwarz Function

Ch. Fassnacht
University of California, Davis

Ch. Keeton
Rutgers University

and

Dmitry Khavinson
University of South Florida

July, 2007
Figure 2: Setup of a gravitational lens situation: The lens $L$ located between source $S$ and observer $O$ produces two images $S_1$ and $S_2$ of the background source.
**Gravitational Microlensing**

- $n$ co-planar point-masses (e.g. condensed galaxies, black holes, etc.) in *lens plane* or *deflector plane*.

- Consider a light source in the plane parallel to the lens plane (*source plane*) and perpendicular to the line of sight from the observer.

- Due to deflection of light by masses multiple images of the source are formed. This phenomenon is known as *gravitational microlensing*. 
Gravitational Lens
Galaxy Cluster 0024+1654

HST • WFPC2

PRC96-10 • ST ScI OPO • April 24, 1996

W.N. Colley (Princeton University), E. Turner (Princeton University),
J.A. Tyson (AT&T Bell Labs) and NASA
Lens Equation for Co-Planar Point Masses

- Light source is located in the position \( w \) in the source plane.

- The lensed image is located at the position \( z \) in the lens plane.

- The masses are located at the positions \( z_j \) in the lens plane.

\[
w = z - \sum_{1}^{n} \sigma_j / (\overline{z} - \overline{z_j}),
\]

where \( \sigma_j \neq 0 \) are real constants. Letting \( r(z) = \sum_{1}^{n} \sigma_j / (z - z_j) + \overline{w} \), the lens equation becomes

\[
z - r(z) = 0, \quad \text{deg } r = n.
\]

The number of solutions = the number of “lensed” images.
**History**

- First calculations of the deflection angle based on Newton’s corpuscular theory of light and the gravitational law (H. Cavendish and Reverend J. Michell circa 1784, P. Laplace - 1796, J. Soldner, 1804 - the first published calculation).

- $n = 1$ (one mass) A. Einstein (circa ’33), either two images or the whole circle (“Einstein ring”).

- H. Witt (’90) For $n > 1$ the maximum number of observed images is $\leq n^2 + 1$.

- S.H. Rhie (’01) conjectured the upper bound for the number of lensed images for an $n$-lens is $5n - 5$. 
Solution

- (Mao - Petters - Witt, ’97) The maximum is $\geq 3n + 1$

- $n = 2, 3$ (’97-'03)(Mao, Petters, Witt, Rhie) - the maximum is 5, 10 respectively.

- $n = 4$, is the maximum 15 or 17?

**Theorem 1.** (G. Neumann-DK, ’06). The number of lensed images by an $n$-mass lens cannot exceed $5n - 5$ and this bound is sharp (Rhie, ’03). Moreover, it follows from the proof that the number of images is even when $n$ is odd and vice versa.
Quadratic vs. Linear Numbers of Images

A model problem: Let \( p(z) := a_n z^n + \cdots + a_0, a_n \neq 0 \) be a polynomial of degree \( n > 1 \).

Question. Estimate \( \# \{ z : z - \bar{p}(z) = 0 \} \), or more generally,
\[
\# \{ (x, y) : A(x, y) + iB(x, y) = 0 \},
\]
where \( A, B \) are real polynomials of degree \( \leq n \).

Bezout’s theorem implies
\[
\# \{ (x, y) : A = B = 0 \} \leq n^2.
\]

Conjecture 1. (T. Sheil-Small - A. Wilmshurst, ’92)
\[
\# \{ z : p(z) - \bar{z} = 0, n > 1 \} \leq 3n - 2.
\]
Results

- In the 1990s D. Sarason and B. Crofoot and, independently, D. Bshouty, A. Lyzzaik and W. Hengartner verified it for $n = 2, 3$.

- In 2001, G. Swiatek and DK proved Conjecture 1 for all $n > 1$.

- In 2003-2005 L. Geyer showed that $3n - 2$ bound is sharp for all $n$. 
Examples

• One-point mass lens with source at $w = 0$.

$$z - \frac{c}{z-a} = 0.$$ 

Two images for $a \neq 0$, a circle (“Einstein ring”) for $a = 0$, i.e., when the observer, the lens and the source coalesce.

• One-point lens with the tidal perturbation (a “shear”) from a far away galaxy, a Chang-Refsdal lens.

$$z - \frac{c}{z} - \gamma z = w.$$ 

The equation reduces to a quadratic and Bezout’s theorem yields a bound of at most 4 images. Curves cannot occur!
Continuous Mass Distribution

For a continuous real-valued mass-distribution $\mu$ in a region $\Omega$ in the plane the lens equation with shear takes form

$$z - \int_{\Omega} \frac{d\mu(\zeta)}{z - \zeta} - \gamma\bar{z} = w.$$

- $\mu = n > 1$ non-overlapping radially symmetric masses. The number of images “outside” the masses $\leq 5n - 5$ if $\gamma = 0$ and $\leq 5n$ if $\gamma \neq 0$ (DK-G. Neumann '06, refinements by J. H. An and N. W. Evans, '06).

- $\mu = \text{uniform-mass distribution inside a quadrature domain } \Omega \text{ of order } n$, i.e. $\Omega = \phi(D)$, $\phi$ is a rational function with $n$ poles univalent in $D := \{|z| < 1\}$. The number of images outside $\Omega$ is $\leq 5n - 5$ (DK-GN, '06).
Smooth Mass Distributions

(W. L. Burke’s Theorem, ’81). The number of images is always odd. \((\gamma = 0)\).

Take \(w = 0\). Let \(n_+\) be the number of sense preserving images and \(n_-\) - the number of sense reversing images. Argument principle yields

\[ 1 = n_+ - n_- \]

so the total number of images

\[ N = n_+ + n_- = 2n_- + 1. \]
Einstein Rings are Ellipses

Theorem 2. (CF-CK-DK, ’07) For any lens \( \mu \), if the lensing produces an image “curve” surrounding the lens, it is either a circle when the shear \( \gamma = 0 \), or an ellipse.

For an illustration assume the shear \( \gamma = 0 \). If the lens produces an image which is a curve \( \Gamma \), then

\[
\bar{z} = \int_{\Omega} \frac{d\mu(\zeta)}{z - \zeta} \quad \text{on} \quad \Gamma
\]

The integral is an analytic function in \( \mathbb{C} \setminus \Omega \) vanishing at \( \infty \). Hence \( |\bar{z}|^2 \) matches on \( \Gamma \) a bounded analytic function in \( \mathbb{C} \setminus \Omega \) and must be a constant.

Remark 1. Using P. Divé’s converse to the Newton’s “no gravity in the ellipsoidal cavity” theorem, we can extend the above result to higher dimensions.
Einstein Ring Gravitational Lenses

_Hubble Space Telescope_ • Advanced Camera for Surveys

NASA, ESA, A. Bolton (Harvard-Smithsonian CfA), and the SLACS Team

STScI-PRC05-32
Ellipsoidal Lens

Lens $\Omega = \{x^2/a^2 + y^2/b^2 \leq 1, a > b > 0\}$, with constant density. $c^2 = a^2 - b^2$.

$$\bar{z} - \frac{1}{\pi} \int_{\Omega} \frac{dA(\zeta)}{z - \zeta} - \gamma z = \bar{w},$$

or, using complex Green’s formula,

$$\bar{z} - \frac{1}{2\pi i} \int_{\partial \Omega} \frac{\bar{\zeta} d\zeta}{z - \zeta} - \gamma z = \bar{w}.$$
The Schwarz Function of the Ellipse

The Schwarz function $S(\zeta) = \bar{\zeta}$ of $\partial\Omega$.

\[ S(\zeta) = \frac{a^2 + b^2}{c^2} \zeta - \frac{2ab}{c^2} (\zeta - \sqrt{\zeta^2 - c^2}) \]
\[ = \frac{a^2 + b^2 - 2ab}{c^2} \zeta + \frac{2ab}{c^2} (\zeta - \sqrt{\zeta^2 - c^2}) \]
\[ = S_1(\zeta) + S_2(\zeta), \]

where $S_1$ analytic inside $\Omega$, $S_2$ - outside $\Omega$, $S_2(\infty) = 0$. This is the Plemelj-Sokhotsky decomposition of the Schwarz function of $\partial\Omega$. 

16
• For $z$ outside $\Omega$ the lens equation then reduces to
\[
\bar{z} + \frac{2ab}{c^2}(z - \sqrt{z^2 - c^2}) - \gamma z = \bar{w},
\]
that may have at most 4 solutions by Bezout’s theorem.

• For $z$ inside $\Omega$ the lens equation reduces to a linear equation giving at most one solution.

**Theorem 3.** *(CF-CK-DK)* An elliptic galaxy $\Omega$ with a uniform mass density may produce at most 4 “bright” lensing images of a point light source outside $\Omega$, and at most one “dim” image inside $\Omega$, i.e., at most 5 lensing images altogether.
Confocal Ellipses

MacLaurin’s mean value theorem concerning potentials of confocal ellipsoids readily yields

**Corollary 1.** An elliptic galaxy $\Omega$ with mass density that is constant on ellipses confocal with $\Omega$, may produce at most 4 “bright” lensing images of a point light source outside $\Omega$. 
“Isothermal” Elliptical Lenses

- Density, inversely proportional to the distance from the origin, is constant on ellipses $\Gamma_t := \{x^2/a^2 + y^2/b^2 = t\}$ homothetic with $\partial \Omega$.

- Lens equation becomes transcendental:

$$z = \text{const} \int_0^1 \frac{dt}{\sqrt{z^2 - c^2 t^2}} - \gamma \bar{z} = w.$$

- There are no more than 5 images (4 + 1) observed as of today.

- In 2000 Ch. Keeton, S. Mao and H. J. Witt constructed models with a strong tidal perturbation (shear) having 9, (8 bright + 1 dim), images.
Remarks

- An isothermal sphere with a shear is covered by ’06 K-N theorem (cf. also ’06 paper by An - Evans on Chang-Refsdal lens) and may produce at most 4 images (observed).

- A rigorous proof that an isothermal elliptical lens may only produce finitely many images is still missing.
Critical Curves and Caustics

• Jacobian of the lens map \( L(z) = z - \overline{p(z)} = w \) with potential \( p(z) \)

\[
J(z) = 1 - |p'(z)|^2.
\]

• Critical Curve \( C := \{z : J = 0\} \).

• Caustic \( C' = L(C) \).

• \( J(z) \) is the area distortion factor. Its reciprocal expresses the ratio of the apparent solid angle covering the lensed images \( z \) to that of the original source \( w \), called magnification.

• Caustics indicate positions for the source where magnification tends to infinity.
Remarks

- Critical curves are *lemniscates*, caustics and their pre-images, “pre-caustics”, $L^{-1}(C')$ are much more complicated.

- Geometry of critical curves and caustics especially for 2, 3 and 4 point lenses was modeled and studied by astrophysicists An, Evans, Keeton, Mao, Petters, Rhie, Witt to name just a few and, independently, by Bshouty, Hengartner, Lyzzaik, Neumann, Ortel, Suez, Suffridge, Wilmshurst. G. Neumann’s thesis ’03 has a variety of deep, novel geometric results.

- (K-N ’06, conjectured by Rhie ’01). The total number of “positive” ($J \geq 0$) images produced by an $n > 1$-point mass lens in absence of a tidal perturbation is $\leq 2n - 2$. Further refinements can be found in ’06 work of An and Evans.
Isothermal Ellipsoid
QUESTIONS

Lensing by a uniform mass in a Q. D..

- Geometric interplay between critical curve(s) vs. the boundary of the q.d.

- Estimate the number of “dim” images inside the q.d. = in-depth study of the algebraic part of the Schwarz function.

- Valence of algebraic vs. transcendental harmonic mappings (cf. G. Neumann’s papers ’05, ’07).

  Model Problem: Sharp estimate for \( \#\{z : \bar{z}^m - p(z) = 0\} \), \( n := \deg p >> m \). Wilmshurst (’94) conjectured the upper bound \( m(m - 1) + 3n - 2 \).

- Estimate the number of bright images for a polynomial mass density.
Elliptic Lenses

- Maximal number of images for the isothermal elliptical lens.

- Elliptical lens with a polynomial (rational) mass density
  (i) Maximal number of images
  (ii) Critical curves and pre-caustics
  (iii) Anomalies related to arbitrary continuous mass-densities

- Lensing by several elliptical masses (observed so far 2 galaxies lens giving 5 images and 3 galaxies lens with 6).
Three-Dimensional Lensing

The 3-dimensional lens equation with mass-distribution $dm(y)$ with source at $\vec{w}$ becomes

$$\vec{x} - \nabla_x \left( \int \frac{dm(y)}{|x - y|} \right) = \vec{w}.$$  

1. $dm = \sum_{1}^{n} c_j \delta y_j$. There are rough estimates for the maximal number of images (Petters, ’90s) based on Morse theory.

2. A difficult Maxwell’s problem concerns a number of stationary points of the Newtonian potential of $n$ point-masses (conjectured $\leq (n - 1)^2$). Most recent progress due to Eremenko, Gabrielov, D. Novikov, B. Shapiro.
• Ellipsoidal mass densities.

• Critical surfaces, caustics and pre-caustics of the lens map.

(CF-CK-DK, ’07: “Einstein” surfaces can only be either spherical in absence of a shear, or ellipsoidal.)

• Other mass-densities???
THANK YOU!