WEAK AND LOCAL VERSIONS OF MEASURABILITY

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Abstract. Local versions of measurability have been around for a long time. Roughly, one splits the notion of $\mu$-completeness into pieces, and asks for a uniform ultrafilter over $\mu$ satisfying just some piece of $\mu$-completeness.

Analogue local versions of weak compactness are harder to come by, since weak compactness cannot be defined by using a single ultrafilter. We deal with the problem by restricting just to a subset $P$ of all the partitions of $\mu$ into $<\mu$ classes and asking for some ultrafilter $D$ over $\mu$ such that no partition in $P$ disproves the $\mu$-completeness of $D$. By making $P$ vary in appropriate classes, one gets both measurability and weak compactness, as well as possible intermediate notions of “weak measurability”.

We systematize the above procedures and combine them to obtain variants of measurability which are at the same time weaker and local. Of particular interest is the fact that the notions thus obtained admit equivalent formulations through topological, model theoretical, combinatorial and Boolean algebraic conditions. We also hint a connection with Katětov order on filters.

§1. Introduction.

1.1. Local forms of measurability. Local versions of measurability have been considered by many authors, among them Chang [Cha67], Prikry [Pri73], Silver [Sil74], just to state some.

To cast our introduction into a general framework, let us recall some definitions. If $D$ is an ultrafilter over $\mu$, then $D$ is $\lambda$-decomposable if there is a partition of $\mu$ into $\lambda$ many classes in such a way that no union of $<\lambda$ classes belongs to $D$. Equivalently (2.5(1)), $D$ is $\lambda$-decomposable if there is a function $f : \mu \rightarrow \lambda$ such that $f(D)$ is uniform over $\lambda$, that is, every member of $f(D)$ has cardinality $\lambda$. Here $f(D)$ is the ultrafilter over $\lambda$ defined by $Y \in f(D)$ if and only if $f^{-1}(Y) \in D$. The relation induced by this “quotient” operation is usually called the Rudin-Keisler (pre-)order, thus an ultrafilter $D$ is $\lambda$-decomposable if and only if there is some ultrafilter uniform over $\lambda$ and $\leq D$ modulo the Rudin-Keisler order. Each of the above points of view—through quotients and through
partitions—has its own advantages, as we shall discuss in details. We say that \( D \) is \( \lambda \)-indecomposable if it is not \( \lambda \)-decomposable.

If \( \mu \) is measurable, then trivially there is a uniform ultrafilter \( D \) over \( \mu \) which is \( \lambda \)-indecomposable for every \( \lambda < \mu \). Chang [Cha67] first noticed that the study of \( \lambda \)-decomposability is not trivial, even for small cardinals. Actually, Chang dealt with the notion of \( \lambda \)-descending incompleteness, which is nevertheless equivalent to \( \lambda \)-decomposability, for \( \lambda \) regular. See [MR] for this and many other equivalences, and for an exhaustive list of references to the subject, which includes works by the mentioned authors and by A. Adler, A. W. Apter, S. Ben-David, M. Benda, G. V. Ćudnovskii, D. V. Ćudnovskii, H.-D. Donder, M. Foreman, J. M. Henle, M. Huberich, T. Jech, R. B. Jensen, M. Jørgensen, A. Kanamori, J. Keisler, J. Ketonen, B. J. Koppelberg, K. Kunen, M. Magidor, M. Sheard, A. D. Taylor and W. H. Woodin.

In fact, an uncountable cardinal \( \mu \) is measurable \textit{if and only if} it carries a uniform ultrafilter which is \( \lambda \)-indecomposable for every \( \lambda < \mu \). More generally, an ultrafilter is \( \mu \)-complete \textit{if and only if} it is \( \lambda \)-indecomposable for every \( \lambda < \mu \). See Remark 2.5(2) here, or [MR] for further details. Thus ultrafilters enjoying various degrees of indecomposability furnish weaker "local" analogues of measurability. The existence of a \( \lambda \)-indecomposable uniform ultrafilter over \( \mu \) admits many equivalent formulations in terms of topology, model-theory (both first order and extended), infinite combinatorics, Boolean algebras. See [CS, UT, CM, CP, CO] for examples. In a sense, the situation is similar to the one described in the classical paper [KT63/64] by H. J. Keisler and A Tarski, where a big deal of conditions equivalent to a cardinal being measurable have been worked out. The existence of a \( \lambda \)-indecomposable uniform ultrafilter over \( \mu \), as a local version of the measurability of \( \mu \), and just as measurability itself, can be expressed in equivalent forms in a varied sets of mathematical frameworks. Indeed, the mentioned topological, model theoretical, etc., characterizations of the existence of a \( \lambda \)-indecomposable uniform ultrafilter over \( \mu \) can be easily generalized in order to provide, for every set \( \Lambda \) of cardinals, characterizations of the existence of a uniform ultrafilter over \( \mu \) which for no \( \lambda \in \Lambda \) is \( \lambda \)-decomposable. When \( \Lambda \) is the set of all cardinals \( < \mu \) one usually recovers exactly the conditions from [KT63/64].

1.2. Weak compactness with and without inaccessibility. The mentioned paper by Keisler and Tarski contains also many topological, model-theoretical, etc., equivalent formulations of all the large cardinals known at that time, in particular, also of weak compactness. It is not apparent how to get "local" versions of such results in a sense parallel to the above local versions of measurability. In this case, the situation is made even more difficult by the fact that there are many definitions of weak compactness which turn out to be equivalent only under the
assumption that the cardinals at hand are inaccessible (by *inaccessible* we shall always mean, as nowadays usual, *strongly inaccessible*). Though conventional wisdom has suggested that inaccessibility should be included right in the definition of weak compactness, we believe that in such a way the richness of many interesting phenomena gets lost.

We refer, for example, to earlier topological and measure-theoretical studies by Mrówka [Mró66, Mró70], followed by Čudnovskij [Čud72], or to a homological theorem by A. Mekler mentioned in Eklof [Ekl77, Theorem 1.6]. All these results are partially trivialized by the assumption of inaccessibility. Also the *tree property* is equivalent to weak compactness only under the assumption of inaccessibility, and even a successor cardinal may satisfy it, Mitchell [Mit73]. Variants of the tree property have received a notable attention in recent years. See, e. g., Fontanella [Fon13], Viale and Weiß [VW11], Weiß [Wei12] and further references there. However the properties we are considering here differ from the tree property and from its variants in that they do imply weak inaccessibility. That strong inaccessibility is effectively necessary for the equivalence of most variants of weak compactness has been verified by Boos [Boo76], who constructed models in which many such variants do not imply (strong) inaccessibility. At the same time, in [Boo76, Theorem 2.4] Boos also finds further equivalent conditions for weak compactness without inaccessibility.

Subsequently in [CS] we showed that weak compactness without inaccessibility deeply affects the study of extended logics. In the present note logics will enter the scene mostly as examples. Roughly, the reader might think of a *logic* as an extension of first order logic which satisfies all the properties common both to infinitary logics and to logics with added quantifiers, e. g., the quantifier $Q_{\alpha}$ saying “there are $\aleph_{\alpha}$ many”. An extensive review of the subject can be found in [BF85]; a concise survey of what happened in the last years can be found in [She12]. Exactly in the same way as weakly compact cardinals can be defined as cardinals for which the corresponding infinitary logic satisfies compactness, we showed in [CS] that, for every logic $\mathcal{N}$, the first cardinal $\kappa$ such that $\mathcal{N}$ is $(\kappa, \kappa)$-compact is weakly compact in the weaker sense that $L_{\kappa, \omega}$ is $(\kappa, \kappa)$-compact. In case $\kappa > \omega$ is inaccessible, this is one among the many possible definitions of weak compactness. Thus a cardinal $\kappa$ is weakly compact (in the above weaker sense, and including $\omega$) if and only if it is the first cardinal such that some logic is $(\kappa, \kappa)$-compact. Recall that $L_{\kappa, \nu}$ is like first order logic, except that conjunctions and disjunctions of $< \kappa$ sentences are allowed, as well as simultaneous universal or existential quantification over sets of $< \nu$ variables. A logic is $(\kappa, \kappa)$-compact if every $\kappa$-satisfiable set $\{\sigma_\alpha \mid \alpha \in \kappa\}$ of $\kappa$ sentences is satisfiable. A set is $\kappa$-satisfiable if every subset of cardinality $< \kappa$ has a model. The results from [CS] not only support the conviction that weak compactness
without inaccessibility is interesting and deserves further study, but also show that, say, the Betelgeusians would have arrived at the very same notions of weak and strong compactness as ours, even had they started by considering entirely different logics. E.g., the first weakly compact (in the above weaker sense) cardinal \( \kappa \) is the first cardinal such that \( \mathcal{L}(Q_0) \) is \((\kappa, \kappa)-\)compact.

1.3. **Local versions of weak compactness.** As we mentioned, it is not apparent how to introduce local versions of weak compactness analogue to the local versions of measurability described in Subsection [1.1] since weak compactness cannot be defined by using a single ultrafilter. We originally devised a model-theoretical approach to the problem in [CS], again motivated by extended logics. However, subsequently we found an ostensibly simpler method which uses the following observation, bearing some similarity with Mrówka ideas from [Mró66, Mró70]. Clearly, an ultrafilter \( D \) over \( \mu \) is \( \mu \)-complete if and only if whenever we partition \( \mu \) into \( < \mu \) classes, one of these classes is in \( D \). Hence checking \( \mu \)-completeness of some ultrafilter over \( \mu \) amounts to check it for all the \( 2^\mu \) partitions of \( \mu \) into \( < \mu \) classes. We can pick a subset \( P \) of all such partitions and only ask that there is some ultrafilter \( D \) such that none of the above partitions is a witness for the \( \mu \)-incompleteness of \( D \) (the \( \mu \)-incompleteness of \( D \) might be or might be not disproved by some partition outside \( P \)).

Taking \( P \) to be the set of all the above partitions gives back the notion of measurability, while making \( P \) vary among sets of cardinality \( \mu \) gives an equivalent formulation of weak compactness. If GCH fails at \( \mu \), we can consider all \( P \)'s of cardinality \( \nu \), for \( \nu \) strictly between \( \mu \) and \( 2^\mu \). This provides weak versions of measurability and Schanker [Sch11], using an equivalent formulation, recently proved that there are models in which such intermediate notions are actually distinct both from measurability and from weak compactness. See also [CGHS].

Pursuing further the idea that \( \mu \)-completeness is “composed” of pieces of \( \lambda \)-indecomposability for \( \lambda < \mu \), as described in Subsection [1.1], and using the observations in the above paragraph, we can introduce a form of “local weak compactness at \( \lambda \)” by asking that \( \kappa \) many partitions of \( \mu \) are not enough to witness that every uniform ultrafilter over \( \mu \) is \( \lambda \)-decomposable. Let us denoted by \( \mu \not\Rightarrow^\kappa \lambda \) the above statement, intended to mean that it is not the case that “uniformity on \( \mu \) implies (by means of \( \kappa \) many partitions) \( \lambda \)-decomposability”. Correspondingly, the above statement between quotes will be denoted by \( \mu \Rightarrow^\kappa \lambda \). Notice that it might happen that there is indeed a \( \lambda \)-indecomposable uniform ultrafilter over \( \mu \) (a local version of measurability at \( \lambda \)), in which case \( \mu \not\Rightarrow^\kappa \lambda \) for every \( \kappa \). On the other hand, it is possible that every uniform ultrafilter over \( \mu \) is \( \lambda \)-decomposable (“local measurability” at \( \lambda \) fails) but perhaps all of the \( 2^\mu \) partitions of \( \mu \) into \( \lambda \) pieces are necessary to witness this (or, at
least, $2^\mu$ many partitions are needed). In this case $\mu \Rightarrow 2^\mu \lambda$, but $\mu \neq^\kappa \lambda$ for every $\kappa < 2^\mu$. We also may have intermediate cases. We interpret $\mu \neq^\mu \lambda$ as a local version of weak compactness at $\lambda$, while if $\kappa$ grows larger in $\mu \Rightarrow^\kappa \lambda$ we go closer and closer to (local versions) of measurability. We sometimes find it convenient to work with functions rather than with partitions (cf. the two equivalent definitions of $\lambda$-decomposability given at the beginning). The definition of $\mu \Rightarrow^\lambda \kappa$ by means of functions is given in Definition 2.1; in Lemma 3.3(2) it is proved equivalent to the definition in terms of partitions sketched above.

In the above discussion we can consider a set $\Lambda$ of cardinals in place of $\lambda$, and introduce a similar principle $\mu \Rightarrow^\Lambda \kappa$ which says that $\lambda$-decomposability is witnessed for at least one $\lambda \in \Lambda$ (Definition 2.3). When we take $\Lambda = \text{Card}_{<\mu}$, the set of all infinite cardinals $< \mu$, then $\mu \neq^\mu \Lambda$ corresponds exactly to the measurability of $\mu$, while $\mu \neq^\mu \Lambda$ corresponds to weak compactness of $\mu$ (provided we assume either that $\mu$ is inaccessible, or that we are dealing with weaker versions of weak compactness, as described in the previous subsection). The situation is represented in the following table, where we write $C_{<\lambda}$ in place of $\text{Card}_{<\lambda}$ to save space.

| LOCAL VERSIONS | (GOING MORE AND MORE GLOBAL) | GLOBAL VERSIONS |
|----------------|-----------------------------|-----------------|
| (a $\lambda$-indecomp. ultrafilter over $\mu$) $\mu \neq^\mu \lambda$ | $\lambda$ getting larger (going closer to meas.) $\Rightarrow^\kappa \lambda$ | $\mu \Rightarrow^\kappa \Lambda$ (going closer to measurability) $\Rightarrow^\kappa \lambda$ |
| $\kappa$ increasing (going closer to local measurability at $\lambda$) | (mu weakly measurable) $\Rightarrow \mu \neq^\mu \lambda$ (mu locally weakly compact at $\lambda$) | (mu weakly compact if inaccessible) $\Rightarrow \mu \neq^\mu \lambda$ |
| (mu weakly compact at $\lambda$) $\mu \neq^\mu \lambda$ | (going closer to weak compactness) $\Rightarrow \mu \neq^\mu \lambda$ (going more and more global) | $\mu \neq^\mu \Lambda$ (mu weakly compact if inaccessible) |

1.4. Topological and model-theoretical equivalences. There are pleasant topological characterizations of the relations $\mu \Rightarrow^\kappa \lambda$ and $\mu \Rightarrow^\kappa \Lambda$. A topological space $X$ is $\mu$-compact if every subset of cardinality $\mu$ has a complete accumulation point. Clearly, if $|X| < \mu$ then $X$ is vacuously $\mu$-compact. On the other hand, if $X$ is a product of cardinals and $|X| \geq \mu$, then $\mu$-compactness of $X$ depends on the relations introduced above.

For example, if $\lambda$ is regular and $|\lambda^\kappa| \geq \mu$ then $\mu \neq^\kappa \lambda$ if and only if $\lambda^\kappa$ is $\mu$-compact. Here $\lambda$ is considered as a topological space endowed with the order topology, a base of which consists of all open intervals, including intervals of the form $[0, \alpha)$. Powers and products are always endowed with the Tychonoff topology, the coarser topology which makes the projections continuous.
More generally, under similar cardinality and regularity assumptions, \( \mu \not\Rightarrow^\kappa \Lambda \) if and only if \( \prod_{\lambda \in \Lambda} \lambda^\kappa \) is \( \mu \)-compact. We can draw again a diagram.

\[
\mu^\kappa \not\Rightarrow \Lambda \quad \text{(all } \lambda \text{'s regular, } \kappa \geq \omega \text{ and all spaces of cardinality } \geq \mu) \]

| Powers of one factor | (More choices for the factors) | Products of many factors |
|----------------------|--------------------------------|--------------------------|
| (all powers of } \lambda \text{ } \mu \)-compact) | \( \mu^\kappa \not\Rightarrow \lambda \quad \text{getting larger} \rightarrow \mu \not\Rightarrow \Lambda \) | (all products of } \mu \text{ } \mu \)-compact) |
| \( \kappa \text{ increasing} \) | \( \mu \not\Rightarrow \lambda \) | \( \mu \not\Rightarrow \Lambda \) |
| \( \lambda^\kappa \mu \)-compact) | \( \mu \not\Rightarrow \lambda \) | \( \mu \not\Rightarrow \Lambda \) (\( \prod_{\lambda \in \Lambda} \lambda^\kappa \mu \)-compact) |

Model theoretical equivalents are dealt with in Section 5.

1.5. Note. This is a preliminary version. More results and observations are planned to be added in the future. Results related to the present work have been presented, proved or announced in the previously quoted papers and, e. g., in [EL, CF, SC], sometimes in equivalent formulations, or with slightly different notations.

§2. Basic definitions. Our notation is standard and, if not mentioned otherwise, follows [Jec78/03]. Throughout, \( \alpha, \beta, \gamma \) are ordinals, \( \lambda, \mu, \nu, \xi \) are infinite cardinals, \( \kappa \) and \( \theta \) are nonzero cardinals, \( J \) is a nonempty set and \( \Lambda, \Upsilon \) are nonempty sets of infinite cardinals. When there is no risk of ambiguity we shall write, say, \( \lambda \geq \mu, \nu \) as a shorthand for \( \lambda \geq \sup \{ \mu, \nu \} \).

Card and Reg denote, respectively, the class of all infinite cardinals and of all infinite regular cardinals. Card\( [\lambda, \mu] \) is \( \{ \nu \mid \lambda \leq \nu \leq \mu \} \), and Card\( (\lambda, \mu) \), Card\( [\lambda, \mu] \) have a similar meaning. We write Card\( [\lambda, \mu] \) in place of simply writing \( [\lambda, \mu] \) both in order to avoid possible confusion with other notions (e. g., compactness of topological spaces or of logics) and to make clear that members of Card\( [\lambda, \mu] \) are cardinals, rather than, say, ordinals. Reg\( [\lambda, \mu] = Card[\lambda, \mu] \cap Reg, \) and similarly for Reg\( [\lambda, \mu] \). We sometimes write Card<\( \mu \) in place of Card\( [\omega, \mu] \) and Reg<\( \mu \) in place of Reg\( [\omega, \mu] \).

Definition 2.1. We denote by \( \mu \Rightarrow (\lambda_j)_{j \in J} \) the following statement.

(*) There is a sequence \( (f_j)_{j \in J} \) of functions \( f_j : \mu \rightarrow \lambda_j \) for \( j \in J \), such that for every uniform ultrafilter \( D \) over \( \mu \) there is \( j \in J \) such that \( f_j(D) \) is uniform over \( \lambda_j \).
Every uniform ultrafilter over \(\mu\) is \(\mu\)-decomposable, for some \(\lambda \in \Lambda\).

We write \(\mu \Rightarrow \lambda\) when \(|J| = \kappa\) and all the \(\lambda_j\)'s in \((*)\) are equal to \(\lambda\).

The negations of the above principles shall be denoted by \(\mu \not\Rightarrow (\lambda_j)_{j \in J}\) and \(\mu \not\Rightarrow ^\kappa \lambda\) respectively.

Trivial facts about \(\mu \Rightarrow (\lambda_j)_{j \in J}\) are that the notion is invariant under a permutation of the indices, and that it is preserved by taking subsequences. By this we mean that if \(J' \subseteq J\) and the subsequence \((f_j)_{j \in J'}\) witnesses \(\mu \Rightarrow (\lambda_j)_{j \in J'}\), then the sequence \((f_j)_{j \in J}\) witnesses \(\mu \Rightarrow (\lambda_j)_{j \in J}\).

Moreover, notice that if \(|f_j(\mu)| < \lambda_j\) then \(f_j(D)\) is not uniform over \(\lambda_j\); in particular, if \(\mu < \lambda_j\) then \(f_j(D)\) is not uniform over \(\lambda_j\). From this we get the following facts.

**Fact 2.2.** (1) The sequence \((f_j)_{j \in J}\) witnesses \(\mu \Rightarrow (\lambda_j)_{j \in J}\) if and only if the subsequence \((f_j)_{j \in J'}\) witnesses \(\mu \Rightarrow (\lambda_j)_{j \in J'}\), where \(J' = \{j \in J \mid |f_j(\mu)| = \lambda_j\}\).

(2) In particular, \(\mu \Rightarrow (\lambda_j)_{j \in J}\) if and only if \(\mu \Rightarrow (\lambda_j)_{j \in J''}\), where \(J'' = \{j \in J \mid \mu \geq \lambda_j\}\).

In view of Fact 2.2(2) it is no loss of generality if in Definition 2.1 we assume that \(\mu \geq \lambda_j\), for every \(j \in J\). On the other hand, if some \(\lambda_j\) equals \(\mu\), then \(\mu \Rightarrow (\lambda_j)_{j \in J}\) is trivially true, as witnessed by taking \(f_j\) to be the identity function, or just any injective function. In conclusion, the principle \(\mu \Rightarrow (\lambda_j)_{j \in J}\) is interesting only when \(\mu > \lambda_j\), for every \(j \in J\).

We are soon going to show that a cardinal is measurable if and only if \(\mu \not\Rightarrow (\lambda_j)_{j \in J}\), where each cardinal \(\lambda < \mu\) appears \(2^\mu\) times in the sequence \((\lambda_j)_{j \in J}\). The above observation is better proved after (and justifies) the introduction of some further more compact notation.

**Definition 2.3.** If \(\Lambda\) is a set of infinite cardinals, we write \(\mu \Rightarrow ^\kappa \Lambda\) if \(\mu \Rightarrow (\lambda_j)_{j \in J}\) holds in case each \(\lambda \in \Lambda\) appears exactly \(\kappa\) times in the sequence \((\lambda_j)_{j \in J}\).

More formally, if \(\Lambda = \{\lambda_h \mid h \in H\}\) then \(\mu \Rightarrow ^\kappa \Lambda\) means \(\mu \Rightarrow (\lambda_j)_{j \in J}\), where \(J = H \times \kappa\) and \(\lambda_j = \lambda_h\), whenever \(j = (h, \gamma), h \in H, \gamma \in \kappa\).

Notice that \(\mu \Rightarrow ^\kappa \{\lambda\}\) is the same as \(\mu \Rightarrow ^\kappa \lambda\).

By convention, \(\mu \Rightarrow ^\kappa \emptyset\) is always considered to be false (this will occur infrequently and only as the basis of some induction).

Trivially, as above, if \(\mu \Rightarrow ^\kappa \Lambda, \theta \geq \kappa\) and \(\Psi \supseteq \Lambda\) then \(\mu \Rightarrow ^\theta \Psi\).

**Fact 2.4.** The following conditions are equivalent.

1. \(\mu \Rightarrow 2^\mu \Lambda\).
2. \(\mu \Rightarrow ^\kappa \Lambda\), for some (equivalently, all) \(\kappa \geq 2^\mu\).
3. Every uniform ultrafilter over \(\mu\) is \(\lambda\)-decomposable, for some \(\lambda \in \Lambda\).
4. Every \(\mu\)-decomposable ultrafilter (over any set) is \(\lambda\)-decomposable, for some \(\lambda \in \Lambda\).
Proof. By Fact 2.2(2), without loss of generality $\mu \geq \lambda$, for every \(\lambda \in \Lambda\). Then the equivalence of (1)-(3) is trivial from the definitions, since for every $\lambda \in \Lambda$ there are exactly $\lambda^\mu = 2^\mu$ functions from $\mu$ to $\lambda$.

(4) $\Rightarrow$ (3) is trivial.

If (3) holds, and $D$ is a $\mu$-decomposable ultrafilter over, say, $I$, then there is some function $g : I \to \mu$ such that $g(D)$ is uniform over $\mu$. By (3) $g(D)$ is $\lambda$-decomposable, for some $\lambda \in \Lambda$. This is witnessed by some $f : \mu \to \lambda$, thus $g \circ f$ witnesses the $\lambda$-decomposability of $D$. ⊥

By Fact 2.4, the only interesting cases in $\mu \Rightarrow \kappa \Lambda$ are when $\kappa \leq 2^\mu$. As $\kappa$ grows larger in $\mu \Rightarrow \kappa \Lambda$, we get a weaker notion, but at the point $\kappa = 2^\mu$ we already get the minimum strength.

The equivalence of (1) and (2) in Fact 2.4 justifies the notation $\mu \Rightarrow^\infty \Lambda$ used in in [SC, Section 6] in place of $\mu \Rightarrow^{2^\mu} \Lambda$. In [SC, Theorem 6.3 and Corollary 6.6] we have listed many results about $\mu \Rightarrow^{2^\mu} \Lambda$. All these results can be appropriately generalized to the relation $\mu \Rightarrow \kappa \Lambda$, but in a few cases we have not yet written down full details.

Notice also that, as discussed in [MR, p. 343] in a parallel situation, while perhaps Condition (3) in 2.4 might appear simpler and more intuitive than Condition (4), the latter is very useful. For example, using (4) one immediately gets that $\mu \Rightarrow^{2^\mu} \lambda$ and $\lambda \Rightarrow^{2^\nu} \nu$ imply $\mu \Rightarrow^{2^\mu} \nu$. This is not immediately obvious using (3). The above observation shall be expanded in Proposition 2.6.

Remarks 2.5. (1) It is elementary to see that the two definitions of $\lambda$-decomposability given in the introduction are equivalent. Indeed, if $\lambda$-decomposability is witnessed by some partition, then any enumeration of its classes produces a function with the desired properties. Conversely, any function witnessing $\lambda$-decomposability naturally gives rise to a partition, which satisfies the corresponding conditions.

Similarly, we shall show in Lemma 3.2 that $\mu \Rightarrow (\lambda_j)_{j \in J}$ can be reformulated in terms of partitions.

(2) As mentioned in the introduction, a cardinal $\mu$ is measurable if and only if there is a uniform ultrafilter $D$ over $\mu$ which is $\lambda$-indecomposable, for every $\lambda < \mu$. Indeed, a $\mu$-complete ultrafilter is trivially $\lambda$-indecomposable for every $\lambda < \mu$. For the other direction, and in contrapositive form, if $D$ over $I$ is not $\mu$-complete, then, using the maximality of $D$, there is a partition of $I$ into $< \mu$ sets such that no member of the partition is in $D$. If we take such a partition of minimal cardinality $\lambda$, then no union of $< \lambda$ classes of the partition is in $D$, thus $D$ is $\lambda$-decomposable.

(3) By (2) and as a particular case of Fact 2.4, a cardinal $\mu$ is measurable if and only if $\mu \not\Rightarrow^{2^\mu} \text{Card}_{<\mu}$ where $\text{Card}_{<\mu}$ denotes the set of all infinite cardinals $< \mu$. 

The just introduced arrow notions satisfy some trivial but very useful transitivity properties. The idea is similar to the proof of the equivalence of (3) and (4) in Fact 2.4. In the statement of the next proposition we shall assume that all the sets under consideration are nonempty.

**Proposition 2.6.** If μ ⇒ (λ_j)_{j ∈ J} and λ_j ⇒ (v_{j,h})_{h ∈ H_j} for every j ∈ J, then μ ⇒ (v_{j,h})_{j ∈ J, h ∈ H_j}.

(2) If μ ⇒^κ λ and λ ⇒^θ Υ then μ ⇒^κ-θ Υ

(3) If μ ⇒^κ Λ, κ ≥ ω, |Λ| and λ ⇒^κ Υ_λ for every λ ∈ Λ, then μ ⇒^κ \bigcup_{λ ∈ Λ} Υ_λ

(4) If κ is infinite, μ ⇒^κ Λ, v ∈ Λ and v ⇒^κ Λ \setminus \{v\} then μ ⇒^κ Λ \setminus \{v\}.

(5) More generally, if κ ≥ ω, |Υ|, μ ⇒^κ Λ, Υ ⊆ Λ and v ⇒^κ Λ \setminus Υ for every v ∈ Υ, then μ ⇒^κ Λ \setminus Υ.

**Proof.** (1) Just consider the compositions of the functions given by μ ⇒ (λ_j)_{j ∈ J} and λ_j ⇒ (v_{j,h})_{h ∈ H_j}.

(2) and (3) are immediate from (1).

(4) Apply (1) by using the trivial relation λ ⇒^κ λ for every λ ≠ v. That is, write μ ⇒^κ Λ as μ ⇒ (λ_j)_{j ∈ J} and, for λ_j = λ ≠ v, take H_j a singleton \{h_j\} and v_{j,h_j} = λ_j.

(5) Same as (4), by using λ ⇒^κ λ for every λ ∉ Υ.

**Theorem 2.7.** (1) μ ⇒^1 cf μ.

(2) If μ is regular then μ^+ ⇒^μ^+ μ.

(3) More generally, if μ is regular then μ^+n ⇒^μ^+n μ.

(4) If λ is regular and cf μ = λ^+n then μ ⇒^cf^μ λ.

(5) If μ is singular then μ^+ ⇒^μ^+ \{cf μ\} ∪ Λ, for every Λ ⊆ Reg_≤μ cofinal in μ.

**Proof.** (1) If μ is regular, this is trivial. If μ is singular, let (μ_β)_{β ∈ cf μ} be an increasing cofinal sequence in μ, and consider f : μ → cf μ defined by f(α) = inf{β ∈ cf μ | α ≠ μ_β}.

See [CS, UT] for proofs of (2) and (5), though with slightly different notations.

(3) follows from Proposition 2.6(2) and iterated applications of (2). Similarly, (4) follows from (1), (3) and Proposition 2.6(2).

More properties of μ ⇒^κ λ can be obtained from Theorem 2.7 and Proposition 2.6, for example combining 2.7(3) and (5).

**Corollary 2.8.** Suppose that μ ⇒^κ Λ and let Λ' = \{λ ∈ Λ | λ ≠^κ Λ ∩ Card_≤Λ\}. (1) If κ ≥ ω, |Λ \setminus Λ'| then μ ⇒^κ Λ'. Moreover, (2) if λ ∈ Λ' then (a) if λ is singular then cf λ ∉ Λ; (b) if λ = ξ^+ and ξ is regular, then ξ ∉ Λ. (c) if λ = ξ^+ and ξ is singular then either cf ξ ∉ Λ or Card(ξ', ξ) ∩ Λ = ∅, for some ξ' < ξ.
Proof. (1) By Fact 2.2(2) without loss of generality \( \mu \geq \sup \Lambda \).

We first prove (1) under the additional assumption that \( \mu \notin \Lambda \). Fix any \( \kappa \geq \omega \), suppose by contradiction that (1) fails with respect to that \( \kappa \), and consider a counterexample in which \( \mu \) is of minimal cardinality. Thus \( \mu \not\Rightarrow^\kappa \Lambda, \mu \not\Rightarrow^\kappa \Lambda' \) and \( \kappa \geq |\Lambda \setminus \Lambda'| \). Let \( \Upsilon = \Lambda \setminus \Lambda' = \{ \lambda \in \Lambda \mid \lambda \not\Rightarrow^\kappa \Lambda \cap \text{Card}_{<\lambda} \} \), thus \( \Lambda' = \Lambda \setminus \Upsilon \).

Suppose that \( \upsilon \in \Upsilon \), hence \( \upsilon < \mu \), since \( \mu \notin \Lambda \). By definition, \( \upsilon \Rightarrow^\kappa \Lambda \cap \text{Card}_{<\upsilon} \). Let \( \Lambda_1 = \Lambda \cap \text{Card}_{<\upsilon} \) and consider the statement of the corollary with \( \upsilon \) in place of \( \mu \) and \( \Lambda_1 \) in place of \( \Lambda \). Clearly, again by Fact 2.2(2), \( \Lambda_1' = \Lambda' \cap \text{Card}_{<\upsilon} \), thus \( |\Lambda_1 \setminus \Lambda_1'| \leq |\Lambda \setminus \Lambda'| \leq \kappa \). By the minimality of \( \mu \), and since \( \upsilon \notin \Lambda_1 \), we can apply (1) thus getting \( \upsilon \Rightarrow^\kappa \Lambda_1' \). Since \( \Lambda_1' \subseteq \Lambda' = \Lambda \setminus \Upsilon \), we get \( \upsilon \Rightarrow^\kappa \Lambda \setminus \Upsilon \).

In the previous paragraph we have proved that if \( \upsilon \in \Upsilon \) then \( \upsilon \Rightarrow^\kappa \Lambda \setminus \Upsilon \). Since \( \mu \Rightarrow^\kappa \Lambda \), by Proposition 2.6(5) we get \( \upsilon \Rightarrow^\kappa \Lambda \setminus \Upsilon \), a contradiction, since \( \Lambda \setminus \Upsilon = \Lambda' \). We have proved (1) under the assumption that \( \mu \notin \Lambda \).

Now suppose that \( \mu \in \Lambda \). If \( \mu \in \Lambda' \) then trivially \( \mu \Rightarrow^\kappa \Lambda' \). Otherwise \( \mu \notin \Lambda' \), then, by definition, \( \mu \Rightarrow^\kappa \Lambda \cap \text{Card}_{<\mu} \). Letting \( \Lambda_2 = \Lambda \cap \text{Card}_{<\mu} \), and since \( \mu \notin \Lambda_2 \), we can apply the already proved particular case, getting \( \mu \Rightarrow^\kappa \Lambda_2' \). But \( \Lambda_2' \subseteq \Lambda' \), thus \( \mu \Rightarrow^\kappa \Lambda' \). The proof of (1) is complete.

(2) is immediate from Theorem 2.7. \( \square \)

§3. Filters, partitions. The principle \( \mu \Rightarrow (\lambda_j)_{j \in J} \) admits a characterization in terms of filters (not necessarily ultra). By a filter we shall always mean a proper filter, that is \( \emptyset \notin F \). If \( F \) is a filter over some set \( I \) and \( f : I \to H \) is a function, we denote by \( f(F) \) the filter over \( H \) defined by \( Y \in f(F) \) if and only if \( f^{-1}(Y) \in F \). This extends the notation introduced for ultrafilters and is connected with the Katetov order, as we shall briefly discuss in Section 3. For every cardinal \( \mu \), we shall denote by \( F_\mu \) the filter consisting of all subsets \( A \) of \( \mu \) such that \( |\mu \setminus A| < \mu \). A filter \( F \) over \( \mu \) is uniform if all members of \( F \) have cardinality \( \mu \). We say that \( F \) is strongly uniform if \( F \supseteq F_\mu \). Equivalently, \( F \) is strongly uniform if and only if every filter extending \( F \) is uniform (since we are taking into account only proper filters). Notice that an ultrafilter over \( \mu \) is uniform if and only if it is strongly uniform. All the above definitions and properties hold also when dealing with a field of subsets of \( \mu \) containing \( F_\mu \).

Proposition 3.1. For every sequence of functions \( f_j : \mu \to \lambda_j \ (j \in J) \), the following conditions are equivalent.

1. The \( f_j \)'s witness \( \mu \Rightarrow (\lambda_j)_{j \in J} \), that is, for every uniform ultrafilter \( D \) over \( \mu \) there is \( j \in J \) such that \( f_j(D) \) is uniform over \( \lambda_j \).
2. For every strongly uniform filter \( F \) over \( \mu \) there is \( j \in J \) such that \( f_j(F) \) is uniform over \( \lambda_j \).
(3) For every family \((B_j)_{j \in J}\) such that \(B_j \subseteq \lambda_j\) and \(|B_j| < \lambda_j\), for \(j \in J\), there is a finite set \(N \subseteq J\) such that \(|\bigcap_{j \in N} f_j^{-1}(B_j)| < \mu\).

**Proof.** (2) \(\Rightarrow\) (1) is trivial.

(1) \(\Rightarrow\) (2) Suppose that (1) holds, and that \(F\) is a strongly uniform filter over \(\mu\). Extend \(F\) to an ultrafilter \(D\); thus \(D\) is uniform over \(\mu\), hence by (1) \(f_j(D)\) is uniform over \(\lambda_j\), for some \(j \in J\). Since \(f_j(F) \subseteq f_j(D)\), then also \(f_j(F)\) is uniform over \(\lambda_j\). Of course here we are heavily using the Axiom of Choice, at least in its weaker incarnation as the Prime Ideal Theorem.

(2) \(\Leftrightarrow\) (3) We shall prove the equivalence of the negations. The negation of (2) means that there is a strongly uniform filter \(F\) over \(\mu\) such that for every \(j \in J\) the filter \(f_j(F)\) is not uniform over \(\lambda_j\), that is, there is \(B_j \in f_j(F)\) such that \(|B_j| < \lambda_j\). A filter \(F\) as in the previous sentence exists if and only if there is a family \((B_j)_{j \in J}\) such that \(B_j \subseteq \lambda_j\), \(|B_j| < \lambda_j\), for \(j \in J\), and \(\{f_j^{-1}(B_j) \mid j \in J\} \cup F\mu\) has the finite intersection property. This is exactly the negation of (3).

We can also equivalently state \(\mu \Rightarrow (\lambda_j)_{j \in J}\) in an “internal way on \(\mu\)” by using partitions. In view of Fact 2.3 this generalizes Remark 2.5(1).

**Lemma 3.2.** Suppose that \(\mu \geq \lambda, \lambda_j\), for \(j \in J\).

1. \(\mu \Rightarrow (\lambda_j)_{j \in J}\) if and only if: there is a sequence \((\pi_j)_{j \in J}\) of partitions of \(\mu\) such that each \(\pi_j\) has \(\lambda_j\) classes and, for every uniform ultrafilter \(D\) over \(\mu\), there is \(j \in J\) such that no union of \(< \lambda_j\) classes of \(\pi_j\) belongs to \(D\).

2. In particular, \(\mu \Rightarrow^\kappa \lambda\) if and only if there is a sequence \((\pi_\gamma)_{\gamma \in \kappa}\) of partitions of \(\mu\) into \(\lambda\) classes such that, for every uniform ultrafilter \(D\) over \(\mu\), there is \(\gamma \in \kappa\) such that no union of \(< \lambda\) classes of \(\pi_\gamma\) belongs to \(D\).

**Proof.** (1) Let \(\mu \Rightarrow (\lambda_j)_{j \in J}\) be witnessed by \((f_j)_{j \in J}\). By Fact 2.2 then \((f_j)_{j \in J')\) witnesses \(\mu \Rightarrow (\lambda_j)_{j \in J'}\), where \(J' = \{j \in J \mid |f_j(\mu)| = \lambda_j\}\). To each \(f_j\) \((j \in J')\) there is naturally associated a partition \(\pi_j\) of \(\mu\) into \(\lambda_j\) classes in such a way that the sufficient condition is satisfied with \(J'\) in place of \(J\). Letting \(\pi_j\) be arbitrary with \(|\lambda_j|\) many classes for \(j \in J \setminus J'\) we get the condition.

Conversely, given \((\pi_j)_{j \in J}\) partitions as in the sufficient condition, enumerate the classes of each \(\pi_j\) and consider the corresponding functions \(f_j: \mu \to \lambda_j\). These witness \(\mu \Rightarrow (\lambda_j)_{j \in J}\).

(2) follows trivially from the definition of \(\mu \Rightarrow^\kappa \lambda\).
"inside $\mu$", as we shall do in the next propositions. Of course, this remark is nothing but a variant on the old story of viewing homomorphic images as corresponding to equivalence relations.

For $(\pi_j)_{j \in J}$ a sequence of partitions of $\mu$ such that each $\pi_j$ has $\lambda_j$ classes, let $\mathcal{F}_\mu(\pi_j)_{j \in J}$ be the smallest field of subsets of $\mu$ which contains $F_\mu$ and which, for every $j \in J$, contains all unions of $< \lambda_j$ classes of $\pi_j$.

**Proposition 3.3.** Suppose that $(\pi_j)_{j \in J}$ is a sequence of partitions of $\mu$ such that each $\pi_j$ has $\lambda_j$ classes. Then the following conditions are equivalent.

1. The $\pi_j$’s witness $\mu \Rightarrow (\lambda_j)_{j \in J}$ that is for every uniform ultrafilter $D$ over $\mu$ there is $j \in J$ such that no union of $< \lambda_j$ classes of $\pi_j$ belongs to $D$.
2. For every strongly uniform filter $F$ over $\mu$ there is $j \in J$ such that no union of $< \lambda_j$ classes of $\pi_j$ belongs to $F$.
3. For every choice of subsets $A_j$ of $\mu$, one for each $j \in J$, such that each $A_j$ is a union of $< \lambda_j$ classes of $\pi_j$, there is a finite set $N \subseteq J$ such that $\bigcap_{j \in N} A_j < \mu$.
4. For every strongly uniform filter $F$ over $\mathcal{F}_\mu(\pi_j)_{j \in J}$ there is $j \in J$ such that no union of $< \lambda_j$ classes of $\pi_j$ belongs to $F$.
5. For every uniform ultrafilter $D$ over $\mathcal{F}_\mu(\pi_j)_{j \in J}$ there is $j \in J$ such that no union of $< \lambda_j$ classes of $\pi_j$ belongs to $D$.

**Proof.** The equivalences of (1)-(3) and of (4)-(5) are entirely similar to the proof of Proposition 3.1 (the assumption that $\mathcal{F}_\mu(\pi_j)_{j \in J} \supseteq F_\mu$ is used in (4) $\iff$ (5)).

Then notice that (2) and (4) are equivalent since both conditions are actually evaluated in $\mathcal{F}_\mu(\pi_j)_{j \in J}$, that is, a filter $F$ over $\mu$ satisfies (2) if and only if $F \cap \mathcal{F}_\mu(\pi_j)_{j \in J}$ satisfies (4).

§4. Topological equivalents. If $X$ is a topological space and $Y$ is an infinite subset of $X$, a point $x \in X$ is a complete accumulation point of $Y$ if $|Y \cap U| = |Y|$, for every neighborhood $U$ of $x$. The space $X$ is $\mu$-compact if every subset of cardinality $\mu$ has a complete accumulation point. In the literature $\mu$-compactness has also been given various other disparate names, such as $\text{CAP}_\mu$, $C(\mu, \mu)$, $[\mu, \mu]$-compactness in the sense of accumulation points, etc.

It is convenient to introduce also a slight modification dealing with sequences rather than subsets. If $(x_\alpha)_{\alpha \in \mu}$ is a sequence (possibly with repetitions) of elements of $X$, a point $x \in X$ is a $\mu$-complete accumulation point of $Y$ if $\{|\alpha \in \mu \mid x_\alpha \in U\} = \mu$, for every neighborhood $U$ of $x$. The space $X$ is $\mu^*$-compact if every sequence $(x_\alpha)_{\alpha \in \mu}$ of elements of $X$ has a $\lambda$-complete accumulation point.
The above notions are connected by the following easy proposition, which shows that the compactness notions are distinct only when $\mu$ is singular. See [CM, VI, Proposition 1] or [SC, Proposition 3.3] for details. The latter proposition is stated in a more general framework, the present case is when $\mathcal{F}$ is the set of all singletons of $X$.

**Proposition 4.1.** (1) If $\mu$ is regular then a topological space is $\mu^*$-compact if and only if it is $\mu$-compact.

(2) A topological space is $\mu^*$-compact if and only if it is both $\mu$-compact and cf $\mu$-compact.

**Theorem 4.2.** Suppose that each $\lambda_j$ is a regular cardinal, endowed either with the order topology or with the left order topology. The space $\prod_{j \in J} \lambda_j$ is $\mu^*$-compact if and only if either (a) $|\prod_{j \in J} \lambda_j| < \mu$, or (b) $|\prod_{j \in J} \lambda_j| \geq \mu$ and $\mu \not\prec (\lambda_j)_{j \in J}$.

In particular, if $|\prod_{j \in J} \lambda_j| \geq \mu$, then $\prod_{j \in J} \lambda_j$ is $\mu$-compact if and only if it is $\mu^*$-compact.

**Proof.** If (a) holds then $\prod_{j \in J} \lambda_j$ is vacuously $\mu$-compact.

If (b) holds then $\prod_{j \in J} \lambda_j$ is $\mu^*$-compact by Theorem 4.2, hence $\mu$-compact by Proposition 4.1.

Conversely, suppose that $\prod_{j \in J} \lambda_j$ is $\mu$-compact and $|\prod_{j \in J} \lambda_j| \geq \mu$. Since each $\lambda_j$ occurs infinitely many times in the sequence, then $\prod_{j \in J} \lambda_j$ is homeomorphic to $Y = \prod_{j \in J} \lambda_j \times \prod_{j \in J} \lambda_j$, in particular, $Y$ is $\mu$-compact. Let $(x_\alpha)_{\alpha \in \mu}$ be a sequence of elements of $\prod_{j \in J} \lambda_j$, and let $(y_\alpha)_{\alpha \in \mu}$ be a sequence of distinct elements of $\prod_{j \in J} \lambda_j$. Such a sequence exists since $|\prod_{j \in J} \lambda_j| \geq \mu$. Then all elements of the sequence $(x_\alpha, y_\alpha)_{\alpha \in \mu}$ in $Y$ are distinct. By $\mu$-compactness of $Y$, the sequence has an accumulation point, say $(x, y)$. Then $x$ is a $\lambda$-accumulation point of $(x_\alpha)_{\alpha \in \mu}$ in (the first copy of) $\prod_{j \in J} \lambda_j$. In conclusion, $\prod_{j \in J} \lambda_j$ is $\mu^*$-compact, hence $\mu \not\prec (\lambda_j)_{j \in J}$ by Theorem 4.2.

**Corollary 4.3.** Suppose that $\Lambda$ is a set of regular cardinals and $\kappa$ is infinite. Then $\mu \not\prec^* \Lambda$ if and only if $\prod_{\lambda \in \Lambda} \lambda^\kappa$ is $\mu^*$-compact. If, in
addition, \(|\prod_{\lambda \in \Lambda} \lambda^\kappa| \geq \mu\) then the above conditions hold if and only if \(\prod_{\lambda \in \Lambda} \lambda^\kappa\) is \(\mu\)-compact.

§5. Model-theoretical equivalents. The main idea from [CS] for applying “classical” model theory to extended logics was the introduction of the notion of a \(\mu\)-nonstandard element (said to bound \(\mu\) in [CS]) in models with a fixed order of type \(\mu\)—without loss of generality we can take this order to be \(\mu\) itself. The principle corresponding to \(\mu \Rightarrow^\kappa \lambda\), for \(\mu, \lambda\) regular, was then introduced there (with the same notation when \(\kappa > \mu\) and with \(\kappa\) omitted when \(\kappa = \mu\)) by asserting that every model with a \(\mu\)-nonstandard element has some \(\lambda\)-nonstandard element, modulo a theory of cardinality \(\kappa\).

In details, let \(\mathfrak{A} = \langle \mu, <, \alpha, \ldots \rangle_{\alpha \in \mu}\). If \(\mathfrak{B}\) is elementarily equivalent to \(\mathfrak{A}\), in symbols \(\mathfrak{B} \equiv \mathfrak{A}\), an element \(b\) of \(B\) is \(\mu\)-nonstandard if \(\alpha < b\) holds in \(B\), for every \(\alpha \in \mu\). Of course, in case \(\mu = \omega\) we get the usual notion of a nonstandard element. Compare also [Cha67, pp. 116–118]. Similarly, for \(\lambda < \mu\), an element \(c\) of \(B\) is \(\lambda\)-nonstandard if \(c < \lambda\) and \(\beta < c\) hold in \(B\), for every \(\beta \in \lambda\).

**Theorem 5.1.** If \(\mu\) is regular, \(\Lambda\) is a set of regular cardinals and \(\kappa \geq \mu \geq \sup \Lambda\), then \(\mu \Rightarrow^\kappa \Lambda\) if and only if there is an expansion \(\mathfrak{A}\) of \(\langle \mu, <, \alpha, \ldots \rangle_{\alpha \in \mu}\) with at most \(\kappa\) new symbols (equivalently, symbols and sorts) such that for every \(\mathfrak{B} \equiv \mathfrak{A}\), if \(\mathfrak{B}\) has a \(\mu\)-nonstandard element then \(\mathfrak{B}\) has a \(\lambda\)-nonstandard element, for some \(\lambda \in \Lambda\).

**Proof.** The proof is an immediate generalization of [CP, Theorem 4]. Full details appeared in [CM].

There are possible generalizations of Theorem 5.1 to singular cardinals, but they are involved, quite technical and, so far, appear to be of little practical use. We address the interested reader to [CM].

§6. The trace of Katětov order.

**Definition 6.1.** Suppose that \(F\) is a filter over \(I\) and, for each \(j \in J, G_j\) is a filter over \(H_j\). We write \(F \Rightarrow (G_j)_{j \in J}\) in case the following statement holds.

There is a sequence \((f_j)_{j \in J}\) of functions such that \(f_j : I \to H_j\) for \(j \in J\), and such that for every ultrafilter \(D\) over \(I\) containing \(F\) there is \(j \in J\) such that \(f_j(D)\) contains \(G_j\).

More generally, if \(K \subseteq \mathcal{P}(J)\), we let \(F \Rightarrow_K (G_j)_{j \in J}\) mean the following.

There is a sequence \((f_j)_{j \in J}\) of functions such that \(f_j : I \to H_j\) for \(j \in J\), and such that \(\{j \in J \mid f_j(D) \supseteq G_j\} \in K\) for every ultrafilter \(D\) over \(I\) containing \(F\).
Thus $F \Rightarrow (G_j)_{j \in J}$ is $F \Rightarrow_K (G_j)_{j \in J}$ when $K$ is the set of all one-element subsets of $J$. As in Definition 2.1, $F \Rightarrow G$ denotes the case when $|J| = \kappa$ and the $G_j$’s are all equal to $G$. The negations of the principles are denoted by $F \not\Rightarrow (G_j)_{j \in J}$ and $F \not\Rightarrow^\kappa G$, everything possibly with the subscript $K$ added.

We can also consider variants of the above definitions in which only $\kappa$-complete ultrafilters $D$ are taken into account; we shall denote such modified notions as $F \Rightarrow (G_j)_{j \in J}$ ($\kappa$-complete), $F \Rightarrow^\kappa_G$ ($\kappa$-complete) and so on.

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