The density and pseudo-phase-space density profiles of cold dark matter haloes

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ABSTRACT

Cosmological N-body simulations indicate that the spherically averaged density profiles of cold dark matter (CDM) haloes are accurately described by Einasto profiles, where the logarithmic slope is a power law of adjustable exponent, \( \gamma \equiv d \ln \rho / d \ln r \propto r^\alpha \). The pseudo-phase-space density (PPSD) profiles of CDM haloes also show remarkable regularity, and are well approximated by simple power laws, \( Q(r) \equiv \rho / \sigma^3 \propto r^{-\chi} \). As reported in earlier work, this is because Jeans’ equations imply, for values of \( \alpha \) typical of CDM haloes, that the PPSD profiles of Einasto haloes should resemble power laws over a wide radial range. Significant deviations from a power-law \( Q \) profile are nevertheless expected near the centre of Einasto haloes. Conversely, density profiles must deviate from a simple Einasto form if a power-law \( Q(r) \) profile holds at all radii. We use an ensemble of haloes drawn from the Millennium-II Simulation to study which of these two descriptions describes best the mass profile of CDM haloes. Our analysis indicates that, at the resolution of the best available simulations, both Einasto and power-law PPSD profiles (with adjustable exponents \( \alpha \) and \( \chi \), respectively) provide equally acceptable fits to the simulations. Although we are unable to discriminate between these two models, our results confirm the need for a ‘shape’ parameter, like \( \alpha \) or \( \chi \), to specify fully the mass profile of a CDM halo. Understanding what determines this shape parameter in the case of individual haloes would help us gain insight into what drives departures from self-similarity in CDM halo structure and how they correlate with evolutionary history, environment or initial conditions.

Key words: methods: numerical – galaxies: haloes – dark matter.

1 INTRODUCTION

The large dynamic range probed by current simulations of structure formation allows for robust measurements of the internal structure of large samples of dark matter haloes spanning a wide range of masses. The mass profile of cold dark matter (CDM) haloes holds particular interest, mainly because of its direct connection with key observational probes of halo structure, such as disc galaxy rotation curves, gravitational lensing measurements, and more recently because of the possibility of observing the dark matter directly through its self-annihilation signal, or in laboratory detectors.

Navarro, Frenk & White (1996, 1997, hereafter, NFW) argued that the simple 2-parameter formula

\[
\rho(r) = \frac{\rho_c}{r/r_s(1 + r/r_s)^2},
\]

could be scaled to provide a good fit to the density profiles of simulated haloes. The two physical scaling parameters are \( r_s \), the radius where the logarithmic slope of the density profile, \( \gamma \equiv d \log \rho / d \log r \), equals \(-2\) (the isothermal value) and the characteristic density, \( \rho_c \). As discussed by NFW, the parameters, \( r_s \) and \( \rho_c \), are not independent but rather reflect the formation history of a given halo (see also, Kravtsov, Klypin & Khokhlov 1997; Avila-Reese et al. 1999; Jing 2000; Bullock et al. 2001; Eke, Navarro & Steinmetz 2001; Klypin et al. 2001).
More recently, it has become clear that the mass profiles of CDM haloes are not strictly self-similar, as originally suggested by NFW, but that a third parameter is actually required to describe their shape accurately (Navarro et al. 2004; Merritt et al. 2005, 2006; Gao et al. 2008; Hayashi & White 2008; Navarro et al. 2010). These studies have shown convincingly that profiles where the logarithmic slope is a simple power-law of radius, \( \gamma = -2 (r/r_2)^{2} \), provide excellent fits to simulated halo profiles when the shape parameter \( \alpha \) is allowed to vary. This implies a density profile of the form

\[
\ln \left( \frac{\rho}{\rho_{-2}} \right) = -2 \left( \frac{r}{r_{-2}} \right)^{\alpha} - 1,
\]

which we call the ‘Einasto profile’ for short (Einasto 1965). The Einasto and NFW parameters are simply related by \( r_{-2} = r_{e} \) and \( \rho_{-2} = \rho_{e}/4 \).

A second result that has received widespread attention has been the fact that the pseudo-phase-space density (PPSD) profiles of simulated dark matter haloes follow simple power laws with radius:

\[
Q(r) = \frac{\rho}{\sigma^{3}} = \frac{\rho_{0}}{\sigma_{0}^{3}} \left( \frac{r}{r_{0}} \right)^{-\alpha}.
\]

This was originally reported by Taylor & Navarro (2001), and has been confirmed by a number of subsequent studies, albeit with some debate over the actual value of the best-fitting power-law exponent (see, e.g. Rasia, Tormen & Moscardini 2004; Dehnen & McLaughlin 2005; Faltenbacher et al. 2007; Vass et al. 2009; Ludlow et al. 2010). Although originally shown to hold for the total velocity dispersion profile, equation (3) also holds when \( \sigma \) is replaced with \( \sigma_{c} \), the radial velocity dispersion (e.g. Dehnen & McLaughlin 2005; Navarro et al. 2010). (We hereafter use \( Q \) when referring to the total PPSD profile, and \( Q_{c} \) for its radial analog, \( \rho/\sigma_{c}^{3} \)).

As discussed by Taylor & Navarro (2001), the power-law nature of \( Q \) does not fully determine the halo density profile. Indeed, a wide range of different density profiles are consistent with this constraint, even for spherically symmetric, isotropic systems in dynamical equilibrium. Well-behaved solutions, however, have only two possible asymptotic inner behaviours, one where the central density diverges like \( \rho \propto r^{2(\alpha+\gamma)} \) (the singular isothermal sphere is an example, for \( \gamma = 2 \)), and another ‘critical’ solution where the cuspy asymptotic slope is much shallower: \( \gamma \to -2 \chi/5 \) as \( r \to 0 \). Taylor & Navarro (2001) showed that the ‘critical’ solution for \( \chi = 1.875 \) (the value predicted by the self-similar secondary infall solution studied by Bertschinger 1985) closely resembles the NFW profile over the radial range resolved by their simulations.

The inner asymptotic behaviour predicted by equations (2) and (3) for the density profile is therefore different (Einasto profiles have a finite-density core rather than a cusp), implying that simulations should be able to discriminate between the two. Indeed, the \( Q(r) \) profiles of Einasto haloes are not in general power laws (e.g. Barnes et al. 2006; Graham et al. 2006; Ma, Chang & Zhang 2009), which implies that haloes cannot satisfy both equations (2) and (3) at once and that departures from either Einasto fits or power-law \( Q \) profiles should become apparent in simulations of adequate resolution.

 Deviations from power laws in the outer \( (r \geq r_{-2}) \) \( Q \) profiles of simulated haloes have already been reported by Ludlow et al. (2010), who argue that such departures are actually expected in regions near the virial boundary separating the inner equilibrium region from the unrelaxed infalling envelope of the halo. Whether similar deviations are also present in the inner regions of haloes has not yet been studied in detail.

We address these issues here using a sample of equilibrium CDM haloes selected from the Millennium-II Simulation (MS-II; Boylan-Kolchin et al. 2009). We begin in Section 2 with a brief description of our simulations and analysis techniques. Our main results are presented in Section 3; density profiles are presented in Section 3.1 and \( Q \) profiles in Section 3.2. Since both Einasto and power-law PPSD profiles are incomplete dynamical models unless the velocity anisotropy profile is specified we analyse this in Section 3.3. In Sections 3.4 and 3.5 we compare the Einasto and power-law PPSD models and directly assess their ability to accurately describe the mass profiles of simulated haloes. We end with a brief summary of our main conclusions in Section 4.

2 NUMERICAL SIMULATIONS

2.1 Numerical simulations

Our analysis uses group and cluster haloes identified in the MS-II, a \( \sim 10^{10} \) particle cosmological simulation of the evolution of dark matter in a 100 \( h^{-1} \) Mpc box. The run adopted a standard CDM cosmogony with the same parameters as the Millennium Simulation presented by Springel et al. (2005): \( \Omega_{M} = 0.25, \Omega_{\Lambda} = 1 - \Omega_{M} = 0.75, n_{s} = 1, \sigma_{8} = 0.9 \) and a Hubble constant \( H_{0} \equiv H(0) = 100 \) km s \(^{-1} \) Mpc \(^{-1} \) = 73 km s \(^{-1} \) Mpc \(^{-1} \). The particle mass is \( m_{p} = 6.885 \times 10^{8} h^{-1} M_{\odot} \), and is the same for all particles. The Plummer-equivalent gravitational softening, \( \epsilon = 1 h^{-1} \) kpc, remains fixed in co-moving coordinates. Interested readers may find further details in Boylan-Kolchin et al. (2009).

2.2 Halo selection

We base our results on a sample of well-resolved equilibrium haloes in MS-II. We focus on haloes where \( N_{200} \), the number of particles within the virial radius, \( r_{200} \), exceeds \( 5 \times 10^{9} \). This corresponds to group and cluster haloes with virial masses above \( \sim 3.44 \times 10^{12} h^{-1} M_{\odot} \).

In addition we require our haloes to satisfy the set of relaxation criteria introduced by Neto et al. (2007) in order to minimize the impact of transient departures from equilibrium on halo profiles. These criteria include limits on (i) the fraction of a halo’s virial mass in self-bound substructures, \( f_{sub} = M_{sub}(r < r_{200})/M_{200} < 0.07 \); (ii) the offset between a halo’s centre of mass and its true centre (defined by the cusp with the minimum potential energy), \( d_{off} = |r_{CM} - r_{cen}|/r_{200} < 0.05 \); and (iii) the virial ratio of kinetic to potential energies, \( 2K/\Phi < 1.3 \). All dynamical quantities have been computed in the halo rest frame. With these selection criteria our sample consists of 440 haloes, more than half of the total number of haloes (790) in the same mass range.

In addition to these haloes we also analyse the level-1 Aquarius simulation, the highest resolution simulation of an individual dark matter halo to date (Springel et al. 2008). This run has the same cosmological parameters as the MS-II, but a particle mass of \( 1.25 \times 10^{7} h^{-1} M_{\odot} \) and a Plummer-equivalent gravitational softening of \( \epsilon = 14 h^{-1} \) pc for all high-resolution particles.

2.3 Analysis

Our analysis deals with the spherically averaged density, \( \rho \), and velocity dispersion, \( \sigma \), profiles of each of our 440 dark matter haloes.

\(^{1}\) The virial radius, \( r_{200} \), defines the mass of a halo, \( M_{200} \), as that of a sphere, centred at the potential minimum, whose enclosed density is 200 times the critical density for closure, \( \rho_{crit} = 3H_{0}^{2} / 8 \pi G \). \( V_{200} = (GM_{200}/r_{200})^{1/2} \) is the halo’s virial velocity.
Each profile is built out of 25 spherical shells equally spaced in \(\log_{10} r\) spanning the range \(r_{\text{conv}} \leq r \leq r_{200}\). Here \(r_{\text{conv}}\) is the ‘convergence radius’ defined by Power et al. (2003), where circular velocities converge to better than \(\sim 10\) per cent. (For haloes in our MS-II sample, \(r_{\text{conv}}\) ranges from roughly 0.5 per cent to 1.5 per cent of \(r_{200}\), for the highest and lowest mass haloes, respectively.) Each spherical shell is centred at the particle with the minimum potential energy, which we identify with the halo centre.

For each radial bin we estimate the dark matter mass-density by dividing the total mass of the shell by its volume; the velocity dispersion is defined such that \(\sigma^2\) is two times the specific kinetic energy in the shell, measured in the halo rest frame. Analogously, we compute the radial velocity dispersion, \(\sigma_r^2\), in terms of the kinetic energy in radial motions within each shell. The velocity anisotropy parameter is defined by \(\beta = 1 - \sigma_{\text{tan}}^2/2\sigma_r^2\), where \(\sigma_{\text{tan}}^2 = \sigma^2 - \sigma_r^2\). The \(\rho\) and \(\sigma\) values in each radial shell are also used to estimate the PPSD profile, \(Q_1(r) = \rho/\sigma^3\). We choose to focus our analysis on the radial \(Q_1\) profiles (rather than the total \(Q = \rho/\sigma^3\)) as this simplifies the analysis of Jeans’ equations, in which the \(\sigma_i\) and \(\beta(r)\) terms separate.

The spherically averaged profiles of the dark matter haloes presented here are likely to be modified in non-trivial ways by the presence of baryons. None the less, a detailed understanding of the equilibrium structure of dissipationless CDM haloes is a prerequisite for understanding the modifications induced by the baryonic assembly of the galaxy (see, e.g. Blumenthal et al. 1986; Gnedin et al. 2004; Abadi et al. 2010; Schulz, Mandelbaum & Padmanabhan 2010; Tissera et al. 2010).

### 3 RESULTS

#### 3.1 Density profiles

Fig. 1 shows the spherically averaged density profiles for haloes in our sample, with the residuals from best-fitting Einasto profiles shown in the middle panels. All profiles have been plotted from \(r_{200}\) down to the innermost resolved radius, \(r_{\text{conv}}\). Densities have been multiplied by \(r^2\) in order to enhance the dynamic range of the graph and to highlight halo-to-halo differences. Radii have been scaled to \(r_{\text{conv}}\), the radius at which the logarithmic slope is equal to the isothermal value, \(-2\), and densities by \(\rho_{\text{conv}} \equiv \rho(r_{\text{conv}})\).

We shall restrict all model fits to the radial range \(r_{\text{conv}} < r < 3r_{\text{conv}}\), so that their parameters are not unduly influenced by regions where crossing times are long, and dynamical equilibrium may not hold (Ludlow et al. 2010). Since \(r_{\text{conv}}\) is itself a fit parameter, we adopt an iterative procedure, where \(r_{\text{conv}}\) is first estimated by fitting an Einasto profile over the entire range radial \(r_{\text{conv}} < r < r_{200}\). The resulting estimate of \(r_{\text{conv}}\) is then used to restrict our fit range to \(r_{\text{conv}} < r < 3r_{\text{conv}}\); these fits are used in the remainder of the paper. Halo concentrations obtained in this way are in the range \(5 \lesssim \rho_{\text{conv}}r_{\text{conv}} < 12.5\).

We assess the quality of fits to the density profiles using the following figure-of-merit function,

\[
y^2 = \frac{1}{N_{\text{bin}}} \sum_{i=1}^{N_{\text{bin}}} (\ln \rho_i - \ln \rho_{i,\text{model}})^2.
\]

3.1 Density profiles

The density and PPSD profiles of CDM haloes are shown here for understanding the modifications induced by the baryonic presence of baryons. None the less, a detailed understanding of the equilibrium structure of dissipationless CDM haloes is a prerequisite for understanding the modifications induced by the baryonic assembly of the galaxy (see, e.g. Blumenthal et al. 1986; Gnedin et al. 2004; Abadi et al. 2010; Schulz, Mandelbaum & Padmanabhan 2010; Tissera et al. 2010).

### Figure 1

Spherically averaged density profiles of all haloes in our sample. All profiles are plotted over the radial range \(r_{\text{conv}} < r < r_{200}\). Radii have been scaled by \(r_{\text{conv}}\); densities by \(\rho_{\text{conv}} \equiv \rho(r_{\text{conv}})\). Density estimates have been multiplied by \(r^2\) in order to increase the dynamic range of the plot so as to highlight differences between haloes. The full sample of haloes has been divided into three subsamples according to the best-fitting value of the Einasto parameter \(\alpha\): red curves (left-hand panels) correspond to haloes with \(0.161 < \alpha \leq 0.195\); green curves (middle panels) to haloes with \(0.161 < \alpha \leq 0.195\) and blue curves (right-hand panels) to haloes with \(\alpha > 0.195\). There are equal numbers of haloes in each subsample. To illustrate the role of the shape parameter, \(\alpha\), Einasto profiles with \(\alpha = 0.132\) (dashed), 0.178 (solid) and 0.230 (dot–dashed) are shown in the top panels. The middle panels are residuals from the best-fitting Einasto profile with adjustable \(\alpha\); bottom panels are the residuals from the best-fitting power-law \(Q_1\) ‘critical’ model (see Section 3.4). In all cases fits are carried out over the radial range \(r_{\text{conv}} < r < 3r_{\text{conv}}\). Solid black curves with error bars show the mean residuals and 1σ scatter.
The best fits are those that minimize $\psi$, and define its minimum value, $\psi_{\text{min}}$. All haloes shown in Fig. 1 have $\psi_{\text{min}} < 0.1$.

In order to highlight the differences in profile shape we have split our halo sample into three subsamples, each containing an equal number of haloes; haloes with $\alpha \leq 0.161$ are shown in red, green curves show those with $0.161 < \alpha \leq 0.195$, and blue curves are used for the rest. We adopt the same colour coding and halo samples in subsequent plots. For illustration, we also show three Einasto profiles with different values of the shape parameter, corresponding to the mean value of each halo subsample: $\alpha = 0.132$ (dashed curve), $\alpha = 0.178$ (solid black curve) and $\alpha = 0.230$ (dot-dashed curve).

The small residuals from best Einasto fits (typically less than 10 per cent throughout the fitted region; see middle panels of Fig. 1) confirm that equation (2) provides an excellent description of the density profiles of CDM haloes down to the innermost resolved radius. Further, Fig. 1 also demonstrates that there is genuine variation in profile shape from halo to halo: profiles of different haloes ‘curve’ differently, yielding best-fitting $\alpha$ parameter values that range between roughly 0.1 and 0.25. The mass profiles of CDM haloes are therefore not strictly self-similar, and $\alpha$ is a structural ‘shape’ parameter genuinely needed to describe accurately the mass profile of CDM haloes (see Navarro et al. 2010, for a full discussion).

### 3.2 PPSD profiles

Fig. 2 shows the (radial) PPSD profiles, $Q_t$, of the haloes in our sample, grouped as in Fig. 1. In order to take out the halo mass dependence, $Q_t$ profiles have been scaled radially by $r_{\text{r}_2}$, and vertically by $\rho_{\text{r}_2}/v_{\text{r}_2}^3$, where $v_{\text{r}_2} = \sqrt{G \rho_{\text{r}_2} r_{\text{r}_2}^2}$ is the characteristic velocity implied by the scale parameters $\rho_{\text{r}_2}$ and $r_{\text{r}_2}$. Our results confirm the striking power-law nature of $Q_t$ profiles previously reported in the literature. For reference, we also show the power law, $Q_t \propto r^{-1.875}$ (shown by a black dotted line), predicted by the self-similar solution of Bertschinger (1985).

In order to emphasize the differences between haloes, we show in the middle panels of Fig. 2 the residuals from the Bertschinger power law normalized to the mean value of $\rho_{\text{r}_2}/v_{\text{r}_2}^3$ for all haloes in each sample. Although the residuals are in general small, the ‘curving’ shape of their radial dependence suggests that the $Q_t$ profiles of simulated CDM haloes deviate systematically from a simple power law of fixed exponent $\chi$.

As discussed by Ludlow et al. (2010), the outer ($r > r_{\text{r}_2}$) upturn in the residuals is most likely associated with the transition from the inner, relaxed parts, to the unrelaxed outer parts, where infalling material has not yet had time to phase-mix with the main body of the halo. Such an upturn is present also in the self-similar solution of Bertschinger (1985), and may be a general feature of the outer $Q_t$ profiles of CDM haloes. Because of this, we have chosen the radial range $r_{\text{cosm}} < r < 3r_{\text{r}_2}$ for all the fits we report here.

Interestingly, the inner $Q_t$ profiles ($r < 0.1 r_{\text{r}_2}$) also deviate from the $r^{-1.875}$ power law in a way that clearly depends on $\alpha$. Note, for example, that at the innermost point the residuals of the lowest-$\alpha$ haloes (leftmost middle panel in Fig. 2) are substantially larger than those of the largest-$\alpha$ haloes (rightmost middle panel). We have explicitly verified that this $\alpha$-dependence is not caused by

![Figure 2](https://academic.oup.com/mnras/article-abstract/415/4/3895/1750178/3898)
grouping haloes of different mass, nor by differences in numerical resolution. It is also unlikely to be due to anisotropies in the velocity distribution, since, as we show below, all haloes in our sample are nearly isotropic at radii this close to the centre. We have also checked that this $\alpha$ dependence is not specific to our choice of $Q_i$; the total PPSD profiles, $Q = \rho/r^2$, follow similar trends with $\alpha$.

These results imply that no single power law can reproduce the radial dependence of the PPSD; if $Q_i$ does indeed follow a power law of radius, then the exponent $\chi$ must vary from halo to halo. We show this explicitly in the bottom panels of Fig. 2, where we plot the residuals from the best-fitting power law when the exponent $\chi$ is allowed to vary. The small residuals over the fitted radial range ($r_{\text{conv}} < r < 3r_{\text{vir}}$) indicate that a power-law with adjustable $\chi$ provides a remarkably accurate description of the inner $Q_i$ profiles of CDM haloes. The Bertschinger law best fits, for example, yield the following average values: $\langle \psi_{\text{min}} \rangle = 0.159$, $0.139$ and $0.126$ for haloes with $\alpha = 0.132$, $0.178$ and $0.230$, respectively. When allowing $\chi$ to vary, however, we obtain $\langle \psi_{\text{min}} \rangle = 0.096, 0.085$ and $0.089$, for the same samples.

Do Einasto profiles provide a better description of the spherically averaged structure of CDM haloes than power-law $Q_i$ profiles or vice versa? Because of different dimensionality, we cannot compare directly the goodness of fits to the $\rho$ and $Q_i$ profiles shown in Figs 1 and 2. One way to make progress is to compute the PPSD profiles corresponding to Einasto haloes, or, alternatively, to compute the density profiles of power-law PPSD models and compare them with the simulations. This may be accomplished by assuming dynamical equilibrium and solving Jeans’ equations to link the $\rho$ and $Q_i$ profiles, a procedure that, however, requires an assumption regarding the radial dependence of the velocity anisotropy, $\beta(r)$. We turn our attention to that issue next.

### 3.3 Velocity anisotropy

Velocity anisotropy profiles for all haloes in our sample are shown in Fig. 3, after rescaling all radii to $r_{\text{vir}}$. Haloes have been grouped according to the value of the best-fitting $\alpha$ parameter, as in Fig. 1. Solid lines with error bars show the median $\beta(r)$ profile of each group and the associated 1σ scatter. The velocity anisotropy profiles exhibit a characteristic shape; they tend to isotropy near the centre, but become increasingly radial with increasing distance before levelling off or even declining in the outskirts.

Fig. 4 shows the logarithmic slope-velocity anisotropy ($\gamma$ versus $\beta$) relation for the median profiles of each halo subsample using the same colour coding as previous plots. The dotted line shows the linear $\beta(\gamma)$ relation proposed by Hansen & Moore (2006),

$$\beta(\gamma) = -0.2(\gamma + 0.8),$$

which reproduces fairly well our simulation data within the scale radius $r_{\text{vir}}$ (i.e., for $\gamma > -2$).

The data in Fig. 4 also suggest that the $\beta(\gamma)$ relation deviates from the Hansen & Moore fit in the outer regions (where $\beta$ tends to be lower), but becomes consistent near the centre.

Figure 4. Median density slope-velocity anisotropy relation for the three groups of haloes shown in previous figures. The linear $\beta-\gamma$ relation proposed by Hansen & Moore (2006) is shown as a dotted line; a relation of the form given in equation (6) is shown separately for each sample using dot–dashed curves. There is some evidence that the mean $\beta-\gamma$ relation departs from the Hansen & Moore (2006) result in the outer regions of our haloes. See text for further discussion.

**Figure 3.** Velocity anisotropy profiles, $\beta = 1 - v^2/2\sigma^2$, for all haloes in our sample. Profiles have been grouped according to the best-fitting Einasto parameter, as in Fig. 1. Solid black lines with error bars show the median anisotropy profile and 1σ dispersion for each halo subsample. The dot-dashed curves shown in each panel show the radial dependence of $\beta(r)$ expected from equation (6) assuming the average Einasto density profile; dotted curves show $\beta(r)$ expected from Hansen & Moore (2006) for the same $\rho(r)$.
to a constant rather than the steadily increasing radial anisotropy predicted by Hansen & Moore). This dependence is captured well by the function
\[
\beta(\gamma) = \frac{\beta_\infty}{2} \left(1 + \text{erf}(\ln[(\gamma \alpha)^2])\right),
\]
as shown by the dot–dashed curves in Fig. 4. The values of \(\beta_\infty\) and \(A\) depend weakly but systematically on \(\alpha\), and are listed in Table 1. The radial \(\beta\) profiles corresponding to equation (6) are also shown with dot–dashed curves in each panel of Fig. 3, assuming that \(\gamma(r)\) is given by the average Einasto profile of each grouping.

### 3.4 Density profiles from power-law PPSD profiles

Once the radial dependence of the velocity anisotropy has been specified, equilibrium density profiles consistent with a power-law \(Q_i(r)\) profile may be obtained from Jeans’ equation. Following Dehnen & McLaughlin (2005) (see also, Austin et al. 2005), we write Jeans’ equation as
\[
\left(y' - \frac{6}{5} \beta'\right) + \frac{2}{\gamma + x + \frac{3}{2}} \left(y + \frac{2}{5} x + \frac{6}{5} \beta\right) = -\frac{3}{5} \kappa - 2 x^{2-2x/3} y^{1/3},
\]
where \(y = \rho/\rho_{-2}\); \(x = r/r_{-2}\); \(\kappa_{-2} = 4\pi G \rho_{-2} r_{-2}^2 / \sigma_{-2}^2\) is a measure of the velocity dispersion in units of the ‘natural’ velocity scale of the halo at \(r_{-2}\), and we have assumed that \(Q_i(r)\) is a power law of exponent \(\chi\). The logarithmic slope, \(d \ln \rho / d \ln r\) is denoted by \(\gamma\), and primes indicate derivatives with respect to \(\ln r\). Once \(\beta(r)\) is specified, the solution set of equation (7) for given \(\chi\) is fully characterized by \(\kappa_{-2}\).

It is instructive to consider the special case \(\beta(r) = 0\), for which three central asymptotic behaviours are possible: (i) a steep central cusp, \(\gamma \to 2 \chi - 6\); (ii) a central ‘hole’ where \(y(0) = 0\); or (iii) a ‘critical’ solution with a shallow central cusp, \(\gamma \to -2x/5\). The latter may be thought of as the limiting case where the radius of the central ‘hole’ solution goes to zero, and corresponds to a maximally mixed state for given halo binding energy and mass (Taylor & Navarro 2001).

Given that steep central cusps such as those in (i) are firmly ruled out by the data (see, e.g. Navarro et al. 2010), the ‘critical’ solutions (iii) are the only viable density profiles consistent with the power-law PPSDs constraint. Although these considerations are strictly true only for isotropic systems, similar conclusions apply to anisotropic systems provided that \(\beta \to 0 \text{ as } r \to 0\). This is especially true considering that haloes are very nearly isotropic close to the centre (see Section 3.3). We shall hereafter adopt the ‘critical’ solutions (evaluated numerically) as the density profiles corresponding to a power-law PPSD profile for a given value of \(\chi\).

The left-hand panel of Fig. 5 shows (in blue) the ‘critical’ density profiles for three different values of \(\chi\), and compares them to Einasto profiles. The values of \(\alpha\) of the three Einasto profiles shown have been chosen to match as closely as possible the profiles corresponding to the PPSD models.

Clearly, for every value of \(\chi\) it is possible to choose a value of \(\alpha\) that matches the resulting density structure over a very wide range in radius. Deviations are only seen in the very inner regions, at less...
than 0.1 per cent of the scale radius and therefore well outside the convergence region of any published halo simulation to date. For example, the black solid line in Fig. 5 shows the density profile of the billion-particle Aq-A-1 halo, whose convergence radius is roughly 0.01 $r_{-2}$ (Navarro et al. 2010). An Einasto profile with $\alpha = 0.17$ and a ‘critical’ solution for $\chi = 1.911$ match the profile of this halo indistinguishably well.

3.5 PPSD profiles from Einasto profiles

PPSD profiles corresponding to Einasto models can also be obtained by solving Jeans’ equation, which may be written as follows:

$$\frac{d \ln \psi}{d \ln r} - \chi = \frac{d \ln \sigma_r^2}{d \ln r} + 2\beta,$$

where $\sigma_r^2 = V_r^2/\langle r^2 \rangle$. With $\beta(r)$ set by equation (6), the right-hand side of this equation is fully specified by $r_{-2}$, $\rho_{-2}$ and $\alpha$. Solutions may therefore be found after choosing the value of $\zeta_i$ (or its logarithmic derivative) at some fiducial radius, such as $\zeta_{-2} = \zeta(r_{-2})$. The shape of the $Q_r$ profile is dictated by $\zeta_{-2}$, and it is not generally a power law (Graham et al. 2006; Ma et al. 2009).

Again, insight may be gained by considering the limiting behaviour of the isotropic solutions. Because Einasto profiles have finite central densities, $\sigma_i$ must approach a constant at the centre. The central velocity dispersion may be found numerically, and turns out to be quite insensitive to $\zeta_{-2}$ (for given $\rho_{-2}$, $r_{-2}$ and $\alpha$), provided that the $\zeta_{-2}$ is greater than about a tenth. (All of our haloes are comfortably in that regime; indeed, the median value of $\zeta_{-2}$ for all of our haloes is 1.274.) The finite value of the central velocity dispersion and its near invariance with $\zeta_{-2}$ also imply that the shape of the inner velocity dispersion profile is quite insensitive to the actual value of $\zeta_{-2}$.

Provided $\beta \to 0$ as $r \to 0$, similar conclusions also apply to the anisotropic solutions. There is, therefore, in practice, a unique $Q_i$ profile that corresponds to an Einasto profile of given $\alpha$, which we identify with the single solution that is well behaved at all radii. This may be found by setting $d \ln \psi_i/d \ln r = \alpha$ at $r = \infty$; for $\alpha = 0.175$, for example, this implies $\zeta_{-2} = 1.265$.

The right-hand panel of Fig. 5 shows (in red) the PPSD profiles of Einasto haloes, for three different values of $\alpha$. For $\alpha = 0.1$ and 0.17 the corresponding Einasto profiles are very well approximated by power laws over the whole plotted radial range. Only for larger values of $\alpha$, such as 0.3, are clear deviations from a power law noticeable. Even in this case, however, these are only evident in regions well inside 1 per cent of the scale radius $r_{-2}$, and therefore outside the converged region of the highest resolution simulation of a CDM halo published to date, the Aq-A-1 halo (shown by a solid black line). Deciding whether power-law PPSD models match CDM haloes better than Einasto profiles, or vice versa, seems to require simulations of even better resolution than Aq-A-1.

3.6 Power-law PPSD versus Einasto density profile fits

The results above suggest that both Einasto profiles and power-law $Q_i$ models provide equally good representations of the spherically averaged structure of simulated CDM haloes, in spite of making very different predictions for their asymptotic inner structure.

This conclusion may be verified quantitatively by fitting the ‘critical’ solutions introduced in Section 3.4 to the density profiles of all haloes in our sample and comparing them with Einasto fits. Residuals from the best fits obtained after varying $\chi$ are shown in the bottom panels of Fig. 1, and are quite clearly indistinguishable from those obtained by fitting Einasto laws with adjustable $\alpha$ (shown in the middle panels of the same figure).

Further quantitative evidence is provided in Fig. 6, where we plot the figure of merit, $\psi_{min}$, of the best Einasto fits compared with that obtained from the best-fitting critical solution. Open and filled symbols in this figure are used to denote cases where, respectively, either isotropic or anisotropic (i.e., $\beta$ given by equation 6) critical solutions have been used in the fitting procedure. The close resemblance of the results obtained with either assumption implies that our conclusions are largely insensitive to our assumption about the radial dependence of the velocity anisotropy. For example, the rms scatter about the best linear fit in the values of $\psi_{min}$ are $5.41 \times 10^{-3}$ and $5.07 \times 10^{-3}$ for the isotropic and anisotropic solutions, respectively.

Finally, our results imply a strong correlation between the best-fitting Einasto’s $\alpha$ and ‘critical’ $\chi$ parameters for any given halo. We show this in Fig. 7, which suggests that the two parameters are essentially equivalent measures of the diversity in halo CDM density profiles, and linked by a simple linear relation, $\chi = 2.1 - 1.16 \alpha$. At the resolution of current simulations, it does not seem possible to decide whether Einasto or power-law PPSDs provide a better description of the spherically averaged structure of simulated CDM haloes.

4 SUMMARY AND CONCLUSIONS

We use a sample of well-resolved equilibrium systems selected from the MS-II simulation in order to study the density and PPSD
profiles of CDM haloes. In particular, we explore the relation between Einasto profiles (often used to parametrize $\rho(r)$) and power-law PPSD profiles, which are often used to construct equilibrium models of CDM haloes.

We solve Jeans’ equations to show that the PPSD profiles of Einasto haloes are close to power laws, and that, conversely, the density profiles of power-law PPSD models can be fitted by Einasto laws over a wide radial range. The two descriptions differ, however, in their inner asymptotic behaviour, although significant differences are only expected at radii well inside 1 per cent of the scale radius, $r_{-2}$, and are therefore beyond the reach of current simulations.

Our analysis of the MS-II halo sample shows that, over the resolved radial range, Einasto and power-law PPSD models provide an equally good description of the spherically averaged structure of simulated CDM haloes, provided that the Einasto parameter $\alpha$ and the power-law exponent $\chi$ are allowed to float freely when fitting their radial structure. The strong correlation between best-fitting $\alpha$ and $\chi$ parameters implies that they constitute, in practice, equivalent measures of the shape of the mass profile.

The spherically averaged mass profiles of equilibrium CDM haloes thus seem to be fully specified by three parameters: two physical scalings, $r_{-2}$ and $\rho_{-2}$, and a shape parameter, $\alpha$ or $\chi$. Where CDM haloes lie in this three-dimensional parameter space is likely linked to each system’s evolutionary history. Exploring the details of these relations remains a pending issue, but one that can be profitably addressed through large-scale numerical efforts such as the Millennium Simulation series. We expect to report progress in this area in the near future.

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REFERENCES

Abadi M. G., Navarro J. F., Fardal M., Babul A., Steinmetz M., 2010, MNRAS, 407, 435
Austin C. G., Williams L. L. R., Barnes E. I., Babul A., Dalcanton J. J., 2005, ApJ, 634, 756
Avila Reese V., Firmani C., Klypin A., Kravtsov A. V., 1999, MNRAS, 310, 527
Barnes E. I., Williams L. L. R., Babul A., Dalcanton J. J., 2006, ApJ, 643, 797
Bertschinger E., 1985, ApJS, 58, 39
Blumenthal G. R., Faber S. M., Flores R., Primack J. R., 1986, ApJ, 301, 27
Boylan-Kolchin M., Springel V., White S. D. M., Jenkins A., Lemson G., 2009, MNRAS, 398, 1150
Bullock J. S., Kolatt T. S., Sigad Y., Somerville R. S., Kravtsov A. V., Klypin A. A., Primack J. R., Dekel A., 2001, MNRAS, 321, 559
Dhenen W., McLaughlin D. E., 2005, MNRAS, 363, 1057
Einasto J., 1965, Trudy Inst. Astrof. Alma-Ata, 51, 87
Eke V. R., Navarro J. F., Steinmetz M., 2001, ApJ, 554, 114
Faltenbacher A., Hoffman Y., Gottlöber S., Yepes G., 2007, MNRAS, 376, 1327
Gao L., Navarro J. F., Cole S., Frenk C. S., White S. D. M., Springel V., Jenkins A., Neto A. F., 2008, MNRAS, 387, 536
Gnedin O. Y., Kravtsov A. V., Klypin A. A., Nagai D., 2004, ApJ, 616, 16
Graham A. W., Merritt D., Moore B., Diemand J., Terzić B., 2006, AJ, 132, 2701
Hansen S. H., Moore B., 2006, New Astron., 11, 333
Hayashi E., White S. D. M., 2008, MNRAS, 388, 2
Jing Y. P., 2000, ApJ, 535, 30
Klypin A., Kravtsov A. V., Bullock J. S., Primack J. R., 2001, ApJ, 554, 903
Kravtsov A. V., Klypin A. A., Khokhlov A. M., 1997, ApJS, 111, 73
Ludlow A. D., Navarro J. F., Springel V., Vogelsberger M., Wang J., White S. D. M., Jenkins A., Frenk C. S., 2010, MNRAS, 406, 137
Ma C., Chang P., Zhang J., 2009, ArXiv e-prints
Merritt D., Navarro J. F., Ludlow A., Jenkins A., 2005, ApJ, 624, L85
Merritt D., Graham A. W., Moore B., Diemand J., Terzić B., 2006, AJ, 132, 2685
Navarro J. F., Frenk C. S., White S. D. M., 1996, ApJ, 462, 563 (NFW)
Navarro J. F., Frenk C. S., White S. D. M., 1997, ApJ, 490, 493 (NFW)
Navarro J. F. et al., 2004, MNRAS, 349, 1039
Navarro J. F. et al., 2010, MNRAS, 402, 21
Neto A. F. et al., 2007, MNRAS, 381, 1450
Power C., Navarro J. F., Jenkins A., Frenk C. S., White S. D. M., Springel V., Stadel J., Quinn T., 2003, MNRAS, 338, 14
Rasia E., Tormen G., Moscardini L., 2004, MNRAS, 351, 237
Schulz A. E., Mandelbaum R., Padmanabhan N., 2010, MNRAS, 408, 1463
Springel V. et al., 2005, Nat, 435, 629
Springel V. et al., 2008, MNRAS, 391, 1685
Taylor J. E., Navarro J. F., 2001, ApJ, 563, 483
Tissera P. B., White S. D. M., Pedrosa S., Scannapieco C., 2010, MNRAS, 406, 922
Vass I. M., Valluri M., Kravtsov A. V., Kazantzidis S., 2009, MNRAS, 395, 1225

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