Entanglement entropy and $C_T$ for monodromy defects of fields on odd–dimensional spheres

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The effect of a spherical monodromy defect on the entanglement entropy and central charge, $C_T$, of a free, conformal scalar field propagating on an odd–dimensional sphere is investigated. As on even spheres, the central charge becomes negative for a range of the flux parameter, $\delta$, with possible implications for reflection positivity. The work is mostly numerical but closed forms are found for the $\mathbb{Z}_2$ monodromy ($\delta = 1/2$) yielding an explicit $d \to \infty$ limit, $C_T \to -6/\pi$. 

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1. Introduction.

This brief report presents the results of a computation of the entanglement entropy and central charge on odd–dimensional spheres, for free scalar fields, in the presence of a spherical monodromy.

The even–dimensional case has been treated before, [1,2], (relevant references can be found in these works) and is the easier calculation in that the expressions are all polynomial, related to the conformal anomaly. In odd dimensions the expressions are transcendental and involve the full effective action.

There are a number of different (but ultimately equivalent) calculational procedures that can be invoked. Two approaches, based on the eigenvalue spectrum, will be combined to effect the evaluation of the entanglement entropy and also of the central charge, $C_T$, as functions of the monodromy.

I do not give much background or technical detail as the analysis has mostly been exposed in the references cited above and below.

2. The effective action or free energy

In odd dimensions, the effective action itself (‘free energy’) can be taken as the universal quantity, rather than the logarithmic coefficient (the conformal anomaly) that appears in even dimensions. Thus the quantity that one aims to compute is the log det of the relevant propagating operator, or, equivalently, the value of the derivative of the corresponding ζ–function at the origin, $\zeta'(s)|_{s=0}$.

There are several ways of manipulating the eigenvalue sum that appears in the ζ–function in order to make the necessary analytic continuation. The one employed in [1,2] reduces the problem to a combination of Barnes ζ–functions. Another rewrites the ζ–function definition using a Bessel transform, [3,4], to yield a contour representation. \(^{2}\)

The spherical monodromy spectrum of the conformal Laplacian on a conically deformed sphere was determined in [5] where both approaches were employed and I will simply take an expression for the effective action that appears there, based on the Bessel transform (for odd $d$),

$$
A(d, q, \delta) = -\frac{1}{2^{d-2}} \text{Re} \int_{0}^{\infty} dx \frac{\cosh q(1 - 2\delta)z \cosh z}{z \sinh qz \sinh^{d-1} z},
$$

(1)

\(^{2}\) This has the form of a ‘character integral’ and is of more general application than the Barnes ζ–function one.
where \( z = x + iy \) with \( 0 < y < \pi/q \). The expression is valid for all real \( q \). For the orbifolded sphere, \( S^d/\mathbb{Z}_q \), \( q \) is an integer. The integral converges in the relevant range of \( \delta \) \((0 \leq \delta \leq 1)\) and is extended beyond this range using the necessary periodicity, \( \delta \equiv 1 + \delta \).

2. The entanglement entropy

The integral, (1), can be used numerically for all \( q \) and \( \delta \). I begin by investigating the entanglement entropy, given, using replicas, by,

\[
\mathcal{S}(d, \delta) = \frac{\partial}{\partial q} A(d, q, \delta) \bigg|_{q=1} - A(d, 1, \delta),
\]

(2)

since, in particular, the first (derivative) term is easy to find from (1).

The integral is independent of the value of \( y \), so long as this sits between 0 and \( \pi/q \). Choosing \( y = \pi/2 \), the integral and its \( q \)-derivatives at \( q = 1 \) (the round sphere) take on simpler forms when \( \delta = 0 \) or \( \delta = 1/2 \). A numerical integration is still necessary for general \( \delta \).

The graph below plots the difference, \( \Delta \mathcal{S}(d, \delta) \equiv \mathcal{S}(d, \delta) - \mathcal{S}(d, 0) \), for \( d = 3 \) and \( d = 5 \) against the flux \( \delta \).

The values at the midpoint, \( \delta = 1/2 \), can be given as closed forms. Those of the free energy difference have already been evaluated and some results, in dimensions 3 and 5, appear in [7]. A more extensive list, and a general formula, is contained

\footnote{Related computations already exist in [5] and [6].}
in [8] obtained using the form of the effective action derived, [5], from the Barnes $\zeta$–function approach, equivalent, but different, to (1). 4

The first term in (2) vanishes for $\delta = 0$ (an old result), while, at $\delta = 1/2$, it is easily found from (1) as the integrand then reduces to the difference of two powers of $\text{sech} \ z$.

Thus, for the defect entanglement entropy at the midpoint, I find,

$$\Delta S(d, 1/2) = (-1)^{(d-1)/2} \frac{\Gamma((d-1)/2)^2}{2d \Gamma(d-1)} - \Delta A(d, 1, 1/2). \quad (3)$$

As a typical example of the final term,

$$\Delta A(7, 1, 1/2) = \frac{\log 2}{512} - \frac{133\zeta(3)}{6080\pi^2} - \frac{79\zeta(5)}{3072\pi^4} - \frac{127\zeta(7)}{2048\pi^6} \approx -0.000663482.$$

The first term in (3) evaluates to approximately, $-0.00238095$ at $d = 7$.

A list of the, rapidly decreasing, midpoint values of $\Delta S$ for $d = 3, 5, 7, 9$ and $11$ is $-0.233384$, $0.021747$, $-0.003044$, $0.0005025$, $-0.0000908$.

3. The central charge $C_T$

The central charge, $C_T$, involves the second derivative of the free energy at $q = 1$, the formula being,\footnote{It is possible to get these explicit forms from (1), when $q = 1$, by means of a complex contour transformation of the integral, after a little labour, in the manner expounded in [9], cf also [4].} using the Perlmutter factor for odd dimensions, [10],

$$C_T(d, \delta) = (-1)^{(d+1)/2} \frac{2^{2d}(d+1)!((d-1)/2)!^2}{\pi^2(d-1)(d-1)!^2} \left( 2 \frac{\partial}{\partial q} + \frac{\partial^2}{\partial q^2} \right) A(d, q, \delta) \bigg|_{q=1}. \quad (4)$$

The first derivative term has already been encountered in the calculation of the entanglement entropy. It vanishes when $\delta = 0$. The second derivative again follows easily from the integral representation, (1), and the results of a purely numerical evaluation are shown in the graph below giving the variation of $C_T$ with $\delta$.

\footnote{The variable $n = 1/q$ could be used but I prefer $q$.}
Just as in even dimensions, $C_T$ exhibits a sign change,\(^6\) for example, for $d = 3$, at $\delta \approx 0.2349278$ and for $d = 11$ at $\delta \approx 0.2452683$. This appears to violate reflection positivity.

As a simple check of the algebra and numerics, the integral can be done analytically for no flux, $\delta = 0$, and yields the standard central charge formula,

$$C_T(d, 0) = \frac{2d}{d - 1},$$

for a complex scalar.

The integral giving the second term in (4) can also be found at the midpoint, $\delta = 1/2$, and, since the first term is known as well, a closed form for the midpoint central charge ensues. Thus,

$$C_T(d, 1/2) = -\frac{2^{2d-4}d(d-1)(3d-5)\Gamma((d-1)/2)^4}{\pi^2 \Gamma(d)^2}.$$

In the limit of infinite dimensions this tends to,\(^7\)

$$C_T(\infty, 1/2) = -\frac{6}{\pi} \approx -1.9098593.$$

For comparison, $C_T(11, 1/2) = -1.9806581$.

\(^6\) This behaviour is the result of a competition between the two terms in (4).

\(^7\) Incidentally, this provides very inefficient rational approximations to $\pi$. 

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4. Discussion and conclusion

As expected, the results are very similar to those for even–dimensional spheres. The sign change in $C_T$ may indicate a problem with the field theory.

Simple modifications of the integral representation (1) allow the analysis to be extended to Dirac fields and higher derivative propagation and also to an interpolation between odd and even $d$.

The defect considered here is of the spherical type. Other defects are available, e.g. [11,12], and would give different answers.

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