Existence, boundedness and integrability of global solutions to delay integro-differential equations of second order

Sizar Abid Mohammed

Department of Mathematics, College of Basic Education, Duhok, Iraq

ABSTRACT
The paper deals with the problems of the global existence, boundedness and integrability, and solutions of an integro-differential equation (IDE) of second order with n-constant retardations. Three new theorems, Theorems 2.1–2.3 and a corollary, Corollary 2.1, have been proved, which consist of new sufficient criteria on global existence, stability, asymptotic stability, boundedness of solutions and square integrability of the first derivatives of solutions of the considered nonlinear retarded IDE of second order (RIDE). The techniques used in the proofs are based on the construction of a suitable Lyapunov–Krasovskii functional (LK functional). The results of this work include and get better results that are available in the literature. The given criteria have very simple forms, and they can be easily verified with applications. A numerical example is given and the orbits of the exact solutions are drawn with assigned initial conditions to demonstrate the effectiveness of the obtained results.

1. Introduction
An integro-differential equation (IDE) is said to have a delay when the rate of variation in the equation state depends on past states. In that case such an IDE is called a time-delay IDE. In fact, in many real-world engineering systems, the fields of modern control theory, physical and biological phenomena, the rate of variation in the equation state depends on past states [1–43].

When we look for available scientific works related to the structure analysis of solutions of RIDEs and IDEs of first order in the relative literature such as stability, uniformly stability, asymptotically stability, boundedness, convergence, integrability, and continuability of solutions, we can find many interesting papers and a few effective books on these properties of solutions for various RIDEs and IDEs of first order since 1970 until now. See, for example, [1–32] and some works can be found in the references of these sources, where the second method of Lyapunov based on function(s) or functional(s) is used as a main tool to work on the related results in [1–32].

Although there are so many interesting results concerning the various qualitative investigations of solutions of RIDEs and IDEs of first order, to the best of knowledge, there are a few papers on stability, global existence, boundedness and integrability solutions of RIDEs and IDEs of second order. See, for example, [33–35], where the second method of Lyapunov is used as a main tool to prove the results in [33–35].

It is known from the second method of Lyapunov that the basic and important idea is to construct or define a nice Lyapunov function or LK functional for a given RIDE or IDE. This problem is a very difficult task for RIDEs and IDEs of higher order. According to the existing situation, the possible reason for the existence of a limited number of results on the qualitative properties of solutions of RIDEs and IDEs of second order is that the construction of appropriate Lyapunov function(s) and LK functional(s) is a key and difficult problem in the qualitative theory of RIDEs and IDEs of higher order. This problem remains as an unsolved open problem in the literature up to now, and therefore it is also interesting and important field of research in itself for RIDEs and IDEs of higher order by this time.

At the same time, the investigation of the various qualitative behaviours of solutions to ODEs, IDEs and RIDEs is a basic and important problem in the theory and applications of ODEs, IDEs and RIDEs. In fact, many dynamical systems have the property aftereffect, i.e. the future states depend not only on the present time, but also on the past history, see [36–43]. At the same time, in applied sciences, some applied problems deal with engineering technique fields, economy, control theory, physics, chemistry, population dynamics, medicine, atomic energy, information theory, mechanics and electromagnetic theory, life sciences, etc. and are associated with these type of IDEs and RIDEs. In particular, some applications on the mentioned concepts can be found.
in the books of [3,7–9,12–14,29,32,36–43]. Therefore, it is worth and useful to maintain the discussion of the structure analysis of solutions of RIDEs of higher order.

We should now mention some related papers on the mentioned topics for RIDEs and IDEs of second order. First, among the mentioned researches, Napoles Valdes [33] considered the following IDE:

\[
\frac{d^2x}{dt^2} + a(t)f\left(t, x, \frac{dx}{dt}\right) + \frac{dx}{dt} + g\left(t, x, \frac{dx}{dt}\right) = \int_0^t C(t, \tau) \frac{dx}{d\tau} d\tau + h(x(t))
\]

(1)

The author studied the continuability and the problem of integrability of the first derivative of solutions of IDE (1). Here Napoles Valdes [33] proved that the solutions of IDE (1) globally exist and bounded under sufficient conditions. He also obtained a result for all solutions of IDE (1) to satisfy \( x' \in L^2(0, \infty) \). The approach in [33] is to construct a Lyapunov function for IDE (1).

Later, Graef and Tunç [34] handled the following RIDE of second order with \( n \)-multiple and different retardations, \( \tau_i \):

\[
\frac{d^2x}{dt^2} + a(t)f\left(t, x, \frac{dx}{dt}\right) + g\left(t, x, \frac{dx}{dt}\right) + h(x(t)) + \sum_{i=1}^n h_i(x(t - \tau_i)) = \int_0^t C(t, \xi) \frac{dx}{d\xi} d\xi.
\]

(2)

Graef and Tunç [34] studied the concepts in [33] for the global continuability and boundedness of solutions to RIDE (2), by using an LK type functional.

Furthermore, Zhao and Meng [35] handled the following both RIDEs including a constant delay, \( \tau \):

\[
\frac{d^2x}{dt^2} + f\left(t, x, \frac{dx}{dt}\right) + \frac{dx}{dt} + g\left(t, x, \frac{dx}{dt}\right) + h(x(t - \tau)) = p(t, x(t)) \int_0^t q\left(s, \frac{dx}{ds}\right) ds
\]

(3)

and

\[
\frac{d^2x}{dt^2} + f\left(t, x, \frac{dx}{dt}\right) + \frac{dx}{dt} + g\left(t, x, \frac{dx}{dt}\right) + h(x) = p(x(t - \tau)) \int_0^t q\left(s, \frac{dx}{ds}\right) ds.
\]

(4)

In [35], the stability of zero solutions for RIDEs (3) and (4) is analysed and some new results are presented on the subject. By constructing an LK functional, the authors give the corresponding sufficient conditions on stability of zero solutions for RIDEs (3) and (4). Moreover, an illustrative example is considered to support the results in [35].

In this paper, motivated by the works in [33–35] and that in the references [1–32,36–43], we consider the below RIDE with constant retardations, \( \tau_i \):

\[
\frac{d^2x}{dt^2} + a(t)f\left(t, x, \frac{dx}{dt}\right) + \frac{dx}{dt} + b(t)p(x) + \sum_{i=1}^n h_i(x(t - \tau_i)) = \int_0^t K(t, \xi) h\left(\xi, \frac{dx}{d\xi}\right) d\xi,
\]

(5)

where \( \tau_i (> 0) \) are fixed constant retardations and let \( \tau^* = \max\{\tau_1, \tau_2, \ldots, \tau_n\} \); the primes in RIDE (5) represent differentiation in \( t, t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty) \); \( a, b \in C(\mathbb{R}^+, \mathbb{R}^+) \) such that the derivative \( b'(t) = (d/dt)b(t) \) exists; \( f, g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}) \), \( \mathbb{R} = (-\infty, \infty) \), \( h_i, h \in C(\mathbb{R}^2, \mathbb{R}) \) and \( p \in C(\mathbb{R}^2, \mathbb{R}) \) with \( g(t, x, 0) = 0, p(0) = 0 \), \( p(x) \neq 0 \) if \( x \neq 0 \), \( h_i(0, 0) = 0 \) and \( h(t, 0) = 0 \) and \( K(t, \xi) \in C(\mathbb{R}^+, \mathbb{R}) \) with \( 0 \leq \xi \leq t < \infty \). The continuity of these functions enables the existence of the solutions of RIDE (5). Next, to hold the uniqueness of solutions of RIDE (5), it is assumed that \( f, g, p, h_i, h \) satisfy a Lipschitz condition in \( x, x' \) and \( x(t - \tau_i) \), \( i = 1, 2, \ldots, n \). Furthermore, we assume that the derivatives \( \partial h_i(x, x')/\partial x' \equiv h_i(x, x') \) exist and are continuous. Throughout this paper, when we write \( x(t) \) and \( x'(t) \) without any argument, this means that their argument is \( t \). It is clear that IDE (1) and RIDE (2) are particular cases of RIDE (5). In addition, if \( p(x) = 1 \) and \( p(x(t - \tau)) = 1 \), then RIDE (5) includes and improves RIDEs (3) and (4), respectively.

It follows that RIDE (5) can be expressed in the following form:

\[
\frac{dx}{dt} = y,
\]

\[
\frac{dy}{dt} = -a(t)f(t, x, y) - b(t)p(x) - \sum_{i=1}^n h_i(x(t), y(t)) = \int_0^t K(t, \xi) h(\xi, y(\xi)) d\xi.
\]

(6)

The results of this paper are unique and contribute to the field of qualitative theory of ODEs, IDEs and RIDEs. These results might also be beneficial for the authors, who are interested in the qualitative theory of the structure of solutions of ODEs, IDEs, RIDEs and neutral IDEs of any order.

### 2. Qualitative analysis of solutions

We first present basic assumptions related to the qualitative analysis of RIDE (5).
2.1. Assumptions

(A1) Let $b_0, p_0, g_0, h_0, c_i$ and $D_i$ be positive constants. We assume that the following hypotheses are satisfied:

$b(t) \geq b_0, b'(t) \leq 0$ for $t \in \mathbb{R}^+$,

$p(0) = 0, x^{-1} p(x) \geq p_0, (x \neq 0)$ for $x \in \mathbb{R}$,

g(t, x, 0) = 0, y^{-1} g(t, x, y) \geq g_0, (y \neq 0)$ for $t \in \mathbb{R}^+$,

$\forall x, y \in \mathbb{R},$

$|h(t, y)| \leq h_0 |y|$ for $t \in \mathbb{R}^+$, $y \in \mathbb{R},$

$x^{-1} h_i(x, y) \geq c_i, (x \neq 0), \forall x, y \in \mathbb{R},$

$\left| \frac{\partial h_i(x, y)}{\partial x} \right| \leq D_i \forall x, y \in \mathbb{R},$

$\int_0^x h_i(\eta, y) \, d\eta \to \infty$ as $|x| \to \infty$;

(A2) Let $R$ and $D$ be positive constants such that the following assumptions hold:

$max \left\{ \int_0^t |K(t, \xi)| \, d\xi, h_0 \int_0^\infty |K(u, \xi)| \, du \right\} < R,$

$-a(t)f(t, x, y) + g_0 + D\tau \leq -R$ for $t \in \mathbb{R}^+$, $\forall x, y \in \mathbb{R},$

where

$$D = \sum_{i=1}^n D_i, \tau = \max\{\tau_1, \tau_2, \ldots, \tau_n\}. $$

The first main result, which studies the qualitative analysis of solutions of RIDE (5), is below.

**Theorem 2.1:** If (A1) and (A2) are available, then the solutions of RIDE (5) globally exist and are bounded.

**Proof:** To proceed with the proof, a new LK functional $V(t) = V(t, x_t, y_t)$ is defined by

$$V(t) = \sum_{i=1}^n \int_0^x h_i(\eta, y) \, d\eta + b(t) \int_0^x p(\eta) \, d\eta + \frac{1}{2} y^2 + \sum_{i=1}^n \lambda_i \int_{\tau_i}^t \int_{t+\tau}^t y^2(\theta) \, d\theta \, ds + \frac{1}{2} \int_0^t \int_{t}^{\infty} |K(u, \xi)| \, du \, h^2(\xi, y(\xi)) \, d\xi, \tag{7}$$

where $\lambda_i \in \mathbb{R}, \lambda_i > 0,$ which will be fixed later.

We observe that $V(t) = V(t, x_t, y_t) = 0$ if and only if $x = y = 0.$ By considering the LK functional in (7) and assumption (A1), it is derived that

$$V(t) = \sum_{i=1}^n \int_0^x h_i(\eta, y) \, d\eta + b(t) \int_0^x p(\eta) \, d\eta + \frac{1}{2} y^2 + \sum_{i=1}^n \lambda_i \int_{\tau_i}^t \int_{t+\tau_i}^t y^2(\theta) \, d\theta \, ds + \frac{1}{2} \int_0^t \int_{t}^{\infty} |K(u, \xi)| \, du \, h^2(\xi, y(\xi)) \, d\xi,$$

where $|x| \to \infty$ as $|x| \to \infty$.

Suppose that $(x(t), y(t))$ is a solution of RIDE system (6). From this solution, from functional $V(t)$ and RIDE system (6), the time derivative implies that

$$\frac{dV}{dt} = -a(t)f(t, x, y)y^2 + y \sum_{i=1}^n \int_{t-\tau_i}^t y(s) \frac{\partial h_i(x(s), y(s))}{\partial x} \, ds - yg(t, x, y)$$

$$+ y \int_0^t K(t, \xi) h_i(\xi, y(\xi)) \, d\xi + b'(t) \int_0^t p(\eta) \, d\eta$$

$$+ \sum_{i=1}^n (\lambda_i \tau_i) y^2 - \sum_{i=1}^n \lambda_i \int_{t-\tau_i}^t y^2(s) \, ds$$

$$+ \frac{1}{2} h^2(t, y) \int_{t}^{\infty} |K(u, \xi)| \, du$$

$$- \frac{1}{2} \int_0^t |K(t, \xi)| h^2(\xi, y(\xi)) \, d\xi. \tag{9}$$

By assumptions (A1) and (A2) of Theorem 2.1 and the inequality $2|x||y| \leq \omega^2 + v^2$, we can derive the following estimates:

$$-yg(t, x, y) \leq -g_0y^2,$$

$$\left( \frac{d}{dt} b(t) \right) \int_0^x p(\eta) \, d\eta \leq 0,$$

$$\int_0^t K(t, \xi) h_i(\xi, y(\xi)) \, d\xi \leq |y| \int_0^t |K(t, \xi)| |h_i(\xi, y(\xi))| \, d\xi$$

$$\leq \frac{1}{2} \int_0^t |K(t, \xi)| (y^2(t) + h^2(\xi, y(\xi))) \, d\xi$$

$$= \frac{1}{2} \int_0^t |K(t, \xi)| y^2(t) \, d\xi,$$
By considering the above inequalities in the derivative \( \frac{dV}{dt} \), we arrive at

\[
\frac{dV}{dt} \leq - \left( f(t, x, y) \alpha(t) - g_0 - \sum_{i=1}^{n} (\lambda_i + \frac{1}{2} D_i) \right) y^2 + \frac{1}{2} y^2 \int_{0}^{t} |K(t, \xi)| \, d\xi \\
+ \frac{1}{2} \int_{0}^{t} |K(t, \xi)| \, d\xi + \frac{1}{2} \int_{0}^{t} |K(t, \xi)| h^2(\xi, y(\xi)) \, d\xi
\]

It is now notable that the unique possible way for a solution \((x(t), y(t))\) of RIDE system (6) can fail for being defined at the time \(T\) if the condition

\[
\lim_{t \to T^-} (x^2(t) + y^2(t)) = +\infty \tag{10}
\]

is satisfied.

Let \((x(t), y(t))\) be a solution of RIDE system (6) with initial condition \((x_0, y_0)\). We can observe the fact that the LK functional \(V(t, x_t, y_t)\) is positive definite, that is, \(V(t, x_t, y_t) \geq 0\). In addition, this LK functional \(V(t, x_t, y_t)\) decreases through the paths of RIDE system (6), that is, \((d/dt)V(t) \leq 0\). Hence, it is verified that the LK functional \(V(t)\) is bounded on the interval \([0, T]\).

Briefly, from the above discussion, we can derive

\[
\beta(x^2 + y^2) \leq \alpha x^2 + \frac{1}{2} y^2 \leq V(t)
\]

and

\[
\frac{dV}{dt}(t) \leq 0,
\]

where \(\beta = \min(\alpha, 2^{-1})\).

If we integrate the derivative \(dV/dt\) on the interval \([t_0, T]\), then it is notable that

\[
V(T) \leq V(t_0).
\]

Next, this inequality with the above discussion leads to

\[
\beta(x^2 + y^2) \leq V(t) \leq V(T) \leq V(t_0), \quad \forall t \geq T.
\]

Suppose that

\[
V(t_0) = \rho, \rho > 0, \rho \in \mathbb{R}.
\]

Next, it follows that

\[
\rho = \frac{\rho}{\beta}.
\]

Hence, we can derive

\[
|x(t)| \leq \Delta, |y(t)| \leq \Delta, \Delta = \frac{\rho}{\sqrt{\beta}}, \beta > 0, \forall t \geq T > t_0 \geq 0.
\]

This results shows that the idea given by (10) is not possible. By considering the above last inequalities, we can derive that the solutions of RIDE system (6) are bounded on \([0, T]\). Hence, the solutions \(x(t)\) of RIDE (5) and their derivatives are bounded. Hence, the proof of Theorem 2.1 is completed.

**Corollary 2.1:** If assumptions (A1) and (A2) hold, then null solution of RIDE (5) is stable. Moreover, it is asymptotically stable.

**Proof:** From Theorem 2.1, it is known that

\[
V(t, x_t, y_t) \geq \alpha x^2 + \frac{1}{2} y^2 + \sum_{i=1}^{n} \lambda_i \int_{-\kappa}^{0} \int_{t-s}^{t} y^2(\theta) \, d\theta \, ds
\]

and

\[
\int_{0}^{t} \int_{t-s}^{t} |C(u, \xi)| \, du \, h^2(\xi, y(\xi)) \, d\xi.
\]
\[ \frac{d}{dt} V(t, x_t, y_t) \leq -\delta y^2. \]

These two results imply and guarantee that the null solution RIDE (5) is stable, see [4,9,43].

As far as the asymptotic stability of the null solution of RIDE (5), we now consider a set \( I_5 \) defined by

\[ I_5 \equiv \{ (t, x, y) : \frac{d}{dt} V(t, x_t, y_t) = 0 \} \]

Let now apply LaSalle’s invariance principle. In fact, we have that \( (d/dt)V(t, x_t, y_t) \leq -\delta y^2 \), \( \delta > 0, \delta \in \mathbb{N} \).

Hence, we observe that \( (d/dt)V(t, x_t, y_t) = 0 \) necessarily implies that \( y = 0 \). That is, \( y = dx/dt = 0 \). By integrating this equality, we have \( x = \xi, \xi \neq 0, \xi \in \mathbb{N} \). From the invariance principle, if we now write \( y = 0 \) in RIDE system (6), then we get

\[ -g(t, x, 0) - b(t)p(x) - \sum_{i=0}^{n} h_i(x, 0) + \int_0^t K(t, \xi) h(\xi, 0) d\xi = 0. \]

In this case, by the assumptions \( g(t, x, 0) = 0, h_i(x, 0) = 0 \) and \( h(t, 0) = 0 \), we arrive at 
\[ -b(t)p(x) = 0. \]

Since \( b(t) \geq b_0 > 0 \) and \( p(0) = 0, p(x) \neq 0 \) if \( x \neq 0 \), it follows \( p(x) = 0 \). However, \( p(x) = 0 \) if and only if \( x = \xi = 0 \). Hence, it is clear that \( x = y = 0 \). Hence, the last inequality verifies that the largest invariant set contained in \( I_5 \) is \( I(t, 0, 0) \in I_5 \). Therefore, we can arrive that the zero solution of nonlinear RIDE (5) is asymptotically stable. Hence, we complete the proof of Corollary 2.1.

**Theorem 2.2:** We suppose that assumptions (A1) and (A2) hold. If \( (x(t), y(t)) \) is a solution of RIDE system (6) with initial condition \((x_0, y_0)\), then \( y(t) \in L^2[0, \infty) \), that is, the first derivative of the solution \( x(t) \) of RIDE (5) is square integrable.

**Proof:** Let \((x(t), y(t))\) be a solution RIDE system (6) with initial condition \((x_0, y_0)\). By means of assumptions (A1) and (A2), we can derive

\[ \frac{d}{dt} V(t, x_t, y_t) \leq -\delta y^2. \]

Integrating this inequality from \( 0 \) to \( t \), we obtain

\[ 0 \leq V(t, x_t, y_t) \leq V(0, x_0, y_0) - \int_0^t y^2(\xi) d\xi. \]

Next, it follows that

\[ \alpha \int_0^t y^2(\xi) d\xi \leq \alpha \int_0^t y^2(\xi) + V(t, x_t, y_t) \leq V(0, x_0, y_0). \]

Let \( V(0, x_0, y_0) = \rho, (\rho > 0, \rho \in \mathbb{N}) \). From this relation, it is clear that

\[ \int_0^\infty y^2(\xi) d\xi \leq \alpha^{-1} \rho < \infty. \]

We can now conclude that \( y^2(t) \in L^2[0, \infty) \). Thus, we can complete the proof of Theorem 2.2.

**Theorem 2.3:** In addition to the assumptions (A1) and (A2), let

\[ | -a(t)f(t, .) y - g(t, .) - b(t)p(x) | \]

be bounded when \( x \) and \( y \) are bounded. If \((x(t), y(t))\) is a solution of RIDE system (6), then \( y(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

**Proof:** We can derive from Theorems 2.1 and 2.2 that the solutions \( x(t) \) and \( y(t) \) are bounded. In addition, we can get \( y(t) \in L^2[0, \infty) \). Since the boundedness of (11) is satisfied, then from RID system (6) we can derive that \( y'(t) \) is bounded and \( y(t) \in L^2[0, \infty) \). Hence, from both \( y'(t) \) is bounded and \( y(t) \in L^2[0, \infty) \), we can derive that \( y(t) \rightarrow 0 \) as \( t \rightarrow \infty \). By these results, we complete the proof of Theorem 2.3.

3. Numerical example

Consider the below nonlinear RIDE with constant delay, \( \tau = \tau^* = 1 \):

\[ x'' + (11 + t + x^2 + (x')^2)x' + (tx' + 2x' + x' \sin x) + (2 + \exp(-t))(x + x^3) + 2x(t - 1) + x \exp(-x^2) \]

\[ = \int_0^t \exp(-t + \xi)x'(\xi) d\xi, t \geq \xi \geq 0, \]

which is a particular case of RIDE (5).

Hence, we write RIDE (12) as the given RIDE system:

\[ x' = y, \]

\[ y' = -(11 + t + x^2 + y^2)y - (ty + 2y + y \sin x) - (2 + \exp(-t))(x + x^3) - 2x - x \exp(-y^2) + 2 \int_0^t (s) ds + \int_0^t \exp(-t + \xi)y(\xi) d\xi. \]

When we compare RIDE system (13) with RIDE system (6), we follow the following relations, respectively:

\[ a(t) = 1, \forall t \in \mathbb{R}^+, \]

\[ f(t, x, y) = 11 + t + x^2 + y^2 \geq 11, \forall t \in \mathbb{R}^+ \text{ as } x, y \in \mathbb{R}, \]

\[ g(t, x, y) = ty + 2y + y \sin x, \]

\[ g(t, x, 0) = 0, y^{-1}g(t, x, y) = 2 + t + \sin x \geq 1 = g_0, \]

\[ \forall t \in \mathbb{R}^+, \forall y \neq 0 \text{ as } x, y \in \mathbb{R}, \]

\[ \exp(-t) + 2 = b(t) \geq 2 = b_0, \]

\[ b'(t) = -\exp(-t) \leq 0, \forall t \in \mathbb{R}^+, \]

\[ p(x) = x + x^3, p(0) = 0, \]

\[ x^{-1}p(x) = 1 + x^2 \geq 1 = p_0, \forall x \neq 0 \text{ as } x \in \mathbb{R}, \]

\[ h(t, y) = 2y, |h(t, y)| \leq 2|y|, h_0 = 2, \]

\[ \forall t \in \mathbb{R}^+ \text{ as } \forall y \in \mathbb{R}, \]

\[ h_1(x, y) = 2x + x \exp(-y^2), h_1(0, 0) = 0, \]
\[ \begin{align*}
x^{-1} h_1(x, y) &= 2 + \exp(-y^2) \geq 2 = c_1, \\
\forall x \neq 0 \text{ as } x, y \in \mathbb{R}, \\
\frac{\partial h_1(x, y)}{\partial x} &= 2 + \exp(-y^2), \\
\left| \frac{\partial h_1(x, y)}{\partial x} \right| &= 2 + \exp(-y^2) \leq 3 = D_1 \forall x, y \in \mathbb{R},
\end{align*} \]

\[ \int_0^x h_1(\eta, y) \, d\eta = \int_0^x [2\eta + \exp(-y^2)] \, d\eta = x^2 + \frac{1}{2} x^2 \exp(-y^2), \]

\[ x^2 + \frac{1}{2} x^2 \exp(-y^2) \to \infty \text{ as } |x| \to \infty, \]

namely

\[ \int_0^\infty h_1(\eta, y) \, d\eta \to \infty \text{ as } |x| \to \infty, \]

\[ \exp(-t + \xi) = K(t, \xi), t \geq \xi \geq 0, \]

\[ \exp(-u + \xi) = K(u, \xi), u \geq \xi \geq 0, \]

\[ \int_t^\infty \exp(-t + \xi) \, d\xi = 1 - \exp(-t) = \int_t^\infty |K(t, \xi)| \, d\xi, \]

\[ 4 \int_t^\infty \exp(-u + \xi) \, du = h_2^2 \int_t^\infty |K(u, \xi)| \, du \leq 4, \]

\[ \max\left\{ \int_0^t |K(t, \xi)| \, d\xi, h_2^2 \int_t^\infty |K(u, \xi)| \, du \right\} = \max\{1 - \exp(-t), 4\} = 4 < 5 = R, \]

\[ a(t)f(t, x, y) - g_0 - D \tau \star = a(t)f(t, x, y) - g_0 - D \tau \]

\[ \geq 11 - 1 - 3 = 7 > 5 = R, \]

\[ \forall t \in \mathbb{R}^+, \forall x, y \in \mathbb{R}. \]

For the above particular case, the LK functional \( V(t) \) given by (7) reduces to the following LK functional:

\[ V_1(t) = x^2 + \frac{1}{2} x^2 \exp(-y^2) \]

\[ + \frac{1}{4} (\exp(-t) + 2)(2x^2 + x^4) + \frac{1}{2} y^2 \]

\[ + \frac{3}{2} \int_0^t \int_{t+s}^\infty y^2(\theta) \, d\theta \, ds \]

\[ + \int_0^t \int_{t+s}^\infty \exp(-u + \xi) \, d\xi \, du \cdot y^2(\xi) \cdot d\xi. \quad (14) \]

Then, from LK functional (14), we obtain

\[ V_1(t) \geq x^2 + \frac{1}{2} (2x^2 + x^4) + \frac{1}{2} y^2 \]

\[ + \frac{3}{2} \int_0^t \int_{t+s}^\infty y^2(\theta) \, d\theta \, ds \]

\[ + \frac{3}{2} \int_0^t \int_{t+s}^\infty \exp(-u + \xi) \, d\xi \, du \cdot y^2(\xi) \cdot d\xi \]

\[ + \int_0^t \int_{t+s}^\infty \exp(-u + \xi) \, d\xi \, du \cdot y^2(\xi) \cdot d\xi \]

\[ \geq 2x^2 + \frac{1}{2} x^4 + \frac{1}{2} y^2 \geq \frac{1}{2} (x^2 + x^4 + y^2). \]

If we calculate the derivative the LK functional \( V(t) \) given by (14) along solutions of RIDE system (13), then it gives:

\[ \frac{dV_1}{dt} = -(11 + t + x^2 + y^2)y^2 - (2 + t + \sin x)y^2 \]

\[ + y \int_{t-1}^t y(s) (2 + \exp(-y^2(s))) \, ds \]

\[ + 2y \int_0^t y(\xi) \exp(-t + \xi) \, d\xi \]

\[ - \exp(-t) \int_0^t (\eta + \eta^3) \, d\eta \]

\[ + \frac{3}{2} y^2 - \frac{3}{2} \int_t^\infty y^2(s) \, ds \]

\[ + 2y^2 \int_t^\infty \exp(-u + \xi) \, du \]

\[ - 2 \int_0^t \exp(-t + \xi)y^2(\xi) \, d\xi. \]

Next, by using some elementary inequalities, it can be seen that

\[ \frac{d}{dt} V_1(t) \leq -(11 + t + x^2 + y^2)y^2 \]

\[ - (2 + t + \sin x)y^2 + 3|y| \int_{t-1}^t |y(s)| \, ds \]

\[ - (2 + t + \sin x)y^2 + 3|y| \int_{t-1}^t |y(s)| \, ds \]

\[ - \exp(-t) \left( \frac{1}{2} x^2 + \frac{1}{4} x^4 \right) + \frac{3}{2} y^2 \]

\[ - 2 \int_0^t \exp(-t + \xi)y^2(\xi) \, d\xi \]

\[ \leq -(11 + t + x^2 + y^2)y^2 - (2 + t + \sin x)y^2 \]

\[ + \frac{3}{2} \int_{t-1}^t (y^2(t) + y^2(s)) \, ds \]

\[ + \frac{3}{2} \int_{t-1}^t (\frac{1}{2} y^2(t) + 2y^2(\xi)) \exp(-t + \xi) \, d\xi \]

\[ + \frac{3}{2} y^2 - \frac{3}{2} \int_{t-1}^t y^2(s) \, ds \]

\[ + 2y^2 \int_{t-1}^\infty \exp(-u + \xi) \, du \]

\[ - 2 \int_0^t \exp(-t + \xi)y^2(\xi) \, d\xi \]

\[ \leq -(11 + t + x^2 + y^2)y^2 - (2 + t + \sin x)y^2 \]

\[ + \frac{3}{2} y^2 + \frac{3}{2} \int_{t-1}^t y^2(s) \, ds \]

\[ \leq -(11 + t + x^2 + y^2)y^2 - (2 + t + \sin x)y^2 \]
Figure 1. Graph of the exact solution $x(t)$ of RIDE system (12) $\tau = \tau^* = 1$.

Hence, we follow that all the assumptions of Theorems 2.1, 2.2 and Corollary 2.1, that is, (A1) and (A2) hold. In this case, we can conclude that the solutions of RIDE (11) globally exist, stable, asymptotically stable, bounded and square integrable. The same results are valid for the RIDE system (12). We would also like to mention that the graphs given by Figures 1 and 2 show behaviours of solutions of the RIDE system (12), which have been solved by MATLAB-Simulink.

### 4. Conclusion

A nonlinear RIDE of second order was considered. The stability, asymptotic stability, global existence, boundedness of solutions and square integrability of first derivatives of solutions were investigated. The established assumptions are known as sufficient conditions.
and they can be expressed in terms of the considered nonlinear RIDE of second order. The proofs of the results are based on the construction of a nice new LK functional and its usage in the proofs. We obtain three new results and a corollary on the topics. It can be followed that the obtained results extend, include and get better some former results, which are available in the literature. To the best of our knowledge, the considered RIDE (5) was not investigated in the literature to date. This work has contributed to the qualitative theory of ODEs, IDEs and RIDEs of higher order.

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ORCID

Sizar Abid Mohammed http://orcid.org/0000-0002-6239-3693

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