ON THE THOMAS-FERMI APPROXIMATION FOR BOSE
CONDENSATES IN TRAPS

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Abstract

Thomas-Fermi theory for Bose condensates in inhomogeneous traps is revisited. The phase-space distribution function in the Thomas-Fermi limit is $f_0(R, p) \propto \delta(\mu - H_{cl})$ where $H_{cl}$ is the classical counterpart of the self-consistent Gross-Pitaevskii Hamiltonian. No assumption on the large $N$-limit is introduced and, e.g. the kinetic energy is found to be in good agreement with the quantal results even for low and intermediate particle numbers $N$. The attractive case yields conclusive results as well.
I. INTRODUCTION

The recent discovery of the Bose-Einstein condensation of magnetically trapped atoms has spurred a huge amount of theoretical investigations. Most of them are based on the Gross-Pitaevskii equation (GPE) [1] which is the mean-field equation for the condensate wave function (order parameter). The experimental conditions are such that the atomic gas is at very low density and therefore the mean-field approximation gives indeed excellent results [2–5]. Since the number N of atoms involved is generally large, it is natural that also the Thomas-Fermi (TF) approximation is applied quite extensively. This has the advantage of yielding in most cases explicit analytical results of great physical transparency. However, with respect to Fermi statistics, the TF-approach to Bose-Einstein condensation shows some peculiarities which in the past have, in our opinion, not fully been born out. It is the purpose of the present paper to further elaborate on the TF-approach to inhomogeneous Bose systems. We deliberately restrict ourselves here to the TF-limit of the GPE at zero temperature. Finite temperature as well as more elaborated theories like the Bogoliubov approach may be the subject of future work.

In detail the paper is organized as follows. In Section 2 the theoretical aspects of the TF approximation to the GPE are presented in detail. In Section 3 the numerical results obtained with the TF method are compared with the ones coming from the GPE. The last Section is devoted to discussions and an outlook.

II. GROSS-PITAEVSKII EQUATION. THOMAS-FERMI LIMIT

As mentioned in the introduction, the basic equation for Bose condensed atoms confined by magnetic traps is, in the low density limit, given by the GPE for the wave function of the condensate:

\[
\left(-\frac{\hbar^2}{2m} \Delta + V_{ex} + g|\psi|^2\right)\psi = \mu \psi
\]  

(1)

where \(V_{ex}\) is the external potential which for simplicity we have considered to be a spherical harmonic oscillator (for non-spherical geometry see remarks at the end of the paper). The coupling constant is given by \(g = 4\pi\hbar^2a/m\) with \(m\) the atomic mass and \(a\) the scattering
length. The chemical potential $\mu$ is identical with the lowest eigenvalue of the self-consistent potential

$$V = V_{ex} + g|\psi|^2.$$  \hfill (2)

It is useful to note that eq. (1) can also be rewritten as an equation for the density $\rho = |\psi|^2$:

$$\frac{\hbar^2}{2m} \frac{1}{4} \left[ \frac{(\nabla \rho)^2}{\rho^2} - 2 \frac{\Delta \rho}{\rho} \right] + V_{ex} + g\rho - \mu = 0$$  \hfill (3)

In the large $N$ limit one can drop in (1) the kinetic energy or, equivalently, in (3) the gradients terms of the density. This leads to $\rho = (\mu - V_{ex})/g$ what is known in literature \cite{3–5} as the TF solution of the GPE. However, for moderate particle numbers the kinetic energy is not negligible and there is no reason for dropping it in the TF limit. Also in the attractive case the kinetic energy is of crucial importance to avoid collapse.

As it is well known from the case of the Fermi systems \cite{6}, the TF-approximation is based on the assumption of a slowly varying potential so that its gradients can be neglected to lowest order. The TF approximation of the density matrix corresponding to a single wave function is, as a matter of fact, well known \cite{7} and given by:

$$\lim_{\hbar \to 0} \left[ \psi(r) \psi(r') \right]_W = Nc\delta(\mu - H_c) + O(\hbar^2)$$  \hfill (4)

where the index ”$W$” stands for the Wigner transform \cite{3} and $H_c$ is the classical hamiltonian: $H_c = \frac{p^2}{2m} + V_{ex}(R) + g\rho(R)$. The constant $c(\mu)$ of dimension of an inverse of energy is determined from the condition that the wave function must be normalized leading to:

$$\frac{1}{c} = \int \frac{dRdp}{(2\pi\hbar)^3} \delta(\mu - H_c)$$  \hfill (5)

As a consequence the chemical potential $\mu$ must be determined from an independent quantization condition (see below). We will, however, see that in the limit $N \to \infty$ the expression for $\mu$ coincides with the usual.

It is clear from (3) that the kinetic energy has been properly included (as usual in TF theory). One way to understand expression (4) is to write eq.(1) in the form $(\mu - H)\hat{\rho} = 0$ with $\hat{\rho} = \langle \psi | \psi \rangle$ the density matrix. Wigner transforming this equation and remembering that to lowest order in $\hbar$ the Wigner transform of a product of operators is the product of their Wigner transforms \cite{3}, one arrives with $x\delta(x) = 0$ at eq.(4).
A. Self-consistent solution. Repulsive case ($g > 0$)

Let us first consider the solution of the self-consistent problem at the TF level defined by eq.(4) for the repulsive case i.e $g > 0$. From (4) we obtain for the density:

$$\rho(R) = \int \frac{dp}{(2\pi\hbar)^3} f_0(R, p) = \frac{Ncm}{2\pi^2\hbar^3} p_0(R)$$ (6)

where the local momentum is given by:

$$p_0(R) = \sqrt{2m(\mu - V_{ex} - g\rho)}$$ (7)

The self-consistency between eqs.(6,7) is easy to solve analytically and we obtain:

$$\rho = -\frac{KgN^2}{2} + \sqrt{\left(\frac{KgN^2}{2}\right)^2 + KN^2(\mu - V_{ex})}$$ (8)

where

$$K = \frac{2m}{\hbar^2}(\frac{cm}{2\pi^2\hbar^2})^2$$ (9)

It is to be noted that $\rho(R)$ is defined only within the classical region limited by $\mu - V_{ex}(R) = 0$. It is straightforward to expand $\rho$ in the repulsive case $g > 0$ for large values of $N$:

$$\rho \approx \frac{1}{g}(\mu - V_{ex}) - \frac{1}{Kg^3N^2}(\mu - V_{ex})^2 + ....$$ (10)

It is satisfying to see that to leading order one recovers the result corresponding to the total neglect of kinetic energy in (1) (see introduction).

The normalization is directly determined from eq.(5)

$$\frac{1}{c} = \frac{m}{2\pi^2\hbar^3} \int dR p_0(R)$$ (11)

or equivalently

$$N = \int dR \rho(R)$$ (12)

Explicitly one obtains from this equation:

$$1 = 4\pi \left(\frac{2}{m\omega^2}\right)^{3/2} \left\{ -\frac{5KgN}{48}\mu^{3/2} - \frac{K^2g^3N^3}{64} - \mu^{1/2} \right. $$

$$+ \frac{\sqrt{K}}{8} \left(\frac{Kg^2N^2}{4} + \mu\right)^2 \arcsin \left(\frac{\mu}{Kg^2N^2 + \mu}\right)$$ (13)
Using in (12) the asymptotic expansion (11) yields

$$1 = 4\pi \left( \frac{2}{m\omega^2} \right)^{3/2} \frac{2}{15} \frac{\mu^{5/2}}{gN} \left( 1 - \frac{4}{7} \frac{\mu}{Kg^2N^2} \right)$$

which also can directly be derived in expanding (13).

From Eqs. (13) or (14) we can determine the normalization constant $c$ as a function of $\mu$. From (14) we see that to lowest order the normalization constant $c$ drops out and thus in this limit the chemical potential is, as usual, determined by the particle number condition. However, as we will see, via the quantization condition $\mu$ depends on $c$ and thus we can consider (13) as determining the normalization in any case.

**B. Chemical potential and quantization**

The semiclassical density matrix (4) corresponds to a single wave function. In such a case the energy must be determined independently from a quantization condition. This is in fact well known [8]. Formally in our case this is necessary, since we have two open constants $\mu$ and $K$. The equation for the chemical potential $\mu = dE/dN$ is also of no help, since it is equivalent to the particle number condition (12). The standard semiclassical quantization procedure is given by the WKB method. However, in order to have a more explicit formula, we here also apply a slightly simpler method, applicable to the lowest state in a single particle potential [9]. To this end we calculate the smooth accumulated level density (number of states) in TF approximation:

$$N_{\text{TF}}(E) = \int \frac{dR dp}{(2\pi\hbar)^3} \Theta(E - H_c)$$

(15)

For a spherical harmonic oscillator (H.O.) this gives:

$$N_{\text{H.O.}}^{\text{TF}} = \frac{1}{6} \left( \frac{E}{\hbar\omega} \right)^3$$

(16)

Taking for $E$ in (16) the H.O. eigenvalues

$$E \rightarrow E_K = (K + \frac{3}{2})\hbar\omega$$

(17)

with $K = 2n + l$ and inserting (14,17) in the left hand side of (13) yields a semiclassical quantization rule which becomes exact in the 3-D spherical H.O. case. It represents an
approximate quantization relation for an arbitrary potential where the quantized energies very well reproduce the centroid of major shells. This has been tested numerically on a potential of Woods-Saxon type of nuclear dimensions \[9\]. It is evident that in the 1-D case the same procedure leads to the exact WKB quantization rule. For the 3-D case this modified quantization prescription is slightly less accurate than WKB for the lowest eigenvalue but has the advantage to be easier and to be readily applicable also to the deformed case \[9\]. In the present problem the eigenvalue $\mu$ is then determined by:

\[
\frac{27}{8} = \frac{1}{\pi^2\hbar^3} \int dR p_0^3(R)
= \frac{32}{\pi^2\hbar^5} \left\{- \frac{27}{80} \left(\frac{K g^2 N^2}{4}\right)^{1/2} \mu^{5/2} - \frac{19}{24} \left(\frac{K g^2 N^2}{4}\right)^{3/2} \mu^{3/2} - \frac{7}{16} \left(\frac{K g^2 N^2}{4}\right)^{5/2} \mu^{1/2} + \frac{3}{8} \frac{K g^2 N^2}{4} \left(\frac{K g^2 N^2}{4} + \mu\right)^2 \arcsin \sqrt{\frac{\mu}{K g^2 N^2}} + \mu \right\}
+ \frac{1}{16} \left(\frac{K g^2 N^2}{4} + \mu\right)^3 \arcsin \sqrt{\frac{\mu}{K g^2 N^2}} + \mu \right\}.
\]

where we have used $p_0(R)$ from \[7\] with \[8\]. To second order we obtain from \[18\]:

\[
\frac{27}{8} = \frac{4}{\pi\hbar^5} \left(\sqrt{\frac{2m}{K g N}}\right)^3 \left(\frac{2}{m\omega^2}\right)^{3/2} \mu^{9/2} \left(1 - 48 \frac{\mu}{11 K g^2 N^2}\right). \tag{19}
\]

To leading order in the large $N$ limit we obtain from \[13\]-\[14\] and \[18\]-\[19\]:

\[
\mu_0 = \left(\frac{15}{8\pi}\right)^{2/5} \left(\frac{m\omega^2}{2}\right)^{3/5} \left(\frac{gN}{2}\right)^{2/5}
K_0 = \frac{4096}{8505\pi\hbar^4\omega^3} \frac{1}{(gN)^2\mu_0^3}. \tag{20}
\]

This completes the solution in the large $N$ limit.

For later comparison let us also give the standard WKB quantization rule \[10\] which we want to evaluate to leading order :

\[
\frac{\pi}{2} = \sqrt{\frac{2m}{\hbar^2}} \int_{r_1}^{r_2} dr \left[\mu - V(r) - \frac{\hbar^2}{2m} \frac{1}{r^2}\right]. \tag{21}
\]

with (see eq.\[10\])

\[
\mu - V(r) = \frac{1}{K g^2 N^2} (\mu - V_{ex}(r))^2. \tag{22}
\]

The classical turning points $r_1$ and $r_2$ are determined from the solution of the cubic equation:

\[
\frac{1}{\sqrt{K g^2 N^2}} (\mu - V_{ex}) = -\frac{1}{2} \sqrt{\frac{\hbar^2}{2m r}}. \tag{23}
\]

With \[22\] and \[23\] eq.\[21\] can be solved for $\mu$. 

6
C. Kinetic energy

One of the main difficulties with the standard $N \to \infty$ TF limit treated in the literature (see introduction) consists in the inability to calculate the kinetic energy \[3\]. In our approach this does not cause any particular problem and one directly obtains

$$E_{\text{kin}} = \int \frac{dR dp}{(2\pi\hbar)^3 2m} p^2 f_0(R, p) = \frac{N c}{4\pi^2\hbar^2} \int dR p_0^3(R) \tag{24}$$

For example to lowest order we obtain from \((7,10)\)

$$p_0 = \frac{1}{\sqrt{K_0 g N}}(\mu_0 - V_{\text{ex}})\Theta(\mu_0 - V_{\text{ex}}) \tag{25}$$

what yields for the kinetic energy per particle, using \((20)\)

$$\frac{E_{\text{kin}}^{(0)}}{N} = \frac{27}{32} \frac{2\pi^2\hbar^3}{m} \sqrt{\frac{K_0}{2m}} = \frac{27}{32} c_0 \tag{26}$$

This simple result must be contrasted with the usual statement that in the $N \to \infty$ TF limit the kinetic energy can not be evaluated, since it diverges \[3\].

Indeed one can write the kinetic energy as:

$$E_{\text{kin}} = \frac{\hbar^2}{2m} \int dR |\nabla \psi|^2 \tag{27}$$

In the $N \to \infty$ limit we have:

$$\psi_{N \to \infty} = \sqrt{\frac{1}{g}}(\mu_0 - V_{\text{ex}}) \tag{28}$$

and one can readily verify that with \((28)\) $E_{\text{kin}}$ of \((27)\) diverges logarithmically. This result obviously is in contradiction with \((28)\) and we shortly want to elucidate the underlying reason. To this end we first rewrite \((27)\) in a different but obviously equivalent way:

$$E_{\text{kin}} = \frac{\hbar^2}{2m} \int dR |\nabla \psi|^2 = \int \frac{dR dp}{(2\pi\hbar)^3 2m} p^2 \tilde{f}_0(R, p) \tag{29}$$

with $\tilde{f}_0$ given by the Wigner transform of the density matrix corresponding to \((28)\):

$$\tilde{f}_0(R, p) = \int ds e^{-i p s / \hbar} \psi_{N \to \infty}(R + \frac{s}{2}) \psi_{N \to \infty}(R - \frac{s}{2}) \tag{30}$$

Since $\tilde{f}_0 \neq f_0$ we argue that \((30)\) is not the correct $\hbar \to 0$ limit of the distribution function because it is not solution of the $\hbar \to 0$ limit of the Schrödinger equation \((1)\)
\[(H_c - \mu)f_0 = 0 \quad (31)\]

Only (4) is the correct solution of this equation which yields for large \(N\):

\[
f_{0N\to\infty} = Nc\delta\left(\frac{1}{K_0 g^2 N^2} (\mu_0 - V_{ee})^2 - \frac{p^2}{2m}\right) \quad (32)
\]

One checks that upon projection onto \(r\)-space (32) gives the correct lowest order expression for the density (see eq.(10)). Therefore both Wigner functions (30) and (32) yield the same leading order density. However, in spite of being a very suggestive non-local generalization of the lowest order local density expression, eq.(30) has to be rejected on the above given grounds and the divergency of (27) is an artifact. On the contrary the lowest order contribution to the kinetic energy is given by (24). Via (14,19) it is straightforward to calculate the next to leading order correction to (26). It should, however, be remembered that \(1/N\) correction do not go in parallel with powers in \(\hbar\) and that \(1/N\) corrections also can come from \(\hbar^2\) corrections to (3) which involve second order gradients of the potential. In any case the Wigner-Kirkwood expansion of the density matrix is an asymptotic expansion which in no way can recover the nonanalytic behavior in \(\hbar\) of the quantal solution. In the present problem the nonanalyticity in \(\hbar\) of the quantal solution entails a nonanalytic behavior in \(1/N\) (see eq.(16) of ref. [3]) and therefore a WK-expansion can never recover the quantal behavior in \(1/N\). It is well known that an asymptotic expansion has to be stopped at a point where the difference to the exact solution is minimal. Afterwards the expansion starts to diverge again. In this work we do not intend to develope a systematic expansion simultaneously in \(\hbar\) and \(1/N\). We rather want to give a complete solution to lowest order in \(\hbar\), i.e. on the TF level.

**D. The attractive case \((g < 0)\)**

It seems that recently, Bose-Einstein condensation has been observed also for the case of negative scattering length \((^{11}\text{Li atoms})\) [11].

For \(g < 0\) the Gross-Pitaevskii approach leads to metastability for particle numbers \(N \leq 1400\) [2]. For large particle numbers the system collapses. For the attractive case \((a < 0)\) the correct treatment of the kinetic energy is crucial in the TF-limit, since otherwise
no stability can be achieved. Formally the TF solution for the density is the same as in (8) with, however, the sign of the first member reversed:

\[ \rho = \frac{K|g|N^2}{2} + \sqrt{(\frac{KgN^2}{2})^2 + KN^2(\mu - V_{ex})} \]  

(33)

Contrary to the repulsive case no large \( N \) expansion is possible here. Therefore the TF solution has to be considered in full. In the next section comparison with quantal results will be given.

III. NUMERICAL RESULTS

In this section we proceed to a detailed numerical comparison of the semiclassical approximations with the exact quantum mechanical results. Along this section energies and lengths are given in harmonic oscillator units: \( \hbar \omega \) and \( a_{HO} = \sqrt{\hbar/2m\omega} \), respectively. First in Table 1 we present the chemical potential (\( \mu \)) and the kinetic (\( e_{kin} \)), harmonic oscillator (\( e_{HO} \)) and the self-interaction (\( e_\rho \)) energies per particle calculated quantally and in the full Thomas-Fermi approximation (8,13,18) as a function of the number of atoms enclosed in the trap. We have considered Cs atoms (as was done in Ref. [12]), the frequency of the harmonic oscillator has been chosen to be \( \omega = 20\pi s^{-1} \) and the scattering length to be \( a = 3.2 \times 10^{-9} m \).

In Table 2 we present the results for the chemical potential and the kinetic energy per particle number beyond 20000. In addition to the quantum mechanical and the full Thomas-Fermi results, we also include the results for the large \( N \) limit (26) and those obtained using the WKB quantization rule (in the large \( N \) limit). Notice that in the large \( N \) limit, the WKB chemical potential coincides with the TF one and the kinetic energy is given also by (26) but with \( c_0 \) replaced by the one calculated from \( K \) via eq. (21). From Tables 1, 2 we see that for instance the results of the full TF solution are in very satisfying agreement with the quantal results over the whole range of particle numbers.

The numbers presented in Table 2 indicate that the asymptotic values of the chemical potential and the kinetic energy are obtained only for very large number of particles (\( N \simeq 10^5 - 10^6 \)). We also realise that the WKB quantization rule yields quite similar results though in fact slightly worse ones than our simpler TF-quantization rule (18,19). Though globally the semiclassical results of Tables 1 and 2 are quite satisfactory, one nevertheless remarks
some unexpected features. For instance the kinetic energy in the TF approximation is larger than the exact values for small numbers of particles whereas it undershoots the quantum values quite considerably in the large $N$ limit. We will come back to a more detailed analysis of this behavior in the discussion section.

Next let us compare in Fig.1 and 2 the densities (normalized to unity) in TF-approximation and calculated exactly for small (200) and large (200000) particle numbers. As expected, the TF densities almost agree with the quantal ones for very large particle numbers. In view of the still quite reasonable expectation values shown in Table 1 for $N=200$, the strong deviation of the TF density from the quantal result is somewhat a surprise. However, one always should remember that the TF-solution for the densities is to be understood as a distribution (see eq.4) which for expectation values of ”slowly varying” operators can still yield very reasonable values in spite of the fact that the detailed shape may only be a charicature of the exact one.

In Figs.3 and 4 we show the self-consistent potentials $V = V_{ex} + g\rho$ corresponding to the densities of Figs.1 and 2. Not astonishingly $V$ deviates from the harmonic oscillator only slightly for $N = 200$, both quantally and semiclassically. On the contrary for $N = 200000$, the potential $V$ deviates strongly from $V_{ex}$ being practically a constant equal to $\mu$ up to the classical turning point from where the harmonic oscillator takes over quite abruptly. Again, both quantal and TF-solution are in close agreement. Fig.4 also teaches us why the TF-approximation (4) to the quantal distribution function is very good for large $N$. The distribution function corresponds to a wavefunction with very large energy $\mu$. In phase space it therefore is very much concentrated around the surface of the hypersphere with radius $\mu$.

Let us now present the attractive case for the same atoms and external potential with , however, the scattering length $a = -1.0 \times 10^{-9}m$. In Table 3 we again show chemical potential and kinetic, harmonic oscillator and self-interaction energies per particle as a function of the particle number in TF and quantal calculation. For small particle numbers ($N \leq 1000$) the agreement of TF with the quantal case is of similar quality as in the repulsive case. However for $N \geq 1000$ the agreement quickly deteriorates, indicating that the whole mean field approximation breaks down. Indeed even quantally the solution of the GPE (1) becomes unstable for $N > 1500$ for Cs atoms.
In Figs. 5, 6, 7 and 8 we also show the densities (normalized to unity) and self-consistent potentials for the particle numbers $N = 250$ and $N = 1500$. We see that, whereas the case $N = 250$ is not dissimilar to the corresponding one with $a > 0$, for $N = 1500$ the situation becomes quite unfavorable for the TF approximation. This is for instance manifest in looking at the graph for the densities. In the attractive case TF and quantal solutions diverge with increasing $N$ whereas in the repulsive case they converge.

In Figs. 9 and 10 we plot the kinetic energy density per particle ($\tau/N$) calculated quantally and in the TF-approximation for $N = 200000$ in the repulsive case and for $N = 250$ in the attractive case. In these Figures the quantal kinetic energy density is given by:

$$\tau = |\nabla \psi|^2 - \frac{1}{4} \Delta \rho \approx \frac{1}{4} \left[ \frac{(\nabla \rho)^2}{\rho} - \Delta \rho \right]$$

(34)

in order to compare with the TF one according to Ref. [6].

For large number of particles, when the density profile has a relatively flat region at the interior (see Fig.2), the quantal kinetic energy density is peaked at the surface (see eq. (34)) whereas the TF one is rather a bulk term (see eq. (24)). Inspite of the rather different form of the quantal and TF kinetic energy densities in this case, the corresponding integrals are in good agreement (see Tables II). This fact points again to the distribution character of the TF-kinetic energy density. However, if the number of particles is small, the quantal and TF kinetic energy density profiles are quite similar. This is due to the fact that the particle density fall-off abruptly from $R = 0$ (see Fig.5) and consequently its derivatives contribute in all the range of $R$.

**IV. DISCUSSION AND OUTLOOK**

In the preceding sections we have derived the Thomas-Fermi approximation i.e. the $\hbar \to 0$ limit of the density matrix corresponding to the wavefunction of the Bose condensate of atoms confined by magnetic traps. We have pointed out some misconceptions on this point which appeared in the past in the literature which for instance prevented the direct calculation of the kinetic energy in the large $N$ limit. On the contrary with our Thomas-Fermi approach the evaluation of the kinetic energy causes no problem and the results are globally in quite satisfactory agreement with the quantal solution of the Gross-Pitaevskii
equation. However from Table 2 we see that the kinetic energy in the TF limit does not have the correct asymptotic behavior as a function of particle number. As was shown by Pitaevskii and Stringari [3], this is due to nonanalytic (logarithmic) corrections which cannot be accounted for by a pure TF approach and needs a partial resummation of all orders in $\hbar$. However, for particle numbers where the kinetic energy represents a significant fraction of the total energy the TF expression yields very satisfying results for $E_{\text{kin}}$. This example shows again that the semiclassical approximations are a powerful tool but not devoid of subtleties and pitfalls. As a matter of fact also in this paper we, for simplicity, avoided to develop the full complexity of the theory. One major simplification resides in the fact that we assume a spherical trap. This results in an isotropic momentum distribution $f_0 \propto \delta(\mu - H_c)$ where $H_c = \frac{p^2}{2m} + V$ is the classical Hamiltonian. Deforming the trap leads to a nontrivial modification of the theory, since squeezing the condensate wavefunction in one direction and relaxing in the other entails in turn a deformation of the momentum distribution which is opposite to the spatial one, i.e. momenta are strongest in the squeezed direction and lowest in the long direction of the deformation [13]. Our TF approach can also be useful for the evaluation of collective excitations of droplets of small or intermediate sizes. Such a situation is in fact well known from the zero sound giant quadrupole vibrations of finite nuclei where to first approximation the restoring force of the vibration is given by the energy stored in the deformation of the Fermi sphere [6]. On the other hand the deformation of the momentum distribution of condensed atoms has also been revealed experimentally in directly measuring the momentum distribution of the expanding particles, once the deformed trap has been turned off. The detailed determination of the anisotropy of the momenta (which may be position dependent) is theoretically a not completely trivial task in the general case and we will elaborate on this in future work. In the present case, however, there exists an evident first guess of the momentum deformation which results from a scaling argument of the harmonic oscillator coordinates. Assuming a prolate quadrupole deformation in the $z$-direction we have to replace the classical Hamiltonian in (4) by:

$$\tilde{H}_c = \frac{1}{2m} \left[ \frac{\omega_0^2}{\omega_z^2} p_x^2 + \frac{\omega_0^2}{\omega_z^2} p_y^2 \right] + V \left( \frac{\omega_x}{\omega_0} x, \frac{\omega_y}{\omega_0} y, \frac{\omega_z}{\omega_0} z \right)$$

(35)

where the ratios $\frac{\omega_z}{\omega_0}$ and $\frac{\omega_x}{\omega_0}$ are the frequency relations in $z$ and $x, y$ with respect to the spherical case ($\omega_0$). From (33) one easily calculates the so-called aspect ratio in the TF-
approximation

\[ \sqrt{\frac{p_z^2}{p_x^2}} = \frac{\omega_z}{\omega_x} \]  

(36)
a result which has been given previously [3]. One other important consequence of the momentum deformation is that with (35) the moment of inertia of the condensate becomes equal to the irrotational flow value [14]. On the contrary using (3) with the isotropic momentum distribution the rigid momentum of inertia results. Consequently the deformed case needs more detailed studies which we reserve to future work. It is also evident that the present TF approach can be extended to finite temperature and to the Bogoliubov theory.

Another interesting subject of a more formal aspect is the evaluation of the \( \hbar \)-correction to the present lowest order theory. In principle this can easily be performed in posing in (3)
\[ \rho = \rho_0 + \hbar^2 \rho_2 \] and
\[ \mu = \mu_0 + \hbar^2 \mu_2 \] and properly sorting out different powers in \( \hbar \). However the proper elimination of divergencies and handling the normalization (12) and quantization (18) are slightly subtle problems.

Investigations on the above mentioned directions are in progress.

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TABLE I. Chemical potential ($\mu$), kinetic energy ($e_{\text{kin}}$), harmonic oscillator energy ($e_{\text{HO}}$) and self-interaction energy ($e_{\rho}$) per particle in harmonic oscillator units ($\hbar\omega$) calculated quantally (QM) and in the Thomas-Fermi approximation (TF) for several numbers of atoms in the traps. The frequency of the harmonic oscillator is $\omega=20\pi s^{-1}$ and the scattering length is $a=3.2x10^{-9}$ m.

TABLE II. Chemical potential ($\mu$) and kinetic energy ($e_{\text{kin}}$) per particle for large number of atoms in the trap (N). The chemical potential is calculated quantally (QM), with the full Thomas-Fermi approach (TF) and with the asymptotic formula for large N (TF$_{N\rightarrow\infty}$). The kinetic energy is obtained quantally, with the exact TF approximation, with the asymptotic TF for large N and using the WKB quantization in the limit of large number of atoms. The frequency of the harmonic oscillator and the scattering length are the same as in Table I.

TABLE III. The same as in Table I but with a scattering length $a=-1.0x10^{-9}$ m.
| $N$  | $\mu$ (QM) | $\mu$ (TF) | $e_{\text{kin}}$ (QM) | $e_{\text{kin}}$ (TF) | $e_{\text{HO}}$ (QM) | $e_{\text{HO}}$ (TF) | $e_{\rho}$ (QM) | $e_{\rho}$ (TF) |
|------|------------|------------|-----------------------|-----------------------|----------------------|----------------------|----------------|----------------|
| 200  | 1.688      | 1.642      | 0.696                 | 0.700                 | 0.811                | 0.804                | 0.080          | 0.069          |
| 400  | 1.806      | 1.766      | 0.654                 | 0.661                 | 0.865                | 0.851                | 0.144          | 0.127          |
| 600  | 1.927      | 1.877      | 0.622                 | 0.630                 | 0.912                | 0.894                | 0.196          | 0.177          |
| 800  | 2.036      | 1.978      | 0.597                 | 0.603                 | 0.955                | 0.934                | 0.242          | 0.220          |
| 1000 | 2.134      | 2.071      | 0.575                 | 0.581                 | 0.944                | 0.970                | 0.282          | 0.260          |
| 1200 | 2.225      | 2.157      | 0.557                 | 0.561                 | 1.031                | 1.005                | 0.319          | 0.296          |
| 1400 | 2.310      | 2.238      | 0.541                 | 0.544                 | 1.065                | 1.037                | 0.352          | 0.329          |
| 1600 | 2.389      | 2.315      | 0.528                 | 0.528                 | 1.065                | 1.068                | 0.382          | 0.359          |
| 1800 | 2.464      | 2.388      | 0.515                 | 0.514                 | 1.127                | 1.097                | 0.411          | 0.388          |
| 2000 | 2.535      | 2.457      | 0.503                 | 0.502                 | 1.158                | 1.125                | 0.437          | 0.415          |
| 4000 | 3.112      | 3.025      | 0.431                 | 0.417                 | 1.395                | 1.356                | 0.643          | 0.626          |
| 6000 | 3.550      | 3.461      | 0.390                 | 0.369                 | 1.577                | 1.536                | 0.792          | 0.778          |
| 8000 | 3.914      | 3.825      | 0.363                 | 0.336                 | 1.729                | 1.687                | 0.911          | 0.901          |
| 10000| 4.231      | 4.142      | 0.343                 | 0.312                 | 1.862                | 1.820                | 1.013          | 1.005          |
| 12000| 4.513      | 4.426      | 0.327                 | 0.293                 | 1.981                | 1.939                | 1.103          | 1.097          |
| 14000| 4.770      | 4.684      | 0.314                 | 0.277                 | 2.089                | 2.047                | 1.184          | 1.180          |
| 16000| 5.007      | 4.921      | 0.303                 | 0.265                 | 2.189                | 2.147                | 1.258          | 1.255          |
| 18000| 5.228      | 5.143      | 0.294                 | 0.254                 | 2.282                | 2.240                | 1.326          | 1.324          |
| 20000| 5.435      | 5.350      | 0.285                 | 0.244                 | 2.369                | 2.328                | 1.390          | 1.389          |
### TABLE II

| $N$   | $\mu$(QM) | $\mu$(TF) | $\mu$(TF$_{N\to\infty}$) | $e_{kin}$(QM) | $e_{kin}$(TF) | $e_{kin}$(TF$_{N\to\infty}$) | $e_{kin}$(WKB$_{N\to\infty}$) |
|-------|-----------|-----------|---------------------------|---------------|---------------|-------------------------------|-------------------------------|
| 20000 | 5.435     | 5.350     | 5.196                     | 0.285         | 0.244         | 0.256                         | 0.238                         |
| 30000 | 6.322     | 6.242     | 6.111                     | 0.255         | 0.210         | 0.218                         | 0.202                         |
| 40000 | 7.051     | 6.973     | 6.856                     | 0.236         | 0.187         | 0.194                         | 0.180                         |
| 50000 | 7.677     | 7.603     | 7.496                     | 0.222         | 0.173         | 0.177                         | 0.165                         |
| 100000| 10.231    | 9.972     | 9.891                     | 0.182         | 0.133         | 0.134                         | 0.125                         |
| 150000| 11.763    | 11.701    | 11.633                    | 0.162         | 0.113         | 0.114                         | 0.106                         |
| 200000| 13.170    | 13.112    | 13.051                    | 0.149         | 0.101         | 0.102                         | 0.095                         |
| 250000| 14.381    | 14.326    | 14.270                    | 0.140         | 0.093         | 0.093                         | 0.087                         |

### TABLE III

| $N$ | QM  | TF  | QM  | TF  | QM  | TF  | QM  | TF  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     | 250 | 250 | 500 | 500 | 1000| 1000| 1500| 1500|
| $\mu$ | 1.424 | 1.437 | 1.338 | 1.369 | 1.120 | 1.212 | 0.691 | 1.009 |
| $e_{kin}$ | 0.788 | 0.774 | 0.815 | 0.802 | 0.926 | 0.872 | 1.240 | 0.974 |
| $e_{HO}$ | 0.723 | 0.726 | 0.691 | 0.702 | 0.613 | 0.646 | 0.472 | 0.579 |
| $e_{\rho}$ | -0.039 | -0.032 | -0.084 | -0.070 | -0.209 | -0.153 | -0.511 | -0.272 |
FIGURE CAPTIONS

Figure 1. Density (normalized to unity) of 200 atoms in a spherical trap in $a_H^{-3}$ units) as a function of the distance (in $a_H$ units) in the repulsive case calculated from the solution of the GPE (solid line) and using the TF approach described in the text (dashed line).

Figure 2. The same as Figure 1 but with 200000 atoms in the trap.

Figure 3. Self-consistent potential (in $\hbar \omega$ units) corresponding to a spherical trap containing 200 atoms as a function of the distance ($a_{HO}$ units) in the repulsive case calculated from the solution of the GPE (solid line) and using the TF approach described in the text (dashed line).

Figure 4. The same as Figure 3 but with 200000 atoms in the trap.

Figure 5. The same as Figure 1 but with 250 atoms in the trap in the attractive case.

Figure 6. The same as Figure 1 but with 1500 atoms in the trap in the attractive case.

Figure 7. The same as Figure 3 but with 250 atoms in the trap in the attractive case.

Figure 8. The same as Figure 3 but with 1500 atoms in the trap in the attractive case.

Figure 9. Kinetic energy density per particle of a spherical trap (in $\hbar \omega a_H^{-3}$ units) containing 200000 atoms as a function of the distance (in $a_{HO}$ units) in the repulsive case calculated from the solution of the GPE (solid line) and using the TF approach described in the text (dashed line).

Figure 10. The same as Figure 9 but with 250 atoms in the trap in the attractive case.