Point Symmetric 2-Structures

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Dedicated to Heinrich Wefelscheid

Abstract. We show that every symmetric 2-structure \((P, G_1, G_2, K)\) of the class (III) [cf. Karzel H et al. (Result. Math., submitted)] is point symmetric, i.e. any two orthogonal chains \(A, B \in K\) intersect in exactly one point and that any two points \(a, b \in P\) have exactly one midpoint \(m := a \ast b\) (with \(\bar{m}(a) = b\) where \(\bar{m}\) is the unique symmetry in the point \(m\)). \(\bar{P} := \{\bar{p} \mid p \in P\}\) is invariant, i.e. \(\forall a, b \in P : \bar{a} \circ \bar{b} \circ \bar{a} \in \bar{P}\). Therefore the pair \((P, \bar{P})\) is an invariant regular involution set and the loop derivation in a point \(o \in P\) gives a K-loop \((P, +)\) uniquely 2-divisible.

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1. Introduction and Notations

This paper is a continuation of our investigations on symmetric 2-structures [5]. This is a generalization of double symmetric 2-structures (cf. [4]) which are closely related with sharply 2-transitive permutation groups [1,6]. We will use the same definitions and notations as in [5] and \(\Sigma := (P, G_1, G_2, K)\) will denote a symmetric 2-structure. We recall that for any point \(p \in P\) and any \(i \in \{1, 2\}\) \([p]_i\) denotes the generator of \(G_i\) passing through the point \(p\) and that two points \(a, b \in P\) are called parallel if there is an \(i \in \{1, 2\}\) such that \([a]_i = [b]_i\) and not parallel otherwise. With \(P^{(2)}\) we denote the set of all pairs of not parallel points and then (by definition of a 2-structure) to any pair \((a, b) \in P^{(2)}\) there exists exactly one chain \(K \in K\)—which we denote by \(\overline{a, b}\)—with \(a, b \in K\). We collect some properties of \(\Sigma\):

Theorem 1.1. \(\forall A, B \in K: \overline{A(B)} \in K\) and \(\overline{\overline{A(B)}} = \overline{A \circ B \circ A}\) hence \(\tilde{K} := \{\tilde{K} \mid K \in K\}\) is invariant.
In [5] we presented a classification of symmetric 2-structures based on the cardinality of the set $(p, K) := \{ L \in \mathcal{K} \mid p \in L \cap L \cup K \}$ with $p \in K$ of all chains passing through the point $p$ and which are orthogonal to the chain $K$. The symmetric 2-structures split into the three classes (cf. [5, Theorem 3.10]):

(I) There is a pair $(p, K) \in P \times \mathcal{R}$ with $p \in K$ and $|(p \perp K)| > 1$.

(II) There is a pair $(p, K) \in P \times \mathcal{R}$ with $p \in K$ and $(p \perp K) = \emptyset$.

(III) There is a pair $(p, K) \in P \times \mathcal{R}$ with $p \in K$ and $|(p \perp K)| = 1$.

An automorphism $\alpha$ of 2-structure $(P, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R})$ is called point reflection if $\alpha$ is involutory, has exactly one fixpoint $p$ and $\alpha(B) = B$ for any chain or generator $B \in \mathcal{B} := \mathcal{K} \cup \mathcal{G}_1 \cup \mathcal{G}_2$ passing through $p$ (cf. [5, Definition 2.1.]). In this paper we assume that each symmetric 2-structure $(P, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R})$ is of class (III). This condition is equivalent to one of the following (cf. [5, Theorem 3.13]):

1. For each pair $(x, X) \in P \times \mathcal{R}$ with $x \in X : |(x \perp X)| = 1$.
2. For every $x \in P$ there is exactly one reflection in the point $x$.

We denote the only chain of the set $(p \perp K)$ by the same symbol and here we can form $(p \perp \perp K) := (p \perp (p \perp K))$. By (2) to each point $p \in P$ there corresponds exactly one reflection $\tilde{p}$ in $p$. By [5, Theorem 3.4 and Corollary 3.5] we have the first statement of the following theorem:

**Theorem 1.2.** If $K \in \mathcal{K}$ and $p \in P$ then:

1. $p \in K \Leftrightarrow \tilde{p}(K) = K \Leftrightarrow \tilde{K} \circ \tilde{p}$ is involutory $\Leftrightarrow \exists L \in \mathcal{R} : \tilde{K} \circ \tilde{p} = \tilde{L}$.
2. If $p \notin K$ and $L \in \mathcal{R}$ with $\tilde{K} \circ \tilde{p} = \tilde{L}$ then $L = (p \perp K)$.
3. If $p \notin K$ then $|(p \perp K)| = 1$.
4. $\forall (a, b) \in P^{(2)} : \langle a \perp b \rangle \cap \langle b \perp a \rangle = \emptyset$.
5. $\forall a, b \in P : \tilde{\alpha}(b) = \tilde{\alpha} \circ \tilde{b} \circ \tilde{\alpha}$ hence $\tilde{P} := \{ \tilde{p} \mid p \in P \}$ is invariant.

**Proof.** (2): Since $\tilde{K} \circ \tilde{L} = \tilde{p} \neq id, L \neq K$ and there is a $x \in L \setminus K$ hence $\tilde{L}(x) = x, \tilde{K}(x) \neq x$. Moreover $\tilde{L}(\tilde{K}(x)) = \tilde{K}(\tilde{L}(x)) = \tilde{K}(x)$ hence $L = x, \tilde{K}(x) \perp K$ and $\tilde{L}(p) = \tilde{K} \circ \tilde{p}(p) = \tilde{K}(p) = p$, i.e. $p \in L$ and so $L = (p \perp K)$.

(3) Since $p \notin K, q := \tilde{K}(p) \neq p$ and $(p, q) \in P^{(2)}$ hence $(p \perp K) = \{ \overline{p, q} \}$.

(5) is a consequence of (1) and Theorem 1.1. \( \Box \)

A symmetric 2-structure of class (III) is called point symmetric if $|K \cap L| = 1$ for any $K, L \in \mathcal{R}$ with $K \perp L$. If $a, b \in P$ are two given points then a point $m$ is called midpoint of $a$ and $b$ if $\tilde{m}(a) = b$. Let $a * b$ denote the set of all midpoints of $a$ and $b$.

In [5] we proved that any two parallel points $a$ and $b$ have exactly one midpoint (cf. [5, Theorem 3.15]).

In this paper we construct for any two points $x, y$ (not necessarily parallel) the (uniquely determined) midpoint $x * y$ (cf. Theorem 2.2). From this it follows that each symmetric 2-structure of class (III) is already point symmetric (cf. Theorem 2.4).
Let \( \tilde{P} := \{ \tilde{p} \mid p \in P \} \) and \( \sim : P \to \tilde{P}; \ x \mapsto \tilde{x} \). From Theorem 1.1 follows that \((P, \sim)\) is an invariant involution set and from Theorem 2.4 that \( \tilde{P} \) acts regularly on \( P \) hence \((P, \sim)\) is a point reflection structure and by [2, page 33 6.1.(3)] we have:

**Theorem 1.3.** If \( o \in P \) is fixed, \( p' := o \ast p, p^+ := \tilde{p}' \circ o \) for \( p \in P \) and if we set for \( a, b \in P, a + b := a^+ (b) \), then \((P, +)\) is a \( K \)-loop uniquely \( 2 \)-divisible.

Moreover we have (cf. Theorem 4.1.(4) for statement (1))

**Theorem 1.4.** Let \( a, b \in P \) and \( K \in \mathcal{K} \) then:

1. \( \tilde{a} \circ \tilde{b} = \tilde{ab} \)
2. If \([a]_1 = [o]_1 \) and \([b]_2 = [o]_2 \) then \( a + b = b + a \).
3. \( \tilde{K} \) is an involutory automorphism of the invariant involution set \((P, \sim)\) (in particular if \( m := a \ast b \) then \( \tilde{K} (m) = \tilde{K} (a) \ast \tilde{K} (b) \)) and an antiautomorphism of \((P, \boxdot)\).
4. If \( o \in K \) then \( \tilde{K} \) is an involutory automorphism of the loop \((P, +)\) interchanging the “axes” \( X := [o]_1 \) and \( Y := [o]_2 \).

### 2. Each Symmetric 2-Structure of Class (III) is Point Symmetric

**Lemma 2.1.** Let \((a, b) \in P^{(2)}, c \in P \) and \( \tau := \tilde{a} \circ \tilde{b} \) then \([c]_i \neq [\tau (c)]_i \) for \( i \in \{1, 2\} \).

**Proof.** (1) Let \( A := (a \perp a, b) \), \( B := (b \perp a, b) \) and \( K = \tilde{a}, b \) then by Theorem 1.2 \( \tilde{a} = \tilde{A} \circ \tilde{K}, \tilde{b} = \tilde{K} \circ \tilde{B}, \tau = \tilde{A} \circ \tilde{B}, \tau (K) = K \) and \( \text{Fix} \tau = \text{Fix} (\tilde{A} \circ \tilde{B}) = A \cap B = 0 \) (by Theorem 1.2.(4)). \( K \cap [c]_i \) consists of exactly one point \( k \). Assume \([c]_i = [\tau (c)]_i \) then \( \tau (k) = \tau (K \cap [c]_i) = \tau (K) \cap \tau ([c]_i) = K \cap [c]_i = k \), a contradiction to \( \text{Fix} \tau = \emptyset \). \( \square \)

**Theorem 2.2.** Let \( \Sigma := (P, \mathcal{G}_1, \mathcal{G}_2, \mathcal{K}) \) be a symmetric 2-structure of class (III) and let \((a, b) \in P^{(2)} \). Then:

1. \( a \) and \( b \) have exactly one midpoint.
2. If \( m = a \ast b \) then \( ma = a \ast ba \) and \( am = a \ast ab \) and moreover \( \tilde{m} (ab) = ba \).
3. If \( m_1 = a \ast ab \) and \( m_2 = ab \ast b \) then \( m_2 m_1 = a \ast b \).
4. For \( x, y \in P \) we have the formulas:
   \( x \ast y = y \ast x \), \( (x \ast y) x = x \ast (yx) \), \( y (x \ast y) = y \ast (yx) \)
   \( x \ast y = (xy) \ast (yx) = (x \ast y) (y \ast yx) \).
5. \([ab]_1 = \{ \tilde{x} (b) \mid x \in [a]_1 \}, [ab]_2 = \{ \tilde{x} (b) \mid x \in [a]_2 \} \) and \([b \ast a]_i = \{ b \ast x \mid x \in [a]_i \} \).

**Proof.** By [5, Theorem 3.15.], each of the pairs of parallel points \( a \) and \( ab, a \) and \( ba, b \) and \( ba, \) and \( b \) and \( ab \) has exactly one midpoint \( m_1, m_2, m'_1 \) and \( m'_2 \). If \( C := a, b \) and \( D := \tilde{ab}, \tilde{ba} \) then by the proof of [5, Theorem 3.15.], \( \tilde{C} (m_1) = m_2, \)
\[ \tilde{C}(m'_1) = m'_2 \] and \( \tilde{D}(m_1) = m'_2 \). \( \tilde{D}(m_2) = m'_1 \). Thus \( m_2m_1, m'_2m'_1 \in C \) and \( m'_2m_1, m_2m'_1 \in D \). We consider the map \( \tau := m'_1 \circ \tilde{m}_1 \) and obtain:

\[ \tau([a]_2) = \tilde{m}'_1([ab]_2) = \tilde{m}'_1([b]_2) = [ba]_2 = [a]_2. \]

Since \((a, b) \in P^{(2)}\) we have \([m_1]_1 = [a]_1 \neq [b]_1 = [m'_1]_1\). Assume \([m_1]_2 \neq [m'_1]_2\) then \((m_1, m'_1) \in P^{(2)}\) and we get a contradiction with Lemma 2.1 since \(\tau([a]_2) = [a]_2\). Therefore \([m_1]_2 = [m'_1]_2\) and in the same way, \([m_2]_1 = [m'_2]_1\). This gives us \(m := m_2m_1 = m'_2m'_1 = m'_2m_1 = m'_2m'_1 \in C \cap D\), i.e. \(m\) is the midpoint of \(a\) and \(b\) and by [5, Theorem 3.14.(2)] \(m\) is unique. Moreover \(\tilde{m}(ab) = \tilde{m}([a] \cap [b]_2) = [\tilde{m}(a)]_1 \cap [\tilde{m}(b)]_2 = [b]_1 \cap [a]_2 = ba\). Hence (1), (2) and (3) are proved. (4) and (5) are consequences of the previous items.

**Corollary 2.3.** Let \(K \in \mathfrak{R}, a, b \in K, a \neq b\) and \(i \in \{1, 2\}\). Then:

1. If \(L = (a \perp K), \{c\} = [b]_i \cap L\) and if \(m\) is the midpoint of \(b, c\) then \([m]_{3-i} = [a]_{3-i}\).
2. If \(m = ba\) or \(m = ab\) and \(c = \tilde{m}(b)\) then \(K \perp a, c\).

**Proof.** (1) Let \(b' := \tilde{a}(b)\) and \(i = 1\). Then \(c = bb'\) and \(L = \overline{bb', b'b}\). Hence \(m = ba\), by Theorem 2.2, i.e. \([m]_2 = [a]_2\).

(2) We consider the case \(m = ba\). Let \(L = (a \perp K), \{c'\} = [b]_1 \cap L\) and \(m'\) is the midpoint of \(b, c'\). Then, by (1), \(m' = ba = m\) hence \(L = \overline{a, c}\).

**Theorem 2.4.** Each symmetric 2-structure of class (III) is point symmetric.

**Proof.** Let \((A, B) \in \mathfrak{R}^{2\perp}\) and \(a \in A \setminus B\). By \(A \perp B\), \(b := \overline{B}(a) \in A\) hence \(A = a, \overline{b}, B = ab, ba\) and by Theorem 2.2 there is exactly one midpoint \(m\) of \(a\) and \(b\) and also of \(ab\) and \(ba\). Therefore \(\tilde{m}(A) = A, \tilde{m}(B) = B\) and so \(m \in A \cap B\). Thus \((A, B) \in \mathfrak{R}^{2\perp}\).

**Corollary 2.5.** Let \((P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})\) be a symmetric 2-structure of class (III). If \((a, b) \in P^{(2)}\), \(C := a, \overline{b}, D = ab, ba\) and \(m := a \ast b\) then \([m] = C \cap D = (ab \perp C) \cap C = (a \perp D) \cap D\), \(\tilde{C}(ab) = ba\) and \(\tilde{D}(a) = b\).

### 3. Reflections in Generators

Let \(\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})\) be a point symmetric 2-structure. For each \(a \in P\) we have the 1-projection and the 2-projection

\[ \Pi_{a_1} : P \to [a]_2; x \mapsto xa \] and \( \Pi_{a_2} : P \to [a]_1; x \mapsto ax \)

and the generator reflections

\[ \overline{[a]}_1 : P \to P; x \mapsto \overline{a}x(x) \] and \( \overline{[a]}_2 : P \to P; x \mapsto \overline{a}x(x) \)

which are involutory permutations of \(P\) fixing exactly the elements of the generators \([a]_1\) and \([a]_2\), respectively. By definition, \([a]_1\) is a 2-map and from Theorem 2.2(5) follows that \(\overline{[a]}_1\) takes any 1-generator into a 1-generator,
hence $[a]_1$ is an automorphism of $(P, \mathcal{G}_1, \mathcal{G}_2)$. $[a]_2$ is an automorphism too and a 1-map. Now let $K \in \mathcal{R}$ and $\{p\} = K \cap [a]_1$ then, by Corollary 2.3, $[a]_1(K) = (p \perp K)$. Hence $[a]_1$ is also an automorphism of $(P, \mathcal{R})$ and together with [5, Theorems 3.7 and 3.8.] we have the result:

**Theorem 3.1.** Let $\Sigma := (P, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R})$ be a point symmetric 2-structure and let $a \in P, K \in \mathcal{R}$ and $i \in \{1, 2\}$. Then:

1. If $\{p\} = K \cap [a]_i$ then $[a]_i(K) = (p \perp K)$ hence each reflection in a generator is involutory and contained in $\Gamma^+(\mathcal{R})$, more precisely, $[a]_i \in \Gamma_{3-i}(\mathcal{R})$ and we have $[a]_i(K) \perp K$.

2. $\forall a \in P : [a]_1 \circ [a]_2 = [a]_2 \circ [a]_1 = \tilde{a}$. 

3. Let $A, B \in \mathcal{R}$ with $A \perp B$ and $\{p\} = A \cap B$ then $[p]_i(A) = B$.

4. Let $A, B \in \mathcal{R}$ with $A \perp B$ and $\{p\} = A \cap B$ and let $\alpha \in \Gamma_i(\mathcal{R})$ with $\alpha(A) = B$ then $\alpha = [p]_{3-i}$.

5. If $A \in \mathcal{R}, \{p\} = A \cap [a]_1, \{q\} = A \cap [a]_2, B := (p \perp A)$ and $C := (q \perp A)$ then $BA \circ \tilde{A} = A \circ AB = [a]_1$ and $\widetilde{AC} \circ \tilde{A} = \tilde{A} \circ CA = [a]_2$.

6. If $A, B \in \mathcal{R}$ with $A \perp B$ and $\{c\} := A \cap B$ then $AB \in \Gamma^-(\mathcal{R})$ and $\widetilde{AB} \circ \tilde{A} = \tilde{A} \circ \tilde{B} = \tilde{c}$ is the reflection in the point $c$.

7. If $[a]_i = [b]_i$ then $[\tilde{a} \circ \tilde{b}] \in \Gamma_i(\mathcal{R})$.

8. Any involution of $\Gamma_i(\mathcal{R})$ is the reflection in a generator of $\mathcal{G}_{3-i}$.

**Proof.** (2) Follows from [5, Theorem 3.8.(3)].

(3) Since $B = (p \perp A)$ we have by (1), $[p]_i(A) = B$.

(4) By (1), $[p]_{3-i} \in \Gamma_i(\mathcal{R})$ and $[p]_{3-i}(A) = B$. Therefore $[p]_{3-i}$ and also $\alpha$ induce the perspectivity $[A \overset{i}{\longrightarrow} B]$ or in other words, $[p]_{3-i}$ and $\alpha$ are extensions of the perspectivity $[A \overset{i}{\longrightarrow} B]$ and so by [3, Theorem 2.4.], $\alpha = [p]_{3-i}$.

(5) Follows from (3) and [3, Theorem 2.9.(4),(5)].

(6) Follows from (2) and (5).

(7) By (2), $\tilde{a} = \sigma_1 \circ \sigma_2$ and $\tilde{b} = \sigma_2 \circ \sigma_3$ where $\sigma_1, \sigma_2$ and $\sigma_3$ are reflections in the generators $[a]_{3-i}, [a]_i = [b]_i$ and $[b]_{3-i}$ respectively. Hence $\tilde{a} \circ \tilde{b} = \sigma_1 \circ \sigma_3 \in \Gamma_i(\mathcal{R})$. (8) Let $\alpha \in \Gamma_i(\mathcal{R})$ be an involution, let $x \in P \setminus \text{Fix} \alpha, y := \alpha(x)$ and $m := x \ast y$. Then $\alpha(m) = m$ and since $\alpha \in \Gamma_i(\mathcal{R}), \alpha$ fixes all elements of the generator $[m]_{3-i}$. Thus $\alpha = [m]_{3-i}$. \qed

Let $\mathfrak{B} := \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{R}$ be the set of all blocks. Then $(P, \mathfrak{B})$ is an incidence space, i.e. any two distinct points $a, b \in P$ can be joined by exactly one block of $\mathfrak{B}$ also denoted by $a \overline{b}$. An automorphism $\delta \in \Gamma(\mathcal{R})$ is called a

- **latation** if for $x \in P$ with $\delta(x) \neq x : \delta(x, \delta(x)) = \overline{x, \delta(x)}$
- **dilatation** if for $B \in \mathfrak{B} : \delta(B) = B$ or $\delta(B) \cap B = \emptyset$
translation if \( \delta \) is the identity or if \( \delta \) is a latation and a dilatation without fixed points.

If \( \sigma \) is an involutory automorphism of \((P, \mathcal{B})\) then \( \sigma \) is a latation.

**Corollary 3.2.** If \([a]_i = [b]_i\) then \( \tau_i = \tilde{a} \circ \tilde{b} \) is an i-map and a translation.

**Proof.** By the proof of Theorem 3.1 (7), \( \tau_i = \tilde{a} \circ \tilde{b} = \overline{[a]_{3-i} \circ [b]_{3-i}} \in \Gamma_i(\mathfrak{K}) \)
\( \) hence \( \tau \) is an i-map and so also a latation. If \( K \in \mathfrak{K} \) then by Theorem 3.1 (1),
\[ K \perp \overline{[b]_{3-i}(K)} \perp \overline{[a]_{3-i} \circ [b]_{3-i}(K)} = \tilde{a} \circ \tilde{b}(K). \]
\( \) Hence by Theorem 1.2(4), \( \tilde{a} \circ \tilde{b}(K) = K \) or \( \tilde{a} \circ \tilde{b}(K) \cap K = \emptyset \), i.e. \( \tau \) is a translation. \( \square \)

4. Midpoint Configurations

For two subsets \( X,Y \subseteq P \) let \( X \ast Y := \{ x \ast y \mid x \in X, \ y \in Y \} \) and for \( a \in P \) let \( a \ast X := \{ a \ast x \} \).

**Theorem 4.1.** Let \( a,b,c,p \in P, X,Y \in \mathfrak{S}_i, \ i \in \{1,2\} \) and \( A \in \mathfrak{K} \) with \( p \notin A \). Then:

1. \( p \ast X, \ X \ast Y \in \mathfrak{S}_i \) hence if \( x \in X, \ y \in Y \) then \( p \ast X = [p \ast x]_i \) and \( X \ast Y = [x \ast y]_i \).

2. \( \tilde{p}(X) = Y \Leftrightarrow p \in X \ast Y \).

3. If \([a]_i = [b]_i\) then \([\tilde{a}(c)]_i = [\tilde{b}(c)]_i\) and so \( \tau := \tilde{a} \circ \tilde{b} \) is an i-map.

4. \( \tilde{a} \ast \tilde{b} = ba \ast \tilde{b} \) is a 1-map and \( \tilde{a} \circ \tilde{b} = \tilde{a} \circ \tilde{b} \) is a 2-map.

5. If \( a_i := [p]_i \cap A \) and \( \{p'\} := (p \perp A) \cap A \) then \( p \ast p' \in p \ast a_1 \ast p \ast a_2 \).

6. Let \([a]_i = [b]_i, a \neq b\) and \( x \in [a \ast b]_{3-i} \setminus [a \ast b] \) then \( \tilde{a}, x \perp \tilde{b}, x \).

**Proof.** (1) Let for instance \( X \in \mathfrak{S}_1, \{p'\} := [p]_2 \cap X \) and \( x \in X \) then \( xp = p' \) and so by Theorem 2.2(2), \( (p \ast x)p = p \ast (xp) = p \ast p' \) implying \( p \ast X = [p \ast x]_1 = [p \ast p']_1 \in \mathfrak{S}_1 \). If \( p, q \in Y \) hence \( [p]_1 = [q]_1 = Y \) and if \( \{q'\} := [q]_2 \cap X \) then \( [p']_1 = [q']_1 \) and \( X = p \ast X = [p \ast p']_1 \cap [q \ast q']_1 \). Again by Theorem 2.2(2), \( [p \ast p']_1 = [p \ast q']_1 = [q \ast q']_1 \). Therefore \( Y \ast X = [p \ast p']_1 \in \mathfrak{S}_1 \).

(2) \( \Rightarrow \)” If \( x \in X \) then \( y := \tilde{p}(x) \in \tilde{p}(X) = Y \), hence \( p = x \ast y \in X \ast Y \).

(3) “\( \Leftarrow \)” Let \( x \in X, \ y \in Y \) with \( p = x \ast y \), i.e. \( \tilde{p}(x) = y \). Then \( \tilde{p}(X) = \tilde{p}([x]_i) = [\tilde{p}(x)]_i = [y]_i = Y \).

(3) By (1) and assumption we have \( b \in [a]_i = [c]_i \ast [\tilde{a}(c)]_i \) and so by (2), \( [\tilde{b}(c)]_i = \tilde{b}([c]_i) = \tilde{a}([c]_i) = [\tilde{a}(c)]_i \).

(4) Since \([ab]_1 = [a]_1 \) and \([ba]_2 = [a]_2\), by (3), \( \tilde{a} \circ \tilde{b} \) and \( \tilde{b} \circ \tilde{a} \) are 1-maps, and \( \tilde{a} \circ \tilde{b} \) and \( \tilde{b} \circ \tilde{a} \) are 2-maps. Therefore for \( x \in P \) we have \( \tilde{a} \circ \tilde{b}([x]_i) = ba \circ \tilde{b}([x]_i) \ast [x]_1 \) and \( \tilde{b}([x]_2) = \tilde{b}([x]_2) \ast [a]_2 \) implying \( \tilde{a} \circ \tilde{b}([x]_2) = ba \circ \tilde{b}([x]_2) \).

Together with \( x = [x]_1 \ast [x]_2 \), we obtain \( \tilde{a} \circ \tilde{b}(x) = ba \circ \tilde{b}(x) \). Thus \( \tilde{a} \circ \tilde{b} = ba \circ \tilde{b} \) and in the same way \( \tilde{a} \circ \tilde{b} = ba \circ \tilde{b} \).
(5) Since \( p = a_1 a_2 \) and so \( \tilde{A}(p) = a_2 a_1 \) we have by Corollary 2.5 \( p' = a_2 * a_1 \) and then by Theorem 2.2.(4), \( p' = (a_2 * a_1 a_2) \square (a_1 * a_2) = (a_2 * p) \square (a_1 * p) \) and so \( p' * p = (a_2 * p) * (a_1 * p) \) hence \( p * p' \in p * a_1, p * a_2 \).

(6) Follows from Theorem 3.1.(1) since \( a = [a * b]_{1-i}(b) \). \( \square \)

**Theorem 4.2.** If \( \Sigma := (P, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R}) \) is a point symmetric 2-structure and if we consider on \( P \) the binary operations \( a \square b := [a]_1 \cap [b]_2 \), \( a * b \) (= midpoint of \( a \) and \( b \)) and \( a \circ b := \tilde{a}(b) \) for \( a, b \in P \) then \( (P, \square, *, \circ) \) satisfies the following rules (let \( a, b, c \in P \)):

1. \((P, \square), (P, *) \) and \((P, \circ) \) are idempotent, i.e. \( a = aa = a * a = a \circ a \).
2. \((P, \square) \) is associative moreover \( a(bc) = (ab)c = ac \).
3. \((P, *) \) is commutative.
4. \((a * b) \circ a = b \) and \( a * (b \circ a) = b \) hence \((P, \circ) \) and \((P, *) \) are quasigroups.
5. \( a(b * c) = (ab) * (ac) \), \( (a * b)c = (ac) * (bc) \), \( a(b \circ c) = (ab) \circ (ac) \), \( (a \circ b)c = (ac) \circ (bc) \) and \( a \circ (b * c) = (a \circ b) * (a \circ c) \).

For each \( B \in \mathcal{B}, B \) is a substructure of \((P, *, \circ) \) and \((B, \circ) \), \((B, *) \) are idempotent quasigroups, if \( A, B \in \mathcal{B} \) then \((A, *, \circ) \) and \((B, *, \circ) \) are isomorphic.

**Proof.** (5) By Theorem 2.2.(4) we have:

\(* \) \( a(a * b) = a \ast (ab) \)

and so also by using (2), \( (ab) * (ac) = ((ac)b) * (ac) = (ac)(b * ac) = (ab)(b * (ac)) = a(b(b * (ac))) = a(b*bc) = ab(b * c) = a(b * c) \). From \( b = c * (b \circ c) \) and \( a(b * c) = (ab) * (ac) \) it follows that \( ab = (ac) * (a(b \circ c)) \). Hence \( ab \ast (ac) = a((b \circ c) \) By Theorem 1.4.(3), \( \tilde{a}(b * c) = \tilde{a}(b) \circ \tilde{a}(c) \), i.e. \( a \circ (b * c) = (a \circ b) * (a \circ c) \).

Now let \( A, B \in \mathcal{B} \). If \( B \in \mathcal{R} \) and \( a \in P \) then \( \pi_{a,1} : B \to [a]_1 ; b \mapsto ab \) and \( \pi_{a,2} : B \to [a]_2 ; b \mapsto ba \) are bijections and for \( b, c \in B \) we have by (5):

\( \pi_{a,1}(b * c) = a(b * c) = (ab) * (ac) = \pi_{a,1}(b) \ast \pi_{a,1}(c), \pi_{a,2}(b * c) = (b * c)a = \pi_{a,2}(b) \ast \pi_{a,2}(c), \pi_{a,1}(b \circ c) = a(b \circ c) = (ab) \circ (ac) = \pi_{a,1}(b) \circ \pi_{a,1}(c) \) and \( \pi_{a,2}(b \circ c) = (b \circ c)a = \pi_{a,2}(b) \circ \pi_{a,2}(c) \) hence \( \pi_{a,1} \) and \( \pi_{a,2} \) are isomorphisms from \((B, *, \circ) \) onto \( ([a]_1, *, \circ) \) and onto \( ([a]_2, *, \circ) \). Consequently \((A, *, \circ) \) and \((B, *, \circ) \) are isomorphic. \( \square \)

5. Products of i-maps and the Corresponding K-loop

For \( i \in \{1, 2\} \) let \( T_i := \{ \tilde{a} \circ b \mid a, b \in P \ \text{with} \ [a]_i = [b]_i \} \). Then by Theorem 4.1.(3), \( T_i \) is a set of i-maps.

By [5, Theorem 3.16] and [2, 6.1] we have

**Theorem 5.1.** Let \( \Sigma := (P, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R}) \) be a point symmetric 2-structure, let \( o \in P \) be fixed, for \( p \in P \) let \( p' := o * p \) and \( p^+ := \tilde{p}' \circ \tilde{o} \). If we set for \( a, b \in P, a + b := a^+(b) \), then \((P, +) \) is a K-loop uniquely 2-divisible.
(\(P, +\)) is the loop derivation in the point \(o\) (cf. [2, Definition 1]). For any two points \(o_1, o_2\) the reflection in the midpoint \(o_1 \ast o_2\) establishes an isomorphism between the loops derived in the points \(o_1\) and \(o_2\). For the following let \(\Sigma\) be a point symmetric 2-structure, let \(o \in P\) be fixed, let \((P, +)\) be the K-loop derived in \(o\) and let \(X := [o]_1\), \(Y := [o]_2\).

**Theorem 5.2.** Let \(\tilde{X} \circ \tilde{X} := \{\tilde{a} \circ \tilde{b} : a, b \in X\}, \tilde{Y} \circ \tilde{Y} := \{\tilde{a} \circ \tilde{b} : a, b \in Y\}, X^+ := \{x^+ : x \in X\}\) and \(Y^+ := \{y^+ : y \in Y\}\) then:

1. \(X^+ \subseteq T_1 = \tilde{X} \circ \tilde{X}\) and \(Y^+ \subseteq T_2 = \tilde{Y} \circ \tilde{Y}\), more precisely, if \(a, b \in P\) with \([a]_1 = [b]_1\) then \(\tilde{a} \circ \tilde{b} = \tilde{o} \circ \tilde{o}\).
2. \(X\) and \(Y\) are subloops of \((P, +)\) and by Theorem 4.1.(3) \(\forall p \in \Pi \exists (x, y) \in X \times Y : p = x + y\).
3. \(\forall (x, y) \in X \times Y : x + y = y + x\) even \(x^+ \circ y^+ = y^+ \circ x^+\).
4. If \(o \in E \in \mathfrak{K}\) then the loops \((E, +), (X, +), (Y, +)\), are isomorphic.

**Proof.** (1) By the definitions of \(X, Y, X^+\) and \(Y^+\) and by Theorem 4.1.(3) we have \(X^+ \subseteq \tilde{X} \circ \tilde{X} \subseteq T_1\) and \(Y^+ \subseteq \tilde{Y} \circ \tilde{Y} \subseteq T_2\). Now let \(a, b \in P\) with \([a]_1 = [b]_1\) then \(\tilde{o} \circ \tilde{o} \circ \tilde{a} \circ \tilde{b} = \tilde{o} \circ \tilde{o} \circ \tilde{a} \circ \tilde{b}\) hence by Theorem 4.1.(4), \(\tilde{a} \circ \tilde{b} = \tilde{a} \circ \tilde{a} \circ \tilde{o} \circ \tilde{b} = \tilde{o} \circ \tilde{o} \circ \tilde{a} \circ \tilde{b} \in \tilde{X} \circ \tilde{X} \) hence \(T_1 = \tilde{X} \circ \tilde{X}\).

(2), (3) Let \(a \in X\). Since \(o \in X, a' = o \ast a \in X\) and so \(\tilde{o}(X) = \tilde{o}'(X) = X\) implying \(a + X = a^+(X) = X\). Moreover if \(a, b \in X\) then \(x_1 := \tilde{o} \circ \tilde{o}'(b) \in \tilde{o} \circ \tilde{o}'(X) = X\) is the solution of the equation \(a + x = b \) contained in \(X\) and since the midpoint \(m := \tilde{o}(a) \ast b \in X \) is contained in \(X\) also \(x_2 := \tilde{o}(b) \in X\) thus \(x_2 + a = x_2 ' \circ \tilde{o}(a) = \tilde{m} \circ \tilde{o}(a) = b\) and so the solution \(x_2\) of \(x + a = b\) is contained in \(X\). Therefore \(X\) and \(Y\) are subloops of \((P, +)\).

If we set \(\{x\} := X \cap [p], \{y\} := Y \cap [p]\) then \(o = xy, p = yx\) and by Theorem 4.1, \([p \ast y] = p \ast p\) \(= o \ast x\) \(= yx\) hence by (1), \(x^+ = x^+ \circ \tilde{o} = p \ast x \circ \tilde{y}\) and we have \(x + y = x^+(y) = p \ast y \circ \tilde{y}(y) = p \ast y(y) = p\) and since \(x^+ \circ y = o\) we have by Theorem 4.1.(4), \(x^+ \circ \tilde{o} \circ \tilde{y} = y^+ x^+ \circ \tilde{o} \circ \tilde{y}\) hence \(x^+ \circ y^+ = y^+ + x^+\). If there are \(a \in X, b \in Y\) with \(p = a + b\) then \(p = x + y = x^+ \circ \tilde{o}(y^+(o)) = y^+ x^+(o) = b' a'(o)\) hence \(p' = o \ast p = y^+ x^+ = b' a'\) and so \(y' = b'\) and \(x' = a\), i.e. \(y = \tilde{y}(o) = \tilde{b'(o)} = b\) and in the same way, \(x = a\).

(4) We have \(a + b = (a \ast o) \circ (o \circ b)\). Hence, by the proof of the last statement of Theorem 4.2, \(\pi_{\eta_1}\) and \(\pi_{\eta_2}\) are isomorphisms from \((E, +)\) onto \((X, +)\) and onto \((Y, +)\). □

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