Two remarks on $C^\infty$ Anosov diffeomorphisms

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Abstract. Let $M$ be a closed oriented $C^\infty$ manifold and $f$ a $C^\infty$ Anosov diffeomorphism on $M$. We show that if $M$ is the two torus $T^2$, then $f$ is conjugate to a hyperbolic automorphism of $T^2$, either by a $C^\infty$ diffeomorphism or by a singular homeomorphism. We also show that for general $M$, if $f$ admits an absolutely continuous invariant measure $\mu$, then $\mu$ is a $C^\infty$ volume. The proofs are concatenations of well known results in the field.

1. Conjugacy

Let $f$ be a $C^\infty$ Anosov diffeomorphism on the two torus $T^2$. Then

$$A = f_* \in \text{Aut}(H_1(T^2, \mathbb{Z})) = \text{SL}(2, \mathbb{Z})$$

defines a hyperbolic automorphism of the abelian Lie group $T^2$, and $f$ is isotopic to $A$. It is known [F, M] that $f$ is conjugate to $A$ by a homeomorphism $h$ which is isotopic to the identity: $h \circ A = f \circ h$. It is well known that the conjugacy $h$ is a bi-Hölder homeomorphism. Also it is easy to show that $h$ is unique. Let us denote by $m$ the normalized Haar measure of $T^2$. A homeomorphism $h$ of $T^2$ is said to be singular if there is an $m$-conull Borel set $E$ such that $h(E)$ is $m$-null. Our first result is the following.

Theorem 1. The conjugacy $h$ is either a $C^\infty$ diffeomorphism or a singular homeomorphism.

Proof. Let $TT^2 = E^u \oplus E^s$ be the hyperbolic splitting associated with $f$. By a dimensional reason, it is a $C^1$ splitting [MM1]. Fix a translation invariant $C^\infty$ Riemannian metric $g$ on $T^2$. The derivative of $f$ along $E^u$ (resp. $E^s$) measured with respect to $g$ is denoted by $J^u f$ (resp. $J^u f$). These are $C^1$ functions. The Gibbs measure $\mu_+$ for the potential $-\log|J^u f|$ (resp. $\log|J^s f|$) is denoted by $\mu_+$ (resp. $\mu_-$).

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Let $f$, $A$ and $h$ be as above. First consider the case where $f$ does not admit an a. c. i. m. (absolutely continuous invariant measure). Then the $f$-invariant measure $h_*m$ is singular to $m$.

To show this, notice that $h_*m$ is decomposed into two parts; one absolutely continuous and the other singular. Since $h$ is a $C^\infty$ diffeomorphism, it leaves each part invariant. But the absolutely continuous part must be zero since by the assumption there is no a. c. i. m. for $f$.

Thus $h$ maps the measure $m$ to a singular measure $h_*m$. Since $h$ is a $C^\infty$ diffeomorphism, it leaves each part invariant. But the absolutely continuous part must be zero since by the assumption there is no a. c. i. m. for $f$.

Next consider the case where $f$ admits an a. c. i. m. $\mu$. Then we have

\[
\mu = \mu_+ = \mu_- \text{ (Proof of Corollary 1 of [S], Corollary 4.13 of [B]).}
\]

In particular an a. c. i. m. $\mu$ is unique and ergodic. The induced measure $h_*m$ is also ergodic. Therefore either $\mu$ and $h_*m$ are mutually singular or coincide. In the former case, we argue just as before, to conclude that the conjugacy $h$ is a singular homeomorphism.

Finally assume that $\mu_+ = \mu_- = h_*m$. These are the Gibbs measures of three potentials, $-\log|J^u f|$, $\log|J^s f|$ and a constant. By Section 3.4 of [S], these three functions, with the identical Gibbs measure, are mutually cohomologous modulo constant. That is, there are continuous functions $v_1$, $v_2$ and constants $c_1$, $c_2$ such that

\[
-\log|J^u f| = v_1 \circ f - v_1 + c_1,
\]

\[
\log|J^s f| = v_2 \circ f - v_2 + c_2.
\]

This shows that the Lyapunov exponents of all periodic orbits are the same. By Theorem 1 of [MM2], the conjugacy $h$ is a $C^\infty$ diffeomorphism. The proof of Theorem 1 is complete.

\[\square\]

2. Absolutely continuous invariant measure

Let $M$ be a closed oriented $n$-dimensional $C^\infty$ manifold and $f$ a $C^\infty$ Anosov diffeomorphism on $M$. Let $g$ be a $C^\infty$ Riemannian metric on $M$, and $m$ the normalized measure given by the volume form associated with $g$.

**Theorem 2.** Assume $f$ admits an a. c. i. m. $\mu$ with density $\varphi$: $\mu = \varphi m$, $\varphi \in L^1(m)$. Then the density $\varphi$ is a positive $C^\infty$ function.

**Proof.** Let $TM = E^u \oplus E^s$ be the hyperbolic splitting associated with $f$. Denote the Jacobian along $E^u$ (resp. $E^s$) measured with respect to $g$ by $J^u f$ (resp. $J^s f$). The total Jacobian measured with respect to $g$ is denoted by $Jf$. All these are continuous real valued functions on $M$.

Define another continuous Riemannian metric $g'$ by $g' = g|_{E^u} \oplus g|_{E^s}$. Thus $E^u$ and $E^s$ are perpendicular with respect to $g'$. Let $m'$ be the normalized measure given by the volume form associated with $g'$. We have $m' = e^a m$ for a continuous function $a$. 
Denote by $J'$ the total Jacobian with respect to $g'$. Then we have

$$
(2.1) \quad \log |J'f| = \log |J^u f| + \log |J^s f|.
$$

By [S] [B], we have $\mu = \mu_+ = \mu_-$, where $\mu_+$ (resp. $\mu_-$) is the Gibbs measure for the potential $-\log |J^u f|$ (resp. $\log |J^s f|$). Then by [S], $\log |J^u f| + \log |J^s f|$ is cohomologous to a constant. Thus by (2.1), we have

$$
(2.2) \quad \log |J'f| = b \circ f - b + C.
$$

for a continuous function $b$ and a constant $C$.

On the other hand, by the invariance of the a. c. i. m. $\mu = \varphi e^{-a} \mu'$, we have $\mu$-almost everywhere

$$
(2.3) \quad \log |J'f| = (a - \log \varphi) \circ f - (a - \log \varphi)
$$

Now by (2.3), we have $\mu(\log |J'f|) = 0$. This implies that $C = 0$ in (2.2). Then (2.2) implies the invariance of the measure $e^{-b} \mu' = e^{-b+a} \mu$. Moreover, adding an appropriate constant to $b$, we may assume that $e^{-b+a} \mu$ is a probability measure. By the uniqueness of the a. c. i. m., we have $\mu = e^{-b+a} \mu$. That is, the density of $\mu$ is positive and continuous. Then by Corollary 2.1 of [LMM], we obtain that $e^{-b+a}$ is a $C^\infty$ function. The proof is complete. □

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