A Fourier approach to pizza inequity

ABSTRACT

Let $n$ be an odd number greater than 1. We slice a circular pizza into $2n$ slices, making cuts from a noncentral interior point of the circle. We estimate the difference between the total area of the even numbered slices and the total area of the odd numbered slices.

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0. Introduction

To cut a circular pizza into 2n slices, one normally takes 2n rays which begin at the center $C$ of the circle, such that the angle between any two adjacent rays is $2\pi/2n$. The slices are the regions bounded by two adjacent rays and the circle. We number the slices counterclockwise, and so we have $n$ odd numbered slices and $n$ even numbered slices. The so-called pizza theorem asserts, somewhat surprisingly, that even if the 2n rays begin at the same noncentral interior point $P$ of the circle, and 2n is divisible by 4 and greater than 4, then the total area of the even numbered slices equals the total area of the odd numbered slices. According to the Wikipedia pizza theorem article, this fact originated as a problem in [3], and the first published proof was given in [2].

Here we consider the case that 2n is not a multiple of 4. In this case, the pizza theorem does not hold; see the interesting and intricate paper [1]. We show here, however, that the theorem is often almost true. We suppose without loss of generality that the circle has radius 1, and we let $a$ be the distance from $P$ to $C$. Thus $0 < a < 1$. Let $\alpha$ denote the angle between the first ray and the line joining $C$ to $P$. Let $g(\alpha, a, n)$ denote the total area of the even numbered slices minus the total area of the odd numbered slices. We show in Corollary 1.5 that
\[
|g(\alpha, a, n)| < a^n/2(1-a^2)(1-a^{2n}),
\]
so the “pizza inequity” is on the order of $a^n$ when $a$ is bounded away from 1. This inequity is too small, one imagines, to be noticed by hungry pizza eaters.

Our key idea is to express an appropriate step function on $[0, 2\pi]$ as a Fourier series. This leads to Theorem 1.2, in which $g(\alpha, a, n)$ is given as an infinite linear combination of functions $\sin m\alpha$, where $m$ ranges over the positive odd multiples of $n$. The coefficient of $\sin m\alpha$ is a power series in $a$. The (numerical) coefficients in this power series all have the same sign. We evaluate these power series coefficients explicitly in terms of binomial coefficients. We also find an exact expression for the maximum value of $|g(\alpha, a, n)|$ when $a$ and $n$ are fixed.

1. Results

**Notational conventions.** In this paper, $n$ denotes a fixed odd integer greater than 1, and $a$ denotes a real number in the interval $(0,1)$. The notation $\sum_m$ means that $m$ ranges over all positive odd multiples of $n$. For a real number $x$, $e(x)$ denotes $e^{ix}$. If $r$ is a positive integer or $r = 1/2$, and $s$ is an integer we define $C(r,s)$ to be the binomial coefficient “$r$ choose $s$”. As usual, this is defined to be zero if $s < 0$ or if $r$ is an integer and $s > r$. If $r$ and $s$ are integers, then $\delta_{r,s}$ is the Kronecker delta.

Consider the circle $(x-a)^2 + y^2 = 1$, with center $(a,0)$ and radius 1. Thus $(a,0)$ and $(0,0)$ respectively play the roles of $C$ and $P$ in the introduction. Fix an initial angle $\alpha \in [0, 2\pi)$ and consider the 2n rays starting from the origin and making angles $\alpha, \alpha + \pi/n, \ldots, \alpha + (2n-1)\pi/n$ with the positive $x$-axis. These rays divide the pizza into 2n slices. We number them counterclockwise, so the first slice is bounded by the $\alpha$ ray, the $\alpha + \pi/n$ ray, and the circle. The second
slice is bounded by the $\alpha + \pi/n$ ray, the $\alpha + 2\pi/n$ ray, and so on. Let $A$ denote the union of the $n$ even numbered intervals $[\alpha + \pi/n, \alpha + 2\pi/n], \ldots$ and let $B$ denote the union of the $n$ odd numbered intervals $[\alpha, \alpha + \pi/n], \ldots$.

In polar coordinates, the circle has equation $r(\theta) = a \cos \theta + \sqrt{1 - a^2 \sin^2 \theta}$; taking the negative sign in the quadratic formula would cause $r$ to be negative.

We define the pizza inequity function $g(\alpha, a, n)$ to be the total area of the even numbered slices minus the total area of the odd numbered slices. Hence

$$2g(\alpha, a, n) = \int_A r(\theta)^2 d\theta - \int_B r(\theta)^2 d\theta.$$ 

Squaring our expression for $r(\theta)$ yields

$$2g(\alpha, a, n) = \int_A 1 + a^2 \cos 2\theta + 2a \cos \theta \sqrt{1 - a^2 \sin^2 \theta} d\theta - \int_B 1 + a^2 \cos 2\theta + 2a \cos \theta \sqrt{1 - a^2 \sin^2 \theta} d\theta.$$ 

We remark that if we replaced our circle $(x - a)^2 + y^2 = 1$ by the circle with center $(aR, 0)$ and radius $R$, then the function $r(\theta)$ would be multiplied by $R$ and so the integrands in the last two integrals would be multiplied by $R^2$. Hence the ratio of the pizza inequity to the area of the circle would be unchanged. Thus we lose no generality by working with a circle of radius 1.

**Lemma 1.1** Let $n, a, \alpha, A,$ and $B$ be as above. Let $s(\theta)$ denote the step function with values $+1$ on $A$ and $-1$ on $B$; the values of $s(\theta)$ at the endpoints of the $2n$ intervals don’t matter. Let $m$ be an integer. Then

$$\int_0^{2\pi} s(\theta) e(-m\theta) d\theta = 0$$

unless $m$ is an odd multiple of $n$. If $m$ is an odd multiple of $n$, then

$$\int_0^{2\pi} s(\theta) e(-m\theta) d\theta = (-4n/m)e(-ma).$$

**Proof.** We may assume that $m \neq 0$. Since $-e(-m\theta)/mi$ is an antiderivative of $e(-m\theta)$, we see that $\int_A e(-m\theta) d\theta$ equals

$$-(1/mi)[e(-m(\alpha + 2\pi/n)) + e(-m(\alpha + 4\pi/n)) + \ldots + e(-ma)]$$

$$+(1/mi)[e(-m(\alpha + \pi/n)) + e(-m(\alpha + 3\pi/n)) + \ldots + e(-m(\alpha + (2n - 1)\pi/n))].$$

Let $\zeta = e(2\pi/n)$ and we let $\eta = e(\pi/n)$. The foregoing becomes

$$-(e(-ma)/mi)[\zeta^{-m} + \zeta^{-2m} + \ldots + 1]$$

$$+(e(-ma)/mi)[\eta^{-m}(1 + \zeta^{-m} + \ldots + \zeta^{-(n-1)m})].$$

If $n$ does not divide $m$, then the map $\zeta \mapsto \zeta^{-m}$ defines a nontrivial character of the multiplicative group generated by $\zeta$, and so both expressions in brackets
vanish. If $2n$ divides $m$, then both expressions in brackets equal $n$ and again the integral vanishes. Suppose next that $m$ is an odd multiple of $n$. Then $\eta^{-m} = -1$ and so $\int_A e(-m\theta)d\theta = (-2n/mi)e(-ma)$.

A similar computation shows that $\int_B e(-m\theta)d\theta$ equals

$$-(e(-ma)/mi)[\eta^{-m}(1 + \zeta^{-m} + \zeta^{-2m} + \ldots + \zeta^{-(n-1)m})]$$

$$+ (e(-ma)/mi)[\zeta^{-m} + \zeta^{-2m} + \ldots + 1].$$

As before, we see that $\int_B e(-m\theta)d\theta$ vanishes unless $m$ is an odd multiple of $n$, in which case $\int_B e(-m\theta)d\theta = +(2n/mi)e(-ma)$. Since

$$\int_0^{2\pi} s(\theta)e(-m\theta)d\theta = \int_A e(-m\theta)d\theta - \int_B e(-m\theta)d\theta,$$

the conclusion of the lemma follows.

**Theorem 1.2** For $n > 1$ odd, $0 < a < 1$, and $\alpha \in [0, 2\pi)$, let $g(\alpha, a, n)$ be the pizza inequity function defined above. Let $f(\alpha, a, n) = g(\alpha, a, n)/a$. Then

$$f(\alpha, a, n) = \sum_m (4n/\pi m)(P_m(\alpha)) \sin ma,$$

where $P_m(x)$ is the power series $\sum_{j=1}^\infty c_{2j}(m)x^{2j}$, with

$$c_{2j}(m) = (-1)^j C(1/2, j) \int_0^{2\pi} \cos \theta \cos m\theta \sin^{2j}\theta d\theta.$$

**Proof.** The step function $s(\theta)$ of Theorem 1.1 has Fourier series $\sum_{m=0}^\infty a_m \cos m\theta + \sum_{k=1}^\infty b_k \sin k\theta$. Since $\int_0^{2\pi} s(\theta)d\theta = 0$, we have $a_0 = 0$. For $m > 0$,

$$a_m = (1/\pi) \int_0^{2\pi} s(\theta) \cos m\theta d\theta = (1/2\pi) \int_0^{2\pi} s(\theta)(e(-m\theta) + e(m\theta))d\theta.$$

By Lemma 1.1, if $m$ is an odd multiple of $n$, then

$$\int_0^{2\pi} s(\theta)(e(-m\theta) + e(m\theta))d\theta = (-4n/mi)e(-ma) + (4n/mi)e(ma)$$

$$= (4n/mi)(e(ma) - e(-ma)) = (8n/m)(1/2i)(e(ma) - e(-ma)) = (8n/m) \sin ma.$$

Hence $a_m = (4n/\pi m) \sin ma$. If $m$ is not an odd multiple of $n$, then $a_m = 0$ by Lemma 1.1. We won’t bother to compute $b_k$, since we will soon see that its value doesn’t matter.

Now

$$2g(\alpha, a, n) = \int_0^{2\pi} s(\theta) \left[1 + a^2 \cos 2\theta + 2a \cos \theta \sqrt{1 - a^2 \sin^2 \theta}\right] d\theta.$$
The functions $1$ and $\cos 2\theta$ are orthogonal on $[0, 2\pi]$ to $\cos m\theta$, whenever $m$ is an odd multiple of $n$. The functions $1$ and $\cos 2\theta$ are also orthogonal to any function $\sin k\theta$. Hence

$$g(\alpha, a, n) = a \int_0^{2\pi} s(\theta) \cos \theta \sqrt{1 - a^2 \sin^2 \theta} d\theta.$$  

If $k$ is an integer, then

$$\int_0^{2\pi} \sin k\theta \cos \theta \sqrt{1 - a^2 \sin^2 \theta} d\theta = \int_{-\pi}^{\pi} \sin k\theta \cos \theta \sqrt{1 - a^2 \sin^2 \theta} d\theta = 0,$$

since the last integrand is an odd function of $\theta$. Hence

$$f(\alpha, a, n) = \sum_m (4n/\pi m) \left[ \int_0^{2\pi} \cos \theta \cos m\theta \sqrt{1 - a^2 \sin^2 \theta} d\theta \right] \sin m\alpha.$$  

The binomial series

$$(1 - x)^{1/2} = \sum_{j=0}^{\infty} C(1/2, j)(-x)^j = \sum_{j=0}^{\infty} (-1)^j C(1/2, j)x^j$$

converges for $|x| < 1$. Note that all coefficients in this series, except the constant term, are negative. The series $\sum_{j=0}^{\infty} (-1)^j C(1/2, j)(a^2 \sin^2 \theta)^j$ converges uniformly in $\theta$ to $\sqrt{1 - a^2 \sin^2 \theta}$. Thus

$$\int_0^{2\pi} \cos \theta \cos m\theta \sqrt{1 - a^2 \sin^2 \theta} d\theta = \sum_{j=0}^{\infty} (-1)^j C(1/2, j) \left[ \int_0^{2\pi} \cos \theta \cos m\theta \sin^{2j} \theta d\theta \right] a^{2j}.$$  

Since $m \geq n > 1$, we have $\int_0^{2\pi} \cos \theta \cos m\theta d\theta = 0$, so we can start the summation at $j = 1$. The conclusion of the theorem follows.

**Proposition 1.3** Let $P_m(x) = \sum_{j=1}^{\infty} c_{2j}(m)x^{2j}$ as in Theorem 1.2. Then

$$c_{2j}(m) = (-1)^{(m+1)/2}(\pi/2j)|C(1/2, j)|[C(2j, (2j-m+1)/2) - C(2j, (2j-m-1)/2)].$$

All nonzero coefficients $c_{2j}(m)$ have the same sign, namely $(-1)^{(m+1)/2}$. The leading coefficient of $P_m(x)$ is

$$c_{m-1}(m) = (-1)^{(m+1)/2}(\pi/2^{m-1})|C(1/2, (m-1)/2)|$$

For all $m$ and $j$ we have

$$|c_{2j}(m)| \leq |c_2(3)| = \pi/8.$$  

**Proof.** Since $(-1)^j C(1/2, j) < 0$ for $j \geq 1$, Theorem 1.2 implies that the first assertion amounts to saying that $\int_0^{2\pi} \cos \theta \cos m\theta \sin^{2j} \theta d\theta$ equals

$$(-1)^{(m+1)/2}(\pi/2^{2j})[C(2j, (2j-m+1)/2) - C(2j, (2j-m-1)/2)].$$  

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Now \( \int_0^{2\pi} \cos \theta \cos m\theta \sin^{2j} \theta d\theta \) equals
\[
i^{2j} 2^{-(2j+2)} \int_0^{2\pi} (e(\theta) + e(-\theta))(e(m\theta) + e(-m\theta))(e(\theta) - e(-\theta)^2) d\theta.
\]

Since \( i^{2j} = (-1)^j \), this equals
\[
2^{-(2j+2)} \sum_{k=0}^{2j} (-1)^{j+k} C(2j, k) I_k,
\]

with
\[
I_k = \int_0^{2\pi} [e((m+1)\theta) + e((-m-1)\theta) + e((m-1)\theta) + e((1-m)\theta)] e((2k-2j)\theta) d\theta.
\]

By orthogonality of exponential functions,
\[
I_k = 2\pi [\delta_{2j-2k,m+1} + \delta_{2j-2k,m-1} + \delta_{2j-2k,-m-1} + \delta_{2j-2k,-m+1}].
\]

If \( 2j - 2k = \pm (m+1) \), then \( k = (2j \mp (m+1))/2 \) and
\[
(-1)^{j+k} C(2j, k) = (-1)^{(m+1)/2} C(2j, (2j - m - 1)/2).
\]

If \( 2j - 2k = \pm (m-1) \), then \( k = (2j \mp (m-1))/2 \) and
\[
(-1)^{j+k} C(2j, k) = (-1)^{(m-1)/2} C(2j, (2j - m + 1)/2).
\]

It follows that
\[
\sum_{k=0}^{2j} (-1)^{j+k} C(2j, k) I_k = 4\pi (-1)^{(m-1)/2} [C(2j, (2j - m + 1)/2) - C(2j, (2j - m - 1)/2)]
\]

which, when multiplied by \( 2^{-(2j+2)} \), gives the desired value for our integral.

Since \( (2j - m - 1)/2 < (2j - m + 1)/2 < j \), the second assertion of the proposition follows immediately from the first. Since \( (2j - m + 1)/2 < 0 \) if \( 2j < m - 1 \), it follows that \( c_{m-1}(m) \) is the leading coefficient of \( P_m(x) \). Since \( C(2j, 0) = 0 = 1 \), the third assertion of the proposition follows.

To prove the final assertion, first suppose that \( j \geq 3 \). For a positive integer \( t \), let \( M(t) \) denote the maximum ratio \( C(t, u)/2^t \) for \( u = 0, 1, \ldots, t \). If \( 0 \leq v \leq t+1 \), then \( C(t+1, v) = C(t, v-1) + C(t, v) \). It follows easily that \( C(t+1, v)/2^{t+1} \leq M(t) \), and so \( M(t+1) \leq M(t) \). In particular, if \( t \geq 6 \), then \( M(t) \leq M(6) = 5/16 \).

Hence the factor
\[
2^{-2j}[C(2j, (2j - m + 1)/2) - C(2j, (2j - m - 1)/2)]
\]
of \( c_{2j}(m) \) is at most \( 5/16 \), and so
\[
|c_{2j}(m)| \leq (5\pi/16)|C(1/2, 3)| = (5\pi/16)(1/16) < \pi/8.
\]
If \( j = 2 \), then \( c_{2j} = 0 \) or \( m \) is 3 or 5. One computes that \(|c_4(5)| = \pi/128\) and \(|c_4(3)| = 3\pi/128\), both less than \( \pi/8 \). If \( j = 1 \), then \( c_{2j} = 0 \) or \( m = 3 \). One computes that \(|c_2(3)| = \pi/8\), as desired.

**Remark.** Our bound for \(|c_{2j}(m)|\) could be improved considerably if one assumes, say, that \( n \geq 5 \). We leave this to the interested reader.

**Lemma 1.4** For \( \alpha, a, \) and \( n \) as above, let \( f_a(\alpha) = f(\alpha, a, n) \). Then \( f_a \) is an odd function of \( \alpha \) and \( f_a \) is periodic with period \( 2\pi/n \).

**Proof.** This is immediate from Theorem 1.2.

**Corollary 1.5** With notation as above, let \( M_a = \sum_m (4n/\pi m)|P_m(a)| \). Then \( f_a(\pi/2n) = (-1)^{(m+1)/2}M_a \) and \( f_a(-\pi/2n) = (-1)^{(m-1)/2}M_a \). The maximum and minimum values of \( f_a \) are \( M_a \) and \( -M_a \), respectively. We have

\[
0 < M_a < a^{n-1}/2(1-a^2)(1-a^{2n}).
\]

Consequently, for all \( \alpha \), the true pizza inequity function \( g(\alpha, a, n) \) satisfies

\[
|g(\alpha, a, n)| < a^n/2(1-a^2)(1-a^{2n}).
\]

**Proof.** Note that \(|P_m(a)| = \pm P_m(a)\) by Proposition 1.3. By Theorem 1.2,

\[
f_a(\pi/2n) = (4/\pi)P_n(a) - (4/3\pi)P_{3n}(a) + (4/5\pi)P_{5n}(a) - \ldots
\]

First suppose that \( n \equiv 1 \) (mod 4). By Proposition 1.3, all nonzero coefficients in \( P_n(x) \) are negative, all nonzero coefficients in \( P_{3n}(x) \) are positive, all nonzero coefficients in \( P_{5n}(x) \) are negative, and so on. It follows that \( f_a(\pi/2n) = -M_a \).

If \( n \equiv 3 \) (mod 4), similar reasoning shows that \( f_a(\pi/2n) = +M_a \). By Lemma 1.4, \( f_a(-\pi/2n) = -f_a(\pi/2n) \). By the triangle inequality, \(|f_a(\alpha)| \leq M_a \) for all \( \alpha \). Hence we have found the maximum and minimum values of \( f_a \).

We now estimate \( M_a \). By Proposition 1.3, we have \(|c_{2j}(m)| \leq \pi/8\) for all \( j \) and \( m \). Since \( c_{2j} = 0 \) for \( 2j < m - 1 \), comparison with a geometric series shows that

\[
|P_m(a)| \leq (\pi/8)a^{m-1}/(1-a^2)
\]

for all \( m \). Hence

\[
M_a \leq \sum_m (4n/\pi m)(\pi/8)a^{m-1}/(1-a^2) < \sum_m a^{m-1}/2(1-a^2).
\]

But the last series is geometric with ratio \( a^{2n} \) and first term \( a^{n-1}/2(1-a^2) \). Hence \( M_a < a^{n-1}/2(1-a^2)(1-a^{2n}) \), as desired. Since \( g(\alpha, a, n) = af(\alpha, a, n) \), the final assertion of the corollary follows.
References

[1] P. Deiermann and R. Mabry, Of cheese and crust: a proof of the pizza conjecture and other tasty results, Amer. Math. Monthly (2009), 423-438.
[2] M. Goldberg, Divisors of a circle, Math. Mag. 41(1968), 46.
[3] L. J. Upton, Problem 660, Math. Mag. 40(1967), 163.