Analytical methods of theoretical physics in the modeling of building structures

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Abstract. In this paper, on the basis of the variational principle of the least Hamilton action, the field-theoretic Lagrange equations of the 2nd kind are derived in the presence of a dependence of the density of the Lagrange function on the higher derivatives. On the basis of the obtained equations, written in terms of variables that most fully reflect hypotheses and used assumptions, the equations of continuum mechanics are obtained and the Timoshenko, Rayleigh and Euler-Bernoulli beam models are calculated. It is shown that the method used leads to a more efficient method of obtaining partial differential equations describing the dynamics of deflection of beams. The possibility of taking into account the distributed load and the elastic base in the Lagrangian formalism is noted. The possibility of further development of analytical and theoretical-physical methods in application to mathematical modeling of building structures is analyzed.

1. Introduction

Lagrangian mechanics is a reformulation of classical mechanics based on Newton’s laws and the Galilean principle of relativity. The Lagrangian formalism makes it easy to easily obtain the equations of a mechanical system, for this it is necessary to know the form of the kinetic and potential energies, the Lagrange function will be built from their difference, and the mechanical system will be characterized by generalized coordinates, which greatly simplifies posed problem.

Hamilton’s variational principle of least action is the most fundamental principle of modern physics, which is used in the derivation of all its equations. It is a generalization of the Fermat principle - the principle of the least time of propagation of a light signal, which will be discussed in connection with wave-particle duality. The essence of the principle is as follows: if you fix the start and end points of the trajectory, the system will move along one of the possible trajectories for which a certain function of this trajectory, the so-called ”action”, is minimal. (In principle, Fermat time is minimal). Before the introduction of this principle into theoretical physics, all its equations were written phenomenologically, that is, as it were, from the ceiling. The principle of least action is closely related to the principles of maximum found in mathematics and economics. The dynamic equations of the described mechanical system can be obtained from Hamilton’s principle. They are called Lagrange equations of the 2nd kind. In theoretical mechanics the Lagrange equations have the form:

$$\frac{\partial L}{\partial q^{(i)}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^{(i)}} \right) = 0,$$

where $L$ — Lagrange function, $q^{(i)}$ — set of generalized coordinates. In classical field theory,
Einstein’s special theory of relativity imposes the requirement of equality of time and spatial coordinates, the equations become covariant. In addition, any field is distributed in space, and not only varies in time. The role of generalized coordinates is now played by the values of the field functions $t, x, y, z$ at all points of a certain region of space, so the field is actually a system with an infinite number of degrees of freedom. The principle of least action and Lagrangian formalism are modified and will be discussed in the next paragraph.

2. Principles and Methods

Considering the field in the form of a mechanical system with an infinite number of degrees of freedom, we can construct a field theory similarly to the classical mechanics of a point. The field will be characterized by a field functions $A^{(i)}$, that corresponds to an infinite number of degrees of freedom. In the classical field theory $L$ is Lagrangian, which define as the integral over the time and $L$ is density of the Lagrangian and will determine as the integral over four-dimensional space-time.

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_V dV \mathcal{L} \left( A^{(i)}, \frac{\partial A^{(i)}}{\partial x^\mu}, \frac{\partial^2 A^{(i)}}{\partial x^\mu \partial x^\nu} \right)$$

(1)

Where $S$ is action functional.

We introduce the concept of Hamilton’s variational principle of least action

$$\delta S = 0 \Rightarrow \frac{\delta \mathcal{L}}{\delta A^{(i)}} = 0$$

(2)

Imagine a variational derivative, where all field functions $A^{(i)}(x, y, z, t)$ are given such that the action will be extreme:

$$0 = \delta S = \int_{t_1}^{t_2} dt \int_V dV \delta \mathcal{L} \left( A^{(i)}, \frac{\partial A^{(i)}}{\partial x^\mu}, \frac{\partial^2 A^{(i)}}{\partial x^\mu \partial x^\nu} \right) = \int_{t_1}^{t_2} dt \int_V dV \left( \frac{\partial \mathcal{L}}{\partial A^{(i)}} \delta A^{(i)} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{(i)})} \delta (\partial_\mu A^{(i)}) \right)$$

(3)

As usual, a member written using identical indices is understood as summed over all values of this index, also called the Einstein summation rule.

By virtue of the fact that the operators $\partial_\mu$ and $\delta$ are permutable, this means that equality $\delta \partial_\mu = \partial_\mu \delta$ will be performed

$$= \int_{t_1}^{t_2} dt \int_V dV \left( \frac{\partial \mathcal{L}}{\partial A^{(i)}} \delta A^{(i)} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{(i)})} \partial_\mu \left( \delta A^{(i)} \right) \right)$$

$$= \int_{t_1}^{t_2} dt \int_V dV \left[ \frac{\partial \mathcal{L}}{\partial A^{(i)}} \delta A^{(i)} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{(i)})} \right) \delta A^{(i)} + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{(i)})} \delta A^{(i)} \right) \right]$$

$$= \int_{t_1}^{t_2} dt \int_V dV \left[ \frac{\partial \mathcal{L}}{\partial A^{(i)}} \delta A^{(i)} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{(i)})} \right) \delta A^{(i)} + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^{(i)})} \delta A^{(i)} \right) \right]$$

(4)
By virtue of the validity of the Ostrogradsky-Gauss theorem

\[ = \int_{t_1}^{t_2} dt \int_{V} dV \left[ \frac{\partial L}{\partial A^{(i)}_t} - \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} A^{(i)})} \right) + \partial_{\nu} \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\nu} \partial_{\mu} A^{(i)})} \right) \right] \delta A^{(i)} + \\
+ \oint \left[ \frac{\partial L}{\partial (\partial_{\mu} A^{(i)})} \delta A^{(i)} - \partial_{\nu} \left( \frac{\partial L}{\partial (\partial_{\nu} \partial_{\mu} A^{(i)})} \right) \delta A^{(i)} \right] dS_{\mu} = *** \]  

By condition of variation \( \delta A^{(i)} = 0 \) and \( \partial_{\nu} (\delta A^{(i)}) = 0 \) at \( t_1, t_2 \) and at the volume boundary \( V \)

\[ = *** \int_{t_1}^{t_2} dt \int_{V} dV \left[ \frac{\partial L}{\partial A^{(i)}} - \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} A^{(i)})} \right) + \partial_{\nu} \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\nu} \partial_{\mu} A^{(i)})} \right) \right] \delta A^{(i)} \]  

By virtue of arbitrariness \( \delta u^i \) the equation will take the form:

\[ \frac{\partial L}{\partial A^{(i)}} - \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} A^{(i)})} \right) + \partial_{\nu} \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\nu} \partial_{\mu} A^{(i)})} \right) = 0 \]  

(7)

So we get the Lagrange equation for field theory.

### 3. Continuum mechanics

The Lagrangian density is represented as the difference between densities of kinetic and potential energies

\[ L = T - U, \]  

where \( T, U \) is densities of kinetic and potential energies, which can be represented as [Landau, Volume 7]

\[ T = \frac{\rho \dot{u}^2}{2}, \]  

\[ U = \frac{\lambda \varepsilon_{ik} \varepsilon_{ik}}{2} + \mu \varepsilon_{ik} \varepsilon_{ik}, \]  

\( \lambda \) and \( \mu \) (shear modulus) is the coefficients of Lame. And the values of strain tensors can be represented as:

\[ \varepsilon_{ik} = \frac{1}{2} \left( \partial_i u^k + \partial_k u^i \right) \]  

\[ \varepsilon_{ii} = \partial_i u^i = \text{div} \vec{u} \]  

(11)

According to the equation (7), where the first term is equal to zero, because independent of potential. The second term decompose in the sum, when \( \mu = 0 = t \) and \( \mu = 1, 2, 3 = x, y, z \), so we get the following equation

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial (\partial_i u^i / \partial t)} \right) + \partial_{ik} \left( \frac{\partial L}{\partial (\partial_k u^i)} \right) = 0 \]  

(13)

\[ \frac{\partial (\varepsilon_{lm} \varepsilon_{lm})}{\partial (\partial_k u^i)} = 2 \varepsilon_{lm} \frac{\partial \varepsilon_{lm}}{\partial (\partial_k u^i)} = 2 \varepsilon_{lm} \frac{1}{2} \left( \delta_i^l \delta_{kl} + \delta_{kl} \delta_i^l \right) = 2 (\varepsilon_{ki} + \varepsilon_{ik}) \frac{1}{2} = 2 \varepsilon_{ik} \]  

(14)

\[ \frac{\partial L}{\partial (\partial_k u^i)} = -\lambda \varepsilon_{ik} \delta_i^k - 2 \mu \varepsilon_{ik} \]  

(15)

\[ \rho \dot{u}^i - \lambda \partial_i \varepsilon_{ij} - 2 \mu \partial_k \varepsilon_{ik} = 0 \]  

(16)
\[
\partial_k \varepsilon_{ik} = \frac{1}{2} \left( \partial_i \partial_k u^k + \partial_k \partial_i u^i \right) = \frac{1}{2} \left( \partial_i \text{div} \vec{u} + \Delta u^i \right)
\]  
(17)

\[
\rho \ddot{u}^i - \lambda \partial_i \text{div} \vec{u} + \mu \partial_i (\text{div} \vec{u}) + \mu \Delta u^i = 0
\]  
(18)

\[
\rho \ddot{\vec{u}} = (\lambda + \mu) \text{grad} (\text{div} \vec{u}) + \mu \Delta \vec{u}
\]  
(19)

So we got a field equation for continuum mechanics. Further specific (simplified) models will be used.

Now we can introduce the value of \(E\), which is called Young’s modulus (modulus of elasticity) and the value of \(K\), which is called bulk modulus. Expressing the value of bulk modulus from presented below equation, we can find connection between the Lame coefficients

\[
E = \frac{9K \mu}{3K + \mu} = 2\mu(1 + \sigma)
\]  
(20)

\[
K = \frac{\mu E}{3(3\mu - E)} = \frac{E}{3(1 - 2\sigma)}
\]  
(21)

\[
\lambda = K - \frac{2}{3} \frac{\mu}{\mu - E} = \frac{E\sigma}{(1 - 2\sigma)(1 + \sigma)}
\]  
(22)

4. Timoshenko beam

Tymoshenko’s model takes into account shear deformation and rotational bends, therefore this model is often used to calculate thick beams in engineering calculations.

According to the Timoshenko beam model, the beam cross sections remain flat during deformation, however, the slope angle of the section \(\psi\) is not connected in any way with the beam bending, and it is an independent dynamic variable by which it will be necessary to vary the Lagrangian and write the Lagrange equation.

\[ u_x = -z \tan \psi (x, t) = -z \psi (x, t) \]  
(23)

Figure 1. Comparison of Timoshenko and Euler-Bernoulli beam deformation
The beam is deformed only in the plane $Oxz$:

$$u_y = 0$$  \hspace{1cm} (24)$$

The deflection of the beam axis is described by the function:

$$u_z = w(x, t)$$  \hspace{1cm} (25)$$

For the strain tensor, we obtain:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{xy} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0,$$

$$\varepsilon_{yz} = 0, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0, \quad \varepsilon_{zx} = \frac{1}{2} \kappa \left( \frac{\partial w}{\partial x} - \psi \right)$$  \hspace{1cm} (26)$$

Since the shear deformations of the beam in reality are not the same throughout the section, we artificially add a correction factor $\kappa$

Consider the derivation of the equation for the bending of the Timoshenko beam under dynamic load using the Lagrangian formalism.

The Lagrangian method allows us to immediately write the Lagrangian in terms (as a function) of the simplified set of functions that make up this model, in the Euler-Lagrange equations to write a variational derivative on these functions. Here there is a situation similar to the situation with generalized coordinates in theoretical mechanics, when geometric quantities are chosen as generalized coordinates, which greatly simplify the description of the mechanical system.

$$\mathcal{T} = \frac{\rho}{2} \left( \ddot{\mathbf{u}} \right)^2 = \frac{\rho}{2} \left( \dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2 \right) = \frac{\rho}{2} \left( \dot{z}^2 \psi^2 + \dot{w}^2 \right)$$  \hspace{1cm} (28)$$

When calculating the potential energy, it must be borne in mind that, in spite of the neglect of displacements along the $z$ axis in our model, it is necessary to take into account the energy spent on transverse strains due to a change in the transverse dimensions of the body during its longitudinal deformations and the relationship between them given by Poisson’s ratio $\sigma$. Then $\varepsilon_{yy} = \varepsilon_{zz} = -\sigma \varepsilon_{xx}$

$$\mathcal{U} = \frac{\lambda}{2} \varepsilon_{ll}^2 + \mu \varepsilon_{ik} \varepsilon_{ik} - q u_z = \frac{\lambda}{2} \left( \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \right)^2 + \mu \left( \varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2 + 2 \varepsilon_{xz}^2 \right) - q w =$$

$$= \left[ \frac{\lambda}{2} \left( 1 - 2 \sigma \right)^2 + \mu \left( 1 + 2 \sigma \right)^2 \right] \varepsilon_{xx}^2 + 2 \mu \varepsilon_{xz}^2 - q w =$$

$$\mu \left[ \frac{\sigma}{1 - 2 \sigma} \left( 1 - 2 \sigma \right)^2 + \left( 1 + 2 \sigma \right)^2 \right] \varepsilon_{xx}^2 + 2 \mu \varepsilon_{xz}^2 - q w = \mu (1 + \sigma) \varepsilon_{xx}^2 + 2 \mu \varepsilon_{xz}^2 - q w =$$

$$= \frac{E}{2} \varepsilon_{xx}^2 + 2 \mu \varepsilon_{xz}^2 - q w = \frac{E}{2} \varepsilon_{xx}^2 + 2 \mu \varepsilon_{xz}^2 + \frac{\mu}{2} \kappa \left( \frac{\partial w}{\partial x} - \psi \right)^2 - q w$$  \hspace{1cm} (29)$$

The last term in potential energy is associated with the interaction of a beam with a distributed load $q(x, t)$.

The density of the Lagrangian will take the form:

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{\rho}{2} \left( \dot{z}^2 \psi^2 + \dot{w}^2 \right) - \frac{E}{2} \varepsilon_{xx}^2 \left( \frac{\partial \psi}{\partial x} \right)^2 - \frac{\mu}{2} \kappa \left( \frac{\partial w}{\partial x} - \psi \right)^2 + q w$$  \hspace{1cm} (30)$$
Taking into account the equation:

$$\frac{\partial L}{\partial \psi} = \mu \kappa \left( \frac{\partial w}{\partial x} - \psi \right)$$

(31)

$$\partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi)} \right) = -\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \psi} \right) - \nabla \cdot \frac{\partial L}{\partial \nabla \psi} =$$

$$= - \rho z^2 \ddot{\psi} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial (\partial_\psi/\partial x)} \right) = - \rho z^2 \ddot{\psi} + E z^2 \frac{\partial^2 \psi}{\partial x^2} \tag{32}$$

Writing down the variational principle we get the equation:

$$\frac{\delta L}{\delta \psi} = \frac{\partial L}{\partial \psi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi)} \right) = \mu \kappa \left( \frac{\partial w}{\partial x} - \psi \right) - \rho z^2 \ddot{\psi} + E z^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{33}$$

By varying with respect to $w$ we obtain the second Lagrange equation:

$$\frac{\partial L}{\partial w} = q \tag{34}$$

$$\partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi)} \right) = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \psi} \right) + \nabla \cdot \left( \frac{\partial L}{\partial \nabla \psi} \right) \tag{35}$$

$$\frac{\delta L}{\delta w} = \frac{\partial L}{\partial w} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi)} \right) = q - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \psi} \right) - \nabla \cdot \left( \frac{\partial L}{\partial \nabla \psi} \right) =$$

$$= q - \rho \ddot{w} + \frac{\partial}{\partial x} \left[ \mu \kappa \left( \frac{\partial w}{\partial x} - \psi \right) \right] \tag{36}$$

$$\rho \ddot{w} - q = \frac{\partial}{\partial x} \left[ \mu \kappa \left( \frac{\partial w}{\partial x} - \psi \right) \right] \tag{37}$$

Integration $z$ over the cross section, we get the second moment of area.

$$\int dA = A, \quad \int z^2 dA = I \tag{38}$$

Let’s consider eqs. (33) and (37). These eqs. are equivalent to the following

$$\rho A \frac{\partial^2 w}{\partial t^2} - q = \frac{\partial}{\partial x} \left[ \kappa A \mu \left( \frac{\partial w}{\partial x} - \psi \right) \right] \tag{39}$$

$$\kappa A \mu \left( \frac{\partial w}{\partial x} - \psi \right) - \rho I \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial x} \left( E I \frac{\partial \psi}{\partial x} \right) = 0 \tag{40}$$

From the eq. (39)

$$\frac{\partial \psi}{\partial x} = - \frac{\rho}{\kappa \mu} \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + \frac{q}{\kappa A \mu} \tag{41}$$
Differentiating the eq. (39) twice with respect to \( t \), we obtain
\[
\frac{\partial^2 q}{\partial t^2} = \rho A \frac{\partial^4 w}{\partial t^4} - \kappa A \mu \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\partial^3 \psi}{\partial x \partial t^2} \right)
\]
(42)

Differentiating the eq. (41) twice with respect to \( x \), we acquire
\[
\frac{\partial^3 \psi}{\partial x^3} = -\frac{\rho}{\kappa \mu} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial x^4} + \frac{1}{\kappa A \mu} \frac{\partial^2 q}{\partial x^2}
\]
(43)

Taking the first derivative with respect to \( x \) from eq. (40), we get
\[
\frac{\partial^3 \psi}{\partial x \partial t^2} = \frac{E}{\rho} \frac{\partial^3 \psi}{\partial x^3} + \frac{\kappa A \mu}{\rho I} \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right)
\]
(44)

Substituting eq. (44) into eq. (42), we obtain
\[
\frac{\partial^2 q}{\partial t^2} = \rho A \frac{\partial^4 w}{\partial t^4} - \kappa A \mu \left[ \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{E}{\rho} \frac{\partial^3 \psi}{\partial x^3} \right] - \frac{\kappa A \mu}{\rho I} \left( \frac{\rho}{\kappa \mu} \frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right)
\]
(45)

Substituting eq. (41) into eq. (45), we gain
\[
\frac{\partial^2 q}{\partial t^2} = \rho A \frac{\partial^4 w}{\partial t^4} - \kappa A \mu \left[ \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{E}{\rho} \frac{\partial^3 \psi}{\partial x^3} - \frac{\kappa A \mu}{\rho I} \left( \frac{\rho}{\kappa \mu} \frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) \right]
\]
(46)

Substituting eq. (43) into eq. (46), we acquire
\[
\frac{\partial^2 q}{\partial t^2} - \kappa A \mu E \left( -\frac{\rho}{\kappa \mu} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial x^4} + \frac{1}{\kappa A \mu} \frac{\partial^2 q}{\partial x^2} \right) =
\rho A \frac{\partial^4 w}{\partial t^4} - \kappa A \mu \left[ \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{\kappa A \mu}{\rho I} \left( \frac{\rho}{\kappa \mu} \frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) \right]
\]
(47)

After some transformations, we get the Tymoshenko dynamic beam equation
\[
\frac{\rho I \partial^2 q}{\kappa A \mu \partial t^2} = \frac{\rho^2 I \partial^4 w}{\kappa \mu \partial t^4} - \rho \frac{\partial^4 w}{\kappa \mu \partial x^2 \partial t^2} - \rho E \frac{\partial^4 w}{\kappa \mu \partial x^2 \partial t^2} + E I \frac{\partial^4 w}{\kappa A \mu \partial x^2} + E I \frac{\partial^2 q}{\kappa A \mu \partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} - q
\]
(48)

Having received the equation for a dynamic beam, we can derive the equation for the static case, it will take the form:
\[
0 = E I \frac{\partial^4 w}{\partial x^4} + E I \frac{\partial^2 q}{\kappa A \mu \partial x^2} - q (x)
\]
(49)

5. Rayleigh Beam

The Rayleigh Beam model is a simplification of the Timoshenko beam model, in which shear deformation is neglected, so all cross sections remain flat and perpendicular to the curved axis of the beam during deformation (Bernoulli hypothesis), and shear modulus can be directed to infinity

According [7]
\[
\varepsilon_{ik} = \frac{1}{9k} \delta_{ik} \sigma_{ll} + \frac{1}{2 \mu} \left( \sigma_{ik} - \frac{1}{3} \delta_{ik} \sigma_{ll} \right)
\]
(50)

where \( \sigma_{ik} \) — stress tensor. Assuming that the shear modulus of the beam tends to infinity \((\mu \to \infty)\) and then according (20) the Poisson’s ratio tends to \(-1\) \((\sigma \to -1)\).
We get the following equation
\[ \varepsilon_{ik} = \frac{1}{9K} \delta_{ik} \sigma_{ll} \] (51)

Taking into account the equation
\[ K = \frac{E}{9} \] (52)

Then the value of the strain tensor will take the form:
\[ \varepsilon_{ik} = \frac{1}{E} \delta_{ik} \sigma_{ll} \] (53)

It can be seen that it is diagonal, and all its diagonal elements are equal to each other. This is due to the -1 Poisson ratio. For his track, we can write:
\[ \varepsilon_{ll} = \frac{3}{E} \sigma_{ll} \] (54)

Beam bending is described by the function
\[ u_z = w(x, t) \] (55)

From the hypothesis of Bernoulli follows:
\[ \frac{\partial w}{\partial x} = \tan \psi = -\frac{u_x}{z}, \] (56)

hence
\[ u_x = -z \frac{\partial w}{\partial x} \] (57)

Due to the equality of the diagonal elements of the strain tensor (due to the equality of -1 of the Poisson’s ratio at longitudinal deformation, transverse deformations equal to it in modulus arise \( \varepsilon_{yy} = \varepsilon_{zz} = -\sigma \varepsilon_{xx} = \varepsilon_{xx} \)):
\[ \varepsilon_{ll} = 3\varepsilon_{xx} = 3 \frac{\partial u_x}{\partial x} = -3z \frac{\partial^2 w}{\partial x^2} = -3zw'' \] (58)

Densities of potential and kinetic energies take the form:
\[ T = \frac{\rho}{2} \dot{z}^2 = \frac{\rho}{2} \left( z^2 \left( \dot{w}' \right)^2 + \dot{w}^2 \right) \] (59)
\[ U = \frac{\sigma_{ik} \varepsilon_{ik}}{2} - qu_z = \frac{\sigma_{ik} \delta_{ik}}{2E} \sigma_{ll} - qw = \left( \frac{\sigma_{ll}}{2E} \right)^2 - qw = \frac{E}{18} \left( \varepsilon_{ll} \right)^2 = \frac{E}{2} z^2 \left( w'' \right)^2 - qw \] (60)

Then the value of the Lagrangian density takes the form:
\[ L = T - U = \frac{\rho}{2} \left( z^2 \left( \dot{w}' \right)^2 + \dot{w}^2 \right) - \frac{E}{2} z^2 \left( w'' \right)^2 + qw \] (61)

Here for the first time the Lagrangian depends on the second derivatives and we will have to use modified Lagrangian field theoretic equations containing the second derivatives (7):
\[ \frac{\delta L}{\delta w} = \frac{\partial L}{\partial w} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial (\dot{w})} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial L}{\partial (w'')} \right) + \frac{\partial^2}{\partial x \partial t} \left( \frac{\partial L}{\partial (\dot{w}'')} \right) = 0 \] (62)
Calculating derivatives and substituting here, we get the Rayleigh dynamic beam equation:

$$EI \frac{\partial^4 w}{\partial x^4} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} + q$$  \hspace{1cm} (63)

6. Euler-Bernoulli beam

The Euler-Bernoulli beam model differs from the Rayleigh model in that, along with shear deformation, the inertia of the beam during rotation is also neglected. Therefore, in statics, these two models of beams completely coincide.

In the Lagrangian of the Rayleigh beam (61), according to the assumptions made by the model, it is necessary to neglect the term $\frac{E}{2} z^2 (\dot{w})^2$ in kinetic energy:

$$\mathcal{L} = T - U = \frac{\rho}{2} \dot{w}^2 - \frac{E}{2} z^2 (w'')^2 + qw$$  \hspace{1cm} (64)

The Lagrange equations will be used in the form:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial w} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial (\dot{w})} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial \mathcal{L}}{\partial (w'')} \right) = 0$$  \hspace{1cm} (65)

As a result, we obtain the dynamic Euler-Bernoulli beam equation:

$$EI \frac{\partial^4 w}{\partial x^4} = -\rho A \frac{\partial^2 w}{\partial t^2} + q$$  \hspace{1cm} (66)

7. Bending of beams on an elastic foundation

In the case of beams lying on an elastic base, the Winkler hypothesis is used: the reaction of the base is proportional to the deflection at a point and is directed against it:

$$p = -kw$$  \hspace{1cm} (67)

The Lagrangian formalism allows to take into account the influence of the base on the statics and dynamics of the beam by including in the potential energy density one more member: $\frac{kw^2}{2}$.

8. Conclusion

In addition to Lagrangian, there is also a Hamiltonian formalism arising from Lagrangian, although Hamiltonian mechanics can be formulated without using Lagrangian, but with the involvement of symplectic and Poisson’s manifolds. From Lagrangian formalism we can pass to Hamiltonian by analogy with Lagrange and Hamilton mechanics. The field function here acts as generalized coordinate. Accordingly, it’s also necessary to determine the generalized momentum conjugate to this coordinate by the formula:

$$\pi (t, x, y, z) = \frac{\partial \mathcal{L} (A^{(i)}, \partial_t A^{(i)})}{\partial A^{(i)} (t, x, y, z)}$$  \hspace{1cm} (68)

Then the density of the Hamiltonian will be equal to:

$$\mathcal{H} = \sum_i \pi^{(i)} \dot{A}^{(i)} - \mathcal{L}$$  \hspace{1cm} (69)

Then the equations of motion with the Hamiltonian approach take the form:

$$\dot{A}^{(i)} = \frac{\partial \mathcal{H}}{\partial \pi^{(i)}}, \quad \dot{\pi}^{(i)} = -\frac{\partial \mathcal{H}}{\partial A^{(i)}}$$  \hspace{1cm} (70)
In the future, a similar use of the covariant Hamiltonian field theory is possible, which is a real analogue of the Lagrangian classical field theory, where the canonical moments correspond to the derivatives of the fields with respect to all space-time coordinates, and not just with respect to time. Hamiltonian field theory is being developed in versions of the Hamilton-De Donder polysymplectic, multisymplectic, and k-symplectic formalisms. Perhaps the use of the Hamilton-Jacobi formalism. Lagrangian formalism allows assumptions for the case of non-Cartesian, already curvilinear spatial coordinates. In this case, it is necessary to replace the usual derivatives in Lagrange equations with covariant ones using affine connectivity coefficients. This can be applied in the calculation of beams and plates having a curved non-deformable shape. It is possible to take into account the effect of temperature (through thermodynamic potentials). In the future, the use of the calibration approach is allowed.

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