Research Article

The Centrosymmetric Matrices of Constrained Inverse Eigenproblem and Optimal Approximation Problem

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In this paper, a kind of constrained inverse eigenproblem and optimal approximation problem for centrosymmetric matrices are considered. Necessary and sufficient conditions of the solvability for the constrained inverse eigenproblem of centrosymmetric matrices in real number field are derived. A general representation of the solution is presented for a solvable case. The explicit expression of the best approximations solution.

1. Introduction

Inverse eigenproblems arise in a remarkable variety of applications, including control theory [1, 2], vibration theory [3, 4], structural design [5], molecular spectroscopy [6], in developing numerical methods, and the ordinary and partial differential equation solving [7, 8]. Centrosymmetric matrices are applied in information theory, linear system theory, and numerical analysis theory [9]. A class of unconstrained matrices’ inverse eigenproblems has been obtained [15–18], and the constrained inverse eigenproblems have been discussed [19–22], but only when the eigenvalues are real or imaginary numbers. For general real matrices, the eigenvalues are not necessarily real or imaginary numbers, so when the eigenvalue is complex, it is difficult to find the constraint solution. In this paper, we will use the real Schur decomposition theorem and the similar decomposition theorem and introduce a new norm to get the corresponding expression of the best approximation solution.

Throughout the paper, we denote the set of real $n \times m$ matrices, real $n \times n$ orthogonal matrices, $n \times n$ centrosymmetric matrices, and real numbers, respectively, by $\mathbb{R}^{n \times m}$, $\mathbb{O}^{n \times n}$, $\mathbb{CSR}^{n \times n}$, and $R$. $A^T$, $A^*$, rank $(A)$, and $\|A\|_{F}$ denote the transpose, the Moore–Penrose generalized inverse, the rank, and the Frobenius norm of a matrix $A$, respectively. $I$ is the identity matrix. $\lambda(A)$ denotes the set of eigenvalues of the matrix $A$. $\lambda(A)/\lambda(B)$ denotes the set of the difference of $\lambda(A)$ and $\lambda(B)$. $D_r$ is a closed disc with radius $r$ and center origin, $[A]_{D_r}$ denotes the square matrix $A$ with all of its eigenvalues located in the closed disc $D_r$, and $\mathbb{R}^{m\times n}_{D_r}$ denotes the set of $n \times m$ matrices with their eigenvalues located in the disc $D_r$. The notation $A_{11} \oplus A_{22} \oplus \cdots \oplus A_{kk}$ denotes the direct sum of the matrices $A_{11}, A_{22}, \ldots, A_{kk}$, where $A_{jj} \in \mathbb{R}^{n_j \times m_j}$. Let $e_i$ be the $i$th standard unit vector, and the matrix $S_n = (e_{n-1}, e_n, \ldots, e_1)$. $[x]$ represents the largest integer less than or equal to $x$. The following definition is given in [23].

Definition 1. $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called centrosymmetric matrix if $a_{ij} = a_{n+1-j,n+1-i}$, $i, j = 1, 2, \ldots, n$.

Clearly, each eigenvalue of matrix $A \in \mathbb{CSR}^{n \times n}$ is either a real number or a complex number, if $A$ has complex eigenvalues, and they must occur in complex conjugate pairs, note that $\lambda_j = \tilde{a}_j + \tilde{b}_j$, $\tilde{\lambda}_j = \tilde{a}_j - \tilde{b}_j$ ($\tilde{b}_j \neq 0$, $j = 1, 2, \ldots, k$) are eigenvalues of $A$, $\chi_j = \xi_j + \eta_j$ and $\Xi_j = \xi_j - \eta_j$ ($j = 1, 2, \ldots, k$) are eigenvectors associated with $\lambda_j$ and $\tilde{\lambda}_j$, respectively; also, we have

$$A[\xi_j \eta_j] = [\xi_j \eta_j]\begin{pmatrix} \tilde{a}_j & \tilde{b}_j \\ -\tilde{b}_j & \tilde{a}_j \end{pmatrix}, \quad j = 1, 2, \ldots, k, \tag{1}$$

where let
Approximation Problem and give a numerical example of Optimal Approximation Problem.

2. The Solvability Conditions and General Solution of Constrained Inverse Eigenproblem

Firstly, let \( k = \lfloor n/2 \rfloor \) and characterize the set of all centrosymmetric matrices as follows. When \( n = 2k \), let

\[
D = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ S_k & -S_k \end{bmatrix}.
\]

When \( n = 2k + 1 \), let

\[
D = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{bmatrix}.
\]

Clearly, \( D \) is an orthogonal matrix for all of the \( n \).

Secondly, given \( X \in \mathbb{R}^{n \times m} \), denote

\[
D^T X = \begin{bmatrix} X_{D_1} \\ X_{D_2} \end{bmatrix}, \quad X_{D_1} \in \mathbb{R}^{(n-k) \times m}, \quad X_{D_2} \in \mathbb{R}^{k \times m}.
\]

Decomposing the matrices \( X, X_{D_1}, \) and \( X_{D_2} \) by the SVD, we have

\[
X = U \begin{bmatrix} 0 & \Delta \\ \Gamma & 0 \end{bmatrix} V^T = U_1 \Delta V_1^T,
\]

where \( \bar{\Delta} = [\bar{\bar{\Delta}}_1, \bar{\bar{\Delta}}_2] \in \mathbb{R}^{r \times r}, \bar{\bar{\Delta}}_1 = [\bar{\bar{\Delta}}_{11}, \bar{\bar{\Delta}}_{12}], \bar{\bar{\Delta}}_2 = \bar{\bar{\Delta}}_{22}, \) and \( \Gamma = \text{diag} (\gamma_1, \gamma_2, \ldots, \gamma_r), \gamma_i > 0, 1 \leq i \leq r_0 \).

\[
X_{D_1} = U \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} V^T = U_1 \Gamma V_1^T,
\]

where \( U = [U_1, U_2] \in \mathbb{R}^{r \times (r-k) \times (n-k)}, V = [V_1, V_2] \in \mathbb{R}^{r \times m}, r_0 = \text{rank}(X), \) and

\[
\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n), \delta_i > 0, 1 \leq i \leq r_0.
\]

\[
X_{D_2} = P \left[ \begin{array}{cc} \Sigma & 0 \\ 0 & 0 \end{array} \right] Q^T = P \Sigma Q_1^T,
\]

where \( P = [P_1, P_2] \in \mathbb{R}^{r \times k}, Q = [Q_1, Q_2] \in \mathbb{R}^{m \times r}, r_2 = \text{rank}(X_{D_2}), \) and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \sigma_i > 0, 1 \leq i \leq r_2 \).

Lemma 1 (see [9], Lemma 2). \( A \in \mathbb{CSR}^{n \times m} \) if and only if \( A \) can be expressed as

\[
A = D \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} D^T, \quad B \in \mathbb{R}^{(n-k) \times (n-k)}, \quad C \in \mathbb{R}^{k \times k},
\]

where, if \( n = 2k \), then \( D \) is the form of equation (8), and if \( n = 2k + 1 \), then \( D \) is the form of (9).
Lemma 2 (see [15], Theorems 7.2 and 7.3). Given $X \in \mathbb{R}^{m \times m}$, $A \in \mathbb{R}^{m \times m}$, and $X$ decomposed as equation (11), let
\[ U^T A U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \] (16)
where $A_{ij} = U_i^T A U_j$ $(i, j = 1, 2)$. Equation $AX = XA$ is consistent if and only if
\[ XAV_2 = 0. \] (17)
If the condition is satisfied, the general solution can be expressed as
\[ A = XAX^* + G\tilde{U}_2^T, \] (18)
and $G = \tilde{U} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \in \mathbb{R}^{n \times (n-r)}$ is arbitrary.

Lemma 3. Given $X \in \mathbb{R}^{m \times m}$ and $A \in \mathbb{R}^{m \times m}$. Equation $AX =XA$ is consistent with $A \in \mathbb{C}S\mathbb{R}^{m \times m}$ if and only if
\[ X_{D_1}AV_2 = 0, \]
\[ X_{D_2}AQ_2 = 0, \] (19)
where $X_{D_1}, X_{D_2}, V_2,$ and $Q_2$ is the same as equations (10)–(14). Let
\[ U^T B \tilde{U} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \] \[ P^T C P = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \] (20)
where $B_{ij} = U_i^T B \tilde{U}_j$ and $C_{ij} = P_i^T C P_j$ $(i, j = 1, 2)$, and the set of the general solution can be expressed as
\[ S = \left\{ A | A = D \begin{bmatrix} X_{D_1}AX_{D_1}^* + G_1U_2^T & 0 \\ 0 & X_{D_2}AX_{D_2}^* + G_2D_2^T \end{bmatrix} | D \right\}, \] (21)
where $G_1 = U \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} \in \mathbb{R}^{n \times (n-k-r)}$ and $G_2 = P \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} \in \mathbb{R}^{k \times (k-r)}$ are arbitrary.

Proof. If $S$ is nonempty, by Lemma 1, we have
\[ A = D \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} D^T, \] \[ B \in \mathbb{R}^{(n-k) \times (n-k)}, C \in \mathbb{R}^{k \times k}. \] (22)
Using equation (10), equation $AX = XA$ is equivalent to
\[ BX_{D_1} = X_{D_1}A, \]
\[ CX_{D_2} = X_{D_2}A. \] (23)
It follows from Lemma 2, and we can obtain $AX = XA$ is solvable if and only if equation (19) holds, and the solution set can be expressed as equation (21) easily.

Remark 2. Equation (21) implies that $\lambda(A)/\lambda(A) \in \mathbb{S}_{D_2}$ if and only if arbitrary matrices $\lambda(B_{ij}) \in \mathbb{S}_{D_2}$ and $\lambda(C_{ij}) \in \mathbb{S}_{D_2}$. From Lemmas 1–3, we have the following theorem.

Theorem 1. Constrained Inverse Eigenproblem is solvable if and only if
\[ X_{D_1}AV_2 = 0, \]
\[ X_{D_2}AQ_2 = 0. \] (24)
The general solution can be expressed as
\[ A = D \begin{bmatrix} X_{D_1}AX_{D_1}^* + U \begin{bmatrix} B_{12} \\ [B_{22}]_{D_1} \end{bmatrix} & 0 \\ 0 & X_{D_2}AX_{D_2}^* + P \begin{bmatrix} C_{12} \\ [C_{22}]_{D_2} \end{bmatrix} \end{bmatrix} D^T, \] (25)
where $B_{ij} \in \mathbb{R}^{n \times (n-k-r)}$, $C_{ij} \in \mathbb{R}^{n \times (n-k-r)}$, and $[B_{22}]_{D_1}, [C_{22}]_{D_2}$ are arbitrary matrices.

3. The Solution of Optimal Approximation Problem

Lemma 5. Given a matrix
\[ B_{ij} = \begin{bmatrix} c_{ij} & d_{ij} \\ -d_{ij} & c_{ij} \end{bmatrix}, \]
\[ d_{ij} > 0, \]
\[ \lambda(B_{ij}) = \sqrt{c_{ij}^2 + d_{ij}^2} = s_j > r > 0, \] (26)
By similarity invariant, we can derive an invertible matrix $Y_{B_{ij}}$ and $\|Y_{B_{ij}}\|_F = 1$ such that
\[ B_{ij} = Y_{B_{ij}}^{-1} B_{ij} Y_{B_{ij}}^{-1}, \] (27)
where $B_{jj} = \begin{bmatrix} c_{jj} & d_{jj} \\ -d_{jj} & c_{jj} \end{bmatrix}$ and $\|\lambda(B_{jj})\|^2 = c_{jj}^2 + d_{jj}^2 = \|\lambda(B_{jj})\|^2$.

Lemma 6. Given $B_{jj} = \begin{bmatrix} c_{jj} & d_{jj} \\ -d_{jj} & c_{jj} \end{bmatrix}$ and $\|\lambda(B_{jj})\|^2 = \sqrt{c_{jj}^2 + d_{jj}^2} = s_j > r > 0$, then, there exists a unique matrix:
\[ A_{jj} = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}, \sqrt{a_j^2 + b_j^2} = r, \] (28)
such that
\[ \|A_{jj} - B_{jj}\|_F = \min, \] (29)
\[ A_{jj} = \frac{r}{\lambda(B_{jj})} B_{jj}. \] (30)

Proof. From
\[ \|A_{jj} - B_{jj}\|_F^2 = \min, \] (31)
we may obtain
Lemma 8. Given matrix $A^* \in R^{m \times n}$, $D^T A^* D$ is decomposed as equation (35), the orthogonal matrices $U$ and $P$ are given in equations (13) and (14), respectively; then, there exist orthogonal matrices $Q^*_1$ and $Q^*_2$ and inverse matrices $Y_1$ and $Y_2$, such that

$$\begin{pmatrix}
\Lambda_{11} & \cdots & \cdots & * \\
0 & \ddots & \ddots & \\
& \ddots & \ddots & \\
0 & \cdots & 0 & \lambda_{n-k-r_1-2s}
\end{pmatrix}
\begin{pmatrix}
Y_1^T
\end{pmatrix}
\begin{pmatrix}
Q^*_1
\end{pmatrix},
$$

where

$$\Lambda_{jj} = \begin{pmatrix}
I_j & f_{jj} \\
-f_{jj} & I_j
\end{pmatrix}, j = 1, 2, \ldots, s.$$

$$\begin{pmatrix}
\Gamma_{11} & \cdots & \cdots & * \\
& \ddots & \ddots & \\
& \ddots & \ddots & \\
& \cdots & 0 & \mu_{k-r_2-2t}
\end{pmatrix}
\begin{pmatrix}
Y_2^T
\end{pmatrix}
\begin{pmatrix}
Q^*_2
\end{pmatrix},
$$

where

$$\Gamma_{jj} = \begin{pmatrix}
g_j & h_j \\
-h_j & g_j
\end{pmatrix}, j = 1, 2, \ldots, t.$$

Proof. For Lemma 7, there exist orthogonal matrices $Q^*_1 \in OR(n-k-r_1 \times (n-k-r_1))$ and $Q^*_2 \in OR(k-r_2 \times (k-r_2))$ such that

$$U^T_2 A_1^* U_2 = Q^*_1,$$

$$P^T_2 A_2^* P_2 = Q^*_2,$$

where

$$\begin{pmatrix}
\Lambda_{11} & \cdots & \cdots & * \\
0 & \ddots & \ddots & \\
& \ddots & \ddots & \\
0 & \cdots & 0 & \lambda_{n-k-r_1-2s}
\end{pmatrix},$$

$$\begin{pmatrix}
\Gamma_{11} & \cdots & \cdots & * \\
& \ddots & \ddots & \\
& \ddots & \ddots & \\
& \cdots & 0 & \mu_{k-r_2-2t}
\end{pmatrix}.$$
Then, Optimal Approximation Problem has a unique solution which can be expressed as
\[
\hat{A} = D \left( \tilde{B} 0 \right) \tilde{C} D^T, \tag{42}
\]
where
\[
\tilde{B} = U \left( \sum_1 V_1^2 A_1 \Sigma_1^{-1} U_1^T A_{12} U_2 \right) U^T, \tag{43}
\]
\[
\tilde{C} = P \left( \sum_2 Q_2^T A_2 \Sigma_2^{-1} P_{22}^T A_{22}^2 P_2 \right) P^T.
\]

Proof. Let \( A \in S_D \), by Definition 2, we have
\[
\| A - A^* \|_f^2 = \| Y^{-1} AY - Y^{-1} A^* Y \|_f^2 = \min_{\forall A \in S_D} \tag{44}
\]

The expression of \( A \) is the same as equation (25) and \( D^T A^* D \) is decomposed as equation (35); then, equation (44) is equivalent to
\[
\left\| Y^{-1} Q_1^T A_1 U_1 Q_1^T Y_1 - Y^{-1} Q_1^T A_1^* U_1 Q_1^T Y_1 \right\|_f^2 = \min, \tag{45}
\]

Obviously, we may derive that equation (44) has the solution \( \hat{A} \) if and only if
\[
\left\| Y^{-1} Q_1^T A_1 U_1 Q_1^T Y_1 - Y^{-1} Q_1^T \left[ B_{22} \right]_{D_r} Q_1^T Y_1 \right\|_f^2 = \min, \tag{46}
\]
\[
\left\| Y^{-1} Q_2^T P_{22}^T A_{22}^2 P_2 Q_2^T Y_2 - Y^{-1} Q_2^T \left[ C_{22} \right]_{D_r} Q_2^T Y_2 \right\|_f^2 = \min, \tag{47}
\]
\[
\left\| U_1^T A_{11}^* U_2 Q_1^T Y_1 - B_{12} Q_1^T Y_1 \right\|_f = \min, \tag{48}
\]
\[
\left[ B_{22} \right]_{D_r} = Q_1^T Y_1 \left( \begin{array}{ccccc} \bf{\bar{A}}_{11} \bf{\bar{A}}_{12} \cdots \bf{\bar{A}}_{1r} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{array} \right) \left( \begin{array}{c} \bf{\bar{A}}_{r1} \bf{\bar{A}}_{r2} \cdots \bf{\bar{A}}_{rr} \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \left( \begin{array}{c} 0 \cdots \cdots \cdots \ \end{array} \right). \tag{50}
\]
\begin{align*}
\mathbf{[X_{ij}]_{D_i}} &= \begin{cases} 
\frac{r}{\lambda(A_{jj})} \mathbf{\bar{X}_{ij}}, & \lambda(A_{jj}) > r, \\
\mathbf{\bar{X}_{ij}}, & \lambda(A_{jj}) \leq r,
\end{cases} \\
\mathbf{[X_{ij}]_{D_i}} &= \begin{cases} 
\text{sign}(\lambda_i) r, & |\lambda_i| > r, \\
\lambda_i, & |\lambda_i| \leq r,
\end{cases} \\
&\quad j = 1, 2, \ldots, s, i = s + 1, \ldots, n - k - r_1 - 2t.
\end{align*} 

By equations (48) and (49), we have

\begin{align*}
B_{12} &= U_1^T A_{11} U_2, \\
C_{12} &= P_1^T A_{22}^* P_2.
\end{align*} 

Note

\begin{align*}
\mathbf{[C_{22}]_{D_i}} &= Q_2^r \mathbf{Y}_2 \\
&= \begin{pmatrix} 
[\Gamma_{11}]_{D_i} & \ast & \cdots & \cdots & \cdots & \ast \\
0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \ast & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \ddots & \ddots \\
\end{pmatrix} Y_2^{-1} Q_2^r,
\end{align*} 

where

\begin{align*}
\mathbf{[\Gamma_{ij}]_{D_i}} &= \begin{cases} 
\frac{r}{\lambda(A_{ij})} \mathbf{\bar{\Gamma}_{ij}}, & \lambda(A_{ij}) > r, \\
\mathbf{\bar{\Gamma}_{ij}}, & \lambda(A_{ij}) \leq r,
\end{cases} \\
\mathbf{[\bar{\mu}_{ij}]_{D_i}} &= \begin{cases} 
\text{sign}(\mu_i) r, & |\mu_i| > r, \\
\mu_i, & |\mu_i| \leq r,
\end{cases} \\
&\quad j = 1, \ldots, t, i = t + 1, \ldots, n - k - r_2 - 2t.
\end{align*} 

Equations (55), (56), and (25) imply equation (42). \qed

4. Numerical Example

Based on Theorems 1 and 2, we propose the following algorithm for solving Constrained Inverse Eigenproblem and Optimal Approximation Problem (Algorithm 1).

In this section, we will give a numerical example to illustrate our results. All the tests are performed by MATLAB 6.5.

The matrices \(X, \Lambda, \) and \(A^*\) and the radius \(r\) are given by following:
We verify that partial eigenvalues of the matrix must be located in a given closed disc or interval. For this purpose, we need to find a matrix $A$ that is a centrosymmetric matrix and the eigenvalues of $\lambda(A)$ are lost in the disc $D_r$.

### Data Availability

All data generated or analyzed during this study are included in this article.

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### Conflicts of Interest

The authors declare that they have no conflicts of interest.

\[ A^* = \begin{pmatrix} 0.1450 & 0.1242 & -0.0729 & -0.2928 & -0.2391 & 0.8318 & 0.3028 & 0.4966 \\ 0.0832 & 0.2522 & 0.3522 & 0.5442 & 0.2411 & -0.5450 & 0.3983 & 0.5450 \\ 0.5450 & 0.3983 & -0.5450 & 0.2411 & 0.5442 & 0.3522 & 0.2522 & 0.0832 \\ 0.6977 & 0.0205 & -0.3526 & -0.2391 & -0.2928 & -0.0729 & 0.1242 & 0.1450 \\ -0.1464 & -0.3727 & 0.2079 & -0.0476 & -0.0179 & -0.1473 & -0.0060 & 0.2569 \\ -0.0710 & 0.0071 & 0.0782 & -0.1080 & 0.0469 & -0.1728 & 0.2884 & 0.4539 \end{pmatrix} \]

\[ \tilde{A} = \begin{pmatrix} 0.4539 & 0.2884 & -0.1728 & 0.0469 & -0.1080 & 0.0782 & 0.0071 & -0.0710 \\ 0.2569 & -0.0060 & -0.1473 & -0.0179 & -0.0476 & 0.2079 & -0.3727 & -0.1464 \\ 0.1450 & 0.1242 & -0.0729 & -0.2928 & -0.2391 & -0.3526 & 0.0205 & 0.6977 \\ 0.0832 & 0.2522 & 0.3522 & 0.5442 & 0.2411 & -0.5450 & 0.3983 & 0.5450 \\ 0.5450 & 0.3983 & -0.5450 & 0.2411 & 0.5442 & 0.3522 & 0.2522 & 0.0832 \\ 0.6977 & 0.0205 & -0.3526 & -0.2391 & -0.2928 & -0.0729 & 0.1242 & 0.1450 \\ -0.1464 & -0.3727 & 0.2079 & -0.0476 & -0.0179 & -0.1473 & -0.0060 & 0.2569 \\ -0.0710 & 0.0071 & 0.0782 & -0.1080 & 0.0469 & -0.1728 & 0.2884 & 0.4539 \end{pmatrix} \]
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