Semi-direct Galois covers of the affine line

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Abstract. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( G \) be a semi-direct product of the form \((\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}\) where \( b \) is a positive integer and \( \ell \) is a prime distinct from \( p \). In this paper, we study Galois covers \( \psi : Z \to \mathbb{P}^1_k \) ramified only over \( \infty \) with Galois group \( G \). We find the minimal genus of a curve \( Z \) which admits a covering map of this form and we give an explicit formula for this genus in terms of \( \ell \) and \( p \). The minimal genus occurs when \( b \) equals the order of \( \ell \) modulo \( b \) and we also prove that the number of curves \( Z \) of this minimal genus which admit such a covering map is at most \((p-1)/a\) when \( p \) is odd.

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1 Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. In sharp contrast with the situation in characteristic 0, there exist Galois covers $\psi : \mathcal{Z} \to \mathbb{P}^1_k$ ramified only over infinity. By Abhyankar’s Conjecture \[2\], proved by Raynaud and Harbater \[2\], \[4\], a finite group $G$ occurs as the Galois group of such a cover $\psi$ if and only if $G$ is quasi-$p$, i.e., $G$ is generated by $p$-groups. This result classifies all the finite quotients of the fundamental group $\pi_1(\mathbb{A}^1_k)$. It does not, however, determine the profinite group structure of $\pi_1(\mathbb{A}^1_k)$ because this fundamental group is an infinitely generated profinite group.

There are many open questions about Galois covers $\psi : \mathcal{Z} \to \mathbb{P}^1_k$ ramified only over infinity. For example, given a finite quasi-$p$ group $G$, what is the smallest integer $g$ for which there exists a cover $\psi : \mathcal{Z} \to \mathbb{P}^1_k$ ramified only over infinity with $Z$ of genus $g$? As another example, suppose $G$ and $H$ are two finite quasi-$p$ groups such that $H$ is a quotient of $G$. Given an unramified Galois cover $\phi$ of $\mathbb{A}^1_k$ with group $H$, under what situations can one dominate $\phi$ with an unramified Galois cover $\psi$ of $\mathbb{A}^1_k$ with Galois group $G$? Answering these questions will give progress towards understanding how the finite quotients of $\pi_1(\mathbb{A}^1_k)$ fit together in an inverse system. These questions are more tractible for quasi-$p$ groups that are $p$-groups since the maximal pro-$p$ quotient $\pi_p^0(\mathbb{A}^1_k)$ is free (of infinite rank) \[10\].

In this paper, we study Galois covers $\psi : \mathcal{Z} \to \mathbb{P}^1_k$ ramified only over infinity whose Galois group is a semi-direct product of the form $\left(\mathbb{Z}/(\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}\right)$, where $\ell$ is a prime distinct from $p$. Such a cover $\psi$ must be a composition $\psi = \phi \circ \omega$ where $\omega : \mathcal{Z} \to \mathcal{Y}$ is unramified and $\phi : \mathcal{Y} \to \mathbb{P}^1_k$ is an Artin-Schreier cover ramified only over infinity. The cover $\phi$ has an affine equation $y^p - y = f(x)$ for some $f(x) \in k[x]$ with degree $s$ prime-to-$p$. The $\ell$-torsion $\text{Jac}(\mathcal{Y})[\ell]$ of the Jacobian of $\mathcal{Y}$ is isomorphic to $\left(\mathbb{Z}/(\ell\mathbb{Z})^2\right)^g$. When $f(x) = x^a$, we determine how an automorphism $\tau$ of $\mathcal{Y}$ of order $p$ acts on $\text{Jac}(\mathcal{Y})[\ell]$. This allows us to construct a Galois cover $\psi_a : \mathcal{Y}_a \to \mathbb{P}^1_k$ ramified only over infinity which dominates $\phi$, such that the Galois group of $\psi_a$ is $(\mathbb{Z}/(\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z})$, where $a$ is the order of $\ell$ modulo $p$ (Section 3). We prove that the genus of $\mathcal{Y}_a$ is minimal among all natural numbers that occur as the genus of a curve $\mathcal{Z}$ which admits a covering map $\psi : \mathcal{Z} \to \mathbb{P}^1_k$ ramified only over infinity with Galois group of the form $\left(\mathbb{Z}/(\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}\right)$. We also prove that the number of curves $\mathcal{Z}$ of this minimal genus which admit such a covering map is at most $(p - 1)/a$ when $p$ is odd (Section 4).

2 Quasi-$p$ semi-direct products

We recall which groups occur as Galois groups of covers of $\mathbb{P}^1_k$ ramified only over infinity.

**Definition 2.1** A finite group is a quasi $p$-group if it is generated by all of its Sylow $p$-subgroups.

It is well-known that there are other equivalent formulations of the quasi-$p$ property, such as the next result.

**Lemma 2.2** A finite group is a quasi $p$-group if and only if it has no nontrivial quotient group whose order is relatively prime to $p$.

The importance of the quasi-$p$ property is that it characterizes which finite groups occur as Galois groups of unramified covers of the affine line.

**Theorem 2.3** A finite group occurs as the Galois group of a Galois cover of the projective line $\mathbb{P}^1_k$ ramified only over infinity if and only if it is a quasi-$p$ group.
Theorem This is a special case of Abhyankar’s Conjecture \cite{2} which was jointly proved by Harbater \cite{4} and Raynaud \cite{9}.

We now restrict our attention to groups \( G \) that are semi-direct products of the form \( (\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z} \). The semi-direct product action is determined by a homomorphism \( \iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut} ((\mathbb{Z}/\ell \mathbb{Z})^b) \).

Lemma 2.4 Suppose a quasi-p group \( G \) is a semi-direct product of the form \((\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}\) for a positive integer \( b \).

1. Then \( G \) is not a direct product.
2. Moreover, \( b \geq \text{ord}_p(\ell) \) where \( \text{ord}_p(\ell) \) is the order of \( \ell \) modulo \( p \).

Proof Part (1) is true since \((\mathbb{Z}/\ell \mathbb{Z})^b \) cannot be a quotient of the quasi-p group \( G \). For part (2), the structure of a semi-direct product \((\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}\) depends on a homomorphism \( \iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut} ((\mathbb{Z}/\ell \mathbb{Z})^b) \). By part (1), \( \iota \) is an inclusion. Thus \( \text{Aut} ((\mathbb{Z}/\ell \mathbb{Z})^b) \cong \text{GL}_b((\mathbb{Z}/\ell \mathbb{Z})) \) has an element of order \( p \). Now

\[
|\text{GL}_b((\mathbb{Z}/\ell \mathbb{Z}))| = (\ell^b - 1)(\ell^b - \ell) \cdots (\ell^b - \ell^{b-1}).
\]

Thus \( \ell^b \equiv 1 \mod p \) for some positive integer \( \beta \leq b \) which implies \( b \geq \text{ord}_p(\ell) \).

Lemma 2.5 If \( a = \text{ord}_p(\ell) \), then there exists a semi-direct product of the form \((\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}\) which is quasi-p. It is unique up to isomorphism.

Proof If \( a = \text{ord}_p(\ell) \), then there is an element of order \( p \) in \( \text{Aut} ((\mathbb{Z}/\ell \mathbb{Z})^a) \) and so there is an injective homomorphism \( \iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut} ((\mathbb{Z}/\ell \mathbb{Z})^a) \). Thus there exists a non-abelian semi-direct product \( G \) of the form \((\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}\). To show that \( G \) is quasi-p, suppose \( N \) is a normal subgroup of \( G \) whose index is relatively prime to \( p \). Then \( N \) contains an element \( \tau \) of order \( p \). By \cite{1} 5.4, Thm. 9, since \( G \) is not a direct product and \((\mathbb{Z}/\ell \mathbb{Z})^a \) is normal in \( G \), the subgroup \( \langle \tau \rangle \) is not normal in \( G \). Thus \( \langle \tau \rangle \) is a proper subgroup of \( N \). It follows that \( \ell \) divides \( |N| \) and so \( N \) contains an element \( h \) of order \( \ell \) by Cauchy’s theorem. Recall that \( \text{Aut} ((\mathbb{Z}/\ell \mathbb{Z})^\beta) \) contains no element of order \( p \) for any positive integer \( \beta < a \). Thus the group generated by the conjugates of \( h \) under \( \tau \) has order divisible by \( \ell^a \). Thus \( N = G \) and \( G \) has no non-trivial quotient group whose order is relatively prime to \( p \). By Lemma 2.4, \( G \) is quasi-p.

The uniqueness follows from \cite{8} Lemma 6.6.

3 Explicit construction of \((\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}\)-Galois covers of \( \mathbb{A}_k^1 \)

In this section, we give concrete examples of Galois covers \( \psi : Z \to P^1_k \) ramified only over \( \infty \) with Galois group of the form \((\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}\). To compute the genus of the covering curve \( Z \), we will need to determine the higher ramification groups of \( \psi \).

Definition 3.1 Let \( L/K \) be a Galois extension of function fields of curves with Galois group \( G \) and let \( P, P' \) be primes of \( K \) and \( L \) such that \( P'|P \). Let \( \nu_{P'} \) and \( \mathcal{O}_{P'} \) be the corresponding valuation function and valuation ring for \( P' \). For any integer \( i \geq -1 \), the \( i \)th ramification group of \( P'|P \) is

\[
I_i(P'|P) = \{ \sigma \in G \mid \nu_{P'}(\sigma(z) - z) \geq i + 1, \forall z \in \mathcal{O}_{P'} \}.\]

Lemma 3.2 Suppose \( f(x) \in k[x] \) is a polynomial of degree \( s \) for a positive integer \( s \) prime to \( p \). Let \( \phi : Y \to P^1_k \) be the cover of curves corresponding to the field extension

\[
k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x)).\]
1. Then \( \phi : Y \to \mathbb{P}^1_k \) is a Galois cover with Galois group \( \mathbb{Z}/p\mathbb{Z} \) ramified only at the point \( P_\infty \) over \( \infty \).

2. The \( i \)th ramification group at \( P_\infty \) satisfies

\[
I_i = \begin{cases} 
\mathbb{Z}/p\mathbb{Z} & \text{if } i \leq s \\
0 & \text{if } i > s.
\end{cases}
\]

3. The genus \( g_Y \) of \( Y \) is equal to

\[
g_Y = (p - 1)(s - 1)/2.
\]

**Proof** For part (1), note that the extension \( k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x)) \) is cyclic of degree \( p \), with Galois group generated by the automorphism \( \tau : y \mapsto y + 1 \) of order \( p \). Let \( P \) be a finite prime of \( k(x) \) and let \( \nu_P \) be the corresponding valuation. Then \( \nu_P(f(x)) \geq 0 \), hence \( P \) is unramified by [12 Prop. III.7.8(b)]. For the infinite prime \( \infty \) with corresponding valuation \( \nu_\infty \), we have

\[
\nu_\infty(f(x) - (z^p - z)) \leq 0
\]

for all \( z \in k[x] \) thus \( P_\infty \) is totally ramified by [12 Prop. III.7.8(c)].

To prove part (2), we note that furthermore

\[
v_{P_\infty}(y^p - y) = v_{P_\infty}(f(x)) = v_{P_\infty}(x^s) = -sp,
\]

which implies that

\[
v_{P_\infty}(y) = -s.
\]

Now let \( \hat{\theta} \) be the completion of the valuation ring of \( k(x)[y]/(y^p - y - f(x)) \) at \( P_\infty \), and let \( \pi_\infty \) be a generator of the unique prime in \( \hat{\theta} \). Then write \( y = \pi_\infty^{-s}u \), where \( u \) is a unit in \( \hat{\theta} \). Since \( k \) is algebraically closed, \( \sqrt{u} \in \hat{\theta} \), and so \( \sqrt{y} \in \hat{\theta} \). After possibly changing \( \pi_\infty \), we can assume without loss of generality that \( \sqrt[y]{y} = \pi_\infty^{-1} \). Recalling that \( \tau \) acts on \( y \) by \( \tau(y) = y + 1 \), we have

\[
\tau(\pi_\infty) = \tau(1/y)^{1/s} = (\pi_\infty^s/(1 + \pi_\infty^s))^{1/s} \\
= \pi_\infty(1 - \pi_\infty^s + \pi_\infty^{2s} - \ldots)^{1/s} \\
= \pi_\infty - (1/s)\pi_\infty^{s+1} + a_2\pi_\infty^{s+2} - \ldots.
\]

Thus \( v_{P_\infty}(\tau(\pi_\infty) - \pi_\infty) = s + 1 \), which completes the proof of part (2).

To find the genus \( g_Y \) of \( Y \) for part (3), we make use of the Riemann-Hurwitz formula

\[
2g_Y - 2 = p(-2) + \sum_{i=0}^{\infty} (|I_i| - 1),
\]

where \( I_i \) denotes the \( i \)th ramification group at \( P_\infty \), [5 Thms. 7.27 & 11.72]). From part (2), we then obtain that \( g_Y = (p - 1)(s - 1)/2 \). \( \square \)

Recall the following facts about the \( p \)th cyclotomic polynomial \( \Phi_p(t) := t^{p-1} + \ldots + 1 \), which is the minimal polynomial over \( \mathbb{Q} \) of a primitive \( p \)th root of unity \( \zeta_p \). Now \( \mathbb{Q}(\zeta_p) \) is a Galois extension of \( \mathbb{Q} \), unramified over \( \ell \) since \( \ell \neq p \), and all primes over \( \ell \) have the same residue field degree. The irreducible factors of \( \Phi_p(t) \) modulo \( \ell \) are in one-to-one correspondence with the primes of \( \mathbb{Z}[\zeta_p] \) over \( \ell \), and each of their degrees is equal to the residue field degree of the corresponding prime over \( \ell \). The latter equals the order \( a = \text{ord}_p(\ell) \) of \( \ell \) modulo \( p \) [3 Ch. 12.2, Exercise #20].

We shall soon explicitly construct a cover of \( \mathbb{P}^1_k \) ramified only over \( \infty \) with Galois group \( (\mathbb{Z}/\ell\mathbb{Z})^a \cong \mathbb{Z}/p\mathbb{Z} \). But before we do so, we start with a specific example.
Example 3.3 Let $p$ be an odd prime. Consider the Artin-Schreier cover $\phi : Y_2 \to \mathbb{P}^1_k$ corresponding to the field extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - x^3)$. By Lemma 3.2(3), the genus of $Y_2$ is $g_Y = (p - 1)/2$.

Let $\text{Jac}(Y)$ be the Jacobian of $Y$. The automorphism $\tau$ of $Y$ given by $\tau(y) = y + 1$ defines an automorphism of $\text{Jac}(Y)$ of order $p$.

Now we describe the action of $\tau$ on the subgroup $\text{Jac}(Y)[2]$ of 2-torsion points of $\text{Jac}(Y)$ explicitly. Note that since $2g_Y = (p - 1)$, then $\text{Jac}(Y)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{p-1}$ by [17, pg. 64]. Thus we can represent $\tau$ as an element of $\text{GL}_{p-1}(\mathbb{Z}/2\mathbb{Z})$.

For $0 \leq i \leq p - 1$, let $P_i$ denote the closed point of $Y$ at which the function $y - i$ vanishes. For each $i$, the divisors $P_i$ and $D_i = P_i - P_\infty$ on $Y$ can be identified with elements of $\text{Jac}(Y)$. Let $O$ be the identity element of $\text{Jac}(Y)$, i.e., the linear equivalence class of principal divisors. Then the divisor $2D_i$ is equivalent to $O$ since $\text{div}(y - i) = 2D_i$. Moreover since $\text{div}(x) = D_0 + D_1 + \cdots + D_{p-1}$ is equivalent to 0, we have $D_i \in \text{Jac}(Y)[2]$ with the only relation $D_{p-1} = -(D_0 + D_1 + \cdots + D_{p-2})$. In particular, $D_0, \ldots, D_{p-2}$ form a basis of $\text{Jac}(Y)[2]$. With respect to this basis, the action of $\tau$ can be represented by the $(p - 1) \times (p - 1)$-matrix

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & -1 \\
1 & 0 & \ldots & 0 & -1 \\
0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -1
\end{pmatrix}.
\]

The characteristic polynomial of $\tau$ is $\Phi_\tau(t) = 1 + t + \cdots + t^{p-1} \in (\mathbb{Z}/2\mathbb{Z})[t]$, which factors into irreducible polynomials each of degree equaling the order of 2 modulo $p$. In particular, $\tau$ acts irreducibly on $\text{Jac}(Y)[2]$ if and only if 2 is a primitive root modulo $p$, i.e., if and only if $p$ is an Artin prime.

For example, if $p = 3$, then $\tau$ acts irreducibly on $\text{Jac}(Y)[2]$ with minimal polynomial $\Phi_\tau(t) = t^2 + t + 1$. If $p = 7$, then $\tau$ has order 3 modulo 7 and the factorization of $\Phi_\tau(t)$ into irreducible polynomials is $\Phi_\tau(t) \equiv (x^3 + x^2 + 1)(x^3 + x + 1)$ modulo 2. Thus the action of $\tau$ on $\text{Jac}(Y)[2]$ can be represented by the $6 \times 6$-matrix

\[
\begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
\]

where $A_1$ and $A_2$ are the irreducible 3-dimensional companion matrices of $x^3 + x^2 + 1$ and $x^3 + x + 1$ respectively.

For the rest of the paper, let $\phi_s : Y_s \to \mathbb{P}^1_k$ be the Artin-Schreier cover corresponding to the field extension

\[
k(x) \hookrightarrow k(x)[y]/(y^p - y - x^s).
\]

We show that $\phi_s$ can be dominated by a Galois cover of $\mathbb{P}^1_k$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \times \mathbb{Z}/p\mathbb{Z}$ for $a$ equal to the order of $\ell$ modulo $p$.

Proposition 3.4 Let $s$ and $\ell$ be primes distinct from $p$. Let $\phi_s : Y_s \to \mathbb{P}^1_k$ be the Artin-Schreier cover with affine equation $y^p - y = x^s$. Let $a = \text{ord}_p(\ell)$ be the order of $\ell$ modulo $p$. Then there exists an unramified Galois cover $\psi : Z_a \to Y_s$ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a$ such that $\psi_a = \phi_s \circ \omega : Z_a \to \mathbb{P}^1_k$ is a Galois cover of $\mathbb{P}^1_k$ ramified only over $\infty$ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$.

Proof By Lemma 3.2(1), $\phi_s : Y_s \to \mathbb{P}^1_k$ is a Galois cover with Galois group $\mathbb{Z}/p\mathbb{Z}$ ramified only at the point $P_\infty$ over $\infty$. The genus $g_s$ of $Y_s$ is $(p - 1)(s - 1)/2$. Consider
two commuting automorphisms of $Y_s$ defined by
\[
\tau : \begin{cases} x \mapsto x, \\ y \mapsto y + 1, \end{cases} \quad \sigma : \begin{cases} x \mapsto \zeta_s x, \text{ where } \zeta_s \text{ is a primitive } s \text{th root of unity,} \\ y \mapsto y. \end{cases}
\]

Let $Jac(Y_s)$ be the Jacobian of $Y_s$. Then $\tau$ and $\sigma$ define commuting automorphisms of $Jac(Y_s)$ of orders $p$ and $s$ respectively. Therefore, $End(Jac(Y_s))$ contains a ring isomorphic to $\mathbb{Z}[\zeta_p, \zeta_s] \cong \mathbb{Z}[\zeta_{ps}]$, which is a $\mathbb{Z}$-module of rank $\phi(ps) = (p - 1)(s - 1) = 2g_s$. Then $\mathbb{Q}(\zeta_{ps})$ is contained in $End(Jac(Y_s)) \otimes \mathbb{Q}$. In other words, $Jac(Y_s)$ has complex multiplication by $\mathbb{Q}(\zeta_{ps})$.

For a prime $\ell$ distinct from $p$, the automorphism $\tau$ induces an action on the subgroup $Jac(Y_s)[\ell]$ of $\ell$-torsion points of $Jac(Y_s)$. Recall that there is a bijection between $\ell$-torsion points $D$ of $Jac(Y_s)$ and unramified $(\mathbb{Z}/\ell\mathbb{Z})$-Galois covers $\omega_D : Z_D \to Y_s$ [6 Prop. 4.11]. Also $D$ has order $\ell$ if and only if $Z_D$ is connected. This induces a bijection between orbits of $\tau$ on the set of unramified $(\mathbb{Z}/\ell\mathbb{Z})$-Galois covers $\omega_D : Z_D \to Y_s$ and on the set of $\ell$-torsion points of $Jac(Y_s)$. For a point $D$ of order $\ell$ of $Jac(Y_s)$, consider the compositum $\omega : Z \to Y_s$ of all of the conjugates $\omega_{\tau^j(D)} : Z_{\tau^j(D)} \to Y_s$ for $0 \leq j \leq p - 1$:

\[
\begin{array}{cccc}
Z_D & \xrightarrow{Z_{\tau(D)}} & \xrightarrow{Z_{\tau^2(D)}} & \cdots & \xrightarrow{Z_{\tau^{p-1}(D)}} \\
(\mathbb{Z}/\ell\mathbb{Z}) & \xrightarrow{(\mathbb{Z}/\ell\mathbb{Z})} & \xrightarrow{(\mathbb{Z}/\ell\mathbb{Z})} & \cdots & \xrightarrow{(\mathbb{Z}/\ell\mathbb{Z})} \\
Y_s & \xrightarrow{(\mathbb{Z}/\ell\mathbb{Z})} & \xrightarrow{(\mathbb{Z}/\ell\mathbb{Z})} & \cdots & \xrightarrow{(\mathbb{Z}/\ell\mathbb{Z})}
\end{array}
\]

Then $Z$ is invariant under $\tau$ and so $\phi_s \circ \omega : Z \to \mathbb{P}_k^1$ is Galois. Moreover, $\phi_s \circ \omega$ is the Galois closure of $\phi_s \circ \omega_D : Z_D \to \mathbb{P}_k^1$.

Suppose there is a non-trivial one-dimensional $\tau$-invariant subspace of $Jac(Y_s)[\ell]$ with eigenvalue 1; i.e. $\tau$ acts trivially on this subgroup of order $\ell$. This yields a cover $\psi_s \circ \omega_1 : Z_1 \to Y_s \to \mathbb{P}_k^1$. Since the action of $\tau$ is trivial, $\psi_s \circ \omega_1$ is Galois, ramified only over $\infty$, with abelian Galois group $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. This contradicts Lemma 2.4.

Since $\tau$ has order $p$, the minimal polynomial $m_\tau(t)$ of $\tau$ divides $t^p - 1 = (t - 1)(t^{p-1} + \cdots + 1)$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$. From the preceding paragraph, there is no non-trivial one-dimensional $\tau$-invariant subspace of $Jac(Y_s)[\ell]$ with eigenvalue 1. This implies that $m_\tau(t)$ divides the $p$th cyclotomic polynomial $\Phi_p(t) = t^p - 1 + \cdots + 1$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$. The irreducible factors of $\Phi_p(t)$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$ all have degree $a$. Thus the degree of $m_\tau(t)$ equals $a$.

Since $2g_s = (p - 1)(s - 1)$, we have $Jac(Y_s)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{(p-1)(s-1)}$, so we can represent $\tau$ as an element of $GL_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$. We can choose a basis of $Jac(Y_s)[\ell]$ such that $\tau$ is represented as an element of $GL_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$ in block form. The first irreducible subrepresentation of $\tau$ has dimension $a$. Moreover, since $\mathbb{Q}(\zeta_{ps})$ is a Galois extension of $\mathbb{Q}$, the block form of $\tau$ consists entirely of irreducible blocks of the same size. In particular, the number of irreducible blocks is $(p - 1)(s - 1)/a$. In other words, $\tau$ can be represented by an element of $GL_{(s-1)(p-1)}(\mathbb{Z}/\ell\mathbb{Z})$ of the form
\[
\begin{pmatrix}
A_1 & 0 \\
A_2 & \ddots \\
0 & A_{(p-1)(s-1)/a}
\end{pmatrix},
\]
where $A_1$ is an $a \times a$ matrix representing an $a$-dimensional irreducible subrepresentation of $\tau$ on $\mathrm{Jac}(Y_a)[\ell]$. 

Using the bijection between orbits of $\mathrm{Jac}(Y_a)[\ell]$ and orbits of $(\mathbb{Z}/\ell\mathbb{Z})$-covers of $Y_a$ under $\tau$ and the above observation for the action of $\tau$ on $\mathrm{Jac}(Y_a)[\ell]$, there exists an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover $\omega : Z_a \to Y_a$ such that $\psi_a = \phi_a \circ \omega : Z_a \to \mathbb{P}^1_k$ is a Galois cover of $\mathbb{P}^1_k$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. Also $\psi_a$ is ramified only over infinity since $\phi_a$ is ramified only over $\infty$ and since $\omega$ is unramified. 

4 Minimal genus of $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$-Galois covers of $\mathbb{A}^1_k$

In this section, we find the minimal genus of a curve $Z$ that admits a covering map $\psi : Z \to \mathbb{P}^1_k$ ramified only over $\infty$, with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. The minimal genus depends only on $\ell$ and $p$. We consider the cases $p$ odd and $p = 2$ separately. We also prove that the number of curves $Z$ of this minimal genus which admit such a covering map is at most $(p - 1)/a$ when $p$ is odd and at most $\ell + 1$ when $p = 2$. The following lemma will be useful.

**Lemma 4.1** Let $G$ be a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ where $\ell$ is a prime distinct from $p$. If $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over $\infty$ with Galois group $G$, then the subcover $\omega : Z \to Y$ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^b$ is unramified.

**Proof** The quotient of $G$ by the normal subgroup $N = (\mathbb{Z}/\ell\mathbb{Z})^b$ is $\mathbb{Z}/p\mathbb{Z}$. Thus the cover $\psi$ is a composition $\psi = \phi \circ \omega$ where $\phi : Y \to \mathbb{P}^1_k$ has Galois group $\mathbb{Z}/p\mathbb{Z}$ and is totally ramified at the unique point $P_\infty$ over $\infty$ and where $\omega : Z \to Y$ has Galois group $N$ and is branched only over $P_\infty$. Then $\omega$ is a prime-to-$p$ abelian cover of $Y$. Let $g$ be the genus of $Y$. Then by [13 XIII, Cor. 2.12], the prime-to-$p$ fundamental group of $Y - \{P_\infty\}$ is isomorphic to the prime-to-$p$ quotient $\Gamma$ of the free group on generators $\{a_1, b_1, \ldots, a_g, b_g, c\}$ subject to the relation $\prod_{i=1}^g [a_i, b_i] = c^{-1}$. The cover $\omega$ corresponds to a surjection of $\Gamma$ onto $N$ where $c$ maps to the canonical generator of inertia $\gamma$ of a point of $Z$ over $P_\infty$. Thus $N$ is generated by elements $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma\}$ subject to the relation $\prod_{i=1}^g [\alpha_i, \beta_i] = \gamma^{-1}$. Then $\gamma = 1$ since $N$ is abelian and so $\omega$ is unramified. 

**Theorem 4.2** Let $p$ be an odd prime. Let $\ell$ be a prime distinct from $p$ and let $a$ be the order of $\ell$ modulo $p$. Then:

1. There exists a Galois cover $\psi_a : Z_a \to \mathbb{P}^1_k$ ramified only over $\infty$ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ such that $g_{Z_a} = 1 + \ell^a(p - 3)/2$.
2. The integer $g_{Z_a}$ is the minimal genus of a curve $Z$ which admits a covering map $\psi : Z \to \mathbb{P}^1_k$ ramified only over $\infty$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ for any positive integer $b$.
3. There are at most $(p - 1)/a$ isomorphism classes of curves $Z$ which admit a Galois covering map as in part (1) with minimal genus $g_{Z_a}$.

**Proof** By the construction in Proposition [3.4] there exists a Galois cover $\psi_a : Z_a \to \mathbb{P}^1_k$ ramified only over $\infty$ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. We compute the genus of the curve $Z_a$. Recall that $\psi_a$ is a composition $\psi = \phi_a \circ \omega$ where $\omega : Z \to Y_2$ is an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover and $\phi_a : Y_2 \to \mathbb{P}^1_k$ has Artin-Schreier equation $y^p - y = x^2$. Then $Y_2$ has genus $g_{Y_2} = (p - 1)/2$ by Lemma [3.2]. By the Riemann-Hurwitz formula, $2g_{Z_a} - 2 = \ell^a(2g_{Y_2} - 2) = \ell^a(p - 3)$, i.e., $g_{Z_a} = 1 + \ell^a(p - 3)/2$. This completes part (1).
For part (2), suppose $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over $\infty$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \times \mathbb{Z}/p\mathbb{Z}$. If $g$ is the genus of $Z$, we will show that $g \geq g_Z$. As described in the proof of Lemma 4.1, the cover $\psi$ is a composition $\psi = \phi \circ \omega$ where $\phi : Y \to \mathbb{P}^1_k$ has Galois group $\mathbb{Z}/p\mathbb{Z}$ and is ramified only over $\infty$ and where $\omega$ is unramified with group $(\mathbb{Z}/\ell\mathbb{Z})^b$. By the Riemann-Hurwitz formula, $2g - 2 = \ell^b(2g_Y - 2)$.

By Artin-Schreier theory, $\phi$ is given by an equation $y^p - y = f(x)$ where $f \in k[x]$ has degree $s$ for some integer $s$ relatively prime to $p$. Since the genus $g_Y$ of $Y$ is $(p-1)(s-1)/2$ by Lemma 2.2(3), we should make $s$ as small as possible. The value $s = 1$ is impossible since then $Y$ is a projective line and there do not exist Galois covers of the projective line ramified only over one point with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Thus $s = 2$ yields the smallest possible value for $g_Y$, namely $(p-1)/2$. Recall that $b \geq a$ by Lemma 2.4. Thus $g \geq 1 + \ell^a(p-3)/2 = g_Z$.

For part (3), suppose $Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over $\infty$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \times \mathbb{Z}/p\mathbb{Z}$ and the genus of $Z$ satisfies $g_Z = 1 + \ell^a(p-3)/2$. As in part (2), $\psi$ factors as $\phi \circ \omega$ where $\omega : Z \to Y$ is an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover, where $\phi : Y \to \mathbb{P}^1_k$ is an Artin-Schreier cover ramified only over $\infty$, and where $Y$ has genus $(p-1)/2$. By Lemma 2.2(3), $Y$ has an affine equation $y^p - y = a_2x^2 + a_1x + a_0$ for some $a_0, a_1, a_2 \in k$. Since $p$ is odd and $k$ is algebraically closed, it is possible to complete the square and write $a_2x^2 + a_1x + a_0 = x_1^2 + \epsilon$ where $x_1 = \sqrt{a_2}x + a_1/2\sqrt{a_2}$. After modifying by an automorphism of the projective line, specifically by the affine linear transformation $x \mapsto x_1$, the equation for $Y$ can be rewritten as $y^p - y = x_1^2 + \epsilon$. Since $\ell$ is algebraically closed, there exists $\delta \in k$ such that $\delta^p - \delta = \epsilon$. Let $y_1 = y - \delta$. After the change of variables $y \mapsto y_1$, the curve $Y$ is isomorphic to the curve $Y_2$ with affine equation $y_1^p - y_1 = x_1^2$. Thus there is a unique possibility for the isomorphism class of the curve $Y$. From the proof of Proposition 3.4 there is a bijection between $\tau$-invariant connected unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois covers of $Y_2$ and orbits of $\tau$ on points $D$ of order $\ell$ on $\text{Jac}(Y_2)$. The action of $\tau$ on $\text{Jac}(Y_2)[\ell]$ decomposes into $(p-1)/a$ irreducible subrepresentations. Each of these is distinct, because the irreducible factors of $\Phi_p(t) \in (\mathbb{Z}/\ell\mathbb{Z})[t]$ are distinct. Thus there are $(p-1)/a$ choices for a $\tau$-invariant unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover of $Y_2$. Thus there are at most $(p-1)/a$ isomorphism classes of curves $Z$ which admit a Galois covering map as in part (1) with minimal genus $g_Z$.

We note that the set of curves which are unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois covers of $Y_2$ may contain fewer than $(p-1)/a$ isomorphism classes of curves.

**Theorem 4.3** Let $p = 2$ and let $\ell$ be an odd prime. Then:

1. There exists a Galois cover $\psi : Z \to \mathbb{P}^1_k$ ramified only over $\infty$ with Galois group of the form $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2. The minimal genus of a curve $Z$ which admits a covering map as in part (1) is $g_Z = 1$.
3. There are at most $\ell + 1$ isomorphism classes of curves $Z$ which admit a Galois covering map as in part (1) with minimal genus $g_Z = 1$.

**Proof** Note that the order of $\ell$ modulo 2 is $a = 1$. For part (1), Lemma 2.5 shows that there exists a semi-direct product of the form $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ which is quasi-2. The result is then immediate from Theorem 2.3.

Suppose $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over $\infty$ with Galois group as in part (1). As before, $\psi$ factors as a composition $\phi \circ \omega$. where $\omega : Z \to Y$ has Galois group $\mathbb{Z}/\ell\mathbb{Z}$ and $\phi : Y \to \mathbb{P}^1_k$ is an Artin-Schreier extension with affine equation $y^2 = f(x)$ for some $f(x) \in k[x]$ of odd degree $s$. By Lemma 4.1 $\omega$ is unramified.
The minimal genus for $Z$ will thus occur when $s$ is as small as possible. As before, $s = 1$ is impossible, and so $s = 3$ is the smallest choice. In this case, by Lemma 3.2(3), $g_Y = 1$, i.e., $Y$ is an elliptic curve. By the Riemann-Hurwitz formula, the minimal genus for $Z$ is $g_Z = 1 + \ell(g_Y - 1) = 1$, which completes part (2).

For part (3), since $k$ is algebraically closed, we can complete the cube of $f(x)$ and make the corresponding change of variables, which is a scaling and translation of $x$. So we can assume that $Y$ has affine equation $y^2 - y = x^3 + a_1 x + a_0$ for some $a_0, a_1 \in k$. Then it follows from [11, Appendix A, Prop. 1.1c] that the $j$-invariant of $Y$ is $j(Y) = 0$ and that the discriminant is $\Delta(Y) = (-1)^3 = 1$. Since $k$ is algebraically closed, by [11, Appendix A, Prop. 1.2b], all elliptic curves $Y$ with $j(Y) = 0$ are isomorphic over $k$. Thus there is a unique choice for $Y$ up to isomorphism. Without loss of generality, we may assume that $Y = Y_3$ has affine equation $y^2 - y = x^3$.

From the proof of Proposition 3.4, the action of $\tau$ on $\text{Jac}(Y_3)[\ell]$ decomposes into the direct sum of two 1-dimensional subrepresentations. In other words, the action of $\tau$ is diagonal with both eigenvalues equal to $-1$. The number of non-trivial $\tau$-invariant subgroups of $\text{Jac}(Y_3)[\ell]$ is the number of subgroups of order $\ell$ in $(\mathbb{Z}/\ell \mathbb{Z})^2$, which is $\ell + 1$. As in Theorem 4.2, this implies that there are at most $\ell + 1$ isomorphism classes of curves $Z$ which admit a Galois covering map as in part (1) with minimal genus $g_Z = 1$.

We note that the set of curves which are unramified $\mathbb{Z}/\ell \mathbb{Z}$-Galois covers of $Y_3$ may contain fewer than $\ell + 1$ isomorphism classes of curves.

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