THE ANALYTIC CLASS NUMBER FORMULA FOR ORDERS IN PRODUCTS OF NUMBER FIELDS

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Abstract. We derive an analytic class number formula for an arbitrary order in a product of number fields.

1. Introduction

Let \( \mathcal{O} \) be an order in a product of \( m \) number fields for some nonnegative integer \( m \). The 1-dimensional scheme \( \text{Spec} \mathcal{O} \) has \( m \) irreducible components; in particular, it is irreducible if and only \( m = 1 \). The scheme \( \text{Spec} \mathcal{O} \) is regular if and only if \( \mathcal{O} \) is the product of the full rings of integers of the \( m \) number fields. If \( \text{Spec} \mathcal{O} \) fails to be regular at some point, we call that point a singularity and say that \( \text{Spec} \mathcal{O} \) is singular. Let \( \zeta_{\mathcal{O}}(s) \) be the zeta function of \( \mathcal{O} \) (see Section 3).

In the case where \( \mathcal{O} \) is the ring of integers of a number field, that is, the case in which \( \text{Spec} \mathcal{O} \) is regular and irreducible, Dedekind [Dir94, Supplement XI, §184, IV], generalizing work of Dirichlet, proved an analytic class number formula for the leading term of the Laurent series of \( \zeta_{\mathcal{O}}(s) \) at \( s = 1 \) (see also Hilbert’s Zahlbericht [Hil97, Theorem 56]). The generalization to the regular and reducible case is immediate. In this paper we generalize further by proving an analytic class number formula for an arbitrary order in a product of number fields, thereby extending Dedekind’s result to orders \( \mathcal{O} \) for which \( \text{Spec} \mathcal{O} \) is singular. We conclude by verifying our formula in an example with a singularity: the fiber product of rings \( \mathbb{Z} \times_{\mathbb{F}_p} \mathbb{Z} \).

2. Orders in products of number fields

If \( F \) is a number field with ring of integers \( \mathcal{O}_F \), classical algebraic number theory defines the following invariants: the number of real embeddings \( r_1(F) \), the number of pairs of complex embeddings \( r_2(F) \), the discriminant \( \text{Disc} \mathcal{O}_F \), the regulator \( R(\mathcal{O}_F) \), the class number \( h(\mathcal{O}_F) \) defined as the order of the Picard group \( \text{Pic} \mathcal{O}_F \), and the number \( w(\mathcal{O}_F) \) of roots of unity in \( \mathcal{O}_F \). All of these invariants occur in the analytic class number formula. In this section we extend each of these definitions to the case of an arbitrary order in a product of number fields.

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2.1. Orders. Let $K$ be a finite étale $\mathbb{Q}$-algebra; in other words, $K = K_1 \times \cdots \times K_m$ for some number fields $K_i$. Let $\mathcal{O} \subseteq K$ be an order, i.e., a subring of $K$ finitely generated as a $\mathbb{Z}$-module such that $\mathbb{Q}\mathcal{O} = K$.

Equivalently, let $\mathcal{O}$ be a reduced ring that is free of finite rank as a $\mathbb{Z}$-module, let $K := \mathcal{O} \otimes \mathbb{Q}$, and let $m := \# \text{Spec } K$.

2.2. The invariants $n$, $r_1$, $r_2$, and $r$ of $K$. Let $n := [K : \mathbb{Q}]$, so $\mathcal{O}$ has rank $n$ over $\mathbb{Z}$. Define $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ by $K \otimes \mathbb{R} \simeq \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}$. Thus $r_1$ is the number of ring homomorphisms $K \rightarrow \mathbb{R}$, and $2r_2$ is the number of ring homomorphisms $K \rightarrow \mathbb{C}$ whose image is not contained in $\mathbb{R}$. Let $r = r(K) := r_1 + r_2 - m$.

2.3. Roots of unity, the Picard group $\text{Pic } \mathcal{O}$, the class number $h(\mathcal{O})$, and the discriminant $\text{Disc } \mathcal{O}$. Let $\mu(\mathcal{O})$ be the torsion subgroup of $\mathcal{O}^\times$, so $\mu(\mathcal{O})$ is the group of roots of unity in $\mathcal{O}^\times$. Let $w(\mathcal{O}) := \# \mu(\mathcal{O})$.

Let $X := \text{Spec } \mathcal{O}$. Then $\text{Pic } \mathcal{O} := \text{Pic } X = H^1(X, \mathcal{O}_X^\times)$ [Har77, Exercise III.4.5]. Let $h(\mathcal{O}) := \# \text{Pic } \mathcal{O}$.

Let $\text{Tr}_{K/\mathbb{Q}} : K \rightarrow \mathbb{Q}$ be the trace map. Let $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis of $\mathcal{O}$. As usual, the discriminant is defined by

$$\text{Disc } \mathcal{O} := \det (\text{Tr}_{K/\mathbb{Q}}(e_i e_j))_{1 \leq i, j \leq n} \in \mathbb{Z}.$$ 

2.4. The normalization. Let $\bar{\mathcal{O}}_i$ be the ring of integers in $K_i$. Let $\bar{\mathcal{O}}$ be the normalization of $\mathcal{O}$. Since $\mathcal{O}$ is a finite $\mathbb{Z}$-module, the normalization of $\mathcal{O}$ equals the normalization of $\mathbb{Z}$ in $K_i$; thus $\bar{\mathcal{O}} = \prod_{i=1}^m \bar{\mathcal{O}}_i$ in $K$. Also, $\bar{\mathcal{O}}$ is finite as an $\mathcal{O}$-module, so $\bar{\mathcal{O}}$ is another order in $K$. Thus $\#(\bar{\mathcal{O}}/\mathcal{O}) < \infty$. The invariants $n$, $r_1$, $r_2$, and $r$ depend only on $K$, so they are the same for $\mathcal{O}$ as for $\bar{\mathcal{O}}$.

Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}$. Localizing the $\mathcal{O}$-module $\bar{\mathcal{O}}$ at $\mathfrak{p}$ yields a semilocal ring $\bar{\mathcal{O}}_\mathfrak{p}$. The quotient $\bar{\mathcal{O}}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \simeq (\bar{\mathcal{O}}/\mathcal{O})_\mathfrak{p}$ is finite, and is trivial for all $\mathfrak{p}$ except the finitely many corresponding to singularities of $\text{Spec } \mathcal{O}$. Each maximal ideal $\mathfrak{P}$ of $\bar{\mathcal{O}}_\mathfrak{p}$ lies above the maximal ideal $\mathfrak{p} \bar{\mathcal{O}}_\mathfrak{p}$ of $\bar{\mathcal{O}}_\mathfrak{p}$. Therefore, given $a \in \bar{\mathcal{O}}_\mathfrak{p}$, saying that $a$ lies outside every $\mathfrak{P}$ is the same as saying that $a$ lies outside $\mathfrak{p} \bar{\mathcal{O}}_\mathfrak{p}$; since $\bar{\mathcal{O}}_\mathfrak{p}$ is semilocal and $\mathcal{O}_\mathfrak{p}$ is local, this means that $a \in \bar{\mathcal{O}}_\mathfrak{p}$ if and only if $a \in \mathcal{O}_\mathfrak{p}^\times$. In other words, the map of sets $\bar{\mathcal{O}}_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times \rightarrow \bar{\mathcal{O}}_\mathfrak{p} / \mathcal{O}_\mathfrak{p}$ is injective. Hence $\bar{\mathcal{O}}_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times$ is finite too, and trivial for all but finitely many $\mathfrak{p}$. Injectivity of

$$\bar{\mathcal{O}} / \mathcal{O} \rightarrow \bigoplus_{\mathfrak{p}} \bar{\mathcal{O}}_\mathfrak{p} / \mathcal{O}_\mathfrak{p}$$

implies that if $a, a^{-1} \in \bar{\mathcal{O}}$ are such that their images in $\bar{\mathcal{O}}_\mathfrak{p}$ land in $\mathcal{O}_\mathfrak{p}$ for every $\mathfrak{p}$, then $a, a^{-1} \in \mathcal{O}$. Thus

$$\bar{\mathcal{O}}^\times / \mathcal{O}^\times \rightarrow \bigoplus_{\mathfrak{p}} \bar{\mathcal{O}}_\mathfrak{p}^\times / \mathcal{O}_\mathfrak{p}^\times$$

is injective. Hence $\bar{\mathcal{O}}^\times / \mathcal{O}^\times$ is finite.

**Proposition 1** (Dirichlet unit theorem for orders). The unit group $\mathcal{O}^\times$ is a finitely generated abelian group of rank $r$. 


Proof. If $O$ is the ring of integers in a number field, this is the Dirichlet unit theorem. In general, $\mathcal{O}$ is a product of such rings of integers, so the result holds for $\mathcal{O}$. Since $O^\times$ is of finite index in $\mathcal{O}^\times$, the result holds for $O$ too. \qed

2.5. The logarithmic embedding and the regulator $R(O)$. For $x \in \mathbb{R}^\times$, let $\lambda_{\mathbb{R}}(x) = \ln |x|$. For $x \in \mathbb{C}^\times$, let $\lambda_{\mathbb{C}}(x) = 2 \ln |x|$. Let 

$$(K \otimes \mathbb{R})^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \xrightarrow{\lambda} \mathbb{R}^{r_1 + r_2}$$

be the homomorphism that applies $\lambda_{\mathbb{R}}$ or $\lambda_{\mathbb{C}}$ coordinate-wise, as appropriate. Let $\phi$ be the composition

$$O^\times \longrightarrow K^\times \longrightarrow (K \otimes \mathbb{R})^\times \xrightarrow{\lambda} \mathbb{R}^{r_1 + r_2}.$$ 

Since $\ker \lambda$ is bounded in $K \otimes \mathbb{R}$, $\ker \phi$ is finite; on the other hand, the codomain of $\phi$ is torsion-free; thus $\ker \phi = \mu(O)$.

Suppose that $K$ is a field. The proof of the classical Dirichlet unit theorem shows that the image $\phi(\mathcal{O}^\times)$ is a full lattice in the hyperplane in $\mathbb{R}^{r_1 + r_2}$ where the coordinates sum to 0. Under the projection to $\mathbb{R}^r = \mathbb{R}^{r_1 + r_2 - 1}$ defined by forgetting one coordinate, the hyperplane maps isomorphically to $\mathbb{R}^r$, and $\phi(\mathcal{O}^\times)$ maps to a full lattice in $\mathbb{R}^r$; the covolume of this lattice is called the regulator, $R(\mathcal{O})$.

In the general case, $\phi(\mathcal{O}^\times)$ is a direct product of lattices in $\prod_{i=1}^m \mathbb{R}^{r_i(K_i)} = \mathbb{R}^r$. As proved in Section 2.4, $O^\times$ is of finite index in $\mathcal{O}^\times$, so $\phi(O^\times)$ is again a full lattice $L(O)$ in $\mathbb{R}^r$; its covolume is denoted $R(O)$.

3. The zeta function

Retain the notation of the previous section. In what follows, $p$ ranges over prime ideals of $O$ with finite residue field. Since $O$ is finitely generated as a $\mathbb{Z}$-algebra, these prime ideals are the same as the maximal ideals of $O$, which correspond to the closed points of Spec $O$. Define $N_p := \#O/p$. Since Spec $O$ is of finite type over $\mathbb{Z}$, it has a zeta function defined as an Euler product, as in [Ser65, p. 83]:

$$\zeta_O(s) = \prod_p (1 - Np^{-s})^{-1}.$$ 

Work of Hecke implies that $\zeta_O(s)$ has a meromorphic continuation to the entire complex plane, and that $\zeta_O(s)$ has a pole at $s = 1$ of order $m$. The analytic class number formula proposed below gives the leading term of $\zeta_O(s)$ at $s = 1$.

**Theorem 2** (Analytic class number formula for orders). Let $O$ be an order in a product of number fields $K = K_1 \times \cdots \times K_m$. Then

$$\lim_{s \to 1} (s - 1)^m \zeta_O(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w(O)\sqrt{|\text{Disc}\mathcal{O}|}} h(O) R(O).$$

In the classical case when $O$ is the ring of integers of a number field, $\zeta_O$ is the Dedekind zeta function, and Theorem 2 was proved by Dedekind, as mentioned already in Section 1. Each factor in $\zeta_O$ is multiplicative if $O$ is a product of rings, so Theorem 2 holds for any product of rings of integers, and in particular for the normalization $\mathcal{O}$ of any $O$. To prove Theorem 2 for a general order $O$, we will relate the formulas for $O$ and $\mathcal{O}$. 3
4. Relating the invariants for $\mathcal{O}$ and $\tilde{\mathcal{O}}$

Let $X = \text{Spec} \mathcal{O}$ and $\tilde{X} = \text{Spec} \tilde{\mathcal{O}}$. The inclusion $\mathcal{O} \hookrightarrow \tilde{\mathcal{O}}$ induces a morphism $\pi : \tilde{X} \to X$ that is an isomorphism above the complement of a finite subset $Z \subseteq X$. For maximal ideals $p \subseteq \mathcal{O}$ and $\mathfrak{p} \subseteq \mathcal{O}$, we write $\mathfrak{p}|p$ when $p = \mathfrak{p} \cap \mathcal{O}$, i.e., when $\pi$ maps the closed point $\mathfrak{p}$ to the closed point $p$.

4.1. The zeta functions of $\mathcal{O}$ and $\tilde{\mathcal{O}}$.

Proposition 3. We have

$$\lim_{s \to 1} \frac{\zeta_{\tilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \prod_p \frac{\prod_{\mathfrak{p}|p} (1 - N\mathfrak{p}^{-1})^{-1}}{(1 - Np^{-1})^{-1}}.$$  

Proof. By definition,

$$\frac{\zeta_{\tilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \prod_p \frac{\prod_{\mathfrak{p}|p} (1 - N\mathfrak{p}^{-s})^{-1}}{(1 - Np^{-s})^{-1}},$$

where, for all but finitely many $p$, the fraction on the right is $1$; cf. [Jen69, Theorem]. □

4.2. The discriminants of $\mathcal{O}$ and $\tilde{\mathcal{O}}$.

Proposition 4. We have

$$\frac{\text{Disc} \tilde{\mathcal{O}}}{\text{Disc} \mathcal{O}} = \left( \frac{\# \mathcal{O}}{\# \mathcal{O}} \right)^{-2}.$$  

Proof. This is standard: Let $A \in \text{M}_2(\mathbb{Z})$ be the change-of-basis matrix expressing the $\mathbb{Z}$-basis of $\mathcal{O}$ in terms of the $\mathbb{Z}$-basis of $\tilde{\mathcal{O}}$. Then $\#(\mathcal{O}/\mathcal{O}) = \det A$. On the other hand, the matrix whose determinant is $\text{Disc} \mathcal{O}$ is obtained from the matrix whose determinant is $\text{Disc} \tilde{\mathcal{O}}$ by multiplying by $A$ on the right and $A^T$ on the left, so $\text{Disc} \mathcal{O} = (\det A)^2 \text{Disc} \tilde{\mathcal{O}}$. □

4.3. The regulators of $\mathcal{O}$ and $\tilde{\mathcal{O}}$.

Proposition 5.

$$\frac{R(\tilde{\mathcal{O}})}{R(\mathcal{O})} \cdot \frac{\mathcal{O}^\times}{\tilde{\mathcal{O}}^\times} = \frac{w(\mathcal{O})}{w(\tilde{\mathcal{O}})}.$$  

Proof. Let $L = L(\mathcal{O})$ be as in Section 2.3 and let $\tilde{L} = L(\tilde{\mathcal{O}})$; these are lattices in the same $\mathbb{R}^r$. Applying the snake lemma to

$$1 \longrightarrow \mu(\mathcal{O}) \longrightarrow \mathcal{O}^\times \longrightarrow L \longrightarrow 0$$

yields an exact sequence

$$1 \rightarrow \mu(\mathcal{O}) \rightarrow \mathcal{O}^\times \rightarrow \tilde{L} \rightarrow 0$$

of finite groups, the last of which has order $R(\mathcal{O})/R(\tilde{\mathcal{O}})$. □
4.4. Relating $\text{Pic }\mathcal{O}$ and $\text{Pic }\tilde{\mathcal{O}}$ via the Leray spectral sequence. View the abelian group $\tilde{\mathcal{O}}_p^\times / \mathcal{O}_p^\times$ as a skyscraper sheaf on $X$ supported at $p$; it is trivial for $p \notin \mathbb{Z}$. We have an exact sequence of sheaves on $X$

$$0 \to \mathcal{O}_X^\times \to \pi_* \mathcal{O}_X^\times \to \bigoplus_p \tilde{\mathcal{O}}_p^\times / \mathcal{O}_p^\times \to 0.$$  

The corresponding long exact sequence in cohomology is

$$0 \to \mathcal{O}^\times \to \tilde{\mathcal{O}}^\times \to \bigoplus_p \tilde{\mathcal{O}}_p^\times / \mathcal{O}_p^\times \to \text{Pic }X \to H^1 \left( X, \pi_* \mathcal{O}_X^\times \right) \to 0.$$  

Lemma 6. We have $H^1(X, \pi_* \mathcal{O}_X^\times) \simeq \text{Pic }\tilde{X}$.

Proof. From the Leray spectral sequence

$$H^p \left( X, R^q \pi_* \mathcal{F} \right) \Rightarrow H^{p+q} \left( \tilde{X}, \mathcal{F} \right)$$

with $\mathcal{F} = \mathcal{O}_X^\times$ we extract an exact sequence

$$0 \to H^1 \left( X, \pi_* \mathcal{O}_X^\times \right) \to \text{Pic }\tilde{X} \to H^0 \left( X, R^1 \pi_* \mathcal{O}_X^\times \right).$$

Lemma 7 below completes the proof. □

Lemma 7. The sheaf $R^1 \pi_* \mathcal{O}_X^\times$ on $X$ is 0.

Proof. By [Har77, Proposition III.8.1], its stalk $(R^1 \pi_* \mathcal{O}_X^\times)_x$ at a closed point $x$ of $X$ is $\lim_{\to U} \text{Pic }\pi^{-1}U$, where $U$ ranges over open neighborhoods of $x$ in $X$. Since $\pi^{-1}(x)$ is finite, every line bundle on $\pi^{-1}U$ becomes trivial on $\pi^{-1}U'$ for some smaller neighborhood $U'$ of $x$ in $X$. Thus $\lim_{\to U} \text{Pic }\pi^{-1}U = 0$. □

Substituting the isomorphism of Lemma 6 into (2) yields an exact sequence of finite groups

$$0 \to \tilde{\mathcal{O}}^\times / \mathcal{O}^\times \to \bigoplus_p \tilde{\mathcal{O}}_p^\times / \mathcal{O}_p^\times \to \text{Pic }X \to \text{Pic }\tilde{X} \to 0.$$  

Remark 8. For a more elementary derivation of (3), at least in the case where $\mathcal{O}$ is an integral domain; see [Neu99, Proposition I.12.9].

Next we compute the order of the second term in (3). Fix a nonzero ideal $c$ of $\tilde{\mathcal{O}}$ such that $c \subseteq \mathcal{O}$; one possibility is $c = n\tilde{\mathcal{O}}$, where $n := (\tilde{\mathcal{O}} : \mathcal{O})$. (In fact, there is a largest $c$—the sum of all of them—called the conductor of $\mathcal{O}$.)

Lemma 9. The natural map

$$\frac{\tilde{\mathcal{O}}_p^\times}{\mathcal{O}_p^\times} \to \frac{\left( \tilde{\mathcal{O}}_p / \mathcal{O}_p \right)^\times}{\left( \mathcal{O}_p / \mathcal{O}_p \right)^\times}$$

is an isomorphism.
Proof. Case 1: \( \mathfrak{c}_p = \mathcal{O}_p \). Then \( 1 \in \mathfrak{c}_p \), so \( \mathfrak{c}_p = \tilde{\mathcal{O}}_p \) too; thus both sides are trivial.

Case 2: \( \mathfrak{c}_p \neq \mathcal{O}_p \). Then \( \mathfrak{c}_p \subseteq p\mathcal{O}_p \subseteq \mathfrak{p} \) for every maximal ideal \( \mathfrak{p} \) of \( \tilde{\mathcal{O}}_p \). If an element \( \tilde{a} \in (\tilde{\mathcal{O}}_p/\mathfrak{c}_p)^\times \) is lifted to an element \( a \in \tilde{\mathcal{O}}_p \), then \( a \) lies outside each \( \mathfrak{p} \), so \( a \in \tilde{\mathcal{O}}_p^\times \). Thus \( \tilde{\mathcal{O}}_p^\times \rightarrow (\tilde{\mathcal{O}}_p/\mathfrak{c}_p)^\times \) is surjective. Similarly, \( \mathcal{O}_p^\times \rightarrow (\mathcal{O}_p/\mathfrak{c}_p)^\times \) is surjective. Both surjections have the same kernel \( 1 + \mathfrak{c}_p \), so the result follows. \( \square \)

Lemma 10. If \( \mathfrak{c}_p \neq \mathcal{O}_p \), then

\[
\# \left( \tilde{\mathcal{O}}_p/\mathfrak{c}_p \right)^\times = \# \left( \tilde{\mathcal{O}}_p/\mathfrak{c}_p \right) \prod_{\mathfrak{p} \mid \mathfrak{p}} \left( 1 - N\mathfrak{p}^{-1} \right),
\]

\[
\# (\mathcal{O}_p/\mathfrak{c}_p)^\times = \# (\mathcal{O}_p/\mathfrak{c}_p) (1 - Np^{-1}).
\]

Proof. The maximal ideals of \( \tilde{\mathcal{O}}_p/\mathfrak{c}_p \) are the ideals \( \mathfrak{p}\tilde{\mathcal{O}}_p \) for \( \mathfrak{p} \mid \mathfrak{p} \). An element of \( \tilde{\mathcal{O}}_p/\mathfrak{c}_p \) is a unit if and only if it lies outside each maximal ideal. The probability that a random element of the finite group \( \tilde{\mathcal{O}}_p/\mathfrak{c}_p \) lies outside \( \mathfrak{p}\tilde{\mathcal{O}}_p \) is \( 1 - N\mathfrak{p}^{-1} \), and these events for different \( \mathfrak{p} \) are independent by the Chinese remainder theorem, so the first equation follows. The second equation is similar (but easier). \( \square \)

Lemma 11. We have

\[
\prod_p \frac{\#(\tilde{\mathcal{O}}_p/\mathfrak{c}_p)^\times}{\#(\mathcal{O}_p/\mathfrak{c}_p)^\times} = \frac{\#(\tilde{\mathcal{O}}/\mathfrak{c})}{\#(\mathcal{O}/\mathfrak{c})} \prod_p \frac{\prod_{\mathfrak{p} \mid \mathfrak{p}} (1 - N\mathfrak{p}^{-1})}{1 - Np^{-1}}.
\]

Proof. By Lemmas 9 and 10

\[
\#(\tilde{\mathcal{O}}_p/\mathfrak{c}_p)^\times = \#(\tilde{\mathcal{O}}_p/\mathfrak{c}_p) \prod_{\mathfrak{p} \mid \mathfrak{p}} (1 - N\mathfrak{p}^{-1});
\]

this holds even if \( \mathfrak{c}_p = \mathcal{O}_p \) since both sides are 1 in that case. Now take the product of both sides and use the isomorphism of finite groups

\[
\tilde{\mathcal{O}} \simeq \prod_p \tilde{\mathcal{O}}_p.
\]

\( \square \)

Proposition 12.

\[
\#(\tilde{\mathcal{O}}/\mathfrak{c}) = \frac{h(\tilde{\mathcal{O}})}{h(\mathcal{O})} \cdot \#(\tilde{\mathcal{O}}/\mathfrak{c}) \prod_p \frac{\prod_{\mathfrak{p} \mid \mathfrak{p}} (1 - N\mathfrak{p}^{-1})}{1 - Np^{-1}}.
\]

Proof. Take the alternating product of the orders of the groups in (3) and use Lemma 11. \( \square \)

4.5. Conclusion of the proof. To complete the proof of Theorem 2 we compare (1) for \( \tilde{\mathcal{O}} \) to (1) for \( \mathcal{O} \). The ratio of the left side of (1) for \( \tilde{\mathcal{O}} \) to the left side of (1) for \( \mathcal{O} \) is

\[
\lim_{s \to 1} \frac{\zeta(\tilde{\mathcal{O}})(s)}{\zeta(\mathcal{O})(s)}.
\]

The ratio of the right sides is

\[
\left| \frac{\text{Disc } \tilde{\mathcal{O}}}{\text{Disc } \mathcal{O}} \right|^{-1/2} \left( \frac{w(\tilde{\mathcal{O}})}{w(\mathcal{O})} \right)^{-1} \frac{h(\tilde{\mathcal{O}})}{h(\mathcal{O})} \cdot \frac{R(\tilde{\mathcal{O}})}{R(\mathcal{O})}.
\]
By Propositions 3, 4, 5, and 12, both ratios equal
\[
\prod_p \frac{\prod_{\mathfrak{p}|p} (1 - N_{\mathfrak{p}}^{-1})^{-1}}{(1 - N_p^{-1})^{-1}}.
\]

5. AN EXAMPLE WITH Spec O SINGULAR: A FIBER PRODUCT

Consider the ring
\[O := \mathbb{Z} \times_{\mathbb{F}_p} \mathbb{Z} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{p}\}.
\]
The normalization \(\widetilde{O}\) of \(O\) is the ring \(\mathbb{Z} \times \mathbb{Z}\); inverting all non-zerodivisors of \(O\) gives its ring of fractions \(K = \mathbb{Q} \times \mathbb{Q}\). The scheme \(X := \text{Spec} \ O\) consists of two copies of the “curve” \(\text{Spec} \ Z\) crossing at the point \((p) \in \text{Spec} \ Z\). The scheme \(\tilde{X} := \text{Spec} \ \widetilde{O}\) is a disjoint union of two copies of \(\text{Spec} \ Z\). The conductor of \(O\) is the \(\widetilde{O}\)-ideal
\[p := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \equiv 0 \pmod{p}\}.
\]
Above the prime \(p\) of \(O\) there are two primes of \(\widetilde{O}\), the two copies of \((p)\).

Proposition 13. We have
\[
\lim_{s \to 1} (s - 1)^2 \zeta_O(s) = 1 - p^{-1}.
\]

Proof. The Riemann zeta function \(\zeta_Z(s)\) has a pole of order 1 at \(s = 1\) with residue 1. Since the one ideal \(p\) of norm \(p\) in \(O\) is replaced by two ideals of norm \(p\) in \(\widetilde{O}\),
\[
\zeta_O(s) = (1 - p^{-s}) \zeta_{\widetilde{O}}(s)
\]
\[
\lim_{s \to 1} (s - 1)^2 \zeta_O(s) = (1 - p^{-1}) \lim_{s \to 1} (s - 1)^2 \zeta_{\widetilde{O}}(s)
\]
\[
= (1 - p^{-1}) \left(\lim_{s \to 1} (s - 1)\zeta_Z(s)\right)^2
\]
\[
= 1 - p^{-1}. \quad \Box
\]

Proposition 14. We have
\[
\frac{2^{r_1}(2\pi)^{r_2}}{w(O)\sqrt{|\text{Disc} \ O|}} h(O) R(O) = 1 - p^{-1}.
\]

Proof. First, \(r_1 = 2, r_2 = 0\), and \(r = 2 + 0 - 2 = 0\), so \(R(O) = 1\).

The trace map on \(\widetilde{O}\) or \(O\) sends \((a, b)\) to \(a + b\). The elements \((1, 1)\) and \((p, 0)\) form a basis of \(O\), so
\[
\text{Disc} \ O = \det \begin{pmatrix} 2 & p \\ p & p^2 \end{pmatrix} = p^2.
\]
Inside \(\widetilde{O}^\times = Z^\times \times Z^\times = \pm 1 \times \pm 1\) we have
\[
O^\times = \begin{cases} \pm(1, 1) & \text{if } p \text{ is odd}, \\
\pm 1 \times \pm 1 & \text{if } p = 2,
\end{cases}
\]
so
\[
w(O) = \begin{cases} 2 & \text{if } p \text{ is odd}, \\
4 & \text{if } p = 2.
\end{cases}
\]
By Lemma 9, the exact sequence (3) is

\[
1 \to \frac{\tilde{O}^\times}{O^\times} \to \frac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times \times \mathbb{F}_p^\times} \to \text{Pic } O \to \text{Pic } \tilde{O}.
\]

The image of \( \frac{\tilde{O}^\times}{O^\times} \) in \( \frac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times \times \mathbb{F}_p^\times} \simeq \mathbb{F}_p^\times \) is \( \pm 1 \), even when \( p = 2 \) in which case these groups are trivial. On the other hand, \( \text{Pic } \tilde{O} = \text{Pic } \mathbb{Z} \times \text{Pic } \mathbb{Z} = 0 \). Thus (4) yields \( \text{Pic } O \simeq \mathbb{F}_p^\times / \pm 1 \), and

\[
h(O) = \begin{cases} (p - 1)/2 & \text{if } p \text{ is odd,} \\ 1 & \text{if } p = 2. \end{cases}
\]

Combining the above calculations yields

\[
\frac{2^{r_1} (2\pi)^{r_2}}{w(O) \sqrt{|\text{Disc } O|}} h(O) R(O) = \begin{cases} \frac{2^2}{2\sqrt{p^2}} \cdot \frac{p - 1}{2} \cdot 1 & \text{if } p \text{ is odd} \\ \frac{2^2}{4\sqrt{2^2}} \cdot 1 \cdot 1 & \text{if } p = 2 \end{cases} = 1 - p^{-1}. \quad \square
\]

Propositions 13 and 14 verify Theorem 2 for \( O \).

**Remark 15.** Fiber products such as \( O \) arise as integral Hecke algebras of elliptic modular forms. For example, the integral Hecke algebra \( T^\ast \) (cf. [Maz77, p. 37]) for modular forms of weight 2 for the congruence subgroup \( \Gamma_0(11) \) is \( \mathbb{Z} \times \mathbb{F}_5 \mathbb{Z} \).

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