SECTIONAL-HYPERBOLIC LYAPUNOV STABLE SETS

Serafin Bautista* and Yeison Sánchez

Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá, Colombia

(Communicated by Shaobo Gan)

Abstract. In hyperbolic dynamics, a well-known result is: every hyperbolic Lyapunov stable set, is attracting; it’s natural to wonder if this result is maintained in the sectional-hyperbolic dynamics. This question is still open, although some partial results have been presented. We will prove that all sectional-hyperbolic transitive Lyapunov stable set of codimension one of a vector field $X$ over a compact manifold, with unique singularity Lorenz-like, which is of boundary-type, is an attractor of $X$.

1. Introduction. In the theory of hyperbolic sets we have properties established for both flows and diffeomorphisms. Classical examples of these sets are Smale’s horseshoe and the suspension of diffeomorphisms that exhibit hyperbolic sets. Later a more general class of sets appeared and they were called sectional-hyperbolic sets containing the hyperbolic sets and non-hyperbolic sets like the geometric Lorenz attractor [7]. Then a natural problem is to ask what results of the hyperbolic theory are still valid in the sectional-hyperbolic theory. The isolated non-trivial hyperbolic sets have periodic orbits by the shadowing lemma [10], but not every sectional-hyperbolic set has periodic orbits as the cherry flow in the torus with a strong contraction (See [4] page 27). It was recently proved that the sectional-hyperbolic Lyapunov stable sets contain a non-trivial homoclinic class [3].

On the other hand, we know that every hyperbolic Lyapunov stable set is an attracting. The proof of this is based on the fact that the points in a hyperbolic set have an unstable manifold. In the sectional-hyperbolic set we do not have that unstable manifold guaranteed, except in its singularities and periodic orbits. In this paper we prove that every sectional-hyperbolic Lyapunov stable set of codimension one with a unique singularity Lorenz-like of boundary-type, is attracting. For this, we need to use the particular case of the connecting lemma for sectional-hyperbolic sets in codimension one presented in [8]. Below we will specify the definitions and results over sectional-hyperbolic dynamics we will work with.

Hereafter $M$ will be a compact manifold possibly with nonempty boundary endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ an induced norm $\| \cdot \|$. Given $X$ a $C^1$ vector field $F$...
field, inwardly transverse to the boundary (if nonempty) we call \( X_t \) its induced flow over \( M \). Define the maximal invariant set of \( X \) as:

\[
M(X) = \bigcap_{t \geq 0} X_t(M).
\]

The orbit of a point \( p \in M(X) \) is defined by \( \mathcal{O}(p) = \{X_t(p) : t \in \mathbb{R} \} \). A singularity will be a zero \( q \) of \( X \), i.e. \( X(q) = 0 \) (or equivalently \( \mathcal{O}(q) = \{q\} \)) and a periodic orbit is an orbit \( \mathcal{O}(p) \) such that \( X_T(p) = p \) for some minimal \( T > 0 \) and \( \mathcal{O}(p) \neq \{p\} \).

By a closed orbit we mean a singularity or a periodic orbit.

Given \( p \in M \) we define the omega-limit set, \( \omega_X(p) = \{x \in M : x = \lim_{n \to \infty} X_{t_n}(p), \text{ for some sequence } t_n \to \infty \} \), if \( p \in M(X) \), define the alpha-limit set \( \alpha_X(p) = \{x \in M : x = \lim_{n \to -\infty} X_{t_n}(p), \text{ for some sequence } t_n \to -\infty \} \).

A compact subset \( \Lambda \) of \( M \) is called invariant if \( X_t(\Lambda) = \Lambda \) for all \( t \in \mathbb{R} \); transitive if \( \Lambda = \omega_X(p) \) for some \( p \in \Lambda \). A compact invariant set \( \Lambda \) is attracting if there is a neighborhood \( U \) such that

\[
\Lambda = \bigcap_{t \geq 0} X_t(U),
\]

and is attractor of \( X \), if it is an attracting set \( \Lambda \) which is transitive. On the other hand, a compact invariant set \( \Lambda \) is Lyapunov stable, if for all neighborhood \( U \) of \( \Lambda \), there exists a neighborhood \( W \) such that: \( X_t(p) \in U \) for every \( t \geq 0 \) and \( p \in W \).

**Definition 1.1.** A compact invariant set \( \Lambda \subseteq M(X) \) is hyperbolic if there are positive constants \( K, \lambda \) and a continuous \( DX_t \)-invariant splitting of the tangent bundle \( T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda \), such that for every \( x \in \Lambda \) and \( t \geq 0 \):

1. \( \|DX_t(x)v_x^s\| \leq Ke^{-\lambda t}\|v_x^s\|, \quad \forall v_x^s \in E^s_x; \)
2. \( \|DX_t(x)v_x^u\| \geq K^{-1}e^{\lambda t}\|v_x^u\|, \quad \forall v_x^u \in E^u_x; \)
3. \( E^s_x = \langle X(x) \rangle. \)

If \( E^s_x \neq 0 \) and \( E^u_x \neq 0 \) for all \( x \in \Lambda \) we will say that \( \Lambda \) is a saddle-type hyperbolic set. A closed orbit is hyperbolic, if as a compact invariant set of \( X \) is hyperbolic.

The invariant manifold theory [9] asserts that if \( H \subseteq M \) is hyperbolic set of \( X \) and \( p \in H \), then the topological sets:

\[
W^{ss}(p) = \{q \in M : \lim_{t \to \infty} d(X_t(q), X_t(p)) = 0 \}
\]

and

\[
W^{uu}(p) = \{q \in M(X) : \lim_{t \to -\infty} d(X_t(q), X_t(p)) = 0 \}
\]

are \( C^1 \) manifolds in \( M \), so called strong stable and unstable manifolds, tangent at \( p \) to the subbundles \( E^s_p \) and \( E^u_p \) respectively. Saturating them with the flow we obtain the stable and unstable manifolds \( W^s(p) \) and \( W^u(p) \) respectively, which are invariant. If \( p, p' \in H \), we have that \( W^s(p) \) and \( W^s(p') \) are the same or they are disjoint (similarly for \( W^u \)).

An homoclinic class \( H_X(p) \) associated to a hyperbolic periodic point \( p \) of \( X \) is the closure of the transverse intersections between \( W^u_X(p) \) and \( W^s_X(p) \), i.e.

\[
H_X(p) = CL(W^u_X(p) \cap W^s_X(p)).
\]

We say that \( H \subseteq M \) is a homoclinic class of \( X \) if \( H = H_X(p) \) for some hyperbolic periodic point \( p \) of \( X \).

**Definition 1.2.** A compact invariant set \( \Lambda \subseteq M(X) \) is sectional-hyperbolic if every singularity in \( \Lambda \) is hyperbolic (as invariant set) and there are a continuous \( DX_t \)-invariant splitting of the tangent bundle \( T_\Lambda M = F^s_\Lambda \oplus F^u_\Lambda \), and positive constants \( K, \lambda \) such that for every \( x \in \Lambda \) and \( t \geq 0 \):
1. $\|DX_t(x)w^s_x\| \leq Ke^{-\lambda t}\|w^s_x\|$, $\forall w^s_x \in F_x^s$.
2. $\|DX_t(x)v^c_x\| \leq Ke^{-\lambda t}\|DX_t(x)v^c_x\| \cdot \|v^c_x\|$, $\forall v^c_x \in F_x^c$, $\forall v^c_x \in F_x^c$.
3. $\|DX_t(x)u^c_x, DX_t(x)v^c_x\|_{X_t(x)} \geq K^{-1}e^{\lambda t}\|u^c_x, v^c_x\|_{X_t(x)}$, $\forall u^c_x, v^c_x \in F_x^c$.

Where $\|\cdot,\cdot\|_x$ it is induced 2-norm by the Riemannian metrics $\langle \cdot, \cdot \rangle_x$ of $T_x\Lambda$, given by

$$\|v_x, u_x\|_x = \sqrt{(v_x, v_x)_x \cdot (u_x, u_x)_x} - \langle v_x, u_x \rangle^2_x$$

for all $x \in \Lambda$ and every $u_x, v_x \in T_x\Lambda$.

The third condition guarantees the exponential increase of the area of parallelograms in the central subbundle $F^c$. Since $X(x) \in F^c_x$ for all $x \in \Lambda$ (see Lemma 4 in [4]), we will require that the dimension of the central subbundle must be greater than or equal to 2. In the particular case where the $\dim(F^c_x) = 2$ we will say that $\Lambda$ is a sectional-hyperbolic set of codimension 1.

Also the invariant manifold theory [9] asserts that through any point $x$ of a sectional-hyperbolic set $\Lambda$ it passes a strong stable manifold $F^s(x)$, tangent at $x$ to the subbundle $F^c_x$, which induces a foliation over $\Lambda$; saturating them with the flow we obtain the invariant manifold $F^s(x)$.

Unlike hyperbolic sets, the sectional-hyperbolic sets can have regular orbits which accumulate singularities. We have:

**Lemma 1.3.** If $\Lambda \subseteq M(X)$ is sectional-hyperbolic set, and $\sigma$ is a singularity in $\Lambda$ then:

$$F^s(\sigma) \cap \Lambda = \{\sigma\}$$

**Proof.** See corollary 2 in [4].

Every singularity $\sigma$ in an sectional-hyperbolic set, is hyperbolic, so it’s invariant manifolds $W^{uu}(\sigma)$ and $W^{ss}(\sigma)$ are well defined. The strong stable manifold $F^{ss}(\sigma)$ is a submanifold of $W^{ss}(\sigma)$, with respect to your dimension, exists two possibilities:

1. $\dim(W^{ss}(\sigma)) = \dim(F^{ss}(\sigma))$, in this case $W^{ss}(\sigma) = F^{ss}(\sigma)$.
2. $\dim(W^{ss}(\sigma)) = \dim(F^{ss}(\sigma)) + 1$, in this case, we say that the singularity is Lorenz-like.

Every singularity Lorenz-like is a type-saddle hyperbolic set with at least two negative eigenvalues, one of which is a real eigenvalue $\lambda$ with multiplicity one such that the real part of the other eigenvalues is outside the closed interval $[\lambda, -\lambda]$.

Over a Lorenz-like singularity $\sigma \in \Lambda$, we have $F^{ss}(\sigma)$ is tangent to the subspace associated the eigenvalues with real part less than $\lambda$, and divides $W^{ss}(\sigma)$ in two connected components. If $\Lambda$ intersect just one connected component of $W^{ss}(\sigma) \setminus F^{ss}(\sigma)$, we say that the singularity Lorenz-like is of boundary-type.

On the other hand, we have:

**Lemma 1.4.** Let $\Lambda \subseteq M(X)$ be a sectional-hyperbolic set and let be $\sigma$ a singularity which is not Lorenz-like in $\Lambda$. If there exists a sequence $x_n \in \Lambda$ of regular points, such that $x_n \to \sigma$, then $x_n \in W^{uu}(\sigma)$ for $n$ large enough.

**Proof.** Since $\sigma$ it’s not Lorenz-like, we have that $F^{ss}(\sigma) = W^{ss}(\sigma)$ and by Lemma 1.3, $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$, then $x_n \notin W^{ss}(\sigma)$, for all $n$. Suppose that there exists a subsequence $X_{n_k}$, such that $x_{n_k} \notin W^{uu}(\sigma)$, then there exists a regular point $y \in M$, such that:

$$y \in Cl\left(\bigcap_{k=1}^{\infty} \sigma^-(x_{n_k}) \right) \cap (W^{ss}(\sigma) \setminus \{\sigma\})$$
but as $x_{n_k} \in \Lambda$, by the compactness of $\Lambda$, $y \in \Lambda \cap W^{ss}(\sigma) \setminus \{\sigma\}$. This is a contradiction, thus $x_n \in W^{ss}(\sigma)$ for $n$ large enough. \hfill \square

Then, the only singularities that can be accumulated by positive orbits of regular points in a sectional-hyperbolic set, are Lorenz-like.

**Corollary 1.** If $\Lambda \subseteq M(X)$ is a sectional-hyperbolic set, $\Lambda$ it’s not a singularity and there exists $q \in M$ such that $\Lambda = \alpha_X(q)$ or $\Lambda = \omega_X(q)$, then every singularity in $\Lambda$ is Lorenz-like.

We say that a cross section $\Sigma$ of $X$ is associated to a Lorenz-like singularity $\sigma$ in a sectional-hyperbolic set $\Lambda$, if $\Sigma$ is very close to $\sigma$, $\Sigma \cap \Lambda \neq \emptyset$ and one of the connected components of $W^{ss}(\sigma) \setminus F^{ss}(\sigma)$ contains a point in $int(\Sigma)$.

Another important result about the sectional-hyperbolic set, is the hyperbolic lemma (see Lemma 9 in [4]), which asserts that any invariant subset $H$ without singularities of a sectional-hyperbolic set $\Lambda$, is hyperbolic. In this case, we have that $F^s_H = E^s_H$ and $F^u_H = E^u_H$, so $W^{ss}(p) = F^{ss}(p)$ for all $p \in H$.

Observe that the closed orbits of a sectional-hyperbolic set always have a hyperbolic structure, the periodic orbits by the hyperbolic lemma and the singularities by definition. The next definitions apply to sets whose closed orbits are hyperbolic.

**Definition 1.5.** A compact invariant set $\Lambda$ of a vector field $X$ over $M$, has the property $(P)$ if for all periodic orbit $O$ in $\Lambda$ exists a singularity $\sigma$ also in $\Lambda$ such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$.

**Definition 1.6.** A sectional-hyperbolic set $\Lambda \in M(X)$, has the property $(S)$ if for every Lorenz-like singularity $\sigma$ in $\Lambda$ there exists a periodic orbit $p$ also in $\Lambda$ such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$.

2. Lyapunov stable attracting sets. In this section we will establish some sufficient conditions for a sectional-hyperbolic Lyapunov stable set to be attracting.

**Lemma 2.1.** Let $\Lambda$ a Lyapunov stable set of a vector field $X$ over $M$ then:

1. $\{y \in M : d(X_{-t}(x), X_{-t}(y)) \to 0 \text{ when } t \to \infty\} \subseteq \Lambda, \forall x \in \Lambda$.

2. $\Lambda$ is a attracting of $X$ if and only if there is a neighborhood $U$ of $\Lambda$ such that $\omega_X(x) \subseteq \Lambda$ for all $x \in U$.

**Proof.** See Lemma 2.25 pag 35 and Lemma 2.26 pag 36 en [1]. \hfill \square

**Theorem 2.2.** Every sectional-hyperbolic Lyapunov stable set $\Lambda$ of a vector field $X$ over $M$, that satisfies the property $(S)$ and whose Lorenz-like singularities are of boundary type, it is an attracting set of $X$.

**Proof.** Denote by $T_\Lambda M = F^u_\Lambda \oplus F^s_\Lambda$ the sectional splitting of $\Lambda$, which we can extend to $T_U M = F^s_U \oplus F^u_U$ where $U$ is a neighborhood of $\Lambda$ in $M$; this extension is continuous for $F^s_U$ and integrable for $F^u_U$. In what follows we will hold the neighborhood $U$ of $\Lambda$.

To prove that $\Lambda$ is an attracting, we will use the item $(2)$ of the Lemma 1.3, so, it suffices to prove that: if $x_n \in M$ is a sequence converging to $p \in \Lambda$, then $\omega_X(x_n) \subseteq \Lambda$ for $n$ large. We have two possible cases:

**Case 1:** $Sing(X) \cap \omega_X(p) = \emptyset$. So, by the hyperbolic lemma, $\omega_X(p)$ is a saddle-type hyperbolic set. Choose $y \in \omega_X(p)$, then $W^{uu}(y)$ is well defined and by the item $(1)$ of Lemma 1.3 we have that $W^{uu}(y) \subseteq \Lambda$. Let $\Sigma \subseteq U$ a cross section of $X$ with $y \in int(\Sigma)$. Denote by $\mathcal{F}_\Sigma$ the vertical foliation of $\Sigma$. 


obtained by projecting $F^{ss}$ into $\Sigma$ along the flow of $X$, (i.e. $F^s(x, \Sigma)$ is the leaf in $\Sigma$ obtained by projecting the leaf $F^{ss}(x)$ in $\Sigma$ along the flow of $X$, for all $x \in \Sigma$). By choosing $\Sigma$ small in size, we have that $W^{uu}(y) \cap F^s(x, \Sigma) \neq \emptyset$ for all $x \in \Sigma$.

As $y \in \omega_X(p)$ we have that the positive orbit of $p$ intersects $\Sigma$, which implies that, for $n$ large, the positive orbit of $x_n$ also intersects $\Sigma$ in a point $x_n'$, since $x_n \to p$; then, there exists $z_n \in F^s(x_n', \Sigma) \cap W^{uu}(p) \subseteq \Lambda$, such that $x_n' \in F^{ss}(z_n)$. So, $\omega_X(z_n) = \omega_X(x_n) \subseteq \Lambda$.

**Case 2:** $\text{Sing}(X) \cap \omega_X(p) \neq \emptyset$. Let $\sigma \in \omega_X(p)$, then $\sigma$ is accumulated by the positive orbit of $p \in \Lambda$ and therefore is Lorenz-like singularity. By the property (S), there exists a hyperbolic periodic orbit $O \subseteq \Lambda$ such that $z \in W^s(O) \cap W^h(\sigma)$ for some point $z \in M$. As $\Lambda$ is a Lyapunov stable set, then $z \in W^s(O) \subseteq \Lambda$. Since $F^{ss}(\sigma) \cap \Lambda = \{\sigma\}$, so $z \in W^{ss}(\sigma) \setminus F^{ss}(\sigma)$.

Let $\Sigma \subseteq U$ be a cross section associated a $\sigma$ which is a singularity of boundary-type, there exists $z' \in \text{Int}(\Sigma) \cap O(z)$. Denote by $F^v_\Sigma$ the vertical foliation of $\Sigma$ and by $\partial^v \Sigma$ and $\partial^h \Sigma$ the vertical and horizontal boundary of $\Sigma$ respectively. We will assume that the vertical boundary $\partial^v \Sigma$ is formed by leaves of the foliation $F^v_\Sigma$, and $\partial^h \Sigma$ is transverse to $F^v_\Sigma$. Given that $F^{ss}(\sigma) \cap \Lambda = \{\sigma\}$, $\Sigma$ can be choose such that $\partial^h \Sigma \cap \Lambda = \emptyset$. In addition, every orbit that accumulates to $\sigma$ necessarily, accumulates the leaf $F^s(z', \Sigma)$ in $\Sigma$.

Now, since $z' \in \text{Int}(\Sigma) \cap W^u(O) \cap W^s(\sigma) \subseteq \Lambda$, then we can choose $\Sigma$ small in size such that:

$$F^s(x, \Sigma) \cap \Lambda \neq \emptyset,$$

for all $x \in \text{Int}(\Sigma)$ next to $F^s(z', \Sigma)$. As $\sigma \in \omega_X(p)$ and $\sigma$ is of boundary-type, necessarily the positive orbit of $p$ intersects $F^s(z', \Sigma)$ or intersects infinite times $\Sigma$ accumulating to $F^s(z', \Sigma)$, then for $n$ large enough we have the positive orbit of $x_n$ intersects to $\text{Int}(\Sigma)$, next to $F^s(z', \Sigma)$. From which it is concluded that, for $n$ large, there exists $z_n \in \Lambda$ such that $x_n \in F^s(z_n)$. Then, $\omega_X(z_n) = \omega_X(x_n) \subseteq \Lambda$.

**Corollary 2.** Every sectional-hyperbolic Lyapunov stable set $\Lambda$ of a vector field $X$ over $M$, without Lorenz-like singularities, is an attracting of $X$.

**Corollary 3.** Every sectional-hyperbolic Lyapunov stable set $\Lambda$ of a vector field $X$ over $M$, that satisfies the property $(P)$ with a unique singularity Lorenz-like, which is of boundary type, is an attracting of $X$.

**Proof.** Let $\sigma$ be the only Lorenz-like singularity in $\Lambda$. By theorem 1.1 in [3], $\Lambda$ has a nontrivial homoclinic class, and therefore a periodic point $q$. Since $\Lambda$ satisfies the property $(P)$, then there exists $\sigma^* \in \Lambda \cap \text{Sing}(X)$ such that $W^u(q) \cap W^s(\sigma^*) \neq \emptyset$, as $W^u(q) \subseteq \Lambda$ for being $\Lambda$ Lyapunov stable, we have that $\sigma^*$ is Lorenz-like, so $\sigma^* = \sigma$, then $\Lambda$ satisfies the property $(S)$ and by the theorem 2.2, $\Lambda$ is an attracting.

In the case that the Lyapunov stable sectional-hyperbolic set is of codimension one, we can replace, the hypothesis that the property $(P)$ is fulfilled in the corollary 3, by transitivity. Let $p, q \in M$ we will say that $p \prec q$ if for all $\epsilon > 0$ there is a trajectory from a point $\epsilon$-close to $p$ to a point $\epsilon$-close to $q$.

**Theorem 2.3 (Main).** Let $\Lambda$ be a sectional-hyperbolic Lyapunov stable set $\Lambda$ of a vector field $X$ over $M$ of codimension 1, with a unique singularity Lorenz-like, which is of boundary type. If $\Lambda = \omega_X(q)$ (or $\Lambda = \alpha_X(q)$) for some point $q \in M$, then $\Lambda$ is an attracting of $X$. 
Proof. Let us verify that under the assumptions of the theorem, $\Lambda$ satisfies the property $(S)$. Let $\sigma$ the only Lorenz-like singularity in $\Lambda$, by theorem 1.1 in [3], $\Lambda$ has a nontrivial homoclinic class, and therefore a periodic point $p$. If there exists $q \in M$ such that $\alpha_X(q) = \Lambda$ or $\omega_X(q) = \Lambda$, by the corollary 1, every singularity in $\Lambda$ is Lorenz-like, then $\sigma$ is the unique singularity of $\Lambda$. Also, using the orbit of $q$ we have that $p \prec \sigma$, and since $\Lambda$ satisfies the conditions of the theorem 10 in [8], so there exists $x \in \Lambda$ such that $\alpha_X(x) = \alpha_X(p)$ and $\omega_X(x)$ is a singularity, and this case, necessarily, $\omega_X(x) = \{\sigma\}$, that is, $W^u(p) \cap W^s(\sigma) \neq \emptyset$. Then by Theorem 2.2, $\Lambda$ is an attracting of $X$.

As a direct consequence of the main theorem we have that:

**Corollary 4.** Let $\Lambda$ be a sectional-hyperbolic transitive Lyapunov stable set of a vector field $X$ over $M$ of codimension 1, with a unique singularity Lorenz-like, which is of boundary type, then $\Lambda$ is an attractor of $X$.

**Acknowledgments.** We would like to thank the referees very much for their valuable comments and suggestions.

**REFERENCES**

[1] V. Araujo and M. J. Pacifico, *Three-Dimensional Flows*, A Series of Modern Surveys in Mathematics, 53. Springer, Heidelberg, 2010.

[2] A. Arbieto and C. A. Morales, A dichotomy for higher-dimensional flows, *Proc. Amer. Math. Soc.*, 141 (2013), 2817–2827.

[3] A. Arbieto, A. M. Lopez Barragán and C. Morales, Homoclinic classes for sectional-hyperbolic sets, *Kyoto Journal of Mathematics*, 56 (2016), 531–538.

[4] S. Bautista and C. Morales, *Lectures on Sectional-Anosov Flows*, Preprint IMPA Serie D 84, 2011.

[5] S. Bautista and C. Morales, A sectional-Anosov connecting lemma, *Ergodic Theory Dynam. Systems*, 30 (2010), 339–359.

[6] S. Bautista and C. Morales, Characterizing omega-limit sets which are closed orbits, *J. Differential Equations*, 245 (2008), 637–652.

[7] S. Bautista and C. Morales, Recent progress on sectional-hyperbolic systems, *Symmetrical Systems: An international Journal*, 30 (2015), 369–382.

[8] S. Bautista, V. Sales and Y. Sánchez, Sectional connecting lemma, preprint, arXiv:1804.00646.

[9] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[10] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[11] C. A. Morales and M. J. Pacifico, A dichotomy for three-dimensional vector fields, *Ergodic Theory Dynam. Systems*, 23 (2003), 1575–1600.

Received April 2018; revised August 2019.

E-mail address: sbautistad@unal.edu.co
E-mail address: yasanchezr@unal.edu.co