Existence of infinitely many solutions for the fractional Schrödinger-Maxwell equations

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Abstract

In this paper, by using variational methods and critical point theory, we shall mainly study the existence of infinitely many solutions for the following fractional Schrödinger-Maxwell equations

\begin{align*}
(-\Delta)^\alpha u + V(x)u + \phi u &= f(x, u), \text{ in } \mathbb{R}^3, \\
(-\Delta)^\alpha \phi &= K_\alpha u^2 \text{ in } \mathbb{R}^3
\end{align*}

where $\alpha \in (0, 1]$, $K_\alpha = \frac{\pi^{-\alpha}\Gamma(\alpha)}{\pi^{-(3-2\alpha)/2}\Gamma((3-2\alpha)/2)}$, $(-\Delta)^\alpha$ stands for the fractional Laplacian. Under some more assumptions on $f$, we get infinitely many solutions for the system.

Key words Fractional Laplacian, Schrödinger-Maxwell equations, infinitely many solutions.

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1 Introduction and the Main Result

In this paper, we study the fractional Schrödinger-Maxwell equations

\begin{align*}
(-\Delta)^\alpha u + V(x)u + \phi u &= f(x, u), \text{ in } \mathbb{R}^3, \\
(-\Delta)^\alpha \phi &= K_\alpha u^2 \text{ in } \mathbb{R}^3
\end{align*}
where \( u, \phi : \mathbb{R}^3 \to \mathbb{R} \), \( f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \), \( \alpha \in (0, 1] \), \( K_\alpha = \frac{1}{\pi^{-(3-2\alpha)/2}} \frac{\pi^{-\alpha}\Gamma(\alpha)}{\Gamma((3-2\alpha)/2)} \), \((-\Delta)^\alpha\) stands for the fractional Laplacian. Here the fractional Laplacian \((-\Delta)^\alpha\) with \( \alpha \in (0, 1] \) of a function \( \phi : \mathbb{R}^3 \to \mathbb{R} \) is defined by:

\[
\mathcal{F}((-\Delta)^\alpha \phi)(\xi) = |\xi|^{2\alpha} \mathcal{F}(\phi)(\xi), \ \forall \alpha \in (0, 1],
\]

where \( \mathcal{F} \) is the Fourier transform, i.e.,

\[
\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp\{-2\pi i \xi \cdot x\} \phi(x) dx.
\]

If \( \phi \) is smooth enough, \((-\Delta)^\alpha\) can also be computed by the following singular integral :

\[
(-\Delta)^\alpha \phi(x) = c_{3,\alpha} \text{P.V.} \int_{\mathbb{R}^3} \frac{\phi(x) - \phi(y)}{|x - y|^{3+2\alpha}} dy.
\]

Here P.V. is the principal value and \( c_{3,\alpha} \) is a normalization constant. Such a system \((1.1)\) is called Schrödinger-Maxwell equations or Schrödinger-Poisson equations which is obtained while looking for existence of standing waves for the fractional nonlinear Schrödinger equations interacting with an unknown electrostatic field. For a more physical background of system \((1.1)\), we refer the reader to [1, 2] and the references therein.

When \( \alpha = 1 \), system \((1.1)\) was first introduced by Benci and Fortunato in [1], and it has been widely studied by many authors; The case \( V \equiv 1 \) or being radially symmetric, has been studied under various conditions on \( f \) in [3]-[9]; When \( V(x) \) is not a constant, the existence of infinitely many large solutions for \((1.1)\) has been considered in [10]-[14] via the fountain theorem (cf. [15, 16].)

In system \((1.1)\), we assume the following hypotheses on potential \( V \) and nonlinear term \( f \):

\((\forall)\) \( V \in C(\mathbb{R}^3, \mathbb{R}) \), \( \inf_{x \in \mathbb{R}^3} V(x) \geq a_1 > 0 \), where \( a_1 \) is a positive constant. Moreover,

\[
\lim_{|x| \to \infty} V(x) = +\infty.
\]

\((H_1)\) \( f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}) \), and there exist \( c_1, c_2 > 0, p \in (4, 2^*_\alpha) \) such that

\[
|f(x,u)| \leq c_1 |u| + c_2 |u|^{p-1}, \ \forall \ x \in \mathbb{R}^3, \ u \in \mathbb{R},
\]

where, \( 2^*_\alpha = \frac{6}{3-2\alpha}, \ \alpha > \frac{3}{4} \), \( f(x,u)u \geq 0 \) for \( u \geq 0 \).

\((H_2)\) \( \lim_{|u| \to \infty} \frac{F(x,u)}{u^p} = +\infty \) uniformly for \( x \in \mathbb{R}^3 \), here \( F(x,u) = \int_0^u f(x,t) dt \).

\((H_3)\) Let \( G(x,u) = \frac{1}{2} f(x,u)u - F(x,u) \), there exist \( a_0 > 0 \), and \( g(x) \geq 0 \) such that \( \int_{\mathbb{R}^3} g(x) dx < +\infty, G(x,u) \geq -a_0 g(x), \ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R} \).

\((H_4)\) \( f(x,-u) = -f(x,u) \ \forall \ x \in \mathbb{R}^3, \ u \in \mathbb{R} \).

Now, we are ready to state the main result of this paper.
Remark 1.1. Assume that $(\mathcal{V})$ and $(\mathcal{H}_1) - (\mathcal{H}_4)$ satisfy. Then system (1.1) possesses infinitely many nontrivial solutions.

**Theorem 1.1.** Assume that $(\mathcal{V})$ and $(\mathcal{H}_1) - (\mathcal{H}_4)$ satisfy. Then system (1.1) possesses infinitely many nontrivial solutions.

**Remark 1.1.** (i) : There are functions \( f \) satisfying the assumptions $$(\mathcal{H}_1) - (\mathcal{H}_4)$$, for example (1) : \( f(x, u) = 4u^3 \ln(u^2 + 1) + \frac{2u^5}{u^2 + 1} \), then \( a_0 = 0 \), $$(\mathcal{H}_3)$$ is satisfied; (2) : \( f(x, u) = e^{-\sum_{i=1}^{3} |x_i|} u + |u|^{p-2} u, \ p \in (4, 2^*_\alpha), \ \alpha > \frac{3}{4} \), then \( a_0 = \frac{\alpha}{4}, g(x) = e^{-\sum_{i=1}^{3} |x_i|}, r_0 = \left( \frac{p}{p-4} \right)^{1/(p-2)} + 1 \), $$(\mathcal{H}_3)$$ is satisfied.

(ii) : the assumption $$(\mathcal{H}_3)$$ is weaker than the assumptions $$(f_4)$$ in paper [12] and $(f3')$ in paper [14].

2 Variational settings and preliminary results

Now, let's introduce some notations. For any \( 1 \leq r < \infty \), \( L^r(\mathbb{R}^3) \) is the usual Lebesgue space with the norm

\[
\|u\|_{L^r} = \left( \int_{\mathbb{R}^3} |u(x)|^r \, dx \right)^{\frac{1}{r}}.
\]

The fractional order Sobolev space:

\[
H^\alpha(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2\alpha} \hat{u}^2 + \hat{u}^2) \, d\xi < \infty \right\},
\]

where \( \hat{u} = \mathcal{F}(u) \), The norm is defined by

\[
\|u\|_{H^\alpha(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} (|\xi|^{2\alpha} \hat{u}^2 + \hat{u}^2) \, d\xi \right)^{\frac{1}{2}}.
\]

The spaces \( D^\alpha(\mathbb{R}^3) \) is defined as the completion of \( C_0^\infty(\mathbb{R}^3) \) under the norms

\[
\|u\|_{D^\alpha(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} (|\xi|^{2\alpha} \hat{u}^2 + \hat{u}^2) \, d\xi \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

Note that, by Plancherel’s theorem we have \( \|u\|_2 = \|\hat{u}\|_2 \), and

\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 \, dx = \int_{\mathbb{R}^3} ((-\Delta)^{\frac{\alpha}{2}} \hat{u}(\xi))^2 \, d\xi = \int_{\mathbb{R}^3} (|\xi|^{\alpha} \hat{u}(\xi))^2 \, d\xi = \int_{\mathbb{R}^3} |\xi|^{2\alpha} \hat{u}^2 \, d\xi < \infty, \ \forall u \in H^\alpha(\mathbb{R}^3).
\]

It follows that

\[
\|u\|_{H^\alpha(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \left( |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + u^2 \right) \, dx \right)^{\frac{1}{2}}.
\]

In our problem, we work in the space defined by

\[
E := \left\{ u \in H^\alpha(\mathbb{R}^3) \mid \left( \int_{\mathbb{R}^3} \left( |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + V(x) u^2 \right) \, dx \right)^{\frac{1}{2}} < \infty \right\}.
\]
Thus, $E$ is a Hilbert space with the inner product
\[ (u, v)_E := \int_{\mathbb{R}^3} ((-\Delta)^{\frac{\alpha}{2}} u(x) \cdot (-\Delta)^{\frac{\alpha}{2}} v(x) + V(x) uv) \, dx. \]
and its norm is $\|u\| = (u, u)^{\frac{1}{2}}$. Obviously, under the assumptions $(\mathcal{V})$, $\|u\|_E \equiv \|u\|_{H^\alpha}$.

**Lemma 2.1** (see [17] Lemma 2.2 and [18]). $H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2\alpha^*_\alpha];$ and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2\alpha^*_\alpha)$ where $2\alpha^*_\alpha = \frac{6}{3 - 2\alpha}$. Therefore, there exists a positive constant $C_p$ such that
\[ \|u\|_p \leq C_p \|u\|_{H^\alpha(\mathbb{R}^3)}. \]

**Lemma 2.2** (see [19]). Under the assumption $(\mathcal{V})$, the embedding $E$ is compactly embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2\alpha^*_\alpha)$.

**Lemma 2.3** (see [20]). For $1 < p < \infty$ and $0 < \alpha < N/p$, we have
\[ \|u\|_{L^\frac{pN}{N-\alpha'}(\mathbb{R}^N)} \leq B \|((-\Delta)^{\alpha/2} u)\|_{L^p(\mathbb{R}^N)} \tag{2.2} \]
with best constant
\[ B = 2^{-\alpha} \pi^{-\alpha/2} \frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha)/2)} \left( \frac{\Gamma(N)}{\Gamma(N/2)} \right)^{\alpha/N}. \]

**Lemma 2.4.** For any $u \in H^\alpha(\mathbb{R}^N)$ and for any $h \in D^{-\alpha}(\mathbb{R}^N)$, there exists a unique solution $\phi = ((-\Delta)^{\alpha} + u^2)^{-1} h \in D^\alpha(\mathbb{R}^N)$ of the equation
\[ (-\Delta)^{\alpha} \phi + u^2 \phi = h, \]
(being $D^{-\alpha}(\mathbb{R}^N)$ the dual space of $D^\alpha(\mathbb{R}^N)$). Moreover, for every $u \in H^\alpha(\mathbb{R}^N)$ and for every $h, g \in D^{-\alpha}(\mathbb{R}^N)$,
\[ \langle h, ((-\Delta)^{\alpha} + u^2)^{-1} g \rangle = \langle g, ((-\Delta)^{\alpha} + u^2)^{-1} h \rangle \tag{2.3} \]
where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $D^{-\alpha}(\mathbb{R}^N)$ and $D^\alpha(\mathbb{R}^N)$.

**Proof.** If $u \in H^\alpha(\mathbb{R}^N)$, then by Hölder inequality and (2.2)
\[ \int_{\mathbb{R}^N} u^2 \phi^2 \, dx \leq \|u\|^2_{2p} \|\phi\|^2_{2q} \leq B^2 \|u\|^2_{2p} \|\phi\|^2_{D^\alpha}, \tag{2.4} \]
where $\frac{1}{p} + \frac{1}{q} = 1$, $q = \frac{N}{N-2\alpha}$, $2q = 2\alpha^*_\alpha$. Thus $\left( \int |((-\Delta)^{\alpha/2} \phi)|^2 + \int u^2 \phi^2 \right)^{1/2}$ is a norm in $D^\alpha(\mathbb{R}^N)$ equivalent to $\|\phi\|_{D^\alpha}$. Hence, by the application of Lax-Milgram Lemma, we
obtain the existence part. For every \( u \in H^\alpha(\mathbb{R}^N) \) and for every \( h, g \in D^{-\alpha}(\mathbb{R}^N) \), we have \( \phi_g = ((-\Delta)^\alpha + u^2)^{-1} g, \phi_h = ((-\Delta)^\alpha + u^2)^{-1} h. \) Hence,

\[
\langle h, ((-\Delta)^\alpha + u^2)^{-1} g \rangle = \int h ((-\Delta)^\alpha + u^2)^{-1} g dx = \int h \phi_g dx = \int ((-\Delta)^\alpha + u^2)^{-1} h, \phi_g dx = \int ((-\Delta)^\alpha + u^2)^{-1} g, \phi_h dx = \int g ((-\Delta)^\alpha + u^2)^{-1} h dx = \langle g, ((-\Delta)^\alpha + u^2)^{-1} h \rangle.
\]

So, we get (2.3). \( \square \)

**Lemma 2.5** (see [21]). Let \( f \) be a function in \( C_0^\infty(\mathbb{R}^N) \) and let \( 0 < \alpha < n. \) Then, with \( c_{n-\alpha} = \pi^{-\alpha/2} \Gamma(-\alpha/2), \)

\[
c_{n-\alpha} \int_{\mathbb{R}^n} |x - y|^{2n - 2\alpha} f(y) dy.
\]

**(2.5)**

**Lemma 2.6.** For every \( u \in H^\alpha \) there exists a unique \( \phi = \phi(u) \in D^\alpha \) which solves equation (1.2). Furthermore, \( \phi(u) \) is given by

\[
\phi(u)(x) = \int_{\mathbb{R}^3} |x - y|^{2\alpha - 3} u^2(y) dy.
\]

**(2.7)**

As a consequence, the map \( \Phi : u \in H^\alpha \mapsto \phi(u) \in D^\alpha \) is of class \( C^1 \) and

\[
[\Phi(u)]'(v)(x) = 2 \int_{\mathbb{R}^3} |x - y|^{2\alpha - 3} u(y)v(y) dy, \quad \forall u, v \in H^\alpha.
\]

**(2.8)**

**Proof.** The existence and uniqueness part follows by Lemma 2.4. By Lemma 2.5 and the Fourier transform of equation (1.2), the representation formula (2.7) holds for \( u \in C_0^\infty(\mathbb{R}^3); \) by density it can be extended for any \( u \in H^\alpha. \) The representation formula (2.8) is obvious. \( \square \)

System (1.1) and (1.2) are the Euler-Lagrange equations corresponding to the functional \( J : H^\alpha(\mathbb{R}^3) \times D^\alpha(\mathbb{R}^3) \to \mathbb{R} \) is

\[
J(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + V(x)u^2 - \frac{1}{2} |(-\Delta)^{\frac{\alpha}{2}} \phi(x)|^2 + K_{\alpha} \phi^2 \right) dx - \int_{\mathbb{R}^3} F(x, u) dx,
\]

where \( F(x, t) = \int_0^t f(x, s) ds, \quad t \in \mathbb{R}. \)
Evidently, the action functional $J$ belongs to $C^1(H^\alpha(\mathbb{R}^3) \times D^\alpha(\mathbb{R}^3), \mathbb{R})$ and the partial derivatives in $(u, \phi)$ are given, for $\xi \in H^\alpha(\mathbb{R}^3)$ and $\eta \in D^\alpha(\mathbb{R}^3)$, by
\[
\left\langle \frac{\partial J}{\partial u}(u, \phi), \xi \right\rangle = \int_{\mathbb{R}^3} \left( (-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} \xi(x) + V(x)u\xi + K_{\alpha}\phi u\xi \right) \, dx - \int_{\mathbb{R}^3} f(x, u)\xi \, dx,
\]
\[
\left\langle \frac{\partial J}{\partial \phi}(u, \phi), \eta \right\rangle = \frac{1}{2} \int_{\mathbb{R}^3} \left( (-\Delta)^{\frac{\alpha}{2}} \phi(x)(-\Delta)^{\frac{\alpha}{2}} \eta(x) + K_{\alpha}u^2\eta \right) \, dx.
\]
Thus, we have the following result:

**Proposition 2.1.** The pair $(u, \phi)$ is a weak solution of system (1.1) and (1.2) if and only if it is a critical point of $J$ in $H^\alpha(\mathbb{R}^3) \times D^\alpha(\mathbb{R}^3)$.

So, we can consider the functional $J : H^\alpha(\mathbb{R}^3) \to \mathbb{R}$ defined by $J(u) = J(u, \phi(u))$.

After multiplying (1.2) by $\phi(u)$ and integration by parts, we obtain
\[
\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} \phi(u)|^2 \, dx = K_{\alpha} \int_{\mathbb{R}^3} \phi(u)u^2 \, dx.
\]
Therefore, the reduced functional takes the form
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + V(x)u^2 \, dx + \frac{1}{4} K_{\alpha} \int_{\mathbb{R}^3} u^2 \phi(u) \, dx - \int_{\mathbb{R}^3} F(x, u) \, dx. \tag{2.9}
\]

**Lemma 2.7.** Assume that there exist $c_1, c_2 > 0$ and $p > 1$ such that
\[
|f(s)| = c_1 |s| + c_2 |s|^{p-1}, \quad \forall s \in \mathbb{R}. \tag{2.10}
\]

Then the following statements are equivalent:

i) $(u, \phi) \in (H^\alpha \cap L^p) \times D^\alpha$ is a solution of the system (1.1) – (1.2);

ii) $u \in H^\alpha \cap L^p$ is a critical point of $J$ and $\phi = \phi(u)$.

**Proof.** By the assumption (2.10), the Nemitsky operator $u \in H^\alpha \cap L^p \mapsto F(x, u) \in L^1$ is of class $C^1$. Hence, by Lemma 2.6 for every $u, v \in H^\alpha$
\[
J'(u)[v] = \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} v(x) \, dx + \int_{\mathbb{R}^3} V(x)uv \, dx + \frac{1}{2} K_{\alpha} \int_{\mathbb{R}^3} uv \int_{\mathbb{R}^3} |x-y|^{2\alpha-3} u^2(y) \, dy \, dx
\]
\[
+ \frac{1}{2} K_{\alpha} \int_{\mathbb{R}^3} u^2 \int_{\mathbb{R}^3} |x-y|^{2\alpha-3} u(y)v(y) \, dy \, dx - \int_{\mathbb{R}^3} f(x, u)vdx
\]
\[
= \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} v(x) \, dx + \int_{\mathbb{R}^3} V(x)uv \, dx + K_{\alpha} \int_{\mathbb{R}^3} uv \phi(u) \, dx - \int_{\mathbb{R}^3} f(x, u)vdx.
\]

By Fubini-Tonelli’s Theorem, we can obtain the conclusion. \qed

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If $1 \leq p < \infty$ and $a, b \geq 0$, then

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$  \hspace{1cm} (2.11)$$

From (1.2) and (2.2), for any $u \in E$ using Hölder inequality we have

$$\|\phi(u)\|^2_{L^p} = K_\alpha \int_{\mathbb{R}^3} \phi(u)u^2dx \leq K_\alpha \|\phi(u)\|_q\|u\|^{2p}_{2p} \leq C\|\phi(u)\|_{L^p}\|u\|^2_{2p}.$$  

where $\frac{1}{p} + \frac{1}{q} = 1$, $q = 2_\alpha = \frac{6}{3-2\alpha}$, $\alpha > \frac{3}{4}$. Here and subsequently, $C$ denotes an universal positive constant. This and lemma 2.2 implies that

$$\|\phi(u)\|_{L^p} \leq C\|u\|^2_{2p} \leq C\|u\|^2_E,$$  \hspace{1cm} (2.12)$$

$$\int_{\mathbb{R}^3} \phi(u)u^2dx \leq C\|u\|^4_{2p} \leq C\|u\|^4_E.$$  \hspace{1cm} (2.13)$$

**Lemma 2.8.** Assume that a sequence $\{u_n\} \subset E$, $u_n \rightharpoonup u$ in $E$ as $n \to \infty$ and $\{u_n\}$ be a bounded sequence. Then

$$\left|\int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)(u_n - u)dx\right| \to 0, \text{ as } n \to \infty.$$  

**Proof.** Let $\{u_n\}$ be a sequence satisfying the assumptions $u_n \rightharpoonup u$ in $E$ as $n \to \infty$ and $\{u_n\}$ is bounded. Lemma 2.2 implies that $u_n \to u$ in $L^r(\mathbb{R}^3)$, where $2 \leq r < 2_\alpha^*$, and $u_n \to u$ for a.e. $x \in \mathbb{R}^3$. Hence $\sup_{n \in \mathbb{N}} \|u_n\|_r < \infty$ and $\|u\|_r$ is finite. By Hölder inequality, (2.11), (2.12) and (2.14)

$$\left|\int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)(u_n - u)dx\right|$$

$$\leq \left(\int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)^2dx\right)^\frac{1}{2} \left(\int_{\mathbb{R}^3} (u_n - u)^2dx\right)^\frac{1}{2}$$

$$\leq \left\{2 \int_{\mathbb{R}^3} (|\phi(u_n)u_n|^2 + |\phi(u)u|^2)dx\right\}^{\frac{1}{2}} \|u_n - u\|_2$$

$$\leq C(\|u_n\|^6_E + \|u\|^6_E)^\frac{1}{2} \|u_n - u\|_2 \to 0, \text{ as } n \to \infty.$$  

3. **Proof of Theorem 1.1**

We say that $J \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$-condition if any sequence $\{u_n\}$ such that

$$J(u_n) \to c, \quad \|J'(u_n)\|(1 + \|u_n\|) \to 0$$

has a convergent subsequence, where $X$ is a Banach space.
Lemma 3.1. Assume that (V) and (H1) – (H4) satisfy. Then any sequence \( \{u_n\} \subset E \) satisfying
\[
J(u_n) \to c > 0, \quad \langle J'(u_n), u_n \rangle \to 0,
\]
is bounded in \( E \). Moreover, \( \{u_n\} \) contains a converge subsequence.

Proof. To prove the boundedness of \( \{u_n\} \), arguing by contradiction, suppose that \( \|u_n\| \to \infty \) as \( n \to \infty \). By (H3) for sufficiently large \( n \in \mathbb{N} \)
\[
c + 1 \geq J(u_n) - \frac{1}{4} \langle J'(u_n), u_n \rangle = \frac{1}{4} \|u_n\|^2 + \int_{\mathbb{R}^3} G(x, u_n)dx \\
\geq \frac{1}{4} \|u_n\|^2 - a_0 \int_{\mathbb{R}^3} g(x)dx \to +\infty.
\]
Thus \( \sup_{n \in \mathbb{N}} \|u_n\| < \infty \), i.e. \( \{u_n\} \) is a bounded sequence.

Now we shall prove \( \{u_n\} \) contains a subsequence, without loss of generality, by Eberlein-Shmulyan theorem (see for instance in [22]), passing to a subsequence if necessary, there exists a \( u \in E \) such that \( u_n \rightharpoonup u \) in \( E \), again by Lemma [22], \( u_n \to u \) a.e. \( x \in \mathbb{R}^3 \). By (H1) and using Hölder inequality we have
\[
\left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx \right| \\
\leq \int_{\mathbb{R}^3} |c_1(|u_n| + |u|) + c_2(|u_n|^{p-1} + |u|^{p-1})| |u_n - u|dx \\
\leq c_1(\|u_n\|_2 + \|u\|_2)\|u_n - u\|_2 + c_2(\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p \\
\to 0, \quad \text{as} \quad n \to \infty.
\]
Since \( J \in C^1(E) \), we have \( J'(u_n) \rightharpoonup J'(u) \) in \( E^* \). i.e.
\[
\langle J'(u_n) - J'(u), u_n - u \rangle \to 0, \quad \text{as} \quad n \to \infty.
\]
This together with Lemma 2.8 implies
\[
\|u_n - u\|^2 = \langle J'(u_n) - J'(u), u_n - u \rangle - K_n \int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)(u_n - u)dx \\
+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx \to 0, \quad \text{as} \quad n \to \infty.
\]
That is \( u_n \to u \) in \( E \).

Lemma 3.2. Suppose that assumptions (V), (H1) and (H2) satisfy, for any finite dimensional subspace \( \bar{E} \subset E \), there holds
\[
J(u) \to -\infty, \quad \|u\| \to \infty, \quad u \in \bar{E}.
\] (3.1)
Proof. Arguing indirectly, assume that for some sequence \( \{ u_n \} \subset \tilde{E} \) with \( \| u_n \| \to \infty \), there is \( M > 0 \) such that \( J(u_n) \geq -M, \forall n \in \mathbb{N} \). Set \( v_n = \frac{u_n}{\| u_n \|} \), then \( \| v_n \| = 1 \). Passing to a subsequence, we may assume that \( v_n \rightharpoonup v \) in \( E \). Since \( \dim E < \infty \), then \( v_n \to v \in \tilde{E} \), \( v_n(x) \to v(x) \) a.e. on \( x \in \mathbb{R}^3 \), and so \( \| v \| = 1 \). Let \( \Omega := \{ x \in \mathbb{R}^3 : v(x) \neq 0 \} \), then \( \text{meas}(\Omega) > 0 \) and for a.e. \( x \in \Omega \), we have \( \lim_{n \to \infty} | u_n(x) | \to \infty \).

It follows from (2.9), (2.13) that
\[
\lim_{n \to \infty} \frac{4}{\| u_n \|^4} \int_{\mathbb{R}^3} F(x, u_n) \, dx = \lim_{n \to \infty} \frac{2\| u_n \|^2 + K \alpha \int_{\mathbb{R}^3} \phi(u_n) u_n^2 \, dx - 4J(u_n)}{\| u_n \|^4} \leq C. \tag{3.2}
\]

But by the non-negative of \( F \), (\((H_2)\) and Fadous Lemma, for large \( n \) we have
\[
\lim_{n \to \infty} \frac{4}{\| u_n \|^4} \int_{\mathbb{R}^3} F(x, u_n) \, dx \geq \lim_{n \to \infty} \int_{\Omega} \frac{4F(x, u_n) v_n^4}{u_n^4} \, dx
\geq \liminf_{n \to \infty} \int_{\Omega} \frac{F(x, u_n) v_n^4}{u_n^4} \, dx
= \int \liminf_{n \to \infty} \frac{F(x, u_n)}{u_n^4} [\chi_{\Omega}(x)] v_n^4 \, dx \to \infty, \quad n \to \infty.
\]
This contradicts to (3.2). \( \Box \)

Corollary 3.1. Under assumptions \((\mathcal{V})\), \((H_1)\) and \((H_2)\), for any finite dimensional subspace \( \tilde{E} \subset E \), there is \( R = R(\tilde{E}) > 0 \) such that
\[
J(u) \leq 0, \quad \forall u \in \tilde{E}, \quad \| u \| \geq R. \tag{3.3}
\]

Let \( \{ e_j \} \) is an orthonormal basis of \( E \) and define \( X_j = \mathbb{R} e_j \),
\[
Y_k = \oplus_{j=1}^{k} X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{N}. \tag{3.4}
\]

Lemma 3.3. Under assumptions \((\mathcal{V})\), for \( 2 \leq r < 2^* \), we have
\[
\beta_k(r) = \sup_{u \in Z_k, \| u \| = 1} \| u \|_r \to 0, \quad k \to \infty. \tag{3.5}
\]

Proof. Since the embedding from \( E \) into \( L^r(\mathbb{R}^3) \) is compact, then Lemma 3.3 can be proved by a similar way as Lemma 3.8 in \([15]\).

By Lemma 3.3, we can choose an integer \( m \geq 1 \) such that
\[
\| u \|^2 \leq \frac{1}{2c_1} \| u \|^2, \quad \| u \|^p \leq \frac{p}{4c_2} \| u \|^p, \quad \forall u \in Z_m. \tag{3.6}
\]
Lemma 3.4. Suppose that assumptions (V) and (H$_1$) are satisfied, there exist constants $\rho, \delta > 0$ such that $J|_{\partial B_\rho \cap Z_m} \geq \delta > 0$.

Proof. By (H$_1$), we have

$$F(x, u) \leq \frac{c_1}{2} u^2 + \frac{c_2}{p} |u|^p, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}.$$  

Hence, by (2.9) and (3.6), we have

$$J(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} K_\alpha \int_{\mathbb{R}^3} \phi(u) u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx$$

$$\geq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^3} F(x, u) dx$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{c_1}{2} \|u\|^2 - \frac{c_2}{p} \|u\|^p$$

$$\geq \frac{1}{4} (\|u\|^2 - \|u\|^p).$$

Hence for any given $0 < \rho < 1$, let $\delta = \frac{1}{4}(\rho^2 - \rho^p)$, then $J|_{\partial B_\rho \cap Z_m} \geq \delta > 0$. This complete the proof.

Lemma 3.5 (see[23]). Let $X$ be an infinite dimensional Banach space, $X = Y \oplus Z$, where $Y$ is finite dimensional. If $J \in C^1(X, \mathbb{R})$ satisfies (C)$_c$-condition for all $c > 0$, and

(J1) $J(0) = 0$, $J(-u) = J(u)$ for all $u \in X$;

(J2) there exist constants $\rho, \delta > 0$ such that $J|_{\partial B_{\rho} \cap Z_m} \geq \delta > 0$;

(J3) for any finite dimensional subspace $\widetilde{E} \subset E$, there is $R = R(\widetilde{E}) > 0$ such that $J(u) \leq 0$, $\forall u \in \widetilde{E} \setminus B_{R}$;

then $J$ possesses an unbounded sequence of critical values.

Proof of Theorem 1.1. Let $X = E, Y = Y_m$ and $Z = Z_m$. By Lemmas 3.2 3.3 and Corollary 3.1 all conditions of Lemma 3.5 are satisfied. Thus, problem (1.1) and (1.2) possesses infinitely many nontrivial solutions.

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