Concatenate and Boost for Multiple Measurement Vector Problems

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Abstract

Multiple measurement vector (MMV) problem addresses the recovery of a set of sparse signal vectors that share common non-zero support, and has emerged an important topics in compressed sensing. Even through the fundamental performance limit of recoverable sparsity level has been formally derived, conventional algorithms still exhibit significant performance gaps from the theoretical bound. The main contribution of this paper is a novel concatenate MMV and boost (CoMBo) algorithm that achieves the theoretical bound. More specifically, the algorithm concatenates MMV to a larger dimensional SMV problem and boosts it by multiplying random orthonormal matrices. Extensive simulation results demonstrate that CoMBo outperforms all existing methods and achieves the theoretical bound as the number of measurement vector increases.

Index Terms

Compressed sensing, multiple measurement vector, S-OMP, ReMBo, QR factorization, concatenate and boost

I. INTRODUCTION

Compressed sensing (CS) theory [1–3] addresses accurate recovery of unknown sparse signals from underdetermined linear measurements, and has become one of the main research topics in signal processing community. Most of the compressed sensing researches have been developed to address the following single measurement vector (SMV) problem [1–3]:

\[(P0): \min_{x} \|x\|_0, \text{ subject to } y = Ax,\]  

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where $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, and $\|\mathbf{x}\|_0$ denotes the number of non-zero elements in the vector $\mathbf{x}$. Since (P0) is a combinatorial optimization problem, alternative approaches such as greedy method [4, 5] or basis pursuit [2, 6] has been widely used. Specifically, the basis pursuit addresses the following $l_1$ minimization:

$$(P1) : \min_{\mathbf{x}} \|\mathbf{x}\|_1, \text{ subject to } \mathbf{y} = \mathbf{A} \mathbf{x}, \quad (2)$$

where $\|\cdot\|_1$ denotes an $l_1$ norm. The beauty of compressed sensing is that (P1) provides the exactly same solution as (P0) if the restricted isometry property (RIP) is satisfied [3]. More specifically, let $k$ denote the sparsity of the solution vector $\mathbf{x}$, i.e. $\|\mathbf{x}\|_0 = k$. Then, the mapping $\mathbf{A}$ satisfies the RIP if there exist a constant $\epsilon_k$ such that

$$(1 - \epsilon_k)\|\mathbf{v}\|_2 \leq \|\mathbf{A} \mathbf{v}\|_2 \leq (1 + \epsilon_k)\|\mathbf{v}\|_2, \quad (3)$$

for all nonzero vector $\mathbf{v}$ such that $\|\mathbf{v}\|_0 \leq k$. It has been shown that for many class of random matrices, the RIP is satisfied with extremely high probability if the number of measurement is given by $m \geq C_1 k \log(n/k)$ for some constant $C_1$ [3].

Another important area of compressed sensing research is so called multiple measurement vector problem (MMV) [5, 7–9]. The MMV problem addresses the recovery of a set of sparse signal vectors that share common non-zero support. Specifically, let $\mathcal{R}(\mathbf{X})$ denote the number of rows that have non-zero elements. Then, MMV problem addresses the following:

$$\min_{\mathbf{x}} \mathcal{R}(\mathbf{X}), \text{ subject to } \mathbf{Y} = \mathbf{A} \mathbf{X}, \quad (4)$$

where $\mathbf{Y} \in \mathbb{R}^{m \times r}$, $\mathbf{X} \in \mathbb{R}^{n \times r}$, and $r$ denotes the number of measurement vectors. In MMV, thanks to the common sparse support, it is easily expected that the number of recoverable sparsity level $\mathcal{R}(\mathbf{X})$ may increase with the increasing number of measurement vectors. This has been formally proven in the deterministic setup [5, 7]. More specifically, Eq. (4) has the unique solution if

$$\mathcal{R}(\mathbf{X}) \leq \frac{1}{2} (\text{spark}(\mathbf{A}) + \text{rank}(\mathbf{Y}) - 1), \quad (5)$$

where $\text{spark}(\mathbf{A})$ is the smallest possible number such that there exists a subgroup of $\text{spark}(\mathbf{A})$ columns from $\mathbf{A}$ that are linearly dependent [10], and $\text{rank}(\mathbf{Y})$ denotes the rank of $\mathbf{Y}$ that may increase with the number of measurement vectors. Note that Eq. (5) is a deterministic bound that guarantees 100% successful reconstruction. As a reconstruction algorithm for MMV problem, greedy algorithm such as S-OMP [5, 11], $l_1$ convex relaxation method [12, 13], or M-FOCUSS [7] have been proposed. Recently, the novel algorithm called REduce MMV and BOost (ReMBo) has been proposed that outperforms the...
conventional methods [8]. However, the performance improvement of the existing MMV algorithms are still limited and there still exist significant performance gaps from the deterministic bound in Eq. (5).

The main contribution of this paper is to overcome the drawbacks of the existing method and to achieve the optimal performance. The new algorithm is based on the observation that QR factorization of the nonzero sparse matrix of $X$ provides a sparse coefficient matrix. Then, by exploiting the sparsity of coefficient matrix, we can convert the MMV problem into a higher dimensional SMV problem, which can be solved using $l_1$ minimization methods to estimate the common sparse support. Since a QR factorization of the solution matrix is unknown a priori, we boost the MMV problem by multiplying a set of random orthonormal matrices. As will be demonstrated later, the new algorithm can achieve the optimal performance as the number of measurement vectors increases.

II. CONVENTIONAL MMV ALGORITHMS

Simultaneous orthogonal matching pursuit (S-OMP) is a simple extension of OMP algorithm for MMV problem [5, 11]. More specifically, S-OMP chooses the maximum correlation between the residual and atoms:

**Algorithm S-OMP**

1) Set the residual matrix $R_0 = Y$, the index set $\Lambda_0 = \emptyset$, and the iteration count $t = 0$.

2) Find the index $i_t$ that solves the optimization problem:

$$\max_i ||a_i^H R_t||_p$$

where $||\cdot||_p$ denotes the $l_p$ norm for $1 \leq p \leq 2$.

3) Set $\Lambda_t = \Lambda_{t-1} \cup \{i_t\}$.

4) Determine the orthogonal projector $P_t = A_{\Lambda_t} (A_{\Lambda_t}^H A_{\Lambda_t})^{-1} A_{\Lambda_t}^H$ onto the span of the atoms indexed in $\Lambda_t$.

5) Calculate the new residual:

$$R_t = (I - P_t) Y$$

6) Increase the step count $t$ and go to step 2) if $||R_t||_F \geq \epsilon$.

Even though S-OMP exploits the MMV structure, the performance dependency on the number of measurement vector has not been proven in the deterministic setup [5]. Recently, Gribonval et al [14], Eldar and Rauhut [15] prove the performance improvement using an average case analysis.

In contrast, REduce MMV and Boost (ReMBo) addresses the MMV problem by reducing it to an SMV problem [8]. More specifically, MMV problem is first converted into an SMV problem by applying...
a random vector \( w \) to the right hand side of Eq. (4) (reducing MMV). The resultant SMV problem is solved by high performance SMV algorithm such as basis pursuit. This procedure is repeated with other random vectors (boosting) until the termination condition is satisfied.

**Algorithm ReMBo**

1) Initialization: set iteration count \( t = 0 \) and threshold value \( \epsilon \).

2) While \( (t \leq MaxIter) \) do

   a) Draw a unit norm random vector \( w \in \mathbb{R}^r \).

   b) Reduce MMV to SMV problem by multiplying \( w \):

      \[
      y = Yw = AXw = Ax .
      \]  

   c) Solve Eq. (8) using SMV technique to get \( \hat{x} \). Let \( \Lambda \) denote the nonzero support of \( \hat{x} \).

   d) if \( (\|\hat{x}\|_0 \leq k) \) and \( (\|y - A\hat{x}\|_2 \leq \epsilon) \) then stop, else \( t = t + 1 \).

3) \( \hat{X}^\Lambda \) denotes the submatrix of \( \hat{X} \) by collecting rows whose indices belong to \( \Lambda \). Then, the final MMV solution is given by:

\[
\begin{cases}
\hat{X}^\Lambda = A_\Lambda^\dagger Y \\
\hat{X}^i = 0, i \notin \Lambda.
\end{cases}
\]  

Using extensive numerical simulation, ReMBo has been shown to overperform other existing MMV algorithms [8]. The theory behind performance improvement of ReMBo has been recently suggested by van der Berg et al [9]. More specifically, based on the polytope geometry, they showed that the recovery performance of SMV can be improved by limiting the sign of the unknown solution \( x \) to optimal patterns [9]. In ReMBo, the unknown solution \( x \) can be obtained from \( X \) by multiplying a random vector \( w \), i.e. \( x = Xw \). Hence, we have more chance to choose better sign patterns for \( x \). The probability of choosing the optimal sign pattern should increase with the number of measurement vector. Hence, we can expect the performance improvement of ReMBo with increasing number of measurement vectors [9].

**III. CONCATENATE MMV AND BOOST ALGORITHM**

In practice, the recovery performance of S-OMP and ReMo are often saturated after a certain number of measurement vectors and there usually exists a significant performance gap from the deterministic bound, as illustrated in Fig. 1(a)(b). Such performance saturation of ReMBo has been suggested by van der Berg et al [9], who observed the difficulty in visiting all desired sign patterns by drawing a finite number of random vectors \( w \). In fact, some specific sign patterns are more dominantly observed [9].
Fig. 1. Recovery rates of (a) SOMP, and (b) ReMBo-BP. The vertical lines denote the deterministic upper limit of recoverable sparsity (Eq. (5)) \((m = 20, n = 30)\).

Therefore, rather than reducing the MMV into SMV, we are interested in directly exploiting the structure of matrix \(X\) for MMV recovery.

Suppose the unknown matrix \(X \in \mathbb{R}^{n \times r}\) with \(R(X) = k\) assumes the following form:

\[
X = \begin{bmatrix}
    K^T \\
    O_{(n-k) \times r}
\end{bmatrix},
\]  

(10)

where \(K \in \mathbb{R}^{r \times k}\) denotes non-zero submatrix, and \(O \in \mathbb{R}^{(n-k) \times r}\) is \((n-k) \times r\) zero matrix, respectively. Now, consider a QR decomposition of matrix \(K\). If \(r \leq k\), the resultant QR decomposition is given by

\[
K = QR = \begin{bmatrix}
    q_1 & \ldots & q_r
\end{bmatrix}
\]  

(11)

where \(Q = \begin{bmatrix}
    q_1 & \ldots & q_r
\end{bmatrix}\) denotes an \(r \times r\) orthonormal matrix and \(R \in \mathbb{R}^{r \times k}\) is a coefficients matrix from Gram-Schmidt orthonormalization, respectively. From Eq. (11), we can easily see that the number of non-zero elements in the coefficient matrix \(R\) is at most \(kr - r(r-1)/2\). Second, if \(r > k\), the
corresponding QR decomposition is given by

$$K = QR = \begin{bmatrix} q_1 & \cdots & q_r \end{bmatrix}$$

where the coefficient matrix $R \in \mathbb{R}^{r \times k}$ has at most $k(k+1)/2$ non-zero elements in this case.

In general, MMV problem can be represented as

$$Y = AX = ASQ^T$$

where $Q$ is the $r \times r$ orthonormal matrix from QR decomposition, and sparse matrix $S$ is given by

$$S = \begin{bmatrix} R^T \\ O^{(n-k)\times r} \end{bmatrix}$$

Now, multiplying $Q$ to the both sides of Eq. (13) provides us

$$YQ = AS.$$  \hspace{1cm} (15)

Let $\text{VEC}\{Z\}$ the vector of concatenated columns of a matrix $Z$. Using this, we can expand MMV problem to an SMV problem:

$$YQ = AS \iff (I^{r \times r} \otimes A)\text{VEC}\{S\} = \text{VEC}\{YQ\},$$  \hspace{1cm} (16)

where $I^{r \times r}$ denotes the $r \times r$ identity matrix. Furthermore, the structure of $R$ matrix in Eq. (11) and Eq. (12) tell us that the total sparsity of $\text{VEC}\{S\}$ is upper limited by:

$$\|\text{VEC}\{S\}\|_0 \leq \begin{cases} r \left( k - \frac{r-1}{2} \right), & \text{if } 1 \leq r \leq k; \\
\frac{k(k+1)}{2}, & \text{if } r > k. \end{cases}$$  \hspace{1cm} (17)

The same analysis can be applied even for more general situation where the location of non-zero rows are not clustered. Specifically, for general matrix $X$ with $\mathcal{R}(X) = k$, there always exist a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that

$$X = P \begin{bmatrix} K^T \\ O^{(n-k)\times r} \end{bmatrix} = PSQ^T.$$  \hspace{1cm} (18)

Hence, the corresponding SMV problem can be easily obtained similar to Eq. (16):

$$(I^{r \times r} \otimes A)\text{VEC}\{PS\} = \text{VEC}\{YQ\}. \hspace{1cm} (19)$$
Since the permutation does not change the sparsity level, we can easily see that the total sparsity of $\text{VEC}\{\mathbf{PS}\}$ is also upper limited by Eq. (17). Of course, the original MMV problem can be concatenated to a high dimension SMV without multiplying $\mathbf{Q}$ matrix as

$$(\Gamma^{\times r} \otimes \mathbf{A})\text{VEC}\{\mathbf{X}\} = \text{VEC}\{\mathbf{Y}\}. \tag{20}$$

However, in this case the total sparsity of $\text{VEC}\{\mathbf{X}\}$ is upper bounded by $kr$. Figs. 2(a)(b) illustrates the total and average sparsity versus the number of measurement vectors for the concatenated SMV problems Eq. (16) and Eq. (20). Average sparsity is defined as the total sparsity divided by the number of measurement vectors. We can easily see that the formulation Eq. (20) has no gain whereas the average sparsity keeps reduced with increasing $r$ for the case of Eq. (16).

![Graphs showing total and average sparsity versus number of measurement vectors.](image)

Fig. 2. (a) Total sparsity and (b) average sparsity versus the number of measurement vectors when $k = 10$.

However, the main problem of the proposed approach is that the orthonormal matrix $\mathbf{Q}$ is not known a priori since the computation of $\mathbf{Q}$ requires the exact knowledge of the unknown matrix $\mathbf{X}$. Therefore, we randomly generate multiple candidates of orthonormal matrices and choose the one that yields best reconstruction performance. First, note that the sparsity level does not increase by multiplying $\mathbf{Q}$. Furthermore, the resultant sparsity level should be in between that of Eq. (16) and Eq. (20) as illustrated in Fig. 2(a)(b), since random $\mathbf{Q}$ matrices approximates the orthonormal matrix in QR decomposition of unknown $\mathbf{X}$. The pseudo-code implementation of the proposed algorithm is then as follows:

Algorithm CoMBBo

1) Initialization: set iteration count $t = 0$ and $\Lambda = \{1, 2, \ldots, n\}$.
2) While ($t \leq \text{MaxIter}$) do
a) Draw a random orthonormal matrix $Q \in \mathbb{R}^{r \times r}$.

b) Convert MMV to SMV problem by multiplying $Q$ and concatenating the columns of $YQ$:

$$\text{VEC}\{YQ\} = (I_{r} \otimes A)\text{VEC}\{S\}$$

(21)

c) Solve Eq. (21) using SMV technique to get the estimate $\hat{S}$. Let $\hat{\Lambda}$ denote the set of indices of the row vectors of $\hat{S}$ whose $l_2$ norm is among the $m$ largest.

d) $\hat{\Lambda} \leftarrow$ the set of non-zero row indices of $(A_{\hat{\Lambda}})^{\dagger}Y$.

e) If $|\hat{\Lambda}| \leq |\Lambda|$, then $\Lambda \leftarrow \hat{\Lambda}$. Set $t = t + 1$.

3) Let $\hat{X}^{\Lambda}$ denotes the submatrix of $\hat{X}$ by collecting rows whose indices belong to $\Lambda$. Then, the final MMV solution is calculated as:

$$\begin{cases}
\hat{X}^{\Lambda} = A_{\Lambda}^{\dagger}Y \\
\hat{X}^i = 0, i \notin \Lambda
\end{cases}$$

(22)

Note the similarity of ReMBo and CoMBo pseudo-code implementation. In CoMBo, rather than reducing the MMV to SMV, we expand an MMV to a higher dimensional SMV problem by multiplying random orthonormal matrix $Q$ and concatenates the resulting column vectors. Besides the slight differences in termination condition, the remaining steps of CoMBo are basically same as those of ReMBo. Hence, we conjecture that the role of random $Q$ may have the same role to assign the best sign pattern. Specifically, CoMBo may exploit the possible combination of all sign patterns simultaneously by concatenating to a higher dimensional vector rather than reducing to the same dimensional vector.

IV. SIMULATION RESULTS

In order to quantify the performance, we declare an algorithm is successful, if the following condition is satisfied:

$$\|X - \hat{X}\|_2 \leq 10^{-5}\|X\|_2,$$

(23)

where $X$ denotes the ground-truth and $\hat{X}$ is its estimate, respectively. The simulation parameters are as following: $m = 20, n = 30$ and the number of measurement vectors are $r = 1, 2, 8, 16, k = 1 \sim 20$, respectively. For ReMBo, we set $\epsilon = 10^{-5}$. The $MaxIter$ is set to 20 for ReMBo and 5 for CoMBo, respectively. The SMV algorithm used in both ReMBo and CoMBo is $l_1$-MAGIC [16]. All elements of sensing matrix $A$ and the nonzero elements of $X$ are drawn from the zero mean unit variance Gaussian distribution. Then, each column of $A$ are normalized to have an unit norm, and the indices of nonzero values of $X$ were chosen randomly. The success rate are averaged for 500 experiments.
Figs. 1(a)(b) illustrate the recovery performance of S-OMP and ReMBo along with the deterministic bound in Eq. (5), respectively. Even though we can observe the performance improvement of S-OMP and ReMBo with increasing number of measurement vectors, there still exist significant performance gaps from the deterministic bound especially for large $r$. However, such performance gap can be significantly reduced using the CoMBo algorithm as shown in Fig. 3(a), which clearly show that the recovery performance of CoMBo algorithm approaches the bounds as $r$ grows. Furthermore, with high probabilities, CoMBo even outperforms the deterministic bound of Eq. (5) for the case of $r \geq 8$. Similar average performance behavior has been observed in SMV problems [17], which shows that the sparsity more than $\text{spark}(A)/2$ can be still recovered with high probability. Fig. 3(b) illustrates that CoMBo outperforms other conventional MMV algorithms. Here, we also illustrate the performance of simple concatenation in Eq. (20). Thanks to the concatenation, the average performance is slightly better than ReMBo. However, the deterministic and average performance of CoMBo outperform these simple concatenation, which confirms the importance of boosting step.

![Graph](image)

Fig. 3. (a) Recovery rates of CoMBo, and (b) comparison of various MMV algorithms when $r = 5$. The vertical lines of (a) denotes the deterministic upper limit of recoverable sparsity (Eq. (5)).

V. CONCLUSION

The conventional MMV algorithms such as S-OMP and ReMBo do not achieve the fundamental performance bound even with the increasing number of measurement vectors. In order to overcome the drawbacks, this paper proposed a novel CoMBo algorithm that concatenates MMV to a larger
dimensional SMV problem and boosts it by multiplying random orthonormal matrix. The CoMBo algorithm is a process of approximating the QR decomposition of the unknown solution matrix that allows reduction of average sparsity as the number of measurement vector increases. We compared the similarity between ReMBo and our CoMBo algorithm and showed that CoMBo is a generalization of ReMBo. Extensive simulation results demonstrate that CoMBo outperforms all existing methods and its performance approaches the deterministic limit as the measurement vector increases.

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