Enveloping $\sigma$-$C^*$-algebra of a smooth Frechet algebra crossed product by $\mathbb{R}$, $K$-theory and differential structure in $C^*$-algebras

SUBHASH J BHATT

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India
E-mail: subhashbhaib@yahoo.co.in

MS received 9 August 2004

Abstract. Given an $m$-tempered strongly continuous action $\alpha$ of $\mathbb{R}$ by continuous $^*$-automorphisms of a Frechet $^*$-algebra $A$, it is shown that the enveloping $\sigma$-$C^*$-algebra $E(S(\mathbb{R},A^\infty, \alpha))$ of the smooth Schwartz crossed product $S(\mathbb{R},A^\infty, \alpha)$ of the Frechet algebra $A^\infty$ of $C^\infty$-elements of $A$ is isomorphic to the $\sigma$-$C^*$-crossed product $C^*(\mathbb{R},E(A), \alpha)$ of the enveloping $\sigma$-$C^*$-algebra $E(A)$ of $A$ by the induced action. When $A$ is a hermitian $Q$-algebra, one gets $K$-theory isomorphism $RK_\ast(S(\mathbb{R},A^\infty, \alpha)) = K_\ast(C^*(\mathbb{R},E(A), \alpha))$ for the representable $K$-theory of Frechet algebras. An application to the differential structure of a $C^*$-algebra defined by densely defined differential seminorms is given.

Keywords. Frechet $^*$-algebra; enveloping $\sigma$-$C^*$-algebra; smooth crossed product; $m$-tempered action; $K$-theory; differential structure in $C^*$-algebras.

1. Introduction

Given a strongly continuous action $\alpha$ of $\mathbb{R}$ by continuous $^*$-automorphisms of a Frechet $^*$-algebra $A$, several crossed product Frechet algebras can be constructed [1,4,11]. They include the smooth Schwartz crossed product $S(\mathbb{R},A, \alpha)$, the $L^1$-crossed products $L^1(\mathbb{R},A, \alpha)$ and $L^1_{\sigma}(\mathbb{R},A, \alpha)$, and the $\sigma$-$C^*$-crossed product $C^*(\mathbb{R},A, \alpha)$. Let $E(A)$ denote the enveloping $\sigma$-$C^*$-algebra of $A$ [4]; and $(A^\infty, \tau)$ denote the Frechet $^*$-algebra consisting of all $C^\infty$-elements of $A$ with the $C^\infty$-topology $\tau$ [14, Appendix I]. The following theorem shows that for a smooth action, the enveloping algebra of smooth crossed product is the continuous crossed product of the enveloping algebra.

**Theorem 1.** Let $\alpha$ be an $m$-tempered strongly continuous action of $\mathbb{R}$ by continuous $^*$-automorphisms of a Frechet $^*$-algebra $A$. Let $A$ admit a bounded approximate identity which is contained in $A^\infty$ and which is a bounded approximate identity for the Frechet algebra $A^\infty$. Then $E(S(\mathbb{R},A^\infty, \alpha)) \cong E(L^1_{\sigma}(\mathbb{R},A^\infty, \alpha)) \cong C^*(\mathbb{R},E(A), \alpha)$. Further, if $\alpha$ is isometric, then $E(L^1(\mathbb{R},A, \alpha)) \cong C^*(\mathbb{R},E(A), \alpha)$.

Notice that neither $L^1(\mathbb{R},A, \alpha)$ nor $S(\mathbb{R},A^\infty, \alpha)$ need be a subalgebra of $C^*(\mathbb{R},E(A), \alpha)$. A particular case of Theorem 1 when $A$ is a dense subalgebra of $C^*$-algebra has been treated in [4]. Let $RK_\ast$ (respectively $K_\ast$) denote the representable $K$-theory functor (respectively $K$-theory functor) on Frechet algebras [10]. We have the following isomorphism of $K$-theory, obtained without direct appeal to spectral invariance.
**Theorem 2.** Let $A$ be as in the statement of Theorem 1. Assume that $A$ is hermitian and a $Q$-algebra. Then $RK_* (S(\mathbb{R}, \mathcal{A}_\alpha^\sigma), \alpha) \cong K_* (\mathcal{C}^*(\mathbb{R}, E(A), \alpha))$. Further if the action $\alpha$ is isometric on $A$, then $RK_* (L^1(\mathbb{R}, A, \alpha)) \cong K_* (\mathcal{C}^*(\mathbb{R}, E(A), \alpha))$.

We apply this to the differential structure of a $C^*$-algebra. Let $\alpha$ be an action of $\mathbb{R}$ on a $C^*$-algebra $A$ leaving a dense $^*$-subalgebra $\mathcal{A}$ invariant. Let $T \sim (T_0)^\infty$ be a differential $^*$-seminorm on $\mathcal{A}$ in the sense of Blackadar and Cuntz [5] with $T_0(x) = \| \cdot \|$ the $C^*$-norm from $A$. Let $T$ be $\alpha$-invariant. Let $\mathcal{A}_k$ be the completion of $\mathcal{A}$ in the submultiplicative $^*$-norm $p_k(x) = \sum_{i=0}^k T_i(x)$. The differential Frechet $^*$-algebra defined by $T$ is $\mathcal{A}_T = \lim_inj \mathcal{A}_k$, the inverse limit of Banach $^*$-algebras $\mathcal{A}_k$.

Now consider $\mathcal{A}$ to be the $\alpha$-invariant smooth envelope of $\mathcal{A}$ defined to be the completion of $\mathcal{A}$ in the collection of all $\alpha$-invariant differential $^*$-seminorms. Notice that neither $\mathcal{A}_T$ nor $\mathcal{A}$ is a subalgebra of $A$, though each admits a continuous surjective $^*$-homomorphism onto $A$ induced by the inclusion $\mathcal{A} \to A$. There exists actions of $\mathbb{R}$ on each of $\mathcal{A}_T$ and $\mathcal{A}$ induced by $\alpha$. The following is a smooth Frechet analogue of Connes' analogue of Thom isomorphism [7]. It supplements an analogues result in [11].

**Theorem 3.**

(a) $RK_* (S(\mathbb{R}, \mathcal{A}_\alpha^\sigma), \alpha) = K_{s+1}(A)$.

(b) Assume that $\mathcal{A}$ is metrizable. Then $RK_* (S(\mathbb{R}, \mathcal{A}, \alpha)) = K_{s+1}(A)$.

2. Preliminaries and notations

A Frechet $^*$-algebra $(A, t)$ is a complete topological involutive algebra $A$ whose topology $t$ is defined by a separating sequence $\{ \| \cdot \|_n; n \in \mathbb{N} \}$ of seminorms satisfying $\| xy \|_n \leq \| x \|_n \| y \|_n$, $\| x^* \|_n = \| x \|_n$, $\| x \|_n \leq \| x \|_{n+1}$ for all $x, y$ in $A$ and all $n$ in $\mathbb{N}$. If each $\| \cdot \|_n$ satisfies $\| x^* x \|_n = \| x \|_n^2$ for all $x$ in $A$, then $A$ is a $\sigma$-$C^*$-algebra [9]. $A$ is called a $Q$-algebra if the set of all quasi-regular elements of $A$ is an open set. For each $n$ in $\mathbb{N}$, let $A_n$ be the Hausdorff completion of $(A, \| \cdot \|_n)$. There exists norm decreasing surjective $^*$-homomorphisms $\pi_n: A_{n+1} \to A_n$, $\pi_n(x + \ker \| \cdot \|_{n+1}) = x + \ker \| \cdot \|_n$ for all $x$ in $A$. Then the sequence

$$A_1 \leftarrow \pi_1 A_2 \leftarrow \pi_2 A_3 \leftarrow \pi_3 \ldots \leftarrow \pi_{n-1} A_n \leftarrow \pi_n A_{n+1} \leftarrow \ldots$$

is an inverse limit sequence of Banach $^*$-algebras and $A = \lim_{\longleftarrow} A_n$, the inverse limit of Banach $^*$-algebras. Let $\text{Rep}(A)$ be the set of all $^*$-homomorphisms $\pi: A \to B(H_\pi)$ of $A$ into the $C^*$-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces $H_\pi$. Let

$$\text{Rep}_n(A) := \{ \pi \in \text{Rep}(A) : \text{ there exists } k > 0 \text{ such that } \| \pi(x) \| \leq k \| x \|_n \text{ for all } x \}. $$

Then $|x|_n := \sup\{ \| \pi(x) \| : \pi \in \text{Rep}_n(A) \}$ defines a $C^*$-seminorm on $A$. The star radical of $A$ is

$$\text{srad}(A) = \{ x \in A : |x|_n = 0 \text{ for all } n \in \mathbb{N} \}.$$ 

The enveloping $\sigma$-$C^*$-algebra $(E(A), \tau)$ of $A$ is the completion of $A/\text{srad}(A)$ in the topology $\tau$ defined by the $C^*$-seminorms $\{ | \cdot |_n : n \in \mathbb{N} \}$, $|x + \text{srad}(A)|_n = |x|_n$ for $x$ in $A$. 

Let $\alpha$ be a strongly continuous action of $\mathbb{R}$ by continuous $^*$-automorphisms of $A$. The $C^\infty$-elements of $A$ for the action $\alpha$ are

$$A^\infty := \{ x \in A : t \rightarrow \alpha_t(x) \text{ is a } C^\infty\text{-function} \}.$$ 

It is a dense $^*$-subalgebra of $A$ which is a Frechet algebra with the topology defined by the submultiplicative $^*$-seminorms

$$\|x\|_{k,n} = \|x\|_n + \sum_{j=0}^k (1/j!) \|\delta^j x\|_n, \quad n \in \mathbb{N}, \ k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$$

where $\delta$ is the derivation $\delta(x) = (d/dr) \alpha_t(x)|_{t=0}$. By Theorem A.2 of [14], $\alpha$ leaves $A^\infty$ invariant and each $\alpha_t$ restricted to $A^\infty$ gives a continuous $^*$-automorphism of the Frechet algebra $A^\infty$. The action $\alpha$ is smooth if $A^\infty = A$.

2.1 Smooth Schwartz crossed product [14]

Assume that $\alpha$ is $m$-tempered in the sense that for each $n \in \mathbb{N}$, there exists a polynomial $P_n$ such that $\|\alpha_t(x)\|_n \leq P_n(r) \|x\|_n$ for all $r \in \mathbb{R}$ and all $x \in A$. Let $S(\mathbb{R})$ be the Schwartz space. The completed (projective) tensor product $S(\mathbb{R}) \otimes A = S(\mathbb{R}, A)$ consisting of $A$-valued Schwartz functions on $\mathbb{R}$ is a Frechet algebra with the twisted convolution

$$(f * g)(r) = \int_{\mathbb{R}} f(s) \alpha_t(g(r-s)) \, ds$$

called the smooth Schwartz crossed product by $\mathbb{R}$ denoted by $S(\mathbb{R}, A, \alpha)$. The algebra $S(\mathbb{R}, A^\infty, \alpha)$ is a Frechet $^*$-algebra with the involution $f^*(r) = \alpha_t(f(-r)^*)$ (Corollary 4.9 of [14]) whose topology $\tau_\alpha$ is defined by the seminorms

$$\|f\|_{n,l,m} = \sum_{l+j=n} \int_{\mathbb{R}} (1+|r|)^l \|f^{(j)}(r)\|_{l,m} \, dr, \quad n \in \mathbb{Z}^+, l \in \mathbb{Z}^+, m \in \mathbb{N}$$

where

$$\|f^{(j)}(r)\|_{l,m} = \sum_{k=0}^j \|1/k!\| \|\delta^k(\alpha_t((d^l/dr^l)f(r)))\|_{s=0} \|m\|$$

(Theorem 3.1.7 of [14], [11]). These seminorms are submultiplicative if $\alpha$ is isometric on $A$ in the sense that $\|\alpha_t(x)\|_n = \|x\|_n$ for all $n \in \mathbb{N}$ and all $x \in A$.

2.2 $L^1$-crossed products [11,14]

Let $F_d$ be the set of all functions $f : \mathbb{R} \rightarrow A$ for which

$$\|f\|_{d,m} := \int_{\mathbb{R}} (1+|r|)^d \|f(r)\|_{m} \, dr < \infty$$

for all $m \in \mathbb{N}$. Here $\int$ denotes the upper integral. Let $L_d$ be the closure in $F_d$ of the set of all measurable simple functions $f : \mathbb{R} \rightarrow A$ in the topology on $F_d$ given by the seminorms $\{\|\cdot\|_{d,m} : m \in \mathbb{N}\}$. Let $N_d = \cap\{\ker\|\cdot\|_{d,m} : m \in \mathbb{N}\}$. Then $N_d = N_{d+1}$; $L_d := L_d/N_d$ is complete in $\{\|\cdot\|_{d,m} : m \in \mathbb{N}\}$ and $L_{d+1} \rightarrow L_d$ continuously. The space of $|\cdot|\text{-rapidly}$
vanishing $L^1$-functions from $\mathbb{R}$ to $A$ is $L^1((\mathbb{R},A,\alpha) := \cap \{L_d: d \in \mathbb{Z}^+\}$, a Frechet algebra with the topology given by the seminorms $\{||\cdot||_{d,m}: m \in \mathbb{N}, d \in \mathbb{Z}^+\}$ and with twisted convolution. Assume that $\alpha$ is isometric on $(A,\{||\cdot||_n\})$. Then the completed projective tensor product $L^1(\mathbb{R}) \otimes_A L^1(\mathbb{R},A)$ is a Frechet $^*$-algebra with twisted convolution and the involution $f \to f^*$. This $L^1$-crossed product is denoted by $L^1(\mathbb{R},A,\alpha)$. Notice that $\alpha$ is isometric on $(A^\infty,\{||\cdot||_{r,m}\})$ also, so that the Frechet $^*$-algebra $L^1(\mathbb{R},A^\infty,\alpha)$ is defined; and then the induced actions $(\alpha,f)(s) = \alpha_r(f(s))$ on $L^1(\mathbb{R},A^\infty,\alpha)$ and on $L^1(\mathbb{R},A,\alpha)$ are also isometric.

2.3 $\sigma$-$C^*$-crossed product

Assume that $\alpha$ is isometric. We define the $\sigma$-$C^*$-crossed product $C^*(\mathbb{R},A,\alpha)$ of $A$ by $\mathbb{R}$ to be the enveloping $\sigma$-$C^*$-algebra $E(L^1(\mathbb{R},A,\alpha))$ of $L^1(\mathbb{R},A,\alpha)$.

3. Technical lemmas

Lemma 3.1. Let $\alpha$ be $m$-tempered on $A$. Then $\alpha$ extends as a strongly continuous isometric action of $\mathbb{R}$ by continuous $^*$-automorphisms of the $\sigma$-$C^*$-algebra $E(A)$.

Proof. By the $m$-temperedness of $\alpha$, for each $n \in \mathbb{N}$, there exists a polynomial $P_n$ such that for all $x \in A$ and all $r \in \mathbb{R}$, $||\alpha_r(x)||_n \leq P_n(r)||x||_n$. Let $r \in \mathbb{R}$. Let $x \in \text{Rad}(A)$. Then for all $\pi \in \text{Rep}(A)$, $\pi(x) = 0$, so that $\sigma(\alpha_r(x)) = 0$ for all $\sigma \in \text{Rep}(A)$, hence $\alpha_r(x) \in \text{Rad}(A)$. Thus $\alpha_r(\text{Rad}(A)) \subseteq \text{Rad}(A)$, and the map

\[ \tilde{\alpha}_r : A/\text{Rad}(A) \to A/\text{Rad}(A), \quad \tilde{\alpha}_r([x]) = [\alpha_r(x)] \]

where $[x] = x + \text{Rad}(A)$, is a well-defined $^*$-homomorphism. Further, let $\tilde{\alpha}_r[x] = 0$. Then $\alpha_r(x) \in \text{Rad}(A)$. Hence $x = \alpha_{-r}(\alpha_r(x)) \in \text{Rad}(A)$, $[x] = 0$. Thus $\tilde{\alpha}_r$ is one-to-one, which is clearly surjective. Now, for each $n \in \mathbb{N}$, and for all $x \in A$,

\[ ||\tilde{\alpha}_r[x]||_n = ||\alpha_r(x)||_n \leq ||\alpha_r(x)||_n \leq P_n(r)||x||_n. \]

Since, by definition, $||\cdot||_n$ is the greatest $C^*$-seminorm on $A/\text{Rad}(A)$ satisfying that for some $k_n > 0$, $||z||_n \leq k_n||z||_n$ for all $z \in A$, it follows that $||\tilde{\alpha}_r[x]||_n \leq ||x||_n$ for all $x \in A$. Hence

\[ ||x||_n \leq ||\alpha_{-r}(\alpha_r(x))||_n = ||\alpha_{-r}(\alpha_r(x))||_n \leq ||\alpha_r(x)||_n = ||\alpha_r(x)||_n \]

showing that $||\tilde{\alpha}_r[x]||_n = ||x||_n$ for all $x \in A$, $r \in \mathbb{R}, n \in \mathbb{N}$. It follows that $\tilde{\alpha}_r$ extends as a $^*$-automorphism $\tilde{\alpha}_r : E(A) \to E(A)$ satisfying $||\tilde{\alpha}_r(z)||_n = ||z||_n$ for all $z \in A$ and all $n \in \mathbb{N}$; and $\tilde{\alpha} : \mathbb{R} \to \text{Aut}^*(E(A))$, $r \to \tilde{\alpha}_r$ defines an isometric action of $\mathbb{R}$ on $E(A)$. We verify that $\tilde{\alpha}$ is strongly continuous. Let $z \in E(A)$. It is sufficient to prove that the map $f : \mathbb{R} \to E(A)$, $f(r) = \tilde{\alpha}_r(z)$ is continuous at $r = 0$. Choose $z_n = [x_n]$ in $A/\text{Rad}(A)$ such that $z_n \to z$ in $E(A)$. Fix $k \in \mathbb{N}$, $\varepsilon > 0$. Choose $n_0$ in $\mathbb{N}$ such that $||z_{n_0} - z||_k < \varepsilon / 3$ with $z_{n_0} = [x_{n_0}]$. Then for all $r \in \mathbb{R}$, $||\tilde{\alpha}_r(z) - \tilde{\alpha}_r(z_{n_0})||_k = ||z - z_{n_0}||_k < \varepsilon / 3$. Since $\alpha$ is strongly continuous, there exists a $\delta > 0$ such that $|r| < \delta$ implies that $||\alpha_r(x_0) - x_0||_k < \varepsilon / 3$. Then for all such $r$, $|\tilde{\alpha}_r(z) - z||_k < \varepsilon$ showing the desired continuity of $f$. This completes the proof. \[ \square \]
Notation. Henceforth we denote the action $\tilde{\alpha}$ by $\alpha$.

A covariant representation of the Frechet algebra dynamical system $(\mathbb{R}, A, \alpha)$ is a triple $(\pi, U, H)$ such that

(a) $\pi: A \to B(H)$ is a $^*$-homomorphism;
(b) $U: \mathbb{R} \to \mathcal{U}(H)$ is a strongly continuous unitary representation of $\mathbb{R}$ on $H$; and
(c) $\pi(\alpha_t(x)) = U_t \pi(x) U_t^*$ for all $x \in A$ and all $t \in \mathbb{R}$.

The following is an analogue of Proposition 7.6.4, p. 257 of [12] which can be proved along the same lines. Let $C_c^\infty(\mathbb{R}, A^\infty) = C_c^\infty(\mathbb{R}) \otimes A^\infty$ (completed projective tensor product) be the space of all $A^\infty$-valued $C^\infty$-functions on $\mathbb{R}$ with compact supports.

Lemma 3.2. Let $A$ have a bounded approximate identity $(e_i)$ contained in $A^\infty$ which is also a bounded approximate identity for the Frechet algebra $A^\infty$. (In particular, let $A$ be unital.)

(a) If $(\pi, U, H)$ is a covariant representation of $(\mathbb{R}, A^\infty, \alpha)$, then there exists a non-degenerate $^*$-representation $(\pi \times U, H)$ of $S(\mathbb{R}, A^\infty, \alpha)$ such that

$$(\pi \times U)y = \int_{\mathbb{R}} \pi(y(t))U_t dt$$

for every $y$ in $C_c^\infty(\mathbb{R}, A^\infty)$. The correspondence $(\pi, U, H) \to (\pi \times U, H)$ is bijective onto the set of all non-degenerate $^*$-representations of $S(\mathbb{R}, A^\infty, \alpha)$.

(b) Let $\alpha$ be isometric. Then the above gives a one-to-one correspondence between the covariant representations of $(\mathbb{R}, A, \alpha)$ and non-degenerate $^*$-representations of each of $L^1(\mathbb{R}, A^\infty, \alpha)$ and $L^1(\mathbb{R}, A, \alpha)$.

Lemma 3.3. $E(A^\infty) = E(A)$; and for all $k$ in $\mathbb{Z}^+$, $n$ in $\mathbb{N}$, $\|x\|_n = \|x\|_n$.

Proof. Consider the inverse limit $A = \lim A_n$ as in the Introduction. Since $\alpha$ satisfies $\|\alpha(x)\|_n \leq P_n(r)\|x\|_n$ for all $x \in A$, all $r \in A$ and all $n \in \mathbb{N}$, it follows that for each $n$, $\alpha$ *extends* uniquely as a strongly continuous action $\alpha^{(n)}$ of $\mathbb{R}$ by continuous $^*$-automorphisms of the Banach $^*$-algebra $A_n$. Let $(A_{n,m}, \|\cdot\|_{n,m})$ be the Banach algebra consisting of all $C^m$-elements $y$ of $A_n$ with the norm $\|y\|_{n,m} = \|y\|_n + \sum_{i=1}^m (1/i!)\|\delta^i(y)\|_n$. Let $(A^m_n, \{\|\cdot\|_{m,n}; m \in \mathbb{Z}^+\})$ be the Frechet algebra consisting of all $C^m$-elements of $A_n$ for the action $\alpha^{(n)}$. Then

$$A^\infty = \lim A_n^\infty = \lim \lim A_{n,m,n} = \lim A_{n,m,n}.$$ 

By Theorem 2.2 of [15], each $A_{m,n}$ is dense and spectrally invariant in $A_n$. Hence each $A_{n,m}$ is a $Q$-normed algebra in the norm $\|\cdot\|_n$ of $A_n$.

Let $\pi: A^\infty \to B(H)$ be a $^*$-representation of $A$ on a Hilbert space $H$. Since the topology of $A^\infty$ is determined by the seminorms

$$\|x\|_{n,n} = \|x\|_n + \sum_{j=1}^n \frac{1}{j!}\|\delta^j(x)\|_n, \quad n \in \mathbb{N}$$

it follows that for some $k > 0$, $\|\pi(x)\| \leq k\|x\|_{n,n}$ for all $a \in A^\infty$. Hence $\pi$ defines a $^*$-homomorphism $\pi: (A_{n,m}, \|\cdot\|_{n,m}) \to B(H)$ satisfying $\|\pi(x)\| \leq k\|x\|_{n,m}$ for all $x$ in $A_{n,m}$. 
Since \((A, \| A \|)\) is a \(Q\)-normed \(\ast\)-algebra, this map \(\pi\) is continuous in the norm \(\| \cdot \|_n\) on \(A_n\). In fact, for all \(x \in A^\infty\),

\[
\| \pi(x) \|^2 = \| \pi(x^\ast x) \| = r_{B(H)}(\pi(x^\ast x)) \leq r_{A_n}(\pi(x^\ast x + \ker \| n,n)) \\
\leq \| x^\ast x + \ker \|_n = \| x \|^2.
\]

Thus \(\| \pi(x) \| \leq \| x \|_n\) for all \(x \in A^\infty\). Since \(A^\infty\) is dense in \(A\), \(\pi\) can be uniquely extended as a \(\ast\)-representation \(\pi: A \rightarrow B(H)\) satisfying that \(\| \pi(x) \| \leq \| x \|_n\) for all \(x \in A\). Then by the definition of the \(C^\ast\)-seminorm \(\| \cdot \|_n\) on \(A, \pi\) extends as a continuous \(\ast\)-homomorphism \(\tilde{\pi}: E(A) \rightarrow B(H)\) such that \(\| \tilde{\pi}(x) \| \leq \| x \|_n\) for all \(x \in E(A)\). This also implies that \(E(A^\infty) = E(A)\) and \(\| \cdot \|_n = \| \cdot \|_n\) for all \(n,m\).

**Lemma 3.4.** Let \(B\) be a \(\sigma\)-\(C^\ast\)-algebra. Let \(j: A \rightarrow E(A)\) be \(j(x) = x + \text{srad}(A)\). Let \(\pi: A \rightarrow B\) be a \(\ast\)-homomorphism. Then there exists a unique \(\ast\)-homomorphism \(\tilde{\pi}: E(A) \rightarrow B\) such that \(\pi = \tilde{\pi} \circ j\).

This follows immediately by taking \(B = \lim_{\downarrow n} B_n\), where \(B_n\)'s are \(C^\ast\)-algebras, and by the universal property of \(E(A)\).

**4. Proof of Theorem 1**

**Step 1.** \(\text{Rep}(S(\mathbb{R}, A^\infty, \alpha)) = \text{Rep}(S(\mathbb{R}, E(A), \alpha)) = \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))\) up to one-to-one correspondence.

By Lemma 3.1, the Frechet algebras \(S(\mathbb{R}, E(A), \alpha)\) and \(L^1(\mathbb{R}, E(A), \alpha)\) are \(\ast\)-algebras with the continuous involution \(y \mapsto y^\ast, y^\ast(t) = \alpha_t(y(-t))\). By Lemma 3.2, \(\text{Rep}(S(\mathbb{R}, E(A), \alpha)) = \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))\) each identified with the set of all covariant representations. Let \(\rho: S(\mathbb{R}, A^\infty, \alpha) \rightarrow B(H)\) be in \(\text{Rep}(S(\mathbb{R}, A^\infty, \alpha))\). There exists \(c > 0\) and appropriate \(n, l, m\) such that for all \(y\),

\[
\| \rho(y) \| \leq c \| y \|_{n, l, m} = c \sum_{i+j=n} \int_R (1 + |r|)^i |y^j(r)| |_{i, m} dr.
\]

By Lemma 3.2, there exists a covariant representation \((\pi, U, H)\) of \((\mathbb{R}, A^\infty, \alpha)\) on \(H\) such that \(\rho = \pi \times U\). Thus \(\pi: A^\infty \rightarrow B(H)\) is a \(\ast\)-homomorphism and \(U: \mathbb{R} \rightarrow \mathcal{B}(H)\) is a strongly continuous unitary representation such that

(i) \(\rho(f) = \int_R \pi(f(t)) U dt\) for all \(f \in S(\mathbb{R}, A^\infty, \alpha)\),

(ii) \(\pi(\alpha_t(x)) = U_t \pi(x) U_t^\ast\) for all \(x \in A^\infty, t \in \mathbb{R}\),

(iii) there exists \(K > 0\) such that \(\| \pi(x) \| \leq k \| x \|_{i, m}\) for all \(x \in A^\infty\).

The \(l, m\) in (iii) are the same as in (1). Let \(\{ \| \cdot \|_{i, m} : i \in \mathbb{Z}^\ast, m \in \mathbb{N}\}\) be the sequence of \(C^\ast\)-seminorms on \(A^\infty\) (and also on \(E(A^\infty)\) via \(\text{srad}A^\infty\)) which are defined by the submultiplicative \(\ast\)-seminorms \(\| \cdot \|_{i, m}\) on \(\mathbb{Z}^\ast, m \in \mathbb{N}\). Then \(\| \cdot \|_{i, m}\) is the greatest \(C^\ast\)-seminorm on \(A^\infty\) satisfying that there exists \(M = M_{i, m} > 0\) such that \(\| \cdot \|_{i, m} \leq M \| \cdot \|_{i, m}\). Hence by (iii) above, \(\pi\) can be uniquely extended as a continuous \(\ast\)-homomorphism \(\tilde{\pi}: E(A^\infty) \rightarrow B(H)\) such that \(\tilde{\pi} \circ j = \pi(x)\) for all \(x \in A^\infty\); and

\[
\| \tilde{\pi}(x) \| \leq \| x \|_{i, m}\) for all \(x \in E(A^\infty)\).
Here $j$ is the map $j: A^\infty \to E(A^\infty)$. Let $I$ denote $\max(I,M)$. Then we have
\[
\|\rho(y)\| \leq c\|y\|_{n,I} \quad \text{for all } y \in S(\mathbb{R}, A^\infty, \alpha);
\|\pi(x)\| \leq k\|x\|_{n,I} \quad \text{for all } x \in A^\infty;
\|\tilde{\pi}(z)\| \leq |z|_{n,I} \quad \text{for all } z \in E(A^\infty).
\] (5)

By Lemma 3.3, $\tilde{\pi}: E(A) \to B(H)$ is a *-representation satisfying $\|\tilde{\pi}(x)\| \leq |x|_I$ for all $x$ in $E(A)$. We have the following commutative diagram.

Now, let $\alpha: \mathbb{R} \to \text{Aut}^*E(A)$ be the action on $E(A)$ induced by $\alpha$ as in Lemma 3.1 satisfying
\[
\alpha(j(x)) = j(\alpha(x)) \quad \text{for all } x \in A.
\] (6)

Then $(\tilde{\pi}, U, H)$ is a covariant representation of $(\mathbb{R}, E(A), \alpha)$. Indeed, let $x \in A^\infty, y = j(x)$. Then for all $t \in \mathbb{R}$,
\[
\tilde{\pi}(\alpha(y)) = \tilde{\pi}(\alpha(j(x))) = \tilde{\alpha}(j(\alpha(x))) = \pi(\alpha(x)) = U_t \pi(x) U_t^* = U_t \pi(j(x)) U_t^* = U_t \pi(y) U_t^*.
\]

By the continuity of $\pi$ and $\alpha$, it follows that $\pi(\alpha(y)) = U_t \pi(y) U_t^*$ for all $y \in E(A)$ and all $t \in \mathbb{R}$. Hence by Lemma 3.2, $\tilde{\rho} = \tilde{\pi} \times U$ is a non-degenerate *-representation of each of $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$ satisfying, for some constants $c$ and $c'$, the following (using (5)):

(iv) For all $f$ in $L^1(\mathbb{R}, E(A), \alpha)$, $\|\rho(f)\| \leq c|f|_I = c \int_{\mathbb{R}} |f(t)| dt$.
(v) For all $f$ in $S(\mathbb{R}, E(A), \alpha)$, $\|\rho(f)\| \leq c'|f|_{n,I,m}$.

(7)

Thus given a *-representation $\rho$ of $S(\mathbb{R}, A^\infty, \alpha)$, there is canonically associated a *-representation $\tilde{\rho}$ of each of $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$.

Conversely, given $\tilde{\rho}$ in $\text{Rep}(S(\mathbb{R}, E(A), \alpha))$, $\rho = \pi \times U$ for a covariant representation $(\pi, U)$ of $(\mathbb{R}, E(A), \alpha)$, $\pi \circ j$ is a covariant representation of $A$, and then $(\pi \circ j) \times U$ is in $\text{Rep}(S(\mathbb{R}, A^\infty, \alpha))$.

**Step II.** The $\sigma$-$C^*$-algebra $C^*(\mathbb{R}, E(A), \alpha)$ is universal for the *-representations of the Frechet algebra $S(\mathbb{R}, A^\infty, \alpha)$. 

Let \( \tilde{j} : S(\mathbb{R}, A^\omega, \alpha) \to L^1(\mathbb{R}, E(A), \alpha) \) be the map

\[
\tilde{j}(f) = j \circ f = \tilde{f} \quad \text{(say, i.e.,)} \quad \tilde{j}(f)(r) = j(f(r)) = f(r) + \text{srad}(A)^\omega \quad \text{for all } r \in \mathbb{R}.
\]

(8)

Notice that the map \( \tilde{j} \) is defined and is continuous; because \( (S(\mathbb{R}, A^\omega, \alpha)) \subset L^1(\mathbb{R}, A^\omega, \alpha) \subset L^1(\mathbb{R}, A, \alpha) \), and for \( n \in \mathbb{N} \) and \( m \) in \( \mathbb{Z}^+ \), all \( f \) in \( S(\mathbb{R}, A^\omega, \alpha) \),

\[
|\tilde{j}(t)|_n \leq \|f(t)\|_n \leq M\|f(t)\|_{m,n}, \quad \text{and hence}
\]

\[
\int_\mathbb{R} |\tilde{j}(t)|_n \, dt \leq \int_\mathbb{R} \|f(t)\|_{m,n} \, dt < \infty
\]

so that \( f \in L^1(\mathbb{R}, E(A), \alpha) \). Let \( j_1 : L^1(\mathbb{R}, E(A), \alpha) \to C^*(\mathbb{R}, E(A), \alpha) \) be the natural map \( j_1(f) = f + \text{srad}(L^1(\mathbb{R}, E(A), \alpha)) \). This gives the continuous \( * \)-homomorphism

\[
J : j_1 \circ \tilde{j} : S(\mathbb{R}, A^\omega, \alpha) \to C^*(\mathbb{R}, E(A), \alpha).
\]

(9)

Let \( \rho \in \text{Rep}(S(\mathbb{R}, A^\omega, \alpha)), \rho = \pi \times U \) in usual notations with \( \pi : A^\omega \to B(H) \) in \( \text{Rep}(E(A)) \) such that \( \pi = \tilde{\pi} \circ j \). Let \( \tilde{\rho} : L^1(\mathbb{R}, E(A), \alpha) \to B(H) \) be \( \tilde{\rho} = \tilde{\pi} \times U \). Then for all \( f \) in \( S(\mathbb{R}, A^\omega, \alpha) \),

\[
\tilde{\rho}(\tilde{j}(f)) = (\tilde{\pi} \times U)(\tilde{j}(f)) = \int_\mathbb{R} \tilde{\pi}(\tilde{j}(f)(t))U_t \, dt = \int_\mathbb{R} \tilde{\pi}(j \circ f)(t)U_t \, dt
\]

\[
= \int_\mathbb{R} \tilde{\pi}(j(f(t)))U_t \, dt = \int_\mathbb{R} \tilde{\pi}(f(t) + \text{srad}(A))U_t \, dt
\]

\[
= \int_\mathbb{R} \pi(f(t))U_t \, dt = \rho(f).
\]

Thus \( \tilde{j} \circ \tilde{\rho} = \rho \); and hence \( J \circ \tilde{\rho} = \rho \), where \( J = j_1 \circ \tilde{j} \) and \( \tilde{\rho} \in \text{Rep}(C^*(\mathbb{R}, E(A), \alpha)) \) is defined by \( j_1 \circ \tilde{\rho} = \tilde{\rho} \) in view of \( C^*(\mathbb{R}, E(A), \alpha) = E(L^1(\mathbb{R}, E(A), \alpha)) \).

Step III. Given a \( * \)-homomorphism \( \rho : S(\mathbb{R}, A^\omega, \alpha) \to B \) from \( S(\mathbb{R}, A^\omega, \alpha) \) to a \( \sigma \)-\( C^* \)-algebra \( B \), there exists \( * \)-homomorphisms \( \tilde{\rho} : L^1(\mathbb{R}, E(A), \alpha) \to B, \tilde{\rho} : C^*(\mathbb{R}, E(A), \alpha) \to B \) such that \( \rho = \tilde{\rho} \circ \tilde{j} = \tilde{\rho} \circ J \) and \( \tilde{\rho} = \tilde{\rho} \circ j_1 \).

This follows by applying Step II to each of the factor \( C^* \)-algebra \( B_n \) in the inverse limit decomposition of \( B \).

Step IV. \( C^*(\mathbb{R}, E(A), \alpha) = E(S(\mathbb{R}, A^\omega, \alpha)) \) up to homeomorphic \( * \)-isomorphism.
Let $k: S(\mathbb{R}, E(A), \alpha) \rightarrow E(S(\mathbb{R}, A^\omega, \alpha))$ be $k(f) = f + \text{srad } S(\mathbb{R}, A^\omega, \alpha)$. Then there exists a $^*$-homomorphism $\tilde{k}: C^*(\mathbb{R}, E(A), \alpha) \rightarrow E(S(\mathbb{R}, A^\omega, \alpha))$ such that $\tilde{k} \circ J = k$. We show that $\tilde{k}$ is the desired homeomorphic $^*$-isomorphism making the following diagram commutative.

\[
\begin{array}{ccc}
S(\mathbb{R}, A^\omega, \alpha) & \xrightarrow{J} & C^*(\mathbb{R}, E(A), \alpha) \\
& & \xleftarrow{\tilde{k}} \\
& & E(S(\mathbb{R}, A^\omega, \alpha)).
\end{array}
\]  

By the universal property of $E(S(\mathbb{R}, A^\omega, \alpha))$, there exists a $^*$-homomorphism $\tilde{j}: E(S(\mathbb{R}, A^\omega, \alpha)) \rightarrow C^*(\mathbb{R}, E(A), \alpha)$ such that $\tilde{j} \circ k = J$. We claim that $[\tilde{k}]_{\text{im}(J)}$ is injective. Indeed, let $f \in S(\mathbb{R}, A^\omega, \alpha)$ be such that $\tilde{k}(J(f)) = 0$. Hence $k(f) = 0$, so that $f \in \text{srad}(S(\mathbb{R}, A^\omega, \alpha))$. Thus, for all $\alpha \in \text{Rep}(S(\mathbb{R}, A^\omega, \alpha))$, $\rho(f) = 0$. Therefore, by Step I, $\sigma(\tilde{j}(f)) = 0$ for all $\sigma \in \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))$. (Recall that $\tilde{j} = j \circ f = \tilde{j}(f)$.) Hence $\tilde{j}(f)$ is in $\text{srad}(L^1(\mathbb{R}, E(A), \alpha))$, and so $j_1(\tilde{j}(f)) = 0$. Therefore $J(f) = 0$. It follows that $\tilde{k}$ is injective on $\text{im}(J)$.

Now by (10) and the injectivity of $\tilde{k}$ on $\text{im}(J)$, $J \circ k = J$. Hence $J = J \circ k \circ J$, and so $J \circ \tilde{k} = \text{id}$ on $\text{im}(J)$. Similarly $\tilde{k} \circ J(f) = \tilde{k}(J(f)) = k(f)$, hence $\tilde{k} \circ \tilde{j} = \text{id}$ on $\text{im}(k)$. Thus $k$ is a homeomorphic $^*$-isomorphism from the dense $^*$-subalgebra $J(S(\mathbb{R}, A^\omega, \alpha))$ of $C^*(\mathbb{R}, E(A), \alpha)$ on the dense $^*$-subalgebra $\tilde{k}(S(\mathbb{R}, A^\omega, \alpha))$ of $E(S(\mathbb{R}, A^\omega, \alpha))$. It follows that $C^*(\mathbb{R}, E(A), \alpha)$ is homeomorphically $^*$-isomorphic to $E(S(\mathbb{R}, A^\omega, \alpha))$.

**Step V**. $E(L^1(\mathbb{R}, A^\omega, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$.

Let $\mathbb{R}$ act on $L^1(\mathbb{R}, A, \alpha)$ by $xf(y) = f(x - y)$. For this action, $(L^1(\mathbb{R}, A, \alpha))^\omega = S(\mathbb{R}, A, \alpha)$ by Theorem 2.1.7 of [12]. Thus $S(\mathbb{R}, A^\omega, \alpha) = (L^1(\mathbb{R}, A, \alpha))^\omega$. Hence by Lemma 3.4, $E(L^1(\mathbb{R}, A^\omega, \alpha))^\omega = E(S(\mathbb{R}, A^\omega, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$. This completes the proof of Theorem 1.

5. **Proof of Theorem 2**

Let the Frechet algebra $A$ be hermitian and a $Q$-algebra. Hence $A$ is spectrally bounded, i.e., the spectral radius $r(x) = r_A(x) < \infty$ for all $x \in A$. Let $s_A(x) := r(x^*x)^{1/2}$ be the Ptak’s spectral function on $A$. By Corollary 2.2 of [11], $E(A)$ is a $C^*$-algebra, the complete $C^*$-norm of $E(A)$ being defined by the greatest $C^*$-seminorm $p_\infty(\cdot)$ (automatically continuous) on $A$. Now for any $x \in A$,

\[
p_\infty(x)^2 = p_\infty(x^*x) = \|x^*x + \text{srad}(A)\|
= r_{E(A)}(x^*x + \text{srad}(A)) \leq r_A(x^*x) = s_A(x)^2.
\]

Hence $p_\infty(x) \leq s_A(x)$ for all $x \in A$. By the hermiticity and $Q$-property, $s_A(\cdot)$ is a $C^*$-seminorm (Theorem 8.17 of [8]), hence $p_\infty(\cdot) = s(\cdot) \geq r(\cdot)$. In this case, $\text{rad}(A) = \text{srad}(A)$. Let $A_q = A/\text{rad}(A)$ which is a dense $^*$-subalgebra of the $C^*$-algebra.
Let \( E(A) \) and is also a Frechet \( Q \)-algebra with the quotient topology \( t_q \). Let \( [x] = x + \text{rad}(A) \) for all \( x \in A \). Since the spectrum

\[
\text{sp}_A(x) = \text{sp}_{A_q}([x]), \quad r_A(x) = r_{A_q}([x]), \quad s_A(x) = s_{A_q}([x]),
\]

and so \( r_{A_q}([x]) \leq s_{A_q}([x]) = ||[x]||_\infty \). Hence \( \| \cdot \|_\infty \) is a spectral norm on \( A_q \), i.e., \( (A_q, \| \cdot \|_\infty) \) is a \( Q \)-algebra. Thus \( A_q \) is spectrally invariant in \( E(A) \). Hence by Corollary 7.9 of [10], \( K_*(A_q) = RK_*(A_q) = K_*(E(A)) \).

Now consider the maps

\[
A \xrightarrow{j} A_q \xrightarrow{id} E(A)
\]

and, for each positive integer \( n \), the induced maps

\[
M_n(A) \xrightarrow{j_n = j \otimes \text{id}_0} M_n(A_q) = [M_n(A)]_q \xrightarrow{id} M_n(E(A)) = E(M_n(A)).
\]

By the spectral invariance of \( A \) in \( A_q \) via the map \( j, j(\text{inv}(A)) = \text{inv}(A_q) \), where \( \text{inv}(K) \) denotes the group of invertible elements of \( K \). Let \( \text{inv}_0(\cdot) \) denote the principle component in \( \text{inv}(\cdot) \). We use the following.

**Lemma 5.1.** Let \( B \) be a Frechet \( Q \)-algebra or a normed \( Q \)-algebra. Then \( \text{inv}_0(B) \) is the subgroup generated by the range \( \exp B \) of the exponential function.

The Frechet \( Q \)-algebra case follows by adapting the proof of the corresponding Banach algebra result in Theorem 1.4.10 of [11]. If \( (B, \| \cdot \|) \) is a \( Q \)-normed algebra, then \( (B, \| \cdot \|) \) is advertebly complete in the sense that if a Cauchy sequence \( (x_n) \) converges to an element \( x \in \text{inv}(B^*) \) \( (B^* \) being the completion of \( B \), then \( x \in B \). Hence the exponential function is defined on \( B \); and then the Banach algebra proof can be adapted.

We use the above lemma to verify the following:

**Claim.** \( j_n(\text{inv}_0(M_n(A))) = \text{inv}_0(M_n(A_q)) \).

Take \( n = 1 \). It is clear that \( j(\text{inv}_0(A)) \subseteq \text{inv}_0(A_q) \). Let \( y \in \text{inv}_0(A_q) \). Hence \( y = \Pi \exp(z_i) \) for finitely many \( z_i = [x_i] = x_i + \text{rad}(A) \) for some \( x_i \) in \( A \). Then \( y = [\Pi \exp(x_i)] \). Hence \( y \in j(\text{inv}_0(A)) \). Thus \( j(\text{inv}_0(A)) = \text{inv}_0(A_q) \). Now take \( n > 1 \). As \( A \) is spectrally invariant in \( A_q \), it follows from Theorem 2.1 of [11] that the Frechet \( Q \)-algebra \( M_n(A) \) is spectrally invariant in \( M_n(A_q) \) via \( j_n \). Also, \( M_n(A_q) = (M_n(A))_q \) is a \( Q \)-algebra in both the quotient topology as well as the \( C^* \)-norm induced from \( M_n(E(A)) = E(M_nA) \). Applying arguments analogous to above, it follows that \( j_n(\text{inv}_0(M_n(A))) = \text{inv}_0(M_n(A_q)) \).

Now consider the surjective group homomorphisms

\[
\text{inv}(M_n(A)) \xrightarrow{j_n^{-1}} \text{inv}(M_n(A_q)) \xrightarrow{j} \text{inv}(M_n(A_q))/\text{inv}_0(M_n(A_q)).
\]

It follows that \( \text{ker}(J \circ j_n) = \text{inv}_0(M_n(A_q)) \), with the result, the group \( \text{inv}(M_n(A))/\text{inv}_0(M_n(A_q)) \) is isomorphic to the group \( \text{inv}(M_n(A_q))/\text{inv}_0(M_n(A_q)) \). Hence by the definition of the \( K \)-theory group \( K_1 \),

\[
K_1(A) = \lim_{n \to \infty} (\text{inv}(M_n(A))/\text{inv}_0(M_n(A)))
= \lim_{n \to \infty} (\text{inv}(M_n(A_q))/\text{inv}_0(M_n(A_q)))) = K_1(A_q).
\]
For $B$ to be $A$ or $A_1$, let the suspension of $B$ be

$$SB = \{ f \in C([0,1],B) : f(0) = f(1) = 0 \} \cong C_0(\mathbb{R},B).$$

We use the Bott periodicity theorem $K_0(B) = K_1(SB)$ to show that $K_0(A) = K_0(A_1)$. It is standard that rad$(SA) = rad(C_0(\mathbb{R},A)) \cong C_0(\mathbb{R},rad(A))$. Hence

$$SA_1 = C_0(\mathbb{R},A_1) = C_0(\mathbb{R},A)/rad(A) = C_0(\mathbb{R},A)/rad(C_0(\mathbb{R},A)) = SA/rad(A).$$

Hence

$$K_0(A_1) = K_1(SA_1) = K_1(SA/rad(SA)) = K_0(A).$$

Thus we have

$$K_*(A) = K_*(A_1) = K_*(E(A)) = RK_*(A) = RK_*(A_1).$$

Now $A^\infty$ is spectrally invariant in $A$ (Theorem 2.2 of [15]); and the action $\alpha$ on $A^\infty$ is smooth (Theorem A.2 of [14]). Then applying the Phillips–Schweitzer analogue of Thom isomorphism for smooth Frechet algebra crossed product (Theorem 1.2 of [11]) and Connes analogue of Thom isomorphism for $C^*$-algebra crossed product [7], it follows that

$$RK_*(S(\mathbb{R},A^\infty,\alpha)) = RK_{n+k}(A^\infty) = RK_{n+k+1}(A) = RK_{n+k+1}(E(A))$$

$$= RK_*(C^*(\mathbb{R},E(A),\alpha)) = RK_*(C^*(\mathbb{R},E(A),\alpha)).$$

When $\alpha$ is isometric, Theorem 1.3.4 of [11] implies that $RK_*(S(\mathbb{R},A^\infty,\alpha)) = RK_*(L^1(\mathbb{R},A,\alpha))$. This completes the proof. \qed

6. An application to the differential structure in $C^*$-algebras

Let $\mathcal{U}$ be a unital $*$-algebra. Let $\| \cdot \|$ be a $C^*$-norm on $\mathcal{U}$. Let $(A,\| \cdot \|)$ be the completion of $(\mathcal{U},\| \cdot \|)$. Following [5], a map $T : \mathcal{U} \to l^1(\mathbb{N})$ is a differential seminorm if $T(x) = (T_k(x))_{k=0}^\infty \in l^1(\mathbb{N})$ satisfies the following:

(i) $T_k(x) \geq 0$ for all $k$ and for all $x$.

(ii) For all $x,y$ in $\mathcal{U}$ and scalars $\lambda$, $T(x+y) \leq T(x) + T(y), T(\lambda x) = |\lambda|T(x)$.

(iii) For all $x,y$ in $\mathcal{U}$, for all $k$,

$$T_k(xy) \leq \sum_{i+j=k} T_i(x)T_j(y).$$

(iv) There exists a constant $c > 0$ such that $T_0(x) \leq c\|x\| \ \forall x \in \mathcal{U}.$

By (ii), each $T_k$ is a seminorm. We say that $T$ is a differential $*$-seminorm if addition-
ally:

(v) $T_k(x^*) = T_k(x)$ for all $x$ and for all $k$.

Further $T$ is a differential norm if $T(x) = 0$ implies $x = 0$. Throughout we assume that $T_0(x) = \|x\|, x \in \mathcal{U}$. The total norm of $T$ is $T_0(x) = \sum_{k=0}^\infty T_k(x), x \in \mathcal{U}$. Given $T$,
the differential Frechet *-algebra defined by $T$ is constructed as follows. For each $k$, let $p_k(x) = \sum_{i=0}^{k} T_i(x), x \in \mathcal{U}$. Then each $p_k$ is a submultiplicative *-norm; and on $\mathcal{U}$, we have

$$p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_k \leq p_{k+1} \leq \cdots$$

and $(p_k)^{\mathcal{U}}_n$ is a separating family of submultiplicative *-norms on $\mathcal{U}$. Let $\tau$ be the locally convex *-algebra topology on $\mathcal{U}$ defined by $(p_k)^{\mathcal{U}}_n$. Let $\mathcal{U}_{\tau} = (\mathcal{U}, \tau)^{\mathcal{U}}$ the completion of $\mathcal{U}$ in $\tau$ and let $\mathcal{U}_{p_k} = (\mathcal{U}, p_k)^{\mathcal{U}}$ the completion of $\mathcal{U}$ in $p_k$. Then $\mathcal{U}_{\tau}$ is a Frechet locally $m$-convex *-algebra, $\mathcal{U}_{p_k}$ is a Banach *-algebra. Let $\mathcal{U}_{T}$ be the completion of $(\mathcal{U}, T_{\mathcal{U}})$. Then the Banach *-algebra $\mathcal{U}_{T} = \{ x \in \mathcal{U}_{\tau}: \sup_n p_n(x) < \infty \}$, the bounded part of $\mathcal{U}_{\tau}$. By the definitions, there exists continuous surjective *-homomorphisms $\phi_k: \mathcal{U}_{p_k} \to A, \phi: \mathcal{U}_{\tau} \to A$. The identity map $\mathcal{U} \to \mathcal{U}$ extends uniquely as continuous surjective *-homomorphisms $\phi_k: \mathcal{U}_{(k+1)} \to \mathcal{U}_{p_k}$ such that

$$\mathcal{U}_{(0)} \leftarrow \mathcal{U}_{(1)} \leftarrow \mathcal{U}_{(2)} \leftarrow \mathcal{U}_{(3)} \leftarrow \cdots$$

is a dense inverse limit sequence of Banach *-algebras and $\mathcal{U}_{T} = \lim_n \mathcal{U}_{p_k}$.

**Lemma 6.1.** Let $(\mathcal{U}, \| \cdot \|)$ be a C*-normed algebra. Let $A$ be the completion of $\mathcal{U}$. Let $B$ denote $\mathcal{U}_{p_k}$ or $\mathcal{U}_{T}$ with respective topologies. Then the following hold:

(i) $B$ is a hermitian Q-algebra.
(ii) $E(B) = A$.
(iii) $K_0(B) = K_0(A) = RK_s(B)$.

The $K$-theory result follows from the following.

**Lemma 6.2.** Let $A$ be a Frechet algebra in which each element is bounded. Let $A$ be spectrally invariant in $E(A)$. Then $K_0(A) = K_s(E(A))$.

Now let $\alpha$ be an action of $\mathbb{R}$ on $A$ leaving $\mathcal{U}$ invariant. Let $T$ be $\alpha$-invariant, i.e., $T_k(\alpha(x)) = T_k(x)$ for all $k$ and for all $x$. Then $\alpha$ induces isometric actions of $\mathbb{R}$ on each of $\mathcal{U}_{p_k}, \mathcal{U}_{\tau}$ and $\mathcal{U}_{T}$. Let $B$ be as above. Hence the crossed product Frechet *-algebras $L^1(\mathbb{R}, B^c, \alpha), L^1(\mathbb{R}, B, \alpha), S(\mathbb{R}, B, \alpha)$ and $S(\mathbb{R}, B^c, \alpha)$ are defined. Theorem 2 and Lemma 6.1 give the following, which is Theorem 3(a).

**COROLLARY 6.3.**

$$RK_s(S(\mathbb{R}, B^c, \alpha)) = RK_s(S(\mathbb{R}, B, \alpha)) = RK_s(C^*(\mathbb{R}, A, \alpha)) = K_{s+1}(A).$$

Now let $\hat{\mathcal{U}}$ be the completion of $\mathcal{U}$ in the family $\mathcal{P}$ of all $\alpha$-invariant differential *-norms on $\mathcal{U}$. Then $\hat{\mathcal{U}}$ is a complete locally $m$-convex *-algebra admitting a continuous surjective *-homomorphism $\Psi: \hat{\mathcal{U}} \to A$. This $\alpha$-invariant smooth envelope $\hat{\mathcal{U}}$ is different from the smooth envelope defined in [5], and it need not be a subalgebra of $A$.

**Lemma 6.4.** Assume that $\hat{\mathcal{U}}$ is metrizable. Then $\hat{\mathcal{U}}$ is a hermitian Q-algebra, $E(\hat{\mathcal{U}}) = A$, and $K_0(\hat{\mathcal{U}}) = K_0(A)$.

This supplements a comment in p. 279 of [5] that $K_0(A) = K_0(\mathcal{U}_1)$ where $\mathcal{U}_1$ is the completion of $\mathcal{U}$ in all, not necessarily $\alpha$-invariant nor closable, differential semi-norms.
Proof. Since $\tilde{\mathcal{U}} = \lim_{\to} \mathcal{U}_x$, we have $E(\tilde{\mathcal{U}}) = \lim_{\to} E(\mathcal{U}_x) = A$; and $\tilde{\mathcal{U}}$ admits greatest continuous $C^*$-seminorm, say $p_\infty(\cdot)$. It is easily seen that for any $x \in \tilde{\mathcal{U}}$, the spectral radius in $\tilde{\mathcal{U}} r(x) \leq p_\infty(x)$; and $\mathcal{U}$ is a hermitian $Q$-algebra. This implies, in view of $E(\tilde{\mathcal{U}}) = A$, that the spectrum in $\tilde{\mathcal{U}} sp(x) = sp_A(j(x))$ for all $x \in \mathcal{U}$, where $j(x) = x + srad \mathcal{U}$.

It follows from Lemma 6.2 that $K_*(A) = K_*(E(A))$. Hence Lemma 6.4 follows. $\square$

Now the action $\alpha$ induces an isometric action of $\mathbb{R}$ on $\tilde{\mathcal{U}}$, with the result that the crossed product algebras $S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$ and $L^1(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$ are defined and are complete locally $m$-convex $\ast$-algebras with a $C^*$-enveloping algebras satisfying

$$E(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) = E(L^1(\mathbb{R}, \tilde{\mathcal{U}}, \alpha))$$

$$= C^*(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$$

$$= C^*(\mathbb{R}, A, \alpha).$$

Theorem 2 quickly gives the following which is Theorem 3(b).

**COROLLARY 6.5.**

Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $RK_*(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) = K_{s+1}(A)$.

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