Closed-form expressions for the sketching approximability of (some) symmetric Boolean CSPs

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Abstract

A Boolean maximum constraint satisfaction problem, $\text{Max-CSP}(f)$, is a maximization problem specified by a constraint function $f : \{-1, 1\}^k \to \{0, 1\}$. An instance of $\text{Max-CSP}(f)$ consists of $n$ variables and $m$ constraints, where each constraint is $f$ applied on a tuple of “literals” of $k$ distinct variables chosen from the $n$ variables. $f$ is said to be symmetric if $f(x)$ depends only on $\sum_{i=1}^{k} x_i$, where $x = (x_1, \ldots, x_k)$. Chou, Golovnev, and Velusamy [CGV20] obtained explicit constants characterizing the streaming approximability of all symmetric $\text{Max-2CSP}$s. More recently, Chou, Golovnev, Sudan, and Velusamy [CGSV21a] proved a general dichotomy theorem showing tight approximability of Boolean $\text{Max-CSP}$s with respect to sketching algorithms. For every $f$, they showed that there exists an optimal approximation ratio $\alpha(f) \in (0,1]$ such that for every $\epsilon > 0$, $\text{Max-CSP}(f)$ is $(\alpha(f) - \epsilon)$-approximable by a linear sketching algorithm in $O(\log n)$ space, but any $(\alpha(f) + \epsilon)$-approximation sketching algorithm for $\text{Max-CSP}(f)$ requires $\Omega(\sqrt{n})$ space. While they show that $\alpha(f)$ is computable to arbitrary precision in $\text{PSPACE}$, they do not give a closed-form expression.

In this work, we build on the [CGSV21a] dichotomy theorem and give closed-form expressions for the sketching approximation ratios of multiple families of symmetric Boolean functions. These include $k\text{AND}$ and $\text{Th}_{k-1}^k$ (the “weight-at-least-(k−1)” threshold function on $k$ variables). In particular, letting $\alpha'_k = 2^{-(k-1)}(1 - k^{-2})^{(k-1)/2}$, we show that for odd $k \geq 3$, $\alpha(k\text{AND}) = \alpha'_k$; for even $k \geq 2$, $\alpha(k\text{AND}) = 2\alpha'_{k+1}$; and for even $k \geq 2$, $\alpha(\text{Th}_{k-1}^k) = \frac{k}{2} \alpha'_{k-1}$. We also resolve the ratio for the “weight-exactly-$\frac{k}{2}$” function for odd $k \in \{3, \ldots, 51\}$ as well as fifteen other functions. We stress here that the closed-form expressions need not have existed just given the [CGSV21a] dichotomy, and our analysis involves identifying structural “saddle-point” properties of the dichotomy. For arbitrary threshold functions, we also give optimal “bias-based” approximation algorithms generalizing [CGV20] while simplifying [CGSV21a]. Finally, we investigate the streaming lower bounds in [CGSV21a]. We show that for $\text{Th}_3^4$, the [CGSV21a] streaming and sketching lower bounds match (previously only known for $2\text{AND}$), while for $3\text{AND}$, their streaming lower bound is provably weaker.

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1 Introduction

In this work, we consider the streaming approximability of various Boolean constraint satisfaction problems, and we begin by defining these terms. See [CGSV21a, §1.1-2] for more details on the definitions.

1.1 Setup: The streaming approximability of Boolean CSPs

Boolean CSPs. Let \( f : \{-1,1\}^k \rightarrow \{0,1\} \) be a Boolean function. In an \( n \)-variable instance of the problem Max-CSP(\( f \)), a constraint is a pair \( C = (b,j) \), where \( j = (j_1, \ldots, j_k) \in [n]^k \) is a \( k \)-tuple of distinct indices, and \( b = (b_1, \ldots, b_n) \in \{-1,1\}^n \) is a negation pattern.

For Boolean vectors \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \in \{-1,1\}^n \), let \( a \odot b \) denote their coordinate-wise product \( (a_1b_1, \ldots, a_nb_n) \). An assignment \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1,1\}^n \) satisfies \( C \) iff \( f(b \odot \sigma|_j) = 1 \), where \( \sigma|_j \) is the \( k \)-tuple \( (\sigma_{j_1}, \ldots, \sigma_{j_k}) \) (i.e., \( \sigma \) satisfies \( C \) iff \( f(b_1\sigma_{j_1}, \ldots, b_k\sigma_{j_k}) = 1 \)).

An instance \( \Psi \) of Max-CSP(\( f \)) consists of a list of constraints; the value \( \text{val}_\Psi(\sigma) \) of an assignment \( \sigma \) to \( \Psi \) is the fraction of constraints in \( \Psi \) satisfied by \( \sigma \); and the value \( \text{val}_\Psi \) of an instance \( \Psi \) is the maximum value of any assignment \( \sigma \in \{-1,1\}^n \).

Approximations to CSPs. For \( \alpha \in [0,1] \), we consider the problem of \( \alpha \)-approximating Max-CSP(\( f \)). In this problem, the goal of an algorithm \( A \) is to, on input an instance \( \Psi \), output an estimate \( A(\Psi) \) such that with probability at least \( \frac{2}{3} \), \( \alpha \cdot \text{val}_\Psi \leq A(\Psi) \leq \text{val}_\Psi \). For \( \beta < \gamma \in [0,1] \), we also consider the closely related \( (\gamma, \beta) \)-approximation to Max-CSP(\( f \)) problem (denoted \( (\gamma, \beta)\text{-Max-CSP}(f) \) for short). In this problem, the algorithm’s input instance \( \Psi \) is promised to either have value \( \leq \beta \) or value \( \geq \gamma \), and the goal is to decide which is the case with probability at least \( \frac{2}{3} \). By standard arguments, the minimum approximation ratio \( \alpha(f) \) for Max-CSP(\( f \)) equals the infimum of \( \frac{\beta}{\gamma} \) over all \( (\gamma, \beta) \) such that Max-CSP(\( f \)) is \( (\gamma, \beta) \)-approximable (see [CGSV21a, Proposition 2.10] for details).

Streaming and sketching algorithms for CSPs. For various Boolean functions \( f \), we consider algorithms which attempt to approximate Max-CSP(\( f \)) instances in the (single-pass, insertion-only) space-\( s \) streaming setting. Such algorithms can only use space \( s \) (which is ideally small, such as \( O(\log n) \), where \( n \) is the number of variables in an input instance), and, when given as input a CSP instance \( \Psi \), can only read the list of constraints in a single, left-to-right pass.

We also consider a (seemingly) weak class of streaming algorithms for CSPs called space-\( s \) sketching algorithms, which are composable in the following sense. After seeing each constraint in the stream, the algorithm’s state (a string in \( \{0,1\}^s \)) is a sketch. If \( C \) is the set of possible stream elements, the algorithm must provide a compression function \( \text{Comp} : C^* \rightarrow \{0,1\}^s \) encoding stream elements to “sketches” and a combination function \( \text{Comb} : \{0,1\}^s \times \{0,1\}^s \rightarrow \{0,1\}^s \) for combining pairs of sketches such that:

1. Given current state \( x \in \{0,1\}^s \) and new stream element \( c \in C^* \), the new state of the streaming algorithm is \( \text{Comb}(x, \text{Comp}(c)) \).

2. For two streams \( \sigma_1, \sigma_2 \in C^* \), we have \( \text{Comb}(\text{Comp}(\sigma_1), \text{Comp}(\sigma_2)) = \text{Comp}(\sigma_1 \sigma_2) \) (where \( \sigma_1 \sigma_2 \) denotes concatenation).

Requirement (2) represents the key feature of sketching algorithms: \( \text{Comb} \) can be used to “compose” the results of the streaming algorithm run on \( \sigma_1 \) and \( \sigma_2 \) separately. (\( \text{Comp} \) and \( \text{Comb} \) can be designed jointly using underlying shared randomness.) A special case of sketching algorithms...
are the linear sketches, where each sketch (i.e., element of \( \{0,1\}^s \)) encodes an element of a vector space and Comb performs vector addition.

This paper. The main goal of this paper is to explicitly determine closed-form expressions for the optimal sketching approximation ratios for various Max-CSPs of interest such as Max-kAND.

1.2 Prior work and motivations

1.2.1 Prior results on streaming and sketching Max-CSP\( (f) \)

We first give a brief review of what is already known about the streaming and sketching approximability of Max-CSP\( (f) \). For \( f : \{−1,1\}^k \to \{0,1\} \), denote \( \rho(f) = \Pr_{b \sim \{−1,1\}^k}[f(b) = 1] \). Note that the Max-CSP\( (f) \) problem has a trivial \( \rho(f) \)-approximation given by simply outputting \( \rho(f) \).

We refer to a function \( f \) as approximation-resistant for some class of algorithms (e.g., streaming or sketching algorithms with some space bound) if it cannot be \( (\rho(f) + \epsilon) \)-approximated for any constant \( \epsilon > 0 \) (equivalently, the algorithms cannot solve the \( (\rho(f) + \epsilon, 1−\epsilon) \)-Max-CSP\( (f) \) problem for any constant \( \epsilon > 0 \)). Otherwise, we refer to \( f \) as approximable for the class of algorithms.

The first two CSPs whose \( o(\sqrt{n}) \)-space streaming approximabilities were resolved were Max-2XOR and Max-2AND. Kapralov, Khanna, and Sudan [KKS15] and Kogan and Krauthgamer [KK15] concurrently showed that Max-2XOR is approximation-resistant to \( o(\sqrt{n}) \)-space streaming algorithms. Later, Chou, Golovnev, and Velusamy [CGV20], building on earlier work of Gurusswami, Velusamy, and Velingker [GVV17], gave an \( O(\log n) \)-space linear sketch which \( \left(\frac{4}{9}−\epsilon\right) \)-approximates Max-2AND for every \( \epsilon > 0 \) and showed that \( \left(\frac{4}{9}+\epsilon\right) \)-approximations require \( \Omega(\sqrt{n}) \) space, even for streaming algorithms.

In two recent works [CGSV21a; CGSV21b], Chou, Golovnev, Sudan, and Velusamy proved so-called dichotomy theorems for sketching CSPs. [CGSV21a] dealt with CSPs over Boolean alphabets, while [CGSV21b] dealt with the more general case of CSPs over finite alphabets.\(^1\)

[CGSV21a] is most relevant for our purposes, as it concerns Boolean CSPs. For a fixed constraint function \( f : \{−1,1\}^k \to \{0,1\} \), [CGSV21a]'s main result is a dichotomy theorem in the following sense: For any \( 0 \leq \gamma < \beta \leq 1 \), either

1. \((\gamma,\beta)\)-Max-CSP\( (f) \) has an \( O(\log n) \)-space linear sketching algorithm, or
2. For all \( \epsilon > 0 \), sketching algorithms for \((\gamma−\epsilon,\beta+\epsilon)\)-Max-CSP\( (f) \) require \( \Omega(\sqrt{n}) \) space.

We will defer stating the technical condition which distinguishes cases (1) and (2) until Section 2.1 (see also the discussion in Section 1.4.1), but do mention that [CGSV21a] extends the lower bound (case 2) to streaming algorithms when special objects called padded one-wise pairs exist (whose definition we also defer). The padded one-wise pair case is sufficient to recover all previous streaming approximability results for Boolean functions (i.e., [KK15; KKS15; CGV20]), and prove several new ones. In particular, [CGSV21a] proves that if \( f : \{−1,1\}^k \to \{0,1\} \) has the property that there exists \( a \in \{−1,1\}^k \) such that \( f(a) = f(−a) = 1 \) (which they term “supporting one-wise independence”), then Max-CSP\( (f) \) is streaming approximation-resistant. [CGSV21b] uses analogous tools to recover streaming approximation-resistance of Max-UniqueGames (proven earlier by [GT19]) and prove approximation-resistance of several new problems, e.g., Max-qColoring.\(^2\)

\(^1\) More precisely, [CGSV21a] and [CGSV21b] both consider the more general case of CSPs defined by families of functions of a specific arity. We do not need this generality for the purposes of our paper, and therefore omit it.

\(^2\) Indeed, Chou, Golovnev, Sudan, Velingker, and Velusamy [CGSV+21] recently proved that some of these “nice” functions are streaming approximation-resistant even in \( o(n) \) space, building on Kapralov and Krachun’s work [KK19] for Max-2XOR.
CGSV21a nor CGSV21b explicitly analyze any new approximable problems, since Max-2AND’s $\frac{4}{9}$-approximability had already been established by [GVV17; CGV20].

### 1.2.2 Questions from previous work

In this work, we address several major questions which CGSV21a leaves unanswered:

1. Can we use CGSV21a’s dichotomy theorem to find closed-form sketching approximability ratios for approximable problems beyond 2AND?
2. CGSV21a implies the following “trivial upper bound” on streaming approximability: for all $f$, $\alpha(f) \leq 2\rho(f)$, as observed later in [CGSV+21, §1.3]. How tight is this upper bound?
3. Does CGSV21a’s streaming lower bound — i.e., the “padded one-wise pair” criterion — suffice to resolve the streaming approximability of every function?
4. CGSV21a, Proposition 2.10] gives an $(\alpha - \epsilon)$-approximation algorithm for Max-CSP$(f)$, where $\alpha$ is the infimum of $\frac{4}{9}$ such that the $(\beta - \epsilon', \gamma + \epsilon')$-Max-CSP$(f)$ distinguishing problem is hard to sketch for every $\epsilon' > 0$. However, this approximation algorithm requires running a “grid” of $O(1/\epsilon^2)$ distinguishers for $(\beta, \gamma)$-Max-CSP$(f)$ distinguishing problems in parallel. Can we give more simple and useful $(\alpha - \epsilon)$-approximation algorithms?

### 1.3 Results

We study the questions in Section 1.2.2 through the lens of CSPs defined by symmetric Boolean functions. A set $S \subseteq [k]$ defines a symmetric function $f_{S,k}: \{-1,1\}^k \to \{0,1\}$, which on input $b \in \{-1,1\}^k$ is the indicator for $\text{wt}(b) \in S$ (where $\text{wt}(b)$ is $b$’s Hamming weight, i.e., its number of 1’s). The simplest symmetric functions are $k\text{AND} = f_{\{\}}$ and the threshold functions $\text{Th}_{i,k} = f_{\{i,i+1,\ldots,k\}}$.

The sketching approximability of Max-$k\text{AND}$. [CGV20] showed that $\alpha(2\text{AND}) = \frac{4}{9}$ (which holds even for streaming algorithms), but for $k \geq 3$, nothing was known prior to this work.

We give a closed-form resolution of the sketching approximability of Max-$k\text{AND}$ for every $k$. For odd $k \geq 3$, define the constant

$$\alpha'_k = \left(\frac{(k-1)(k+1)}{4k^2}\right)^{(k-1)/2} = 2^{-(k-1)} \cdot \left(1 - \frac{1}{k^2}\right)^{(k-1)/2}.$$  

In Section 4, we prove the following theorem:

**Theorem 1.1.** For odd $k \geq 3$, $\alpha(k\text{AND}) = \alpha'_k$, and for even $k \geq 2$, $\alpha(k\text{AND}) = 2\alpha'_{k+1}$.

For instance, $\alpha(3\text{AND}) = \alpha'_3 = \frac{2}{9}$. Theorem 1.1 also has the following corollary, recalling that $\rho(k\text{AND}) = 2^{-k}$:

**Corollary 1.2.** $\lim_{k \to \infty} \frac{\alpha(k\text{AND})}{2\rho(k\text{AND})} = 1$.

Interestingly, [CGSV21a]’s trivial upper bound shows that $\alpha(f) \leq 2\rho(f)$; [CGSV+21] later improved this hardness to the $o(n)$-space streaming setting (which is optimal up to logarithmic factors). Hence Corollary 1.2 implies that $k\text{AND}$ is an “asymptotically optimally streaming-approximable” function and [CGSV21a]’s trivial upper bound is tight for a function family of interest.
The sketching approximability of other symmetric functions. In Section 5, we resolve the streaming approximability of a number of other symmetric Boolean functions. Specifically, in Section 5.1, we resolve the approximability of the functions \(\Theta_k^{k-1}\) for even \(k\):

**Theorem 1.3.** For even \(k \geq 2\), \(\alpha(\Theta_k^{k-1}) = \frac{k}{2} - 1\).

We also provide partial results for \(f_{\{\frac{k+1}{2}\}}\), where \(k\) is odd in Section 5.2, including closed forms for small \(k\) and an asymptotic result:

**Theorem 1.4** (Informal version of Theorem 5.9). For odd \(k \in \{3, \ldots, 51\}\), \(\alpha(f_{\{\frac{k+1}{2}\}})\) is the root of a quadratic in \(k\).

**Corollary 1.5.** For odd \(k\), the limit of \(\alpha(f_{\{\frac{k+1}{2}\}})\) is \(\rho(f_{\{\frac{k+1}{2}\}})\) as \(k \to \infty\).

Finally, in Section 5.3, we explicitly resolve fifteen other cases (e.g., \(f_{\{2,3\},3}\) and \(f_{\{4\},5}\)).

**Simple approximation algorithms for threshold functions.** [CGV20]'s optimal \((\frac{1}{2} - \epsilon)\)-approximation for 2AND, like [GVV17]'s earlier \((\frac{3}{2} - \epsilon)\)-approximation, is based on measuring a quantity called the bias of an instance \(\Psi\), denoted \(\text{bias}(\Psi)\), which is defined as follows: The bias \(\text{bias}_i(\Psi)\) of variable \(x_i\) is the absolute value of the difference between the number of positive and negative appearances of \(x_i\) in \(\Psi\), and \(\text{bias}(\Psi) \equiv \frac{1}{km} \sum_{i=1}^{n} \text{bias}_i(\Psi) \in [0, 1]\).

In the sketching setting, \(\text{bias}(\Psi)\) can be estimated using standard \(\ell_1\)-norm sketching algorithms [Ind06; KNW10] (see Theorem 2.10 below).

In Section 3, we show that when \(f_{S,k}\) is a threshold function, Max-CSP\((f)\) has a very simple bias-based approximation:

**Theorem 1.6.** Let \(f_{S,k} = \Theta_k^i\) be a threshold function. Then for every \(\epsilon > 0\), there exists a piecewise linear function \(\gamma : [-1, 1] \to [0, 1]\) and a constant \(\epsilon' > 0\) such that the following is a sketching \((\alpha(f_{S,k}) - \epsilon)\)-approximation for Max-CSP\((f_{S,k})\): On input \(\Psi\), compute an estimate \(\hat{b}\) for \(\text{bias}(\Psi)\) up to a multiplicative \((1 \pm \epsilon')\) error and output \(\gamma(b)\).

Our construction generalizes [CGV20]'s analysis for 2AND to all threshold functions, and is also a simplification, since [CGV20]'s algorithm computes a more complicated function of \(\hat{b}\) (see Observation 3.3 below).

For all CSPs we study in this paper (Sections 4 and 5.3), we apply an analytical technique which we term the “max-min method;” see the discussion in Section 1.4 below. For these functions, our algorithm has interesting implications beyond the sketching setting for the classical problem of outputting an approximately optimal assignment (instead of simply deciding whether one exists). Indeed, we describe a simple linear-time algorithm for this problem achieving the same approximation factor as our sketching algorithm:

**Corollary 1.7** (Informal version of Corollary 3.4). Let \(f_{S,k}\) be a function for which the max-min method applies, such as \(k\)AND (for any \(k \geq 2\)) or \(\Theta_k^{k-1}\) (for any even \(k \geq 2\)). Then there exists a constant \(p^* \equiv \{0, 1\}\) such that following algorithm, on input \(\Psi\), outputs an assignment which is \(\alpha(f_{S,k})\)-approximately optimal in expectation: Assign every variable to 1 if it occurs more often positively than negatively, and -1 otherwise, and then flip each variable’s assignment independently with probability \(p^*\).

This algorithm can be derandomized using universal hash families (following the recent argument of Biswas and Raman [BR21] for [CGV20]'s Max-\(k\)SAT algorithm).
Sketching vs. streaming approximability. In Section 6, we show that [CGSV21a]'s techniques cannot resolve the streaming approximability of Max-3AND. That is, while Theorem 1.1 implies $\alpha(3\text{AND}) = \frac{2}{3}$, [CGSV21a] cannot show that streaming algorithms cannot outperform this limitation. However, they do give an almost-tight bound:

**Theorem 1.8** (Informal version of Theorem 6.1 + Observation 6.5). [CGSV21a]'s padded one-wise pair criterion is not strong enough to show that there is no $\alpha(\sqrt{n})$-space streaming $(\frac{2}{3} - \epsilon)$-approximation for $3\text{AND}$ for any $\epsilon > 0$; however, it does rule out $0.2363$-approximations.

Separately, Theorem 1.3 implies that $\alpha(\text{Th}_3^3) = \frac{4}{9}$, and the padded one-wise pair criterion can be used to show that $(\frac{4}{9} + \epsilon)$-approximating Max-CSP($\text{Th}_3^3$) requires $\Omega(\sqrt{n})$ space in the streaming setting (see Observation 5.5 below).

### 1.4 Techniques: the max-min method

Next, we give more background on the technical aspects of [CGSV21a]'s dichotomy theorem and the novel aspects of our analysis which allow us to obtain closed-form expressions for $\alpha(f)$ for various functions $f$ of interest.

#### 1.4.1 Background from [CGSV21a]

Fix a constraint function $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ and let $\Delta(\{-1, 1\}^k)$ denote the set of all distributions on the set $\{-1, 1\}^k$. An element $D \in \Delta(\{-1, 1\}^k)$ can be viewed as a weighted instance of Max-CSP($f$) on $k$ variables (where the constraint with negation pattern $b$ has weight $Pr[a \sim D[a = b]]$). For a distribution $D \in \Delta(\{-1, 1\}^k)$, let $\mu(D) = (E_{a \sim D}[a_1], \ldots, E_{a \sim D}[a_k]) \in [-1, 1]^k$ denote $D$'s vector of marginals.

Morally, [CGSV21a]'s dichotomy theorem states that “all sketching algorithms can do is measure the marginals of their input instances, and to design algorithms it suffices to reduce to the $k$-variable case.” Indeed, for any pair $0 \leq \gamma < \beta \leq 1$, [CGSV21a] defines a set of weighted “satisfiable” instances $S_Y^Y(f) \subseteq \Delta(\{-1, 1\}^k)$ and a set of “unsatisfiable” instances $S_N^N(f) \subseteq \Delta(\{-1, 1\}^k)$.

The [CGSV21a] dichotomy theorem then states that if there exists $D_Y \in S_Y^Y(f)$ and $D_N \in S_N^N(f)$ such that $\mu(D_Y) = \mu(D_N)$, $(\gamma, \beta)$-Max-CSP($f$) cannot be solved by $o(\sqrt{n})$-space sketching algorithms, and otherwise, $(\gamma + \epsilon, \beta - \epsilon)$-Max-CSP($f$) can be solved by $O(\log n)$-space linear sketching algorithms. (See Definition 2.1 below for the definitions of $S_Y^Y(f)$ and $S_N^N(f)$, and Theorem 2.2 for the formal statement of the dichotomy theorem.) Thus, [CGSV21a]'s dichotomy theorem implies that Max-CSP($f$) can be $(\alpha(f) - \epsilon)$-approximated by $O(\log n)$-space linear sketches, but not $(\alpha(f) + \epsilon)$-approximated by $o(\sqrt{n})$-space sketches, where

$$
\alpha(f) \overset{\text{def}}{=} \inf_{\beta < \gamma \in [0, 1]} \{ \exists D_N \in S_N^N(f), D_Y \in S_Y^Y(f) \text{ s.t. } \mu(D_N) = \mu(D_Y) \left\{ \frac{\beta}{\gamma} \right\} \}.
$$

(1.9)

Let $\Delta_2 \subseteq \Delta(\{-1, 1\}^2)$ denote the set of symmetric distributions over $\{-1, 1\}^2$. In order to find a closed form for $\alpha(2\text{AND})$, [CGSV21a, Example 1] makes the observation that since $2\text{AND}$ is symmetric, it suffices WLOG to only consider symmetric distributions $D_Y, D_N \in \Delta_2$. Distributions $D \in \Delta_2$ can be represented using 3 instead of 4 probabilities and have scalar marginals $\mu(D) \in [-1, 1]$, reducing the dimension of the optimization problem. Thus, [CGSV21a] considers the following more convenient optimization problem:

$$
\alpha(2\text{AND}) = \inf_{\mu \in [-1, 1]} \left\{ \frac{\beta_{2\text{AND}}(\mu)}{\gamma_{2\text{AND}}(\mu)} \right\},
$$

(1.11)

where $\beta_{2\text{AND}}(\mu)$ and $\gamma_{2\text{AND}}(\mu)$ capture the bounds for symmetric distributions.
Theorem 1.1

This yields a novel technique, which we call the “max-min method”, for resolving the

\[ \text{Eq. (1.10)} \]

\[ \text{Eq. (1.11)} \]

below). Thus, for \( \alpha \), in order to give a closed form for \( \beta_{\text{AND}}(\mu) \) can be written as

\[
\beta_{\text{AND}}(\mu) = \inf_{D_N \in \Delta_{\text{AND}}(\mu)} \left\{ \sup_{p \in [0,1]} \{ \lambda_{\text{AND}}(D_N, p) \} \right\}
\]

where \( \lambda_{\text{AND}}(D_N, p) \) is a multivariate polynomial which is linear in the probabilities of \( D_N \) and quadratic in \( p \). Finally, [CGSV21a] calculates a closed form for \( \beta_{\text{AND}}(\mu) \) using the quadratic formula and uses this to determine \( \alpha(\text{AND}) \).

1.4.2 Our contribution: The max-min method

When \( f_{S,k} \) is a general symmetric function, applying similar ideas to [CGSV21a]’s 2AND analysis leads us to consider the set \( \Delta_k \) of symmetric distributions over \( \{-1,1\}^k \); each element \( D \in \Delta_k \) is described by \( k + 1 \) instead of \( 2^k \) probabilities and has a scalar marginal \( \mu_k(D) \in [-1,1] \). We can analogously define \( \gamma_{S,k}(\mu) \) and \( \beta_{S,k}(\mu) \). We calculate that \( \gamma_{S,k}(\mu) \) is a piecewise linear function of \( \mu \) (Lemma 2.8 below), while \( \beta_{S,k}(\mu) \) now involves a supremum over \( p \in [0,1] \) of a function \( \lambda_{S,k}(D_N, p) \) which is linear in \( D_N \) and degree-\( k \) in \( p \) (Lemma 2.9 below). Thus, for \( k \geq 3 \), to the best of our knowledge the “[CGSV21a]-style analysis” of explicitly calculating \( \beta_{S,k}(\mu) \) becomes impractical (as it involves working reasoning about the maxima of a generic degree-\( k \) polynomial).

Instead, we introduce a slightly different formulation of calculating \( \alpha \), parametrized now by \( D_N \):

\[
\alpha(f_{S,k}) = \inf_{D_N \in \Delta_k} \left\{ \frac{\beta_{S,k}(D_N)}{\gamma_{S,k}(\mu_k(D_N))} \right\}, \text{ where } \beta_{S,k}(D_N) = \sup_{p \in [0,1]} \{ \lambda_{S,k}(D_N, p) \}. \tag{1.10}
\]

We view optimizing directly over \( D_N \in \Delta_k \) as an important conceptual switch. In particular, our formulation emphasizes the calculation of \( \beta_{S,k}(D_N) \) as the centrally difficult feature (as opposed to first calculating \( \beta_{S,k}(\mu) \) and then optimizing over all \( \mu \)), yet we can still take advantage of the easiness of calculating \( \gamma_{S,k}(\mu) \).

A priori, calculating \( \beta_{S,k}(D_N) \) still involves maximizing a degree-\( k \) polynomial. To get around this difficulty, we have a crucial insight, which was not noticed by [CGSV21a] even in the 2AND case. If \( D_N^* \) minimizes the right-hand side of Eq. (1.10), and \( p^* \) maximizes \( \lambda_{S,k}(D_N^*, \cdot) \), the max-min inequality gives

\[
\alpha(f_{S,k}) \geq \inf_{D_N \in \Delta_k} \left\{ \frac{\lambda_{S,k}(D_N, p^*)}{\gamma_{S,k}(\mu_k(D_N))} \right\}. \tag{1.11}
\]

The right-hand side of Eq. (1.11) is relatively easy to calculate, being a ratio of a linear and piecewise linear function of \( D_N \). Our insight is that, in a wide variety of cases, the quantity on the right-hand side of Eq. (1.11) serendipitously equals \( \alpha(f_{S,k}) \); that is, \((D_N^*, p^*) \) is a saddle point of \( \lambda_{S,k}(\cdot, \cdot) \).\(^4\) This yields a novel technique, which we call the “max-min method”, for resolving the sketching approximability of \( f_{S,k} \): find \( D_N^* \) and \( p^* \), and then show that \( \lambda_{S,k}(\cdot, \cdot) \) has a saddle point at \((D_N^*, p^*) \). For instance, in Section 4, in order to give a closed form for \( \alpha(k\text{AND}) \) for odd \( k \) (i.e., the odd case of Theorem 1.1), we construct \( D_N^* \) by placing all the probability mass on strings of Hamming weight \( \frac{k+1}{2} \) (all of which are equally likely), set \( p^* = \frac{k+1}{2k} \), and prove \( \alpha(k\text{AND}) = \alpha^*_k \) by analyzing the right-hand side of the appropriate instantiation of Eq. (1.11). While we initially

\(^4\)This term comes from the optimization literature; such points are also said to satisfy the “strong max-min property” (see, e.g., [BV04, pp. 115, 238]). The saddle-point property is guaranteed by von Neumann’s minimax theorem for functions which are concave and convex in the first and second arguments, respectively, but this theorem and the generalizations we are aware of do not apply even to \( \lambda_{[3,3]} \).
found this pattern for $D_N^*$ by numerically investigating small odd $k$, in Section 4, we use the max-min method to provide an analytical proof for all odd $k$. We use similar techniques to the cases of $k$AND for even $k$ (also Theorem 1.1), $Th_k^{k-1}$ for even $k$ (Theorem 1.3, proved in Section 5.1), and several other cases in Section 5.

In all of these cases, the $D_N^*$ we construct is supported on at most two distinct Hamming weights, which is the property which makes finding $D_N^*$ tractable (using computer assistance). However, this technique is not a “silver bullet”: it is not the case that the sketching approximability of every symmetric Boolean CSP can be exactly calculated by finding the optimal $D_N^*$ supported on two elements and using the max-min method. Indeed, (as mentioned in Section 5) we verify using computer assistance that this is not the case for $f_{\{3,4\}}$.

Finally, we remark that the saddle-point property is precisely what defines the value $p^*$ required for our simple classical algorithm for outputting approximately optimal assignments for Max-CSP($f_{S,k}$) where $f_{S,k} = Th_k^1$ is a threshold function (see Corollary 3.4 below).

1.5 Related work

The classical approximability of Max-$k$AND has been the subject of intense study, both in terms of algorithms [GW95; FG95; Zwi98; Tre98a; TSSW00; Has04; Has05] and hardness-of-approximation [Has01; Tre08b; ST98; ST00; EHO08; ST09]. Currently, the best results appear to be as follows: Hast [Has05] constructed a $\Omega(k/\log k \cdot 2^{-k})$-approximation to Max-$k$AND; Engebretsen and Holmerin [EH08] proved that it is NP-hard to $2^{\sqrt{2k-2+1/2-k}}$-approximate Max-$k$AND, and this was improved by Samorodnitsky and Trevisan [ST09] to $(k+1)2^{-k}$ under the unique games conjecture, matching [Has05]’s algorithm up to logarithmic factors.

Interestingly, recalling that $\alpha(k\text{AND}) \to \rho(k\text{AND}) = 2^{-(k-1)}$ as $k \to \infty$, in the large-$k$ limit our simple sketching algorithm (given by Theorem 1.6) matches the performance of Trevisan [Tre98a]’s parallelizable LP-based algorithm for $k$AND, which (to the best of our knowledge) was the first work on the general $k$AND problem! (The subsequent works [Has04; Has05] superseding [Tre98a] used more complex techniques involving SDPs and random restrictions.)

1.6 Future directions

In this paper, we introduce the max-min method and use it to resolve the streaming approximability of a wide variety of symmetric Boolean CSPs (including multiple infinite families). However, these techniques are in a sense “ad hoc,” as they require numerically solving the intended optimization problem with computer assistance. We conjecture that the max-min method applies for all symmetric Boolean CSPs. We also hope to develop new techniques for finding $D_N^*$ and $p^*$ in a wider variety of cases (including those where $D_N^*$ is not supported on two elements).

Separately, Theorem 6.1 proves that [CGSV21a]’s streaming-hardness classification is incomplete and establishes resolving the streaming approximability of Max-3AND as a significant frontier problem.

Code

Our Mathematica code, which we use for calculations primarily in Sections 5 and 6, is available online on Github at https://gist.github.com/singerng/48f1e28e1dc671319ad75578fb45c0f0.
2 Preliminaries

2.1 Definitions and results from [CGSV21a]
Let us begin by defining the sets $S^Y_\gamma(f)$ and $S^N_\beta(f)$. For $f : \{-1,1\}^k \rightarrow \{0,1\}$, let

$$
\lambda_f(D, p) \overset{\text{def}}{=} \mathbb{E}_{a \sim D, b \sim \text{Bern}(p)}[f(a \odot b)]
$$

where $\text{Bern}(p)$ is 1 with probability $1-p$ and $-1$ with probability $p$, i.e., $\lambda_f(D, p)$ is the probability that a “$p$-noisy” random assignment from $D$ satisfies $f$. Then we have:

**Definition 2.1** (The sets $S^Y_\gamma(f)$ and $S^N_\beta(f)$). Let $f : \{-1,1\}^k \rightarrow \{0,1\}$ and $0 \leq \beta < \gamma \leq 1$. Then

$$
S^Y_\gamma(f) \overset{\text{def}}{=} \{D_Y \in \Delta(\{-1,1\}^k) : \lambda_f(D_Y,0) \geq \gamma\}
$$

and

$$
S^N_\beta(f) \overset{\text{def}}{=} \{D_N \in \Delta(\{-1,1\}^k) : \forall p \in [0,1], \lambda_f(D_N,p) \leq \beta\}.
$$

The main result of [CGSV21a] is a dichotomy theorem for sketching approximations to CSPs based on the marginals of distributions in $S^Y_\gamma(f)$ and $S^N_\beta(f)$:

**Theorem 2.2** (Sketching dichotomy theorem, [CGSV21a, Theorem 2.3]). For every function $f : \{-1,1\}^k \rightarrow \{0,1\}$ and for every $0 \leq \beta < \gamma \leq 1$, the following hold:

1. If there exist $D_Y \in S^Y_\gamma(f), D_N \in S^N_\beta(f)$ such that $\mu(D_Y) = \mu(D_N)$, then for every $\epsilon > 0$, every sketching algorithm for $(\gamma - \epsilon, \beta + \epsilon)$-$\text{Max-CSP}(f)$ requires $\Omega(\sqrt{n})$ space.

2. If not, then the $(\gamma, \beta)$-$\text{Max-CSP}(f)$ admits a linear sketching algorithm that uses $O(\log n)$ space.

[CGSV21a] also defines the following condition on pairs $(D_N, D_Y)$, stronger than $\mu(D_N) = \mu(D_Y)$, which implies hardness of $(\gamma, \beta)$-$\text{Max-CSP}(f)$ for streaming algorithms:

**Definition 2.3** (Padded one-wise pairs, [CGSV21a, §2.3]). A pair of distributions $(D_Y, D_N) \in \Delta(\{-1,1\}^k)$ forms a padded one-wise pair if there exists $\tau \in [0,1]$ and distributions $D_0, D'_Y$, and $D'_N$ such that (1) $\mu(D'_Y) = \mu(D'_N) = 0^\tau$ and (2) $D_Y = \tau D_0 + (1-\tau) D'_Y$ and $D_N = \tau D_0 + (1-\tau) D'_N$.

**Theorem 2.4** (Streaming lower bound for padded one-wise pairs, [CGSV21a, Theorem 2.11]). For every function $f : \{-1,1\}^k \rightarrow \{0,1\}$ and for every $0 \leq \beta < \gamma \leq 1$, if there exists a padded one-wise pair of distributions $D_Y \in S^Y_\gamma(f)$ and $D_N \in S^N_\beta(f)$ then, for every $\epsilon > 0$, $(\gamma - \epsilon, \beta + \epsilon)$-$\text{Max-CSP}(f)$ requires $\Omega(\sqrt{n})$ space in the streaming setting.

2.2 Setup for the symmetric case

Recall, we denote by $f_{S,k} : \{-1,1\}^k \rightarrow \{0,1\}$ the symmetric function which is the indicator for its input $b$ having Hamming weight $\text{wt}(b) \in S$, and $\Delta_k \subseteq \Delta(\{-1,1\}^k)$ denotes the set of symmetric distributions on $\{-1,1\}^k$. When studying the approximability of $\text{Max-CSP}(f)$, we restrict WLOG to the case where every element of $S$ is larger than $\frac{k}{2}$, since if $S$ contains elements $s \leq \frac{k}{2}$ and $t \geq \frac{k}{2}$, not necessarily distinct, then $f_{S,k}$ supports one-wise independence and is therefore streaming approximation-resistant.

Given $D \in \{-1,1\}^k$, we define its symmetrization $\text{Sym}(D)$ as the symmetric distribution given by randomly permuting a random element of $D$. Then following proposition lets us restrict to examining symmetric distributions in $S^Y_\gamma(f)$ and $S^N_\beta(f)$ for the purposes of determining the sketching and streaming approximability of symmetric functions.
**Proposition 2.5.** Let \( f = f_{S,k} \) be a symmetric function. For \( \beta < \gamma \in [0,1] \), suppose that there exists \( D_N \in S^N_\beta(f) \), \( D_Y \in S^Y_\gamma(f) \) with \( \mu(D_N) = \mu(D_Y) \). Then there exists symmetric \( \text{Sym}(D_N) \in S^N_\beta(f), \text{Sym}(D_Y) \in S^Y_\gamma(f) \) with \( \mu(\text{Sym}(D_N)) = \mu(\text{Sym}(D_Y)) \). Moreover, if \((D_Y, D_N)\) is a padded one-wise pair, then so is \((\text{Sym}(D_Y), \text{Sym}(D_N))\).

We typically write a distribution \( D \in \Delta_k \) as a vector \((p_0, \ldots, p_k)\) (where each \( p_i \) is the total probability mass on strings of Hamming weight \( i \)), which we refer to as the “mass on level \( i \)”, but use \( b \sim D \) to indicate an element drawn from \([-1, 1]^k\) according to the induced distribution.

The following proposition encapsulates the optimization problem arising from calculating the sketching approximability of a symmetric function according to \[CGSV21a\]:

**Proposition 2.6.** Let \( S \subseteq [k] \) be such that every element is larger than \( \frac{k}{\gamma} \). Then

\[
\alpha(f_{S,k}) \overset{\text{def}}{=} \inf_{D_N \in \Delta_k} \{\alpha_{S,k}(D_N)\}
\]

where

\[
\alpha_{S,k}(D_N) \overset{\text{def}}{=} \frac{\beta_{S,k}(D_N)}{\gamma_{S,k}(\mu_k(D_N))}
\]

and where for \( D \in \Delta_k \),

\[
\mu_k(D) = \mathbb{E}_{b \sim D}[b_1] = \cdots = \mathbb{E}_{b \sim D}[b_k];
\]

for \( \mu \in [-1, 1] \),

\[
\gamma_{S,k}(\mu) \overset{\text{def}}{=} \sup_{D_Y \in \Delta_k : \mu(D_Y) = \mu} \{\gamma(D_Y)\}, \text{ where } \gamma(D_Y) = \mathbb{E}_{a \sim D_Y}[f_{S,k}(a)];
\]

for \( D_N \in \Delta_k \),

\[
\beta(D_N) \overset{\text{def}}{=} \sup_{p \in [0,1]} \{\lambda_{S,k}(D_N, p)\};
\]

and for \( D \in \Delta_k \) and \( p \in [0,1] \),

\[
\lambda_{S,k}(D_N, p) \overset{\text{def}}{=} \mathbb{E}_{a \sim D_N, b \sim \text{Bern}(p)}[f_{S,k}(a \odot b)].
\]

Moreover, we have the following explicit formulae for \( \mu_k, \gamma_{S,k}, \) and \( \lambda_{S,k} \):

**Lemma 2.7.** For any \( D = (p_0, \ldots, p_k) \in \Delta_k \),

\[\mu_k(D) = \sum_{i=0}^{k} \epsilon_{i,k} p_i\]

where \( \epsilon_{i,k} \overset{\text{def}}{=} -1 + \frac{2i}{k} \) for each \( i \in [k] \).

**Proof.** Use linearity of expectation; the contribution of weight-\( i \) vectors to \( \mu_k(D) = \mathbb{E}_{b \sim D}[b_1] \) is \( p_i \cdot \frac{1}{k}(i \cdot 1 + (k - i) \cdot (-1)) = \epsilon_{i,k} p_i \).

**Lemma 2.8.** Let \( S \subseteq [k] \), and let \( s \) be its smallest element and \( t \) its largest element (they need not be distinct). Then for \( \mu \in [-1, 1] \),

\[
\gamma_{S,k}(\mu) = \begin{cases} 
1 + \mu & \mu \in [-1, \epsilon_{s,k}] \\
1 & \mu \in [\epsilon_{s,k}, \epsilon_{t,k}] \\
1 - \mu & \mu \in [\epsilon_{t,k}, 1]
\end{cases}
\]

(which also equals \( \min \left\{ \frac{1+\mu}{1+\epsilon_{s,k}}, 1, \frac{1-\mu}{1-\epsilon_{t,k}} \right\} \)).
Case 2: $\mu \in [-1, \epsilon_{s,k}]$. Our strategy is to reduce to $\mathcal{D}$ being supported on $\{0, s\}$ while preserving the marginal $\mu$ and (possibly weakly) increasing the value of $\gamma$.

Consider the following operation on distributions: For $u < v < w \in [k]$, increase $p_u$ by $p_u \frac{w-v}{w-u}$, increase $p_w$ by $p_w \frac{w-u}{w-v}$, and set $p_v$ to zero. Note that this results in a new distribution with the same marginal, since

$$p_v \frac{w-v}{w-u} \epsilon_{u,k} + p_u \frac{v-u}{w-u} \epsilon_{w,k} = p_v \epsilon_{v,k}.$$ 

Given an initial distribution $\mathcal{D}$, we can apply this operation to zero out $p_v$ for $v \in \{1, \ldots, s-1\}$ by redistributing to $p_0$ and $p_s$, preserving the marginal and only increasing the value of $\gamma$ (since $v \not\in S$ while $s \in S$). Similarly, we can redistribute $p_v \in \{t+1, \ldots, k-1\}$ to $p_t$ and $p_k$, and $p_v \in \{s+1, \ldots, t-1\}$ to $p_s$ and $p_t$. Thus, we need only consider $\mathcal{D}$ supported on $\{0, s, t, k\}$, which we assume WLOG are distinct.

We have

$$\mu(\mathcal{D}) = -p_0 + p_s \left(1 + \frac{2s}{k}\right) + p_t \left(-1 + \frac{2t}{k}\right) + p_k \leq 1 + \frac{2s}{k}.$$ 

Substituting $p_s = 1 - p_0 - p_t - p_k$ and multiplying through by $\frac{k}{2}$, we have

$$kp_k - sp_k - sp_t + tp_t \leq 0;$$ 

defining $\delta = p_t(\frac{k}{2} - 1) + p_k(\frac{k}{2} - 1)$, we can rearrange to get $p_0 \geq \delta$. Then given $\mathcal{D}$, we can zero out $p_t$ and $p_k$, decrease $p_0$ by $\delta$, and correspondingly increase $p_s$ by $p_t + p_k + \delta$. This fixes the marginal since

$$(\delta + p_t + p_k) \epsilon_{s,k} = -\delta + p_t \epsilon_{t,k} + p_k$$

and can only increase $\gamma$.

Thus, it suffices to consider $\mathcal{D}$ supported only on $\{0, s\}$; setting $p_0 = \frac{\epsilon_{s,k} - \mu}{\epsilon_{s,k} + 1}$ and $p_s = \frac{1 + \mu}{\epsilon_{s,k} + 1}$, which has marginal $\mu(\mathcal{D}) = \mu$ and the desired $\gamma(\mathcal{D}) = p_s$.

Case 2: $\mu \in [\epsilon_{s,k}, \epsilon_{t,k}]$. We simply construct $\mathcal{D} = (p_0, \ldots, p_k)$ with $p_s = \frac{\epsilon_{t,k} - \mu}{\epsilon_{s,k} - \epsilon_{t,k}}$ and $p_t = \frac{\mu - \epsilon_{s,k}}{\epsilon_{s,k} - \epsilon_{t,k}}$; we have $\mu(\mathcal{D}) = \mu$ and $\gamma(\mathcal{D}) = 1$.

Case 3: $\mu \in [\epsilon_{t,k}, 1]$. Following the symmetric logic to Case 1, we consider $\mathcal{D}$ supported on $\{t, k\}$ and set $p_t = \frac{1 - \mu}{1 - \epsilon_{t,k}}$ and $p_k = \frac{\mu - \epsilon_{t,k}}{1 - \epsilon_{t,k}}$, yielding $\mu(\mathcal{D}) = \mu$ and $\gamma(\mathcal{D}) = p_t$. 

Lemma 2.9. For any $\mathcal{D} = (p_0, \ldots, p_k)$ and $p \in [0, 1]$, we have

$$\lambda_{S,k}(\mathcal{D}, p) = \sum_{s \in S} \sum_{i=0}^{k} \left( \sum_{j=\max(0, s-(k-i))}^{\min\{i,s\}} \binom{i}{j} \binom{k-i}{s-j} q^{s+i-2j} p^{k-s-i+2j} \right) p_i$$

where $q \overset{\text{def}}{=} 1 - p$.

Proof. By linearity of expectation and symmetry, it suffices to fix $s$ and $i$ and consider, given a fixed string $a = (a_1, \ldots, a_k)$ of Hamming weight $i$ and a random string $b = (b_1, \ldots, b_k) \sim \text{Bern}(p)^k$, the probability of the event that $\text{wt}(a \otimes b) = s$. 

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Let \( A = \text{supp}(a) = \{ t \in [k] : a_t = 1 \} \) and similarly \( B = \text{supp}(b) \). We have \( |A| = i \) and
\[
s = \text{wt}(a \odot b) = |A \cap B| + |([k] \setminus A) \cap ([k] \setminus B)|.
\]

Let \( j = |A \cap B| \), and consider cases based on \( j \).

Given fixed \( j \), we must have \( |A \cap B| = j \) and \( |([k] \setminus A) \cap ([k] \setminus B)| = s - j \). Thus if \( j \) satisfies \( j \leq i, s - j \leq k - i, j \geq 0, j \leq s \), we have \( \binom{i}{j} \) choices for \( A \cap B \) and \( \binom{k-j}{s-j} \) choices for \( ([k] \setminus A) \cap ([k] \setminus B) \); together, these completely determine \( B \). Moreover \( \text{wt}(b) = |B| = |B \cap A| + |B \cap ([k] \setminus A)| = j + (k - i) - (s - j) = k - s - i + 2j \), yielding the desired formula.

### 2.3 \( \ell_1 \)-norm estimation

For our simple sketching algorithms for Max-CSP(\( f \)) where \( f \) is a threshold function (Section 3 immediately below), we use \( \ell_1 \)-norm sketching as a subroutine (as used also in [GVV17; CGV20; CGSV21a]):

**Theorem 2.10 ([Ind06; KNW10]).** For \( \epsilon > 0 \), there exists an \( O(\log n/\epsilon^2) \)-space randomized sketching algorithm for the following problem: The input is a stream \( S \) of updates of the form \((i, v) \in [n] \times \{1, -1\}\), and the goal is to output the \( \ell_1 \)-norm of the vector \( x \in [n]^n \) defined by \( x_i = \sum_{(i, v) \in S} v \), up to a multiplicative factor of \( 1 \pm \epsilon \).

### 3 Simple sketching algorithms for threshold functions

Let \( f_{S,k} \) be a threshold function. First, we define \( \beta_{S,k} \) and \( \gamma_{S,k} \) values for a marginal \( \mu \), recalling [CGSV21a]’s formulation of finding \( \alpha \) by minimizing \( \frac{\beta_{S,k}(\mu)}{\gamma_{S,k}(\mu)} \) over \( \mu \in [-1, 1] \). (\( \gamma_{S,k}(\mu) \) was defined already in Proposition 2.6; we repeat its definition here for clarity.) We let
\[
\beta_{S,k}(\mu) \overset{\text{def}}{=} \inf_{\mathcal{D}_N \in \Delta_k \text{ s.t. } \mu_k(\mathcal{D}_N) = \mu} \{ \beta_{S,k}(\mathcal{D}_N) \} \text{ and } \gamma_{S,k}(\mu) \overset{\text{def}}{=} \sup_{\mathcal{D}_Y \in \Delta_k \text{ s.t. } \mu_k(\mathcal{D}_Y) = \mu} \{ \gamma_{S,k}(\mathcal{D}_Y) \},
\]
where \( \beta_{S,k}(\mathcal{D}_N) \) and \( \gamma_{S,k}(\mathcal{D}_Y) \) are defined as in Proposition 2.6.

Given an instance \( \Psi \) of Max-CSP(\( f \)), for \( i \in [n] \), let \( \text{diff}_i(\Psi) \) denote the difference between the number of occurrences of \( x_i \) and \( \overline{x}_i \) in \( \Psi \). Let \( \mu(\Psi) = \frac{1}{km} \sum_{i=1}^n \text{diff}_i(\Psi) \) and \( \text{bias}(\Psi) = \frac{1}{km} \sum_{i=1}^n \text{diff}_i(\Psi) \). Also recall that for \( a \in \{-1, 1\}^n \), [CGSV21a] defined \( \Psi^a \) as flipping the sign of each variable \( x_i \) according to \( a_i \). Note that for all \( a \), \( \text{bias}(\Psi) = \text{bias}(\Psi^a) \). Also, since \( f_{S,k} \) is a threshold function, \( \gamma_{S,k}(\cdot) \) is monotone (e.g., by inspection from Lemma 2.8).

For an \( n \)-variable instance \( \Psi \), let \( \text{Sym}(\Psi) \) denote the weighted symmetric \( k \)-variable instance (or alternatively, distribution in \( \Delta_k \)) given by randomly selecting a constraint in \( \Psi \) and randomly permuting it. Note that \( \mu(\Psi) = \mu(\text{Sym}(\Psi)) \) by linearity (and \( \mu(\text{Sym}(\Psi)) = \mu_k(\text{Sym}(\Psi)) \) as we defined for distributions in \( \Delta_k \) in Proposition 2.6).

We make the following two claims:

**Lemma 3.1.** \( \text{val}_{\Psi} \leq \gamma_{S,k}(\text{bias}(\Psi)) \).

**Lemma 3.2.** \( \text{val}_{\Psi} \geq \beta_{S,k}(\text{bias}(\Psi)) \).

These two lemmas imply that outputting \( \alpha(f_{S,k})\gamma(\text{bias}(\Psi)) \) gives an \( \alpha \)-approximation to Max-CSP(\( f \)) for every \( \epsilon > 0 \). We get a small-space sketching algorithm since \( \text{bias}(\Psi) \) is measurable using \( \ell_1 \)-sketching algorithms (Theorem 2.10):
Proof of Theorem 1.6. To get an \((\alpha - \epsilon)\)-approximation to \(\text{val}_\Psi\), let \(\delta > 0\) be small enough s.t. 
\[
\frac{(1-\delta)\alpha(f_{S,k})}{1+\delta} \geq \alpha(f_{S,k}) - \epsilon.
\]
We claim that calculating an estimate \(\hat{b}\) for \(\text{bias}(\Psi)\) using \(\ell_1\)-norm sketching (Theorem 2.10) up to a multiplicative \(\delta\) factor and outputting \(\hat{v} = \alpha(f_{S,k})\gamma_{S,k}(\frac{\hat{b}}{1+\delta})\) is sufficient.

Indeed, suppose \(\hat{b} \in [(1-\delta)\text{bias}(\Psi), (1+\delta)\text{bias}(\Psi)]\); then \(\hat{b} \in \left[\frac{1-\delta}{1+\delta}\text{bias}(\Psi), \text{bias}(\Psi)\right]\). We claim that \(\gamma_{S,k}(\frac{\hat{b}}{1+\delta}) \in \left[\frac{1-\delta}{1+\delta}\gamma_{S,k}(\text{bias}(\Psi)), \gamma_{S,k}(\text{bias}(\Psi))\right]\); \(\gamma_{S,k}(\frac{\hat{b}}{1+\delta}) \leq \gamma_{S,k}(\text{bias}(\Psi))\) follows immediately from monotonicity, and on the other hand we have

\[
\gamma_{S,k}\left(\frac{\hat{b}}{1+\delta}\right) \geq \gamma_{S,k}\left(\frac{1-\delta}{1+\delta}\text{bias}(\Psi)\right) = \min\left\{\frac{1 + \frac{\delta}{1+\delta}\text{bias}(\Psi)}{1 + \epsilon_{s,k}}, 1\right\} \\
\geq \frac{1 + \frac{\delta}{1+\delta}}{1 - \delta} \min\left\{\frac{1 + \text{bias}(\Psi)}{1 + \epsilon_{s,k}}, 1\right\} = \frac{1 + \delta}{1 - \delta} \gamma_{S,k}(\text{bias}(\Psi)).
\]

Then by Lemmas 3.1 and 3.2 and the fact that \(\beta_{S,k}(\mu) \geq \alpha(f_{S,k})\gamma_{S,k}(\mu)\) (by definition), we conclude

\[(\alpha(f_{S,k}) - \epsilon)\text{val}_\Psi \leq \frac{(1-\delta)\alpha(f_{S,k})}{1+\delta} \gamma_{S,k}(\text{bias}(\Psi)) \leq \hat{v} \leq \alpha(f_{S,k})\gamma_{S,k}(\text{bias}(\Psi)) \leq \beta_{S,k}(\text{bias}(\Psi)) \leq \text{val}_\Psi,
\]
as desired. \(\square\)

It remains to prove the lemmas.

Proof of Lemma 3.1. Let \(\text{opt} \in \{-1, 1\}^n\) denote the optimal assignment for \(\Psi\). Then

\[\text{val}_\Psi = \text{val}_{\Psi_{\text{opt}}}(1^n) = \text{val}_{\text{Sym}(\Psi_{\text{opt}})}(1^n) \leq \gamma_{S,k}(\mu(\text{Sym}(\Psi_{\text{opt}}))) = \gamma_{S,k}(\mu(\Psi_{\text{opt}})).\]

Since \(f\) is a threshold function, \(\gamma_{S,k}(\cdot)\) is monotone (by Lemma 2.8), so

\[\text{val}_\Psi \leq \gamma_{S,k}(\text{bias}(\Psi_{\text{opt}})) = \gamma_{S,k}(\text{bias}(\Psi)),\]
as desired. \(\square\)

Proof of Lemma 3.2. Let \(\text{maj} \in \{-1, 1\}^n\) denote the assignment assigning \(x_i\) to 1 if \(\text{diff}_i(\Psi) \geq 0\) and \(-1\) otherwise, so that in \(\Psi_{\text{maj}}\) every variable occurs at least as often positively as negatively; thus,

\[\text{bias}(\Psi) = \text{bias}(\Psi_{\text{maj}}) = \mu(\Psi_{\text{maj}})\].

Now

\[\text{val}_\Psi = \text{val}_{\Psi_{\text{maj}}} \geq \max_{p \in [0,1]} \left\{ \mathbb{E}_{a \sim \text{Bern}(\mu)^n} \left[ \text{val}_{\Psi_{\text{maj}}}(a) \right] \right\} = \max_{p \in [0,1]} \left\{ \mathbb{E}_{a \sim \text{Bern}(\mu)^k} \left[ \text{val}_{\text{Sym}(\Psi_{\text{maj}})}(a) \right] \right\} \geq \beta_{S,k}(\mu(\text{Sym}(\Psi_{\text{maj}}))) = \beta_{S,k}(\mu(\Psi_{\text{maj}})) = \beta_{S,k}(\text{bias}(\Psi)),\]
as desired. \(\square\)

Remark. Why do we require that \(f\) is symmetric and monotone (i.e., a threshold function)? If \(f\) weren’t symmetric, we could measure a coordinate-wise \(\text{bias}\) vector in \([0,1]^k\) (whose entries, for \(j \in [k]\), would be “\(\sum_i\) of absolute value of difference in positive and negative appearances of variable \(i\) in the \(j\)-th coordinate in a constraint”), and correspondingly define a coordinate-wise \(\mu\) vector in \([-1,1]^k\). But in the proof of Lemma 3.2, we couldn’t say that the bias and \(\mu\) vectors of \(\Psi_{\text{maj}}\) are equal. If \(f\) weren’t monotone, we couldn’t invoke the monotonicity of \(\gamma\) in the proofs of Lemma 3.1 and Theorem 1.6.
Observation 3.3. For 2AND, \[CGV20\] almost contains our analysis. \[CGV20, \S 3.2\] shows that \(\text{val}_\Psi \leq \frac{1}{2}(1 + \text{bias}(\Psi))\) (which is the 2AND version of Lemma 3.1), as well as \(\text{val}_\Psi \geq \frac{2}{3}(1 + \text{bias}(\Psi))\) in the regime \(\text{bias}(\Psi) \leq \frac{1}{3}\). This can easily be extended to the full 2AND version of Lemma 3.2 (for all values of \(\text{bias}(\Psi)\)) by setting \(\gamma\) to \(\frac{1}{8}\). This can be seen as an instance of the max-min method.

Finally, we explore another consequence of Lemma 3.1—a simple algorithm for outputting approximately-optimal assignments when the max-min method applies.

Corollary 3.4. Let \(f_{S,k} = \text{Th}_k^i\) be a threshold function and \(p^* \in [0, 1]\) be such that the max-min method applies, i.e.,

\[
\alpha(f_{S,k}) = \inf_{D_N \in \Delta_k} \left\{ \frac{\lambda_{S,k}(D_N, p^*)}{\gamma_{S,k}(\mu_k(D_N))} \right\}.
\]

Then the following algorithm, on input \(\Psi\), outputs an assignment which is \(\alpha(f_{S,k})\)-approximately optimal in expectation: Assign every variable to 1 if it occurs more often positively than negatively, and −1 otherwise, and then flip each variable’s assignment independently with probability \(p^*\).

Proof of Corollary 3.4. Suppose that \(\alpha(f_{S,k}) = \inf_{D_N \in \Delta_k} \left\{ \frac{\lambda_{S,k}(D_N, p^*)}{\gamma_{S,k}(\mu_k(D_N))} \right\}\). Following the proof of Lemma 3.1, we have

\[
\mathbb{E}_{a \sim \text{Bern}(p^*)} [\text{val}_\Psi^\text{maj}(a)] = \lambda_{S,k}(\text{Sym}(\Psi^\text{maj}), p^*) \geq \alpha(f_{S,k})\gamma_{S,k}(\mu_k(\text{Sym}(\Psi^\text{maj}))) \geq \alpha(f_{S,k})\text{val}_\Psi,
\]

where the last two steps follows from the max-min assumption and Lemma 3.1, respectively. Thus, outputting the \(p^*\)-noisy majority assignment gives an \(\alpha(f_{S,k})\)-approximation to Max-CSP\((f_{S,k})\) in expectation.

\section{Analysis of optimal ratio for Max-kAND}

| \(k\) | \(\alpha\) | \(D_N^*=\) |
|-----|-----|-----|
| 2   | \(\frac{4}{9}\) | \(0, \frac{1}{3}, \frac{1}{3}\) |
| 3   | \(\frac{2}{9}\) | \(0, 0, 1, 0\) |
| 4   | \(\frac{72}{625}\) | \(0, 0, \frac{9}{13}, \frac{4}{13}, 0\) |
| 5   | \(\frac{36}{625}\) | \(0, 0, 0, 1, 0, 0\) |

Table 1: Exact \(\alpha\) values for Max-kAND, for some values of \(k\).

In this section, we prove Theorem 1.1 (on the sketching approximability of Max-kAND). We first deduce from Lemma 2.8 and Lemma 2.9 that for \(D_N = (p_0, \ldots, p_k)\), \(\gamma_{(k), k}(\mu_k(D_N)) = \sum_{i=0}^k \frac{i}{k}p_i\) and \(\lambda_{(k), k}(D_N, p) = \sum_{i=0}^k q^{k-i}p^ip_i\) where \(q = 1 - p\). Theorem 1.1 then follows immediately from the following two lemmas:

Lemma 4.1. For all odd \(k \geq 3\), \(\alpha(k\text{-AND}) \leq \alpha_k\). For all even \(k \geq 2\), \(\alpha(k\text{-AND}) \leq 2\alpha_{k+1}\).

\footnote{In \[CGV20\]'s notation, \(\gamma\) is used to denote \(p - \frac{1}{2}\). \[CGV20\] sets \(\gamma\) to minimize the quadratic \(\lambda_{(2), 2}(D_N, \cdot)\) and achieves a stronger lower bound, but it still only gives a \(\frac{1}{2}\)-approximation.}

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Lemma 4.2. For all odd $k \geq 3$, $\alpha(k\text{AND}) \geq \alpha'_k$. For all even $k \geq 2$, $\alpha(k\text{AND}) \geq 2\alpha'_{k+1}$.

We prove Lemma 4.1 directly:

Proof of Lemma 4.1. Consider the case where $k$ is odd. It suffices to construct $D^*_N$ and $p^*$ such that $p^*$ maximizes $\lambda_{(k),k}(D^*_N,\cdot)$ and $\lambda_{(k),k}(\mu_k(D^*_N)) = \alpha'_k$. Let $D^*_N$ put all mass on level $\frac{k+1}{2}$ and let $p^* = \frac{1}{2} + \frac{k}{2k}$. To show $p^*$ maximizes $\lambda_{(k),k}(D^*_N,\cdot)$, we calculate its derivative:

$$\frac{d}{dp} \left[ (1 - p)^{k-1}p^{k+1} \right] = -(1 - p)^{k-1}p^{k+1} \left( kp - \frac{k+1}{2} \right),$$

which has zeros only at 0, 1, $p^*$. Thus, $\lambda_{(k),k}(D^*_N,\cdot)$ has critical points only at 0, 1, $p^*$ and, being positive only at $p^*$, is maximized at $p^*$. Moreover,

$$\frac{\lambda_{(k),k}(D^*_N,p^*)}{\gamma_{(k),k}(\mu_k(D^*_N))} = \left( \frac{\frac{k}{2} - \frac{1}{2k}}{\frac{5}{2}} \right) \left( \frac{\frac{k}{2} + \frac{1}{2k}}{\frac{5}{2} + \frac{1}{2k}} \right) = \alpha'_k,$$

as desired.

Similarly, consider the case where $k$ is even; here, we define $D^*_N$ with mass $\frac{(\frac{k}{2}+1)^2}{(\frac{k}{2})^2+(\frac{k}{2}+1)^2}$ on level $\frac{k}{2}$ and mass $\frac{(\frac{k}{2})^2}{(\frac{k}{2})^2+(\frac{k}{2}+1)^2}$ on level $\frac{k}{2} + 1$, and set $p^* = \frac{1}{2} + \frac{1}{2(k+1)}$. Calculating the derivative of $\lambda_{(k),k}(D^*_N,\cdot)$ yields

$$\frac{d}{dp} \left[ \frac{(\frac{k}{2}+1)^2}{(\frac{k}{2})^2+(\frac{k}{2}+1)^2} (1 - p)^{\frac{k}{2}p^{\frac{k}{2}}} + \frac{(\frac{k}{2})^2}{(\frac{k}{2})^2+(\frac{k}{2}+1)^2} (1 - p)^{\frac{k}{2}p^{\frac{k}{2}+1}} \right]$$

$$= -\frac{k}{2+2k+2k^2}(1 - p)^{\frac{k}{2}-2}p^{\frac{k}{2}-1} \left( \frac{k}{2} + 1 - 2p \right) \left( (k+1)p - \left( \frac{k}{2} + 1 \right) \right),$$

so $\lambda_{(k),k}(D^*_N,\cdot)$ has critical points at 0, 1, $p^*$, $\frac{1}{2} + \frac{k}{2k}$; $p^*$ is the only critical point in the interval [0, 1] for which $\lambda_{(k),k}(D^*_N,\cdot)$ is positive, and hence is its maximum. Finally, it can be verified algebraically that $\frac{\lambda_{(k),k}(D^*_N,p^*)}{\gamma_{(k),k}(\mu_k(D^*_N))} = 2\alpha'_{k+1}$, as desired. \[ \square \]

We prove Lemma 4.2 using the max-min method:

Proof of Lemma 4.2. First, suppose $k \geq 3$ is odd. Set $p^* = \frac{1}{2} + \frac{k}{2k} = \frac{k}{k+2}$. Using the max-min inequality, we want to show that

$$\alpha'_k \leq \inf_{D_N \in \Delta_k} \frac{\lambda_{(k),k}(D_N,p^*)}{\gamma_{(k),k}(\mu_k(D_N))} = \inf_{(p_0,\ldots,p_k) \in \Delta_k} \frac{\sum_{i=0}^{k}(1-p^*)^{k-i}(p^*)^i p_i}{\sum_{i=0}^{k} i p_i}.$$

Since the numerator and denominator are both linear, it suffices to check that for all $i \in \{0\} \cup [k]$,

$$(1-p^*)^{k-i}(p^*)^i \geq \alpha'_k \cdot \frac{i}{k}.$$ 

Note that $\alpha'_k = (1-p^*)^{\frac{k+1}{2}}(p^*)^{\frac{k-1}{2}}$, defining $r = \frac{p^*}{1-p^*} = \frac{k+1}{k-1}$ (so that $p^* = r(1-p^*)$), factoring out $(1-p^*)^k$, and simplifying, we can rewrite our desired inequality as

$$\frac{1}{2}(k-1)r^{i-\frac{k+1}{2}} \geq i$$

(4.3)
for each \( i \in \{0\} \cup [k] \). When \( i = \frac{k+1}{2} \) or \( \frac{k-1}{2} \), we have equality in Eq. (4.3). We extend to the other values of \( i \) by induction. Indeed, when \( i \geq \frac{k+1}{2} \), then “\( i \) satisfies Eq. (4.3)” implies “\( i + 1 \) satisfies Eq. (4.3)” because \( ri \geq i + 1 \), and when \( i \leq \frac{k-1}{2} \), then “\( i \) satisfies Eq. (4.3)” implies “\( i - 1 \) satisfies Eq. (4.3)” because \( \frac{i}{2} \geq i - 1 \).

Similarly, in the case where \( k \geq 2 \) is even, we set \( p^* = \frac{1}{2} + \frac{1}{2(k+1)} \) and \( r = \frac{p^*}{1-p^*} = \frac{k+2}{k} \). In this case, for \( i \in \{0\} \cup [k] \) the following analogue of Eq. (4.3) can be derived:

\[
\frac{1}{2} k r^{i - \frac{k}{2}} \geq i,
\]

and these inequalities follow from the same inductive argument. \( \square \)

5 Sketching approximability of other symmetric functions

5.1 \( \text{Th}^{k-1}_k \) for even \( k \)

In this subsection, we prove Theorem 1.3 (on the sketching approximability of \( \text{Th}^{k-1}_k \) for even \( k \geq 2 \)). It is necessary and sufficient to prove the following two lemmas:

Lemma 5.1. For all even \( k \geq 2 \), \( \alpha(\text{Th}^{k-1}_k) \leq \frac{k}{2} \alpha'_k \).

Lemma 5.2. For all even \( k \geq 2 \), \( \alpha(\text{Th}^{k-1}_k) \geq \frac{k}{2} \alpha'_k \).

Firstly, we give explicit formulas for \( \gamma_{(k-1,k),k} \) and \( \lambda_{(k-1,k),k} \). For \( D_N = (p_0, \ldots, p_k) \), Lemma 2.8 gives

\[
\gamma_{(k-1,k),k}(\mu_k(D_N)) = \min \left\{ \sum_{i=0}^{k} \frac{i}{k-1} p_i, 1 \right\}.
\]

Next, we calculate \( \lambda_{(k-1,k),k}(D_N, p) \) with Lemma 2.9. Let \( q = 1 - p \), and let us examine the coefficient on \( p_i \). For \( s = k \) contributes \( q^{k-i} p^k \). In the case \( i \leq k - 1 \), \( s = k - 1 \) contributes \( (k - i)q^{k-i-1}p^{i+1} \) for \( j = i \), and in the case \( i \geq 1 \), \( s = k - 1 \) contributes \( iq^{k-i+1}p^{i-1} \) for \( j = i - 1 \). Thus, altogether we can write

\[
\lambda_{(k-1,k),k}(D_N, p) = \sum_{i=0}^{k} q^{k-i-1}p^{i-1} ((k - i)p^2 + pq + iq^2) p_i.
\]

Now, we prove Lemmas 5.1 and 5.2.

Proof of Lemma 5.1. As in the proof of Lemma 4.1, it suffices to construct \( D^*_N \) and \( p^* \) such that \( p^* \) maximizes \( \lambda_{(k-1,k),k}(D^*_N, \cdot) \) and \( \frac{\lambda_{(k-1,k),k}(D^*_N, p^*)}{\gamma_{(k-1,k),k}(\mu_k(D_N))} = \frac{k}{2} \alpha'_k \).

We again \( p^* = \frac{1}{2} + \frac{1}{2(k-1)} \), but let \( D^*_N \) put mass \( \frac{(\frac{k}{2})^2}{\frac{k}{2} + (\frac{k}{2} - 1)} \) on level \( \frac{k}{2} \) and mass \( \frac{(\frac{k}{2} - 1)^2}{(\frac{k}{2})^2 + (\frac{k}{2} - 1)} \) on level \( \frac{k}{2} + 1 \). The derivative of \( \lambda_{(k-1,k),k}(D^*_N, \cdot) \) is now

\[
\frac{d}{dp} \left[ \frac{(\frac{k}{2})^2}{\frac{k}{2} + (\frac{k}{2} - 1)} (1 - p) \frac{k}{2} - p \frac{k}{2} - 1 \left( \frac{k}{2} p^2 + pq + \frac{k}{2} q^2 \right) + \frac{(\frac{k}{2} - 1)^2}{(\frac{k}{2})^2 + (\frac{k}{2} - 1)} (1 - p) \frac{k}{2} - 2 \frac{k}{2} \left( \left( \frac{k}{2} - 1 \right) p^2 + pq + \left( \frac{k}{2} + 1 \right) q^2 \right) \right]
\]

\[
= -\frac{1}{8(k^2 - 2k + 2)} (1 - p)^{\frac{k}{2} - 3} p^{\frac{k}{2} - 2} (-k + (2(k - 1)p) \xi(p),
\]

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where $\xi(p)$ is the cubic

$$
\xi(p) = -8k(k - 1)p^3 + 2(k^3 + k^2 + 6k - 12)p^2 - 2(k^3 - 4)p + k^2(k - 2).
$$

Thus, $\lambda_{\{k-1,k\},k}$'s critical points on the interval $[0, 1]$ are $0, 1, p^*$ and any roots of $\xi$ in this interval. We claim that $\xi$ has no additional roots in the interval $(0, 1)$. This can be verified directly by calculating roots for $k = 2, 4$, so assume WLOG $k \geq 6$.

Suppose $\xi(p) = 0$ for some $p \in (0, 1)$, and let $x = \frac{1}{p} - 1 \in (0, \infty)$. Then $p = \frac{1}{x + 1}$; plugging this in for $p$ and multiplying through by $(x + 1)^3$ gives the new cubic

$$
(-2k^2 + k^3)x^3 + (k^3 - 6k^2 + 8)x^2 + (k^3 - 4k^2 + 12k - 8)x + (k^3 - 8k^2 + 20k - 16) = 0 \tag{5.3}
$$

whose coefficients are cubic in $k$. It can be verified by calculating the roots of each coefficient of $x$ in Eq. (5.3) that all coefficients are positive for $k \geq 6$. Thus, Eq. (5.3) cannot have roots for positive $x$, a contradiction. Hence $\lambda_{\{k-1,k\},k}(D^*_N, \cdot)$ is maximized at $p^*$. Finally, it can be verified that $\frac{\lambda_{\{k-1,k\},k}(D^*_N, \cdot)}{\gamma_{\{k-1,k\},k}(\mu_k(D^*_N))} = \frac{k}{2} \alpha'_{k-1}$, as desired. \hfill $\square$

**Proof of Lemma 5.2.** Define $p^* = \frac{1}{2} + \frac{1}{2(2k-1)}$. Following the proof of Lemma 4.2 and using the lower bound $\gamma_{\{k-1,k\},k}(\mu_k(D_N)) \leq \sum_{i=0}^{k} \frac{i}{k-1} p_i$, it suffices to show that

$$
\frac{k}{2} \alpha'_{k-1} \leq \inf_{(p_0, \ldots, p_k) \in \Delta_k} \sum_{i=0}^{k} (1 - p)^{k-i-1}(p^*)^{i-1}((k - i)(p^*)^2 + p^*(1 - p^*) + i(1 - p^*)^2)p_i
$$

for which it, in turn, suffices to prove that for each $i \in \{0\} \cup [k]$,

$$
\frac{k}{2} \alpha'_{k-1} \frac{i}{k-1} \leq (1 - p^*)^{k-i-1}(p^*)^{i-1}((k - i)(p^*)^2 + p^*(1 - p^*) + i(1 - p^*)^2).
$$

We again observe that $\alpha'_{k-1} = (1-p^*)^{\frac{k}{2}-1}(p^*)^{\frac{k}{2}-1}$, define $r = \frac{p^*}{1-p^*} = \frac{k}{k-2}$, and factor out $(1-p^*)^{k-1}$, which simplifies our desired inequality to

$$
\frac{1}{2} r^{i-\frac{k}{2}-1} \frac{k - 2}{k - 1} (i + r + (k - i)r^2) \geq i. \tag{5.4}
$$

for each $i \in \{0\} \cup [k]$. Again, we assume $k \geq 6$ WLOG; the bases cases $i = \frac{k}{2} - 1, \frac{k}{2}$ can be verified directly, and we proceed by induction. If Eq. (5.4) holds for $i$, and we seek to prove it for $i + 1$, it suffices to cross-multiply and instead prove the inequality

$$
r(i + 1 + r + (k - (i + 1))r^2)i \geq (i + 1)(i + r + (k - i)r^2),
$$

which simplifies to

$$(k - 2i)(k - 1)(k^2 - 4i - 4) \leq 0,$$

which holds whenever $i \in \left[\frac{k}{2}, \frac{k^2-4}{4}\right]$ (and $\frac{k^2-4}{4} \geq k$ for all $k \geq 6$). The other direction (where $i \leq \frac{k}{2} - 1$ and we induct downwards) is similar. \hfill $\square$

**Observation 5.5.** For $\text{Th}_4^3$ the optimal $D^*_N = (0, 0, \frac{4}{5}, \frac{1}{5}, 0)$ does participate in a padded one-wise pair with $D^*_N = (0, \frac{4}{9}, 0, \frac{1}{15}, 0)$ (given by $D_0 = (0, 0, 0, 1, 0)$ and $\tau = \frac{1}{5}$) so we can rule out streaming $(\frac{4}{9} + \epsilon)$-approximations to Max-CSP($\text{Th}_4^3$).
5.2 \( f_{\frac{k+1}{2}, k} \) for odd \( k \)

In this section, we prove bounds on the sketching approximability of \( f_{\frac{k+1}{2}, k} \) for odd \( k \). To be precise, let \( D_{0,k} \) put mass \( \frac{k-1}{2k} \) on level 0 and mass \( \frac{k+1}{2k} \) on level \( k \).

We prove the following two lemmas:

**Lemma 5.6.** For all odd \( k \geq 3 \), \( \alpha(f_{\frac{k+1}{2}, k}) \leq \lambda_{\frac{k+1}{2}, k}(D_{0,k}, p'_k) \), where \( p'_k \) is defined as \( 3k - k^2 + \sqrt{4k^2 - 2k^3 + k^4} \).

**Lemma 5.7.** For odd \( k \in \{3, \ldots, 51\} \), for all \( p \in [0,1] \), \( \alpha(f_{\frac{k+1}{2}, k}) \) is minimized at \( D_{0,k} \).

Again, we begin by writing an explicit formula for \( \lambda_{\frac{k+1}{2}, k} \). For \( D_N = (p_0, \ldots, p_k) \), Lemma 2.9 gives

\[
\lambda_{\frac{k+1}{2}, k}(D_N, p) = \sum_{i=0}^{k} \left( \min\{i, \frac{k+1}{2}\} \left( \frac{i}{j} \right) \left( \frac{k}{k+1} - j \right) (1 - p) \frac{k+1}{2} - i + 1 \right) p_i.
\]

For \( i \leq \frac{k}{2} \), the sum over \( j \) goes from 0 to \( i \), and for \( i \geq \frac{k+1}{2} \), it goes from \( i - \frac{k}{2} \) to \( \frac{k+1}{2} \). Thus, in particular,

\[
\lambda_{\frac{k+1}{2}, k}(D_{0,k}, p) = \left( \frac{k}{k+1} \right) \left( \frac{k-1}{2k} (1 - p) \frac{k+1}{2} - i + 1 \right) \left( \frac{k}{k+1} - \frac{k+1}{2} \right) p_i.
\]

hence Lemmas 5.6 and 5.7 imply the following theorem:

**Theorem 5.9.** For odd \( k \in \{3, \ldots, 51\} \),

\[
\alpha(f_{\frac{k+1}{2}, k}) = \left( \frac{k}{k+1} \right) \left( \frac{k-1}{2k} (1 - p') \frac{k+1}{2} + \frac{k+1}{2k} (1 - p') \frac{k+1}{2} \right),
\]

where \( p'_k = \frac{3k - k^2 + \sqrt{4k^2 - 2k^3 + k^4}}{4k} \) as in Lemma 5.6.

Moreover, recalling that \( p(f_{\frac{k+1}{2}, k}) = \left( \frac{k}{k+1} \right) 2^{-k} \), it can be verified that

\[
\lim_{k \text{ odd} \to \infty} 2^k \left( \frac{k-1}{2k} (1 - p'_k) \frac{k+1}{2} + \frac{k+1}{2k} (1 - p'_k) \frac{k+1}{2} \right) = 1,
\]

proving Corollary 1.5.

We remark that Lemma 5.7 is stronger than what we were able to prove for \( k\text{AND} \) (Lemma 4.2) and \( \text{Th}_{k-1} \) (Lemma 5.2) as the inequality holds regardless of \( p \) (which is fortunate for us, as the optimal \( p^* \) from Lemma 5.6 is rather messy).\(^6\) It remains to prove Lemmas 5.6 and 5.7.

**Proof of Lemma 5.6.** Taking the derivative with respect to \( p \) of Eq. (5.8) yields

\[
\frac{d}{dp} \left[ \lambda_{\frac{k+1}{2}, k}(D_{0,k}, p) \right] = -\frac{1}{4k} \left( \frac{k}{k+1} \right) (pq) \frac{k+1}{2} (4kp^2 + (2k^2 - 6k)p + (-k^2 + 2k - 1)),
\]

\(^6\)The analogous statement is false for e.g. \( 3\text{AND} \), where we had \( D_N = (0,0,1,0) \), but

\[
\frac{\lambda_{3,3}(0,\frac{1}{2},\frac{1}{2},0)}{\gamma_{3,3}(\mu_3(0,\frac{1}{2},\frac{1}{2},0))} = \frac{3}{16} \leq \frac{27}{128} = \frac{\lambda_{3,3}(0,0,1,0,\frac{1}{2})}{\gamma_{3,3}(\mu_3(0,0,1,0))}.
\]
where \( q = 1 - p \). Thus, \( \lambda_{\{k+1\},k}(D_{0,k},\cdot) \) has critical points at \( p = 0, 1, p_k', \frac{3k^2 - \sqrt{4k + k^2 - 2k^2 + k^2}}{4k} \), and \( \frac{3k^2 - \sqrt{4k + k^2 - 2k^2 + k^2}}{4k} \leq 0 \) for all \( k \geq 0 \) (since \( (3k - k^2)^2 - (4k + k^2 - 2k^2 + k^2)^2 = -4k(k - 1)^2 \)). Finally, we observe that \( \mu_k(D_{0,k}) = \frac{1}{k} = \epsilon_{\{k+1\},k} \) and so \( \gamma_{\{k+1\},k}(\mu_k(D_{0,k})) = 1 \) and \( \alpha_{\{k+1\},k}(D_{0,k}) = \beta_{\{k+1\},k}(D_{0,k},p_k') \), as desired.  

To prove Lemma 5.7, we first reflect on our earlier lower bounds (i.e., Lemmas 4.2 and 5.2). Let \( D_i \in \Delta_k \) put mass 1 on level \( i \). We view the lower bound for \( k \text{AND} \) (Lemma 4.2) as invoking the fact that \( \frac{\lambda_{\{k\},k}(\cdot,p')}{\gamma_{\{k\},k}(\mu_k(\cdot))} \) is concave over its domain \( \Delta_k \) (since it is a ratio of linear functions); thus, since every \( \Delta_{\gamma} \in \Delta_k \) can be written as a convex combination of the distributions \( D_i \) for \( i \in \{0\} \cup [k] \) (which are the extreme points of \( \Delta_k \)), it is sufficient (and necessary) to check that

\[
\frac{\lambda_{\{k\},k}(D_i,p')}{\gamma_{\{k\},k}(\mu_k(D_i))} \geq \frac{\lambda_{\{k\},k}(D_{\gamma},p')}{\gamma_{\{k\},k}(\mu_k(D_{\gamma}))}
\]

for each \( i \in \{0\} \cup [k] \). On the other hand, in the lower bound for \( Th_k^{\{k\}} \) (Lemma 5.2), \( \frac{\lambda_{\{k-1\},k}(\cdot,p')}{\gamma_{\{k-1\},k}(\mu_k(\cdot))} \) is not concave over \( \Delta_k \), because the denominator \( \gamma_{\{k-1\},k}(\mu_k(p_0, \ldots, p_n)) = \min \{\sum_{i=0}^{k-1} p_i, 1\} \) is not convex over \( \Delta_k \). However, we can upper-bound the denominator with the convex function \( \gamma'(p_0, \ldots, p_k) = \sum_{i=0}^{k-1} \frac{1}{k-1} p_i \), and thus in the proof of Lemma 5.2 we check the sufficient condition that the inequality

\[
\frac{\lambda_{\{k-1\},k}(D_i,p')}{\gamma'(D_i)} \geq \frac{\lambda_{\{k-1\},k}(D_{\gamma},p')}{\gamma_{\{k-1\},k}(\mu_k(D_{\gamma}))}
\]

holds for each \( i \in \{0\} \cup [k] \). Unfortunately, no similar “convex upper-bounding” technique appears to work for \( f_{\{k+1\},k} \).

Instead, we use the fact that \( \alpha_{\{k+1\},k} \) is piecewise concave:

**Proof of Lemma 5.7.** Recalling that \( \epsilon_{\{k+1\},k} = \frac{1}{k} \), let \( \Delta_k^+ = \{ D \in \Delta_k : \mu_k(D) \leq \frac{1}{k} \} \) and \( \Delta_k^- = \{ D \in \Delta_k : \mu_k(D) \geq \frac{1}{k} \} \). Note that \( \Delta_k^+ \cup \Delta_k^- = \Delta_k \), and restricted to either \( \Delta_k^+ \) or \( \Delta_k^- \), \( \gamma_{\{k+1\},k}(\mu_k(\cdot)) \) is linear and thus \( \frac{\lambda_{\{k+1\},k}(\cdot,p)}{\gamma_{\{k+1\},k}(\mu_k(\cdot))} \) is concave.

Let \( D_{i,j} \in \Delta_k \), for \( i < \frac{k+1}{2}, j > \frac{k+1}{2} \), put mass \( \frac{2j-(k+1)}{2(j-i)} \) on level \( i \) and mass \( \frac{(k+1)-2i}{2(j-i)} \) on level \( j \). Note that \( \mu_k(D_{i,j}) = \frac{1}{k} \) for each \( i, j \). We claim that \( \{D_{i,j} \}_{i<\frac{k+1}{2}} \cup \{D_{i,j} \}_{j>\frac{k+1}{2}} \) are the extreme points of \( \Delta_k^- \), or more precisely, that every distribution \( D \in \Delta_k^- \) can be represented as a convex combination of these distributions. Indeed, this follows constructively from the procedure which, given a distribution \( D \), subtracts mass from levels below \( \frac{k+1}{2} \) (adding it to the coefficient of the appropriate \( D_i \)) until the marginal of the (renormalized) distribution is \( \frac{1}{k} \), and then subtracts mass from paired levels (with \( i < \frac{k+1}{2} \) and \( j > \frac{k+1}{2} \), adding it to the coefficient of the appropriate \( D_{i,j} \)) until no mass remains. Similarly, every distribution \( D \in \Delta_k^+ \) can be represented as a convex combination of the distributions \( \{D_{i,j} \}_{i<\frac{k+1}{2}} \cup \{D_{i,j} \}_{j>\frac{k+1}{2}} \). Thus, it is sufficient (and necessary) to check that

\[
\frac{\lambda_{\{k+1\},k}(D,p)}{\gamma_{\{k+1\},k}(\mu_k(D))} \geq \frac{\lambda_{\{k+1\},k}(D_{0,k},p)}{\gamma_{\{k+1\},k}(\mu_k(D_{0,k}))}
\]

for each \( D \in \{D_i \} \cup \{D_{i,j} \} \). Treating \( p \) as a variable, for each odd \( k \in \{3, \ldots, 51\} \) we produce a list of \( O(k^2) \) degree-\( k \) polynomial inequalities in \( p \) which we verify using Mathematica.  

\[\square\]
5.3 Other functions

In Table 2 below, we list four more symmetric Boolean functions whose sketching approximability (i.e., $\alpha$-value) we have analytically resolved using the “max-min method”. These values were calculated using two functions in the Mathematica code, estimateAlpha — which numerically or symbolically estimates the $D_N$, with a given support, which minimizes $\alpha$ — and testMinMax — which, given a particular $D_N$, calculates $p^*$ for that $D_N$ and checks analytically whether lower-bounding by evaluating $\lambda$ at $p^*$ proves that $D_N$ is minimal.

| $S$     | $k$ | $\alpha$                          | $D_N^*$         |
|---------|-----|-----------------------------------|-----------------|
| $\{2,3\}$ | 3   | $\frac{1}{2} + \frac{3\sqrt{7}}{15} \approx 0.5962$ | $(0, \frac{1}{2}, 0, \frac{1}{2})$ |
| $\{4,5\}$ | 5   | root$_R(P_1) \approx 0.2831$   | $(0, 0, 1 - \text{root}_R(P_2), \text{root}_R(P_2), 0, 0)$ |
| $\{4\}$   | 5   | root$_R(P_3) \approx 0.2394$   | $(0, 0, 1 - \text{root}_R(P_4), \text{root}_R(P_4), 0, 0)$ |
| $\{3,4,5\}$ | 5   | $\frac{1}{2} + \frac{3\sqrt{5}}{125} \approx 0.5537$ | $(0, \frac{1}{2}, 0, 0, 0, \frac{1}{2})$ |

Table 2: Symmetric functions for which we have analytically calculated exact $\alpha$ values using the “max-min method”. For a polynomial $P : \mathbb{R} \to \mathbb{R}$ with a unique positive real root, let root$_R(p)$ denote that root, and define the polynomials $P_1(z) = -72 + 4890z - 108999z^2 + 800000z^3$, $P_2(z) = -908 + 5021z - 9001z^2 + 5158z^3$, $P_3(z) = -60 + 5745z - 183426z^2 + 1953125z^3$, $P_4(z) = -344 + 1770z - 3102z^2 + 1811z^3$. (We note that in the $f_{\{4\},5}$ and $f_{\{4,5\},5}$ calculations, we were required to check equality of roots numerically (to high precision) instead of analytically).

We remark that two of the cases in Table 2 (as well as $k$AND), the optimal $D_N$ is rational and supported on two coordinates. However, in the other two cases in Table 2, the optimal $D_N$ involves roots of a cubic.

Recall that in Section 5.2, we showed that $D_N^*$ which puts mass $\frac{k-1}{2k}$ on level 0 and mass $\frac{k+1}{2k}$ on level $k$ is optimal for $f_{\{\frac{k+1}{2}\}}$, $k$ for odd $k \in \{3, \ldots, 51\}$. Using the same $D_N^*$, we are also able to resolve 11 other cases in which $S$ is “close to” $\{\frac{k+1}{2}\}$; for instance, $S = \{5,6\}$, $\{5,6,7\}$, $\{5,7\}$ for $k = 9$. (We have omitted the values of $\alpha$ and $D_N$ because they are defined using the roots of polynomials of degree up to 8).

In all previously-mentioned cases, the condition “$D_N^*$ has support size 2” was helpful, as it makes the optimization problem over $D_N^*$ essentially univariate; however, we have confirmed analytically in two other cases ($S = \{3\}$, $k = 4$ and $S = \{3,5\}$, $k = 5$) that “max-min method on distributions with support size two” does not suffice for tight bounds on $\alpha$ (see testDistsWithSupportSize2 in the Mathematica code). However, using the max-min method with $D_N$ supported on two levels still achieves decent (but not tight) bounds on $\alpha$. For $S = \{3\}$, $k = 4$, using $D_N = (\frac{1}{4}, 0, 0, 0, \frac{3}{4})$, we get the bounds $\alpha(f_{\{3,4\}}) \in [0.3209, 0.3295]$ (the difference being 2.67%). For $S = \{3,5\}$, $k = 5$, using $D_N = (\frac{1}{4}, 0, 0, 0, \frac{3}{4}, 0)$, we get $\alpha(f_{\{3,5\},5}) \in [0.3416, 0.3635]$ (the difference being 6.42%).

Finally, we have also analyzed cases where we get numerical solutions which are very close to tight (but we lack analytical solutions because they likely involve roots of high-degree polynomials). For instance, in the case $S = \{4,5,6\}$, $k = 6$, setting $D_N = (0, 0, 0, 0.930013, 0, 0, 0.069987)$ gives $\alpha(f_{\{4,5,6\},6}) \in [0.44409972, 0.44409973]$, differing only by 0.000003%. (We conjecture here that $\alpha = \frac{4}{7}$.) For $S = \{6,7,8\}$, $k = 8$, using $D_N = (0, 0, 0, 0, 0.699501, 0.300499)$, we get the bounds
\(\alpha(f_{\{6,7,8\},8}) \in [0.20848, 0.20854]\) (the difference being 0.02%).

### 6 Streaming lower bounds for Max-3AND

The goal of this section is to prove that [CGSV21a]'s dichotomy theorem fails to rule out streaming \((\frac{2}{3} + \epsilon)\)-approximations to Max-3AND:

**Theorem 6.1.** [CGSV21a]'s streaming hardness results cannot show that streaming algorithms cannot \((\frac{2}{3} + \epsilon)\)-approximate Max-3AND for every \(\epsilon > 0\). That is, there is no infinite sequence \((D_Y^{(1)}, D_N^{(1)}), (D_Y^{(2)}, D_N^{(2)}), \ldots\) of padded one-wise pairs on \(\Delta_3\) such that

\[
\lim_{t \to \infty} \frac{\beta_{\{3\},3}(D_N^{(t)})}{\gamma_{\{3\},3}(\mu_3(D_Y^{(t)}))} = \frac{2}{9}.
\]

**Proof outline.** Recall that since \(k = 3\) is odd, to prove Theorem 1.1 we showed that \(D_N^* = (0,0,1,0)\) minimizes \(\alpha_{\{3\},3}(\cdot)\). We have \(\mu_3(D_N^*) = \frac{1}{3}\) and \(\gamma_{\{3\},3}(\frac{1}{3}) = \frac{2}{3}\), which is achieved by \(D_Y^* = (\frac{1}{3},0,0,\frac{2}{3})\) (the minimizer of \(\gamma\) is in general unique). \((D_N^*, D_Y^*)\) are not a padded one-wise pair.

Hence, we first furthermore show that \(D_N^*\) is the unique minimizer of \(\alpha_{\{3\},3}(\cdot)\). For this purpose, the max-min method is not sufficient because \(\frac{\lambda_{\{3\},3}(\cdot,\frac{2}{3})}{\gamma_{\{3\},3}(\mu_3(\cdot))}\) is not uniquely minimized at \(D_N^*\).

Intuitively, this is because \(p^* = \frac{2}{3}\) is not a good enough estimate for the maximizer of \(\lambda_{\{3\},3}(D_N,\cdot)\). To remedy this, we observe that \(\lambda_{\{3\},3}((1,0,0,0),\cdot), \lambda_{\{3\},3}((0,1,0,0),\cdot), \lambda_{\{3\},3}((0,0,1,0),\cdot)\) and \(\lambda_{\{3\},3}((0,0,0,1),\cdot)\) are minimized at 0, \(\frac{1}{3}, \frac{2}{3}\), and 1, respectively. Hence, when \(D_N = (p_0,p_1,p_2,p_3)\), and instead lower-bounding \(\beta_{\{3\},3}(D_N,\cdot)\) by evaluating at \(\frac{1}{3}p_1 + \frac{2}{3}p_2 + p_3\). The theorem then follows from continuity arguments.

To formalize the proof, we begin with a few lemmas.

**Lemma 6.2.** For \((p_0,p_1,p_2,p_3) \in \Delta_3\), the expression

\[
\frac{\lambda_{\{3\},3}((p_0,p_1,p_2,p_3),\frac{1}{3}p_1 + \frac{2}{3}p_2 + p_3)}{\gamma_{\{3\},3}(\mu_3(p_0,p_1,p_2,p_3))}
\]

is minimized uniquely at \((0,0,1,0)\), with value \(\frac{2}{3}\).

**Proof.** Letting \(p = \frac{1}{3}p_1 + \frac{2}{3}p_2 + p_3\) and \(q = 1 - p\), by Lemmas 2.7 to 2.9 the expression expands to

\[
\frac{p_0p^3 + p_1p^2(1-p) + p_2p(1-p)^2 + p_3(1-p)^3}{\frac{1}{3}(1-p - \frac{1}{3}p_1 + \frac{2}{3}p_2 + p_3)}.
\]

The expression’s minimum, and its uniqueness, are confirmed analytically in the Mathematica code.

**Lemma 6.3.** Let \(X\) be a compact topological space, \(Y \subseteq X\) a closed subset, \(Z\) another topological space, and \(\alpha : X \to Z\) a continuous map. Let \(x^* \in X, z^* \in Z\) be such that \(x^*\) is the unique point mapped to \(z^*\) by \(f\). Let \(\{x_i\}_{i \in \mathbb{N}}\) be a sequence of points in \(Y\) such that \(\{z_i = f(x_i)\}_{i \in \mathbb{N}}\) converges to \(z^*\). Then \(x^* \in Y\).

\(\text{\footnote{Interestingly, in this latter case, we get bounds differing by 2.12\% using } D_N = (0,0,0,0,\frac{2}{3},\frac{1}{3},0,0,0)\text{ in an attempt to continue the pattern from } f_{\{7,8\},8}\text{ and } f_{\{8\},8}\text{ (where we set } D_N^* = (0,0,0,0,\frac{2}{3},\frac{1}{3},0,0,0)\text{ and } (0,0,0,0,\frac{2}{3},\frac{1}{3},0,0,0)\text{ in Section 5.1 and Section 4, respectively).}}\)
Proof. By compactness of $X$, there is a subsequence $\{x_{j_i}\}_{i \in \mathbb{N}}$ which converges to a limit $\tilde{x}$. By closure of $Y$, $\tilde{x} \in Y$. But by continuity, $f(\tilde{x}) = z^*$, so $\tilde{x} = x^*$.

Lemma 6.4. Let $M \subset \Delta_3 \times \Delta_3 \subset \mathbb{R}^8$ denote the space of pairs of distributions with matching marginals, and define the function

$$\alpha : M \to \mathbb{R} \cup \{\infty\} : (D_N, D_Y) \mapsto \frac{\beta_{3,3}(D_N)}{D_Y,3},$$

where $D_Y,3$ denotes the level-3 mass in $D_Y$. Then $\alpha$ is continuous.

Proof. It suffices to check that $\beta_{3,3}(\cdot)$ is continuous, as a ratio of continuous functions is continuous; a single-variable supremum of a continuous function over a compact interval is in general continuous in the other variables.

Finally, we have:

Proof. By Lemma 6.2, $\alpha_{3,3}$ is minimized uniquely at $D_N^* = (0, 0, 1, 0)$. Finally, we rule out the possibility of an infinite sequence of padded one-wise pairs which achieve ratios arbitrarily close to $\frac{2}{9}$ using topological properties. Let $P \subset M$ denote the space of padded one-wise pairs. Note that $P$ is closed. Since $\alpha$ is continuous (by Lemma 6.4), if there were a sequence of padded one-wise pairs $\{(D_N^{(i)}, D_Y^{(i)}) \in P\}_{i \in \mathbb{N}}$ such that $\alpha(D_N^{(i)}, D_Y^{(i)})$ converges to $\frac{2}{9}$ as $i \to \infty$, Lemma 6.3 implies that $(D_N^*, D_Y^*) \in P$, a contradiction.

We conclude with a related observation:

Observation 6.5. The padded one-wise pair $D_N = (0, 0.45, 0.45, 0.1), D_Y = (0.45, 0, 0, 0.55)$ (discovered by numerical search) does prove a streaming approximability upper bound of $\approx .2362$ for 3AND, which is still quite close to $\alpha(3\text{AND}) = \frac{2}{9}$.

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