REMARKS ON THE WDW EQUATION

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Abstract. We show a kind of converse to some results of Hall and Reginatto on exact uncertainty related to the Schrödinger and Wheeler-deWitt equations. Some survey material on statistical geometrodynamics is also sketched.

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1. INTRODUCTION

We abbreviate SE for the Schrödinger equation and WDW for the Wheeler-deWitt equation.

2. EXACT UNCERTAINTY AND THE SE

The exact uncertainty principle of Hall and Reginatto is discussed at length in [9, 24, 25, 26, 45]. Basically following e.g. [24, 26] one defines Fisher information via $F_x = \int dx P(x) [\partial_x \log(P(x))]^2$ and a Fisher length by $\delta x = F_x^{-1/2}$ where $P(x)$ is a probability density for a 1-D observable $x$. The Cramer-Rao inequality says $Var(x) \geq F_x^{-1}$ or simply $\Delta x \geq \delta x$. For a quantum situation with $P(x) = |\psi(x)|^2$ and $\psi$ satisfying a SE
finds immediately

\begin{equation}
F_X = \int dx |\psi|^2 \left[ \frac{\psi'}{\psi} + \frac{\psi'}{\psi} \right]^2 \ dx =
\end{equation}

\[= 4 \int dx \psi' \psi' + \int dx |\psi|^2 \left[ \frac{\psi'}{\psi} - \frac{\psi'}{\psi} \right]^2 = \frac{4}{\hbar^2} \left[ < p^2 > - < p^2 > \right] \]

where \(p_{cl} = (\hbar/2i) (\psi'/\psi) - (\psi'/\bar{\psi})\) is the classical momentum observable conjugate to \(x\) \((\sim S_X\) for \(\psi = R e^{iS/\hbar}\)). Setting now \(p = p_{cl} + p_{nc}\) one obtains after some calculation \((\bigstar)\) \(F_X = (4/\hbar^2)(\Delta p_{nc})^2 = 1/(\delta x)^2 \Rightarrow \delta x\Delta p_{nc} = \hbar/2\) as a relation between nonclassicality and Fisher information.

We recall also that from (2.1) \(F_X\) is proportional to the difference of a quantum and a classical kinetic energy. Thus \((\hbar^2/4)F_X(1/2m) = (1/2m) < p^2 > - (1/2m) < p^2 > \) and \(E_F = (\hbar^2/8m)F_X\) is added to \(E_{cl}\) to get \(E_{quant}\). By deBroglie-Bohm (dBB) theory there is a quantum potential

\begin{equation}
Q = \frac{\hbar^2}{8m} \left[ \left( \frac{P'}{P} \right)^2 - 2 \frac{P''}{P} \right]; \ P = |\psi|^2
\end{equation}

and evidently \((\bigstar)\) \(< Q > = \int PQ dx = (\hbar^2/8m)F_X\) (upon neglecting the boundary integral term at \(\pm \infty\) - i.e. \(P' \to 0\) at \(\pm \infty\)).

Now the exact uncertainty principle (cf. \([24, 26, 45]\)) looks at momentum fluctuations \((\bigstar)\) \(p = \nabla S + f\) with \(< f >= \bar{f} = 0\) and replaces a classical ensemble energy \(< E >_{cl}\) by \((P \sim |\psi|^2)\)

\begin{equation}
< E > = \int dx P \left[ (2m)^{-1} |\nabla S + f|^2 + V \right] = < E >_{cl} + \int dx P \frac{\bar{f} \cdot f}{2m}
\end{equation}

Upon making an assumption of the form \((\bigstar)\) \(\bar{f} \cdot f = \alpha(x, P, S, \nabla P, \nabla S, \cdots)\) one looks at a modified Hamiltonian \((\bigcdot \cdot)\) \(\tilde{H}_q[P, S] = \tilde{H}_{cl} + \int dx P (\alpha/2m)\).

Then, assuming

1. Causality - i.e. \(\alpha\) depends only on \(S, P\) and their first derivatives
2. Independence for fluctuations of noninteracting uncorrelated ensembles
3. \(f \to L^T f\) for invertible linear coordinate transformations \(x \to L^{-1} x\)
4. Exact uncertainty - i.e. \(\alpha = \bar{f} \cdot f\) is determined solely by uncertainty in position

one arrives at

\begin{equation}
\tilde{H}_q = \tilde{H}_{cl} + c \int dx \frac{\nabla P \cdot \nabla P}{2mP}
\end{equation}
and putting $h = 2\sqrt{c}$ with $\psi = \sqrt{P} \exp(iS/h)$ a SE is obtained (cf. Sections 4 and 5 for more detail).

As pointed out in [10] in the SE situation with $Q$ as in (2.2), in 3-D one has

\begin{equation}
\int PQd^3x \sim -\frac{\hbar^2}{8m} \int \left[2\Delta P - \frac{1}{P}(\nabla P)^2\right] d^3x = \frac{\hbar^2}{8m} \int \frac{1}{P}(\nabla P)^2 d^3x
\end{equation}

since $\int_\Omega \Delta Pd^3x = \int_{\partial\Omega} \nabla P \cdot \mathbf{n}d\Sigma$ can be assumed zero for $\nabla P = 0$ on $\partial\Omega$. Hence (cf. Section 5 for more precision)

**THEOREM 2.1.** Given that any quantum potential for the SE has the form (2.2) (with $\nabla P = 0$ on $\partial\Omega$) it follows that the quantization can be identified with momentum fluctuations of the type studied in [26] and thus has information content as described by the Fisher information.

3. WDW

The same sort of arguments can be applied for the WDW equation (cf. [10, 24, 25, 42, 45, 50]). Thus take an ADM situation

\begin{equation}
ds^2 = -(N^2 - h^{ij}N_iN_j) + 2N_i dx^i dt + h^{ij} dx^i dx^j
\end{equation}

and assume dynamics generated by an action (♦♦) $A = \int dt [\tilde{H} + \int \mathcal{D}h \partial_t S]$. One will have equations of motion (☆☆) $\partial_t P = \delta \tilde{H}/\delta S$ and $\partial_t S = -\delta \tilde{H}/\delta P$ (cf. [25]). A suitable “classical” Hamiltonian is

\begin{equation}
\tilde{H}_c[P, S] = \int \mathcal{D}h P \left[ h_{ij}, \frac{\delta S}{\delta h_{ij}} \right];
\end{equation}

\begin{equation}
H_0 = \int dx \left[ N \left( \frac{1}{2} G^{ij\ell} \pi^{ij} \pi^{\ell} + V(h_{ij}) \right) - 2N_i \nabla_j \pi^{ij} \right]
\end{equation}

where $G^{ij\ell}$ is the deWitt (super)metric (♣♣) $G^{ij\ell} = (1/\sqrt{h})(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell})$ and $V \sim \sqrt{h}(3\Lambda - 3R)$. Then thinking of $\pi^{ij} = \delta S/\delta h_{ij} + f^{ij}$ and e.g. $\tilde{H}_q = \tilde{H}_c + (1/2) \int \mathcal{D}h \int dx N G^{ij\ell} f^{ij} f^{k\ell}$ one arrives via exact uncertainty at a Fisher information contribution (cf. [10, 20])

\begin{equation}
\tilde{H}_q[P, S] = \tilde{H}_c + \frac{c}{2} \int \mathcal{D}h \int dx N G^{ij\ell} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{k\ell}}
\end{equation}

with $h = 2\sqrt{c}$ and $\psi = \sqrt{P} \exp(iS/h)$ resulting in (for $N = 1$ and $N_i = 0$)

\begin{equation}
\left[ -\frac{\hbar^2}{2} \frac{\delta}{\delta h_{ij}} G^{ij\ell} \frac{\delta}{\delta h_{k\ell}} + V \right] \psi = 0
\end{equation}
with a sandwich ordering \((G_{ijkl} \text{ in the middle})\). In general there are also constraints

\[ \frac{\delta \psi}{\delta N} = \frac{\delta \psi}{\delta N_i} = \partial_i \psi = 0; \quad \nabla_j \left( \frac{\delta \psi}{\delta h_{ij}} \right) = 0 \]

We note here (keeping \(N = 1\) with \(N_i = 0\))

\[ \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta}{\delta h_{kl}} \sqrt{P} e^{iS/h} \right) = \left[ \delta G_{ijkl} \left( \frac{1}{2} P^{-1/2} \frac{\delta P}{\delta h_{kl}} + \frac{i P^{1/2}}{h} \frac{\delta S}{\delta h_{kl}} \right) + \right. \]

\[ + G_{ijkl} \left\{ \frac{1}{4} P^{-3/2} \frac{\delta P}{\delta h_{kl}} \frac{\delta P}{\delta h_{ij}} + \frac{1}{2} P^{-1/2} \frac{\delta^2 P}{\delta h_{kl} \delta h_{ij}} - \frac{P^{1/2}}{h^2} \frac{\delta S}{\delta h_{kl}} \frac{\delta S}{\delta h_{ij}} + \right. \]

\[ + \frac{i}{2h} P^{-1/2} \frac{\delta S}{\delta h_{kl}} \frac{\delta h_{ij}}{\delta h_{kl}} \frac{\delta^2 P}{\delta h_{kl} \delta h_{ij}} + \frac{i P^{1/2}}{2} \left( \frac{\delta P}{\delta h_{kl}} \frac{\delta h_{ij}}{\delta h_{kl}} \frac{\delta P}{\delta h_{ij}} + \frac{i P^{1/2}}{2} \frac{\delta^2 S}{\delta h_{kl} \delta h_{ij}} \right) \right] e^{iS/h} \]

Therefore writing out the WDW equation gives

\[ \frac{-\hbar^2}{4P} \frac{\delta}{\delta h_{ij}} \left[ G_{ijkl} \frac{\delta P}{\delta h_{kl}} \right] + \]

\[ + \frac{\hbar^2}{8P^2} G_{ijkl} \frac{\delta P}{\delta h_{kl}} \frac{\delta P}{\delta h_{ij}} + G_{ijkl} \left[ \frac{\hbar^2}{8P} \frac{\delta^2 P}{\delta h_{kl} \delta h_{ij}} + \frac{1}{4} P \frac{\delta S}{\delta h_{kl}} \frac{\delta S}{\delta h_{ij}} \right] + V = 0; \]

\[ 2P \frac{\delta G}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} + G \left( \frac{\delta S}{\delta h_{kl}} \frac{\delta h_{ij}}{\delta h_{kl}} + \frac{\delta P}{\delta h_{kl}} \frac{\delta h_{ij}}{\delta h_{kl}} \right) + 2PG \frac{\delta^2 S}{\delta h_{kl} \delta h_{ij}} = 0 \]

It is useful here to compare with \(-(\hbar^2/2m)\psi'' + V \psi = 0\) which for \(\psi = R \exp(iS/h)\) yields

\[ \frac{1}{2m} \frac{\delta^2 R^2}{\delta x^2} + V + Q = 0; \quad Q = -\frac{\hbar^2}{4m} \frac{R''}{R} = \frac{\hbar^2}{8m} \left[ \frac{2P''}{P} - \left( \frac{P'}{P} \right)^2 \right] \]

along with \(\partial (R^2 S') = \partial (PS') = 0\) (leading to (2.5)). The analogues here are then in particular

\[ \frac{1}{2m} \frac{\delta^2 S}{\delta x^2} \sim \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{kl}} \frac{\delta S}{\delta h_{ij}}; \quad Q \sim -\frac{\hbar^2}{4P} \frac{\delta}{\delta h_{ij}} \left[ G_{ijkl} \frac{\delta P}{\delta h_{kl}} \right] + G_{ijkl} \left\{ \frac{\hbar^2}{8P^2} \frac{\delta S}{\delta h_{kl}} \frac{\delta S}{\delta h_{ij}} + \frac{\hbar^2}{4P} \frac{\delta^2 P}{\delta h_{kl} \delta h_{ij}} \right\} \]

We note that the \(Q\) term arises directly from

\[ Q = -\frac{\hbar^2}{2} P^{-1/2} \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{kl}} \right) \]
and hence

\[ \int \mathcal{D}f \, PQ = -\frac{\hbar^2}{2} \int \mathcal{D}f \, P^{1/2} \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{kl}} \right) \]

But from \( \int \mathcal{D}f \delta [ \ ] = 0 \) one has (cf. (4.3))

\[ \int \mathcal{D}f P^{1/2} \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{kl}} \right) = -\int \mathcal{D}f \frac{\delta P^{1/2}}{\delta h_{ij}} \delta h_{ij} G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{kl}} \]

This suggests heuristically (see Section 4 for more details of proof and Section 5 for more precision)

**THEOREM 3.1.** Given a WDW equation of the form (3.4) with associated quantum potential given via (3.10) (or (3.9)) it follows that the quantum potential can be expressed via momentum fluctuations as in (3.3) (for \( N = 1 \)).

4. SOME FUNCTIONAL CALCULUS

We go here to [9, 25, 28, 37] and will first sketch the derivation of (3.4) following [24, 25] (cf. also [9]). The relevant functional calculus goes as follows. One defines a functional \( F \) of fields \( f \) and sets

\[ \delta F = F[f + \delta f] - F[f] = \int dx \frac{\delta F}{\delta f_x} \delta f_x \]

Here e.g. \( dx \sim d^4x \) and in the space of fields there is assumed to be a measure \( \mathcal{D}f \) such that \( \int \mathcal{D}f = \int \mathcal{D}f' \) for \( f' = f + h \) (cf. [8, 25]). Then evidently (♠♠) \( \int \mathcal{D}f (\delta F/\delta f) = 0 \) when \( \int \mathcal{D}f F[f] < \infty \). Indeed

\[ 0 = \int \mathcal{D}f (F[f + \delta f] - F[f]) = \int dx \delta f_x \left( \int \mathcal{D}f \frac{\delta F}{\delta f_x} \right) \]

and this provides an integration by parts formula

\[ \int \mathcal{D}f \, P \left( \frac{\delta F}{\delta f} \right) = -\int \mathcal{D}f \left( \frac{\delta P}{\delta f} \right) F \]

for \( P[f] \) a probability density functional. Classically a probability density functional arises in discussing an ensemble of fields and conservation of probability requires

\[ \partial_t P + \sum_a \int dx \frac{\delta}{\delta f^a_x} \left( P \frac{\delta H}{\delta g^a_x} \right) = 0 \]

where \( g^a_x \) is the momentum corresponding to \( f^a_x \) and one assumes a motion equation

\[ \partial_t S + H \left( f, \frac{\delta S}{\delta f}, t \right) = 0 \]
The equations of motion here are then

\[
\partial_t P = \frac{\Delta \tilde{H}}{\Delta S}; \quad \partial_t S = -\frac{\tilde{H}}{\Delta P}
\]

where \((\bullet \bullet \bullet)\) \(\tilde{H}(P, S, t) = \langle H \rangle = \int \mathcal{D}f PH(f, \delta S/\delta f, t)\). The variational theory here involves functionals \(I[F] = \int \mathcal{D}f \xi (F, \delta F/\delta f)\) and one can write

\[
\Delta I = I[F + \Delta F] - I[F] = \int \mathcal{D}f \left[ \frac{\partial \xi}{\partial F} \Delta F + \int dx \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \frac{\delta (\Delta F)}{\delta f_x} \right] = \int \mathcal{D}f \left[ \frac{\partial \xi}{\partial F} - \int dx \frac{\delta}{\delta f_x} \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \right] \Delta F + \int dx \int \mathcal{D}f \frac{\delta}{\delta f_x} \left[ \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \delta F \right]
\]

Assuming the term \(\int \mathcal{D}f \frac{\delta}{\delta f_x} \Delta F\) is finite the last integral vanishes and one obtains \((\bullet\bullet\bullet)\) \(\Delta I = \int \mathcal{D}f (\Delta I/\Delta F) \Delta F\), thus defining a variational derivative

\[
\frac{\Delta I}{\Delta F} = \frac{\partial \xi}{\partial F} - \int dx \frac{\delta}{\delta f_x} \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right)
\]

In the Hamiltonian theory one can work with a generating function \(S\) such that \((\star\star\star)\) \(g = \delta S/\delta f\) and \(\partial_t S + H(f, \delta S/\delta f, t) = 0\) (HJ equation) and solving this is equivalent to \(\partial_t f = \delta H/\delta g\) and \(\partial_t g = -\delta H/\delta f\) (cf. [25]). Once \(S\) is specified the momentum density \(g\) is determined via \(g = \delta S/\delta f\) and an ensemble of fields is specified by a probability density functional \(P[f]\) (and not by a phase space density functional \(\rho[f, g]\)). In the HJ formulation one writes \((\star\star\star\star)\) \(V_x[f] = \partial f_x/\partial t = (\delta H/\delta g)|_{g=\delta S/\delta f}\) and hence the associated continuity equation \(\partial_t \int \mathcal{D}f P\) is

\[
\partial_t P + \int dx \frac{\delta}{\delta f_x}[PV_x] = 0
\]

provided \(<V_x>\) is finite.

Now after proving (2.4) one proceeds as follows to produce a SE. The Hamiltonian formulation gives \((\bigstar\bigstar\bigstar)\) \(\partial_t P = \Delta \tilde{H}/\Delta S\) and \(\partial_t S = -\Delta \tilde{H}/\Delta P\) where the ensemble Hamiltonian is

\[
\tilde{H} = \tilde{H}(P, S, t) = \langle H \rangle = \int \mathcal{D}f PH[f, \delta S/\delta f, t]
\]

where \(P\) and \(S\) are conjugate variables. The equations \((\bigstar\bigstar\bigstar)\) arise from \(\Delta \tilde{A} = 0\) where \(\tilde{A} = \int dt [-\tilde{H} + \int \mathcal{D}f S \partial_t P].\) One specializes here to quadratic...
Hamiltonian functions

(4.11) \[ H_c[f, g, t] = \sum_{a,b} dx K^{ab}_x[f] g^a_x g^b_x + V[f] \]

and to this is added a term as in (2.4) to get \( \tilde{H} \) (which does not depend on \( S \)). Hence from (♠♠♠) with \( \partial_t f_x = \delta H_c / \delta g_x \) one obtains following (4.9)

(4.12) \[ \partial_t P + \int dx \frac{\delta}{\delta f_x} \left[ P \frac{\delta H}{\delta g_x} \right]_{g=\delta S/\delta f} = 0 \]

(cf. 4.8)). The other term in \( \tilde{H} \) is simply

(4.13) \[ \left( \frac{\hbar^2}{4} \right) \int f \int PK^{ab}_x(\delta P/\delta f^a_x)(\delta P/\delta f^b_x)(1/P^2) \]

and this provides a contribution to the HJ equation via \( \partial_t S = -\Delta \tilde{H}/\Delta P \) which will have the form

(4.14) \[ Q = -\frac{\hbar^2}{4} P^{-1/2} \int dx \frac{\delta}{\delta f^2_x} \left( K^{ab}_x \frac{\delta P^{1/2}}{\delta f^b_x} \right) \]

corresponding to (3.10). We note further then from (3.12)

(4.15) \[ Q \sim \frac{\hbar^2}{2} \int dx G_{ijk\ell} \frac{\delta P^{1/2}}{\delta h_{ij}} \frac{\delta P^{1/2}}{\delta h_{k\ell}} \sim \frac{\hbar^2}{8} \int dx G_{ijk\ell} \frac{1}{P} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{k\ell}} \]

as in (3.3). Hence Theorem 3.1 is established under the hypotheses indicated concerning \( \mathcal{D} f \) etc.

5. ENTROPY AND FISHER INFORMATION

We recall first (cf. [7, 9, 11]) that the relation between the SE and the quantum potential (QP) is not 1-1. The QP \( Q \) depends on the wave function \( \psi = R \exp(iS/\hbar) \) via \( Q = -(\hbar^2/2)(\Delta R/R) \) for the SE and thus the solution of a quantum HJ equation, involving \( S \) and \( R \) (via \( Q \)), requires the companion “continuity” equation to determine \( S \) and \( R \) (and hence \( \psi \)). There is some lack of uniqueness since \( Q \) determines \( R \) only up to uniqueness for solutions of \( \Delta R + (2m/\hbar^2)QR = 0 \) and even then the HJ equation \( S_t + \cdots = 0 \) could introduce still another arbitrary function (cf. [9, 11]). Thus to indicate precisely what is said in Theorems 2.1 and 3.1 we rephrase this in the form

**THEOREM** 5.1. In Theorem 2.1 we see that given a SE described via a probability distribution \( P = |\psi|^2 \) one can identify this equation as a quantum model arising from a classical Hamiltonian \( \tilde{H}_{cl} \) perturbed by a Fisher information term as in (2.4). Thus the quantization involves an information content with entropy significance (cf. here [10, 40]) for entropy connections). This suggests that any quantization of \( \tilde{H}_{cl} \) arises (or can
arise) through momentum perturbations related to Fisher information and it also suggests that 
\( P = |\psi|^2 \) (with \( \int P d^3x = 1 \)) should be deemed a requirement for any solution \( \psi \) of the related SE (note \( \int P d^3x = 1 \) eliminates many putative counterexamples). Thus once \( P \) is specified as a probability distribution for a wave function \( \psi = \sqrt{P} \exp(iS/\hbar) \) arising from a SE corresponding to a quantization of \( \tilde{H}_{cl} \), then \( Q \) can be expressed via Fisher information. Similarly given \( Q \) as a Fisher information perturbation of \( \tilde{H}_{cl} \) (arising from momentum fluctuations involving \( P \) as in (2.4)) there is a unique wave function \( \psi = \sqrt{P} \exp(iS/\hbar) \) satisfying the corresponding SE.

**THEOREM 5.2.** For Theorem 3.1 let us assume there exists a suitable \( \mathcal{D}f \) as in Section 4, which is a measure in the (super)space of fields \( h \). Then there is an integration by parts formula (4.3) which removes the need for considering surface terms in integrals \( \int d^4x \) (cf. [18] for cautionary remarks about Green’s theorem, etc.). Consequently given a WDW equation of the form (3.4) with corresponding \( Q \) as in (3.10) (and \( \psi = \sqrt{P} \exp(iS/\hbar) \)), one can show that the equation can be modelled on a perturbation of a classical \( \tilde{H}_c \) via a Fisher information type perturbation as in (3.3) (cf. here [9] [19] [20]). Here \( P \) represents a probability density of fields \( h_{ij} \) which determine \( G_{ijk\ell} \) (and \( V \) incidentally) and the very existence of a quantum equation (i.e. WDW) seems to require entropy type input via Fisher information fluctuation of fields. This suggests that quantum gravity requires a statistical spacetime (an idea that has appeared before - cf. [9]).

We sketch now some material from [12] [13] supporting the idea of a statistical geometrodynamics (SGD). Here one builds a model of SGD based on (i) Positing that the geometry of space is of statistical origin and is explained in terms of the distinguishability Fisher-Rao (FR) metric and (ii) Assuming the dynamics of the geometry is derived solely from principles of inference. There is no external time but an intrinsic one à la [5]. A scale factor \( \sigma(x) \) is required to assign a Riemannian geometry and it is conjectured that it can be chosen so that the evolving geometry of space sweeps out a 4-D spacetime. The procedure defines only a conformal geometry but that is entirely appropriate d’après [60]. One uses the FR metric in two ways, one to distinguish neighboring points and the other to distinguish successive states. Consider then a “cloud” of dust with coordinate values \( y^i \) \((i = 1, 2, 3)\) and estimates \( x^i \) with \( p(y|x)dy \) the probability that the particle labeled \( x^i \) should have been labeled \( y^i \) (the FR metric encodes the use of probability distributions - instead of structureless points). One writes

\[
(5.1) \quad \frac{p(y|x + dx) - p(y|x)}{p(y|x)} = \frac{\partial \log[p(y|x)]}{\partial x^i} dx^i
\]
\[ (5.2) \quad d\lambda^2 = \int d^4y p(y|x) \frac{\partial \log p(y|x)}{\partial x^i} \frac{\partial \log p(y|x)}{\partial x^j} dx^i dx^j = \gamma_{ij} dx^i dx^j \]

and \(d\lambda^2 = 0 \iff dx^i = 0\). The FR metric \(\gamma_{ij}\) is the only local Riemannian metric reflecting the underlying statistical nature of the manifold of distributions \(p(y|x)\) and a scale factor \(\sigma\) giving a metric \(g_{ij}(x) = \sigma(x)\gamma_{ij}(x)\) is needed for a Riemannian metric (cf. [12, 13]). Also the metric \(d\lambda^2\) is related to the entropy of \(p(y|x+dx)|p(y|x)\), namely

\[ (5.3) \quad S[p(y|x+dx)|p(y|x)] = -\int d^3y p(y|x+dx) \log \frac{p(y|x+dx)}{p(y|x)} = -\frac{1}{2} d\lambda^2 \]

and maximizing the relative entropy \(S\) is equivalent to minimizing \(d\lambda^2\). One thinks of \(d\lambda\) as a spatial distance in specifying that the reason that particles at \(x\) and \(x+dx\) are considered close is because they are difficult to distinguish. To assign an explicit \(p(y|x)\) one assumes the relevant information is given via \(\langle y^i \rangle = x^i\) and the covariance matrix \(\langle (y^i - x^i)(y^j - x^j) \rangle = C^{ij}(x)\); this leads to

\[ (5.4) \quad p(y|x) = \frac{C^{1/2}}{(2\pi)^{3/2}} \exp \left[ -\frac{1}{2} C_{ij}(y^i - x^i)(y^j - x^j) \right] \]

where \(C^{ik}C_{kj} = \delta^i_j\) and \(C = \det(c_{ij})\). Subsequently to each \(x\) one associates a probability distribution

\[ (5.5) \quad p(y|x, \gamma) = \frac{\gamma^{1/2}}{(2\pi)^{3/2}} \exp \left[ -\frac{1}{2} \gamma_{ij}(x)(y^i - x^i)(y^j - x^j) \right] \]

where \(\gamma_{ij}(x) = C^{ij}(x)\) (extreme curvature situations are avoided here). One deals with a conformal geometry described via \(\gamma_{ij}\) and a scale factor \(\sigma(x)\) will be needed to compare uncertainties at different points; the choice of \(\sigma\) should then be based on making motion “simple”.

Thus define a macrostate via

\[ (5.6) \quad P[y|\gamma] = \prod_x p(y|x, \gamma_{ij}(x)) = \]

\[ = \prod_x \gamma^{1/2} \exp \left[ -\frac{1}{2} \sum_x \gamma_{ij}(x)(y^i - x^i)(y^j - x^j) \right] \]

Once a dust particle in an earlier state \(\gamma\) is identified with the label \(x\) one assumes that this particle can be assigned the same label \(x\) as it evolves into the later state \(\gamma + \Delta\gamma\) (equilocal comoving coordinates). Then the change between \(P[y|\gamma + \Delta\gamma]\) and \(P[y|\gamma]\) is denoted by \(\Delta\ell\) and is measured
via their relative entropy (this is a form of Kullback-Liebler entropy - cf. \cite{9})

\[ S[\gamma + \Delta \gamma] = - \int \left( \prod_x dy(x) \right) P[y|\gamma] \log \frac{P[y|\gamma + \Delta \gamma]}{P[y|\gamma]} = -\frac{1}{2} \Delta \ell^2 \]

Since \( P[y|\gamma] \) and \( P[y|\gamma + \Delta \gamma] \) are products one can write

\[ S[\gamma + \Delta \gamma, \gamma] = \sum_x S[\gamma(x) + \Delta \gamma(x), \gamma(x)] = -\frac{1}{2} \sum_x \Delta \ell^2(x) = g^{ijkl} \Delta \gamma_{ij}(x) \Delta \gamma_{k\ell}(x) \]

where, using (5.5)

\[ g^{ijkl} = \int d^3y \frac{\partial \log[p(y|x,\gamma)]}{\partial \gamma_{ij}} \frac{\partial \log[p(y|x,\gamma)]}{\partial \gamma_{k\ell}} = \frac{1}{4} (\gamma^{ik} \gamma^{ji} + \gamma^{i\ell} \gamma^{jk}) \]

Then \( \Delta \ell^2 = \sum_x \Delta \ell^2(x) \) can be written as an integral if we note that the density of distinguishable distributions is \( \gamma^{1/2} \). Thus the number of distinguishable distributions, or distinguishable points in the interval \( dx \) is \( dx \gamma^{1/2} (dx \sim d^3x) \) and one has

\[ \Delta \ell^2 = \int dx \gamma^{1/2} \Delta \ell^2 = \int dx \gamma^{1/2} g^{ijkl} \Delta \gamma_{ij} \Delta \gamma_{k\ell} \]

Thus the effective number of distinguishable points in the interval \( dx \) is finite (due to the intrinsic fuzziness of space). Now to describe the change \( \Delta \gamma_{ij}(x) \) one introduces an arbitrary time parameter \( t \) along a trajectory

\[ \Delta \gamma_{ij} = \gamma_{ij}(t + \Delta t, x) - \gamma_{ij}(t, x) = \partial_t \gamma_{ij} \Delta t \]

Thus \( \partial_t \gamma_{ij} \) is the “velocity” of the metric and (5.10) becomes

\[ \Delta \ell^2 = \int dx \gamma^{1/2} g^{ijkl} \partial_t \gamma_{ij} \partial_t \gamma_{k\ell} \Delta t^2 \]

Now go to an arbitrary coordinate frame where equilocal points at \( t \) and \( t + \Delta t \) have coordinates \( x^i \) and \( \vec{x}^i = x^i - \beta^i(x) \Delta t \). Then the metric at \( t + \Delta t \) transforms into \( \tilde{\gamma}_{ij} \) with

\[ \gamma_{ij}(t + \Delta t, x) = \tilde{\gamma}_{ij}(t + \Delta t, x) - (\nabla_i \beta_j + \nabla_j \beta_i) \Delta t \]

where \( \nabla_i \beta_j = \partial_i \beta_j - \Gamma^k_{ij} \beta_k \) is the covariant derivative associated to the metric \( \gamma_{ij} \). In the new frame, setting \( \tilde{\gamma}_{ij}(t + \Delta t, x) - \gamma_{ij}(t, x) = \Delta \gamma_{ij} \) one
(5.14) \[ \Delta \beta_{ij} = \Delta \gamma_{ij} - (\nabla_i \beta_j + \nabla_j \beta_i) \Delta t \sim \Delta_j \gamma_{ij} = \dot{\gamma}_{ij} \Delta t \]

leading to

(5.15) \[ \Delta \beta L^2 = \int dx \gamma^{1/2} \frac{1}{2} g^{ij} \dot{\gamma}_{ij} \dot{\gamma}_{kl} \Delta t^2 \]

Next one addresses the problem of specifying the best matching criterion, i.e., what choice of \( \beta^i \) provides the best equilocality match. This is treated as a problem in inference and asks for minimum \( \Delta \beta L^2 \) over \( \beta \). Hence one gets

(5.16) \[ \delta(\Delta \beta L^2) = 2 \int dx \gamma^{1/2} \frac{1}{2} g^{ij} \dot{\gamma}_{ij} \dot{\gamma}_{kl} \Delta t^2 = 0 \Rightarrow \]

\[ \Rightarrow \nabla_k (2 g^{ij \ell} \dot{\gamma}_{ij}) = 0 = \nabla \gamma_{ij} \dot{\gamma}_{kl} = 0 \]

(using (5.9) and \( \dot{\gamma}_{ij} = \partial_t \gamma_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \)). These equations determine the shifts \( \beta^i \) giving the best matching and equilocality for the geometry \( \gamma_{ij} \) and alternatively they could be considered as constraints on the allowed change \( \Delta \gamma_{ij} = \partial_t \gamma_{ij} \Delta t \) for given shifts \( \beta^i \). In describing a putative entropic dynamics one assumes now e.g. continuous trajectories with each factor in \( P[y|\gamma] \) evolving continuously through intermediate states labeled via \( \omega(x) = \omega \xi(x) \) where \( \xi(x) \) is a fixed positive function and \( 0 < \omega < \infty \) is a variable parameter (some kind of many fingered time à la Schwinger, Tomonaga, Wheeler, et al). It is suggested that they dynamics be determined by an action

(5.17) \[ J = \int_{t_i}^{t_f} dt \int dx \gamma^{1/2} \left[ g^{ij \ell} \dot{\gamma}_{ij} \dot{\gamma}_{kl} \Delta t^2 \right]^{1/2} \]

The similarities to “standard” geometrodynamics are striking.

5.1. INFORMATION DYNAMICS. We go here to [14, 15] and consider the idea of introducing some kind of dynamics in a reasoning process. One looks at the Fisher metric defined by

(5.18) \[ g_{\mu\nu} = \int_X d^4 x p_\theta(x) \left( \frac{1}{p_\theta(x)} \frac{\partial p_\theta(x)}{\partial \theta^\mu} \right) \left( \frac{1}{p_\theta(x)} \frac{\partial p_\theta(x)}{\partial \theta^\nu} \right) \]

and constructs a Riemannian geometry via

(5.19) \[ \Gamma^\sigma_{\lambda\nu} = \frac{1}{2} g^{\nu\sigma} \left( \frac{\partial g_{\mu\nu}}{\partial \theta^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial \theta^\mu} - \frac{\partial g_{\mu\lambda}}{\partial \theta^\nu} \right) ; \]

\[ R^\lambda_{\mu\nu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial \theta^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial \theta^\nu} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta} \]
Then the Ricci tensor is \( R_{\mu \kappa} = R^\lambda_{\mu \lambda \kappa} \) and the curvature scalar is \( R = g^{\mu \nu} R_{\mu \nu} \). The dynamics associated with this metric can then be described via functionals

\[
J[g_{\mu \nu}] = -\frac{1}{16\pi} \int \sqrt{g(\theta)} R(\theta) d^4 \theta
\]

leading upon variation in \( g_{\mu \nu} \) to equations

\[
R^{\mu \nu}(\theta) - \frac{1}{2} g^{\mu \nu}(\theta) R(\theta) = 0
\]

Contracting with \( g_{\mu \nu} \) gives then the Einstein equations \( R_{\mu \nu}(\theta) = 0 \) (since \( R = 0 \)). \( J \) is also invariant under \( \theta \to \theta + \epsilon(\theta) \) and variation here plus contraction leads to a contracted Bianchi identity. Constraints can be built in by adding terms \( (1/2) \int \sqrt{g} \Gamma^{\mu \nu \rho} g_{\mu \nu} d^4 \theta \) to \( J[g_{\mu \nu}] \). If one is fixed on a given probability distribution \( p(x) \) with variable \( \theta \) attached to give \( p_{\theta}(x) \) then this could conceivably describe some gravitational metric based on quantum fluctuations for example. As examples a Euclidean metric is produced in 3-space via Gaussian \( p(x) \) and complex Gaussians will give a Lorentz metric in 4-space.

6. OTHER FORMS OF WDW

In general there are many approaches to WDW and we cite in particular [1, 2, 3, 4, 5, 6, 9, 10, 16, 18, 21, 22, 23, 27, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]. In particular (for \( \phi \) a matter field) the theory of [43, 44] leads to a Bohmian form

\[
\left\{ -\hbar^2 \left[ \kappa G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} \right] + \frac{1}{2} \hbar^{-1/2} \frac{\delta^2}{\delta \phi^2} \right\} \psi(h_{ij}, \phi) = 0; \]

\[
V = \hbar^{1/2} \left[ -\kappa^{-1} (3 R - 2 \Lambda) + \frac{1}{2} \hbar^{ij} \partial_i \partial_j \phi \right] + U(\phi)
\]

involving (for \( A^2 \sim P \))

\[
\kappa G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} + \frac{1}{2} \hbar^{-1/2} \left( \frac{\delta S}{\delta \phi} \right)^2 + V + Q = 0;
\]

\[
Q = -\frac{\hbar^2}{A} \left( \kappa G_{ijkl} \frac{\delta^2 A}{\delta h_{ij} \delta h_{kl}} + \frac{\hbar^{-1/2} \delta^2 A}{2 \delta \phi^2} \right)
\]

where the unregularized \( Q \) above depends on the regularization and factor ordering prescribed for the WDW equation. In addition to (6.2) one has

\[
\kappa G_{ijkl} \frac{\delta}{\delta h_{ij}} \left( A^2 \frac{\delta S}{\delta h_{kl}} \right) + \frac{\hbar^{-1/2} \delta}{2 \delta \phi} \left( A^2 \frac{\delta S}{\delta \phi} \right) = 0
\]
Other Bohmian situations are indicated in [9] and we are preparing a survey article.
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