Emergent Gauge Fields in Holographic Superconductors

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Abstract

Holographic superconductors have been studied so far in the absence of dynamical electromagnetic fields, namely in the limit in which they coincide with holographic superfluids. It is possible, however, to introduce dynamical gauge fields if a Neumann-type boundary condition is imposed on the AdS-boundary. In 3 + 1 dimensions, the dual theory is a 2 + 1 dimensional CFT whose spectrum contains a massless gauge field, signaling the emergence of a gauge symmetry. We study the impact of a dynamical gauge field in vortex configurations where it is known to significantly affect the energetics and phase transitions. We calculate the critical magnetic fields $H_{c1}$ and $H_{c2}$, obtaining that holographic superconductors are of Type II ($H_{c1} < H_{c2}$). We extend the study to 4 + 1 dimensions where the gauge field does not appear as an emergent phenomenon, but can be introduced, by a proper renormalization, as an external dynamical field. We also compare our predictions with those arising from a Ginzburg-Landau theory and identify the generic properties of Abrikosov vortices in holographic models.
1 Introduction

The AdS/CFT correspondence has become a powerful tool to study strongly-coupled systems in different environments. Very recently its applicability has been also extended to condensed matter systems. In Ref. [1] a gravitational description of a superconductor was proposed whose properties have been extensively studied in the last couple of years [2]. In these models a gauge U(1) symmetry is broken by a scalar field that turns on near the black-hole horizon. This corresponds, in the dual description, to a global U(1) broken by a scalar condensate. Therefore, strictly speaking, these models describe either a superfluid [3] or a superconductor in the gauge-less limit.

The reason for the absence of a dynamical gauge field in previously studied holographic superconductors is the chosen AdS-boundary condition for the U(1) gauge field. In most of these studies the gauge field was chosen to be frozen at the AdS-boundary by imposing a Dirichlet boundary condition. One can make, however, the gauge field dynamical if one instead imposes a Neumann-type boundary condition at the AdS-boundary. In this article we will make use of this option to study the role of dynamical gauge fields in holographic superconductors.

In a 3 + 1 dimensional AdS space it is known that we can impose either a Dirichlet or a Neumann AdS-boundary condition to quantize a gauge theory, being both related by an S-duality [4]. In the Neumann case, one finds a massless gauge field in the spectrum that, by means of the AdS/CFT correspondence, can be considered to arise from a 2 + 1 dimensional CFT. It is therefore an emergent phenomenon. In 4 + 1 dimensions, however, a Neumann AdS-boundary condition for the gauge field is not well-defined since it leads to a non-finite Hamiltonian. This will require, as we will show, to regularize the theory and absorb the divergencies in local counterterms. In this case, the gauge field will not be an emergent phenomenon but just an external dynamical gauge field coupled to a 3 + 1 CFT [5].

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1 Neumann boundary conditions have been previously considered in holographic superconductors to study systems at fixed charge density. In these cases, however, the studied systems are homogeneous and therefore the dynamical electric field vanishes.
An alternative way to understand the distinction between a gauge field in a 2+1 and 3+1 CFT is to look at the zero mode of the Kaluza-Klein expansion of the gauge field in the holographic superconductor model at temperatures $T$ bigger than the critical temperature $T_c$. For a stationary vector potential $a_i$, we find, after integrating over the extra dimension, a kinetic term given by $\int d^d x F_{ij}^2/(4e_0^2)$, where $F_{ij} = \partial_i a_j - \partial_j a_i$ and

$$\frac{1}{e_0^2} = \frac{3}{4\pi g^2 T} \quad \text{for } d = 2 + 1, \quad \frac{1}{e_0^2} = -\frac{L}{g^2} \ln (z\pi T)|_{z=0} \quad \text{for } d = 3 + 1. \quad (1)$$

Here $g$ is the gauge coupling in the AdS model and $L$ is the AdS radius. In $2+1$ dimensions this massless gauge boson mode has a finite norm and therefore remains in the spectrum, while for $d = 3 + 1$ this is a non-normalizable mode and disappears from the set of dynamical degrees of freedom. To keep this mode in $3+1$ dimensions we must then make its norm finite, for example by adding local counterterms.

The impact of a dynamical gauge field in superconductors is expected to be important in inhomogeneous configurations. For this reason we will concentrate here on the vortex configurations of the holographic models. We will explicitly analyze the cases $d = 2 + 1$ and $d = 3 + 1$, introducing, when studying the $d = 3 + 1$ holographic superconductor, local counterterms to render the norm of $a_i$ finite. This will allow us to explicitly see that the dynamical magnetic field $B$ plays an important role in reproducing some of the known features of superconductor vortices, such as the exponential damping of $B$ inside the superconductor. We will also show that the dynamical gauge field changes the vortex configurations of the holographic models, making the energy and, correspondingly, the first critical magnetic field $H_{c1}$ independent of the sample size, as expected from a true Abrikosov vortex. The properties of the new configurations are qualitatively similar to those arising from a Ginzburg-Landau (GL) theory [6], although we find important quantitative differences in the size of the vortex core, the profile of $B$ flowing through the vortex and $H_{c1}$.

The effect of an external magnetic field on holographic superconductors has been considered in [7, 8] and vortex solutions have been studied before for $d = 2 + 1$ in Refs. [9, 10]. In all these cases there was no dynamical electromagnetic (EM) field, and therefore the vortices were not true Abrikosov configurations but just superfluid vortices. Only Ref. [11] showed, for $d = 2 + 1$, a non-trivial profile for the magnetic field of the form of a vortex magnetic tube; it is however unclear the origin of this magnetic field and its relation with Abrikosov configurations.

The organization of the paper is as follows. In Section 2 we describe, similarly in spirit to Ref. [12], effective field theories from which we can obtain model-independent properties of superconductors and superfluids, and their vortex configurations. This will also help us to make contact with the GL predictions[2]. In Section 3 we present the holographic model and explain how to introduce dynamical gauge fields. We then focus on the holographic superfluid and superconductor (Abrikosov) vortex in $d = 2 + 1$ and $d = 3 + 1$ dimensions, comparing them with those of the GL theory. We calculate the energy of these configurations to find the critical magnetic fields $H_{c1}$ and $H_{c2}$ and show that the holographic superconductors are always of Type II. In Section 4 we present a summary of the results and other concluding remarks.

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2What we mean with GL theory in the case of superfluids is the limit of frozen magnetic fields in the GL theory for superconductors.
2 Effective theories of superfluids and superconductors

We are interested in the effective theory for time-independent configurations of a U(1) gauge field \( a_\mu = (a_0, a_i) \), where \( i, j = 1, \ldots, d - 1 \), and a scalar field \( \Phi_{cl} \) whose non-zero value will be responsible for the U(1) breaking. The effective action at finite temperature \( T \) for \( a_i \) and the order parameter \( \Phi_{cl} \), obtained after integrating out all the other fields of the theory, depends on an effective Lagrange density constructed from gauge-invariant operators:

\[
\Gamma = \beta \int d^{d-1}x \mathcal{L}_{\text{eff}} , \quad \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(F_{ij}^2, |D_i \Phi_{cl}|^2, |\Phi_{cl}|,...),
\]

where \( D_j = \partial_j - i a_j \) and \( \beta = 1/T \). Eq. (2) is defined in some renormalization scheme. We will be assuming that this theory depends only on two mass-scales, \( \mu \) (that later we will associate with a chemical potential) and the temperature \( T \).

The generic effective theory given by Eq. (2) simplifies in two limits. In the limit of small fields (as compared to \( \mu \) and \( T \)), this theory approximates to the GL theory

\[
\Gamma_{\text{GL}} = \beta \int d^{d-1}x \left\{ \frac{1}{4e_0^2} F_{ij}^2 + |D_i \Phi_{GL}|^2 + V_{GL}(|\Phi_{GL}|) \right\}.
\]

The GL field \( \Phi_{GL} \) is defined to be canonically normalized, \( \Phi_{GL} = \sqrt{h_0} \Phi_{cl} \), where \( h_0 \) is a positive constant, and

\[
V_{GL} = -\frac{1}{2e_{GL}^2} |\Phi_{GL}|^2 + b_{GL} |\Phi_{GL}|^4.
\]

This approximation becomes reliable, for example, close to the critical temperature \( T \lesssim T_c(\mu) \) where the “condensate” \( \Phi_{cl} \) has a small value. The other useful limit corresponds to slowly varying fields, which implies that \( D_i \Phi_{cl} \) and \( F_{ij} \) are small and

\[
\Gamma \simeq \beta \int d^{d-1}x h(|\Phi_{cl}|) \left\{ \frac{1}{4e^2(|\Phi_{cl}|)} F_{ij}^2 + |D_i \Phi_{cl}|^2 + W(|\Phi_{cl}|) \right\},
\]

where \( h \), \( W \) and \( e \) are generic functions of \( |\Phi_{cl}|^2 \). In the limit of small fields, we obtain the GL theory: \( h(|\Phi_{cl}|) \to h(0) = h_0 \), \( W(|\Phi_{cl}|) \to V_{GL}(|\Phi_{GL}|)/h_0 \) and \( e^2(|\Phi_{cl}|) \to e^2(0) = h_0 e_0^2 \). When Eq. (3) and/or Eq. (5) are applicable they can be used to extract model independent features of superconductors and superfluids.

Consider first the case where we are at large temperatures \( T > T_c(\mu) \); here the condensate \( \Phi_{cl} \) is zero, corresponding to the “normal” phase. By decreasing the temperature, \( T < T_c(\mu) \) at zero magnetic field, the modulus of the scalar field \( \psi_{cl} = |\Phi_{cl}| \) will get a nonzero constant value \( \psi_{\infty} \). For this homogeneous configuration the effective action in Eq. (5) is obviously a good approximation of the theory \( ^3 \), and the value of \( \psi_{\infty} \) is determined by the minimum of the potential \( V = hW \):

\[
\frac{\partial V}{\partial \psi_{cl}}(\psi_{\infty}) = 0.
\]

\(^3\)Notice however that, generically, this is not the case for the GL theory in Eq. (3).
This configuration corresponds to the superfluid/superconduct or phase. Two important parameters describing these systems are $\xi$ and $\lambda$, defined as

$$\frac{1}{\xi^2} = \frac{1}{2h(\psi_\infty)} \frac{\partial^2 V}{\partial \psi_{cl}^2}(\psi_\infty) > 0, \quad \lambda = \frac{1}{\sqrt{2e(\psi_\infty)} \psi_\infty}.$$  

These quantities exactly correspond to the inverse mass of the scalar $\psi_{cl}$ and $a_i$ respectively.

In this work we will be considering time-independent vortex configurations with cylindrical symmetry as the main example of our theoretical framework. We define $(r, \phi)$ as the polar coordinates restricted to $0 \leq r \leq R, 0 \leq \phi < 2\pi$. We will always consider the case $\xi \ll R$ and, in the superconductor case, also $\lambda \ll R$. We take the Ansatz

$$a_\phi = a_\phi(r), \quad \Phi_{cl} = e^{in\phi} \psi_{cl}(r),$$

where $n$ is an integer, and all other gauge components are set to zero. For $n \neq 0$, the fields $a_\phi(r)$ and $\psi_{cl}(r)$, satisfying the equations of motion from the Lagrangian in (2), describe a straight vortex line centered at $r = 0$. If we insert the Ansatz (8) into the action in (5) we obtain

$$\Gamma \simeq 2\pi V^{d-3} \beta \int_0^R drr h(\psi_{cl}) \left\{ \frac{1}{2e^2(\psi_{cl}) r^2} (\partial_r a_\phi)^2 + (\partial_r \psi_{cl})^2 + \frac{1}{r^2} (n - a_\phi)^2 \psi_{cl}^2 + W(\psi_{cl}) \right\},$$

where $V^{d-3}$ is the volume of the space orthogonal to the plane $(r, \phi)$. Here the current is given by

$$J_\phi = -\frac{1}{\beta} \frac{\delta \Gamma}{\delta a_\phi} = 2h(\psi_{cl})(n - a_\phi)\psi_{cl}^2 + r\partial_r \left( \frac{h(\psi_{cl})}{e^2(\psi_{cl}) r} \partial_r a_\phi \right).$$

In the vortex case ($n \neq 0$) $\psi_{cl}$ goes to $\psi_\infty$ far away from the vortex center; this corresponds to the physical fact that a vortex line destroys superfluidity/superconductivity only in a region close to its center. The details of the vortex configurations depend on whether the field $a_\phi$ is dynamical as in the superconductor case, or just a non-dynamical background as in a superfluid system. We consider the two cases in turn.

### 2.1 Superfluid vortex

For superfluids the modulus and phase of $\Phi_{cl}$ are respectively associated with the density $n_s$ and velocity $v_i$ of the superfluid. In the limit of slowly varying fields, Eq. (5), we define them as

$$n_s(\Phi_{cl}) = 2|\Phi_{cl}|^2 h(|\Phi_{cl}|), \quad v_i = \partial_i \text{Arg}[\Phi_{cl}].$$

The field $a_\phi$ is not dynamical; it just represents an external angular velocity performed on the superfluid. This is implemented by working in a rotating frame with a constant angular velocity $\Omega = a_\phi/r^2$. In going from the static to the rotating frame the angular velocity of the superfluid is changed accordingly: $v_\phi \to v_\phi - \Omega r^2$. The current is then given by $J_\phi = n_s(v_\phi - \Omega r^2)$.

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4 Notice that we have defined the current to include the kinetic term of the gauge field. This is done in order to facilitate our treatment for both, dynamical and non-dynamical gauge fields.

5 Right dimensions are obtained by putting appropriate powers of the “boson” mass causing the superfluidity.
Superfluid dynamics coincides with those of a superconductor in the limit in which the EM field is frozen to certain values. This is achieved by taking the limit \( e \to 0 \) while keeping the external magnetic field \( B = \partial_r a_\phi / r \) constant. In this limit the correspondence between the superfluid and the superconductor systems is given by

\[
\Omega \leftrightarrow B/2 \ , \quad L_\perp \leftrightarrow 2M ,
\]

where \( L_\perp \) is the angular momentum and \( M \) the magnetization of the system in the direction perpendicular to the \((r, \phi)\) plane. In the rest of this section we will use the superconductor notation.

Vortices correspond to configurations with \( n \neq 0 \) where \( \psi_{\text{cl}} \) varies from zero (at \( r = 0 \)) to \( \psi_\infty \) at large \( r \). The exact solution depends on the specific effective action and therefore it is very model dependent. Nevertheless, we can obtain the behavior of \( \psi_{\text{cl}} \) in the limit \( r \to 0 \) and for large \( r \). Indeed, for \( r \to 0 \), the condensate goes to zero and the GL action can be applied to obtain

\[
\psi_{\text{cl}} \propto r^{|n|} .
\]

(13)

For large \( r \), we can use Eq. (9) to obtain, in the absence of rotation \((B = 0)\),

\[
\psi_{\text{cl}} \simeq \psi_\infty \left[ 1 - n^2 \frac{\xi^2}{r^2} \left( 1 + \frac{\psi_\infty}{2h(\psi_\infty)} \frac{\partial h}{\partial \psi_{\text{cl}}}(\psi_\infty) \right) \right] ,
\]

(14)

showing that \( \sim \xi \) gives the size of the vortex core radius. For \( B = 0 \) the free energy per unit of volume \( V^{d-3} \), \( F_n \), is dominated by the third term of Eq. (9):

\[
F_n - F_0 \simeq 2\pi \int_0^R \frac{dr}{r} h(\psi_{\text{cl}}) n^2 \psi_{\text{cl}}^2 \simeq \pi n_s(\psi_\infty) n^2 \int_\xi^R \frac{dr}{r} = \pi n_s(\psi_\infty) n^2 \ln \left( R/\xi \right) ,
\]

(15)

that depends logarithmically on the size of the superfluid sample \( R \). This shows that superfluid vortices are not finite-energy configurations in the limit \( R \to \infty \). For \( B \neq 0 \), we have to consider the free energy as a function of the angular velocity, obtaining\(^6\)

\[
F_n(B) = F_n(0) - \int_0^B M_n(B) dB ,
\]

(16)

where \( M_n(B) \) is the magnetization (angular momentum from Eq. (12)) of the \( n \)-vortex configuration:

\[
M_n = \pi \int dr \ r J_\phi .
\]

(17)

The value of \( M_n \) is approximately given by

\[
M_n \simeq \pi n_s(\psi_\infty) \int_\xi^R drr \left( n - \frac{r^2}{2B} \right) \simeq \pi n_s(\psi_\infty) \left( \frac{nR^2}{2} - \frac{R^4}{8B} \right) ,
\]

(18)

\(^6\)In the superfluid case, this is the correct expression for the energy calculated in the co-rotating system with respect to the container.
that leads to

$$F_n(B) \simeq F_0(B) + \pi n_s(\psi_\infty) \left( n^2 \ln (R/\xi) - \frac{1}{2} n R^2 B \right) .$$

(19)

From this formula we can easily calculate the critical angular velocity $B_{c1}$ above which the vortex configuration is energetically favorable. This is given by the $B$ field at which $F_1 = F_0$:

$$B_{c1} \simeq \frac{2}{R^2} \ln (R/\xi) .$$

(20)

We observe that $B_{c1} \to 0$ when the size of the sample goes to infinity, that is $R \to \infty$.

By increasing $B$, more and more vortices are formed up to a critical value $B_{c2}$ at which the normal phase is favorable. In the limit $B \to B_{c2}$ the condensate goes to zero, and the GL theory can be applied. One obtains, with a standard textbook derivation,

$$B_{c2} = \frac{1}{2 \xi_{GL}^2} .$$

(21)

2.2 Superconductor vortex

For superconductors, $a_i$, and correspondingly the magnetic field $B$, are dynamical fields. Superconductor vortex configurations are therefore described by two fields, $\psi_{cl}$ and $a_\phi$. The value of $a_\phi$ varies from zero (at $r = 0$) to $n$ at infinity, canceling the logarithmic divergence in Eq. (15) and making the vortex energy finite in the limit $R \to \infty$.

Like in the superfluid case, we can obtain the behavior of the fields at small and large $r$ in a model independent way. At small $r$, the condensate drops to zero and the GL action can be used; in this limit one can derive

$$\psi_{cl} \propto r^{|n|}, \quad a_\phi \propto r^2 .$$

(22)

At large $r$ the situation is more complicated. If one uses Eq. (9) it is possible to show that the fields have the following large $r$ behavior

$$\psi_{cl} \simeq \psi_\infty + \frac{\psi_1}{\sqrt{r}} e^{-r/\xi'}, \quad a_\phi \simeq n + a_1 \sqrt{r} e^{-r/\lambda'},$$

(23)

with $\xi' = \xi$, $\lambda' = \lambda$ and $\psi_1$ and $a_1$ being constants. Nevertheless, Eq. (23) shows that higher derivatives are not negligible with respect to the first and second derivatives that we included in Eq. (9). Indeed, we have

$$\frac{\partial^n a_\phi}{\mu^n} \sim \frac{1}{(\lambda \mu)^{n-1} \mu} \frac{\partial^n a_\phi}{\mu} \sim \frac{\partial^n a_\phi}{\mu} ,$$

(24)

where we have assumed, based on dimensional grounds, that the scale $\mu$ suppresses the higher-dimensional operators, and that $\lambda$ is of order $1/\mu$. A similar situation happens for $\psi_{cl}$. We are therefore led to the conclusion that we cannot neglect higher-derivative terms to describe the large $r$ behavior of the fields. Including them, the equations of motion can (formally) be written as

$$\mathcal{M}(\Box)\psi_{cl} \simeq \frac{1}{\xi^2} (\psi_{cl} - \psi_\infty) , \quad \mathcal{N}(\Box)a_\phi \simeq \frac{1}{\lambda^2} (a_\phi - n) ,$$

(25)

In this case we call the dynamical magnetic field $B$, while we keep $H$ for the external magnetic field.
where $M$ and $N$ are unknown functions and the box operator acts on $\psi_{cl}$ and $a_\phi$ as

$$\Box \psi_{cl} = \frac{1}{r} \partial_r \left( r \partial_r \psi_{cl} \right), \quad \Box a_\phi = r \partial_r \left( \frac{1}{r} \partial_r a_\phi \right).$$  \hfill (26)

Fortunately, the solutions to the equations above are also of the form of Eq. (23) but with $\lambda'$ and $\xi'$ generically different from $\lambda$ and $\xi$. In other words, the effect of the higher-derivative terms is just to change the values of $\lambda'$ and $\xi'$. From this large $r$ behavior we can see that the radius size of the vortex core and the radius size of the magnetic tube passing through the vortex (the penetration length) are, respectively, characterized by $\xi'$ and $\lambda'$.

To calculate the external magnetic field $H$ at which the vortex configuration is energetically favorable we must obtain the Gibbs free energy. This is given in terms of the free energy $F$ by

$$G[J_{ext}] = F - \int d^{d-1}x a_i J_{i,ext}^i,$$  \hfill (27)

where $J_{i,ext}$ is an external current coupled to the gauge field $a_i$. We can relate $J_{ext}$ to the external magnetic field $\vec{H}$ that it produces, through

$$\nabla \times \vec{H} = \frac{e_0}{\epsilon_0} J_{ext}. \hfill (28)$$

Then we end up with the following Gibbs free energy per unit of volume $V^{d-3}$ of the vortex configuration

$$G_n[H] = F_n - \frac{1}{e_0^2} \int r dr d\phi BH = F_n - \frac{2\pi n}{e_0^2} H,$$  \hfill (29)

where we have used the magnetic flux condition $\int r dr d\phi B = 2\pi n$ and assumed that $H$ is constant. The critical $H_{c1}$ is defined as the value of $H$ at which $G_1 = G_0$ that corresponds to

$$H_{c1} = \frac{e_0^2}{2\pi} (F_1 - F_0).$$  \hfill (30)

The exact value of $F_1 - F_0$ depends strongly on the model and therefore $H_{c1}$ can only be calculated once the model is specified.

The minimum value of $H$ for which the energetically favorable phase is the normal phase is also, as in the superfluid case, $H_{c2} = 1/(2\xi_{GL}^2)$. The superconductors that have energetically favorable vortex solutions, that is $H_{c1} < H_{c2}$, are called Type II superconductors, while the others are called Type I. When the external field is slightly smaller than $H_{c2}$ the condensate has a small value and the GL theory can be applied to predict that Type II superconductors present a triangular lattice of vortices [13]. Superfluids can be considered as deep Type II superconductors and therefore they also present a triangular lattice of vortices. We will show that holographic superconductors are of Type II.

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8In this Maxwell equation of the external field we use $e_0$, defined as the electric charge in the normal phase ($\psi_{cl} = 0$), to guarantee that when $T \to T_c$, the magnetic field $B$ approaches $H$. 
3 Holographic superfluids and superconductors

The holographic theory that we want to study is defined \[1, 14\] by a charged scalar $\Psi$ coupled to a U(1) gauge field $A_\alpha$ in $d + 1$ dimensions ($\alpha, \beta = 0, 1, \ldots, d$) and an action given by

$$S = \int d^{d+1}x \sqrt{-G} \left\{ \frac{1}{16\pi G_N} (R - \Lambda) + \frac{1}{g^2} \mathcal{L} \right\}, \quad \text{with} \quad \mathcal{L} = -\frac{1}{4} F_{\alpha\beta}^2 - \frac{1}{L^2} |D_\alpha \Psi|^2. \quad (31)$$

$G_N$ is the gravitational Newton constant and the cosmological constant $\Lambda$ defines the asymptotic AdS radius $L$ via the relation $\Lambda = -d(d-1)/L^2$; moreover we introduced $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $D_\alpha = \partial_\alpha - i A_\alpha$. For simplicity, we have not added any potential for the scalar. We will later discuss the implications of including these terms. We will work in the limit $G_N \to 0$ and $g \to 0$ taken such that the gravitational effect of $L/g^2$ can be neglected. In this limit the metric is given by an AdS-Schwarzschild black hole (BH):

$$ds^2 = \frac{L^2}{z^2} \left[ -f(z) dt^2 + dy^2 \right] + \frac{L^2}{z^2 f(z)} dz^2, \quad f(z) = 1 - \left( \frac{z}{z_h} \right)^d, \quad (32)$$

where $t$ is time, $z$ is the holographic direction such that the AdS-boundary occurs at $z = 0$, while the BH horizon is at $z = z_h$ and $dy^2$ stands for the $d - 1$ dimensional flat metric. Since we are interested in the theory at finite temperature, we will perform the Euclidean continuation with compact time $it \in [0, 1/T]$ where $T = d/(4\pi z_h)$.

3.1 The AdS/CFT correspondence and dynamical gauge fields

This $d + 1$ dimensional theory has a dual interpretation in terms of a $d$ dimensional CFT at nonzero temperature. The AdS/CFT dictionary relates the properties of the AdS gravitational theory with those of the CFT. In particular, the fields $A_\mu$ and $\Psi$ evaluated on the AdS-boundary correspond to fields external to the CFT:

$$a_\mu = A_\mu|_{z=0}, \quad s = \Psi|_{z=0}. \quad (33)$$

They are coupled to CFT operators through the interaction terms $a_\mu \hat{J}^\mu + s \mathcal{O}$. The operator $\hat{J}^\mu$ corresponds to the U(1) current of the CFT theory, while $\mathcal{O}$ is a CFT operator charged under the U(1) with Dim[$\mathcal{O}] = 3(4)$ for $d = 3(4)$. Having chosen a nonzero mass for the scalar $\Psi$ in Eq. (31), would have corresponded to take another dimensionality for $\mathcal{O}$. We do not expect however any important qualitative difference for other choices of the mass. The dual CFT theory, if it exists, is supposed to be strongly coupled and the limit $g \to 0$ in the AdS theory corresponds to be working at the planar level in the CFT.

Integrating over the CFT fields, one can obtain the free energy $F[a_\mu, s]$ from which the vacuum expectation values (VEV) of the CFT operators can be extracted. In the gravity side, $F[a_\mu, s]$ is obtained from the $d + 1$ dimensional AdS Euclidean action $S_E[a_\mu, s]$ evaluated with all bulk fields on-shell restricted to Eq. (33):

$$F[a_\mu, s] = T S_E[a_\mu, s], \quad (34)$$

from which we obtain the VEVs of the currents

$$\langle \hat{J}^\mu \rangle = \frac{L^{d-3}}{g^2} z^{3-d} f_{z\mu}|_{z=0}, \quad \langle \mathcal{O} \rangle = \frac{L^{d-3}}{g^2} z^{1-d} D_z \Psi^*|_{z=0}. \quad (35)$$
The matching with the effective theory of Section 2 is straightforward: the gauge field $a_i$ of Eq. (33) is identified with that in Eq. (2), while $\langle \hat{J}_i \rangle$ and $\langle O \rangle$ of Eq. (35) are identified respectively with $-\beta^{-1} \delta \Gamma / \delta a^i$ and $\Phi_{cl}$ when renormalized in the same scheme.

In the AdS/CFT correspondence, the external fields in Eq. (34) are considered to be frozen background fields. This is suited for holographic superfluids, but not for superconductors that require the presence of dynamical gauge fields coupled to the CFT. It is easy however to promote the external $a_\mu$ field to a dynamical field. This corresponds to integrating over it in the path integral:

$$G[s, J_{ext}] = -T \ln \int Da \ e^{-\beta F[a_\mu, s]+\int d^d x \left[ -\frac{1}{e_b^2} F_{\mu \nu}^2 + a_\mu J_{ext}^\mu \right]} ,$$

(36)

where, for generality, we have added to $F[a_\mu, s]$ a “bare” kinetic term for $a_\mu$ ($e_b$ denotes the bare electric charge), and have coupled it to a background external current $J_{ext}^\mu$ to define a Gibbs energy. Working in the semiclassical approximation, Eq. (36) leads to the Maxwell equation for the gauge field $a_\mu$:

$$\langle \hat{J}_\mu \rangle + \frac{1}{e_b^2} \partial_\nu F_{\nu \mu} + J_{ext}^\mu = 0 ,$$

(37)

where we have used that $\langle \hat{J}_\mu \rangle = -\delta F / \delta a_\mu$. Let us see how the above procedure can be implemented in the gravity side. Using Eq. (35), we can write Eq. (37) as the following AdS-boundary condition:

$$\frac{L^{d-3}}{g^2} z^{3-d} F_{\mu} \bigg|_{z=0} + \frac{1}{e_b^2} \partial_\nu F_{\nu \mu} \bigg|_{z=0} + J_{ext}^\mu = 0 .$$

(38)

This is a boundary condition of Neumann type that, in order to be consistent with the variational principle, requires the AdS model to include the following extra terms on the AdS-boundary:

$$\int d^d x \left[ -\frac{1}{4 e_b^2} F_{\mu \nu}^2 + A_\mu J_{ext}^\mu \right]_{z=0} .$$

(39)

Therefore the Gibbs free energy is given by the AdS Euclidean action $S_E$ including the additional terms Eq. (39) evaluated on-shell with the bulk fields restricted to the AdS-boundary condition Eq. (38).

In the particular case of $d = 2 + 1$, and in the limit where $e_b / g \to \infty$ (not adding a kinetic term for the gauge field on the AdS-boundary), one can show that the theory defined by Eq. (36) preserves conformal symmetry. In this case the original CFT and the $a_\mu$ can be considered as part of a new CFT. Another way to understand this result is given in Ref. [4]. There it was shown that there are two ways to quantize a gauge field in the four dimensional AdS. We can either impose a Dirichlet or a Neumann boundary condition at $z = 0$. Each option is associated with a different CFT, S-dual to each other, with different global U(1). While in the first option (Dirichlet boundary condition) the gauge field $a_\mu$ is a background field, in the second one (Neumann boundary condition) the gauge field is truly dynamical [4]. In this latter case the gauge field arises from the CFT as a composite state, as shows the fact that its kinetic term is induced by the AdS bulk dynamics. In other words, this local U(1) appears as an emerging phenomenon. As emphasized in Ref. [4], this CFT, which includes a dynamical gauge field, has also a global U(1)

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9 The semiclassical approximation is valid in the limit $g \to 0$ and $e_b \to 0$. 

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with an associated conserved current given by $\tilde{J}^\mu = \epsilon^{\mu\nu\sigma} \partial_\nu a_\sigma$, and should not be confused with the emerging local U(1). Here we also observe that the emergence of the dynamical U(1) can be understood without using conformal invariance. Even for a warped space different from AdS, the massless zero-mode of a gauge field in $3 + 1$ dimensions has finite norm, corresponding then to a composite state in the dual $d = 2 + 1$ theory. This is related to the fact that the gauge interaction in $d = 2 + 1$ is a relevant operator and therefore is dominated by IR physics. It is thus possible to send $e_b/g$ to infinity in this case.

For $d = 3 + 1$, the situation is different. The current $\langle \tilde{J}_\mu \rangle$ contains a logarithmically divergent piece given by (in the gauge $A_z = 0$)

$$
\left. \frac{1}{z} \partial_z A_\mu \right|_{z=0} = -\partial^\nu F_{\nu\mu} \ln z \bigg|_{z=0} + \ldots .
$$

(40)

The appearance of the logarithmic divergence was already expected from the calculation of the kinetic term of $a_\mu$ in Eq. (1). This can also be understood by looking at the dual CFT interpretation of the gravitational theory. Indeed, at short distances (smaller than $1/T$) this dual theory is a $3 + 1$ dimensional relativistic theory charged under a U(1). At the quantum level an external $a_\mu$ gauging this U(1) receives corrections to its self-energy that in momentum space go as

$$
\Pi(p^2) \simeq p^2 \ln \left( \frac{p^2}{\Lambda_b^2} \right) ,
$$

(41)

where $p^2$ is the 4-dimensional momentum of the gauge field and $\Lambda_b$ is a momentum cut-off that regularizes a logarithmic divergence. Therefore $a_\mu$ is a state of infinite norm. If our intention is to keep the external gauge field in the theory we must renormalize it. A possible way to do so is to place a UV-brane at finite $z > 0$ as in Randall-Sundrum models [15]. Alternatively, we can absorb the divergence in the local counterterm of Eq. (39), i.e., defining the bare coupling $e_b$ as

$$
\frac{1}{e_b^2} = \frac{1}{e_0^2} + \frac{L}{g^2} \ln z |_{z=0} + \text{finite terms} ,
$$

(42)

where $e_0$ denotes here and thereafter our renormalized (physical) electric charge at the normal phase ($\psi_{cl} = 0$). Contrary to the $d = 2 + 1$ case, the presence of the gauge field $a_\mu$ breaks conformal invariance; therefore the gauge field cannot be considered an emerging phenomenon but just a new external state coupled to the CFT [5]. The same is true for any $d > 4$.

We are now ready to study models of holographic superconductors. We are interested in vortex configurations where, as we said, the effects of dynamical gauge fields are important. We will however present first the holographic superfluid vortex configurations for both $d = 2 + 1$ and $d = 3 + 1$. This will be useful to clarify previous results in the literature [9], showing that these holographic vortices fulfill the expectations of Section 2 for configurations without dynamical gauge fields. Then, we will present our main result: the Abrikosov superconducting vortex.

3.2 The vortex Ansatz

For both the superfluid and the superconductor we will demand

$$
s = 0 , \quad a_0 = \mu ,
$$

(43)
where $\mu$ is a constant. We fix $s = 0$ since we are only interested in the case in which the U(1) symmetry is broken dynamically by the VEV of $O$. The constant $\mu$ plays the role of a chemical potential. As shown in Ref. [11], a nonzero $\mu$ is necessary in order to induce, at temperatures smaller than some critical temperature $T_c(\mu)$, a nonzero value for $\langle O \rangle$ and to have the system in a superfluid/superconductor phase. This critical temperature is given by $T_c(\mu) \simeq 0.03(0.05) \mu$ for $d = 3(4)$. (44)

Notice that $\mu \neq 0$ breaks the conformal symmetry of the system and, together with $T$, set the scales of the model.

To obtain vortex solutions we take the Ansatz [9, 11]

$$\Psi = \psi(z, r) e^{i n \phi}, \quad A_0 = A_0(z, r), \quad A_\phi = A_\phi(z, r),$$

and the other components of $A_\alpha$ set equal to zero. As in the previous section, $n$ is an integer and a vortex corresponds to $n \neq 0$. We will be working in polar coordinates ($dy^2 = dr^2 + r^2 d\phi^2$) for $d = 2 + 1$ and in cylindrical coordinates ($dy^2 = dr^2 + r^2 d\phi^2 + dz^2$) for $d = 3 + 1$. The equations of motion for the Ansatz (45) are given by

$$z^{d-1} \partial_z \left( f \frac{1}{z^{d-1}} \partial_z \psi \right) + \frac{1}{r} \partial_r (r \partial_r \psi) + \left( \frac{A_0^2}{f} - \frac{(A_\phi - n)^2}{r^2} \right) \psi = 0,$$

$$z^{d-3} \partial_z \left( f \frac{1}{z^{d-3}} \partial_z A_\phi \right) + r \partial_r \left( \frac{1}{r} \partial_r A_\phi \right) - \frac{2(A_\phi - n)}{z^2} \psi^2 = 0,$$

$$z^{d-3} \partial_z \left( \frac{\partial_z A_0}{z^{d-3}} \right) + \frac{1}{r f} \partial_r (r \partial_r A_0) - \frac{2 A_0}{z^2 f} \psi^2 = 0.$$  

(46)

We will impose regularity to our solutions. This requires at $z = z_h$:

$$- \frac{d}{z_h} \partial_z \psi + \frac{1}{r} \partial_r (r \partial_r \psi) \quad \frac{(A_\phi - n)^2}{r^2} \psi = 0,$$

$$- \frac{d}{z_h} \partial_z A_\phi + r \partial_r \left( \frac{1}{r} \partial_r A_\phi \right) \quad \frac{2(A_\phi - n)}{z_h^2} \psi^2 = 0,$$

$$A_0 = 0,$$

(47)

while at $r = 0$ we must have

$$\partial_r A_0 = 0, \quad A_\phi = 0,$$

$$\partial_r \psi = 0 \quad \text{for} \; n = 0, \quad \psi = 0 \quad \text{for} \; n \neq 0.$$  

(48)

For a superfluid, as we explained before, the energy of the vortices is sensitive to the size of the sample $R$. We will therefore limit $r \leq R$, where $R$, as we commented before, is taken much bigger than the vortex radius.
3.3 Holographic superfluid vortices

For a vortex superfluid configuration $a_\phi$ is fixed:

$$a_\phi = A_\phi|_{z=0} = \frac{1}{2} Br^2,$$

(49)

where the constant $B$ represents the external rotation (or, equivalently, the external magnetic field for a superconductor in a situation in which the magnetic field can be considered frozen). This corresponds to a Dirichlet boundary condition at $z = 0$.

Also we impose the following boundary conditions at $r = R$:

$$\partial_r \psi = 0, \quad \partial_r A_0 = 0, \quad A_\phi = \frac{1}{2} BR^2.$$

(50)

These conditions are consistent with the variational principle which is used to derive the equations of motion from the action. The first two conditions represent the physical requirement that, far away from the vortex center, the solution should reduce to the superconducting/superfluid phase, which is independent of $r$, while the third one is a simple option compatible with (49).

We have solved numerically Eqs. (46) with the boundary conditions Eqs. (47), (48), (43), (49) and (50) by using the COMSOL 3.4 package [16]. In Fig. 1 we present the order parameter and the current as functions of $r$ for the $n = 1$ vortex solution obtained from such numerical analysis. Our solutions have the right behavior at $r \to 0$ and $r \to \infty$ as predicted in Eqs. (13) and (14) respectively. We notice however that, unexpectedly, the order parameter $\langle O \rangle$ develops a small bump at around $r \sim 12/\mu$, especially for the $d = 2 + 1$ case.

It is interesting to know whether our results deviate from those of the simple GL theory. For this purpose, we must first specify the input parameters, $\xi_{GL}$ and $b_{GL}$, of the GL model. We fit these two parameters from two predictions of the holographic model: $B_{ct}$ and $\langle \dot{J}_\phi \rangle$ at large $r$. The value of $B_{ct}$ is determined in the holographic model as the value of $B$ at which $\langle O \rangle$ reduces to zero everywhere in space. With this value and Eq. (21) we can obtain $\xi_{GL}$. From the value of $\langle \dot{J}_\phi \rangle$ at large $r$ as given in Fig. 1 we match with the corresponding current in the GL model, which, for $B = 0$, reads $J_{\phi}^{GL}(r \to \infty) = 2n|\Phi_{GL}(r \to \infty)|^2$; this allows to obtain $b_{GL}$. We find

$$\xi_{GL} \simeq 1.1 (0.9) \mu^{-1}, \quad b_{GL} \simeq 3.3 \mu (12.4),$$

(51)

for $d = 3 (4)$ at $T/T_c = 0.3$. Once $\xi_{GL}$ and $b_{GL}$ are determined, we can obtain the prediction of the GL model for the condensate and the current as functions of $r$ in the $n = 1$ vortex configuration. We show these in Fig. 1. We can appreciate that the holographic vortex differs significantly from that of the GL theory. In particular, the radius size of the vortex core in the holographic model is considerably bigger than that in the GL theory, namely $\xi > \xi_{GL}$. For $T \simeq T_c$, however, the holographic model, like any other model of superfluidity, should reduce to the GL theory. We have checked numerically that the holographic prediction for $\langle O \rangle$ and $\langle \dot{J}_\phi \rangle$ approaches that of the GL theory when the temperature is very close to $T_c$.

Next, we calculate the vortex free energy $F_n$ of the $d$ dimensional superfluid. It is then possible to verify that $B_{ct}$ behaves as predicted by the effective theory approach, Eq. (20), and also that $F_n$ follows, to a very good approximation, Eq. (19). Similarly, the results from the effective theory of
Figure 1: The modulus of $\langle O \rangle$ and $\langle \hat{J}_\phi \rangle$ (up to a factor $L^{d-3}/g^2$) as functions of $r$ from the holographic model in the $n = 1$ superfluid vortex solution for $d = 2 + 1$ (solid lines on the left) and $d = 3 + 1$ (solid lines on the right). In this plot we chose $T/T_c = 0.3$ and $B = 0$. The dashed lines are the corresponding profiles in the GL model. Presented in units of $\mu = 1$.

Section 2, can explain the results obtained in Ref. [9]. Indeed, taking the value of $n_s(\psi_\infty) = 0.28\sqrt{\rho}$ as in Ref. [9], we obtain, from Eqs. (18) and (19),

$$M_n \simeq 0.4nR^2\sqrt{\rho} - 0.1R^4\sqrt{\rho}B , \quad \frac{F_1 - F_0}{\sqrt{\rho}} = 0.9\ln(R/\xi) - 0.4R^2B , \quad (52)$$

that agrees [10] as well as $B_{c1}$ in Eq. (20), with the numerical values obtained in Ref. [9].

The value of $B_{c2}$ as a function of $T$, that, as explained before, coincides with that of a superconductor ($B_{c2} = H_{c2}$), will be presented in Section 3.4 As discussed in Section 2.2, any superfluid can be considered as a deep Type II superconductor and therefore, when $B$ is slightly smaller than $B_{c2}$, presents a triangular vortex lattice. This property has been checked in Ref. [17] for a holographic superfluid for $d = 2 + 1$. Here we stress that the same remains valid also for bigger values of $d$ as it uniquely comes from the fact that, when the condensate is small, the theory is well approximated by a GL theory. In the next section we will show that our holographic superconductor is a Type II superconductor and therefore is also characterized by a triangular lattice of vortices for $H$ slightly smaller than $H_{c2}$.

[10]Here we point out a missprint in the value of $\beta_n$ given in Eq. (24) of Ref. [9]: the correct one is $\beta_n \simeq 0.1(0.2)R^4\sqrt{\rho}$. 

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3.4 Holographic superconductor vortices

To model an Abrikosov vortex we consider stationary configurations that do not possess a dynamical electric field but only a dynamical magnetic field. Therefore at $z = 0$ we will impose the boundary condition Eq. (43) for $A_0$ and Eq. (38) for $A_i$ that, in polar coordinates, reads

$$L^{d-3}g^2z^3-d\partial_z A_\phi \bigg|_{z=0} + \frac{1}{e_b^2} r\partial_r \left( \frac{1}{r} \partial_r A_\phi \right) \bigg|_{z=0} = 0,$$

where we have taken $J_{\mu}^{ext} = 0$. At $r = R \to \infty$ we impose that

$$\partial_r \psi = 0, \quad \partial_r A_0 = 0, \quad A_\phi = n.$$

From the set of equations (46) and boundary conditions Eqs. (47), (48), (43), (53) and (54), we can numerically obtain the superconductor vortex configurations. The profile for the condensate $\langle O \rangle$ and the magnetic field $B(r) = \partial_r A_\phi|_{z=0}/r$ are given as functions of $r$ in Fig. 2. We have chosen $T/T_c = 0.3$ and $e_b/g \to \infty$ for $d = 2 + 1$, while, for $d = 3 + 1$, we have taken $e_b$ to satisfy $e_b^{-2}(T = T_c) \approx 1.7L/g^2$. We observed that the fields have the expected behavior at small and large $r$ given by Eqs. (22) and (23) respectively. Indeed, the vortex profile at large $r$ has changed from the behavior of Eq. (14) to that of Eq. (23) as expected in an Abrikosov vortex with dynamical
Figure 3: $\lambda$ and $\lambda'$ as functions of $T$ from our holographic model for $d = 2 + 1$ (on the left) and $d = 3 + 1$ (on the right). Presented in units of $\mu = 1$.

EM fields. Similar to the superfluid case, however, the order parameter $\langle O \rangle$ shows an unexpected slight increase at around $r \sim 12/\mu$. In Fig. 3 we show $\lambda$ and $\lambda'$, defined respectively in Eqs. (7) and (23), as functions of the temperature. For $T \to T_c$ both quantities diverge as expected, since in this limit we have $\psi_\infty \to 0$ and therefore $\lambda' \to \lambda \to \infty$. As $T \to 0$, however, we observe that $\lambda$ and $\lambda'$ differ considerably, with $\lambda'$ increasing its value at $T/T_c \simeq 0.3 - 0.4$. A priori, this would indicate that the magnetic flux tube becomes broader as $T$ goes to zero, since the penetration length $\lambda'$ grows. Nevertheless, we find that the situation is more complex; as $T \to 0$ the magnetic flux develops two cores, one of size $\sim 1/\mu$ while the other $\sim \lambda'$. This unexpected behavior deserves further studies.

In Fig. 2 we also provide the corresponding curves in the GL theory; the parameters $\xi_{\text{GL}}$ and $b_{\text{GL}}$ in the GL potential are fixed as in the superfluid case, Eq. (51), while the electric charge $e_0$ appearing in the GL action is determined by using the second definition in Eq. (7) applied to the GL case, that is $\lambda_{\text{GL}} = 1/(\sqrt{2}e_0|\Phi_{\text{GL}}(r \to \infty)|)$, and by requiring $\lambda_{\text{GL}}$ to be equal to the value of $\lambda'$ of the holographic superconductor. Again, as in the superfluid case, we observe that the radius size of the vortex core is bigger in the holographic model than in the GL theory. As expected, we find these differences disappear as $T \to T_c$.

The free energy per unit of volume $V^{d-3}$ of the vortex configuration is, after taking into account the kinetic term in Eq. (36), given by

$$F_n = \frac{T}{V^{d-3}} S_E + 2\pi \int dr r^2 \frac{(\partial_r a_\phi)^2}{2e_0^2} \tag{55}$$

where $S_E$ is calculated with the appropriated boundary conditions already stated. Contrary to the superfluid case, we have checked that $F_1 - F_0$ is finite for $R \to \infty$ thanks to the presence of the gauge field.

To calculate the critical magnetic field $H_{c1}$ we follow Eq. (30). We find that $H_{c1} < H_{c2}$ for any real value of $e_0$. This implies that for these holographic superconductors there is always a range of values of $H$ for which vortex solutions are energetically favorable; the superconductors are always of Type II. In Fig. 4 we show $H_{c1}$ and $H_{c2}$ as functions of the temperature for the same values of $e_b$ as in Fig. 2. Notice that $H_{c1}$ approaches zero as $T \to 0$. This is due to our normalization of $H$ in Eq. (28) that makes $H_{c1} \propto e_0^2$, which goes to zero as $T \to 0$. We can, however, derive $H_{c2}/H_{c1} \to \infty$ as $T \to 0$ independently of such normalization. This is a generic prediction of
Figure 4: $H_{c1}$ and $H_{c2}$ as functions of $T$ from our holographic model for $d = 2 + 1$ (solid lines on the left) and $d = 3 + 1$ (solid lines on the right). The dashed lines are the corresponding predictions for $H_{c1}$ from the GL theory. Presented in units of $\mu = 1$.

superconducting CFT. Finally, we compare our results with those arising from the GL theory of superconductors. We observe that $H_{c1}$ deviates from the GL prediction for temperatures smaller than $T_c$.

From the discussion given in Section 2.2 and the fact that our superconductors are of Type II, we know that the energetically favorable configuration when $H$ is slightly smaller than $H_{c2}$ is a triangular lattice of vortices.

4 Conclusions

We have shown how to introduce a dynamical gauge field in holographic superconductors to study vortex configurations. In $d = 2 + 1$ this gauge field is part of the CFT spectrum and therefore can be considered to be an emergent phenomenon, instead of an external field. We have shown that vortex configurations, in the presence of this gauge field, follow the expected properties of finite-energy Abrikosov vortices where the magnetic field drops exponentially at distances larger than the vortex core. We have calculated the energy of the vortices and the critical magnetic fields $H_{c1}$ and $H_{c2}$ that determine the intermediate (Shubnikov) phase. In all cases we have found that $H_{c1} < H_{c2}$ indicating that holographic superconductors are of Type II. For comparison, we have also calculated the vortex configurations in the absence of dynamical fields, corresponding
to superfluid vortices, and described their properties.

The vortex configurations found here differ considerably from those arising from a GL theory. In particular, the vortex size comes out to be larger, $H_{c1}$ has a different $T$ dependence, and, more importantly, the penetration length of the magnetic field differs significantly as $T \to 0$.

We have extended the study to $d = 3 + 1$ where a dynamical gauge field has to be introduced by a proper renormalization of the AdS-boundary terms. In this case, the gauge field does not respect conformal symmetry and is external to the CFT. In spite of this, the vortex properties are found to be very similar to the $d = 2 + 1$ case.

Although we have focused on vortex solutions, the method described here to introduce a dynamical gauge field is general and can be used in other situations. For example, one could study the behavior of the EM field near the surface of a finite size superconductor or in the junction between two superconducting samples in the presence of the Josephson effect [18]. Moreover, it would be interesting to extend the present analysis to 3+1 dimensional gauge fields sourced by a 2+1 dimensional CFT; this would allow to study interactions between the EM field and layered superconductors. Our studies can also be extended to $p$-wave holographic superconductors [19] and to holographic models that are dual to non-relativistic scale-invariant theories [20].

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