FUBINI-TONELLI TYPE THEOREM FOR NONPRODUCT MEASURES IN A PRODUCT SPACE

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Abstract. I prove a theorem about iterated integrals for non-product measures in a product space. The first task is to show the existence of a family of measures on the second space, indexed by the points on of the first space (outside a negligible set), such that integrating the measures on the index against the first marginal gives back the original measure (see Theorem 2.1). At the end, I give a simple application in Optimal Transport.

1. Introduction

The Fubini-Tonelli theorem states that the integral of a function defined on a product space, against a measure which is a product of measures on the factor spaces, can be obtained by iterated integration, i.e. integrating one variable (against its marginal measure) at the time.

If a measure on a product space is not a product measure, is it still possible to decompose the measure and evaluate the integral using iterated integration? To better understand the problem, imaging we are dealing with a measure \( \zeta \) which is absolutely continuous with respect to the product measure \( \mu \otimes \nu \), i.e. there is a function \( \delta : X \times Y \to [0, \infty] \), such that for all measurable set \( C \subseteq X \times Y \),

\[
\zeta(C) = \int_C \delta(x, y) \mu \otimes \nu (dx, dy).
\]

Then, using the classical Fubini-Tonelli theorem, we can decompose this integral into

\[
\zeta(C) = \int_X \left( \int_{C_x} \delta(x, y) \nu(dy) \right) \mu(dx),
\]

where \( C_x := \{y \in Y; (x, y) \in C\} \) is the slice of \( C \) at \( X \). Accordingly, \( \nu \) decomposes into the measures

\[
\nu_x(dy) := \delta(x, y) \nu(dy),
\]

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which integrates against $\mu$ to give $\zeta$. Symbolically,

$$\zeta(dx, dy) = \nu_x(dy) \mu(dx).$$

With this decomposition the order of integration is not interchangeable, since the first measure depends on the second variable. To interchange the order of integration, we must decompose $\mu$ in a similar way,

$$\mu_y(dx) := \delta(x, y) \mu(dx),$$

which integrates against $\nu$ to give $\zeta$. Symbolically,

$$\zeta(dx, dy) = \mu_y(dx) \nu(dy).$$

In this paper, the existence of this kind of decomposition is established for arbitrary Borel probability measures on the product of two complete, separable, locally compact, metric spaces (see Theorem 2.1). I restricted myself to the case of probability measures to simplify the discourse, although the results stay valid for $\sigma$–finite measures.

In the best of my knowledge, there is nothing of the kind in the literature on foundations of measure theory that describes similar results. Nonetheless, this question is natural and I believe it may provide a useful calculation and/or analytical tool as much as the classical Fubini-Tonelli theorem does.

Optimal Transport, for example, deals with fixed marginal probability measures and a minimal cost is seek among all the couplings of the given marginal probabilities, i.e. among all the probabilities on the product space, such that the marginal measures are the ones given. It would be a nice research project to look for a new characterization of the optimal transport plans in terms of the measures along the “fibers”, obtained from the decomposition described in Theorem 2.1. In section 4, I give a simple application of Theorem 2.1 showing that a pair of competitive price functions, whose integral with respect to some transference plan matches the transport cost, are conjugate to each other almost surely. This complements the Kantorovich duality theorem on the nature, regarding convexity/concavity, of pair of competitive prices maximizing the profit. See Villani’s book [4], page 70, for a very detailed discussion of the Kantorovich theorem.

Another interesting project is the application of Theorem 2.1 to the study of measures on the Tangent bundle of Riemannian manifolds. Indeed, the local charts of the tangent bundle are Cartesian products of Euclidean open sets. Using local charts, we can transport the measure to this product to be decomposed and then sent back the family.
of measures fiber-wise. In the literature, the measures on tangent bundles are a kind of product measures, as is the volume obtained from the Sasaki [3] metric, or the measure on the unit sphere on the tangent space, integrated against the volume element of the base manifold. I believe Theorem 2.1 is a tool that could help exploring general integration on tangent bundles.

2. Main theorem

**Theorem 2.1.** Let \( X \times Y \) be the product of two complete, separable, locally compact, metric spaces. We equip \( X, Y \), and \( X \times Y \) with their Borel \( \sigma \)-algebras, denoted by \( B_X, B_Y, \) and \( B_{X \times Y} \) respectively.

Let \( \zeta \) be a probability measure on \( B_{X \times Y} \), and denote \( \mu \) and \( \nu \) the marginal probabilities on \( B_X, B_Y \) respectively. i.e.

\[
\forall A \in B_X, \ \mu(A) = \zeta(A \times Y)
\]

and

\[
\forall B \in B_Y, \ \nu(B) = \zeta(X \times B).
\]

Then, outside an exceptional \( \mu \)-negligible set \( E_1 \in B_X \) (\( \mu(E_1) = 0 \)), for all \( x \in X \setminus E_1 \), there is a measure \( \nu_x \) defined on \( B_Y \), such that for all \( C \in B_{X \times Y} \), the function

\[
(2.1) \quad x \in X \setminus E_1 \rightarrow \nu_x(C_x),
\]

where \( C_x = C \cap (\{x\} \times Y) \), is \( B_X \)-measurable and

\[
(2.2) \quad \zeta(C) = \int_X \nu_x(C_x) \mu(dx).
\]

In particular,

\[
\forall B \in B_Y, \ \nu(B) = \int_X \nu_x(B) \mu(dx),
\]

Moreover, for all positive \( B_{X \times Y} \)-measurable function, \( f : X \times Y \rightarrow \mathbb{R} \),

\[
(2.3) \quad x \in X \setminus E_1 \rightarrow \int_Y f(x, y) \nu_x(dy)
\]

is \( B_X \)-measurable and

\[
(2.4) \quad \int_{X \times Y} f(x, y) \zeta(dx, dy) = \int_X \left( \int_Y f(x, y) \nu_x(dy) \right) \mu(dx).
\]

Likewise, there is a \( \nu \)-negligible set \( E_2 \in B_Y \), such that for every \( y \in Y \setminus E_2 \) there is a measure \( \mu_y \) on \( B_X \), such that for all \( C \in B_{X \times Y} \), the function

\[
y \in Y \setminus E_2 \rightarrow \mu_y(C_y),
\]
where $C_y = C \cap (X \times \{y\})$, is $\mathcal{B}_Y$–measurable and

$$
\zeta(C) = \int_X \mu_y(C_y) \mu(dx).
$$

In particular,

$$
\forall A \in \mathcal{B}_X, \quad \mu(A) = \int_Y \mu_y(A) \nu(dx).
$$

Moreover, for all positive $\mathcal{B}_{X \times Y}$–measurable function, $f : X \times Y \to \mathbb{R}$,

$$
y \in Y \setminus E_2 \to \int_X f(x,y) \mu_y(dx)
$$

is $\mathcal{B}_Y$–measurable and

$$
\int_{X \times Y} f(x,y) \zeta(dx,dy) = \int_Y \left( \int_X f(x,y) \mu_y(dx) \right) \nu(dy).
$$

As a consequence, given a $\mathcal{B}_{X \times Y}$–measurable function, $f : X \times Y \to \mathbb{R}$, the following affirmations are equivalent

(1) $f : X \times Y \to \mathbb{R}$ is $\zeta$–integrable.

(2) $x \in X \setminus E_1 \to \int_Y |f(x,y)| \nu_x(dy)$ is $\mu$–integrable.

(3) $y \in Y \setminus E_2 \to \int_X |f(x,y)| \mu_y(dx)$ is $\nu$–integrable.

And

$$
\int_{X \times Y} f(x,y) \zeta(dx,dy) = \int_X \left( \int_Y f(x,y) \nu_x(dy) \right) \mu(dx).
$$

$$
= \int_Y \left( \int_X f(x,y) \mu_y(dx) \right) \nu(dy).
$$

3. Proof of Theorem 2.1

Note about the notation: We will use $x$ and $y$ to denote generic points in $X$ and $Y$ respectively. In this way, $B_r(x)$ automatically refers to a ball in $X$, of center $x$ and radius $r$, while $B_r(y)$ represents a ball in $Y$ (different space, different metric).

The proof of Theorem 2.1 will be given in several steps.
3.1. **Definition of \( l_x \).** Let \( Y \) be a dense subset of \( Y \). Denote by \( \mathcal{B} \) the set of open balls \( B_r(y) \) with center \( y \in Y \) and radius \( r \in \mathbb{Q} \), i.e.

\[
\mathcal{B} := \{ B_r(y) ; y \in Y, \ r \in \mathbb{Q} \}.
\]

Consider also the complement of the closed balls,

\[
\mathcal{B}_c := \{ Y \setminus \overline{B}_r(y) ; B_r(y) \in \mathcal{B} \}.
\]

Finally, let \( \mathcal{L} \) be the set of finite unions of finite intersections of elements of \( \mathcal{B} \cup \mathcal{B}_c \) (note that \( \emptyset \in \mathcal{L} \)).

For each \( O \in \mathcal{L} \), define the measure

\[
A \in \mathfrak{B}_X \rightarrow \mu_O (A) = \zeta (A \times O).
\]

Since \( \mu_O (A) \leq \mu (A) \), \( \mu_O \) is absolutely continuous with respect to \( \mu \).

By Radon-Nikodym’s Theorem, there is a density function \( \frac{d\mu_O}{d\mu} \), defined \( \mu \)-almost surely, such that \( \mu_O \) is represented as an integral of this density against \( \mu \).

To obtain a common exceptional \( \mu \)-negligible set outside which \( \frac{d\mu_O}{d\mu} \) is well defined (by a formula) for all \( O \in \mathcal{L} \), we choose the version of \( \frac{d\mu_O}{d\mu} \) given by the limit of the quotient of balls. To avoid talking about measurability issues, we fix once and for all a sequence \( \rho_k \) decreasing to 0. Given \( O \in \mathcal{L} \), define

\[
\overline{\mathcal{L}}_x (O) := \limsup_{k \to \infty} \frac{\mu_O (B_{\rho_k}(x))}{\mu (B_{\rho_k}(x))},
\]

and

\[
\underline{\mathcal{L}}_x (O) := \liminf_{k \to \infty} \frac{\mu_O (B_{\rho_k}(x))}{\mu (B_{\rho_k}(x))},
\]

It is well known, by a generalization of Lebesgue differentiation theorem (see for example Federer [2], section 2.9), that \( \overline{\mathcal{L}}_x \) and \( \underline{\mathcal{L}}_x \) are versions of \( \frac{d\mu_O}{d\mu} \). i.e. For all \( A \in \mathfrak{B}_X \),

\[
(3.2) \quad \mu_O (A) = \int_A \overline{\mathcal{L}}_x (O) \mu (dx) = \int_A \underline{\mathcal{L}}_x (O) \mu (dx).
\]

In particular,

\[
\overline{\mathcal{L}}_x (O) = \underline{\mathcal{L}}_x (O), \quad \mu \text{ - a.s.}
\]

Let \( E_O \) be the exceptional set where the limit does not exist. i.e.

\[
E_O := \{ x \in X ; \overline{\mathcal{L}}_x (O) - \underline{\mathcal{L}}_x (O) > 0 \} \in \mathfrak{B}_X
\]

Put

\[
E := \bigcup_{O \in \mathcal{L}} E_O.
\]
Since \( \mu(E_O) = 0 \) for all \( O \in \mathcal{L} \), and \( \mathcal{L} \) is numerable, we have
\[
\mu(E) = 0.
\]

For \( O \in \mathcal{L} \), and \( x \in X \setminus E \), put
\[
l_x(O) = l_x(O) = \mathcal{L}_x(O).
\]

Recapitulating, for all \( O \in \mathcal{L} \) and every \( A \in \mathcal{B}_Y \), by (3.2) we have
\[
(3.3) \quad \zeta(A \times O) = \int_A l_x(O) \mu(dx).
\]

3.2. **Outer measure** \( \nu_x^* \). Changing the standpoint, we fix \( x \in X \setminus E \) and consider the set function
\[
O \in \mathcal{L} \rightarrow l_x(O).
\]

For future reference, observe that \( l_x \) has the following properties:

- **Finite additivity**
  \[
  (3.4) \quad l_x(O) + l_x(\tilde{O}) = l_x(O \cup \tilde{O}) + l_x(O \cap \tilde{O}),
  \]

- **Finite subadditivity**
  \[
  (3.5) \quad l_x(O \cup \tilde{O}) \leq l_x(O) + l_x(\tilde{O})
  \]

- **Monotonicity**
  \[
  (3.6) \quad O \subseteq \tilde{O} \Rightarrow l_x(O) \leq l_x(\tilde{O})
  \]

We need a measure on \( \mathcal{B}_Y \), capable of fulfilling the role of \( l_x \) in equation (3.3). Let us start by defining the outer measure
\[
(3.7) \quad C \subseteq Y \rightarrow \nu_x^*(C) := \inf \sum_{i=1}^{\infty} l_x(O_i),
\]
where the infimum is taken over all the covers \( \{O_i\}_{i \in \mathbb{N}} \subseteq \mathcal{L} \) of \( C \). i.e.
\[
C \subseteq \bigcup_{i=1}^{\infty} O_i, \text{ and } \forall i \in \mathbb{N}, O_i \in \mathcal{L}
\]

It is well known that \( \nu_x^* \), restricted to the set of \( \nu_x^* \)-measurable sets, is a measure (denoted \( \nu_x \)). What we need to prove are: Firstly, that every Borel subset of \( Y \) (i.e. in \( \mathcal{B}_Y \)) is \( \nu_x^* \)-measurable and secondly that the integration property (3.3) is preserved when \( l_x \) is replaced by \( \nu_x \) (and therefore valid for any set in \( \mathcal{B}_Y \)).
Unfortunately, \( l_x \) is not countably subadditive, as a result, \( \nu_x^* \) is not the extension of \( l_x \), hardening our task a little bit. Indeed, for all \( O \in \mathcal{L} \), we clearly have \( \nu_x^*(O) \leq l_x(O) \), but the inverse inequality may fail, as the following example shows.

**Example:** Let \( X = Y = [0, 1] \). Let \( \mu = \nu \) be the Lebesgue measure on \([0, 1]\) and \( \zeta \) the normalized length on the diagonal \( \{(x,x) ; x \in [0, 1]\} \).

Observe that for every \( x \in ]0,1[ \) and \( \rho_k \) small enough,
\[
\frac{\mu_{[0,x]}(B_{\rho_k}(x))}{\mu(B_{\rho_k}(x))} = \frac{1}{2}.
\]
So, \( l_x([0,x]) = 1/2 \), while \( \nu_x^*([0,x]) = 0 \). In fact, we can cover \([0,x]\) with a sequence of intervals \([0,x_n]\], where \( x_n \in \mathbb{Q} \) increases to \( x \). Each one of the intervals \([0,x_n]\) verify
\[
\frac{\mu_{[0,x_n]}(B_{\rho_k}(x))}{\mu(B_{\rho_k}(x))} = 0,
\]
for all \( \rho_k \) small enough. Therefore, for all \( n \in \mathbb{N} \), \( l_x([0,x_n]) = 0 \) and
\[
\nu_x^*([0,x]) \leq \sum_{i=1}^{\infty} l_x([0,x_n]) = 0.
\]

3.3. **Borel subsets of \( Y \) are \( \nu_x^* \)-measurable.** Let’s prove first that any open ball \( B_r(y) \in \mathcal{L} \) is \( \nu_x^* \)-measurable. To this end, fix \( C \subseteq Y \) and a cover \( \{O_i\}_{i \in \mathbb{N}} \subseteq \mathcal{L} \) of \( C \). We must show that
\[
\nu_x^*(C \cap B_r(y)) + \nu_x^*(C \setminus B_r(y)) \leq \sum_{i=1}^{\infty} l_x(O_i).
\]
Since \( O_i \cap B_r(y) \in \mathcal{L} \) and \( \{O_i \cap B_r(y)\}_{i \in \mathbb{N}} \) is a covering of \( C \cap B_r(y) \),
\[
\nu_x^*(C \cap B_r(y)) \leq \sum_{i=1}^{\infty} l_x(O_i \cap B_r(y)).
\]
Now, let \( \alpha_i < 1, \alpha_i \in \mathbb{Q} \). Then, \( O_i \setminus \overline{B}_{\alpha_i r}(y) \in \mathcal{L} \) and \( \{O_i \setminus \overline{B}_{\alpha_i r}(y)\}_{i \in \mathbb{N}} \) is a covering of \( C \setminus B_r(y) \). So,
\[
\nu_x^*(C \setminus B_r(y)) \leq \sum_{i=1}^{\infty} l_x(O_i \setminus \overline{B}_{\alpha_i r}(y)).
\]
By (3.4) and (3.6), we have
\[
l_x(O_i \cap B_r(y)) + l_x(O_i \setminus \overline{B}_{\alpha r}(y)) \leq l_x(O_i) + l_x(B_r(y) \setminus \overline{B}_{\alpha r}(y)).
\]
Adding (3.9) and (3.10), and using (3.11), we obtain

\[ \nu^*_x (C \cap B_r(y)) + \nu^*_x (C \setminus B_r(y)) \leq \sum_{i=1}^{\infty} l_x (O_i) + \sum_{i=1}^{\infty} l_x (B_r(y) \setminus \overline{B}_{\alpha_i r}(y)). \]

The result follows if we can make the second sum as small as we want. Unfortunately, for a fixed \( x \in E \), we might fail to do so, even though

\[ \bigcap_{i=1}^{\infty} B_r(y) \setminus \overline{B}_{\alpha_i r}(y) = \emptyset, \quad \text{for any sequence } \alpha_i \to 1. \]

In fact, we can not switch the limits in

\[ \lim_{i \to \infty} \lim_{k \to \infty} \zeta (B_{p_k}(x) \times (B_r(y) \setminus \overline{B}_{\alpha_i r}(y))). \]

So, we need to look back at what happens for \( x \) variable and check whether we can solve the problem by throwing away a few more points (meaning to enlarge \( E \)).

By (3.3) and (3.12), and \( \gamma < 1 \),

\[ \int_X l_x (B_r(y) \setminus \overline{B}_{\gamma r}(y)) \mu (dx) = \nu (B_r(y) \setminus \overline{B}_{\gamma r}(y)) \to \gamma \to 1 \quad 0. \]

Therefore,

\[ l_x (B_r(y) \setminus \overline{B}_{\gamma r}(y)) \to \gamma \to 1 \quad 0, \quad \mu - a.s. \]

Now, fix an increasing sequence \( \{\gamma_j\}_{j \in \mathbb{N}} \subseteq \mathbb{Q}, \gamma_j \to 1 \). Define, for all \( y \in \mathcal{Y} \) and \( r \in \mathbb{Q} \),

\[ E_{r,y} := \left\{ x \in X \setminus E; \liminf_{j \to \infty} l_x (B_r(y) \setminus \overline{B}_{\gamma_j r}(y)) > 0 \right\}. \]

By (3.13), \( \mu (E_{r,y}) = 0 \). Since

\[ E_1 := E \cup \bigcup_{r \in \mathbb{Q}, y \in \mathcal{Y}} E_{r,y} \]

is a countable union of sets of \( \mu \)-measure 0, we have

\[ \mu (E_1) = 0. \]

For all \( x \in X \setminus E_1 \), and every \( \epsilon > 0 \), we can choose a subsequence \( \{\alpha_i\}_{i \in \mathbb{N}} \) of \( \{\gamma_j\}_{k \in \mathbb{N}} \), such that

\[ \sum_{i=1}^{\infty} l_x (B_r(y) \setminus \overline{B}_{\alpha_i r}(y)) < \epsilon. \]
This completes the proof of (3.8).

Consequently, for all \( x \in X \setminus E_1 \), \( \nu_x \) is a measure defined at least in the \( \sigma \)-field generated by \( B \), i.e. \( \mathfrak{B}_Y \).

### 3.4. Measurability and integrability for compact sets

Our task now is to prove that given \( B \in \mathfrak{B}_Y \), the function

\[
(3.14) \quad x \in X \setminus E_1 \mapsto \nu_x (B)
\]

is measurable and, for all \( A \in \mathfrak{B}_X \),

\[
(3.15) \quad \zeta (A \times B) = \int_A \nu_x (B) \, \mu(dx).
\]

Let’s consider first a finite intersection of closed, compact balls

\[
\overline{B} = \overline{B}_{r_1} (y_1) \cap \cdots \cap \overline{B}_{r_n} (y_n),
\]

with \( y_1, \ldots, y_n \in Y \) and \( r_1, \ldots, r_n \in \mathbb{Q} \). By the measurability of \( l_x \) and (3.3), the properties (3.14) and (3.15) are proven at once if we show

\[
(3.16) \quad \nu_x (\overline{B}) = 1 - l_x (Y \setminus \overline{B}), \; \mu \text{-a.s.}
\]

(We use \( l_x (Y \setminus \overline{B}) \) just because \( l_x \) is not defined for \( \overline{B} \).

Let \( O_1, O_2, \cdots, O_m \subseteq L \) be a covering of \( \overline{B} \). We can assume the covering is finite, since \( \overline{B} \) is compact and the sets in \( L \) are open.

Clearly, for all \( x \in X \setminus E \),

\[
\lim_{k \to \infty} \frac{\zeta (B_{r_k} (x) \times \overline{B})}{\mu (B_{r_k} (x))} = \lim_{k \to \infty} \left( 1 - \frac{\zeta (B_{r_k} (x) \times (Y \setminus \overline{B}))}{\mu (B_{r_k} (x))} \right) = 1 - l_x (Y \setminus \overline{B}).
\]

Since the covering is finite, by (3.5),

\[
\lim_{k \to \infty} \frac{\zeta (B_{r_k} (x) \times \overline{B})}{\mu (B_{r_k} (x))} \leq l_x \left( \bigcup_{i=1}^n O_i \right) \leq l_x (O_1) + \cdots + l_x (O_m).
\]

Then,

\[
1 - l_x (Y \setminus \overline{B}) \leq \nu_x (\overline{B}).
\]

On the other hand, given \( \eta > 1, \eta \in \mathbb{Q} \), and denoting

\[
B_{\eta} = B_{\eta r_1} (y_1) \cap \cdots \cap B_{\eta r_n} (y_n),
\]

we have

\[
\nu_x (\overline{B}) \leq l_x (B_{\eta}).
\]

Consequently, taking any sequence \( \eta_k \searrow 1, \eta_k \in \mathbb{Q} \), we have

\[
1 - l_x (Y \setminus \overline{B}) \leq \nu_x (\overline{B}) \leq \lim_{k \to \infty} l_x (B_{\eta_k}).
\]
Since the functions at the left and at the right of the above inequalities are $\mathcal{B}_X$-measurable, and equal between them $\mu$-almost-surely, using (3.3), we have proved (3.16) and, a fortiori, (3.14) and (3.15), at least for a finite intersection of compact balls.

3.5. Measurability and integrability for Borel sets. Let $\mathcal{M}$ be the collection of sets $B \in \mathcal{B}_Y$ actually verifying (3.14) and (3.15).

Clearly, $\emptyset \in \mathcal{M}$ and $Y \setminus B \in \mathcal{M}$, whenever $B \in \mathcal{M}$.

Now, take a disjoint sequence $B_1, B_2, \cdots \in \mathcal{M}$ ($B_i \cap B_j = \emptyset$, $i \neq j$).

Since, for all $x \in X \setminus E_1$, $\nu_x$ is a measure on $\mathcal{B}_Y$,

\begin{equation}
\nu_x \left( \bigcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \nu_x (B_i).
\end{equation}

By (3.14), $x \to \nu_x \left( \bigcup_{i=1}^{\infty} B_i \right)$ is $\mathcal{B}_X$-measurable, being a countable sum of $\mathcal{B}_X$-measurable functions.

By (3.17), the monotone convergence theorem, and (3.15) (remember $B_i \in \mathcal{M}$, for all $i \in \mathbb{N}$), given $A \in \mathcal{B}_X$,

\[
\int_A \nu_x \left( \bigcup_{i=1}^{\infty} B_i \right) \mu \left( dx \right) = \int_A \sum_{i=1}^{\infty} \nu_x (B_i) \mu \left( dx \right)
\]

\[
= \sum_{i=1}^{\infty} \int_A \nu_x (B_i) \mu \left( dx \right)
\]

\[
= \sum_{i=1}^{\infty} \zeta (A \times B_i)
\]

\[
= \zeta \left( \bigcup_{i=1}^{\infty} A \times B_i \right)
\]

\[
= \zeta \left( A \times \bigcup_{i=1}^{\infty} B_i \right).
\]

Then,

\[
\bigcup_{i=1}^{\infty} B_i \in \mathcal{M}.
\]

Since $\mathcal{M}$ contains all the finite intersections of compact balls with center in $Y$ and rational radius, by the $\pi - \lambda$ theorem (see [1], page 36),

$$\mathcal{M} = \mathcal{B}_Y.$$
3.6. **Proof of the theorem.** We have proven so far (2.1) and (2.2) for sets of the form \( C = A \times B \), with \( A \in \mathcal{B}_X \) and \( B \in \mathcal{B}_Y \).

Using a similar argument as before, let \( \tilde{M} \) denote the collection of sets \( C \in \mathcal{B}_{X \times Y} \) verifying (2.1) and (2.2). We readily see that \( \emptyset \in M \).

Now, let \( C \in \tilde{M} \). Since \( ((X \times Y) \setminus C)_x = Y \setminus C_x \), the function
\[
x \to \nu_x ((X \times Y) \setminus C_x) = 1 - \nu_x (C_x)
\]
is \( \mathcal{B}_X \)-measurable, and
\[
\int_X \nu_x ((X \times Y) \setminus C_x) \mu (dx) = 1 - \int_X \nu_x (C_x) \mu (dx)
\]
\[
= 1 - \zeta (C)
\]
\[
= \zeta ((X \times Y) \setminus C).
\]

Then, for all \( C \in \tilde{M} \), we have \( (X \times Y) \setminus C \in \tilde{M} \).

Finally, let \( C_1, C_2, \ldots \in \tilde{M} \) a sequence of disjoint sets \( (C_i \cap C_j = \emptyset \), for all \( i \neq j \)). Since \( \nu_x \) is a measure on \( \mathcal{B}_Y \),
\[
\nu_x \left( \left( \bigcup_{i=1}^\infty C_i \right)_x \right) = \nu_x \left( \bigcup_{i=1}^\infty (C_i)_x \right) = \sum_{i=1}^\infty \nu_x ((C_i)_x).
\]

Then, \( x \to \nu_x \left( \left( \bigcup_{i=1}^\infty C_i \right)_x \right) \) is \( \mathcal{B}_X \)-measurable, being a sum of \( \mathcal{B}_X \)-measurable functions, since \( C_i \in \tilde{M} \), for all \( i \in \mathbb{N} \).

By (3.18), the monotone convergence theorem, and (2.2) \( (C_i \in \tilde{M}) \),
\[
\int_X \nu_x \left( \left( \bigcup_{i=1}^\infty C_i \right)_x \right) \mu (dx) = \int_X \sum_{i=1}^\infty \nu_x ((C_i)_x) \mu (dx)
\]
\[
= \sum_{i=1}^\infty \int_X \nu_x ((C_i)_x) \mu (dx)
\]
\[
= \sum_{i=1}^\infty \zeta (C_i)
\]
\[
= \zeta \left( \bigcup_{i=1}^\infty C_i \right)
\]

Then,
\[
\bigcup_{i=1}^\infty C_i \in \tilde{M}.
\]

Since
\[
\{ A \times B; A \in \mathcal{B}_X, B \in \mathcal{B}_Y \} \subseteq \tilde{M},
\]
by the \( \pi - \lambda \) theorem,
\[
\hat{M} = \mathcal{B}_{X \times Y}.
\]
The remaining of the proof is standard. Assume \( f : X \times Y \to \mathbb{R} \) is a positive, \( \mathcal{B}_{X \times Y} \)-measurable functions. Then, \( f \) can be approached by an increasing sequence of simple, \( \mathcal{B}_{X \times Y} \)-measurable functions (linear combinations of characteristic functions)
\[
f_k(x, y) = \sum_{i=1}^{n_k} \lambda_{k,i} C_{k,i},
\]
where \( \lambda_{k,i} \in \mathbb{R} \) and \( C_{k,i} \in \mathcal{B}_{X \times Y} \) \((i = 1, \ldots, n_k, k \in \mathbb{N})\).

By linearity, for each \( f_k \), properties \((2.3)\) and \((2.4)\) are a direct consequence of \((2.1)\) and \((2.2)\). Passing to the limit as \( k \to \infty \), equations \((2.3)\) and \((2.4)\) are conserved, therefore valid for any positive function like \( f \).

To establish the integrability equivalence, we apply the preceding result to the positive and negative parts of the given function. In this way, the proof of Theorem \(2.1\) is complete.

### 4. An application to Optimal Transport

In this section we use Theorem \(2.1\) to show that a pair of competitive price functions, whose integral with respect to some transference plan equals the transport cost, are conjugate to each other almost surely, complementing Kantorovich’s duality theorem on the nature of a pairs of competitive prices maximizing the profit. See Villani’s book [1], page 70, for a very detailed discussion on Kantorovich’s theorem, in particular Theorem 5.1, part \((ii)\), item \((d)\). For this result, we do not need lower semicontinuity of the cost and other assumptions used to prove Kantorovich’s theorem, so we state the following lemma in its simplest form, using the notations in Theorem \(2.1\) by the way.

**Lemma 4.1.** Let \( c : X \times Y \to \mathbb{R} \cup \{+\infty\} \) be a \( \mathcal{B}_{X \times Y} \)-measurable, cost function, and \( \psi : X \to \mathbb{R} \cup \{+\infty\} \) and \( \phi : Y \to \mathbb{R} \cup \{-\infty\} \) be a pair of competitive prices, i.e.
\[
\forall (x, y) \in X \times Y, \quad \phi(y) - \psi(x) \leq c(x, y).
\]
Let \( \pi \) be a probability measure on \( \mathcal{B}_{X \times Y} \), with marginal measures \( \mu \) on \( \mathcal{B}_X \) and \( \nu \) on \( \mathcal{B}_Y \). (i.e. \( \pi \) is a transference plan between \( \mu \) and \( \nu \).) Assume that \( \phi - \psi \) and \( c \) are \( \pi \)-integrable and
\[
\phi(y) - \psi(x) = c(x, y), \quad \pi \text{-a.s.}
\]
Then

\[ (4.1) \quad \psi(x) = \sup_{y \in Y} (\phi(y) - c(x,y)), \quad \mu - \text{a.s.} \]

and

\[ (4.2) \quad \phi(y) = \inf_{x \in X} (\psi(y) + c(x,y)), \quad \nu - \text{a.s.} \]

**Proof:** Since \( \psi + c - \phi = 0, \pi - \text{a.s.} \),

\[ \int_{X \times Y} (\psi(x) + c(x,y) - \phi(y)) \pi(dx,dy) = 0. \]

Using the decomposition given by Theorem 2.1, equation (2.4),

\[ (4.3) \quad \int_X \left( \int_Y (\psi(x) + c(x,y) - \phi(y)) \nu_x(dy) \right) \mu(dx) = 0. \]

Since \( \psi + c - \phi \geq 0, \)

\[ (4.4) \quad \psi(x) \geq \sup_{y \in Y} (\phi(y) - c(x,y)) \]

and, for all \( x \) where it is defined, the function

\[ x \mapsto \int_Y (\psi(x) + c(x,y) - \phi(y)) \nu_x(dy) \]

is nonnegative. By (4.3),

\[ \int_Y (\psi(x) + c(x,y) - \phi(y)) \nu_x(dy) = 0, \quad \mu - \text{a.s.} \]

Then,

\[ (4.5) \quad \psi(x) = \int_Y (\phi(y) - c(x,y)) \nu_x(dy) \leq \sup_{y \in Y} (\phi(y) - c(x,y)), \quad \mu - \text{a.s.} \]

Combining (4.4) and (4.5), we obtain (4.1). Equation (4.2) is validated in a similar way.

**References**

1. Patrick Billingsley, *Probability and measure*, Second edition, John Willey & Sons, 1986.
2. Herbert Federer, *Geometric measure theory*, Springer-Verlag, Berlin - Heidelberg - NewYork, 1969.
3. Shigeo Sasaki, *On differential geometry of tangent bundles of riemannian manifolds*, Tohoku Math. J. (2) 10, Number 3 (1958), 338–354.
4. Cédric Villani, *Optimal transport, old and new*, Springer, Berlin, 2008.
