Can differently prepared mixed states be distinguished?

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Abstract

A measurement has been proposed by B. d’Espagnat that would distinguish from one another some ensembles that are differently prepared but correspond to the same density matrix. Here, the idea is modified so that it becomes applicable to much more general situations. The method is illustrated in simple examples. Some matter of concern might then be that information could be transmitted by methods based on our idea in the EPR kind of experiments. A simple proof is given that this is impossible.

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1 Introduction

In chapters 7 and 8 of his book [1], B. d’Espagnat discusses the quantum mechanical notion of mixed states in terms of ensembles related to preparation of such states. He presents examples of mixed ensembles prepared by mixing a great number $N$ of polarized spin 1/2 particles in several different ways, all of them leading, however, to the unpolarized beam.

It is a well known fact that the values of all measured quantities are uniquely determined if the density operator of the system is given. Hence there is no operational method to distinguish between physically different realizations of the ensembles, provided they lead to “states” with the same density operator. In his examples $\rho = \frac{1}{2} \text{Id}(2)$, where Id(2) is the $2 \times 2$ unit matrix.

In order to test the difference between the ensembles experimentally, B. d’Espagnat presents on p. 122 of [1] a proposal: ‘Can we produce such a difference? Yes, we can, provided we treat these “ensembles” as what, after all, they physically are, that is, as systems of (noninteracting) particles.’ The aim of this paper is to generalize and expand this proposal. Our motivation and concern is the well-known fact (see, e.g., [2], p. 170) that, in the EPR experiment, superluminal signals could be sent if such differences were measurable in general. However, we show in Sect. 3 that applying to the context of the EPR experiment the here considered generalization of d’Espagnat’s method does not lead to the possibility of transmitting information.

2 Ensembles and mixed states

In quantum mechanics a separable Hilbert space $\mathcal{H}$ is associated to each physical system. Its states are represented by the density operators. We assume that no superselection rules are operating, hence the correspondence between states and density operators is one–to–one. The family $\mathcal{S}$ of density operators is naturally closed under convex combination. The projectors onto one–dimensional subspaces belong to $\mathcal{S}$ and represent the pure states of the system. They are the extremal elements of $\mathcal{S}$. For more details, see [3].

In the notation of [1] a mixed ensemble $\hat{E}$ consisting of a large number $N = \sum_{\alpha=1}^{r} N_\alpha$ particles of which $N_\alpha$, $\alpha = 1, \ldots, r$, belong to pure states $|\phi_\alpha\rangle$, is described by the density operator

$$\rho = \sum_{\alpha=1}^{r} \frac{N_\alpha}{N} |\phi_\alpha\rangle\langle\phi_\alpha|.$$  

If $|\phi_\alpha\rangle$ are pairwise orthogonal, then the decomposition of $\rho$ is unique if and only if there is no degeneracy, i.e. $N_\alpha \neq N_\beta$ for $\alpha \neq \beta$. But if there is degeneracy, or if the
pure states $|\phi_\alpha\rangle$ are not pairwise orthogonal, then there are infinitely many convex decompositions of $\rho$.

The ensembles I and II are described on p. 121 of [1]: ‘Let us consider two methods that can be used to prepare an unpolarized beam of spin 1/2 particles. Method I consists in mixing, by means of suitable magnets, two fully polarized beams of equal intensity, one polarized along the $O_z$ direction, the other along the opposite direction. Method II is identical to method I except that the $\pm O_x$ directions replace the $\pm O_z$ directions.’

Our attitude is operational, based on the usual statistical interpretation of quantum mechanics. Thus the non-unique decomposability of mixtures tells us that the prepared state cannot be physically distinguished from results of infinitely many other preparation procedures. For instance, a state obtained by mixing up (without phase correlations) equal numbers of spin up and spin down particles (along $O_z$ axis, method I) cannot be physically distinguished from a state obtained by a similar mixture of spin left and spin right particles (along $O_x$ axis, method II). This fact is embodied in the formula for the expectation value of an observable $A$ in the state $\rho$,

$$\langle A \rangle_{\rho} = Tr(\rho A) = \sum_{\alpha=1}^{N} \frac{N_{\alpha}}{N} \langle \phi_{\alpha} | A | \phi_{\alpha} \rangle .$$

For a detailed discussion of this question see [4].

3 Ensembles of beams

Let us modify the proposal of [1] in order to apply it to EPR situations. One should consider an ensemble of particles as being also an ensemble of beams, i.e. ensemble of systems of $N$ noninteracting particles. These ensembles can in principle be prepared along the general lines of methods I or II (only, dropping the condition that the two fully polarized beams mentioned in d’Espagnat’s above quoted sentence should be of same intensity since this condition is incompatible with our requirement of complete randomness, see below). In this way, the same preparation procedure can, if repeated sufficient number of times, be interpreted as also preparing an $N$-particle state for any fixed positive integer $N$. Such beams can therefore be repeatedly prepared and then subjected to measurements. In this way, the proposal can also be formulated without the notion of ensemble.

Let us consider a system of $N$ spin 1/2 particles with the Hilbert space being the $N$-fold tensor product of two–dimensional complex vector spaces (equipped with the standard inner products)

$$\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 .$$
To prepare the beam according to method I, we take the set of its $2^N$ basis vectors in the form

$$\{e_{i_1} \otimes \cdots \otimes e_{i_N}\}_{i_1,\ldots,i_N=\pm 1},$$

where

$$e_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

are normalized eigenvectors of the Pauli matrix $\sigma_z$.

For method II another basis

$$\{f_{i_1} \otimes \cdots \otimes f_{i_N}\}_{i_1,\ldots,i_N=\pm 1}$$

is appropriate, where

$$f_{+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

are normalized eigenvectors of the Pauli matrix $\sigma_x$.

Now, we specify the preparation in more detail: it should satisfy the requirement of complete randomness. A random mixture corresponding to ensemble I of $N$-particle beams is described by the density operator where all $2^N$ projectors on basis states contribute with equal weights,

$$\rho^I_N = \frac{1}{2^N} \sum_{i_1,\ldots,i_N=\pm 1} E_{i_1} \otimes \cdots \otimes E_{i_N},$$

where $E_{\pm 1}$ denote orthogonal projectors on vectors $e_{\pm 1}$.

This can also be seen in more detail using elementary combinatorics. Imagine a chopping of a big random ensemble of spin 1/2 particles or, equivalently, of a very long sequence of randomly distributed ±1’s, into a sequence of sections of length $N$ or $N$-beams. It is clear that, in each $N$-beam, some spins (say $m$), are up and the remaining $N-m$ are down. The number of all possible distinct configurations of an $N$-beam with $m$ spins up is given by the combination number

$$N_m = \binom{N}{m},$$

hence the total number of all possible configurations in $N$-beams is the sum

$$\sum_{m=0}^{N} N_m = \sum_{m=0}^{N} \binom{N}{m} = 2^N.$$

Now to each configuration the representative projection operator in $\mathcal{H}$ is associated, e.g.

$$(E_{+1})^{\otimes m} \otimes (E_{-1})^{\otimes N-m}$$

corresponds to the configuration $\{+1,\ldots,+1,-1,\ldots,-1\}$.

\begin{footnote}{We disregard quantum statistics to keep the argument simple. A justification is given at the end of Sect. 4.}\end{footnote}
For fixed $m$, a random mixture is obtained by assigning equal probabilities $1/N_m$ to all $N_m$ projectors corresponding to distinct configurations of $\pm 1$'s:

$$\rho_{Nm}^I = \frac{1}{N_m}[(E_{+1})^{\otimes m} \otimes (E_{-1})^{\otimes N_m} + \cdots].$$

If, however, a completely random mixture including all $m = 0, \ldots, N$ is to be formed, the density matrices $\rho_{Nm}^I$ should be mixed with relative weights $N_m/\sum_{m=0}^{N} N_m$ proportional to numbers of configurations contributing to each $\rho_{Nm}^I$:

$$\rho_N^I = \sum_{m=0}^{N} \frac{N_m}{\sum_{m=0}^{N} N_m} \rho_{Nm}^I = \frac{1}{2N} \sum_{m=0}^{N} \left( \begin{array}{c} N_m \\ m \end{array} \right) \sum_{i_1, \ldots, i_N = \pm 1} E_{i_1} \otimes \cdots \otimes E_{i_N}.$$

But this is just the above formula for $\rho_N^I$.

Since the result can be written as

$$\rho_N^I = \frac{1}{2} \sum_{i_1 = \pm 1} E_{i_1} \otimes \cdots \otimes \frac{1}{2} \sum_{i_N = \pm 1} E_{i_N},$$

we obtain the density matrix of a completely unpolarized beam,

$$\rho_N^I = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \cdots \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),$$

or

$$\rho_N^I = \frac{1}{2N} \text{Id}(2^N)$$

in a suitable basis, where $\text{Id}(n)$ is the $n \times n$ unit matrix.

Following the same calculation with basis vectors $e_i$ replaced by $f_i$ and projectors $E_i$ by projectors $F_i$ on eigenvectors of $\sigma_x$, the same density matrix is obtained for case II, $\rho_N^{II} = \rho_N^I$.

The $N$-particle observable of twice the $z$-component of total spin

$$\Sigma_z = \sigma_z \otimes I \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes \sigma_z$$

has in both cases I and II the zero expectation value

$$\langle \Sigma_z \rangle_N^{I,II} = Tr(\rho_N^{I,II} \Sigma_z) = 0$$

because $Tr \sigma_z = 0$.

The dispersion $\sigma_N^I$ of $\Sigma_z$ on basis states I is obviously zero, since the basis I consists of eigenvectors of $\Sigma_z$. Let us compute it, however, on the mixed state $\rho_N^I$:

$$\langle \sigma_N^I \rangle^2 = \langle \Sigma_z^2 \rangle - \langle \Sigma_z \rangle^2 = \frac{1}{2N} \sum_{i_1, \ldots, i_N = \pm 1} \langle e_{i_1} \otimes \cdots \otimes e_{i_N} | \Sigma_z^2 | e_{i_1} \otimes \cdots \otimes e_{i_N} \rangle.$$
The result is the weighted sum of eigenvalues of $\Sigma^2_z$,
\[
(\sigma^I_N)^2 = \frac{1}{2^N} \sum_{i_1,\ldots,i_N=\pm1} (i_1 + \cdots + i_N)^2 = N.
\]
This sum is equal to $N$ as can be easily proved e.g. by induction. Our result in fact concerns not only case I but also case II because their density operators coincide. Hence
\[
\sigma^I_N = \sigma^{II}_N = \sqrt{N}.
\]
All ensembles that can be obtained in the EPR experiment are completely random, see, e.g., [2], p. 148. Since $\rho^I_N = \rho^{II}_N$, measurements of arbitrary $N$-particle observables yield the same results for both preparations I and II. We conclude that these preparations cannot be distinguished even as ensembles of $N$-beams (for arbitrary $N$) and the superluminal signals are not available.

4 Correlations within a beam

In the previous section, we have worked with purely random methods I and II. However, if the two methods are modified so that correlations are built in among particles of one beam, then d’Espagnat’s idea can be made use of. Let us formulate the idea so that each appearance of paradox is removed. Indeed, d’Espagnat does not claim that states described by the same statistical operator are distinguishable (this would be the paradox). His idea is that one and the same ensemble can be considered as representing a one-particle state as well as an $N$-particle state, where $N$ is, in principle, arbitrary. There can then be two different ensembles (I and II) such that the corresponding one-particle states are described by the same statistical operators while the corresponding $N$-particle ones have different statistical operators. If the measurements are restricted to one-particle quantities only, no difference can of course ever be found between the two ensembles. However, measurements of $N$-particle observables will find differences. We are going to give some examples in this section.

In order to have definite correlations, we have to assume that the states of individual particles within a beam are distinguishable and can be labeled in a unique way. This can be achieved e.g. by separating individual particle preparations by a suitable time interval $\tau$, say, and utilizing the position degree of freedom of the particle in addition to its spin. We assume that the individual pure states are well-separated wave packets of the same profile. This may still be considered as one definite preparation method of a one-particle state, applied at different times $t$, $t+\tau, \ldots, t+N\tau, \ldots, t+MN\tau, \ldots$ and, simultaneously, as a method of preparing an
\(N\)-particle beam state applied at times \(t, t + N\tau, \ldots, t + M N\tau, \ldots\); \(M\) and \(N\) are positive integers. We have further to assume that the whole arrangement leads to a dynamics that is time-translation invariant so that it is sensible to speak of “the same state” at different times.

As an example of a strict correlation, let us now additionally specify method I (II) as follows: the \(n\)-th one-particle state is prepared at the time \(t + n\tau\) with spin component \(\frac{1}{2}(-1)^n\) into \(Oz\) (\(Ox\)) axis.

If this is viewed as a preparation of a one-particle state, \(N = 1\), then both resulting states are again described by the same density matrices
\[
\rho_1^I = \rho_1^{II} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

However, if considered as a preparation of a two-particle state, \(N = 2\), then the result is the pure state \(e_{+1}(t) \otimes e_{-1}(t - \tau)\) for method I and \(f_{+1}(t) \otimes f_{-1}(t - \tau)\) for II. In this way, the two preparation methods give two identical one-particle states (mixed) but two different two-particle ones (pure). An easy calculation gives
\[
\langle \Sigma_z \rangle_2^I = \langle \Sigma_z \rangle_2^{II} = 0, \quad \langle \Sigma_z^2 \rangle_2^I = 0, \langle \Sigma_z^2 \rangle_2^{II} = 2,
\]
hence
\[
\sigma_2^I = 0, \quad \sigma_2^{II} = \sqrt{2},
\]
in agreement with [1].

Thus in this (rather special) case of preparation, the numbers that agree with [1] are obtained. Now consider the described methods I and II as preparations of a three-particle state, \(N = 3\). Then, two different pure states are mixed with equal frequencies: for example, in method I the states
\[
e_{+1}(t) \otimes e_{-1}(t - \tau) \otimes e_{+1}(t - 2\tau)
\]
and
\[
e_{-1}(t) \otimes e_{+1}(t - \tau) \otimes e_{-1}(t - 2\tau)
\]
are mixed with equal weights \(\frac{1}{2}\). Clearly, the three-particle beams can be described in \(2^3 = 8\)-dimensional Hilbert space with basis \(\{e_{i_1} \otimes e_{i_2} \otimes e_{i_3}\}_{i_k = \pm 1}\). The 8 basis vectors can be labeled, via the correspondence \(i_k = (-1)^{\alpha_k}\), by 3-digit binary numbers \((\alpha_1, \alpha_2, \alpha_3)\) in increasing order 0, 1, \ldots, 7. Then the density matrix of the mixed 3-beam is diagonal,
\[
\rho_3^I = \text{diag}(0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0),
\]
and
\[
\Sigma_z = \text{diag}(3, 1, 1, -1, 1, -1, -1, -3), \quad \Sigma_z^2 = \text{diag}(9, 1, 1, 1, 1, 1, 1, 9).
\]
This gives
\[ \langle \Sigma_z \rangle_3^I = 0, \quad \langle \Sigma^2_z \rangle_3^I = 1, \]
hence
\[ \sigma_3^I = 1. \]

In method II we can work with the basis \( \{ f_{i_1} \otimes f_{i_2} \otimes f_{i_3} \}_{i_k = \pm 1} \). Then the density matrix has the same diagonal form
\[ \rho_3^I = \text{diag}(0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0), \]
but \( \Sigma_z \) and \( \Sigma^2_z \) are non-diagonal. They can be explicitly found by using the action \( \sigma_z f_{\pm 1} = f_{\mp 1} \). In this way we obtain
\[ \langle \Sigma_z \rangle_3^{II} = 0, \quad \langle \Sigma^2_z \rangle_3^{II} = 3, \]
hence
\[ \sigma_3^{II} = \sqrt{3}. \]

We see that the 3-beams are also distinguishable. However, the corresponding dispersions \( \sigma_3^I \) and \( \sigma_3^{II} \) do not agree with those of [1] (being both non-zero).

For general \( N > 3 \) we have to distinguish the cases when \( N \) is even or odd. For even \( N = 2K \) we have a pure \( (2K) \)-particle state with binary labels 0101\ldots 01 and obtain
\[ \langle \Sigma_z \rangle_{2K}^I = \langle \Sigma_z \rangle_{2K}^{II} = 0, \quad \langle \Sigma^2_z \rangle_{2K}^I = 0, \quad \langle \Sigma^2_z \rangle_{2K}^{II} = 2K, \]
hence
\[ \sigma_{2K}^I = 0, \quad \sigma_{2K}^{II} = \sqrt{2K} \]
in agreement with [1]. For odd \( N = 2K + 1 \) we have a mixture of two pure \( (2K + 1) \)-particle states with \( 2K + 1 \) binary labels 0101\ldots 10 and 1010\ldots 01, respectively, and equal weights \( \frac{1}{2} \). Straightforward computation now results in
\[ \langle \Sigma_z \rangle_{2K+1}^I = \langle \Sigma_z \rangle_{2K+1}^{II} = 0, \quad \langle \Sigma^2_z \rangle_{2K+1}^I = 1, \quad \langle \Sigma^2_z \rangle_{2K+1}^{II} = 2K + 1, \]
i.e.
\[ \sigma_{2K+1}^I = 1, \quad \sigma_{2K+1}^{II} = \sqrt{2K + 1}. \]

We conclude that in the above examples the \( N \)-beams contain more information than the one-particle state prepared by the same procedure.\(^2\)

\(^2\)Note that e.g. the beams with even \( N = 2K \) need not consist of sequences of the special type 010101\ldots 01, but any sequence will do the job provided that the numbers of 0’s and 1’s remain the same for all beams composing the ensemble. So the interesting cases in which the physical ensembles prepared by methods I and II are theoretically distinguishable are not as special as the given examples would suggest [5].
Our final remark concerns the quantum statistics. Since we are working with particles of spin 1/2, the states of an N-particle beam must be antisymmetric in the particle names. In the notation of tensor products (of states and operators), the particle names are the order numbers of the factors in each such product taken, say, from left to right. For fermions, these products must, therefore, be antisymmetrized and provided by a normalization factor. As an example, consider the two-particle state \( e_{+1}(t) \otimes e_{+1}(t - \tau) \). A fermion state obtained from it is
\[
\frac{1}{\sqrt{2}} \left[ e_{+1}(t_1) \otimes e_{+1}(t_2 - \tau) - e_{+1}(t_2) \otimes e_{+1}(t_1 - \tau) \right].
\]

In the previous section we could also work with one-particle states distinguished by their times in the way described in this section. For all our purposes, it has been satisfactory to consider only tensor-product states containing one-particle ones in the order of decreasing time, as in the above example. Then a bijective linear mapping of our ordered states onto the antisymmetric fermion states is well defined. This map preserves the number of states and their inner products. Since all our arguments are based on counting the elements of state bases, they are preserved by this map.

It is also clear that everything could be done, in an analogous manner, with photons, or with other discrete degrees of freedom than spins.

5 Concluding remarks

Let us return to the quotations from [1] in Sects. 1 and 2. On pp. 122-123 of [1] there follow the results on \( N = 2K \)-beams formed as random mixtures characterized by zero total spin, or by fixed \( m = K \) in our notation:

‘Ensembles of these beams should therefore be considered. Such ensembles may in principle be prepared and can therefore be subjected to statistical measurements. For example, the fluctuations of the quantity
\[
\Sigma_z = \sum_{n=1}^{N} \sigma_{z,n}
\]
where \( \sigma_{z,n} \) is twice the \( z \) component of the spin of the \( n \)th particle in the beam and \( N \) is the number of such particles, can be experimentally measured on ensembles of beams prepared by any of the two methods (I and II).’ ... ‘We then observe that these fluctuations are different in the two cases. In fact they are characterized by the standard deviations
\[
\sigma^I = 0, \quad \sigma^{II} = \sqrt{N}.
\]
So that beams prepared by methods I and II can in principle be distinguished from each other. In fact, such possibility was the reason for our query, whether information could be transmitted in this way in the qualitatively different context of the EPR experiment. Therefore in Sect. 3 we considered an idea inspired from d’Espagnat’s one, applied it to EPR situations and found that in these situations information cannot be transmitted by using $N$-beams.

Finally we note that there is a general point of view implicit in our paper. In Sect. 3 the ensemble of $N$-beams was defined by giving very partial information about it and led to statistical predictions to be verified by experiment. Then in Sect. 4 we defined the ensemble of $N$-beams by giving more detailed information about it, implying entanglement between the particles. The considered cases can be compared with general situations in quantum mechanics involving an ensemble of entangled particles in spatially separated regions $A$ and $B$. Very partial information is obtained by observing only the particles in region $A$, neglecting partners in region $B$. It is well known that this situation is described by a density matrix which is not a pure state. However, when the information is obtained in both regions $A$ and $B$, then, because of entanglement, the experimentalist finds results different from the predictions in the former case.

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**References**

[1] B. d’Espagnat, *Veiled Reality. An Analysis of Present-Day Quantum Mechanical Concepts* (Addison-Wesley Publ. Co., Reading (Mass.), 1995).

[2] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1993).

[3] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1932).
[4] E. G. Beltrametti, in *Symposium on the Foundations of Modern Physics. 50 Years of Einstein–Podolsky–Rosen Gedankenexperiment*. Editors P. Lahti and P. Mittelstaedt (World Scientific Publishers, Singapore, 1985).

[5] B. d’Espagnat, private communication.