Physical Interpretation of Cylindrically Symmetric Static Gravitational Fields

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The explicit relationship is determined between the interior properties of a static cylindrical matter distribution and the metric of the exterior space-time according to Einstein gravity for space-time dimensionality larger or equal to four. This is achieved through use of a coordinate system isotropic in the transverse coordinates. As a corollary, similar results are obtained for a spherical matter distribution in Brans-Dicke gravity for dimensions larger than or equal to three. The approach used here leads to consistency conditions for those parameters characterizing the exterior metric.

It is shown that these conditions are equivalent to the requirement of hydrostatic equilibrium of the matter distribution (generalized Oppenheimer-Volkoff equations). These conditions lead to a consistent Newtonian limit where pressures and the gravitational constant go to zero at the same rate.

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I. INTRODUCTION

Cylindrically symmetric space-times were first investigated in the framework of Einstein’s general relativity by Levi-Civita [1] and Weyl [2] in 1917-1919. Despite its age, the issue is still of much interest and under active research, especially since the introduction of cosmic strings [3] by Kibble [4], Zel’dovich [5] and Vilenkin [6].

The spacetime which is generated around a static cylindrically-symmetric source is described by a line element of the form (\(\phi\) is a radial coordinate; plain \(r\) is reserved for future use):

\[
ds^2 = N_0(\varrho)dt^2 - N_1(\varrho)d\varrho^2 - N_2(\varrho)d\phi^2 - N_3(\varrho)dz^2.\]

(1.1)

The functional form of the metric components depends on the choice of coordinate system (gauge). This freedom is the relic of the original diffeomorphism symmetry. The two most popular gauges in the literature [1] are:

1) Weyl - Levi-Civita gauge: \(N_1(\tilde{r}) = N_3(\tilde{r})\) which gives

\[
ds^2 = (k\tilde{r})^{2p}dt^2 - (k\tilde{r})^{2p(p-1)}(d\tilde{r}^2 + dz^2) - \gamma^2(k\tilde{r})^{-2p}\tilde{r}^2d\phi^2,
\]

(1.2)

where \(p\) and \(\gamma\) are free parameters and \(k\) sets the length scale. A variant of the Weyl - Levi-Civita gauge is the Thorne gauge [1] which is defined by \(N_1 = N_0\).

2) Kasner gauge \(N_1(\tilde{r}) = 1\) which gives

\[
ds^2 = (k\tilde{r})^{2a}dt^2 - (k\tilde{r})^{2b}dz^2 - d\tilde{r}^2 - \gamma^2(k\tilde{r})^{2(b-1)}\tilde{r}^2d\phi^2,
\]

(1.3)

where \(a, b, c\) satisfy the Kasner conditions:

\[
a + b + c = a^2 + b^2 + c^2 = 1.
\]

(1.4)

\(k\) and \(\gamma\) are identical in both parametrizations only up to a multiplicative constant. In what follows we will use only those that appear in the Kasner metric.

It is obvious from the above forms of the line-element that the general static cylindrically-symmetric vacuum solution of Einstein equations is characterized by two free parameters. A question of fundamental interest is then the interpretation of these two parameters and the connection between them and the internal properties of the matter distribution.

A large amount of progress in this subject originated from the intensive studies of cosmic strings, although the first results were obtained earlier [1, 3]. It is well-known by now that the parameter \(\gamma\) describes a conic angular deficit which is also related to the mass distribution of the source [3, 13]. Around a so-called "gauge string" (i.e., one with \(T_0^0 = T_3^3\) as the only non-vanishing components of the energy-momentum tensor), a simple relation between the angular deficit \(\delta \phi = 2\pi(1 - \gamma)\) and the "inertial mass" (per unit length) \(\tilde{m}\), was found

\[
\delta \phi = 8\pi G\tilde{m},
\]

(1.5)

first [3] in the linearized approximation assuming also an infinitesimally thin source, and then [12, 13] by solving the full non-linear Einstein equations around a uniform source (constant \(T_0^0\)) with a finite radius. The same relation was also derived [14] for a non-uniform source. Since the space-time around a gauge string is locally flat \((p = 0\) in [12] or \(a = c = 0, b = 1\) in [13]), this angular deficit is the only geometrical evidence of its existence.
Further study of the subject involved a more realistic model (i.e. the abelian Higgs model) for the gauge strings and the analysis of the full coupled field equations for the gravitational field and matter (scalar + vector) fields.

A major contribution to the understanding of the "relation between gravitational mass, angular deficit and internal structure" was supplied by Frolov, Israel and Unruh (FIU) who, without the use of any specific model for the matter distribution, obtained several relations between the parameters in the Kasner metric and integrals of the components of the energy-momentum tensor.

In the present work we take a model-independent approach. We consider an arbitrary smeared matter distribution with cylindrical symmetry and a finite radius $r_b$. We start by introducing a line element which is isotropic in the transverse coordinates $r$, i.e. we take $N_2(r) = r^2 N_1(r)$ in (1.1). The line element is:

$$ds^2 = e^A dt^2 - e^B dx^i dx^j - e^C dz^2$$

where $i, j$ label the transverse coordinates, and $A$, $B$ and $C$ are functions of the radial coordinate $r$ only.

This metric is similar to the one used by Vilenkin, although more general. It has the advantage that the Einstein equations can be integrated by quadrature, giving the functions $A$, $B$ and $C$ in terms of integrals of the components of the energy-momentum tensor. These functions are subject to a consistency condition in the general cylindrically symmetric case, which is a manifestation of the hydrostatic equilibrium of the source. The consistency condition is studied in the Newtonian limit and it turns out that the pressures and the gravitational constant go to zero at the same rate with the masses kept fixed. An exception is a gauge string, where the consistency condition is trivially satisfied because the string tension is equal to the mass density. Since it is both straightforward and instructive to generalize the discussion to higher dimensions, we analyze the case of $(D + 1)$-Einstein gravity with $D > 3$. We furthermore find analogous consistency conditions in spherically symmetric space-time in $D$-dimensional Brans-Dicke gravity, with $D \geq 3$. We finally use our results to show that $D = 3$ Brans-Dicke gravity has a Newtonian limit (unlike Einstein gravity). This last point may be regarded as a generalization of an earlier study \[13\].

We also establish a connection between some of our results and those of FIU [16], correcting on the way some misprints which make their results difficult to use.

The outline of the paper is the following: In sec. II we consider cylindrically symmetric solutions of the Einstein equations for four-dimensional space-time ($D = 3$). In sec. III we discuss the interpretation of these solutions, focussing on cosmic strings and the Newtonian limit. In sec. IV we generalise to Einstein gravity in $D + 1$ dimensions, $D > 3$, and in sec. V to Brans-Dicke gravity. Our results are summarized in sec. VI.
\[(B' + \frac{2}{r})(A' + C') + A'C' = 32\pi G e^B p_r. \quad (2.9)\]

In vacuum the right-hand-sides of these equations vanish and the first three of them are trivially integrated. In this way we may get a line element equivalent to (1.2) and (1.3). The line element contains in the new version also two free parameters. However, in order to gain some insight as to the meaning of the parameters and the relations between them, we need to take into account, while solving the Einstein equations, the existence of the source. One way to do it is to solve Einstein equations inside the source \((r \leq r_b)\) and join the interior solution with the exterior vacuum solution on the boundary \((r = r_b)\) using the formalism of junction conditions (20). This strategy requires some assumptions concerning the matter in the source (i.e. its energy-momentum tensor). However, by the special choice we made here for the coordinate system, the Ricci tensor components have a form which is ready for integration irrespective of the internal structure of the source, as seen from (2.6), (2.7) and (2.8). We therefore exploit this fact and integrate the field equations in a way that gives a physical meaning to the parameters in the exterior metric.

For that purpose we define:

\[M(r) = 2\pi \int_0^r r'dr' e^{\frac{A+B+C}{2}} (\rho + p_r + p_\perp + p_z), \quad (2.10)\]

\[W(r) = -2\pi \int_0^r r'dr' e^{\frac{A+B+C}{2}} (\rho - p_r - p_\perp - p_z). \quad (2.11)\]

\[X(r) = -2\pi \int_0^r r'dr' e^{\frac{A+B+C}{2}} (\rho - p_r + p_\perp + p_z), \quad (2.12)\]

We integrate (2.6), (2.7) and (2.8) to obtain:

\[rA'e^{\frac{A+C}{2}} = 4G M, \quad (2.13)\]

\[(rB' + 2)e^{\frac{A+C}{2}} - 2 = 4GW \quad (2.14)\]

\[rC'e^{\frac{A+C}{2}} = 4GX \quad (2.15)\]

that by insertion into (2.14) lead to the equation

\[\frac{1}{2} + GW)(M + X) + GMX = 2\pi r^2 p_r e^{A+B+C}. \quad (2.16)\]

The physical origin of (2.16) is the requirement of hydrostatic equilibrium of the source which is necessary for having a static solution. It is essentially a first integral of the equation of energy-momentum conservation as we can readily see by calculating the derivatives of both sides, using the other three field equations and getting rid of as many metric components as possible, using (2.13), (2.14) and (2.15). We can then cast the conservation equation into the form:

\[e^{\frac{A+C}{2}} \frac{dp_r}{dr} + \frac{p_r - p_\perp}{r} = -\frac{2G}{r}(M\rho - Xp_z + (M + W + X)p_r - Wp_\perp). \quad (2.17)\]

This equation is the condition for hydrostatic equilibrium with general-relativistic corrections. It may deserve the name Oppenheimer-Volkoff equation (see e.g. MTW [24] p. 605) for cylindrical symmetry.

Outside the source \((r \geq r_b)\) we can integrate (2.13), (2.14) and (2.17) analytically, since \(M, W\) and \(X\) here are constant. We find:

\[A(r) = \frac{2M}{M + X} \ln \left(1 + 2\mathcal{G}(M + X)\ln \frac{r}{r_0}\right), \quad (2.18)\]

\[B(r) = \frac{2W + \mathcal{G}}{M + X} \ln \left(1 + 2\mathcal{G}(M + X)\ln \frac{r}{r_0}\right) - 2\ln \frac{r}{r_0}, \quad (2.19)\]

\[C(r) = \frac{2X}{M + X} \ln \left(1 + 2\mathcal{G}(M + X)\ln \frac{r}{r_0}\right), \quad (2.20)\]

where \(r_0\) is an arbitrary length scale. Having determined the functions \(A, B\) and \(C\) outside the source we have a third form of the line element of a cylindrically symmetric space-time which again depends on two parameters. The angular deficit is not described in this coordinate system by a parameter which multiplies the \(dr\) term in the line element (like \(\gamma\) in (1.3)). It is rather hidden now in \(W\) as we will see shortly.

The logarithms have the argument

\[1 + 2\mathcal{G}(M + X)\ln \frac{r}{r_0}.\]

Here we could get rid of the term 1 by a \(\mathcal{G}\)-dependent redefinition of the scale \(r_0\). This redefinition would, however, destroy the possibility of obtaining a Newtonian limit.

Note that \(M, W\) and \(X\) are not independent for \(r > r_b\), but obey according to (2.16) the consistency condition

\[\left(\frac{1}{2} + \mathcal{G}W\right)(M + X) + GMX = 0. \quad (2.21)\]

The four equations (2.18), (2.19), (2.20) and (2.21), summarize the relation between the exterior metric and the matter distribution of the source, or stated somewhat differently, describe how much information about the source can be inferred from the exterior geometry. Since the exterior metric contains two independent parameters it is clear that a distant observer will have to be satisfied with only two quantities in order to characterize the source. One such quantity is the “Tolman mass” (per unit length) \(M\). As the second parameter we suggest the parameter \(X\), which is the corresponding quantity associated with the \(z\)-direction. Notice that the right-hand sides of (2.18) and (2.20) go into each other under the interchange of \(M\) and \(X\). The left-hand sides
Adding this last equation with FIU’s (16) and the correct identity with (2.23).

The constants a, b, c, γ and k in the Kasner metric (1.3) are expressed by the new parameters:

\[ a = \frac{M}{M + W + X + 1/(2G)}, \]
\[ b = \frac{W + 1/(2G)}{M + W + X + 1/(2G)}, \]
\[ c = \frac{X}{M + W + X + 1/(2G)}. \]
\[ \gamma = 1 + 2G(M + W + X). \]
\[ (2.24) \]

The parameters a, b and c obey the Kasner conditions (2.3). One is obvious, while the other is a consequence of (2.21).

This last issue, namely the connection between the Kasner parameters and the matter distribution of the source, has been considered already by FIU [10]. Their results were obtained using a somewhat more geometric approach, based on an identity involving the extrinsic curvature and the Ricci tensor. It is, however, difficult to use their results due to some typographical errors in the relevant equations, as well as the fact that they are not given in an explicit form which clarifies the full dependence of the Kasner parameters on the quantities M, W and X. For the sake of completeness we give here the correct form of eqs. (17)-(19) of FIU using their notation.

- eq. (17). The right hand side should read \( \frac{1}{2}k(c - b). \)
- eq. (18) should be:

\[ \int_{\Delta z = 1} (T^p_p - T^\phi_\phi - T^z_z + T^t_t)(-g)^{1/2}d^3x = \frac{1}{2}k b \]
- eq. (19) should be:

\[ \int_{\Delta z = 1} (T^\rho_\rho + T^\phi_\phi)(-g)^{1/2}d^3x = \frac{1}{4}k(1 - a) \]

Note the difference in notation: FIU use dimensionless coordinates, the parameters a, b and c differ by a cyclic permutation, and their k is our \( \gamma \). It should be stressed that FIU’s k has an implicit dependence on the integrals of \( T^\rho_\rho \). This dependence can be unveiled by use of a third independent relation for the Kasner parameter a which is absent from FIU’s paper:

\[ \int_{\Delta z = 1} (T^p_p - T^\phi_\phi + T^z_z + T^t_t)(-g)^{1/2}d^3x = \frac{1}{2}(ka - 1) \]

Adding this last equation with FIU’s (16) and the correct form of (18) yields an expression for FIU’s k which is identical with (2.24). Their other relations are contained in (2.23).

### III. INTERPRETATION, COSMIC STRINGS AND NEWTONIAN LIMIT

In order to get some feeling of the physics of the solutions we first look for the familiar limits of the Minkowski space-time and the gauge string. The Minkowski metric should be obtained in the absence of source, namely: \( M = W = X = 0 \). In this case we may either solve the field equations again or take the appropriate limits in (2.18)-(2.24). Either way one finds immediately \( A = B = C = 0 \).

The gauge string is a somewhat less trivial example. It is characterized by \( p_z = -\rho \) while \( p_r = p_\perp = 0 \). Thus we still have \( A = C = 0 \). However, W is now arbitrary (and yet the condition (2.21) is satisfied), so by taking the limit \( M \to 0, X \to 0 \) in (2.19) we find that \( B = 4GW/\ln \frac{r_0}{r} \). Therefore a gauge string is described in our coordinate system by:

\[ ds^2 = dt^2 - dz^2 - \frac{r_0}{r} - 4GW(dr^2 + r^2d\phi^2) \]
\[ (3.1) \]

This line element can be brought into a locally flat form by the transformation \( \frac{r}{r_0} = \frac{1}{1 + 2GW (\frac{r}{r_0})^{1 + 2GW}} \) which is a special case of (2.22). This transformation gives:

\[ ds^2 = dt^2 - dz^2 - dt^2 - (1 + 2GW)^2 r^2 d\phi^2 \]
\[ (3.2) \]

which is the standard metric for a conical space-time with an angular deficit of

\[ \delta \phi = -4\pi GW. \]
\[ (3.3) \]

Using (2.11) for the gauge string one gets Vilenkin’s result (1.3), with:

\[ \tilde{m} = 2\pi \int_{r_0}^{r_*} r' dr' e^{\frac{A_{2, p, 0} + C}{2} r_0^2} \rho. \]
\[ (3.4) \]

This is also consistent with calculating the Kasner parameters a, b, c and \( \gamma \) by (2.21) and (2.24).

Actually, it follows from this analysis that a conical space-time is generated in more general circumstances than by a gauge string. The source may also have non-vanishing \( p_r \) and \( p_\perp \) provided their sum vanishes. In this case we are dealing with a cosmic string which is characterized by two parameters, \( \tilde{m} \) and say,

\[ \tilde{p}_r = 2\pi \int_{r_0}^{r_*} r' dr' e^{\frac{A_{2, p, 0} + C}{2} r_0^2} \rho. \]
\[ (3.5) \]

The angular deficit is still given by (3.3), but using here (2.11) for this special case we obtain a generalization of Vilenkin’s result, namely:

\[ \delta \phi = 8\pi GW (\tilde{m} - \tilde{p}_r). \]
\[ (3.6) \]

If we give up any restriction on \( p_r \) and \( p_\perp \), but keep \( p_z = -\rho \) we have \( X = M \), but not necessarily vanishing. This makes (2.24) reduce to
\[(1 + 2G/W)M + GM^2 = 0. \tag{3.7}\]

One of the two solutions of (3.7) is \(M = 0\) that by (2.13) and (2.11) again gives a conical space outside the source, with the deficit angle given by eq. (3.6). If we choose the other possibility, viz. \(X = M \neq 0, W = -\frac{1}{2}(M + \frac{1}{2})\), the line element will still be symmetric under boosts in the \(z\)-direction:

\[
ds^2 = (1 + 4GM \ln \frac{r}{r_0}) (dt^2 - dz^2) - \left(4GM \ln \frac{r}{r_0}\right)^{-1/2} (r_0 \frac{r}{r_0})^2 (dr^2 + r^2 d\phi^2) \tag{3.8}\]

This spacetime has the peculiar property that asymptotic azimuthal circles have vanishing circumference. However, the matter distribution which generates this solution is perfectly reasonable, so we may interpret this solution as representing gravitationally collapsed cylindrical matter distribution which is totally disconnected from the external space. Actually, this solution has the same asymptotic behavior as the Melvin universe [2]. A special case of this situation is a Higgs model cosmic string, which was discussed by Garfinkle [15] who obtained a consistent way of defining a Newtonian limit is to keep masses fixed and let \(G\) be small, with the pressures of order \(G\). From (2.21) we get in this limit the consistency statement:

\[
\frac{M + X}{2G} \simeq -MX, \tag{3.10}\]

while (2.18) implies:

\[
V \simeq \frac{1}{2} A \simeq 2GM \ln \frac{r_0}{r} \tag{3.11}\]

which is recognized as the standard expression of the Newtonian potential of a cylindrically symmetric mass distribution. It is curious that the logarithmic form is valid in a wider context than we are interested in: It is enough to take \(M + X \to 0\) in order to obtain (3.11).

To check the consistency of this limiting procedure we consider the requirement of hydrostatic equilibrium (2.16), or equivalently (2.17). Keeping in (2.17) only terms of order \(G\), where pressures according to what was said above are already of order \(G\), we get

\[
\frac{dp_r}{dr} + \frac{p_r - p_\bot}{r} \simeq -\frac{2GM \rho}{r} \tag{3.12}\]

which is the familiar condition for hydrostatic equilibrium in Newtonian gravity.

\[\text{IV. HIGHER DIMENSIONALITY}\]

In the previous sections an analysis of the static solutions of the Einstein equations in 3+1 dimensions for a general matter distribution with cylindrical symmetry was carried out. The analysis was facilitated by the use of a coordinate system where the metric tensor is isotropic in the transverse coordinates. It is natural to extend this analysis to higher dimensionalities, in order to check which of the results we found still hold.

The metric tensor is still chosen isotropic in the transverse coordinates, so the line element has the following form:

\[
ds^2 = e^A dt^2 - e^B \delta_{ij} dx^i dx^j - e^C (dx^D)^2, \quad i, j = 1, \cdots D - 1 \quad (4.1)\]

with \(A\), \(B\) and \(C\) functions of the radial coordinate \(r\). We define:

\[
\Upsilon = e^{\frac{A + (D - 2)p + B + C}{2}} \tag{4.2}\]

The Ricci tensor has the following components:

\[
\mathcal{R}_{00} = -\frac{1}{2} e^{A-B} \frac{1}{r^{D-2}} \Upsilon^{-1} \frac{d}{dr} (r^{D-2} A' \Upsilon), \tag{4.3}\]

\[
\mathcal{R}_{DD} = \frac{1}{2} e^{C-B} \frac{1}{r^{D-2}} \Upsilon^{-1} \frac{d}{dr} (r^{D-2} C' \Upsilon). \tag{4.4}\]
\[
R_{ij} = \delta_{ij} \frac{1}{2r D-2} \frac{1}{r} \left( \frac{d}{dr} (r^{D-2} B') Y' + \frac{1}{r} Y^{-1} Y' \right) + \frac{x^i x^j}{r^2} Y^{-1} Y'' + \frac{1}{2r} \frac{1}{r} \left( \frac{d}{dr} (r^{D-2} B') Y' \right) - \frac{1}{2} (D - 2) (A' + C') B' - \frac{1}{4} (D - 2) (D - 3) (B')^2 - \frac{1}{2} A' C'.
\] (4.5)

The energy-momentum tensor for a cylindrical distribution of energy and pressure has the components:

\[
T_{00} = \rho e^A \\
T_{DD} = p D e^C \\
T_{ij} = (p_r \frac{x^i x^j}{r^2} + p_1 \delta_{ij}) e^B
\] (4.6)

whence is found \( T = \rho - p_r - (D - 2) p_\perp - p_D, \rho, p_D, p_r \) and \( p_\perp \) are functions of the radial coordinate \( r \) only.

Einstein equations in \( D + 1 \) dimensions imply:

\[
R_{\mu\nu} = -\kappa_{D+1} (T_{\mu\nu} - \frac{1}{D-1} g_{\mu\nu} T)
\] (4.7)

where

\[
\kappa_{D+1} = \frac{D-1}{D-2} \Omega_{D-1} G_{D+1}
\] (4.8)

with \( \Omega_n = \frac{2 \pi^{n+1}}{\Gamma(n+1)} \) being the area of the unit \( n \)-sphere and \( G_{D+1} \) Newton’s constant in \( D + 1 \) dimensions. The 00 and DD components of (4.7) are:

\[
e^{-B} \frac{1}{Y^{-1}} \frac{d}{dr} r^{D-2} A' Y
= \frac{2\kappa_{D+1}}{D-1} ((D - 2) \rho + p_r + (D - 2) p_\perp + p_D),
\] (4.9)

\[
e^{-B} \frac{1}{Y^{-1}} \frac{d}{dr} r^{D-2} C' Y
= -\frac{2\kappa_{D+1}}{D-1} (\rho + (D - 2) p_D - p_r - (D - 2) p_\perp),
\] (4.10)

The angular part is obtained from (4.3):

\[
e^{-B} \frac{1}{Y^{-1}} \frac{d}{dr} r^{D-2} B' Y + \frac{2}{r} Y^{-1} \frac{d}{dr} Y
= -\frac{2\kappa_{D+1}}{D-1} (\rho - p_r + p_\perp - p_D),
\] (4.11)

By linear combination of (4.9), (4.10) and (4.11) we obtain:

\[
\frac{1}{r^{D-3}} \frac{d}{dr} r^{D-5} \frac{d}{dr} Y = \kappa_{D+1} r^{D-2} e^{-B} Y (p_r + p_\perp)
\] (4.12)

which in combination with the radial part of (4.7) is used to give as a fourth equation the following:

\[
\frac{2(D - 2)}{r} Y' + \frac{1}{2} (A' + C')^2 + 2(\frac{Y'}{Y})^2 + \frac{1}{2} A'C' = 2\kappa_{D+1} e^B p_r.
\] (4.13)

Now define the quantities:

\[
U(r) = \frac{\kappa_{D+1}}{2(D - 3)} \int_0^r (r')^{2D-5} e^B Y (p_r + p_\perp) dr',
\] (4.14)

\[
V(r) = -\frac{\kappa_{D+1}}{2(D - 3)} \int_r^\infty (r')^{2D-5} e^B Y (p_r + p_\perp) dr'.
\] (4.15)

Eq. (4.12) can be formally solved in terms of \( U(r) \) and \( V(r) \):

\[
\Upsilon(r) = \frac{1}{1 - r^{2(3-D)} U(r)} + V(r),
\] (4.16)

where also the boundary condition \( \lim_{r \to \infty} \Upsilon = 1 \) is used.

Defining furthermore the two quantities

\[
M(r) = \Omega_{D-2} \int_0^r (r')^{D-2} e^B Y (\rho + \frac{p_D + p_r}{D - 2} + p_\perp) dr',
\] (4.17)

\[
X(r) = -\Omega_{D-2} \int_0^r (r')^{D-2} e^B Y (\rho - p_r - p_\perp) dr',
\] (4.18)

we can solve (4.19) and (4.11) since they imply

\[
A' = \frac{\Omega_{D-1} G_{D+1} M}{\Omega_{D-2} r^{D-2} Y},
\] (4.19)

\[
C' = \frac{\Omega_{D-1} G_{D+1} X}{\Omega_{D-2} r^{D-2} Y}.
\] (4.20)

Inserting (4.10), (4.17) and (4.18) into (4.13) one gets:

\[
2(D - 2)(D - 3) U(1 + V) + \left[ \frac{\Omega_{D-1} G_{D+1}}{\Omega_{D-2}} \right]^2 \left( \frac{1}{2} \frac{D - 2}{D - 3} (M + X)^2 + M X \right) = \kappa_{D+1} r^{2D-4} e^B Y^2 p_r.
\] (4.21)

As in the \( D = 3 \) case, this equation is equivalent to a generalized Oppenheimer-Volkoff equation. After some manipulations on the same line as in sec. Ii one gets the following differential equation (cf. (2.17)):

\[
\frac{d}{dr} p_r + \frac{D - 2}{r} (1 + r^{2(3-D)} U + V) (p_r - p_\perp)
= -\frac{\Omega_{D-1} G_{D+1}}{\Omega_{D-2}} r^{2D-4} (M p_r - X p_D)
- \frac{M + X}{D - 3} (p_r - (D - 2) p_\perp)
\] (4.22)
The right-hand side of this equation is obtained from the right-hand side of (2.17) using the substitution $W \rightarrow -\frac{D-3}{D}(M + X)$.

An explicit expression for the metric tensor can again be obtained for $r \geq r_h$ (outside the source), where $M$, $X$ and $U$ are constant, and we introduce $r^2_h = U^{1/2}(D-3)$. These three quantities obey according to (4.22):

$$\left(\frac{\Omega_{D-1}}{\Omega_{D-2}^2} G_{D+1}^{D+1}\right)^2 \left(-\frac{1}{2} \frac{D-2}{D-3}(M + X)^2 + MX\right)
+ 2(D - 2)(D - 3)r_h^{2(D-3)} = 0.$$ (4.23)

This consistency condition is the higher-dimensional generalization of (2.22). From (4.16), (4.19) and (4.20) we get

$$\Upsilon(r) = 1 - \left(\frac{r_h}{r}\right)^{2(D-3)},$$ (4.24)

$$A(r) = \frac{\Omega_{D-1}}{(D-3)\Omega_{D-2}}^{-1} M \ln \left(1 - \left(\frac{r_h}{r}\right)^{D-3}\right),$$ (4.25)

$$C(r) = \frac{\Omega_{D-1}X}{(D-3)\Omega_{D-2}}^{-1} M \ln \left(1 - \left(\frac{r_h}{r}\right)^{D-3}\right)$$ (4.26)

that correspond to (2.18) and (2.20) in the $D = 3$ case. The solution describes a black string with the horizon at $r_h$. The line element is again characterized by the line element of a black hole in $D$ dimensions. Schwarschild coordinates are obtained by the coordinate transformation

$$\hat{r} = \frac{1}{r}(r^{D-3} + r_h^{D-3})^{\frac{1}{D-3}}.$$ (4.28)

In terms of $\hat{r}$ the line element is the Taubnerlini solution

$$ds^2 = \left(1 - \left(\frac{r_h}{\hat{r}}\right)^{D-3}\right) d\hat{r}^2 - \frac{d\hat{r}^2}{1 - \left(\frac{r_h}{\hat{r}}\right)^{D-3}} + \hat{r}^2 d\Omega_{D-1}^2.$$ (4.29)

In sec. III a thorough discussion of conical space-times for $D = 3$ was carried out. A parallel investigation for $D > 3$ is trivial: From (4.23) and (4.26) follows that $A = C = 0$ requires $M = X = 0$, and from the consistency condition (4.23) then follows $r_h = 0$, i.e. there is no angular deficit.

A consistent Newtonian limit must respect the consistency condition (4.23). Also in this case this is obtained by letting pressures and the gravitational constant go to zero at the same rate, with the mass kept fixed. In this limit (4.22) reduces to:

$$\frac{dp_r}{dr} + \frac{D - 2}{r}(p_r - p_\perp) \simeq \frac{G_{D+1}M\rho}{r^{D-2}}$$ (4.30)

which is recognized as the condition of hydrostatic equilibrium. The Newtonian potential is then obtained from (4.25):

$$V \simeq \frac{1}{2} A = \frac{\Omega_{D-1}G_{D+1}M \left(\frac{r_h}{r}\right)^{D-3}}{(D-3)\Omega_{D-2}}.$$ (4.31)

V. BRANS-DICKE GRavity

Brans-Dicke gravity in $D$ dimensions is obtained from Einstein gravity in $D + 1$ dimensions through the following steps:

1. the $D$-coordinate is eliminated by dimensional reduction,

2. the combination

$$\frac{1}{G_{D+1}} \dot{\Phi},$$

is identified with a scalar field,

3. a conformal transformation is carried out on the $D$-dimensional metric.

This procedure leads to the following action:

$$S = \int d^Dx \sqrt{|g|} \left[\phi R + \omega \frac{\nabla^\mu \phi \nabla_\mu \phi}{\phi} + \frac{2(D-2)}{D-3} G_{D-2} L \right].$$ (5.1)

where the Lagrangian density of $D$-dimensional matter $L$ has been added to the action. $\omega$ is a free parameter and $\phi$ a scalar field [20].

Because of this connection between the two theories we expect an analogy between the cylindrically symmetric solutions of Einstein gravity in $D + 1$ dimensions discussed in the previous sections and spherically symmetric solutions of Brans-Dicke gravity in $D$ dimensions. Especially, results on $D = 3$ Brans-Dicke gravity can be obtained in this way. To use this connection, we take again the line element in the isotropic form:
\[ ds^2 = e^A dt^2 - e^B \delta_{ij} dx^i dx^j \]  

(5.2)

with \( A \) and \( B \) functions of the radial coordinate \( r \) only.

For \( D = 3 \) the matter term is singular. For ordinary Einstein gravity, this difficulty has been discussed by Cornish and Frankel \[3\]. They suggest that it should be circumvented by use of a redefined gravitational coupling constant \( G_D = G_3/(D-3) \) sacrificing the Newtonian limit. In Brans-Dicke gravity this corresponds to using a redefined scalar field \( \Phi = (D-3)\phi \) with the difference that the Newtonian limit will be kept. In the following sections, which means that it has components:

\[ T_{00} = \rho e^A \]
\[ T_{ij} = (p_r + \frac{1}{r^2} + p_\perp \delta_{ij}) e^B \]  

(5.3)

whence \( T = \rho - p_r - (D - 2)p_\perp \). All quantities are functions of the radial coordinate \( r \) only.

The resulting field equations are, as expected according to the procedure sketched above for obtaining Brans-Dicke gravity, very similar to those obtained for Einstein gravity in secs. II and IV, with some minor modifications. For this reason we omit many details, giving only the important results.

The energy-momentum tensor for a spherical matter distribution has the same structure as those of the previous sections, which means that it has components:

\[ T_{\alpha\beta} = \left( \begin{array}{cc}
\rho & -e^{-B}(\phi\Xi) \\
-p_r & -e^{-B}(\phi\Xi) + e^{-2B}\phi \end{array} \right) \]

(5.4)

with

\[ \Xi = e^{\frac{A+2(D-3)B}{2}}. \]  

(5.8)

To proceed from here, one has to distinguish between the cases \( D > 3 \) and \( D = 3 \). First we consider \( D > 3 \). To obtain formal solutions of the differential equations we define the quantities:

\[ \mathcal{M}(r) = \Omega_{D-2} \int_0^r (r')^{D-2} e^B \Xi \times \]

\[ (\rho - \frac{T(\omega + 1)}{(D - 2)\omega + D - 1}) dr', \]  

(5.9)

\[ \mathcal{X}(r) = -\frac{\Omega_{D-2}}{(D - 2)\omega + D - 1} \int_0^r (r')^{D-5} e^B \Xi dr', \]  

(5.10)

\[ \mathcal{U}(r) = \frac{(D - 2)\Omega_{D-2}}{2(D - 3)^2} \int_r^{r_b} r' e^B \Xi (p_r + p_\perp) dr'. \]  

(5.11)

The solutions are then obtained from:

\[ \phi \Xi(r) = \phi_0 - r^{2(3-D)} \mathcal{U}(r) + \mathcal{V}(r) \]  

(5.13)

with \( \phi_0 \) an integration constant, as well as

\[ A' = \frac{2(D - 2)}{D - 3} \frac{\mathcal{M}}{r^{D-2}\phi \Xi}, \]  

(5.14)

\[ \phi' = \frac{D - 2}{D - 3} \frac{\mathcal{X}}{r^{D-2}}, \]  

(5.15)

Corresponding to (5.21) one gets from (5.7):

\[ 2(\phi_0 + 2(3-D) \mathcal{U}(\phi_0 + \mathcal{V}) + \frac{(D - 2)\mathcal{U}}{D - 3} (\mathcal{M} + \mathcal{X})^2 + \mathcal{M} \mathcal{X} - \frac{\omega}{2} \mathcal{X}^2) = \frac{(D - 2)\Omega_{D-2}}{D - 3} r^{2D-4} e^B \phi \Xi^2 p_r \]  

(5.16)

from which by differentiation a generalized Oppenheimer-Volkov equation is obtained:

\[ \frac{\phi \Xi dp_r}{dr} + \frac{D - 2}{r} \phi_0 + r^{2(3-D)} \mathcal{U}(p_r + \mathcal{V}) = - \frac{D - 2}{D - 3} r^{2-D} (\mathcal{M} - \mathcal{M} + \mathcal{X}) p_r \]  

(5.17)

For \( r > r_b \) the solution is given by:
\( \phi \Xi = \phi_0 (1 - (r_h/r)^2(D-3)) \), \( (5.18) \)

\[ A = \frac{(D-2)M}{2\phi_0(D-3)^2r_h^{D-3}} \ln \left( \frac{1 - (r_h/r)^{D-3}}{1 + (r_h/r)^{D-3}} \right), \quad (5.19) \]

\[ e^{2\Phi} = \frac{(D-2)\mathcal{X}}{2\phi_0(D-3)^2r_h^{D-3}} \ln \left( \frac{1 - (r_h/r)^{D-3}}{1 + (r_h/r)^{D-3}} \right), \quad (5.20) \]

where the horizon radius \( r_h = \left( \frac{M}{\phi_0} \right)^{1/(D-3)} \) is always real for \( \omega > -\frac{D-1}{D-2} \) as a consequence of \( (5.16) \).

The gravitational constant is here proportional to \( 1/\phi_0 \), and it is seen from \( (5.16) \) that a consistent Newtonian limit is obtained by taking the pressures to zero at the same rate as \( 1/\phi_0 \), keeping the mass fixed. The Newtonian potential is then determined from \( (5.19) \):

\[ V \approx \frac{1}{2} A \left( 1 - \frac{(D-2)M}{2\phi_0(D-3)^2} \right) \ln \frac{r}{r_h} = \frac{1}{2} A \left( 1 - \frac{(D-2)M}{2\phi_0(D-3)^2} \right) \ln \frac{r}{r_h}. \quad (5.21) \]

In the Newtonian limit \( (5.13) \) reduces to the condition of hydrostatic equilibrium.

At \( D = 3 \) we introduce \( \Phi = (D-3)\phi \) and obtain in analogy to \( (2.4) \), \( (2.13) \), \( (2.14) \) and \( (2.15) \) in sec. II the following four equations:

\[ (B' + \frac{2}{r}(A' + 2\Phi'/\Phi) + A' + 2\Phi'/\Phi - 2\omega(\Phi'/\Phi)^2 = \frac{16\pi e^B p_r}{\Phi}. \quad (5.22) \]

\[ rA'\Phi e^{\Phi} = 2M, \quad (5.23) \]

\[ (rB' + 2\Phi e^{\Phi} - 2\Phi(0) = 2W, \quad (5.24) \]

\[ r\Phi' e^{\Phi} = \mathcal{X}, \quad (5.25) \]

with

\[ \mathcal{M}(r) = 2\pi \int_0^r r'dr' e^{\frac{\mathcal{X} + 2\Phi}{\Phi}} (\rho - \mathcal{T}\frac{\omega + 1}{\omega + 2}), \quad (5.26) \]

\[ \mathcal{W}(r) = -2\pi \int_0^r r'dr' e^{\frac{\mathcal{X} + 2\Phi}{\Phi}} (p_{\perp} + \mathcal{T}\frac{\omega + 1}{\omega + 2}), \quad (5.27) \]

and

\[ \mathcal{X}(r) = -2\pi \int_0^r r'dr' e^{\frac{\mathcal{X} + 2\Phi}{\Phi}} \mathcal{T}/(\omega + 2). \quad (5.28) \]

Inserting \( (5.23) \), \( (5.24) \) and \( (5.25) \) into \( (5.22) \) it becomes

\[ (\mathcal{W} + \frac{1}{2}\Phi(0))(\mathcal{M} + \mathcal{X}) + \mathcal{M}\mathcal{X} - \frac{1}{2}\omega\lambda^2 = \pi r^2 e^{A+B}p_r, \quad (5.29) \]

that is equivalent to the generalized Oppenheimer-Volkoff equation:

\[ e^{2\Phi} \frac{d\rho_r}{dr} + \Phi(0) \frac{d\mathcal{X}_r}{dr} - \frac{2}{r}(\mathcal{M}\rho + (\mathcal{M} + \mathcal{W})p_r - \mathcal{W}p_{\perp}) \]

\[ (5.30) \]

Comparing this result with \( (5.17) \), we see that the substitution \( \mathcal{W} \to -\frac{2}{r}(\mathcal{M} + \mathcal{X}) \) gives the correct right-hand side of \( (5.17) \). At \( r \geq r_h \) (outside the matter distribution) eq. \( (5.29) \) gives rise to a consistency condition.

The solutions for \( A, \mathcal{B}, \mathcal{X} \) and \( \Phi \) are here, as seen from \( (5.23), (5.24) \) and \( (5.25) \):

\[ A(r) = \frac{2M}{\mathcal{M} + \mathcal{X}} \ln \left( 1 + \frac{\mathcal{M} + \mathcal{X}}{\Phi_0} \ln \frac{r}{r_0} \right), \quad (5.31) \]

\[ B(r) = \frac{2\mathcal{W} + \Phi(0)}{\mathcal{M} + \mathcal{X}} \ln \left( 1 + \frac{\mathcal{M} + \mathcal{X}}{\Phi_0} \ln \frac{r}{r_0} \right) - 2\ln \frac{r}{r_0}, \quad (5.32) \]

\[ \ln \frac{\Phi(r)}{\Phi_0} = \frac{X}{\mathcal{M} + \mathcal{X}} \ln \left( 1 + \frac{\mathcal{M} + \mathcal{X}}{\Phi_0} \ln \frac{r}{r_0} \right), \quad (5.33) \]

where \( \Phi_0 \) is a constant of integration that will play the role of inverse gravitational constant (up to a constant of proportionality).

A few special cases are worth considering. If we take \( \mathcal{M}_0 = \mathcal{X} = 0, \mathcal{W} \) can be arbitrary, and we obtain a locally flat conical space characterized by an angular deficit:

\[ \delta \phi = -2\pi \frac{\mathcal{W} + \Phi(0) - \Phi_0}{\Phi_0}. \quad (5.34) \]

For \( \mathcal{M} \neq 0, \mathcal{X} = 0 \) the solution is the Kawai solution \cite{22,23}.

A consistent Newtonian limit is obtained according to \( (5.29) \) by letting pressures get small at the same rate as \( 1/\phi_0 \), keeping the mass fixed. Requiring a flat-space metric in this limit, we see from \( (5.32) \) that also \( 1/\Phi_0 \) must get small in the limit such that:

\[ \frac{\Phi(0)}{\Phi_0} \to 1. \quad (5.35) \]

Using this observation, we get from \( (5.33) \) the Newtonian gravitational potential \( V \):

\[ V \approx \frac{1}{2} A \simeq \frac{M}{2\Phi_0} \ln \frac{r}{r_0}. \quad (5.36) \]

Eq. \( (5.30) \) reduces in this limit to the equation of hydrostatic equilibrium.

**VI. CONCLUSION**

In this work we have studied in detail the relation between the gravitational field and the matter distribution...
of a cylindrically-symmetric source in Einstein as well as a spherically-symmetric source in Brans-Dicke gravity.

We have taken a model-independent approach and obtained relations between the two free parameters in the general cylindrically-symmetric solution of Einstein gravity and the integrals of components of the energy-momentum tensor. The two parameters that characterize the source may be taken as the Tolman mass and the corresponding quantity associated with the axial direction.

We further report on consistency conditions, relating the masses and pressures in cylindrically symmetric space-time in Einstein gravity and interpret them as a manifestation of the hydrostatic equilibrium of the source. This was achieved through use of a coordinate system isotropic in the transverse coordinates. In this coordinate system the line element is symmetric by interchange of time and the direction of the axial coordinate.

We generalized the discussion from four dimensions (\(D = 3\) in our notation) to higher dimensions, and analyzed the case of \((D + 1)\)-dimensional Einstein gravity. We furthermore found analogous consistency conditions in spherically symmetric space-time in \(D\)-dimensional Brans-Dicke gravity. The consistency conditions were studied in the Newtonian limit and it turned out that the pressures and the gravitational constant (appropriately defined for each of the theories) go to zero at the same rate with the masses kept fixed. A remarkable detail is the fact that the role of inverse gravitational constant is played by an integration constant.

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