The Yagita Invariant of Symplectic Groups of Large Rank

Cornelia M. Busch and Ian J. Leary

Abstract. Fix a prime $p$, and let $\mathcal{O}$ denote a subring of $\mathbb{C}$ that is either integrally closed or contains a primitive $p$th root of 1. We determine the Yagita invariant at the prime $p$ for the symplectic group $\text{Sp}(2n, \mathcal{O})$ for all $n \geq p - 1$.

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1. Introduction

The Yagita invariant $p^\circ(G)$ of a discrete group $G$ is an invariant that generalizes the period of the $p$-local Tate–Farrell cohomology of $G$, in the following sense: it is a numerical invariant defined for any $G$ that is equal to the period when the $p$-local cohomology of $G$ is periodic. Yagita considered finite groups [6], and Thomas extended the definition to groups of finite vcd [5]. In [3] the definition was extended to arbitrary groups and $p^\circ(G)$ was computed for $G = \text{GL}(n, \mathcal{O})$ for $\mathcal{O}$ any integrally closed subring of $\mathbb{C}$ and for sufficiently large $n$ (depending on $\mathcal{O}$).

In [2], one of us computed the Yagita invariant for $\text{Sp}(2(p+1), \mathbb{Z})$. Computations from [3] were used to provide an upper bound, and computations with finite subgroups and with mapping class groups were used to provide a lower bound [4]. The action of the mapping class group of a surface upon the first homology of the surface gives a natural symplectic representation of the mapping class group of a genus $p+1$ surface inside $\text{Sp}(2(p+1), \mathbb{Z})$. In the current paper, we compute $p^\circ(\text{Sp}(2n, \mathcal{O}))$ for each $n \geq p - 1$ for each $\mathcal{O}$ for which $p^\circ(\text{GL}(n, \mathcal{O}))$ was computed in [3]. By using a greater range of finite subgroups, we avoid having to consider mapping class groups.

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Throughout the paper, we fix a prime $p$. Before stating our main result, we recall the definitions of the symplectic group $\text{Sp}(2n, R)$ over a ring $R$, and of the Yagita invariant $p^\circ (G)$, which depends on the prime $p$ as well as on the group $G$. The group $\text{Sp}(2n, R)$ is the collection of invertible $2n \times 2n$ matrices $M$ over $R$ such that

$$M^T J M = J, \text{ where } J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

Here, $M^T$ denotes the transpose of the matrix $M$, and as usual $I_n$ denotes the $n \times n$ identity matrix. Equivalently, $M \in \text{Sp}(2n, R)$ if $M$ defines an isometry of the antisymmetric bilinear form on $R^{2n}$ defined by $\langle x, y \rangle := x^T J y$. If $C$ is cyclic of order $p$, then the group cohomology ring $H^*(C; \mathbb{Z})$ has the form

$$H^*(C; \mathbb{Z}) \cong \mathbb{Z}[x] / (px), \quad x \in H^2(C; \mathbb{Z}).$$

If $C$ is a cyclic subgroup of $G$ of order $p$, define $n(C)$ a positive integer or infinity to be the supremum of the integers $n$ such that the image of $H^*(G; \mathbb{Z}) \to H^*(C; \mathbb{Z})$ is contained in the subring $\mathbb{Z}[x^n]$. Now, define

$$p^\circ (G) := \text{lcm}\{2n(C) : C \leq G, \ |C| = p\}.$$ 

It is easy to see that if $H \leq G$, then $p^\circ (H)$ divides $p^\circ (G)$ [3, Prop. 1].

2. Results

In the following theorem statement and throughout the paper, we let $\zeta_p$ be a primitive $p$th root of 1 in $\mathbb{C}$ and we let $\mathcal{O}$ denote a subring of $\mathbb{C}$ with $F \subseteq \mathbb{C}$ as its field of fractions. We assume that either $\zeta_p \in \mathcal{O}$ or that $\mathcal{O}$ is integrally closed in $\mathbb{C}$. We define $l := |F[\zeta_p] : F|$, the degree of $F[\zeta_p]$ as an extension of $F$. For $t \in \mathbb{R}$ with $t \geq 1$, we define $\psi(t)$ to be the largest integer power of $p$ less than or equal to $t$.

**Theorem 1.** With notation as above, for each $n \geq p - 1$, the Yagita invariant $p^\circ (\text{Sp}(2n, \mathcal{O}))$ is equal to $2(p-1)\psi(2n/l)$ for $l$ even and equal to $2(p-1)\psi(n/l)$ for $l$ odd.

By the main result of [3], the above is equivalent to the statement that $p^\circ (\text{Sp}(2n, \mathcal{O})) = p^\circ (\text{GL}(2n, \mathcal{O}))$ when $l$ is even and $p^\circ (\text{Sp}(2n, \mathcal{O})) = p^\circ (\text{GL}(n, \mathcal{O}))$ when $l$ is odd. By definition $\text{Sp}(2n, \mathcal{O})$ is a subgroup of $\text{GL}(2n, \mathcal{O})$ and there is an inclusion $\text{GL}(n, \mathcal{O}) \to \text{Sp}(2n, \mathcal{O})$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix},$$

and so for any $n$, $p^\circ (\text{GL}(n, \mathcal{O}))$ divides $p^\circ (\text{Sp}(2n, \mathcal{O}))$, which in turn divides $p^\circ (\text{GL}(2n, \mathcal{O}))$.

Before we start, we recall two standard facts concerning symplectic matrices that will be used in the proof of Corollary 3: if $M$ is in the symplectic group, then $\det(M) = 1$ and $M$ is conjugate to the inverse of its transpose $(M^{-1})^T = (M^T)^{-1}$. We shall use the notation $\mathbb{F}_p^\times$ to denote the multiplicative group of units in the field $\mathbb{F}_p$. 
Proposition 2. Let $f(X)$ be a polynomial over the field $\mathbb{F}_p$ and suppose that 0 is not a root of $f$, but that $f$ factors as a product of linear polynomials over $\mathbb{F}_p$. If there is a polynomial $g$ and an integer $n$ so that $f(X) = g(X^n)$, then $n$ has the form $n = mp^q$ for some $m$ dividing $p - 1$ and some integer $q \geq 0$. If $p$ is odd and for each $i \in \mathbb{F}_p^\times$, the multiplicity of $i$ as a root of $f$ is equal to that of $-i$, then $m$ is even.

Proof. The only part of this that is not contained in [3, Prop. 6] is the final statement. Since $(1 - iX)(1 + iX) = 1 - i^2X^2$ is a polynomial in $X^2$, the final statement follows. For the benefit of the reader, we sketch the rest of the proof. If $n = mp^q$ where $p$ does not divide $m$, then $g(X^n) = g(X^m)p^q$, so we may assume that $q = 0$. If $g(Y) = 0$ has roots $y_i$, then the roots of $g(X^m) = 0$ are the roots of $y_i - X^m = 0$. Since $p$ does not divide $m$, these polynomials have no repeated roots; since their roots are assumed to lie in $\mathbb{F}_p$, it is now easy to show that $m$ divides $p - 1$. □

Corollary 3. With notation as in Theorem 1, let $G$ be a subgroup of $\text{Sp}(2n, F)$. Then the Yagita invariant $p^\sigma(G)$ divides the number given for $p^\sigma(\text{Sp}(2n, O))$ in the statement of Theorem 1.

Proof. As in [3, Cor. 7], for each $C \leq G$ of order $p$, we use the total Chern class to give an upper bound for the number $n(C)$ occurring in the definition of $p^\sigma(G)$. If $C$ is cyclic of order $p$, then $C$ has $p$ distinct irreducible complex representations, each one dimensional. If we write $H^*(C; \mathbb{Z}) = \mathbb{Z}[x]/(px)$, then the total Chern classes of these representations are $1 + ix$ for each $i \in \mathbb{F}_p$, where $i = 0$ corresponds to the trivial representation. The total Chern class of a direct sum of representations is the product of the total Chern classes, and so when viewed as a polynomial in $\mathbb{F}_p[x] = H^*(C; \mathbb{Z}) \otimes \mathbb{F}_p$, the total Chern class of any faithful representation $\rho: C \to \text{GL}(2n, \mathbb{C})$ is a non-constant polynomial of degree at most $2n$ all of whose roots lie in $\mathbb{F}_p^\times$. Now, let $F$ be a subfield of $\mathbb{C}$ with $l = [F[G]: F]$ as in the statement. The group $C$ has $(p - 1)/l$ non-trivial irreducible representations over $F$, each of dimension $l$, and the total Chern classes of these representations have the form $1 - ix^l$, where $i$ ranges over the $(p - 1)/l$ distinct $l$th roots of unity in $\mathbb{F}_p$. In particular, the total Chern class of any representation $\rho: C \to \text{GL}(2n, F) \leq \text{GL}(2n, \mathbb{C})$ is a polynomial in $x^l$ whose $x$-degree is at most $2n$. If $\rho$ has image contained in $\text{Sp}(2n, \mathbb{C})$, then it factors as $\rho = \iota \circ \tilde{\rho}$ with $\tilde{\rho}: C \to \text{Sp}(2n, \mathbb{C})$ and $\iota$ is the inclusion of $\text{Sp}(2n, \mathbb{C})$ in $\text{GL}(2n, \mathbb{C})$. In this case, the matrix representing a generator for $C$ is conjugate to the transpose of its own inverse; in particular, it follows that the multiplicities of the irreducible complex representations of $C$ with total Chern classes $1 + ix$ and $1 - ix$ must be equal for each $i$. Hence in this case, if $p$ is odd, the total Chern class of the representation $\rho = \iota \circ \tilde{\rho}$ is a polynomial in $x^2$. If $p = 2$ (which implies that $l = 1$), then the total Chern class of any representation $\rho: C \to \text{GL}(2n, \mathbb{C})$ has the form $(1 + x)^i$, where $i$ is equal to the number of non-trivial irreducible summands. Since $\text{Sp}(2n, \mathbb{C}) \leq \text{SL}(2n, \mathbb{C})$, it follows that for symplectic representations $i$ must be even, and so for $p = 2$ the total Chern class is a polynomial in $x^2$.

In summary, let $\tilde{\rho}$ be a faithful representation of $C$ in $\text{Sp}(2n, F)$. In the case when $l$ is odd, then the total Chern class of $\tilde{\rho}$ is a non-constant
polynomial \( f(y) = f(x) \) in \( y = x^{2l} \) such that \( f(x) \) has degree at most \( 2n \), \( \tilde{f}(y) \) has degree at most \( n/l \), and all roots of \( f, \tilde{f} \) lie in \( \mathbb{F}_p^\times \). In the case when \( l \) is even, the total Chern class of \( \rho \) is a non-constant polynomial \( \tilde{f}(y) = f(x) \) in \( y = x^l \) such that \( f(x) \) has degree at most \( 2n \), \( \tilde{f}(y) \) has degree at most \( 2n/l \), and all roots of both lie in \( \mathbb{F}_p^\times \). By Proposition 2, it follows that each \( n(C) \) is a factor of the number given for \( p^\circ(\text{Sp}(2n, \mathcal{O})) \), and hence the claim. \( \square \)

**Lemma 4.** Let \( H \leq G \) with \( |G : H| = m \), and let \( \rho \) be a symplectic representation of \( H \) on \( V = \mathcal{O}^{2n} \). The induced representation \( \text{Ind}^G_H(\rho) \) is a symplectic representation of \( G \) on \( W := \mathcal{O}G \otimes_{\mathcal{O}H} V \cong \mathcal{O}^{2mn} \).

**Proof.** Let \( e_1, \ldots, e_n, f_1, \ldots, f_m \) be the standard basis for \( V = \mathcal{O}^{2n} \), so that the bilinear form \( \langle v, w \rangle := v^T J w \) on \( V \) is given by

\[
\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle, \quad \langle e_i, f_j \rangle = -\langle f_i, e_j \rangle = \delta_{ij}.
\]

The representation \( \rho \) is symplectic if and only if each \( \rho(h) \) preserves this bilinear form.

Let \( t_1, \ldots, t_m \) be a left transversal to \( H \) in \( G \), so that \( \mathcal{O}G = \oplus_{i=1}^m t_i \mathcal{O}H \) as right \( \mathcal{O}H \)-modules. Define a bilinear form \( \langle \ , \ \rangle_W \) on \( W \) by

\[
\left\langle \sum_{i=1}^m t_i \otimes v^i, \sum_{i=1}^m t_i \otimes w^i \right\rangle_W := \sum_{i=1}^m \langle v^i, w^i \rangle_W.
\]

To see that this bilinear form is preserved by the \( \mathcal{O}G \)-action on \( W \), fix \( g \in G \) and define a permutation \( \pi \) of \( \{1, \ldots, m\} \) and elements \( h_1, \ldots, h_m \in H \) by the equations \( gt_i = t_{\pi(i)}h_i \). Now for each \( i, j \) with \( 1 \leq i, j \leq m \),

\[
\langle \text{Ind}(\rho(g))t_i \otimes v, \text{Ind}(\rho(g))t_j \otimes w \rangle_W = \langle t_{\pi(i)} \otimes \rho(h_i)v, t_{\pi(j)} \otimes \rho(h_j)w \rangle_W = \delta_{\pi(i)\pi(j)} \langle \rho(h_i)v, \rho(h_j)w \rangle = \delta_{ij} \langle \rho(h_i)v, \rho(h_i)w \rangle = \delta_{ij} \langle v, w \rangle = \langle t_i \otimes v, t_j \otimes w \rangle_W.
\]

To see that \( \langle \ , \ \rangle_W \) is symplectic, define basis elements \( E_1, \ldots, E_{mn}, F_1, \ldots, F_{mn} \) for \( W \) by the equations

\[
E_{n(i-1)+j} := t_i \otimes e_j, \quad \text{and} \quad F_{n(i-1)+j} := t_i \otimes f_j, \quad \text{for} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.
\]

It is easily checked that for \( 1 \leq i, j \leq mn \)

\[
\langle E_i, E_j \rangle_W = 0 = \langle F_i, F_j \rangle_W, \quad \langle E_i, F_j \rangle_W = -\langle F_i, E_j \rangle_W = \delta_{ij},
\]

and so with respect to this basis for \( W \), the bilinear form \( \langle \ , \ \rangle_W \) is the standard symplectic form. \( \square \)

**Proposition 5.** With notation as in Theorem 1, the Yagita invariant \( p^\circ(\text{Sp}(2n, \mathcal{O})) \) is divisible by the number given in the statement of Theorem 1.

**Proof.** To give lower bounds for \( p^\circ(\text{Sp}(2n, \mathcal{O})) \), we use finite subgroups. Firstly, consider the semidirect product \( H = C_{p} \rtimes C_{p-1} \), where \( C_{p-1} \) acts faithfully on \( C_{p} \); equivalently, this is the group of affine transformations of the line over \( \mathbb{F}_p \). It is well known that the image of \( H^*(G; \mathbb{Z}) \) inside \( H^*(C_{p}; \mathbb{Z}) \cong \mathbb{Z}[x]/(px) \)
is the subring generated by $x^{p-1}$. It follows that $2(p - 1)$ divides $p^\circ(G)$ for any $G$ containing $H$ as a subgroup. The group $H$ has a faithful permutation action on $p$ points, and hence a faithful representation in $GL(p - 1, \mathbb{Z})$, where $\mathbb{Z}^{p-1}$ is identified with the kernel of the $H$-equivariant map $\mathbb{Z}\{1, \ldots, p\} \to \mathbb{Z}$. Since $GL(p - 1, \mathbb{Z})$ embeds in $Sp(2(p - 1), \mathbb{Z})$, we deduce that $H$ embeds in $Sp(2n, \mathcal{O})$ for each $\mathcal{O}$ and for each $n \geq p - 1$.

To give a lower bound for the $p$-part of $p^\circ(\text{Sp}(2n, \mathcal{O}))$, we use the extraspecial $p$-groups. For $p$ odd, let $E(p, 1)$ be the non-abelian $p$-group of order $p^3$ and exponent $p$, and let $E(2, 1)$ be the dihedral group of order 8. (Equivalently in each case $E(p, 1)$ is the Sylow $p$-subgroup of $GL(3, \mathbb{F}_p)$.) For $m \geq 2$, let $E(p, m)$ denote the central product of $m$ copies of $E(p, 1)$, so that $E(p, m)$ is one of the two extraspecial groups of order $p^{2m+1}$. Yagita showed that $p^\circ(E(p, m)) = 2p^m$ for each $m$ and $p \mid [6]$. The centre and commutator subgroup of $E(p, m)$ are equal and have order $p$, and the abelianization of $E(p, m)$ is isomorphic to $C_p^{2m}$. The irreducible complex representations of $E(p, m)$ are well understood: there are $p^2m$ distinct one-dimensional irreducibles, each of which restricts to the centre as the trivial representation, and there are $p - 1$ faithful representations of dimension $p^m$, each of which restricts to the centre as the sum of $p^m$ copies of a single (non-trivial) irreducible representation of $C_p$. The group $G = E(p, m)$ contains a subgroup $H$ isomorphic to $C_p^{m+1}$, and each of its faithful $p^m$-dimensional representations can be obtained by inducing up a one-dimensional representation $H \to C_p \to GL(1, \mathbb{C})$.

According to Bürgisser, $C_p$ embeds in $Sp(2l, \mathcal{O})$ (resp. in $Sp(l, \mathcal{O})$ when $l$ is even) provided that $\mathcal{O}$ is integrally closed in $\mathbb{C}$ [1]. Here as usual, $l := |F[\zeta_p], F|$ and $F$ is the field of fractions of $\mathcal{O}$. If instead $\zeta_p \in \mathcal{O}$, then $l = 1$ and clearly $C_p$ embeds in $GL(1, \mathcal{O})$ and hence also in $Sp(2, \mathcal{O}) = Sp(2l, \mathcal{O})$. Taking this embedding of $C_p$ and composing it with any homomorphism $H \to C_p$, we get a symplectic representation $\rho$ of $H$ on $\mathcal{O}^{2l}$ for any $l$ (resp. on $\mathcal{O}^l$ for $l$ even). For a suitable homomorphism, we know that $\text{Ind}_H^G(\rho)$ is a faithful representation of $G$ on $\mathcal{O}^{2lp^m}$ (resp. on $\mathcal{O}^{lp^m}$ for $l$ even) and by Lemma 4 we see that $\text{Ind}_H^G(\rho)$ is symplectic. Hence, we see that $E(m, p)$ embeds as a subgroup of $Sp(2lp^m, \mathcal{O})$ for any $l$ and as a subgroup of $Sp(lp^m, \mathcal{O})$ in the case when $l$ is even. Since $p^\circ(E(m, p)) = 2p^m$, this shows that $2p^m$ divides $p^\circ(\text{Sp}(2lp^m, \mathcal{O}))$ always and that $2p^m$ divides $p^\circ(\text{Sp}(lp^m, \mathcal{O}))$ in the case when $l$ is even.

Corollary 3 and Proposition 5 together complete the proof of Theorem 1.

We finish by pointing out that we have not computed $p^\circ(\text{Sp}(2n, \mathcal{O}))$ for general $\mathcal{O}$ when $n < p - 1$; to do this one would have to know which meta-cyclic groups $C_p \rtimes C_k$ with $k$ coprime to $p$ admit low-dimensional symplectic representations.

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