Abstract

Bödeker has recently shown that the high temperature sphaleron rate, which measures baryon number violation in the hot standard model, receives logarithmic corrections to its leading parametric behavior; \( \Gamma = \kappa' \log \left( \frac{m_D}{g^2 T} \right) + O(1) \left( \frac{g^2 T^2}{m_D^2} \right) \alpha_W^5 T^4 \). After discussing the physical origin of these corrections, I compute the leading log coefficient numerically; \( \kappa' = 10.8 \pm 0.7 \). The log is fairly small relative to the \( O(1) \) “correction;” so nonlogarithmic contributions dominate at realistic values of the coupling.

1 Introduction

It has been known for some time now that baryon number is not a conserved quantity in the minimal standard model \[1\]. It is violated nonperturbatively because of the anomaly, the chiral couplings of fermions to SU(2) weak, and the topologically nontrivial vacuum structure of SU(2). However, as is characteristic of a nonperturbative process in a weakly coupled theory, the rate of violation is so tiny that it is completely irrelevant phenomenologically. Certainly, if baryon number is also violated due to high dimension operators descended from some GUT, baryon number violating decay rates due to the GUT mechanism will greatly exceed the electroweak rate, of order \( m_W^3 \exp (-4\pi/\alpha_W) < 10^{-170} \text{ GeV} \) even before accounting for additional suppression from powers of small CKM matrix elements and a high power of \( (m_{\text{proton}}/m_W) \).

However, as first realized by Kuzmin, Rubakov, and Shaposhnikov in 1985, the efficiency of standard model baryon number violation is very much higher at finite temperature \[2\]. A
perturbative estimate [3] based on a saddlepoint expansion about Klinkhamer and Manton’s sphaleron [4] indicated that the rate is more than enough to erase any relic abundance of baryon number left over from the GUT scale, unless the combination \( B - L \), baryon minus lepton number, is nonzero; this quantity is preserved by electroweak physics. It also may be that the baryon number abundance in the universe was produced by electroweak physics, which motivates the more careful study of electroweak baryon number violation at finite temperature.

Recently our understanding of thermal baryon number violation has improved, though it is not complete. We understand that the efficiency of baryon number violation can in most relevant circumstances be related by a fluctuation dissipation argument to the diffusion constant for Chern-Simons number [5, 6, 7]. This diffusion constant, called the sphaleron rate, is defined as

\[
\Gamma \equiv \lim_{V \to \infty} \lim_{t \to \infty} \frac{\langle (N_{CS}(t) - N_{CS}(0))^2 \rangle}{V t},
\]

where the expectation values refer to a trace over the equilibrium thermal density matrix. The quantity \( \Gamma \) is the topological susceptibility of the electroweak sector at finite temperature, in Minkowski time.

We know that, when the electroweak phase transition is first order, the value of \( \Gamma \) jumps discontinuously between the phases; it is quite small in the broken phase and much larger in the symmetric phase. An old power counting argument says that the symmetric phase rate should be of order \( \alpha^4 W T^4 \) with an order unity coefficient. This argument relied, correctly, on the natural nonperturbative length scale in the hot plasma being \( 1/(\alpha W T) \). One then assumes that the natural time scale is the same; on dimensional grounds the spacetime rate of \( N_{CS} \) diffusion must then be of order \( (\alpha W T)^3 \alpha W T \). However, Arnold, Son, and Yaffe have shown that at leading parametric order, the natural time scale is **not** \( 1/(\alpha W T) \), but \( 1/\alpha^2 W T \) [8], up to possible logarithmic corrections, which the authors did not consider. More recently, Bödeker has demonstrated that logarithmic corrections to their argument do occur [9].

This says nothing about the numerical value of the sphaleron rate; it could be parametrically \( \alpha^5 W T^4 \) but numerically irrelevantly small. Ambjørn and Krasnitz presented numerical evidence that it was large, by considering classical, thermal Yang-Mills theory on the lattice [10]. Their definition of \( N_{CS} \) was not topological and could therefore suffer from potentially severe lattice artifacts, but Turok and I studied the same system with a topological definition of \( N_{CS} \) and verified that \( \Gamma \) is substantial [11]. The sphaleron rate we found, expressed in physical units, was lattice spacing dependent, which turns out to be a prediction of the arguments of Arnold, Son, and Yaffe; the extra power of \( \alpha W \) arises from the interaction between the infrared fields and ultraviolet excitations, as I will discuss more below, and on the lattice it becomes one power of the lattice spacing \( a \).

In fact the lattice spacing dependence of \( \Gamma \) for pure classical lattice Yang-Mills theory only fits \( \Gamma \propto a \) if there are substantial corrections to scaling, which has led some to call into question whether the Arnold, Son, and Yaffe’s analysis is correct. In this paper I will assume that it is, as seems justified on theoretical grounds and numerical evidence from classical Yang-Mills theory “enhanced” with added degrees of freedom which reproduce the hard thermal loop effects [13, 14]. I will return to the large corrections to scaling in the classical lattice theory in subsection 3.3.
Because the interactions between infrared and ultraviolet excitations are important to setting the sphaleron rate, it is somewhat difficult to actually extract $\Gamma$ at leading order in $\alpha_W$ for the continuum quantum theory, at the physical value of $\alpha_W$ or even in the parametric small $\alpha_W$ limit. Hu and Müller proposed a technique based on including the UV physics, lost to the lattice regulation, by introducing new degrees of freedom which influence the IR fields in the same way [12]. They implemented and applied the technique jointly with me [13]. As I will discuss later, this technique still suffers from some poorly controlled systematics, which are related to the logarithmic corrections discovered by Bödeker; in fact, beyond the leading log the infrared physics the technique will simulate is not rotationally invariant.

A good first step to answering the remaining questions about the sphaleron rate is to determine it at leading logarithmic order in $\alpha_W$. Bödeker has demonstrated that this can be done within an effective theory which is completely UV well behaved; in fact it is nothing but the Langevin equation for 3-D Yang-Mills theory [9]. The leading log behavior is probably not very useful by itself, for estimating $\Gamma$ at the physical value of $\alpha_W$. Leading log expansions often miss large constant corrections; we know for instance that the $O(g^2T)$ contribution to the Debye mass has a much larger constant contribution than the leading log might suggest [14, 15]. However, it is still useful to know the leading log; for instance, its size is related to the severity of the systematic problems with the method of Hu and Müller, and it might in principle be useful for extrapolating lattice results which correspond to an inappropriate value of $\alpha_w$ back to the correct value.

The purpose of this paper is to determine the coefficient of the leading log behavior of $\Gamma$; namely, to find $\kappa'$ defined through

$$\Gamma = \kappa' \left( \log \frac{m_D}{g^2T} + O(1) \right) g^2T^2 \frac{m_D^5}{\alpha^5_W T^4} + \text{(higher order)}. \quad (2)$$

(My logs are always natural logs.) Neglecting the $O(1)$ means that $\log(1/g)$ is treated as much larger than any order unity constant, a rather extreme interpretation of the perturbative expansion. Probably this expansion is completely unjustified, but as I said there are still important things to be learned from making it. Until Section 3 I will not worry about whether the expansion in $\log(1/g) \gg 1$ is justified; the goal is simply to determine $\kappa'$. I will also work in Yang-Mills theory, which is appropriate at leading order only for temperatures well above the equilibrium temperature. I will mention how to include the Higgs field in Section 5.

A summary of the paper is as follows. Section 2 will discuss in an intuitive, physical, but nontechnical level why the rate has the parametric form I show; where the extra $\alpha$ comes from and particularly why there is a log. The section provides two apparently different arguments, one in terms of conductivities and scattering processes for hard particles and one in terms of hard thermal loops (HTL’s) and Wilson lines; the two are of course equivalent. Some salient details about the Wilson line are in Section 3, which is more technical. This section also shows how the numerical model of Hu, Müller and Moore fails beyond leading log order. Section 4 studies Bödeker’s effective theory for extracting the leading log, numerically. Since the Langevin dynamics are simple but numerically costly, the emphasis is on controlling systematic errors. Many of the details appear either in previous papers or in the appendix. For the reader’s ease I present the answer now: $\kappa' = 10.8 \pm 0.7$. The dominant error here is
statistical; systematic errors are completely under control. In Section I discuss the meaning of the result, and try to estimate what the sphaleron rate is for the realistic values of $m_D^2$ and $g^2$ by using this result to extrapolate the results of Hu, Müller, and Moore to the right value of the log. My estimate is $\Gamma \simeq 20 a_5^5 W T^4$ for $m_D^2 = (11/6) g^2 T^2$ and $g^2 = 0.4$; however there are uncontrolled systematics which may be as large as 30%. I also discuss corrections to the approximation that $g \ll 1$; the largest of these will be of order 10%. The conclusion concludes. There is also a technical appendix which discusses the match between lattice and continuum Langevin time scales; the match between continuum and Langevin time scales is computed at $O(a)$ and the match between lattice Langevin and heat bath time scales is determined by a measurement.

2 The physics behind $\alpha_5^5 \log(m_D/g^2 T)$

2.1 note on the classical approximation

To a very good approximation the behavior of infrared fields in thermal Yang-Mills or Yang-Mills Higgs theory at weak coupling is that of classical fields. The “old” parametric estimate, $\Gamma \propto \alpha_4^4 W T^4$, relies on the fact that the only length scale available in classical Yang-Mills theory is $1/g^2 T$. It is also assumed that the only time scale is the same, in which case any essentially infrared spacetime rate must go as $g^8 T^4$.

This argument relies on a decoupling between the infrared and ultraviolet fields, since the ultraviolet fields (by which I mean fields of wave number $k \sim T$) do not behave classically. It is known that this decoupling is very accurate for thermodynamic variables, except that the $A_0$ component of the gauge field receives a Debye mass. In fact the validity of the decoupling is equivalent to the quality of the dimensional reduction approximation \[16\], which has been discussed extensively \[17, 18\].

However, the decoupling does not extend to dynamics; the generalization of the Debye mass to unequal times are the hard thermal loops, which significantly affect the infrared dynamics. The hard thermal loops are precisely that set of diagrams which are linearly divergent within the classical theory. This linear divergence is cut off at the ultraviolet scale $k \sim \pi T$ where the theory ceases to behave classically; so the size of these effects depends essentially on the way the IR classical theory is regulated. Thus it is only correct to say that the IR fields behave classically if we mean that they behave like the IR fields of a classical theory regulated in some way which correctly reproduces the hard thermal loops. In nature that regulator is quantum mechanics, but we might be able to find some other appropriate regulator in a numerical setting.

2.2 Argument in terms of conductivity and scatterings

Here I will give an argument for the $\alpha_5^5 \log(1/g) T^4$ law based on Lenz’s Law and the conductivity of the plasma. The argument has very recently been made quantitative \[19\], but I will present it at the qualitative, intuitive level.

\[1\] I should mention parenthetically that the classical theory also contains quadratic and cubic divergences in the energy density, but these do not affect the IR dynamics responsible for baryon number violation.
To see how hard thermal loops influence the sphaleron rate, I first make the point that the sphaleron rate is set by the evolution of very soft infrared fields, where by very soft I mean fields with wave number \( k \sim g^2 T \). Parametrically shorter wave lengths do not contribute appreciably because the probability for nonperturbative physics to occur at such scales is exponentially suppressed. According to standard sphaleron type arguments, the contribution from the scale \( k \sim g^2 - \delta T \) is suppressed by of order \( \exp\left(-\frac{k}{g^2 T}\right) = \exp(-g^{-\delta}) \). Even the scale \( k \sim g^2 T \log(1/g) \) gives a contribution suppressed by a power of \( g \).

The second point is that it is physics in the transverse sector which matters, and in particular, diffusion of \( N_{CS} \) requires the evolution of magnetic fields. To see this, first go back to the definition of \( N_{CS} \):

\[
N_{CS}(t_2) - N_{CS}(t_1) = \int_{t_1}^{t_2} dt \int d^3 x \frac{g^2}{8\pi^2} F_i^a B_a^i(x). \tag{3}
\]

Now the \( B \) field is always transverse, meaning that \( D \cdot B = 0 \), by the Bianchi identity; and so only the transverse part of the \( E \) field contributes. The Bianchi identity also states that \( \vec{D} \times \vec{E} = -[D_i, \vec{B}] \). Since the relevant part of the electric field is transverse, it will in general have nonzero covariant curl. For \( E \) to be nonzero and to remain the same sign for long enough to give a nontrivial contribution to \( \int E_i^a B_a^i dt \), there must then be time evolution of infrared magnetic fields.

At this point it is useful to recall how infrared magnetic fields evolve in the abelian theory, on wave lengths longer than the Debye screening length. The answer is familiar plasma physics; the plasma is very conducting, and a conducting medium resists changes in magnetic fields by Lenz’s Law. In the limit of infinite conductivity the magnetic fields are perfectly frozen; for finite conductivity the time scale for their evolution scales with conductivity. A magnetic field of wave number \( k \), with well more than its mean thermal energy density but much less energy density than is contained in the bulk plasma, decays according to

\[
[D_0, A_i] = \frac{dA_i}{dt} = -\frac{1}{\sigma(k, \omega \ll k)} (k^2 \delta_{ij} - k_i k_j) A_j, \tag{4}
\]

in the parametric limit that the decay time is well longer than \( 1/k \), which is satisfied at all length scales parametrically longer than the Debye screening length. The characteristic decay time of a magnetic field in the plasma is then \( \tau = \sigma(k, \omega \ll k)/k^2 \). Of course I have only written the dissipative part of the magnetic field evolution equation; there must also be a noise term which is uniquely specified by the requirement that the thermodynamics of the IR magnetic fields are correct.

Note that the conductivity is wave number dependent. It has a good infrared limit which is achieved for length scales larger than the mean scattering length \( l_{\text{free}} \) of a current carried by hard particles. In the abelian theory a particle’s charge is preserved when it undergoes a scattering, so this length scale is the mean length for large angle scattering, parametrically \( l_{\text{free}} \sim 1/\alpha^2 T \) times logarithmic corrections. If every particle had the same free path and a scattering perfectly randomized its momentum, the conductivity on scales longer than \( l_{\text{free}} \)

\[\text{5}\]

\[\text{2}\] Although I set the speed of light \( c = 1 \), I typically write expressions noncovariantly with a positive space metric, which is convenient in the finite temperature context because the thermal bath establishes a preferred frame.
would be
\[
\sigma = \frac{m_D^2}{3} l_{\text{free}}, \quad k \ll 1/l_{\text{free}}. \tag{5}
\]
When scattering processes are more complicated this formula defines an effective value of \(l_{\text{free}}\). For length scales well between the Debye screening length and \(l_{\text{free}}\), where scatterings of the charge carriers can be neglected, the conductivity is related to the Debye length through
\[
\sigma = \frac{\pi m_D^2}{4k}, \quad m_D \gg k \gg 1/l_{\text{free}}. \tag{6}
\]
(In both expressions the Debye length \(m_D\), which is \(O(gT)\), is just keeping track of the number density, charge, and \(\langle 1/E \rangle\) of the particles. The derivations of each expression assume ultrarelativistic dispersion relations for the charge carriers.) This expression follows from the form of the transverse self-energy and the fact that the conductivity we are discussing in this case is just a special case of the HTL self-energy, \(\sigma(k, \omega \ll k) = \text{Im} \Pi_T(k, \omega)/\omega\). This is the connection between this “conductivity” picture and the hard thermal loops.

The behavior of the electroweak gauge fields for scales parametrically between the non-perturbative scale \(k = g^2 T\) and the Debye scale \(k = g T\) is the same as in the abelian theory at leading parametric order. Since the mean free path of a hard excitation to undergo any scattering is \(\sim 1/(g^2 T \log(1/g))\), the conductivity for \(k = g^{2-\delta} T\), with \(0 < \delta < 1\), is \(\sigma = \pi m_D^2/4k \sim g^{\delta} T\), and the time constant associated with the decay of a magnetic field is \(\tau \sim 1/(g^{1-3\delta} T)\). Although the scale \(k \sim g^2 T\) does not fit within the range of validity of this argument it cannot be that the decay rate for a magnetic field with \(k \sim g^2 T\) differs from the \(\delta \to 0\) limit by any nonzero power of \(g\). Hence the relevant time scale for the dynamics of nonperturbative infrared magnetic fields in weakly coupled, hot Yang-Mills theory is \(O((g^4 T)^{-1})\), up to corrections at most logarithmic in \(g\). This is a paraphrase of the argument of Arnold, Son, and Yaffe, who however neglected the possibility of logarithmic corrections.

Logarithmic corrections do in fact occur. While in the abelian theory the electrical conductivity only reaches a long wave length limit at a scale set by a hard particle’s free path for large angle scattering, in the nonabelian theory that limit is set by the free path for any scattering. The reason is that when a colored particle undergoes a scattering, however small the transfer momentum, its color is changed. This degrades the color current even if it does not degrade the momentum carried by the particle substantially. The total rate for any scattering to occur is twice the damping rate, which has been computed for hard particles at leading log by Pisarski. The damping rate for an adjoint charged particle of any spin is
\[
\gamma = \frac{Ng^2 T}{4\pi} \log \frac{m_D}{g^2 T} + O(1), \tag{7}
\]
and for a fundamental representation particle it is the same with \(N \to (N^2 - 1)/2N\). Note the log, which arises from an integral over exchange momenta, running from the \(g^2 T\) to the

\footnote{Some years ago Ambjørn and Olesen argued that nonabelian fields obey an anti-Lenz’s Law, apparently in contradiction to the argument presented here. Their work refers to the nonabelian interactions between \(W\) and \(Z\) fields in the presence of strong (electromagnetism) magnetic fields at zero temperature. It may have some bearing on the mutual interactions of the fields at the \(g^2 T\) scale, but the response of the harder modes to the very soft fields is at leading order the Lenz law type behavior seen in the abelian theory.}
\(g^2T\) scale; also note that the result is independent of the particle’s momentum, provided it is harder than the \(gT\) scale. A collision largely but incompletely randomizes a particle’s charge, and so on scales longer than \(\sim 1/(g^2T\log(1/g))\) the electrical conductivity of the plasma is \(\sim m^2_D/3\gamma\), up to nonlogarithmic corrections. To find the numerical constant one must determine how thoroughly a scattering randomizes a particle’s charge, which depends on the representation of the particle. Also, the particle’s charge is not destroyed, just transferred to another particle; one must check whether this induces any important currents. (It turns out not to, because a particle is as likely to scatter from a charge carrier moving in one direction as in the exact opposite direction.) The calculation is quite nontrivial but it has been done recently by Arnold, Son, and Yaffe [19], who show that, at leading log, the conductivity is simply \(m^2_D/3\gamma\), with \(\gamma\) given in Eq. (7). This is independent of the group representation of the particles carrying the current.

Performing an extreme parametric expansion, \(\log(1/g) \gg 1\), the scale set by particle mean free paths and the nonperturbative scale are well separated, and the fields with \(k \sim g^2T\) see a \(k\) independent conductivity. Hence the relevant infrared dynamics for transverse modes is, at leading logarithmic order,

\[
[D_0, A_i] = \frac{3Ng^2T\log(1/g)}{4\pi m^2_D} D_j F_{ji} + \text{noise},
\]

where the transverse part of the noise is fixed by the requirement that the thermodynamics come out right. The longitudinal part of the noise generates time dependent gauge rotations of the \(A\) fields, which are irrelevant to Chern-Simons number; so we may choose the amplitude of the longitudinal part of the noise to be whatever we want. It is most convenient to choose it to be of the same magnitude as the transverse part, in which case we reproduce the effective theory of Bödeker, which is also the Langevin equation for 3-D Yang-Mills theory.

While this derivation has presented the ideas in an intuitive way it is scarcely rigorous, so I will now approach the problem a little more formally by looking at the hard thermal loop effective theory.

### 2.3 HTL effective theory and the Wilson line

Now I will go through the argument for Bödeker’s effective theory also from the point of view of the effective HTL theory for the infrared modes. The idea of the HTL effective infrared theory is that one can construct an effective theory for the modes with \(k \ll T\), valid at leading order in \(g\), by integrating out all degrees of freedom with \(k \sim T\). The procedure for separating the degrees of freedom and regulating the effective theory, cut off at some scale between \(gT\) and \(T\), is left unspecified and does not affect the result at leading order.

The resulting effective theory is a classical theory for the remaining modes, but with a nonlocal "HTL" effective action correction. The nonlocal effective action was first derived in [22]. In the current context, since in the long term we have in mind a position space regulation of the IR theory, it is most convenient to write the effective action in real space, as was first done by Huet and Son [23]:

\[
- [D_0, E_i]^a(x, t) + [D_j, F_{ ji}]^a(x, t) = \xi_i^a(x, t) + \frac{m^2}{4\pi} \int d^3y \frac{y_i y_j}{y^4} U^{ab}((x, t), (x + y, t - y)) \times E_j^b(x + y, t - y). \]

(9)
of the Wilson line
Ultrarelativistic

dispersion relations
Most important
to generation
Responsible for the log in the decay

Classical approximation
is good
Evolves slowly
HTL effects
O(HTL) IR
Nonperturbative

Figure 1: “It’s quite simple, really . . .” A scorecard of the scales involved in the problem and the approximations which are valid in each. None of these scales are distinct if we do not take $g \ll 1$.

Here $U^{ab}((x, t), (x + y, t - y))$ is the adjoint parallel transporter along the straight, lightlike path between the points $(x, t)$ and $(x + y, t - y)$. Note that the electric field on the right hand side is at the retarded time $t - y$. The noise $\xi^a_i$ is Gaussian with a nonvanishing two point function for lightlike separated points,

$$\langle \xi^a_i(x, t)\xi^b_j(x + y, t') \rangle = \frac{m_2^2 T}{4\pi} \delta(|t - t'| - |y|) \frac{y_i y_j}{y^4} U^{ab}((x, t), (x + y, t')).$$

For $k \ll gT$, the effective action simplifies somewhat. The coefficient of the term on the right in Eq. (9), which contains one time derivative (since $E_i = [D_0, A_i]$), is large enough that the time evolution is determined by this term and we can neglect the $[D_0, E]$ term on the left. Further, the time scale associated with the fields’ evolution is longer than the time retardation appearing in the nonlocal HTL action, and we are justified to neglect that retardation. The result is

$$[D_j, F_{ji}]^a(x) = \xi^a_i(x) + \frac{m_2^2 T}{4\pi} \int d^3y \frac{y_i y_j}{y^4} U^{ab}(x, x + y)E^b_j(x + y),$$

and the noise correlator is simplified by dropping the retardation there as well (though to get its magnitude right we must remember that there are two contributions from $\delta(|t - t'| - |y|)$).

If in addition $k \gg g^2 T \log(1/g)$ then it is possible to choose a gauge such that the parallel transporter is close to the identity and at leading order it can be ignored. In this case we can recover Eq. (8) by expanding $[D_j, F_{ji}]$ to leading order in $A$ and Fourier transforming.

Huet and Son argue that, to model the $O(g^2 T)$ modes alone, we can use Eq. (11), interpreted as an expression for fields at the $g^2 T$ scale alone with all shorter wave length scales integrated out. The reasoning is that the higher modes only appear in the equation in $[D_j, F_{ji}]$. But $[D_j, F_{ji}]$ is a thermodynamical quantity depending only on the transverse fields, and as already stated, the UV causes perturbatively small corrections to this sector,
which can be ignored at leading order. The power counting arguments are laid out explicitly in \[24\].

However, this misses one key issue. Can we neglect the influence of the modes with \(g^2 T \ll k \ll gT\) on the adjoint parallel transporter \(U^{ab}(x, x + y)\) when \(y \sim 1/(g^2 T)\)? The answer is, no. When evaluating an adjoint parallel transporter for a path of length \(\sim 1/(g^2 T)\), essential contributions arise from all scales intermediate between \(gT\) and \(g^2 T\). This problem has recently been addressed by Arnold and Yaffe, in the context of studying \(O(g^2 T)\) corrections to the Debye screening length \[25\]. They show that, in SU(\(N\)) pure gauge theory, any two point correlator of equal time adjoint fields at separation \(y\), connected by a straight adjoint Wilson line, falls off at least as rapidly as

\[
\exp\left(-\frac{y}{\lambda}\right), \quad \lambda^{-1} = \frac{Ng^2 T}{4\pi} \left(\log \frac{m_{\text{reg}}}{g^2 T} + K\right), \tag{12}
\]

where \(m_{\text{reg}}\) is a UV regulator and \(K\) is a constant of order unity. The constant is evaluated for SU(2) in \[13\]; for the regulation appropriate to determining the Debye mass, \(m_{\text{reg}} = m_D\) and \(K = 6.7 \pm 3\). Nonlogarithmic corrections are large in this case; I will return to this point later. The value of the constant \(K\) includes nonperturbative physics at the \(g^2 T\) scale, but the logarithm arises perturbatively from scales intermediate between \(g^2 T\) and \(m_{\text{reg}}\). Thus, to study physics on the length scale \(1/g^2 T\), we must include the influence of scales between \(gT\) and \(g^2 T\) on the parallel transporter in Eq. (12).

Note the sneaky way the modes with \(g^2 T \ll k \ll gT\) have entered the dynamics of the \(g^2 T\) modes. Their direct influence on the interactions between the \(g^2 T\) modes is indeed small, as Son showed \[24\]; but they change the way that the \(g^2 T\) modes interact with the hard modes, in a way which turns out to be important. Bödeker emphasizes this viewpoint in his original derivation of the effective theory for the leading log \[4\].

Using the correction to the Wilson line from the intermediate momentum modes, I can now establish Bödeker’s effective theory for the \(k \sim g^2 T\) modes. The modes with \(k \gg g^2 T\) change quickly compared to the \(g^2 T\) scale modes, so the \(g^2 T\) fields see an average over the more UV scales. In particular the parallel transporter relevant for the \(g^2 T\) modes’ evolution is the average of the parallel transporter over realizations of the \(g^2 T \ll k \ll gT\) fields. As I demonstrate in subsection 3.1, averaging over realizations of \(k \gg g^2 T\) modes leads to exponential damping of the parallel transporter, for \(y\) of order \(1/(g^2 T \log(m_D/g^2 T))\). In the Coulomb gauge \footnote{The use of Coulomb gauge becomes problematic when considering length scales \(l \geq 1/g^2 T\), and for considering any unequal time correlator if the total volume of space considered is \(V \gg (g^2 T)^{-3}\); however this is not relevant because technically we are only applying Coulomb gauge to modes with \(k \gg g^2 T\) in order to integrate them out and establish an effective theory. The gauge fixing of the IR effective theory, ie of the problematic \(k \sim g^2 T\) modes, has not been specified. Also, at leading log the same results would be obtained in Landau gauge.}

\[
U^{ab}_{\text{Coulomb}}(x, x + y) \simeq \delta^{ab} \exp\left(-\frac{y}{\lambda}\right), \tag{13}
\]

with \(\lambda\) the same as in Eq. (12). If we are permitted to expand in \(\log(m_D/g^2 T) \gg 1\), then the integrand on the right hand side of Eq. (14) has already fallen away before \(y\) comes on order \(1/(g^2 T)\); therefore the approximation which gives Eq. (13) is valid, up to log corrections, throughout the range of \(y\) which dominates the contribution to the integral. The effective
theory for the $g^2 T$ modes (fixing the gauge freedom on scales more UV than $g^2 T$ to Coulomb gauge) is therefore

$$\xi^a_i + \int d^3 y e^{-(y/\lambda)} \frac{y \cdot y_j}{y^4} E^a_i(x + y) = [D_j, F_{ ji}]^a(x).$$

(14)

Since the integral is dominated by $y \sim 1/(g^2 T \log(m_D/g^2 T))$, and for the IR fields of interest $E$ varies only on the $1/g^2 T$ scale, it is permissible at leading log order to pull the $E$ field out of the integral. The integral is then quite simple; performing it gives

$$[D_j, F_{ ji}] = \frac{2 T m^2_D}{3} \lambda \delta(x - y) \delta(t - t') \delta_{ij} \delta_{ab},$$

(15)

where the form for the noise correlator also follows from the approximation for the parallel transporter. (Alternately, one can always recover the form of $\xi$ by insisting that the thermodynamics come out correctly.) This is Bödeker’s effective theory, though it remains to establish that $m_{\text{reg}}$ should be $m_D$. I discuss this more in the next section.

## 3 Wilson line, more carefully

Now it is time to look more carefully at the Wilson line appearing in the last section, first to verify the claims there, second to show the connection to the argument involving scatterings of the hard particles, and third because it is relevant to the analysis of the results of Hu, Müller, and Moore.

### 3.1 Exponential decay of the Wilson line

What we want to know about is the Wilson line between lightlike separated points a spatial distance $l$ apart, with $l \gg 1/m_D$ but less than $1/(g^2 T)$ by at least a logarithmic factor. Actually, we want to know the average of the Wilson line over realizations of the modes with $k \gtrsim g^2 T \log(1/g)$, since these fields fluctuate faster than the $O(g^2 T)$ fields, which therefore see the average over realizations, up to corrections subdominant in $1/\log(1/g)$.

The Wilson line is given by

$$U = P \exp \int_0^l i g (A_0 + A_z)(z, t = z) dz,$$

(16)

with $A$ in the adjoint representation. I will only consider the transverse contributions here, the longitudinal ones are subdominant. I use Coulomb gauge, in which the $A_0$ contribution arises entirely from the longitudinal modes and the $A_z$ only arises from the transverse modes. Further, I will work at leading parametric order, by which I mean that higher point correlators and vertex insertions are ignored, and combinations of $A$ fields are evaluated assuming $A$ is Gaussian by applying Wick’s theorem. Of course I will include the hard thermal loop corrections to the $A$ field propagators. These approximations are justified at leading log down to $k \sim g^2 T \log(1/g)$, which is all we need. (If I were interested in lengths $l \sim 1/g^2 T$ rather than logarithmically shorter, most of the approximations I make would break down completely.)
All terms with odd powers of $A$ vanish on averaging over realizations, while even terms look like (applying Wick’s theorem between lines 1 and 2)

\[
\langle U \rangle = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( \frac{g^2}{2} \right)^n \int dz_1 \ldots dz_{2n} \langle A_z(z_1) \ldots A_z(z_{2n}) \rangle
\]

so the average over realizations is the identity times the exponential of the two point contribution. (Note that for any given realization the Wilson line has unit modulus. But the average over realizations does not, its modulus falls exponentially with distance.)

To evaluate this we need the two point correlator,

\[
\langle A_i(x, t) A_j(y, t') \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \rho(k, \omega) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{ik \cdot (t' - t)} e^{-i\omega(y-x)}.
\]

Here $\rho(k, \omega)$ is the spectral density, which is the magnitude of the discontinuity in the propagator $1/\left(\omega^2 - k^2 - \Pi_T(k, \omega)\right)$ across the real $\omega$ axis on analytic continuation from Euclidean (imaginary) $\omega$,

\[
\rho(k, \omega) = 2\text{Im} \left( (\omega + i\epsilon)^2 - k^2 - \Pi_T(k, \omega + i\epsilon) \right)^{-1},
\]

and $T/\omega$ is the classical approximation for $1 + n(\omega)$, with $n(\omega)$ the Bose distribution function.

So far I have suppressed group indices, but when they and the integrals over $z$ are evaluated we get

\[
\log \frac{\text{Tr} \langle U \rangle}{\text{Tr} 1} = - \frac{Ng^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \rho(k, \omega) \frac{k_x^2 + k_y^2}{k^2} \frac{4\sin^2(k_z - \omega)l/2}{(k_z - \omega)^2}.
\]

The equal time correlator of the $A$ field goes as $T/k^2$ plus subleading corrections, for all $k \gg g^2 T$; so

\[
\int \frac{d\omega}{2\pi} \frac{T \rho(k, \omega)}{\omega} = \frac{T}{k^2}.
\]

What matters now is where $\rho$ is concentrated. In the regime $k \ll m_D$, almost all the contribution to Eq. (21) is from $|\omega| \ll k$, see (21). This is just the statement that these modes evolve on time scales slower than $1/k$, as I have already discussed. Hence, in evaluating the low $k$ contribution to Eq. (21) I can set $k_z - \omega \approx k_z$, and then perform the integral over $\omega$, giving

\[
- \frac{Ng^2 T}{2} \int |k| \leq m_D \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \frac{k_x^2 + k_y^2}{k^2} \frac{4\sin^2 k_z l/2}{k_z}.
\]

The $k_z$ integral is completely well behaved; in fact the large $l$ limit of $4\sin^2(k_z l/2)/k_z^2$ is $2\pi l \delta(k_z)$. This represents the fact that only modes with $k_z < 1/l$ have $A$ of the same phase all along the Wilson line; other modes’ contributions destructively interfere in the integral.
along the Wilson line. The integral over the other two directions is dominated by a logarithm arising from scales intermediate between \( k_\perp \sim 1/l \) and \( k_\perp = m_D \), where the approximation for \( \rho \) breaks down. Performing the integral over \( k_\perp \) first gives

\[
- \frac{Ng^2lT}{4\pi^2} \left( \log \frac{m_D}{l} + O(1) \right) \int \frac{\sin^2(kz/l)}{(k_\perp^2/2)^2}dk_\perp/2 = - \frac{Ng^2lT}{4\pi} \left( \log \frac{m_D}{l} + O(1) \right). \tag{23}
\]

In the opposite limit, \( k \gg m_D \), the excitations obey normal vacuum ultrarelativistic dispersion relations to good approximation, so the spectral density is approximately

\[
\rho(k \gg m_D, \omega) \simeq \frac{\pi}{k} (\delta(k - \omega) - \delta(k + \omega)). \tag{24}
\]

The large \( k \) contribution is then

\[
- \frac{Ng^2}{2} \int_{m_D} d^3k \frac{k^2 + k_\perp^2}{k^2} \frac{T}{|k| - k_\perp} \frac{4\sin^2((|k| - k_\perp)l/2)}{(1 - x^2)^2} \sim - \frac{Ng^2T}{2\pi^2m_D} (\log(lm_D) + O(1)), \tag{25}
\]

Now the term with \( \sin^2 \) in it is forcing \( |k| = k_\perp \); only modes propagating along the direction of the Wilson line keep in phase, others destructively interfere. But the polarization vector of such a mode is close to orthogonal to the Wilson line; because \( k \gg 1/l \), for any \( k \) for which \( |k| - k_\perp < 1/l \), the factor \( k_\perp^2/k^2 \) will be near zero. Thus, the only modes which avoid destructive phase interference are polarized in the wrong direction to contribute significantly.

Continuing to carry out the integral by defining \( x = k_\perp/k \), we get

\[
- \frac{Ng^2T}{2\pi^2} \int_{-1}^{1} dx \int_{m_D} dk(1 - x^2) \frac{4\sin^2(k(1-x)/2)}{(1 - x^2)} \sim - \frac{Ng^2T}{2\pi^2m_D} (\log(lm_D) + O(1)), \tag{26}
\]

which is \( O(g) \). The hard modes do not contribute at leading order to the Wilson line along a lightlike path.

I have not treated the modes with \( k \sim m_D \), which are more complicated because this is where \( \rho \) does not fit into either limiting category. But they turn out to give a result smaller by a logarithm than that from the modes with \( k \ll m_D \). The final result is that, on averaging over realizations of the modes with \( k \) greater than \( g^2T \) by at least a logarithm, the Wilson line in Coulomb gauge is

\[
\langle U \rangle = 1 \exp(-l/\lambda), \quad \lambda^{-1} = \frac{Ng^2T}{4\pi} \left( \log \frac{m_D}{g^2T} + O(1) \right). \tag{27}
\]

### 3.2 relation to scattering

Now look at Eq. (20) again. Remember that the Wilson line is representing the trajectory of a hard particle, with \( p \) much greater than any momentum scale which gives a leading order contribution to the integral. We want to interpret Eq. (20) as \( l \) times the rate for the particle to undergo a scattering involving the transfer of a soft field, times a (representation dependent) group theory factor which tells how thoroughly the scattering randomizes the
particle charge. To see the relation, take the large $l$ limit. Then $4 \sin^2((k_z - \omega)/2)/(k_z - \omega)^2 = 2\pi l \delta(k_z - \omega)$, and the decay rate per unit length of the Wilson line is
\[
\frac{N g^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \frac{T \rho(k,\omega)}{\omega} \frac{k_x^2 + k_y^2}{k^2} 2\pi \delta(k_z - \omega).
\] (28)

Recall that the hard particle starts out with $p_z \gg |k|$, $p_x = p_y = 0$. If it emitted a particle of wave number $k$, its momentum would change to $\vec{p} - \vec{k}$ and its energy would change by $-k \cdot \hat{p} = -k_z$ plus a correction of order $k^2/|p|$, which is negligible by assumption. The delta function is just the energy conserving delta function appearing in the expression for the rate of the process shown in Figure 2 (a), the emission of a soft gluon by an adjoint charged, hard mode.

The rate for this process has been considered by Pisarski [21]; his expression (4.4) does not look quite the same as Eq. (28) for the following reasons. First, his definition of the spectral density differs from the one used here by a factor of $2\pi$. Second, he has allowed the hard particle to be a little off shell, which is important in the part of the integral with $k \sim g^2 T \log(1/g)$ but not for higher $k$. (Note however that my large $l$ approximation breaks down in exactly this regime.) For momenta $k \gg g^2 T \log(1/g)$ we may integrate over $\omega$ in his expression and obtain Eq. (28). The difference caused by $k \sim g^2 T \log(1/g)$ does not change the leading log, but would be important in investigating corrections to the leading log. Also note that Pisarski considers the contribution of longitudinal momenta and finds that they do not give a log, only a constant times $Ng^2 T$.

The process considered involves the emission of a soft mode with $|k| > |\omega|$. The reason that such emission is possible at all is because of the hard thermal loop correction to the gauge propagator. The rate is the imaginary part obtained by cutting the one loop self-energy diagram, Fig. 2(b). The cut gives a nonzero result when it goes through a HTL self-energy insertion, which can be interpreted as diagram (c) in the figure. Hence the Wilson loop calculation is proportional to the rate for scatterings of one hard particle off all other hard particles by exchange of soft intermediaries.

Pisarski calculates the total rate for a particle to undergo a collision, and because of the conventional definition of the damping rate, his damping rate is actually half of the scattering rate. Also, his result does depend on the representation of the particle undergoing
the scattering. The rate the Wilson line calculation determines is actually the rate of color randomization, not of collisions; there is a representation dependent correction between the two, which depends purely on color factors at the vertex where the particle of interest interacts with the soft background field. The total collision rate of a particle is proportional to the group factor

$$\frac{\text{Tr} T^a T^a}{\text{Tr} 1}$$

(29)

where $T^a$ is in whatever representation the particle is in. The original disturbance of the particle distribution from equilibrium is caused by an electric field, which is an adjoint object; the disturbance to the single particle density matrix is proportional to $E^b T^b$. The color randomization per collision is

$$1 - \frac{\text{Tr} 1}{\text{Tr} T^a T^a} T^a T^b T^a T^b$$

(30)

where the second term tells how much the color after the collision is aligned with the color before. Multiplying by the total collision rate gives a color randomization rate $\propto N/2$, independent of representation.

### 3.3 subleading corrections in the method of Hu and Müller

Now I will discuss the relation between the quantum theory and the technique proposed by Hu and Müller, refined and implemented jointly with myself, concentrating on whether the behavior is the same at next to leading logarithmic order.

First I should explain why the technique is necessary. Traditionally people have tried to determine the sphaleron rate by studying classical Yang-Mills (or Yang-Mills Higgs) theory regulated on a spatial lattice. However, if we just study the classical theory on the lattice, using the lattice spacing as the UV cutoff, the hard lattice modes generate HTL effects which do not look like Eq. (9) and in fact are not rotationally invariant [26]. Even in the leading log approximation this is a problem, because it means that, where Eq. (15) has $E_i$, we will get $E$ rescaled by a rotationally non-invariant factor, determined in [27].

A proposal by Arnold to fix this problem, staying within lattice classical theory, by making the dispersion relations turn up very steeply [27], does not work because the hard modes are then Landau damped [27]. The only alternatives I am aware of involve adding new degrees of freedom which influence the IR classical fields in a way equivalent to correct hard thermal loops. Two such proposals exist in the literature. One is due to Bödeker, McLerran, and Smilga [26], more recently discussed by Iancu [28]. I will not discuss it since no one has yet specified a complete discrete numerical implementation. The other idea was proposed by Hu and Müller [12]; the details of the implementation were worked out and applied jointly with me [13].

We also simulated the classical IR physics by studying nonperturbatively the classical system regulated on a spatial lattice, thereby treating the left hand side of Eq. (9) fully.
nonperturbatively. To include the HTL effective action, the right hand side of the expression, we added to the classical lattice system a large number of adjoint charged classical particles. They take coordinate positions in the continuous space in which the lattice fields sit, obey ultrarelativistic dispersion relations, and interact with the lattice fields when they cross the dual planes to lattice links. There are two parts to the interaction with the lattice fields.

First, the particles “kick” lattice electric fields, and their momenta receive a similar “kick”. The kicks and the approximately random distribution of the charges performs the noise, and a correlation between past fields and the “kick” the gauge field receives, arising from a change in the particle trajectory from the “kick” it received, accounts for the nonlocal term. The size of each kick is proportional to a charge $Q$ which is made small so the particles individually interact weakly.

Second, the particles’ adjoint charges are parallel transported by the gauge field connection. Since the gauge fields are only defined on the lattice links, the Wilson line used for the parallel transportation of a particle is the sequence of lattice links which maintains the shortest distance to the actual path of the particle, as illustrated in Figure 3. The exact rule used to choose the sequence of links is that a link is used if the Wilson line penetrates the face of the dual lattice corresponding to that link. Parallel transporting the particle’s charge ensures that we are including the parallel transporter in the right hand side of Eq. (9); it is also absolutely necessary to make the update rule manifestly gauge covariant. The full update is described in excruciating detail in [13].

The Debye mass depends on the density $\langle n \rangle$ and charge $Q$ of the added particles as $m_D^2 \propto Q^2 \langle n \rangle$, so by changing the number of particles one can tune the HTL strength to be whatever is needed while keeping $Q$ small. If there were only IR fields, meaning fields with $k \ll 1/a$ (a the lattice spacing) then the behavior would correctly reproduce Eq. (9), at least in the limit $Q \to 0$, $\langle n \rangle \to \infty$ with $m_D^2$ fixed. This is also discussed in [13].

The lattice theory involves two scales, $a$ and $m_D$; to get the leading parametric behavior
we must seek the limits \( a \to 0, m_D \to \infty \) (if we think of the length scale \( 1/(g^2 T) \) as remaining fixed). There are two ways we could go about doing this. One corresponds to using \( a \) as a cutoff between the \( g T \) and the \( T \) scales, meaning that we maintain \( a \ll 1/m_D \) but \( a \gg g^2 T/m_D^2 \). The latter condition is essential to make sure that “bad” hard thermal loops arising from modes with \( k \sim 1/a \), which have the wrong dependence on \( k, \omega \) and in particular are not rotationally invariant \[^{[26]}\] , are subdominant to “good” hard thermal loops due to the particles. The other way of taking the limits is to make \( m_D \gg 1/a \), so the wave number \((1/a)\) falls between the \( g T \) and \( g^2 T \) scales.

In each approach, the model correctly generates the effective HTL dynamics up to power corrections in \((g^2 a T), (Q/g^2 a T), g^2 T/m_D \), and \((g^2 T)/(am_D^2)\), except perhaps for the behavior of the parallel transporter. The contribution to the parallel transporter from IR fields is correct because for a gauge field of wave number \( k \ll 1/a \) the replacement of the straight line path with the “jagged” path actually used (see again Figure \( 3 \)) gives the right behavior up to corrections suppressed by \( O(k^2 a^2) \). The question is, how does the UV contribute to the parallel transporter?

In the case where we make \( m_D \gg 1/a \), all of the lattice degrees of freedom have over-damped evolution. None of them propagate and there are no problems from hard thermal loops arising from classical lattice degrees of freedom with \( k \sim 1/a \). Except for the parallel transporter, the systematics are then best under control. However the log appearing in the parallel transporter is obviously cut off by the inverse lattice spacing, \( \log(m_D/g^2 T) \) becomes \( C + \log(1/g^2 a T) \) with \( C \) a coefficient to be determined by matching. Since we know that the lattice regulator is not rotationally invariant we expect that \( C \) will be direction dependent, leading to a rotationally non-invariant correction. However, this correction is only at next to leading order in an expansion in \( \log(1/g^2 a T) \).

In the other case, \( m_D \ll 1/a \), all of the physics which sets both upper and lower limits on the \( \log \) in Eq. (20) lie at \( k \ll 1/a \), and naively we should get the correct behavior at leading order in \( g \). We do not, however, because the most UV lattice modes have dispersion relations which “turn over,” \( k > \omega \). Diagram \((a)\) from Figure 2 can occur with the emitted gluon on shell. In fact this is just Čerenkov radiation, which is permitted because the group velocity of the most UV lattice modes is subluminal, while the hard particles move at the speed of light. Hence, while there is no contribution to the parallel transporter from modes with \( 1/a \gg k \gg m_D \), there is a contribution from \( k \sim 1/a \), and again we get a rotationally non-invariant correction, \( \log(m_D/g^2 T) \) becomes \( C + \log(m_D/g^2 T) \).

To determine the value of \( C \) we need to repeat the arguments leading to Eq. (20) but using lattice gauge fields multiplied along a “jagged” path which stays closest to the straight line path under consideration. For a Wilson line of length \( l = La \) in a direction with unit vector \( p_i \), with \( p_i \) in the first octant, the equivalent of Eq. (20) is

\[
\log \frac{\text{Tr}(U)}{\text{Tr}1} = - \frac{N g^2}{2} \int_{-\pi/a}^{\pi/a} \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} T \rho(k, \omega) F(k, \omega),
\]

\[
F(k, \omega) = \sum \alpha_x \sum_{n=1}^{[L p_n + x]} \exp i \left( \omega \frac{n_x - x}{p_x} - k_x(n_x - .5) - k_y \left[ y + \frac{n_x - x}{p_x} p_y \right] - k_z \left[ z + \frac{n_x - x}{p_x} p_z \right] \right) + \left( \sum \sum \text{similar} \right)^2,
\]
and the values in other octants follow from cubic symmetry. Here the sum over $\epsilon$ is over a basis of the two unit vectors satisfying $\sum_i \epsilon_i \sin(k_i a/2) = 0$. The integral over $(x, y, z)$ appearing in the definition of $F$ averages over starting positions for the Wilson line within a lattice cell, and square brackets always mean the argument is rounded down to an integer. The rounding down to an integer makes it very difficult to evaluate the expression analytically. It is also tricky to find $\rho(k, \omega)$ on the lattice, away from the $m_D \gg 1/a$ or $m_D \ll 1/a$ limits.

In the infrared, meaning $k \ll 1/a$, the sum in $F$ is well approximated by an integral, recovering Eq. (20). In the UV, while Eq. (31) is manifestly cubic symmetric, it is not spherically symmetric: the $k$ integral is over a cubic range, and neither $F(k, \omega)$ nor $\rho(k, \omega)$ are rotationally invariant. The integral has been computed by Arnold and Yaffe in the large $L$ limit for the special case that $p_i$ lies in a lattice direction and the spectral density is concentrated at $\omega = 0$ [25], although they did not present their calculation in this context. I will now compute it in the opposite limit, $m_D \ll 1/a$, but still along a lattice direction. If $L$ is large and $p_i$ lies in a lattice direction (say, the $z$ direction) then $F(k, \omega) \simeq 2\pi l \delta(k_z - \omega \mod (2\pi/a))$. For $m_D \ll 1/a$ the dispersion relation is the free lattice dispersion relation $\omega^2 a^2 = \sum_i (2 - 2\cos(k_i a))$ and the spectral density $\rho$ all lies on shell, and the integral reduces to

$$\frac{N g^2 l T}{2} \int_{-\pi/a}^{\pi/a} \frac{d^3 \tilde{k}}{(2\pi)^3} \frac{\tilde{k}_x^2 + \tilde{k}_y^2}{(\tilde{k}^2)^2} 2\pi \delta \left( k_z - \frac{\sqrt{\tilde{k}^2}}{a} \mod \frac{2\pi}{a} \right),$$

(33)

where $\tilde{k}_i = 2 \sin(k_i a/2)/a$. Evaluating numerically gives $0.0466539 N g^2 l T$, which is $0.586$ times the coefficient of $\log(m_D/g^2 T)$. This demonstrates that there are extra UV contributions arising from $k \sim 1/a$ in the case $m_D \ll 1/a$, which though not terribly large are not negligible.

The upshot is that however $m_D$ is tuned, the UV lattice degrees of freedom will introduce a rotationally non-invariant contribution to the radiative correction to the Wilson line which determines the parallel transport of a particle’s charge. This spoils the rotational invariance of the IR HTL effective theory at next to leading order in $\log(1/g)$. The best we could do would be to average the value of Eq. (31) over angles. Then an appropriate choice of lattice spacing $a$ in the regime with $m_D \gg 1/a$ could match the value of the $\log(m_D/g^2 T)$ as closely as possible to the quantum theory value. This is the best option I see for determining the sphaleron rate beyond leading order, but it does not eliminate all systematic errors even at leading parametric order in $g$.

I have not shown that the same problem will arise for other possible lattice implementations of HTL effects; but since such implementations must generally involve parallel transportation on the lattice I expect that the problem discussed here is general. Certainly, when proposing some other numerical implementation of HTL’s, the burden of proof must lie on the side of showing that problems from rotational non-invariance of Wilson lines do not arise.

### 4 Numerics

Now that I have discussed the establishment of the effective theory, Eq. (13), I will discuss how to make a lattice model of that effective theory and how I compute the leading log
coefficient of the sphaleron rate by using that model. First, define a Langevin time \( \tau \) related to the time appearing in Eq. \( (15) \) via

\[
d\tau \equiv \frac{3}{m_D^2 \lambda} dt = \frac{3N g^2 T \log(m_D/g^2 T)}{4 \pi m_D^2} dt.
\]

Note that \( \tau \) has dimensions of length squared, not length. The effective infrared theory then has the familiar form

\[
[D_\tau, A_i]_a(x, \tau) = [D_j, F_{ji}]_a(x, \tau) + \xi_i^a(x, \tau), \quad \langle \xi_i^a(x, \tau) \xi_j^b(y, \tau') \rangle = 2T \delta(x - y) \delta(\tau - \tau'),
\]

which is a Langevin equation. Studying it, I will find the diffusion constant for Chern-Simons number

\[
\Gamma_{\text{Langevin}} = \lim_{V \to \infty} \lim_{\tau \to \infty} \frac{\langle (N_{\text{CS}}(\tau) - N_{\text{CS}}(0))^2 \rangle}{V \tau},
\]

numerically. To do that I find the diffusion constant per lattice site per \( a^2 \) of Langevin time, and multiply by \( a^{-5} \). On dimensional grounds \( \Gamma_{\text{Langevin}} \) must be of order \( \alpha_5^5 W T^5 \). Using Eq. \( (34) \), the relation between \( \Gamma_{\text{Langevin}} \) and \( \kappa' \) defined in Eq. \( (2) \) is

\[
\kappa' = \frac{3N \Gamma_{\text{Langevin}}}{4\pi \alpha_5^5 W T^5}.
\]

Before getting further it is worth commenting that the IR behavior of Eq. \( (35) \) is insensitive to the UV regulation and the limit in which that regulation is removed, exists. Though this statement appears banal, it makes Bödeker’s effective theory completely different from the classical Hamiltonian dynamics. The essential difference is that there are no long time scale correlations for the UV fields in the Langevin evolution; a mode with wave number \( k \) gets randomized in Langevin time \( \tau \sim 1/k^2 \), which is much faster than the natural time scale for the evolution of IR fields. Hence the IR fields see the average over all excitations of the UV fields. The influence of the UV must be purely thermodynamic, and we know from the super-renormalizability of 3-D Yang-Mills theory that the thermodynamic influence of the UV on the transverse sector is well behaved. For a more rigorous presentation of the argument see \cite{13}, who show that any purely dissipative update algorithm will give a good continuum limit. For the Hamiltonian system, on the other hand, UV modes are propagating; the unequal time correlator behaves like \( \cos(kt) \) rather than like \( \exp(-k^2 \tau) \). The existence of long time scale correlations of the UV fields is what makes the HTL effects important to the IR dynamics. Because of this difference, we can expect a good small lattice spacing \( a \) limit to exist for the Langevin time dynamics, and it is worth it to try to control systematic errors.

I discuss the continuum \( \tau \), spatial lattice implementation of Eq. \( (35) \) in the appendix; here I will just mention how I discretize the time update. I define a time step \( \Delta \tau = a^2 \Delta \), \( \Delta \) a pure number much less than 1. The fields will be well defined at times \( n \Delta \tau \), \( n \) an integer. The noise is constant in each interval \( [n \Delta \tau, (n+1) \Delta \tau) \); its value at each point, direction, and Lie algebra direction is drawn from the Gaussian distribution with mean value \( \sqrt{2T/a^3 \Delta \tau} \); its value at each point, direction, and Lie algebra direction, and in each time interval, is independent. To determine the fields at time \( (n+1) \Delta \tau \) from the fields at time \( n \Delta \tau \) I use
Table 1: Results for $\kappa'$ at two lattice spacings and two lattice volumes. The results show excellent spacing and volume independence.

| lattice spacing $a$ | Volume | Langevin time | $\kappa' \pm$ statistical error |
|---------------------|--------|---------------|---------------------------------|
| $2/3 g^2 T$        | $(8/g^2 T)^3$ | $290000 a^2$  | $10.44 \pm 0.23$ |
| $2/3 g^2 T$        | $(16/g^2 T)^3$ | $49500 a^2$   | $10.30 \pm 0.21$ |
| $2/5 g^2 T$        | $(16/g^2 T)^3$ | $21000 a^2$   | $10.70 \pm 0.67$ |
| $2/7 g^2 T$        | $(16/g^2 T)^3$ | $42000 a^2$   | $10.26 \pm 0.79$ |

Unfortunately the above update is quite inefficient. However there is a much more efficient algorithm for dissipatively updating the fields, the heat bath algorithm. Rather than applying a very small step of Langevin update to each lattice link in parallel, the idea is to go through the links of the lattice at random, performing a complete heat bath update of each link. The relation between Langevin time and the number of links updated is discussed in Appendix A; in particular it is possible to make a very accurate match between the Langevin and heat bath time scales by measuring the autocorrelations of some IR observable. I apply the small $\Delta$ limit as part of the matching, so that is taken care of.

Appendix A together with previous work [29], shows how to control lattice spacing systematics so they first appear at $O(a^2)$. Also, it is possible to define a lattice measurable to use for $N_{CS}$ which is topological and will eliminate systematic errors in $\Gamma$ due to the definition of $N_{CS}$. In fact, two fairly efficient techniques are available [11,30]; here I will use the method developed in [30]. It remains to take the large volume and time limits. It was shown in [30] that finite volume systematics are negligible on cubic toroidal lattices larger than $8/g^2 T$ on a side. To be doubly sure, I have used a lattice $16/g^2 T$ on a side; as a check I measure $\Gamma$ also on a lattice of half this size to check that the result is the same. Taking the infinite time limit is tied up with the problem of converting a Langevin time series for $N_{CS}$ into a measurement of $\Gamma$. I use the same analysis techniques as [11].

To verify good control of lattice spacing systematics I have made measurements of $\Gamma$ at three lattice spacings, $a = 2/3 g^2 T$ ($\beta_L = 6$), $a = 2/5 g^2 T$ ($\beta = 10$), and $a = 2/7 g^2 T$ ($\beta_L = 14$). The results are presented in Table 1. Finite volume and spacing systematics are under control. In particular, varying the lattice spacing by over a factor of two leads to corrections smaller than the statistical errors. This makes large $a$ extrapolation unnecessary, which is very important, since numerical cost rises as $(1/a)^5$.

If I had used naive rather than radiatively corrected relations between lattice and continuum parameters, then the value of $\kappa'$ at the three lattice spacings would be 17.7, 15.0, and
13.3 respectively. Such strong dependence is because converting $\Gamma$ from lattice to continuum units involves the 5th power of $a$, and the radiative corrections are at leading order a shift in the meaning of $a$ from the naive value, by roughly 10%, 6%, and 4% for the three lattice spaces used. At two loops there is a further shift, estimated to be of order (and probably less than) the square of the first order shift [29]; around 1%, .36%, and .2% respectively. The latter two are negligible compared to the statistical errors, even after taking account of the 5th power dependence. It is less clear how to estimate the importance of $O(a^2)$ nonrenormalizable operators; but if I estimate all $O(a^2)$ errors by using the three lattice spacings to extrapolate to $a = 0$, assuming errors proportional to $a^2$, the result lies within the error bars of the two finer lattice data, and the error in the extrapolation is dominated by the error in the finer lattice data. Hence I will adopt the middle lattice spacing result and its statistical error bars as the best estimate.

5 Interpretation and systematic corrections

The numerical result is that

$$\Gamma = (10.8 \pm 0.7) \left( \log \frac{m_D}{g^2 T} + O(1) \right) \frac{g^2 T^2}{m_D} \alpha^5 W T^4 + \text{(higher order in } g), \quad (38)$$

but it remains to determine or estimate the rate at the realistic standard model value of $m_D$, $m_D^2 = (11/6)g^2 T^2$ with $g^2 \simeq 0.4$. For this value, $\log(m_D/g^2 T) \simeq 1.5$, and the $O(1)$ correction may be quite important.

In fact we might expect that the log needs to be quite large before it dominates the $O(1)$ “correction”. The leading log behavior is based on the hard particles propagating only a short distance before undergoing a collision which randomizes their charge. This “short” distance is $4\pi/(Ng^2 T \log(m_D/g^2 T))$, with $N = 2$ since we are in SU(2) theory. The leading log contribution to the free path is $\sim (6/g^2 T)/\log(m_D/g^2 T)$. For comparison, above I confirm Ambjørn and Krasnitz’ result [10] that a lattice only $8/g^2 T$ across is already large enough to give continuum like behavior for the sphaleron rate. The nonperturbative length scale characterizing baryon number violating processes must be shorter than this, perhaps by a factor of two. Hence the log will need to be quite big before the “short” distance really is short compared to the scale which is setting the physics. This supports the expectation that there will be large corrections to the leading log.

5.1 estimate using Laine and Philipsen’s results

As I noted already in subsection 2.3, the log arises from the behavior of a Wilson line, and the same Wilson line appears in the definition of the Debye mass beyond leading order. In this context the value for the $O(1)$ correction to the leading log behavior has been found by Laine and Philipsen. In that case, $[\log(m_D/g^2 T) + O(1)]$ has $O(1) = 6.7$. If the same number held for the sphaleron rate, then using the standard model value of $m_D$ to evaluate the log, the leading behavior would be $\Gamma = 89(g^2 T^2/m_D)\alpha^5 W T^4 = 48\alpha^5 W T^4$. This is a crude way of estimating the nonleading corrections, though, and I do not take it too seriously. In particular there is evidence that the length scale relevant for baryon number violating
processes is longer than the \( \simeq 1/g^2T \) Laine and Philipsen find for the \( O(g^2T) \) correction to Debye mass; the baryon number violation rate on a cubic toroidal lattice \( 3/g^2T \) across is over 1000 times slower than for a large volume \([30]\), so physics must be going on involving lengths at least half as long as \( 3/g^2T \).

5.2 estimate using Hu, Müller, and Moore’s results

Another way of trying to determine the subleading corrections is to use the value of the leading log coefficient to correct my results with Hu and Müller. There, we used a technique which included hard thermal loops, but in a way which does not correctly reproduce the subtleties of the Wilson line responsible for the logarithmic dependence of \( \Gamma \) on \( m_D \). As discussed in subsection \([33]\), the log arises because excitations more UV than the \( g^2T \) scale make the Wilson line effectively randomize the charge of a propagating particle, and there is a log in the reciprocal length for randomization. The best approach would be to compute the angle averaged value of the reciprocal length, in the quantum theory and in the lattice theory actually studied. Unfortunately, so far I can only compute the reciprocal length in the lattice theory in one direction, and only in the cases \( m_D \ll 1/a \) or \( m_D \gg 1/a \). The data in \([33]\) are taken at \( m_D \sim 1/a \), not \( \gg 1/a \), so it is at least reasonable do the match using the small \( m_D a \) approximation. I will make do with the log evaluated in the one direction where I can do the integral, which gives a difference in logs between the two theories of

\[
\log \left( \frac{m_D \text{ (lattice)}}{m_D \text{ (continuum)}} + 0.59 \right).
\]

I will use the datapoint from that paper taken with the largest value of \( m_D \), because the “wrong” lattice mode induced HTL’s really are strongly subdominant to the “right” particle induced HTL’s for this case. Redoing the match between the lattice and continuum time scales, which was performed wrongly there because we did not have the \( O(a) \) calculation performed in Appendix \([A]\) of this paper, revises the result from \( \Gamma = 53 \pm 5(g^2T^2/m_D^2)\alpha_W^5 T^4 \) down to \( 50 \pm 5(g^2T^2/m_D^2)\alpha_W^5 T^4 \). This result was obtained at \( m_D \simeq 4g^2T \), so the difference of logs between the lattice theory where this number was computed and the quantum theory at the physical value of \( m_D^2 \) is about .63 due to the values of \( m_D \) plus .59 due to UV contributions present on the lattice but not in the continuum theory. Using the determined coefficient of the leading log term, I should correct the diffusion constant we found downwards by \( 1.22 \times 10.8 \), giving 37 ± 5, with only the statistical error bar shown. (Substituting in \( m_D^2 = 11g^2T^2/6 \) gives \( \Gamma = 20 \pm 3\alpha_W^5 T^4 \)).

Using \( \kappa' \) to correct the old data in this way assumes that

\[
\frac{md}{dm_D} \left( \frac{m_D^2}{g^2T^2} \Gamma \right) = \kappa'\alpha_W^5 T^4. \tag{39}
\]

We really only know that this true in the large \( (m_D/g^2T) \) limit, where the leading log expansion is valid. It may have quite nonnegligible corrections at realistic values of \( m_D \), which would appear in a systematic expansion in \( \log(m_D/g^2T) \) as inverse powers of the log; the \( O(1) \) correction to the leading log behavior would really be \( \tilde{C}_1 + C_2(\log(m_D/g^2T))^{-1} + \ldots \). Intuitively I expect that the real \( m_D \) dependence will be weaker than the leading log suggests (meaning \( C_2 > 0 \)), though I cannot give a cogent argument to show this is so. In this case I have performed an overcorrection, and the real rate would be higher. For now I will accept
the corrected answer as the best current guess, but I take a systematic error bar of order 30% to cover both the rotational non-invariance not handled correctly in the correction, and the fact that the correction may be an overestimate at realistic $m_D$.

### 5.3 corrections which are formally parametrically suppressed

There is a further cause of systematic error in the determination of the sphaleron rate, arising from corrections to the $g \ll 1$ approximation. I will mention the two such corrections which I think are the largest; fortunately they have opposite sign and the optimistic can hope that they largely cancel.

One problem is that the parametric argument for the importance of hard thermal loops assumes $m_D \gg g^2 T$, and it is not clear realistically whether this is obtained. There is evidence that it is not. In particular, while the sphaleron rate in classical Yang-Mills theory depends on the lattice spacing in a way consistent with the Arnold-Son-Yaffe prediction $\Gamma \propto m_D^2 \propto a$, the corrections to the linear dependence are large. Of course, some dependence is expected, since we know now that the scaling behavior should be not $\Gamma \propto a$ but $\Gamma \propto a \log(1/a) + O(1)$; but we can determine the coefficient of the log$(1/a)$ term by using the results of this paper. According to Arnold, we can relate the classical lattice results to continuum ones, approximately, by replacing $m_D^2 \simeq 0.684 g^2 T/a$. In this case the $a$ dependence of $\Gamma$ in classical, lattice Yang-Mills theory should be

$$\Gamma_{\text{on the lattice}} = 0.465 \kappa' \left( \log \left( (g^2 aT)^{-1/2} \right) + O(1) \right) \frac{g^2 aT}{4} \alpha_W^4 T^4,$$

(40)

with $\kappa'$ the same as the one we find but the $O(1)$ correction different. This formula makes it possible to correct the data in [11] to remove the logarithmic dependence on $a$, for instance by adjusting the data so they all correspond to $g^2 aT = 1/4$, which is the value for the finest lattice used there. I have done so, and the result is plotted in Figure 4.

I fit the data to the form $\Gamma/a = C_1 + C_2 a$ to find the corrections to the small spacing limit which do not arise from the log. The fit is very good, see Figure 4, but the coefficient $C_2$ is quite substantial. The physical value of HTL strength, using $m_D = (11/6)g^2 T^2$ and $g^2 = 0.4$, corresponds to $g^2 aT/4 = 0.037$, where the correction $C_{2a}$ accounts for an 11% shift from the small $a$ limit, with the actual value falling below the large $m_D$ extrapolation. The correction reduces the sphaleron rate. Given the other systematics in play it is probably not fair to call this a measurement of the correction to the $m_D \gg g^2 T$ limit; rather I will call it an estimate to tell how large the systematic error is.

Another correction which is parametrically suppressed but not necessarily very small arises from QCD scatterings of quarks. Quarks are responsible for almost half of $m_D^2$, and hence almost half of all HTL effects. But quarks scatter strongly. While a strong scattering does not disturb a quark’s electroweak charge, it does change its direction of flight, whereas the calculation of the HTL's is made assuming particles maintain straight line trajectories. If the free path for strong scattering were $\leq 1/g^2 T$ this would make an $O(1)$ correction to the quarks’ influence on IR physics. Actually the quark free path for large angle scattering is parametrically order $(\alpha_s^2 \log(1/g_s) T)^{-1}$ and the correction is formally $O(\alpha_s^2/\alpha_W) \sim g^2$; but numerically this might not be small. The effect of this correction is to increase the sphaleron rate. It might be possible to include this correction within the context of the “particles”
Figure 4: Small lattice spacing extrapolation of data in pure classical lattice Yang-Mills theory, taken from [11]. The data have been corrected to absorb the leading log dependence on $m^2_D \propto 1/a$ determined here. They show a substantial linear correction to the predicted $\Gamma \propto a$ behavior. This is evidence of nonnegligible corrections to the parametric $m_D \gg g^2 T$ limit.

approach by calculating more precisely the size and frequency of quark scatterings and adding them to the dynamics of the particle degrees of freedom.

5.4 including the Higgs

I should also mention that the evaluation of the leading log coefficient made here was within pure SU(2) Yang-Mills theory, without a Higgs boson. In the context of baryogenesis we actually need to know the sphaleron rate in the presence of at least one Higgs boson, in the symmetric phase and roughly at the equilibrium temperature. The Higgs field’s evolution is not overdamped, because the hard thermal loops for a scalar field are nothing but a mass squared correction. Hence the $k \sim g^2 T$ modes of the Higgs fields evolve on the time scale $1/g^2 T$, which is parametrically faster than the gauge fields. Therefore, on the $1/g^4 T$ time scale, the gauge fields only see the thermodynamic average over all Higgs fields in the fixed gauge field background. This could be simulated by including the Higgs field in the Hamiltonian, and evolving it with heat bath dynamics, but updating the Higgs fields much more often than the gauge fields. Then we would have to extract the limit as the Higgs field update is made infinitely faster than the gauge field update. I have not yet attempted to do this, but I anticipate that for parameters which make the phase transition strongly first order, the effect should be a slight reduction of the leading log coefficient for the sphaleron rate.

\footnote{Dam Son pointed this out to me.}
6 Conclusions

In the formal small $g$ limit, in which an expansion in $\log(1/g) \gg 1$ is justified, the sphaleron rate in SU(2) Yang-Mills theory is

$$\Gamma = (10.8 \pm 0.7) \left( \log \frac{m_D}{g^2T} + O(1) \right) \frac{g^2T^2}{m_D^2} \alpha_W^5 T^4. \quad (41)$$

The value $10.8 \pm 0.7$ is clean; the errors are dominated by statistics, systematics are well under control.

However, interpreting this result to get the sphaleron rate at the realistic values of $m_D^2 = (11/6)g^2T^2$ and $g^2 \simeq 0.4$ is very problematic, because the $O(1)$ correction is not subdominant. The reason is that the expansion in large $\log(1/g)$ corresponds to treating the length scale $2\pi/[g^2T \log(m_D/g^2T)]$ as much shorter than the scale relevant for baryon number violating processes, which is around $\sim 4/g^2T$. The best current estimate for the $O(1)$ correction comes from using the result of this paper to correct previous results of Hu, Müller, and myself, which effectively were using the wrong value for the log. The result at the physical values for $m_D$ and $g^2$ is $\Gamma \simeq (20 \pm 3)\alpha_W^5 T^4$, with only statistical errors quoted. Unfortunately the correction procedure is not well under control; I estimate that the systematic errors should be taken to be at least twice as large as the statistical ones. There are also corrections to the $g \ll 1$ approximation, probably at around the 10% level.

Besides making it possible to correct previous results, the leading log approximation and Bödeker’s effective theory provide a clean test-bed for determining how good an approximation it is to treat the symmetric phase of Yang-Mills Higgs theory using just Yang-Mills theory. Checking how much difference the Higgs field makes is an interesting project for the future. It would also be interesting to study the dependence of $\kappa'$ on the number of colors, since the sphaleron rate in the strong sector is also phenomenologically interesting for baryogenesis.

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A $O(a)$ lattice time scale renormalization

In this appendix I compute at $O(a)$ the lattice to continuum correction to the Langevin time scale, and the relation between the continuum Langevin time scale and the amount of heat bath applied. The calculation of the Langevin time scales will be quite technical and will depend to some extent on previous work relating the lattice and continuum theories at the thermodynamic level, found in [29]. I will relate the heat bath depth to the continuum Langevin time scale by making a nonperturbative lattice measurement comparing it to the lattice Langevin time scale, and then using the analytic relation between this and the continuum Langevin time scale. It would also be possible in principle to make the connection directly, but the analysis is difficult and I am lazy.
A.1 lattice and continuum Langevin time scales

Reference [29] explains how the lattice to continuum relation for thermodynamical properties of 3-D gauge or gauge-Higgs theory can be studied as an expansion in $g^2 a T$, where $a$ is the lattice spacing and $g^2$ is the coupling in 4-D notation. The combination $g^2 T$ is the coupling constant of the 3-D theory. Since it is dimensionful, a perturbative matching between continuum and lattice theories must be an expansion in $g^2 a T$ on dimensional grounds.

In Yang-Mills Higgs theory with fundamental or adjoint scalar fields the leading terms can behave as $1/a$ because the theory contains dimension 2 operators, but in strict Yang-Mills theory the lowest order operator is dimension 4 and the leading corrections are $O(a)$. Further, the only $O(a)$ correction is a rescaling of the coupling, equivalent to a rescaling of the physical length scale, and it only arises from one loop diagrams, and has been computed [29].

Arnold, Son, and Yaffe have recently demonstrated that, because of super-renormalizability and general considerations worked out by Zinn-Justin and Zwanziger [31], the same applies to the dynamics under Langevin dynamics [19]. In particular, any high dimension corrections which can appear in Eq. (15) would change the dynamics at $O(a^2)$, simply because there are no gauge invariant, P even dimension 5 operators in Yang-Mills theory. The only possible $O(a)$ correction, besides the thermodynamic one already mentioned, is a rescaling of the Langevin time scale, which will arise exclusively at one loop. This opens the possibility of computing the $O(g^2 a T)$ corrections between the continuum and lattice Langevin dynamics, by performing a one loop match. It also means the Langevin dynamics can be replaced by heat bath dynamics at the cost of another $O(g^2 a T)$ shift, which I will measure, rather than compute, in the next subsection.

In the continuum theory and in temporal gauge the Langevin evolution is

$$ \frac{dA_i^a(x)}{d\tau} = -\frac{\partial H}{\partial A_i^a(x)} + \xi_i^a(x), \quad \langle \xi_i^a(x, \tau) \xi_j^b(y, \tau') \rangle = 2T \delta_{ij} \delta_{ab} \delta^3(x-y) \delta(\tau-\tau'). \quad (42) $$

The Langevin equation for the lattice theory must be written in terms of the unitary parallel transporter matrices $U_i(x)$ and their derivatives. By definition the matrix $U_i(x) \in SU(N)$ is the matrix such that, if $\Phi$ is a fundamental representation object transforming as an object at point $x + a \hat{i}$, $U_i(x) \Phi$ is the parallel transport to point $x$; thus $U_i(x)$ “lives on” the link between the site $x$ and the site $x + a \hat{i}$. Writing $D_L^a$ for the left acting derivative,

$$ D_L^a U = ig a T^a U, \quad D_L^a F(U) = F(U \to U + D_L^a U) - F(U), \quad (43) $$

the Langevin equation for $U$ is

$$ \frac{dU_i(x)}{d\tau} = -D_L^a U_i(x) (\beta_L T D_L^a H_{KS} + \xi_i^a(x)), \quad \langle \xi_i^a(x) \xi_j^b(y) \rangle = \frac{8 \delta_{ij} \delta_{ab} \delta_{xy}}{g^2 a^4 \beta_L} \delta(\tau-\tau'). \quad (44) $$

Here $T^a$ is a fundamental representation Lie algebra generator with the standard normalization; $H_{KS}$ is the Kogut Susskind Hamiltonian, the sum over elementary plaquettes of the trace of the plaquette,

$$ H_{KS} = \sum_\square 1 - \frac{1}{2} \text{Tr} U_\square; \quad (45) $$
and $\beta_L$ is the inverse temperature in lattice units. At leading perturbative order $\beta_L = 4/(g^2 aT)$, but it receives radiative corrections, computed in [29], which shift it by a constant. What I write here as $\beta_L$ is $\beta_{\text{naive}}$ in the notation of [29], but in the body of the paper I have always used the $O(a)$ improved definitions when I refer to $a$ or $\beta_L$. The combination $\beta_L H_{\text{KS}}$ equals $H/T$ of the continuum theory, up to radiative corrections and high dimension operators which correct infrared physics at $O(a^2)$. The radiative corrections are absorbed up to errors which are $O(a^2)$ by the one loop radiative correction to $\beta_L$, which I will present in due course.

Our task is to compute to $O(a)$ the relation between the Langevin time scale $\tau$ in the continuum and lattice cases. The correction at loop order $l$ will be $O(g^2 a T^l)$, with the $g^2 T$ from loop counting and the $a$ to balance dimensions; so we need only go to one loop order. Part of the correction is from the shift in $\beta$ already mentioned, but there are also corrections from the relation between $U$ and $A$ and from radiative differences between the lattice $D^a$ and the continuum derivative. I will be satisfied to perform the calculation in a particular gauge, strict Coulomb gauge, without checking for the gauge independence of the result. By strict Coulomb gauge I mean that at every Langevin time the 3-D configuration is in 3-D Landau gauge. I should also fix a global time dependent gauge ambiguity; but this is irrelevant at the level of perturbation theory.

In Coulomb gauge the $U$ matrices are all close to the identity matrix, with a departure of order $\beta_L^{-1/2}$; so it makes sense to define a lattice gauge field $A^a_i$ through

$$U_i(x) \equiv \exp(i g a T a^a_i).$$

(46)

the $A$ field so defined has the same normalization as the continuum $A$ field at tree level, but there are radiative corrections. The value of the radiative correction to $A$ is obtained in the theory with an added scalar field by matching the one loop values of the gauge-scalar three point vertex at small transfer momentum. There is a contribution from one loop vertex corrections and from one loop scalar wave function corrections, and neither depends on the number of scalar particles in the theory, so the answer is the same in the pure gauge theory. The one loop correction was computed as a byproduct in [29] for SU(2). The result, in Coulomb gauge but for SU($N$) gauge theory, is

$$A^a_i(\text{continuum}) = \left[1 + \left(\frac{N g^2 a T}{4}\right) \left(\frac{1}{18} \frac{\Sigma}{4\pi} + \frac{3}{4\pi} \xi \right)\right] A^a_i(\text{lattice}).$$

(47)

The numerical constants $\Sigma$ and $\xi$ were first defined in [32], and their numerical values are $\Sigma = 3.175911536$ and $\xi = 0.152859325$.

Next I need to find the relation between applying the lattice and continuum derivatives. For this purpose it is actually more convenient to write the update rule in terms of center acting derivatives,

$$D^a_i U = U^{1/2}(i g a T a^a_i) U^{1/2},$$

(48)

because the formulation will then be parity symmetric. Naively one would expect that if we set $dU_i/d\tau = E^a_i D^a_i U$, that $dA^a_i/d\tau = E^a_i$, and indeed this is correct at leading order in $A$. But beyond leading order there are corrections; expanding both sides gives

$$\frac{dU_i}{d\tau} = \frac{d}{d\tau} \exp(i g a T a^a_i) = i g a T a^a_i \frac{dA^a_i}{d\tau} - \frac{g^2 a^2 T^b T^c}{2} \left( A^a_i \frac{dA^b_i}{d\tau} + (b \leftrightarrow c)\right)$$

26
The above correction is strictly a lattice effect and the equivalent continuum relation is same coordinate position. To save some writing I left off marking that all $A, E$ above are in position space and at the same coordinate position. The above correction is strictly a lattice effect and the equivalent continuum relation is $dA^a_i/d\tau = E^a_i$.

Naively I should now equate $E^a_i$ with $-\beta_L D^a_C H_{KS} + \xi$, evaluate the (UV dominated) mean value of the correction between the lattice and continuum relations, and thereby determine the rescaling of the Langevin time. This would not be strictly correct, though, since the Langevin equation, Eq. (14), gives time evolution in temporal gauge. The temporal gauge evolution breaks the Coulomb gauge condition, and if we are to do the calculation in Coulomb gauge we must make a time dependent gauge change to maintain the Coulomb condition at all Langevin times. There is some danger that the gauge changing will also lead to a radiative correction to the time relation; it turns out it does not, but I will go through the calculation anyway to show that it does not, since the cancellations may be special to strict Coulomb gauge.

The Coulomb gauge condition is $D_L \cdot A^a(x) = 0$, which means

$$\sum_i A^a_i(x) - A^a_i(x - a\hat{t}) = 0.$$  

This does not look cubic invariant because our labeling associates $U_i$, and hence $A_i$, on the link between $x$ and $x + a\hat{t}$ with the site $x$; it might better be associated with the point $x + a\hat{t}/2$, which would make the cubic invariance more obvious.

To maintain this Coulomb condition, we must apply a time dependent gauge transformation at each site. The difference between Coulomb gauge and temporal gauge satisfying the Coulomb gauge condition at $\tau = 0$ will be a gauge transformation by $\Lambda = \mathcal{T} \exp \int igT^a G^a(\tau) d\tau$, where the $\mathcal{T}$ means the exponential should be time ordered with respect to $\tau$. The value of $G$ is fixed by the requirement that the Coulomb condition remain true; we must choose $G$ so the departure from the gauge condition due to evolution of the fields and due to the action of $G$ cancel. In an infinitesimal time interval $d\tau$ the gauge change alters $U$ through

$$\delta U_i(x) = (1 - igT^a G^a(x)\delta\tau)U_i(x)(1 + igT^b G^b(x + a\hat{t})\delta\tau) - U_i(x).$$

Expanding $U$ on each side, the $G$ contribution to the time evolution of $A$ is

$$\frac{dA^a_i}{d\tau} \text{(from $G$)} = \frac{G^a(x + a\hat{t}) - G^a(x)}{a} \left( \delta_{ad} - \frac{g^2 a^2}{24} f_{abc} f_{cde} A^b_i A^e_i + O(a^3) \right).$$
\[-g f_{\alpha\beta\gamma} A^\alpha_i(x) \frac{G^\gamma(x + \hat{a} \hat{i}) + G^\gamma(x)}{2}. \tag{54}\]

Now we need to substitute this expression, and the relation between \(E\) and \(dA/d\tau\), into Eq. (52) to determine the relation between \(E\) and \(G\). The result, Fourier transformed to momentum space, is

\[
0 = i\tilde{k}_i E^\alpha_i(k) - \frac{ig^2 a^2}{24} \tilde{k}_i f_{\alpha\beta\gamma} \int_{lm} A^\beta_i(l) A^\gamma_i(m) E^d_i(k - l - m) - \\
-\tilde{k}^2 G(k) - ig f_{\alpha\beta\gamma} \tilde{k}_i \int_l \cos(\alpha_i/2) A^\beta_i(k - l) G^d_i(l) - \\
-\frac{g^2 a^2}{12} \tilde{k}_i \tilde{k}_j f_{\alpha\beta\gamma} \int_{lm} A^\beta_i(l) A^\gamma_i(m) E^d_i(k - l - m) G^d_i(m). \tag{55}\]

Here summation over vector indices is implied in terms where the index appears at least 2 times. I have used the conventional shorthand \(\tilde{k}_i = (2/a) \sin(k_i a/2), \tilde{k}^2 = \sum_i \tilde{k}_i^2,\) and \(f_i = \int d^3l/(2\pi)^3,\) where the range of the integration is \([-\pi/a, \pi/a]^3\). Indices are summed whenever the index appears an even number of times, which can be more than twice in cubic symmetric but rotationally nonsymmetric expressions; but if an index appears an even number of times on each side of a + or − sign I am re-using it, it is an independent index in each expression. To get the continuum version of this expression, drop the \(O(a^2)\) terms and set \(\tilde{k} = k, \cos(ka/2) = 1.\)

We can determine \(G\) perturbatively in \(g\) by expanding Eq. (55) in powers of \(g\). The result to \(O(g^2)\) is

\[
G^\alpha(k) = \frac{i \tilde{k}_i E^\alpha_i}{k^2} + g f_{\alpha\beta\gamma} \int_l \frac{\tilde{k}_i \tilde{l}_j}{k^2 l^2} \cos(\tilde{l}_j a/2) A^\beta_i(k - l) E^\gamma_j(l) - \\
-ig^2 f_{\alpha\beta\gamma} \int_{lm} \frac{\tilde{k}_i \tilde{l}_j \tilde{m}_k}{k^2 l^2 m^2} \cos(\tilde{l}_j a/2) \cos(\tilde{m}_k a/2) A^\beta_i(k - l) A^\gamma_i(l - m) E^d_i(m) - \\
-\frac{ig^2 a^2 \tilde{k}_i}{24 k^2} f_{\alpha\beta\gamma} \int_{lm} A^\beta_i(l) A^\gamma_i(m) E^d_i(k - l - m) - \\
-\frac{ig^2 a^2}{12} f_{\alpha\beta\gamma} \int_{lm} \frac{\tilde{k}_i \tilde{k}_j \tilde{m}_k}{k^2 m^2} A^\beta_i(l) A^\gamma_i(k - l - m) E^d_i(m). \tag{56}\]

This in turn must be substituted into Eq. (54) to find the true value of \(dA/d\tau\) in Coulomb gauge,

\[
\frac{dA^\alpha_i(k)}{d\tau} = \left( \delta_{ij} - \frac{\tilde{k}_i \tilde{k}_j}{k^2} \right) E^\alpha_j + ig f_{\alpha\beta\gamma} \int_l \frac{l_k}{l^2} \left( \delta_{ij} - \frac{\tilde{k}_i \tilde{l}_j}{k^2} \cos(l_j a/2) \right) A^\beta_j(k - l) E^\gamma_k(l) + \\
+g^2 a^2 f_{\alpha\beta\gamma} \int_{lm} \left[ -\frac{1}{24} \left( \delta_{ij} - \frac{\tilde{k}_i \tilde{k}_j}{k^2} \right) A^\beta_j(l) A^\gamma_i(m) E^d_j(k - l - m) - \\
-\frac{1}{12} \left( \tilde{m}_k \tilde{m}_l \delta_{ij} - \frac{\tilde{k}_i \tilde{k}_j \tilde{m}_k}{k^2 m^2} \right) A^\beta_j(k - l - m) A^\gamma_i(l) E^d_k(m) - \\
-\frac{l_k \tilde{m}_l}{l^2 \tilde{m}^2} \left( \delta_{ij} - \frac{\tilde{k}_i \tilde{l}_j}{k^2} \right) \cos(l_j a/2) \cos(m_k a/2) \times \\
A^\beta_j(k - l) A^\gamma_i(l - m) E^d_i(m) \right]. \tag{57}\]
Below I will be interested in the case where $E$ is uncorrelated with $A$, in which case the mean value of the term linear in $A$ vanishes and we can substitute the leading order Landau gauge $A$ field correlator,

$$
\langle A_a^i(k) A_j^b(l) \rangle = \delta_{ab} \delta(k + l) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) (k^2)^{-1},
$$

into the remaining terms. The contributions from the terms with $(1/12)$ and $(1)$ in front vanish, only the term with the leading $(1/24)$ coefficient contributes. It requires that we perform the integral

$$
a^2 \int \frac{1}{l^2} = a \frac{\Sigma}{4\pi},
$$

which is the definition of the constant $\Sigma$ which appeared earlier. We also need the integral

$$
a^2 \int \frac{\tilde{l}_1^2}{(\tilde{l}^2)^2} = a \frac{\Sigma}{34\pi}.
$$

The result is

$$
\frac{dA^a_i(k)}{d\tau} \text{(lattice)} = \left( \delta_{ij} - \frac{\tilde{k}_i \tilde{k}_j}{\tilde{k}^2} \right) \left[ 1 + \frac{1}{9} \frac{Ng^2 aT}{4} \frac{\Sigma}{4\pi} \right] E^a_j(k),
$$

valid when $E$ is uncorrelated with gauge fields. The first factor projects out the transverse component of $E$ and is responsible for maintaining Coulomb gauge. The continuum expression is the same but with $a$ set to zero.

Using the previously established relation between lattice and continuum gauge field normalization, the relation for the continuum normalized gauge field is

$$
\frac{dA^a_i(k)}{d\tau} \text{(lattice)} = \left[ 1 + \frac{Ng^2 aT}{4} \left( \frac{1}{6} \frac{\Sigma}{4\pi} + 3 \frac{\xi}{4\pi} \right) \right] \left( \delta_{ij} - \frac{\tilde{k}_i \tilde{k}_j}{\tilde{k}^2} \right) E^a_j(k),
$$

valid for the gauge field response to the random force part of the Langevin equation. I will not attempt to study the response of $A$ from the Hamiltonian gradient part of the Langevin equation, since it would involve understanding the radiative corrections to $D^a H$ and would lead to an $E$ which might be correlated with $A$ fields. Since we know that the Langevin equation correctly thermalizes the lattice system when we use the radiatively corrected value of $\beta_L$, it is sufficient to study the response to the random force alone to determine the rescaling of the Langevin time scale.

Note that the correction appearing in Eq. (62) is precisely one quarter of the radiative wave function correction for an adjoint scalar field in 3-D lattice gauge theory when the scalar self-coupling vanishes, see [29]. This is perhaps not too surprising. In the real time theory, the time evolution is generated by the electric fields, which appear in the thermodynamics as the $A_0$ field, an adjoint scalar with zero self-coupling. In [14] Turok and I speculated incorrectly that the time scale correction would not contain any large tadpole corrections; but Eq. (62) is one quarter the adjoint scalar wave function correction, which does contain tadpoles. Our incorrect guess was based on analyzing the abelian theory, where the $A_0$ field
turns out to be a free field. The correction found here vanishes in the abelian case, for compact or noncompact implementations.

Now I will finish the relation between time scales. I have just shown that the response of the $A$ field to the random force, shifting its normalization to correspond to the continuum theory normalization, is

$$\frac{dA^a_i(x)}{d\tau(\text{latt})} = (1 + \text{corr})\xi,$$

where $(1 + \text{corr})$ is the quantity in brackets in Eq. (62). But the autocorrelator of $\xi$ is not the same as it would be in the continuum, because of the radiative corrections to $\beta_L$. The mean square change in $A^a_i$ over a Langevin time $\delta\tau(\text{latt})$ is, using Eq. (44),

$$\langle (\delta A^a_i(x))^2 \rangle = (1 + \text{corr})^2 \frac{2T\delta\tau(\text{latt})}{a^3} \frac{4}{g^2aT\beta_L}.$$  (64)

However, $\beta_L = 4/g^2aT$ only at leading order in $\beta_L$. Beyond leading order, in the pure gauge theory, it is \cite{29}

$$\frac{\beta_L g^2 a T}{4} - 1 \equiv Z_g^{-1} - 1 = \frac{g^2 a T}{4} \left( \frac{N^2 - 2}{3N} + \frac{37N}{3} \frac{\xi}{4\pi} \right).$$  (65)

The continuum theory Langevin equation would cause a mean square change to $A$ of

$$\langle (\delta A^a_i(x))^2 \rangle = \frac{2T\delta\tau(\text{contin})}{a^3},$$

so the relation between time scales is

$$\frac{\delta\tau(\text{contin})}{\delta\tau(\text{latt})} = (1 + \text{corr})^2 Z_g = 1 + \frac{g^2 a T}{4} \left( \frac{N}{3} \frac{\Sigma}{4\pi} - \frac{19}{3} \frac{\xi}{4\pi} - \frac{N^2 - 2}{3N} \right).$$  (67)

The numerical value of this expression for $N = 2$ is $1 - 0.3189(g^2aT/4)$.

I have now related the lattice and continuum Langevin time scales at $O(a)$. It is also easy to show that, for the Hamiltonian system, the correction between the lattice and continuum time scales is exactly half as large. However this is less useful in that case because, while this correction is technically correct for determining the time falloff of correlators over very short time scales, the IR dynamics on longer time scales receive HTL corrections which depend on the lattice spacing as $1/a + O(1)$. For the technique of Hu and Müller, HTL’s are included by adding “particle” degrees of freedom. In \cite{13} we work out the correction for time scales in the limit $m_D^2 \gg g^2T/a$, in an approximation which corresponds to setting $(1 + \text{corr}) = 1$.

The correction found here changes our result there from being $t_{\text{latt}}/t_{\text{contin}} = Z_g^{-2}$ to being $Z_g^{-2}(1 + \text{corr})^{-2}$. I used this correction in Section \[3\].

### A.2 lattice Langevin time and depth of heat bath

The effect of the heat bath algorithm on the infrared degrees of freedom is equivalent to Langevin evolution. I will show this at tree level, which gives a tree relation between Langevin time and the number of heat bath updates applied. The only $O(a)$ correction to this relation
possible is an $O(a)$ shift in time scales between the algorithms; to find the magnitude of the shift I make a direct numerical measurement. I do not attempt a perturbative calculation of the relation between continuum Langevin time and amount of heat bath applied, beyond leading order.

First I will show that the effect of the heat bath algorithm on the IR degrees of freedom is equivalent to the Langevin algorithm, and I find the relation between the Langevin time scale to the number of links updated by heat bath, at leading order. The way the heat bath algorithm works is as follows:

1. pick a link on the lattice at random.
2. replace the connection $U$ on the link with the one which minimizes the Hamiltonian.
3. multiply this link by a random SU(2) element chosen from a distribution centered on the identity, with a weight function dependent only on the arc length from the identity, not the direction. The weight is chosen to correctly reproduce the thermal ensemble on this link holding others fixed; it is approximately but not exactly Gaussian.

This is a heat bath update; for a more precise description see [33]. The first part, the quench of the link, serves to perform the $-\partial H / \partial A$ part of the Langevin update, and the multiplication by a random SU(2) element reproduces the noise part of the Langevin update.

To see the relation between the heat bath and the Langevin update at leading order it is sufficient to consider the linearized theory, that is, to expand the Hamiltonian to quadratic order in the gauge field $A$. On an $N \times N \times N$ toroidal lattice the relation between the connections $U$, the gauge field $A_i(x)$, and the Fourier transform of the gauge field $A^a(k, s)$ ($s$ a polarization index) is (writing all Lorentz and group index sums explicitly, there is no implicit summation convention in what follows)

$$
U_i(x) = \exp \left( \sum_a i g T^a A^a_i(x) \right) \simeq 1 + \sum_a i g T^a A^a_i(x) - \frac{g^2 a^2}{8} \sum_a A^a_i(x) A^a_i(x), \quad (68)
$$

$$
A^a(k, s) = N^{-3/2} \sum_{x,i} \epsilon_i(s, k) A^a_i(x) \exp \left( -i k \cdot (x + a \hat{i}/2) \right), \quad (69)
$$

$$
A^a_i(x) = \text{(longitudinal piece)} + N^{-3/2} \sum_{k,s} \epsilon_i(k, s) A^a_i(k, s) \exp \left( i k \cdot (x + a \hat{i}/2) \right), \quad (70)
$$

where $\epsilon_i(s, k)$ is a transverse polarization vector, satisfying

$$
\sum_i \epsilon_i(s, k) \epsilon_i(s', k) = \delta_{s,s'}, \quad (71)
$$

and

$$
\sum_i \epsilon_i(s, k) \tilde{k}_i = 0; \quad (72)
$$

there are two such states for each $k$. The sum over $k$ includes all $k$ of form $(2\pi/aN)\vec{n}$, with $\vec{n}$ a triple of integers each in the range $0 \leq n_i < N$. Only the transverse fields are of interest here, at the order we are working the longitudinal part is pure gauge. Its behavior depends on our choice of gauge fixing, and has no influence on the transverse fields (or on physics).
Now we should compute the size of $\xi_k$, s

The (quadratic order, the noise is Gaussian, of amplitude set by the size of the quadratic in equation.

which I will call the “cross-talk” term, are responsible for the noise term in the Langevin Eqs. (69) and (76), we find

$$A_{k,s} = \frac{T}{k^2}$$ (73)

Now let us analyze how the fields change when the heat bath algorithm is applied to a link $(x, i)$. The terms in the Hamiltonian containing the link $x, i$ are

$$H = \frac{1}{a^2 T} \sum_{a,j \neq i} \left[ \frac{1}{2} \left( A_j^a(x) + A_j^a(x + a\hat{j}) - A_j^a(x + a\hat{i}) - \xi_j^a \right)^2 + \frac{1}{2} \left( -A_j^a(x - a\hat{j}) + A_j^a(x - a\hat{i}) + A_j^a(x - a\hat{i} + \hat{j}) - A_j^a(x) \right)^2 \right]$$ (74)

and “quench” part of the heat bath algorithm will replace $A_j^a(x)$ with the value which minimizes this expression,

$$A_j^a(x, \text{after}) = \frac{1}{4} \sum_{j \neq i} \left( A_j^a(x) + A_j^a(x + a\hat{j}) - A_j^a(x + a\hat{i}) - A_j^a(x - a\hat{i} + \hat{j}) \right)$$ (75)

Using Eq. (71) and Eq. (72), and adding a term $\xi^a$ to represent the noise which will be added, this is

$$A_j^a(x)(\text{after}) = A_j^a(x, \text{before}) + \xi^a - \sum_{k,s} a^2 T \frac{\hat{k}^2}{N^3/2} \epsilon_i(s, k) A^a(k, s) \exp(i k \cdot (x + i a/2))$$ (76)

Now we should compute the size of $\xi^a$. Because the Hamiltonian is expanded only to quadratic order, the noise is Gaussian, of amplitude set by the size of the quadratic in $A_i(x)$ term in $H$, which from Eq. (74) is $(2/a^2 T) \sum_a A_i^a(x) A_i^a(x)$. The amplitude of the noise $\xi^a$ is then (no sum on $a$) $\langle \xi^a \xi^a \rangle = a^2 T/4$.

Next we will see what impact this update has had on the Fourier mode $A(k, s)$. Combining Eqs. (75) and (76), we find

$$A^a(k, s, \text{after}) = A^a(k, s, \text{before}) \left( 1 - \frac{\epsilon_i^2(s, k) a^2 T}{4 N^3} \right) + \frac{\epsilon_i(s, k)}{N^{3/2}} \exp(-i k \cdot (x + \hat{i}a/2)) \xi^a - \sum_{(s', l) \neq (s, k)} a^2 T \frac{\hat{k}^2 \epsilon_i(s, k) \epsilon_i(s', l)}{4 N^3} A^a(l, s', \text{before}) \exp(i (k - l) \cdot (x + a\hat{i}/2))$$ (77)

The $(k, s)$ term in the sum is removed and included instead in the first term. It is responsible for the damping term in the Langevin equation. Both the noise term $\xi$ and the final term, which I will call the “cross-talk” term, are responsible for the noise term in the Langevin equation.

To measure the magnitude of the Langevin damping term, we must compute the correlator $\langle A^a(k, s, \text{before}) A(k, s, \text{after}) \rangle$. Because $\langle A(k, s) A(l, s') \rangle = 0 = \langle A(l, s') \xi \rangle$, we get

$$\langle A^a(k, s, \text{before}) A(k, s, \text{after}) \rangle = \langle A(k, s, \text{before})^2 \rangle \left( 1 - \frac{a^2 T \epsilon_i^2(s, k)}{4 N^3} \right)$$ (78)
It is also important to make sure that the mean square value of $A^a(k, s)$ is unchanged by the update, which is the requirement that the noise have the right amplitude. Here we get a little surprise; squaring Eq. (77),

$$
\langle (A^a(k, s, \text{after}))^2 \rangle = \left(1 - \frac{a^2 k^2 \epsilon_i^2(s, k)}{4N^3}\right)^2 \langle (A^a(k, s, \text{before}))^2 \rangle + \frac{\epsilon_i^2(s, k)}{N^3} \langle (\xi^a)^2 \rangle
$$

$$
+ \frac{1}{N^6} \sum_{(l, s') \neq (k, s)} \langle A^a(l, s') A^a(l, s') \rangle \epsilon_i^2(s, k) \epsilon_j^2(s', l) \left(\frac{a^2 l^2}{4}\right)^2.
$$

(79)

Using Eq. (73) and taking $N^3 \gg 1$, this becomes

$$
\langle (A^a(k, s, \text{after}))^2 \rangle = \langle (A^a(k, s, \text{before}))^2 \rangle + \frac{a^2 T \epsilon_i^2(s, k)}{N^3} \left(-\frac{1}{2} + \frac{1}{4} + \frac{1}{16N^3} \sum_{l, s'} \epsilon_i^2(s', l) a^2 l^2\right).
$$

(80)

At leading order in large $N$, the sum gives $4N^3$. Therefore the mean size of $A^a(k, s)$ is unchanged, which means that we have the correct amount of noise. We see that fully half of the noise actually arises from “cross-talk” between the mode of interest and extremely UV modes, with the other half arising from the noise explicitly appearing in the algorithm.

To be Langevin the noise must have zero unequal time correlation. This is the case for $\xi$ by explicit construction, but we need to check it for the “cross-talk” noise. For the heat bath algorithm to act like a Langevin algorithm, the “cross-talk” contribution to $A^a(k, s)$ from updating the $(x, i)$ link must be independent of that from the $(y, j)$ link, at least on averaging over the choice of $(y, j)$ (which is indeed chosen randomly in the algorithm I use). This is the case; from Eq. (77), the correlation between the “cross-talks” is

$$
\sum_{(l, s') \neq (k, s)} \sum_{(m, s'') \neq (k, s)} \left(\frac{a^4 l^2 \tilde{m}^2}{16N^6}\right) \epsilon_i(s, k) \epsilon_j(s, k) \epsilon_i(s', l) \epsilon_j(s'', m)
$$

$$
\times \langle A^a(l, s') A^a(m, s'') \rangle \exp \left(ik \cdot (x - y + a(\hat{i} - \hat{j})/2)\right)
$$

$$
\times \exp \left(-il \cdot (x + a\hat{i}/2) + im \cdot (y + a\hat{j}/2)\right)
$$

$$
= \sum_{(l, s') \neq (k, s)} \frac{a^4 l^2}{16N^6} \epsilon_i(s, k) \epsilon_j(s, k) \epsilon_i(s', l) \epsilon_j(s', l) \exp \left(i(k - l) \cdot (x - y + a(\hat{i} - \hat{j})/2)\right),
$$

(81)

which suffers from a rapidly oscillating phase. The expression is typically smaller in magnitude by $N^{-3/2}$ compared to the corresponding term in Eq. (79) and its average over $(y, j)$ is strictly zero. Hence there is no unequal time correlation in the “cross-talk” part of the noise. Note also that the cross-talk is very strongly UV dominated, which means that there will be no hidden correlations in the IR effective evolution because of it, at least at $O(a)$ and probably higher; it is also fortunate because the UV is most quickly randomized.

What we have shown is that applying the heat bath update is equivalent to damping the $A$ fields and applying noise. In particular, its influence on the IR degrees of freedom is equivalent to that of Langevin dynamics. Applying the heat bath algorithm to many links in succession, the rate at which a mode is damped (and the amount of noise it receives) is given
by Eq. (78) (and Eq. (80)) after averaging over the direction $i$ (since each direction is bathed with equal frequency). Using $\sum_i \epsilon_i^2(s, k) = 1$, we find that it takes $12N^3$ heat bath updates to perform the equivalent of $a^2$ of Langevin update. This gives the tree relation between the algorithms. We can then define a heat bath time in terms of the number of links we have updated, $\tau(\text{heatbath}) = a^2(\text{number of links updated})/(12N^3)$. At leading order in a weak field expansion this is the same as Langevin time but we expect subleading corrections.

Now we must push the analysis beyond tree level. Since the influence of the heat bath algorithm on the IR degrees of freedom (in fact, all degrees of freedom) “looks like” Langevin dynamics, the analysis of [31] applies; up to high dimension operator corrections, which by power counting appear first at $O(a^2)$, the algorithms are related, in the presence of interactions, by a rescaling of all parameters. Since each algorithm gives correct thermodynamic behavior (after the $O(a)$ correction already discussed is applied), the only remaining correction would be a rescaling of the time scale, which must be at worst $O(a)$ since it vanishes as $g^2 \rightarrow 0$ (in which limit the calculation just presented is exact), and on dimensional grounds any $O(g^2)$ correction must be $O(g^2a T)$.

We could in principle determine this $O(a)$ correction by an analytic computation, extending the one just presented to second order in $g$. Instead I compute the subleading effects by the following strategy. I choose some infrared measurable $\mathcal{O}$, and measure it at each lattice point at a closely spaced series of Langevin times. I do the same using the heat bath algorithm. Then I compare the unequal time correlator or autocorrelator $C(\tau - \tau') = \langle \mathcal{O}(x, \tau)\mathcal{O}(x, \tau') \rangle$, where the average is over the ensemble of Langevin trajectories, or in practice over coordinates and times in a single very long Langevin trajectory. To match the time scales, we see what rescaling of the heat bath time scale is needed to make the autocorrelator match the autocorrelator for the Langevin case. Any IR measurable will do because we know that the algorithms both behave as Langevin algorithms on the IR degrees of freedom, so the only $O(a)$ difference would be a time rescaling which will be of the same amplitude for any unequal time observable.

I should explain the reason this is worth doing at all is that, first, the heat bath algorithm is much faster and does not suffer from step size errors like the Langevin algorithm, and second, there are infrared measurables other than the topological density, for which the autocorrelation statistics improve much more quickly. If the latter were not true we would spend as much computer time making the match between techniques as it would take to do the measurement of $\Gamma$ by the Langevin method.

The measurable I choose is a fundamental representation Wilson loop after some amount of cooling, specifically a $4 \times 4$ square Wilson loop after $\tau = 3.125a^2$ of gradient flow cooling under $H_{KS}$. This is an infrared measurable because such a large Wilson loop samples mostly the infrared gauge fields, and because the cooling removes most of the UV fluctuations anyway.

Incidentally, it is not too hard to compute the leading order perturbative prediction for this quantity. The mean trace of an $l \times l$ Wilson loop in SU($N$) gauge theory after a length $\tau$ of gradient flow cooling, neglecting lattice artifacts, is

$$N - \text{Tr}U_{\text{ext}} = (N^2 - 1)\frac{g^2T}{4} \int \frac{d^3k}{(2\pi)^3} \frac{16\sin^2(k_x l/2)\sin^2(k_y l/2)}{k_x^2k_y^2} \frac{k_x^2 + k_y^2}{k^2} \exp(-2k^2\tau) + O(g^4),$$

(82)
In the $\tau \to 0$ limit the integral has logarithmic UV divergences but for finite $\tau$ it has a well defined value and is dominated by the infrared, $k \lesssim \tau^{-1/2}$. Of course, for the Wilson loops under consideration here, perturbation theory will be unreliable because the length scales involved are close to the scale where perturbation theory breaks down completely. It might be interesting to see whether the infrared fields are stronger or weaker than at leading order in perturbation theory, though.

I measured the same site, unequal time Wilson loop correlator by measuring each $4 \times 4$ Wilson loop in an even sublattice every $a^2/2$ of Langevin time, for a series of Langevin trajectories each about $200a^2$ long, with $50a^2$ Langevin time between trajectories to eliminate correlations between trajectories. For each trajectory I determined the autocorrelator averaged over volume and time. The autocorrelation function $C(\Delta \tau)$ looks something like an exponential tail but is not well fit by one; there is some small amount of much longer time scale correlation caused by the slow evolution of the most infrared gauge fields. To compare the Langevin and heat bath time scales I averaged $C(\Delta \tau)$ over data sets for each update method and determined the rescaling of the Langevin time which minimized the difference between the results,

$$\chi^2 = \int_0^{\Delta \tau_{\text{max}}} (C_{\text{Langevin}}(\Delta \tau) - r_C C_{\text{heat}}(r_{\tau} \Delta \tau))^2 d\Delta \tau,$$

where I allow a rescaling both of the autocorrelation time and of the overall magnitude of $C$. I chose $\Delta \tau_{\text{max}}$ to be enough that the autocorrelator has fallen about $1 - e^{-1}$ of the way to its large $\Delta \tau$ limit, but the result is quite insensitive to the specific choice. The coefficient $r_{\tau}$ at the extremum of $\chi^2$ gives the rescaling of the Langevin time scale. The multiplicative rescaling of $C$ is necessary because there are small, very long time scale correlations in the measurable which can effectively shift one data set somewhat with respect to the other. As a check I compared the first half of the Langevin data I took with the second half. I find that $r_{\tau} = 1$ within a small tolerance the same size as the jackknife error bars, but $r_C$ differs from 1 by a few percent.

For $(4/g^2aT) = 6$ I ran each update procedure on a $24^3$ lattice for $\tau \simeq 9000a^2$. The rescaling of the time scales was

$$\Delta \tau_{\text{heat}} = 1.098 \pm 0.007 \Delta \tau_{\text{Langevin}},$$

with the error bar determined by the jackknife method.

The Langevin step size used here was $\Delta = 0.05$. The definition of $\Delta$ and the second order algorithm I used are in Section 4 in the body of the paper. To check for step size errors I evolved a trajectory for half as much Langevin time, using $\Delta = 0.025$. The rescaling between this trajectory and the heat bath was $1.099 \pm 0.010$ and the rescaling between Langevin evolutions with the two step sizes was $1.001 \pm 0.009$, so Langevin step size errors are negligible at $\Delta = 0.05$.

I also measured $N_{CS}$ during the $\Delta = 0.05$ Langevin trajectory; the ratio of the diffusion constants for the heat bath and Langevin algorithms was

$$\frac{\Gamma_{\text{heat}}}{\Gamma_{\text{Langevin}}}(\text{unrescaled}) = 1.114 \pm 0.058,$$
which is compatible with the measured difference in time scales, but with much larger error bars. The autocorrelator of the Wilson loop develops good statistics more quickly than the diffusion constant for $N_{CS}$.

Incidentally the mean value of the Wilson loop trace was $\langle 2 - \text{Tr} U_{4 \times 4} \rangle = .261$, while the leading order perturbative prediction is .096. The infrared of Yang-Mills theory has more excitation than leading order perturbation theory predicts, by quite a bit on the scale of a square Wilson loop $8/(3g^2T)$ on a side.

For $(4/g^2aT) = 10$ I used a $40^4$ lattice but only $\tau = 2000a^2$. The rescaling of the time scales was smaller as expected, 1.065 ± .016. Assuming the subleading correction to be purely $O(a)$, we would have guessed from the $\beta_L = 6$ result that the rescaling factor would be $1.059 \pm .004$, which is within error. The rescaling at both lattice spacings are within error of being $\tau_{\text{Langevin}}/\tau_{\text{heat}} = Z_g$, and I speculate that this is the correct analytic relation at $O(a)$.

For the finest lattice spacing, I have simply extrapolated the medium spacing data assuming a pure $O(a)$ form for the correction.

The deviation of the mean value of the Wilson loop measurable from perturbation theory is also smaller on the finer lattice; the value is .110, while the perturbative estimate is .058. This is also expected, since a $4 \times 4$ Wilson loop is smaller in physical units in this case, and perturbation theory works better as the length scale becomes smaller.

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