On Data-Driven Prescriptive Analytics with Side Information: A Regularized Nadaraya-Watson Approach

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Abstract

We consider generic stochastic optimization problems in the presence of side information which enables a more insightful decision. The side information constitutes observable exogenous covariates that alter the conditional probability distribution of the random problem parameters. A decision maker who adapts her decisions according to the observed side information solves an optimization problem where the objective function is specified by the conditional expectation of the random cost. If the joint probability distribution is unknown, then the conditional expectation can be approximated in a data-driven manner using the Nadaraya-Watson (NW) kernel regression. While the emerging approximation scheme has found successful applications in diverse decision problems under uncertainty, it is largely unknown whether the scheme can provide any reasonable out-of-sample performance guarantees. In this paper, we establish guarantees for the generic problems by leveraging techniques from moderate deviations theory. Our analysis motivates the use of a variance-based regularization scheme which, in general, leads to a non-convex optimization problem. We adopt ideas from distributionally robust optimization to obtain tractable formulations. We present numerical experiments for newsvendor and wind energy commitment problems to highlight the effectiveness of our regularization scheme.

Keywords: stochastic optimization; side information; Nadaraya-Watson estimator; moderate deviation principles; large deviation principles; distributionally robust optimization

1 Introduction

In the presence of uncertainty, decisions can often be improved by taking into account the side information, such as weather conditions, interest rates, exchange rates, past prices and demands, volatility indices, etc., that provides a more accurate description of the uncertain problem parameters. In the stochastic optimization setting, the side information corresponds to observable exogenous covariates \((\gamma_1, \ldots, \gamma_p)\) that may reshape the conditional probability distribution of the random problem parameters \((\xi_1, \ldots, \xi_q)\). A decision maker prescribed with full knowledge about the joint distribution of the random vectors \(\tilde{\gamma} := (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p)\) and \(\tilde{\xi} := (\tilde{\xi}_1, \ldots, \tilde{\xi}_q)\) endeavors to

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solve the following stochastic optimization problem with side information:

$$\min_{x \in \mathcal{X}} \left\{ \mathbb{E}_\gamma[\ell(x, \xi)] : = \mathbb{E}[\ell(x, \xi) \mid \tilde{\gamma} = \gamma] \right\}.$$  \hspace{1cm} (SO)

Here, the vector $x \in \mathbb{R}^d$ comprises all decision variables, while the objective function is specified through the conditional expectation of the random cost $\ell(x, \xi)$ given the side information $\gamma$.

For instance, in the context of portfolio optimization—which aims to maximize the expected portfolio return—the loss function is defined as $\ell(x, \xi) := -\xi^\top x$, where $\xi \in \mathbb{R}^q$ ($d = q$ in this case) and $x$ correspond respectively to the vectors of random asset returns and allocated investments. If short selling is prohibited, then the feasible set $\mathcal{X}$ of the allocation vector $x$ is described by the unit $q$-simplex. In this problem, the exogenous covariate vector $\tilde{\gamma}$ may additionally comprise the firms’ market capitalizations, book-to-market ratios, past returns, and also include other market indicators such as the volatility indices and financial news indicators \cite{Brandt, Bazier-Matte, Delage}. We now illustrate the importance of side information through the following example.

**Example 1 (A Three-Asset Portfolio).** Consider a stylized three-asset portfolio optimization problem, in which the decision maker allocates a total wealth of $\$1$. The return of asset $i$ is

$$\tilde{\xi}_i(\tilde{\gamma}) = \begin{cases} 
0.5 - \tilde{\gamma}^2 + 0.1 \cdot \tilde{\epsilon}_i & \forall i = 1, 2, \\
0 & i = 3,
\end{cases}$$

where the side information/covariate $\tilde{\gamma} \in \mathbb{R}$ is governed by a uniform distribution on the interval $[-1, 1]$. The random variables $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are assumed to be bivariate normally distributed with zero means and unit variances and are perfectly negatively correlated.

Under this setting, the unconditional expected returns of the risky assets (i.e., $i = 1, 2$) are equal to $\mathbb{E} [0.5 - \tilde{\gamma}^2 + 0.1 \cdot \tilde{\epsilon}_i] = 1/6$. Thus, in the absence of any side information, the optimal expected portfolio return is $1/6$, which can be obtained by allocating the entire wealth into any convex combination of the risky assets.

However, suppose that the value of the side information $\gamma$ is revealed before the decision is made. In this case, the conditional expected return of each risky asset is $\frac{1}{2} - \gamma^2$. Hence, when $\gamma^2 < 1/2$, it is optimal to allocate the entire wealth into any convex combination of the risky assets; otherwise, it is optimal to allocate the entire wealth into the risk free asset. Since $\tilde{\gamma}$ follows a uniform distribution on $[-1, 1]$, the optimal expected return of this strategy is given by

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{2} \left( \frac{1}{2} - \gamma^2 \right) d\gamma = \frac{2}{3} \left( \frac{1}{2} \right)^{3/2} = \frac{1}{3\sqrt{2}}.$$  \hspace{1cm} \text{(1)}

The above calculations show that the expected return deteriorates by $(1/(3\sqrt{2}) - 1/6)/(1/(3\sqrt{2})) \approx 29\%$ if the portfolio manager ignores the side information.

The example highlights the critical benefits of exploiting the side information in our decision making processes, when such information is available.
1.1 Literature Review

In the ideal case, solving (SO) exactly allows us to make optimal decisions with side information. However, in most situations of practical interest, the joint distribution of \((\hat{\gamma}, \hat{\xi})\) is unknown, and only past historical data \(\{(\gamma_1, \xi_1), \ldots, (\gamma_n, \xi_n)\}\) is available to infer the conditional distribution of \(\hat{\xi}\) and to estimate the conditional expectation in (SO). In recent years, there has been a focus on developing integrated learning and optimization frameworks to approximate the optimal solution for (SO) with statistical guarantees on their performances. Bertsimas and Kallus [2020] consider different machine learning approaches to construct empirical conditional expectations that well approximate the conditional expectation in (SO). They further establish that the resulting approximations are asymptotically consistent, meaning that the approximations converge to the true conditional expectation as the sample size grows. Bertsimas and McCord [2019] extend the result of Bertsimas and Kallus [2020] to the multistage setting under the assumption that covariates evolve according to a Markov process. The resulting data-driven decision is shown to be consistent and asymptotically optimal, and finite-sample guarantees are developed for k-nearest neighbors (KNN)-based approaches. Solutions to their proposed formulations, however, exhibit an optimistic bias if the sample size is small.

To mitigate this overfitting effect, Hanasusanto and Kuhn [2013] propose a robust version that minimizes a worst-case empirical conditional expectation in view of the most adverse weight vector that is close to the nominal one generated by the Nadaraya-Watson (NW) estimator. Bertsimas et al. [2019] incorporate side information into robust dynamic programming problems and establish that the solution is asymptotically optimal for multi-period stochastic programs. Bertsimas and Van Parys [2017] propose an alternative robust scheme whose solutions enjoy a limited disappointment on the bootstrap data. Esteban-Pérez and Morales [2020] construct a framework using trimmings of probability distributions, which they prove to be connected with the partial mass transportation problem and show that the approach naturally produces distributionally robust optimization (DRO) extensions of formulations with some nonparametric regression techniques.

There exist other powerful and interesting approaches that solve (SO) under more specific settings. For example, Sen and Deng [2018] and Ban et al. [2019] first consider regression models with additive residual terms to model and generate scenarios for \(\hat{\xi}\) given side information \(\gamma\). Inspired by their work and the sample average approximation scheme for classical stochastic optimization problems, Kannan et al. [2020a] propose a formulation based on a regression model that assumes \(\hat{\xi}\) to be modeled in terms of \(\hat{\gamma}\) as \(\hat{\xi} = f(\hat{\gamma}) + \hat{\epsilon}\), where \(f(\gamma) = \mathbb{E}[\xi | \hat{\gamma} = \gamma]\) and \(\hat{\epsilon}\) are mean zero errors. This formulation, however, relies on the crucial assumption that the distribution of the errors \(\hat{\epsilon}\) is independent of the covariates \(\hat{\gamma}\), which allows them to formulate the problem as a sample average approximation problem that assigns an equal weight of \(1/n\) to each observation. With the idea of obtaining better out-of-sample performances on problems with limited data, the authors incorporate their residual-based formulation into a DRO framework [Kannan et al., 2020b] and also consider extensions where they relax the homoscedasticity assumption on the residuals [Kannan et al., 2021]. In a similar spirit, Elmachtoub and Grigas [2021] propose a smart “Predict,
Building upon the above ideas, Sim et al. [2021] propose a robustness optimization counterpart for the robust satisficing framework. In this paper, we focus on the setting without assuming the regression models for $\tilde{\xi}$.

Despite the practical significance of the stochastic optimization problem ($SO$), there is an incomplete picture of the properties of the existing solution schemes. Although the NW approximation is shown to be asymptotically consistent [Bertsimas and Kallus, 2020], it is unknown whether the scheme could provide out-of-sample performance guarantees for solutions to the generic problems. An alternative method that optimizes over parametric decision rules, such as linear or quadratic functions in $\gamma$, can generate finite-sample performance bounds [Bertsimas and Kallus, 2020, Ban and Rudin, 2018, Bazier-Matte and Delage, 2020]. In Brandt et al. [2009], the portfolio optimization with side information model is solved in view of linear decision rules (LDR) where one seeks for the best linear policy in the exogenous covariates that maximizes the empirical return. An $\ell_2$-regularized version of the linear decision rule approximation is studied in Bazier-Matte and Delage [2020]. The decision rules scheme, however, is less attractive because it is not asymptotically consistent, meaning that we cannot produce results that would parallel those of sample-average approximation in the classical setting of stochastic optimization without side information [Kleywegt et al., 2002, Shapiro et al., 2009]. In Ban and Rudin [2018], the authors apply both the NW and decision rule approximations to the single-item newsvendor problem and derive finite-sample performance guarantees for the solutions. Unfortunately, the bound for the NW approximation inconveniently relies on an optimal solution to the corresponding linear decision rule problem. An alternative bound derived in Bertsimas and Van Parys [2017] holds only for the bootstrap data, that is generated via resampling from the empirical distribution. Although encouraging, their bound does not provide a complete understanding on its out-of-sample performance.

1.2 Our Contributions

This paper focuses on the approximation scheme using the popular NW kernel regression estimator [Nadaraya, 1964, Watson, 1964]. By leveraging techniques from large and moderate deviations theory, we derive for the first time out-of-sample performance guarantees for the empirical conditional expectation minimization model. Our result indicates that the out-of-sample errors of the approximation scale with $O(\sqrt{1/(nh^p)})$, where $h > 0$ is the bandwidth parameter that is used for the kernel function in our proposed model. In contrast to the result in Ban and Rudin [2018] for a single-item newsvendor problem, our guarantees hold independently of optimal solutions to the corresponding linear decision rule problems and conform with the best bandwidth parameter scaling $h = O(1/n^{1/(p+4)})$ suggested in the literature. As a byproduct of our new theoretical result, we identify a suitable regularization term in empirical conditional standard deviation. If this term is small, then our guarantees imply that the out-of-sample errors are of the lower rate $\sim O(1/(nh^p))$. Thus, the regularization term will encourage an optimal solution that yields small generalization errors. We devise a solution scheme for this variance regularized formulation based on a distri-
butionally robust optimization (DRO) problem. Numerical results in the context of newsvendor and wind energy commitment problems demonstrate the superiority of our new regularized NW approximation over the linear decision rule scheme and a state-of-the-art DRO framework proposed by Kannan et al. [2020].

We summarize below the main contributions of the paper:

1. Leveraging techniques from large and moderate deviations theory, we derive generalization bounds for the NW estimator. Unfortunately, typical for settings where kernel functions are used, the bound suffers from the curse of dimensionality, which becomes prominent when the side information vector $\gamma$ is high-dimensional. We propose a dimensionality reduction scheme based on principal component analysis (PCA) that strengthens our obtained bounds for the case where the intrinsic dimensionality of $\gamma$ is small, even though the dimensionality of the ambient space may be large.

2. Our generalization bound motivates the use of a variance-based regularization scheme, where in addition to the empirical conditional expectation specified by the NW estimator, we minimize a penalty term that corresponds to empirical conditional standard deviation of the random cost function in the objective. Furthermore, we derive suboptimality bounds for the optimal solution $x^*$ obtained for this variance-regularized formulation.

3. In general, a variance-based regularization scheme leads to a non-convex formulation and, therefore, is intractable. We derive an exact mixed-integer second-order cone programming (MISOCP) reformulation for the case when the loss function $\ell(x, \xi)$ is piecewise linear in $x$ for all $\xi \in \Xi$, which can be solved using off-the-shelf optimization solvers. Furthermore, we show that the problem reduces to an efficiently solvable second-order cone program (SOCP) if $\ell(x, \xi)$ is linear in $x$ for all $\xi \in \Xi$ and the solution set $\mathcal{X}$ is second-order conic representable.

4. Adapting ideas from Duchi and Namkoong [2019] proposed in the context of empirical risk minimization problems, we develop a DRO formulation for the case when the loss function is a general convex function of $x$ for all $\xi \in \Xi$. Furthermore, we establish the equivalence of our variance regularized formulation and the DRO formulation for large sample sizes. For a convex loss function that is quadratic or piecewise linear in $x$, the DRO formulation reduces to a SOCP if $\mathcal{X}$ is second-order conic representable.

The remainder of the paper is organized as follows. In Section 2 we provide a background on the Nadaraya-Watson kernel regression estimator as well as introduce the large and moderate deviations theory on which our main results are based. In Section 3 we derive the generalization bound for the NW approximation using results from moderate deviations theory and present the PCA-based dimensionality reduction scheme. Section 4 develops a regularization scheme that is motivated by the generalization bound and derives the suboptimality bound for the proposed method. The section also develops an exact reformulation for the regularized problem based on piecewise linear convex loss functions and presents an application from portfolio management. In
Section 5, we propose a distributionally robust optimization formulation for general convex loss functions. In Section 6, we provide computational results for inventory management and wind energy commitment problems. Finally, we provide concluding remarks in Section 7. For clarity of exposition, lengthy and technical proofs are deferred to the appendix.

Notation and terminology We use bold letters for vectors, while scalars are printed in regular font. We denote by $\mathbf{e}$ the vector of all ones. Random variables are designated by tilde signs (e.g., $\tilde{\xi}$), while their realizations are represented by the same symbols without tildes (e.g., $\xi$). For any $n \in \mathbb{N}$, we define $[n]$ as the index set $\{1, \ldots, n\}$. For any matrix $A$, the operator norm $\|A\|_2$ represents the largest singular value of $A$ and its Frobenius norm is defined as $\|A\|_F = \left(\sum_{ij} A_{ij}^2\right)^{1/2}$. We define by $\text{SOC}(n + 1) \subseteq \mathbb{R}^{n+1}$ the standard second-order cone: $v \in \text{SOC}(n + 1) \iff \|(v_1, \ldots, v_n)^T\| \leq v_{n+1}$. The probability simplex in $\mathbb{R}_+^n$ is denoted as $\Delta^n = \{w \in \mathbb{R}_+^n : e^T w = 1\}$ and the Dirac distribution which assigns unit mass on $\xi$ is denoted by $\delta_\xi$. For any $x \in \mathbb{R}$, we define $(x)_+ = \max(x, 0)$.

For asymptotic analysis, we use standard notations like $o$ and $O$ to represent rates of convergence. We use $\tilde{O}$ notation to denote the $O$ notation that suppresses multiplicative terms with logarithmic dependence on $n$.

2 Background

In this section, we provide the preliminaries of Nadaraya-Watson (NW) approximation and large and moderate deviations theory that are necessary for the development of the main results in this paper.

2.1 Nadaraya-Watson Kernel Regression

To approximate $\mathbf{SO}$, we apply the NW kernel regression which estimates the conditional expectation with

$$\hat{E}_\gamma[\ell(x, \tilde{\xi})] = \frac{\sum_{i=1}^n K\left(\frac{\gamma - \gamma_i}{h}\right) \ell(x, \xi_i)}{\sum_{i=1}^n K\left(\frac{\gamma - \gamma_i}{h}\right)},$$

(NW$_\text{est}$)

where $K$ is a prescribed kernel function and $h > 0$ is the bandwidth parameter. In this paper, we consider the exponential kernel function given by [Genton, 2001]

$$K(\theta) = \frac{1}{Z} \exp(-\|\theta\|_2),$$

(1)

with $Z = \int_{\mathbb{R}^p} \exp(-\|\theta\|_2) \, d\theta$ a normalization constant.

The estimator (NW$_\text{est}$) encapsulates a popular model in data-driven analytics. Indeed, an extremely large value of the bandwidth parameter $h$ means that the approximation (NW$_\text{est}$) reduces to the unconditional sample-average approximation $\frac{1}{n} \sum_{i=1}^n \ell(x, \xi_i)$. On the other hand, a very small bandwidth implies that most of the probability mass is assigned to the sample point closest to $\gamma$. 

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The choice $h = O(1/n^{1/(p+4)})$ provides the best balance between bias and variance that yields the minimum expected error [Györfi et al. 2006].

Using the estimator ($NW_{\text{est}}$), we arrive at the following approximation to the stochastic optimization problem ($SO$):

$$\min_{x\in\mathcal{X}} \mathbb{E}_{\gamma}[\ell(x, \hat{\xi})]. \quad (NW)$$

This approximation is first developed by [Hannah et al. 2010].

### 2.2 Large and Moderate Deviations Theory

Large deviations theory studies the tail behavior of sequences of random variables. It characterizes the exponential decay rate of the probability that a random variable in the sequence realizes on any particular rare event. Formally, we say that the sequence of random variables $\{\tilde{z}_n\}_{n\in\mathbb{N}}$ satisfies a large deviation principle with speed $\nu_n$ and rate function $I: \mathbb{R} \to [0, +\infty]$ if

$$\liminf_{n\to\infty} \frac{1}{\nu_n} \log \mathbb{P}(\tilde{z}_n \in \mathcal{O}) \geq -\inf_{y \in \mathcal{O}} I(y) \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{\nu_n} \log \mathbb{P}(\tilde{z}_n \in \mathcal{C}) \leq -\inf_{y \in \mathcal{C}} I(y),$$

for every open subset $\mathcal{O}$ and closed subset $\mathcal{C}$ of $\mathbb{R}$, respectively. If the random variable is defined as the average $\tilde{z}_n = \frac{1}{n} \sum_{i=1}^{n} \tilde{r}_i$ of i.i.d. random variables $\tilde{r}_i$, $i \in \mathbb{N}$, with a finite logarithmic moment generating function $\Lambda(t) = \mathbb{E}[\exp(t\tilde{r}_1)] < +\infty$, then we obtain the Cramer’s theorem which states that the sequence $\{\tilde{z}_n\}_{n\in\mathbb{N}}$ obeys a large deviation principle with speed $\nu_n = n$ and rate function $I(y) = \sup_{t \geq 0} (ty - \Lambda(t))$. The inequalities in (2) thus imply that for large enough $n$ the probability that $\tilde{z}_n$ takes value within the rare event set $\{z: z \geq y\}$, with $y > \mathbb{E}[\tilde{r}_1]$, is roughly equal to $\exp(-nI(y))$. That is, it decays exponentially fast in $n$ at the rate $I(y)$. Note that the rate function depends on the particular distribution of the random variable $\tilde{r}_1$. From the central limit theorem, however, we know that the distribution of the renormalized average $\sqrt{n}\tilde{z}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{r}_i$ is asymptotically normal, which admits a succinct description through the first and second-order moments of $\tilde{r}_1$.

Moderate deviations theory delineates the intermediate cases between the two extremes of large deviations theory and central limit theorem. The theory studies situations where the sequence $\{a_n\tilde{z}_n\}_{n\in\mathbb{N}}$ obeys a large deviation principle with the same rate function for a certain range of renormalization parameters $a_n \to \infty$. The theory often provides a result that combines both large deviations theory and central limit theorem. Analogous to the central limit behavior, the rate function in a moderate deviation principle is typically analytical, requiring only limited information about the distribution, such as the variance. However, we also observe an exponential decay rate characteristic of results in large deviations theory. In the case of i.i.d. random variables, we find that if $a_n^2/n \to 0$ as $n \to \infty$ then the sequence $\{a_n\tilde{z}_n\}_{n\in\mathbb{N}}$ obeys a large deviation principle with speed $n/a_n^2$ and analytical rate function $I(y) = \frac{1}{2} y^2 / \sigma^2$, where $\sigma^2$ is the variance of the random variable $\tilde{r}_1$ [Dembo and Zeitouni 1998, Theorem 3.7.1]. We refer the reader to the references [Dembo and Zeitouni 1998, Eichelsbacher and Löwe 2003] for a more detailed account on large and moderate deviations theory.
3 Generalization Bounds via Moderate Deviation Principles

In this section, we first derive generalization bounds on the approximation \((NW)\) for a fixed decision \(x\). The result leverages the following moderate deviations theory of the NW estimator by Mokkadem et al. [2008] in the setting of exponential kernel functions. To apply this theorem, in this paper we assume the following mild regularity conditions:

(A1) The support \(\Xi\) of the random vector \(\tilde{\xi}\) is compact and the loss function \(\ell(x, \xi)\) takes value in the interval \([0,1]\) for all \(x \in X\) and \(\xi \in \Xi\).

(A2) The density function \(f(\gamma, \xi)\) is twice differentiable with continuous and bounded partial derivatives. In addition, the marginal density \(f_{\tilde{\gamma}}(\gamma)\) is non-zero at the given side information vector \(\gamma\).

(A3) The bandwidth parameter \(h\) for the kernel function \(K_h\) is scaled such that \(\lim_{n \to \infty} h_n = 0\) and \(\lim_{n \to \infty} nh_n^p = \infty\).

The assumptions about the support set and the loss function in (A1) are typical in the literature. Here, we do not impose any restriction on the size and structure of the support set other than its compactness. If the loss function is bounded, then one can simply apply scaling and translation so that it takes value in the interval \([0,1]\). The assumptions about the density function in (A2) are standard regularity conditions in kernel density and kernel regression estimations. They ensure that the conditional distribution of \(\tilde{\xi}\) given the side information \(\gamma\) can be inferred reasonably well using the historical observations. The assumption about the bandwidth parameter \(h\) in (A3) ensures that the estimator \((NW_{\text{est}})\) is asymptotically consistent [Györfi et al., 2006, Silverman, 1986].

**Theorem 1. [Moderate Deviation Principles]** Let the density function \(f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}\) satisfy assumption (A2). Consider a function \(L : \mathbb{R}^q \to \mathbb{R}\) that satisfies the following conditions:

1. The function \(t \to \int_{\mathbb{R}^q} L(\xi)^2 f(t, \xi) d\xi\) is continuous at \(t = \gamma\).

2. For every \(u \in \mathbb{R}\), the function \(t \to \int_{\mathbb{R}^q} \exp(uL(\xi)) f(t, \xi) d\xi\) is bounded and continuous at \(t = \gamma\).

3. The function \(t \to \int_{\mathbb{R}^q} L(\xi) f(t, \xi) d\xi\) is twice differentiable on \(\mathbb{R}^p\), with continuous and bounded partial derivatives at \(t = \gamma\).

Then, for any positive sequence \(\{a_n\}_{n \in \mathbb{N}}\) such that

\[
\lim_{n \to \infty} a_n = \infty, \quad \lim_{n \to \infty} \frac{a_n^2}{nh_n^2} = 0, \quad \text{and} \quad \lim_{n \to \infty} a_n h_n^2 = 0,
\]

the sequence \(\{a_n(\mathbb{E}_\gamma[L(\xi)] - \hat{\mathbb{E}}_\gamma[L(\xi)])\}_{n \in \mathbb{N}}\) satisfies a large deviation principle with speed \(\nu_n = nh_n^p/a_n^2\) and rate function

\[
I_\gamma(y) = \frac{y^2 g(\gamma)}{\mathbb{V}_\gamma[L(\xi)]},
\]
where \( g(\gamma) = f_\gamma(\gamma) / (2 \int_{\mathbb{R}^p} K^2(\theta) d\theta) \) is the scaled marginal density of \( \gamma \) and \( \mathbb{V}_\gamma[L(\hat{\xi})] = \mathbb{V}[L(\hat{\xi}) | \hat{\gamma} = \gamma] \) is the conditional variance of \( L(\hat{\xi}) \) given the side information \( \gamma \). That is, we have

\[
\lim_{n \to \infty} \frac{1}{\nu_n} \log \mathbb{P} \left( a_n \left( \mathbb{E}_\gamma[L(\hat{\xi})] - \hat{\mathbb{E}}_\gamma[L(\hat{\xi})] \right) \in 0 \right) \geq \inf_{y \in 0} I_\gamma(y) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{\nu_n} \log \mathbb{P} \left( a_n \left( \mathbb{E}_\gamma[L(\hat{\xi})] - \hat{\mathbb{E}}_\gamma[L(\hat{\xi})] \right) \in C \right) \leq \inf_{y \in C} I_\gamma(y),
\]

for every open subset \( 0 \) and closed subset \( C \) of \( \mathbb{R} \), respectively.

### 3.1 Generalization Bounds

Using Theorem 1, we arrive at our first main result whose proof can be found in Appendix B.

**Proposition 1.** For any fixed \( x \in X \), we have

\[
\mathbb{P} \left( \left| \mathbb{E}_\gamma[\ell(x, \hat{\xi})] - \hat{\mathbb{E}}_\gamma[\ell(x, \hat{\xi})] \right| \geq \epsilon \right) = \exp \left( -n h_n^2 \frac{g(\gamma)(1 + o(1))}{\mathbb{V}_\gamma[\ell(x, \hat{\xi})]} \right).
\]

(5)

Proposition 1 asserts that, as the sample size grows, the probability that the NW approximation deviates by at least \( \epsilon \) from the true conditional expectation decays exponentially fast in \( n h_n^2 \). Setting the right-hand side of (5) to \( \delta \), we arrive at the following guarantee on the out-of-sample errors.

**Corollary 1** (Generalization Bound). For any fixed \( x \in X \), we have

\[
\left| \mathbb{E}_\gamma[\ell(x, \hat{\xi})] - \hat{\mathbb{E}}_\gamma[\ell(x, \hat{\xi})] \right| \leq \sqrt{\frac{\mathbb{V}_\gamma[\ell(x, \hat{\xi})]}{n h_n^2 g(\gamma)(1 + o(1))}} \log \left( \frac{1}{\delta} \right) = O \left( \sqrt{\frac{1}{n h_n^2}} \right),
\]

with probability at least \( 1 - \delta \).

**Remark 1.** With minor modifications, it is possible to derive a similar generalization bound when the popular Gaussian kernel is used in NW\textsubscript{est} instead of the exponential kernel.

The bound in (6) degrades if the scaled density \( g(\gamma) \) is small or if the conditional variance \( \mathbb{V}_\gamma[\ell(x, \hat{\xi})] \) is large. In the limit where \( g(\gamma) \downarrow 0 \), there are fewer historical samples close to the given side information, implying that the NW estimator constitutes a poor approximation of the true conditional expectation. On the other hand, a smaller variance indicates that few data points are sufficient to accurately describe the conditional distribution of \( \hat{\xi} \) given \( \gamma \).

Using the best bandwidth parameter scaling \( h_n = O(1/n^{1/(p+4)}) \) for the multivariate NW estimator [Györfi et al., 2006, Chapter 5.2], we find that the error bound in (6) diminishes at the rate of \( O(1/n^{2/(p+4)}) \). Note that we have a dependence on the dimension \( p \), which suggests that the estimator suffers from the curse of dimensionality. In general, such a result is quite typical for settings where kernels are used; it has also been observed in other works such as Kannan et al. [2020a]. In Section 3.2, we propose a dimensionality reduction scheme based on principal
component analysis that allows us to obtain tighter bounds when the intrinsic dimensionality of \( \gamma \) is considerably smaller than the dimensionality \( p \) of the ambient space.

So far, we have obtained the generalization bound for a fixed \( \mathbf{x} \in \mathcal{X} \). In the following theorem, we extend the result in Corollary 1 to obtain uniform generalization bounds for all \( \mathbf{x} \in \mathcal{X} \), under the assumption that the feasible set \( \mathcal{X} \) consists of finitely many points.

**Theorem 2** (Generalization Bound for a Finite Set \( \mathcal{X} \)). Suppose that \( \mathcal{X} \) is a finite set. Then, we have

\[
\mathbb{E}_\gamma [\ell (\mathbf{x}, \tilde{\xi})] \leq \hat{\mathbb{E}}_\gamma [\ell (\mathbf{x}, \tilde{\xi})] + \sqrt{\frac{V_v [\ell (\mathbf{x}, \tilde{\xi})]}{n h_n^2 g(\gamma) (1 + o(1))} \log \left( \frac{1}{\delta} \right)} \quad \forall \mathbf{x} \in \mathcal{X}
\]

with probability at least \( 1 - \delta \).

**Proof.** The proof follows from a straightforward application of the union bound to the result obtained in Corollary 1. \( \square \)

Note that the bound (2) grows only logarithmically in the cardinality of the feasible set \( \mathcal{X} \) and, hence, at most linearly in the dimension of the decision vector \( \mathbf{x} \).

In our analysis for Theorem 2, we assumed that the feasible set \( \mathcal{X} \) is finite. In what follows, we show that under additional mild assumptions on the loss function, the result can be extended to the setting where \( \mathcal{X} \) is a continuous and bounded set.

**Theorem 3** (Generalization Bound for a Continuous and Bounded Set \( \mathcal{X} \)). Suppose \( \mathcal{X} \subset \mathbb{R}^d \) is a bounded subset with finite diameter \( D = \sup_{\mathbf{x}, \mathbf{x}' \in \mathcal{X}} \| \mathbf{x} - \mathbf{x}' \| \). Assume that the loss function \( \ell (\mathbf{x}, \tilde{\xi}) \) is \( M \)-Lipschitz continuous in \( \mathbf{x} \), i.e., there exists a constant \( M > 0 \) such that

\[
|\ell (\mathbf{x}, \xi) - \ell (\mathbf{x}', \xi)| \leq M \| \mathbf{x} - \mathbf{x}' \| \quad \forall \mathbf{x}, \mathbf{x}' \in \mathcal{X} \forall \xi \in \Xi.
\]

(7)

Fix a tolerance level \( \eta > 0 \). Then, with probability at least \( 1 - \delta \), we have

\[
\mathbb{E}_\gamma [\ell (\mathbf{x}, \tilde{\xi})] \leq \hat{\mathbb{E}}_\gamma [\ell (\mathbf{x}, \tilde{\xi})] + \sqrt{\frac{V_v [\ell (\mathbf{x}, \tilde{\xi})]}{n h_n^2 g(\gamma) (1 + o(1))} \log \left( \frac{1}{\delta} \right)} + M \eta \left( 1 + \sqrt{\frac{\log \left( \frac{|\mathcal{X}_\eta|}{\delta} \right)}{n h_n^2 g(\gamma) (1 + o(1))}} \right) \quad \forall \mathbf{x} \in \mathcal{X},
\]

where \( |\mathcal{X}_\eta| = O(1)(D/\eta)^d \).

We defer the proof of the above theorem to Appendix C.

**Remark 2.** An alternative way to construct an empirical estimator for conditional expectation is by using the \( k \)-nearest neighbors regression (KNN), which assigns equal weight \( 1/k \) to the \( k \) nearest points of \( \gamma \). Bertsimas and McCord [2019] derive a generalization bound of the scheme. They prove that under more restrictive assumptions, such as \( \tilde{\gamma} \) is supported on a subset \( \Gamma \) of \( [0,1]^p \) and there exists a constant \( g > 0 \) such that \( \mathbb{P} (\| \tilde{\gamma} - \gamma \| \leq \epsilon) \geq g \) for all \( \gamma \in \Gamma \), the generalization bound of the scheme decays at the rate of \( \tilde{O}(1/n^{1/(2p)}) \). However, unlike our bounds in Theorems 2 and
their bound is independent of the variance (or risk) of the decisions. Therefore, designing an effective regularization scheme for the KNN-based approach is not immediately obvious.

3.2 Extension to High-Dimensional $\gamma$

In this section, we extend our analysis to the setting where the side information $\gamma \in \mathbb{R}^p$ is high-dimensional, i.e., where $p$ is large. From the result obtained in Corollary 1, we observe that the generalization bound decays at the rate $O(n^{-\frac{2}{p+4}})$, which is slow for decision-making problems with large $p$.

In real-world settings, however, data often lies on a low-dimensional subspace or manifold. In other words, the intrinsic dimensionality of the data is much smaller than the dimensionality of the ambient space. To take this into consideration, we consider the setting where the side information vector $\gamma$ is drawn from a sub-gaussian probability distribution with sub-gaussian parameter $\sigma$ and lies approximately in a low-dimensional linear subspace $S$ where $\text{dim}(S) = p' \ll p$. Here, we make the assumption that $\gamma^S$—the component of $\gamma$ that lies in the subspace $S$—corresponds to the signal and influences the random cost parameter vector $\tilde{\xi}$, while its orthogonal component $\gamma^{S^\perp}$ corresponds to the noise term, which, given $\gamma^S$, does not provide any information about $\tilde{\xi}$. In other words, the random vector $\tilde{\xi}$ is conditionally independent of $\gamma^{S^\perp}$ given $\gamma^S$, i.e., $(\xi \perp \gamma^{S^\perp})|\gamma^S$.

Thus, the conditional distribution satisfies

$$f(\xi|\gamma) := f(\xi|\gamma^S, \gamma^{S^\perp}) = f(\xi|\gamma^S).$$

The sub-gaussian assumption on $\gamma$ is also non-restrictive and encompasses a wide class of probability distributions, including all multivariate Gaussian distributions and distributions with bounded support. We mention here that a setup similar to ours has been considered in Xu et al. [2016] for robust optimization problems in high-dimensions.

Under the conditional independence assumption, the optimization problem $(SO)$ is equivalent to

$$\min_{x \in \mathcal{X}} \left\{ \mathbb{E}_\rho[\ell(x, \tilde{\xi})] := \mathbb{E}[\ell(x, \tilde{\xi}) \mid \hat{\rho} = \rho] \right\},$$

$(SO_{\text{reduced}})$

where $\hat{\rho} = \text{proj}_S(\gamma)$ is the projection of the random vector $\gamma$ onto the subspace $S$. As discussed in Section 1, the exact conditional distribution $f(\xi|\rho)$ is usually not known. If the exact subspace $S$ is known, the historical data $\{(\rho^1, \xi^1), \ldots, (\rho^n, \xi^n)\}$ can be obtained by projecting the realizations $\gamma^i$ onto the subspace $S$. Similar to the $(NW)$ formulation developed before for the stochastic optimization problem $(SO)$, we propose to approximate the problem $(SO_{\text{reduced}})$ using the Nadaraya-Watson kernel regression estimator, as follows:

$$\min_{x \in \mathcal{X}} \left\{ \mathbb{E}_\rho[\ell(x, \tilde{\xi})] := \frac{\sum_{i=1}^n K\left( \frac{\rho - \rho^i}{h} \right) \ell(x, \xi^i)}{\sum_{i=1}^n K\left( \frac{\rho - \rho^i}{h} \right)} \right\}. \quad (8)$$

$^1$We refer the reader to Vershynin [2010], Wainwright [2019] for more details about sub-gaussian random vectors.
In general, however, the exact subspace $S$ may also be unknown. Therefore, we develop a
dimensionality reduction procedure based on principal component analysis (PCA) that allows us
to construct an estimate $\hat{S}$ of the true subspace $S$ in a data-driven manner. Our approach is
based on the idea of sample splitting, which has been previously proposed in the literature to
obtain tighter bounds for high-dimensional problems in other contexts [Chaudhuri et al. 2009,
Srivastava et al. 2019] Yan and Sarkar 2020]. The main idea is to randomly split the observations
in the data matrix $\Gamma = (\gamma^1, \ldots, \gamma^n)^T$ into two disjoint parts, $\Gamma_1$ and $\Gamma_2$, with the observations in
the corresponding parts indexed by sets $I_1$ and $I_2$ with cardinalities $|I_1| = n_1$ and $|I_2| = n_2$,
respectively. Using the observations in $\Gamma_2$, we construct the sample covariance matrix $\Sigma_2 = \frac{1}{n_2} \sum_{i \in I_2} (\gamma^i - \bar{\gamma})(\gamma^i - \bar{\gamma})^T$ where $\bar{\gamma} = \frac{1}{n_2} \sum_{i \in I_2} \gamma^i$ and compute its top $p'$ eigenvectors $\hat{U} = [\hat{u}_1, \ldots, \hat{u}_{p'}]^T \in \mathbb{R}^{p' \times p}$, which form a basis for the estimated subspace $\hat{S} := \text{span}(\hat{U})$. Once $\hat{S}$ is
determined, the observations in $\Gamma_1$ are projected on to the subspace to obtain their projections $\hat{\Pi}_1 = \Gamma_1 \hat{U}^T$. Sample splitting ensures that the projected points are independent of each other, which is
required for the application of moderate deviations theory to obtain the theoretical guarantees. In
practice, however, this step can be usually skipped and the subspace $\hat{S}$ can be estimated from the
entire data matrix. Next, we let
\[
\hat{\mathbb{E}}_\rho[\ell(x, \xi)] := \frac{\sum_{i \in I_2} \mathcal{K}\left(\frac{\hat{\rho} - \hat{\rho}^i}{h}\right) \ell(x, \xi^i)}{\sum_{i \in I_1} \mathcal{K}\left(\frac{\hat{\rho} - \hat{\rho}^i}{h}\right)},
\]  
(AW$_\text{est}$) to denote the NW estimator defined in (8) based on the dimensionality reduction procedure detailed
above. We delineate the generalization bound for the reduced estimator in the following proposition
whose proof is deferred to Appendix D.

**Proposition 2** (Generalization Bound for $\mathbb{AW}_{\text{est}}$ with Finite Set $\mathcal{X}$). Suppose $\mathcal{X}$ is a finite set,
$n_1$ and $n_2$ are sufficiently large and $n_2^{-1/2}/h_{n_1} < 1$. Then, we have
\[
|\mathbb{E}_\gamma[\ell(x, \xi)] - \hat{\mathbb{E}}_\rho[\ell(x, \xi)]| \leq \sqrt{\frac{\mathbb{V}_\gamma[\ell(x, \xi)]}{n_1 h_{n_1} p' g(\rho)(1 + o(1))}} \log \left(\frac{5n_1 |\mathcal{X}|}{\delta}\right)
+ \frac{8}{h} \frac{4\sigma C}{\lambda_{p'} - \lambda_{p'+1}} \left(\frac{p'}{n_2} \log \left(\frac{10n_1 |\mathcal{X}|}{\delta}\right) \left(\sqrt{p} + \sqrt{\frac{1}{2} \log \left(\frac{5n_1 |\mathcal{X}|}{\delta}\right)}\right)\right) \quad \forall x \in \mathcal{X}
\]
with probability at least $1 - \delta$. Here, $C > 0$ is a constant that depends on the sub-gaussian parameter
$\sigma$, $\lambda_{p'}$ is the $p'$-th largest eigenvalue of the true covariance matrix $\Sigma$ of $\bar{\gamma}$.

From the proposition, we see that if we choose $h_{n_1} = O(1/n_1^{1/(p'+4)})$ and the sizes of $\Gamma_1$ and $\Gamma_2$
such that $n_1 = \alpha n$ and $n_2 = (1 - \alpha)n$ for some $0 < \alpha < 1$, then the requirement $n_2^{-1/2}/h_{n_1} < 1$
holds for sufficiently large $n$, and the generalization bound decays at the rate $O(n^{-\frac{1}{2\sigma^2 + 4} \log(n_1)})$.
Thus, by adopting the proposed dimensionality reduction procedure, the generalization bound no
longer depends on the original dimension $p$ of the ambient space. Instead, it is a function of only the intrinsic dimensionality $p'$ of the side information vector $\gamma$. Hence, the adverse impact on the generalization bound associated with the curse of dimensionality is mitigated.

When $\tilde{\gamma}$ is bounded, i.e., $\|\tilde{\gamma}\| \leq \gamma_{\text{max}}$ almost surely, we obtain a sharper bound without the $\log(n_1)$ factor. In this case, the error decays at a faster rate $O\left(n^{-\frac{p'}{2}}\right)$.

**Corollary 2** (Generalization Bound for $NW_{\text{est}}$ with bounded covariates). Consider the same setting as in Proposition 2 and assume that $\gamma$ is a bounded random variable where $\|\tilde{\gamma}\| \leq \gamma_{\text{max}}$ almost surely. Then, we have

$$|E_{\gamma}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]| \leq \sqrt{\frac{\nabla_{\gamma}[\ell(x, \tilde{\xi})]}{n_1 h_{n_1}(\rho)(1 + o(1))}} \log \left( \frac{2|X|}{\delta} \right)$$

$$+ \frac{8}{h} \frac{C}{\lambda_{p'} - \lambda_{p'+1}} \frac{p'}{n_2} \log \left( \frac{4|X|}{\delta} \right) \gamma_{\text{max}} \quad \forall x \in X$$

with probability at least $1 - \delta$. Here, $C > 0$ is a constant that depends on the sub-gaussian parameter $\sigma$, $\lambda_{p'}$ is the $p'$-th largest eigenvalue of the true covariance matrix $\Sigma$ of $\tilde{\gamma}$.

### 4 A Conditional Standard Deviation Regularization Scheme

The generalization bounds obtained in Theorems 2 and 3 imply that the out-of-sample errors are negligible if the conditional standard deviation $\sqrt{\nabla_{\gamma}[\ell(x, \tilde{\xi})]}$ is small. This suggests that a regularization scheme involving the term would ensure a solution with a strong generalization bound. However, as we do not have access to the true conditional variance, we propose to utilize the empirical conditional variance as a surrogate

$$\hat{\nabla}_{\gamma}[\ell(x, \tilde{\xi})] := \hat{E}_{\gamma}[\ell(x, \tilde{\xi})] - \hat{E}_{\gamma}[\ell(x, \tilde{\xi})]^2 = \hat{E}_{\gamma}[\ell(x, \tilde{\xi})^2] - \hat{E}_{\gamma}[\ell(x, \tilde{\xi})]^2.$$

This setting gives rise to the regularized NW approximation

$$\min_{x \in X} \hat{E}_{\gamma}[\ell(x, \tilde{\xi})] + \lambda \sqrt{\hat{\nabla}_{\gamma}[\ell(x, \tilde{\xi})]}, \quad (RNW)$$

where $\lambda \geq 0$ is a tuning parameter that controls the degree of regularization. We point out here that a similar formulation on the variance-based regularization scheme has been previously proposed and analyzed in the empirical risk minimization literature [Maurer and Pontil, 2009, Duchi and Namkoong, 2019] for the unconditional setting, where the true (unconditional) probability distribution is approximated by the empirical distribution.

#### 4.1 Suboptimality Bounds

In this section, we aim to establish the properties of the optimal solutions to problem (RNW). We first show that replacing the true conditional variance with its empirical estimate (9) does not
significantly weaken the generalization bound derived in Section 3.

Proposition 3. Fix a tolerance level \( \tau > 0 \). For any fixed \( x \in \mathcal{X} \), we have

\[
\left| \sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} - \sqrt{\hat{\mathbb{V}}_\gamma[\ell(x, \tilde{\xi})]} \right| \leq \tau + \frac{\log \left( \frac{1+2/\tau}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))},
\]

(10)

with probability at least \( 1 - \delta \).

The proof of Proposition 3 can be found in Appendix E. We remark that the tolerance level \( \tau \) can be made small without significantly increasing the square root term on the right-hand side of (10) as the latter displays merely a logarithmic dependence in \( \tau \).

We next analyze the suboptimality bound resulting from solving the regularized problem \((\text{RNW})\).

We first assume that the feasible set \( \mathcal{X} \) is finite even though its cardinality can be exponential in the problem dimensions. Let \( \hat{x} \) be a minimizer of the regularized problem and \( x^* \) be a minimizer of the true stochastic optimization problem \((\text{SO})\).

Theorem 4 (Suboptimality Bound for a Finite Set \( \mathcal{X} \)). Fix a tolerance level \( \tau > 0 \). Then, for some scaling of the regularization parameter \( \lambda = O \left( \frac{1}{\sqrt{nh_n^p g(\gamma)}} \right) \), we have

\[
\mathbb{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(x^*, \tilde{\xi})] + \left( \sqrt{\mathbb{V}_\gamma[\ell(x^*, \tilde{\xi})]} + \tau \right) \sqrt{\frac{4 \log \left( \frac{6|\mathcal{X}|}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))}} + \frac{2 \log \left( \frac{6|\mathcal{X}|(1+2/\tau)}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))},
\]

(11)

with probability at least \( 1 - \delta \).

The proof of the theorem is deferred to Appendix F. Theorem 4 asserts that if there is an optimal solution \( x^* \) of the stochastic problem \((\text{SO})\) that yields a cost with negligible conditional variance, then the regularized solution \( \hat{x} \) will converge to this optimal solution at a rate of \( O \left( \frac{1}{\sqrt{nh_n^p}} \right) \).

In our analysis for Theorem 4 we assumed that the feasible set \( \mathcal{X} \) is finite. In the next theorem, under the assumption of a Lipschitz loss function, we extend the result to obtain a similar suboptimality bound for the case where the solution set \( \mathcal{X} \) is continuous and bounded.

Theorem 5 (Suboptimality Bound for a Continuous and Bounded Set \( \mathcal{X} \)). Suppose \( \mathcal{X} \) is a bounded subset of \( \mathbb{R}^d \) with finite diameter \( D = \sup_{x, x' \in \mathcal{X}} \|x - x'\| \) and the cost function \( \ell \) is Lipschitz continuous in \( x \), i.e., it satisfies condition (7). Then, for some scaling of the regularization parameter \( \lambda = O \left( \frac{1}{\sqrt{nh_n^p g(\gamma)}} \right) \) and any \( \tau, \eta > 0 \), we have

\[
\mathbb{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(x^*, \tilde{\xi})] + (2 + \lambda) M \eta + \left( \sqrt{\mathbb{V}_\gamma[\ell(x^*, \tilde{\xi})]} + \tau \right) \sqrt{\frac{4 \log \left( \frac{O(1)(D/\eta)^d}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))}} + \frac{2 \log \left( \frac{(1+2/\tau)O(1)(D/\eta)^d}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))},
\]

with probability at least \( 1 - \delta \).
4.2 A Mixed-Integer Second-Order Cone Programming Formulation

In general, the exact problem \(\mathcal{R}NW\) is intractable because of the non-convexity of the regularization term in the objective function. In this section, we consider the case where the loss function is piecewise linear convex and \(\mathcal{X}\) is second-order conic representable, and we derive a mixed-integer SOCP formulation for \(\mathcal{R}NW\). Although the problem remains hard to solve, reasonably large instances of the problem can be solved using off-the-shelf solvers such as Gurobi and CPLEX. Based on our derivation, we also show that, particularly for the case where the loss function is linear, the problem is efficiently solvable as a SOCP.

Proposition 4. Suppose the loss function \(\ell(x, \xi) = \max_{j \in [m]} a_j(x)^T \xi + b_j\) is piecewise linear convex in \(x\) and the feasible set \(\mathcal{X}\) is second-order conic representable. Let \(w_i = K(\gamma - \gamma_i h) / \sum_{i=1}^{n} K(\gamma - \gamma_i h)\) denote the kernel weight associated with the \(i\)-th observation, then the problem \(\mathcal{R}NW\) is solvable as the following mixed-integer second-order cone program:

\[
\begin{align*}
\min & \quad \mathbf{w}^T \mathbf{\nu} + \lambda \rho \\
\text{s.t.} & \quad \left( \sqrt{\mathbf{w}_1 (\nu_1 - t)}, \ldots, \sqrt{\mathbf{w}_n (\nu_n - t)} , \rho \right) \in \text{SOC}(n + 1), \\
& \quad a_j(x)^T \xi^i + b_j \leq \nu_i \quad \forall i \in [n] \forall j \in [m], \\
& \quad a_j(x)^T \xi^i + b_j + M(1 - z_{ij}) \geq \nu_i \quad \forall i \in [n] \forall j \in [m], \\
& \quad \sum_{j \in [m]} z_{ij} = 1 \quad \forall i \in [n], \\
& \quad x \in \mathcal{X}, \ \nu \in \mathbb{R}^n, \ \rho \in \mathbb{R}, \ t \in \mathbb{R}, \ z \in \{0, 1\}^{n \times m}.
\end{align*}
\]

where \(M > 0\) is a sufficiently large constant. Under the assumption that \(\ell(x, \xi)\) takes value in the interval \([0, 1]\), it is sufficient to set \(M = 1\).

Proof. To obtain the formulation, we first introduce the epigraphical variable \(\rho\) to \(\mathcal{R}NW\) to bring the conditional standard deviation term into the constraint:

\[
\begin{align*}
\min & \quad \hat{\mathbb{E}}_{\gamma} [\ell(x, \xi)] + \lambda \rho \\
\text{s.t.} & \quad \sqrt{\hat{\mathbb{V}}_{\gamma} [\ell(x, \xi)]} \leq \rho, \\
& \quad x \in \mathcal{X}, \ \rho \in \mathbb{R}.
\end{align*}
\]

Then, we have that the above formulation is equivalent to

\[
\begin{align*}
\min & \quad \left( \sum_{i=1}^{n} \mathbf{w}_i \cdot \ell(x, \xi^i) \right) + \lambda \rho \\
\text{s.t.} & \quad \left( \sum_{i=1}^{n} \mathbf{w}_i \cdot (\ell(x, \xi^i) - t) \right)^2 \leq \rho, \\
& \quad x \in \mathcal{X}, \ \rho \in \mathbb{R}, \ t \in \mathbb{R}.
\end{align*}
\]
where (as in the proof of Proposition 3) we make use of the fact that for any random variable \( \tilde{\chi} \), \( \hat{\gamma} \tilde{\chi} = \arg\min_{t \in \mathbb{R}} \hat{\gamma}(\tilde{\chi} - t)^2 \). Next, we introduce the auxiliary variables \( \nu_i = \ell(x, \xi^i) = \max_{j \in [m]} a_j(x)^\top \xi^i + b_j \) for all \( i \in [n] \). Using the Big-M notation, we can linearize the resulting non-convex constraints to obtain the final formulation

\[
\min \quad \nu^\top w + \lambda \rho \\
\text{s.t.} \quad \sqrt{\sum_{i=1}^{n} w_i \cdot (\nu_i - t)^2} \leq \rho, \\
\quad a_j(x)^\top \xi^i + b_j \leq \nu_i \quad \forall i \in [n], j \in [m], \\
\quad a_j(x)^\top \xi^i + b_j + M(1 - z_{ij}) \geq \nu_i \quad \forall i \in [n], j \in [m], \\
\quad \sum_{j \in [m]} z_{ij} = 1 \quad \forall i \in [n], \\
\quad x \in \mathcal{X}, \ \nu \in \mathbb{R}^n, \ \rho \in \mathbb{R}, \ t \in \mathbb{R}, \ z \in \{0,1\}^{n \times m}.
\]

This completes the proof.

Due to the binary decision variables \( z \in \{0,1\}^{n \times m} \), the above formulation is a mixed-integer second-order cone program (MISOCP), provided that \( \mathcal{X} \) is second-order conic representable with binary/integer variables. If the loss function \( \ell(x, \xi) \) is simply a linear function of \( x \), i.e., \( m = 1 \), then the formulation reduces to a second-order cone program (SOCP), which is efficiently solvable in polynomial time using interior-point methods. We state this result formally in the following corollary.

**Corollary 3.** Suppose the loss function \( \ell(x, \xi) \) is a linear function of \( x \) and the feasible set \( \mathcal{X} \) is second-order conic representable, then the problem \( \text{(RNW)} \) can equivalently be reformulated as the second-order cone program

\[
\min \quad \nu^\top w + \lambda \rho \\
\text{s.t.} \quad (\sqrt{w_1}(\nu_1 - t), \ldots, \sqrt{w_n}(\nu_n - t), \rho) \in \mathcal{SOC}(n + 1), \\
\quad a(x)^\top \xi^i + b = \nu_i \quad \forall i \in [n], \\
\quad x \in \mathcal{X}, \ \nu \in \mathbb{R}^n, \ \rho \in \mathbb{R}, \ t \in \mathbb{R}.
\]

Next, based on our discussion above, we obtain the SOCP formulation for the generic portfolio optimization problem with side information and present the results of a small example.

### 4.3 A Portfolio Optimization Example

In this section, we investigate the performance of our proposed regularized NW approximation on the portfolio optimization problem described in Example 1. We compare the performances of the LDR approach and our regularization scheme. As a direct application of Corollary 3, our regularization scheme can be reformulated as a SOCP. For both the proposed regularization scheme and the LDR approach, the details of the formulations are provided in Appendix K.
Figure 1: Out-of-sample portfolio returns of different approaches over 300 $\gamma$’s for each $n$. The black dot line is the optimal expected return with the side information $\gamma$ given. The black solid line is the optimal expected return without considering side information. The blue and red solid lines are the average returns of our proposed model and the LDR formulation, respectively. The shaded region for each color records the returns between the 10th and 90th percentile of the returns of the corresponding approach.

Example 2. [A Three-Asset Portfolio] Consider the portfolio optimization problem in Example 1. We compare our regularized NW approximation from Corollary 3 with the state-of-the-art linear decision rule (LDR) formulation for the problem proposed by Brandt et al. [2009] and Bazier-Matte and Delage [2020]. We first empirically test the proposed regularized NW approximation and the LDR formulation, and see how they perform against these optimal returns.

Figure 1a shows the out-of-sample returns of the two approaches, as well as the optimal expected portfolio returns with and without consideration of the side information, respectively. We find that our proposed approach substantially outperforms LDR in terms of both return and risk. Even though the two approaches attempt to exploit the side information when generating their portfolios, the NW approach is more effective in capitalizing the information as it consistently generates higher expected returns. We also observe that the NW returns have significantly lower variability. This is not entirely surprising because the regularization term encourages a portfolio with lower standard deviation. Figure 1b depicts the out-of-sample returns for a fixed covariate $\gamma = 0$. In this case, the conditional expected return of each risky asset is 0.5 and investing in any convex combination of the two risky assets yields the optimal expected portfolio return. Since Asset 1 and Asset 2 have perfect negative correlation, the NW approach tends to allocate an equal weight to both assets so that the individual noise terms $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are neutralized in the resulting portfolio.

As expected, the returns of the NW approximation converge fast to the best expected portfolio return as the data size grows. On the other hand, we observe that LDR disappointingly performs as if it were oblivious to the side information, even with large data size. This phenomenon can be explained analytically as follows. For any fixed parameters $x_1, x_2$, and $y$, the expected portfolio...
return is given by

\[
E \left[ \sum_{i=1}^{2} \tilde{\xi}_i(\tilde{\gamma})(x_i + \tilde{\gamma} \cdot y) \right] = E \left[ \left( \frac{1}{2} - \tilde{\gamma}^2 + 0.1 \cdot \tilde{\epsilon}_1 \right) (x_1 + \tilde{\gamma} \cdot y) + \left( \frac{1}{2} - \tilde{\gamma}^2 + 0.1 \cdot \tilde{\epsilon}_2 \right) (x_2 + \tilde{\gamma} \cdot y) \right] \\
= E \left[ \left( \frac{1}{2} - \tilde{\gamma}^2 \right) (x_1 + x_2) + 2 \left( \frac{1}{2} \tilde{\gamma} \cdot y - \tilde{\gamma}^3 \cdot y \right) \right] \\
= \left( \frac{1}{2} - \frac{1}{3} \right) (x_1 + x_2) + 0 = \frac{x_1 + x_2}{6},
\]

where the second equality holds because the random variables \( \tilde{\epsilon}_1 \) and \( \tilde{\epsilon}_2 \) are independent of \( \tilde{\gamma} \) and have mean zero, while the penultimate equality follows from the identities \( E[\tilde{\gamma}^2] = 1/3 \) and \( E[\tilde{\gamma}] = E[\tilde{\gamma}^3] = 0 \). Since the constraint \( x_1 + x_2 \leq 1 \) is imposed in the formulation, the LDR approach will never generate an expected portfolio return greater than \( 1/6 \). This result affirms our observation that LDR indeed performs as poorly as the model that disregards the side information.

From the above example, we demonstrate that the LDR approach could fail miserably at exploiting the side information, even on a simple setting. On the other hand, the proposed regularized NW approximation is highly effective at leveraging the side information and can generate a remarkably higher average return with minimal risks.

## 5 Connections with Distributionally Robust Optimization

In this section, we consider the setting where the loss function \( \ell(x, \xi) \) is a general (not necessarily a piecewise linear) convex function of \( x \) for all \( \xi \in \Xi \). Leveraging ideas from Duchi and Namkoong [2019], we obtain a distributionally robust optimization (DRO) formulation, which is a tractable approximation for our proposed variance regularization scheme. In the following proposition, we derive the DRO formulation and show that for large \( n \), the DRO formulation is equivalent to the proposed variance regularized formulation.

**Proposition 5.** Let \( \bar{w}_i = K_h(\gamma - \gamma^i)/\sum_{j=1}^{n} K_h(\gamma - \gamma^j) \), \( i \in [n] \), denote the empirical weights obtained from NW regression, and \( \hat{P}_\gamma = \sum_{i=1}^{n} \bar{w}_i \delta_{\xi_i} \) be the empirical conditional distribution. For any \( x \in \mathcal{X} \), we have

\[
\hat{\mathbb{E}}_{\gamma}[\ell(x, \hat{\xi})] + \left( \lambda \sqrt{\hat{\mathbb{V}}_{\gamma}[\ell(x, \hat{\xi})]} - \lambda^2 \right) + \leq \max_{P \in \mathcal{P}_\lambda(\hat{P}_\gamma)} \mathbb{E}_P[\ell(x, \xi)] \leq \hat{\mathbb{E}}_{\gamma}[\ell(x, \hat{\xi})] + \lambda \sqrt{\hat{\mathbb{V}}_{\gamma}[\ell(x, \hat{\xi})]},
\]

where

\[
\mathcal{P}_\lambda(\hat{P}_\gamma) = \left\{ P = \sum_{i=1}^{n} w_i \delta_{\xi_i} : \sum_{i=1}^{n} \frac{(w_i - \bar{w}_i)^2}{\bar{w}_i} \leq \frac{\lambda^2}{2}, w \in \Delta^n \right\} \tag{13}
\]

is a modified \( \chi^2 \) ambiguity set constructed around the empirical conditional distribution. In partic-
ular, if \( \hat{\nabla}_\gamma [\ell(x, \bar{\xi})] \geq \lambda^2 \), then

\[
\max_{P \in \mathcal{P}_\lambda(P_X)} \mathbb{E}_P[\ell(x, \bar{\xi})] = \hat{\mathbb{E}}_\gamma[\ell(x, \bar{\xi})] + \lambda \sqrt{\hat{\nabla}_\gamma [\ell(x, \bar{\xi})]}.
\]

As stated in Proposition 6, if \( \hat{\nabla}_\gamma [\ell(x, \bar{\xi})] \geq \lambda^2 \) then the DRO model is equivalent to the proposed regularization scheme. Although \( \hat{\nabla}_\gamma [\ell(x, \bar{\xi})] \) is a random quantity, it should be close to \( \nabla_\gamma [\ell(x, \bar{\xi})] \) with high probability when \( n \) is sufficiently large. In addition, Theorem 5 suggests the scaling \( \lambda = O\left(1/\sqrt{nh_n^p g(\gamma)}\right) \), which converges to 0 as \( n \to \infty \). Based on these observations, we derive the condition under which the two models are equivalent with high probability.

**Proposition 6.** Suppose

\[
\sqrt{\nabla_\gamma [\ell(x, \bar{\xi})]} \geq \frac{C_\lambda}{\sqrt{nh_n^p g(\gamma)}} + \tau + \sqrt{\frac{\log(|X_\eta|)}{nh_n^p g(\gamma)(1 + o(1))} \log(1 + \frac{1}{\sigma})} + 2M \eta \quad \forall x \in \mathcal{X},
\]

for some constants \( C_\lambda, \tau, \eta \in \mathbb{R}^+ \), \( \delta \in (0, 1) \), and \( |X_\eta| = O(1) \). Then, with probability at least \( 1 - \delta \),

\[
\max_{P \in \mathcal{P}_\lambda(P_X)} \mathbb{E}_P[\ell(x, \bar{\xi})] = \hat{\mathbb{E}}_\gamma[\ell(x, \bar{\xi})] + \lambda \sqrt{\hat{\nabla}_\gamma [\ell(x, \bar{\xi})]},
\]

where the ambiguity set \( \mathcal{P}_\lambda(P_X) \) is defined in (13) and the regularization parameter \( \lambda \) is set to \( C_\lambda/\sqrt{nh_n^p g(\gamma)} \).

Proposition 6 provides a technical condition (14) for which, with high probability, the DRO model is equivalent to the proposed regularization scheme, which is in general intractable. We emphasize that the condition (14) should hold for sufficiently large \( n \) if \( \nabla_\gamma [\ell(x, \bar{\xi})] > 0 \) for all \( x \in \mathcal{X} \). In particular, by carefully choosing the bandwidth \( h_n \) (accordingly, \( \tau, \eta, \delta \)), the right-hand side converges to 0 as \( n \to \infty \). For example, suppose the bandwidth \( h_n = C_h/n^{1/(p+4)} \) is used with some constant \( C_h > 0 \). Then, one can show that for sufficiently large \( n \), the right-hand side of (14) becomes

\[
\sqrt{\frac{2}{n^{2/(p+4)} g(\gamma)(1 + o(1))} + \frac{2}{\exp\left(C_h^p n^{2/(p+4)}\right) - 1} + \frac{2MD}{\exp\left(C_h^p n^{2/(p+4)}/2d\right)}} \to 0 \quad \text{as } n \to \infty,
\]

and the DRO model is equivalent to the proposed regularization scheme with probability at least \( 1 - C_X \exp\left(-\frac{C_h^p n^{2/(p+4)}}{2d}\right) \) for some constant \( C_X \). We provide the details and the associated corollary of Proposition 6 in Appendix I.

**Remark 3.** Assume that \( \mathcal{X} \) is a convex set and \( \ell(x, \xi) \) is convex in \( x \) for all \( \xi \in \Xi \). Then, the DRO problem

\[
\min_{x \in \mathcal{X}} \max_{P \in \mathcal{P}_\lambda(P_X)} \mathbb{E}_P[\ell(x, \bar{\xi})] \quad \text{(DRO)}
\]
can be formulated as the convex optimization problem given by

\[
\begin{align*}
\min & \quad \alpha - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \frac{\lambda}{\sqrt{2}} \nu \\
\text{s.t.} & \quad \alpha \geq \ell(x, \xi^i) + \frac{\beta_i}{\sqrt{w_i}} \\
& \quad x \in \mathcal{X}, \; \alpha \in \mathbb{R}, \; (\beta, \nu) \in \text{SOC}(n+1).
\end{align*}
\]

(15)

Thus, the \textbf{DRO} problem is efficiently solvable as a second-order cone program provided that \( \mathcal{X} \) is second-order conic representable and \( \ell(x, \xi) \) is either a convex quadratic or a piecewise linear convex function of \( x \) for all \( \xi \).

6 Numerical Experiments

We evaluate the performance of the distributionally robust model (\textbf{DRO}) in the context of inventory management and wind energy commitment applications. All the experiments were run on a 2.2 GHz Intel Core i7 CPU laptop with 8 GB RAM and solved using MOSEK 9.2.

6.1 Inventory Management

We first consider the classical newsvendor problem with side information. Faced with an uncertain demand \( \tilde{\xi} \), the vendor is interested in determining the order quantity \( q \) that minimizes the overall cost. The vendor incurs a cost, which includes two components: holding cost and stock-out cost. Associated with order quantity \( q \), the cost function assumes the following form:

\[
\ell(q, \xi) = h(q - \xi)_{+} + b(\xi - q)_{+},
\]

(16)

where \( b \) and \( h \) denote respectively the per unit stock-out and holding costs. We assume that the random side information vector \( \tilde{\gamma} = (\tilde{t}, \tilde{p}) \) consists of two components: \( \tilde{t} \in [0, 15] \), which represents the time of the day and \( p \in [0, 10] \), which is a measure of the popularity of the product at any given time. We assume that the demand varies according to the conditional distribution \( \tilde{\xi} \sim U(\psi(\gamma) - 10, \psi(\gamma) + 10) \), which is uniform with mean

\[
\psi(\gamma) = 50 + 20 \cdot \sin \left( \frac{t}{\pi/3} \right) + 5p.
\]

(17)

In this equation, the first constant term represents a baseline demand for the product at any given time. The second term, which is a sinusoidal function of \( t \), aims to capture the fluctuations in demand based on time, while the final term represents a linear relationship in the popularity \( p \) of the product and its mean demand. Based on the derivation in \cite{15}, we obtain the following DRO
formulation for the newsvendor problem:

$$\begin{align*}
\text{min} & \quad \alpha - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \frac{\lambda}{\sqrt{2}} \nu \\
\text{s.t.} & \quad \alpha \geq z_i + \frac{\beta_i}{\sqrt{w_i}} \quad \forall i \in [n], \\
& \quad z_i \geq h s_i^+ + b s_i^-, \\
& \quad s_i^+ \geq q - \xi^i \quad \forall i \in [n], \\
& \quad s_i^- \geq \xi^i - q \quad \forall i \in [n], \\
& (\beta, \nu) \in \text{SOC}(n + 1), \\
& s^+, s^- \in \mathbb{R}_+^n, \ q \in \mathbb{R}_+, \\
& \alpha \in \mathbb{R}, \ z \in \mathbb{R}^n. 
\end{align*}$$

(18)

We measure the quality of the optimal solution $q^*$ obtained by solving the formulation (18) in terms of the out-of-sample loss for SOC formulation, which represents the true stochastic optimization problem with side information. Since we do not have access to the true conditional expectation of the loss function, we generate 500 samples of $\tilde{\xi}$ to approximate the conditional loss and solve the sample average approximation problem at each of the side information covariates $\gamma$ of interest.

In our problem setup, we set the parameters for the newsvendor problem to $b = 10$ and $h = 6$. We assume that the side information vector $\gamma$ has a bivariate normal distribution $\mathcal{N}(\mu, \Sigma)$, with mean $\mu = [7.5, 5]^T$ and covariance matrix $\Sigma = \text{Diag}([2, 1])$. For our experiments, we conduct 10 simulation runs for each side information covariate $\gamma$ and sample size $n$ of interest. In each simulation, we generate a training dataset $\{(\gamma_i, \xi_i^i)\}_{i=1}^n$ consisting of $n$ samples, solve the newsvendor DRO formulation, and evaluate the out-of-sample loss at each $\gamma$ of interest.

Figure 2 shows the results obtained. From the figure, we note that at points $\gamma = (1.5, 2.5)$ and $\gamma = (3, 2.5)$, where the density function values $g(\gamma)$ for the bivariate normal are much smaller, the regularization scheme is quite effective and the average out-of-sample loss decreases significantly with the increase in regularization parameter $\lambda$. On the other hand, for points close to the mean $\mu = [7.5, 5]^T$, for example, $\gamma = (4.5, 2.5)$ and $\gamma = (4.5, 5)$, the unregularized ($\lambda = 0$) version of the formulation perform much better. This is quite intuitive since, in regions of high density, the NW estimator forms a good approximation to the true conditional expectation even for small sample sizes. By contrast, in the regions where the density values are smaller, the regularization term seeks to control the amount of overfitting to the limited data available. Another important observation that we make is that, in regions of moderate density values, for example, for points $\gamma = (1.5, 5)$ and $\gamma = (3, 5)$, the regularization helps in the setting where the sample sizes are smaller ($n = 10$ and $n = 25$). This is consistent with the generalization bound obtained in Corollary 1.
Figure 2: Effect of regularization parameter $\lambda$ on the average out-of-sample loss for different sizes of training datasets at different side information covariates $\gamma$. 

(a) $\gamma = (1.5, 2.5)$
(b) $\gamma = (3, 2.5)$
(c) $\gamma = (4.5, 2.5)$
(d) $\gamma = (4.5, 5)$
(e) $\gamma = (1.5, 5)$
(f) $\gamma = (3, 5)$
6.2 Wind energy commitment

We next apply our DRO formulation to the wind energy problem considered in [Hannah and Dunson 2011] and [Kim and Powell 2011]. At the beginning of day $t$, a wind energy producer determines the wind energy commitment levels $x \in \mathbb{R}^{24}$ for the next 24 hours. The day-ahead prices $\pi_t \in \mathbb{R}^{24}$ are known to the decision maker. However, the hourly amounts of wind energy $\xi_t \in \mathbb{R}^{24}$ generated for the next 24 hours are uncertain. If the actual production falls short of the commitment level, there is a penalty of twice the respective day-ahead price for each unit of unsatisfied demand. As the wind energy is generally highly correlated to the past data, we consider the side information vector $\gamma = \xi_{t-1}$ in the implementation. Based on the derivation in (15), we arrive at the following DRO formulation for the wind energy commitment problem:

$$
\begin{align*}
\min \quad & \alpha - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \frac{\lambda}{\sqrt{2}} \nu \\
\text{s.t.} \quad & \alpha \geq -x^\top \pi_i + 2 \sum_{j=1}^{24} \pi_j \max\{x_j - \xi_j, 0\} + \frac{\beta_i}{\sqrt{w_i}} \quad \forall i \in [n], \\
& x \in \mathbb{R}_{+}^{24}, \quad \alpha \in \mathbb{R}, \quad (\beta, \nu) \in \text{SOC}(n+1).
\end{align*}
$$

(19)

In the experiment, we obtain the hourly wind energy data from North American Land Data Assimilation System\(^2\) from 2002 to 2011 at the following locations: Rhode Island (41.8252N, 71.4188W) and North Carolina (33.9375N, 77.9375W). The hourly day-ahead prices are downloaded from the publicly available PJM market dataset\(^3\). As the wind energy and day-ahead prices are closely related to seasons, we divide each year’s data into four parts according to different seasons and conduct out-of-sample tests on each of them separately. In each season, we assume the decision maker has access to the first $n+1$ days of data, and plans for the commitment levels for the next day. To incorporate side information, the historical data is then rearranged to $n$ samples of the form $\{(\gamma^i, \xi^{i+1})\}_{i=1}^{n}$, where we set $\gamma^i = \xi^i$ to be the covariate vector comprising of the wind energy productions on day $i$. As $\gamma^i$ is a 24-dimensional vector with high correlations between its components, we adopt the dimensionality reduction procedure described in Section 3.2 to determine a 3-dimensional subspace that explains more than 90% of the variability of the historical observations. The NW kernel weights $w_i$ are consequently computed using the projected data. We solve problem (19) to obtain the optimal commitment levels and evaluate its true profit using the next day’s data. We then drop the first day’s data and include the data of the $(n+2)$th day, and move on to the planning for the $(n+3)$th day. We repeat this process $N$ times, and compute the total profit for these $N$ days as one trial’s result. As there are 40 seasons in 10 years, we have 40 trials in total.

We then benchmark our Regularized Nadaraya-Watson (RNW) method with sample average approximation (SAA), the unregularized Nadaraya-Watson (NW) [Hannah and Dunson 2011] and the residual-based distributionally robust optimization (ERDRO) [Kannan et al., 2020b] methods.

\(^2\)https://climatedataguide.ucar.edu/climate-data/nldas-north-american-land-data-assimilation-system

\(^3\)http://dataminer2.pjm.com/feed/da_hrl_lmps/definition
in out-of-sample experiments. We also implemented the regularized linear decision rule (LDR) method [Bazier-Matte and Delage 2020]; however, the method performs poorly and thus we do not report the results. LDR fails in this experiment because the wind energy data is nonlinear and very complicated; such a parameterized regression model cannot fit it well and thus yields poor predictions. The ERDRO method assumes $\hat{\xi}$ can be modeled in terms of $\hat{\gamma}$ as $\hat{\xi} = f(\hat{\gamma}) + \tilde{\epsilon}$, where $f(\hat{\gamma}) = E[\hat{\xi}|\hat{\gamma}]$ is the regression function while $\tilde{\epsilon}$ are mean zero errors. For the same reason with LDR, we adopt the nonparametric Nadaraya-Watson regression model to predict $\hat{\xi}$ conditioned on the side information $\gamma$, and we solve for the best commitment level in view of the worst-case distribution from within a modified $\chi^2$ ambiguity set. With these settings, we find that the ERDRO model performs really well for this particular problem.

In the experiment, we set $n = 14$ and $N = 25$. The radius of the ambiguity set $\lambda$ and the bandwidth parameter $C_h$ are determined following a cross-validation procedure. In each trial, we split the first $2/3$ of the training set into a sub-training set and keep the remaining samples as a sub-validation set. Then we set the radius $\lambda$ to zero, and collect the total return of different bandwidth parameters $C_h \in [5 \times 10^2, 5 \times 10^4]$ on a logarithm searching grid with 9 equidistant points. Next, we fix the best $C_h$ obtained in the previous procedure and tune for the best radius $\lambda \in [10^{-2}, 10^2]$ on a logarithm searching grid with 17 equidistant points.

| Site | Statistic | NW | ERDRO | RNW |
|------|-----------|----|-------|-----|
| RI   | Mean      | 55.5 | 96.1 | 110.0 |
|      | 20th prct. | -6.3 | 13.1 | 45.7 |
|      | 80th prct. | 116.5 | 163.6 | 164.8 |
| NC   | Mean      | 64.2 | 69.1 | 79.1 |
|      | 20th prct. | -7.9 | -2.8 | 7.0 |
|      | 80th prct. | 179.1 | 192.5 | 189.4 |

Table 1: Statistics of improvements over SAA (%)
7 Concluding remarks

The NW approximation has recently garnered an increasing interest due to its significance in the context of decision-making under uncertainty with side information. The scheme, however, has so far resisted any sensible result about its out-of-sample performance. In this paper, we established for the first time a complete, comprehensive theoretical result on the performance guarantees of the approximation. The new result inspired us to design a novel regularization scheme that can better mitigate the overfitting effects. In contrast to the popular $L_2$ regularization scheme which attempts to minimize the norm of the decision vector and may pointlessly encourage an optimal solution that is close to the origin, our proposed regularization scheme is directly constructed using the conditional standard deviation term appearing in the theoretical bounds and can faithfully prioritize an optimal solution that generalizes well. In the future, it would be interesting to extend the model to the multi-stage setting, and devise a tractable solution procedure with similar performance guarantees for dynamic stochastic optimization problems.

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Appendices

A  Proof of Theorem 1

Proof. To prove the desired result, we define a random variable $\tilde{L} = L(\tilde{\xi})$ and $\hat{f}(\gamma, L)$ to be the joint density function of $(\hat{\gamma}, \tilde{L})$. Then, we show that our setting is eligible to apply Theorem 2 in Mokkadem et al. [2008], which requires the following conditions to hold.

(A) The kernel function $K: \mathbb{R}^p \to \mathbb{R}$ is a bounded and integrable function that satisfies

$$\int_{\mathbb{R}^p} K(\theta)d\theta = 1 \text{ and } \lim_{\|\theta\| \to \infty} K(\theta) = 0.$$ 

(B) For any $u \in \mathbb{R}$, the function $t \to \int_{\mathbb{R}} \exp(uL)\hat{f}(t, L)dL$ is continuous at $t = \gamma$ and bounded.

(C) For any $u \in \mathbb{R}$, the functions $t \to \int_{\mathbb{R}} u^2 L^2 \hat{f}(t, L)dL$ and $t \to \int_{\mathbb{R}} uL \hat{f}(t, L)dL$ is continuous at $t = \gamma$, and the marginal density $\hat{f}_\gamma(\gamma) \neq 0$, where $\gamma$ is the fixed side information of interest in (SO).

(D) The sequence $\{a_n\}_{n \in \mathbb{N}}$ is chosen such that

$$\lim_{n \to \infty} a_n = \infty \text{ and } \lim_{n \to \infty} \frac{a_n^2}{nh_n^p} = 0.$$ 

(E) There exists an integer $S \geq 2$ such that

(i) For $\forall s \in [S-1]$, $\forall j \in [p]$,

$$\int_{\mathbb{R}^p} \theta_j^s K(\theta)d\theta = 0 \text{ and } \int_{\mathbb{R}^p} |\theta_j^S K(\theta)|d\theta < \infty.$$ 

(ii) The chosen sequence $\{a_n\}_{n \in \mathbb{N}}$ satisfies $\lim_{n \to \infty} a_nh_n^S = 0$.

(iii) Both functions $\hat{f}_\gamma(\gamma)$ and $\int_{0}^{1} L \cdot \hat{f}(\gamma, L)dL$ are $S$-times differentiable on $\mathbb{R}^p$, and their differentials of order $S$ are bounded and continuous at $\gamma$.

In what follows, we will show that Assumption [A2] and all conditions in the theorem imply the conditions (A)-(E) above. We first notice that condition (A) holds for our choice of exponential kernel [1]. To show that condition (B) holds, we note that

$$E[\exp(uL(\xi)) | \hat{\gamma} = \gamma] = \frac{\int_{\mathbb{R}} \exp(uL)\hat{f}(\gamma, L)dL}{\hat{f}_\gamma(\gamma)} = \frac{\int_{\mathbb{R}^q} \exp(uL(\xi))f(\gamma, \xi)d\xi}{\hat{f}_\gamma(\gamma)}. \quad (A.1)$$

Since $\hat{f}_\gamma(\gamma) = f_\gamma(\gamma)$ and $\int_{\mathbb{R}} \exp(uL)\hat{f}(\gamma, L)dL = \int_{\mathbb{R}^q} \exp(uL(\xi))f(\gamma, \xi)d\xi$, therefore by condition 2 stated in the theorem, condition (B) holds. Following the same argument as above in (A.1) and
the conditions 1 and 3 stated in the theorem, the first part of condition (C) holds. Condition (D) holds as stated in the statement of the theorem. For condition (E), we show that it holds in our setting for $S = 2$. Firstly, we note that $\int_{\mathbb{R}^p} \theta_j K(\theta) d\theta = 0$, $\forall j \in [p]$, since the expectation of the distribution (1) is 0 due to the symmetry of this distribution. Moreover,

$$\int_{\mathbb{R}^p} |\theta_j^2 K(\theta)| d\theta = \frac{1}{Z} \int_{\mathbb{R}^p} |\theta_j^2 \exp(-\|\theta\|_2)| d\theta \leq \frac{1}{Z} \int_{\mathbb{R}^p} |\theta_j^2 \exp(-\|\theta\|_1/p)| d\theta < \infty,$$

where the last inequality holds because

$$\int_{\mathbb{R}} |\exp(-\theta_i/p)| d\theta_i < \infty \quad \text{and} \quad \int_{\mathbb{R}} |\theta_j^2 \exp(-\theta_j/p)| d\theta_j < \infty.$$

Thus, part (i) of condition (E) holds. Part (ii) of condition (E) also holds with $S = 2$ as stated in the statement of the theorem. Following the same argument that we used in (A.1) to show that condition (B) is satisfied, part (iii) of condition (E) holds from condition 3 in the statement of the theorem.

Therefore, we can apply Theorem 2 in Mokkadem et al. [2008] which implies that the sequence

$$\{a_n(\mathbb{E} \gamma[L(\tilde{\xi})] - \mathbb{E} \gamma[L(\tilde{\xi})])\}_{n \in \mathbb{N}}$$

obeys a large deviation principle with speed $nh_n^2/a_n^2$ and rate function

$$I_\gamma(y) = \frac{y^2 f_\gamma(y)}{2V_\gamma[L(\tilde{\xi})] \int_{\mathbb{R}^p} K^2(\theta) d\theta}.$$  \hfill (A.2)

### B Proof of Proposition 1

**Proof.** We set the function in (4) to $L(\xi) = \ell(x, \xi)$ and verify that the conditions in Theorem 1 are satisfied. To establish continuity of $\int_{\Xi} \ell(x, \xi)^2 f(\gamma, \xi) d\xi$ at $\gamma$, we fix $\epsilon > 0$ and show that there exists $\delta > 0$ such that

$$\|\gamma - \gamma'\| \leq \delta \implies \left| \int_{\Xi} \ell(x, \xi)^2 f(\gamma, \xi) d\xi - \int_{\Xi} \ell(x, \xi)^2 f(\gamma', \xi) d\xi \right| \leq \epsilon. \hfill (B.1)$$

Let $\mu(\Xi)$ be the Lebesgue measure of the support set $\Xi$. By assumption (A1), the following chain of inequalities hold:

$$\left| \int_{\Xi} \ell(x, \xi)^2 f(\gamma, \xi) d\xi - \int_{\Xi} \ell(x, \xi)^2 f(\gamma', \xi) d\xi \right| \leq \sup_{\xi \in \Xi} |\ell(x, \xi)^2| \sup_{\xi \in \Xi} |f(\gamma, \xi) - f(\gamma', \xi)| \mu(\Xi) \leq \sup_{\xi \in \Xi} |f(\gamma, \xi) - f(\gamma', \xi)| \mu(\Xi).$$

We now show that there exists $\delta > 0$ such that

$$\|\gamma - \gamma'\| \leq \delta \implies \sup_{\xi \in \Xi} |f(\gamma, \xi) - f(\gamma', \xi)| \leq \epsilon/\mu(\Xi). \hfill (B.2)$$
which is sufficient to prove the claim. Suppose for the sake of contradiction the implication \( (B.2) \)
do not hold. That is, for any \( \delta > 0 \), there exist \( \gamma'_\delta \) with \( \| \gamma - \gamma'_\delta \| \leq \delta \) and \( \xi_\delta \in \Xi \) such that \( |f(\gamma, \xi_\delta) - f(\gamma'_\delta, \xi_\delta)| > \epsilon/\mu(\Xi) \). By construction, we have \( \lim_{\delta \to 0} \gamma'_\delta = \gamma \). Let \( \xi^* \) be a limit point of the sequence \{\( \xi_\delta \)\} as \( \delta \to 0 \). By the compactness of the support set in assumption \((A1)\) we have \( \xi^* \in \Xi \). The continuity of the density function in assumption \((A2)\) then implies that

\[
\frac{\epsilon}{\mu(\Xi)} \leq \lim_{\delta \to 0} |f(\gamma, \xi_\delta) - f(\gamma'_\delta, \xi_\delta)| = |f(\gamma, \xi^*) - f(\gamma, \xi^*)| = 0,
\]

which is a contradiction because \( \epsilon/\mu(\Xi) > 0 \). We may thus conclude that the first condition in
Theorem \([1]\) is indeed satisfied.

By following the same argument, one can show that \( \int_{\mathbb{R}^q} \exp(u\ell(x, \xi)) f(\gamma, \xi) d\xi \) is continuous at \( \gamma \). The boundedness of the expression holds because \( \sup_{\xi \in \Xi} \exp(u\ell(x, \xi)) \leq \exp(u) \) for every \( u \in \mathbb{R} \). Thus, the second condition in Theorem \([1]\) is also satisfied. Finally, by the Leibniz’s rule we have

\[
\frac{\partial}{\partial \gamma_i} \int_{\Xi} \ell(x, \xi) f(\gamma, \xi) d\xi = \int_{\Xi} \ell(x, \xi) \frac{\partial}{\partial \gamma_i} f(\gamma, \xi) d\xi \quad \forall i \in [p] \quad \text{and}
\]

\[
\frac{\partial^2}{\partial \gamma_i \partial \gamma_j} \int_{\Xi} \ell(x, \xi) f(\gamma, \xi) d\xi = \int_{\Xi} \ell(x, \xi) \frac{\partial^2}{\partial \gamma_i \partial \gamma_j} f(\gamma, \xi) d\xi \quad \forall i, j \in [p].
\]

Thus, in view of our assumption that \( \partial f(\gamma, \xi)/\partial \gamma_i \) and \( \partial^2 f(\gamma, \xi)/(\partial \gamma_i \partial \gamma_j) \) are continuous and bounded, we may apply the same argument to conclude that the third condition in Theorem \([1]\) is also satisfied.

Next, let the closed and the open sets in \([1]\) be defined as \( C = (-\infty, -\epsilon'] \cup [\epsilon', \infty) \) and \( O = (-\infty, -\epsilon] \cup (\epsilon, \infty) \), respectively. The function \( I_\gamma(y) \) is a convex quadratic function centered at 0, which implies that \( \inf_{y \in C} I_\gamma(y) = \inf_{y \in O} I_\gamma(y) = I_\gamma(\epsilon') \). Thus, we obtain

\[
-I_\gamma(\epsilon') \leq \liminf_{n \to \infty} \frac{1}{\nu_n} \log \mathbb{P} \left( a_n \left( \mathbb{E}_\gamma[\ell(x, \hat{\xi})] - \tilde{\mathbb{E}}_\gamma[\ell(x, \hat{\xi})] \right) \in C \right)
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{\nu_n} \log \mathbb{P} \left( a_n \left( \mathbb{E}_\gamma[\ell(x, \hat{\xi})] - \tilde{\mathbb{E}}_\gamma[\ell(x, \hat{\xi})] \right) \in C \right) \leq -I_\gamma(\epsilon'),
\]

which gives rise to the stronger result

\[
\frac{1}{\nu_n} \log \mathbb{P} \left( \left| a_n \left( \mathbb{E}_\gamma[\ell(x, \hat{\xi})] - \tilde{\mathbb{E}}_\gamma[\ell(x, \hat{\xi})] \right) \right| \geq \epsilon' \right) = -I_\gamma(\epsilon') + o(1).
\]

Multiplying both sides of the inequality with \( \nu_n \), taking exponential, and substituting the definition of \( G_\gamma(\epsilon) \) yield

\[
\mathbb{P} \left( \left| a_n \left( \mathbb{E}_\gamma[\ell(x, \hat{\xi})] - \tilde{\mathbb{E}}_\gamma[\ell(x, \hat{\xi})] \right) \right| \geq \epsilon' \right) = \exp \left( -\frac{(\epsilon')^2 \nu_n g(\gamma)}{\mathbb{V}_\gamma[\ell(x, \hat{\xi})]} + o(\nu_n) \right).
\]
Since $\nu_n = nh_n^p/a_n^2$, we have
\[
P\left(\left| a_n \left( \mathbb{E}_\gamma[\ell(x, \xi)] - \hat{\mathbb{E}}_\gamma[\ell(x, \xi)] \right) \right| \geq \epsilon' \right) = \exp \left( -\frac{(\epsilon')^2 (nh_n^p/a_n^2)g(\gamma)}{\sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]}} + o(nh_n^p/a_n^2) \right).\]

We consider $a_n$'s that are strictly positive (there exists such $a_n$'s. For example, $a_n = \log n$) and denote $\epsilon = \epsilon a_n$ for some constant $\epsilon$, we obtain
\[
P\left(\left| \mathbb{E}_\gamma[\ell(x, \xi)] - \hat{\mathbb{E}}_\gamma[\ell(x, \xi)] \right| \geq \epsilon \right) = \exp \left( -\frac{\epsilon^2 (nh_n^p)g(\gamma)}{\sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]}} + o(nh_n^p/a_n^2) \right).
\]

Since $g(\gamma)$ and $\mathbb{V}_\gamma[\ell(x, \xi)]$ are constants for fixed $x$ and $\gamma$, the above is equivalent to
\[
P\left(\left| \mathbb{E}_\gamma[\ell(x, \xi)] - \hat{\mathbb{E}}_\gamma[\ell(x, \xi)] \right| \geq \epsilon \right) = \exp \left( -nh_n^p \epsilon^2 g(\gamma)(1 + o(1/a_n^2)) \right).
\]

Since $\lim_{a_n \to \infty} a_n^2 = \infty$, we complete the proof. \qed

\section{Proof of Theorem 3}

Before we prove Theorem 3, we first obtain some useful results for Lipschitz continuous loss functions in the following lemma.

\textbf{Lemma 1.} Assume that the loss function $\ell(x, \xi)$ is $M$-Lipschitz continuous in $x$, i.e., there exists a constant $M > 0$ such that
\[
|\ell(x, \xi) - \ell(x', \xi)| \leq M \|x - x'|| \quad \forall x, x' \in \mathcal{X}, \xi \in \Xi. \tag{C.1}
\]

Then, for any $\gamma \in \mathbb{R}^p$, we have
\[
\begin{align*}
|\mathbb{E}_\gamma[\ell(x, \xi)] - \mathbb{E}_\gamma[\ell(x', \xi)]| & \leq M \|x - x'|| \quad \forall x, x' \in \mathcal{X}, \\
|\hat{\mathbb{E}}_\gamma[\ell(x, \xi)] - \hat{\mathbb{E}}_\gamma[\ell(x', \xi)]| & \leq M \|x - x'|| \quad \forall x, x' \in \mathcal{X}, \\
|\sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]} - \sqrt{\mathbb{V}_\gamma[\ell(x', \xi)]}| & \leq M \|x - x'|| \quad \forall x, x' \in \mathcal{X}, \\
|\sqrt{\hat{\mathbb{V}}_\gamma[\ell(x, \xi)]} - \sqrt{\hat{\mathbb{V}}_\gamma[\ell(x', \xi)]}| & \leq M \|x - x'|| \quad \forall x, x' \in \mathcal{X}.
\end{align*}
\]

\textbf{Proof.} The first two inequalities above can be verified by directly applying (C.1). To show that the third inequality holds, we use the observation used in the proof of Proposition 3 that for any $x \in \mathcal{X}$,
\[
\sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]} = \min_{t \in [0, 1]} \sqrt{\mathbb{E}_\gamma[(\ell(x, \xi) - t)^2]}.
\]

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Without any loss of generality, we assume \( \sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]} \geq \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} \), and we obtain
\[
\sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]} - \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} = \min_{t \in [0,1]} \sqrt{\mathbb{E}_\gamma[(\ell(x, \xi) - t)^2]} - \min_{t \in [0,1]} \sqrt{\mathbb{E}_\gamma[(\ell(x', \tilde{\xi}) - t')^2]}
\leq \sqrt{\mathbb{E}_\gamma[(\ell(x, \xi) - t')^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x', \tilde{\xi}) - t')^2]},
\]
where \( t' = \arg \min_{t \in [0,1]} \sqrt{\mathbb{E}_\gamma[(\ell(x', \tilde{\xi}) - t')^2]} \).

Next, we note that the function \( \sqrt{\mathbb{E}_\gamma[\ell(\cdot)^2]} \) constitutes a semi-norm, which gives us that
\[
\sqrt{\mathbb{V}_\gamma[\ell(x, \xi)]} - \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} \leq \sqrt{\mathbb{E}_\gamma[(\ell(x, \xi) - t')^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x', \tilde{\xi}) - t')^2]}
\leq \sqrt{\mathbb{E}_\gamma[(\ell(x, \xi) - t')^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x', \tilde{\xi}) - t')^2]}
\leq M \| x - x' \|.
\]
Here the inequality \( (i) \) follows from the reverse triangle inequality while inequality \( (ii) \) is obtained by noting that \( \ell(x, \xi) \) is \( M \)-Lipschitz continuous for all \( \xi \in \Xi \). Thus, we have verified the third inequality. Using the same argument, we can show that the fourth inequality also holds.

**Proof of Theorem 3.** From Corollary 1 we have that for a fixed \( x \in \mathcal{X} \),
\[
\mathbb{E}_\gamma[\ell(x, \tilde{\xi})] \leq \hat{\mathbb{E}}_\gamma[\ell(x, \tilde{\xi})] + \sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} \leq \hat{\mathbb{E}}_\gamma[\ell(x, \tilde{\xi})] + \sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} \leq \hat{\mathbb{E}}_\gamma[\ell(x, \tilde{\xi})] + M \| x - x' \| \) (C.2)
\]
with probability \( 1 - \delta \). Next, define a finite set of points \( \mathcal{X}_\eta \subseteq \mathcal{X} \) such that for any \( x \in \mathcal{X} \), there exists some \( x' \in \mathcal{X}_\eta \) such that \( \| x - x' \| \leq \eta \). From Shapiro and Nemirovski [2005], we know that \( |\mathcal{X}_\eta| = O(1/d \eta) \). Since the loss function \( \ell(x, \tilde{\xi}) \) is \( M \)-Lipschitz continuous in \( x \), from Lemma 1 we have that for any \( x \in \mathcal{X} \), there exists some \( x' \in \mathcal{X}_\eta \), such that \( \| x - x' \| \leq \eta \) and the following condition holds:
\[
\mathbb{E}_\gamma[\ell(x, \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(x', \tilde{\xi})] + M \eta. \) (C.3)
\]
In addition, from Corollary 1 we note that for a fixed \( x' \in \mathcal{X}_\eta \)
\[
\mathbb{E}_\gamma[\ell(x', \tilde{\xi})] \leq \hat{\mathbb{E}}_\gamma[\ell(x', \tilde{\xi})] + \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} \leq \hat{\mathbb{E}}_\gamma[\ell(x', \tilde{\xi})] + \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} \leq \hat{\mathbb{E}}_\gamma[\ell(x', \tilde{\xi})] + M \eta \) (C.4)
\]
with probability at least \( 1 - \delta \). Applying union bound, we get that for all \( x' \in \mathcal{X}_\eta \)
\[
\mathbb{E}_\gamma[\ell(x', \tilde{\xi})] \leq \hat{\mathbb{E}}_\gamma[\ell(x', \tilde{\xi})] + \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} \leq \hat{\mathbb{E}}_\gamma[\ell(x', \tilde{\xi})] + \sqrt{\mathbb{V}_\gamma[\ell(x', \tilde{\xi})]} \leq \hat{\mathbb{E}}_\gamma[\ell(x', \tilde{\xi})] + M \eta \) (C.5)
\]
with probability at least \( 1 - \delta \). Combining the bounds in (C.3) and (C.5), we get that for any
\( x \in \mathcal{X} \), there exists some \( x' \in \mathcal{X}_\eta \), such that \( \|x - x'\| \leq \eta \) and

\[
\mathbb{E}_\gamma[\ell(x, \xi)] \leq \hat{\mathbb{E}}_\gamma[\ell(x', \xi)] + \sqrt{\frac{\mathbb{V}_\gamma[\ell(x', \xi)]}{n h_n^p g(\gamma)(1 + o(1))}} \log \left( \frac{|\mathcal{X}_\eta|}{\delta} \right) + M \eta
\]

with probability \( 1 - \delta \). Again, using the Lipschitz continuity of \( \ell(x, \xi) \), from Lemma 1 we get

\[
\mathbb{E}_\gamma[\ell(x, \xi)] \leq \hat{\mathbb{E}}_\gamma[\ell(x', \xi)] + \sqrt{\frac{\mathbb{V}_\gamma[\ell(x', \xi)]}{n h_n^p g(\gamma)(1 + o(1))}} \log \left( \frac{|\mathcal{X}_\eta|}{\delta} \right) + M \eta \left( 1 + \sqrt{\frac{\log \left( \frac{|\mathcal{X}_\eta|}{\delta} \right)}{n h_n^p g(\gamma)(1 + o(1))}} \right)
\]

with probability at least \( 1 - \delta \).

\[\square\]

### D Proofs of Proposition 2 and Corollary 2

Before we prove the result in Proposition 2, we first present below the statement of the Davis-Kahan Theorem and some useful results about sub-gaussian random vectors.

**Theorem 6** (Davis-Kahan Theorem (Theorem 2 in \cite{Yu et al., 2014})). Let \( \Sigma, \hat{\Sigma} \in \mathbb{R}^{p \times p} \) be symmetric with eigenvalues \( \lambda_1 \geq \ldots \geq \lambda_p \) and \( \hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_p \), respectively. Fix \( 1 \leq s \leq r \leq p \) and assume that \( \min(\lambda_{s-1} - \lambda_s, \lambda_r - \lambda_{r+1}) > 0 \), where \( \lambda_0 := \infty \) and \( \lambda_{p+1} := -\infty \). Let \( d = r - s + 1 \), and let \( U = [u_s, u_{s+1}, \ldots, u_r] \in \mathbb{R}^{p \times d} \) and \( \hat{U} = [\hat{u}_s, \hat{u}_{s+1}, \ldots, \hat{u}_r] \in \mathbb{R}^{p \times d} \) have orthonormal columns satisfying \( \Sigma u_j = \lambda_j u_j \) and \( \hat{\Sigma} \hat{u}_j = \hat{\lambda}_j \hat{u}_j \) for \( j = s, s+1, \ldots, r \). Then, there exists an orthogonal matrix \( O \in \mathbb{R}^{d \times d} \) such that

\[
\|UO - \hat{U}\|_F \leq \frac{2^{3/2} d^{1/2} \|\Sigma - \hat{\Sigma}\|_2}{\min(\lambda_{s-1} - \lambda_s, \lambda_r - \lambda_{r+1})}. \tag{D.1}
\]

**Lemma 2** (Covariance Estimation for Sub-Gaussian distributions (Corollary 5.50 in \cite{Vershynin, 2010})). Consider a sub-gaussian probability distribution in \( \mathbb{R}^p \) with true covariance matrix \( \Sigma \) and sample covariance matrix \( \hat{\Sigma} \) constructed from \( n \) i.i.d. observations. Let \( \epsilon \in (0, 1) \) and \( t \geq 1 \). Then, with probability at least \( 1 - 2 \exp(-t^2 p) \), we have

\[
\|\hat{\Sigma} - \Sigma\|_2 \leq \epsilon \text{ provided } n \geq C \left( \frac{t}{\epsilon} \right)^2 p. \tag{D.2}
\]

Here, \( C \) is a constant that depends only on the sub-gaussian parameter \( \sigma \) for the distribution.

**Lemma 3** (Theorem 1.19 in \cite{Rigollet and Hutter}). Let \( \tilde{\gamma} \in \mathbb{R}^d \) be a sub-gaussian random vector with sub-gaussian parameter \( \sigma \). Then, with probability \( 1 - \delta \), we have

\[
\|\tilde{\gamma}\| \leq 4\sigma \sqrt{d} + 2\sigma \sqrt{2 \log \left( \frac{1}{\delta} \right)}
\]

for some \( \delta \in (0, 1) \).
Lemma 4. Consider a sub-gaussian random vector \( \tilde{\gamma} \in \mathbb{R}^p \) with sub-gaussian parameter \( \sigma \). Let \( \Sigma, \hat{\Sigma} \in \mathbb{R}^{p \times p} \) denote respectively the true covariance matrix for \( \tilde{\gamma} \) and the sample covariance matrix estimated from \( n \) i.i.d. observations of \( \tilde{\gamma} \). Let \( U \) and \( \hat{U} \) be the matrices whose columns comprise the top \( p' \) eigenvectors of these covariance matrices. Suppose \( \tilde{\rho}, \hat{\rho} \in \mathbb{R}^{p'} \) denote respectively the true and estimated projections of \( \tilde{\gamma} \in \mathbb{R}^p \) onto the subspace spanned by the columns of \( U \) and \( \hat{U} \). Then, with probability at least \( 1 - 2\delta \), we have under some basis coordinate system

\[
\| \tilde{\rho} - \hat{\rho} \|_2 \leq \frac{C}{\lambda_{p'} - \lambda_{p'+1}} \sqrt{\frac{p'}{n} \log \left( \frac{2}{\delta} \right)} \gamma_{\max},
\]

where \( C > 0 \) is a constant that depends on the sub-gaussian parameter \( \sigma \), \( \lambda_{p'} \) is the \( p' \)-th largest eigenvalue of the true covariance matrix \( \Sigma \), and \( \gamma_{\max} := 4\sigma \left( \sqrt{p} + \frac{1}{2} \log \left( \frac{1}{\delta} \right) \right) \).

Proof. Based on the definition of \( \tilde{\rho} \) and \( \hat{\rho} \), we have

\[
\| \tilde{\rho} - \hat{\rho} \|_2 = \| (UO - \hat{U})\tilde{\gamma} \|_2 \\
\leq \| UO - \hat{U} \|_2 \| \tilde{\gamma} \|_2 \\
\leq \| UO - \hat{U} \|_F \| \tilde{\gamma} \|_2 \\
(i) \leq \sqrt{\frac{3}{2} \frac{p}{n} \log \left( \frac{2}{\delta} \right)} \| \tilde{\gamma} \|_2 \\
(ii) \leq \sqrt{\frac{3}{2} \frac{p}{n} \log \left( \frac{2}{\delta} \right)} \| \tilde{\gamma} \|_2,
\]

with probability at least \( 1 - \delta \). Here, \( O \) is an orthogonal (change-of-basis) matrix and \( C' > 0 \) is a constant that depends on the sub-gaussian parameter \( \sigma \). Inequality (i) follows from the application of the Davis-Kahan Theorem whose statement is detailed in Theorem 6. We obtain inequality (ii) by noting that \( \tilde{\gamma} \) is a sub-gaussian random vector and, therefore, Lemma 2 applies to our setting. Putting \( \epsilon = \sqrt{\frac{C'}{n} \log \left( \frac{1}{\delta} \right)} \) and \( t = \frac{1}{p} \log \left( \frac{1}{\delta} \right) \frac{1}{2} \) in Lemma 2, we get that \( \| \Sigma - \hat{\Sigma} \|_2 \leq \sqrt{\frac{C'}{n} \log \left( \frac{1}{\delta} \right)} \) with probability at least \( 1 - \delta \).

Next, we define \( \gamma_{\max} := 4\sigma \left( \sqrt{p} + \frac{1}{2} \log \left( \frac{1}{\delta} \right) \right) \). From Lemma 3, we get \( \| \tilde{\gamma} \| \leq \gamma_{\max} \) with probability at least \( 1 - \delta \). Therefore, by applying union bound to (D.4), we obtain that with probability at least \( 1 - 2\delta \),

\[
\| \rho - \hat{\rho} \|_2 \leq \frac{C}{\lambda_{p'} - \lambda_{p'+1}} \sqrt{\frac{p'}{n} \log \left( \frac{2}{\delta} \right)} \gamma_{\max}.
\]

Substituting \( C = 2^{3/2} \sqrt{C'} \), we obtain the desired result. \( \square \)

Proof of Proposition 3. To make the dependence of the bandwidth \( h \) explicit, we define the kernel function as

\[
k_h(x) = \frac{1}{Z} \exp \left( -\frac{x^2}{h^2} \right),
\]

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Next, we consider the second term in Equation (D.5), which gives us

\[ 1 = \sum_{i \in I} k_h(\eta_i) \| x, \xi_i \| - \sum_{j \in I_1} k_h(\eta_j) \| x, \xi_j \|. \]

Similarly, for each \( \eta^i \in \mathbb{R}^d \), its projection is expressed as \( \rho^i \in \mathbb{R}^{p'} \). Since the true subspace is not known, we estimate the projection matrix by \( \hat{U} \) obtained using principal component analysis (PCA), and denote the estimated projections of \( \rho \) and \( \rho^i \) by \( \hat{\rho} \) and \( \hat{\rho}^i \) respectively.

Let \( \eta_i = \| \rho - \rho^i \| \) and \( \eta'_i = \| \hat{\rho} - \hat{\rho}^i \| \). Using the result from Lemma 4 and applying union bound, we get that with probability at least \( 1 - 2(n_1 + 1)\delta \), \( \| \rho - \hat{\rho} \| \leq \tau \) and \( \| \rho^i - \hat{\rho}^i \| \leq \tau \) for all \( i \in I_1 \), where we set \( \tau = \frac{C}{\log(\frac{3}{\delta})} \gamma_{\max} \) and \( \gamma_{\max} = 4\sigma \left( \sqrt{p} + \sqrt{\frac{1}{2} \log \left( \frac{1}{\delta} \right)} \right) \) in the bound obtained from Equation (D.3). Therefore, by using reverse triangle and triangle inequalities, we get that \( |\eta_i - \eta'_i| \leq \| (\rho - \rho^i) - (\hat{\rho} - \hat{\rho}^i) \| \leq \| \rho - \hat{\rho} \| + \| \rho^i - \hat{\rho}^i \| \leq 2\tau \) with probability at least \( 1 - 2(n_1 + 1)\delta \).

We are now in a position to obtain a bound for \( |E_{\rho}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]| \) for a fixed \( x \in \mathcal{X} \). From triangle inequality, we first note that

\[ |E_{\rho}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]| \leq |E_{\rho}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]| + |E_{\rho}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]|. \] (D.5)

Next, we obtain high probability bounds for each term in the right hand side of the above expression. By applying Corollary 4 we get that with probability at least \( 1 - \delta \), the first term is upper-bounded as

\[ |E_{\rho}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]| \leq \sqrt{\frac{\mathbb{V}_{\rho}[\ell(x, \tilde{\xi})]}{n_1 h_{n_1} g(\rho)(1 + o(1))} \log \left( \frac{1}{\delta} \right)} \]

Next, we consider the second term in Equation (D.5), which gives us

\[ |\hat{E}_{\rho}[\ell(x, \tilde{\xi})] - \hat{E}_{\rho}[\ell(x, \tilde{\xi})]| = \left| \sum_{i \in I_1} \frac{k_h(\eta_i)}{k_h(\eta_j)} \sum_{j \in I_1} k_h(\eta_j) \ell(x, \xi_i) - \sum_{j \in I_1} \frac{k_h(\eta_j)}{k_h(\eta_j)} \sum_{i \in I_1} k_h(\eta_i) \ell(x, \xi_j) \right| \]

\[ \leq \left( \max_{i} |\ell(x, \xi_i)| \right) \sum_{i \in I_1} \left| \frac{k_h(\eta_i)}{k_h(\eta_j)} \sum_{j \in I_1} k_h(\eta_j) - \frac{k_h(\eta_j)}{k_h(\eta_j)} \sum_{j \in I_1} k_h(\eta_i) \right| \]

\[ \leq \sum_{i \in I_1} \left| \frac{k_h(\eta_i)}{k_h(\eta_j)} - \frac{k_h(\eta_j)}{k_h(\eta_i)} \right|. \] (D.6)

Here, the last inequality follows from the Assumption (A3) that \( \ell(x, \xi) \) takes values between 0 and 1.
for all $x \in \mathcal{X}$ and $\xi \in \Xi$. Next, we obtain a bound for each term within the summation below.

\[
\left| \frac{k_h(\eta)}{\sum_{j \in I_1} k_h(\eta_j)} - \frac{k_h(\eta')}{\sum_{j \in I_1} k_h(\eta'_j)} \right| \\
= \max \left\{ \frac{k_h(\eta)}{\sum_{j \in I_1} k_h(\eta_j)} - \frac{k_h(\eta')}{\sum_{j \in I_1} k_h(\eta'_j)}, \frac{k_h(\eta')}{\sum_{j \in I_1} k_h(\eta'_j)} - \frac{k_h(\eta)}{\sum_{j \in I_1} k_h(\eta_j)} \right\} \\
\leq \max \left\{ \frac{k_h(\eta)}{\sum_{j \in I_1} k_h(\eta_j)} - \frac{k_h(\eta - 2\tau)}{\sum_{j \in I_1} k_h(\eta_j)}, \frac{k_h(\eta - 2\tau)}{\sum_{j \in I_1} k_h(\eta_j)} - \frac{k_h(\eta)}{\sum_{j \in I_1} k_h(\eta_j)} \right\} \\
= \max \left\{ \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} - \frac{\exp(-\eta + 2\tau/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)}, \frac{\exp(-\eta + 2\tau/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} - \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} \right\} \\
\overset{(i)}{=} \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} \max \{ 1 - \exp(-4\tau/h), \exp(4\tau/h) - 1 \} \\
\leq \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} (\exp(4\tau/h) - 1).
\]

Here, we obtain equality $(i)$ by considering each of the two terms within the max operator in the previous expression separately. We first evaluate the first term

\[
\frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} - \frac{\exp(-\eta + 2\tau/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} = \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} (1 - \exp(-4\tau/h)).
\]

Next, we consider the second term

\[
\frac{\exp(-\eta + 2\tau/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} - \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} = \frac{\exp(-\eta/h)}{\sum_{j \in I_1} \exp(-\eta_j/h)} (\exp(4\tau/h) - 1).
\]

Combining the results from (D.6) and (D.7), we get

\[
\left| \sum_{i \in I_1} \frac{k_h(\eta)}{\sum_{j \in I_1} k_h(\eta_j)} \ell(x, \xi^i) - \sum_{i \in I_1} \frac{k_h(\eta')}{\sum_{j \in I_1} k_h(\eta'_j)} \ell(x, \xi^i) \right| \leq \exp(4\tau/h) - 1 \\
\overset{(i)}{\leq} \frac{4\tau}{h} + \left( \frac{4\tau}{h} \right)^2 \\
\overset{(ii)}{\leq} \frac{8\tau}{h}.
\]

Note that $h$ is scaled with $n_1$ such that $n_2^{-1/2}/h < 1$. Thus, $4\tau/h < 1$ for sufficiently large $n_1$ and $n_2$, and inequalities $(i)$ and $(ii)$ then follow from the fact that $e^x \leq 1 + x + x^2$ and $x^2 \leq x$ for $x \leq 1$. Therefore, we have that with probability at least $1 - 2(n_1 + 1)\delta - \delta \geq 1 - 5n_1\delta$, for a fixed $x \in \mathcal{X}$,
we have

\[
\left| \mathbb{E}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right| \leq \sqrt{\frac{\mathbb{V}_\rho [\ell(x, \tilde{\xi})]}{n_1 h_{n_1}^p g(\rho)(1+o(1))}} \log \left( \frac{1}{\delta} \right) + \frac{8\tau}{h}
\]

\[
= \sqrt{\frac{\mathbb{V}_\rho [\ell(x, \tilde{\xi})]}{n_1 h_{n_1}^p g(\rho)(1+o(1))}} \log \left( \frac{1}{\delta} \right) + \frac{8\sigma C}{h \lambda_{p'} - \lambda_{p'+1}} \sqrt{\frac{p'}{n_2 \log \left( \frac{2|\mathcal{X}|}{\delta} \right)}} \sqrt{p} + \sqrt{\frac{1}{2} \log \left( \frac{|\mathcal{X}|}{\delta} \right)}
\]

Therefore, by applying union bound, we get that with probability at least 1 \(- 5n_1 \delta\),

\[
\left| \mathbb{E}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right| \leq \sqrt{\frac{\mathbb{V}_\rho [\ell(x, \tilde{\xi})]}{n_1 h_{n_1}^p g(\rho)(1+o(1))}} \log \left( \frac{|\mathcal{X}|}{\delta} \right) + \frac{8\sigma C}{h \lambda_{p'} - \lambda_{p'+1}} \sqrt{\frac{p'}{n_2 \log \left( \frac{2|\mathcal{X}|}{\delta} \right)}} \sqrt{p} + \sqrt{\frac{1}{2} \log \left( \frac{|\mathcal{X}|}{\delta} \right)} \quad \forall x \in \mathcal{X}.
\]

The result then follows by performing the change of variable \(\delta \leftarrow 5n_1 \delta\), and by noting that \(\mathbb{E}_\gamma [\ell(x, \tilde{\xi})] = \mathbb{E}_\rho [\ell(x, \tilde{\xi})]\) and \(\mathbb{V}_\gamma [\ell(x, \tilde{\xi})] = \mathbb{V}_\rho [\ell(x, \tilde{\xi})]\).

Proof of Corollary \ref{cor:bounded_gammas}[With bounded \(\gamma\)]. We consider the same setup as in the proof of Proposition \ref{prop:bounded_gammas}, where we define the kernel function as

\[ k_h(x) = \frac{1}{Z} \exp \left( -\frac{x}{h} \right), \]

with \(Z\) a normalization constant. As before, for any generic \(\gamma \in \mathbb{R}^p\), we denote its projection on the low-dimensional space as \(\rho \in \mathbb{R}^{p'}\). Similarly, for each \(\gamma^i \in \mathbb{R}^p\), its projection is expressed as \(\rho^i \in \mathbb{R}^{p'}\). Since the true subspace is not known, we estimate the projection matrix by \(\hat{\mathbf{U}}\) obtained using PCA, and denote the estimated projections of \(\rho\) and \(\rho^i\) by \(\hat{\rho}\) and \(\hat{\rho}^i\), respectively.

Let \(\eta_i = \|\rho - \rho^i\|\) and \(\eta'_i = \|\hat{\rho} - \hat{\rho}^i\|\). Using the result from the proof of Lemma \ref{lem:bounded_gammas}, if \(\|\gamma\| \leq \gamma_{\text{max}}\) almost surely, then we get that with probability at least \(1 - \delta\), \(\|\rho - \hat{\rho}\| \leq \tau\) and \(\|\rho^i - \hat{\rho}^i\| \leq \tau\) for all \(i \in \mathcal{I}_1\), where we set \(\tau = \frac{C}{\lambda_{p'} - \lambda_{p'+1}} \sqrt{\frac{p'}{n_2 \log \left( \frac{2}{\delta} \right)}} \gamma_{\text{max}}\) in the bound obtained from Equation (D.3).

The rest of this proof proceeds as that of Proposition \ref{prop:bounded_gammas}. In particular, from triangle inequality, we first note that

\[
\left| \mathbb{E}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right| \leq \left| \mathbb{E}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right| + \left| \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right|
\]

where the first term in the right hand side is upper-bounded as

\[
\left| \mathbb{E}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right| \leq \sqrt{\frac{\mathbb{V}_\rho [\ell(x, \tilde{\xi})]}{n_1 h_{n_1}^p g(\rho)(1+o(1))}} \log \left( \frac{1}{\delta} \right)
\]

with probability at least \(1 - \delta\). The second term gives us

\[
\left| \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] - \hat{\mathbb{E}}_\rho [\ell(x, \tilde{\xi})] \right| \leq \frac{8\tau}{h}
\]
Therefore, we have that with probability at least $1 - \delta - \delta = 1 - 2\delta$, for a fixed $x \in \mathcal{X}$, we have

$$|E_\rho[\ell'(x,\xi)] - \hat{E}_\rho[\ell'(x,\xi)]| \leq \sqrt{\frac{V_\rho[\ell'(x,\xi)]}{n_1h'_n g(\rho)(1 + o(1))}} \log \frac{1}{\delta} + \frac{8\tau}{h}$$

$$= \sqrt{\frac{V_\rho[\ell'(x,\xi)]}{n_1h'_n g(\rho)(1 + o(1))}} \log \frac{1}{\delta} + \frac{8}{h} \frac{C}{\lambda_{\rho'} - \lambda_{\rho' + 1}} \sqrt{\frac{p'}{n_2} \log \frac{2}{\delta}} \gamma_{\max}.$$  

By applying union bound, we get that with probability at least $1 - \delta$, 

$$|E_\rho[\ell'(x,\xi)] - \hat{E}_\rho[\ell'(x,\xi)]| \leq \sqrt{\frac{V_\rho[\ell'(x,\xi)]}{n_1h'_n g(\rho)(1 + o(1))}} \log \frac{2|\mathcal{X}|}{\delta} + \frac{8}{h} \frac{C}{\lambda_{\rho'} - \lambda_{\rho' + 1}} \sqrt{\frac{n_2}{p'} \log \frac{4|\mathcal{X}|}{\delta}} \gamma_{\max} \quad \forall x \in \mathcal{X}.$$  

The result then follows by noting that $E_\gamma[\ell'(x,\xi)] = E_\rho[\ell'(x,\xi)]$ and $V_\gamma[\ell'(x,\xi)] = V_\rho[\ell'(x,\xi)].$  

### E Proof of Proposition 3

To prove Proposition 3 we rely on the following lemma.

**Lemma 5.** For any fixed $x \in \mathcal{X}$ and $t \in [0, 1]$, we have

$$\sqrt{E_\gamma[(\ell(x,\xi) - t)^2]} - \sqrt{\hat{E}_\gamma[(\ell(x,\xi) - t)^2]} \leq \sqrt{\frac{\log \left( \frac{1}{\delta} \right)}{nh_n^\gamma g(\gamma)(1 + o(1))}},$$  

(E.1)

with probability at least $1 - \delta$.

**Proof.** By applying Theorem 1 to the input function $(\ell(x,\xi) - t)^2$, which also satisfies all conditions in the theorem, we obtain that with a probability at least $1 - \delta$

$$\left| E_\gamma[(\ell(x,\xi) - t)^2] - \hat{E}_\gamma[(\ell(x,\xi) - t)^2] \right| \leq \sqrt{\frac{V_\gamma[(\ell(x,\xi) - t)^2]}{nh_n^\gamma g(\gamma)(1 + o(1))}} \log \left( \frac{1}{\delta} \right)$$

$$\leq \sqrt{\frac{E_\gamma[(\ell(x,\xi) - t)^2]}{nh_n^\gamma g(\gamma)(1 + o(1))}} \log \left( \frac{1}{\delta} \right).$$

Here, the last inequality follows from

$$V_\gamma[(\ell(\xi) - t)^2] = E_\gamma[(\ell(\xi) - t)^4] - E_\gamma[(\ell(\xi) - t)^2]^2$$

$$\leq E_\gamma[(\ell(\xi) - t)^4]$$

$$\leq E_\gamma[(\ell(\xi) - t)^2],$$

where the final inequality holds because the random variable $(\ell(x,\xi) - t)^2$ is supported on a subset
of \([0, 1]\). Next, expanding the absolute value term yields the following two cases:

\[
\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2] - \mathbb{E}_\gamma[(\ell(x, \xi) - t)^2] \leq \sqrt{\frac{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]}{nh_n^p g(\gamma)(1 + o(1))} \log \left( \frac{1}{\delta} \right)}.
\]

and

\[
\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2] - \mathbb{E}_\gamma[(\ell(x, \xi) - t)^2] \leq \sqrt{\frac{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]}{nh_n^p g(\gamma)(1 + o(1))} \log \left( \frac{1}{\delta} \right)}.
\]

(E.2)

From the first case, we obtain

\[
\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2] - \sqrt{\frac{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]}{nh_n^p g(\gamma)(1 + o(1))} \log \left( \frac{1}{\delta} \right)} \leq \mathbb{E}_\gamma[(\ell(x, \xi) - t)^2],
\]

which is equivalent to

\[
\left( \sqrt{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]} - \frac{1}{2} \sqrt{\frac{\log \left( \frac{1}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))}} \right)^2 \leq \frac{1}{4 nh_n^p g(\gamma)(1 + o(1))} + \mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2].
\]

Taking square root on both sides then yields

\[
\sqrt{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]} - \frac{1}{2} \sqrt{\frac{\log \left( \frac{1}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))}} \leq \frac{1}{2} \sqrt{\frac{\log \left( \frac{1}{\delta} \right)}{nh_n^p g(\gamma)(1 + o(1))}} + \mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2] - \frac{1}{4 nh_n^p g(\gamma)(1 + o(1))} + \sqrt{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]},
\]

(E.3)

where the last inequality follows from the relation \(\sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2}\). Next, the second case in (E.2) yields

\[
\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2] \leq \mathbb{E}_\gamma[(\ell(x, \xi) - t)^2] + \sqrt{\frac{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]}{nh_n^p g(\gamma)(1 + o(1))} \log \left( \frac{1}{\delta} \right)} + \mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2] - \frac{1}{4 nh_n^p g(\gamma)(1 + o(1))} \log \left( \frac{1}{\delta} \right).
\]

Finally, taking square root on both sides and combining with the inequality in (E.3), we conclude that the bound in (E.1) indeed holds. This completes the proof.

Using this lemma, we prove the bound of the error introduced by the empirical conditional standard deviation.

**Proof of Proposition 3** We first show that the function \(\sqrt{\mathbb{E}_\gamma[(\ell(x, \hat{\xi}) - t)^2]}\) is Lipschitz continuous
in $t$ with constant $1$. Indeed, by the reverse triangle inequality, we have

$$
|\sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t)^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t')^2]}| \leq \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t - \ell(x, \tilde{\xi}) + t')^2]} = |t - t'|,
$$

(E.4)

where the inequality holds because the function $\sqrt{\mathbb{E}_\gamma[(\cdot)^2]}$ constitutes a semi-norm. One can similarly show that the function $\sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t)^2]}$ is Lipschitz continuous in $t$ with constant $1$. We next observe that

$$
\sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} = \min_{t \in [0,1]} \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t)^2]} \quad \text{and} \quad \sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} = \min_{t \in [0,1]} \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t)^2]},
$$

which follows from the fact that the minimizers of these scalar optimization problems are respectively given by the mean $\mathbb{E}_\gamma[\ell(x, \tilde{\xi})]$ and the empirical mean $\mathbb{E}_\gamma[\ell(x, \hat{\xi})]$. Consider now a finite subset $T = \{0, \tau/2, \tau, 3\tau/2, \ldots, 1\}$ of $[0,1]$ with cardinality $|T| = 1 + 1/(\tau/2) = 1 + 2/\tau$. Let $t^*$ and $\hat{t}^*$, respectively, be the minimizers of the above optimization problems over the subset $T$ instead of $[0,1]$. By the Lipschitz continuity of the objective functions, we can guarantee that

$$
|\sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} - \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t^*)^2]}| \leq \tau/2 \quad \text{and} \quad |\sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} - \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - \hat{t}^*)^2]}| \leq \tau/2.
$$

Thus, to ensure that the bound $|\sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]} - \sqrt{\mathbb{V}_\gamma[\ell(x, \tilde{\xi})]}| \leq \epsilon$ holds, we require the sufficient condition

$$
|\sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t^*)^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - \hat{t}^*)^2]}| \leq \epsilon - \tau.
$$

Note that the left-hand side expression is upper bounded by the largest error

$$
\max_{t \in T} \left| \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t)^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t)^2]} \right|.
$$

Thus, applying the union bound to (E.1) over $t \in T$ yields an upper bound on left-hand side expression, as follows

$$
\left| \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - t^*)^2]} - \sqrt{\mathbb{E}_\gamma[(\ell(x, \tilde{\xi}) - \hat{t}^*)^2]} \right| \leq \frac{\log \left( \frac{|T|}{\delta} \right)}{n h^p_{\ell^p}(\gamma)(1 + o(1))}.
$$

The result then follows by equating the right hand side with $\epsilon - \tau$. \hfill \Box

\section{Proof of Theorem 4}

Using the result in Proposition 3, we first obtain a new generalization bound in view of the empirical conditional standard deviation.
Lemma 6. Fix a tolerance level \( \tau > 0 \). Then, for any \( x \in X \), we have

\[
|E_\gamma[\ell(x, \tilde{\xi})] - \hat{E}_\gamma[\ell(x, \tilde{\xi})]| \leq \left( \sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} + \tau \right) \sqrt{\frac{\log \left( \frac{2}{\delta} \right)}{n h_n g(\gamma)(1 + o(1))}} + \frac{\sqrt{\log \left( \frac{2(1 + 2/\tau)}{\delta} \right) \log \left( \frac{2}{\delta} \right)}}{n h_n g(\gamma)(1 + o(1))},
\]

with probability at least \( 1 - \delta \).

Proof. The bounds in (6) and (10) yield

\[
|E_\gamma[\ell(x, \tilde{\xi})] - \hat{E}_\gamma[\ell(x, \tilde{\xi})]| \leq \left( \sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} + \sqrt{V_\gamma[\ell(x, \tilde{\xi})]} \right) \sqrt{\frac{\log \left( \frac{2}{\delta} \right)}{n h_n g(\gamma)(1 + o(1))}} + \frac{\sqrt{\log \left( \frac{1 + 2/\tau}{\delta} \right) \log \left( \frac{2}{\delta} \right)}}{n h_n g(\gamma)(1 + o(1))}.
\]

The above inequality holds with probability at least \( 1 - 2\delta \), which completes the proof. \( \square \)

The above lemma shows that the errors introduced by replacing the conditional variance term with its empirical estimates diminish at the faster rate of \( O\left( \frac{1}{n h_n g(\gamma)} \right) \), and become negligible when the sample size is large.

Proof of Theorem 4. Applying the union bound to (F.1) over \( x \in X \), we find that with probability at least \( 1 - \delta \),

\[
|E_\gamma[\ell(x, \tilde{\xi})] - \hat{E}_\gamma[\ell(x, \tilde{\xi})]| \leq \left( \sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} + \tau \right) \sqrt{\frac{\log \left( \frac{2|X|}{\delta} \right)}{n h_n g(\gamma)(1 + o(1))}} + \frac{\sqrt{\log \left( \frac{2|X|(1 + 2/\tau)}{\delta} \right) \log \left( \frac{2}{\delta} \right)}}{n h_n g(\gamma)(1 + o(1))},
\]

\( \forall x \in X \).

Thus, for \( x = \hat{x} \) we get

\[
E_\gamma[\ell(\hat{x}, \tilde{\xi})] \leq \hat{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] + \left( \sqrt{\hat{V}_\gamma[\ell(\hat{x}, \tilde{\xi})]} + \tau \right) \sqrt{\frac{\log \left( \frac{2|X|}{\delta} \right)}{n h_n g(\gamma)(1 + o(1))}} + \frac{\sqrt{\log \left( \frac{2|X|(1 + 2/\tau)}{\delta} \right) \log \left( \frac{2}{\delta} \right)}}{n h_n g(\gamma)(1 + o(1))},
\]

where the second inequality holds because \( \hat{x} \) is suboptimal for the regularized problem \( (\hat{R}N\hat{W}) \).
Next, applying the bound (6) for \( \hat{E}_\gamma[\ell(x^*, \tilde{\xi})] \) and the bound (10) for \( \sqrt{\hat{V}_\gamma[\ell(x^*, \tilde{\xi})]} \), we obtain

\[
\mathbb{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(x^*, \tilde{\xi})] + \sqrt{\frac{\mathbb{V}_\gamma[\ell(x^*, \tilde{\xi})]}{n h_n^p g(\gamma)(1 + o(1)) \log \left( \frac{3}{\delta} \right)}} \\
+ \left( \sqrt{\mathbb{V}_\gamma[\ell(x^*, \tilde{\xi})]} + 2\tau + \frac{\log \left( \frac{3 \gamma}{\delta} \right)}{n h_n^p g(\gamma)(1 + o(1))} \right) \sqrt{\frac{\log \left( \frac{6|X|}{\delta} \right)}{n h_n^p g(\gamma)(1 + o(1))}} \\
+ \frac{\log \left( \frac{6|X|(1+2/\gamma)}{\delta} \right)}{n h_n^p g(\gamma)(1 + o(1))}.
\]

Finally, after performing further algebraic simplifications, we arrive at the desired bound. This completes the proof. \(\square\)

G Proof of Theorem 5

Proof. Recall that \( x^* \in X \) minimize the true conditional expectation \( \mathbb{E}_\gamma[\ell(x, \tilde{\xi})] \) over all \( x \in X \). Next, consider a fixed parameter \( \eta > 0 \). As before, similar to the proof of Theorem 3, we define a finite set of points \( X_\eta \subset X \) such that \( x^* \in X_\eta \) and for any \( x \in X \), there exists \( x' \in X_\eta \) such that \( \|x - x'\| \leq \eta \). From Shapiro and Nemirovski [2005], we know that the cardinality for the set \( |X_\eta| = O(1)(D/\eta)^d \).

Let \( \hat{x} \in X \) and \( \hat{x}' \in X_\eta \) denote the minimizers of \( \hat{E}_\gamma[\ell(x, \tilde{\xi})] + \lambda \sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} \) over \( X \) and \( X_\eta \) respectively. Next, consider a solution \( \hat{x}'' \in X_\eta \) such that \( \|\hat{x} - \hat{x}''\| \leq \eta \). Using the result obtained in Lemma 1 by Lipschitz continuity of \( \hat{E}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + \lambda \sqrt{\hat{V}_\gamma[\ell(\hat{x}'', \tilde{\xi})]} \), we have

\[
\hat{E}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + \lambda \sqrt{\hat{V}_\gamma[\ell(\hat{x}'', \tilde{\xi})]} \leq \hat{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] + \lambda \sqrt{\hat{V}_\gamma[\ell(\hat{x}, \tilde{\xi})]} + M''||\hat{x}'' - \hat{x}|| \\
\leq \hat{E}_\gamma[\ell(\hat{x}', \tilde{\xi})] + \lambda \sqrt{\hat{V}_\gamma[\ell(\hat{x}', \tilde{\xi})]} + M'\eta \\
\leq \hat{E}_\gamma[\ell(x^*, \tilde{\xi})] + \lambda \sqrt{\hat{V}_\gamma[\ell(x^*, \tilde{\xi})]} + M'\eta, \tag{G.1}
\]

where \( M' = (1 + \lambda)M \). Furthermore, since \( X_\eta \) is finite, we can apply the same approach as used in the proof of Theorem 4 to obtain the following result:

\[
\left| \mathbb{E}_\gamma[\ell(x, \tilde{\xi})] - \hat{E}_\gamma[\ell(x, \tilde{\xi})] \right| \leq \left( \sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} + \tau \right) \sqrt{\frac{\log \left( \frac{2|X_n|}{\delta} \right)}{n h_n^p g(\gamma)(1 + o(1))}} + \frac{\log \left( \frac{2|X_n|(1+2/\gamma)}{\delta} \right)}{n h_n^p g(\gamma)(1 + o(1))},
\]

with probability at least \( 1 - \delta \). Substituting \( x = \hat{x}'' \) and using the same approach as discussed in
the proof of Theorem 4 we obtain

\[ \mathbb{E}_\gamma[\ell(\hat{x}'', \tilde{\xi})] \leq \hat{\mathbb{E}}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + \left( \sqrt{\hat{\mathbb{V}}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + \tau} \right) \left( \frac{\log \left( \frac{2|X_0|}{\delta} \right) + \log \left( \frac{2|X_0| (1 + 2/\gamma)}{\delta} \right)}{nh_\mathcal{B}^3 g(\gamma)(1 + o(1))} \right), \]

where the second inequality follows from (G.1). Following the steps in Theorem 4, we obtain

\[ \mathbb{E}_\gamma[\ell(\hat{x}'', \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + M' \eta + \left( \sqrt{\mathbb{V}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + \tau} \right) \left( \frac{4 \log \left( \frac{6|X_0|}{\delta} \right) + 2 \log \left( \frac{6|X_0| (1 + 2/\gamma)}{\delta} \right)}{nh_\mathcal{B}^3 g(\gamma)(1 + o(1))} \right), \]

with probability at least \( 1 - \delta \). Furthermore, from Lipschitz continuity of \( \ell \), we get that

\[ \mathbb{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(\hat{x}'', \tilde{\xi})] + M \eta. \]

This gives us the final bound below

\[ \mathbb{E}_\gamma[\ell(\hat{x}, \tilde{\xi})] \leq \mathbb{E}_\gamma[\ell(\hat{x}', \tilde{\xi})] + (M + M') \eta + \left( \sqrt{\mathbb{V}_\gamma[\ell(\hat{x}', \tilde{\xi})] + \tau} \right) \left( \frac{4 \log \left( \frac{6|X_0|}{\delta} \right) + 2 \log \left( \frac{6|X_0| (1 + 2/\gamma)}{\delta} \right)}{nh_\mathcal{B}^3 g(\gamma)(1 + o(1))} \right), \]

\( \square \)

### H Proof of Proposition 5

**Proof.** The proof of this proposition follows and generalizes the approach discussed in [Duchi and Namkoong 2019](#). To simplify the notation, we define a random variable \( \tilde{z} = \ell(\hat{x}, \tilde{\xi}) \) and a vector \( z \in \mathbb{R}^n \) where \( z_i = \ell(x, \xi^i) \). We denote

\[ z = \hat{\mathbb{E}}_\gamma[\ell(\hat{x}, \tilde{\xi})] = \sum_{i=1}^n w_i \cdot \ell(x, \xi^i) \quad \text{and} \quad s = \hat{\mathbb{V}}_\gamma[\ell(\hat{x}, \tilde{\xi})] = \sum_{i=1}^n w_i \left( \ell(x, \xi^i) - \frac{z}{2} \right)^2. \]

The DRO problem \( \max_{\mathbb{P} \in \mathcal{P}_h(\hat{x}_\gamma)} \mathbb{E}_\mathbb{P}[\ell(\hat{x}, \tilde{\xi})] \) can be equivalently written as

\[ \max_w \left\{ w^\top z : \sum_{i=1}^n \frac{1}{2w_i} (w_i - \tilde{w}_i)^2 \leq \rho, \ w^\top e = 1, \ w \in \mathbb{R}_+^n \right\}, \]

where \( \rho = \frac{\lambda^2}{\tau} \). By change of variable \( u = w - \tilde{w} \), the above problem is equivalent to

\[ \max_u \left\{ z + u^\top (z - z \cdot e) : \|u\|_Q^2 \leq \rho, \ u^\top e = 0, \ u + \tilde{w} \geq 0 \right\}, \]
where \( \| u \|_W := \sqrt{\sum_{i=1}^{n} \frac{1}{2\overline{w}_i} (u_i)^2} \) is defined to be a weighted norm. We further define its dual norm \( \| u \|_{W^{-1}} := \sqrt{\sum_{i=1}^{n} 2\overline{w}_i (u_i)^2} \), and the upper bound of the above optimization problem is

\[
\overline{z} + u^\top (z - \overline{z} \cdot e) \leq \overline{z} + \| u \|_W \| z - \overline{z} \cdot e \|_{W^{-1}} \leq \overline{z} + \sqrt{\rho \| z - \overline{z} \cdot e \|_{W^{-1}} = \overline{z} + \sqrt{2s}}
\]

where the last equality holds because

\[
\| z - \overline{z} \cdot e \|_{W^{-1}} = \left\| \sum_{i=1}^{n} 2\overline{w}_i (z_i - \overline{z})^2 = \sqrt{2\hat{V}_\gamma[\tilde{z}]} \right\|
\]

The above upper bound can be achieved by selecting

\[ u_i = \frac{\sqrt{2\rho \overline{w}_i (z_i - \overline{z})}}{\sqrt{s}}. \]

The above choice of \( u \) satisfies the constraints \( \| u \|_W^2 \leq \rho \) and \( u^\top e = 0 \). Therefore, such \( u \) is feasible as long as

\[ \frac{2\rho 1}{s} \leq 1 \iff s \geq 2\rho \iff \sqrt{2\rho s} \geq 2\rho. \]

Thus, if \( s - 2\rho \geq 0, u \) is a feasible solution. On the other hand, \( u = 0 \) is another feasible solution for this problem. Thus

\[
\overline{z} + \left( \sqrt{2\overline{\hat{V}_\gamma[\tilde{z}]} - 2\rho} \right) \leq \max_{w \in \Delta^n} \left\{ w^\top z : \sum_{i=1}^{n} \frac{1}{2\overline{w}_i} (w_i - \overline{w}_i)^2 \leq \rho \right\} \leq \overline{z} \sqrt{2\rho \hat{V}_\gamma[\tilde{z}]}.
\]

By letting \( \lambda = \sqrt{2\rho} \), we complete this proof.

I Proof of Proposition 6 and its Corollary

Proof. We show that for any fixed tolerance level \( \tau > 0 \), we have

\[
\sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} \geq \sqrt{\hat{V}_\gamma[\ell(x, \tilde{\xi})]} - \tau - \sqrt{\log \left( \frac{|X_\eta|}{\delta} \right) + \log(1 + 2/\tau)} - 2M\eta \quad \forall x \in \mathcal{X} \quad (I.1)
\]

with probability at least \( 1 - \delta \). Here, \( |X_\eta| = O(1)(D/\eta)^d \). Then, the claim of this proposition should immediately follow as both (14) and (I.1) together imply that \( \hat{V}_\gamma[\ell(x, \tilde{\xi})] \geq \lambda^2 \), which is sufficient to establish that the regularization scheme is equivalent to the DRO model (cf. Proposition 5).
From (10), we have that for any fixed $x \in \mathcal{X}$, with probability at least $1 - \delta$, the following lower bound holds:

$$\sqrt{V_\gamma[\ell(x, \xi)]} \geq \sqrt{V_\gamma[\ell(x, \bar{\xi})]} - \tau - \sqrt{\frac{\log \left(\frac{1+2/\tau}{\eta}\right)}{nh_n^p g(\gamma)(1 + o(1))}}. \quad (I.2)$$

Next, we define a finite solution set $\mathcal{X}_\eta \subset \mathcal{X}$ such that, for any $x \in \mathcal{X}$, there exists some $x' \in \mathcal{X}_\eta$ such that $||x - x'|| \leq \eta$. From Shapiro and Nemirovski [2005], we know that $|\mathcal{X}_\eta| = O(1)(D/\eta)^d$.

Since the loss function $\ell(x, \xi)$ is $M$-Lipschitz continuous in $x$, we have that for any $x \in \mathcal{X}$, there exists some $x' \in \mathcal{X}_\eta$, such that $||x - x'|| \leq \eta$ and the following condition holds:

$$\left|\sqrt{V_\gamma[\ell(x, \xi)]} - \sqrt{V_\gamma[\ell(x', \bar{\xi})]}\right| \leq M\eta. \quad (I.3)$$

Next, consider fixed $x \in \mathcal{X}$ and $x' \in \mathcal{X}_\eta$ such that $||x - x'|| \leq \eta$. Combining (I.2) and (I.3), we have that

$$\sqrt{V_\gamma[\ell(x', \bar{\xi})]} \geq \sqrt{V_\gamma[\ell(x', \bar{\xi})]} - \tau - \sqrt{\frac{\log \left(\frac{|X_\eta|}{\eta}\right) + \log(1 + 2/\tau)}{nh_n^p g(\gamma)(1 + o(1))}} - M\eta$$

with probability at least $1 - \delta$. Next, by applying union bound, we get that for all $x' \in \mathcal{X}_\eta$,

$$\sqrt{V_\gamma[\ell(x', \bar{\xi})]} \geq \sqrt{V_\gamma[\ell(x', \bar{\xi})]} - \tau - \sqrt{\frac{\log \left(\frac{|X_\eta|}{\eta}\right) + \log(1 + 2/\tau)}{nh_n^p g(\gamma)(1 + o(1))}} - 2M\eta$$

with probability at least $1 - \delta$. Thus, from above, we get that for all $x \in \mathcal{X}$, the following bound holds with a probability of at least $1 - \delta$:

$$\sqrt{V_\gamma[\ell(x, \bar{\xi})]} \geq \sqrt{V_\gamma[\ell(x, \bar{\xi})]} - \tau - \sqrt{\frac{\log \left(\frac{|X_\eta|}{\eta}\right) + \log(1 + 2/\tau)}{nh_n^p g(\gamma)(1 + o(1))}} - M\eta$$

$$\geq \sqrt{V_\gamma[\ell(x, \bar{\xi})]} - \tau - \sqrt{\frac{\log \left(\frac{|X_\eta|}{\eta}\right) + \log(1 + 2/\tau)}{nh_n^p g(\gamma)(1 + o(1))}} - 2M\eta.$$

Thus, we proved (I.1). \(\square\)

**Corollary 4.** Fix a parameter $\omega > 0$ such that $n^{1-\omega}h_n^p$ is increasing in $n$. Then, for all $x \in \mathcal{X}$, we have

$$\sqrt{V_\gamma[\ell(x, \bar{\xi})]} \geq \sqrt{V_\gamma[\ell(x, \bar{\xi})]} - \sqrt{\frac{2}{n^\omega g(\gamma)(1 + o(1))}} - \frac{2}{\exp(n^{1-\omega}h_n^p)} - 1 - \frac{2MD}{\exp(n^{1-\omega}h_n^p/2d)}$$

with probability at least $1 - C\mathcal{X} \exp\left(-\frac{n^{1-\omega}h_n^p}{2}\right)$ for some constant $C\mathcal{X}$. In particular, if the bandwidth $h_n = C_n/n^{1/(p+4)}$ is used with some constant $C_n$, then by setting $\omega = 2/(p+4)$, we obtain $n^{1-\omega}h_n^p = \ldots$. 

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$C_h^p n^{2/(p+4)}$, and so
\[
\sqrt{\hat{V}_g[\ell(x, \xi)]} \geq \sqrt{V_g[\ell(x, \xi)]} - \sqrt{\frac{2}{n^{2/(p+4)} g(\gamma)(1 + o(1))}} - \exp \left( \frac{2}{C_h^p n^{2/(p+4)}} \right) - \frac{2}{\exp \left( \frac{2MD}{2d} \right)}
\]
with probability at least $1 - C_X \exp \left( - \frac{C_h^p n^{2/(p+4)}}{2} \right)$ for some constant $C_X$.

**Proof.** From Proposition 6, for any $\tau > 0$, we have
\[
\sqrt{\hat{V}_g[\ell(x, \xi)]} \geq \sqrt{V_g[\ell(x, \xi)]} - \tau - \sqrt{\log \left( \frac{|X_\eta|}{\delta} \right) + \log(1 + 2/\tau)} - \frac{2M\eta}{\log(\gamma)(1 + o(1))} - 2M\eta,
\]
with probability at least $1 - \delta$. Here, $|X_\eta| = O(1)(D/\eta)^d$. Suppose we let
\[
\eta = \exp \left( \frac{n^{1-\omega h_n^p}}{2d} \right)
\]
for some $\omega > 0$. Then for sufficiently large $n$ (that is, when $\eta$ is sufficiently small), we have $|X_\eta| = C_X(D/\eta)^d$, where $C_X$ is some constant. We then let
\[
\delta = \frac{C_X}{\exp \left( \frac{n^{1-\omega h_n^p}}{2} \right)} \quad \text{and} \quad \tau = \frac{2\delta}{C_X \left( \frac{D}{\eta} \right)^d} = \frac{2}{\exp \left( n^{1-\omega h_n^p} \right) - 1}.
\]
With the above specified parameters, we have
\begin{align*}
1. \quad -\tau - 2M\eta &= -\frac{2}{\exp \left( n^{1-\omega h_n^p} \right) - 1} - \frac{2MD}{\exp \left( \frac{n^{1-\omega h_n^p}}{2d} \right)}.
\end{align*}
\begin{align*}
2. \quad \log \left( \frac{|X_\eta|}{\delta} \right) + \log(1 + 2/\tau) &= 2 \log \left( \frac{|X_\eta|}{\delta} \right).
\end{align*}
\begin{align*}
3. \quad \log \left( \frac{|X_\eta|}{\delta} \right) &= \log \left( \frac{C_X(D/\eta)^d}{\delta} \right) = \frac{n^{1-\omega h_n^p}}{n h_n^p} = n^{-\omega}.
\end{align*}
By combining the above three results, we obtain the desired result. \[\square\]
J  Proof of Remark 3

Proof. We first consider the inner maximization problem, which given a feasible solution \(x \in \mathcal{X}\), yields the distribution with the worst-case expected loss as given below

\[
\max_{P \in \mathcal{P}_X(\mathcal{P}_\gamma)} \mathbb{E}_P[\ell(x, \tilde{\xi})].
\]

To simplify the notation, we first let \(\eta = \frac{\lambda^2}{2}\). In addition, for a fixed \(x \in \mathcal{X}\), we define the vector \(z \in \mathbb{R}^n\) whose \(i\)-th component \(z_i = \ell(x, \xi^i)\) denotes the loss function evaluated for the \(i\)-th data point. We then have the following formulation for the worst-case expected loss:

\[
\max \sum_{i=1}^{n} z_i w_i \\
\text{s.t.} \quad e^T w = 1, \\
\quad \sum_{i=1}^{n} \left(\frac{w_i - \overline{w}_i}{\sqrt{w_i}}\right)^2 \leq \eta, \\
\quad w \in \mathbb{R}_+^n.
\] (J.1)

We note that the above formulation can be equivalently expressed in terms of the second-order cone constraints below:

\[
\max \quad z^T w \\
\text{s.t.} \quad e^T w = 1, \\
\quad \left[\frac{(w_1 - \overline{w}_1)}{\sqrt{w_1}}, \ldots, \frac{(w_n - \overline{w}_n)}{\sqrt{w_n}}, \sqrt{\eta}\right]^T \in \mathcal{SO}_{n+1}, \\
\quad w \in \mathbb{R}_+^n.
\] (J.2)

From the theory of conic programming duality, we know that the second-order cone is self-dual. We then introduce the dual variables \(\alpha \in \mathbb{R}\) and \((\beta, \nu) \in \mathcal{SO}(n+1)\) corresponding to the first two constraints, and write out the Lagrange function in terms of the dual variables, as follows:

\[
\mathcal{L}(w, \alpha, \beta, \nu) = z^T w + \alpha (1 - e^T w) + \sum_{i=1}^{n} \frac{(w_i - \overline{w}_i)}{\sqrt{w_i}} \beta_i + \nu \sqrt{\eta} \\
= \alpha + \sum_{i=1}^{n} \left(z_i - \alpha + \frac{\beta_i}{\sqrt{w_i}}\right) w_i - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \sqrt{\eta} \nu.
\] (J.3)
Thus, we have that the associated Lagrangian dual function is given by

\[ g(\alpha, \beta, \nu) = \max_{w \geq 0} \left( \alpha + \sum_{i=1}^{n} \left( z_i - \alpha + \frac{\beta_i}{\sqrt{w_i}} \right) w_i - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \sqrt{\eta} \nu \right) \]

\[ = \alpha - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \sqrt{\eta} \nu + \sum_{i=1}^{n} \max_{w_i \geq 0} \left( z_i - \alpha + \frac{\beta_i}{\sqrt{w_i}} \right) w_i. \]

(J.4)

From above, we have

\[ g(\alpha, \beta, \nu) =\begin{cases} 
\alpha - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \sqrt{\eta} \nu & \text{if } z_i + \frac{\beta_i}{\sqrt{w_i}} \leq \alpha \\
\infty & \text{otherwise}. 
\end{cases} \]

(J.5)

Therefore, the dual problem can be written as:

\[ \min \alpha - \sum_{i=1}^{n} \sqrt{w_i} \beta_i + \sqrt{\eta} \nu \]

\[ \text{s.t. } \alpha \geq \ell(x, \xi^i) + \frac{\beta_i}{\sqrt{w_i}} \forall i \in [n], \]

\[ x \in X, \alpha \in \mathbb{R}, (\beta, \nu) \in \text{SOC}(n+1). \]

(J.6)

\(\square\)

### K Details of Example 2

The proposed regularized NW approximation of the portfolio optimization problem is given by

\[ \max_{x \in X} \hat{E}_{\gamma} [\xi^T x] - \lambda \sqrt{\hat{V}_{\gamma} [\xi^T x]}. \]

By applying Corollary 3, the above problem can equivalently be reformulated as the second-order cone program

\[ \max \left( \sum_{i=1}^{n} \overline{w}_i (\xi_i^T x) \right) - \lambda \rho \]

\[ \text{s.t. } x \in X, t \in \mathbb{R}, \]

\[ \left( \sqrt{\overline{w}_1} ((\xi_1)^T x - t), \ldots, \sqrt{\overline{w}_n} ((\xi_n)^T x - t), \rho \right) \in \text{SOC}(n+1), \]

where \( \overline{w}_i = \frac{\kappa_i(\gamma - \gamma^i)}{\sum_{j=1}^{n} \kappa_j(\gamma - \gamma^j)}, i \in [n]. \) In this example, we select \( \lambda \) such that the model provides the best out-of-sample performance.

On the other hand, the LDR approach seeks for the best parameters \( x_1, x_2, y \in \mathbb{R} \) so that the decision of investing \( \$(x_i + \gamma \cdot y) \) in asset \( i \), for \( i = 1, 2 \), and \( \$(1 - x_1 - x_2 - 2\gamma \cdot y) \) in asset 3 generates the highest empirical return. The optimal portfolio allocation thus constitutes an affine function
in $\gamma$. To find these parameters, we solve the following regularized empirical maximization problem:

$$\max \frac{1}{n} \sum_{i=1}^{n} \left( \xi_i^T \left[ x + (\gamma_i \cdot y) \cdot e \right] \right) - \lambda (x_1^2 + x_2^2 + y^2)$$

s.t. $x_1, x_2, y \in \mathbb{R}$,

$$x_1 + \gamma_i \cdot y \geq 0, \quad x_2 + \gamma_i \cdot y \geq 0, \quad x_1 + x_2 + 2\gamma_i \cdot y \leq 1 \quad \forall i \in [n].$$

The constraints of this problem prohibit short selling and ensure that the total allocation does not exceed $\$1$. In this example, similar to the NW approximation, we select $\lambda$ such that the this model provides the best out-of-sample performance.