Ordering uniform supertrees by their spectral radii

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Abstract

A connected and acyclic hypergraph is called a supertree. In this paper we mainly focus on the spectral radii of uniform supertrees. Li, Shao and Qi determined the first two k-uniform supertrees with large spectral radii among all the k-uniform supertrees on n vertices [H. Li, J. Shao, L. Qi, The extremal spectral radii of k-uniform supertrees, arXiv:1405.7257v1, May 2014]. By applying the operation of moving edges on hypergraphs and using the weighted incidence matrix method we extend the above order to the fourth k-uniform supertree.

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1 Introduction

Let G be an ordinary graph, and A(G) be its adjacency matrix. Denote by ρ(G) the spectral radius of graph G, i.e., the largest eigenvalue of A(G). As usual, denote by S_n, P_n, the star on n vertices, the path on n vertices, respectively.

We will take some notation from [9] and [11]. We denote the set {1, 2, · · · , n} by [n]. Hypergraph is a natural generalization of an ordinary graph (see [1]). A hypergraph H = (V (H), E(H)) on n vertices is a set of vertices say V (H) = {1, 2, · · · , n} and a set of edges, say E(H) = {e_1, e_2, · · · , e_m}, where e_j = {i_1, i_2, · · · , i_l}, i_j ∈ [n], j = 1, 2, · · · , l. If |e_i| = k for any i = 1, 2, · · · , m, then H is called k-uniform hypergraph. A vertex v is said to be incident to an edge e if v ∈ e. The degree d(i) of vertex i is defined as d(i) = |{e_j : i ∈ e_j ∈ E(H)}|. A vertex of degree one is called a pendent vertex. For a k-uniform hypergraph H, an edge e ∈ E(H) is called a pendent edge if e contains exactly k − 1 pendent vertices.

An order k dimension n tensor A = (A_{i_1i_2···i_k}) ∈ C^{n×n×···×n} is a multidimensional array with n^k entries, where i_j ∈ [n] for each j = 1, 2, · · · , k. To study the properties of uniform hypergraph by algebraic methods, adjacency matrix of an ordinary graph is naturally generalized to adjacency tenor (it is called adjacency hypermatrix in [9]) of a hypergraph (see [5] [16]).

Definition 1 Let H = (V (H), E(H)) be a k-uniform hypergraph on n vertices. The adjacency tensor of H is defined as the k-th order n-dimensional tensor A(H) whose (i_1 · · · i_k)-entry is:

\[ A(H)_{i_1i_2···i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, i_2, · · · , i_k\} \in E(H) \\ 0 & \text{otherwise.} \end{cases} \]

The following general product of tensors, is defined in [17] by Shao, which is a generalization of the matrix case.

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Definition 2 Let $A$ and $B$ be order $m \geq 2$ and $k \geq 1$ dimension $n$ tensors, respectively. The product $AB$ is the following tensor $C$ of order $(m-1)(k-1)+1$ and dimension $n$ with entries:

$$C_{\alpha_1\cdots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n]} A_{i_2 \cdots i_m} B_{i_2 \alpha_1} \cdots B_{i_m \alpha_{m-1}}.$$ 

(1)

Where $i \in [n], \alpha_1, \cdots, \alpha_{m-1} \in [n] \times \cdots \times [n]$.

Let $A$ be an order $k$ dimension $n$ tensor, let $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension $n$. Then by (1) $Ax$ is a vector in $\mathbb{C}^n$ whose $i$th component is as the following

$$(Ax)_i = \sum_{i_2, \ldots, i_k = 1}^n A_{i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}.$$ 

Let $x^{[k]} = (x_1^k, \ldots, x_n^k)^T$. Then (see [2] [16]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $A$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$Ax = \lambda x^{[k-1]},$$

and in this case, $x$ is called an eigenvector of $A$ corresponding to eigenvalue $\lambda$.

Let $A$ be a $k$th-order $n$-dimensional nonnegative tensor. The spectral radius of $A$ is defined as

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$ 

In this paper we call $\rho(A(H))$ the spectral radius of uniform hypergraph $H$, denoted by $\rho(H)$. For more details on the eigenvalues of a uniform hypergraph one can refer to [13] [7] and [14].

In [6], the weak irreducibility of nonnegative tensors was defined. It was proved in [6] and [18] that a $k$-uniform hypergraph $H$ is connected if and only if its adjacency tensor $A(H)$ is weakly irreducible.

Theorem 3 [2] If $A$ is a nonnegative tensor, then $\rho(A)$ is an eigenvalue with a nonnegative eigenvector $x$ corresponding to it. If furthermore $A$ is weakly irreducible, then $x$ is positive, and for any eigenvalue $\lambda$ with nonnegative eigenvector, $\lambda = \rho(A)$. Moreover, the nonnegative eigenvector is unique up to a constant multiple.

By Theorem 3 for a $k$th-order weakly irreducible nonnegative tensor $A$, it has a unique positive eigenvector $x$ corresponding to $\rho(A)$ with $\|x\|_1 = 1$ and it is called the principal eigenvector of $A$ ([11]).

Definition 4 [11] A supertree is a hypergraph which is both connected and acyclic.

A characterization of acyclic hypergraph has been given in Berge’s textbook [1], and we just state a version for uniform hypergraphs.

Proposition 5 [7] If $H$ is a connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, then it is acyclic if and only if $m(k-1) = n-1$.

The concept of power hypergraphs was introduced in [9]. Let $G = (V, E)$ be an ordinary graph. For every $k \geq 3$, the $k$th power of $G$, $G^k := (V^k, E^k)$ is defined as the $k$-uniform hypergraph with the edge set

$$E^k := \{e \cup \{i_e,1, \cdots, i_e,k-2\} | e \in E\}$$

and the vertex set

$$V^k := V \cup \cup_{e \in E}\{i_e,1, \cdots, i_e,k-2\}.$$ 

The $k$th power of an ordinary tree was called a $k$-uniform hypertree ([9] [11]). The following observations are clear. Any $k$-uniform hypertree is a supertree. A $k$-uniform supertree $T$ with at least two edges is a $k$-uniform hypertree if and only if each edge of $T$ contains at most two non-pendent vertices.
The $k$th power of $S_n$, denoted by $S_n^k$, is called hyperstar in [9]. Let $S(a,b)$ be the tree on $a+b+2$ vertices obtained from an edge $e$ by attaching a pendent edge to one end vertex of $e$, and attaching $b$ pendent edges to the other end vertex of $e$. Let $S^k(a,b)$ be the $k$th power of $S(a,b)$.

In [13], it was proved that the hyperstar $S_n^k$ attains uniquely the maximum spectral radius among all $k$-uniform supertrees on $n$ vertices, and $S^k(1,n'-3)$ attains uniquely the second largest spectral radius among all $k$-uniform supertrees on $n$ vertices (where $n' = \frac{n-1}{k-1} + 1$).

Suppose that $m = \frac{n-1}{k-1}$, now we introduce a special class of supertrees with $m$ edges, which are not hypertrees. Let $1 \leq t_1 \leq t_2 \leq t_3$ be three integers such that $t_1+t_2+t_3 = m-1$. Denote by $T(t_1,t_2,t_3)$ the $k$-uniform supertree containing exactly three non-pendent vertices, say $u_1, u_2, u_3$, incident to one edge, and $d(u_i) = t_i+1$ holding for each $i = 1, 2, 3$.

In this paper, we will determine the third and the fourth $k$-uniform supertree with the large spectral radii among all $k$-uniform supertrees on $n$ vertices.

**Theorem 6** Let $T$ be a $k$-uniform supertree on $n$ vertices (with $m = n'-1$ edges, where $n' = \frac{n-1}{k-1} + 1 \geq 5$). Suppose that $T \notin \{S_n^k, S^k(1,n'-3)\}$. Then we have

$$\rho(T) \leq \rho(S^k(2,n'-4)),$$

with equality holding if and if $T \cong S^k(2,n'-4)$.

**Theorem 7** Let $T$ be a $k$-uniform supertree on $n$ vertices (with $m = n'-1$ edges, where $n' = \frac{n-1}{k-1} + 1 \geq 5$). Suppose that $T \notin \{S_n^k, S^k(1,n'-3), S^k(2,n'-4)\}$. Then we have

$$\rho(T) \leq \rho(T(1,1,m-3)),$$

with equality holding if and if $T \cong T(1,1,m-3)$.

The operation of moving edges on hypergraphs introduced by Li, Shao and Qi ([11]) and the weighted incidence matrix method introduced by Lu and Man ([13]) are crucial for our proofs. In Section 2 we will show them and other useful tools. In Section 3, we will give the proofs of our main results.

## 2 Several tools to compare spectral radii

A novel method (we call it weighted incidence matrix method) for computing (or comparing) the spectral radii of hypergraphs was raised by Lu and Man.

**Definition 8** [13] A weighted incidence matrix $B$ of a hypergraph $H = (V,E)$ is a $|V| \times |E|$ matrix such that for any vertex $v$ and any edge $e$, the entry $B(v,e) > 0$ if $v \in e$ and $B(v,e) = 0$ if $v \notin e$.

**Definition 9** [13] A hypergraph $H$ is called $\alpha$-normal if there exists a weighted incidence matrix $B$ satisfying

1. $\sum_{e \in E} B(v,e) = 1$, for any $v \in V(H)$.
2. $\prod_{e \in E} B(v,e) = \alpha$, for any $e \in E(H)$.

Moreover, the weighted incidence matrix $B$ is called consistent if for any cycle $v_0 e_1 v_1 e_2 \cdots v_l e_l v_0$

$$\prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$

**Definition 10** [13] A hypergraph $H$ is called $\alpha$-subnormal if there exists a weighted incidence matrix $B$ satisfying

1. $\sum_{e \in E} B(v,e) \leq 1$, for any $v \in V(H)$. 

3
(2). \( \prod_{e \in E} B(v, e) \geq \alpha, \) for any \( e \in E(H). \)

Moreover, \( H \) is called strictly \( \alpha \)-subnormal if it is \( \alpha \)-subnormal but not \( \alpha \)-normal.

**Definition 11** [13] A hypergraph \( H \) is called \( \alpha \)-supernormal if there exists a weighted incidence matrix \( B \) satisfying

1. \( \sum_{v \in e} B(v, e) \geq 1, \) for any \( v \in V(H). \)
2. \( \prod_{v \in e} B(v, e) \leq \alpha, \) for any \( e \in E(H). \)

Moreover, \( H \) is called strictly \( \alpha \)-supernormal if it is \( \alpha \)-supernormal but not \( \alpha \)-normal.

For a fixed \( k \)-uniform hypergraph \( H, \rho(H) \) defined here times constant factor \((k - 1)!\) is the value of \( \rho(H) \) defined in [13]. While this is not essential. Remembering this difference we modify Lemma 3 and Lemma 4 of [13] as the following Theorem 12.

**Theorem 12** [13] Let \( H \) be a \( k \)-uniform hypergraph.

1. If \( H \) is strictly \( \alpha \)-subnormal, then we have \( \rho(H) < \alpha^{\frac{k}{2}}. \)
2. If \( H \) is strictly and consistently \( \alpha \)-supernormal, then \( \rho(H) > \alpha^{\frac{k}{2}}. \)

The following result reveals the numerical relationship between \( \rho(G^k) \) and \( \rho(G) \), where \( G^k \) is the \( k \)-th power of an ordinary graph \( G \).

**Theorem 13** [10] Let \( G^k \) be the \( k \)th power of an ordinary graph \( G \). Then we have
\[
\rho(G^k) = (\rho(G))^k.
\]

Let \( F_n \) \((n \geq 5)\) be the tree obtained by coalescing the center of the star \( S_{n-4} \) and the center of the path \( P_5 \). Ordering the trees on \( n \) vertices according to their spectral radii was well studied in [10], [4] and [12]. We outline parts of the work in [10] as follows.

**Theorem 14** [10] Let \( T \) be a tree on \( n \) vertices \((n \geq 5)\) and \( T \notin \{S_n, S(1, n-3), S(2, n-4), F_n\}\). Then we have
\[
\rho(S_n) > \rho(S(1, n-3)) > \rho(S(2, n-4)) > \rho(F_n) > \rho(T).
\]

Combining Theorems [13] and [14] we have the following corollary.

**Corollary 15** Let \( T^k \) be the \( k \)th power of an ordinary tree \( T \). Suppose that \( T^k \) has \( n \) vertices, and \( n' = \frac{n-1}{k-1} + 1 \geq 5 \). Suppose \( T \notin \{S_{n',} S(1, n'-3), S(2, n'-4), F_{n'}\}\). Then we have
\[
\rho(S^k_{n'}) > \rho(S^k(1, n'-3)) > \rho(S^k(2, n'-4)) > \rho(F^k_{n'}) > \rho(T^k).
\]

**Definition 16** [11] Let \( r \geq 1, G = (V, E) \) be a hypergraph with \( u \in V \) and \( e_1, \ldots, e_r \in E, \) such that \( u \notin e_i \) for \( i = 1, \ldots, r \). Suppose that \( v_i \in e_i \) and write \( e'_i = (e_i \setminus \{v_i\}) \cup \{u\} \) for \( i = 1, \ldots, r \). Let \( G' = (V', E') \) where \( E' = (E \setminus \{e_1, \ldots, e_r\}) \cup \{e'_1, \ldots, e'_r\} \). Then we say that \( G' \) is obtained from \( G \) by moving edges \((e_1, \ldots, e_r)\) from \((v_1, \ldots, v_r)\) to \( u \).

The effect on \( \rho(G) \) of moving edges was studied by Li, Shao and Qi (see Theorem 17). The following fact was pointed out in [11]. If \( G \) is acyclic and there is an edge \( e \in E(G) \) containing all the vertices \( u, v_1, \ldots, v_r \), then the graph \( G' \) defined as above contains no multiple edges.

**Theorem 17** [11] Let \( r \geq 1, G \) be a connected hypergraph, \( G' \) be the hypergraph obtained from \( G \) by moving edges \((e_1, \ldots, e_r)\) from \((v_1, \ldots, v_r)\) to \( u \), and \( G' \) contain no multiple edges. If \( x \) is the principal eigenvector of \( A(G) \) corresponding to \( \rho(G) \) and suppose that \( x_u \geq \max_{1 \leq i \leq r} \{x_{v_i}\} \), then \( \rho(G') > \rho(G) \).
Denote by $N_2(T)$ the number of non-pendent vertices of $T$. By using Theorem 17 (or modifying parts of the proof of Theorem 21 of [11]), we have the following observation.

**Lemma 18** Let $T$ be a $k$-uniform supertree on $n$ vertices with $N_2(T) \geq 2$. Then there exists a $k$-uniform supertree $T'$ on $n$ vertices with $N_2(T') = N_2(T) - 1$ and $\rho(T') > \rho(T)$.

**Lemma 19** Let $a, b, c, d$ be nonnegative integers with $a + b = c + d$. Suppose that $a \leq b$, $c \leq d$ and $a < c$, then we have $\rho(S^k(a, b)) > \rho(S^k(c, d))$.

**Lemma 20** Let $1 \leq t_1 \leq t_2 \leq t_3$ be three integers with $t_1 + t_2 + t_3 = m - 1$. Then we have

$$\rho(T(1, 1, m - 3)) \geq \rho(T(t_1, t_2, t_3)),$$

with equality holding if and only if $t_2 = 1$.

**Proof** If $t_2 = 1$, the result is obvious. Now we suppose $t_2 > 1$, thus $t_3 > 1$. Let $u_1, u_2$ and $u_3$ be the (only) three non-pendent vertices of $T(t_1, t_2, t_3)$ with $d(u_i) = t_i + 1$, $i = 1, 2, 3$. It is easy to see that $u_i$ is incident to $t_i$ pendent edges, $i = 1, 2, 3$. Let $x$ be the principal eigenvector of $A(T(t_1, t_2, t_3))$ corresponding to $\rho(T(t_1, t_2, t_3))$. Without loss of generality we suppose that $x_{u_i} = \max_{1 \leq i \leq 3} \{x_{u_i}\}$. Let $G$ be obtained from $T(t_1, t_2, t_3)$ by moving $t_1 - 1$ pendent edges from $u_1$ to $u_3$, and moving $t_2 - 1$ pendent edges from $u_2$ to $u_3$. Then $G$ is isomorphic to $T(1, 1, m - 3)$. Noting that $t_2 > 1$, by Theorem 17 we have $\rho(T(1, 1, m - 3)) > \rho(T(t_1, t_2, t_3))$.

By Theorem 18 we know that $\rho(S^k(2, n' - 4))$ is determined by $\rho(S(2, n' - 4))$, and $\rho(F^k_n')$ is determined by $\rho(F_n')$. We will use the weighted incidence matrix method to compare $\rho(T(1, 1, m - 3))$ with $\rho(S^k(2, n' - 4))$ and $\rho(F^k_n')$.

**Lemma 21** Suppose that $n' = \frac{n - 1}{k} + 1$, $m = n' - 1 \geq 4$. We have

$$\rho(S^k(2, n' - 4)) > \rho(T(1, 1, m - 3)) > \rho(F^k_n').$$

**Proof** Denote by $u_1, u_2$ and $u_3$ three non-pendent vertices of $T(1, 1, m - 3)$. Label the $m$ edges of $T(1, 1, m - 3)$ as follows. The unique non-pendent edge (the edge containing $u_1, u_2$ and $u_3$) is numbered $e_0$, the pendent edge containing $u_1$ is numbered $e_1$, the pendent edge containing $u_2$ is numbered $e_2$, and the pendent edges containing $u_3$ are numbered $e_3, \cdots, e_{m - 1}$. Now we construct an $n \times m$ matrix $B$. For any vertex $v$ and any edge $e$ of $T(1, 1, m - 3)$, let $B(v, e) = 0$ if $v \notin e$. For any pendent vertex $v$ in an edge $e$, let $B(v, e) = 1$. For the non-pendent vertices $u_1, u_2$ and $u_3$, let $B(u_1, e_1) = \alpha$, $B(u_1, e_0) = 1 - \alpha$; $B(u_2, e_2) = \alpha$, $B(u_2, e_0) = 1 - \alpha$; and let $B(u_3, e_i) = \alpha$, for $i = 3, \cdots, m - 1$, $B(u_3, e_0) = 1 - (m - 3)\alpha$. According to the above rules, we say that for any vertex $v$ of $T(1, 1, m - 3)$ we have

$$\sum_{e : v \in e} B(v, e) = 1. \tag{2}$$

For the pendent edge $e_i$ ($i = 1, 2, \cdots, m - 1$), we have

$$\prod_{v : v \in e_i} B(v, e_i) = \alpha. \tag{3}$$

For the unique non-pendent edge $e_0$ we have

$$\prod_{v : v \in e_0} B(v, e_0) = (1 - \alpha)^2[1 - (m - 3)\alpha],$$

and then

$$\prod_{v : v \in e_0} B(v, e_0) - \alpha = -(m - 3)\alpha^3 + (2m - 5)\alpha^2 - m\alpha + 1. \tag{4}$$
(1). Write \( \rho = \rho(S(2, n' - 4)) \) for short. By Theorem 13 we have \( \rho(S^k(2, n' - 4)) = \rho^k \). It is easy to check that the tree \( S(2, n' - 4) \) contains \( m \) edges and the value \( \rho \) satisfies
\[
\rho^4 - m\rho^2 + 2(m - 3) = 0. \tag{5}
\]
As we all know that
\[
\rho > \sqrt{\Delta(S(2, n' - 4))} = \sqrt{n' - 3} = \sqrt{m - 2},
\]
where \( \Delta(S(2, n' - 4)) \) is the maximum degree of the tree \( S(2, n' - 4) \).

Take \( \alpha = \frac{1}{\rho^2} \). Then \( \alpha < \frac{1}{m-2} \) and
\[
1 - \alpha \geq 1 - (m - 3)\alpha > 1 - \frac{m - 3}{m - 2} > 0.
\]
So \( B(v, e) > 0 \) for any vertex \( v \) and any edge \( e \) of \( T(1, 1, m - 3) \) when \( v \in e \), i.e., the matrix \( B \) is a weighted incidence matrix of the supertree \( T(1, 1, m - 3) \) according to Definition 8. Now we will show \( T(1, 1, m - 3) \) is strictly \( \alpha \)-subnormal with \( \alpha = \frac{1}{\rho^2} \).

Combining (4) and (5), we only need to show
\[
\prod_{v: v \in e_0} B(v, e_0) > \alpha.
\]
In fact by (4) and (5) we have
\[
\prod_{v: v \in e_0} B(v, e_0) - \alpha = -(m - 3)\alpha^3 + (2m - 5)\alpha^2 - m\alpha + 1
\]
\[
= \frac{1}{\rho^6} [\rho^6 - m\rho^4 + (2m - 5)\rho^2 - (m - 3)]
\]
\[
= \frac{1}{\rho^6} [\rho^2 - (m - 3)]
\]
\[
> 0.
\]
So for the unique non-pendent edge \( e_0 \) we have
\[
\prod_{v: v \in e_0} B(v, e_0) > \alpha. \tag{6}
\]

By (1) of Theorem 12 we have
\[
\rho(T(1, 1, m - 3)) < \alpha^{-\frac{1}{k}} = \rho^\frac{k}{k} = \rho(S^k(2, n' - 4)).
\]

(2). Write \( \rho = \rho(F_{n'}) \) for short. By Theorem 13 we have \( \rho(F^k_{n'}) = \rho^k \). It is easy to see that the tree \( F_{n'} \) contains \( m \) edges and the value \( \rho \) satisfies
\[
\rho^4 - (m - 1)\rho^2 + (m - 4) = 0, \tag{7}
\]
and
\[
\rho > \sqrt{\Delta(F_{n'})} = \sqrt{n' - 3} = \sqrt{m - 2},
\]
where \( \Delta(F_{n'}) \) is the maximum degree of the tree \( F_{n'} \).

Take \( \alpha = \frac{1}{\rho^2} \). Then \( \alpha < \frac{1}{m-2} \) and
\[
1 - \alpha \geq 1 - (m - 3)\alpha > 1 - \frac{m - 3}{m - 2} > 0.
\]
So \( B(v, e) > 0 \) for any vertex \( v \) and any edge \( e \) of \( T(1, 1, m - 3) \) when \( v \in e \), i.e., the matrix \( B \) is a weighted incidence matrix of the supertree \( T(1, 1, m - 3) \). Now we will show \( T(1, 1, m - 3) \) is strictly
Clearly, the weighted incidence matrix $B$ So for the unique non-pendent edge is acyclic. By (2) of Theorem 12, we have (4) and (7) we have

\[
\prod_{v: e \in e_0} B(v, e_0) < \alpha. \quad (8)
\]

Clearly, the weighted incidence matrix $B$ of $T(1, 1, m - 3)$ is consistent, since the supertree $T(1, 1, m - 3)$ is acyclic. By (2) of Theorem 12 we have

\[
\rho(T(1, 1, m - 3)) > \alpha^{-\frac{1}{2}} = \rho(T_{\alpha}).
\]

The proof is complete. $\square$

3 The proofs of the main results

Suppose that $n' = \frac{m - 1}{k - 1} + 1$, and $m = n' - 1$. Recall that $N_2(T)$ is the number of non-pendent vertices of a supertree $T$. For a $k$-uniform supertree $T$ on $n$ vertices we have the following observations.

1. $N_2(T) = 1$ if and only if $T \cong S_{n'}^k$;
2. $N_2(T) = 2$ if and only if $T \cong S^k(a, b)$ for some integers $a, b$, where $b \geq a \geq 1$ and $a + b = n' - 2$;
3. $N_2(T) = 3$ and three non-pendent vertices incident to one edge if and only if $T \cong T(t_1, t_2, t_3)$ for some integers $t_1, t_2, t_3$, where $t_1 + t_2 + t_3 = m - 1$.

4. $N_2(T) = 3$ and three non-pendent vertices not incident to one edge, if and only if $T \cong T_k$ for some ordinary tree $T$ and $T$ containing three non-pendent vertices.

**Proof of Theorem 6** Since $T \not\cong S_{n'}^k$, we have $N_2(T) \geq 2$.

If $N_2(T) = 2$, then $T \cong S^k(a, b)$ for some integers $a, b$, where $b \geq a \geq 1$ and $a + b = n' - 2$. Since $T \not\cong S^k(1, n' - 3)$, by Lemma 19 we have

\[
\rho(T) < \rho(S^k(2, n' - 4)),
\]

with equality holding if and if $T \cong S^k(2, n' - 4)$.

If $N_2(T) = 3$ and $T \cong T(t_1, t_2, t_3)$, then combining Lemmas 20 and 21 we have

\[
\rho(T) < \rho(T(1, 1, m - 3)) < \rho(S^k(2, n' - 4)).
\]

If $N_2(T) = 3$ and $T \cong T_k$ for some ordinary tree $T$, then $T$ contains three non-pendent vertices and then $T \not\in \{S_{n'}, S(a, b)\}$. From Corollary 15 we have

\[
\rho(T) < \rho(S^k(2, n' - 4)).
\]
If $N_2(T) \geq 4$, then there exists a $k$-uniform supertree $T'$ with $N_2(T') = 3$ and $\rho(T') > \rho(T)$ by Lemma 18. Thus we have

$$\rho(T) < \rho(T') < \rho(S^k(2, n' - 4)).$$

The proof is complete. □

Proof of Theorem 7 Since $T \not\cong S^k_n$, we have $N_2(T) \geq 2$.

If $N_2(T) = 2$, then $T \cong S^k(a, b)$ for some integers $a, b$, where $b \geq a \geq 1$, and $a + b = n' - 2$. Since $T \not\in \{S^k(1, n' - 3), S^k(2, n' - 4)\}$, by Lemma 19, Corollary 15 and Lemma 21, we have

$$\rho(T) \leq \rho(S^k(3, n' - 5)) < \rho(F^k_{n'}) < \rho(T(1, 1, m - 3)).$$

If $N_2(T) = 3$ and $T \cong T(t_1, t_2, t_3)$, then from Lemma 20 we have

$$\rho(T) \leq \rho(T(1, 1, m - 3)),$$

with equality holding if and only if $T \cong T(1, 1, m - 3)$.

If $N_2(T) = 3$ and $T \cong T^k$, then $T \not\in \{S^k_n, S(a, b)\}$. From Corollary 15 Lemma 21 we have

$$\rho(T) \leq \rho(F^k_{n'}) < \rho(T(1, 1, m - 3)).$$

If $N_2(T) \geq 4$, then there exists a $k$-uniform supertree $T'$ with $N_2(T') = 3$ and $\rho(T') > \rho(T)$ by Lemma 18. Thus we have

$$\rho(T) < \rho(T') \leq \rho(T(1, 1, m - 3)).$$

The proof is complete. □

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