POINCARÉ POLYNOMIALS OF MODULI SPACES OF STABLE BUNDLES OVER CURVES

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Abstract. Given a curve over a finite field, we compute the number of stable bundles of not necessarily coprime rank and degree over it. We apply this result to compute the virtual Poincaré polynomials of the moduli spaces of stable bundles over a curve. A similar formula for the virtual Hodge polynomials and motives is conjectured.

1. Introduction

Let $X$ be a smooth projective curve of genus $g$ over $\mathbb{C}$. Let $\mathcal{M}(n,d)$ be the moduli space of stable vector bundles of rank $n$ and degree $d$ on $X$. The problem of computation of the Betti numbers of $\mathcal{M}(n,d)$ has attracted the interest of many people. In the case of coprime $n$ and $d$, a recursive formula for the Poincaré polynomial of $\mathcal{M}(n,d)$ was given by Harder, Narasimhan, Desale and Ramanan [18, 11] by using the Weil conjectures. The same recursive formula was obtained later by Atiyah and Bott [3] using the gauge theory. Another method to prove the same recursive formula was found by Bifet, Ghione and Letizia [6]. The method of Atiyah and Bott was adapted later by Earl and Kirwan [14] to give a similar recursive formula for the Hodge polynomial of $\mathcal{M}(n,d)$. The method of Bifet, Ghione and Letizia was adapted by del Baño [9] to give an analogous formula for the motive of $\mathcal{M}(n,d)$. The recursive formula for the Poincaré polynomials was solved by Zagier [27]. As a result, he got a nice explicit formula, which is much simpler than the initial recursion (e.g. it does not contain infinite sums in contrast to the recursive formula). The formula of Zagier was further generalized by Laumon and Rapoport [20]. The formula of Zagier can be easily adapted for the cases of Hodge polynomials and motives (see [9] and Section 7).

In the case of not necessarily coprime rank and degree only some partial results are known. The Betti and Hodge numbers of $H^i(\mathcal{M}(n,d))$ were computed for $i < 2(n-1)g - (n-1)(n^2 + 3n + 1) - 7$ by Arapura and Sastry [1] and for $i < 2(n-1)(g-1)$ by Dhillon [12].

In this paper we go in a somewhat different direction and determine the virtual Poincaré polynomial (see [8] and Remark 4.10) of $\mathcal{M}(n,d)$ for arbitrary $n$ and $d$. We use the approach by Harder, Narasimhan, Desale and Ramanan and compute first the number of points of the corresponding moduli space over finite fields and then use the Weil conjectures to obtain the virtual Poincaré polynomial of this moduli space. Given a curve $X$ over a finite field $k$, a recursive formula has been proved in [18, 11] for a weighted sum

$$r_{(n,d)}(k) = \sum_{M \in \mathcal{M}(n,d)} \frac{1}{\# \text{Aut} M},$$
where \( \mathcal{E}^X_{n,d}(n,d) \) denotes the set of representatives of isomorphism classes of semistable sheaves over \( X \) having rank \( n \) and degree \( d \). We show that the numbers \( r_{n,d}(K) \) (for all possible \( n, d \) and finite field extensions \( K/k \)) can be used to compute the numbers \( a_{n,d}(k) \) of absolutely stable (see Section 2.3) sheaves over \( X \) of rank \( n \) and degree \( d \). The last number is precisely the number of the \( k \)-rational points of the moduli space \( \mathcal{M}(n,d) \).

The same approach was used in [23] to relate the weighted sums over semistable representations of a quiver over finite fields to the numbers of absolutely stable representations. There the analogues of \( r_{n,d}^q(\mathbb{F}_q) \) and \( a_{n,d}^q(\mathbb{F}_q) \) (here \( \mathbb{F}_q \) is a finite field extension of \( k \)) are rational functions (respectively, polynomials) in \( q \) and it is possible to apply the machinery of \( \lambda \)-rings to the usual \( \lambda \)-ring structure on \( \mathbb{Q}(q) \) (see Section 2.4) to get the relation between the corresponding rational functions and polynomials. In the case of sheaves over a curve, the numbers \( r_{n,d}(\mathbb{F}_q) \) and \( a_{n,d}(\mathbb{F}_q) \) are not, in general, rational functions in \( q \) and we cannot use the formalism of \( \lambda \)-rings. To pass around this problem, we introduce a new \( \lambda \)-ring, called the ring of \( \mathbb{c} \)-sequences (see Section 4) such that the systems of rational numbers \( r_{n,d}(K) = (r_{n,d}(K))_{K/k} \) and \( a_{n,d}(K) = (a_{n,d}(K))_{K/k} \), where \( K \) runs through the finite field extensions of \( k \), are elements of this ring. Actually, for any algebraic variety \( X \) over \( k \), the system \( (\#X(K))_{K/k} \) is a \( \mathbb{c} \)-sequence. We use the \( \lambda \)-ring of \( \mathbb{c} \)-sequences as an analogue of the Grothendieck ring of motives in the case of a finite field. We prove a formula relating the \( \mathbb{c} \)-sequences \( r_{n,d}(K) \) and \( a_{n,d}(K) \) using the formalism of \( \lambda \)-rings in a similar way as it was done for representations of quivers in [23].

With any \( \mathbb{c} \)-sequence, we associate its Poincaré function. For any algebraic variety over \( k \), its virtual Poincaré polynomial coincides with the Poincaré function of the associated \( \mathbb{c} \)-sequence. The relation between the \( \mathbb{c} \)-sequences \( r_{n,d}(K) \) and \( a_{n,d}(K) \) induces the relation between the corresponding Poincaré functions. The virtual Poincaré polynomial of \( \mathcal{M}(n,d) \) is precisely the Poincaré function of the \( \mathbb{c} \)-sequence \( a_{n,d}(K) \). The Poincaré function of \( r_{n,d}(K) \) can be computed using the recursive formula of Harder, Narasimhan, Desale and Ramanan or using the explicit formula due to Zagier.

Concerning the virtual Hodge polynomials (see [8]), we give a conjectural formula that relates the virtual Hodge polynomials of \( \mathcal{M}(n,d) \) and certain rational functions, that should be understood as analogues of the Poincaré function of \( r_{n,d}(K) \). These rational functions where used by Earl and Kirwan [14] to compute the Hodge polynomial of \( \mathcal{M}(n,d) \) for coprime \( n \) and \( d \).

We also give a conjecture for the motive of \( \mathcal{M}(n,d) \). In [4], Behrend and Dhillon found a formula for the motives of the stacks of all bundles on a curve. One can relate the motives of the stacks of semistable bundles (which are analogues of \( r_{n,d}(K) \)) and the motives of the stacks of all bundles using the Harder–Narasimhan filtration and get a formula similar to the recursive formula of Harder, Narasimhan, Desale and Ramanan. This was done, using a different language, by del Baño [9]. The solution of the recursive formula due to Zagier can also be applied here. We give a conjectural formula relating the motives of stacks of semistable bundles and motives of moduli spaces of stable bundles. To formulate it we use the standard \( \lambda \)-ring structure on the Grothendieck ring of motives, where the \( \sigma \)-operations are given by symmetric products.
The paper is organized as follows. In Section 2 we gather some preliminary material: basic operations in complete \( \lambda \)-rings, the Galois descent for the sheaves on a variety over a finite field, indecomposable and stable sheaves, some results on the Hall algebra of the category of sheaves on a curve over a finite field. In Section 3 we discuss solutions of the recursive formulas that occur from Harder-Narasimhan filtrations. In Section 4 we introduce the \( \lambda \)-ring of c-sequences and prove its basic properties. In Section 5 we discuss formulas for c-sequences \( r_{(n,d)} \) and its Poincaré functions. As a corollary, we give a formula for the virtual Poincaré polynomials of the moduli spaces of stable vector bundles on a curve. In Section 7 we give conjectural formulas for the virtual Hodge polynomials and for the motives of the moduli spaces of stable vector bundles on a curve.

I would like to thank Markus Reineke for many helpful and encouraging discussions. The suggestion that our methods from [23] could be applied in the case of stable bundles is due to him.

### 2. Preliminaries

#### 2.1. Basic operations in complete \( \lambda \)-rings. Any \( \lambda \)-ring \( R \) (see e.g. [16, 21]) has three families of operations: \( \lambda \)-operations, \( \sigma \)-operations and Adams operations. Our \( \lambda \)-rings are always algebras over \( \mathbb{Q} \). Under this condition, the \( \lambda \)-ring structure is uniquely determined by any of the above three families of operations. We will usually use the Adams operations \( \psi_n : R \rightarrow R, n \geq 1 \), to describe the \( \lambda \)-ring structure. Given a \( \lambda \)-ring \( R \), we endow the ring \( R[\{x_1, \ldots, x_r\}] \) with a structure of a \( \lambda \)-ring by the formula

\[
\psi_n(ax^\alpha) = \psi_n(a)x^{n\alpha}, \quad a \in R, \alpha \in \mathbb{N}^r.
\]

In the same way we endow the ring \( R[\{x_1, \ldots, x_r\}] \) with a structure of a \( \lambda \)-ring. Let \( R \) be a \( \lambda \)-ring and let \( S \) be a multiplicatively closed subset of \( R \) that is closed under Adams operations. Then we can endow the localization \( S^{-1}R \) with a structure of a \( \lambda \)-ring by the formula

\[
\psi_n(a/s) = \psi_n(a)/\psi_n(s), \quad a \in R, s \in S.
\]

If otherwise not stated, we always endow our rings with \( \lambda \)-ring structures in the above described way.

A complete filtered \( \lambda \)-ring \( R \) is a complete filtered ring endowed with a \( \lambda \)-ring structure respecting the filtration, that is \( \psi_n(F^mR) \subset F^{mn}R \), where \( R \supset F^1R \supset F^2R \supset \ldots \) is a filtration of \( R \). For example, if \( R \) is a \( \lambda \)-ring, then \( R[\{x_1, \ldots, x_r\}] \) is a complete filtered \( \lambda \)-ring with the filtration induced by total grading. Let \( R \) be a complete filtered \( \lambda \)-ring. Define the map \( \text{Exp} : F^1R \rightarrow 1 + F^1R \) by the formula

\[
\text{Exp}(f) = \sum_{k \geq 0} \sigma_k(f) = \exp \left( \sum_{k \geq 1} \frac{1}{k} \psi_k(f) \right).
\]

We have \( \text{Exp}(f + g) = \text{Exp}(f) \text{Exp}(g) \), for \( f, g \in F^1R \). The map \( \text{Exp} \) has an inverse \( \text{Log} : 1 + F^1R \rightarrow F^1R \) (see [16, 21]) given by

\[
\text{Log}(f) = \sum_{k \geq 1} \frac{\mu(k)}{k} \psi_k(\log(f)),
\]
where $\mu$ is a Möbius function. Define the map $\text{Pow} : (1 + F^1R) \times R \to 1 + F^1R$ by the formula

$$\text{Pow}(f, g) = \text{Exp}(g \log(f)), \quad f \in 1 + F^1R, g \in R.$$ 

Define the usual power map by $f^n = \exp(g \log(f))$.

**Lemma 2.1** (see [21, Lemma 22]). Let $R$ be a complete filtered $\lambda$-ring, $f \in 1 + F^1R$ and $g \in R$. Define the elements $g_k \in R$, $k \geq 1$, inductively by the formula $\sum_{k|n} kg_k = \psi_n(g)$, $n \geq 1$. Then we have

$$\text{Pow}(f, g) = \prod_{k \geq 1} \psi_k(f)^{g_k}.$$ 

For any $m \geq 0$, define $[\infty, m]_v = \prod_{i=1}^m (1 - v^i)^{-1} \in \mathbb{Q}(v)$.

**Lemma 2.2** (Heine formula, see [22]). We have

$$\text{Exp} \left( \frac{x}{1 - v} \right) = \sum_{m \geq 0} [\infty, m]_v x^m$$

in $\mathbb{Q}(v)[[x]]$.

### 2.2. Chern character.

Let $X$ be a smooth projective curve over a field $k$. The map

$$\text{ch} : K_0(\text{Coh} X) \to \mathbb{Z}^2$$

given by $[F] \to (\text{rk } F, \deg F)$ is called the Chern character. By the Riemann-Roch formula [15, Ch. 15], we have

$$\chi(F, G) = \dim \text{Hom}(F, G) - \dim \text{Ext}^1(F, G)$$

$$= \text{rk } F \deg G - \deg F \text{rk } G + (1 - g) \text{rk } F \text{rk } G.$$ 

For any elements $\alpha = (n, d)$, $\beta = (n', d')$ in $\mathbb{Z}^2$, define

$$\langle \alpha, \beta \rangle = \text{nd}' - \text{dn}' + (1 - g)\text{nn}' = \alpha \beta',$$

where $C = \begin{pmatrix} 1 & -g \\ 1 & 0 \end{pmatrix}$. Then

$$\chi(F, G) = \langle \text{ch } F, \text{ch } G \rangle.$$ 

### 2.3. Galois descent.

Let $X$ be an algebraic variety over a finite field $k$ and let $K/k$ be a finite field extension. The group $\Gamma = \text{Gal}(K/k)$ acts on $X_K = X \otimes_k K$ in a natural way. For any coherent sheaf $M$ over $X_K$ and for any $\sigma \in \Gamma$, there is defined a direct image sheaf $\sigma_* M$. A $\Gamma$-equivariant coherent sheaf $M$ on $X_K$ is a coherent sheaf endowed with a system of isomorphisms $\sigma_M : M \to \sigma_* M$, $\sigma \in \Gamma$, satisfying $\sigma(\tau_M) \circ \sigma_M = (\sigma \tau)_M$ for any $\sigma, \tau \in \Gamma$. Note that the map $f : M \to \sigma_* M$ of coherent sheaves is given by the map $\overline{f} : M \to \text{Ab}$ of abelian sheaves, satisfying locally $\overline{f}(am) = \sigma(a) \overline{f}(m)$. It follows that if $\sigma \in \Gamma$ is a generator, then the structure of a $\Gamma$-equivariant sheaf on $M$ is given by the map $f : M \to \sigma_* M$ such that $(\overline{f})^\sharp = 1$.

We denote by $\text{Coh}_\Gamma X_K$ the category of $\Gamma$-equivariant coherent sheaves on $X_K$. For any coherent sheaf $M$ on $X$, the sheaf $M_K = M \otimes_k K$ on $X_K$ is a $\Gamma$-equivariant sheaf in a canonical way. The following result is a version of the Galois descent (see, e.g., [17]).

**Proposition 2.3.** The functor $\text{Coh}_\Gamma X \to \text{Coh}_\Gamma X_K$, sending a sheaf $M$ to the $\Gamma$-equivariant sheaf $M_K$ is an equivalence of categories. The inverse functor is given by $N \mapsto N^\Gamma$, the $\Gamma$-invariant part of $N$. 

2.4. Indecomposable sheaves. Let $X$ be a projective scheme over a finite field $k$. Then all the homomorphism spaces between coherent sheaves on $X$ are finite dimensional and the category $\text{Coh} X$ is Krull-Schmidt \cite{1}. We say that a sheaf $M \in \text{Coh} X$ is absolutely indecomposable if, for any finite field extension $K/k$, the sheaf $M_K$ on $X_K$ is indecomposable. For any sheaf $M \in \text{Coh} X$, we define the splitting field of a sheaf $M$ to be the minimal finite field extension $K/k$ such that all the indecomposable direct summands of $M_K$ are absolutely indecomposable (it will be shown that such a field is unique). Given a finite field extension $K/k$ and a sheaf $M \in \text{Coh} X_K$, we define the minimal field of definition of $M$ to be the minimal field extension $L/k$ contained in $K$ such that there exists a sheaf $M' \in \text{Coh} X_L$ with $M'_K \simeq M$ (it will be shown that such a field is unique). For any coherent sheaf $M \in \text{Coh} X_K$, we define $k(M) = \text{End} M / \text{rad}(\text{End} M)$.

Lemma 2.4. For any finite field extension $L/K$, we have $k(M_L) = k(M) \otimes_K L$.

Proof. It is enough to show that, for any finite-dimensional algebra $A$ over $K$, we have $\text{rad}(A_L) = (\text{rad} A)_L$. This follows from \cite{2} Cor. 7.2.2 as $L/K$ is a separable field extension.

Lemma 2.5. Let $M, N \in \text{Coh} X$ and assume that there exists some finite extension $K/k$ such that $M_K \simeq N_K$. Then $M \simeq N$.

Proof. Consider the finite etale morphism $\pi : X_K \to X$. Then $\pi_*(M_K) \simeq M^n$ and $\pi_*(N_K) \simeq N^n$, where $n = [K : k]$. As $\text{Coh} X$ is a Krull-Schmidt category, we get $M \simeq N$. □

This lemma implies that any two $\text{Gal}(K/k)$-equivariant structures on the sheaf on $X_K$ descend to isomorphic sheaves on $X$. The following proposition should be compared with \cite{19} Lemma 3.4

Proposition 2.6. Let $K/k$ be a finite field extension and let $M \in \text{Coh} X_K$ be a coherent sheaf. Then

(1) $M$ is indecomposable if and only if $k(M)$ is a field.
(2) $M$ is absolutely indecomposable if and only if $k(M) = K$.
(3) If $M$ is indecomposable then the splitting field of $M$ is unique and equals $k(M)$.
(4) If $M$ is indecomposable then the minimal field of definition of $M$ is unique and equals $K^\Gamma$, where $\Gamma = \{\sigma \in \text{Gal}(K/k) \mid \sigma_* M \simeq M\}$.
(5) If $N \in \text{Coh} X$ is indecomposable and $K$ is its splitting field, then there exists an absolutely indecomposable sheaf $M \in \text{Coh} X_K$ with a minimal field of definition $K$ such that $N_K \simeq \bigoplus_{\sigma \in \text{Gal}(K/k)} \sigma_* M$ and both $\text{Gal}(K/k)$-equivariant sheaves descend to $N$.
(6) If $M$ is indecomposable and has the minimal field of definition $K$, then the $\text{Gal}(K/k)$-equivariant sheaf $\bigoplus_{\sigma \in \text{Gal}(K/k)} \sigma_* M$ descends to the indecomposable sheaf over $X$ having the splitting field $k(M)$.

Proof. 1) If $M$ is indecomposable then $\text{End} M$ is a local algebra and $k(M)$ is a division algebra over a finite field $K$. Thus, $k(M)$ is a field. If $M \simeq \bigoplus_{i=1}^n N_i^{r_i}$, where $N_i$ are pairwise non-isomorphic sheaves then $k(M) \simeq \prod_{i=1}^n M(r_i, k(N_i))$. This is a field if and only if $n = 1$ and $r_1 = 1$.
2) If $k(M) = K$ and $L/K$ is a finite field extension then $k(M_L) \simeq k(M) \otimes_K L \simeq L$ and therefore $M_L$ is indecomposable. If $M$ is absolutely indecomposable and
\( k(M) = L \), then \( k(M_L) \simeq L \otimes_K L \simeq \prod_{i=1}^{[L : K]} L \). As \( M_L \) is indecomposable, we have \( [L : K] = 1 \).

3) Let \( L = k(M) \) and let \( L' \) be some splitting field of \( M \). It is known \cite{7} Proposition 8.3 that \( k(M_L') \simeq L \otimes_K L' \simeq \prod_{i=1}^{n} L_i \) is a product of fields. The corresponding decomposition of unit into the sum of idempotents can be lifted to \( \text{End}(M_L') \) and we get a decomposition \( M_L' = \bigoplus_{i=1}^{n} M_i \), where \( k(M_i) \simeq L_i \). It follows that the sheaves \( M_i \) are indecomposable and therefore absolutely indecomposable. This implies that \( L_i = L' \). It is known \cite{7} Proposition 8.3 that every \( L_i \) is a composite field of the fields \( L \) and \( L' \). Therefore \( L \) is contained in \( L_i = L' \). To show that \( L \) is itself a splitting field, we note that \( k(M_L) \simeq L \otimes_K L \simeq \prod_{i=1}^{[L : K]} L \) by \cite{7} Proposition 8.4. The corresponding decomposition of unit into the sum of idempotents can be lifted to \( \text{End}(M_L) \) and we get a decomposition \( M_L = \bigoplus_{i=1}^{[L : K]} M_i \), where \( k(M_i) \simeq L \). This implies that \( M_i \) are absolutely indecomposable.

4) If \( L \subset K \) is a field of definition of \( M \) then, for every \( \sigma \in \text{Gal}(K/L) \), we have \( \sigma_* M \simeq M \) and therefore \( \sigma \in \Gamma \). This implies that \( K^\Gamma \subset L \). Let us show that \( K^\Gamma \) is a field of definition of \( M \). Let \( L = K^\Gamma \), \( \sigma \) be some generator of \( \Gamma = \text{Gal}(K/L) \) and let \( n = [K : L] \). To show that \( M \) is defined over \( L \), we have to define a \( \text{Gal}(K/L) \)-equivariant structure on \( M \). We will need to enlarge the field \( K \) first. By assumption, there exists an isomorphism \( f : M \to \sigma_* M \). We will write \( f^n \) to denote the composition \( M \to \sigma_* M \to \sigma_*^2 M \to \ldots \to \sigma_*^n M \). Then \( f^n : M \to M \) is an automorphism of \( M \). The group \( \text{Aut}(M) \) is finite and there exists some \( n' \geq 1 \) such that \( f^{n'} = 1 \). Let \( K'/K \) be a field extension of degree \( n' \). Let \( \sigma' \in \text{Gal}(K'/L) \) be some generator that is mapped to \( \sigma \in \text{Gal}(K/L) \). We can extend the map \( f : M \to \sigma_* M \) to the map \( f' : M_{K'} \to \sigma'_* M_{K'} \) by \( f'(m \otimes a) = f(m) \otimes \sigma'(a) \). This map satisfies \( (f')^{n'n} = 1 \) and defines a structure of \( \text{Gal}(K'/L) \)-equivariant sheaf on \( M_{K'} \). By Galois descent, there exists some \( N \in \text{Coh} X_L \) with \( N_{K'} \simeq M_{K'} \). It follows from the above lemma that \( N_K \simeq M \).

5) Let \( A = \text{End}(N) \) and let \( \Gamma = \text{Gal}(K/k) \). It is known \cite{12} Proposition 8.4 that there exists an isomorphism of \( k \)-algebras \( K \otimes_k K \to \prod_{\sigma \in \Gamma} K_{\sigma}, \) where \( K_{\sigma} \) denotes a copy of \( K \) and the projection \( K \otimes_k K \to K_{\sigma} \) is given by \( x \otimes y \mapsto x \sigma(y) \). By the Wedderburn-Malcev theorem \cite{13} Theorem 6.2.1, there exists a section \( \bar{K} \to K \) of the projection \( A \to K \). Applying \( A \otimes K \) to the isomorphism \( K \otimes_k K \to \prod_{\sigma \in \Gamma} K_{\sigma} \), we get an isomorphism \( A \otimes_k K \to \prod_{\sigma \in \Gamma} A_{\sigma} \), where \( A_{\sigma} \) denotes a copy of \( A \) and the projection \( A \otimes_k K \to A_{\sigma} \) is given by \( x \otimes y \mapsto x \sigma(y) \). Let \( 1 = \sum_{\sigma \in \Gamma} e_{\sigma} \) be the corresponding decomposition of the unit in \( A \otimes_k K \). Then we have \( \tau(e_{\sigma}) = e_{\tau \sigma^{-1}} \), where the action of \( \Gamma \) on \( A \otimes_k K \) is given by the action on \( K \). We get a decomposition \( N_K = \bigoplus_{\sigma \in \Gamma} M_\sigma \), where the direct summand \( M_\sigma \) corresponds to the idempotent \( e_{\sigma} \). It follows from \( k(N_K) \simeq K \otimes_k K \simeq \prod_{\sigma \in \Gamma} K_{\sigma} \) that \( k(M_\sigma) = K \) and the sheaves \( M_\sigma \) are pairwise non-isomorphic. It follows from \( \tau(e_{\sigma}) = e_{\tau \sigma^{-1}} \) that \( \tau_* M_\sigma \simeq M_{\tau \sigma} \). Let \( M = M_1 \). Then we have \( N_K \simeq \bigoplus_{\sigma \in \text{Gal}(K/k)} \sigma_* M \). By the above lemma these \( \text{Gal}(K/k) \)-equivariant sheaves descend to the same sheaf \( N \) on \( X \). It follows from \( k(M) = K \) that \( M \) is absolutely indecomposable. It follows from the condition \( \sigma_* M \simeq M_{\sigma} \neq M \) for every \( \sigma \neq 1 \) that \( M \) has minimal field of definition \( K \).

6) Let \( \bigoplus_{\sigma \in \text{Gal}(K/k)} \sigma_* M \) descends to the sheaf \( N \) over \( X \). If \( N' \) is a direct summand of \( N \) then the set of isomorphism classes of indecomposable summands of \( N'_K \) contains all \( \sigma_* M, \sigma \in \text{Gal}(K/k) \). They are pairwise non-isomorphic and therefore
For any finite field extension $K/k$, $N_K' = N_K$ and $N' = N$. This implies that $N$ is indecomposable. We have

$$k(N_K) = k\left( \bigoplus_{\sigma \in \text{Gal}(K/k)} \sigma_* M \right) = \prod_{\sigma \in \text{Gal}(K/k)} k(M).$$

This implies that $\dim_k k(N) = \dim_k k(M)$ and therefore $k(N) = k(M)$. \qed

2.5. Stable bundles. Let $X$ be a curve over a finite field $k$. Define the slope of $\alpha = (n, d) \in \mathbb{N}^* \times \mathbb{Z}$ to be $\mu(\alpha) = d/n$. For any vector bundle $M$ on $X$ (we identify vector bundles with locally free sheaves), define its slope $\mu(M) = \mu(\text{ch} M)$.

**Definition 2.7.** We say that a bundle $M$ on $X$ is semistable (respectively, stable) if, for any proper nonzero subbundle $N \subset M$, we have $\mu(N) \leq \mu(M)$ (respectively, $\mu(N) < \mu(M)$). We say that a stable bundle $M$ is absolutely stable if it stays stable after any finite field extension of $k$. A semistable bundle is called polystable if it is a direct sum of stable bundles. For any $\alpha \in \mathbb{N}^* \times \mathbb{Z}$, let $\mathcal{E}_X(\alpha)$ be the set of representatives of isomorphism classes of vector bundles over $X$ having character $\alpha$. Let $\mathcal{E}^s_X(\alpha)$ (respectively, $\mathcal{E}^s_X(\alpha)$) be the subset of $\mathcal{E}_X(\alpha)$ consisting of semistable (respectively, stable) bundles. For any $\mu \in \mathbb{Q}$, we define

$$\mathcal{E}^s_X(\mu) = \bigcup_{\mu(\alpha) = \mu} \mathcal{E}^s_X(\alpha), \quad \mathcal{E}_X^s(\mu) = \bigcup_{\mu(\alpha) = \mu} \mathcal{E}_X(\alpha).$$

For any finite field extension $K/k$, we define the sets $\mathcal{E}_{X_K}(\alpha)$, $\mathcal{E}^s_{X_K}(\alpha)$, $\mathcal{E}^s_{X_K}(\alpha)$, $\mathcal{E}^s_{X_K}(\mu)$, $\mathcal{E}^s_{X_K}(\mu)$ in the same way as above.

It is known that, for any $\alpha \in \mathbb{N}^* \times \mathbb{Z}$, the set $\mathcal{E}^s_X(\alpha)$ is finite. For any $\mu \in \mathbb{Q}$, we define the category $\text{Coh}_{\mu} X$ to be the subcategory of $\text{Coh} X$ consisting of semistable bundles having slope $\mu$. This is an abelian category whose simple objects are precisely the stable bundles from $\mathcal{E}^s_X(\mu)$. For any finite field extension $K/k$, define $a_{\alpha}(K)$ to be the number of absolutely stable bundles in $\mathcal{E}^s_{X_K}(\alpha)$. For $r \geq 1$, define $s_{\alpha,r}(K)$ to be the number of bundles in $\mathcal{E}_{X_K}^s(\alpha)$ with $\text{End} K \text{End} M = r$.

**Lemma 2.8.** A stable bundle $M$ over $X$ is absolutely stable if and only if $\text{End} M = k$.

**Proof.** If $M \in \text{Coh} X$ is stable then, for any finite extension $K/k$, the sheaf $M_K$ is polystable. Indeed, the relative socle (in the category $\text{Coh}_{\mu} X$, $\mu = \mu(M)$) of $M_K$ is $\text{Gal}(K/k)$-stable and determines therefore some $N \subset M$ with $\mu(N) = \mu$. It follows from stability of $M$ that $N = M$ and therefore the socle of $M_K$ is the whole $M_K$. This implies that $M_K$ is polystable. If $\text{End} M = k$ then $\text{End} M_K = K$ and therefore $M_K$ is stable. On the other hand, if $M$ is absolutely stable then it is absolutely indecomposable and we have $\text{End}(M) = k(M) = k$. \qed

It follows that $a_{\alpha}(K) = s_{\alpha,1}(K)$ for any finite field extension $K/k$. We will prove a more strong result now.

**Proposition 2.9.** Let $K/k$ be a finite field extension and let $\Gamma = \text{Gal}(K/k)$. For any absolutely stable sheaf $M$ over $X_K$ with the minimal field of definition $K$, the sheaf $\bigoplus_{\sigma \in \Gamma} \sigma_* M$ descends to a stable sheaf over $X$ having endomorphism ring isomorphic to $K$. The induced map between the corresponding sets is a quotient map with respect to the free action of $\Gamma$ on the first set.
Proof. All the sheaves $\sigma_*M$ are stable and therefore $\bigoplus_{\sigma \in \Gamma} \sigma_*M$ is semistable, as also its descend $N$ over $X$. Assume that there exists some nonzero $U \subset N$ with $\mu(U) = \mu(N)$. Then $U$ is semistable as also $U_K \subset \bigoplus_{\sigma \in \Gamma} \sigma_*M$. Any projection $U_K \to \sigma_*M$ is either surjective or zero. One of them is surjective and the inclusion $U_K \hookrightarrow \bigoplus_{\sigma \in \Gamma} \sigma_*M$ is $\Gamma$-equivariant. This implies that $U_K = \bigoplus_{\sigma \in \Gamma} \sigma_*M$ and $U = N$. Thus, $N$ is stable. It follows from Proposition 2.6 that $\text{End}(N) = k(N) = k(M) = K$.

Assume now that $N$ is some stable sheaf over $X$ with $\text{End}(N) = K$. Then $N$ is indecomposable and its splitting field equals $k(N) = \text{End}(N) = K$. Applying Proposition 2.6 we can find an absolutely indecomposable sheaf $M$ with a minimal field of definition $K$ such that $N_K \simeq \bigoplus_{\sigma \in \Gamma} \sigma_*M$. As $N$ is stable, the sheaf $N_K$ is polystable. It follows that $M$ is stable and therefore absolutely stable. It is clear that if $M'$ is a different absolutely stable sheaf over $X_K$ such that $N_K \simeq \bigoplus_{\sigma \in \Gamma} \sigma_*M'$ then $M' \simeq \sigma_*M$ for some $\sigma \in \Gamma$. This implies the last assertion of the proposition. □

Remark 2.10. If $\bigoplus_{\sigma \in \Gamma} \sigma_*M$ descends to the sheaf $N$ then $\text{ch}(N) = [K : k]\text{ch}(M)$. It follows from the proposition that $s_{\alpha,r}(k) \neq 0$ implies $\alpha/r \in \mathbb{Z}_2$.

Lemma 2.11. Let $\mathbb{F}_q/k$ be a finite field extension, $\alpha \in \mathbb{N}^* \times \mathbb{Z}$ and $n \geq 1$. Then we have

$$a_\alpha(\mathbb{F}_q^n) = \sum_{r | n} rs_{r \alpha,r}(\mathbb{F}_q).$$

Proof. Without loss of generality, we may assume that $k = \mathbb{F}_q$. Any absolutely stable sheaf over $\mathbb{F}_q^n$ has the minimal field of definition of the form $\mathbb{F}_q^r$ with $r | n$. It follows from Proposition 2.6 that the number of absolutely stable sheaves over $\mathbb{F}_q^r$ having character $\alpha$ and minimal field of definition $\mathbb{F}_q^r$ equals $rs_{r \alpha,r}(\mathbb{F}_q)$. Now the statement of the lemma is obvious. □

The goal of this paper is to determine the numbers $a_\alpha(\mathbb{F}_q)$ and the Poincaré polynomials of the corresponding moduli spaces.

2.6. Hall algebra. Let $X$ be a smooth projective curve over a finite field $k = \mathbb{F}_q$. Let $\mathcal{H}$ be the Hall algebra [24] of the category $\text{Coh} X$ and let $\mathcal{H}_+$ be its subalgebra generated by the sheaves having nonnegative degree. We endow $\mathcal{H}_+$ with an $\mathbb{N}$-grading, where an isomorphism class $[M]$ has degree $\text{rk} M + \text{deg} M$. Let $\mathcal{H}_+$ be the corresponding completion. Let us endow the algebra $\mathbb{Q}[x_1, x_2]$ with a new product given by the formula

$$x^\alpha \circ x^\beta = q^{-\langle \alpha, \beta \rangle} x^{\alpha + \beta}.$$

This new algebra is denoted by $\mathbb{Q}^{\text{tw}}[x_1, x_2]$ or $\mathbb{Q}^{\text{tw}}_q[x_1, x_2]$.

Lemma 2.12 (see [24] Lemma 3.3). The map $f : \mathcal{H}_+ \to \mathbb{Q}^{\text{tw}}[x_1, x_2]$ given by

$$[M] \mapsto \frac{1}{\# \text{Aut} M} x^{\text{ch} M}$$

is a homomorphism of algebras. It induces a homomorphism of completions $f : \hat{\mathcal{H}}_+ \to \mathbb{Q}^{\text{tw}}_q[x_1, x_2]$. 


3. Solution of the HN-recursion

Let $\Gamma$ be a commutative semigroup equipped with a total preorder $\leq$ on $\Gamma$ (e.g. a preorder given by some stability on $\mathbb{N}^I \setminus \{0\}$, see [23]) such that

$$\min\{\alpha, \beta\} \leq \alpha + \beta \leq \max\{\alpha, \beta\}, \quad \alpha, \beta \in \Gamma.$$  

We will write $\alpha < \beta$ if $\alpha \leq \beta$ but $\beta \not\leq \alpha$.

**Definition 3.1.** For any sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of elements in $\Gamma$, we define $|\lambda| = \sum \lambda_i$ and $l(\lambda) = k$. We will say that a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ is stable if $\sum_{i=1}^j \lambda_i < |\lambda|$ for $1 \leq j < k$. For any $\alpha \in \Gamma$, let $P_\alpha$ be the set of sequences $\lambda$ in $\Gamma$ such that $|\lambda| = \alpha$. Let $P_\alpha^s \subset P_\alpha$ be a subset consisting of stable sequences and let $P_\alpha^i \subset P_\alpha$ be a subset consisting of strictly increasing sequences.

The following result is an easy generalization of [24, Theorem 5.1].

**Theorem 3.2.** Assume that, for any $\alpha \in \Gamma$, the set $P_\alpha^s$ is finite. Let $A$ be an associative ring and let $(a_\alpha)_{\alpha \in \Gamma}$, $(b_\alpha)_{\alpha \in \Gamma}$ be two systems of elements in $A$ related by the formula

$$b_\alpha = \sum_{\lambda \in P_\alpha^i} a_{\lambda_1} \cdots a_{\lambda_{l(\lambda)}}.$$  

Then we have

$$a_\alpha = \sum_{\lambda \in P_\alpha^s} (-1)^{k-1} b_{\lambda_1} \cdots b_{\lambda_{l(\lambda)}}.$$  

**Proof.** We use the induction on the number of elements in $P_\alpha^s$. Given a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$, we say that a set of positive integers $S = (s_1 < \cdots < s_{r-1} < s_r = k)$ is $\lambda$-admissible if the sequences $\mu^i = (\lambda_{s_{i-1}+1}, \ldots, \lambda_{s_i})$, $i = 1, \ldots, r$ (we set $s_0 = 0$) are stable and $|\mu^1| < \cdots < |\mu^r|$. If there exists any $\lambda$-admissible set, then $\lambda$ is stable. Let $A(\lambda)$ be the set of all $\lambda$-admissible sets. For any sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$, we define $a_\lambda = a_{\lambda_1} \cdots a_{\lambda_k}$ and $b_\lambda = b_{\lambda_1} \cdots b_{\lambda_k}$. Then

$$a_\alpha + \sum_{\lambda \in P_\alpha^s} (-1)^{l(\lambda)} b_\lambda = a_\alpha - b_\alpha + \sum_{\lambda \in P_\alpha^i, l(\lambda) > 1} (-1)^{l(\lambda)} b_\lambda$$

$$= - \sum_{\mu \in P_\alpha^i, l(\mu) > 1} a_\mu + \sum_{\lambda \in P_\alpha^s, l(\lambda) > 1} (-1)^{l(\lambda)} b_\lambda$$

$$= - \sum_{\lambda \in P_\alpha^s} \sum_{s \in A(\lambda)} (-1)^{l(\lambda) - \#S} b_\lambda + \sum_{\lambda \in P_\alpha^i, l(\lambda) > 1} (-1)^{l(\lambda)} b_\lambda$$

$$= \sum_{\lambda \in P_\alpha^i, l(\lambda) > 1} (-1)^{l(\lambda) - 1} b_\lambda \sum_{S \in A(\lambda)} (-1)^{\#S}.$$  

It is proved in [24, Lemma 5.4] that for any sequence $\lambda$ of length $l(\lambda) > 1$

$$\sum_{S \in A(\lambda)} (-1)^{\#S} = 0.$$  

This implies the theorem. \qed
The case that we are actually interested in in this paper is \( \Gamma = \mathbb{N}^* \times \mathbb{Z} \) with the preorder given by \( \alpha \leq \beta \) if \( \mu(\alpha) \leq \mu(\beta) \). Unfortunately, the above theorem can not be applied in this situation, as \( P^*_\alpha \) (and also \( P^*_1 \)) are not finite sets. Even in the complete topological rings, the convergence of the sums \( \sum \) do not imply the convergence of the sums \( \sum \). Still, under certain conditions, there exists a nice solution of the recursive formula due to Zagier. Given a topological space \( A \) and a system of elements \( (f_\lambda)_{\lambda \in P^*_1} \), we say that the sum \( \sum_{\lambda \in P^*_n} f_\lambda \) converges if there exists a limit

\[
\lim_{n \to \infty} \sum_{\lambda \in P^*_n, \mu(\lambda(\alpha)) < n} f_\lambda.
\]

Theorem 3.3 (Zagier, [27, Theorem 3]). Let \( A \) be a complete normed ring and let \( t \in A \) be an invertible element with \( |t| < 1 \). Let \( (a_\alpha)_{\alpha \in \Gamma} \) and \( (b_n)_{n \geq 1} \) be two systems of elements in \( A \) related by

\[
b_n = \sum_{\lambda \in P^*_n} t^{\sum_{i<j}(\lambda_i, \lambda_j)} a_{\lambda_1} \cdots a_{\lambda(\alpha)}, \quad \alpha = (n, d) \in \Gamma,
\]

where the pairing \( \langle \cdot, \cdot \rangle \) on \( \mathbb{Z}^2 \) is given by \( \langle (n, d), (n', d') \rangle = nd' - nd' \). Then

\[
a_\alpha = \sum_{\substack{n_1, \ldots, n_k > 0 \not= 1 \ldots 1 \ldots n_k+n_k=n}} \Phi_{n_\alpha, d}(t) b_{n_1} \cdots b_{n_k}, \quad \alpha = (n, d) \in \Gamma,
\]

where \( n_\alpha = (n_1, \ldots, n_k) \),

\[
\Phi_{n_\alpha, d}(t) = (-1)^{k-1} \prod_{i=1}^{k-1} \frac{t^{\ell(n_i+n_{i+1})\{n_1+\cdots+n_k\}d/n}}{1-t^{n_i+n_{i+1}}},
\]

and \( \{x\}_+ = \max\{x - |x|, 1\} \).

Remark 3.4. Assume that \( A \) is a commutative complete normed ring, \( t \in A \) is invertible with \( |t| > 1 \), and \( (a_\alpha)_{\alpha \in \Gamma}, (b_n)_{n \geq 1} \) are two systems of elements in \( A \) satisfying

\[
b_n = \sum_{\lambda \in P^*_n} t^{-\sum_{i<j}(\lambda_i, \lambda_j)} a_{\lambda_1} \cdots a_{\lambda(\alpha)}, \quad \alpha = (n, d) \in \Gamma.
\]

Then

\[
a_\alpha = \sum_{\substack{n_1, \ldots, n_k > 0 \not= 1 \ldots 1 \ldots n_k+n_k=n}} \Psi_{n_\alpha, d}(t) b_{n_1} \cdots b_{n_k}, \quad \alpha = (n, d) \in \Gamma,
\]

where \( n_\alpha = (n_1, \ldots, n_k) \),

\[
\Psi_{n_\alpha, d}(t) = \prod_{i=1}^{k-1} \frac{t^{\ell(n_i+n_{i+1})\{n_1+\cdots+n_k\}d/n}}{1-t^{n_i+n_{i+1}}},
\]

and \( \{x\} = x - |x| \). This follows from the above theorem and the equality \( \Phi_{n_\alpha, d}(t^{-1}) = \Psi_{n_\alpha, d}(t) \), where \( n_\alpha = (n_k, \ldots, n_1) \).
4. \(\lambda\)-ring of c-sequences

Let \(S\) be the set of sequences \(s = (s_k)_{k \geq 1}\) of complex numbers. For any two sequences \(s = (s_k)_{k \geq 1}, s' = (s'_k)_{k \geq 1}\), define their sum and product by
\[
s + s' = (s_k + s'_k)_{k \geq 1}, \quad ss' = (s_k s'_k)_{k \geq 1}.
\]
This endows \(S\) with a structure of a ring. Define the Adams operations on \(S\) by \(\psi_{r}(s) = (s_{rk})_{k \geq 1}, r \geq 1\). This endows \(S\) with a structure of a \(\lambda\)-ring.

**Remark 4.1.** For any commutative \(\mathbb{Q}\)-algebra \(R\), we can define the \(\lambda\)-ring \(S(R)\) of sequences in \(R\) in the same way as above. There is a commutative diagram of \(\lambda\)-ring isomorphisms

\[
\begin{align*}
W(R) & \xrightarrow{a} S(R) \\
& \downarrow \bigwedge \downarrow \\
\Lambda(R) & \xrightarrow{c}
\end{align*}
\]
where \(W(R)\) is the ring of Witt vectors over \(R\), \(\Lambda(R) = 1 + tR[t]\) is Grothendieck’s \(\lambda\)-ring (see, e.g., [\(\mathbb{Q}\) V.2.3]), and the maps \(a, b,\) and \(c\) are given by

\[
\begin{align*}
a : (x_n)_{n \geq 1} & \mapsto \left(\sum_{d|n} dx_d^{n/d}\right)_{n \geq 1}, \\
b : (x_n)_{n \geq 1} & \mapsto \prod_{n \geq 1} (1 - x_n t^n)^{-1}, \\
c : (s_n)_{n \geq 1} & \mapsto \exp \left(\sum_{n \geq 1} s_n t^n / n\right).
\end{align*}
\]

Let \(\mathbb{F}_q\) be some finite field with \(q\) elements. Define the subgroups of \(\mathbb{C}^*\)
\[
G_0 = \{\lambda \in \mathbb{C}^* \mid |\lambda| = q^{n/2}, n \in \mathbb{Z}\}, \quad G_1 = \{q^{n/2} \mid n \in \mathbb{Z}\}.
\]
We endow the group algebra \(\mathbb{Q}[G_0]\) with a structure of a \(\lambda\)-ring by the formula \(\psi_{t}(\sum_{\lambda \in G_0} a_{\lambda}\lambda) = \sum_{\lambda \in G_0} a_{\lambda}\lambda^{t}\). Consider the map \(\mathbb{Q}[G_0] \to S\), given by
\[
\mathbb{Q}[G_0] \ni \sum_{\lambda \in G_0} a_{\lambda}\lambda \mapsto \left(\sum_{\lambda \in G_0} a_{\lambda}\lambda^{k}\right)_{k \geq 1} \in S.
\]

**Proposition 4.2.** The map \(\mathbb{Q}[G_0] \to S\) is an injective \(\lambda\)-ring homomorphism. If \(s \in S\) is the image of \(\sum a_{\lambda}\lambda \in \mathbb{Z}[G_0]\) then
\[
Z_s(t) := \exp \left(\sum_{k \geq 1} \frac{s_k t^k}{k}\right) = \prod_{\lambda \in G_0} (1 - t\lambda)^{-a_{\lambda}}
\]
in \(\mathbb{C}[t]\).

**Proof.** It is clear that \(\mathbb{Q}[G_0] \to S\) is a \(\lambda\)-ring homomorphism. To prove the formula, we note that
\[
\log \left(\prod_{\lambda \in G_0} (1 - t\lambda)^{-a_{\lambda}}\right) = \sum_{\lambda \in G_0} a_{\lambda} \log \left(\frac{1}{1 - t\lambda}\right) = \sum_{k \geq 1} \sum_{\lambda \in G_0} a_{\lambda} (t\lambda)^k / k = \sum_{k \geq 1} s_k t^k / k.
\]

If the map \(\mathbb{Q}[G_0] \to S\) is not injective, then we can find some nonzero element \(\sum a_{\lambda}\lambda \in \mathbb{Z}[G_0]\) in the kernel. It follows that \(\prod_{\lambda \in G_0} (1 - t\lambda)^{-a_{\lambda}} = 1\) and therefore all \(a_{\lambda}\) are zeros, contradicting the assumption. \(\square\)
Definition 4.3. Define the ring $S^0_c$ of effective c-sequences to be the image of $\mathbb{Q}[G_0]$ in $S$. Define the ring of special c-sequences to be the image of $\mathbb{Q}[G_1]$ in $S$. Define the ring $S_c$ of c-sequences to be localization of $S^0_c$ with respect to special c-sequences that are invertible in $S$. The ring $S_c$ is a $\lambda$-subring of $S$.

Remark 4.4. A sequence $s \in S$ is an effective c-sequence if and only if there exists some integer $r \geq 1$ such that $Z_s(t)^r = \exp(\sum_{k \geq 1} s_k t^k / k)^r$ is a rational function with zeros and poles having absolute values of the form $q^{n/2}$, $n \in \mathbb{Z}$.

Remark 4.5. If $X$ is a scheme over $\mathbb{F}_q$ then, by the Weil conjectures, the sequence $(\#X(\mathbb{F}_q^n))_{k \geq 1}$ is an effective c-sequence.

Remark 4.6. The special c-sequence $L = (q^k)_{k \geq 1}$ is called the Lefschetz c-sequence. This is a c-sequence associated to the affine line $\mathbb{A}^1_{\mathbb{F}_q}$. The ring of special c-sequences is generated by $L$ and $L^{-1}$.

Remark 4.7. There exists a structure of a $\lambda$-ring on the Grothendieck ring of algebraic varieties over $\mathbb{C}$ (or, more precisely, on the Grothendieck ring of the category of motives over $\mathbb{C}$), see [16]. In this structure the sigma-operations on algebraic varieties are given by the symmetric products. In the case of a finite field $\mathbb{F}_q$, we define the $\lambda$-ring structure not on the Grothendieck ring of algebraic varieties over $\mathbb{F}_q$ or $\mathbb{F}_{q^m}$ but rather on the ring of c-sequences of algebraic varieties. The Adams operation $\psi_n$ applied to the c-sequence of the algebraic variety $X$ over $\mathbb{F}_q$ gives the c-sequence of the algebraic variety $X_{\mathbb{F}_{q^n}}$ over $\mathbb{F}_{q^n}$.

Definition 4.8. For any $\lambda \in G_0$, define its weight $w(\lambda)$ to be the integer such that $|\lambda| = q^{w(\lambda)/2}$. This defines a group homomorphism $w : G_0 \to \mathbb{Z}$, which induces a $\lambda$-ring homomorphism $P : S^0_c \simeq \mathbb{Q}[G_0] \to \mathbb{Q}[v, v^{-1}]$, $\sum a_\lambda \lambda \mapsto \sum a_\lambda v^{w(\lambda)}$, called the Poincaré polynomial. It induces the map $P : S_c \to \mathbb{Q}(v)$, called the Poincaré function.

Remark 4.9. If $s \in S^0_c$ is an effective c-sequence and $r \geq 1$ is an integer such that $Z_s(t)^r$ is a rational function, say $\prod_{\lambda \in G_0} (1 - \lambda t)^{-a_\lambda}$, then the Poincaré polynomial of $s$ equals $P(s, v) = \frac{1}{r} \sum_{\lambda \in G_0} a_\lambda v^{w(\lambda)}$.

Remark 4.10. Let $X$ be a scheme over $\mathbb{F}_q$ and let $l$ be a prime number, coprime with $q$. There is a weight filtration $(W_n)_{n \in \mathbb{Z}}$ defined on the cohomology groups $H^i_c(X, \mathbb{Q}_l)$. One defines the virtual Betti numbers

$$b_n(X) = \sum_{i \geq 0} (-1)^i \dim(\text{Gr}^W_n H^i_c(X, \mathbb{Q}_l))$$

and the virtual Poincaré polynomial $P(X, v) = \sum_{n \in \mathbb{Z}} b_n(X) v^n$. If $s \in S^0_c$ is a c-sequence associated to $X$ as in Remark 4.5 then $P(X, v) = P(s, v)$.

5. C-SEQUENCES OF SEMISTABLE BUNDLES

Let $X$ be a curve of genus $g$ over a finite field $k = \mathbb{F}_q$. We introduce the pairing $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}^2$ as in Section 2.2. Let $\Gamma = \mathbb{N}^* \times \mathbb{Z}$ be endowed with the total preorder by the rule $\alpha \leq \beta$ if $\mu(\alpha) \leq \mu(\beta)$. For any $\alpha \in \Gamma$ and any finite field extension $K/k$, we define

$$r_\alpha(K) = \frac{1}{\# \text{Aut} \ M}, \quad m_\alpha(K) = \sum_{M \in E_{X,K}(\alpha)} \frac{1}{\# \text{Aut} \ M}.$$
Moreover, (see [18, 11]) Theorem 5.1 Narasimhan filtration, we can write introduce the formula for its Poincaré function due to Zagier.

If we could apply the map would get the relation between the numbers \( r_k \) (Theorem 5.1). Define the elements \( r \) are well defined in some completion of the Hall algebra of \( \text{Coh} X \). Using the Harder-Narasimhan filtration, we can write

\[
M_\alpha = \sum_{M \in E_\lambda(\alpha)} [M], \quad R_\alpha = \sum_{M \in E'_\lambda(\alpha)} [M]
\]

are well defined in some completion of the Hall algebra of \( \text{Coh} X \). Using the Harder-Narasimhan filtration, we can write

\[
M_\alpha = \sum_{\lambda \in P_n} R_{\lambda_1} \cdots R_{\lambda_i(\lambda)}
\]

If we could apply the map \( f : \mathcal{H}_+ \to \mathbb{Q}^{tw} \{ x_1, x_2 \} \) to the above formula then we would get the relation between the numbers \( m_\alpha(k) \) and \( r_\alpha(k) \).

**Theorem 5.1** (see [18, 11]). For any \( \alpha = (n, d) \in \Gamma \), we have

\[
m_\alpha(\mathbb{F}_q) = \sum_{\lambda \in P_n} q^{-\sum_{i<j}(\lambda_i, \lambda_j)} r_{\lambda_1}(\mathbb{F}_q) \cdots r_{\lambda_i(\lambda)}(\mathbb{F}_q).
\]

Moreover,

\[
m_\alpha(\mathbb{F}_q) = m_n(\mathbb{F}_q) = \frac{\prod_{i=1}^{2g} (1 - \omega_i)}{q-1} q^{(n^2-1)(g-1)} Z_X(q^{-2}) \cdots Z_X(q^{-n}),
\]

where

\[
Z_X(t) = \frac{\prod_{i=1}^{2g} (1 - \omega_i t)}{(1-t)(1-qt)}
\]

is a zeta-function of \( X \) and \(|\omega_i| = q^{1/2}\).

**Corollary 5.2.** For any \( n \geq 1 \), \( m_n = (m_\alpha(\mathbb{F}_q))_{k \geq 1} \) is a c-sequence with the Poincaré function

\[
P(m_n, v) = (v^{2n} - 1) \prod_{i=1}^{n} \frac{1 - v^{2i-1})^{2g}}{(1 - v^{2i})^{2}}.
\]

The sum in the above theorem is infinite, so we can not deduce directly that \( r_\alpha \) is a c-sequence.

**Theorem 5.3.** For any \( \alpha = (n, d) \), we have

\[
r_\alpha(\mathbb{F}_q) = \sum_{n_1, \ldots, n_k > 0, n_i = n} q^{(g-1) \sum_{i<j} n_i n_j} \Psi_{n_1, \ldots, n_k}(q) m_{n_1}(\mathbb{F}_q) \cdots m_{n_k}(\mathbb{F}_q)
\]

where

\[
\Psi_{n_1, \ldots, n_k}(t) = \prod_{i=1}^{k-1} \frac{t^{(n_i+n_{i+1})}}{1 - t^{n_i+n_{i+1}}}.
\]

**Proof.** Let \( a_\alpha = q^{n^2(1-g)/2} r_\alpha(\mathbb{F}_q) \), \( \alpha = (n, d) \in \Gamma \) and let \( b_n = q^{n^2(1-g)/2} m_n(\mathbb{F}_q) \), \( n \geq 1 \). Then the above theorem implies

\[
b_n = \sum_{\lambda \in P_n} q^{-\sum_{i<j}(\lambda_i, \lambda_j)} a_{\lambda_1} \cdots a_{\lambda_i(\lambda)}, \quad \alpha = (n, d) \in \Gamma,
\]
where \( \langle (n, d), (n', d') \rangle' = nd' - dn' \). Applying Remark 3.4, we get

\[
a_\alpha = \sum_{n_1, \ldots, n_k > 0} \Psi_{n, d}(q) b_{n_1} \ldots b_{n_k}, \quad \alpha = (n, d) \in \Gamma,
\]

which implies the theorem. \( \square \)

**Corollary 5.4.** For any \( \alpha = (n, d) \), we have

\[
r_\alpha = \sum_{n_1, \ldots, n_k > 0} L(g-1) \sum_{i<j} n_i n_j \Psi_{n, d}(L) m_{n_1} \ldots m_{n_k}.
\]

**Corollary 5.5.** For any \( \alpha = (n, d) \in \Gamma \), the element \( r_\alpha \in S \) is a c-sequence and its Poincaré function equals

\[
P(r_\alpha, v) = \sum_{n_1, \ldots, n_k > 0} v^{(g-1)} \sum_{i<j} n_i n_j \Psi_{n, d}(v^2) P_{n_1}(v) \ldots P_{n_k}(v),
\]

where

\[
P_n(v) = (v^2 n - 1) \prod_{i=1}^n \frac{(1 - v^{2i-1})^2}{(1 - v^{2i})^2}.
\]

### 6. Number of stable bundles

In the last section we have seen how to compute the Poincaré function of the c-sequence \( a_\alpha \). In this section we will prove a formula relating the Poincaré functions of \( a_\alpha \) and \( r_\alpha \). Without loss of generality, we may assume that \( \alpha \) has nonnegative second coordinate or, equivalently, nonnegative slope \( \mu = \mu(\alpha) \).

Recall, that with any curve \( X \) over a finite field \( k = \mathbb{F}_q \), we have associated the numbers \( a_\alpha(K) \) and \( s_{\alpha,r}(K) \), where \( K/k \) is a finite field extension and \( r \geq 1 \). The totality of elements \( a_\alpha(K) \) (respectively, \( s_{\alpha,r}(K) \)) for finite field extensions \( K/k \) defines the element \( a_\alpha \) (respectively, \( s_{\alpha,r} \)) in \( S \).

**Lemma 6.1.** We have

\[
\psi_n(a_\alpha) = \sum_{k | n} k s_{k\alpha,k}
\]

in the \( \lambda \)-ring \( S \).

**Proof.** Let \( k = \mathbb{F}_q \). The \( r \)-th component of \( \psi_n(a_\alpha) \) equals \( a_\alpha(\mathbb{F}_{q^r}) \) and by Lemma 2.11 we have

\[
a_\alpha(\mathbb{F}_{q^r}) = \sum_{k | n} k s_{k\alpha,k}(\mathbb{F}_{q^r}).
\]

\( \square \)

We can apply Lemma 2.1 to the result of the above lemma in the \( \lambda \)-ring \( S[x] \).

**Corollary 6.2.** For any element \( f \in 1 + xS[x] \), we have

\[
\text{Pow}(f, a_\alpha) = \prod_{k \geq 1} \psi_k(f)^{s_{k\alpha,k}}.
\]
Remark 6.3. For any \( \alpha \in \mathbb{N}^* \times \mathbb{Z} \), one can construct the moduli space \( \mathcal{M}(\alpha) \) of stable bundles over \( X_k \) having character \( \alpha \), see [18]. This moduli space is defined over some finite field extension of \( k \) (let us assume that over \( k \) itself). The number of \( \mathbb{F}_q \)-rational points of \( \mathcal{M}(\alpha) \) equals \( a_\alpha(\mathbb{F}_q^r) \). This implies that the virtual Poincaré polynomial of \( \mathcal{M}(\alpha) \) equals the Poincaré polynomial of the c-sequence \( a_\alpha \).

If \( X \) is a curve over complex numbers then we can find its form over some ring finitely generated over \( \mathbb{Z} \). This form can be reduced to finite fields \( \mathbb{F}_p \) so that, for almost all \( p \), the reduction is again a smooth projective curve. The virtual Poincaré polynomials of the moduli spaces of stable vector bundles over these curves coincide with the virtual Poincaré of the moduli space of stable bundles on a complex curve. This means that the last invariants are given by the Poincaré functions of the c-sequences \( a_\alpha \). In this section we prove a formula for these c-sequences and their Poincaré polynomials.

For any \( \alpha \in \mathbb{N}^* \times \mathbb{Z} \), there are only finitely many isomorphism classes of semistable bundles on \( X \) having character \( \alpha \). Define the elements

\[
R_\alpha(K) = \sum_{[M] \in \mathcal{E}_{X_K}^s(\alpha)} [M]
\]

and

\[
R_\mu(K) = \sum_{M \in \mathcal{E}_{X_K}^s(\mu)} [M] = [0] + \sum_{\mu(\alpha) = \mu} R_\alpha(K)
\]

in \( \widehat{\mathcal{H}}_+(X_K) \). The elements \( R_\mu(K) \) are invertible in \( \widehat{\mathcal{H}}_+(X_K) \).

Lemma 6.4. (see [25, Lemma 3.4]) We have

\[
R_\mu(K)^{-1} = \sum_{M \in \mathcal{E}_{X_K}^s(\mu)} \gamma_M [M],
\]

where

\[
\gamma_M = \begin{cases} 
0 & \text{if } M \text{ is not polystable,} \\
\prod_{S \in \mathcal{E}_{X_K}^s(\mu)} (-1)^{m_S} \left( \# \text{ End } S \right)^{m_S} & \text{if } M = \bigoplus_{S \in \mathcal{E}_{X_K}^s(\mu)} S^{m_S}.
\end{cases}
\]

Define the generating functions

\[
r_\mu(K) = 1 + \sum_{\mu(\alpha) = \mu} r_\alpha(K)x^\alpha, \quad a_\mu(K) = \sum_{\mu(\alpha) = \mu} a_\alpha(K)x^\alpha
\]

in \( \mathbb{Q}[x_1, x_2] \). It is clear that \( r_\mu(K) = \int R_\mu(K) \). The totality of the elements \( r_\mu(K) \) and \( a_\mu(K) \) for finite field extensions \( K/\mathbb{F}_q \) defines the elements \( r_\mu, a_\mu \in S[x_1, x_2] \).

We can write also

\[
r_\mu = 1 + \sum_{\mu(\alpha) = \mu} r_\alpha x^\alpha, \quad a_\mu = \sum_{\mu(\alpha) = \mu} a_\alpha x^\alpha.
\]

Recall, that the element \( L \in S \) is defined by \( L = (q^k)_{k \geq 1} \). This is a c-sequence with a Poincaré polynomial \( P(L, v) = v^2 \). Define a new multiplication on \( S[x_1, x_2] \) by the formula

\[
x^\alpha \circ x^\beta = L^{-(\alpha, \beta)} x^{\alpha + \beta}.
\]

The new algebra is denoted by \( S_{\text{tw}}^L[x_1, x_2] \).
Theorem 6.5. We have
\[ r_\mu \circ \Exp \left( \frac{a_\mu}{1 - L} \right) = 1 \]
in \( S_{L_1}^w[[x_1, x_2]] \).

Proof. Consider some finite field extension \( K \) of \( \mathbb{F}_q \). The corresponding component of \( r_\mu \) is given by \( r_\mu(K) = \int R_\mu(K) \). Its inverse in \( Q_K^w[[x_1, x_2]] \) is given, in view of Lemma 6.4 and Lemma 2.12, by the formula

\[
\int R_\mu(K)^{-1} = \sum_{(m_S) \in E^*} \prod_{S \in E} \left( \frac{(-1)^{m_S} (\# \text{End } S)^{(m_S)}}{\# \text{GL}_{m_S}(\text{End } S)} \right)^{\chi_{m_S} \text{ch } S}
\]

where by \( E^* \) we denote the set \( E^*_K(\mu) \). For any stable bundle \( S \), the ring of endomorphisms \( \text{End } S \) is a finite field, say \( F_s \). We have

\[
\frac{(-1)^{m_S} (\# \text{GL}_m(F_s))}{\# \text{GL}_m(F_s)} = \prod_{i=1}^{m} (1 - s_i)^{-1} = [\infty, m]_s.
\]

This implies

\[
\int R_\mu(K)^{-1} = \prod_{S \in E^*} \left( \sum_{m \geq 0} [\infty, m]_v x^m \text{ch } S \bigg|_{v = \# \text{End } S} \right)^{\chi_{m_S} \text{ch } S}
\]

It follows that we have in \( S_{L_1}^w[[x_1, x_2]] \)

\[
r_\mu^{-1} = \prod_{m \geq 0} \psi_r([\infty, m]_v x^m |_{v = L})^{s_{a,r}} = \prod_{m \geq 0} \psi_r([\infty, m]_v x^{ma/r} |_{v = L})^{s_{a,r}}
\]

where we have used the formula

\[
\Exp \left( \frac{x}{1 - L} \right) = \sum_{m \geq 0} [\infty, m]_v x^m |_{v = L}
\]
in \( S[[x]] \). It is obtained from Lemma 2.2 as follows. Let \( R \) be a localization of \( \mathbb{Q}[v] \) with respect to \( v \) and \((1 - v^k)^{-1}, k \geq 1\). It is a \( \lambda \)-subring of \( \mathbb{Q}(v) \) and all the components of the Heine formula are contained in \( R \). Now we apply the \( \lambda \)-ring homomorphism \( R \to S, v \mapsto L \).

\[ \square \]
Let us endow the algebra $\mathbb{Q}(v)[x_1, x_2]$ with a new multiplication

$$x^\alpha \circ x^\beta = v^{-2(\alpha, \beta)} x^{\alpha + \beta}.$$ 

The new algebra is denoted by $\mathbb{Q}(v)^{tw}[x_1, x_2]$.

**Corollary 6.6.** We have

$$\left(1 + \sum_{\mu(\alpha) = \mu} P(r_\alpha, v)x^\alpha\right) \circ \text{Exp} \left(\sum_{\mu(\alpha) = \mu} P(a_\alpha, v)x^\alpha \over 1 - v^2\right) = 1$$

in $\mathbb{Q}(v)^{tw}[x_1, x_2]$.

This formula allows us to compute the Poincaré polynomial of $a_\alpha$ using the formula for the Poincaré function of $r_\alpha$ (see Corollary 5.5). By Remark 6.3 this gives us the virtual Poincaré polynomials

$$P(M(\alpha), v) = P(a_\alpha, v)$$

of the moduli spaces of stable sheaves on a complex curve.

**Remark 6.7.** For any $\alpha \in \Gamma$, let $\overline{M}(\alpha)$ denote the moduli space of semistable sheaves on $X$ having character $\alpha$. Then we have

$$1 + \sum_{\mu(\alpha) = \mu} P(\overline{M}(\alpha), v)x^\alpha = \text{Exp} \left(\sum_{\mu(\alpha) = \mu} P(M(\alpha), v)x^\alpha\right).$$

**Remark 6.8.** Note that the series $a_\mu$ and $r_\mu$ and their Poincaré functions can be considered as series in one variable. Indeed, there exists a unique $\gamma \in \mathbb{N}^* \times \mathbb{Z}$ having coprime components and satisfying $\mu(\gamma) = \mu$. We can write

$$r_\mu = \sum_{k \geq 0} r_{k\gamma}x^{k\gamma}, \quad a_\mu = \sum_{k \geq 1} a_{k\gamma}x^{k\gamma}$$

and analogously for their Poincaré functions.

**Remark 6.9.** In order to determine $P(a_\mu, v)$, we have to invert $P(r_\mu, v)$ in the ring $\mathbb{Q}(v)^{tw}[x_1, x_2]$. This can be reduced to the inversion in the usual ring of power series in the following way. Consider the map $\mathbb{Q}(v)[x] \rightarrow \mathbb{Q}(v)^{tw}[x_1, x_2]$

$$x^k \mapsto v^{-\langle \gamma, \gamma \rangle} k^2 x^{k\gamma},$$

which is obviously a homomorphism of algebras. The element $P(r_\mu, v)$ is contained in the image of this homomorphism and we just have to invert the preimage of $P(r_\mu, v)$ in $\mathbb{Q}(v)[x]$.

7. **On the virtual Hodge polynomial and motive of moduli space**

As we mentioned in introduction, the recursive formula of Harder, Narasimhan, Desale and Ramanan (see Theorem 5.1) for the Poincaré polynomials of the moduli spaces $M(n, d)$ (with coprime $n$ and $d$) of stable bundles on a curve was extended by Earl and Kirwan [14] to the recursive formula for the Hodge numbers of $M(n, d)$ for coprime $n$ and $d$. Let us recall their result. We endow the semigroup $\Gamma = \mathbb{N}^* \times \mathbb{Z}$ with a total preorder as in Section 3.
Theorem 7.1. Let $X$ be a curve of genus $g$ over $\mathbb{C}$ and, for any $\alpha \in \Gamma$, let $\mathcal{M}(\alpha)$ be the moduli space of stable bundles on $X$ having character $\alpha$. Define the rational functions $R_\alpha \in \mathbb{Q}(u,v)$, $\alpha = (n,d) \in \Gamma$, inductively by the formula

$$
\sum_{\lambda \in \mathcal{P}_\alpha} (uv)^{-\lambda} R_{\lambda_1} \ldots R_{\lambda_{|\lambda|}} = (u^n v^n - 1) \prod_{i=1}^n \frac{(1 - u^i v^{i-1})^g (1 - v^i u^{i-1})^g}{(1 - u^i v^i)^2}.
$$

Then the Hodge polynomial of $\mathcal{M}(n,d)$, for coprime $n$ and $d$, equals

$$
(uv - 1) R_{(n,d)}(u,v).
$$

Remark 7.2. The formula in [14] is actually slightly different. Namely, they define the rational functions $F_\alpha(u,v)$, $\alpha \in \Gamma$, inductively by the formula

$$
\sum_{\lambda \in \mathcal{P}_\alpha} (uv)^{-\lambda} F_{\lambda_1} \ldots F_{\lambda_{|\lambda|}} = (1 - u^n v^n) \prod_{i=1}^n \frac{(1 - u^i v^{i-1})^g (1 - v^i u^{i-1})^g}{(1 - u^i v^i)^2}
$$

and prove that the Hodge polynomial of $\mathcal{M}(n,d)$, for coprime $n$ and $d$, equals

$$(1 - uv) F_{(n,d)}(u,v).$$

Using the Poincaré duality, these two formulas can be shown to be equivalent.

Using the Zagier formula (see Theorem 3.3), we get an explicit formula

Corollary 7.3. Let $X$ be a curve of genus $g$ over $\mathbb{C}$ and, for any $\alpha \in \Gamma$, let $\mathcal{M}(\alpha)$ be the moduli space of stable bundles on $X$ having character $\alpha$. Define the rational functions $R_\alpha \in \mathbb{Q}(u,v)$, $\alpha = (n,d) \in \Gamma$, by the formula

$$
R_\alpha(u,v) = \sum_{n_1 \geq \ldots \geq n_k > 0} (uv)^{(g-1)\sum_{i<j} n_i n_j} \Psi_{n,*}(uv) P_{n_1} \ldots P_{n_k},
$$

where

$$
P_n(u,v) = (u^n v^n - 1) \prod_{i=1}^n \frac{(1 - u^i v^{i-1})^g (1 - v^i u^{i-1})^g}{(1 - u^i v^i)^2},
$$

$$
\Psi_{n,*}(t) = \prod_{i=1}^{k-1} \frac{t^{(n_{i} + n_{i+1})((n_{i} + \ldots + n_{k})/d)}}{1 - t^{n_{i} + n_{i+1}}},
$$

Then the Hodge polynomial of $\mathcal{M}(n,d)$, for coprime $n$ and $d$, equals

$$(uv - 1) R_{(n,d)}(u,v).$$

We can formulate now the conjectural formula for the virtual Hodge polynomials of the moduli spaces $\mathcal{M}(n,d)$. Denote by $\mathbb{Q}(u,v)[x_1, x_2]$ the ring obtained from $\mathbb{Q}(u,v)[x_1, x_2]$ by changing the multiplication

$$
x^\alpha \circ x^\beta = (uv)^{-\langle \alpha, \beta \rangle} x^{\alpha + \beta}. 
$$

Conjecture 7.4. With the notations as above, let $A_\alpha \in \mathbb{Q}[u,v]$ be the virtual Hodge polynomial of the moduli space $\mathcal{M}(\alpha)$, $\alpha \in \Gamma$. For any $\mu \in \mathbb{Q}$, define the generating functions $A_\mu = \sum_{\mu(\alpha) = \mu} A_\alpha x^\alpha$, $R_\mu = 1 + \sum_{\mu(\alpha) = \mu} R_\alpha x^\alpha$ in $\mathbb{Q}(u,v)[x_1, x_2]$. Then we have

$$
R_\mu \circ \text{Exp} \left( \frac{A_\mu}{1 - uv} \right)
$$

in $\mathbb{Q}(u,v)[w][x_1, x_2]$. 


Let $S$ be the Grothendieck ring of motives over $\mathbb{C}$, completed and localized in some appropriate way. It is a $\lambda$-ring, with $\sigma$-operations given by symmetric products [16]. For any $\alpha = (n, d) \in \Gamma$, we can define the motive $M_\alpha$ (respectively, $R_\alpha$) of the stack of vector bundles (respectively, stack of semistable vector bundles) on $X$ having Chern character $\alpha$. We define $A_\alpha$ to be the motive of $M(\alpha)$. It is proved in [4] that

$$M_\alpha = \left( Z(C, t)(1 - t)(1 - \mathbb{L} t) \right) \bigg|_{t=1} \mathbb{L} \left( n^2 - 1 \right)^{g-1} \prod_{i=2}^{n} Z(C, \mathbb{L}^{-i}),$$

where $Z(C, t)$ is a motivic zeta-function of $C$, given by $\text{Exp}(\lfloor C \rfloor t)$. The following result is due to del Baño [9], although he uses a different terminology

**Theorem 7.5.** For any $\alpha = (n, d) \in \Gamma$, we have

$$M_\alpha = M_n = \sum_{\lambda \in P_n} \mathbb{L}^{-\sum_{i<j} (\lambda_i, \lambda_j)} R_{\lambda_1} \cdots R_{\lambda_l}(\lambda)$$

and

$$R_\alpha = \sum_{\sum_{i=1}^{n} n_i = n} \mathbb{L}^{(g-1)\sum_{i<j} n_i n_j} \Psi_{n, d}(\mathbb{L}) M_{n_1} \cdots M_{n_k}.$$ 

**Conjecture 7.6.** We have

$$R_\mu \circ \text{Exp} \left( \frac{A_\mu}{1 - \mathbb{L}} \right)$$

in $S^w_L[x_1, x_2]$, where $R_\mu = 1 + \sum_{\mu(\alpha)=\mu} R_\alpha x^\alpha$ and $A_\mu = \sum_{\mu(\alpha)=\mu} A_\alpha x^\alpha$.

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20

Sergey Mozgovoy

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