DETECTING FOURIER SUBSPACES

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Abstract. Let $G$ be a finite abelian group. We examine the discrepancy between subspaces of $l^2(G)$ which are diagonalized in the standard basis and subspaces which are diagonalized in the dual Fourier basis. The general principle is that a Fourier subspace whose dimension is small compared to $|G| = \dim(l^2(G))$ tends to be far away from standard subspaces. In particular, the recent positive solution of the Kadison-Singer problem shows that from within any Fourier subspace whose dimension is small compared to $|G|$ there is standard subspace which is essentially indistinguishable from its orthogonal complement.

The purpose of this note is to describe a simple application of the recent solution of the Kadison-Singer problem \[6, 7\] to a question in harmonic analysis and signal analysis.

Let $G$ be a finite abelian group equipped with counting measure, and for each $g \in G$ let $e_g \in l^2(G)$ be the function which takes the value 1 at $g$ and is zero elsewhere. Then $\{e_g : g \in G\}$ is an orthonormal basis of $l^2(G)$; call it the standard basis.

Another nice basis of $l^2(G)$ comes from the dual group $\hat{G}$, the set of characters of $G$, i.e., homomorphisms from $G$ into the circle group $\mathbb{T}$. Every character has $l^2$ norm equal to $|G|^{-1/2}$, where $|G|$ is the cardinality of $G$, and the normalized set $\{\hat{e}_\phi = |G|^{-1/2} \phi : \phi \in \hat{G}\}$ is also an orthonormal basis of $l^2(G)$. We call this the Fourier basis. (Note that when every element of $G$ has order 2, then the Fourier basis forms the rows of a Hadamard matrix \[11\]. The method of this paper applies to bases of this type as well even if they don’t arise from a Fourier transform.)

Say that a subspace of $l^2(G)$ is standard if it is the span of some subset of the standard basis, and Fourier if it is the span of some subset of the Fourier basis. Now each $\hat{e}_\phi$ is as far away from the standard basis as possible in the sense that $|\langle \hat{e}_\phi, e_g \rangle| = |G|^{-1/2}$ for all $g \in G$ and $\phi \in \hat{G}$. However, Fourier subspaces can certainly intersect standard subspaces — trivially, $l^2(G)$ is itself both a standard subspace and a Fourier subspace.

A more interesting question is whether Fourier subspaces whose dimensions are “small” compared to $|G|$ can intersect standard subspaces which are small in the same sense. This could be of interest in relation to signal analysis, say, if we are trying to detect a signal by measuring a relatively small number of frequencies.

(By interchanging the roles of a group and its dual group, we see that the problem of detecting standard subspaces using Fourier subspaces is equivalent to the problem of detecting Fourier subspaces using standard subspaces. But in keeping with the signal analysis perspective, we will stick with the first formulation.)

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The basic obstacle to having small standard and Fourier subspaces which intersect is the uncertainty principle for finite abelian groups \([5, 9, 10]\). According to this principle, if a nonzero function \(f \in l^2(G)\) is supported on a set \(S \subseteq G\) and its Fourier transform is supported on \(T \subseteq \hat{G} - \hat{G}\), then \(|S| \cdot |T| \geq |G|\), meaning that \(\langle f, \hat{e}_\phi \rangle \neq 0\) only for \(\phi \in T\). In the special case where \(G\) has prime order \(p\), we have the much stronger inequality \(|S| + |T| \geq p + 1\), and this inequality is absolutely sharp \([9]\). Intuitively, if \(f\) is very localized with respect to the standard basis then it must be “spread out” with respect to the Fourier basis. In terms of subspaces, the result can be stated as follows:

**Proposition 0.1.** Let \(G\) be a finite abelian group and let \(E\) and \(F\) respectively be standard and Fourier subspaces of \(l^2(G)\). If \(\dim(E) \cdot \dim(F) < |G|\) then \(E \cap F = \{0\}\). If \(|G|\) is prime and \(\dim(E) + \dim(F) \leq |G|\) then \(E \cap F = \{0\}\).

This follows immediately from the uncertainty principles described above because the dimensions of \(E\) and \(F\) equal the number of basis elements which span them, so that the support of any element of \(E\) (respectively, \(F\)) has cardinality at most \(\dim(E)\) (respectively, \(\dim(F)\)). (Uncertainty principles are related to signal reconstruction in a different way in \([4]\) and related papers.)

So standard and Fourier subspaces must intersect only in \(\{0\}\) if both are sufficiently small. However, according to the multiplicative bound in the last proposition, both dimensions could be as small as \(|G|^{1/2}\), which is small compared to \(|G|\) when \(|G|\) is large. This means that the multiplicative bound does not prevent standard and Fourier subspaces whose dimensions are small compared to \(|G|\) from intersecting, although the additive bound when \(|G|\) is prime certainly does. Indeed, there are easy examples of intersecting standard and Fourier subspaces whose dimensions are both equal to \(|G|^{1/2}\).

**Example 0.2.** Let \(G = \mathbb{Z}/n^2\mathbb{Z}\), where \(|G| = n^2\) and the characters have the form \(\phi_b : a \mapsto e^{2\pi iab/n^2}\) for \(a, b \in \mathbb{Z}/n^2\mathbb{Z}\). Here the function \(f = \sum_{b=0}^{n-1} \phi_{nb} \in l^2(G)\) satisfies

\[
    f(a) = \sum_{b=0}^{n-1} e^{2\pi iab/n} = \begin{cases} 
        n & \text{if } a \equiv 0 \pmod{n} \\
        0 & \text{if } a \not\equiv 0 \pmod{n},
    \end{cases}
\]

so that \(f\) belongs both to an \(n\)-dimensional Fourier subspace (directly from its definition) and to an \(n\)-dimensional standard subspace (by the preceding calculation).

From the point of view of signal analysis, however, we are probably not so interested in intersecting a single standard subspace. If we do not know where the signal we want to detect is supported, we would presumably want to intersect, if not every standard subspace, at least every standard subspace whose dimension is greater than some threshold value. But unless the relevant dimensions are large, this is impossible for elementary linear algebra reasons. We have the following simple result:

**Proposition 0.3.** Let \(G\) be a finite abelian group and let \(F\) be a Fourier subspace of \(l^2(G)\). Then there exists a standard subspace \(E\) with \(\dim(E) = |G| - \dim(F)\) such that \(E \cap F = \{0\}\).

The proof is easy. Starting with the standard basis \(B = \{e_g : g \in G\}\) of \(l^2(G)\) and the basis \(B' = \{\hat{e}_\phi : \phi \in T\}\) of \(F\), we can successively replace distinct elements of \(B\) with elements of \(B'\) in such a way that the set remains linearly
Nonetheless, we can still get something from simply counting dimensions. Namely, the standard subspace makes the problem much more interesting and difficult. Merely asking that the Fourier subspace contain unit vectors which are close to does not intersect. But requiring subspaces to intersect is a very strong condition. Thus, even a single Fourier basis vector can “detect” arbitrary standard subspaces to the extent that those subspaces have dimension comparable to $|G|$. Therefore, we have the following theorem:

**Theorem.**

For any Fourier basis vector and $E$ a standard subspace, the orthogonal projection onto $E$ is defined as $P = \text{proj}_E f$. Thus, we can write $P \hat{\varphi} = \sum_{\varphi \in \mathcal{S}} \langle \hat{\varphi}, \varphi \rangle \varphi$. The surprising result is that, by this measure, so long as the dimension of a Fourier subspace is small compared to $|G|$, there are standard subspaces which it detects only marginally better than a single Fourier basis vector. We have the following theorem:

Thus, even a single Fourier basis vector can “detect” arbitrary standard subspaces to the extent that those subspaces have dimension comparable to $|G|$. Can Fourier subspaces do better? The relevant gauge here is the quantity

$$\|PQ\| = \|QP\| = \sup\{\|Qv\| : v \in E, \|v\| = 1\}$$

where $P$ and $Q$ are the orthogonal projections onto a standard subspace $E$ and a Fourier subspace $F$, respectively. It effectively measures the minimal angle between $E$ and $F$. The surprisingly strong result is that, by this measure, so long as the dimension of a Fourier subspace is small compared to $|G|$, there are standard subspaces which it detects only marginally better than a single Fourier basis vector does.

One might suspect that a randomly chosen standard subspace would demonstrate that claim. Maybe this technique would work for most Fourier subspaces, but it does not in general, even for Fourier subspaces whose dimension is small compared to $|G|$. However, in the next example, we will show that this is possible.

**Example 0.4.** Recall Example 0.2 where an $n$-dimensional Fourier subspace $F$ intersected an $n$-dimensional standard subspace $E$. Here $\dim(F)/|G| = 1/n$, which can be as small as we like. Now consider the group $G' = \mathbb{Z}/n^2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ where $N$ is large compared to $n$. We have a natural identification $l^2(G') \cong l^2(\mathbb{Z}/n^2\mathbb{Z}) \otimes l^2(\mathbb{Z}/N\mathbb{Z})$, under which identification $F \otimes l^2(\mathbb{Z}/N\mathbb{Z})$ is a Fourier subspace. The ratio of its dimension to the cardinality of $G'$ is still $1/n$. But for sufficiently large $N$, a randomly chosen subset of $G'$ will contain at least one element of $S \times \mathbb{Z}/N\mathbb{Z}$ with high probability, where $E = \text{span}\{e_g : g \in S\}$. One can even say this of a randomly chosen subset of $G'$ of cardinality $|G'|/n$. This means that a randomly chosen standard subspace of dimension $|G'|/n$ will intersect the Fourier subspace $F \otimes l^2(\mathbb{Z}/N\mathbb{Z})$ with high probability — depending on the value of $N$, with probability as close to 1 as we like.

Nonetheless, if a Fourier subspace $F$ has relatively small dimension, there will always exist standard subspaces of arbitrary dimension which are barely closer to $F$ than they are to a single Fourier basis vector. We have the following theorem:
Theorem 0.5. Let $G$ be a finite abelian group, let $F$ be a Fourier subspace of $l^2(G)$, and let $Q$ be the orthogonal projection onto $F$. Then for any $k \leq |G|$ there is a set $S \subseteq G$ with $|S| = k$ and such that

$$\|PQ\|^2 \leq \frac{k}{|G|} + O(\sqrt{\epsilon}),$$

where $P$ is the orthogonal projection onto $\text{span}\{e_g : g \in S\}$ and $\epsilon = \dim(F)/|G|$. 

This result is strengthened further by Theorem 0.6 below, but Theorem 0.5 has an easier proof which is of independent interest. We include this proof in the appendix.

The strengthened version of Theorem 0.5 simultaneously asks the same question about the complementary standard subspace. If our goal is detection, then the worst that could happen here is that a Fourier subspace $F$ does essentially no better than a single Fourier basis vector at detecting either some standard subspace $E$ or its orthocomplement $E^\perp$. In fact, this worst-case scenario is realized: we can show that every Fourier subspace of small dimension relative to $|G|$ fails to do significantly better than a single Fourier basis vector at detecting both a sequence of standard subspaces of varying dimension and their orthocomplements.

In this case, constructing the undetectable standard subspaces is no longer merely a matter of controlling the largest eigenvalue of $QPQ$ (which suffices because $\|QPQ\| = \|PQ\|^2$). Now we also have to control the largest eigenvalue of $Q(1-P)Q$, and this is a Kadison-Singer type setup. Although the problem we consider here is not as general as the full Kadison-Singer problem, the core difficulty is clearly already present. Thus, one should not expect any easier proof than the ones that appear in [6], [7] or the remarkable generalization in [3].

Theorem 0.6. Let $G$ be a finite abelian group, let $F$ be a Fourier subspace of $l^2(G)$, and let $Q$ be the orthogonal projection onto $F$. Then for any $k \leq |G|$ there is a set $S \subseteq G$ with $|S| = k$ and such that both

$$\|PQ\|^2 \leq \frac{k}{|G|} + O(\sqrt{\epsilon})$$

and

$$\|(I-P)Q\|^2 \leq \frac{|G|-k}{|G|} + O(\sqrt{\epsilon})$$

where $P$ is the orthogonal projection onto $\text{span}\{e_g : g \in S\}$ and $\epsilon = \dim(F)/|G|$.

Proof. For each $g \in G$ let $u_g = Qe_g \in F$. Then the rank one operators $u_g u_g^* : f \mapsto \langle f, u_g \rangle u_g$ satisfy

$$\sum_{g \in G} u_g u_g^* f = \sum_{g \in G} \langle f, Qe_g \rangle \cdot Qe_g = Q \left( \sum_{g \in G} \langle Qf, e_g \rangle e_g \right) = Qf$$

for all $f \in l^2(G)$; that is, $\sum_{g \in G} u_g u_g^* = Q$. We also have $\|u_g\|^2 = \|Qe_g\|^2 = \epsilon$ for all $g$. So by (1, comment following Corollary 1.2), for any $k \leq |G|$ there is a set $S \subseteq G$ such that

$$\left\| \sum_{g \in S} u_g u_g^* - \frac{k}{|G|} Q \right\| = O(\sqrt{\epsilon}).$$
Letting $P$ be the orthogonal projection onto $\text{span}\{e_g : g \in S\}$, we have

$$P = \sum_{g \in S} e_g e_g^*,$$

and it follows that

$$\left\| QPQ - \frac{k}{|G|} Q \right\| = O(\sqrt{\epsilon}),$$

which also implies

$$\left\| Q(I - P)Q - \frac{|G| - k}{|G|} Q \right\| = O(\sqrt{\epsilon}).$$

We therefore have

$$\|PQ\|^2 = \|QPQ\| \leq k + O(\sqrt{\epsilon})$$

and

$$\|(I - P)Q\|^2 = \|Q(I - P)Q\| \leq \frac{|G| - k}{|G|} + O(\sqrt{\epsilon}),$$

as desired. The set $S$ might not have cardinality exactly $k$, but it cannot have cardinality greater than $k + O(\sqrt{\epsilon})$ or less than $k - O(\sqrt{\epsilon})$, so it can be adjusted to have cardinality $k$ without affecting the order of the estimate, if needed. \hfill \Box

In particular, taking $k \approx |G|/2$ in Theorem 0.6, we get

$$\|PQ\| \approx \|(1 - P)Q\| \approx \frac{1}{\sqrt{2}}.$$ 

This implies that every nonzero vector in $\text{ran}(P)$ is roughly $45^\circ$ away from $F$, and the same is true of every nonzero vector in $\text{ran}(P)^\perp$. That is, if $\epsilon$ is small then from within $F$ the two subspaces are essentially indistinguishable. Another way to say this is that every vector in $F$ has roughly half of its $l^2$ norm supported on $S$ and roughly half supported on $G \setminus S$.

**Appendix A.**

We prove Theorem 0.5. The argument is a straightforward application of the spectral sparsification technique introduced in Srivastava’s thesis [8].

Let $\{u_1, \ldots, u_n\}$ be a finite set of vectors in $\mathbb{C}^m$, each of norm $\sqrt{\epsilon}$, satisfying

$$\sum_{i=1}^n u_i u_i^* = I$$

where $u_i u_i^*$ is the rank one operator on $\mathbb{C}^m$ defined by $u_i u_i^* : v \mapsto \langle v, u_i \rangle u_i$ and $I$ is the identity operator on $\mathbb{C}^m$. Note that $\text{Tr}(u_i u_i^*) = \text{Tr}(u_i^* u_i) = \|u_i\|^2 = \epsilon$; since $\text{Tr}(I) = m$, it follows that $n \epsilon = m$.

Let $k < n$. As in [8], we will build a sequence $u_{i_1}, \ldots, u_{i_k}$ one element at a time. The construction is controlled by the behavior of the operators $A_j = \sum_{d=1}^j u_i u_i^*$ using the following tool. For any positive operator $A$ and any $a > \|A\|$, define the upper potential $\Phi^a(A)$ to be

$$\Phi^a(A) = \text{Tr}((aI - A)^{-1});$$

then, having chosen the vectors $u_{i_1}, \ldots, u_{i_{j-1}}$, the plan will be to select a new vector $u_{i_j}$ so as to minimize $\Phi^a_j(A_j)$, where the $a_j$ are an increasing sequence of upper bounds. This potential function disproportionately penalizes eigenvalues which are
close to \(a_j\) and thereby controls the maximum eigenvalue, i.e., the norm, of \(A_j\). The key fact about the upper potential is given in the following result.

**Lemma A.1.** ([8], Lemma 3.4) Let \(A\) be a positive operator on \(\mathbb{C}^m\), let \(a, \delta > 0\), and let \(v \in \mathbb{C}^m\). Suppose \(|A| < a\). If

\[
\frac{\langle ((a + \delta)I - A)^{-2}v, v \rangle}{\Phi^a(A) - \Phi^{a + \delta}(A)} + \frac{\langle ((a + \delta)I - A)^{-1}v, v \rangle}{1 - \delta} \leq 1
\]

then \(|A + vv^*| < a + \delta\) and \(\Phi^{a + \delta}(A + vv^*) \leq \Phi^a(A)\).

The proof relies on the Sherman-Morrison formula, which states that if \(A\) is positive and invertible then \((A + vv^*)^{-1} = A^{-1} - \frac{A^{-1}vv^*A^{-1}}{1 + (A^{-1}v)v}\).

We also require a simple inequality.

**Lemma A.2.** Let \(a_1 \leq \cdots \leq a_m\) and \(b_1 \geq \cdots \geq b_m\) be sequences of positive real numbers, respectively increasing and decreasing. Then \(\sum a_i b_i \leq \frac{1}{m} \sum a_i \sum b_i\).

**Proof.** Let \(M = \frac{1}{m} \sum b_i\). We want to show that \(\sum a_i b_i \leq \sum a_i M\), i.e., that \(\sum a_i (b_i - M) \leq 0\). Since the sequence \((b_i)\) is decreasing, we can find \(j\) such that \(b_i \geq M\) for \(i \leq j\) and \(b_i < M\) for \(i > j\). Then \(\sum_{i=1}^j a_i (b_i - M) \leq a_j \sum_{i=1}^j (b_i - M)\) (since the \(a_i\) are increasing and the values \(b_i - M\) are positive) and \(\sum_{i=j+1}^m a_i (b_i - M) \leq a_j \sum_{i=j+1}^m (b_i - M)\) (since the \(a_i\) are increasing and the values \(b_i - M\) are negative). So

\[
\sum_{i=1}^m a_i (b_i - M) \leq a_j \sum_{i=1}^m (b_i - M) = 0,
\]

as desired. \(\square\)

**Theorem A.3.** Let \(m \in \mathbb{N}\) and \(\epsilon > 0\), and suppose \(\{u_1, \ldots, u_n\}\) is a finite sequence of vectors in \(\mathbb{C}^m\) satisfying \(\|u_i\|^2 = \epsilon\) for \(1 \leq i \leq n\) and \(\sum_{i=1}^n u_i u_i^* = I\). Then for any \(k \leq n\) there is a set \(S \subseteq \{1, \ldots, n\}\) with \(|S| = k\) such that

\[
\left\| \sum_{i \in S} u_i u_i^* \right\| \leq \frac{k}{n} + O(\sqrt{\epsilon}).
\]

**Proof.** Define \(a_j = \sqrt{\epsilon} + \frac{1}{1 - \sqrt{\epsilon}} \cdot \frac{n}{k}\) for \(0 \leq j \leq k\). We will find a sequence of distinct indices \(i_1, \ldots, i_k\) such that the operators \(A_j = \sum_{d=1}^j u_{i_d} u_{i_d}^*, 0 \leq j \leq k\), satisfy \(\|A_j\| < a_j\) and \(\Phi^{a_k}(A_0) \geq \cdots \geq \Phi^{a_k}(A_k)\). Thus

\[
\|A_k\| < \sqrt{\epsilon} + \frac{1}{1 - \sqrt{\epsilon}} \cdot \frac{k}{n} = \frac{k}{n} + O(\sqrt{\epsilon}),
\]

yielding the desired conclusion. We start with \(A_0 = 0\), so that \(\Phi^{a_0}(A_0) = \Phi^{\sqrt{\epsilon}}(0) = \text{Tr}(\sqrt{I})^{-1} = m/\sqrt{\epsilon}\).

To carry out the induction step, suppose \(u_{i_1}, \ldots, u_{i_j}\) have been chosen. Let \(\lambda_1 \leq \cdots \leq \lambda_m\) be the eigenvalues of \(A_j\). Then the eigenvalues of \(I - A_j\) are \(1 - \lambda_j \geq \cdots \geq 1 - \lambda_m\) and the eigenvalues of \((a_{j+1} I - A_j)^{-1}\) are \(\frac{1}{a_{j+1} - \lambda_1} \leq \cdots \leq \frac{1}{a_{j+1} - \lambda_m}\). Thus by Lemma A.2

\[
\text{Tr}((a_{j+1} I - A_j)^{-1} (I - A_j)) = \sum_{i=1}^m \frac{1}{a_{j+1} - \lambda_i} (1 - \lambda_i)
\]
Thus

\[
\begin{align*}
\leq & & \frac{1}{m} \sum_{i=1}^{m} \frac{1}{a_{j+1} - \lambda_i} \sum_{i=1}^{m} (1 - \lambda_i) \\
= & & \frac{1}{m} \text{Tr}((a_{j+1}I - A_j)^{-1}) \text{Tr}(I - A_j) \\
= & & \frac{1}{m} \Phi^{a_{j+1}}(A_j) \text{Tr}(I - A_j) \\
\leq & & \frac{1}{m} \Phi^{a_j}(A_j) \text{Tr}(I - A_j) \\
\leq & & \frac{1}{m} \Phi^{a_0}(A_0) \text{Tr}(I - A_j) \\
= & & \frac{1}{\sqrt{\epsilon}} \text{Tr}(I - A_j).
\end{align*}
\]

Next, \(a_{j+1} - a_j = \frac{1}{1 - \sqrt{\epsilon}} \cdot \frac{1}{n}\), so we can estimate

\[
\Phi^{a_j}(A_j) - \Phi^{a_{j+1}}(A_j) = \text{Tr}((a_jI - A_j)^{-1} - (a_{j+1}I - A_j)^{-1}) \\
= \frac{1}{1 - \sqrt{\epsilon}} - \frac{1}{n} \text{Tr}((a_jI - A_j)^{-1} - (a_{j+1}I - A_j)^{-1}) \\
> \frac{1}{1 - \sqrt{\epsilon}} - \frac{1}{n} \text{Tr}((a_jI - A_j)^{-2})
\]

since each of the eigenvalues \(\frac{1}{a_j - \lambda_i}, \frac{1}{a_{j+1} - \lambda_i}\) of the operator \((a_jI - A_j)^{-1}(a_{j+1}I - A_j)^{-1}\) is greater than the corresponding eigenvalue \(\frac{1}{a_{j+1} - \lambda_i}\) of the operator \((a_{j+1}I - A_j)^{-2}\). Combining this with Lemma A.2 yields

\[
\text{Tr}((a_{j+1}I - A_j)^{-2}(I - A_j)) \leq \frac{1}{m} \text{Tr}((a_{j+1}I - A_j)^{-2}) \text{Tr}(I - A_j) \\
< \frac{1}{\epsilon} (1 - \sqrt{\epsilon}) (\Phi^{a_j}(A_j) - \Phi^{a_{j+1}}(A_j)) \text{Tr}(I - A_j).
\]

Thus

\[
\frac{\text{Tr}((a_{j+1}I - A_j)^{-2}(I - A_j))}{\Phi^{a_j}(A_j) - \Phi^{a_{j+1}}(A_j)} \leq \frac{1}{\epsilon} (1 - \sqrt{\epsilon}) \text{Tr}(I - A_j).
\]

Now let \(S' \subseteq \{1, \ldots, n\}\) be the set of indices which have not yet been used. Observe that \(\langle Au, u \rangle = \text{Tr}(A uu^*)\) and that \(\sum_{i \in S'} u_i u_i^* = I - \sum_{d=1}^j u_{i_d} u_{i_d}^* = I - A_j\). Thus

\[
\sum_{i \in S'} \left( \frac{\langle (a_{j+1}I - A_j)^{-2} u_i, u_i \rangle}{\Phi^{a_j}(A_j) - \Phi^{a_{j+1}}(A_j)} + \langle (a_{j+1}I - A_j)^{-1} u_i, u_i \rangle \right) \\
= \frac{\text{Tr}((a_{j+1}I - A_j)^{-2}(I - A_j))}{\Phi^{a_j}(A_j) - \Phi^{a_{j+1}}(A_j)} + \text{Tr}((a_{j+1}I - A_j)^{-1}(I - A_j)) \\
\leq \frac{1}{\epsilon} (1 - \sqrt{\epsilon}) \text{Tr}(I - A_j) + \frac{1}{\sqrt{\epsilon}} \text{Tr}(I - A_j) \\
= \frac{1}{\epsilon} \text{Tr}(I - A_j).
\]

But

\[
\frac{1}{\epsilon} \text{Tr}(I - A_j) = \frac{1}{\epsilon} (m - \text{Tr}(A_j)) = n - j
\]

is exactly the number of elements of \(S'\). So there must exist some \(i \in S'\) for which

\[
\frac{\langle (a_{j+1}I - A_j)^{-2} u_i, u_i \rangle}{\Phi^{a_j}(A_j) - \Phi^{a_{j+1}}(A_j)} + \langle (a_{j+1}I - A_j)^{-1} u_i, u_i \rangle \leq 1.
\]
Therefore, by Lemma A.1, choosing \( u_{i+1} = u_i \) allows the inductive construction to proceed.

Theorem 0.5 follows by taking \( m = \dim(F) \), \( \epsilon = m/|G| \), and \( u_i = Qe_i \) for \( 1 \leq i \leq n = |G| \), and identifying \( F \) with \( \mathbb{C}^m \). Letting \( P \) be the orthogonal projection onto \( \text{span}\{e_i : i \in S\} \), we then have \( \|PQ\|^2 = \|QPQ\| = \|\sum_{i \in S} u_i u_i^*\| \).

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