Cohomology of Commuting Varieties of Connected Compact Reductive Lie Groups

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1 Abstract

We calculate the rational cohomology of the commuting variety $X_{G,n}$ consisting of $n$-tuples of commuting elements of a compact reductive group $G$. This is done by studying a map from a related variety $Y_{G,n}$, which has easily calculated cohomology. The proof studies the fibers of the map and uses the Vietoris-Begle theorem to prove that the induced map on rational cohomology is an isomorphism.

2 Introduction

Let $G$ be a connected compact reductive group. Then the $n$th commuting variety $X_{G,n}$ is the variety consisting of all $n$-tuples $(g_1, g_2, ..., g_n) \in G \times G$ that pairwise commute (i.e. $g_i g_j = g_j g_i$ for all $i, j$). Let $T$ be a maximal torus, and let $N_G(T)$ denote the normalizer of $T$ in $G$. Then let $Y_{G,n} := (G \times T^n)/N_G(T)$, where $N_G(T)$ acts by right-multiplication on $G$ and by conjugation on $T$. Let $f : Y_{G,n} \to X_{G,n}, f(g, t'_1, t'_2, ..., t'_n) = (gt'_1 g^{-1}, gt'_2 g^{-1}, ..., gt'_n g^{-1})$; note that $f$ is a $G$-equivariant map where $G$ acts on $Y_{G,n}$ by acting on the factor of $G$ by left-multiplication, and on $X_{G,n}$ by simultaneous conjugation.

**Theorem 1** (Main Theorem). The map $f$ induces an isomorphism on rational cohomology, i.e. $H \left( X_{G,n}, \mathbb{Q} \right) \overset{f^*}{\longrightarrow} H \left( Y_{G,n}, \mathbb{Q} \right)$

This theorem is a generalization of two already-known theorems.

**Theorem 2.** Let $G$ be a connected compact reductive group, $T$ a maximal torus, and $N_G(T)$ the normalizer of $T$ in $G$. Then $G/N_G(T)$ has trivial rational cohomology.

**Theorem 3.** Let $G$ be a connected compact reductive group, $T$ a maximal torus, and $N_G(T)$ the normalizer of $T$. Then $f : (G \times T)/N_G(T) \to G$ induces an isomorphism on rational cohomology.
Proofs of both of these can be found in [1] (in the proof of Proposition 1). These can be seen as the $n = 0$ and $n = 1$ case of the main theorem, respectively.

This theorem allows relatively simple computation of the cohomology of the $n$th commuting variety. $X_{G,n}$ can be rewritten as $(G/T \times T^n)/W$, where $W$ is the Weyl group of $G$. The action of $W$ on $G/T$ is free, so the action of $W$ on $G/T \times T^n$ is free. Therefore, the cohomology of $X_{G,n}$ can be given by the $W$-invariants in the cohomology of $G/T \times T^n$. As the cohomology of $G/T$ is known (if the grading is ignored, it is the regular representation of $W$), and the cohomology of $T$ is isomorphic to the exterior algebra on the reflection representation of $W$), the cohomology is easy to calculate.

3 Proof of Main Theorem

We rely on a theorem from algebraic topology to reduce the question to studying the fibers of $f$.

Theorem 4 (Vietoris-Begle Theorem). Let $f : Y \rightarrow X$ be a surjective map of compact metric spaces such that for all $x \in X$, $f^{-1}(x)$ is cohomologically trivial (with respect to some cohomology theory). Then $f$ induces an isomorphism on cohomology (for the same cohomology theory).

As all elements of a compact group are diagonalizable, any commuting $n$-tuple is contained in some maximal torus. All maximal tori are conjugate, so $f$ is surjective. By Theorem 4 using rational cohomology, we only need to prove the following lemma:

Lemma 5. For any commuting $n$-tuple $(g_1, g_2, ..., g_n)$, the set $\{(g, t_1', t_2', ..., t_n') \in G \times T^n | \forall i gt_i' g_i^{-1} = g_i\}/N_G(T)$ has trivial rational cohomology.

The rest of the paper will prove this lemma by rewriting this set until it is in a form known to have trivial rational cohomology.

We can assume without loss of generality that the commuting $n$-tuple $(g_1, g_2, ..., g_n)$ is contained in our chosen maximal torus $T$. Change notation so that our commuting $n$-tuple is $(t_1, t_2, ..., t_n)$. Let $X = \{(g, t_1', t_2', ..., t_n') | \forall i gt_i' g_i^{-1} = t_i\}$; then $f^{-1}(t_1, t_2, ..., t_n) = X/N_G(T)$.

Lemma 6. Let $G$ be a (not necessarily connected) reductive group. The $G$-orbit of an $n$-tuple of elements $(t_1, t_2, ..., t_n) \in T^n$ meets $T^n$ in exactly the $N_G(T)$-orbit of $(t_1, t_2, ..., t_n)$. In other words, if $g(t_1, t_2, ..., t_n)g^{-1} = (t_1', t_2', ..., t_n') \in T^n$, then there is some $g' \in N_G(T)$ with $g'(t_1, t_2, ..., t_n)g'^{-1} = (t_1', t_2', ..., t_n')$.

Proof. We first reduce to the case that $G$ is connected. Let $g \in G$ such that $g(t_1, t_2, ..., t_n)g^{-1} = (t_1', t_2', ..., t_n') \in T^n$. Let $T' = gTg^{-1}$. As all maximal tori are conjugate by an element of the connected component of the identity $G_0$, there is some $g_0 \in G_0$ such that $g_0Tg_0^{-1} = T'$. Then let $g_1 = g_0g^{-1}$; an easy calculation shows that $g_1 \in N_G(T)$. As such, $g_1t_0'g_1^{-1} \in T''$, so let $t_i'' = g_1t_i'g_1^{-1}$. We then have that $g_0(t_1, t_2, ..., t_n)g_0^{-1} = (t_1', t_2', ..., t_n')$. If
the theorem is true for connected $G$, then there is some $g_2 \in N_{G_0}(T)$ with $g_2(t_1, t_2, ..., t_n)g_2^{-1} = (t_1', t_2', ..., t_n')$. Let $g' = g_1^{-1}g_2$; then $g'((t_1, t_2, ..., t_n))g_1^{-1} = g_1^{-1}(t_1', t_2', ..., t_n')g_11 = (t_1', t_2', ..., t_n')$. We therefore only need to prove this in the case that $G$ is connected.

The $n = 1$ case is a consequence of Chevalley’s theorem. We prove this in the $n = 2$ case; the general case is similar, and works by induction. The general strategy is to reduce to the case that $t_i' = t_i$ for $i > 1$ by the inductive assumption, and then to reduce to the $n = 1$ case for a subgroup of $G$.

Assume $g(t_1, t_2)g^{-1} = (t_1', t_2') \in T^2$. Then $g_1t_2g_1^{-1} = t_1'$, so by the $n = 1$ case, there is some $g_0 \in N_G(T)$ with $g_0t_2g_0^{-1} = t_1'$.

Let $g_1 = g_0^{-1}g$; then $g_1t_2g_1^{-1} = g_0^{-1}gt_2g_0^{-1}g_0 = g_0^{-1}t_2g_0 = t_2$, so $g_1$ is in the centralizer $Z_G(t_2)$. The centralizer is a reductive group with maximal torus $T$. Let $t_1' = g_0^{-1}t_1g_0 = g_1t_1g_1^{-1}$. As the centralizer is a reductive group (although not necessarily connected), we can apply the $n = 1$ case again to get some element $g_2 \in N_{Z_G(t_2)}(T)$ with $g_2t_1g_2^{-1} = t_1'$. But through some rearrangement of the definition,

$$N_{Z_G(t_2)}(T) = \{ n \in Z_G(t_2) | nTn^{-1} = T \} = \{ n \in G | nt_2n^{-1} = t_2, nTn^{-1} = T \} = N_G(T) \cap Z_G(t_2)$$

Let $g' = g_0g_2$; an easy calculation shows that $g'(t_1, t_2)g'^{-1} = (t_1', t_2')$, and as $g_0, g_2$ are both in $N_G(T)$, the lemma is proven. □

Define $X' = \{ g | \forall i, gt_ig^{-1} = t_i \} \subset X$; then $N_G(T) \cap Z_G(t_1, t_2, ..., t_n)$ acts on $X'$. There is an obvious map $X'/(N_G(T) \cap Z_G(t_1, t_2, ..., t_n)) \to X/N_G(T)$. Lemma 6 allows us to construct an inverse map, as it implies that any element of $X/N_G(T)$ has some representative in $X'$, so the two are isomorphic.

Therefore, we can rewrite $f^{-1}(t_1, t_2, ..., t_n) = \{(g, t_1, t_2, ..., t_n) | \forall i, gt_ig^{-1} = t_i \}/(N_G(T) \cap Z_G(t_1, t_2, ..., t_n))$. As the $n$-tuple in the numerator is now constant, this is isomorphic to $Z_G(t_1, t_2, ..., t_n)/N_{Z_G(t_1, t_2, ..., t_n)}(T)$.

We now have that for each $x \in X$, the fiber is isomorphic to the quotient of a reductive group by the normalizer of its maximal torus. By the same trick as in the beginning of lemma 6, this is isomorphic to the quotient of a connected reductive group (the connected component of the identity of the original group) by the normalizer of its maximal torus. This is exactly the situation referred to in Theorem 2 so the fiber has trivial rational cohomology. This proves the theorem.

References

[1] Brion, M.: Equivariant cohomology and equivariant intersection theory. arXiv:math/9802063