LIMIT POINTS FOR BROWDER SPECTRUM OF OPERATOR MATRICES

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ABSTRACT. Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$, where $X$ and $Y$ are Banach spaces, and let $M_C$ be an operator acting on $X \oplus Y$ given by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. We investigate the limit point set of the Browder spectrum of $M_C$. It is shown that

$$\text{acc}_b(M_C) \cup W_{\text{acc}_b} = \text{acc}_b(A) \cup \text{acc}_b(B)$$

where $W_{\text{acc}_b}$ is a subset of $\text{acc}_b(B) \cap \text{acc}_b(A)$ and a union of certain holes in $\text{acc}_b(M_C)$. Furthermore, several sufficient conditions for $\text{acc}_b(M_C) = \text{acc}_b(A) \cup \text{acc}_b(B)$ hold for every $C \in \mathcal{B}(Y, X)$ are given.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $X$ and $Y$ denote infinite dimensional complex Banach spaces, and $\mathcal{B}(X, Y)$ denotes the complex algebra of all bounded linear operators from $X$ to $Y$. When $Y = X$ we simply write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$. For $T \in \mathcal{B}(X, Y)$ we use $R(T)$ and $N(T)$ to denote the range and the null space of $T$, respectively. Write $\alpha(T) = \dim N(T)$ and $\beta(T) = \text{codim} R(T)$. Sets of upper semi-Fredholm operators, lower semi-Fredholm operators, left semi-Fredholm operators and right semi-Fredholm operators are defined respectively as $\Phi_+(X) = \{ T \in \mathcal{B}(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed} \}$, $\Phi_-(X) = \{ T \in \mathcal{B}(X) : \beta(T) < \infty \}$, $\Phi_l(X) = \{ T \in \mathcal{B}(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed and complemented subspace of } X \}$ and $\Phi_r(X) = \{ T \in \mathcal{B}(X) : \beta(T) < \infty \text{ and } N(T) \text{ is a complemented subspace of } X \}$. Sets of semi-Fredholm and Fredholm operators are defined as $\Phi_{\pm}(X) = \Phi_+(X) \cup \Phi_-(X)$ and $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

The ascent of $T$ is defined as $\text{asc}(T) = \inf\{ n \in \mathbb{N} : N(T^n) = N(T^{n+1}) \}$ and the descent of $T$ is defined as $\text{des}(T) = \inf\{ n \in \mathbb{N} : R(T^n) = R(T^{n+1}) \}$. Where $\inf \emptyset = \infty$. Sets of upper semi-Browder operators, lower semi-Browder operators, left semi-Browder operators and right semi-Browder operators are defined respectively $B_+(X) = \{ T \in \Phi_+(X) : \text{asc}(T) < \infty \}$, $B_-(X) = \{ T \in \Phi_-(X) : \text{dec}(T) < \infty \}$, $B_l(X) = \{ T \in \Phi_l(X) : \text{asc}(T) < \infty \}$ and $B_r(X) = \{ T \in \Phi_r(X) : \text{dec}(T) < \infty \}$. Sets of semi-Browder and Browder operators are defined as $B_{\pm}(X) = B_+(X) \cup B_-(X)$ and $B(X) = B_+(X) \cap B_-(X)$. For $T \in \mathcal{B}(X)$, the essential spectrum, the upper semi-Browder spectrum, the lower semi-Browder spectrum, the semi-Browder spectrum, the semi-Browder spectrum, the left semi-Browder spectrum, the right
there exists a non-zero complex polynomial $P$ such that

There exists a non-zero complex polynomial $P$ such that $P(T) = 0$. Denote by $\lambda = \Phi(X)$ the concept of Drazin invertible to generalized Drazin invertible, in fact operators are exactly Fredholm Drazin invertible operators. Koliha [7] generalized reader to ([3], [7], [13] and [14]). It is common knowledge that the spectrum generalized Drazin spectrum. For more details about those inverses we refer the reader to ([9], [17], [18] and [19]).

Evidently, $\sigma_{gD}(T) = \sigma_{b_+}(T^*)$ and $\sigma_{b_+}(T) = \sigma_{gD}(T^*)$, where $T^* \in \mathcal{B}(X^*)$ the adjoint operator of $T$ on the dual space $X^*$ and $\sigma_{b}(T) = \sigma_{gD}(T) \cup \sigma_{b_+}(T)$ consequently $\sigma_{b}(T) = \sigma_{gD}(T^*)$.

Recall that an operator $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of $\lambda_0 \in \mathbb{C}$, the only analytic function $f : U \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f = 0$. Denote by $S(T)$ the open set of $\lambda \in \mathbb{C}$ where $T$ fails to have the SVEP at $\lambda$. An operator $T$ is said to have the SVEP if $T$ has the SVEP at every $\lambda \in \mathbb{C}$, in this case $S(T) = \emptyset$. According to [1] Theorem 3.52] we have

$$\sigma_{b}(T) = \sigma_{gD}(T^*) \cup S(T^*) = \sigma_{b_+}(T) \cup S(T).$$

For a compact subset $K$ of $\mathbb{C}$, let $\text{acc} K$, $\text{int} K$, $\text{iso} K$, $K^c$, $\partial K$, $\overline{K}$ and $\eta(K)$ be the set of all accumulation points, the interior set, the set of isolated points, the complement, the boundary, the closure and the polynomially convex hull of $K$ respectively.

An operator $T$ is called Drazin invertible if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and} \quad TST - T \quad \text{is nilpotent,}$$

it is well known that $S$ exists if and only if $p = \text{asc}(T) = \text{des}(T)$. The set $\sigma_D(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{is not Drazin invertible} \}$ is the Drazin spectrum. Although Browder operators are exactly Fredholm Drazin invertible operators. Koliha [7] generalized the concept of Drazin invertible to generalized Drazin invertible, in fact $T \in \mathcal{B}(X)$ is generalized Drazin invertible if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and} \quad TST - T \quad \text{is quasi-nilpotent,}$$

(i.e. $\sigma(TST - T) = \{0\}$). The former author realized that $T$ is generalized Drazin invertible if and only if $0 \notin \text{acc}(T)$. The set $\sigma_{gD}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{is not generalized Drazin invertible} \} = \text{acc} \sigma(T) = \text{acc} \sigma_D(T)$ is the generalized Drazin spectrum. For more details about those inverses we refer the reader to ([3], [7], [13] and [14]). It is common knowledge that the spectrum $\sigma_+ = \sigma_{b+} \cup \sigma_{b_{rb}} \cup \sigma_{b_+} \cup \sigma_{b_{rb}}$ or $\sigma_{rb}$ is a compact subset of the complex plan $\mathbb{C}$, and $\sigma_D(T)$ or $\sigma_{gD}(T)$ are closed subsets of $\mathbb{C}$ possibly empty.

We can easily show that

$$\text{acc} \sigma_{b}(T) = \text{acc} \sigma_+(T) \cup \text{acc} \sigma_D(T) = \text{acc} \sigma_+(T) \cup \sigma_{gD}(T).$$

The set $\text{acc} \sigma_{b}(T)$ may be empty, for example when $T$ is polynomially Riesz (i.e. there exists a non-zero complex polynomial $P$ such that $P(T)$ is a Riesz operator).
Then $P(\sigma_b(T)) = \sigma_b(P(T)) = \{0\}$, as a result $\sigma_b(T) = P^{-1}\{0\}$ is a finite set which has no accumulation point.

If $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ then $M_C \in \mathcal{B}(X \oplus Y)$ represents a bounded linear operator on Banach space $X \oplus Y$ given by:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

it called upper triangular operator matrix. It is well know that in the case of infinite dimensional, the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, may be strict. This attracts the attention of many mathematicians to study the defect $(\sigma_*(A) \cup \sigma_*(B)) \setminus \sigma_*(M_C)$ where $\sigma_*$ runs over different type of spectra.

In [15] S.Zhang et al, gave a description of the set $\bigcap_{C \in \mathcal{B}(X,Y)} \sigma_b(M_C)$. They showed the following theorem which we are going to need in the sequel.

**Theorem 1.1.** [15] For given $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ the following holds:

$$\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B) \cup W_0(A, B)$$

Where $W_0(A, B) = \{ \lambda \in \mathbb{C} : N(A - \lambda) \times N(B - \lambda) \text{ is not isomorphic to } X/R(A - \lambda) \times Y/R(B - \lambda) \}$.

In [16], authors investigate the filling-in-holes problem of $2 \times 2$ upper triangular operator matrices for Browder spectrum, they showed the following theorem.

**Theorem 1.2.** [16] Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then

$$\sigma_b(M_C) \cup W_b = \sigma_b(A) \cup \sigma_b(B)$$

where $W_b$ is the union of certain holes in $\sigma_b(M_C)$, which happen to be subsets of $\sigma_b(A) \cap \sigma_b(B)$.

The next lemma has been demonstrated in [13]:

**Lemma 1.1.** [13] For given $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ if $M_C$ is Drazin invertible for some $C \in \mathcal{B}(Y, X)$ then:

(i) $\text{des}(B) < \infty$ and $\text{asc}(A) < \infty$.

(ii) $\text{des}(A^*) < \infty$ and $\text{asc}(B^*) < \infty$.

The purpose of this paper is to study the relationship between $\text{acc}\sigma_b(M_C)$ and $\text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B)$. We investigate the local spectral theory to prove the equality

$$\text{acc}\sigma_b(M_C) \cup [S(A^*) \cap S(B)] = \text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B).$$

Also, we show that the passage from $\text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B)$ to $\text{acc}\sigma_b(M_C)$ can be described as follows:

$$\text{acc}\sigma_b(M_C) \cup W_{\text{acc}\sigma_b} = \text{acc}\sigma_b(M_0) = \text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B)$$

where $W_{\text{acc}\sigma_b}$ is the union of certain holes in $\text{acc}\sigma_b(M_C)$, which happen to be subsets of $\text{acc}\sigma_b(A) \cap \text{acc}\sigma_b(B)$. Finally we give sufficient conditions on $A$ and $B$ to ensure the equality $\text{acc}\sigma_b(M_C) = \text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B)$.
2. Main results and proofs

In order to state precisely the relationship between $accσ_b(\lambda I_C)$ and $accσ_b(A) \cup accσ_b(B)$, we began this section by the following two lemmas which will be widely used in the sequel.

**Lemma 2.1.** Let $(A, B) \in B(X) \times B(Y)$ and $C \in B(Y, X)$. Then

$$accσ_b(\lambda I_C) \subseteq accσ_b(M_0) = accσ_b(A) \cup accσ_b(B)$$

**Proof.** We have $σ_b(M_0) = σ_b(A) \cup σ_b(B)$, it is clear that $accσ_b(M_0) = accσ_b(A) \cup accσ_b(B)$. Now without lose generality let $0 \notin accσ_b(A) \cup accσ_b(B)$ then, there exists $ε > 0$ such that $A - λI$ and $B - λI$ are Browder for every $λ$, $0 < |λ| < ε$, according to [4, Lemma 2.4], $M_ε - λI$ is Browder for every $λ$, $0 < |λ| < ε$. Thus $0 \notin accσ_b(M_0)$. □

The inclusion, $accσ_b(\lambda I_C) \subseteq accσ_b(A) \cup accσ_b(B)$, may be strict as we can see in the following example.

**Example 1.** Let $A, B, C \in B(l^2)$ defined by:

$$Ae_n = e_{n+1}.$$  

$$B = A^*.$$  

$$C = e_0 \otimes e_0.$$  

where $\{e_n\}_{n \in \mathbb{N}}$ is the orthonormal basis of $l^2$. We have $σ_b(A) = \{λ \in \mathbb{C}; |λ| \leq 1\}$, then $accσ_b(A) = \{λ \in \mathbb{C}; |λ| \leq 1\}$. Since $M_C$ is unitary, then $accσ_b(M_C) \subseteq \{λ \in \mathbb{C}; |λ| = 1\}$. So $0 \notin accσ_b(M_C)$, but $0 \in accσ_b(A) \cup accσ_b(B)$. Notes that $A^* = B$ has not the SVEP.

**Definition 2.1.** Let $T \in B(X)$. We said that $T$ has the property $(aB)$ at $λ \in \mathbb{C}$ if $λ \notin accσ_b(T)$.

**Lemma 2.2.** If two of $M_C$, $A$ and $B$ have the property $(aB)$ at 0, then the third have also the property $(aB)$ at 0.

**Proof.** (1) If $A$ and $B$ have the property $(aB)$ at 0, by lemma 2.1 $M_C$ has the property $(aB)$ at 0.

(2) If $M_C$ and $A$ have the property $(aB)$ at 0, that is $0 \notin accσ_b(M_C)$ and $0 \notin accσ_b(A)$, then there exists $ε > 0$ such that $M_C - λI$ and $A - λI$ are Browder operators for every $λ$, $0 < |λ| < ε$. Thus according to [6, Corollary 5] and [8, Lemma 2.3] we have $B - λI$ is Browder for every $λ$, $0 < |λ| < ε$, i.e $0 \notin accσ_b(B)$.

(3) If $B$ and $M_C$ have the property $(aB)$ at 0, then $A$ has the property $(aB)$ at 0, the proof is similar to (ii). □

Now we are in position to prove our first main result.

**Theorem 2.1.** For $A \in B(X)$, $B \in B(Y)$ and $C \in B(Y, X)$ we have

$$accσ_b(M_C) \cup [S(A^*) \cap S(B)] = accσ_b(A) \cup accσ_b(B).$$
Lemma 2.3. Let $M$ and $N$ be two bounded subsets of complex plane $\mathbb{C}$. Then the following statements hold:

(i) $\text{acc} M \cup \text{acc} N = \text{acc}(M \cup N)$;
(ii) $\text{iso}(M \cup N) \subseteq \text{iso}M \cup \text{iso}N$;
(iii) If $M$ is closed, then $\partial(\text{acc} M) \cup \text{iso}M = \partial M$;
(iv) If $M$ is closed, then $\text{iso}(\partial M) = \text{iso}M$.

Proof. It follows from lemma [23] that $\text{acc} \sigma_b(M_C) \subseteq \text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B)$. Also, it is known that $S(A^*) \cap S(B) \subseteq \text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B)$. Hence $\text{acc} \sigma_b(M_C) \cup |S(A^*) \cap S(B)| \subseteq \text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B)$. For the contrary inclusion, it is sufficient to prove that $(\text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B)) \setminus \text{acc} \sigma_b(M_C) \subseteq S(A^*) \cap S(B)$.

Let $\lambda \in (\text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B)) \setminus \text{acc} \sigma_b(M_C)$ we can assume without lose of generality that $\lambda = 0$. Then $0 \notin \text{acc} \sigma_b(M_C)$, hence there exists $\varepsilon > 0$ such that $M_C - \mu I$ is Browder for every $0 < |\mu| < \varepsilon$, so $A - \mu I \in \Phi_+(X)$, $B - \mu I \in \Phi_-(Y)$ for every $0 < |\mu| < \varepsilon$. Moreover it follows from lemma [11] that $\text{des}(B - \mu I) < \infty$ and $\text{asc}(A - \mu I) < \infty$ for every $0 < |\mu| < \varepsilon$, so $0 \notin \text{acc}_{b_+}(A) \cup \text{acc}_{b_-}(B)$. For the sake of contradiction assume that $0 \notin S(A^*) \cap S(B)$.

Case 1 $0 \notin S(A^*)$: If $0 \in \sigma_b(A^*)$ we have $\sigma_b(A^*) = \sigma_{b_-}(A^*) \cup S(A^*)$, then $0 \in \sigma_{b_-}(A^*)$. Since $0 \notin \text{acc} \sigma_{b_-}(A) = \text{acc} \sigma_{b_-}(A^*)$, it follows that 0 is an isolated point of $\sigma_{b_-}(A^*)$. On the other hand $\overline{S(A^*)} \subseteq \sigma_b(A^*) = \sigma_{b_-}(A^*) \cup S(A^*)$, thus $\partial S(A^*) \subseteq \sigma_{b_-}(A^*)$. As $\sigma_b(A^*) = \sigma_{b_-}(A^*) \cup S(A^*)$ and $0 \notin \text{iso} \sigma_{b_-}(A^*)$ then 0 is an isolated point of $\sigma_b(A) = \sigma_b(A^*)$. Hence $0 \notin \text{acc} \sigma_b(A)$, according to Lemma [2,2] $0 \notin \text{acc} \sigma_b(B)$ but this is impossible. Now if $0 \notin \sigma_b(A^*)$ then $0 \notin \text{acc} \sigma_b(A^*) = \text{acc} \sigma_b(A)$. Hence $0 \notin \text{acc} \sigma_b(B)$, and this is contradiction.

Case 2 $0 \notin S(B)$: If $0 \in \sigma_b(B)$ we have $0 \notin \text{acc} \sigma_b(B)$. Indeed $\sigma_b(B) = \sigma_{b_-}(B) \cup S(B)$ then $0 \in \sigma_{b_-}(B)$ but $0 \notin \text{acc} \sigma_{b_-}(B)$ thus $0 \notin \text{iso} \sigma_{b_-}(B)$ it follows that $0 \in \text{iso} \sigma_b(B)$ i.e $0 \notin \text{acc} \sigma_b(B)$. Since $0 \notin \text{acc} \sigma_b(M_C)$ then $0 \notin \text{acc} \sigma_b(A)$, contradiction. Now if $0 \notin \sigma_b(B)$ then $0 \notin \text{acc} \sigma_b(B)$ since $0 \notin \text{acc} \sigma_b(M_C)$ thus $0 \notin \text{acc} \sigma_b(A)$, contradiction. As a result

$$\text{acc} \sigma_b(M_C) \cup |S(A^*) \cap S(B)| = \text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B).$$

From theorem [23], we obtain immediately the following corollary.

Corollary 2.1. Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ in $S(A^*) \cap S(B) = \emptyset$, then for every $C \in \mathcal{B}(Y, X)$ we have

$$\text{acc} \sigma_b(M_C) = \text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B).$$

In particular if $A^*$ or $B$ have he SVEP, then the last equality hold.

Example 2. Let $U$ be the simple unilateral shift on $l^2([\mathbb{N}])$, set $S = U \oplus U^*$ the operator defined on $l^2([\mathbb{N}]) \oplus l^2([\mathbb{N}])$. It follows that $\sigma_b(S) = \sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Consider the operators $A$, $B$ defined by $A = S + I$ and $B = S - I$, we have

$$S(A) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda - 1| < 1\}$$

$$S(B) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda + 1| < 1\}$$

So, $S(A^*) \cap S(B) = \emptyset$. Consequently $\text{acc} \sigma_b(M_C) = \text{acc} \sigma_b(A) \cup \text{acc} \sigma_b(B)$.

The following lemma summarizes some well-known facts which will be used frequently.

Lemma 2.3. Let $M$ and $N$ be two bounded subsets of complex plane $\mathbb{C}$. Then the following statements hold:

(i) $\text{acc} M \cup \text{acc} N = \text{acc}(M \cup N)$;
(ii) $\text{iso}(M \cup N) \subseteq \text{iso}M \cup \text{iso}N$;
(iii) If $M$ is closed, then $\partial(\text{acc} M) \cup \text{iso}M = \partial M$;
(iv) If $M$ is closed, then $\text{iso}(\partial M) = \text{iso}M$. 
The following two lemmas are the key of our second main result.

**Lemma 2.4.** Let $E$, $F$ be two compact subsets of $C$, such that $E \subseteq F$ and $\partial F \subseteq E$, then

$$\partial \text{acc} F \subseteq \text{acc} E.$$ 

**Proof.** For any $\lambda \in \partial (\text{acc} F)$ then either $\lambda \in \text{acc}(\partial \text{acc} F)$ or $\lambda \in \text{iso}(\partial \text{acc} F)$.

**Case 1:** If $\lambda \in \text{acc}(\partial \text{acc} F)$, then there exist $\lambda_n \in \partial \text{acc} F$, $n = 1, 2, \ldots$ such that $\lim_{n \to \infty} \lambda_n = \lambda$, since

$$\partial \text{acc} F \subseteq \partial F \subseteq E$$

it follows that $\lambda_n \in E$, $n = 1, 2, \ldots$; thus we have $\lambda \in \text{acc} E$.

**Case 2:** If $\lambda \in \text{iso}(\partial \text{acc} F)$, then we get $\lambda \in \text{iso}(\text{acc} F)$ from Lemma 2.3, that is $\text{iso}(\partial \text{acc} F) = \text{iso}(\text{acc} F)$.

Thus there exists $\varepsilon > 0$ such that $\tau \notin \text{acc} F$ for every $\tau$, $0 < |\lambda - \tau| < \varepsilon$, but $\lambda \in \text{acc} E$ then there exist $\mu_n \in \partial F$, $n = 1, 2, \ldots$ such that $\lim_{n \to \infty} \mu_n = \lambda$ and $\mu_n \neq \lambda$ for every $n = 1, 2, \ldots$ i.e there exists $N \in \mathbb{N}^*$ such that $0 < |\mu_n - \lambda| < \varepsilon$ for every $n \geq N$.

Now let $\lambda_n = \mu_{N+n+1}$, then $\lambda_n \in \text{iso} F$, $n = 1, 2, \ldots$ and $\lim_{n \to \infty} \lambda_n = \lambda$. It follows from Lemma 2.3 that

$$\text{iso} F \subseteq \partial F \subseteq E$$

so $\lambda_n \in \text{iso} F \subseteq E$ for every $n = 1, 2, \ldots$ and $\lim_{n \to \infty} \lambda_n = \lambda$, therefore $\lambda \in \text{acc} E$.

Then for both cases $\partial \text{acc} F \subseteq \text{acc} E$ is true. □

**Lemma 2.5.** Let $E$, $F$ be two compact subsets of $C$ such that $E \subseteq F$ and $\eta(E) = \eta(F)$ then $\eta(\text{acc} E) = \eta(\text{acc} F)$.

**Proof.** Since $E \subseteq F$ then $\text{acc} E \subseteq \text{acc} F$, we need to show that $\partial(\text{acc} F) \subseteq \partial(\text{acc} E)$ from the maximum module theorem. But since $\text{int}(\text{acc} E) \subseteq \text{int}(\text{acc} F)$ it suffices to show that $\partial(\text{acc} E) \subseteq (\text{acc} F)$ which is always verified by lemma 2.3. □

The following theorem says that the passage from $\text{acc}_b(A) \cup \text{acc}_b(B)$ to $\text{acc}_b(M_C)$ is the punching of some set in $\text{acc}_b(A) \cap \text{acc}_b(B)$.

**Theorem 2.2.** Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then

$$\text{acc}_b(M_C) \cup W_{\text{acc}_b} = \text{acc}_b(A) \cup \text{acc}_b(B)$$

where $W_{\text{acc}_b}$ is the union of certain holes in $\text{acc}_b(M_C)$, which happen to be subsets of $\text{acc}_b(A) \cap \text{acc}_b(B)$.

**Proof.** We first claim that, for every $C \in \mathcal{B}(Y, X)$ we have

$$\left(\text{acc}_b(A) \cup \text{acc}_b(B)\right) \setminus \left(\text{acc}_b(A) \cap \text{acc}_b(B)\right) \subseteq \text{acc}_b(M_C).$$

(1) to see this suppose that $\lambda \in (\text{acc}_b(A) \cup \text{acc}_b(B)) \setminus (\text{acc}_b(A) \cap \text{acc}_b(B))$ then either $\lambda \in \text{acc}_b(A) \setminus \text{acc}_b(B)$ or $\lambda \in \text{acc}_b(B) \setminus \text{acc}_b(A)$.

(1) If $\lambda \in \text{acc}_b(A) \setminus \text{acc}_b(B)$ it follows that $A$ has not the property $(aB)$ at $\lambda$ and $B$ has the property $(aB)$ at $\lambda$, thus $\lambda \in \text{acc}_b(M_C)$, for it were not so, then $\lambda \notin \text{acc}_b(M_C)$ and hence according to lemma 2.2 $\lambda \notin \text{acc}_b(A)$ which is impossible.

(2) If $\lambda \in \text{acc}_b(B) \setminus \text{acc}_b(A)$, similarly as in (1) we can show that $\lambda \in \text{acc}_b(M_C)$. 

6 A. TAJMOUATI, M. KARMOUNI AND S. ALAOUI CHRIFI
Moreover, from the proof of Theorem 1.2 we have
\[ \partial(\sigma_{b}(A) \cap \sigma_{b}(B)) \subseteq \sigma_{b}(M_{C}). \]
Applying lemma 2.4 and lemma 2.6 we have
\[ \eta(\text{acc}\sigma(M_{C})) = \eta(\text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B)). \]
Therefore 2 says that the passage from \( \text{acc}\sigma(M_{C}) \) to \( \text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B) \) is the filling in certain of the holes in \( \text{acc}\sigma(M_{C}) \). But \( (\text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B)) \setminus \text{acc}\sigma_{b}(M_{C}) \) is contained in \( \text{acc}\sigma_{b}(A) \cap \text{acc}\sigma_{b}(B) \) by 1 it follows that the filling in certain of the holes in \( \text{acc}\sigma_{b}(M_{C}) \) should occur in \( \text{acc}\sigma_{b}(A) \cap \text{acc}\sigma_{b}(B) \).

\[ \square \]

**Corollary 2.2.** Let \((A, B) \in B(X) \times B(Y)\). If \( \text{acc}\sigma_{b}(A) \cap \text{acc}\sigma_{b}(B) \) has no interior points, then for every \( C \in B(Y, X) \) we have
\[ \text{acc}\sigma_{b}(M_{C}) = \text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B). \]

From theorem 1.2 and theorem 2.2 we have the following.

**Theorem 2.3.** Let \((A, B) \in B(X) \times B(Y)\) and \( C \in B(Y, X) \). Then the following assertions are equivalent

1. \( \sigma_{b}(M_{C}) = \sigma_{b}(A) \cup \sigma_{b}(B) \),
2. \( \text{acc}\sigma_{b}(M_{C}) = \text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B) \).

**Proof.** First we show that
\[ W_{b} \subseteq W_{\text{acc}\sigma_{b}}. \]
Indeed, if \( \lambda \in W_{b} \) then according to Theorem 1.2 we have \( \lambda \in (\sigma_{b}(A) \cup \sigma_{b}(B)) \setminus \sigma_{b}(M_{C}) \), then \( \lambda \notin \sigma_{b}(M_{C}) \) consequently \( \lambda \notin \text{acc}\sigma_{b}(M_{C}) \), it is enough to show that \( \lambda \in \text{acc}(\sigma_{b}(A) \cup \sigma_{b}(B)) \), if it was not then \( \lambda \notin \text{acc}(\sigma_{b}(A) \cup \sigma_{b}(B)) \) but \( \lambda \in \sigma_{b}(A) \cup \sigma_{b}(B) \) thus
\[ \lambda \in \text{iso}(\sigma_{b}(A) \cup \sigma_{b}(B)) \subseteq \text{iso}(\sigma_{b}(A)) \cup \text{iso}(\sigma_{b}(B)) \] (Lemma 2.3)
\[ \subseteq \partial\sigma_{b}(A) \cup \partial\sigma_{b}(B) \] (Lemma 2.3)
\[ \subseteq \sigma_{rb}(A) \cup \sigma_{rb}(B) \] (Theorem 1.1)
\[ \subseteq \sigma_{b}(M_{C}) \]

Hence \( \lambda \in \sigma_{b}(M_{C}) \), contradiction. Therefore
\[ \lambda \in (\text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B)) \setminus \text{acc}\sigma_{b}(M_{C}) \]
by theorem 2.2 we have \( \lambda \in W_{\text{acc}\sigma_{b}} \), so \( W_{b} \subseteq W_{\text{acc}\sigma_{b}} \), according to this inclusion the following implication is hold
\[ \text{acc}\sigma_{b}(M_{C}) = \text{acc}\sigma_{b}(A) \cup \text{acc}\sigma_{b}(B) \Rightarrow \sigma_{b}(M_{C}) = \sigma_{b}(A) \cup \sigma_{b}(B) \]
Conversely, if \( \sigma_{b}(M_{C}) = \sigma_{b}(A) \cup \sigma_{b}(B) \) then
\[ \text{acc}(\sigma_{b}(M_{C})) = \text{acc}(\sigma_{b}(A) \cup \sigma_{b}(B)) \]
\[ = \text{acc}(\sigma_{b}(A)) \cup \text{acc}(\sigma_{b}(B)). \]

\[ \square \]

The following example shows that the inclusion used in the proof of theorem 2.3 may be strict in general.
Example 3. Define $U, V \in \mathcal{B}(l^2)$ by

$$Ue_n = e_{n+1}$$
$$V_{e_n+1} = e_n$$

where $\{e_n\}_{n \in \mathbb{N}}$ is the orthonormal basis of $l^2$. Let us introduce an operator $P : l^2 \to l^2$ as:

$$P(x_1, x_2, x_3, ...) = (x_1, 0, 0, ...), \quad (x_1, x_2, x_3, ...) \in l^2.$$ 

consider the operator $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : l^2 \oplus l^2 \oplus l^2 \to l^2 \oplus l^2 \oplus l^2$, where $A = U$, $B = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} : l^2 \oplus l^2 \to l^2 \oplus l^2$ and $C = (P, 0) : l^2 \oplus l^2 \to l^2$.

We have $\sigma_{\rho}(M_C) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \cup \{ 0 \}$, $\sigma_{\rho}(A) = \sigma_{\rho}(B) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \};$ then $\text{acc}_{\rho}(M_C) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}, \text{acc}_{\rho}(A) = \text{acc}_{\rho}(B) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$. Consequently

$$W_{\sigma_{\rho}} = \{ \lambda \in \mathbb{C} : 0 < |\lambda| < 1 \}, \quad W_{\text{acc}_{\rho}} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}.$$ 

Thus $W_{\sigma_{\rho}} \neq W_{\text{acc}_{\rho}}$.

Nevertheless, we have the following theorem.

Theorem 2.4. Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. If $\text{iso}\partial W_{\sigma_{\rho}} = \emptyset$ then

$$W_{\sigma_{\rho}} = W_{\text{acc}_{\rho}}.$$ 

Proof. According to Lemma 2.3, and since $\text{iso}\partial W_{\sigma_{\rho}} = \emptyset$, we obtain

$$\text{iso}_{\rho}(M_C) = \text{iso}_{\rho}(A) \cup \text{iso}_{\rho}(B) \subseteq \text{iso}_{\rho}(A) \cup \text{iso}_{\rho}(B).$$

Let $\lambda \in \text{iso}_{\rho}(M_C)$, then either $\lambda \in \text{iso}_{\rho}(A)$ or $\lambda \in \text{iso}_{\rho}(B)$. If $\lambda \in \text{iso}_{\rho}(A)$, then $A$ has the property $(A\rho)$ at $\lambda$ and $\lambda I - A$ is not Browder, by Lemma 2.2, $B$ has also the property $(A\rho)$ at $\lambda$. Thus $\lambda \in \text{iso}_{\rho}(B)$ or $\lambda \in \rho_{\theta}(A)$. In contrast, if $\lambda \in \text{iso}_{\rho}(B)$ we obtain similarly that $\lambda \in \text{iso}_{\rho}(B)$ or $\lambda \in \rho_{\theta}(A)$.

In addition, due to theorem 1.1 and Lemma 2.3, we have

$$\text{iso}_{\rho}(A) \cap \text{iso}_{\rho}(B) \cup (\text{iso}_{\rho}(A) \cap \rho_{\theta}(B)) \cup (\rho_{\theta}(A) \cap \text{iso}_{\rho}(B)) \subseteq \text{iso}_{\rho}(A) \cup \text{iso}_{\rho}(B) \subseteq \partial \sigma_{\rho}(A) \cup \partial \sigma_{\rho}(B)$$

$$\subseteq \sigma_{\rho}(A) \cup \sigma_{\rho}(B) \subseteq \sigma_{\rho}(M_C),$$

and from Lemma 2.2 we have

$$\text{iso}_{\rho}(A) \cap \text{iso}_{\rho}(B) \cup (\text{iso}_{\rho}(A) \cap \rho_{\theta}(B)) \cup (\rho_{\theta}(A) \cap \text{iso}_{\rho}(B)) \subseteq \text{acc}_{\rho}(M_C).$$

it follows that

$$(\text{iso}_{\rho}(A) \cap \text{iso}_{\rho}(B)) \cup (\text{iso}_{\rho}(A) \cap \rho_{\theta}(B)) \cup (\rho_{\theta}(A) \cap \text{iso}_{\rho}(B)) \subseteq \text{iso}_{\rho}(M_C).$$

Consequently, $(\text{iso}_{\rho}(A) \cap \text{iso}_{\rho}(B)) \cup (\text{iso}_{\rho}(A) \cap \rho_{\theta}(B)) \cup (\rho_{\theta}(A) \cap \text{iso}_{\rho}(B)) = \text{iso}_{\rho}(M_C).$ but $\text{iso}_{\rho}(M_C) \cap (\text{acc}_{\rho}(A) \cup \text{acc}_{\rho}(B)) = \emptyset$.

From Lemma 2.2 it follows that

$$(\text{iso}_{\rho}(A) \cup \text{iso}_{\rho}(B)) \setminus \text{iso}_{\rho}(M_C) = (\text{iso}_{\rho}(A) \setminus \text{iso}_{\rho}(M_C)) \cup (\text{iso}_{\rho}(B) \setminus \text{iso}_{\rho}(M_C)) \subseteq \text{acc}_{\rho}(B) \cup \text{acc}_{\rho}(A).$$
These imply that
\[ \sigma_b(A) \cup \sigma_b(B) = (\text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B)) \cup (\text{iso}\sigma_b(A) \cup \text{iso}\sigma_b(B)) \]
\[ = \text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B) \cup \text{iso}\sigma_b(M_C) \cup (\text{iso}\sigma_b(A) \cup \text{iso}\sigma_b(B)) \setminus \text{iso}\sigma_b(M_C) \]
\[ \subseteq (\text{acc}\sigma_b(A) \cup \text{acc}\sigma_b(B)) \cup \text{iso}\sigma_b(M_C) \]
\[ \subseteq \text{acc}\sigma_b(M_C) \cup W_{\text{acc}\sigma_b} \cup \text{iso}\sigma_b(M_C). \]

However
\[ \sigma_b(M_C) \cap W_{\text{acc}\sigma_b} = (\text{acc}\sigma_b(M_C) \cup \text{iso}\sigma_b(M_C)) \cap W_{\text{acc}\sigma_b} \]
\[ = (\text{acc}\sigma_b(M_C) \cap W_{\text{acc}\sigma_b}) \cup (\text{iso}\sigma_b(M_C) \cap W_{\text{acc}\sigma_b}) \]
\[ = \emptyset. \]

On the other hand, \( \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C) \cup W_{\sigma_b} = (\text{acc}\sigma_b(M_C) \cup \text{iso}\sigma_b(M_C)) \cup W_{\sigma_b} \) and \( \sigma_b(M_C) \cap W_{\sigma_b} = \emptyset \), this highlight that \( W_{\sigma_b} = W_{\text{acc}\sigma_b}. \)

\[\square\]

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