Abstract

We extend the Kreiss–Majda theory of stability of hyperbolic initial-boundary-value and shock problems to a class of systems, notably including the equations of magnetohydrodynamics (MHD), for which Majda’s block structure condition does not hold: namely, simultaneously symmetrizable systems with characteristics of variable multiplicity, satisfying at points of variable multiplicity either a “totally nonglancing” or a “nonglancing and linearly splitting” condition. At the same time, we give a simple characterization of the block structure condition as “geometric regularity” of characteristics, defined as analyticity of associated eigenprojections. The totally nonglancing or nonglancing and linearly splitting conditions are generically satisfied in the simplest case of crossings of two characteristics, and likewise for our main physical examples of MHD or Maxwell equations for a crystal. Together with previous analyses of spectral stability carried out by Gardner–Kruskal and Blokhin–Trakhinin, this yields immediately a number of new results of nonlinear inviscid stability of shock waves in MHD in the cases of parallel or transverse magnetic field, and recovers the sole previous nonlinear result, obtained by Blokhin–Trakhinin by direct “dissipative integral” methods, of stability in the zero-magnetic field limit. Our methods apply also to the viscous case.

* MAB Université de Bordeaux I, 33405 Talence Cedex, France; metivier@math.u-bordeaux.fr, partially supported by European network HYKE, HPRN-CT-2002-00282.
† Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: K.Z. thanks the University of Bordeaux I for its hospitality during the visit in which this work was carried out. Research of K.Z. was partially supported under NSF grants number DMS-0070765 and DMS-0300487.
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1 Introduction

There are two cases where the analysis of hyperbolic boundary value problems is well developed: first, when the system is symmetric and the boundary conditions are dissipative $[F_1, F_2, F_3]$; second, when the system is hyperbolic with constant multiplicities and the boundary conditions satisfy a Lopatinski conditions $[Sal, Sa2, Kn, MaOs, Me3]$. In both cases, the main energy estimate is proved using symmetrizers and integrations by parts. In the first case, the $L^2$ estimate is immediate: the symmetrizer is given and the dissipativity property is assumed. In the second case, the construction of symmetrizers is delicate. It was achieved first by Kreiss (see also Chazarain-Piriou $[ChP]$) in the strictly hyperbolic case. Kreiss’ construction was next extended by Majda-Osher and Majda to systems satisfying a “block structure condition”, which holds in particular as soon as the eigenvalues of the system are semi-simple and have constant multiplicity (see $[Me3]$). Kreiss’s symmetrizers are tangential pseudo-differential operators. For symmetrizable systems with dissipative boundary conditions, a trivial, constant symmetrizer is available in the form $SA_d$, where $S$ is the symmetrizer for the initial value problem, and $A_d$ is the normal coefficient matrix; see discussion above Theorem 5.6.

The constant multiplicity assumption is satisfied in many applications such as the linearized Euler equations or isotropic Maxwell’ equation, but it is not satisfied in other interesting example such as MHD or Maxwell’s equation in crystals. On the other hand, the symmetry condition is commonly satisfied as a consequence of the existence of a conserved energy. For instance, the last two examples are symmetric. But in the symmetric case, the boundary conditions are not always dissipative, while the sharp criterion of stability is given by a Lopatinski condition (see examples in Majda-Osher). In particular, for shock waves considered as transmission problems, the Rankine-Hugoniot transmission conditions are not dissipative in general, and Majda’s analysis of shock waves relies on the use of Kreiss symmetrizers. Therefore, with the goal of studying shock waves for MHD, it is natural to extend the construction of Kreiss symmetrizers to cases where the eigenvalues may have variable multiplicity.

Such an extension is given below: for symmetric systems, we extend the construction of symmetrizers to a framework which contains MHD. First, we classify the multiple eigenvalues into categories: algebraically regular, geometrically regular and nonregular. We show that Majda’s block structure condition is equivalent to geometric regularity. This indicates clearly where Kreiss’ construction stops. Next, we extend the notion of being nonglancing
to multiple eigenvalues. Finally, using the symmetry of the system, we extend the construction of Kreiss’ symmetrizers to the case of eigenvalues that are not geometrically regular provided that they are “totally nonglancing” or else “linearly splitting” (defined in Section 3); see Theorem 4.2.

For applications to nonconstant coefficient or quasilinear systems, it is important that symmetrizers vary smoothly with respect to frequency, i.e., that the choice of symmetrizer be robust under changes in the various parameters. The basic construction for the totally nonglancing case is smooth. However, the one in the linearly splitting case is not; thus, it is useful mainly in the constant coefficient case. A second possibility, under sufficiently strong structural assumptions, would be to perform a second microlocalization about the locus of variable multiplicity; however, we do not pursue this direction, lacking a motivating physical example.

Instead, we pursue a different course, investigating in Section 6 the conditions under which one may obtain a smooth symmetrizer in the original frequency variables, without further microlocalization. This turns out to have a simple and pleasing answer, which also clarifies the relation between Kreiss’ theory and the earlier Friedrichs theory of symmetrizable systems with dissipative boundary conditions. Namely, one may define at points of variable multiplicity an appropriate reduced system, with reduced boundary condition, involving only the modes that change multiplicity. Then, necessary and sufficient conditions for existence of a smooth symmetrizer are that the full system satisfy the uniform Lopatinski condition, and the reduced problem admit a constant symmetrizer, or equivalently be symmetrizable with maximally dissipative boundary condition; see Theorems 5.6–6.10. In the simplest case of a double root, for which the reduced system is $2 \times 2$, it turns out that the Lopatinski conditions is equivalent to symmetrizability/maximal dissipativity (Appendix D; see also [MaOs]). Thus, we find that existence of a smooth symmetrizer requires only the Lopatinski condition and nothing further; see Theorem 6.19.

Finally, in Section 7, we apply the tools we have developed to the motivating problem of shock stability in MHD. In this case, it turns out that all characteristics are at least algebraically regular. More, they are either geometrically regular or totally nonglancing; see calculations, Appendix A. Thus, our results yield existence of smooth symmetrizers, and consequent linearized and nonlinear stability, provided the uniform Lopatinski condition is satisfied, thus reducing the stability problem as in the constant-multiplicity case to a (nontrivial!) linear algebraic calculation. This immediately gives several new results and recovers the sole existing result, obtained by Blokhin and Trakhinin [BT1, BT4] by direct methods, of stability...
in the zero-magnetic field limit; see Corollary 7.6.

Though we concentrate here on inviscid, i.e., first-order problems, the
methods introduced extend in straightforward fashion to the viscous case
considered in [Z1, Z3, GWMZ1, GWMZ2, GWMZ3, GWMZ4]; see Appendix
E. In particular, this extends the results obtained in [Z2, Z3] for compressible
Navier–Stokes equations to the case of (viscous) MHD.

2 Multiple eigenvalues of hyperbolic systems

In this section, we recall several basic definitions and start a classification
of multiple eigenvalues.

2.1 Basic definitions

Consider an $N \times N$ first order system with symbol

$$L(p, \tau, \xi) = \tau \text{Id} + A(p, \xi) = \tau \text{Id} + \sum_{j=1}^{d} \xi_j A_j(p).$$

The characteristic polynomial is $\Delta(p, \tau, \xi) = \det L(p, \tau, \xi)$. Recall the fol-
lowing definition.

**Definition 2.1.** (i) The homogenous polynomial $\pi(\eta)$ is hyperbolic in the
real direction $\nu$ if and only if $\pi(\nu) \neq 0$ and for all real $\eta' \notin \mathbb{R}\nu$, all the roots
in $z \in \mathbb{C}$ of $\pi(z\nu + \eta') = 0$ are real.

(ii) The system (2.1) is hyperbolic in the direction $\nu$ if its characteristic
polynomial is.

(iii) $L$ is symmetric hyperbolic in the direction $\nu$ in the sense of Friedrichs
if there is a smooth self adjoint matrix $S(p)$, called a Friedrichs symmetrizer,
such that all the matrices $S(p)A_j(p)$ are self adjoint and $S(p)L(p, \nu)$ is def-
inite positive. This implies hyperbolicity.

Below, we always assume that $L$ is hyperbolic in the direction $dt =
(1, 0, \ldots, 0)$. This means that all the eigenvalues of $A(p, \xi)$ are real when $\xi$
is real.

Suppose that $\Delta(p, \tau, \xi) = 0$. Then $-\tau$ is an eigenvalue of $A(p, \xi)$. Its
algebraic multiplicity $m$ is the multiplicity of $-\tau$ as a root of the polynomial
$\Delta(p, \cdot, \xi)$. Its geometric multiplicity is $m_g = \dim \ker L(p, \tau, \xi)$. The eigen-
value is semi-simple when $m_g = m$. Recall that for symmetric hyperbolic
systems, the eigenvalues are always semi-simple.
The simplest cases of multiple roots are described in the following definition.

**Definition 2.2.** Consider a root \((p, \tau, \xi)\) of \(\Delta(p, \tau, \xi) = 0\), of algebraic multiplicity \(m\) in \(\tau\).

i) \((p, \tau, \xi)\) is algebraically regular, if on a neighborhood \(\omega\) of \((p, \xi)\) there are \(m\) smooth real functions \(\lambda_j(p, \xi)\), analytic in \(\xi\), such that for \((p, \xi) \in \omega:\)

\[
\Delta(p, \tau, \xi) = e(p, \tau, \xi) \prod_{j=1}^{m} \left(\tau + \lambda_j(p, \xi)\right)
\]

where \(e\) is a polynomial in \(\tau\) with smooth coefficients such that \(e(p, \tau, \xi) \neq 0\).

ii) \((p, \tau, \xi)\) is geometrically regular if in addition there are \(m\) smooth functions \(e_j(p, \xi)\) on \(\omega\) with values in \(\mathbb{C}^N\), analytic in \(\xi\), such that

\[
A(p, \xi) e_j(p, \xi) = \lambda_j(p, \xi) e_j(p, \xi),
\]

and the \(e_1, \ldots, e_m\) are linearly independent.

**Remarks 2.3.**

a) Simple roots (i.e. \(m = 1\)), are geometrically regular.

b) Roots of constant multiplicity are algebraically regular. In this case all the \(\lambda_j\) are equal. They are geometrically regular if and only if in addition they are semi-simple: the \(m = m_g\) vectors \(e_j\) form a smooth basis of the eigenspace.

**Examples 2.4.**

a) For MHD, the multiple eigenvalues are algebraically regular, but some are not geometrically regular; see Appendix A.

b) For Maxwell’s equation in a biaxial crystal, the multiple eigenvalues are not algebraically regular; see Appendix B.

Systems with only geometrically regular multiple eigenvalues play an important role, since they provide exactly the class of boundary value problems satisfying the block structure condition (see Theorem 3.4 below). To treat examples such as MHD or non isotropic Maxwell Equations, we have to go beyond this class.

### 2.2 Block reduction

Consider a root \((p, \tau, \xi)\) of \(\Delta(p, \tau, \xi) = 0\), of algebraic multiplicity \(m\) in \(\tau\).

When \((p, \xi)\) is close to \((p, \xi)\), \(\Delta(p, \cdot, \xi)\) has exactly \(m\) roots, counted with
their multiplicity, close to $\tau$ and there is a smooth block reduction of $A(p, \xi)$:

$$U^{-1}(p, \xi)A(p, \xi)U(p, \xi) = \begin{pmatrix} A^\flat(p, \xi) & 0 \\ 0 & \tilde{A}(p, \xi) \end{pmatrix}$$

with $A^\flat$ of dimension $m$ and $\tilde{A}$ has no eigenvalue in a neighborhood of $\tau$. Moreover,

$$A^\flat(p, \xi) = -\tau \text{Id} \quad \text{if} \quad \tau \text{ is semisimple.}$$

Denoting by $V$ [resp. $W$] the $m$ first columns of $U$ [resp. $m$ first rows of $U^{-1}$], there holds,

$$A(p, \xi)V(p, \xi) = V(p, \xi)A^\flat(p, \xi), \quad W(p, \xi)A(p, \xi) = A^\flat(p, \xi)W(p, \xi)$$

$$A^\flat(p, \xi) = W(p, \xi)A(p, \xi)V(p, \xi) \quad W(p, \xi)V(p, \xi) = \text{Id.}$$

Note that when $L$ is symmetric hyperbolic, then $A^\flat$ is symmetric since we can choose $U(p, \xi)$ such that $U^{-1}(p, \xi) = U^*(p, \xi)S(p)$ implying that

$$W(p, \xi) = V^*(p, \xi)S(p), \quad A^\flat(p, \xi) = V^*(p, \xi)S(p)A(p, \xi)V(p, \xi).$$

There holds

$$\Delta(p, \tau, \xi) = e(p, \tau, \xi) \det \left( \tau \text{Id} + A^\flat(\tau, \xi) \right)$$

with $e(p, \tau, \xi) \neq 0$.

### 2.3 Examples and linearly splitting eigenvalues

We consider first the example of double roots for symmetric systems. In this case, the reduced matrix $A^\flat$ has the form

$$A^\flat(p, \xi) = \lambda(p, \xi)\text{Id} + \begin{pmatrix} a(p, \xi) & b(p, \xi) \\ b(p, \xi) & -a(p, \xi) \end{pmatrix}$$

with $\lambda$, $a$ and $b$ smooth and homogeneous of degree 1 in $\xi$. Double roots occur on the set

$$\mathcal{M} = \{ (p, \xi) : a(p, \xi) = b(p, \xi) = 0 \}$$

and $(p, \xi) \in \mathcal{M}$. Several cases can be considered.
1) $a = b \equiv 0$; this is the constant multiplicity case.

2) $\mathcal{M}$ is a smooth manifold of codimension 1, of equation $\{ \phi(p, \xi) = 0 \}$. This means that $a$ and $b$ vanish on $\mathcal{M}$, and one can factor out $\phi$, or powers of $\phi$ in $a$ and $b$. Assume for instance that

$$a = \phi^k \tilde{a}, \quad b = \phi^k \tilde{b}, \quad \text{with} \ (\tilde{a}, \tilde{b}) \neq (0, 0) \ \text{at} \ (p, \xi).$$

This occurs with $k = 1$ for some eigenvalues of MHD. Then the eigenvalues are $\lambda + \phi^k \tilde{\lambda}_j$ where the $\tilde{\lambda}_j$ are the smooth and distinct eigenvalues of

$$\tilde{A} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & -\tilde{a} \end{pmatrix}.$$

The eigenvectors of $A^b$ are those of $\tilde{A}$. We are in the geometrically regular case.

3) $\mathcal{M}$ is a smooth manifold of codimension 2 and more precisely

$$d\xi a(p, \xi) \wedge d\xi b(p, \xi) \neq 0.$$ 

Then $a$ and $b$ can be taken as independent coordinates near $(p, \xi)$, transversal to $\mathcal{M}$. The eigenvalues are $\lambda \pm \sqrt{a^2 + b^2}$. $(p, \xi)$ is not an algebraically regular point. This situation occurs for Maxwell’s equations in bi-axial crystals.

4) We give now an example of a more degenerate situation, which occurs for MHD. $\mathcal{M}$ is a manifold of codimension 2, given by the equations

$$\mathcal{M} = \{ \phi = \psi = 0 \}, \quad d\phi \wedge d\psi \neq 0.$$ 

Moreover, $a$ and $b$ vanish at the second order on $\mathcal{M}$, and there is a smooth function $c$ such that $a^2 + b^2 = c^2$. For instance, this holds when

$$a = \phi^2 - \psi^2, \quad b = 2\phi\psi, \quad c = \phi^2 + \psi^2.$$ 

In this case the eigenvalues are smooth, equal to $\lambda \pm c$. $(p, \xi)$ is an algebraically regular point. But it is not geometrically regular, since the eigenvectors are

$$\begin{pmatrix} \phi \\ -\psi \end{pmatrix}, \quad \begin{pmatrix} -\psi \\ \phi \end{pmatrix}$$

which have no limit as $(\phi, \psi) \to (0, 0)$. 


These examples can be generalized to higher order roots. Suppose that we are given a smooth conic manifold $\mathcal{M}$ of codimension $\nu$ and a smooth function $\lambda$ on $\mathcal{M}$ such that for all $(p, \xi) \in \mathcal{M}$, $\lambda(p, \xi)$ is a semi-simple eigenvalue of $A(p, \xi)$ of multiplicity $m$. Suppose that $(\underline{p}, \underline{\xi}) \in \mathcal{M}$ and $\tau + \lambda(\underline{p}, \underline{\xi}) = 0$.

1) The trivial case $\nu = 0$ corresponds to constant multiplicity.

2 a) Suppose that there are no parameter $p$ and $d = 2$. By homogeneity, $\nu$ is at most one. When $\nu = 1$, on the unit sphere $|\xi| = 1$, $\mathcal{M}$ consists of isolated points, and $A^b(\xi)$ is an analytic family depending on one parameter. If $L$ is symmetric, then so does $A^b$, and therefore $A^b$ is analytically diagonalizable: $(\tau, \xi)$ is a geometrically regular characteristic point.

2 b) Suppose that $\nu = 1$. Then $\mathcal{M}$ is given by an equation $\phi = 0$, and, extending $\lambda$ outside $\mathcal{M}$, $A^b - \lambda \text{Id}$ vanishes when $\phi = 0$. Suppose that

$$A^b(p, \xi) = \lambda(p, \xi) \text{Id} + \phi^k \tilde{A}(p, \xi)$$

where $\tilde{A}(p, \xi)$ has distinct eigenvalues. This framework extends (2.12); again $(\tau, \xi)$ is a geometrically regular characteristic point.

3) Suppose that $\nu \geq 1$ and $\mathcal{M}$ is given by equations $\phi_1 = \ldots = \phi_\nu = 0$ with $d_\xi \phi_1, \ldots, d_\xi \phi_\nu$ linearly independent. Then

$$A^b(p, \xi) = \lambda(p, \xi) \text{Id} + \sum_{j=1}^{\nu} \phi_j \tilde{A}_j(p, \xi).$$

Extending (2.14), consider the case where

$$\tilde{A}(\theta) := \sum_j \theta_j \tilde{A}_j(p, \xi)$$

is strictly hyperbolic,

meaning that for all $\theta \in \mathbb{R}^\nu \setminus \{0\}$, $\tilde{A}(\theta)$ has only real and simple eigenvalues. Then, for $|\theta| = 1$ and $(p, \xi)$ close to $(\underline{p}, \underline{\xi})$, $\sum \theta_j \tilde{A}_j(p, \xi)$ has distinct simple eigenvalues $\tilde{\lambda}_j(p, \xi, \theta)$, with eigenvectors $e_j(p, \xi, \theta)$. The eigenvalues of $A^b$ are

$$\lambda_j(p, \xi) = \lambda(p, \xi) + |\phi| \tilde{\lambda}_j(p, \xi, \frac{\phi}{|\phi|}).$$

The eigenvalues are thus simple away from $\mathcal{M}$ and, in general, $(\underline{p}, \underline{\tau}, \underline{\xi})$ is not algebraically regular.

In the next section, we give an intrinsic formulation of (2.18). We will refer to this example as the linearly splitting case.
2.4 The tangent systems at semi-simple multiple eigenvalues

First, we recall the following result about multiple roots of hyperbolic polynomials.

**Proposition 2.5.** Assume that \( \pi(\tau, \xi) \) is hyperbolic in the direction \( dt = (1, 0, \ldots, 0) \). Consider \( (\tau, \xi) \neq 0 \) and assume that \( \tau \) is a root of multiplicity \( m \geq 1 \) of \( \pi(\cdot, \xi) = 0 \). Then

\[
\pi(\tau + \tau, \xi + \xi) = \pi^{(m)}(\tau, \xi) + O(|\tau, \xi|^{m+1})
\]

where \( \pi^{(m)} \) is homogeneous of degree \( m \) in \( (\tau, \xi) \) and hyperbolic in the direction \( dt \).

This means that all terms of degree smaller that \( m \) in the Taylor expansion of \( \pi \) with respect to all variables, vanish.

Consider a system (2.1) and a root \( (p, \tau, \xi) \) of \( \Delta \), of multiplicity \( m \). Denote by \( A^b \) the associated \( m \times m \) block as in (2.4) and by \( W \) and \( V \) the matrices defined in (2.6) (2.7). Following Proposition 2.5, we also introduce the \( m \)-order term in the Taylor expansion of \( \Delta(p, \cdot) \) at \( (\tau, \xi) \), that we denote by \( \Delta^m \).

**Proposition 2.6.** Assume that \( -\tau \) is a semi-simple eigenvalue of \( A(p, \xi) \) of order \( m \). Then

\[
A^b(p, \xi + \xi) = -\tau \text{Id} + A'(\xi) + O(|\xi|^2)
\]

with

\[
A'(\xi) = \sum_j \xi_j W A_j(p) V, \quad W = W(p, \xi), \quad V = V(p, \xi)
\]

Moreover, there is \( \xi \neq 0 \), such that

\[
\det (\tau \text{Id} + A'(\xi)) = e \Delta^m(\tau, \xi).
\]

**Proof.** Differentiating the relations (2.6) (2.7) and using that \( A(p, \xi) = -\tau \text{Id} \), yields (2.21).

The determinant \( \Delta^b(p, \tau, \xi) \) is a factor of \( \Delta(p, \tau, \xi) \), as in (2.9). By (2.21),

\[
\det \left( (\tau + \tau) \text{Id} + A^b(\xi + \xi) \right) = \det (\tau \text{Id} + A'(\xi)) + O(|\tau, \xi|^{m+1}).
\]

Comparing with (2.20), implies (2.23).
We will refer to $L' = \partial_t + A'(\partial_x)$ as the tangent system to $L$ at $(p,\tau,\xi)$.

**Remark 2.7.** The operator $L'$ is well known in the analysis of the propagation singularities or of wave packets

$$u(t,x) = e^{i (\tau \cdot x + p \cdot \xi)/\varepsilon} \left( a_0(t,x) + \varepsilon a_1(t,x) + \ldots \right)$$

solutions to $L(p,\partial_t,\partial_x)u = 0$. The propagation is given by the laws of geometric optics. The principal term $a_0$ satisfies

$$(2.24) \quad a_0 = V a_0^\flat, \quad L'(\partial_t,\partial_x) a_0^\flat = 0$$

(see [La], [JMR]; see also [Te1], [Te2] for (2.23)). This means that the high frequency oscillations are transported by $L'(\partial_t,\partial_x)$.

**Remark 2.8.** By homogeneity, $A'(\xi) = -\tau \Id$ and $(-\tau,\xi)$ is always a characteristic direction for $L'$.

**Example 2.9.** If $\tau + \lambda(\xi) = 0$ with $\lambda$ an eigenvalue of $A(\xi)$ of constant multiplicity $m$ near $\xi$, then

$$(2.25) \quad L' = (\partial_t + v \cdot \partial_x) \Id, \quad \text{with } v = \partial_t \lambda(\xi),$$

$$\Delta(\tau,\xi) = \beta (\tau + v \cdot \xi)^m, \quad \beta = \frac{1}{m!} \partial^m_\tau \Delta(p,\tau,\xi).$$

$L'$ is the transport field at the group velocity $v$. In this case, (2.24) means that the amplitudes $a_0$ are transported along the rays of geometric optics, which are the integral curve of the transport field $\partial_t + v \cdot \partial_x$.

**Example 2.10.** Suppose that $(\tau,\xi)$ is geometrically regular. With $\lambda_j$ and $e_j$ as in (2.2), (2.3), in the basis $e_j(\xi)$,

$$(2.26) \quad L' = \text{diag}(\partial_t + v_j \cdot \partial_x), \quad v_j = \partial_t \lambda_j(\xi),$$

$$\Delta(\tau,\xi) = \beta \prod (\tau + v_j \cdot \xi)^m, \quad \beta = \frac{1}{m!} \partial^m_\tau \Delta(p,\tau,\xi).$$

**Example 2.11.** Suppose that $A^\flat$ has the form (2.17), with $d_\xi \phi_1, \ldots, d_\xi \phi_\nu$ linearly independent. Using the notation $\hat{A}$ as in (2.18), the symbol of $L'$ is

$$(2.27) \quad L'(\tau,\xi) = (\tau + v \cdot \xi) \Id + \hat{A}(\Phi \xi)$$

with $v = \partial_t \lambda(p,\xi)$ and $\Phi = \partial_t \phi(p,\xi)$. $\Phi \xi = 0$ if and only if $\xi$ is tangent to $M_p := \{ \phi_1(p,\cdot) = \ldots = \phi_\nu(p,\cdot) = 0 \}$. Thus, the condition (2.18) is equivalent to

$$(2.28) \quad \text{for all } \xi \notin T_\xi M_p, \quad A'(\xi) \text{ has only real and simple eigenvalues.}$$

This motivates the following definition.
Definition 2.12. A multiple root \((p, \tau, \xi)\) is called linearly splitting transversally to a smooth manifold \(M = \{\phi_1 = \ldots = \phi_\nu = 0\}\) with \(\phi_1, \ldots, \phi_\nu\) analytic in \(\xi\) and \(d_\xi \phi_1, \ldots, d_\xi \phi_\nu\) linearly independent, if \(M\) contains \((p, \xi)\) and

i) there is a smooth real valued function \(\lambda(p, \xi)\), analytic in \(\xi\), such that for all \((p, \xi) \in M\), \(\lambda(p, \xi)\) is a semi-simple eigenvalue of \(A(p, \xi)\) of constant multiplicity \(m\),

ii) the condition (2.28) is satisfied.

Remark 2.13. In this case, the manifold

\[
(2.29) \quad \tilde{M} = \{(p, \tau, \xi) : (p, \xi) \in M, \tau = -\lambda(p, \xi)\}
\]

is a smooth submanifold of the characteristic variety of \(L\) which contains \((p, \xi)\). The corresponding block

\[
(2.30) \quad L'(p, \tau, \xi) := \tau \text{Id} + A^b(p, \xi)
\]

satisfies \(L'(p, \tau, \xi) = 0\) on \(\tilde{M}\). Therefore,

\[
(2.31) \quad L'(\tau, \xi) = 0, \quad \text{for } (\tau, \xi) \in T_{\tau, \xi} \tilde{M}.
\]

This means that \(L'\) only depends on frequency variables which are transversal to \(\tilde{M}\). Introduce the quotient \(\hat{E} = \mathbb{R}^{1+d}/T_{\tau, \xi} \tilde{M}^p\), and denote by \(\varpi\) the projection from \(\mathbb{R}^{1+d}\) onto \(\hat{E}\). By (2.31), there is \(\hat{L}\) on \(\hat{E}\) such that

\[
(2.32) \quad L'(\tau, \xi) = \hat{L}(\varpi(\tau, \xi))
\]

Because \(dt\) is transversal to \(\tilde{M}\), \(\varpi dt \neq 0\), and the condition (2.28) is equivalent to

\[
(2.33) \quad \hat{L}(\theta), \theta \in \hat{E}, \text{ is strictly hyperbolic in the direction } \varpi dt.
\]

This approach gives a completely intrinsic definition of linear splitting. In particular, it shows that one can replace \(dt\) by any hyperbolic direction.

Alternatively, consider \(E\) such that \(\mathbb{R}^{1+d} = E \oplus T_{\tau, \xi} \tilde{M}^p\) and a basis in \(E\). Identifying \(E\) and \(\hat{E}\), the operator \(\hat{L}\) has the form \(\hat{L} = \sum_j \hat{A}_j \partial_{y_j}\). One can choose \(E\) and the basis such that \(dt = dy_0\); in this case (2.33) or (2.28) means that \(\hat{L}\) is strictly hyperbolic in the direction \(dy_0\). For example, for Maxwell’s equation in a bi-axial crystal, the multiplicity is two and \(L'\) is a \(2 \times 2\), strictly hyperbolic system in space dimension 2, equivalent to a 2-D wave equation (see Appendix B). That one space dimension (at least) is lost when passing from \(L\) to \(L'\) is a general fact that follows from Remark 2.8.
Remark 2.14. If $\mathcal{M}$ is of codimension one, then (2.16) holds with $k = 1$, and (2.28) implies that $\tilde{A}(p, \xi)$ has distinct eigenvalues. This shows that if $(\tilde{p}, \bar{\tau}, \bar{\xi})$ is linearly splitting transversally to a smooth manifold $\mathcal{M}$ of codimension 1, then $(\tilde{p}, \bar{\tau}, \bar{\xi})$ is geometrically regular.

3 The boundary block analysis

In this section we start the investigation of noncharacteristic boundary value problems for hyperbolic systems. In particular, we propose an extension of the definition of glancing to multiple eigenvalues with variable multiplicity.

We consider a planar boundary. Changing notations and calling $(y, x)$ the spatial components, the boundary is $\{x = 0\}$. We assume that the boundary is not characteristic, that is

$$\det A_d \neq 0.$$  

(3.1)

The spatial Fourier frequency variables are denoted by $(\eta, \xi)$. and $\tau - i\gamma$ is the complex time Fourier-Laplace frequency. Given a hyperbolic system $L$ (2.1), we consider

$$G(p, \zeta) = A_d(p)^{-1}((\tau - i\gamma)\text{Id} + \sum_{j=1}^{d-1} \eta_j A_j(p))$$  

(3.2)

with $\zeta = (\tau, \eta, \gamma) \in \mathbb{R}^{d+1}$. We denote by $\mathbb{R}_{+}^{d+1}$ the half space $\{\gamma > 0\}$. By homogeneity, we can restrict attention to $\zeta$ in the unit sphere $S^d$; we denote by $S^d_+$ the open half sphere $S^d \cap \{\gamma > 0\}$ and by $\overline{S^d_+}$ its closure $S^d \cap \{\gamma \geq 0\}$.

Denoting by $\Delta$ the characteristic polynomial of $L$, there holds

$$\Delta(p, \tau - i\gamma, \eta, \xi) = \det A_d(p) \det (\xi \text{Id} + G(p, \zeta)).$$  

(3.3)

The consequence of hyperbolicity, is that for $\gamma \neq 0$, $G(p, \zeta)$ has no real eigenvalue.

3.1 Block reduction

The goal is to construct symmetrizers for $G(p, \zeta)$. This is done locally, near each point $(p, \zeta)$ with $\zeta \in \overline{S^d_+}$ using a diagonal block reduction of $G(p, \zeta)$.

Accordingly, consider an eigenvalue $-\xi$ of $G(p, \zeta), \zeta \in \overline{S^d_+}$. Denote by $m'$ its algebraic multiplicity. There are $N \times m'$, $m' \times m'$ and $m' \times N$ matrices,
by hyperbolicity det $A_d$ is real and det$(\xi \text{Id} + G)$ has real coefficients when $\gamma = 0$. If $\xi$ is real, then for $(p, \tau, \eta, 0)$ close to $(p_0, \zeta)$, the roots close to $\xi$ go by conjugated pair, implying that det$(\xi \text{Id} + G^2(p, \tau, \eta, 0))$ has real coefficients.

By definition of $G^b$ and $G^d$, both determinant in (3.7) are equal to $\Delta$ up to a nonvanishing factor near $(p, \tau, \eta, \xi)$.

Moreover, 
\[
V(p, \eta, \xi) \ker \left( (\tau - i\gamma)\text{Id} + A^b(p, \eta, \xi) \right) = \ker \left( (\tau - i\gamma)\text{Id} + A(p, \eta, \xi) \right) \\
= \ker \left( \xi \text{Id} + G(p, \zeta) \right) = V_b(p, \zeta) \ker \left( \xi \text{Id} + G^b(p, \zeta) \right).
\]
This implies (3.7). If $-\tau$ is semi-simple, then $E := \ker (\tau \text{Id} + A(p, \eta, \xi))$ has dimension $m$ and is the image of $V(p, \eta, \xi)$. It is equal to $\ker (\xi \text{Id} + G(p, \zeta))$ thus contained in the invariant space which is the image of $\tilde{V}_b(p, \zeta)$. This implies that $m' \geq m$ and that $W_b(p, \eta)$ is injective on $E$.

3.2 The block structure condition

It was pointed out by A. Majda, [Maj] (see also [MaOs]) that Kreiss’ construction of symmetrizers can be carried out if $G$ satisfies the following block structure condition.

**Definition 3.3.** A matrix $G(p, \zeta)$ has the block structure property near $(p, \zeta)$ if there exists a smooth matrix $V$ on a neighborhood of that point such that $V^{-1}GV = \text{diag}(G_k)$ is block diagonal, with blocks $G_k$ of size $v_k \times v_k$, having one the following properties:

i) The spectrum of $G_k(p, \zeta)$ is contained in $\{\text{Im} \mu \neq 0\}$.

ii) $v_k = 1$, $G_k(p, \zeta)$ is real when when $\gamma = 0$, and $\partial_\gamma G_k(p, \zeta) \neq 0$.

iii) $v_k > 1$, $G_k(p, \zeta)$ has real coefficients when $\gamma = 0$, there is $\mu_k \in \mathbb{R}$ such that

$$(3.8) \quad G_k(p, \zeta) = \mu_k \text{Id} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \end{bmatrix},$$

and the lower left hand corner of $\partial_\gamma G_k(p, \zeta)$ does not vanish.

Consider a block $G^\flat$ associated to an eigenvalue $-\xi$ of $G(p, \zeta)$. If the block is elliptic, that is if $\xi$, then $G^\flat$ satisfies property i) near $(p, \zeta)$, and this finishes the analysis when $\gamma > 0$.

**Theorem 3.4.** Consider $\zeta = (\tau, \eta, 0)$ and $-\xi$ a real eigenvalue of $G(p, \xi)$. Then, the associated block $G^\flat(p, \zeta)$ satisfies Majda’s block structure condition on a neighborhood of $(p, \xi)$ if and only if $(\tau, \eta, \xi)$ is geometrically regular for $L$ is the sense of Definition 2.2.

The if part was proved by Kreiss for strictly hyperbolic systems. It was next extended to hyperbolic systems with semi-simple eigenvalues of constant multiplicity in [M3]. This proof, which only uses the factorization (2.2) and the existence of smooth eigenvectors (2.3), is easily extended to the geometrically regular case. Surprisingly, the condition of geometric regularity is also necessary. We postpone the independent proof of this theorem to Appendix C.
3.3 Glancing modes

We want to extend the definition of *glancing* to general eigenvalues. Recall first the case of a simple eigenvalue of $A(p, \eta, \xi)$ or of an eigenvalue of constant multiplicity $m$. Suppose that the real eigenvalue $-\xi$ of $G(p, \zeta)$ satisfies

\[(3.9) \quad \tau + \lambda(p, \eta, \xi) = 0.\]

Following [Kr] and [Me], the analysis of $G^\parallel$ depends on the multiplicity of $\xi$ as root of (3.9). *Hyperbolic modes* correspond to simple roots and *glancing modes* to multiple roots. Only the latter yield non trivial Jordan blocks in (3.8). The glancing condition reads

\[(3.10) \quad \partial_\xi \lambda(p, \eta, \xi) = 0.\]

Introducing the transport field (2.25) and the speed $v = (v_1, \ldots, v_d) = \partial_\eta, \xi \lambda(p, \eta, \xi)$, (3.10) is equivalent to

\[(3.11) \quad v_d = 0.\]

In this case, the rays which are the integral curves of the transport fields are tangent to the boundary. According to Example 2.9, the conditions (3.10) or (3.11) are satisfied if and only if the boundary is characteristic for $\Delta^{(m)}$, the principal term of $\Delta$ at $(p, \tau, \eta, \xi)$. This motivates the following definition.

**Definition 3.5.** A root $(p, \tau, \eta, \xi)$ of $\Delta$, of multiplicity $m$ in $\tau$ is nonglancing, if and only if the $m$-th order term in the Taylor expansion of $\Delta(p, \cdot)$ at $(\tau, \eta, \xi)$ satisfies

\[(3.12) \quad \Delta^{(m)}(dx) \neq 0\]

where $dx = (0, \ldots, 0, 1)$ is the (space-time) conormal to the boundary.

Also recall that in the constant multiplicity analysis, the two cases $v_d > 0$ and $v_d < 0$ are different. In the first case, the rays launched at the boundary $\{x = 0\}$ at time $t = 0$ enter the domain $\{x > 0\}$ for $t > 0$; the mode is said *incoming*. In the second case, the rays go off the domain, and the mode is *outgoing*. This has a natural extension to general eigenvalues. By Proposition 2.5, $\Delta^{(m)}$ is hyperbolic in the direction $dt$. Introduce the component of $dt$ in $\Delta^{(m)} > 0$. It is an open convex cone $\Gamma_+$ and $\Delta^{(m)}$ is hyperbolic in any direction in $\Gamma_+$. Its dual cone $\hat{\Gamma}_+ = \{(t, y, x) : \forall (\tau; \eta, \xi) \in \Gamma_+, \quad t\tau + y\eta + x\xi \geq 0\}$ is the forward cone of propagation (see e.g. [Hor]).
the constant multiplicity case, \( \Gamma^+ = \{ \tau + v' \eta + v_d \xi > 0 \} \) and \( \hat{\Gamma}^+ = \mathbb{R}_+(1, v) \). Thus, \( v_d > 0 \) if and only if to one of the following two equivalent conditions hold:

\[
\begin{align*}
(3.13) & \quad dx \in \Gamma^+, \\
(3.14) & \quad \hat{\Gamma}^+ \setminus \{0\} \subset \{ x > 0 \}.
\end{align*}
\]

Similarly, the outgoing nonglancing condition \( v_d < 0 \) is equivalent to one of the following conditions.

\[
\begin{align*}
(3.15) & \quad -dx \in \Gamma^+, \\
(3.16) & \quad \hat{\Gamma}^+ \setminus \{0\} \subset \{ x < 0 \}.
\end{align*}
\]

This leads to the following extension.

**Definition 3.6.** A root \((p, \tau, \eta, \xi)\) of \( \Delta \), of multiplicity \( m \) in \( \tau \), is said to be totally incoming if one of the two equivalent conditions \((3.13) \ (3.14)\) is satisfied. It is totally outgoing if the equivalent conditions \((3.13) \ (3.14)\) hold. It is totally nonglancing if it is either totally incoming or outgoing.

**Example 3.7.** We have already seen the example of constant multiplicity \((3.9)\). Consider now the case of an algebraically regular root as in Example \((2.10)\). Then

\[
\Delta = \beta \prod_j (\tau + v'_j \eta + v_{j,d} \xi),
\]

see \((2.26)\). Each mode can be glancing, incoming or outgoing depending on the sign of \( v_{j,d} \). According to Definition \((3.5)\), the mode is nonglancing when all the \( v_{j,d} \) are different from 0. It is totally incoming [resp., outgoing] if all the \( v_{j,d} \) are positive [resp., negative]. This explains the terminology totally incoming or totally outgoing.

When \(-\tau\) is a semi-simple eigenvalue of \( A(p, \eta, \xi) \) of multiplicity \( m \), we have introduced the tangent system \( L'(\tau, \eta, \xi) \) at \((p, \tau, \eta, \xi)\). With notations as in \((2.22)\), it reads

\[
(3.17) \quad L'(\tau, \eta, \xi) = \tau \text{Id} + \sum_{j=1}^{d-1} A_{j} \eta_j + A_d \xi, \quad A_j = WA_j(p)V.
\]

**Lemma 3.8.** Assume that \(-\tau\) is a semi-simple eigenvalue of \( A(p, \eta, \xi) \). Then
i) \( p, \tau, \eta, \xi \) is nonglancing if and only if the boundary is noncharacteristic for \( L' \), which means that \( A_d \) is invertible,

ii) \( p, \tau, \eta, \xi \) is totally incoming [resp., outgoing] if and only \( L' \) is hyperbolic in the direction \( dx \) normal to the boundary and \( dx \) [resp. \( -dx \)] is in the same component as \( dt \); this holds if and only if the spectrum of \( A_d \) is positive [resp., negative].

**Proof.** By (2.23), \( \Delta \) is the characteristic polynomial of \( L' \), up to a nonvanishing factor. This implies i).

Since \( \Delta \) is hyperbolic in the time direction, the eigenvalues of \( A_d \) are real. In addition, the spectrum of \( A_d \) is positive if and only if the matrix \( \alpha \Id + (1-\alpha)A_d \) is invertible for all \( \alpha \in [0,1] \), thus, by definition and convexity of \( \Gamma^+ \), if and only if \( dx \) belongs to this cone. In the totally outgoing case, the proof is similar.

When \(-\tau\) is semi-simple and nonglancing, there is a simple link between the first order Taylor expansion of the reduced symbols \( \tau \Id + A^0(p, \eta, \xi) \) and \( G^0(p, \zeta) + \xi \Id \) defined at (2.7) and (3.5) respectively. Similarly to (2.21), we introduce \( G'(\zeta) \) the first order variation of \( G^0 \) at the base point:

\[
G^0(p, \zeta + \zeta) = G^0(p, \zeta) + G'(\zeta) + O(|\zeta|^2).
\]

**Proposition 3.9.** Suppose that \(-\tau\) is a semi-simple eigenvalue of \( A(p, \eta, \xi) \) of multiplicity \( m \) and \( (p, \tau, \eta, \xi) \) is nonglancing. Then \(-\xi\) is a semi-simple eigenvalue of \( G(p, \zeta) \) of multiplicity \( m \). Moreover, one can choose bases such that

\[
G'(\zeta) + \xi \Id = (A_d)^{-1}\left((\tau - i\gamma)\Id + A'(\eta, \xi)\right).
\]

**Proof.** a) We fix \( p = p \) and we omit the parameter \( p \) in the notations below. There holds

\[
\Delta(\tau + \tau, \eta, \eta, \xi + \xi) = e(\tau, \eta, \xi)\Delta'(\tau, \eta, \xi)
\]

with \( e(\tau, \eta, \xi) \neq 0 \) and

\[
\Delta'(\tau, \eta, \xi) = \tau^m + \sum_{j=1}^{m} a_{m-j}(\eta, \xi)\tau^j, \quad a_{m-j}(\eta, \xi) = O(|\eta, \xi|^m). \]

Thus,

\[
\Delta(\tau, \eta, \xi) = e(0, 0, 0)\left(\tau^m + \sum_{j=1}^{m} a_{m-j}(\eta, \xi)\tau^j\right)
\]
where $a^{(m-j)}(\eta, \tau)$ denotes the homogeneous term of degree $m - j$ in the Taylor expansion of $a_{m-j}$. The nonglancing condition implies that

$$a_{m}^{(m)}(0, \xi) = \alpha_m \xi^m \quad \text{with} \quad \alpha_m \neq 0.$$  

Thus,

$$\Delta(\tau, \eta, \xi + \xi) = e_0(0,0,0) a_{m}^{(m)} \xi^m + O(\xi^{m+1})$$

showing that $\xi$ is a root of exact order $m$ of $\Delta(\tau, \eta, \cdot) = 0$. Thus, $-\xi$ is an eigenvalue of $G(\zeta)$ of algebraic multiplicity $m$.

b) Introduce the splittings

$$C^N = E(\zeta) \oplus F(\zeta), \quad C^N = E_b(\zeta) \oplus F_b(\zeta)$$

where $E$ and $F$ [resp. $E_b$ and $F_b$] denote the invariant spaces of $A$ [resp. $G$] associated to eigenvalues near and away from $-\tau$ [resp. $-\xi$]. We have shown that $E(\eta, \xi) = E_b(\zeta)$. Thus, one can choose bases such that the matrices $V$ and $V_b$ occurring in (2.6) and (3.4) respectively, satisfy

$$V = V(\eta, \xi) = V_b(\zeta).$$

The matrix $W$ [resp. $W_b$] vanishes on $F$ [resp. $F_b$] and is equal to the inverse of $V$ [resp. $V_b$ on $E$ [resp. $E_b$]. Since $-\tau$ and $-\xi$ are semi-simple,

$$F_b(\zeta) = \text{Range}(G(\zeta) + \xi \text{Id}) = A_d^{-1} \text{Range}(\tau \text{Id} + A(\eta, \xi)) = A_d^{-1} F(\eta, \xi)$$

Since $A_d = W A_d(p) V$ is invertible by Lemma 3.8, this implies that

$$W_b := W_b(\zeta) = (A_d)^{-1} W A_d \quad \text{and} \quad W_b V = \text{Id}.$$  

Indeed, the matrix in the right hand side vanishes on $F_b(\zeta)$ and multiplying it by $V$ on the right gives $\text{Id}$.

Differentiating (3.24) and using that $G(\zeta) = -\xi \text{Id}$, implies that

$$G'(\zeta) = W_d G(p, \zeta) V.$$  

With (3.22), (2.22) and (3.26) the identity (3.19) follows.
3.4 Linearly splitting modes

We now study the structure of $G^\flat$ when $(p, \tau, \eta, \xi)$ is a real root of $\Delta$, of multiplicity $m \geq 2$, linearly splitting transversally to $\mathcal{M}$ and nonglancing. We use the notations of Definition 2.12. We first assume that

$$(3.28) \quad dx \notin T_{\eta, \xi}\mathcal{M}_p.$$  

With (2.22) and (2.28), this is equivalent to the condition that $A_d$ has distinct eigenvalues, or, since $m \geq 2$,

$$(3.29) \quad A_d \notin \mathbb{R}\text{Id}.$$  

In this case, there are coordinates $(\eta_1, \ldots, \eta_{d-1})$ in $\mathbb{R}^{d-1}$ such that $\mathcal{M}$ is parametrized by $q = (p, \eta')$, with $\eta' = (\eta_1, \ldots, \eta_{d-\nu})$, close to $q = (p, \eta')$: $\mathcal{M}$ is locally given by

$$(3.30) \quad \eta'' = \phi(q), \quad \xi = -\mu(q),$$

with $\eta'' = (\eta_{d-\nu+1}, \ldots, \eta_{d-1})$ and $\phi, \mu$ smooth. From now on, we work in such coordinates. With some abuse of notations, we write

$$(3.31) \quad \lambda(q) := \lambda(p, \eta', \phi(q), -\mu(q)).$$

Introduce the following notations

$$(3.32) \quad q = (p, \eta'), \quad \tilde{\eta} = \eta'' - \phi(q), \quad \tilde{\xi} = \xi + \mu(q),$$

$$\tilde{\tau} = \tau + \lambda(q), \quad \tilde{\gamma} = \gamma, \quad \tilde{\zeta} = (\tilde{\tau}, \tilde{\eta}, \tilde{\gamma}),$$

By assumption, $\lambda(q)$ is a semi-simple eigenvalue of $A(p, \eta', \phi(q), -\mu(q))$. Therefore,

$$(3.33) \quad A^\flat(q, \eta, \xi) = \psi(q)\text{Id} + \tilde{A}(q, \tilde{\eta}, \tilde{\xi}), \quad \text{with} \quad \tilde{A}(q, 0, 0) = 0.$$  

Moreover, Proposition 3.9 implies that $\mu(q)$ is a semi-simple eigenvalue with multiplicity $m$ of $G(p, \lambda(q), \eta', \phi(q'), 0)$, hence

$$(3.34) \quad G^\flat(p, \zeta) = \mu(q)\text{Id} + \tilde{G}(q, \tilde{\zeta}), \quad \text{with} \quad \tilde{G}(q, 0) = 0.$$  

To blow up the singularity near $\mathcal{M}$, we introduce the matrices $\tilde{G}(q, \rho, \tilde{\zeta})$ and $\tilde{A}(q, \rho, \tilde{\tau}, \tilde{\xi})$ such that for $\rho \tilde{\zeta}$ and $\rho \tilde{\xi}$ small enough:

$$(3.35) \quad \tilde{G}(q, \rho \tilde{\zeta}) = \rho \tilde{G}(q, \rho \tilde{\zeta}), \quad \tilde{A}(q, \rho \tilde{\eta}, \rho \tilde{\xi}) = \rho \tilde{A}(q, \rho \tilde{\eta}, \rho \tilde{\xi}).$$
Proposition 3.10. Suppose that \((p, \tau, \eta, \xi)\) is a real root of \(\Delta\), of multiplicity \(m\), linearly splitting transversally to the manifold \(\mathcal{M}\), nonglancing and satisfying \((3.28)\). Then, with the notations as above, for \(q\) in a neighborhood of \(p\) and \(\rho\.\xi\) small,

\[
\text{(3.36)} \quad \det (\xi \text{Id} + \tilde{G}(q, \rho, \tilde{\gamma})) = e(q, \rho\.\xi, \rho\.\xi) \det ((\tilde{\tau} - i\tilde{\gamma}) \text{Id} + \tilde{A}(q, \rho, \tilde{\eta}, \tilde{\xi})),
\]

\[
\text{(3.37)} \quad \ker ((\tilde{\tau}u - i\tilde{\gamma}) \text{Id} + \tilde{A}(q, \rho, \tilde{\eta}, \tilde{\xi})) = E(q, \rho\.\xi, \rho\.\xi) \ker (\xi \text{Id} + \tilde{G}(q, \rho, \tilde{\xi})),
\]

with \(e(q, 0, 0) \neq 0\), and \(E(q, 0, 0) = \text{Id}\). Moreover, the polynomial in \(\xi\), \(\det (\xi \text{Id} + \tilde{G}(q, \rho, \tilde{\xi}))\), has real coefficients when \(\tilde{\gamma} = 0\), and \(\tilde{A}(q, \rho, \tilde{\eta}, \tilde{\xi})\) has only simple real eigenvalues when \(\tilde{\xi}\) is real and \((\tilde{\eta}, \tilde{\xi}) \neq 0\).

Proof. The identities \((3.36)\) and \((3.37)\) follow directly from Lemma \(3.2\) and \((3.28)\), through the change of variables \((3.32)\). Similarly, that \(\det (\xi \text{Id} + \tilde{G}(q, \rho, \tilde{\xi}))\) is real when \(\tilde{\gamma} = 0\) follows from Lemma \(3.2\).

Since the mode is linearly splitting, the condition \((2.28)\) implies that \(\tilde{A}(q, 0, \tilde{\eta}, \tilde{\xi})\) has only real and simple eigenvalues when \(|(\tilde{\eta}, \tilde{\xi})| = 1\) and \(\tilde{\xi}\) is real. By continuity this extends to \(\tilde{A}(q, \rho, \tilde{\eta}, \tilde{\xi})\) for \(q\) close to \(p\) and \(\rho\) small and \(|(\tilde{\eta}, \tilde{\xi})| = 1\). Since

\[\tilde{A}(q, \rho, \tilde{\eta}, \tilde{\xi}) = \frac{1}{\rho} \tilde{A}(q, \rho \tilde{\eta}, \rho \tilde{\xi}) = |(\tilde{\eta}, \tilde{\xi})| \tilde{A}(q, \rho |(\tilde{\eta}, \tilde{\xi})|, (\tilde{\eta}, \tilde{\xi})/|(\tilde{\eta}, \tilde{\xi})|),\]

\(\tilde{A}(q, \rho, \tilde{\eta}, \tilde{\xi})\) has only simple real eigenvalues when \(\tilde{\xi}\) is real and \(\rho |(\tilde{\eta}, \tilde{\xi})|\) is small enough.

Next, we consider the case where

\[
\text{(3.38)} \quad \text{dx} \in T_{\tilde{\eta}, \tilde{\xi}} \widetilde{\mathcal{M}}_{\rho}.\]

Equivalently, this means that

\[
\text{(3.39)} \quad \tilde{A}_{\text{d}} = a \text{Id}.
\]

The nonglancing hypothesis implies that \(a \neq 0\). In particular, the mode is totally incoming when \(a > 0\) and totally outgoing when \(a < 0\).

As in Remark \(2.13\) introduce the manifold \(\mathcal{M}\) of points \((p, \tau, \eta, \xi)\) with \((p, \eta, \xi) \in \mathcal{M}\) and \(\tau = \lambda(p, \eta, \xi)\). Since \(L'(\tilde{\tau}, \tilde{\eta}, \tilde{\xi})\) vanishes when \((\tilde{\tau}, \tilde{\eta}, \tilde{\xi}) \in T_{\tilde{\eta}, \tilde{\xi}} \widetilde{\mathcal{M}}_{\rho}\), the nonglancing hypothesis implies that \(\tilde{A}_{\text{d}} = L'(\text{dx}) \neq 0\), thus

\[
\text{(3.40)} \quad \text{dx} \notin T_{\tilde{\eta}, \tilde{\xi}} \widetilde{\mathcal{M}}_{\rho}.
\]
where \( dx \) is considered in the now considered as the space-time conormal to the boundary. However, \((3.39)\) implies that \( L'(dt + adx) = 0 \). By \((2.18)\), if \( \theta \) is not tangent to \( \mathcal{M}_\mathcal{p} \), the dimension of \( \ker L'(\theta) \) is at most 1, thus \( dt + adx \) must be tangent to \( \mathcal{M}_\mathcal{p} \). This shows that, in contrast with the previous case, we cannot take \( \tau \) and \( \xi \) as independent variables transversal to \( \mathcal{M} \). However, \((3.40)\) implies that there are coordinates \((\eta_1, \ldots, \eta_{d-1})\) in \( \mathbb{R}^{d-1} \) such that \( \mathcal{M} \) is parametrized by \((q, \tau, \eta)\), with \( q = (p, \eta_1, \ldots, \eta_{d-\nu-1}) \), and locally given by

\[
(3.41) \quad \eta'' = \phi(q, \tau), \quad \xi = -\mu(q, \tau),
\]

with \( \eta'' = (\eta_{d-\nu+1}, \ldots, \eta_{d-1}) \) and \( \phi, \mu \) smooth, analytic in \( \tau \). Denote by \( \mathcal{M}_b \) the manifold \( \{\eta'' = \phi(q, \tau)\} \).

Proposition \(3.9\) implies that \( -\xi \) is a semi-simple eigenvalue with multiplicity \( m \) of \( G(p, \tau, \eta, 0) \) if \((p, \tau, \eta, \xi) \in \mathcal{M} \). Thus,

\[
(3.42) \quad G^\flat(p, \tau, \eta, 0) = \mu(q, \tau)\text{Id} + \tilde{G}(q, \tau, \tilde{\eta}) \quad \text{with} \quad \tilde{\eta} = \eta'' - \phi(q, \tau)
\]

and \( \tilde{G}(q, \tau, 0) = 0 \). Thus, there are matrices \( \tilde{G}(q, \rho, \tau, \tilde{\eta}) \) such that for \( \rho \tilde{\eta} \) small enough:

\[
(3.43) \quad \tilde{G}(q, \tau, \rho\tilde{\eta}) = \rho\tilde{G}(q, \rho, \tau, \tilde{\eta}).
\]

**Proposition 3.11.** Suppose that \((p, \tau, \eta, \xi)\) is a real root of \( \Delta \), of multiplicity \( m \), linearly splitting transversally to the manifold \( \mathcal{M} \), nonglancing and satisfying \((3.28)\). Then, with the notations as above, for \((q, \tau)\) in a neighborhood of \((q, \tau)\) and \( \rho \tilde{\eta} \) small, \( \tilde{G}(q, \rho, \tau, \tilde{\eta}) \) has only real and simple eigenvalues when \( \tilde{\eta} \neq 0 \). Moreover, \( \partial_\gamma G^\flat(p, \zeta) = -ia^{-1}\text{Id} \)

**Proof.** By \((3.39)\), we are in a totally nonglancing framework, implying that \( \tilde{\xi} + G'(p, \tau, \eta, 0) \) is hyperbolic in the direction \( dx \). Taking \( (\tilde{\eta}, \xi) \) as variables transversal to \( \mathcal{M} \), \((2.14)\), implies that \( \tilde{\xi} + G'(p, \tau, \eta', 0) = \tilde{\xi} + \tilde{G}(q, 0, \tau, \tilde{\eta}) \) is strictly hyperbolic in the direction \( dx \). This shows that \( \tilde{G}(q, 0, \tau, \tilde{\eta}) \) has only real and simple eigenvalues when \( \tilde{\eta} \neq 0 \). This is preserved for \((q, \tau)\) close to \((q, \tau)\), \( \tilde{\eta} \neq 0 \) and \( \rho \) such that \( |\rho\tilde{\eta}| \) is small enough.

By \((3.19)\),

\[
\partial_\gamma G^\flat(p, \zeta) = -i\partial_\zeta G^\flat(p, \zeta) = -iA^{-1}_d = -ia^{-1}\text{Id}.
\]

\[\square\]
4 The Lopatinski condition and symmetrizers

4.1 Maximal estimates

Consider a system
\[ L(p, \partial_t, \partial_y, \partial_x) = A_d(p)(\partial_x + G(p, \partial_t, \partial_y)), \]
hyperbolic in the direction \( dt \) such that the boundary \( \{x = 0\} \) is not characteristic. The classical plane wave analysis yields the equations

\[ \partial_x u + iG(p, \zeta)u = f, \]

with \( \zeta = (\tau - i\gamma, \eta) \), identified with \( (\tau, \eta, \gamma) \). We supplement the operator \( L \) with boundary conditions, which read, after Laplace-Fourier transform,

\[ M(p, \zeta)u_{|x=0} = g. \]

We assume that \( M \) depends smoothly on \( p \) and \( \zeta \in \mathbb{R}^{d+1}\setminus\{0\} \) and is homogeneous of degree 0 in \( \zeta \). We allow \( M \) to depend on \( \zeta \) for two reasons: first, it is so in the shock problem and second, the block reduction process leads us to consider such cases. By homogeneity, we can restrict attention to \( \zeta \in S_d^+ = \{||\zeta|| = 1; \gamma > 0\} \).

Recall that the hyperbolicity condition implies that for \( \zeta \in S_d^+ \), the eigenvalues of \( G(p, \zeta) \) are nonreal. The \( L^2 \) solutions of \( (\partial_x + iG)u = 0 \) consist of all \( u(x) = e^{-ixG}h \) with \( h \in E_-(p, \zeta) \), where \( E_-(p, \zeta) \) denotes the invariant space of \( G(p, \zeta) \) associated to the spectrum in \( \{\text{Im} \mu < 0\} \). Recall that

\[ \dim E_-(p, \zeta) = N_+ \]

is the number of positive eigenvalues of \( A_d \). We assume that we have the correct number of boundary conditions, that is

\[ \dim \ker M(p, \zeta) = N_- = N - N_+, \]

the number of negative eigenvalues of \( A_d \). For simplicity, we also assume that \( M \) is surjective, that is that \( M \) is an \( N_+ \times N \) matrix.

Following Kreiss and Majda, we are interested in the maximal estimates for solutions of (4.1) (4.2):

\[ \gamma \|u\|_{L^2(\mathbb{R}_+)^2} + |u(0)|^2 \lesssim \frac{1}{\gamma} \|f\|_{L^2(\mathbb{R}_+)}^2 + |g|^2, \]

where \( \lesssim \) means that the left hand side is smaller than \( C \) times the right hand side where \( C \) is independent of \( u, f, g \) and \( \zeta \in S_d^+ \). A necessary condition is that

\[ |u(0)| \lesssim |M(p, \zeta)u(0)|. \]
for all \( u \in L^2 \) such that \( \partial_x u + iGu = 0 \). Thus, a necessary condition for (4.5) is that for all \( \zeta \in S^d_+ \) there holds

(4.6) \[ \forall h \in \mathbb{E}_- (p, \zeta), \quad |h| \leq C |M(p, \zeta)h|. \]

**Definition 4.1.** The Lopatinski determinant associated with \( L, M \) is defined for \( \gamma > 0 \) as

(4.7) \[ D(p, \zeta) = \det \left( \mathbb{E}_- (p, \zeta), \ker M(p, \zeta) \right). \]

We say that \( L, M \) satisfy the uniform Lopatinski condition on \( \omega \) if there is a constant \( c > 0 \) such that

(4.8) \[ \forall (p, \zeta) \in \omega \times S^d_+: \quad |D(p, \zeta)| \geq c \]

In (4.7), the determinant is obtained by taking orthonormal bases in each space \( \mathbb{E}_- \) and \( \ker M \) (note that \( \dim \mathbb{E}_- + \dim \ker M = N \)) and independent of this choice. It depends on the scalar product used in \( \mathbb{C}^N \) to form orthonormal bases, but the condition (4.8) is independent of this scalar product. Denoting by \( \pi M\perp \) the orthogonal projection on \( \ker M\perp \), then \( D(p, \zeta) \) is the determinant of the restriction of \( \pi M\perp \) to \( \mathbb{E}_- \) (in orthonormal bases). Thus, using that \( M \) is an isomorphism from \( \ker M\perp \) to \( \mathbb{C}^N \), uniformly bounded with uniformly bounded inverse for \( \zeta \in S^d_+ \), we see that the condition

(4.9) \[ |D(p, \zeta)| \geq c, \]

implies (4.6) with \( C \) depending only on \( c \). Conversely, if (4.6) holds, then (4.9) is satisfied for some \( c > 0 \) which depends only on \( C \).

This shows that the uniform Lopatinski condition is necessary for (4.5), see [Kr]. Conversely, there holds:

**Theorem 4.2 (First Main Theorem).** Suppose that \( L \) is symmetric and that the roots of the characteristic equations are either geometrically regular or totally nonglancing, or nonglancing and linearly splitting transversally to smooth manifolds. Then, the uniform Lopatinski condition at \( p \) implies the maximal estimate (4.5) for \( p \) in a neighborhood of \( p \).

This was established by Kreiss ([Kr]; see also [Ral, ChP]) in the strictly hyperbolic case by an ingenious construction of symmetrizers. It was extended to systems with block structure by Majda and Osher [Maj, MaOs]. By Theorem 3.3 geometric regularity implies block structure. In the totally nonglancing case, we extend Kreiss’s construction of smooth symmetrizers...
using the symmetry of $L$ (Theorem 5.6). In the presence of nonglancing linearly splitting modes, we extend further the construction of smooth symmetrizers when there is at most one linearly splitting mode (for each $\zeta$) corresponding to a double root (Theorem 6.19). In the general case of nonglancing linearly splitting modes, we prove Theorem 4.2 using nonsmooth symmetrizers. For applications to nonconstant coefficients and nonlinear equations, the use of nonsmooth symmetrizers seems very delicate. In this respect, Theorems 5.6 and 6.19 below are more important than Theorem 4.2.

Remark 4.3. When the vector bundle $\mathbb{E}_-(p, \zeta)$ has a continuous extension to $\omega \times \overline{S}_+^d$, for some neighborhood $\omega$ of $\underline{p}$, then $D$ extends as a continuous function on $\omega \times \overline{S}_+^d$, and the uniform Lopatinski condition holds, on a possibly smaller neighborhood of $\underline{p}$, if and only if

$$\forall \zeta \in \overline{S}_+^d, \quad D(p, \zeta) \neq 0,$$

or, equivalently, if and only if

$$\forall \zeta \in \overline{S}_+^d, \quad \mathbb{E}_-(p, \zeta) \cap \ker M(p, \zeta) = \{0\}.$$

The continuous extendability of $\mathbb{E}_-$ to boundary values $\gamma = 0$, is true for strictly hyperbolic systems ([Kr]), or when the block structure condition is satisfied ([Maj]), or, more generally, when the nongeometrically regular modes are totally nonglancing (see Theorem 5.6). In the other cases, one can introduce the set of limits of sequences of vectors $\mathbb{E}_-(p_n, \zeta_n)$ as $(p_n, \zeta_n)$ tends to $(p, \zeta)$ and $\gamma_n > 0$:

$$\hat{\mathbb{E}}_-(p, \zeta) = \limsup_{(p', \zeta') \to (p, \zeta), \gamma' > 0} \mathbb{E}_-(p', \zeta').$$

This is a closed cone in $\mathbb{C}^N$, equal to $\mathbb{E}_-(p, \zeta)$ if $\gamma > 0$ and equal to the continuous extension of $\mathbb{E}_-$ when such an extension exists. If the uniform estimate (4.6) holds for $p$ close to $\underline{p}$ and $\zeta \in S_+^d$, then

$$\forall \zeta \in \overline{S}_+^d, \forall h \in \hat{\mathbb{E}}_-(p, \zeta), \quad |h| \leq C |M(p, \zeta)h|$$

Conversely, this implies (4.6) on a neighborhood of $\underline{p}$, with a possibly larger constant $C$. By homogeneity, it is sufficient to check (4.13) for $h$ in the unit sphere, and by compactness this shows that this condition is equivalent to the following analogue of (4.11):

$$\forall \zeta \in \overline{S}_+^d, \quad \hat{\mathbb{E}}_-(p, \zeta) \cap \ker M(p, \zeta) = \{0\}.$$
4.2 Symmetrizers

We recall first the method of symmetrizers. We fix a neighborhood $\omega$ of $p$.

**Definition 4.4.** A bounded symmetrizer for $G(p, \zeta)$ on $\Omega = \omega \times U$, $U \subset S^d_+$, is a smooth matrix $\Sigma(p, \zeta)$ on $\Omega$, such that, for some $C, c > 0$, there holds for all $(p, \zeta) \in \Omega$,

\begin{align}
(4.15) & \quad \Sigma(p, \zeta) = \Sigma^*(p, \zeta), \\
(4.16) & \quad |\Sigma(p, \zeta)| \leq C, \\
(4.17) & \quad \text{Im} \Sigma(p, \zeta)G(p, \zeta) \geq c\gamma \text{Id},
\end{align}

It is a Kreiss symmetrizer for $(G, M)$ if in addition

\begin{equation}
(4.18) \quad \Sigma(p, \zeta) + CM^*(p, \zeta)M(p, \zeta) \geq c \text{Id}.
\end{equation}

The symmetrizer is smooth, if it extends smoothly to $\omega \times \overline{U} \subset \omega \times S^d_+$.

Taking the scalar product of the equation (4.1) with $\Sigma u$ and integrating by parts, immediately implies the following.

**Lemma 4.5.** If there exists a bounded Kreiss symmetrizer for $(L, M)$ on $\omega \times S^d_+$ then the maximal estimate (4.15) holds for all $(p, \zeta) \in \omega \times S^d_+$.

**Remark 4.6.** (4.18) implies that (4.19) $\Sigma(p, \zeta) \geq c \text{Id}$ on $\ker M(p, \zeta)$.

Conversely, if this inequality holds, then

\begin{equation}
(\Sigma h, h) + C_1|\pi_{\ker M^\bot} h|^2 \geq C_2|h|^2
\end{equation}

if $C_1 \geq C + c/2 + 2C^2/c$ and $C \geq |\Sigma|$ as in (4.16). With $C_2$ such that

\begin{equation}
|\pi_{\ker M^\bot} h| \leq C_2 \|Mh\|,
\end{equation}

this shows that (4.19) implies (4.18) with $c$ replaced by $c/2$ and $C$ by $C_1 C_2^2$. This shows that one can replace the condition (4.18) by (4.19).

**Remark 4.7.** If $S(p)$ is a symmetrizer for $L$, that is $S(p) = S^*(p)$ is positive definite and all the $S(p)A_j(p)$ are self adjoint, then $\Sigma(p) = -S(p)A_d(p)$ is a symmetrizer for $G(p, \zeta)$ : the properties (4.15), (4.16) and (4.17) are satisfied. When the property (4.18) or equivalently (4.19) holds, the boundary conditions are said to be strictly dissipative. In this case, the maximal estimates (4.15) are satisfied. For the theory of dissipative boundary value problems, we refer to [2, 3, 12, 14].
We recall the following general fact about symmetrizers.

**Lemma 4.8.** Let $\Sigma$ be a symmetrizer for $G$. Then, for $\gamma > 0$, $\Sigma(p, \zeta)$ is negative definite on the stable subspace $E_-(p, \zeta)$ of $G(p, \zeta)$.

**Proof.** Let $v$ satisfy $\partial_x v + iGv = 0$, $v(0) = u \in E_-$, $u \neq 0$. Then, $v \to 0$ exponentially fast as $x \to +\infty$. On the other hand, using symmetry of $\Sigma$, we have

\[
\frac{d}{dx}(\Sigma v, v) = 2\text{Re} \langle \Sigma \partial_x v, v \rangle = -2\text{Re} \langle iGv, v \rangle.
\]

Since $\langle v, \Sigma v \rangle \to 0$ and is integrable as $x \to +\infty$, we must have, therefore

\[
\langle u, \Sigma u \rangle \leq -c\gamma \|v\|_L^2 < 0.
\]

as claimed. \qed

In the analysis of [Kr, Maj], there is an intermediate step in the construction of symmetrizers: one construct first a family of symmetrizers $\Sigma^\kappa$, which satisfy (4.15) (4.16) and (4.17), such that the negative cone of $S^\kappa$ is an arbitrarily small conic neighborhood of $E_-$. Next, one uses the uniform Lopatinski condition: because $\ker M$ does not intersect $E_-$, it is contained in the positive cone of $\Sigma^\kappa$ for $f \kappa$ large enough implying (4.18).

**Definition 4.9.** Consider a family $\Sigma^\kappa$ of bounded [resp. smooth] symmetrizers for $G$ on $\Omega^\kappa = \omega^\kappa \times U^\kappa$. It is a K-family of bounded [resp. smooth] symmetrizers, if there are $C$ and projectors $\Pi_-(p, \zeta)$ on $E_-(p, \zeta)$ such that

\[
\forall \kappa \geq 1, \forall (p, \zeta) \in \Omega^\kappa, \quad |\Pi_-(p, \zeta)| \leq C,
\]

\[
\forall \kappa \geq 1, \forall (p, \zeta) \in \Omega^\kappa, \quad \Sigma^\kappa(p, \zeta) \geq m(\kappa)\Pi_+^* \Pi_+ - \Pi_-^* \Pi_-.
\]

where $\Pi_+ = \text{Id} - \Pi_-$ and $m(\kappa) \to +\infty$ as $\kappa \to +\infty$.

**Proposition 4.10.** Suppose that $\Sigma^\kappa$ is a K-family of symmetrizers for $G$ on $\Omega^\kappa$. Suppose that $M$ satisfies the uniform Lopatinski condition. Then, for $\kappa$ large enough, $\Sigma^\kappa$ is a Kreiss symmetrizer on $\Omega^\kappa$ for $(G, M)$.

**Proof.** The Lopatinski condition implies that there is a constant $C$ such that (4.16) holds. Therefore,

\[
|\Pi_- h| \leq C|Mh| + C|M||\Pi_+ h|.
\]

Thus, there are $C_1$ and $C_2$ such that

\[
|h|^2 \leq 2|\Pi_- h|^2 + 2|\Pi_+ h|^2 \leq C_1|Mh|^2 + C_2|\Pi_+ h|^2 - |\Pi_- h|^2.
\]

Therefore, if $m(\kappa) \geq C_2$, (4.18) follows. \qed
Remark 4.11. The choice of the projector $\Pi_-$ is irrelevant, as long as the uniform bound holds. If $\Pi_-$ is another projector on $E_-$ satisfying with a constant $\tilde{C}$, then $\tilde{\Pi}_+\Pi_- = 0$ and

$$|	ilde{\Pi}_+ h| = |	ilde{\Pi}_+\Pi_- h| \leq \tilde{C}|\Pi_- h|,$$
$$|\Pi_- h| \leq C(|\tilde{\Pi}_+ h| + |\Pi_- h|).$$

Thus,

$$m(\kappa)|\Pi_+ h|^2 - |\Pi_- h|^2 \geq \left( \frac{m(\kappa)}{\tilde{C}^2} - 2C^2 \right) |\tilde{\Pi}_+ h|^2 - 2C^2 |\Pi_- h|^2.$$

Therefore, changing $\Sigma^\kappa$ to $C^{-2}\Sigma^\kappa$ we see that for $\Sigma^\kappa$ and $\Pi_\pm$ implies similar estimates for $\tilde{\Sigma}^\kappa$ and $\tilde{\Pi}_\pm$, with $m(\kappa) = m(\kappa)/C^2\tilde{C}^2 - 2$. In particular, we can always choose $\Pi_- (p, \zeta)$ to be the orthogonal projector on $E_-(p, \zeta)$.

4.3 Localization and block reduction

We collect here several remarks concerning the construction of symmetrizers. First, the construction is local.

Lemma 4.12. Suppose that for all $(p, \zeta) \in \bar{\omega} \times S^d_+$, there are neighborhoods $\Omega^\kappa$ of $(p, \zeta)$ and self-adjoint matrices $\Sigma^\kappa (p, \zeta)$ for $(p, \zeta) \in \Omega^\kappa_+ = \Omega^\kappa \cap \{ \gamma > 0 \}$ such that the $\Sigma^\kappa$ form a $K$-family of symmetrizers for $G$ on $\Omega^\kappa_+$. Then there exists neighborhoods $\omega^\kappa$ of $\pi$ and $K$-families of symmetrizers $\tilde{\Sigma}^\kappa$ on $\omega^\kappa \times S^d_+$.

If the local $\Sigma^\kappa$ are smooth, then the global $\tilde{\Sigma}^\kappa$ can be chosen smooth.

Proof. By Remark we can assume that the local symmetrizers $\Sigma^\kappa (p, \zeta)$ on $\Omega^\kappa_+$ satisfy with $\Pi_- = 0$ and the orthogonal projector on $E_-$. Relabeling the families, we can also assume that they satisfy with $m(\kappa) = \kappa$.

For all $\kappa$, we can find a finite covering of $\bar{\omega} \times S^d_+$ by open sets $\Omega^\kappa_j$, such that a $K$-family of symmetrizers $\Sigma^\kappa_j$ exists on $\Omega^\kappa_j$. Consider a a partition of unity $1 = \sum_j \chi^\kappa_j$ on $\bar{\omega} \times S^d_+$, with $\chi^\kappa_j$ supported in $\Omega^\kappa_j$. Let

$$\Sigma^\kappa (p, \zeta) = \sum_j \chi_j (\zeta) \Sigma^\kappa_j (p, \zeta).$$

Because the covering is finite, we can choose uniform constants $C$ and $c$ in and the lemma follows.

There is an analogous result for Kreiss symmetrizers.
Lemma 4.13. Suppose that for all \((p, \zeta) \in \mathcal{W} \times \mathbb{S}_+^d\), there is a neighborhood \(\Omega\) of \((p, \zeta)\) and a bounded [resp. smooth] Kreiss symmetrizer for \((G, M)\) on \(\Omega_+ = \mathcal{W}^\kappa \cap \{\gamma > 0\}\). Then there is a neighborhood \(\omega'\) of \(\mathcal{W}\) and a bounded [resp. smooth] symmetrizer on \(\omega' \times \mathbb{S}_+^d\).

Next, we consider a smooth diagonal block reduction of \(G\) on a neighborhood \(\Omega\) of \((p, \zeta)\):

\[
U^{-1}(p, \zeta)G(p, \zeta)U(p, \zeta) = \text{block diag}(G_k(z, \zeta)).
\]

(4.26)

For instance, we can consider the distinct eigenvalues \(\mu_k, k \in \{1, \ldots, k\}\), of \(G(z, \zeta)\), small discs \(D_k\) centered at \(\mu_k\) that do not intersect each other, and for \((p, \zeta)\) close to \((p, \zeta)\), the diagonal block reduction where the spectrum of \(G_k\) is contained in \(D_k\).

Equivalently, taking appropriate blocks in \(U\) and \(U^{-1}\), one can introduce smooth matrices \(V_k\) and \(W_k\) such that

\[
G = \sum V_k G_k W_k, \quad W_k V_j = \delta_{j,k} \text{Id}.
\]

(4.27)

For \(\gamma > 0\), we denote by \(E_k\) the invariant subspace of \(G_k\) associated to eigenvalues in \(\{\text{Im } \mu < 0\}\). Thus,

\[
E_{\kappa}(p, \zeta) = \bigoplus_k V_k(p, \zeta)E_k,\kappa(p, \zeta).
\]

(4.28)

Given symmetrizers \(\Sigma_k\) for \(G_k\) on \(\Omega\),

\[
\Sigma = U^{-1}\text{diag}(\Sigma_k)U^{-1} = \sum W_k^* \Sigma_k W_k.
\]

is a symmetrizer for \(G\).

Lemma 4.14. Suppose that for all \(k\), \(\Sigma_k^\kappa\) is a \(K\)-family of symmetrizers for \(G_k\) on \(\Omega^\kappa\), then there are \(K\)-families of symmetrizers \(\Sigma^\kappa\) for \(G\) on \(\Omega^\kappa\).

If the \(\Sigma_k^\kappa\) are smooth, then \(\Sigma^\kappa\) can be chosen smooth.

Proof. Consider the symmetrizers \(\Sigma_k^\kappa\) associated by (4.29) to the \(\Sigma_k^\kappa\). By Remark 4.11 we can assume that the \(\Sigma_k^\kappa\) satisfy (4.24) with \(m(\kappa) = \kappa\) and \(\Pi_{k,-}\) equal to the orthogonal projection on \(E_{k,-}\). Therefore, for \(h = \sum V_k h_k\), there holds

\[
(\Sigma^\kappa h, h) = \sum (\Sigma_k^\kappa h_k, h_k) \geq \kappa \sum |\Pi_{k,+} h_k|^2 - \sum |\Pi_{k,-} h_k|^2.
\]

Let \(\Pi_- = \sum V_k \Pi_{k,-} W_k\). It is a projector on \(E_-\) and

\[
|\Pi_+ h|^2 \leq |U|^2 \sum |\Pi_{k,+} h_k|^2, \quad \sum |\Pi_{k,-} h_k|^2 \leq |U^{-1}|^2 |\Pi_- h|^2.
\]

Therefore, \(\tilde{\Sigma}^\kappa = |V^{-1}|^{-2} \Sigma^\kappa\) satisfies (4.24) with \(m(\kappa) = \kappa/|U|^2|U^{-1}|^2\).
There are no simple analogue of Lemma 4.14 for Kreiss symmetrizers, since the boundary conditions \( M \) do not split, in general, into boundary independent conditions \( M_k \) for each block \( G_k \). However, we give in Section 6 an interesting result in this direction.

We end this section with the following elementary remark.

**Remark 4.15.** If \( \Sigma \) is a symmetrizer for \( G \), then \( \Sigma_k = V_k^* \Sigma V_k \) is a symmetrizer for \( G_k \), since \( \Sigma_k G_k = V_k^* \Sigma G V_k \). In particular, if the original system \( L \) is hyperbolic symmetric, with symmetrizer \( S(p) \), then \( \Sigma(p) = -S(p)A_d(p) \) is a symmetrizer for \( G \) and \( \Sigma_k(p, \zeta) = -V_k^*(p, \zeta)S(p)A_d(p)V_k(p, \zeta) \) is a smooth symmetrizer for \( G_k(p, \zeta) \).

## 5 Proof of the maximal estimates

In this section we prove Main Theorem 4.2. Because of its special interest for MHD, we also point out in Theorem 5.6 below, the particular case where only geometrically regular and totally nonglancing modes are present. In this case, smooth Kreiss symmetrizers are available, while they are not in the more general situation of Theorem 4.2. The construction of smooth symmetrizers is studied in more details in the next section.

We fix \( \zeta \in S^d \). With \( \mu_k \) denoting the distinct eigenvalues of \( G(p, \zeta) \), we consider on a neighborhood \( \Omega \) of \( (p, \zeta) \), the block reduction (4.26) of \( G \) such that the spectrum of \( G_k(p, \zeta) \) is \( \{\mu_k\} \). In this section, give different constructions of K-families of symmetrizers, depending on the properties of the blocks \( G_k \). Be denote by \( N_k \) the dimension of the block \( G_k \), that is the algebraic multiplicity of \( \mu_k \). When \( \gamma > 0 \), \( G_k(p, \zeta) \) has no real eigenvalue, we denote by \( E_{k,-}(p, \zeta) \) the invariant space of \( G_k \) associated to eigenvalues in \( \{\text{Im} \mu < 0\} \) and, in accordance with (4.3), we set

\[
(5.1) \quad \dim E_{k,-}(p, \zeta) = N_{k,+} \quad \gamma > 0.
\]

### 5.1 Kreiss’ Theorem
We first recall Kreiss’ construction with Majda’s extension.

**Proposition 5.1.** Suppose that \( G_k \) satisfies the block structure condition at \((p, \zeta)\). There are spaces \( E_{k,+} \) and \( E_{k,-} \), neighborhoods \( \Omega^\kappa \) of \((p, \zeta)\) and a smooth family of symmetrizers \( \Sigma_k^\kappa(p, \zeta) \) on \( \Omega^\kappa_+ = \Omega^\kappa \cap \{\gamma \geq 0\} \) such that:

\[
(5.2) \quad C_m^k = E_{k,+} \oplus E_{k,-}, \quad \dim E_{k,-} = N_{k,+},
\]

\[
(5.3) \quad \Sigma_k^\kappa(p, \zeta) \geq \kappa \Pi_{k,+} \Pi_{k,+} - \Pi_{k,-} \Pi_{k,-}.
\]
where $\Pi_\pm$ is the projector on $E_\pm$ in the decomposition (5.2).

The block structure condition covers two cases:

1) **Elliptic modes.** If $\mu_k$ is not real, then $G_k$ satisfies the block structure condition. Then $E_{k,-} = E_k$ [resp. $E_{k,-} = \{0\}$] if $\text{Im} \mu_k < 0$ [resp. $\text{Im} \mu_k > 0$]; there is a symmetrizer $\Sigma$ such that

\[
\text{Im} \Sigma_k G_k \geq c > 0
\]

and $\Sigma$ is definite negative [resp. positive] when $\text{Im} \mu_k < 0$ [resp. $\text{Im} \mu_k > 0$]; one chooses $\Sigma^\kappa = \Sigma$ [resp. $\Sigma^\kappa = \kappa \Sigma$].

Elliptic modes are the only possibility when $\gamma \neq 0$.

2) **Geometrically regular modes.** If $\gamma = 0$ and $(p, \tau, \eta, -\mu_k)$ is geometrically regular then, by Theorem 3.4, $G_k$ satisfies the block structure condition. The symmetrizers $\Sigma^\kappa_k$ constructed in [Kr] (see also [ChP]) satisfy

\[
\text{Im} (\Sigma^\kappa_k G_k) = \gamma E_k^\kappa, \quad E_k \geq \kappa \text{Id},
\]

which implies (4.17). This improvement is useful for applications to variable coefficients: it allows one to use standard Garding’s inequalities for the operator with symbol $E_k$ (see [ChP, Mok, Mé2, MéZ1]).

When the block structure condition is satisfied, the spaces $E_{k,-}(p, \zeta)$ have limits as $\gamma \to 0$ (see [Kr, ChP]). It is trivial for elliptic modes. For geometrically regular modes, it is proved in [MéZ2] that the existence of symmetrizers as in Proposition 5.1 implies that

\[
E_- = \lim_{(p, \zeta) \to (p, \zeta') \gamma \to 0} E_-(p, \zeta)
\]

From its definition, geometric regularity is a local property, and remains satisfied in a neighborhood of the given point. Therefore, for $(p, \zeta)$ in a neighborhood of $(p, \zeta)$ with $\gamma \geq 0$, the $E_{k,-}(p', \zeta')$ have limits $E_{k,-}(p, \zeta)$ as $(p', \zeta') \to (p, \zeta)$ with $\gamma' > 0$. Arguing as in [MéZ2], this implies that the vector bundle $E_{k,-}$ has a continuous extension to $\gamma = 0$ on a neighborhood of $(p, \zeta)$.

Therefore, with notations as in Proposition 5.1 there is a smooth splitting

\[
\mathbb{C}^N_k = E_+ \oplus E_{k,-}(p, \zeta)
\]
for \((p, \zeta)\) close to \((\bar{p}, \bar{\zeta})\). With projectors associated to this decompositions, by continuity, there holds locally

\[
\Sigma_k^\zeta(p, \zeta) \geq \frac{\kappa}{2} \Pi^*_k, + \Pi^*_k, - 2 \Pi^*_k, - \Pi^*_k, - .
\]

This shows \(\Sigma_k^\zeta(p, \zeta)\) is a smooth K-family of symmetrizers for \(G_k\). Summing up, we have proved:

**Corollary 5.2.** Suppose that \(G_k\) satisfies the block structure condition at \((p, \zeta)\). Then,

i) there is a neighborhood \(\Omega\) of \((p, \zeta)\) such that the vector bundle \(E^k, -\) has a continous extension to \(\Omega \cap \{\gamma \geq 0\}\);

ii) there are K-families of smooth symmetrizers \(\Sigma_k^\zeta(p, \zeta)\) near \((p, \zeta)\).

### 5.2 Totally nonglancing modes

It remains to consider the case where \(\gamma = 0\) and the blocks \(G_k\) are associated with non-geometrically regular real roots of the characteristic equations. We first consider totally nonglancing roots.

**Proposition 5.3.** ii) If \((p, \tau, \eta, \xi)\) is totally incoming, then, for \((p, \zeta)\) in a neighborhood of \((p, \zeta)\), \(E^-(p, \zeta) = E(p, \zeta)\)

iii) If \((p, \tau, \eta, \xi)\) is totally outgoing, then, for \((p, \zeta)\) in a neighborhood of \((p, \zeta)\), \(E^-(p, \zeta) = \{0\}\).

**Proof.** We omit the parameter \(p\) in the notation below. We have to show that when \(\gamma > 0\), \(\Delta(\tau - i\gamma, \eta, \xi)\) has \(m\) roots in \(\xi\) close to \(\bar{\xi}\) in the half space \(\{\text{Im} \xi > 0\}\) [resp., in \(\{\text{Im} \xi > 0\}\) in the totally incoming [resp., outgoing] case. Since \(\Delta\) has no real roots when \(\gamma > 0\), the number of these roots is constant for \(\zeta \) close to \(\bar{\zeta}\) and \(\gamma > 0\), and it is sufficient to count them when \(\tau = \bar{\tau}\) and \(\eta = \eta\).

By (3.21), there holds

\[
\Delta^{(m)}(\tau, 0, \xi) = \sum_{j=0}^{m} a_{m-j} \tau^j \xi^{m-j} = a_0 \prod_{j=1}^{m} (\tau + c_j \xi),
\]

where \(a_0 \neq 0\) and the second equality is the definition of the coefficients \(c_j\). The hyperbolicity of \(\Delta^{(m)}\) implies that all the \(c_j\) are real and the nonglancing condition implies that the \(c_j\) do not vanish. The roots of \(\Delta^{(m)}(-i\gamma, 0, \xi) = 0\) are \(\xi = i\gamma/c_j\). Since

\[
\Delta(\bar{\tau} - i\gamma, 0, \bar{\xi} + \gamma \bar{\xi}) = e(0, 0, 0) \gamma^m \left(\Delta^{(m)}(-i, 0, \bar{\xi}) + O(\gamma)\right),
\]

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we see that the roots of $\Delta(\tau - i\gamma, 0, \xi) = 0$ close to $\xi$ are

\[
\xi = \xi + \frac{i\gamma}{c_j} + o(\gamma).
\]

The assumption is that all the $c_j$ have the same sign. Thus, if the $c_j$ are positive [resp., negative], all the roots are in $\{\text{Im } \xi > 0\}$ [resp., in $\{\text{Im } \xi > 0\}$] for $\gamma > 0$ small.

**Remark 5.4.** When $-\tau$ is semi-simple of multiplicity $m$, the geometric multiplicity of $-\xi$ as an eigenvalue of $G(p, \zeta)$ is $m$. The nonglancing hypothesis is that the algebraic multiplicity is also $m$. In this case, the proof above shows that the condition that all the $c_j$ have the same sign is necessary and sufficient for having $E_- = E$ or $E_- = \{0\}$.

**Proposition 5.5.** Suppose that $L$ is symmetric hyperbolic and $G_k$ is associated to a totally nonglancing root. Then, there are smooth $K$-families of symmetrizers for $G_k$ near $(p, \zeta)$.

**Proof.** By assumption, there is a definite positive matrix $S(p)$ such that the $SA_j$ are symmetric and there is a $N \times m$ matrix $V_k$ such that for $(p, \zeta)$ close to $(p, \zeta)$:

\[
E_k = V_k C^m, \quad V_k G_k = G V_k.
\]

By Remarks 4.7 and 4.10, the symmetric matrices

\[
\Sigma_k(p, \zeta) = -V^*_k(p, \zeta) S(p) A_d(p) V_k(p, \zeta).
\]

are symmetrizers for $G_k$. More precisely, there holds

\[
\text{Im } \Sigma_k G_k = \gamma V^*_k S V_k \geq c_k \gamma \text{Id}, \quad c_k > 0.
\]

With notations as in (2.22), denote by $A_{k,d} = W_k A_d(p) V_k$ the boundary matrix of the tangent system $L'_{k}$ at the real root $(\tau, \eta, \xi_k)$ under consideration. By (2.8), the symmetry implies that $W_k = V^*_k S(p)$, hence

\[
\Sigma_k(p, \zeta) = -A_{k,d}.
\]

By Lemma 3.8, $A_{k,d}$ is definite positive [resp. negative] when the mode is totally incoming [resp. outgoing]. With Proposition 5.3, this implies that

\[
\Sigma_k^\kappa = \begin{cases}
\Sigma_k & \text{in the incoming case}, \\
\kappa \Sigma_k & \text{in the outgoing case}.
\end{cases}
\]

are K-families of symmetrizers for $G_k$. \qed
With Corollary (5.2) and the results of Section 4, this implies the next theorem.

**Theorem 5.6 (Second Main Theorem).** Suppose that $L$ is symmetric hyperbolic in the sense of Friedrichs and that all the real roots $(p, \tau, \eta, \xi)$ of the characteristic equation are either geometrically regular or totally nonglancing. Then, there is a neighborhood $\omega$ of $p$ such that:

i) there are smooth $K$-families of symmetrizers $\Sigma^k$ on $\omega \times S^4$;

ii) the vector bundle $E_{-(p, \zeta)}$ has a continuous extension to $\gamma = 0$;

iii) if in addition the boundary conditions satisfy the uniform Lopatinski, then there are smooth Kreiss symmetrizers and the maximal estimates (4.5) are satisfied.

**Example 5.7.** The assumptions are satisfied by the equations of MHD, under appropriate conditions on the parameters. Indeed, all points are either geometrically regular or algebraically regular and totally nonglancing; see Appendix A.

**Remark 5.8.** In Proposition 5.5 and Theorem 5.6 the symmetry of $L$ is used at only one place, to construct definite (positive or negative) symmetrizers for the blocks which are not geometrically regular. Thus, the symmetry assumption on $L$ can be relaxed, as soon as one can construct such symmetrizers. For instance, this can be done for double linearly splitting eigenvalues, see Section 6.

**Remark 5.9.** The symmetrizers constructed in Corollary 5.2 and Theorem 5.6 are obtained by localization and block reduction. Gluing the properties (5.4) for elliptic modes and (5.5) or (5.9) for real roots, we see that the symmetrizers satisfy the following condition which implies (4.17):

(5.12) \[ \text{Im } \Sigma G = \sum V_j^* E_j V_j, \]

with either $E_j$ definite positive on the support of $V_j$ or $E_j = \gamma E_j^1$ with $E_j^1$ definite positive on the support of $V_j$. This improvement is useful in applications to variables coefficients and nonlinear problems.

### 5.3 Linearly splitting modes

Next we consider a block $G_k$ associated to a linearly splitting semi simple nonglancing eigenvalue: we are given a smooth manifold $\mathcal{M}$ passing through $(p, \eta, \xi_k)$ as in Definition 5.2.
Proposition 5.10. Suppose that $G_k$ is associated to a linearly splitting eigenvalue transversally to a smooth manifold, and assume that it is nonglancing. Then, there are neighborhoods $\Omega^\kappa$ of $(p, \zeta)$ and $K$-families of bounded symmetrizers $\Sigma^\kappa$ on $\Omega^\kappa_+ = \Omega^\kappa \cap \{\gamma > 0\}$.

The main difference with the previous cases is that the symmetrizers $\Sigma^\kappa$ are bounded, but not necessarily smooth, that is, they may have no continuous extensions to $\Omega^\kappa_+$.

Proof. a) We first consider the case where

$$dx \notin T_{\nabla \zeta} \mathcal{M}_{\nu}. \quad (5.13)$$

In local coordinates $(q, \tilde{\zeta})$ as in (3.32), the block $G_k$ has the form

$$G_k = \mu(q) \text{Id} + \tilde{G}_k(q, \tilde{\zeta}), \quad \text{with } \tilde{G}(q, 0) = 0. \quad (5.14)$$

Introduce polar coordinates:

$$\tilde{\zeta} = \rho \xi, \quad \rho = |\tilde{\zeta}|, \quad \xi \in S^\nu_+, \quad (5.15)$$

where $S^\nu$ is the unit sphere in $\mathbb{R}^{\nu+1}$ where $\tilde{\zeta}$ lives, and $S^\nu_+$ is the half sphere $\tilde{\gamma} > 0$. With $\tilde{G}$ as in (3.35), there holds

$$G_k(p, \zeta) = \mu(q) \text{Id} + \rho \tilde{G}_k(q, \rho, \xi). \quad (5.16)$$

Introduce next the matrix $\tilde{A}_k(q, \rho, \tilde{\eta}, \tilde{\xi})$ as in (3.35). By Proposition 3.10 and the strict hyperbolicity of $\tilde{A}$, there holds for $(q, \rho)$ in a neighborhood of $(q, 0)$ and $\tilde{\xi} \in S^\nu$:

$$\det (\xi \text{Id} + \tilde{G}_k(q, \rho, \tilde{\xi}) = e(q, \rho, \tilde{\xi}) \prod ((\tilde{\tau} - i\tilde{\gamma}) + \tilde{\lambda}_j(q, \rho, \tilde{\eta}, \tilde{\xi})), \quad (5.17)$$

where $e(q, 0, 0) \neq 0$ and the $\tilde{\lambda}_j$ are smooth, analytic in $\tilde{\xi}$, and real when $\tilde{\xi}$ is real. Moreover, there are smooth vectors $e_j(q, \rho, \tilde{\eta}, \tilde{\xi})$, analytic in $\xi$ and linearly independent, such that

$$\left(\xi \text{Id} + \tilde{G}_k(q, \rho, \tilde{\tau}, \tilde{\gamma})\right)e_j(q, \rho, \tilde{\eta}, \tilde{\xi}) = 0, \quad \tilde{\tau} - i\tilde{\gamma} = -\tilde{\lambda}_j(q, \rho, \tilde{\eta}, \tilde{\xi}). \quad (5.18)$$

By Theorem 3.3, this shows that $\tilde{G}(q, \rho, \tilde{\xi})$ satisfies the block structure condition near $(q, 0, \tilde{\xi})$, for all $\tilde{\xi} \in S^\nu_+$. Therefore, the Kreiss construction applies to $\tilde{G}$; Lemmas 4.13, 4.14 and Corollary 5.2 imply the following.
Proposition 5.11. With notations as above, there is a neighborhood \( \tilde{\omega} \) of \((q,0)\) such that:

i) the vector bundle of invariant spaces \( \tilde{E}_k \) of \( \tilde{G}_k \) associated to eigenvalues of \( \tilde{G}_k \) in \( \{ \Re \tilde{\mu} < 0 \} \), defined for \( \tilde{\gamma} > 0 \), has a continuous extension to \( \tilde{\omega} \times S'_+ \); its dimension is equal to the number of positive eigenvalues of \( A_{k,d} \);

ii) there are smooth K-families of symmetrizers \( \tilde{\Sigma}^k \) of \( \tilde{G}_k \) on \( \tilde{\omega} \times S'_+ \).

By (5.16), the negative space of \( G_k(p,\zeta) \) is

\[
\mathbb{E}_{k,-}(p,\zeta) = \tilde{\mathbb{E}}_{k,-}(q,|\tilde{\eta}|,\frac{\tilde{\eta}}{|\tilde{\eta}|}).
\]

Since \( \mu(q) \) is real, it follows that

\[
\Sigma^k(p,\zeta) = \tilde{\Sigma}^k(q,|\tilde{\eta}|,\frac{\tilde{\eta}}{|\tilde{\eta}|})
\]

is a K-family of bounded symmetrizers for \( G_k \), for \((p,\zeta)\) close to \((p,\zeta)\) and \( \gamma > 0 \).

b) Next we consider the case where

\[
dx \in T_{\tilde{\eta}_k} M_{\eta}.
\]

By (3.39), the normal matrix of the tangent system at \((q,\tau,\tilde{\eta},\tilde{\xi})\) is

\[
A_{k,d} = a_k \text{Id}, \quad a_k \neq 0.
\]

In local coordinates \((q,\tau,\tilde{\xi})\) as in (3.42), the block \( G_k \) for \( \gamma = 0 \), has the form

\[
G_{k|\gamma=0} = \mu(q,\tau) \text{Id} + \tilde{G}_k(q,\tau,\tilde{\eta}), \quad \text{with } \tilde{G}_k(q,\tau,0) = 0.
\]

We switch to polar coordinates \( \tilde{\eta} = \rho \tilde{\eta} \) with \( \tilde{\eta} \in S^\nu \) and introduce \( \tilde{G}_k \) as in (3.43). By Proposition 3.11 the eigenvalues of \( \tilde{G} \) are real and simple when \( \tilde{\eta} \neq 0 \). Therefore, there exists a neighborhood \( \omega \) of \((q,\tau)\), a constant \( c > 0 \), and a smooth symmetric matrix \( \tilde{\Sigma}(q,\tau,\rho,\tilde{\eta}) \) on \( \omega \times S^\nu \), such that

\[
\tilde{\Sigma}(q,\tau,\rho,\tilde{\eta}) \geq c \text{Id}, \quad \text{Im}(\tilde{\Sigma}G) = 0.
\]
Introduce

\[(5.25) \quad \Sigma_k'(p, \tau, \eta, \gamma) = \begin{cases} \Sigma_k(q, \tau, |\hat{\eta}|, \frac{\hat{\eta}}{|\hat{\eta}|}) & \text{when } \hat{\eta} \neq 0, \\ \text{Id} & \text{when } \hat{\eta} = 0. \end{cases} \]

These matrices are uniformly bounded for \((p, \zeta)\) in a neighborhood of \((p, \zeta)\) and

\[(5.26) \quad \Sigma_k'(q, \zeta) \geq c \text{Id}, \quad \text{Im}(\Sigma_k' G_k)|_{\gamma=0} = 0. \]

Symmetrizer \(\Sigma_k'(q, \zeta)\) is independent of \(\gamma\) and \(G_k\) is smooth; using Proposition 3.11, this implies

\[(5.27) \quad \text{Im}(\Sigma_k' G_k) = -\frac{\gamma}{\alpha} \Sigma_k + O(\gamma|(p, \zeta) - (p, \zeta)|) + O(\gamma^2). \]

Therefore, \(\Sigma_k = -\frac{\alpha}{\gamma} \Sigma_k'\) is a bounded symmetrizer for \(G_k\) on a neighborhood of \((p, \zeta)\) which is positive definite [resp. negative] when \(\alpha\) is negative [resp. positive]. In the first case, by (5.22) and Lemma 3.3, the mode is totally outgoing and by Proposition 5.3, the negative space \(E_{k,-}\) is \(\{0\}\). In the second case, the mode is totally incoming and \(E_{k,-} = E_k\). Thus, \(\Sigma_k^* = \kappa \Sigma_k\) in the outgoing case and \(\Sigma_k^* = \Sigma_k\) in the incoming case provide us with \(K\)-families of bounded symmetrizers for \(G_k\).

**Remark 5.12.** If the mode is totally nonglancing, the space \(E_{k,-}\) is either \(\mathbb{C}^N_k\) or \(\{0\}\); it is trivially continuous at \((p, \zeta)\). If the mode is not totally nonglancing, then the matrix \(A_k\) has positive and negative eigenvalues. In this case, the negative space is given by (5.19): its dimension is positive and smaller then \(N_k\), and it is a smooth function of \(\tilde{\zeta}/|\tilde{\zeta}|\). In general, \(E(p, \zeta)\) is not continuous at \((p, \zeta)\) since the limit depends on the direction of \(\zeta\).

The Main Theorem 4.2 is a consequence of Corollary 5.2, Propositions 5.5 and 5.11, using the general results of Section 4.

## 6 Smooth Kreiss symmetrizers

We next investigate when symmetrizers may be chosen in *smooth fashion*; this is important for the treatment of variable-coefficient or nonlinear problems. Consider again the boundary value problem

\[(6.1) \quad \partial_x u + iG(p, \zeta) u = f, \quad M(p, \zeta) u|_{x=0} = g, \]

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for \((p, \zeta)\) in a neighborhood of \((p, \zeta)\) with \(\gamma = 0\) and \(G\) associated to a hyperbolic system \(L\) as in (3.2).

We suppose that we are given two invariant spaces \(E_0(p, \zeta)\) and \(E_1(p, \zeta)\) for \(G(p, \zeta)\), which depend smoothly on \((p, \zeta)\) and such that \(CN = E_0 \oplus E_1\).

We denote by \(G_k\) the restriction of \(G\) to \(E_k\) and by \(E_{k,-}\) the negative space of \(G_k\), for \(\gamma > 0\). The negative space for \(G\) is \(E_- = E_{0,-} \oplus E_{1,-}\).

**Assumption 6.1.** \(E_{0,-}(p, \zeta)\) has a continuous extension to \(\gamma = 0\) near \((p, \zeta)\). Moreover, \(\dim \ker M = N - \dim E_-\) and

\[
\ker M(p, \zeta) \cap E_{0,-}(p, \zeta) = \{0\}. 
\]

The latter two conditions are implied by the uniform Lopatinski condition.

### 6.1 Reduced boundary value problems

Introduce

\[
F_0(p, \zeta) = M(p, \zeta)E_{0,-}(p, \zeta).
\]

This is a continuous bundle on \(\Omega_+\) where \(\Omega\) is a neighborhood of \((p, \zeta)\) and \(\Omega_+ = \Omega \cap \{\gamma > 0\}\). By Assumption 6.1, \(\dim F_0 = \dim E_{0,-}\). Let \(F_1\) denote a continuous bundle such that

\[
MC^N = F_0 \oplus F_1.
\]

**Definition 6.2.** The reduced boundary conditions for block \(E_j\) is

\[
M_j = \pi_j M|_{E_j}
\]

where \(\pi_j\) denotes the projection on \(F_j\) in the splitting (6.4). Likewise, we define reduced Lopatinski determinants \(\Delta^j := \det \mid_{E_j}(E_{j,-}, \ker M^j)\) and a reduced uniform Lopatinski condition consisting of a uniform lower bound on the modulus of the reduced Lopatinski determinant.

**Remark 6.3.** Note that there is a useful asymmetry in Definition 6.2 in that we do not require continuity of \(E_{1,-}\) at \(\zeta\); indeed, \(E_{1,-}\) is mentioned only in the Lopatinski determinant evaluated for \(\gamma > 0\). This allows us to separate off discontinuous blocks. The continuous block \(E_0\) may be further subdivided into arbitrary many blocks on which the negative subspace is continuous, each with its own reduced boundary condition.
Remark 6.4. The choice of $F_0$ implies that

\[ (6.6) \quad \pi_1 M |_{E_{0,-}} = 0. \]

On the other hand, $\pi_0 M$ is not necessarily 0 on $E_{1,-}$. This is another asymmetry in the role of $E_0$ and $E_1$. It reflects that the boundary conditions do not decouple in the block reduction of $G$.

Proposition 6.5. The uniform Lopatinski condition is satisfied on the full space $E$ if and only if the reduced uniform Lopatinski conditions are satisfied on each $E_j$.

Proof. a) If the reduced problems satisfy the uniform Lopatinski condition on $\Omega_+$, there are constants $C_j$ such that for all $(p, \zeta) \in \Omega_+$,

\[ (6.7) \quad |u_j| \leq C_j |M_j u_j| \quad \text{for} \quad u_j \in E_{j,-}. \]

Let $u = (u_0, u_1) \in E_-$. Then, using (6.6), we have

\[ |u_1| \leq C_1 \pi_1 M u_1 \leq C_1 |\pi_1|M u | \]
\[ |u_0| \leq C_0 |M u_0| \leq C_0 |M u| + C_0 |M| |u_1|, \]

implying that

\[ (6.8) \quad |u| \leq C |M u| \quad \text{for} \quad u \in E_- . \]

b) Conversely, if the uniform Lopatinski condition is satisfied on $\Omega_+$, then, since $M = \pi_0 M$ on $E_{0,-}$, (6.8) restricted to $E_{0,-}$ implies that

\[ |u_0| \leq C |M_0 u_0| = C |M_0 u_0| \quad \text{for} \quad u_0 \in E_{0,-}. \]

verifying the reduced Lopatinski condition on $E^0$. Next, by definition of $F_0$, there is an inverse $R$ of $M$ from $F_0$ to $E_{0,-}$. By continuity of $E_{0,-}$, this inverse is uniformly bounded on a neighborhood of $(p, \zeta)$. For $u_1 \in E_{1,-}$, let $u_0 = R \pi_0 M u_1 \in E_{0,-}$ such that $u = (u_0, u_1) \in E_-$, $\pi_0 M u = 0$ and $M u = \pi_1 M u = M_1 u_1$. Then

\[ (6.9) \quad |u_1| \leq C_2 |u| \leq C_2 C |M u| = C_2 C |M_1 u_1|. \]

where $C_2$ is the norm of the projection from $\mathbb{C}^N$ on $E_0$. This verifies the reduced Lopatinski condition on $E_1$. \qed
6.2 Symmetrizers

We shall seek symmetrizers commuting with the projectors onto $E_j$; we call such a symmetrizer consistent with the decomposition $E = E_0 \oplus E_1$. Note that if $\Sigma$ is a symmetrizer for $G$, then the $\Sigma_j$ defined by restricting the hermitian form $(\Sigma u, v)$ to $E_j$ is a symmetrizer for $G_j$ (see Remark 4.15). Thus, $\tilde{\Sigma} = \Sigma_0 \oplus \Sigma_1$ is a symmetrizer for $G$, consistent with the decomposition.

We have shown in Section 5 that the notion of K-families of symmetrizers is well adapted to a block decomposition: if there are such families $\Sigma^k$ for each block, then $\Sigma^k = \text{diag}(\Sigma^k)$ is a $k$-family for the full system. Such an analogue is not true for Kreiss symmetrizers: a $K$-family depends only on $G$, while the definition of a Kreiss symmetrizer also depends on the boundary conditions, which do no decouple, in general, in the block reduction of $G$. However, we have the following useful partial result.

**Proposition 6.6.** Consider $G$ with boundary condition $M$, and a specified decomposition $E_0, E_1$ in a neighborhood of $(p, \zeta)$ satisfying Assumption 6.1. If there exists a local smooth Kreiss symmetrizer $\Sigma$ respecting this decomposition then

i) the reduced boundary value problem on $E^0$ satisfies the (reduced) uniform Lopatinski condition,

ii) the restriction $\Sigma_1$ of $\Sigma$ to $E_1$ is a smooth Kreiss symmetrizer for the reduced boundary value problem $(G_1, M_1)$.

Conversely, assume ii) and

iii) the reduced boundary value problem on $E^0$ satisfies the uniform Lopatinski condition and there exists a smooth $K$-family of symmetrizers for the reduced problem $(G_0, M_0)$.

Then, there exists a local smooth Kreiss symmetrizer for $(G, M)$.

**Proof. a)** Suppose that $\Sigma = \text{block-diag}\{\Sigma_0, \Sigma_1\}$ is a local smooth Kreiss symmetrizer for $(G, M)$. Then there are $C > 0$ and $\varepsilon > 0$ such that

\begin{equation}
\langle u, \Sigma u \rangle = \langle u_0, \Sigma_0 u_0 \rangle + \langle u_1, \Sigma_1 u_1 \rangle \geq \varepsilon |u|^2 - C |Mu|^2.
\end{equation}

We have recalled in Section 4 that the existence of a Kreiss symmetrizer implies that the uniform Lopatinski condition must hold, and thus, by Proposition 6.5, the uniform reduced Lopatinski condition must hold on $E^0$.

Further, for all $u \in E_1$, there is $\tilde{u}_{0,-} \in E_{0,-}$ such that $\pi_0 M(u^1 + \tilde{u}_{0,-}) = 0$. By Lemma 4.8 and continuity of $E_{0,-}$, $\langle \Sigma_0 u_{0,-}, u_{0,-} \rangle \leq 0$. Moreover, by
Hence, if
\[ M(u_1 + u_{0,-}) = \pi_1 M u_1 = M_1 u_1. \]
Thus,
\[
\langle u_1, \Sigma_1 u_1 \rangle \geq \langle u_1, \Sigma_1 u_1 \rangle + \langle \tilde{u}_{0,-}, \Sigma_0 \tilde{u}_{0,-} \rangle = ((u_1 + \tilde{u}_{0,-}), \Sigma (u_1 + \tilde{u}_{0,-}))
\]
\[
\geq \epsilon (|u_1|^2 + |\tilde{u}_{0,-}|^2) - C |M (u_1 + \tilde{u}_{0,-})|^2
\]
\[
= \epsilon |u^1|^2 - C |M^1 u|^2,
\]
verifying ii).

b) Conversely, let \( \Sigma^0 \) denote a K-family of symmetrizers for \((G_0, M_0)\). Then, by Proposition 4.10, for \((p, \zeta) \in \Omega^*\), the uniform Lopatinski condition implies that
\[
\langle u_0, \Sigma^0 u_0 \rangle + C_0 |M_0 u_0|^2 \geq |u_0|^2 + m(\kappa)|u_{0,+}|^2
\]
where \( m(\kappa) \to +\infty \) as \( \kappa \to +\infty \). By assumption, there is also \( S_1 \) satisfying
\[
\langle u_1, \Sigma_1 u_1 \rangle \geq |u_1|^2 - C_1 |M_1 u_1|^2.
\]
Therefore,
\[
\delta \langle u_0, \Sigma_0 u_0 \rangle + \langle u_1, \Sigma_1 u_1 \rangle + C_0 \delta |M_0 u_0|^2 + C_1 |M_1 u_1|^2 \geq |u_1|^2 + \delta m(\kappa)|u_{0,+}|^2 + \delta |u_0|^2.
\]
Using (6.6), we have for \( u = (u_0, u_1) \):
\[
2|\pi_1 M u|^2 \geq |M_1 u_1|^2 - C_2 |u_{0,+}|^2
\]
\[
2|\pi_0 M u|^2 \geq |M u_0|^2 - C_3 |u_1|^2.
\]
Therefore,
\[
\delta \langle u_0, \Sigma_0 u_0 \rangle + \langle u_1, \Sigma_1 u_1 \rangle + 2C_0 \delta |\pi_0 M u|^2 + 2C_1 |\pi_1 M u|^2 \geq |u_1|^2 + \delta |u_0|^2 + \delta m(\kappa)|u_{0,+}|^2 - C_3 \delta |u_1|^2 - C_2 |u_{0,+}|^2.
\]
Hence, if \( C_3 \delta \leq 1/2 \), \( \delta m(\kappa) \geq C_2 \), \( C \geq 2C_0 \delta |\pi_0|^2 + 2C_1 |\pi_1|^2 \) and \( \Sigma = \text{diag}(\delta \Sigma_0, \Sigma_1) \):
\[
\langle u, \Sigma u \rangle + C |M u|^2 \geq \delta |u|^2.
\]
\[ \square \]

**Remark 6.7.** If \( G_0 \) satisfies the block structure condition, then i) and iii) are equivalent.
6.3 Smooth symmetrizers for $E_1$

We investigate the existence of smooth Kreiss symmetrizers for blocks $E_1$ associated to nonregular eigenvalues. We have in mind the case of linearly splitting eigenvalues. We first give necessary and next sufficient conditions. We assume, as we may, that $G_1$ has a block diagonal form

\[(6.14) \quad G_1(p, \zeta) = \text{block-diag}(G_{1,k}(p, \zeta)) \]

where the $G_{1,k}$ are some of the invariant blocks of $G(p, \zeta)$, for $(p, \zeta)$ close to $(p, \zeta)$ associated to pairwise distinct real eigenvalues $\mu_k = -\xi_k$ of $G(p, \zeta)$.

We further assume that the modes $(p, \tau, \eta, \xi_k)$ are semi-simple and nonlancing. In this case, by Proposition 3.9, the $\mu_k$ are semi-simple eigenvalues of $G$ and

\[(6.15) \quad G_{1,k}(up, \zeta + \zeta) = \mu_k \text{Id} + G'_{1,k}(\zeta) + O(|\zeta|^2). \]

Moreover, denoting by

\[L'_{1,k} = \tau \text{Id} + \sum_{j=1}^{d-1} \eta_j A_{k,j} + \xi A_{k,d} \]

the tangent system at $(p, \tau, \eta, \xi_k)$, there holds

\[(6.16) \quad G_{1,k} = A_{k,d}^{-1}((\tau - i\gamma) \text{Id} + \sum_{j=1}^{d-1} \eta_j A_{k,j}), \]

The Taylor expansion of $G_1$ at the base point reads

\[(6.17) \quad G_1(p, \zeta + \zeta) = G_1(p, \zeta) + G'_1(\zeta) + O(|\zeta|^2). \]

$G_1(p, \zeta)$ and $G'_1$ are block diagonal; we further introduce

\[(6.18) \quad L' = \text{block-diag}(L'_{1,k}) \]

**Proposition 6.8.** Suppose that $\Sigma(p, \zeta)$ is a smooth symmetrizer for $G$. Let $\Sigma_1$ denote the restriction of $\Sigma$ to $E_1$. Then $\Sigma_1$ is a smooth symmetrizer for $G_1$ and

i) $\Sigma_1 := \Sigma_1(p, \zeta)$ is block diagonal and satisfies $\text{Im}(\Sigma_1 G_1) = 0$;

ii) $L'$ is hyperbolic symmetric in the sense of Friedichs.

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Proof. Since \( \Sigma \) is a symmetrizer, \( \text{Im} (\Sigma G) \geq c\gamma \), which by restriction to \( E_1 \), implies that for \( \gamma > 0 \):

\[
(6.19) \quad \text{Im} \Sigma_1 G_1 \geq c\gamma \text{Id}_{E_1}.
\]

Thus, by continuity,

\[
\text{Im} (\Sigma_1 G_1) \geq 0.
\]

Since \( G_1 \) is block diagonal, with real entries \( \mu_j \), this means that for all pair \((k,l)\) with \( k \neq l \), the nondiagonal block entries \( \Sigma_{1,k,l} \) of \( \Sigma_1 \) are such that

\[
\begin{pmatrix}
0 & P \\
P^* & 0
\end{pmatrix}, \quad P = \frac{1}{2i}(\mu_k - \mu_j)\Sigma_{j,k}
\]

is nonnegative. This implies that \( P = 0 \), hence \( \Sigma_{1,k,l} = 0 \) when \( k \neq l \). Thus

\[
(6.20) \quad \text{Im} (\Sigma_1 G_1) = 0.
\]

The diagonal block \( \Sigma_{1,k,k} \) is a symmetrizer for \( G_{1,k} \) by Remark 4.15. Taking the first order term in \( \zeta \) in \( \text{Im} (\Sigma_{1,k,k} G_{1,k}) \), implies that

\[
(6.21) \quad \text{Im} \Sigma_{1,k,k} G'_{1,k}(\zeta) \geq c\gamma.
\]

We have used that the \( G'_{1,k} = \mu_k \text{Id} \) and that the derivatives of \( \Sigma_{1,k,k} \) are self-adjoint. Because \( G'_{1,k} \) is a linear in \( \zeta \), changing \( \zeta \) to \(-\zeta\) implies that \( \text{Im} \Sigma_{1,k,k} G'_{1,k}(\zeta) = 0 \) when \( \gamma = 0 \). Thus, the matrices \( S_{1,k} := -\Sigma_{1,k,k} A_{k,d}^{-1} \) and \( S_{1,k} A_{k,j} \) are self-adjoint. Moreover, (6.21) implies that \( S_{1,k} \) is definite positive. This shows that \( S_{1,k} \) is a Friedrichs symmetrizer for \( L'_{1,k} \), therefore that

\[
(6.22) \quad S_1 = \text{block-diag}(S_{1,k})
\]

is a Friedrichs symmetrizer for \( L' \).

\[\square\]

**Proposition 6.9.** Suppose that \( \Sigma_1 \) is a smooth symmetrizer for \( G_1 \). Then, it is a smooth Kreiss symmetrizer on a neighborhood of \( (p,\zeta) \) if and only if the reduced boundary condition \( M_1(p,\zeta) \) is maximal dissipative for \( L'_1 \) with symmetrizer (6.22).

**Proof.** If \( \Sigma_1 \) is a Kreiss symmetrizer for \((G_1, M_1)\), then

\[
(6.23) \quad \langle \Sigma_1 u_1, u_1 \rangle + C|M_1 u_1|^2 \geq c|u_1|^2.
\]

Specializing at \((p,\zeta)\) implies that \( S_1 A_d = -\Sigma_1 \) is maximal negative on \( \text{ker} M_1(p,\zeta) \), thus that the boundary condition \( M_1(p,\zeta) \) is maximal dissipative for \( L' \) with symmetrizer \( S_1 \).

Conversely, if \( M_1(p,\zeta) \) is maximal dissipative for \( L' \) with symmetrizer \( S_1 \), then (6.23) holds at \((p,\zeta)\), therefore on a neighborhood of that point. \[\square\]
Theorem 6.10 (Third Main Theorem). Suppose that \( L \) is a hyperbolic system and \( M \) are boundary conditions which satisfy the uniform Lopatinski condition. For all \((p, \zeta)\) in a neighborhood of \((p, \zeta)\), denote by \( \mathbb{E}_0(p, \zeta) \) the invariant space of \( G \) corresponding to eigenvalues which are either nonreal, or real and associated to geometrically regular or totally nonglancing modes of \( L \). Denote by \( \mathbb{E}_1(p, \zeta) \) the invariant space associated to the remaining eigenvalues of \( G \). Then, Assumption 6.1 is satisfied and, with notations as in Proposition 6.9, there exists a smooth Kreiss symmetrizer for \((G, M)\) near \((p, \zeta)\) if and only if there is a smooth symmetrizer \( \Sigma_1 \) for \( G_1 \) such that the associated tangent system \( L_1' \) is hyperbolic symmetric in the sense of Friedrichs with symmetrizer \( S_1 \) and the reduced boundary condition is maximally dissipative for \( L_1' \).

Proof. By Corollary 5.2, Propositions 5.3, 5.5 and Lemma 4.12, there are smooth K-families of symmetrizers for \( G_0 \) and Assumption 6.1 is satisfied.

If \( \Sigma \) is a smooth Kreiss symmetrizer for \((G, M)\), then by Proposition 6.6 \( \Sigma_1 \) is a smooth Kreiss symmetrizer for the reduced problem \((G_1, M_1)\) and Proposition 6.9 implies that the reduced boundary condition is maximally dissipative for \( L_1' \).

Conversely, if \( \Sigma_1 \) is a smooth symmetrizer for \( G_1 \) such that the reduced boundary condition is maximally dissipative for \( L_1' \), then \( \Sigma_1 \) is a Kreiss symmetrizer for the reduced problem \((G_1, M_1)\) and Proposition 6.6 implies that there is a smooth Kreiss symmetrizer for the full system \((G, M)\).

This reduces the analysis to the construction of smooth Kreiss symmetrizers for \( G_1 \). When \( G \) is associated to a symmetric hyperbolic system \( L \), with symmetrizer \( S \), then \( \Sigma = -SA_{d} \) is a smooth symmetrizer for \( G \) and the restriction \( \Sigma_1 \) of \( \Sigma \) to \( \mathbb{E}_1 \) is a smooth symmetrizer on a neighborhood of \((p, \zeta)\). For this symmetrizer, the maximal dissipativity condition of the reduced boundary condition holds if and only if \( \Sigma_1(p, \zeta) \) is definite positive on \( \ker M_1(p, \zeta) \), that is if and only if there is \( c > 0 \) such that

\[
(6.24) \quad \forall u \in \mathbb{E}_1, \quad Mu \in M\mathbb{E}_{0,-} \Rightarrow \langle SA_{d}u, u \rangle \geq c|u|^2,
\]

where \( \mathbb{E}_{0,-} \) means evaluation at the base point \((p, \zeta)\). Therefore,

Corollary 6.11. With assumptions and notations as in Theorem 6.10, if the condition \((6.24)\) holds at \((p, \zeta)\), then there is a smooth Kreiss symmetrizer for \((G, M)\) on a neighborhood of \((p, \zeta)\).

Remark 6.12. Suppose that \( L \) is symmetric and split \( G_1 \) in diagonal blocks \( G_{1,k} \) as in \((6.14)\). Then, by Remark 4.15 there are smooth symmetrizers \( \Sigma_{1,k} \).
for $G_{1,k}$, of the form $\Sigma_{1,k} = -V_{1,k}^*(SA_d)V_{1,k}$. Therefore, there are smooth symmetrizers for $G_1$ of the form

\[(6.25) \quad \Sigma_1 = \text{block-diag}(\alpha_k \Sigma_{1,k}), \quad \alpha_k > 0.\]

For these symmetrizers, one can state the analogues of condition (6.24) and Corollary 6.11.

**Remark 6.13.** We have seen that a necessary condition for the existence of a smooth symmetrizer $\Sigma_1$ is that $L'_{1,k}$ must be hyperbolic symmetric in the sense of Friedrichs, with symmetrizer $\tilde{S}_{1,k} = \text{block-diag}(\tilde{S}_{1,k})$, where $\tilde{S}_{1,k} = -\tilde{S}_{1,k}A_{1,k}^{-1}$ and $\tilde{S}_{1,k} = \text{block-diag}(\tilde{S}_{1,k})$. Therefore, in the construction of a smooth Kreiss symmetrizer $\tilde{S}_{1,k}$, the first step is construct a Friedrichs symmetrizer $\tilde{S}_{1,k} = \text{block-diag}(\tilde{S}_{1,k})$ for $L'_{1,k}$ such that $M_1$ is maximal dissipative. This defines $\tilde{S}_{1,k}$, and the second is to determine a smooth symmetrizer for $G_1$ such that $\tilde{S}_{1,k}(p,\zeta) = \tilde{S}_{1,k}$. Since $\tilde{S}_{1,k}$ is block diagonal, it is sufficient to construct smooth symmetrizers for $G_{1,k}$ such that $\tilde{S}_{1,k}(p,\zeta) = \tilde{S}_{1,k}$.

When $L$ is symmetric hyperbolic, the $L'_{1,k}$ have Friedrichs symmetrizers $\tilde{S}_{1,k}$, as in Remark 6.12. In generic cases, these symmetrizers are unique up to a constant, implying that necessarily

\[(6.26) \quad \tilde{S}_{1,k} = \alpha_k \tilde{S}_{1,k}, \quad \alpha_k > 0.\]

Therefore,

\[(6.27) \quad \tilde{S}_{1,k} = \text{block-diag}(\alpha_k \Sigma_{1,k}).\]

In this case, if there is a smooth Kreiss symmetrizer, there is one of the form (6.25), indicating that this form contains almost all the possible choices of smooth symmetrizers.

**Remark 6.14.** More precisely, symmetrizers $\tilde{S}_{1,k}$ are unique up to constants if and only if the reduced tangent systems $L'_{1,k}$ are irreducible in the sense that there do not exist nontrivial constant invariant subspaces of $A_{1,k}(\eta,\xi)$. For, without loss of generality taking $A_{1,k}^d$ diagonal, so that $\tilde{S}_{1,k}$ must be diagonal as well, and expressing $\tilde{S}_{1,k}$ as a block-diagonal matrix with blocks consisting of distinct scalar multiples of the identity, we find that symmetry of $\tilde{S}_{1,k}A_{1,k}^d$ implies holds if and only if each $A_{1,k}^d$ shares the same block-diagonal structure.

In the linearly splitting case, we find by spectral separation that these invariant subspaces must correspond to the limit as $(\eta,\xi) \to 0$ of group
eigenprojections associated to fixed subsets of the associated eigenvalues \( \lambda^j_{1,k}(\eta,\xi) \) of the nearby perturbed system. In particular, this implies that these group eigenprojections are continuous along rays across the origin, which is sufficient to give analyticity separately in coordinates \( \eta_j, \xi \). By Hartog’s Theorem, therefore, they are jointly analytic in \((\eta,\xi)\), and thus there is an analytic decomposition of the perturbed system into two invariant blocks. By an argument similar to that for the geometrically regular case, we find that, at least for the easier nonglancing case, the associated block \( G_{1,k} \) decomposes analytically as well. Thus, in the linearly splitting, nonglancing case, we may always arrange that the tangent system have unique (up to constants) symmetrizers.

### 6.4 The case of double roots

In this section, we assume that \( E_1 \) is the direct sum of two dimensional subspaces \( E_{1,k} \) associated to double linearly splitting characteristic roots:

**Assumption 6.15.** Suppose that \( L \) is symmetric hyperbolic and noncharacteristic with respect to the boundary \( x = 0 \), and that its characteristic roots are either geometrically regular, totally nonglancing, or second order, nonglancing and linearly splitting transversally to a smooth manifold of codimension two.

The following result is proved in Appendix D.

**Proposition 6.16.** Suppose that \( G_{1,k} \) is a \( 2 \times 2 \) block associated to a nonglancing linearly splitting eigenvalue, near \((p,\zeta)\) with \( \gamma = 0 \). Then there exist smooth symmetrizers \( \Sigma_{1,k} \) near \((p,\zeta)\) for \( G_{1,k} \) such that

\[
\tilde{S}_{1,k} := -\Sigma_{1,k}(p,\zeta)A_k^{-1}(p,\zeta)
\]

is a Friedrichs symmetrizer for \( L_{1,k}' \).

Moreover, the symmetrizer for \( L_{1,k}' \) is unique, up to multiplication by a positive constant.

In this case, the only block diagonal symmetrizers of \( L_{1}' \) are

\[
(6.28) \quad \tilde{\Sigma}_1 = \text{block-diag}(\alpha_k \tilde{\Sigma}_{1,k}),
\]

with positive constants \( \alpha_k \). Moreover, given the \( \alpha_k \), (6.28) defines a smooth symmetrizer of \( G_1 \) such that

\[
(6.29) \quad \tilde{\Sigma}_1(p,\zeta) = \text{block-diag}(\alpha_k \Sigma_{1,k}).
\]

Therefore, Theorem 6.10 takes the special form:
Theorem 6.17. Suppose that $L$ satisfies Assumption \ref{assumption:uniform_Lopatinski} and $M$ are boundary conditions which satisfy the uniform Lopatinski conditions. With notations as in Theorem \ref{thm:existence_of_smooth_symmetrizer} there exists a smooth Kreiss symmetrizer for $(G, M)$ if and only if for all $(p, \zeta)$ with $\zeta \in S_d^+$ and $\gamma = 0$, the reduced boundary conditions $M_1(p, \zeta)$ are maximal dissipative for some Friedrichs symmetrizer of $L'$.

Moreover, we have by direct calculation the following useful fact, generalizing a corresponding observation of Majda-Osher in the case of the $2 \times 2$ wave equation. For the proof, see Appendix D and Proposition \ref{prop:uniform_Lopatinski} below.

Proposition 6.18. If $G_1$ itself is a $2 \times 2$ matrix, there is a smooth Kreiss symmetrizer for the reduced problem $(G_1, M_1)$, if and only if it satisfies the uniform Lopatinski condition.

This immediately implies the following

**Theorem 6.19 (Fourth Main Theorem).** Suppose that Assumption \ref{assumption:uniform_Lopatinski} is satisfied and that for all $\zeta \in S_d^+$ with $\gamma = 0$, there is at most one real eigenvalue associated to a mode which is neither geometrically regular nor totally nonglancing.

Then, for boundary conditions $M$ there is a smooth Kreiss symmetrizer for $(G, M)$ if and only if $M$ satisfies the uniform Lopatinski condition.

We conclude by briefly considering in the simple double-root case, under what conditions our structural hypotheses hold.

Proposition 6.20. For a symmetrizable system in dimension $d \leq 2$, geometric regularity holds at any point $\zeta_0 \neq 0$.

**Proof.** Restricting by homogeneity to the unit sphere, we obtain a symmetric matrix perturbation problem in a single parameter, for which eigenprojections necessarily vary analytically \cite{Kat}. \hfill $\square$

**Proposition 6.21.** In dimension $d \geq 3$, a variable multiplicity point of multiplicity two is generically nonglancing, nonregular, and linearly splitting of codimension two. Moreover, a linearly splitting point of multiplicity two is either of codimension two or else geometrically regular of codimension one.

**Proof.** The reduced, symmetrizable system in dimension $d - 1 = 2$ generically has noncharacteristic $\tilde{A}_1$, hence is nonglancing. Algebraic regularity can hold only if the reduced, frozen coefficient system has linearly varying

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eigenvalues, in which case \( \tilde{A}_1 \) and \( \tilde{A}_2 \) commute, a degenerate case. Moreover, if \( \sum_{j=1}^{d-1} \xi_j A_j \) has equal eigenvalues for \( \xi \neq (0,0) \), then by symmetry it is a multiple \( \alpha I \) of the identity, which is three conditions (again taking into account symmetry to eliminate one entry) on the unknowns \( \xi, \alpha \), generically determining a codimension two manifold \( M \) in \( \xi \).

The latter argument shows also that a linearly splitting point can be at most codimension two. Suppose that it is codimension one. Then, by an analytic change of coordinates taking the manifold \( M \) on which eigenvalues agree to the \((\xi_1, \ldots, \xi_{d-1})\) plane, and subtracting off the (analytic) average of the two eigenvalues, we obtain a matrix perturbation problem of form \( \tilde{A}(\xi) = \xi_d D(\xi) \), where \( D(0) \) is diagonalizable with distinct eigenvalues. The eigenvalues and associated eigenprojections of \( D \) are therefore analytic, as are the eigenvalues and eigenprojections for the original problem.

**Proposition 6.22.** In any dimension, an algebraically regular variable multiplicity point of multiplicity two is either geometrically regular or else has characteristics tangent to first order in \( \xi \), in which case nonglancing and totally nonglancing are equivalent.

*Proof.* Denote the two characteristics as \( \tau_1(\xi), \tau_2(\xi) \). If \( \partial(\tau_1 - \tau_2)/\partial \xi_d = 0 \), then one of the last two possibilities must occur. If \( \partial(\tau_1 - \tau_2)/\partial \xi_d \neq 0 \), on the other hand, then by the Implicit Function Theorem, there is a hypersurface \( \xi_d = \phi(\xi') \) on which \( \tau_1 - \tau_2 \equiv 0 \). But symmetrizability implies semisimplicity, so that the reduced matrix \( \tilde{A}(\xi) \) obtained by projection onto the associated two-dimensional total eigenspace of \( \tau_1, \tau_2 \) must be a multiple of the identity; likewise, \( \partial \tilde{A}/\partial \xi_d \) must be diagonalizable, with distinct eigenvalues. The matrix \( \tilde{B}(\xi) := (\tilde{A} - \tau_1 I)(\xi) \) is thus of form \( (\xi_d - \phi(\xi')) D(\xi') + O(|\xi_d - \phi(\xi')|^2) \), analytic in \( \xi \), where \( D \) is diagonalizable with distinct eigenvalues, and it follows that the associated projectors are analytic. The final assertion follows by the same argument applied to each direction \( \xi_j \).

Collecting results, we have the following conclusion.

**Corollary 6.23.** Assumption 6.15 is satisfied always in dimension \( d = 2 \). For multiplicity two crossings in dimension \( d = 3 \), assumption 6.15 is generically satisfied, and at algebraically regular points is always satisfied.
7 Application to MHD

Finally, we discuss our main application, to the shock wave problem in MHD. As is well known, the Euler equations of MHD are symmetrically hyperbolic for a thermodynamically stable equation of state, in particular for an ideal gas law; see [G, Kaw, KSh, MZ, Z2]. However, they do not satisfy the condition of constant multiplicity, and so the standard Kreiss–Majda theory cannot be applied; see calculations, Appendix A. Moreover, the boundary condition that arises through linearization of the shock transmission problem is not dissipative, and so the older theory of dissipative boundary conditions cannot either be applied; see [BT1].

For this reason, and also the inherent complexity of the system, progress on shock stability for MHD has been so far somewhat limited. In particular, the only nonlinear stability result up to now has been obtained by Blokhin and Trakhinin [BT1] [BT4] using the “direct method” introduced by Blokhin [B1] for the study of gas dynamics, in the very special case that magnetic field \( H \) goes to zero: essentially, the fluid-dynamical limit. This consists of constructing a Lyapunov function in the form of certain “dissipative integrals”, from which linearized and nonlinear stability follow directly. The construction, which is rather intricate, depends crucially on symmetrizability of the system; as the authors describe it, it consists of augmenting the original system with various derivatives, chosen in such a way that the (initially nondissipative) boundary condition becomes dissipative. At least as presented, it does not appear easily generalizable to other cases.

On the other hand, with additional symmetry, either in the “transverse” case that the magnetic field be orthogonal to fluid velocity, or the “parallel” case that they are parallel, the Lopatinski condition has been explicitly evaluated even for large \( H \), to yield physical stability conditions analogous to those of Majda [Maj] in the gas-dynamical case; see [GK, BT1, BT2, BT3, BT4]. More generally, the Lopatinski condition has been extensively studied numerically for arbitrary parameters, with interesting results [BTM1, BTM2]. For a detailed description of the current status of the theory, we refer to the excellent survey [BT1]. Up to now, however, the precise relation between the Lopatinski condition and linearized and nonlinear stability was not known.

Using the framework developed in this paper, we can now resolve this issue, at least for the simpler, isentropic case. We expect that similar considerations hold in the general case.
7.1 Equations

The equations of isentropic magnetohydrodynamics (MHD) appear in basic form as

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0 \\
\partial_t (\rho u) + \text{div}(\rho u^t u) + \nabla p + H \times \text{curl}H = 0 \\
\partial_t H + \text{curl}(H \times u) = 0
\end{cases}
\]

(7.1)

\( \text{div}H = 0, \)

where \( \rho \in \mathbb{R} \) represents density, \( u \in \mathbb{R}^3 \) fluid velocity, \( p = p(\rho) \in \mathbb{R} \) pressure, and \( H \in \mathbb{R}^3 \) magnetic field. With \( H \equiv 0 \), (7.1) reduces to the equations of isentropic fluid dynamics.

Equations (7.1) may be put in conservative form using identity

\( H \times \text{curl}H = (1/2)\text{div}(|H|^2 I - 2H^t H)^{\text{tr}} + H\text{div}H \)

(7.3)

together with constraint (7.2) to express the second equation as

\( \partial_t (\rho u) + \text{div}(\rho u^t u) + \nabla p + (1/2)\text{div}(|H|^2 I - 2H^t H)^{\text{tr}} = 0. \)

(7.4)

They may be put in symmetrizable (but no longer conservative) form by a further change, using identity

\( \text{curl}(H \times u) = (\text{div}u)H + (u \cdot \nabla)H - (\text{div}H)u - (H \cdot \nabla)u \)

(7.5)

together with constraint (7.2) to express the third equation as

\( \partial_t H + (\text{div}u)H + (u \cdot \nabla)H - (H \cdot \nabla)u = 0. \)

(7.6)

7.2 Initial boundary value problem with constraint

Constraint (7.2) is preserved by the flow, hence may be regarded as a constraint on the initial data. Thus, a sufficient condition for well-posedness under constraint (7.2) is that for some form of the equations, rewritten modulo (7.2), there hold well-posedness with respect to general initial data. Under the assumptions of the previous section, i.e. noncharacteristicity, symmetrizability, and the property that characteristics everywhere be geometrically regular, totally nonglancing, or linearly splitting of order two, this is equivalent to satisfaction of the uniform Lopatinski condition for the
chosen representation of the equations. This is sufficient for our present needs, and seems to be the general approach followed in the literature.

To obtain sharp (i.e., necessary) conditions, one should rather use the constraint together with the linearized equations to make a pseudodifferential change of coordinates eliminating one variable, arriving at an unconstrained system in one fewer variables. Equivalently, as described in [ChP], one might by a pseudodifferential choice of augmented variables reduce the unconstrained second-order hyperbolic equation from which the system is derived to an unconstrained first-order system with modified boundary conditions. However, the resulting analysis would be special to the system considered, and not of the general form considered by Majda.

In any case, our analysis here concerns sufficient conditions for stability, and so we shall follow the first, simpler approach and consider the equations:

$$\begin{align*}
\rho_t + u \cdot \nabla \rho + \rho \text{div} u &= 0, \\
\rho (\nabla u + u \cdot \nabla u) + \nabla p + H \times \text{curl} H &= 0, \\
\partial_t H + u \nabla H + (\text{div} u) H - H \cdot \nabla u &= 0.
\end{align*}$$

(7.7)

The Rankine–Hugoniot conditions are deduced from the conservative form of the equations. Using notations $(y, x) \in \mathbb{R}^2 \times \mathbb{R}$ for spatial variables as above, consider a shock front

$$x = \varphi(t, y)$$

(7.8)

with space time normal

$$(-\sigma, n), \quad \sigma = \partial_t \varphi, \quad n = (-\partial y_1 \varphi, \ldots, -\partial y_{d-1} \varphi, 1).$$

(7.9)

The jump conditions read, with $u_n := n \cdot u$ and $H_n = n \cdot H$,

$$\begin{align*}
\begin{cases}
\rho(u_n - \sigma) = 0, \\
[\rho u(u_n - \sigma)] + n \left[ p + \frac{1}{2} |H|^2 \right] - [H_n H] = 0, \\
[(u_n - \sigma)H] - [H_n u] = 0, \\
[H_n] = 0.
\end{cases}
\end{align*}$$

(7.10)

The last jump condition comes from the constraint equation (7.2). Apparently this system of 8 scalar equations is too large. However, projecting the third equation in the normal direction yields $\sigma[H_n] = 0$ which is implied by
the last equation. This shows that (7.10) is made of 7 independent equations, as expected. Denoting by \( u_{tg} \) and \( H_{tg} \) the tangential part of \( u \) and \( H \), that is their orthogonal projection on \( n^\perp \), (7.10) is equivalent to

\[
egin{align*}
\left[ \rho (u_n - \sigma) \right] &= 0, \\
\left[ \rho u (u_n - \sigma) \right] + n \left[ p + \frac{1}{2} |H|^2 \right] - [H_n H] &= 0, \\
\left[ (u_n - \sigma) H_{tg} \right] - [H_n u_{tg}] &= 0, \\
[H_n] &= 0.
\end{align*}
\]

(7.11)

The constraint (7.2) is preserved by the equations in the following sense.

**Proposition 7.1.** Suppose that \( U = (\rho, u, H) \) is a smooth solution of (7.7) for \( t \in [0, T] \) on both side of the front \( x = \varphi(t, y) \) and satisfy the jump conditions (7.11). Assume in addition that the traces of \( u_n - \sigma \) on both side never vanish on the front and at least one among the traces \( u_n^- - \sigma \) and \( \sigma - u_n^+ \) is positive. Then, if the initial value of \( H \) satisfies \( \text{div} H \big|_{t=0} = 0 \), then \( \text{div} H = 0 \) for all \( t \in [0, T] \).

**Proof.** The third equation in (7.7) implies that on both side of the front:

\[
\partial_t H + \text{curl}(H \times u) + (\text{div} H) u = 0.
\]

Given a smooth function \( v \) on both side of the front, denote by \( \tilde{v} \) the distribution equal to \( v \) on both side of the front. The jump condition \( [H_n] = 0 \) implies that \( \text{div} H = \text{div} \tilde{H} \). Moreover, the jump conditions imply that

\[
\partial_t \tilde{H} + \text{curl}(\tilde{H} \times \tilde{u}) = (\partial_t H + \text{curl}(H \times u)) = - (\text{div} H) \tilde{u} = -(\text{div} \tilde{H}) \tilde{u}.
\]

Therefore \( \tilde{w} = \text{div} \tilde{H} \) is a bounded and piecewise continuous solution of

\[
\partial_t \tilde{w} + \text{div}(\tilde{w} \tilde{u}) = 0, \quad \tilde{w}|_{t=0} = 0.
\]

In particular, the jumps of \( w = \text{div} H \) satisfy \( [(u_n - \sigma) w] = 0 \), which could have been derived directly from the equations and the jumps conditions. Note that there are no general uniqueness theorem for equations \( \partial_t w + \text{div}(aw) = 0 \) with \( a \in L^\infty \). But here \( a \) is piecewise smooth ant the front is never characteristic for this equation since we assumed that \( u_n - \sigma \neq 0 \). Moreover, the characteristics are impinging the front at least on one side. Therefore, integrating along characteristics, this implies that \( w = 0 \) and thus that \( \text{div} \tilde{H} = 0 \). 

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7.3 The shock wave problem for MHD

Thanks to Proposition 7.1, we consider the equations (7.7) on both side of a front (7.8), plus the transmission conditions (7.11).

Following the standard approach for linearized stability analysis of fluid interfaces, (see [Maj]) we carry the front location as an additional dependent variable, and make the change of (independent) variables

\[ \tilde{x} := x - \varphi(x', t), \]

to fix the shock location in the new variables at \( z \equiv 0 \). In the new coordinates, the problem becomes

\[ \partial_t U + \sum_{j=1}^{2} A_j(U) \partial_{y_j} U + \tilde{A}_3(U, d\varphi) \partial_{\tilde{x}} U = 0, \quad \text{on } \pm \tilde{x} > 0, \]

with

\[ \tilde{A}_3(U, d\varphi) = A_3(U) - \sum_{j=1}^{2} (\partial_{y_j} \varphi) A_j(U) - (\partial_t \varphi) \text{Id}. \]

The explicit form of the matrices \( A_j \) is given in Appendix A, where it is also verified that the system is hyperbolic symmetric in the sense of Friedrichs as soon as \( c^2 = dp/d\rho > 0 \). The unknowns are \( U = (\rho, u, H) \). The equations are supplemented with the transmission conditions (7.11) which read

\[ \partial_t \varphi[F_0(U)] + \sum_{j=1}^{2} \partial_{y_j} \varphi[F_j(U)] = [F_3(U)] \quad \text{on } \tilde{x} = 0. \]

The stability analysis yields to consider the linearized equations

\[ \partial_t \hat{U} + \sum_{j=1}^{2} A_j(U) \partial_{y_j} \hat{U} + \tilde{A}_3(U, d\varphi) \partial_{\tilde{x}} \hat{U} = \hat{f}, \quad \text{on } \pm \tilde{x} > 0, \]

\[ \partial_t \hat{\varphi}[F_0(U)] + \sum_{j=1}^{2} \partial_{y_j} \hat{\varphi}[F_j(U)] = [\tilde{F}_3(U, d\varphi) \hat{U}] + \hat{g} \quad \text{on } \tilde{x} = 0, \]

with

\[ \tilde{F}_3(U, d\varphi) = F_3'(U) - \sum_{j=1}^{2} (\partial_{y_j} \varphi) F_j'(U) - (\partial_t \varphi) F_0'(U). \]
The Majda-Lopatinski determinant is computed as follows. One consider the linearized equations (7.14) (7.15) for \( U^- \) and \( U^+ \) constant and \( d\phi \) constant. This yields constant coefficients system, depending on parameters \( p = (U^-, U^+, d\phi) \) to which we apply the analysis developed in the previous sections. Equations (7.14) (7.15) comprise a slightly nonstandard, constant-coefficient initial-boundary-value problem on the half-open space \( \tilde{x} > 0 \), in the “doubled” variables

\[
\tilde{U}^+(z) := \dot{U}^+(z), \quad \tilde{U}^-(z) := \dot{U}^-(z),
\]

the new feature being the appearance of front variable \( \dot{\phi} \). As pointed out by Majda [Maj], this may be handled within the standard framework, thanks to the cascaded form of the equations.

For, assuming that the shock front is noncharacteristic, which follows from Lax’ shock conditions, taking the Fourier transform in the transverse coordinates \( y \) and the Laplace transform in \( t \), we obtain a system of ODE in variables \( \hat{U}^+, \hat{U}^- \),

(7.16) \[ \partial_\tilde{x} \hat{U}^\pm + G^\pm(p, \zeta) \hat{U}^\pm = \hat{f}^\pm, \quad \text{on } \tilde{x} \geq 0, \]

with \( G^\pm \) as in (3.2), and boundary condition

(7.17) \[ \dot{\phi} X(p, \zeta) = [\hat{F}_3 \hat{U}] + \hat{g}, \quad \text{on } \tilde{x} = 0, \]

with

\[ X(p, \zeta) = (i\tau - \gamma)[F_0(U)] + \sum_{j=1}^{2} \eta_j[F_j(U)]. \]

Noting that \( \dot{\phi} \) does not appear in the first equation, we may convert it to a separate, standard boundary-value problem by introducing a new, decoupled boundary condition of rank \( N - 1 = 6 \), obtained by taking the inner product of (7.17) with a basis of unit vectors in \( X(p, \zeta)^\perp \). This reduces the boundary conditions to a system of the form

(7.18) \[ M(p, \zeta) \dot{U} = \hat{g}', \quad \dot{\phi} = \Phi(p, \zeta) \dot{U} + \hat{g}_1, \]

indicating how the boundary value problem analysis developed in the previous sections applies to the present framework.

More precisely, one introduces for \( \gamma > 0 \), the spaces \( \mathcal{E}^\pm(p, \zeta) \) of initial conditions corresponding to decaying solutions of (7.16) at \( \pm\infty \). They are generated by the generalized eigenspaces of \( G^\pm(p, \zeta) \) located in \( \{\pm \text{Im} \mu < 0\} \).

Lax’ condition implies that

(7.19) \[ \dim \mathcal{E}^- (p, \zeta) + \dim \mathcal{E}^+(p, \zeta) = N - 1 = 6. \]
The stability condition for the boundary condition (7.17) states that the homogeneous equation with \( \hat{g} = 0 \) has no solution \( (\hat{\varphi}, \hat{U}, \hat{\varphi}) \) in \( \mathbb{C} \times \mathbb{E}^- \times \mathbb{E}^+ \).

In accordance with the general Definition 4.1, we obtain Majda's Lopatinski determinant for the full shock problem,

\[
D(p, \zeta) = \det \left( \hat{F}_3' - \mathbb{E}^-, \hat{F}_3^+ + \mathbb{E}^+, \mathbb{C}X \right),
\]

or

\[
D(p, \zeta) = \det \left( r_1^-, \ldots, r_{p-1}^-, r_{p+1}^-, \ldots, r_7^-, X \right),
\]

where \( r_j^- \) and \( r_j^+ \) are bases for \( \mathbb{E}^- \) and \( \mathbb{E}^+ \).

### 7.4 Applications of Theorem 5.6

Following the analysis presented in the previous subsection, we consider the linearized equations (7.14) (7.15) around a planar shock. Without loss of generality, we can assume that the spatial normal to the front is \( n = (0, 0, 1) \) so that the normal matrix is \( \hat{A}_3 = A_3 - \sigma \text{Id} \). The seven eigenvalues of \( A(\eta, \xi) = \eta_1 A_1 + \eta_2 A_2 + \xi A_3 \) are

\[
\begin{align*}
\lambda_0 &= \eta_1 u_1 + \eta_2 u_2 + (u_3 - \sigma)\xi \\
\lambda_{\pm 1} &= \lambda_0 \pm c_s(|\eta|^2 + |\xi|^2)^{1/2} \\
\lambda_{\pm 2} &= \lambda_0 \pm (\eta H_1 + \eta_2 H_2 + \xi H_3)/\sqrt{\rho} \\
\lambda_{\pm 3} &= \lambda_0 \pm c_f(|\eta|^2 + |\xi|^2)^{1/2}
\end{align*}
\]

with

\[
\begin{align*}
c_s^2 &= \frac{1}{2} \left( c^2 + h^2 \right) - \sqrt{(c^2 - h^2)^2 + 4b^2c^2} \\
c_f^2 &= \frac{1}{2} \left( c^2 + h^2 \right) + \sqrt{(c^2 - h^2)^2 + 4b^2c^2}
\end{align*}
\]

with \( c^2 = p'(\rho) \) the sound speed, \( h^2 = |H|^2/\rho \), \( b^2 = |(\hat{\eta}, \hat{\xi}) \times H|^2/\rho \) and \( (\hat{\eta}, \hat{\xi}) = (\eta, \xi)/(|\eta, \xi|) \) (see Appendix A for the computations). The first eigenvalue corresponds to the transport of the constraint. It can be decoupled from the system: there is a smooth one dimensional subspace, \( \mathbb{E}_0 \) such that \( A(\eta, \xi) = \lambda_0 \) on this space and \( \mathbb{E}_0^\perp \) is stable for \( A(\eta, \xi) \). The other eigenvalues are in general simple.

The boundary \( \{\tilde{x} = 0\} \) is noncharacteristic if and only if

\[
u_3 - \sigma \notin \{ 0, \pm H_3/\sqrt{\rho}, \pm c_s(n), \pm c_f(n) \} \]

where \( c_s(n) \) and \( c_f(s) \) are the slow and fast speed computed in the normal direction \( n = (0, 0, 1) \). The next lemma is proved in Appendix A.
Lemma 7.2. Assume that \( 0 < |H|^2 \neq \rho c^2 \). Consider \((\eta, \xi)\) in the unit sphere \(S^2\).

i) When \((\eta, \xi) \cdot H \neq 0\) and \((\eta, \xi) \times H \neq 0\), the eigenvalues are simple.

ii) On the manifold \((\eta, \xi) \cdot H = 0\) the eigenvalues \(\lambda_{\pm 3}\) are simple and the multiple eigenvalue \(\lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2}\) is geometrically regular.

iii) On the manifold \((\eta, \xi) \times H = 0\) the eigenvalues \(\lambda_{\pm 1} = \lambda_{\pm 2}\) are simple, the other eigenvalues \(\lambda_{\pm 3}\) are double, algebraically regular, and totally nonglancing provided that \(u_3 - \sigma \neq \pm H_3/\sqrt{\rho}\).

This shows that the assumptions of Theorem 5.6 are satisfied as soon as the hyperbolicity condition \(c^2 := dp/d\rho > 0\) holds, and \(H \neq 0\) with length \(|H| \neq c\sqrt{\rho}\) and condition (7.23) is satisfied.

Corollary 7.3. Under the generically satisfied conditions \(0 < |H|^2 \neq \rho \pm (c^\pm)^2\) the uniform Lopatinski condition implies linearized and nonlinear stability of noncharacteristic Lax-type MHD shocks.

Proof. Lemma 7.2 and Theorem 5.6 imply the existence of smooth Kreiss symmetrizers. The stability follows through the standard shock stability framework of Majda-Me2 (see the useful Remark 5.9). \(\square\)

7.5 The \(H \to 0\) limit

Using Corollary 7.3, we easily recover the result of Blokhin and Trakhinin in the isentropic case, by a simple perturbation argument. When \(H = 0\), the system (7.7) reduces to isentropic Euler’s equations and (7.11) to the corresponding Rankine Hugoniot condition. Denote by \(D_0(p_0, \zeta)\) the Majda-Lopatinski determinant of this problem, where the parameters are now \(p_0 = (U_0^-, U_0^+, d\varphi)\) with \(U_0 = (\rho, u)\). Note that because we have everywhere either geometric regularity or total nonglancing, the stable subspace \(E_-\), and thus the Lopatinski determinants have continuous extensions to \(\zeta \in \mathbb{S}_+\) (see Theorem 5.6).

When \(H = 0\), the eigenvalues are

\[
\begin{align*}
\lambda_0 &= \lambda_{\pm 1} = \lambda_{\pm 2} = \eta_1 u_1 + \eta_2 u_2 + \xi (u_3 - \sigma), \\
\lambda_{\pm 3} &= \lambda_0 \pm c(|\eta, \xi|).
\end{align*}
\]

The boundary is noncharacteristic when

\[
(7.25) \quad u_3 - \sigma \notin \{-c, 0, +c\}.
\]
Lemma 7.4. Suppose that $\underline{U} = (\rho, u, 0)$ and $\sigma$ satisfy (A.26) holds. Then the multiple eigenvalue $\lambda_0$ is totally nonglancing at $(\underline{U}, \eta, \xi)$ for all $(\eta, \xi) \neq 0$ and the eigenvalues $\lambda_{\pm3}$ are simple.

The proof is given in the Appendix A. By Theorem 5.6, this implies that the negative spaces depend continuously of $\tilde{H}$ for $\tilde{H}$ close to 0. Therefore, the Lopatinski determinant $D(p, \zeta)$ is a continuous function of $\zeta \in S^3_+$ and $p = (U^-, U^+, d\varphi)$, with $U^\pm = (p^\pm, u^\pm, H^\pm)$, when $U^\pm$ satisfy (7.25) and $H^\pm$ are small. Therefore:

Corollary 7.5. Suppose that $p_0$ is a Lax shock for the insentropic Euler’s equation. Then, as $(H^-, H^+) \to 0$, there holds $D(p, \zeta) \to D_0(p_0, \zeta)$, uniformly for $\zeta \in S^3$, with $\gamma \geq 0$.

Corollary 7.6. In the $H \to 0$ limit, Lax-type MHD shocks approaching a noncharacteristic limiting fluid-dynamical shock satisfy Assumption 6.15 so that the Lopatinski condition reduces to (4.10); moreover, they satisfy the Lopatinski condition if and only if it is satisfied by the limiting fluid-dynamical shock (i.e., satisfies the physical stability conditions of Erpenbeck–Majda [Er, Maj], in which case the MHD shocks are linearly and nonlinearly stable. In particular, for an ideal gas equation of state, they are always stable in the $H \to 0$ limit.

Likewise, using Corollary 7.8, we may immediately convert the results of [GK], [BT1]–[BT4], etc. on satisfaction of the Lopatinski condition to full results of linearized and nonlinear stability. More generally, Corollary 7.8 gives a useful framework for the systematic study of MHD shock stability and other variable-multiplicity PDE problems through the investigation of the linear-algebraic Lopatinski condition.

Remark 7.7. Corollary 7.6 guarantees stability for sufficiently small but nonzero values $H^\pm$. But since the system is symmetric hyperbolic for all $H$, a direct application of Theorem 5.6 provides us with Kreiss symmetrizers which are smooth in $p = (U^-, U^+, d\varphi)$ for $|H^\pm|$ small. Therefore, we have maximal stability estimates for the linearized equations which are uniform in $H^\pm$ for $H^\pm$ small. This implies uniform nonlinear stability, since the time of existence depends only on the estimates (see [Maj], [M2]). This is likely to imply the continuity of the shock solution as $H \to 0$.

Remark 7.8. Calculations as in [M2, Z2] show that Lax-type MHD shocks are stable in the small-amplitude limit $|U| \to 0$, again, for $H_\pm \neq 0$. 

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A Appendix A. The symbolic structure of MHD

A.1 Multiple eigenvalues

The first order term of the linearized equations of (7.7) about \((u, H)\) is

\[
\begin{cases}
D_t\dot{\rho} + \rho \text{div} \dot{u} \\
D_t \dot{u} + \rho^{-1} c^2 \nabla \dot{\rho} + \rho^{-1} H \times \text{curl} \dot{H} \\
D_t \dot{H} + (\text{div} \dot{u}) H - H \cdot \nabla \dot{u}
\end{cases}
\]

with \(D_t = \partial_t + u \cdot \nabla\) and \(c^2 = dp/d\rho\). This system is hyperbolic symmetric with symmetrizer \(S = \text{block-diag}(c^2, \rho \text{Id}, \text{Id})\). The associated symbol is

\[
\begin{cases}
\tilde{\tau} \dot{\rho} + \rho (\xi \cdot \dot{u}) \\
\tilde{\tau} \dot{u} + \rho^{-1} c^2 \dot{\rho} \xi + \rho^{-1} H \times (\xi \times \dot{H}) \\
\tilde{\tau} \dot{H} + (\xi \cdot \dot{u}) H - (H \cdot \xi) \dot{u}
\end{cases}
\]

with \(\tilde{\tau} = \tau + u \cdot \xi\). We use here the notation \(\xi = (\xi_1, \xi_2, \xi_3)\) for the spatial frequencies and

\[\xi = |\xi| \hat{\xi}, \quad u_\| = \hat{\xi} \cdot u, \quad u_\bot = u - u_\| \hat{\xi} = -\hat{\xi} \times (\hat{\xi} \times u).\]

We write (A.2) in the general form \(\tau \text{Id} + A(U, \xi)\) with parameters \(U = (\rho, u, H)\). The eigenvalue equation \(A(U, \xi) \dot{U} = \lambda \dot{U}\) reads

\[
\begin{cases}
\tilde{\lambda} \dot{\rho} = \rho \dot{u}_\| \\
\rho \tilde{\lambda} \dot{u}_\| = c^2 \dot{\rho} + H_\bot \cdot \dot{H}_\bot, \\
\rho \tilde{\lambda} \dot{u}_\bot = -H_\| \dot{H}_\bot, \\
\tilde{\lambda} \dot{H}_\| = \dot{\rho} H_\bot - \dot{H}_\bot \dot{u}_\bot, \\
\tilde{\lambda} \dot{H}_\bot = 0
\end{cases}
\]

with \(\tilde{\lambda} = \lambda - (u \hat{\xi})\). The last condition decouples. On the space

\(\mathbb{E}_0(\xi) = \{ \dot{\rho} = 0, \dot{u} = 0, \dot{H}_\| = 0 \}\),

\(A\) is equal to \(\lambda_0 := u \cdot \xi\). From now on we work on \(\mathbb{E}_0^\bot = \{ \dot{H}_\| = 0 \}\) which is invariant by \(A(p, \xi)\).
Consider \( v = H/\sqrt{\rho}, \dot{v} = \dot{H}/\sqrt{\rho} \) and \( \dot{\sigma} = \dot{\rho}/\rho \). The characteristic system reads:

\[
\begin{aligned}
\dot{\lambda} \sigma &= \dot{u}_\parallel \\
\dot{\lambda} \dot{u}_\parallel &= c^2 \dot{\sigma} + v_\perp \cdot \dot{v}_\perp \\
\dot{\lambda} \dot{u}_\perp &= -v_\perp \dot{v}_\perp \\
\dot{\lambda} \dot{v}_\perp &= \dot{u}_\parallel v_\perp - v_\parallel \dot{u}_\perp
\end{aligned}
\]  
(A.5)

Take a basis of \( \xi^\perp \) such that \( v_\perp = (b, 0) \) and let \( a = v_\parallel \). In such a basis, the matrix of the system reads

\[
\dot{\lambda} - \tilde{A} :=
\begin{bmatrix}
\tilde{\lambda} & -1 & 0 & 0 & 0 \\
-c^2 & \tilde{\lambda} & 0 & 0 & -b \\
0 & 0 & \tilde{\lambda} & 0 & a \\
0 & 0 & 0 & \tilde{\lambda} & 0 \\
0 & -b & a & 0 & \tilde{\lambda} \\
0 & 0 & 0 & a & \tilde{\lambda}
\end{bmatrix}
\]  
(A.6)

The characteristic roots satisfy

\[
(\tilde{\lambda}^2 - a^2)((\tilde{\lambda}^2 - a^2)(\tilde{\lambda}^2 - c^2) - \tilde{\lambda}^2 b^2) = 0.
\]  
(A.7)

Thus, either

\[
\begin{aligned}
\tilde{\lambda}^2 &= a^2 \\
\tilde{\lambda}^2 &= c_f^2 := \frac{1}{2} \left( c^2 + h^2 \right) + \sqrt{(c^2 - h^2)^2 + 4b^2c^2} \\
\tilde{\lambda}^2 &= c_s^2 := \frac{1}{2} \left( c^2 + h^2 \right) - \sqrt{(c^2 - h^2)^2 + 4b^2c^2}
\end{aligned}
\]  
(A.8) (A.9) (A.10)

with \( h^2 = a^2 + b^2 = |H|^2/\rho \).

With \( P(X) = (X - a^2)(X - c^2) - b^2 X \), \( \{ P \leq 0 \} = [c_s^2, c_f^2] \) and \( P(X) \leq 0 \) for \( X \in [\min(a^2, c^2), \max(a^2, c^2)] \). Thus,

\[
\begin{aligned}
c_f^2 &\geq \max(a^2, c^2) \geq a^2 \\
c_s^2 &\leq \min(a^2, c^2) \leq a^2
\end{aligned}
\]  
(A.11) (A.12)

1. The case \( v_\perp \neq 0 \) i.e. \( w = \dot{\xi} \times v \neq 0 \). Thus, the basis such that \( \text{(A.6)} \) holds is smooth in \( \xi \). In this basis, \( w = (0, b), b = |v_\perp| > 0 \).

1.1 The spaces

\[
E_\pm(\hat{\xi}) = \{ \dot{\sigma} = 0, \dot{u}_\parallel = 0, \dot{v}_\perp \in C(\hat{\xi} \times v), \dot{u}_\perp = \mp \dot{v}_\perp \}
\]

are invariant for \( \tilde{A} \) and

\[
\tilde{A} = \pm a \quad \text{on} \quad E_\pm.
\]  
(A.13)
1.2 In $(E_+ \oplus E_-)^\perp$, which is invariant, the matrix of $\tilde{A}$ is

(A.14) $\tilde{A}_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ c^2 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & -b & a & 0 \end{bmatrix}$

Since $P(c^2) = -b^2c^2 < 0$, there holds $c^2_s < c^2 < c^2_f$.

1.2.1 Suppose that $a \neq 0$. Then, $P(a^2) = -a^2c^2 < 0$ and $c^2_s < a^2 < c^2_f$. Thus, all the eigenvalues are simple. Moreover, $c^2_s c^2_f = a^2c^2$ and $c^2_f > 0$. The space

$$F_\lambda = \{ \dot{\sigma} = \tilde{\lambda} \dot{u}_\parallel, \, \dot{u}_\parallel = \frac{\tilde{\lambda} bv_1}{\lambda^2 - c^2}, \, \dot{u}_1 = \frac{-av}{\tilde{\lambda}}, \, v_1 \in C \}$$

is an eigenspace associated to the eigenvalue $\tilde{\lambda}$ when $\lambda = \pm c_f$ and $\lambda = \pm c_s$. Here $\dot{u}_1$ and $\dot{v}_1$ denote the first component of $\dot{u}_\perp$ and $\dot{v}_\perp$ respectively in the basis $(v_\perp, w)$.

1.2.1 Suppose that $a$ is close to 0. Since $c^2_f > c^2 > 0$, the spaces $F_{\pm c_f}$ are still eigenspaces associated to the eigenvalues $\tilde{\lambda} = \pm c_f$.

By direct computations:

$$c^2_s = \frac{c^2 a^2}{c^2 + h^2} + O(a^4).$$

Therefore,

$$\frac{c^2_s}{a^2} \to \frac{c^2}{c^2 + h^2} > 0 \quad \text{as } a \to 0.,$$

Therefore, consider $\tilde{c}_s = \frac{a}{|a|} c_s$ is an analytic function of $a$ (and $b \neq 0$) near $a = 0$ and

$$\tilde{F}_{\pm, \tilde{c}_s}(a, b) = F_{\pm \tilde{c}_s}$$

are analytic determinations of Eigenspaces, associated to the eigenvalues $\pm \tilde{c}_s$. Moreover, the values at $a = 0$ are

$$\tilde{F}_{\pm, \tilde{c}_s}(0, b) = \{ \dot{\sigma} = 0, \, \dot{u}_\parallel = 0, \, \dot{u}_\perp = \frac{\mp cv}{\sqrt{c^2 + b^2}} \}$$

and $\tilde{F}_{+, \tilde{c}_s} \cap \tilde{F}_{-, \tilde{c}_s} = \{0\}$, thus we still have an analytic diagonalization of $\tilde{A}_0$. 

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2. Suppose now that \( b \) is close to zero. At \( b = 0 \), the eigenvalues of \( \tilde{A} \) are \( \pm c \) (simple) and \( \pm h \) (double). **Assume that \( c^2 \neq h^2 \).** Note that when \( b = 0 \), then \(|a| = h\) and

\[
\text{when } c^2 > h^2 : \quad c_f = c, \quad c_s = h,
\]
\[
\text{when } c^2 < h^2 : \quad c_f = h, \quad c_s = c.
\]

2.1 The eigenvalues close to \( \pm c \) remain simple.

2.2 We look for the eigenvalues close to \( h \). The characteristic equation implies that

\[
\begin{align*}
\lambda^2 - a^2 &= \tilde{\lambda} \tilde{u}_\perp \cdot \tilde{v}_\perp \\
(\tilde{\lambda}^2 - c^2) \tilde{u}_\perp &= \tilde{\lambda} \tilde{v}_\perp \cdot \tilde{v}_\perp
\end{align*}
\]
Eliminating \( \tilde{u}_\perp \), we are left with the \( 4 \times 4 \) system in \( \xi_\perp \times \xi_\perp \):

\[
\begin{align*}
\tilde{\lambda} \tilde{u}_\perp &= -a \tilde{v}_\perp \\
\tilde{\lambda} \tilde{v}_\perp &= -a \tilde{u}_\perp + \frac{\tilde{\lambda}}{\lambda^2 - c^2} (v_\perp \otimes v_\perp) \tilde{v}_\perp
\end{align*}
\]
Thus,

\[
(\tilde{\lambda}^2 - a^2) \tilde{v}_\perp = \frac{\tilde{\lambda}^2}{\lambda^2 - c^2} (v_\perp \otimes v_\perp) \tilde{v}_\perp.
\]
Recall that \(|v_\perp| = b\) is small. We recover 4 smooth eigenvalues

\[
\pm a, \quad \pm \sqrt{a^2 + O(b^2)} = \pm (a + O(b^2)).
\]
(remember that \( a = \pm h + O(b^2) \). However, the eigenspaces are not smooth in \( v \), since they are \( \mathbb{R}v_\perp \) and \( \mathbb{R} \xi \times v_\perp \) and have no limit at \( v_\perp \to 0 \).

Summing up, we have proved the following.

**Lemma A.1.** Assume that \( c^2 = dp/d\rho < 0 \). The eigenvalues of \( A(U, \xi) \) are

\[
\begin{align*}
\lambda_0 &= \xi \cdot u \\
\lambda_{\pm 1} &= \lambda_0 \pm c_s(\tilde{\xi}) |\xi| \\
\lambda_{\pm 2} &= \lambda_0 \pm (\xi \cdot H)/\sqrt{\rho} \\
\lambda_{\pm 3} &= \lambda_0 \pm c_f(\tilde{\xi}) |\xi|
\end{align*}
\]

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with $\hat{\xi} = \xi/|\xi|$ and

(A.19) \[ c_f^2(\hat{\xi}) := \frac{1}{2} \left( c^2 + h^2 + \sqrt{(c^2 - h^2)^2 + 4b^2c^2} \right) \]

(A.20) \[ c_s^2(\hat{\xi}) := \frac{1}{2} \left( c^2 + h^2 - \sqrt{(c^2 - h^2)^2 + 4b^2c^2} \right) \]

where $h^2 = |H|^2/\rho$, $b^2 = |\hat{\xi} \times H|^2/\rho$.

Lemma A.2. Assume that $0 < |H|^2 \neq \rho c^2$ where $c^2 = dp/d\rho > 0$.

i) When $\xi \cdot H \neq 0$ and $\xi \times H \neq 0$, the eigenvalues of $A(U,\xi)$ are simple.

ii) On the manifold $\xi \cdot H = 0$, $\xi \neq 0$, the eigenvalues $\lambda_{\pm 3}$ are simple and the multiple eigenvalue $\lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2}$ is geometrically regular.

iii) On the manifold $\xi \times H = 0$, $\xi \neq 0$, $\lambda_0$ is simple. When $|H|^2 < \rho c^2$ [resp. $|H|^2 > \rho c^2$], $\lambda_{\pm 3}$ [resp. $\lambda_{\pm 1}$] are simple; $\lambda_{+2} \neq \lambda_{-2}$ are double, equal to $\lambda_{\pm 1}$ [resp. $\lambda_{\pm 3}$] depending on the sign of $\xi \cdot H$, algebraically regular, but not geometrically regular.

A.2 Glancing conditions for MHD

Write the space variables $(y, x)$, $y \in \mathbb{R}^2$, $x \in \mathbb{R}$ and consider the planar front $\{x = \sigma t\}$. As indicated in Section 7, we perform the change of variables $\tilde{x} = x - \sigma t$, obtaining the linearized system (7.14) with reads

(A.21) \[ \partial_t - \sigma \partial_x + A(U, \partial_y, \partial_x) = \partial_t + A(\tilde{U}, \partial_y, \partial_x) \]

with $\tilde{U} = (\rho, \tilde{u}, H)$, $\tilde{u} = (u_1, u_2, u_3 - \sigma)$. This shows that the eigenvalues of the symbol are

(A.22) \[ \tilde{\lambda}(U, \eta, \xi) = \lambda(\tilde{U}, \eta, \xi) \]

where $(\eta, \xi)$ denote the tangential Fourier frequencies dual to $(y, \tilde{x})$. Therefore, Lemma A.1 implies that the eigenvalues are given by (7.22).

In particular, the eigenvalues of the boundary matrix are

(A.23) \[ u_3 - \sigma, \quad u_3 - \sigma \pm H_3/\sqrt{\rho}, \quad u_3 - \sigma \pm c_f(n), \quad u_3 - \sigma \pm c_s(n) \]

with $n = (0, 0, 1)$. The boundary is noncharacteristic if they are different from zero, that we assume from now on. The next result finishes the proof of Lemma 7.2.

Lemma A.3. Suppose that $0 < |H|^2 \neq \rho c^2$ and the front is noncharacteristic. On the manifold $(\eta, \xi) \times H = 0$, $(\eta, \xi) \neq 0$, the double eigenvalues $\lambda_{+2}$ and $\lambda_{-2}$ are totally nonglancing.
Proof. Consider

(A.24) \[ \lambda_2 = \eta_1 (u_1 + v_1) + \eta_2 (u_2 + v_2) + \xi (u_3 + v_3 - \sigma), \]

with \( v = H/\sqrt{\rho} \). We have shown at (A.17) that the eigenvalue is algebraically regular, and the nearby eigenvalue is

\[ \lambda_2 + O(|\xi \times H|^2). \]

It is tangent to \( \lambda_2 \) to second order. Therefore, the nonglancing condition reduces to

\[ \frac{\partial \lambda_2}{\partial \xi} = u_3 + v_3 - \sigma \neq 0, \]

that is automatically satisfied when the front is noncharacteristic.

We now pass to the proof of Lemma 7.4. When \( H = 0 \), the eigenvalues are

(A.25) \[ \lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2} = \eta_1 u_1 + \eta_2 u_2 + 2(\xi (u_3 - \sigma), \] \[ \lambda_{\pm 3} = \lambda_0 \pm c|\eta, \xi|. \]

The boundary is noncharacteristic when

(A.26) \[ u_3 - \sigma \notin \{-c, 0, +c\}. \]

Lemma A.4. Suppose that \( \bar{U} = (\bar{\rho}, \bar{u}, 0) \) and \( \sigma \) satisfy (A.26) holds. Then the multiple eigenvalue \( \lambda_0 \) is totally nonglancing at \( (\bar{U}, \eta, \xi) \) for all \( (\eta, \xi) \neq 0 \).

Proof. Note that \( \lambda_0 \) has constant multiplicity 5 in \( (\eta, \xi) \) for \( H = 0 \), and is linear in \( (\eta, \xi) \). Therefore, the tangent system at \( (\bar{U}, \eta, \xi) \) is \( \lambda_0 \text{Id} \) in dimension 5. In particular, with notations as in Section 3, the boundary matrix of this system is \( A_3 = (u_3 - \sigma)\text{Id} \), implying that the eigenvalue is totally nonglancing.

B. Maxwell’s equations in a bi-axial crystal

We give here an example where geometric regularity fails. The consequence is that the tangent operator is no longer a transport operator along a single direction, but rather a system of hyperbolic equations of wave equation type. This example permits the description of the phenomenon of conical refraction (cf [Land, Tay, BW]).
Maxwell’s equations in the absence of exterior charge may be written as

\[
\begin{align*}
\frac{\partial t}{t} B + \text{curl} E &= 0, \quad \text{div} B = 0, \\
\frac{\partial t}{t} D - \text{curl} H &= 0, \quad \text{div} D = 0,
\end{align*}
\]

with

\[
D = \mathcal{E} E, \quad B = \mu H
\]

where \(\mathcal{E}\) and \(\mu\) are \(3 \times 3\) positive definite matrices. In the case of a bi-axial crystal, \(\mu\) is scalar and \(\mathcal{E}\) has three distinct eigenvalues. We change variables so that \(\mu = 1\) and choose coordinate axes so that \(\mathcal{E}\) is diagonal:

\[
\mathcal{E}^{-1} = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}
\]

with \(\alpha_1 > \alpha_2 > \alpha_3\). Ignoring the divergence conditions, the characteristic equation and the polarization conditions are obtained as solutions of system

\[
L(\tau, \xi) \begin{pmatrix} B \\ E \end{pmatrix} := \begin{pmatrix} \tau B + \xi \times E \\ \tau E - \mathcal{E}^{-1}(\xi \times B) \end{pmatrix} = 0.
\]

For \(\eta \neq 0\), \(\tau = 0\) is a double eigenvalue, with eigenspace generated by \((\eta, 0)\) and \((0, \eta)\). These modes are incompatible with the divergence conditions. The nonzero eigenvalues are given as solutions of

\[
E = \mathcal{E}^{-1}(\frac{\xi}{\tau} \times B), \quad (\tau^2 + \Omega(\xi)\mathcal{E}^{-1}\Omega(\xi))B = 0
\]

where \(i\Omega(\xi)\) is the symbol of the operator curl. We have

\[
A(\xi) := \Omega(\xi)\mathcal{E}^{-1}\Omega(\xi) = \begin{pmatrix}
-\alpha_2\xi_3^2 & \alpha_3\xi_1\xi_2 & \alpha_2\xi_1\xi_3 \\
\alpha_3\xi_1\xi_2 & -\alpha_1\xi_3^2 - \alpha_3\xi_1^2 & \alpha_1\xi_2\xi_3 \\
\alpha_2\xi_1\xi_3 & \alpha_1\xi_2\xi_3 & -\alpha_1\xi_2^2 - \alpha_2\xi_1^2
\end{pmatrix},
\]

\[
\det(\tau^2 + A(\xi)) = \tau^2(\tau^4 - \Psi(\xi)\tau^2 + |\xi|^2\Phi(\xi))
\]

with

\[
\Psi(\xi) = (\alpha_1 + \alpha_2)\xi_3^2 + (\alpha_2 + \alpha_3)\xi_1^2 + (\alpha_3 + \alpha_1)\xi_2^2,
\]

\[
\Phi(\xi) = \alpha_1\alpha_2\xi_3^2 + \alpha_2\alpha_3\xi_1^2 + \alpha_3\alpha_1\xi_2^2.
\]
The equation $\det(\tau^2 + A(\xi)) = 0$ yields $\tau = 0$ and an equation of second order in $\tau^2$, of which the discriminant is
$$\Psi^2(\xi) - 4|\xi|^2 \Phi(\xi) = P^2 + Q$$
with
$$P = (\alpha_1 - \alpha_2)\xi_3^2 + (\alpha_3 - \alpha_2)\xi_1^2 + (\alpha_3 - \alpha_1)\xi_2^2$$
$$Q = 4(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\xi_3^2 \xi_2^2 \geq 0.$$  

The root is double if and only if
(B.5) $\xi_2 = 0$, $\alpha_1 \xi_3^2 + \alpha_3 \xi_1^2 = \alpha_2(\xi_1^2 + \xi_3^2) = \tau^2$.

Let $(\tau, \xi)$ be a solution of (B.5). For $\theta$ such that $\alpha_3 \cos^2 \theta + \alpha_1 \sin^2 \theta = \alpha_2$, we have
(B.6) $\xi = \lambda b_1$, $\tau = \pm \lambda \sqrt{\alpha_2}$.

where we have introduced the basis of $\mathbb{R}^3$

$$b_1 = \begin{pmatrix} \cos \theta \\ 0 \\ \sin \theta \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} -\sin \theta \\ 0 \\ \cos \theta \end{pmatrix}$$

In (B.6), suppose now that $\tau = \lambda \sqrt{\alpha_2}$, the other case being similar. The kernel of $L(\tau, \xi)$ is of dimension two, generated by
(B.7)
$$u_2 := \begin{pmatrix} b_2 \\ e_2 \end{pmatrix}, \quad u_3 := \begin{pmatrix} b_3 \\ e_3 \end{pmatrix} \quad \text{with} \quad e_2 = \begin{pmatrix} -\delta_1 \sin \theta \\ 0 \\ \delta_3 \cos \theta \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ -\delta_2 \\ 0 \end{pmatrix}$$

and $\delta_j := \alpha_j / \sqrt{\alpha_2}$. As $L(\tau, \xi)$ is self-adjoint, $(u_2, u_3)$ also form a basis of ker $L^* (\tau, \xi)$. The tangent system $L'$ at $(\tau, \xi)$ is obtained by multiplying on the left by a basis for the left kernel and on the right by a basis for the right kernel of $L(\tau, \xi)$, see Remark 2.7. We obtain a $2 \times 2$ system with symbol
(B.8) $\begin{pmatrix} \tau - (\xi_1 \delta_3 \cos \theta + \xi_3 \delta_1 \sin \theta) & \xi_2(\delta_1 - \delta_3) \cos \theta \sin \theta \\ \xi_2(\delta_1 - \delta_3) \cos \theta \sin \theta & \tau - (\xi_1 \delta_2 \cos \theta + \xi_3 \delta_2 \sin \theta) \end{pmatrix}$.

Compare to equations (2.25) and (2.26) obtained in the semisimple and geometrically regular cases. By a change of variables, we arrive at the form
(B.9) $\begin{pmatrix} \tilde{\tau} - \xi_1 & \xi_2 \\ \xi_2 & \tilde{\tau} + \xi_1 \end{pmatrix}$.

This shows that the eigenvalue is linearly splitting.

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Appendix C. The block structure condition

We consider the system

\[ \partial_x + iG(p, \zeta) \]

With \( \zeta = (\tau - i\gamma, \eta) \). Assume that

**Assumption C.1.** The characteristic polynomial in \( \xi \), \( \Delta(p, \zeta, \xi) = \det(\xi \text{Id} + G(p, \zeta)) \), has real coefficients when \( \gamma = 0 \) and has no real roots when \( \gamma > 0 \).

**Definition C.2.** \( G \) has the block structure property near \((p, \zeta)\) if there exists a smooth matrix \( V \) on a neighborhood of that point such that \( V^{-1}GV = \text{diag}(G_k) \) is block diagonal, with blocks \( G_k \) of size \( \nu_k \times \nu_k \), holomorphic in \( \tau - i\gamma \), having one of the following properties:

i) the spectrum of \( G_k(p, \zeta) \) is contained in \( \{\text{Im} \mu \neq 0\} \),

ii) \( \nu_k = 1 \), \( G_k(p, \zeta) \) is real when \( \gamma = 0 \), and \( \partial_\gamma G_k(p, \zeta) \neq 0 \),

iv) \( \nu_k > 1 \), \( G_k(p, \zeta) \) has real coefficients when \( \gamma = 0 \), there is \( \mu_k \in \mathbb{R} \) such that

\[ G_k(p, \zeta) = \mu_k \text{Id} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ \cdots & \cdots & \cdots & 0 \end{bmatrix}, \]

and the lower left hand corner of \( \partial_\gamma Q_k(p, \zeta) \) does not vanish.

**Theorem C.3.** The block structure condition is always satisfied when \( \gamma > 0 \). When \( \gamma = 0 \), it holds if and only if for all real root \( \xi \) of \( \Delta(p, \zeta, \xi) = 0 \),

i) there are smooth functions \( \lambda_j(p, \eta, \xi) \), analytic in \( \xi \), real when \( \xi \) is real, near \((p, \eta, \xi)\) such that

\[ \Delta(p, \zeta, \xi) = e(p, \zeta, \xi) \prod_{j=1}^2 (\tau - i\gamma + \lambda_j(p, \eta, \xi)) \]

with \( e(p, \zeta, \xi) \neq 0 \) and \( \lambda_j(p, \eta, \xi) + \tau = 0 \),

ii) there are smooth vectors \( e_j(p, \eta, \xi) \) near \((p, \eta, \xi)\), analytic in \( \xi \), linearly independent, such that

\[ (\xi \text{Id} + G^{(j)}(p, \eta, \xi)) e_j(p, \eta, \xi) = 0 \]

where \( G^{(j)} \) is the matrix \( G \) evaluated at \( \tau - i\gamma = -\lambda_j(p, \eta, \xi) \).
Proof. The first statement is clear from Assumption 1. Moreover, one can always make a first block diagonal reduction with blocks \( G_k \) associated to distinct eigenvalues of \( G(p, \zeta) \). Non real roots always satisfy condition \( i \) of Definition 2. So it is sufficient to consider separately each block, associated to one real eigenvalue. Thus, it is sufficient to prove the theorem when

\[
\xi \text{Id} + G(p, \zeta) = 0, \quad \xi \in \mathbb{R}.
\]

\( a) \) Suppose that the block structure condition holds. Then

\[
\Delta(p, \zeta, \xi) = \prod \Delta_k(p, \zeta, \xi), \quad \Delta_k(p, \zeta, \xi) := \det (\xi \text{Id} + G_k(p, \zeta)).
\]

The form \( 2 \) of \( G_k \), implies that

\[
\det (\xi + G_k(p, \zeta - i\gamma, \eta)) = (-1)^{\nu_k-1} c\gamma + O(\gamma^2)
\]

where \( c \) is the lower left hand corner of \( \partial_\tau Q_k(p, \zeta) \). Since \( c \neq 0 \) and \( Q_k \) is holomorphic in \( \tau - i\gamma \), this implies that

\[
\partial_\tau \Delta_k(p, \zeta, \xi) \neq 0.
\]

Since \( \Delta_k \) is real when \( \xi \) and \( \tau - i\gamma \) are real, the implicit function theorem implies that there is a smooth function \( \lambda_k(p, \eta, \xi) \), analytic in \( \xi \) such that

\[
\Delta_k(p, \zeta, \xi) = e(p, \zeta, \xi)(\tau - i\gamma + \lambda_k(p, \eta, \xi))
\]

on a neighborhood of \( (p, \zeta, \xi) \), with \( e(p, \zeta, \xi) \neq 0 \).

Decompose the matrix \( V \) into \( (V_1, \ldots, V_k) \) where \( V_k \) is of size \( N \times \nu_k \) and denote by \( V_k^{(k)}(p, \eta, \xi) \) the matrix \( V_k \) evaluated at \( \tau - i\gamma = -\lambda_k(p, \eta, \xi) \). At the base point

\[
V_k^{(k)}(p, \eta, \xi) = V_k(p, \xi).
\]

If \( \nu_k = 1 \), then the coefficient \( \xi + Q_k(p, \zeta) \) vanishes at when \( \tau - i\gamma = -\lambda_k \) and \( r_k = V_k^{(k)} \) is a smooth element in the kernel of \( \xi + G_k(p, \eta, \xi) \).

If \( \nu_k > 1 \), consider next the first \( \nu_k - 1 \) rows of the eigenvector equation \((\xi \text{Id} + G_k)u = 0\). Denoting by \( v_1 \) the first component and by \( v' \) the remaining \( \nu_k - 1 \), these equations read

\[
Q(p, \zeta, \xi)v' = v_1 h(p, \zeta, \xi)
\]
with \( Q(p, \zeta, \xi) = \text{Id} \) and \( h(p, \zeta, \xi) = 0 \). Therefore, one can solve \( v' = v_1 Q^{-1} h \). When \( \tau - i \gamma = -\lambda_k (p, \eta, \xi) \), the determinant of \( \xi \text{Id} + G_k^{(k)} (p, \eta, \xi) \) vanishes. This shows that the kernel of \( \xi \text{Id} + G_k^{(k)} (p, \eta, \xi) \) has dimension one and is spanned by the smooth vector \( e_k = t(1, Q^{-1} h) \) where the functions are evaluated at \( \tau - i \gamma = -\lambda_k (p, \eta, \xi) \). This implies that \( r_k := V(k) e_k \) is a smooth vector in the kernel of \( \xi \text{Id} + G(k) \). By \((C.8)\), the vectors \( r_k \) are linearly independent at \((p, \eta, \xi)\), and thus remain independent in the vicinity of that point.

\( b) \) Conversely, suppose that \((C.4)\) and \((C.5)\) are satisfied. By \((C.5)\), there is only one real eigenvalue \(-\xi\) at \((p, \zeta)\). For all \( j \), there is an integer \( \nu_j \geq 1 \) such that

\[
\partial^{\nu_j} \lambda_j = \ldots = \partial^{\nu_j-1} \lambda_j = 0, \quad \partial^{\nu_j} \lambda_j \neq 0 \quad \text{at} \quad (p, \eta, \xi).
\]

Repeating the analysis in \([M\text{e}3]\), this implies that

\[
\tau - i \gamma + \lambda_j (p, \eta, \xi) = e_j (p, \zeta, \xi) \Delta_j (p, \zeta, \xi)
\]

with \( e_j (p, \zeta, \xi) \neq 0 \) and \( \Delta_j (p, \zeta, \xi) \) a monic polynomial in \( \xi \), of degree \( \nu_j \), with real coefficients when \( \gamma = 0 \), such that

\[
\Delta_j (p, \zeta, \xi) = (\xi - \xi^{(j)}).
\]

Suppose that \( \nu_j = 1 \). Then, \( \Delta_j = \xi - \mu_j (p, \zeta) \) with \( \mu_j \) smooth and \( \tilde{e}_j (p, \zeta) = e_j (p, \eta, \mu_j) \) is a smooth eigenvector of \( G(p, \zeta) \) with eigenvalue \( \mu_j \). Comparing with \((C.11)\), implies that \( \partial_\gamma \mu_j (p, \zeta) \neq 0 \). Thus, the one dimensional space \( \mathbb{C} \tilde{e}_j \) provides us with a one dimensional block which satisfies property \( ii) \) of Definition \((C.2)\).

Suppose next that \( \nu_j > 1 \). Differentiate the equation \((C.4)\) with respect to \( \xi \), at \((p, \eta, \xi)\). Since \( G^{(j)}(p, \eta, \xi) = G(p, \zeta) + O( (x - \xi)^{\nu_j} ) \), this implies that

\[
e_{j,k} = \frac{(-1)^k}{k!} \left( \partial_\xi e_j \right)(p, \eta, \xi)
\]

satisfy

\[
(x + G(p, \zeta)) e_{j,0} = 0,
\]

\[
(x + G(p, \zeta)) e_{j,k} = -e_{j,k-1}, \quad k = 1, \ldots, \nu_j - 1.
\]

The definition of the \( e_{j,k} \) is extended to \((p, \zeta)\) in a neighborhood of \((p, \zeta)\), as in \([M\text{e}3]\):

\[
e_{j,k} (p, \zeta) = \frac{(-1)^k (\nu_j - k - 1)!}{2 \pi i \nu_j !} \int_{|\xi - \xi| = r} \frac{\partial^{k+1} \Delta_j (p, \zeta, \xi)}{\Delta_j (p, \zeta, \xi)} e_j (p, \eta, \xi) d\xi.
\]
where \( r > 0 \) is such that \( \Delta_j(p, \zeta, \xi) \) has no root on the circle \( \{ |\xi - \xi| = r \} \). By (C.12), the two definitions agree at \( (p, \zeta) \), so that \( e_{j,k}(p, \zeta) = e_{j,k} \). Repeating the analysis of \( \text{Mc3} \), one shows that the space \( E_j(p, \zeta) \) spanned by the \( e_{j,k}(p, \zeta) \) is invariant by \( G(p, \zeta) \). It is smooth and of dimension \( \nu_j \).

Moreover, using (C.14) and the independence of the \( e_{j,k}(p, \zeta) \), one shows that \( e_{j,k} \), for \( k \in \{ 0, \ldots, \nu_j - 1 \} \) and \( j \in \{ 1, \ldots, \tilde{j} \} \) are linearly independent.

Therefore, this provides a smooth decomposition of \( \mathbb{C}^N \) into invariant subspaces of \( G \), thus a block decomposition of \( G \). Moreover the characteristic polynomial of \( Q_j \) is \( \Delta_j \). The property (C.2) is stated in (C.14).

The proof goes on as in \( \text{Mc3} \). For each block, one can perform an additional change of bases such that

\[
Q_k = Q_k(p, \zeta) + \begin{bmatrix}
q_1(z) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
q_{\nu}(z) & 0 & \cdots & 0
\end{bmatrix}.
\]

(see [Rai]). Thus,

\[
\Delta_j(p, \zeta, \xi)(\xi - \xi)\nu + \sum_{l=1}^{\nu} (-1)^l q_l(z)(\xi - \xi)\nu - l
\]

Since \( \Delta_j \) has real coefficients when \( \gamma = 0 \), this implies that the \( q_l(z) \) and therefore \( Q_k(p, \zeta) \) is real when \( \gamma = 0 \). In addition, by (C.11),

\[
\frac{\partial \Delta_j}{\partial \gamma}(p, \zeta, \xi) = \frac{\partial q_\nu}{\partial \gamma}(p, \zeta) \neq 0.
\]

Therefore, \( \tilde{Q}(z) \) satisfies the property iv) of Definition C.2 and the proof of Theorem C.3 is complete.

D Appendix D. 2 \times 2 sytems

Consider the 2 \times 2 complex system in space dimension 2:

(D.1) \( \tau \text{Id} + \xi A(p) + \eta B(p) \)

Assumption D.1. The system is strictly hyperbolic in the direction \( dt \) and noncharacteristic and nonhyperbolic in the direction \( dx \).
D.1 Reductions

In particular, the eigenvalues of $A$ are real and distinct. Thus, there is a smooth change of basis such that $A$ is diagonal, with nonvanishing real diagonal coefficients $a_1 \neq a_2$. One can perform changes of variables which preserve the directions $dt$ and $dx$:

(D.2) \[ \tau' = \tau + \alpha \eta, \quad \xi' = \xi + \beta \eta, \quad \eta' = \eta. \]

This transform (D.1) to

(D.3) \[ \tau' \text{Id} + \xi' A(p) + \eta' B'(p). \]

With appropriate choices of $\alpha(p)$ and $\beta(p)$, one can cancel the the diagonal terms in $B'$:

(D.4) \[ B' = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \]

Strict hyperbolicity implies that $bc$ is real and positive. By a change of basis, one can assume that $b = c$ is real, so that the system reads

(D.5) \[ \tau' \text{Id} + \xi' \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \eta' \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \]

The assumption implies that $a_1 a_2 < 0$. With $a = (-a_1/a_2)^{1/2} > 0$ and

(D.6) \[ \tau'' = \tau', \quad \xi'' = \frac{a}{a_1} \xi', \quad \eta'' = b \eta, \]

we obtain the reduced form

(D.7) \[ \tau'' \text{Id} + \xi'' \begin{pmatrix} a & 0 \\ 0 & -1/a \end{pmatrix} + \eta'' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

It remains only one parameter $a = a(p) > 0$.

D.2 Negative spaces

Dropping the $''$ we consider (D.7), and more precisely

(D.8) \[ \xi \text{Id} + G(p, \tau, \eta) := \xi \text{Id} + \begin{pmatrix} \tau/a & \eta/a \\ -a \eta & -a \tau \end{pmatrix} \]

The eigenvalue equation for $G$ is

(D.9) \[ \mu^2 + a \tau \mu - \frac{1}{a} \tau \mu = \tau^2 - \eta^2. \]
If $\mu + a\tau \neq 0$, the corresponding eigenspace is

\[(D.10) \quad E = Cr, \quad r = \left( -a\eta/(\mu + a\tau) \right). \]

For $\text{Im} \tau < 0$, there is one eigenvalue $\mu$ in $\text{Im} \mu < 0$. We denote by $E_-(\tau, \eta)$ and $r(\tau, \eta)$ the associated eigenspaces and eigenvectors.

Let $z = \eta/(\mu + a\tau)$. Then, for $z \neq 0$, \[(D.11) \quad (a + 1/a)\tau = \eta(z + 1/z). \]

In particular,

\[(D.12) \quad (a + 1/a)\text{Im} \tau = \eta\text{Im} z \left( 1 - \frac{1}{|z|^2} \right). \]

When $\text{Im} \tau < 0$ and $\eta \neq 0$, $\text{Im} (\mu + a\tau) < 0$ and $\text{Im} \eta z > 0$, and \[(D.12) \quad \text{implies that } |z| < 1. \]

Conversely, for $|z| < 1$, $\text{Im} z \neq 0$, we can choose $\eta = \pm 1$ such that $\eta\text{Im} z > 0$, implying that $\tau$ defined by \[(D.11) \quad \text{satisfies } \text{Im} \tau < 0. \]

Then $\mu = -a\tau + \eta/z$ is a root of \[(D.9) \quad \text{Im} \mu = -a\text{Im} \tau - \frac{\eta\text{Im} z}{|z|^2} = -\frac{\eta\text{Im} z (1 + a^2|z|^2)}{1 + a^2} < 0. \]

This shows that the mapping $(\tau, \eta) \mapsto z$ sends $\{\text{Im} \tau < 0; \eta \neq 0\}$ onto $\{|z| < 1; \text{Im} z \neq 0\}$. By continuity argument, or by direct computation, this implies the following.

**Lemma D.2.** The union of the complex lines $E_-(\tau, \eta)$ for $\text{Im} \tau \leq 0$ and $(\tau, \eta) \neq 0$ is the cone $\Gamma \subset \mathbb{C}^2$ of vectors $u = (u_1, u_2)$ such that $a|u_2| \leq |u_1|$.

**Corollary D.3.** The boundary condition $Mu = u_1 - cu_2 = 0$ satisfies the Lopatinski condition, if and only if $a|c| < 1$.

### D.3 Symmetrizers

In the form \[(D.7) \quad \text{or } (D.5) \]

there is a unique symmetrizer (up to a scalar factor) which is the identity matrix. Tracing back to the original system \[(D.1) \]

yields a unique (up to scalars) symmetrizer $S(p) = V^*(p)V(p)\Sigma$ where $V$ is the change of basis. The symmetrizer for $G$ is $\Sigma = SA$.

**Proposition D.4.** The boundary condition $Mu = 0$ satisfies the Lopatinski condition if and only if it is maximal dissipative.
Proof. This is invariant by the change of basis. For systems (D.7), maximal dissipative and uniform Lopatinski conditions are such that \( \ker M = \{ u_1 = cu_2 \} \). Since

\[
(Su, u) = a|u_1|^2 - \frac{1}{a}|u_2|^2,
\]

the boundary condition is maximal dissipative if and only if this form is definite negative on \( \ker M \), that is if and only if \( a^2|c|^2 < 1 \). \( \square \)

### D.4 2 × 2 linearly splitting blocks

We are given a \( 2 \times 2 \) matrix \( G(q, \tau, \eta, \gamma) \), \( \zeta = (\tau, \eta, \gamma) \) in a neighborhood of 0 in \( \mathbb{R}^3 \), and \( q \) a set of parameters in a neighborhood of \( \underline{q} \) with the property that

\[
(D.13) \quad G(q, \zeta) = \phi(q) \text{Id} + \tilde{G}(q, \zeta), \quad \tilde{G}(q, 0) = 0.
\]

**Assumption D.5.**

i) The characteristic polynomial in \( \xi \), \( \Delta(q, \zeta, \xi) = \det (\xi \text{Id} + G(q, \zeta)) \), has real coefficients when \( \gamma = 0 \).

ii) The first order expansion in \( \zeta \) of \( \tilde{G} \) at \( \underline{q} \)

\[
(D.14) \quad \tilde{G}'(\hat{\zeta}) = (\hat{\tau} - i\hat{\gamma})G_0 + \hat{\eta}G_1
\]

where \( \det G_0 < 0 \) and \( \tau \text{Id} + G_0^{-1}(\xi + \eta G_1) \) is strictly hyperbolic.

**Proposition D.6.** There is a smooth self adjoint matrix \( \Sigma(p, \zeta) \) on a neighborhood of \( (q, 0) \) such that

i) \(-\text{Im}(\Sigma G) = \gamma E\), with \( E(q, 0) \) positive definite,

ii) with \( \Sigma = \Sigma(q, 0) \), \( \Sigma G_0 \) and \( \Sigma G_1 \) are self adjoint.

**Proof.**

a) We can assume that \( G_0 \) is diagonal. Introduce the notations

\[
(D.15) \quad \tilde{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad G_0 = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.
\]

The assumption implies that

\[
(D.16) \quad a_0 = \partial_\tau a(q, 0) \text{ and } d_0 = \partial_\tau b(q, 0) \text{ are real and } a_0d_0 < 0,
\]

\[
\begin{align*}
b_1 &= \partial_\tau b(q, 0), & c_1 &= \partial_\tau c(q, 0), & b_1c_1 \text{ is real and } b_1c_1 < 0.
\end{align*}
\]

By assumption, there holds

\[
(D.17) \quad a + d \in \mathbb{R}, \quad ad - bc \in \mathbb{R} \quad \text{when } \gamma = 0.
\]
We look for a symmetrizer

(D.18) \( \Sigma = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}, \quad \alpha, \delta \in \mathbb{R}, \beta \in \mathbb{C}. \)

b) We show that one can choose such that

(D.19) \( \text{Im}(\Sigma G) = 0 \) when \( \gamma = 0. \)

In this part, we assume that \( \gamma = 0. \) (D.19) is equivalent to the conditions

(D.20) \( \text{Im}(\alpha a + \beta c) = 0, \)
(D.21) \( \text{Im}(\beta b + \delta d) = 0, \)
(D.22) \( (\pi - d)\beta = \alpha b - \delta \bar{c}. \)

Because of (D.17), \( \sigma := \pi - d = \text{Re}(a - d) \) is real and by (D.16), \( \partial_\tau \sigma(q, 0) \neq 0. \) Therefore, we can take \( (q, \sigma, \eta) \) as local coordinates near \( (q, 0, 0), \) and we first solve the equation

(D.23) \( \alpha b - \delta \bar{c} = 0, \quad \text{on } \sigma = \gamma = 0. \)

By (D.23), there holds

(D.24) \( b|_{\sigma=0,\gamma=0} = \eta b_1 + O(\eta^2), \quad c|_{\sigma=0,\gamma=0} = \eta c_1 + O(\eta^2) \)

Moreover, when \( \sigma = 0, \) \( ad = |a|^2 \) and by (D.17), \( bc \) is real. Therefore, factoring out \( \eta \) in (D.23), we see that there is a smooth local solution \( \alpha, \delta, \) such that

(D.25) \( \alpha(q, \zeta) \in \mathbb{R}, \quad \delta(q, \zeta) \in \mathbb{R}, \quad \alpha(q, 0) = |c_1|^2, \quad \delta(q, 0) = b_1 c_1. \)

By (D.23), one can factor out \( \sigma \) in \( \alpha b - \delta \bar{c} \) restricted to \( \gamma = 0 \) and there is a smooth function \( \beta \) such that (D.22) holds when \( \gamma = 0. \) In particular, since \( b \) and \( c \)

(D.26) \( \beta(q, 0) = 0. \)

For \( \gamma = 0 \) and \( \sigma \neq 0, \)

\[ \text{Im}(\alpha a + \beta c) = \frac{\alpha}{\sigma} \text{Im}(a \sigma + bc) = \frac{\alpha}{\sigma} \text{Im}(bc - ad) \]

vanishes by (D.17). Thus, (D.20) and similarly, (D.21) are satisfied on \( \gamma = 0. \) Therefore, we have constructed a smooth solution \( \Sigma(q, \zeta) \) of (D.19).
Moreover, differentiating this equation with respect to \( \tau \) and \( \eta \), implies that

\[(D.27) \quad \text{Im} (\Sigma G_0) = 0, \quad \text{Im} (\Sigma G_1) = 0.\]

The first equality, \((D.15)\) and \((D.16)\), imply that \( \beta(q,0) = 0 \). Thus, with \((D.25)\), we see that

\[(D.28) \quad \Sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \quad \text{with} \quad \alpha \delta < 0.\]

c) Changing \( \Sigma \) to \( -\Sigma \) if necessary, we can assume that \( \alpha a_0 > 0 \), implying that \( \delta d_0 > 0 \), hence that \( \Sigma G_0 \) is definite positive.

The identity \((D.19)\) implies that \( \text{Im} (\Sigma G) = -\gamma E \) with \( E \) depending smoothly on \((q, \zeta)\) in a neighborhood of \((q,0)\). Moreover, at \( q = q, \tau = \eta = 0 \), there holds

\[\text{Im} (\Sigma G) = \text{Im} (-i\gamma \Sigma G_0) + O(\gamma^2)\]

showing that \( E(q,0) = \Sigma G_0 \) is definite positive. \( \square \)

**Theorem D.7.** Suppose that Assumption \( D.5 \) is satisfied and that \( M(q,\zeta) \) is a \( 1 \times 2 \) matrix such that the equation

\[(D.29) \quad \partial_x u + iG(q,\zeta)u = f, \quad M(q,\zeta)u|_{x=0} = g\]

satisfies the uniform Lopatinski condition near \((q,0)\). Then, there is a smooth symmetrizer \( \Sigma(q,\zeta) \) near \((q,0)\) and there are constants \( c > 0 \) and \( C \), such that

\[(D.30) \quad -\text{Im} (\Sigma G) \geq c\gamma \text{Id}, \quad \text{when} \quad \gamma \geq 0,\]

\[(D.31) \quad -\Sigma + CM^*M \geq c\text{Id}.\]

**Proof.** The symmetrizer \( \Sigma \) has been constructed in Proposition \( D.6 \). To prove that \( D.31 \) holds on a neighborhood of \((q,0)\), it is sufficient to prove that it holds at \((q,0)\), thus it is sufficient to check that \( -\Sigma \) is definite positive (or \( \Sigma \) definite negative) on \( \ker M \), where \( M = M(q,0) \).

In addition, since \( \Sigma G_0 \) is self adjoint and definite positive, and \( \Sigma G_1 \) is self adjoint, \( \Sigma \) is a symmetrizer for \( \mathcal{G}' = (\tau - i\gamma)G_0 + \eta G_1 \). Thus, by Proposition \( D.4 \), it is sufficient to prove that the boundary condition \( M \) satisfies the uniform Lopatinski condition for the operator \( \partial_x + i\mathcal{G}' \).

This follows from the analysis in Section 4. We have shown that the negative space for \( G(q,\zeta) \) with \( \gamma > 0 \), is the negative space for a matrix
\( \hat{G}(q, \rho, \hat{\zeta}) \) with \( \rho = |\zeta| \) and \( \hat{\zeta} = \zeta/|\zeta| \). Moreover, \( \hat{G}(q, 0, \hat{\zeta}) = G'(\hat{\zeta}) \). The Lopatinski condition implies that

\[
(D.32) \quad |u| \leq C|M(q, \rho \hat{\zeta})u|
\]

for all \( u \in E_{-}(q, \rho, \hat{\zeta}) \). As \( \rho \) tends to zero, this implies that

\[
(D.33) \quad |u| \leq C|M(q, 0)u|
\]

for all \( u \in E_{-}(q, 0, \hat{\zeta}) \) and \( |\hat{\zeta}| = 1 \) with \( \hat{\gamma} > 0 \). This means that \( M \) does satisfy the uniform Lopatinski condition for \( \partial_x + iG' \).

\[ \Box \]

### Appendix E. The viscous case

In conclusion, we briefly discuss the case of a hyperbolic system with viscous regularization, that is, an \( N \times N \) second order system with symbol

\[
L_{\nu}(p, \tau, \xi) = \tau \text{Id} + A(p, \xi) + \nu B(p, \xi)
\]

\[ (E.1) \]

\[
= \tau \text{Id} + \sum_{j=1}^{d} \xi_j A_j(p)s - i\nu \sum_{j,k=1}^{d} \xi_j \xi_k B_{j,k}(p)s
\]

We make the assumptions of symmetrizability of the first-order part, i.e., existence of positive definite symmetric \( S \) such that \( SA_j \) is symmetric for all \( j \), and dissipativity of the second-order part, i.e. \( B(p, \xi) = \sum_{j,k} \xi_j \xi_k SB_{j,k} \geq 0 \), with no eigenvector of \( A(p, \xi) \) lying in the kernel of \( B(p, \xi) \) (the “genuine coupling” condition of [Kaw]).

**The small-viscosity limit.** In [McZ1, GWMZ2, GWMZ3, GMWZ4], there was considered, under the additional assumption of constant multiplicity of the first-order part, the problem of obtaining maximal estimates generalizing those of (4.5), where \( \nu \) is the coefficient of viscosity (“Reynolds number”) in (E.1) and \( \zeta = (\tau, \eta) \) as in previous sections denotes Laplace–Fourier transform frequencies, for \( \nu|\zeta| \) sufficiently small. Such small-frequency
estimates, together with analogous intermediate- and high-frequency estimates not depending on the constant-multiplicity assumption, were used to verify, respectively, existence of viscous boundary and shock layers in the \( \nu \to 0 \) limit, with rigorous convergence to associated formal asymptotic series.

A consequence of the matrix perturbation analysis of [MeZ1] is that the reduced equations \( G^1_{\text{viscous}} \) for the viscous system for a hyperbolic mode associated with basis \( V \) satisfies the intuitively appealing relation

\[
(E.3) \quad V G^1_{\text{viscous}} = GV = (G^{\text{inviscid}} + \nu B(p, \xi)) V,
\]

where \( V = R(p, \xi) \) is a matrix of right eigenvectors of \( A(p, \xi) \) associated with the single eigenvalue \( \tau \), and thus \( V^* S = L(p, \xi) \) is a matrix of left eigenvectors, where \( S \) is the symmetrizer of the system.

A well-known consequence of the dissipativity/genuine coupling assumption (see especially [Kaw, KSh]) is that \( LBR(p, \xi) = V^* SBV \) is uniformly positive definite. From this fact, it is straightforward to see that, under the structural hypotheses of Assumption 6.15, the smooth hyperbolic symmetrizer \( \Sigma = V^* S \) constructed in Section 6 near points of variable multiplicity serves also as a symmetrizer for the viscous system, yielding the desired maximal estimate (E.2). This extends the results of the above papers to the variable-multiplicity case under Assumption 6.15 in particular yielding existence of boundary and shock layers for MHD under the uniform Lopatinski (Evans function) condition.

We remark that a continuity argument like that in Section 7 shows that the uniform Evans function condition reduces in the vanishing-magnetic field \((H \to 0)\) limit to the corresponding condition on the limiting fluid-dynamical shock. The uniform Evans function condition is always satisfied for sufficiently small-amplitude shocks in either gas dynamics or MHD [PZ, FS].

**Long-time stability of viscous shock waves.** In [Z1, Z2, Z3, GWMZ2], there was considered under the constant-multiplicity assumption together with an additional technical hypothesis (satisfied for gas dynamics and MHD) that the glancing set have a certain “foliated structure”, the related problem of time-asymptotic \( L^1 \cap H^s \to L^2 \cap H^s \) stability, for \( s \) sufficiently large, of planar viscous shock profiles, for which the relevant symbol is \( (E.1) \) with constant coefficients and \( \nu = 1 \). Again, the constant-multiplicity hypothesis was used only in the small-frequency regime, this time to obtain maximal \( L^1 \to L^p \) stability estimates, \( p \geq 2 \). Away from the glancing set, the only way in which the constant-multiplicity assumption was used was to show that the real part of “slow”, or “hyperbolic” eigenvalues was bounded
below by multiples of $\gamma + |\zeta|$. But, it is a straightforward exercise to show that this is also implied by the existence of a symmetrizer $\Sigma G^1 \geq C^{-1}(\gamma + |\zeta|)$. 

This extends the results of the above papers to the variable-multiplicity case under Assumption 6.15, in particular yielding long-time stability of Lax- and overcompressive type shock waves for MHD under the refined Lopatinski (Evans function) condition defined therein.

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