3-Dimensional Homogenization Finite Element Analysis of Foamed Rubber

Abstract. In this study, 3-dimensional homogeneous finite element analysis of a truncated octahedral unit cell which consisted of hyperelastic beams was conducted to predict macroscopic mechanical characteristics of the foam rubbers considering its microscopic structure. The homogenization theory was applied to finite element analysis code and the unit cells for simulation were assumed to have the periodic boundary condition. An original finite-strain homogenization FEM code for hyperelastic material and truncated octahedral unit cell were developed. The relative density of the homogenization analysis was adjusted by width of the beams in unit cell model. The rubber matrix was assumed to be represented by the nearly-incompressible hyperelasticity. The Mooney-Rivlin model and nearly-incompressible condition were applied to the hyperelasticity. The material parameters of the Mooney-Rivlin model for foamed rubber matrix were identified by the tensile loading test results. The developed code showed enough stability to predict the mechanical property of foamed rubber in the large strain region.

1. Introduction

Foam rubbers such as polyurethane foam have been used widely in transport equipment, sports equipment, living ware, and so on. The advantage reasons of which the foam rubber are used, are large deformable performance, soft stiffness, durability for cyclic deformations and manufacturing formability. The foam rubber includes numerous microscopic cavities inside of the solid body. So, the mechanical characteristics of the foam rubber are strongly depending on mechanical properties of rubber matrix and its microstructure. The relative density is important material parameter which defined mechanical calculated by the volume and density of foam rubber. The relative density which is calculated by the ratio of rubber matrix in foam rubber are used as parameter which present material characteristics. Material design using numerical simulation considering the microstructure would be required to improve quality, size and productivity of foam rubber products. In this study, 3-dimensional homogeneous finite element analysis of a truncated octahedral unit cell which consisted of hyperelastic beams was conducted to predict macroscopic mechanical characteristics of foam rubbers (shown in Fig. 1). The homogenization theory was applied to original finite element analysis code of hyperelasticity and the unit cells for simulation were assumed to have the periodic boundary condition. An original finite-strain homogenization FEM code for hyperelastic material and truncated octahedral unit cell models for the homogenization analysis were newly developed. The relative density of the homogenization analysis was adjusted by width of the beams in the unit cell models. The rubber matrix was assumed to be represented by the nearly-incompressible hyperelasticity. The Mooney-Rivlin model and the nearly-incompressible condition were applied to the hyperelasticity. The material parameters of the Mooney-Rivlin model for foamed rubber matrix were identified by the tensile loading test results. Homogenization FE-analysis were conducted to verify the applicability of developed code.
2. Homogenization finite element analysis

In this study, the microstructure of the foam rubber like polyurethane foam was assumed to have the periodic structure for analysis. The formulation of homogenization analysis for large deformation was described with two different coordinates $X$ and $Y$. The macroscopic behavior was defined in $X$-coordinate, and the microscopic behavior was defined in $Y$-coordinate. The two coordinates were related by the scale ratio $\varepsilon$ as follows:

$$ Y = X / \varepsilon $$(1)

When the scale of the microstructure was much smaller than the scale of whole structure, the scale ratio $\varepsilon$ was a very small value. Therefore, macroscopic characteristics, such as stiffness, stress and strain were calculated from the average of microscopic characteristics using the homogenization theory. The all microscopic structure deformed uniformly, and microscopic periodicity was kept under finite deformation. The total position of microscopic structure $\mathbf{y}$ was divided into macroscopic displacement $\mathbf{Y}$ and microscopic periodical displacement $\mathbf{w}$ as follows:

$$ \mathbf{y} = \mathbf{\bar{F}} \mathbf{Y} + \mathbf{w} $$ (2)

where $\mathbf{\bar{F}}$ is the macroscopic deformation gradient tensor. The deformation gradient tensor $\mathbf{\bar{F}}$ was given as the boundary condition of numerical simulation. The microscopic deformation gradient tensor $\mathbf{F}$ was calculated as follows:

$$ \mathbf{F} = \nabla \mathbf{y} = \mathbf{\bar{F}} + \frac{\partial \mathbf{w}}{\partial \mathbf{Y}} $$ (3)

The macroscopic deformation gradient tensor $\mathbf{\bar{F}}$ was supposed to be constant and calculated as average of volume integration of microscopic deformation gradient tensor $\mathbf{F}$, so $\mathbf{\bar{F}}$ was calculated as follows:

$$ \mathbf{\bar{F}} = \frac{1}{V} \int_{Y_0} \mathbf{F} dY $$ (4)

Also, the average of volume integration of microscopic deformation gradient $\mathbf{F}$ was calculated using Equation 4 as follows:

$$ \frac{1}{V} \int_{Y_0} \mathbf{F} dY = \frac{1}{V} \int_{Y_0} \left( \mathbf{\bar{F}} + \frac{\partial \mathbf{w}}{\partial \mathbf{Y}} \right) dY = \mathbf{\bar{F}} + \frac{1}{V} \int_{Y_0} \left( \frac{\partial \mathbf{w}}{\partial \mathbf{Y}} \right) dY $$ (5)

From Equation 5, it is necessary to satisfy the following equation. By giving $\mathbf{w}$ a periodic boundary condition, the following equation was satisfied automatically.

$$ \frac{1}{V} \int_{Y_0} \left( \frac{\partial \mathbf{w}}{\partial \mathbf{Y}} \right) dY = 0 $$ (6)

In order to accurate evaluation in consideration of nearly-incompressibility of the hyperelastic material, the displacement/pressure mixing method with the Langrangian multiplier $p$ was applied to the FEM formulation. Therefore, the strain energy function of nearly incompressible hyperelastic material was defined as follows:

$$ W(\mathbf{\bar{C}}, p) = \tilde{W}(\mathbf{\bar{C}}) + p(f - 1) - U_C(p) $$ (7)
Here, $p$ is the Lagrangian multiplier which is correspond to the hydrostatic pressure, $\bar{W}(\bar{C})$ is the elastic potential function, $C$ is the right Cauchy-Green deformation tensor, $\bar{C}$ is the modified right Cauchy-Green deformation tensor removing the volume component, $U_c(p)$ is the complementary strain energy expressing the relation between volumetric strain and pressure. $U_c(p)$ was expressed using the bulk modulus $k$ as follows:

$$U_c(p) = \frac{p^2}{2k}$$

(8)

Also, $C$ and $J$ were expressed as follows:

$$C = F^T F$$

(9)

$$J = \det(F)$$

(10)

where, $J$ is the determinant of the deformation gradient tensor and represented the volumetric deformation. The elastic potential $\bar{W}(\bar{C})$ was expressed using the modified first principal invariant $\tilde{I}_1$ and the modified second principal invariant $\tilde{I}_2$ of $\bar{C}$ as follows:

$$\bar{W}(\bar{C}) = \bar{W}(\tilde{I}_1, \tilde{I}_2)$$

(11)

Here, the modified deformation gradient tensor $\bar{F}$ that doesn’t contain volume components, $\bar{C}$, $\tilde{I}_1$ and $\tilde{I}_2$ were expressed as follows:

$$\bar{C} = \bar{F}^T \bar{F} = J^{-2/3} F^T F$$

(12)

The second Piola-Kirchhoff stress tensor $S$ of hyperelastic material is given by the partial differentiation of the strain energy function $W$ with respect to the right Cauchy-Green deformation tensor $C$ as follows:

$$S = 2 \frac{\partial W}{\partial C}$$

(13)

Fig. 1. FEM model based on a truncated octahedron
The periodic boundary condition was set on the boundary surface of FEM model. The macroscopic deformation gradient tensor $\tilde{F}$ was given by multiplying the deformation gradient tensor $F_0$ by the rotation tensor $R$ as follows:

$$\tilde{F} = R \cdot F_0 \cdot R^T$$  \hspace{1cm} (14)

### 3. Material modeling and simulated results

To verify the applicability of the homogenization analysis to foam rubber, compression tests of polyurethane foam and tensile loading tests of polyurethane matrix were conducted.

In Fig. 2, compression stress-strain behavior of polyurethane foam specimens were shown. The relative density of compression specimens were 3.73%, 4.82% and 5.94%, respectively.

In order to identify the material parameters of polyurethane matrix, tensile loading tests on 3-solid specimens were conducted. The solid specimens were filled with the polyurethane foam matrix. Material parameters of the hyperelasticity were identified by using the stress-strain relationship of the solid specimens shown in Fig. 3.

The truncated octahedral unit-cell for foam rubber like polyurethane were constructed. As shown in Fig. 4, it is three-dimensional models based on a truncated octahedron. Relative densities for the homogenization analysis were adjusted based on the width of the beams in unit cell. Uniform macroscopic deformation $\tilde{F}$ was given as analysis condition and periodic microscopic displacement $w$ was calculated by homogenization FEM analysis. Homogenization analysis of foam rubbers were conducted to evaluate applicability of developed code.

![Fig. 2. Stress-strain relationships of compression tests of the polyurethane foam](image-url)
The relative densities of the homogenization analysis were adjusted to 3.7%, 4.8% and 5.9%. The stress-strain relationships are shown in Fig. 5. The stress-strain relationships calculated by the homogenization analysis shows good agreement with the experimental results shown in Fig. 3.

The diagrams of the compression deformation of the 5.94%-relative density unit cell are shown in Fig. 6. Vertical deformation was applied to the truncated octahedron unit cell under the periodic boundary condition which is shown in Fig. 1. The lateral stress was constrained to be zero by the penalty method. The simulated results had good stability with the compression deformation to a strain of 0.5.

![Stress-strain relationships of tensile test of solid specimens which was filled with polyurethane matrix](image)

**Fig. 3.** Stress-strain relationships of tensile test of solid specimens which was filled with polyurethane matrix
Fig. 4. Definition of load direction for analysis, $\phi$ is rotation angle around Z axis, $\theta$ is rotation angle around Y axis.

Fig. 5. The stress-strain relationships calculated by the homogenization method.
4. Conclusion
An evaluation method for foam rubber that considers its microstructure was investigated in this study. In order to reproduce the foam rubber’s microstructure, three-dimensional homogeneous FEM analysis of truncated octahedral unit cells were conducted. The microstructure of foam rubber like polyurethane was assumed to have the periodic structure. The polyurethane matrix was assumed to be represented by the nearly-incompressible hyperelasticity. The developed FEM code with homogenization theory shows enough stability to represent the mechanical characteristics of foam rubbers.

Acknowledgment
This work was supported by JSPS KAKENHI Grant Number 15K05671.

References
[1] Zhang J, Kikuchi N, Li V, Yee A and Nusholtz G 1998 Constitutive Modeling of Polymer Foam Material Subjected to Dynamic Crash Loading International J. Impact Engine. 21 5 pp 369-386
[2] Youssef S Marie E and Gaertner R 2005 Finite Element Modelling of the Actual Structure of Cellular Materials Determined by X-ray Tomography Acta Materials 53 pp 719-730
[3] Gibson L J and Ashby M F 1997 Cellular Solids: Structure and Properties (Cambridge Solid State Science Series) (Cambridge: Cambridge University Press) Chapter 3 pp 175-231
[4] Shimazu R, Yautaka H, Nomoto A and Matsuda A 2015 3-dimensional Homogenization FEM Analysis of Hyperelastic Low Density Polymer Foams European Conference on Constitutive Models for Rubbers IX pp 675-681

[5] Shinoda Y and Matsuda A 2013 Homogenization Analysis of Porous Polymer Considering Microscopic Structure Procedia Engine. 60 pp 343 – 348

[6] Terada K, Hori M and Kyoya T and Kikuchi N 2000 Simulation of the Multi-scale Convergence in Computational Homogenization Approaches Int. J. Solids Structures 37 pp 2285-2311

[7] Treloar L R G 2005 The Physics of Rubber Elasticity (Oxford Classic Series)(Oxford: Oxford University Press) Chapter 10 pp 211-229