Large Rank-Based Models with Common Noise

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Abstract. For large systems of Brownian particles interacting through their ranks introduced in (Banner, Fernholz, Karatzas, 2005), the empirical cumulative distribution function satisfies a porous medium PDE. However, when we introduce a common noise, the limit is no longer deterministic. Instead, we show that this limit is a solution of a stochastic PDE related to this porous medium PDE. This stochastic PDE is somewhat similar to the equations developed for conservation laws with rough stochastic fluxes (Lions, Perthame, Souganidis, 2013).

1. Introduction

We study interacting particle systems $X^{(n)}_1(t), \ldots, X^{(n)}_n(t), t \geq 0,$ on the real line, governed by the following system of stochastic differential equations:

$$dX^{(n)}_i(t) = b(F^{(n)}(t)(X^{(n)}_i(t))) dt + \sigma(F^{(n)}(t)(X^{(n)}_i(t))) dB^{(n)}_i(t) + \gamma(t, \rho^{(n)}(t)) dW(t), \quad i = 1, \ldots, n; \tag{1.1}$$

$$\rho^{(n)}(t) := \frac{1}{n} \sum_{i=1}^n \delta_{X^{(n)}_i(t)}.$$ 

Here, $F_\mu$ is the cumulative distribution function of a probability measure $\mu,$ and $b : [0, 1] \to (0, \infty),$ $\sigma : [0, 1] \to (0, \infty)$ are given functions. We fix a $p > 1$ and denote by $\mathcal{P}_p$ the space of all probability measures on $\mathbb{R}$ with finite $p$th moment, so $\gamma : [0, \infty) \times \mathcal{P}_p(\mathbb{R}) \to \mathbb{R}$ is another given function. Finally, $W, B^{(n)}_1, B^{(n)}_2, \ldots, B^{(n)}_n,$ are i.i.d. standard Brownian motions.

In the absence of the common noise: $\gamma \equiv 0,$ the system (1.1) reduces to a rank-based model. Originally, rank-based model appeared as a special case in the context of the piecewise linear filtering problem in [BP] where weak uniqueness is established (weak existence being a consequence of the general result in [SV, Exercise 12.4.3]). Rank-based models have attracted a lot of attention recently since their appearance in stochastic portfolio theory where they are used to model stock prices in large equity markets ([FK, section 13], [BFK], [IPS], [CP]). Let $S_i(t)$ denote the market capitalization (the number of shares, multiplied by the share price) of the $i$th company, $i = 1, \ldots, n,$ listed in any of the major stock exchanges. Then $(\log S_1(t), \ldots, \log S_n(t))$ is modeled as a rank-based model. A limitation of these rank-based models used for large equity markets is that independent Brownian motions drive stock prices (even though rank-based particles themselves do not evolve independently). A richer model, with correlated Brownian motions driving the rank-based particles, would probably better capture the characteristics of a large equity market. The model in (1.1) is a step in this direction.

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In this paper, we are concerned with the hydrodynamic limit \((n \to \infty)\) of the particle system in (1.1). This would give us an understanding of the behavior of the whole market under the assumption that the number of firms operating in the market is large. It turns out that understanding the limiting behaviour \((n \to \infty)\) of the rank-based particles is paramount to deriving the hydrodynamic limit of the particle system in (1.1). Keeping this in mind, we will state some known results about the limiting behavior of the rank-based particles. Under suitable regularity conditions, it was shown in [JR, Proposition 2.1] that the measure-valued processes \(\mu(n)\), \(n \in \mathbb{N}\) (see below) converge (in a certain functional space, see below) to a deterministic limit \(\mu\). In [DSVZ, Theorem 1.4], it was shown that the empirical cumulative distribution function around its limit are governed by a suitable SPDE; see [Gi, Definition 3], where

\[ H \]

A central limit theorem type result was obtained in [KoS, Theorem 1.2]: fluctuations of the empirical cumulative distribution function around its limit are governed by a suitable SPDE; and a large deviations result was obtained in [DSVZ, Theorem 1.4]

Since there is common noise in (1.1), we cannot hope that as \(n \to \infty\), all noise will be canceled. The limit \(\rho\), in other words, will be stochastic: it satisfies the SPDE (1.5) below. Let \(\mathcal{H}^\beta(\mathbb{R})\) denote the Hölder space, see [LSU, Pg. 7]. For a \(p \geq 1\) and a metric space \(E\), \(\mathcal{P}_p(E)\) is the space of probability measures on \(E\) with finite \(p\)th moment, equipped with Wasserstein distance \(\mathcal{W}_p\), which is defined for \(\mu, \nu \in \mathcal{P}_p(\mathbb{R})\) as follows:

\[ \mathcal{W}_p(\mu, \nu) = \inf_{(Y_1, Y_2)} \mathbb{E}[|Y_1 - Y_2|^p]^{1/p}, \]

with the inf over random vectors \((Y_1, Y_2)\) such that \(Y_1 \sim \mu\) and \(Y_2 \sim \nu\). Also, \(C([0, T], E)\) is the space of continuous functions \([0, T] \to E\) with sup-distance, and \(D([0, T], E)\) stands for the Skorohod space of right-continuous functions with left limits from \([0, T]\) to \(E\).

We fix a \(p > 1\) and a time horizon \(T > 0\).

**Assumption 1.1.** (a) The functions \(b, \sigma\) are differentiable, and \(b', \sigma' \in \mathcal{H}^\beta(\mathbb{R})\) for a \(\beta > 0\).
(b) The function \(\sigma\) is bounded away from zero: \(\min_{a \in [0, 1]} \sigma(a) > 0\).
(c) The function \(\gamma\) is bounded and Lipschitz with respect to \(\mathcal{W}_1\):

\[ |\gamma(t, \nu_1) - \gamma(t, \nu_2)| \leq L_\gamma \mathcal{W}_1(\nu_1, \nu_2), \quad \text{for all } t \in [0, T] \text{ and } \nu_1, \nu_2 \in \mathcal{P}_1. \]

(d) There exists a measure \(\lambda \in \mathcal{P}_p\) such that \(X_1^{(n)}(0), X_2^{(n)}(0), \ldots, X_n^{(n)}(0) \sim \lambda\) i.i.d.
(e) The cumulative distribution function of \(\lambda\) satisfies \(F_\lambda(\cdot) \in \mathcal{H}^{2+\beta}(\mathbb{R})\).

We are now ready to state the main result of the paper.

**Theorem 1.2.** Under Assumption 1.1, for every \(n\) the system (1.1) has a unique solution in the weak sense, unique in law. For every \(T > 0\) and \(q \in [1, p]\), the sequence \((\rho^{(n)})_{n \geq 1}\) of random elements in \(C([0, T], \mathcal{P}_q)\) weakly converges to \(\rho\), a unique solution in \(C([0, T], \mathcal{P}_1)\) to the following functional equation:

\[ F_{\rho(t)}(x) = R(t, x - \Gamma(t)), \quad \Gamma(t) := \int_0^t \gamma(\rho(s))dW(s). \]
The function \( G(t, \cdot) := F_{\rho(t)}(\cdot) \) solves the following SPDE:

\[
(1.5) \quad dG = \left[ -B(G)_x + \Sigma(G)_{xx} + \frac{1}{2} G_{xx}(t, \rho(t)) \right] dt - \gamma(t, \rho(t)) G_x dW(t).
\]

Remark 1.3. The SPDE in (1.5) is very closely related to stochastic scalar conservation laws introduced by Lions, Perthame and Souganidis, [LPS]. They introduced the notion of pathwise entropy solutions to stochastic scalar conservation laws and this theory was extended to a certain class of problems in [GS]. In particular, if \( b = 0 \) and \( \gamma = 1 \), the SPDE in (1.5) reduces to the SPDE in [GS Equation 1.1] with \( F(x) = x \) and \( A(u) = \frac{1}{2} \sigma(u)^2 \) and in this case \( G(t, x) = R(t, x - W(t)) \) is a solution of the SPDE. This observation opens the door to further research on stochastic scalar conservation laws from the perspective of rank-based models with common noise.

Remark 1.4. The functional equation in (1.4) admits an explicit representation when the function \( \gamma(t, \nu) = f(t) \), where \( f \) is any continuous function defined on \([0, \infty)\). Another case of special interest is the function \( \gamma(t, \nu) = \int_{\mathbb{R}} f(x) \nu(dx) \), where the function \( f \) defined on \( \mathbb{R} \) is differentiable and has a bounded derivative. Integrating by parts, we get:

\[
(1.6) \quad \int_{\mathbb{R}} f(x) \nu(dx) = -\int_{\mathbb{R}} f'(x) F_{\nu}(x) dx.
\]

Using representation of \( W_1 \) from [SW p.64], we prove the Lipschitz property of \( \gamma \):

\[
|\gamma(t, \nu_1) - \gamma(t, \nu_2)| = \left| \int_{\mathbb{R}} f'(x) [F_{\nu_1}(x) - F_{\nu_2}(x)] dx \right| \\
\leq \sup_{x \in \mathbb{R}} |f'(x)| \int_{\mathbb{R}} |F_{\nu_1}(x) - F_{\nu_2}(x)| dx = \sup_{x \in \mathbb{R}} |f'(x)| \cdot W_1(\nu_1, \nu_2).
\]

2. Proof of Theorem 1.2

2.1. Overview of the proof. For notational convenience, we assume \( \gamma \) does not depend on the time variable. We split the proof into four sections. In the first, we simply prove existence and uniqueness for the finite system (1.1). In the second section, we prove tightness of the sequence \((\rho^{(n)})\) of empirical measures in \(D([0, T], \mathcal{P}_q)\). In the third section, we prove that every weak limit point \( \rho \) has the cumulative distribution function \( F_{\rho(t)} \) solving (1.4). In the fourth section, we derive (1.5) from (1.4). Finally, in the fifth section we prove uniqueness for solutions of (1.4).

2.2. Existence and uniqueness of the finite system. First, let us show weak existence and uniqueness in law of the system (1.1). Define

\[
(2.1) \quad Y_i^{(n)}(t) = X_i^{(n)}(t) - \int_0^t \gamma(\rho^{(n)}(s)) dW(s), \quad \mu^{(n)}(t) := \frac{1}{n} \sum_{i=1}^n \delta_{Y_i^{(n)}(t)}.
\]

It is straightforward to check that

\[
(2.2) \quad F_{\rho^{(n)}(t)}(x) = F_{\mu^{(n)}(t)}(x - \int_0^t \gamma(\rho^{(n)}(s)) dW(s)); \quad F_{\rho^{(n)}(t)}(X_i^{(n)}(t)) = F_{\mu^{(n)}(t)}(Y_i^{(n)}(t)).
\]

Therefore, (2.1) satisfy the system of equations similar to (1.1), but with \( \gamma = 0 \). This is our key observation. The classic system of competing Brownian particles \( Y^{(n)} = (Y_1^{(n)}, \ldots, Y_n^{(n)}) \)
exists in the weak sense and is unique in law. We can rewrite (2.1) as follows:

\begin{equation}
X_i^{(n)}(t) = Y_i^{(n)}(t) + \int_0^t \gamma(\rho^{(n)}(s))dW(s), \quad 0 \leq t \leq T.
\end{equation}

Define the space \( \mathcal{X} \) of all random elements on our filtered probability space with values in the metric space \( C([0,T], \mathcal{P}_1(\mathbb{R})) \), and with finite second moment. Define the mapping \( \Phi : \mathcal{X} \to \mathcal{X} \) as follows: Fix \( \mu \in \mathcal{X} \) and fix some realization \( \mu(\omega) \). For \( t \in [0,T] \), \( \Phi(\mu)(t) \) is the empirical distribution of \( n \) particles

\[ X_i^{(n,\mu)}(t) := Y_i^{(n)}(t) + \int_0^t \gamma(\mu(s))dW(s), \quad i = 1, \ldots, n, \]

Then (2.3) is equivalent to saying that \( \rho^{(n)} \) is a fixed point of the mapping \( \Phi \). Couple \( \Phi(\mu)(t) \) and \( \Phi(\nu)(t) \) as the uniform measure on the set \( \{(X_i^{(n,\mu)}(t), X_i^{(n,\nu)}(t)) \mid i = 1, \ldots, n \} \) (if some of these points coincide, we count them twice in this measure). From properties of the Itô integral and Lipschitz condition (1.3),

\[ \mathbb{E} \mathcal{W}_t^2(\Phi(\mu)(t), \Phi(\nu)(t)) \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} |X_i^{(n,\mu)}(t) - X_i^{(n,\nu)}(t)|^2 \]

\begin{equation}
\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left| \int_0^t \{\gamma(\mu(u)) - \gamma(\nu(u))\} dW(u) \right|^2 \leq \int_0^t \mathbb{E} (\gamma(\mu(u)) - \gamma(\nu(u)))^2 du \leq L_2 \int_0^t \mathbb{E} \mathcal{W}_s^2(\mu(s), \nu(s)) ds.
\end{equation}

Iterating \( \Phi \) and integrating by parts, we get similarly to [KS Section 5.2.B, (2.19)]

\[ \mathbb{E} \sup_{0 \leq s \leq t} \mathcal{W}_s^2(\Phi^k(\mu)(s), \Phi^k(\nu)(s)) \leq \frac{(4L_2^2)^k}{k!} \mathbb{E} \sup_{0 \leq s \leq t} \mathcal{W}_s^2(\mu(s), \nu(s)). \]

The rest of the proof is the standard argument, see [KS Section 5.2.B, (2.19)]: We fix \( \nu_0 \in \mathcal{P}_1 \), use the Borel-Cantelli lemma to prove convergence of the sequence \( (\Phi^n(\nu_0)) \) a.s. in \( C([0,T], \mathcal{P}_1) \) to some \( \nu \). This limit satisfies \( \Phi \nu = \nu \). From (2.4), we get uniqueness of the fixed point for \( \Phi \). This completes the proof of existence and uniqueness.

2.3. Tightness. Let us show that the sequence \( (\rho^{(n)})_{n \geq 1} \) is tight in \( D([0,T], \mathcal{P}_q) \) for \( q < p \). We follow the proof of [ILS Lemma 7.4]. Apply Itô’s formula to \( (\rho_i^{(n)}, f) \) for \( f \in C^2_b(\mathbb{R}) \) (the space of \( C^2 \) functions \( \mathbb{R} \to \mathbb{R} \) bounded together with their first and second derivatives):

\[ d(\rho_i^{(n)}, f) = \frac{1}{n} \sum_{k=1}^n f'(X_i^{(n)}(t))b(X_i^{(n)}(t))dt \]

\begin{equation}
+ \frac{1}{n} \sum_{k=1}^n f'(X_i^{(n)}(t))\sigma(X_i^{(n)}(t))dB_i(t) + \frac{1}{n} \sum_{k=1}^n f'(X_i^{(n)}(t))\gamma(\rho_i^{(n)})dW(t) \\
+ \frac{1}{2n} \sum_{i=1}^n f''(X_i^{(n)}(t))\sigma^2(X_i^{(n)}(t))dt + \frac{1}{2n} \sum_{i=1}^n f''(X_i^{(n)}(t))\gamma^2(\rho_i^{(n)})dt.
\end{equation}

Since \( f', f'', b, \sigma, \gamma \) are bounded, the equation (2.5) has the form

\[ d(\rho_i^{(n)}, f) = \alpha_n(t)dt + \theta_n(t)dB_n(t), \quad n = 1, 2, \ldots; \quad t \in [0,T], \]
for uniformly bounded $\alpha_n$ and $\theta_n$, and for Brownian motions $\tilde{B}_n$. Thus application of standard tools gives us tightness of $(\rho_{i(n)}^n, f)_{n \geq 1}$ in $C[0,T]$, and therefore in $D[0,T]$. From the Burkholder-Davis-Gundy inequality [KS, Theorem 3.28] and boundedness of $b, \sigma, \gamma$, we get:

$$\mathbb{E}\left[\max_{0 \leq t \leq T} |X_i^{(n)}(t)|^p\right] \leq C < \infty.$$  

Thus, the sequence of measure-valued processes $(\rho^{(n)})$ satisfies for $f_p(x) := |x|^p$:

$$\mathbb{E}\sup_{0 \leq t \leq T} \left(\rho_{i(n)}^n, f_p\right) \leq C.$$  

Take any $\eta > 0$, and consider the subset $\mathcal{K} := \{\nu \in \mathcal{P}_q \mid (\nu, f_p) \leq C/\eta\}$, which is compact in $\mathcal{P}_q$ by [ILS, Lemma 2.2]. From the standard Markov inequality, we have:

$$\mathbb{P}\left[\rho_{i(n)}^n \notin \mathcal{K} \quad \forall t \in [0,T]\right] > 1 - \eta.$$  

Next, take the algebra $\mathfrak{A}$ in $C_b(\mathcal{P}_q)$, the space of bounded continuous functions $\mathcal{P}_q \to \mathbb{R}$, generated by $\mathfrak{M} := \{(., f) \mid f \in C^2_b\}$. This set $\mathfrak{M}$ separates points: for every $\nu'$ and $\nu''$ in $\mathcal{P}_q$, there exists an $f \in C^2_b$ such that $(\nu', f) \neq (\nu'', f)$. This set $\mathfrak{M}$ also contains 1, because $f_0 = 1 \in C^2_b$. By the Stone-Weierstrass theorem [F, Section 4.7], the algebra $\mathfrak{A}$ is dense in $C_b(\mathcal{P}_q)$ in the topology of uniform convergence on compact subsets. Note that $(\rho_{i(n)}^n, f)$ is uniformly bounded for $f \in C^2_b(\mathbb{R})$. Therefore, for every collection $g_1, \ldots, g_m \in C^2_b(\mathbb{R})$, the following sequence is tight in $C[0,T]$ (and therefore in $D[0,T]$, [Bi, Section 13]):

$$(\rho_{i(n)}^n, g_1)(\rho_{i(n)}^n, g_2) \cdots (\rho_{i(n)}^n, g_m); \quad n = 1, 2, \ldots$$  

Therefore, for every $\Phi \in \mathfrak{A}$, the following sequence is tight in $D[0,T]$:

$$\Phi(\rho_{i(n)}^n), t \in [0,T]; \quad n = 1, 2, \ldots$$  

Apply criteria of relative compactness: [EK, Proposition 3.9.1], and complete the proof.

2.4. Characterization of weak limits. In this step, we will characterize any weak limit point $\rho = (\rho_t, 0 \leq t \leq T)$ of $(\rho^{(n)})$. We shall think in terms of cumulative distribution functions: Let $(\rho^{(n)})$ be any subsequence weakly converging in $D([0,T], \mathcal{P}_q)$ to $\rho$, and let $F_k(t, \cdot)$ be the cumulative distribution function of $\rho_{i(n)}^{(m_k)}$. Without loss of generality, by the Skorohod representation theorem we can assume convergence a.s. in $D([0,T], \mathcal{P}_q)$.

The standard approach to derive the limit is to adapt the arguments in [Jo, Lemma 1.5]; however, the arguments in [IR] cannot be extended to prove uniqueness of solutions for (1.5). We adopt a different and a much simpler approach to derive the limit and to prove uniqueness of limits. We again use the idea from subsection 2.1: Reduce the particle system in (1.1) to the rank-based system, with $\gamma = 0$. We use the notation from there. The arrow $\xrightarrow{p}$ indicates convergence in probability.

Under Assumption (1.3), the Cauchy problem (1.2) admits a unique solution $R$ (in the distributional sense) with distributional derivative $R_x$, which is a classic function, and

$$C_* := \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} |R_x(t, x)|.$$  

We claim that:

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \left| F_k(t, x) - R\left(t, x - \int_0^t \gamma(\rho(s))dW(s)\right)\right| \xrightarrow{p} 0, \quad k \to \infty.$$
Similarly to (1.4), we use the following notation for shorthand:

$$\Gamma(t) := \int_0^t \gamma(\rho(s)) \, dW(s), \quad \Gamma_k(t) := \int_0^t \gamma(\rho^{(n_k)}(s)) \, dW(s)$$

In view of (2.1) and (2.2), we need only to show convergence in probability:

$$(2.9) \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |F_{\mu^{(n_k)}(t)}(x - \Gamma_k(t)) - R(t, x - \Gamma(t))| \xrightarrow{p} 0.$$ 

We apply the triangle inequality to bound from above the left-hand side of (2.9):

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |F_{\mu^{(n_k)}(t)}(x - \Gamma_k(t)) - R(t, x - \Gamma(t))|$$

$$+ \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |R(t, x - \Gamma_k(t)) - R(t, x - \Gamma(t))|$$

$$\leq \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |F_{\mu^{(n_k)}(t)}(x) - R(t, x)| + C_* \sup_{t \in [0, T]} |\Gamma_k(t) - \Gamma(t)|.$$ 

The claim and its proof in proposition [KoS, Equation 5.17, Proposition 5.1] imply:

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |F_{\mu^{(n_k)}(t)}(x) - R(t, x)| \xrightarrow{p} 0.$$ 

It remains to show that

$$(2.11) \sup_{t \in [0, T]} |\Gamma_k(t) - \Gamma(t)| \xrightarrow{p} 0.$$ 

In light of Assumption 1.1 (c), we can estimate

$$(2.12) \langle \Gamma_k - \Gamma \rangle_T = \int_0^T (\gamma(\rho_{n_k})(t)) - \gamma(\rho(t))) \, dt \leq L_\gamma^2 \int_0^T \mathcal{W}_1^2(\rho^{(n_k)}(t), \rho(t)) \, dt.$$ 

We have the following convergence in law, and therefore in probability:

$$(2.13) \int_0^T \mathcal{W}_1^2(\rho^{n_k}(s), \rho(s)) \, ds \leq T \cdot \sup_{0 \leq t \leq T} \mathcal{W}_1^2(\rho^{n_k}(s), \rho(s)) \to 0$$ 

Combining (2.12) and (2.13) with [KS, Chapter 1, Problem 5.25], we prove (2.11). This completes the proof of (2.9). Next,

$$(2.14) \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |F_k(t, x) - F_{\mu(t)}(x)| \, dx = \sup_{0 \leq t \leq T} \mathcal{W}_1(\rho^{(n_k)}(t), \rho(t)) \to 0$$

in law (and therefore in probability) as $k \to \infty$. Combining (2.9) with (2.14), we get: for any bounded interval $I \subseteq \mathbb{R}$, almost surely,

$$\sup_{0 \leq t \leq T} \int_I |F_{\mu(t)}(x) - R(t, x - \int_0^t \gamma(\rho(s)) \, dW(s))| \, dx = 0.$$ 

Thus almost surely for every $(t, x) \in [0, T] \times I$ we get (1.4). Representing the real line as a countable union of such intervals $I$, and noting that intersection of countably many almost sure events is also almost sure, we prove (1.4) a.s. for all $(t, x) \in [0, T] \times \mathbb{R}$.
2.5. **Derivation of the SPDE.** Assumption 1.1(e) with [JR] Lemma 2.7 yield classical regularity for $R$. Apply Itô’s formula to $G(t, x) = F_{\rho(t)}(x)$ in (1.4):

$$
\begin{align*}
    dG &= \frac{\partial R}{\partial t} (t, x - \Gamma(t)) \, dt - \frac{\partial R}{\partial x} (t, x - \Gamma(t)) \gamma(\rho(t)) \, dW(t) + \frac{1}{2} \frac{\partial^2 R}{\partial x^2} (t, x - \Gamma(t)) \gamma^2(\rho(t)) \, dt \\
    &= -B(R(t, x - \Gamma(t))) \, dt + \Sigma(R(t, x - \Gamma(t))) \, dW(t) \\
    &\quad + \frac{1}{2} \frac{\partial^2 R}{\partial x^2} (t, x - \Gamma(t)) \gamma^2(\rho(t)) \, dt,
\end{align*}
$$

where the last equality is a consequence of (1.2). Noting that $G_x(t, x) = R_x(t, x - \Gamma(t))$ and $G_{xx}(t, x) = R_{xx}(t, x - \Gamma(t))$, we obtain the SPDE (1.5). In the fourth and final section, we prove that the solution to this functional equation (1.4) is unique. Taken together, all of this proves Theorem 1.2 with convergence we obtain the SPDE (1.5). In the fourth and final section, we prove that the solution to this functional equation (1.4) is unique. Taken together, all of this proves Theorem 1.2 with convergence in the Skorohod space instead of the uniform convergence. Since the corresponding measure-valued process $\rho$ is a.s. continuous with respect to time $t$, it is an element of $C([0, T], \mathcal{P}_q)$. The same can be said about the pre-limit processes $\rho^{(n)}$, and thus the convergence takes place in $C([0, T], \mathcal{P}_q)$: See [Bi, Chapter 12].

2.6. **Uniqueness of the limit.** Let $\rho_1$ and $\rho_2$ be in $C([0, T], \mathcal{P}_p(\mathbb{R}))$ with continuous cumulative distribution functions $F_i(t, x) := F_{\rho_i(t)}(x)$, $i = 1, 2$, satisfying (1.4). Denote $\Gamma_i(t) := \int_0^t \gamma(\rho_i(s)) \, dW(s)$ for $i = 1, 2$, we get:

$$
\begin{align*}
    F_1(t, x) - F_2(t, x) &= R(t, x - \Gamma_1(t)) - R(t, x - \Gamma_2(t)) \\
    &= \int_0^1 R_x \left[ t, x - \Gamma_1(t) \theta - \Gamma_2(t)(1 - \theta) \right] \, d\theta \cdot (\Gamma_1(t) - \Gamma_2(t)).
\end{align*}
$$

We can represent the Wasserstein distance between $\rho_1$ and $\rho_2$ as follows [SW] p.64:

$$
\begin{align*}
    W_1(\rho_1(t), \rho_2(t)) &= \int_{\mathbb{R}} |F_1(t, x) - F_2(t, x)| \, dx.
\end{align*}
$$

Applying (2.7) above to (2.15) with (2.16) and interchanging integrations by Fubini’s theorem, we obtain

$$
\begin{align*}
    W_1(\rho_1(t), \rho_2(t)) &\leq \int_{\mathbb{R}} R_x(t, x) \, dx \cdot \left| \int_0^t \gamma(\rho_1(s)) - \gamma(\rho_2(s)) \, dW(s) \right|.
\end{align*}
$$

Note that $R_x(t, \cdot)$ is the probability density function, which integrates to 1. We square both sides in (2.17), take expectation, and apply the Doob’s martingale inequality:

$$
\begin{align*}
    \mathbb{E} \left[ W_1^2(\rho_1(t), \rho_2(t)) \right] &\leq 4 \mathbb{E} \int_0^t \left| \gamma(\rho_1(s)) - \gamma(\rho_2(s)) \right|^2 \, ds \\
    &\leq 4L^2 \int_0^t \mathbb{E} W_1^2(\rho_1(s), \rho_2(s)) \, ds \leq 4L^2 \int_0^t \mathbb{E} W_1^2(\rho_1(s), \rho_2(s)) \, ds.
\end{align*}
$$

Gronwall’s lemma implies uniqueness.
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