THE FIBER DIMENSION OF A GRAPH

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Abstract. Graphs on integer points of polytopes whose edges come from a set of allowed differences are studied. It is shown that any simple graph can be embedded in that way. The minimal dimension of such a representation is the fiber dimension of the given graph. The fiber dimension is determined for various classes of graphs and an upper bound in terms of the chromatic number is stated.

1. Introduction

The study of geometric properties of graphs is a key ingredient in understanding their algorithmic behaviour and combinatorial structure [17, 15]. In [11], the dimension of a graph was introduced, which is the smallest \( n \in \mathbb{N} \) such that the graph can be embedded in \( \mathbb{R}^n \) with every edge having unit length. Recently, isometrically embeddings of graphs into discrete objects like hypercubes or lattices were paid a lot of attention in the literature and lead to a lot of variations like the isometric dimension [12], the lattice dimension [10, 14], or the Fibonacci dimension [4] of a graph.

In this paper, a new notion is added to the list of graph dimensions (Remark 2.6). Our concept has its origin in algebraic statistics, an emerging field which explores statistical questions with algebraic tools [8, 9, 13, 1]. A main task there is to construct connected graphs on integer points of polytopes in order to draw samples by performing a random walk [16, Chapter 5]. For a given polytope \( P \subset \mathbb{Q}^d \) and a symmetric set \( \mathcal{M} \subset \mathbb{Z}^d \), a graph on \( P \cap \mathbb{Z}^d \) is given by connecting two nodes \( u \) and \( v \) by an edge if \( u - v \in \mathcal{M} \). Graphs which can be obtained in that way are often referred to as fiber graphs in the literature and can be understood as a discrete analogue of unit distance graphs [3].

At first glance, it seems that fiber graphs are distinguished graphs with their own rich structure. However, as it turns out, every graph can be represented as a fiber graph (Proposition 2.3). This motivates the question for the smallest dimension in which a graph \( G \) can be represented as a fiber graph, the fiber dimension of \( G \) (Definition 2.5). We explore general properties of this dimension and state upper bounds in terms of the chromatic number (Theorem 3.5) in the spirit of [11]. We then determine the fiber dimension for a variety of graphs. The fiber dimension of a cycle of length \( n \) depends on Euler’s totient function and we show that \( \text{fdim}(C_n) = 1 \) if and only if \( n \in \mathbb{N} \setminus \{3, 4, 6\} \). Cycles whose length is one of the exceptional cases \( n \in \{3, 4, 6\} \) have fiber dimension 2 (Proposition 4.4). Its proof uses the well-known fact that Euler’s totient function of \( n \in \mathbb{N} \) is 2 if and only if \( n \in \{3, 4, 6\} \). We also determine the fiber dimension of complete graphs and show that it is logarithmic in the
number of nodes (Theorem 5.5). In the end, a connection to distinct pair-sum polytopes [5] is established and it is shown how the fiber dimension leads to relations between the number of vertices and the dimension of the ambient space of these polytopes.

Conventions and Notations. The natural numbers are denoted by \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and for \( n \in \mathbb{N} \) with \( n > 0 \), \([n] := \{1, 2, \ldots, n\} \). All graphs that appear in this paper are simple, i.e. all edges are undirected and they do not have loops or multiple edges. For any graph \( G \), the set of nodes is denoted by \( V(G) \) and its chromatic number is \( \chi(G) \). The unit vectors of \( \mathbb{Q}^d \) are denoted by \( e_1, \ldots, e_d \). For any \( n \in \mathbb{N} \), \( K_n \) and \( C_n \) denote the complete graph and the cycle graph on \( n \) nodes respectively.

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2. Fiber graphs

A polytope \( P \subset \mathbb{Q}^d \) is a lattice polytope if all its vertices are in \( \mathbb{Z}^d \). A finite set \( \mathcal{M} \subset \mathbb{Z}^d \setminus \{0\} \) is a set of moves if \( \mathcal{M} = -\mathcal{M} \) and if for all \( \lambda \in \mathbb{N} \) with \( \lambda \geq 2 \) and all \( m \in \mathcal{M} \), \( \lambda \cdot m \notin \mathcal{M} \).

Definition 2.1. Let \( P \subset \mathbb{Q}^d \) be a lattice polytope and let \( \mathcal{M} \subset \mathbb{Z}^d \) be a set of moves. The fiber graph \( P(\mathcal{M}) \) is the graph on \( P \cap \mathbb{Z}^d \) where two nodes \( v \) and \( u \) are adjacent if \( u - v \in \mathcal{M} \).

The set of moves \( \mathcal{M} \) is minimal for \( P \) if every move in \( \mathcal{M} \) contributes an edge. A minimal set of moves \( \mathcal{M} \) is a Markov basis of \( P \) if \( P(\mathcal{M}) \) is connected.

Remark 2.2. The notions Markov bases and fiber graphs come from algebraic statistics [9]. The goal there is to run irreducible Markov chains on fibers of a Matrix \( A \in \mathbb{Z}^{m \times d} \), i.e. the sets \( A^{-1}b := \{u \in \mathbb{N}^d : Au = b\} \) for \( b \in \mathbb{Z}^m \). With tools from commutative algebra [8, 9], a universal set of moves \( \mathcal{M} \subset \ker \mathbb{Z}(A) \) can be computed such that the fiber graphs on all fibers of \( A \) are connected simultaneously.

![Figure 1. A fiber graph in \( \mathbb{Q}^2 \).](image)

Proposition 2.3. Every simple graph is isomorphic to a fiber graph.

Proof. Let \( G = (\{v_1, \ldots, v_n\}, E) \) be graph and let \( P := \{x \in \mathbb{Q}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\} \) be the \( (n-1) \)-dimensional simplex, then \( P \cap \mathbb{Z}^n = \{e_1, \ldots, e_n\} \). Consider \( \mathcal{M} := \{e_i - e_j : \{v_i, v_j\} \in E\} \), then \( P(\mathcal{M}) \) is isomorphic to \( G \). \( \square \)
The restriction on graphs without loops is necessary, since, in a fiber graph, either every node has a loop, or none. The next lemma states that every fiber graph can be written as a fiber graph in a full dimensional polytope.

**Lemma 2.4.** Let $P \subset \mathbb{Q}^m$ be a $d$-dimensional polytope and $\mathcal{M} \subset \mathbb{Z}^m$ a set of moves. There exists a $d$-dimensional polytope $P' \subset \mathbb{Q}^d$ and moves $\mathcal{M}' \subset \mathbb{Z}^d$ such that $P(\mathcal{M}) \cong P'(\mathcal{M}')$.

**Proof.** We can assume that $d < m$. Translation of $P$ does not change the graph structure of $P(\mathcal{M})$ and thus we can assume that $P \subset \mathbb{Q}^m_{\geq 0}$. Since $P$ is a rational polytope and since $d < m$, there exists a matrix $A \in \mathbb{Z}^{n \times m}$ with $\dim \ker Z(A) = 0$ and $n \geq m$ and $b \in \mathbb{Z}^n$ such that $P = \{x \in \mathbb{Q}^m_{\geq 0} : Ax \leq b\}$. Consider the injective and affine map

$$\phi : \mathbb{Q}^m \to \mathbb{Q}^{m+n}, x \mapsto \left(\frac{x}{b-Ax}\right).$$

Then $\phi(P) = \{(x, y)^T \in \mathbb{Q}^{m+n}_{\geq 0} : Ax + y = b\}$ and $\dim(\phi(P)) = \dim(P) = d$. The set $\mathcal{N} := \{(m, Am)^T : m \in \mathcal{M}\}$ is a set of moves and the graphs $\phi(P)(\mathcal{N})$ and $P(\mathcal{M})$ are isomorphic. Hence, it suffices to show the statement for $d$-dimensional polytopes of the form $P = \{x \in \mathbb{Q}^{k}_{\geq 0} : Bx = b\}$ for some $b \in \mathbb{Z}^n$, a matrix $B \in \mathbb{Z}^{n \times k}$, and a set of moves $\mathcal{M} \subset \ker Z(B)$ with the property that $k \geq n$, rank$(B) = n$, and $\dim(\ker Z_B) = k - n \geq d$. We can add rows to $B$ without changing $P$ such that $\dim(\ker Z_B) = k - n = d$. First, we transform $B$ into its Hermite normal form, that is, we write $B = (H, 0) \cdot C$ for an unimodular matrix $C \in \mathbb{Z}^{k \times k}$ and a matrix $H \in \mathbb{Z}^{n \times n}$ of full rank. Let $H^{-1} \in \mathbb{Q}^{n \times n}$ and $C^{-1} \in \mathbb{Q}^{k \times k}$ be the inverse matrices of $H$ and $C$ respectively. Since $C$ is unimodular, $C^{-1} \in \mathbb{Z}^{k \times k}$ and thus let $C_1 \in \mathbb{Z}^{k \times n}$ and $C_2 \in \mathbb{Z}^{k \times d}$ such that $C^{-1} = (C_1, C_2)$ and consider the affine map

$$\psi : \mathbb{Z}^d \to \mathbb{Z}^k, x \mapsto C^{-1} \left(\frac{H^{-1}b}{x}\right).$$

Clearly, $\psi$ is injective and the image of the polytope $P' := \{v \in \mathbb{Q}^d : C_2 \cdot v \leq C_1 H^{-1}b\} \subset \mathbb{Q}^d$ is $P$. Since $P \cap \mathbb{Z}^k \neq \emptyset$, $H^{-1}b \in \mathbb{Z}^n$ (see [7, Theorem 2.3.6]) and since $C$ is unimodular, integer points of $P'$ get mapped to integer points of $P$. Thus $\dim(P') = \dim(P) = d$. That is, $P'$ is full dimensional in $\mathbb{Q}^d$. Consider

$$\mathcal{M}' := \{\psi^{-1}(v) - \psi^{-1}(u) : v, u \in P \cap \mathbb{Z}^k, v - u \in \mathcal{M}\},$$

then $\mathcal{M}' = -\mathcal{M}'$ and $\mathcal{M}'$ cannot contain multiples. Let $\psi(v') = v$ and $\psi(u') = u$ for $v', u' \in P' \cap \mathbb{Z}^d$, then $v' - u' \in \mathcal{M}'$ if and only if $v - u \in \mathcal{M}$. Thus, all edges in $P'(\mathcal{M}')$ are mapped bijective to edges in $P(\mathcal{M})$ under $\psi$, which proves that these graphs are isomorphic. \qed

**Definition 2.5.** Let $G$ be a graph. The *fiber dimension* $\text{fdim}(G)$ of $G$ is the smallest $d \in \mathbb{N}$ such that there exists a full dimensional lattice polytope $P \subset \mathbb{Q}^d_{\geq 0}$ and a set of moves $\mathcal{M} \subset \mathbb{Z}^d$ with $G \cong P(\mathcal{M})$.

**Remark 2.6.** In general, the fiber dimension of a graph $G$ is different from its *dimension* $\text{dim}(G)$ as defined in [11]. For example, the complete graph $K_5$ can be realized as fiber graph in $\mathbb{Q}^3$ (see Theorem 5.5 and Figure 4), in contrast to $\text{dim}(K_5) = 4$. 


Remark 2.7. Proposition 2.3 and Lemma 2.4 imply that the fiber dimension of any graph with $n$ nodes is bounded from above by $n - 1$. All graphs with at most one vertex have fiber dimension 0 and all graphs on at least two nodes without edges have fiber dimension 1.

Remark 2.8. Let $P$, $\mathcal{M}$, and $G$ as in the proof of Proposition 2.3 and consider the integer matrix $A = (1, \ldots, 1) \in \mathbb{Z}^{1 \times n}$. If $G$ is connected, than it is easy to show that $\mathcal{M}$ is a Markov basis for all polytopes of the form $\{u \in \mathbb{Q}_{\geq 0}^n : Au = b\}$ with $b \in \mathbb{Q}$. In particular, $\mathcal{M}$ is a Markov basis of the matrix $A$ in the sense of [9, Definition 1.1.12] (see also Remark 2.2).

Proposition 2.9. Let $P \subset \mathbb{Q}^d$ be a lattice polytope and $\mathcal{M} \subset \mathbb{Z}^d$ be a Markov basis of $P$ with $2 \cdot \dim(P) = |\mathcal{M}|$, then $P(\mathcal{M})$ is bipartite.

Proof. Let $k := \dim(P) \leq d$. Since $\mathcal{M}$ is a Markov basis of $P$, $\dim(P) = \dim(\mathbb{Q} \cdot \mathcal{M})$ and thus we can write $\mathcal{M} = \{m_1, -m_1, \ldots, m_k, -m_k\}$. The assumption on the dimension says that $\{m_1, \ldots, m_k\}$ is linear independent. Let $v \in P \cap \mathbb{Z}^d$ and let $v + \sum_{i=1}^{k} \lambda_i m_i + \sum_{i=1}^{k} -\mu_i m_i = v$ be a cycle in $P(\mathcal{M})$ of length $r = \sum_{i=1}^{k} (\lambda_i + \mu_i)$. The linear independence gives that $\lambda_i = \mu_i$ for all $i \in [k]$ and thus $r$ is even. \qed

Remark 2.10. The converse of Proposition 2.9 is false in general since the 8-cycle can be minimally embedded in $\mathbb{Q}^1$ (Proposition 4.4) with a Markov basis consisting of 4 moves.

Remark 2.11. Any Markov basis $\mathcal{M} \subset \mathbb{Z}^d$ of a polytope $P \subset \mathbb{Q}^d$ fulfills $|\mathcal{M}| \geq \dim(\mathbb{Q} \cdot \mathcal{M}) = \dim(P)$. Thus, Proposition 2.9 yields a lower bound on the number of moves in an embedding of non-bipartite graphs: If $G$ is a graph with $\chi(G) > 2$, then any embedding as a fiber graph needs strictly more than $2d$ moves.

Remark 2.12. Every 1-dimensional lattice polytope in $P \subset \mathbb{Q}^1$ has a Markov basis of size $|\mathcal{M}| = 2 \cdot \dim(P)$, namely $\mathcal{M} = \{-1, 1\}$. This equation fails to be true already in $\mathbb{Q}^2$: All Markov bases $\mathcal{M} \subset \mathbb{Z}^2$ of the polytope $P \subset \mathbb{Q}^2$ shown in Figure 2 have more than 6 elements, i.e. $2 \cdot \dim(\mathbb{Q} \cdot \mathcal{M}) = 4 < |\mathcal{M}|$.

![Figure 2](image.png)  

Figure 2. A polytope without a Markov basis with fewer than 6 moves.

3. Bounds on the fiber dimension

In this section, we explore upper bounds on the fiber dimension. Our first observation is that the cartesian products of graphs behaves nicely with cartesian products of polytopes.

Proposition 3.1. Let $G_1, \ldots, G_n$ be graphs, then $\text{fdim}(\times_{i=1}^n G_i) \leq \sum_{i=1}^n \text{fdim}(G_i)$.

Proof. It suffices to prove the inequality for $n = 2$. Let $P_1, P_2, \mathcal{M}_1, \mathcal{M}_2$ such that $G_i \cong P_i(\mathcal{M}_i)$. The cartesian product $P := P_1 \times P_2$ is a polytope of dimension $\dim(P_1) + \dim(P_2)$. Additionally, let $\mathcal{M} := \{(m, 0)^T : m \in \mathcal{M}_1\} \cup \{(0, m)^T : m \in \mathcal{M}_2\}$. It is straight-forward to check that $P(\mathcal{M}) = P_1(\mathcal{M}_1) \times P_2(\mathcal{M}_2)$. Hence, $\text{fdim}(G_1 \times G_2) \leq \text{dim}(P)$. \qed
Remark 3.2. The inequality given in Proposition 3.1 is sharp for $K_2 \times K_2 = C_4$ (see Theorem 5.5 and Proposition 4.4).

Proposition 3.3. Let $G$ be a graph and $v \in V(G)$, then $\text{fdim}(G) \leq \text{fdim}(G-v) + 1$.

Proof. Write $V(G) = \{v_0, \ldots, v_n\}$ with $v_0 := v$. Let $d := \text{fdim}(G-v)$ and let $\phi : G-v \to P(M)$ a graph isomorphism that embeds $G-v$ in dimension $d$ for a polytope $P \subset \mathbb{Q}^d$ and a set of moves $M \subset \mathbb{Z}^d$. Let $P' := \text{conv}_\mathbb{Q}(\{0, (1, \phi(v_1)), \ldots, (1, \phi(v_n))\}) \subset \mathbb{Q}^{1+d}$, then $\text{dim}(P') = d+1$ and $P' \cap \mathbb{Z}^{d+1} = \{0, \phi(v_1), \ldots, \phi(v_n)\}$. Let $N \subseteq \{v_1, \ldots, v_n\}$ be the neighborhood of $v$ in $G$ and consider $M' = \{(0, m) : m \in M\} \cup \{\phi(v_i) : v_i \in N\}$. Then $G \cong P'(M')$. \qed

As in [11], we obtain an upper bound on the dimension in terms of the chromatic number of the graph. Our strategy the following: First, we construct sets of integer points which represent the color-classes of the graph in such a way that we can freely assign moves within them. In a second step, we map the vertices of the graph on these sets and construct the set of moves accordingly. For this method to work, the constructed integer points must be the integer points of a polytope.

Definition 3.4. A finite set $F \subset \mathbb{Z}^d$ is normal if $\text{conv}_\mathbb{Q}(F) \cap \mathbb{Z}^d = F$.

Theorem 3.5. Let $G$ be a graph with a $k$-coloring in which $r$ color-classes have cardinality 1, then $\text{fdim}(G) \leq 2 \cdot k - r - 1$.

Proof. Write $V(G) = V_1 \cup \cdots \cup V_k$ and set $n_i := |V_i|$. Define for $i \in [k]$

$$W_i := \{(e_i, j \cdot e_i)^T : j \in [n_i]\} \subset \mathbb{N}^{2k}$$

and let $W := \bigcup_{i=1}^k W_i$ and $P := \text{conv}_\mathbb{Q}(W)$. To show that $P \cap \mathbb{Z}^{2k} = W$, let $u \in P \cap \mathbb{Z}^{2k}$. Since every $W_i$ is normal, there exists $w_i \in W_i$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{Q}$ with $0 \leq \lambda_i \leq 1$ for all $i \in [k]$ and $\sum_{i=1}^k \lambda_i = 1$ such that $u = \sum_{i=1}^k \lambda_i w_i$. The projection of $W$ onto the first $k$ coordinates is the set of integer points of the standard simplex and thus normal. The projection of $u$ onto its first $k$ coordinates is $e_i$ for some $i \in [k]$. In particular, the only choice to build an integer vector is thus $\lambda_j = 0$ for $j \neq i$ and $\lambda_i = 1$, i.e., $u = w_i \in W_i$.

Let us now construct a graph on $P \cap \mathbb{Z}^{2k}$ which is isomorphic to $G$. For that, let $\phi : \bigcup_{i=1}^k V_i \to \bigcup_{i=1}^k W_i$ be any bijection which maps elements from $V_i$ to $W_i$ and consider the set of moves $M = \{\phi(v) - \phi(w) : \{v, w\} \in E(G)\}$. By construction of $M$, $\phi$ is a graph homomorphism from $G$ to $P(M)$. Since edges in $G$ do only connect nodes from different color classes, the first $k$ coordinates of any element in $M$ do only contain elements from $\{-1, 0, 1\}$ and thus $M$ cannot contain multiples. Next, let $s \in [n_i]$ and $t \in [n_j]$ such that $(e_i, se_i)^T - (e_j, te_j)^T = \phi(u) - \phi(v) \in M$ with $v, w \in V(G)$. It follows immediately that $\phi(v) = (e_j, te_j)^T$ and hence $\phi(u) = (e_i, se_i)^T$. Thus, $\phi$ maps edges from $G$ to $P(M)$ bijectively and hence $\text{fdim}(G) \leq \text{dim}(P)$. The vertices of the polytope $P$ are $\{(e_1,e_1)^T, (e_1,n_1e_1)^T, \ldots, (e_k,e_k)^T, (e_k,n_ke_k)^T\}$ and since $n_i = 1$ for $r$ indices $i \in [k]$, $\text{dim}(P) \leq 2k - r - 1$. \qed

Corollary 3.6. For any graph $G$, $\text{fdim}(G) \leq 2 \cdot \chi(G) - 1$.

Remark 3.7. By the Four Color Theorem [2], the fiber dimension of every planar graph is at most 7.
4. Fiber dimension one

The class of connected graphs of fiber dimension 1 consists of far more than path graphs. To see it, let us first specialize Definition 2.1 to the 1-dimensional case:

**Definition 4.1.** Let \( n \in \mathbb{N}_{\geq 1} \) and let \( D \subseteq [n-1] \) be a finite set such that for all distinct \( d, d' \in D \) neither \( d|d' \) nor \( d'|d \). The graph \( G_D^n \) has nodes \([n]\) where \( i \) and \( j \) are adjacent if \( |i-j| \in D \). A graph \( G \) which is isomorphic to \( G_D^n \) is a difference graph.

**Proposition 4.2.** A graph has fiber dimension 1 if and only if it is a difference graph.

**Lemma 4.3.** Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and \( D \subseteq [n] \). If \( G_D^n \) is connected, then \( \gcd(D) = 1 \).

**Proof.** Since \( n \geq 2 \), there exists a path between 1 and 2 in \( G_D^n \). Let \( d_1, \ldots, d_k \in D \) be the distinct integers that appear in that path and write \( 1 + \sum_{i=1}^k \lambda_i d_i = 2 \) for \( \lambda_1, \ldots, \lambda_k \in \mathbb{Z} \setminus \{0\} \). Then \( \gcd(d_1, \ldots, d_k) \) divides 1.

**Proposition 4.4.** For any \( n \in \mathbb{N} \) with \( n \geq 3 \),

\[
\text{fdim}(C_n) = \begin{cases} 
1, & \text{if } n \notin \{3,4,6\} \\
2, & \text{if } n \in \{3,4,6\} 
\end{cases}
\]

**Proof.** Let \( n \geq 3 \) with \( n \in \mathbb{N} \setminus \{3,4,6\} \). We first show that there exists an integer \( k \in \mathbb{N} \) with \( 2 \leq k < \frac{n}{2} \) such that \( \gcd(k,n) = 1 \). Let \( \phi : \mathbb{N} \to \mathbb{N} \) be Euler’s totient function. Since \( n \in \mathbb{N} \setminus \{3,4,6\} \) and \( n \geq 3 \), \( \phi(n) \geq 4 \) and we have for all \( k \in [n] \), \( \gcd(k,n) = 1 \) if and only if \( \gcd(n-k,n) = 1 \). In particular, coprime elements of \( n \) come in pairs \((k,n-k)\) with \( k < n-k \). Thus, since \( \phi(n) \geq 4 \), there must exists \( k \in [n] \) with \( 1 < k < \frac{n}{2} \) such that \( \gcd(k,n) = 1 \). We now show that \( G_{\{k,n-k\}}^n \) is a cycle of length \( n \). Clearly, \( n-k \) is not a multiple of \( k \) since this would imply that \( n \) is a multiple of \( k \) as well which in turn would contradict \( \gcd(n,k) = 1 \) since \( k > 1 \). Any node in \( G_{\{k,n-k\}}^n \) has degree 2 and hence it suffices to prove that this graph is connected. Since \( k \) and \( n \) are coprime, \( \langle k+n\mathbb{Z} \rangle = \mathbb{Z}_n \). Now, take distinct \( i,j \in [n] \), then there exists \( s \in \mathbb{N} \) such that \( j+n\mathbb{Z} = i+sk+n\mathbb{Z} \) in \( \mathbb{Z}_n \). For any \( r \in [s] \), let \( i_r \in [n] \) such that \( i_r+n\mathbb{Z} = i+rk+n\mathbb{Z} \). Either \( i_r+k \) or \( i_r-(n-k) \) are in \([n]\) and since their congruence classes in \( \mathbb{Z}_n \) coincide, \( i_r \) and \( i_{r-1} \) are adjacent in \( G_{\{k,n-k\}}^n \). Since \( i_k = j, i \) and \( j \) are connected. It follows that \( C_n = G_{\{k,n-k\}}^n \).

Conversely, let \( n \in \{3,4,6\} \). Clearly, \( \text{fdim}(C_n) \leq 2 \) due to Proposition 3.3 since a path has fiber dimension 1. Hence, it suffices to show that \( \text{fdim}(C_n) > 1 \) for \( n \in \{3,4,6\} \). If \( n = 3 \), then \( C \cong K_3 \) and the claim follows from Theorem 5.5. If \( n \in \{4,6\} \), assume that there exists \( D \subseteq [n-1] \) such that \( C_n \cong G_D^n \) is a difference graph. It is easy to see that \( |D| = 2 \) since if \( |D| \geq 3 \) or \( |D| = 1 \), the node 1 has either degree greater than 3 or is a leaf respectively. Thus, we can write \( D = \{d_1, d_2\} \). Since \( G_D^n \) is connected, \( \gcd(d_1, d_2) = 1 \) by Lemma 4.3. Hence, the only possible choices for \( \{d_1, d_2\} \) are \( \{2,3\} \) if \( n = 4 \) and \( \{2,3\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \) if \( n = 6 \). However, in all these cases, \( G_{\{d_1,d_2\}}^n \) is not a cycle. \( \square \)
3. Complete graphs

Proposition 5.1. Let \( n \geq 3 \), then \( \text{fdim}(K_{1,n}) = 2 \).

Proof. Let \( v \in V(K_{1,n}) \) be the vertex with maximal degree \( n \). Removing \( v \) from \( K_{1,n} \) gives a graph on \( n \geq 3 \) nodes without edges, i.e., the fiber dimension of this graph is 1. Proposition 3.3 then yields that \( \text{fdim}(K_{1,n}) \leq 2 \). Conversely, assume that \( \text{fdim}(K_{1,n}) = 1 \) and let \( D \subset [n] \) such that \( K_{1,n} \cong G_{D}^{n+1} \) is a difference graph. The graph isomorphism maps \( v \) to some \( j \in \{1, \ldots, n+1\} \). Since \( j \) must be adjacent to all vertices in \( \{1, \ldots, n+1\} \setminus \{j\} \), \( 1 \in D \). The constraints on \( D \) imply already that \( D = \{1\} \) and thus \( G_{D}^{n+1} \) is a path. \( \square \)

Proposition 5.2. For any \( n_1, \ldots, n_r \in \mathbb{N} \),

\[
\text{fdim}(K_{n_1, \ldots, n_r}) \leq \lceil \log_2 r \rceil + \lceil \log_2 \max\{n_i : i \in [r]\} \rceil.
\]

Proof. First, decompose the vertex set of \( K_{n_1, \ldots, n_r} \) into its color classes \( V_1, \ldots, V_r \) such that \( |V_i| = n_i \). Let \( s := \lceil \log_2 r \rceil \) and \( m := \lceil \log_2 \max\{n_i : i \in [r]\} \rceil \). We prove the upper bound by realizing \( K_{n_1, \ldots, n_r} \) as fiber graph. For any \( i \in [r] \), we can choose a set \( W_i \subset \{0,1\}^m \) of size \( n_i \) since \( n_i \leq 2^m \). Similarly, choose a set \( C := \{c_1, \ldots, c_r\} \subset \{0,1\}^s \) of size \( r \). The set

\[
\bigcup_{i=1}^{r} \{c_i\} \times W_i \subseteq \{0,1\}^{s+m}
\]

has cardinality \( n_1 + \cdots + n_r \) and is normal since all subsets of \( \{0,1\}^{s+m} \) are normal. Let \( \mathcal{P} \) be its convex hull and let \( \phi : \bigcup_{i=1}^{r} V_i \to \mathcal{P} \cap \mathbb{N}^{s+m} \) be bijective map which maps nodes from \( V_i \) to \( W_i \) (here, it doesn’t matter which node of color \( i \) gets mapped to which node in \( W_i \) since the colour classes are independent sets). Let us now construct the Markov basis. Let \( \mathcal{M} := \{c_i - c_j : i \neq j\} \times \{-1,0,1\}^m \). The first \( s \) coordinates of every element in \( \mathcal{M} \) are non-zero and hence there are no edges within \( \{c_i\} \times W_i \) for any \( i \in [r] \) in \( \mathcal{P}(\mathcal{M}) \). Since for distinct \( i \) and \( j \), all elements in \( \{c_i\} \times W_i \) and \( \{c_j\} \times W_j \) are adjacent, \( \mathcal{P}(\mathcal{M}) \) is isomorphic to \( K_{n_1, \ldots, n_r} \) and with Lemma 2.4, the claim follows. \( \square \)

Lemma 5.3. Let \( \mathcal{L} \subset \mathbb{Z}^n \) be a lattice of full rank and let \( P \subset \mathbb{Q}^d \) be a set such that for any distinct \( v, w \in P \cap\mathbb{Z}^d \), \( v - w \notin \mathcal{L} \). Then \( |P \cap \mathbb{Z}^d| \leq |\mathbb{Z}^d / \mathcal{L}| \).

Proof. Let \( P \cap \mathbb{Z}^d = \{v_1, \ldots, v_n\} \) and consider the linear map \( \phi : \mathbb{Z}^d \to \mathbb{Z}^d / \mathcal{L}, \phi(v) = v + \mathcal{L} \). By assumption, \( \phi(v_i - v_j) \neq 0 \) in \( \mathbb{Z}^d / \mathcal{L} \) for all \( i, j \in [n] \) with \( i \neq j \). Assume that there are \( i, j \in [n-1] \) with \( i \neq j \) such that \( \phi(v_n - v_i) = \phi(v_n - v_j) \). Then \( \phi(v_i - v_j) = \phi(v_i - v_n + v_n - v_j) = \phi(v_i - v_n) - \phi(v_j - v_n) = 0 \), a contradiction. Thus, \( \phi(v_n - v_i) \neq \phi(v_n - v_j) \) for all distinct \( i, j \in [n-1] \). That is, \( |\{\phi(v_n - v_i) : i \in [n-1]\}| = n - 1 \). The proposition follows from \( n - 1 = |\{\phi(v_n - v_i) : i \in [n-1]\}| \leq |\mathbb{Z}^d / \mathcal{L}| - 1 \). \( \square \)

Remark 5.4. Since \( |\mathbb{Z}^n / \mathcal{L}| = \det(\mathcal{L}) \), Lemma 5.3 can be seen as a discrete analogon of Blichfeldt’s theorem [7, Theorem 2.4.1].
**Theorem 5.5.** For any \( n \in \mathbb{N} \), \( \text{fdim}(K_n) = \lceil \log_2 n \rceil \).

**Proof.** Due to Proposition 5.2, the fiber dimension of \( K_n \) is bounded from above by \( \lceil \log_2 n \rceil \). Let \( d := \text{fdim}(K_n) \) and \( P \subset \mathbb{Q}^d \) a \( d \)-dimensional polytope and \( \mathcal{M} \subset \mathbb{Z}^d \) a Markov basis such that \( K_n \cong P(\mathcal{M}) \). Assume there are \( v, w \in P \cap \mathbb{Z}^d \) such that \( v - w \in 2 \cdot \mathbb{Z}^d \). Since \( (v + w)_i \) is even for all \( i \in [d] \), \( (v + w)_i \) is even for all \( i \in [d] \) and thus \( v + w \in 2\mathbb{Z}^d \). In particular, then \( \frac{1}{2}(v + w) \in \mathbb{Z}^d \) and since \( P \) is normal, \( \frac{1}{2}(v + w) \in P \cap \mathbb{Z}^d \). This, however, implies that \( v - w \in \mathcal{M} \) and \( \frac{1}{2}(v - w) \in \mathcal{M} \). Thus, \( v - w \notin 2\mathbb{Z}^d \). Due to Lemma 5.3, \( n = |P \cap \mathbb{Z}^d| \leq 2^d \) and thus \( d \geq \lceil \log_2 n \rceil \). \( \Box \)

![Figure 4. Fiber graph embeddings of \( K_5, K_6, \) and \( K_7 \) in \( \mathbb{Q}^3 \).](image)

**Remark 5.6.** Proposition 3.3 yields the trivial upper bound for complete graphs, that is \( \text{fdim}(K_n) \leq n - 1 \) for \( n \geq 2 \). This bound is strict for the first time for \( n = 4 \).

6. DISTINCT PAIR-SUM POLYTOPES

For the remainder, we investigate an universal upper bound on the fiber dimension by generalizing our embedding in Proposition 2.3 into the simplex. A priori, a move in a Markov basis give rise to distinguished edges in a fiber graph. The next definition states a property of polytopes assuring that Markov moves lead to precisely one edge in the graph.

**Definition 6.1.** A lattice polytope \( P \subset \mathbb{Q}^d \) with \( n := |P \cap \mathbb{Z}^d| \) is a distinct pair-sum polytope if \( |P \cap \mathbb{Z}^d + P \cap \mathbb{Z}^d| = \binom{n}{2} + n \).

**Remark 6.2.** Let \( P \subset \mathbb{Q}^d \) be a distinct pair-sum polytope and write \( P \cap \mathbb{Z}^d = \{v_1, \ldots, v_n\} \), then all the possible sums \( 2v_1, \ldots, 2v_n, v_1 + v_2, v_1 + v_3, \ldots, v_{n-1} + v_n \) are pairwise distinct. We refer to [5, 6] for more on distinct pair-sum polytopes.

The next proposition states that distinct pair-sum polytopes allow embeddings of all possible graphs whose number of nodes equals the number of lattice points of the polytope.

**Proposition 6.3.** Let \( P \subset \mathbb{Q}^d \) be a distinct pair-sum polytope with \( n := |P \cap \mathbb{Z}^d| \). For any graph \( G \) on \( n \) nodes, there exists a set of moves \( \mathcal{M} \subset \mathbb{Z}^d \) such that \( G \cong P(\mathcal{M}) \).

**Proof.** Pick an arbitrary bijection \( \phi : V(G) \to P \cap \mathbb{Z}^n \) and define \( \mathcal{M} := \{\phi(u) - \phi(v) : u \text{ and } v \text{ adjacent in } G\} \).

We claim that \( G \cong P(\mathcal{M}) \). First, we need to show that \( \mathcal{M} \) does not contain multiples. Assume, there are \( m, m' \in \mathcal{M} \) and \( k \in \mathbb{N} \) with \( k \geq 2 \) such that \( m = km' \). Let \( v, w \in P \cap \mathbb{Z}^d \) with \( v - w = m \). Then \( w + km' = v \). Then \( w, w + m', w + 2m' \in P \cap \mathbb{Z}^d \) are disjoint elements that fulfill \( (w + m') + (w + m') = w + (w + 2m') \), that is, \( |P \cap \mathbb{Z}^d + P \cap \mathbb{Z}^d| < n(n - 1) + 1 \) since two different sums lead to the same element. Clearly, every edge in \( G \) get mapped to
an edge in $P(M)$. Conversely, let $v, w \in P \cap \mathbb{Z}^d$ such that $v - w \in M$. Then there exists adjacent nodes $v', w' \in V(G)$ with $\phi(v') - \phi(w') = v - w$. We have to prove that $\phi(v') = v$ and $\phi(w') = w$. If not, then $\phi(v') + w = \phi(w') + v$ implies that two different sums yield the same element in $P \cap \mathbb{Z}^d + P \cap \mathbb{Z}^d$ which again gives a contraction. 

In [5], a distinct pair-sum polytope in $\mathbb{Q}^n$ on $2^n$ lattice points was constructed for any $n \in \mathbb{N}$. This gives rise to the following result.

**Proposition 6.4.** Let $G$ be a graph on $2^n$ nodes, then $\text{fdim}(G) \leq n$.

**Proof.** This is [5, Theorem 3] together with Proposition 6.3. 

Lower bounds on the fiber dimension can be translated to relations between the number of lattice points and the dimension of the ambient space of distinct pair-sum polytopes. The next proposition demonstrates this for complete graphs and redisCOVERS a bound which was already proven in [5, Theorem 2].

**Proposition 6.5.** Let $P \subset \mathbb{Q}^d$ be a distinct pair-sum polytope, then $|P \cap \mathbb{Q}^d| \leq 2^d$.

**Proof.** Let $n := |P \cap \mathbb{Q}^d|$. According to Proposition 6.3, there exists a Markov basis $M \subset \mathbb{Z}^d$ such that $K_n \cong P(M)$. By the definition of the fiber dimension and Theorem 5.5, $\lceil \log_2 n \rceil = \text{fdim}(K_n) \leq d$, i.e., $n \leq 2^d$. 

**Remark 6.6.** For any $n \in \mathbb{N}$, there exists a distinct pair-sum polytope on $n$ lattice points, for example, take the $(n-1)$-dimensional simplex in $\mathbb{Q}^{n-1}$. Thus, for fixed $n \in \mathbb{N}$, we can ask for the smallest natural number $d \in \mathbb{N}$ such that there exists a distinct pair-sum polytope $P \subset \mathbb{Q}^d$ on $n$ lattice points. Then $d \leq n - 1$ and Proposition 6.5 on the other hand gives $\lfloor \log_2 n \rfloor \leq d$. Given such a minimal $d$, Proposition 6.3 implies the fiber dimension of any graph on $n$ nodes is bounded from above by $d$. However, the embedding of graphs into distinct pair-sum polytopes is far from optimal. For instance, a path has fiber dimension 1 and thus this bound can be made arbitrarily bad. However, we think its an interesting question for which class beside complete graphs this bound is (asymptotically) tight.

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