RIGIDITY OF SMALL DELAUNAY TRIANGULATIONS OF THE HYPERBOLIC PLANE

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Abstract. We show that a small and regular triangulation of the hyperbolic plane is rigid under the discrete conformal change, extending previous rigidity results on the Euclidean plane. Our result is a discrete analogue of the conformal rigidity of the hyperbolic plane.

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1. Introduction

It is well-known that a conformal diffeomorphism between two hyperbolic planes must be an isometry. In this paper we discretize the hyperbolic plane by small geodesic triangulations and prove an analogous discrete rigidity result.

In this paper we use the Poincaré disk model to represent the hyperbolic plane $H^2$, and assume $H^2 = (D,g)$ where $D = \{ z \in \mathbb{C} : |z| < 1 \}$ is the unit open disk in the complex plane and $g(z) = \frac{4|dz|^2}{(1-|z|^2)^2}$.

Given two points $z_1, z_2$ in $D$, $d(z_1, z_2)$ denotes the hyperbolic distance between $z_1, z_2$. Assume $T = (V, E, F)$ is an (infinite) simplicial topological triangulation of the hyperbolic plane $H^2$, where $V$ is the set of vertices, $E$ is the set of edges and $F$ is the set of faces. Given a subcomplex $T_0 = (V_0, E_0, F_0)$ of $T$, denote $|T_0|$ as the underlying space of $T_0$. A mapping $\phi : |T_0| \rightarrow H^2$ is called hyperbolic geodesic if

(a) $\phi$ maps each edge of $T_0$ to a hyperbolic geodesic arc in $H^2$, and
(b) $\phi$ maps each triangle of $T_0$ to a hyperbolic geodesic triangle in $H^2$.

A hyperbolic geodesic mapping $\phi$ of $T_0$ naturally induces an edge length $l = l(\phi) \in \mathbb{R}^{E_0}$ on $T_0$ by letting $l_{ij} = d(\phi(i), \phi(j))$.

Bobenko-Pinkall-Springborn [BPS15] introduced the following notion of hyperbolic discrete conformality, extending Luo’s notion of Euclidean discrete conformality in [Luo04].
Definition 1.1 (Bobenko-Pinkall-Springborn [BPS15]). Given \( l, l' \in \mathbb{R}^{E_0} \), they are (hyperbolic) discretely conformal if there exists some \( u \in \mathbb{R}^{V_0} \) such that for any edge \( ij \in E_0 \)
\[
\sinh \frac{l'_{ij}}{2} = e^{\frac{1}{2}(u_i + u_j)} \sinh \frac{l_{ij}}{2}.
\]
In this case, \( u \) is called a discrete conformal factor, and we denote \( l' = u *_{h} l \).

A hyperbolic geodesic mapping \( \phi \) is called Delaunay if \( \phi(k') \) is not in the open circumdisk of \( \phi(\triangle ijk) \) for any pair of adjacent triangles \( \triangle ijk \) and \( \triangle ijk' \) in \( T_0 \). The main result of the paper is the following.

Theorem 1.2. Suppose \( \epsilon > 0 \) and \( \phi, \psi \) are Delaunay hyperbolic geodesic mappings of \( T \), such that
(a) \( \phi, \psi \) are homeomorphisms from \( |T| \) to \( \mathbb{H}^2 \), and
(b) \( l(\phi), l(\psi) \) are discretely conformal, and
(c) all the inner angles in \( \phi(\triangle ijk), \psi(\triangle ijk) \) are at least \( \epsilon \) for all \( \triangle ijk \in F \), and
(d) \( \| l(\phi) \|_{\infty} \) and \( \| l(\psi) \|_{\infty} \) are both less than \( \epsilon^3/8192 \).
Then \( l(\phi) = l(\psi) \).

The rigidity of triangulations of the Euclidean plane under Luo's notion of discrete conformality was studied in [WGS15] [DGM18] [LSW20] [Wu22]. In particular, [Wu22] relates the Euclidean discrete conformality with the hyperbolic conformality. We will use the tools developed in [Wu22] to prove our hyperbolic rigidity results.

1.1. Other Related Works. Such notion of discrete conformality, proposed by Luo [Luo04] for the Euclidean case and then by Bobenko-Pinkall-Springborn [BPS15] for the hyperbolic case, proved to have rich math theory and useful applications. Researchers have been studied various problems such as rigidity, convergence and discrete geometric flows. One may refer to [WGS15] [GLW19] [WZ20] [LSW20] [LWZ21a] [DGM22] [GLSW18] [GGL+18] [SWGL15] [Spr19] [LW19] [Wu14]. Other works on deformations of triangle meshes could be found in [LWZ21b] [LWZ21c] [LWZ22].

1.2. Organization of the Paper. In Section 2 we introduce necessary properties and tools for the proof of the main theorem. The proof of Theorem 1.2 is given in Section 3.

2. Preparations for the Proof

It is elementary to verify that for all \( x \in [0, 1] \),
\[
\frac{x}{2} \leq \sin x \leq x \leq \sinh x \leq e^x - 1 \leq 2x.
\]

Lemma 2.1. Suppose \( z_1, z_2 \in D \).
(a) If \( d(z_1, z_2) \leq 1 \) then
\[
\frac{|z_2 - z_1|}{1 - |z_2|} \leq 2d(z_1, z_2).
\]
(b) If \( |z_1| \leq |z_2| \) then
\[
\frac{1}{2} d(z_1, z_2) \leq \frac{|z_2 - z_1|}{1 - |z_2|}.
\]
Lemma 2.4

Proof: (a) Suppose $\Gamma \subseteq \mathbb{H}^2$ is the hyperbolic circle containing $z_1, z_2$ such that $d(z_1, z_2)$ is the hyperbolic diameter of $\Gamma$. By a rotation we may assume that $[a, b] \subseteq (-1, 1)$ is a diameter of $\Gamma$ and $b \geq |a|$. Then

\[
1 + 2d(z_1, z_2) \geq e^{d(z_1, z_2)} = \frac{e^{d(a, b)}}{(1 - b)(1 + a)} \geq \frac{1 - a}{1 - b} = 1 + b - a = 1 + \frac{|z_2 - z_1|}{1 - |z_2|}.
\]

(b) Assume $\gamma(t) = tz_2 + (1 - t)z_1$. Then

\[
|\gamma(t)| \leq t|z_2| + (1 - t)|z_1| \leq |z_2|
\]

for all $t \in [0, 1]$ and $d(z_1, z_2)$ is at most equal to the hyperbolic length of $\gamma([0, 1])$, which is equal to

\[
\int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt \leq \frac{2|z_2 - z_1|}{1 - |z_2|^2} \leq \frac{2|z_2 - z_1|}{1 - |z_2|}.
\]

\[\square\]

Lemma 2.2. Suppose $z_1, z_2$ are two distinct points in $D$ with $d(z_1, z_2) \leq 1$. Let $\gamma$ and $\gamma'$ be the hyperbolic and Euclidean geodesic arcs between $z_1, z_2$ respectively. Then the intersecting angle between $\gamma, \gamma'$ is less than $2d(z_1, z_2)$.

Proof. If $\gamma$ is straight in the Euclidean background geometry, then the intersecting angle is just $0$. So we may assume $\gamma$ is a Euclidean circular arc orthogonal to $\{|z| = 1\}$. Denote $z_*$ as the center of the circle containing $\gamma$. Then the intersecting angle of $\gamma, \gamma'$ is equal to

\[
\frac{1}{2} \angle z_1z_*z_2 = \sin^{-1} \frac{|z_2 - z_1|}{2|z_* - z_1|} \leq \frac{|z_2 - z_1|}{|z_* - z_1|} < \frac{|z_2 - z_1|}{1 - |z_1|} \leq 2d(z_1, z_2).
\]

Here we used the fact that $\sin^{-1}(x) \leq 2x$ for all $x \in [0, 1]$.

\[\square\]

The following local maximum principle is indeed implied in [Wu22].

Lemma 2.3. Let $i \in V$ and $T_0 = (V_0, E_0, F_0)$ be the 1-ring neighborhood of $i$. Suppose $\phi, \psi$ are Delaunay hyperbolic geodesic embeddings of $T_0$. Assume $l(\psi) = u \ast_h l(\phi)$, then $u_i^h < 0$ implies that

\[
u_i > \min_{j: i \in E} u_j.
\]

The hyperbolic discrete conformal change is related with the Euclidean discrete conformal change as follows.

Lemma 2.4 ([Wu22]). Suppose $z_1, z_2, z_1', z_2' \in D$ and $u_1, u_2, u_1^h, u_2^h \in \mathbb{R}$ are such that

\[
u_i^h = u_i + \log \frac{1 - |z_i|^2}{1 - |z_i'|^2}
\]

for $i = 1, 2$. Then

\[
|z_1' - z_2'| = e^{\frac{1}{2}(u_1 + u_2)}|z_1 - z_2|
\]

if and only if

\[
sinh \frac{d(z_1', z_2')}{2} = e^{\frac{1}{2}(u_1^h + u_2^h)} \sinh \frac{d(z_1, z_2)}{2}.
\]
Corollary 2.5. Suppose \( b \in \mathbb{R} \) and \( T_0 = (V_0, E_0, F_0) \) is a subcomplex of \( T \). If \( \psi, \tilde{\psi} \) are both hyperbolic geodesic maps of \( T_0 \) such that \( \tilde{\psi}(i) = e^b \psi(i) \) for all \( i \in V_0 \), then
\[
\ell(\tilde{\psi}) = u \ast_h \ell(\psi)
\]
where
\[
u_i = b + \log \frac{1 - |\psi(i)|^2}{1 - |\psi(i)|^2}.
\]

3. Proof of Theorem 1.2

Assume \( \ell(\psi) = u \ast_h \ell(\phi) \). We will prove \( u_i \geq 0 \) for all \( i \in V \), and then by symmetry \( u_i \leq 0 \) for all \( i \in V \) and we are done. Let us prove by contradiction. Suppose \( a \in V \) and \( u_a < 0 \). Denote \( z_i = \phi(i) \) and \( z'_i = \psi(i) \) for all \( i \in V \). Without loss of generality, we may assume \( z_a = z'_a = 0 \).

Notice that \( 2d(z'_i, z'_j) = l_{ij}(\psi) \leq \epsilon/4 \) for all \( ij \in E \). By Lemma 2.2, there exists a map \( \psi_E : [T] \rightarrow D \) such that
(a) \( \psi_E(i) = z'_i \) for all \( i \in V \),
(b) \( \psi_E(\triangle ijk) \) is a Euclidean triangle in \( D \) for all \( \triangle ijk \in E \), and
(c) \( \psi_E \) is an embedding restricted on the 1-ring neighborhood of \( j \) for any \( j \in V \), and
(d) all the inner angles in \( \psi_E(\triangle ijk) \) are at least \( \epsilon/2 \) for all \( \triangle ijk \in E \), and
(e) \( \psi_E \circ \psi^{-1} \) is topologically positively oriented on \( \psi(\triangle ijk) \) for all \( \triangle ijk \in D \).

Pick \( b \in (0, -u_a) \cap (0, 1/10000) \) and denote \( \hat{\psi}_E \) as \( e^b \psi_E \). Denote \( z''_i = \hat{\psi}_E(i) = e^b z'_i \) for all \( i \in V \). Given \( z_1, z_2 \in \mathbb{C} \), denote \( d(z_1, z_2) = \infty \) if \( z_1 \notin D \) or \( z_2 \notin D \). Let
\[
E_0 = \{ ij \in E : d(z''_i, z''_j) \leq \frac{\epsilon}{16} \}
\]
and
\[
V_0 = \{ i \in V : i \in e \text{ for some } e \in E_0 \}
\]
and
\[
F_0 = \{ \triangle ijk \in F : i, j, k \in E_0 \}.
\]
Then \( T_0 = (V_0, E_0, F_0) \) is a finite subcomplex of \( T \), and it is elementary to verify that \( a \in V_0 \). By Lemma 2.2, there exists a hyperbolic Delaunay geodesic map \( \tilde{\psi} \) from \( T_0 \) to \( D \) such that
(a) \( \tilde{\psi}(i) = z''_i \), and
(b) \( \tilde{\psi} \) is an embedding restricted on the 1-ring neighborhood of \( i \) if \( ij \in E_0 \) for all neighbors \( j \) of \( i \).

Denote \( \ell = \ell(\phi) \) and \( \ell' = \ell(\tilde{\psi}) \) and by Corollary 2.5 we have
\[
\ell' = \tilde{u} \ast_h \ell
\]
on \( T_0 \) where
\[
\tilde{u}_i = u_i + b + \log \frac{1 - |z''_i|^2}{1 - |z''_i|^2}.
\]
Then \( \tilde{u}_a = u_a + b < 0 \). Let \( i \in V_0 \) be such that
\[
u_i = \min_{j \in V_0} u_j < 0.
\]
Then by Lemma 2.3 there exists an edge \( ik \in E \) not in \( E_0 \). Furthermore, there exists a triangle \( \triangle ijk \) such that \( ij \in E_0 \) and \( ik \notin E_0 \).

Then
\[
\ell'_{ij} \leq \frac{\epsilon}{16}.
\]
and by Lemma 2.1
\[
\frac{|z''_j - z''_k|}{1 - |z''_j|} \leq \frac{\epsilon}{8},
\]
and by the Euclidean sine law
\[
\frac{|z''_j - z''_k|}{1 - |z''_j|} \leq \frac{\epsilon}{8 \sin(\epsilon/2)} \leq \frac{\epsilon}{8(\epsilon/2)/2} = \frac{\epsilon}{2},
\]
and
\[
1 - |z''_j| \geq 1 - |z''_k| - |z''_j - z''_k| \geq 1 - |z''_j| = \frac{1}{2}(1 - |z''_j|),
\]
and by Lemma 2.1 and the Euclidean sine law
\[
l'_{ik} \leq \frac{2|z''_k - z''_j|}{1 - |z''_j|} \leq \frac{4|z''_j - z''_k|}{(1 - |z''_j|) \sin(\epsilon/2)} \leq \frac{16}{\epsilon} \cdot \frac{|z''_j - z''_k|}{1 - |z''_j|} \leq \frac{32}{\epsilon} l_{ij} \leq 2.
\]
For the same reason \( l'_{ik} \leq 2 \). On the other hand \( l'_{ik} \geq \epsilon/16 \) since \( ik \notin E_0 \). Then we can derive a contradiction using the hyperbolic sine law as the following.

\[
1 > \epsilon^2 = \left( \frac{\sinh \frac{l'_{ik}}{2}}{\sinh \frac{l_{ik}}{2}} \cdot \frac{\sinh \frac{l'_{ij}}{2}}{\sinh \frac{l_{ij}}{2}} \right) \left( \frac{\sinh \frac{l'_{jk}}{2}}{\sinh \frac{l_{jk}}{2}} \right) \geq \frac{1}{8} \frac{l'_{ik}}{l_{ij}} \frac{l'_{ij}}{l_{jk}} \sinh l_{ik} \geq \frac{1}{8} \cdot \frac{\epsilon/16}{l_{ij}} \frac{\epsilon}{32} \sin \epsilon \geq \frac{\epsilon^3}{8192 l_{ij}} \geq 1.
\]

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