Quantum integrable systems and representations of Lie algebras

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Introduction

A quantum many particle system on the line with an interaction potential \( U(x) \) is defined by the Hamiltonian

\[
H = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + K \sum_{1 \leq i < j \leq N} U(x_i - x_j), \quad K \in \mathbb{R}.
\]

Such systems were first considered by Calogero for \( U(x) = x^{-2} \) [Ca] and Sutherland for \( U(x) = \sinh(x)^{-2} \) [Su].

Any differential operator commuting with \( H \) is called a quantum integral of the system. One says that two quantum integrals are in involution if they commute as differential operators.

The system defined by \( H \) is called completely integrable if it has \( N \) quantum integrals in involution, \( L_1, ..., L_N \), which are algebraically independent: every polynomial \( P \in \mathbb{R}[z_1, ..., z_N] \) such that \( P(L_1, ..., L_N) = 0 \) is identically zero.

For \( N > 2 \), the system defined by \( H \) is not always completely integrable. However, for special choices of \( U(x) \) it is known that it is completely integrable for any \( K \). For example, this is proved for \( U(x) = x^{-2} \) (rational case), \( U(x) = \sinh(x)^{-2} \) (trigonometric case), and \( U(x) = \wp(x|\tau) \), where \( \wp \) is the Weierstrass elliptic function:

\[
\wp(x|\tau) = \frac{1}{x^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left[ \frac{1}{(x-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} \right]
\]

(elliptic case) (see [OP] and references therein). These results come from an explicit construction of quantum integrals via the Lax matrix.

In this paper we propose a new construction of the quantum integrals for \( H \) in the trigonometric and elliptic cases using representation theory of Lie algebras, and give a new proof of the complete integrability theorem based on this construction.

In the trigonometric case, we construct a homomorphism from the center of the universal enveloping algebra of \( \mathfrak{gl}_N \) to the algebra of differential operators in \( N \) variables. We show that the image of the second-order Casimir under this homomorphism is a multiple of the Sutherland operator, which implies that the images of higher Casimirs are its quantum integrals and thus proves the integrability. In the elliptic case, we apply a similar method to the center of the (completed) universal enveloping algebra of the affine \( \mathfrak{gl}_N \) at the critical level \( k = -N \), and thus produce quantum integrals of the Hamiltonian (1) with the elliptic potential \( U = K\wp \).

If \( U(x) \) is one of the above three choices, we call \( H \) a Calogero-Sutherland (CS) operator. If \( H \) is a CS operator then for any set of complex numbers \( \Lambda_i \) the system of differential equations

\[
L_i \psi_i = \Lambda_i \psi_i, \quad 1 \leq i \leq N,
\]

is integrable.
is consistent and has \( N! \) linearly independent solutions. This system is called the eigenvalue problem for \( H \). The solutions of (3) for the rational and trigonometric cases (unlike the elliptic case) have been studied in many papers [8] and are rather well understood. We will give a new construction of these eigenfunctions for the trigonometric case as normalized traces of intertwiners between representations of \( \mathfrak{gl}_N \).

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1. Diagonalization of the Sutherland operator

The Sutherland operator is the following differential operator:

\[
H = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq i < j \leq N} \frac{K}{\sinh^2(x_i - x_j)}, \quad K \in \mathbb{C}
\]

The quantum integrals for this operator and their eigenfunctions (zonal spherical functions) are known [OP, HO]. In this section we will give a new description of the quantum integrals and eigenfunctions using representation theory of \( \mathfrak{gl}_N \).

Let \( \mathfrak{gl}_N \) denote the Lie algebra of complex \( N \times N \) matrices. If \( A \in \mathfrak{gl}_N \), we will denote by \( A_{ij} \) the entry at the intersection of the \( i \)-th row and \( j \)-th column of \( A \). Also, \( E_{ij} \) will denote the elementary matrices: \((E_{ij})_{mn} = \delta_{im}\delta_{jn}\). For brevity we denote \( E_{ii} \) by \( h_i \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N \), and let \( M_\lambda \) be the Verma module over \( \mathfrak{gl}_N \) with highest weight \( \lambda \), i.e. the module generated by a highest weight vector \( v_\lambda \) satisfying the defining relation

\[
A v_\lambda = \sum_{j=1}^{N} \lambda_j A_{jj} v_\lambda,
\]

whenever \( A \) is an upper triangular matrix. Let \( M_\lambda^* \) denote the restricted dual module to \( M_\lambda \) — the direct sum of dual spaces to the weight subspaces in \( M_\lambda \) with the action of \( \mathfrak{gl}_N \) defined by duality.

Let \( \mu \in \mathbb{C} \). Define a module \( V_\mu \) over \( \mathfrak{gl}_N \) as follows. As a vector space, \( V_\mu \) is the space of functions of the form \( f(x) = (\prod_{j=1}^{m} \xi_j)^{\mu} p(\frac{\xi_1}{\xi_2}, \ldots, \frac{\xi_{N-1}}{\xi_N}) \), where \( p \in \mathbb{C}[y_1^{\pm 1}, \ldots, y_N^{\pm 1}] \) is a Laurent polynomial, and the action \( \phi \) of \( \mathfrak{gl}_N \) in \( V_\mu \) is described by the formula \( \phi(E_{ij}) = \xi_i \frac{\partial}{\partial \xi_j} - \mu \delta_{ij} \).

It is obvious that the set of weights of \( V_\mu \) is \( \{ (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N : \sum \lambda_j = 0 \} \), and every weight occurs with multiplicity 1.

Consider the completed tensor product \( M_\lambda \hat{\otimes} V_\mu = \text{Hom}_\mathbb{C}(M_\lambda^*, V_\mu) \). It has a natural structure of a \( \mathfrak{gl}_N \)-module.

**Proposition 1.1.** If \( M_\lambda \) is irreducible then there exists a unique up to a factor nonzero intertwining operator \( \Phi_\lambda : M_\lambda \rightarrow M_\lambda \hat{\otimes} V_\mu \).

**Proof.** We need to prove that the module \( \text{Hom}_\mathbb{C}(M_\lambda^*, V_\mu) \) contains a singular vector \( v_{\lambda, \mu} \) of weight \( \lambda \) (a vector satisfying relation (1.2)), and that such a vector is
unique. Since $w_\lambda$ is invariant under the Lie subalgebra $n^+$ of strictly upper triangular matrices, it is uniquely determined by its value on the lowest weight vector $v_\lambda^*$ of the module $M_\lambda^*$. Because $M_\lambda$ is irreducible, $M_\lambda^* = U(n^+)v_\lambda^*$, and therefore $w_\lambda(v_\lambda^*)$ can be any zero weight vector in $V_\mu$. Since the zero weight subspace of $V_\mu$ is one dimensional, the vector $w_\lambda$ exists and is unique up to a factor, Q.E.D. ■

We fix the normalization of the operator $\Phi_\lambda$ under which $<v_\lambda^*, \Phi v_\lambda^* >= \prod \xi^\mu_j \in V_\mu$.

Let $\rho = (\rho_1, ..., \rho_N), \rho_j = \frac{N+1}{2} - j$. Consider the function of $N$ variables

\[
\Psi_\lambda(z_1, ..., z_N) = \frac{\text{Tr}_{M_\lambda}(\Phi_\lambda z_1^{h_1} ... z_N^{h_N})}{\text{Tr}_{M_{-\rho}}(z_1^{h_1} ... z_N^{h_N})}.
\]

This function takes values in $V_\mu$ and has weight zero, hence

\[
\Psi_\lambda = \psi_\lambda \prod \xi^\mu_j,
\]

where $\psi_\lambda$ is a complex-valued function.

Let us introduce new coordinates $x_i$ such that $z_i = e^{2x_i}$, and regard the function $\psi_\lambda$ as a function of $(x_1, ..., x_N)$

If $a, b \in \mathbb{C}^N$, let $<a, b> = \sum_{j=1}^{N} a_j b_j$, and $a^2 = <a, a>$. Also, for brevity we use the notation $Z = z_1^{h_1} ... z_N^{h_N}$, and write $\text{Tr}_{\lambda}$ for $\text{Tr}_{M_\lambda}$.

**Theorem 1.2.** If $K$ in formula (1.1) equals $-2\mu(\mu + 1)$ then the function $\psi_\lambda$ is an eigenfunction of the differential operator $H$ defined by (1.1) with the eigenvalue $4(\lambda + \rho)^2$.

**Proof.** The Casimir element

\[
C_2 = \sum_{i,j=1}^{N} E_{ij} E_{ji}
\]

acts in $M_\lambda$ by multiplication by $<\lambda, \lambda + 2\rho>$. Therefore,

\[
\frac{\text{Tr}_{\lambda}(\Phi_\lambda C_2 Z)}{\text{Tr}_{-\rho}(Z)} = <\lambda, \lambda + 2\rho> \Psi_\lambda(z_1, ..., z_N).
\]

On the other hand, the left hand side of (1.6) can be written in the form $\sum_{i,j=1}^{N} \Psi^{ij}_\lambda$, where

\[
\Psi^{ij}_\lambda = \frac{\text{Tr}_{\lambda}(\Phi_\lambda E_{ij} E_{ji} Z)}{\text{Tr}_{-\rho}(Z)}.
\]

Let us express the terms $\Psi^{ij}_\lambda$ in terms of $\Psi_\lambda$. First of all, consider the function $F(z_1, ..., z_N) = \text{Tr}_{-\rho}(Z)$. This function is the character of the module $M_{-\rho}$, therefore

\[
F(z_1, ..., z_N) = \prod_{j=1}^{N} z_j^{-\rho_j} \prod_{i>j} \left(1 - \frac{z_i}{z_j}\right)^{-1} = \left(\prod_{j=1}^{N} z_j \right) \left(\prod_{i>j} (z_i - z_j)\right)^{-1},
\]

\[
z_i \frac{\partial F}{\partial z_i} = \left(\frac{N-1}{2} - \sum_{i>j} z_i \frac{1}{z_i - z_j}\right).
\]
We consider two cases.

1) \( i = j \). Then, using (1.8), we obtain

\[
\Psi_{ij}^\lambda = \frac{1}{F} \left( z_i \frac{\partial}{\partial z_i} \right)^2 (F \Psi_\lambda) = 
\]

(1.9) \( \left( z_i \frac{\partial}{\partial z_i} \right)^2 \Psi_\lambda + \left( (N-1)z_i - \sum_{j \neq i} \frac{2z_i^2}{z_i - z_j} \right) \frac{\partial \Psi_\lambda}{\partial z_i} + \frac{\left( z_i \frac{\partial}{\partial z_i} \right)^2 F}{F} \Psi_\lambda. \)

2) \( i \neq j \). Then we have

\[
\Psi_{ij}^\lambda = F^{-1} \text{Tr}_\lambda (\Phi_\lambda E_{ij} E_{ji} Z) = 
\]

\[
F^{-1} \text{Tr}_\lambda (E_{ij} \Phi_\lambda E_{ji} Z) + F^{-1} E_{ij} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z) = 
\]

\[
F^{-1} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z E_{ij}) + F^{-1} E_{ij} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z) = 
\]

\[
\frac{z_i}{z_j} F^{-1} \text{Tr}_\lambda (\Phi_\lambda E_{ij} E_{ji} Z) + \frac{z_i}{z_j} F^{-1} \text{Tr}_\lambda (\Phi_\lambda (E_{jj} - E_{ii}) Z) 
\]

\[
+ F^{-1} E_{ij} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z). \]

(1.10) \( \frac{z_i}{z_j} \Psi_{ij}^\lambda + F^{-1} \frac{z_i}{z_j} \left( z_j \frac{\partial}{\partial z_j} - z_i \frac{\partial}{\partial z_i} \right) (F \Psi_\lambda) + F^{-1} E_{ij} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z). \)

Formula (1.10) is a linear equation on \( \Psi_{ij}^\lambda \). Solving this equation, we obtain

(1.11) \( \Psi_{ij}^\lambda = F^{-1} \frac{z_i}{z_j - z_i} \left( z_j \frac{\partial}{\partial z_j} - z_i \frac{\partial}{\partial z_i} \right) (F \Psi_\lambda) + \frac{F^{-1} z_j}{z_j - z_i} E_{ij} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z). \)

It remains to compute \( \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z) \). We have

\[
\text{Tr}_\lambda (\Phi_\lambda E_{ji} Z) = \text{Tr}_\lambda (E_{ji} \Phi_\lambda Z) + E_{ji} \text{Tr}_\lambda (\Phi_\lambda Z) = \text{Tr}_\lambda (\Phi_\lambda Z E_{ji}) + FE_{ji} \Psi_\lambda = 
\]

(1.12) \( \frac{z_j}{z_i} \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z) + FE_{ji} \Psi_\lambda, \)

which implies

(1.13) \( \text{Tr}_\lambda (\Phi_\lambda E_{ji} Z) = \frac{z_i}{z_i - z_j} FE_{ji} \Psi_\lambda. \)

Combining (1.11) and (1.13), we deduce

(1.14) \( \Psi_{ij}^\lambda = F^{-1} \frac{z_i}{z_j - z_i} \left( z_j \frac{\partial}{\partial z_j} - z_i \frac{\partial}{\partial z_i} \right) (F \Psi_\lambda) - \frac{z_i z_j}{(z_i - z_j)^2} E_{ij} E_{ji} \Psi_\lambda. \)

It is easy to see that the operator \( E_{ij} E_{ji} \) acts in the zero weight subspace of \( V_\mu \) by multiplication by \( \mu(\mu + 1) \). Therefore, (1.14) can be rewritten as follows:

\[
\Psi_{ij}^\lambda = \frac{z_i}{z_j - z_i} \left( z_j \frac{\partial}{\partial z_j} - z_i \frac{\partial}{\partial z_i} \right) \Psi_\lambda \]

(1.15) \( + \left[ \frac{z_i}{z_j - z_i} \left( z_j \frac{\partial}{\partial z_j} - z_i \frac{\partial}{\partial z_i} \right) \right] \log F - \mu(\mu + 1) \frac{z_i z_j}{(z_i - z_j)^2} \Psi_\lambda. \)
Now, summing up equations (1.9) for all $i$ and equations (1.14) for all $i \neq j$, and using (1.8) and the identity

\[(1.16) \quad \sum_{i \neq j} \frac{z_i}{z_j - z_i} \left( \frac{z_j}{z_j - z_i} - \frac{z_i}{z_i - z_j} \right) = \sum_{i \neq j} \left( \frac{2z_i^2}{z_i - z_j} - (N - 1)z_i \right) \frac{\partial}{\partial z_i}, \]

we obtain

\[(1.17) \quad \frac{\text{Tr}_\lambda(\Phi_\lambda C_2 Z)}{\text{Tr}_{-\rho}(Z)} = \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 \Psi_\lambda + \frac{N}{4} \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 F^{-1} \Psi_\lambda - 2 \sum_{i=1}^{N} \mu(\mu + 1) \frac{z_i z_j}{(z_i - z_j)^2} \Psi_\lambda. \]

From formula (1.8) it follows that

\[(1.18) \quad F^{-1}(z_1, ..., z_N) = \det(z_i^{-\rho_j}). \]

Using this identity, we find

\[(1.19) \quad \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 F^{-1} = <\rho, \rho> F^{-1}. \]

Substituting (1.19) into (1.17), we get

\[(1.20) \quad \frac{\text{Tr}_\lambda(\Phi_\lambda C_2 Z)}{\text{Tr}_{-\rho}(Z)} = \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} \right)^2 \Psi_\lambda - <\rho, \rho> \Psi_\lambda - 2 \sum_{i<j} \mu(\mu + 1) \frac{1}{z_i + z_j - 2} \Psi_\lambda. \]

Rewriting equation (1.20) in the new variables $x_i$ such that $z_i = e^{2x_i}$, we obtain

\[(1.21) \quad \frac{\text{Tr}_\lambda(\Phi_\lambda C_2 Z)}{\text{Tr}_{-\rho}(Z)} = \frac{1}{4} \sum_{i=1}^{N} \frac{\partial^2 \Psi_\lambda}{\partial x_i^2} - <\rho, \rho> \Psi_\lambda - \frac{\mu(\mu + 1)}{2} \sum_{i<j} \frac{1}{\sinh^2(x_i - x_j)} \Psi_\lambda. \]

Comparing (1.6) and (1.21), we finally get

\[(1.22) \quad \sum_{i=1}^{N} \frac{\partial^2 \psi_\lambda}{\partial x_i^2} - 2\mu(\mu + 1) \sum_{i<j} \frac{1}{\sinh^2(x_i - x_j)} \psi_\lambda = 4(\lambda + \rho)^2 \psi_\lambda. \]

The theorem is proved. ■

2. Quantum integrals of the Sutherland operator

Let us now find the quantum integrals of the Sutherland operator. For this purpose we will use the higher Casimir elements of the Lie algebra $\text{gl}_N$.

Let $Y$ be any element of the universal enveloping algebra $U(\text{gl}_N)$. Then we can consider the following $V_\mu$-valued function:

\[(2.1) \quad \Psi_\lambda(Y|z_1, ..., z_N) = \frac{\text{Tr}_{M_\lambda}(\Phi_\lambda Y z_1^{h_1} \cdots z_N^{h_N})}{\text{Tr}_{-\rho}(z_1^{h_1} \cdots z_N^{h_N})}. \]
Proposition 2.1. There exists a differential operator $\mathcal{L}_Y$ in $z_1, ..., z_N$ whose coefficients are $U(\mathfrak{gl}_N)$-valued functions, such that

\begin{equation}
\Psi_\lambda(Y|z_1, ..., z_N) = \mathcal{L}_Y \Psi_\lambda(z_1, ..., z_N).
\end{equation}

Proof. We say that $Y$ is of order $\leq m$ if it is a sum of monomials including $\leq m$ factors from $\mathfrak{gl}_N$. The proof is by induction in the order of $Y$. For order 0 the statement is obvious. If $Y$ is of a positive order $m$ then we may assume that $Y$ is a monomial of the form $Y = X E_{ij}$. Since $X$ is a monomial of a lower order than $Y$, we may assume that the operator $L_X$ is already defined.

Again, we consider two cases. If $i = j$ then

\begin{equation}
\Psi_\lambda(Y|z_1, ..., z_N) = F^{-1} z_i \frac{\partial}{\partial z_i} (F \Psi_\lambda(X|z_1, ..., z_N)),
\end{equation}

so we can set $\mathcal{L}_Y = F^{-1} \circ z_i \frac{\partial}{\partial z_i} \circ F \circ \mathcal{L}_X$.

If $i \neq j$ then let $U = [X, E_{ij}]$. Clearly, the order of $U$ is $\leq m - 1$, so we can assume that $L_U$ has been defined. Then we have

\begin{equation}
\Psi_\lambda(Y) = \Psi_\lambda(E_{ij} X) + \Psi_\lambda(U) = E_{ij} \Psi_\lambda(X) + \frac{z_i}{z_j} \Psi_\lambda(Y) + L_U \Psi_\lambda = (E_{ij} \mathcal{L}_X + L_U) \Psi_\lambda + \frac{z_i}{z_j} \Psi_\lambda(Y),
\end{equation}

which implies

\begin{equation}
\Psi_\lambda(Y) = \frac{z_j}{z_j - z_i} (E_{ij} \mathcal{L}_X + L_U) \Psi_\lambda.
\end{equation}

This proves the step of induction Q.E.D. \[\Box\]

Let $Y \in U(\mathfrak{gl}_N)$ be of weight zero, i.e. such that $[h_i, Y] = 0$ for all $i$. Then $\Psi_\lambda(Y|z_1, ..., z_N) = \psi_\lambda(Y|x_1, ..., x_N) \prod_{i=1}^N \xi_i^{\mu}$, where $\psi$ is a complex-valued function.

Proposition 2.2. If $Y$ is of weight zero then there exists a unique scalar-valued differential operator $D_Y$ in the variables $x_1, ..., x_N$ whose coefficients are rational functions of $e^{2x_i}$, $1 \leq i \leq N$ such that

\begin{equation}
\psi_\lambda(Y|x_1, ..., x_N) = D_Y \psi_\lambda(x_1, ..., x_N)
\end{equation}

for a generic $\lambda$.

Proof. If $Y$ is of weight 0 then $\mathcal{L}_Y$ preserves weight, so its coefficients map the zero weight subspace in $V_\mu$ to itself. Since the zero weight subspace is one-dimensional, the coefficients of $\mathcal{L}_Y$ become scalars when restricted to this space. In this way the operator $\mathcal{L}_Y$ turns into a scalar differential operator $D_Y$ which satisfies (2.6).

To establish the uniqueness, it is enough to prove the following:

Lemma 2.3. Any differential operator $D$ annihilating $\psi_\lambda$ for a generic $\lambda$ whose coefficients are rational functions in $z_i = e^{2x_i}$, $1 \leq i \leq N$ must be identically equal to 0.
Proof. Since the functions $\psi_\lambda$ are homogeneous in $z_1, \ldots, z_N$, any homogeneous component of $D$ also annihilates them. Therefore, we may assume that $D$ is homogeneous. We can also assume that $D$ is of degree 0 (otherwise we can multiply it by a suitable power of $z_1$) and that its coefficients are polynomials in $y_i = z_i/z_{i+1}$ (otherwise we can multiply $D$ by the common denominator of its coefficients). Let us write $D$ in the form

$$D = \sum_{r_j \geq 0} y_1^{r_1} \cdots y_{N-1}^{r_{N-1}} D_{r_1, \ldots, r_{N-1}},$$

where $D_{r_1, \ldots, r_{N-1}} = p_{r_1, \ldots, r_{N-1}} \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}$, where $p_{r_1, \ldots, r_{N-1}}$ are polynomials. Let $k$ be the smallest integer for which there exist $r_1, \ldots, r_{N-1}$ with $\sum r_j = k$ such that $D_{r_1, \ldots, r_{N-1}} \neq 0$. Since $\psi_1 \in \prod e^{2x_1(\lambda_j + \rho_j)} \mathbb{C}[[y_1, \ldots, y_{N-1}]]$, for this set of $r_j$ we have

$$D_{r_1, \ldots, r_{N-1}} e^{2\mathbf{x}, \lambda + \rho} = 0, \quad \mathbf{x} = (x_1, \ldots, x_N).$$

This implies that $p_{r_1, \ldots, r_{N-1}} (2(\lambda + \rho)) = 0$ for a generic $\lambda$, which means that $p_{r_1, \ldots, r_{N-1}}$ is identically zero – a contradiction. This completes the proof of the lemma and the proposition. \[\Box\]

It is well known that the center of $U(\mathfrak{gl}_N)$ is freely generated by the Casimir elements

$$C_m = \sum_{j_1, \ldots, j_m = 1}^N E_{j_1j_2} \cdots E_{j_{m-1}j_m} E_{j_mj_1}, \quad 1 \leq m \leq N.$$

Define the differential operators

$$L_j = D C_j, \quad 1 \leq j \leq N$$

**Proposition 2.4.**

(i) $L_1 = \frac{1}{2} \sum_{j=1}^N \frac{\partial}{\partial x_j}$, $L_2 = \frac{H}{4} + \rho, \rho >$.

(ii) For any $1 \leq j \leq N$ the function $\psi_\lambda$ is an eigenfunction of $L_j$ with an eigenvalue $p_j(\lambda + \rho)$, where $p_j$ is a symmetric polynomial of degree $j$.

(iii) $[L_i, L_j] = 0$ for all $i, j$.

(iv) The symbol of $L_m$ is $2^{-m} \sum_{i=1}^N \frac{\partial^m}{\partial x_i^m}$.

(v) The operators $L_j$ and the polynomials $p_j$ are algebraically independent.

(vi) The operators $L_i$ are invariant under permutations of $x_1, \ldots, x_N$.

*Proof. (i) The formula for $L_1$ is obvious. The formula for $L_2$ follows from Theorem 1.2.

(ii) The element $C_i$ acts in $M_\lambda$ by multiplication by a scalar polynomially dependent on $\lambda$. This scalar is a symmetric polynomial in $\lambda + \rho$, since the action of $C_i$ is the same in $M_\lambda$ and $M_{\sigma(\lambda + \rho)} - \rho$ for any permutation $\sigma$ (proof: the first module contains the second one when $\lambda$ is a dominant integral weight, and any two polynomials coinciding on dominant integral weights coincide identically). Denote this symmetric polynomial by $p_i$. Then (2.10) implies that $L_i \psi_\lambda = p_i(\lambda + \rho) \psi_\lambda$.

(iii) Statement (ii) implies that $[L_i, L_j] \psi_\lambda = 0$ for a generic $\lambda$. According to Lemme 2.3, this means that $[L_i, L_j] = 0$.\[\Box\]
(iv) It is easy to show that

\begin{equation}
C_m = \sum_{j=1}^{N} h_j^m + \sum_{i<j} X_{ij} E_{ij} + Y,
\end{equation}

where \(X_{ij}, Y\) are elements of \(U(\mathfrak{gl}_N)\) of order \(\leq m - 1\). This fact combined with the proof of Proposition 2.1 shows that the symbol of \(L_m\) will be the same as the symbol of \(D_{\sum_{j=1}^{N} h_j^m}\) (since the rest of the terms in (2.11) will give lower order contributions), i.e. it will equal \(2^{-m} \sum_{i=1}^{N} \frac{\partial^m}{\partial x_i^m}\).

(v) Since the symbols of \(L_j\) are algebraically independent (they are elementary symmetric functions), so are \(L_j\) themselves. Therefore, so are \(p_j\): if there existed a nonzero polynomial \(Q\) such that \(Q(p_1, ..., p_N)\) is identically zero, then \(Q(L_1, ..., L_N)\) would annihilate \(\psi_\lambda\) for a generic \(\lambda\), which, by Lemma 2.3, would mean that this operator is zero, so that from the algebraic independence of \(L_j\) one gets \(Q = 0\).

(vi) Let \(\sigma\) be any transposition. Then for any \(Y\) one has \(L_{\sigma Y} \psi^{-1} = \sigma L_Y \sigma^{-1}\), where \(L_Y^\sigma\) is defined by the formula:

\begin{equation}
(L^\sigma f)(x_{\sigma(1)}, ..., x_{\sigma(N)}) = L f(x_{\sigma(1)}, ..., x_{\sigma(N)}).
\end{equation}

Therefore, \(D_{\sigma Y} \psi^{-1} = D_Y^\sigma\). On the other hand, \(\sigma C_m \sigma^{-1} = C_m\), which implies that \(L_m^{\sigma} = L_m\).

We can now consider the eigenvalue problem (3) associated with the collection of operators \(\{L_j\}\). Recall that this problem is to find a basis of the \(N!\)-dimensional space of solutions of the system of differential equations \(L_i \psi = \Lambda_i \psi, \ 1 \leq i \leq N\), where \(\Lambda_i\) are some given complex numbers.

**Proposition 2.5.** Let \(\lambda\) be a solution of the system of algebraic equations

\begin{equation}
p_i(\lambda + \rho) = \Lambda_i, \ 1 \leq i \leq N.
\end{equation}

Suppose that \(\lambda_i + \rho_i - \lambda_j - \rho_j\) is not an integer for any \(i \neq j\). Then the functions \(\{\psi_{\sigma(\lambda + \rho) - \rho}, \ \sigma \in S_N\}\), (where \(S_N\) is the symmetric group) form a basis in the space of solutions of system (3).

**Proof.** First of all, all the functions \(\{\psi_{\sigma(\lambda + \rho) - \rho}\}\) are defined since all the modules \(M_{\sigma(\lambda + \rho) - \rho}\) are irreducible. Next, the weights \(\sigma(\lambda + \rho) - \rho\) are different for different \(\sigma\), hence the functions \(\{\psi_{\sigma(\lambda + \rho) - \rho}\}\) are linearly independent (this follows from the asymptotics \(\psi_\lambda \sim \prod z_i^{\lambda_i + \rho_i}, \ z_i / z_{i+1} \to 0, 1 \leq i \leq N - 1\)). And finally, all the functions \(\{\psi_{\sigma(\lambda + \rho) - \rho}\}\) are solutions of (3): \(\psi_\lambda\) is a solution of (3) by the definition of \(\lambda\), and the rest are solutions because the operators \(L_i\) and the polynomials \(p_i\) are symmetric and therefore \(\lambda^\sigma = \sigma(\lambda + \rho) - \rho\) satisfies (2.13) for any \(\sigma \in S_N\).

**Remark.** In 1976 I.Frenkel computed the Laplace’s operator in the space of functions on \((SU(N) \times SU(N))/SU(N)_{\text{diag}}\) with values in a symmetric power of the fundamental representation of \(SU(N)\) equivariant with respect to the left diagonal action of \(SU(N)\), and found that it was equal to the Sutherland operator (with sin instead of sinh) and \(K = -2m(m+1)\), where \(m\) is an integer (I.Frenkel, unpublished work). Proposition 2.5 can be regarded as a generalization of this result.
3. Affine Lie algebras, vertex operators, correlation functions, and regular expressions

Recall the definition of affine Lie algebras.

The affine Lie algebra $\widehat{\mathfrak{gl}_N}$ is defined as follows: as a vector space $\widehat{\mathfrak{gl}_N} = \mathfrak{gl}_N \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, and the commutator is given by

$$[A(t) + \alpha c, B(t) + \beta c] = [A(t), B(t)] + \text{Res}_{t=0}(t^{-1}\text{Tr}(A'(t)B(t))),$$

where $A(t), B(t)$ are matrix-valued trigonometric polynomials.

The principal gradation on $\widehat{\mathfrak{gl}_N}$ assigns degree $j - i + Nm$ to the element $E_{ij}t^m$, and degree zero to $c$. The degree of an element $a$ is denoted by $\text{deg}(a)$.

Define the Lie subalgebra $\mathfrak{g}_0$ in $\widehat{\mathfrak{gl}_N}$ to be the Lie algebra of all elements $A(t) + \alpha c \in \widehat{\mathfrak{gl}_N}$ such that $\text{Tr}(A(t))$ is a constant (does not depend on $t$). This algebra is isomorphic to the direct sum of the affine Lie algebra $\mathfrak{sl}_N$ and the one-dimensional abelian Lie algebra. It inherits the grading from $\widehat{\mathfrak{gl}_N}$. We denote by $\mathfrak{g}_0^\pm$ the subalgebras in $\mathfrak{g}_0$ spanned by the elements of positive (respectively negative) degree.

We will work with two kinds of modules over $\mathfrak{g}_0$ – Verma modules and evaluation modules.

Let $\lambda \in \mathbb{C}^N$ and $V$ be a $\mathfrak{g}_0$-module. We say that a nonzero vector $v \in V$ is of weight $\lambda$ if $h_i v = \lambda_i v$. We say that a nonzero element $a \in U(\mathfrak{g}_0)$ is of weight $\lambda$ if $[h_i, a] = \lambda_i a$.

Let $\lambda \in \mathbb{C}^N$ and $k \in \mathbb{C}$. The Verma module $M_{\lambda, k}$ over $\mathfrak{g}_0$ is defined by a single generator $v_{\lambda, k}$ and the relations:

$$[A(t) + \alpha c]v_{\lambda, k} = \left( \sum_{j=1}^N A_{jj}(0)\lambda_j + \alpha k \right) v_{\lambda, k}, \quad A(t) + \alpha c \in \mathfrak{g}_0$$

if $A(t)$ is regular at $t = 0$, and $A(0)$ is upper triangular. This module is a highest weight module which is irreducible for generic $\lambda$ and $k$.

The principal gradation operator $\partial$ on $M_{\lambda, k}$ is defined by the relations:

$$\partial v_{\lambda, k} = 0, \quad [\partial, a] = na \text{ if } a \in \mathfrak{g}_0, \quad \text{deg}(a) = n.$$
Now let us define evaluation modules. Let \( \mu \in \mathbb{C} \). Define the module \( V_\mu \) over \( \mathfrak{g}_0 \) to be the pullback of the module \( V_\mu \) over \( \mathfrak{gl}_N \) defined in Section 1 with respect to the evaluation map \( \pi : \mathfrak{g}_0 \to \mathfrak{gl}_N \) given by \( \pi(A(t) + \alpha c) = A(1) \).

Now let us introduce vertex operators. Throughout this paper by a vertex operator we mean a \( \mathfrak{g}_0 \)-intertwining operator \( \Phi : M_{\lambda,k} \to M_{\lambda,k} \otimes V_\mu \), where \( M_{\lambda,k} \otimes V_\mu \) denotes the space of formal sums of the form \( \sum_{j=0}^{\infty} u_j \otimes v_j \), \( u_j \in M_{\lambda,k}[-j] \), \( v_j \in V_\mu \).

**Proposition 3.1.** If \( M_{\lambda,k} \) is irreducible then there exists a unique nonzero vertex operator up to a scalar factor.

**Proof.** In order to construct \( \Phi \), we first need to construct the vector \( w = \Phi v_{\lambda,k} \in M_{\lambda,k} \otimes V_\mu \). This vector has to be annihilated by the subalgebra \( \mathfrak{g}_0^+ \) and satisfy the condition \( \lambda_i w = \lambda_i w \). To construct such a vector is the same as to construct a map \( w : M'_{\lambda,k} \to V_\mu \) which commutes with the action of \( \mathfrak{g}_0^+ \) and maps \( v'_{\lambda,k} \) to a zero weight vector. Since \( M_{\lambda,k} \) is irreducible, \( M'_{\lambda,k} \) is freely generated by \( v'_{\lambda,k} \) over \( U(\mathfrak{gl}_N) \), which implies that the map \( w \) is uniquely determined once we know the image of \( v'_{\lambda,k} \) under this map. Because the space of zero weight vectors in \( V_\mu \) is one-dimensional, \( w \) is defined uniquely up to a factor.

Finally, since \( \Phi \) is an intertwiner, and \( M_{\lambda,k} \) is freely generated by \( v_{\lambda,k} \) over \( U(\mathfrak{g}_0) \), \( \Phi \) is uniquely determined once we know the image of \( v_{\lambda,k} \) under it, i.e. once we know \( w \). Therefore, \( \Phi \) is unique up to a factor. \( \blacksquare \)

Let \( A : M_{\lambda,k} \to \hat{M}_{\lambda,k} \) be any linear operator. It is obvious that we can uniquely represent the operator \( A \) as an infinite sum

\[
A = \sum_{m \in \mathbb{Z}} A[m],
\]

where \( A[m] = \oplus_j A[m]_j \), and \( A[m]_j : M_{\lambda,k}[j] \to M_{\lambda,k}[j + m] \) are linear operators: for a homogeneous vector \( u \) of degree \( j \) in \( M_{\lambda,k} \) the vector \( A[m]_j u \) is defined as the degree \( j + m \) component of \( Au \).

Let \( z_j, 1 \leq j \leq N \), and \( q \) be formal variables. Let \( z = (z_1, ..., z_N) \). Then we can consider the formal series

\[
F_{\lambda,k}(A, z, q) = \text{Tr} \left|_{M_{\lambda,k}} \left( Aq^{-\vartheta} \prod_{j=1}^{N} z_j^{h_j} \right) \right| = \sum_{m \geq 0} q^m \text{Tr} \left|_{M_{\lambda,k}[m]} \left( A[0]_m \prod_{j=1}^{N} z_j^{h_j} \right) \right|.
\]

Every coefficient of this formal series is a function of the form \( \prod_{j=1}^{N} z_j^{\lambda_j} \cdot p(z_1, ..., z_N) \), where \( p \) is a Laurent polynomial.

Now let us define the (modified) expectation value (or 1-point correlation function) of an operator \( A \) as follows:

\[
\langle A \rangle_{\lambda,k}^\rho(z) = \frac{F_{\lambda,k}(A, z, q)}{F_{-\rho,k}(1d, z, q)}, \quad \rho = (\rho_1, ..., \rho_N), \quad \rho_j = \frac{N+1}{2} - j.
\]

Then \( \langle A \rangle_{\lambda,k}^\rho(z) \in \mathbb{C}[[z_1/\pm 1, ..., z_N/\pm 1]] \) for any \( A \).

Let \( \theta : V_\mu \to \mathbb{C} \) be a zero weight linear function. This function is unique up to a factor. Let the operator \( \Phi_0 : M_{\lambda,k} \to \hat{M}_{\lambda,k} \) be defined by the condition: \( \Phi_0 u = (1d \otimes \theta)(\Phi u), \ u \in M_{\lambda,k} \). The main object of our study will be the 1-point correlation function of this operator:

\[
\langle \Phi_0 \rangle_{\lambda,k}^\rho(z) = \langle \Phi \rangle_{\lambda,k}^\rho(z).
\]
Finally, let us define the algebra of regular expressions, $U(\mathfrak{g}_0)_{\text{reg}}$.

Let $X_\pm(j)$ be an eigenbasis for $\{h_i\}$ and $\partial$ in $\mathfrak{g}_0^\pm$, respectively, such that the eigenvalue of $\partial$ on $X_\pm(j)$ is a monotonic function of $j$. Let us call an expression of the form

$$Y = X_-(m)^{n_m} \ldots X_-(1)^{n_1} h_1^{p_1} \ldots h_N^{p_N} X_+(1)^{r_1} \ldots X_+(s)^{r_s} c^M \in U(\mathfrak{g}_0)$$

a standard monomial. Let us call the number

$$\text{ord}(Y) = \sum_{j=1}^m n_j + \sum_{j=1}^N p_j + \sum_{j=1}^s r_j + M$$

the order of the standard monomial $Y$, and the number

$$\text{sdeg}(Y) = \sum_{j=1}^s r_j \text{deg}(X_j^+)$$

the subsidiary degree of $Y$. According to the Poincaré-Birkhoff-Witt theorem, the standard monomials form a basis in $U(\mathfrak{g}_0)$.

**Definition.**

1) A sum $\sum_{m=1}^r b_m Y_m$, where $b_m \in \mathbb{C}^*$ and $Y_m$ are distinct standard monomials, is called a regular expression of order $\leq n$ if $\text{ord}(Y_m) \leq n$ for all $m$.

2) A series $\sum_{m=1}^\infty b_m Y_m$, where $b_m \in \mathbb{C}^*$ and $Y_m$ are distinct standard monomials, is called a regular expression of order $\leq n$ if
   
   (i) $\text{ord}(Y_m) \leq n$ for all $m \geq 1$;
   
   (ii) $\lim_{m \to \infty} \text{sdeg}(Y_m) = \infty$.

3) A regular expression is of order $n$ if it is of order $\leq n$ but not of order $\leq n-1$.

It is easy to show that regular expressions form an associative algebra. This algebra is a completion of $U(\mathfrak{g}_0)$ and is denoted by $U(\mathfrak{g}_0)_{\text{reg}}$.

The main property of the algebra $U(\mathfrak{g}_0)_{\text{reg}}$ is that the action of $U(\mathfrak{g}_0)$ in Verma modules over $\mathfrak{g}_0$ can be naturally extended to $U(\mathfrak{g}_0)_{\text{reg}}$: if $R = \sum_{m=1}^\infty b_m Y_m$ is a regular expression then for any $w \in M_{\lambda,k}$ $Rw = \sum_{m=1}^n b_m Y_m w$ if $n$ is large enough.

4. The mapping $\chi$

Let $\mathcal{DO}_N$ be the algebra of differential operators in $z_1, \ldots, z_N$ with Laurent polynomial coefficients (over $\mathbb{C}$). Let $DO_N = \mathcal{DO}_N[[q]]$. Let $A = \text{End}(V_\mu) \otimes DO_N$.

We will consider expressions of the form $< \Phi Y >_{\lambda,k}$, where $Y$ is a linear combination of standard monomials. Such expressions are (formal) functions of $z, q$ with values in $V_\mu$. For the sake of brevity, we will write $< A >$ instead of $< A >_{\lambda,k}$.

**Theorem 4.1.** Let $Y$ be a finite linear combination of standard monomials of order $\leq n$, and let $k$ be a complex number. Then:

1) There exists a differential operator $L_Y(k) \in A$ such that for all $\lambda$ for which $M_{\lambda,k}$ is irreducible

$$< \Phi Y >= L_Y(k) < \Phi >,$$

satisfying the following conditions:

2) The operator $L_Y(k)$ is of order $\leq n$, as a differential operator.
3) The operator \( L_Y(k) \) is a polynomial in \( k \) of degree \( \leq n \);

4) If \( Y \) is a standard monomial and \( \text{sd}(Y) = s \) then \( L_Y = q^s \tilde{L}_Y \), where \( \tilde{L}_Y \in A \).

**Proof.** The proof of the theorem is by induction in the order of \( Y \). More precisely, we will construct \( L_Y \) for \( Y \) of order \( n \) from \( L_J \) for standard monomials \( J \) of lower orders.

For order 0, \( Y = 1 \), and we can take \( L = 1 \). Assume that \( Y \) is a standard monomial of order \( n > 0 \). Consider the last factor \( X \in \mathfrak{g}_0 \) of this monomial, so that \( Y = Y'X \), where \( Y' \) is a standard monomial of order \( n - 1 \). Since \( \text{ord}(Y') = n - 1 \), we already know that there exists an operator \( L_{Y'} \), of order \( \leq n - 1 \) which is a polynomial in \( k \) of degree \( \leq n - 1 \), such that \( < \Phi Y' > = L_{Y'} < \Phi > \).

There are three possibilities for \( X \).

1. \( X = c \). In this case we have \( < \Phi Y > = k < \Phi Y' > \). Then we can set \( L_Y = kL_{Y'} \). This operator satisfies (4.1). It also satisfies statements 2,3,4 of the theorem because so does \( L_{Y'} \).

2. \( X = X_{\pm}[j] \). In this case let \( T = [Y'X] \). It is clear that \( T \) is a finite linear combination of standard monomials of order \( \leq n - 1 \), so we may assume that the operator \( L_T \) is already defined. Then, using the intertwining property of \( \Phi \) (\( \Phi X = (X \otimes 1 + 1 \otimes X)\Phi \)), we can write

\[
< \Phi Y > = < \Phi Y'X > = < \Phi XY' > + < \Phi T > = < X\Phi Y' > + \pi(X) < \Phi Y' > + < \Phi T > .
\]

On the other hand, we have

\[
< X\Phi Y' > = \frac{\text{Tr} |_{M_{\lambda,k}} (X\Phi Y'q^{-\partial} \prod_{j=1}^{N} z_j^{h_j})} {F_{-\rho,k}} = \frac{\text{Tr} |_{M_{\lambda,k}} (\Phi Y'q^{-\partial} \prod_{j=1}^{N} z_j^{h_j}X)} {F_{-\rho,k}} = q^{-\text{deg}(X)} \prod_{i=1}^{N} z_i^{a_i} < \Phi Y > .
\]

(4.3)

where \( a_i \) are integers defined by \([h_i, X] = a_i X \). Relations (4.2), (4.3) show that

\[
< \Phi Y > = q^{-\text{deg}(X)} \prod_{i=1}^{N} z_i^{a_i} < \Phi Y > + \pi(X)L_{Y'} + L_T < \Phi > ,
\]

(4.4)

which implies that

\[
< \Phi Y > = \frac{1}{1 - q^{-\text{deg}(X)} \prod_{i=1}^{N} z_i^{a_i}} (\pi(X)L_{Y'} + L_T) < \Phi > ,
\]

(4.5)

where \( \frac{1}{1 - q^{-m}x} \) denotes \( \sum_{p \geq 0} q^{mp}x^p \) if \( m > 0 \), and \( - \sum_{p \geq 1} q^{-mp}x^{-p} \) if \( m < 0 \). This means that the differential operator

\[
L_Y = \frac{1}{1 - q^{-\text{deg}(X)} \prod_{i=1}^{N} z_i^{a_i}} (\pi(X)L_{Y'} + L_T)
\]

(4.6)

satisfies (4.1). The fact that it satisfies statements 2,3,4 follows from the validity of these statements for \( L_{Y'} \) and \( L_T \).
3. $X = h_i$. Then we have

\[(4.7) \quad < \Phi Y > = \frac{z_i}{F_{-\rho,k}} \frac{\partial}{\partial z_i} (F_{-\rho,k} < \Phi Y' >),\]

which implies that the operator

\[(4.8) \quad L_Y = \frac{z_i}{F_{-\rho,k}} \frac{\partial}{\partial z_i} F_{-\rho,k} L_{Y'},\]

satisfies (4.1). The fact that it also satisfies statements 2, 3, 4, as before, follows from the validity of these statements for $L_{Y'}$.

The theorem is proved. ■

**Theorem 4.2.** Let $Y$ be a regular expression of order $\leq n$ and weight 0 (i.e. commuting with $h_i$ for all $i$), and let $k$ be a complex number ($k \notin \mathbb{Q}$). Then:

1) There exists a unique differential operator $D_Y(k) \in \text{DO}_N$ such that for all $\lambda$ for which $M_{\lambda,k}$ is irreducible

\[(4.9) \quad < \Phi_0 Y > = D_Y(k) < \Phi_0 >,

satisfying the following conditions:

2) The operator $D_Y(k)$ is of order $\leq n$ as a differential operator;

3) The operator $D_Y(k)$ is a polynomial in $k$ of degree $\leq n$.

4) If $Y$ is a standard monomial and $\text{sdeg}(Y) = s$ then $D_Y = q^s \bar{D}_Y$, where $\bar{D}_Y \in \text{DO}_N$.

**Proof.** First assume that $Y$ is a finite linear combination of standard monomials. Since $Y$ is of weight 0, the operator $L_Y$ preserves weight in $V_\mu$. Let $D_Y$ denote the restriction of $L_Y$ to the zero weight subspace in $V_\mu$. Obviously, $D_Y \in \text{DO}_N$, and it satisfies properties 2, 3, 4. In particular, because it satisfies property 4, the definition of $D_Y$ can be extended to infinite regular expressions: if $Y = \sum_{m=1}^{\infty} b_m Y_m$, where $Y_m$ are standard monomials, then we can set

\[(4.10) \quad D_Y = \sum_{m=1}^{\infty} b_m D_{Y_m},\]

and this series will be convergent as a formal series in $q$ since any fixed power of $q$ occurs in only a finite number of its terms.

It remains to prove that $D_Y$ satisfying (4.9) is unique for any $Y$. Suppose $D_Y^{(1)}$ and $D_Y^{(2)}$ both satisfy (4.9), and let $D = D_Y^{(1)} - D_Y^{(2)}$. Then $D < \Phi >_{\lambda,k} = 0$ for all $\lambda$. Assume that $D \neq 0$. Let $D_m q^m$, $D_m \in \text{DO}_N$, be the leading term in the $q$-expansion of $D$. Since $< \Phi >_{\lambda,k} | q = 0 = \prod_{j=1}^{N} z_j^{(j)} - k/N$, we have: $D_m (\prod_{j=1}^{N} z_j^{(j)}) = 0$ for generic complex $\lambda_1, ..., \lambda_N$. This implies that $D_m = 0$, which contradicts the assumption that $D_m$ is the leading coefficient. Therefore, $D = 0$, and hence $D_Y^{(1)} = D_Y^{(2)}$. ■

Now we are in a position to define the mapping $\chi$. From now on the notation $A^0$ will mean "the zero weight part of $A$ (where $A$ is a subquotient of $U(g)$)."
First we define the linear mapping $\tilde{\chi} : U(g_0)^0_{\text{reg}} \to DO_N$ which acts according to the formula:

\begin{equation}
\tilde{\chi}(Y) = D_Y (-N).
\end{equation}

It is easy to see that $D_{cY} = kD_Y$, which implies that $\tilde{\chi}$ kills the ideal $(c + N)U(g_0)^0_{\text{reg}}$, and therefore descends to a mapping

\begin{equation}
\chi : U(g_0)^0_{\text{reg}}/(c + N)U(g_0)^0_{\text{reg}} \to DO_N.
\end{equation}

This mapping will be the main tool in our proof of the integrability theorem.

For brevity we denote the algebra $U(g_0)^0_{\text{reg}}/(c + N)U(g_0)^0_{\text{reg}}$ by $U_{\text{crit}}$. This notation is suggested by the physical terminology: representations of $g_0$ with $c = -N$ in which $U_{\text{crit}}$ acts are called critical level representations.

The mapping $\chi$ has the following important property:

**Lemma 4.3.** Let $C \in U_{\text{crit}}$ be a central element: $[C, X] = 0$ for any $X \in U_{\text{crit}}$. Then

\begin{equation}
\chi(CY) = \chi(Y)\chi(C), \ Y \in U_{\text{crit}}^0.
\end{equation}

**Proof.** Let $\hat{Y}, \hat{C}$ be representatives of $Y, C$ in $U(g_0)^0_{\text{reg}}$. Then for any $X \in g_0$ $[\hat{C}, X] = (c + N)X'$, $X' \in U(g_0)^0_{\text{reg}}$. Therefore, it follows from the construction of the operators $D_Y$ (cf. proofs of Theorems 4.1, 4.2) that

\begin{equation}
D_{\hat{C}\hat{Y}} < \Phi > = D_{\hat{Y}} < \Phi \hat{C} > = (k + N)E < \Phi > = (k + N)E + (k + N)E < \Phi >, \ E \in DO_N,
\end{equation}

i.e. (by the uniqueness of $D_Y$)

\begin{equation}
D_{\hat{C}\hat{Y}}(k) = D_{\hat{Y}}D_{\hat{C}}(k) + (k + N)E(k).
\end{equation}

Therefore,

\begin{equation}
D_{\hat{C}\hat{Y}}(-N) = D_{\hat{Y}}D_{\hat{C}}(-N),
\end{equation}

which implies (4.13), Q.E.D. ■

Lemma 4.3 implies

**Theorem 4.4.** Let $Z$ be the center of $U_{\text{crit}}$. Then the linear mapping $\chi : Z \to DO_N$ is a homomorphism of algebras, and hence $\chi(Z)$ is a commutative subalgebra in $DO_N$. 

5. Sugawara operators and the integrability theorem

It is well known that the algebra $U_{crit}$ has a big center $Z$. This center is generated by the so-called Sugawara operators. Explicit construction of these operators is very technical, but for the proof of the integrability theorem we will need a very limited amount of information about them.

Let $\xi_1,...,\xi_{N-1}$ is any basis of the Cartan subalgebra of $\mathfrak{sl}_N$ orthonormal with respect to the inner product $<A,B> = \text{Tr}AB$. We assume that $\xi_i t^m$, $m \neq 0$, are among the basis vectors $X^\pm_i$. Also, if $a \in \mathfrak{sl}_N$, we denote $at^m$ as $a[m]$.

Sugawara operators for $g_0$ have orders $1, 2, ..., N$. We denote the Sugawara operator of order $j$ by $\Omega_j$. The explicit expressions for $\Omega_1$ and $\Omega_2$ are very simple:

$$\Omega_1 = \sum_{i=1}^{N} h_i,$$

$$\Omega_2 = \sum_{i=1}^{N} h_i^2 + \sum_{i \neq j} E_{ij} E_{ji} +$$

$$2 \sum_{m=1}^{\infty} \sum_{i=1}^{N-1} \xi_i[-m] \xi_i[m] + 2 \sum_{m=1}^{\infty} \sum_{i \neq j} E_{ij}[-m] E_{ji}[m].$$

About higher order Sugawara operators we need to know the following:

**Theorem 5.1.** ([*], Proposition 3.3; see also [*]) There exist elements $T_j \in Z$, $3 \leq j \leq N$, of order $j$, which can be written in the following form:

$$T_j = S_j + P_j(S_1,...,S_{j-1}) + E_j, \quad S_m = \sum_{i=1}^{N} h_i^m$$

where $P$ is a polynomial and $E_j$ contains only standard monomials of positive subsidiary degree.

**Corollary 5.2.** There exist degree zero elements $\Omega_j \in Z$ $3 \leq j \leq N$, of order $j$, which have the following form:

$$\Omega_j = S_j + E'_j,$$

where $E'_j$ contains only standard monomials of positive subsidiary degree.

**Proof.** Define the elements $T_j^{(1)}$ of $Z$ as follows:

$$T_j^{(1)} = T_j - P_j(T_1,...,T_{j-1}).$$

Then we have

$$T_j^{(1)} = S_j + P_j^{(1)}(S_1,...,S_{j-2}) + E_j^{(1)},$$

where $P_j^{(1)}$ is a polynomial, and all standard monomials in $E_j^{(1)}$ have positive subsidiary degree.
Next, define $T_j^{(2)} \in Z$ by:

\begin{equation}
T_j^{(2)} = T_j^{(1)} - P_j(T_1^{(1)}, \ldots, T_{j-1}^{(1)}).
\end{equation}

Then we have

\begin{equation}
T_j^{(2)} = S_j + P_j^{(1)}(S_1, \ldots, S_{j-3}) + E_j^{(2)},
\end{equation}

where $P_j^{(2)}$ is a polynomial, and all standard monomials in $E_j^{(2)}$ have positive subsidiary degree.

Continuing this procedure, we will get to the $j-1$-th step, and define $T_j^{(j-1)}$, which will have the decomposition $S_j + E_j^{(j-1)}$. Therefore, we can set $\Omega_j = T_j^{(j-1)}$, Q.E.D. ■

Let us now assume that $z_j, q$ are complex numbers such that $z_j \neq 0$, $0 < |q| < 1$, and introduce new variables $x_j \in \mathbb{C}$ and $\tau \in \mathbb{C}^+$ such that $e^{2\pi i x_j} = z_j$, $e^{2\pi i \tau / N} = q$.

For the proof of the integrability theorem we need one more result:

**Theorem 5.3.**

\begin{equation}
\chi(\Omega_2) = -\frac{\hat{H}}{4\pi^2} + \text{const},
\end{equation}

where $\hat{H}$ is defined by the formula

\begin{equation}
H = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - 2\mu(\mu + 1) \sum_{1 \leq i < j \leq N} \phi(x_i - x_j - \frac{j - i}{N}\tau|\tau).
\end{equation}

**Proof.** It is shown in [EK, Proposition 4.2] that $\psi_{\lambda,k}$ satisfies the following parabolic differential equation:

\begin{equation}
-4\pi^2(k + N)q \frac{\partial \psi_{\lambda,k}}{\partial q} = (\hat{H} + c)\psi_{\lambda,k},
\end{equation}

where $c$ is a constant. But we know that $(k + N)q \frac{\partial \psi_{\lambda,k}}{\partial q} = <\Phi_0(k + N)\phi = <\Phi_0\Omega_2 = D_{\Omega_2} \Phi_0 = D_{\Omega_2} \psi_{\lambda,k}$, which implies that $D_{\Omega_2} = (4\pi^2)^{-1}\hat{H} + \text{const}$, Q.E.D.

Now we are ready to prove the complete integrability theorem for the elliptic case.

**Theorem 5.4. ([OP])** The Hamiltonian $H$ given by (1) with $U = K\phi$ is completely integrable.

**Proof.** Let $\mu$ be such that $K = -2\mu(mu + 1)$. Consider the differential operators $D_{\Omega_1}, \ldots, D_{\Omega_N}$. They are pairwise commutative, and, thanks to Corollary 5.2,

\begin{equation}
D_{\Omega_j} = (2\pi \subset)^{-j} \sum_{i=1}^{N} \frac{\partial^j}{\partial x_j} + \text{lower order terms}
\end{equation}

Let us make a change of variable $\tilde{x}_j = x_j + \frac{j}{N}\tau$, and let $L_j$ be the image of $(2\pi \subset)^{j}D_{\Omega_j}$ under this change of variable. Then $[L_j, L_m] = 0$ for all $j, m$, $L_2 = H + \text{const}$ because of Theorem 3.3, and $L_j = \sum_i \left( \frac{\partial}{\partial \tilde{x}_i} \right)^j + \text{lower order terms}$, which implies that $\{L_i\}$ are algebraically independent. The theorem is proved.
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