SUP NORMS OF MAASS FORMS ON SEMISIMPLE GROUPS

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Abstract. We prove a power saving over the local bound for the $L^\infty$ norm of a Hecke-Maass form on a class of groups that includes split classical groups and complex groups.

1. Introduction

Let $M$ be a compact Riemannian manifold of dimension $n$, and $\psi$ a function on $M$ satisfying $\Delta \psi = \lambda^2 \psi$ and $\|\psi\|_2 = 1$. A classical theorem of Avacumović [1] and Levitan [16] states that

(1) \[ \|\psi\|_\infty \ll \lambda^{(n-1)/2}, \]

that is, the pointwise norm of $\psi$ is bounded in terms of its Laplace eigenvalue. This bound is sharp on the round sphere $S^n$, or on a surface of revolution that is diffeomorphic to $S^2$, but is far from the truth on flat tori. It is an interesting problem in semiclassical analysis to find conditions on $M$ under which (1) can be strengthened, and such conditions often take the form of a non-recurrence assumption for the geodesic flow on $M$. One result of this kind is due to Bérard [2], who proves that if $M$ has negative sectional curvature (or has no conjugate points if $n = 2$) then we have

(2) \[ \|\psi\|_\infty \ll \frac{\lambda^{(n-1)/2}}{\sqrt{\log \lambda}}. \]

See also [21, 24] for other theorems bounding $\|\psi\|_\infty$ under assumptions on the geodesic flow of $M$. The problem of strengthening (1) for negatively curved $M$ is an interesting one, because for generic $M$ we expect that $\|\psi\|_\infty \ll_\epsilon \lambda^\epsilon$, whereas the strongest upper bound that is known in general is (2).

In [15], Iwaniec and Sarnak introduced a different condition on $M$ and $\psi$ which allows them to deduce quite a strong bound for $\|\psi\|_\infty$. They assume that $M$ is a congruence hyperbolic manifold, such as the quotient of $\mathbb{H}^2$ by the group of units in an order in a quaternion division algebra over $\mathbb{Q}$, and that $\psi$ is an eigenfunction of the Hecke operators on $M$. They then prove that $\|\psi\|_\infty \ll_\epsilon \lambda^{5/12+\epsilon}$. Moreover, one expects that the assumption on $\psi$ is not necessary because the spectral multiplicities of negatively curved manifolds are always observed to be bounded (and what multiplicities there are always arise from a group of isometries of $M$). This bound is the strongest that is known for the supremum norm of an eigenfunction on a negatively curved surface, with the next strongest being (2).

We are interested in extending the methods of Iwaniec and Sarnak to higher dimensional manifolds, which requires considering eigenfunctions on general locally symmetric spaces. We

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shall only consider spaces of noncompact type, although the method of proof would apply
equally well to spaces of compact type. We make this restriction partly for convenience, and
partly because the multiplicities of the Laplace spectrum on such manifolds are expected
to be bounded as in the hyperbolic case. Although these manifolds have zero sectional
curvature in certain directions, their eigenfunctions are expected to exhibit essentially the
same chaotic behaviour that is observed on negatively curved manifolds.

We recall that locally symmetric spaces of noncompact type are constructed by taking a
semisimple real Lie group $G$, a maximal compact subgroup $K \subset G$, and a lattice $\Gamma \subset G$, and
defining $Y = \Gamma \backslash G / K$. We do not assume that $Y$ is compact. We let $n$ and $r$ be the dimension
and rank of $Y$. We consider functions $\psi \in L^2(Y)$ that are eigenfunctions of the full ring of
invariant differential operators, which is isomorphic to a finitely generated polynomial ring
in $r$ variables. This ring contains $\Delta$, and we continue to denote $\Delta \psi = \lambda^2 \psi$.

If $\Omega \subseteq Y$ is compact, Sarnak proves in [20] that $\psi$ satisfies

\[ \| \psi \|_\Omega \ll \lambda^{(n-r)/2}. \]  

The analogous problem to the one solved by Iwaniec and Sarnak for $\mathbb{H}^2$ is to improve
the exponent in this bound, under the assumptions that $\Gamma$ is congruence arithmetic, and that $\psi$
is an eigenfunction of the ring of Hecke operators. (Note that when $r \geq 2$, $\Gamma$ is automatically
arithmetic by a theorem of Margulis.) This is often referred to as the problem of giving a
subconvex, or sub-local, bound for the sup norm of a Maass form in the eigenvalue aspect.
Besides the original work of Sarnak and Iwaniec, the pairs $\Gamma \subset G$ for which it has previously
been solved are $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$ by Blomer, Harcos, and Milićević [3], $Sp_4(\mathbb{Z}) \subset Sp_4(\mathbb{R})$
by Bomer and Pohl [9], $SL_3(\mathbb{Z}) \subset PGL_3(\mathbb{R})$ by Holowinsky, Ricotta, and Royer [10], and
$SL_n(\mathbb{Z}) \subset PGL_n(\mathbb{R})$ for any $n$ by Blomer and Mága [5, 6]. There are also results bounding
eigenfunctions on the round spheres $S^2$ and $S^3$ equipped with Hecke algebras [7, 8].

We note that much work has been done on variants of the sup-norm problem. One may
consider Maass forms of varying level and eigenvalue as in [4, 13, 14, 22, 23]. There are also
results bounding the $L^2$ norm of the restriction of $\psi$ to a submanifold of positive dimension
[18, 19].

In this paper, we shall solve the sup-norm problem for a large class of Lie groups. Let $G$
be a semisimple algebraic group of adjoint type over $\mathbb{Q}$. We introduce the following two
conditions on $G$.

**Complex:** There exists a CM extension $K/K^+$, and a semisimple group $\underline{G}^0 / K^+$ that is
compact at all real places, such that $\underline{G} = \text{Res}_{K/\mathbb{Q}} \underline{G}^0$.

**$K$-small:** Let $\underline{K} \subset \underline{G}$ be a connected subgroup defined over $\mathbb{R}$ such that $K(\mathbb{R})$ is a
maximal connected compact subgroup of $\underline{G}(\mathbb{R})$. If we let $T_K \subseteq T$ be maximal tori in $\underline{K}$ and
$\underline{G}$ and define the functions

\[ \| \mu \|_*^* = \max_{w \in W} \langle w \mu, \rho \rangle \]
\[ \| \mu \|_K^* = \max_{w \in W_K} \langle w \mu, \rho_K \rangle \]
on $X_*(T)$ and $X_*(T_K)$ respectively, then we have
∥μ∥∗ − 2∥μ∥_K > 0
for all nonzero μ ∈ X_*(T_K). Here, W and W_K are the respective Weyl groups, and ρ and ρ_K the half-sums of some system of positive roots.

In the complex case, if we define K = Res_{K'/Q}G_0, then G(R) ≃ K(C), and K(R) is a maximal compact subgroup of G(R).

Note that the condition of being K-small is independent of the choice of K. We also have

**Theorem 1.1.** Suppose G(R) is isogenous to a product of split classical groups or the split form of G_2. Then G satisfies K-small.

In all cases we have checked, K-small is in fact equivalent to G(R) being split.

Let Γ ⊂ G(Q) be an arithmetic congruence lattice, which we shall consider as embedded in G(R). Let K/R be a maximal connected compact subgroup of G(R), and define Y = Γ\G(R)/K(R). Let ψ ∈ L^2(Y) be a joint eigenfunction of the ring of invariant differential operators and the Hecke operators on Y. We shall assume that ∥ψ∥_2 = 1. If a is a split real Cartan subalgebra of G(R), we now let λ ∈ a^* ×_R C denote the spectral parameter of ψ. There is a constant C_1(G) such that ∆ψ = (C_1(G) + ∥λ∥^2)ψ. We assume that λ ∈ a^*, so that ψ satisfies the Ramanujan conjectures at infinity. Let Ω_Y ⊆ Y be compact. Our main theorem is that (3) may be strengthened by a power when G satisfies either of the conditions we have introduced.

**Theorem 1.2.** Suppose G satisfies either complex or K-small. Then there exists δ = δ(G) and C = C(G, Ω_Y) such that

(7) ∥ψ|_{Ω_Y}∥_∞ ≤ C∥λ∥^{(n-r)/2−δ}

In particular, the bound (7) holds when G = Res_{F/Q}PGL_n/F for any totally real or CM field F. To see this in the CM case, let F/F+ be a CM extension and Φ be a Hermitian form with respect to F/F+ that is definite at all real places. We then have Res_{F/Q}PGL_n/F ≃ Res_{F/Q}PU(F/F+, Φ), so that the result falls under the case complex. Note that the bound we shall actually prove is given by (10) below, which is stronger than (7) when λ is near the singular set, but is equivalent to (7) for regular λ.

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2. **Proof in the K-small case**

2.1. **Notation.** Let G be a semisimple algebraic group of adjoint type over Q, with Lie algebra g/Q. In this section, all implied constants will be assumed to depend on G. We begin by proving that G has a maximal connected compact subgroup defined over a number field.

**Lemma 2.1.** There is a number field L with at least one real embedding u, and a connected subgroup K ⊂ G defined over L, such that if K_u = K ×_u R then K_u(R) is a maximal connected compact subgroup of G(R).
Proof. Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{g} \times _\mathbb{Q} \mathbb{R}$ be the real Lie algebra of a maximal connected compact subgroup of $G(\mathbb{R})$. Choose a basis $X_1, \ldots, X_r$ of $\mathfrak{h}_\mathbb{R}$, and $Y_1, \ldots, Y_s$ of $\mathfrak{g}$, and define $a_{ij} \in \mathbb{R}$ and $b_{ijk} \in \mathbb{R}$ by

$$X_i = \sum_{j=1}^s a_{ij} Y_j,$$

$$[X_i, X_j] = \sum_{k=1}^r b_{ijk} X_k.$$  

Let $R$ be the $\mathbb{Q}$-subalgebra of $\mathbb{R}$ generated by $a_{ij}$ and $b_{ijk}$. It comes with an embedding $\iota : R \to \mathbb{R}$. We let $\mathfrak{h}$ be the Lie algebra over $R$ defined by $\mathfrak{g} \times \mathbb{R}$, so that $\mathfrak{h}_\mathbb{R} \simeq \mathfrak{h} \times _\iota \mathbb{R}$. Let $V / \mathbb{Q}$ be the variety associated to $R$. $V$ comes together with a point $p_i \in V(\mathbb{R})$ corresponding to $\iota$, and by repeatedly replacing $V$ with its singular set we may assume that $p_i$ is a regular point of $V$. This implies that $V(\mathbb{R})$ is a real manifold of the same dimension as $V$ near $p_i$, and it follows that there are points of $V(\mathbb{R} \cap \mathbb{Q})$ arbitrarily close to $p_i$. Let $p \in V(\mathbb{R} \cap \mathbb{Q})$ be close to $p_i$, and let $L$ be its field of definition. The point $p$ gives a map $\mathfrak{h} \to \mathfrak{g} \times _\iota L$, and we let $\mathfrak{h}_p$ be its image. If $p$ is chosen sufficiently close to $p_i$ then $\mathfrak{h}_p \times _L \mathbb{R}$ will be isomorphic to $\mathfrak{h}_\mathbb{R}$, and the restriction of the Killing form to $\mathfrak{h}_p \times _L \mathbb{R}$ will be negative definite. It follows that $\mathfrak{h}_p \times _L \mathbb{R}$ is the Lie algebra of a maximal connected compact subgroup, so if we define $K$ to be the subgroup of $G$ with Lie algebra $\mathfrak{h}_p$ then $K$ has the desired properties.

Let $L$ and $K$ be as in Lemma 2.1. We choose a number field $F / L$ with ring of integers $\mathcal{O}$ and an integer $D$ with the following properties.

1. $G$ and $K$ are defined over $\mathcal{O}[1/D]$.
2. $G$ and $K$ contain maximal tori $T$ and $T_K$, and Borel subgroups $T \subseteq B$ and $T_K \subseteq B_K$, that are defined over $\mathcal{O}[1/D]$ and satisfy $T_K \subseteq T, B_K \subseteq B$.
3. $G(\mathbb{Z}_p)$ and $K(\mathcal{O}_{K,v})$ are hyperspecial maximal compact subgroups of $G(\mathbb{Q}_p)$ and $K(K_v)$ for $p, v \nmid D$.

Let $\Delta$ be the set of roots of $T$ in $G$, and let $\Delta^+$ be the set of positive roots corresponding to $B$. We define

$$X_*(T)^+ = \{ \mu \in X_*(T) | \langle \mu, \alpha \rangle \geq 0, \alpha \in \Delta^+ \}.$$ We define the functions $\| \cdot \|_*$ and $\| \cdot \|_K^*$ on $X^*(T)$ and $X^*(T_K)$ as in (1) and (5). By applying the definition of $K$-smallness to the group $K_n$ obtained by embedding $L$ in $\mathbb{R}$, we see that if we define

$$\kappa = \min_{0 \neq \mu \in X_*(T_K)} \| \mu \|_* - 2 \| \mu \|_K^*,$$

then $\kappa > 0$.

2.1.1. Local fields. If $v$ is a place of $F$, we let $F_v$ be the completion of $F$, and $\mathcal{O}_v$ be its ring of integers. If $v \nmid D$, and $\pi \in \mathcal{O}_v$ is a uniformiser, we have an isomorphism $X_*(T) \simeq T(F_v)/T(\mathcal{O}_v)$ via the map $\mu \mapsto \mu(\pi)$, and this is independent of the choice of $\pi$. We let
be the function supported on $G$. Conditions (2) and (3) imply that $G(F_v)$ has a Cartan decomposition

$$G(F_v) = \prod_{y \in Y^+} G(O_v)yG(O_v)$$

with respect to $G(O_v)$ and $T(F_v)$ when $v \nmid D$.

We let $\mathcal{P}$ be the set of rational primes that do not divide $D$, and that have a split prime of $F$ lying over them. For each $p \in \mathcal{P}$, we fix a split place $v$ of $F$ above $p$. For $\mu \in X_*(T)$ and $p \in \mathcal{P}$, we will often think of $\mu(p) \in G(F_v)$ as lying in $G(Q_p)$ under the natural isomorphism $G(F_v) \simeq G(Q_p)$.

2.1.2. Metrics. We fix a $Q$-embedding $\rho : G \to SL_d$. For $\gamma \in G(Q)$, let $\|\gamma\|_f$ be the LCM of the denominators of the matrix entries of $\rho(\gamma)$. Fix a left-invariant Riemannian metric on $G(R)$. Let $d(\cdot, \cdot)$ be the associated distance function. We define $d(x, y) = \infty$ when $x$ and $y$ are in different connected components of $G(R)$ with the topology of a real manifold.

2.1.3. Adelic groups. Let $\mathbb{A}$ be the adeles of $Q$. We choose a compact subgroup $K = \bigotimes_{v \leq \infty} K_v$ of $G(\mathbb{A})$ satisfying $K_\infty = K_\infty(R)$ and $K_p = G(Z_p)$ for $p \nmid D$. For each prime $p$, let $dg_p$ be the Haar measure on $G(Q_p)$ that assigns $K_p$ measure 1. Choose a Haar measure $dg_\infty$ on $G_\infty$, and let $dg = \otimes_{v \leq \infty} dg_v$. All convolutions on $G$ will be defined with respect to these measures. We define $Y = G(Q) \backslash G(\mathbb{A}) / K$. Choose compact sets $\Omega_Y \subseteq Y$ and $\Omega = \bigotimes_{v \leq \infty} \Omega_v \subseteq G(\mathbb{A})$ such that the projection of $\Omega$ to $Y$ contains $\Omega_Y$. We assume that $\Omega_p = K_p$ for $p \nmid D$.

2.1.4. Hecke algebras. Let $\mathcal{H}$ be the algebra of functions on $G(\mathbb{A}_{f})$ that are compactly supported and bi-invariant under $K_f$. For each $p \nmid D$, we let $\mathcal{H}_p$ be the algebra of functions on $G(Z_p)$ that are compactly supported and bi-invariant under $K_p$, which we will often identify with a subalgebra of $\mathcal{H}$ in the natural way. For $p \in \mathcal{P}$ and $\mu \in X_*(T)$, we define $T(p, \mu)$ to be the function supported on $G(Z_p)\mu(p)\mathcal{G}(Z_p)$ and taking the value $p^{-\|\mu\|^*}$ there.

2.1.5. Real Lie algebras. Let $\mathfrak{g} \times_Q \mathbb{R} = \mathfrak{t} + \mathfrak{p}$ be the Cartan decomposition associated to $K$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a Cartan subalgebra. We let $W_\mathfrak{a}$ be the Weyl group $N_G(R)(\mathfrak{a}) / Z_{G(R)}(\mathfrak{a})$. We let $\Delta_\mathfrak{a}$ be the roots of $\mathfrak{a}$ in $\mathfrak{g}$, and let $\Delta_\mathfrak{a}^+$ be a choice of positive roots. For $\alpha \in \Delta_\mathfrak{a}$, we let $m(\alpha)$ denote the dimension of the corresponding root space. We denote the Killing form on $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle \cdot, \cdot \rangle$. For $\lambda \in \mathfrak{a}^*$, we define

$$D(\lambda) = \prod_{\alpha \in \Delta^+} (1 + |\langle \alpha, \lambda \rangle|)^{m(\alpha)}.$$ 

2.1.6. Maass forms. Let $\psi \in L^2(Y)$ be an eigenfunction of the ring of invariant differential operators on $Y$ and the Hecke algebras $\mathcal{H}_p$. We assume that $\|\psi\|_2 = 1$, and that the spectral parameter $\lambda$ of $\psi$ lies in $\mathfrak{a}^*$. One may easily prove the bound

$$\|\psi|_{\Omega_Y}\|_\infty \ll_{\Omega_Y} D(\lambda)^{1/2}$$

which is equivalent to (3) when $\lambda$ lies in a regular cone in $\mathfrak{a}^*$, but is stronger for singular $\lambda$. We shall in fact prove the bound

$$\|\psi|_{\Omega_Y}\|_\infty \leq CD(\lambda)^{1/2-\delta},$$

(10)
subject to the same conditions and notation as Theorem [1.2]

2.2. Amplification. The amplifier we shall apply to $\psi$ has an infinite and a finite component. The finite component is given by the following proposition.

Proposition 2.2. There exists a constant $R = R(G) > 0$ such that for all $p \in \mathcal{P}$, there exists $T_p \in \mathcal{H}_p$ depending on $\psi$ such that

\begin{equation}
T_p(\psi) = 1.
\end{equation}

Moreover, we have the expansions

\begin{align}
T_p &= \sum_{\mu \in X^*(T)^+} a(p, \mu)T(p, \mu), \\
T_p T_p^* &= \sum_{\mu \in X^*(T)^+} b(p, \mu)T(p, \mu),
\end{align}

where $a(p, \mu)$ and $b(p, \mu)$ satisfy

\begin{align}
a(p, \mu) &\ll 1 \text{ and } a(p, 0) = 0, \\
b(p, \mu) &\ll 1.
\end{align}

Proof. We apply Proposition [4.6] to the group $G$ to obtain $C > 0$, $\delta > 0$, and a finite set $0 \notin \mathcal{X} \subset X^*(T)$ such that if $p \in \mathcal{P}$ satisfies $p > C$, then there exists $\mu = \mu(\psi, p) \in \mathcal{X}$ such that $T(p, \mu)\psi = C(\psi, p)\psi$ and $|C(\psi, p)| > \delta$. We shrink $\mathcal{P}$ if necessary so that we have $p > C$ for all $p \in \mathcal{P}$. For $p \in \mathcal{P}$, we define $T_p = T(p, \mu)/C(\psi, p)$, so that (11) is satisfied. If we let $R$ be such that $\|\mu\| < R/2$ for all $\mu \in \mathcal{X}$, then (12) and (14) will be satisfied. Note that the condition $a(p, 0) = 0$ follows from the fact that $0 \notin \mathcal{X}$. Formulas (13) and (15) follow from the formulas (34) and (35) for the Satake transform.

Let $N$ be a positive integer and $Q \subseteq \mathcal{P}$ a subset, both to be chosen later, and define

$\mathcal{T} = \sum_{p \leq N, p \in Q} T_p$.

This is the finite part of our amplifier.

To define the infinite part, choose a function $h \in C_0^\infty(a^*)$ of Paley-Wiener type that is real, nonnegative, and satisfies $h(0) = 1$. Let

\begin{equation}
h_\lambda(\nu) = \sum_{w \in W} h(w\nu - \lambda),
\end{equation}

and let $k^0_\lambda$ be the $K_\infty$-invariant function on $G(\mathbb{R})$ with Harish-Chandra transform $h_\lambda$. It is of compact support independent of $\lambda$ by the Paley-Wiener theorem of [11]. We define $k_\lambda = k^0_\lambda * \tilde{k}^0_\lambda$. If $\varphi_\mu$ are the elementary spherical functions on $G(\mathbb{R})$, Theorem 2 of [9] gives
Lemma 2.3. \( \varphi(x) \ll C (1 + \|\mu\|d(x, K_\infty))^{-1/2} \)

for \( \mu \in \mathfrak{a}^* \) and \( x \) in a compact set \( C \subset \mathcal{G}(\mathbb{R}) \). By inverting the Harish-Chandra transform as in Proposition 2.6 of [17], it follows that

\[
(17) \quad k_\lambda(x) \ll D(\lambda)(1 + \|\lambda\|d(x, K_\infty))^{-1/2}.
\]

We let \( B \subset \mathcal{G}(\mathbb{R}) \) be a compact set that contains the support of \( k_\lambda \) for all \( \lambda \), and let \( 1_B \) be the characteristic function of \( B \). For \( 1 > \delta > 0 \) to be chosen later, let \( 1_\delta \) be the characteristic function of the set \( \{g \in \mathcal{G}(\mathbb{R})|d(g, K_\infty) < \delta\} \). We shall use the following amplification inequality, which is often referred to as a pre-trace formula.

**Lemma 2.3.**

\[
(18) \quad |(k_\lambda^0 \mathcal{T} \ast \psi(x)|^2 \leq \sum_{\gamma \in \mathcal{G}(\mathcal{Q})} (\mathcal{T} \ast \mathcal{T}^*)(x^{-1} \gamma x)k_\lambda(x^{-1} \gamma x).
\]

**Proof.** By definition of the Hecke operators, we have

\[
(k_\lambda^0 \mathcal{T} \ast \psi(x) = \int_{\mathcal{G}(\mathcal{A})} \psi(g)(k_\lambda^0 \mathcal{T})(g^{-1}x)dg.
\]

Folding the sum over \( \mathcal{G}(\mathcal{Q}) \) gives

\[
(k_\lambda^0 \mathcal{T} \ast \psi(x) = \int_{\mathcal{G}(\mathcal{Q}) \setminus \mathcal{G}(\mathcal{A})} \psi(g) \sum_{\gamma \in \mathcal{G}(\mathcal{Q})} (k_\lambda^0 \mathcal{T})(g^{-1} \gamma^{-1}x)dg.
\]

If we apply Cauchy-Schwartz and expanding the square, we obtain

\[
|\sum_{\gamma \in \mathcal{G}(\mathcal{Q})} (k_\lambda^0 \mathcal{T})(g^{-1} \gamma^{-1}x)|^2 \leq \int_{\mathcal{G}(\mathcal{Q}) \setminus \mathcal{G}(\mathcal{A})} \sum_{\gamma \in \mathcal{G}(\mathcal{Q})} (k_\lambda^0 \mathcal{T})(g^{-1} \gamma^{-1}x)dg.
\]

Unfolding again gives

\[
(19) \quad |(k_\lambda^0 \mathcal{T} \ast \psi(x)|^2 \leq \sum_{\gamma \in \mathcal{G}(\mathcal{Q})} \int_{\mathcal{G}(\mathcal{A})} (k_\lambda^0 \mathcal{T})(g^{-1} \gamma x)(k_\lambda^0 \mathcal{T})(g^{-1}x)dg.
\]

Because \( (k_\lambda^0 \mathcal{T})(g^{-1}x) = (k_\lambda^0 \mathcal{T}^*)(x^{-1}g) \), we have

\[
\int_{\mathcal{G}(\mathcal{A})} (k_\lambda^0 \mathcal{T})(g^{-1} \gamma x)(k_\lambda^0 \mathcal{T})(g^{-1}x)dg = \int_{\mathcal{G}(\mathcal{A})} (k_\lambda^0 \mathcal{T}^*)(x^{-1}g)(k_\lambda^0 \mathcal{T})(g^{-1} \gamma x)dg
\]

\[
= (k_\lambda^0 \mathcal{T}^*) (x^{-1} \gamma x)(k_\lambda^0 \mathcal{T})(g^{-1} \gamma x)
\]

Inserting this into (19) completes the proof. \( \Box \)
2.3. Estimating Hecke returns. We apply the inequality (18) for $x \in \Omega$, and break up the sum depending on whether $x^{-1}\gamma x$ is within $\delta$ of $K_{\infty}$ to obtain

$$|(k^0_\lambda T) \ast \psi(x)|^2 \leq \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (T \ast T^\ast)(x^{-1}\gamma x)k_\lambda(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x)$$

$$+ \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (T \ast T^\ast)(x^{-1}\gamma x)k_\lambda(x^{-1}\gamma x)(1_B - 1_\delta)(x^{-1}\gamma x).$$

Because $x \in \Omega$, we have

$$\# \{ \gamma \in \mathcal{G}(\mathbb{Q}) | (x^{-1}\gamma x)_{\infty} \in B, (x^{-1}\gamma x)_f \in \text{supp}(T \ast T^\ast) \} \ll_\Omega N^{A_1}$$

for some $A_1 = A_1(G)$. When combined with equation (17) this gives

$$\sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (T \ast T^\ast)(x^{-1}\gamma x)k_\lambda(x^{-1}\gamma x)(1_B - 1_\delta)(x^{-1}\gamma x) \ll_\Omega D(\lambda)(1 + \|\lambda\|\delta)^{-1/2}N^{A_1}.$$  

By Proposition 2.2 part (12), if $x^{-1}\gamma x$ lies in the support of $T \ast T^\ast$ then $\|\gamma\|_f \leq C_1N^{A_2}$ for some $C_1 = C_1(G, \Omega)$ and $A_2 = A_2(G)$. Apply Proposition 4.1 with the data $(\mathcal{G}, \mathcal{H}, B, B_H)$ chosen to be $(\mathcal{G}, K, \Omega_{\infty}, K_\infty)$, and the auxiliary data $\rho$ and $d(\cdot, \cdot)$ the ones chosen above. This gives constants $C = C(G, \Omega) > 0$ and $M = M(G) > 0$ such that if $\delta \leq C(C_1N^{A_2})^{-M}$ then there exists a Galois extension $E/\mathbb{Q}$ with $|E : \mathbb{Q}| \ll 1$, $Q \in \mathbb{Z}$ with $\ll_\Omega \log N$ prime factors, and $y \in \mathcal{G}(\mathcal{O}_E[1/DQ])$ depending on $x$ such that if we define

$$L = \bigcap_{\sigma \in \text{Gal}(E/\mathbb{Q})} (yKy^{-1})^\sigma$$

then

$$\sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (T \ast T^\ast)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x) = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (T \ast T^\ast)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x).$$

We shall assume for the remainder of the proof that $\delta \leq C(C_1N^{A_2})^{-M}$. We now choose the subset $\mathcal{Q} \subseteq \mathcal{P}$ to consist of those primes not dividing $Q$. Bounding (21) requires estimating the sizes of the sets

$$\mathcal{M}(p, \mu) = \{ \gamma \in L(\mathbb{Q}) | T(p, \mu)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x) \neq 0 \}$$

$$\mathcal{M}(p, \mu, q, \nu) = \{ \gamma \in L(\mathbb{Q}) | T(p, \mu) \ast T(q, \nu)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x) \neq 0 \}$$

for $\mu, \nu \in X_*(T)$ and $p, q \in \mathcal{Q}$. We also define

$$\mathcal{L}(p, \mu) = \#(L(\mathbb{Q}_p)K_p \cap K_p\mu(p)K_p)/K_p$$

for $p \in \mathcal{P}$ and $\mu \in X_*(T)$.

**Lemma 2.4.** We have the bounds
\[ |\mathcal{M}(p, \mu)| \ll \Omega \mathcal{L}(p, \mu), \]
\[ |\mathcal{M}(p, \mu, q, \nu)| \ll \Omega \mathcal{L}(p, \mu)\mathcal{L}(q, \nu). \]

**Proof.** We shall only prove the first inequality, as the proof of the second is identical. Because \( x \in \Omega \), our assumptions on \( \Omega \) and \( \mathcal{P} \) imply that \( x_p \in K_p \). It follows that if \( \gamma \in \mathcal{M}(p, \mu) \), then \( \gamma_p \in L(\mathbb{Q}_p) \cap K_p \mu(p)K_p \). This gives a map
\[ \mathcal{M}(p, \mu) \to (L(\mathbb{Q}_p)K_p \cap K_p \mu(p)K_p)/K_p. \]

We shall show that the fibers of this map have bounded size. Suppose \( \gamma_1 \) and \( \gamma_2 \in \mathcal{M}(p, \mu) \) lie in the same coset \( g_pK_p \). Then \( \gamma_1^{-1}\gamma_2 \) must lie in a compact set \( C \subset \mathcal{G}(\mathbb{A}) \) depending only on \( \Omega \), and the result now follows from the fact that \( \mathcal{G}(\mathbb{Q}) \cap C \) is finite.

\[ \square \]

Proposition 5.1 now implies

**Proposition 2.5.** Let \( p, q \in \mathbb{Q} \). \( \mathcal{M}(p, \mu, q, \nu) \) is empty if \( W_\mu \) or \( W_\nu \) do not intersect \( X^*_s(\mathbb{T}_K) \), and likewise for \( \mathcal{M}(p, \mu) \). If \( \mu, \nu \in X^*_s(\mathbb{T}_K) \) we have
\[ |\mathcal{M}(p, \mu)| \ll p^{2\|\mu\|_K}, \]
\[ |\mathcal{M}(p, \mu, q, \nu)| \ll p^{2\|\mu\|_K}q^{2\|\nu\|_K}. \]

If \( \kappa \) is as in (9), Proposition 2.5 implies that
\[ \sum_{\gamma \in L(\mathbb{Q})} T(p, \mu)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x) \ll p^{-\kappa} \]
for all \( p \in \mathbb{Q} \) and nonzero \( \mu \in X^*_s(\mathbb{T}) \). We also have
\[ \sum_{\gamma \in L(\mathbb{Q})} T(p, 0)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x) \ll 1 \]

We expand \( \mathcal{T} \ast \mathcal{T}^* \) in the sum (21) above. By applying (13), we see that the terms of the form \( \mathcal{T}_p \ast \mathcal{T}_p^* \) contribute
\[ \sum_{p \in \mathbb{Q}} \sum_{\mu \in X^*_s(\mathbb{T})} \sum_{\gamma \in L(\mathbb{Q})} b(p, \mu)T(p, \mu)(x^{-1}\gamma x)1_\delta(x^{-1}\gamma x), \]
and by combining (15), (22), and (23) we see this is
\[ \ll \sum_{p \in \mathbb{Q}} 1 \leq N. \]

The terms \( \mathcal{T}_p \ast \mathcal{T}_q^* \) contribute

\[ 9 \]
\[
\sum_{\substack{p,q \in \mathbb{Q} \mid \mu,\nu \in \mathbb{X}^r(T) \\gamma \in L(\mathbb{Q})}} \sum_{\nu \leq N, \|\nu\| \leq \kappa} a(p,\mu)a(q,\nu) T(p,\mu) * T(q,\gamma) (x^{-1} \gamma x) \eta(x^{-1} \gamma x).
\]

Equation (14) implies that only nonzero \(\mu\) and \(\nu\) contribute to the sum, so by Proposition 2.5 this is

\[
(25) \quad \ll \sum_{\substack{p,q \in \mathbb{Q} \mid \kappa}} (pq)^{-\kappa} \ll N^{2-2\kappa}.
\]

Combining (24) and (25) gives

\[
\sum_{\gamma \in L(\mathbb{Q})} (T * T^\gamma) (x^{-1} \gamma x) \eta(x^{-1} \gamma x) \ll \max\{N, N^{2-2\kappa}\}.
\]

Combining this with the bound \(k_\lambda \ll D(\lambda)\) from (17), and equation (20), gives

\[
|\langle k_\lambda^x \rangle \psi(x) \rangle|^2 \ll D(\lambda) \max\{N, N^{2-2\kappa}\} + D(\lambda)(1 + \|\lambda\|\delta)^{-1/2} N^{\lambda_1}.
\]

Applying \(T_p(\psi) = 1\) and the bounds \(|\mathbb{Q}| \gg_{\Omega, \epsilon} N^{1-\epsilon}\) and \(h_\lambda(\lambda) \geq 1\) gives

\[
(26) \quad |\psi(x)|^2 \ll_{\Omega, \epsilon} D(\lambda) \max\{N^{1+\epsilon}, N^{2-2\kappa+\epsilon}\} + D(\lambda)(1 + \|\lambda\|\delta)^{-1/2} N^{\lambda_1}.
\]

It may be seen that there are constants \(\sigma = \sigma(G) > 0\) and \(A = A(G) > 0\) such that we may choose \(\delta = \|\lambda\|^{-1+\epsilon} \text{ and } N \sim_{\Omega} \|\lambda\|^{\sigma/4} A\) while satisfying the inequality \(\delta \leq C(C_1 N^{A_2})^{-M}\). With these choices, the inequality (26) becomes \(\psi(x) \ll_{\Omega} D(\lambda)^{1/2-\epsilon}\) for some \(\epsilon = \epsilon(G)\), which completes the proof.

3. Proof in the complex case

3.1. Notation. Let \(L / L^+\) be a CM extension of number fields. Let \(G^0\) be a connected semisimple algebraic group of adjoint type over \(L^+\), and assume that \(G^0\) is compact at all real places of \(L^+\). All implied constants in this section will be assumed to depend on these data. Let \(K = \text{Res}_{K^+/\mathbb{Q}} G^0\) and \(G = \text{Res}_{K^+/\mathbb{Q}} G^0\). Our definitions imply that \(K(\mathbb{R})\) is compact and \(G(\mathbb{R}) \simeq K(\mathbb{C})\), so that \(G(\mathbb{R})\) has the structure of a complex Lie group and \(K(\mathbb{R})\) is a maximal compact subgroup of \(G(\mathbb{R})\).

We choose a number field \(F\) with ring of integers \(\mathcal{O}\) and an integer \(D\) with the following properties.

1. \(K\) and \(G\) are defined over \(\mathbb{Z}[1/D]\).
2. \(K \otimes_{\mathbb{Z}[1/D]} \mathcal{O}[1/D]\) contains a maximal torus and Borel subgroup \(T_K \subset B_K\) defined over \(\mathcal{O}[1/D]\).
3. There exists an isomorphism \(\iota : (K \subset G) \otimes_{\mathbb{Z}[1/D]} \mathcal{O}[1/D] \rightarrow (K_\Delta \subset K \times K) \otimes_{\mathbb{Z}[1/D]} \mathcal{O}[1/D]\), where \(K_\Delta\) denotes the diagonal subgroup.
4. \(K(\mathbb{Z}_p)\) and \(G(\mathbb{Z}_p)\) are hyperspecial maximal compact subgroups of \(K(\mathbb{Q}_p)\) and \(G(\mathbb{Q}_p)\) for \(p \nmid D\).

Combining (2) and (3), we see that \(G \otimes_{\mathbb{Z}[1/D]} \mathcal{O}[1/D]\) contains a torus \(T \simeq T_K \times T_K\), and that \(T\) contains \(T_K \subset K \otimes_{\mathbb{Z}[1/D]} \mathcal{O}[1/D]\) as its diagonal subgroup. We continue to use the notation of Section 2.1 applied to the choices of data we have made.
3.2. Amplification. The advantage of working with the groups we have defined, and in particular those satisfying condition (3) above, is that we may choose the Hecke operators in our amplifier to be supported on double cosets of the form $K_p \mu(p) K_p$ with $\mu \in X_*(T) \simeq X_*(T_K) \times X_*(T_K')$ of the form $\nu \times 0$. In other words, if we apply the isomorphism $G \simeq K \times K$, then the Hecke operators are supported ‘entirely on the first factor’. On the other hand, this isomorphism maps $K \subset G$ to the diagonal, and we may use the fact that the diagonal meets the first factor only in the identity to show that there are few returns. The variant of Proposition 2.2 we shall apply in this context is:

**Proposition 3.1.** There exists a constant $R = R(G) > 0$ such that for all $p \in \mathcal{P}$, there exists $T_p \in \mathcal{H}_p$ depending on $\psi$ such that

1. $T_p(\psi) = 1$.
2. $\|T_p\|_2 \ll 1$.
3. $T_p$ is supported on double cosets $K_p(\mu \times 0)(p)K_p$ with $\mu \in X_*(T_K)$, $\|\mu\| \leq R$, and $\mu \neq 0$.

**Proof.** We apply Proposition 6.6 with the data $F, D$, and the group $K \times \mathbb{Z}[1/D]\mathcal{O}[1/D]$, and let the data obtained be $C > 0$, $\delta > 0$, and a finite set $0 \not\in X \subset X_*(T_K)$. We enlarge $D$ if necessary to ensure that $p > C$ if $p \nmid D$. If $p \in \mathcal{P}$, the Proposition implies that there is $\mu \in X$ such that $T(p, \mu \times 0)\psi = C_\psi \psi$ with $|C_\psi| > \delta$, and so we define $T_p$ to be $T(p, \mu \times 0)/C_\psi$. Condition (1) is therefore satisfied, and condition (3) is equivalent to saying that $X$ is finite. Condition (2) follows from the Plancharel theorem, or by direct calculation. 

We define the finite component

$$T = \sum_{\substack{p \leq N \\text{prime} \\text{factors}}} T_p,$$

and the infinite components $k_0^\infty$ and $k_\infty$, of our amplifier as before. We must bound the RHS of (13). The term involving $(1_B - 1_{\delta})(y^{-1}\gamma y)$ may be dealt with in the same way, so that we are left with bounding

$$\sum_{\gamma \in G(\mathbb{Q})} (T \ast T^*)(y^{-1}\gamma y)k_0(y^{-1}\gamma y)k_\infty(y^{-1}\gamma y)1_{\delta}(y^{-1}\gamma y).$$

We apply Proposition 4.4 with the data $(G, B, B_H)$ chosen to be $(G, K, \Omega_\infty, K(\mathbb{R}))$, and the auxiliary data $\rho$ and $d(\cdot, \cdot)$ the ones chosen above. This gives constants $C = C(G, \Omega) > 0$ and $M = M(G) > 0$ such that if $\delta \leq C(C_1 N^{A_2})^{-M}$ then there exists a Galois extension $E/\mathbb{Q}$ with $|E : \mathbb{Q}| \ll 1 \times Q \in \mathbb{Z}$ with $\ll \log N$ prime factors, and $y \in G(\mathcal{O}_E[1/DQ])$ depending on $x$ such that if we define

$$L = \bigcap_{\sigma \in \text{Gal}(E/\mathbb{Q})} (yK_y^{-1})^\sigma,$$

then

$$\sum_{\gamma \in G(\mathbb{Q})} (T \ast T^*)(y^{-1}\gamma y)1_{\delta}(y^{-1}\gamma y) = \sum_{\gamma \in L(\mathbb{Q})} (T \ast T^*)(y^{-1}\gamma y)1_{\delta}(y^{-1}\gamma y).$$
We shall assume for the remainder of the proof that \( \delta \leq C(C_1 N^{-A_2})^{-M} \). We now choose the subset \( \mathcal{Q} \subseteq \mathcal{P} \) to consist of those primes not dividing \( Q \).

**Proposition 3.2.** If \( p \in \mathcal{Q} \) and \( \mu \in X_*(\mathbb{T}_K) \) is nonzero, then

\[
L_p K_p \cap K_p(\mu \times 0)(p) K_p = \emptyset
\]

**Proof.** Let \( v \) be a split place of \( F \) above \( p \). Under the embedding \( v \), \( \iota \) gives an isomorphism \( \iota : (\mathbb{K} \subset \mathcal{G}) \otimes_v \mathbb{Z}_p \to (\mathbb{K}_\Delta \subset \mathbb{K} \times \mathbb{K}) \otimes_v \mathbb{Z}_p \). We abbreviate \( \mathbb{K} \times \mathbb{K} \) to \( \mathbb{K}^\times 2 \). If we let \( L \) be the image of \( L \) under this map, then \( \iota(L_p K_p) = L((\mathbb{Q}_p)K^\times 2(\mathbb{Z}_p)) \) and

\[
\iota(K_p(\mu \times 0)(p) K_p) = K^\times 2(\mathbb{Z}_p)(\mu \times 0)(p) K^\times 2(\mathbb{Z}_p).
\]

Choose a place \( w \) of \( E \) above \( v \). Let \( E_w \) be the completion of \( E \) at \( w \), and \( \mathcal{O}_w \) its ring of integers. Extend \( \iota \) to an isomorphism over \( \mathcal{O}_w \). The definition of \( L \) implies that \( L((\mathbb{Q}_p) \subset K^\times 2(\mathbb{Q}_p)) \cap \iota(y)K_\Delta(E_w)\iota(y)^{-1} = \emptyset \).

Because \( p \in \mathcal{Q} \), we have \( \iota(y) \in K^\times 2(\mathcal{O}_w) \), and the result follows from considering the Cartan decomposition in \( K^\times 2 \).

Expanding the RHS of (27) gives

\[
\sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (\mathcal{T} \ast \mathcal{T}^*)(y^{-1} \gamma y)_{1\delta(y^{-1} \gamma y)} = \sum_{p, q \leq N} \sum_{\gamma \in \mathcal{L}(\mathbb{Q})} (\mathcal{T}_p \ast \mathcal{T}_q^*)(y^{-1} \gamma y)_{1\delta(y^{-1} \gamma y)}.
\]

Proposition 3.2 implies that the inner sum is empty when \( p \neq q \), and when \( p = q \) we only need to consider the identity term in \( \mathcal{T}_p \ast \mathcal{T}_p^* \). This is equal to \( \|\mathcal{T}_p\|^2_{1K_\gamma} \), and Proposition 3.1 part (2) gives

\[
\sum_{\gamma \in \mathcal{L}(\mathbb{Q})} (\mathcal{T}_p \ast \mathcal{T}_p^*)(y^{-1} \gamma y)_{1\delta(y^{-1} \gamma y)} \ll \sum_{\gamma \in \mathcal{L}(\mathbb{Q})} 1_{K_\gamma}(y^{-1} \gamma y)_{1\delta(y^{-1} \gamma y)} \ll \mathfrak{c}_1.
\]

This gives

\[
(28) \quad \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} (\mathcal{T} \ast \mathcal{T}^*)(y^{-1} \gamma y)_{1\delta(y^{-1} \gamma y)} \ll N,
\]

and Theorem 1.2 now follows as before.

### 4. Diophantine approximation

This section establishes the existence of the group \( L \) used in Sections 2.3 and 3.2. It may be read independently from the rest of the paper, and is based on unpublished notes of Peter Sarnak and Akshay Venkatesh. Let \( \mathcal{G} \) be an affine algebraic group defined over \( \mathbb{Z}[1/D] \). Let \( F \subset \mathbb{R} \) be a number field with ring of integers \( \mathcal{O} \), and let \( \mathcal{H} \subset \mathcal{G} \) be a subgroup defined over \( \mathcal{O}[1/D] \). Let \( \mathcal{O}_\mathcal{G} \) and \( \mathcal{O}_\mathcal{H} \) be the \( F \)-algebras of functions on \( \mathcal{G} \times \mathbb{Q} F \) and \( \mathcal{H} \). We let \( \rho : \mathcal{G} \rightarrow SL_n \) be a \( \mathbb{Q} \)-embedding. For \( \gamma \in \mathcal{G}(\mathbb{Q}) \), define \( \|\gamma\|_f \) to be the LCM of the denominators of the entries of \( \rho(\gamma) \). We fix a left-invariant Riemannian metric on \( \mathcal{G}(\mathbb{R}) \), and
let $d(\cdot, \cdot)$ be the associated distance function. We define $d(x, y) = \infty$ when $x$ and $y$ are in different connected components of $\mathcal{G}(\mathbb{R})$ with the topology of a real manifold. Fix compact sets $B \subseteq \mathcal{G}(\mathbb{R})$ and $B_H \subseteq H(\mathbb{R})$. For $x \in \mathcal{G}(\mathbb{R})$ and $T, \delta > 0$ we define

$$\mathcal{M}(x, \delta, T) = \{\gamma \in \mathcal{G}(\mathbb{Q}) | ||\gamma||_f \leq T, d(x^{-1}\gamma x, B_H) < \delta\}.$$  

In other words, $\mathcal{M}(x, \delta, T)$ is roughly the set of $\gamma \in \mathcal{G}(\mathbb{Q})$ that are within $\delta$ of $xB_Hx^{-1}$ and have denominators bounded by $T$. Our main result is that, if $\delta$ is small, all such $\gamma$ lie in a group $\mathcal{L}/\mathbb{Q}$ that is stably conjugate to a subgroup of $H$.

**Proposition 4.1.** There exists $M = M(\mathcal{G}, H) > 0$ and $C = C(\mathcal{G}, H, B, B_H) > 0$ such that if $x \in B$ and $\delta \leq CT^{-M}$, there exists a Galois extension $E/F$ with $|E : F| \ll C$, $Q \in \mathbb{Z}$ with $\ll_{\mathcal{G}, H, B, B_H} \log T$ prime factors, and $y \in \mathcal{G}(\mathcal{O}_F[1/DQ])$ such that if we define

$$\mathcal{L} = \bigcap_{\sigma \in \text{Gal}(E/F)} (ghy^{-1})^\sigma$$

then $\mathcal{M}(x, \delta, T) \subseteq \mathcal{L}(\mathbb{Q})$. The data $F$, $Q$, and $y$ depend on $x$, $\delta$, and $T$.

For any subset $S \subseteq \mathcal{M}(x, \delta, T)$, define the variety $X(S)/F \subseteq \mathcal{G}$ by $X(S) = \{g \in \mathcal{G} | S \subset gHg^{-1}\}$. We define $X = X(\mathcal{M}(x, \delta, T))$. Proving Proposition 4.1 is roughly equivalent to showing that $X \neq \emptyset$. We begin by showing that there is a finite set $S$ such that $X = X(S)$.

**Lemma 4.2.** Let $Z$ be an irreducible algebraic variety of dimension $d$, $I$ an arbitrary set, and let $\{Z_\alpha | \alpha \in I\}$ be a set of subvarieties of $Z$. For $S \subseteq I$ we set $Z(S) = \cap_{\alpha \in S} Z_\alpha$. Let $A : \mathbb{N} \rightarrow \mathbb{N}$ be a function and suppose that the number of irreducible components of $Z(S)$ is bounded above by $A(|S|)$. Then there exists $S \subseteq I$ with $|S| \leq (1 + A(1))(1 + A(2)) \ldots (1 + A(d))$, and $Z(S) = Z(I)$.

**Proof.** We may assume without loss of generality that there exists $\alpha_0 \in I$ such that $Z_{\alpha_0} \neq Z$. If $Y$ is an irreducible component of $Z_{\alpha_0}$, we may define the collection of subvarieties $\{Y_\alpha | \alpha \in I\}$ by $Y_\alpha = Y \cap Z_\alpha$. The collection $(Y_\alpha)$ has the same property as $(Z_\alpha)$, with $A$ replaced by the function $n \mapsto A(n+1)$. By induction, there exists a set $S'$ with $|S'| \leq (1 + A(2)) \ldots (1 + A(d))$ and such that $Z(S') \cap Y = Z(I) \cap Y$.

Applying this to each of the components of $Z_{\alpha_0}$, we see that there are sets $S_1, \ldots, S_k$ with $k \leq A(1)$ and $|S_i| \leq (1 + A(2)) \ldots (1 + A(d))$ such that $\cap_{i=1}^k Z(S_i) \cap Z_{\alpha_0} = Z(I)$, which completes the proof.

**Lemma 4.3.** There exists an integer $N = N(\mathcal{G}, H)$ and $S \subseteq \mathcal{M}(x, \delta, T)$ with $|S| \leq N$ such that $X(S) = X$

**Proof.** By Lemma 4.2 it suffices to show that there exists a function $A : \mathbb{N} \rightarrow \mathbb{N}$ depending only on $\mathcal{G}$ and $H$ such that the number of connected components of $X(S)$ is at most $A(|S|)$. $X(S)$ is defined by $|S|$ conditions of the form $\gamma \in gHg^{-1}$, and each such condition is encoded in a set of equations whose cardinality and degrees are bounded in terms of $\mathcal{G}$ and $H$. The existence of the function $A$ then follows from Theorem 7.1 of [25].

We now show that if $S \subseteq \mathcal{M}(x, \delta, T)$ is finite, and $\delta$ is sufficiently small in terms of $T$ and $|S|$, then $X(S)$ is nonempty.
Proposition 4.4. Let \( l > 0 \). There are \( C = C(\mathbf{G}, \mathbf{H}, B, B_H, l) > 0 \) and \( M = M(\mathbf{G}, \mathbf{H}, B, B_H, l) > 0 \) such that if \( x \in B, \delta < CT^{-M} \), and \( S \subseteq \mathcal{M}(x, \delta, T) \) has \( |S| = l \), then \( X(S) \neq \emptyset \).

Proof. Let \( \pi : \mathcal{H}^l \times \mathcal{G} \to \mathcal{G}^l \) be given by

\[
\pi(h_1, \ldots, h_l, g) = (gh_1 g^{-1}, \ldots, gh_l g^{-1})
\]

Chevalley’s theorem implies that the image of \( \pi \) is a constructable subset of \( \mathcal{G}^l \) defined over \( F \), which we denote \( \mathcal{C}(l) \). This implies that there is a finite decreasing chain of subvarieties \( \mathcal{C}^l = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{2a} \supseteq V_{a+1} = \emptyset \) of \( \mathcal{G} \) defined over \( F \) and such that

\[
\mathcal{C}(l) = \bigcup_{i=1}^{a} V_{2i-1} \setminus V_{2i},
\]

where \( V_{2a} \) may be empty. For each \( 0 \leq i \leq 2a + 1 \), we let \( \{ f_{i,j} \in (\mathcal{O}_G)^{\otimes l} | 1 \leq j \leq D(i) \} \) be a finite collection of equations defining \( V_i \). We have \( X_i := \pi^{-1}(V_{2i}) = \pi^{-1}(V_{2i+1}) \) for all \( 0 \leq i \leq a \), and so if we define the ideals \( N_i = \langle \pi^* f_{i,j} | 1 \leq j \leq D(i) \rangle \subseteq (\mathcal{O}_H)^{\otimes l} \otimes_F \mathcal{O}_G \) we see that \( X_i \) is the vanishing set of both \( N_{2i} \) and \( N_{2i+1} \). The Nullstellensatz then gives the existence of an integer \( L = L(\mathbf{G}, \mathbf{H}, l) > 0 \) and elements

\[
\{ p(i, j, k) \in (\mathcal{O}_H)^{\otimes l} \otimes_F \mathcal{O}_G \otimes_F \mathbb{C} | 0 \leq i \leq a, 1 \leq j \leq D(2i+1), 1 \leq k \leq D(2i) \}
\]

such that

\[
(\pi^* f_{2i+1,j})^L = \sum_{k=1}^{D(2i)} p(i, j, k) \pi^* f_{2i,k}.
\]

We let \( C_1 = C_1(\mathbf{G}, \mathbf{H}, B, B_H, l) > 0 \) be an upper bound for all \( |p(i, j, k)| \) on \( B_H^l \times B \). Fix a compact set \( B_1 \subseteq \mathcal{G}(\mathbb{R})^l \) containing both \( \pi(B_H^l \times B) \) and the set

\[
\{ g \in \mathcal{G}(\mathbb{R})^l | d(g, BB_H B^{-1}) \leq 1 \}.
\]

There is a constant \( \kappa = \kappa(\mathbf{G}, \mathbf{H}, B, B_H, l) > 0 \) such that

\[
|f_{i,j}(\gamma) - f_{i,j}(\gamma')| \leq \kappa \max d(g, g')
\]

for all \( i \) and \( j \), and all \( g, g' \in B_1 \). Choose a subset \( S = \{ \gamma_1, \ldots, \gamma_l \} \subseteq \mathcal{M}(x, \delta, T) \), and let \( \gamma = (\gamma_1, \ldots, \gamma_l) \in \mathcal{G}(\mathbb{Q})^l \). We wish to show that \( \gamma \in \mathcal{C}(l) \), which implies that \( X(S) \) is nonempty. If \( \gamma \notin \mathcal{C}(l) \), there must be some \( 0 \leq r \leq a \) such that \( \gamma \in V_r \), but \( \gamma \notin V_{r+1} \). As \( \gamma \notin V_{2r+1} \), there must be some \( s \) such that \( f_{2r+1,s}(\gamma) \neq 0 \). If \( v \) is an infinite place of \( F \), the valuation \( |f_{2r+1,s}(\gamma)|_v \) is bounded in terms of the polynomial \( f_{2r+1,s} \) and the sets \( B \) and \( B_H \).

The condition that \( |\gamma||_v \leq T \) for all \( i \) then implies that there exist \( C_2 = C_2(\mathcal{G}, \mathcal{H}, B, B_H, l) \) and \( M_1 = M_1(\mathcal{G}, \mathcal{H}, l) \) such that \( f_{2r+1,s}(\gamma) \geq 2C_2T^{-M_1} \).

Because \( \gamma_i \in \mathcal{M}(x, \delta, T) \), there exists \( \bar{g} \in \pi(B_H^l \times x) \) with \( d(g, \gamma_i) < \delta \) for all \( i \). The Lipschitz bound \( \ref{eq:lipschitz} \) then implies that \( |f_{2r+1,s}(\bar{g})| \geq 2C_2T^{-M_1} - \kappa \delta \). By decreasing \( C \) and increasing \( M \) if necessary, we may assume that \( |f_{2r+1,s}(\bar{g})| \geq C_2T^{-M_1} \). Let \( \bar{g} = \pi^{-1}(\bar{g}) \cap (B_H^l \times x) \). Combining the bound \( |\pi^* f_{2r+1,s}(\bar{g})| \geq CT^{-M} \) with \( \ref{eq:lipschitz} \) gives
\[ \sum_{k=1}^{D(2i)} p(r, s, k) \pi^* f_{2r, k}(\tilde{g}) \geq C_2^L T^{-LM_1}, \]

and so there must be some \( t \) such that

\[ |f_{2r, t}(|\gamma|) \geq C_2^L T^{-LM_1} \]

Combined with (30) this gives

\[ |f_{2r, t}(\gamma)| \geq C_2^L T^{-LM_1} \]

By shrinking \( C \) and increasing \( M \) if necessary, this implies that \( f_{2r, t}(\gamma) \neq 0 \), which contradicts \( \gamma \in V_{2i} \).

\[ \square \]

**Proof of Proposition 4.1.** Let \( N \) be the integer provided by Lemma 4.3 and apply Proposition 4.4 for every \( l \) between 1 and \( N \). This gives \( C = C(G, H, B, B_H) > 0 \) and \( M = M(G, H) > 0 \) such that \( X \neq \emptyset \) if \( x \in B \) and \( \delta < CT^{-M} \), which is equivalent to saying that \( \mathcal{M}(x, \delta, T) \) is contained in a conjugate of \( H(\mathbb{Q}) \) over \( \mathbb{Q} \).

\( X \) is defined by at most \( N \) equations of the form \( \gamma \in gHg^{-1} \). This means that \( X \) is cut out by a set of polynomials over \( F \) whose cardinality is bounded in terms of \( G \) and \( H \), and our assumption that \( \|\gamma\|_f \leq T \) implies that the heights of the coefficients of these polynomials are bounded by \( CT' \) for some \( C' = C'(G, H, B, B_H) \) and \( M' = M'(G, H) \). This implies that \( X \) has a point \( y \) over some Galois extension \( E/F \) whose degree is bounded in terms of \( G \) and \( H \), and moreover that we can choose \( y \) to be defined over \( \mathcal{O}_F[1/DQ] \) for some \( Q \) with \( \ll G, H, B, B_H \log T \) prime factors. The definition of \( X \) then gives

\[ \mathcal{M}(x, T, \delta) \subseteq yHy^{-1}(E), \]

and because \( \mathcal{M}(x, T, \delta) \subseteq \mathcal{G}(\mathbb{Q}) \) we may descend \( yHy^{-1} \) to the group \( L/\mathbb{Q} \).

\[ \square \]

5. **Estimating intersections in buildings**

This section contains the proof of Proposition 4.5 and may be read independently from the rest of the paper. Let \( H \subset G \) be a pair of connected, reductive, split, algebraic groups over \( \mathbb{Q}_p \). We assume that both \( H \) and \( G \) are the general fibers of group schemes over \( \mathbb{Z}_p \) (denoted in the same way) with reductive special fibers. We let \( T \subset B \subset G \) and \( T_H \subset B_H \subset H \) be maximal tori contained in Borel subgroups of \( G \) and \( H \), all defined over \( \mathbb{Z}_p \). These assumptions imply that \( \mathcal{G}(\mathbb{Q}_p) \) has a Cartan decomposition with respect to \( T \) and \( \mathcal{G}(\mathbb{Z}_p) \), and likewise for \( H \). We assume that \( T_H \subset T \) and \( B_H \subset B \). We define the Weyl groups \( W = N_G(T)/T \) and \( W_H = N_H(T_H)/T_H \). We let \( \Delta \subset X^*(T) \) be the roots of \( G \), and \( \Delta^+ \) the positive roots corresponding to \( B \), and define \( \Delta_H \) and \( \Delta_H^+ \) similarly for \( H \). We let \( 2\rho \) and \( 2\rho_H \) be the sums of \( \Delta^+ \) and \( \Delta_H^+ \) respectively. If \( \mu \in X_*(T_H) \), we define

\[ \|\mu\|_H^* = \max_{w \in W_H} \langle w\mu, \rho_H \rangle. \]
Let $F$ be an unramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We let $k_p$ and $k_q$ be the residue fields of $\mathbb{Z}_p$ and $\mathcal{O}$. We let $\Gamma = \text{Gal}(F/\mathbb{Q}_p) \simeq \text{Gal}(k_q/k_p)$. Let $y \in G(\mathcal{O})$, and define

$$L = \bigcap_{\sigma \in \Gamma} (y H y^{-1})^\sigma$$

We let $K_p = G(\mathbb{Z}_p)$, and define

$$\mathcal{L}(\mu) = \#(L(\mathbb{Q}_p)K_p \cap K_p \mu(p)K_p)/K_p$$

for $\mu \in X_*(T)$. This can be thought of as the size of the intersection of $L$ with the sphere of radius $\mu$ in the building of $G$. The main result of this section is a bound for this size.

**Proposition 5.1.** If $W_\mu$ does not intersect $X_*(T_H)$, then $\mathcal{L}(\mu) = 0$. If $\mu \in X_*(T_H)$, we have $\mathcal{L}(\mu) \ll p^{2\|\mu\|_H}$, where the implied constant depends only on the degree $|F: \mathbb{Q}_p|$ and the dimension of $G$, but not $p$.

We begin by reducing to the case of $\mu \in X_*(T_H)$.

**Lemma 5.2.** $\mathcal{L}(\mu) = 0$ unless $W_\mu$ intersects $X_*(T_H)$.

**Proof.** Suppose that the intersection $L(\mathbb{Q}_p) \cap K_p \mu(p)K_p$ is nonempty. We have $L(\mathbb{Q}_p) = (y H(F)y^{-1}) \cap G(\mathbb{Q}_p)$, so there must be some $x \in H(F)$ such that $yxy^{-1} \in K_p \mu(p)K_p$. The Cartan decomposition on $H$ gives $\nu \in X_*(T_H)$ such that $x \in H(\mathcal{O})\nu(p)H(\mathcal{O})$, and because $y \in G(\mathcal{O})$ this gives $yxy^{-1} \in G(\mathcal{O})\nu(p)G(\mathcal{O})$. The uniqueness of the Cartan decomposition implies that $\nu \in W_\mu$ as required. \(\square\)

We may therefore assume that $\mu \in X_*(T_H)$. We define

$$\mathcal{P}_\mu = G(\mathcal{O}) \cap \mu(p)G(\mathcal{O})\mu(p)^{-1}$$

$$\mathcal{P}(\mu, k) = \mathcal{P}_\mu G(1 + p^k \mathcal{O})$$

and

$$\mathcal{F}_\mu = G(\mathcal{O})/\mathcal{P}_\mu$$

$$\mathcal{F}(\mu, k) = G(\mathcal{O})/\mathcal{P}(\mu, k),$$

and denote the analogous objects for $H$ with a superscript $H$. Note that that $\mathcal{P}_\mu^H = \mathcal{P}_\mu \cap H(\mathcal{O})$. It may be seen that $|\mathcal{F}_\mu^H| = p^{2\|\mu\|_H}$. $\Gamma$ stabilizes $\mathcal{P}_\mu$, and so acts on all of the sets $\mathcal{P}(\mu, k)$. We define

$$\mathcal{S}_\mu = \{h \in \mathcal{F}_\mu^H | yh \in \mathcal{F}_\mu^\Gamma\}$$

$$\mathcal{S}(\mu, k) = \{h \in \mathcal{F}(\mu, k)^H | yh \in \mathcal{F}(\mu, k)^\Gamma\}.$$

**Lemma 5.3.** We have $\mathcal{L}(\mu) \leq |\mathcal{S}_\mu|$. 

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Proof. We have \( L(Q_p) = (yH(F)y^{-1}) \cap G(Q_p) \), so
\[
L(\mu) = \#((yH(F)y^{-1})K_p \cap K_p\mu(p)K_p)/K_p.
\]
Because the map \( G(Q_p)/K_p \to G(F)/G(O) \) is an injection, we have
\[
L(\mu) \leq \#((yH(F)y^{-1})G(O) \cap K_p\mu(p)G(O))/G(O).
\]
This becomes
\[
\Gamma \text{ acts on } G(F)/G(O), \text{ and the cosets in } K_p\mu(p)G(O) \text{ must be fixed by } \Gamma. \text{ This gives}
\]
\[
L(\mu) \leq \#\{gG(O) \in yH(F)G(O) \cap G(O)\mu(p)G(O)|\sigma(gG(O)) = gG(O), \sigma \in \Gamma\},
\]
and if we replace \( g \) with \( y^{-1}g \) this becomes
\[
L(\mu) \leq \#\{gG(O) \in H(F)G(O) \cap G(O)\mu(p)G(O)|\sigma(ygG(O)) = (ygG(O)), \sigma \in \Gamma\}.
\]
We have \( H(F)G(O) \cap G(O)\mu(p)G(O) = H(O)\mu(p)G(O) \).
\[
L(\mu) \leq \#\{gG(O) \in H(O)\mu(p)G(O)|\sigma(ygG(O)) = (ygG(O)), \sigma \in \Gamma\}.
\]
It may be checked that the map
\[
F^H_\mu = H(O)/P^H_\mu \to H(O)\mu(p)G(O)/G(O)
\]
\[
k \mapsto k\mu(p)G(O)
\]
is a bijection, and the condition that the coset \( yk\mu(p)G(O) \) is fixed by \( \Gamma \) is just that \( yk \) is fixed by \( \Gamma \) as an element of \( F_\mu \). This completes the proof.

For \( k \geq 0 \), define
\[
\Delta^-(\mu, k) = \{ \alpha \in \Delta | \langle \alpha, \mu \rangle \leq k \}
\]
\[
\Delta^-(H, \mu, k) = \{ \alpha \in \Delta_H | \langle \alpha, \mu \rangle \leq k \}
\]
\[
\delta(\mu, k) = |\Delta_H \setminus \Delta^-(H, \mu, k)|.
\]

**Lemma 5.4.** \( |S(\mu, 1)| \ll p^{\delta(\mu, 0)}. \)

**Proof.** Let \( P_\mu/Z_p \) be the parabolic subgroup of \( G \) generated by \( T \) and the one-parameter subgroups corresponding to the roots in \( \Delta^-(\mu, 0) \). Let \( P^H_\mu = P_\mu \cap H \) be the corresponding parabolic in \( H \). Let \( Q_\mu \) and \( Q^H_\mu \) be the flag varieties \( G/P_\mu \) and \( H/P^H_\mu \).

We have \( F(\mu, 1)^H = H(k_q)/P^H_\mu(k_q) \subseteq Q^H_\mu(k_q) \), and \( F(\mu, 1)^F = Q_\mu(k_p) \). It therefore suffices to show that \( |yQ^H_\mu(k_q) \cap Q_\mu(k_p)| \ll p^{\delta(\mu, 0)} \). Because \( \dim Q^H_\mu = \delta(\mu, 0) \) and all the varieties involved have complexity bounded in terms of the dimension of \( G \), this follows from e.g. the Lang-Weil bound.

**□**

**Lemma 5.5.** If \( k \geq 1 \), the fibers of \( S(\mu, k + 1) \to S(\mu, k) \) have size at most \( p^{\delta(\mu, k)} \).

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Proof. The fibers of the map \( \mathcal{F}(\mu, k + 1)^{H} \rightarrow \mathcal{F}(\mu, k)^{H} \) may be identified with

\[
\mathcal{H}(1 + p^{k}O)\mathcal{P}_{\mu}^{H}/\mathcal{H}(1 + p^{k+1}O)\mathcal{P}_{\mu}^{H}.
\]

Every coset \( h\mathcal{H}(1 + p^{k+1}O)\mathcal{P}_{\mu}^{H} \) contains a representative \( h \in \mathcal{H}(1 + p^{k}O) \), and this gives an identification of sets

\[
\mathcal{H}(1 + p^{k}O)\mathcal{P}_{\mu}^{H}/\mathcal{H}(1 + p^{k+1}O)\mathcal{P}_{\mu}^{H} \simeq \mathcal{H}(1 + p^{k}O)/\mathcal{H}(1 + p^{k+1}O)(\mathcal{H}(1 + p^{k}O) \cap \mathcal{P}_{\mu}^{H}).
\]

We define

\[
V = G(1 + p^{k}O)/G(1 + p^{k+1}O)(G(1 + p^{k}O) \cap \mathcal{P}_{\mu})
\]

\[
V_{H} = H(1 + p^{k}O)/H(1 + p^{k+1}O)(H(1 + p^{k}O) \cap \mathcal{P}_{\mu}^{H}).
\]

If \( \mathfrak{g} \) and \( \mathfrak{h} \) are the Lie algebras of \( G \) and \( H \) over \( \mathbb{Z}_{p} \), the exponential map gives isomorphisms

\[
G(1 + p^{k}O)/G(1 + p^{k+1}O) \simeq \mathfrak{g}(k_{q})
\]

\[
H(1 + p^{k}O)/H(1 + p^{k+1}O) \simeq \mathfrak{h}(k_{q}).
\]

It follows that

\[
V \simeq \mathfrak{g}(k_{q})/ \bigoplus_{\alpha \in \Delta_{-}(\mu, k)} \mathfrak{g}_{\alpha}(k_{q})
\]

\[
V_{H} \simeq \mathfrak{h}(k_{q})/ \bigoplus_{\alpha \in \Delta_{\mu}(\mu, k)} \mathfrak{h}_{\alpha}(k_{q}),
\]

and in particular that \( V \) and \( V_{H} \) have the structure of vector spaces over \( k_{q} \). We have \( \dim V_{H} = \delta(\mu, k) \). It may be seen that the action of \( \Gamma \) on \( \mathcal{G}(O) \) reduces to actions on \( V \) and \( V_{H} \). If \( g \in \mathcal{P}_{\mu} \) then \( g \) acts on \( V \) by conjugation, and we denote this action by \( \text{Ad}_{g} \).

Let \( h \in \mathcal{S}(\mu, k) \). We need to bound the number of \( h_{1} \in V_{H} \) that satisfy \( \sigma(yh_{1}) \in yhh_{1}\mathcal{G}(1 + p^{k+1}O)\mathcal{P}_{\mu} \) for all \( \sigma \in \Gamma \). Our assumption that \( h \in \mathcal{S}(\mu, k) \) means that \( \sigma(yh) = yh\delta(\sigma)p(\sigma) \) for some \( \delta(\sigma) \in \mathcal{G}(1 + p^{k}O) \) and \( p(\sigma) \in \mathcal{P}_{\mu} \). This gives

\[
\sigma(yh_{1}) = yh\delta(\sigma)p(\sigma)\sigma(h_{1})
\]

\[
= yh\delta(\sigma)\text{Ad}_{p(\sigma)}\sigma(h_{1})p(\sigma),
\]

so that \( \sigma(yh_{1}) \in yhh_{1}\mathcal{G}(1 + p^{k+1}O)\mathcal{P}_{\mu} \) is equivalent to

\[
\delta(\sigma)\text{Ad}_{p(\sigma)}\sigma(h_{1}) \in h_{1}\mathcal{G}(1 + p^{k+1}O)\mathcal{P}_{\mu}.
\]

Because \( \delta(\sigma)\text{Ad}_{p(\sigma)}\sigma(h_{1}) \) and \( h_{1} \) lie in \( \mathcal{G}(1 + p^{k}O) \), this is equivalent to

\[
\delta(\sigma)\sigma(h_{1})p(\sigma) \in h_{1}\mathcal{G}(1 + p^{k+1}O)(\mathcal{G}(1 + p^{k}O) \cap \mathcal{P}_{\mu}).
\]

Taking the quotient by \( \mathcal{G}(1 + p^{k+1}O)(\mathcal{G}(1 + p^{k}O) \cap \mathcal{P}_{\mu}) \), we obtain the linear equation

\[
(31) \quad \delta(\sigma) + \text{Ad}_{p(\sigma)}(\sigma(h_{1})) = h_{1}
\]
We must show that (31) has at most $p^\delta(\mu,k)$ solutions. Let $h_0$ be any fixed solution. We must have

$$\text{Ad}_{p(\sigma)}(\sigma(h_1 - h_0)) = h_1 - h_0$$

for all $\sigma \in \Gamma$. If we define $T_\sigma = \text{Ad}^{-1}_{p(\sigma)}$, we must therefore bound the number of $v \in V_H$ such that $\sigma(v) = T_\sigma(v)$ for all $\sigma \in \Gamma$. Let $A$ be the set of all such $v$. It is a $k_p$-subspace of $V_H$, and we let $v_1, \ldots, v_r$ be a basis. Suppose that there exist $\alpha_i \in k_q$, not all 0, such that

$$\sum_{i=1}^r \alpha_i v_i = 0.$$

Let $\beta \in k_q$. This gives

$$\sum_{i=1}^r \sigma(\beta \alpha_i v_i) = 0$$

$$\sum_{i=1}^r \sigma(\beta \alpha_i) T_\sigma(v_i) = 0$$

$$T_\sigma \left( \sum_{i=1}^r \sigma(\beta \alpha_i) v_i \right) = 0$$

$$\sum_{i=1}^r \sigma(\beta \alpha_i) v_i = 0.$$

Summing this over all $\sigma$, we have

$$\sum_{i=1}^r \text{tr}_{k_q/k_p}(\beta \alpha_i) v_i = 0.$$

There must be some $\beta$ and $i$ such that $\text{tr}_{k_q/k_p}(\beta \alpha_i) \neq 0$, which contradicts the independence of the $v_i$ over $k_p$. Therefore the $v_i$ are linearly independent over $k_q$, which means that $r \leq \dim V_H = \delta(\mu,k)$ and $|A| \leq p^\delta(\mu,k)$ as required.

Proposition 5.1 now follows by combining Lemmas 5.4 and 5.5, and the identity

$$\sum_{k=0}^\infty \delta(\mu,k) = 2\|\mu\|_H^2.$$

6. Constructing an amplifier

This section contains the proof of Propositions 2.2 and 3.1, and may be read independently from the rest of the paper. We begin with the following lemma, which follows immediately from the Peter-Weyl theorem.
Lemma 6.1. Let $U$ be a compact real Lie group. Given $u \in U$, there exists a nontrivial irreducible character $\chi$ of $U$ such that $\chi(u) \neq 0$.

6.1. Complex groups. Let $G$ be a complex connected reductive algebraic group and $T \subseteq B \subseteq G$ a torus contained in a Borel subgroup. Let $X^*(T)$ and $X_*(T)$ denote the character and cocharacter lattice of $T$, and denote the pairing between them by $\langle \cdot, \cdot \rangle$. Let $\Delta$ and $\hat{\Delta}$ be the set of roots and co-roots respectively, and $\Delta^+$ and $\hat{\Delta}^+$ be the set of positive roots and co-roots corresponding to $B$. We let $\Phi$ and $\tilde{\Phi}$ be the simple roots and co-roots. Let $2\rho = \sum_{\alpha \in \Delta^+} \alpha \in X^*(T)$. We define

$$X_*(T)^+ = \{ \mu \in X_*(T) | \langle \mu, \alpha \rangle \geq 0, \alpha \in \Delta^+ \}$$

and $X^*(T)^+$ likewise. If $\mu \in X^*(T)^+$, we let $\chi_\mu$ be the character of the representation of $G$ with highest weight $\mu$. We let $W$ be the Weyl group $N_G(T)/T$.

If $L$ is the standard Levi subgroup associated to a set $\theta \subseteq \Phi$, we let $T_L \subseteq T$ be the connected component of $\cap_{\alpha \in \theta} \ker \alpha$. We let $T_L T^L \subseteq T$ be the torus generated by $\{ \alpha^\vee | \alpha \in \theta \}$. We have $T = T_L T^L$. We let $W_L \subseteq W$ be the group generated by the reflections in elements of $\theta$. Let $\Delta_L^+ = \Delta^+ \cap \Delta_L$, and let $2\rho_L = \sum_{\alpha \in \Delta_L^+} \alpha \in X^*(T)$. Let $\lambda_L = 2\rho - 2\rho_L$.

$$X^*(T)_L^+ = \{ \mu \in X^*(T) | \langle \mu, \alpha^\vee \rangle \geq 0, \alpha \in \theta \}$$

If $\mu \in X^*(T)_L^+$, we let $\chi_{L, \mu}$ be the character of the associated representation of $L$.

Lemma 6.2. $\lambda_L$ has the following properties:

1. $\lambda_L \in X^*(T/T^L)$.
2. $\lambda_L \in X^*(T)^+$.
3. Given $\mu \in X^*(T)_L^+$, there is $k \in \mathbb{Z}$ such that $\mu + k\lambda_L \in X^*(T)^+$.
4. $\text{Stab}_W(\lambda_L) = W_L$.

Proof. If $\alpha \in \theta$, the set $\Delta^+ \setminus \Delta_L^+$ is invariant under the reflection in $\alpha$ so we have $\langle \lambda_L, \alpha^\vee \rangle = 0$. This implies (1). If $\alpha \notin \theta$, we have $\langle \rho, \alpha^\vee \rangle > 0$ and $\langle \rho_L, \alpha^\vee \rangle < 0$ so that $\langle \lambda_L, \alpha^\vee \rangle > 0$. This implies the remaining assertions.

Lemma 6.3. If $w \in W \setminus W_L$, we have

$$\lambda_L - w\lambda_L = \sum_{\alpha \in \Phi} n_{\alpha} \alpha$$

where $n_{\alpha} \geq 0$, and there is $\alpha \notin \theta$ such that $n_{\alpha} > 0$.

Proof. Because $\lambda_L \in X^*(T)^+$, we have

$$\chi(w) := \lambda_L - w\lambda_L = \sum_{\alpha \in \Phi} n_{\alpha} \alpha$$

with $n_{\alpha} \geq 0$ for all $\alpha$. If $n_{\alpha} = 0$ for all $\alpha \notin \theta$, then $\chi(w)$ is trivial on $T_L$. This implies that
Proposition 6.4. Let $G$ be a complex connected reductive algebraic group, and let $T \subseteq G$ be a maximal torus. Let $\Omega \subseteq T$ be compact. There exists a finite set $0 \notin \mathcal{X} \subset X^*(T)$ and $\delta > 0$ such that for all $x \in \Omega$ we have

$$\max_{\mu \in \mathcal{X}} \{|\chi_{\mu}(x)|\} > \delta$$

Proof. As $\Omega$ is compact, it suffices to prove that for each $x \in T$ there exists $0 \neq \mu \in X^*(T)$ with $\chi_{\mu}(x) \neq 0$. Fix $x \in T$. After applying an element of $W$, we may assume without loss of generality that $|\alpha(x)| \geq 1$ for $\alpha \in \Delta^+$. Let $\theta = \{\alpha \in \Phi| |\alpha(x)| = 1\}$, and let $L$ be the corresponding Levi subgroup. Decompose $x = x_L x^L$ with $x_L \in T_L$ and $x^L \in T^L$. If $\alpha \in \theta$, we have $\alpha(x_L) = 1$ and $|\alpha(x)| = 1$, so $|\alpha(x^L)| = 1$. As $x^L$ lies in the torus generated by $\{\alpha^\vee| \alpha \in \theta\}$, this implies that $|\alpha(x^L)| = 1$ for all $\alpha \in \Delta$. We therefore have

$$|\alpha(x^L)| \geq 1,$$

with equality iff $\alpha \in \theta$.

and $x^L$ lies in the maximal compact subgroup $T_c$ of $T$. If we let $U$ be a compact real form of $L$ containing $x^L$, we may apply Lemma 6.1 to produce $0 \neq \mu \in X^*(T)_L^+$ such that $\chi_{L,\mu}(x^L) \neq 0$, and hence $\chi_{L,\mu}(x) \neq 0$. We shall prove that there exists $k \in \mathbb{Z}$ such that if we define $\mu' = \mu + k\lambda_L$ then $\chi_{L,\mu'}(x) \neq 0$.

By Lemma 6.2 part (3), we may first assume that $k$ is chosen to be sufficiently positive that $\mu' \in X^*(T)_L^+$. The Weyl character formula then gives

$$\chi_{\mu'}(x) = \lim_{y \to x} \chi_{\mu'}(y) = \frac{\sum_{w \in W} (-1)^{l(w)}(w(\mu' + \rho) - \rho)(y)}{\prod_{\alpha \in \Delta_+} (1 - \alpha^{-1}(y))}.$$

As $\alpha(x) \neq 1$ when $\alpha \notin \Delta_L$, it suffices to prove that

$$\lim_{y \to x} \frac{\sum_{w \in W} (-1)^{l(w)}(w(\mu' + \rho) - \rho)(y)}{\prod_{\alpha \in \Delta_L} (1 - \alpha^{-1}(y))} \neq 0.$$

Let $\Xi(y) = \prod_{\alpha \in \Delta_L^+} (1 - \alpha^{-1}(y))$. For $w \in W$, define $\nu(w) = w(\mu' + \rho) - \rho$. Because $\rho - \rho_L$ is fixed by $W_L$, this implies that

$$w_L(\nu(w) + \rho_L) = w_Lw(\mu' + \rho) - \rho$$

for any $w_L \in W_L$. Choose coset representatives $\{w_i| 1 \leq i \leq |W : W_L|\}$ for $W_L \backslash W$ such that $\nu(w_i) \in X^*(T)_L^+$. Our assumption that $\mu' \in X^*(T)_L^+$ implies that $e \in W$ is the representative of the identity coset. If $y \in T$ is regular, we then have

$$0 = \sum_{w_L \in W_L} w_L \chi(w) = \sum_{w_L \in W_L} \lambda_L - w_L w \lambda_L$$

Because all the terms in the sum are non-negative, we must have $\lambda_L - w_L w \lambda_L = 0$ for all $w_L \in W_L$. Therefore $\lambda_L = w \lambda_L$, which contradicts our assumption that $w \notin W_L$. \qed
\[
\Xi(y)^{-1} \sum_{w \in W} (-1)^{(w)}(w(\mu' + \rho) - \rho)(y) = \Xi(y)^{-1} \sum_{w_i \in W \setminus W} \sum_{w_L \in W_L} (-1)^{(w_i w_L)}(w_L(\nu(w_i) + \rho_L) - \rho_L)(y)
\]

\[
= \sum_{w_i \in W_L \setminus W} (-1)^{(w_i)} \chi_{L,\nu(w_i)}(y).
\]

It therefore suffices to prove that \( \sum_{w_i \in W_L \setminus W} (-1)^{(w_i)} \chi_{L,\nu(w_i)}(x) \neq 0 \). Applying our decomposition \( x = x_Lx^L \) gives

\[
\sum_{w_i \in W_L \setminus W} (-1)^{(w_i)} \chi_{L,\nu(w_i)}(x) = \sum_{w_i \in W_L \setminus W} (-1)^{(w_i)} \nu(w_i)(x_L) \chi_{L,\nu(w_i)}(x^L).
\]

If we separate the identity coset and define \( C_i = (-1)^{(w_i)}(w_i(\mu + \rho) - \rho)(x_L) \), we have

\[
(-1)^{(w_i)} \nu(w_i)(x_L) \chi_{L,\nu(w_i)}(x^L) = C_i(kw_i\lambda_L)(x_L) \chi_{L,\nu(w_i)}(x^L)
\]

so that

\[
(33) \sum_{w_i \in W_L \setminus W} (-1)^{(w_i)} \nu(w_i)(x_L) \chi_{L,\nu(w_i)}(x^L) = (k\lambda)(x_L) \left( \mu(x_L) \chi_{L,\mu}(x^L) + \sum_{w_i \in W_L \setminus W, w_i \neq e} C_i(kw_i\lambda - k\lambda)(x_L) \chi_{L,\nu(w_i)}(x^L) \right).
\]

Lemma \( 6.3 \) and equation \( (32) \) then imply that there is some \( C > 1 \) such that \( (\lambda_L - w_i\lambda_L)(x_L) \geq C \) for all \( w_i \neq e \). As \( (kw_i\lambda_L - k\lambda_L)(x_L) \leq C^{-k} \) and \( |\chi_{L,\nu(w_i)}(x^L)| \ll k^A \) for some \( A \), we see that \( (33) \) will be nonzero for \( k \) large as required.

\[
\square
\]

6.2. Groups over a number field. Let \( F \) be a number field, \( D \) an integer, and \( G \) a connected split reductive algebraic group of adjoint type defined over \( \mathcal{O}[1/D] \). Let \( T \subset B \subset G \) be a maximal torus contained in a Borel subgroup, both defined over \( \mathcal{O}[1/D] \), and let \( N \) be the unipotent radical of \( B \).

Let \( X^*(\underline{T}) \) and \( X_*(\underline{T}) \) be the character and cocharacter lattice of \( \underline{T} \). Let \( \Delta \) and \( \hat{\Delta} \) be the set of roots and co-roots of \( \underline{T} \) in \( G \), and let \( \Delta^+ \) and \( \hat{\Delta}^+ \) be the set of positive roots and co-roots corresponding to \( \underline{B} \). Let \( \Phi \) and \( \hat{\Phi} \) be the simple roots and co-roots. Let \( X_*(\underline{T})^+ \) be the set

\[
\{ \mu \in X_*(\underline{T}) | (\mu, \alpha) \geq 0, \alpha \in \Delta^+ \}.
\]

We define \( 2\rho \in X^*(\underline{T}) \) to be the sum of the elements of \( \Delta^+ \).

If \( v \) is a place of \( F \), let \( F_v \) denote the corresponding completion with ring of integers \( \mathcal{O}_v \). Let \( \pi \in \mathcal{O}_v \) be a uniformiser, and let \( q = |\mathcal{O}/\pi\mathcal{O}| \). We assume that \( G(\mathcal{O}_v) \) is hyperspecial in \( G(F_v) \) for all \( v \nmid D \). If \( v \nmid D \) we have an isomorphism \( X_*(\underline{T}) \simeq \underline{T}(F_v)/\underline{T}(\mathcal{O}_v) \) via the map \( \mu \mapsto \mu(\pi) \), and this is independent of the choice of \( \pi \). We let \( Y = \underline{T}(F_v)/\underline{T}(\mathcal{O}_v) \), and
$Y^+$ be the subset corresponding to $X_*(\mathcal{L})^+$. Our assumptions imply that we have a Cartan decomposition

$$G(F_v) = \prod_{y \in Y^+} G(O_v)yG(O_v).$$

We let $\mathcal{H}$ be the algebra of functions on $G(F_v)$ that are compactly supported and bi-invariant under $G(O_v)$. For $\mu \in X_*(\mathcal{L})^+$, we let $\omega_\mu$ be the function supported on $G(O_v)\mu(\pi)G(O_v)$ and equal to $q^{-\langle \rho, \mu \rangle}$ there.

Let $\widehat{G}$ be the complex dual group of $G$. We fix a maximal torus and Borel subgroup $\hat{T} \subset \hat{B} \subset \hat{G}$. There is an isomorphism $X^*(\hat{T}) \simeq X_*(\mathcal{L})$ which takes the positive roots for $\hat{B}$ to the positive co-roots for $\hat{B}$, and so we shall think of $\hat{\Delta}$ and $\hat{\Delta}^+$ as the set of roots, resp. positive roots, for $\widehat{G}$. We define

$$\hat{T}^+ = \{ x \in \hat{T} \mid |\alpha(x)| \geq 1, \alpha \in \hat{\Delta}^+ \}.$$ 

We let $X^*(\hat{T})^+$ correspond to $X_*(\mathcal{L})^+$. We have $\hat{T} \simeq \text{Hom}(X_*(\mathcal{L}), \mathbb{C}^\times) \simeq \text{Hom}(Y, \mathbb{C}^\times)$. We shall often apply the notation of Section 6.1 to $\widehat{G}$.

Let $\mathcal{H}_T$ be the algebra of compactly supported $W$-invariant functions on $X_*(\mathcal{L})$. We define the Satake transform $\mathcal{S} : \mathcal{H} \to \mathcal{H}_T$ by

$$\mathcal{S} f(\lambda) = q^{\langle \mu, \rho \rangle} \int_{\hat{\Delta}(F_v)} f(n\lambda(\pi)) dn.$$

We shall often think of $\mathcal{S}$ as an isomorphism between $\mathcal{H}$ and $R(\widehat{G})$, the representation ring of $\widehat{G}$, via the isomorphism $X_*(\mathcal{L}) \simeq X^*(\hat{T})$ and the isomorphism between $R(\widehat{G})$ and $W$-invariant functions on $X^*(\hat{T})$ that takes a representation to its character. We have

$$\mathcal{S} \omega_\mu = \chi_\mu + \sum_{\lambda < \mu} C(\lambda, \mu, q) \chi_\lambda$$

and it is known that

$$C(\lambda, \mu, q) \ll_{\lambda, \mu} q^{-1},$$

see for instance Sections 3 and 4 of [12], in particular the fact that the Kazhdan-Lusztig polynomials have degree strictly less than $\langle \lambda - \mu, \rho \rangle$ in $q$. Note that this is the point at which we require our assumption that $G$ was of adjoint type.

We say that an admissible representation $\pi$ of $G(F_v)$ is spherical if it contains a vector fixed by $G(O_v)$. The spherical admissible representations are in bijection with $\hat{T}/W$, via the map that takes $\alpha \in \hat{T}$ first to the corresponding character $\chi \in \text{Hom}(Y, \mathbb{C}^\times)$, and then to the unique spherical subquotient of the representation induced from $\chi$ on $B(F_v)$. We denote this representation by $\pi(\alpha)$. If $f \in \mathcal{H}$ and $v_0$ is the unique spherical vector up to scaling in the space of $\pi(\alpha)$, we then have $\pi(\alpha)(f)v_0 = \mathcal{S} f(\alpha)v_0$.

**Proposition 6.5.** Let $\hat{T} \subseteq L \subseteq \hat{G}$ be the standard Levi subgroup corresponding to a subset $\theta \subseteq \hat{\theta}$. Given $C_1 > 1$, there exists $C_2 > 1$, $C_3 > 0$, $\delta > 0$, and a finite set $0 \notin \mathcal{X} \subset X_*(\mathcal{L})^+$ such that if $q > C_3$ and $x \in \hat{T}^+$ satisfies
\begin{align*}
|\alpha(x)| & < C_1, \quad \alpha \in \theta, \\
|\alpha(x)| & > C_2, \quad \alpha \in \bar{\Phi} \setminus \theta
\end{align*}

we have

$$
\max_{\mu \in X} \{|S_{\omega \mu}(x)|\} > \delta
$$

\textbf{Proof.} By applying Proposition 6.4 to \(L/\hat{T}_L\), the torus \(\hat{T}/\hat{T}_L\), and the set

\[
\Omega_L = \{ x \in \hat{T}^+ \hat{T}_L/\hat{T}_L | |\alpha(x)| < 2C_1, \alpha \in \theta \},
\]

we obtain a finite set \(0 \not\in \mathcal{X}_L \subset X^*(\hat{T}/\hat{T}_L)\) and \(\delta > 0\) such that for all \(x \in \Omega_L\) we have

$$
\max_{\mu \in \mathcal{X}_L} \{|\chi(x,\mu)|\} > \delta
$$

By Lemma 6.2 parts (1) and (3), we may lift each element of \(\mathcal{X}_L\) to \(X^*(\hat{T})\) and twist by a character trivial on \(\hat{T}_L\) to obtain a subset \(0 \not\in X \subset X^*(\hat{T})\). Define

\[
\Omega = \{ x \in \hat{T}^+ | |\alpha(x)| < 2C_1, \alpha \in \theta \}.
\]

Let \(x \in \Omega\), and decompose \(x = x_L x^L\). Then \(x^L\) projects to \(\Omega_L\), and must lie in a bounded subset of \(T^L\) depending on \(C_1\). By shrinking \(\delta\) we therefore have

$$
\max_{\mu \in X} \{|\chi_L(x^L,\mu)|\} > \delta.
$$

Let \(\lambda \in X^*(\hat{T})^+\). We have \(\chi_\lambda(x) = O_\lambda(\lambda(x)) = O_{\lambda,C_1}(\lambda(x_L))\), and combined with (34) and (35) this gives

\[
S_{\omega \lambda}(x) = \chi_\lambda(x) + O_{\lambda,C_1}(q^{-1}\lambda(x_L)).
\]

We divide \(\chi_\lambda\) into those \(\mu\) with \(\mu|_{\bar{T}_L} = \lambda|_{\bar{T}_L}\) and the remainder, which gives

\[
\chi_\lambda(x) = \chi_{L,\lambda}(x) + \sum_{\alpha \not\in \theta} \sum_{\mu \leq \lambda - \alpha \atop W \mu \leq \lambda} C(\alpha, \lambda, \mu) \mu(x)
\]

\[
= \lambda(x_L) \left( \chi_{L,\lambda}(x^L) + \sum_{\alpha \not\in \theta} \sum_{\mu \leq \lambda - \alpha \atop W \mu \leq \lambda} C(\alpha, \lambda, \mu)(\mu - \lambda)(x_L) \mu(x^L) \right).
\]

We have

\[
\sum_{\alpha \not\in \theta} \sum_{\mu \leq \lambda - \alpha \atop W \mu \leq \lambda} C(\alpha, \lambda, \mu)(\mu - \lambda)(x_L) \mu(x^L) \ll_{\lambda,C_1} \min_{\alpha \not\in \theta} |\alpha(x_L)|^{-1} \leq C_2^{-1},
\]

and combined with (36) this gives

\[
S_{\omega \lambda}(x) = \lambda(x_L) \left( \chi_{L,\lambda}(x^L) + O_{\lambda,C_1}(C_2^{-1}) + O_{\lambda,C_1}(q^{-1}) \right).
\]

(37)
We may assume that (37) applies uniformly to all $\mu \in \mathcal{X}$. By choosing $C_2$ and $C_3$ sufficiently large, we see that there exists $\mu \in \mathcal{X}$ such that $|S_\omega \mu(x)| > \delta/2$, which completes the proof. 

\[ \Box \]

**Proposition 6.6.** There exists a finite set $0 \notin \mathcal{X} \subset X_*(T)^+$, $C > 0$ and $\delta > 0$ such that if $x \in \hat{T}$ and $q > C$ we have

$$\max_{\mu \in \mathcal{X}} |S_\omega \mu(x)| > \delta.$$ 

**Proof.** We define two collections of constants $C_1(\theta)$ and $C_2(\theta)$ for $\theta \subseteq \hat{\Phi}$. If $C_1(\theta)$ has been defined, we let $C_2(\theta)$ be the value of $C_2$ produced by applying Proposition 6.5 to $\theta$ with $C_1 = C_1(\theta)$. We let $C_1(\emptyset) = 1$, and if $\theta \neq \emptyset$ we define

$$C_1(\theta) = \max_{\theta' \subseteq \theta} \{ C_1(\theta'), C_2(\theta') \}.$$ 

Apply Proposition 6.5 to each $\theta \subseteq \hat{\Phi}$ with $C_1 = C_1(\theta)$, and let $\mathcal{X}_\theta$, $C_3(\theta)$ and $\delta(\theta)$ be the sets of characters and constants produced. Define

$$\mathcal{X} = \bigcup_{\theta \subseteq \Phi} \mathcal{X}_\theta, \quad C = \max_{\theta \subseteq \Phi} C_3(\theta), \quad \delta = \min_{\theta \subseteq \Phi} \delta(\theta).$$ 

We claim that these data satisfy the conditions of the proposition. Let $x \in \hat{T}$, and let $\theta_x$ be maximal subject to the condition that $|\alpha(x)| \leq C_1(\theta_x)$ for $\alpha \in \theta_x$. We claim that $|\alpha(x)| > C_2(\theta_x)$ for $\alpha \notin \theta_x$. If not, there is some $\beta \notin \theta_x$ such that $|\beta(x)| \leq C_2(\theta_x)$. We have

$$C_1(\theta_x \cup \{\beta\}) \geq \max\{C_1(\theta_x), C_2(\theta_x)\}$$

so that $|\alpha(x)| \leq C_1(\theta_x \cup \{\beta\})$ for $\alpha \in \theta_x \cup \{\beta\}$, contradicting the maximality of $\theta_x$. We may therefore apply Proposition 6.5 to the set $\theta_x$ and the constant $C_1(\theta_x)$, which completes the proof. 

\[ \Box \]

7. Verification of $K$-smallness

We now prove Theorem 1.1. As $K$-smallness is invariant under isogeny, it suffices to establish it for the split real classical groups and the split form of $G_2$.

7.1. $SL_n(\mathbb{R})$. As $K$-smallness is unchanged under extension of scalars, we may let $K$ be the split special orthogonal group preserving the standard anti-diagonal quadratic form.

First suppose that $n = 2k$ is even, so we are considering $SO(k, k) \subset SL_n(\mathbb{R})$. We choose $T_K$ and $T$ to be the standard diagonal tori, and consider a cocharacter

$$\mu : t \mapsto \begin{pmatrix} t^{x_1} \\
\vdots \\
t^{x_k} \\
t^{-x_k} \\
\vdots \\
t^{-x_1} \end{pmatrix}$$

25
of $T_K$. To calculate $\|\mu\|^*$, we must take the numbers $\pm x_1, \ldots, \pm x_k$, arrange them in decreasing order, and sum them with the weights $2k, 2k - 2, \ldots, 1$. To calculate $\|\mu\|^*_K$, we must take $x_1, \ldots, x_k$, make an even number of sign changes so that the set of negative terms is either empty or consists of the term with $|x_i|$ smallest, arrange them in decreasing order, and sum them with the weights $2k - 2, 2k - 4, \ldots, 0$. Both $\|\mu\|^*$ and $\|\mu\|^*_K$ are unchanged if we replace $x_1, \ldots, x_k$ with $|x_1|, \ldots, |x_k|$, so we shall assume that all $x_i \geq 0$. We may also assume that they are in decreasing order. We then have

$$\|\mu\|^* = (4k - 2)x_1 + (4k - 6)x_2 + \ldots + 2x_k$$

$$\|\mu\|^*_K = (2k - 2)x_1 + (2k - 4)x_2 + \ldots + 2x_{k-1}.$$  

This gives

$$\|\mu\|^* - 2\|\mu\|^*_K = 2x_1 + 2x_2 + \ldots + 2x_k,$$

so that $\|\mu\|^* \geq 2\|\mu\|^*_K$ with equality iff $\mu$ is 0, as required.

Now suppose $n = 2k + 1$ is odd, so we are considering $SO(k, k + 1) \subset SL_n(\mathbb{R})$. We again choose $T_K$ and $T$ to be diagonal, and let

$$\mu : t \mapsto \begin{pmatrix} t^{x_1} & \cdots & t^{x_k} \\ & 1 \\ & \ddots \\ & & t^{-x_1} \end{pmatrix}$$

be a cocharacter of $T_K$. We obtain $\|\mu\|^*$ by arranging the numbers $\pm x_1, \ldots, \pm x_k, 0$ in decreasing order and summing with the weights $2k, 2k - 2, \ldots, -2k$. We obtain $\|\mu\|^*_K$ by arranging $|x_1|, \ldots, |x_k|$ in decreasing order and summing with the weights $2k - 1, 2k - 3, \ldots, 1$. Both quantities are unchanged by replacing $x_1, \ldots, x_k$ with a decreasing rearrangement of $|x_1|, \ldots, |x_k|$, and after doing this we have

$$\|\mu\|^* = 4kx_1 + (4k - 4)x_2 + \ldots + 4x_k$$

$$\|\mu\|^*_K = (2k - 1)x_1 + (2k - 3)x_2 + \ldots + x_k.$$  

This gives

$$\|\mu\|^* - 2\|\mu\|^*_K = 2x_1 + 2x_2 + \ldots + 2x_k,$$

so that $\|\mu\|^* \geq 2\|\mu\|^*_K$ with equality iff $\mu$ is 0, as required.

7.2. $Sp_{2n}(\mathbb{R})$. We shall let $Sp_{2n}(\mathbb{R})$ be the group preserving the form

$$\begin{pmatrix} I_n \\ -I_n \end{pmatrix}.$$  

After extending scalars to $\mathbb{C}$, the embedding $U(n) \subset Sp_{2n}(\mathbb{R})$ is conjugate to the embedding $GL_n(\mathbb{R}) \subset Sp_{2n}(\mathbb{R})$ given by
\[ A \mapsto \begin{pmatrix} A & tA^{-1} \end{pmatrix}, \]

and so we shall work with this embedding instead. We choose \( T_K \) and \( T \) to be diagonal, and let \( \mu \in X_*(T_K) \) map \( t \) to the diagonal matrix \((t^{x_1}, \ldots, t^{x_n})\). We obtain \( \|\mu\|_K^* \) by taking a decreasing rearrangement of \( x_1, \ldots, x_n \) and summing with weights \( n - 1, n - 3, \ldots, 1 - n \). We obtain \( \|\mu\|^* \) by taking a decreasing rearrangement of \(|x_1|, \ldots, |x_n|\) and summing with weights \( 2n, 2(n - 1), \ldots, 2 \).

Let the sequence \( y_1, \ldots, y_n \) be a decreasing rearrangement of \(|x_1|, \ldots, |x_n|\). We let \( z_1, \ldots, z_j \) and \( w_1, \ldots, w_{n-j} \) be the subsequences of terms for which \( x_i \geq 0 \) and \( x_i < 0 \) respectively, so that \( z_1, \ldots, z_j, -w_{n-j}, \ldots, -w_1 \) is the decreasing rearrangement of \( x_1, \ldots, x_n \). We have

\[ \|\mu\|^* = 2ny_1 + 2(n - 1)y_2 + \ldots + 2y_n, \]

and

\[ \|\mu\|_K = (n - 1)z_1 + (n - 3)z_2 + \ldots + (n - 2j + 1)z_j - (n - 2j - 1)w_{n-j} - \ldots - (1 - n)w_1. \]

We therefore see that \( \|\mu\|_K^* \) is given by adding up \( y_1, \ldots, y_n \) with the weights \( n - 1, \ldots, n + 1 - 2j \) and \( n - 1, \ldots, n + 1 - 2(n - j) \) in some order. This is maximised by choosing \( j = \lfloor n/2 \rfloor \), so that

\[ mu\|_K^* \leq (n - 1)(y_1 + y_2) + (n - 3)(y_3 + y_4) + \ldots + (n - 2j - 1)(y_{2j-1} + y_{2j}). \]

This gives

\[ \|\mu\|^* - 2\|\mu\|_K^* \geq 2y_1 + 2y_3 + \ldots + 2y_{2j-1} \geq 0, \]

with equality iff \( \mu = 0 \).

7.3. \( SO(2k, 2k) \). We may switch to the embedding \( SO(k, k) \times SO(k, k) \subset SO(2k, 2k) \). We realise \( SO(2k, 2k) \) as the group preserving the standard anti-diagonal quadratic form, and the two copies of \( SO(k, k) \) as the subgroups embedded as the central \( 2k \times 2k \) block and the four \( k \times k \) corner blocks. We let \( T \) and \( T_K \) be diagonal, and let \( \mu \in X_*(T_K) \) map \( t \) to the diagonal matrix \((t^{x_1}, \ldots, t^{x_k}, t^{y_1}, \ldots, t^{y_k}, t^{-y_k}, \ldots, t^{-y_1}, t^{-x_k}, \ldots, t^{-x_1})\). We obtain \( \|\mu\|_K^* \) by rearranging \( x_1, \ldots, x_k \) in decreasing order and changing an even number of signs so that \( x_{k-1} + x_k \geq 0 \), and likewise for \( y_1, \ldots, y_k \), and then letting

\[ \|\mu\|_K = (2k - 2)x_1 + \ldots + 2x_{k-1} + (2k - 2)y_1 + \ldots + 2y_{k-1}. \]

We obtain \( \|\mu\|^* \) by rearranging and making sign changes to the concatenation of \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \) in the same way, and then summing with weights \( 4k - 2, 4k - 4, \ldots, 2, 0 \).

We may assume without loss of generality that \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \) are already in decreasing order, and satisfy \( x_{k-1} + x_k \geq 0 \) and \( y_{k-1} + y_k \geq 0 \). We let \( z_1, \ldots, z_{2k} \) be the sequence obtained by concatenating \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \) and then taking the decreasing rearrangement. We have

\[ \|\mu\|^* \geq (4k - 2)z_1 + (4k - 4)z_2 + \ldots + 2z_{2k-1} \]

and
\[ \|\mu\|_K^* \leq (2k - 2)(z_1 + z_2) + (2k - 4)(z_3 + z_4) + \ldots + 2(z_{2k-3} + z_{2k-2}), \]
which gives
\[ \|\mu\| - 2\|\mu\|_K^* \geq 2z_1 + 2z_3 + \ldots + 2z_{2k-1}. \]
If \( z_{2k-1} \geq 0 \), then we are done. If \( z_{2k-1} < 0 \), we may change the sign of both \( z_{2k-1} \) and \( z_2 \) to obtain
\[ \|\mu\| - 2\|\mu\|_K^* \geq 2z_1 + \ldots + 2z_{2k-3} - 2z_{2k-1}, \]
which gives
\[ \|\mu\| - 2\|\mu\|_K^* \geq 2z_1 + \ldots + 2z_{2k-3} - 2z_{2k-1}. \]
As \( z_{2k-3} \geq 0 \), this gives \( \|\mu\| - 2\|\mu\|_K^* > 0 \) and we are done.

7.4. \( SO(2k, 2k+1) \). We switch to \( SO(k, k) \times SO(k, k+1) \subset SO(2k, 2k+1) \), where \( SO(2k, 2k+1) \) preserves the anti-diagonal form, \( SO(k, k+1) \) is embedded as the central \( (2k+1) \times (2k+1) \) block, and \( SO(k, k) \) as the corner blocks. We let \( \mu \in X_*(\mathcal{L}_K) \) send \( t \) to the diagonal matrix \( (t^{x_1}, \ldots, t^{x_k}, t^{y_1}, \ldots, t^{y_k}, 1, t^{-y_k}, \ldots, t^{-y_1}, t^{-x_k}, \ldots, t^{-x_1}) \). We obtain \( \|\mu\|_K^* \) by rearranging \( x_1, \ldots, x_k \) in decreasing order and changing an even number of signs so that \( x_{k-1} + x_k \geq 0 \), rearranging \( y_1, \ldots, y_k \) in decreasing order and changing signs so that all terms are positive, and taking
\[ \|\mu\|_K^* = (2k - 2)x_1 + \ldots + 2x_{k-1} + (2k - 1)y_1 + \ldots + y_k. \]
We obtain \( \|\mu\|^* \) by concatenating \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \), making all terms positive and rearranging in decreasing order, and summing with weights \( 4k - 1, 4k - 3, \ldots, 1 \).

We may assume without loss of generality that \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \) are already in decreasing order, and satisfy \( x_{k-1} + x_k \geq 0 \) and \( y_k \geq 0 \). We let \( z_1, \ldots, z_{2k} \) be the sequence obtained by concatenating \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \) and then taking the decreasing rearrangement. We have
\[ \|\mu\|^* \geq (4k - 1)z_1 + (4k - 3)z_2 + \ldots + z_{2k} \]
and
\[ \|\mu\|^*_K \leq (2k - 1)z_1 + (2k - 2)z_2 + \ldots + z_{2k-1}, \]
which gives
\[ \|\mu\|^* - 2\|\mu\|^*_K \geq z_1 + z_2 + \ldots + z_{2k}. \]
If \( z_{2k} \geq 0 \), we are done. If \( z_{2k} < 0 \), we may replace \( z_{2k} \) with \( -z_{2k} \), which gives
\[ \|\mu\|^* \geq (4k - 1)z_1 + (4k - 3)z_2 + \ldots + z_{2k-1} - z_{2k} \]
7.5. $SO(2k+1, 2k+1)$. We switch to $SO(k, k+1) \times SO(k+1, k) \subset SO(2k+1, 2k+1)$. Let $V_1$ and $V_2$ be real quadratic spaces of signature $(k, k+1)$ and $(k+1, k)$ respectively, and let $V = V_1 \oplus V_2$. Let $\mu \in X_*(T_K)$. We may choose bases of $V_1$ and $V_2$ so that $\mu$ maps $t$ to the pair of diagonal matrices with entries $(t^{x_1}, \ldots, t^{x_k}, 1, t^{-x_k}, \ldots, t^{-x_1})$ and $(t^{y_1}, \ldots, t^{y_k}, 1, t^{-y_k}, \ldots, t^{-y_1})$ respectively. If $x_1', \ldots, x_k'$ and $y_1', \ldots, y_k'$ are decreasing rearrangements of $|x_1|, \ldots, |x_k|$ and $|y_1|, \ldots, |y_k|$, we have

$$\|\mu\|^*_K = (2k-1)x_1' + \ldots + x_k' + (2k-1)y_1' + \ldots + y_k'.$$

When we map $\mu$ into $X_*(T)$, we are free to change an arbitrary number of signs of the $x_i$ and $y_i$ due to the presence of the two 1's. We may therefore let $z_1', \ldots, z_{2k}'$ be a decreasing rearrangement of $|x_1|, \ldots, |x_k|, |y_1|, \ldots, |y_k|$, and obtain

$$\|\mu\|^* = 4kz_1' + (4k - 2)z_2' + \ldots + 2z_{2k}'. $$

We have

$$\|\mu\|^*_K \leq (2k-1)(z_1' + z_2') + \ldots + (z_{2k-1}' + z_{2k}'),$$

so that

$$\|\mu\|^* - 2\|\mu\|^*_K \geq 2z_1' + 2z_3' + \ldots + 2z_{2k-1}'$$

as required.

7.6. $SU(2) \times SU(2)/\{\pm I\} \subset G_2(\mathbb{R})$. Choose tori $T_1$ and $T_2$ in each copy of $SU(2)$, and let $T_K = T = T_1 \times T_2$. Choose isomorphisms $T_1 \simeq T_2 \simeq \mathbb{G}_m$ over $\mathbb{C}$. By switching factors if necessary, the weights $\Delta_t$ of the adjoint representation of $K$ on $t$ are $(\pm 2, 0)$ and $(0, \pm 2)$, and the weights $\Delta_p$ on $p$ are $(\pm 1, \pm 1)$ and $(\pm 1, \pm 3)$ with respect to the standard basis of $X^*(T)$. Let $\mu \in X_*(T)$. Because $W_K \subset W$, we may assume without loss of generality that $\mu = (x_1, x_2)$ with $x_1, x_2 \geq 0$, in which case $\|\mu\|^*_K = 2x_1 + 2x_2$. Acting on $\mu$ by $W$ is equivalent to choosing different systems of positive roots. Two possible choices are

$$\Delta^+ = \{(2, 0), (0, 2), (1, \pm 1), (1, \pm 3)\},$$

$$\Delta^+ = \{(2, 0), (0, 2), (1, \pm 1), (1, 3), (-1, 3)\},$$

which give $\|\mu\|^* \geq 6x_1 + 2x_2$ and $\|\mu\|^* \geq 4x_1 + 8x_2$ respectively. Averaging these gives $\|\mu\|^* \geq 5x_1 + 5x_2 \geq 2\|\mu\|^*_K$, with equality iff $\mu = 0$.

REFERENCES

[1] G. V. Avacumović: Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten, Math. Z. 65 (1956), 327-344.
[2] P. H. Bérard: On the wave equation on a compact manifold without conjugate points, Math. Z. 155 (1977), 249-276.
[3] V. Blomer, G. Harcos, D. Miličević: Bounds for eigenforms on arithmetic hyperbolic 3-manifolds, preprint.
[4] V. Blomer, R. Holowinsky: Bounding sup-norms of cusp forms of large level, Invent. Math. 179 (2010), no. 3, 645-681.
[5] V. Blomer, P. Mága: The sup-norm problem for PGL(4), preprint.
[6] V. Blomer, P. Mága: Subconvexity for sup-norms of automorphic forms on PGL(n), preprint.
V. Blomer, P. Michel: *Sup-norms of eigenfunctions on arithmetic ellipsoids*, IMRN 2011 no.21 (2011), 4934-4966.

V. Blomer, P. Michel: *Hybrid bounds for automorphic forms on ellipsoids over number fields*, J. Inst. Math. Jussieu 12 (2013), 727-758.

V. Blomer, A. Pohl: *The sup-norm problem on the Siegel modular space of rank two*, preprint.

R. Holowinsky, G. Ricotta, E. Royer: *On the sup norm of an $SL(3)$ Hecke-Maass cusp form*, preprint.

R. Gangolli: *On the plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. 93, no. 1 (1971), 105-165.

R. Gross: *On the Satake isomorphism*, in: Galois representations in arithmetic algebraic geometry. LMS Lecture Notes, 254, (1998), 223-238.

G. Harcos, N. Templier: *On the sup-norm of Maass cusp forms of large level. II*, Int. Math. Res. Not. (2011).

G. Harcos, N. Templier: *On the sup-norm of Maass cusp forms of large level. III*, Math. Ann., 356 no.1 (2013) 209-216.

H. Iwaniec, P. Sarnak: $L^\infty$ norms of eigenfunctions of arithmetic surfaces, Ann. of Math. (2) 141 (1995), 301-320.

B. M. Levitan: *On the asymptotic behavior of the spectral function of a self-adjoint differential equation of second order*, Isv. Akad. Nauk SSSR Ser. Mat. 16 (1952), 325-352.

S. Marshall: $L^p$ norms of higher rank eigenfunctions and bounds for spherical functions, preprint.

S. Marshall: *Geodesic restrictions of arithmetic eigenfunctions*, preprint.

S. Marshall: *Restrictions of $SL_3$ Maass forms to maximal flat subspaces*, preprint.

P. Sarnak: *Letter to Morawetz*, available at [http://www.math.princeton.edu/sarnak/](http://www.math.princeton.edu/sarnak/)

C. Sogge, S. Zelditch: *Riemannian manifolds with maximal eigenfunction growth*, Duke Math J.

N. Templier: *On the sup-norm of Maass cusp forms of large level*, Selecta Math. 16 vol. 3 (2010), 501-531.

N. Templier: *Hybrid sup-norm bounds for Hecke-Maass cusp forms*, J. Eur. Math. Soc, to appear.

J. Toth, S. Zelditch: *Riemannian manifolds with uniformly bounded eigenfunctions*, Duke Math. J. 111 (2002), 97-132.

N. Wallach: *On a theorem of Milnor and Thom*, Progress in Nonlinear Differential Equations, 20 (1996), 331-348.

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