Chromatic number and minimum degree of $K_r$-free graphs

Vladimir Nikiforov

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152
email: vnikifrv@memphis.edu

January 13, 2010

Abstract

Let $\delta(G)$ be the minimum degree of a graph $G$. A number of famous results about triangle-free graphs determine the maximum chromatic number of graphs of order $n$ with $\delta(G) > n/3$. In this note these results are extended to $K_{r+1}$-free graphs of order $n$ with $\delta(G) > (1 - 2/(2r - 1))n$. In particular:

(a) there exist $K_{r+1}$-free graphs of order $n$ with $\delta(G) > (1 - 2/(2r - 1))n - o(n)$ and arbitrary large chromatic number;

(b) if $G$ is a $K_{r+1}$-free graph of order $n$ with $\delta(G) > (1 - 2/(2r - 1))n$, then $\chi(G) \leq r + 2$;

(c) the structure of the $(r + 1)$-chromatic $K_{r+1}$-free graphs of order $n$, with $\delta(G) > (1 - 2/(2r - 1))n$ is found.

Keywords: $K_r$-free graph; minimum degree; chromatic number; Andrásfai graph; Hajnal graph.

1 Introduction and main results

In notation we follow [3]. In 1962 Andrásfai [2] introduced the function

$$\psi(n, r, h) = \max \{\delta(G) : G \text{ is a } K_{r+1}\text{-free graph of order } n \text{ with } \chi(G) \geq h\},$$

which has been widely studied during the years. One of the first contributions were the famous theorem and example of Andrásfai, Erdős and Sós [1] showing that for every $r \geq 2$,

$$\left(1 - \frac{3}{3r - 1}\right)n + O(1) \leq \psi(n, r, r) \leq \left(1 - \frac{3}{3r - 1}\right)n. \quad (1)$$

Another milestone along this road is an example of Hajnal [9] showing that $\psi(n, 2, h) > n/3 - o(n)$ for every $h \geq 3$; for an updated version of Hajnal’s example see Example 8 below. Thus, for $r = 2$, this example leaves unanswered only one simple, yet tricky question: how large can be $\chi(G)$ of a $K_3$-free graph $G$ of order $n$ with $\delta(G) > n/3$.

Erdős and Simonovits [9] conjectured that all $K_3$-free graphs of order $n$ with $\delta(G) > n/3$ are 3-chromatic, but this was disproved by Haggkvist [10], who described for every $k \geq 1$ a $10k$-regular,
4-chromatic, \( K_3 \)-free graph of order \( 29k \), for a description see Example \[10\] below. The example of H"aggkvist is based on the Mycielski graph \( M_3 \), also known as the Gr"otzsch graph, which is a 4-chromatic \( K_3 \)-free graph of order 11. We shall see later that \( M_3 \) is a true landmark in this area, but let us first recall the Mycielski graphs given in \[15\]: a sequence \( M_1, M_2, \ldots \) of \( K_3 \)-free graphs with \( \chi (M_i) = i + 1 \), constructed as follows:

Set \( M_1 = K_2 \). To obtain \( M_{i+1} \): write \( v_1, \ldots, v_n \) for the vertices of \( M_i \); choose \( n+1 \) other vertices \( u_1, \ldots, u_{n+1} \); for every \( i \in [n] \) join \( u_i \) precisely to the neighbors of \( v_i \); join \( u_{n+1} \) to \( u_1, \ldots, u_n \).

Other graphs crucial in the study of \( \psi (n, r, h) \) are the \( K_3 \)-free 3-chromatic Andrásfai graphs \( A_1, A_2, \ldots \), first described in \[2\]:

Set \( A_1 = K_2 \) and for every \( i \geq 2 \) let \( A_i \) be the complement of the \( (i - 1) \)’th power of \( C_{3i-1} \).

In \[11\], Jin proved the following theorem, generalizing the case \( r = 2 \) of the theorem of Andrásfai, Erdős and Sós and a result of H"aggkvist from \[10\].

**Theorem A** Let \( 1 \leq k \leq 9 \), and let \( G \) be a \( K_3 \)-free graph of order \( n \). If

\[
\delta (G) > \frac{k + 1}{3k + 2} n,
\]

then \( G \) is homomorphic to \( A_k \).

Note that this result is tight: taking the graph \( A_{k+1} \), and blowing it up by a factor \( t \), we obtain a \( K_3 \)-free graph \( G \) of order \( n = (3k + 2) t \) vertices, with \( \delta (G) = (k + 1) n / (3k + 2) \), which is not homomorphic to \( A_k \). Note also that all graphs satisfying the premises of Theorem A are 3-chromatic. Addressing this last issue, Jin \[12\], and Chen, Jin and Koh \[8\] gave a finer characterization of \( K_3 \)-free graphs with \( \delta > n/3 \).

**Theorem B** Let \( G \) be a \( K_3 \)-free graph of order \( n \), with \( \delta (G) > n/3 \). If \( \chi (G) \geq 4 \), then \( M_3 \subseteq G \). If \( \chi (G) = 3 \) and

\[
\delta (G) > \frac{k + 1}{3k + 2} n,
\]

then \( G \) is homomorphic to \( A_k \).

Later Brandt and Pisanski \[4\] found an infinite family of 4-chromatic, \( K_3 \)-free graphs with \( \delta (G) > n/3 \). All these interesting results shed some light on the structure of dense \( K_3 \)-free graphs, but could not answer the original question of Erdős and Simonovits. The answer was given by Brandt and Thomassé \[7\] in the following ultimate result, culminating the series \[6\], \[12\] and \[16\].

**Theorem C** Let \( G \) be a \( K_3 \)-free graph of order \( n \). If \( \delta (G) > n/3 \), then \( \chi (G) \leq 4 \).

This theorem essentially concludes the study of \( \psi (n, 2, h) \). Although there are still unsettled questions about 4-chromatic \( K_3 \)-free graphs \( G \) of order \( n \) with \( \delta (G) > n/3 \), the broad picture is already fixed. The goal of this paper is to conclude likewise the study of \( \psi (n, r, h) \) for \( r \geq 3 \).

To this end, in Section 3.1, we extend the example of Hajnal and show that

\[
\psi (n, r, h) = (1 - 2 / (2r - 1)) n - o (n).
\]  (2)
for every $h > r + 2$. This is to say, for every $\varepsilon$ there exists a $K_{r+1}$-free graph of order $n$ with

$$\delta(G) > \left(1 - \frac{2}{2r-1} - \varepsilon\right)n$$

and arbitrary large chromatic number, provided $n$ is sufficiently large. We believe that for $r \geq 3$ this extension is not widely known, although its main idea is the same as for $r = 2$.

Thus, from this point on, we are concerned mainly with the question: how large can be $\chi(G)$ of a $K_{r+1}$-free graph $G$ of order $n$ with $\delta(G) > (1 - 2/(2r - 1))n$. To give the reader an immediate clue we first state an extension of Theorem C.

**Theorem 1** Let $r \geq 2$ and $G$ be a $K_{r+1}$-free graph of order $n$. If

$$\delta(G) > \left(1 - \frac{2}{2r-1}\right)n,$$

then $\chi(G) \leq r + 2$.

This theorem leaves only two cases of $\chi(G)$ to investigate, viz., $\chi(G) = r + 1$ and $\chi(G) = r + 2$. As one can expect, when $\delta(G)$ is sufficiently large, we have $\chi(G) = r + 1$. The precise statement extends Theorem A as follows.

**Theorem 2** Let $r \geq 2$, $1 \leq k \leq 9$, and let $G$ be a $K_{r+1}$-free graph of order $n$. If

$$\delta(G) > \left(1 - \frac{2k-1}{(2k-1)r-k+1}\right)n$$

then $G$ is homomorphic to $A_k + K_{r-2}$.

As a corollary, under the premises of Theorem 2 we find that $\chi(G) \leq r + 1$. Also Theorem 2 is best possible in the following sense: for every $k$ and $n$ there exists a $(r + 1)$-chromatic $K_{r+1}$-free $G$ of order $n$ with

$$\delta(G) \geq \left(1 - \frac{2k-1}{(2k-1)r-k+1}\right)n - 1$$

that is not homomorphic to $A_k + K_{r-2}$. This example is given in Section 3.3.

We also generalize the example of Häggkvist, constructing for every $n$ an $(r + 2)$-chromatic, $K_{r+1}$-free graph $G$ with

$$\delta(G) \geq \left(1 - \frac{19}{19r-9}\right)n - 1,$$

which shows that the conclusion of Theorem 2 does not necessarily hold for $k \geq 10$. This example is given in Section 3.2.

To give some further structural information, we extend Theorem C as follows.
Theorem 3  Let \( r \geq 2 \) and \( G \) be a \( K_{r+1} \)-free graph of order \( n \) with

\[
\delta (G) > \left( 1 - \frac{2}{2r-1} \right) n.
\]

If \( \chi (G) \geq r + 2 \), then \( M_3 + K_{r-2} \subset G \). If \( \chi (G) \leq r + 1 \) and

\[
\delta (G) > \left( 1 - \frac{2k - 1}{(2k - 1)r - k + 1} \right) n
\]

then \( G \) is homomorphic to \( A_k + K_{r-2} \).

This result is best possible in view of the examples described prior to Theorem 3.

For \( r \geq 3 \) we obtain the following summary for \( \psi (n, r, h) \):

\[
(1 - 3/ (3r - 1)) n - 1 \leq \psi (n, r, r + 1) \leq (1 - 3/ (3r - 1)) n
\]

\[
(1 - 19/ (19r - 9)) n - 1 \leq \psi (n, r, r + 2) \leq (1 - 19/ (19r - 9)) n
\]

\[
\psi (n, r, h) = (1 - 2/ (2r - 1)) n - o(n) \text{ for all } h > r + 2.
\]

About the proof method

We deduce the proofs of Theorems 1, 2 and 3 by induction on \( r \) from Theorems C, A and B respectively. Although this method seems simple and natural, to our best knowledge none of Theorems 1, 2 and 3 has been mentioned in the existing literature. The same is true for the extensions of the examples of Hajnal, H"aggkvist and Andrásfai.

The induction step, carried uniformly in all the three proofs, is based on the crucial Lemma 4.

This lemma can be applied immediately to extend other results about triangle-free graphs, but we leave these extensions to the interested reader.

2 Proofs

For a graph \( G \) and a vertex \( u \in V(G) \) we write \( \Gamma (u) \) for the set of neighbors of \( u \), and \( d_G (u) \) for \( |\Gamma (u)| \). If \( U \) is a subgraph of \( G \), we set- \( \Gamma (U) = \cap_{x \in U} \Gamma (x) \) and \( d (U) = |\Gamma (U)| \); note that this is not the usual definition.

Our main proof device is the following lemma.

Lemma 4  Let \( r \geq 3 \) and \( G \) be a maximal \( K_{r+1} \)-free graph of order \( n \). If

\[
\delta (G) > \left( 1 - \frac{2}{2r-1} \right) n,
\]

then \( G \) has a vertex \( u \) such that \( e (G'_u) = 0 \).
Proof For short, set $\delta = \delta (G)$ and $V = V (G)$. Write $\delta' (H)$ for the minimum nonzero vertex degree in a graph $H$, and set $\delta' = \min \{ \delta' (G_u) : u \in V \}$.

We start with two facts which will be used several times throughout the proof. First, for every set $S$ of $s \leq r - 1$ vertices, we have

$$d (S) \geq \sum_{v \in S} d (v) - (s - 1) n \geq s \delta - (s - 1) n \geq (r - 1) \delta - (r - 2) n$$

$$\geq (r - 1) (1 - 2 / (2r - 1)) n - (r - 2) n = n / (2r - 1) > 0;$$

thus, every clique is contained in an $r$-clique.

Second, since $G$ is a maximal $K_{r+1}$-free graph, there is an $(r - 1)$-clique in the common neighborhood $\Gamma (uw)$ of every two nonadjacent vertices $u$ and $w$.

Now, for a contradiction, assume the conclusion of the lemma false: let $E (G_u) \neq \emptyset$ for every $u \in V (G)$. For convenience the first part of the proof is split into separate claims.

Claim 5 For every edge $uw \in E (G)$ we have $d (uw) \leq (r - 2) (n - \delta) - \delta'$.

Proof Let $uw \in E (G)$, select an $(r - 1)$-clique $R$ containing $uw$, and let $w \in \Gamma (R)$. Since $G$ is $K_{r+1}$-free, $\Gamma (R)$ is an independent set, and so $\Gamma (R) \subset G'_w$. It is easy to see that $G'_w$ contains at least $\delta'$ vertices which do not belong to $\Gamma (R)$. Indeed, by assumption $G'_w$ contains edges. If some of these edges contains a vertex $t \in \Gamma (R)$, we have $\Gamma_{G'_w} (t) \cap \Gamma (R) = \emptyset$, and the assertion follows since $d_{G'_w} (t) \geq \delta'$. If no edge of $G'_w$ is incident to $\Gamma (R)$, there are at least $\delta' + 1$ vertices of $G'_w$ which do not belong to $\Gamma (R)$. Hence,

$$d (R) \leq n - d (w) - \delta' \leq n - \delta - \delta'. $$

Letting $S = R - u - v$, we have

$$d (S) \geq (r - 3) \delta - (r - 4) n,$$

and so

$$n - \delta - \delta' \geq d (R) \geq d (uv) + d (S) - n \geq d (uv) - (r - 3) (n - \delta),$$

as claimed. \qed

Claim 6 For every $u \in V$ and $w \in V (G'_w)$ we have

$$d (uw) \geq d (u) + d (w) - (r - 1) (n - \delta).$$

Proof Since $G$ is a maximal $K_{r+1}$-free graph, we can select an $(r - 1)$-clique $R \subset \Gamma (uw)$. Since $G$ is $K_{r+1}$-free we have $\Gamma (u) \cap \Gamma (R) = \emptyset$ and $\Gamma (w) \cap \Gamma (R) = \emptyset$, implying that $\Gamma (u) \cup \Gamma (w) \subset V \setminus \Gamma (R)$, and so,

$$d (uw) = |\Gamma (u) \cap \Gamma (w)| = |\Gamma (u)| + |\Gamma (w)| - |\Gamma (u) \cup \Gamma (w)|$$

$$\geq d (u) + d (w) - |V \setminus \Gamma (R)| = d (u) + d (w) - n + d (R)$$

$$\geq d (u) + d (w) - (r - 1) (n - \delta),$$

completing the proof. \qed
Claim 7 For every vertex \( u \in V \) the graph \( G'_u \) is triangle-free.

Proof Assume the opposite: let \( u \in V \) and \( T \) be a triangle in \( G'_u \) with vertices \( v_1, v_2, v_3 \). Using Claim 6 we have

\[
d(T) = \vert \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \vert \geq \vert \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) \cap \Gamma(u) \vert \\
\geq d(v_1u) + d(v_2u) + d(v_3u) - 2d(u) \\
\geq 3(d(u) - (r - 1)(n - \delta)) + d(v_1) + d(v_2) + d(v_3) - 2d(u) \\
\geq d(u) + 3r\delta - 3(r - 1)n \\
\geq (3r + 1)\delta - 3(r - 1)n.
\]

Select an \( r \)-clique \( R \) containing \( T \), and let \( S = R - v_1 - v_2 - v_3 \). We have

\[
d(S) \geq (r - 3)\delta - (r - 4)n,
\]

and hence,

\[
d(R) \geq d(T) + d(S) - n > (3r + 1)\delta - 3(r - 1)n + (r - 3)\delta - (r - 3)n \\
\geq (4r - 2)\delta - (4r - 6)n > (4r - 2)\left(\frac{2r - 3}{2r - 1}\right)n - (4r - 6)n = 0.
\]

Therefore \( d(R) > 0 \), and so, \( K_{r+1} \subseteq G \), a contradiction completing the proof of the claim. \( \square \)

Let \( u \in V \) and \( vw \in E(G'_u) \) be such that \( \delta' = d_{G'_u}(v) \). Since \( G'_u \) is triangle-free, the set \((\Gamma(v) \cap \Gamma(w)) \setminus \Gamma(u) \) is empty, and so

\[
d_{G'_u}(v) + d_{G'_u}(w) = \vert \Gamma(v) \setminus \Gamma(u) \vert + \vert \Gamma(w) \setminus \Gamma(u) \vert = \vert \Gamma(u) \cup \Gamma(v) \setminus \Gamma(u) \vert \\
\geq \vert \Gamma(u) \cup \Gamma(w) \vert - d(u) = d(v) + d(w) - \vert \Gamma(v) \cap \Gamma(w) \vert - d(u) \\
\geq 2\delta - \vert \Gamma(v) \cap \Gamma(w) \vert - d(u).
\]

Now, estimating \( \vert \Gamma(v) \cap \Gamma(w) \vert \) by Claim 5 we find that

\[
\delta' + d_{G'_u}(w) \geq 2\delta - (r - 2)(n - \delta) + \delta' - d(u),
\]

and so,

\[
d_{G'_u}(w) \geq r\delta - (r - 2)n - d(u).
\]

On the other hand, using Claim 6 to estimate \( d(uw) \), we see that

\[
d_{G'_u}(w) = \vert \Gamma(w) \setminus \Gamma(u) \vert = d(w) - d(uw) \\
\leq d(w) - d(u) - d(w) + (r - 1)(n - \delta) \\
= -d(u) + (r - 1)(n - \delta).
\]

Therefore,

\[
-d(u) + (r - 1)(n - \delta) \geq d_{G'_u}(w) \geq r\delta - (r - 2)n - d(u),
\]
and so
\[(2r - 3) n \geq (2r - 1) \delta,\]
a contradiction, completing the proof of the lemma. \(\square\)

**Proof of Theorem 1** We shall show that \(G = H + G_0\), where \(G_0\) is an \((r - 2)\)-partite graph, and \(H\) is a \(K_3\)-free graph with \(\delta (H) > |H|/3\). We shall prove this assertion by induction on \(r\). For \(r = 2\) there is nothing to prove, so assume that the assertion holds for \(r' < r\). Add some edges to make \(G\) maximal \(K_{r+1}\)-free; \(\delta (G)\) can only increase, and \(G\) remains \(K_{r+1}\)-free. Lemma \(4\) implies that there is a vertex \(u \in V (G)\) such that \(G_u\) is empty. This means that \(G\) is homomorphic to \(G_u + N\), where \(N\) is the graph induced by the neighbors of \(u\), which is obviously \(K_r\)-free. We also have
\[
\delta (N) \geq \delta + d (u) - n > d (u) - \frac{2}{2r - 1} n > d (u) - \frac{2}{2r - 3} \delta
\]
\[
\geq \left(1 - \frac{2}{2r - 3}\right) |N|.
\]
By the induction hypothesis, \(N\) is a join of an \((r - 3)\)-partite graph \(N_0\), and a \(K_3\)-free graph \(H\) with \(\delta (H) > |H|/3\). Thus \(G = H + (N_0 + G_u)\), completing the induction step and the proof of the assertion. Since \(\chi (H) \leq 4\), it follows that \(\chi (G) \leq \chi (H) + r - 2 \leq r + 2\), completing the proof. \(\square\)

**Proof of Theorem 2** In this case we shall show that \(G = H + G_0\), where \(G_0\) is an \((r - 2)\)-partite graph, and \(H\) is a \(K_3\)-free graph with
\[
\delta (H) > \frac{k + 1}{2k + 1} |H|.
\]
This assertion follows as in the proof of Theorem \(1\). The only difference is given in the following calculation
\[
\delta (N) \geq \delta + d (u) - n > d (u) - \frac{2k - 1}{(2k - 1) r - k + 1} n
\]
\[
\geq d (u) - \frac{1}{(2k - 1) (r - 1) - k + 1} \delta
\]
\[
\geq \left(1 - \frac{1}{(2k - 1) (r - 1) - k + 1}\right) |N|.
\]
According to Theorem A, \(H\) is homomorphic to \(A_k\), and so \(G\) is homomorphic to \(A_k + K_{r-2}\), completing the proof. \(\square\)

The proof of Theorem \(3\) is the same as of Theorem \(2\) so we shall omit it.

### 3 Extension of some basic examples

Below we construct three types of graphs by the same simple method: we take the join of a known \(K_3\)-free graph and the \((r - 2)\)-partite Turán graph. Choosing appropriately the order of the two graphs, the resulting graph can be made almost regular.
3.1 Extending the example of Hajnal

In this section we shall construct, for every \( r \geq 2, h > 1, \varepsilon > 0 \) and \( n \) sufficiently large, a \( K_{r+1} \)-free graph \( G \) of order \( n \) with

\[
\delta(G) > \left(1 - \frac{2}{2r-1} - \varepsilon\right)n
\]

and \( \chi(G) > h \).

We start by an updated version of the example of Hajnal, reported in [9]: a \( K_3 \)-free graph \( G \) of order \( n \) with arbitrary large chromatic number and \( \delta(G) > n/3 - o(n) \).

Let \( K_{2m+h}^{(m)} \) be a Kneser graph: its vertices are the sets \( S \subset [2m+h] \) of size \( |S| = m \); two vertices \( S_1 \) and \( S_2 \) are joined if \( S_1 \cap S_2 = \emptyset \). Clearly if \( m > h \), the graph \( K_{2m+h}^{(m)} \) is \( K_3 \)-free. Kneser [13] conjectured and Lovász [14] proved that \( \chi(K_{2m+h}^{(m)}) = h+2 \).

**Example 8** Let \( K \) be a copy of \( K_{2m+h}^{(m)} \) and let \( S_1, \ldots, S_t \) be its vertices, where \( t = \binom{2m+h}{m} \). Let \( n \geq 3m+h+\binom{2m+h}{m} \), and set

\[
n_1 = n - \binom{2m+h}{m} \quad \text{and} \quad k = \left\lfloor \frac{n_1}{3m+h} \right\rfloor.
\]

Add additional \( n_1 \) vertices to \( K \) in the following way: add a set \( A \) of \( (2m+h)k \) vertices, indexed for convenience as \( v_{ij}, i \in [2m+h], j \in [k] \), and add a set \( B \) of additional \( n_1 - (2m+h)k \) vertices. Now join every vertex of \( A \) to every vertex of \( B \), and join every vertex \( v_{ij} \in A \) to every vertex \( S_l \) such that \( i \in S_l \). Write \( H(n,m,h) \) for the resulting graph.

We immediately see that \( v(H(n,m,h)) = n \) and that

\[
\chi(H(n,m,h)) \geq \chi(K_{2m+h}^{(m)}) = h+2.
\]

Let us check that \( H(n,m,h) \) is \( K_3 \)-free. Since no vertex in \( B \) is connected to a vertex in \( K \), and \( A \) and \( B \) are independent, after a brief inspection, we see that a triangle in \( H(n,m,h) \) must have an edge \( S_i S_j \) in \( K \) and a vertex \( v_{pq} \in A \); thus \( p \in S_i \) and \( p \in S_j \), and so \( S_i \cap S_j \neq \emptyset \), contrary to the assumption that \( S_i S_j \) is an edge in \( K \). Hence, \( H(n,m,h) \) is \( K_3 \)-free.

To estimate \( \delta(H(n,m,h)) \) observe that every set \( S_i \in V(K) \) is joined to \( mk \) vertices of \( A \); every vertex from \( B \) is joined to \( (2m+h)k \) vertices of \( A \) and every vertex of \( A \) is joined to \( n_1 - (2m+h)k \geq mk \) vertices of \( B \). Therefore, selecting \( m \) sufficiently large with respect to \( h \), we see that

\[
\delta(H(n,m,h)) \geq mk = m \left\lfloor \left(n - \binom{2m+h}{m}\right)/(3m+h) \right\rfloor = n/3 + o(n).
\]

Therefore \( H(n,m,h) \) has the required properties.

For \( r \geq 3 \) we construct our graph \( G \) as a join of a properly selected graph \( H(n',m',h') \) and an \((r-2)\)-partite Turán graph.
Example 9 Let $h > r$; select $n_0$ and $m$ such that, for $n_1 \geq n_0$, we have

$$\delta(H(n_1, m, h - r)) > n_1 \left( \frac{1}{3} - \frac{\varepsilon}{3} \right).$$

Assume that

$$n > \frac{2r - 1}{3}n_0;$$

set

$$G_1 = T_{r-2} \left( \left\lfloor \frac{2r - 4}{2r - 1}n \right\rfloor \right)$$

$$G_2 = H \left( n - \left\lfloor \frac{2r - 4}{2r - 1}n \right\rfloor, m, h - r \right),$$

and let $G = G_1 + G_2$.

Let us show that $G$ satisfies the requirements. Since $G_1$ is $K_{r-1}$-free and $G_2$ is triangle-free, we see that $G$ is $K_{r+1}$-free. Also, we have

$$\chi(G) \geq \chi(G_2) + r - 2 \geq h.$$

For every $v \in G_1$,

$$d(v) = d_{G_1}(v) + |G_2| \geq \left\lfloor \frac{r - 3}{r - 2} \left\lfloor \frac{2r - 4}{2r - 1}n \right\rfloor \right\rfloor + n - \left\lfloor \frac{2r - 4}{2r - 1}n \right\rfloor \geq n - \frac{2}{2r - 1}n - 1.$$

On the other hand, for every $v \in G_2$,

$$d(v) = d_{G_2}(v) + |G_1| \geq \left\lfloor \frac{2r - 4}{2r - 1}n \right\rfloor + \left( n - \left\lfloor \frac{2r - 4}{2r - 1}n \right\rfloor \right) \left( \frac{1}{3} - \frac{\varepsilon}{3} \right) \geq \frac{2r - 4}{2r - 1}n - 1 + \frac{3r}{2r - 1}n \left( \frac{1}{3} - \frac{\varepsilon}{3} \right) = \left( 1 - \frac{2}{2r - 1} - \frac{r\varepsilon}{2r - 1} - \frac{1}{n} \right)n$$

$$> \left( 1 - \frac{2}{2r - 1} - \varepsilon \right)n.$$

Hence, (3) also holds, and thus, $G$ has the required properties.

From our construction and Theorem 1 it follows that for all $h > r + 2$,

$$\psi(n, r, h) = (1 - 2/(2r - 1))n - o(n).$$
3.2 Extending the example of H"aggkvist

As mentioned in the introduction, H"aggkvist\cite{10} constructed for every $k \geq 1$, a 4-chromatic, $10k$-regular graph of order $29k$. For completeness we describe this example.

**Example 10** Partition $V (G) = [n]$ into 11 sets $[n] = A_1 \cup ... \cup A_5 \cup B_1 \cup ... \cup B_5 \cup C$ such that $|A_1| = \cdots = |A_5| = 3k$, $|B_1| = \cdots = |B_5| = 2k$, $|C| = 4k$; join $u \in A_i$ to $v \in A_j$ if $i - j = \pm 1 \mod 5$; join $u \in A_i$ to $v \in B_j$ if $i - j = \pm 1 \mod 5$; join all vertices of $C$ to all vertices of $\bigcup_{i=1}^{5} B_i$.

Write $H (k)$ for the resulting graph.

Observe that $H (k)$ contains the Mycielski graph $M_3$, which is $K_3$-free and 4-chromatic. In fact, $H (k)$ is homomorphic to $M_3$; hence, it is $K_3$-free and 4-chromatic itself. It is obvious that $\delta (H (k)) = 10k$.

Now we shall construct for every $r \geq 3$ and every $n > 19r - 9$ a $K_{r+1}$-free, $(r + 2)$-chromatic graph of order $n$ with

$$\delta > \left(1 - \frac{19}{19r - 9}\right) n - 1.$$  \hspace{1cm} (4)

**Example 11** Assume that $n > 19r - 9$; set

$$G_1 = T_{r-2} \left( n - 29 \left\lfloor \frac{n}{19r - 9} \right\rfloor \right)$$

$$G_2 = A_k \left( \left\lfloor \frac{n}{19r - 9} \right\rfloor \right),$$

and let $G = G_1 + G_2$.

We shall show that $G$ satisfies the requirements. Since $G_1$ is $K_{r-1}$-free and $G_2$ is $K_3$-free, we see that $G$ is $K_{r+1}$-free. Also, we have

$$\chi (G) = \chi (G_2) + r - 2 = r + 2.$$

For every $v \in G_2$,

$$d (v) = d_{G_2} (v) + |G_1| \geq n - 29 \left\lfloor \frac{n}{19r - 9} \right\rfloor + 10 \left\lfloor \frac{n}{19r - 9} \right\rfloor$$

$$\geq \left(1 - \frac{19}{19r - 9}\right) n.$$
On the other hand, for every $v \in G_1$ we have

\[
d(v) = d_{G_1}(v) + |G_2| \geq \delta \left( T_{r-2} \left( n - 29 \left\lceil \frac{n}{19r - 9} \right\rceil \right) \right) + 29 \left\lceil \frac{n}{19r - 9} \right\rceil
\]

\[
= \left\lceil \frac{r - 3}{r - 2} \left( n - 29 \left\lceil \frac{n}{19r - 9} \right\rceil \right) \right\rceil + 29 \left\lceil \frac{n}{19r - 9} \right\rceil
\]

\[
= \left\lceil \frac{r - 3}{r - 2} n + \frac{29}{r - 2} \left\lceil \frac{n}{19r - 9} \right\rceil \right\rceil.
\]

Suppose that $n = (19r - 9)k + s$, where $k \geq 0$ and $0 \leq s \leq 19r - 10$ are integers. Then

\[
\left\lceil \frac{r - 3}{r - 2} n + \frac{29}{r - 2} \left\lceil \frac{n}{19r - 9} \right\rceil \right\rceil = n + \left\lceil \frac{(19 - 9)k + s}{r - 2} + \frac{29k}{r - 2} \right\rceil
\]

\[
= n - 19k + \left\lfloor \frac{s}{r - 2} \right\rfloor > n - 19k - \frac{s}{r - 2} - 1
\]

\[
= \left( 1 - \frac{19}{19r - 9} \right) n - 1.
\]

Hence, (4) also holds, and $G$ has the required properties. Note that if $19r - 9$ divides $n$, then

\[
\delta(G) = \left( 1 - \frac{19}{19r - 9} \right) n.
\]

### 3.3 Extending the Andrásfai graphs

Let $A_k$ be the $k$’th Andrásfai graph, which is a $k$-regular graph of order $3k - 1$. Write $A_k(t)$, for the blow-up of $A_k$ by factor $t$, i.e., $A_k(t)$ is obtained by replacing each vertex $u \in V(A_k)$ with a set $V_u$ of size $t$ and each edge $uv \in E(H)$ with a complete bipartite graph with vertex classes $V_u$ and $V_v$. Note that $A_k(t)$ is $K_3$-free, 3-chromatic $kt$-regular graph of order $(3k - 1)t$.

We shall construct for every $r \geq 3$ and every $n > (2k - 1)r - k + 1$ a $K_{r+1}$-free, $(r + 1)$-chromatic graph of order $n$ with

\[
\delta > \left( 1 - \frac{2k - 1}{(2k - 1)r - k + 1} \right) n - 1.
\]

#### Example 12

Assume that $n > (2k - 1)r - k + 1$; set

\[
G_1 = T_{r-2} \left( n - (3k - 1) \left\lceil \frac{n}{(2k - 1)r - k + 1} \right\rceil \right)
\]

\[
G_2 = A_k \left( \left\lceil \frac{n}{(2k - 1)r - k + 1} \right\rceil \right),
\]

and let $G = G_1 + G_2$. 

11
We shall show that $G$ satisfies the requirements. Since $G_1$ is $K_{r-1}$-free and $G_2$ is $K_3$-free, we see that $G$ is $K_{r+1}$-free. Also, we have

$$\chi(G) = \chi(G_2) + r - 2 = r + 1.$$  

For every $v \in G_2$,

$$d(v) = |G_1| + d_{G_2}(v) = n - (3k - 1) \left\lfloor \frac{n}{(2k - 1) r - k + 1} \right\rfloor + k \left\lfloor \frac{n}{(2k - 1) r - k + 1} \right\rfloor \geq \left(1 - \frac{2k - 1}{(2k - 1) r - k + 1}\right)n.$$

On the other hand, for every $v \in G_1$ we have

$$d(v) = d_{G_1}(v) + |G_2| \geq \delta\left(T_{r-2} \left(n - (3k - 1) \left\lfloor \frac{n}{(2k - 1) r - k + 1} \right\rfloor \right)\right) + (3k - 1) \left\lfloor \frac{n}{(2k - 1) r - k + 1} \right\rfloor \geq \left[\frac{r - 3}{r - 2} n + \frac{3k - 1}{r - 2} \left\lfloor \frac{n}{(2k - 1) r - k + 1} \right\rfloor\right].$$

Suppose that $n = ((2k - 1) r - k + 1) t + s$, where $t \geq 1$ and $0 \leq s < (2k - 1) r - k + 1$ are integers. Then

$$\left\lfloor \frac{r - 3}{r - 2} n + \frac{3k - 1}{r - 2} \left\lfloor \frac{n}{(2k - 1) r - k + 1} \right\rfloor\right\rfloor = n + \left[-\frac{(2k - 1) r - k + 1) t + s}{r - 2} + \frac{(3k - 1) t}{r - 2}\right]$$

$$= n - (2k - 1) t + \left[-\frac{s}{r - 2}\right]$$

$$> n - (2k - 1) t - \frac{s}{r - 2} - 1$$

$$= \left(1 - \frac{2k - 1}{(2k - 1) r - k + 1}\right)n - 1.$$  

Hence, (5) also holds, and $G$ has the required properties. Note that if $(2k - 1) r - k + 1$ divides $n$, then

$$\delta(G) = \left(1 - \frac{2k - 1}{(2k - 1) r - k + 1}\right)n.$$  

Acknowledgement This research has been supported in part by NSF Grant # DMS-0906634.

References

[1] B. Andrásfai, P. Erdős, V.T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, Discrete Math. 8 (1974), 205–218.
[2] B. Andrásfai, Über ein Extremalproblem der Graphentheorie, *Acta Math. Acad. Sci. Hungar.* **13** (1962), 443–455.

[3] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, **184**, Springer-Verlag, New York (1998), xiv+394 pp.

[4] S. Brandt, T. Pisanski, Another infinite sequence of dense triangle-free graphs, *Electron. J. Combin.* **5** (1998).

[5] S. Brandt, On the Structure of Dense Triangle-Free Graphs, *Combin. Probab. Comput.* **8** (1999), 237-245.

[6] S. Brandt, A 4-colour problem for dense triangle-free graphs, *Discrete Math.* **251** (2002), 33–46.

[7] S. Brandt, S. Thomassé, Dense triangle-free graphs are four-colorable: A solution to the Erdős-Simonovits problem, to appear in *J. Combin Theory Ser B*.

[8] C.C. Chen, G.P. Jin, K.M. Koh, Triangle-free graphs with large degree, *Combin. Probab. Comput.* **6** (1997), 381–396.

[9] P. Erdős, M. Simonovits, On a valence problem in extremal graph theory, *Discrete Math.*, **5** (1973), 323-334.

[10] R. Häggkvist, Odd cycles of specified length in nonbipartite graphs, *Graph theory* (Cambridge, 1981), pp. 89–99, North-Holland Math. Stud., **62**, North-Holland, Amsterdam-New York, 1982.

[11] G.P. Jin, Triangle-free graphs with high minimal degrees, *Combin. Probab. Comput.* **2** (1993), 479–490.

[12] G.P. Jin, Triangle-free four-chromatic graphs, *Discrete Math.* **145** (1995), 151–170.

[13] M. Kneser, Aufgabe 360, *Jahresbericht Deutchen Math. Ver.* **58** (2) (1955-56), 27.

[14] L. Lovász, Kneser’s conjecture, chromatic number, and homotopy, *J. Combin. Theory Ser. A* **25** (1978), 319–324.

[15] J. Mycielski, Sur le coloriage des graphs (in French), *Colloq. Math.* **3** (1955), 161–162.

[16] C. Thomassen, On the chromatic number of triangle-free graphs of large minimum degree, *Combinatorica*, **22** (2002), 591–596.