Gane Samb LO, Pape Djiby Mergane, Thilabola Atozou Kpanzou, Mohamed Cheikh Haidara

Weak Convergence (IIIA)
Asymptotic Representations of Statistics in the Functional Empirical process: A portal and some applications

Statistics and Probability African Society (SPAS) Research Monographs Series. Calgary, Alberta. 2018.
SPAS Research Monographs Series

Advisers

List of published books
Library of Congress Cataloging-in-Publication Data

Main Author : Gane Samb LO, 1958-

Weak Convergence (IIIB). Asymptotic Representations of Statistics in the Functional Empirical process : A portal and some applications.

SPAS Research Monographs Series, 2018.
Author: Gane Samb LO

Emails:
gane-samb.lo@ugb.edu.sn, ganesamblo@ganesamblo.net.

Url’s:
www.ganesamblo@ganesamblo.net
www.statpas.net/cva.php?email.ganesamblo@yahoo.com.

Affiliations.
Main affiliation: University Gaston Berger, UGB, SENEGAL.
African University of Sciences and Technology, AUST, ABuja, Nigeria.
Affiliated as a researcher to: LSTA, Pierre et Marie Curie University, Paris VI, France.

Teaches or has taught at the graduate level in the following universities:
Saint-Louis, Senegal (UGB)
African University of Sciences and Technology (AUST), Abuja, Nigeria
Banjul, Gambia (TUG)
Bamako, Mali (USTTB)
Ouagadougou - Burkina Faso (UJK)
African Institute of Mathematical Sciences, Mbour, SENEGAL, AIMS.
Franceville, Gabon

Dedication.

To my mother (1927-2011)

Acknowledgment of Funding.

The author acknowledges continuous support of the World Bank Excellence Center in Mathematics, Computer Sciences and Intelligence Technology, CEA-MITIC. His research projects in 2014, 2015 and 2016 are funded by the University of Gaston Berger in different forms and by CEA-MITIC.
Author: Tchilabalo A. Kpanzou

Dr Tchilabalo holds a PhD from the University of Stellenbosch, South Africa (2011).

Emails:
kpanzout@gmail.com, kpanzout@yahoo.fr

Url’s:
https://sites.google.com/a/aims.ac.za/tchilabalo
http://univi.net/spas/cvf.php?email=kpanzout@yahoo.fr

Affiliations.
Main affiliation: University of Kara, Kara, TOGO.

Teaches or has taught at the graduate level in the following universities:
University of Kara (UK), TOGO
University of Lomé (UL), TOGO
Ecole Normale Supérieure (ENS), TOGO
University of Abomey-Calavi (UAC), BENIN
Author: Pape Djiby Mergane

Dr Pape Djiby Mergane holds a PhD from Gaston Berger University of Saint-Louis (2014).

Emails:
merganedjiby@gmail.com.

Url’s:
https://arxiv.org/find/all/1/all:+AND+djiby+mergane/0/1/0/all/0/1

Affiliations.
LERSTAD, Gaston Berger University (UGB), Saint-Louis, SENEGAL.
Alioune Diop University of Bambey (UADB), Bambey (SENEGAL)

Teaches or has taught at:
Alioune Diop University of Bambey (UADB), Bambey (SENEGAL)
Gaston Berger University (UGB), Saint-Louis, SENEGAL.
National Institute of Statistics and Demography, Dakar, SENEGAL.
Author: Mohamed Cheikh Haidara
Dr Mohamed Cheikh Haidara holds a PhD from Gaston Berger University of Saint-Louis (2012).

Emails:
chheikhh@yahoo.fr, mcheikhhaidara@gmail.com

Urls:
http://univi.net/spas/cva.php?email=chheikhh@yahoo.fr
https://arxiv.org/find/all/1/all:+AND+haidara+AND+mohamed+cheikh/0/1/0/all/0/1

Affiliations.
LERSTAD, Gaston Berger University (UGB), Saint-Louis, SENEGAL.

Teaches or has taught at:
Cheikh Anta Diop University (UCAD), Dakar, SENEGAL
Weak Convergence (IIIA). Asymptotic Representations of Statistics in the Functional Empirical process: A portal and some applications
Abstract. (Short Abstract) In this research monograph, we deal with a very
general asymptotic representation for statistics named GRI expressed in the
functional empirical process, both one-dimensional and multidimensional, and
another call residual empirical process. Most of statistics in form of combina-
tion of L-statistics are covered by the asymptotic theory dealt here. We also
treat three important exaples as show cases.

Keywords. Empirical process; Functional empirical process; Empirical Residual
process; Gaussian Field; Asymptotic Representations of Statics; spatial and
temporal study of statistics; Joint Asymptotic distributions; Copula

AMS 2010 Classification Subjects : 60XXX; 62G30
ABSTRACT (English) In this research monograph, we deal with a very general asymptotic representation for statistics named GRI expressed in the functional empirical process, both one-dimensional and multidimensional, and another call residual empirical process. Most of statistics in form of combination of L-statistics are covered by the asymptotic theory dealt here. This treatise is conceived to be a kind of spaceship on which modules are hanged. The spaceship is a functional Gaussian process and each module is the asymptotic representation of one statistic in terms of that Gaussian process. In that way, it is possible to navigate from one module to another, that is, to find the joint distribution of any pair of statistics, to compare them with respect to the areas and the times. In order to be able to do so, we should have a broad conception at the beginning. Within the constructed frame, the asymptotic joint law of any finite number of other statistics is automatically given as well as the joint distribution of its spatial variation or temporal variation, in absolute or relative values. We also deal with the general problem of decomposability of statistics by comparing statistical decomposability, a new view we introduce, versus functional decomposability. A general result only based on the GRI is provided.

This monograph is also the portal of a handbook of GRI that will cover the largest number possible of statistics. In prevision of that, we treat three important examples as show cases.

It is expected that this portal and the handbook will attract the attention of researchers working in the asymptotic area and will furnish useful tools to scientists who are interested in application of asymptotic tests, completed by computer packages.

RESUMÉ (Français) Dans cette monographie de recherche, nous traitons d’une représentation générale asymptotique pour des statistiques exprimée par rapport au processus empirique fonctionnel, à la fois unidimensionnel et multidimensionnel, et un autre processus empirique appelé résiduel. La plupart des statistiques sous forme de combinaison de L-statistiques sont couvertes par la théorie asymptotique traitée ici. Ce traité est conçu pour être une sorte de vaisseau spatial sur lequel les modules sont accrochés. Le vaisseau spatial est un processus gaussien fonctionnel et chaque module est la représentation asymptotique d’une statistique en fonction de ce processus gaussien. De cette manière, il est possible de naviguer d’un module à un autre, c’est-à-dire de trouver la distribution conjointe de n’importe quelle paire de
statistiques, de les comparer par rapport spatialement et temporellement. Pour pouvoir le faire, nous devrions avoir une conception large au début. À l'intérieur du cadre construit, la loi conjointe asymptotique d'un nouvel élément avec un nombre fini d'autres statistiques est automatiquement donnée ainsi que la distribution conjointe de sa variation spatiale ou variation temporelle, en valeurs absolues ou relatives. Nous traitons également du problème général de la décomposabilité des statistiques en comparant la décomposabilité statistique, une nouvelle notion que nous introduisons, par rapport à la décomposabilité fonctionnelle. Un résultat général basé uniquement sur la représentation GRI est fourni.

Cette monographie est également annonciatrice d'un recueil de représentations GRI qui couvrira le plus grand nombre possible de statistiques. En prévision de cela, nous traitons aussi de trois cas spécifiques importants.

Nous espérons que ce portail et le recueil attireront l'attention de tous ceux qui travaillent dans le domaine des lois asymptotiques et fourniront aux spécialistes des domaines appliqués des outils de travail qui seront complétés par des programmes informatiques.
# Contents

General Preface

1

General Preface of Our Series of Weak Convergence

3

General Introduction

7

**Part 1. The Gateway**

Chapter 1. Introduction and Notation

15

1. The empirical Process

17

2. The General and Simple Method of Using the *fep* for Asymptotic laws deriving

21

3. Notations and Probability Space

23

4. The residual empirical process

29

5. General handling

35

Chapter 2. Statistical decomposability of indices

41

1. Introduction

41

2. General Statistical Decomposition Theorem

44

3. Proof of the Theorem

47

Chapter 3. Asymptotic Laws of indices, of their absolute and relative variation of indices

63

1. Asymptotic Laws of indices

63

2. Asymptotic Laws of variations of an index

64

3. Asymptotic Laws of relative variations of an index

67

Chapter 4. Mutual Asymptotic Influence between indices

69

1. Mutual influence of two simple indices

71

2. Mutual influence of variations of indices

72

3. Mutual influence of Relative Variations of indices

73

**Part 2. Applications and Examples**

Introduction to Part II

77

Chapter 5. Moments Estimation of moments

79
General Preface

This textbook is the first of series whose ambition is to cover broad part of Probability Theory and Statistics. These textbooks are intended to help learners and readers, both of all levels, to train themselves.

As well, they may constitute helpful documents for professors and teachers for both courses and exercises. For more ambitious people, they are only starting points towards more advanced and personalized books. So, these texts are kindly put at the disposal of professors and learners.

Our textbooks are classified into categories.

A series of introductory books for beginners. Books of this series are usually accessible to student of first year in universities. They do not require advanced mathematics. Books on elementary probability theory and descriptive statistics are to be put in that category. Books of that kind are usually introductions to more advanced and mathematical versions of the same theory. The first prepare the applications of the second.

A series of books oriented to applications. Students or researchers in very related disciplines such as Health studies, Hydrology, Finance, Economics, etc. may be in need of Probability Theory or Statistics. They are not interested by these disciplines by themselves. Rather, the need to apply their findings as tools to solve their specific problems. So adapted books on Probability Theory and Statistics may be composed to on the applications of such fields. A perfect example concerns the need of mathematical statistics for economists who do not necessarily have a good background in Measure Theory.

A series of specialized books on Probability theory and Statistics of high level. This series begin with a book on Measure Theory, its counterpart of probability theory, and an introductory book on
topology. On that basis, we will have, as much as possible, a coherent presentation of branches of Probability theory and Statistics. We will try to have a self-contained, as much as possible, so that anything we need will be in the series.

Finally, research monographs close this architecture. The architecture should be so large and deep that the readers of monographs booklets will find all needed theories and inputs in it.

We conclude by saying that, with only an undergraduate level, the reader will open the door of anything in Probability theory and statistics with Measure Theory and integration. Once this course validated, eventually combined with two solid courses on topology and functional analysis, he will have all the means to get specialized in any branch in these disciplines.

Our collaborators and former students are invited to make live this trend and to develop it so that the center of Saint-Louis becomes or continues to be a renown mathematical school, especially in Probability Theory and Statistics.
General Preface of Our Series of Weak Convergence

The series Weak convergence is an open project with three categories.

The special series Weak convergence I consists of texts devoted to the core theory of weak convergence, each of them concentrated on the handling of one specific class of objects. The texts will have labels A, B, etc. Here are some examples.

(1) Weak convergence of Random Vectors (IA).

(2) Weak convergence of stochastic processes and empirical processes (IB).

(3) Weak convergence of random measures (IC).

(4) Weak convergence of random measures (ID).

(5) etc.

The special series Weak convergence II consists of texts related to the theory of weak convergence, each of them concentrated on one specialized field using weak convergence. Usually, these subfields are treated apart in the literature. Here, we want to put them in our general frame as continuations of the Weak Convergence Series I. Some examples are the following.

(1) Weak laws of sums on independent random variables.

(2) Weak laws of sums on associated random variables.
(3) Univariate Extreme values Theory.

(4) Multivariate Extreme values Theory.

(5) Etc.

**The special series Weak convergence III** consists of texts focusing on statistical applications of Parts of the Weak Convergence Series I and Weak Convergence Series II. Examples:

The present book falls in the category III of our series devoted to weak convergence. It constitutes a portal to a handbook of Gaussian Asymptotic Distributions Using the Functional Empirical Process as defined and introduced here.

Here, we establish a general representation for a large class of statistics and indexes. Since these type of indexes are very recurrent in a significant number of disciplines, it seemed important to us to gather their asymptotic treatment in a unified approach and specifically deal with important issues in the same Gaussian field (a frame we lay out in the monograph) like:

(1) A general asymptotic representation for individual statistics.

(2) Asymptotic representations for temporal absolute or relative variation of statistics.

(3) Spatial Asymptotic representations for statistics.

(4) Estimation of decomposability default for statistics.

These points are important for any statistics and pay important roles in Applications. In the field of socio-economic studies, the important of last point quite significant for example.

The importance of this monograph resides in the fact that, virtually, the asymptotic theory of a significant number of statistics is implicitly done in this monograph even if they do not exist yet. Better than that, their asymptotic theory are placed in an already existing Gaussian field that allow to see get at one their interaction with other statistics whose representations are already available. A none less important feature is
that the frame allows to make the interaction possible for statistics with different dimensions.

Once this portal settled, the monograph ay be extended by hanging on it a list of individual representations to form a handbook.
General Introduction

Some of my students, my collaborators and myself have spent more than one decade to contribute on the asymptotic theory of welfare indices. A list of the papers we wrote is at the appendix of this introduction. Some papers are published in indexed papers, other in non-indexed ones, others are posted in Arxiv (arxiv.org).

The main reason which justifies such a monograph is two-fold.

(a) One side, we concluded that using the function empirical process \( fep \) to achieve the results is powerful and efficient.

At the beginning, we tried to use the real empirical process and the non less powerful tools of Hungarian constructions ([Komlós et al. (1980)], [Csörgö et al. (1986)]). When passing to the functional approach, everything became almost easy. However, the price has been paid for acquiring the technology of this wonderful theory of \( fep \), which has been popularized by [van der Vaart and Wellner (1996)], and based on the developments of many authors, for example [Dudley R.M. (1984)], [Pollard (1984)], Gaenssler [Gaenssler (1983)], [Billingsley (1968)], [Pollard (1984)], etc.

(b) On the other side, we discovered that behavior that asymptotic behavior of the indices, and by the way a large number of statistics, depend on two functions \( h \) and \( \ell \) in the following general asymptotic representation

\[
G_{n,(1)}(h) + \int_0^1 G_{n,(1)}(\tilde{f}_s)\ell(s) \, ds, \quad (GRI)
\]

where \( \tilde{f}_s \) is a function of \( s \in (0, 1) \) that will be precised later and \( G_{n,(1)} \) is the \( fep \) in dimension one, based on a sample of size \( n \geq 1 \).
From there comes the idea to share our experience in using the fep and, by this, to devote one single broad study on the origin, the properties and the application of the representation (GRI), in which the main notation and terminology would be precised.

Once this frame fixed, we open a king of spaceship on which we may attach modules, each module being the (GRI) formula of new statistics. An open handbook containing that spaceship and modules, will be the next step of this monograph.

The monograph deals with the fep which does not make differences between dimensions of the space since only the metrical topology is used. This allows the treatment of multivariate statistics.

Actually, the fep treats one-dimensional and multidimensional statistics in the same way. This allows to have a unique conception of the study. In that conception, for any statistic which is added to the vessel, its asymptotic joint law of any finite number of other statistics in the vessel is automatically known. As well, even the joint distribution of its spatial variation or temporal variation (in absolute or relative values) with other statistics is already established.

To allow passing from one dimension to higher dimensions, we adopt notation in form of subscripts that clear indicated the dimension associated with the use of the natural projections. At first sight, this may be an over-notation. But at the end, it allows to keep the constructions and its use clear and unequivocal.

We introduce and justified the notion of Gaussian field within the strict scope of the study.

While the theoretical aspects are pretty well surrounded, the variances and covariances, might seem complicated. But nowadays, computers take care of such questions, and there is nothing to worry about. We already have a number of own packages that work well. May be, my collaborators will be able to design an $R$ project in that sense.

Before we announce the organization of the book, we wish to point out that researchers outside of Mathematics circles, will not find unavoidable difficulties to understand and to use the tools presented. The main reason is that most of the techniques are based on convergence of multivariate random variables, for with the book [Lo et al. (2016)]
Here is how is organized the monograph.

The first part, the gateway, concerns the intrinsic results. It includes four Chapters.

In Chapter 1, we give the main notation on the $fep$ and its properties. Next, we explain the General Representation of Indices (GRI), its origin, its conditions and its potential applications. Three approaches are studied: Fixed-time, partial and time evolution.

In Chapter 2, we address the general problem of decomposability of statistics. We introduce the notion of statistical decomposability versus functional decomposability. The results are also general and may be applied to any statistic for which the (GRI) is admissible.

In Chapter 3, we show how to find the asymptotic laws of the variation (absolute and relative) of an index for a time to another, and the joint distribution of variations of two indices.

In Chapter 4, the joint law of two statistics admitting the GRI is given, having in mind potential applications to the pro-poor and anti-poor growth in Welfare analysis.

In the second part, we provide first constituents of the announced handbook. We applied our techniques to important Welfare indices, as show-cases on how they work.

**What next?** Computational resources will be gathered under an independent release. Also, a handbook of the applications of the method to as many as possible statistics is open.
List of papers of the authors of the monograph and co-authors.

1 - The asymptotic theory of the poverty intensity in view of Extreme value theory for two simple cases, (2007), Afrika Statistika, 41-55, (2). (With Serigne Touba Sall)

2 - Estimation Asymptotique des Indices de Pauvret : Modlisation Continue et Analyse spatio-temporelle de la pauvret au Sngal (Asymptotic estimation of poverty indices : continuous modelling and, time and space analysis of poverty in Senegal), (2009), Journal Africain des Sciences de la Communication et des Technologies, 341-377, (3).

3 - The asymptotic theory of the Kakwani class of poverty measures, (2009), African Diaspora Journal of Mathematics, 54-67, 1. (With Serigne Touba Sall)

4 - Une thorie Gnrale Asymptotique des Mesures de Pauvret (A general theory of the asymptotics poverty measures), (2009), C. R. Math. Rep. Acad. Sci. Canada, 45-52, 31 (2). (With Serigne Touba Sall and Cheikh Tidiane Seck)

5 - Uniform Convergence of the Non-Weighted Poverty Measures, (2009), Commun. Stat., Theory Methods 38, No. 20, 3697-3704 (2009). (With Cheikh Tidiane Seck). (Zbl pre05648823).

6 - Uniform weak convergence of the time-dependent poverty measures for continuous longitudinal data, Brazilian Journal of Probability and Statistics, 2010, Vol. 24, No. 3, 457-467 (avec Serigne Touba Sall).

7 - A Simple Note on some Empirical Stochastic Process as a Tool in Uniform L-Statistics Weak Laws. Afrika Statistika, Special volume (5) : Proceedings of the International Workshop on Multiple Risks and Copula, Biskra 2010, pp. 245-251. Ed. Abdelhakim Necir.

8 - Asymptotic Representation Theorems for Poverty Indices. Afrika Statistika, Special Volume (5) : Proceedings of the International Workshop on Multiple Risks and Copula, Biskra 2010, pp. 238-244. Ed. Abdelhakim Necir. (With serigne Touba Sall)

9 - On the General Poverty Index. (2013). Far East Journal of Theoretical Statistics. Volume 42. (1), 1-22
10 - On the influence of the Theil-like inequality measure on the growth (2013). arXiv:1210.3190. Applied Mathematics, 2013, 4, 986-1000 doi:10.4236/am.2013.47136. (With Pape Djiby Mergane)

11 - Functional Weak Laws for the Weighted Mean Losses or Gains and Applications Applied Mathematics Vol.6 No.5. (with Serigne Touba Sall, Pape Djiby Mergane)

12 - Asymptotic Confidence Bands for Copulas Based on the Local Linear Kernel Estimator Applied Mathematics. 2015. 6 (12), 2077-2095 (with Diam Ba, Cheikh Tidiane Seck) http://dx.doi.org/10.4236/am.2015.612183

14 - Robust ordering of two income distributions by means of poverty indices. Fast East Journal of Theoretical Statistics. 50 (3), 2015, pages 203-230. http://dx.doi.org/10.1765/FJTSMay2015203_230 (With Cheikh Tidiane Seck)

15. Asymptotic inference in poverty indices: An empirical processes approach. Communications in Statistics - Theory and Methods, 46:12, 6192-6212, DOI: 10.1080/03610926.2015.1122060 (with Cheikh Tidiane Seck and J. Ngatchou).

16. Asymptotic inference in poverty indices: an empirical processes approach. Asymptotic Theory and Statistical Decomposability gap Estimation for Takayama’s Index. arXiv:1701.04735 (With Pape Djiby Mergane, Cheikh Mohamed Haidara, Cheikh Tidiane Seck).

17. Sur la décomposabilité empirique des indicateurs de pauvret. arXiv:1701.02649. (With Cheikh Mohamed Haidara)
Part 1

The Gateway
CHAPTER 1

Introduction and Notation

This chapter opens the gateway and may be considered as a portal of all the parts of ongoing A handbook of Asymptotic Representations of Statistics in the Functional Empirical process and Applications, as we explained earlier. Its gives the main aspects of the functional empirical process (fep) which is the tool on which depend all the results in the remainder of the book and the quoted handbook.

As mentioned in the introduction, the monograph deals with asymptotic normality results and their applications. But, as we know, there are so many of such results, which may be combined in a great number of ways. But how many times did we have, for example, two asymptotic normality results of two different statistics based on the same data, or such that one of them is based on some sub-data of the other, and we cannot see how to combine them to have the the joint asymptotic laws. The same situation may occur with one statistic which is observed in different areas or over different times. To find the joint asymptotic distributions of two or more statistics, combined with areas or periods of time, we are frequently obliged to do the work anew. The famous delta method, even if it is very powerful, requires new computations each time we have new situations.

In many fields, we already have working and existing statistics. New ones are regularly found. It would be better to have a kind of spaceship on which modules are hanged. In our situations, the spaceship is a functional Gaussian process and each module is the asymptotic representation of one statistic in terms of that Gaussian process. We may call that spaceship a Gaussian field in which the asymptotic laws of the statistics are expressed. In that way, it is possible to navigate from one module to another, that is, to find the joint distribution of any pair of statistics, to compare them with respect to the areas and the times. In order to be able to do so, we should have a broad conception at the beginning. This chapter constitutes that construction.
We begin by some general facts on the empirical process, in its real and functional forms. Next, we present in details the functional form.

It is amazing that we will not need all the sophisticated and extremely complicated aspects of uniform convergence and tightness we necessarily have to deal with when working on weak convergences in the space of bounded functions on some space $T$ ($T = \mathbb{R}^k$, here). There are some circumstances where they are useful and handy. But for the needs of our study, the finite-distributional convergence will be enough and then the multivariate central limit theorem is just needed. The readers who are interested in detailed results in the theory of empirical processes are directed to [Billingsley (1968)], [Gaenssler (1983)], [Pollard (1984)], [van der Vaart and Wellner (1996)], etc. For the needs for the finite-distributions scheme are, we will back on [Lo et al. (2016)].

Before we proceed, we point out that a similar enterprise has been done in [Barrett and Donald (2000)], but using real empirical processes. As we will see latter, a huge part of the limitations due to the use of real valued empirical processes are lifted by the functional empirical process to, the most important one of them being non-linearity.
1. The empirical Process

I - The real empirical process.

Let \( X, X_1, X_2, \ldots \) be a sequence of independent and identically distributed following a real-valued cumulative distribution function \( F \) and all defined on the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\). For each \( n \geq 1 \), we may define the empirical distribution function associated with \( X_1, X_2, \ldots, X_n \):

\[
\mathbb{R} \ni x \mapsto F_n(x) = \frac{1}{n} \text{Card}\{j, 1 \leq j \leq n, \ X_j \leq x\}
\]

The empirical process associated with \( X_1, X_2, \ldots, X_n \) is defined as follows

\[
\alpha_n(x) = \sqrt{n}(F_n(x) - F(x)), \ x \in \mathbb{R}.
\]

In the real case, we have the two following keys results.

**The Glivenko-Cantelli Law**:

\[
\|F_n - F\|_\infty = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0 \ a.s.\ as \ n \to +\infty.
\]

**The Donsker Law**. The sequence of stochastic processes \((\alpha_n(x), x \in \mathbb{R})\) weakly convergences on \( \ell^\infty(\mathbb{R}) \) - the space of bounded real-valued function defined on \( \mathbb{R} \) - to a re-scaled Brownian bridge \((B(F(x)), x \in \mathbb{R})\), denoted as

\[
(\alpha_n(x), x \in \mathbb{R}) \rightsquigarrow (B(F(x)), x \in \mathbb{R}) \ \text{in} \ \ell^\infty(R) \ \text{as} \ n \to +\infty,
\]

where \((B(t), t \in [0, 1])\) is by definition the Brownian bridge, which is a centered Gaussian process of variance-covariance function

\[
\Gamma(s, t) = \min(s, t) - st, \ (s, t) \in [0, 1]^2.
\]

In many occasions, we do not need the full version of the Donsker Theorem as we will see in the sequel. We usually only need the finite-distributional version, which is readily proved by using multinomial probabilities, and which is stated as below.
The finite-distribution weak law of the empirical process. For any finite number \( k \geq 1 \), and for any real numbers \( x_1 < \ldots < x_k \), we have the following weak convergence on \( \mathbb{R}^k \)
\[
(\alpha_n(x_1), \ldots, \alpha_n(x_k))^t \Rightarrow (B(F(x_1)), \ldots, B(F(x_1)))^t.
\]

where, throughout the monograph, \( x^t \) stands for the transpose of a matrix, column or line and we consider elements of \( \mathbb{R}^d, d \geq 1 \), as columns.

The real empirical process has been deeply investigated, mainly in the Skorohod topology in \( D(0, 1) \), the space of real-valued functions defined on \([0, 1]\) with at most a countable number of discontinuity points which are all of the first kind (see [Billingsley (1968)], as a main reference). But for a long time, the direct approach, which by the way a counting one, had hidden the linearity of this fundamental object. And linearity brings more powerful tools from the functional analysis prospective. Define for any \( x \in \mathbb{R} \)
\[
f_x = 1_{]-\infty, x]},
\]
we get for any fixed \( n \geq 1 \),
\[
\alpha_n(x) = G_n(f_x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \{f_x(X_j) - Ef_x(X_j)\},
\]
and any real numbers \( x_1, \ldots, x_k \) and any \( a_1, \ldots, a_k \), we have for any fixed \( n \geq 1 \)
\[
G_n \left( \sum_{h=1}^{k} a_h f_{x_h} \right) = \sum_{h=1}^{k} a_h G_n \left( f_{x_h} \right).
\]

This properties renders much easier the study of the empirical process. This leads to the functional approach.

II - The Functional Empirical Process.

Let \( Z_1, Z_2, \ldots \) be a sequence of independent copies of a random variable \( Z \) defined on the same probability space with \((\Omega, \mathcal{A}, \mathbb{P})\) values on some metric space \((S, d)\). The mathematical expection symbol with respect to \( \mathbb{P} \) is denoted by \( \mathbb{E} \) and \( \mathbb{P}_Z = \mathbb{P} \circ Z^{-1} \) is the probability measure image of \( \mathbb{P} \) by a measurable mapping \( Z \). Define for each \( n \geq 1 \), the functional
empirical process by

\[ G_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f(Z_j) - \mathbb{E}f(Z_j)) \]

where \( f \) is a real and measurable function defined on \( \mathbb{R} \) such that

\[(1.1)\quad \mathbb{V}_Z(f) = \int (f(x) - \mathbb{P}_Z(f))^2 \, d\mathbb{P}_Z(x) < \infty,\]

which entails

\[(1.2)\quad \mathbb{P}_Z(|f|) = \int |f(x)| \, d\mathbb{P}_Z(x) < \infty.\]

Let us denote by \( \mathcal{F}(S) - \mathcal{F} \) for short - the class of real-valued measurable functions that are defined on \( S \) such that (1.1) holds. The space \( \mathcal{F} \), when endowed with the addition and the external multiplication by real scalars, is a linear space. Next, it is remarkable that \( G_n \) is linear on \( \mathcal{F} \), that is for \( f \) and \( g \) in \( \mathcal{F} \) and for \( (a, b) \in \mathbb{R}^2 \), we have

\[ aG_n(f) + bG_n(g) = G_n(af + bg). \]

We have this result

**Lemma 1.** Given the notation above, then for any finite number of elements \( f_1, ..., f_k \) of \( S, k \geq 1 \), we have

\[ (G_n(f_1), ..., G_n(f_k))^t \sim \mathcal{N}_k(0, \Gamma(f_i, f_j)_{1 \leq i, j \leq k}), \]

where

\[ \Gamma(f_i, f_j) = \int (f_i - \mathbb{P}_Z(f_i)) (f_j - \mathbb{P}_Z(f_j)) \, d\mathbb{P}_Z(x), 1 \leq i, j \leq k. \]

**Proof.** It is enough to use the Cramér-Wold Criterion (see for example [Billingsley (1968)], page 45, or [Lo et al. (2016)], Chapter one), that is to show that for any \( a =^t (a_1, ..., a_k) \in \mathbb{R}^k \), by denoting \( T_n =^t (G_n(f_1), ..., G_n(f_k)) \), we have \( \langle a, T_n \rangle \sim \langle a, T \rangle \) where \( T \) follows the \( \mathcal{N}_k(0, \Gamma(f_i, f_j)_{1 \leq i, j \leq k}) \) law and \( \langle \cdot, \cdot \rangle \) stands for the usual
product scalar in $\mathbb{R}^k$. But, by the standard central limit theorem in $\mathbb{R}$, we have

$$<a, T_n> = \mathbb{G}_n \left( \sum_{i=1}^{k} a_i f_i \right) \rightsquigarrow \mathcal{N}(0, \sigma^2_\infty),$$

where, for $g = \sum_{1 \leq i \leq k} a_i f_i$,

$$\sigma^2_\infty = \int (g(x) - \mathbb{P}_Z(g))^2 \, d\mathbb{P}_Z(x)$$

and this easily gives

$$\sigma^2_\infty = \sum_{1 \leq i,j \leq k} a_i a_j \Gamma(f_i, f_j),$$

so that $\mathcal{N}(0, \sigma^2_\infty)$ is the law of $<a, T>$. The proof is finish.

This functional approach leads to an almost universal method for finding the asymptotic laws of multidimensional statistics.

We first give, as an application of the delta method, an easy way to find simple asymptotic laws.
2. The General and Simple Method of Using the \textit{fep} for Asymptotic laws deriving

We usually work with usual asymptotic statistics on $\mathbb{R}^k$. Once we have our sample $Z_1, Z_2, \ldots$ as random variables defined in the same probability space with values in $\mathbb{R}^k$, the studied statistics, say $T_n$, is usually a combination of expressions of the form

$$H_n = \frac{1}{n} \sum_{j=1}^{k} H(Z_j)$$

for $H \in \mathcal{F}$. We use this simple expansion, for $\mu(H) = \mathbb{E}H(Z)$,

(2.1) \hspace{1cm} H_n = \mu(H) + n^{-1/2} \mathcal{G}_n(H).

We have that $\mathcal{G}_n(H)$ is asymptotically bounded in probability since $\mathcal{G}_n(H)$ weakly converges to, say $M(H)$ and then by the continuous mapping theorem $\|\mathcal{G}_n(H)\| \sim \|M(H)\|$. Since all the $\mathcal{G}_n(H)$ are defined on the same probability space, we get for all $\lambda > 0$, by the assertion of the Portmanteau Theorem for concerning open sets,

$$\limsup_{n \to \infty} P(\|\mathcal{G}_n(H)\| > \lambda) \leq P(\|M(H)\| > \lambda)$$

and then

$$\liminf_{\lambda \to \infty} \limsup_{n \to \infty} P(\|\mathcal{G}_n(H)\| > \lambda) \leq \limsup_{\lambda \to \infty} P(\|M(H)\| > \lambda) = 0.$$

From this, we use the big $O_p$ notation, that is $\mathcal{G}_n(H) = O_p(1)$. Formula (2.1) becomes

$$H_n = \mu(H) + n^{-1/2} \mathcal{G}_n(H) = \mu(H) + O_p(n^{-1/2})$$

and we will be able to use the delta method. Indeed, let $g : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable on a neighborhood of $\mu(H)$. The mean value theorem leads to

(2.2) \hspace{1cm} g(H_n) = g(\mu(H)) + g'(\mu_n(H)) n^{-1/2} \mathcal{G}_n(H)

where

$$\mu_n(H) \in [(\mu(H) + n^{-1/2} \mathcal{G}_n(H)) \wedge \mu(H), (\mu(H) + n^{-1/2} \mathcal{G}_n(H)) \vee \mu(H)]$$

so that

$$|\mu_n(H) - \mu(H)| \leq n^{-1/2} \mathcal{G}_n(H) = O_p(n^{-1/2}).$$
Then $\mu_n(H)$ converges to $\mu(H)$ in probability (denoted $\mu_n(H) \to_{\mathbb{P}} \mu(H)$). But the convergence in probability to a constant is equivalent to the weak convergence. Then $\mu_n(H) \rightsquigarrow \mu(H)$. Using again the continuous mapping theorem, $g'(\mu_n(H)) \rightsquigarrow g'(\mu(H))$ which in tern yields $g'(\mu_n(H)) \to_{\mathbb{P}} g'(\mu(H))$ by the characterization of the weak convergence to a constant. Now (2.2) becomes
\[
g(H_n) = g(\mu(H)) + (g'(\mu(H) + o_P(1)) n^{-1/2} \mathbb{G}_n(H)
\]
\[
= g(\mu(H)) + g'(\mu(H) \times n^{-1/2} \mathbb{G}_n(H) + o_P(1)) n^{-1/2} \mathbb{G}_n(H)
\]
\[
= g(\mu(H)) + n^{-1/2} \mathbb{G}_n(g'(\mu(H)H) + o_P(n^{-1/2})
\]
We arrive at the final expansion
(2.3) \[
g(H_n) = g(\mu(H)) + n^{-1/2} \mathbb{G}_n(g'(\mu(H)H) + o_P(n^{-1/2}).
\]
The method consists in using the expansion (2.3) as many times as needed and next to do some algebra on these expansions. By using the same techniques as above, we have the following three formulas

**Lemma 2.** Let $(A_n)$ and $(B_n)$ be two sequences of real valued random variables defined on the same probability space holding the sequence $Z_1, Z_2, \ldots$. Let $A$ and $B$ be two real numbers and let $L(z)$ and $H(z)$ be two real-valued functions of $z \in S$. Suppose that $A_n = A + n^{-1/2} \mathbb{G}_n(L) + o_P(n^{-1/2})$ and $A_n = B + n^{-1/2} \mathbb{G}_n(H) + o_P(n^{-1/2})$. Then
\[
A_n + B_n = A + B + n^{-1/2} \mathbb{G}_n(L + H) + o_P(n^{-1/2}),
\]
\[
A_n B_n = AB + n^{-1/2} \mathbb{G}_n(BL + AH)
\]
and if $B \neq 0$,
\[
\frac{A_n}{B_n} = \frac{A}{B} + n^{-1/2} \mathbb{G}_n(\frac{1}{B} L - \frac{A}{B^2} H) + o_P(n^{-1/2})
\]
By putting together all the described steps in a smart way, the methodology will lead us to a final result of the form
\[
T_n = T + n^{-1/2} \mathbb{G}_n(h) + o_P(n^{-1/2}),
\]
where
\[
h = \frac{1}{B} L - \frac{A}{B^2} H,
\]
which entails the following weak convergence
3. Notations and Probability Space

In this Subsection, we complete the notations we already gave and precise our probability space.

**Univariate frame.** We are going to describe the general Gaussian field in which we present our results. Indeed, we use a unified approach when dealing with the asymptotic theories of the welfare statistics. It is based on the Functional Empirical Process (fep) and its Functional Brownian Bridge (fbb) limit. It is laid out as follows.

When we deal with the asymptotic properties of one statistic or index at a fixed time, we suppose that we have a non-negative random variable of interest which may be the income or the expense \(X\) whose probability law on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\), the Borel measurable space on \(\mathbb{R}\), is denoted by
\( \mathbb{P}_X \). We consider the space \( \mathcal{F}_{(1)} \) of measurable real-valued functions \( f \) defined on \( \mathbb{R} \) such that

\[
V_X(f) = \int (f - \mathbb{E}_X(f))^2 d\mathbb{P}_X = \mathbb{E}(f(X) - \mathbb{E}(f(X)))^2 < +\infty,
\]

where

\[
\mathbb{E}_X(f) = \mathbb{E} f(X).
\]

On this functional space \( \mathcal{F}_{(1)} \), which is endowed with the \( L_2 \)-norm

\[
\|f\|_2 = \left( \int f^2 d\mathbb{P}_X \right)^{1/2},
\]

we define the Gaussian process \( \{ \mathcal{G}_{(1)}(f), f \in \mathcal{F}_{(1)} \} \), which is characterized by its variance-covariance function

\[
\Gamma_{(1)}(f, g) = \int (f - \mathbb{E}_X(f))(g - \mathbb{E}_X(g))d\mathbb{P}_X, (f, g) \in \mathcal{F}_{(1)}^2.
\]

This Gaussian process is the asymptotic weak limit of the sequence of functional empirical processes (fep) defined as follows. Let \( X_1, X_2, ... \) be a sequence of independent copies of \( X \). For each \( n \geq 1 \), we define the functional empirical process associated with \( X \) by

\[
\mathcal{G}_{n,(1)}(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f(X_j) - \mathbb{E} f(X_j)), f \in \mathcal{F}_{(1)},
\]

and denote the integration with respect to the empirical measure by

\[
\mathbb{P}_{n,(1)}(f) = \frac{1}{n} \sum_{i=1}^{n} f(X_i), f \in \mathcal{F}_{(1)}.
\]

Let us denote by \( \ell^\infty(T) \) the space of real-valued bounded functions defined on \( T = \mathbb{R} \) equipped with its uniform topology. In the terminology of the weak convergence theory, the sequence of objects \( \mathcal{G}_{n,(1)} \) weakly converges to \( \mathcal{G}_{(1)} \) in \( \ell^\infty(\mathbb{R}) \), as stochastic processes indexed by \( \mathcal{F}_{(1)} \), whenever it is a Donsker class. The details of this highly elaborated theory may be found in [Billingsley (1968)], [Pollard (1984)], [van der Vaart and Wellner (1996)] and similar sources.

We only need the convergence in finite distributions which is a simple consequence of the multivariate central limit theorem, as described in Chapter 3 in [Lo et al. (2016)].
We will use the Renyi’s representation of the random variable $X_i$’s of interest by means ($cdf$) $F_{(1)}$ as follows

$$X = d F_{(1)}^{-1}(U),$$

where $U$ is a uniform random variable on $(0, 1)$, $=d$ stands for the equality in distribution and $F_{(1)}^{-1}$ is the generalized inverse of $F_{(1)}$, defined by

$$F_{(1)}^{-1}(s) = \inf\{x, F_{(1)}(x) \geq s\}, \ s \in (0, 1).$$

Based on these representations, we may and do assume that we are on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding a sequence of independent $(0, 1)$-uniform random variables $U_1, U_2, \ldots$, and the sequence of independent observations of $X$ are given by

$$X_1 = F_{(1)}^{-1}(U_1), \ X_2 = F_{(1)}^{-1}(U_2), \ etc.$$ (3.2)

For each $n \geq 1$, the order statistics of $U_1, \ldots, U_n$ and of $X_1, \ldots, X_n$ are denoted respectively by $0 \equiv U_{0,n} < U_{1,n} \leq \cdots \leq U_{n,n} < U_{n+1,n} \equiv 1$ and $X_{1,n} \leq \cdots \leq X_{n,n}$.

To the sequences of $(U_n)_{n \geq 1}$, we also associate the sequence of real empirical functions

$$\mathbb{U}_{n,(1)}(s) = \frac{1}{n} \#\{j, 1 \leq j \leq n, \ U_j \leq s\}, \ s \in (0, 1) \ n \geq 1$$ (3.3)

and the sequence of real uniform quantile functions

$$\mathbb{V}_{n,(1)}(s) = U_{1,n} 1_{s=0} + \sum_{j=1}^{n} U_{j,n} 1_{((j-1)/n < s \leq (j/n))}, \ s \in (0, 1), \ n \geq 1$$ (3.5)

and next, the sequence of real uniform empirical processes

$$\alpha_{n,(1)}(s) = \sqrt{n}(\mathbb{U}_{n,(1)} - s),$$ (3.6)

for $s \in (0, 1)$ and $\geq 1$, and the sequence of real uniform quantile processes
(3.7) \[ \gamma_{n,(1)}(s) = \sqrt{n}(s - V_{n,(1)}), \quad s \in (0,1), \quad n \geq 1. \]

The same can be done for the sequence \((X_n)_{n \geq 1}\), and we obtain the associated sequence of real empirical processes a

(3.8) \[ G_{n,r,(1)}(x) = \sqrt{n} \left( \mathbb{F}_{n,(1)}(x) - F_{(1)}(x) \right), \quad x \in \mathbb{R}, \quad n \geq 1 \]

where

(3.9) \[ \mathbb{F}_{n,(1)}(x) = \frac{1}{n} \# \{ j, 1 \leq j \leq n, \ X_j \leq x \}, \quad x \in \mathbb{R} \quad n \geq 1 \]

is the associated sequence of empirical functions. We also have the associated sequence of quantile processes

(3.10) \[ Q_{n,(1)}(x) = \sqrt{n} \left( \mathbb{F}^{-1}_{n,(1)}(s) - F^{-1}_{(1)}(s) \right), \quad s \in (0,1), \quad n \geq 1 \]

where, for \( n \geq 1 \),

(3.11) \[ \mathbb{F}^{-1}_{n,(1)}(s) = X_{1,n,1}(0 \leq s \leq 1/n) + \sum_{j=1}^{n} X_{j,n,1}(j-1/n \leq s \leq j/n), \quad s \in (0,1), \]

is the associated sequence of quantile processes.

By passing, we recall that \( \mathbb{F}^{-1}_{n,(1)} \) is actually the generalized inverse of \( \mathbb{F}_{(n),(1)} \) and for the uniform sequence, we have

(3.12) \[ V_{n,(1)} = U^{-1}_{n,(1)} \]

In virtue of Representation (3.2), we have the following remarkable relations

(3.13) \[ G_{n,r,(1)}(x) = \alpha_{n,(1)}(F_{(1)}(x)), \quad x \in \mathbb{R} \]

and

(3.14) \[ Q_{n,(1)}(x) = \sqrt{n} \left( F^{-1}_{(1)}(V_{n,(1)}(s)) - F^{-1}_{(1)}(s) \right), \quad s \in (0,1), \quad n \geq 1, \]

We also have the following relations between the empirical functions and quantile functions
\[ (3.15) \quad \mathbb{F}_{n,(1)}(x) = \mathbb{U}_{n,(1)}(F(1)(x)), \quad x \in \mathbb{R} \]

and

\[ (3.16) \quad \mathbb{F}_{n,(1)}^{-1}(s) = F_{(1)}^{-1}(\mathbb{V}_{n,(1)}(s)), \quad s \in (0, 1), \quad n \geq 1. \]

As well, the real and functional empirical processes are related as follows: for \( n \geq 1, \)

\[ (3.17) \quad \mathbb{G}_{n,r,(1)}(x) = \mathbb{G}_{n,(1)}(f^*_x), \quad \alpha_{n,(1)}(s) = \mathbb{G}_{n,(1)}(\tilde{f}_s), \quad s \in (0, 1), \quad x \in \mathbb{R}, \]

where for any \( x \in \mathbb{R}, \) \( f^*_x = 1_{-\infty,x} \) is the indicator function of \( ]-\infty,x] \) and for \( s \in (0, 1), \) \( f_s = 1_{[0,s]} \) and \( \tilde{f}_s = 1_{[-\infty, F_{(1)}^{-1}(s)]} \).

To finish the description, a result of Kiefer-Bahadur (See [Bahadur (1966)]) that says that the addition of the sequences of uniform empirical processes and quantiles processes (3.6) and (3.7) is asymptotically, and uniformly on \([0,1]\), zero in probability, that is

\[ (3.18) \quad \sup_{s \in [0,1]} |\alpha_{n,(1)}(s) + \gamma_{n,(1)}(s)| = o_p(1) \quad \text{as} \quad n \to +\infty. \]

This result is a powerful tool to handle the rank statistics when our studied statistics are \( L \)-statistics.

**Bivariate frame.** As to the bivariate case, we use the Sklar’s theorem (See [Sklar (1959)]). We can also refer to [Lo (2018)] for a quick proof of Sklar’s Theorem. Let us begin to define a copula in \( \mathbb{R}^2 \) as bivariate probability distribution function \( C(u, v), \quad (u, v) \in \mathbb{R}^2 \) with support \([0,1]^2\) and with \([0,1]\)-uniform margins, that is

\[ C(u, v) = 0 \quad \text{for} \quad (u, v) \in [0\times\mathbb{R}. \]

Let us denote by \( F_{(2)} \) the bivariate distribution function of our random couple \( Y = (X^{(1)}, X^{(2)}) \) and by \( F_{(21)} \) and \( F_{(22)} \) its margins, which are the \( cdf \) of \( X^{(1)} \) and \( X^{(2)} \) respectively. The Sklar’s theorem ([Sklar (1959)]) says that there exists a copula \( C_{(2)} \) such that we have

\[ (3.19) \quad F_{(2)}(x, y) = C_{(2)}(F_{(21)}(x), F_{(22)}(y)), \quad \text{for any} \quad (x, y) \in \mathbb{R}^2. \]
This copula is unique if the marginal cdf’s are continuous. In this paper, we will suppose that the marginal cdf’s are continuous and then \( C(2) \) is unique and fixed for once. By the Kolmogorov Theorem, there exists a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) holding a sequence of independent random couples \((U_n^{(1)}, U_n^{(2)})\), \(n \geq 1\), of common bivariate distribution function \(C(2)\). On that space the random couples \((F_{(21)}^{-1}(U_n^{(1)}), F_{(22)}^{-1}(U_n^{(2)}))\) are independent and have a common bivariate distribution function equal to \(C(2)\), since

\[
\mathbb{P}(F_{(21)}^{-1}(U_i^{(1)}) \leq x_1, F_{(22)}^{-1}(U_i^{(2)}) \leq x_2) = \mathbb{P}(U_i^{(1)} \leq F_{(21)}(x_1), U_i^{(2)} \leq F_{(22)}(x_2)) = C(2)(F_{(21)}(x_1), F_{(22)}(x_2)) = F(2)(x_1, x_2),
\]

by (3.19), and where we applied the general formula for generalized inverses functions for a cdf:

\[
F^{-1}(s) \leq y \Leftrightarrow s \leq F(x), \text{ for } (s, x) \in [0, 1] \times \mathbb{R}.
\]

For more on interesting properties of generalized inverses of monotone functions, see [Lo et al. (2016)], Chapter 4.

Based on this remark, we place ourselves on the probability space holding the sequence of independent random couples \((U^{(1)}, U^{(2)}), (U_n^{(1)}, U_n^{(2)}), n \geq 2\), with common distribution function \(C(2)\), and the observations from \( Y = (X^{(1)}, X^{(2)}) = (F_{(21)}^{-1}(U^{(1)}), F_{(22)}^{-1}(U^{(2)})) \), are generated as follows:

\[
Y_n = (F_{(21)}^{-1}(U_n^{(1)}), F_{(22)}^{-1}(U_n^{(2)})), \ n \geq 1.
\]

We may directly study the empirical process

\[
\mathbb{G}_{n,(2)}(h) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( h(X_j^{(1)}, X_j^{(2)}) - \mathbb{P}(X^{(1)}, X^{(2)})(h) \right).
\]

where \( h \in L_2(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathbb{P}(X^{(1)}, X^{(2)})).\)

In this setting, we rather use the bidimensional functional empirical process based on \( \{(U_i^{(1)}, U_i^{(2)})\}_{i=1,...,n} \) and defined by
\[(3.22) \quad G_{n,u,(2)}(\tilde{h}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( \tilde{h}(U_{j}^{(1)}, U_{j}^{(2)}) - P( (U^{(1)}, U^{(2)}) \cdot \tilde{h}) \right), \]

whenever \( \tilde{h} \) is a function of \((u, v) \in [0, 1]^2 \) such that \( \mathbb{E}(\tilde{h}(U^{(1)}, U^{(2)})^2) \) is finite.

For any Donsker class \( \mathcal{F}(2)([0, 1]^2) \), the stochastic process \( G_{n,u,(2)} \) converges to a Gaussian process \( \mathbb{T} \) with variance-covariance function, for \((f, g) \in L^2_2([0, 1]^2, \mathbb{P}(U^{(1)}, U^{(2)})) \), denoted by \( \tilde{\Gamma}_{(2)}(f, g) \), is given the following formula we name (GammaStar)

\[
\int_{[0,1]^2} \left( f(u,v) - \mathbb{P}(U^{(1)}, U^{(2)}) (f) \right) \left( g(u,v) - \mathbb{P}(U^{(1)}, U^{(2)}) (g) \right) dC(u,v)
\]

with

\[
\mathbb{P}(U^{(1)}, U^{(2)}) (f) = \mathbb{E}( f(U^{(1)}, U^{(2)})) = \int_{[0,1]^2} f(u,v) dC(u,v)
\]

and the same is true for \( g \). So, by using the transform

\[(3.23) \quad \tilde{h}(s,t) = h \left( F_{(2),1}^{-1}(s), F_{(2),2}^{-1}(t) \right), \quad (s, t) \in [0, 1]^2, \]

and the representation \( (3.20) \), we get the remarkable following relation for any \( h \), whenever one of the members makes sense,

\[(3.24) \quad G_{n,(2)}(h) = G_{n,u,(2)}(\tilde{h}). \]

All the needed notation are now complete and will allow the expression of the asymptotic theory we undertake here.

4. The residual empirical process

(A) - The origin.

There is a considerable class of statistics which are combinations of one dimensional statistics of the form
where $q_0$ is some measurable mapping, $c(\circ, n)$ a function of $j \in \{1, \cdots, n\}$ and $(d_n)_{n \geq 1}$ is a sequence of real numbers. If $F_{(1)}$ is continuous, we may use the rank statistics $(R_{1,n}, \cdots, R_{n,n})$ defined by

$$
\forall 1 \leq i \leq n, \forall 1 \leq j \leq n, \; R_{j,n} = i \iff X_{i,n} = X_j.
$$

Thus, for $n \geq 1$, $L_n$ becomes

$$
L_n = \sum_{1 \leq j \leq n} F_{n,(1)}q_0(X_j).
$$

But it happens that for any $n \geq 1$, for any $1 \leq j \leq n$,

$$
\frac{R_{j,n}}{n} = F_{n,(1)}(X_j),
$$

and this leads to

$$
L_n = \frac{1}{n} \sum_{1 \leq j \leq n} \left( n d_n c \left( n F_{n,(1)}(X_j) \right) \right) q_0(X_j), \; n \geq 1.
$$

Fortunately, in many cases, there exists a measurable mapping $g$ such that $E(|X|) < +\infty$ and

$$
L_n = \frac{1}{n} \sum_{1 \leq j \leq n} q_1(F_{(1)}(X_j))q_0(X_j)
$$

$$
+ \frac{1}{n} \sum_{1 \leq j \leq n} \left( n d_n c \left( n F_{n,(1)}(X_j) \right) - q_1(F_{(1)}(X_j)) \right) q_0(X_j)
$$

and that, by means of the mean value theorem,
4. THE RESIDUAL EMPIRICAL PROCESS

\[ \frac{1}{n} \sum_{1 \leq j \leq n} \left( nd_n c \left( n \mathbb{F}_{n,(1)}(X_j) \right) - q_1 \left( F_{(1)}(X_j) \right) \right) h(X_j) \]
\[ = \frac{1}{n} \sum_{1 \leq j \leq n} \left( \mathbb{F}_{n,(1)}(X_j) - F_{(1)}(X_j) \right) q_3(X_j) q_1(X_j) + o_P(n^{-1/2}). \]

Upon specific conditions to be checked, we arrive at the form

\[ L_n = \frac{1}{n} \sum_{1 \leq j \leq n} h(X_j) + \frac{1}{n} \sum_{1 \leq j \leq n} \left( \mathbb{F}_{n,(1)}(X_j) - F_{(1)}(X_j) \right) q(X_j) + o_P(n^{-1/2}). \]

We conclude that, in our effort to asymptotically represent \( L_n \) as an application of the empirical measure to some function \( h \), that is \( \mathbb{P}_n(h) \), we still have a residual term in the form of

\[ Re_n(\ell) = \frac{1}{n} \sum_{j=1}^{n} \left( \mathbb{F}_{n,(1)}(X_j) - F_{(1)}(X_j) \right) q(X_j). \]

This made [Lo (2010)] to name it a residual empirical process and proceeded to its independent study.

Now let us describe deeper this stochastic process.

(B) - Residual empirical processes.

A residual empirical process is any stochastic process of the form

\[ Re_n(\ell) = \frac{1}{n} \sum_{j=1}^{n} \left( \mathbb{F}_{n,(1)}(X_j) - F_{(1)}(X_j) \right) q(X_j). \]

where \( q \) is a measurable function from \([0,1]\) to \( \mathbb{R} \) and

\[ \ell(s) = q(F_{(1)}^{-1}(s)), \ s \in (0, 1) \]

and

\[ \Delta_n(s) = \left( \ell \left( \mathbb{V}_{n,(1)}(s) \right) - \ell(s) \right), \ s \in (0, 1). \]
We stress that the function $\ell$ depends on the cdf $F(1)$ and should have been denoted $\ell(\circ) = \ell(F(1), \circ)$. This warning is important in the situation of spatial analysis, as we will see it.

4.1. General result.

Theorem 1. If the following two assertions:

1. (CRe1) $\mathbb{E}q(X) < +\infty$

and,

2. and, as $n \to +\infty$,

$$\int_0^1 \sqrt{n} \left( s - \mathbb{V}_{n(1)}(s) \right) \Delta_n(s) \, ds \to 0 \quad (CRe2)$$

holds, we have the representation

$$\sqrt{n}R_{n}(\ell) = \int_0^1 G_{n(1)}(\tilde{f}_s) \ell(s) \, ds + o_p(1),$$

Proof. By using Formulas (3.3) and (3.5), we get

$$R_{n} = \sum_{j=1}^{n} \int_{j-1/n}^{j} \left\{ \mathbb{F}_{n(1)}(\mathbb{F}_{n(1)}^{-1}(s)) - F(1)(\mathbb{F}_{n(1)}^{-1}(s)) \right\} q\left( \mathbb{F}_{n(1)}^{-1}(s) \right) \, ds,$$

and hence

$$R_{n} = \int_0^1 \left\{ F_{n(1)}(F_{n(1)}^{-1}(s)) - F(1)(F_{n(1)}^{-1}(s)) \right\} q\left( F_{n(1)}^{-1}(s) \right) \, ds. \tag{4.1}$$

By using Formulas (3.12), (3.13) and (3.14), we get

$$\sqrt{n}R_{n} = -\int_0^1 \sqrt{n} \left\{ \mathbb{U}_{n(1)}(\mathbb{V}_{n(1)}(s)) - \mathbb{V}_{n(1)}(s) \right\} q\left( F(1)(\mathbb{V}_{n(1)}(s)) \right) \, ds$$

$$= -\int_0^1 \sqrt{n} \left( s - \mathbb{V}_{n(1)}(s) \right) q\left( F(1)(\mathbb{V}_{n(1)}(s)) \right) \, ds$$

$$-\int_0^1 \sqrt{n} \left( \mathbb{U}_{n(1)}(\mathbb{V}_{n(1)}(s)) - s \right) q\left( F(1)(\mathbb{V}_{n(1)}(s)) \right) \, ds$$

$$=: R_{n}(1) + R_{n}(2).$$
From [Shorack and Wellner (1995)] (page 585), we have

$$\sup_{0 \leq s \leq 1} \left| \mathbb{U}_{n,(1)} \left( \mathbb{V}_{n,(1)}(s) \right) - s \right| \leq \frac{1}{n}.$$ 

We get

$$|Re_n(2)| \leq \frac{1}{\sqrt{n}} \int_0^1 q \left( F_{(1)}^{-1} \left( \mathbb{V}_{n,(1)}(s) \right) \right) ds$$

$$= \frac{1}{\sqrt{n}} \left( \frac{1}{n} \sum_{j=1}^n q(X_j) \right), \quad (CRe0)$$

which is an $o_p(n^{-1/2})$ whenever $\mathbb{E}q(X)$ is finite. Under the Assumption $(Re2)$, we may replace $q \left( F_{(1)}^{-1} \left( \mathbb{V}_{n,(1)}(s) \right) \right)$ by $q \left( F_{(1)}^{-1}(s) \right)$, to get

$$\sqrt{n}Re_n = - \int_0^1 \sqrt{n} \left( s - \mathbb{V}_{n,(1)}(s) \right) F_{(1)}^{-1} \left( \mathbb{V}_{n,(1)}(s) \right) ds + o_p(1)$$

(4.2) $$= - \int_0^1 \gamma_{n,(1)}(s) q \left( F_{(1)}^{-1}(s) \right) ds + o_p(1),$$

and by using the Bahadur’s representation (See Formula 3.18) and by applying Formula 3.17, we arrive at

$$\sqrt{n}Re_n = \int_0^1 \mathbb{G}_{n,(1)}(\tilde{f}_s) \ell(s) ds + o_p(1),$$

whenever

$$\mathbb{E}(\ell(X)) = \int_0^1 q(F_{(1)}^{-1}(s)) ds < +\infty.$$ 

This concludes the proof.
4.2. Checking the Conditions (Re1) and (Re2). We preferred to state Theorem 1 with general the condition (Re2) and not to enter in detailed forms based on convergence theorems. Instead, in each case, we will check whether or not they hold. Let us give here some general more specific conditions based on properties of the empirical process. Let us go back to the place where we apply (Re2) in the proof, that is, in Formula (4.2). First, we replace $\gamma_{n,(1)}$ by the uniform empirical process $G_{n,(1)} \gamma_{n,(1)}$ to have

$$\sqrt{n}R_n = \int_0^1 \mathbb{G}_{n,(1)}(s) \ell(\mathbb{V}_{n,(1)}(s)) \, ds$$

$$- \int_0^1 (\gamma_{n,(1)}(s) + \mathbb{G}_{n,(1),r}(s)) \ell(\mathbb{V}_{n,(1)}(s)) \, ds + o_p(1).$$

$$=: R_n(3) + R_n(4) + o_p(1).$$

The exact rate of convergence in the Bahadur-Kiefer Theorem (See [Shorack and Wellner (1995)], p.620) is $a_n = n^{-1/2}(\log \log n)^{1/4}, n > e$, that is

$$\lim \sup_{n \to +\infty} \sup_{0 \leq s \leq 1} a_n |\gamma_{n,(1)}(s) + \mathbb{G}_{n,(1),r}(s)|/a_n = 1/2, \ a.s.\]$$

A condition that $R_n(3) = o_p(1)$ is

$$\lim \sup_{n \to +\infty} a_n \int_0^1 \ell(\mathbb{V}_{n,(1)}(s)) \, ds. \]$$

This is obviously true if $q$ is bounded, which will be the case in many situation. Next, we may write

$$R_n(4) = \int_0^1 \mathbb{G}_{n,(1),r}(s) \ell(s) \, ds + R_n(5),$$

with, for $\nu$ fixed such that $0 < \nu < 1$,

$$R_n(5) = \int_0^1 \mathbb{G}_{n,(1),r}(s) \Delta_n(s) ds$$

$$\leq \int_0^1 (s(1-s))^{1-\nu} \sup_{0 \leq s \leq 1} \left| \frac{\mathbb{G}_{n,(1),r}(s)}{(s(1-s))^{1-\nu}} \right| |\Delta_n(s)| \, ds$$

But we have
\[ \Delta_n = \sup_{0 \leq s \leq 1} \left| \frac{G_{n,(1),r}(s)}{(s(1 - s))^{1-\nu}} \right| = O_P(1), \text{ as } n \to +\infty. \]

(See for instance [Csörgö et al. (1986)], Formulas 2.7, 2.8, 4.2.18, third and fourth formulas in page 69, first formula in page 70). Now, a condition that \( R(e_n)(5) = o_P(1) \) is

\[ \int_0^1 (s(1 - s))^{1-\nu} \ell(V_{n,(1)}(s)) \, ds = O_P(1), \text{ as } n \to +\infty, \]

which, by \((Rc1)\), is obviously obtained whenever

\[ \int_0^1 (s(1 - s))^{1-\nu} \delta_n(s) = o_P(1), \text{ as } n \to +\infty, \quad (CRe4). \]

which is obtained if \( \ell \), for instance, \( \ell \) is continuous, and hence uniformly, on \((0, 1)\).

Remind that \( (Cre1) \) was used first in Formula \((CRe0)\) above. In reality, the conclusion was obtained if

\[ \frac{1}{\sqrt{n}} \left( \frac{1}{n} \sum_{j=1}^{n} q(X_j) \right) = o_p(1), \text{ as } n \to +\infty, \quad (CRe3). \]

In conclusion, the result in Theorem 1 is still valid if the latter formulas \((CRe3)\) and \((CRe4)\) hold.

### 5. General handling

Let us show how works the methodology.

**Part A : Fixed time scheme.**

For a number of statistics, the representation of the form \((2.4)\) is possible by directly applying the method of Section 2.

Unfortunately, most of the statistics, used in Welfare analysis, use the rank statistics so that the statistics is sum of terms that are products of a function of the ordered statistic \( X_{j,n} \) by a function of the rank \( j \). In such a case, it is usually possible, as we described in the lines above and as we will see it in the examples, to express the current statistic \( I_n \) into a sum of two terms such that:
(a) the first is a functional empirical probability $P_{n,(1)}(h)$,

(b) the second of the form:

$$Re_n(\ell) = \frac{1}{n} \sum_{j=1}^{n} (\mathbb{P}_{n,(1)}(X_j) - F_{(1)}(X_j)) q(X_j).$$

where $\ell(s) = q(F^{-1}_{(1)}(s))$, $s \in [0,1]$. [Lo and Sall (2010)] called this process as a residual one. Among results, it is shown in the cited paper that, under smooth assumptions on $q$ (see the cited reference), the Bahadur representation exploitation leads to

$$\sqrt{n}R_n(\ell) = \int_0^1 G_{n,(1)}(\tilde{f}_s) \ell(s) \, ds + o_{\mathbb{P}}(1),$$

which in turn, leads to

$$\sqrt{n}(I_n - \mathbb{E}h(X)) = G_{n,(1)}(h) + \beta_{n,(1)}(\ell) + o_{\mathbb{P}}(1) \quad (GRI)$$

The ordered pair $(G_{n,(1)}(h), \beta_{n,(1)}(\ell))$ is constructed such that it inherits the weak convergence $G_{n,(1)}$ to $G_{(1)}$, which entails the convergence of that couple to a Gaussian bivariate random variable $(G_{(1)}(h), \beta_{(1)}(\ell))$. With the proper handling, as we will do in Section 1 in Chapter 3, we will have no difficulty to have the general law:

$$\sqrt{n}(I_n - \mathbb{E}h(X)) \sim \mathcal{N}(0, \sigma^2)$$

where $\Gamma = \gamma_1 + 2\gamma_3$, with

$$\Gamma_{(1)}(h,h) = \int (h(x) - \mathbb{E}(h(X)))^2 dF_{(1)}(x)$$

and

$$\gamma_1 = \Gamma_{(1)}(h,h), \quad \gamma_2 = \int_0^1 \int_0^1 \Gamma_{(1)}(f_s,f_t) \, ds \, dt \text{ and } \gamma_3 = \int_0^1 \Gamma_{(1)}(h,f_s) \, ds.$$
When dealing, in a fixed time, with a family \( (I_n(\lambda), \lambda \in \Lambda) \) of welfare indices based on the real-valued variables \( X > 0 \), we may represent them by the family of their representations

\[
G_{n,(1)}(h_\lambda) + \beta_{n,(1)}(\ell_\lambda), \; \lambda \in \Lambda.
\]
Part B : Spatial scheme.

Suppose that we are monitoring the same index $I$ over a population divided on $K$ subgroups or areas and the particular value of the index in the $i$-th area, denoted $S_i$, is named as $I^{(i)}$, $i = 1, ..., K$. Let $X$ be the random variable which composes $I$ and let $F^{(i)}_i$ be the cdf of $X$ on $S_i$, denoted $X^{(i)}$, and $F$ be the cdf of $X$ on the global population. Suppose that we perform independent studies on each area $S_i$ with respective samples of sizes $n_i$ for $X^{(i)}$. We get:

(a) For each $i$, a representation of the empirical index $I^{(i)}_h$ in the form

$$G^{(i)}_{n_i,1}(h) + \beta^{(i)}_{n_i,1}(\ell_i) + o_P(n_i^{-1/2}).$$

where $\ell_i(.) = q(F^{(i)}_i^{-1}(\circ))$ and $G^{(i)}_{n_i,1}$ is the fep based on the sample sample $X^{(i)}$ with common cdf $F^{(i)}$.

(b) It is important to see that the function $h$ may depend on the cdf. Thus, the function $h$ may vary with $i$.

From these two points, finding the laws of aggregated indices from the $I^{(i)}$'s are readily obtained. Interesting questions may also be treated if the sub-samples are not independent. For examples, the decomposability gap may be estimated in a purely random drawing in the whole population (See Chapter 2 below). It is remarkable that, in Formula (5.2), the function $h$ is constant for over the areas since if depends on the mathematical form of the index.

If more than one index is monitored with respect to areas, we still may label them with $\lambda$ and use the results of Part A.
Part C : Time Evolution Scheme.

To be simple, suppose that we monitor the same index $I$ over two periods $t = 1$ and $s = 2$ and we name as $I^{(i)}$ at the period $i = 1, 2$, and by $I_n^{(i)}$ their empirical counterparts. Let $X = (X^{(1)}, X^{(2)})$ be the vector of the two incomes from time 1 to time 2. How do we set the frame in which the evolution of the index $I$ is easily handled, at least in the theoretical way?

It will be enough to use the joint $fep$ and next to use projections in the notations introduced in Section 3. Suppose that

$$\sqrt{n}(I_n - \mathbb{E}h(X)) = G_n(h) + \beta_{n,(1)}(\ell) + o_P(1)$$

is the general representation of $I$ at a fixed time. It is important to see that this form depends only on the mathematical form of $I$ and on the $cdf$ through $\ell$. As a reminder, $G_{n,(2)}$ is the $fep$ based on the observations $(X_1^{(1)}, X_1^{(2)}), \ldots, (X_n^{(1)}, X_n^{(2)})$. Denote:

$$h^{(1)}(x, y) = h_1(x), \ h^{(2)}(x, y) = h_2(y), \ (x, y) \in \mathbb{R}^2,$$

$$f_s^{(1)}(x, y) = 1_{(x \leq F_{(1)}^{-1}(s))}, \ f_s^{(2)}(x, y) = 1_{(y \leq F_{(2)}^{-1}(s))}, \ s \in [0, 1] \ and \ (x, y) \in \mathbb{R}^2.$$

and

$$\ell^{(i)}(s) = q(F_{(i)}^{-1}(s)), \ i = 1, 2$$

We have

$$I_n^{(i)} = I^{(i)} + n^{-1/2} (G_{n,(2)}(h^{(i)}) + \beta_{n,(2)}(\ell^{(i)})) + o_P(n^{-1/2}), \ i = 1, 2.$$  

where

$$\beta_{n,(2)}(\ell) = \int_0^1 G_{n,(2)}(\tilde{f}_s^{(i)}) \ell^{(i)}(s) \, ds + o_P(1)$$

Here again, we conclude as follows:
The asymptotic probability law of \((I_n^{(1)}, I_n^{(2)})\) is readily obtained through Formula (5.3), allowing any kind of comparison or evolution study.

The frame we have set allows to express all needed variances or covariances.

The generalization to \(k\) times and then to behavior of \(k\) \((I_n^{(1)}, ..., I_n^{(k)})\) is straightforward, even if the notation become heavier.

We will only describe it below.
CHAPTER 2

Statistical decomposability of indices

1. Introduction

One of the most desired axiom of a welfare measure is the decomposability one. Let us begin explain that concept.

Suppose that we are monitoring some index $I$ over a given population of size $N$. When $I$ is applied to the whole population, we may use the notation $I = I_N$. In a large population subjected to a number of inequalities between areas and in which there are groups with specific features at the exclusion of the others, public policy efficiency usually requires to target disadvantaged areas or groups and to implement therein strong strategies aimed at improving the status of this group in relation to a given pattern (for example poverty, health covering, education level, etc.), monitored by the index $I$. In such a case, the population is divided into sensitive $K$ subgroups of interest $S_1, \ldots, S_K$ of respective sizes $N_i, i \in \{1, \ldots, K\}$, and the studied behavior is followed up by an index, say $I$, taking the values $I^{(i)} = I^{(i)}_N$ in each subgroup $S_i, i \in \{1, \ldots, K\}$.

The index $I$ is said to be decomposable if we may express the global index on the whole population with respect to the partial indices at the subgroup level as follows, that is

\begin{equation}
I_N = \sum_{1 \leq i \leq K} \frac{N_i}{N} I_N(i).
\end{equation}

Formula (1.2) offers the practical and comfortable latitude to work at the local level with the possibility to recompose the global index at the global level. This explains why decomposable indices are so preferred, in particular the Foster-Greer-Thorbecke ([Foster et al.(1984)]) index of parameter $\alpha \geq 0$,

\begin{equation}
FGT_n(\alpha) = \frac{1}{n} \sum_{1 \leq j \leq n} \max \left( \frac{Z - X_j}{Z}, 0 \right)^\alpha, \quad \alpha \geq 0.
\end{equation}
The problem is that some of the most interesting measures are not decomposable, in particular the weighted ones. Indeed, successful policies require to target disadvantaged or vulnerable groups. For example, suppose that we are dealing with poverty. A measure that counts all poor individuals with the same weight is less interesting than another that puts bigger weights to poorer individuals. A variation of such an index in the good direction tends to be negligible if the less poor individual behave better, and to be noticeable if the poorer individuals among the poor become better off.

Our problematic is to keep using weighted measures like the ones of Sen (1976), Kakwani (1980), Shorrocks (1995), Takayama (1979), to cite a few, and yet, to have a quick approach to report the global situation.

The solution resides certainly in the estimation of the decomposability gap:

\[ g_N = I_N - \sum_{1 \leq i \leq K} \frac{N_i}{N} I_{N_i}. \]

We will see later that we will be able to estimate this gap. Then we will be able to work at a local level and to report the global index in accurate confidence interval.

Recently, Haidara and Lo (2012) motivated the estimation of decomposability gap of non-decomposable measures in the sense described above. Their results seem to be the first of that kind. The original work of Haidara and Lo concerned the general poverty index GPI Lo (2013). But, these results implicitly include their extensions to any indice admitting the indice’s general representation (GRI) in Section 5, Chapter 1.

In the sequel, we suppose that we are working with indices satisfying the (GRI) representation. Let us precise the statistical problem.

We already described the decomposability in a non-random context. We are going to describe it in the random frame.

Suppose that the population is divided into \( K \) subgroups \( S_1, \ldots, S_K \) and for each \( i \in \{1, \ldots, K\} \), let us denote the subset of the random sample
\{X_1, \ldots, X_n\} coming from \(S_i\) by \(\mathcal{E}_i = \{X_{i,1}, \ldots, X_{i,n_i^*}\}\) and then put \(I_n^{(i)} = I(X_{i,1}, \ldots, X_{i,n_i^*})\) the random value of the index \(I\) under study on the \(i^{th}\) subgroup. We denote by \(F_{i,(1)}\) the cdf of \(X\) on \(S_i\). Let \(I_n = I(X_1, \ldots, X_n)\) be the observed index on the whole sample. The empirical decomposability gap is defined by

\[
gd_n = I_n - \frac{1}{n} \sum_{i=1}^{K} n_i^* I_n^{(i)}.\]

At this step, we have to precise our random drawing. We are going to use a probability space in the form \((\Omega_1 \times \Omega_2, \mathcal{P}(\Omega_1) \otimes \mathcal{A}_2, \mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)})\), with \(\Omega_1 = \{1, 2, \ldots, K\}\), \(\mathcal{P}(\Omega_1)\) is the power set of \(\Omega_1\) and \(\mathbb{P}^{(1)}\) is the discrete uniform probability on \(\Omega_1\) such that \(\mathbb{P}^{(1)}(\{i\}) = p_i, 1 \leq i \leq K\). We draw the observations in the following way. In each trial \(j\), we draw a subgroup according to \(\mathbb{P}^{(1)}\). We define

\[
\pi_{i,j}(\omega_1) = 1_{\{\text{the } i^{th} \text{ subgroup is drawn at the } j^{th} \text{ trial}\}}(\omega_1),
\]

where, \(1 \leq i \leq K, 1 \leq j \leq n\). Now, given that the \(i^{th}\) subgroup is drawn at the \(j^{th}\) trial, we pick one individual in this subgroup, according to \(\mathbb{P}^{(2)}\), and observe its income \(X_j(\omega_1, \omega_2)\). We then have the observations

\[
\{X_j(\omega_1, \omega_2), 1 \leq j \leq n\}.
\]

Here, \(\mathbb{P}^{(2)}\) is the probability in Section 3 of Chapter 1. We denote \(\mathbb{P} = \mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)}\), while keeping in mind that the representations in \((\text{GRI})\) in Section 4 of Chapter 1 are valid with respect to \(\mathbb{P}^{(2)}\).

We have these simple facts. First, for \(1 \leq i \leq K\).

\[
(1.3) \quad n_i^* = \sum_{j=1}^{n} \pi_{i,j}.
\]

Let us denote the distribution of \(X_j\) given \((\pi_{i,j} = 1)\), by \(F_{i,(1)}\) that is

\[
\mathbb{P}(X_j \leq y / \pi_{i,j} = 1) = F_{i,(1)}(x).
\]
2. STATISTICAL DECOMPOSABILITY OF INDICES

We simply put, in some places, \( F_{i,(1)}(x) = F_{i,(1)} \), \( y \in \mathbb{R} \), to keep the notation simple. Then we have

\[
\forall (x \in \mathbb{R}), \mathbb{P}(X_j \leq y) = \sum_{i=1}^{K} \mathbb{P}(\pi_{i,j} = 1) \mathbb{P}(X_j \leq y / \pi_{i,j} = 1) = \sum_{i=1}^{K} p_i F_{i,(1)}(x).
\]

We conclude that \( \{X_1, ..., X_n\} \) is an independent sample drawn from \( F_{(1)}(x) = \sum_{i=1}^{K} p_i F_{i,(1)}(x) \), which is the mixture of the distribution functions of the subgroups incomes.

The formula above ensures that for any real-valued function \( h \) such that the \( h(X^{(i)}) \)'s are integrable, we have

\[
Eh(X) = \sum_{1 \leq i \leq K} p_i F^{(i)}_{(1)}.
\]

Finally, we readily see that conditionally on \( n^* \equiv (n_1^*, n_2^*, ..., n_K^*) = (n_1, n_2, ..., n_K) \equiv \pi \) with \( n_1 + n_2 + ... + n_K = n \), \( \{X_{i,j}, 1 \leq j \leq n_i\} \) are independent random variables with distribution function \( F_{i,(1)} \).

2. General Statistical Decomposition Theorem

We suppose that the indice’s general representation (GRI) in Section 4, Chapter 1, with

\[
\ell(s) = q \left( F_{(1)}^{-1}(s) \right), \quad s \in (0, 1).
\]

We already knew that the function \( h \) in the (GRI) formula may depend on the cdf \( F_{i,(1)} \) on each subgroup to become \( h_i \) and denote accordingly

\[
\ell_i(s) = q_i \left( F_{i,(1)}^{-1}(s) \right), \quad s \in (0, 1).
\]

Let us introduce the constants:

\[
A_1 = \sum_{i=1}^{K} p_i \left\{ \int_0^1 (h - h_i)^2 (F_{i,(1)}^{-1}(t)) dt - \left( \int_0^1 (h - h_i) (F_{i,(1)}^{-1}(t)) dt \right)^2 \right\},
\]
\[ A_2 = \sum_{i=1}^{K} P_i \int_0^1 \int_0^1 (s \wedge t - st)(p_i \ell - \ell_i)(s)(p_i \ell - \ell_i)(s)dsdt, \]

\[ A_{31} = \sum_{i=1}^{K} p_i^2 \sum_{h \neq i}^{K} p_h \int_0^1 \int_0^1 \left[ F_h(F^{-1}_{i,(1)}(s)) \wedge F_h(F^{-1}_{j,(1)}(t)) \right. \]
\[ \left. - F_h(F^{-1}_{i,(1)}(s))F_h(F^{-1}_{j,(1)}(t)) \right] \ell(s)q(F^{-1}_{i,(1)}(t))ds dt, \]

\[ A_{32} = \sum_{i=1}^{K} p_i \sum_{j \neq i}^{K} p_j \sum_{h \neq \{i,j\}}^{K} p_h \int_0^1 \int_0^1 \left[ F_h(F^{-1}_{i,(1)}(s)) \wedge F_h(F^{-1}_{j,(1)}(t)) \right. \]
\[ \left. - F_h(F^{-1}_{i,(1)}(s))F_h(F^{-1}_{j,(1)}(t)) \right] \ell(s)\ell(s)ds dt, \]

\[ B_1 = \sum_{i=1}^{K} p_i \int_0^1 \left\{ \int_0^s (h - h_i)(F^{-1}_{i,(1)}(t))dt \right. \]
\[ \left. - s \int_0^1 (h - h_i)(F^{-1}_{i,(1)}(t))dt \right\} (p_i \ell - \ell_i)(s)ds, \]

\[ B_2 = \sum_{j=1}^{K} p_j \sum_{i \neq j}^{K} p_i \int_0^1 \int_0^1 \left[ s \wedge F_{i,(1)}(F^{-1}_{j,(1)}(t)) - sF_{i,(1)}(F^{-1}_{j,(1)}(t)) \right], \]
\[ \times (p_i \ell - \ell_i)(s)\ell(s)ds dt, \]

\[ B_3 = \sum_{j=1}^{K} p_j \sum_{i \neq j}^{K} p_i \int_0^1 \left\{ \int_0^1 (h - h_i)(F^{-1}_{i,(1)}(t))dt \right. \]
\[ \left. - F_{i,(1)}(F^{-1}_{j,(1)}(s)) \times \int_0^1 (h - h_i)(F^{-1}_{i,(1)}(t))dt \right\} \ell(s) ds, \]

\[ gd = I - \sum_{i=1}^{K} p_i I^{(i)} \]

and, finally,

\[ gd_{0,n} = I - \sum_{i=1}^{K} (n_i^*/n)I^{(i)}. \]

We will need the following components of our variances. First, define for \( i = 1, ..., K \)
2. STATISTICAL DECOMPOSABILITY OF INDICES

\[ L_i = \mathbb{E} h(X^i) - I_i + \sum_{\alpha=1}^{K} p_\alpha \mathbb{E} F_{i,(1)}(X^{(\alpha)})q(X^{(\alpha)}), \]

and

\[ M_i = \mathbb{E} h(X^i) + \sum_{\alpha=1}^{K} p_\alpha \mathbb{E} F_{i,(1)}(X^{(\alpha)})q(X^{(\alpha)}). \]

Next, define

\[ \vartheta_1^2 = A_1 + A_2 + A_3 + 2(B_1 + B_2 + B_3) \]

and

\[ \vartheta_2^2 = \sum_{i=1}^{K} L_i^2 p_i - \left( \sum_{i=1}^{K} L_i p_i \right)^2 \]

and

\[ \vartheta_3^2 = \sum_{\alpha=1}^{K} M^2_\alpha p_\alpha - \left( \sum_{\alpha=1}^{K} M_\alpha p_\alpha \right)^2 \]

Here is the general decomposability result.

2.1. The theoretical result. We have the following result.

**Theorem 2.** Let \( \mathbb{E} X^2 < \infty \), \( \mathbb{E} (X^{(i)})^2 < \infty \). Let us suppose also that \( F_{(1)} \) and each \( F_{i,(1)}, 1 \leq i \leq K \) are increasing so that they are invertible. Let assume also the conditions (FHEP1) for the validity of the (GRI) representations of the indices holds on each subgroup and at the whole area.

Then we have

\[ gd_{n,0}^* = \sqrt{n}(gd_n - gd_{0,n}) \sim N(0, \vartheta_1^2 + \vartheta_3^2) \]

and

\[ gd_n^* = \sqrt{n}(gd_n - gd) \sim N(0, \vartheta_1^2 + \vartheta_2^2) \]
A particular version of this theorem has already been proved in [Haidara and Lo (2012)], for specific welfare indices. A more general proof based only on the GRI is proposed below.

3. Proof of the Theorem

From the assumptions, we write

$$\sqrt{n}(I_n - I) = \mathcal{G}_{n,(1)}(h) + \beta_{n,(1)}(\ell) + o_p(1),$$

with

$$\beta_{n,(1)}(\ell) = \int_0^1 \mathcal{G}_{n,(1)}(f_s)\ell(s)ds,$$

and for $i = 1, \ldots, k$, for non-random sizes $n_i$ becoming infinitely large,

$$\sqrt{n}(I_{n_i}^{(i)} - I^{(i)}) = \mathcal{G}_{n_i,(1)}(h_i) + \beta_{n_i,(1)}(\ell_i) + o_p(1),$$

with

$$\beta_{n_i,(1)}(\ell_i) = \mathcal{G}_{n_i,(1)}(f_s)\ell_i(s)ds.$$

Here we have simplified the notation and used $\mathcal{G}_{n_i,(1)}$ instead of $\mathcal{G}_{n_i,(1)}^{(i)}$ which is the functional empirical process based on observations from the random variable $X^{(i)}$. We think that there will be no confusion because of the subscript $n_i$ that will remind us that we are on the $i^{th}$ subgroup.

To begin the proof, we remark that $n^*(\omega_1) = (n^*_1(\omega_1), \ldots, n^*_K(\omega_1)) \rightarrow_p \{+\infty\}^K$ as $n = n^*_1(\omega_1) + \ldots + n^*_K(\omega_1) \rightarrow \infty$.

We then get

(3.1) $$\sqrt{n}(I_n - I) = \mathcal{G}_{n,(1)}(h) + \beta_{n,(1)}(\ell) + o_p(1) := \gamma_n + o_p(1)$$

and for any $1 \leq i \leq K$,

(3.2) $$\sqrt{n^*_i}(I_{n_i}^{(i)} - I^{(i)}) = \mathcal{G}_{n_i,(1)}(h_i) + \beta_{n_i,(1)}(\ell_i) + o_p(1) := \gamma_{i,n^*_i} + o_p(1)$$
Now we use the intermediate centering coefficient

\[ gd_{0,n} = I - \sum_{i=1}^{K} \frac{n^*_i}{n} I^{(i)} \]

and, after some direct manipulations based on (3.1) and (3.2), to find

\[
\left| \sqrt{n}(gd_n - gd_{0,n}) - \left\{ \gamma_n - \sum_{j=1}^{K} \left( \frac{n^*_j}{n} \right)^{1/2} \gamma_{i,n_i} \right\} \right| (\omega_1, \omega_2) = o_{P_1 \otimes P_2}(1),
\]

as \( n \to \infty \). Then, we have that \( S^*_n \) is equal to

\[
\gamma_n - \sum_{j=1}^{K} \left( \frac{n^*_j}{n} \right)^{1/2} \gamma_{i,n_i}^*
= \mathbb{G}_{n,(1)}(h) - \sum_{j=1}^{K} \left( \frac{n^*_j}{n} \right)^{1/2} \mathbb{G}_{n^*_i,(1)}(h) + \beta_{n,(1)}(\ell) - \sum_{j=1}^{K} \left( \frac{n^*_j}{n} \right)^{1/2} \beta_{n^*_i,(1)}(\ell_i).
\]

We use Formula (1.4) and remark that

\[
\mathbb{G}_{n,(1)}(h) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (h(X_j) - \mathbb{E}h(X)) = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^{n} h(X_j) - \mathbb{E}h(X) \right)
= : \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^{n} h(X_j) - \sum_{i=1}^{K} \frac{n^*_i}{n} \mathbb{E}h(X^{(i)}) \right) + D^*(n, 1),
\]

with

\[
D^*(n, 1) = \sum_{i=1}^{K} \frac{n^*_i - np_i \mathbb{E}h(X^{(i)})}{\sqrt{np_i}} \sqrt{p_i}.
\]

When conditioning on \( n^* = n \), we denote

\[
D(n, 1) = \sum_{i=1}^{K} \frac{n_i - np_i \mathbb{E}h(X^{(i)})}{\sqrt{np_i}} \sqrt{p_i},
\]

This leads to

\[
S^*_n = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^{n} h(X_j) - \sum_{i=1}^{K} \frac{n^*_i}{n} \mathbb{E}h(X^{(i)}) \right) - \sum_{j=1}^{K} \left( \frac{n^*_j}{n} \right)^{1/2} \mathbb{G}_{n^*_i,(1)}(h_i)
\]

2. STATISTICAL DECOMPOSABILITY OF INDICES
Now, by denoting
\[ C^*(n, 1) = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^{n} h(X_j) - \sum_{i=1}^{K} \frac{n_i^*}{n} \mathbb{E} h(X^{(i)}) \right) - \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right)^{1/2} \mathbb{G}_{n_i^*, (1)}^* (h_i), \]
we have
\[
C^*(n, 1) = \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right)^{1/2} \left\{ \frac{1}{\sqrt{n_i^*}} \sum_{j=1}^{n_i^*} \left\{ (h - h_i) (X_{ij}) - \mathbb{E} (h - h_i) (X^{(i)}) \right\} \right\}.
\]
We get
\[
(3.3) \quad S_n^* = C^*(n, 1) + D^*(n, 1) + \beta_{n,(1)}(\ell) - \sum_{j=1}^{K} \left( \frac{n_j^*}{n} \right)^{1/2} \beta_{n_j^*, (1)}(\ell_j).
\]
Further, we have
\[
(3.4) \quad \sum_{j=1}^{K} \left( \frac{n_j^*}{n} \right) \beta_{n_j^*, (1)}(\ell_j) = \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \mathbb{G}_{n_i^*, (1)}^* (X_{ij}) - F_{i,(1)} (X_{ij}) \right] q_i(X_{ij}).
\]
But
\[
F_{i}(X_{ij}) = \sum_{h=1}^{K} p_h F_{h,(1)} (X_{ij}),
\]
and for \( x \in \mathbb{R} \)
\[
\mathbb{G}_{n,r,(1)}(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{(x \leq x)} = \frac{1}{n} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} 1_{(x_{ij} \leq x)}
\]
\[
= \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right) \frac{1}{n_i^*} \sum_{j=1}^{n_i^*} 1_{(x_{ij} \leq x)} = \sum_{i=1}^{K} \frac{n_i^*}{n} \mathbb{G}_{n_i^*, (1)}(x).
\]
Thus
\[
\beta_{n,(1)}(\ell) = \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( \frac{n_h^*}{n} \right) G_{n_h^*, r,(1)} (X_{ij}) - p_h F_{h,(1)} (Y_{ij}) \right] q(X_{ij}).
\]
From this, we put and subtract \( \sum_{h=1}^{k} \left( \frac{n_i^*}{n} \right) F_{h,(1)}(X_{ij}) \) to have

\[
\beta_{n,(1)}(\ell) = \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( \frac{n_h^*}{n} \right) G_{h,n_i^*}(X_{ij}) - \sum_{h=1}^{K} \left( \frac{n_h^*}{n} \right) F_{h,(1)}(X_{ij}) \right] q(X_{ij})
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( \frac{n_h^*}{n} - p_h \right) F_{h,(1)}(X_{ij}) \right] q(X_{ij})
\]

(3.5) \[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( \frac{n_h^*}{n} \right) \left( G_{h,n_i}(X_{ij}) - F_{h,(1)}(X_{ij}) \right) \right] q(X_{ij})
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( \frac{n_h^*}{n} - p_h \right) F_{h,(1)}(X_{ij}) \right] q(X_{ij}).
\]

Now we put together (3.4) and (3.5), while separating the two cases \( h = i \) and \( h \neq i \) in (3.5) to get

\[
\beta_{n,(1)}(\ell) - \sum_{j=1}^{K} \left( \frac{n_i^*}{n} \right) \beta_{n_i^*,(1)}(\ell_i)
\]

\[
= \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right)^{1/2} \left\{ \frac{1}{\sqrt{n_i^*}} \sum_{j=1}^{n_i^*} \left( G_{i,n_i^*}(X_{ij}) - F_{i,(1)}(X_{ij}) \right) \left( \frac{n_i^*}{n} q - q_i \right) (X_{ij}) \right\}
\]

\[
+ \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right)^{1/2} \sum_{h \neq i} \left( \frac{n_h^*}{n} \right) \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( G_{h,n_i^*}(X_{ij}) - F_{h,(1)}(X_{ij}) \right) \right] q(X_{ij})
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{K} \sum_{j=1}^{n_i^*} \left[ \sum_{h=1}^{K} \left( \frac{n_h^*}{n} - p_h \right) F_{h,(1)}(X_{ij}) \right] q(X_{ij})
\]

(3.6) \[
=: C^*(n, 2) + C^*(n, 3) + D^*(n, 2),
\]

with

\[
C^*(n, 2) = \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right)^{1/2} \left\{ \frac{1}{\sqrt{n_i^*}} \sum_{j=1}^{n_i^*} \left( G_{i,n_i^*}(X_{ij}) - F_{i,(1)}(X_{ij}) \right) \left( \frac{n_i^*}{n} q - q_i \right) (X_{ij}) \right\},
\]
and

\[ C^*(n, 3) \quad \text{(c3)} \]

\[ = \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right)^{1/2} \sum_{h \neq i}^{K} \frac{n_h^*}{n} \left\{ \frac{1}{\sqrt{n_i^*}} \sum_{j=1}^{n_i^*} \{ G_{h,n_i^*} (X_{ij}) - F_{h,(1)} (X_{ij}) \} q(X_{ij}) \right\} . \]

We arrive, by comparing (3.3) and (3.6), at

(3.7) \[ S_n^* = C^*(n, 1) + C^*(n, 2) + C^*(n, 3) + D^*(n, 1) + D^{**}(n, 2). \]

Let us have a look at

\[ D^{**}(n, 2) = \sqrt{n} \sum_{h=1}^{K} \left( \frac{n_h^*}{n} - p_h \right) \left\{ \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right) \frac{1}{n_i^*} \sum_{j=1}^{n_i^*} F_{h,(1)} (X_{ij}) q(X_{ij}) \right\} . \]

By the weak law of large numbers

\[ \left\{ \sum_{i=1}^{K} \left( \frac{n_i^*}{n} \right) \frac{1}{n_i^*} \sum_{j=1}^{n_i^*} F_{h,(1)} (X_{ij}) q(X_{ij}) \right\} \rightarrow \sum_{i=1}^{K} p_i E F_{h,(1)} (X_i) q(X_i) = H_h. \]

That is

\[ D^{**}(n, 2) = \sum_{h=1}^{K} \left( \frac{n_h^* - n p_h}{\sqrt{n p_h}} \right) H_h \sqrt{p_h} + o_P(1). \]

\[ =: D^*(n, 2) + o_P(1). \]

Finally, we have for all \( n \geq 1, \)

(3.8) \[ g d_n^* = S_n^* + \sqrt{n} (g d_{0,n} - g d). \]

Hence

\[ g d_n^* = C^*(n, 1) + C^*(n, 2) + C^*(n, 3) \]

\[ + D^*(n, 1) + D^*(n, 2) - \sum_{i=1}^{K} \left( \frac{n_i^* - n p_i}{\sqrt{n p_i}} \right) I^{(i)} \sqrt{p_i} + o_P(1), \]

(3.9) \[ =: C^*(n) + D^*(n) + o_P(1), \]

with

(3.10) \[ C^*(n) = C^*(n, 1) + C^*(n, 2) + C^*(n, 3) \]
and
\[
D^*(n) = D^*(n, 1) + D^*(n, 2) - \sum_{i=1}^{K} \left( \frac{n^*_i - np_i}{\sqrt{np_i}} \right) I(i) \sqrt{p_i}
\]
\[
= \sum_{i=1}^{K} \left( \frac{n^*_i - np_i}{\sqrt{np_i}} \right) (H_i + \mathbb{E} h(X^{(i)}) - I^{(i)}) \sqrt{p_i}
\]
\[
= \sum_{i=1}^{K} \left( \frac{n^*_i - np_i}{\sqrt{np_i}} \right) F_i \sqrt{p_i}.
\]

We have now to prove that \( gd^*_n = \sqrt{n}(gd_n - gd) \) weakly converges to a \( N(0, \vartheta_1^2 + \vartheta_2^2) \) random variable. For this it suffices, based on (3.9), to prove that \( S^*_{\pi} = C^*(n) + D^*(n) \) converges to \( N(0, \vartheta_1^2 + \vartheta_2^2) \). Now put
\[
\mathbb{N}(K) = \{ \pi = (n_1, \ldots n_K), n_i \geq 0, n_1 + \ldots + n_K = n \}.
\]

Since \( n^* = (n^*_1, \ldots n^*_K) \rightarrow_{p_1} \{ \infty \}^K \), we find for a fixed \( \varepsilon > 0 \), \( K \) positive numbers \( N_i \) (\( 1 \leq i \leq K \)) such that for \( n_i \geq N_i \) (\( 1 \leq i \leq K \)), which implies that \( n \geq N = N_1 + \ldots + N_K \),
\[
\mathbb{P}(\exists (1 \leq i \leq K), n^*_i < N_i) < \varepsilon.
\]

Let
\[
\mathbb{N}(K, 1) = \mathbb{N}(K) \cap \{ \pi = (n_1, \ldots n_K), \exists (1 \leq i \leq K), n_i < N_i \}
\]

and \( \mathbb{N}(K, 2) = \mathbb{N}(K) \backslash \mathbb{N}(K, 1) \). We remark that conditionally on \( (n^* = \pi) \), \( C^*(n) \) becomes \( C(n) \), does not depend on \( \omega_1 \) and only include the independent random variables \( \{X_{i,j}, 1 \leq j \leq n_i, 1 \leq i \leq K \} \). From Lemma 3 below, we have
\[
C(n) \rightarrow \mathcal{N}(0, \vartheta_1^2).
\]

Also conditionally on \( (n^* = \pi) \), \( D^*(n) \) becomes \( D(n) \) and we denote it \( D(n) \). Now for \( h^2 = -1 \),
\[
\psi_{S^*_n}(t) = \mathbb{E}(\exp(htS^*_n))
\]
\[
= \sum_{\pi \in \mathbb{N}(K)} \mathbb{P}(n^* = \pi) \mathbb{E}(\exp(htC^*(n) + htD^*(n)) / (n^* = \pi))
\]
\[
= \sum_{\pi \in \mathbb{N}(K)} \mathbb{P}(n^* = \pi) \mathbb{E}(\exp(htD(n)) \mathbb{E}(\exp(htC^*(n)) / (n^* = \pi))).
\]
Recall that, by the classical limiting law of the multinomial $K$-vector,

$$D^*(n) \to D = \sum_{i=1}^{K} Z_i F_i \sqrt{p_i},$$

where $(Z_1, ..., Z_K)^t$ is a Gaussian vector with $Var(Z_i) = 1 - p_i$ and $Cov(Z_i, Z_j) = -\sqrt{p_i p_j}$, for $i \neq j$. Then

$$D^*(n) \to \mathcal{N}(0, \vartheta_2^2),$$

with

$$\vartheta_2^2 = \sum_{h=1}^{K} F_h^2 p_h (1 - p_h) - \sum_{1 \leq h \neq k \leq K} F_h F_k p_h p_k$$

$$= \sum_{h=1}^{K} F_h^2 p_h - \left( \sum_{h=1}^{K} F_h p_h \right)^2.$$

We remark that this is the variance of the function $F_h$ of $h \in [1, K]$ with respect to the probability measure $\sum_{1 \leq h \leq K} p_h \delta_h$.

Put now

$$N(K, 1) = N(K) \cap \{\pi = (n_1, ... n_K), \exists (1 \leq i \leq K), n_i < N_i\}$$

and $N(K, 2) = N(K) \setminus N(K, 1)$. Then

$$\sum_{\pi \in N(K)} \exp(htD(n)) P(n^* = \pi) \mathbb{E}(\exp(htC(n))) = B(n, 1) + B(n, 2)$$

with

$$|B(n, 1)| = \left| \sum_{\pi \in N(K, 1)} \exp(htD(n)) P(n^* = \pi) \mathbb{E}(\exp(htC(n))) \right|$$

$$\leq \mathbb{P}(\exists (1 \leq i \leq K), n_i^* < N_i) \to 0,$$

and

$$|B(n, 2) - \sum_{\pi \in N(K, 2)} \exp(-(\vartheta_1 t)^2/2) \exp(htD(n)) P(n^* = \pi)|$$
\[ \leq \varepsilon \sum_{\pi \in \mathbb{N}(K,2)} \mathbb{P}(n^* = \pi) \leq \varepsilon. \]

Finally, for
\[ (3.13) \quad B^*(n, 2) = \sum_{\pi \in \mathbb{N}(K)} \exp(-\vartheta_1 t^2/2) \exp(htD(n))\mathbb{P}(n^* = \pi), \]
we are able to use (3.13) and to get
\[ \limsup_{n \to \infty} \left| B^*(n, 2) - \sum_{\pi \in \mathbb{N}(K)} \exp(htD(n))\mathbb{P}(n^* = \pi)\mathbb{E}(\exp(-\vartheta_1 t^2/2)) \right| = 0. \]

But
\[ (3.14) \quad \mathbb{E}\exp(thD^*(n)) = \sum_{\pi \in \mathbb{N}(K)} \exp(htD^*(n)/(n^* = \pi))\mathbb{P}(n^* = \pi) \quad \rightarrow \exp(-\vartheta_2 t^2/2)) \]

By putting together the previous formulas, and by letting \( \varepsilon \downarrow 0 \), we arrive at
\[ \psi_{d^*_s}(t) \rightarrow \exp(-\vartheta_1^2 + \vartheta_2^2 t^2/2). \]

This proves the asymptotic normality of \( dg_{n*}^* \) of the theorem corresponding to \( S_n^{**} \). That of \( dg_{n,0}^* \) corresponds to \( S_n^* \). This latter is achieved by omitting the term \( \sqrt{n} \sum_{i=1}^{K} (n^*_i/n - p_i)I^{(i)} \) in (3.8). This leads to \( M_{th} \) obtained from \( F_h \) by dropping \( I^{(i)} \). This completes the proofs.

We now prove this lemma used in the proof.

**Lemma 3.** Let \( C(n) = C(n, 1) + C(n, 2) + C(n, 3) \), where the \( C(n, i) \) are respectively defined in Formula (c1) (page 49), Formula (c2) (page 50) and Formula (c3) (page 51) for \( i = 1, 2, 3 \). Then, as \( n \to +\infty \),
\[ C(n) \sim \mathcal{N}(0, \vartheta_i^2). \]

Recall that
\[ (3.15) \quad C(n) = C(n, 1) + C(n, 2) + C(n, 3). \ (ca) \]

At this step, new tools are introduced and the attention of the reader is drawn. Using the continuity of the the cdf \( F_{i,(1)} \)'s, we are
going to use the functional empirical process based on the independent and \((0, 1)\)-uniform random variables \(\{F_{i,1}(X_{i,j}), 1 \leq i \leq n_i\}\) for each \(1 \leq i \leq K\). The remainder of the proof uses this frame. So let \(G_{n_i}(i, f), U_{n_i}(i, \circ)\) and \(V_{n_i}(i, \circ)\), be the functional empirical process, the empirical cdf and quantile functions based on \(\{F_{i,1}(X_{i,j}), 1 \leq i \leq n_i\}\) for each \(1 \leq i \leq K\). Similarly we define the functional empirical process, the empirical cdf and quantile functions based on the whole sample \(\{F_{1}(X_{i,j}), 1 \leq i \leq K, 1 \leq i \leq n_i\}\) by dropping the label \(i\) in the definition relative to group \(i \in \{1, ..., K\}\).

We will consider the three terms in Formula (ca) (page 54), that is the \(C(n, i), 1 \leq i \leq 3\), defined in Formula (c1) (page 49), Formula (c2) (page 50) and in Formula (c3) (page 51), and prove that each of them converges to a random variable \(C(i)\) depending on the limiting Gaussian processes \(G(i, \cdot)\) of \(G_{n_i}(i, \cdot)\). This is enough to prove the asymptotic normality. The variance \(\vartheta^2_i\) will be nothing else but that of \(C(1) + C(2) + C(3)\). Firstly, we treat \(C(n, 1)\). Remark that conditionally on \((n^* = \overline{n})\), the random sequences \(\{X_{i,j}, 1 \leq i \leq n_i, 1 \leq i \leq K\}\) are independent and only depend on the \(\omega_2 \in \Omega_2\). We have

\[
\sum_{i=1}^{K} \left(\frac{n_i}{n}\right)^{1/2} \mathbb{G}_{n_i^*,(1)}(h_i) = \frac{1}{\sqrt{n}} \left[ \sum_{i=1}^{K} \sum_{j=1}^{n_i} h_i(X_{i,j}) - \sum_{i=1}^{K} n_i \mathbb{E}(h_i(X^{(i)})) \right]
\]

\[
= \sqrt{n} \left[ \frac{1}{n} \sum_{i=1}^{K} \sum_{j=1}^{n_i} h_i(X_{i,j}) - \sum_{i=1}^{K} \left(\frac{n_i}{n}\right) \mathbb{E}(h_i(X^{(i)})) \right],
\]

and

\[
\alpha_n(h, 1) = \sqrt{n} \left( \frac{1}{n} \sum_{j=1}^{n} h(X_j) - \sum_{i=1}^{K} \left(\frac{n_i}{n}\right) \mathbb{E}(h(X^{(i)})) \right)
\]

\[
= \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{K} \sum_{j=1}^{n_i} h(X_{i,j}) - \sum_{i=1}^{K} \left(\frac{n_i}{n}\right) \mathbb{E}(h(X^{(i)})) \right).
\]

Then, by Formula (c1) (page 49) and replacing \(n^*_i\) by \(n_i\), \(i = 1, ..., K\), we get

\[
C(n, 1) = \alpha_n(h, 1) - \sum_{i=1}^{K} \left(\frac{n_i}{n}\right) \mathbb{G}_{n_i^*,(1)}(h_i)
\]
\[ (3.16) \]
\[
= \sum_{i=1}^{K} \left( \frac{n_i}{n} \right)^{1/2} \left\{ \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} \left\{ \left( h - h_i \right) X_{ij} - \mathbb{E} \left( h - h_i \right) (X^{(i)}) \right\} \right\}. 
\]

This implies that
\[
C(n, 1) = \sum_{i=1}^{K} \left( \frac{n_i}{n} \right)^{1/2} \mathbb{G}_{n_i} \left( i, (h - h_i) F_{i,(1)}^{-1} \right). 
\]

We finally have that
\[
C(n, 1) \to C(1) = \sum_{i=1}^{K} p_i^{1/2} \mathbb{G}(i, (h - h_i) F_{i,(1)}^{-1}). 
\]

Since the \( \mathbb{G} \left( i, (h - h_i) F_{i,(1)}^{-1} \right) \) are independent, centered and Gaussian, we get that
\[
A_1 = \mathbb{E} C^2(1) = \sum_{i=1}^{K} p_i \mathbb{E} \mathbb{G}^2(i, (h - h_i) F_{i,(1)}^{-1}) 
\]
\[
= \sum_{i=1}^{K} p_i \left\{ \mathbb{E} (h - h_i)^2(X^{(i)}) - (\mathbb{E}(h - h_i)(X^{(i)}))^2 \right\}. 
\]

Then we arrive
\[
A_1 = \sum_{i=1}^{K} p_i \left\{ \int_{0}^{1} (\bar{h} - h_i)^2(F_{i,(1)}^{-1}(t))dt - \left( \int_{0}^{1} (\bar{h} - h_i)(F_{i,(1)}^{-1}(t))dt \right) ^2 \right\}. 
\]

Secondly, one has
\[
C(n, 2) = \sum_{i=1}^{K} \left( \frac{n_i}{n} \right)^{1/2} \left\{ \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} \left\{ \mathbb{G}_{n_i,(1)}(X_{ij}) - F_{i,(1)}(X_{ij}) \right\} \left( \frac{n_i}{n} q - q_i \right) (X_{ij}) \right\}. 
\]

We have
\[
\frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} \left\{ \mathbb{G}_{n_i}(X_{ij}) - F_{i,(1)}(X_{ij}) \right\} \left( \frac{n_i}{n} q - q_i \right) (X_{ij}) \]
\[
= \int_{0}^{1} -\varepsilon_{n_i}(i, s)(p_i q - q_i)(F_{i,(1)}^{-1}(s))ds + o_P(1) 
\]
3. Proof of the Theorem

\[ = \int_0^1 \mathbb{G}_{n_1}(i, s)(p_i q - q_i)(F_{i,1}^{-1}(s))ds + o_P(1) \]

\[ \rightarrow \int_0^1 \mathbb{G}(i, s)(p_i q - q_i)(F_{i,1}^{-1}(s))ds, \]

and thus

\[
(3.17) \quad C(n, 2) \rightarrow C(2) = \sum_{i=1}^{K} p_i^{1/2} \int_0^1 \mathbb{G}(i, s)(p_i q - q_i)(F_{i,1}^{-1}(s))ds.
\]

Finally, we have

\[
C(n, 3) = \sum_{i=1}^{K} \left( \frac{n_i}{n} \right)^{1/2} \sum_{h \neq i}^{K} \frac{n_h}{n} \left\{ \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} \{ G_{h,n_h}(X_{ij}) - F_{h,1}(X_{ij}) \} q(X_{ij}) \right\}.
\]

But, for each fixed \( i \in \{1, .., K\} \),

\[
= \int_0^1 \sqrt{n_i} \left\{ G_{h,n_h}(F_{i,1}^{-1}(V_{n_i}(i, s))) - F_{h,1}(F_{i,1}(V_{n_i}(i, s))) \right\} \times q(F_{i,1}(V_{n_i}(i, s)))ds.
\]

By the assumptions, the functions \( q \) and \( F_{1}(1) \) are continuous on such compact sets. Thus

\[
= \sqrt{n_i} \int_0^1 \mathbb{G}_{n_h}(h, F_{i,1}(F_{i,1}(V_{n_i}(i, s)))) \times q(F_{i,1}(V_{n_i}(i, s)))ds
\]

\[ = \sqrt{n_i} \int_0^1 \mathbb{G}_{n_h}(h, F_{h,1}(F_{i,1}(V_{n_i}(i, s)))) \times q(F_{i,1}(V_{n_i}(i, s)))ds + o_P(1) \]

\[ = \sqrt{n_i} \int_0^1 \mathbb{G}_{n_h}(h, F_{h,1}(F_{i,1}(s))) \times q(F_{i,1}(s))ds + R_n + o_P(1), \]

with

\[ R_n = \int_0^1 \left\{ \mathbb{G}_{n_h}(h, F_{h,1}(F_{i,1}(V_{n_i}(i, s)))) - \mathbb{G}_{n_h}(h, F_{h,1}(F_{i,1}(s))) \right\} \times q(F_{i,1}(s))ds. \]
Based on the assumption that, for any \((i, h) \in \{1, ..., K\}^2\),

\[
\sup_{s \leq 1} \left| F_{h,(1)}(F_{i,(1)}^{-1}(V_n(i, s))) - F_{h,(1)}(F_{i,(1)}^{-1}(s)) \right| = a_n \to 0.
\]

We obtain here a continuous modulus of the uniform empirical process (see [Shorack and Wellner (1995)], page 531) and then

\[
\sup_{0 \leq s \leq 1} \left| \{ G_{n,h}(h, F_{h,(1)}(F_{i,(1)}^{-1}(V_n(i, s)))) - G_{n,h}(h, F_{i,(1)}^{-1}(s)) \} \right| = O(\sqrt{-a_n \log a_n}).
\]

We finally get

\[
R_n = O\left(\sqrt{-a_n \log a_n}\right) \int_0^1 q(F_{i,(1)}^{-1}(s))ds \to 0
\]

and we arrive at

\[
(3.18) \quad C(n, 3) \to C(3)
\]

\[
= \sum_{i=1}^K p_i \sum_{h \neq i} \sqrt{p_h} \int_0^1 G(h, F_{h,(1)}(F_{i,(1)}^{-1}(s))) \times q(F_{i,(1)}^{-1}(s))ds.
\]

Now, we are going to compute the variance \(\sigma^2 \) based on the independent functional Brownian bridges \(G(i, \cdot)\) which are limits of the functional empirical process \(G_n(i, \cdot)\) respectively associated with \(\{F_{i,(1)}(X_{ij}), 1 \leq i \leq n_i\}, i = 1, .., K\). Straightforward calculations give what comes.

First

\[
A_1 = \mathbb{E}C^2(1) = \sum_{i=1}^K p_i \mathbb{E}G^2(i, (h - h_i)F_{i,(1)}^{-1}).
\]

In order to lessen the expressions, we write for \(i \in \{1, \cdots, K\},

\[
h_i^*(\phi) = (h - h_i)(F_{i,(1)}^{-1}(\phi)), \text{ and } c_i(\phi) = (p_iq - q_i)\left(F_{i,(1)}^{-1}(\phi)\right).
\]

Next for

\[
C'(2) = \sum_{i=1}^K p_i^{1/2} \int_0^1 G(i, s)(p_iq - q_i)(F_{i,(1)}^{-1}(s))ds
\]

we have

\[
A_2 = \mathbb{E}(C^2(2)) = \sum_{i=1}^K p_i \int_0^1 \int_0^1 (s \wedge t - st)c_i(t)c_i(s)dsdt
\]
3. Proof of the Theorem

\[ \sum_{i} p_i \int_{0}^{1} \int_{0}^{1} (s \land t - st)(p_i q_i - q_i)(F_{i,(1)}^{-1}(s))(p_i q_i - q_i)(F_{i,(1)}^{-1}(t))dsdt, \]

* Now for

\[ C(3) = \sum_{i=1}^{K} p_i \sum_{h \neq i} \sqrt{p_h} \int_{0}^{1} G(h, F_{h,(1)}(F_{i,(1)}^{-1}(s))) \times q(F_{i,(1)}^{-1}(s))ds, \]

we have

\[ A_3 = \mathbb{E}(C^2(3)) \]

\[ = \mathbb{E} \left\{ \sum_{i=1}^{K} p_i^2 \left( \sum_{h \neq i} K_{i,h} \right)^2 + \sum_{i=1}^{K} \sum_{j \neq i} p_i p_j \left( \sum_{h \neq i} K_{i,h} \right) \left( \sum_{h' \neq j} K_{j,h'} \right) \right\}. \]

Put

\[ K_{i,h} = \sqrt{p_h} \int_{0}^{1} G(h, F_{h,(1)}(F_{i,(1)}^{-1}(s))) \times q(F_{i,(1)}^{-1}(s))ds, \]

Let us split \( A_3 \) into

\[ A_{31} = \mathbb{E} \left( \sum_{i=1}^{K} p_i^2 \left( \sum_{h \neq i} K_{i,h} \right)^2 \right) \]

and

\[ A_{32} = \mathbb{E} \left( \sum_{i=1}^{K} \sum_{j \neq i} p_i p_j \left( \sum_{h \neq i} K_{i,h} \right) \left( \sum_{h' \neq j} K_{j,h'} \right) \right). \]

Now by using the independence of the centered stochastic process \( G(h, \cdot) \) for different values of \( h \in \{1, \ldots, K\} \), one gets

\[ A_{31} = \mathbb{E} \left( \sum_{i=1}^{K} p_i^2 \left( \sum_{h \neq i} K_{i,h} \right)^2 \right) \]

and then

\[ A_{31} = \sum_{i=1}^{K} p_i^2 \sum_{h \neq i} p_h \int_{0}^{1} \int_{0}^{1} \left[ F_{h,(1)}(F_{i,(1)}^{-1}(s)) \land F_{h,(1)}(F_{i,(1)}^{-1}(t)) - F_{h,(1)}(F_{i,(1)}^{-1}(s)) F_{h,(1)}(F_{i,(1)}^{-1}(t)) \right] q(F_{i,(1)}^{-1}(s))q(F_{i,(1)}^{-1}(t))dsdt. \]
Next, one has

\[ A_{32} = \mathbb{E} \sum_{i=1}^{K} p_i \sum_{j \neq i}^{K} p_j \sum_{h \neq i}^{K} p_h \sum_{h' \neq j}^{K} p_{h'} \int_{0}^{1} \int_{0}^{1} \]

\[ \mathcal{G}(h, F_{h,(1)}(F_{i,(1)}^{-1}(s))) \mathcal{G}(h', G_{h'}(F_{j,(1)}^{-1}(t))) q(F_{i,(1)}^{-1}(s)) q(F_{j,(1)}^{-1}(t)) dt ds \]

\[ = \sum_{i=1}^{K} p_i \sum_{j \neq i}^{K} p_j \sum_{h \notin \{i,j\}}^{K} p_h \int_{0}^{1} \int_{0}^{1} \left[ F_{j,(1)}(F_{i,(1)}^{-1}(s)) \wedge F_{h,(1)}(F_{i,(1)}^{-1}(s)) \right] q(F_{i,(1)}^{-1}(s)) q(F_{j,(1)}^{-1}(t)) ds \ dt. \]

Now we have

\[ C(1)C(2) = \left( \sum_{i=1}^{K} p_i^{1/2} \mathcal{G}(i, h_i^*) \right) \left( \sum_{i=1}^{K} p_i^{1/2} \int_{0}^{1} \mathcal{G}(i, s) c_i(s) ds \right) \]

\[ = \sum_{i=1}^{K} p_i^{1/2} \sum_{j=1}^{K} p_j^{1/2} \int_{0}^{1} \mathcal{G}(i, s) c(s) \mathcal{G}(j, h_j^*) c_i(s) ds. \]

And we get

\[ B_1 = \mathbb{E} C(1) C(2) = \sum_{i=1}^{K} p_i \int_{0}^{1} \mathbb{E}(\mathcal{G}(i, s) \mathcal{G}(i, \ell_i) c_i(s) ds \]

\[ = \sum_{i=1}^{K} p_i \int_{0}^{1} \left\{ \int_{-\infty}^{F_{i,(1)}^{-1}(s)} (h - h_i(y)) dF_{i,(1)}(y) - s \mathbb{E}(h - h_i(X^{(i)})) \right\} c_i(s) ds \]

\[ = \sum_{i=1}^{K} p_i \int_{0}^{1} \left\{ \int_{0}^{s} (h - h_i)(F_{i,(1)}^{-1}(t)) dt \ight. \]

\[ - s \int_{0}^{1} (h - h_i)(F_{i,(1)}^{-1}(t)) dt \left\} (p_i q - q_i) (F_{i,(1)}^{-1}(s)) ds. \]

We have next

\[ C(2)C(3) = \left( \sum_{i=1}^{K} p_i^{1/2} \int_{0}^{1} \mathcal{G}(i, s) c_i(s) ds \right) \]

\[ \times \left( \sum_{i=1}^{K} p_i \sum_{h \neq i}^{K} p_{h}^{1/2} \int_{0}^{1} \mathcal{G}(h, F_{h,(1)}(F_{i,(1)}^{-1}(s))) q(F_{i,(1)}^{-1}(s)) ds \right) \]

\[ = \sum_{i=1}^{K} p_i^{1/2} \sum_{j=1}^{K} p_j \sum_{h \neq j}^{K} p_{h}^{1/2} \int_{0}^{1} \int_{0}^{1} \mathcal{G}(i, s) \mathcal{G}(h, F_{h,(1)}(F_{j,(1)}^{-1}(t))) c_i(s) q(F_{j,(1)}^{-1}(t))) ds dt. \]
It is derived from what above that

\[ B_2 = \mathbb{E}C(2)C(3) = \sum_{j=1}^{K} p_j \sum_{i \neq j}^{K} p_i \int_{0}^{1} \int_{0}^{1} \left[ s \wedge F_{i,(1)}(F_{j,(1)}^{-1}(t)) - s F_{i,(1)}(F_{j,(1)}^{-1}(t)) \right] \times (p_i q - q_i) \{ F_{i,(1)}^{-1}(s) \} q(F_{j,(1)}^{-1}(t)) \, ds \, dt. \]

Now finally for

\[ C(1)C(3) = \left( \sum_{i=1}^{K} p_i^{1/2} \mathcal{G}(i, \ell_i) \right) \times \left( \sum_{i=1}^{K} p_i \sum_{h \neq i}^{K} p_h^{1/2} \int_{0}^{1} \mathcal{G}(h, F_{h,(1)}(F_{i,(1)}^{-1}(s)) \times q(F_{i,(1)}^{-1}(s)) \, ds \right) \]

\[ = \sum_{i=1}^{K} p_i^{1/2} \sum_{j=1}^{K} p_j \sum_{h \neq j}^{K} p_h^{1/2} \int_{0}^{1} \mathcal{G}(h, F_{h,(1)}(F_{j,(1)}^{-1}(s)) \mathcal{G}(i, h_i^*) \times q(F_{j,(1)}^{-1}(s)) \, ds, \]

where the \( h_i^* \)'s are defined in (3), we have

\[ B_3 = \mathbb{E}C(1)C(3) \]

\[ = \sum_{j=1}^{K} p_j \sum_{i \neq j}^{K} p_i \int_{0}^{1} \mathbb{E} \left\{ \mathcal{G}(i, h_i^*) \mathcal{G}(i, F_{i,(1)}(F_{j,(1)}^{-1}(s)) \right\} \times q(F_{i,(1)}^{-1}(s)) \, ds \]

\[ = \sum_{j=1}^{K} p_j \sum_{i \neq j}^{K} p_i \int_{0}^{1} \left\{ \int_{0}^{1} (h - h_i)(F_{i,(1)}^{-1}(t)) \, dt \right\} q(F_{i,(1)}^{-1}(s)) \, ds. \]

We have now finished the variance computation, that is

\[ \vartheta_1^2 = A_1 + A_2 + A_3 + 2(B_1 + B_2 + B_3) \]
CHAPTER 3

Asymptotic Laws of indices, of their absolute and relative variation of indices

In all this chapter, we use limiting results on variance-covariances of finite linear combinations of the margins of a same sequences of stochastic processes whose finite-distributions converge to those of a Gaussian processes.

1. Asymptotic Laws of indices

Suppose we deal with an index $I$. Suppose that general representation (GRI) in Section 4 in Chapter 1 holds for the sampled indice $I_n$, that is $h(X)$ is square integrable and that conditions (Re1) and (Re2) of Theorem 1 (Section 4 in Chapter 1) also are satisfied for $\ell$. We refer to these conditions as (HFEP1).

**Theorem 3. (General law of Indice)** Suppose that Assumptions (HFEP1) hold. Then we have as $n \to +\infty$,

$$I_n^* = \sqrt{n}(I_n - I) \rightarrow N(0, \Gamma),$$

where $\Gamma = \gamma_1 + \gamma_2 + 2\gamma_3$, with

$$\Gamma(1)(h, h) = \int (h(x) - \mathbb{E}(h(X)))^2 dF(1)(x)$$

and

$$\gamma_1 = \Gamma(1)(h, h), \quad \gamma_2 = \int_0^1 \int_0^1 \Gamma(1)(f_s, f_t) ds dt \quad \text{and} \quad \gamma_3 = \int_0^1 \Gamma(1)(h, f_s) ds.$$

**Remark.** Later, we will deal with different indices. In that situation the variance $\Gamma$ for the specific index $I$ will be denoted

$$\Gamma^{(I)} = \Gamma^{(I)}(h, \ell). \quad (\text{Var-I})$$

**Proof.** The proof easily comes from the preliminaries in Chapter 1, especially in Section 3. We simply say that under the assumption and
the (GRI) representation that \( I_n^* = \sqrt{n}(I_n - I) \) weakly converges to a Gaussian variable and, by using Formula 3.1 and straightforward computations, we have that the asymptotic variance is
\[
\Gamma = \gamma_1 + \gamma_2 + 2\gamma_3,
\]
where
\[
\gamma_1 = \Gamma^{(1)}(h, h), \quad \gamma_2 = \int_0^1 \int_0^1 \Gamma^{(1)}(f_s, f_t)dsdt \quad \text{and} \quad \gamma_3 = \int_0^1 \Gamma^{(1)}(h, f_s)ds. \quad \blacksquare
\]

2. Asymptotic Laws of variations of an index

Let us place ourselves in the bidimensional space created Section 3, Chapter 1. Let us suppose that the index \( I \) is measured for from a sample of observations of the couple \( Y = (X^{(1)}, X^{(2)}) \). We get the statistics \( I_n^{(i)} \) for times \( t = 1 \) and \( t = 2 \). We are interested in finding the asymptotic laws of the variation \( \Delta I_n = I_n^{(2)} - I_n^{(1)} \) of \( I_n \) from times \( t = 1 \) and \( t = 2 \).

Let us begin to suppose that the square integrability conditions required for the convergence of the empirical processes based on \( X^{(1)} \) and \( X^{(2)} \), and that conditions (Re1) and (Re2) of Theorem 1 (Section 4 in Chapter 1) based on \( X^{(1)} \) and \( X^{(2)} \) and the appropriate function \( \ell \) hold. We refer to these conditions by (HFEP2). So we may write the indice’s general representation (GRI) in Section 5 in Chapter 1 for both times to get (GR1):
\[
\sqrt{n}(I_n^{(i)} - E h_i(X)) = G_{n,(1),(i)}(h_i) + \beta_{n,(1),(i)}(\ell_i) + o_p(1), \quad i = 1, 2
\]
where \( G_{n,(1),(i)} \) and \( \beta_{n,(1),(i)} \) are respectively the one dimensional fep and residual empirical process based on the \( n \)-sized sample from \( X^{(i)} \). To simplify, we drop the subscript in \( \beta_{n,(1),(i)} \) to only write \( \beta_{n,(1)} \), and where \( \ell_i(s) = q_i(F_{(2)}^{-1}(s)), \quad s \in (0, 1) \). Denote
\[
h^{(1)}(x, y) = h_1(x) \quad \text{and} \quad h^{(2)}(x, y) = h_2(y), \quad (x, y) \in \mathbb{R}^2;
\]
\[
f_s^{(i)}(x, y) = 1_{(x \leq F_{(2),(1)}^{-1}(s))}, \quad s \in (0, 1),
\]
and
2. ASYMPTOTIC LAWS OF VARIATIONS OF AN INDEX

\[ f_s^{(2)}(x, y) = 1_{y \leq F_{(2);2}^{-1}(s)}, \quad s \in (0, 1). \]

We will use the following transform for any function \( g \) of \((x, y) \in \mathbb{R}^2:\)

\[ (2.1) \quad \tilde{g}(s, t) = g\left(F_{(2);1}^{-1}(s), F_{(2);2}^{-1}(t)\right), \quad (s, t) \in [0, 1]^2. \]

But we may express (GRIS) using the bi-dimensional fep based on the \( n \)-sized sample from \( Y = (X^{(1)}, X^{(2)}) \) through (GRI2)

\[ \sqrt{n}(I_n^*(1) - \mathbb{E}h(X)) = \mathbb{G}_{n,(2),(i)}(h^{(i)}) + \int_0^1 \mathbb{G}_{n,(1)}(f_s^{(i)}) \ell_1(s) \, ds + o_p(1), \quad (GRIS) \]

\[ i = 1, 2, \] which, by the notations in Section 3, is (GRI2):

\[ I_n^*(i) = \sqrt{n}(I_n^{(i)} - \mathbb{E}h_i(X)) = \mathbb{G}_{n,u,(2),(i)}(\tilde{h}^{(i)}) + \int_0^1 \mathbb{G}_{n,u,(1)}(\tilde{f}_s^{(i)}) \ell(s) \, ds + o_p(1), \quad (GRIS) \]

\[ i = 1, 2. \] Let us remark that

\[ \tilde{f}_s^{(i)} = 1_{[0,s]}, \quad s \in (0, 1), \quad i = 1, 2. \]

The asymptotic covariance \( \Gamma_{12} \) between \((I_n^*(1) \text{ and } I_n^*(2))\) is obtained from the combination between Formula (GRIS) just above and Formula (GammaStar) (page 29) in Section 3 in 1 following these notations.

\[ \tilde{\gamma}^{(12)}(s, t) = \Gamma^*(\tilde{f}_s^{(1)}, \tilde{f}_s^{(2)}) = \int_0^s \int_0^t dC(u, v) \, dudv - st, \]

\[ \tilde{\gamma}^{(12)}(s, t) = C(s, t) - st, \]

We also need

\[ \tilde{\gamma}^{(1)}(s) = \Gamma_{(2)}(\tilde{h}^{(1)}, \tilde{f}_s^{(2)}) = \int_0^s \tilde{h}^{(1)}(u, v) dC(u, v) \, dudv - s \int_0^1 h_1(F_{(2);1}^{-1}(u)) \, du, \]

\[ \tilde{\gamma}^{(2)} = \Gamma_{(2)}(\tilde{f}_s^{(1)}, \tilde{h}^{(2)}) = \int_0^s \tilde{h}^{(2)}(u, v) dC(u, v) \, dudv - s \int_0^1 h_2(F_{(2);2}^{-1}(u)) \, du. \]

Then the asymptotic co-variance \( \Gamma = (\Gamma_{ij}, \quad 1 \leq i \leq 2, \quad 1 \leq i \leq 2) \) of \((I_n^*(1), I_n^*(2))\) is given by:
3. ABSOLUTE AND RELATIVE VARIATIONS

\[ \gamma_{11} = \tilde{\Gamma}_{(2)}(\tilde{h}^{(1)}, \tilde{h}^{(2)}) \]

\[ \gamma_{22} = \int_0^1 \int_0^1 \tilde{\gamma}^{(12)}(s, t) \ell_1(s) \ell_2(t) ds dt = \int_0^1 \int_0^1 (C(s, t) - st) \ell_1(s) \ell_2(t) ds dt \]

and

\[ \gamma_{12} = \int_0^1 \tilde{\gamma}^{(1)} \ell_2(s) ds \quad \text{and} \quad \gamma_{21} = \int_0^1 \tilde{\gamma}^{(2)} \ell_1(s) ds. \]

By using the product of factors in (GRIS) for \( i = 1, 2 \) and by using the function \( \Gamma^* \), we arrive at

\[ \Gamma^{(12)} = \sum_{1 \leq i, j \leq 2} \gamma_{ij}. \]

As to the asymptotic variances of \( I_n^{(i)} \), \( i = 1, 2 \), we find it as in Theorem 3, by

\[ \Gamma^{(i)} = \gamma_{1}^{(i)} + \gamma_{2}^{(i)} + 2 \gamma_{3}^{(i)}, \]

with

\[ \Gamma^{(i)}_{(1)}(h^{(i)}, h^{(i)}) = \int (h^{(i)}(x) - \mathbb{E}(h^{(i)}(X)))^2 dF_{(2),i}(x), \]

\[ \gamma_{1}^{(i)} = \Gamma^{(i)}_{(1)}(h^{(i)}, h^{(i)}), \quad \gamma_{2}^{(i)} = \int_0^1 \int_0^1 \Gamma^{(i)}_{(1)}(f_s^{(i)}, f_t^{(i)}) ds dt, \]

and

\[ \gamma_{3}^{(i)} = \int_0^1 \Gamma^{(i)}_{(1)}(h^{(i)}, f_s^{(i)}) ds. \]

for \( i = 1, 2 \).

With these notations, we are able to give the general result:
3. Asymptotic Laws of Relative Variations of an Indice

**Theorem 4. (General law of Variation of Indices)** Suppose that the Assumptions (HFEP2) hold and denote \( \Delta I = I_2 - I_1 \). Then we have as \( n \to +\infty \)

\[
\Delta I^*_n = \sqrt{n}(\Delta I_n - \Delta I) \sim \mathcal{N}(0, \Delta \Gamma),
\]

where \( \Delta \Gamma = \Gamma^{(1)} + \Gamma^{(2)} + 2\Gamma^{(12)} \).

**Proof.** The proof follows the same lines as in the proof of Theorem 3, by remarking that \( \Delta I^*_n = \sqrt{n}(\Delta I_n - \Delta I) \) is still a finite linear combinations of the margins of a same sequences of stochastic processes whose finite-distributions converge to those of a Gaussian processes. The remainder is a matter of computations which are featured above.

3. Asymptotic Laws of relative variations of an indice

Following the results of the previous section, we use the Delta method and the same principles of finite linear combinations of the margins of a same sequences of stochastic processes whose finite-distributions converge to those of a Gaussian processes to get the law of the relative variation of \( I \)

\[
\Delta RI_n = \frac{I_n^{(2)} - I_n^{(1)}}{I_n^{(1)}}.
\]

We have

**Theorem 5. (General law of Relative Variation of Indices)** Suppose that the Assumptions (HFEP2) hold and \( \Delta I = (I_2 - I_1)/I_1 \) and

\[
\gamma_4 = 1/I_1 \text{ and } \gamma_5 = \Delta I/I_1^2.
\]

Then we have, as \( n \to +\infty \),

\[
\Delta RI^*_n = \sqrt{n}(\Delta RI_n - \Delta RI) \sim \mathcal{N}(0, \Delta RI\Gamma^2),
\]

where \( \Delta RI\Gamma = \gamma_5(\gamma_5\Gamma^{(1)} - 2\gamma_4) + \gamma_4^2\Delta RI\Gamma \).

By using the delta method (see for Chapter 4 in [Lo et al. (2016)], for example), we have that

\[
\Delta RI^*_n = \sqrt{n}(\Delta RI_n - \Delta RI) = \frac{1}{I_1} \Delta I^*_n - \frac{\Delta I}{I_1^2} \sqrt{n}(I_n^{(1)} - I_1) + o_P(1).
\]
We already denote $\gamma_4 = 1/I_1$ and $\gamma_5 = \Delta I/I_1^2$. The computations of the variance-covariances imply that the asymptotic variance of $\Delta R_1$ is
\[
\gamma_5^2 \Gamma^{(1)} + \gamma_4^2 \Delta R \Gamma - 2\gamma_4 \gamma_5 (\Gamma_{(12)} - (\Gamma^{(1)})^2)
\]
which is
\[
\gamma_5 (\gamma_5 \Gamma^{(1)} - \gamma_4) + \gamma_4^2 \Delta R \Gamma.
\]

Let us finish by emphasizing the importance of knowing the law of $\Delta R_1$. It is useful to check whether a Millennium Development Goals (MDG) is achieved. For example, the poverty reduction MDG is expressed as to have a poverty measure $I$ to be reduced by a fixed rate $r$ from a time $t = 1$ to a $t = 2$. For poverty, $r$ was set to 50% at 2015. One has to check that
\[
\Delta R_1 \leq -r.
\]
A way to answer to this requirement is to find cover of $\Delta R_1$, say at 95% of the form
\[
P(\Delta R_1 \leq A) \geq 95\%
\]
and
\[
A \leq r.
\]
Of course, the exact law of $\Delta R_1$ allows a precise answer to the problem. Since we do not know it, we may try a use an approximated solution from the asymptotic law of $\Delta R_1$. 
CHAPTER 4

Mutual Asymptotic Influence between indices

Here, we face the question of mutual influence between two indices. Usually, this question may be of interest if we want to know if a growth, in Economics, is fair or not. Fairness means here that all the population concerned by the growth, of the worst off of them, make benefice of that grow, what we call pro-poor growth. But in general, given two indices based on the same set of variables, we may also see if they evolve together in the same direction or not, and how much they evolve relatively each other.

We are going see in the lines below the influence of two different indices based on the same random variable between them at a fixed time and that of their absolute and/or relative variations. To begin, suppose that we have two indices $I$ and $J$.

In a one-dimensional frame, we consider their measures $I_n$ and $J_n$ from the $n$-size sample $X_1, \ldots, X_n$, $n \geq 1$, with underlying cdf $F(1)$. We suppose that Assumptions (HFEP1) holds for both indices so that we have for them, the indice’s general representation (GRI) in Section 4 in Chapter 1 in the from :

$$\sqrt{n}(I_n - I) = \mathbb{G}_{n,(2)}(h) + \int_0^1 \mathbb{G}_{n,(1)}(f) \ell(s) \, ds + o_p(1), \quad (GRI - I)$$

and

$$\sqrt{n}(J_n - J) = \mathbb{G}_{n,(2)}(g) + \int_0^1 \mathbb{G}_{n,(1)}(f) \nu(s) \, ds + o_p(1), \quad (GRI - J)$$

where for there exist two measurable function $p(x)$ and $q(x)$ of $x \in \mathbb{R}$ such that $\ell(s) = q(F^{-1}_{(1)}(s))$ and $\nu(s) = p(F^{-1}_{(1)}(s))$, for $s \in (0, 1)$.
In a two-dimensional frame, we still use the created Section 3 in Chapter 1. Assuming Assumptions (HEFP2) hold for both $I$ and $J$ by using the notations in Chapter 3 and in Formulas (GRI-I) and (GRI-J) above, we have for time $i = 1$ and time $i = 2$,

$$\sqrt{n}(I_n^{(i)} - I^i) = \mathbb{G}_{n,(2)}(h^i) + \int_0^1 \mathbb{G}_{n,(1)}(\hat{f}^i_s), ds + o_P(1), \ i = 1, 2 (GRIS-I)$$

$$\sqrt{n}(J_n^{(i)} - J^i) = \mathbb{G}_{n,(1)}(h^{(i)}) + \int_0^1 \mathbb{G}_{n,(2)}(\hat{f}_s^{(i)}) \nu_s(s) ds + o_p(1), \ (GRIS-J)$$

In the sequel, the full details of the computations will not be given. Once the representations are given, we suppose the reader will be able to make some direct and easy computations to derive the results. The most essential arguments and notations are Chapter 1.
1. Mutual influence of two simple indices

Theorem 6. Suppose Assumptions (HFEP1) are satisfied for two indices $I$ and $J$, then we have as $n \to +\infty$,

$$(I^*_n, J^*_n) \rightsquigarrow \mathcal{N} \left( 0, \begin{pmatrix} \Gamma^{(I)} & \Gamma^{(I,J)} \\ \Gamma^{(I,J)} & \Gamma^{(J)} \end{pmatrix} \right)$$

where $\Gamma^{(I)}$ and $\Gamma^{(J)}$ are described in Formula in $(\text{Var} - I)$ in Chapter 3, and $\Gamma^{(I,J)} = \Gamma^{(1)}(h,g) + \int_0^1 \int_0^1 \Gamma^{(1)}(f_s, f_t) \ell(s) \nu(t) ds \ dt$

$$+ \int_0^1 \Gamma^{(1)}(h, f_t) \nu(t) ds + \int_0^1 \Gamma^{(1)}(f_s, g) \ell(s) ds$$
2. Mutual influence of variations of indices

**Theorem 7.** Suppose Assumptions (HFEP2) are satisfied for two indices \( I \) and \( J \), then we have as \( n \to +\infty \),

\[
(\Delta I_n^*, \Delta J_n^*) \rightsquigarrow N\left(0, \begin{pmatrix} \Delta \Gamma^{(I)} & \Delta \Gamma^{(I,J)} \\ \Delta \Gamma^{(I,J)} & \Delta \Gamma^{(J)} \end{pmatrix}\right)
\]

where \( \Delta \Gamma^{(I)} \) and \( \Delta \Gamma^{(J)} \) are described in Theorem 4 in Chapter 3, and

\[
\Delta \Gamma^{(I,J)} = \Delta \Gamma^{(I,J)}_{11} + \Delta \Gamma^{(I,J)}_{22} - \Delta \Gamma^{(I,J)}_{12} - \Delta \Gamma^{(I,J)}_{21},
\]

where for \( i = 1, 2 \),

\[
\Gamma^{(I,J)}_{ii} = \tilde{\Gamma}_{(2)}(\tilde{h}^{(i)}, \tilde{g}^{(i)}) + \int_0^1 \int_0^1 (C(s, t) - st)\ell_i(s)\nu_i(t)ds \, dt
\]

\[
+ \int_0^1 \nu_i(s) \left( \int_0^s \tilde{h}^{(i)}(t) - \mathbb{E}\tilde{h}^{(i)}(X^{(i)}) \, dt \right) \, ds
\]

\[
+ \int_0^1 \ell_i(s) \left( \int_0^s \tilde{g}^{(i)}(t) - \mathbb{E}\tilde{g}^{(i)}(X^{(i)}) \, dt \right) \, ds
\]

\[
\Gamma^{(I,J)}_{12} = \tilde{\Gamma}_{(2)}(\tilde{h}^{(1)}, \tilde{g}^{(2)}) + \int_0^1 \int_0^1 \tilde{\Gamma}_{(2)}(\tilde{f}_s^{(1)}, \tilde{g}_s^{(2)})\ell_1(s)\nu_2(t)ds \, dt
\]

\[
+ \int_0^1 \tilde{\Gamma}_{(2)}(\tilde{h}^{(1)}, \tilde{f}_s^{(2)})\nu_2(s)ds + \int_0^1 \tilde{\Gamma}_{(2)}(\tilde{g}^{(2)}, \tilde{f}_s^{(1)})\ell_1(s) \, ds
\]

and

\[
\Gamma^{(I,J)}_{21} = \tilde{\Gamma}_{(2)}(\tilde{g}^{(2)}, \tilde{h}^{(1)}) + \int_0^1 \int_0^1 \tilde{\Gamma}_{(2)}(\tilde{f}_s^{(1)}, \tilde{g}_s^{(2)})\ell_2(s)\nu_1(t)ds \, dt
\]

\[
+ \int_0^1 \tilde{\Gamma}_{(2)}(\tilde{h}^{(2)}, \tilde{f}_s^{(1)})\nu_1(s)ds + \int_0^1 \tilde{\Gamma}_{(2)}(\tilde{g}^{(1)}, \tilde{f}_s^{(2)})\ell_2(s) \, ds
\]
3. Mutual influence of Relative Variations of indices

Let us denote as previously

\[ \tilde{\gamma}_{4,I} = \frac{1}{I_1}, \tilde{\gamma}_{5,I} = \frac{\Delta I}{I_1}, \tilde{\gamma}_{4,J} = \frac{1}{J_1}, \text{ and } \tilde{\gamma}_{5,J} = \frac{\Delta J}{J_1}. \]

As in the proof of Theorem 5, we have

\[ \Delta R I_n^* = \tilde{\gamma}_{4,I} \Delta I_n^* - \tilde{\gamma}_{5,I} \sqrt{n} (I^{(1)}_n - I^{(2)}) + o_P(1). \]

and

\[ \Delta R J_n^* = \tilde{\gamma}_{4,J} \Delta J_n^* - \tilde{\gamma}_{5,J} \sqrt{n} (J^{(1)}_n - J^{(1)}) + o_P(1). \]

Doing the right the computations leads to

**Theorem 8.** Suppose Assumptions (HFEP2) are satisfied for two indices $I$ and $J$, then we have as $n \to +\infty$,

\[ (\Delta R I_n^*, \Delta R J_n^*) \rightsquigarrow \mathcal{N} \left( 0, \begin{pmatrix} \Delta R \Gamma^{(I)} & \Delta R \Gamma^{(I,J)} \\ \Delta R \Gamma^{(I,J)} & \Delta R \Gamma^{(J)} \end{pmatrix} \right) \]

where $\Delta R \Gamma^{(I)}$ and $\Delta R \Gamma^{(J)}$ are described in Theorem 5 in Chapter 3, and

\[ \Delta R \Gamma^{(I,J)} = \tilde{\gamma}_{4,I} \tilde{\gamma}_{4,J} \Delta \Gamma^{(I,J)} - \tilde{\gamma}_{4,I} \tilde{\gamma}_{5,J} (\Gamma^{(I,J)}_{21} - \Gamma^{(I,J)}_{11}) - \tilde{\gamma}_{4,J} \tilde{\gamma}_{5,I} (\Gamma^{(I,J)}_{12} - \Gamma^{(I,J)}_{11}) + \tilde{\gamma}_{5,I} \tilde{\gamma}_{5,J} \Gamma^{(I,J)}_{11}. \]

As announced, we include in this portal a second part with the aim to show how to apply the results of the gateway in some important example before we move to the handbook.
Part 2

Applications and Examples
Introduction to Part II

In this part, we will give some examples of GRI’s of noticeable statistics. Some will be reports of existing results and hence given without proofs. Others will be proved here. The results given here will be consigned and will be used by coming works. So we want to begin by the most basic statistics which are moments statistics.

The examples given here are:

(a) The moments estimators and the normalized moments estimators.

(b) The general poverty index in Welfare Analysis.

(c) The Takayama poverty index in Welfare Analysis.

We make profit of this introduction to present a technical result which has been proved to be useful in many situations and which may be useful to check condition Condition (CRe2) in page 32. Here is the lemma.

**Lemma 4.** Let $(A_n)_{n \geq 1}$ and $(B_n(\eta))_{(n \geq 1, \eta \in T)}$, where $T \neq \emptyset$ be two families of non-negative real-valued random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that:

\[
\forall \varepsilon > 0, \exists \eta_0 \in T, \exists n_0 \geq 1, \forall n \geq n_0, \mathbb{P}(A_n > B_n(\eta_0)) \leq \varepsilon
\]

and, as $n \to +\infty$,

\[
\forall \eta \in T, \quad B_n(\eta) \to_{\mathbb{P}} 0 \quad \text{or} \quad \mathbb{E}B_n(\eta) \to 0.
\]

Then $A_n \to_{\mathbb{P}} 0$, $n \to +\infty$. 
**proof.** Assume that the hypotheses of the lemma hold. Fix $0 \delta > 0$ and $0 < \varepsilon < \delta$. Then there exists $\eta_0$ such that $B_n(\eta) \to \mathbb{P} 0$ as $n \to +\infty$ and $\mathbb{P}(A_n > B_n(\eta_0)) \leq \varepsilon$ for $n$ large enough. Hence

\[
\mathbb{P}(A_n > \delta) = \mathbb{P}((A_n > \delta) \cup (B_n(\eta_0) \leq \varepsilon)) + \mathbb{P}((A_n > \delta) \cup (B_n(\eta) \leq \varepsilon)) \\
\leq \mathbb{P}((A_n > \delta) \cup (B_n(\eta_0) \leq \varepsilon)) + \mathbb{P}((A_n > \delta) \cup (B_n(\eta) \leq \varepsilon)) \\
\leq \mathbb{P}(A_n > B_n(\eta_0)) + \mathbb{P}(B_n(\eta) \leq \varepsilon) \\
\leq \varepsilon + \mathbb{P}(B_n(\eta_0) \leq \varepsilon).
\]

Hence for all any $0 \varepsilon \in ]0, \delta[,$ we have

\[
\lim_{n \to +\infty} \sup \mathbb{P}(A_n > \delta) \leq \varepsilon.
\]

The proof of the lemma is finished by letting $\varepsilon \searrow 0$. \qed
CHAPTER 5

Moments Estimation of moments

1. Asymptotic representations of the empirical moments

We are going to provide asymptotic representations of the non-centered moments

\[ m_\ell = \mathbb{E}(X^\ell), \]

with \( m_1 \equiv m \) and the centered moments

\[ \mu_\ell = \mathbb{E}(X - m_1)^\ell, \]

where \( \ell \geq 1 \) whenever they exist, in the Gaussian field described in the Gateway. Their plug-in estimators are respectively

\[ m_{n,\ell} = \sum_{i=1}^{n} X_i^\ell, \quad \ell \geq 1. \]

and

\[ \mu_{n,\ell} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^\ell, \quad \ell \geq 1. \]

Let us put \( \mu_2 = \sigma^2 \) and \( m_1 = m \) and \( h_\ell(x) = x^\ell, x \in \mathbb{R} \) and the following functions:

\[(1.1) \quad A(\ell) = h_\ell + \sum_{p=0}^{\ell-1} C_\ell^p (-1)^{\ell-p} \left( m_1^{\ell-p} h_p + (\ell - p)m_1^{\ell-p-1} m_p h_1 \right), \]

\[(1.2) \quad B(p) = \sigma^{-2p-1} \left( A(2p-1) - \frac{1}{2} (2p-1) \sigma^{-2} \mu_{2p-1} A(2) \right) \]

and

\[(1.3) \quad C(p) = \sigma^{-2p} \left( A(2p) - p \sigma^{-2} \mu_{2p} A(2) \right) \]

we have the following results which were proved first in [Lo et al. (2015)].
Theorem 9. Let $\ell \geq 1$ and assume that $\int x^{2\ell} dF_1(x) < \infty$, then
\begin{align*}
\sqrt{n} (\mu_{n,\ell} - \mu_\ell) &= \mathcal{G}_n (A(\ell)) + o_p(1) \\
&= \mathcal{G} (A(\ell)) \sim \mathcal{N}(0, \text{Var}(A(\ell)(X))).
\end{align*}

Proof. we have
\begin{align*}
\mu_{n,\ell} &= \sum_{p=0}^{\ell} C_p^\ell (-X)^{\ell-p} \left( \frac{1}{n} \sum_{i=1}^{n} X_i^p \right) \\
&= \sum_{p=0}^{\ell} C_p^\ell (-1)^{\ell-p} \left( m_1 + \mathcal{G}_n(h_1) \sqrt{n} \right)^{\ell-p} \left( m_p + \mathcal{G}_n(h_p) \sqrt{n} \right) \\
&= \left( m_\ell + \frac{\mathcal{G}_n(h_\ell) \sqrt{n}}{\sqrt{n}} \right) + \sum_{p=0}^{\ell-1} C_p^\ell (-1)^{\ell-p} \left( m_1^{\ell-p} + (\ell - p)m_1^{\ell-p-1} \mathcal{G}_n(h_1) \sqrt{n} \right) + o_p(n^{-1/2}) \\
&\times \left( m_p + \frac{\mathcal{G}_n(h_p) \sqrt{n}}{\sqrt{n}} \right) \\
&= m_\ell + h_\ell + \sum_{p=0}^{\ell-1} C_p^\ell (-1)^{\ell-p} \left( m_1^{\ell-p} m_p + \frac{\mathcal{G}_n(A_\ell) \sqrt{n}}{\sqrt{n}} \right) + o_p(n^{-1/2}),
\end{align*}

where $A(\ell)$ is defined in (1.1) and where we used that the linearity of the empirical functional process. By observing that $\mu_\ell = \sum_{p=0}^{\ell} C_p^\ell (-m_1)^{\ell-p} (m_p)$, we finally obtain
\begin{equation}
\sqrt{n} (\mu_{n,\ell} - \mu_\ell) = \mathcal{G}_n (A(\ell)) + o_p(1).
\end{equation}

Now, we may do some algebra to find estimators of normalized moments including skewness and kurtosis.

2. Estimation of normalized moments

This section is an example of what can be done once we have established a GRI. We are going to combine the obtained representations to represent the normalized centered empirical moments (NCM), defined by,
\begin{equation}
b_p = \frac{\mathbb{E}((X - m)^{2p-1})}{\sigma^{(2p-1)}},
\end{equation}

and
2. ESTIMATION OF NORMALIZED MOMENTS

(2.2) \[ a_p = \frac{\mathbb{E}((X - m)^{2p})}{\sigma^{2p}}, \]

where \( p \geq 2 \) whenever they exist, and consider their plug-in estimators called normalized centered empirical moments (NCEM),

(2.3) \[ b_{n,p} = \frac{\mu_{n,2}^{(2p-1)/2}}{\mu_{n,2}^{(2p-1)/2}} \quad \text{and} \quad a_{n,p} = \frac{\mu_{n,2}^{p}}{\mu_{n,2}^{p}}, \quad p \geq 2, \]

We have the following results below.

**Theorem 10.** Let \( p \geq 1 \) and assume that \( \int x^{2k}dG(x) < \infty \), then

(2.4) \[ \sqrt{n}((b_{n,p} - b_p), (a_{n,2} - a_p)) = (G_n(B(p)), G_n(C(p))) + o_P(1). \]

**Proof.** This proof is a continuation of that of 9. Then the law of \( b_{n,p} \) is given by

\[
\sqrt{n} (b_{n,p} - b_p) = \frac{1}{\mu_{n,2}^{(2p-1)/2}} \sqrt{n} (\mu_{n,2}^{2p-1} - \mu_{2p-1}) \\
- \frac{\mu_{2p-1}}{\mu_{n,2}^{(2p-1)/2}} \sqrt{n} \left( \mu_{n,2}^{(2p-1)/2} - \mu_{2}^{(2p-1)/2} \right). 
\]

By the delta-method, we have

\[
\mu_{n,2}^{(2p-1)/2} = \left( \mu_2 + \frac{G_n(A(2))}{\sqrt{n}} \right)^{2p-1} + o_p(n^{-1/2}). \\
= \frac{2p-1}{2} \mu_2 \frac{G_n(A(2))}{\sqrt{n}} + o_p(n^{-1/2}). 
\]

and then

\[
\sqrt{n} \left( \mu_{n,2}^{(2p-1)/2} - \mu_2^{(2p-1)/2} \right) = \left( \frac{2p-1}{2} \right) \mu_2 \frac{G_n(A(2))}{\sqrt{n}} + o_p(1), 
\]

and next, by noticing, by the Weak law of Large numbers, that \( \mu_{n,\ell} \to \mu_\ell \), for all \( \ell \leq 2k \), whenever the \((2k)^{th}\) moment of the \( X_i \)'s are finite, we have

\[
\sqrt{n} (b_{n,p} - b_p) 
\]
\[ G_n \left( \sigma^{-(2p-1)} A(2p - 1) - \frac{1}{2} (2p - 1) \sigma^{-(2p+1)} \mu_{2p-1} A(2) \right) + o_p(1). \]

\[ G_n (B(p)) + o_p(1) \to G(B(p)), \]

where \( B(p) \) is given in (1.2). By the very same methods, we have

\[ \sqrt{n} (a_{n,p} - a_p) = G_n (C(p)) + o_p(1), \]

Important applications of these laws concern extension of the Jarque-Berra test for normality to almost any distribution function provided that the moment exist at the dimension we want to work on. Such an extension has been done first in [Lo et al. (2015)]. It will be further developed in [Lo et al. (2018)].
CHAPTER 6

The General Poverty Index

0.1. Representation of the GPI. In this paper, we use the GPI in a unified approach that leads to an asymptotic representation for a large class of indices classified in three kinds. We are entering into the details of the poverty theory nor in the general description of the poverty indexes (See [Lo (2013)] for details on those questions). We are just giving the general description of the indexes and provide their a unified GRI.

Here the observed random variable $X$ is non-negative and represents an income or an expense. $Z > 0$ is a fixed number and considered as a threshold and $Q_n = n S_n(1)(Z)$ is the number of individual in the sample whose value $X$ is below the $Z$. $\mu_1, \mu_2, \mu_3, \mu_4$ are constants.

Let us suppose given measurable mapping $A(p, q, z)$, $w(t)$, and $d(t)$ of $p, q \in N$, and $z, t \in R$ and

$$B(Q_n, n) = \sum_{i=1}^{q} w(i).$$

The General Poverty Index proposed by [Lo et al. (2006)] and [Lo (2013)] is of the form

(0.1)

$$GPI_n = \frac{A(Q_n, n, Z)}{n B(Q_n, n)} \sum_{j=1}^{Q_n} w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4) d\left(\frac{Z - X_{j,n}}{Z}\right), \ n \geq 1.$$

This class of indices contains among others:

(1) The Foster-Greer-Thorbecke (FGT) index of parameter [Foster et al.(1984)] defined for $\alpha \geq 0,$
6. THE GENERAL POVERTY INDEX

(0.2) \[ FGT_n(\alpha) = \frac{1}{n} \sum_{j=1}^{Q_n} \left( \frac{Z - X_{j,n}}{Z} \right)^\alpha, \quad n \geq 1. \]

(2) The Sen poverty measure ([Sen (1976)]):

(0.3) \[ P_{Sen} = \frac{2}{n(Q_n + 1)} \sum_{j=1}^{Q_n} (Q_n - j + 1) \left( \frac{Z - X_{j,n}}{Z} \right), \quad n \geq 1. \]

(3) The Kakwani ([Kakwani (1980)]) class of poverty measures:

(0.4) \[ P_{KAK,n}(k) = \frac{Q_n}{n\Phi_k(Q_n)} \sum_{j=1}^{Q_n} (Q_n - j + 1)^k \left( \frac{Z - X_{j,n}}{Z} \right), \quad n \geq 1. \]

where

\[ \Phi_k(Q_n) = \sum_{j=1}^{Q_n} j^k = B(Q_n, n) \]

(4) The Shorrock ([Shorrocks (1995)]) index

(0.5) \[ P_{SH,n} = \frac{1}{n^2} \sum_{j=1}^{Q_n} (2n - 2j + 1) \left( \frac{Z - X_{j,n}}{Z} \right), \]

(5) The Thon ([Thon (1979)]) proposed the following measure

\[ P_{Th} = \frac{2}{n(n+1)} \sum_{j=1}^{Q_n} (n - j + 1) \left( \frac{Z - X_{j,n}}{Z} \right), \quad n \geq 1. \]

In [Lo et al. (2006)] and [Lo (2013)], a GRI Formula has been given under the following conditions.

First we consider the threshold condition:

(H1) There exist \( \beta > 0 \) and \( 0 < \xi < 1 \) such that,

\[ 0 < \beta < F_{(1)}(Z) < \xi < 1. \]
Next we have form conditions (on the indices):

(H2a) There exist a function $h(p, q)$ where $(p, q) \in \mathbb{N}^2$ and a function $c(s, t)$ where $(s, t) \in (0, 1)^2$ such that, when $n \to +\infty$,
\[
\max_{1 \leq j \leq Q} \left| A(n, Q)h^{-1}(n, Q)w(\mu_1 n + \mu_2 Q - \mu_3 j + \mu_4) - c(Q/n, j/n) \right| = o_P(n^{-1/2});
\]

(H2b) There exists a function $\pi(s, t)$ with $(s, t) \in \mathbb{R}^2$ such that, when $n \to +\infty$,
\[
\max_{1 \leq j \leq Q} \left| w(j)h^{-1}(n, Q) - \frac{1}{n}\pi(Q/n, j/n) \right| = o_P(n^{-3/2}).
\]

Further we need regularity conditions on $c$ and $\pi$:

(H3) The functions $c(\cdot)$ and $\pi(\cdot)$ have uniformly continuous partial derivatives, that is
\[
\lim_{(k, l) \to (0, 0)} \sup_{(x, y) \in (0, 1)^2} \left| \frac{\partial c}{\partial y}(x + l, y + k) - \frac{\partial c}{\partial y}(x, y) \right| = 0
\]
and
\[
\lim_{(k, l) \to (0, 0)} \sup_{\beta \leq x \leq \xi, y \in (0, 1)} \left| \frac{\partial c}{\partial x}(x + l, y + k) - \frac{\partial c}{\partial x}(x, y) \right| = 0;
\]

(H4) The functions $y \to \frac{\partial c}{\partial y}(x, y)$ and $y \to \frac{\partial c}{\partial y}(x, y)$ are monotonous.

(H5) The distribution function $F_{(1)}$ is increasing.

(H6) There exist $H_0 > 0$ and $H_\infty < +\infty$ such that
\[
H_0 < H_c(F_{(1)}) = \int_0^{+\infty} c(F_{(1)}(Z), F_{(1)}(y))\gamma(y)dF_{(1)}(y) < H_\infty,
\]
and
\[
H_0 < H_\pi(F_{(1)}) = \int_0^{+\infty} \pi(F_{(1)}(Z), F_{(1)}(y))e(y)dF_{(1)}(y) < H_\infty
\]
where
\[
\gamma(x) = d\left(\frac{Z - x}{Z}\right) I_{(x \leq Z)} \text{ and } e(x) = I_{(x \leq Z)} \text{ for } x \in \mathbb{R}.
\]

Based on these hypotheses, we put
\[ J(F_1) = H_c(F_1)/H_\pi(F_1), \]

\[ h(\cdot) = H^{-1}_\pi(F_1)h_c(\cdot) - H_c(F_1)H^{-2}_\pi(F_1)h_\pi(\cdot) + K(F_1)e(\cdot), \]

* with

\[ h_c(\cdot) = c(F_1(Z), F_1(\cdot))\gamma(\cdot), \quad h_\pi(\cdot) = \pi(F_1(Z), F_1(\cdot))e(\cdot), \]

\[ K(F_1) = H^{-1}_\pi(F_1)K_c(F_1) - H_c(F_1)H^{-2}_\pi(F_1)K_\pi(F_1) \]

where

\[ K_c(F_1) = \int_0^1 \frac{\partial c}{\partial x}(F_1(Z), s)\gamma(F_1^{-1}(s))ds, \]

\[ K_\pi(F_1) = \int_0^1 \frac{\partial \pi}{\partial x}(F_1(Z), s)e(F_1^{-1}(s))ds, \]

\[ q(\cdot) = H^{-1}_\pi(F_1)q_c(\cdot) - H_c(F_1)H^{-2}_\pi(F_1)q_\pi(\cdot), \]

and

\[ q_c(\cdot) = \frac{\partial c}{\partial y}(F_1(Z), F_1(\cdot))\gamma(\cdot), \quad q_\pi(\cdot) = \frac{\partial \pi}{\partial y}(F_1(Z), F_1(\cdot))e(\cdot). \]

and \[ \ell(s) = q(F_1^{-1}(s), s \in (0, 1). \]

We have the following GRI Formulas.

**Theorem 11.** Suppose that (H1)-(H6) are true, then we have the following representation

\[ \sqrt{n}(J_n(F_1) - J(F_1)) = \mathcal{G}_{n(1)}(h) + \beta_{n(1)}(\ell) + o_{\mathbb{P}}(1). \]

where

\[ h_s(y) = \left\{ 2 \left[ \left( 1 - \frac{F_1(y)}{F_1(Z)} \right) \left( \frac{Z - y}{Z} \right) - \left( \frac{F_1(y)}{F_1(Z)} \right) + \frac{J_s(F_1)}{F_1(Z)} \right] K_s(F_1) \right\} \mathbb{I}_{(y \leq Z)}, \]

and

\[ q_s(y) = -\frac{2}{F_1(Z)} \left[ \left( \frac{Z - y}{Z} \right) + \frac{J_s(F_1)}{F_1(Z)} \right] \mathbb{I}_{(y \leq Z)}. \]
Table 1. Specific functions of the poverty measures

| Measure | $h$                              | $q$                              |
|---------|----------------------------------|----------------------------------|
| Shorrocks | $2 \left(1 - F(1)(y)\right) \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)} - 2 \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)}$ |                                  |
| Thon    | $2 \left(1 - F(1)(y)\right) \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)} - 2 \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)}$ |                                  |
| Sen     | $h_s$                            | $q_s$                            |
| Kakwani | $h_k$                            | $q_k$                            |

with

$$J_s(F(1)) = 2 \int_0^{F(1)(Z)} \left(1 - \frac{s}{F(1)(Z)}\right) \left(\frac{Z - F^{-1}(s)}{Z}\right) ds,$$

$$K_s(F(1)) = 2 \left(1 - \frac{1}{ZF(1)(Z)}\right) \int_0^{F(1)(Z)} \frac{F^{-1}(s)}{F(1)(s)} ds + \frac{J_s(F(1))}{F(1)(Z)}.$$

And

$$h_k(y) = \left\{(k+1) \left[\left(1 - \frac{F(1)(y)}{F(1)(Z)}\right)^k \left(\frac{Z-y}{Z}\right)\right] - \frac{J_k(F(1))}{F(1)(Z)} \left(\frac{F(1)(y)}{F(1)(Z)}\right)^k + K_k(F(1))\right\} \mathbb{I}_{(y \leq Z)},$$

and

$$q_k(y) = -\frac{k(k+1)}{F(1)(Z)} \left[\left(1 - \frac{F(1)(y)}{F(1)(Z)}\right)^{k-1} \left(\frac{Z-y}{Z}\right)\right] + \frac{J_k(F(1))}{F(1)(Z)} \left(\frac{F(1)(y)}{F(1)(Z)}\right)^{k-1} \mathbb{I}_{(y \leq Z)}.$$

where

$$J_k(F(1)) = (k+1) \int_0^{F(1)(Z)} \left(1 - \frac{s}{F(1)(Z)}\right)^k \left(\frac{Z - F^{-1}(s)}{Z}\right) ds,$$
and

\[ K_k(F_{(1)}) = \frac{k(k + 1)}{F_{(1)}(Z)} \int_0^{F_{(1)}(Z)} \left( 1 - \frac{s}{F_{(1)}(Z)} \right)^{k-1} \left( \frac{Z - F_{(1)}^{-1}(s)}{Z} \right) ds + \frac{J_k(F_{(1)})}{F_{(1)}(Z)}. \]

Notice that the functions are indexed by \( k \) for the Kakwani measure. For the FGT measure of index \( \alpha \), we have that \( q = 0 \) and

\[ h(x) = \max(0, (Z - x)/Z)\alpha. \]
CHAPTER 7

Asymptotic Representation of Takayama’s statistics

The one-dimensional Takayama statistic ([Takayama (1979)]) is originally defined for a non-negative random variable $X$. Here, the not defined notation are supposed to be already done in Chapter 1 (page 15). The Takayama welfare measure is given, for $n \geq 1$, by

$$T_n = 1 + \frac{1}{n} + \frac{1}{n^2 \mu_n(1)} \sum_{1 \leq j \leq nF_n(Z)} (n_j + 1)d(X_{n-j+1,n}),$$

where $\mu_n(1)$ is the empirical mean for a sample of size $n \geq 1$, $d(x)$ is some measurable function of $x \in \mathbb{R}_+$. Originally $d$ is the identity function. But we will treat the general case. We have that $T_n$ is composed of the statistics $\mu_n(1)$ with

$$C_n = \frac{1}{n^2} \sum_{1 \leq j \leq nF_n(Z)} (n_j + 1)d(X_{n-j+1,n})$$

$$\mu = \mathbb{E}X \in \mathbb{R}$$

We will need the following conditions :

(C1) $0 < \mathbb{E}d(X) \in \mathbb{R}$.

(C2) $0 < F_{(1)}(Z) < 1$.

(C3) For all $0 < H < u_e(F_{(1)})$, the measurable function $q$ is continuous on $[0, H]$.

The GRI of the Takayama is given as follows.

**Theorem 12.** Under conditions (C1), (C2) and (C3), we have:
(1) For $Id(x) = x$ for $x \in \mathbb{R}$,
\[ \mu_n(1) = \mu + n^{-1/2}G_{n,1}(Id) + o_P, \quad n \geq 1, \]

(2) For
\[ C = \int_0^Z (1 - F_{(1)}(x)) \, dF_{(1)}(x), \]
\[ h_c(x) = (1 - F_{(1)}(x)) \, d(x)1_{(x \leq Z)}, \quad x \in \mathbb{R}_+ \]
and
\[ q(x) = -d(x)1_{(x \leq Z)}, \quad x \in \mathbb{R}_+ \] and $\ell(s) = q\left(F_{(1)}^{-1}(s)\right), \quad s \in (0, 1),$
we have
\[ \sqrt{n}(C_n - C) = G_{n,1}(h_c) + \int_0^1 G_{n,1}(\tilde{f}) \, \ell(s) \, ds + o_P(1). \]

(3) For
\[ T = \frac{1}{\mu} \int_0^Z (1 - F_{(1)}(x)) \, dF_{(1)}(x), \]
and
\[ h(x) = \mu^{-1}(h_c - C\mu^{-1})Id), \quad x \in \mathbb{R}, \]
We have
\[ \sqrt{n}(T_n - T) = G_{n,1}(h) + \int_0^1 G_{n,1}(\tilde{f}_s) \, \ell(s) \, ds + o_P(1). \]

We already know (see Chapter 5, page 79) that $\mu_n(1)$ has the $GRI$ given in Point (1) of the Theorem.

Before we come to establishing the $GRI$ of $C_n$, we remark that condition (C3) implies that for any $0 < H < \text{up}(F_{(1)}),$ 
\[ \varpi(q, \delta, H) = \sup (x, y) \in [0, H]^2 : |x - y| < \delta |q(x) - q(y)| \to 0, \text{ as } \delta \to 0, \]
where $\varpi(q, \delta, H)$ is the $\delta$-uniform continuity modulus of $q$ on $[0, H]$ and for $0 < h < 1,$
7. ASYMPTOTIC REPRESENTATION OF TAKAYAMA’S STATISTICS

\[ \zeta(d, h) = \sup_{0 \leq s \leq h} |d \left( F_{(1)}^{-1}(s) \right) | < +\infty. \]

Let us establish GRI for

\[ C_n = \frac{1}{n^2} \sum_{1 \leq j \leq n} (n - j + 1)d(X_{n-j+1,n}), \quad n \geq 1. \]

We suppose that the underlying cdf \( F_{(1)} \) is continuous. Hence, by using the rank statistics in the lines in Section 4 in Chapter 1, we have, for \( n \geq 1, \)

\[ C_n = \frac{1}{n^2} \sum_{1 \leq j \leq n} (n - R_{j,n} + 1)d(X_j) \]

\[ = \frac{1}{n} \sum_{1 \leq j \leq n} \left( 1 - F_{n,(1)}(X_j) + \frac{1}{n} \right) d(X_j)1_{x \leq Z}. \]

Based on the finiteness of the mathematical expectation of \( d(X) \) and the boundedness of the \( F_{n,(1)}(\cdot) \)'s, and by using the law of large numbers, we easily see that

\[ C_n \to C = \int (1 - F_{(1)}(x)) \left( a \right) dF_{(1)}(x), \quad \text{as } n \to +\infty. \]

and

\[ C_n = \frac{1}{n} \sum_{1 \leq j \leq n} \left( 1 - F_{n,(1)}(X_j) \right) d(X_j)1_{x \leq Z} + o_P(n^{-1}). \]

We get for \( n \geq 1, \)

\[ C_n = \frac{1}{n} \sum_{1 \leq j \leq n} \left( 1 - F_{(1)}(X_j) \right) d(X_j)1_{x \leq Z} \]

\[ - \frac{1}{n} \sum_{1 \leq j \leq n} \left( F_{(1)}(X_j) - F_{(1)}(X_j) \right) d(X_j)1_{x \leq Z} + o_P(n^{-1}). \]

By denoting

\[ h(x) = \left( 1 - F_{(1)}(x) \right) d(x)1_{x \leq Z}, \quad x \in \mathbb{R}_+ \]

and
\[ q(x) = -d(x)1_{(x \leq Z)}, \quad x \in \mathbb{R}_+ \text{ and } \ell(s) = q\left( F_{(1)}^{-1}(s) \right), \quad s \in (0, 1), \]

and hence, we reach half on the way, that is

\[(0.1) \quad n^{1/2} (A_n - A) = G_{n, (1)}(h) + R_{n}(\ell) + o_p(1). \quad (RT01)\]

To do the other half way, we have to check \( \text{(CRe1)} \) and \( \text{(CRe2)} \) in page 32. As to \( \text{(CRe1)} \), we have

\[ \mathbb{E}q(X) = \int_0^Z d(x)F_1(x)\, dF_1(x). \]

Condition \( \text{(CRe2)} \) is checked by showing that

\[ A_n = \int_0^1 \sqrt{n} \left( (s - \mathbb{V}_{n, (1)}(s)) \Delta_n(s) ds \right) \rightarrow 0 \quad (CRe2) \]

where

\[ \Delta_n(s) = \left( \ell \left( \mathbb{V}_{n, (1)}(s) \right) - \ell(s) \right), \quad s \in (0, 1). \]

We have, for \( n \geq 1 \) and \( s \in (0, 1), \)

\[ \Delta_n(s) = d\left( F_{n, (1)}^{-1}(s) \right) - d\left( F_{(1)}^{-1}(s) \right) 1_{(s \leq F_{n, (1)}(Z))} \]
\[ + \quad d\left( F_{(1)}^{-1}(s) \right) \left( 1_{(s \leq F_{n, (1)}(Z))} - 1_{(s \leq F_{(1)}(Z))} \right) \]
\[ =: \quad \Delta_n(1, s) + \Delta_n(2, s) \]

But, since \( F_{(1)}(Z) < 1 \), there exists \( \eta > 0 \), such that \( F_{(1)}(Z) + \eta < 1 \). By the uniform convergence of the uniform quantile process, for any \( \varepsilon > 0 \) there exists \( n_0 \) such that for any \( n \neq n_0 \)

\[ \mathbb{P}(\left( \Delta_n \leq \eta \right) \cup (F_{n, (1)}(Z) \leq F_{(1)}(Z) + \eta)) \leq \varepsilon. \]

where

\[ D_n = \sup_{s \in (0, 1)} \left| \left( \mathbb{V}_{n, (1)}(s) - s \right) \right|, \quad n \geq 1. \]

Let us split \( A_n \) into
\[ A_n = \int_0^1 \sqrt{n} (s - \mathbb{V}_{n,(1)}(s)) \Delta_n(1, s) + \int_0^1 \sqrt{n} (s - \mathbb{V}_{n,(1)}(s)) \Delta_n(2, s) \]

+ \( A_n(1) + A_n(2) \)

Denote \( \Omega_n = (\Delta_n \leq \eta) \cup (\mathcal{F}_{n,(1)}(Z) \leq F(1)(Z) + \eta), n \geq 1 \). On \((\Delta_n \leq \eta), n \geq n_0\), we have

\[
|A_n(1)| = \int_{s \in (0,1), s \leq \mathcal{F}_{n,(1)}(Z)} \left| \sqrt{n} \left(s - \mathbb{V}_{n,(1)}(s)\right) \Delta_n(1, s) \right| ds \\
+ \int_{s \in (0,1), s > \mathcal{F}_{n,(1)}(Z)} \left| \sqrt{n} \left(s - \mathbb{V}_{n,(1)}(s)\right) \Delta_n(1, s) \right| ds \\
\leq \varpi(d, D_n, Z) \int_{(0,1)} \left| \sqrt{n} \left(s - \mathbb{V}_{n,(1)}(s)\right) \right| ds \\
+ \varsigma(q, G(Z) + \eta) (\mathcal{F}(1)(Z) - \mathcal{F}_{n,(1)}(Z))^+ \\
=: B_n(1, 1) + B_n(1, 2).
\]

where \( x^+ = \max(0, x) \) for any \( x \in \mathbb{R} \). By classical results on uniform empirical processes, we have

\[
\int_0^1 \left| \sqrt{n} \left(s - \mathbb{V}_{n,(1)}(s)\right) \right| ds \rightarrow \int_0^1 |B(s)| ds \equiv Y,
\]

where \((B(s), s \in (0,1))\) is a standard Brownian Bridge so that \( Y \) has a finite expectation and by then is finite a.e and next,

\[
\int_0^1 \left| \sqrt{n} \left(s - \mathbb{V}_{n,(1)}(s)\right) \right| ds
\]

is bounded in probability. As a result, we have \( B_n(1, 1) \rightarrow_{\mathbb{P}} 0 \) since \( \varpi(d, D_n, Z) \rightarrow 0 \), as \( n \rightarrow 0 \). As well, \( B_n(1, 2) \rightarrow_{\mathbb{P}} 0 \) since \( b(q, G(Z) \) is bounded. We also have

\[
|A_n(2)| \leq \left| (\mathcal{F}(1)(Z) - \mathcal{F}_{n,(1)}(Z)) \right| \int_{(0,1)} \left| \sqrt{n} \left(s - \mathbb{V}_{n,(1)}(s)\right) \right| ds
\]

which goes to zero in probability for the same reasons given before. In Total, for \( B_n = A_n(1, 1) + A_n(1, 2) + A_n(2) \), we have for \( n \geq n_0 \)

\[
\mathbb{P}(|A_n| > B_n) \leq \varepsilon,
\]
7. ASYMPTOTIC REPRESENTATION OF TAKAYAMA’S STATISTICS

with $0 \leq B_n \rightarrow \mathbb{P} 0$ as $n \rightarrow +\infty$. Thus, we may and do apply Lemma 4 in Chapter 2 (See page 77) to conclude that

$$A_n(\eta) \rightarrow \mathbb{P} 0, \text{ as } n \rightarrow 0.$$ 

which closes the proof of Point (b).

As to point (c), it is enough to use the techniques provided in the proof of Lemma 2 in Chapter 1, page 22. The proof of the GRI is done. □.
CHAPTER 8

Conclusion

We have set a frame has been set up for establishing General (Asymptotic) Representations for Indices $GRI$ for a large class of statistics. In the Gaussian field we have described, we are able to study asymptotic joint distributions of different statistics including temporal (longitudinal) and spatial configurations. As well, in the spatial case, the statistical estimation of the default of decomposability is handle based on the $GRI$ formula.

Based on these results, we are going to open two important project:

(a) The handbook of $GRI$’s is open. Any contributor will present a specific or class of statistics and establish the $GRI$ along with the full proof. The contribution has to respect the notation given in this portal in order to be coherently included. The contribution in for a chapter will be assigned a digital object identifier and cite as an independent publication. The authors of such contribution will be allowed to use and adapt the packages described below.

(b) Since all the results described above which also will be extended to new $GRI$ depend only on functions $h$ and $\ell$, a package of computer programs has to be done in different languages. Actually, this package exists. It should be done again in a detailed writing and extended to other language. An R package is schedule.
Bibliography

[Barrett and Donald (2000)] Barrett G.F. and Donald S.G. (2000). Statistical inference with generalized Gini indices of inequality and poverty. Discussion paper 2002/01. Sydney: School of Economics, University of New South Wales.

[Bahadur (1966)] Bahadur, R.R. (1966). A note on quantiles in large samples, Ann. Math. Stat. 37, pp. 577–580. MR :32:6522
ZL : 0147.18805. doi:10.1214/aoms/1177699450. euclid.aoms/1177699450

[Billingsley (1968)] Billingsley, P.(1968). Convergence of Probability measures. John Wiley, New-York.

[Csörgő et al. (1986)] Csörgő, H., Csörgő, M., Mason D.M. and Horváth, L.(1986) Weighted empirical process and quantile process. Ann. Probab. 14 (1), 31-85.

[Dudley R.M.(1984)] Dudley R.M.(1984) A course on empirical Processes (École dété de Probabilités de Saint-Flour XII-1982). Lecture Notes in Mathematics 1097, 2-141 (ed. P.L. hennequin). Springer-Verlag, New-York.

[Foster et al.(1984)] Foster, J., Greer, J. and Thorbecke, E.(1984) A class of decomposable poverty measures. Econometrica, 3 (52), pp. 761766. doi:10.2307/1913475.

[Haidara and Lo(2012)] Haidara M.C and LO G.S.(2012) Statistical Estimation of Gap of Decomposability of the General Poverty Index. International Journal of Statistics and Probability, 1 (2), doi:10.5539/ijsp.v1n2p211

[Gaenssler (1983)] Gaenssler, P. (1983). Empirical processes. IMS Lecture Notes - Monograph Series, Vol. 3.

[Kakwani (1980)] Kakwani, N.(1980). On a Class of Poverty Measures. Econometrica, 48, 437-446. (MR0560520). http://dx.doi.org/10.2307/191106.

[Komlós et al. (1980)] Komlós, J. Májor, M. and Tusnády, G. (1975). Weak convergence and embedding. In : Colloquia Math. Soc. Janos. Boglai. Limit theorems of probability Theory, 149-165. Amsterdam, North-Holland.

[Lo et al. (2006)] Lo, G. S., Sall. S. and Seck, C. T.(2006). Une Théorie asymptotique des indicateurs de pauvreté. C. R. Math. Acad. Sci. R. Can. 31 (2009), no. 2, 45-52. (MR2535867), (2010m:91167)

[Lo and Sall (2010)] Lo G. S. and Sall, S.T.(2010). Asymptotic Representation Theorems for Poverty Indices. Afrika Statistika., 5, pp.238-244. (MR2920300)

[Lo (2010)] Lo G. S.(2010). A simple note on some empirical stochastic process as a tool in uniform L-statistics weak laws. Afrika Statistika, 5, pp. 437-446. (MR2920301)

[Lo (2013)] Lo, G. S.(2013). The Generalized Poverty Index. far East Journal of Theoretical Statistics Vol. 42, No. 1, pp. 1-22. Available online at
[Lo et al. (2015)] Lo, G.S., Thiam O. and Haidara C.M.(2015) High Moments Jarque-Bera Tests for Arbitrary Distribution Functions. *Applied Mathematics*, Vol. 6, pp. 706-717. DOI: 10.4236/am.2015.64066.

[Lo et al. (2016)] Lo, G.S., Ngom M. and Kpanzou T. A.(2016). Weak Convergence (IA). Sequences of random vectors. SPAS Books Series.(2016). Doi : 10.16929/sbs/2016.0001. Arxiv : 1610.05415

[Lo (2018)] Lo, G.S.(2018) A simple proof of the theorem of Sklar and its extension to distribution functions. Arxiv : 1803.00409.

[Lo et al. (2018)] Lo, G.S., Kpanzou, T.A., Haidara C.M.(2018) Chi-square Jarque-Bera Tests for Arbitrary Distribution Functions. To appear.

[Nelsen (2006)] Nelsen, R. B. (2006). An Introduction to Copulas. Springer.

[Pollard (1984)] Pollard D.(1984). Convergence of Stochastic Processes. Springer-Verlag, Berlin.

[Sall and Lo (2007)] Sall, S.T. and Lo, G.S., (2007). The Asymptotic Theory of the Poverty Intensity in View of Extreme Values Theory For Two Simple Cases. *Afrika Statistika*, vol 2 (n^31), p.41-55

[Sall and Lo (2010)] Sall, S. T. and Lo, G. S., (2010). Uniform Weak Convergence of the time-dependent poverty Measure for Continuous Longitudinal Data. *Braz. J. Probab. Stat.*, 24, (3), 457-467. (MR2719696)

[Sen (1976)] Sen Amartya K.(1976). Poverty: An Ordinal Approach to Measurement. *Econometrica*, 44, 219-231.

[Shorack and Wellner (1995)] Shorack G.R. and Wellner J.A. (1986). Empirical Processes with Applications to Statistics, wiley-Interscience, New-York.

[Shorrocks (1995)] Shorrocks, A. (1995). Revisiting the Sen Poverty Index. *Econometrica*, 63, 1225-1230. (doi:10.2307/2171728)

[Sklar (1959)] Sklar A.(1959) Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist Univ Paris*, 8:229-231.

[Takayama (1979)] Takayama, N.(1979). Poverty, Income Inequality, and Their Measures: Professor Sen’s Axiomatic Approach Reconsidered, Econometrica, 47, 747-759.

[van der Vaart and Wellner (1996)] van der Vaart, A. W. and Wellner J. A.(1996) *Weak Convergence and Empirical Processes: With Applications to Statistics*, Springer-Verlag New-York.

[Thon (1979)] Thon, D.(1979). On Measuring Poverty. Review of Income and Wealth 25, 429-440.