Abstract

We estimate the decay rates of $\eta_c \to 2\gamma$, $\eta'_c \to 2\gamma$, and $J/\psi \to e^+e^-$, $\psi' \to e^+e^-$, by taking into account both relativistic and QCD radiative corrections. The decay amplitudes are derived in the Bethe-Salpeter formalism. The Bethe-Salpeter equation with a QCD-inspired interquark potential are used to calculate the wave functions and decay widths for these $c\bar{c}$ states. We find that the relativistic correction to the ratio $R \equiv \Gamma(\eta_c \to 2\gamma)/\Gamma(J/\psi \to e^+e^-)$ is negative and tends to compensate the positive contribution from the QCD radiative correction. Our estimate gives $\Gamma(\eta_c \to 2\gamma) = (6 - 7) \text{ keV}$ and $\Gamma(\eta'_c \to 2\gamma) = 2 \text{ keV}$, which are smaller than their nonrelativistic values. The hadronic widths $\Gamma(\eta_c \to 2g) = (17 - 23) \text{ MeV}$ and $\Gamma(\eta'_c \to 2g) = (5 - 7) \text{ MeV}$ are then indicated accordingly to the first order QCD radiative correction, if $\alpha_s(m_c) = 0.26 - 0.29$. The decay widths for $b\bar{b}$ states are also estimated. We show that when making the assumption that the quarks are on their mass shells our expressions for the decay widths will become identical with that in the NRQCD theory to the next to leading order of $v^2$ and $\alpha_s$.

Charmonium physics is in the boundary domain between perturbative and nonperturbative QCD. Charmonium decays may provide useful information on understanding the nature of interquark forces and decay mechanisms. Both QCD radiative corrections and relativistic corrections are important for charmonium decays, because for charmonium the strong coupling constant $\alpha_s(m_c) \approx 0.3$ [defined in the $\overline{\text{MS}}$ scheme (the modified minimal subtraction scheme)] and the velocity squared of the quark in the meson rest frame $v^2 \approx 0.3$, both are not small. Decay rates of heavy quarkonia in the nonrelativistic limit with QCD radiative corrections have been studied (see, e.g., refs.[1,2,3]). However, the decay rates of many processes are subject to substantial
relativistic corrections. In the present paper, we will investigate relativistic corrections to the pseudoscalar quarkonium decays such as $\eta_c \rightarrow 2\gamma$ and $\eta'_c \rightarrow 2\gamma$, and give an estimate of their widths by taking into account both relativistic and QCD radiative corrections. (For a brief report on this result, see also ref.[4].) For comparison we will also study the leptonic decays of the vector charmonium such as $J/\psi \rightarrow e^+e^-$ and $\psi' \rightarrow e^+e^-$. 

These pseudoscalar charmonium decays are interesting. Experimentally, the branching ratio of $\eta_c \rightarrow 2\gamma$ may provide an independent determination of $\alpha_s$ at the charm quark mass, but the measured $\Gamma(\eta_c \rightarrow 2\gamma)$ ranges from 6 keV to 28 keV [5], and the measured $\eta_c$ total width is also uncertain. As for the $\eta'_c$, its existence needs to be confirmed, and its two gamma decay mode is being searched for by the E835 experiment at the Fermi Lab $p\bar{p}$ collider, and its hadronic decay modes are being studied by BES Collaboration at BEPC.

A lot of theoretical work have been done on charmonium and, in particular, on these pseudoscalar charmonium decays[4,9-14]. Nonrelativistic quark model gives $\Gamma(\eta_c \rightarrow 2\gamma) = 8.5\text{keV}$ (using the observed $J/\psi$ leptonic width as input), while the QCD sum rule approach predicts a value of $4.6 \pm 0.4\text{keV}$[12].

Recently, there have been significant progresses in the study of heavy quarkonium decays based on a more fundamental approach of the NRQCD (nonrelativistic QCD) effective theory[15,16]. The factorization theorem was further discussed, and some important issues (e.g., the infrared divergences in the P-wave state decay rates) were clarified in this study. The NRQCD theory combined with nonperturbative lattice simulations have achieved many interesting results on heavy quarkonium spectrum and decays[17,18,19].

In this paper, we will use the Bethe-Salpeter (BS) formalism[20] to derive the decay amplitudes and to calculate the decay widths of heavy quarkonium. The meson will be treated as a bound state consist of a pair of constituent quark and antiquark (i.e., higher Fock states such as $|Q\bar{Q}g>$ and $|Q\bar{Q}gg>$ are neglected, which may be justified to the first order relativistic corrections of $S$ wave heavy quarkonium decays) and described by the BS wavefunction which satisfies the BS equation. A phenomenological QCD-inspired interquark potential will be used to solve for the wavefunctions and to calculate the decay widths. Both relativistic and QCD radiative corrections to next-to-leading order will be considered based on the factorization assumption for the long distance and short distance effects.

We first consider the $\eta_c \rightarrow 2\gamma$ decay. This process proceeds via the $c\bar{c}$ annihilation. In the Bethe-Salpeter (BS) formalism the annihilation matrix element can be written as follows

$$\langle 0 | \overline{Q}IQ | P \rangle = \int d^4q \text{Tr}[I(q,P)\chi_P(q)], \quad (1)$$

where $|P\rangle$ represents the heavy quarkonium state, $P(q)$ is the total (relative) momentum of the $Q\bar{Q}$, $\chi_P(q)$ is its four dimensional BS wave function, and where $I(q,P)$ is the interaction vertex of the $Q\bar{Q}$ with other fields (e.g., the photons or gluons) which, in general, may also depend on the variable $q^0$ (the time-component of the relative
momentum). If $I(q, P)$ is independent of $q^0$ (e.g., if quarks are on their mass-shells in the annihilation), this equation can be written as

$$
\langle 0 \mid \bar{Q}I Q \mid P \rangle = \int d^3 q \text{Tr}[I(\vec{q}, P)\Phi_P(\vec{q})],
$$

(2)

where

$$
\Phi_P(\vec{q}) = \int d^3 q^0 \chi_P(q)
$$

(3)
is the three dimensional BS wave function of the $Q\bar{Q}$ meson. Note that in this approximation the decay amplitude is greatly simplified and only the three dimensional BS wave function is needed (but this does not necessarily require the interquark interaction to be instantaneous). In the BS formalism in the meson rest frame, where $\vec{p}_1 = -\vec{p}_2 = \vec{q}$, $P = (M, 0)$, and $p_1(p_2)$ is the quark(antiquark) momentum, $M$ is the meson mass, we have

$$
\Phi^0_\pm(\vec{q}) = \Lambda^1_\pm(\vec{q}) \gamma^0 (1 + \gamma^0) \gamma^5 \Lambda^2_\pm (-\vec{q}) \varphi(\vec{q}),
$$

$$
\Phi^1_\pm(\vec{q}) = \Lambda^1_\pm(\vec{q}) \gamma^0 (1 + \gamma^0) \phi^0 \Lambda^2_\pm (-\vec{q}) f(\vec{q}),
$$

(4)

where $\Phi^0_\pm(\vec{q})$, and $\Phi^1_\pm(\vec{q})$ represent the three dimensional wave functions of $0^-$ and $1^-$ mesons respectively, $\phi = e_\mu \gamma^\mu$, $e_\mu$ is the polarization vector of $1^-$ meson, $\varphi$ and $f$ are scalar functions which can be obtained by solving the BS equation for $0^-$ and $1^-$ mesons, and $\Lambda_\pm(\Lambda_-)$ are the positive (negative) energy projector operators

$$
\Lambda^1_\pm(\vec{q}) = \Lambda_\pm(\vec{p}_i) = \frac{1}{2E}(E + \gamma^0 \vec{\gamma} \cdot \vec{p}_i + m\gamma^0),
$$

$$
\Lambda^2_\pm(-\vec{q}) = \Lambda_\pm(\vec{p}_2) = \frac{1}{2E}(E - \gamma^0 \vec{\gamma} \cdot \vec{p}_2 - m\gamma^0),
$$

$$
E = \sqrt{\vec{q}^2 + m^2}.
$$

(5)

For process $\eta_c \rightarrow 2\gamma$ with the photon momenta and polarizations $q_1, \epsilon_1$ and $q_2, \epsilon_2$, the decay amplitude can be written as

$$
T = \langle 0 \mid \sigma \Gamma_{\mu\nu}(q)c \mid \eta_c \epsilon_1^\mu(\lambda_1)\epsilon_2^\nu(\lambda_2) \rangle + \langle 0 \mid \sigma \Gamma'_{\mu\nu}(q)c \mid \eta_c \epsilon_2^\mu(\lambda_2)\epsilon_1^\nu(\lambda_1) \rangle,
$$

(6)

where $p_1(p_2)$ is the charm quark(antiquark) momentum, $p = p_1 - q_1$, $p' = p_1 - q_2$, $m$ and $M$ represent the masses of $c$ quark and $\eta_c$ meson respectively, and where

$$
\Gamma_{\mu\nu}(q) = \gamma_\mu \frac{e_\nu e_Q^2}{\vec{p} - m} \gamma_\nu, \quad \Gamma'_{\mu\nu}(q) = \gamma_\mu \frac{e_\nu e_Q^2}{\vec{p}' - m} \gamma_\nu,
$$

(7)

$e_Q = \frac{2}{3}$ for $Q = c$. Since $p_1^0 + p_2^0 = M$, as usual we take[1,13]

$$
p_1^0 = p_2^0 = \frac{M}{2}.
$$

(8)
Thus, \( p^0 = \frac{1}{2} M - q_1^0 = 0, \) \( p'^0 = \frac{1}{2} M - q_2^0 = 0, \) the amplitude \( T \) becomes independent of \( q_0. \) Employing Eqs. (2) and (4), we get

\[
    T = B e^{\rho_{\sigma} \mu} q_1 \rho q_2 \sigma \epsilon_{1 \nu} (\lambda_1) \epsilon_{2 \nu} (\lambda_2) e^2 e_Q^2 - B' e^{\rho_{\sigma} \mu} q_1 \rho q_2 \sigma \epsilon_{1 \nu} (\lambda_1) \epsilon_{2 \nu} (\lambda_2) e^2 e_Q^2,
\]

where

\[
    B = B', \quad B = i \frac{2m}{M} \int d \vec{q} \frac{\sqrt{-\vec{q}^2 + m^2}}{\left( \vec{q}^2 + m^2 \right) \left( \vec{q}_1^2 + m^2 - 2 \vec{q} \cdot \vec{q}_1 \right)} \varphi(\vec{q}).
\]

Using \( q_1 \cdot \epsilon_1 = 0 \) and \( q_2 \cdot \epsilon_2 = 0, \) it is easy to get the decay width

\[
    \Gamma(\eta_c \rightarrow 2\gamma) = 3 M^3 \pi \alpha^2 e_Q^4 |B|^2,
\]

In the nonrelativistic (NR) limit \( (M \approx 2m, \vec{q}^2 \rightarrow 0) \)

\[
    |B|^2 = \frac{1}{2m^5} |\psi(0)|^2,
\]

where we have used the relation

\[
    \int \varphi(\vec{q}) d \vec{q} = \frac{\sqrt{M}}{2} \psi(0),
\]

where \( \psi(0) \) is the Schrödinger wave function at origin in coordinate space. Substituting (12) into (11), we get

\[
    \Gamma^{NR}(\eta_c \rightarrow 2\gamma) = 12 \pi \alpha^2 e_Q^4 |\psi(0)|^2 / m^2,
\]

where \( \Gamma^{NR}(\eta_c \rightarrow 2\gamma) \) represents the decay width of \( \eta_c \rightarrow 2\gamma \) in the nonrelativistic limit, which is consistent with that given in ref.[1]. The QCD radiative correction to this process has been given in ref.[1]. Recently, in the framework of NRQCD the factorization formulas for the long distance and short distance effects were found to involve a double expansion in the quark relative velocity \( v \) and in the QCD coupling constant \( \alpha_s[15]. \) To the next to leading order in both \( v^2 \) and \( \alpha_s, \) as an approximation, we may write

\[
    \Gamma(\eta_c \rightarrow 2\gamma) = 3 M^3 \pi \alpha^2 e_Q^4 |B|^2 \left( 1 - \frac{3.4 \alpha_s(m_c)}{\pi} \right),
\]

where the strong coupling constant \( \alpha_s(m_c) \) is defined in the \( \overline{MS} \) scheme (the modified minimal subtraction scheme). By expanding \( B \) in (10) in terms of \( \vec{q}^2 / m^2, \) to the next to leading order of \( v^2 \) we have

\[
    B = \frac{16i}{M(M^2 + 4m^2)} \int d \vec{q} \varphi(\vec{q})(1 - \frac{11}{24} \frac{\vec{q}^2}{m^2}).
\]
We see that the relativistic kinematic effect is to suppress the $\eta_c \to 2\gamma$ decay width. For comparison with the process $J/\psi \to e^+e^-$, we also give the decay amplitude for the $Q\bar{Q}$ annihilate into an electron with momentum $k_1$ and helicity $r_1$ and a positron with momentum $k_2$ and helicity $r_2$. Here the interaction vertex $I(P, q) = -ie\gamma_\mu$, which is independent of $q^0$, and the amplitude can be written as

$$T = e^2 e_Q \langle 0 \mid \overline{c} \gamma_\mu c \mid J/\psi \rangle \overline{\nu}_{r_1}(k_1) \gamma^\mu \nu_{r_2}(k_2) \frac{1}{M^2}. \quad (17)$$

Define the decay constant $f_V$ by

$$f_V e_\mu \equiv \langle 0 \mid \overline{c} \gamma_\mu c \mid J/\psi \rangle = \int d \vec{q} \, T_r[\gamma_\mu \Phi_\rho(\vec{q})], \quad (18)$$

where $e_\mu$ is the polarization vector of $J/\psi$ meson. Then with (4) we find

$$f_V = \frac{2\sqrt{3}}{M} \int d \vec{q} \, \frac{m + E}{E} \left( \frac{\vec{q}^2}{3E^2} \right) f(\vec{q}), \quad (19)$$

where $E = \sqrt{\vec{q}^2 + m^2}$. Summing over the polarizations of the final states and averaging over that of the initial states, it is easy to get the decay width

$$\Gamma(J/\psi \to e^+e^-) = \frac{4}{3} \pi \alpha^2 e_Q f_V^2 / M. \quad (20)$$

In the nonrelativistic limit it is reduced to the well known result

$$\Gamma_{NR}(J/\psi \to e^+e^-) = 16\pi \alpha^2 e_Q^2 |\psi(0)|^2 / M^2. \quad (21)$$

Including also the QCD radiative correction[1], we will get

$$\Gamma(J/\psi \to e^+e^-) = \frac{4}{3} \pi \alpha^2 e_Q^2 f_V^2 (1 - \frac{5.3\alpha_s(m_c)}{\pi}). \quad (22)$$

To the next to leading order of $v^2$, $f_V$ is expressed as

$$f_V = -\frac{4\sqrt{3}}{M} \int d \vec{q} \, f(\vec{q}) \left( 1 - \frac{5}{12} \frac{\vec{q}^2}{m^2} \right). \quad (23)$$

Again, the relativistic kinematic correction is to reduce the leptonic decay width. Comparing the two photon width with the leptonic width, we get

$$R \equiv \frac{\Gamma(\eta_c \to 2\gamma)}{\Gamma(J/\psi \to e^+e^-)} = \frac{9}{4} M_{\eta_c}^3 M_{J/\psi} e^2_Q |B|^2 f_V^2 / M^3 (1 + 1.96 \frac{\alpha_s(m_c)}{\pi}), \quad (24)$$

and in the nonrelativistic limit it becomes

$$R_{NR} \equiv \frac{\Gamma_{NR}(\eta_c \to 2\gamma)}{\Gamma_{NR}(J/\psi \to e^+e^-)} = \frac{4}{3} (1 + 1.96 \frac{\alpha_s(m_c)}{\pi}). \quad (25)$$
In fact, there are two sources of relativistic corrections: 1) the correction of relativistic kinematics which appears explicitly in the decay amplitudes; 2) the correction due to inter-quark dynamics (e.g. the well known Breit-Fermi interactions), which mainly causes the correction to the bound state wave functions. In general, due to the attractive spin-spin force induced by one gluon exchange for the $0^-$ meson, the $\eta_c$ wave function at origin becomes larger than its nonrelativistic value, one might expect the width of $\eta_c \rightarrow 2\gamma$ to be enhanced after taking relativistic corrections into account. However, because the kinematic relativistic correction to the decay rates is in the opposite direction and can be even larger, the overall relativistic correction to the decay width of $\eta_c$ is found to be negative.

To calculate the decay widths, we need to know the wavefunctions $\varphi(\vec{q})$ for the $0^-$ meson and $f(\vec{q})$ for the $1^-$ meson, which are determined mainly by the long distance interquark dynamics. In the absence of a deep understanding for quark confinement at present, we will follow a phenomenological approach by using QCD inspired interquark potentials including both spin-independent and spin-dependent potentials, which are supported by both lattice QCD calculations and heavy quark phenomenology, as the interaction kernel in the BS equation. We begin with the bound state BS equation[20] in momentum space

$$\left(\not{q}_1 - m_1\right)\chi_P(q_1)(\not{q}_2 + m_2) = \frac{i}{2\pi} \int d^4k G(P,q_1 - k)\chi_P(k),$$  \hspace{1cm} (26)

where $q_1$ and $q_2$ represent the momenta of quark and antiquark respectively, $G(P,q - k)$ is the interaction kernel which dominates the interquark dynamics. In solving Eq.(26), we will employ the instantaneous approximation since for heavy quarks the interaction is dominated by instantaneous potentials. Meanwhile, we will neglect negative energy projectors in the quark propagators which are of even higher orders. We then get the reduced Salpeter equation[20] for the three dimensional BS wavefunction $\Phi_P(\vec{q})$ defined in (6)

$$\Phi_P(\vec{q}) = \frac{1}{P_0 - E_1 - E_2} \Lambda^1 \gamma_0 \int d^3k G(P,\vec{q} - \vec{k})\Phi_P(\vec{k})\gamma_0 \Lambda^2,$$ \hspace{1cm} (27)

where $G(P,\vec{q} - \vec{k})$ represents the instantaneous potential.

We employ the following interquark potentials including a long-ranged confinement potential (Lorentz scalar) and a short-ranged one-gluon exchange potential (Lorentz vector)[21]

$$V(r) = V_S(r) + \gamma_\mu \otimes \gamma^\mu V_V(r),$$

$$V_S(r) = \lambda r \frac{1 - e^{-\alpha r}}{\alpha r},$$

$$V_V(r) = -\frac{4}{3} \alpha_s(r) \frac{e^{-\alpha r}}{r},\hspace{1cm} (28)$$
where the introduction of the factor $e^{-\alpha r}$ is to regulate the infrared divergence and also to incorporate the color screening effects of the dynamical light quark pairs on the $Q\bar{Q}$ linear confinement potential[22]. In momentum space the potentials become[21]

$$G(\vec{p}) = G_S(\vec{p}) + \gamma_\mu \otimes \gamma^\mu G_V(\vec{p}),$$

$$G_S(\vec{p}) = -\frac{\lambda}{\alpha} \delta^3(\vec{p}) + \frac{\lambda}{\pi^2} \frac{1}{(\vec{p}^2 + \alpha^2)^2},$$

$$G_V(\vec{p}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\vec{p})}{\vec{p}^2 + \alpha^2},$$

(29)

where $\alpha_s(\vec{p})$ is the quark-gluon running coupling constant and is assumed to become a constant of $O(1)$ as $\vec{p}^2 \to 0$

$$\alpha_s(\vec{p}) = \frac{12\pi}{27} \frac{1}{\ln(a + \vec{p}^2 / \Lambda_{QCD}^2)}. \tag{30}$$

The constants $\lambda$, $\alpha$, $a$ and $\Lambda_{QCD}$ are the parameters that characterize the potential.

Substituting (4) and (29) into Eq.(27), one derives the equation for the $0^-$ meson wavefunction $\varphi(\vec{q})$ in the meson rest frame

$$M\varphi_1(\vec{q}) = (E_{q_1} + E_{q_2})\varphi_1(\vec{q})$$

$$= \frac{1}{4E_{q_1}E_{q_2}} \left( -(E_{q_1}E_{q_2} + m_1m_2 + \vec{q}^2) \int d^3k (G_S(\vec{q} - \vec{k}) - 4G_V(\vec{q} - \vec{k}))\varphi_1(\vec{k}) \right.$$  

$$- (E_{q_1}m_2 + E_{q_2}m_1) \int d^3k (G_S(\vec{q} - \vec{k}) + 2G_V(\vec{q} - \vec{k})) \frac{m_1 + m_2}{E_{k_1} + E_{k_2}} \varphi_1(\vec{k})$$  

$$+ (E_{q_1} + E_{q_2}) \int d^3k G_S(\vec{q} - \vec{k})(\vec{q} \cdot \vec{k}) \frac{m_1 + m_2}{E_{k_1}m_2 + E_{k_2}m_1} \varphi_1(\vec{k})$$  

$$+ (m_1 - m_2) \int d^3k (G_S(\vec{q} - \vec{k}) + 2G_V(\vec{q} - \vec{k})) (\vec{q} \cdot \vec{k}) \frac{E_{k_1} - E_{k_2}}{E_{k_1}m_2 + E_{k_2}m_1} \varphi_1(\vec{k}), \tag{31}$$

where $E_{qi} = \sqrt{\vec{q}^2 + m_i^2}$, $E_{ki} = \sqrt{\vec{k}^2 + m_i^2}$, ($i = 1, 2$), and

$$\varphi_1(\vec{q}) = \frac{(m_1 + m_2 + E_{q_1} + E_{q_2})(E_{q_1}m_2 + E_{q_2}m_1)}{4E_{q_1}E_{q_2}(m_1 + m_2)} \varphi(\vec{q}). \tag{32}$$

The normalization condition $\int d^3q Tr\{\Phi^\dagger(\vec{q})\Phi(\vec{q})\} = (2\pi)^{-3}2M$ for the BS wavefunction leads to[21]

$$\int d^3q \frac{(m_1 + E_{q_1})(m_2 + E_{q_2})}{8E_{q_1}E_{q_2}} |\varphi(\vec{q})|^2 = \frac{M}{(4\pi)^3}. \tag{33}$$
For the $1^-$ meson we have

$$M f_1(q) = (E_{q1} + E_{q2}) f_1(q)$$

$$-\frac{E_{q1} + m_1 + E_{q2} + m_2}{4 E_{q1} E_{q2}[3(E_{q1} + m_1)(E_{q2} + m_2) + q^2]} \frac{3((E_{q1} E_{q2} + m_1 m_2 + q^2) \times}

$$\int d^3 k (G_S(q - k) - 2G_V(q - k)) \frac{3(E_{k1} + m_1)(E_{k2} + m_2) + \vec{k}^2}{E_{k1} + E_{k2} + m_1 + m_2} f_1(k)$$

$$-2q^2 \int d^3 k (G_S(q - k) - 2G_V(q - k)) \frac{E_{k2 m_1} + E_{k1 m_2}}{m_1 + m_2} f_1(k)$$

$$+(E_{q1 m_2} + E_{q2 m_1}) \int d^3 k G_S(q - k) \frac{3(E_{k1} + m_1)(E_{k2} + m_2) - \vec{k}^2}{E_{k1} + E_{k2} + m_1 + m_2} f_1(k)$$

$$-(m_1 + m_2) \int d^3 k G_S(q - k) E_{k1 + m_1 + E_{k2} + m_2} f_1(k)$$

$$-2(E_{q1} - E_{q2}) \int d^3 k (G_S(q - k) - 2G_V(q - k))(\vec{q} \cdot \vec{k}) f_1(k)$$

$$+ \frac{E_{k1 + m_1}}{E_{k1} + m_1 + E_{k2} + m_2} f_1(k)$$

$$-2(q - m_2) \int d^3 k G_S(q - k) E_{k1 - E_{k2}} \frac{E_{k1 - E_{k2}}}{m_1 + m_2} f_1(k)$$

$$+(E_{q1} + 3E_{q2}) \int d^3 k G_S(q - k) (\vec{q} \cdot \vec{k}) f_1(k)$$

$$-(6E_{q1} + 2E_{q2}) \int d^3 k G_V(q - k) \vec{q} \cdot \vec{k} f_1(k)$$

(34)

where

$$f_1(q) = \frac{E_{q1} + m_1 + E_{q2} + m_2}{4 E_{q1} E_{q2}} f(q).$$

The normalization condition $\int d^3 q Tr\{\Phi^\dagger(q)\Phi(q)\} = (2\pi)^{-3}2M$ for the BS wavefunction leads to[21]

$$\int d^3 q \frac{(m_1 + E_{q1})(m_2 + E_{q2})}{8 E_{q1} E_{q2}} |f(q)|^2 = \frac{M}{(4\pi)^3}.$$

(36)

To the leading order in the nonrelativistic limit, Eqs.(31) and (34) are just the ordinary nonrelativistic Schrödinger equation with simply a spin-independent linear plus Coulomb potential. To the first order of $v^2$, Eqs.(31) and (34) become the well known Breit equations for the $0^-$ and $1^-$ mesons with both spin-independent and spin-dependent potentials from vector (one-gluon) exchange and scalar (confinement) exchange.

For the heavy quarkonium $c\bar{c}$ and $b\bar{b}$ systems, $m_1 = m_2 = m$, Eqs.(31) and (34) become much simpler. By solving these equations we can find the wave functions for the $0^-$ and $1^-$ mesons. Here not only the ground state (1S) wave functions but also
the first radial excitation wave functions (2S) are obtained. They are shown in Fig.1
and Fig.2.

Substituting the obtained BS wave functions into (10), (15), and (19), (22), respectively, we then get the decay widths for both 0− and 1− charmonium states. In
the calculation following parameters have been chosen

\[ m_c = 1.5 GeV, \quad \lambda = 0.23 GeV^2, \quad \Lambda_{QCD} = 0.18 GeV, \]
\[ \alpha = 0.06 GeV, \quad a = e = 2.7183. \] (37)

With these parameters the 2S − 1S spacing and J/ψ − ηc splitting are required to fit
the data. We then get

\[ \Gamma(\eta_c \to 2\gamma) = 6.2 keV, \quad \Gamma(\eta'_c \to 2\gamma) = 1.8 keV, \]
\[ \Gamma(J/\psi \to e^+e^-) = 5.6 keV, \quad \Gamma(\psi' \to e^+e^-) = 2.7 keV. \] (38)

Our results are satisfactory, as compared with the Particle Data Group experimental
values[5] \( \Gamma(\eta_c \to 2\gamma) = 7.0^{+2.0}_{-1.7} keV, \quad \Gamma(J/\psi \to e^+e^-) = 5.36 \pm 0.29 keV, \quad \Gamma(\psi' \to e^+e^-) = 2.14 \pm 0.21 keV. \) Here in above calculations the value of \( \alpha_s(m_c) \) in the QCD
radiative correction factor in (15) and (22) is chosen to be 0.29 (re fs.[3,14]), which is
also consistent with our determination from the ratio of \( B(J/\psi \to 3g) \) to \( B(J/\psi \to e^+e^-) \) (see refs.[4,9]).

In order to see the sensitivity of the decay widths to the parameters especially the
charm quark mass, we have also used other two sets of parameters

\[ m_c = 1.4 GeV, \quad \lambda = 0.24 GeV^2; \]
\[ m_c = 1.6 GeV, \quad \lambda = 0.22 GeV^2; \] (39)

with other potential parameters (\( \Lambda_{QCD}, \alpha, a \)) unchanged, and found

\[ \Gamma(\eta_c \to 2\gamma) = 7.0(5.5) keV, \quad \Gamma(\eta'_c \to 2\gamma) = 1.7(1.5) keV \] (40)

for \( m_c = 1.4 (1.6) GeV, \) where the experimental value of \( \Gamma(J/\psi \to e^+e^-) = 5.36 keV \)
as input) and the calculated ratio \( R \) in (24) are used to give predictions for the pseudoscalar decay widths.

We see that for smaller charm quark masses \( \Gamma(\eta_c \to 2\gamma) \) gets enhanced. This
tendency is in line with the QCD sum rule result[12]. Our estimate that \( \Gamma(\eta_c \to 2\gamma) = (6-7) keV \) is consistent with the CLEO data[6] \( \Gamma(\eta_c \to 2\gamma) = (5.9^{+1.1}_{-1.2} \pm 1.9) keV, \) and the E760 data[7] \( 7 \pm 3 keV, \) and slightly smaller than the L3 data[8] \( (8.0 \pm 2.3 \pm 2.4) keV. \) Our results for \( \eta_c \) and \( \eta'_c \) distinguish them from the nonrelativistic values, which can be obtained by using the ratio \( R^{NR} \) (25) and the experimental values of \( \Gamma[J/\psi(\psi') \to e^+e^-] \)

\[ \Gamma^{NR}(\eta_c \to 2\gamma) = 8.5 keV, \quad \Gamma^{NR}(\eta'_c \to 2\gamma) = 3.4 keV. \] (41)

In particular, our prediction \( \Gamma(\eta'_c \to 2\gamma) = 2 keV \) is significantly smaller than its
nonrelativistic value.
We may further use these results to give an estimate for the total widths of $\eta_c$ and $\eta'_c$. Note the branching ratio

$$B(P \to 2\gamma) \approx \frac{\Gamma(P \to 2\gamma)}{\Gamma(P \to 2g)} = \frac{9\alpha_s^2 e_Q^4}{2\alpha_s^2} \times \left(1 - \frac{3.4\alpha_s}{\pi}\right)$$  \hspace{1cm} (42)

is free of the relativistic correction. Using the calculated two gamma widths $\Gamma(P \to 2\gamma) = 6.2(1.8)\text{keV}$ for $P = \eta_c(\eta'_c)$ and the strong coupling constant at the mass of the charm quark $\alpha_s(m_c) = 0.26 - 0.29[3,9,14]$, we will get

$$\Gamma_{\text{tot}}(\eta_c) = 17 - 23\text{ MeV},$$

$$\Gamma_{\text{tot}}(\eta'_c) = 5.0 - 6.7\text{ MeV}. \hspace{1cm} (43)$$

This is the prediction for the total widths up to the next to leading order of QCD radiative corrections, but higher order corrections may further modify this result. With the present Particle Data Group values[5] $\Gamma_{\text{tot}}(\eta_c) = 10.3^{+3.5}_{-3.0}\text{MeV}$ and $\Gamma(\eta_c \to 2\gamma) = 7.0^{+2.0}_{-1.7}\text{keV}$, however, a value of $\alpha_s(m_c) \approx 0.20$ will be indicated, which is significantly lower than expected from other experiments and theoretical studies on the QCD scale parameter. Therefore, it will be very interesting to see the accuracy of the experiment or to take the higher order QCD radiative correction more seriously.

Moreover, for the $b\bar{b}$ states, with $m_b = 4.9\text{GeV}$ and other potential parameters unchanged, we find

$$\Gamma(\eta_b \to 2\gamma) = 0.46\text{keV}, \hspace{1cm} \Gamma(\eta'_b \to 2\gamma) = 0.21\text{keV},$$

$$\Gamma(\Upsilon \to e^+e^-) = 1.36\text{keV}, \hspace{1cm} \Gamma(\Upsilon' \to e^+e^-) = 0.78\text{keV}. \hspace{1cm} (44)$$

Here $\alpha_s(m_b) = 0.20[3,9,14]$ is used in the QCD radiative corrections. We see that the relativistic corrections become smaller for $b\bar{b}$ than for $c\bar{c}$ states.

We finally discuss the relation between our approach and the NRQCD theory. In fact, our decay widths can be written in terms of the standard Schrödinger wavefunction (with relativistic corrections) $\psi_{\text{Sch}}(\vec{q})$, which is related to $\varphi(\vec{q})$ (or $f(\vec{q})$) through the normalization condition (33) (or (36)) which leads to

$$\psi_{\text{Sch}}(\vec{q}) = \frac{1}{\sqrt{M}} \left(\frac{m + E}{E}\right) \varphi(\vec{q}),$$

$$\int d^3 q \, \psi^*_{\text{Sch}}(\vec{q}) \psi_{\text{Sch}}(\vec{q}) = 1. \hspace{1cm} (45)$$

For the above discussed pseudoscalar($P$) and vector($V$) heavy quarkonium decays (see (10), (15), and (19), (22)) to the next to leading order in $v^2$ and $\alpha_s$ we then have

$$\Gamma(P \to 2\gamma) = \frac{192\pi \alpha_s^2 e_Q^4 M^2}{(M^2 + 4m^2)^2} \left(1 - \frac{3.4\alpha_s(m_Q)}{\pi}\right) \int d^3q \left(1 - \frac{2q^2}{3m^2}\right) \psi_{\text{Sch}}(\vec{q})^2, \hspace{1cm} (46)$$

$$\Gamma(V \to e^+e^-) = \frac{16\pi \alpha_s^2 e_Q^2 M^2}{3m^2} \left(1 - \frac{16\alpha_s(m_Q)}{3\pi}\right) \int d^3q \left(1 - \frac{q^2}{6m^2}\right) \psi_{\text{Sch}}(\vec{q})^2. \hspace{1cm} (47)$$
In above expressions $M$ is the mass of the meson. In previous calculations we have taken $M$ as their observed values for $\eta_c$ and $J/\psi$. However, we may take the on-shell condition, which assumes the quark and antiquark to be on the mass shell (see (8))

$$q_i^0 = q_2^0 = M/2 = E = \sqrt{m^2 + q^2},$$

(48)

to replace the observed value of the meson mass $M$ then (46) and (47) will become

$$\Gamma(P \to \eta^\prime) = \frac{12\pi\alpha^2 e_Q^4}{m^2}(1 - \frac{3.4\alpha_s(m_Q)}{\pi})\int d^3 q(1 - \frac{2q^2}{3m^2})|\psi_{Sch}(q)|^2,$$

(49)

$$\Gamma(V \to e^+e^-) = \frac{4\pi\alpha^2 e_Q^2}{m^2}(1 - \frac{16\alpha_s(m_Q)}{3\pi})\int d^3 q(1 - \frac{2q^2}{3m^2})|\psi_{Sch}(q)|^2,$$

(50)

It is easy to see that to the first order of $v^2$, in coordinate space (49) and (50) can be expressed as

$$\Gamma(P \to \eta^\prime) = \frac{3\alpha^2 e_Q^4}{m^2}(1 - \frac{3.4\alpha_s(m_Q)}{\pi})[|R(0)|^2 + \frac{4}{3m^2} Re(R^*(0)\nabla^2 R(0))],$$

(51)

$$\Gamma(V \to e^+e^-) = \frac{\alpha^2 e_Q^2}{m^2}(1 - \frac{16\alpha_s(m_Q)}{3\pi})[|R(0)|^2 + \frac{4}{3m^2} Re(R^*(0)\nabla^2 R(0))],$$

(52)

where $R(0)$ is the Schrödinger radial wavefunction at the origin of the $P$ ($P = \eta_c, \eta_c'$) or $V$ ($V = J/\psi, \psi'$) meson. These expressions are exactly the same as that given in ref.[15] with the NRQCD effective theory, if we identify our bound state wavefunctions with their regularized operator matrix elements, i.e.:

$$R(0) = \sqrt{\frac{2\pi}{3}}\epsilon \cdot <0|\lambda^i \sigma \psi|V >,$$

(53)

$$\nabla^2 R(0) = -\sqrt{\frac{2\pi}{3}}\epsilon \cdot <0|\lambda^i \sigma \left(-\frac{i}{2} \hat{D}\right)^2 \psi|V > [1 + O(v^2/c^2)].$$

(54)

In the NRQCD theory, the expectation values of the quark operators are well defined[15] and can be calculated with lattice simulations, which is a more fundamental method for describing nonperturbative dynamics than the quark potential model. In our approach the wavefunctions (and their derivatives) are estimated on the basis of the QCD-inspired potential model by solving the BS equation. Although this is not a first principle theory and it is difficult to control the systematic accuracy within the potential model, it may provide a rather useful estimate of the decay rates, since not only the zeroth order spin-independent potential but also the first order spin-dependent potential i.e. the Breit-Fermi Hamiltonian, which stems from one gluon exchange and has a good theoretical and phenomenological basis, are considered in the calculation, and different quark masses are also chosen to estimate the uncertainties in the calculation. In fact, the potentials are required to reproduce the observed
mass difference between $\eta_c$ and $J/\psi$ and the $J/\psi$ leptonic decay width, and then give predictions for the pseudoscalar mesons. This may reduce the uncertainty in the calculation of pseudoscalar decay widths. Nevertheless, for more reliable estimates we hope that these decay widths of heavy quarkonium can be eventually calculated from more fundamental theoretical methods, e.g., the lattice QCD simulations. It will be interesting to see the numerical results in the NRQCD approach and compare them with our results.

In summary, we have estimated the photonic widths and hadronic widths for pseudoscalar heavy quarkonium states, and the leptonic widths for vector heavy quarkonium states as well, by taking into account both relativistic and QCD radiative corrections. The photonic widths of $\eta_c$ and $\eta'_c$ tend to take lower values than the nonrelativistic result. We hope that experiments especially the E835 and BES experiments or experiments at the Tau-Charm Factory in the future will be able to have more accurate measurements on the branching ratios and the total widths for the $\eta_c$ and $\eta'_c$ particles. This will provide the basis for testing theoretical predictions.

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