The power of entangled quantum channels

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All communication channels are at bottom quantum mechanical. Quantum mechanics contributes both obstacles to communication in the form of noise, and opportunities in the use of intrinsically quantum representations for information. This paper investigates the trade-off between power and communication rate for coupled quantum channels. By exploiting quantum correlations such as entanglement, coupled quantum channels can communicate at a potentially higher rate than unentangled quantum channels given the same power. In particular, given the same overall power, $M$ coupled, entangled quantum channels can send $M$ bits in the same time it takes a single channel to send a single bit, and in the same time it takes $M$ unentangled channels to send $\sqrt{M}$ bits.

As communication technologies push down to the quantum level, a considerable effort has been made to uncover the physical limits to the communication process. Much progress has been made in investigating the physical limits to the bosonic communications channel (1-22) (e.g., quantum optical communication (1)) and to quantum communications in general (23-26). Quantum systems can be correlated with each other in ways that classical systems cannot, a feature known as entanglement. It has been speculated that entanglement might be used to enhance the capacity of quantum channels as compared to classical channels (24-28). Perhaps the best known example of the use of entanglement to enhance communication capacity is that of super-dense coding (25), in which two parties who initially possess an entangled state can send two bits of classical information by sending a single quantum bit. In addition, shared prior entanglement may enhance the transmission capacity of quantum channels in the presence of noise (28). In general, the enhancement in channel capacity that can be attained using entanglement is not known. This letter investigates the situation in which channels are coupled via a nonlinear dynamics to induce an entangled state in the process of transmission. For fixed power, I show that $M$ coupled,
entangled quantum channels can transmit information at a rate $\sqrt{M}$ times greater than $M$ uncoupled, unentangled channels.

To see that quantum mechanics can in principle attain large gains in channel capacity for power-limited parallel channels by entangling them, first review the case of unentangled parallel quantum channels. In particular, it is well established (1-22) that the broadband bosonic channel with power $P$ can transmit no more than $C_1 = \alpha \sqrt{P/\hbar}$ bits per second, where $\alpha = \sqrt{\pi/3(1/\ln 2)}$. As a consequence (1), if the power is spread amongst $M$ unentangled broadband bosonic channels, each with power $P/M$, the rate of communication is $C_M^C = \sqrt{MC_1}$. By contrast, as I now show, $M$ entangled quantum channels using power $P$ can send information from A to B at a rate $C_M^Q = \beta M \sqrt{P/\hbar} \approx MC_1 = \sqrt{MC_M^C}$, where $\beta = \sqrt{2/\pi(1-2^{-M})}$. That is, for a fixed power, entangled channels can in principle outperform unentangled channels by a wide margin for large $M$. Perhaps more remarkably, $M$ entangled channels can send $M$ bits in the same time and using the same overall power that it takes a single channel to send a single bit. A summary of these results is shown in figure 1. Not surprisingly, producing the necessary entangling dynamics for $M$ quantum channels is likely to prove difficult. As will be shown, however, simple demonstrations of the power of a small number of entangled channels can be performed using existing techniques of quantum information processing. And since the potential gains are large, the attempt to entangle $M$ channels may be worth the effort.

Quantum channels are physical systems linking sender and receiver. A quantum channel can be a fiber-optic cable, a wire connecting two bits in a computer, or a tunneling barrier connecting two quantum bits in a quantum computer. To calculate their limits requires a treatment of the channels’ dynamics, together with their interaction with sender and receiver. The calculation of the limits to the bosonic channel is detailed and relies on the physics of quantum electrodynamics (1). Here, a simpler and easily generalizable channel model is analyzed — the ‘qubit’ channel.

The qubit channel transmits a quantum bit from A to B. Suppose that A and B each possess a two-state quantum system, or ‘qubit.’ A’s qubit holds the quantum state $|\psi\rangle$ which is to be transmitted to B, whose qubit is initially in the state $|0\rangle$. The two states $|0\rangle$ and $|1\rangle$ of the qubits are assumed to be degenerate, so that no energy is required to store the qubit. The qubit channel can be used either to transmit classical information $|\psi\rangle = |0\rangle$ or $|1\rangle$ or to transmit quantum information $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. The dynamics of the channel should transfer the information from A’s qubit to B’s qubit. After the transfer has taken place, B’s qubit is in the state $|\psi\rangle$ and A’s qubit is in a standard state such as $|0\rangle$. The sole restriction placed on dynamics of the channel is that it obeys the rules of quantum mechanics: the time evolution of A’s qubit and B’s qubit, together with their environment
and whatever interaction they use to transfer the information, is a unitary, Hamiltonian dynamics. (Note that an arbitrary strictly positive time evolution can be embedded in such dynamics (29).) This requirement alone allows the establishment of a bound for the rate of information transmission down the qubit channel given limited power.

To derive this bound, note that the average overlap $|\langle 0|\psi\rangle|^2$ of A and B’s initial state with their final state is $1/2$. Accordingly, each time the channel is used, the total state of A and B together with environment must evolve by an average angle in Hilbert space of at least $\pi/2$. That is, to transmit a 0 requires no transformation of B’s qubit, but to transmit a 1, B’s qubit must be rotated by $\pi$. The Margolus-Levitin theorem then implies that when B’s qubit is rotated by $\pi$, the average energy of the complete system above its ground state is $E \geq \pi\hbar/2\Delta t$, where $\Delta t$ is the time over which the transfer takes place (30). If a 1 is sent half the time, the average power associated with applying energy $E$ for time $\Delta t$ is $P = E/\Delta t \geq \pi\hbar/4\Delta t^2$. Consequently, the rate at which a bit can be reliably transferred from A to B is

$$C_{1q} = 1/\Delta t = (2/\sqrt{\pi})\sqrt{P/\hbar}.$$  \hfill (1)

Equation (1) applies to the reliable transmission of a single bit. If one accepts unreliable or noisy transfer, then the transfer time can be less than $\Delta t = \pi\hbar/4E$, as one does not have to rotate the state of A and B by the full angle $\pi$ to transform them into an orthogonal state. This feature, together with the use of error correcting codes, can be used to enhance the rate of transfer of information for a given energy (31). In addition, if one is willing to send less than a full bit of information by sending a 0 with a higher probability, one can decrease the energy per transmission time $\Delta t$.

The first obvious point to note about equation (1) is that the power-limited capacity of the qubit channel is very similar to the power-limited capacity of the broadband bosonic channel. The capacity of the qubit channel and the bosonic channel differ by only a constant of order unity. At first, this might seem surprising. After all, the broadband bosonic channel has an infinite number of temporal modes, each of which in turn has an infinite number of states. In contrast, the qubit channel has only two qubits, each of which has only two states. The capacity of the bosonic channel is attained in the limit of infinite time by preparing energy eigenstates with a thermal distribution that satisfies the integrated power constraint. In contrast, as will be seen below, the qubit channel operates by sending one bit dynamically in time $\Delta t$ while respecting the power constraint. The two channels – bosonic and qubit – could hardly look more different. Why then do they exhibit almost the same channel capacity, including the same dependence on power?
The answer is that both of these channel capacities are attained by creating distributions of states with the maximum spread in energy given the overall power constraint — they are both broadband systems (1, 30). In addition, the bosonic channel attains its maximum capacity in the limit where approximately one bit is transmitted for each quantum sent down the channel: at its capacity limit, the bosonic channel closely resembles the qubit channel. To see broadband nature of the qubit channel, examine how the channel capacity limit can be attained.

A simple example of an interaction that attains the qubit capacity limit is the application of a ‘swap’ operation:

\[ S = \sum_{ij=0}^{1} |ij\rangle_{AB} \langle ji| \]

is a unitary transformation that swaps the quantum information in A’s qubit with the quantum information in B’s qubit. Note that \( S^2 = 1 \): two applications of \( S \) returns the bits to their original states. Consequently, \( S \) is Hermitian and has eigenvalues \( \pm 1 \). Note also that \( e^{-i\theta S} = \cos \theta - i \sin \theta S \).

Now analyze the power needed to apply a swap operation. Apply the Hamiltonian \( \tilde{S} = \pi \hbar (1 - S) / 2 \Delta t \) to A and B’s qubits for time \( \Delta t \). It is straightforward to verify that \( e^{-i\tilde{S} \Delta t / \hbar} = S \). So applying \( \tilde{S} \) for time \( \Delta t \) swaps the qubits. The average energy of A and B and the apparatus that swaps the qubits is

\[ E = (\langle \psi | A \langle 0 | B \rangle \tilde{S} (|0\rangle_A |\psi\rangle_B) = \pi \hbar (1 - |\langle 0 | \psi \rangle|^2) / 2 \Delta t. \]  

(2)

Averaging over states \( |\psi\rangle \) gives an energy \( E \) that saturates the Margolus-Levitin bound. Swap attains the power/capacity limit of equation (1) above.

The ‘swap’ picture of quantum information transmission assumes a direct transfer of A’s qubit to B. A similar picture holds in which A’s and B’s qubits are coupled by an intervening chain of qubits \( A_1 B_1 A_2 B_2 \ldots A_n B_n \), where A has access to \( A_1 \) and B has access to \( B_n \). Here, quantum information can be sent along the chain by swapping \( A_i \) with \( B_i \) over a time \( \Delta t / 2 \), then swapping \( B_i \) with \( A_{i+1} \), and repeating until the qubit has been moved from A to B. In this case, the time taken to send a qubit from A to B is \( n \Delta t \), and the average energy employed is \( 2nE \), giving an average power of \( 2P = 2E \Delta t \). The rate at which information is sent down the channel is still 1 bit in time \( \Delta t \). Accordingly, the transmission of information from A to B by repeated swapping down a chain of qubits comes within \( \sqrt{2} \) of the the power/capacity limit of (1). In fact, a variety of Hamiltonians (e.g., \( H = S_{A_1 B_1} + S_{B_1 A_2} + \ldots S_{A_n B_n} \)) can be used to propagate ‘spin waves’ down the qubit chain at rates on the order of the power/capacity limit (1). The similarity of the power/capacity tradeoff for chains of qubits and for bosons should not be surprising as the physics of spin wave propagation is closely related to the physics of wave propagation in the multi-mode bosonic channel.
Now investigate the case of multiple quantum channels. It is here that entanglement leads to a significant enhancement in power-limited transmission rate. Clearly, $M$ parallel quantum channels can transmit information at a rate $\sqrt{M}$ greater than a single quantum channel using the same power $P$ merely by dividing the power equally amongst the channels (1). Each channel now transmits at a rate $(2/\sqrt{\pi})\frac{\sqrt{P/M}}{\sqrt{\pi}}$ giving an overall rate of transmission $C_M^C = \sqrt{MP/\pi} = \sqrt{MC_1^C}$. This rate enhancement for parallel unentangled channels holds for both the bosonic channel and for the qubit channel. Because of the square root dependence of transmission rate on power, both the qubit and broadband bosonic channel are more efficient at a lower power. As a result, one improves performance by dividing up information and power among the different channels.

If one is able to apply operations that entangle the channels, one can do even better, as I now show. The goal of the $M$-channel transfer is to enact the $2M$-qubit analog of the swap above:

$$S_{1...M} = \sum_{i_1j_1...i_Mj_M=0}^1 |i_1j_1\rangle_{AB_1}\langle j_1i_1| \otimes \ldots \otimes |i_Mj_M\rangle_{AB_M}\langle j_Mi_M|. \quad (3)$$

The $2M$ qubit swap $S_{1...M}$ swaps A’s $M$ qubits with B’s $M$ qubits and has the same properties as the 2-qubit swap above (Hermitian, squares to one, etc.). As above, define the Hamiltonian $\tilde{S}_{1...M} = \pi \hbar (1 - S_{1...M})/2\Delta t$. Applying the Hamiltonian $\tilde{S}_{1...M}$ for a time $\Delta t$ then swaps A’s qubit string with B’s qubits. The average energy during the $M$-qubit swap is $E = \pi \hbar (1 - |\langle 0|\psi\rangle|^2)/2\Delta t$, as in equation (2) above. Now, however, $|\langle 0|\psi\rangle|^2 = 1/2^M$ for a randomly selected $|\psi\rangle$. Accordingly, the time taken to transfer A’s bit to B using power $P$ is given by

$$1/\Delta t = \sqrt{2(1 - 2^{-M})P/\pi \hbar}, \quad (4)$$

which is the $M$-channel analog of the limit (1). The time taken to perform the transfer using power $P$ attains the limit (4), but now for the transfer of $M$ bits rather than a single bit.

Application of the Hamiltonian $S_{1...M}$ transfers $M$ bits down $M$ parallel qubit channels using essentially the same energy $E \approx \hbar/\Delta t$, the same power $P \approx \hbar/\Delta t^2$, and in the same time $\Delta t$ it takes a single channel to transmit a single bit. Similar results hold for the transmission of $M$ qubits down $M$ chains of $n$ qubits, as above: the transmission time in this case is $n$ times as long, but the power is the same as the single bit case, while the number of bits per second is $M$ times the single qubit channel rate.

It is easy to verify that during the transfer, the $M$ qubit channels are mutually entangled. For example, if A’s input state is $|b_M\rangle = |b_1...b_M\rangle$, then at time $\Delta t/2$ (halfway
through the controlled flipping operation) A and B’s qubits are in the state
\[ (e^{-i\pi/4}/\sqrt{2})(|b_1 \ldots b_M\rangle_A|00 \ldots 0\rangle_B + i|00 \ldots 0\rangle_A|b_1 \ldots b_M\rangle_B). \] (5)
The fact that the channels are entangled during the course of transmission has implications for the sensitivity of the entangled channel to noise. On the one hand, entangled states of the form (5) are typically \(\sqrt{M}\) times more sensitive to decoherence than unentangled states. On the other hand, this sensitivity to decoherence can actually be an advantage in sending information. Suppose that A never sends the state 00\ldots0. If the multiple channel is decohered in the course of transmission, then B either receives 00\ldots0 or the correct message \(b_1 \ldots b_M\). If he receives 00\ldots0, he just waits and measures again until he receives \(b_1 \ldots b_M \neq 00 \ldots 0\). That is, decoherence on its own gives no errors for the transmission of classical information down the entangled channels. In fact, as noted in (31), decoherence can actually enhance the rate of transmission for a fixed power.

Transferring \(M\) bits down \(M\) unentangled quantum channels using the same power as a single qubit channel takes \(\sqrt{M}\) times longer than transferring the information down coupled, entangled channels. Unentangled transfer corresponds to the application of \(M\) two-qubit swap operations with Hamiltonian \(\tilde{S}_1 + \ldots + \tilde{S}_M\) as opposed to the \(2M\)-qubit swap Hamiltonian \(\tilde{S}_1 \ldots \tilde{S}_M\), and takes \(\sqrt{M}\) times the energy of the entangled swap. As a result, the coupled, entangled channels have a capacity of at least \(\sqrt{M}\) times the capacity of the uncoupled, unentangled channels.

Perhaps the most remarkable aspect of this result is that the use of entanglement allows the transfer of \(M\) bits in the same time and using the same power that it takes to transfer a single bit. Does entanglement truly allow one to get ‘something for nothing’ as this result suggests? What is the catch? After all, previous investigations of the use of entanglement in uncoupled quantum channels have found at best modest increases in channel capacity (23-27) (with the possible exception of noisy quantum channels (28)).

In fact, entanglement does indeed allow the capacity increase derived above. To find the absolute upper bound on the capacity of coupled quantum channels will require the detailed application of Kholevo’s theorem (23, 1). However, the Margolus-Levitin theorem (30) is very general, and suggests that the absolute limit on coupled channel capacity differs from (3) by at most a constant of order unity. In a certain sense, that it is just as easy in terms of power and energy to rotate \(2M\) bits from one state to another as it is to rotate 2 bits from one state to another should not be surprising: no two states in Hilbert space are more than angle of \(\pi\) apart. Accordingly, if one can effect arbitrary evolutions on the \(M\)-qubit channel Hilbert space, \(M\) bits can be transferred using the same power and time as one bit. The situation is summarized in figure 2. Effectively, the coupling
between the channels allows them to transmit information in the form a ‘super-boson’ with $2^M$ internal states. The $\sqrt{M}$ enhancement afforded by exploiting entanglement is typical of quantum information processing and arises from essentially the same source as the $\sqrt{M}$ enhancements in quantum search (32) and quantum positioning (33).

The catch is that enacting the necessary Hamiltonian $\hat{S}_{1...M}$ is likely to prove experimentally difficult. Even for two qubit channels, enacting the Hamiltonian of equation (2) involves entangling four quantum bits, a difficult action using current technologies. One might hope to be able to build up this Hamiltonian time evolution using elementary quantum logic operations on two quantum bits at a time, but in this case most of the power advantage is lost, as the net angle rotated in Hilbert space becomes larger than $\pi$. To attain the $\sqrt{M}$ enhancement of channel capacity allowed by entanglement, an $M$-qubit entangling operation must be used. Such operations correspond to interaction operators of the form $\sigma^1_x \sigma^2_x \ldots \sigma^M_x$ for spin qubits and $a_{A1} a_{B1}^\dagger \ldots a_{AM} a_{BM}^\dagger + H.C.$ for particle modes. That is, $M'$th order nonlinear interactions are required to attain entanglement-enhanced channel capacity. Such interactions are hard to enact experimentally, although it is possible to use simple quantum logic and quantum communication devices to perform proof-of-principle demonstrations of entanglement-enhanced capacity for small $M$ (34-36). For example, suppose that A wants to use microwaves or light to load two bits onto the nuclear spins or hyperfine levels of B’s two atoms. The results derived above show that if A is able to manipulate entangling interactions between the two spins or atoms, the two bits can be loaded using $\sqrt{2}$ less power than in the case that the spins or atoms remain unentangled. Such a proof-of-principle experiment could be performed using nuclear resonance on two spins in a molecule, or using optical resonance on two interacting atoms in a trap. For larger $M$, enacting the proper entangling coupling is likely to prove experimentally difficult, but if such coupling can be enacted, substantial gains in quantum channel capacity can be obtained. Whether or not the potential gains afforded by entanglement can be realized in experimentally feasible quantum optical systems acting over significant distances remains an open question.

This paper has derived power/speed limits for the problem of transferring $M$ qubits reliably from A to B. By use of entanglement, the $M$ qubits can be transferred $\sqrt{M}$ times more rapidly for the same power as $M$ unentangled qubits. The absolute capacity of $M$ coupled, entangled qubit channels will have to be derived by the sophisticated application of Holevo’s theorem (27). Open questions include the fully relativistic treatment of entangling channels, the possibility of significant but lesser communication enhancements by the use of partial entanglement, and the effect of noise on entangling channels. But as this paper shows, entanglement in principle gives a significant increase in channel capacity.
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Figure captions:

Figure 1: Summary of the power/capacity tradeoff for entangled and unentangled channels.

1a) A single channel can transmit information at a rate $\approx \sqrt{P/\hbar}$, where $P$ is the power.

1b) Because the channels are more efficient at low power, $M$ unentangled channels can send information at a rate $\approx M \sqrt{P/M} \sqrt{\hbar} = \sqrt{M} \sqrt{P/\hbar}$.

1c) If the channels are coupled to entangle them in the course of transmission, they can send information at a rate $\approx M \sqrt{P/\hbar}$.

Figure 2: Explanation of the $\sqrt{M}$ enhancement from entanglement. No two states in Hilbert space are more than an angle $\pi$ apart. As a result, to send a state $|\psi\rangle$ from A to B in time $\Delta t$ requires energy $E \approx \hbar/\Delta t$, independent of the number of bits $M$ in $|\psi\rangle$. The direct path through Hilbert space passes through states in which the $M$ channels are maximally entangled. In contrast, the transmission path corresponding to unentangled states is $\sqrt{M}$ times longer.
a) \[ A \longrightarrow B \]

b) \[ A_1 \longrightarrow B_1 \]
\[ A_2 \longrightarrow B_2 \]
\[ \vdots \]
\[ A_M \longrightarrow B_M \]

c) \[ A_1 \overset{\text{loop}}{\longrightarrow} B_1 \]
\[ A_2 \overset{\text{loop}}{\longrightarrow} B_2 \]
\[ \vdots \]
\[ A_M \overset{\text{loop}}{\longrightarrow} B_M \]
