THE MODULI SPACE OF SMOOTH AMPLE HYPERSURFACES IN ABELIAN VARIETIES

FABRIZIO CATANESE AND YONGNAM LEE

ABSTRACT. We give a characterization of smooth ample hypersurfaces in Abelian Varieties and also describe the connected component of the moduli space containing such hypersurfaces of a given polarization type: we show that this component is irreducible and that it consists of these hypersurfaces, plus the smooth iterated univariate coverings of normal type (of the same polarization type).

The above manifolds yield also a connected component of the open set of Teichmüller space consisting of Kähler complex structures.

1. INTRODUCTION

Since the seminal work of Kodaira and Spencer [15] (see also [14]) it is clear that the main problem in the classification theory of complex manifolds is to describe compact complex manifolds and the relation of deformation equivalence among them. For instance, any deformation of a complex torus is again a complex torus (see [1], [7]).

Via Teichmüller theory [8], this problem can be seen as the description of the connected components in the space of complex structures on a given oriented differentiable manifold.

In this paper we focus on the case of (smooth ample) hypersurfaces in Abelian varieties, which yield points of the moduli space of canonically polarized manifolds introduced by Viehweg [23], [24] (or, for complex dimension 2, of the Gieseker moduli space [12] of surfaces of general type).

We know that these moduli spaces have in general a lot of distinct connected components [11], which we cannot hope at the moment to
fully understand; unless the manifolds in question have special topological properties, e.g. if they have a large fundamental group.

For instance, in this paper we show that there is an irreducible moduli space parametrizing all the Kähler complex manifolds which are deformations of ample hypersurfaces in an Abelian variety. And we ask whether these are all the possible complex structures on the differentiable manifolds underlying ample hypersurfaces in an Abelian variety.

Before we turn to a more detailed description of our present results, let us recall the definition of Inoue type varieties, an attempt to make precise the above vague notion of ‘special topological properties’.

The first author and Ingrid Bauer [4] defined, quite generally, an Inoue-Type Variety to be the quotient \( X = X'/G \) of an ample smooth hypersurface \( X' \) in a projective classifying space \( Z \) by a free action of a finite group \( G \). They showed in many special cases (see a few instances in [3], [4], [5]) that for such varieties the topology determines the moduli space (see also [9] for a general treatment of this phenomenon). But essentially, up to now, in the theory of Inoue-type varieties one could describe their moduli spaces explicitly only in the case where the morphism \( \phi_0 \) to the classifying variety had necessarily degree one onto its image.

In the case where the group \( G \) is non-trivial one main difficulty is to describe the possible actions of \( G \) on \( Z \) or on \( X' \) (as in [4] or [9]): then one reduces the investigation to the study of the \( G \)-action on the moduli space of the hypersurfaces \( X' \), i.e., to the case where the group \( G \) is trivial.

However, even the simplest case with \( G \) trivial, the one where \( Z \) is an Abelian Variety (the projective classifying space of an Abelian group), remained open.

In this paper we make use of a powerful result proven in [10], characterizing the deformations of morphisms to hypersurface embeddings. This result is particularly suitable in order to analyze when does the Albanese map deform to a hypersurface embedding: because it allows us to resort to existing deformation techniques (see [21], [13], [22]).

Thus we can study the moduli space of compact Kähler manifolds diffeomorphic to ample hypersurfaces in Abelian varieties, essentially showing that we get a connected component of the moduli space once we add to the Hypersurfaces of a given dimension \( n \), and of a given polarization type, the iterated univariate coverings of normal type.

More precisely, we have the following main results: theorem 1.1 and theorem 2.2 characterizing smooth ample hypersurfaces in Abelian varieties.

**Theorem 1.1.** Let \( n \geq 2 \) and \( \bf{d} := (d_1, d_2, \ldots, d_{n+1}) \) be a polarization type for complex Abelian Varieties.
Then, for \( n \geq 3 \), the smooth hypersurfaces of type \( \overline{d} \) in some complex Abelian variety and the smooth Iterated Univariate Coverings of Normal Type and of type \( \overline{d} \) form an irreducible connected component of the moduli space of canonically polarized manifolds, and also an irreducible connected component of the Kähler-Teichmüller space (of any such smooth hypersurface \( X \)).

For \( n = 2 \) we need also to include the minimal resolutions of such surfaces which have only Rational Double Points as singularities.

For \( d_1 = 1 \) there is only this connected component of the Kähler-Teichmüller space if
- \( n = 2 \) or if
- we restrict ourselves to compact Kähler manifolds \( Y \) with \( K_Y^n = (n + 1)!d_1d_2 \ldots , d_{n+1} \), or if
- we restrict ourselves to compact Kähler manifolds \( Y \) with \( p_g(Y) = d_1d_2 \ldots , d_{n+1} + n \).

The Kähler-Teichmüller space is the open set of Teichmüller space corresponding to the Kähler complex structures (see [8] for general facts about Teichmüller space). In special situations (see [7]) it is a connected component of Teichmüller space: the same should hold also in the present case, but our arguments are not yet complete (see the proof of the main Theorem 1.1).

The results of this paper lend themselves to generalizations and extensions to the case of more general Inoue type varieties; but, for the sake of clarity, we have confined ourselves to the single crucial case of hypersurfaces in Abelian varieties.

2. Hypersurfaces in Abelian Varieties

Let \( A \) be an Abelian variety of dimension \( n + 1 \), and let \( X \subset A \) be a smooth and ample divisor, whose Chern class is a polarization of type \((d_1, d_2, \ldots , d_n, d_{n+1})\), where \( d_i | d_{i+1}, \forall i = 1, \ldots , n \).

We assume throughout that \( n = \text{dim}(X) \geq 2 \), so that Lefschetz' theorem says that

1. \( \pi_1(X) \cong \pi_1(A) \cong \mathbb{Z}^{2n+2} =: \Gamma; \)
2. \( \Lambda^i(\Gamma) = H^i(A, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z}) \) is an isomorphism for \( i \leq n - 1 \), and is injective for \( i = n \);
3. \( H_i(X, \mathbb{Z}) \rightarrow H_i(A, \mathbb{Z}) \) is an isomorphism for \( i \leq n - 1 \), and is surjective for \( i = n \).

We consider now a projective manifold which is diffeomorphic to \( X \), actually some weaker hypotheses are sufficient:
- (a) Assume that \( Y \) is a complex projective manifold, or
- (a') Assume that \( Y \) is a cKM = compact Kähler Manifold, and that
- (b) \( Y \) is homotopically equivalent to \( X \), or
• (b1) there is an isomorphism \( \alpha_Y : \pi_1(Y) \to \Gamma \) and an isomorphism of algebras \( \psi : H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \) such that, letting \( \alpha_X : \pi_1(X) \to \Gamma \) the analogous isomorphism, then \( \psi \circ H^*(\alpha_Y) = H^*(\alpha_X) \); i.e., \( \psi \) commutes with the homomorphisms to \( H^*(\Gamma, \mathbb{Z}) \) induced by the classifying maps for \( \alpha_Y, \alpha_X \) respectively, or

• (b2) the same occurs for homology: there are isomorphisms \( \phi_i : H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z}) \) commuting with \( H_i(\alpha_Y), H_i(\alpha_X) \), or

• (b') there is an isomorphism \( \alpha_Y : \pi_1(Y) \to \Gamma = \mathbb{Z}^{2n+2} \) such that, denoting by \( a_Y \) the corresponding classifying map,

• (b'1): \( (a_Y)_*[^Y] \) is dual to a polarization of type \( d_1, \ldots, d_{n+1} \), and

• (b'2): for \( n \geq 3 \), \( H_2(a_Y, \mathbb{Z}) \) is an isomorphism.

Observe that Hypothesis (a) implies (a'), Hypothesis (b) implies (b1), (b1) implies (b2) by Poincaré duality, and (b2) implies (b'), (b'1) and (b'2).

**Proposition 2.1.** Assume Hypotheses (a'), (b'), (b'1) and (b'2) above.

Then

I) the Albanese map \( a_Y : Y \to \text{Alb}(Y) =: A' \) has image \( \Sigma \) which is an ample hypersurface, indeed \( (a_Y)_*(Y) = \text{deg}(a_Y)\Sigma \) is the dual class of a polarization of type \( (d_1, \ldots, d_{n+1}) \).

II) a) holds, i.e. \( Y \) is a projective manifold,

III) if \( n \geq 3 \), then \( a_Y \) is a finite map.

**Proof.** I) By assumption (b'1) the class of \( (a_Y)_*(Y) = \text{deg}(a_Y)\Sigma \) is the dual to a polarization of type \( (d_1, \ldots, d_{n+1}) \).

II) Since \( \Sigma \) is an ample hypersurface in \( A' \), then \( A' \) is projective: hence \( \Sigma \) is projective too, and the algebraic dimension of \( Y \) equals \( n \). By Moishezon’s theorem [13], a cKM \( Y \) with algebraic dimension equal to \( \text{dim}(Y) = n \) is projective.

III) Assume that \( n \geq 3 \) and that \( a_Y \) is not a finite map: then there is a curve \( C \) such that \( a_Y(C) \) is a point. Then \( (a_Y)_*([C]) = 0 \in H_2(A', \mathbb{Z}) = H_2(\Gamma, \mathbb{Z}) \): by Lefschetz’ theorem and (b'2) follows that the homology class \([C]\) is zero: this is impossible on a compact Kähler manifold.

□

**Remark 2.1.** When \( n = \text{dim}(Y) = 2 \) it can indeed happen that \( a_Y \) is not finite: since we may take \( \Sigma \) to be a hypersurface with Rational Double Points, and by Brieskorn-Tyurina’s theorem, the minimal resolution of singularities \( Y \) is diffeomorphic to a smooth deformation \( X \) of \( \Sigma \).

The following characterization of smooth ample hypersurfaces in Abelian varieties is a refinement of a theorem obtained with Ingrid Bauer [4]: in particular here the hypothesis that \( K_Y \) is ample is removed:
Theorem 2.2. Assume that $X$ is a smooth ample hypersurface in an Abelian variety, of dimension $n \geq 2$.

Assume that $Y$ is a compact Kähler manifold which satisfies the topological conditions $(b')$, $(b'1)$ and $(b'2)$ above.

Moreover, for $n \geq 3$, assume either:

(I) $K^n_Y = K^n_X = d_1 \ldots d_{n+1}(n+1)!$, or

(II) $p_g(Y) = p_g(X) = d_1 \ldots d_{n+1} + n$.

Whereas, for $n = 2$, assume either the topological condition $(b1)$, or (I) above.

Denote the image of $a_Y$ by $W$, and assume either that

(i) the class of $X$ is indivisible (i.e., $d_1 = 1$), or the following consequence:

(ii) the degree of the map $a_Y : Y \to W$ equals 1

Then, for $n \geq 3$, the Albanese map $a_Y$ yields an isomorphism:

$$ a_Y : Y \cong W. $$

Whereas, for $n = 2$, $a_Y$ is the minimal resolution of singularities of a canonical surface, i.e., a surface with Rational Double Points as singularities, and with ample canonical divisor.

Proof. Since the class $(a_Y)_*(\mathcal{I}[Y]) = \text{deg}(a_Y)(W)$ is indivisible, it follows from (i) that $a_Y : Y \to W$ is a birational morphism, i.e. (ii) holds.

Moreover $Y$ is of general type with $p_g \geq n + 1$, since

$$ H^0(A', \Omega^n_{A'}) \to H^0(Y, \Omega^n_Y) = H^0(Y, \mathcal{O}_Y(K_Y)) $$

induces a generically finite map to $\mathbb{P}^n$ (see for instance [20]).

If $n = 2$, $Y \to W$ factors through the minimal model $Y'$ of $Y$, and indeed through the canonical model $\Xi$ of $Y$, so that we have $\pi : Y \to \Xi$, $f : \Xi \to W$.

We have

$$ K^2_\Xi = K^2_{Y'} \geq K^2_Y, $$

( $\Xi$ has hypersurface singularities which impose no adjoint conditions: these are precisely the Rational Double Points, see [2]) equality holding iff $Y = Y'$, that is, iff $Y$ is the minimal resolution of singularities of $\Xi$.

Since $K_\Xi$ is ample, we can argue as in [4]: $K_\Xi = f^*(K_W) - A$, where the adjoint divisor $A$ is effective. Since $W$ is ample, and $K_W = W|_W$, $f^*(K_W)$ is nef and big, hence:

$$ K^2_X = K^2_W = f^*(K_W)^2 = f^*(K_W)(K_\Xi + A) \geq f^*(K_W)(K_\Xi) = K^2_\Xi + K_\Xi A \geq K^2_\Xi, $$
equality holding if and only if $A = 0$.

If (I) holds, we have

$$ K^2_\Xi = K^2_X \geq K^2_\Xi \geq K^2_Y, $$

where equality holds if and only if $Y = Y'$ and $A = 0$ : hence $\Xi = W$, as claimed.
If instead we use hypothesis (b1), it implies (since we have an isomorphism of algebras) that the positivity index $b^+$ of the intersection form is the same for $Y$ and for $X$.

Recall moreover that, for any algebraic surface $Y$, $K^2_Y$ is a topological invariant, equal to $(6b^+ - 4b_1 + 4 - b_2)$, hence (b1) implies that $K^2_Y = K^2_X$, i.e., (I) holds and we are done.

Before we pass to the case $n \geq 3$, recall that, on an Abelian variety $A$ of dimension equal to $g$:

1. If $L$ is a line bundle, the index theorem for complex tori states:
   $$H^i(A, L) = 0$$
   for $i \notin [\nu(L), g - p(L)]$, where $p(L)$ is the positivity of the Chern form of $L$, and $\nu(L)$ is the negativity.
2. If $L$ is effective, then there is a morphism $f : A \rightarrow B$, where $B$ is another Abelian variety, and an ample line bundle $\delta$ on $B$ such that $L = f^*(\delta)$.
3. In particular, if $L$ is effective, then $L$ is nef, $\nu(L) = 0$, and $p(L) = \text{dim}(B)$.
4. If $H$ is an ample divisor on $A$, and $H = D + L$, where both $D, L$ are effective, then $H^g \geq L^g$, equality holding iff $D = 0$.
5. Assume that $H = D + L$ as in (4): then
   $$H^g \geq HL^{g-1},$$
   again equality holding iff $D = 0$.

Proof for items 4), 5):

Since the Chern form of $H$ is strictly positive definite, we can simultaneously diagonalize the Chern forms for $H$ and $L$. Dividing by $H^g$, we may assume that $H$ corresponds to the identity matrix, while $L$ corresponds to a matrix $\text{diag}(\lambda_1, \ldots, \lambda_g)$.

Since $L, D$ are effective, $0 \leq \lambda_i \leq 1 \forall i = 1, \ldots, g$.
Then $L^g/H^g = \lambda_1 \ldots \lambda_g \leq 1$, equality iff $\lambda_i = 1 \forall i = 1, \ldots, g$.
Similarly $HL^{g-1}/H^g = \frac{1}{g} \sum_i \lambda_1 \ldots \lambda_i \ldots \lambda_g \leq 1$, equality iff $\lambda_i = 1 \forall i = 1, \ldots, g$.

A second more general proof is that if $H$ is an ample divisor on a $g$-dimensional smooth variety, and $H = D + L$, with $D, L$ nef and $D$ effective, then

$$H^g = HL^{g-1}(L + D) \geq H^{g-1}L \geq H^{g-2}L^2 \geq \cdots \geq HL^{g-1} \geq L^g,$$
where the first equality holds if and only if $D = 0$.

For $n \geq 3$, since $a_Y$ is finite, this map is the normalization of $W$, and it suffices to show that $W$ is normal: then $a_Y : Y \rightarrow W$ is an isomorphism and $W$ is smooth.

---

1 Valery Alexeev pointed out that one should rather use this more general argument
Let us now prove the normality of \( W \).

By the Lefschetz theorem and hypothesis (b’2) (observe that since \( H^1(Y, \mathbb{Z}) \) is free, \( H^2(Y, \mathbb{Z}) \) has no torsion) it follows that the canonical divisor \( K_Y \) is a pull-back from \( A' := \text{Alb}(Y) \). Hence we can write \( K_Y = a_Y^*(K_W) - C \), where \( C \) is the conductor divisor, and it is a pull-back from a divisor \( D \) on \( A' \).

We want to show that the conductor divisor \( C \) is zero, whence the normality of \( W \) follows.

Step 1: the divisor \( D \) is effective.

Proof. Consider the exact sequence

\[
0 \to \mathcal{O}_{A'}(D - W) \to \mathcal{O}_{A'}(D) \to \mathcal{O}_W(D) \to 0.
\]

Since the conductor is an effective divisor on \( W \), \( H^0(\mathcal{O}_W(D)) \neq 0 \).

Since \( K_Y \) is the pull back of \( W - D \), and \( K_Y \) is big, hence \( W - D \) has positivity at least \( n \), whence \( D - W \) has negativity at least \( n \) and \( H^1(\mathcal{O}_{A'}(D - W)) = 0 \), \( H^0(\mathcal{O}_W(D - W)) = 0 \).

Therefore also \( H^0(\mathcal{O}_{A'}(D)) \neq 0 \).

Let us first make assumption (II). We have, by the property of the conductor ideal,

\[
p_g(Y) = h^0(Y, \mathcal{O}_Y(K_Y)) = h^0(Y, \mathcal{O}_Y(a_Y^*(W - D))) = h^0(W, \mathcal{O}_W(W - D)).
\]

We know that \( p_g(X) = h^0(W, \mathcal{O}_W(W)) \), and we are going to show that \( p_g(X) = p_g(Y) \) if and only if \( D = 0 \). In fact, first of all we must have \( h^0(W, \mathcal{O}_W(D)) = 1 \), and therefore also \( h^0(\mathcal{O}_{A'}(D)) = 1 \).

Consider now the exact sequences

\[
0 \to \mathcal{O}_{A'} \to \mathcal{O}_{A'}(W) \to \mathcal{O}_W(W) \to 0,
\]

\[
0 \to \mathcal{O}_{A'}(-D) \to \mathcal{O}_{A'}(W - D) \to \mathcal{O}_W(W - D) \to 0.
\]

Passing to cohomology, since \( W \) is ample, \( H^1(A', \mathcal{O}_{A'}(W)) = 0 \), and

\[
h^0(W, \mathcal{O}_W(W)) = h^0(A', \mathcal{O}_{A'}(W)) + 1.
\]

We have in general

\[
h^0(W, \mathcal{O}_W(W - D)) \leq h^1(A', \mathcal{O}_{A'}(W - D)) + h^1(A', \mathcal{O}_{A'}(-D)).
\]

Now, \( D \) is effective, hence by the index theorem for complex tori,

\[
h^1(A', \mathcal{O}_{A'}(-D)) = 0
\]

unless \( -D \) has negativity one, which means that \( D \) pulls back from a divisor of degree \( d \) on an elliptic curve \( E \). In the second case, since \( h^0(\mathcal{O}_{A'}(D)) = 1 \), it follows that \( d = 1 \), hence \( h^1(A', \mathcal{O}_{A'}(-D)) = 1 \).

In both cases, since in the second case \( h^1(A', \mathcal{O}_{A'}(-D)) = 1 \), it follows that

\[
h^0(W, \mathcal{O}_W(W - D)) \leq h^0(A', \mathcal{O}_{A'}(W - D)) + 1 \leq h^0(A', \mathcal{O}_{A'}(W)) + 1
\]

and we get a contradiction to \( p_g(X) = p_g(Y) \).

Let us make now assumption (I), namely, that \( K^p = K^p_W (:= K^p_W) \).
Step 2: the divisor $L := W - D$ is effective or $D^2 = 0$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_{A'}(-D) \to \mathcal{O}_{A'}(W - D) \to \mathcal{O}_W(W - D) \to 0,$$

where $h^0(\mathcal{O}_W(W - D)) \geq n + 1$ since, as we have already observed at the beginning of the proof, $h^0(\mathcal{O}_W(W - D)) = p_g(Y)$, and, by assumption (II), $p_g(Y) = p_g(X) \geq n + 1$.

It follows that $W - D$ is effective on $A'$ provided $h^1(A', \mathcal{O}_{A'}(-D)) = 0$.

In the contrary case, $D$ is again a pull-back from an elliptic curve hence $D^2 = 0$.

If $W - D$ is effective we apply then item (5):

$$K^n_X = K^n_W = W^{n+1} \geq W(W - D)^n = K^n_Y.$$

Since equality holds by assumption I, it follows then that $D = 0$, i.e., $W$ is normal and isomorphic to $Y$.

Otherwise, $D^2 = 0$ and $W(W - D)^n = W^{n+1} - nW^nD \leq W^{n+1}$, with equality holding if and only if $D = 0$.

\[\square\]

Remark 2.3. If the dimension $n \geq 3$, then the geometric genus of a compact complex manifold is not a differentiable invariant, as shows the case of Jacobian Blanchard-Calabi manifolds, cf. remark 7.3 of [7].

Similarly, Le Brun [17] gave examples of complex threefolds for which $K^3_X$ is not a differentiable invariant.

Moreover, Kotschick has proven in [16] that, for smooth projective manifolds $X$ of dimension $n \geq 3$, $K^n_X$ is not a differentiable invariant.

Remark 2.4. An essential step in the above proof is to show the normality of $W$: for a hypersurface in a smooth manifold, normality is equivalent to the condition that the singular locus of $W$ has codimension at least 2.

In turn, this is equivalent to asking that a general curve $C$ obtained as the intersection of $W$ with $n - 1$ very ample hypersurfaces $H_1, \ldots, H_{n-1}$ is smooth. One could for instance take these Hypersurfaces to lie in the class of $3W$.

Since the system $|\omega_Y^*(3W)|$ is base point free on $Y$, by Bertini’s theorem the inverse image of $C$ in $Y$ is a smooth curve $D$. Since $C$ is reduced, and $f : D \to C$ is birational, the cokernel of the pull back map $\mathcal{O}_C \to f_*\mathcal{O}_D$ is a skyscraper sheaf $\mathcal{F}$: if $D$ and $C$ were to be shown to have the same arithmetic genus $p(C) = p(D)$, then $\chi(\mathcal{F}) = 0$, hence $\mathcal{F} = 0$ and we could conclude that $D \cong C$, hence $C$ is smooth.

$p(C) = p(D)$ follows if one knows that the class of the canonical divisor of $Y$ corresponds to the class of the canonical divisor of $X$: since
then, by adjunction and by hypothesis \((b'')\), the genus of \(D\) equals the one of the analogous intersection curve \(C' \subset X\), which in turn equals the arithmetic genus of \(C\).

A similar argument was given in lemma 1.5 of [19]: but, as we have shown, this argument in our case requires a stronger condition than \(K^n_Y = K^n_X\), namely that \(\psi([K_Y]) = [K_X]\) (see hypothesis \((b1)\), where the isomorphism \(\psi\) was introduced).

### 3. Iterated univariate coverings of normal type = ITUNCONTR

For the reader’s benefit, we recall the following definitions, introduced in [10].

i) Given a complex space (or a scheme) \(X\), a **univariate covering** of \(X\) is a hypersurface \(Y\), contained in a line bundle over \(X\), and defined there as the zero set of a monic polynomial.

In more abstract wording, \(Y = \text{Spec}(\mathcal{R})\), where \(\mathcal{R}\) is the quotient algebra of the symmetric algebra over an invertible sheaf \(L\),

\[
\text{Sym}(\mathcal{L}) = \oplus_{i \geq 0} \mathcal{L}^\otimes i,
\]

by the principal ideal generated by a monic (univariate) polynomial \(P\).

Indeed, \(\text{Sym}(\mathcal{L})\) is a sheaf of univariate (one variable) polynomials in \(w\), where \(w\) is the tautological linear form on \(\text{Spec}(\text{Sym}(\mathcal{L}))\), a line bundle over \(X\). In more concrete terms,

\[
\mathcal{R} := \text{Sym}(\mathcal{L})/(P), P = w^m + a_1(x)w^{m-1} + a_2(x)w^{m-2} + \cdots + a_m(x).
\]

Here \(a_j \in H^0(X, \mathcal{L}^\otimes j)\).

Without loss of generality, over \(\mathbb{C}\) one may assume the covering to be in Tschirnhausen form, that is, with \(a_1 \equiv 0\).

The univariate covering is said to be **smooth** if both \(X\) and \(Y\) are smooth.

ii) An **iterated univariate covering** \(W \rightarrow X\) is a composition of univariate coverings

\[
f_{k+1} : X_{k+1} := W \rightarrow X_k, f_k : X_k \rightarrow X_{k-1}, \ldots, f_1 : X_1 \rightarrow X =: X_0,
\]

whose associated line bundles are denoted \(\mathcal{L}_k, \mathcal{L}_{k-1}, \ldots, \mathcal{L}_1, \mathcal{L}_0\).

It is said to be smooth if all \(f_j\) are smooth.

iii) In the case where \(X \subset Z\) is a (smooth) hypersurface, we say that the iterated univariate covering is of **normal type** if

- all the line bundles \(\mathcal{L}_j\) are pull back from \(X\) of a line bundle of the form \(\mathcal{O}_X(m_j X)\) and moreover
- \(1 = m_0 | m_1 | m_2 | \ldots | m_k | m_{k+1} := m\), and the degree of \(f_j\) equals \(\frac{m_j}{m_{j-1}}\).
- we say that the iterated covering is **normally induced** if moreover all the coefficients \(a_j(x)\) of the polynomials
\[ Q_j(w_0, \ldots, w_{j-1}, x) = \sum_{I} a_I(x) w^I \]

describing the intermediate extensions are sections of a line bundle \( \mathcal{O}_X(r(I)X) \) coming from \( H^0(Z, \mathcal{O}_Z(r(I)X)) \).

**Remark 3.1.** (i) The property that the iterated univariate covering \( W \to X \) is normally induced clearly means that it is the restriction to \( X \) of an iterated univariate covering of \( Z \).

(ii) For simplicity of notation we shall use the same notation \( L_j, j = 0, \ldots, k + 1 \), to denote the pull back from \( X \) of the line bundle \( \mathcal{O}_X(m_jX) \) via any iterated intermediate covering.

(iii) As already done in the heading of this section, we shall use the acronym ITUNCONT to refer to the iterated univariate coverings of normal type.

This said, we proceed to construct iterated univariate coverings of hypersurfaces in an Abelian variety.

**Definition 3.2.** Given a polarization type \( \overline{d} = (d_1, d_2, \ldots, dn, d_{n+1}) \) (here, as before, \( d_i|d_{i+1}, \forall i = 1, \ldots, n \)) and a sequence of positive integers \( \overline{m} = (m_0 = 1, m_1m_2| \ldots |m_{k+1}), \) such that \( m_{k+1} \) divides \( d_1 \), we define the family \( F_{\overline{d}, \overline{m}} \) of iterated univariate coverings of normal type, and of numerical-bitype \( \overline{d}, \overline{m} \), of hypersurfaces in an Abelian variety, as the family of the following complete intersections \( W \):

- there is an Abelian variety \( A \) with a polarization \( L_0 \), of type \( \frac{1}{m_{k+1}} \overline{d} \), and a smooth divisor \( X \) in \( |L_0| \)
- there are line bundles \( L_0, \ldots, L_k \) defined as \( L_j := \mathcal{O}_A(m_jX) \) for \( j = 0, \ldots, k \),
- \( W \) is a complete intersection in the vector bundle associated to \( L_0 \oplus \cdots \oplus L_k \), defined by equations of the following standard form:

\[
\begin{align*}
\sigma(z) &= w_0t_0 \\
Q_1(w_0, z) &= w_1t_1 \\
&\quad \cdots \cdots \\
Q_k(w_0, \ldots, w_{k-1}, z) &= w_kt_k \\
Q_{k+1}(w_0, \ldots, w_k, z) &= 0.
\end{align*}
\]  

(3.1)

- where \( t_0, t_1, \ldots, t_k \in \mathbb{C}, \sigma \in H^0(A, L_0), div(\sigma) = X, \) and \( Q_j(w_0, \ldots, w_{j-1}, z) = \)

\[
w_{j-1}^{m_j} + a_2(w_0, \ldots, w_{j-2}, z)w_{j-1}^{m_j-2} + \cdots + a_{m_j/m_{j-1}}(w_0, \ldots, w_{j-2}, z)
\]

- so that for \( t_0 = t_1 = \cdots = t_k = 0 \) the projection of \( W_0 \) onto \( X \)
  is a normally induced iterated smooth univariate covering,
- whereas for \( t_0 \neq 0, t_1 \neq 0, \ldots, t_k \neq 0 \), we get a hypersurface.
Theorem 3.3. 1) The family $\mathcal{F}_{\vec{d},\vec{m}}$ of iterated univariate coverings of normal type, and of numerical-bitype $\vec{d},\vec{m}$, of hypersurfaces in an Abelian variety is a family with smooth base which is locally complete, i.e., such that at each point $W$ it maps to an open set of the base $Def(W)$ of the Kuranishi family of $W$.

2) The family includes all smooth hypersurfaces of polarization type $\vec{d}$.

Proof. Abelian varieties with a given polarization type are parametrized by the Siegel upper half space $\mathcal{H}_{n+1}$, and form a family $\mathcal{A}_{\vec{d},\vec{m}}$ over $\mathcal{H}_{n+1}$, endowed with a universal bundle $\mathcal{L}_0$ yielding on each fibre a polarization of type $\frac{1}{m_k+1}d$. Moreover, for an ample line bundle $\mathcal{L}$ on an Abelian variety $A$, the dimension $h^0(\mathcal{L})$ only depends on its polarization type: therefore we take as base space for $\mathcal{F}_{\vec{d},\vec{m}}$ a smooth complex manifold $\mathcal{B}_{\vec{d},\vec{m}}$, a dense open set in a vector bundle over the family $\mathcal{A}_{\vec{d},\vec{m}}$ (the latter parametrizes the line bundles of a given polarization type as points in the fibres of $\mathcal{A}_{\vec{d},\vec{m}}$ transform $\mathcal{L}_0$ via translation).

If $W$ is a variety in the family $\mathcal{F}_{\vec{d},\vec{m}}$, then by definition of the family $W$ is smooth and we consider the deformation of the map $f : W \to A$. The map $f$ is the Albanese map of $W$: for hypersurfaces $W \subset A$ this follows from the theorem of Lefschetz.

For the other ITUNCONT′s (see remark 3.1) we use that the Hodge number $h^{1,0}(W)$ and the first homology group are deformation invariants.

A deformation of the complex manifold $W_s, s \in \mathcal{S}$, of $W$ yields a morphism

$$\Phi : \mathcal{W} \to \mathcal{A}, \quad A_s = Alb(W_s)$$

that gives the family of Albanese maps: since the image of $W_s$ has dimension $n$, automatically we get a deformation $A_s$ of the Albanese variety $A$ of $W$, together with an ample divisor in $A_s$. Hence we see that we get necessarily a deformation of $A$ as a polarized Abelian variety.

Since $\mathcal{B}_{\vec{d},\vec{m}}$ maps submersively onto Siegel space, it suffices to show the completeness of the deformations with fixed target $A$.

In this case the tangent space to the deformations (with fixed target) is given by $H^0(W, N_f)$, where $N_f$ is the normal sheaf to the map $f$, defined by Horikawa [13] as the cokernel of

$$0 \to T_W \to f^*T_A = \mathcal{O}_W^{n+1} \to N_f \to 0.$$

The exact cohomology sequence yields ($0 = H^0(T_W)$ since $W$ is of general type):

$$0 \to \mathbb{C}^{n+1} \to H^0(N_f) \to H^1(T_W) \to H^1(\mathcal{O}_W)^{n+1} = H^1(T_A),$$

where the first map corresponds to the action of $A$ via translations, and the last map yields the deformation of the Abelian variety $A$. 


We want to show that it suffices to show 1) for the case where \( W \) corresponds to the choice \( t_0 = t_1 = \cdots = t_k = 0 \) of the parameters \( t_j \).

In passing, we shall prove 2).

2) holds: this is elementary, it suffices to fix a hypersurface of equation \( \Phi(z) = 0 \), and in the above equations to take all \( t_j \neq 0 \), and finally fixing the constant term in the polynomial \( a_{m_{k+1}/m_k}(w_0, \ldots, w_{k-1}, z) \) so that, replacing inductively \( w_j \) by \( \frac{1}{t_j}Q_j(w_0, \ldots, w_{k-1}, z) \) we get from \( Q_{k+1} \) the desired equation \( \Phi(z) \).

Similarly, for each \( t_j \neq 0 \), we can eliminate \( w_j \) and reduce to an iterated cover with a smaller number \( k \) of iterated coverings. Hence, it suffices to prove 1) in the case where all \( t_j = 0 \).

As already observed, since \( W \) is of general type, \( H^0(T_W) = 0 \); moreover the image of \( H^0(f^*A) = \mathbb{C}^{n+1} \) accounts for deforming the class of the line bundle \( L_0 \) in \( Pic(A) \) (via translations on \( A \)).

Therefore it suffices to look at the deformation of the map \( f \) into the vector bundle associated to \( L_0 \oplus \cdots \oplus L_k \).

Here, we shall just show that the subfamily where we vary only the equations \( Q_j \) surjects onto the normal bundle of \( W \), which is indeed the restriction of \( L_0 \oplus \cdots \oplus L_k \oplus L_{k+1} \) to \( W \) since \( W \) is a complete intersection.

For this, consider the chain of maps
\[
f_{k+1} : X_{k+1} := W \to X_k, f_k : X_k \to X_{k-1}, \ldots, f_1 : X_1 \to X,
\]
and set \( \frac{m_j}{m_{j-1}} =: h_j \).

We have \( (f_{k+1})_*(\mathcal{O}_W) = \oplus_{i=0}^{h_{k+1}-1} \mathcal{O}_{X_k}(-iL_k) \), hence
\[
H^0(W, L_j) = H^0(X_k, L_j)
\]
for \( j < k \) and
\[
H^0(W, L_k) = H^0(X_k, L_k) \oplus \mathbb{C}.
\]

Proceeding inductively, we see in a similar way that
\[
H^0(W, L_j) = H^0(X_j, L_j) \oplus \mathbb{C}.
\]

And by the same argument
\[
H^0(X_j, L_j) = \{Q_j(w_0, \ldots, w_{j-1}, x) = \sum_i a_I(x)w^I \},
\]
where \( a_I(x) \in H^0(\mathcal{O}_X(r(I)X)) \), whereas \( t_jw_j \) accounts for the summand \( \mathbb{C} \).

It suffices finally to show that these sections come from \( H^0(Z, \mathcal{O}_Z(r(I)X)) \).

The surjectivity of \( H^0(A, \mathcal{O}_A(iX)) \to H^0(X, \mathcal{O}_X(iX)) \) for \( i \geq 2 \) is implied by \( H^1(A, \mathcal{O}_A(iX)) = 0, \forall i \geq 1 \), as follows from the exact sequence
\[
0 \to \mathcal{O}_A(iX) \to \mathcal{O}_A((i+1)X) \to \mathcal{O}_X((i+1)X) \to 0.
\]
We do not need the surjectivity for $i = 1$, as we simply recall that all the sections of the normal bundle give a deformation which, after an automorphism replacing $w_j$ with $w_j - \frac{1}{h_j}a_{1,j}(x)$, can be put in Tschirnhausen form, i.e., with $a_{1,j}(x) \equiv 0$.

\[\square\]

4. DEFORMATIONS TO HYPERSURFACE EMBEDDINGS

**Definition 4.1.** A 1-parameter deformation to hypersurface embedding consists of the following data:

1. a one dimensional family of smooth projective varieties of dimension $n$ (i.e., a smooth projective holomorphic map $p : W \to T$ where $T$ is a germ of a smooth holomorphic curve at a point $0 \in T$) mapping to another family $\pi : Z \to T$ of smooth projective varieties of dimension $n + 1$ via a relative map $\Phi : W \to Z$ such that $\pi \circ \Phi = p$ and such that moreover
2. for $t \neq 0$ in $T$, $\Phi_t$ is an embedding of $W_t := p^{-1}(t)$ in $Z_t$,
3. the restriction of the map $\Phi$ on $W_0$ is a generically finite morphism of degree $m$, so that the image of $\Phi|_{W_0}$ is the cycle $\Sigma_0 := mX$ where $X$ is a reduced hypersurface in $Z_0$, defined by an equation $X = \{\sigma = 0\}$.

The following is the first part of the main theorem of [10] (without the converse part):

**Theorem 4.2.** (A) Suppose we have a 1-parameter deformation to hypersurface embedding and assume that $K_{W_0}$ is ample.

Then we have:

(A1) $X$ is smooth,

(A2) There are line bundles $\mathcal{L}_0, \ldots, \mathcal{L}_k$ on $Z$, such that $\mathcal{L}_j|_{Z_0} = \mathcal{O}_{Z_0}(m_jX)$ for $j = 0, \ldots, k$, with $1 = m_0|m_1|m_2|\cdots|m_k|m_{k+1} := m$ (here $m$ is the degree of the morphism $\Phi_0 : W_0 \to X$), and such that $W_0$ is a complete intersection in the vector bundle associated to $\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_k|_{Z_0}$, with $\Phi_0$ a normally induced iterated smooth univariate covering.

(A3) $W$ is obtained from $\Sigma := \Phi(W)$ by a finite sequence of blow-ups.

Moreover the local equations of $W$ are of the following standard form

\[
\begin{align*}
\sigma(z) &= w_0t \\
Q_1(w_0, z) &= w_1t \\
& \quad \ldots \quad \\
Q_k(w_0, \ldots, w_{k-1}, z) &= w_kt \\
Q_{k+1}(w_0, \ldots, w_k, z, t) &= 0.
\end{align*}
\]

The following intermediate result played an important role in the proof.
Lemma 4.3. Suppose we have a one dimensional smooth family $p : \mathcal{W} \to T$ of smooth projective varieties of dimension $n$ mapping to another flat family $q : \mathcal{Y} \to T$ of projective varieties of the same dimension via a relative map $\Psi : \mathcal{W} \to \mathcal{Y}$ over a smooth holomorphic curve $T$ such that $q \circ \Psi = p$.

Assume that

1. $\mathcal{Y}$ is normal and Gorenstein,
2. $\Psi$ is birational,
3. for $t \neq 0$ in $T$, $\Psi$ induces an isomorphism,
4. $K_{\mathcal{W}_0}$ is ample.

Then we have that $\Psi$ is an isomorphism, in particular $\mathcal{W}_0 \cong \mathcal{Y}_0$.

Proof. We have $K_{\mathcal{W}} = \Psi^*(K_{\mathcal{Y}}) + \mathcal{B}$. Since we assume that $\Psi$ induces an isomorphism for $t \neq 0$ in $T$, the support of the Cartier divisor $\mathcal{B}$ is contained in $W_0$, which is irreducible.

Now $Y_0$ has dimension $n$ and the morphism $\Psi_0 : W_0 \to Y_0$ is generically finite, hence we conclude that $\mathcal{B} = 0$. In particular, $K_{\mathcal{W}_0} = \Psi^*(K_{\mathcal{Y}_0})$; restricting to the special fibre, we obtain $K_{\mathcal{W}_0} = \Psi_0^*(K_{\mathcal{Y}_0})$.

Since by assumption $K_{\mathcal{W}_0}$ is ample, we obtain that $\Psi_0$ is finite, hence also $\Psi$ is finite, whence an isomorphism in view of the normality of $\mathcal{Y}$. \hfill \Box

Remark 4.1. Without the assumption that $K_{\mathcal{W}_0}$ is ample one can only assert that $Y_0$ is normal with at most canonical singularities.

Proposition 4.2. Suppose we have a 1-parameter deformation to hypersurface embedding, where $W_t$ is a hypersurface in an Abelian variety, and $W_0$ is Kähler.

If $n \geq 3$, then necessarily $K_{W_0}$ is ample.

Proof. By proposition 2.1 it follows first of all that $W_0$ is projective.

Rerunning the proof of theorem 1.2 we construct, by a finite sequence of blow-ups starting from $\Sigma := \Phi(\mathcal{W})$, a family $q : \mathcal{Y} \to T$ as in lemma 4.3 and with relatively ample canonical divisor, such that the local equations of $\mathcal{Y}$ are of the following standard form

\[
\left\{
\begin{array}{l}
\sigma(z) = w_0 t \\
Q_1(w_0, z) = w_1 t \\
\ldots \\
Q_k(w_0, \ldots, w_{k-1}, z) = w_k t \\
Q_{k+1}(w_0, \ldots, w_k, z, t) = 0.
\end{array}
\right.
\]

(4.2)

Since $n \geq 3$, again proposition 2.1 guarantees that $\Psi_0$ is finite. Hence, as in the proof of lemma 4.3, $K_{W_0} = \Psi_0^*(K_{Y_0})$ is ample. \hfill \Box
5. Proof of the main theorem 1.1

For the first assertion, theorem 3.3 ensures that ITUNCONT’s yield an open set in Teichmüller space; this open set is irreducible because the dense open set corresponding to smooth hypersurfaces in Abelian varieties is irreducible.

We would like now to show that this set is closed.

We observe that every variety $X_0$ in the closure will be the limit of a 1-parameter deformation.

First of all, we can use some results of [7], namely lemma 2.4 and corollary 2.5, asserting that $X_0$ will have a very good Albanese variety, and that the image of the Albanese map will be a hypersurface. Hence the algebraic dimension of $X_0$ is equal to its dimension.

At this point we use the assumption that $X_0$ is Kähler for $n \geq 3$: for $n = 2$ it follows automatically since $X_0$ is then of general type.

In dimension $n \geq 3$, as observed in proposition 4.2, the canonical bundle will be ample since the Albanese map is finite.

We can then apply Theorem 4.2 to conclude.

For $n = 2$ we work directly with the canonical models of our surfaces, and everything works verbatim, by remark 4.1.

The case $d_1 = 1$ is just a restatement of theorem 2.2: observe just that homeomorphism implies that the Betti numbers and $b^+$ are the same, hence also $p_g, K^2$.

6. Final question

It is natural to ask whether our irreducible connected component is the unique one. To prove this, it would of course be sufficient to show that the Albanese map can be deformed to an embedding.

A second question is whether, for $n \geq 3$, we really get a connected component of Teichmüller space: for this it would be sufficient to show that the Albanese map for a limit variety $X_0$ is finite (equivalently, that $X_0$ is projective).

Acknowledgements: the first author would like to thank Stefan Schreieder for an interesting conversation and especially Ingrid Bauer for spotting a mistake in the first version of theorem 2.2 and for several very useful discussions. Thanks to the referee for careful reading of the manuscript, which helped remove a few inaccuracies.

References

[1] [AS-60] A. Andreotti and W. Stoll, Extension of holomorphic maps. Ann. of Math. (2) 72 1960 312–349.
[2] [Ar66] M. Artin, On isolated rational singularities of surfaces. Amer. J. Math. 88 (1966) 129–136.
[3] [BC11] I. Bauer and F. Catanese, Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces. Classification of algebraic varieties, 49–76, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, (2011).
[4] [BC12] I. Bauer and F. Catanese, Inoue type manifolds and Inoue surfaces: a connected component of the moduli space of surfaces with $K^2 = 7, p_g = 0$. Geometry and arithmetic, 23–56, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, (2012).

[5] [BCF15] I. Bauer, F. Catanese, D. Frapporti, Generalized Burniat type surfaces and Bagnera-de Franchis varieties. J. Math. Sci. Univ. Tokyo 22 (2015), no. 1, 55–111.

[6] [Cat88] F. Catanese, Moduli of algebraic surfaces, ‘Theory of moduli’, Lecture Notes in Math., 1337, Springer, Berlin, 1–83 (1988).

[7] [Cat04] F. Catanese, Deformation in the large of some complex manifolds. I. Ann. Mat. Pura Appl. (4) 183, No. 3, 261–289 (2004).

[8] [Cat13] F. Catanese, A superficial working guide to deformations and moduli. Handbook of moduli. Vol. I, 161–215, Adv. Lect. Math. (ALM), 24, Int. Press, Somerville, MA, (2013).

[9] [Cat15] F. Catanese, Topological methods in moduli theory. Bull. Math. Sci. 5, no. 3, 287–449. (2015).

[10] [CL18] F. Catanese, Y. Lee, Deformation of a generically finite map to a hypersurface embedding. J. Math. Pures Appl. (2018), (1-14) https://doi.org/10.1016/j.matpur.2018.06.024.

[11] [CW07] F. Catanese, B. Wajnryb, Diffeomorphism of simply connected algebraic surfaces. J. Differential Geom. 76 (2007), no. 2, 177–213.

[12] [Gies77] D. Gieseker, Global moduli for surfaces of general type. Invent. Math. 43 (1977), no. 3, 233–282.

[13] [Hor73] E. Horikawa, On deformations of holomorphic maps. I. J. Math. Soc. Japan 25 (1973), 372–396.

[14] [Hor75] E. Horikawa, On deformations of quintic surfaces. Invent. Math. 31, 43-85, (1975).

[15] [KS58] K. Kodaira and D. C. Spencer On deformations of complex analytic structures, I- II. Ann. of Math. (2) 67, 328–466, (1958).

[16] [Kot12] D. Kotschick Topologically invariant Chern numbers of projective varieties. Adv. Math. 229 (2012), no. 2, 1300–1312.

[17] [LB99] D. LeBrun Topology versus Chern numbers for complex 3-folds. Pacific J. Math. 191 (1999), no. 1, 123–131.

[18] [Mois66] B. Moishezon On n-dimensional compact complex manifolds having $n$ algebraically independent meromorphic functions, I. Izv. Akad. Nauk SSSR Ser. Mat. 30 , 133–174, (1966).

[19] [Ot-Sch18] J.C. Ottem and S. Schreieder On deformations of quintic and septic hypersurfaces arXiv: 1810. 12711.

[20] [Ran84] Z. Ran, The structure of Gauss-like maps. Compositio Math. 52, no. 2, 171–177, (1984).

[21] [Ser75] E. Sernesi, Small deformations of global complete intersections. Boll. Un. Mat. Ital. (4) 12 , no. 1–2, 138–146, (1975).

[22] [Ser06] E. Sernesi, Deformations of algebraic schemes. Grundlehren der Mathematischen Wissenschaften, 334. Springer-Verlag, Berlin, 2006. xii+339 pp.

[23] [Vieh89-90] E. Viehweg, Weak positivity and the stability of certain Hilbert points., I , II Invent. Math. 96 (1989), no. 3, 639–667, Invent. Math. 101 (1990), no. 1, 191–223.

[24] [Vieh95] E. Viehweg, Quasi-projective moduli for polarized manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 30. Springer-Verlag, Berlin, (1995). viii+320 pp.
Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth, (and Korea Institute for Advanced Study, Hoegiro 87, Seoul, 133-722, Korea)

E-mail address: Fabrizio.Catanese@uni-bayreuth.de

Department of Mathematical Sciences, KAIST, 291, Daehak-ro, Yuseong-gu, Daejeon, 34141, Korea

E-mail address: ynlee@kaist.ac.kr