Beyond Hypertree Width: Decomposition Methods Without Decompositions

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Abstract. The general intractability of the constraint satisfaction problem has motivated the study of restrictions on this problem that permit polynomial-time solvability. One major line of work has focused on structural restrictions, which arise from restricting the interaction among constraint scopes. In this paper, we engage in a mathematical investigation of generalized hypertree width, a structural measure that has up to recently eluded study. We obtain a number of computational results, including a simple proof of the tractability of CSP instances having bounded generalized hypertree width.

1 Introduction

The constraint satisfaction problem (CSP) is widely acknowledged as a convenient framework for modelling search problems. Instances of the CSP arise in a variety of domains, including artificial intelligence, database theory, algebra, propositional logic, and graph theory. An instance of the CSP consists of a set of constraints on a set of variables; the question is to determine if there is an assignment to the variables satisfying all of the constraints. Alternatively, the CSP can be cast as the fundamental algebraic problem of deciding, given two relational structures A and B, whether or not there is a homomorphism from A to B. In this formalization, each relation of A contains the tuples of variables that are constrained together, which are often called the constraint scopes, and the corresponding relation of B contains the allowable tuples of values that the variable tuples may take.

It is well-known that the CSP, in its general formulation, is NP-complete; this general intractability has motivated a large and rich body of research aimed at identifying and understanding restricted cases of the CSP that are polynomial-time tractable. The restrictions that have been studied can, by and large, be placed into one of two categories, which—due to the homomorphism formulation of the CSP—have become known as left-hand side restrictions and right-hand side restrictions. From a high level view, left-hand side restrictions, also known as structural restrictions, arise from prespecifying a class of relational structures A from which the left-hand side structure A must come, while right-hand side
restrictions arise from prespecifying a class of relational structures \( B \) from which the right-hand side structure \( B \) must come. As this paper is concerned principally with structural restrictions, we will not say more about right-hand side restrictions than that their systematic study has origins in a classic theorem of Schaefer [21], and that recent years have seen some extremely exciting results on them (for instance [4,5]).

The structural restrictions studied in the literature can all be phrased as restrictions on the hypergraph \( H(A) \) naturally arising from the left-hand side relational structure \( A \), namely, the hypergraph \( H(A) \) with an edge \( \{a_1, \ldots, a_k\} \) for each tuple \( (a_1, \ldots, a_k) \) of \( A \). Let us briefly review some of the relevant results that have been obtained on structural tractability. The tractability of left-hand side relational structures having bounded treewidth was shown in the constraint satisfaction literature by Dechter and Pearl [9] and Freuder [10]. Later, Dalmau et al. [8] building on ideas of Kolaitis and Vardi [19,20] gave a consistency-style algorithm for deciding the bounded treewidth CSP. For our present purposes, it is worth highlighting that although the notion of bounded treewidth is defined in terms of tree decompositions, which can be computed efficiently (under bounded treewidth), the algorithm given by Dalmau et al. [8] does not compute any form of tree decomposition. Dalmau et al. also identified a natural expansion of structures having bounded treewidth that is tractable—namely, the structures homomorphically equivalent to those having bounded treewidth. The optimality of this latter result, in the case of bounded arity, was demonstrated by Grohe [15], who proved—roughly speaking—that if the tuples of \( A \) are of bounded arity and \( A \) gives rise to a tractable case of the CSP, then it must fall into the natural expansion identified by Dalmau et al. [8].

A number of papers, including [17,16,13,14,11,7], have studied restrictions that can be applied to relational structures of unbounded arity. (Note that any class of relational structures of unbounded arity cannot have bounded treewidth.) In a survey [13], Gottlob et al. show that the restriction of bounded hypertree width [11] is the most powerful structural restriction for the CSP in that every other structural restriction studied in the literature is subsumed by it. Since this work [11,13], whether or not there is a more general structural restriction than bounded hypertree width that ensures tractability, has been a tantalizing open question.

In this paper, we study generalized hypertree width, a structural measure for hypergraphs defined in [12] that is a natural variation of hypertree width; we call this measure coverwidth. Coverwidth is trivially upper-bounded by hypertree width, and so any class of hypergraphs having bounded hypertree width has bounded coverwidth. We define a combinatorial pebble game that can be played on any CSP instance, and demonstrate that this game is intimately linked to coverwidth (Theorem 13). Our study of coverwidth is conceptually simple, mathematically elegant, and relatively compact; we believe that this hints that

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1 One way to define what we mean by treewidth here is the treewidth of the graph obtained from \( H(A) \) by drawing an edge between any two vertices that are in the same hyperedge.
coverwidth is in fact a natural and robust mathematical concept that may find further applications. Overall, the investigation we perform takes significant inspiration from methods, concepts, and ideas developed by Kolaitis, Vardi, and coauthors [19,20,8,2] that link together CSP consistency algorithms, the existential k-pebble games of Kolaitis and Vardi [18], and bounded treewidth.

Using the pebble game perspective, we are able to derive a number of computational results. One is that the structural restriction of \textit{bounded coverwidth} implies polynomial-time tractability; this result generalizes the tractability of bounded hypertree width. It has been independently shown by Adler et al. that the hypertree width of a hypergraph is linearly related to the coverwidth [1]. This result can be used in conjunction with the tractability of bounded hypertree width to derive the tractability of bounded coverwidth. However, we believe our proof of bounded coverwidth tractability to be \textit{considerably simpler} than the known proof of bounded hypertree width tractability [11], even though our proof is of a more general result.

To describe our results in greater detail, it will be useful to identify two computational problems that every form of structural restriction gives rise to: a \textit{promise} problem, and a \textit{no-promise} problem. In both problems, the goal is to identify all CSP instances obeying the structural restriction as either satisfiable or unsatisfiable. In the promise problem, the input is a CSP instance that is \textit{guaranteed} to obey the structural restriction, whereas in the no-promise problem, the input is an \textit{arbitrary} CSP instance, and an algorithm may, on an instance not obeying the structural restriction, decline to identify the instance as satisfiable or unsatisfiable. Of course, CSPs arising in practice do \textit{not} come with guarantees that they obey structural restrictions, and hence an algorithm solving the no-promise problem is clearly the more desirable. Notice that, for any structural restriction having a polynomial-time solvable promise problem, if it is possible to solve the \textit{identification problem} of deciding whether or not an instance obeys the restriction, in polynomial time, then the no-promise problem is also polynomial-time solvable. For bounded hypertree width, both the identification problem and the no-promise problem are polynomial-time solvable. In fact, the survey by Gottlob et al. [13] only considers structural restrictions for which the identification problem is polynomial-time solvable, and thus only considers structural restrictions for which the no-promise problem is polynomial-time solvable.

One of our main theorems (Theorem 20) is that the promise problem for bounded coverwidth is polynomial-time tractable, via a general consistency-like algorithm. In particular, we show that, on an instance having bounded coverwidth, our algorithm detects an inconsistency if and only if the instance is unsatisfiable. Our algorithm, like the consistency algorithm of Dalmau et al. [8] for bounded treewidth, can be applied to \textit{any} CSP instance to obtain a more constrained instance; our algorithm does \textit{not} need nor compute any form of decomposition, even though the notion of coverwidth is defined in terms of decompositions!
Intriguingly, we are then able to give a simple algorithm for the no-promise problem for bounded coverwidth (Theorem 21) that employs the consistency-like algorithm for the promise problem. The behavior of this algorithm is reminiscent of self-reducibility arguments in computational complexity theory, and on an instance of bounded coverwidth, the algorithm is guaranteed to either report a satisfying assignment or that the instance is unsatisfiable. We believe that our results offer a direct challenge to the view of structural tractability advanced in the Gottlob et al. survey [13], since we are able to give a polynomial-time algorithm for the bounded coverwidth no-promise problem without explicitly showing that there is a polynomial-time algorithm for the bounded coverwidth identification problem.

Returning to the promise problem, we then show that the tractability of bounded coverwidth structures can be generalized to yield the tractability of structures homomorphically equivalent to those having bounded coverwidth (Theorem 22). This expansion of bounded coverwidth tractability is analogous to the expansion of bounded treewidth tractability carried out in [8].

In the last section of this paper, we use the developed theory as well as ideas in [6] to define a tractable class of quantified constraint satisfaction problems based on coverwidth.

We emphasize that none of the algorithms in this paper need or compute any type of decomposition, even though all of the structural restrictions that they address are defined in terms of decompositions.

**Definitions.** In this paper, we formalize the CSP as a relational homomorphism problem. We review the relevant definitions that will be used. A relational signature is a finite set of relation symbols, each of which has an associated arity. A relational structure \( A \) (over signature \( \sigma \)) consists of a universe \( A \) and a relation \( R^A \) over \( A \) for each relation symbol \( R \) (of \( \sigma \)), such that the arity of \( R^A \) matches the arity associated to \( R \). We refer to the elements of the universe of a relational structure \( A \) as \( A \)-elements. When \( A \) is a relational structure over \( \sigma \) and \( R \) is any relation symbol of \( \sigma \), the elements of \( R^A \) are called \( A \)-tuples. Throughout this paper, we assume that all relational structures under discussion have a finite universe. We use boldface letters \( A, B, \ldots \) to denote relational structures.

A homomorphism from a relational structure \( A \) to another relational structure \( B \) is a mapping \( h \) from the universe of \( A \) to the universe of \( B \) such that for every relation symbol \( R \) and every tuple \( (a_1, \ldots, a_k) \in R^A \), it holds that \( (h(a_1), \ldots, h(a_k)) \in R^B \). (Here, \( k \) denotes the arity of \( R \).) The constraint satisfaction problem (CSP) is to decide, given an ordered pair \( A, B \) of relational structures, whether or not there is a homomorphism from the first structure, \( A \), to the second, \( B \). A homomorphism from \( A \) to \( B \) in an instance \( A, B \) of the CSP is also called a satisfying assignment, and when a satisfying assignment exists, we will say that the instance is satisfiable.
2 Coverwidth

This section defines the structural measure of hypergraph complexity that we call coverwidth. As we have mentioned, coverwidth is equal to generalized hypertree width, which was defined in [12]. We begin by defining the notion of hypergraph.

Definition 1. A hypergraph is an ordered pair \((V, E)\) consisting of a vertex set \(V\) and a hyperedge set \(E\). The elements of \(E\) are called hyperedges; each hyperedge is a subset of \(V\).

Basic to the measure of coverwidth is the notion of a tree decomposition.

Definition 2. A tree decomposition of a hypergraph \((V, E)\) is a pair \((T = (I, F), \{X_i\}_{i \in I})\) where

1. \(T = (I, F)\) is a tree, and
2. each \(X_i\) (with \(i \in I\)) is called a bag and is a subset of \(V\),

such that the following conditions hold:

1. \(V = \bigcup_{i \in I} X_i\).
2. For all hyperedges \(e \in E\), there exists \(i \in I\) with \(e \subseteq X_i\).
3. For all \(v \in V\), the vertices \(T_v = \{i \in I : v \in X_i\}\) form a connected subtree of \(T\).

Tree decompositions are generally applied to graphs, and in the context of graphs, the measure of treewidth has been heavily studied. The treewidth of a graph \(G\) is the minimum of the quantity \(\max_{i \in I} |X_i| - 1\) over all tree decompositions of \(G\). In other words, a tree decomposition is measured based on its largest bag, and the treewidth is then defined based on the “lowest cost” tree decomposition.

The measure of coverwidth is also based on the notion of tree decomposition. In coverwidth, a tree decomposition is also measured based on its “largest” bag; however, the measure applied to a bag is the number of hyperedges needed to cover it, called here the weight.

Definition 3. A \(k\)-union over a hypergraph \(H\) (with \(k \geq 0\)) is the union \(e_1 \cup \ldots \cup e_k\) of \(k\) edges \(e_1, \ldots, e_k\) of \(H\).

The empty set is considered to be the unique 0-union over a hypergraph.

Definition 4. Let \(H = (V, E)\) be a hypergraph. The weight of a subset \(X \subseteq V\) is the smallest integer \(k \geq 0\) such that \(X \cap (\bigcup_{e \in E} e)\) is contained in a \(k\)-union over \(H\).

We measure a tree decomposition according to its heaviest bag, and define the coverwidth of a hypergraph according to the lightest-weight tree decomposition.
Definition 5. The weight of a tree decomposition of $H$ is the maximum weight over all of its bags.

Definition 6. The coverwidth of a hypergraph $H$ is the minimum weight over all tree decompositions of $H$.

It is straightforward to verify that the coverwidth of a hypergraph is equal to the generalized hypertree width of a hypergraph [12]. Since the generalized hypertree width of a hypergraph is always less than or equal to its hypertree width, coverwidth is at least as strong as hypertree width.

There is another formulation of tree decompositions that is often wieldy, see for instance [3].

Definition 7. A scheme of a hypergraph $H = (V, E)$ is a graph $(V, F)$ such that

– $(V, F)$ has a perfect elimination ordering, that is, an ordering $v_1, \ldots, v_n$ of its vertices such that for all $i < j < k$, if $(v_i, v_k)$ and $(v_j, v_k)$ are edges in $F$, then $(v_i, v_j)$ is also an edge in $F$, and
– the vertices of every hyperedge of $E$ induce a clique in $(V, F)$.

It is well known that the property of having a perfect elimination ordering is equivalent to being chordal. The following proposition is also well-known.

Proposition 8. Let $H$ be a hypergraph. For every tree decomposition of $H$, there exists a scheme such that each clique of the scheme is contained in a bag of the tree decomposition. Likewise, for every scheme of $H$, there exists a tree decomposition such that each bag of the tree decomposition is contained in a clique of the scheme.

Let us define the weight of a scheme (of a hypergraph $H$) to be the maximum weight (with respect to $H$) over all of its cliques. The following proposition is immediate from Proposition 8 and the definition of coverwidth, and can be taken as an alternative definition of coverwidth.

Proposition 9. The coverwidth of a hypergraph $H$ is equal to the minimum weight over all schemes of $H$.

We now define the hypergraph associated to a relational structure. Roughly speaking, this hypergraph is obtained by “forgetting” the ordering of the $A$-tuples.

Definition 10. Let $A$ be a relational structure. The hypergraph associated to $A$ is denoted by $H(A)$; the vertex set of $H(A)$ is the universe of $A$, and for each $A$-tuple $(a_1, \ldots, a_k)$, there is an edge $\{a_1, \ldots, a_k\}$ in $H(A)$.

We will often implicitly pass from a relational structure to its associated hypergraph, that is, we simply write $A$ in place of $H(A)$. In particular, we will speak of $k$-unions over a relational structure $A$. 
3 Pebble Games

We now define a class of pebble games for studying the measure of coverwidth. These games are essentially equivalent to the existential \( k \)-pebble games defined by Kolaitis and Vardi and used to study constraint satisfaction \cite{KolaitisVardi1992,KolaitisVardi1994}. The pebble game that we use is defined as follows. The game is played between two players, the \textit{Spoiler} and the \textit{Duplicator}, on a pair of relational structures \( A, B \) that are defined over the same signature. Game play proceeds in rounds, and in each round one of the following occurs:

1. The Spoiler places a pebble on an \( A \)-element \( a \). In this case, the Duplicator must respond by placing a corresponding pebble, denoted by \( h(a) \), on a \( B \)-element.
2. The Spoiler removes a pebble from an \( A \)-element \( a \). In this case, the corresponding pebble \( h(a) \) on \( B \) is removed.

When game play begins, there are no pebbles on any \( A \)-elements, nor on any \( B \)-elements, and so the first round is of the first type. We assume that the Spoiler never places two pebbles on the same \( A \)-element, so that \( h \) is a partial function (as opposed to a relation). The Duplicator wins the game if he can always ensure that \( h \) is a \textit{projective homomorphism} from \( A \) to \( B \); otherwise, the Spoiler wins. A \textit{projective homomorphism} (from \( A \) to \( B \)) is a partial function \( h \) from the universe of \( A \) to the universe of \( B \) such that for any relation symbol \( R \) and any tuple \( (a_1, \ldots, a_k) \in R_A \) of \( A \), there exists a tuple \( (b_1, \ldots, b_k) \in R_B \) where \( h(a_i) = b_i \) for all \( a_i \) on which \( h \) is defined.

As we mentioned, the definition of this game is based on the existential \( k \)-pebble game introduced by Kolaitis and Vardi \cite{KolaitisVardi1992,KolaitisVardi1994}. In the existential \( k \)-pebble game, the number of pebbles that the Spoiler may use is bounded by \( k \), and the Duplicator need only must ensure that \( h \) is a \textit{partial homomorphism}. A close relationship between this game and bounded treewidth has been identified \cite{Courcelle1990}.

\textbf{Theorem 11}. \cite{Courcelle1990} Let \( A \) and \( B \) be relational structures. For all \( k \geq 2 \), the following are equivalent.

\begin{itemize}
  \item There is a winning strategy for the Duplicator in the existential \( k \)-pebble game on \( A, B \).
  \item For all relational structures \( T \) of treewidth \( < k \), if there is a homomorphism from \( T \) to \( A \), then there is a homomorphism from \( T \) to \( B \).
\end{itemize}

To relate the game that we have defined with coverwidth, we are interested in parameterized versions of the game where the \textit{weight} of the pebbles that the Spoiler has in play, is bounded by a constant \( k \). (Here, by “weight” we are using Definition \ref{def:weight}). That is, the weight of the \( A \)-elements that have pebbles, is bounded by the constant \( k \). We call this the \textit{existential \( k \)-cover game}. We now formalize the notion of a \textit{winning strategy} for the Duplicator in the existential \( k \)-cover game. Note that when \( h \) is a partial function, we use \( \text{dom}(h) \) to denote the domain of \( h \).
Definition 12. A winning strategy for the Duplicator in the existential $k$-cover game on relational structures $A, B$ is a non-empty set $H$ of projective homomorphisms (from $A$ to $B$) having the following two properties.

1. (the “forth” property) For every $h \in H$ and $A$-element $a \notin \text{dom}(h)$, if $\text{dom}(h) \cup \{a\}$ has weight $\leq k$, then there exists a projective homomorphism $h' \in H$ extending $h$ with $\text{dom}(h') = \text{dom}(h) \cup \{a\}$.
2. The set $H$ is closed under subfunctions, that is, if $h \in H$ and $h$ extends $h'$, then $h' \in H$.

We have the following analog of Theorem 11.

Theorem 13. Let $A$ and $B$ be relational structures. For all $k \geq 1$, the following are equivalent.

- There is a winning strategy for the Duplicator in the $k$-cover game on $A, B$.
- For all relational structures $T$ of coverwidth $\leq k$, if there is a homomorphism from $T$ to $A$, then there is a homomorphism from $T$ to $B$.

Theorem 13 can be easily applied to show that in an instance $A, B$ of the CSP, if the left-hand side structure has coverwidth bounded by $k$, then deciding if there is a homomorphism from $A$ to $B$ is equivalent to deciding the existence of a Duplicator winning strategy in the existential $k$-cover game.

Theorem 14. Let $A$ be a relational structure having coverwidth $\leq k$, and let $B$ be an arbitrary relational structure. There is a winning strategy for the Duplicator in the $k$-cover game on $A, B$ if and only if there is a homomorphism from $A$ to $B$.

We will use this theorem in the next section to develop tractability results. Although we use Theorem 13 to derive this theorem, we would like to emphasize that the full power of Theorem 13 is not needed to derive it, as pointed out in the proof.

Proof. If there is a homomorphism from $A$ to $B$, the Duplicator can win by always setting pebbles according the homomorphism. The other direction is immediate from Theorem 13 (note that we only need the forward implication and $T = A$).

4 The Algorithmic Viewpoint

The previous section introduced the existential $k$-cover game. We showed that deciding a CSP instance of bounded coverwidth is equivalent to deciding if the Duplicator has a winning strategy in the existential $k$-cover game. In this section, we show that the latter property—the existence of a Duplicator winning strategy—can be decided algorithmically in polynomial time. To this end, it will be helpful to introduce the notion of a compact winning strategy.
Definition 15. A compact winning strategy for the Duplicator in the existential k-cover game on relational structures $A, B$ is a non-empty set $H$ of projective homomorphisms (from $A$ to $B$) having the following properties.

1. For all $h \in H$, $\text{dom}(h)$ is a $k$-union (over $A$).
2. For every $h \in H$ and for every $k$-union $U$ (over $A$), there exists $h' \in H$ with $\text{dom}(h') = U$ such that for every $v \in \text{dom}(h) \cap \text{dom}(h')$, $h(v) = h'(v)$.

Proposition 16. In the existential k-cover game on a pair of relational structures $A, B$, the Duplicator has a winning strategy if and only if the Duplicator has a compact winning strategy.

Proof. Suppose that the Duplicator has a winning strategy $H$. Let $C$ be the set containing all functions $h \in H$ such that $\text{dom}(h)$ is a $k$-union. We claim that $C$ is a compact winning strategy. Clearly $C$ satisfies the first property of a compact winning strategy, so we show that it satisfies the second property. Suppose $h \in C$ and let $U$ be a $k$-union. By the subfunction property of a winning strategy, the restriction $r$ of $h$ to $\text{dom}(h) \cap U$ is in $H$. By repeated application of the forth property, there is an extension $e$ of $r$ that is in $H$ and has domain $U$, which serves as the desired $h'$.

Now suppose that the Duplicator has a compact winning strategy $C$. Let $H$ be the closure of $C$ under subfunctions. We claim that $H$ is a winning strategy. It suffices to show that $H$ has the forth property. Let $h \in H$ and suppose that $a$ is an $A$-element where $\text{dom}(h) \cup \{a\}$ has weight $\leq k$. Let $U$ be a $k$-union such that $\text{dom}(h) \cup \{a\} \subseteq U$. By definition of $H$, there is a function $e \in C$ extending $h$. Apply the second property of a compact winning strategy to $e$ and $U$ to obtain an $e' \in C$ with domain $U$ such that for every $v \in \text{dom}(e) \cap \text{dom}(e')$, $e(v) = e'(v)$. Notice that $\text{dom}(h) \subseteq \text{dom}(e) \cap \text{dom}(e')$. Thus, the restriction of $e'$ to $\text{dom}(h) \cup \{a\}$ is in $H$ and extends $h$. \qed

We have just shown that deciding if there is a winning strategy, in an instance of the existential k-cover game, is equivalent to deciding if there is a compact winning strategy. We now use this equivalence to give a polynomial-time algorithm for deciding if there is a winning strategy.

Theorem 17. For all $k \geq 1$, there exists a polynomial-time algorithm that, given a pair of relational structures $A, B$, decides whether or not there is a winning strategy for the Duplicator in the existential k-cover game on $A, B$.

Proof. By Proposition 16, it suffices to give a polynomial-time algorithm that decides if there is a compact winning strategy. It is straightforward to develop such an algorithm based on the definition of compact winning strategy. Let $H$ be the set of all functions $h$ such that $\text{dom}(h)$ is a $k$-union (over $A$) and such that $h$ is a projective homomorphism from $A$ to $B$. Iteratively perform the following until no changes can be made to $H$: for every function $h \in H$ and every $k$-union $U$, check to see if there is $h' \in H$ such that the second property (of compact winning strategy) is satisfied; if not, remove $h$ from $H$. Throughout the algorithm, we have maintained the invariant that any compact winning strategy
must be a subset of $H$. Hence, if when the algorithm terminates $H$ is empty, then there is no compact winning strategy. And if $H$ is non-empty when the algorithm terminates, $H$ is clearly a compact winning strategy.

The number of $k$-unions (over $A$) is polynomial in the number of tuples in $A$. Also, for each $k$-union $U$, the number of projective homomorphisms $h$ with $\text{dom}(h) = U$ from $A$ to $B$ is polynomial in the number of tuples in $B$. Hence, the size of the original set $H$ is polynomial in the original instance. Since in each iteration an element is removed from $H$, the algorithm terminates in polynomial time. □

The algorithm we have just described in the proof of Theorem 17 may appear to be quite specialized. However, we now show that essentially that algorithm can be viewed as a general inference procedure for CSP instances in the vein of existing consistency algorithms. In particular, we give a general algorithm called projective $k$-consistency for CSP instances that, given a CSP instance, performs inference and outputs a more constrained CSP instance having exactly the same satisfying assignments as the original. On a CSP instance $A, B$, the algorithm might detect an inconsistency, by which we mean that it detects that there is no homomorphism from $A$ to $B$. If it does not, then it is guaranteed that there is a winning strategy for the Duplicator.

**Definition 18.** The projective $k$-consistency algorithm takes as input a CSP instance $A, B$, and consists of the following steps.

- Create a new CSP instance $A', B'$ as follows. Let the universe of $A'$ be the universe of $A$, and the universe of $B'$ be the universe of $B$. Let the signature of $A'$ and $B'$ contain a relation symbol $R_U$ for each $k$-union $U$ over $A$. For each $k$-union $U$, the relation $R^A_U$ is defined as $(u_1, \ldots, u_k)$, where $u_1, \ldots, u_k$ are exactly the elements of $U$ in some order; and $R^B_U$ is defined as the set of all tuples $(b_1, \ldots, b_k)$ such that the mapping taking $u_i \to b_i$ is a projective homomorphism from $A'$ to $B'$.

- Iteratively perform the following until no changes can be made: remove any $B'$-tuple $(b_1, \ldots, b_k)$ that is not a projective homomorphism. We say that a $B'$-tuple $(b_1, \ldots, b_k) \in R^B_U$ is a projective homomorphism if, letting $(u_1, \ldots, u_k)$ denote the unique element of $R^A_U$, the function taking $u_i \to b_i$ is a projective homomorphism from $A'$ to $B'$.

- Report an inconsistency if there are no $B'$-tuples remaining.

**Theorem 19.** For each $k \geq 1$, the projective $k$-consistency algorithm, given as input a CSP instance $A, B$:

- runs in polynomial time,
- outputs a CSP instance $A', B'$ that has the same satisfying assignments as $A, B$, and
- reports an inconsistency if and only if the Duplicator does not have a winning strategy in the existential $k$-cover game on $A, B$. 
Proof. The first property is straightforward to verify. For the second property, observe that, each time a tuple is removed from \( B' \), the set of satisfying assignments is preserved. For the third property, observe that, associating \( B' \)-tuples to functions as in Definition 18, the behavior of the projective \( k \)-consistency algorithm is identical to the behavior of the algorithm in the proof of Proposition 10.

By using the results presented in this section thus far, it is easy to show that CSP instances of bounded coverwidth are tractable. Define the coverwidth of a CSP instance \( A, B \) to be the coverwidth of \( A \). Let \( \text{CSP}[\text{coverwidth} \leq k] \) be the restriction of the CSP to all instances of coverwidth less than or equal to \( k \).

**Theorem 20.** For all \( k \geq 1 \), the problem \( \text{CSP}[\text{coverwidth} \leq k] \) is decidable in polynomial time by the projective \( k \)-consistency algorithm. In particular, on an instance of \( \text{CSP}[\text{coverwidth} \leq k] \), the projective \( k \)-consistency algorithm reports an inconsistency if and only if the instance is not satisfiable.

Proof. Immediate from Theorem 14 and the third property of Theorem 19.

Note that we can derive the tractability of CSP instances having bounded hypertree width immediately from Theorem 20.

Now, given a CSP instance that is promised to have bounded coverwidth, we can use projective \( k \)-consistency to decide the instance (Theorem 20). This tractability result can in fact be pushed further: we can show that there is a generic polynomial-time that, given an arbitrary CSP instance, is guaranteed to decide instances of bounded coverwidth. Moreover, whenever an instance is decided to be a “yes” instance by the algorithm, a satisfying assignment is constructed.

**Theorem 21.** For all \( k \geq 1 \), there exists a polynomial-time algorithm that, given any CSP instance \( A, B \),

1. outputs a satisfying assignment for \( A, B \),
2. correctly reports that \( A, B \) is unsatisfiable, or
3. reports “I don’t know”.

The algorithm always performs (1) or (2) on an instance of \( \text{CSP}[\text{coverwidth} \leq k] \).

Proof. The algorithm is a simple extension of the projective \( k \)-consistency algorithm. First, the algorithm applies the projective \( k \)-consistency algorithm; if an inconsistency is detected, then the algorithm terminates and reports that \( A, B \) is unsatisfiable. Otherwise, it initializes \( V \) to be the universe \( A \) of \( A \), and does the following:

- If \( V \) is empty, terminate and identify the mapping taking each \( a \in A \) to the \( B \)-element in \( R^B \), as a satisfying assignment.
- Pick any variable \( v \in V \).
- Expand the signature of \( A, B \) to include another symbol \( R_v \) with \( R^A_v = \{(v)\} \).
Try to find a $B$-element $b$ such that when $R^B_v$ is set to $\{(b)\}$, no inconsistency is detected by the projective $k$-consistency algorithm on the expanded instance.

- If there is no such $B$-element, terminate and report “I don’t know”.
- Otherwise, set $R^B_v$ to such a $B$-element, remove $v$ from $V$, and repeat from the first step using the expanded instance.

If the procedure terminates from $V$ being empty in the first step, the mapping that is output is straightforwardly verified to be a satisfying assignment.

Suppose that the algorithm is given an instance of $\text{CSP}[\text{coverwidth} \leq k]$. If it is unsatisfiable, then the algorithm reports that the instance is unsatisfiable by Theorem 20. So suppose that the instance is satisfiable. We claim that each iteration preserves the satisfiability of the instance. Let $A, B$ denote the CSP instance at the beginning of an arbitrary iteration of the algorithm. If no inconsistency is detected after adding a new relation symbol $R_v$ with $R^A_v = \{(v)\}$ and $R^B_v = \{(b)\}$, there must be a satisfying assignment mapping $v$ to $b$ by Theorem 20. Note that adding unary relation symbols to a CSP instance does not change the coverwidth of the instance.

We now expand the tractability result of Theorem 20, and show the tractability of CSP instances that are homomorphically equivalent to instances of bounded coverwidth. Formally, let us say that $A$ and $A'$ are homomorphically equivalent if there is a homomorphism from $A$ to $A'$ as well as a homomorphism from $A'$ to $A$. Let $\text{CSP}[\mathcal{H}(\text{coverwidth} \leq k)]$ denote the restriction of the CSP to instances $A, B$ where $A$ is homomorphically equivalent to a relational structure of coverwidth less than or equal to $k$.

**Theorem 22.** For all $k \geq 1$, the problem $\text{CSP}[\mathcal{H}(\text{coverwidth} \leq k)]$ is decidable in polynomial time by the projective $k$-consistency algorithm. In particular, on an instance of $\text{CSP}[\mathcal{H}(\text{coverwidth} \leq k)]$, the projective $k$-consistency algorithm reports an inconsistency if and only if the instance is not satisfiable.

**Proof.** Let $A, B$ be a CSP instance where $A$ is homomorphically equivalent to a relational structure $A'$ of coverwidth $\leq k$. The following conditions are equivalent; after stating each condition, we indicate how to show equivalence with the previous condition.

- There is a homomorphism from $A$ to $B$.
- There is a homomorphism from $A'$ to $B$ (straightforward).
- The Duplicator has a winning strategy in the existential $k$-cover game on $A', B$ (Theorem 14).
- The Duplicator has a winning strategy in the existential $k$-cover game on $A, B$ (Theorem 13).
- The projective $k$-consistency algorithm does not report an inconsistency on $A, B$ (Theorem 10).
5 Quantified Constraint Satisfaction

We now sketch how the ideas given in this paper on constraint satisfaction can be combined with the ideas in [6] to yield results on quantified constraint satisfaction. Specifically, we define a notion of coverwidth for QCSPs, and show that bounded coverwidth QCSPs are tractable.

Definitions. We first briefly define the QCSP and relevant associated notions. A quantified relational structure is a pair \((p, A)\) where \(A\) is a relational structure and \(p\) is a quantifier prefix, an expression of the form \(Q_1v_1 \ldots Q_nv_n\) where each \(Q_i\) is a quantifier (either \(\exists\) or \(\forall\)) and \(v_1, \ldots, v_n\) are exactly the elements of the universe of \(A\). The quantified constraint formula \(\phi_{(p,A)}\) associated to a quantified relational structure \((p, A)\) (where \(A\) is over signature \(\sigma\)) is defined to be the formula \(pC_A\), where \(C_A\) is the conjunction of all atomic formulas in the set \(\{R(a_1, \ldots, a_k) : R \in \sigma, (a_1, \ldots, a_k) \in R^A\}\). We say that there is a homomorphism from \((p, A)\) to \(B\) if \(B \models \phi_{(p,A)}\). We define the QCSP as the problem of deciding, given a quantified relational structure \((p, A)\) and a relational structure \(B\), if there is a homomorphism from \((p, A)\) to \(B\).

A quantifier prefix \(p = Q_1v_1 \ldots Q_nv_n\) can be viewed as the concatenation of quantifier blocks where quantifiers in each block are the same, and consecutive quantifier blocks have different quantifiers. For example, the quantifier prefix \(\forall v_1 \forall v_2 \exists v_3 \forall v_4 \forall v_5 \exists v_6 \exists v_7 \forall v_8\), consists of four quantifier blocks: \(\forall v_1 \forall v_2\), \(\exists v_3\), \(\forall v_4 \forall v_5\), and \(\exists v_6 \exists v_7 \forall v_8\). We say that a variable \(v_j\) comes after a variable \(v_i\) in \(p\) if they are in the same quantifier block, or \(v_j\) is in a quantifier block following the quantifier block of \(v_i\). Equivalently, the variable \(v_j\) comes after the variable \(v_i\) in \(p\) if one of the following conditions holds: (1) \(j \geq i\), or (2) \(j < i\) and all of the quantifiers \(Q_j, \ldots, Q_i\) are of the same type.

Coverwidth. We now define a notion of coverwidth for quantified relational structures. This can be viewed as a generalization of the definition of coverwidth in terms of schemes, given in Proposition [9].

Definition 23. A scheme of a quantified relational structure \((p, A)\) is a scheme \((V, F)\) of the hypergraph \(H(A)\) (in the sense of Definition [9]) such that \((V, F)\) has a perfect elimination ordering \(v_1, \ldots, v_n\) respecting the quantifier prefix \(p\) in that if \(i < j\), then \(v_j\) comes after \(v_i\) in \(p\).

Definition 24. The coverwidth of a quantified relational structure \((p, A)\) is equal to the minimum weight (with respect to \(A\)) over all schemes of \((p, A)\).

The quantified \(k\)-cover game. We can naturally extend the \(k\)-cover game, making use of ideas from [9], to define the quantified \(k\)-cover game. We describe the quantified \(k\)-cover game by defining the notion of a winning strategy for the Duplicator.

Definition 25. A winning strategy for the Duplicator in the quantified \(k\)-cover game on \((p, A)\) and \(B\) is a non-empty set \(H\) of projective homomorphisms (from \(A\) to \(B\)) having the following properties.
1. For every \( h \in H \) and every existentially quantified \( A \)-element \( a \notin \text{dom}(h) \) coming after all elements of \( \text{dom}(h) \), if \( \text{dom}(h) \cup \{a\} \) has weight \( \leq k \), then there exists a projective homomorphism \( h' \in H \) extending \( h \) with \( \text{dom}(h') = \text{dom}(h) \cup \{a\} \).

2. For every \( h \in H \), every \( B \)-element \( b \), and every universally quantified \( A \)-element \( a \notin \text{dom}(h) \) coming after all elements of \( \text{dom}(h) \), if \( \text{dom}(h) \cup \{a\} \) has weight \( \leq k \), then there exists a projective homomorphism \( h' \in H \) extending \( h \) with \( \text{dom}(h') = \text{dom}(h) \cup \{a\} \) and \( h'(a) = b \).

3. The set \( H \) is closed under subfunctions, that is, if \( h \in H \) and \( h \) extends \( h' \), then \( h' \in H \).

We have the following analog of Theorem 14.

**Theorem 26.** Let \((p, A)\) be a quantified relational structure having coverwidth \( \leq k \), and let \( B \) be an arbitrary relational structure. There is a winning strategy for the Duplicator in the quantified \( k \)-cover game on \((p, A), B\) if and only if there is a homomorphism from \((p, A)\) to \( B \).

**Tractability.** We let \( \text{QCSP}[\text{coverwidth} \leq k] \) denote the restriction of the QCSP to all instances \((p, A), B\) where \((p, A)\) has coverwidth less than or equal to \( k \). We have the following tractability result.

**Theorem 27.** For all \( k \geq 1 \), the problem \( \text{QCSP}[\text{coverwidth} \leq k] \) is decidable in polynomial time.

The algorithm for Theorem 27 is similar to projective \( k \)-consistency, but in addition to removing tuples that are not projective homomorphisms, it removes further tuples, as follows. Let \( A_1 \ldots A_m \) denote the quantifier blocks of the prefix \( p \), and let \( A_i \) be an existential quantifier block. Take a tuple from the right-hand side structure, view it as a mapping \( h : U \rightarrow B \), and consider its restriction \( h' \) to \( A_1 \cup \ldots \cup A_i \). Let \( Y \) be the set of all universally quantified variables in \( U \cap (A_{i+1} \cup \ldots \cup A_m) \). If there exists any extension of \( h' \) to \( \text{dom}(h') \cup Y \) that is not a projective homomorphism, then the tuple is removed. After the procedure terminates, if no inconsistency is detected, then the projective homomorphisms \( h \) of the new instance where the weight of \( \text{dom}(h) \) is \( \leq k \), is a winning strategy for the Duplicator in the quantified \( k \)-cover game on the original instance.

We can expand the result of Theorem 27 by \( \mathcal{Q} \)-homomorphic equivalence, defined in [6]. Let \( \text{QCSP}[\mathcal{H}(\text{coverwidth} \leq k)] \) denote the restriction of the QCSP to all instances \((p, A), B\) where \((p, A)\) is \( \mathcal{Q} \)-homomorphically equivalent to a quantified relational structure that has coverwidth less than or equal to \( k \).

**Theorem 28.** For all \( k \geq 1 \), the problem \( \text{QCSP}[\mathcal{H}(\text{coverwidth} \leq k)] \) is decidable in polynomial time.

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A Proof of Theorem 13

Proof. ($\Rightarrow$) Let $H$ be a winning strategy for the Duplicator in the $k$-cover game on $A$ and $B$, let $T$ be any structure of coverwidth $\leq k$, let $f$ be any homomorphism from $T$ to $A$, let $G = (T, F)$ be a scheme for $T$ of weight $\leq k$, and let $v_1, \ldots, v_m$ be a perfect elimination ordering of $G$.

We shall construct a sequence of partial mappings $g_0, \ldots, g_n$ from $T$ to $B$ such that for each $i$:

1. $\text{dom}(g_i) = \{v_1, \ldots, v_i\}$, and
2. for every clique $L \subseteq \{v_1, \ldots, v_i\}$ in $G$, there exists a projective homomorphism $h \in H$ with domain $f(L)$ in the winning strategy of the Duplicator, such that for every $v \in L$, $h(f(v)) = g_i(v)$.

We define $g_0$ to be the partial function with empty domain. For every $i \geq 0$, the partial mapping $g_{i+1}$ is obtained by extending $g_i$ in the following way. As $v_1, \ldots, v_n$ is a perfect elimination ordering, the set

$$L = \{v_{i+1}\} \cup \{v_j : j < i + 1, (v_j, v_{i+1}) \in F\}$$

is a clique of $G$. Define $L'$ as $L \setminus \{v_{i+1}\}$. By the induction hypothesis, there exists $h \in H$ such that for every $v \in L'$, $h(f(v)) = g_i(v)$. Let us consider two cases.

If $f(v_{i+1}) = f(v_j)$ for some $v_j \in L'$ then we set $g_{i+1}(v_{i+1})$ to be $g_i(v_j)$. Note that in this case property (2) is satisfied, as every clique in $G$ containing $v_{i+1}$ is contained in $L$ and $h$ serves as a certificate. (For any clique not containing $v_{i+1}$, we use the induction hypothesis.)

Otherwise, that is, if $f(v_{i+1}) \neq f(v_j)$ for all $v_j \in L'$, we do the following. First, since the weight of $L$ is bounded above by $k$ and $f$ defines an homomorphism from $T$ to $A$ then the weight of $f(L)$ is also bounded by $k$. Observe that $f(L) = \text{dom}(h) \cup \{f(v_{i+1})\}$. By the forth property of winning strategy there exists an extension $h' \in H$ of $h$ that is defined over $v_{i+1}$. We set $g_{i+1}(v_{i+1})$ to be $h'(f(v_{i+1}))$. Note that $h'$ certifies that property (2) is satisfied for very clique containing $v_{i+1}$; again, any clique not containing $v_{i+1}$ is covered by the induction hypothesis.

Finally, let us prove that $g_n$ indeed defines an homomorphism from $T$ to $B$. Let $R$ be any relation symbol and let $(t_1, \ldots, t_l)$ be any relation in $R^T$. We want to show that $(g_n(t_1), \ldots, g_n(t_l))$ belongs to $R^B$. Since $G$ is an scheme for $T$, $(t_1, \ldots, t_l)$ constitutes a clique of $G$. By property (2) there exists $h \in H$ such that $h(f(t_i)) = g(t_i)$ for all $i$. Observing that as $f$ is an homomorphism from $T$ to $A$, we can have that $(f(t_1), \ldots, f(t_l))$ belongs to $R^A$. Finally, as $h$ is a projective homomorphism from $A$ to $B$, the tuple $(h(f(t_1)), \ldots, h(f(t_l)))$ must be in $B$.

$(\Leftarrow)$ We shall construct a winning strategy $H$ for the Duplicator. We need a few definitions. Fix a sequence $a_1, \ldots, a_m$ of elements of $A$. A valid tuple for $a_1, \ldots, a_m$ is any tuple $(T, G, v_1, \ldots, v_m, f)$ where $T$ is a relational structure, $G$ is an scheme of weight $k$ for $T$, and $\{v_1, \ldots, v_m\}$ is a clique of $G$, and $f$ is an homomorphism from $T$, $v_1, \ldots, v_m$ to $A, a_1, \ldots, a_m$. (By a homomorphism from
Although we do not need the equality in our proof). For a structure $T$, $v_1, \ldots, v_m$ to $A$, $a_1, \ldots, a_m$, we mean a homomorphism from $T$ to $A$ that maps $v_i$ to $a_i$ for all $i$. By $S(T, G, v_1, \ldots, v_m, f)$ we denote the set of all mappings $h$ with domain $\{a_1, \ldots, a_m\}$ such that there is an homomorphism from $T, v_1, \ldots, v_m$ to $B, h(a_1), \ldots, h(a_m)$. We are now in a situation to define $H$. $H$ contains for every subset $a_1, \ldots, a_m$ of weight at most $k$, every partial mapping $h$ that is contained in all $S(T, G, v_1, \ldots, v_m, f)$ where $(T, G, v_1, \ldots, v_m, f)$ is a valid tuple for $a_1, \ldots, a_m$.

Let us show that $H$ is indeed a winning strategy. First, observe that $H$ is nonempty, as it contains the partial function with empty domain. Second, let us show that $H$ contains only projective homomorphisms. Indeed, let $h$ be any mapping in $H$ with domain $a_1, \ldots, a_m$, let $R$ be any relation symbol and let $(c_1, \ldots, c_l)$ be any tuple in $R^A$. Let us define $T$ to be the substructure (not necessarily induced) of $A$ with universe $\{a_1, \ldots, a_k, c_1, \ldots, c_l\}$ containing only the tuple $(c_1, \ldots, c_l)$ in $R^T$. It is easy to verify that the graph $G = (\{a_1, \ldots, a_k, c_1, \ldots, c_l\}, F)$ where $F = \{(a_i, a_j) : i \neq j\} \cup \{(c_i, c_j) : i \neq j\}$ is an scheme of $T$ of weight $\leq k$. Consequently, $(T, G, a_1, \ldots, a_m, id)$ is a valid tuple for $a_1, \ldots, a_m$ and therefore there exists an homomorphism $g$ from $T$ to $B$, and hence satisfying $(g(c_1), \ldots, g(c_l)) \in R^B$, such that $g(a_i) = h(a_i)$ for all $i = 1, \ldots, k$.

To show that $H$ is closed under subfunctions is rather easy. Indeed, let $h'$ be any mapping in $H$ with domain $a_1, \ldots, a_m$. We shall see that the restriction $h$ of $h'$ to $\{a_1, \ldots, a_{m-1}\}$ is also in $H$. Let $(T, G, v_1, \ldots, v_{m-1}, f)$ be any valid tuple for $a_1, \ldots, a_k$. We construct a valid tuple $(T', G', v_1, \ldots, v_m, f')$ for $a_1, \ldots, a_m$ in the following way: $v_m$ is a new (not in the universe of $T$) element, $T'$ is the structure obtained from $T$ by adding $v_m$ to the universe of $T$ and keeping the same relations. $f'$ is the extension of $f$ in which $v_m$ is map to $a_m$, and $G'$ is the scheme of $T$ obtained by adding to $G$ an edge $(v_j, v_m)$ for every $j = 1, \ldots, m-1$. Since $(T', G', v_1, \ldots, v_m, f')$ is a valid tuple for $a_1, \ldots, a_m$ and $h' \in H$, there exists an homomorphism $g'$ from $T', v_1, \ldots, v_m$ to $B, h'(a_1), \ldots, h'(a_m)$. Observe then that the restriction $g$ of $g'$ to $\{a_1, \ldots, a_{m-1}\}$ defines then an homomorphism from $T, v_1, \ldots, v_{m-1}$ to $B, h(a_1), \ldots, h(m)$.

Finally, we shall show that $H$ has the forth property. The proof relies in the following easy properties of the valid tuples. Let $a_1, \ldots, a_m$ be elements of $A$ and let $(T_1, G_1, v_1, \ldots, v_m, f_1)$ and $(T_2, G_2, v_1, \ldots, v_m, f_2)$ be valid tuples for $a_1, \ldots, a_m$ such that $T_1 \cap T_2 = \{v_1, \ldots, v_m\}$, let $T$ be $T_1 \cup T_2$ (that is, the structure $T$ whose universe is the union of the universes of $T_1$ and $T_2$, and in which $R^T = R^{T_1} \cup R^{T_2}$ for all relation symbols $R$), $G = G_1 \cup G_2$ and let $f$ be the mapping from the universe $T$ of $T$ to $B$ that sets $a$ to $f_1(a)$ if $a \in T_1$ and to $f_2(a)$ if $a \in T_2$ (observe that $f_1$ and $f_2$ coincide over $\{v_1, \ldots, v_m\}$). Then $(T, G, v_1, \ldots, v_m, f)$ is a valid tuple for $a_1, \ldots, a_m$. We call $(T, G, v_1, \ldots, v_m, f)$, the union of $(T_1, G_1, v_1, \ldots, v_m, f_1)$ and $(T_2, G_2, v_1, \ldots, v_m, f_2)$. Furthermore, $S(T, G, v_1, \ldots, v_m, f) \subseteq S(T_1, G_1, v_1, \ldots, v_m, f_1) \cap S(T_2, G_2, v_1, \ldots, v_m, f_2)$ (in fact, $S(T, G, v_1, \ldots, v_m, f) = S(T_1, G_1, v_1, \ldots, v_m, f_1) \cap S(T_2, G_2, v_1, \ldots, v_m, f_2)$, although we do not need the equality in our proof).
Let $h$ be any mapping in $H$, let $\{a_1, \ldots, a_{m-1}\}$ be its domain, and let $a_m$ be any element in the universe of $A$ such that $\{a_1, \ldots, a_m\}$ has weight $\leq k$. Let us assume, towards a contradiction, that there is not extension $h'$ of $h$ in $\mathcal{H}$. Then there exists a finite collection $\{(T_i, G_i, v_1, \ldots, v_m, f_i) : i \in I\}$ of valid tuples for $a_1, \ldots, a_m$ such that the intersection $\bigcap_{i \in I} S(T_i, G_i, v_1, \ldots, v_m, f_i)$ does not contain any extension of $h$. We can rename the elements of the universes so that for every different $i, j \in I$ we have that $T_i \cap T_j = \{v_1, \ldots, v_m\}$.

Let $(T, G, v_1, \ldots, v_m, f)$ be the union of $(T_i, G_i, v_1, \ldots, v_m, f_i), i \in I$, which is a valid tuple for $a_1, \ldots, a_m$. Since

$$S(T, G, v_1, \ldots, v_m, f) \subseteq \bigcap_{i \in I} S(T_i, G_i, v_1, \ldots, v_m, f_i)$$

we can conclude that $S(T, G, v_1, \ldots, v_m, f)$ does not contain any extension of $h$. We are almost at home. It is only necessary to observe that $(T, G, v_1, \ldots, v_{m-1}, f)$ is a valid tuple for $a_1, \ldots, a_{m-1}$ and since $S(T, G, v_1, \ldots, v_m, f)$ does not contain any extension of $h$, $S(T, G, v_1, \ldots, v_{m-1}, f)$ cannot contain $h$, in contradiction with $h \in H$. $\square$