Vibration frequencies and mode shapes of CCFF rectangular plates simply supported at the free corner. Application to large vibration amplitudes.

Ahmed Babahammou and Rhali Benamar
University Mohammed V in Rabat, E.M.I, BP 765, Rabat, Morocco
E-mail: ahmedbabahammou@gmail.com
E-mail: rhali.benamar@gmail.com

Abstract. The linear flexural vibration of a thin isotropic rectangular plate simply supported at the free corner has been investigated by the symplectic superposition method by Rui.Li and al. The purpose of the present study was to establish the efficiency of the Rayleigh-Ritz method (RRM) applied to similar plate problems. Due to its numerical and systematic character, the RRM has many practical advantages compared to the classical laborious analytical approaches and can be extended to investigate the case of geometrically nonlinear vibrations. The trial plate functions used were obtained as products of beam functions with appropriate end conditions in each direction and the point support was modeled by a translational spring with an infinite stiffness. Different cases have been dealt with corresponding to different edge conditions, different aspect ratios, and for several modes. The comparisons made of the solutions obtained with the known solutions of isotropic rectangular plates with one or more point supports show a satisfactory agreement. The nonlinear vibration of various plates with a with point support at one corner have been also examined. The backbone curves based on the SMA have been calculated by Matlab code. The systematic and straightforward method presented can be readily extended to plates with inside point corners.

1. Introduction
Rectangular plates are commonly used as structural components in many engineering fields. The knowledge of their natural frequencies and mode shapes is necessary to determine their response under the working loads and to estimate properly the induced strains and stresses in order to make an optimal design. In spite of the very large number of research work devoted to this topic, due to the variety of edge conditions, there are still numerous situations which are not yet covered by the literature, particularly in the non-linear regime. Free vibration of rectangular plates punctually supported at a corner has been studied by numerous researches using various laborious methods, such as [1], [2], but yet the studies were restricted to linear vibration. The present work investigated the vibration of plates clamped at two adjacent edges and free at the two other edges with a point support at the free corner, denoted as (CCFF). The Rayleigh-Ritz method (RRM) has been used to study the plate linear vibrations for various values of the aspect ratio. Benamar’s method (BM) [3], [4] has then been applied to investigate the nonlinear vibration for large vibration amplitudes, leading to the plotted backbone curves.
2. General formulation:
Consider the rectangular plate shown in Figure 1, it is clamped at two adjacent edges and punctually supported at the opposite corner whose adjacent edges are free. It has a length $a$, a weight $b$, a thickness $H$, a mass per unit volume $\rho$, a Young’s modulus $E$ and a Poisson modulus $\nu$. The transverse displacement function $W(x,y,t)$ of the current mid-plane point $(x,y)$ is expanded in the form of a series of basic functions:

$$W(x,y,t) = \sum_{k=1}^{N} a_{k} w_{i}(x,y) \sin(\omega t)$$ (1)

In which $\omega$ is the natural frequency and $a_{ij}$ is the contribution coefficient of the $k^{th}$ plate trial function $w_{k}$. The RRM plate trial functions $w_{k}$ used, for $k=1$ to $N = n^{2}$, are obtained as products of $n$ linear clamped free (CF) beam mode shapes $X_{i}$ in the x-and y-directions:

$$w_{k}(x,y) = X_{i}(x) \cdot X_{j}(y) \quad k = n(i-1) + j$$ (2)

$n$ is the number of beam functions used. The clamped-free (CF) beam function is expressed by:

$$X_{i}(x) = C_{1} \cos(\lambda_{i}x) + C_{2} \sin(\lambda_{i}x) + C_{3} \cosh(\lambda_{i}x) + C_{4} \sinh(\lambda_{i}x)$$ (3)

The CF beam frequency parameters $\lambda$ such as $\lambda_{2} = \frac{L_{i}}{2} \sqrt{\frac{m}{EI}} \omega_{b}$, summarized in Table 1, are classically obtained from solution of the fourth order differential equation of beam vibration. $m$ is the beam mass per unit length, $\omega_{b}$ is the beam natural frequency and $I$ is the beam second moment of inertia of cross-section.

| $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ | $\lambda_{6}$ | $\lambda_{7}$ | $\lambda_{8}$ | $\lambda_{9}$ | $\lambda_{10}$ |
|----------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0.59686        | 1.49418      | 2.5          | 3.5          | 4.5          | 5.5          | 6.5          | 7.5          | 8.5          | 9.5          |

Figure 1. A plate clamped at two adjacent edges and free at the other edges with a point support at the free corner.
The point support is modeled by a translational spring which has a very big stiffness. The kinetic energy of the plate can be expressed as [5],[6]:

\[ T = \frac{1}{2} \rho H \int_S \left( \frac{\partial W}{\partial t} \right)^2 dS \] (4)

where \( D = \frac{EH^3}{12(1-\nu^2)} \) is the bending stiffness. The plate total strain energy, which is the sum of the strain energy due to the bending \( V_b \), of the membrane strain energy \( V_m \) induced by the large vibration amplitudes and the strain energy stored by the point support modeled by a spring which has an infinity stiffness \( V_{SP} \): \( V_T = V_b + V_m + V_{SP} \). The energy expressions are given by [7]:

\[ V_b = \frac{D}{2} \int_S \left( \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 W}{\partial x \partial y} \cdot \frac{\partial^2 W}{\partial x \partial y} - \frac{\partial^2 W}{\partial x^2} \cdot \frac{\partial^2 W}{\partial y^2} \right) dS \] (5)

\[ V_m = \frac{3D}{2H^2} \int_S \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right]^2 dS \] (6)

\[ V_{SP} = \frac{1}{2} K_{SP} W^2(a, b) \] (7)

The discretization of the energy expressions in Eqs. (4), (5), (6) and (7) leads to:

\[ T = \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2(\omega t); \quad V_b = \frac{1}{2} a_i a_j k_{ij}^b \sin^2(\omega t) \] (8)

\[ V_{SP} = \frac{1}{2} a_i a_j k_{ij}^{SP} \sin^2(\omega t); \quad V_m = \frac{1}{2} a_i a_j a_k a_l b_{ijkl} \sin^4(\omega t) \] (9)

\( m_{ij} , k_{ij}^b , b_{ijkl} \) are the mass, the linear rigidity and the non-linear geometrical rigidity tensors respectively. \( k_{ij}^{SP} \) is the rigidity tensor associated to the elastic energy stored in the point support which is modeled by a spring. For very large values of \( K_{SP} \), \( W(a, b) \) tends to zero. These tensors are given by [6], [7]:

\[ m_{ij} = \rho H \int_S w_i w_j dS \] (10)

\[ k_{ij}^b = \int_S D \left( \frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} \right) \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} \right) + 2(1-\nu) \left( \frac{\partial^2 w_i}{\partial x \partial y} \cdot \frac{\partial^2 w_j}{\partial x \partial y} - \frac{\partial^2 w_i}{\partial x^2} \cdot \frac{\partial^2 w_j}{\partial y^2} \right) dS \] (11)

\[ b_{ijkl} = \frac{3D}{H^2} \int_S \left( \frac{\partial w_i}{\partial x} \cdot \frac{\partial w_j}{\partial x} + \frac{\partial w_i}{\partial y} \cdot \frac{\partial w_j}{\partial y} \right) \left( \frac{\partial w_k}{\partial x} \cdot \frac{\partial w_l}{\partial x} + \frac{\partial w_k}{\partial y} \cdot \frac{\partial w_l}{\partial y} \right) dS \] (12)

\[ k_{ij}^{SP} = K_{SP} w_i(a, b) w_j(a, b) \] (13)

The vibration problem is governed by Hamilton’s principle [7]:

\[ \delta \int_0^{\frac{2\pi}{\omega}} (V_T - T) dt = 0 \] (14)
After the integration of the time functions over the range \([0, 2\pi/\omega]\), and calculation of the derivatives with respect to the \(a_i\)'s and after putting the rigidity tensor \(k_{ij} = k_{ij}^{v} + k_{ij}^{SP}\), one gets a non-linear eigen value problem, written in a matrix form as \([7], [6]\).

\[
2 [K] \{A\} + 3 [B (A)] \{A\} - 2\Omega^2 [M] \{A\} = 0
\] (15)

\(\{A\}\) is the column vector of the basic function contribution coefficients. \([M], [K] [B (A)]\) are the matrices associated to the tensors defined above, \(\Omega = \omega a^2 \sqrt{\rho H/D}\) is a CCFF plate frequency parameter to be calculated. A computer program has been written to calculate numerically these parameters. If the non-linear term is neglected, Eq 15 reduces to the classical eigen value problem corresponding to the Rayleigh-Ritz formulation of linear vibrations. This is treated in section 3 below.

### 3. Linear vibration

By neglecting \([B (A)]\) in Eq 15 one gets the classical eigen value problem:

\[
[K] \{A\} - \Omega^2 [M] \{A\} = 0
\] (16)

The results obtained from solution of Eq. 16 are summarized in Table 2 giving the first seven frequency parameters of a CCFF plate for various values of aspect ratio \(\alpha = a/b\), together with those mentioned by Li in [2]. The difference does not exceed 0.1%. The corresponding first four mode shapes are plotted in figure 2 for \(\alpha = 0.6\). The mode sections at the plate diagonal \((x/a = y/b)\) are shown in Figure 3.

**Table 2. First ten frequency parameters of a CCFF plate for different values of the aspect ratio**

| \(\alpha\) | \(\Omega_1\) | \(\Omega_2\) | \(\Omega_3\) | \(\Omega_4\) | \(\Omega_5\) | \(\Omega_6\) | \(\Omega_7\) | \(\Omega_8\) | \(\Omega_9\) | \(\Omega_{10}\) |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1       | 15.216      | 24.001      | 39.421      | 54.339      | 62.946      | 77.362      | 85.858      | 105.766     | 121.823     | 126.473     |
|         | 15.165\(^{a}\) | 23.902      | 39.388      | 54.083      | 62.705      | 77.321      | 85.695      | 105.290     | 121.340     | 126.250     |
| 2/3     | 9.222       | 18.277      | 26.365      | 31.747      | 44.248      | 53.838      | 59.279      | 68.392      | 75.605      | 84.771      |
|         | 9.197\(^{a}\) | 18.195      | 26.294      | 31.683      | 44.142      | 53.637      | 59.021      | 68.346      | 75.467      | 84.464      |
| 0.5     | 6.568       | 14.090      | 20.119      | 25.922      | 30.746      | 36.883      | 46.326      | 53.196      | 58.582      | 65.305      |
|         | 6.555\(^{a}\) | 14.035      | 20.026      | 25.897      | 30.637      | 36.829      | 46.133      | 53.048      | 58.292      | 65.254      |
| 0.4     | 5.362       | 10.494      | 17.018      | 21.822      | 24.953      | 30.177      | 33.815      | 40.433      | 47.181      | 53.011      |
|         | 5.355\(^{a}\) | 10.461      | 16.933      | 21.735      | 24.931      | 30.105      | 33.711      | 40.342      | 46.937      | 52.833      |
| 1/3     | 4.739       | 8.305       | 13.700      | 18.763      | 23.189      | 24.645      | 28.549      | 33.129      | 36.414      | 42.877      |
|         | 4.735\(^{a}\) | 8.284       | 13.642      | 18.657      | 23.142      | 24.583      | 28.512      | 32.994      | 36.333      | 42.681      |
| 1/3.5\(^{a}\) | 4.382      | 6.965       | 11.088      | 16.102      | 20.120      | 23.221      | 25.516      | 27.369      | 31.634      | 35.083      |
|         | 4.379\(^{a}\) | 6.951       | 11.048      | 16.009      | 20.015      | 23.206      | 25.430      | 27.317      | 31.555      | 34.911      |
| 0.25    | 4.160       | 6.104       | 9.282       | 13.490      | 17.813      | 21.412      | 23.062      | 25.418      | 27.626      | 29.917      |
|         | 4.158\(^{a}\) | 6.093       | 9.253       | 13.422      | 17.687      | 21.319      | 23.047      | 25.369      | 27.513      | 29.859      |
| 1/4.5   | 4.014       | 5.522       | 8.025       | 11.440      | 15.478      | 19.083      | 22.468      | 23.109      | 24.959      | 27.624      |
|         | 4.012\(^{a}\) | 5.514       | 8.003       | 11.388      | 15.371      | 18.939      | 22.415      | 23.058      | 24.924      | 27.522      |
| 0.2     | 3.912       | 5.114       | 7.126       | 9.911       | 13.365      | 17.078      | 20.162      | 22.582      | 23.830      | 24.750      |
|         | 3.910\(^{a}\) | 5.108       | 7.108       | 9.871       | 13.284      | 16.914      | 20.033      | 22.568      | 23.751      | 24.694      |

\(^{a}\) Results given by LI Ref [2]
Figure 2. The first four mode shapes of CCFF with $\alpha = 2/3$

Figure 3. Cross-sections of the four modes corresponding to the plate diagonal ($x/a = y/b$) and corresponding to middle plate $x/a = 0.5$ $\alpha = 2/3$
4. Nonlinear vibration

The nonlinear geometrical rigidity tensor \( [B(A)] \) is taken now into account in Eq.15 in order to apply Benamar’s method to determine the amplitude dependent non-linear fundamental frequency parameters and associated mode shapes, for several values of the plate aspect ratio. From equation 15, it is possible to calculate the frequency by pre-multiplying both sides of the equation by \( \{A\}^T \) which gives [5]:

\[
\Omega_{NL}^2 = \frac{a_1 a_j m_{ij} + 1.5 a_1 a_j a_k a_l b_{ijkl}}{a_i a_j k_{ij}}. \tag{17}
\]

The single mode approach (SMA) applied in the neighborhood of the fundamental nonlinear mode consists on neglecting all modes except the single resonant one i.e.: \( a_i = 0 \) for \( i \neq 1 \). Applying the SMA to Eq. 17 gives in the modal functions basis (MFB) [8]:

\[
\Omega_{NL}^2 = \frac{k_{11}}{m_{11}} + a^2 \frac{3b_{1111}}{2 m_{11}}. \tag{18}
\]

In which \( m_{11}, k_{11} \) and \( b_{1111} \) are the plate parameters expressed in the MFB.

The SMA, leading to the simple expression 18, is known to lead to good estimates of the amplitude frequency dependence, (the error does not exceed 6% for \( W_{max}^* = 0.8 \)). However, the SMA does not lead to any information concerning the changes in the mode shapes due to the non-linearity. In order to investigate this effect, the so-called second formulation (SF) was used in this work to solve Eq. 15 and find the \( a_i \)'s, \( i = 2 \) to \( N \), for various values of \( a_1 \), and consequently the amplitude dependent non-linear mode shapes. The bases of this approximate method, developed by El Kadiri–Benamar, are detailed in Ref [8] for the non-linear free vibration problem formulated in the MFB (see Appendix B in Ref [8]), leading to \( (n^2-1) \) equations, the \( i^{th} \) equation is:

\[
-\Omega_{NL}^2 \overline{a_s} \overline{m_{si}} + \overline{a_s} \overline{k_{si}} + 1.5 \overline{a_u} \overline{a_v} \overline{a_s} \overline{a_v} \overline{b_{suv}} = 0 \quad i = 2, 3, 4, \ldots n^2 \tag{19}
\]

Examples of numerical results of Eq. 19 obtained by the SF are listed in Table 3 corresponding to a CCFF plate for \( \alpha = 0.6 \), and in Table 4 for \( \alpha = 1/3, 2/3, 1 \) and for several values of \( a_{11} \). Figure 4 gives the backbone curves of CCFF plate by the SF with various aspect ratios. Figure 5 shows the influence of the aspect ratio on the first nonlinear mode shape. The corresponding normalized cross-sections of the amplitude dependent non-linear fundamental mode, for increasing values of \( a_{11} \), at the plate diagonal \( y/b = x/a \) and at the plate middle line \( y = b/2 \) are given in Figure 6.
Figure 4. Backbone curves of CCFF by SF with various aspect ratio $\alpha$

Table 3. $\Omega_{NL}/\Omega$, $W_{max}$ and $a_{ij}$ as functions of $a_{11}$ for $\alpha = 2/3$ ($\Omega = 9.248$)

| $W_{Max}$ | $\Omega_{NL}/\Omega$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | $a_{31}$ | $a_{32}$ | $a_{33}$ | $a_{34}$ | $a_{41}$ | $a_{42}$ | $a_{43}$ | $a_{44}$ |
|-----------|----------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0.3496    | 1.0331               | 0.0000000014 | 4.142E-02 | 7.744E-04 | 9.596E-03 | 7.324E-03 | 7.687E-05 | 7.5535E-05 |
| 0.5064    | 1.0720               | 0.150000002 | 1.193E-01 | 2.232E-03 | 2.483E-03 | 2.147E-03 | 5.362E-03 | 5.3118E-04 |
| 0.6499    | 1.1229               | 0.200000004 | 1.530E-01 | 2.996E-03 | 3.104E-03 | 3.206E-03 | 3.380E-04 | 3.380E-03 |
| 0.7789    | 1.1839               | 0.250000007 | 1.890E-01 | 3.809E-03 | 5.620E-03 | 5.906E-03 | 5.331E-04 | 5.331E-03 |
| 0.8967    | 1.2532               | 0.300000003 | 2.217E-01 | 4.929E-03 | 7.562E-03 | 7.737E-03 | 4.357E-03 | 4.357E-03 |
| 1.0164    | 1.3297               | 0.350000005 | 2.508E-01 | 5.801E-03 | 6.364E-03 | 9.756E-03 | 2.508E-04 | 6.496E-03 |
| 1.1158    | 1.4123               | 0.400000004 | 2.854E-01 | 8.039E-03 | 1.210E-02 | 2.956E-02 | 9.073E-03 | 9.073E-03 |
| 1.2163    | 1.5001               | 0.450000003 | 3.168E-01 | 1.562E-02 | 1.094E-02 | 1.459E-02 | 1.209E-02 | 1.209E-02 |

Table 4. $\Omega_{NL}/\Omega$ and $W_{max}$ as functions of $a_{11}$ for several values of aspect ratio $\alpha$

| $\alpha = 1/3$ ($\Omega = 4.747$) | $\alpha = 2/3$ ($\Omega = 9.248$) | $\alpha = 1$ ($\Omega = 15.27$) |
|-----------------------------------|-----------------------------------|-----------------------------------|
| $a_{11}$ | $W_{max}$ | rap | $W_{max}$ | rap | $W_{max}$ | rap | $W_{max}$ | rap | $W_{max}$ | rap |
| 0.05    | 0.2033   | 1.0184 | 0.1791   | 1.0085 | 0.1077   | 1.0035 | 0.1077   | 1.0035 |
| 0.1     | 0.3977   | 1.0715 | 0.3496   | 1.0331 | 0.2153   | 1.0138 | 0.2153   | 1.0138 |
| 0.2     | 0.7371   | 1.2503 | 0.6489   | 1.1229 | 0.4303   | 1.0545 | 0.4303   | 1.0545 |
| 0.3     | 1.0069   | 1.5146 | 0.8987   | 1.2532 | 0.6446   | 1.1206 | 0.6446   | 1.1206 |
| 0.4     | 1.2287   | 1.8026 | 1.1158   | 1.4123 | 0.8586   | 1.2098 | 0.8586   | 1.2098 |
Figure 5. Fundamental nonlinear mode shape of a CCFF plate with various aspect ratios $\alpha = 1/3, 2/3, 1$.

Figure 6. Normalized cross-sections of the amplitude dependent non-linear fundamental mode of a CCFF plate for increasing values of $a_{11}$; (a) at the plate diagonal $y/b = x/a$, (b) at $y = b/2$. 
5. Conclusion
First, the linear vibration of rectangular plates clamped at two adjacent edges and free at the two other edges with a point support at the free corner has been investigated using the RRM with appropriate plate functions. The calculated frequency parameters, compared with the available references, permitted to establish the efficiency of the RRM applied to this problem and the possibility of using the mode shapes obtained in the non-linear analysis. The nonlinear study was dealt with by BM. The ratio of the nonlinear frequency parameter $\Omega_{NL}$ to the linear frequency parameter $\Omega$ for various values of the maximum displacement $W$ has been calculated by the SMA and plotted. The amplitude dependent fundamental non-linear mode shape, obtained via the multi-mode second formulation was determined and plotted for several values of the plate aspect ratio $\alpha = a/b$ and various values of the vibration amplitude.

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