Cubic couplings in $D = 6, \mathcal{N} = 4b$ supergravity on $\text{AdS}_3 \times S^3$

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Abstract

We determine the AdS exchange diagrams needed for the computation of 4–point functions of chiral primary operators in the SCFT$_2$ dual to the $D = 6, \mathcal{N} = 4b$ supergravity on the $\text{AdS}_3 \times S^3$ background and compute the corresponding cubic couplings. We also address the issue of consistent truncation.
1 Introduction

The AdS/CFT correspondence provides information about the strong coupling behaviour of some conformal field theories by studying their supergravity (string) duals \([1]–[4]\). In particular, the AdS/CFT duality relates type IIB string theory on \(AdS_3 \times S^3 \times M^4\), where \(M^4\) is either \(K3\) or \(T^4\), to a certain \(\mathcal{N} = (4,4)\) supersymmetric two-dimensional conformal field theory (CFT) living on the boundary of \(AdS_3\). A two–dimensional sigma model with the target space being a deformation of the orbifold symmetric product \(S^N(M^4) = (M^4)^N/S_N\), \(N \to \infty\), is believed to provide an effective description of this CFT \([5]\).

An important class of operators in the supersymmetric CFT are the Chiral Primary Operators (CPOs) since they are annihilated by \(1/2\) of the supercharges and in two dimensions their highest weight components of the R–symmetry group form a ring. On the gravity side the CPOs correspond to Kaluza–Klein (KK) modes of the type IIB supergravity compactification. Recently, using the orbifold technique developed in \([6]\) the three–point functions of scalar CPOs were computed \([7]\) in the CFT on the symmetric product \(S^N(M^4)\) and, on the other hand, in \(AdS_3 \times S^3\) supergravity \([8]\), and were found to disagree\(^1\). On the other hand, computations of quantities that are stable under deformations of the orbifold CFT, like the spectrum of the CPOs and the elliptic genus, were found to be in complete agreement \([10]\). Obviously this supports the expectation that the \(AdS_3 \times S^3\) background may correspond to some deformation of the target space \(S^N(M^4)\) of the boundary CFT. However, even though one presently does not have an explicit sigma model formulation of the boundary CFT (see also \([11]\) for recent developments), one may proceed to study the CFT by using directly the gravity dual description and the AdS/CFT correspondence \([12]\).

In this paper we study the \(AdS_3 \times S^3\) compactification of the \(D = 6\) \(\mathcal{N} = 4\) \(b\) supergravity coupled to \(n\) tensor multiplets. In particular, the case \(n = 21\) corresponds to the theory obtained by dimensional reduction of type IIB supergravity on \(K3\). Our final aim will be to find the 4–point correlation functions of the scalar CPOs in the supergravity approximation. This program was successfully performed for the \(\mathcal{N} = 4\) SYM\(_4\) which is related to the \(AdS_5 \times S^5\) compactification of type IIB supergravity and led to an understanding of the structure of the Operator Product Expansion in the field theory at strong coupling \([13]–[16]\). As a first necessary step in this direction we derive the effective gravity action on \(AdS_3\) that contains all cubic couplings involving at least two gravity fields corresponding to CPOs in the boundary CFT.

Since the supergravity we consider is a chiral theory it suffers from the absence of a simple Lagrangian formulation. In principle, one may approach the problem of computing correlation functions by using the Pasti–Sorokin–Tonin formulation of the six–dimensional supergravity action, where the manifest Lorentz covariance is achieved by introducing an auxilliary scalar field \(a\) \([17]\). However, to

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\(^1\) Computing the 2– and 3–point functions of CPOs in the supergravity approximation by using the prescription \([8]\), which is known to be compatible with the Ward identities \([9]\), one finds a result different from \([8]\). This however does not remove the disagreement between CFT and gravity calculations.
obtain the action for physical fields one needs to fix the gauge symmetries, in particular the additional symmetry associated with the field \(a\). This breaks the manifest Lorentz covariance and makes the problem of solving the noncovariant constraints imposed by gauge fixing unfeasible. Thus, we prefer to start with the covariant equations of motion of chiral six–dimensional supergravity \[18\] and obtain the quadratic, cubic and so on corrections to the equations of motion for physical fields by decomposing the original equations near the \(AdS_3 \times S^3\) background and partially fixing the gauge (diffeomorphism) symmetries. The equations obtained in this way are in general non–Lagrangian with higher derivative terms and we perform the nonlinear field redefinitions to remove higher derivative terms \[13\] and bring the equations to the Lagrangian form.

The spectrum of the \(AdS_3 \times S^3\) compactification of the \(D = 6\) \(\mathcal{N} = 4\) \(b\) supergravity coupled to \(n\) tensor multiplets was found in \[19\] and it is governed by the supergroup \(SU(1,1|2)_L \times SU(1,1|2)_R\). Since we are interested in the quadratic and ultimately in cubic corrections to the equations of motion for the gravity fields we reconsider the derivation of the linearized equations of motion and recover the spectrum of \[19\]. According to \[19\] the scalar CPOs are divided into two classes. The first class contains CPOs \(\sigma\) that are singlets with respect to the internal symmetry group \(SO(n)\). The corresponding gravity fields are mixtures of the trace of the graviton on \(S^3\) and the sphere components of the self–dual form. The second class comprise the CPOs \(s^r\) transforming in the fundamental representation of \(SO(n)\) and the corresponding gravity fields are mixtures of \(n\) scalars from the coset space \(SO(5,n)/SO(5) \times SO(n)\) and the sphere components of the \(n\) antiself–dual forms.

As follows from our study, the fields appearing in the exchange diagrams involving at least two CPOs may contain, in addition to the CPOs themselves, also other scalars or vectors, either in the singlet or in the vector representation of \(SO(n)\), and symmetric 2nd rank (massive) tensor fields. We find the corresponding cubic couplings. By using the factorization property of the Maxwell operator in odd dimensions we diagonalize the equations for the vector fields which originate from components of the second order Einstein equation and the first order self–duality equation. This diagonalization is helpful to identify the vector fields propagating in the AdS exchange diagrams. To ensure the wider applicability of our results we keep \(n\) unspecified.

The cubic couplings exhibit the same vanishing property in the extremal case (e.g. for three scalar fields \(\sigma_k\), where \(k\) denotes a Kaluza–Klein mode, the extremality condition is \(k_1 + k_2 = k_3\) and permutations thereof) as the cubic couplings found in the compactification of type IIB supergravity on \(AdS_5 \times S^5\) \[13\].

In addition to the cubic couplings involving CPOs we also compute certain cubic couplings of vector fields, which allows us to check the consistency of the KK truncation of the three–dimensional action to the massless graviton multiplet. Recall that the bosonic part of this multiplet contains the

\[\text{Except } n = 21 \text{ another case of interest is } n = 5.\] Dimensional reduction of type IIB supergravity on \(T^4\) produces the non–chiral \(D = 6\) \(\mathcal{N} = 8\) theory, for which the equations of motion for the metric, the scalar fields and the two–forms are the same as for \(D = 6\) \(\mathcal{N} = 4\) with \(n = 5\).
graviton and the $SU(2)_L \times SU(2)_R$ gauge fields, all of them carrying non–propagating (topological) degrees of freedom. Since the other multiplets contain the propagating modes, the graviton multiplet should admit a consistent truncation and we show that this is indeed the case. The truncated action coincides with the topological Chern–Simons action constructed in [20]. We also consider the problem of the KK truncation to the sum of two multiplets, one of them the massless graviton multiplet, whereas the second involves the fields corresponding to the lowest weight CPOs. Surprisingly, we have found indications that the sum of the massless graviton multiplet and the special spin–$\frac{1}{2}$ multiplet containing the lowest mode scalar CPOs may admit a consistent truncation. This situation reminds of, but is different from the $AdS_5 \times S^5$ compactification, where the lowest weight CPOs occur in the stress tensor multiplet, which on the gravity side corresponds to the massless graviton multiplet, allowing a consistent truncation [13]. In the $AdS_3$ case the gauge degrees of freedom encoded in the graviton multiplet give rise to the $\mathcal{N} = (4, 4)$ superconformal algebra of the boundary CFT [21].

The paper is organized as follows. In section 2 we recall the covariant equations of motion for the bosonic sector of $D = 6 \mathcal{N} = 4b$ supergravity [18], introduce notation and represent our results. In section 3 we review the linearized equations of motion and recover the spectrum found in [13]. In section 4 we discuss the structure of the quadratic corrections to the linearized equations of motion and explain the necessary steps to reduce the equations of motion to a Lagrangian form. In Appendix A we give the results for the expansion of the covariant equations of motion up to the second order, both in spacetime and coset metric perturbations and in Appendix B we establish the formulae for various integrals involving spherical harmonics of different kinds.

2 The cubic effective action in $AdS_3$

Cubic couplings of chiral primaries may be derived from the quadratic corrections to the covariant equations of motion for $D = 6 \mathcal{N} = 4b$ supergravity coupled to $n$ tensor multiplets [18]. All the bosonic fields — the graviton, the two–form potentials $B_{MN}^I$, $I = 1, \ldots, 5 + n$ and the scalar sector — provide relevant contributions to the quadratic corrections. The scalar sector constitutes a sigma model over the coset space $SO(5, n)/SO(5) \times SO(n)$ with vielbein $(V_I, \bar{V}_j)$, $i = 1, \ldots, 5$, $r = 1, \ldots, n$ which is parameterized by $5n$ scalar fields. The index $I$ transforms under global $SO(5, n)$ transformations and is raised and lowered with the $SO(5, n)$ invariant metric $\eta = \text{diag}(1_{5 \times 5}, -1_{n \times n})$, whereas the indices $(i, r)$ transform under local composite $SO(5) \times SO(n)$ transformations. We use the following indices: $M, N$ for $D = 6$, $\mu, \nu$ for $AdS_3$ and $a, b$ for $S^3$ coordinates.

Defining

$$dVV^{-1} = \begin{pmatrix} Q^{ij} & \sqrt{2}P^{is} \\ \sqrt{2}P_{ij} & Q^{rs} \end{pmatrix},$$

(2.1)
the covariant derivative in the scalar sector is found by the Cartan–Maurer equation to be

\[ D_M P^r_N = \nabla_M P^r_N - Q^j_M P^r_N - P^{is} Q^r_N \]  

(2.2)

and the equations of motion for the bosonic sector of \( D = 6 \) \( \mathcal{N} = 4 \) supergravity are:

\[ R_{MPQ} = H^i_{MPQ} H^i_{N} + H^r_{MPQ} H^r_{N} + 2P^{ir}_{N}, \]  

(2.3)

\[ D_M P^r_M = \frac{\sqrt{2}}{3} H^i_{MPQ} H^r_{MP}, \]  

(2.4)

\[ *H^i = H^i, \quad *H^r = -H^r, \]  

(2.5)

where

\[ H^i = G^i V^i, \quad H^r = G^r V^r \text{ and } G^I = dB^I. \]  

(2.6)

In units where the radius of \( S^3 \) is set to unity, the \( AdS_3 \times S^3 \) background solution is

\[ ds^2 = \frac{1}{x^2_0} (dx_0^2 + \eta_{ij} dx^i dx^j) + d\Omega_3^2, \]  

(2.7)

where \( \eta_{ij} \) is the 2–dimensional Minkowski metric. One of the self–dual field strengths is singled out and set equal to the Levi–Cevita tensor, while all others vanish:

\[ H^i_{\mu
u\rho} = \delta^i_{\delta^\mu\nu\rho}, \quad H^i_{abc} = \delta^i_{\delta^a\beta\gamma}, \quad H^r_{MNP} = 0. \]  

(2.8)

Here \( \varepsilon_{\mu\nu\rho} \) and \( \varepsilon_{abc} \) are the volume forms on \( AdS_3 \) and \( S^3 \), respectively, so that \( \varepsilon_{\mu\nu\rho} \varepsilon_{abc} \) is the volume form in six dimensions. The \( SO(5,n) \) background vielbein is taken to be constant and by a global \( SO(5,n) \) rotation it can be set to unity.

To construct the Lagrangian equations of motion we represent the fields as

\[ g_{MN} = \bar{g}_{MN} + h_{MN}, \]  

\[ G^I = \bar{G}^I + g^I, \quad g^I = db^I, \]  

(2.9)

(2.10)

and

\[ V^i_I = \delta^i_I + \phi^{ir} \delta{r}^I + \frac{1}{2} \phi^{ir} \phi^{js} \delta{r}^J, \]  

(2.11)

\[ V^r_I = \delta^r_I + \phi^{ir} \delta^I + \frac{1}{2} \phi^{ir} \phi^{is} \delta^J. \]  

(2.12)

The gauge symmetry of the equations of motion allows one to impose the de Donder–Lorentz gauge\(^3\):

\[ \nabla^a h_{\mu \alpha} = \nabla^a h_{ab} = \nabla^a b_{Ma}^I = 0. \]  

(2.13)

\(^3\)From now on all the covariant derivatives are understood to be in the background geometry.
where we have represented 

\[ h_{ab} = h_{(ab)} + \frac{1}{3} g_{ab} h_c, \quad g^{ab} h_{(ab)} = 0. \]  

(2.14)

The various spherical harmonics transform in the following irreducible representations of \( SO(4) \simeq SU(2)_L \times SU(2)_R \):

Scalar spherical harmonics \( Y^I \): \((\frac{k}{2}, \frac{k}{2}), k \geq 0 \),

Vector spherical harmonics \( Y^I_a \) = \( Y^I_a^+ + Y^I_a^- \): \((\frac{1}{2}(k + 1), \frac{1}{2}(k - 1)) \oplus (\frac{1}{2}(k - 1), \frac{1}{2}(k + 1)), k \geq 1 \),

Tensor spherical harmonics \( Y^I_{(ab)} = Y^I_{(ab)}^+ + Y^I_{(ab)}^- \): \((\frac{1}{2}(k + 2), \frac{1}{2}(k - 2)) \oplus (\frac{1}{2}(k - 2), \frac{1}{2}(k + 2)), k \geq 2 \).

The upper index enumerates a basis in a given irreducible representation of \( SO(4): I_1 = 1, \ldots, (k + 1)^2, k \geq 0; I_3 = 1, \ldots, (k + 1)^2 - 1, k \geq 1; I_5 = 1, \ldots, (k + 1)^2 - 4, k \geq 2 \). The action of the Laplacian is \([22]\):

\[
\begin{align*}
\nabla^2 Y^I_1 &= - \Delta Y^I_1, \\
\nabla^2 Y^I_{a}^{\pm} &= (1 - \Delta) Y^I_{a}^{\pm}, \quad \nabla^a Y^I_{a}^{\pm} = 0, \\
\nabla^2 Y^I_{(ab)}^{\pm} &= (2 - \Delta) Y^I_{(ab)}^{\pm}, \quad \nabla^a Y^I_{(ab)}^{\pm} = 0, \quad g^{ab} Y^I_{(ab)}^{\pm} = 0,
\end{align*}
\]

(2.15)

where \( \Delta \equiv k(k + 2) \). The vector spherical harmonics \( Y^I_{a}^{\pm} \) are also eigenfunctions of the operator \((\star \nabla)^c_a Y^I_{a}^{\pm} = \varepsilon_{a}^{bc} \nabla_b \) :

\[
(\star \nabla)^c_a Y^I_{a}^{\pm} = \pm (k + 1) Y^I_{a}^{\pm}. \quad (2.16)
\]

\[\text{Here and in what follows we use normalized spherical harmonics, i.e. } \int Y^I_1 Y^J_1 = \delta_{I_1 J_1}, \int g^{ab} Y^I_{a}^{\pm} Y^J_{b}^{\pm} = \delta_{I_3 J_3}, \int g^{a_{(ab)}} Y^I_{(ab)}^{\pm} Y^J_{(cd)}^{\pm} = \delta_{I_5 J_5}.\]
We also need to make a number of field redefinitions, the simplest ones, required to diagonalize the linearized equations of motion, are

\[
\begin{align*}
\phi^5_I &= 2ks'_I + 2(k + 2)t'_I, \quad U^5_I = s'_I - t'_I, \\
\pi_I &= -6k\sigma_I + 6(k + 2)\tau_I, \quad U^5_I = \sigma_I + \tau_I, \\
h_{\mu\nu} &= \varphi_{\mu\nu} + \nabla_\mu \nabla_\nu \zeta_I + g_{\mu\nu} \eta_I, \\
\zeta_I &= \frac{4}{k + 1} (\tau_I - \sigma_I), \\
\eta_I &= \frac{2}{k + 1} (k(k - 1)\sigma_I - (k + 2)(k + 3)\tau_I), \\
h^\pm_{\mu I} &= \frac{1}{2}(C^\pm_{\mu I} - A^\pm_{\mu I}), \quad Z^\pm_{\mu I} = \pm \frac{1}{4}(C^\pm_{\mu I} + A^\pm_{\mu I}).
\end{align*}
\]

Here \(s'_I\) and \(\sigma_I\) are scalar chiral primaries \([13]\). Note also that we use an off–shell shift for \(h_{\mu\nu}\) that first appeared in \([23]\). It differs from the on–shell shift used in \([13]\) by higher order terms. We recall the origin of the off–shell shift in section 3.3.

The field redefinitions that are needed to make the equations of motion Lagrangian and to remove terms with two and four derivatives from the quadratic corrections to the equations of motion will be discussed in section 4.

### 2.1 Cubic couplings of chiral primaries

To compute four–point functions involving only chiral primary operators in the boundary conformal field theory one needs the quartic couplings giving rise to contact diagrams and cubic couplings involving at least two chiral primaries, which contribute to the AdS exchange diagrams. Here we confine ourselves to the problem of determining the corresponding cubic couplings.

Obviously, fields like \(\phi^{5r}\), \(r = 1, \ldots, 4\) that transform as vectors under the \(SO(4)\) R–Symmetry cannot contribute to these couplings. Therefore, we can set all these fields to zero and, to simplify the notation, we denote e.g. \(\phi^{5r}\) as \(\phi^r\), etc.

Then the action for the chiral primaries \(s^r\) and \(\sigma\) may be written in the form

\[
S(s^r, \sigma) = \frac{N}{(2\pi)^3} \int d^3x \sqrt{-g_{AdS^3}} \left( L_2(s^r) + L_2(t^r) + L_2(\sigma) + L_2(\tau) + L_2(\varphi^\pm) + L_2(Z^\pm_{\mu}) + L_2(A^\pm_{\mu}, C^\pm_{\mu}) + L_2(\varphi_{\mu\nu}) + L_3^\sigma(\sigma) + L_3^\tau(\tau) + L_3^\varphi(\varphi^\pm) + L_3^Z(Z^\pm_{\mu}) + L_3^A(A^\pm_{\mu}, C^\pm_{\mu}) + L_3^C(A^\pm_{\mu}, C^\pm_{\mu}) + L_3^\varphi(\varphi_{\mu\nu}) \right).
\]

[13]: Reference to a book or paper
[23]: Reference to another book or paper
The quadratic terms for the various scalar fields are

\[ L_2(s^\pm) = \sum 16(k+1) \left( -\frac{1}{2} \nabla^\mu s_I^\pm \nabla_\mu s_I^\pm - \frac{1}{2} m_s^2 s_I^\pm s_I^\pm \right) \]  
(2.24)

\[ L_2(\sigma) = \sum 16(k-1) \left( -\frac{1}{2} \nabla^\mu \sigma_I \nabla_\mu \sigma_I - \frac{1}{2} m_\sigma^2 (\sigma_I)^2 \right) \]  
(2.25)

\[ L_2(t^\pm) = \sum 16(k+1)(k+2) \left( -\frac{1}{2} \nabla^\mu t_I^\pm \nabla_\mu t_I^\pm - \frac{1}{2} m_t^2 (t_I^\pm)^2 \right) \]  
(2.26)

\[ L_2(\tau) = \sum 16(k+2)(k+3) \left( -\frac{1}{2} \nabla^\mu \tau_I \nabla_\mu \tau_I - \frac{1}{2} m_\tau^2 (\tau_I)^2 \right) \]  
(2.27)

\[ L_2(\phi^\pm) = \sum \left( -\frac{1}{4} \nabla^\mu \phi_I^\pm \nabla_\mu \phi_I^\pm - \frac{1}{4} m_\phi^2 (\phi_I^\pm)^2 \right) \]  
(2.28)

with masses

\[ m_s^2 = m_\sigma^2 = k(k-2), \quad m_t^2 = m_\tau^2 = (k+2)(k+4), \quad m_\phi^2 = \Delta. \]  
(2.29)

The quadratic Lagrangians for the vector fields can be written as

\[ L_2(Z_{\mu}^{\pm}) = \sum 16(k+1) \left( \mp \frac{1}{4} \varepsilon^{\mu \nu} Z_{\mu I}^{\pm} \partial_\nu Z_{\rho I}^{\pm} + \frac{1}{4} m_Z Z_{\mu I}^{\pm} Z_{\rho I}^{\pm} \right) \]  
(2.30)

for the fields \( Z_{\mu}^{\pm} \) with mass \( m_Z = k+1 \) and

\[ L_2(A_{\mu}^{\pm}, C_{\mu}^{\pm}) = L_2(A_{\mu}^{\pm}) + L_2(C_{\mu}^{\pm}) + L_2^{\text{cross}}(A_{\mu}^{\pm}, C_{\mu}^{\pm}) \]  
(2.31)

for the fields \( A_{\mu}^{\pm} \) and \( C_{\mu}^{\pm} \), where

\[ L_2(A_{\mu}^{\pm}) = \sum \left( -\frac{1}{8} F_{\mu \nu I}^{\pm} F_I^{\mu \nu}(A_{\mu}^{\pm}) - \frac{1}{4}(k+1)(k-1)A_{\mu I}^{\pm} A_{\nu I}^{\pm} \mp \frac{1}{2} \varepsilon^{\mu \nu \rho} A_{\mu I}^{\pm} \partial_\nu A_{\rho I}^{\pm} \right) \]  
(2.32)

\[ L_2(C_{\mu}^{\pm}) = \sum \left( -\frac{1}{8} F_{\mu \nu I}^{\pm} F_I^{\mu \nu}(C_{\mu}^{\pm}) - \frac{1}{4}(k+1)(k+3)C_{\mu I}^{\pm} C_{\nu I}^{\pm} \pm \frac{1}{2} \varepsilon^{\mu \nu \rho} C_{\mu I}^{\pm} \partial_\nu C_{\rho I}^{\pm} \right) \]  
(2.33)

\[ L_2^{\text{cross}}(A_{\mu}^{\pm}, C_{\mu}^{\pm}) = \sum \left( \frac{1}{4} F_{\mu \nu I}^{\pm} F_I^{\mu \nu}(C_{\mu}^{\pm}) - \frac{1}{2}(k-1)(k+3)A_{\mu I}^{\pm} C_{\nu I}^{\pm} \right. \]  
\[ \left. \pm \frac{1}{2}(k+1) \varepsilon^{\mu \nu \rho} (A_{\mu I}^{\pm} \partial_\nu C_{\rho I}^{\pm} + C_{\mu I}^{\pm} \partial_\nu A_{\rho I}^{\pm}) \right). \]  
(2.34)

Here \( F_{\mu \nu I}(V) = \partial_\mu V_{\nu I} - \partial_\nu V_{\mu I} \) and we have introduced the first order operators

\[ (P_{\mu I}^\pm)^\lambda_\mu = \varepsilon_\mu^{\rho \lambda} \nabla_\nu \pm m_\rho^\lambda = (\ast \nabla)^\lambda_\mu \pm m_\rho^\lambda. \]  
(2.35)
Some comments are in order. Since the quadratic action for the vector fields $Z^\tau \pm$ is of the Chern–Simons form it vanishes on–shell and, therefore, to compute 2–point functions in the boundary CFT we have to add certain boundary terms \cite{23}. It is also worthwhile to note that the equations of motion for the vector fields $A^\mu_\pm$, $C^\mu_\pm$ are not the Proca equations, rather they are Proca–Chern–Simons equations containing both the usual and the topological mass terms. Indeed, the equations for $A^\mu_\pm$ and $C^\mu_\pm$ following directly from (2.31) are nondiagonal and both are of the second order. Adding them produces an equation of the first order (a constraint) that relates the fields $A^\mu_\pm$ and $C^\mu_\pm$:

$$P_{k-1}^\pm(A^\pm)_\mu I + P_{k+3}^\pm(C^\pm)_\mu I = 0.$$  \hspace{1cm} (2.36)

This constraint is then used to obtain the closed Proca–Chern–Simons equations for the vector fields, e.g.

$$\nabla^\nu F_{\nu\mu I}(A^\pm) - (k-1)(k+1)A^\pm_\mu I + 2\varepsilon_\mu^{\nu\rho} \partial_\nu A^\pm = P_{k+1}^\pm P_{k-1}^\pm(A^\pm)_\mu I = 0.$$  \hspace{1cm} (2.37)

Thus, intrinsically one has a second order equation for one of the vector fields and a constraint on the second one. The number of physical degrees of freedom described by a massive pair $A_\mu I$, $C_\mu I$ is then three and this is in agreement with the discussion in \cite{19}. The original equations being components of the second order Einstein equation and the first order self–duality equation are related to (2.36) and (2.37) by simple linear transformations of the fields $A^\mu_\pm$ and $C^\mu_\pm$ (c.f. (2.22)). Note that the conformal dimensions of the operators in the boundary CFT dual to $A^\mu_\pm$ and $C^\mu_\pm$ are $k$, $k+2$ and $k+4$.

Finally, the quadratic Lagrangian for the symmetric second rank tensor field $(\varphi \equiv \varphi^\mu_\pm)$ is:

$$L_2(\varphi_{\mu\nu}) = \sum \left( -\frac{1}{4} \nabla_\mu \varphi_{\nu\rho} \nabla^\mu \varphi^\nu_\rho + \frac{1}{2} \nabla_\mu \varphi^I_\nu \nabla^\rho \varphi_{\rho I} - \frac{1}{2} \nabla_\mu \varphi_I \nabla^\nu \varphi^\nu_I + \frac{1}{4} \nabla_\mu \varphi_I \nabla^\mu \varphi_I + \frac{1}{4} (2 - \Delta)(\varphi_{\mu\nu})^2 + \frac{1}{4} \Delta(\varphi_I)^2 \right).$$  \hspace{1cm} (2.38)

The cubic couplings of scalar fields are

$$L_3^\sigma(\psi) = V_{I_1 I_2 I_3}^{s,\psi} s_{I_1}^r t_{I_2}^r \psi_{I_3}, \quad L_3^{\sigma,\sigma}(\psi) = V_{I_1 I_2 I_3}^{\sigma,\sigma} \sigma_{I_1} \sigma_{I_2} \psi_{I_3}, \quad L_3^{\sigma,\tau}(\psi) = V_{I_1 I_2 I_3}^{\sigma,\tau} s_{I_1}^r t_{I_2}^r \sigma_{I_3},$$  \hspace{1cm} (2.39)

with $\psi \in \{ \sigma, \tau, \varphi^\pm \}$ and the vertices (the notation is explained in eqs. (2.61) and (2.62))

$$V_{I_1 I_2 I_3}^{s,\psi} = -\frac{2^4(\Sigma - 2)\Sigma(\Sigma + 2)\alpha_1 \alpha_2 \alpha_3}{k_3 + 1} a_{I_1 I_2 I_3},$$  \hspace{1cm} (2.40)

$$V_{I_1 I_2 I_3}^{s,\sigma} = \frac{2^6(\Sigma + 2)\alpha_1 (\Sigma + 2)\alpha_3}{k_3 + 1} a_{I_1 I_2 I_3},$$  \hspace{1cm} (2.41)

$$V_{I_1 I_2 I_3}^{s,\tau} = -\frac{2^3(\Sigma + 2)(\Sigma + 2)\alpha_1 \alpha_2 \alpha_3}{3(k_1 + 1)(k_2 + 1)(k_3 + 1)} (k_1^2 + k_2^2 + k_3^2 - 2) a_{I_1 I_2 I_3},$$  \hspace{1cm} (2.42)

$$V_{I_1 I_2 I_3}^{s,\sigma \tau} = \frac{2^5(\Sigma + 2)(\alpha_1 + 1)(\alpha_2 + 1)\alpha_3 (\alpha_3 - 1)(\alpha_3 - 2)}{(k_1 + 1)(k_2 + 1)(k_3 + 1)} (k_1^2 + k_2^2 + (k_3 + 2)^2 - 2) a_{I_1 I_2 I_3},$$  \hspace{1cm} (2.43)
\[ V_{\mu\nu}^{s\sigma} = \frac{2^7(\Sigma + 2)(\alpha_1 + 1)\alpha_2(\alpha_2 - 1)(\alpha_3 + 1)}{(k_3 + 1)} a_{I_1I_2I_3}, \]  
\[ (2.44) \]
\[ V_{\mu\nu}^{ss\pm} = 2^2(\alpha_3 - 1)p_{I_1I_2I_3}^\pm, \]  
\[ (2.45) \]
\[ V_{\mu\nu}^{\pm} = \frac{2\Sigma(\alpha_3 - 1)}{(k_1 + 1)(k_2 + 1)}(k_1^2 + k_2^2 - (k_3 + 1)^2 - 1)p_{I_1I_2I_3}^\pm. \]  
\[ (2.46) \]

In our notation, the vertices (2.40) and (2.42) are precisely the ones found in [8].

Cubic terms involving two chiral primaries and the vector fields \( A_\mu^\pm, C_\mu^\pm \) can be represented as

\[ \mathcal{L}^s_3(A_\mu^\pm, C_\mu^\pm) = V_{I_1I_2I_3}^{sA_\mu^\pm} \nabla^\mu s_1^I A_\mu^\pm + V_{I_1I_2I_3}^{sC_\mu^\pm} \nabla^\mu s_2^I C_\mu^\pm \]  
\[ \pm W_{I_1I_2I_3}^{sA_\mu^\pm} (P_{k_3-1}(s_1^I \nabla s_2^I)^\mu A_\mu^\pm - P_{k_3+3}(s_1^I \nabla s_2^I)^\mu C_\mu^\pm), \]  
\[ (2.47) \]
\[ \mathcal{L}^\sigma_3(A_\mu^\pm, C_\mu^\pm) = V_{I_1I_2I_3}^{sA_\mu^\pm} \sigma_1 I_1 \nabla^\mu \sigma_2 I_2 A_\mu^\pm + V_{I_1I_2I_3}^{sC_\mu^\pm} \sigma_1 I_1 \nabla^\mu \sigma_2 I_2 C_\mu^\pm \]  
\[ \pm W_{I_1I_2I_3}^{sA_\mu^\pm} (P_{k_3-1}(\sigma_1 I_1 \nabla \sigma_2 I_2)^\mu A_\mu^\pm - P_{k_3+3}(\sigma_1 I_1 \nabla \sigma_2 I_2)^\mu C_\mu^\pm), \]  
\[ (2.48) \]

whereas the interaction of \( s^r \) and \( \sigma \) with the fields \( Z_\mu^\pm \) is found to be

\[ \mathcal{L}^s_3(Z_\mu^\pm) = \pm V_{I_1I_2I_3}^{sZ_\mu^\pm} \sigma_1 I_1 \nabla^\mu s_2^I Z_\mu^\pm. \]  
\[ (2.49) \]

These expressions describe the minimal interactions of vector fields with two scalars in three dimensions. Here the couplings are

\[ V_{I_1I_2I_3}^{sA_\mu^\pm} = -2(\Sigma + 1)(\Sigma - 1)t_{I_1I_2I_3}^\pm, \]  
\[ (2.50) \]
\[ V_{I_1I_2I_3}^{sC_\mu^\pm} = 2(2\alpha_3 - 1)(2\alpha_3 - 3)t_{I_1I_2I_3}^\pm, \]  
\[ (2.51) \]
\[ W_{I_1I_2I_3}^{sA_\mu^\pm} = 2(k_3 + 1)t_{I_1I_2I_3}^\pm, \]  
\[ (2.52) \]
\[ V_{I_1I_2I_3}^{sA_\mu^\pm} = \frac{(\Sigma + 1)(\Sigma - 1)}{(k_1 + 1)(k_2 + 1)}(k_1^2 + k_2^2 - k_3^2 - 1)t_{I_1I_2I_3}^\pm, \]  
\[ (2.53) \]
\[ V_{I_1I_2I_3}^{sC_\mu^\pm} = \frac{(2\alpha_3 - 1)(2\alpha_3 - 3)}{(k_1 + 1)(k_2 + 1)}(k_1^2 + k_2^2 - (k_3 + 2)^2 - 1)t_{I_1I_2I_3}^\pm, \]  
\[ (2.54) \]
\[ W_{I_1I_2I_3}^{sA_\mu^\pm} = 2(k_3 + 1)(k_1 - 1)(k_2 - 1)(k_1 + 1)(k_2 + 1)t_{I_1I_2I_3}^\pm, \]  
\[ (2.55) \]
\[ V_{I_1I_2I_3}^{sZ_\mu^\pm} = \frac{2^4(\Sigma + 1)(2\alpha_3 - 1)(k_3 + 1)}{k_1 + 1}t_{I_1I_2I_3}^\pm. \]  
\[ (2.56) \]

Finally, the interaction of chiral primaries with symmetric tensors of the 2nd rank are

\[ \mathcal{L}_3^s(\varphi_{\mu\nu}) = V_{I_1I_2I_3}^{ss\varphi} \left( \nabla^\mu s_1^I \nabla^\nu s_2^I \varphi_{\mu\nu}I_3 - \frac{1}{2} \left( \nabla^\mu s_1^I \nabla_\mu s_2^I + \frac{1}{2} (m_1^2 + m_2^2 - \Delta_3) s_1^I s_2^I \right) \varphi_{I_3} \right), \]  
\[ (2.57) \]
\[ \mathcal{L}_3^s(\varphi_{\mu\nu}) = V_{I_1I_2I_3}^{ss\varphi} \left( \nabla^\mu \sigma_1 I_1 \nabla^\nu \sigma_2 I_2 \varphi_{\mu\nu}I_3 - \frac{1}{2} \left( \nabla^\mu \sigma_1 I_1 \nabla_\mu \sigma_2 I_2 + \frac{1}{2} (m_1^2 + m_2^2 - \Delta_3) \sigma_1 I_1 \sigma_2 I_2 \right) \varphi_{I_3} \right), \]  
\[ (2.58) \]

where

\[ V_{I_1I_2I_3}^{ss\varphi} = 2^2(\Sigma + 2)\alpha_3 a_{I_1I_2I_3}, \]  
\[ (2.59) \]
\[ V_{I_1I_2I_3}^{ss\varphi} = \frac{2(\Sigma + 2)\alpha_3}{(k_1 + 1)(k_2 + 1)}(k_1^2 + k_2^2 - (k_3 + 1)^2 - 1)a_{I_1I_2I_3}. \]  
\[ (2.60) \]
Above the summation over $I_1$, $I_2$, $I_3$ and $r$ is assumed and we have defined

$$\Sigma \equiv k_1 + k_2 + k_3, \quad \alpha_i \equiv \frac{1}{2}(k_l + k_m - k_i), \quad l \neq m \neq i \neq l, \quad (2.61)$$

and (c.f. also Appendix B)

$$a_{I_1 I_2 I_3} \equiv \int Y_{I_1} Y_{I_2} Y_{I_3}, \quad t_{I_1 I_2 I_3}^{\pm} \equiv \int \nabla^a Y_{I_1} Y_{I_2} Y_{a I_3}^{\pm}, \quad p_{I_1 I_2 I_3}^{\pm} \equiv \int \nabla^a Y_{I_1} \nabla^b Y_{I_2} Y_{(ab) I_3}^{\pm}. \quad (2.62)$$

To summarize, we have the following types of cubic vertices:

![Diagram of cubic vertices]

Table 1: Cubic vertices containing two supergravity fields dual to CPOs.

In particular we see that all possible cubic invariants under $SO(n) \times SO_R(4)$ containing at least two supergravity fields dual to CPOs are present.

### 2.2 Cubic couplings at extremality

With the cubic couplings at hand the problem of computing the 3–point correlation functions of two CPOs with an operator associated to another gravity field entering the cubic vertex becomes straightforward. One needs to determine the on–shell value of the corresponding cubic action, which amounts to computing certain integrals over the AdS space, where for the latter problem a well–developed technique is available [9]. Generally the AdS integrals diverge for some “extremal” values of conformal dimensions (masses) of the fields involved and this is an indication that the corresponding supergravity coupling should vanish, otherwise the correlation function would be ill–defined [9], [13]. For example, the AdS integral corresponding to the 3–point correlation function of scalar fields with conformal dimensions $\Delta_1$, $\Delta_2$ and $\Delta_3$ is ill–defined if $\Delta_1 + \Delta_2 = \Delta_3$ (or any relation obtained from this by permutation of indices). Inspection shows that the cubic couplings we found do indeed vanish at extremality, i.e. when the accompanying AdS integrals diverge. The only case where this property
can not be seen straightforwardly is for the couplings of scalar fields with vector fields $A^\pm_\mu$ and $C^\pm_\mu$. Below we present the analysis making the property of vanishing at extremality manifest.

Recall that due to (2.36) the fields $A^\pm_\mu$ and $C^\pm_\mu$ do not describe independent degrees of freedom. Regarding, e.g., $A^\pm_\mu$ as independent variables we first consider the solution of eq. (2.37) satisfying

$$P^\pm_{k-1}(A^\pm)_{\mu I} = 0.$$  

Then the constraint (2.38) gives $P^\pm_{k+3}(C^\pm)_{\mu I} = 0$. Clearly, the last two equations imply the Maxwell equations

$$\nabla^\nu F^\nu_{\mu I} (A^\pm) - (k - 1)^2 A^\pm_{\mu I} = 0, \quad \nabla^\nu F^\nu_{\mu I} (C^\pm) - (k + 3)^2 C^\pm_{\mu I} = 0. \quad (2.63)$$

Therefore, the masses of the vector fields $A^\pm_\mu$ and $C^\pm_\mu$ are $m_A = k - 1$ and $m_C = k + 3$. Recalling the formula for the conformal weight $\Delta_V$ of an operator dual to a vector field $V_\mu$ with mass $m$ in $AdS_{d+1}$ (see, e.g. [4]) we find $\Delta_A = k$ and $\Delta_C = k + 4$. It is worthwhile to note that $\Delta_A$ has the same conformal dimension as the scalar CPOs. The corresponding CFT operators are the vector CPOs in the spin 2 tower of supermultiplets [19].

The evaluation of the 3–point functions of CPOs with vector fields requires the knowledge of the following AdS integral

$$\int \frac{d^3 \omega}{\omega_0^3} K_{\Delta_1}(\omega, x_1) \nabla^\mu K_{\Delta_2}(\omega, x_2) G_{\mu I}(x_2) G_{\nu I}(x_3) = \frac{R_{123}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_V} |x_{13}|^{\Delta_1 + \Delta_2 - \Delta_V - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_V - \Delta_1} Z_i Z_i, \quad (2.64)$$

where the coordinate dependence on the r.h.s. is completely fixed by the conformal symmetry. Here $x_i$ are the positions of the operators in the correlation function of the boundary CFT, $x_{ij} = x_i - x_j$,

$$Z_i = \frac{(x_{13})_i}{x_{13}^2} - \frac{(x_{23})_i}{x_{23}^2}, \quad Z^2 = Z_i Z_i \quad \text{and } K_{\Delta}(\omega, x), G_{\mu I}(x), G_{\nu I}(x) \text{ are the scalar and vector bulk–to–boundary propagators respectively. Applying the inversion method [8] one finds for } R_{123} \text{ the following answer:}$$

$$R_{123} = \frac{1}{\pi^2} \frac{\Gamma\left(\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_V + 1)\right) \Gamma\left(\frac{1}{2}(\Delta_1 + \Delta_V - \Delta_2 + 1)\right) \Gamma\left(\frac{1}{2}(\Delta_2 + \Delta_V - \Delta_1 + 1)\right)}{\Gamma(\Delta_1 - 1) \Gamma(\Delta_2 - 1) \Gamma(\Delta_V)} \times \Gamma\left(\frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_V - 1)\right). \quad (2.65)$$

$R_{123}$ is ill–defined in several cases. First we consider the case when

$$\Delta_1 + \Delta_2 - \Delta_V + 1 = 0. \quad (2.65)$$

For CPOs with $\Delta = k$ this equation becomes $k_1 + k_2 - \Delta_V + 1 = 0$ and, therefore, for $C^\pm_\mu$ it reads as

$$k_1 + k_2 - \Delta_C + 1 = k_1 + k_2 - k_3 - 3 = 0,$$

i.e., $R_{123}$ is also divergent for $\Delta_1 + \Delta_2 - \Delta_V + 1$ a negative integer, but in that cases $t_i^\pm t_j t_j = 0$. 

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i.e., $\alpha_3 = 3/2$. But the couplings $V^{ssC}_{l_1l_2l_3}$ and $V^{\sigma\sigma C}_{l_1l_2l_3}$ (see (2.47) and (2.48)) contain the factor $2\alpha_3 - 3$ and, therefore, vanish.\(^6\) Computing the correlation functions involving the fields $A_{\mu}^\pm$ a divergence arises when

$$k_1 + k_2 - \Delta_A + 1 = k_1 + k_2 - k_3 + 1 = 0,$$

i.e., when $\alpha_3 = -1/2$. However, the couplings $V^{ssA}_{l_1l_2l_3}$ and $V^{\sigma\sigma A}_{l_1l_2l_3}$ contain the tensors $t^\pm_{l_1l_2l_3}$ that are non–vanishing only if $k_1 + k_2 \geq k_3 + 1$ (and relations obtained by permutation of the indices). Hence, the divergence is irrelevant since the couplings are zero due to the vanishing of $t^\pm_{l_1l_2l_3}$.

Moreover, $R_{123}$ also diverges when

$$\Delta_1 + \Delta_V - \Delta_2 + 1 = 0. \quad (2.66)$$

For $C^\pm_\mu$ this gives $k_1 + k_3 = k_2 - 5$, i.e., $\alpha_2 = -5/2$. On the other hand, non–vanishing of $t^\pm_{l_1l_2l_3}$ requires the inequality $k_1 + k_3 \geq k_2 + 1$, so that for the case under consideration $t^\pm_{l_1l_2l_3}$ again vanish. For $A^\pm_\mu$ eq. (2.66) gives $k_1 + k_3 = k_2 - 1$, i.e., $\alpha_2 = -1/2$, and the couplings vanish by the same reason as for $C^\pm_\mu$.

Equation (2.37) has another solution obeying $P^\pm_{k+1}(A^\pm)_{\mu l} = 0$, which we now consider. Perform the shift

$$C^\pm_\mu = C'^\pm_\mu - \frac{k}{k + 2} A^\pm_\mu,$$  \hspace{1cm} (2.67)

where $A^\pm_\mu$ is not arbitrary, rather it solves $P^\pm_{k+1}(A^\pm)_{\mu l} = 0$. Then the linear constraint (2.36) turns into

$$P^\pm_{k+3}(C'^\pm_{\mu l}) + \frac{2}{k + 2} P^\pm_{k+1}(A^\pm)_{\mu l} = P^\pm_{k+3}(C'^\pm_{\mu l}) = 0.$$ 

Thus, $C'^\pm_\mu$ decouple from $A^\pm_\mu$. The fields $A^\pm_\mu$ then correspond to operators with $\Delta_A = k + 2$. The divergence (2.63) now gives $k_1 + k_2 = k_3 + 1$, i.e., $\alpha_3 = 1/2$. The coupling of two scalars with the vector fields $A^\pm_\mu$ corrected by the shift (2.67) (we again integrate the terms in (2.47) and (2.48) proportional to $W^{ss\pm}_{l_1l_2l_3}$ and to $W^{\sigma\sigma\pm}_{l_1l_2l_3}$ by parts and use the equations of motion for $A^\pm_\mu$ and $C'^\pm_\mu$) reads

$$V^{\sigma\sigma A}_{l_1l_2l_3} \equiv V^{\sigma\sigma A}_{l_1l_2l_3} + V^{\sigma\sigma C}_{l_1l_2l_3} + 4k_3 W^{\sigma\sigma}_l_{1l_2l_3}$$  \hspace{1cm} (2.68)

and analogously for $s^r$. The explicit results are given by

$$V^{ssA}_{l_1l_2l_3} = -8(k_3 + 1)(2\alpha_3 - 1)t^\pm_{l_1l_2l_3}, \quad (2.69)$$

$$V^{\sigma\sigma A}_{l_1l_2l_3} = -4(k_3 + 1)(2\alpha_3 - 1)\frac{(k_1 + 1)(k_1 + k_3) + (k_2 + 1)(k_2 + k_3) - 4(k_3 + 1)}{(k_1 + 1)(k_2 + 1)}t^\pm_{l_1l_2l_3} \quad (2.70)$$

and vanish at extremality. The AdS integral is also divergent for (2.66), i.e. for $\alpha_1 = -3/2$. However, in this case $t^\pm_{l_1l_2l_3}$ is zero.

Thus, we have shown that all the cubic couplings we determined vanish in the extremal cases.

\(^6\)The terms in (2.47) and (2.48) proportional to $W^{ss\pm}_{l_1l_2l_3}$ and to $W^{\sigma\sigma\pm}_{l_1l_2l_3}$ vanish after integrating by parts and taking into account the equations of motion for $A^\pm_\mu$ and $C'^\pm_\mu$.\]
2.3 Truncation to the graviton multiplet

The bosonic part of the Lagrangian density for the three-dimensional supergravity based on the $SU(1,1|2)_L \times SU(1,1|2)_R$ supergroup is [20]

$$
\mathcal{L} = R + 2 - \varepsilon^{\mu\nu\rho} (A_\mu^{ij} \partial_\nu A_\rho^{ji} + \frac{2}{3} A_\mu^{ij} A_\nu^{jk} A_\rho^{ki} ) + \varepsilon^{\mu\nu\rho} (A_\mu^{ij} \partial_\nu A_\rho^{ji} + \frac{2}{3} A_\mu^{ij} A_\nu^{jk} A_\rho^{ki} ) ,
$$

where $A_\mu^{ij} = -A_\mu^{ji}$, $A_\mu^{ij} = -A_\mu^{ji}$ are the $SO(3)$ gauge fields and according to our conventions we have set the cosmological constant to $C$ and therefore it is consistent to set the fields $S$ and therefore the $A$ where

$$
\mathcal{L} = R + 2 - \varepsilon^{\mu\nu\rho} A_\mu^{ij} \partial_\nu A_\rho^{ji} + \frac{2}{3} A_\mu^{ij} A_\nu^{jk} A_\rho^{ki} ,
$$

We now demonstrate that the lowest modes of the vector fields $A_\mu^\pm$ obey the first order Chern-Simons equations, although generically the equations of motion are of second order. Thus, we consider the self–interaction of the vector fields $A_\mu^\pm$ and restrict ourselves to the case where two of the three fields, say $A_{\mu 2}^\pm$, $A_{\mu 3}^\pm$ come from the massless graviton multiplet, i.e. their equations of motion are

$$
P_0 (A_\mu^\pm) = \varepsilon_\mu^{\nu\rho} \partial_\nu A_\rho^\pm = 0 \iff \nabla_\mu A_\mu^\pm = \nabla_\nu A_\nu^\pm .
$$

Then the quadratic corrections to the linear constraint (2.36) can be written as

$$
P_{k_1-1}^\pm (A_\mu^\pm)_{\mu I_1} + P_{k_1+3}^\pm (C^\pm)_{\mu I_1} = \pm \varepsilon_\mu^{\nu\rho} A_\nu^\pm A_\rho^\pm \int \varepsilon^{abc} Y_{aI_1}^\pm Y_{bI_2}^\pm Y_{cI_3}^\pm .
$$

Since both vector fields on the r.h.s. transform in the $(1,0)$ of $SU(2)_L \times SU(2)_R$ (or $(0,1)$ respectively), $Y_{bI_2}^\pm Y_{cI_3}^\pm$ transform as

$$(1,0) \otimes (1,0) = (0,0) \oplus (1,0) \oplus (2,0); \quad (0,1) \otimes (0,1) = (0,0) \oplus (0,1) \oplus (0,2),
$$

and therefore the $S^3$ integral is nonzero only if $k_1 = 1$. In this case we have

$$
P_0 (A_\mu^\pm)_{\mu I_1} + P_4^\pm (C^\pm)_{\mu I_1} = \pm \varepsilon_\mu^{\nu\rho} A_\nu^\pm A_\rho^\pm \int \varepsilon^{abc} Y_{aI_1}^\pm Y_{bI_2}^\pm Y_{cI_3}^\pm .
$$

On the other hand, it is easy to show that there is no coupling of $C_\mu^\pm$ with two massless vector fields and therefore it is consistent to set the fields $C_\mu^\pm$ to zero.

Since the $S^3$ integral is completely antisymmetric in $I_1$, $I_2$ and $I_3$ (and the $I_i$ run from 1 to 3) it is proportional to $\varepsilon_{I_1I_2I_3}$ and can be represented as $\mp 2 C_{I_1I_2I_3}^{ij} C_{\mp 1}^{kij}$, where $C_{\mp 1}^{ij} = -C_{\pm 1}^{ij}$. Defining

$$
A_\mu^{ij} = C_\mp 1 A_\mu^{I^+} , \quad A_\mu^{ij} = C_\pm 1 A_\mu^{I^-}
$$

the equation of motion for $A_\mu^{ij}$ reads

$$
\varepsilon_\mu^{\nu\rho} \partial_\nu A_\rho^{ij} = -\varepsilon_\mu^{\nu\rho} A_\nu^{ki} A_\rho^{kj}
$$

and analogously for $A_\mu^{ij}$. These are precisely the equations of motion following from eq.(2.71).
Now we address the issue of the consistency of the KK truncation to the sum of two multiplets, one of them naturally the massless graviton multiplet and a second one containing lowest mode scalar CPOs. Surprisingly, all the cubic couplings we computed involving two fields from the sum of the massless graviton multiplet and the special spin–\(\frac{1}{2}\) multiplet\(^7\) and one field belonging to any other multiplet vanish\(^8\). Recall that the spin–\(\frac{1}{2}\) multiplet contains the scalar modes \(s^r\) with \(k = 1\) and \(\phi^{2r}\) with \(k = 0\), and spin–\(\frac{1}{2}\) states \(\chi^r\) \[^9\]. All the operators in the boundary CFT dual to the gravity fields from the spin–\(\frac{1}{2}\) multiplet are either relevant or marginal. Based on the analysis presented here, one cannot exclude that a consistent truncation to the sum “massless graviton multiplet + special spin–\(\frac{1}{2}\) multiplet” does exist. Of course, only on the basis of the cubic vertices considered here, this issue cannot be decided. It is worthwhile to note that \(s^r\) with \(k = 1\) correspond in the boundary CFT to the scalar CPOs with the lowest conformal dimension.

Another natural example to consider is the lowest level of the spin 1 \(SO(n)\) singlet multiplet, containing \(\sigma\) with \(k = 2\). Here, however, the consistent truncation is not possible. Indeed, the cubic coupling of two CPOs and one symmetric second rank (massive) tensor

\[
V_{\sigma\sigma\phi}^{I_1I_2I_3} \sim (\Sigma + 2)\alpha_3 (k_1^2 + k_2^2 - (k_3 + 1)^2 - 1)a_{I_1I_2I_3}
\]

(2.78)
does not vanish if \(k_1 = k_2 = k_3 = 2\). Note also that the CFT multiplet dual to the \(SO(n)\) singlet discussed above contains irrelevant operators.

In the following two sections we describe how to obtain the cubic action (2.23) from the covariant equations of motion (2.3).

### 3 The linearized equations of motion

In this section we discuss the linearized equations of motion for physical fields. We also keep track of the corrections \(Q\) to the linearized equations though for the sake of clarity we do not give their explicit expressions. They can be found in \[^23\]. The structure of the quadratic corrections will be discussed in section 4.

#### 3.1 Scalar fields

Scalar fields arise from two sources: from the equations of motion for the scalars \(\phi^r\) and the 2–forms \(B^r\), where \(r = 1, \ldots, n\), and from the sphere components of the Einstein equation and the equation for the 2–form \(B\). We start with the first category and obtain

\[
(\nabla_{\mu}^2 + \nabla_{\bar{\mu}}^2)\phi^r + 4(\nabla_{\mu}^2 + \nabla_{\bar{\mu}}^2)U^r = 4\nabla^\mu(\nabla_\mu U^r - X^r_\mu) + Q^r_1,
\]

(3.1)

\(^7\)Generically the multiplets in the vector representation of \(SO(n)\) involve fields with spin 1. However at the lowest level these are absent \[^19\].

\(^8\)For the cubic couplings with vector fields see section 2.2.
from the equation of motion for $\phi^r$,

$$
\nabla_a (X^r_\mu - \nabla_\mu U^r) + \varepsilon_{a}^{\ b c} \nabla_b \phi^r_{\mu c} - \varepsilon_{\mu}^{\ \nu \rho} \nabla_\nu \phi^r_{\rho a} = Q^r_{2a\mu} \tag{3.2}
$$

and

$$
(\nabla^2 + \nabla^2_\alpha) U^r + 2\phi^r = \nabla^\mu (\nabla_\mu U^r - X^r_\mu) + Q^r_3 \tag{3.3}
$$

from the ($\mu\nu\alpha$) and ($abc$) components of the antiself–duality equation for $B^r$. $Q^r_i$ denote higher order corrections to the linearized equations.

First we expand all the fields in spherical harmonics on $S^3$. Equation (3.2) contains both the transversal terms proportional to $Y^r_{I3\pm}$ and the longitudinal ones proportional to $\nabla_a Y^r_{I1}$. Projecting (3.2) on the longitudinal part by multiplying it with $\nabla_a$ one gets a linear constraint. One then uses this constraint to diagonalize the system (3.1) and (3.3). The resulting equations are

$$
4(k + 1) \left(\nabla^2_\mu - k(k - 2)\right) s_{rI_1} = Q^r_{sI_1},
$$

$$
4(k + 1) \left(\nabla^2_\mu - (k + 2)(k + 4)\right) t_{rI_1} = Q^r_{tI_1}, \tag{3.4}
$$

where

$$
Q^r_{sI_1} = Q^r_{1I_1} + 2 \frac{\nabla^\mu \nabla^\alpha}{k} Q^r_{2a\mu} + 2k Q^r_{3I_1}, \quad Q^r_{tI_1} = Q^r_{1I_1} - 2 \frac{\nabla^\mu \nabla^\alpha}{k + 2} Q^r_{2a\mu} - 2(k + 2) Q^r_{3I_1} \tag{3.5}
$$

and

$$
S_{rI_1} = \frac{1}{4(k + 1)} (\phi^r_{I_1} + 2(k + 2) U^r_{I_1}), \tag{3.6}
$$

for the chiral primary $s^r$ and

$$
T_{rI_1} = \frac{1}{4(k + 1)} (\phi^r_{I_1} - 2k U^r_{I_1}) \tag{3.7}
$$

for the scalar $t^r$ (c.f. (2.17)).

For the second category we obtain

$$
- \frac{1}{6} (\nabla^2_\mu + \nabla^2_\alpha - 8) h^b_b - 4 \nabla^2_\alpha - \frac{1}{6} \nabla^2 (h^\mu_\mu + \frac{1}{3} h_\mu^\mu) = Q_1 \tag{3.8}
$$

and

$$
- \frac{1}{2} (\nabla^2_\mu + \nabla^2_\alpha - 2) h_{(ab)} - \frac{1}{2} \nabla_{(a} \nabla_{b)} (h^\mu_\mu + \frac{1}{3} h^\mu_\mu) + \nabla_{(a} \nabla^\mu h_{b)\mu} = Q_{2(abc)} \tag{3.9}
$$

from the ($ab$) components of the Einstein equation together with

$$
\nabla_a (X_\mu + \nabla_\mu U) - \varepsilon_a^{\ b c} \nabla_b \phi^r_{\mu c} - \varepsilon_\mu^{\ \nu \rho} \nabla_\nu \phi^r_{\rho a} - h_{a\mu} = Q_{3a\mu} \tag{3.10}
$$
and
\[(\nabla^2_\mu + \nabla^2_a) U - \frac{2}{3} h_b^b + \frac{1}{2} (h^\mu_b + \frac{1}{3} h_\mu_b) - \nabla^\mu (X_\mu + \nabla_\mu U) = Q_4\] (3.11)
from the \(\mu\nu a\) and \(abc\) components of the self-duality equation for \(B\).

Again, expanding the fields in spherical harmonics, solving the longitudinal constraints (3.9) (i.e. multiplying (3.9) by \(\nabla^a \nabla^b\)) and (3.10) and diagonalizing the resulting system of equations we obtain
\[
2(k + 1) \left( \nabla_\mu^2 - k(k - 2) \right) \sigma_I^I = Q_\sigma^I,
\]
\[
2(k + 1) \left( \nabla^2_\mu - (k + 2)(k + 4) \right) \tau_I^I = Q_\tau^I,
\]
where
\[
Q_\sigma^I = Q_1^I + \frac{1}{2k(k - 1)} \nabla^a \nabla^b Q_{2(ab)}^{I_1} - \frac{1}{k} \nabla^\mu \nabla^a Q_{3a\mu}^{I_1} + (k + 2) Q_4^{I_1},
\]
\[
Q_\tau^I = -Q_1^I - \frac{1}{2(k + 2)(k + 3)} \nabla^a \nabla^b Q_{2(ab)}^{I_1} - \frac{1}{k + 2} \nabla^\mu \nabla^a Q_{3a\mu}^{I_1} + k Q_4^{I_1}
\]
and the chiral primary field \(\sigma_I\) and the scalar \(\tau_I\) are defined as (c.f. (2.18)):
\[
\sigma_I = \frac{1}{12(k + 1)} (6(k + 2) U_I - \pi_I),
\]
\[
\tau_I = \frac{1}{12(k + 1)} (6k U_I + \pi_I).
\]

Finally, the equations of motion for the scalars \(\phi^{I_5^\pm}\) originate from the transverse traceless part of eq.(3.9)
\[
-\frac{1}{2} \left( \nabla_\mu^2 - \Delta \right) \phi^{I_5^\pm} = Q_2^{I_5^\pm}.
\]
(3.17)
We summarize the results for the relevant scalar modes (c.f. Table 1) in Fig. 1:
The lowest mode for $\sigma_{I_1}$ is actually $k = 2$, since the $k = 1$ mode can be gauged away by residual shift transformations [19]. This can also be seen from the action (2.23), where this mode is simply absent.

### 3.2 Vector fields

The equations of motion for the vector fields on $AdS_3$ are obtained from the (anti)self–duality equation for the three–form field strength with indices $(\mu\nu\alpha)$ and from the $(\mu\alpha)$ component of the Einstein equation.

We start with the vector fields $Z^{r\pm}_{\mu}$ transforming in the fundamental representation of $SO(n)$. The transverse part of (3.2) is

$$P_{k+1}^{\pm}(Z^{r\pm})_{\mu I_3} = -Q_{2\mu I_3}^{\pm},$$

and it automatically implies that the vector fields $Z^{r\pm}_{\mu}$ are transverse at the linearized order. Clearly, the linearized equation for $Z^{r}_{\mu}$ follows from the action (2.23).

Consider the vector fields being mixtures of the spin one components of the graviton and the 2–form potential $B$. Their equations of motion have their origin in the transverse parts of the $(\mu\alpha)$ component of the Einstein equation and the $(\mu\nu\alpha)$ component of the self–duality equation and we find

$$-\frac{1}{4}\text{Max}(h)^{\pm}_{\mu I_3} + \frac{1}{4}(\Delta + 1)h^{\pm}_{\mu I_3} - P_{k+1}^{\pm}(Z^{r\pm})_{\mu I_3} = \frac{Q^{\pm}_{3\mu I_3}}{4},$$

and

$$P_{k+1}^{\pm}(Z^{r\pm})_{\mu I_3} + h^{\pm}_{\mu I_3} = -Q_{3\mu I_3}^{\pm},$$

where

$$\text{Max}(V)_{\mu} = \nabla^{\nu} V_{\mu} - \nabla^{\nu} \nabla_{\mu} V_{\nu},$$

which satisfies $(\ast \nabla)^2 = \text{Max}$.

Equation (3.19) can be factorized as follows

$$\left( P_{k+1}^{\pm} \right)_{\mu}^{\nu} \left( P_{k+1}^{\pm}(h^{\pm})_{\nu I_3} + 4Z^{r\pm}_{\nu I_3} \right) = -4Q_{3\mu I_3}^{\pm},$$

Therefore, one set of solutions is found by diagonalizing the system of first order equations

$$P_{k+1}^{\pm}(h^{\pm})_{\mu I_3} + 4Z^{r\pm}_{\mu I_3} = 0,$$

$$P_{k+1}^{\pm}(Z^{r\pm})_{\mu I_3} + h^{\pm}_{\mu I_3} = 0.$$
satisfy the following equations

\[ P_{k-1}^\pm (A^\pm)_{\mu I_3} = 0, \quad P_{k+3}^\pm (C^\pm)_{\mu I_3} = 0 \]  

(3.26)

at the linearized order.

On the other hand, solving the linear equation (3.20) and substituting into (3.19), introducing the canonical fields \( A^\pm_\mu \) and \( C^\pm_\mu \) and using \( [P_m, P_{m'}] = 0 \), one obtains

\[ P^\pm_{k+1} P^\pm_{k+3} (C^\pm)_{\mu I_3} = -4 Q^\pm_{3\mu I_3} \]  

(3.27)

\[ P^\pm_{k+1} P^\pm_{k+3} (C^\pm)_{\mu I_3} = -4 Q^\pm_{3\mu I_3} + 2 P^\pm_{k+1} (Q^\pm)_{3\mu I_3}. \]  

(3.28)

It is straightforward to write down an action that leads to the second order equations for the fields \( A^\pm_\mu \) and \( C^\pm_\mu \) but this is not what we need, since these fields are related by the linear constraint (3.24), that in terms of the canonical fields reads as

\[ P^\pm_{k-1} (A^\pm)_{\mu I_3} + P^\pm_{k+3} (C^\pm)_{\mu I_3} = \mp 4 Q^\pm_{3\mu I_3}. \]  

(3.29)

By using equation (3.20), (3.19) can be reduced to the form

\[ \text{Max}(h^\pm)_{\mu I_3} - (k - 1)(k + 3)h^\pm_{\mu I_3} - 8(h^\pm_{\mu I_3} \pm (k + 1)Z^\pm_{\mu I_3}) = 4(Q^\pm_{3\mu I_3} - Q^\pm_{5\mu I_3}). \]  

(3.30)

The equations (3.20) and (3.30) can be derived from the Lagrangian

\[ L^\pm = -\frac{1}{4} F_{\mu \nu} (h^\pm) F^{\mu \nu} (h^\pm) - \frac{1}{2} (k - 1)(k + 3) h^\pm_{\mu I_3} h^\pm_{\mu I_3} - 4(h^\pm_{\mu I_3} \pm (k + 1)Z^\pm_{\mu I_3})^2 + 4(k + 1) \epsilon^{\mu \nu \rho} Z^\pm_{\mu I_3} \partial_\nu Z^\pm_{\rho}. \]  

(3.31)

After substituting the canonical fields this Lagrangian turns into the one given in section 2, equation (2.31).

The results for the relevant vector modes (c.f. Table 1) are summarized in Fig. 2:
The Yang–Mills states $A_{\mu k=1}^{I_3 \pm}$ transform in the adjoint representation of $SU(2)_L \times SU(2)_R$; they are pure gauge and are components of the massless graviton multiplet, which also contains the non–propagating graviton and 4 gravitini.

### 3.3 Symmetric tensor fields of second rank

In principle, to find the equation of motion for the massive gravitons on $AdS_3$ one has to consider the Einstein equation (2.3) not only with indices ($\mu \nu$), but also with the indices ($\mu a$) and ($a b$). The reason for this is that the equations for $\nabla^\mu \varphi_\mu \nu$ and $\varphi_\mu \equiv \varphi_\mu^I$ are constraints and do not follow from (2.3) if one considers only the ($\mu \nu$) components. Moreover, the Einstein equation involves both the gravitons and the scalar fields already at the linearized level and, therefore, the procedure of constructing the quadratic Lagrangian becomes highly non–trivial. However, analogously to the case of type IIB supergravity on $AdS_5 \times S^5$ [23], [15], we can replace the Einstein equation (2.3) by a new equivalent equation from which the true equation for massive gravitons follows from the ($\mu \nu$) components. “Equivalent” means that the original and the new equations coincide on–shell, i.e., when one takes into account the (anti)self–duality equation for the field strength $H$.

To find the new equation we will use the shift of the graviton given in equations (2.19), (2.20) and (2.21).

At linear order we have \[ R_{\mu I}^{(1)} = R_{\mu I}^{(1)}(\varphi) + \frac{8}{k+1} \nabla_\mu \nabla_\nu (\sigma_I - \tau_I) + 4 g_{\mu I} \left( \frac{k^2(k-1)}{k+1} \sigma_I + \frac{(k+2)^2(k+3)}{k+1} \tau_I \right) \]

\[ + g_{\mu I} \left( - \frac{k(k-1)}{k+1} (\nabla_\rho^2 - m^2_\sigma) \sigma_I + \frac{(k+2)(k+3)}{k+1} (\nabla_\rho^2 - m^2_\tau) \tau_I \right) , \] \hspace{1cm} (3.32)

where \[ R_{\mu I}^{(1)}(\varphi) \equiv - \frac{1}{2} (\nabla_\rho^2 - \Delta + 6) \varphi_{\mu I} - \nabla_\mu \nabla_\rho \varphi_{\rho I} - \nabla_\nu \nabla_\rho \varphi_{\rho I} + \nabla_\mu \nabla_\nu \varphi_I + g_{\mu \nu} \varphi_I . \] \hspace{1cm} (3.33)

Introducing the notation $H_M H_N \equiv H_{MPQ} H_{NPQ}$ we obtain \[ (H_M H_N)^{(1)} = 2 g_{\mu \nu} \varphi_I - 2 \varphi_{\mu \nu I} + \frac{8}{k+1} \nabla_\mu \nabla_\nu (\sigma_I - \tau_I) + 4 g_{\mu I} \left( \frac{k^2(k-1)}{k+1} \sigma_I \right. \]

\[ \frac{(k+2)^2(k+3)}{k+1} \tau_I ) + 4 g_{\mu I} \left( k \frac{k-1}{k+1} (\nabla_\rho^2 - m^2_\sigma) \sigma_I + \frac{k+3}{k+1} (\nabla_\rho^2 - m^2_\tau) \tau_I \right) . \] \hspace{1cm} (3.34)

The first correction to the total curvature is found to be \[ R^{(1)}_I = \nabla^\mu \nabla^\nu \varphi_{I \mu \nu} - (\nabla_\mu^2 - \Delta - 2) \varphi_I \]

\[ - 2 \left( \frac{k(k-1)}{k+1} (\nabla_\mu^2 - m^2_\sigma) \sigma_I - \frac{(k+2)(k+3)}{k+1} (\nabla_\mu^2 - m^2_\tau) \tau_I \right) . \] \hspace{1cm} (3.35)

---

9 We loosely refer to symmetric tensor fields coming from the $AdS_3$ components of the metric as to massive gravitons.

10 To simplify the notation throughout this section we use $I \equiv I_1$. 

19
Now we consider the combination \( (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - H_\mu H_\nu)^{(1)} \), substitute the previous results and get
\[
(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - H_\mu H_\nu)^{(1)} = \mathcal{E}_{\mu\nu I}(\varphi) - 2g_{\mu\nu}\varphi_I
- \frac{4}{k+1}g_{\mu\nu}((k-1)(\nabla^2 - m_\sigma^2)\sigma_I + (k+3)(\nabla^2 - m_\tau^2)\tau_I).
\]
(3.36)

Here
\[
\mathcal{E}_{\mu\nu}(\varphi) \equiv -\frac{1}{2}((\nabla^2 - \Delta + 2)\varphi_{\mu\nu} - \nabla_\mu\nabla^\rho\varphi_{\rho\nu} - \nabla_\nu\nabla^\rho\varphi_{\rho\mu} + g_{\mu\nu}\nabla^\rho\nabla_\lambda\varphi_{\rho\lambda}
+ \nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}(\nabla^2 - \Delta)\varphi).
\]
(3.37)

Next we use
\[
(H^2)^{(1)}_I = 6\varphi_I + 12\left(\frac{k}{k+1}(\nabla_\mu^2 - m_\sigma^2)\sigma_I + \frac{k+3}{k+1}(\nabla_\mu^2 - m_\tau^2)\tau_I \right)
\]
(3.38)
and find that
\[
\left( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - (H_\mu H_\nu - \frac{1}{3}g_{\mu\nu}H^2) \right)^{(1)} = \mathcal{E}_{\mu\nu}(\varphi)
\]
(3.39)
holds off–shell, i.e. without using the linearized equations of motion for \( \sigma_I \) and \( \tau_I \). Therefore the proper equation for the massive gravitons is (c.f. (2.3))
\[
R_{MN} - \frac{1}{2}g_{MN}R = H^I_M H^I_N - \frac{1}{3}g_{MN}H^I_{M_1 M_2 M_3}H^I M_1 M_2 M_3
+ 2P^I_{ir} P^I N - g_{MN}P^I_{ir} P^I L.
\]
(3.40)

At the linearized level the equation for massive gravitons is \( \mathcal{E}_{\mu\nu I}(\varphi) \equiv \mathcal{E}_{\mu\nu}(\varphi^I_{\rho\lambda}) = 0 \). For \( k = 0 \) this reduces to the known equation for (massless) gravitons. Note that the first line of (3.40) cannot be obtained from
\[
S \sim \int \sqrt{-g}(R - \frac{1}{3}H^2)
\]
(3.41)
since this would give a coefficient 1/6 in front of \( H^2 \). However, recalling that the field strengths are (anti)self–dual we see that on–shell the term \( H^2 \) is zero and, therefore, its coefficient in the Einstein equation remains arbitrary and may be fixed to any desired value. Of course this just reflects the fact that there is no simple covariant action when (anti)self–dual field strengths are involved.

Taking the trace of \( \mathcal{E}_{\mu\nu}(\varphi) \) one obtains
\[
\mathcal{E}^\mu_{\mu}(\varphi) = -\frac{1}{2}\nabla_\mu(\nabla_\nu\varphi^{\mu\nu} - \nabla^{\mu}\varphi) - (\Delta + 1)\varphi.
\]
(3.42)

On the other hand, the divergence of \( \mathcal{E}_{\mu\nu}(\varphi) \) is
\[
\nabla^\mu\mathcal{E}_{\mu\nu}(\varphi) = \frac{\Delta}{2}(\nabla^\mu\varphi_{\mu\nu} - \nabla_\nu\varphi).
\]
(3.43)

Thus for \( k \neq 0 \) on–shell massive gravitons are transverse and traceless at the linear order.
The result for the tensor modes (c.f. Table 1) is summarized in Fig. 3:

![Mass spectrum of symmetric tensors.](image)

Fig. 3: Mass spectrum of symmetric tensors.

4 Cubic couplings

The cubic couplings involving at least two chiral primaries were listed in section 2.1. In this section we sketch their derivation\(^\text{11}\). As usual, the quadratic corrections involve higher derivative terms and are in general non–Lagrangian. To make the couplings Lagrangian and to remove the higher derivative terms one performs nonlinear field redefinitions \(^\text{13}\). To determine the cubic couplings it is sufficient to derive them from the equation of motion of one of the fields involved. However, the problem of finding 4–point functions requires to consider all equations of motion. The reason is that the field redefinitions mentioned above will induce contributions to quartic couplings \(^\text{15}\).

4.1 Cubic couplings with scalar fields

The cubic couplings of scalar fields corresponding to CPOs were already obtained in \(^\text{8}\). As an example we just consider the self–interaction of \(\sigma\). Keeping only the quadratic contributions one finds the following structure\(^\text{12}\):

\[
(\nabla^2_\mu - m_\sigma^2)\sigma_1 = A_{123}\sigma_2\sigma_3 + B_{123}\nabla_\mu\sigma_2\nabla^\mu\sigma_3 + C_{123}\nabla_\mu\nabla_\nu\sigma_2\nabla^\mu\nabla^\nu\sigma_3.
\]

(4.1)

To remove higher derivative terms one should make the field redefinition

\[
\sigma_1 \to \sigma_1 + J_{123}\sigma_2\sigma_3 + L_{123}\nabla_\mu\sigma_2\nabla^\mu\sigma_3.
\]

(4.2)

\(^{11}\)To simplify the notation we drop the index \(I\) that specifies a basis of an irreducible representation and denote \(\sigma^I\) as \(\sigma_1\) and similarly for the other fields.

\(^{12}\)We do not present the explicit values of the coefficients here and below because they are not very instructive. They are explicitly given in \(^\text{21}\).
where

\[ 2L_{123} = C_{123}, \quad 2J_{123} + L_{123}(m_2^2 + m_3^2 - m_1^2 - 4) = B_{123}. \]  

(4.3)

Then (4.1) takes the form

\[ (\nabla^2_{\mu} - m_\sigma^2) \sigma_1 = -\frac{3}{k_1^2} V_{123}^{\sigma\sigma\sigma} \sigma_2 \sigma_3 + \cdots, \]

(4.4)

where \( \cdots \) denote cubic terms induced by the field redefinitions, \( \kappa_\sigma = 16k(k-1) \) and \( V_{123}^{\sigma\sigma\sigma} \) as in (2.42).

### 4.2 Cubic couplings with vector fields

We begin with the coupling \( s^r \sigma Z^{r\pm}_{\mu} \) and consider only the quadratic corrections to the equations of motion for \( Z^{r\pm}_{\mu} \). One finds (c.f. (3.18))

\[ P_{m_{z2}}^{r}(Z_{r\pm})_{3\mu} = A_{123} \nabla_{\mu}(\sigma_1 s^r_2) + B_{123} \sigma_1 \nabla_{\mu} s^r_2 + C_{123} \nabla_{\mu} \nabla_{\nu} \sigma_1 \nabla_{\nu} s^r_2. \]  

(4.5)

To simplify the notation, here and below we suppress the label \( \pm \) on the coefficients describing the quadratic corrections. After performing the field redefinition

\[ Z_{r\pm}^{3\mu} \rightarrow Z_{r\pm}^{3\mu} \pm \nabla_{\mu} \Lambda_3^r + L_{123} P_{k_{3+1}}^{r}(\sigma_1 \nabla s^r_2), \]

(4.6)

where

\[ 2L_{123} = -C_{123}, \quad (k_3 + 1) \Lambda_3^r = (m_2^2 L_{123} - A_{123}) \sigma_1 s^r_2 + L_{123} \nabla_{\mu} \sigma_1 \nabla_{\nu} s^r_2. \]  

(4.7)

equation (4.5) becomes

\[ P_{m_{z2}}^{r}(Z_{r\pm})_{3\mu} = \frac{1}{8(k_3 + 1)} V_{123}^{\sigma Z_{r\pm}} \sigma_1 \nabla_{\mu} s^r_2. \]

(4.8)

For the vector fields \( A^\pm_\mu \) and \( C^\pm_\mu \) we restrict ourselves to the interaction with \( \sigma \); the case of \( s^r \) is analogous.

Consider the quadratic corrections to the equations of motion for \( A^\pm_\mu \). Keeping only terms quadratic in \( \sigma \) we find the following structure:

\[ P_{k_{3+1}}^{r} P_{k_{3-1}}^{r}(A^\pm)_{3\mu} = \nabla_{\mu} \Lambda_3 + (V1)_{123} \nabla_{\mu} \sigma_1 \sigma_2 + (V2)_{123} \nabla_{\nu} \nabla_{\mu} \sigma_1 \nabla_\nu \sigma_2 + (V3)_{123} \nabla_{\nu} \nabla_{\lambda} \sigma_1 \nabla_\nu \nabla_\lambda \sigma_2 \]

\[ \pm \varepsilon_\mu^{\nu\lambda} \nabla_{\nu} ((W0)_{123} \nabla_{\lambda} \sigma_1 \sigma_2 + (W2)_{123} \nabla_{\lambda} \sigma_1 \nabla_\rho \sigma_2), \]

(4.9)

where

\[ \Lambda_3 = (\Lambda 0)_{123} \sigma_1 \sigma_2 + (\Lambda 2)_{123} \nabla_{\mu} \sigma_1 \nabla^\mu \sigma_2. \]

(4.10)
After the shift

\[ A_{3\mu}^\pm \rightarrow A_{3\mu}^\pm + \nabla_\mu \bar{\Lambda}_3 + J_{123} \nabla_\mu \sigma_1 \sigma_2 + L_{123} \nabla_\nu \nabla_\mu \sigma_1 \sigma_2^\prime \pm G_{123} \varepsilon_\mu^\nu \varepsilon_\nu^\lambda \nabla_\lambda (\nabla_\chi \sigma_1 \sigma_2) \]  

(4.11)

with

\[ 2L_{123} = (V3)_{123}, \quad 2G_{123} = (V3)_{123} + (W2)_{123}, \]

\[ 2J_{123} = (V2)_{123} + 4G_{123} - L_{123}(m_1^2 + m_2^2 - 6 - (k + 1)(k - 1)), \]  

(4.12)

\[-(k_3 + 1)(k_3 - 1)\bar{\Lambda}_3 = \Lambda_3 + \frac{1}{2}(m_1^2 - m_2^2)((J_{123} - L_{123} - 2G_{123})\sigma_1 \sigma_2 + L_{123} \nabla_\mu \sigma_1 \nabla_\mu \sigma_2) \]

we can represent (4.9) as

\[ \frac{1}{2} P_{k_3+1}^\pm P_{k_3-1}^\pm (A^\pm)_3 \mu = V_{123}^\sigma A_3^\pm \nabla_\mu \sigma_1 \sigma_2 \pm W_{123}^\sigma A_3^\pm \nabla_\mu (\nabla_\sigma_1 \sigma_2)_\mu. \]  

(4.13)

Analogously one finds for \( C_\mu^\pm \)

\[ \frac{1}{2} P_{k_3+1}^\pm P_{k_3+3}^\pm (C^\pm)_3 \mu = V_{123}^\sigma C_3^\pm \nabla_\mu \sigma_1 \sigma_2 \pm W_{123}^\sigma C_3^\pm \nabla_\mu (\nabla_\sigma_1 \sigma_2)_\mu. \]  

(4.14)

The coefficients \( W_{123}^\sigma A^\pm \) and \( W_{123}^\sigma C^\pm \) are

\[ W_{123}^\sigma A^\pm = \frac{(\Sigma + 1)}{(k_1 + 1)(k_2 + 1)}((k_1 - 1)(k_1 - k_3) + (k_2 - 1)(k_2 - k_3))t_{123}^\pm, \]  

(4.15)

\[ W_{123}^\sigma C^\pm = \frac{(2\alpha_3 - 1)}{(k_1 + 1)(k_2 + 1)}((k_1 + 1)(k_1 + k_3) + (k_2 + 1)(k_2 + k_3) - 4(k_3 + 1))t_{123}^\pm. \]  

(4.16)

From the equation of motion for \( \sigma \) one gets

\[ (\nabla_\mu - m_\sigma^2)\sigma_1 = A_{123}^{A_\mu} \nabla_\mu \sigma_2 A_3^{3\mu} + B_{123}^{A_\mu} \nabla_\mu \nabla_\nu \sigma_2 A_3^{3\nu} + A_{123}^{C_\mu} \nabla_\mu \sigma_2 C_3^{3\mu} + B_{123}^{C_\mu} \nabla_\mu \nabla_\nu \sigma_2 C_3^{3\nu} \]

\[ + D_{123} \nabla_\mu \sigma_2 (P_{k_3+3}^\pm (C^\pm)_3 \mu - P_{k_3-1}^\pm (A^\pm)_3 \mu), \]  

(4.17)

where \( A_{123}^{A_\mu}, B_{123}^{A_\mu}, A_{123}^{C_\mu}, B_{123}^{C_\mu} \) and \( D_{123} \) are known coefficients [25].

Performing the field redefinition

\[ \sigma_1 \rightarrow \sigma_1 + \frac{1}{2} B_{123}^{A_\mu} \nabla_\mu \sigma_2 A_3^{3\mu} + \frac{1}{2} B_{123}^{C_\mu} \nabla_\mu \sigma_2 C_3^{3\mu} \]  

(4.18)

we represent the result in the following form:

\[ (\nabla_\mu - m_\sigma^2)\sigma_1 = -\frac{2}{\kappa_1^{\sigma}} \left( V_{123}^\sigma A_{123}^{A_\mu} \nabla_\mu \sigma_2 A_3^{3\mu} + V_{123}^\sigma C_{123}^{C_\mu} \nabla_\mu \sigma_2 C_3^{3\mu} \right) \]

\[ + \nabla_\mu \sigma_2 \left( (W_{123}^{A_\mu} + \alpha_3^{A_\mu})P_{k_3-1}^\pm (A^\pm)_3 \mu + (W_{123}^{C_\mu} + \alpha_3^{C_\mu})P_{k_3+3}^\pm (C^\pm)_3 \mu \right) \]

\[ + ((d\Omega)_{123} + \frac{2}{\kappa_1^{\sigma}} \alpha_{123}^{A_\mu}) \nabla_\mu \sigma_2 (P_{k_3-1}^\pm (A^\pm)_3 \mu + P_{k_3+3}^\pm (C^\pm)_3 \mu). \]  

(4.19)
Here
\[ B_{123}^\pm + D_{123}^\pm = \frac{2}{\kappa_1^2} W_{123}^{\sigma A^\pm} - (d\Omega)_{123}, \quad B_{123}^\pm + D_{123}^\pm = -\frac{2}{\kappa_1^2} W_{123}^{\sigma C^\pm} + (d\Omega)_{123} \tag{4.20} \]
and \( \alpha_{123}^\pm \) are yet to be determined \( SO(4) \) tensors that are antisymmetric in 1 and 2. The last line in (4.19) is proportional to the linear constraint (2.33) and, therefore, only contributes to the next order. The tensors \( \alpha_{123}^\pm \) are fixed as follows. Equation (4.19) can be derived from
\[ \mathcal{L}_3^\sigma (A_\mu^\pm, C_\mu^\pm) = V_{123}^{\sigma A^\pm} \sigma_{1i} \nabla^\mu \sigma_{12} A_{\mu i}^\pm + V_{123}^{\sigma C^\pm} \sigma_{1i} \nabla^\mu \sigma_{12} C_{\mu i}^\pm \]
\[ \pm (W_{123}^{\sigma A^\pm} + \alpha_{123}^\pm) P_{k3-1}^\pm \sigma_{1i} \nabla^\mu \sigma_{12} A_{\mu i}^\pm \pm (W_{123}^{\sigma C^\pm} + \alpha_{123}^\pm) P_{k3+3} \sigma_{1i} \nabla^\mu \sigma_{12} C_{\mu i}^\pm. \tag{4.21} \]

Varying this Lagrangian with respect to \( A_\mu^\pm \) and \( C_\mu^\pm \) we can write down the quadratic corrections to the equations of motion for the vector fields and compare them with our results (4.13) and (4.14). Then we find that
\[ \alpha_{123}^\pm = -\frac{1}{2} (W_{123}^{\sigma A^\pm} + W_{123}^{\sigma C^\pm}). \tag{4.22} \]
Substituting the shifts of \( A_\mu \) and \( C_\mu \) we finally represent (4.19) as
\[ \kappa_1^\sigma (\nabla^2 - m_\sigma^2) \sigma_1 = -V_{123}^{\sigma A^\pm} (2\nabla^\mu \sigma_2 A_{3\mu}^\pm + \sigma_2 \nabla^\mu A_{3\mu}^\pm) \pm W_{123}^{\sigma A^\pm} (2\nabla^\mu \sigma_2 P_{k3-1}^\pm (A^\pm)_{3\mu} \pm (k_3 - 1) \sigma_2 \nabla^\mu A_{3\mu}^\pm) \]
\[ - V_{123}^{\sigma C^\pm} (2\nabla^\mu \sigma_2 C_{3\mu}^\pm + \sigma_2 \nabla^\mu C_{3\mu}^\pm) \pm W_{123}^{\sigma C^\pm} (2\nabla^\mu \sigma_2 P_{k3+3} (C^\pm)_{3\mu} \pm (k_3 + 3) \sigma_2 \nabla^\mu C_{3\mu}^\pm) \]
\[ + \text{cubic}, \tag{4.23} \]
where
\[ W_{123}^{\sigma A^\pm} = \frac{1}{2} (W_{123}^{\sigma A^\pm} - W_{123}^{\sigma C^\pm}). \tag{4.24} \]
Note that the terms on the r.h.s. quadratic in fields follow from action (2.23), while “cubic” denotes the cubic terms relevant only for determining the quartic couplings.

### 4.3 Cubic couplings with massive gravitons

Here the most complicated part is to derive the interaction of \( \sigma \) with massive gravitons from the equation of motion for \( \varphi_{\mu \nu} \), because \( \sigma \) is constructed by using the metric itself. Since the analysis of the interaction of \( \varphi_{\mu \nu} \) with \( s^r \) (although much simpler) proceeds along the same lines we will omit it. After careful computation one finds the following structure:
\[ \mathcal{E}_{1\mu
u}(\varphi) = (A2)_{123} \nabla_\mu \sigma_2 \nabla_\nu \sigma_3 + (A4)_{123} \nabla_\nu \sigma_2 \nabla_\mu \sigma_3 + (A6)_{123} \nabla_\nu \nabla_\lambda \sigma_2 \nabla^\mu \sigma_2 \nabla^\nu \nabla^\lambda \nabla_\nu \sigma_3 + (B2)_{123} \nabla_\nu (\sigma_2 \nabla^\nu \sigma_3) + (B4)_{123} \nabla_\nu (\nabla_\rho \sigma_2 \nabla^\rho \nabla_\nu \sigma_3) + g_\mu \nu C_1. \tag{4.25} \]
where
\[ C_1 = (C0)_{123}\sigma_2\sigma_3 + (C2)_{123}\nabla_\mu\sigma_2\nabla^\mu\sigma_3 + (C4)_{123}\nabla_\mu\nabla_\nu\sigma_2\nabla^\mu\nabla^\nu\sigma_3 - \frac{1}{2}(A6)_{123}\nabla_\mu\nabla_\rho\sigma_2\nabla^\mu\nabla^\nu\nabla^\rho\sigma_3. \] (4.26)

Performing the shift of the graviton
\[ \varphi_{1\mu\nu} = \varphi'_{1\mu\nu} + \nabla_\mu\xi_{1\nu} + \nabla_\nu\xi_{1\mu} + g_{\mu\nu}\eta_1 + K_{123}\nabla_\mu\sigma_2\nabla_\nu\sigma_3 + L_{123}\nabla_\rho\nabla_\nu\sigma_2\nabla^\rho\nabla^\nu\sigma_3, \] (4.27)

where
\[ L_{123} = -\bar{(A6)}_{123}, \quad K_{123} = -(A4)_{123} + \frac{1}{2}(A6)_{123}(m_2^2 + m_3^2 - \Delta_1 - 10), \quad \frac{\Delta_1}{2}\xi_{1\mu} = \frac{1}{4}\nabla_\mu\eta_1 + M_{123}\sigma_2\nabla_\mu\sigma_3 + N_{123}\nabla_\nu\sigma_2\nabla^\nu\nabla_\mu\sigma_3 \] (4.28)

and \( M_{123}, N_{123} \) are given by
\[ M_{123} = (B2)_{123} - \frac{3}{2}m_2^2(A6)_{123} - \frac{1}{2}m_2^2K_{123}, \quad N_{123} = (B4)_{123} + \frac{1}{2}(m_2^2 - 1)(A6)_{123} \] (4.29)

equation (4.25) takes the form
\[ \mathcal{E}_{1\mu\nu}(\varphi') = V^{\sigma\varphi}_{123}\nabla_\mu\sigma_2\nabla_\nu\sigma_3 + g_{\mu\nu}\tilde{C}_1. \] (4.30)

Here
\[ \tilde{C}_1 = (\bar{C}0)_{123}\sigma_2\sigma_3 + (\bar{C}2)_{123}\nabla_\mu\sigma_2\nabla^\mu\sigma_3 + (\Delta_1 + 1)\eta_1. \] (4.31)

Thus only \( \eta \) has not been fixed yet. In fact, a change of \( \eta \) (with a simultaneous change of \( \xi \) according to (4.28)) amounts only to a change of the interaction of the trace of the massive gravitons with the chiral primaries. In particular, it is possible to choose \( \eta \) in such a way that only the traceless part of \( \varphi_{\mu\nu} \) interacts with \( \sigma \). But as was pointed out in [15], for the case of AdS\(_5\) this choice leads to the appearance of quartic couplings with six derivatives, which are absent only if \( \eta \) is chosen such that \( \varphi'_{1} = (T0)_{123}\sigma_2\sigma_3 \). We expect that this will also be the case for AdS\(_3\) and therefore follow this approach.

Taking the trace and divergence of (4.30) and representing
\[ \eta_1 = A_{123}\sigma_2\sigma_3 + B_{123}\nabla_\mu\sigma_2\nabla^\mu\sigma_3, \] (4.32)

where
\[ (\Delta_1 + 1)A_{123} = -(\bar{C}0)_{123} - \frac{1}{4}(m_2^2 + m_3^2 - \Delta_1)V^{\sigma\varphi}_{123}, \quad (\Delta_1 + 1)B_{123} = -(\bar{C}2)_{123} - \frac{1}{2}V^{\sigma\varphi}_{123}, \] (4.33)

we find that
\[ \varphi'_{1} = \frac{V^{\sigma\varphi}_{123}}{4\Delta_1(\Delta_1 + 1)}\left((m_2^2 - m_3^2)^2 + 2\Delta_1(m_2^2 - m_3^2) - 3\Delta_1^2\right)\sigma_2\sigma_3. \] (4.34)
Then the final result for the interaction of $\varphi_{\mu\nu}$ with $\sigma$ has the nice form

$$E_{1\mu\nu}(\varphi') = V^{\sigma\sigma}_{123} (\nabla_{\mu}\sigma_2 \nabla_{\nu}\sigma_3 - \frac{1}{2} g_{\mu\nu} (\nabla_{\rho}\sigma_2 \nabla^\rho\sigma_3 + \frac{1}{2} (m_2^2 + m_3^2 - \Delta_1) \sigma_2 \sigma_3)), \quad (4.35)$$

which is the natural generalization of the interaction of the massless graviton with scalar fields.

On the other hand from the equation of motion for $\sigma$ one obtains

$$\left(\nabla^2_{\mu} - m^2_{\sigma}\right) \sigma_1 = \frac{2}{\kappa_1} V^{\sigma\sigma}_{123} \nabla^\mu \nabla^\nu \varphi_{23\mu\nu} + K^{\varphi}_{123} \nabla^\mu \nabla^\nu \varphi_{3\mu\nu} + T^{\varphi}_{123} \nabla^\mu \nabla^\nu \varphi_{3\mu\nu}. \quad (4.36)$$

Substituting the shift $(4.27)$ of the graviton we represent the equation as

$$\kappa_1 (\nabla^2_{\mu} - m^2_{\sigma}) \sigma_1 = V^{\sigma\sigma}_{123} \left(2 \nabla^\mu (\nabla^\nu \varphi_{2\mu' \nu} - \nabla^\mu (\nabla_{\rho} \sigma_2 \varphi_{3\mu\nu}) + \frac{1}{2} (m_1^2 + m_2^2 - \Delta_3) \sigma_2 \varphi_{3\mu} \sigma_3 \right) + \text{cubic}. \quad (4.37)$$

Again the terms on the r.h.s. quadratic in fields follow from action $(2.23)$, while "cubic" again denotes the cubic terms relevant only for determining the quartic couplings.

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**Appendix A**

Here we give the results for the expansion of the Einstein equation $(2.3)$, the equations of motion for the scalar fields $(2.4)$ and the (anti)self–duality equation $(2.5)$ up to the second order both in spacetime and coset metric perturbations. The various quantities defined in $(2.1)$ and $(2.6)$ are given by:

$$Q^{ij}_M = \frac{1}{2} (\phi^{ir} \nabla_M \phi^{jr} - \nabla_M \phi^{ir} \phi^{jr}), \quad (A.1)$$

$$Q^{rs}_M = \frac{1}{2} (\phi^{jr} \nabla_M \phi^{is} - \nabla_M \phi^{jr} \phi^{is}), \quad (A.2)$$

$$P^{ir}_M = \frac{1}{\sqrt{2}} \nabla_M \phi^{ir}, \quad (A.3)$$

$$H^i = \delta^i_5 \varepsilon + g^i + \phi^{ir} g^r + \frac{1}{2} \varepsilon \phi^{ir} \phi^5, \quad (A.4)$$

$$H^r = g^r + \varepsilon \phi^{5r} + g^i \phi^{ir}. \quad (A.5)$$
Decomposing the Einstein equation (2.3) up to the second order in $h_{MN}$ one finds

$$R^{(1)}_{MN} + R^{(2)}_{MN} = H^I_{MPQ}H^I_{NPQ} - 2(h^{KL} - h^{2KL})H^I_{MKQ}H^I_{NLQ} + h^{K_1L_1}h^{K_2L_2}H^I_{MK_1K_2}H^I_{NL_1L_2} + \nabla_M\phi^{ijr}\nabla_N\phi^{ijr}. \quad (A.6)$$

Here

$$R^{(1)}_{MN} = \nabla_K h^K_{MN} - \frac{1}{2}\nabla_M\nabla_N h, \quad (A.7)$$

$$R^{(2)}_{MN} = -\nabla_K (h^K_{LM}h^L_{MN}) + \frac{1}{4}\nabla_M\nabla_N (h^{KL}h_{KL}) + \frac{1}{2}h^K_{MN}\nabla_K h - h^K_{ML}h^L_{KN}, \quad (A.8)$$

where

$$h^K_{MN} \equiv \frac{1}{2}(\nabla_M h^K_N + \nabla_N h^K_M - \nabla^K h_{MN}), \quad h \equiv h^K_K. \quad (A.9)$$

Decomposing the equations of motion for the scalar fields $\phi^{ir}$ (2.4) up to the second order we obtain

$$(\nabla^2 + \nabla^2_a)\phi^{ijr} = \nabla_K\phi^{ijr} \left(\nabla_L h^{KL} - \frac{1}{2}\nabla^K h\right) + h^{KL}\nabla_K\nabla_L\phi^{ijr} + \frac{2}{3}H^{KLM} \left(H^{rKLM} - 3h^K_S H^{rSLM}\right). \quad (A.10)$$

Finally, the (anti)self–duality equation (2.5), expanded to second order in $h_{MN}$ is:

$$H \mp *H \pm T^{(1)} \mp T^{(2)} = 0, \quad (A.11)$$

where we have introduced the following notation:

$$T^{(1)}_{M_1M_2M_3} \equiv \frac{1}{2}h(*H)_{M_1M_2M_3} - 3h^K_{[M_1}(*H)_{M_2M_3]K}, \quad (A.12)$$

$$T^{(2)}_{M_1M_2M_3} \equiv \frac{3}{2}hh^K_{[M_1}(*H)_{M_2M_3]K} - \left(\frac{1}{8}h^2 + \frac{1}{4}h^{KL}h_{KL}\right)(*H)_{M_1M_2M_3} - 3h^K_{[M_1}h^K_{M_2}(*H)_{M_3]KL}. \quad (A.13)$$

Recall that the operation $*$ is with respect to the background metric. Equations (A.12) and (A.13) have to be combined with (A.4) and (A.5).

**Appendix B**

Following [13] we establish the formulae for integrals of spherical harmonics on $S^3$ used in the paper.

In deriving the equations of motion for the various scalar fields we encountered a number of integrals of scalar spherical harmonics, all of which can be reduced to $a_{I_1I_2I_3}$ (c.f. (2.64)). In terms of
\[ \Delta_i \equiv k_i(k_i + 2) \] we find:

\[ \int_{S^3} \nabla^a Y^{I_1} \nabla_a Y^{I_2} = \Delta_1 \delta^{I_1 I_2}, \]  
\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla^b Y^{I_2} = \frac{2}{3} \Delta_1(\Delta_1 - 3) \delta^{I_1 I_2} \]  
(B.1)

and

\[ \int_{S^3} \nabla^a Y^{I_1} \nabla_a Y^{I_2} Y^{I_3} = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3) a_{I_1 I_2 I_3}, \]  
\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla_a Y^{I_2} Y^{I_3} = \left( \frac{1}{4}(\Delta_1 + \Delta_2 - \Delta_3)(\Delta_1 + \Delta_2 - \Delta_3 - 4) - \frac{1}{3} \Delta_1 \Delta_2 \right) a_{I_1 I_2 I_3}, \]  
(B.2)

\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla_a Y^{I_2} Y^{I_3} = \left( \frac{1}{4}(\Delta_1 + \Delta_3 - \Delta_2)(\Delta_2 + \Delta_3 - \Delta_1) + \frac{1}{6} \Delta_3(\Delta_1 + \Delta_2 - \Delta_3) \right) a_{I_1 I_2 I_3}. \]

In the computation of interaction vertices involving vector fields one needs the following integrals:

\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla_a Y^{I_2} Y^{I_3} = \frac{1}{2}(\Delta_1 - \Delta_2 + \Delta_3 - 3) t^+_{I_1 I_2 I_3}, \]  
\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla_a Y^{I_2} Y^{I_3} = \frac{1}{2} \left( \frac{1}{3} \Delta_1 + \Delta_2 - \Delta_3 \right) t^+_{I_1 I_2 I_3}. \]  
(B.3)

Finally, for tensor spherical harmonics \( Y^{I_3}_{(ab)} \) we used

\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla_a (\nabla^c Y^{I_2} Y^{I_3}_{(bc)}) = \frac{2}{3}(\Delta_1 - 3) p^+_{I_1 I_2 I_3}, \]  
\[ \int_{S^3} \nabla^2 \nabla^a Y^{I_1} \nabla^c Y^{I_2} \nabla_c Y^{I_3}_{(ab)} = \frac{1}{2}(\Delta_1 - \Delta_2 - \Delta_3 - 4) p^+_{I_1 I_2 I_3}. \]  
(B.4)

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