On the geometric distance between quantum states with positive partial transposition and private states

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Abstract

We prove an analytic positive lower bound for the geometric distance between entangled positive partial transpose (PPT) states of a broad class and any private state that delivers one secure key bit. Our proof holds for any Hilbert space of finite dimension. Although our result is proven for a specific class of PPT states, we show that our bound nonetheless holds for all known entangled PPT states with non-zero distillable key rates whether or not they are in our special class.

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Key words: Bound entangled state, private state, trace-norm distance, positive partial transposition (PPT).

1 Introduction

Whereas quantum entanglement is a non-local quantum correlation among distinct systems providing various useful applications such as quantum teleportation and dense coding [1][2], it is known that there are two different
types of entanglement. One of them is free entanglement, which can be distilled into pure entanglement by means of Local operations and Classical Communications (LOCC). Otherwise, the entanglement is said to be bounded, if it is undistillable [3].

As most applications of quantum entanglement are based on the use of pure entangled states, it is clear that having free entanglement assures all of the tasks are possible. In quantum cryptography, especially in quantum key distribution (QKD), secure key distillation in QKD protocols has been considered to be closely related with the distillation of pure entanglement [4]. Moreover, the existence of entanglement (whether it is free entanglement or not) in a given state is known to be necessary for any protocol to distill secure key from the state [5, 6].

Although the class of bound entangled states cannot be converted into a maximally entangled state even in an asymptotic sense, bound entanglement can be useful, in a catalytic way, in some quantum information processing (QIP) [7, 8]. However the use of bound entanglement as a resource in most QIP still seems limited and strongly doubtful.

For any bipartite quantum systems, there is a simple necessary condition for a given state to be separable. This is due to the property that any separable state has Positive Partial Transposition (PPT) [9], and PPT is also known to be sufficient for separability in $2 \otimes 2$ or $2 \otimes 3$ quantum systems [10]. For entangled states, PPT is sufficient to guarantee bound entanglement: Any PPT entangled state is a bound entangled state [3], whereas the existence of a bound entangled state with Negative Partial Transposition (NPT) is still an open question.

In quantum cryptography, it has been recently shown that the class of quantum states from which a secure key can be obtained just by performing local measurement and classical communication is much wider than that of maximally entangled states. Quantum states of this class are called private states [11], and surprisingly, it has also been shown that there do exist some classes of bound entangled states that can be asymptotically approximated to private states [11, 12, 13, 14]. In other words, even though the distillation of pure entanglement from bound entangled states is not feasible, some bound entangled states can be used as the resource in QKD at least asymptotically. However, most examples of bound entangled states with non-zero distillable key rates [11, 12, 13, 14] generally require very large dimensional Hilbert space, in fact infinite dimensional quantum system, to be approximated to private states.
Here, we address the question of bound entanglement as being a resource for finite dimensional quantum systems by showing that a broad class of bound entangled states are never arbitrarily close to private states for any finite dimensional case. By using trace norm value as the geometric distance between quantum states, we provide an analytic positive lower bound of the distance between a set of PPT states and the set of private states in terms of the dimension of the quantum system. We conjecture that this lower-bounded separation of private states from bounded entangled states holds for all bounded entangled states, and verify this conjecture for all known cases of bound entangled states with positive distillable secure key rate [11, 12, 13, 14, 15].

This paper is organized as follows. In Sec. 2 we recall the definition of private states and some related propositions. In Sec. 3 we propose an analytic lower bound of the trace norm distance for a class of PPT states from the set of private states. In Sec. 4 we show that the proposed bound holds for every PPT bound entangled state with non-zero distillable key rates given in [11, 12, 13, 14, 15], and we finally summarize our results in Sec. 5.

2 Private States and Distillable Key Rate

A private state (or pbit) [11, 12] \( \gamma_{ABA'B'} \) in \( \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \) is defined as

\[
\gamma_{ABA'B'} = \frac{1}{2} \sum_{k,l=0}^{1} |kk\rangle_{AB} \langle ll| \otimes U_{kk} \rho_{A'B'} U_{ll}^\dagger,
\]

where \( U_{jj} \)’s and \( \rho_{A'B'} \) are arbitrary unitary matrices and a quantum state in subsystems \( A'B' \), and \( d \) is the dimension of the subsystems \( A' \) and \( B' \). (We may assume that \( A \) and \( B \) have same dimension, otherwise we can take the larger one)

The set of private states is known to be the most general class of quantum states that contains perfectly secure key [11]. In other words, by performing local measurements on subsystems \( A \) and \( B \) of \( \gamma_{ABA'B'} \) in Eq. (1), one can obtain one-bit secure key between \( A \) and \( B \). Conversely, if there is a quantum state from which one can obtain one-bit secure key just by performing local measurements on some subsystems (possibly whole system) of the state, it has to be in forms of a private state.

Because private states define all possible quantum states containing one-
bit perfectly secure key, a quantum state having high fidelity with a private state can also be expected to behave similarly as a private state does: It would deliver a secure key, although the key itself might not be perfectly secure. This intuitive expectation is also shown to be true concerned with the relation between fidelity of quantum states and their geometric distance. A state $\rho$ has a non-zero distillable key rate, $K_D(\rho) > 0$, if it is close enough to a private state by means of trace norm distance [12]. Moreover, $\rho$ itself does not need to be close enough to a private state. Instead, if sufficiently many copies of $\rho$ can be transformed into a state $\rho'$ by LOCC, and $\rho'$ is close enough to a private state, it can be easily shown that not only $\rho'$ but $\rho$ as well have non-zero distillable key rates [13].

The trace norm distance between a state $\rho$ and private states was also shown to have the following analytic characterizations [12].

**Proposition 1.** If the state $\rho_{ABA'B'} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ written in the form

$$
\rho_{ABA'B'} = \sum_{i,j,k,l=0}^1 |ij\rangle_{AB} \langle kl| \otimes A_{ijkl}
$$

$$
= \begin{bmatrix}
A_{0000} & A_{0001} & A_{0010} & A_{0011} \\
A_{0100} & A_{0101} & A_{0110} & A_{0111} \\
A_{1000} & A_{1001} & A_{1010} & A_{1011} \\
A_{1100} & A_{1101} & A_{1110} & A_{1111}
\end{bmatrix},
$$

fulfills $\|\rho_{ABA'B'} - \gamma_{ABA'B'}\| \leq \varepsilon$ for some pbit $\gamma_{ABA'B'}$ and $0 < \varepsilon < 1$, then there holds $\|A_{0011}\| \geq 1/2 - \varepsilon$. Here, $\| \cdot \|$ is the trace norm defined as $\|A\| = \text{tr} \sqrt{A^\dagger A}$, for any operator $A$.

**Proposition 2.** If the state $\rho_{ABA'B'} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ written in the form $\rho_{ABA'B'} = \sum_{i,j,k,l} |ij\rangle_{AB} \langle kl| \otimes A_{ijkl}$ fulfills $\|A_{0011}\| \geq 1/2 - \varepsilon$ for $0 < \varepsilon < 1$, then there exists a pbit $\gamma_{ABA'B'}$ such that $\|\rho_{ABA'B'} - \gamma_{ABA'B'}\| \leq \delta(\varepsilon)$ with $\delta(\varepsilon)$ vanishing, when $\varepsilon$ approaches zero.

With respect to the trace norm value of a certain block $A_{0011}$ as well as its difference from 1/2 in the propositions above, let us consider a purification $|\psi\rangle_{ABA'B'E}$ of $\rho_{ABA'B'}$ such that

$$
\text{tr}_E(|\psi\rangle_{ABA'B'E}\langle\psi|) = \rho_{ABA'B'}.
$$
By straightforward calculation [12], we have

\[ \|A_{0011}\| = \sqrt{p_{00} p_{11}} F(\rho^E_{00}, \rho^E_{11}). \] 

(4)

where \( p_{ii} = \text{tr}[(|ii\rangle_{AB}\langle ii| \otimes I_{A'B'E}) |\psi\rangle_{ABA'B'E}\langle \psi|] \) is the probability of the outcome state \(|ii\rangle_{AB}\) for \(i = 0, 1\) by the local measurement of \(A\) and \(B\). \(\rho^E_{ii}\) is the resulting state on the subsystem \(E\) corresponding to the outcome \(|ii\rangle_{AB}\) on \(AB\), that is,

\[ \rho^E_{ii} = \text{tr}_{ABA'B'}[(|ii\rangle_{AB}\langle ii| \otimes I_{A'B'E}) |\psi\rangle_{ABA'B'E}\langle \psi|] / p_{ii}, \] 

(5)

and \( F(\rho^E_{00}, \rho^E_{11}) \) is the fidelity of \(\rho^E_{00}\) and \(\rho^E_{11}\), defined as

\[ F(\rho^E_{00}, \rho^E_{11}) = \text{tr} \sqrt{\sqrt{\rho^E_{00}} \sqrt{\rho^E_{11}}}. \] 

(6)

By assuming the worst case scenario, that is, the eavesdropper Eve has the purification of \(\rho_{ABA'B'}\), \(\|A_{0011}\| = 1/2\) happens if and only if \(p_{00} = p_{11} = 1/2\), and, at the same time, \( F(\rho^E_{00}, \rho^E_{11}) = 1\). In other words, the only possible outcomes are \(|00\rangle_{AB}\) and \(|11\rangle_{AB}\) with the same probability, which provide perfect correlation between \(A\) and \(B\), and this correlation is independent of Eve because \(\rho^E_{00}\) and \(\rho^E_{11}\) are identical. This implies that \(\rho_{ABA'B'}\) contains one-bit perfectly secure key, and thus it is a private state [11].

Moreover, Proposition[11] together with Proposition[2] give us a quantitative relation between the distance of a quantum state \(\rho_{ABA'B'}\) from a private state and the trace norm value \(\|A_{0011}\|\). Thus, the lower bound of the trace norm distance between a quantum state and private states can be now rephrased as an upper bound of \(\|A_{0011}\|\).

3 PPT states and the lower bound of the distance

Before we provide an analytic lower bound, let us consider the negative eigenvalues of bipartite pure states that might arise after partial transposition. For any pure state \(|\psi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^d\) with its Schmidt decomposition

\[ |\psi\rangle_{AB} = \sum_{i=0}^{d-1} a_i |ii\rangle_{AB}, \quad a_i \geq 0, \quad \sum_{i=0}^{d-1} a_i^2 = 1, \] 

(7)
the negative eigenvalues of the partially transposed state \( (|\psi\rangle_{AB}\langle\psi|)^\Gamma \) can be \(-a_ia_j\) for \( i \neq j \) \cite{16}. Thus, the largest negativity \cite{17} (the sum of the absolute values of all negative eigenvalues) can be achieved when \( a_i = 1/\sqrt{d} \) for all \( i = 0, \ldots, d-1 \), and it is \( \sum_{i<j} 1/d = (d-1)/2 \). In this case, the sum of positive eigenvalues is \( (d+1)/2 \), since \( \text{tr}((|\psi\rangle_{AB}\langle\psi|)^\Gamma) = \text{tr}(|\psi\rangle_{AB}\langle\psi|) = 1 \).

Thus, we have

\[
\text{tr} \left( (|\psi\rangle_{AB}\langle\psi|)^\Gamma \right) = P - N, \tag{8}
\]

where \( P \) and \( N \) are the sum of the absolute values of positive and negative eigenvalues of \( (|\psi\rangle_{AB}\langle\psi|)^\Gamma \) respectively, and

\[
P \leq \frac{d+1}{2}, \quad N \leq \frac{d-1}{2}. \tag{9}
\]

Now, let \( |\phi^\pm\rangle \) and \( |\psi^\pm\rangle \) be the Bell states in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \); then we have the following lemma.

**Lemma 1.** For a state \( \rho_{AB'A'B'} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \),

\[
\rho_{AB'A'B'} = |\phi^+\rangle_{AB}\langle\phi^+| \otimes \sigma_0 + |\phi^-\rangle_{AB}\langle\phi^-| \otimes \sigma_1
\]

\[
+ (|\psi^+\rangle_{AB}\langle\psi^+| + |\psi^-\rangle_{AB}\langle\psi^-|) \otimes \sigma_2, \tag{10}
\]

with arbitrary states \( \sigma_i, i = 0, \ldots, 2 \) on subsystem \( A'B' \), if \( \rho \) has PPT, then \( \|\sigma_0 - \sigma_1\| \) has an upper bound depending on the dimension \( d \),

\[
\|\sigma_0 - \sigma_1\| \leq 1 - \frac{1}{d+1}. \tag{11}
\]

Here, we note that any bipartite state can be transformed into the class of quantum states in Lemma 1 by applying local depolarization and mixing operations \cite{18} on its two-qubit subsystems \( AB \). Thus, any PPT state can also be transformed into this class with PPT property preserved. Furthermore, \( \rho_{AB'A'B'} \) in Eq. (10) has the matrix form,

\[
\rho_{AB'A'B'} = \frac{1}{2} \begin{bmatrix}
\sigma_0 + \sigma_1 & 0 & 0 & \sigma_0 - \sigma_1 \\
0 & 2\sigma_2 & 0 & 0 \\
0 & 0 & 2\sigma_2 & 0 \\
\sigma_0 - \sigma_1 & 0 & 0 & \sigma_0 + \sigma_1
\end{bmatrix}, \tag{12}
\]

where the trace norm value of its upper-right block \( A_{0011} \) is

\[
\|A_{0011}\| = \frac{\|\sigma_0 - \sigma_1\|}{2}. \tag{13}
\]
If $\rho_{A'B'B'}$ has PPT, then Lemma 1 implies that Eq. (13) is bounded above by $1/2 - 1/[2(d + 1)]$. Thus, by Proposition 1, the trace norm distance between $\rho_{A'B'B'}$ and any private state is bounded below by $1/[2(d + 1)]$.

Proof of Lemma 1. $(\sigma_0 - \sigma_1)^\Gamma$ is hermitian so it can have a diagonal representation as

$$
(\sigma_0 - \sigma_1)^\Gamma = \sum_j \lambda_j |x_j\rangle\langle x_j|,
$$

where $\lambda_j$ are the eigenvalues (not necessarily non-negative) with corresponding eigenvectors $|x_j\rangle$, then we have

$$
\sigma_0 - \sigma_1 = \sum_j \lambda_j (|x_j\rangle\langle x_j|)^\Gamma.
$$

As $\rho$ has PPT, $\rho^\Gamma \geq 0$, which is equivalent to $2 \sigma_2^\Gamma \pm (\sigma_0 - \sigma_1)^\Gamma \geq 0$ [13], we have

$$
\sum_j |\lambda_j| = \| (\sigma_0 - \sigma_1)^\Gamma \| \leq 2 \| \sigma_2^\Gamma \| = 2 \| \sigma_2 \|.
$$

Now, suppose $\| \sigma_0 - \sigma_1 \| > 1 - \frac{1}{d+1}$, then we have

$$
1 - \frac{1}{d+1} < \| \sigma_0 + \sigma_1 \| = 1 - 2 \| \sigma_2 \|,
$$

and thus, $\sum_j |\lambda_j| \leq 2 \| \sigma_2 \| < \frac{1}{d+1}$. Hence

$$
\| \sigma_0 - \sigma_1 \| = \| \sum_j \lambda_j (|x_j\rangle\langle x_j|)^\Gamma \|
\leq \sum_j |\lambda_j| \| (|x_j\rangle\langle x_j|)^\Gamma \|
= \sum_i |\lambda_i| \| (|P\rangle + |N\rangle) \|
\leq \sum_j |\lambda_j| d
< \| \sigma_0 - \sigma_1 \|,
$$

where the second inequality is due to Eq. (9). However, this is a contradiction, and thus,

$$
\| \sigma_0 - \sigma_1 \| \leq 1 - \frac{1}{d+1}.
$$

\[\square\]
Even though partial transposition (as well as full transposition) of an operator strongly depends on the basis of its matrix representation, the eigenvalues of the partially transposed operator are independent from the choice of basis. Furthermore, the proof of Lemma 1 is only concerned with the eigenvalues of the partially transposed quantum state. For this reason, the result of Lemma 1 is true not only for the private states with a certain basis, but for any private state regardless of the basis choice as well.

Now, we propose a more general class of PPT states, and prove that the lower bound of the trace norm distance obtained in Lemma 1 is still valid for this class.

**Theorem 1.** For any state \( \rho_{ABA'B'} \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \) with

\[
\rho_{ABA'B'} = \sum_{i,j,k,l=0}^{1} |i,j\rangle\langle k,l| \otimes A_{ijkl}
\]

\[
= \begin{bmatrix}
A_{0000} & A_{0001} & A_{0010} & A_{0011} \\
A_{0100} & A_{0101} & A_{0110} & A_{0111} \\
A_{1000} & A_{1001} & A_{1010} & A_{1011} \\
A_{1100} & A_{1101} & A_{1110} & A_{1111}
\end{bmatrix},
\]

if \( \rho_{ABA'B'} \) has PPT and its upper-right block \( A_{0011} \) is hermitian, then there exists a positive lower bound of the trace norm distance between \( \rho_{ABA'B'} \) and any private state \( \gamma_{ABA'B'} \),

\[
\| \rho_{ABA'B'} - \gamma_{ABA'B'} \| \geq \frac{1}{2(d+1)}.
\]

**Proof.** As \( A_{0011} \) is hermitian, we have \( A_{1100} = A_{0011}^\dagger = A_{0011} \). By applying depolarization and mixing operations, \( \rho \) can be transformed to

\[
\tilde{\rho} = \frac{1}{2} \begin{bmatrix}
A_{0000} + A_{1111} & 0 & 0 & 2A_{0011} \\
0 & A_{0101} + A_{1010} & 0 & 0 \\
0 & 0 & A_{0101} + A_{1010} & 0 \\
2A_{0011} & 0 & 0 & A_{0000} + A_{1111}
\end{bmatrix}.
\]

Because local depolarization and mixing operations are LOCC, \( \tilde{\rho} \) still has PPT. Furthermore, now \( \tilde{\rho} \) is of the form in Eq. (12); therefore \( \| A_{0011} \| \) has an upper bound \( \frac{1}{2} - \frac{1}{2(d+1)} \) by Lemma 1. Thus, by Proposition 1, there is a lower bound \( \frac{1}{2(d+1)} \) of the trace norm distance between \( \rho \) and any private state. \( \square \)
In fact, it can be easily seen that the hermitian condition of $A_{0011}$ in Theorem 1 is equivalent to that $\rho$ can be transformed into $\rho'$, a Bell-diagonal block matrix, with $A_{0011}$ being untouched. Thus, Theorem 1 deals with the most general class of quantum states where Lemma 1 can be directly applied.

4 PPT bound entangled states with non-zero distillable key rates

In this section, we consider all known examples of PPT bound entangled states of bipartite quantum systems with non-zero distillable key rates [11, 12, 13, 14, 15], and provide a positive lower bound of the trace norm distance from private states.

Example 1. (Horodecki et al [11, 12])

We first consider the PPT states with $K_D > 0$ presented in [11, 12]. Let

$$\rho = \frac{1}{N} \begin{bmatrix}
    [p(\tau_1 + \tau_0)]^{\otimes m} & 0 & 0 & [p(\tau_1 - \tau_0)]^{\otimes m} \\
    0 & [(1 - 2p)\tau_0]^{\otimes m} & 0 & 0 \\
    0 & 0 & [(1 - 2p)\tau_0]^{\otimes m} & 0 \\
    [p(\tau_1 - \tau_0)]^{\otimes m} & 0 & 0 & [p(\tau_1 + \tau_0)]^{\otimes m}
\end{bmatrix},$$

(23)

where $N = 2(2p)^m + 2(1 - 2p)^m$, $\tau_0 = \varrho_s^{\otimes l}$, $\tau_1 = [(\varrho_a + \varrho_s)/2]^{\otimes l}$, $\varrho_s = 2P_{\text{sym}}/(d^2 + d)$ and $\varrho_a = 2P_{\text{as}}/(d^2 - d)$ with the antisymmetric projector $P_{\text{as}}$ and symmetric projector $P_{\text{sym}}$ in $\mathbb{C}^d \otimes \mathbb{C}^d$ system. For sufficiently large $l$, $m$ and $d$, $\rho$ is known to have a non-zero distillable key rate $K_D(\rho) > 0$. At the same time, $\rho$ can be also shown to have PPT with a choice of $p \in [0, 1/3]$, thus $\rho$ is a PPT bound entangled state with $K_D(\rho) > 0$.

It can be directly checked that the upper-right block of $\rho$ in Eq. (23)

$$A_{0011} = \frac{1}{N} [p(\tau_1 - \tau_0)]^{\otimes m}$$

(24)

is hermitian since both $\tau_0$ and $\tau_1$ are hermitian operators. Thus, $\rho$ satisfies the condition of Theorem 1. There is a positive lower bound for the trace norm distance between $\rho_{ABA'B'}$ and any private state in finite dimensional quantum system, that is, for any finite $l$, $m$ and $d$ in Eq. (23).

Example 2. (Chi et al. [15])


There were two different classes of PPT bound entangled states with non-zero distillable key rates proposed in [13].

First, let us consider a quantum state \( \rho_{ABA'\prime} \in \mathcal{B} \left( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \right) \) such that

\[
\rho_{ABA'\prime} = \frac{1}{2} \begin{bmatrix}
\sigma_0 + \sigma_1 & 0 & 0 & \sigma_0 - \sigma_1 \\
0 & 2\sigma_2 & 0 & 0 \\
0 & 0 & 2\sigma_2 & 0 \\
\sigma_0 - \sigma_1 & 0 & 0 & \sigma_0 + \sigma_1
\end{bmatrix},
\]

(25)

where

\[
\sigma_0 = p \left( |\phi^+\rangle \langle \phi^+| + |01\rangle \langle 01| \right), \quad \sigma_1 = p \left( |\phi^-\rangle \langle \phi^-| + |10\rangle \langle 10| \right),
\]

\[
\sigma_2 = \frac{p}{\sqrt{2}} \left( |01\rangle \langle 01| + |10\rangle \langle 10| \right) + q |00\rangle \langle 00| + r |11\rangle \langle 11|,
\]

(26)

and \( |\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \).

For \( p = [1 - 2(q + r)]/(4 + 2\sqrt{2}) \), \( q > 0, r > 0 \), and \( 0 \leq q + r < (2 - \sqrt{2})/8 \), \( \rho \) is has PPT. Furthermore, by using an entanglement distillation protocol [19] on the subsystem \( AB \), \( n \) copies of \( \rho_{ABA'\prime} \) can be transformed to a quantum state \( \rho_{ABA'\prime}' \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \) with \( d = 2^n \) such that

\[
\rho_{ABA'\prime}' = \frac{1}{N} \begin{bmatrix}
(\sigma_0 + \sigma_1)^{\otimes n} & 0 & 0 & (\sigma_0 - \sigma_1)^{\otimes n} \\
0 & (2\sigma_2)^{\otimes n} & 0 & 0 \\
0 & 0 & (2\sigma_2)^{\otimes n} & 0 \\
(\sigma_0 - \sigma_1)^{\otimes n} & 0 & 0 & (\sigma_0 + \sigma_1)^{\otimes n}
\end{bmatrix},
\]

(27)

where \( N = 2^{n+1} \left[ (2p)^n + (\sqrt{2}p + 2q + 2r)^n \right] \) is the normalization factor.

\( \rho_{ABA'\prime}' \) was shown to be approximated to a private state as \( n \) is getting larger, so that it has a non-zero distillable key rate. Furthermore, \( \rho_{ABA'\prime}' \) still has PPT since the entanglement distillation protocol is LOCC, and \( \rho_{ABA'\prime}' \) is transformed from many copies of PPT state \( \rho_{ABA'\prime} \).

However, the upper-right block of \( \rho_{ABA'\prime}' \) is

\[
A_{0011} = \frac{1}{N} (\sigma_0 - \sigma_1)^{\otimes n},
\]

(28)

which is a hermitian operator. Thus, \( \rho_{ABA'\prime}' \) satisfies the condition of Theorem [1] and this class of PPT states with non-zero distillable key rates has a lower bound of the trace norm distance for any finite number of \( n \).
Chi et al.'s other class of PPT states with non-zero distillable key rates was constructed as

\[
\rho_{ABA'B'} = |\phi^+\rangle\langle\phi^+| \otimes \sigma_0 + |\phi^-\rangle\langle\phi^-| \otimes \sigma_1 + |\psi^+\rangle\langle\psi^+| \otimes \sigma_2 + |\psi^-\rangle\langle\psi^-| \otimes \sigma_3,
\]

(29)

where

\[
\sigma_0 = \rho \left( |\phi^+\rangle\langle\phi^+| + |01\rangle\langle01| \right), \quad \sigma_1 = \rho \left( |\phi^-\rangle\langle\phi^-| + |10\rangle\langle10| \right),
\]

\[
\sigma_2 = \sqrt{2p} |x_0\rangle\langle x_0| + q |00\rangle\langle00|, \quad \sigma_3 = \sqrt{2p} |x_1\rangle\langle x_1| + q |00\rangle\langle00|,
\]

(30)

and some orthonormal states \(|x_0\rangle\) and \(|x_1\rangle\) such that

\[
|x_0\rangle\langle x_0| + |x_1\rangle\langle x_1| = |01\rangle\langle01| + |10\rangle\langle10|.
\]

(31)

For \(p = (1 - 2q)/(4 + 2\sqrt{2})\) and \(0 < q < (2 - \sqrt{2})/8\), it was shown that \(\rho_{ABA'B'}\) has PPT, and it can be approximated to a private state by using the entanglement distillation protocol [19]. Similarly with the class of Eq. (25), it can be checked that the approximation in any finite dimensional quantum system still has a lower bound of the trace norm distance.

Later, there was another class of PPT bound entangled states with non-zero distillable key rates was proposed [14] based on the similar construction with [11, 12]. It can also be directly checked that the states in this class satisfy the condition of Theorem [1] and thus, they have a positive lower bound of the trace norm distance from any private state.

Now, let us consider the examples of PPT bound entangled states proposed in [15].

Example 3. (Horodecki et al. [15]) For any quantum state \(\rho_{ABA'B'}\) in \(\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d)\) such that

\[
\rho_{ABA'B'} = \frac{1}{2} \begin{bmatrix} pX_1X_1^\dagger & 0 & 0 & pX_1 \\
0 & (1 - p)\sqrt{X_2X_2^\dagger} & (1 - p)X_2 & 0 \\
0 & (1 - p)X_2^\dagger & (1 - p)\sqrt{X_2^\dagger X_2} & 0 \\
pX_1^\dagger & 0 & 0 & p\sqrt{X_1^\dagger X_1} \end{bmatrix},
\]

(32)
where \( X_1 \) and \( X_2 \) are arbitrary operator with trace norm one, and \( 0 \leq p \leq 1 \), it is shown that the distillable key rate \( K_D(\rho_{ABA'B'}) \) fulfills \( K_D(\rho_{ABA'B'}) \geq 1 - h(p) \) where \( h(p) \) is the binary entropy of distribution \( \{p, 1 - p\} \). To construct a PPT state of the form Eq. (32), let

\[
X_1 = \frac{1}{\|W_U\|} W_U, \quad X_2 = \frac{W_T}{\|W_T\|},
\]

where

\[
W_U = \sum_{i,j} u_{ij} |ij\rangle\langle ji|,
\]

\( u_{ij} \) are the elements of a unitary matrix \( U \) on \( \mathbb{C}^d \), and \( W_T \) is the partially transposed matrix of \( W_U \). We also let the probabilities be

\[
p = \frac{\|W_U\|}{\|W_U\| + \|W_T\|}, \quad 1 - p = \frac{\|W_T\|}{\|W_U\| + \|W_T\|};
\]

then, by direct observation, \( \rho_{ABA'B'} \) is invariant under partial transposition and obviously PPT. Furthermore, we have

\[
\frac{p}{1 - p} = \frac{\|W_U\|}{\|W_T\|} = \frac{\sum_{i,j} |u_{ij}|}{d}.
\]

Thus, a non-zero distillable key rate of \( \rho_{ABA'B'} \) that is equivalent to \( p \neq \frac{1}{2} \) can be obtained from any choice of the unitary matrix \( U \) satisfying \( \sum_{i,j} |u_{ij}| > d \).

Now let us consider the trace norm distance of \( \rho_{ABA'B'} \) from private states. Unfortunately, \( \rho_{ABA'B'} \) does not satisfy the condition of Theorem 1, since its upper-right block \( A_{0011} = \frac{1}{2}pX_1 \) is not hermitian. However, in this case, we can directly evaluate an upper bound of \( \|A_{0011}\| \). By noticing that \( \|X_1\| = 1 \) and \( \|W_T\| = d \), we have

\[
\|A_{0011}\| = \|\frac{1}{2}pX_1\| = \frac{1}{2} - \frac{\|W_T\|}{2(\|W_U\| + \|W_T\|)} = \frac{1}{2} - \frac{d}{2(\|W_U\| + d)},
\]

and thus, \( \|A_{0011}\| \) attains its maximum value when \( \|W_U\| \) is the largest. We also have

\[
\|W_U\| = \text{tr} \sqrt{W_U W_U^\dagger} = \sum_{i,j} |u_{ij}|;
\]
therefore, by simple calculus, we have \( \|W_U\| \leq d\sqrt{d} \), and

\[
\|A_{0011}\| \leq \frac{1}{2} - \frac{1}{2(\sqrt{d} + 1)} \leq \frac{1}{2} - \frac{1}{2(d + 1)},
\]

(39)

where the second inequality indicates the upper bound obtained in Lemma 1. Thus, the trace norm distance between \( \rho_{ABA'B'} \) and private states is still satisfied by the lower bound obtained in Theorem 1,

\[
\|\rho_{ABA'B'} - \gamma_{ABA'B'}\| \geq \frac{1}{2(\sqrt{d} + 1)} \geq \frac{1}{2(d + 1)},
\]

(40)

for any private state \( \gamma_{ABA'B'} \) in \( \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \).

In this section, we have considered all known examples of PPT bound entangled states in bipartite quantum systems with non-zero distillable key rates. We have remarked that most of the examples belong to the class proposed in Theorem 1 and the only exceptional case \[15\] is also shown to have the same bound. (In fact, a coarser bound). Here, we conjecture that the lower bound of the trace norm distance proposed in Theorem 1 is true for any PPT state.

5 Conclusions

We have provided an analytic positive lower bound of the distance for a class of PPT states from private states in terms of the dimension of quantum systems, and we have further shown that this lower bound holds for all known examples of PPT bound entangled states with non-zero distillable key rates.

Our result implies that most PPT bound entangled states, including all the examples of non-zero distillable key rates, are geometrically separated from private states in any finite dimensional quantum system. This is a strong clue for our conjecture that every PPT state is distantly separated from private states in any finite dimensional Hilbert space. The conjecture directly leads to the answer for an important question: Whether or not,
the operational meaning of bound entanglement in QKD is only asymptotic. Thus, our result and conjecture will play a central role to clarify the essential difference between free and bound entangled states in quantum cryptography in both operational and analytical ways. Besides quantum cryptography, our result will also provide a rich reference with useful tools in studying the geometric structure of bounded operators with certain properties in Hilbert space.

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**References**

[1] Bennett, C.H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. **70**, 1895–1899 (1993)

[2] Bennett, C.H., Wiesner, S.J.: Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. Phys. Rev. Lett. **69**, 2881–2884 (1992)

[3] Horodecki, M., Horodecki, P., Horodecki, R.: Mixed-State Entanglement and Distillation: Is there a “Bound” Entanglement in Nature? Phys. Rev. Lett. **80**, 5239–5242 (1998)

[4] Shor, P.W., Preskill, J.: Simple Proof of Security of the BB84 Quantum Key Distribution Protocol. Phys. Rev. Lett **85**, 441–444 (2000)

[5] Curty, M., Lewenstein, M., Lütkenhaus, N.: Entanglement as a Precondition for Secure Quantum Key Distribution. Phys. Rev. Lett. **92**, 217903 (2004)

[6] Curty, M., Gühen, O., Lewenstein, M., Lütkenhaus, N.: Detecting two-party quantum correlations in quantum-key-distribution protocols. Phys. Rev. A **71**, 022306 (2005)
[7] Horodecki, P., Horodecki, M., Horodecki, R.: Bound Entanglement Can Be Activated. Phys. Rev. Lett. 82, 1056 (1999)

[8] Masanes, L.: All Bipartite Entangled States Are Useful for Information Processing. Phys. Rev. Lett. 96, 150501 (2006)

[9] Peres, A.: Separability criterion for density matrices. Phys. Rev. Lett. 77, 1413 (1996)

[10] Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A 223, 1–8 (1996)

[11] Horodecki, K., Horodecki, M., Horodecki, P., Oppenheim, J.: Secure Key from Bound Entanglement. Phys. Rev. Lett 94, 160502 (2005)

[12] Horodecki, K., Horodecki, M., Horodecki, P., Oppenheim, J.: General paradigm for distilling classical key from quantum states. IEEE Trans. Inf. Theory, in press; arXiv:quant-ph/0506189 (2005)

[13] Chi, D.P., Choi, J.W., Kim, J.S., Kim, T., Lee, S.: Bound entangled states with nonzero distillable key rate. Phys. Rev. A 75, 032306 (2007)

[14] Horodecki, P., Augusiak, R.: On quantum cryptography with bipartite bound entangled states. Quantum Information Processing: From Theory to Experiment, D.G. Angelakis et al. (eds.), NATO Science Series III, vol. 199, 19–29, IOS Press, Amsterdam, 2006; arXiv:0712.3999.

[15] Horodecki, K., Pankowski, L., Horodecki, M., Horodecki, P.: Low dimensional bound entanglement with one-way distillable cryptographic key. IEEE Trans. Inf. Theory 54, 2621 (2008)

[16] Kim, J.S., Das, A., Sanders, B.C.: Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extended negativity. Phys. Rev. A 79, 012329 (2009)

[17] Vidal, G., Werner, R.F.: Computable measure of entanglement. Phys. Rev. A 65, 032314 (2002)

[18] Dür, D., Cirac, J.I., Lewenstein, M., Bruß, D.: Distillability and partial transposition in bipartite system. Phys. Rev. A 61, 062313 (2000)
[19] Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A 54, 3824 – 3851 (1996)