ANALYTIC NEWVECTORS FOR $\text{GL}_n(\mathbb{R})$

SUBHAJIT JANA AND PAUL D. NELSON

Abstract. We relate the analytic conductor of a generic irreducible representation of $\text{GL}_n(\mathbb{R})$ to the invariance properties of vectors in that representation. The relationship is an analytic archimedean analogue of some aspects of the classical non-archimedean newvector theory of Casselman and Jacquet–Piatetski-Shapiro–Shalika. We illustrate how this relationship may be applied in trace formulas to majorize sums over automorphic forms on $\text{PGL}_n(\mathbb{Z}) \backslash \text{PGL}_n(\mathbb{R})$ ordered by analytic conductor.

1. Introduction

Let $F$ be a local field, and let $\pi$ be a generic irreducible representation of $\text{GL}_n(F)$. Suppose first that $F$ is non-archimedean, with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and residue field cardinality $q = \#\mathfrak{o}/\mathfrak{p}$. The newvector theory developed by Casselman and Jacquet–Piatetski-Shapiro–Shalika (see [11, 23] and [29]) then gives a precise relationship between the conductor $C(\pi)$ of $\pi$ and the existence of vectors $v$ in $\pi$ invariant under certain compact open subgroups. Here we define $C(\pi)$ in terms of the local $\gamma$-factor with respect to an unramified additive character $\psi$ of $F$ by the relation

$$\left| \gamma(s, \pi, \psi) \right| = C(\pi)^{1/2-s},$$

so that if $c(\pi) \in \mathbb{Z}_{\geq 0}$ denotes the conductor exponent, then $C(\pi) = q^{c(\pi)}$. The main theorem of newvector theory says that $C(\pi)$ is the smallest element $X = q^a \geq 1$ of the value group of $F$ for which $\pi$ contains a nonzero vector $v$ invariant under the group

$$K_1(X) := \left\{ g \in \text{GL}_r(\mathfrak{o}) : \begin{array}{c} |g_{nj}| \leq 1/X \text{ for } 1 \leq j < n, \\ |g_{nn} - 1| \leq 1/X \end{array} \right\} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ p^n & \cdots & p^n & 1 + p^n \end{pmatrix} \cap \text{GL}_r(\mathfrak{o})$$

or equivalently, transforming under the group

$$K_0(X) := \left\{ g \in \text{GL}_r(\mathfrak{o}) : |g_{nj}| \leq 1/X \text{ for } 1 \leq j < n \right\} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ p^n & \cdots & p^n & 0 \end{pmatrix} \cap \text{GL}_r(\mathfrak{o})$$
by the character \( g \mapsto \omega_g(g_{nn}) \), with \( \omega_g \) the central character of \( \pi \). The vector \( v \), called the newvector (or, in the language of [23], the essential vector), is then uniquely determined up to a scalar.

Suppose now that \( F = \mathbb{R} \). (A similar discussion should apply when \( F = \mathbb{C} \), but we focus in this paper on the real case.) The analytic conductor [22] of \( \pi \) is then typically defined, for the sake of concreteness, by the formula \( C(\pi) := \prod_j (1 + |\mu_j|) \), where \( \{\mu_1, \ldots, \mu_n\} \) is the set of the parameters of \( \pi \) characterized by the relation \( L(s, \pi) = \prod_j \Gamma(s + \mu_j) \). \( \Gamma(s) := \pi - s/2 \Gamma(s/2) \). In practice, minor variants of this definition serve the same purpose; for instance, one occasionally sees the factors \( 1 + |\mu_j| \) replaced by \( 3 + |\mu_j| \), so that \( \log C(\pi) \) is bounded uniformly away from zero. By Stirling’s formula, the identity (1.1) holds asymptotically for bounded values of \( s \), and may be understood as an asymptotic characterization of \( C(\pi) \) and its mild variants.

The classical non-archimedean newvector theory has been applied towards many problems in the analytic theory of automorphic forms. For instance, to apply trace formulas (or relative trace formulas, integral representations, ...) to study averages over a family of automorphic forms of various quantities of interest (Fourier coefficients, Hecke eigenvalues, \( L \)-values, ...), one needs a test function that approximately projects to that family, so that the spectral side of the trace formula localizes to that family. For a level aspect family of forms having spectral parameter bounded and finite conductor dividing a given large natural number, one can use the (twisted) characteristic functions of the congruence subgroups (1.2) and (1.3) to construct suitable projectors. We mention the works [1, 2, 3, 7, 37, 30] which make (often implicit) use of such ideas.

By comparison, we are not aware of an existing comparably simple way to project (approximately) onto (say) the family of cusp forms on \( \text{PGL}_n(\mathbb{Z}) \setminus \text{PGL}_n(\mathbb{R}) \) whose analytic conductor at the real place is bounded from above by a certain quantity. One approach would be to describe that family in terms of local parameters, partition it into subfamilies according to the approximate values of those parameters, and then sum the projectors associated to the subfamilies; see [10, §8]. However, this approach is significantly less direct than in the non-archimedean case, and does not immediately clarify the shape of the test function defining the projection.

This discrepancy motivates looking for an archimedean analogue to classical newvector theory, that is to say, an interpretation of the analytic conductor \( C(\pi) \) in terms of the invariance properties of vectors in a generic representation \( \pi \) of \( \text{GL}_n(\mathbb{R}) \). One might hope for such an interpretation to be useful in analytic problems in which \( \pi \) varies in archimedean aspects in much the same way that newvector theory has been useful in non-archimedean aspects.

An exact analogue is clearly too ambitious: the group \( \text{GL}_n(\mathbb{R}) \) has no compact open subgroups, and very few vectors exactly invariant by any open subset of \( \text{GL}_n(\mathbb{R}) \). One can ask instead for approximate analogues.

The main purpose of this paper is to introduce, for generic irreducible representations \( \pi \) of \( \text{GL}_n(\mathbb{R}) \), a relationship between the analytic conductor \( C(\pi) \) and the existence of vectors
satisfying a form of approximate invariance under certain subsets closely related to the subgroups \( K_1(X) \) and \( K_0(X) \) arising in the non-archimedean case. We hope that our local results will be useful as a starting point for further global analysis of the sort indicated below in §1.4.

1.1. The case \( n = 1 \). As warmup for the more complicated statements given below, let us consider in detail what “analytic newvector theory” looks like in the simplest case \( n = 1 \).

We recall first the non-archimedean story, which we elect to present classically in terms of Dirichlet characters (rather than the closely-related characters of \( \text{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times \)). Let \( \chi \) be a Dirichlet character. There are then two equivalent ways to define the conductor \( q = q(\chi) \):

- (in terms of the invariance) \( \chi \) has conductor \( q \) if it is well-defined on \( \mathbb{Z}/q\mathbb{Z} \) and if the following conditions on a divisor \( d \) of \( q \) are equivalent:
  - \( d = q \)
  - For all integers \( n \) with \( n - 1 \equiv 0 \pmod{d} \), we have \( \chi(n) = 1 \).

- (analytically) \( q = C(\chi) \), defined using local \( \gamma \)-factors (or Gauss sums) as above.

Consider now a character \( \chi \) of \( \text{GL}_1(\mathbb{R}) \), say \( \chi = |t|^i \) for some real number \( t \). As noted above, the analytic conductor is defined by \( C(\chi) = (1 + |t|) \), and admits an asymptotic analytic characterization via the local \( \gamma \)-factors. How should one interpret \( C(\chi) \) in terms of invariance, by analogy to the second of the two characterizations of \( q(\chi) \) above? A natural interpretation is that \( \chi \) is approximately invariant under group elements \( y \in \text{GL}_1(\mathbb{R}) \) of the form \( y = 1 + o(1/C(\chi)) \), but not in general under those of the form \( y = 1 + O(1/C(\chi)) \).

More precisely, we have the following asymptotic characterization of the analytic conductor (1 + |t|) up to bounded multiplicative error:

**Toy Theorem.** For a sequence of real numbers \( t_j \) tending off to \( \infty \) and a corresponding sequence of positive real scaling parameters \( X_j \), the following are equivalent:

- \( X_j/(1 + |t_j|) \to \infty \).
- For all sequences \( y_j \in \text{GL}_1(\mathbb{R}) \) with \( |y_j - 1| < 1/X_j \), we have \( |y_j|^{it_j} \to 1 \).

The straightforward proof is left to the reader. Note the similarity between these conditions and those appearing above in the characterizations of the conductor of a Dirichlet character, with \( d \) playing the role of \( X_j \) and \( n \) that of \( y_j \).

1.2. Notation and preliminaries. We now prepare to describe our main results. We denote the identity element of \( \text{GL}_r(\mathbb{R}) \) by \( 1_r \). When the dimension \( r \) is clear from context, we abbreviate \( 1 := 1_r \).

1.2.1. Standard congruence subsets. Let \( X \geq 1 \) (thought of as tending off to \( \infty \)) and let \( \tau \in (0, 1) \) (thought of as small but fixed, or perhaps very slowly tending to \( 0 \)). We define the following archimedean analogues of the standard \( p \)-adic congruence subgroup (1.3):

\[
K_0(X, \tau) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_{n+1}(\mathbb{R}) \middle| a \in \text{GL}_n(\mathbb{R}), d \in \text{GL}_1(\mathbb{R}), \begin{array}{c} |a - 1_n| < \tau, |b| < \tau, \\ |c| < \frac{\tau}{X}, |d - 1| < \tau \end{array} \right\}.
\]
Here the various \(|.|\) denote arbitrary fixed norms on the various spaces of matrices. We define \(K_1(X, \tau)\) similarly, but with the stronger constraint \(|d - 1| < \tau/X\).

While the sets \(K_\star(X, \tau)\) are not groups, they have some group-like properties in the \(\tau \to 0\) limit. For instance, it is easy to see that if \(\tau'\) is small enough with respect to \(\tau\), then \(g_1g_2 \in K_0(X, \tau)\) for all \(g_1, g_2 \in K_0(X, \tau')\). Moreover, these subsets enjoy the following Følner-type property, whose verification we leave to the reader.

**Lemma 1.1.** Let \(* \in \{0, 1\}\). For all \(\tau_0, \delta \in (0, 1)\) there exists \(\tau_1 > 0\) so that for all \(X \geq 1\) and all \(g \in K_\star(X, \tau_1)\), the set \(A := K_\star(X, \tau_0)\) enjoys the following approximate invariance property under translation by \(g\):

\[
\frac{\text{vol}(gA \triangle A)}{\text{vol}(A)} < \delta.
\]

Here \(A \triangle B := (A \setminus B) \cup (B \setminus A)\) denotes the symmetric difference and \(\text{vol}\) is taken with respect to any fixed Haar measure.

**1.2.2. \(\theta\)-temperedness.** Let \(\theta \geq 0\). By the Langlands classification, we know that any unitary irreducible representation \(\pi\) of \(\text{GL}_n(\mathbb{R})\) is a Langlands quotient of an isobaric sum of the form

\[
\sigma_1 \otimes |\det|^{s_1} \boxplus \cdots \boxplus \sigma_r \otimes |\det|^{s_r},
\]

where the underlying Levi of the above induction is attached to a partition of \(n\) by 2’s and 1’s. Here each \(\sigma_i\) is either a discrete series of \(\text{GL}_2(\mathbb{R})\) or a character of \(\text{GL}_1(\mathbb{R})\) of the form \(\text{sgn}^{\delta_1}|\mu_1^\mu\) for some \(\delta \in \{0, 1\}\) and \(\mu_i \in i\mathbb{R}\). We say that \(\pi\) is \(\theta\)-tempered if all such \(s_i\) have real parts in \([-\theta, \theta]\). By \([28, 4]\), the local component at any real place of any cuspidal automorphic representation of \(\text{GL}(n)\) over a number field is \(\theta\)-tempered with \(\theta = 1/2 - 1/(1 + n^2) < 1/2\).

**1.3. Main results.** Theorem 1 gives a simple sense in which the analytic conductor controls the invariance properties of vectors. Theorem 2 is a more powerful yet more technical result with additional features that we expect to be useful in applications.

**Theorem 1.** Fix \(n \in \mathbb{Z}_{\geq 1}\) and \(\theta \in [0, 1/2)\). For each \(\delta > 0\) there exists \(\tau > 0\) with the following property: For each generic irreducible \(\theta\)-tempered unitary representation \(\Pi\) of \(\text{GL}_{n+1}(\mathbb{R})\), there exists a unit vector \(v \in \Pi\) such that for all \(g \in K_0(C(\Pi), \tau)\),

\[
||\Pi(g)v - \omega_\Pi(d_g)v||_\Pi < \delta.
\]

Here \(\omega_\Pi\) denotes the central character of \(\Pi\), and \(d_g\), the lower-right entry of \(g\).

We fix a generic additive character \(\tilde{\psi}\) of the standard maximal unipotent subgroup of \(\text{GL}_{n+1}(\mathbb{R})\), consisting of upper-triangular unipotent matrices. We choose \(\tilde{\psi}\) to be defined in a similar way as in \([5, 1]\). We denote by \(W(\Pi, \tilde{\psi})\) the Whittaker model of a generic irreducible representation \(\Pi\) of \(\text{GL}_{n+1}(\mathbb{R})\) (see \([3, 2]\) for details).

**Theorem 2.** Fix \(n \in \mathbb{Z}_{\geq 1}\) and \(\theta \in [0, 1/2)\), let \(\Omega\) be a bounded open subset of \(\text{GL}_n(\mathbb{R})\), and let \(\iota > 0\) be small enough in terms of \(n\) and \(\Omega\). For each \(\delta > 0\) there exists \(\tau > 0\) with the following property:
For each generic irreducible \( \theta \)-tempered unitary representation \( \Pi \) of \( \text{GL}_{n+1}(\mathbb{R}) \), there exists an element \( V \in \mathcal{W}(\Pi, \tilde{\psi}) \) of its Whittaker model satisfying

- the normalization \( \|V\|_{\mathcal{W}(\Pi, \tilde{\psi})} = 1 \), with the norm taken in the Kirillov model (§3.2),
- the lower bound \( V \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) \geq 1 \) for all \( h \in \Omega \), and
- the invariance properties:
  1. for all \( g \in K_0(C(\Pi), \tau) \),
     \[ \|\Pi(g)V - \omega_\Pi(d_g)V\|_{\mathcal{W}(\Pi, \tilde{\psi})} < \delta, \]
  2. and for \( h \in \Omega \),
     \[ \|V \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) g - \omega_\Pi(d_g)V \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) \| < \delta. \]

Here \( \omega_\Pi \) and \( d_g \) are as in Theorem 1.

Informally, Theorem 2 asserts that if \( \tau \) is small enough, then there are nonzero vectors in \( \Pi \) satisfying a form of approximate invariance under \( K_0(C(\Pi), \tau) \), both in the sense of norm and as quantified by the Whittaker functional.

**Remark 1.** It may be instructive to record the formulation of Theorem 2 in terms of sequences: for each bounded open \( \Omega \subseteq \text{GL}_n(\mathbb{R}) \) and sequence \( \Pi_j \) of generic irreducible \( \theta \)-tempered unitary representations of \( \text{GL}_{n+1}(\mathbb{R}) \) there is a corresponding sequence \( V_j \in \mathcal{W}(\Pi_j, \tilde{\psi}) \) of Whittaker model elements satisfying

- \( \|V_j\| = 1 \),
- \( \inf_{h \in \Omega} \inf_{j} |V_j(h)| > 0 \), and
- for any sequence \( \tau_j \) of positive numbers tending to zero and any sequence of group elements \( g_j \in K_0(C(\Pi_j), \tau_j) \),
  \[ \lim_{j \to \infty} \|\Pi(g_j)V_j - \omega_\Pi(d_{g_j})V_j\| = 0 \]
and
  \[ \lim_{j \to \infty} \inf_{h \in \Omega} |V_j(hg_j) - \omega_\Pi(d_{g_j})V_j(h)| = 0. \]

**Remark 2.** The proof is constructive (see §2), and shows that we may take \( V \) to be a fixed bump function in the Kirillov model.

**Remark 3.** The assumption of \( \theta \)-temperedness (or even unitarity) may seem artificial, since it is not required in the non-archimedean setting. It is used in the proof to ensure that \( \gamma(1/2 - s, \Pi) \) remains holomorphic for \( R(s) \geq 0 \) during a contour shift argument (see §2). This assumption is satisfied in our intended applications.

**Remark 4.** The new vector defined by Jacquet–Piatetski-Shapiro–Shalika in [23] is defined differently from ours. They have defined a Whittaker function \( V \) on \( \text{GL}_{n+1}(F) \) to be the
newvector (up to a scalar) if for any spherical representation \( \pi \) of \( \text{GL}_n(F) \) containing the spherical vector \( W_0 \) with \( W_0(1) = 1 \), the local zeta integral (see \( \S \) 3.5) of \( V \) and \( W_0 \) equals the \( L \)-factor of the Rankin-Selberg convolution \( \Pi \otimes \pi \). One can also define newvectors at an archimedean place as test vectors of the Rankin-Selberg zeta integral; Popa [33] has introduced such a theory for \( \text{GL}_2(\mathbb{R}) \), while the case of \( \text{GL}_n(\mathbb{R}) \) is the subject of an ongoing work of P. Humphreys. Such test vectors can be thought as \textit{algebraic} analogues at the archimedean place of the classical newvectors in [23]. The analytic newvectors considered here are “analytic test vectors” (i.e., the zeta integral enjoys a quantitative lower bound rather than merely nonvanishing) for “analytically unramified” representations (i.e., those whose analytic conductor is sufficiently small). The source of this dichotomy between algebraic and analytic is related to the question: what is the analogue inside \( \text{GL}_n(\mathbb{R}) \) of \( \text{GL}_n(\mathbb{Z}_p) \subseteq \text{GL}_n(\mathbb{Q}_p) \)? An algebraic analogue is \( O(n) \) (a maximal compact subgroup), while an analytic analogue is a small balanced neighborhood of the identity. Algebraic newvectors transform nicely under \( O(n) \), while analytic newvector transform nicely under suitable neighborhoods of the identity.

\textbf{Remark 5.} We expect, by analogy to the non-archimedean theory, that Theorems 1 and 2 enjoy a “converse” to the effect that their conclusion fails if one replaces \( K_0(C(\Pi), \tau) \) by \( K_0(X, \tau) \) for \( X \) substantially smaller than \( C(\Pi) \). To make these assertions precise, let \( f_X \) be an \( L^1 \)-normalized smoothened characteristic function of \( K_0(X, \tau) \). Since \( K_0(X, \tau) \) behaves like a group, we may assume that \( f_X \) is a self-convolution, so that the integral operator \( \Pi(f_X) \) is positive-definite. The trace \( d_X(\Pi) \) of that operator is then an analytic proxy for “the dimension of the space of \( V \in \Pi \) approximately invariant by \( K_0(X, \tau) \).” (In some applications an alternative proxy \( J_X(\Pi) \) defined using the Bessel distribution is more relevant, see \( \S \) 7 for details.) Theorem 1 implies that \( d_X(\Pi) \gg 1 \) for \( X \geq C(\Pi) \) and \( \tau > 0 \) small but fixed. Conversely, we expect that \( d_X(\Pi) \ll_N (C(\Pi)/X)^{-N} \) for \( X \ll C(\Pi) \), for any fixed \( N \), thus \( d_X(\Pi) \) is small when \( X \) is substantially smaller than \( C(\Pi) \). In the transition regime \( X \approx C(\Pi) \) our estimates show that such an upper bound is sharp if true, suggesting an analogue of the “multiplicity one” property of non-archimedean newvectors as well as an asymptotic characterization of the analytic conductor in terms of invariance properties of vectors.

\textbf{1.4. Application.} As noted above, we hope that the local technology developed in this paper will serve as a useful starting point for global problems involving averages (of Fourier coefficients, \( L \)-values, ...) over automorphic forms on (say) \( \text{PGL}_n(\mathbb{Z}) \backslash \text{PGL}_n(\mathbb{R}) \) ordered by analytic conductor. To illustrate how we hope our results will be applied, we record a very simple application.

\textbf{Theorem 3.} For \( X > 1 \), we have

\[ \sum_{\pi: C(\pi) \leq X} \frac{1}{L(1, \pi, \text{Ad})} \ll X^{n-1}, \]
where the sum is over cuspidal automorphic representations \( \pi \subseteq L^2(\text{PGL}_n(\mathbb{Z})\backslash \text{PGL}_n(\mathbb{R})) \) and \( L(s, \pi, \text{Ad}) \) denotes the adjoint \( L \)-function.

The proof of Theorem 3 gives an asymptotic formula for a smoothly truncated and weighted variant of the LHS of (1.5), together with the corresponding contribution from the continuous spectrum. The analogous estimate for forms on \( \text{GL}_n(\mathbb{Z})\backslash \text{GL}_n(\mathbb{R}) \) having a given central character may be proved in the same way. Such estimates are standard when \( n = 2 \) [21, §14.10, §16.5]. Some variants have been established for \( n = 3 \) in [11, 16, 18], for \( n = 4 \) in [18]. We note that Brumley [5, Theorem 3] established lower bounds for \( L(1, \pi, \text{Ad}) \) much sharper than those that follow from (1.5), and Brumley–Miličević [10] have recently obtained a Weyl law on adelic quotients, such as \( \text{PGL}_n(\mathbb{Q})\backslash \text{PGL}_n(\mathbb{A}) \), for cuspidal automorphic representations ordered by the product of the analytic conductors at each place.

While the estimate (1.5) appears to be new, it should not be regarded as analytically difficult. Indeed, by applying the Kuznetsov formula to a test function that projects onto representations having parameters in a ball of essentially bounded size, it should be possible to establish the analogue of (1.5) over substantially smaller families. Our point is to record how a soft proof follows naturally from Theorem 2 as illustration of a technique that we hope will be useful also in more analytically challenging problems. We are motivated in particular by the problem of studying higher moments of \( L \)-functions over such families.

1.5. Organization of the paper. In §2 we give a sketch of the proof of Theorem 2. We recommend the reader to go through it first to understand various technicalities in the proof. In §3 we record and prove most of the definitions and auxiliary lemmas which we need at the various stages of the proofs. We recommend that readers should skim over this section for first reading and come back when some Lemma is recalled. In §4 and §5 we reduce the proof of Theorem 2 to Proposition 4.1 and Proposition 4.1 to Proposition 5.1, respectively. Finally, in §6 we prove Proposition 5.1 which is the most technical part of the paper. Lastly, in §7 we prove Theorem 3.

Acknowledgements. We would like to thank Jack Buttcane for encouragement and several helpful discussions regarding the decomposition of the spherical Whittaker function. We also thank Djordje Miličević, Farrell Brumley, and Valentin Blomer for feedbacks and comments on an earlier draft of this paper.

2. Sketch for the proof of Theorem 2

In this section we assume \( \Pi \) to be tempered (instead of \( \theta \)-tempered) and has trivial central character. We construct the vector \( V \in \Pi \) by specifying that it be given by a fixed bump function in the Kirillov model (see (4.1)). The key step in the proof of Theorem 2 is to verify that \( V(g) \) approximates \( V(1) \) for all \( g \in K_n(C(\Pi), \tau) \) with \( \tau \) small; the remaining assertions are deduced from this one fairly easily. The main difficulties in the proof are present in the special case that \( g \) is lower-triangular unipotent, so for the purposes of this sketch we restrict to that case. It will suffice then to prove the following quantitative refinement of the
conclusion of Theorem 2 for all small $1 \times n$ row vectors $c$,

$$(2.1) \quad V \left[ \begin{pmatrix} 1 \\ c/C(\Pi) \\ 1 \end{pmatrix} \right] - V(1) \ll |c|$$

To that end, we first expand the LHS of (2.1) using the Whittaker-Plancherel formula (3.2). We then apply the $\text{GL}(n+1) \times \text{GL}(n)$ local functional equation (3.4) and attempt to analyze the resulting integral. We will indicate how this goes first in the simplest case $n = 1$, and then describe the modifications necessary for general $n$, along with technical difficulties in those cases.

2.1. **Proof for $n = 1$.** In this case $\Pi$ is a representation of $\text{GL}_2(\mathbb{R})$. We define the Whittaker model $W(\Pi)$ using the additive character of the unipotent radical in $\text{GL}_2(\mathbb{R})$ defined as in (3.1). We recall that, by the theory of the Kirillov model, there is for each $f \in C_c^\infty(\mathbb{R} \times)$ a unique element $V \in W(\pi)$ of the Whittaker model of $\Pi$ for which

$$(2.2) \quad V \left[ \begin{pmatrix} y \\ 1 \end{pmatrix} \right] = f(y)$$

for all $y \in \mathbb{R}^\times$. We recall the local functional equation (3.5) for $\text{GL}_2 \times \text{GL}_1$: for $s \in \mathbb{C}$, the zeta integral

$$\int_{\mathbb{R}^\times} V \left[ \begin{pmatrix} t \\ 1 \end{pmatrix} \right] |t|^s \, dt$$

converges absolutely for $\Re(s) > -1/2$ and extends to a meromorphic function on the complex plane, where it satisfies the relation

$$\int_{\mathbb{R}^\times} V \left[ \begin{pmatrix} t \\ 1 \end{pmatrix} \right] |t|^s \, dt = \frac{1}{\gamma(\pi, 1/2 + s)} \int_{\mathbb{R}^\times} V \left[ \begin{pmatrix} 1 \\ t \end{pmatrix} w \right] |t|^s \, dt.$$

Here $w := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ denotes the Weyl element and $\gamma$ the local $\gamma$-factor, whose properties we recall in greater detail in §3.5. The meromorphic function

$$\Theta(s, \Pi) := \frac{C(\pi)^{-s}}{\gamma(1/2 + s, \pi)}$$

is holomorphic for $\Re(s) > -1/2$ and non-vanishing for $\Re(s) < 1/2$. The normalization is such that $\Theta(s, \Pi)$ is approximately of size 1 for bounded $s$, uniformly in $\Pi$; more precisely, we have

$$(2.3) \quad (1 + |\Re(s)|)^{-2|\Re(s)|} \ll \Theta(s, \Pi) \ll (1 + |\Re(s)|)^{2|\Re(s)|}$$

for $s$ of bounded real part and a fixed positive distance away from the poles of $\Theta(s, \Pi)$, and with the implied constant uniform in $\Pi$ (see Lemma 3.1).

Let us assume in this sketch that $\Pi$ belongs to the discrete series, so that in the Kirillov model of $\Pi$, the subspace of functions that vanish off the group $\mathbb{R}^\times_+$ of positive reals is invariant by the group of positive-determinant elements of $\text{GL}_2(\mathbb{R})$; this allows us to simplify the exposition slightly because the character group of $\mathbb{R}^\times_+$ is a bit simpler than that of $\mathbb{R}^\times$. 
We fix a test function \( f \in C^\infty_c(\mathbb{R}^\times) \) satisfying the normalization \( \|f\|_2 = 1 \). We extend \( f \) by zero to an element of \( C^\infty_c(\mathbb{R}^\times) \). We construct \( V \) using the theory of the Kirillov model by requiring that (2.2) hold for this choice of \( f \). We aim then to verify the estimate (2.1). To achieve this, we first apply Mellin inversion, giving for any \( c \in \mathbb{R} \) the identity

\[
V \left( \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix} \right) = \int_{(0)} \left( \int_{t \in \mathbb{R}_+^\times} V \left( \begin{pmatrix} t & 1 \\ c & 1 \end{pmatrix} \right) |t|^s d^\times t \right) \frac{ds}{2\pi i}.
\]

We then apply the local functional equation to the inner integral; after some matrix multiplication and appeal to the left-\( N \)-equivariance of \( V \), we obtain

\[
V \left( \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix} \right) = \int_{(0)} \frac{1}{\gamma(\pi, 1/2 + s)} \left( \int_{t \in \mathbb{R}_+^\times} e(-c/t)V \left( \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \right) |t|^s d^\times t \right) \frac{ds}{2\pi i}.
\]

We then substitute \( c \mapsto c/C(\Pi) \), apply the change of variables \( t \mapsto t/C(\Pi) \), and subtract the corresponding identity for \( c = 0 \), giving

\[
V \left( \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix} \right) - V(1) = \int_{(0)} \Theta(s, \Pi) \left( \int_{t \in \mathbb{R}_+^\times} (e(-c/t) - 1)V \left( \begin{pmatrix} C(\Pi) & 1 \\ t & 1 \end{pmatrix} \right) |t|^s d^\times t \right) \frac{ds}{2\pi i}.
\]

We claim now that if \( t \) is small, then \( V \left( \begin{pmatrix} C(\Pi) & 1 \\ t & 1 \end{pmatrix} \right) \) is negligible; more precisely, we claim that for any fixed integers \( M, N \geq 0 \),

\[
(t\partial_t)^N V \left( \begin{pmatrix} C(\Pi) & 1 \\ t & 1 \end{pmatrix} \right) \ll \min(1, t^M).
\]

From the claim it follows that \( e(-c/t) \approx 1 \) on the “essential support” of the inner integral, leading eventually to the required estimate (2.1).

We focus on the case \( N = 0 \) of the claim, and suppose that \( t \) is small; we must show then that

\[
V \left( \begin{pmatrix} C(\Pi) & 1 \\ t & 1 \end{pmatrix} \right) \ll t^M
\]

for any fixed \( M \). To that end, we appeal once again to the local functional equation, which gives

\[
V \left( \begin{pmatrix} C(\Pi) & 1 \\ t & 1 \end{pmatrix} \right) = \int_{(0)} t^s \Theta(s, \Pi) \tilde{f}(s) \frac{ds}{2\pi i},
\]

with \( \tilde{f}(s) := \int_{t \in \mathbb{R}_+^\times} f(t)t^{-s} d^\times t \). By the construction of \( f \), its Mellin transform \( \tilde{f} \) is entire and of rapid decay in vertical strips. The crux of the argument is to now shift the integration in (2.6) to the line \( \Re(s) = M \) for some fixed, large and positive \( M \); the properties of \( \Theta \) summarized above imply that

\[
\Theta(s, \Pi) \tilde{f}(s) \ll |s|^{-2}
\]
(say) for such $s$, leading to the required estimate (2.5).

In summary, the proof in the case $n = 1$ follows readily from two applications of the local functional equation and a straightforward Paley–Wiener type analysis of the Mellin integral representation (2.6).

2.2. Difficulties in generalizing to $n \geq 2$. We choose $V$ in a similar manner to the $n = 1$ case; that is $V \left[ \begin{pmatrix} 1 & g \\ 1 & 1 \end{pmatrix} \right]$ is given by a bump function an unipotent equivariant bump function on $GL_n(\mathbb{R})$ (see (4.1) for details) and aim to show as before that for small $1 \times n$ row vectors $c$,

$$V \left[ \begin{pmatrix} 1 & c/C(\Pi) \\ 1 & 1 \end{pmatrix} \right] - V(1) \ll |c|.$$

The matrix entries here are written in the evident block form. We first appeal to the local functional equation much like in the $n = 1$ case, reducing in this way to proving estimates slightly more general than the following generalization of (2.5) (see Proposition 6.1 for details): if $a = \text{diag}(a_1, \ldots, a_n)$ is a diagonal matrix with positive entries and $a_1$ small, then

$$(2.8) 
V \left[ \begin{pmatrix} C(\Pi) \\ a \end{pmatrix} w \right] \ll_N \delta^{1/2}(a) a_1^N. \tag{2.8}$$

Here $\delta(a) := \prod_{j<k} |a_j/a_k|$ is the modular character of the upper-triangular Borel in $GL_n$. There are some similarities between the present task and what is accomplished in the non-archimedean analogue by [23, §5 Lemme].

For the proof of (2.8), we begin as in the $n = 1$ case by expanding the function $GL_n(\mathbb{R}) \ni h \mapsto V \left[ \begin{pmatrix} C(\Pi) \\ h \end{pmatrix} w \right]$ using the Whittaker-Plancherel formula for $GL_n(\mathbb{R})$ and applying the local functional equation for $GL_{n+1} \times GL_n$ (see (6.6)). We arrive in this way at the following generalization of (2.6):

$$(2.9) 
V \left[ \begin{pmatrix} C(\Pi) \\ a \end{pmatrix} w \right] = \int_{(0)^n} W_\mu(a) \Theta(\mu, \Pi) \langle f, W_\mu \rangle \frac{d\mu}{|c(\mu)|^2}. \tag{2.9}$$

Here

- $\int_{(0)^n}$ denotes an integral over $\mu \in \mathbb{C}^n$ with $\Re(\mu_1) = \cdots = \Re(\mu_n) = 0$,
- $W_\mu$ is the spherical Whittaker function normalized so that $W_\mu(1) \asymp 1$ (see 3.11 for details),
- $\Theta(\mu, \Pi) := C(\Pi)^{-\mu_1 - \cdots - \mu_n} \gamma(\Pi \otimes \tilde{\pi}_\mu, 1/2)$ is holomorphic for $\Re(\mu_i) > -1/2$ and has properties analogous to those of $\Theta(s, \mu)$ mentioned above in the $n = 1$ case, and
- $c(\mu)$ is a product of $\Gamma$-functions, related to the Plancherel density.
- $\langle f, W_\mu \rangle := \int_{N \setminus GL_n(\mathbb{R})} f(g) W_\mu(g) dg$, where $N$ is the unipotent subgroup of upper triangular matrices in $GL_n(\mathbb{R})$. 
The product \( \Theta(\mu, \Pi) \langle f, W_\mu \rangle \) enjoys strong estimates analogous to (2.7): it extends to a meromorphic function in \( \mu \), holomorphic in \( \Re(\mu_i) > -1/2 \) and of rapid decay in vertical strips, uniformly in \( \Pi \).

As in any Paley-Wiener type statement, we use the rapid decay of \( \langle f, W_\mu \rangle \) in \( \mu \) to ensure the convergence of the integral (2.9), and our source of the decay in the \( a_1 \) direction (i.e. as \( a_1 \to 0 \)) is contour shifts. However, the Paley-Wiener type argument as simply as in \( n = 1 \) case can not be directly applied in this case. In fact, it turns out that any shift of the \( \mu \) contour that avoids polar hyperplanes is insufficient to achieve the required bound \( \ll \min(1, a_1^N) \).

The reason for this obstruction is that \( W_\mu \) has asymptotic expansion of the form
\[
\delta^{1/2}(a) \sum_{w \in S_n} a^{w\mu} M_{w\mu}(a), \quad a^\mu = \prod_{i=1}^n a_i^{i\mu}.
\]
Here \( S_n \) is the Weyl group of \( \GL(n) \) and \( M(w\mu, a) \), which we informally call as \( M \)-Whittaker function, is an infinite series of the form
\[
\sum_{k \in \mathbb{Z} \geq 0} c_{w\mu}(k) \prod_{i=1}^{n-1} (a_i/a_{i+1})^{2k_i}.
\]

For example, on \( \GL(2) \) the spherical Whittaker function \( W_\mu(a) \) is given by the \( K \)-Bessel function as
\[
(a_1/a_2)^{1/2}(a_1a_2)^{(\mu_1+\mu_2)/2}K_{(\mu_1-\mu_2)/2}(2\pi a_1/a_2)
\]
\[
= a_1^{\mu_1+1/2} a_2^{\mu_2-1/2} \sum_{k=0}^{\infty} c_{\mu}(k)(a_1/a_2)^{2k} + a_1^{\mu_2+1/2} a_2^{\mu_1-1/2} \sum_{k=0}^{\infty} d_{\mu}(k)(a_1/a_2)^{2k}.
\]
for some complex coefficients \( c_{\mu} \) and \( d_{\mu} \). The infinite sums are essentially \( I \)-Bessel functions, and are exponentially increasing in \( a_1/a_2 \). If we shift the contour of \( \mu_1 \) to right side e.g. to \( \Re(\mu_1) = N \) for some large positive \( N \) we do not cross any pole. If \( a_2 \) is very large compared to \( a_1 \) (e.g. \( a_2 = a_1^{-1} \) and \( a_1 \to 0 \)) this contour shift does not yield the required bound \( \ll a_1^{-N} \). That is why we first need to decompose the Whittaker functions into finitely many \( M \)-Whittaker functions, (see Lemma 6.6), and for each summand \( M \)-Whittaker function we shift contour to the relevant direction and get the required bound. For instance, in this case in the first summand we shift \( \mu_1 \) and in the second we shift \( \mu_2 \) to the right side.

However, there is still one technical issue. Each \( M \)-Whittaker function, like the \( I \)-Bessel functions, is exponentially increasing in the positive roots. That is why we can effectively apply the technique of contour shifting only when the positive roots are bounded. For example, on \( \GL(2) \) we decompose \( W_\mu(\diag(a_1, a_2)) \) into relevant \( M \)-Whittaker function and shift contour only when \( a_1 < a_2 \), as in the discussion in the previous paragraph.

On the other hand, \( W_\mu(a) \) decays rapidly as \( a_2/a_1 \to 0 \). It is tempting to imagine that we should decompose only when \( a \) is in the positive Weyl chamber. If \( a \) does not lie in a positive Weyl chamber then at least one root is large, and we may expect the rapid decay
of the Whittaker function will save the day. Although this works when \( n = 2 \) this fails for general \( n \); for e.g., there are diagonal elements in \( \text{GL}(n) \) which barely fail to be in a positive Weyl chamber. For example, in \( \text{GL}(3) \) the element \( Y := \text{diag}(y, -y/\log y, e^{1/y}) \) as \( y \to 0 \) logarithmically fails to be in the positive chamber. For this element the rapid decay estimate of the Whittaker function yields only a logarithm decay
\[ W(Y) \ll |\log y|^{-N}, \]
not a polynomial decay \( \ll y^N \), so does not meet our requirement.

To deal with this issue we need to treat the elements like \( Y \) as if they are in the positive chamber. To do that we divide the set of diagonal matrices into two classes whether they satisfy a property \( \text{pop} \) or not (see Definition 6.1). The \( \text{pop} \) refers to whether the tuple \( (a_1, \ldots, a_n) \) of a diagonal element \( a = \text{diag}(a_1, \ldots, a_n) \) has a partial ordering of the form all of \( a_1, \ldots, a_s \) are smaller than all of \( a_{s+1}, \ldots, a_n \). For instance, the element \( Y \) above has \( \text{pop} \) property for \( s = 2 \), as \( y \to 0 \).

In Lemma 6.1 we showed for the elements \( a \) which do not satisfy \( \text{pop} \) the rapid decay of the Whittaker function implies the required bound. Rest of the section §6 is devoted to the case when \( a \) satisfies \( \text{pop}(s) \) for some \( s \). In this case we decompose the Whittaker function \( W \) into the \( M \)-Whittaker functions. However, we cannot do a full decomposition as we described above in the \( \text{GL}(2) \) case, because \( a \) may not lie in the positive Weyl chamber and \( M \) might exponentially blow up. To make sure we have control on the exponential increment we only partially decompose, so that the \( M \)-Whittaker functions have exponential increment only in the roots of the form \( a_i/a_j \) with \( 1 \leq i \leq s \) and \( s + 1 \leq j \leq n \); hence \( M \) does not blow up as \( a \) satisfies \( \text{pop}(s) \). Loosely speaking, a partial decomposition is corresponding to the Levi in \( \text{GL}(n) \) attached to the partition of \( n \) of the form \( n = s + 1 + \cdots + 1 \), and a full decomposition is the same with \( s = 1 \). Such full decompositions of the spherical Whittaker function have appeared in the literature (see for instance [19, 6]).

2.3. Sketch of the proof for \( \text{PGL}_{n+1}(F) \), where \( F \) is non-archimedean. We re-establish the invariance result [23, §5 Théorême] along the exact same lines we prove in the archimedean case. In [23] it is, instead, proved using the test vector property of the newvector. However, the proof in the \( p \)-adic case is, still, simpler due to existence of simpler explicit algebraic formulas. Also the corresponding \( M \)-Whittaker functions do not exponentially increase in the non-archimedean case.

We will only concentrate on proving the analogue of the crucial part, Proposition 5.1. Let \( F \) be a non-archimedean local field with ring of integers \( \mathfrak{o} \) and uniformizer \( \varpi \). Let \( \Pi \) be a generic irreducible tempered unitary representation of \( \text{GL}_{n+1}(F) \). Let \( V \) be the vector in \( \mathcal{W}(\Pi, \tilde{\psi}) \) given in the Kirillov model by
\[ V \left( \begin{pmatrix} nak & \lambda \\ 1 & \lambda \end{pmatrix} \right) = \psi(n) \text{char}_{\mathfrak{o}^\times}(a), \]
where \( n, a := \text{diag}(a_1, \ldots, a_n) \) are the unipotent and diagonal elements, respectively, and \( k \in \text{GL}_n(\mathfrak{o}) \).
Proposition 2.1. Let $w$ be the long Weyl element in $GL(n+1)$. Then
\[ V \left[ \left( C(\Pi) \right)_{a} \right] w \neq 0 \implies |a_1| \gg 1. \]

We emphasize the similarity of this proposition with [23, §5 Lemme]. For simplicity in the sketch of the proof we assume that $\Pi$ is supercuspidal.

**Sketch of the proof.** Our point of departure is, as in the archimedean case, the $p$-adic Kontorovich-Lebedev-Whittaker transform (for a proof in the case of $F = Q_p$ see [17], for general Whittaker-Plancherel formula we refer to [13]) and $GL(n+1) \times GL(n)$ local functional equation. We obtain
\[ V \left[ \left( C(\Pi) \right)_{a} \right] w = \int_{\pi} W_{\pi}(a) \gamma(1/2, \Pi \otimes \bar{\pi}) \omega_{\pi}^{-1}(C(\Pi)) \langle \text{char}_{\pi}, W_{\pi} \rangle d\mu_{p}(\pi). \]
Here $\pi$ runs over the spherical tempered dual of $GL_n(F)$ and $d\mu_{p}$ is the Plancherel measure on it. $W_{\pi}$ is the spherical Whittaker function of $\pi$ described by Shintani’s formula [34] below. Let $m \in \mathbb{Z}^n$ and $a = \text{diag}(\varpi^m)$, i.e. $a_i = \varpi^{m_i}$. Let $\alpha \in (S^1)^n$ be the Langlands parameters of $\pi$.

\[
W_{\pi}(a) = \begin{cases} 
\delta^{1/2}(a) \frac{\det((\alpha_j^{m_i+n-i})_{i,j})}{\prod_{i<j} (\alpha_i - \alpha_j)}, & \text{if } m_1 \geq \ldots \geq m_n, \\
0, & \text{if otherwise}.
\end{cases}
\]

Thus we may restrict $a$ to be of the form $\text{diag}(\varpi^m)$ with $m_1 \geq \ldots \geq m_n$. Inserting this formula for $W_{\pi}$ and explicating the Plancherel density (see [17]), we may rewrite
\[ V \left[ \left( C(\Pi) \right)_{a} \right] w \] as
\[
\delta^{1/2}(a) \int_{(S^1)^n} \det((\alpha_j^{m_i+n-i-1})_{i,j}) \gamma(1/2, \Pi \otimes \bar{\pi}) \omega_{\pi}^{-1}(C(\Pi)) \prod_{i>j} (\alpha_i - \alpha_j) d\alpha.
\]
First we note that when $\pi$ is unitary and unramified
\[ \omega_{\pi}^{-1}(C(\Pi)) \gamma(1/2, \Pi \otimes \bar{\pi}) \]
is independent of $\pi$ and is bounded. To see this, recall that if $\pi$ has Langlands parameters $\{\alpha_i\}_{i=1}^{n}$ then
\[ \gamma(1/2, \Pi \otimes \bar{\pi}) = \prod_{i=1}^{n} \gamma(1/2 - \alpha_i, \Pi). \]

As $\Pi$ is supercuspidal,
\[ \gamma(1/2 - \alpha_i, \Pi) = |\alpha_i|^{C(\Pi)} \epsilon(1/2, \Pi), \]
where $\epsilon$ denotes the epsilon factor of $\Pi$. From the above formula and unitarity of $\epsilon(1/2, \Pi)$ the claim is immediate.

We will now proceed along the sketch we have provided in the previous subsection in the archimedean case. The analogue of “decomposing the spherical Whittaker function...
into finitely many \( M \)-Whittaker function” is expanding the determinant \( \det((\alpha_j^{m_1+n-i})_{i,j}) \) into various monomials depending on \( \alpha_i \); i.e., the analogue of \( M \)-Whittaker function is a monomial of the form \( \prod_j \alpha_j^{n_j} \). We distribute the integral over this decomposition. A generic term in this decomposition will look like

\[
\delta^{1/2}(a) \int_{(S^1)^n} \alpha_j^{m_1} H_j(\alpha) d\alpha,
\]

where \( H_j(\alpha) \) is a meromorphic function in \( \{ \alpha \mid |\alpha_j| \leq 1 \}^n \) with poles at most at \( \alpha_i = 0 \), such that order of the pole at \( \alpha_j = 0 \) is bounded (i.e. does not depend on \( m \)). Thus making \( m_1 \) sufficiently positive we compute the \( \alpha_j \) integral to be zero. Hence we conclude. \( \square \)

3. Basic Notations and Auxiliary Lemmata

In the section we will recall some notations which we will use frequently. We will also need some some well-known tools from representation theory of \( \text{GL}_n(\mathbb{R}) \), which we will describe in the next few subsections.

We use the Iwasawa decomposition of \( G = NAK \) with \( N \) being the maximal unipotent subgroup of upper triangular matrices, \( A \) being the subgroup of the positive diagonal matrices, and \( K = \text{O}(n) \). We fix Haar measure \( dg, dn, dk \) on \( G, N, K \), respectively such that the volume of \( K \) is one, and

\[
dg = dn \frac{da}{\delta(a)} dk, \quad g = nak,
\]

where \( da \) on \( A \ni \text{diag}(a_1, \ldots, a_n) \) is given by \( \prod_i d^x a_i \) and \( \delta \) is the modular character on \( NA \). We will use similar Haar measures on \( \text{GL}(r) \) (and its subgroups) for any \( r \) without mentioning explicitly.

We introduce a modified Vinogradov notation. Let \( \epsilon > 0 \) be a fixed small quantity (say, \( < n^{-10} \)). In this article we abbreviate the inequality

\[
\varphi_1(a, \ldots) \ll_{\epsilon, \ldots} \varphi_2(a, \ldots) \prod_{i=1}^n (a_i + a_i^{-1})^\epsilon
\]

by

\[
\varphi_1(a, \ldots) \preceq_{\ldots} \varphi_2(a, \ldots),
\]

where \( \varphi_i \) are some functions on \( a, \ldots \).

3.1. Additive character. Recall the maximal unipotent \( N < G \). We fix an additive character \( \psi \) of \( N \), which is given by

(3.1) \[
\psi(n(x)) = e \left( \sum_{i=1}^{n-1} x_{i,i+1} \right), \quad n(x) := (x_{i,j}) \in N.
\]

where \( e(z) := \exp(2\pi iz) \). By \( \tilde{\psi} \) we will denote the similarly defined character of \( N_{n+1} \) which is the maximal unipotent subgroup of \( \text{GL}_{n+1}(\mathbb{R}) \).
3.2. **Whittaker and Kirillov models.** For details of this subsection we refer to [26 Chapter 3]. Let \( \Pi \) be a generic irreducible unitary representation of \( \text{GL}_{n+1}(\mathbb{R}) \). Let \( \tilde{\psi} \) be the character of \( N_{n+1} < \text{GL}_{n+1}(\mathbb{R}) \) as in the previous subsection. Recall that \( \Pi \) is generic if

\[
\text{Hom}_{\text{GL}_{n+1}(\mathbb{R})}(\Pi, \text{Ind}_{N_{n+1}(\mathbb{R})}^\text{GL}_{n+1}(\mathbb{R}) \tilde{\psi}) \neq 0.
\]

It is known that if \( \Pi \) is generic then the above space is one dimensional. Let \( \lambda \) be a nonzero element in this Hom-space. Then the Whittaker model \( W(\Pi, \tilde{\psi}) \) of \( \Pi \) is the image of \( \lambda \). One writes

\[
W_v(g) = \lambda(\Pi(g)v), \quad g \in \text{GL}_{n+1}(\mathbb{R}), \, v \in \Pi.
\]

If \( \Pi \) is unitary then the corresponding unitary structure on \( W(\Pi, \tilde{\psi}) \) is given by

\[
\langle W_1, W_2 \rangle_{W(\Pi, \tilde{\psi})} = \int_{N \backslash G} W_1\left(\begin{pmatrix} g & \varepsilon \\ 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} g & \varepsilon \\ 1 \end{pmatrix}\right) dg.
\]

for \( W_1, W_2 \in W(\Pi, \tilde{\psi}) \).

Let \( C_c^\infty(N \backslash G, \tilde{\psi}) \) be the set of smooth functions on \( G \) compactly supported mod \( N \) such that

\[
\phi(ng) = \psi(n)\phi(g), \quad n \in N, \, g \in G.
\]

The theory of Kirillov model states that [26 Proposition 5] there exists a unique \( W_\phi \in W(\Pi, \tilde{\psi}) \) such that

\[
W_\phi\left(\begin{pmatrix} g & \varepsilon \\ 1 \end{pmatrix}\right) = \phi(g),
\]

and the map \( \phi \mapsto W_\phi \) is continuous.

3.3. **Langlands parameters.** For a representation \( \pi \) of \( \text{GL}_m(\mathbb{R}) \) let us write its \( L \)-factor

\[
L(s, \pi) = \prod_{i=1}^m \Gamma_\mathbb{R}(s + \mu_i(\pi)),
\]

where \( \mu_i(\pi) \in \mathbb{C} \) are the **Langlands Parameters** attached to \( \pi \). In this case we define the analytic conductor of \( \pi \) to be

\[
C(\pi) = \prod_{i=1}^m (1 + |\mu_i(\pi)|).
\]

By \( \mu \) we will denote a complex \( n \)-tuple \((\mu_1, \ldots, \mu_n)\). We define a quantity

\[
c(s, \mu) := \prod_{i<j} \Gamma_\mathbb{R}(s + \mu_i - \mu_j).
\]

We abbreviate \( c(0, \mu) \) as \( c(\mu) \).
3.4. **Whittaker-Plancherel formula.** For general discussion on the Whittaker-Plancherel theorem we refer to [38, Chapter 15]. Let \( \hat{G} \) be the set of isomorphism classes of generic irreducible tempered unitary representations of \( G \). Let \( \hat{G}_0 \subseteq \hat{G} \) be the isomorphism classes of spherical representations (by spherical representation we mean a representation which contains a right \( K \)-invariant vector). One can write down the general Whittaker-Plancherel formula for \( G \) as follows. Let \( F \in L^2(N \backslash G, \psi) \). Then,

\[
F(g) = \int_{\hat{G}} \sum_{W \in \mathcal{B}(\pi)} W(g) \langle F, W \rangle d\mu_p(\pi),
\]

where \( d\mu_p \) is the Plancherel measure on \( \hat{G} \), and

\[
\langle F, W \rangle := \int_{N \backslash G} F(g) \overline{W(g)} dg.
\]

Here \( \mathcal{B}(\pi) \) is an orthonormal basis of \( \pi \). The above sum does not depend on a choice of \( \mathcal{B}(\pi) \).

We may choose a basis

\[
\mathcal{B}(\pi) := \bigcup_{\tau \in K} \{ W_\tau^i \mid 1 \leq i \leq n_\tau \}
\]

consisting of \( K \)-isotypic vectors. Here \( \{ W_\tau^i \}_{i=1}^{n_\tau} \) is an orthonormal basis of \( \tau \)-type. We also know that \( n_\tau = 1 \) if \( \tau \) is the trivial representation. We will now produce a rather simplified version of the Whittaker-Plancherel formula for spherical functions i.e. functions which are right \( K \)-invariant. We first note that, if \( F \in L^2(N \backslash G, \psi)^K \) then,

\[
\langle F, W \rangle = 0, \quad \text{for all } W \in \mathcal{B}(\pi) \setminus \pi^K.
\]

Therefore for spherical \( F \) only the spherical representations will contribute to the right hand side of (3.2). Let \( W_\pi \in \pi^K \) with \( \| W_\pi \| = 1 \). Then (3.2) reduces to

\[
(3.3) \quad F(g) = \int_{\hat{G}_0} W_\pi(g) \langle F, W_\pi \rangle d\mu_p(\pi).
\]

3.5. **Local functional equation.** Let \( \omega_\pi \) be the central character of \( \pi \). Then for any \( W \in \mathcal{W}(\pi, \tilde{\psi}) \) the local functional equation is (see [24, 25])

\[
(3.4) \quad \int_{N \backslash G} V \left[w \begin{pmatrix} g^{-1} & \\ 1 \end{pmatrix}\right] W(w'g^{-t}) |\det(g)|^{-s} dg
= \omega_\pi(-1)^n \gamma(1/2 + s, \Pi \otimes \tilde{\pi}) \int_{N \backslash G} V \left[\begin{pmatrix} g & \\ 1 \end{pmatrix}\right] W(g) |\det(g)|^s dg,
\]

where \( w, w' \) are long Weyl elements of GL_{n+1} and \( G \) respectively, and

\[
\gamma(s, \Pi \otimes \tilde{\pi}) := \epsilon(s, \Pi \otimes \tilde{\pi}) \frac{L(1 - s, \tilde{\Pi} \otimes \tilde{\pi})}{L(s, \Pi \otimes \pi)},
\]

where \( \epsilon(s, \Pi \otimes \tilde{\pi}) \) is the epsilon factor.
and $\epsilon(s,\cdot)$ is the epsilon factor. Changing variable $g \mapsto w'g^{-t}w'$ we can also rewrite (3.4) as

$$\int_{N\setminus G} V \left[ \begin{pmatrix} 1 \\ g \end{pmatrix} w \right] W(gw') |\det(g)|^s dg$$

$$= \omega \pi (-1)^n \gamma(1/2 + s, \Pi \otimes \pi) \int_{N\setminus G} V \left[ \begin{pmatrix} g \\ 1 \end{pmatrix} \right] W(g) |\det(g)|^s dg.$$

3.6. Spherical tempered dual. One can parametrize $\hat{G}_0$ by

$$\{ \mu := (\mu_1, \ldots, \mu_n) \mid \mu_i \in i\mathbb{R}, 1 \leq i \leq n \},$$

where a purely imaginary $n$-tuple $\mu$ corresponds with the induced representation

$$\pi_\mu := \text{Ind}_{B_\chi}^G \chi, \quad \chi(na) = \prod_{i=1}^n |a_i|^\mu_i, \quad n \in \mathbb{N},$$

where $a := \text{diag}(a_1, \ldots, a_n)$, because any tempered spherical representation of $G$ is of the above form. For later purpose, for $\mu \in \mathbb{C}^n$ we define the quantity

$$d(\pi_\mu) := d(\mu) := 1 + \sum_{j=1}^n |\Im(\mu_j)|^2.$$

3.7. Conductors and gamma-factors. Recall the definition of $\gamma$-factor from §3.5 and the definition of Conductor.

Lemma 3.1. Let $\Pi$ and $\pi$ be generic irreducible unitary representations of $\text{GL}_{n+1}(\mathbb{R})$ and $G$, respectively. Then

1. For $s \in \mathbb{C}$ of bounded real part and a fixed positive distance away from any pole of $\gamma(1/2 - s, \pi)$, we have

$$\gamma(1/2 - s, \pi) \asymp C(\pi \otimes |\det|^3(s))^{\Re(s)}.$$

2. $\frac{C(\Pi)^n}{C(\pi)^{n+1}} \leq C(\Pi \otimes \pi) \leq C(\Pi)^n C(\pi)^{n+1}$.

Proof. (1) is standard and follows from the Stirling approximation of the $L$-factors, for e.g., (see [5]). The second inequality of (2) is obtained in the [20 Appendix A]. The first inequality can be proved in the very same way as the other one. As in the appendix of [20 Appendix A] one can appeal to Langlands classification of the admissible dual and reduce to the case of representations $\Phi$ and $\phi$ of the Weil-Deligne group of $\mathbb{R}$. For instance, let both $\Phi$ and $\phi$ are one dimensional representations with Langlands parameters $(\mu, 0)$ and $(\nu, 0)$, respectively, then the parameter of $\Phi \otimes \phi$ can be given by $(\mu + \nu, 0)$. Then the first inequality follows from

$$(1 + |\mu|) \leq (1 + |\mu + \nu|)((1 + |\nu|).$$

Rest of the cases follow similarly. \qed
For Brevity, we define $\Theta : \hat{G} \to \mathbb{C}$ by

$$\pi \mapsto \Theta(\pi, \Pi) := \omega_\pi^{-1}(C(\Pi))\gamma(1/2, \Pi \otimes \bar{\pi}),$$

where $\omega_\pi$ is the central character of $\pi$. If $\pi$ is the spherical representation $\pi_\mu$ for some $\mu \in \mathbb{C}^n$ we, by abuse of notation, denote $\Theta(\pi_\mu, \Pi)$ by $\Theta(\mu, \Pi)$. We record that it follows from Lemma 3.1

$$\Theta(\mu + 2M, \Pi) \ll_M \prod_{i=1}^n (1 + |\mu_i|)^O_M(1).$$

for $M \in \mathbb{Z}_{\geq 0}$ fixed and $\mu \in \mathbb{C}^n$ with $0 \leq R(\mu) \ll 1$. We also note that if $\Pi$ is $\theta$-tempered for $0 \leq \theta < 1/2$ then $\Theta(\mu, \Pi)$ is holomorphic for $R(\mu_i) \geq 0$.

3.8. **Explicit Plancherel measure.** We describe the Plancherel measure explicitly in the case of the spherical Whittaker-Plancherel transform (3.3). From (see [15]) we get that if $\pi_\mu \in \hat{G}_0$ for some $\mu \in i\mathbb{R}^n$ then

$$d\mu_\mu(\pi_\mu) = \left|\frac{c(1, \mu)}{c(0, \mu)}\right|^2 d\mu_1 \ldots d\mu_n,$$

where $d\mu_i$ are the Lebesgue measures on $i\mathbb{R}$ normalized by $2\pi i$.

3.9. **Differential operator and Sobolev norm.** Let $\{X_i\}$ be a basis of the $\mathfrak{g} := \text{Lie}(G)$. We define, for each $M \geq 0$, a second order differential operator by

$$D_M := M + 1 - \sum_{i=1}^{n^2} (X_i^2).$$

We abbreviate $D_0$ as $D$. We define a Sobolev norm on the space of $\pi \in \hat{G}$ by

$$S_d(v) := \|D^d v\|_\pi,$$

A similar sort of Sobolev norm has been used in [30].

**Lemma 3.2.** Let $D_M$ be the differential operator in (3.8).

1. $D_M$ is self-adjoint and positive definite on unitary representations of $G$. Eigenvalues of $D_M$ are at least $M + 1$.
2. If $C_G$ and $C_K$ denote the Casimir elements for the groups $G$ and $K$, respectively then,

$$D_M = M - (1 + C_G) + 2(1 + C_K).$$

3. $C_G$ acts on $\pi_\mu$ by the scalar $\lambda(\pi_\mu) := -T + \|\mu\|^2$, where $T > 0$, and $\|\mu\|^2 := \sum_{i=1}^n |\mu_i|^2$ is an absolute constant depending only on $n$.
4. Eigenvalues of $D_M$ on $\pi_\mu$ are of size $\gg 1 + \|\mu\|^2$. 

Proof. (1) is standard and follows from the fact that \( \sum_{i=1}^{n^2} X_i^2 \) is self-adjoint and negative definite. This can be found in [31]. To prove (2) note that \( g = p + \mathfrak{k} \) where \( p \) and \( \mathfrak{k} \) are Lie algebras of \( NA \) and \( K \), respectively. Thus from the definition of the standard Cartan involution and inner product we get that

\[
C_G = -\sum_{X_i \in \mathfrak{k}} X_i^2 + \sum_{Y_i \in p} Y_i^2, \quad C_K = -\sum_{X_i \in \mathfrak{k}} X_i^2.
\]

Thus from the definition of (3.2) we get that

\[
D_M = M + 1 - \sum_{X_i \in \mathfrak{k}} X_i^2 - \sum_{Y_i \in p} Y_i^2 = M - (1 + C_G) + 2(1 + C_K).
\]

(3) is standard. In (4), \( C_K \) will act trivially on \( \pi_\mu \) as \( \pi_\mu \) is spherical. Then the result follows from (3). □

Lemma 3.3. Let \( S_d \) be the Sobolev norm defined in (3.9). Then for \( d_1, d_2 > 0 \) there exists \( L := L(d_1, d_2) > 0 \) such that

\[
\int_G C(\pi)^{d_1} \sum_{W \in \mathcal{B}(\pi)} S_{d_2}(W)S_{-L}(W)d\mu_p(\pi)
\]

is convergent. Here \( \mathcal{B}(\pi) \) is an orthonormal basis of \( \pi \) consisting of eigenvectors of \( D_+ \).

Proof. Let \( \mathcal{B}(\pi) = \{W_i\}_{i \in \mathbb{N}} \) with eigenvalues \( \{\lambda_i\}_{i \in \mathbb{N}} \), correspondingly. From (1) of Lemma 3.2 we get that \( \lambda_i \geq 1 \). Thus,

\[
\sum_{W \in \mathcal{B}(\pi)} S_{d_2}(W)S_{-L}(W) = \text{Trace}_\pi (D^{d_2-L}).
\]

There exists an element \( P_{d_1} \) in the center of the universal enveloping algebra of \( G \) such that

\[
C(\pi)^{d_1} \ll \lambda_\pi(P_{d_1}),
\]

where \( \lambda_\pi(P) \) is the scalar by which \( P \) acts on \( \pi \). Thus the integral in the question can be bounded by

\[
\int_G \text{Trace}_\pi (P_{d_1}D^{d_2-L})d\mu_p(\pi).
\]

From [32] §8.5, Lemma 2] we know that for large enough \( A \) the operator \( P_{d_1}D^{-A} \) is bounded. Finally, the integral \( \int_G \text{Trace}_\pi (D^{-B})d\mu_p(\pi) \) is convergent for sufficiently large \( B \). A proof of this result can be found in [32] §A.4.2, Lemma (ii)]. We conclude our proof by making \( L \) large enough. □
3.10. **Spherical Whittaker functions.** In this subsection we will work out some relevant analysis of the spherical Whittaker function on $G$. The general references for spherical Whittaker functions are [14, Chapter 5], [35, 36], and Jacquet’s work [27]. Let $\mu \in \mathbb{C}^n$. We call $\pi_\mu$ to be the spherical principal series representation with the Langlands parameters $\mu$. Let $W_\mu$ be the spherical vector in $\pi_\mu$ defined by the following normalization of the Jacquet’s integral.

\[
W_\mu(g) := c(1, \mu) d_n \int_N I_\mu(wng) \psi(n) dn, \quad g \in G, \Re(\mu_i - \mu_{i+1}) > 0;
\]

where

\[
I_\mu(nak) := \delta^{1/2}(a) \prod_{i=1}^n a_i^{\mu_i}, \quad n \in N, a \in A, k \in K.
\]

Jacquet in [27] showed that $W_\mu$ has a analytic continuation to $\mathbb{C}^n$ and is invariant under action of the Weyl group on $\mu$. Here $d_n$ is an absolute constant such that when $\mu$ is purely imaginary,

\[
\|W_\mu\|^2 = |c(1, \mu)|^2,
\]

by Stade’s formula [36, Theorem 1.1]. We record two type of bounds of $W_\mu$ we will use at various stages of the proofs.

**Lemma 3.4.** Let $\mu$ be purely imaginary. Then for any $M \in \mathbb{Z}_{\geq 0}^{n-1}$

\[
\frac{W_\mu(a)}{c(1, \mu)} \prec_M \delta^{1/2}(a) d(\mu)^{O_M(1)} \prod_{j=1}^{n-1} \min(1, a_{j+1}/a_j)^{M_j},
\]

where $O_M(1)$ denotes a bounded quantity depending on $M$.

**Proof.** This result is already proved in a similar form in [3, Theorem 1]. However, as we are happy with a polynomial dependency on $\mu$ we can infer from more general result in Lemma 5.2. \qed

**Lemma 3.5.** Let $\mu \in \mathbb{C}^n$ such that $\Re(\mu_i)$ are non-negative distinct and small enough (say $< 1/100$). Then for any $k \in \mathbb{Z}_{\geq 0}^n$

\[
W_{\mu+k}(a^\sigma) \prec c(1, -\sigma(\mu + k)) \delta^{1/2}(a^\sigma) \prod_{i=1}^n a_i^{\Re(\sigma(\mu + k))i},
\]

where $\sigma \in S_n$ such that $\Re(\mu_{\sigma(1)} + k_{\sigma(1)}) \leq \ldots \leq \Re(\mu_{\sigma(n)} + k_{\sigma(n)})$.

**Proof.** Let $\mu' := \mu + k$. Using the Weyl group invariance of $W_{\mu'}$ we may assume, without loss of generality, that $\Re(\mu'_i) \geq \ldots \geq \Re(\mu'_n)$. Also note that the assumption on the real parts of $\mu$ forces the above ordering to be strict. In fact, there is an $\epsilon > 0$ such that
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\( n \) \(( \mathbb{R} )^2 \) \( \min \{ \Re(\mu'_i - \mu'_j) \mid i < j \} \geq \epsilon \). Now we do a change of variable int the integral of (3.10) to see that

\[ W_{\mu'}(a) = d_n c(1, \mu') \delta^{1/2}(a) \prod_{i=1}^{n} a_i^{(\mu_i')_i} \int_{N} I_{\mu'}(wn) \psi(-ana^{-1})dn \]

\[ \ll |c(1, \mu')| \delta^{1/2}(a^s) \prod_{i=1}^{s} a_i^{\Re((\mu_i')_i)}. \]

The last integral is absolutely convergent, as \( \min \{ \Re(\mu'_i - \mu'_j) \mid i < j \} \geq \epsilon \) (see [27]). Moreover, we can bound the last integral uniformly in \( \mu' \). A proof of this is essentially done in the proof of the absolute convergence of the Jacquet’s integral in [14, Chapter 5]. From that proof it can be inductively seen that

\[ \int_{N} |I_{\mu'}(wn)|dn \ll \int_{N} I_\epsilon(wn)dn \ll \epsilon^{-1}, \]

where \( I_\epsilon \) is the spherical section in the principal series with real parameters \( \nu \) such that \( \min \{ (\nu_i - \nu_j) \mid i < j \} \geq \epsilon \).

We record a rapid decay estimate of the Whittaker transform of the test function \( f \), as follows, which we will need later in the proof.

**Lemma 3.6.** Let \( \mu \in \mathbb{C}^n \) with \( \Re(\mu_i) \geq 0 \) and \( \sum_{i=1}^{n} \Re(\mu_i) \leq R \) for all \( i \) and for some \( R \geq 0 \). Let \( p \) be a fixed sufficiently large natural number. Then for each fixed \( f \in C_\infty(N \backslash G, \psi) \),

\[ \langle f, W_{\mu} \rangle := \int_{N \backslash G} f(g) \overline{W_{\mu}(g)} dg \ll_{R, p} d(\mu)^{-p} |c(1, \Im(\mu))|. \]

**Proof.** Recall from Lemma 3.2 that \( D_R = R - C_G + 2C_K \). As \( W_{\mu} \) is right \( K \)-invariant, \( C_K W_{\mu} = 0 \). Thus

\[ D_RW_{\mu} = (R + T - \sum_{j=1}^{n} \mu_j^2)W_{\mu}. \]

We check that

\[ |R + T - \sum_{j=1}^{n} \mu_j^2| \geq T + \sum_{j=1}^{n} |\Im(\mu_j)|^2 \leq d(\mu). \]

Let \( Z \) denote the center of \( G \); we identify it with \( \mathbb{R}^\times \) in the usual way. Integrating by parts with respect to \( D^p \), we obtain

\[ \langle f, W_{\mu} \rangle = (R + T - \sum_{j=1}^{n} \mu_j^2)^{-p} \int_{Z} W_{\mu}(h) \int_{Z} D^p f(zh) |z|^{\sum_{i=1}^{n} \mu_i} d^x z dh. \]
We apply Cauchy-Schwartz on the outer integral and use sphericity of $W_\mu$ to obtain that the last integral is
\[ \leq_{p,R,f} d(\mu)^{-p} \int_{\mathbb{Z}N\backslash G} |W_\mu(h)|^2 dh \asymp d(\mu)^{-p} \prod_{i,j} \Gamma_R(1 + \mu_i + \mu_j). \]

Here the last estimate follows from [36, Theorem 1.1]. We use Stirling’s estimate to obtain that
\[ \prod_{i,j} \Gamma_R(1 + \mu_i + \mu_j) \asymp R \prod_{i \neq j} |\Im(\mu_i) - \Im(\mu_j) + \Re(\mu_i) + \Re(\mu_j)| \]
\[ \asymp_{R} \prod_{i \neq j} |\Im(\mu_i) - \Im(\mu_j)| R \Gamma_R(1 + \Re(\mu_i) - \Re(\mu_j)) \ll_{R,p} d(\mu)^R |c(1, \Im(\mu))|^2, \]
where $R'$ is a bounded constant depending on $R$ and $n$. Making $p$ sufficiently large we conclude the proof.

\section{Reduction of the proof of the main results}

We adopt the standard convention from analytic number theory of writing $\epsilon$ for a small positive fixed quantity, whose precise value we allow to change from one line to the next.

Let $\Omega \subseteq \text{GL}_n(\mathbb{R})$ be the bounded neighborhood of the identity element and $\iota$ be as in Theorem 2. Recall $\psi$ from (3.1). We first construct $V \in W(\Pi, \tilde{\psi})$ using the theory of the Kirillov model. We denote by $C^\infty(\mathbb{N}\backslash G, \psi)$ the space of smooth functions $f : G \rightarrow \mathbb{C}$ satisfying $f(ng) = \psi(n) f(g)$ for all $n \in \mathbb{N}$ and for which the support of $f$ has compact image in $\mathbb{N}\backslash G$. We choose an element $f \in C^\infty_c(\mathbb{N}\backslash G, \psi)$ with the following properties:

- $f$ is right $K$-invariant
- $f(h) \geq \iota$ for all $h \in \Omega$
- $\int_{\mathbb{N}\backslash G} |f|^2 dg = 1$

We now define $V$ by requiring that
\begin{equation}
V \left[ \begin{pmatrix} g \\ 1 \end{pmatrix} \right] := f(g),
\end{equation}

We remark that the sphericality assumption of $f$ is not essential. We refer to the discussion in §6.2 for details.

\begin{proposition}
Let $\Pi$ be as in Theorem 2. For every $\delta > 0$ and $h$ in a fixed bounded neighbourhood around the identity in $G$ there exists a $\tau > 0$ such that
\[ \left| V \left[ \begin{pmatrix} h \\ c/C(\Pi) \end{pmatrix} \right] - V \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \right| < \delta, \]
for $c \in \mathbb{R}^n$ with $|c| < \tau$. Here $V$ is the vector chosen in (4.1).
\end{proposition}
We will now assume Proposition 4.1 and prove Theorems 2. In the next two sections we will prove Proposition 4.1. In the following Lemma we first prove a weaker version of Theorem 2.

Lemma 4.1. For every $\delta > 0$ and $h$ in a fixed bounded neighbourhood around the identity in $G$ there exists a $\tau > 0$ such that

- the normalization $\| V \|_{W(\Pi, \tilde{\psi})} = 1$, with the norm taken in the Kirillov model (§3.2),
- the lower bound $V \left( \begin{array}{c} h \\ 1 \end{array} \right) \geq \iota$ for all $h \in \Omega$, and
- for all $g \in K_0(C(\Pi), \tau)$, and for $h \in \Omega$,

$$\left| V \left( \begin{array}{c} h \\ 1 \end{array} \right) g - \omega_{\Pi}(d_g)V \left( \begin{array}{c} h \\ 1 \end{array} \right) \right| < \delta.$$ 

Here $V$ is the vector chosen in (4.1) and $\omega_{\Pi}$ and $d_g$ are as in Theorem 1.

Proof. We choose $V$ as in (4.1). We note that first two requirements of $V$ are automatically satisfied by the choice (4.1). To prove the invariance property of $V$ we claim the following.

For every $\delta > 0$ and $h$ in a given fixed bounded set there exists $\tau > 0$ such that for all $(g' c_1) \in K_0(C(\Pi), \tau)$,

$$\left| V \left( \begin{array}{c} h g' \\ c \\ 1 \end{array} \right) - V \left( \begin{array}{c} h \\ 1 \end{array} \right) \right| < \delta.$$ 

This claim is sufficient. To see that first we note the following Iwahori-type decomposition that

$$GL_{n+1}(\mathbb{R}) \ni \left( \begin{array}{cc} A & b \\ c & d \end{array} \right) = d \left( \begin{array}{cc} 1_n & b/d \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} A/d - bc/d^2 & -bc/d \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1_n & b \\ 0 & 1 \end{array} \right).$$

Hence we can assume that any $g \in K_0(C(\Pi), \tau)$ is of the form

$$g = d_g \left( \begin{array}{cc} 1_n & b \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} g' c_1 \\ 0 \end{array} \right),$$

with $g' \in G$ such that $\|g' - 1\| \ll \tau$ and $|b|, |d - 1|, C(\Pi)|c| \ll \tau$. Therefore,

$$V \left( \begin{array}{c} h g' \\ c \\ 1 \end{array} \right) = \omega_{\Pi}(d_g)\tilde{\psi} \left( \begin{array}{c} 1_n \\ h b \\ 1 \end{array} \right) \omega_{\Pi}(d_g)\tilde{\psi} \left( \begin{array}{c} 1_n \\ h g' \\ c \\ 1 \end{array} \right).$$

Therefore using unitarity of $\omega_{\Pi}$ and $\tilde{\psi}$,

$$\left| V \left( \begin{array}{c} h \\ 1 \end{array} \right) g - \omega_{\Pi}(d_g)V \left( \begin{array}{c} h \\ 1 \end{array} \right) \right| = \left| (e(h b, c_n) - 1) V \left( \begin{array}{c} h g' \\ c \\ 1 \end{array} \right) + V \left( \begin{array}{c} h g' \\ c \\ 1 \end{array} \right) - V \left( \begin{array}{c} h \\ 1 \end{array} \right) \right|. $$
Now using the claim we get that all
\[ V \left( \begin{pmatrix} hg' c & 1 \\ 1 & 1 \end{pmatrix} \right) \ll 1, \]
and hence the invariance property by making \( b \) small enough.

Now we turn to prove the claim. We note that from (4.1), for \( h \) in a compact set in \( G \) there exists \( \tau \) small enough with \( \|g' - 1\| < \tau \) such that
\[ \left| V \left( \begin{pmatrix} hg' c & 1 \\ 1 & 1 \end{pmatrix} \right) - V \left( \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} \right) \right| < \delta. \]
Now applying triangle inequality in the following
\[ V \left( \begin{pmatrix} hg' c & 1 \\ 1 & 1 \end{pmatrix} \right) - V \left( \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} \right) = V \left( \begin{pmatrix} hg' c & 1 \\ 1 & 1 \end{pmatrix} \right) - V \left( \begin{pmatrix} hg' c & 1 \\ 1 & 1 \end{pmatrix} \right) + V \left( \begin{pmatrix} hg' c & 1 \\ 1 & 1 \end{pmatrix} \right) - V \left( \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} \right), \]
along with Proposition 4.1 we prove the claim.

\[ \square \]

Proof of Theorem 2 assuming Proposition 4.1. First of all we can use similar technique as in Lemma 4.1 and reduce to the case \( g \in K_1(C(\Pi), \tau) \) using the unitarity of \( \omega_\Pi \), as follows,
\[ |\Pi(g)V - \omega_\Pi(dg)V| = |\Pi(g/dg)V - V|. \]
Set \( \delta_0 := \min(\delta, 1) \omega/2 \) where \( \delta \) and \( \omega \) are as in Theorem 2. Let \( V_0 \in W(\Pi, \tilde{\psi}) \) and \( \tau_0 > 0 \) be as in (4.1), so that
\[ V_0 \left( \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} \right) \geq \omega \]
and for all \( g \in K_1(C(\Pi), \tau_0), h \in \Omega, \)
\[ \left| V_0 \left( \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} g \right) - V_0 \left( \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} \right) \right| \leq \delta_0. \]
These follow from Lemma 4.1 and the toy theorem.

Let \( \xi \) be the \( L^1 \)-normalized characteristic function of \( K_1(C(\Pi), \tau_1) \). Note that there exists \( \tau_2 > 0 \) such that for \( g \in K_1(C(\Pi), \tau_2), \)
\[ \|g \ast \xi - \omega_\Pi(dg)\xi\|_{L^1} \leq \|g \ast \xi - \xi\| + |\omega_\Pi(dg) - 1| \leq \delta_0, \]
which follows from Lemma 4.1 and the toy theorem. Here \( g \ast \xi(h) := \xi(g^{-1}h), \) so that \( \Pi(g)\pi(\xi) = \Pi(g \ast \xi) \). Set \( V_1 := \pi(\xi)V_0 \). Since \( \|V_0\| = 1 \), we then have by the triangle inequality that for \( g \in K_1(C(\Pi), \tau_1), \)
\[ \|\Pi(g)V_1 - \omega_\Pi(dg)V_1\| \leq \|g \ast \xi - \omega_\Pi(dg)\xi\|_{L^1(G)} \leq \delta_0, \]
and for \( \tau_1 \) sufficiently small in terms of \( \tau_0 \)

\[
\left| V_1 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} g \right] - V_1 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] \right| \\
\leq \int_{K_1(C(\Pi), \tau_1)} \xi(t) \left| V_0 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} g t \right] - V_0 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} t \right] \right| \, dg \leq 2\delta_0.
\]

Also, for \( h \in \Omega \), we have

\[
V_1 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] - V_0 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] = \int_{K_1(C(\Pi), \tau_0)} \xi(t) \left( V_0 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} t \right] - V_0 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] \right) \, dt,
\]

hence

\[
\left| V_1 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] - V_0 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] \right| \leq \iota/2,
\]

so in particular

\[
V_1 \left[ \begin{pmatrix} h & 1 \\ 1 & 0 \end{pmatrix} \right] \geq \iota - \iota/2 = \iota/2 > 0.
\]

It follows that the vector \( V_1/\|V_1\| \in \mathcal{W}(\Pi, \tilde{\psi}) \) and its image in \( \Pi \) satisfy the required conclusions of Theorem 2 and Theorem 1, respectively. \( \square \)

5. Proof of Proposition 4.1

To prove Proposition 4.1 we need an uniform bound of an Weyl element shifted newvector. This is actually the heart and the most difficult part of the article.

Proposition 5.1. Let \( l \in \mathbb{N} \). Let \( g = ak \) with \( a := \text{diag}(a_1, \ldots, a_n) \in A, k \in K \). Then

\[
\mathcal{D}_M^l V \left[ \begin{pmatrix} C(\Pi) \\ g \end{pmatrix} w \right] \preceq \mathcal{D} \delta^{1/2}(a) \min(a_1^{N}, 1),
\]

where \( w \) is the long Weyl element in \( GL_{n+1}(\mathbb{R}) \).

We will prove the above proposition in the next section. For now we will see how to derive Proposition 4.1 from Proposition 5.1.

Lemma 5.1. Let \( p \in \mathbb{N} \) and \( \mathcal{D} \) as in (3.8). Then

1. \( \mathcal{D}^p |\det(g)|^\sigma \ll_p |\det(g)|^\sigma \),
2. \( \mathcal{D}^p (e(cw'h^{-1}e_1) - 1) \ll_p \sum_{r=1}^{2p+1} |c|^r |h^{-1}e_1|^r \).

Proof. Recall the definition of \( \mathcal{D} \). It is straightforward to check that

\[
\mathcal{D}^p |\det(g)|^\sigma = (1 - 4\sigma^2)^p |\det(g)|^\sigma,
\]

which proves (1).

Let \( x := 2\pi icw' \). So

\[
e(cw'h^{-1}e_1) - 1 = \sum_{r=1}^{\infty} \frac{(xh^{-1}e_1)^r}{r!}.
\]
Let \( f_r := \frac{(xh^{-1}e_1)^r}{r!} \). Then it is straightforward to check that
\[
D f_r = f_r - 2(xh^{-1}e_1)f_{r-1} - |x|^2|h^{-1}e_1|^2 f_{r-2}.
\]
One can also easily check that
\[
D(|x|^2|h^{-1}e_1|^2) = -4|x|^2|h^{-1}e_1|^2, \quad D(xh^{-1}e_1) = -xh^{-1}e_1.
\]
Therefore inducting on \( p \) and summing over \( r \) we conclude (2).

**Lemma 5.2.** Let \( \pi \) be a tempered representation of \( G \) and \( W \in W(\pi, \psi) \) be any \( L^2 \)-normalized vector. Let \( a = \text{diag}(a_1, \ldots, a_n) \in A \), \( y_i = a_i/a_{i+1} \), and \( k \in K \) and \( L > 0 \). For any \( \eta \) small enough and \( M \) large enough
\[
D^{-L}W(ak) \ll_{\eta, N} \delta^{1/2-\eta}(a) \prod_{i=1}^{n-1} \min(1, (a_{i+1}/a_i)^M)S_{p-L}(W),
\]
for \( p \) depends on \( M \) and \( G \) only.

**Proof.** As \( W(ak) = \pi(k)W(a) \) and \( S_p(W) \ll_{K} S_p(\pi(k)W) \) it is enough to prove the Lemma for \( k = 1 \). We will prove this inducting on \( n \). Note that for \( n = 2 \) this is proved in [30, Proposition 3.2.3]. We will generalize their idea of proof for general \( n \). First note that there is a \( Y \in g \) such that
\[
d\pi(Y)W(a) = (a_{n-1}/a_n)W(a).
\]
Let \( \omega_\pi \) be the central character of \( \pi \).

For a generic irreducible tempered representation \( \pi' \) of \( GL_{n-1}(\mathbb{R}) \) let \( B(\pi') := \{ W' \} \) be an orthonormal basis of \( \pi' \) consisting of eigenvectors of \( D' \) by diagonalizing it, where \( D' \) is the analogous differential operator on \( GL_{n-1}(\mathbb{R}) \) as in [33, 32]. We obtain using Whittaker-Plancherel formula (3.2) on \( GL(n-1) \)
\[
(a_{n-1}/a_n)^N D^{-L}W(a) = \omega_\pi(a_n)D^{-L}d\pi(Y^N)W(a/a_n)
\]
\[
= \omega_\pi(a_n) \int_{GL_{n-1}} \sum_{W' \in B(\pi')} W'(a'/a_n)Z(W')d\mu_p(\pi'),
\]
where \( a' := \text{diag}(a_1, \ldots, a_{n-1}) \), and
\[
Z(W') := \int_{N_{n-1}\backslash GL_{n-1}} D^{-L}d\pi(Z^N)W \begin{pmatrix} h & \cdot \\ 1 & 1 \end{pmatrix} W'(h)dh.
\]
We change variable \( \pi' \mapsto \pi' \otimes |\det|^s \) to obtain
\[
(a_{n-1}/a_n)^N D^{-L}W(a) = \omega_\pi(a_n) \int_{GL_{n-1}} \sum_{W' \in B(\pi')} W'(a'/a_n)|\det(a'/a_n)|^s Z(W' \otimes |\det|^s)d\mu_p(\pi').
\]
Using induction hypothesis we get
\[
W'(a'/a_n) \ll_{\eta, N} \delta^{1/2-\eta}(a') \prod_{i=1}^{n-2} (1, (a_{i+1}/a_i)^N)S_p(W').
\]
From the local functional equation (3.5) for $\text{GL}(n) \times \text{GL}(n-1)$ we confirm that $Z(W' \otimes |\det|^s)$ is holomorphic for $\Re(s) < 1/2$ (as both $\pi$ and $\pi'$ are tempered), and the defining integral of $Z$ is absolutely convergent. We choose $s = 1/2 - \eta$ for some small $\eta > 0$. We note that 
\[ |\det(a'/a_n)|^{1/2-\eta}d^{1/2-\eta}(a') = d^{1/2-\eta}(a). \]

Now in the integral defining $Z(W' \otimes |\det|^{1/2-\eta/2})$ we do integration by parts sufficiently many times using $D'$. Applying Lemma 3.3 we conclude. □

Proof of Proposition 4.1 assuming Proposition 2.1 Recall that $\Pi$ is $\theta$-tempered as in Theorem 2. For every $\pi \in G$ we choose an orthonormal basis $\mathcal{B}(\pi) := \{W\}$ of $\pi$ consisting of eigenvectors of $D$ by diagonalizing it. Applying the Whittaker-Plancherel formula (3.2) of $G$ we get, for $0 < \sigma < 1/2 - \theta$, that

\[
|\det(g)|^{-\sigma}V \left[ \begin{pmatrix} g \\ c \\ 1 \end{pmatrix} \right] = |\det(g)|^{-\sigma} \Pi \left[ \begin{pmatrix} 1 \\ n \\ c \\ 1 \end{pmatrix} \right] V \left[ \begin{pmatrix} g \\ 1 \end{pmatrix} \right]
\]

\[
= \int_{\mathcal{G}} \sum_{W \in \mathcal{B}(\pi)} W(g) \int_{N \setminus G} \Pi \left[ \begin{pmatrix} 1 \\ n \\ c \\ 1 \end{pmatrix} \right] V \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] |\det(h)|^{-\sigma} d\mu_p(\pi)
\]

We apply the $\text{GL}(n+1) \times \text{GL}(n)$ local functional equation (3.5) and the $N$-equivariance for $V$ to obtain the inner integral above equals to

\[
\gamma(1/2 - \sigma, \Pi \otimes \bar{\pi})^{-1} \int_{N \setminus G} e(cw'h^{-1}e_1)V \left[ \begin{pmatrix} 1 \\ h \end{pmatrix} \right] W(hw') |\det(h)|^{-\sigma} dh.
\]

Changing variable $h \mapsto C(\Pi)^{-1}h$ in the latter integral we conclude that

\[
V \left[ \begin{pmatrix} g \\ c/C(\Pi) \\ 1 \end{pmatrix} \right] - V \left[ \begin{pmatrix} g \\ 1 \end{pmatrix} \right]
\]

\[
= \int_{\mathcal{G}} C(\Pi)^{n\sigma} \gamma(1/2 - \sigma, \Pi \otimes \bar{\pi})^{-1} \sum_{W \in \mathcal{B}(\pi)} W(g) |\det(g)|^{\sigma} \int_{N \setminus G} (e(cw'h^{-1}e_1) - 1)V \left[ \begin{pmatrix} C(\Pi) \\ h \end{pmatrix} \right] W(hw') |\det(h)|^{-\sigma} d\mu_p(\pi).
\]

From Lemma 3.3

\[
C(\Pi)^{n\sigma} \gamma(1/2 - \sigma, \Pi \otimes \bar{\pi})^{-1} \ll C(\pi)^{(n+1)\sigma}.
\]

In the last integral of (5.1) we integrate by parts by $D$ as

\[
\int_{N \setminus G} D^L \left( (e(cw'h^{-1}e_1) - 1)V \left[ \begin{pmatrix} C(\Pi) \\ h \end{pmatrix} \right] |\det(h)|^{-\sigma} \right) D^{-L} W(hw') dh.
\]

We write $N \setminus G$ with $(a, k)$ coordinates. Using Lemma 5.1 Proposition 5.1, and Lemma 5.2 we estimate the last integral by

\[
\ll S_p(L(W) \sum_{1 \leq r \leq L} |c|^r \int_A a_1^{-r} \prod_{i=1}^{n-1} \min(1, (a_{i+1}/a_i)^M) \min(1, a_i^{N_1}) \frac{d^\sigma a}{|\det(a)|^\sigma},
\]
where \( p \) depends on \( M \) and \( n \). One can check that the above \( A \)-integral is absolutely convergent and hence (5.1) is bounded by

\[
\ll_p |c| \int_G C(\pi)^{(n+1)\sigma} \sum_{W \in B(\pi)} |W(g)||\text{det}(g)|^{\sigma} S_{p-L}(W) d\mu(\pi).
\]

As \( g \) varies in a compact set, we use Lemma 5.2 to bound

\[
W(g)||\text{det}(g)||^{\sigma} \ll S_q(W),
\]

for all \( W \in B(\pi) \), for some fixed \( q \). Finally, making \( L \) sufficiently large and appealing to Lemma 3.9 we conclude. \( \square \)

6. Proof of Proposition 5.1

For \( 1 \leq s \leq n \) we define a property \( \text{pop}(s) \), which stands for \textit{Partial Ordering at the Pivot} \( s \), of the elements \( a \in A \).

**Definition 6.1.** We say an element \( a \in A \) satisfies the property \( \text{pop}(s) \) for some \( 1 \leq s \leq n \) if

\[
\max\{a_1, \ldots, a_s\} \leq \min\{1, \min\{a_{s+1}, \ldots, a_n\}\}.
\]

The proof of Proposition 5.1 will be proved in two different cases, depending whether \( a \) satisfies \( \text{pop}(s) \) for some \( s \) or not. When \( a \) does not satisfy \( \text{pop}(s) \) for any \( s \) the proof would be relatively easier which can be seen at the end of this section. Here we prove the required bound of \( W_\mu \) on the elements \( a \) which do not satisfy \( \text{pop}(s) \) for any \( s \).

**Lemma 6.1.** Let \( a \in A \) does not satisfy \( \text{pop}(s) \) for any \( 1 \leq s \leq n \) then

\[
\frac{W_\mu(a)}{c(1, \mu)} \ll_N \delta^{1/2}(a) \min(1, a_1^N) d(\mu)^{O_N(1)},
\]

for \( N \) large enough.

**Proof.** If \( a_1 \geq 1 \) Lemma 3.4 with \( M = 0 \) immediately implies this lemma. So we will assume that \( a_1 < 1 \). Note that, as we do not have \( \text{pop}(1) \) there exists \( 1 < l' \leq n \), such that \( a_{l'} < a_1 \). Also note that, as we do not have \( \text{pop}(n) \) there exists \( 1 < r' \leq n \) such that \( a_{r'} > 1 \). Let

\[
l := \max\{l' \mid a_{l'} < a_1\}, \quad r := \min\{r' \mid a_{r'} > 1\}.
\]

If \( r = n \) then we have \( \text{pop}(n-1) \), so \( r < n \). If \( l > r \) then from Lemma 3.4 we estimate that

\[
\frac{W_\mu(a)}{c(1, \mu)} \ll_N \delta^{1/2}(a) d(\mu)^{O_N(1)} \prod_{j=r}^{l-1} (a_{j+1}/a_j)^N \leq \delta^{1/2}(a) a_1^N d(\mu)^{O_N(1)},
\]

and we are done. Thus we may assume that \( l < r \) (note that \( l \neq r \) as \( a_l < a_1 < 1 < a_r \)). Now we define

\[
a_L := \max\{a_1, \ldots, a_l\}, \quad a_R := \min\{a_r, \ldots, a_n\}.
\]
Thus we may now assume that \( a_L \leq a_R \). Hence, we have \( 1 < l < r < n \) such that

\[
\max\{a_1, \ldots, a_l\} \leq \min\{1, \min(a_r, \ldots, a_n)\}.
\]

Now we define

\[
l_1 := \max\{l' \mid a_{l'} \leq a_L\}, \quad r_1 := \min\{r' \mid a_{r'} \geq a_R\}.
\]

Note that, if \( l_1 \geq l \) clearly; in fact \( l_1 > l \), otherwise we will have \( \text{pop}(l) \). Similarly, \( r_1 \leq r \) clearly; in fact \( r_1 < r \), otherwise we will have \( \text{pop}(r-1) \). Also \( l_1 \neq r_1 \) as \( a_{l_1} \leq a_L < a_R \leq a_{r_1} \). But if \( l_1 > r_1 \) then

\[
\frac{W_\mu(a)}{c(1, \mu)} \lesssim_N \delta^1/2(a)d(\mu)^{O_N(1)} \prod_{j=L}^{l-1} (a_{j+1}/a_j)^N \prod_{i=r}^{l_1-1} (a_{i+1}/a_i)^N \prod_{k=r}^{R-1} (a_{i+1}/a_i)^N
\]

\[
\leq \delta^1/2(a_1^N d(\mu)^{O_N(1)}).
\]

Thus we may assume that \( l_1 < r_1 \), as \( l_1 = r_1 \) would have \( \text{pop}(l_1 - 1) \). We now define

\[
a_{L_1} := \max\{a_1, \ldots, a_{l_1}\}, \quad a_{R_1} := \min\{a_{r_1}, \ldots, a_n\}.
\]

Note that, if \( a_{L_1} = a_{l_1} \) then we will have \( \text{pop}(l_1) \), so \( L_1 < l_1 \). Similarly, if \( a_{R_1} = a_{r_1} \) then we will have \( \text{pop}(r_1 - 1) \), so \( R_1 > r_1 \). Also note that, if \( a_{L_1} \geq a_{R_1} \) then

\[
\frac{W_\mu(a)}{c(1, \mu)} \lesssim_N \delta^1/2(a_1^N d(\mu)^{O_N(1)} \prod_{j=L}^{l-1} (a_{j+1}/a_j)^N \prod_{i=r}^{l_1-1} (a_{i+1}/a_i)^N \prod_{k=r}^{R-1} (a_{i+1}/a_i)^N
\]

\[
\leq \delta^1/2(a_1^N d(\mu)^{O_N(1)}).
\]

So we may assume that \( a_{L_1} < a_{R_1} \). Thus we have got nested pairs \( 1 < l < l_1 < r_1 < r < n \) such that

\[
\max\{a_1, \ldots, a_l\} \leq \min\{1, \min(a_r, \ldots, a_n)\}.
\]

Proceeding this way we will eventually, as there are only finitely, say \( P \), many steps, we will eventually land in \( l_P = r_P \) or \( l_P = r_P - 1 \) with similar properties as in (6.1) or (6.2). In either case, we will arrive at \( \text{pop} \), thus a contradiction. \( \square \)

### 6.1. Decomposition of the spherical Whittaker function

In the rest of this section we will prove the required decomposition of \( W_\mu \) into \( M \) Whittaker functions and prove the required bounds of them. Here by \( N_r, A_r, K_r \ldots \) etc., we will denote the maximal unipotent subgroup of upper triangular matrices, the positive diagonal subgroup, the maximal compact \( O(r) \ldots \) in \( GL_r(\mathbb{R}) \), respectively. For \( \mu \in \mathbb{C}^r \) by \( W_\mu \) (suppressing \( r \)) we will denote the spherical Whittaker function (with our chosen normalization as in (3.11)) on \( GL_r(\mathbb{R}) \) with
parameters $\mu$. By $a^\nu$ we will denote the element diag($a_1, \ldots, a_r$) $\in A_r$. Let us denote $(\alpha)^s := (\alpha_1, \ldots, \alpha_s)$ for $s \leq n$ and $\sum \alpha := \alpha_1 + \ldots + \alpha_n$, for any $\alpha \in \mathbb{C}^n$. We will abbreviate the condition $\Re(\alpha_i) \geq 0$ (resp. $> 0$) for all $i$ by $\Re(\alpha) \geq 0$ (resp. $> 0$). By a regular $\alpha$ we mean the coordinates $\alpha_j$ are distinct. By $S_r$ we will denote the symmetric group of $r$ letters, which is isomorphic to the Weyl group of GL($r$).

We record that the residue of $\Gamma_R(s)$ at $s = -2n$ for any $n \in \mathbb{Z}_{\geq 0}$ is $2(-\pi)^n / (2n+2)!$. Let $\nu \in \mathbb{C}^r$ and $\nu' \in \mathbb{C}^{r'}$ with $r > r'$. Let $\{\nu_i - \nu'_j \mid 1 \leq i \leq r, 1 \leq j \leq r'\} = A(\nu, \nu') \cup B(\nu, \nu')$, where $A(\nu, \nu')$ is the set of elements which are of form $2\mathbb{Z}_{\leq 0}$ i.e. a possible pole of $\Gamma_R$ and $B(\nu, \nu')$ is the compliment. We define

$$L(\nu, \nu') := \prod_{\alpha \in A(\nu, \nu')} \text{res}_{s=a} \Gamma_R(s) \prod_{b \in B(\nu, \nu')} \Gamma_R(b).$$

We will use these notations in the rest of the section.

We first note that using (3.3), (3.11), and (3.7) along with the description of the tempered spherical dual of $G$ we write for $F \in L^2(N \setminus G, \psi)^K$ as

$$F(g) = \int_{(0)^n} W_\mu(a) \int_{N \setminus G} F(t) \overline{W_\mu(t) dt} \frac{d\mu}{|c(\mu)|^2}. \tag{6.4}$$

Applying the Whittaker-Plancherel formula (3.12) and the GL($n+1) \times$ GL($n$) local functional equation (3.5) we get that

$$V \left( \begin{pmatrix} C(\Pi) \\ g \end{pmatrix} \right) w = \int_G \omega_\pi(-1)^n \Theta(\pi, \Pi) \sum_{W \in \mathcal{B}(\pi)} W(gw') \langle f, W \rangle d\mu_p(\pi),$$

We choose $\mathcal{B}(\pi) := \{W\}$ containing an ONB of $\pi$ of $K$-types. Now noting the fact that we have chosen $f$ to be spherical (see (4.1)) we conclude that for all $W \in \mathcal{W}(\pi, \psi)$ with non-trivial $K$-type $\langle f, W \rangle = 0$. Applying $\mathcal{D}$ we obtain that

$$\mathcal{D}_M^t V \left( \begin{pmatrix} C(\Pi) \\ g \end{pmatrix} \right) w = \int_{\tilde{G}_0} \Theta(\pi, \Pi) \mathcal{D}_M^t W_\pi(g) \langle f, W_\pi \rangle d\mu_p(\pi), \tag{6.5}$$

where $W_\pi$ is an $L^2$-normalized spherical vector in $\pi$. Also we conclude that $V \left( \begin{pmatrix} C(\Pi) \\ g \end{pmatrix} \right) w$ is spherical. Hence it is enough to prove Proposition 5.1 for $g = a \in A$. Finally, using (6.4) we rewrite (6.5) as

$$\mathcal{D}_M^t V \left( \begin{pmatrix} C(\Pi) \\ a \end{pmatrix} \right) w = \int_{(0)^n} \Theta(\mu, \Pi) \mathcal{D}_M^t W_\mu(a) \langle f, W_\mu \rangle \frac{d\mu}{|c(\mu)|^2}. \tag{6.6}$$

We record following integral representation of the spherical Whittaker function which is a corollary of Stade’s formula.
Lemma 6.2. Let $\nu \in \mathbb{C}^{r+1}$ with $\Re(\nu_i) \geq 0$. Then
\[
W_\nu(a^r) = \kappa_r a_r^{\sum \nu} \int_{(0)^{r-1}} W_{\nu'}(a^{r-1}/a_r) \frac{L(\frac{1}{2}, \pi_\nu \otimes \pi_{-\nu'})}{c(\nu')c(-\nu')} \, dv',
\]
for some absolute constant $\kappa_r$ (depending only on $r$).

The unspecified constant appears because of our chosen normalization in (3.11). From now on we will denote any unspecified constant which only depends on $n,...$ by $\kappa_{n,...}$, i.e. $\kappa_{n,...}$ may vary from line to line.

Proof. Using (6.4) we can write
\[
W_\nu(a^r) = a_r^{\sum \nu} W_\nu \left( \left( a^{r-1}/a_r \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = a_r^{\sum \nu} \int_{(0)^{r-1}} W_{\nu'}(a^{r-1}/a_r) \int_{N_{r-1}\setminus GL_{r-1}} W_\nu \left( \left( \begin{pmatrix} t \\ 1 \end{pmatrix} \right) \right) \frac{W_{\nu'}(t)dt}{|c(\nu')|^2} \, dv'.
\]

We conclude the proof noting that the inner integral is a constant multiple of $L(\frac{1}{2}, \pi_\nu \otimes \pi_{-\nu'})$ by Stade’s formula [35, Theorem 3.4].

We abbreviate $\delta^{-1/2} W$, for any GL$_r$ Whittaker function $W$, by $W'$. Note that if $\Re(\nu) > 0$ then shifting contour $\nu' \mapsto \nu' + 1/2$ in the $\nu'$ integral in Lemma 6.2 (without crossing any pole) we obtain that
\[
W'_\nu(a^r) = a_r^{\sum \nu} \int_{(0)^{r-1}} W_{\nu'}(a^{r-1}/a_r) \frac{L(\nu, \nu')}{c(\nu)c(-\nu')} \, dv'.
\]

From now on we will work with $W'$ instead of $W$ (This is because if we work with $W$ the modular character will appear in every equation, somewhat unimportantly).

We fix $1 \leq s \leq n$ from now on. Let $\Re(\mu) > 0$ be small enough. In (6.4) we shift the contours of $\nu$ integrals to some positive quantity so that the integrand does not cross any polar hyperplanes. For instance, we may choose $\Re(\nu) > 0$ such that $\max_j \Re(\nu_j) < \min_i \Re(\mu_i)$. We obtain
\[
W'_\mu(a) = \kappa_n a_n^{\sum \mu} \int_{(0)^{n-1}} W_{\nu'}(a^{n-1}/a_n) \frac{L(\mu, \nu)}{c(\nu)c(-\nu')} \, dv',
\]
where the contours are the vertical lines with real parts as said above. In this section, we will, mostly, not specify this type of contours explicitly. If the contours are unspecified then we will implicitly assume that the contours are vertical lines on the left of all possible poles and very close to the contours with real parts being zero, as described above. In the RHS we expand $W'_\nu$ using (6.7) exactly same as before and obtain
\[
W'_\mu(a) = \kappa_n a_n^{\sum \mu} \left( \frac{a_{n-1}}{a_n} \right)^{\sum \nu} \int_{(0)^{n-1}} W_{\nu'}(a^{n-2}/a_{n-1}) \frac{L(\nu, \nu')}{{c(\nu)c(-\nu')}} \, dv' \, dv.
\]
Proceeding in this way we get an iterated integral representation of $W'$ as following.

\[(6.8)\quad W'_{\mu}(a) = \kappa_n a_{\infty}^\mu \int \left( \frac{a_{n-1}}{a_n} \right)^{\sum \nu} \frac{L(\mu, \nu)}{c(\nu)c(-\nu)} \ldots \int W'_{\tau}(a^{s+1}/a_{s+2}) \frac{L(\gamma, \tau)}{c(\tau)c(-\tau)} d\tau d\gamma \ldots d\nu.\]

Now we start preparing for the decomposition of the Whittaker function. We define some power series which are analogue of the $M$-Whittaker function (analogue of $I$-Bessel function on $\text{GL}(2)$; see the relevant discussion in [222]. Let $\tau \in \mathbb{C}^{s+1}$ with $\Re(\tau) > 0$ small enough, and $k \in \mathbb{Z}_{\geq 0}$. We define

\[(6.9)\quad P_k(\tau) := \frac{L(\tau, (\tau)^s + 2k)}{c(\tau)c(-\tau)c((\tau)^s + 2k)c(-((\tau)^s - 2k))},\]

and

\[(6.10)\quad M_{\tau}(a^{s+1}) := \sum_{k \in \mathbb{Z}_{\geq 0}} P_k(\tau)W'_{(\tau)^s + 2k}(a^s/a_{s+1}).\]

In the next four lemmata we prove the decomposition of $W'$ into $M$, inductively (in Lemma 6.6). For ease of the reader we describe the themes of these technical lemmata. In Lemma 6.3 we prove the base case of the induction. We start with the inner most integral in (6.8) and shift all the contours to infinity. The integrand will cross finitely many families of infinitely many poles. We will collect the residues and construct the some $M$-Whittaker function as power series. We do the similar contour shifting process in the second inner most integral of (6.8), and thus we obtain similar $M$-Whittaker functions, inductively, which we define in (6.11). Finally, we prove the inductive step of the decomposition in the proof of Lemma 6.6.

We note that a similar decomposition in the case of $s = 1$ appeared in [19][6]. In the both articles the authors proved the results by the method of differential equation, while we prove it by the spectral analysis and the zeta integrals.

**Lemma 6.3.** For $\tau \in \mathbb{C}^{s+1}$ with $\Re(\tau) > 0$ and small enough. Then,

\[\frac{1}{c(\tau)c(-\tau)}W'_{\tau}(a^{s+1}) = \kappa_{s}a_{s+1}^\tau \sum_{\sigma \in \mathcal{S}_{s+1}} M_{\sigma}(a^{s+1}),\]

such that $M_{\tau}$ are entire in $\tau$.

**Proof.** We will prove the equality for regular $\tau$ so that the result will follow by analyticity. Note that using (6.7) we write

\[W'_{\tau}(a^{s+1}) = \kappa_{s}a_{s+1}^\tau \int_{(0)^s} W'_{z}(a^s/a_{s+1}) \frac{L(\tau, z)}{c(z)c(-z)} dz.\]

We want to shift the contour of $z_1$ to $\infty$. We claim that this is possible, i.e. we first shift $z_1$ contour to $\Re(z_1) = 2N + 1$ collect residues at the poles and estimate the following shifted integral

\[\int_{(0)^s-1} \int_{(2N+1)} W'_{z_1, z'}(a^s/a_{s+1}) \frac{L(\tau, z')L(\tau, z_1)}{c(z')c(-z')} \prod_{i=2}^{s} \Gamma_{\Re}(z_1 - z_i)\Gamma_{\Re}(z_s - z_1) dz_1 dz'.\]
We perturb the \( z' \) contour a little bit so that we can apply Lemma 3.5 to obtain

\[
W'_{z_1, z'} \ll_a |c(1, -z')| \prod_{i=2}^s |\Gamma_\mathbb{R}(1 + z_1 - z_s)|.
\]

We note that, using Stirling’s approximation

\[
\frac{c(1, -z')}{c(-z')} \prod_{i=2}^s \frac{\Gamma_\mathbb{R}(1 + z_1 - z_s)}{\Gamma_\mathbb{R}(z_1 - z_s)} \ll \prod_{i=2}^s (1 + |z_i|)^{O(1)} \prod_{i=2}^s |z_i - z_i|^{O(1)}.
\]

On the other hand,

\[
L(\tau, z_1) \ll (N!)^{-s-1} \prod_{i=1}^{s+1} \Gamma_\mathbb{R}(\tau_j - 1/2 - \Im(z_1)).
\]

Writing \( z_1 \) with \( \Re(z_1) = 2N + 1 \) as \( z_1 + 2N + 1 \) with \( \Re(z_1) = 0 \) we obtain, by Stirling’s estimate, that the integral is bounded by

\[
\ll_{\tau} \frac{N^{O(1)}}{(N!)^{s+1}} \int \frac{E_\tau(z)}{L(\tau, z_1)} \prod_{i=2}^s (1 + |z_i|)^{O(1)} \prod_{k=1}^N |z_i - z_1 - k|,
\]

where

\[
E_\tau(z) := \exp \left[ - \sum_{i,j} |\Im(\tau_i - z_j)| + \sum_{i \neq j} |\Im(z_i - z_j)| \right] \ll_{\tau} \exp \left[ - \sum_i |\Im(z_i)| \right].
\]

We estimate

\[
(1 + |z_i|)^{O(1)} |z_1 - z_i|^{O(1)} \prod_{k=1}^N |z_i - z_1 - k| \ll z_1^{O(1)} z_i^{O(1)} \sum_{k_i \leq N; r} k_1 \ldots k_r |z_1 - z_i|^{N-r}.
\]

Upon integrating the above against \( \exp [- \sum_i |\Im(z_i)|] \) we obtain that the integral is bounded by \( \frac{N^{O(1)}}{(N!)^{s+1}} \prod_{i=1}^s (N + O(1))! \ll \frac{N^{O(1)}}{N^s} \), which tends to zero as \( N \to \infty \).

Now we shift the \( z_1 \) contour to infinity. We cross poles and gather the corresponding residues to obtain the following:

\[
\int_{(0)} W'_z(a^s/a_{s+1}) \frac{L(\tau, z)}{c(z)c(-z)} dz
\]

\[
= \int_{(0)^{s+1}} \prod_{i=1}^{s+1} \prod_{j=2}^s \frac{\Gamma_\mathbb{R}(\tau_i - z_j)}{|c(z_2, \ldots, z_s)|^2} \int_{(0)} W'_z(a^s/a_{s+1}) \prod_{j=2}^{s+1} \frac{\Gamma_\mathbb{R}(\tau_i - z_j)}{\Gamma_\mathbb{R}(z_1 - z_j) \Gamma_\mathbb{R}(z_j - 1)} dz_1 dz_2 \ldots dz_s
\]

\[
= \sum_{i=1}^{s+1} \int_{(0)^{s+1}} \prod_{i=1}^{s+1} \prod_{j=2}^s \frac{\Gamma_\mathbb{R}(\tau_i - z_j)}{|c(z_2, \ldots, z_s)|^2} \sum_{k_{i=0}}^{\infty} W'_{\tau_i + 2k_1, z_{i+1}, \ldots, z_s} (a^s/a_{s+1}) L(\tau, \tau_i + 2k_1) \prod_{j=2}^s \frac{\Gamma_\mathbb{R}(\tau_i - z_j + 2k_1)}{\Gamma_\mathbb{R}(z_j - \tau_i - 2k_1)} dz_2 \ldots dz_s.
\]
We check that each factor in the last expression of $P$ we obtain that for any $\tau$ such that $(\tau, \tau')$ are holomorphic.

Hence in the $i$’th integral above, any $z_2$ only has family of poles at $\tau_j$ for $j \neq i$. Now we shift the $z_2$ contour to $\infty$ (upon similar justification as in the case of $z_1$). Proceeding in this way we obtain that for any $s$-tuple $\tau'$ consisting of distinct elements from $\{\tau_1, \ldots, \tau_{s+1}\}

\int_{(0)^{s}} W'_{\bar{z}}(a^s/a_{s+1}) \frac{L(\tau, z)}{c(z)c(-z)} dz = \sum_{\tau'} \sum_{k \in Z_{\geq 0}} C_k(\tau) W'_{\tau'+2k}(a^s/a_{s+1}),

where

$C_k(\tau, \tau') := \frac{L(\tau, \tau' + 2k)}{c(\tau' + 2k)c(-\tau' - 2k)}$.

Noting that, if $\sigma \in S_{s+1}$ such that $(\sigma\tau)^s = \tau'$ then

$P_k(\sigma\tau) = \frac{C_k(\tau, \tau')}{c(\tau)c(-\tau)}$.

Thus we conclude proof of the decomposition. To check that $M_{\tau}$ is holomorphic it is enough to check that $P_k(\tau)$ are holomorphic ($W'$ is entire in its parameters, see [27]) and the series defining $M_{\tau}$ is locally uniformly convergent. Here we only check that $P_k(\tau)$ are holomorphic. Later in a different lemma we will estimate $P_k$ and $W$ which will imply locally uniform convergence. To check holomorphy we first note

$c(\tau)c(-\tau) = c((\tau)^s)c(-(\tau)^s) \prod_{j=1}^{s} \Gamma_{\mathbb{R}}(\tau_{s+1} - \tau_j) \Gamma_{\mathbb{R}}(\tau_j - \tau_{s+1}).$

and thus $P_k(\tau)$ equals to

$$\prod_{j=1}^{s} (-\pi)^{kj} k_j! \prod_{j>i} \Gamma_{\mathbb{R}}(\tau_i - \tau_j - 2k_j) \prod_{i<j} \Gamma_{\mathbb{R}}(\tau_j + 2k_j - \tau_i - 2k_i) \Gamma_{\mathbb{R}}(\tau_i + 2k_i - \tau_j - 2k_j).$$

We check that that each factor in the last expression of $P_k(\tau)$ is holomorphic, thus we conclude. □
Now using Lemma 6.3 we can expand the inner most integral in (6.8) and rewrite (6.8) as following.

\[ W'_\mu(a) = \kappa_{s,n} a_n \sum_\mu \int \left( \frac{a_{n-1}}{a_n} \right)^{\sum_\nu} \frac{L(\mu, \nu)}{c(\nu)c(-\nu)} \cdots \]
\[ \cdots \int (0)^{s+1} \left( \frac{a_{s+1}}{a_{s+2}} \right)^{\sum_\tau} L(\gamma, \tau) \sum_{\sigma \in S_{s+1}} M_{\sigma \tau}(a^{s+1}) d\tau d\gamma \cdots d\nu. \]

We interchange the innermost integral with the finite sum over \( S_{s+1} \). We change variable to \( \sigma \tau \mapsto \tau \) and rewrite as following.

\[ W'_\mu(a) = \kappa_{s,n} a_n \sum_\mu \int \left( \frac{a_{n-1}}{a_n} \right)^{\sum_\nu} \frac{L(\mu, \nu)}{c(\nu)c(-\nu)} \cdots (s+1)! \int (0)^{s+1} \left( \frac{a_{s+1}}{a_{s+2}} \right)^{\sum_\tau} \]
\[ \times \int (0)^{s+1} \left( \frac{a_{s+1}}{a_{s+2}} \right)^{\sum(\tau)^s} L(\gamma_1, (\tau)^s) M_{\tau}(a^{s+1}) d\tau d\gamma \cdots d\nu. \]

Now we shift contours of the last integral to \( \infty \). The family of the poles will occur at \( (\tau)^j = \gamma_i + 2Z_{\geq 0} \) for all \( 1 \leq j \leq s \) and some \( 1 \leq i \leq s+1 \). The residues will be of the form

\[ \left( \frac{a_{s+1}}{a_{s+2}} \right)^{\sum \gamma' + 2l} L(\gamma, \gamma' + 2k) M_{\gamma' + 2l, \tau_{s+1}}, \]

for \( l \in Z_{\geq 0} \) and \( \gamma' \in C^s \) with \( \gamma'_j \in \{ \gamma_1, \ldots, \gamma_{s+2} \} \). However, some of these residues do not occur in the asymptotic expansion of \( W' \), for instance, the residues with \( \gamma' \) with \( \gamma'_1 = \gamma'_2 \). In fact, the terms in the asymptotic expansion come from the residues where the coordinates of \( \gamma' \) are distinct\(^1\). But, thanks to Lemma 6.4 where we show that the residues other than

\[ \left( \frac{a_{s+1}}{a_{s+2}} \right)^{\sum (\sigma \gamma)^s + 2l} L(\gamma, (\sigma \gamma)^s + 2k) M_{(\sigma \gamma)^s + 2l, \tau_{s+1}}, \]

for \( \sigma \in S_{s+2} \), will vanish identically. In other words, the family of poles (of form \( \gamma + 2Z \)) of the integrand only occur at \( \gamma \) which has distinct coordinates.

We follow the same method of collecting residues and construct the power series. We continue this in (6.8) until the outer most integral. We recursively define the following.

\[ M^1_\tau(a^{s+1}) := M_\tau(a^{s+1}), \]

\(^1\)We may assume that \( \gamma \) is regular.
for $r \geq 1$, and $\alpha \in \mathbb{C}^{s+r}$

\begin{equation}
M'_\alpha(a^{s+r}) := \int_{(0)^{r-1}} \left( \frac{a_{s+r-1}}{a_{s+r}} \right)^{\sum z} L(\alpha, z) \sum_{l \in \mathbb{Z}_{\geq 0}^s} \left( \frac{a_{s+r-1}}{a_{s+r}} \right)^{\sum (\alpha)^s + 2l} \frac{L(\alpha, (\alpha)^s + 2l)}{c(\alpha)c(-\alpha)} M_r^{-1}(a^{s+r-1})dz.
\end{equation}

In each stage of contour shifting we need to prove that the integrand is holomorphic at the non-regular points, as discussed above. We show this, inductively, in Lemma 6.4, which is the base case, and in Lemma 6.5 where we prove the inductive step. These lemmata can be thought as a higher rank analogues of the fact that $I_n = I_{-n}$ for any natural number $n$, where $I$ denotes the classical $I$-Bessel function.

**Lemma 6.4.** Let $\tau \in \mathbb{C}^{s+1}$ such that $\Re(\tau) \geq 0$ and $\tau_a \equiv \tau_b \mod 2\mathbb{Z}$ for $1 \leq a \neq b \leq s$. Then $M_\tau$ is identically zero.

**Proof.** Let $1 \leq a < b \leq s$. Expanding out the definition (6.9) of $P_k$ gives

\[
P_k(\tau) = \frac{\Gamma_{\mathbb{R}}(\tau_a - \tau_b - 2k_b)\Gamma_{\mathbb{R}}(\tau_b - \tau_a - 2k_a) \text{res}_{s=-2k_a} \Gamma_{\mathbb{R}}(s) \text{res}_{s=-2k_b} \Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(\tau_a - \tau_b)\Gamma_{\mathbb{R}}(\tau_b - \tau_a)\Gamma_{\mathbb{R}}(\tau_a - \tau_b + 2k_a - 2k_b)\Gamma_{\mathbb{R}}(\tau_b - \tau_a + 2k_b - 2k_a)} \times \prod_{j \neq a,b} L(\tau, \tau_j + 2k_j) \prod_{i \neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_a - 2k_a) \Gamma_{\mathbb{R}}(\tau_i - \tau_b - 2k_b) \Gamma_{\mathbb{R}}(\tau_i - \tau_j + 2k_i - 2k_j).
\]

Computing the residues we obtain that the above equals to

\[
\frac{\pi^{-4}(\tau_a - \tau_b)(\tau_b - \tau_a + 2k_b - 2k_a)}{\Gamma_{\mathbb{R}}(2k_a + 2)\Gamma_{\mathbb{R}}(2k_b + 2)\Gamma_{\mathbb{R}}(\tau_a + 2k_a - \tau_b + 2)\Gamma_{\mathbb{R}}(\tau_b + 2k_b - \tau_a + 2)} \times Q_k(\tau) \times \prod_{j \neq a,b} L(\tau, \tau_j + 2k_j) \prod_{i \neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_j),
\]

where

\[
Q_k(\tau) := \frac{\prod_{i \neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_a - 2k_a) \Gamma_{\mathbb{R}}(\tau_i - \tau_b - 2k_b)}{\prod_{i \neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_a + 2k_b - 2k_a)}.
\]

Suppose, without loss of generality, that $\tau_a - \tau_b = 2l \geq 0$. Note that $\Gamma_{\mathbb{R}}(\tau_a - \tau_b + 2k_a + 2) = 0$ for $k_b < l$. Now changing $k_b \mapsto k_b + l$ and replacing $\tau_b + 2l = \tau_a$ only in the first two factors we obtain from (6.10)

\[
M_\tau = \sum_{k \in \mathbb{Z}_{\geq 0}^s} W' \cdots W_{a+2k_a} \cdots W_{a+2k_b} \cdots \times Q_k(\tau_a, \ldots, \tau_a, \ldots) \times \prod_{j \neq a,b} L(\tau, \tau_j + 2k_j) \prod_{i \neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_j) \times \frac{\pi^{-4}(2l)(k_b - k_a)}{\Gamma_{\mathbb{R}}(2k_a + 2)\Gamma_{\mathbb{R}}(2k_b + 2l + 2)\Gamma_{\mathbb{R}}(2k_a + 2l + 2)\Gamma_{\mathbb{R}}(2k_b + 2)}.
\]
Let \( \sigma_{ab} \) be the element in the Weyl group which transposes the \( a \)'th and \( b \)'th elements and fixes everything else. Then doing a similar calculation we can check that

\[
M_{\sigma_{ab}\tau} = \sum_{k \in \mathbb{Z}_{\geq 0}} W'_{\ldots, \tau_a + 2k_a, \ldots, \tau_b + 2k_b, \ldots} \times Q_k(\ldots, \tau_a, \ldots, \tau_a, \ldots) \times \frac{\prod_{j \neq a, b} L(\tau, \tau_j + 2k_j)}{\prod_{(i,j) \neq (a,b)} \Gamma(\tau_i - \tau_j)} 
\times \frac{\pi^{-4}(-2l)(k_b - k_a)}{\Gamma(2k_a + 2l + 2)\Gamma(2k_b + 2)\Gamma(2k_a + 2)\Gamma(2k_b + 2l + 2)}
\]

\[
= -M_r.
\]

On the other hand it can be easily checked from (6.9) that

\[
P_{\sigma_{ab}k}(\sigma_{ab}\tau) = P_k(\tau).
\]

Thus using the fact that Whittaker function is invariant under Weyl group action on its parameters we can also conclude that

\[
M_r = M_{\sigma_{ab}\tau},
\]

hence the conclusion. \( \square \)

**Lemma 6.5.** Let \( \alpha \in \mathbb{C}^{s+r} \) such that \( \Re(\alpha) \geq 0 \) and \( \alpha_a \equiv \alpha_b \mod 2\mathbb{Z} \) for \( 1 \leq a \neq b \leq s \). Then \( M^r_\alpha \) is identically zero.

**Proof.** We prove by inducting on \( r \). Note that the base case \( r = 1 \) is proved in Lemma 6.4. We assume the claim is true for \( r \geq 1 \). We consider the inner sum of \( M^{r+1} \), which is

\[
\sum_{l \in \mathbb{Z}_{\geq 0}} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{\sum(\alpha)^{s+2l}} \frac{L(\alpha, (\alpha)^s + 2l)}{c(\alpha)c(-\alpha)} M_r^{(\alpha)^{s+2l}}(a^{s+r-1}).
\]

We take a similar proof path as in Lemma 6.4. Suppose that \( \sigma_{ab} \) be the element in the which transposes \( a \)'th and \( b \)'th elements. Clearly, \( M_{\sigma_{ab}\alpha} = M_\alpha \). We will show that when \( \alpha_a - \alpha_b \in 2\mathbb{Z} \) then \( M_{\sigma_{ab}\alpha} = -M_\alpha \) which will yield the claim. We write the coefficient

\[
\frac{L(\alpha, (\alpha)^s + 2l)}{c(\alpha)c(-\alpha)}
\]

\[
= \frac{4\pi^{-2}\Gamma(\alpha_a - \alpha_b - 2l_b)\Gamma(\alpha_b - \alpha_a - 2l_a)(-1)^{l_a + l_b} \prod_{i,j \neq (a,b)} L(\alpha_i, \alpha_j + 2l_j)}{\Gamma(\alpha_a - \alpha_b)\Gamma(\alpha_b - \alpha_a)\Gamma(2l_a + 2)\Gamma(2l_b + 2) \prod_{(e \neq f) \neq (a,b), (b,a)} \Gamma(\alpha_e - \alpha_f)}
\times \prod_{i \neq a,b} \Gamma(\alpha_i - \alpha_a - 2l_a)\Gamma(\alpha_i - \alpha_a - 2l_b) \prod_{j \neq a,b} \Gamma(\alpha_a - \alpha_j - 2l_j)\Gamma(\alpha_b - \alpha_j - 2l_j).
\]
Here $i$ and $j$ are varying over $1, \ldots, r+s$ and $1, \ldots, s$, respectively. Using the functional equation for $\Gamma_R$ we obtain the above is

$$= \frac{2\pi^{-3}(\alpha_b - \alpha_a)\Gamma_R(\alpha_b - \alpha_a - 2l_a)(-1)^{l_a}}{\Gamma_R(\alpha_b - \alpha_a + 2l_a + 2)\Gamma_R(2l_a + 2)\Gamma_R(2l_b + 2)\prod_{(e \neq f) \neq (a,b)}(\alpha_e - \alpha_f)} \prod_{i \neq a,b} \Gamma_R(\alpha_i - \alpha_a - 2l_a) \prod_{j \neq a,b} \Gamma_R(\alpha_i - \alpha_j - 2l_j) \Gamma_R(\alpha_a - \alpha_j - 2l_j).$$

Without loss of generality, suppose that $a < b$ and $\alpha_a - \alpha_b = 2t \geq 0$, and $t \in \mathbb{Z}$. As $M^r_a = 0$ by induction hypothesis we note by the above computation that the summand vanishes for $l_b < t$. So in the sum in consideration we change variable $l_b \mapsto l_b + t$ and obtain the sum equals to

$$\sum_{t \in \mathbb{Z} \geq 0} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{2t + \sum(\alpha) + 2t} \times Q_t(\alpha) \times \frac{2\pi^{-3}(\alpha_b - \alpha_a)(-1)^{l_a}}{\Gamma_R(2l_b + 2)\Gamma_R(2l_a + 2)\Gamma_R(2l_b + 2t + 2)\prod_{i \neq a,b} \Gamma_R(\alpha_i - \alpha_a - 2l_a) \prod_{j \neq a,b} \Gamma_R(\alpha_i - \alpha_j - 2l_j)} \lim_{\alpha_a - \alpha_b \to 2t} \Gamma_R(\alpha_b - \alpha_a - 2l_a) M^r_{\alpha_a+2l_a,\ldots,\alpha_a+2l_b+2t,\ldots,z}(a^{s+r-1}),$$

where

$$Q_t(\alpha) := \frac{\prod_{i,j \neq \{a,b\}} L(\alpha_i, \alpha_j + 2l_j)}{\prod_{(e \neq f) \neq (a,b),(b,a)} \Gamma_R(\alpha_e - \alpha_f)} \prod_{j \neq a,b} \Gamma_R(\alpha_i - \alpha_j - 2l_j) \prod_{i \neq a,b} \Gamma_R(\alpha_i - \alpha_a - 2l_a) \Gamma_R(\alpha_i - \alpha_a - 2l_b).$$

We compute the last limit. Let $\beta := \alpha_b - \alpha_a + 2t$. Then the last limit equals to

$$\text{res}_{s = -2t - 2l_a} \Gamma_R(s) \lim_{\beta \to 0} \beta^{-1} M^r_{\alpha_a+2l_a,\ldots,\alpha_a+\beta+2l_b,\ldots,z}(a^{s+r-1})$$

$$= \frac{2(-1)^{l_a+4t}}{\pi \Gamma_R(2l_a + 2t + 2)} \lim_{\beta \to 0} \beta^{-1} M^r_{\alpha_a+2l_a,\ldots,\alpha_a+\beta+2l_b,\ldots,z}(a^{s+r-1}).$$

Thus we obtain that the sum of $M_a$ in consideration equals to

$$\sum_{t \in \mathbb{Z} \geq 0} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{2t + \sum(\alpha) + 2t} \times \frac{4\pi^{-4}(-1)^{4t}Q_t(\alpha)}{\Gamma_R(2l_b + 2)\Gamma_R(2l_a + 2)\Gamma_R(2l_b + 2t + 2)\Gamma_R(2l_a + 2t + 2)} \times (\alpha_b - \alpha_a) \lim_{\beta \to 0} \beta^{-1} M^r_{\alpha_a+2l_a,\ldots,\alpha_a+2l_b+\beta,\ldots,z}(a^{s+r-1}).$$
Then doing a similar computation we obtain that the relevant sum of $M_{\sigma_{ab},\alpha}$ in consideration equals to

$$
\sum_{l \in \mathbb{Z}_{\geq 0}} \left( \frac{a_s + r}{a_s + r + 1} \right)^{2t + \sum(\alpha)^s + 2l} \times \frac{4\pi^{-4}(-1)^t Q_l(\alpha)}{\Gamma(2l_b + 2) \Gamma(2l_a + 2) \Gamma(2l_b + 2t + 2) \Gamma(2l_a + 2t + 2)}
\times (\alpha_a - \alpha_b) \lim_{\beta \to 0} \beta^{-1} M^r_{\ldots, \alpha_a + 2l_b - \beta, \ldots, \alpha_a + 2l_b, \ldots, z}(a^{s+r-1}).
$$

Thus the proof will be complete if we can show that

$$
\lim_{\beta \to 0} \beta^{-1} M^r_{\ldots, \alpha_a + 2l_b - \beta, \ldots, \alpha_a + 2l_b, \ldots, z}(a^{s+r-1}) = \lim_{\beta \to 0} \beta^{-1} M^r_{\ldots, \alpha_a + 2l_a, \ldots, \alpha_a + 2l_b + \beta, \ldots, z}(a^{s+r-1}).
$$

The above follows from twisting $M_a$ by $|\det|^\beta$ and applying induction hypothesis on $M^r$. $\square$

Finally, we prove the inductive step of the decomposition of $W'$ in to $M$-Whittaker function, whose base case is proved in Lemma 6.3.

**Lemma 6.6.** Let $\mu \in E_n(\epsilon)$ for some $\epsilon > 0$. Then there exists an absolute constant $\kappa_{s,n}$ such that,

$$
\frac{1}{c(\mu)c(-\mu)} W'_\mu(a) = \kappa_{s,n} a_n^{\sum \mu} \sum_{\sigma \in S_n} M^\mu_{\sigma_{\mu}}(a),
$$

and $M_\mu$ is holomorphic in $\Re(\mu) \geq 0$.

**Proof.** From the definition (6.11) and Lemma 6.5 holomorphicity of $M_\mu$ is clear. To prove the first relation we will induct on $r$. The base case $r = 1$ is proved in Lemma 6.3. At $r$'th intermediate stage expression of $W'_\mu(a)$ looks like

$$
\frac{1}{c(\mu)c(-\mu)} W'_\mu(a)
= \kappa_{s,r,n} a_n^{\sum \mu} \int \left( \frac{a_{n-1}}{a_n} \right)^{\nu} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{\sum \theta} \frac{L(\mu, \nu)}{c(\mu)c(-\mu)} \frac{L(\eta, \theta)}{c(\eta)c(-\eta)} M^\mu_{\sigma_{\mu}}(a^{s+r}) d\theta \ldots d\nu
= \kappa_{s,r,n} a_n^{\sum \mu} \int \left( \frac{a_{n-1}}{a_n} \right)^{\nu} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{\sum \theta} \frac{L(\mu, \nu)}{c(\mu)c(-\mu)} \frac{L(\eta, \theta)}{c(\eta)c(-\eta)} M^\mu_{\sigma_{\mu}}(a^{s+r}) d\theta \ldots d\nu
\times \int_{(0)^s} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{\sum(\theta)^s} \frac{L(\eta, (\theta)^s)}{c(\eta)c(-\eta)} M^\mu_{\sigma_{\mu}}(a^{s+r}) d(\theta)^s d\theta' \ldots d\nu,
$$
where $\theta' := (\theta_{s+1}, \ldots, \theta_{s+r})$. Now we shift contours to infinity in the last integral. Thus, employing Lemma 6.5, collecting residues we obtain for some constant $d' = d_{r,s}$ that

$$\frac{1}{c(\mu)c(-\mu)} W'_\mu(a)$$

$$= \kappa_{s,r,n} \sum_{\sigma} \left( \frac{a_{n-1}}{a_n} \right)^{\nu} L(\mu, \nu) \cdots \int_{(0)} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{\theta'} L(\eta, \theta')$$

$$\sum_{\sigma \in S_{s+r}} \sum_{l \in \mathbb{Z}_{\geq 0}} \left( \frac{a_{s+r}}{a_{s+r+1}} \right)^{\sum(\sigma)n + 2l} L(\eta, (\sigma\eta)^s + 2l) \frac{1}{c(\eta)c(-\eta)} M_{(\sigma\eta)^s + 2l, \theta'}(a^{s+r}) d\theta' \cdots d\nu.$$  

Noting the symmetries of the variables inside the integrals and recalling (6.11) we can conclude. 

Now we will estimate the function $M^{n-s}$ inductively. This lemma will be used to prove the required bound in Proposition 3.1 for those $a \in A$ which are in the complementary case of what we considered in Lemma 6.1, i.e. $a$ satisfies pop(s). We loosely mention that $M^{n-s}$ Whittaker functions will be exponentially increasing in $a_i/a_j$ for $1 \leq i \leq s$ and $s+1 \leq j \leq n$ (see §2.2). This is exactly where we will be using pop(s) to control the increment.

**Lemma 6.7.** Let $1 \leq s \leq n$ and $a$ satisfies pop(s). Also let, $\mu \in \mathbb{C}^n$ such that $\Re(\mu) > 0$ small enough. We define $\mu' := (\mu_1 + 2N, \ldots, \mu_s + 2N, \mu_{s+1}, \ldots, \mu_n)$ for some fixed large integer $N$. Then

$$M_{\mu'}^{(n-s)}(a) \ll_{N, \varepsilon} \frac{d(\mu)^{O(1)}}{c(3(\mu))} a_1^{2N} := \Gamma(s-k) \ll |\Gamma(s)|.$$  

Here $O(1)$ in the exponent of $d(\mu)$ denotes a bounded constant depending on $N$ and $n$.

**Proof.** We prove by induction on $r$ using the inductive definition in (6.11). Note that it is enough to prove the required bound of $M'_\mu$ for $\alpha := (\mu_1 + 2N_1, \ldots, \mu_s + 2N_s, \alpha')$ where $N_i \geq N$ and $\alpha' \in \mathbb{C}^s$ with small positive real parts. For $\alpha \in \mathbb{C}^s$, by $\iota(\alpha)$ we will denote the reordering of the coordinates of $\alpha$ such that $\Re(\iota(\alpha)_1) \leq \ldots \leq \Re(\iota(\alpha)_s)$. We will frequently use Stirling approximation, and also for $s \in \mathbb{C}$ having small real part and $k \geq 0$

$$|\Gamma(s-k)| \ll |\Gamma(s)|.$$  

Let us first prove the claimed bound for $M^1$. From the definition (6.9), for $\tau \in \mathbb{C}^{s+1}$ with $\Re(\tau_j) \geq 2N$ for $1 \leq j \leq s$, we estimate

$$c(1, -\iota((\tau)^s + 2k)) P_k(\tau) := \frac{L(\tau, (\tau)^s + 2k)c(1, -\iota((\tau)^s + 2k))}{c(\tau)c(-\tau)c((\tau)^s + 2k)c(-\tau)^s - 2k)\cdots} \times \prod_{i=1}^{s} \prod_{k_i}^{\Gamma_{\Re}(\tau_{s+1} - \tau_i - 2k_i)\Gamma_{\Re}(\tau_i - \tau_{s+1})} \prod_{i,j}^{\iota(\tau_i - \tau_j - 2k_j)\Gamma_{\Re}(\tau_i - \tau_{s+1})} \prod_{i\neq j}^{\Gamma_{\Re}(\tau_j - \tau_i)\Gamma_{\Re}(\tau_i - \tau_{s+1})} \prod_{i,j}^{\Gamma_{\Re}(\tau_i - \tau_{s+1})} c(1, -\iota((\tau)^s + 2k))c(-\iota((\tau)^s + 2k)).}$$
We can do similar estimate for each sub-factor in the third factor. If we define
\[ \| \tau \| \]
then we have obtained that
\[ c(1, -\ell((\tau)^s + 2k)) P_k(\tau) \ll \| c(\tau) \|^{-1} \prod_{i=1}^s \frac{\pi^{k_i}}{k_i!} (1 + \| k \|)^O(1) d(\tau)^O(1). \]

Finally using (6.10) and Lemma 3.5 we estimate
\[ M_\tau(a^{s+1}) = \sum_{k \in \mathbb{Z}_{\geq 0}} P_k(\tau) W_{\tau + 2k}(a^s/a_{s+1}) \]
\[ \ll \frac{d(\tau)^O(1)}{\tilde{c}(\tau)} \frac{a_1^{2N}}{a_{s+1}^{\frac{\left\| \tau \right\|}{s}}} \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{\pi^{k_i}}{k_i!} (1 + \| k \|)^O(1) \]
\[ \ll \frac{d(\tau)^O(1)}{\tilde{c}(\tau)} \frac{a_1^{2N}}{a_{s+1}^{\frac{\left\| \tau \right\|}{s}}}, \]
where in the first inequality we have employed \( \text{pop}(s) \) assumption.

Now we make an inductive hypothesis that for \( r \geq 1 \)
\[ M_\alpha(a^{s+r}) \ll \frac{d(\alpha)^O(1)}{\tilde{c}(\alpha)} \frac{a_1^{2N}}{a_{s+r}^{\frac{\left\| \alpha \right\|}{s+r}}}, \]
where motivated by the previous computation we define
\[ \tilde{c}(\alpha) := \prod_{i=1}^s \prod_{j=1}^r \Gamma(\alpha_i - \alpha_{s+j}) \prod_{1 \leq i < j \leq r} \Gamma(\alpha_{s+i} - \alpha_{s+j}) \prod_{1 \leq i \neq j \leq s} \min\{|\Gamma(\alpha_i - \alpha_j)|, |\Gamma(\alpha_j - \alpha_i)|\}. \]

We start with (6.11) and integrate term by term. If we prove that
\[ \frac{L(\alpha, (\alpha)^s + 2l)}{c(\alpha)c(-\alpha)} \int_0^1 \left( \frac{a_{s+r}}{a_{s+r+1}} \right) \sum_{z} L(\alpha, z) d((\alpha)^s + 2l)^O(1) d(z)^O(1) \]
\[ \ll \frac{a_1^{2N}}{a_{s+r}^{\frac{\left\| \alpha \right\|}{s+r} + 2l}} \tilde{c}(\alpha)^{-1} \prod_{i=1}^s \pi^{l_i} (1 + \| l \|)^O(1) d((\alpha)^s)^O(1) d(\alpha_{s+1}, \ldots, \alpha_{s+r+1})^O(1), \]
employing the inductive hypothesis we can yield the proof. Note that,

\[
\frac{L(\alpha, (\alpha)^s + 2l)L(\alpha, z)}{c(\alpha)c(-\alpha)\tilde{c}((\alpha)^s + 2l, z)} \prod_{i=1}^{s} \prod_{j=1}^{r+1} \Gamma_{\mathbb{R}}(\alpha_i - \alpha_{s+j}) \prod_{1 \leq i < j \leq r+1} \Gamma_{\mathbb{R}}(\alpha_{s+i} - \alpha_{s+j}) \times \prod_{i=1}^{s} \frac{\prod_{i=1}^{r+1} \prod_{j=1}^{r} \Gamma_{\mathbb{R}}(\alpha_{s+i} - z_j)}{\prod_{i=1}^{r+1} \prod_{j=1}^{r} \Gamma_{\mathbb{R}}(\alpha_i - z_j)} \\
\times \prod_{1 \leq i \neq j \leq s} \Gamma_{\mathbb{R}}(\alpha_i - \alpha_j - 2l_j) \prod_{i=1}^{r+1} \prod_{j=1}^{r} \frac{\Gamma_{\mathbb{R}}(\alpha_{s+i} - \alpha_j - 2l_j)}{c((\alpha)^s)\tilde{c}((\alpha)^s + 2l) \prod_{i=1}^{r+1} \prod_{j=1}^{r} \Gamma_{\mathbb{R}}(\alpha_i - \alpha_j - 2l_j)}.
\]

Note that, as in the \(r = 1\) case, the fifth and third factors are \(\ll 1\), and the fourth factor is \(\ll \tilde{c}((\alpha)^s)\). Thus it will be enough to prove that the integral

\[
\int_{(0)^r} \left| \frac{\prod_{i<j}^{r+1} \prod_{j=1}^{r} \Gamma_{\mathbb{R}}(\alpha_{s+i} - z_j)}{c(z) \prod_{1 \leq i < j \leq r+1} \Gamma_{\mathbb{R}}(\alpha_{s+j} - \alpha_{s+i})} \right| d(z)^{(1)} dz \ll d(\alpha_{s+1}, \ldots \alpha_{s+r+1})^{(1)},
\]

as this along with

\[
d((\alpha)^s + 2l)^{(1)} \ll d((\alpha)^s)^{(1)}(1 + \|l\|)^{(1)},
\]

proves the claim. To see the claim we follow the same path as in [1, Proposition 1]. We may assume that \(0 < \Re(\alpha_{s+j}) < \epsilon\). We write \(\alpha_{s+j} = \Re(\alpha_{s+j}) + i\beta_j\) and \(z_j = it_j\). We use Stirling approximation to obtain that the integral is bounded by

\[
d(\beta)^{(1)} \int_{\mathbb{R}^r} \prod_{i,j} (1 + |\beta_i - t_j|)^{-1/2 + O(\epsilon)} \prod_{i \neq j} |t_i - t_j|^{1/2} d(t)^{(1)} E(t, \beta) dt,
\]

where \(E\) is the exponential factor given by

\[
E(t, \beta) := \exp \left[ -\frac{\pi}{4} \left( \sum_{i,j} |\beta_i - t_j| - \sum_{i<j} |t_i - t_j| - \sum_{i<j} |\beta_i - \beta_j| \right) \right].
\]

By fixing an order among \(\beta_i\) it can be checked elementarily that (as in the proof of [1, Proposition 1]) the quantity inside exp is always non-positive. Thus the essential supports of \(t\) in this integral are bounded by polynomials in \(\beta\). The integrand, other than the exponential factor, also being a polynomial in \(t\) and \(\beta\), the integral is bounded by some polynomial in \(\beta\), thus is \(d(\beta)^{(1)}\).

Finally, we conclude the proof by noting that

\[
\tilde{c}(\mu) \gg c(3(\mu))d(\mu)^{O(1)},
\]

where \(O(1)\) in the exponent means a fixed non-negative exponent depending on \(N\).

We finally have all the ingredients we need to prove Proposition 5.11. We will prove this by dividing the argument into two cases: whether \(a\) has the property \(\text{pop}(s)\) for some \(1 \leq s \leq n\), or not.
Proof of Proposition 5.1. Main ingredient of the proof is to shift contours in the integral of (6.6). We will divide the proof into two cases.

Case I: We assume that \( a \) does not satisfy \( \text{pop}(s) \) for any \( 1 \leq s \leq n \). Let \( d_0(\mu) \) be the eigenvalue of \( W_\mu \) under \( D_0 \) (recall Definition (3.8)). We see that, using (3.6) for \( M = 0 \), Lemma 3.6 for \( R = 0 \), and Lemma 6.1 the RHS of (6.6) is bounded by

\[
\mathcal{D}_M^I V \left[ \left( C(\Pi) \right)_a w \right] \prec_{N,p} \delta^{1/2}(a) \min(1, a_N) \int_{\mathbb{R}^n} \frac{|d_0(it)|^4}{d(it)^p} \frac{|c(1, it)|^2}{|c(it)|^2} dt.
\]

We have \( d_0(it) \ll d(it) \), and using Stirling, \( \frac{c(1, it)}{c(it)} \ll d(it)^{l'} \) for some absolute \( l' \). Thus the integral in the RHS above is convergent if \( p \) is sufficiently large. Hence proof of this case concludes.

Case II: We assume that \( a \) satisfies \( \text{pop}(s) \) for some given \( s \). Let \( d_M(\mu) \) be the eigenvalue of \( W_\mu \) under \( D_M \). We use Lemma 6.6 in the RHS of (6.6). We exchange the finite sum with integral over \( \mu \) and change variable \( w_\mu \mapsto \mu \). We obtain that for some explicit constant \( c_n \)

\[
\mathcal{D}_M^I V \left[ \left( C(\Pi) \right)_a w \right] = c_n \delta^{1/2}(a) \sum_{\mu} \Theta(\mu, \Pi) d_M(\mu) \langle f, W_\mu \rangle M_\mu^{-s}(a) d\mu.
\]

Now we shift contour of \( \mu_i \mapsto \mu_i + 2N \) for \( 1 \leq i \leq s \) for some natural number \( N \). Note that the integrand does not cross any pole. Employing Lemma 6.7 bound of \( \Theta \) in (3.6), and Lemma 3.6 with \( R = nN + 1 \) we conclude that

\[
\mathcal{D}_M^I V \left[ \left( C(\Pi) \right)_a w \right] \prec_{N,p} \delta^{1/2}(a) a_1^{2N} \int_{\mathbb{R}^n} \frac{d(it)^{O(1)} |c(1, it)|}{d(it)^p |C(it)|} dt,
\]

where \( O(1) \) in the exponent of \( d(it) \) depends at most on \( M, N, n \). We argue as in the previous case and thus conclude. \( \square \)

6.2. Remarks on the sphericality assumption of the chosen newvector. Here we take the opportunity to say a few words about the choice of the newvector in (4.1). It is mostly motivated by the newvector in [23, 29]. We note that in the non-archimedean case the newvectors are spherical in the Kirillov model, i.e., \( \text{GL}_n(\mathbb{Z}_p) \)-invariant. We analogously choose our analytic newvectors to be \( O(n) \)-invariant, but this feature of our construction is not essential. The main purpose of this assumption is to make the presentations and proof of Proposition 5.1 a little simpler. In fact, any \( f \in C_c^\infty(N \backslash G, \psi) \) will serve the purpose as in (4.1). We give a brief description about the essential modification one needs to carry out in the case when \( f \) is not spherical.

One only needs to modify the proof of Proposition 5.1 because the sphericality of \( f \) has been used only in this proof. If we do not choose \( f \) in (4.1) to be spherical then for Whittaker-Plancherel expansion we have to use (3.2) instead of (3.3); and hence, (6.6) will change to (say, for \( l = 0 \))

\[
(6.12) \quad \int_G \langle \lambda(g)f, J_\pi \rangle d\mu_p(\pi),
\]
where \( \lambda(g)f(h) := f(hg) \), and \( J_\pi \) is the relative character of \( \pi \), also known as the (long Weyl) Bessel distribution attached to \( \pi \), defined as a distribution on \( G \) by

\[
J_\pi(g) := \sum_{W \in \mathcal{B}(\pi)} W(g)W(1).
\]

Here \( \mathcal{B} \) is an orthonormal basis of the Whittaker model of \( \pi \). A good reference on the Bessel distribution can be found in [12, 26, 9].

It can be proved using spectral analysis that \( J_\pi \) satisfies the following recursion (compare with (6.7)):

\[
J_\pi \left( \begin{pmatrix} 1 \\ g \end{pmatrix} \right) w = \int_{\text{GL}_n-1(\mathbb{R})} \gamma(1/2, \pi \otimes \overline{\sigma}) \omega_\sigma(-1)^{n-1} J_\sigma(gw')d\mu_\sigma(\sigma),
\]

where \( \omega_\sigma \) is the central character of \( \sigma \), and \( w, w' \) are the long Weyl elements of \( \text{GL}(n) \) and \( \text{GL}(n-1) \), respectively. \( J_\pi \) is the Bessel distribution attached to \( \sigma \). A decomposition of \( J_\pi \) analogous to the decomposition of the spherical Whittaker function can be obtained using (6.13) (for \( \text{GL}(2) \) see, for instance, [12, chapter 6]). We should also point out that the Plancherel density in this case is not holomorphic unlike the spherical case. So while we shift contour we will likely to cross some polar hyperplanes coming from the Plancherel density. However, the residues will cancel with some part of the integral over \( \pi \). For instance, in \( \text{GL}(2) \) the Plancherel integral over \( \hat{\text{GL}}(2) \) can be decomposed as a sum of contour integrals over principal series and sum over discrete series. It can be checked that such residues from the integral over principal series will cancel out some summands in the sum over discrete series.

7. Proof of Theorem

Our proof uses a pre-Kuznetsov type formula on \( \text{PGL}_n(\mathbb{R}) \). Let \( f_X \) be a smoothened, \( L^1 \)-normalized, supported on \( K_0(X, \tau) \) (see [14]) and sufficiently concentrated near 1 (i.e. \( \tau \) is sufficiently small). We also assume that \( f_X \) takes non-negative values. Let \( F_X \) be the self-convolution of \( f_X \) defined by

\[
F_X(g) := \int_{\text{PGL}_n(\mathbb{R})} f_X(h)f_X(gh)dh.
\]

Note that \( F_X \) is also a smoothened, \( L^1 \)-normalized, characteristic function of \( K_0(X, \tau) \). We check that \( \text{Vol}(K_0(X, \tau)) \propto X^n \) which implies that \( \int_{N_a} F_X(x)\psi(x)dx \ll X^{n-1} \).

We obtain a spectral decomposition of

\[
\sum_{\gamma \in \text{PGL}_n(\mathbb{Z})} F_X(x_1^{-1}\gamma x_2) = \int_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} \pi(F_X)\varphi(x_1)\overline{\varphi(x_2)}d\mu_{\text{aut}}(\pi),
\]

where \( \mathcal{B}(\pi) \) is an orthonormal basis of \( \pi \) and we integrate over \( \pi \) in the automorphic spectrum of \( \text{PGL}_n(\mathbb{Z})/\text{PGL}_n(\mathbb{R}) \) with respect to the automorphic Plancherel measure \( d\mu_{\text{aut}} \) (see [14, Chapter 11.6] for details).
Let $N$ be the maximal unipotent of the upper triangular matrices in $\text{PGL}_n(\mathbb{R})$ and $\Gamma_\infty := N \cap \text{PGL}_n(\mathbb{Z})$. It is easy to see that for a sufficiently small neighborhood $U \subset \text{PGL}_n(\mathbb{R})$ around the identity,

(7.2) \quad NU \cap \text{PGL}_n(\mathbb{Z}) = \Gamma_\infty.

Recall $\psi$, which is an additive character of $N$, defined in (3.1). Let $W_\varphi$ be the Whittaker function of $\varphi$, i.e.

\[
\int_{\Gamma_\infty \setminus N} \varphi(xg)\overline{\psi(x)}dx = W_\varphi(g).
\]

We recall that there exists a positive constant $c_\pi$ such that

(7.3) \quad \|\varphi\|^2_\pi = c_\pi \|W_\varphi\|^2_{W(\pi, \psi)}.

We note that when $\pi$ is cuspidal then $c_\pi \asymp L(1, \pi, \text{Ad})$ where the underlying constant in $\asymp$ is absolute (coming from the residue of a maximal Eisenstein series at 1, see [3, p. 617]).

We define (cf. (6.12))

(7.4) \quad J_X(\pi) := \int_G J_\pi(g)F_X(g)dg = \sum_{W \in \mathcal{B}(\pi)} \pi(F_X)W(1)\overline{W(1)} = \sum_{W \in \mathcal{B}(\pi)} |\pi(f_X)W(1)|^2.

We claim that $J_X$ is well-defined, i.e., that the sum defining it converges absolutely. To see that we note that by integration by parts with $D$ (see (3.8)) one has

\[
\pi(f_X)W(1) \ll_{X, d} S_{-d}(W).
\]

Then convergence follows from Lemma 3.3. For non-generic $\pi$ we define $J_X(\pi)$ to be identically zero. We fix a basis $\mathcal{B}(\pi)$ containing a Whittaker newvector $V$, i.e. $V$ as in (4.1).

We now take $x_i \in \Gamma_\infty \setminus N$ and multiply both sides of (7.1) by $\psi(x_2)\overline{\psi(x_1)}$ and integrate with respect to $x_i$. Thus we automatically get rid of the non-generic part of the spectrum. Using (7.2), and (7.4) we rewrite (7.1) we obtain

\[
\int_{\mathcal{F}_n} J_X(\pi)e_\pi^{-1}d\mu_{\text{aut}}(\pi) = \sum_{\gamma \in \Gamma_\infty} \int_{\Gamma_\infty \setminus N} \int_{\Gamma_\infty \setminus N} F_X(x_1^{-1}\gamma x_2)\psi(x_2 - x_1)dx_1dx_2
\]

\[
= \int_{\Gamma_\infty \setminus N} \int_{N} F_X(x_1^{-1}x_2)\psi(x_2 - x_1)dx_1dx_2
\]

\[
= \int_{N} F_X(x)\psi(x) \ll X^{n-1}.
\]

Note that $J_X$ is non-negative on the automorphic spectrum $\mathcal{F}_n$. We first drop everything other than the cuspidal spectrum from the LHS. It is known that (see [28, 4]) that cuspidal automorphic representations are $\theta$-tempered for $0 \leq \theta \leq \frac{1}{2} - \frac{1}{1+n}$. From Theorem 2 we get if $\pi$ is cuspidal with $C(\pi) < X$ then

\[
\pi(f_X)W(1) \gg 1.
\]

Hence $J_X(\pi) \gg 1$ for cuspidal $\pi$ with $C(\pi) < X$. 

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ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland.
E-mail address: subhajit.jana@math.ethz.ch, paul.nelson@math.ethz.ch