On the Asymptotic Normality of the Conditional Maximum Likelihood Estimators for the Truncated Regression Model and the Tobit Model

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Abstract

In this paper, we study the asymptotic normality of the conditional maximum likelihood (ML) estimators for the truncated regression model and the Tobit model. We show that under the general setting assumed in his book, the conjectures made by Hayashi (2000) about the asymptotic normality of the conditional ML estimators for both models are true, namely, a sufficient condition is the nonsingularity of $x_t'x_t$.

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1see page 516, and page 520 of Hayashi (2000).
1 Introduction

The truncated regression model and the Tobit model (censored regression model) are two important basic models appearing in many applications in economics and other areas. The method of conditional maximum likelihood (ML) can be used to estimate the parameters in both models. In order to apply this method, the consistency and asymptotic normality of the estimator have to be verified. In the following, these two models and their conditional ML estimators are introduced in the exactly same way as what Hayashi (2000) did.

1.1 Truncated Regression Model

For the truncated regression model, the following assumptions are adopted:

Assumption 1. Suppose that \( \{y_t, x_t\} \) is i.i.d satisfying
\[
y_t = x_t' \beta_0 + \epsilon_t, \tag{1.1}
\]
\[
\epsilon_t | x_t \sim N(0, \sigma_0^2), \quad t = 1, 2, ..., n, \tag{1.2}
\]
where \( x_t \) and \( \beta_0 \) are both vectors with \( K \) components.

Assumption 2. The truncation rule is: \( y_t > c \) where \( c \) is a known constant. Only those observations satisfying the truncation rule are included in the sample.

Since \( y_t | x_t \sim N(x_t' \beta_0, \sigma_0^2) \), it can be established that
\[
E(y_t | x_t, y_t > c) = x_t' \beta_0 + \sigma_0 \lambda \left( \frac{c - x_t' \beta_0}{\sigma_0} \right), \tag{1.3}
\]
\[
Var(y_t | x_t, y_t > c) = \sigma_0^2 \left[ 1 - \lambda \left( \frac{c - x_t' \beta_0}{\sigma_0} \right) \right] \left[ \lambda \left( \frac{c - x_t' \beta_0}{\sigma_0} \right) - \frac{c - x_t' \beta_0}{\sigma_0} \right], \tag{1.4}
\]
where \( \lambda \left( \frac{c - x_t' \beta_0}{\sigma_0} \right) \equiv \frac{\phi \left( \frac{c - x_t' \beta_0}{\sigma_0} \right)}{1 - \Phi \left( \frac{c - x_t' \beta_0}{\sigma_0} \right)} \) with \( \phi \) as the density of \( N(0, 1) \) and \( \Phi \) as the cumulative distribution function of \( N(0, 1) \). \( \lambda \) is also called the inverse Mill's ratio.

The log conditional likelihood for observation \( t \) is:
\[
\log f(y_t | x_t; \beta, \sigma^2) = \left\{ -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma^2) - \frac{1}{2} \left( \frac{y_t - x_t' \beta}{\sigma} \right)^2 \right\} - \log \left[ 1 - \Phi \left( \frac{c - x_t' \beta}{\sigma} \right) \right], \tag{1.5}
\]
where \( (\beta, \sigma^2) \) are the hypothetical values of \( (\beta_0, \sigma_0^2) \) , and \( \Phi \) is the cumulative distribution function of \( N(0, 1) \).

For simplification, the following reparameterization is used:
\[
\delta = \beta / \sigma, \quad \gamma = 1 / \sigma. \tag{1.6}
\]
The reparameterized log conditional likelihood is
\[
\log ˜f(y_t|x_t; δ, γ) = -\frac{1}{2} \log(2\pi) + \log(γ) - \frac{1}{2}(γy_t - x'_tδ)^2 - \log[1 - Φ(γc - x'_tδ)].
\]
(1.7)

The objective function in ML estimation is the average log conditional likelihood of the sample. The conditional ML estimator \(( ˆδ, ˆγ)\) of \((δ_0, γ_0)\) is the \((δ, γ)\) that maximizes the objective function.

Hayashi (2000) gave the following expressions of the score and the Hessian for observation \(t\):
\[
s(w_t; δ, γ) = \left[\frac{1}{γ} - (γy_t - x'_tδ)y_t\right] + λ(v_t) \left[-x_t\right],
\]
(1.8)
\[
H(w_t; δ, γ) = -\left[x_tx'_t - y_tx'_t\right] + λ(v_t)\left[λ(v_t) - v_t\right] \left[x_tx'_t - cx'_t\right],
\]
(1.9)
where \(s(w_t; δ, γ)\) is a vector of dimension \((K + 1) \times 1\), \(H(w_t; δ, γ)\) is a square matrix of dimension \((K + 1) \times (K + 1)\) with \(K\) as the number of regressors, \(w_t = (y_t, x'_t)'\), \(λ(v_t) ≡ \frac{φ(v_t)}{1 - Φ(v_t)}\) with \(v_t ≡ γc - x'_tδ = c - x'_tβ\).

By verifying that the conditions of Proposition 1.1 (see below) are satisfied, Hayashi (2000) proved that the ML estimator \(( ˆδ, ˆγ)\) of \((δ_0, γ_0)\) is consistent under the nonsingularity of \(E(x_tx'_t)\).

As to asymptotic normality, Hayashi (2000) pointed out that \(E[s|x_t] = 0\) and the conditional information equality holds, i.e. \(E[ss'|x_t] = -E[H|x_t]\). So condition 3 of Proposition 1.3 is satisfied. However conditions 4 and 5 of proposition 1.3 are not verified. For the case where \(\{x_t\}\) is a sequence of fixed constants, Sapra (1992) showed asymptotic normality under the assumption that \(x_t\) is bounded, \(\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} x_tx'_t\) is nonsingular and observations are serially correlated.

Hayashi (2000) conjectured that for the case where \(x_t\) is random as in the current setting (i.e., Assumption 1 and 2 are satisfied), a sufficient condition for asymptotic normality is the nonsingularity of \(E(x_tx'_t)\).

1.2 Tobit Model

For the Tobit model, the following assumption is adopted:

\(^2\)see page 516 of Hayashi (2000).
Assumption 1’. Suppose that \{y_t, x_t\} is i.i.d satisfying
\[
y_t^* = x_t' \beta_0 + \epsilon_t, \tag{1.10}
\]
\[
\epsilon_t|x_t \sim N(0, \sigma_0^2), \quad t = 1, 2, \ldots, n, \tag{1.11}
\]
\[
y_t = \begin{cases} 
y_t^* & \text{if } y_t^* > c, \\
c & \text{if } y_t^* \leq c,
\end{cases} \tag{1.12}
\]
where \(x_t\) and \(\beta_0\) are both vectors with \(K\) components, \(c\) is a known constant. Different from the truncated regression model in above, here the observations for which the value of the dependent variable \(y_t^*\) doesn’t meet the rule \(y_t^* > c\) are included in the sample. Another way to write the Tobit model is
\[
y_t = \max\{x_t' \beta_0 + \epsilon_t, c\}. \tag{1.13}
\]

The log conditional likelihood for observation \(t\) is:
\[
\log f(y_t|x_t; \beta, \sigma^2) = (1 - D_t)\log \left[ \frac{1}{\sigma} \phi \left( \frac{y_t - x_t' \beta}{\sigma} \right) \right] + D_t \log \Phi \left( \frac{c - x_t' \beta}{\sigma} \right), \tag{1.14}
\]
where \((\beta, \sigma^2)\) are the hypothetical values of \((\beta_0, \sigma_0^2)\), \(\phi\) is the density of \(N(0, 1)\) and \(\Phi\) is the cumulative distribution of \(N(0, 1)\), and the dummy variable \(D_t\) is defined as
\[
D_t = \begin{cases} 
0 & \text{if } y_t > c \text{ (i.e., } y_t^* > c) \\
1 & \text{if } y_t = c \text{ (i.e., } y_t^* \leq c) .
\end{cases} \tag{1.15}
\]

The objective function in ML estimation is the average log conditional likelihood of the sample.

As in the truncation regression model in above, to make analysis easier, the reparameterization (1.6) is used and the reparameterized log conditional likelihood is:
\[
\log \tilde{f}(y_t|x_t; \delta, \gamma) = (1 - D_t)\left\{ -\frac{1}{2} \log(2\pi) + \log(\gamma) - \frac{1}{2} \left( \gamma y_t - x_t' \delta \right)^2 \right\} - D_t \log \Phi(\gamma c - x_t' \delta). \tag{1.16}
\]

Hayashi (2000) gave the following expressions of the score and the Hessian for observation \(t\):
\[
s(w_t; \delta, \gamma) = (1 - D_t) \left[ \frac{\gamma y_t - x_t' \delta}{\gamma} x_t - D_t \lambda(-v_t) \begin{bmatrix} -x_t \\ c \end{bmatrix} \right], \tag{1.17}
\]
\[
H(w_t; \delta, \gamma) = - (1 - D_t) \left[ \frac{x_t x_t'}{y_t x_t'} - \frac{y_t x_t}{y_t^2 + y_t^2} \right] - D_t \lambda(-v_t) \left[ \lambda(-v_t) + v_t \right] \begin{bmatrix} x_t x_t' - c x_t \\ -c x_t' c^2 \end{bmatrix}, \tag{1.18}
\]
where $s(w_t; \delta, \gamma)$ is a vector of dimension $(K + 1) \times 1$, $H(w_t; \delta, \gamma)$ is a square matrix of dimension $(K + 1) \times (K + 1)$, $K$ is the number of regressors, $w_t = (y_t, x_t')'$, $\lambda(-v_t) \equiv \frac{\phi(-v_t)}{1 - \Phi(-v_t)}$ with $v_t \equiv \gamma c - x_t' \delta = c - x_t' \beta \sigma$.

For consistency, Hayashi (2000) pointed out that the relevant consistency theorem for the Tobit Model is Proposition 1.2 (see below). He also mentioned that when $\{y_t, x_t\}$ is ergodic stationary but not necessary i.i.d, the conditional ML estimator $(\hat{\delta}, \hat{\gamma})$ of $(\delta_0, \gamma_0)$ is consistent.

For the case where $\{x_t\}$ is a sequence of fixed constants, Amemiya (1973) proved the consistency and asymptotic normality of the conditional ML estimator for the Tobit model under the assumption that $x_t$ is bounded and $\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^n x_t x_t'$ is nonsingular.

For the Tobit model, Hayashi (2000) conjectured that for the case where $x_t$ is random as in the current setting (i.e., Assumption 1' is satisfied), a sufficient condition for asymptotic normality is the nonsingularity of $E(x_t x_t')$.

In this paper, we show that Hayashi’s conjectures for the asymptotic normality of both models are true, i.e. a sufficient condition for the asymptotic normality for both models is the nonsingularity of $E(x_t x_t')$.

The content of this paper is organized as follows. First in below we cite three propositions from Hayashi (2000), which will be used to show the main results. Then in Section 2, we show that a sufficient condition for the asymptotic normality for the truncated regression model is the nonsingularity of $E(x_t x_t')$. In Section 3, we show that a sufficient condition for the asymptotic normality for the Tobit model is the nonsingularity of $E(x_t x_t')$.

Now we present three propositions from Hayashi (2000).

**Proposition 1.1** (Consistency of conditional ML with compact parameter space):

Let $\{y_t, x_t\}$ be ergodic stationary with conditional density $f(y_t|x_t; \theta_0)$ and let $\hat{\theta}$ be the conditional ML estimator, which maximizes the average log conditional likelihood (derived under the assumption that $\{y_t, x_t\}$ is i.i.d.):

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^n \log f(y_t|x_t; \theta).$$

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3see page 520 of Hayashi (2000)
4see exercise 3 on page 521 of Hayashi (2000)
5see page 520 of Hayashi (2000).
6see Proposition 7.5 on page 464 of Hayashi (2000).
Suppose the model is correctly specified so that $\theta_0$ is in $\Theta$. Suppose that (i) the parameter space $\Theta$ is a compact subset of $\mathbb{R}^p$, (ii) $f(y_t|x_t; \theta)$ is continuous in $\theta$ for all $(y_t, x_t)$, and (iii) $f(y_t|x_t; \theta)$ is measurable in $(y_t, x_t)$ for all $\theta \in \Theta$ (so $\hat{\theta}$ is a well-defined random variable). Suppose, further that

1. (identification) $\text{Prob}[f(y_t|x_t; \theta) \neq f(y_t|x_t; \theta_0)] > 0$ for all $\theta \neq \theta_0$ in $\Theta$,

2. (dominance) $E[\sup_{\theta \in \Theta} |\log f(y_t|x_t; \theta)|] < \infty$ (note: the expectation is over $y_t$ and $x_t$).

Then $\hat{\theta} \to_p \theta_0$.

Proposition 1.2 (Consistency of conditional ML without compactness): Let $(y_t, x_t)$ be ergodic stationary with conditional density $f(y_t|x_t; \hat{\theta}_0)$ and let $\hat{\theta}$ be the conditional ML estimator, which maximizes the average log conditional likelihood (derived under the assumption that $(y_t, x_t)$ is i.i.d.):

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \log f(y_t|x_t; \theta).$$

Suppose the model is correctly specified so that $\theta_0$ is in $\Theta$. Suppose that (i) the true parameter vector $\theta_0$ is an element of the interior of a convex parameter space $\Theta$ ($\subset \mathbb{R}^p$), (ii) $\log f(y_t|x_t; \theta)$ is concave in $\theta$ for all $(y_t, x_t)$, and (iii) $\log f(y_t|x_t; \theta)$ is measurable in $(y_t, x_t)$ for all $\theta \in \Theta$. (For sufficiently large $n$, $\hat{\theta}$ well-defined). Suppose, further that

1. (identification) $\text{Prob}[f(y_t|x_t; \theta) \neq f(y_t|x_t; \theta_0)] > 0$ for all $\theta \neq \theta_0$ in $\Theta$,

2. $E[|\log f(y_t|x_t; \theta)|] < \infty$ (i.e., $E[\log f(y_t|x_t; \theta)]$ exists and is finite) for all $\theta \in \Theta$ (note: the expectation is over $y_t$ and $x_t$).

Then as $n \to \infty$, $\hat{\theta}$ exists with probability approaching 1 and $\hat{\theta} \to_p \theta_0$.

Proposition 1.3 (Asymptotic normality of conditional ML): Let $w_t$ ($\equiv (y_t, x_t)'$) be i.i.d. Suppose the conditions of either Proposition 1.1 or Proposition 1.2 are satisfied, so that $\hat{\theta} \to_p \theta_0$. Suppose, in addition, that

1. $\theta_0$ is in the interior of $\Theta$,

2. $f(y_t|x_t; \theta)$ is twice continuously differentiable in $\theta$ for all $(y_t, x_t)$,

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7 see Proposition 7.6 on page 464-465 of Hayashi (2000).

8 see Proposition 7.9 on page 475 of Hayashi (2000).
3. \( E[s(w_t; \theta_0)] = 0 \) and \(-E[H(w_t; \theta_0)] = E[s(w_t; \theta_0)s(w_t; \theta_0)']\), where \( s \) and \( H \) functions are the score and the Hessian for observation \( t \).

4. (local dominance condition on the Hessian) for some neighborhood \( N \) of \( \theta_0 \),
\[
E[\sup_{\theta \in N} \|H(w_t; \theta)\|] < \infty,
\]
so that for any consistent estimator \( \hat{\theta} \), \( \frac{1}{n} \sum_{t=1}^n H(w_t; \hat{\theta}) \to_p E[H(w_t; \theta_0)] \).

5. \( E[H(w_t; \theta_0)] \) is nonsingular.

Then \( \hat{\theta} \) is asymptotic normal with \( \text{Avar}(\hat{\theta}) \) given by the following:
\[
\text{Avar}(\hat{\theta}) = -\{E[H(w_t; \theta_0)]\}^{-1} = \{E[s(w_t; \theta_0)s(w_t; \theta_0)']\}^{-1}.
\]

2 Asymptotic Normality of the Conditional ML of the Truncated Regression Model

As we mentioned in above, for the truncated regression model introduced in subsection 1.1, Hayashi (2000) showed that the conditional ML estimator satisfies the conditions of Proposition 1.1 under the nonsingularity of \( E(x_t x_t') \). Therefore, by Proposition 1.1, the conditional ML estimator \((\hat{\delta}, \hat{\gamma})\) of \((\delta_0, \gamma_0)\) is consistent when \( E(x_t x_t') \) is nonsingular. For asymptotic normality, he mentioned that condition 3 of Proposition 1.3 is satisfied. It is easy to see that conditions 1 and 2 of Proposition 1.3 are satisfied.

In this section, we show the following theorem holds.

**Theorem 2.1** For the truncated regression model satisfying Assumptions 1 and 2, if \( E(x_t x_t') \) is nonsingular, then conditions 4 and 5 of Proposition 1.3 are satisfied.

**Proof.** First we show that condition 4 of Proposition 1.3 is satisfied.

Define \( A \) as the matrix
\[
\begin{bmatrix}
  x_t x_t' & -y_t x_t^2 \\
  -y_t x_t' & \frac{1}{\gamma^2} + y_t^2
\end{bmatrix},
\]
and define \( B \) as the matrix
\[
\lambda(v_t)[\lambda(v_t) - v_t] \begin{bmatrix}
  x_t x_t' & -c x_t^2 \\
  -c x_t' & c^2
\end{bmatrix}.
\]
By the expression of the Hessian $H(w_t; \delta, \gamma)$ in (1.9),

$$H(w_t; \delta, \gamma) = A + B.$$  \hfill (2.3)

Therefore,

$$\|H(w_t; \delta, \gamma)\| \leq \|A\| + \|B\|,$$  \hfill (2.4)

where $\| \cdot \|$ is the Euclidean norm of a matrix, which is defined as the square root of the sum of squares of the elements of the matrix.

It is easy to see that

$$\|A\|^2 = \|x_t x_t'\|^2 + 2\|y_t x_t\|^2 + (1/\gamma^2 + y_t^2)^2.$$  \hfill (2.5)

Since $x_t$ is a vector of $K$ components, we write it as $(x_{t1}, x_{t2}, ..., x_{tK})'$. We have

$$\|y_t x_t\|^2 \leq \sum_{i=1}^{K} \frac{1}{2}(y_i^4 + x_i^4) \leq \frac{K}{2} y_t^4 + \frac{1}{2}\|x_t x_t'\|^2.$$  \hfill (2.6)

Thus

$$\|A\|^2 \leq \|x_t x_t'\|^2 + 2\frac{K}{2} y_t^4 + \|x_t x_t'\|^2 + 2/\gamma^4 + 2 y_t^4$$

$$\leq 2\|x_t x_t'\|^2 + (K + 2) y_t^4 + 2/\gamma^4,$$  \hfill (2.7)

which implies

$$\|A\| \leq \sqrt{2}\|x_t x_t'\| + \sqrt{K + 2} y_t^2 + \sqrt{2}/\gamma^2.$$  \hfill (2.8)

By (1.1) and (1.6), we have

$$y_t^2 \leq \frac{2}{\gamma_0} (x_t' \delta_0)^2 + 2 \epsilon_t^2.$$  \hfill (2.9)

Since $\delta$ and $\delta_0$ are vectors of $K$ components, we write them as $(\delta_1, ..., \delta_K)'$ and $(\delta_{01}, ..., \delta_{0K})'$. Define the neighborhood $N$ of $(\delta_0, \gamma_0) (\equiv \theta_0)$ as

$$\{ (\delta, \gamma) : \max_{i=1,...,K} |\delta_i - \delta_{0i}| < C_1, |\gamma - \gamma_0| < C_2, \text{ with } C_2 \text{ satisfies } 0 < -C_2 + \gamma_0 \},$$

where $C_1$ and $C_2$ are positive constants.

Since $\gamma_0 = 1/\sigma_0$ and $\sigma_0$ is finite, $\gamma_0 \neq 0$. So we can always find a small positive $C_2$ such that $0 < -C_2 + \gamma_0$. This implies for any $(\delta, \gamma) \in N$, $1/\gamma \leq \frac{1}{-C_2 + \gamma_0}$. Combining this with (2.8) and (2.9), we have, for any $(\delta, \gamma) \in N$,

$$\sup_{(\delta, \gamma) \in N} \|A\| \leq \sqrt{2}\|x_t x_t'\| + \sqrt{K + 2} \tilde{C}\|x_t x_t'\| + 2\sqrt{K + 2} \epsilon_t^2 + \sqrt{2} \frac{1}{(-C_2 + \gamma_0)^2}$$

$$\leq (\sqrt{2} + \sqrt{K + 2} \tilde{C})\|x_t x_t'\| + 2\sqrt{K + 2} \epsilon_t^2 + \sqrt{2} \frac{1}{(-C_2 + \gamma_0)^2}.$$  \hfill (2.10)
where \( \tilde{C} = \frac{2}{\gamma_0} \max(\delta_0^2, \ldots, \delta_0^2) \).

Next we look at the matrix \( B \) defined in (2.2).

It is well known that as the derivative of \( \lambda(v_t) \), \( \lambda'(v_t) \) satisfies

\[
\lambda'(v_t) = \lambda(v_t)(\lambda(v_t) - v_t),
\]

and \( \lambda(v_t) \) is between 0 and 1. Therefore

\[
\|B\|^2 \leq \|x_t x'_t\|^2 + 2\|c x_t\|^2 + c^4.
\]

Since \( \|c x_t\|^2 \) satisfies

\[
\|c x_t\|^2 \leq \sum_{i=1}^{K} \frac{1}{2}(c^4 + x_t^4) \leq \frac{K}{2} c^4 + \frac{1}{2}\|x_t x'_t\|^2,
\]

which implies

\[
\|B\| \leq \sqrt{2}\|x_t x'_t\| + \sqrt{K + 1}c^2.
\]

Combining this with (2.10) and (2.4), we have

\[
\sup_{(\delta, \gamma) \in N} \|H(w_t; \delta, \gamma)\| \leq (2\sqrt{2} + \sqrt{K + 2} \tilde{C})\|x_t x'_t\| + 2\sqrt{K + 1}c^2 + \sqrt{2} \frac{1}{\epsilon_0} + \sqrt{K + 1}c^2.
\]

Since \( E[\epsilon_0^2 | x_t] = \sigma_0^2 = \frac{1}{\gamma_0} \),

\[
E[\sup_{(\delta, \gamma) \in N} \|H(w_t; \delta, \gamma)\|] \leq (2\sqrt{2} + \sqrt{K + 2} \tilde{C})E[\|x_t x'_t\|] + 2\sqrt{K + 1}c^2 + \sqrt{2} \frac{1}{\gamma_0} + \sqrt{2} \frac{1}{(-c_2 + \gamma_0)^2}.
\]

We know that \( E[\|x_t x'_t\|] < \infty \) if \( E[x_t x'_t] \) exists and is finite, and \( E[x_t x'_t] \) exists and is finite if \( E[x_t x'_t] \) is nonsingular. Therefore, when \( E[x_t x'_t] \) is nonsingular,

\[
E[\sup_{(\delta, \gamma) \in N} \|H(w_t; \delta, \gamma)\|] < \infty,
\]

namely, condition 4 of Proposition 1.3 is satisfied.

Next we show that condition 5 of Proposition 1.3 is satisfied.

Since condition 3 of Proposition 1.3 is satisfied,

\[
- E[H(w_t; \delta_0, \gamma_0)] = E[s(w_t; \delta_0, \gamma_0)s(w_t; \delta_0, \gamma_0)'].
\]
It is clear that $s(\mathbf{w}_t; \delta_0, \gamma_0)s(\mathbf{w}_t; \delta_0, \gamma_0)'$ is positive semidefinite. This implies that $E[s(\mathbf{w}_t; \delta_0, \gamma_0)s(\mathbf{w}_t; \delta_0, \gamma_0)']$ is positive semidefinite. Therefore $E[H(\mathbf{w}_t; \delta_0, \gamma_0)]$ is negative semidefinite.

Let $\mathbf{z} \equiv (\mathbf{z}_1, ..., \mathbf{z}_K, \mathbf{z}_{K+1})' \in \mathbb{R}^{K+1}$ be a solution to the equation

$$\mathbf{z}'E[H(\mathbf{w}_t; \delta_0, \gamma_0)]\mathbf{z} = 0. \quad (2.20)$$

We know that if $\mathbf{z} = (0, ..., 0)$ is the only solution to the above equation, then $E[H(\mathbf{w}_t; \delta_0, \gamma_0)]$ is nonsingular.

By (2.19), (2.20) is equivalent to

$$\mathbf{z}'E[s(\mathbf{w}_t; \delta_0, \gamma_0)s(\mathbf{w}_t; \delta_0, \gamma_0)']\mathbf{z} = 0, \quad (2.21)$$

namely,

$$E[\mathbf{z}'s(\mathbf{w}_t; \delta_0, \gamma_0)s(\mathbf{w}_t; \delta_0, \gamma_0)']\mathbf{z} = 0, \quad (2.22)$$

because $\mathbf{z}$ is not random.

In terms of the expression of $s$ in (1.8), (2.22) is equivalent to

$$E \left[ \sum_{i=1}^{K} \left( \gamma_0 y_t - \mathbf{x}'_i \delta_0 - \lambda(v_{0t}) \right) \mathbf{x}_i \mathbf{z}_i 1_{\{y_t > c\}} + \left( \frac{1}{\gamma_0} - (\gamma_0 y_t - \mathbf{x}'_0 \delta_0)y_t + \lambda(v_{0t})c \right) \mathbf{z}_K + 1_{\{y_t > c\}} \right]^2 = 0, \quad (2.23)$$

where $v_{0t} \equiv \gamma_0 y_t - \mathbf{x}'_0 \delta_0$.

This implies

$$\sum_{i=1}^{K} \left( \gamma_0 y_t - \mathbf{x}'_i \delta_0 - \lambda(v_{0t}) \right) \mathbf{x}_i \mathbf{z}_i 1_{\{y_t > c\}} + \left( \frac{1}{\gamma_0} - (\gamma_0 y_t - \mathbf{x}'_0 \delta_0)y_t + \lambda(v_{0t})c \right) \mathbf{z}_K + 1_{\{y_t > c\}} = 0 \quad \text{a.e.,} \quad (2.24)$$

where a.e. means almost everywhere.

The left side of (2.24)

$$= \sum_{i=1}^{K} \left( \gamma_0 y_t - \mathbf{x}'_i \delta_0 \right) \mathbf{x}_i \mathbf{z}_i 1_{\{y_t > c\}} - \sum_{i=1}^{K} \lambda(v_{0t}) \mathbf{x}_i \mathbf{z}_i 1_{\{y_t > c\}}$$

$$+ \left( \frac{1}{\gamma_0} - (\gamma_0 y_t - \mathbf{x}'_0 \delta_0) \right) \gamma_0 \left( \gamma_0 y_t - \mathbf{x}'_0 \delta_0 + \mathbf{x}'_0 \delta_0 \right) + \lambda(v_{0t})c \mathbf{z}_K + 1_{\{y_t > c\}}$$

$$= \sum_{i=1}^{K} \left( \gamma_0 y_t - \mathbf{x}'_i \delta_0 \right) \mathbf{x}_i \mathbf{z}_i 1_{\{y_t > c\}} - \frac{1}{\gamma_0} \left( \gamma_0 y_t - \mathbf{x}'_0 \delta_0 \right) \mathbf{x}'_0 \delta_0 \mathbf{z}_K + 1_{\{y_t > c\}}$$

$$- \frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}'_0 \delta_0)^2 \mathbf{z}_K + 1_{\{y_t > c\}} - \sum_{i=1}^{K} \lambda(v_{0t}) \mathbf{x}_i \mathbf{z}_i 1_{\{y_t > c\}} + \left( \frac{1}{\gamma_0} + \lambda(v_{0t})c \right) \mathbf{z}_K + 1_{\{y_t > c\}}$$

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Thus (2.24) is equivalent to

\[ \epsilon_t f(x_t, z, \delta_0, \gamma_0)1_{\{\epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0\}} - \gamma_0 \epsilon_t^2 z_{K+1}1_{\{\epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0\}} + g(x_t, z, \delta_0, \gamma_0)1_{\{\epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0\}} = 0 \text{ a.e.,} \]

namely,

\[ -\epsilon_t^2 z_{K+1}1_{\{\epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0\}} + \epsilon_t f(x_t, z, \delta_0, \gamma_0)1_{\{\epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0\}} + \frac{1}{\gamma_0} g(x_t, z, \delta_0, \gamma_0)1_{\{\epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0\}} = 0 \text{ a.e..} \]

Suppose \( z_{K+1} \neq 0 \). When \( \epsilon_t > c - \frac{1}{\gamma_0} x'_t \delta_0 \) holds, (2.28) is a quadratic equation of \( \epsilon_t \). If the quadratic equation has solutions, then the solutions are

\[ \epsilon_t = \frac{-f(x_t, z, \delta_0, \gamma_0) \pm \sqrt{f^2(x_t, z, \delta_0, \gamma_0) + \frac{4z_{K+1}^2}{\gamma_0} g(x_t, z, \delta_0, \gamma_0)}}{-2z_{K+1}} \text{ a.e..} \]

But this contradicts the fact that \( \epsilon_t | x_t \sim N(0, \sigma_0^2) \) (see (1.2)).

Therefore \( z_{K+1} \) must be 0. This means that

\[ z'E[H(w_t; \delta_0, \gamma_0)]z = 0 \implies z = (z_1, ..., z_K, 0)' \text{.} \]
By (1.9),
\[
(z_1, ..., z_K, 0)E[H(w_t; \delta_0, \gamma_0)](z_1, ..., z_K, 0)'
= - (z_1, ..., z_K)E\left[ (1 - \lambda(v_{0t})[\lambda(v_{0t}) - v_{0t}])x_t x_t' (z_1, ..., z_K) \right]'
= - E\left[ (z_1, ..., z_K) (1 - \lambda(v_{0t})[\lambda(v_{0t}) - v_{0t}])x_t x_t' (z_1, ..., z_K) \right],
\]
(2.31)
We know that for any positive constant $\tau \in \mathbb{R},$
\[
E\left[ (z_1, ..., z_K) (1 - \lambda(v_{0t})[\lambda(v_{0t}) - v_{0t}])x_t x_t' (z_1, ..., z_K) \right]'
\geq E\left[ (z_1, ..., z_K) 1_{\{|v_{0t}| \leq \tau\}} (1 - \lambda(v_{0t})[\lambda(v_{0t}) - v_{0t}])x_t x_t' (z_1, ..., z_K) \right]
\geq C E\left[ (z_1, ..., z_K) 1_{\{|v_{0t}| \leq \tau\}} x_t x_t' (z_1, ..., z_K) \right],
\]
(2.32)
where $C$ is a positive constant depending on $\tau.$ Here we used the fact that $\lambda(v_{0t})[\lambda(v_{0t}) - v_{0t}]$ is between 0 and 1, and asymptotes to 0 as $v_{0t} \to -\infty$ and to 1 as $v_{0t} \to \infty.$

Therefore, by (2.30), (2.31) and (2.32),
\[
z' E[H(w_t; \delta_0, \gamma_0)] z = 0 \implies (z_1, ..., z_K)E[1_{\{|v_{0t}| \leq \tau\}} x_t x_t'] (z_1, ..., z_K)' = 0.
\]
(2.33)

It is easy to see that if $E[x_t x_t']$ is nonsingular, then for large enough $\tau,$ $E[1_{\{|v_{0t}| \leq \tau\}} x_t x_t']$ is also nonsingular. Thus when $E[x_t x_t']$ is nonsingular, for large enough $\tau,$ \{z_i = 0, \ i = 1, ..., K\} is the only solution satisfying
\[
(z_1, ..., z_K)E[1_{\{|v_{0t}| \leq \tau\}} x_t x_t'] (z_1, ..., z_K)' = 0.
\]
(2.34)
This means that when $E[x_t x_t']$ is nonsingular,
\[
z' E[H(w_t; \delta_0, \gamma_0)] z = 0 \implies z = (0, ..., 0, 0)'.
\]
(2.35)
Thus when $E[x_t x_t']$ is nonsingular, $E[H(w_t; \delta_0, \gamma_0)]$ is nonsingular, i.e. condition 5 of Proposition 1.3 holds.

Theorem 2.1 implies that when $E[x_t x_t']$ is nonsingular, under Assumptions 1 and 2, the conditional ML estimator $(\hat{\delta}, \hat{\gamma})$ is asymptotic normal with $Avar(\hat{\delta}, \hat{\gamma})$ given by the following:
\[
Avar(\hat{\delta}, \hat{\gamma}) = -\{E[H(w_t; \delta_0, \gamma_0)]\}^{-1} = \{ E[s(w_t; \delta_0, \gamma_0)]^{-1} \}
\]
(2.36)
To recover original parameters and obtain the asymptotic variance of $(\hat{\beta}, \hat{\sigma}^2),$ the delta method can be applied. (see page 517 of Hayashi (2000))
3 Asymptotic Normality of the Conditional ML of the Tobit Model

In this section, we show that for the Tobit Model introduced in subsection 1.2, the conditions of Proposition 1.3 are satisfied, thereafter the asymptotic normality of the conditional ML is verified.

It is easy to see that conditions 1 and 2 of Proposition 1.3 are satisfied.

We know that in the expression of $H(w_t; \delta, \gamma)$ of (1.18), $1 - D_t$ is either 1 or 0, so bounded, and $\lambda(-v_t)[\lambda(-v_t) + v_t]$ is between 0 and 1, so bounded. Thus by the same argument as in (2.1) through (2.18) for the truncated regression model, we can show that condition 4 of Proposition 1.3 is satisfied.

In the following, we show conditions 3 and 5 of Proposition 1.3 are satisfied.

Before we move on to the main results, we need some preliminary results on the conditional moments of $\gamma_0 y_t - x'_t \delta_0$ which is conditioning on $\{x_t, y_t > c\}$.

Lemma 3.1 For the Tobit model, the following equalities hold:

$$E[\gamma_0 y_t - x'_t \delta_0 | x_t, y_t > c] = \lambda(v_{0t}), \quad (3.1)$$

$$E[(\gamma_0 y_t - x'_t \delta_0)^2 | x_t, y_t > c] = v_{0t}^2 \lambda(v_{0t}) + 1, \quad (3.2)$$

$$E[(\gamma_0 y_t - x'_t \delta_0)^3 | x_t, y_t > c] = v_{0t}^3 \lambda(v_{0t}) + 2v_{0t} \lambda(v_{0t}), \quad (3.3)$$

$$E[(\gamma_0 y_t - x'_t \delta_0)^4 | x_t, y_t > c] = v_{0t}^4 \lambda(v_{0t}) + 3[v_{0t} \lambda(v_{0t}) + 1]. \quad (3.4)$$

Proof.

$$E[\gamma_0 y_t - x'_t \delta_0 | x_t, y_t > c] = E\left[\frac{y_t - x'_t \beta_0}{\sigma_0} \mid x_t, \frac{y_t - x'_t \beta_0}{\sigma_0} > \frac{c - x'_t \beta_0}{\sigma_0}\right]$$

$$= \int_{c - x'_t \beta_0}^{\infty} y_0 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \int_{c - x'_t \beta_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} d\gamma$$

$$= -\int_{c - x'_t \beta_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} d\gamma$$

$$= \phi\left(\frac{c - x'_t \beta_0}{\sigma_0}\right)$$

$$= \lambda(v_{0t}).$$
This shows that (3.1) holds.

\[
E[(\gamma_0 y_t - x'_t \delta_0)^2 \mid x_t, y_t > c] = E \left[ \left( \frac{y_t - x'_t \beta_0}{\sigma_0} \right)^2 \mid x_t, \frac{y_t - x'_t \beta_0}{\sigma_0} > \frac{c - x'_t \beta_0}{\sigma_0} \right]
\]

\[
= \int_{-\infty}^{\infty} \bar{y}^2 \frac{\phi(\bar{y})}{1 - \Phi(\frac{c - x'_t \beta_0}{\sigma_0})} \, d\bar{y}
\]

\[
= \int_{-\infty}^{\infty} \bar{y}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} \, d\bar{y}
\]

\[
= - \int_{-\infty}^{\infty} \bar{y}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} \, d\bar{y}
\]

\[
= \left( c - x'_t \beta_0 \right)^2 \frac{ \phi(\frac{c - x'_t \beta_0}{\sigma_0}) }{1 - \Phi(\frac{c - x'_t \beta_0}{\sigma_0})} + 2 \int_{-\infty}^{\infty} \bar{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} \, d\bar{y}
\]

\[
= v_{0t}(v_{0t}) + 1.
\]

Thus (3.2) holds.

\[
E[(\gamma_0 y_t - x'_t \delta_0)^3 \mid x_t, y_t > c] = E \left[ \left( \frac{y_t - x'_t \beta_0}{\sigma_0} \right)^3 \mid x_t, \frac{y_t - x'_t \beta_0}{\sigma_0} > \frac{c - x'_t \beta_0}{\sigma_0} \right]
\]

\[
= \int_{-\infty}^{\infty} \bar{y}^3 \frac{\phi(\bar{y})}{1 - \Phi(\frac{c - x'_t \beta_0}{\sigma_0})} \, d\bar{y}
\]

\[
= \int_{-\infty}^{\infty} \bar{y}^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} \, d\bar{y}
\]

\[
= - \int_{-\infty}^{\infty} \bar{y}^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} \, d\bar{y}
\]

\[
= \left( c - x'_t \beta_0 \right)^3 \frac{ \phi(\frac{c - x'_t \beta_0}{\sigma_0}) }{1 - \Phi(\frac{c - x'_t \beta_0}{\sigma_0})} + 2 \int_{-\infty}^{\infty} \bar{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} \, d\bar{y}
\]

\[
= v_{0t}^2 \lambda(v_{0t}) + 2 \lambda(v_{0t}), \quad \text{ (by (3.1))}
\]

This shows that (3.3) holds.
Proof. Thus (3.4) holds. \qed

Next we claim

**Theorem 3.2** For the Tobit Model satisfying Assumption 1', if \( \mathbf{E}(\mathbf{x}, \mathbf{x}') \) is nonsingular, then condition 3 of Proposition 1.3 is satisfied.

**Proof.**

First we show that \( E[s(\mathbf{w}_i; \delta_0, \gamma_0)|\mathbf{x}_i] = 0. \)

In terms of the expression of \( s(\mathbf{w}_i; \delta, \gamma) \) in (1.17),

\[
E[s(\mathbf{w}_i; \delta_0, \gamma_0)|\mathbf{x}_i] = \begin{bmatrix}
E[(1 - D_t)(\gamma_0 y_t - \mathbf{x}'_t \delta_0)|\mathbf{x}_t] + D_t \lambda(-v_{0t})(-\mathbf{x}_t)|\mathbf{x}_t]

E[(1 - D_t)[\frac{1}{\gamma_0} - (\gamma_0 y_t - \mathbf{x}'_t \delta_0)y_t] + D_t \lambda(-v_{0t})c|\mathbf{x}_t]
\end{bmatrix},
\]

(3.5)

where

\[
E[(1 - D_t)(\gamma_0 y_t - \mathbf{x}'_t \delta_0)|\mathbf{x}_t] + D_t \lambda(-v_{0t})(-\mathbf{x}_t)|\mathbf{x}_t]
\]

\[
= E[(\gamma_0 y_t - \mathbf{x}'_t \delta_0)|\mathbf{x}_t, y_t > c] \text{Prob}[y_t > c|\mathbf{x}_t] + E[\lambda(-v_{0t})(-\mathbf{x}_t)|\mathbf{x}_t, y_t = c] \text{Prob}[y_t = c|\mathbf{x}_t]
\]

\[
= E[(\gamma_0 y_t - \mathbf{x}'_t \delta_0)|\mathbf{x}_t, y_t > c](\mathbf{x}_t) \text{Prob}[y_t > c|\mathbf{x}_t] - \lambda(-v_{0t})(\mathbf{x}_t) \text{Prob}[y_t = c|\mathbf{x}_t]
\]
\[ E(\gamma y_t - \delta_0, \gamma_0) = (1 - D_t)\left(\frac{1}{70} - \gamma_0 y_t - \delta_0, y_t > c\right) \] 

\[ = \lambda(\gamma_0 y_t - \delta_0) + D_t\lambda(-\gamma_0)c \]

\[ = \lambda(\gamma_0 y_t - \delta_0) + D_t\lambda(-\gamma_0)c \]

**Therefore** \( E[\mathbf{s}(\mathbf{w}_t; \delta_0, \gamma_0)|x_t] = 0 \).

Next we claim \( E[\mathbf{s}(\mathbf{w}_t; \delta_0, \gamma_0)\mathbf{s}(\mathbf{w}_t; \delta_0, \gamma_0)'|x_t] = -E[H(\mathbf{w}_t; \delta_0, \gamma_0)|x_t] \).

In terms of the expression of \( \mathbf{s}(\mathbf{w}_t; \delta, \gamma) \) in (1.17), we write the matrix \( \mathbf{s}(\mathbf{w}_t; \delta_0, \gamma_0)s(\mathbf{w}_t; \delta_0, \gamma_0)' \) as

\[
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix},
\]

where

\[ M_{11} = (1 - D_t)(\gamma_0 y_t - \delta_0)^2 x_t x_t' + D_t\lambda^2(-\gamma_0)c \]

\[ M_{12} = (1 - D_t)(\gamma_0 y_t - \delta_0) x_t \left[ \frac{1}{70} - (\gamma_0 y_t - \delta_0) y_t \right] + D_t\lambda^2(-\gamma_0)(-x_t c), \]

\[ M_{21} = M_{12}', \]

\[ M_{22} = (1 - D_t) \left[ \frac{1}{70} - (\gamma_0 y_t - \delta_0) y_t \right] ^2 + D_t\lambda^2(-\gamma_0)c^2. \]
In terms of the expression of $H(w_t; \delta, \gamma)$ in (1.18), we write the matrix $-H(w_t; \delta, \gamma_0)$ as

$$
\begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix},
$$

where

$$
N_{11} = (1 - D_t) x_t x_t' + D_t \lambda(-v_0|\gamma) + v_0|x_t x_t',
$$

$$
N_{12} = (1 - D_t)(-y_t x_t) + D_t \lambda(-v_0|\gamma) + v_0|(-c x_t),
$$

$$
N_{21} = N_{21}',
$$

$$
N_{22} = (1 - D_t)[\frac{1}{\gamma_0} + y_t^2] + D_t \lambda(-v_0|\gamma) + v_0|c^2.
$$

If we can show $E[N_{ij}|x_t] = E[M_{ij}|x_t]$, $i, j = 1, 2$, then the claim is true. The verification is as follows.

Since

$$
E[M_{11}|x_t] = E[(\gamma_0 y_t - x_t' \delta_0)^2|x_t, y_t > c] x_t x_t' \text{Prob}[y_t > c|x_t] + \lambda^2(-v_0|x_t x_t' \text{Prob}[y_t = c|x_t]
$$

$$
= \left\{ [v_0 \lambda(v_0) + 1] [1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] + \lambda^2(-v_0) \Phi(\frac{c-x_t' \delta_0}{\sigma_0}) \right\} x_t x_t' (\text{by (3.2)})
$$

$$
= \left\{ v_0 \lambda(v_0) \left[ 1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0}) \right] + \left[ 1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0}) \right] \right\} x_t x_t'
$$

$$
= \left\{ v_0 \lambda(v_0) + \left[ 1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0}) \right] \right\} x_t x_t',
$$

and

$$
E[N_{11}|x_t] = x_t x_t' \text{Prob}[y_t > c|x_t] + \lambda(-v_0|x_t x_t' \text{Prob}[y_t = c|x_t]
$$

$$
= \left\{ [1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] + [\lambda^2(-v_0) + \lambda(-v_0) v_0 \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] \right\} x_t x_t'
$$

$$
= \left\{ [1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] + [\lambda^2(-v_0) + \lambda(-v_0) v_0 \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] \right\} x_t x_t'
$$

$$
= \left\{ [1 - \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] + [\lambda^2(-v_0) \Phi(\frac{c-x_t' \delta_0}{\sigma_0})] + v_0 \Phi(\frac{c-x_t' \delta_0}{\sigma_0}) \right\} x_t x_t',
$$

( by the fact that $\lambda(-v_0) \Phi(\frac{c-x_t' \delta_0}{\sigma_0}) = \phi(\frac{c-x_t' \delta_0}{\sigma_0})$)

we have $E[M_{11}|x_t] = E[N_{11}|x_t]$.

As to $E[M_{12}|x_t]$ and $E[N_{12}|x_t]$,

$$
E[M_{12}|x_t]
$$

$$
= E[(\gamma_0 y_t - x_t' \delta_0) x_t [\frac{1}{\gamma_0} - (\gamma_0 y_t - x_t' \delta_0) y_t] | x_t, y_t > c] \text{Prob}[y_t > c|x_t] + \lambda^2(-v_0)(-x_t c)
$$

$$
\cdot \text{Prob}[y_t = c|x_t]
$$

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\[ E \left[ (\gamma_0 y_t - \mathbf{x}_t \delta_0) \mathbf{x}_t \left[ \frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}_t \delta_0) (\gamma_0 y_t - \mathbf{x}_t \delta_0) - \frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}_t \delta_0) \mathbf{x}_t \delta_0 \right] | \mathbf{x}_t, y_t > c \right] \\
\cdot \text{Prob}[y_t > c | \mathbf{x}_t] + \lambda^2(-v_{0t})(-\mathbf{x}_t c) \text{Prob}[y_t = c | \mathbf{x}_t] \\
= E \left[ (\gamma_0 y_t - \mathbf{x}_t \delta_0) \frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}_t \delta_0) - \frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}_t \delta_0) \mathbf{x}_t \delta_0 | \mathbf{x}_t, y_t > c \right] \mathbf{x}_t \text{Prob}[y_t > c | \mathbf{x}_t] \\
+ \lambda^2(-v_{0t})(-\mathbf{x}_t c) \text{Prob}[y_t = c | \mathbf{x}_t] \\
= \left\{ \frac{1}{\gamma_0} \lambda(v_{0t}) - \frac{1}{\gamma_0} \left[ v_{0t}^2 \lambda(v_{0t}) + 2\lambda(v_{0t}) \right] \right\} \mathbf{x}_t \text{Prob}[y_t > c | \mathbf{x}_t] \\
+ \lambda^2(-v_{0t})(-\mathbf{x}_t c) \text{Prob}[y_t = c | \mathbf{x}_t] \\
= \left\{ \frac{1}{\gamma_0} \lambda(v_{0t}) - \frac{1}{\gamma_0} \left[ v_{0t} \lambda(v_{0t}) + 1 \right] (\gamma_0 c - v_{0t}) \right\} \mathbf{x}_t \text{Prob}[y_t > c | \mathbf{x}_t] \\
+ \lambda^2(-v_{0t})(-\mathbf{x}_t c) \text{Prob}[y_t = c | \mathbf{x}_t] \\
= \left\{ -\frac{1}{\gamma_0} \lambda(v_{0t}) - [v_{0t} \lambda(v_{0t}) + 1] c + \frac{1}{\gamma_0} v_{0t} \right\} \mathbf{x}_t \left[ 1 - \Phi\left( \frac{c-x_0^{2}}{\sigma} \right) \right] \\
+ \lambda^2(-v_{0t})(-\mathbf{x}_t c) \Phi\left( \frac{x_0^{2}}{\sigma} \right) \\
= \left\{ -\frac{1}{\gamma_0} \lambda(v_{0t}) - c + \frac{1}{\gamma_0} v_{0t} \right\} \mathbf{x}_t \left[ 1 - \Phi\left( \frac{c-x_0^{2}}{\sigma} \right) \right] - v_{0t} \lambda(v_{0t}) c \mathbf{x}_t \left[ 1 - \Phi\left( \frac{c-x_0^{2}}{\sigma} \right) \right] \\
+ \lambda^2(-v_{0t})(-\mathbf{x}_t c) \Phi\left( \frac{x_0^{2}}{\sigma} \right) \\
= \left\{ -\frac{1}{\gamma_0} \lambda(v_{0t}) - c + \frac{1}{\gamma_0} v_{0t} \right\} \mathbf{x}_t \left[ 1 - \Phi\left( \frac{c-x_0^{2}}{\sigma} \right) \right] - v_{0t} \lambda(v_{0t}) c \mathbf{x}_t \Phi\left( \frac{-x_0^{2}}{\sigma} \right) \\
+ \lambda^2(-v_{0t})(-\mathbf{x}_t c) \Phi\left( \frac{-x_0^{2}}{\sigma} \right) \\
= \left\{ -\frac{1}{\gamma_0} \lambda(v_{0t}) - c + \frac{1}{\gamma_0} v_{0t} \right\} \mathbf{x}_t \left[ 1 - \Phi\left( \frac{-x_0^{2}}{\sigma} \right) \right] \\
+ \lambda(-v_{0t}) [\lambda(-v_{0t}) + v_{0t}][v_{0t} c] \Phi\left( \frac{-x_0^{2}}{\sigma} \right), \quad (3.20) \\
\text{and} \\
E[N_{12} | \mathbf{x}_t] \\
= E[-y_t \mathbf{x}_t | \mathbf{x}_t, y_t > c] \text{Prob}[y_t > c | \mathbf{x}_t] + \lambda(-v_{0t}) [\lambda(-v_{0t}) + v_{0t}][v_{0t} c] \text{Prob}[y_t = c | \mathbf{x}_t] \\
= E[-\frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}_t \delta_0 + \mathbf{x}_t \delta_0) \mathbf{x}_t | \mathbf{x}_t, y_t > c] \text{Prob}[y_t > c | \mathbf{x}_t] \\
+ \lambda(-v_{0t}) [\lambda(-v_{0t}) + v_{0t}][v_{0t} c] \text{Prob}[y_t = c | \mathbf{x}_t] \\
= E[-\frac{1}{\gamma_0} (\gamma_0 y_t - \mathbf{x}_t \delta_0) - \frac{1}{\gamma_0} (\gamma_0 c - v_{0t}) \mathbf{x}_t | \mathbf{x}_t, y_t > c] \mathbf{x}_t \text{Prob}[y_t > c | \mathbf{x}_t] \\
+ \lambda(-v_{0t}) [\lambda(-v_{0t}) + v_{0t}][v_{0t} c] \text{Prob}[y_t = c | \mathbf{x}_t] \\
= \left\{ -\frac{1}{\gamma_0} \lambda(v_{0t}) - c + \frac{1}{\gamma_0} v_{0t} \right\} \mathbf{x}_t \left[ 1 - \Phi\left( \frac{-x_0^{2}}{\sigma} \right) \right] \\
+ \lambda(-v_{0t}) [\lambda(-v_{0t}) + v_{0t}][v_{0t} c] \Phi\left( \frac{-x_0^{2}}{\sigma} \right), \quad (3.21) \\
\text{Therefore } E[M_{12} | \mathbf{x}_t] = E[N_{12} | \mathbf{x}_t]. \text{ Since } M_{21} = M'_{12} \text{ and } N_{21} = N'_{12}, E[M_{21} | \mathbf{x}_t] = E[N_{21} | \mathbf{x}_t] \text{ also holds.} \]
Finally for $E[M_{22}|x_t]$ and $E[N_{22}|x_t]$,

$$E[M_{22}|x_t] = E\left[\frac{1}{\gamma_0} - (\gamma_0 y_t - x_t'\delta_0)y_t \right] | x_t, y_t > c \right] \Pr[y_t > c|x_t] + \lambda^2(-v_0)c^2 \Pr[y_t = c|x_t]$$

$$= E\left[\frac{1}{\gamma_0} - (\gamma_0 y_t - x_t'\delta_0)(\gamma_0 y_t - x_t'\delta_0) - \frac{1}{\gamma_0}(\gamma_0 y_t - x_t'\delta_0)x_t'\delta_0 \right]^2 | x_t, y_t > c \right] \Pr[y_t > c|x_t]$$

$$+ \lambda^2(-v_0)c^2 \Pr[y_t = c|x_t]$$

$$= E\left[\frac{1}{\gamma_0} \left[1 - (\gamma_0 y_t - x_t'\delta_0)^2\right] - 2\frac{1}{\gamma_0} [1 - (\gamma_0 y_t - x_t'\delta_0)^2] \right] (\gamma_0 y_t - x_t'\delta_0)x_t'\delta_0$$

$$+ \frac{1}{\gamma_0}(\gamma_0 y_t - x_t'\delta_0)^2(x_t'\delta_0)^2 | x_t, y_t > c \right] \Pr[y_t > c|x_t] + \lambda^2(-v_0)c^2 \Pr[y_t = c|x_t]$$

$$= E\left[\frac{1}{\gamma_0} \left[1 - (\gamma_0 y_t - x_t'\delta_0)^2\right] | x_t, y_t > c \right] \Pr[y_t > c|x_t]$$

$$- E \left[2\frac{1}{\gamma_0} \left[1 - (\gamma_0 y_t - x_t'\delta_0)^2\right] \right] (\gamma_0 y_t - x_t'\delta_0)x_t'\delta_0 | x_t, y_t > c \right] \Pr[y_t > c|x_t]$$

$$+ E \left[\frac{1}{\gamma_0} (\gamma_0 y_t - x_t'\delta_0)^2(x_t'\delta_0)^2 | x_t, y_t > c \right] \Pr[y_t > c|x_t]$$

$$+ \lambda^2(-v_0)c^2 \Pr[y_t = c|x_t], \quad (3.22)$$

where

$$E\left[\frac{1}{\gamma_0} \left[1 - (\gamma_0 y_t - x_t'\delta_0)^2\right] | x_t, y_t > c \right] = \frac{1}{\gamma_0} - \frac{2}{\gamma_0} E \left[(\gamma_0 y_t - x_t'\delta_0)^4 | x_t, y_t > c \right]$$

$$= \frac{1}{\gamma_0} - \frac{2}{\gamma_0} [v_0c + 1] + \frac{1}{\gamma_0} [v_0^2c + v_0]$$

$$= \frac{1}{\gamma_0} \left[v_0c + v_0 \right], \quad (3.23)$$

$$E \left[2\frac{1}{\gamma_0} \left[1 - (\gamma_0 y_t - x_t'\delta_0)^2\right] \right] (\gamma_0 y_t - x_t'\delta_0)x_t'\delta_0 | x_t, y_t > c \right]$$

$$= \frac{2}{\gamma_0} E \left[(\gamma_0 y_t - x_t'\delta_0) \right] (x_t'\delta_0) \right] - \frac{2}{\gamma_0} E \left[(\gamma_0 y_t - x_t'\delta_0)^3 | x_t, y_t > c \right] (x_t'\delta_0)$$

$$= \frac{2}{\gamma_0} \lambda(v_0c)(x_t'\delta_0) - \frac{2}{\gamma_0} \left[v_0^2c + v_0 \right] (x_t'\delta_0)$$

$$= \frac{2}{\gamma_0} \left[v_0c + v_0 \right], \quad (3.24)$$

and

$$E \left[\frac{1}{\gamma_0} (\gamma_0 y_t - x_t'\delta_0)^2(x_t'\delta_0)^2 | x_t, y_t > c \right]$$

$$= \frac{1}{\gamma_0} [v_0c + v_0] \left[v_0^2c + v_0 \right]. \quad (3.25)$$
Thus

\[ E[M_{22}|x_t] \]
\[ = \frac{1}{\gamma_0} [v_{0t}^2 \lambda(v_{0t}) + v_{0t} \lambda(v_{0t}) + 2] \text{Prob}[y_t > c|x_t] \]
\[ + \frac{2}{\gamma_0} [v_{0t} \lambda(v_{0t}) + \lambda(v_{0t})] (\gamma_0 c - v_{0t}) \text{Prob}[y_t > c|x_t] \]
\[ + \frac{1}{\gamma_0} [v_{0t} \lambda(v_{0t}) + 1] (\gamma_0 c - v_{0t})^2 \text{Prob}[y_t > c|x_t] \]
\[ + \lambda^2 (-v_{0t})^2 \text{Prob}[y_t = c|x_t] \]
\[ = \frac{1}{\gamma_0} [v_{0t}^2 \lambda(v_{0t}) + v_{0t} \lambda(v_{0t}) + 2] \text{Prob}[y_t > c|x_t] \]
\[ + \frac{1}{\gamma_0} \left\{ 2v_{0t}^2 \lambda(v_{0t}) + 2 \lambda(v_{0t}) + [v_{0t} \lambda(v_{0t}) + 1] \gamma_0 c - v_{0t} \right\} (\gamma_0 c - v_{0t}) \text{Prob}[y_t > c|x_t] \]
\[ + \lambda^2 (-v_{0t})^2 \text{Prob}[y_t = c|x_t] \]

(3.26)

On the other hand,

\[ E[N_{22}|x_t] \]
\[ = E \left[ \frac{1}{\gamma_0} + y_t^2 | x_t, y_t > c \right] \text{Prob}[y_t > c|x_t] + \lambda(-v_{0t})[\lambda(-v_{0t}) + v_{0t}] c^2 \text{Prob}[y_t = c|x_t] \]
\[ = E \left[ \frac{1}{\gamma_0} + \frac{1}{\gamma_0} [(\gamma_0 y_t - x_t \delta_0 + x_t \delta_0)^2] | x_t, y_t > c \right] \text{Prob}[y_t > c|x_t] + \lambda(-v_{0t})[\lambda(-v_{0t}) + v_{0t}] c^2 \]
\[ \cdot \text{Prob}[y_t = c|x_t] \]
Proof. Since condition 3 of Proposition 1.3 is satisfied,

\[ E\left[ \frac{1}{\pi} + \frac{1}{\pi} (\gamma_0 y_t - x_t \delta_0)^2 + \frac{2}{\pi} (\gamma_0 y_t - x_t \delta_0)(x_t \delta_0) + \frac{1}{\pi} (x_t \delta_0)^2 \mid x_t, y_t > c \right] \Pr[y_t > c \mid x_t] \\
+ \lambda(-v_{ot})[\lambda(-v_{ot}) + v_{ot}]c^2 \Pr[y_t = c \mid x_t] \\
= \frac{1}{\pi} \left\{ 1 + [v_{ot}\lambda(v_{ot}) + 1] + 2\lambda(v_{ot})(x_t \delta_0) + (x_t \delta_0)^2 \right\} \Pr[y_t > c \mid x_t] \\
+ \lambda(-v_{ot})[\lambda(-v_{ot}) + v_{ot}]c^2 \Pr[y_t = c \mid x_t] \\
= \frac{1}{\pi} \left\{ 1 + [v_{ot}\lambda(v_{ot}) + 1] + 2\lambda(v_{ot})(\gamma_0 c - v_{ot}) + (\gamma_0 c - v_{ot})^2 \right\} \left[ 1 - \Phi\left( \frac{c-x_t \delta_0}{\sigma_0} \right) \right] \\
+ \lambda(-v_{ot})[\lambda(-v_{ot}) + v_{ot}]c^2 \Phi\left( \frac{c-x_t \delta_0}{\sigma_0} \right) \\
= \frac{1}{\pi} \left\{ 2 + 2\lambda(v_{ot})\gamma_0 c - v_{ot}\lambda(v_{ot}) + \gamma_0^2 c^2 - 2\gamma_0 c v_{ot} + v_{ot}^2 \right\} \left[ 1 - \Phi\left( \frac{c-x_t \delta_0}{\sigma_0} \right) \right] \\
+ \lambda(-v_{ot})[\lambda(-v_{ot}) + v_{ot}]c^2 \Phi\left( \frac{c-x_t \delta_0}{\sigma_0} \right). \tag{3.27} \]

Comparing (3.27) with (3.26), we can see that \( E[M_{22} \mid x_t] = E[N_{22} \mid x_t]. \)

Since \( E[M_{ij} \mid x_t] = E[N_{ij} \mid x_t], \ i = 1, 2, j = 1, 2, \ E[s(w_t; \delta_0, \gamma_0) \mid s(w_t; \delta_0, \gamma_0) \mid x_t] \)

\[ = -E[H(w_t; \delta_0, \gamma_0) \mid x_t]. \]

Therefore, condition 3 of Proposition 1.3 is satisfied. \[ \square \]

Next we claim

**Theorem 3.3** For the Tobit Model satisfying Assumption 1', if \( E(x_t x'_t) \) is nonsingular, then condition 5 of Proposition 1.3 is satisfied.

**Proof.** Since condition 3 of Proposition 1.3 is satisfied,

\[ -E[H(w_t; \delta_0, \gamma_0)] = E[s(w_t; \delta_0, \gamma_0)s(w_t; \delta_0, \gamma_0)']. \tag{3.28} \]

Clearly \( s(w_t; \delta_0, \gamma_0)s(w_t; \delta_0, \gamma_0)' \) is positive semidefinite. This means that \( E[s(w_t; \delta_0, \gamma_0) \cdot s(w_t; \delta_0, \gamma_0)'] \) is positive semidefinite. Thus \( E[H(w_t; \delta_0, \gamma_0)] \) is negative semidefinite.

Let \( z \equiv (z_1, ..., z_K, z_{K+1})' \in \mathbb{R}^{K+1} \) be a solution to the equation

\[ z' E[H(w_t; \delta_0, \gamma_0)] z = 0. \tag{3.29} \]

We know that if \( z = (0, ..., 0) \) is the only solution to the above equation, then \( E[H(w_t; \delta_0, \gamma_0)] \) is nonsingular.

Define \( \hat{A} \) as the matrix

\[ \begin{bmatrix} x_t x'_t & -y_t x_t \\ -y_t x'_t & \frac{y_t^2}{\sigma_t^2} + y_t^2 \end{bmatrix}. \tag{3.30} \]

Define \( \hat{B} \) as the matrix

\[ \begin{bmatrix} x_t x'_t & -c x_t \\ -c x'_t & c^2 \end{bmatrix}. \tag{3.31} \]

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Then in terms of the expression $H(w_t; \delta, \gamma)$ in (1.18),

$$H(w_t; \delta_0, \gamma_0) = -(1 - D_t)\tilde{A} - D_t\lambda(-v_{0t})[\lambda(-v_{0t}) + v_{0t}]\tilde{B}. \quad (3.32)$$

It is easy to see that

$$\tilde{A} = \begin{bmatrix} x_t \\ -y_t \end{bmatrix} \begin{bmatrix} x'_t & -y_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0' & \gamma_0 \end{bmatrix}. \quad (3.33)$$

and

$$\tilde{B} = \begin{bmatrix} x_t \\ -c \end{bmatrix} \begin{bmatrix} x'_t & -c \end{bmatrix} \quad (3.34)$$

Thus combining these with the fact that $1 - D_t \geq 0$ and $D_t\lambda(-v_{0t})[\lambda(-v_{0t}) + v_{0t}] \geq 0$, we can see that $H(w_t; \delta_0, \gamma_0)$ is negative semidefinite.

Suppose $z_{K+1} \neq 0$, then

$$z'\tilde{A}z = \left[ \sum_{i=1}^K x_i z_i - y_t z_{K+1} \right]^2 + z_{K+1}^2 \frac{1}{\gamma_0^2} \geq z_{K+1}^2 \frac{1}{\gamma_0^2} > 0. \quad (3.35)$$

In terms of the expression $D_t$ in (1.15) and the expression $y_t^*$ in (1.10), we have

$$\text{Prob}(1 - D_t = 1) = \text{Prob}(y_t^* > c) = \text{Prob}(\epsilon_t > c - \frac{1}{\gamma_0}x'_t\delta_0). \quad (3.36)$$

Since $\epsilon_t|x_t \sim N(0, \sigma_0^2)$ (by (1.11)), $\text{Prob}(\epsilon_t > c - \frac{1}{\gamma_0}x'_t\delta_0) > 0$. Thus $\text{Prob}(1 - D_t = 1) > 0$. Combining this with (3.35), we have

$$E[(1 - D_t)z'\tilde{A}z] > 0, \quad (3.37)$$

when $z_{K+1} \neq 0$.

Since $\tilde{B}$ is positive semidefinite and $D_t\lambda(-v_{0t})[\lambda(-v_{0t}) + v_{0t}] \geq 0$,

$$E[D_t\lambda(-v_{0t})[\lambda(-v_{0t}) + v_{0t}]z'\tilde{B}z] \geq 0. \quad (3.38)$$

\footnote{see hint of exercise 3 on page 521 of Hayashi (2000).}
Thus, when $z_{K+1} \neq 0$,

$$
z' E[H(w_t; \delta_0, \gamma_0)] z = E[z' H(w_t; \delta_0, \gamma_0) z]
= E[(1 - D_t) z' \tilde{A} z] + E[D_t \lambda(-v_{0t}) \lambda(-v_{0t}) + v_{0t}] z' \tilde{B} z > 0,
$$
(3.39)

But this contradicts the assumption $z' E[H(w_t; \delta_0, \gamma_0)] z = 0$ (see (3.29)). Therefore $z_{K+1}$ must be 0. This means that $z = (z_1, ..., z_K, 0)$, thereafter,

$$
z E[H(w_t; \delta_0, \gamma_0)] z' = (z_1, ..., z_K, 0) E[H(w_t; \delta_0, \gamma_0)] (z_1, ..., z_K, 0)'
= -(z_1, ..., z_K) E[(1 - D_t + D_t \lambda(-v_{0t}) \lambda(-v_{0t}) + v_{0t}) x_t x_t'] (z_1, ..., z_K)'
= - E [(z_1, ..., z_K) (1 - D_t + D_t \lambda(-v_{0t}) \lambda(-v_{0t}) + v_{0t}) x_t x_t' (z_1, ..., z_K)']
= - E [(z_1, ..., z_K) (1 - D_t + D_t \lambda(-v_{0t}) \lambda(-v_{0t}) + v_{0t}) x_t x_t' (z_1, ..., z_K)']
= - E [(z_1, ..., z_K) (1 - D_t + D_t \lambda(-v_{0t}) \lambda(-v_{0t}) + v_{0t}) x_t x_t' (z_1, ..., z_K)'],
$$
(3.40)

It is clear that for any positive constant $\overline{v} \in \mathbb{R}$, there exists a positive constant $\overline{C}$ which depends on $\overline{v} \in \mathbb{R}$, such that on the set $\{|v_{0t}| \leq \overline{v}\}, 1 - D_t + D_t \lambda(-v_{0t}) \lambda(-v_{0t}) - v_{0t} \geq \overline{C}$. This implies that

$$
E [(z_1, ..., z_K) (1 - \lambda(v_{0t}) \lambda(v_{0t}) - v_{0t}) x_t x_t' (z_1, ..., z_K)']
\geq E [(z_1, ..., z_K) 1_{\{|v_{0t}| \leq \overline{v}\}} (1 - D_t + D_t \lambda(-v_{0t}) \lambda(-v_{0t}) - v_{0t}) x_t x_t' (z_1, ..., z_K)']
\geq \overline{C} E [(z_1, ..., z_K) 1_{\{|v_{0t}| \leq \overline{v}\}} x_t x_t' (z_1, ..., z_K)']
$$
(3.41)

The remaining argument follows the same line as we did in the truncation regression model.

By (3.29), (3.40) and (3.41),

$$
z' E[H(w_t; \delta_0, \gamma_0)] z = 0 \implies (z_1, ..., z_K) E[1_{\{|v_{0t}| \leq \overline{v}\}} x_t x_t'] (z_1, ..., z_K)' = 0.
$$
(3.42)

It is easy to see that if $E[x_t x_t']$ is nonsingular, then for large enough $\overline{v}$, $E[1_{\{|v_{0t}| \leq \overline{v}\}} x_t x_t']$ is also nonsingular. Thus when $E[x_t x_t']$ is nonsingular, for large enough $\overline{v}$, $\{z_i = 0, i = 1, ..., K\}$ is the only solution satisfying

$$
(z_1, ..., z_K) E[1_{\{|v_{0t}| \leq \overline{v}\}} x_t x_t'] (z_1, ..., z_K)' = 0.
$$
(3.43)

This means that

$$
z' E[H(w_t; \delta_0, \gamma_0)] z = 0 \implies z = (0, ..., 0, 0)'.
$$
(3.44)

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Therefore when $E[x_t'x_t]$ is nonsingular, $E[H(w_t; \delta_0, \gamma_0)]$ is nonsingular, i.e. condition 5 of Proposition 1.3 holds.

Theorem 3.2 and Theorem 3.3 imply that the conditional ML estimator $(\hat{\delta}, \hat{\gamma})$ is asymptotic normal with $A\text{var}(\hat{\delta}, \hat{\gamma})$ given by the following:

$$A\text{var}(\hat{\delta}, \hat{\gamma}) = -\left\{E[H(w_t; \delta_0, \gamma_0)]\right\}^{-1} = \left\{E[s(w_t; \delta_0, \gamma_0)s(w_t; \delta_0, \gamma_0)']\right\}^{-1}. \quad (3.45)$$

To recover original parameters and obtain the asymptotic variance of $(\hat{\beta}, \hat{\sigma}^2)$, the delta method can be applied. (see page 520 and page 521 of Hayashi (2000).)

References

[1] Amemiya, T., 1973, "Regression Analysis When the Dependent Variable is Truncated Normal", *Econometrica*, 41, 997-1016.

[2] Hayashi, F., 2000, *Econometrics*, Princeton: Princeton University Press.

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