Emergent universe with exotic matter

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Received 31 May 2006, in final form 18 September 2006
Published 20 October 2006
Online at stacks.iop.org/CQG/23/6927

Abstract
A general framework for an emergent universe scenario has been given which makes use of an equation of state. The general features of the model have also been studied and some possible primordial compositions of the universe have been suggested.

PACS numbers: 04.20.Jb, 98.80.Cq, 98.80.Jk

1. Introduction
The possibilities of an emergent universe have been studied recently in a number of papers [1–4] in which one looks for a universe which is ever-existing and large enough so that the spacetime may be treated as classical entities. There is no timelike singularity. In these models, the universe in the infinite past is in an almost static state but it eventually evolves into an inflationary stage. These ideas are in conformity with the Lemaître–Eddington concepts forwarded in the early days of modern cosmology, although the details and the context are different now. An emergent universe model, if developed in a consistent way, is capable of solving the well-known conceptual problems of the big-bang model. A model of an ever-existing universe, which eventually enters at some stage into the standard big-bang epoch and has features precisely known to us, will be of considerable interest. The purpose of this paper is to examine the possibilities of such a scenario.

We mention here three models of the emergent universe which are relevant here.

(1) A closed universe containing radiation and a cosmological constant, given by Harrison [5], which contains the scale factor

\[ a(t) = a_0 \left[ 1 + \exp \left( \frac{\sqrt{2}t}{a_0} \right) \right]^{1/2}. \]
As \( t \to -\infty \), the model goes over asymptotically to an Einstein static universe. Although ever inflating, at any time \( t_o \gg a_i \), the expansion is given by a finite number of e-folds,

\[
N_o = \ln \left( \frac{a_o}{a_i} \right) \sim \frac{t_o}{\sqrt{2}} a_i. \tag{2}
\]

(2) The second example has been studied by Ellis and Maartens [1] and Ellis et al [2]. They considered a closed universe containing a minimally coupled scalar field \( \phi \), which has a self-interaction given by a special potential function \( V(\phi) \). This potential looks similar to what one obtains in an \( R + \alpha R^2 \) theory [6–13] after the conventional conformal transformation and identifying the scalar field \( \phi \) as \( \phi = -\sqrt{3} \ln(1 + 2\alpha R) \) with \( \alpha \) negative. Although the solution is not obtained analytically, the model exhibits features expected in an emergent universe. Making use of loop quantum cosmology, Mulryne et al [3] have recently shown the existence and stability of the resulting classical emergent universe.

(3) A third example has been provided by Mukherjee et al [4], where it was shown that the Starobinsky model has a solution which can describe an emergent universe. Here, one considers the semiclassical Einstein equation,

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = -8\pi G \langle T_{\mu \nu} \rangle, \tag{3}
\]

where \( \langle T_{\mu \nu} \rangle \) is the vacuum expectation value of the energy momentum tensor of the fields. Assuming only free, massless and conformally invariant fields, and a Robertson–Walker metric, one obtains, for a spatially flat universe, the following equation:

\[
H^2 \left( \frac{1}{K_3} - H^2 \right) = -\frac{6K_1}{K_3} (2H \dot{H} + 6H^2 \dot{H} - H^2), \tag{4}
\]

where the constant \( K_3 \) is determined by the species and number of fields and \( K_1 \) is a constant which may be chosen freely. It has been shown that the equation permits a solution which describes an emergent universe, with a scale factor

\[
a(t) = a_i (\beta + e^{\alpha t})^{2/3}, \tag{5}
\]

where \( \alpha = \frac{3}{2} \sqrt{\frac{1}{K_3}} \) and \( K_1 = -\frac{2}{27} K_3 \), \( \beta \) is an integration constant. The general features of the model have been given in [4].

These examples indicate that solutions describing an emergent universe occur in different contexts. It will, therefore, be interesting to see if solutions describing an emergent universe can be classified and studied in a general way. A simple approach will be to look for the equation of state (EOS) which leads to such solutions. In the next section we obtain such an EOS and study the general features of the relevant solutions, without referring to the actual source of the energy density. In section 3, we consider various combinations of radiation and matter, normal or exotic. We discuss our results in the last section.

2. The equation of state (EOS) for emergent universe

Cosmological models usually consider a linear equation of state (EOS), namely \( p = \omega \rho \), where \( \omega \) is a constant, depending on the nature of the constituents. A notable exception is the case of a scalar field. For a homogeneous scalar field \( \phi \) interacting with a potential \( V(\phi) \),

\[
\omega = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}
\]
where \( \omega \) may vary between \(-1\) and \(+1\). Tachyonic condensates provide another case of varying \( \omega \), e.g.

\[
\omega = -(1 - \dot{\phi}^2),
\]

where \( \omega \) is always negative.

Recent astronomical data when interpreted in the context of the big-bang model have provided some interesting information about the composition of the universe. The total energy density has three components. While big-bang nucleosynthesis data suggest that baryonic matter can account for only about 4% of the total energy density, the cold dark matter (CDM) is about 23% and the third part, called dark energy, constitutes the remaining 73%. The CDM has an almost dustlike EOS and it is considered to be responsible for clustering on galactic or supergalactic scales. The dark energy on the other hand provides a negative pressure which may explain the recent acceleration in the expansion of the universe in the context of a closed universe. The behaviour of dark energy is very similar to that of a cosmological constant.

In looking for a model of the emergent universe, we assume the following features for the universe:

1. The universe is isotropic and homogeneous at large scales.
2. It is spatially flat, as indicated by WMAP results, which give a total density parameter, \( \Omega_1 = 1.02 \pm 0.02 \) [14].
3. It is ever existing. There is no singularity.
4. The universe is always large enough so that a classical description of spacetime is adequate.
5. The matter or, in general, the source of gravity has to be described by quantum field theory.
6. The universe may contain exotic matter so that energy conditions may be violated.
7. The universe is accelerating as suggested by recent measurements [15] of distances of high redshift type Ia supernovae.

The assumptions made above make our model somewhat different from that of Ellis et al [1, 2], who consider a closed universe containing a minimally coupled self-interacting scalar field. The universe in this case spends a period in the Einstein static state before an early inflation takes over. In our model we look for an analytic solution of the Einstein equations, which evolves from a static state in the infinite past to be eventually dominated by a cosmological constant. The universe we look for is spatially flat and accelerating. To achieve this we need to revise our concepts about the primordial composition of the universe and the EOS need some generalization. In the following, we consider the EOS

\[
p(\rho) = A\rho - B\rho^{\frac{3}{2}},
\]

where \( A \) and \( B \) are the constants. The energy density \( \rho \) may have different components, each satisfying its own equation of state. Thus, the choice of equation (7) may be looked upon as a mathematical tool for generating solutions for an emergent universe. Of course in exceptional cases, this may give the EOS of a single source, as in the Starobinsky model [4].

The Einstein equations for a flat universe in RW metric are given by

\[
\rho = \frac{3}{8\pi} \frac{\dot{a}^2}{a^2},
\]

\[
p = \frac{-2\dot{a}}{a} - \frac{\ddot{a}}{a^2}.
\]
Making use of equations (7)–(9), we get the equation, with $\dot{a} > 0$,

$$2 \frac{\ddot{a}}{a} + (3A + 1) \frac{\dot{a}^2}{a^2} - \sqrt{3} B \frac{\dot{a}}{a} = 0$$

(10)

which can be integrated once to give

$$\dot{a} a \frac{\dot{a}}{a} = \kappa e^{-\frac{2}{\sqrt{3}} B t}$$

(11)

leading to the solution

$$a(t) = \left( \frac{3 \kappa (A + 1)}{2} \left( \sigma + \frac{2}{\sqrt{3} B} e^{\frac{2}{\sqrt{3}} B t} \right) \right)^{\frac{2}{3(A+1)}}$$

(12)

where $\kappa$ and $\sigma$ are two constants of integration. We note the following:

1. If $B < 0$, the solution has a singularity and it is not of interest to us here.
2. If $B > 0$, the solution describes an emergent universe if $A > -1$. The solution in this case can be written as

$$a(t) = a_i (\beta + e^{\alpha t})^{\omega},$$

(13)

where $a_i$ and $\beta$ are the constants, $\alpha = \frac{\sqrt{3}}{2} B$, and $\omega = \frac{2}{3(A+1)}$. The Hubble parameter and its derivatives are given by

$$H = \frac{\alpha \omega e^{\alpha t}}{\beta + e^{\alpha t}}, \quad \dot{H} = \frac{\beta \omega e^{\alpha t} (\beta - e^{\alpha t})}{(\beta + e^{\alpha t})^2}, \quad \ddot{H} = \frac{\beta \omega e^{\alpha t} (\beta - e^{\alpha t})^2}{(\beta + e^{\alpha t})^3}.$$  

(14)

Here $H$ and $\dot{H}$ are both positive, but $\ddot{H}$ changes sign at $t = \frac{1}{\alpha} \ln \beta$. Thus $H$, $\dot{H}$ and $\ddot{H}$ all tend to zero as $t \to -\infty$. On the other hand as $t \to \infty$ solution gives asymptotically a de Sitter universe. $\beta$ can be determined if the time when $\dot{H}$ is a maximum, can be fixed from observational data.

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(3) The solution with the Starobinsky model, obtained by Mukherjee et al [4], appears to be a special solution with $A = 0, B > 0$. However, the Starobinsky model is based on a semi-classical Einstein equation and in the initial stage, we have no matter and the vacuum energy of the fields acts as the source of gravitation. It is indeed a solution of a different equation and in that sense it does not belong to the class of solutions considered here.

3. Composition of the emergent universe

To study the possible composition of the emergent universe, we first study the dependence of the energy density on the scale factor. Consider the energy conservation equation

$$\dot{\rho} + 3(\rho + p) \frac{\dot{a}}{a} = 0.$$  

(15)

Making use of the EOS, equation (7), and integrating we obtain the relation

$$\rho(a) = \frac{1}{(A + 1)^2} \left[ B + \frac{K}{a^{3/2(A+1)}} \right]^2.$$  

(16)

where $K$ is an integration constant. Since $a$ is a monotonically increasing function of $t$ in the model, one may use $a$ to study the evolution of the universe also.

It may be pointed out that a minimally coupled scalar field alone cannot give rise to the emergent universe of the type we are considering here (spatially flat, expanding, accelerating
Emergent universe with exotic matter

Table 1. Composition of universal matter for some values of A. The value of $\omega_2$ defines the exotic matter with density $\rho_2$ relevant in each case.

| $A$  | $\omega_2$ in unit $K_B$ | $\omega_2$ in unit $(K_B)^2$ | $\omega_3$ | Composition                  |
|------|--------------------------|-------------------------------|------------|-----------------------------|
| $\frac{1}{3}$ | $\frac{0}{K_B}$          | $\frac{0}{(K_B)^2}$           | $\frac{1}{3}$ | dark energy, exotic matter and radiation |
| $\frac{1}{3}$ | $\frac{0}{K_B}$          | $\frac{0}{(K_B)^2}$           | $\frac{1}{3}$ | dark energy, exotic matter and cosmic strings |
| $1$  | $\frac{1}{K_B}$          | $\frac{1}{(K_B)^2}$           | $1$        | dark energy, exotic matter and stiff matter |
| $0$  | $\frac{2}{K_B}$          | $\frac{1}{(K_B)^2}$           | $0$        | dark energy, exotic matter and dust |

and singularity free). To see this we note that the conservation equation (15) leads to the field equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0,$$

(17)

where $V(\phi)$ is the self-interaction potential of the field $\phi$. However, $\rho + p = \dot{\phi}^2$ must be positive. We, therefore, require $\dot{\rho} < 0$. But we have $\dot{\rho} = 6H\dot{H}$ which is always positive in our model. Inclusion of a cosmological constant will not change the conclusion. Note that the solutions of [1–3] were obtained for a closed universe where the above results need not apply. If the recent observations of spatial features and the presence of both dark matter and dark energy are confirmed, one should look for alternative sources and the possibility of an emergent universe containing exotic matter cannot be ruled out.

Equation (16) provides us with some indications about the components of energy density that lead to an emergent universe. We can rewrite the equation as

$$\rho = \frac{B^2}{(A + 1)^2} + \frac{2KB}{(A + 1)^2} \frac{1}{a^{3/2(A+1)}} + \frac{K^2}{(A + 1)^2} \frac{1}{a^{3(A+1)}} = \rho_1 + \rho_2 + \rho_3.$$

(18)

The pressure $p$, given by equation (7), can also be expressed similarly,

$$p = -\frac{B^2}{(A + 1)^2} + \frac{BK(A - 1)}{(A + 1)^2} \frac{1}{a^{3/2(A+1)}} + \frac{AK^2}{(A + 1)^2} \frac{1}{a^{3(A+1)}} = p_1 + p_2 + p_3.$$

(19)

One may now try to identify the components from the EOS of individual components:

(a) The first term behaves like a cosmological constant and may account for the dark energy.

(b) The interpretation of the second and third terms depend on the choice of the parameters. Note that since $K$ is negative, $\rho_2$ has to be negative. This makes the interpretation of $\rho_2$ difficult, at least in terms of known matter. We will call these exotic matters. The component $\rho_3$, however, can be identified from its EOS. In table 1, we have shown the relevant EOS of the three components. The value of $\omega_2$ gives the EOS for the exotic matter with density $\rho_2$ in each case. Not all the possible exotic matters may be physically relevant but even if one of them is permitted in nature, we will have a realistic model of an emergent universe. Of particular interest in this connection is the case with $A = \frac{1}{3}$ where cosmic strings may occur naturally. Another interesting possibility is the case $A = \frac{1}{3}$, where we have dark energy, radiation and an exotic matter with negative energy density $\rho_2$, which has the same EOS as cosmic strings. Cosmic strings have already been studied extensively in connection with the structure formation in the early universe and it is interesting to note that an emergent universe
can accommodate this exotic source. The composition may change, as in the standard big-bang cosmology, due to non-gravitational interactions.

We have four parameters in this theory \( A, B, a_i \) and \( \beta \). As the present observational data indicate that the total energy density has three components, three of these parameters can be determined. The measurement of the scale factor at any time or when the universe is quasi-static with \( a_i \sim a_\beta \), will determine the fourth parameter.

4. Discussion

We have shown in this paper that emergent universe scenarios are not isolated solutions and they may occur for different combinations of radiation and matter. The recipe for an emergent universe for a given cosmological constant (dark energy), \( \Lambda = \left( \frac{\rho}{\rho_c} \right)^2 \), has been given in table 1. The exotic matters mentioned in the table, which have an EOS, \( p_2 = A\rho_2 \), are not yet known. Perhaps non-gravitational interactions relevant here may provide an explanation for the negative values of \( \rho_2 \). The possibility of cosmic strings playing a role in the evolution of the universe has been studied previously in detail [16–18] and it is interesting to note that these topological defects may be a part of the material composition of the early universe. Cosmic strings in particular could serve as seeds for galaxy formation and larger scale structure formation. These should be observable through their gravitational lensing and also from studies of anisotropy in the microwave background radiation and the background gravitational waves. However, the scenario of the phase transition of the relevant scalar field leading to these topological defects remains to be worked out in this model. It will be interesting to try to develop an evolutionary scenario of the emergent universe and this is presently under consideration. One interesting observation in this context is that the inclusion of bulk viscosity [19–21], following Eckart’s theory, converts Einstein equation for a perfect fluid to a form similar to equation (10). Thus with an EOS \( p = \omega \rho \), one gets for the effective pressure, \( p_{\text{eff}} = p + \Pi \), where \( \Pi = -3\xi H \), and \( \xi \) is the constant coefficient of bulk viscosity. The resulting Einstein equation is similar to equation (10), if one substitutes \( B = \sqrt{3\xi} \). Additional solutions of the same type exist in a higher derivative theory in the presence of bulk viscosity [22]. The presence of a bulk viscosity is conceivable in different stages of evolution and the existence of solutions similar to equation (13), may facilitate the construction of models to describe the later evolution of the emergent universe.

Finally, an important issue which has not been discussed here is the stability of the solution under both homogeneous and inhomogeneous perturbations. The stability of the Einstein static universe under inhomogeneous perturbations has recently been studied in [23–24]. Einstein static solutions are usually saddle points in the phase space of the system and hence unstable. Mulryne et al [3] used loop quantum cosmology to partially remedy the situation by showing the existence of a static solution which dynamically is a centre equilibrium point and constructed an emergent universe model about such an initial static state. The situation with our model, which is spatially flat and has no fine tuning, needs to be examined separately and this is presently under investigation.

Acknowledgments

SM and BCP would like to thank the University of Zululand and the University of KwaZulu-Natal, South Africa for hospitality during their visit when a part of the work was done. They would also like to thank IUCAA, Pune and IUCAA Reference Centre, North Bengal University for providing facilities.
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