Short-Time Loschmidt Gap in Dynamical Systems with Critical Chaos

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We study the Loschmidt echo $F(t)$ for a class of dynamical systems showing critical chaos. Using a kicked rotor with singular potential as a prototype model, we found that the classical echo shows a gap (initial drop) $1 - F_0$ where $F_0$ scales as $F_0(\alpha, \epsilon, \eta) = f_0(\chi_{cl}) \equiv \eta^{3-\alpha}/\epsilon$; $\alpha$ is the order of singularity of the potential, the spread of the initial phase space density and $\epsilon$ is the perturbation strength. Instead, the quantum echo gap is insensitive to $\alpha$, described by a scaling law $F_q = f_q(\chi_q = \eta^{2}/\epsilon)$ which can be captured by a Random Matrix Theory modeling of critical systems. We trace this quantum-classical discrepancy to strong diffraction effects that dominate the dynamics.

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The study of systems with a phase transition was always a fruitful subject of study for many areas of theoretical and experimental physics. Specifically in the field of disordered metals, the celebrated Anderson Metal-Insulator Transition (MIT)\textsuperscript{6} has been an exciting subject of research for more than fifty years. On the other hand, the field of quantum chaos brought up a connection between quantized chaotic systems and localization ideas emerging from solid-state physics\textsuperscript{2}. It has been shown that quantum suppression of classical diffusion is a result of wave interference phenomena of similar nature as the ones responsible for Anderson localization in disordered metals. Quite recently the connection between the two fields was further strengthened with the observation that certain non-KAM dynamical systems exhibiting classically anomalous diffusion, can have statistical properties resembling the ones of disordered metals at MIT\textsuperscript{3,4}. This phenomenon is referred to as critical chaos. Some of these properties include, quantum anomalous diffusion\textsuperscript{3}, multifractal wavefunctions\textsuperscript{4}, and critical spectral statistics\textsuperscript{5}. Many of these intriguing statistical properties can be exactly derived using non-conventional ensembles of random matrices with variance decaying from the diagonal in a power-law fashion\textsuperscript{3}, which in turn model a large variety of physical systems\textsuperscript{6}.

Most of the above studies discuss the stationary properties of critical systems. On the other hand this knowledge is often not sufficient for a complete description of the dynamics. This need led us in recent years to focus on new measures that efficiently probe the complexity of quantum time evolution. One such measure is the so-called Loschmidt Echo (LE) which probes the sensitivity of quantum dynamics to external perturbations (for recent reviews see\textsuperscript{10}). The recent literature on the subject is quite vast and ranges in areas as diverse as atomic optics\textsuperscript{11,12,13}, microwaves\textsuperscript{14}, elastic waves\textsuperscript{15}, quantum information\textsuperscript{16}, and quantum chaos\textsuperscript{17,18,19,20,21,22,24,25}. Formally, the LE $F(t)$, is defined as:

\begin{equation}
F(t) = |\langle \psi_0 | e^{iH_0 t} e^{-iH \epsilon t} | \psi_0 \rangle |^2; \quad \hbar = 1
\end{equation}

where $H_\epsilon = H_0 + \epsilon V$ is a one-parameter family of Hamiltonians, $H_0$ is the unperturbed Hamiltonian, $V$ represents a perturbation of strength $\epsilon$ and $| \psi_0 \rangle$ is an initial state. The focus of the presentation is on the non-perturbative regime, the decay of the LE depends on the strength of the perturbation parameter $\epsilon$. Three regimes have been identified: the standard perturbative, the Fermi Golden Rule, and the non-perturbative regime. The first two can be described by Linear Response Theory leading to a decay which depends on the perturbation strength $\epsilon$ as $F(t) \sim e^{-(\alpha t)^2}$ and $F(t) \sim e^{-\epsilon^2 t^2}$, respectively\textsuperscript{18,21}. In the non-perturbative regime, the LE initially follows a Lyapunov decay $F(t) \sim e^{-\lambda t}$, with a rate given by the Lyapunov exponent $\lambda$ of the underlying classical system\textsuperscript{17,18}, whereas for longer times (beyond the so-called Ehrenfest time) the LE decays in accordance with the classical autocorrelation function\textsuperscript{22}. This behavior matches the decay of the classical echo according to the correspondence principle.

In this letter we make the first step in understanding the echo decay of dynamical systems with critical chaos. The focus of the presentation is on the non-perturbative regime where our study revealed a novel result (see Fig. 1): we have discovered the appearance of an echo gap (initial drop of LE) $1 - F_q$ for initial states which are distributed in the parts of phase-space where the Hamiltonian function exhibits singularities. We have found that at the shortest classical time scale (mean free time between singular scattering events) $F_q$ scales as

\begin{equation}
F_q(\alpha, \epsilon, \eta) = \begin{cases} 
  f_0(\chi_{cl}) & \text{where } \chi_{cl} = C \cdot \frac{2^{3-\alpha}}{\epsilon^3} \\
  f_q(\chi_q) & \text{where } \chi_q = C \cdot \frac{\eta^2}{\epsilon}
\end{cases}
\end{equation}

where the sub-indices $q/\text{cl}$ indicate the quantum/classical scaling function for $F_q$. In Eq. (2), $\alpha$ is the order of singularity of a non-analytical potential, $\eta$ is the characteristic spread of the phase space density of the initial classical
or quantum state (e.g., its Wigner function) and the constant C only depends on details of $H_0$. Moreover, we found that the scaling function $f(\chi)$ behaves asymptotically as $f(\chi \to 0) \sim \chi$. The above scaling laws were derived based on the analysis of classical dynamics and confirmed nicely by numerical simulations.

The apparent deviation of the quantum echo behavior from its classical counterpart (see Eq. (2)), can be seen as a violation of the quantum-classical correspondence; the latter being confirmed in all previous fidelity studies. We have found that the origin of this anomalous behavior is due to strong diffraction effects which dictate the wave dynamics for the class of dynamical systems we investigate in this Letter.

Below we consider a class of parametric Kicked Rotors (KR) defined by the time-dependent Hamiltonian

$$H_0 = \frac{p^2}{2} + K_0 V(q) \sum_n \delta(t - nT)$$

where $(p, q)$ is a pair of canonical variables, $T$ and $K_0$ are the period and the strength of the kicking potential respectively. The class of KRs that we will study below have a potential which is given by

$$V(q) = \begin{cases} |q|^\alpha & \text{for } \alpha \neq 0, \\ \log |q| & \text{for } \alpha = 0. \end{cases}$$

The $\alpha = 0$ case corresponds to MIT, and will be investigated below in detail. The classical dynamics is described by the following map:

$$p_{n+1} = p_n - K_0 V'(q_n); \quad q_{n+1} = q_n + Tp_{n+1}.$$  \hspace{1cm} (5)

where all variables are calculated immediately after one map iteration and $V' = \partial V(q)/\partial q$. The domain of $q$ is within the interval $-\pi < q < \pi$. The map can be studied on a cylinder $p \in (-\infty, \infty)$, which can also be closed to form a torus of length $2\pi L$, where $L$ is an integer. For $K_0 > 0$ the motion is chaotic (see inset of Fig. 2) with a (local) Lyapunov exponent given by $\lambda(q) = 2 \log(1 + K_0/(2q^2)) + \sqrt{K_0/q^2 + [K_0/(2q^2)]^2}$ for $q \neq 0$.

The quantum evolution is described by a one-step unitary operator $\hat{U}_0$ acting on the wavefunction $\psi(q)$:

$$\hat{U}_0 = \exp(-i\tau \hat{n}^2/2) \exp(-iK_0 \hat{V}(q)),$$  \hspace{1cm} (6)

where $\hat{n} = -i\partial/\partial q, \quad -N/2 \leq n \leq N/2, \quad \tau = (2\pi L/N)T$ and $K = (N/2\pi L)K_0$. Optionally, we define an effective Planck constant $\hbar_{\text{eff}} = 2\pi L/N$. The classical limit corresponds to $N \to \infty$. Without loss of generality we will assume below that $T = 1$. It was shown in [1], that for $-0.5 \leq \alpha \leq 0.5$ the eigenfunctions of the above unitary operator are multifractal while the levels resemble the statistical properties of disordered systems at MIT.
For the echo calculation, we perturbed our system with the following (smooth) potential \( W(q) = \cos q \). Correspondingly the perturbed quantum kicking parameter is \( \sigma = \epsilon/h_{\text{eff}} \). Quantum mechanically, the initial preparation is a Gaussian wavepacket centered along the line of singularity i.e. \((q_0,p_0) = (0,p_0)\). The linear width \(2\eta\) of the packet is taken to be minimal (i.e. \(2\eta = \Delta p = \Delta q = \sqrt{h_{\text{eff}}/2}\)), where we perform an averaging over different \(p_0\)’s in order to eliminate fluctuations. The corresponding classical initial preparation is given by a uniform distribution of trajectories located inside a box of area \(A = 2\eta \times 2\eta \sim h_{\text{eff}}\). We then define the classical LE, \(F_{cl}(t=n)\), as the overlap of the initial area \(A_0\) with the area \(A_t\) obtained by evolving \(A_0\) for \(n\) iterations of the perturbed map and then reversing the evolution for \(n\) iterations with the unperturbed one. We have also checked that the results remain qualitatively the same when we chose an initial classical distribution to be a Gaussian density, equivalent to a Wigner function of the quantum Gaussian wavepacket.

We start our analysis with the classical derivation of Eq. (2). To this end we consider the classical echo dynamics \[26\]. We denote by \(\Phi_0(p,q)\) the forward symplectic map defined in Eq. (5) while by \(\Phi_{\epsilon} = \Phi_0 \circ P_{\epsilon}\) we denote the corresponding perturbed forward map. In this notation \(P_{\epsilon}(p,q) = (p + \epsilon \sin(q),q)\) is a symplectic map generated by the perturbation (perturbation map). In this framework, the \(n\)-step echo map is defined as

\[
\Phi_n = \Phi_{\epsilon}^n \circ \Phi_0^n = \tilde{P}_\epsilon(n-1) \circ \tilde{P}_\epsilon(n-1) \circ \ldots \circ \tilde{P}_\epsilon(0) \tag{7}
\]

where the perturbation map is written in the interaction picture i.e. \(\tilde{P}_\epsilon(n) = \Phi_0^n \circ P_{\epsilon} \circ \Phi_0^n\). Explicitly, 
\[
(p_{n+1}^E, q_{n+1}^E) = \tilde{P}_\epsilon(n)(p_n^E, q_n^E)
\]

where
\[
p_{n+1}^E = p_n^E + \epsilon \sin[\Phi_0^n(p_n^E, q_n^E)] \frac{\partial[\Phi_0^n(p_n^E, q_n^E)]}{\partial q_n^E} q_n^E
\]
\[
q_{n+1}^E = q_n^E - \epsilon \sin[\Phi_0^n(p_n^E, q_n^E)] \frac{\partial[\Phi_0^n(p_n^E, q_n^E)]}{\partial p_n^E} p_n^E \tag{8}
\]

For singular potentials and initial conditions close to singularity, we express the phase space shift produced by the echo map after the second iteration step:

\[
\Delta p_2^E = p_2^E - p_0^E \approx -\epsilon \sin(q_1) K_0 V''(q_0) \tag{9}
\]

provided that the typical echo shift \(\Delta p_2\) is much larger than \(\eta\). The echo probability is just the integral of \(P(q_0)\) over the initial interval \(q_0 \in [-\eta, \eta]\):

\[
F_{cl}^q = \frac{1}{2\pi \epsilon K_0} \int_{-\eta}^{\eta} P(q_0) dq_0 \sim \frac{1}{2\pi \epsilon K_0} \int_{-\eta}^{\eta} dq_0 |V''(q_0)| \tag{11}
\]

For the specific family of singular potentials discussed in this Letter, the above relation gives us:

\[
F_{cl}^q = \frac{1}{2\pi \epsilon K_0} \int_{-\eta}^{\eta} P(q_0) dq_0 \approx \frac{1}{2\pi \epsilon K_0} \int_{-\eta}^{\eta} dq_0 \frac{\eta^2}{V''(q_0)} \tag{12}
\]

Our results in Eq. (12) are nicely confirmed in Fig. 3, where we are plotting the echo gap for various \(\alpha, \epsilon\) and \(\eta\) values by making use of the rescaled variable \(x_{cl}\) given by Eq. (2). Strictly speaking, the above results are applicable only for the case where \(F_{cl}^q \ll 1\). Nevertheless, our numerics indicates that the scaling behavior Eq. (2) continues to apply for values of \(F_{cl}^q \sim 1\).

We have also tested the results of the classical analysis against the quantum echo gap. A complete breakdown of the quantum-classical correspondence (QCC) is observed after the shortest non-trivial time scale (two iteration steps). This is associated with the fact that the Ehrenfest time for our system \(\tau_E \sim \log(h_{\text{eff}})/\lambda(q_0) \rightarrow 0\) when \(q_0 \rightarrow 0\) \[27\]. Weak correspondence is restored for longer times.
when the echo dynamics spreads ergodically resulting in a vanishing measure of the critical line at $q_0 = 0$.

In Fig. 3 we observe that although the $\epsilon$ dependence of the quantum $F_\epsilon$ is captured by the classical calculations, both the $\eta \sim \sqrt{\alpha}$ and the $\alpha$ dependence differ drastically. The latter can be explained by a Random Phase Kicked Rotor (RPKR) with singular potential, which is simply given by Eq. (6) and replacing the eigenvalues of $\tau^2/n^2/2$ by random phases. This indicates that the appearance of a gap is insensitive to classical dynamics and thus can be captured by a Random Matrix Theory (RMT) modeling which preserves the power-law band structure of the evolution operator. Thus the fidelity gap is a universal phenomenon of critical systems described by these RMT models, and can be used as an alternative criterium to level or wavefunction statistics.

We argue that the violation of the QCC is due to dominating diffraction effects that appear as a consequence of the singular potential. This is illustrated in the inset of Figure 2 where we show the Wigner function (computed according to Ref. (28) of the echo map for the first non-trivial time step. One observes the appearance of non-classical regions in the phase space where the Wigner function takes negative values.

In conclusion, we find that the LE of dynamical systems exhibiting critical chaos decays instantaneously with a gap that scales inverse proportionally to the strength of the perturbation. The order of the potential singularity is encoded in the scaling properties of the classical echo gap, while the corresponding quantum gap is insensitive to it and its scaling properties are described by an RMT modeling of critical systems. This deviation is explained in the basis of strong diffraction which is a dominating mechanism of short time echo dynamics.

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