Vibration analysis of structure with uncertainty using two-level Gaussian processes and Bayesian inference

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Abstract. Vibration analysis of structure with uncertainty is computationally costly, especially when the finite element model involved has high dimensionality. In this research a combination of two-level Gaussian processes and Bayesian inference is employed to facilitate the development of an efficient and accurate probabilistic order-reduced model. We first employ the two-level Gaussian processes emulator to integrate together small amount of high-fidelity data from full-scale finite element analysis and large amount of low-fidelity data from order-reduced component mode synthesis (CMS) model to improve the response variation prediction. We then utilize the improved response variation prediction on modal characteristics to update the CMS model in the probabilistic sense. The effectiveness of this method is demonstrated through a case study.

1. Introduction

Real-world structures are subject to uncertainties. Ideally, in analyzing a structure, variations in parameters of concern can be approximately defined in terms of probabilities, based on which Monte Carlo-based analysis is to be performed to generate the response variations. However, when a single run of numerical simulation or physical experiment is too expensive, such Monte Carlo-based analysis will be impractical [1,2]. One way of tackling this issue is to create a more economical model to reduce the computational cost of each single dynamic analysis. A representative class of order-reduction methods in structural dynamic analysis is called the component mode synthesis (CMS) approach. The basic idea of CMS is to decompose the whole structure into a number of substructures and then carry out modal analyses for individual substructures. An order-reduced model can be established by keeping only a few lowest-order modes of individual substructures and also including some additional compensating modes for recovering information in the modes that are truncated [3-6]. Another approach is to develop statistical meta-models to replace the complete Monte Carlo simulation, by using for example the so-called Gaussian processes [7-12].

Fundamentally, both the CMS-based order reduction and the Gaussian processes approach are subject to errors. On one hand, the CMS order-reduction can allow more efficient generation of first principle-based response prediction; on the other hand, the truncation of higher-order modes in each substructure generally results in error as compared with a full-scale finite element analysis. The usage of Gaussian processes as emulators, while being able to yield extremely fast analysis by using only a few data points, inevitably causes error due to its inference nature, even if all the data points involved...
can be considered accurate (i.e., coming from a full-scale finite element model in numerical simulation).

On the other hand, along with the advancement of Gaussian processes approach, a technique of utilizing data at multiple resolutions for inference has been studied [13,14]. It was suggested that the combination of data with different fidelities for Gaussian process emulation can maintain both prediction accuracy and efficiency [7]. As such, it is tempting to apply such technique to structural dynamic analysis. That is, the CMS based order-reduction may be used to generate large amount of low-fidelity, low-cost, first principle-based data, while the full-scale finite element analysis may be used to produce a small amount of high-fidelity data. Nevertheless, in the context of structural dynamic analysis, one is usually interested in a series of data sets, e.g., various responses at different locations under excitations which themselves may vary. Conceptually, it would be much more efficient if one could come up with a computationally low cost and accurate, probabilistic, low-dimension model that can adapt to varying excitations and predict various response variations at arbitrary location(s) of the structure.

Therefore, the objective of this research is to establish a new framework that embeds the two-level Gaussian processes into the Bayesian probabilistic model updating procedure [15,16] to produce a probabilistic CMS model. In the new framework, we plan to use the modal information of the entire structure obtained through two-level Gaussian processes as the evidences, and update the probabilistic distributions of key parameters of the CMS model, i.e., the modal information of individual substructures used in the model. The resultant probabilistic model can then eventually be used to perform response variation analysis.

2. Mathematical formulation of two-level Gaussian processes and Bayesian inference

2.1 Two-level models and data sets

In this research, we consider a structural dynamic system whose finite element-based equation of motion is given as

\[ M \ddot{x} + C \dot{x} + K x = f \]  

(1)

where \( M \), \( C \), and \( K \) are constant mass, damping, and stiffness matrices, and \( x \) and \( f \) are the \( N \)-dimensional displacement response vector and external force vector, respectively.

In a classical CMS procedure, the original, large-scale structure is decomposed into a collection of substructures first, and smaller-size eigenvalue problems are computed for all these separate substructures [17]. Then, a global, order-reduced model is synthesized by combining the reduced-order representations of substructures together with the interface compatibility condition. Without loss of generality, in this research we apply the free-interface CMS procedure, and the order-reduced model can be expressed as

\[ M_r \ddot{z} + C_r \dot{z} + K_r z = F_r \]  

(2)

where the order-reduced mass and stiffness matrices are

\[ M_r = I, \quad K_r = \begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix} = \begin{bmatrix} \Lambda_{AA} + \Phi_{AB}^T (G_A + G_B)^{-1} \Phi_{AB} & -\Phi_{AB}^T (G_A + G_B)^{-1} \Phi_{BB} \\ -\Phi_{AB}^T (G_A + G_B)^{-1} \Phi_{BA} & \Lambda_{BB} + \Phi_{AB}^T (G_A + G_B)^{-1} \Phi_{BA} \end{bmatrix} \]  

(3a,b)

Here we use a two-substructure, \( A \) and \( B \), for illustration. \( \Lambda_k \) is the set of eigenvalues corresponding to the kept modes, \( \Phi_{jk} \) is the subset of the kept modes at interface degrees-of-freedom (DOFs), and \( G \) is the residual flexibility matrix which is essentially the portion of the flexibility matrix contributed by the truncated modes. The transformation relation between the order-reduced space and the DOFs of the original structure can be used to derive the order-reduced damping matrix \( C_r \) and force vector \( F_r \).

In uncertainty quantification, some or all of the above-mentioned coefficient matrices become random due to the uncertainties of the structure. Consequently, the dynamic characteristics and responses of the structure become random even if the forcing function is deterministic. While the uncertainties in the structural system may be described in different ways either through parameterized
or non-parameterized manner, Monte Carlo-type sampling approach is the intuitive and most straightforward technique to extract the response uncertainties. Equations (1) and (2) and the associated eigenvalue problems can be used to generate high- and low-fidelity data.

2.2 Two-level Gaussian processes

This section presents a concise formulation of the two-level Gaussian processes. Let an unknown system be denoted as $f(x)$, where $x$ is an $r$-dimensional input vector. Let the observed value of $f(x)$ be denoted as $y$. Given a set of $s$ observations, described as $\mathcal{D} = \{(y_i, x_i), i = 1, 2, \ldots, s\}$, a single-level Gaussian process regression can be simply implemented to predict the output $y_i$ over target input $x_i$.

In this research, two data sets with different fidelities, i.e., low- and high-fidelity are considered, shown as $\mathcal{D}(1)$ and $\mathcal{D}(2)$. Here superscript $u$ indicates the fidelity level of data, and each $(y_i^{(u)}, x_i^{(u)})$ is called a data point. The goal is to predict the high-fidelity output $y^{(2)}$ at target input $X^{(2)}$ given two observed data sets $\mathcal{D}(1)$ and $\mathcal{D}(2)$. We assume

$$y^{(1)} = \delta^{(1)}, \quad y^{(2)} = \rho^{(1)}\delta^{(1)} + \delta^{(2)}$$

where $\rho^{(1)}$ is a regression parameter. $\delta^{(1)}$ and $\delta^{(2)}$ are modeled as two independent stationary Gaussian processes, i.e.,

$$\delta^{(u)} \sim N(\mu^{(u)}(x), \Sigma^{(u)}(x^{(u)})), \quad u = 1, 2$$

where $\mu$ and $\Sigma$ represent the mean vector and covariance matrix,

$$\mu^{(u)}(x^{(u)}) = \begin{bmatrix} m(x^{(u)}) & m(x_2^{(u)}) & \ldots & m(x_m^{(u)}) \end{bmatrix}^T$$

$$\Sigma^{(u)}(X^{(u)}, X^{(u)}) = \begin{bmatrix} k_{11}^{(u)} & k_{12}^{(u)} & \ldots & k_{1m}^{(u)} \\ k_{21}^{(u)} & k_{22}^{(u)} & \ldots & k_{2m}^{(u)} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m1}^{(u)} & k_{m2}^{(u)} & \ldots & k_{mm}^{(u)} \end{bmatrix}, \quad u = 1, 2$$

where $m$ is the number of $u$-level data points. Here we adopt the linear mean function and squared exponential covariance function,

$$m(x^{(u)}) = h(x^{(u)})\beta^{(u)}$$

$$k^{(u)} = \sigma_f^{(u)} \exp(-h(x^{(u)}) - h(x_i^{(u)})^T (x_i^{(u)} - X^{(u)}))$$

where $h(.)$ is a base function of $X^{(u)}$ [12], and $\beta^{(u)}$ is a vector of unknown parameters of $u$-level data points with respect to the base function. $\varphi^{(u)} = \{h^{(u)}, \sigma_f^{(u)}\}$ is the hyperparameter vector of $u$-level data points associated with the covariance function.

The input data points of the high-fidelity data $X^{(2)}$ follows the relation $X^{(2)} \subseteq X^{(1)}$, which enables the covariance evaluation of two data sets that are either at the same or different fidelity levels. As the sum of independent Gaussians remains in the closed form, we can derive the representation of observed low- and high-fidelity data points, namely

$$\begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix} \sim N \left( \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix}, \begin{bmatrix} \Sigma^{(1)}(X^{(1)}, X^{(1)}) & \rho^{(1)}\Sigma^{(1)}(X^{(1)}, X^{(2)}) \\ \rho^{(1)}\Sigma^{(1)}(X^{(2)}, X^{(1)})^T & \Sigma^{(2)}(X^{(2)}, X^{(2)}) + \rho^{(1)}\rho^{(2)}\Sigma^{(2)}(X^{(2)}, X^{(2)}) \end{bmatrix} \right)$$

where $y^{(1)}$ and $y^{(2)}$ are the respective datasets. Based on such observed/training data, we can utilize the Bayesian approach for parameter estimation by re-writing the Equation (10) in an explicit Bayesian fashion,

$$y \mid X^{(1)}, X^{(2)}, \beta, \varphi \sim N(\mu, \Sigma)$$

The unknown parameters $\beta$ and $\varphi$ are suggested to be optimized by maximizing the marginal likelihood of Equation [7,12]. Here we assume the optimized parameters are expressed as
\[ \mathbf{b} = [\hat{\mathbf{b}}^{(1)}, \hat{\mathbf{b}}^{(2)}] \quad \text{and} \quad \mathbf{\phi} = [\hat{\mathbf{\phi}}^{(1)}, \hat{\mathbf{\phi}}^{(2)}, \hat{\sigma}_j^{(1)}, \hat{\sigma}_j^{(2)}]. \] The target output \( y_j^{(2)} \) over target input \( X_j^{(2)} \) can be simply characterized as the Gaussian distribution with updated mean and covariance function.

2.3 Bayesian inference for probabilistic CMS model updating

Here we formulate a Bayesian inference framework that is tailored towards the probabilistic updating of the CMS model as the eventual outcome of the two-level Gaussian processes. We compute the natural frequencies and mode shapes under the effect of uncertainties employing both the full-scale finite element and CMS, and use the two-level Gaussian processes to obtain the modal response variations. We then use the two-level results as input to the Bayesian inference to update the CMS model. Observing Equation (3b), we can find that the CMS model is fundamentally built upon the natural frequencies of individual components, and the mass matrix of the order-reduced model is an identity matrix. Therefore, here we update the natural frequencies of component natural frequencies corresponding to the kept modes that are contained in the order-reduced stiffness matrix to improve the order-reduced model in the probabilistic sense.

We start from Bayes’ rule [18]

\[ p(\theta | \mathbf{D}) \propto p(\mathbf{D} | \theta) p(\theta) \]  

where the hypothesis \( \theta \) in this case is the vector of model parameters that need to be identified, i.e., the combination of \( \mathbf{A}_{kl}^{(1)} \) and \( \mathbf{A}_{kl}^{(2)} \) shown in Equation (3b). \( \mathbf{D} \) denotes the evidence, which in this research is selected as the improved modal response variation prediction obtained based on the two-level Gaussian processes. The prior PDF \( p(\theta) \) represents the initial distribution of parameters \( \theta \) built upon the empirical knowledge. Here this term can be simply defined as a standard statistical distribution, such as normal and uniform distributions. The posterior PDF \( p(\theta | \mathbf{D}) \) indicates the updated distribution of \( \theta \) conditional on the prior PDF and input data \( \mathbf{D} \). The likelihood PDF \( p(\mathbf{D} | \theta) \) is used to evaluate the agreement between the measurement and the response prediction of the model parameterized by \( \theta \), which is selected as

\[ p(\mathbf{D} | \theta) = \frac{e^{-\frac{1}{2\nu}(\mathbf{D} - \mathbf{D}(\theta))^T\Omega^{-1}(\mathbf{D} - \mathbf{D}(\theta))}}{(2\pi)^{\nu/2} |\Omega|^{1/2}} \]  

It is worth mentioning that in this study the evidence \( \mathbf{D} \) is considered as the response set with the highest probability value at the response distribution predicted by the two-level Gaussian processes emulator. \( \mathbf{D}(\theta) \) is the evaluated response under parameters \( \theta \) using the CMS analysis. \( \Omega \) and \( \nu \) are constants that are respectively related to the mean and covariance. \( \nu \) essentially is used to represent the confidence level for model calibration. For a certain data set, both the natural frequencies and corresponding mode shapes are used to evaluate the likelihood function. Qualitatively, we have \( \mathbf{D} = \{o_1, q_1, \cdots, o_r, q_r\} \) which represents the natural frequencies and the corresponding mode shapes that are used in comparison. \( r \) is the number of modes analyzed through the two-level Gaussian processes. To facilitate the comparison of mode shapes, we introduce the Mode Assurance Criterion [19] as the metric. We then have

\[ p(\mathbf{D} | \theta) = \prod_{i=1}^{r} p(o_i^2 | \theta) p(MAC_i | \theta) \]  

In general, the large number of possible candidates \( \theta \) that is needed to densely span over the high-dimensional parametric space of specified prior PDF lead to significant computational cost even when the order-reduced CMS analysis is employed to characterize the modal response at each single run. In this research, an expedited optimization scheme that aims at reducing the number of analysis runs is integrated into the Bayesian framework. This optimization algorithm is built upon the Metropolis Hasting sampling principle that is conventionally employed in Markov Chain Monte Carlo (Martinez and Martinez, 2007).
3. Case analyses

We consider a benchmark structure that is a plate structure clamped at the left-end \((x = 0)\) with 3 substructures as shown in Figure 1(a). The nominal values of the Young’s Modulus, mass density and Poisson ratio are 2.05 Pa, 7800 kg/m³ and 0.3, respectively. We start from deterministic analysis without uncertainties. As can be seen from Figure 1(b), we construct a full-scale finite element model with 3,600 DOFs as well as its order-reduced CMS representation with a significantly reduced number of DOFs, i.e., 8 DOFs. Specifically, this order-reduced model involves 4 kept modes of Substructure 1 and 2 kept modes of each of Substructure 2 and 3. In this case study, as an illustration we let uncertainty occur in material property, i.e., Young’s modulus at substructure-level. We assume that the Young’s modulus variations of all substructures are subject to a multivariate Gaussian distribution with zero mean and 10% , 7.1% , 3.2% standard deviations, respectively. Based on the specified distribution, 1,000 input samples of Young’s Modulus are parameterized by Latin Hypercube technique, and cast into both the full-scale finite element model and the CMS model to yield the fine (high-fidelity) and coarse (low-fidelity) response distributions, respectively. The results are shown in Figures 2 and 3.
The first step to conduct two-level Gaussian processes emulation is to define its training data points. Earlier studies have suggested using well-spaced training data points [8]. Based on this principle, we choose 10 and 60 sets of input parameters and the related high- and low-fidelity response data points respectively as two-level training data. The Gaussian process regression model is then trained to fit the best relation between the input and output through optimization. The output prediction over input uncertainty space eventually can be estimated. The distributions of the first 4 natural frequencies are not shown here, as they are fairly close to those shown in Figures 2 and 3 since the original CMS is fairly accurate in terms of natural frequency prediction. The associated mode shape MACs are plotted in Figures 4. Overall, this Gaussian process emulation leads to adequately accurate results, i.e., the errors distributed are all under 3%, even in the case that the original low-fidelity 4th mode contains substantial error (Figure 3). Indeed, a very small number of training data employed in this emulation can be efficiently evaluated, which results in nearly 90% computational time reduction as compared with the full-scale finite element-based full Monte Carlo simulation in characterizing the benchmark response distribution.

As mentioned, instead of repeating the two-level Gaussian processes emulator for each response characteristic of concern, here we use the two-level Gaussian processes results to update the CMS order-reduced model which can then be utilized under different scenarios for rapid uncertainty quantification using such as full Monte Carlo simulation. We employ the Bayesian inference in combination with MCMC-based optimization to conduct model updating. The promising aspect of this model updating idea is that we can carry out efficient and accurate characterization of any interested response characteristics based on the updated order-reduced model, even when such response information has not been directly used as the evidence for model updating. Here we further investigate the forced response prediction performance of this updated order-reduced probabilistic model. As the first several low-order modes are the bending modes along the plate thickness direction, we assume the forced responses are only retrieved at the corresponding DOFs in accordance with this bending motion. To compare the forced responses between the order-reduced and full-scale models, we define an index to quantify their discrepancy, where the first term at the right hand side is used to calculate the magnitude ratio that originates from the concept of MSF (modal scale factor), and the second term is the MAC to evaluate the pattern similarity [19]. This index falls into the range [0, 1], where larger value indicates better closeness with the full-scale result, i.e., better accuracy. \( \nu_{\text{CMS}} \) denotes the forced response vector under the updated
order-reduced model with certain parameter uncertainty, while \( \mathbf{u}_{\text{full}} \) denotes the corresponding forced response vector obtained via full finite element analysis.

Each index value indicates the forced response prediction error of one model sample under certain excitation frequency. Hence, the accumulated index value of all model samples at certain excitation frequency can be simply expressed as a summation

\[
Y_j = \sum_{i=1}^{M} e_{ij}, \quad (M \text{ is the number of parameterized model samples; } i \text{ denotes the model sample ID and } j \text{ denotes the excitation frequency ID}).
\]

Here we use the accumulated index value as metric to investigate the forced response prediction accuracy over interested excitation frequency range. Here we examine the performance of frequency response prediction around the 3rd resonant frequency, since the 3rd mode shape prediction has been improved considerably (Figure 4). The frequency response curve at such frequency range, i.e., around 3rd resonant frequency is shown in Figure 5. We can observe that most of the accumulate index values from the updated order-reduced model are higher than those of the original order-reduced model, which verifies the validity of the proposed methodology.

4. Conclusions
A new method of uncertainty quantification for structural dynamic analysis is developed. This method is built upon the two-level Gaussian processes that can integrate together a small amount of high-fidelity data obtained from full-scale finite element analysis with a large amount of low-fidelity data obtained from CMS order-reduced model. The improved response variation prediction on structural modal properties is then used to update the CMS model that eventually facilitates efficient and accurate uncertainty quantification.

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