Majorana corner states in an attractive quantum spin Hall insulator with opposite in-plane Zeeman energy at two sublattice sites

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Higher-order topological superconductors and superfluids host lower-dimensional Majorana corner and hinge states since novel topology exhibitions on boundaries. While such topological nontrivial phases have been explored extensively, more possible schemes are necessary for engineering Majorana states. In this paper we propose Majorana corner states could be realized in a two-dimensional attractive quantum spin-Hall insulator with opposite in-plane Zeeman energy at two sublattice sites. The appropriate Zeeman field leads to the opposite Dirac mass for adjacent edges of a square sample, and naturally induce Majorana corner states. This topological phase can be characterized by Majorana edge polarizations, and it is robust against perturbations on random potentials as long as the edge gap remains open. Our work provides a new possibility to realize a second-order topological superfluid in two dimensions and engineer Majorana corner states.

1. INTRODUCTION AND MOTIVATION

Higher-order topological (HOT) superconductors and superfluids have attracted great attentions in recent years due to their novel exhibitions of topology on the lower-dimensional boundaries including corners and hinges [1–58] and potential applications in topological quantum computations [59–64]. In contrast to conventional (first-order) topological superconductors (SCs) and superfluids (SFs), rth ($r \geq 2$)-order SCs and SFs in d dimensions manifest $(d − r)$D ($d − r$-dimensional) topologically protected Majorana boundary states. For example, second-order topological SCs and SFs in two (three) dimensions host Majorana zero-energy modes at 0D corners (1D hinges).

In previous studies, a variety of HOT SCs and SFs have been proposed based on $s_\pm$-wave [2, 3], p-wave [4], d-wave pairings [3], and the cooperation of multi-pairing orders [65–67]. In particular, an evidence for helical hinge zero modes in a Fe-Based superconductor has been observed [68]. In general, crystalline symmetries play important roles on HOT states [69–74]. The topological classification of HOT states has been made based on spatial symmetries [75, 76]. Recently, to avoid complex pairings and complicated lattice structures, it is proposed that second-order topological SCs and SFs could emerge in a quantum spin-Hall insulator with the $s$-wave pairing and in-plane Zeeman field [77, 78]. In these schemes, the in-plane Zeeman field gives rise to the opposite Dirac masses on adjacent edges, and induces Majorana zero-energy states at corners. Naturally, an interesting questions aries: are there any other tuned parameters to implement Majorana corner states in a QSHI with $s$-wave pairing besides the in-plane Zeeman field? If yes, is there a platform that could realize this HOT phase?

In this paper, we propose that, in addition to the in-plane Zeeman field proposed in previous proposals, the opposite in-plane Zeeman energy (OIPZE) at two sublattice sites could also induce Majorana corner states in a quantum spin-Hall insulator. In recent years, rapid advances in ultracold atom systems provide a controllable platform to simulate topological quantum states. Various parameters, such as the tunneling between sites in optical lattices, Zeeman field, spin-orbit coupling, and the interactions between atoms could be tuned by lasers [79–88]. In particular, some topological models have been realized in this platform, such as Haldane model [89], Weyl semimetals [90] and etc [91–95]. Hence, the ultracold atom platform could be utilized as a potential candidate to reach the second-order topological superfluid.

In this work, we start with an attractive QSHI on a square optical lattice with OIPZE, and derive the ground states by the mean-field theory. Through the energy gap closing and re-opening for edge states versus OIPZE, we deduce that the topological phase transition occurs along $y$, while not along $x$, which implies Majorana corner states would emerge in an appropriate parameter region. By numeric calculations, we verify the second-order topological SF phase and characterize it by the Majorana edge polarizations. To more thoroughly understand the emergence of Majorana corner states, we investigate this system through the low-energy edge theory. It is found that the cooperation of the OIPZE and $s$-wave pairing order could also leads to Dirac mass at adjacent edges possessing opposite signs. However, contrary to the in-plane Zeeman field in Refs. [77, 78], OIPZE determines the sign of Dirac mass along $y$ while not along $x$.

The remainder of this paper is organized as follows. In Sec. II, we introduce an attractive quantum spin-Hall insulator (QSHI) on a square lattice. In Sec. III, we study its ground states by mean-field theory and present its global phase diagram. In Sec. IV, we utilize Majorana edge polarizations to characterize the second-order topological superfluid in the phase diagram. In Sec. V, we explore the origin of the emergence of Majorana corner states by the effective low-energy edge theory. Finally, we draw discussions and conclusions in Sec. VI.
II. CORRELATED QUANTUM SPIN-HALL INSULATOR IN SQUARE LATTICES

We consider a two-component Fermi gas loaded into a square optical lattice as sketched in Fig. 1 (a). Its physics is described by the following effective Hamiltonian as

$$H_0(k) = -2t_1 \cos k_x \beta_x \alpha_0 - 2t_1 \cos k_y \beta_y \alpha_0 + m_0 \beta_z \alpha_0 + 4t_2 \sin k_x \sin k_y \beta_x \alpha_0 + \beta_z (\mathbf{h} \cdot \alpha)$$

under the basis $\hat{C}_k = (\hat{c}_{A,\uparrow,k}, \hat{c}_{B,\uparrow,k}, \hat{c}_{A,\downarrow,k}, \hat{c}_{B,\downarrow,k})^T$. Here, $t_1$ and $t_2$ denote the tunneling strength between nearest-neighbor and next-nearest-neighbor sites. $\alpha$ and $\beta$ represent Pauli matrices acting on spin ($\uparrow$, $\downarrow$) and sublattice ($A$, $B$) degrees of freedom, respectively. $\alpha_0$ denotes a two-by-two identity matrix. $m_0$ represents the staggered on-site potential at two sublattices. $\mathbf{h}$ is an in-plane Zeeman energy that are opposite at two sublattice sites. Hereafter, without loss of generality, we set $\mathbf{h} = (h_0, 0, 0)$ if not specified.

When $h_0 = 0$, the system preserves time reversal symmetry, and describes a quantum spin-Hall insulator in the band inverted region $t_2^4 > m_0^2/16$. This nontrivial phase is characterized by $Z_2$ topological invariant and hosts helical gapless edge states, as shown in Fig. 2 (a) and (b).

After turning on OIPZE $\mathbf{h}$, the time-reversal symmetry is broken for the system. However, the edge states remain gapless along $y$ while are gapped along $x$ as illustrated in Fig. 2 (c) and (d). This phenomenon originates from that OIPZE commutes with the nearest-neighbor term along $y$, but anti-commutes with the nearest-neighbor term along $x$. It would be more explicit from the effective low-energy edge theory. See details in Sec. V.

Consider the on-site attractive interaction for fermions defined by $\hat{H}_U = -U \sum_i n_i \gamma n_i$, and now the Hamiltonian for this correlated system becomes

$$\hat{H} = \sum_k \hat{C}_k H_0(k) \hat{C}_k + \hat{H}_U - \mu \sum_i n_i$$

where $\mu$ denotes the chemical potential, and $n_i = n_{i,\uparrow} + n_{i,\downarrow}$. In the following, we study the ground phases for this system through the mean-field theory and the effective low-energy theory.

III. PHASE DIAGRAM

As the attractive interaction increases, it is expected that fermions would be paired into a superfluid phase if the superfluid gap function exceeds the energy gap of single-particle Hamiltonian $H_0$. Therefore we introduce the mean-field ansatz for the superfluid order parameter as $\Delta_s = -U \langle \hat{c}_i^\dagger \hat{c}_i \rangle$. Through Fourier transformation, the kernel of the Hamiltonian in momentum space is written as

$$H(k) = \kappa_k \beta_z \alpha_0 \gamma_0 - 2t_1 \cos k_x \beta_x \alpha_0 \gamma_0 - 2t_1 \cos k_y \beta_y \alpha_z \gamma_0 + \Delta_s \beta_y \alpha_0 \gamma_0 - m_0 \beta_0 \alpha_0 \gamma_0 + h_0 \beta_z \alpha_0 \gamma_0$$

under the basis $\hat{\Psi}_k = (\hat{C}_k, \hat{C}_k^\dagger)^T$, where $\kappa_k = 4t_2 \sin k_x \sin k_y - m_0$, $\gamma$ represents Pauli matrices acting on particle-hole space, $\beta_0$ and $\gamma_0$ are two-by-two identity matrices. The free energy for the ground state at zero-temperature for the system is given by $F_u = -2 \sum_k (E_{k,+} + E_{k,-}) + 2N_u \Delta_s^2/U$, where $E_{k,\pm} = \sqrt{(\epsilon_k \pm h_0)^2 + (2t_1 \cos kx)^2}$ with $\epsilon_k =$
\( \sqrt{k_x^2 + (2t_x \cos ky)^2} \), and \( N_u \) is the number of unit cells. By minimizing the ground state energy \( F_u \) with respect to the superfluid order \( \Delta_s \), we obtain the self-consistent equation as

\[
\frac{U}{2N_s} \sum_k \left( \frac{\hbar_0/\zeta_k + 1}{E_{k,+}} - \frac{\hbar_0/\zeta_k - 1}{E_{k,-}} \right) = 1 \tag{4}
\]

with \( \zeta_k = \sqrt{k_x^2 + (2t_x \cos ky)^2 + \Delta_s^2} \). Here we consider the case with \( \mu = 0 \) for simplicity but without loss of generality. Through numeric calculations based on Eq. (4), we get the phase diagram for the system, as shown in Fig. 1(b). It shows that as attractive interaction increases from zero, the system remains an insulator below the critical interaction values. When the attractive interaction exceeds critical values, the system enters the superfluid phase.

We next investigate topological properties of superfluid phases. OIPZE plays different roles on the helical edge states (HES) of quantum spin Hall insulators. We would like to see the response of gapless HES to the combination of OIPZE and superfluid order. We first plot energy spectra for a strip under periodic boundary conditions along \( x \) (y) but open along \( y \) (x) with increasing OIPZE and fixed superfluid order parameter \( \Delta_s = 0.2 \), as shown in Fig. 3. It showcases that when \( h_0 = 0 \), the edge states along both \( x \) and \( y \) are gaped by the superfluid order. As illustrated in Figs. 3(a)-(c), the gap for edge states along \( x \) decreases gradually as OIPZE increases, closes at \( h_0 = \Delta_s \) and reopens when \( h_0 > \Delta_s \). However, the gap for edge states along \( y \) remains gaped in this process, as shown in Figs. 3(d)-(f). This implies that a topological phase transition from a trivial to a nontrivial phase may occur for edges states along \( x \) while no topological phase transition occurs for edge states along \( y \).

To demonstrate that the system is in a topological phase when \( h_0 > \Delta_s \), we compute energies of the superfluid on a square lattice under open boundary conditions along \( x \) and \( y \), as presented in the inset of Fig. 5(a). It shows that there are four Majorana zero-energy states in the energy gap. Their density distributions, as shown in Fig. 5(a), indicate that they are localized at four corners of the sample, which implies that this superfluid phase is a second-order topological superfluid (SSF) hosting Majorana corner states. Therefore, to summarize, the SF phase consists of a SSF and a normal SF, as shown in the phase diagram in Fig. 1(b).

**IV. TOPOLOGICAL INVARIANTS**

We utilize Majorana edge polarizations to characterize topological properties for this second-order topological superfluid [77]. We first construct Wilson loop \( W = F_{k_y + 2\pi i} \cdots F_{k_y} F_{k_y - 2\pi i} F_{k_y} \), where \( F_{k_y} \) is a \( 4N_y \)-by-\( 4N_y \) matrix with the matrix element \( \langle F_{k_y} \rangle_{mn} = \langle u^m_{k_y + \Delta k_y} | u^n_{k_y} \rangle \), where \( | u^n_{k_y} \rangle \) is \( n \)-th occupied Bloch wave function with eigen-energy \( E_n \), \( N_u \) is the site number along \( y \), and the momentum interval \( \Delta k_y = 2\pi/N_y \) with \( N_y \) the number of unit cells. The Wannier Hamiltonian for \( W \) is then defined by

\[
\mathcal{H} = \frac{1}{2\pi i} \ln W. \tag{5}
\]

The eigenvalues of \( \mathcal{H} \) are Wannier values \( \nu_k \) corresponding to eigenfunctions \( | \nu_k \rangle \). By using Bloch wave functions, Wannier values and corresponding eigenfunctions, we define Majorana edge polarization along \( y \) as \( \rho_{y}^{edge,x} = ...F_{k_y + 2\pi i} \cdots F_{k_y} F_{k_y - 2\pi i} F_{k_y} \), where \( F_{k_y} \) is a \( 4N_y \)-by-\( 4N_y \) matrix with the matrix element \( \langle F_{k_y} \rangle_{mn} = \langle u^m_{k_y + \Delta k_y} | u^n_{k_y} \rangle \), where \( | u^n_{k_y} \rangle \) is \( n \)-th occupied Bloch wave function with eigen-energy \( E_n \), \( N_u \) is the site number along \( y \), and the momentum interval \( \Delta k_y = 2\pi/N_y \) with \( N_y \) the number of unit cells. The Wannier Hamiltonian for \( W \) is then defined by

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\[
\mathcal{H} = \frac{1}{2\pi i} \ln W. \tag{5}
\]
\[ \sum_{i=1}^{N_y/2} \frac{1}{N_x} \sum_{k_x,m,j} \nu_j^x \left( \langle \nu_j^x | u_{k_x}^m | v_{k_x}^m \rangle \langle v_{k_x}^m | \nu_j^x \rangle \right) . \]  

(6)

Majorana edge polarization \( p_y^{edge,x} \) along \( x \) takes similar formulation as \( p_y^{edge,x} \).

In SSF phase, two quantized Wannier values \( \nu^x = 0.5 \) emerge in the gap of Wannier band \( \nu^x \) while not in the Wannier band \( \nu^y \), as shown in Fig. 4(a) and (b). It shows that edge states along \( x \) exhibit non-trivial topological properties. We next further compute Majorana edge polarizations on each site, as plotted in Fig. 4(c) and (d). Obviously, nontrivial Majorana edge polarization exists for edge states along \( x \) while not for edge states along \( y \). Through numeric calculations, we obtain the topological invariants \( (p_y^{edge,x}, p_y^{edge,y}) = (1/2, 0) \) for SSF phase. While for NSF phase, the topological invariant becomes \( (0, 0) \), as shown in Fig. 4(e). From the perspective of Majorana edge polarizations, the emergence of Majorana corner states could be simply understood as follows. Half quantized Majorana edge polarization along \( x \) implies that edge states along \( x \) are topologically nontrivial. Yet, zero Majorana edge polarization along \( y \) indicates that edge states along \( y \) are topologically trivial. When trivial and nontrivial edge states encounter at a corner, Majorana zero-energy corner state naturally arises from the index theorem. Finally, with the consideration of the topological phase transition in the superfluid phase, we plot the global phase diagram in Fig. 1(b).

To show the robustness of Majorana corner states, we impose random fluctuations on chemical potentials. The energies of a square sample in the presence of random fluctuations with different amplitudes have been shown in 5(b). It showcases Majorana corner states are robust against perturbations as long as the edge gap remains open.

V. EFFECTIVE LOW-ENERGY EDGE THEORY

In this section, we will explore the emergence of Majorana corner states form low-energy edge theory. The low-energy Hamiltonian in the continuum limit around \( K = (\pi/2, \pi/2) \) up to the second order is written as

\[ H_{LE}(k) = (\varepsilon - 2t_2k_x^2 - 2t_2k_y^2) \beta_2 \alpha_0 \gamma_z + 2t_1 k_x \beta_2 \alpha_0 \gamma_z \\
+ 2t_1 k_y \beta_2 \alpha_0 \gamma_z + \Delta_x \beta_0 \alpha_y \gamma_y + h_0 \beta_2 \alpha_x \gamma_z. \]  

(7)

with \( \varepsilon = 4t_2 - m_0 \). As illustrated in Fig. 5(a), the four edges of a square lattice are labeled by I, II, III and IV. For edge I, we replace the momentum operator \( k_x \) by \(-i\partial_x\). The Hamiltonian then becomes \( H_{LE}(k) = H_{M,1} + H_{P,1} \), where the main term \( H_{M,1} \) and perturbation term \( H_{P,1} \) are given by

\[ H_{M,1}(-i\partial_x, k_y) = (\varepsilon + 2t_2 \partial_x^2) \beta_2 \alpha \gamma_z - 2it_1 \partial_x \beta \alpha_0 \gamma_z, \]

\[ H_{P,1}(-i\partial_x, k_y) = -t_1 k_x \beta_2 \alpha_0 \gamma_z + 2t_1 k_y \beta_2 \alpha \gamma_z + \Delta_x \beta_0 \alpha \gamma_y \\
+ h_0 \beta_2 \alpha \gamma_z. \]  

(8)

Here we have assumed that the pairing order compared to the energy gap is relatively small.

In the following, we first solve the main part \( H_{M,1} \), and then derive the effective low-energy edge Hamiltonian for edge I. We assume \( H_{M,1} \) has zero energy solutions \( \psi_n \) localized at edge I. Since \( \{H_{M,1}, \beta_2 \alpha \gamma_z \} = 0 \), \( \beta_2 \alpha \gamma_z \psi_m \) are also eigen-states for \( H_{M,1} \). Therefore we choose eigenvectors \( \chi_m \) with \( \beta_2 \alpha \gamma_z \chi_m = -\chi_m \)

\[ \chi_1 = |\beta_2 = -1 \rangle |\alpha = +1 \rangle |\gamma_z = +1 \rangle, \]

\[ \chi_2 = |\beta_2 = +1 \rangle |\alpha = -1 \rangle |\gamma_z = +1 \rangle, \]

\[ \chi_3 = |\beta_2 = +1 \rangle |\alpha = +1 \rangle |\gamma_z = -1 \rangle, \]

\[ \chi_4 = |\beta_2 = -1 \rangle |\alpha = -1 \rangle |\gamma_z = -1 \rangle. \]  

(9)

Four zero-energy states \( \psi_m = 1, 2, 3, 4 \) localize at edge I in
this basis with \( \psi_m = N \sin \vartheta e^{-i \frac{\epsilon}{2} z} \chi_m \), where \( N \) is a normalization constant, and \( \vartheta = \sqrt{\epsilon/(2t_2) - \tau_1^2/(4t_2^2)} \). In the basis \( \psi_m \), the effective low-energy edge Hamiltonian for the perturbation term \( H_{P,1} \) then becomes

\[
H_{\text{Edge},1} = i t y_\alpha y_0 \partial_x + \Delta_\alpha \alpha y_y + h_0 \alpha \alpha z. \tag{10}
\]

The low-energy effective Hamiltonian at edges II, III, IV takes similar formulations as

\[
H_{\text{Edge},II} = -i t_1 \alpha y_0 \partial_x + \Delta_\alpha \alpha y_y,
\]

\[
H_{\text{Edge},III} = -i t_1 \alpha y_0 \partial_y + \Delta_\alpha \alpha y_y + h_0 \alpha \alpha z,
\]

\[
H_{\text{Edge},IV} = i t_1 \alpha y_0 \partial_x + \Delta_\alpha \alpha y_y. \tag{11}
\]

To summarize, we have

\[
H_{\text{Edge}} = i t_1 \alpha y_0 \partial_x + \Delta_\alpha \alpha y_y + h(m) \alpha \alpha z, \tag{12}
\]

with \( h(m) = h_0, 0, h_0, 0 \) at edges \( m = I, II, III, IV \), respectively. To be more explicit, we rewrite \( H_{\text{Edge}} \) as

\[
H'_{\text{Edge}} = -i t_1 \alpha y_0 \partial_x + \Delta_\alpha \alpha \tau_x + h(m) \alpha \alpha y_0 \tag{13}
\]

in the space \( \eta_1 = |\alpha z = +1\rangle |\gamma_x = +1\rangle, \eta_2 = |\alpha z = +1\rangle |\gamma_x = +1\rangle \). Now \( H'_{\text{Edge}} \) is composed of two 2-by-2 decoupled blocks, one block with Dirac mass \( h(m) + \Delta_\alpha \), and the other with \( h(m) - \Delta_\alpha \). When Dirac masses on adjacent edges have opposite signs, i.e., \( (h_0 + \Delta_\alpha) \Delta_\alpha < 0 \) or \( (h_0 - \Delta_\alpha) \Delta_\alpha < 0 \). Majorana zero-energy state naturally arises at the corner.

If a general OIPZE \( h = (h_x, h_y, 0) \) is applied, the term \( h(m) \alpha x \gamma_0 \), in Eq.(12) would be replaced by another term \( h(m) \tilde{\alpha} \gamma_0 \), where \( \{\alpha, \tilde{\alpha}\} = 0 \) and \( h(m) = h, 0, h, 0 \) with \( h = |h| \) at edges I – IV, respectively. Majorana zero states naturally emerge when \( (h + \Delta_\alpha) \Delta_\alpha < 0 \) or \( (h - \Delta_\alpha) \Delta_\alpha < 0 \). Hence, with the assistance of s-wave superfluid order and appropriate OIPZE at two sublattice sites, Dirac mass of edge states on adjacent edges possesses opposite signs which naturally induces Majorana zero-energy states at corners.

VI. DISCUSSION AND CONCLUSION

The ultracold atom system provides a highly controllable platform with various degrees of freedom. The quantum spin Hall insulator in Eq.(2) could be implemented by a spin-dependent optical lattice [96, 97], and the OIPZE can be reached by another addressing laser. The attractive interaction can be tuned by Feshbach resonance technique [87, 88]. Using the generalized Bragg spectroscopy and the expansion of the atomic cloud [98, 99], one may detect the induced Majorana corner states.

In summary, in contrast to previous schemes, we propose an OIPZE can drive a second-order topological superfluid phase in an attractive quantum spin Hall insulator. Here, the cooperation of OIPZE and the s-wave superfluid order leads to the opposite signs of Dirac mass at adjacent edges. Majorana zero states naturally emerge localized at corners. They are robust against weak perturbations as long as the edge energy gap remains open. This topological phase can be characterized by the Majorana edge polarizations. Our work provides a new route towards Majorana corner states through the cooperation of the OIPZE and the s-wave pairing.

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[1] J. Langbehn, Y. Peng, L. Trifunovic, F. V. Oppen, and P. W. Brouwer, Reflection-Symmetric Second-Order Topological Insulators and Superconductors, Phys. Rev. Lett. 119, 246401 (2017).
[2] Z. Yan, F. Song, and Z. Wang, Majorana corner modes in a high-temperature platform, Phys. Rev. Lett. 121, 096803 (2018).
[3] Y. Wang, C. C. Liu, Y. M. Lu, and F. Zhang, High-Temperature Majorana Corner States, Phys. Rev. Lett. 121, 186801 (2018).
[4] X. Zhu, Tunable Majorana corner states in a two-dimensional second-order topological superconductor in
duced by magnetic fields, Phys. Rev. B \textbf{97}, 205134 (2018).
[5] Y. X. Wang, M. Lin, and T. L. Hughes, Weak-pairing higher order topological superconductors, Phys. Rev. B \textbf{98}, 165144 (2018).
[6] C.-H. Hsu, P. Stano, J. Klinovaja, and D. Loss, Majorana Kramers Pairs in Higher-Order Topological Insulators, Phys. Rev. Lett. \textbf{121}, 196801 (2018).
[7] T. Liu, J. J. He, and F. Nori, Majorana corner states in a two-dimensional magnetic topological insulator on a high-temperature superconductor, Phys. Rev. B \textbf{98}, 245413 (2018).
[8] C. Zeng, T. D. Stanescu, C. Zhang, V. W. Scarola, and S. Tewari, Majorana corner modes with solitons in an attractive Hubbard-Hofstadter model of cold atom optical lattices, Phys. Rev. Lett. \textbf{123}, 060402 (2019).
[9] K. Plekhanov, M. Thakurathi, D. Loss, and J. Klinovaja, Floquet second-order topological superconductor driven via ferromagnetic resonance, Phys. Rev. Research \textbf{1}, 032013(R) (2019).
[10] N. Bultinck, B. A. Bernevig, and M. P. Zaletel, Three-dimensional superconductors with hybrid higher-order topology, Phys. Rev. B \textbf{99}, 125149 (2019).
[11] Y. Peng and Y. Xu, Proximity-induced Majorana hinge modes in antiferromagnetic topological insulators, Phys. Rev. B \textbf{99}, 195431 (2019).
[12] S. A. A. Ghornshi, X. Hu, T. L. Hughes, and E. Rossi, Second-order Dirac superconductors and magnetic field induced Majorana hinge modes, Phys. Rev. B \textbf{100}, 020509(R) (2019).
[13] Y. Volpez, D. Loss, and J. Klinovaja, Second-Order Topological Superconductivity in $\pi$-Junction Rashba Layers, Phys. Rev. Lett. \textbf{122}, 126402 (2019).
[14] R.-X. Zhang, W. S. Cole, and S. Das Sarma, Helical Hinge Majorana Modes in Iron-Based Superconductors, Phys. Rev. Lett. \textbf{122}, 187001 (2019).
[15] S. Franca, D. V. Efremov, and I. C. Fulga, Phase-tunable second-order topological superconductor, Phys. Rev. B \textbf{100}, 075415 (2019).
[16] K. Laubscher, D. Loss, and J. Klinovaja, Fractional topological superconductivity and parafermion corner states, Phys. Rev. Research \textbf{1}, 032017(R) (2019).
[17] M. J. Park, Y. Kim, G. Y. Cho, and S. B. Lee, Higher-Order Topological Superconductivity in Twisted Bilayer Graphene, Phys. Rev. Lett. \textbf{123}, 216803 (2019).
[18] J. Niu, T. Yan, Y. Zhou, Z. Tao, X. Li, W. Liu, L. Zhang, S. Liu, Z. Yan, Y. Chen, D. Yu, Simulation of Higher-Order Topological Phases and Related Topological Phase Transitions in a Superconducting Qubit, \textcolor{red}{arXiv:2001.03933} (2020).
[19] R.-X. Zhang, J. D. Sau, S. Das Sarma, Kitaev Building-block Construction for Higher-order Topological Superconductors, \textcolor{red}{arXiv:2003.02559} (2020).
[20] A. Tiwari, A. Jahin, and Y. Wang, Chiral Dirac superconductors: Second-order and boundary-obstructed topology, Phys. Rev. Research \textbf{2}, 043300 (2020).
[21] A. D. Fedoseev, Corner excitations in the 2D triangle-shaped topological insulator with chiral superconductivity on the triangular lattice, J. Phys.: Condens. Matter \textbf{32}, 40 (2020).
[22] Y.-B. Wu, G.-C. Guo, Z. Zheng, and X.-B. Zou, Effective Hamiltonian with tunable mixed pairing in driven optical lattices, Phys. Rev. A \textbf{101}, 013622 (2020).
[23] Y.-B. Wu, G.-C. Guo, Z. Zheng, and X.-B. Zou, Boundary-Obstructed Topological Superfluids in Staggered Spin-Orbit Coupled Fermi Gases, \textcolor{red}{arXiv:2007.15886} (2020).
[24] B. Roy, Higher-order topological superconductors in $\mathcal{P}$-$\mathcal{T}$-odd quadrupolar Dirac materials, Phys. Rev. B \textbf{101}, 220506(R) (2020).
[25] M. Kheirkhah, Z. B. Yan, Y. Nagai, and F. Marsiglio, First- and Second-Order Topological Superconductivity and Temperature-Driven Topological Phase Transitions in the Extended Hubbard Model with Spin-Orbit Coupling, Phys. Rev. Lett. \textbf{125}, 017001 (2020).
[26] R.-X. Zhang, Y.-T. Hsu, and S. Das Sarma, Higher-order topological Dirac superconductors, Phys. Rev. B \textbf{102}, 094503 (2020).
[27] M. Ezawa, Edge-corner correspondence: Boundary-obstructed topological phases with chiral symmetry, Phys. Rev. B \textbf{102}, 121405(R) (2020).
[28] K. Laubscher, D. ChuhtaI, D. Loss, and J. Klinovaja, Kramers pairs of Majorana corner states in a topological insulator bilayer, Phys. Rev. B \textbf{102}, 195401 (2020).
[29] Y.-T. Hsu, W. S. Cole, R.-X. Zhang, and J. D. Sau, Inversion-Protected Higher-Order Topological Superconductivity in Monolayer WTe$_2$, Phys. Rev. Lett. \textbf{125}, 097001 (2020).
[30] S.-B. Zhang and B. Trauzettel, Detection of second-order topological superconductors by Josephson junctions, Phys. Rev. Research \textbf{2}, 012018(R) (2020).
[31] J. Ahn and B.-J. Yang, Higher-order topological superconductivity of spin-polarized fermions, Phys. Rev. Research \textbf{2}, 012060(R) (2020).
[32] M. Kheirkhah, Z. Yan, Y. Nagai, and F. Marsiglio, First- and second- order topological superconductivity and temperature-driven topological phase transitions in the extended Hubbard model with spin-orbit coupling, Phys. Rev. Lett. \textbf{125}, 017001 (2020).
[33] X. Wu, W. A. Benalcazar, Y. Li, R. Thomale, C.-X. Liu, and J. Hu, Boundary-Obstructed Topological High-$T_c$ Superconductivity in Iron Pnictides, Phys. Rev. X \textbf{10}, 041014 (2020).
[34] K. Laubscher, D. Loss, and J. Klinovaja, Majorana and parafermion corner states from two coupled sheets of bilayer graphene, Phys. Rev. Research \textbf{2}, 013330 (2020).
[35] R. W. Bomantara, Time-induced second-order topological superconductors, Phys. Rev. Research \textbf{2}, 033495 (2020).
[36] M. Kheirkhah, Y. Nagai, C. Chen, and F. Marsiglio, Majorana corner flat bands in two-dimensional second-order topological superconductors, Phys. Rev. B \textbf{101}, 104502 (2020).
[37] M. Kheirkhah, Z. Yan, and F. Marsiglio, Vortex-line topology in iron-based superconductors with and without second-order topology, Phys. Rev. B \textbf{103}, L14052 (2021).
[38] K. Plekhanov, N. Müller, Y. Volpez, D. M. Kennes, H. Schoeller, D. Loss, and J. Klinovaja, Quadrupole spin polarization as signature of second-order topological superconductors, Phys. Rev. B \textbf{103}, L041401 (2021).
[39] A. K. Ghosh, T. Nag, and A. Saha, Floquet generation of a second-order topological superconductor, Phys. Rev. B \textbf{103}, 045424 (2021).
[40] W. B. Rui, Song-Bo Zhang, Moritz M. Hirschmann, Zhen Zheng, Andreas P. Schnyder, Björn Trauzettel, and Z. D. Wang, Higher-order Weyl superconductors with anisotropic Weyl-point connectivity, Phys. Rev. B \textbf{103}, 184510 (2021).
[41] B. Fu, Z.-A. Hu, C.-A. Li, J. Li, and S.-Q. Shen, Chiral Majorana hinge modes in superconducting Dirac materials, Phys. Rev. B 103, L180504 (2021).

[42] Y. B. Wu, G.C. Guo, Z. Zheng, and X.B. Zou, Multi-order topological superfluid phase transitions in a two-dimensional optical superlattice, Phys. Rev. A 104, 013306 (2021).

[43] S. Ikegaya, W. B. Rui, D. Manske, and A. P. Schnyder, Tunable Majorana corner modes in noncentrosymmetric superconductors: Tunneling spectroscopy and edge imperfections, Phys. Rev. Research 3, 023007 (2021).

[44] A. Jahin, A. Tiwari, and Y. Wang, Higher-order topological superconductors from Weyl semimetals, arXiv:2103.05010 (2021).

[45] S. Qin, C. Fang, F.-C. Zhang, J. Hu, Topological Superfluidity in an s-wave Superconductor and Its Implication to Iron-based Superconductors, arXiv:2106.04200 (2021).

[46] Y. Tan, Z. H. Huang, X. J. Liu, Edge geometric phase mechanism for second-order topological insulator and superconductor, arXiv:2106.12507 (2021).

[47] M. Kheirkhah, Z. Y. Zhuang, J. Maciejko, Z. B. Yan, Surface Majorana Cones and Helical Majorana Hinge Modes in Superconducting Dirac Semimetals, arXiv:2107.02811 (2021).

[48] A. Chew, Y. J. Wang, B. A. Bernevig, Z. D. Song, Higher-Order Topological Superconductivity in Twisted Bilayer Graphene, arXiv:2108.05373 (2021).

[49] A. K. Ghosh, T. Nag, and A. Saha, Hierarchy of higher-order topological superconductors in three dimensions, Phys. Rev. B 104, 134508 (2021).

[50] L. Yang, A. Principi, N. R. Walet, Rotating Majorana Zero Modes in a disk geometry, arXiv:2109.03549 (2021).

[51] Y.-J. Wu, X.-W. Luo, J. Hou, and C. Zhang, Majorana corner pairs in a two-dimensional s-wave cold atomic superfluid, Phys. Rev. A 103, 013307 (2021).

[52] B.-X. Li and Z. Yan, Boundary topological superconductors, Phys. Rev. B 103, 064512 (2021).

[53] A. O. Zlotnikov, M. S. Shustin, and A. D. Fedoseev, Aspects of topological superconductivity in 2D systems: noncollinear magnetism, skyrmions, and higher-order topology, J. Supercond. Nov. Magn. 34, 3053 (2021).

[54] Z. Li, S. Qin, and C. Fang, Theory of Topological Superconductivity in Doped IV-VI Semiconductors, arXiv:2108.05780 (2021).

[55] X. Wang, T. Zhou, Fragile topology in nodal-line semimetal superconductors, arXiv:2106.06928 (2021).

[56] X.-J. Luo, X.-H. Pan, and X. Liu, Higher-order topological superconductors based on weak topological insulators, Phys. Rev. B 104, 104510 (2021).

[57] Y.-B. Wu, G.-C. Guo, Z. Zheng, and X.-B. Zou, Multi-order topological superfluid phase transitions in a two-dimensional optical superlattice, Phys. Rev. A 104, 013306 (2021).

[58] Y. Tan, Z.-H. Huang, and X.-J. Liu, Two-particle Berry phase mechanism for Dirac and Majorana Kramers pairs of corner modes, Phys. Rev. B 105, L041105 (2022).

[59] S. D. Sarma, M. Freedman, C. Nayak, Majorana zero modes and topological quantum computation, npj Quantum Inf. 1, 15001 (2015).

[60] T. E. Palhami, M. Sigrist, and A. A. Soluyanov, Braiding Majorana corner modes in a second-order topological superconductor, Phys. Rev. Research 2, 032068(R) (2020).

[61] R. Song, P. Zhang, and N. Hao, Phase-Manipulation-Induced Majorana Mode and Braiding Realization in Iron-Based Superconductor Fe(Se,Te), Phys. Rev. Lett. 128, 016402 (2021).

[62] M. F. Lapa, M. Cheng, Y. Wang, Symmetry-protected gates of Majorana qubits in a high-Tc, higher-order topological superconductor platform, SciPost Phys. 11, 086 (2021).

[63] X.-H. Pan, X.-J. Luo, J.-H. Gao, Xin Liu, Braiding higher-order Majorana corner states through their spin degree of freedom, arXiv:2111.12359 (2021).

[64] M. Amundsen, V. Juričić, Controlling Majorana modes by p-wave pairing in two-dimensional p + id topological superconductors, arXiv:2108.09338 (2021).

[65] R.-X. Zhang, W. S. Cole, X. Wu, and S. Das Sarma, Higher-Order Topology and Nodal Topological Superconductivity in Fe(Se,Te) Heterostructures, Phys. Rev. Lett. 123, 167001 (2019).

[66] Z. Wu, Z. Yan, and W. Huang, Higher-order topological superconductivity: Possible realization in Fermi gases and Sr$_2$RuO$_4$, Phys. Rev. B 99, 020508(R) (2019).

[67] X. Zhu, Second-Order Topological Superconductors with Mixed Pairing, Phys. Rev. Lett. 122, 236401 (2019).

[68] M. J. Gray, J. Frederstein, S. Yang, F. Zhao, R. O’Connor, S. Jenkins, N. Kumar, M. Hoek, A. Kopeck, S. Huh, T. Taniguchi, K. Watanabe, R. Zhong, C. Kim, G. D. Gu and K. S. Burch, Evidence for Helical Hinge Zero Modes in an Fe-Based Superconductor, Nano Lett. 19, 4890 (2019).

[69] H. Shapourian, Y. Wang, and S. Ryu, Topological crystalline superconductivity and second-order topological superconductivity in nodal-loop materials, Phys. Rev. B 97, 094508 (2018).

[70] X.-H. Pan, K.-J. Yang, L. Chen, G. Xu, C.-X. Liu, and X. Liu, Lattice-Symmetry-Assisted Second-Order Topological Superconductors and Majorana Patterns, Phys. Rev. Lett. 123, 156801 (2019).

[71] B. Huang, G. Luo, and N. Xu, Mirror-symmetry-protected topological superfluid and second-order topological superfluid in bilayer fermionic gases with spin-orbit coupling, Phys. Rev. A 100, 023602 (2019).

[72] Y. Peng, Floquet higher-order topological insulators and superconductors with space-time symmetries, Phys. Rev. Research 2, 013124 (2020).

[73] D. D. Vu, R.-X. Zhang, and S. Das Sarma, Time-reversal-invariant C$_2$-symmetric higher-order topological superconductors, Phys. Rev. Research 2, 043223 (2020).

[74] Y.-X. Li and T. Zhou, Rotational symmetry breaking and partial Majorana corner states in a heterostructure based on high-Tc superconductors, Phys. Rev. B 103, 024517 (2021).

[75] E. Khalaf, Higher-order topological insulators and superconductors protected by inversion symmetry, Phys. Rev. B 97, 205136 (2018).

[76] M. Geier, P. W. Brouwer, and L. Trifunovic, Symmetry-based indicators for topological Bogoliubov–de Gennes Hamiltonians, Phys. Rev. B 101, 245128 (2020).

[77] Y. J. Wu, J. P. Hou, Y. M. Li, X. W. Luo, X. Y. Shi, and C. W. Zhang, In-Plane Zeeman-Field-Induced Majorana Corner and Hinge Modes in an s-Wave Superconductor Heterostructure, Phys. Rev. Lett. 124, 227001 (2020).

[78] Y.-J. Wu, T.-B. Gao, N. Li, J. Zhou and S.-P. Kou, Majorana corner modes in an s-wave second order topological superfluid, J. Phys.: Condens. Matter 32, 145601 (2020).
P. Hauke, O. Tieleman, A. Celi, C. Ölschläger, J. Simonet, J. Struck, M. Weinberg, P. Windpassinger, K. Sengstock, M. Lewenstein, and A. Eckardt, Non-abelian gauge fields and topological insulators in shaken optical lattices, Phys. Rev. Lett. 109, 145301 (2012).

M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Realization of the Hofstadter Hamiltonian with ultracold atoms in optical lattices, Phys. Rev. Lett. 111, 185301 (2013).

H. Miyake, G. A. Siviloglou, C. J. Kennedy, W. Cody Burton, and W. Ketterle, Realizing the Harper Hamiltonian with laser-assisted tunneling in optical lattices, Phys. Rev. Lett. 111, 185302 (2013).

A. Celi, P. Massignan, J. Ruseckas, N. Goldman, I. B. Spielman, G. Juzeliūnas, and M. Lewenstein, Synthetic gauge fields in synthetic dimensions, Phys. Rev. Lett. 112, 043001 (2014).

N. Goldman, G. Juzeliūnas, P. Ohberg, and I. B. Spielman, Light-induced gauge fields for ultracold atoms, Rep. Prog. Phys. 77, 12 (2014).

M. Aidelsburger, M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, S. Nascimbène, N. R. Cooper, I. Bloch, N. Goldman, Measuring the Chern number of Hofstadter bands with ultracold bosonic atoms, Nat. Phys. 11, 162 (2015).

L. Huang, Z. Meng, P. Wang, P. Peng, S.-L. Zhang, L. Chen, D. Li, Q. Zhou, and J. Zhang, Experimental realization of two-dimensional synthetic spin-orbit coupling in ultracold Fermi gases, Nature Phys. 12, 540 (2016).

Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Ji, Y. J. Deng, S. Chen, X.-J. Liu and J.-W. Pan, Realization of two-dimensional spin-orbit coupling for Bose-Einstein condensates, Science 354, 83 (2016).

C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Feshbach resonances in ultracold gases, Rev. Mod. Phys. 82, 1225 (2010).

T. Köhler, K. Góral, Production of cold molecules via magnetically tunable Feshbach resonances, Rev. Mod. Phys. 78, 1311 (2006).

G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif and T. Esslinger, Experimental realization of the topological Haldane model with ultracold fermions, Nature 515, 237 (2014).

Z.-Y. Wang, X.-C. Cheng, B.-Z. Wang, J.-Y. Zhang, Y.-H. Lu, C.-R. Yi, S. Niu, Y. Deng, X.-J. Liu, S. Chen, and J.-W. Pan, Realization of an ideal Weyl semimetal band in a quantum gas with 3D spin-orbit coupling, Science 372, 6539 (2021).

N. Goldman, J. C. Budich and P. Zoller, Topological quantum matter with ultracold gases in optical lattices, Nat. Phys. 12, 639 (2016).

Flächner, B. S. Rem, M. Tarnowski, D. Vogel, D.-S. Lührmann, K. Sengstock, and C. Weitenberg, Experimental reconstruction of the Berry curvature in a Floquet Bloch band, Science 352, 1091 (2016).

A. Eckardt, Colloquium: Atomic quantum gases in periodically driven optical lattices, Rev. Mod. Phys. 89, 011004 (2017).

M. Lohse, C. Schweizer, H. M. Price, O. Zilberberg, I. Bloch, Exploring 4D quantum Hall physics with a 2D topological charge pump, Nature 553, 55 (2018).

B. Song, L. Zhang, C. He, T. F. J. Poon, E. Hajiyev, S. Zhang, X.-J. Liu, G.-B. Jo, Observation of symmetry-protected topological band with ultracold fermions, Sci. Adv. 4, eaa04748 (2018).

J.-M. Hou, Hidden-Symmetry-Protected Topological Semimetals on a Square Lattice, Phys. Rev. Lett. 111, 130403 (2013).

Y.-J. Wu, N. Li, J. He and S.-P. Kou, Antiferromagnetic order driven chiral topological spin density waves on the repulsive Haldane–Hubbard model on square lattices, J. Phys.: Condens. Matter 28, 115602 (2016).

N. Goldman, J. Beugnon, and F. Gerbier, Detecting Chiral Edge States in the Hofstadter Optical Lattice, Phys. Rev. Lett. 108, 255303 (2012).

N. Goldman, J. Dalibard, A. Dauphin, F. Gerbier, M. Lewenstein, P. Zoller, and I. B. Spielman, Direct imaging of topological edge states in cold-atom systems, PNAS 110, 6736 (2013).