GRADINGS ON THE KAC SUPERALGEBRA

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ABSTRACT. We describe the group gradings on the $K_{10}$ Jordan superalgebra. There are 21 nonequivalent gradings, two of them fine and 6 nontoral gradings.

1. INTRODUCTION

As a motivation for Lie superalgebras, the notion of supersymmetry in theoretical physics reflects the known symmetry between bosons and fermions. The mathematical structure formalizing this idea is that of supergroup, or $\mathbb{Z}_2$-graded Lie group. As mentioned in [11], their job is that of modelling continuous supersymmetry transformations between bosons and fermions. As Lie algebras consist of generators of Lie groups, the infinitesimal Lie group elements tangent to the identity, so $\mathbb{Z}_2$-graded Lie algebras, otherwise known as Lie superalgebras, consist of generators of (or infinitesimal) supersymmetry transformations. Closely related to these (for instance by the Kantor-Koecher-Tits construction) are the Jordan superalgebras (see [6]). We are primarily interested in $K_{10}$.

The debut of the simple exceptional Jordan superalgebra $K_{10}$ in the mathematical literature was in V. Kac’s classification of finite dimensional simple Jordan superalgebras over fields of characteristic zero [4]. Though one can introduce $K_{10}$ by giving its multiplication table relative to some basis, more conceptual approaches are possible. One of such convenient viewpoints is the given in [1]. Works as recent as [5] and [7] deal with the right definition of the algebra over arbitrary rings of scalars (agreeing with the usual $K_{10}$ when the base ring is an algebraically closed field of characteristic not 2). The Grassmann envelope of the algebra has been studied in [8]. The interest on group gradings of superalgebras seems to start with the work [12]. The paper [3] is essential for our study since it describes the automorphism group of $K_{10}$. Its relevance stems from the fact that, in our setting, gradings are just the simultaneous diagonalization relative to commuting sets of automorphisms.

2. PRELIMINARIES

Although this work the base field $F$ will be an algebraically closed field of zero characteristic. Let $J = J_0 \oplus J_1$ be a superalgebra over $F$. The term grading will always mean group grading, that is, a decomposition in vector subspaces $J = \bigoplus_{g \in G} J^g$ where $G$ is a finitely generated abelian group and the homogeneous spaces verify $J^g J^h \subset J^{gh}$ (denoting by juxtaposition the product in $G$). We assume also that $G$ is generated by the set of all $g$ such that $J^g \neq 0$, usually called the support of the grading, and that the grading is compatible with the grading $J = J_0 \oplus J_1$ of the superalgebra $J$. This means that any homogeneous component $J^i$ splits as $J^i = J_0^i \oplus J_1^i$ where $J_k^i := J^i \cap J_k$ for $k \in \{0, 1\}$.

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To distinguish the $\mathbb{Z}_2$-grading providing the superalgebra structure of $J$ from the rest of its possible gradings, we denote it with subscripts rather than with superscripts.

Given two gradings $J = \oplus_{g \in G} U^g$ and $J = \oplus_{h \in H} V^h$ we shall say that they are isomorphic if there is a group isomorphism $\sigma: G \to H$ and a (superalgebra) automorphism $\varphi: J \to J$ such that $\varphi(U^g) = V^{\sigma(g)}$ for all $g \in G$. We recall that a superalgebra automorphism $\varphi$ is just an automorphism such that the even and the odd part of the superalgebra are $\varphi$-invariant. The above two gradings are said to be equivalent if there are: (1) a bijection $\sigma: I \to I'$ between the supports of the first and second gradings respectively, and, (2) a superalgebra automorphism $\varphi$ of $J$ such that $\varphi(U^g) = V^{\sigma(g)}$ for any $g \in I$.

Consider the 10-dimensional $F$-algebra $K_{10}$ whose basis is $(e, v_1, v_2, v_3, v_4, f, x_1, x_2, y_1, y_2)$ with multiplication table:

| .   | $e$  | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $f$  | $x_1$ | $x_2$ | $y_1$ | $y_2$ |
|-----|------|-------|-------|-------|-------|------|-------|-------|-------|-------|
| $e$ | $e$  | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $f$  | $\frac{1}{2} x_1$ | $\frac{1}{2} x_2$ | $\frac{1}{2} y_1$ | $\frac{1}{2} y_2$ |
| $v_1$ | $v_1$ | 0     | 2$e$  | 0     | 0     | 0    | 0     | 0     | 0     | 0     |
| $v_2$ | $v_2$ | 2$e$  | 0     | 0     | 0     | 0    | 0     | 0     | 0     | 0     |
| $v_3$ | $v_3$ | 0     | 0     | 0     | 2$e$  | 0    | 0     | 0     | 0     | 0     |
| $v_4$ | $v_4$ | 0     | 0     | 0     | 0     | $2e$ | 0     | 0     | 0     | 0     |
| $f$  | 0    | 0     | 0     | 0     | 0     | 0    | $\frac{1}{2} x_1$ | $\frac{1}{2} x_2$ | $\frac{1}{2} y_1$ | $\frac{1}{2} y_2$ |
| $x_1$ | $\frac{1}{2} x_1$ | 0     | $-y_2$ | 0     | $x_2$ | $\frac{1}{2} x_1$ | 0     | $-v_1$ | 0     | 0     |
| $x_2$ | $\frac{1}{2} x_2$ | 0     | $y_1$ | $x_1$ | 0     | $\frac{1}{2} x_2$ | 0     | 0     | $v_4$ | 0     |
| $y_1$ | $\frac{1}{2} y_1$ | $x_2$ | 0     | $y_2$ | 0     | $-\frac{1}{2} y_1$ | 0     | 0     | 0     | $v_2$ |
| $y_2$ | $\frac{1}{2} y_2$ | $-x_1$ | 0     | 0     | $y_1$ | $\frac{1}{2} y_2$ | 0     | 0     | 0     | $-v_2$ |

This is a Jordan superalgebra called the Kac superalgebra, with even part generated by $(e, v_1, \ldots, v_4, f)$ and odd part generated by $(x_1, x_2, y_1, y_2)$. We shall denote it by $K_{10}$ as usual in the mathematical literature. We have used the basis introduced in [10] for this superalgebra, however a more conceptual approach is possible (see for instance [3].) In order to adhere to this alternative approach we shall need the Kaplansky superalgebra. This is the 3-dimensional Jordan superalgebra $K = K_0 \oplus K_1$, with $K_0 = F e$ and $K_1 F x \oplus F y$, and with multiplication given by $e^2 = e$, $e x = \frac{1}{2} x e$, $e y = \frac{1}{2} y e$, $x e = -y e$, $x^2 = y^2 = 0$. Following [3] we define on $K$ the following supersymmetric bilinear form: $(e|e) = \frac{1}{2}, (x|y) = 1, (K_0, K_1) = 0$, (it must be understood that the form is symmetric on $K_0$ and alternating on $K_1$.) Consider now the $F$-vector space over $F \cdot 1 \oplus (K \otimes F K)$ and define on it the product where

$$\langle a \otimes b \rangle (c \otimes d) = (-1)^{bc} (ac \otimes bd) - \frac{3}{4} (a, c)(b, d) \cdot 1$$

for $a, b, c, d \in K$ homogeneous elements, where for any homogeneous element $x \in K_0 \cup K_1$, $\bar{x} = 0$ if $x \in K_0$ and $\bar{x} = 1$ if $x \in K_1$. Then, a result of G. Benkart and A. Elduque states that $K_{10}$ is isomorphic to $F \cdot 1 \oplus (K \otimes K)$ with the above super-product, by means of the isomorphism given in [3, Theorem 2.1, p. 4]. Unless otherwise stated, we shall identify $K_{10}$ with the algebra whose product is (1). Under this identification the even part is $(1, e \otimes e, x \otimes x, x \otimes y, y \otimes x, y \otimes y)$ and the odd one is $(e \otimes x, x \otimes e, e \otimes y, y \otimes e)$.

In this work we shall have the occasion to consider automorphism groups of several Jordan superalgebras. These are, of course, linear algebraic groups whence some aspects of this theory will be used in the sequel. We must take into account that for a Jordan superalgebra $J = J_0 \oplus J_1$, the notation $\text{Aut}(J)$ will mean the group of all automorphisms (as above preserving the homogeneous components.) The gradings on the Kaplansky superalgebra $K$ may be computed rather straightforwardly from scratch but we prefer to develop some
algebraic group tools which can be applied not only to $K$ but to other algebras (alternative, Lie, Jordan, super-Jordan) of more complex nature. At this point there are two key aspects to mention:

- Any grading is induced by a finitely generated abelian subgroup of diagonalizable automorphisms of the automorphism group of the algebra under study. The homogeneous components are the simultaneous eigenspaces relative to the given group of automorphisms.
- Any such subgroup is contained in the normalizer of some maximal torus of the automorphism group of the algebra. This is an algebraic group version of the Borel-Serre theorem for Lie groups. It has been given by V. P. Platonov in Theorem 6 and Theorem 3.15, p. 92 of [9]: A supersoluble subgroup of semisimple elements of an algebraic group $G$ is contained in the normalizer of a maximal torus. Here we must recall that a group is called supersolvable (or supersoluble) if it has an invariant normal series whose factors are all cyclic. Any finitely generated abelian group is supersolvable. Since we are considering gradings on Jordan superalgebras over this class of abelian groups, we may assume that the set of automorphisms inducing the grading (as simultaneous diagonalization) is contained in the normalizer of a maximal torus of its automorphism group.

A special kind of gradings arises when we consider the inducing automorphisms not only in the normalizer of a maximal torus, but in the torus itself.

**Definition 1.** A grading of a superalgebra is said to be toral if it is produced by automorphisms within a torus of the automorphism group of the superalgebra.

Returning to the Kaplansky superalgebra $K$, it is straightforward that any element in $\text{Aut}(K)$ fixes $e$ so that $\text{Aut}(K)$ can be identified with a subgroup of $\text{GL}_2(F)$. Moreover taking into account that $f(x)f(y) = e$, we easily check that $\text{Aut}(K) \cong \text{SL}_2(F)$. We are denoting by $T$ the maximal torus of $\text{Aut}(K)$ (identified once and for all with $\text{SL}_2(F)$) consisting of all its diagonal matrices. This maximal torus is isomorphic to $F^\times$ and a generic element in $T$ will be denoted by $t_\lambda := \text{diag}(\lambda, \lambda^{-1})$ with $\lambda \in F^\times$. The normalizer of $T$ in $\text{SL}_2(F)$ is the quasitorus $\mathcal{N} = T \cup T \sigma$ where $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which corresponds to the automorphism of $K$ given by $x \mapsto y \mapsto -x$ (of order 4.) Thus, it is easy to see that $\mathcal{N} / T \cong \mathbb{Z}_2$ (since $\sigma^2 = t_{-1} \equiv -1 \in T$.)

**Proposition 1.** Any abelian subgroup of $\mathcal{N}$ is toral.

**Proof.** It is essential the fact that $\sigma t = t^{-1} \sigma$ for any $t \in T$. Let $S$ be a nontrivial abelian subgroup of $\mathcal{N}$. It $S \subseteq T$ we are done. On the contrary, since $S \not\subseteq T \sigma$, we must have toral elements $t \in T \cap S$ and also elements $t' \sigma \in T \sigma \cap S$. Since they must commute we can write $t' \sigma = t' \sigma t = t' t^{-1} \sigma$, from which we get $t^2 = 1$ and hence $t = \pm 1$. So $S \cap T \subset \{ \pm 1 \}$. On the other hand, if $S$ contains two elements $t_1 \sigma$ and $t_2 \sigma$, from the fact that they commute one gets $t_1 = \pm t_2$. Thus $S \subset \{ \pm 1, \pm t \sigma \}$ for some $t \in T$. But for any $t \in T$ there is some $t_1 \in T$ such that $t_1 \sigma t_1^{-1} = \sigma$ (take $t_1^2 = t$.) So by conjugating $S$ with $t_1$ we have $t_1 S t_1^{-1} \subset \{ \pm 1, \pm t \sigma \}$, but this set is toral since $t$ is toral: for some $p \in \text{SL}_2(F)$ we have $p \sigma p^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t_1$ (so $p \{ \pm 1, \pm \sigma \} p^{-1} \subset T$.)

**Corollary 1.** All the gradings on the Kaplansky superalgebra $K$ are toral. Up to equivalence the nontrivial ones are the following:

- The $\mathbb{Z}_2$-grading $K = K_0 \oplus K_1$ providing its superalgebra structure.
satisfies

\[ K = K^{-1} \oplus K^0 \oplus K^1 \text{ such that } K^{-1} = Fx, K^0 = Fe \text{ and } K^1 = Fy. \]

Proof. Consider a minimal set \( S \neq \{1\} \) of diagonalizable automorphisms inducing the grading. Since any two maximal tori of the automorphism group are conjugated, we may assume \( S \subset T \). Now, if \( S \) contains some \( t_\lambda \) with \( \lambda \neq \pm 1 \), we get the fine \( \mathbb{Z}_2 \)-grading. On the contrary, \( S = \langle t_{-1} \rangle \) induces the \( \mathbb{Z}_2 \)-grading.

3. Grading on \( K_{10} \)

Now we develop a similar program for the \( K_{10} \) superalgebra. Since our study depends heavily on the knowledge of \( \text{Aut}(K_{10}) \), we must return to the reference [3] which gives full details on this group. Firstly, if we take \( f, g \in \text{Aut}(K) \) then we may define an automorphism of \( K_{10} \) (denoted \( (f, g) \)) such that \( (f, g): x \otimes y \mapsto f(x) \otimes g(y) \). Thus we have a group monomorphism of \( \text{Aut}(K)^2 \) to \( \text{Aut}(K_{10}) \). But as stated in [3] there is an automorphism \( \delta \) of \( K_{10} \) such that \( \delta(x \otimes y) = (-1)^{2y}y \otimes x \). Moreover \( \text{Aut}(K_{10}) \cong \text{Aut}(K)^2 \times \{1, \delta \} \cong \text{SL}_2(F)^2 \cup \text{SL}_2(F)^2 \delta \) where the product in \( \text{SL}_2(F)^2 \) (or in \( \text{Aut}(K)^2 \)) is componentwise and \( (f, g)\delta = \delta(g, f) \). So \( \text{Aut}(K_{10}) \) has two connected components and the component of the unit is \( \text{Aut}(K_{10})_0 = \text{SL}_2(F)^2 \). A maximal torus of \( \text{Aut}(K_{10}) \) is then \( \mathcal{T}^2 = T \times T \) and its normalizer in \( \text{Aut}(K_{10}) \) is \( \mathcal{N}(\mathcal{T}^2) = \mathcal{N}^2 \cup \mathcal{N}^2 \delta \), where we recall that \( \mathcal{N} \) is the normalizer of \( T \) in \( \text{SL}_2(F) \). From Section II we know that the set of diagonalizable automorphisms of \( K_{10} \) producing any grading is (up to conjugacy) contained in \( \mathcal{N}(\mathcal{T}^2) \).

3.1. Toral gradings on \( K_{10} \). Any toral grading on \( K_{10} \) is isomorphic to a grading produced by a subquasitorus of \( \mathcal{T}^2 \). We are denoting \( t_{\lambda, \mu} := (t_\lambda, t_\mu) \in \mathcal{T}^2 \) for any \( \lambda, \mu \in F^\times \). In order to use matrices in our study of gradings we fix the following basis of \( K_{10} \):

\[
B = (1, e \otimes e, x \otimes x, x \otimes y, y \otimes x, y \otimes y, e \otimes x, x \otimes e, e \otimes y, y \otimes e)
\]

in which the first six elements span the even part while the four last elements span the odd part. Taking into account that any \( t_\lambda \) fixes \( e \) and \( t_\lambda(x) = \lambda x, t_\lambda(y) = \lambda^{-1}y \), the matrix of \( t_{\lambda, \mu} \) relative to \( B \) is

\[
\text{diag}(1, 1, \lambda \mu, \frac{1}{\lambda \mu}, \frac{1}{\lambda}, \mu, \frac{1}{\mu}, 1)
\]

Furthermore, as conjugated elements produce isomorphic gradings, we must devote a few lines to the action of the group \( \mathfrak{U} := \mathcal{N}(\mathcal{T}^2)/\mathcal{T}^2 \) on the maximal torus \( \mathcal{T}^2 \). First of all the normalizer \( \mathcal{N}(\mathcal{T}^2) \) acts on \( \mathcal{T}^2 \) by conjugation: there is an action \( \mathcal{N}(\mathcal{T}^2) \times \mathcal{T}^2 \rightarrow \mathcal{T}^2 \) such that \( (f, t) \mapsto f \circ t := ft f^{-1} \), for \( f \in \mathcal{N}(\mathcal{T}^2) \) and \( t \in \mathcal{T}^2 \). Besides the element \( ft f^{-1} \) does not change if we replace \( f \) by \( g \in \mathcal{N}(\mathcal{T}^2) \) such that \( fg^{-1} \in \mathcal{T}^2 \). So the previous action induces an action of \( \mathfrak{U} \) on \( \mathcal{T}^2 \) by conjugation. If \( t, t' \in \mathcal{T}^2 \) are in the same orbit under the action of \( \mathfrak{U} \) we shall write \( t \sim t' \). Thus we have:

\[
t_{\lambda, \mu} \sim t_{\lambda^{-1}, \mu} \sim t_{\lambda, \mu^{-1}} \sim t_{\lambda^{-1}, \mu^{-1}} \sim t_{\mu, \lambda}.
\]

To prove this, note that \( (1, \sigma) \in \mathcal{N}^2 \subset \mathcal{N}(\mathcal{T}^2) \) since \( \mathcal{N} = \mathcal{T} \cup \mathcal{T} \sigma \). But

\[
(1, \sigma) \circ t_{\lambda, \mu} := (1, \sigma)t_{\lambda, \mu}(1, \sigma^{-1}) = (1, \sigma)(t_{\lambda, \mu})(1, \sigma^{-1}) = (t_{\lambda, \sigma^{-1}}, \sigma t_{\mu, \sigma^{-1}}) = (t_{\lambda, \mu^{-1}}, \sigma t_{\mu, \sigma^{-1}}),
\]

and similarly \( (\sigma, 1) \circ t_{\lambda, \mu} = t_{\lambda^{-1}, \mu} \). On the other hand it is easy to check that \( \delta \in \mathcal{N}(\mathcal{T}^2) \) satisfies \( \delta \circ t_{\lambda, \mu} = t_{\mu, \lambda} \) so that \( t_{\lambda, \mu} \sim t_{\mu, \lambda} \).
The first step in our study of toral gradings is to look at those induced by only one toral element $t_{\lambda,\mu} \in T$. It turns out that this kind of gradings provides most of the cases appearing in our classification.

3.1.1. Cyclic gradings. A cyclic grading is a toral grading produced by a single toral element $t_{\lambda,\mu}$. In this case the grading is always equivalent to a grading by a cyclic group, although not necessarily the universal group is cyclic (see [2] for the concept of universal group.) In order to study the grading induced by $t_{\lambda,\mu}$ on $J = K_{10}$, which is the decomposition of $K_{10}$ as a direct sum of eigenspaces of such toral element $t_{\lambda,\mu}$, we define the set of eigenvalues $S := \{1, \lambda, \mu, \lambda \mu, -1, \lambda^{-1}, \mu^{-1}, \lambda^{-1} \mu^{-1}\}$ of $t_{\lambda,\mu}$ and consider the different possibilities for the cardinal $|S|$. In case $|S| = 9$ we get the fine toral grading $J = J^1 \oplus J^\lambda \oplus J^\mu \oplus J^{\lambda \mu - i} \oplus J^{\mu \lambda - i} \oplus J^{\lambda^{-1} \mu - i} \oplus J^{\mu^{-1} \lambda - i} \oplus J^{\lambda^{-1} \mu^{-1} - i}$ where

$$J^1 = \langle 1, e \otimes e \rangle, \quad J^\lambda = \langle x \otimes x \rangle, \quad J^\mu = \langle x \otimes y \rangle, \quad J^{\lambda \mu - i} = \langle y \otimes x \rangle, \quad J^{\mu \lambda - i} = \langle y \otimes y \rangle, \quad J^{\lambda^{-1} \mu - i} = \langle y \otimes e \rangle, \quad J^{\mu^{-1} \lambda - i} = \langle y \otimes e \rangle.$$

This is a $\mathbb{Z} \times \mathbb{Z}$-grading of type $(8, 1)$ (that is, eight 1-dimensional homogeneous components, and one of dimension 2), given by $J = \oplus_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} J^{(n,m)}$ with $J^{(n,m)} := J^{\lambda^n \mu^m}$. Our interest to see this as a $\mathbb{Z} \times \mathbb{Z}$-grading comes from the fact that this is the universal grading group, which presents some advantages when is compared to other groups producing an equivalent grading. In the remaining gradings in this section, we will also prefer to use the superindices to indicate the eigenvalue of the automorphism, instead of the element of the grading group (of course, both notations are closely related).

If $|S| < 9$ we have several possibilities.

- $1 \in \{\lambda, \frac{\lambda}{\mu}, \frac{\mu}{\lambda}, \mu, \lambda, \frac{1}{\lambda}, \frac{1}{\mu}\}$. Then the grading automorphism is $t_{1,1}$, $t_{\lambda,\lambda}$, $t_{1,\lambda}$, or $t_{\lambda,1}$. But $t_{1,1}$ and $t_{1,\lambda}$ are the two cases we have to distinguish several cases. If the elements $\lambda^{-2}, \lambda^{-1}, 1, \lambda, \lambda^2$ are all different we have the grading $J = J^{\lambda^2} \oplus J^{\lambda^{-2}} \oplus J^1 \oplus J^\lambda \oplus J^{\lambda^2}$ where

$$J^{\lambda^2} = \langle y \otimes y \rangle, \quad J^{\lambda^{-2}} = \langle e \otimes y, y \otimes e \rangle, \quad J^1 = \langle 1, e \otimes e, x \otimes y, y \otimes x \rangle, \quad J^\lambda = \langle x \otimes x, x \otimes e \rangle.$$

This is a $\mathbb{Z}$-grading of type $(2, 2, 0, 1)$. If there are coincidences in the set $\{\lambda^{-2}, \lambda^{-1}, 1, \lambda, \lambda^2\}$ then necessarily $\lambda$ is a primitive $n$-th root of the unit for $n = 1, 2, 3, 4$. The case $n = 1$ corresponds to $\lambda = 1$ and thus to the trivial grading given by $J^1 = J$. For $n = 2$ we have $\lambda = -1$ so that we have a $\mathbb{Z}_2$-grading $J = J^1 \oplus J^{-1}$ given by

$$J^1 = \langle 1, e \otimes e, x \otimes y, y \otimes x, x \otimes y, y \otimes x \rangle, \quad J^{-1} = \langle e \otimes y, y \otimes e, e \otimes x, x \otimes e \rangle.$$

This is the $\mathbb{Z}_2$-grading associated to the superalgebra structure of $J$, of type $(0, 0, 1, 0, 1)$. For $n = 3$, there is primitive cubic root of 1, say $\omega$, such that $\lambda = \omega$. Then $\lambda^{-1} = \lambda^2$ so that the grading is $J = J^{\omega^2} \oplus J^1 \oplus J^\omega$ where

$$J^{\omega^2} = \langle x \otimes x, e \otimes y, y \otimes e \rangle, \quad J^1 = \langle 1, e \otimes e, x \otimes y, y \otimes x \rangle, \quad J^\omega = \langle y \otimes y, e \otimes x, x \otimes e \rangle.$$

This is a $\mathbb{Z}_3$-grading of type $(0, 0, 2, 1)$. For $n = 4$ we take $\lambda = i$ the complex unit so that the grading is $J = J^{-1} \oplus J^{-i} \oplus J^1 \oplus J^i$, which is the $\mathbb{Z}_4$-grading of type $(0, 3, 0, 1)$ given by

$$J^{-1} = \langle x \otimes x, y \otimes y \rangle, \quad J^{-i} = \langle e \otimes y, y \otimes e \rangle, \quad J^1 = \langle 1, e \otimes e, x \otimes y, y \otimes x \rangle, \quad J^i = \langle e \otimes x, x \otimes e \rangle.$$
In case the grading is induced by $t_{\lambda,1}$ with $\lambda \neq \pm 1$ we have $J = J^{\lambda^{-1}} \oplus J^1 \oplus J^\lambda$ which is the $\mathbb{Z}$-grading of type $(0, 0, 2, 1)$ given by

$$J^{\lambda^{-1}} = \langle y \otimes x, y \otimes y, y \otimes e \rangle, \quad J^1 = \langle e \otimes e, e \otimes x, e \otimes y \rangle, \quad J^\lambda = \langle x \otimes x, x \otimes y, x \otimes e \rangle.$$

For $\lambda = -1$ we get the coarsening $J = J^{-1} \oplus J^1$ which is a $\mathbb{Z}_2^2$-grading of type $(0, 0, 1, 1)$ given by

$$J^1 = \langle 1, e \otimes e, e \otimes x, e \otimes y \rangle, \quad J^{-1} = \langle x \otimes x, x \otimes y, y \otimes x, y \otimes e \rangle.$$

1. $1 \not\in \{ \lambda \mu, \frac{\lambda}{\mu}, \frac{\mu}{\lambda}, \frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\mu \lambda}, \frac{\mu}{\lambda}, \frac{1}{\mu \lambda}, \frac{\lambda}{\mu}, \frac{1}{\lambda \mu}, \frac{\lambda \mu}{1}, \frac{1}{\lambda \mu} \}$. Now we analyze the possibility $\lambda \mu \in \{ \frac{\lambda}{\mu}, \frac{\mu}{\lambda}, \frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\mu \lambda}, \frac{\mu}{\lambda}, \frac{1}{\mu \lambda}, \frac{\lambda}{\mu}, \frac{1}{\lambda \mu}, \frac{\lambda \mu}{1}, \frac{1}{\lambda \mu} \}$. Modulo the action of the group $\mathfrak{W}$, this gives the following cases:

i) $\lambda \mu = \frac{\lambda}{\mu}$ implying $\mu^2 = 1$. The solution $\mu = 1$ has been considered before.

So we have to deal with the toral element $t_{\lambda,-1}$. For $\lambda \neq \pm 1, \pm i$, this induces the $\mathbb{Z}_2 \times \mathbb{Z}$-grading $J = J^1 \oplus J^\lambda \oplus J^{-1} \oplus J^{-\lambda} \oplus J^{1/\lambda} \oplus J^{-1/\lambda}$ where

$$J^1 = \langle 1, e \otimes e \rangle, \quad J^{-1} = \langle e \otimes x, e \otimes y \rangle, \quad J^\lambda = \langle x \otimes e \rangle,$$

$$J^{-\lambda} = \langle x \otimes x, x \otimes y \rangle, \quad J^{1/\lambda} = \langle y \otimes e \rangle, \quad J^{-1/\lambda} = \langle y \otimes y, y \otimes x \rangle,$$

which is of type $(2, 4)$. For $\lambda = 1$ the grading is the induced by $t_{1,-1} \sim t_{-1,1}$, that is, (9). For $\lambda = -1$ the grading is produced by $t_{-1,-1}$ and this is (5). For $\lambda = i$ we get the grading induced by $t_{i,-1}$, that is: $J = J^1 \oplus J^{-1} \oplus J^\lambda \oplus J^{-\lambda}$.

This is the $\mathbb{Z}_4$-grading of type $(0, 2, 2)$ such that

$$J^1 = \langle 1, e \otimes e \rangle, \quad J^{-1} = \langle e \otimes x, e \otimes y \rangle, \quad J^\lambda = \langle y \otimes e \otimes x, y \otimes x \rangle.$$

For $\lambda = -i$ we have $t_{-1,-1}$, which is in the orbit of $t_{1,-1}$ by $\mathfrak{W}$.

ii) $\lambda \mu = \frac{1}{\mu}$ since the possibility $\lambda \mu = 1$ has been studied above. Thus the grading automorphism is $t_{\lambda,-\frac{1}{\mu}} \sim t_{\lambda,-\lambda}$.

If all the elements in the set $\{ \pm 1, \pm \lambda, \pm \lambda^{-1}, -\lambda^2, -\lambda^{-2} \}$ are different, the automorphism $t_{\lambda,-\lambda}$ induces a $\mathbb{Z}_2 \times \mathbb{Z}$-grading of type $(6, 2)$ given by $J = J^1 \oplus J^{-1} \oplus J^\lambda \oplus J^{-\lambda} \oplus J^{1/\lambda} \oplus J^{-1/\lambda} \oplus J^{-\lambda^2} \oplus J^{1/\lambda^2}$ where

$$J^1 = \langle 1, e \otimes e \rangle, \quad J^{-1} = \langle x \otimes y, y \otimes x \rangle, \quad J^\lambda = \langle x \otimes e \rangle, \quad J^{-\lambda} = \langle e \otimes x \rangle,$$

$$J^{1/\lambda} = \langle y \otimes e \rangle, \quad J^{-1/\lambda} = \langle e \otimes y \rangle, \quad J^{-\lambda^2} = \langle x \otimes x \rangle, \quad J^{1/\lambda^2} = \langle y \otimes y \rangle.$$

If there are coincidences in the above set the possibilities are: $\lambda = \pm 1, \lambda^2 = -1$ or $\lambda^3 = \pm 1$. The first two cases have been previously studied. The possibility $\lambda^3 = 1$ gives $\lambda = \omega$ a primitive cubic root of the unit so that this is a $\mathbb{Z}_6$-grading of type $(2, 4)$ induced by $t_{\omega,-\omega}$ and given by $J = J^1 \oplus J^{-1} \oplus J^{1/2} \oplus J^{-1/2} \oplus J^{-1/3} \oplus J^{1/3}$ and

$$J^1 = \langle 1, e \otimes e \rangle, \quad J^{-1} = \langle x \otimes y, y \otimes x \rangle, \quad J^{1/2} = \langle x \otimes e \rangle,$$

$$J^{-1/2} = \langle y \otimes e \rangle, \quad J^{1/3} = \langle x \otimes x, e \otimes y \rangle, \quad J^{-1/3} = \langle x \otimes y, e \otimes x \rangle.$$

And the possibility $\lambda^3 = -1$ gives $\lambda = -\omega$, but $t_{-\omega,-\omega} \sim t_{\omega,-\omega}$, which has just been studied.

iii) $\lambda \mu = \mu$. This would imply $\lambda = 1$.

iv) $\lambda \mu = \lambda$. This would imply $\mu = 1$.

v) $\lambda \mu = \frac{1}{\lambda}$. Then modulo the action of $\mathfrak{W}$ the grading automorphism is $t_{\lambda,\lambda^2}$.

If all the elements in the set $\{ 1, \lambda, \lambda^{-1}, \lambda^2, \lambda^{-2}, \lambda^3, \lambda^{-3} \}$ are different, we get a $\mathbb{Z}$-grading of type $(4, 3)$ given by

$$J = J^{\lambda^{-3}} \oplus J^{\lambda^{-2}} \oplus J^{\lambda^{-1}} \oplus J^1 \oplus J^\lambda \oplus J^{\lambda^2} \oplus J^{\lambda^3}.$$
where
\begin{equation}
J^{\lambda^3} = \langle y \otimes y \rangle, \quad J^{\lambda^2} = \langle e \otimes y \rangle, \quad J^{\lambda^{-1}} = \langle x \otimes y, y \otimes e \rangle, \\
J^1 = \langle 1, e \otimes e \rangle, \quad J^\lambda = \langle y \otimes x, x \otimes e \rangle, \quad J^{\lambda^2} = \langle e \otimes x \rangle, \quad J^{\lambda^3} = \langle x \otimes x \rangle.
\end{equation}

If there are coincidences in the above set, the possibilities are: \( \lambda = \pm 1, \lambda^3 = 1, \lambda^4 = 1, \lambda^5 = 1 \) or \( \lambda^6 = 1 \). The first possibility has been previously considered. The possibility \( \lambda^3 = 1 \) gives \( \lambda = \omega \) a primitive cubic root of the unit so that the grading is induced by \( t_{\omega, \omega^2} = t_{\omega, \omega^{-1}} \sim t_{\omega, \omega} \), previously studied. The possibility \( \lambda^4 = 1 \) implies \( \lambda \in \pm 1, \pm i \) and so it has also been studied. Let us consider the case \( \lambda^5 = 1 \). Hence \( \lambda = \kappa \) is a primitive fifth root of the unit. We obtain a \( \mathbb{Z}_5 \)-grading of type \( (0, 5) \) induced by \( t_{\kappa, \kappa^2} \), given by \( J = J^{\kappa^4} \oplus J^{\kappa^3} \oplus J^{\kappa^2} \oplus J^{\kappa} \oplus J^1 \) where
\begin{equation}
J^{\kappa^4} = \langle x \otimes y, y \otimes e \rangle, \quad J^{\kappa^3} = \langle x \otimes x, e \otimes y \rangle, \quad J^{\kappa^2} = \langle y \otimes y, e \otimes x \rangle, \\
J^\kappa = \langle y \otimes x, x \otimes e \rangle, \quad J^1 = \langle 1, e \otimes e \rangle.
\end{equation}

Finally, we have to investigate the possibility \( \lambda^6 = 1 \) which gives \( \lambda = -\omega \) a primitive sixth root of the unit. We get the \( \mathbb{Z}_6 \)-grading induced by \( t_{-\omega, -\omega^2} \sim t_{-\omega, -\omega^5} \) and hence equivalent to \( (13) \).

Thus we finish the analysis of the possibility \( \lambda \mu \in \{ \frac{1}{\mu}, \frac{i}{\mu}, -\mu, \lambda, \mu^{-1}, \lambda^{-1} \} \). Therefore we must continue analyzing the cases coming from \( \lambda \mu \in \{ \frac{1}{\mu}, \frac{i}{\mu}, -\mu, \lambda, \mu^{-1}, \lambda^{-1} \} \). However this analysis reveals no new possibilities for cyclic gradings. To summarize the results in this subsection we have:

**Theorem 1.** The nontrivial cyclic gradings on \( K_{10} \) are those described in (3)-(15).

We list now the essential information of cyclic gradings on the following table:

| Number | Group | type | quasitorus |
|--------|-------|------|------------|
| (3)    | \( \mathbb{Z} \times \mathbb{Z} \) | (8, 1) | \( \langle t_{\lambda, \lambda^2} : \lambda \in F^\times \rangle \) |
| (4)    | \( \mathbb{Z} \) | (2, 2, 0, 1) | \( \langle t_{-1, -1} \rangle \) |
| (5)    | \( \mathbb{Z}_2 \) | (0, 0, 0, 1, 0, 1) | \( \langle t_{\omega, \omega^2} \rangle \) |
| (6)    | \( \mathbb{Z}_3 \) | (0, 0, 2, 1) | \( \langle t_{\lambda, i} \rangle \) |
| (7)    | \( \mathbb{Z}_4 \) | (0, 3, 0, 1) | \( \langle t_{\lambda, 1} : \lambda \in F^\times \rangle \) |
| (8)    | \( \mathbb{Z}_6 \) | (0, 0, 1, 0, 1) | \( \langle t_{-1, 1} \rangle \) |
| (9)    | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) | (2, 4) | \( \langle t_{\lambda, -\lambda} : \lambda \in F^\times \rangle \) |
| (10)   | \( \mathbb{Z}_2 \) | (0, 2, 2) | \( \langle t_{-1, -1} \rangle \) |
| (11)   | \( \mathbb{Z} \times \mathbb{Z}_2 \) | (6, 2) | \( \langle t_{\lambda, -\lambda} : \lambda \in F^\times \rangle \) |
| (12)   | \( \mathbb{Z}_6 \) | (2, 4) | \( \langle t_{\omega, \omega} \rangle \) |
| (13)   | \( \mathbb{Z}_5 \) | (0, 5) | \( \langle t_{\kappa, \kappa^2} \rangle \) |
| (14)   | \( \mathbb{Z} \) | (4, 3) | \( \langle t_{\lambda, \lambda^2} : \lambda \in F^\times \rangle \) |

where \( \omega, i, \) and \( \kappa \) are respectively 3rd, 4th and 5th primitive roots of the unit. It should be noted that we have chosen as **quasitorus** in the table the maximal quasitorus \( Q \) containing \( t_{\lambda, \mu} \) but producing the same grading than the automorphism \( t_{\lambda, \mu} \) alone. For that choice, the group of characters \( \chi(Q) = \text{Hom}(Q, F^\times) \) is just the universal group.

Note also that the gradings (6) and (8), and (10) and (13), are not equivalent in spite of being of the same type, because their universal groups are not isomorphic. The gradings (5) and (9) are also nonequivalent, because the automorphisms \( t_{-1, -1} \) and \( t_{-1, 1} \) are
not conjugated in \( \text{Aut}(K_{10}) \). We can discard also the equivalence of these gradings by a
dimensional argument.

### 3.1.2. Noncyclic toral gradings.

To determine the rest of the toral gradings, we have to
continue by studying the possible refinements of cyclic toral gradings of the last section. To
do that, we consider the grading automorphism \( t_{\lambda, \mu} \) in each case and analyze a refinement
to a grading induced by a set of automorphisms \( \{t_{\lambda, \mu}, t_{\alpha, \beta}\} \). Ruling out the fine grading
(3), which cannot be further refined, the rest of the cyclic gradings do not provide cyclic
refinements except in two cases:

1. **Refinements of (5).** The grading is produced by \( t_{-1, -1} \). An analysis as before of all
   proper refinements, yields either cyclic gradings or the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-grading induced
   by \( \{t_{-1, -1}, t_{-1, 1}\} \) and given by
   \[
   J = J_{1,1} \oplus J_{1,-1} \oplus J_{-1,1} \oplus J_{-1,-1}
   \]
   where
   \[
   J_{1,1} = \langle 1, e \otimes e \rangle, \quad J_{1,-1} = \langle x \otimes x, x \otimes y, y \otimes x, y \otimes y \rangle, \\
   J_{-1,1} = \langle e \otimes e, e \otimes y \rangle, \quad J_{-1,-1} = \langle x \otimes e, y \otimes e \rangle.
   \]
   This grading is of type \((0, 3, 0, 1)\). Though its type is the same as (7), both gradings
   are not equivalent because the universal groups are different.

2. **Refinements of (7).** The grading is produced by \( t_{i,i} \) with \( i \) the complex unit. The
   refinement induced by \( \{t_{i,i}, t_{-1,1}\} \) is a \( \mathbb{Z}_4 \times \mathbb{Z}_2 \)-grading given by
   \[
   J = J_{-1,-1} \oplus J_{-i,1} \oplus J_{-i,-1} \oplus J_{1,1} \oplus J_{1,-1} \oplus J_{i,1} \oplus J_{i,-1}
   \]
   where
   \[
   J_{-1,-1} = \langle x \otimes x, y \otimes y \rangle, \quad J_{-i,1} = \langle e \otimes y \rangle, \quad J_{-i,-1} = \langle y \otimes e \rangle, \quad J_{1,1} = \langle 1, e \otimes e \rangle \\
   J_{1,-1} = \langle x \otimes y, x \otimes x \rangle, \quad J_{i,1} = \langle e \otimes x \rangle, \quad J_{i,-1} = \langle x \otimes e \rangle.
   \]
   This grading is of type \((4,3)\), although not equivalent to the grading (14), again
   because they have nonisomorphic universal groups.

So far, we have only detected two noncyclic gradings produced by two toral elements
which refine the cyclic gradings. To complete our study of refinements we must describe
now the possible refinements induced by a set of automorphisms \( \{t_{-1,-1}, t_{-1,1}, t_{e,\gamma}\} \) and
\( \{t_{i,i}, t_{-1,1}, t_{e,\gamma}\} \). But again a straightforward analysis of the different possibilities reveals
the inexistence of new gradings. Thus the unique proper refinements (up to equivalences)
of gradings are the ones given in (16) and (17).

**Theorem 2.** *The nontrivial toral gradings on \( K_{10} \) are those described in (3)-(17).*

The following table contains the relevant information on all toral gradings:
Indeed, if $\Delta$ denotes the centralizer of any other element of $\langle T \rangle$. We refer the reader to Section II and III for notations. Recall also from previous sections the following observations: (i) For $(f, g) \in SL_2(F)^2$, $\delta(f, g) = (g, f)$. (ii) For any $t_\sigma \in T$, $\sigma t_\sigma^{-1} = t_\lambda$, and (iii) For any $t \in T$, there exists $p \in SL_2(F)$ such that $p(t_\sigma^{-1}) \in T$ (which follows from the connectedness of $SL_2(F)$).

As examples of MAD-groups of $Aut(K_{10})$ we have $T^2$ and $\mathcal{M} := \{(t_\lambda, t_\sigma) \times \{1, \delta\} \subset N(T^2)$.

Indeed, if $(f, g) \in SL_2(F)^2$ commutes with $\mathcal{M}$, it also does with $\delta$, and $f = g$ according to ii. But it also commutes with $(t, t) \in T^2$, so $f \in Z_{SL_2(F)}(T) = T$, where $Z_G(H)$ denotes the centralizer of $H$ in $G$. Let us see that, up to conjugacy, these are the only examples.

**Theorem 3.** The MAD-groups of $Aut(K_{10})$ are, up to conjugacy, $T^2$ and $\mathcal{M}$.

Proof. Let $A$ be a MAD-group of $Aut(K_{10})$, which can be taken contained in $N(T^2) = N^2 \cup N^2 \delta$. First suppose that $A \subset N^2 = T^2 \cdot \{1, (\sigma, 1), (1, \sigma), (\sigma, \sigma)\}$.

- If $A \subset T^2$, then $A = T^2$.
- If $A \cap (T^2 \cdot (\sigma, 1)) \neq \emptyset$, there exist $(t, t_\sigma, t_\sigma) \in T$ such that $(t_1 t_\sigma, t_2) \in A$. If $t_2 \neq \pm 1$, $A \subset Z_{N^2}(t_1 t_\sigma, t_2) = \langle (t_1 t_\sigma, t) : t \in T \rangle$. As there is some $p \in SL_2(F)$ such that $p(t_1 t_\sigma p^{-1} = t_3 \in T$, thus $(p, 1) A(p^{-1}, 1) \not\subset T^2$, what contradicts the maximality of $A$. If $t_2 = \pm 1$, then $A \subset Z_{N^2}(t_1 t_\sigma, \pm 1) = \langle (t_1 t_\sigma, t) : t \in T \rangle \cdot \langle (1, \sigma) \rangle$, and $(p, 1) A(p^{-1}, 1) \subset \langle (t_3, t) : t \in T \rangle \cdot \langle (1, \sigma) \rangle$, necessarily containing an element of the form $(t_3, t_4)$ with $n \in \mathbb{N}$ and $t_4 \in T$. Now take $q \in SL_2(F)$ such that $q(t_1 t) q^{-1} \in T$ and so $(p, q) A(p, q^{-1}) \not\subset T^2$, against a contradiction.
- If $A \cap (T^2 \cdot (\sigma, 1)) \neq \emptyset$, there exist $(t, t_\sigma, t_\sigma) \in T$ such that $(t_1 t_\sigma, t_2 \sigma) \in A$. As $\delta(t_1 t_\sigma, t_2 \sigma) \delta^{-1} = (t_2 \sigma, t_1)$, we have $A \Delta \Delta^{-1} \cap (T^2 \cdot (\sigma, 1)) \neq \emptyset$, what, taking into account the previous case, is a contradiction.
- If $A \cap (T^2 \cdot (\sigma, \sigma)) \neq \emptyset$, there exist $(t_1, t_2) \in T$ such that $(t_1 t_2, t_2 \sigma) \in A$. As any other element of $A$ commutes with it, it is of some of the following types:

| Number | Group | type | quasitorus |
|-------|-------|------|-----------|
| (3)   | $\mathbb{Z} \times \mathbb{Z}$ | $(8, 1)$ | $T$ |
| (4)   | $\mathbb{Z}$ | $(2, 2, 0, 1)$ | $\langle t_{\lambda, \lambda} : \lambda \in F^\times \rangle$ |
| (5)   | $\mathbb{Z}_2$ | $(0, 0, 0, 1, 0, 1)$ | $\langle t_{-1,-1} \rangle$ |
| (6)   | $\mathbb{Z}_3$ | $(0, 0, 2, 1)$ | $\langle t_{\omega, \omega} \rangle$ |
| (7)   | $\mathbb{Z}_4$ | $(0, 3, 0, 1)$ | $\langle t_{i,i} \rangle$ |
| (8)   | $\mathbb{Z}$ | $(0, 0, 2, 1)$ | $\langle t_{\lambda,1} : \lambda \in F^\times \rangle$ |
| (9)   | $\mathbb{Z}_2$ | $(0, 0, 0, 1, 0, 1)$ | $\langle t_{-1,1} \rangle$ |
| (10)  | $\mathbb{Z} \times \mathbb{Z}_2$ | $(2, 4)$ | $\langle t_{\lambda,-1} : \lambda \in F^\times \rangle$ |
| (11)  | $\mathbb{Z}_4$ | $(0, 2, 2)$ | $\langle t_{i,-1} \rangle$ |
| (12)  | $\mathbb{Z} \times \mathbb{Z}_2$ | $(6, 2)$ | $\langle t_{\lambda,-\lambda} : \lambda \in F^\times \rangle$ |
| (13)  | $\mathbb{Z}_6$ | $(2, 4)$ | $\langle t_{\omega,-\omega} \rangle$ |
| (14)  | $\mathbb{Z}$ | $(4, 3)$ | $\langle t_{\lambda,2} : \lambda \in F^\times \rangle$ |
| (15)  | $\mathbb{Z}_5$ | $(0, 5)$ | $\langle t_{\omega,\omega} \rangle$ |
| (16)  | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $(0, 3, 0, 1)$ | $\langle t_{-1,-1}, t_{-1,1} \rangle$ |
| (17)  | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $(4, 3)$ | $\langle t_{i,i}, t_{-1,1} \rangle$ |
\( (\pm t_1 \sigma, \pm t_2 \sigma), (\pm 1, \pm t_2 \sigma), (\pm t_1 \sigma, \pm 1) \) or \( (\pm 1, \pm 1) \). Thus taking \( p, q \in SL_2(F) \) such that \( p(t_1 \sigma)p^{-1}, q(t_2 \sigma)q^{-1} \in T \), we find that \( (p, q)A(p, q)^{-1} \nsubseteq T^2 \) and again the contradiction appears.

We have shown that in the case \( A \subseteq \mathcal{N}^2 \), necessarily \( A = T^2 \). Suppose next that \( A \nsubseteq \mathcal{N}^2 \), so that \( A \cap \mathcal{N}^2 \delta \neq \emptyset \). Then there exist \( t_1, t_2 \in T \) such that either \( (t_1, t_2) \delta \in A \), \( (t_1 \sigma, t_2) \delta \in A \), \( (t_1, t_2 \sigma) \delta \in A \) or \( (t_1 \sigma, t_2 \sigma) \delta \in A \). Observe that the third possibility can be reduced to the second one by conjugating with \( \delta \), and that the forth possibility can be reduced to the first one by conjugating with \( (\sigma, 1) \). Besides the second possibility can be reduced to the first one by conjugating with \( (pt_2^{-1}, p) \delta \), for \( p \in SL_2(F) \) such that \( p(t_1 t_2^{-1} \sigma)p^{-1} \in T \).

So we will consider the first possibility. We can suppose that certain \( (1, t_1) \delta \in A \) by conjugating \( (t_1, t_2) \delta \in A \) with \( (t_1^{-1}, 1) \). Take \( s \in T \) such that \( s^2 = t_1 \). Because of the abelian character of \( A \), it is contained in \( Z_{\mathcal{N}(T^2)}((1, t_1) \delta) \). In case \( t_1 \neq \pm 1 \), we have \( Z_{\mathcal{N}(T^2)}((1, t_1) \delta) = \{(t, t) : t \in T \} \cdot \{1, (1, t_1) \delta\} \), and hence \( (s, 1)A(s, 1)^{-1} \subset \{(t, t) : t \in T \} \cdot \{1, (s, s) \delta\} = \mathcal{M} \), so that the equality holds by maximality. In the case \( t_1 = \pm 1 \), one has \( Z_{\mathcal{N}(T^2)}((1, \pm 1) \delta) = \{(t, t) : t \in T \} \cdot \{(1, \pm 1) \delta\} \cdot (\langle \sigma, \sigma \rangle) = \hat{A} \). If \( A \subset \{(t, t) : t \in T \} \cdot \{(1, \pm 1) \delta\} \), as before \( (s, 1)A(s, 1)^{-1} \subset \mathcal{M} \) and is equal to \( \mathcal{M} \) by maximality. Otherwise, there exists \( t_2 \in T \) such that \( (t_2 \sigma, t_2) \delta \in A \). Then \( A \subset Z_\mathcal{H}(t_2 \sigma, t_2 \delta) = \{(1, \pm 1) \delta \} \cdot (\langle t_2 \sigma, t_2 \delta \rangle) \). For \( t_1 = 1 \), take \( p \in SL_2(F) \) such that \( p(t_2 \sigma)p^{-1} \in T \) and so \( (p, p)A(p, p)^{-1} \nsubseteq \mathcal{M} \), what contradicts the maximality of \( A \). For \( t_1 = -1 \), take \( p \in SL_2(F) \) such that \( p(\sigma t_2)p^{-1} \in T \) and check that \( (p \sigma, pt_2^{-1})A(p \sigma, pt_2^{-1})^{-1} \nsubseteq \mathcal{M} \), again a contradiction.

**Corollary 2.** The fine gradings on \( K_{10} \) are, up to equivalence:

1. The toral \( \mathbb{Z} \times \mathbb{Z} \)-grading, \( J^{(0,0)} = \{1, e \otimes e\}, J^{(1,1)} = \{x \otimes x\}, J^{(1,-1)} = \{x \otimes y\}, J^{(-1,1)} = \{y \otimes x\}, J^{(-1,-1)} = \{y \otimes y\}\). This is of type \((8, 1)\).

2. The non-toral \( \mathbb{Z} \times \mathbb{Z} \)-grading, \( J^{(0,0)} = \{1, e \otimes e, x \otimes y - y \otimes x\}, J^{(-1,1)} = \{e \otimes y - y \otimes x\}, J^{(2,1)} = \{x \otimes x\}, J^{(-2,1)} = \{y \otimes y\}, J^{(1,0)} = \{x \otimes x + x \otimes e\}, J^{(-1,0)} = \{e \otimes y + y \otimes e\}\). This is of type \((7, 0, 1)\).

### 3.3. Nontoral gradings.

Recall that the quasitorus \( \mathcal{M} = \{t_{\lambda, \delta} : \lambda \in F^\times\} \cong F^\times \times \mathbb{Z}_2 \) induces a \( \mathbb{Z} \times \mathbb{Z}_2 \)-grading of type \((7, 0, 1)\) with homogeneous components

\[
J^{(1,1)} = \{1, e \otimes e, x \otimes y - y \otimes x\}, \quad J^{(-1,1)} = \{x \otimes x\},
\]

\[
J^{(2,1)} = \{y \otimes y\}, \quad J^{(-2,1)} = \{x \otimes x\},
\]

\[
J^{(1,0)} = \{e \otimes y + y \otimes e\}, \quad J^{(-1,0)} = \{e \otimes y + y \otimes e\},
\]

where the coordinates of the superindex indicate now the eigenvalues of the actions of \( t_{\lambda, \delta} \) and \( \delta \) respectively. Note that any nontoral grading is produced by a subquasitorus \( Q \subset \mathcal{M} \) such that \( Q \cap \{\delta t_{\lambda, \delta} : \lambda \in F^\times\} \neq \emptyset \) (otherwise \( Q \subset T' := \{t_{\lambda, \delta} : \lambda \in F^\times\} \) would be toral.) Note also that, for any \( \delta_{\beta, \beta} \in Q \),

\[
Q = (Q \cap T') \cdot \langle \delta_{\beta, \beta} \rangle.
\]

First suppose that \( Q = \langle \delta_{\beta, \beta} \rangle \). Recall that \( \delta_{\beta, \beta} \) acts in the homogeneous components of \( (18) \) with eigenvalues \( \{1, -\beta^2, -\frac{1}{\beta}, \beta, \beta, -1, -\beta, -\frac{1}{\beta}\} \) respectively. Hence it produces a grading equivalent to \( (18) \) if those numbers are all different, that is, if \( \beta^4 \neq 1 \) and \( \beta^6 \neq 1 \).
Let us see which is the induced grading for any value of \( \beta = \pm 1, \pm i, \pm \omega \), for \( \omega \) a cubic root of unit, taking into account that \( t_{1,-1} (\delta t_{\beta,\beta}) (t_{1,-1})^{-1} = \delta t_{-\beta,-\beta} \), and so \( \langle \delta t_{\beta,\beta} \rangle \) and \( \langle \delta t_{-\beta,-\beta} \rangle \) produce equivalent gradings.

- If \( \beta = 1 \), we get the \( \mathbb{Z}_2 \)-grading produced by \( \delta \), given by

\[
J^1 = \langle 1, e \otimes e, x \otimes y - y \otimes x, e \otimes x + x \otimes e, e \otimes y + y \otimes e \rangle,
\]

of type \( (0,0,0,0,2) \).

- If \( \beta = i \), we get the \( \mathbb{Z}_2 \)-grading produced by \( \delta \), given by

\[
J^1 = \langle 1, e \otimes e, x \otimes y - y \otimes x, x \otimes x, y \otimes y \rangle,
\]

of type \( (1,2,0,0,1) \).

- If \( \beta = \omega \), we get the \( \mathbb{Z}_2 \)-grading produced by \( \delta \), given by

\[
J^1 = \langle 1, e \otimes e, x \otimes y - y \otimes x, x \otimes x, y \otimes y \rangle,
\]

of type \( (3,2,1) \).

In case that \( Q \neq \langle \delta t_{\beta,\beta} \rangle \), since \( t_{\beta,\beta} = (\delta t_{\beta,\beta})^2 \in Q \cap T' \), we get \( \langle t_{\beta,\beta} \rangle \subseteq Q \cap T' \). If there is some \( t_{\lambda,\lambda} \in Q \cap T' \) with \( \lambda_1 \) and \( \lambda_1 \neq 1 \), then the grading induced by \( Q \) is not a coarsening of \( (18) \), because the pair \( (t_{\lambda,\lambda}, \delta t_{\beta,\beta}) \) acts with eigenvalues

\[
\{(1,1), (\lambda_1^2, -\beta^2), (\lambda_1^{-2}, -\beta^{-2}), (\lambda_1, \beta), (\lambda_1^{-1}, -\beta^{-1}), (1,1), (1,1), (1,1), (1,1), (1,1)\},
\]

which are all different (in fact the first coordinates are different, up to the pairs \( (1,1) \) and \( (1,1) \)). Hence we can assume that every element in \( Q \cap T' \) has order divisor of either 3 or 4. As \( Q \cap T' \) is a subgroup of \( T' \cong F^3 \), we conclude that \( Q \cap T' \) equals either \( \langle t_{-1,-1} \rangle \), or \( \langle t_{i,1} \rangle \), or \( \langle t_{0,0} \rangle \).

- If \( Q \cap T' = \langle t_{-1,-1} \rangle \), then \( \beta^2 = 1 \) (with \( \beta^2 \) does not generate \( \{1,-1\} \)) and \( Q = Q \cap T' \cdot \langle \delta t_{\beta,\beta} \rangle = \langle t_{-1,-1}, \delta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), which induces a grading of type \( (0,2,2) \):

\[
J^{(1,1)} = \langle 1, e \otimes e, x \otimes y - y \otimes x \rangle, \quad J^{(1,-1)} = \langle x \otimes y + y \otimes x, x \otimes x, y \otimes y \rangle, \quad J^{(-1,1)} = \langle e \otimes x + x \otimes e, e \otimes y + y \otimes e \rangle, \quad J^{(-1,-1)} = \langle e \otimes x - x \otimes e, e \otimes y - y \otimes e \rangle.
\]

- If \( Q \cap T' = \langle t_{i,1} \rangle \), then \( \beta^2 = \pm 1 \) (\( \beta^2 \) does not generate \( \{1,-1,i,-i\} \)) so that \( \beta \in \{1,-1,i,-i\}, \delta \in Q \) and \( Q = \langle t_{i,1}, \delta \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \), which induces

\[
J^{(1,1)} = \langle 1, e \otimes e, x \otimes y - y \otimes x \rangle, \quad J^{(i,1)} = \langle e \otimes x + x \otimes e \rangle, \quad J^{(-1,1)} = \langle x \otimes x, y \otimes y \rangle, \quad J^{(-1,i)} = \langle e \otimes y + y \otimes e \rangle, \quad J^{(1,-1)} = \langle x \otimes y + y \otimes x \rangle, \quad J^{(i,-1)} = \langle e \otimes x - x \otimes e \rangle, \quad J^{(-1,-1)} = \langle e \otimes y - y \otimes e \rangle
\]

a grading of type \( (5,1,1) \).

- If \( Q \cap T' = \langle t_{0,0} \rangle \), then \( \beta^2 = 1 \) (the only non-generator in \( \mathbb{Z}_3 \)) so that \( \beta = \pm 1 \). Both cases are conjugated, since \( t_{-1,1} \delta (t_{-1,1})^{-1} = \delta t_{-1,1} \). Thus we can assume that \( \delta \in Q \) and \( Q = \langle t_{0,0}, \delta \rangle = \langle \delta t_{0,0} \rangle \), so that we are in the first case.

Summarizing the results in these sections, we claim
Theorem 4. Up to equivalence, the nontrivial $G$-gradings on $J = K_{10}$ (compatible with the superalgebra structure) are the following:

1. $G = \mathbb{Z} \times \mathbb{Z}$, $J^{(0,0)} = \langle 1, e \otimes e \rangle$, $J^{(1,1)} = \langle x \otimes x \rangle$, $J^{(1,-1)} = \langle x \otimes y \rangle$, $J^{(-1,1)} = \langle y \otimes x \rangle$, $J^{(-1,-1)} = \langle y \otimes y \rangle$, $J^{(1,0)} = \langle x \otimes e \rangle$, $J^{(0,1)} = \langle y \otimes e \rangle$, $J^{(-1,0)} = \langle e \otimes x \rangle$, $J^{(0,-1)} = \langle e \otimes y \rangle$, $J^{(-1,1)} = \langle y \otimes x \rangle$.

2. $G = \mathbb{Z}$, $J^{(-2)} = \langle y \otimes y \rangle$, $J^{(-1)} = \langle e \otimes y \rangle$, $J^{(0)} = \langle e \otimes e \rangle$, $J^{(1)} = \langle x \otimes x \rangle$, $J^{(2)} = \langle x \otimes e \rangle$, $J^{(3)} = \langle e \otimes x \rangle$.

3. $G = \mathbb{Z}_2$, $J^0 = \langle 1, e \otimes e, x \otimes y, y \otimes x, x \otimes x, y \otimes y \rangle$, $J^1 = \langle y \otimes e, y \otimes e, e \otimes x, x \otimes e \rangle$.

4. $G = \mathbb{Z}_3$, $J^2 = \langle x \otimes x, e \otimes y, y \otimes e \rangle$, $J^0 = \langle 1, e \otimes e, x \otimes y, y \otimes x \rangle$, $J^1 = \langle y \otimes x, e \otimes e, x \otimes y \rangle$.

5. $G = \mathbb{Z}_4$, $J^2 = \langle x \otimes x, y \otimes y \rangle$, $J^3 = \langle e \otimes y, y \otimes e \rangle$, $J^0 = \langle 1, e \otimes e, x \otimes y, y \otimes x \rangle$, $J^1 = \langle e \otimes x, x \otimes e \rangle$.

6. $G = \mathbb{Z}$, $J^{(-1)} = \langle y \otimes x, y \otimes y, y \otimes e \rangle$, $J^0 = \langle 1, e \otimes e, e \otimes x, e \otimes y \rangle$, $J^1 = \langle x \otimes x, x \otimes y, x \otimes e \rangle$.

7. $G = \mathbb{Z}_2$, $J^0 = \langle 1, e \otimes e, e \otimes x, e \otimes y \rangle$, $J^1 = \langle x \otimes x, x \otimes y, y \otimes x, x \otimes e \rangle$.

8. $G = \mathbb{Z}_2 \times \mathbb{Z}$, $J^{(0,0)} = \langle 1, e \otimes e \rangle$, $J^{(1,0)} = \langle x \otimes y, y \otimes x \rangle$, $J^{(0,1)} = \langle e \otimes e \rangle$, $J^{(1,1)} = \langle e \otimes x \rangle$, $J^{(0,-1)} = \langle y \otimes e \rangle$, $J^{(-1,1)} = \langle e \otimes y \rangle$, $J^{(-1,-1)} = \langle e \otimes x \rangle$, $J^{(0,1)} = \langle x \otimes x \rangle$, $J^{(1,-1)} = \langle x \otimes y \rangle$$\ldots$
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