Recently, we have presented some simple arguments supporting the existence of certain complementarity between thermodynamic quantities of temperature and energy, an idea suggested by Bohr and Heinsenberg in the early days of Quantum Mechanics. Such a complementarity is expressed as the impossibility of performing an exact simultaneous determination of the system energy and temperature by using an experimental procedure based on the thermal equilibrium with other system regarded as a measure apparatus (thermometer). In this work, we provide a simple generalization of this latter approach with the consideration of a thermodynamic situation with several control parameters.

Keywords: Thermodynamic uncertainty relations, Fluctuation theory.

1. Introduction

Bohr and Heisenberg suggested in the past that the thermodynamical quantities of temperature and energy are complementary in the same way as position and momentum in Quantum Mechanics. Their argument was that a definite temperature can be attributed to a system only if it is submerged in a heat bath, in which case energy fluctuations are unavoidable. On the other hand, a definite energy can be assigned only to systems in thermal isolation, thus excluding the simultaneous determination of its temperature. By considering the analogy with Quantum Mechanics, a simple dimensional analysis allows to conjecture the following uncertainty relation:

\[ \Delta U \Delta (1/T) \geq k_B, \]  

(1)
where \( k_B \) is Boltzmann’s constant.

A serious attempt in order to support the expression (1) was provided by Rosenfeld in turn of 1960 within the framework of classical fluctuation theory. However, this approach was performed under special restrictions which meant that the fluctuations of energy and temperature became dependent on each other and were no longer really complementary. Along the years, other formulations of the thermodynamical uncertainty relations were proposed by Mandelbro, Gilmore, Lindhard, Lavenda, Schögl, among other authors. Remarkably, the versions of this relation which have appeared in the literature give different interpretations of the uncertainty in temperature \( \Delta (1/T) \) and often employ widely different theoretical frameworks, ranging from statistical thermodynamics to modern theories of statistical inference. Despite of all devoted effort, this work has not led to a consensus in the literature, as clearly discussed in the most recent review by J. Uffink and J. van Lith. Even nowadays, this intriguing problem is still lively debated.

In our opinion, the underlying difficulties in arriving at a definitive formulation of the energy-temperature complementarity rely on a common and subtle misunderstanding of the temperature concept. Most of previous attempts have tried to justify a complementarity between energetic isolation and thermal contact inspired on the Bohr’s intuitive arguments. A primary idea here is that a system has a definite temperature when it is put in thermal contact with a heat bath (a system having an infinite heat capacity) at that temperature, which is the equilibrium situation associated to the known Gibbs’ canonical ensemble:

\[
p_c(U | \beta) \, dU = Z(\beta)^{-1} \exp \left(-\frac{1}{k_B} \beta U \right) \Omega(U) \, dU,
\]

where \( \beta = 1/T \), \( \Omega(U) \), the states density, and \( Z(\beta) \), the partition function. If one can only attribute a definite value of the system temperature by appealing to the temperature of a second thermodynamic system (the heat bath), this fact necessarily implies that the temperature of an isolated system is a meaningless concept, or at least, it is imperfectly definite. Such an idea is explicitly expressed in some classical books on Statistical Mechanics, as the known Lev Landau and Evgenii Lifshitz treatise (see last paragraph of §112). This latter conclusion is counterfactual, since it could not be possible to talk about a definite value for the temperature of the system acting as a heat bath when it is put in energetic isolation. Besides, the actual meaning of the temperature uncertainty \( \Delta (1/T) \) in Eq. (1) is unclear from this viewpoint, since the temperature \( T \) is just a constant parameter of the canonical probabilistic distribution. This particular question leads to the polemic exchange among Feshbach, Kittel and Mandelbro in the years 1987-1989 in Physics Today, and explains by itself the several interpretations of temperature uncertainty \( \Delta (1/T) \) existing in the literature.

On the contrary, the concept of temperature of a closed isolated system admits a clear and unambiguous definition in terms of the microcanonical temperature
derived from the celebrated Boltzmann’s entropy:

\[ S = k_B \log W \rightarrow \frac{1}{T} = \frac{\partial S}{\partial U}. \] (3)

One can realize after revising the Gibbs’ derivation of canonical ensemble (2) from the microcanonical ensemble that the temperature appearing as a parameter in the canonical distribution (2) is just the microcanonical temperature of the heat bath when its size is sent to the thermodynamic limit \( N \to \infty \). Consequently, such a parameter actually characterizes the internal thermodynamical state of the bath and its thermodynamic influence on the system under consideration. While the differences between the temperature \( T \) appearing in the Gibbs’s canonical distribution (2) and the one associated to the microcanonical ensemble of a closed system (3) is not so relevant in most of practical situations involving large thermodynamic systems driven by short-range forces, this is not the case of small or mesoscopic systems such as molecular and nuclear clusters, or even, the large thermodynamic system driven by long-range interactions such as astrophysical systems. The crucial feature is that only by using the microcanonical temperature (3) it is possible to describe the existence of negative heat capacities \( C < 0 \) and the occurrence of phase transitions in finite systems (16, 17, 18, 19, 20).

By using this latter interpretation, temperature is, at least in a formal viewpoint, a function on the energy of a closed system. Consequently, these quantities seem to be no longer complementary. Indeed, the imposition of the energetic isolation \( \Delta U = 0 \) into Eq. (3) leads to \( \Delta (1/T) = 0 \), which clearly violates the supposed validity of Eq. (1). Apparently, there is not way to support the existence of a complementary relation between the energy and temperature by assuming microcanonical definition (3). Fortunately, this preliminary conclusion is incorrect. The temperature is not a physical quantity with a direct mechanical interpretation as the energy. As the entropy, it is just a thermo-statistical quantity whose physical interpretation is only possible by appealing to the notion of statistical ensemble. Such a nature implies that any practical determination of the temperature of a system is always imprecise. In practice, a temperature mensuration can be only performed, in an indirectly way, through the statistical processing (or the temporal expectation values) of certain physical observables commonly referred as thermometric quantities. Besides, it is necessary to account for the unavoidable affectation produced by the experimental measurements on the internal state of the system under study, a feature which is simply ignored in previous attempts to justify an energy-temperature complementarity.

Recently, we reconsider this old problem of Statistical Mechanics by analyzing the limits of precision involved during the practical determination of the temperature of a given system by using the usual experimental procedure based on the thermal equilibrium with a second system (21, 22), which plays the role of a measure apparatus (thermometer). Our analysis allows to obtain the following result:

\[ \Delta U \Delta \eta \geq k_B, \] (4)
where $\eta = 1/T_A - 1/T$ the inverse temperature difference between the measure apparatus and the system under study, and $\Delta a = \sqrt{\langle \delta a^2 \rangle}$, the square root of the statistical deviation of a physical quantity $a$ undergoing thermal fluctuations.

A crucial difference of result (4) with other attempts to justify the complementary character between energy and temperatures, as the cases of Rosenfeld’s and Mandelbrot’s approaches \[3\] \[5\], is found in the fact that the thermal uncertainties $\Delta U$ and $\Delta \eta$ depend on the nature of the measure apparatus. Thus, an experimentalist has a free will to change the experimental conditions and modify the energy and temperature uncertainties. Clearly, Eq. (4) indicates the impossibility of carry out an exact simultaneous determination of the energy and temperature of a given system by using any experimental procedure based on the consideration of the thermal equilibrium condition with a second thermodynamic system.

As already evidenced, our proposal of energy-temperature complementarity is remarkably simple, and even, it is quite an expected result. Clearly, the physical arguments leading to the uncertainty relation (4) admit a simple extension in order to support a complementary character among other conjugated thermodynamic variables. The analysis of this question is the main goal of this work.

2. Extending the energy-temperature complementarity

2.1. Notation conventions

Conventionally, an equilibrium situation with several thermodynamic variables is customarily described within the Statistical Mechanics in terms of the Boltzmann-Gibbs distributions \[4\]:

$$dp_{BG} (U, X | \beta, Y) = Z^{-1} \exp \left[ - \frac{1}{k_B} \beta (U + Y X) \right] \Omega dU dX,$$

being $\beta = 1/T$, $Z = Z(\beta, Y)$ the partition function and $\Omega = \Omega (U, X)$ the density of states. The quantities $X = (V, M, P, N_i, \ldots)$ represent other macroscopic observables acting in a given application like the volume $V$, the magnetization $M$ and polarization $P$, the number of chemical species $N_i$, etc.; being $Y = (p, -H, -E, -\mu_i, \ldots)$ the corresponding conjugated thermodynamic parameters: the external pressure $p$, magnetic $H$ and electric $E$ fields, the chemical potentials $\mu_i$, etc.

In order to simplify the analysis, we shall adopt the following notation:

$$(U, X) \rightarrow I = (I^1, I^2, \ldots I^n)$$

for the physical observables and

$$(\beta, \xi = \beta Y) \rightarrow \beta = (\beta_1, \beta_2, \ldots, \beta_n)$$

for the thermodynamic parameters. Hereafter, we also assume the Einstein’s summation convention:

$$a^i b_i = \sum_{i=1}^{n} a^i b_i.$$
The above notations allow to rephrase the distribution function (5) as follows:

$$d p_{BG} (I|\beta) = Z(\beta)^{-1} \exp \left[ -\frac{1}{k_B} \beta I \right] \Omega (I) dI.$$  

(9)

2.2. Starting considerations

As elsewhere discussed, the Boltzmann-Gibbs distribution accounts for an equilibrium situation between the system under study is under the influence of a very large surrounding (a heat bath, a particles reservoir, etc.). The thermodynamic variables $(\beta, Y)$ characterizing the internal thermodynamic states of such a surrounding are regarded as constant parameters in distribution (5). Such an approximation follows from the fact that the size of the surrounding is so large that it is possible to dismiss the thermodynamic influence of the system under study.

In general, the above equilibrium situation is inappropriate in order to describe a general equilibrium situation between a system $(S)$ and certain measure apparatus $(A)$. Such an apparatus constitutes a key piece in any experimental setup used to obtain the thermodynamic parameters of the system through the known equilibrium conditions:

$$\beta^S = \beta^A, \ Y^S = Y^A$$  

(10)  

(thermal equilibrium, mechanical equilibrium, chemical equilibrium, etc.). Actually, the size of the interacting part of the apparatus measure should be comparable or smaller than the system under study in order to guarantee that the unavoidable interaction involved in any measurement process does not affect in a significant way the internal thermodynamic state of the system, e.g.: a thermometer always interchanges energy with the system during the thermal equilibration, and hence, it is desirable that its size be small in order to reduce the perturbation of the system temperature.

Let us consider an equilibrium situation where the system and the measure apparatus constitutes a closed system. By admitting only additive physical observables (e.g.: energy, volume, particles number or electric charge) obeying the constrain $I_T = I + I_A$, a simple ansatz for the distribution function is given by:

$$p (I|I_T) dI = \frac{1}{W (I_T)} \Omega_A (I_T - I) \Omega (I) dI,$$  

(11)

where $\Omega (I)$ and $\Omega_A (I_A)$ are the densities of states of the system and the apparatus measure respectively, and $W (I_T)$ the partition function that ensures the normalization condition:

$$\int_{\Gamma} p (I|I_T) dI = 1,$$  

(12)

Here, $\Gamma$ the subset of all admissible values of the physical observables $I$, which constitutes a compact subset of the Euclidean n-dimensional space $R^n$, $\Gamma \subset R^n$.  

where \( n \) is the number of macroscopic observables.\(^*\)

The ansatz (11), however, clearly dismisses some important practical situations where the additive constraint \( I_T = I + I_A \) cannot be ensured for all physical observables involved in the system-apparatus thermodynamic interaction. Significant examples of *unconstrained observables* are the magnetization or the electric polarization associated to magnetic and electric systems respectively. A formal way to overcome such a difficulty is carried out by representing the density of state of the measure apparatus \( \Omega_A \) in terms of the physical observables of the system \( I \) under study and certain set of control parameters \( a \) determining the given experimental setup, \( \Omega_A = \Omega_A (I|a) \). Such an explicit dependence of the density of states of the measure apparatus on the internal state of the system follows from their mutual interaction, which leads to the existence of *correlative effects* between these systems. Thus, our analysis starts from the consideration of the following distribution function:

\[
p(I|a) \, dI = \frac{1}{W(a)} \Omega_A (I|a) \Omega (I) \, dI.
\]

(13)

Obviously, the subset \( \Gamma \) of all admissible values of the physical observables \( I \) shall also depend on the control parameter \( a \), so that, it is more appropriate to denote this subset as \( \Gamma_a \) instead of \( \Gamma \).

Let us denote by \( p_a (I) \equiv p(I|a) \) the distribution function (13) and \( \partial_i A (I) = \partial A (I) / \partial I^i \) the first partial derivatives of a given real function \( A (I) \) defined on \( \Gamma_a \), where the positive integer \( i \in [1, 2, \ldots, n] \). The general mathematical properties of the distribution function \( p_a : \Gamma_a \rightarrow R \) associated to its thermo-statistical relevance are the following:

**C1. Existence**: The distribution function \( p_a \) is a nonnegative, bounded, continuous and differentiable function on \( \Gamma_a \).

**C2. Normalization**: The distribution function \( p_a \) obeys the normalization condition:

\[
\int_{\Gamma_a} p_a (I) \, dI = 1.
\]

(14)

**C3. Boundary conditions**: The distribution function \( p_a \) vanishes with its first partial derivatives \( \{ \partial_i p_a \} \) on the boundary \( \partial \Gamma_a \) of the subset \( \Gamma_a \). Moreover, the distribution function \( p_a \) satisfies the condition:

\[
\lim_{|I| \to \infty} |I|^\alpha p_a (I) = 0 \text{ with } \alpha \leq n + 1
\]

(15)

whenever the boundary \( \partial \Gamma_a \) contains the infinite point \( \{ \infty \} \) of \( R^n \).

The conditions (C1) and (C2) are natural for any distribution function. The validity of condition (C3) accounts for the fact that the product of densities of *We are dismissing in this approach the possible discrete nature of the macroscopic observables \( I \) associated to quantum effects, as example, the quantification of the system energy, which is a suitable approximation in Statistical Mechanics when the systems under consideration are large enough.*
states $\Omega_A(I|a)\Omega(I)$ usually vanishes at a finite point on the boundary $\partial\Gamma_a$ of the set $\Gamma_a$. The applicability of the associated condition shown in Eq. (15) ensures the existence of the averages $\langle I^i \rangle$ and some correlations functions when the boundary $\partial\Gamma_a$ contains the infinite point $\{\infty\}$ of $R^n$.

2.3. Thermodynamic parameters

As elsewhere discussed in any standard book of Statistical Mechanics,[4] thermodynamic parameters can be derived from the known Boltzmann’s entropy:

$$S = k_B \log W,$$

in terms of its first derivatives:

$$\beta = \frac{\partial S(U, X)}{\partial U}, \quad \beta Y = \frac{\partial S(U, X)}{\partial X},$$

being $W = \Omega \delta c$ the coarsed grained volume, and $\delta c$ certain small energy constant. Such parameters provide the thermo-statistical interpretation of the called equilibrium conditions between two interacting separable systems, Eq. (10). Ordinarily, these relations follow from the stationary conditions associated to the most likely macrostate.

The application of this latter argument for the case of distribution function (13) yields:

$$k_B \partial_i \log p_a = \partial_i S_A + \partial_i S_S = 0,$$

which can be rephrased as follows:

$$\beta^S_i = \beta^A_i,$$

after assuming that the conjugated thermodynamic parameters of the system $\{\beta^S_i\}$ and the measure apparatus $\{\beta^A_i\}$ for both constrained as well as unconstrained observables are defined by:

$$\beta^S_i = \frac{\partial S_S(I)}{\partial I^i}, \quad \beta^A_i = -\frac{\partial}{\partial I^i} S_A(I|a).$$

Notice that $\beta^A_i$ becomes equivalent to the standard definition when a physical observable obeying an additive constrain:

$$-\frac{\partial}{\partial I^i} S_A(I_T - I) \equiv \frac{\partial}{\partial I^i_A} S_A(I_A) \equiv \beta^A_i.$$  

As already referred, equilibrium conditions (19) allow the practical determination of the thermodynamic parameters of the system $\beta^S_i$ through the corresponding parameters of the measure apparatus $\beta^A_i$, whose interdependence with certain thermometric quantities (pressure, force, length, electric signals, etc.) with a direct mechanical interpretation is previously known. Our next step in the present discussion is to analyze the limits of precision of such an experimental procedure.
2.4. Derivation of uncertainties relations

Let us introduce the thermodynamic quantity \( \eta_i(I) \) defined by:

\[
\eta_i(I) = -k_B \frac{\partial \log p_a(I)}{\partial I_i} = \beta_i^A - \beta_i^S,
\]

which is just the difference between the \( i \)-th thermodynamic parameters of the measure apparatus \( \beta_i^A \) and the system \( \beta_i^S \) respectively. It is easy to show the validity of the following expectation values:

\[
\langle \eta_i \rangle = \int_\Gamma \eta_i(I) p_a(I) dI = 0,
\]

\[
\langle I^j \eta_i \rangle = \int_\Gamma I^j \eta_i(I) p_a(I) dI = k_B \delta_{ij}.
\]

Derivation of these latter results reads as follows. Let us consider a set \( \{ a_i \} \) of independent constants and conform with them the expectation value \( \langle a_i A \eta_i \rangle \), being \( A = A(I) \) certain function on the physical observables \( I \) of the system under consideration:

\[
\langle a_i A \eta_i \rangle \equiv -k_B \int_\Gamma A(I) a_i \frac{\partial}{\partial I_i} p_a(I) dI.
\]

It is easy to see that the term \( a_i \frac{\partial}{\partial I_i} p_a \) can be rephrased as a divergence of a vector, \( a_i \frac{\partial}{\partial I_i} p_a = \nabla \cdot (p_a \vec{a}) \), which allows to rewrite the above expression as follows:

\[
-k_B \int_{\partial \Gamma_a} A p_a \vec{a} \cdot \vec{n} + k_B \int_{\Gamma_a} p_a \vec{a} \cdot \nabla A dI
\]

by considering the identity \( A \nabla \cdot (p_a \vec{a}) = \nabla \cdot (A p_a \vec{a}) - p_a \vec{a} \cdot \nabla A \). The surface integral vanishes since \( p_a \) vanishes on the boundary \( \partial \Gamma_a \) when \( \Gamma_a \) is a finite subset, or the function \( A(I) \) obeys the condition \( |A(I)| \leq C_0 |I|^2 \) when \( |I| \to \infty \), according to the condition (C3). Thus, we arrive at the following identity:

\[
\langle A a_i \eta_i \rangle \equiv k_B \langle a_i \partial_i A \rangle.
\]

The latter result drops to Eqs. (23) and (24) by considering the independent character of the set of constants \( \{ a^i \} \) and by assuming \( A \equiv 1 \) and \( A(I) \equiv I^j \) respectively.

Eq. (23) is just the usual equilibrium conditions (10), which is expressed now in terms of expectation values:

\[
\langle \beta_i^A \rangle = \langle \beta_i^S \rangle,
\]

or equivalently:

\[
\left\langle \frac{1}{T_S} \right\rangle = \left\langle \frac{1}{T_A} \right\rangle, \quad \left\langle \frac{Y}{T} \right\rangle_S = \left\langle \frac{Y}{T} \right\rangle_A.
\]

It worth to mention that the above interpretation of thermodynamic equilibrium conditions (10) is more general in a thermo-statistical viewpoint than the usual result (19) derived from the argument of the most likely macrostate. In fact, Eq. (29)
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drops to Eq. (19) for large thermodynamical systems, where thermal fluctuations can be disregarded. Eq. (24) can be rephrased as the following fluctuation relation:

\[ \langle \delta I_i \delta \eta_i \rangle = k_B \delta^2_i \]  

(30)

by considering Eq. (23) and the identity \( \langle \delta A \delta B \rangle \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle \). Uncertainty relations follow from Eq. (30) and the application of the Schwarz inequality:

\[ \langle \delta A^2 \rangle \langle \delta B^2 \rangle \geq \langle \delta A \delta B \rangle^2 \]  

(31)

for conjugated thermodynamic quantities \( A = I_i \) and \( B = \eta_i \). By denoting \( \Delta a = \sqrt{\langle \delta a^2 \rangle} \), we finally obtain the following result:

\[ \Delta I_i \Delta \eta_i \geq k_B. \]  

(32)

3. Direct consequences and analogies with Quantum Mechanics

We have shown that the energy-temperature uncertainty relation:

\[ \Delta U \Delta (1/T_A - 1/T_S) \geq k_B, \]  

(33)

can be also extended to other conjugated thermodynamic quantities:

\[ \Delta X \Delta (Y_A/T_A - Y_S/T_S) \geq k_B. \]  

(34)

A simple example is the complementarity between the internal pressure \( p_S \) and volume \( V \) of a fluid system:

\[ \Delta V \Delta (p_A/T_A - p_S/T_S) \geq k_B. \]  

(35)

It is evident that it is impossible to determine the internal pressure \( p_S \) of a fluid system with the help of a measure apparatus (barometer) without involving certain perturbation of its volume \( V \). Any attempt to reduce this perturbation leads to an increasing the thermodynamic fluctuations of the quantity \( \xi = p_A/T_A - p_S/T_S \), which affects the determination of the system pressure \( p_S \) by using its equalization with an external pressure \( p_A \) associated to the measure apparatus (condition of mechanical equilibrium).

It is important to remark that the thermodynamic uncertainty relations and (34) are quite subtle. By considering definitions of Eq. (17), it is clearly evident that \( \Delta (1/T_S) \to 0 \) and \( \Delta (Y_S/T_S) \to 0 \) when \( \Delta U \to 0 \) and \( \Delta X \to 0 \). However the corresponding thermodynamic quantities of measure apparatus \( T_A \) and \( Y_A \) become independent on the state of the system (defined by precise values of \( U \) and \( X \)) at this limit situation (thermal isolation), and hence, the system thermodynamic parameters \( T_S \) and \( Y_S \) cannot be estimated by using \( T_A \) and \( Y_A \). On the other hand, any attempt to change the experimental conditions in order to reduce to zero

\(^b\)In absence of phase coexistence phenomenon, where there exist several local maxima of the distribution function \( p_a \).
\( \Delta \left( \frac{1}{T_A} - \frac{1}{T_S} \right) \rightarrow 0 \) and \( \Delta \left( \frac{Y_A}{T_A} - \frac{Y_S}{T_S} \right) \rightarrow 0 \) leads to a strong perturbation on the state of the system \( \Delta U \rightarrow \infty \) and \( \Delta X \rightarrow \infty \) and its thermodynamic quantities \( \Delta \left( \frac{Y_S}{T_S} \right) \rightarrow \infty \) and \( \Delta \left( \frac{1}{T_S} \right) \rightarrow \infty \), which simultaneously provokes a strong perturbation on the thermodynamic quantities of the measure apparatus \( \Delta \left( \frac{Y_A}{T_A} \right) \rightarrow \infty \) and \( \Delta \left( \frac{1}{T_A} \right) \rightarrow \infty \). Thus, we have arrived at the following:

**Conclusion:** It is impossible to perform an exact simultaneous determination of the conjugated thermodynamic quantities \( I^i \) and \( \beta^S_i = \partial_i S \left( I \right) \) of a given system by using an experimental procedure based on the thermodynamic equilibrium with a second system (measure apparatus).

In practice, we have to admit the existence of small uncertainties in the determination of physical observables \( U, X \) and thermodynamic parameters \( T, Y \), which can be disregarded in the framework of the large thermodynamic systems where \( \Delta U/U \sim 1/\sqrt{N}, \Delta X/X \sim 1/\sqrt{N} \) and \( \Delta \left( 1/T \right) \sim 1/\sqrt{N}, \Delta \left( Y/T \right) \sim 1/\sqrt{N} \), being \( N \) the number of system constituents. However, uncertainty relations (33) and (34) clearly indicate the limited practical utility of some thermo-statistical concepts in systems with few constituents.

One can object that the Boltzmann-Gibbs distributions (5) allow a thermo-statistical description of a given system without mattering about its size, so that, it could be always possible to attribute, as example, a definite temperature to any system by using this conventional description. Basically, this is the viewpoint of Bohr\(^1\), Heisenberg\(^2\), as well as other investigators\(^{14}\). However, such a temperature has nothing to do with the system temperature since it is actually the temperature of the thermostat. As already discussed in the introductory section, the system temperature has a clear and unambiguous definition in terms of the Boltzmann’s entropy (3), which has a definite value only for a closed system in thermodynamic equilibrium (thermal isolation) and can be attributed to systems with few constituents. The limitation associated to the present uncertainty relations is that any mensuration of such a temperature (and other thermodynamic parameters) by using the thermal equilibrium with a second system involves an uncontrollable strong perturbation on the internal thermodynamical macrostate of a small system, which undetermine its initial conditions.

As already evidenced, the measure apparatus plays a role during the experimental study of the internal state of a given system in Statistical Mechanics quite analogous to the one existing in Quantum Mechanics. Surprisingly, this analogy with Quantum Mechanics can be extended to the interpretation of the uncertainty relation (32) in terms of noncommuting operators. In fact, one can easily notice that the thermodynamic quantity \( \eta_i = \eta_i \left( I \right) \) introduced in Eq. (22) can be associated to a differential operator \( \hat{\eta}_i \) defined by:

\[
\hat{\eta}_i = -k_B \partial_i \tag{36}
\]

due to the validity of the mathematical identity:

\[
\hat{\eta}_i p_a \left( I \right) \equiv \eta_i p_a \left( I \right). \tag{37}
\]
Thus, the commutator identity $[\hat{I}_i, \hat{\eta}_j] = k_B \delta_{ij}$, where $\hat{I}_i \equiv I_i$, could be considered as the statistical mechanics counterpart of the quantum relations $[q_i, p_j] = i\delta_{ij}\hbar$.

One can provide other analogies between these physical theories. While Quantum Mechanics is hallmarked by the Ondulatory-Corpuscular Dualism, Statistical Mechanics exhibits another kind of dualism: the one existing between physical observables with a mechanical significance such as energy and other physical observables, and those quantities with a thermo-statistical significance such as temperature and other thermodynamical parameters. Classical Mechanics appears as an asymptotic theory of Quantum Mechanics when $\hbar \to 0$, while Thermodynamics appears as a suitable approximation of Statistical Mechanics when $k_B \to 0$, or equivalently, during the imposition of the thermodynamic limit $1/N \to 0$. Generally speaking, our analysis seems to support the Bohr's idea about the existence of uncertainty relations in any theory with a statistical formulation.

**Epilogue**

The present approach to uncertainty relations of Statistical Mechanics can be considered as an improvement of Rosenfeld's, Gilmore's, and Schöberg's works in the past, which are also based on fluctuation theory. In fact, our primary interest in this thermo-statistical formulation was never related with the justification of uncertainty relations in the framework of Statistical Mechanics, whose long history in literature was simply ignored by us. We accidentally advertise the existence of certain energy-temperature complementarity during an attempt to develop an extension of fluctuation theory compatible with the existence of macrostates with anomalous values in response functions, particularly, the presence of negative heat capacities observed in the thermodynamic description of many nonextensive systems.

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**References**

1. N. Bohr in: *Collected Works*, J. Kalckar, Ed. (North-Holland, Amsterdam, 1985), Vol. 6, pp. 316-330, 376-377.
2. W. Heisenberg, *Der Teil und das Gauze*, Ch. 9. R. Piper, München (1969).
3. L. Rosenfeld, in *Ergodic Theories*, P. Caldirola, Ed. (Academic Press, New York, 1961), pp. 1.
4. L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1980).
5. B. B. Mandelbrot, Ann. Math. Stat. 33 (1962) 1021; J. Math. Phys. 5 (1964) 164.
6. R. Gilmore, Phys. Rev. A 31 (1985) 3237.
7. J. Lindhard, *The Lessons of Quantum Theory* (Edited by J. de Boer, E. Dal and O.Ulfbeck). Elsevier, Amsterdam (1986).
8. B. Lavenda, Int. J. Theor. Phys. 26 (1987) 1069; *Statistical Physics: A Probabilistic Approach* (Wiley, New York, 1991).
9. F. Schölg, J. Phys. Chem. Sol. 49 (1988) 679.
10. J. Uffink and J. van Lith, Found. Phys. 29 (1999) 655.
11. B. H. Lavenda, Found. Phys. Lett. 13 (2000) 487.
12. J. Uffink and J. van Lith, Found. Phys. Lett. 14 (2001) 187.
13. H. Feshbach, Phys. Today 40 (1987) 9.
14. Ch. Kittel, Phys. Today 41 (1988) 93.
15. B. B. Mandelbrot, Phys. Today 42 (1989) 71.
16. L.G. Moretto, R. Ghetti, L. Phair, K. Tso, G.J. Wozniak, Phys. Rep. 287 (1997) 250.
17. M. D’Agostino, F. Gulminelli, P. Chomaz, M. Bruno, F. Cannata, R. Bougault, F. Gramegna, I. Iori, N. Le Neindre, GV. Margagliotti, A. Moroni and G. Vannini, Phys. Lett. B 473 (2000) 219.
18. D. H. E. Gross and M. E. Madjet, Z. Phys. B 104 (1997) 521.
19. D. H. E. Gross, *Microcanonical thermodynamics: Phase transitions in Small systems*, 66 Lectures Notes in Physics (World scientific, Singapore, 2001).
20. T. Dauxois, S. Ruffo, E. Arimondo and M. Wilkens (Eds.), *Dynamics and Thermodynamics of Systems with Long Range Interactions*, Lecture Notes in Physics (Springer, New York, 2002).
21. L. Velazquez and S. Curilef, J. Phys. A: Math. Theor. 42 (2009) 095006.
22. L. Velazquez and F. Curilef, J. Stat. Mech. (2009) P03027.