On the smoothing of point force in the two-way coupling simulation of polymer-laden turbulent flow

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Abstract. Effects of smoothing of point force used in the two-way coupling simulation of particle-laden turbulent flow are examined for one dimensional simple model and for the Eulerian-Lagrangian simulations of polymers in turbulence. We show that the regularization by using Gaussian function yields preferable behavior in evaluating the smoothed point force compared to the conventional methods used in the previous studies. An efficient computational scheme to use the approximated Gaussian function is newly developed for the two-way coupling simulation in which the turbulent field is solved by using the spectral method. This scheme is applied to the Eulerian-Lagrangian simulations for examining polymer-turbulence interactions.

1. Introduction
It is very important to understand interaction between particles and turbulence because the issue is related to various fields in science and engineering [1]. One of the important problems is to understand the mechanism of growth of droplets in cloud [2], where the effect of turbulent mixing on coagulation and coalescence of droplets is crucial [3]. The other example is the drag reduction phenomenon observed in turbulent channel flow of non-Newtonian fluid where the long-chain polymers dispersed in the solvent fluid are stretched by strong local shear in turbulence, leading to the modification of turbulence properties [4, 5].

In many cases the particle size is comparable to or smaller than the dissipation length in turbulence. This suggests that a small particle is approximated by a mass point. When the Stokes drag force is dominant over the forces acting on point particle, the governing equations of motion for point particles (particle radius \(a\); total particle number \(N_p\)) in fluid are given by [6]

\[
\frac{dx_p^i(t)}{dt} = v_p^i(t), \quad \frac{dv_p^i(t)}{dt} = -\zeta \left[ v_p^i(t) - u \left( x_p^i(t), t \right) \right] + F^i \quad (i = 1, \ldots, N_p),
\]

where \(m_p\) is a particle mass and \(\zeta = 6\pi\nu_f\rho_f a\) is Stokes drag coefficient, \(\nu_f\) and \(\rho_f\) being the kinematic viscosity and density of solvent fluid, respectively. The term \(F^i\) represents the remaining forces like the repulsive force among particles, buoyancy force, etc., and \(u(x, t)\) is the velocity field of the solvent fluid. The reaction force due to the suspended point particles acts on the incompressible fluid which obeys the Navier-Stokes (NS) equations. This can be expressed by the ensemble of point force as follows [7].

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho_f} \nabla p + \nu_f \nabla^2 u + \frac{1}{\rho_f} \sum_{i=1}^{N_p} \zeta \left[ v_p^i(t) - u \left( x_p^i(t), t \right) \right] \delta \left( x - x_p^i(t) \right),
\]
with $\nabla \cdot \mathbf{u} = 0$, where $p(x, t)$ is the pressure field and $\delta(x)$ is the Dirac delta function. To obtain the drag force acting on point particle numerically, fluid velocity at the particle position, $\mathbf{u} \left( x_p^i(t), t \right)$, must be evaluated by interpolating the data of fluid velocity on the computational grid points. The evaluated point force is then returned to fluid by distributing it over the surrounded grid points because we cannot deal with $\delta(x)$ in computation. This calculation is called the two-way coupling.

In the past many studies have conducted to clarify the nature of interaction between the ensemble of point particles and turbulence by performing two-way coupling simulations. The modification of the kinetic energy spectrum of isotropic turbulence due to the presence of inertial particles was discussed in [8], where it was observed that the spectral excitation at high wavenumber grows with increasing the Stokes number $S_t = m_p/\zeta \tau$, where $\tau$ is the characteristic time scale in turbulence at small scale, suggesting strong modification of dissipation scale dynamics in turbulence. Our previous studies [9, 10] investigated the effect of polymer additives on isotropic turbulence by performing the Eulerian-Lagrangian simulations, where the long-chain polymer molecule is represented by the beads-spring model. Although these analysis can be performed with reasonable computational cost even when the number of particles becomes much larger, there remains two kinds of problem in two-way coupling calculation: 1) we do not have enough knowledge about interactions between turbulence and ensemble of particles because the flow around particles is not resolved on the computational grid points, 2) the results obtained by two-way coupling calculation depend on interpolation/distribution methods representing the interaction between particles and turbulence.

The purpose of the present study is to clarify the second problem described above. That is, the question is how the details of smoothing method for point force in two-way coupling calculation affect the interaction between the ensemble of particles and turbulent flow. First we consider one dimensional model with three kinds of test function for which the analytical solution is given. This analysis brings insights into the validity/limitation of the method. Then we examine the two-way coupling simulations of polymer-laden isotropic turbulence.

This paper is organized as follows. In Section 2 we investigate three smoothing methods for the test model of point force. The efficiency and accuracy of the methods are discussed in detail. In Section 3 the two-way coupling simulations for polymer-laden isotropic turbulence are performed and examined from the viewpoint same as in Section 2. We present conclusions obtained in the present study in Section 4.

2. Characteristics of smoothed point force: case of the simple model

The problem when performing the two-way coupling simulations of equations (1) and (2) is how the delta function $\delta(x - x_p^i)$ in point force is approximated. From the computational point of view, the sufficient accuracy and reasonable computational cost are simultaneously required when evaluating the point forces with $N_p \gg 1$ case. In this section we investigate this problem by using the simple model.

2.1. Consideration by one-dimensional test problem

In this subsection we discuss the one-dimensional problem represented by

$$T(x) = \frac{L}{N_p} \sum_{i=1}^{N_p} t_i \delta(x - x_i), \quad (3)$$

where $x_i (i = 1, \cdots, N_p)$ indicates the position of particle which locates within $-L/2 \leq x \leq L/2$, and $t_i$ is a physical quantity characterizing a particle at $x_i$. When we evaluate equation (3) numerically, $T(x)$ is discretized as $T(x_j)$ with $x_j = j\Delta x \ (j = -N/2, \cdots, N/2)$, $\Delta x$ is the grid
spacing, so that the delta function $\delta(x)$ must be replaced by some approximated function $\delta_\Delta(x)$ whose support is in the range $|x| \leq \Delta$. Here we choose $\Delta = \Delta x$. We rewrite equation (3) as

$$T_\Delta(x) = \frac{L}{N_p} \sum_{i=1}^{N_p} t_i \delta_\Delta(x - x_i),$$  \hspace{1cm} (4)$$

where

$$\int_{-\infty}^{\infty} \delta_\Delta(x) \, dx = 1, \quad \lim_{\Delta \to 0} \delta_\Delta(x) = \delta(x)$$  \hspace{1cm} (5)$$

must be satisfied. Computational cost when computing (4) comes from 1) evaluating $\delta_\Delta(x)$, 2) product of $t_i$ and $\delta_\Delta(x - x_i)$, and 3) accessing the memory space that the data set of $T(x_j)$ is maintained (array variable for $T(x_j)$). We write the number of times of abovementioned operations as $C_1, C_2$ and $C_3$, respectively. In this study we consider the following three cases of $\delta_\Delta(x)$.

i) Rectangular function: this is defined by

$$\delta_\Delta^R(x) = \begin{cases} \frac{1}{\Delta} & (|x| < \Delta/2), \\ 0 & (|x| \geq \Delta/2). \end{cases}$$  \hspace{1cm} (6)$$

This may be the simplest case for calculation. The evaluation of $T(x_j)$ means $t_i/\Delta$ is put on a grid point that $|x_i - x_j|$ is minimum. In this case, the number of times of the operation is estimated as $C_1 = 0$, $C_2 = 1$ and $C_3 = N_p$.

ii) Triangular function: this is defined by

$$\delta_\Delta^T(x) = \begin{cases} \frac{1}{\Delta} \left(1 - \frac{|x|}{\Delta}\right) & (|x| < \Delta), \\ 0 & (|x| \geq \Delta). \end{cases}$$  \hspace{1cm} (7)$$

Using this function when computing (4) means that $t_i$ with multiplying the weight determined by $\delta_\Delta^R(x_i - x_j)$ or $\delta_\Delta^T(x_{j+1} - x_i)$ is distributed on grid points of $x_j$ and $x_{j+1}$ which satisfy $x_j < x_i < x_{j+1}$. This also corresponds to the method using the linear interpolation scheme in which the interaction between particle and fluid occurs within the surrounded grid points [7, 9].

The number of times of operation when computing equation (7) is $C_1 = 2N_p$ which is mostly dominated by evaluating the distance $|x_i - x_j|$. While the number of times of the product in (4) is $C_2 = 2N_p$, so that the number of times of accessing the memory for $T(x_j)$ is $C_3 = 2N_p$.

iii) Gaussian function: this is defined by

$$\delta_\Delta^G(x) = \frac{1}{\sqrt{2\pi\Delta}} \exp \left( -\frac{x^2}{2\Delta^2} \right).$$  \hspace{1cm} (8)$$

Recently the rigorous form of regularized point force was derived from analyzing the unsteady Stokes equations, in which the regularized point force involves Gaussian function for $\delta_\Delta(x)$ [11]. Because the support of Gaussian function is almost in the range $-3\Delta \leq x \leq 3\Delta$, we evaluate the values of $\delta_\Delta(x)$ on 7 grid points per particle. In this case we have the estimations $C_1 = 7N_p, C_2 = 7N_p$ and $C_3 = 7N_p$. Note that here $C_1$ is the number of times of the calculation for $\exp \left( -x^2/(2\Delta^2) \right)$ whose calculating cost is much higher than the case of $\delta_\Delta^R(x)$.

To test the effect of variation of $\delta_\Delta(x)$ on resulting $T_\Delta(x)$, we examine the test problem that $t_i$ in equation (4) is given by $t_i = \sin x_i$. $T_\Delta(x)$ is defined in $-\pi \leq x \leq \pi$, and supposed to have the periodicity $L = 2\pi$. The computational domain is discretized using $N$ grid points as
Figure 1. Comparison of $T_\Delta(x)$ computed by equation (4). (a) For three cases of $\delta_\Delta(x)$ when $N_p = 2048$, (b) for various $N_p$ in the case of the triangular function. In each figure, the inset shows the enlarged plot of the interval $1 \leq x \leq 2$.

$x_j = j \Delta x$ ($j = -N/2, \cdots, N/2 - 1$), where the grid spacing is $\Delta x = 2\pi/N$. We fix $N = 128$, while the number of particles $N_p$ is varied as $N_p = 1024, 2048, 4096$ and $8192$. In this case the number of particles within an unit width $\Delta x$ is $N_p/N = 8, 16, 32$ and $64$. The position of particles $x_i$ is determined by using the uniform random number in the interval $[-\pi, \pi]$. We compute equation (4) for above three cases of $\delta_\Delta(x)$ and compare the obtained results with the analytical result of $\sin x$ which is given in the limit $N_p \to \infty$.

Figure 1 (a) shows the comparison of obtained $T_\Delta(x)$ for the cases of i)-iii) when $N_p = 2048$. Zig-zag fluctuations around the exact sine curve are observed when the rectangular and triangular functions are used, while the curve obtained by using the Gaussian function is smooth in the whole region and close to the exact one. Figure 1 (b) indicates how $T_\Delta(x)$ depends on $N_p$ in the case for the triangular function. As expected, the amplitude of zig-zag fluctuations decreases with increasing $N_p$, although small fluctuations around the curve of $\sin x$ remain for $N_p = 8192$ ($N_p/N = 64$).

To quantify the deviation of resulting $T_\Delta(x)$ curves from the analytical result, we evaluate the $L^2$ norm

$$\sigma^2 = \frac{1}{L} \int_{-L/2}^{L/2} [T_\Delta(x) - \sin x]^2 \, dx \quad (9)$$

and examine its $N_p$-dependence. Figure 2 shows $\sigma$ against $N_p$ for above three cases of $\delta_\Delta(x)$. Result obtained by using Gaussian function is about one decade smaller than those of rectangular and triangular cases, indicating that the smoothing by Gaussian function leads to the better behavior when compared to that by other two cases. It is worth noting that $\sigma$ decays with $N_p$ in the power-law form $N_p^{-0.4}$ irrespective of the functional form of $\delta_\Delta(x)$. This slow decay suggests that much larger number of $N_p$ is required to obtain well converged $T_\Delta(x)$, especially the case of rectangular function is serious.

2.2. Efficient calculation of smoothed point force using Gaussian function

In the previous subsection, we obtained that the smoothed point force using Gaussian function yields the accurate behavior of the resulting function which is close to the analytical result. However the computational cost to obtain equation (4) when using $\delta_\Delta^G(x)$ was much higher than other methods. Moreover if we consider the problem in three dimension, for example, the number of times of evaluating $\delta_\Delta^G(x)$ is increased by $C_1 = 7^3 N_p$, which is considerable computational cost when $N_p \gg N$ case like the polymer-laden flows is conducted [9]. In this subsection, we newly
propose the approximated but efficient computational method of equation (4) with making use of the Gaussian function when the turbulent motion is solved by Fourier spectral method.

It is interesting to note that the relationship between \( T(x) \) and \( T_\Delta(x) \) is expressed by

\[
\hat{T}_\Delta(k) = \hat{\delta}_\Delta(k) \hat{T}(k)
\]

in Fourier space, where the variable with symbol of top hat \( \hat{g}(k) \) represents the Fourier amplitude of \( g(x) \). The rectangular (6) and Gaussian (8) functions are respectively represented in Fourier space by

\[
\hat{\delta}_\Delta(k) = \frac{\sin(k\Delta/2)}{k\Delta/2}, \quad \hat{\delta}_g^\Delta(k) = \exp \left[ -\frac{1}{2} (k\Delta)^2 \right].
\]

The idea is to find an appropriate function \( \hat{d}_\Delta(k) \) leading to the relationship

\[
\hat{\delta}_\Delta^g(k) \simeq \hat{d}_\Delta(k) \hat{\delta}_\Delta^r(k)
\]

because

\[
\hat{T}_\Delta^g(k) \simeq \hat{d}_\Delta(k) \hat{T}_\Delta^r(k),
\]

where \( \hat{T}_\Delta^g(k) \) and \( \hat{T}_\Delta^r(k) \) are evaluated by using the Gaussian and rectangular functions for \( \delta_\Delta(x) \), respectively. Once we know the function \( \hat{d}_\Delta(k) \), the procedure to compute the approximated \( T_\Delta^g(x) \) is as follows.

1) Evaluating \( T_\Delta^r(x) \) in physical space using equation (4), and transforming it into the Fourier amplitude \( \hat{T}_\Delta^r(k) \) with the aid of the fast Fourier transform (FFT) algorithm.

2) Evaluating \( \hat{d}_\Delta(k) \hat{T}_\Delta^r(k) \) in Fourier space and transforming it into the physical space variable.

If \( N_p \gg N \), most of the computational costs come from accessing an array for \( T_\Delta(x_j) \), indicating that the computational cost for the above scheme is comparable with that of evaluating \( T_\Delta^g(x) \).

The left problem is how \( \hat{d}_\Delta(k) \) is determined. We propose the following Gaussian function as

\[
\hat{d}_\Delta(k) = \exp \left[ -\frac{1}{2} (k\gamma)^2 \right], \quad \gamma = \sqrt{\frac{TT}{12}\Delta}.
\]

The factor \( \sqrt{11/12} \) originates from the condition that the first and second order terms of the Taylor series expansion of \( \hat{d}_\Delta(k) \hat{\delta}_\Delta^r(k) \) with (14) are equalized to those of \( \hat{\delta}_\Delta^g(k) \) in (11). To check the relationship (12) with (14), we compare the approximated form of \( \hat{\delta}_\Delta^g(k) \) with exact one. Figure 3 (a) indicates the resulting curves when \( \Delta = \pi/64 \). Approximated form evaluated
by equation (12) with (14) almost coincides with the result by exact Gaussian function as far as \( k \leq 64 \). Figure 3 (b) indicates \( T_\Delta(x) \) evaluated by equation (4) with Gaussian function and by the approximated method we proposed here when \( N_p = 8192 \). It is confirmed that both curves are almost similar, representing that the approximated method can reproduce the result by the direct method using (4) with much lower computational cost.

3. Two-way coupling simulation for polymer-laden isotropic turbulence

In this section we discuss how the details of smoothing of the point force affect the nature of interactions between polymer and turbulent flow by performing the two-way coupling simulations.

3.1. Fundamental equations of motion

We briefly review the fundamental equations of motion. The motion of long-chain polymer in fluid is described by the beads-spring model [12]. In this study we consider the simplest case that the elastic nature of polymer chain is represented by 2-beads connected by nonlinear spring, called the dumbbell model [13]. This is given by

\[
\frac{dR^{(n)}}{dt} = u_1^{(n)} - u_2^{(n)} - \frac{1}{2\tau} f \left( \frac{|R^{(n)}|}{L_{max}} \right) R^{(n)} + \frac{r_{eq}}{\sqrt{2\tau}} \left( W_1^{(n)} - W_2^{(n)} \right),
\]

\[
\frac{dr_g^{(n)}}{dt} = \frac{1}{2} \left( u_1^{(n)} + u_2^{(n)} \right) + \frac{r_{eq}}{\sqrt{8\tau}} \left( W_1^{(n)} + W_2^{(n)} \right), \quad u_\alpha^{(n)} = u(x_\alpha^{(n)}(t), t),
\]

where \( R^{(n)}(t) \) and \( r_g^{(n)}(t) \) are, respectively, the end-to-end vector and the center-of-mass vector of the \( n \)-th dumbbell. We adopt the finitely extensible nonlinear elastic (FENE) model \( f(z) = 1/(1 - z^2) \) for the elastic force of a dumbbell. In equation (15), \( L_{max} \) is the maximum extension length of the dumbbell because \( f(z \to 1) = \infty \). The term \( W_{1,2}^{(n)}(t) \) indicates a random force representing the Brownian motion of particles in the solvent fluid, which obeys Gaussian statistics with a white-in-time correlation of \( \langle W_\alpha^{(n)}(t) \rangle = 0 \) and \( \langle W_\alpha^{(n)}(t) W_\beta^{(m)}(s) \rangle = \delta_{\alpha\beta} \delta_{ij} \delta_{mn} \delta(t - s) \), where \( \langle \cdots \rangle \) denotes the ensemble average. The subscripts \( \alpha, \beta, i, j, n, \)

![Figure 3. (a) Validation of the proposed form equation (12) with equation (14) (approximated Gaussian function). (b) Comparison of \( T_\Delta(x) \) computed by the direct method equation (4) with that by the approximated method using equations (12) and (14), where the resulting curves in the range \( 0 \leq x \leq \pi \) are shown for simplicity.](image-url)
and \( m \) take the values \((\alpha, \beta) = 1 \) or \(2\), \((i, j) = 1\), \(2\), and \(3\), and \((n, m) = 1\), \(2\), \(\cdots\), \( N_p \), respectively. Moreover, \( \delta_{ij} \) denotes the Kronecker delta. The constants \( \tau \equiv \zeta/(4k) \) and \( r_{eq} = \sqrt{k_BT/k} \), are, respectively, the relaxation time and the equilibrium length of the dumbbell under \( u(x, t) = 0 \). Here, \( k \) is the spring constant, \( k_B \) and \( T \) are the Boltzmann constant and temperature, respectively.

The turbulent velocity field obeys the continuity equation \( \nabla \cdot u = 0 \) for an incompressible fluid and the NS equations

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu_f \nabla^2 u + \nabla \cdot T^p. \tag{17}
\]

Here, \( \rho_f \) is set to unity and is equal to the density of bead \( \rho_p \) representing the polymer. \( T^p(x, t) \) is the polymer stress tensor due to the force acting on the fluid from the dispersed dumbbells and is defined by

\[
T^p_{ij}(x, t) = \frac{\nu_f \eta}{\tau} \left( \frac{L_{box}^3}{N_p} \right) \sum_{n=1}^{N_p} \left[ \frac{P_{i}^{(n)} R_{j}^{(n)}}{N_{eq}^2} - \frac{1}{L_{max}} \right] \delta_{ij} \delta(x - r_{j}^{(n)}), \tag{18}
\]

where \( N_p \) is the total number of dumbbells used in simulations. \( \eta \) is the ratio of the zero-shear viscosity \( \nu_p \) of the polymer to the solution viscosity \( \nu_f \) \((\eta = \nu_p/\nu_f)\), and is the parameter to control the concentration of polymer solution.

### 3.2. Summary of numerical setup

The two-way coupling simulations of equations (15) through (18) are performed in a periodic box with periodicity \( L_{box} = 2\pi \) using the pseudo-spectral method in space and the second-order Runge-Kutta method in time. A total of \( 128^3 \) grid points are set to solve the NS equations. We consider the case of decaying isotropic turbulence, as discussed in our previous studies [9, 10]. The initial velocity field is given by the random solenoidal field obeying Gaussian statistics and yields an energy spectrum \( E(k, 0) = 16\sqrt{2/\pi} (u_0^2/k_0) (k/k_0)^4 \exp\left(-2(k/k_0)^2\right) \) with constants \( u_0 = 1 \) and \( k_0 = 2 \).

The total number of dumbbells in computation is \( N_p = 5 \times 10^8 \). The dumbbells are uniformly and randomly distributed over the computational domain. The initial configuration of each dumbbell is set according to \( \mathbf{R}^{(m)}(0) = \sqrt{3} r_{eq} \mathbf{n}^{(m)} \), where \( \mathbf{n}^{(m)} \) is an unit vector whose orientation is randomly distributed. We fix \( \eta = 0.1 \) and \( \tau = 20 \) whose values correspond to those used for Run D3 in [10].

The computation is parallelized by writing the code using Message Passing Interface (MPI) and Open MP. For details about parallel computation, please refer to the reference [14]. We examine the three cases of smoothing method discussed in the previous section. The approximated method for making use of the Gaussian function, proposed in the previous section, is introduced in the present analysis. For comparison we present the results obtained by our previous scheme [10] (named by WG14), in which we evaluate the polymer stress on the regular and staggered grid points, respectively, with making use of the triangular function and by taking the average of them in the Fourier space. For more details please see appendix in the reference [10].

### 3.3. Results

Figure 4 (a) shows the temporal evolutions of the kinetic energy spectrum which is defined by

\[
E(k, t) = \sum_k \frac{1}{2} |\hat{u}(k, t)|^2 \tag{19}
\]
in terms of the Fourier amplitude \( \hat{u}(k, t) \) where \( \sum'_{k} \) is the summation taken over the spherical shell within \( k - \Delta k/2 < |k| \leq k + \Delta k/2 \) in the wavenumber space. In this figure \( k^2 E(k, t) \) is plotted to stress the behavior in the high wavenumber range. It is seen from this figure that curves of \( E(k, t) \) in the lower wavenumber range collapse well irrespective of the choice of \( \delta_\Delta(x) \), indicating that large-scale motion of turbulence is not affected by the details of smoothing method. However there are significant difference of the behavior of \( k^2 E(k, t) \) in the high wavenumber range for different \( \delta_\Delta(x) \). The worst case is to use the rectangular function because the spectra near the cutoff wavenumber pile up when compared to other cases. This means that the velocity gradient fluctuations like the local energy dissipation are not well resolved on grids. On the other hand the results with the Gaussian function show the rapid decay of spectrum in its spectral tail, suggesting the smooth behavior of fluctuating velocity gradient field. This trend is also directly observed in the spectral behavior of the polymer stress tensor shown in figure 4 (b), where we indicate the spectrum of \( T_{11}(x, t) \), \( P_{11}(k) = \sum_{j} T_{11}(k, t)^2 \), with multiplying \( k^2 \). Note that the results obtained by using the triangular function almost coincide with those by WG14 because the smoothing method in WG14 is based on the triangular function. Difference among them appears for the intermittent fluctuations of polymer stress \( T^p \) in physical space (see figure 12 (b) in reference [10]).

As shown in figure 4 (a), \( k^2 E(k, t) \) obtained by using the smoothed point force with Gaussian function is steeper than that obtained by our previous method (WG14) [10] in which the power-law decay of the form \( E(k, t) \propto k^{-\alpha} \) with \( \alpha = 4 \sim 5 \) is observed in the final period of decaying turbulence. Because the support of Gaussian function in physical space is wider than that of rectangular and triangular functions, it is plausible that the amplitude of a spectrum evaluated by the smoothed point force with Gaussian function damps strongly in the high wavenumber range more than that evaluated by using other functions. This fact suggests if the Fourier modes are excited in the wide range of scale like the power-law decay of \( E(k, t) \), we need great care for the effect of smoothing with a Gaussian function on the resulting spectral behavior. To examine the impact for using \( \delta_\Delta(x) \) on spectral behavior in the low wavenumber range, we perform an additional simulation which has higher resolution with grid points 256\(^3\) but other physical conditions are unchanged. Note that because \( \Delta x \) with \( N = 256^3 \) is half of that with \( N = 128^3 \), the width \( \Delta \) of Gaussian function is also half of that with the case of 128\(^3\). Figure 5 shows the comparison of resulting curves of \( k^2 E(k, t) \). Although \( E(k, t) \) in lower wavenumber

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Temporal evolutions of (a) the kinetic energy and (b) variance of polymer stress \((T_{11}^{(p)})\) spectra evaluated for the four cases of the smoothing method when computing \( T_{11}^{(p)} \) (equation (18)). All spectra are plotted by multiplying \( k^2 \) to stress the behavior in high wavenumber range, and the curves obtained at \( t = 4 \) and \( t = 8 \) are shifted lower for being easy to see.}
\end{figure}
range almost coincides with each other, we observe different behavior at the high wavenumbers where the curves obtained by $256^3$ grid points are shallower than those by $N = 128^3$ and follow the results of our previous method [10]. This result suggests that we must pay attention to the spatial resolution when using the Gaussian function and examining the physical process in the high wavenumber range.

Finally we comment on the computational time required for simulating the present polymer-laden isotropic turbulence. The real computational time required for time development per one time step $\Delta t$ is summarized in table 1. As expected the computational time required for the simulation using the rectangular function is shortest throughout the present runs, and this time is almost comparable to that using the approximated Gaussian function. The computational time of our previous method [10] is about twice of that using Gaussian function, indicating the efficiency of computation using approximated Gaussian function developed in the present study.

4. Conclusion
In this paper we discussed how the details of smoothing method for point force of two-way coupling simulations in polymer-laden turbulent flow affect the turbulence statistics. We considered the three types of functions (rectangular, triangular and Gaussian) to approximate the delta function when evaluating the point force. First we examined the accuracy of the methods by applying to the simple one-dimensional problem for which the analytical form is known. It was found that the rectangular and triangular functions showed zig-zag fluctuations of the resulting curves. On the other hand the results obtained by using the Gaussian function showed the accurate behavior close to the analytical result, suggesting the advantage to use it for two-way coupling problem. We also proposed the efficient computational method to use the Gaussian function when smoothing the point force, and confirmed the validity of our idea. This method was applied to the two-way coupling simulations of polymer-laden isotropic turbulence, where we could reproduce the results obtained by the method in [10] with satisfying accuracy.

| Rectangular | Triangular | Gaussian | WG14 | Gaussian (N = 256$^3$) |
|-------------|------------|----------|------|------------------------|
| time (s)    | 1.85       | 4.95     | 2.18 | 4.65                   |

Table 1. Comparison of mean computational time required for time development per one time step $\Delta t$. “Gaussian” in table means to use the approximated form (12) with (14). Note that $\Delta t$ of the computation using $N = 256^3$ grid points is half of that of $N = 128^3$ case.
and better efficiency of computation.

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