INVERSE OPTICAL IMAGING VIEWED AS A BACKWARD CHANNEL COMMUNICATION PROBLEM

ENRICO DE MICHELI
IBF – Consiglio Nazionale delle Ricerche
Via De Marini, 6 - 16149 Genova, Italy

GIOVANNI ALBERTO VIANO
Dipartimento di Fisica – Università di Genova,
Istituto Nazionale di Fisica Nucleare – Sezione di Genova,
Via Dodecaneso, 33 - 16146 Genova, Italy

Abstract. The inverse problem in optics, which is closely related to the classical question of the resolving power, is reconsidered as a communication channel problem. The main result is the evaluation of the maximum number $M_\varepsilon$ of $\varepsilon$–distinguishable messages ($\varepsilon$ being a bound on the noise of the image) which can be conveyed back from the image to reconstruct the object. We study the case of coherent illumination. By using the concept of Kolmogorov's $\varepsilon$–capacity, we obtain: $M_\varepsilon \sim 2^{S \log(1/\varepsilon)} \xrightarrow{\varepsilon \to 0} \infty$, where $S$ is the Shannon number. Moreover, we show that the $\varepsilon$–capacity in inverse optical imaging is nearly equal to the amount of information on the object which is contained in the image. We thus compare the results obtained through the classical information theory, which is based on the probability theory, with those derived from a form of topological information theory, based on Kolmogorov's $\varepsilon$–entropy and $\varepsilon$–capacity, which are concepts related to the evaluation of the massiveness of compact sets.

1. Introduction

The definition of the resolving power of an optical system is a classical problem of optics with a very long history, which goes back to Lord Rayleigh. It is precisely his criterion for resolution which is a milestone in this theory. As is well–known, however, this criterion remains somehow empirical, and it is sometimes considered a quite arbitrary choice.

According to geometrical optics, the image of a point source provided by an optical instrument is a perfectly sharp point. However, because of diffraction effects, the image of a point is not a point but a small light patch, called the diffraction pattern. Optical instruments, whose diffraction effects are important, are called diffraction–limited imaging systems, and hereafter we shall refer to only this type of optical systems.

E-mail addresses: enrico.demicheli@cnr.it, viano@ge.infn.it.
We assume, for simplicity, that the scalar theory of light can be used. In this theory monochromatic light is represented by a scalar function, which is usually written as a complex–valued function, called the complex amplitude, whose modulus and phase are respectively the amplitude and the phase of the light disturbance. In the case of spatially coherent illumination (for short, coherent illumination) the relative phase of two object points is constant in time, i.e. even if the two phases can vary randomly in time, they vary in an identical fashion.

We consider systems producing real (non virtual) images. We also assume that the system is isoplanatic, i.e., space–invariant. In practice, optical imaging systems are seldom isoplanatic over the whole object field, but it is also possible to divide the object field into regions within which the system is approximately space–invariant. Finally, we assume that the magnification factor of the optical system has been reduced to one by a suitable re–scaling of the space variables of the image plane.

Diffraction–limited imaging systems are usually treated by Fourier methods, and the corresponding theory is called Fourier optics. Assume that \( f(x) \) denote the complex amplitude distribution of a coherently illuminated object; for reasons of simplicity but without loss of generality, we limit ourselves to consider unidimensional objects. The Fourier transform of \( f(x) \),

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} \, dx,
\]

is an entire function in the complex \( \omega \)–plane since \( f(x) \) is space–limited. Then one could argue (as observed by several authors \[1, 2\]) that, even though the knowledge of the function \( F(\omega) \) is limited to the finite interval \( |\omega| \leq \Omega \) since the pupil stops all the waves with \( \omega \) larger than a positive constant \( \Omega \), nevertheless, in view of the uniqueness of the analytic continuation, one could determine uniquely \( F(\omega) \) everywhere. Hence, the object could be reconstructed in all its details, and there should be no loss of information in passing through the optical system: in principle, analytic continuation in the frequency domain will allow for restoration of unlimited details \[1\]. But the uniqueness of the analytic continuation does not imply its stability, namely, a continuous dependence of the solution on the data. The ill–posedness \[3\] of the analytic continuation, and more generally of the inverse problem, was then recognized \[4\], and the theory of the regularization of the ill–posed problems in the sense of Hadamard was extensively applied to Fourier optics \[5\].

The mathematical inverse problem in optics, within the scheme outlined above, can be formulated as follows. Consider an unidimensional object and refer to the conventional optical system depicted in Fig. \[1\]. A plane object, illuminated with coherent light, gives rise to a complex amplitude distribution \( f(x) \) at the front focal plane of the lens \( L_1 \) (see Fig. \[1\]). A real image is formed at the rear focal plane of the lens \( L_2 \). Lenses \( L_1 \) and \( L_2 \) have a common focus at the stop plane or pupil plane. Now, let us return to the Fourier transform \( F(\omega) \); if both lenses \( L_1 \) and \( L_2 \) are assumed to fulfill the sine condition \[6, p. 166\], then \( \omega \) is proportional to the vertical coordinate on the pupil plane (see Fig. \[1\]). As we have already remarked, the pupil–stop blocks all of the contributions that have \( |\omega| \) larger than the positive constant \( \Omega \). As a consequence, on the image plane we will not recover exactly \( f(x) \),
but its band–limited version

\[ g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} F(\omega) e^{i\omega y} d\omega. \]  

Inserting the expression of \( F(\omega) \) given by (1) into (2), and assuming that the object distribution \( f(x) \) vanishes outside the interval \( -X_0/2 \leq x \leq X_0/2 \), we have

\[ g(y) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{i\omega y} d\omega \int_{-X_0/2}^{X_0/2} f(x) e^{-i\omega x} dx = \int_{-X_0/2}^{X_0/2} \frac{\sin[\Omega(x-y)]}{\pi(x-y)} f(x) dx. \]

The image function \( g(y) \) is an entire band–limited function, and the sampling theorem guarantees that it can be reconstructed, without loss of information, when its values are known at a set of sampling points, chosen in arithmetic progression with difference \( \pi/\Omega \); notice that in optics, in the unidimensional situation, the Rayleigh distance \( R \) (also called the Nyquist distance) is \( R = (\pi/\Omega) = \text{resolution distance} \). In particular, the image \( g(y) \) can be reconstructed in the interval \( (-X_0/2, X_0/2) \) from the knowledge of the function on a set of \( S \) points, where \( S = X_0/(\pi/\Omega) = \Omega X_0/\pi \) is the Shannon number of the image \([2, 7]\). It is in this connection that several authors (notably, Toraldo di Francia \([2]\)) argued that an image can be completely determined by \( S \) (complex) numbers, which are called the image degrees of freedom.

Equation (3) can be re–written in operator form as follows

\[ (Af)(y) = \int_{-X_0/2}^{X_0/2} \frac{\sin[\Omega(x-y)]}{\pi(x-y)} f(x) dx = g(y) \quad \left(-\frac{X_0}{2} \leq y \leq \frac{X_0}{2}\right). \]

Then, the problem of object restoration is equivalent to solving the Fredholm integral equation of the first kind \( Af = g \), where \( A \) is a self–adjoint, non–negative and compact operator, \( g \) represents the data (the image), and \( f \) is the unknown (the object distribution). As we remarked above, this problem is ill–posed: the solution to Eq. (4), even if it is unique, does not depend continuously on the data. Small perturbations of the data, due to the noise, produce wide oscillations in the solution, the problem needs regularization.

\[ \text{Figure 1. Schematic of coherent light image formation in a one–dimensional diffraction–limited optical system (see also [2]).} \]
Summarily, we may distinguish two different approaches to regularization:

(a) Methods that require well-defined \emph{a priori} global bounds on the solution, and work in definite functional spaces. These methods could be called \emph{deterministic}, understanding this terminology in a wide sense \cite{8, 9}.

(b) Methods that make use of techniques taken from the theory of probability, which can be called \emph{probabilistic} \cite{10, 11}.

In Section 2 we shall briefly review both these methods, advancing some remarks, in particular about the standard \emph{deterministic} regularization.

In this paper we approach the problem from a new viewpoint. The problem of reconstructing the object from the image is regarded as a communication channel problem. In this context we estimate the messages which can be conveyed back from the data set (the image) to reconstruct the signal (the object). The maximum number of these messages is limited by the noise affecting the image. One could expect that the maximum number of these messages tends to infinity as the noise affecting the image tends to zero. In this way the theory can provide a precise and quantitative dependence of the resolution on the noise.

One of the main purposes of this paper is therefore to connect the regularization methods to information theory. In Section 3 we shall develop a topological information theory, which can be derived without making use of the tools proper of the probability theory. It is rather based on the concepts of \(\varepsilon\)-entropy and \(\varepsilon\)-capacity, introduced by Kolmogorov \cite{12}. The main result which we obtain is summarized by the following formula

\[ M_\varepsilon \simeq 2^{S_{\log(1/\varepsilon)}}, \]

where \(M_\varepsilon\) is the maximum number of \(\varepsilon\)-distinguishable messages which can be conveyed back through the channel from the noisy data set (the image) to recover the object; \(S = \Omega X_0/\pi\) is the Shannon number (introduced above), \(\varepsilon\) is a bound on the noise affecting the image, and \(\log x\) stands here and throughout the paper for the logarithm of \(x\) to the base 2. In Section 4 instead, we develop an approach based on the probabilistic information theory, that is, the information theory which follows from the use of probabilistic methods \cite{13}. This allows us to compare the results obtained by the topological and the probabilistic information theory; in particular, we can interpret the bounds on the information content of the image in terms of spectral distribution of the noise and of the object.

2. Review and remarks on regularization methods

Equation (4) is a Fredholm equation of the first kind, and the operator \(A\) is acting as follows: \(A : X \to Y\), where \(X\) and \(Y\) are the solution and the data space, respectively. We take here, for simplicity and without loss of generality, \(X = Y = L^2(-X_0/2, X_0/2)\). As we said in the Introduction, the operator \(A\) is self-adjoint, non-negative, and compact. Moreover, the unique solution of the equation \(Af = 0\) is \(f = 0\). Then, we can say that the integral operator \(A\) admits a complete set of orthogonal eigenfunctions \(\{\psi_k\}_{k=0}^\infty\) corresponding to a countably infinite set of real positive eigenvalues \(\lambda_0 > \lambda_1 > \lambda_2 > \cdots\); moreover, \(\lim_{k \to +\infty} \lambda_k = 0\). The properties of this integral operator have been already studied by several authors \cite{14, 15, 16, 17}, and the literature on this topic is quite extensive. Suppose that \(S = \Omega X_0/\pi\) is sufficiently large, then the eigenvalues \(\lambda_k\) form a decreasing sequence \(1 > \lambda_0 > \lambda_1 > \cdots > 0\), which enjoys a step-like behavior, i.e., they are approximately
equal to 1 for $k \lesssim S$, and then fall off to zero exponentially (see Fig. 2 and Refs. [2, 17]). Since $A : X \to Y$ is compact then the range $R(A)$ is not closed in the data space $Y$. Therefore, given a data function $g \in Y$, it does not necessarily follow that there exists a solution $f \in X$. Moreover, even if two data functions $g_1$ and $g_2$ belong to $R(A)$ and their distance in $Y$ is small, nevertheless the distance between $A^{-1}g_1$ and $A^{-1}g_2$ can be unlimited large, in view of the fact that the inverse of the compact operator $A$ is not bounded ($X$ and $Y$ being infinite dimensional spaces).

Since there always exists some inherent noise in the data, instead of (4) we have to deal with the following equation

$$Af + n = g \quad (g = g + n),$$

where $n$ denotes the noise. Here we have assumed a purely additive model of noise, and hereafter we suppose that $n$ is a small perturbation of the data function, in order to still have $g \in R(A)$.

2.1. **Deterministic regularization methods.** Several methods of regularization have been proposed (see [8, 9] and references quoted therein); all of them aim at modifying one of the elements of the triplet $\{A, X, Y\}$, where $A$ is the integral operator defined by (3), and $X$ and $Y$ are the solution and data space, respectively (here we continue to assume $X = Y = L^2(-X_0/2, X_0/2)$). Among these methods the procedure which is probably the most popular consists in looking for the solution in a compact subset of the solution space $X$; then continuity of the inverse operator follows from compactness. This restriction of the solution space, which ultimately leads to a compact subset of $X$, is realized by means of suitable a priori bounds that should represent some prior knowledge on the solution. More precisely, in addition to the inequality

$$\|Af - g\|_Y \leq \varepsilon \quad (\varepsilon = \text{constant}),$$

![Figure 2](image-url)  
**Figure 2.** The eigenvalues $\lambda_k$ (filled dots) of the kernel in (3) with Shannon number $S = 12.7$. 
which corresponds to a bound on the noise, one also assume an a priori bound on
the solution of the following form
\[ \|Bf\|_Z \leq E \quad (E = \text{constant}), \]
where \( Z \) denotes the constraint space, and \( B \) is the constraint operator. From
bounds (7) and (8) we are led to determine the minimum of the following functional
\[ \Phi(f) = \|Af - g\|_Y^2 + \mu^2 \|Bf\|_Z^2 \quad (\mu = \frac{\epsilon}{E}), \]
which can be proved to be a regularized solution [18]. Let \( A^* \) denote the adjoint
operator of \( A \). We take as constraint operator \( B \) a self-adjoint operator; moreover,
we assume that \( B^*B \) and \( A^*A \) commute (this assumption does not restrict signif-
icantly the theory and the applications). The space \( Z \) is then composed of those
functions \( f \in L^2(-X_0/2, X_0/2) \) such that
\[ \|Bf\|_Z \text{ is finite, i.e.,} \]
\[ \|Bf\|_Z = \left( \sum_{k=0}^{\infty} \beta_k^2 |f_k|^2 \right)^{1/2} < E \quad (E = \text{constant}), \]
where \( f_k = (f, \psi_k) \) (\( (\cdot, \cdot) \) denoting the scalar product in \( L^2(-X_0/2, X_0/2) \)), \( B^*Bf = \sum_{k=0}^{\infty} \beta_k^2 f_k \psi_k \), \( \beta_k^2 \) being the eigenvalues of \( B^*B \) (i.e., \( B^*B \psi_k = \beta_k^2 \psi_k \)). Moreover,
we require that \( \lim_{k \to \infty} \beta_k^2 = +\infty \), in order to guarantee that the subset of the
solution space, which is composed of those functions satisfying (8), is compact.
Now the functional \( \Phi(f) \) has a unique minimum, given by
\[ f = \frac{A^*g}{A^*A + (\epsilon/E)^2 B^*B}, \]
which, by expanding \( g \) in terms of the functions \( \psi_k \), can be written as follows,
\[ f = \sum_{k=0}^{\infty} \frac{\lambda_k g_k}{\lambda_k^2 + (\epsilon/E)^2 \beta_k^2} \psi_k. \]
Then, the following propositions can be proved. The proofs are given in very
detailed form in [18], which refers to a different physical problem (the antenna
synthesis). Nevertheless, the eigenfunctions used there are the prolate spheroidal
wave functions (as in the present problem), and the deterministic regularization
methods are given in variational form, which is appropriate for our case here.

**Proposition 1.** For any function \( f \) satisfying the bounds (7) and (8), the following
limit holds
\[ \lim_{\epsilon \to 0} \|f - f\|_X = 0 \quad (E = \text{fixed}). \]

**Proof.** See Proposition 12 of [18]. \qed

In actual numerical computation it is often convenient to use truncated approxi-
mations. For instance, the solution (12) leads to define the following approximation
\[ f^{(1)} = \sum_{k=0}^{k_0} \frac{g_k}{\lambda_k} \psi_k, \]
where \( k_0 \) is the largest integer such that
\[ \lambda_k \geq |\beta_k| \frac{\epsilon}{E}. \]
Proposition 2. For any function $f$ satisfying bound (7), the following limit holds
\[
\lim_{\varepsilon \to 0} \| f - f^{(1)} \| = 0 \quad (E = \text{fixed}).
\]

Proof. See Proposition 12 and its Corollary in [18]. □

In several problems a weaker a priori bound on the solution can be used by setting $B = I$, the identity operator. Therefore, instead of bound (8), we have
\[
\| Bf \|_Z \equiv \| f \|_{L^2(-X_0/2,X_0/2)} = \left( \sum_{k=0}^{\infty} |f_k|^2 \right)^{1/2} \leq E \quad (E = \text{constant}).
\]

In this case the unique minimum of functional (9) is given by
\[
f^{(2)} = \sum_{k=0}^{\infty} \frac{\lambda_k g_k}{\lambda_k^2 + (\varepsilon/E)^2} \psi_k,
\]
and, accordingly, the following truncated approximation can be introduced
\[
f^{(3)} = \sum_{k=0}^{k_1} \frac{g_k}{\lambda_k} \psi_k,
\]
where $k_1$ is the largest integer such that
\[
\lambda_k \geq \frac{\varepsilon}{E}.
\]

Both $f^{(2)}$ and $f^{(3)}$ converge to $f$ in the weak sense. In fact, the following propositions can be proved.

Proposition 3. For any function $f$ satisfying bounds (7) and (17), the following limit holds
\[
\lim_{\varepsilon \to 0} |(f - f^{(2)}, v)| = 0 \quad \left( \forall v \in L^2\left(-\frac{X_0}{2}, \frac{X_0}{2}\right); \ E = \text{fixed} \right).
\]

Proof. See Proposition 13 and its Corollary in [18]. □

Proposition 4. For any function $f$ satisfying bound (7) and (17), the following limit holds
\[
\lim_{\varepsilon \to 0} |(f - f^{(3)}, v)| = 0 \quad \left( \forall v \in L^2\left(-\frac{X_0}{2}, \frac{X_0}{2}\right); \ E = \text{fixed} \right).
\]

Proof. See Proposition 14 and its Corollary in [18]. □

Remark 1. These regularization methods are not free from faults. We restrict ourselves to mention just two of them. For reason of simplicity we shall focus on the approximation $f^{(3)}$, but the same considerations hold also for $f^{(1)}$.

(i) Approximation (19) is based on the truncation criterion (20). Put, for simplicity and without loss of generality, $E = 1$. Then formula (20) reads: $\lambda_k \geq \varepsilon$. This means that the values of $\lambda_k$ (i.e., the eigenvalues of the operator $A$ representing the optical instrument) should be compared with the bound on the noise $\varepsilon$. But this approach appears quite unnatural from the viewpoint of the experimental or physical sciences, whose methodology rather suggests to compare the signal with the noise. In other words, the expansions should rather be truncated at the value $k_p$ of $k$ such that for $k > k_p$ the Fourier coefficients $g_k = (g, \psi_k)$ of the noiseless data are smaller or, at most, of the same order of magnitude of $\varepsilon$. In this case, in
fact, it would be impossible to extract information from the corresponding noisy coefficients $g_k = (g, \psi_k)$.

(ii) This second remark is strictly connected to the first one. It is easy to exhibit examples of objects $f$ whose corresponding images $g$ have Fourier components small for low values of $k$, while the significant contributions are carried by those Fourier components which are suppressed by condition (20) (i.e., $\lambda_k < \varepsilon$, $E = 1$). This remark holds also for more refined solutions of the form (11), which correspond to the minimization of functional (9). Indeed, the minimization of this functional works as a low–pass filter, whose action is smoothing the Fourier components $g_k$ for high values of $k$. This latter statement follows by noting that Proposition 1 holds if and only if $\lim_{k \to \infty} \beta_k^2 = +\infty$. In conclusion, it is possible to give examples where the standard deterministic regularization methods fail in spite of their rigorous mathematical correctness, since these procedures do not guarantee that the bulk of the signal (of the object, in our case) has been really recovered (see [11]).

2.2. Probabilistic regularization methods. We now want to reconsider Eq. (6) from a probabilistic point of view. With this in mind, we re–write (6) in the following form

$$A\xi + \zeta = \eta,$$

where $\xi$, $\zeta$, and $\eta$, which correspond to $f$, $n$, and $g$ respectively, are Gaussian weak random variables (w.r.v.) in the Hilbert space $L^2(-X_0/2, X_0/2)$ [19]. A Gaussian w.r.v. is uniquely defined by its mean element and its covariant operator. In the present case we denote by $R_{\xi\xi}$, $R_{\zeta\zeta}$, and $R_{\eta\eta}$ the covariance operators of $\xi$, $\zeta$, and $\eta$ respectively. Next, we make the following assumptions:

(I) $\xi$ and $\zeta$ have zero mean, i.e., $m_\xi = m_\zeta = 0$;
(II) $\xi$ and $\zeta$ are uncorrelated, i.e., $R_{\xi\zeta} = 0$;
(III) $R_{\zeta\zeta}^{-1}$ exists.

The third assumption is the mathematical formulation of the fact that all the components of the data function are affected by noise. As it has been proved by Franklin (see formula (3.11) of [10]), if the signal and the noise satisfy assumptions (I) and (II), then

$$R_{\eta\eta} = AR_{\xi\xi}A^* + R_{\zeta\zeta},$$

and the cross–covariance operator is given by

$$R_{\xi\eta} = R_{\xi\xi}A^*.$$

We also assume that $R_{\zeta\zeta}$ depends on a parameter $\varepsilon$ which tends to zero when the noise vanishes, i.e., we write

$$R_{\zeta\zeta} = \varepsilon^2 \mathbb{H},$$

where $\mathbb{H}$ is a given operator (e.g., $\mathbb{H} = \mathbb{I}$ = the identity operator, in the case of white noise). We can now state the following problem.

**Problem 1.** Given a value $g$ of the w.r.v. $\eta$, find an estimate of the w.r.v. $\xi$.

We first turn Eq. (23) into an infinite sequence of unidimensional equations by means of the orthogonal projections,

$$\lambda_k \xi_k + \zeta_k = \eta_k \quad (k = 0, 1, 2, \ldots),$$
where $\xi_k = (\xi, \psi_k)$, $\zeta_k = (\zeta, \psi_k)$, $\eta_k = (\eta, \psi_k)$ are Gaussian random variables. Equations \cite{26} can be obtained by formal expansions of the w.r.v. $\xi$, $\zeta$, and $\eta$ on the orthonormal basis $\{\psi_k\}_{k=0}^{\infty}$ (which are the eigenfunctions of the operator $A$), i.e., $\xi = \sum_{k=0}^{\infty} \xi_k \psi_k$, $\zeta = \sum_{k=0}^{\infty} \zeta_k \psi_k$, and $\eta = \sum_{k=0}^{\infty} \eta_k \psi_k$. Then, from \eqref{23} we obtain an infinite sequence of equalities of the following form: $(\lambda_k \xi_k + \zeta_k - \eta_k) \psi_k = 0$ ($k = 0, 1, 2, \ldots$) from which Eqs. \cite{26} follow. Let us remark that the expansions of $\xi$, $\zeta$, and $\eta$ are not orthogonal expansions, since their coefficients are statistically interconnected or, in other words, $E\{\xi_m, \xi_n\}$ (and similarly $E\{\zeta_m, \zeta_n\}$ and $E\{\eta_m, \eta_n\}$; $E\{\cdot\}$ denoting the expectation value) does not in general vanish. It amounts to say that the coefficients of these expansions are not statistically independent. Let us indeed remind that, in general, it is not possible to expand the process $\xi$ (or $\zeta$, or $\eta$) in an orthogonal series on a finite interval, except in the limiting situation of a stationary white noise process. This remark is relevant below in connection with the evaluation in the information theory approach.

Next, we can introduce the variances: $\rho_k^2 = (R_{\xi \xi} \psi_k, \psi_k)$, $\varepsilon^2 \nu_k^2 = (R_{\zeta \zeta} \psi_k, \psi_k)$, $\lambda_k^2 \rho_k^2 + \varepsilon^2 \nu_k^2 = (R_{\eta \eta} \psi_k, \psi_k)$.

In view of assumptions (I) and (II), the probability densities for $\xi_k$ and $\zeta_k$ can be written as follows

\begin{equation}
\rho_k^2 \frac{1}{\sqrt{2 \pi \rho_k}} \exp \left( -\frac{x^2}{2 \rho_k^2} \right) \quad (k = 0, 1, 2, \ldots),
\end{equation}

and

\begin{equation}
\nu_k^2 \frac{1}{\sqrt{2 \pi \varepsilon \nu_k}} \exp \left( -\frac{x^2}{2 \varepsilon^2 \nu_k^2} \right) \quad (k = 0, 1, 2, \ldots).
\end{equation}

By the use of Eqs. \cite{26} we can also introduce the conditional probability density $p_{\eta_k}(y|x)$ of the random variable $\eta_k$ for fixed $\xi_k = x$, which reads

\begin{equation}
p_{\eta_k}(y|x) = \frac{1}{\sqrt{2 \pi \varepsilon \nu_k}} \exp \left( -\frac{(y - \lambda_k x)^2}{2 \varepsilon^2 \nu_k^2} \right) = \frac{1}{\sqrt{2 \pi \varepsilon \nu_k}} \exp \left[ -\frac{\lambda_k^2}{2 \varepsilon^2 \nu_k^2} \left( x - \frac{y}{\lambda_k} \right)^2 \right].
\end{equation}

Now, let us apply the Bayes formula that provides the conditional probability density of $\xi_k$ given $\eta_k$ through the following expression \cite{20}

\begin{equation}
p_{\xi_k}(x|y) = \frac{p_{\xi_k}(x)p_{\eta_k}(y|x)}{p_{\eta_k}(y)},
\end{equation}

provided $p_{\eta_k}(y) \neq 0$. Thus, if a realization of the random variable $\eta_k$ is given by $g_k$, formula \eqref{31} becomes

\begin{equation}
p_{\xi_k}(x|g) = A_k \exp \left( -\frac{x^2}{2 \rho_k^2} \right) \exp \left[ -\frac{\lambda_k^2}{2 \varepsilon^2 \nu_k^2} \left( x - \frac{g_k}{\lambda_k} \right)^2 \right].
\end{equation}

Next, we introduce the following sets:

\begin{equation}
\mathcal{I} = \{ k \in \mathbb{N} : \lambda_k \rho_k \geq \varepsilon \nu_k \},
\end{equation}

\begin{equation}
\mathcal{N} = \{ k \in \mathbb{N} : \lambda_k \rho_k < \varepsilon \nu_k \}.
\end{equation}

We can now see that the conditional probability density \cite{32} can be regarded as the product of two Gaussian probability densities: $p_1(x) = A_k^{(1)} \exp(-\frac{x^2}{2 \rho_k^2})$ and $p_2(x) = A_k^{(2)} \exp\left(-\frac{\lambda_k^2}{2 \varepsilon^2 \nu_k^2} (x - \frac{g_k}{\lambda_k})^2 \right)$ with $A_k = A_k^{(1)} A_k^{(2)}$, whose variances are respectively given by $\rho_k^2$ and $(\varepsilon \nu_k / \lambda_k)^2$. Now, if $k \in \mathcal{I}$ the variance associated with
$p_2(x)$ is smaller than the corresponding variance of $p_1(x)$, and vice versa if $k \in \mathcal{N}$. Therefore, it appears reasonable to consider as an acceptable approximation of $\langle \xi_k \rangle$, i.e. the mean value of the random variable $\xi_k$, the mean value associated with the density $p_2(x)$ if $k \in \mathcal{I}$, and the mean value associated with the density $p_1(x)$ if $k \in \mathcal{N}$. We can then write the following approximation

$$\langle \xi_k \rangle = \begin{cases} \frac{g_k}{\lambda_k} & \text{if } k \in \mathcal{I}, \\ 0 & \text{if } k \in \mathcal{N}. \end{cases}$$

Consequently, given the value $g$ of the w.r.v. $\eta$, we are led to consider the following linear estimator of $\xi$

$$T\eta = \sum_{k \in \mathcal{I}} g_k \frac{\lambda_k}{\lambda_k} \psi_k.$$ 

In order to pass from heuristic considerations to rigorous statements, we must prove that the linear estimator (36) leads to a probabilistically regularized solution. For this purpose, we must evaluate the global mean–squared error associated with the linear estimator (36), i.e. $E\{\|\xi - T\eta\|^2\}$, along with $E\{\|\eta\|^2\} = \sum_{k=0}^{\infty} (R_{\xi\xi} \psi_k, \psi_k) = \text{Trace}(R_{\xi\xi})$.

The following propositions can be proved.

**Proposition 5.**

(i) If $\lim_{k \to \infty} (\lambda_k \rho_k / \nu_k) = 0$, then the set $\mathcal{I}$ is finite for any $\varepsilon > 0$. 

(ii) Assuming that the limit stated in (i) holds and, in addition, that $R_{\xi\xi}$ is an operator of trace class, then the following relationship holds

$$E\{\|\xi - T\eta\|^2\} = \sum_{k \in \mathcal{N}} \rho_k^2 + \sum_{k \in \mathcal{I}} \varepsilon^2 \frac{\nu_k^2}{\lambda_k} < \infty.$$

**Proof.** See Proposition 3.3 of [11].

**Proposition 6.** If the covariance operator $R_{\xi\xi}$ is of trace class, and if the set $\mathcal{I}$ is finite (see Proposition 5), then the following limit holds

$$\lim_{\varepsilon \to 0} E\{\|\xi - T\eta\|^2\} = 0,$$

i.e., the linear estimator of $\xi$ given by formula (36) gives a probabilistically regularized solution to Problem 1.

**Proof.** See Proposition 3.5 of [11].

**Remark 2.** As we have already remarked above, the deterministic regularization methods do not guarantee that the bulk of the signal (the object, in our case) has been really recovered. Conversely, the probabilistic regularization methods (i.e., the solution given by formula (36)) can really reconstruct, within a certain degree of approximation, the bulk of the object, once the sets $\mathcal{I}$ and $\mathcal{N}$ have been neatly separated. In fact, as we shall see in Section 4, the Gaussian random variables $\eta_k$, associated with the set $\mathcal{I}$, contain a significant amount of information on the corresponding variables $\xi_k$, whereas in the random variables $\eta_k$, associated with the set $\mathcal{N}$, the noise is prevailing. At this point the problem is how to split the set of the Gaussian random variables $\{\eta_k\}$ into the two sets $\mathcal{I}$ and $\mathcal{N}$. This task can be achieved by computing the correlation function of the random variables $\eta_k$, which are the probabilistic counterpart of the coefficients $g_k$. Let us indeed recall that the coefficients $\eta_k = (\eta, \psi_k)$, obtained by the formal expansion $\eta = \sum_{k=0}^{\infty} \eta_k \psi_k$ are
not statistically independent. These statistical methods require great caution and involve delicate mathematical questions, which have been studied in \[11, 21\], and we do not return on these problems here. In particular, in \[11]\ some explicit examples have been shown, where the deterministic regularization method fails, whereas the statistical one can actually reconstruct the solution of the integral equation considered. Analogous statistical methods have been also used in optics \[21]\, in connection with the object restoration in the case of spatially incoherent illumination. In this case, in particular, a positivity constraint has been incorporated into the probabilistically regularized solution by means of a quadratic programming technique. Several examples were shown, and satisfactory results had been obtained.

As a final remark, we point out that the deterministic and the statistical regularization methods can be used jointly in the following sense. Assuming that we know \textit{a priori} global bounds on the solution such that deterministic regularized solutions can be tried, then their reliability can be tested by using statistical correlation methods along the lines suggested by the probabilistic regularization procedures.

3. Topological information theory: the $\varepsilon$–entropy of the image

Let us return to the deterministic regularization methods, and to the related \textit{a priori} truncation criteria. Consider bound \[20]\ where, for simplicity and without loss of generality, we put $E = 1$. Accordingly, we consider the approximation $f^{(3)} = \sum_{k=0}^{k_l}(g_k/\lambda_k)\psi_k$, where $k_l$ is the largest integer such that $\lambda_k \geq \varepsilon$. As proved in Proposition 4, $f^{(3)}$ converges weakly to $f$ and, consequently, only a \textit{weak} continuity can be guaranteed in the restored solution.

We now make two additional assumptions:

1. We assume that the noise $n$ is moderate enough, namely, it is such that the noisy image belongs to the range of $A$: $g \in R(A)$.

2. We assume that $k_l \simeq k_p$, i.e. the truncation number $k_l$ associated with the approximation $f^{(3)}$ is very close to $k_p$, which is the value of $k$ such that for $k > k_p$ the Fourier components $g_k = (g, \psi_k)$ of the noiseless data are smaller or, at most, of the same order of magnitude of $\varepsilon$ (see Remark 1). It is obvious that in making this assumption we suppose that the modulus of the coefficients $g_k$ decreases for increasing values of $k$. If these assumptions are true, then we can exclude those “pathological” examples in which the bulk of the object is not recovered by the approximation $f^{(3)}$.

In view of the \textit{a priori} bound \[17]\ with $E = 1$, we are led to consider the unit ball in the solution space $X \equiv L^2(-X_0/2, X_0/2)$: i.e., the set $\{f \in X : \|f\|_X \leq 1\}$; the operator $A$ maps the unit ball onto a compact ellipsoid $E \in R(A)$, contained in the data space $Y \equiv L^2(-X_0/2, X_0/2)$, whose semi–axes lengths are the eigenvalues $\lambda_k$ of the operator $A$.

Let us now recall some basic definitions from the information theory \[12]\:

(a) In the theory of information, the unit of a \textit{collection of information} is the amount of information in one binary sign (that is, designating whether it is 0 or 1).

(b) The \textit{entropy} of a collection of possible \textit{communications}, undergoing transmission with a specified accuracy, is defined as the number of binary signs
necessary to transmit an arbitrary one of these communications with a
given accuracy.

(c) The capacity of a transmitting apparatus is defined as the number of binary
signs that it can transmit reliably.

Coming back to the compact ellipsoid $\mathcal{E}$, we recall some basic definitions which give
a numerical estimate of its massiveness \[12\, 22].

(a') A family $Y_0, \ldots, Y_n$ of subsets of $Y$ is an $\varepsilon$–covering of $\mathcal{E}$ if the diameter of
each $Y_k$ does not exceed $2\varepsilon$ and if the sets $Y_k$ cover $\mathcal{E}$: i.e., $\mathcal{E} \subset \bigcup_{k=0}^n Y_k$.

(b') Points $y_0, \ldots, y_m$ of $\mathcal{E}$ are called $\varepsilon$–distinguishable if the distance between
each two of them exceeds $\varepsilon$.

Since $\mathcal{E}$ is compact, then a finite $\varepsilon$–covering exists for each $\varepsilon > 0$, and, moreover,
$\mathcal{E}$ can contain only finite many $\varepsilon$–distinguishable points. For a given $\varepsilon > 0$, the
number of sets $Y_k$ in a covering family depends on the family, but the minimal value of $n$, $N_\varepsilon(\mathcal{E}) \doteq \min n$, is an invariant of the set $\mathcal{E}$, which depends only on $\varepsilon$.
Its logarithm, that is, the function $H_\varepsilon(\mathcal{E}) \doteq \log N_\varepsilon(\mathcal{E})$ is the $\varepsilon$–entropy of the set $\mathcal{E}$,
and gives the length of the binary sequence from which a signal in $\mathcal{E}$ can be
distinguished up to $\varepsilon$ accuracy. Analogously, the number $m$ in definition (b')
above depends on the choice of the points, but its maximum $M_\varepsilon(\mathcal{E}) \doteq \max m$, is
an invariant of the set $\mathcal{E}$, and represents the maximum number of $\varepsilon$–distinguishable
messages that can be conveyed back in the backward channel to reconstruct the
object: i.e., the maximum number of those data which satisfy the inequalities:
$\|g(i) - g(k)\|_Y > \varepsilon$ for all $i \neq k$, $g(i), g(k) \in \mathcal{E}$. Its logarithm, that is, the function $C_\varepsilon(\mathcal{E}) \doteq \log M_\varepsilon(\mathcal{E})$, is the $\varepsilon$–capacity of the set $\mathcal{E}$, and provides the length (in binary
units) of the messages that can be reliably transmitted in the backward channel.

The following inequalities hold \[12\, 23]:

\[ H_\varepsilon(\mathcal{E}) \leq C_\varepsilon(\mathcal{E}) \leq H_{\varepsilon/2}(\mathcal{E}). \] (39)

Then, in order to obtain estimates for the $\varepsilon$–capacity $C_\varepsilon(\mathcal{E})$, our aim is now to look
for a lower bound for $H_\varepsilon(\mathcal{E})$ and an upper bound for $H_{\varepsilon/2}(\mathcal{E})$. For this purpose,
let us consider the finite dimensional subspace $Y_k$ of $Y$, spanned by the first $k_1 + 1$
axes of $\mathcal{E}$, and put $\mathcal{E}_{k_1} = \mathcal{E} \cap Y_{k_1}$. Then, $\mathcal{E}_{k_1}$ is a finite dimensional ellipsoid whose
volume is just $\prod_{k=0}^{k_1} \lambda_k$ times the volume $\Omega_{k_1}$ of the unit ball in $Y_{k_1}$. Since the
volume of an $\varepsilon$–ball in $Y_{k_1}$ is $\varepsilon^{(k_1+1)}\Omega_{k_1}$, we see that in order to cover the ellipsoid
$\mathcal{E}$ by the $\varepsilon$–balls we shall need at least $\prod_{k=0}^{k_1} (\lambda_k/\varepsilon)$ such balls. From this it follows that \[24\, 25]:

\[ \prod_{k=0}^{k_1} \frac{\lambda_k}{\varepsilon} \leq N_\varepsilon(\mathcal{E}), \] (40)

and, therefore, we have the following lower bound for the $\varepsilon$–entropy $H_\varepsilon(\mathcal{E})$:

\[ \sum_{k=0}^{k_1} \log \frac{\lambda_k}{\varepsilon} \leq \log N_\varepsilon(\mathcal{E}) = H_\varepsilon(\mathcal{E}). \] (41)

The determination of an upper bound for $H_{\varepsilon/2}(\mathcal{E})$ is more involved, and we limit
ourselves to report the result \[24\, 25]:

\[ H_{\varepsilon/2}(\mathcal{E}) \leq k_1 \left( \frac{\varepsilon}{4} \right) \left\{ \log \left( \frac{1}{\varepsilon} \right) + \log 6 + \frac{1}{2} \log k_1 \left( \frac{\varepsilon}{4} \right) \right\}, \] (42)
where \( k_1(\varepsilon/4) \) represents the number of terms in the sequence \( \{\lambda_k\}_{k=0}^\infty \) which are larger or equal to \( (\varepsilon/4) \).

Now, we come back to the optical problem, specifically to Eq. (3), and investigate the behavior of the \( \varepsilon \)-entropy \( H_\varepsilon(E) \) in the limit of low level of noise. Assuming that the Shannon number \( S = \Omega X_0/\pi \) is sufficiently large, the eigenvalues \( \lambda_k \) can be approximated with 1 for \( k \leq S \) (see e.g., Fig. 2), whereas, for \( k > S \), the eigenvalues \( \lambda_k \) fall off to zero exponentially \[5\]. Consider now the bound in (41); for \( \varepsilon \) sufficiently small, we have \( k_1(\varepsilon) > S \), and the sum in (41) can be split into two parts:

\[
\sum_{k=0}^{k_1} \log \frac{\lambda_k}{\varepsilon} = \sum_{k=0}^{S-1} \log \frac{\lambda_k}{\varepsilon} + \sum_{k=S}^{k_1} \log \frac{\lambda_k}{\varepsilon},
\]

where the symbol \( \lfloor x \rfloor \) stands for the integral part of \( x \). Since for \( k < S \) we have \( \lambda_k \approx 1 \), the contribution of the first sum on the r.h.s. of (43) is about \( S \log(1/\varepsilon) \). Instead, for \( k \geq S \) we have \( \lambda_k \approx \varepsilon \), so that the second sum on the r.h.s. of (43) is nearly null. Then, from (42) we obtain the following lower bound for the \( \varepsilon \)-entropy:

\[
H_\varepsilon(E) \sim S \log \left( \frac{1}{\varepsilon} \right).
\]

Therefore, we can conclude that the maximum number of \( \varepsilon \)-distinguishable messages, which can be conveyed back from the image to recover the object, at least should be:

\[
M_\varepsilon(E) \gtrsim 2^{S \log(1/\varepsilon)} \quad \varepsilon \to 0 \quad \infty.
\]

Next, we can consider formula (42), which limits superiorly the number of \( \varepsilon \)-distinguishable messages. First we note that the eigenvalues \( \lambda_k \) decrease exponentially for \( k \to \infty \); precisely, we have \[5\]: \( \lambda_k = \text{O}(\exp[-2k \log(k/c)]/k) \), \( c = \text{constant} \). Then, it follows that, for \( \varepsilon \to 0 \), \( k_1(\varepsilon/4) \sim \frac{1}{2} \log(1/\varepsilon) \), and the leading term, for \( \varepsilon \to 0 \), on the r.h.s. of (42) is: \( k_1(\varepsilon/4) \log(1/\varepsilon) \). We thus have:

\[
H_{\varepsilon/2}(E) \sim \frac{1}{2} \log \left( \frac{1}{\varepsilon} \right) \sim \frac{1}{2} \log^2 \left( \frac{1}{\varepsilon} \right).
\]

Summarizing, from (39), (44), and (46) we have, for \( \varepsilon \) sufficiently small:

\[
S \log \left( \frac{1}{\varepsilon} \right) \lesssim C_\varepsilon(E) \lesssim \frac{1}{2} \log^2 \left( \frac{1}{\varepsilon} \right).
\]

These latter inequalities require: \( S < \frac{1}{4} \log \left( \frac{1}{\varepsilon} \right) \), that is, \( \varepsilon < 2^{-2S} \). In other words, this means that as long as the noise level is not too small, i.e. for \( \varepsilon > 2^{-2S} \), the \( \varepsilon \)-capacity is essentially: \( C_\varepsilon(E) \approx S \log (1/\varepsilon) \) (to have a flavor of the numbers, for the operator \( A \) whose eigenvalues are shown in Fig. 2 with \( S = 12.7 \), this approximation of the the \( \varepsilon \)-capacity holds for \( \varepsilon \gtrsim 10^{-7.6} \) or, equivalently, for a signal–to–noise ratio: \( (E/\varepsilon) \lesssim 76 \text{dB} \)). Instead, when the noise gets smaller, i.e. for \( \varepsilon < 2^{-2S} \), the \( \varepsilon \)-capacity may increase faster when \( \varepsilon \to 0 \), remaining (approximately) within the range specified by inequalities (47).
4. Comparing Probabilistic and Topological Information Theory

Let us return now to the probabilistic regularization methods, and evaluate the amount of information on the random variable $\xi_k$, which is contained in the random variable $\eta_k$; we have \cite{20}:

\begin{equation}
J(\xi_k, \eta_k) = -\frac{1}{2} \ln(1 - r_k^2) \quad (k = 0, 1, 2, \ldots),
\end{equation}

($\ln x$ denotes the logarithm of $x$ to the base $e$), where $r_k$ is given by:

\begin{equation}
r_k^2 = \frac{|E \{\xi_k, \eta_k^2\}|^2}{E \{\xi_k^2\} E \{\eta_k^2\}} = \frac{(\lambda_k \rho_k)^2}{(\lambda_k \rho_k)^2 + (\varepsilon \nu_k)^2} \quad (k = 0, 1, 2, \ldots),
\end{equation}

and the equality $R_{\xi\eta} = R_{\xi\xi} A^*$ (see \cite{25}) has been used. From (48) and (49) it follows:

\begin{equation}
J(\xi_k, \eta_k) \leq \frac{1}{2} \ln 2 \quad (k \in \mathcal{N}).
\end{equation}

Let us now consider the sets $\mathcal{I}$ and $\mathcal{N}$, defined in \cite{33} and \cite{54}. We see that, for the random variables $\xi_k$ and $\eta_k$ whose $k$–values belong to the set $\mathcal{N}$, Eq. (50) gives:

\begin{equation}
J(\xi_k, \eta_k) < \frac{1}{2} \ln 2 \quad (k \in \mathcal{N}).
\end{equation}

We can thus say that, in the components $\eta_k$ whose values of $k$ belong to the set $\mathcal{N}$ (for simplicity we write $\eta_k \in \mathcal{N}$), the noise is prevailing and therefore they can be neglected in the approximate reconstruction of the object, in agreement with formula \cite{33}.

Conversely, the components $\eta_k \in \mathcal{I}$ contain a significant amount of information on the corresponding components $\xi_k$. We can thus write, with obvious notation:

\begin{equation}
\mathfrak{J} = \sum_{k \in \mathcal{I}} J(\xi_k, \eta_k) = \sum_{k \in \mathcal{I}} \ln \sqrt{1 + \frac{\lambda_k^2 \rho_k^2}{\varepsilon^2 \nu_k^2}}.
\end{equation}

Remark 3. The quantity $\mathfrak{J}$ in (52) is not the total information $J(\xi, \eta)$. In fact, the pairs $\{\xi_i, \eta_j\}$ ($i \neq j$) are not mutually independent. A linear coordinate transformation could always been chosen in such a way that all the components $\{\xi, \eta\} = \{\xi_0, \xi_1, \ldots, \xi_k, \eta_0, \eta_1, \ldots, \eta_k\}$ (with the exception of the pairs $\{\xi_j, \eta_j\}$, ($j = 0, 1, 2, \ldots, k$)) are mutually independent. But this would imply to introduce a basis $\{\psi_k\}_{k=0}^{\infty}$, which differs from that obtained by the eigenfunctions $\{\psi_k\}_{k=0}^{\infty}$ of the operator $A$ that we used in the derivation of the probabilistic regularization methods. Therefore, we limit ourselves to evaluate $\sum_{k \in \mathcal{I}} J(\xi_k, \eta_k)$, which does not provide the total amount of information $J(\xi, \eta)$ but represents only an approximation of it.

Next, we make the following approximation:

\begin{equation}
\mathfrak{J} = \sum_{k \in \mathcal{I}} \ln \sqrt{1 + \frac{\lambda_k^2 \rho_k^2}{\varepsilon^2 \nu_k^2}} \simeq \sum_{k \in \mathcal{I}} \ln \left| \frac{\lambda_k \rho_k}{\varepsilon \nu_k} \right|,
\end{equation}

which is admissible if $\lambda_k \rho_k \geq \varepsilon \nu_k$: i.e., for the components $\eta_k \in \mathcal{I}$. We now assume that: $\rho_k \sim \nu_k$ for $k \in \mathcal{I}$. Then, from (53) we obtain:

\begin{equation}
\mathfrak{J} = \sum_{k \in \mathcal{I}} J(\xi_k, \eta_k) \simeq \sum_{k \in \mathcal{I}} \ln \frac{\lambda_k}{\varepsilon}.
\end{equation}
In particular, let us note that from the assumption \( \rho_k \sim \nu_k \) (for \( k \in I \)) it follows that the set \( I \) is composed of those components such that \( \lambda_k \geq \varepsilon \), which is precisely the truncation criterion (20) (with \( E = 1 \)) which generates the approximation \( f^{(3)} \). Thus, from (54) we have:

\[
\sum_{k \in I} J(\xi_k, \eta_k) \simeq \sum_{k \in I} \ln \frac{\lambda_k}{\varepsilon} = \sum_{k=0}^{k_1} \ln \frac{\lambda_k}{\varepsilon},
\]

which coincides with the lower bound on the \( \varepsilon \)-capacity (see Eq. (11)) up to an immaterial conversion factor between logarithms to different bases. Again, as we made for obtaining formula (11), we have \( \lambda_k \simeq 1 \) for \( k \leq S \), which finally yields:

\[
\tilde{J} = \sum_{k \in I} J(\xi_k, \eta_k) \simeq S \ln \left( \frac{1}{\varepsilon} \right).
\]

Correspondingly, the maximum number of \( \varepsilon \)-distinguishable messages which can be conveyed back in the backward channel from the image to recover the object, can therefore written as (neglecting the conversion factor between \( \log x \) and \( \ln x \)):

\[
M_\varepsilon(\mathcal{E}) = 2^{C_\varepsilon(\mathcal{E})} \simeq 2^{S \log(1/\varepsilon)} \simeq 2^\tilde{J} = 2^{\{ \sum_{k \in I} J(\xi_k, \eta_k) \}},
\]

which, as expected, tends to infinity as \( \varepsilon \) tends to zero.

Returning to Eq. (53), let us now make the following assumption: \( \lambda_k \rho_k \sim \nu_k \) for \( k \in I \). We have:

\[
\tilde{J} \simeq \sum_{k \in I} \ln \left| \frac{\lambda_k \rho_k}{\varepsilon \nu_k} \right| \simeq \sum_{k \in I} \ln \left( \frac{1}{\varepsilon} \right) = k_1(\varepsilon) \ln \left( \frac{1}{\varepsilon} \right).
\]

Now, recalling that the sequence of eigenvalues \( \lambda_k \) falls off exponentially to zero for \( k \) sufficiently large, from (55) we obtain:

\[
\tilde{J} \simeq k_1(\varepsilon) \ln \left( \frac{1}{\varepsilon} \right) \simeq \frac{1}{2} \ln^2 \left( \frac{1}{\varepsilon} \right),
\]

which coincides with the upper bound on the \( \varepsilon \)-capacity given in (46).

Summarizing, we see that for a given (small) level of noise \( \varepsilon \), the two extremal cases for the maximum number of \( \varepsilon \)-distinguishable data–messages which represent the information that can be sent back through the backward channel to reconstruct the object, are related to the spectral distribution of the noise. The lower limit is obtained when, for \( k \in I \), the spectral distribution of the noise (i.e., \( \nu_k \)) coincides with the distribution of the object (i.e., \( \rho_k \)). The upper bound corresponds to the case when, for \( k \in I \), the spectral distribution of the noise coincides with that of the image (i.e., \( \lambda_k \rho_k \)).

5. Conclusions

Let us start from the classical Whittaker–Kotel’nikov–Shannon sampling theorem [30], which states that a function, whose Fourier transform vanishes outside a certain interval of length \( 2\Omega \), can be reconstructed by a discrete collection of its values, chosen in arithmetic progression with difference \( \pi/\Omega \). Since the image \( g(y) \) is a band–limited function, it could, in principle, be reconstructed by an infinite collection of its samples, taken at equidistant points spaced \( \pi/\Omega \) apart. More realistically, the image \( g(y) \) can be reconstructed in an interval of length \( X_0 \) by a finite collection \( S = \Omega X_0/\pi \) of its samples. The classical Rayleigh resolution distance
$R$ equals the Nyquist distance $\pi/\Omega$, while the Shannon number $S$ turns out to be given by $\text{Trace}(A) = \sum_{k=0}^{\infty} \lambda_k$ [27] [28].

Since both the image $g(y)$ and the Fourier transform of the object $F(\omega)$ are entire functions in the complex variables $y$ and $\omega$ respectively, they can be analytically continued beyond the interval where they are known. Consider, for instance, $F(\omega)$: in principle, it might be possible to extrapolate the function outside the data band $[-\Omega, \Omega]$ by making use of appropriate regularization methods of ill-posed problems, and then to find an estimate of it over a broader band, say, $[-W, W]$. This would imply a better resolution $\pi/W$: this improvement can be called super-resolution.

In fact, it has been shown that whenever the Shannon number is not too large (i.e., not much greater than unity) the behavior of the eigenvalues $\lambda_k$ is not similar to that of a step function (see Fig. 2), and therefore, the extrapolation of $F(\omega)$ out of band is indeed possible [29].

We have focused on aspects of the problem by analyzing the inverse imaging problem from two different viewpoints: the classical information theory based on probabilistic methods, and the Kolmogorov’s $\varepsilon$-capacity (and entropy), which can be thought of as a form of information theory based on topological concepts. The main results obtained, if a few conditions (specified at the beginning of Section 3) are satisfied, can be summarized in the following points:

(a) The $\varepsilon$–capacity of the image data set is essentially given by:

\begin{equation}
C_{\varepsilon}(E) \sim S \log \left( \frac{1}{\varepsilon} \right),
\end{equation}

where $S$ is the Shannon number. Consequently, the maximum number of $\varepsilon$–distinguishable messages which can be conveyed back in the backward channel from the image to reconstruct the object is given by:

\begin{equation}
M_{\varepsilon}(E) \sim 2^{S \log(1/\varepsilon)}.
\end{equation}

(b) For $\varepsilon$ sufficiently small, i.e. $\varepsilon \lesssim 2^{-S}$, the $\varepsilon$–capacity is bounded above by:

\begin{equation}
C_{\varepsilon}(E) \lesssim \frac{1}{2} \log^2 \left( \frac{1}{\varepsilon} \right) \quad \varepsilon \to 0.
\end{equation}

(c) The upper and lower bounds on the information content of the noisy image (i.e., $C_{\varepsilon}(E)$) obtained by the topological information theory may be interpreted within the framework of the probabilistic information theory. In fact, the sum $\mathcal{J}$ of the information contained in the random variables $\eta_k$, which represent the noisy image, on the corresponding random variable $\xi_k$, which represent the object, is given by:

(c1) If, for $k \in \mathcal{I}$, the spectral distribution of the noise is as that of the object, i.e. $\nu_k \sim \rho_k$:

\begin{equation}
\mathcal{J} = \sum_{k \in \mathcal{I}} J(\xi_k, \eta_k) \simeq S \ln \left( \frac{1}{\varepsilon} \right).
\end{equation}

(c2) If, for $k \in \mathcal{I}$, the spectral distribution of the noise is as that of the image, i.e. $\nu_k \sim \lambda_k \rho_k$:

\begin{equation}
\mathcal{J} \simeq \frac{1}{2} \ln^2 \left( \frac{1}{\varepsilon} \right).
\end{equation}
(d) The maximum number of $\varepsilon$–distinguishable messages which can be conveyed back from the image to reconstruct the object is given by:

$$M_\varepsilon(\mathcal{E}) = 2^{C_\varepsilon(\mathcal{E})} \approx 2^{\sum_{k \in \mathcal{I}} J(\xi_k, \eta_k)} \xrightarrow{\varepsilon \to 0} \infty.$$ 

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