A 3+1 formalism for quantum electrodynamical corrections to Maxwell equations in general relativity

J. Pétri

1 Observatoire astronomique de Strasbourg, Université de Strasbourg, CNRS, UMR 7550, 11 rue de l’université, F-67000 Strasbourg, France.

ABSTRACT

Magnetized neutron stars constitute a special class of compact objects harbouring gravitational fields that deviate strongly from the Newtonian weak field limit. Moreover strong electromagnetic fields anchored into the star give rise to non-linear corrections to Maxwell equations described by quantum electrodynamics (QED). Electromagnetic fields close to or above the critical value of $B_Q = 4.4 \times 10^9$ T are probably present in some pulsars and for most of the magnetars. To account properly for emission emanating from the neutron star surface like for instance thermal radiation and its polarization properties, it is important to include general relativistic (GR) effects simultaneously with non-linear electrodynamics. This can be achieved through a 3+1 formalism known in general relativity and that incorporates QED perturbations to Maxwell equations. Starting from the lowest order corrections to the Lagrangian for the electromagnetic field, as given for instance by Born-Infeld or Euler-Heisenberg theory, we derive the non-linear Maxwell equations in general relativity including quantum vacuum effects. We also derive a prescription for the force-free limit and show that these equations can be solved with classical finite volume methods for hyperbolic conservation laws. It is therefore straightforward to include general relativity and quantum electrodynamics in the description of neutron star magnetospheres by using standard classical numerical techniques borrowed from Maxwell and Newton theory. As an application, we show that spin-down luminosity corrections associated to QED effects are negligible with respect to GR corrections.

Key words: gravitation - magnetic fields - plasmas - stars: neutron - methods: analytical

1 INTRODUCTION

As a final stage of the stellar evolution process, neutron stars constitute a special class of compact objects showing strong field effects in both gravitational and electromagnetic interactions. Typical values for their radius lie around $R \approx 12$ km whereas their fiducial mass is about $M = 1.4 M_\odot$ (where $M_\odot$ is the mass of the sun) thus leading to a compactness parameter defined by

$$\Xi = 2 G M / R c^2 \approx 0.34 .$$

This compactness is close to the maximal reachable compactness defining a black hole. $G$ is the gravitational constant and $c$ the speed of light. General relativity is therefore required for an accurate description of the gravitational field in the vicinity of the neutron star. Moreover, from simple estimates of their spin-down luminosity being interpreted as magneto-dipole losses, normal pulsars can harbour magnetic fields of strength comparable to $B \approx 10^8$ T and even higher fields for magnetars, of the order $B \approx 10^{10-11}$ T. About thirty magnetars are known today and compiled in a catalogue described in Olausen & Kaspi (2014). These values are comparable or largely above the critical field value of

$$B_Q = \frac{m_e c^2}{e \hbar} \approx 4.4 \times 10^9 \text{ T}.$$ (2)

$m_e$ is the mass of an electron, $e$ the absolute value of its electric charge and $\hbar$ the reduced Planck constant. At such field strengths, quantum electrodynamical corrections to classical electromagnetism as given by Maxwell equations become significant. Vacuum polarization and electron-positron pair creation are two examples of such QED effects. The perturbations induced by QED can be conveniently expressed in terms of an effective Lagrangian field theory such as the one found empirically by Born-Infeld (1934) or derived directly from a first order expansion of the QED Lagrangian as given by Heisenberg & Euler (1936). Such effective Lagrangians are very fruitful in getting more physical insight

* E-mail: jerome.petri@astro.unistra.fr
and intuition on QED effects and represent very efficient tools to extend classical electrodynamics methods to the realm of quantum physics. Extreme fields around $B_0$ cannot be reach in terrestrial laboratories although the power of current lasers are approaching it in order to do reproducible experiments and check the theory. \cite{King & Di Piazza 2014}. Pulsars and magnetars could be used as extraterrestrial laboratories to investigate QED in strong electromagnetic fields and in curved space-time. The electromagnetic properties of quantum vacuum in special relativity are summarized in \cite{Battesti & Rizzi 2013} and \cite{Ruffini et al. 2010} propose a comprehensive review on electron-positron pair creation/annihilation in the physical and astrophysical contexts. The classical picture of high-energy radiation processes in astrophysics need also to be revised according to QED corrections. This is discussed in \cite{Harding & Lai 2006} with a special emphasize to neutron star interiors and atmosphere. For instance, \cite{Heyl & Hernquist 2004, Heyl 2007} suggested an explanation of the non-thermal emission from magnetar in terms of QED.

Not only radiation is affected by QED but also the behaviour of pair plasmas and electron-ion plasmas as explained in depth in the review by \cite{Uzdensky & Rightley 2014}. The plasma dielectric tensor and therefore also the normal modes of propagation of electromagnetic waves and plasma oscillations are affected. This had led some authors to extend the equations of fluid hydrodynamics to some quantum aspects. For instance \cite{Manfredi & Hass 2001} looked at some of these corrections and \cite{Hass 2003} added the magnetic field into the description leading to quantum plasma magnetohydrodynamic (QMHD). QED corrections to MHD have already been proposed by \cite{Thompson & Blaes 1998} and by \cite{Heyl & Hernquist 1999}. Interestingly, the non-linear nature of electrodynamics in strong fields induces shocks as in the case of flows in hydrodynamics as pointed out by \cite{Heyl & Hernquist 1999}, see also \cite{Mazur & Heyl 2011} for non-linear wave propagation in the magnetospheric plasma. From a more fundamental point of view, QED corrections can be brought to any magnetic multipole, like for instance the QED corrections to the magnetic dipole as investigated by \cite{Heyl & Hernquist 1997a}.

Gravitational effects are usually ignored. Nevertheless \cite{Heyl 2001} applied quantum field theory near a rotating black hole and showed that it could copiously produce electron-positron pairs. Also \cite{Ruffini et al. 2013} studied QED corrections in strong gravity by inspecting spherically symmetric black holes using what they called the Einstein-Euler-Heisenberg theory. \cite{Denisov et al. 2004, Denisov & Svertilov 2005} looked at electromagnetic wave propagation in a strong gravitational field described by the Schwarzschild metric and including QED effects. In another series of papers, \cite{Denisov & Svertilov 2003, Denisov et al. 2014} examined the propagation of light rays, bending and time delay due to vacuum birefringence.

From an observational point of view, X-ray polarization is the key observable to study QED effects in general relativity in the context of neutron star magnetospheres and atmospheres. Such signatures have indeed already been investigated by \cite{Heyl & Shaviv 2002}. They showed that the X-ray polarization degree is largely enhanced compared to classical field theory. \cite{Heyl & Shaviv 2001} also demonstrated a possible phase lag between different wavelengths. \cite{Taverna et al. 2014} showed that phase-resolved polarimetry gives insight into magnetar magnetospheres in the ultraviolet strong field regime already with modest X-ray telescopes. At optical and infrared wavelengths, plasma and vacuum polarization effects compete, leading to possible constraints on the total charge density within the magnetosphere as explained by \cite{Shannon & Hevl 2006}. Moreover because vacuum birefringence in strong magnetic field leads to variable refractive indexes, \cite{Shaviv et al. 1999} predicted strong QED lensing analogue to the gravitational lensing although in QED lensing the strength of the lens depends on the polarization states of the photons. \cite{Dupays et al. 2003, Kim 2012}. Such effects are supposed to help diagnosing the magnetic field structure around neutron stars.

To study the magnetosphere of neutron stars, it is often useful to start with a simple approximation called force-free electrodynamics (FFE). It represents a zero order approximation to compute the plasma response and current density knowing the electromagnetic field topology. FFE has been used in the last decade by many authors to investigate pulsar and magnetar magnetospheres \cite{Timokhin 2004, Spitkovsky 2006, McKinney 2006, Petr 2012, Parfrey et al. 2014, Petr 2015a}. \cite{Presttis & Grafta 2013} showed that quantum force-free electrodynamics can be recast into the 3+1 language as a set of time evolution equations for the electric and magnetic field. The system looks very similar to special-relativistic FFE except for some corrections in the current density.

In the birefringent medium induced by vacuum polarization and in the presence of plasma, vacuum resonances occurs, which translates into a mode conversion from low to high opacity, according to \cite{Lai & Ho 2003a}. But this resonance depends on the energy $E$ of the photon, so it is important to include possible gravitational redshift effects into that picture, especially when the radiation is emitted from the neutron star surface. This vacuum resonance imprints a special signature in the X-ray polarization properties from its surface emission \cite{Lai & Ho 2003a}. Because the vacuum resonance condition includes $E$ it has to be corrected for the strong gravitational field. A 3+1 formalism as the one we developed here might help to better understand such resonance in neutron star magnetospheres. For instance, our framework could be used to extend the Monte Carlo simulations for Compton scattering performed by \cite{Bulik & Miller 1997} for soft gamma repeaters in flat space-time or by \cite{Fernández & Davis 2014} in a Schwarzschild background metric accounting for light ray deflection as proposed by \cite{Heyl et al. 2003}. It could also serve to compute the radiation spectra from vacuum polarization and proton cyclotron resonances as done by \cite{Ozaki 2003} or atmospheres of neutron stars and their resulting spectra as explained by \cite{Ho & Lai 2003}. As a general conclusion, our approach combining gravity and QED in a 3+1 split of space-time will be extremely useful to straightforwardly extend the investigations cited before into the realm of general relativity. This will enable to make accurate predictions about the observational signature of neutron star atmospheres and magnetospheres.
section 3. In section 4 we derive the non-electromagnetic field equations in curved space-time. An application to the spin-down luminosity in strongly magnetized neutron stars is discussed in section 5. Concluding remarks are given in section 6.

2 THE 3+1 FORMALISM IN GENERAL RELATIVITY

In order to apply traditional finite volume schemes and to get more physical insight into electromagnetism in general relativity, we remind for completeness the 3+1 language often used to transform covariant equations into time-dependent hyperbolic systems of spatial vectors in three dimensions. To this aim, we split the four dimensional space-time into a 3+1 foliation such that the metric can be expressed as

\[ ds^2 = \alpha^2 c^2 dt^2 - \gamma_{ab} (dx^a + \beta^a c dt) (dx^b + \beta^b c dt) \tag{3} \]

where \( x^i = (ct, x^a) \), \( t \) is the time coordinate or universal time and \( x^a \) some associated space coordinates. We use the Landau-Lifshitz convention for the metric signature given by \((+,-,-,-)\) (Landau & Lifchitz 1989b). \( \alpha \) is the lapse function, \( \beta^a \) the shift vector and \( \gamma_{ab} \) the spatial metric of absolute space. By convention, latin letters from \( a \) to \( h \) are used for the components of vectors in absolute space (in the range \( \{1,2,3\} \)) whereas latin letters starting from \( i \) are used for four dimensional vectors and tensors (in the range \( \{0,1,2,3\} \)). Our derivation of the 3+1 equations follows the method outlined by Komissarov (2004, 2011) and extensively used by Petri (2013, 2014, 2015a).

Let \( F^{ik} \) and \( *F^{ik} \) be the electromagnetic tensor and its dual respectively. It is useful to introduce the following spatial vectors (\( B, E, D, H \)) such that

\[ B^a = \alpha \ F^{a0} \tag{4a} \]

\[ E_a = \frac{\alpha}{2} \ e_{abc} \, c \, *F^{bc} \tag{4b} \]

\[ D^a = \varepsilon_0 \, c \, \alpha \ F^{a0} \tag{4c} \]

\[ H_a = -\, \frac{\alpha}{2 \, \mu_0} \, e_{abc} \, F^{bc} \tag{4d} \]

where \( \varepsilon_0 \) is the vacuum permittivity and \( \mu_0 \) the vacuum permeability, \( e_{abc} = \sqrt{\gamma} \, \varepsilon_{abc} \) the fully antisymmetric spatial tensor and \( \varepsilon_{abc} \) the three dimensional Levi-Civita symbol. The contravariant analogue is \( e^{abc} = e_{abc} / \sqrt{\gamma} \). \( \gamma \) is the determinant of the spatial metric \( \gamma_{ab} \). The three dimensional vector fields are not independent, they are related by two important constitutive relations, namely

\[ \varepsilon_0 \, E = \alpha \, D + \varepsilon_0 \, c \, \beta \times B \tag{5a} \]

\[ \mu_0 \, H = \alpha \, B - \frac{\beta \times D}{\varepsilon_0 c} \tag{5b} \]

The curvature of absolute space is taken into account by the lapse function factor \( \alpha \) in the first term on the right-hand side and the frame dragging effect is included in the second term, the cross-product between the shift vector \( \beta \) and the fields.

To complete the description of the electromagnetic field in general relativity including QED corrections, we need to derive the field equations. This is done starting from a Lagrangian for the electromagnetic field, as presented in the next section.

3 LAGRANGIAN OF THE ELECTROMAGNETIC FIELD

In classical field theory, the Lagrangian \( \mathcal{L} \) of the electromagnetic field is given according to the two field invariants expressed in covariant form as \( L_1 = F_{ik} \, F^{ik} \) and \( L_2 = F_{ik} \, \star F^{ik} \). To the lowest order in these invariants, the classical Lagrangian (Jackson 2001) is given by

\[ \mathcal{L}_0 = -\frac{1}{4 \, \mu_0} \, F_{ik} \, F^{ik} - I^i \, A_i \tag{6} \]

where \( I^i \) is the four current and \( A_i \) the four potential of the electromagnetic field. In general relativity, the evolution equations for the electromagnetic field follow then from the variational principle expressed by Euler-Lagrange equations (Uzan & Deruelle 2014) such that

\[ \frac{\partial \mathcal{L}}{\partial A_i} - \frac{1}{\sqrt{-g}} \, \partial_k \sqrt{-g} \, \frac{\partial \mathcal{L}}{\partial \partial_k A_i} = 0 \tag{7} \]

where \( g = \alpha \sqrt{\gamma} \) is the determinant of the space-time metric in equation (3). Any non-linear covariant theory of electrodynamics starts from a Lagrangian including higher orders of the invariants \( L_1 \) and \( L_2 \), like for instance the Lagrangian proposed by Heisenberg & Euler (1930) or Born & Infeld (1934). To remain as general as possible in our discussion, we use a parametrized post-Maxwellian description, similar to the parametrized post-Newtonian case used for the gravitational field. The lowest order correction includes two parameters (\( \eta_1, \eta_2 \)) such that the Lagrangian to this order becomes

\[ \mathcal{L} = \mathcal{L}_0 + \eta_1 \, L_1^{\eta_1} + \eta_2 \, L_2^{\eta_2} \tag{8} \]

In the Euler-Heisenberg prescription we have

\[ \eta_1 = \frac{\alpha_{sf}}{180 \pi} \frac{1}{2 \, \mu_0 \, B_Q^2} \tag{9a} \]

\[ \eta_2 = \frac{7}{4} \eta_1 \tag{9b} \]

whereas for the Born-Infeld Lagrangian in the weak field limit we have

\[ \eta_1 = \frac{1}{32 \, \mu_0 \, b^2} \tag{10a} \]

\[ \eta_2 = \eta_1 \tag{10b} \]

with \( \alpha_{sf} \) the fine structure constant and \( b = 9.18 \times 10^{11} \) T the empirical maximal absolute field strength in Born-Infeld theory.

Comparing the relative strength of the classical and QED Lagrangian, we deduce that quantum corrections to \( \mathcal{L} \) are of the order

\[ \frac{\mathcal{L} - \mathcal{L}_0}{\mathcal{L}_0} \approx 4 \, \mu_0 \, \eta_1 \, F_{ik} \, F^{ik} \approx \frac{\alpha_{sf}}{90 \pi} \frac{B^2}{B_Q} \approx 2.6 \times 10^{-5} \frac{B^2}{B_Q^2} \tag{11} \]

and therefore remain small even in the case of magnetic fields close to the critical field \( B_Q \). QED can always be treated as a small non-linear perturbation of Maxwell equations. For magnetic fields well above \( B_Q \) the aforementioned perturbative Lagrangians do not hold any longer but it can be shown that the relative strength between classical and QED Lagrangian remains much less than unity (Heyl & Hernquist 1997; Ruffini & Xu 2004) unless \( B \) reaches unrealistic value of the order \( B_Q e^{\eta_1/\eta_2} \approx 10^{17} \) T because of the logarithmic dependence in \( \ln(B/B_Q) \) of the Lagrangian in this ultra high magnetic field limit (Landau & Lifchitz 1989a).
4 NON LINEAR ELECTRODYNAMICS IN GENERAL RELATIVITY

From the above first order corrections to the Lagrangian $\mathcal{L}$, eq. (5), it is straightforward to get the time evolution of the electromagnetic field, that is the non-linear Maxwell equations in general relativity.

4.1 Maxwell equations

The equation of motion for the fields are given by the expression (7). To write them down in the 3+1 language we note that

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_m} = -\frac{F_{lm}}{\mu_0} + 8 \eta_1 F_{lm} \mathcal{I}_1 + 8 \eta_2 * F_{lm} \mathcal{I}_2$$  (12)

or briefly with the spatial vectors $D$ and $B$

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_m} = -\xi_1 \frac{F_{im}}{\mu_0} - \xi_2 * F_{im}$$  (13a)

$$\xi_1 = 1 - 16 \mu_0 \eta_1 \left( B^2 - \frac{\mu_0}{\varepsilon_0} D^2 \right)$$  (13b)

$$\xi_2 = 32 \eta_2 \frac{D \cdot B}{\varepsilon_0 c}.$$  (13c)

Following the standard 3+1 decomposition used in general relativity we introduce two more auxiliary vector fields denoted by $F$ and $G$ and defined by

$$F = \xi_1 D + \frac{\xi_2}{c} B$$  (14a)

$$G = \xi_1 H - \frac{\xi_2}{c} E.$$  (14b)

Straightforward computation shows that

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -I^i$$  (15a)

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial \dot{A}_0} \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = \nabla_k \left( \frac{\xi_1}{\mu_0} F^{ak} + \xi_2 * F^{ak} \right)$$  (15b)

$$\nabla_k \left( \frac{\xi_1}{\mu_0} F^{ak} + \xi_2 * F^{ak} \right) = \frac{1}{\alpha} \nabla \cdot F$$  (15c)

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial \dot{A}_a} \frac{\partial \mathcal{L}}{\partial \dot{A}_a} = 0$$  (15d)

$$= \frac{1}{\alpha} \nabla \cdot F = \frac{1}{\alpha} \nabla \times \nabla \times F.$$  (15e)

The inhomogeneous Maxwell equations then follow immediately as

$$\nabla \cdot F = \rho$$  (16a)

$$\nabla \times G = J + \frac{1}{\sqrt{-g}} \partial_t \left( \sqrt{-g} F \right).$$  (16b)

Note that $\rho = \alpha I^0 / c$ has to be interpreted as the external charge density and $J^a = \alpha I^a$ as the external current density. Vacuum polarization is treated implicitly through the definition of the vectors $F$ and $G$ as given by equations (13).

The homogeneous Maxwell equations are not modified, they read as in classical general relativity:

$$\nabla \times E = -\frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} B)$$  (16c)

$$\nabla \cdot B = 0.$$  (16d)

It is seen from eq. (10) that the primary fields to be evolved are $B$ and $F$. The other auxiliary fields are then deduced from the constitutive relations. Indeed, $B$ and $F$ being known, we can retrieve $D$ from eq. (14a). Next from $D$ and $B$ we can get $E$ and $H$ through eq. (5). Finally $G$ is obtained from eq. (14b) knowing $E$ and $H$ from the previous calculation. This completes one full time step to advance the primary fields $B$ and $F$.

The source terms $\rho$ and $J$ are left free so far. In quantum vacuum as well as in classical vacuum, they vanish $\rho = 0$ and $J = 0$. In the most general case, source terms have to be specified by some other equations, like the conservation of energy-momentum of the plasma. Nevertheless, in some restricted problems, it is possible to compute the current density from only the knowledge of the electromagnetic field. Such an example, called force-free electrodynamics is presented in the next section.

4.2 Force-free quantum electrodynamics (FFQED)

The source terms have not yet been specified. Having in mind to apply the above equations to pulsar and magnetar magnetospheres, we give expressions for the current density in the limit of a force-free plasma, neglecting inertia and pressure. The force-free condition in covariant form and including QED corrections reads

$$F_{ik} I^k = 0$$  (17)

and in the 3+1 formalism it becomes

$$J \cdot E = 0$$  (18a)

$$\rho E + J \times B = 0.$$  (18b)

which implies $E \cdot B = 0$ and therefore also $D \cdot B = 0$. Thus the parameter $\xi_2$ must vanish and therefore it follows that $F \cdot B = 0$. Applying the usual technique from special relativistic electrodynamics we get the current density including general relativity and QED such that

$$J = \rho \frac{E \wedge B}{B^2} + B \cdot \nabla \times G - F \cdot \nabla \times E \cdot B.$$  (19)

Because $c B^a = * F^{ak} n_k$ and $D^a / \varepsilon_0 = F^{ak} n_k$, $B$ and $D / \varepsilon_0$ can be interpreted as the magnetic and electric field respectively as measured by the fiducial observer whose four velocity is $n_k = (\alpha c, 0)$. Moreover

$$I^k n_k = \rho c^2$$  (20)

thus $\rho$ is the electric charge density as measured by this same observer. Its electric current density $\mathbf{j}$ is given by

$$\alpha \mathbf{j} = \mathbf{J} + \rho c \beta.$$  (21)

Maxwell equations (15), the constitutive relations (5, 14) and the prescription for the source terms set the background system to be solved for any prescribed metric in the low electromagnetic field limit according to the classical Lagrangian correction. As a check of these equations, let us compute their simplified version in the classical general-relativistic as well as in the special-relativistic QED limits.
4.3 Classical general-relativistic limit

If QED corrections are neglected, we have to set $\xi_1 = 1$ and $\xi_2 = 0$. In this limit the new auxiliary fields $F$ and $G$ reduce respectively to the classical fields $D$ and $H$. The inhomogeneous Maxwell equations become

$$\nabla \cdot D = \rho, \quad \nabla \times H = J + \frac{1}{\sqrt{\gamma}} \partial_t (\sqrt{\gamma} D)$$

which are the expressions found by Komissarov (2004, 2011) and by Pétri (2013). The current density then simplifies to

$$J = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mathbf{B} \cdot \nabla \times \mathbf{H} - \mathbf{D} \cdot \nabla \times \mathbf{E}}{B^2}$$

as expected from the above cited works.

4.4 Quantum special-relativistic limit

In the other limit, when the gravitational field becomes negligible, the lapse function is equal to one, $\alpha = 1$, and the shift vector vanishes, $\beta = 0$. The constitutive relations simplify to $\varepsilon_0 \mathbf{E} = \mathbf{D}$ and $\mu_0 \mathbf{H} = \mathbf{B}$. Maxwell equations then resemble those in a medium given by

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{G} = \mathbf{J} + \frac{\partial \mathbf{F}}{\partial t}$$

with the initial conditions on the divergence of the $F$ and $B$ fields such as

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{F} = \rho .$$

To retrieve the more familiar notation, we should substitute $F$ by $D$ and $G$ by $H$. Electrodynamics in the presence of strong electromagnetic fields can be described by non-linear Maxwell equations derived from an effective Lagrangian computed by Euler and Heisenberg. Quantum electrodynamics described vacuum as a polarized and magnetized media without external current density, $\mathbf{J} = 0$, and without any charge density, $\rho = 0$. The quantum vacuum is depicted by a magnetization $\mathbf{M}$ and a polarisation $\mathbf{P}$ such that

$$\mathbf{F} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{G} = \mathbf{B}/\mu_0 - \mathbf{M}.$$

To the first order in the fine structure constant, the Euler and Heisenberg Lagrangian shows that

$$P = \kappa (2 (E^2 - c^2 B^2) E + 7 c^2 (E \cdot B) B), \quad M = -\kappa (2 c^2 (E^2 - c^2 B^2) B - 7 c^2 (E \cdot B) E)$$

with

$$\kappa = \frac{\alpha_{\rm eff}}{45 \pi \mu_0 c^4 B_Q^2} .$$

Consequently, our constitutive relations include both limits, the classical general-relativistic field and QED in Newtonian gravitational field. We refer to it as general-relativistic force-free quantum electrodynamics (GRFFQED).

5 SPIN-DOWN LUMINOSITY OF PULSARS AND MAGNETARS

What kind of change in the spin-down luminosity can we expect from the vacuum polarization? We know already from previous works by Rezzolla & Ahmedov (2004) and Pétri (2013) that general relativity leads to an increase by a factor 2 to 6 of the inferred spin-down losses compared to flat space-time. Nevertheless Maxwell equations remain linear in general relativity. The enhanced luminosity can be interpreted as a combination of magnetic field amplification and gravitational redshift of the angular frequency of the neutron star due to the curvature of space-time. Moreover, from the lowest order corrections brought by QED, Maxwell field equations become non-linear. These non-linearities will perturb the topology of the dipole but they will also generate higher order multipoles like for instance an hexapole due to terms containing products of $(E^2, B^2)$ with $(\mathbf{B}, \mathbf{E})$. For the corrections to the dipole we have

$$\frac{\delta L_{\text{dip}}}{L_{\text{dip}}} = 2 \frac{\delta B}{B} \approx 4 \frac{\alpha_{\text{eff}}}{45 \pi} \frac{B^2}{B_Q^2} \approx 3 \times 10^{-4} \frac{B^2}{B_Q^2}$$

which remains negligible. These conclusions have already been found by Heyl & Hernquist (1997). For the hexapole, using the point multipole formula for the most luminous mode i.e. $m = 2$ given by Pétri (2015b) we find

$$\frac{L_{\text{hex}}}{L_{\text{dip}}} = \frac{243}{25} \left( \frac{R}{r_L} \right)^4 \frac{\delta B^2}{B^2} \approx 10^{-7} \frac{B^4}{B_Q^2}$$

so even less significant than the dipole corrections. We conclude that whatever the period and the magnetic field strength of pulsars and magnetars, QED corrections to the spin-down luminosities of any multipolar component remain meaningless with respect to the corrections brought by general relativity alone.

6 CONCLUSION

In this paper we showed how to include vacuum polarization into the description of a force-free magnetosphere in general relativity according to the weak electromagnetic field limit i.e. in the first order perturbation theory of the Lagrangian for the electromagnetic field. By introducing two new auxiliary fields, it is possible to cast the full set of Maxwell equations into a classical three dimensional picture treating curved space-time and vacuum polarization as two medium with specified constitutive relations for both parts. Our new formalism should help to quantify the merit of both contributions, general relativity and quantum electrodynamics, to the dynamics of neutron star magnetospheres, especially for magnetars and pulsars with high-B fields. As shown in the previous section, on a global length scale related to the size of the magnetosphere and to its rotational braking, QED is irrelevant to account for variation in the spin-down luminosity. Although QED corrections to the electrodynamics of neutron star magnetospheres remain weak, its implications for interpretation of observations like propagation and polarization of electromagnetic waves and pair creation are likely to be important.
ACKNOWLEDGEMENTS

I am grateful to the referee for helpful comments and suggestions. This work has been supported by the French National Research Agency (ANR) through the grant No. ANR-13-JS05-0003-01 (project EMPERE).

REFERENCES

Battesti R., Rizzo C., 2013, Reports on Progress in Physics, 76, 016401
Born M., Infeld L., 1934, Royal Society of London Proceedings Series A, 144, 425
Bulik T., Miller M. C., 1997, MNRAS, 288, 596
Denisov V. I., Denisova I. P., Svertilov S. I., 2004, Theoretical and Mathematical Physics, 140, 1001
Denisov V. I., Sokolov V. A., Vasil’ev M. I., 2014, Physical Review D, 90, 023011
Denisov V. I., Svertilov S. I., 2003, A&A, 399, L39
Denisov V. I., Svertilov S. I., 2005, Physical Review D, 71, 063002
Dupays A., Robilliard C., Rizzo C., Bignami G. F., 2005, Physical Review Letters, 94, 161101
Fernández R., Davis S. W., 2011, ApJ, 730, 131
Freytsis M., Gralla S. E., 2015, ArXiv e-prints
Haas F., 2005, Physics of Plasmas, 12, 062117
Harding A. K., Lai D., 2006, Reports on Progress in Physics, 69, 2631
Heisenberg W., Euler H., 1936, Zeitschrift für Physik, 98, 714
Heyl J. S., 2001, Physical Review D, 63, 064028
Heyl J. S., 2007, Ap&SS, 308, 101
Heyl J. S., Hernquist L., 1997a, Physical Review D, 55, 2449
Heyl J. S., Hernquist L., 1997b, Journal of Physics A Mathematical General, 30, 6475
Heyl J. S., Hernquist L., 1998, Physical Review D, 58, 043005
Heyl J. S., Hernquist L., 1999, Physical Review D, 59, 045005
Heyl J. S., Hernquist L., 2005, MNRAS, 362, 777
Heyl J. S., Shaviv N. J., 2000, MNRAS, 311, 555
Heyl J. S., Shaviv N. J., 2002, Physical Review D, 66, 023002
Heyl J. S., Shaviv N. J., Lloyd D., 2003, MNRAS, 342, 134
Ho W. C. G., Lai D., 2003, MNRAS, 338, 233
Jackson J. D., 2001, Électrodynamique classique. Dunod, 2001
Kim J. Y., 2012, Jour. Cosm. Astro. Phys., 10, 56
King B., Di Piazza A., 2014, European Physical Journal Special Topics, 223, 1063
Komissarov S. S., 2004, MNRAS, 350, 427
Komissarov S. S., 2011, MNRAS, 418, L94
Lai D., Ho W. C., 2003a, Physical Review Letters, 91, 071101
Lai D., Ho W. C. G., 2003b, ApJ, 588, 962
Landau L., Lifchitz E., 1989a, Électrodynamique quantique. Editions MIR Moscow
Landau L., Lifchitz E., 1989b, Théorie des champs. Editions MIR Moscow
Manfredi G., Haas F., 2001, Phys. Rev. B, 64, 075316
Mazur D., Heyl J. S., 2011, MNRAS, 412, 1381
McKinney J. C., 2006, MNRAS, 368, L30
Olausen S. A., Kaspi V. M., 2014, ApJS, 212, 6
Özel F., 2003, ApJ, 583, 402
Parfrey K., Beloborodov A. M., Hui L., 2012, MNRAS, 423, 1416
Pétri J., 2012, MNRAS, 424, 605
Pétri J., 2013, MNRAS, 433, 986
Pétri J., 2014, MNRAS, 439, 1071
Pétri J., 2015a, MNRAS, 447, 3170
Pétri J., 2015b, MNRAS, 450, 714
Rezzolla L., Ahmadov B. J., 2004, MNRAS, 352, 1161
Ruffini R., Vereshchagin G., Xue S.-S., 2010, Phys. Rep., 487, 1
Ruffini R., Wu Y.-B., Xue S.-S., 2013, Physical Review D, 88, 085004
Ruffini R., Xue S.-S., 2006, J.Korean Phys.Soc., 49, S715
Shannon R. M., Heyl J. S., 2006, MNRAS, 368, 1377
Shaviv N. J., Heyl J. S., Lithwick Y., 1999, MNRAS, 306, 333
Spitkovsky A., 2006, ApJL, 648, L51
Taverna R., Muleri F., Turolla R., Soffitta P., Fabiani S., Nobili L., 2014, MNRAS, 438, 1686
Thompson C., Blaes O., 1998, Physical Review D, 57, 3219
Timokhin A. N., 2006, MNRAS, 368, 1055
Uzan J., Deruelle N., 2014, Théories de la Relativité. Belin
Uzdensky D. A., Rightley S., 2014, Reports on Progress in Physics, 77, 036902