CARLEMAN ESTIMATES FOR PARABOLIC EQUATIONS WITH INTERIOR DEGENERACY AND NEUMANN BOUNDARY CONDITIONS

By

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Abstract. We consider a parabolic problem with degeneracy in the interior of the spatial domain and Neumann boundary conditions. In particular, we focus on the well-posedness of the problem and on Carleman estimates for the associated adjoint problem. The novelty of the present paper is that, for the first time, the problem is considered as one with an interior degeneracy and Neumann boundary conditions, so no previous result can be adapted to this situation. As a consequence, new observability inequalities are established.

1 Introduction

The study of degenerate parabolic equations is the subject of numerous articles and books. Indeed, many problems coming from physics, biology, and economics are described by degenerate parabolic equations, whose linear prototype is

\( \frac{\partial u}{\partial t} - Au = h(t, x), \quad (t, x) \in (0, T) \times (0, 1) \)

with the associated desired boundary conditions. Here \( T > 0 \) is given, \( h \) belongs to a suitable Lebesgue space and \( Au = A_1 u := (au_x)_x \) or \( Au = A_2 u := au_{xx} \), where \( a \) is a degenerate function.

In the present paper, we focus on a particular topic related to this field of research, viz, Carleman estimates for the adjoint problem of the previous equation. Numerous papers have been devoted to proving certain forms of these estimates and to discussing applications. For example, it is well known that they are crucial for inverse problems (see, for example, [23]) and for unique continuation properties (see, for example, [22]). In particular, they are a fundamental tool for proving...
observability inequalities, which leads to global null controllability results for the Cauchy problem associated to (1.1) also in the nondegenerate case (see, for instance, [1] - [3], [7] - [10], [14] - [20], [22], [24], and the references therein). For related systems of degenerate equations, we refer, for example, to [2] and [1].

In most of the previous papers, the authors assume that the function $a$ degenerates at the boundary of the space domain, for example $a(x) = x^k(1-x)^\alpha$, $x \in [0, 1]$, where $k$ and $\alpha$ are positive constants, and the degeneracy is regular. The question of Carleman estimates for partial differential systems with nonsmooth coefficients, i.e., the coefficient $a$ is not of class $C^1$ (or even with higher regularity, as is sometimes required) is not fully solved yet. Indeed, the presence of a nonsmooth coefficient introduces several complications, and, in fact, the literature in this context is quite poor also in the nondegenerate case (for more details see [20]). To the best of our knowledge, the first results on Carleman estimates for the adjoint problem of (1.1) with an interior degenerate point are obtained in [19], for a regular degeneracy, and in [20], for a globally nonsmooth degeneracy. We emphasize that in [19] and in [20], the authors consider the problem in divergence form ([19], [20]) or in nondivergence form ([20]). We also refer to [5], where an inverse source problem of a $2 \times 2$ cascade parabolic systems with interior degeneracy is studied. However, in all the previous papers, the authors consider (1.1) only with Dirichlet boundary conditions. Neumann boundary conditions are considered in [3] and in [17], but again the degeneracy is at the boundary of the space domain.

The goal of this paper is to give a full analysis of (1.1) with Neumann boundary conditions in the case that the degeneracy occurs at the interior of the space domain; moreover, the coefficient is allowed to be nonsmooth in the nondivergence case and in the strongly degenerate divergence case. In particular, we consider the problem

$$\begin{cases}
\frac{\partial u}{\partial t} - Au = h(t, x), & (t, x) \in Q_T, \\
u_x(t, 0) = u_\alpha(t, 1) = 0, & t \geq 0, \\
u(0, x) = u_0(x), & x \in (0, 1),
\end{cases}$$

(1.2)

where $Q_T := (0, T) \times (0, 1)$, $Au := A_1u := (a u_x)_x$, or $Au := A_2u := au_{xx}$, $a$ degenerates at $x_0 \in (0, 1)$, $u_0 \in X$, and $h \in L^2(0, T; X)$. Here $X$ denotes the Hilbert space $L^2(0, 1)$, in divergence form, and $L^2_{1/a}(0, 1)$, in nondivergence form (for the precise definition of $L^2_{1/a}(0, 1)$, we refer to Section 3).

**Definition 1.1.** The operators $A_1u := (au)'$ and $A_2u = au''$ are **weakly degenerate** if there exists $x_0 \in (0, 1)$ such that $a(x_0) = 0$, $a > 0$ on $[0, 1] \setminus \{x_0\}$, $a \in W^{1,1}(0, 1)$, and there exists $K \in (0, 1)$ such that $(x - x_0)a' \leq Ka$ a.e. in $[0, 1]$.
Definition 1.2. The operators $A_1u := (au')'$ and $A_2u = au''$ are strongly degenerate if there exists $x_0 \in (0, 1)$ such that $a(x_0) = 0$, $a > 0$ on $[0, 1] \setminus \{x_0\}$, $a \in W^{1,\infty}(0, 1)$, and there exists $K \in [1, 2)$ such that $(x - x_0)a' \leq Ka$ a.e. in $[0, 1]$.

Typical examples of weak and strong degeneracies are, respectively, $a(x) = |x - x_0|^\alpha$, $0 < \alpha < 1$, and $a(x) = |x - x_0|^\alpha$, $1 \leq \alpha < 2$.

The object of this paper is twofold: first we analyze the well-posedness of the problem with Neumann boundary conditions; secondly, we prove Carleman estimates. To this end we have a new approach: first, we use a reflection procedure, and then we employ the Carleman estimates for the analogue of (1.2) with Dirichlet boundary conditions proved in [20]. Finally, using again a reflection procedure, we prove observability inequalities as a consequence of the Carleman estimates. In particular, we prove that there exists a positive constant $C_T$ such that every solution $v$ of

$$
\begin{align*}
\begin{cases}
v_t + Av &= 0, & (t, x) \in Q_T, \\
v_x(t, 0) = v_x(t, 1) &= 0, & t \in (0, T), \\
v(T, x) &= v_T(x) \in X,
\end{cases}
\end{align*}
$$

satisfies, under suitable assumptions, the estimate

$$
\|v(0)\|_X^2 \leq C_T \|v\|_{L^2(0, T; X)}^2.
$$

Here $\chi_\omega$ is the characteristic function of the control region $\omega$, which is assumed to be an interval containing the degeneracy point or an interval lying on one side of the degeneracy point. As an immediate consequence, using a standard technique (e.g., see [22, Section 7.4]), we can prove the null controllability result for the linear degenerate problem: if (1.3) holds, then for every $u_0 \in X$, there exists $h \in L^2(0, T; X)$ such that the solution $u$ of

$$
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - Au &= h(t, x)\chi_\omega(x), & (t, x) \in Q_T, \\
u_x(t, 0) &= u_x(t, 1) = 0, & t \geq 0, \\
u(0, x) &= u_0(x), & x \in (0, 1),
\end{cases}
\end{align*}
$$

is such that $u(T, x) = 0$ for every $x \in [0, 1]$; moreover, $\|h\|_{L^2(0, T; X)}^2 \leq C \|u_0\|_X^2$, for some universal positive constant $C$.

We stress the fact that in the present paper, we consider equations both in divergence form and in nondivergence form, since the last equation cannot be recast in divergence form. For example, the simple equation $u_t = a(x)u_{xx}$ can be written in divergence form as $u_t = (a(x)u_x)_x - a' u_x$, only if $a'$ exists; in addition, as far as
well-posedness is concerned, for the last equation, additional conditions are necessary. For instance, for the prototype \( a(x) = |x-x_0|^K \), well-posedness is guaranteed if \( K \geq 2 \). However, in [20], the authors prove that if \( a(x) = |x-x_0|^K \), the global null controllability fails exactly when \( K \geq 2 \).

The paper is organized as follows. In Sections 2 and 3, we study the well-posedness of the problem and characterize the domain of the operator in some cases. In Sections 4 and 5, we prove Carleman estimates for the problem in divergence form and in nondivergence form. As a consequence, in Section 6, we prove observability inequalities. We conclude the paper with some comments on Carleman estimates.

As for notation, \( C \) and \( C_T \) denote universal positive constants, which are allowed to vary from line to line and depend only on the coefficients of the equation.

2 Well-posedness in the divergence case

In this section, we consider the operator in divergence form, that is, \( A_1 u = (au')' \), and we distinguish, as usual, two cases: the weakly degenerate case and the strongly degenerate case.

2.1 Weakly degenerate operator. Throughout this subsection, we assume that the operator is weakly degenerate.

In order to prove that \( A_1 \), with a suitable domain, generates a strongly continuous semigroup, we introduce, as in [3] or [21], the following weighted spaces:

\[
H^1_\alpha(0,1) : = \{ u \text{ is absolutely continuous in } [0,1] \text{ and } \sqrt{a}u' \in L^2(0,1) \}
\]

with the norm

\[
\| u \|_{H^1_\alpha(0,1)}^2 := \| u \|_{L^2(0,1)}^2 + \| \sqrt{a}u' \|_{L^2(0,1)}^2
\]

and \( H^2_\alpha(0,1) := \{ u \in H^1_\alpha(0,1) : au' \in H^1(0,1) \} \) with

\[
\| u \|_{H^2_\alpha(0,1)}^2 := \| u \|_{H^1(0,1)}^2 + \| (au')' \|_{L^2(0,1)}^2.
\]

Then, we define the operator \( A_1 \) by \( D(A_1) = \{ u \in H^2_\alpha(0,1) : u'(0) = u'(1) = 0 \} \), and, for all \( u \in D(A_1) \), \( A_1 u := (au')' \). As in [21, Lemma 2.1], using the fact that \( u'(0) = u'(1) = 0 \) for all \( u \in D(A_1) \), one can prove the following formula of integration by parts.

**Lemma 2.1.** For all \( (u,v) \in D(A_1) \times H^1_\alpha(0,1) \),

\[
\int_0^1 (au')'v \, dx = -\int_0^1 au'v' \, dx.
\]
Now let us return to the problem (1.2), recalling the following definition.

**Definition 2.1.** For \( u_0 \in L^2(0, 1) \) and \( h \in L^2(Q_T) := L^2(0, T; L^2(0, 1)) \), a function \( u \) is said to be a **weak solution** of (1.2) with \( A = A_1 \) if

\[
u \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1_a(0, 1))
\]

and

\[
\int_0^1 u(T, x)\varphi(T, x) \, dx - \int_0^1 u_0(x)\varphi(0, x) \, dx - \int_{Q_T} u \varphi_t \, dxdt = - \int_{Q_T} a u \varphi_x \, dxdt + \int_{Q_T} h \varphi \, dxdt
\]

for all \( \varphi \in H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^1_a(0, 1)). \)

**Theorem 2.1.** The operator \( A_1 : D(A_1) \to L^2(0, 1) \) is self-adjoint, nonpositive on \( L^2(0, 1) \), and generates an analytic contraction semigroup of angle \( \pi/2 \). Therefore, for all \( h \in L^2(Q_T) \) and \( u_0 \in L^2(0, 1) \), there exists a unique solution

\[
u \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1_a(0, 1))
\]

of (1.2) such that

\[
\sup_{t \in [0, T]} \| u(t) \|^2_{L^2(0, 1)} + \int_0^T \| u(t) \|^2_{H^1_a(0, 1)} \, dt \leq C_T \left( \| u_0 \|^2_{L^2(0, 1)} + \| h \|^2_{L^2(Q_T)} \right),
\]

for some positive constant \( C_T \). Moreover, if \( h \in W^{1, 1}(0, T; L^2(0, 1)) \) and \( u_0 \in H^1_a(0, 1), \) then

\[
u \in C^1([0, T]; L^2(0, 1)) \cap C([0, T]; D(A_1)),
\]

and there exists a positive constant \( C \) such that

\[
\sup_{t \in [0, T]} \left( \| u(t) \|^2_{H^1_a(0, 1)} + \int_0^T \left( \| u_t \|^2_{L^2(0, 1)} + \| (au_x)_x \|^2_{L^2(0, 1)} \right) \, dt \right) \leq C \left( \| u_0 \|^2_{H^1_a(0, 1)} + \| h \|^2_{L^2(Q_T)} \right).
\]

**Proof.** Observe that \( D(A_1) \) is dense in \( L^2(0, 1) \). To show that \( A_1 \) is nonpositive and self-adjoint, it suffices to prove that \( A_1 \) is symmetric, nonpositive, and that \( (I - A_1)(D(A_1)) = L^2(0, 1). \) Following [21], one can prove that \( A_1 \) is symmetric and nonpositive. Now, we prove that \( I - A_1 \) is surjective, since the proof is quite different.
First, observe that $H^1_0(0, 1)$ equipped with the inner product

$$(u, v)_1 := \int_0^1 (uv + au'v')dx,$$

is a Hilbert space. Moreover, $H^1_0(0, 1) \hookrightarrow L^2(0, 1) \hookrightarrow (H^1_0(0, 1))^*$, where $(H^1_0(0, 1))^*$ is the dual space of $H^1_0(0, 1)$ with respect to $L^2(0, 1)$. Now, clearly, for $f \in L^2(0, 1)$, the functional $F : H^1_0(0, 1) \rightarrow \mathbb{R}$ defined as $F(v) := \int_0^1 fv\,dx$ belongs to $(H^1_0(0, 1))^*$. As a consequence, by the Lax–Milgram Lemma, there exists a unique $u \in H^1_0(0, 1)$ such that $(u, v)_1 = \int_0^1 fv\,dx$ for all $v \in H^1_0(0, 1)$. In particular, since $C_c^\infty(0, 1) \subset H^1_0(0, 1)$, the previous equality holds for all $v \in C_c^\infty(0, 1)$, i.e., $\int_0^1 au'v'\,dx = \int_0^1 (f - u)v\,dx$ for all $v \in C_c^\infty(0, 1)$. Thus, the distributional derivative of $au$ is a function in $L^2(0, 1)$, that is, $au' \in H^1(0, 1)$ (recall that $\sqrt{au'} \in L^2(0, 1)$) and $(au')' = u - f$ a.e. in $(0, 1)$. Then $u \in H^1_0(0, 1)$ and, proceeding as in [6, Proposition VIII.16], one can prove that $u'(0) = u'(1) = 0$. In fact, by the Gauss Green Identity and $(u, v)_1 = \int_0^1 fv\,dx$, one has that for all $v \in H^1_0(0, 1)$,

$$(2.7) \quad \int_0^1 (au')'v\,dx = [au'v]_x=1^{x=1} - \int_0^1 au'v'\,dx = [au'v]_x=1^{x=1} - \int_0^1 (f - u)v\,dx.$$

In particular, the previous equality holds for all $v \in C_c^\infty(0, 1)$. Thus, $[au'v]_x=1^{x=0} = 0$ for all $v \in C_c^\infty(0, 1)$, and $(au')' = u - f$ a.e. in $(0, 1)$. Returning to (2.7), we see that it becomes $[au'v]_x=1^{x=0} = 0$, for all $v \in H^1_0(0, 1)$. Since $v(0)$ and $v(1)$ are arbitrary and $a$ does not degenerate in 0 and in 1, one can conclude that $u'(0) = u'(1) = 0$. Hence $u \in D(A(1))$; and, by $(u, v)_1 = \int_0^1 fv\,dx$ and Lemma 2.1, we have $\int_0^1 (u - (au')' - f)v\,dx = 0$. Consequently, $u \in D(A(1))$ and $u - A_1u = f$.

Finally, it is well known (see, e.g., [21]) that since $A_1$ is a nonpositive self-adjoint operator on a Hilbert space, $(A_1, D(A(1)))$ generates a cosine family and an analytic contractive semigroup of angle $\pi/2$ on $L^2(0, 1)$.

Following [20, Theorem 2.1], we now prove (2.4)–(2.6). First, since $A_1$ is the generator of a strongly continuous semigroup on $L^2(0, 1)$, if $u_0 \in L^2(0, 1)$, then the solution $u$ of (1.2) belongs to $C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1_0(0, 1))$, while, if $u_0 \in D(A(1))$ and $h \in W^{1,1}(0, T; L^2(0, 1))$, then $u \in C^1([0, T]; L^2(0, 1)) \cap C([0, T]; H^2_0(0, 1))$ by [4, Proposition 3.3] or [12, Proposition 4.1.6].

Now, we prove (2.5) and (2.6), from which the last required regularity property for $u$ follows by standard linear arguments. First, take $u_0 \in D(A(1))$ and multiply the equation of (1.2) by $u$. By the Cauchy–Schwarz inequality, we obtain

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(0, 1)}^2 + \|\sqrt{a}u_x(t)\|_{L^2(0, 1)}^2 \leq \frac{1}{2} \|u(t)\|_{L^2(0, 1)}^2 + \frac{1}{2} \|h(t)\|_{L^2(0, 1)}^2.$$
for every $t \in (0, T]$, from which it follows that

\begin{equation}
\|u(t)\|_{L^2(0,1)}^2 \leq e^T \left( \|u(0)\|_{L^2(0,1)}^2 + \|h\|_{L^2(Q_T)}^2 \right)
\end{equation}

for every $t \leq T$. From (2.8) and (2.9), we immediately get

\begin{equation}
\int_0^T \|\sqrt{a}u_x(t)\|_{L^2(0,1)}^2 dt \leq C_T \left( \|u(0)\|_{L^2(0,1)}^2 + \|h\|_{L^2(Q_T)}^2 \right)
\end{equation}

for every $t \leq T$ and some universal constant $C_T > 0$. Thus, by (2.9) and (2.10), (2.4) follows if $u_0 \in D(A_1)$. Since $D(A_1)$ is dense in $L^2(0,1)$, the same inequality holds if $u_0 \in L^2(0,1)$.

Now, multiplying the equation by $-(au_x)_x$ and integrating over $(0,1)$, we easily get

\begin{equation}
\frac{d}{dt} \|\sqrt{a}u_x(t)\|_{L^2(0,1)}^2 + \|(au_x)_x(t)\|_{L^2(0,1)}^2 \leq \|h(t)\|_{L^2(0,1)}^2
\end{equation}

for every $t$, so that, as before, we find $C'_T > 0$ such that

\begin{equation}
\|\sqrt{a}u_x(t)\|_{L^2(0,1)}^2 + \int_0^T \|(au_x)_x(t)\|_{L^2(0,1)}^2 dt \leq C'_T \left( \|\sqrt{a}u_x(0)\|_{L^2(0,1)}^2 + \|h\|_{L^2(Q_T)}^2 \right)
\end{equation}

for every $t \leq T$. Finally, from $u_t = (au_x)_x + h$, squaring and integrating, we find

\begin{equation}
\int_0^T \|u(t)\|_{L^2(0,1)}^2 \leq C \left( \int_0^T \|(au_x)_x\|_{L^2(0,1)}^2 + \|h\|_{L^2(Q_T)}^2 \right),
\end{equation}

and, together with (2.11), we find

\begin{equation}
\int_0^T \|u_t(t)\|_{L^2(0,1)}^2 \leq C \left( \|\sqrt{a}u_x(0)\|_{L^2(0,1)}^2 + \|h\|_{L^2(Q_T)}^2 \right).
\end{equation}

In conclusion, (2.8), (2.9), (2.11) and (2.12) give (2.4) and (2.6). Clearly, (2.5) and (2.6) hold also if $u_0 \in H^1_a(0,1)$, since $H^2_a(0,1)$ is dense in $H^1_a(0,1)$.

\section*{2.2 Strongly degenerate operator.}
In this subsection, we assume that the operator is strongly degenerate. Following [3], we introduce the weighted space

\[ H^1_a(0,1) := \{ u \in L^2(0,1) : u \text{ locally absolutely continuous in } [0,x_0) \cup (x_0, 1], \text{ and } \sqrt{a}u' \in L^2(0,1) \} \]

with the norm given in (2.1). Define the operator $A_1$ by

\[ D(A_1) = \{ u \in H^2_a(0,1) | u'(0) = u'(1) = 0 \} \]
and, for any \( u \in D(A_1) \), \( A_1 u := (au)' \), where \( (H^2_a(0, 1), \| \cdot \|_{H^2_a(0, 1)}) \) is defined as before. Since in this case a function \( u \in H^2_a(0, 1) \) is locally absolutely continuous in \([0, 1] \) and not necessarily absolutely continuous in \([0, 1]\) as for the weakly degenerate case, equality (2.3) is not true a priori. Thus, as in [21], we have to prove again the formula of integration by parts. One idea for doing this is to characterize the domain of \( A_1 \).

**Proposition 2.1.** Let

\[ X := \{ u \in L^2(0, 1) : u \text{ locally absolutely continuous in } [0, 1] \setminus \{x_0\}, \quad \sqrt{a}u' \in L^2(0, 1), \, au \text{ is continuous at } x_0, \text{ and } (au)(x_0) = 0 \}. \]

Then \( H^1_a(0, 1) = X \).

**Proof.** Obviously, \( X \subseteq H^1_a \). Now we take \( u \in H^1_a \) and prove that \( au \) is continuous at \( x_0 \) and that \( (au)(x_0) = 0 \), i.e., \( u \in X \). Toward this end, observe that since \( a \in W^{1, \infty}(0, 1) \), \( (au)' = a'u + au' \in L^2(0, 1) \). Thus, for \( x < x_0 \), one has \( au(x) = (au)(0) + \int_0^x (au)'(t)dt \) (observe that \( (au)(0) \in \mathbb{R} \)). This implies that there exists \( \lim_{x \to x_0^-} (au)(x) = (au)(0) + \int_0^{x_0^-} (au)'(t)dt = L \in \mathbb{R} \). As in [21, Proposition 2.3], one can prove that \( L = 0 \). Analogously, \( \lim_{x \to x_0^+} (au)(x) = (au)(x_0) = 0 \). Thus \( (au)(x_0) = 0 \). \( \square \)

Using the previous result, one can prove the following characterization.

**Proposition 2.2.** Let

\[ D := \{ u \in L^2(0, 1) : u \text{ locally absolutely continuous in } [0, 1] \setminus \{x_0\}, \quad au \in H^1(0, 1), \quad au' \in H^1(0, 1), \quad au \text{ is continuous at } x_0, \text{ and } (au)(x_0) = (au')(x_0) = u'(0) = u'(1) = 0 \}. \]

Then \( D(A_1) = D \).

**Proof.** \( D \subseteq D(A_1) \): It is a simple adaptation of the proof of [21, Proposition 2.4] to which we refer. We emphasize that here we use the boundary conditions \( u'(0) = u'(1) = 0 \).

\( D(A_1) \subseteq D \): As in the proof of Proposition 2.1, we can prove that \( au, (au') \in L^2(0, 1) \); thus \( au \in H^1(0, 1) \). Moreover, by Proposition 2.1, \( (au)(x_0) = 0 \). Thus, it suffices to prove that \( (au')(x_0) = 0 \). This follows as in [21, Proposition 2.4]. \( \square \)

We point out the fact that in the proof of the previous characterization, the condition \( 1/a \not\in L^1(0, 1) \) is crucial. Clearly this condition is not satisfied if the
operator is weakly degenerate. Indeed, in [19, Lemma 2.1] it is proved that if the operator is weakly degenerate, then $1/a \in L^1(0, 1)$; on the other hand, if the operator is strongly degenerate, then $1/\sqrt{a} \in L^1(0, 1)$, while $1/a \not\in L^1(0, 1)$.

Proceeding as in [21, Lemma 2.6] and using the previous characterization, we can prove the formula of integration by parts (2.3) also in the strongly degenerate case. Thus, the analogue of Theorem 2.1 holds.

3 Well-posedness in the nondivergence case

Now, we consider the operator $A_2u = au''$ in the weakly and in the strongly degenerate cases and, as in [20, Chapter 2], we consider the Hilbert spaces

$$L^2_{1/a}(0, 1) := \left\{ u \in L^2(0, 1) : \int_0^1 \frac{u^2}{a} \, dx < \infty \right\},$$

$$H^1_{1/a}(0, 1) := L^2_{1/a}(0, 1) \cap H^1(0, 1),$$

$$H^2_{1/a}(0, 1) := \left\{ u \in H^1_{1/a}(0, 1) : u' \in H^1(0, 1) \right\},$$

endowed with the respective norms

$$\|u\|^2_{L^2_{1/a}(0, 1)} := \int_0^1 \frac{u^2}{a} \, dx,$$

$$\|u\|^2_{H^1_{1/a}(0, 1)} := \|u\|^2_{L^2_{1/a}(0, 1)} + \|u'\|^2_{L^2(0, 1)},$$

$$\|u\|^2_{H^2_{1/a}(0, 1)} := \|u\|^2_{H^1_{1/a}(0, 1)} + \|au''\|^2_{L^2_{1/a}(0, 1)}.$$

Indeed, it is a trivial fact that, if $u' \in H^1(0, 1)$, then $au'' \in L^2_{1/a}(0, 1)$, so the norm for $H^2_{1/a}(0, 1)$ is well-defined; we can write in a more appealing way

$$H^2_{1/a}(0, 1) := \left\{ u \in H^1_{1/a}(0, 1) : u' \in H^1(0, 1) \text{ and } au'' \in L^2_{1/a}(0, 1) \right\}.$$

Using the previous spaces, we define the operator $A_2$ by

$$D(A_2) = \{ u \in H^2_{1/a}(0, 1) : u'(0) = u'(1) = 0 \}$$

and, for all $u \in D(A_2)$, $A_2u := au''$.

Proceeding as in [21, Corollary 3.1], one can prove the following characterization.

**Corollary 3.1.** If the operator is weakly degenerate, the spaces $H^1_{1/a}(0, 1)$ and $H^1(0, 1)$ coincide algebraically. Moreover the two norms are equivalent.

In every case, $C^\infty_c(0, 1) \subset H^1_{1/a}(0, 1)$.

As for the divergence form, a crucial tool is the following formula of integration by parts.
Lemma 3.1. For all \((u, v) \in D(A_2) \times H^1_{1/a}(0, 1)\),
\[
(3.1) \quad \int_0^1 u'' v \, dx = -\int_0^1 u' v' \, dx.
\]

Proof. The lemma is trivial, since \(u'(0) = u'(1) = 0\) and both \(u' \in H^1(0, 1)\) and \(v \in H^1(0, 1)\).

We also recall the following definition.

Definition 3.1. Assume that \(h \in L^2_{1/a}(Q_T) := L^2(0, T; L^2_{1/a}(0, 1))\) and \(u_0 \in L^2_{1/a}(0, 1)\). A function \(u\) is said to be a \textbf{weak solution} of (1.2) with \(A = A_2\) if
\[
u \in C([0, T]; L^2_{1/a}(0, 1)) \cap L^2(0, T; H^1_{1/a}(0, 1))
\]
and \(u\) satisfies
\[
\int_0^1 \frac{u(T, x)\varphi(T, x)}{a(x)} \, dx - \int_0^1 \frac{u_0(x)\varphi(0, x)}{a(x)} \, dx - \int_{Q_T} \frac{\varphi(t, x)u(t, x)}{a(x)} \, dx \, dt
\]
\[= -\int_{Q_T} u_x(t, x)\varphi_x(t, x) \, dx \, dt + \int_{Q_T} h(t, x)\frac{\varphi(t, x)}{a(x)} \, dx \, dt
\]
for all \(\varphi \in H^1(0, T; L^2_{1/a}(0, 1)) \cap L^2(0, T; H^1_{1/a}(0, 1))\).

As a consequence of the Lemma 3.1, one has the next theorem, whose proof is similar to that of Theorem 2.1.

Theorem 3.1. The operator \(A_2 : D(A_2) \to L^2_{1/a}(0, 1)\) is self-adjoint, non-positive on \(L^2_{1/a}(0, 1)\), and generates an analytic contraction semigroup of angle \(\pi/2\). Therefore, for all \(h \in L^2_{1/a}(Q_T)\) and \(u_0 \in L^2_{1/a}(0, 1)\), there exists a unique solution
\[
u \in C([0, T]; L^2_{1/a}(0, 1)) \cap L^2(0, T; H^1_{1/a}(0, 1))
\]
of (1.2) such that
\[
(3.2) \quad \sup_{t \in [0, T]} \|u(t)\|_{L^2_{1/a}(0, 1)}^2 + \int_0^T \|u(t)\|_{H^1_{1/a}(0, 1)}^2 \, dt \leq C_T \left( \|u_0\|_{L^2_{1/a}(0, 1)}^2 + \|h\|_{L^2_{1/a}(Q_T)}^2 \right)
\]
for some positive constant \(C_T\). Moreover, if \(h \in W^{1,1}(0, T; L^2_{1/a}(0, 1))\) and \(u_0 \in H^1_{1/a}(0, 1)\), then
\[
u \in C^1([0, T]; L^2_{1/a}(0, 1)) \cap C([0, T]; D(A_2)),
\]
and there exists a positive constant \(C\) such that
\[
(3.4) \quad \sup_{t \in [0, T]} \left( \|u(t)\|_{H^1_{1/a}(0, 1)}^2 + \int_0^T \left( \|u_t\|_{L^2_{1/a}(0, 1)}^2 + \|au_xa_x\|_{L^2_{1/a}(0, 1)}^2 \right) \, dt \right)
\]
\[\leq C \left( \|u_0\|_{H^1_{1/a}(0, 1)}^2 + \|h\|_{L^2(Q_T)}^2 \right).
\]
Lemma 3.1, one has that \( \angle \pi f \) clearly, if \( u \in L^2(0, 1) \), it becomes \( (3.5) \), it becomes 

\[ \text{that (} \]

where \( H^1_1(0, 1) \) is equipped with the natural inner product \( (u, v)_1 := \int_0^1 \left( \frac{u}{a} + u'v' \right) \, dx \). Moreover, it is clear that 

\[ H^1_1(0, 1) \hookrightarrow L^2_1(0, 1) \hookrightarrow (H^1_1(0, 1))^*, \]

where \((H^1_1(0, 1))^*\) is the dual space of \( H^1_1(0, 1) \) with respect to \( L^2_1(0, 1) \). Now, clearly, if \( f \in L^2_1(0, 1) \), the functional \( F : H^1_1(0, 1) \to \mathbb{R} \) defined as \( F(v) := \int_0^1 (fv/a) \, dx \) belongs to \((H^1_1(0, 1))^*\). As a consequence, by the Lax–Milgram Lemma, there exists a unique \( u \in H^1_1(0, 1) \) such that \((u, v)_1 = \int_0^1 (fv/a) \, dx \) for all \( v \in H^1_1(0, 1) \). In particular, since \( C_c^\infty(0, 1) \subset H^1_1(0, 1) \), the previous equality holds for all \( v \in C_c^\infty(0, 1) \), i.e., 

\[ \int_0^1 u'v' \, dx = \int_0^1 \left( \frac{f - u}{a} \right) v \, dx \]

for every \( v \in C_c^\infty(0, 1) \). Thus, the distributional derivative of \( u' \) is a function in \( L^2_1(0, 1) \subset L^2(0, 1) \); hence it is easy to see that \( au'' \in L^2_1(0, 1) \). Thus \( u \in H^2_1(0, 1) \). Proceeding as in Theorem 2.1, one can prove that \( u'(0) = u'(1) = 0 \). In fact, by the Gauss Green Identity and \((u, v)_1 = \int_0^1 (fv/a) \, dx \), one has 

\[ (3.5) \int_0^1 u'' \, dx = [u'v]_{x=0}^{x=1} - \int_0^1 u'v' \, dx = [u'v]_{x=0}^{x=1} - \int_0^1 \left( \frac{f - u}{a} \right) v \, dx \]

for all \( v \in H^1_1(0, 1) \). In particular, (3.5) holds for all \( v \in C_c^\infty(0, 1) \). Thus, \([u'v]_{x=0}^{x=1} = 0\) for all \( v \in C_c^\infty(0, 1) \), and \( u'' = (u - f)/a \) a.e. in \((0, 1)\). Returning to (3.5), it becomes \([u'v]_{x=0}^{x=1} = 0\), for all \( v \in H^1_1(0, 1) \). Again one can conclude that \( u'(0) = u'(1) = 0 \). Thus \( u \in D(A_2) \), and by \((u, v)_1 = \int_0^1 (fv/a) \, dx \) and Lemma 3.1, we have 

\[ \int_0^1 \left( \frac{u - f}{a} - u'' \right) v \, dx = 0. \]

Consequently, \( u \in D(A_2) \) and \( u - A_2u = f \). As in Theorem 2.1, one can conclude that \((A_2, D(A_2))\) generates a cosine family and an analytic contractive semigroup of angle \( \pi/2 \) on \( L^2_1(0, 1) \). The rest of the theorem follows as in [20, Theorem 2.2].

**3.1 Characterizations in the strongly degenerate case.** In this subsection, as in [21], we concentrate on the strongly degenerate case and characterize
the spaces $H^{1}_{1,a}(0,1)$ and $H^{2}_{1,a}(0,1)$. We point out the fact that in nondivergence form, the characterization of the domain of the operator is not important for proving the formula of integration by parts as in divergence form.

First observe that, as in [19, Lemma 2.1], one can prove that $|x-x_0|^2/a(x) \leq C$ for all $x \in [0,1) \setminus \{x_0\}$, where $C := \max\{(x_0)^2/a(0), (1-x_0)^2/a(1)\}$. We have the following characterization.

**Proposition 3.1.** Let $X := \{u \in H^{1}_{1,a}(0,1) : u(x_0) = 0\}$. If $A_2$ is strongly degenerate, then $H^{1}_{1,a}(0,1) = X$ and, for all $u \in X$, $\|u\|_{H^{1}_{1,a}(0,1)}$ is equivalent to $\left( \int_0^1 (u')^2 \, dx \right)^{1/2}$.

The proof of the Proposition 3.1 is a simple adaptation of the proof of [21, Proposition 3.6], to which we refer.

An immediate consequence of Proposition 3.1 is the following result.

**Proposition 3.2.** Let $D := \{u \in H^{1}_{1,a}(0,1) : au'' \in L^{2}_{1,a}(0,1), u' \in H^{1}(0,1) \text{ and } u(x_0) = (au')(x_0) = 0\}$. If $A_2$ is strongly degenerate, then $H^{2}_{1,a}(0,1) = D$.

**Proof.** Obviously, $D \subseteq H^{2}_{1,a}(0,1)$. Now, we take $u \in H^{2}_{1,a}(0,1)$ and prove that $u \in D$. By Proposition 3.1, $u(x_0) = 0$. Thus, it suffices to prove that $(au')(x_0) = 0$. Since $u' \in H^{1}(0,1)$ and $a \in W^{1,\infty}(0,1)$, we have $au' \in C[0,1]$ and $\sqrt{a}u' \in L^{2}(0,1)$. This implies that $\lim_{x \rightarrow x_0} (au')(x) = (au')(x_0) = L \in \mathbb{R}$. Proceeding as in the proof of [21, Proposition 3.6], one can prove that $L = 0$, that is, $(au')(x_0) = 0$. □

4 Carleman estimate for degenerate parabolic problems: the divergence case

In this section, we prove an interesting estimate of Carleman type for the adjoint problem of (1.2) in divergence form

$$
\begin{cases}
v_t + (av_x)_x = h, & (t, x) \in Q_T, \\
v_x(t, 0) = v_x(t, 1) = 0, & t \in (0, T), \\
v(T, x) = v_T(x) \in L^2(0, 1),
\end{cases}
$$

where $T > 0$ is given and $h \in L^2(Q_T)$. As is well known, for the proof of Carleman estimates, the final datum is irrelevant; only the equation and the boundary condi-
tions are important. For this reason, we can consider only the following problem:

\begin{align}
\begin{cases}
  v_t + (av_x)_x = h, & (t, x) \in Q_T, \\
  v_x(t, 0) = v_x(t, 1) = 0, & t \in (0, T).
\end{cases}
\end{align}

To deal with Carleman estimates for (4.1), we introduce the function \( \tilde{a} \), which is a continuation of \( a \) to the interval \(( -x_0, 2 - x_0 )\):

\begin{align}
(4.2) \quad \tilde{a}(x) := \begin{cases}
  a(2 - x), & x \in [1, 2], \\
  a(x), & x \in [0, 1], \\
  a(-x), & x \in [-1, 0].
\end{cases}
\end{align}

This is quite natural for problems with Neumann boundary conditions and, as we can see already by the next assumption, this function plays a very essential role in the rest of the paper.

On the function \( a \), we assume the following hypotheses.

**Hypothesis 4.1.** The function \( a \) is such that

1. the operator \( A_1 \) is weakly degenerate or strongly degenerate;
2. if \( A_1 \) is weakly degenerate, there exist \( B_1 \in (0, x_0), B_2 \in (1, 2 - x_0) \), functions \( g \in L^\infty_\text{loc}((-x_0, 2 - x_0) \setminus \{x_0\}), h(\cdot, B_i) \in W^{1, \infty}_\text{loc}((-x_0, 2 - x_0) \setminus \{x_0\}) \) and strictly positive constants \( g_0, h_0 \) such that

\begin{align}
(4.3) & \frac{\tilde{a}'(x)}{2\sqrt{\tilde{a}(x)}} \left( \int_x^{B_i} g(t) dt + h_0 \right) + \sqrt{\tilde{a}(x)} g(x) = h(x, B_i)
\end{align}

for a.e. \( x \in (-x_0, 2 - x_0) \) with \( i = 1, 2, -x_0 < x < B_1 \text{ or } x_0 < x < B_2 \);
3. if \( A_1 \) is strongly degenerate and \( K > 4/3 \), there exists a constant \( \vartheta \in (0, K] \) such that

\begin{align}
(4.4) & x \mapsto \frac{a(x)}{|x - x_0|^{\vartheta}} \begin{cases}
  \text{is nonincreasing on the left of } x = x_0, \\
  \text{is nondecreasing on the right of } x = x_0.
\end{cases}
\end{align}

In addition, when \( K > 3/2 \), this map is bounded below away from 0 and there exists a constant \( \kappa > 0 \) such that \( |a'(x)| \leq \kappa |x - x_0|^{2\vartheta - 3} \) for a.e. \( x \in [0, 1] \). Here \( K \) is the constant that appears in Definition 1.2.

**Remark 1.** The additional requirements when \( K > 3/2 \) are technical and are introduced in [20, Hypothesis 4.1] to guarantee the convergence of some integrals for this sub-case; see [20, Appendix]. Of course, the prototype \( a(x) = |x - x_0|^\alpha \) satisfies such a condition with \( \vartheta = \alpha \).
Remark 2. Since we require identities (4.3) to be satisfied far from $x_0$, once $a$ is given, it is easy to find $g$, $h$, $g_0$, and $h_0$ with the desired properties. For example, if $a(x) := |x - x_0|^\alpha$, $\alpha \in (0, 1)$, in (4.3) we can take

$$g_0 = \min \left\{ 1, \frac{a(B_1 + 1) + x_0}{x_0}, \frac{aB_2 + 1 - x_0}{1 - x_0} \right\}, \quad h_0 = 1,$$

while

$$g(x) = 1 \text{ in } [0, x_0] \cup [1, 2 - x_0),$$

and

$$g(x) = \frac{a(B_1 + 1) + x_0}{x_0} =: L_1 \text{ in } (-x_0, 0) \quad \text{and} \quad g(x) = \frac{aB_2 + 1 - x_0}{1 - x_0} =: L_2 \text{ in } (x_0, 1).$$

On the other hand,

$$h(x, B_1) = \begin{cases} (x - x_0)^{2-1} \left[ \frac{x}{2} (B_1 + 1 - x) + (x_0 - x) \right], & x \in [0, B_1), \\ (x + x_0)^{2-1} \left[ -\frac{x}{2} (-L_1 x + B_1 + 1) + L_1 (x + x_0) \right], & x \in (-x_0, 0), \end{cases}$$

and

$$h(x, B_2) = \begin{cases} (x - x_0)^{2-1} \left[ -\frac{x}{2} (L_2 (1 - x) + B_2) + L_2 (x - x_0) \right], & x \in (x_0, 1), \\ (2 - x - x_0)^{2-1} \left[ \frac{x}{2} (B_2 + 1 - x) + (2 - x - x_0) \right], & x \in [1, B_2), \end{cases}$$

Clearly, $g \in L_{loc}^\infty((-x_0, 2 - x_0) \setminus \{ x_0 \})$ and $h \in W_{loc}^{1,\infty}((-x_0, 2 - x_0) \setminus \{ x_0 \}).$

By $W_{loc}^{1,\infty}((-x_0, 2 - x_0) \setminus \{ x_0 \})$, we mean the space of functions belonging to $W_1^{1,\infty}((-x_0, 2 - x_0))$ far away from $\{ x_0 \}$.

As in [19] or in [20, Chapter 4], let us introduce the functions $\varphi(t, x) := \Theta(t) \psi(x)$, where

$$\Theta(t) := \frac{1}{[t(T - t)]^4} \quad \text{and} \quad \psi(x) := c_1 \left[ \int_{x_0}^x \frac{y - x_0}{a(y)} dy - c_2 \right],$$

with $c_2 > \max \left\{ \left( \frac{(1 - x_0)^2}{a(1)(2 - K)} \right), \frac{x_0^2}{a(0)(2 - K)} \right\}$ and $c_1 > 0$.

Observe that $\Theta(t) \to +\infty$ as $t \to 0^+, T^-$; and by [19, Lemma 2.1], we have that $-c_1 c_2 \leq \psi(x) < 0$. Therefore, for $A < B$, define

$$\rho_{A,B}(x) := \begin{cases} -r \left[ \int_A^{\frac{x}{\sqrt{\alpha(t)}}} \int_t^B g(s) ds dt + \int_A^{\frac{x}{\sqrt{\alpha(t)}}} h_B dt \right] - c, \quad & \text{in the weakly degenerate case}, \\
\exp^{c_2(x)} - c & \text{in the strongly degenerate case}, \end{cases}$$

in the weakly degenerate case,
where
\[ \zeta(x) = 0 \int_x^B \frac{1}{\tilde{a}(t)} dt, \]
\[ \varrho = \|\tilde{a}\|_{L^\infty(A,B)}, \quad r > 0 \text{ and } c > 0 \text{ are such that } \max_{[A,B]} \rho_{A,B} < 0. \]

Our main result is thus the following theorem.

**Theorem 4.1.** Let \( \omega \subset (0, 1) \) be an open interval containing \( x_0, B_1 \) and \( 2 - B_2 \), or let \( \omega = \omega_1 \cup \omega_2 \), where \( \omega_i = (\lambda_i, \beta_i) \subset (0, 1), \ i = 1, 2, \beta_1 \leq B_1 \) and \( 2 - B_2 \leq \lambda_2 \). Then, under Hypothesis 4.1, there exist two positive constants \( C \) and \( s_0 \) (depending on \( \lambda \)) such that every solution \( \nu \) of (4.1) in \( \mathcal{V} := L^2(0, T; D(A_1)) \cap H^1(0, T; H^1_a(0, 1)) \) satisfies

\[
\int_0^T \int_0^1 \left( s\Theta(a_v)^2 + s^3\Theta^3\left(\frac{(x-x_0)^2}{a}\right)^2 \right) e^{2\nu} dx dt \leq C \int_0^T \int_0^1 \nu^2 dx dt
\]
\[ + C \left( \int_0^T \int_0^1 h^2 e^{2\nu} dx dt + \int_0^T \int_0^{B_1} h^2 e^{2\varphi\theta_1(t-x)} dx dt + \int_0^T \int_{2-B_2}^{1} h^2 e^{2\varphi\theta_2(t-x)} dx dt \right). \tag{4.7} \]

for all \( s \geq s_0 \), where \( \Phi_1(t, x) := \Theta(t)\rho_{-B_1, B_1}(x) \) and \( \Phi_2(t, x) := \Theta(t)\rho_{2-B_2, B_2}(x) \).

**Remark 3.** Observe that an inequality analogous to (4.7) in the nondegenerate case is proved in [22], where the authors show that

\[
\int_0^T \int_0^1 \left( s\Theta(a_v)^2 + s^3\Theta^3\nu^2 \right) e^{2\nu} dx dt \leq C \left( \int_0^T \int_0^1 h^2 e^{2\nu} dx dt + s^3 \int_0^T \int_\omega \Theta^3 \nu^2 e^{2\nu} dx dt \right), \tag{4.8} \]

for a different weight function \( \varphi \) and for a fixed subset \( \omega \) compactly contained in \( (0, 1) \). We emphasize that we do not have \( s^3\Theta^3 \) in the term \( \int_0^T \int_\omega \nu e^{2\nu} dx dt \), and we cannot estimate such an integral by

\[ s^3 \int_0^T \int_0^1 \Theta^3 \left(\frac{(x-x_0)^2}{a}\right) \nu^2 e^{2\nu} dx dt \]

due to the degeneracy term. So (4.7) is a good alternative for (4.8).

The following estimate is crucial for proving Theorem 4.1.

**Theorem 4.2** ([20, Theorem 4.1]). Under Hypothesis 4.1, there exist positive constants \( C \) and \( s_0 \) such that every solution

\[ \nu \in L^2(0, T; H^2_\theta(0, 1)) \cap H^1(0, T; H^1_\theta(0, 1)) \]

of

\[
\begin{cases}
\nu_t + (av_x)_x = h, & (t, x) \in (0, T) \times (0, 1), \\
\nu(t, 0) = \nu(t, 1) = 0, & t \in (0, T)
\end{cases}
\]
satisfies

\[ \int_{Q_T} \left( s^2 a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\phi} \, dx \, dt \leq C \left( \int_{Q_T} h^2 e^{2s\phi} \, dx \, dt + sc_1 \int_0^T \left[ a\Theta e^{2s\phi} (x-x_0)(v_x)^2 \, dt \right]_{x=0}^{x=1} \right), \]

for all \( s \geq s_0 \), where \( c_1 \) is the constant introduced in (4.5). Here

\[ \mathcal{H}_a^1(0, 1) := \{ u \text{ is absolutely continuous in } [0, 1], \]
\[ \sqrt{a}u' \in L^2(0, 1) \text{ and } u(0) = u(1) = 0 \}, \]

in the weakly degenerate case and

\[ \mathcal{H}_a^1(0, 1) := \{ u \in L^2(0, 1) : u \text{ locally absolutely continuous in } [0, x_0) \cup (x_0, 1], \]
\[ \sqrt{a}u' \in L^2(0, 1) \text{ and } u(0) = u(1) = 0 \}

in the strongly degenerate case. In all cases,

\[ \mathcal{H}_a^2(0, 1) := \{ u \in \mathcal{H}_a^1(0, 1) | au' \in H^1(0, 1) \}. \]

We emphasize the fact that in [20], Theorem 4.2 is proved in the weakly degenerate case under weaker assumptions.

**Proof of Theorem 4.1.** We use a technique based on cut-off functions. First, assume that \( x_0, B_1, 2 - B_2 \in \omega \). Then we can fix two subintervals \( \omega_1 = (\lambda_1, \beta_1) \subset (0, x_0) \), \( \omega_2 = (\lambda_2, \beta_2) \subset (x_0, 1) \), with \( \beta_1 = B_1 \) and \( \lambda_2 = 2 - B_2 \), and four points \( \bar{\lambda}_i, \beta_i \in (\lambda_i, \beta_i), i = 1, 2 \), with \( \bar{\lambda}_i < \beta_i \). Consider a smooth function \( \xi : [-1, 2] \to [0, 1] \) such that

\[ \xi(x) = \begin{cases} 0 & x \in [-1, -\bar{\beta}_1] \cup [\bar{\beta}_1, \bar{\lambda}_1] \cup [2 - \bar{\lambda}_2, 2], \\ 1 & x \in [-\bar{\lambda}_1, \bar{\lambda}_1] \cup [\bar{\lambda}_2, 2 - \bar{\lambda}_2], \end{cases} \]

where \( \bar{\lambda}_i = (\bar{\lambda}_i + \beta_i)/2, i = 1, 2 \). Now we consider

\[ W(t, x) := \begin{cases} v(t, -x), & x \in [-1, 0], \\ v(t, x), & x \in [0, 1], \\ v(t, 2 - x), & x \in [1, 2], \end{cases} \]

where \( v \) solves (4.1); \( W \) solves the problem

\[ \left\{ \begin{array}{ll} W_t + (\bar{a}W_x)_x = \bar{h}, & (t, x) \in (0, T) \times (-1, 2), \\ W_x(t, -1) = W_x(t, 2) = 0, & t \in (0, T), \end{array} \right. \]
where

\[
\tilde{h}(t, x) := \begin{cases} 
  h(t, 2 - x), & x \in [1, 2], \\
  h(t, x), & x \in [0, 1], \\
  h(t, -x), & x \in [-1, 0].
\end{cases}
\]  

(4.12)

Observe that \(\tilde{a}\) belongs to \(W^{1,1}(-1, 2)\) in the weakly degenerate case and belongs to \(W^{1,\infty}(-1, 2)\) in the strongly degenerate case. Now \(Z := \xi W\) solves the problems

\[
\begin{cases}
  Z_t + (\tilde{a}Z)_x = H, & (t, x) \in (0, T) \times (-B_1, B_1), \\
  Z(t, -B_1) = Z(t, B_1), & t \in (0, T),
\end{cases}
\]  

(4.13)

and

\[
\begin{cases}
  Z_t + (\tilde{a}Z)_x = H, & (t, x) \in (0, T) \times (2 - B_2, B_2), \\
  Z(t, 2 - B_2) = Z(t, B_2) = 0, & t \in (0, T),
\end{cases}
\]  

(4.14)

with \(H := \xi \tilde{h} + (\tilde{a} \xi)_x W_x + \tilde{a} \xi W_x\). Observe that

\[
Z_x(t, -B_1) = Z_x(t, B_1) = Z_x(t, 2 - B_2) = Z_x(t, B_2) = 0;
\]

and, by the assumption on \(a\) and the fact that \(\xi_x\) is supported in

\([-\beta_1, -\lambda_1] \cup [\lambda_1, \beta_1] \cup [\lambda_2, \bar{\lambda}_2] \cup [2 - \lambda_2, 2 - \bar{\lambda}_2]\),

we have \(H \in L^2((0, T) \times I)\), where \(I := (-B_1, B_1) \cup (2 - B_2, B_2)\).

Thus we can apply the analogue of [20, Theorem 3.1] on \((-B_1, B_1)\) in place of \((A, B)\) and with weight \(\Phi_1\), obtaining the existence of positive constants \(C\) and \(s_0\) (\(s_0\) sufficiently large), such that \(Z\) satisfies

\[
\int_0^T \int_{-B_1}^{B_1} \left( s \Theta(Z_x)^2 + s^3 \Theta^3 Z_x^2 \right) e^{2s\Phi_1} \, dx \, dt \leq C \int_0^T \int_{-B_1}^{B_1} H^2 e^{2s\Phi_1} \, dx \, dt
\]

for all \(s \geq s_0\). Now, as in [20], we can prove that exists a positive constant \(k\) such that

\[
\tilde{a}(x)e^{2s\Phi_1(t, x)} \leq ke^{2s\Phi_1(t, x)}
\]

(4.15)

and

\[
\frac{(x - x_0)^2}{\tilde{a}(x)} e^{2s\Phi_1(t, x)} \leq ke^{2s\Phi_1(t, x)}
\]

(4.16)
for every \((t, x) \in [0, T] \times [-B_1, B_1]\).

Thus, by definitions of \(\zeta\), \(W\) and \(Z\), we have

\[
\int_0^T \int_{-B_1}^{B_1} \left( s\Theta a(v_0)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{\alpha^2} \right) e^{2s\Phi} \, dxdt \\
\leq \int_0^T \int_{-B_1}^{B_1} \left( s\Theta \tilde{a}(Z_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{\tilde{a}} Z^2 \right) e^{2s\Phi} \, dxdt \\
\leq C \int_0^T \int_{-B_1}^{B_1} (s\Theta(Z_x)^2 + s^3 \Theta^3 Z^2) e^{2s\Phi} \, dxdt \\
\leq C \int_0^T \int_{-B_1}^{B_1} H^2 e^{2s\Phi} \, dxdt.
\]

Using again the fact that \(\tilde{\zeta}_x\) is supported in

\([-\tilde{\beta}_1, -\tilde{\lambda}_1] \cup [\tilde{\lambda}_1, \tilde{\beta}_1] \cup [\tilde{\lambda}_2, \tilde{\beta}_2] \cup [2 - \tilde{\lambda}_2, 2 - \tilde{\lambda}_2]\)

and the boundedness of \(\tilde{a}'\) (far away from \(x_0\) in the weakly degenerate case, see (4.3), and since \(a \in W^{1,\infty}(-1, 2)\) in the strongly degenerate case), we have, by the Caccioppoli inequality for the nondegenerate case (see, e.g., [18, Remark 7])

\[
\int_0^T \int_{-B_1}^{B_1} H^2 e^{2s\Phi} \, dxdt = \int_0^T \int_{-B_1}^{B_1} (\tilde{\zeta} \tilde{h} + (\tilde{a}_x W_x + \tilde{a}_x W) e^{2s\Phi} \, dxdt \\
\leq C \left( \int_0^T \int_{-B_1}^{B_1} \tilde{h}^2 e^{2s\Phi} \, dxdt + \int_0^T \left( \int_{\tilde{\beta}_1}^{\tilde{\lambda}_1} \right) (W^2 + W_x^2) e^{2s\Phi} \, dxdt \right) \\
\leq C \left( \int_0^T \int_{-B_1}^{B_1} \tilde{h}^2 e^{2s\Phi} \, dxdt + \int_0^T \left( \int_{\tilde{\lambda}_1}^{B_1} \right) W^2 \, dxdt \right) \\
\leq C \int_0^T \int_{-B_1}^{B_1} \tilde{h}^2 e^{2s\Phi} \, dxdt + C \int_0^T \int_{\omega} v^2 \, dxdt.
\]

Now observe that

\[(4.17) \quad \int_0^T \int_{-B_1}^{B_1} \tilde{h}^2(t, x)e^{2s\Phi}(t, x) \, dxdt \leq C \int_0^T \int_{-B_1}^{B_1} \tilde{h}^2(t, x)e^{2s\Phi}(t, -x) \, dxdt.\]

Indeed, using a change of variable and the definition of \(\tilde{h}\), we obtain

\[
\int_{-B_1}^{B_1} \tilde{h}^2(t, x)e^{2s\Phi}(t, x) \, dx = \int_0^B \tilde{h}^2(t, x)e^{2s\Phi}(t, x) \, dx + \int_0^B \tilde{h}^2(t, x)e^{2s\Phi}(t, x) \, dx \\
= \int_0^B \tilde{h}^2(t, y)e^{2s\Phi}(t, -y) \, dy + \int_0^B \tilde{h}^2(t, y)e^{2s\Phi}(t, x) \, dx \\
\leq \int_0^B \tilde{h}^2(t, y)e^{2s\Phi}(t, -y) \, dy + \int_0^B \tilde{h}^2(t, y)e^{2s\Phi}(t, x) \, dx,
\]
since \( \rho_{-B_1,B_1}(x) \leq \rho_{-B_1,B_1}(-x) \), for all \( x \in [0, B_1] \). Hence, using the definitions of \( \tilde{a}, \tilde{h} \) and \( W \), we have

\[
\int_0^T \int_0^{\tilde{x}_1} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2x\psi} \, dx dt \\
\leq C \left( \int_0^T \int_0^{B_1} h^2 e^{2x\Phi_1(t,-x)} \, dx dt + \int_0^T \int_\omega v^2 \, dx dt \right)
\]

for all \( s \geq s_0 \). Analogously, we can choose \( s_0 \) so large that for all \( s \geq s_0 \) and for a positive constant \( C \),

\[
\int_0^T \int_{\tilde{x}_2} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2x\psi} \, dx dt \\
\leq C \left( \int_0^T \int_{2-B_2} h^2 e^{2x\Phi_2(t,x)} \, dx dt + \int_0^T \int_\omega v^2 \, dx dt \right),
\]

since \( \rho_{2-B_2,B_2}(2-x) \leq \rho_{2-B_2,B_2}(x) \) for all \( x \in [2 - B_2, 1] \). Finally, consider a smooth function \( \eta : [0, 1] \to [0, 1] \) satisfying

\[
\eta(x) := \begin{cases} 
0, & x \in [0, \bar{x}_1] \cup [\bar{x}_2, 1], \\
1, & x \in [\bar{x}_1, \bar{x}_2].
\end{cases}
\]

Then \( \omega := \eta \psi \) solves the problem

\[
\begin{cases} 
\omega_t + (a \omega_x)_x = \tilde{h}, & (t, x) \in (0, T) \times (0, 1), \\
\omega(t, 0) = \omega(t, 1) = 0, & t \in (0, T),
\end{cases}
\]

with \( \tilde{h} := \eta h + (a \eta_x v)_x + a \eta_x v_x \). Hence, by Theorem 4.2 and [20, Proposition 5.2],

\[
\int_0^T \int_{\tilde{x}_1} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2x\psi} \, dx dt \\
\leq C \int_0^T \int_0^{\tilde{x}_1} \left( s \Theta a(w_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} w^2 \right) e^{2x\psi} \, dx dt \\
\leq C \int_0^T \int_0^{\bar{x}_1} (\eta h + (a \eta_x v)_x + a \eta_x v_x)^2 e^{2x\psi} \, dx dt \\
\leq C \left( \int_0^T \int_0^{\bar{x}_1} h^2 e^{2x\psi} \, dx dt + \int_0^T \left( \int_{\tilde{x}_1}^{\bar{x}_1} + \int_{\bar{x}_2}^{\bar{x}_2} \right) (v^2 + v_x^2) e^{2x\psi} \, dx dt \right) \\
\leq C \left( \int_0^T \int_0^{\bar{x}_1} h^2 e^{2x\psi} \, dx dt + \int_0^T \int_\omega v^2 \, dx dt \right).
\]
Hence we can choose $s_0$ so large that for all $s \geq s_0$ and for a positive constant $C$,

\[
\int_0^T \int_0^1 \left( s \Theta a(v_x)^2 + s^3 \Theta_3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\phi} \, dx \, dt \leq C \int_0^T \int_\omega v^2 \, dx \, dt \\
+C \left( \int_0^T \int_0^1 h^2 e^{2s\phi} \, dx \, dt + \int_0^T \int_{B_1} h^2 e^{2s\Phi_1(t,x)} \, dx \, dt + \int_0^T \int_{2-B_2} h^2 e^{2s\Phi_2(t,x)} \, dx \, dt \right).
\]

Nothing changes in the proof if $\omega = \omega_1 \cup \omega_2$ and each of these intervals lies on a different side of $x_0$, as the assumption implies.

We emphasize that in the weakly degenerate case, the boundedness of $a'$ far away from $x_0$ is crucial in the previous proof. Indeed, thanks to it, we are able to estimate the integral $\int_0^T \left( \int_{-\beta_1}^{\beta_1} + \int_{\lambda_1}^{\lambda_1} \right) [(\tilde{a} \xi W_x)]^2 e^{2s\Phi_1} \, dx \, dt$.

5 Carleman estimate for degenerate parabolic problems: the nondivergence case

In this section, we prove the analogue of the Carleman estimate given in Theorem 4.1 for the adjoint problem of (1.2) in the nondivergence case when the degeneracy is either weak or strong:

\[
\begin{cases}
  v_t + av_{xx} = h, & (t, x) \in Q_T, \\
  v_x(t, 0) = v_x(t, 1) = 0, & t \in (0, T).
\end{cases}
\]

Here $h \in L^2_{1/a}(Q_T)$, while on $a$ we make the following assumptions.

**Hypothesis 5.1.** The function $a$ is such that

1. the operator $A_2$ is weakly or strongly degenerate;
2. $(x-x_0)a'(x)/a(x) \in W^{1,\infty}(0, 1)$;
3. in the weakly degenerate case, there exist $B_1 \in (0, x_0), B_2 \in (1, 2-x_0)$, functions $g \in L_1^\infty((-x_0, 2-x_0) \setminus \{x_0\}), h(\cdot, B_i) \in W^{1,\infty}_1((-x_0, 2-x_0) \setminus \{x_0\})$, and strictly positive constants $g_0$ and $h_0$ such that $g(x) \geq g_0$ and

\[
\frac{\tilde{a}'(x)}{2\sqrt{\tilde{a}(x)}} \left( \int_x^{B_i} g(t) \, dt + h_0 \right) + \sqrt{\tilde{a}(x)} g(x) = h(x, B_i)
\]

for a.e. $x \in (-x_0, 2-x_0)$ with $i = 1, 2, -x_0 < x < B_1$ or $x_0 < x < B_2$, and $\tilde{a}$ as in (4.2);
4. if $K \geq 1/2$, then (4.4) holds.

**Remark 4.** We emphasize the fact that in the nondivergence case, the assumptions on $a$ are weaker than in the divergence case: the additional condition
Hypothesis 4.1.3 when $K > 3/2$ is not necessary, since all integrals and integrations by parts are justified by definition of $D(A_2)$.

Moreover, Hypothesis 4.1.3 is replaced with Hypothesis 5.1.4, which is essential for proving [20, Theorem 4.2]; see [20, Lemma 4.3] and [9, Lemma 3.10] or [8, Lemma 5] for the case when the degeneracy occurs at the boundary of the domain.

**Remark 5.** As in Remark 2, we can take $g_0 = \min \left\{ 1, \frac{-\alpha (B_1 + 1) + x_0}{x_0}, \frac{-\alpha B_2 + 1 - x_0}{1 - x_0} \right\}$, $h_0 = 1$.

$g(x) = 1$ in $[0, x_0] \cup [1, 2 - x_0)$, while

$g(x) = \frac{-\alpha (B_1 + 1) + x_0}{x_0} =: L_1$ in $(-x_0, 0)$ and

$g(x) = \frac{-\alpha B_2 + 1 - x_0}{1 - x_0} =: L_2$ in $(x_0, 1)$.

On the other hand

$h(x, B_1) = \begin{cases} (x_0 - x)^{\frac{2}{\alpha} - 1} \left[ \frac{-\alpha}{2} (B_1 + 1 - x) + (x_0 - x) \right], & x \in [0, B_1), \\ (x + x_0)^{\frac{2}{\alpha} - 1} \left[ \frac{\alpha}{2} (\frac{L_1 x + B_1 + 1}{L_1} + (x + x_0)) \right], & x \in (-x_0, 0) \end{cases}$

and

$h(x, B_2) = \begin{cases} (x - x_0)^{\frac{2}{\alpha} - 1} \left[ \frac{\alpha}{2} (L_2 (1 - x) + B_2) + L_2 (x - x_0) \right], & x \in (x_0, 1) \\ (2 - x - x_0)^{\frac{2}{\alpha} - 1} \left[ \frac{\alpha}{2} (B_2 + 1 - x) + (2 - x - x_0) \right], & x \in [1, B_2), \end{cases}$

Again $g \in L^\infty_{\text{loc}} ((-x_0, 2 - x_0) \setminus \{x_0\})$ and $h \in W^{1, \infty}_{\text{loc}} ((-x_0, 2 - x_0) \setminus \{x_0\}; L^\infty(0, 1))$.

To prove an estimate of Carleman type, we proceed as before. To this aim, as in [20, Chapter 4], let us introduce the function

(5.3) $\gamma(t, x) := \Theta(t) \mu(x),$ 

where $\Theta$ is as in (4.5).

(5.4) $\mu(x) := d_1 \left( \int_{x_0}^{x} \frac{y - x_0}{a(y)} e^{R(y-x_0)^2} dy - d_2 \right),$ 

$d_2 > \max \left\{ \frac{\alpha (1 - x_0)^2 e^{R(1-x_0)^2}}{(2 - K)\alpha(1)}, \frac{x_0^2 e^{R x_0^2}}{(2 - K)\alpha(0)} \right\},$

and $R$ and $d_1$ are strictly positive constants. The main result of this section is the following theorem.
\textbf{Theorem 5.1.} Let \( \omega \subset (0, 1) \) be an open interval containing \( x_0, B_1 \) and \( 2 - B_2 \), or let \( \omega = \omega_1 \cup \omega_2 \), where \( \omega_i = (\lambda_i, \beta_i) \subset (0, 1), i = 1, 2, \beta_1 \leq B_1, \) and \( 2 - B_2 \leq \lambda_2 \). Then, under Hypothesis 5.1, there exist positive constants \( C \) and \( s_0 \) such that every solution \( v \) of (5.1) in \( L^2(0, T; H^2_{1/a}(0, 1)) \cap H^1(0, T; H^1_{1/a}(0, 1)) \) satisfies
\[
\int_0^T \int_0^1 \left( s \Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{x - x_0}{a} \right)^2 v^2 \right) e^{2s\psi} \, dx \, dt \leq C \int_0^T \int_\omega v^2 \, dx \, dt + C \left( \int_0^T \int_0^1 \frac{h^2}{a} e^{2s\psi(t,x)} v^2 \, dx \, dt + \int_0^T \int_{B_1} h^2 e^{2s\psi(t,x)} v^2 \, dx \, dt \right)
\]
for all \( s \geq s_0 \).

To prove Theorem 5.1, we use the Carleman estimate given for the analogous problem of (5.1) with Dirichlet boundary conditions.

\textbf{Theorem 5.2} ([20, Theorem 4.2]). Under Hypothesis 5.1, there exist positive constants \( C \) and \( s_0 \) such that every solution \( v \) of
\[
\begin{cases}
vt + av_{xx} = h & (t, x) \in Q_T, \\
v(t, 0) = v(t, 1) = 0 & t \in (0, T),
\end{cases}
\]
satisfies
\[
\int_{Q_T} \left( s \Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{x - x_0}{a} \right)^2 v^2 \right) e^{2s\psi} \, dx \, dt \leq C \left( \int_{Q_T} \frac{h^2}{a} e^{2s\psi} \, dx \, dt + s d_1 \left[ \int_0^T \Theta e^{2s\psi(x - x_0)^2 dt} \right]_{x=1} \right)
\]
for all \( s \geq s_0 \), where \( d_1 \) is the constant introduced in (5.4),
\[
\mathcal{H}^1_{1/a}(0, 1) := L^2_{1/a}(0, 1) \cap H^1_0(0, 1),
\]
and
\[
\mathcal{H}^2_{1/a}(0, 1) := \left\{ u \in \mathcal{H}^1_{1/a}(0, 1) : u' \in H^1(0, 1) \right\}.
\]
The proof is similar to that of Theorem 4.1, so we only sketch it.

**Sketch of proof of Theorem 5.1.** Consider \( \tilde{\lambda}_i, \tilde{\beta}_i, \tilde{\beta}_i (i = 1, 2), \xi, \eta, W, \) and \( Z \) as in the proof of Theorem 4.1. Obviously, \( W \) and \( Z \) solve, respectively, the problems

\[
\begin{aligned}
W_t + aW_{xx} &= \tilde{h}, \quad (t, x) \in (0, T) \times (-1, 2), \\
W_x(t, -1) &= W_x(t, 2) = 0, \quad t \in (0, T),
\end{aligned}
\]

\[
\begin{aligned}
Z_t + aZ_{xx} &= H, \quad (t, x) \in (0, T) \times (-B_1, B_1), \\
Z(t, -B_1) &= Z(t, B_1), \quad t \in (0, T),
\end{aligned}
\]

and

\[
\begin{aligned}
Z_t + aZ_{xx} &= H, \quad (t, x) \in (0, T) \times (2 - B_2, B_2), \\
Z(t, 2 - B_2) &= Z(t, B_2) = 0, \quad t \in (0, T),
\end{aligned}
\]

with \( H := \tilde{\zeta} \tilde{h} + a(\tilde{\zeta}W + 2\tilde{\zeta}_xW_x), \) and \( \tilde{a} \) and \( \tilde{h} \) defined as before. Observe that \( Z_\alpha(t, -B_1) = Z_\alpha(t, B_1) = Z_\alpha(t, 2 - B_2) = Z_\alpha(t, B_2) = 0 \) and, by the assumption on \( a, H \in L^2((0, T); L^2(I)) \), where \( I \) is as before. Thus we can apply the analogue of [20, Theorem 3.2] on \((-B_1, B_1)\) in place of \((0, 1)\) and with weight \( \Phi_1 \), obtaining the existence of positive constants \( C \) and \( s_0 \) (\( s_0 \) sufficiently large), such that

\[
\int_0^T \int_{-B_1}^{B_1} \left(s\Theta(Z_x)^2 + s^3\Theta^3Z^2\right)e^{2s\Phi_1} \, dx \, dt \leq C \int_0^T \int_{-B_1}^{B_1} H^2e^{2s\Phi_1} \, dx \, dt
\]

for all \( s \geq s_0 \). By the definitions of \( \tilde{\xi}, W, Z \) and by (4.17), proceeding as before, we have

\[
\int_0^T \int_0^{\tilde{x}_1} \left(s\Theta(d_x)^2 + s^3\Theta^3\left(\frac{x - x_1}{\tilde{a}}\right)^2\right)e^{2s\gamma_1} \, dx \, dt
\]

\[
\leq \int_0^T \int_{-B_1}^{B_1} \left(s\Theta(Z_x)^2 + s^3\Theta^3\left(\frac{x - x_1}{\tilde{a}}\right)^2\right)Z^2 \, dx \, dt
\]

\[
\leq C \int_0^T \int_{-B_1}^{B_1} s\Theta(Z_x)^2 + s^3\Theta^3Z^2 \, dx \, dt
\]

\[
\leq C \int_0^T \int_{-B_1}^{B_1} H^2e^{2s\Phi_1} \, dx \, dt
\]

\[
\leq C \left(\int_0^T \int_{-B_1}^{B_1} \tilde{h}^2e^{2s\Phi_1} \, dx \, dt + \int_0^T \int_{\tilde{x}_1}^{\tilde{x}_1} W^2 \, dx \, dt\right)
\]

\[
\leq C \left(\int_0^T \int_{-B_1}^{B_1} \frac{\tilde{h}^2e^{2s\Phi_1}}{\tilde{x}} \, dx \, dt + \int_0^T \int_{\tilde{x}_1}^{\tilde{x}_1} \frac{v^2}{\tilde{a}} \, dx \, dt\right),
\]
for all \( s \geq s_0 \). Analogously, we can choose \( s_0 \) so large that for all \( s \geq s_0 \) and for a positive constant \( C \),

\[
\int_0^T \int_{\lambda_2}^{1} \left( s \Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{X - X_0}{a} \right)^2 \right) e^{2s\gamma} \, dx \, dt \\
\leq C \left( \int_0^T \int_{2-B_2}^{2} h^2 e^{2s\Phi_2(t,x)} \, dx \, dt + \int_0^T \int_{2-B_2}^{\frac{\partial}{a}} \frac{v^2}{a} \, dx \, dt \right).
\]

Finally, \( w := \eta v \) satisfies

\[
\begin{cases}
  w_t + a w_{xx} = \bar{h}, & (t, x) \in (0, T) \times (0, 1), \\
  w(t, 0) = w(t, 1) = 0, & t \in (0, T),
\end{cases}
\]

with \( \bar{h} := \eta h + a(\eta \xi v + 2\eta v_x) \). Hence, by Theorem 5.2 and [20, Proposition 5.4], we have

\[
\int_0^T \int_{\lambda_1}^{\lambda_2} \left( s \Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{X - X_0}{a} \right)^2 \right) e^{2s\gamma} \, dx \, dt \\
\leq C \left( \int_0^T \int_{\lambda_1}^{\lambda_2} \frac{h^2}{a} e^{2s\gamma} \, dx \, dt + \int_0^T \left( \int_{\lambda_1}^{\lambda_2} + \int_{\lambda_2}^{\bar{\lambda}_2} \right) \frac{v^2}{a} \, dx \, dt \right) \\
\leq C \left( \int_0^T \int_{\lambda_1}^{\lambda_2} \frac{h^2}{a} e^{2s\gamma} \, dx \, dt + \int_0^T \left( \int_{\lambda_1}^{\lambda_2} + \int_{\lambda_2}^{\bar{\lambda}_2} \right) \frac{v^2}{a} \, dx \, dt \right) \\
\leq C \left( \int_0^T \int_{\lambda_1}^{\lambda_2} \frac{h^2}{a} e^{2s\gamma} \, dx \, dt + \int_0^T \left( \int_{\lambda_1}^{\lambda_2} + \int_{\lambda_2}^{\bar{\lambda}_2} \right) \frac{v^2}{a} \, dx \, dt \right).
\]

Hence, we can choose \( s_0 \) so large that, for all \( s \geq s_0 \) and for a positive constant \( C \),

\[
\int_0^T \int_{\lambda_1}^{\lambda_2} \left( s \Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{X - X_0}{a} \right)^2 \right) e^{2s\gamma} \, dx \, dt \\
\leq C \int_0^T \int_{\lambda_1}^{\lambda_2} \frac{v^2}{a} \, dx \, dt \\
+ C \left( \int_0^T \int_{\lambda_1}^{\lambda_2} \frac{h^2}{a} e^{2s\gamma(t,x)} \, dx \, dt + \int_0^T \int_{\lambda_1}^{\lambda_2} h^2 e^{2s\Phi_1(t,x)} \, dx \, dt + \int_0^T \int_{2-B_2}^{1} h^2 e^{2s\Phi_2(t,x)} \, dx \, dt \right). \square
\]

6 Observability inequalities as applications of Carleman estimates

In this section, we consider problem (1.4). We make the following assumption.

Hypothesis 6.1. The control set \( \omega \) is an interval which contains the degeneracy point or the union of two intervals, each of them lying on one side of the degeneracy point, i.e.,
1. either \( \omega = (\alpha, \beta) \subset (0, 1) \) is such that \( x_0 \in \omega \), or
2. \( \omega = \omega_1 \cup \omega_2 \), where \( \omega_i = (\lambda_i, \beta_i) \subset (0, 1) \), \( i = 1, 2 \), and \( \beta_1 < x_0 < \lambda_2 \).

**Remark 6.** Observe that under Hypothesis 6.1.1, we can find subintervals \( \omega_1 = (\lambda_1, \beta_1) \subset (\alpha, x_0) \) and \( \omega_2 = (\lambda_2, \beta_2) \subset (x_0, \beta) \) such that \( (\omega_1 \cup \omega_2) \subset \omega \setminus \{x_0\} \).

We also make the following assumption.

**Hypothesis 6.2.** Hypothesis 4.1 holds in the divergence case and Hypothesis 5.1 holds in the nondivergence case, both with \( B_1 = \beta_1 \) and \( B_2 = 2 - \lambda_2 \).

Now, we associate to (1.4) the homogeneous adjoint problem

\[
\begin{aligned}
&v_t + Av = 0, \\
&v_x(t, 0) = v_x(t, 1) = 0, \\
&v(T, x) = v_T(x) \in X,
\end{aligned}
\]

where \( T > 0 \) is given. Recall that \( X = L^2(0, 1) \) in the divergence case and \( X = L^{1/a}(0, 1) \) in the nondivergence case. From the Carleman estimates given in Theorems 4.1 and 5.1, we deduce the following observability inequalities for both the weakly and the strongly degenerate cases.

**Proposition 6.1.** Under Hypothesis 6.2, there exists a positive constant \( C_T \) such that every solution \( v \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^{1/a}_a(0, 1)) \) of (6.1) with \( Au = A_1u \) satisfies

\[
\int_0^1 v^2(0, x) \, dx \leq C_T \int_0^T \int_{\omega} v^2(t, x) \, dx \, dt.
\]

**Proposition 6.2.** Under Hypotheses 6.2, there exists a positive constant \( C_T \) such that every solution \( v \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^{1/a}_a(0, 1)) \) of (6.1) with \( Au = A_2u \) satisfies

\[
\int_0^1 \frac{v^2(0, x)}{a} \, dx \leq C_T \int_0^T \int_{\omega} \frac{v^2(t, x)}{a} \, dx \, dt.
\]

**6.1 Sketch of Proof of Proposition 6.1.** In this subsection, we outline the proof of the observability inequality (6.2) as a consequence of the Carleman estimate given in Section 4. We only sketch the proof, because it is similar to that given in [19] or in [20, Proposition 5.1].
To begin, consider the adjoint problem with more regular final-time datum

\[
\begin{cases}
v_t + A_1 v = 0, & (t, x) \in Q_T, \\
v_x(t, 0) = v_x(t, 1) = 0, & t \in (0, T), \\
v(T, x) = v_T(x) \in D(A_1^2),
\end{cases}
\]

(6.4)

where \( D(A_1^2) = \{ u \in D(A_1) : A_1 u \in D(A_1) \} \). Observe that \( D(A_1^2) \) is densely defined in \( D(A_1) \) (see, e.g., [6, Lemma 7.2]) and hence in \( L^2(0, 1) \). As in [9], [8], [17], [19] or [20], letting \( v_T \) vary in \( D(A_1^2) \), we define the class of functions \( \mathcal{W}_1 := \{ v \) is a solution of (6.4)\}. Obviously (see, e.g., [6, Theorem 7.5]),

\[ \mathcal{W}_1 \subset C^1([0, T]; H^2_a(0, 1)) \subset \mathcal{V} \subset \mathcal{U}_1, \]

where \( \mathcal{U}_1 := C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1_a(0, 1)) \). We need the following lemma, which deals with the cases \( x_0 \) is inside the control region \( \omega \) and \( x_0 \) is outside \( \omega \). The statements of the conclusions are the same; however, the proofs, though inspired by the same ideas, are different. For this reason, we divide the proof into two parts.

**Lemma 6.1.** Under Hypotheses 6.1 and 6.2, there exist positive constants \( C \) and \( s_0 \) such that every solution \( v \in \mathcal{W}_1 \) of (6.4) satisfies

\[
\int_0^T \int_0^1 \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx \, dt \leq C \int_0^T \int_0^1 v^2 \, dx \, dt
\]

for all \( s \geq s_0 \), where \( \Theta \) and \( \varphi \) are as in Section 4.

**Proof.** The proof of Lemma 6.1 is divided into two parts to distinguish the cases when \( \omega \) is an interval which contains the degeneracy point or the union of two intervals, each lying on one side of the degeneracy point.

First case: Hypothesis 6.1.1 holds. As in the proof of Theorem 4.1, fix \( \xi, \eta \), \( \beta_i \) (\( i = 1, 2 \)), and smooth functions \( \xi, \eta \) as in (4.9) and (4.19). Define \( w := \eta v \), where \( v \) solves (6.4). Hence, \( w \) solves

\[
\begin{cases}
w_t + (aw_x)_x = (a\eta_x)v_x + \eta_x aw_x =: f, & (t, x) \in (0, T) \times (0, 1), \\
w(t, 0) = w(t, 1) = 0, & t \in (0, T).
\end{cases}
\]

(6.5)

Applying Theorem 4.1, we have positive constants \( C \) and \( s_0 \) such that

\[
\int_0^T \int_0^1 \left( s \Theta a(w_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx \, dt \leq C \left( \int_0^T \int_0^1 w^2 \, dx \, dt \right)
\]

\[
+ C \left( \int_0^T \int_0^1 f^2 e^{2s\varphi} + \int_0^T \int_0^{\beta_1} f^2 e^{2s\varphi} \, dxdt + \int_0^T \int_0^{\beta_2} f^2 e^{2s\varphi} \, dxdt \right)
\]

(6.6)
for all $s \geq s_0$. Then, using the definition of $\eta$, in particular, the fact that in $[0, 1]$ the functions $\eta_x$ and $\eta_{xx}$ are supported in $\tilde{\omega} := [\tilde{\lambda}_1, \tilde{\lambda}_1] \cup [\tilde{\lambda}_2, \tilde{\beta}_2]$, we can write $f^2 = ((a\xi_x v) + \xi_{xx}v_x)^2 \leq C(v^2 + (v_x)^2)\chi_{\tilde{\omega}}$, since the function $a'$ is bounded on $\tilde{\omega}$. Hence, applying the Caccioppoli inequality [19, Proposition 4.2] and (6.6), we obtain

$$
\int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} \left( s\Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\phi} \, dx \, dt \\
\leq \int_0^T \int_{\omega}^1 \left( s\Theta a(w_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) e^{2s\phi} \, dx \, dt \\
\leq C \int_0^T \int_{\omega}^1 v^2 \, dx \, dt + C \int_0^T \int_{\omega}^1 e^{2s\phi} (v^2 + (v_x)^2) \, dx \, dt \\
+ C \int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} e^{2s\Phi_1(t,x)} (v^2 + (v_x)^2) \, dx \, dt \\
+ C \int_0^T \int_{\tilde{\lambda}_2}^{\tilde{\beta}_2} e^{2s\Phi_2(t,x)} (v^2 + (v_x)^2) \, dx \, dt \leq C \int_0^T \int_{\omega}^1 v^2 \, dx \, dt
$$

(6.7)

for some positive constant $C$. Now, define $Z := \xi W$, where $W$ is defined in (4.10). Then $Z$ is the solution of (4.13) and (4.14) with $H := (\tilde{\alpha} \xi_x W)_x + \tilde{\alpha} \eta_x W_x$. Proceeding as in proof of Theorem 4.1 and using the fact that in $[0, 2]$ the functions $\xi_x, \tilde{\xi}_{xx}$ are supported in $[\tilde{\lambda}_1, \tilde{\beta}_1] \cup [\tilde{\lambda}_2, \tilde{\beta}_2] \cup [2 - \tilde{\lambda}_2, 2 - \tilde{\lambda}_2]$, we get

$$
\int_0^T \int_{\tilde{\lambda}_2}^{\tilde{\beta}_2} \left( s\Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\phi} \, dx \, dt \\
\leq \int_0^T \int_{\tilde{\lambda}_2}^{\tilde{\beta}_2} \left( s\Theta \tilde{a}(Z_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 \right) e^{2s\phi} \, dx \, dt \\
\leq \int_0^T \int_{\tilde{\lambda}_2}^{\tilde{\beta}_2} \left( s\Theta (Z_x)^2 + s^3 \Theta^3 Z^2 \right) e^{2s\Phi_2} \, dx \, dt \\
\leq C \int_0^T \int_{\omega}^1 v^2 \, dx \, dt.
$$

(6.8)

Thus (6.7) and (6.8) imply

$$
\int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} \left( s\Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\phi} \, dx \, dt \leq C \int_0^T \int_{\omega}^1 v^2 \, dx \, dt,
$$

(6.9)

for some positive constant $C$. To complete the proof, it suffices to prove a similar inequality on the interval $[0, \tilde{\lambda}_1]$. To this aim, we follow a reflection procedure, as before, considering the problem (4.13). Hence, using the Caccioppoli inequality
\[ \int_0^T \int_{-\beta_1}^{\beta_1} \left( s \Theta \tilde{a}(z_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{\tilde{a}} \right) e^{2s\varphi} \, dx \, dt \]
\[ \leq k \int_0^T \int_{-\beta_1}^{\beta_1} (s \Theta(z_x)^2 e^{2s\Phi_1} + s^3 \Theta^3 Z^2 e^{2s\Phi_1}) \, dx \, dt \]
\[ \leq C \int_0^T \int_{-\beta_1}^{\beta_1} e^{2s\Phi_1} H^2 \, dx \, dt \]
\[ \leq C \int_0^T \int_{-\tilde{\lambda}_1}^{\tilde{\lambda}_1} e^{2s\Phi_1} (W^2 + (W_x)^2) \, dx \, dt \]
\[ + C \int_0^T \int_{\tilde{\lambda}_1}^{\beta_1} e^{2s\Phi_1} (W^2 + (W_x)^2) \, dx \, dt \]
\[ \leq C \int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1} W^2 \, dx \, dt \]
\[ \leq C \int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1} v^2 \, dx \, dt \leq C \int_0^T \int_0^T v^2 \, dx \, dt, \]
\[ \text{for all } s \geq s_0. \]

Hence, by (6.10) and the definitions of \( W \) and \( Z \), we get
\[ \int_0^T \int_0^{\tilde{\beta}_1} \left( s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 + s \Theta a(v_x)^2 \right) e^{2s\varphi} \, dx \, dt \]
\[ \leq \int_0^T \int_{-\tilde{\beta}_1}^{\beta_1} \left( s^3 \Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 + s \Theta \tilde{a}(z_x)^2 \right) e^{2s\varphi} \, dx \, dt \]
\[ \leq C \int_0^T \int_{\tilde{\omega}} v^2 \, dx \, dt, \]
\[ \text{for some positive constant } C. \]

Therefore, by (6.9) and (6.11), Lemma 6.1 follows.

Second case: Hypothesis 6.1.2 holds. Fix \( \tilde{\lambda}_i, \tilde{\beta}_i \in (\lambda_i, \beta_i) \), \( i = 1, 2 \), and smooth functions \( \tilde{\zeta}, \eta \) as before. Then, define \( w := \eta \tilde{\omega} \), where \( \tilde{\omega} \) is any fixed solution of (6.4). Hence \( w \) satisfies (6.5), and \( f^2 \leq C(v^2 + (v_x)^2) \chi_{\tilde{\omega}}, \) where \( \tilde{\omega} \) is as in the first case. Applying Theorem 4.1 to \( w \), we obtain positive constants \( C \) and \( s_0 \) such that
\[ \int_0^T \int_0^1 \left( s \Theta a(w_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx \, dt \leq C \int_0^T \int_{\tilde{\omega}} w^2 \, dx \, dt \]
\[ + C \left( \int_0^T \int_0^1 f^2 e^{2s\varphi} \, dx \, dt + \int_0^T \int_{-\tilde{\beta}_1}^{\beta_1} f^2 e^{2s\Phi_1(t, x)} \, dx \, dt + \int_0^T \int_{\tilde{\lambda}_2}^{\tilde{\lambda}_1} f^2 e^{2s\Phi_2(t, x)} \, dx \, dt \right), \]
for all \( s \geq s_0 \). Hence, using \([19, \text{Proposition 4.2}]\), we find

\[
\int_0^T \int_{\mathcal{O}_1} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq \int_0^T \int_{\mathcal{O}_1} \left( s \Theta a(w_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx \, dt
\]

(6.13)

\[
\leq C \int_0^T \int_{\mathcal{O}_1} v^2 \, dx \, dt + C \int_0^T \int_{\mathcal{O}_1} e^{2s\varphi} (v^2 + (v_x)^2) \, dx \, dt \\
+ C \left( \int_0^T \int_{\mathcal{O}_1} (v^2 + v_x^2) e^{2s\varphi_1(t,x)} \, dx \, dt + \int_0^T \int_{\mathcal{O}_1} (v^2 + v_x^2) e^{2s\varphi_2(t,x)} \, dx \, dt \right)
\]

\[
\leq C \int_0^T \int_{\mathcal{O}_1} v^2 \, dx \, dt.
\]

Finally, define \( Z := \xi W \), where \( W \) is given in (4.10). As before,

(6.14)

\[
\int_0^T \int_{\mathcal{O}_1} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx \, dt \leq C \int_0^T \int_{\mathcal{O}_1} v^2 \, dx \, dt,
\]

for some positive constant \( C \) and \( s \geq s_0 \). To complete the proof it suffices to prove a similar inequality for \( x \in [0, \lambda_1] \). Using a reflection procedure as in the first part of the proof and applying [20, Theorem 3.1], one obtains

(6.15)

\[
\int_0^T \int_{\mathcal{O}_1} \left( s \Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx \, dt \leq C \int_0^T \int_{\mathcal{O}_1} v^2 \, dx \, dt
\]

for a positive constant \( C \) and \( s \) large enough. Therefore, by (6.13), (6.14), and (6.15), the conclusion follows. \( \square \)

We emphasize that in the proof of Lemma 6.1 a crucial role is played by the Carleman estimates stated in [20, Theorem 3.1] for nondegenerate parabolic problems with nonsmooth coefficient. Moreover, for such a result to apply, equation (4.3) is essential.

Using Lemma 6.1, we obtain the following result, which is crucial to proving Proposition 6.1.

**Lemma 6.2.** Under Hypotheses 6.1 and 6.2, there exists a positive constant \( C_T \) such that every solution \( v \in \mathcal{W}_1 \) of (6.4) satisfies (6.2).

The proof is similar to that [20, Lemma 5.3], but we repeat it quickly for the reader’s convenience.
Proof. Multiplying the equation of (6.4) by \( v_t \) and integrating by parts over \((0, 1)\), one obtains

\[
0 = \int_0^1 (v_t + (av_x)_x)v_t \, dx = \int_0^1 (v_t^2 + (av_x)_xv_t) \, dx
\]

\[
= \int_0^1 v_t^2 \, dx + [av_xv_t]_{x=0}^{x=1} - \int_0^1 av_xv_{tx} \, dx
\]

\[
= \int_0^1 v_t^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_0^1 a(v_x)^2 \geq -\frac{1}{2} \frac{d}{dt} \int_0^1 a(v_x)^2 \, dx.
\]

Thus the function \( t \mapsto \int_0^1 a(v_x)^2 \, dx \) is increasing for all \( t \in [0, T] \). In particular,

\[
\int_0^1 av_x(0, x)^2 \, dx \leq \int_0^1 av_x(t, x)^2 \, dx
\]

for every \( t \in [0, T] \). Integrating the last inequality over \([T/4, 3T/4]\) and using Lemma 6.1, we obtain a positive constant \( C \) such that

\[
\int_0^1 a(v_x)^2(0, x) \, dx \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_0^1 a(v_x)^2(t, x) \, dx \, dt
\]

\[
\leq C_T \int_{T/4}^{3T/4} \int_0^1 s \Theta a(v_x)^2(t, x)e^{2sp} \, dx \, dt \leq C \int_0^T \int_0^1 v^2 \, dx \, dt.
\]

Applying the Hardy-Poincaré inequality given in [19, Proposition 2.3] and the previous inequality, one obtains

\[
\int_0^1 \left( \frac{a}{(x-x_0)^2} \right)^{1/3} v^2(0, x) \, dx \leq \int_0^1 \frac{p}{(x-x_0)^2} v^2(0, x) \, dx
\]

\[
\leq C_{HP} \int_0^1 p(v_x)^2(0, x) \, dx
\]

\[
\leq \max \{ C_1, C_2 \} C_{HP} \int_0^1 a(v_x)^2(0, x) \, dx
\]

\[
\leq C \int_0^T \int_0^1 v^2 \, dx \, dt,
\]

for some positive constant \( C \), where

\[
p(x) = \begin{cases} 
(a(x)|x-x_0|^4)^{1/3} & \text{if } K > 4/3, \\
\max_{[0,1]} a|x-x_0|^{4/3} & \text{otherwise,}
\end{cases}
\]

\[
C_1 := \max \left\{ \left( \frac{x_0^2}{a(0)} \right)^{2/3}, \left( \frac{(1-x_0)^2}{a(1)} \right)^{2/3} \right\}, \quad C_2 := \max \left\{ \frac{x_0^{4/3}}{a(0)}, \frac{(1-x_0)^{4/3}}{a(1)} \right\}.
\]
and $C_{HP}$ is the Hardy-Poincaré constant.

By [19, Lemma 2.1], the function $x \mapsto a(x)/(x - x_0)^2$ is nondecreasing on $[0, x_0)$ and nonincreasing on $(x_0, 1]$. Thus

$$
\left( \frac{a(x)}{(x - x_0)^2} \right)^{1/3} \geq C_3 := \min \left\{ \left( \frac{a(1)}{(1 - x_0)^2} \right)^{1/3}, \left( \frac{a(0)}{x_0^2} \right)^{1/3} \right\} > 0.
$$

Hence

$$
C_3 \int_0^1 v(0, x)^2 \, dx \leq C \int_0^T \int_\omega v^2 \, dx \, dt,
$$

and the result follows. \(\square\)

Using Lemma 6.2 and proceeding as in [20, Proposition 5.1], one can prove Proposition 6.1.

### 6.2 Proof of Proposition 6.2

We consider again the adjoint problem (6.4), where the operator $A_1$ is replaced by $A_2$. In this case,

$$
\mathcal{W}_2 := \{ v \text{ is a solution of (6.4), with } A_2 \text{ in place of } A_1 \}
$$

with

$$
\mathcal{W}_2 \subset C^1([0, T]; H_{1/a}^2(0, 1)) \subset \mathcal{S} \subset \mathcal{U}_2,
$$

and

$$
\mathcal{U}_2 := C([0, T]; L_{1/a}^2(0, 1)) \cap L^2(0, T; H_{1/a}^1(0, 1)).
$$

As in [20, Lemma 5.4], one can prove the following lemma.

**Lemma 6.3.** Under Hypotheses 6.1 and 6.2, there exist positive constants $C$ and $s_0$ such that every solution $v \in \mathcal{W}_2$ of (6.4) satisfies

$$
\int_0^T \int_0^1 \left( s\Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{x - x_0}{a} \right)^2 v^2 \right) e^{2s\gamma} \, dx \, dt \leq C \int_0^T \int_\omega v^2 \frac{1}{a} \, dx \, dt
$$

for all $s \geq s_0$, where $\Theta$ and $\gamma$ are as (4.5) and (5.3), respectively.

The proof of Lemma 6.3 is similar to that of Lemma 6.1 with the suitable changes, but we repeat it here for the reader’s convenience. We emphasize that, in this case, a crucial role is played by the Carleman estimates stated in [20, Theorem 3.2] for nondegenerate parabolic problems with nonsmooth coefficient. Again, for such a result to apply, equation (5.2) is essential.

Another important result needed to prove Lemma 6.3 is the following the Caccioppoli inequality for the nondivergence case.
Proposition 6.3 (Caccioppoli’s inequality). Assume that either the function \( a \) is such that the associated operator \( A_2 \) is weakly degenerate and (5.2) holds, or the function \( a \) is such that \( A_2 \) is strongly degenerate. Moreover, let \( I' \) and \( I \) be open subintervals of \((0, 1)\) such that \( I' \subset I \subset (0, 1) \) and \( x_0 \notin I \). Let \( \varphi(t, x) = \Theta(t) \varphi(x) \), where \( \Theta \) is defined in (4.5) and \( \varphi \in C([0, 1], (-\infty, 0))\cap C^1([0, 1] \setminus\{x_0\}, (-\infty, 0)) \) satisfies \( |\varphi_x| \leq c/\sqrt{a} \) in \([0, 1] \setminus\{x_0\}\), for some \( c > 0 \). Then, there exist positive constants \( C \) and \( s_0 \) such that every solution \( v \in \mathcal{W}_2 \) of the adjoint problem (6.4) satisfies

\[
\int_0^T \left( v_s \right)^2 e^{2s} \, dt \leq C \int_0^T \left( v_x \right)^2 a \, dx \, dt
\]

for all \( s \geq s_0 \).

We omit the proof since it is similar to that of [20, Proposition 5.4].

Remark 7. Of course, our prototype for \( \varphi \) is the function \( \mu \) defined in (5.4). Indeed, if \( \mu \) is as in (5.4), then, by [19, Lemma 2.1],

\[
|\mu'(x)| = d_1 \frac{|x - x_0|^2 e^{R(x-x_0)^2}}{a(x)} = d_1 \sqrt{\frac{|x - x_0|^2 e^{2R(x-x_0)^2}}{a(x)}} \frac{1}{\sqrt{a(x)}} \leq c \frac{1}{\sqrt{a(x)}}.
\]

Proof of Lemma 6.3. First assume that \( \omega = (\alpha, \beta) \subset (0, 1) \) is such that \( x_0 \in \omega \). Take \( \tilde{\lambda}_i, \tilde{\lambda}_i, \tilde{\beta}_i \) (\( i = 1, 2 \)), and the smooth functions \( \xi, \eta \) as in the proof of Theorem 4.1. Define \( \omega := \tilde{\omega} \varphi \), where \( \varphi \) solves (6.4), and, in this case, \( A_1 \) is replaced by \( A_2 \). Hence, \( \omega \) solves

\[
\begin{cases}
  w_t + aw_{xx} = a(\xi_x, v + 2\xi_x v_x) =: f, & (t, x) \in (0, T) \times (0, 1), \\
  w(t, 0) = w(t, 1) = 0, & t \in (0, T).
\end{cases}
\]

Applying Theorem 5.1, we have

\[
\int_0^T \int_0^1 \left( s \Theta(w_x)^2 + s^3 \Theta^3 \left( \frac{x - x_0}{a} \right)^2 \frac{w^2}{a} \right) e^{2sx} \, dx \, dt
\]

\[
\leq C \left( \int_0^T \int_0^1 \frac{f^2}{a} e^{2sx(t, x)} \, dx \, dt + \int_0^T \int_0^1 w^2 \, dx \, dt \right)
\]

\[
+ C \left( \int_0^T \int_0^1 \frac{f^2}{a} e^{2sx(t, x)} \, dx \, dt + \int_0^T \int_0^1 \frac{f^2}{a} e^{2sx(t, x)} \, dx \, dt \right),
\]

for all \( s \geq s_0 \) and for a positive constant \( C \). Then, using the definition of \( \eta, \) in particular, the fact that \( \eta_x \) and \( \eta_{xx} \) are supported in \( \tilde{\omega} \), we can write

\[
f^2 = ((a\eta_v)_x + \eta_s a_{xx})^2 \leq (v_x^2 + (v_s)^2)\chi_{\tilde{\omega}},
\]
since $a'$ is bounded on $\tilde{\omega}$. Hence, applying (6.16) and Proposition 6.3 with $I' = \tilde{\omega}$ and $I = (\alpha, \beta_1) \cup (\lambda_2, \beta)$, we get, as in (6.7),

$$
\int_0^T \int_{\tilde{\lambda}_1} \left( s \Theta(v_x)^2 + s^3 \Theta^3 \left( \frac{x - x_0}{a} \right)^2 \frac{v^2}{a} \right) e^{2s\gamma} \, dx \, dt 
\leq C \int_0^T \int_{\omega} v^2 e^{2s\gamma} \, dx \, dt + C \int_0^T \int_I v^2 \frac{1}{a} \, dx \, dt 
\leq C \int_0^T \int_{\omega} v^2 \frac{1}{a} \, dx \, dt
$$

for some positive constant $C$.

The rest of the proof is similar to the last part of the proof of Lemma 6.1. □

Thanks to Lemma 6.3, we have the following observability inequality in the case of a regular final-time datum.

**Lemma 6.4.** Under Hypotheses 6.1 and 6.2, there exists a positive constant $C_T$ such that every solution $v \in W_2$ of (6.4) satisfies (6.3).

The proof of Lemma 6.4 follows as in [20, Lemma 5.5], but we refer the reader also to the proof of Lemma 6.2.

Using Lemma 6.4, one can prove, as in [9] or [8], Proposition 6.2.

## 7 Final comments

We conclude the paper with some comments about the estimates (4.7) and (5.5).

A Carleman estimate similar to (4.7) for the problem in divergence form can follow by [3, Theorem 4.1] at least in the strongly degenerate case and if the initial datum is more regular. Indeed, in this case, given $u_0 \in H^1_{a}(0, 1)$, $u$ is a solution of (1.2) if and only if the restrictions of $u$ to $[0, x_0)$ and to $(x_0, 1]$, $u_{|[0,x_0)}$ and $u_{|_{(x_0,1)}}$, are solutions of

$$
\begin{cases}
    u_t - A_1 u = h(t, x) \chi_\omega(x), & (t, x) \in (0, T) \times (0, x_0), \\
    u(t, 0) = (au_x)(t, x_0) = 0, & t \in (0, T), \\
    u(0, x) = u_0(x)_{|[0,x_0)},
\end{cases}
$$

and

$$
\begin{cases}
    u_t - A_1 u = h(t, x) \chi_\omega(x), & (t, x) \in (0, T) \times (x_0, 1), \\
    u(t, 1) = (au_x)(t, x_0) = 0, & t \in (0, T), \\
    u(0, x) = u_0(x)_{|_{(x_0,1)}},
\end{cases}
$$

(7.1)
respectively. This fact is implied by the characterization of the domain of $A_1$ given in Propositions 2.2 and by the Regularity Theorems 2.1 when the initial datum is more regular. On the other hand, if $u_0$ is only of class $L^2(0,1)$, the solution is not sufficiently regular to verify the additional condition at $(t,x_0)$, and this procedure cannot be pursued.

Moreover, in the weakly degenerate case, even in the case of a regular initial data, the lack of characterization of the domain of $A_1$ prevents us from considering a decomposition of the system into two disjoint systems like (7.1) and (7.2) to apply the results of [3].

Even if the problem is in nondivergence form and the initial data are more regular, the above decomposition does not work. Indeed in this case, using the characterization of the domain of $A_2$, one has that $(a u_0)(t,x_0) = 0$ (this equality holds only in the strongly degenerate case, see Proposition 3.2). But, to our best knowledge, the only result on Carleman estimates in this field is for problems with pure Neumann boundary conditions, in the sense that $u_x(t,x_0) = 0$, and with more regular degenerate functions (see [17]), that we do not have in our hands.

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