ANALYTIC VARIETIES INVARIANT BY FOLIATIONS AND PFAFF SYSTEMS

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Abstract. In this work we shall present a survey on problems and results on singular holomorphic foliations and Pfaff systems on complex manifolds assuming that these objects possess invariant analytic varieties. We will focus on recent results which have been motivated by classical works of Darboux, Poincaré and Painlevé on the problem of algebraic integration of singular polynomial differential equations. We present results on Poincaré and Painlevé problem of bounding the degree and the genus of analytic varieties invariant by holomorphic foliations and Pfaff systems. We shall discuss the general ideas of the theory of integrability characterizing the existence of meromorphic first integrals for complex analytic Pfaff equations.

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1. Introduction

The study of regular distributions and foliations and their integral manifolds was motivated by the classical work due to Pfaff [57, Chapter III]. The study of singular polynomial differential equations on the complex plane was investigated by Poincaré, Darboux, Painlevé. Nowadays, this corresponds to the study of singular holomorphic foliations on complex projective spaces. G. Darboux presented in [49] a theory on the existence of first integrals for polynomial differential equations based on the existence of sufficiently many invariant algebraic curves. In [91] H. Poincaré observed that it would be sufficient to bound the degree of algebraic solutions. P. Painlevé, in [86], stated an integrability problem as follows: Is it possible to recognize the genus of the general solution of an algebraic differential equation in two variables which has a rational first integral? It is worth mentioning that Painlevé won the “Grand Prix des Sciences Mathématiques” of the French Academy of Sciences by his important contributions in this subject, see [97].

These problems are known as Poincaré’s and Painlevé’s Problems. In [75] A. Lins Neto constructed families of foliations on $\mathbb{P}^2$, with fixed degree and local analytic type of the singularities, where foliations with rational first integrals of arbitrarily large degree appear. In other words, such families show that the questions of Poincaré and Painlevé have a negative answer in general. However, one can obtain an affirmative answer provided that some additional hypotheses are assumed.

Several works, such as D. Cerveau and A. Lins Neto [30], M. Carnicer [27], M. Soares [101, 100] and Walcher [106] have stimulated the current interest in the investigation of the Poincaré problem. Many authors have been working on these problems and on some of its generalizations, see for instance the papers by M. Soares [100], M. Brunella and L.G. Mendes [17], E. Esteves and S. Kleiman [55], V. Cavalier and D. Lehmann [28]. This problem also has been considered in other varieties such as: del Pezzo surfaces and K3 surfaces [61], weighted projective spaces [12, 45], multiprojective spaces [44], toric varieties [92], projective manifolds with Picard number equal to one [17], surfaces with trivial Picard group [3], varieties over an algebraically closed field of arbitrary characteristic [55]. It follows from a celebrated theorem by Jouanolou [69] that foliations on projective plane, of degree at least 2, with some invariant algebraic curve are rare. We refer the reader to [46] and references therein, where Coutinho and Pereira provide a generalization of Jouanolou’s result for one dimensional foliations and Pfaff fields on projective manifolds.

The problem of bounding the genus of an invariant curve in terms of the degree of a one-dimensional foliation on $\mathbb{P}^n$ was considered for instance by Campillo, Carnicer and de la Fuente [25]. They showed that, if $C$ is a reduced curve which is invariant by a one-dimensional foliation $\mathcal{F}$, of degree $d$, on $\mathbb{P}^n$, then

\[
\frac{2p_a(C) - 2}{\deg(C)} \leq d - 1 + a,
\]

where $p_a(C)$ is the arithmetic genus of $C$ and $a$ is an integer obtained from the concrete problem of imposing singularities to projective hypersurfaces. For instance, if $C$ has only nodal singularities then $a = 0$, and thus formula (1) follows from [60]. Esteves and Kleiman in [55] have provided bounds for the arithmetic genus of a curve invariant by a foliation by curves, which improve and extend some results of Campillo, Carnicer, and de la Fuente, and of du Plessis and Wall [53]. In [40] we establish upper bounds for the sectional genus of Gorenstein varieties which are invariant by Pfaff systems on projective spaces and in [4] Ballico gave an extension of this result.

The work of J.P. Jouanolou in [69] gives an improvement and generalization of the Darboux theory of integrability characterizing the existence of rational first integrals for Pfaff equations of codimension one on $\mathbb{P}^n_k$, where $k$ is an algebraically closed field of characteristic zero. Namely, consider $\omega \in H^0(\mathbb{P}^n_k, \Omega^1_{\mathbb{P}^n_k}(d + 1))$ a twisted 1-form which induces a codimension one Pfaff system.
Then by [69, Theorem 3.3], we have that $\omega$ admits a rational first integral if and only if it has an infinite number of invariant irreducible hypersurfaces. More generally, Jouanolou proved in [68] that on a complex compact manifold $X$ satisfying certain conditions on its Hodge-to-de Rham spectral sequence, a Pfaff system $\omega \in H^0(X, \Omega^2_X \otimes L)$, where $L$ is a line bundle, admits a meromorphic first integral if and only if it admits an infinite number of invariant irreducible divisors. Moreover, if $\omega$ does not admit a meromorphic first integral, then the number of invariant irreducible divisors is at most

$$\dim_C(H^0(X, \Omega^2_X \otimes L) / \omega \wedge H^0(X, \Omega^1_X)) + \rho(X) + 1,$$

where $\rho(X)$ is the Picard number of $X$. Deschamps provides a proof of this result for projective surfaces in his monograph [52] on Bogomolov’s work [7] on the boundedness of families of curves of fixed geometric genus on a surface of general type with positive Segre class.

E. Ghys in [63] drops the cohomological hypotheses given by Jouanolou showing that this result is valid for all compact complex manifolds. More precisely, the result of [63] states that if a codimension one Pfaff system $\omega \in H^0(X, \Omega^2_X \otimes L)$, does not admit a meromorphic first integral, then the number of invariant irreducible divisors must be smaller than

$$\dim_C[H^0(X, \Omega^2_X \otimes L) / \omega \wedge H^0(X, \Omega^1_{cl})] + \dim_C H^1(X, \Omega^1_{cl}) + 2,$$

where $\Omega^1_{cl}$ denotes the sheaf of closed holomorphic 1-forms on $X$. In the case for holomorphic foliations of dimension one, we presented in [80] a similar result which reads as follows: let $X$ be as above and let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $X$. If $\mathcal{F}$ admits at least invariant irreducible hypersurfaces, then $\mathcal{F}$ admits a meromorphic first integral. Here, $T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$ and $i_{\omega, \mathcal{F}}(\cdot)$ denotes the contraction by the local vector fields inducing the foliation $\mathcal{F}$. A higher codimensional version of these results has been proved in [42]. More precisely, we prove that if $\mathcal{F}$ is a Pfaff system, of codimension $k$, on a compact complex manifold $X$, defined by $\omega \in H^0(X, \Omega^k_X \otimes L)$, such that $\mathcal{F}$ admits

$$\dim_C[H^0(X, \Omega^{k+1}_X \otimes L) / \omega \wedge H^0(X, \Omega^1_X)] + \dim_C H^1(X, \Omega^1_{cl}) + k + 1$$

invariant irreducible analytic hypersurfaces, then $\mathcal{F}$ admits a meromorphic first integral. Versions of Darboux-Jouanolou theorem for polynomial differential forms have been provided in [41, 78, 36, 50].

A discrete dynamical version of Darboux-Jouanolou’s theorem was given by S. Cantat. He proved in [26] that if there exist $N$ irreducible hypersurfaces invariant by an automorphism $f: X \to X$ with

$$N \geq \dim(X) + \dim_C(H^1(X, \Omega^1_X))$$

then $f$ preserves a nontrivial meromorphic fibration.

M. Brunella and M. Nicolau in [18] prove a version of the Darboux-Jouanolou theorem for codimension one foliations in positive characteristic. The authors have improved a previous result due to Kim in [71]. Also, in [18] the authors provide a Darboux-Jouanolou type theorem for non-singular codimension one transversely holomorphic foliations on compact manifolds. In [23] L. Câmara has proved a high dimensional version of Brunella and Nicolau result for transversely holomorphic foliations. Scárdua in [94] shows, under certain conditions, a Darboux-Jouanolou type theorem for codimension one germs of holomorphic foliations on $(\mathbb{C}^n, 0)$. For similar results in dimension 2 we refer to [22, 95]. A version of Darboux-Jouanolou theorem for singular toric varieties was proved in [79] by using Cox’s homogeneous coordinate ring [47].
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2. Pfaff systems, distributions and foliations

2.1. Distributions and foliations. Throughout this survey we denote by $X$ a complex manifold of dimension $n$.

A regular holomorphic foliation $\mathcal{F}$ of codimension $q$ on a complex manifold $X$ is given by an open covering $\{U_\alpha\}_\alpha$ of $X$, holomorphic submersions $f_\alpha : U_\alpha \to \mathbb{C}^q$ and biholomorphisms $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^q \to f_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^q$ such that $f_\alpha = g_{\alpha\beta} \circ f_\beta$. This gives us a holomorphic vector bundle $T\mathcal{F} := \bigcup_\alpha \ker(df_\alpha) \subset TX$ called tangent bundle of $\mathcal{F}$. It follows from Frobenius theorem that a holomorphic subvector bundle $E \subset TX$ (called distribution) of rank $n - q$ is a tangent bundle of a foliation of codimension $q$ if and only if $[E,E] \subset E$.

In this survey we will be interested in holomorphic foliations and distributions with singularities.

Definition 2.1. A singular holomorphic distribution $\mathcal{F}$ of codimension $p$ on $X$ is a nonzero coherent subsheaf $T\mathcal{F} \subset TX$ of generic rank $n - p$ which is saturated, i.e., such that $TX/T\mathcal{F}$ is torsion-free.

We have an exact sequence of sheaves
$$0 \to T\mathcal{F} \to TX \to TX/T\mathcal{F} := N\mathcal{F} \to 0.$$

The sheaves $T\mathcal{F}$ and $N\mathcal{F}$ are called the tangent and the normal sheaves of $\mathcal{F}$, respectively. Now, by taking the double dual of the $p$-th wedge product of the inclusion $N\mathcal{F}^* \to \Omega_X^p$, we get a map
$$(\wedge^p N\mathcal{F}^*)^{**} \to \Omega_X^p.$$

Since $N\mathcal{F}$ and $N\mathcal{F}^*$ are torsion-free, it follows from [72, Proposition 5.6.10] and [72, Proposition 5.6.12] that $(\wedge^p N\mathcal{F}^*)^{**} \simeq \det(N\mathcal{F})^* \simeq \det(N\mathcal{F}^*)^*$. This gives rise to a nonzero twisted holomorphic $p$-form $\omega_\mathcal{F} \in H^0(X, \Omega_X^p \otimes \det(N\mathcal{F}^*)) \simeq H^0(X, \Omega_X^p \otimes \det(N\mathcal{F}))$. The twisted $p$-form $\omega_\mathcal{F}$ is called by the Pfaff system associated to $\mathcal{F}$. The singular set of $\mathcal{F}$ is given by the analytic subset
$$\text{Sing}(\mathcal{F}) = \{x \in X; \omega_\mathcal{F}(x) = 0\} \subset X.$$

For more details we refer to [42, 20].

Definition 2.2. A distribution $\mathcal{F}$ on $X$ whose tangent sheaf is invariant under the Lie bracket is called a foliation, i.e., if $[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$.

Therefore, we have a regular foliation on $X \setminus \text{Sing}(\mathcal{F})$.

2.2. Pfaff systems.

Definition 2.3. A Pfaff system $\mathcal{F}$ of codimension $p$ on $X$ is a nonzero map $\eta : \mathcal{L}^* \to \Omega_X^p$, where $\mathcal{L}$ is a line bundle on $X$. It corresponds to a twisted $p$-form $\omega_\eta \in H^0(X, \Omega_X^p \otimes \mathcal{L})$. The singular set $\text{Sing}(\eta)$ of $\eta$ is the zero set of $\omega_\eta$. 
The scheme structure of Sing($\mathcal{F}$) is defined as follows. Consider the dual morphism $\eta^* : \wedge^p TX \to \mathcal{L}$. Twisting by $\mathcal{L}$ we obtain a morphism $\wedge^p TX \otimes \mathcal{L} \to \mathcal{O}_X$. The image is the ideal sheaf $\mathcal{I}_Z$ of a subscheme $Z$ whose support is Sing($\mathcal{F}$).

Consider a Pfaff system $\mathcal{F}$ of codimension $p$ on $X$ induced by a twisted $p$-form $\omega \in H^0(X, \mathcal{O}_X^p \otimes \mathcal{L})$. Then $\omega$ is determined by the following:

(i) an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of $X$;
(ii) holomorphic $p$-forms $\omega_\alpha \in \Omega^p_{U_\alpha}$ satisfying

$$\omega_\alpha = h_{\alpha,\beta} \omega_\beta \quad \text{on} \quad U_\alpha \cap U_\beta \neq \emptyset,$$

where $h_{\alpha,\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)^*$ determines a cocycle representing $\mathcal{L}$.

**Definition 2.4.** We say that an analytic subvariety $V \subset X$ is invariant by a Pfaff system $\mathcal{F}$ induced by a twisted $p$-form $\omega \in H^0(X, \mathcal{O}_X^p \otimes \mathcal{L})$ if $i^* \omega \equiv 0$, where $i : V \hookrightarrow X$ is the inclusion map.

Let $\mathcal{F}$ be a Pfaff system of codimension $p$ on $X$ induced by a twisted $p$-form $\omega \in H^0(X, \mathcal{O}_X^p \otimes \mathcal{L})$ and $V$ an analytic subvariety of $X$ of pure codimension $k \leq p$. Suppose that for each $\alpha \in \Lambda$ we have

$$V \cap U_\alpha = \{ z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0 \},$$

where $f_{\alpha,1}, \ldots, f_{\alpha,k} \in \mathcal{O}(U_\alpha)$. If $V$ is invariant by $\omega$, then for each $i \in \{1, \ldots, k\}$ there exist holomorphic $(p+1)$-forms $\omega_{\alpha,1}, \ldots, \omega_{\alpha,k} \in \Omega^{p+1}_{U_\alpha}$, such that

$$\omega_\alpha \wedge df_{\alpha,i} = f_{\alpha,1} \omega_{\alpha,1} + \cdots + f_{\alpha,k} \omega_{\alpha,k}.$$

A. G. Aleksandrov in [2] introduced the concept of multiple residues of a logarithmic differential form with poles along a complete intersection which is a generalization of Saito’s residues [93]. Here we recall such theory. Let $U$ be a germ of $n$-dimensional complex manifold and $D$ an analytic reduced hypersurface in $U$ whose decomposition into irreducible components is given by

$$D = D_1 \cup \cdots \cup D_k.$$

Suppose that the analytic subvariety $V = D_1 \cap \cdots \cap D_k$ is reduced and has pure codimension $k$. We assume that

$$V = \{ z \in U : f_1(z) = \cdots = f_k(z) = 0 \},$$

with $f_1, \ldots, f_k \in \mathcal{O}(U)$ and for each $i \in \{1, \ldots, k\}$, $D_i = \{ z \in U : f_i(z) = 0 \}$. Since $V$ is a reduced variety, the $k$-form $df_1 \wedge \cdots \wedge df_k$ does not vanish identically at each irreducible component of $V$. Denote by $\Omega^p_U(D_i)$, $q \geq 1$, the $\mathcal{O}_U$-module of meromorphic differential $q$-forms with simple poles on $D_i = D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_k$, for each $i = 1, 2, \ldots, k$.

We can prove the following result as a consequence of Alexandrov’s theory.

**Proposition 2.5.** [3, Proposition 2.8] Let $\mathcal{F}$ be a Pfaff system of codimension $p$ on a complex manifold $X$ induced by a twisted $p$-form $\omega \in H^0(X, \mathcal{O}_X^p \otimes \mathcal{L})$, and consider $V \subset X$ a reduced local complete intersection subvariety of codimension $k$ which is invariant by $\omega$. Then for all local representations $\omega_\alpha = \omega|_{U_\alpha}$ of $\omega$, and all local expressions of $V$ in $U_\alpha$

$$V \cap U_\alpha = \{ z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0 \},$$

there exist holomorphic function $g_\alpha \in \mathcal{O}(U_\alpha)$, a holomorphic $(p-k)$-form $\xi_\alpha \in \Omega^{p-k}_{U_\alpha}$ and a holomorphic $p$-form $\eta_\alpha \in \Omega^p_{U_\alpha}$, such that

$$g_\alpha \omega_\alpha = df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k} \wedge \xi_\alpha + \eta_\alpha.$$

Moreover, $g_\alpha$ is not identically zero on every irreducible component of $V$ and $\eta_\alpha$ is given by

$$\eta_\alpha = f_{\alpha,1} \eta_{\alpha,1} + \cdots + f_{\alpha,k} \eta_{\alpha,k}.$$
where each \( \eta_{i, j} \in \Omega^p_{\mathbb{U}_x} \) is a holomorphic \( p \)-form.

**Definition 2.6.** If \( \mathcal{F} \) is a Pfaff system on \( X \), a first integral for \( \mathcal{F} \) is a non-constant meromorphic function \( f: X \to \mathbb{P}^1 \), such that the fibers of \( f \) are invariant by \( \mathcal{F} \).

By using the isomorphism \( \Omega^p_X \simeq \wedge^{n-p}TX \otimes \det TX^* \), we have that a Pfaff system \( \omega \in H^0(X, \Omega^p_X \otimes \mathcal{L}) \) induces a global section

\[
\partial \omega \in H^0(\wedge^{n-p}TX \otimes \det TX^* \otimes \mathcal{L})
\]

which is the so-called Pfaff field associated to \( \omega \), see [56, Section 3].

### 2.3. Pfaff systems on complex projective spaces.

Let \( \omega \in H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(r)) \) be a holomorphic Pfaff system of codimension \( k \) on \( \mathbb{P}^n \). Take a generic non-invariant linearly embedded subspace \( i: H \simeq \mathbb{P}^k \hookrightarrow \mathbb{P}^n \). We have an induced non-trivial section

\[
i^*\omega \in H^0(H, \Omega^p_H(r)) \simeq H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(-k - 1 + r)),
\]

since \( \Omega^p_{\mathbb{P}^k} = \mathcal{O}_{\mathbb{P}^k}(-k - 1) \). The tangency set between \( \omega \) and \( H \), denoted by \( Z(i^*\omega) \), is defined as the hypersurface of zeros of \( i^*\omega \) on \( H \). The degree of \( \omega \), denoted by \( \deg(\omega) \), is defined as the degree of \( Z(i^*\omega) \) in \( H \) and, therefore, is given by

\[
\deg(\omega) = -k - 1 + r.
\]

In particular, we have that

\[
\omega \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d + k + 1)) \simeq H^0(\mathbb{P}^n, \wedge^{n-k}T\mathbb{P}^n(d - k + n)),
\]

where \( \deg(\omega) = d \). A Pfaff system of degree \( d \) can be induced by a polynomial differential \( k \)-form on \( \mathbb{C}^{n+1} \) with homogeneous coefficients of degree \( d + 1 \), see for instance [42, 41, 69].

### 3. Characteristic classes and residues

#### 3.1. Baum-Bott Theorem.

In [5] P. Baum and R. Bott developed a residue theory for singular holomorphic foliations on complex manifolds. More precisely, they proved the following:

**Theorem 3.1** (Baum-Bott). Let \( \mathcal{F} \) be a holomorphic foliation of codimension \( k \) on a complex manifold \( X \) and \( \varphi \) be a homogeneous symmetric polynomial of degree \( d \) satisfying \( k < d \leq n \). Let \( Z \) be a compact connected component of the singular set \( \text{Sing}(\mathcal{F}) \). Then there exists a homology class \( \text{Res}_\varphi(\mathcal{F}, Z) \in H_{2(n-d)}(Z; \mathbb{C}) \) such that:

i) \( \text{Res}_\varphi(\mathcal{F}, Z) \) depends only on \( \varphi \) and on the local behavior of the leaves of \( \mathcal{F} \) near \( Z \).

ii) Suppose that \( X \) is compact and denote by \( \text{Res}(\varphi, \mathcal{F}, Z) := \alpha_\ast \text{Res}_\varphi(\mathcal{F}, Z) \), where \( \alpha_\ast \) is the composition of the maps

\[
H_{2(n-d)}(Z; \mathbb{C}) \xrightarrow{i_\ast} H_{2(n-d)}(X; \mathbb{C})
\]

and

\[
H_{2(n-d)}(X; \mathbb{C}) \xrightarrow{P} H^{2d}(X; \mathbb{C})
\]

where \( i_\ast \) is the induced map of the inclusion \( i: Z \to X \) and \( P \) is the Poincaré duality. Then

\[
\varphi(N_{\mathcal{F}}) = \sum_Z \text{Res}(\varphi, \mathcal{F}, Z).
\]

An explicit calculation of the residues is difficult in general, see [5, 29, 9, 19, 32, 38, 10, 39]. If the foliation \( \mathcal{F} \) has dimension one with isolated singularities, Baum and Bott in [6] show that residues can be expressed in terms of a Grothendieck residue, i.e., for each \( x \in \text{Sing}(\mathcal{F}) \) we have

\[
\text{Res}_\varphi(\mathcal{F}, x) = \text{Res}_x \left[ \varphi(Jv) \frac{dz_1 \wedge \cdots \wedge dz_n}{v_1 \cdots v_n} \right],
\]
where \( v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial z_i} \) is a germ of holomorphic vector field at \( x \) tangent to \( \mathcal{F} \) and \( Jv \) is the jacobian of \( v \). In particular, if \( \varphi = \det \), then
\[
\text{Res}_{\det}(\mathcal{F}, x) = \text{Res}_x \left[ \det(Jv) \frac{dz_1 \wedge \cdots \wedge dz_n}{v_1 \cdots v_n} \right] = \mu_p(v),
\]
where \( \mu_x(v) \) is the Milnor number of \( v \) at \( x \). In this situation Baum-Bott theorem is
\[
\int_X c_n(N_{\mathcal{F}}) = \int_X c_n(TX - T\mathcal{F}) = \sum_{x \in \text{Sing}(\mathcal{F})} \mu_x(v).
\]
A similar formula is given in [43] in the context of complex compact orbifolds and these residue formulae have been applied to the Poincaré problem for quasi-homogeneous vector fields [12, 45].

In the context of a hypersurface \( V \) invariant by a one-dimensional foliation \( \mathcal{F} \), we have the following Baum-Bott type theorem \([34, 33]\): if \( \mathcal{F} \) has isolated singularities and \( V \) is a normal crossing divisor, then
\[
(6) \quad \int_X c_n(TX(-\log V) - T\mathcal{F}) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X\setminus V)} \mu_x(\mathcal{F}) + \sum_{x \in \text{Sing}(\mathcal{F}) \cap V} \text{Log}(\mathcal{F}, V, x),
\]
where \( \text{Log}(\mathcal{F}, V, x) \) denotes Aleksandrov’s logarithmic index \([1]\) of \( \mathcal{F} \) at the point \( x \) and \( TX(-\log V) \) is the logarithmic vector bundle associated to \( V \).

### 3.2. GSV-index for holomorphic Pfaff Systems.

The GSV-index for vector fields tangent to hypersurfaces with isolated singularities was introduced by X. Gómez-Mont, J. Seade and A. Verjovsky \([65]\) as a generalization of the the Poincaré-Hopf index. The concept of GSV-index for vector fields tangent to complete intersections has been extended by J. Seade and T. Suwa in \([99, 98]\) and J. -P. Brasselet, J. Seade and T. Suwa in \([11]\).

M. Brunella in \([14]\) introduced the GSV-index for one-dimensional singular foliations in complex surfaces in terms of the germs of 1-forms inducing the foliation and established a relation between the GSV-index with the Khanedani-Suwa variational \([70]\) and Camacho-Sad \([21]\) indices. Let us recall Brunella’s definition of GSV index. Let \( X \) be a complex compact surface and \( \mathcal{F} \) a one-dimensional holomorphic foliation on \( X \). Let \( C \) be a reduced curve on \( X \) invariant by \( \mathcal{F} \). Consider \( \omega \in H^0(X, \Omega^1_X \otimes \mathcal{L}) \) a twisted 1-form inducing the foliation \( \mathcal{F} \). Given a point \( x \in C \), let \( f = 0 \) be a local equation of \( C \) in a neighborhood \( U_\alpha \) of \( x \) and let \( \omega_\alpha \) be the holomorphic 1-form inducing the foliation \( \mathcal{F} \) on \( U_\alpha \). Since \( \mathcal{F} \) is logarithmic along \( C \), it follows from \([93, 74, 104]\) that there are holomorphic functions \( g \) and \( \xi \) defined in a neighborhood of \( x \), that do not vanish identically simultaneously on \( C \), such that
\[
(7) \quad g \frac{\omega_\alpha}{f} = \xi \frac{df}{f} + \eta,
\]
with \( \eta \) being a suitable holomorphic 1-form. In view of this relation, M. Brunella in \([14]\) showed that the GSV-index can be defined as follows:
\[
\text{GSV}(\mathcal{F}, C, x) = \sum_i \text{ord}_x \left( \frac{\xi}{g} |_{C_i} \right),
\]
where \( C_i \subset C \) are irreducible components of \( C \) and \( \text{ord}_x \left( \frac{\xi}{g} |_{C_i} \right) \) denotes the order of vanishing of \( \frac{\xi}{g} |_{C_i} \) at \( x \).
Theorem 3.2 ([14, 13]). Let $\mathcal{F}$ be a one-dimensional holomorphic foliation on a complex compact surface $X$ and $C \subset X$ a reduced curve invariant by $\mathcal{F}$. Then
\[ \sum_{x \in \text{Sing}(\mathcal{F}) \cap C} \text{GSV}(\mathcal{F}, C, x) = L \cdot C - C \cdot C. \]

Cavalier and Lehmann in [28] have studied a GSV index via K-theory for locally complete intersection invariant curves and they prove certain inequalities in the Poincaré problem context. Recently, T. Suwa in [105] gave a new interpretation of the GSV-index as a residue arising from a certain localization of the Chern class of the ambient tangent bundle.

In [31] we introduce a GSV type index for Pfaff systems on projective manifolds and we prove some of its important properties. Consider a Pfaff system of codimension $p$ induced by the twisted $p$-form $\omega \in \mathbb{H}^0(X, \Omega^p_X \otimes \mathcal{L})$ and $V$ a reduced local complete intersection subvariety of pure codimension $k$ invariant by $\omega$. Let us denote $\text{Sing}(\omega, V) := \text{Sing}(\omega) \cap V$. We also assume that the codimension of the system $\omega$ coincides with the codimension of $V$, i.e., $p = k$. Fixed an irreducible component $S_i$ of $\text{Sing}(\omega, V)$, let $\omega_\alpha = \omega|_{U_\alpha}$ be a local representative of $\omega$, such that $U_\alpha \cap S_i \neq \emptyset$.

Suppose $V$ is represented in $U_\alpha$ by
\[ V \cap U_\alpha = \{ z \in U_\alpha : f_\alpha,1(z) = \cdots = f_\alpha,k(z) = 0 \}. \]

and it follows from Proposition 2.5 that
\[ g_\alpha \omega_\alpha = (df_\alpha,1 \wedge \cdots \wedge df_\alpha,k) \xi_\alpha + \eta_\alpha, \]

where $\eta_\alpha = f_\alpha,1 \eta_\alpha,1 + \cdots + f_\alpha,k \eta_\alpha,k, \eta_\alpha,i \in \Omega^i_{U_\alpha}$, and $\xi_\alpha$ being a holomorphic function. Now, we are able to define the GSV-index for Pfaff systems with an invariant reduced local complete intersection subvariety.

Definition 3.3. Suppose that $S := \text{Sing}(\omega, V)$ is a codimension one subvariety of $V$. The GSV-index of $\omega$ relative to $V$ in $S$ is given by
\[ \text{GSV}(\omega, V, S) := \sum_j \text{ord}_S \left( \frac{\xi_\alpha}{g_\alpha} \big|_{V_j} \right), \]

where the sum is taken over all irreducible components $V_j$ of $V$ and $\text{ord}_S \left( \frac{\xi_\alpha}{g_\alpha} \big|_{V_j} \right)$ denotes the order of vanishing of $\frac{\xi_\alpha}{g_\alpha} \big|_{V_j}$ along $S$.

In [31] we prove the following result.

Theorem 3.4 ([31]). Let $X$ be a projective manifold and $V \subset X$ a reduced local complete intersection subvariety of codimension $k$ invariant by a Pfaff system of codimension $k$ induced by a twisted $k$-form $\omega \in \mathbb{H}^0(X, \Omega^k_X \otimes \mathcal{L})$. Then the following hold:

(a) there exists a complex number $\text{GSV}(\omega, V, S_i)$ which depends only on the local representatives of $\omega, V$ and $S_i$;

(b) if $\text{Sing}(\omega, V) := \text{Sing}(\omega) \cap V$ has codimension one in $V$, then
\[ \sum_i \text{GSV}(\omega, V, S_i)[S_i] = c_1([\mathcal{L} \otimes \det(N_{V/X})^{-1}])|_V \sim [V], \]

where $S_i$ denotes an irreducible component of $\text{Sing}(\omega, V)$ and $N_{V/X}$ is the normal sheaf of $V$. 
4. Polar varieties and foliations

In [102] Soares proves several bounds relating polar classes of smooth invariant varieties and the degree of the foliation. In what follows we shall introduce the polar classes associated to a smooth projective variety.

If $\mathcal{F}$ is a one-dimensional foliation on $\mathbb{P}^n$ of degree $d$, then $\mathcal{F}$ is given by a global section of $T\mathbb{P}^n(d-1)$, see (5). Let $D$ be an analytic hypersurface on $\mathbb{P}^n$ defined locally by functions $\{f_{\alpha} \in \mathcal{O}(U_{\alpha})\}_{\alpha \in \Lambda}$, where $\{U_{\alpha}\}_{\alpha \in \Lambda}$ is an open covering of $\mathbb{P}^n$. If $U_{a\beta} := U_{\alpha} \cap U_{\beta} \neq \emptyset$, then there exists $f_{a\beta} \in \mathcal{O}^*(U_{a\beta})$, such that $f_{\alpha} = f_{a\beta}f_{\beta}$. Since $\mathcal{F}$ is given by a section of $T\mathbb{P}^n(d-1)$, then we have collections $\{\{\theta_{\alpha}\}; \{U_{\alpha}\}; \{g_{a\beta} \in \mathcal{O}^*(U_{a\beta})\}\}_{\alpha \in \Lambda}$ on $\mathbb{P}^n$, where $g_{a\beta}$ is the cocycle inducing $T\mathcal{F}^* = \mathcal{O}_{\mathbb{P}^n}(d-1)$. Consider the following functions

$$
\zeta^{(\mathcal{F}, D)} = \theta_{\alpha}(f_{\alpha})|_{D} \in \mathcal{O}(U_{\alpha} \cap D).
$$

If $U_{\alpha} \cap U_{\beta} \cap D \neq \emptyset$, using Leibniz’s rule we get $\zeta^{(\mathcal{F}, D)} = f_{a\beta}g_{a\beta}\zeta^{(\mathcal{F}, D)}$. This yields a global section $\zeta^{(\mathcal{F}, D)}$ of the line bundle $(T\mathcal{F}^* \otimes \mathcal{O}_{\mathbb{P}^n}(D))|_{D}$. The tangency variety of $\mathcal{F}$ with $D$ is given by

$$
T(\mathcal{F}, D) = \{p \in D; \zeta^{(\mathcal{F}, D)}(p) = 0\}.
$$

Definition 4.1. Consider a pencil of hyperplanes $\mathcal{H} = \{H_\lambda\}_{\lambda \in \mathbb{P}^1}$, with base locus $\bigcap_{\lambda \in \mathbb{P}^1} H_\lambda = \mathbb{L}^{n-2}$, where $\mathbb{L}^{n-2}$ is a linear subspace of dimension $n-2$ which is not contained in $\text{Sing}(\mathcal{F})$. The polar divisor of $\mathcal{F}$ with respect $\mathcal{H}$ is

$$
D_{\mathcal{H}} = \bigcup_{\lambda \in \mathbb{P}^1} T(\mathcal{F}, H_\lambda).
$$

Lemma 4.2. [102, Lemma 2.1] $D_{\mathcal{H}}$ is a hypersurface of degree $d+1$.

Let $V \subset \mathbb{P}^n$ be a smooth projective variety, of dimension $n-k$, and $\mathbb{L}^{k+j-2}$ a linear subspace of dimension $k+j-2$. Then the $j$-th polar locus [58] of $V$ is defined by

$$
\mathcal{P}_j(V) = \{p \in V; \text{dim}(\mathcal{T}_p V \cap \mathbb{L}^{k+j-2}) \geq j-1\},
$$

for $0 \leq j \leq n-k$. The $j$-th class $\mathcal{P}_j(V)$ of $V$ is the degree of the cycle $\mathcal{P}_j(V)$ and can be written as

$$
\mathcal{P}_j(V) = \sum_{i=0}^{j}(-1)^i \binom{n-k-i+1}{j-i} \int_V c_i(TV)c_i(\mathcal{O}_V(1))^{n-k-i}.
$$

Lemma 4.3. [102, Lemma 3.1] Let $V$ be a smooth irreducible algebraic variety of dimension $n-k$, invariant by a one-dimensional foliation $\mathcal{F}$ such that $V$ is not contained in $\text{Sing}(\mathcal{F})$. Consider a pencil of hyperplanes $\mathcal{H} = \{H_\lambda\}_{\lambda \in \mathbb{P}^1}$. Then

$$
\mathcal{P}_{n-k}(V) \subset D_{\mathcal{H}} \text{ and } \mathcal{P}_0(V) = V \notin D_{\mathcal{H}}.
$$

Mol in [83] introduces the notion of polar sets associated to the holomorphic distributions.

5. Poincaré and Painlevé problems for foliations and Pfaff systems

In [30] Cerveau and Lins Neto have initiated the study of Poincaré’s problem by showing that if $C$ is a curve of degree $m$ with singularities of normal crossing type, invariant by a foliation by curves $\mathcal{F}$ on $\mathbb{P}^2$, of degree $d$, then

$$
m \leq d + 2.
$$

Moreover, they proved that the equality holds if $\mathcal{F}$ is given by a logarithmic 1-form and $C$ is reducible. If $C$ is irreducible this bound follows from the following equation [30, Proposition 885]

$$
2 - 2g(C) = \sum_i d(\mathcal{F}, C_i) - m(d - 1),
$$

(9)
that the non-negativity of the GSV-indices for Pfaff systems yields
results in $0 \leq \ell = m(d + 2 - m)$, which implies that $m \leq d + 2$.

Carnicer in [27] proved that if the singularities of the foliation along $C$ are of non-dicritical type, then $m \leq d + 2$ as well.

In order to investigate these results in higher dimensions many authors have provided generalizations of the formula (9). Let $\mathcal{F}$ be a one-dimensional foliation on a projective manifold $X$ and $C \subset X$ a curve invariant by $\mathcal{F}$. Esteves and Kleiman in [55, Proposition 5.2] prove the following formula

$$2p_a(C) - 2 - C \cdot T\mathcal{F}^* = \lambda(C) - \deg(C \cap \text{Sing}(\mathcal{F})),$$

where $p_a(C)$ is the arithmetic genus of $C$, $\lambda(C)$ is the colength in the dualizing sheaf of the subsheaf generated by the Kähler differentials. Esteves and Kleiman showed this result in the context of a (possibly singular) algebraic variety over an algebraically closed field of arbitrary characteristic. But, in this survey we will only deal with smooth ambient spaces. The formula (10) is a generalization of those obtained in [76, Prop. 2.7, p. 659.] and [25, Prop. 2.2, p. 60].

If $C \subset \mathbb{P}^n$ is a smooth curve which is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-1}$ invariant by a one-dimensional foliation $\mathcal{F}$ of degree $d$, Soares has provided in [101] the following bound

$$d_1 + \cdots + d_{n-1} \leq d + n - 1.$$

Now, if $C$ has at most ordinary nodes as singularities, Campillo, Carnicer and J. de la Fuente showed in [25], by using a formula of the type of (10), that

$$d_1 + \cdots + d_{n-1} \leq d + n.$$

Esteves in [54] also proved the same bound by employing algebro-geometric techniques.

Let $V$ be a variety of codimension $k$ which is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$. If $V$ is invariant by a Pfaff system $\mathcal{F}$ of degree $d$ and codimension $k$ on $\mathbb{P}^n$, Esteves and Cruz proved in [48, Corollary 4.5] that

$$d_1 + \cdots + d_k \leq \begin{cases} d + k, & \text{if } \rho \leq 0 \\ d + k + \rho, & \text{if } \rho > 0 \end{cases}$$

where $\rho := \sigma + k + 1 - d_1 - \cdots - d_k$, with $\sigma$ denoting the Castelnuovo-Mumford regularity of the singular locus of $V$. In [40] we prove that if $V$ is nonsingular in codimension 1, then one can take $\rho = 1$, regardless of $\sigma$. That is, under this condition we get that

$$d_1 + \cdots + d_k \leq d + k + 1.$$
5.1. The approach of Soares. Soares made substantial contributions to the investigation of the Poincaré problem for one-dimensional foliations with smooth invariant varieties. He has obtained certain bounds via Baum-Bott Theorem and also polar varieties.

Let $\mathcal{F}$ be a one-dimensional foliation on $\mathbb{P}^n$, of degree $d$, with non-degenerate isolated singularities. If $V \subset \mathbb{P}^n$ is a smooth hypersurface of degree $m$ which is invariant by $\mathcal{F}$, then Soares applied the Baum-Bott Theorem to prove that

$$\sum_{i=0}^{n-1} [1 + (-1)^i(m-1)^{i+1}]d^{n-1-i} = \int_V c_{n-1}(TV(d-1)) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap V} \mu_p(v) = \# \text{Sing}(\mathcal{F}) \cap V.$$ 

In order to prove the bound

$$m \leq d + 1$$

Soares studied the following generating function

$$\psi(x) = \sum_{i=0}^{n-1} [1 + (-1)^i x^{i+1}]d^{n-1-i}.$$ 

By Lehmann’s result [73] we have that $\psi(m-1) = \# \text{Sing}(\mathcal{F}) \cap V$ is positive. If $n$ is even, then $\psi(x) \leq 0$ whenever $x \geq d + 1$. Thus $m \leq d + 1$. For the case when $n$ is odd, then

$$\psi(x) \geq \psi(m-1) = \# \text{Sing}(\mathcal{F}) \cap V$$

whenever $x \geq d$ and once again we can conclude that $m \leq d + 1$.

As we have seen, Soares’ approach consists of studying the number of singularities contained in the invariant hypersurface. In [34, Theorem 3] the authors by using the formula (6) prove that if $n$ is odd, then $\text{Sing}(\mathcal{F}) \subset V$ if and only if the Soares’ bound (14) is achieved.

Now, if $V \subset \mathbb{P}^n$ is a smooth irreducible projective variety which is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_r$ invariant by a foliation $\mathcal{F}$, of degree $d$ and with non-degenerate isolated singularities. Soares, by applying Baum-Bott theorem [101], provided the following bound

$$\frac{\theta_{n-r}(V)}{\theta_{n-r-1}(V)} \leq d.$$ 

On the other hand, Soares proved the following bound via polar varieties

$$\frac{\theta_{n-r-j}(V)}{\theta_{n-r-j-1}(V)} \leq d + 1,$$

if $\mathcal{P}_{n-k-j}(V) \subset \mathcal{D}_H$ but $\mathcal{P}_{n-k-j-1}(V) \not\subset \mathcal{D}_H$, for some $0 \leq j \leq n - k - 1$. Observe from Lemma 4.3 that $\mathcal{P}_{n-k}(V) \subset \mathcal{D}_H$ and $V \not\subset \mathcal{D}_H$, then

$$\frac{\theta_{n-r}(V)}{\theta_{n-r-1}(V)} \leq d + 1.$$ 

Therefore, under the assumption that the foliation has non-degenerate singularities the bound (15) is sharper than the one in (17).

5.2. Brunella-Mendes approach. In [17] Brunella and Mendes proved a bound for normal crossing hypersurfaces invariant by Pfaff systems as a consequence of results due to Deligne [51] and Saito [93] on the theory of logarithmic forms. Brunella and Mendes show that a Pfaff system, of codimension $k$, with an invariant hypersurface is a meromorphic logarithmic form, i.e, a global section of

$$\Omega^1_{\mathbb{P}^n}(\log V) \otimes \mathcal{O}_{\mathbb{P}^n}(-V) \otimes \mathcal{L}.$$
They apply Bogomolov’s lemma [8] which states that if there exists a nontrivial global section of $\Omega^1_X(\log V) \otimes N$, then

\begin{equation}
\kappa(X, N^{-1}) \leq k,
\end{equation}

where $\kappa(X, N^{-1})$ denotes the Kodaira-Iitaka dimension of the line bundle $N^{-1}$. Thus, if $X = \mathbb{P}^n$ and $V$ has degree $m$, we have that $N^{-1} = O_{\mathbb{P}^n}(m) \otimes O_{\mathbb{P}^n}(-d - k - 1) = O_{\mathbb{P}^n}(\ell)$ for some $\ell \in \mathbb{Z}$. Since $\ell > 0$ implies that $\kappa(\mathbb{P}^n, O_{\mathbb{P}^n}(\ell)) = n$ we conclude from (18) that

\[ m \leq d + k + 1. \]

5.3. **Poincaré problem and birational geometry of foliations.** The birational classification of foliations on surfaces has been established by Brunella [15, 16], McQuillan [81], Mendes [82].

In [87] Pereira has considered the Poincaré problem by obtaining bounds for the degree of the first integral of a foliation of general type (Kodaira dimens ion equal to 2) in terms of the degree, the birational invariants and the geometric genus of a generic leaf of the foliation. A refinement of this result was given by Pereira and Svaldi in [89]. They prove that if $F$ is a foliation on $\mathbb{P}^2$ such that $F$ is birationally equivalent to a non-isotrivial fibration of genus $g \geq 2$. Then the degree of the general leaf of $F$ is bounded by

\[ \left( \frac{4}{42(2g - 2)} + 1 \right)^2 (4g - 4) \deg(F). \]

In [66] Hacon and Langer prove a result on the effective generation of pluri-canonical linear systems of general type foliations on surfaces which is related to the Poincaré problem. They prove the following: for any integer valued function $P : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ and any integer $g \geq 0$, there exists an integer $\delta > 0$ such that if $(X, F)$ is a weak nef model of a complete foliated surface with

\[ \chi(X, O_X(mT F^*)) = P(m) \]

for all $m \geq 0$, and if $(X, F)$ has a meromorphic first integral whose general fiber has geometric genus $g$, then the general leaf $C$ satisfies

\[ C \cdot T F^* \leq \delta. \]

Such result is also related with the problem of bounding the order of the automorphism group of foliations of general type, see [35, 37, 85, 103]. In the recent paper [67] the authors prove a bound for other birational invariant of a foliation, called the slope, which is related to the Poincaré problem.

5.4. **Zariski-Esteves approach.** Esteves in [54] gave an algebraic proof of Soares’ result. He proved a characterization of all vector fields that leave invariant a smooth hypersurface via the Koszul complex associated to the ideal generated by the partial derivatives of a polynomial defining the hypersurface. Esteves point out in his paper that such proof is inspired by an observation by O. Zariski [77, Part c of Example 7, p. 892].

Let $V = \{ f = 0 \}$ be a smooth hypersurface $f$ is a polynomial of degree $m$ defining the hypersurface $V$ and $v$ a polynomial vector field inducing a foliation on $\mathbb{P}^n$ which leaves $V$ invariant. Esteves in [54] proves that

\begin{equation}
v = \sum_{i<j} P_{i,j} \left( \frac{\partial f}{\partial z_i} \frac{\partial}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i} \right) + \frac{h}{k} \sum_{i=0}^{n-1} z_i \frac{\partial}{\partial z_i},
\end{equation}
with \( P_{i,j}, h \in \mathbb{C}[z_0, \ldots, z_n] \). Now, Soares' bound is a consequence of this normal form. Indeed, since \( v \neq 0 \), we conclude that \( P_{i,j} \neq 0 \) for some \( i, j \). Otherwise, we would have that

\[
v = \frac{h}{k} \sum_{i=0}^{n} z_i \frac{\partial}{\partial z_i},
\]

and such vector field does not define a foliation on \( \mathbb{P}^n \). Thus, since \( v \) has degree \( d \), we have that \( 0 \leq \deg P_{i,j} = d - m + 1 \) which gives us that \( m \leq d + 1 \). It is worth mentioning that Esteves considered this problem in arbitrary characteristic.

**Example 5.1.** Let \( \mathcal{F} \) be the foliation on \( \mathbb{P}^3 \) induced by the polynomial vector field

\[
v = (-z_1^{m-1} - z_2^{m-1} - z_3^{m-1}) \frac{\partial}{\partial z_0} + (z_0^{m-1} - z_2^{m-1} - z_3^{m-1}) \frac{\partial}{\partial z_1} + \]

\[+(z_0^{m-1} + z_2^{m-1} - z_3^{m-1}) \frac{\partial}{\partial z_2} + (z_0^{m-1} + z_1^{m-1} + z_2^{m-1}) \frac{\partial}{\partial z_3}.
\]

The hypersurface \( V = \{z_0^n + z_2^n + z_3^n = 0\} \) is invariant by \( \mathcal{F} \) and \( \text{Sing}(\mathcal{F}) \subset V \). Note that \( \deg(V) = m \) and \( \deg(\mathcal{F}) = m - 1 \).

**5.5. Poincaré problem and GSV-indices.** M. Brunella in [14] showed that the non-negativity of the GSV-index is an obstruction to the solution of the Poincaré problem in complex compact surfaces. We also have that the non-negativity of the GSV-index gives an obstruction to the solution of the Poincaré Problem for Pfaff systems on complex projective space. More precisely, let \( \omega \in H^0(\mathbb{P}^n, \Omega^{k}_{\mathbb{P}^n}(d + k + 1)) \) be a holomorphic Pfaff system of codimension \( k \) and degree \( d \). Let \( V \subset \mathbb{P}^n \) be a reduced complete intersection subvariety, of codimension \( k \) and multidegree \((d_1, \ldots, d_k)\), invariant by \( \omega \). Suppose that \( \text{Sing}(\omega, V) \) has codimension one in \( V \), then by Theorem 3.4 we obtain

\[
\sum_i GSV(\omega, V, S_i) \deg(S_i) = [d + k + 1 - (d_1 + \cdots + d_k)] \cdot (d_1 \cdots d_k),
\]

where \( S_i \) denotes an irreducible component of \( \text{Sing}(\omega, V) \). Therefore, if \( GSV(\omega, V, S_i) \geq 0 \), for all \( i \), we have

\[d_1 + \cdots + d_k \leq d + k + 1.
\]

In [31] we prove the following non-negativity properties for the index:

(i) If \( S_i \cap \text{Sing}(V) = \emptyset \), then \( GSV(\omega, V, S_i) \geq 0 \).

(ii) If \( V \) is smooth, then \( GSV(\omega, V, S_i) > 0 \).

Now, we give an optimal example.

**Example 5.2.** Consider the Pfaff system \( \omega \in H^0(\mathbb{P}^n, \Omega^{k}_{\mathbb{P}^n}(d + k + 1)) \) given by

\[
\omega = \sum_{0 \leq j \leq k} (-1)^j d_j f_j \, df_0 \wedge \cdots \wedge df_j \wedge \cdots \wedge df_k,
\]

where \( f_j \) is a homogeneous polynomial of degree \( d_j \). We can see that

\[d_0 + d_1 + \cdots + d_k = d + k + 1.
\]

Suppose that \( \deg(f_0) = d_0 = 1 \) and that \( V = \{f_1 = \cdots = f_k = 0\} \) is smooth. We have that \( V \) is invariant by \( \omega \) and

\[d_1 + \cdots + d_k = d + k.
\]
5.6. **Lins Neto’s examples.** In general the Poincaré problem has no solution as we can see in the following simple example. Consider the vector field

\[ v_{(p,q)}(x,y) = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y} \]

with \( p \neq q \) integers. We have that the curve \( C_{p,q} = \{ y^p - x^q = 0 \} \) is invariant by \( v_{(p,q)} \) whereas the family of vector fields \( \{ v_{(p,q)} \} \) has degree one. Such family of examples has a singularity which is of dicritical type. Under certain conditions we can give a positive answer which can be found for instance in the works [39, 62, 24]. Lins Neto in [75] has provided non-trivial examples of one-parameter families which show that Poincaré and Painlevé problems have a negative answer. Such family of foliations on \( \mathbb{P}^2 \) has small degree and has algebraic leaves of arbitrarily large degree. Lins Neto’s families of foliations are given, in affine chart, by the following polynomial vector fields

\[ v_1^2 = (4x - 9x^2 + y^2 + \alpha(2y - 4xy))(6y - 12xy + 3\alpha(x^2 - y^2)) \frac{\partial}{\partial x}; \]
\[ v_1^3 = (-x + 2y^2 - 4x^2y + x^4 - \alpha(2y - x^2 + 2xy)) \frac{\partial}{\partial x} + (y(2 - 3xy + x^3) - \alpha(3xy - x^3 + 2y^3)) \frac{\partial}{\partial y}; \]
\[ v_1^4 = ((x^3 - 1)(x - \alpha y^2)) \frac{\partial}{\partial x} + ((y^3 - 1)(x - \alpha x^2)) \frac{\partial}{\partial y}. \]

Lins Neto showed that there exists a countable and dense set of parameters \( E \subset \mathbb{P}^1 \) such that for any \( \alpha \in E \) the induced foliation \( \mathcal{F}_\alpha \) has a rational first integrals whose degrees can be chosen arbitrarily large. In these examples the genus of an algebraic invariant curve is constant and is equal to 1. For other interesting examples we refer to the papers by Picard [90, pg 298-299] and Movasati [84].

6. **Darboux-Jouanolou-Ghys theorem for Pfaff systems**

In this section we will give an idea of the proof of the Darboux-Jouanolou-Ghys theorem for Pfaff systems. Let \( X \) be a complex manifold. We denote by \( \Omega^1_X \) the sheaf of germs of closed holomorphic 1-forms and \( \mathcal{M}(X) \) is the field of meromorphic functions on \( X \).

**Theorem 6.1.** [42, 41, 68, 63, 80] Let \( \mathcal{F} \) be a Pfaff system on a compact, connected, complex manifold \( X \), induced by an \( r \)-form \( \omega \in H^0(X, \Omega^r \otimes \mathcal{L}) \). If \( \mathcal{F} \) admits

\[ \dim_{\mathbb{C}} H^1(X, \Omega^1_X) + \dim_{\mathbb{C}} (H^0(X, \Omega^{r+1} \otimes \mathcal{L}) / \omega \wedge H^0(X, \Omega^1_{cl})) + r + 1 \]

invariant irreducible analytic hypersurfaces, then \( \mathcal{F} \) admits a meromorphic first integral.

Denote by \( \text{Div}(X, \mathcal{F}) \) the abelian group of divisors on \( X \) which are invariant by \( \mathcal{F} \). We have the homomorphism

\[ \text{Div}(X, \mathcal{F}) \rightarrow \text{Pic}(X) \]

\[ \sum \lambda^\alpha L^\alpha \rightarrow \bigotimes_\alpha [L^\alpha]^{\otimes \lambda^\alpha}, \lambda^\alpha \in \mathbb{Z}. \]

Since \( \text{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*) \), logarithmic differentiation defines a homomorphism

\[ H^1(X, \mathcal{O}_X^*) \rightarrow H^1(X, \Omega^1_{cl}) \]

\[ g \mapsto \frac{dg}{g}. \]

Composition of (21) and (22) gives a \( \mathbb{C} \)-linear map

\[ \Psi : \text{Div}(X, \mathcal{F}) \otimes \mathbb{C} \rightarrow H^1(X, \Omega^1_{cl}) \]
which is expressed, in terms of a sufficiently fine open cover \( \{ U_i \}_{i \in \Lambda} \) of \( X \) by: if \( L^\alpha \) is defined by \( f_i^\alpha = 0 \) in \( U_i \) and, in \( U_{ij} = U_i \cap U_j, f_i^\alpha = g_{ij}^\alpha f_j^\alpha \),

\[
\Psi \left( \sum_\alpha \lambda^\alpha L^\alpha \right) = \left[ \sum_\alpha \lambda^\alpha \frac{dg_{ij}^\alpha}{g_{ij}^\alpha} \right].
\]

Consider the kernel of \( \Psi \). Thus, \( \sum_\alpha \lambda^\alpha L^\alpha \in \ker \Psi \) implies that there are closed holomorphic 1-forms \( \omega_i \) such that in \( U_{ij}, \)

\[
\sum_\alpha \lambda^\alpha \frac{dg_{ij}^\alpha}{g_{ij}^\alpha} = \omega_j - \omega_i.
\]

But this says that

\[
\sum_\alpha \lambda^\alpha \frac{df_i^\alpha}{f_i^\alpha} + \omega_i = \sum_\alpha \lambda^\alpha \frac{df_j^\alpha}{f_j^\alpha} + \omega_j.
\]

These glue together to give a global closed meromorphic 1-form \( \eta \) on \( X \), defined up to addition of a global closed holomorphic 1-form \( \rho \).

Since \( L^\alpha \) is \( \omega \)-invariant and hence \( (\omega \wedge df_i^\alpha)_{(f_i^\alpha = 0)} \equiv 0, \omega \wedge \eta \) is a holomorphic \( r + 1 \)-form, defined up to addition of \( \omega \wedge \rho \), with \( \rho \) a global closed holomorphic 1-form. Therefore, the \( \mathbb{C} \)-linear map

\[
\Theta : \ker(\Psi) \rightarrow H^0(X, \Omega^{r+1}_X \otimes \mathcal{L})/\omega \wedge H^0(X, \Omega^1_{cl})
\]

\[
\sum_\alpha \lambda^\alpha L^\alpha \quad \mapsto \quad \omega \wedge (\eta + \rho)
\]

is well-defined.

**Lemma 6.2.** [69, Lemme 3.1.1, p. 102] Let \( \mathcal{M}^1 \) be the sheaf of germs of meromorphic 1-forms on \( X \) and \( \mathcal{D}^1_X := \mathcal{M}^1/\Omega^1_X \). Then, the \( \mathbb{C} \)-linear map

\[
\text{Div}(X, \mathcal{F}) \otimes \mathbb{C} \rightarrow \mathcal{D}^1_X
\]

\[
\sum_\alpha \lambda^\alpha \cdot L^\alpha \quad \mapsto \quad \sum_\alpha \lambda^\alpha \frac{df_i^\alpha}{f_i^\alpha}
\]

is injective provided the divisors have no common factor.

Suppose that \( \mathcal{F} \) admits at least

\[
\dim_{\mathbb{C}} H^1(X, \Omega^1_X) + \dim_{\mathbb{C}} \left( H^0(X, \Omega^{r+1}_X \otimes \mathcal{L})/\omega \wedge H^0(X, \Omega^1_{cl}) \right) + r + 1
\]

invariant irreducible analytic hypersurfaces. From (23) we have

\[
\dim_{\mathbb{C}} \ker(\Psi) \geq \dim_{\mathbb{C}} (H^0(X, \Omega^{r+1}_X \otimes \mathcal{L})/\omega \wedge H^0(X, \Omega^1_{cl})) + r + 1.
\]

Hence \( \dim_{\mathbb{C}} \ker(\Theta) \geq r + 1 \) and \( \ker(\Theta) \) is non trivial. Now, given a non-zero element \( \lambda \in \ker(\Theta) \), we can choose \( L^1, \ldots, L^k \in \text{Div}(X, \mathcal{F}) \) and \( \lambda^1, \ldots, \lambda^k \in \mathbb{C}^* \) such that

i) \( x = \sum_{\alpha=1}^k \lambda^\alpha L^\alpha \in \ker(\Psi) \setminus \{0\} \).

ii) \( \Theta(x) = \sum_{\alpha=1}^k \lambda^\alpha \frac{df_i^\alpha}{f_i^\alpha} \in H^0(X, \Omega^{r+1}_X \otimes \mathcal{L})/\omega \wedge H^0(X, \Omega^1_{cl}) \).

Using (26) we have that there exists \( \mu = (\mu_i) \in \mathcal{D}^0(X, \Omega^1_{cl}) \) such that, in \( U_i, \)

\[
\omega \wedge \left( \omega_i + \sum_{\alpha=1}^k \lambda^\alpha \frac{df_i^\alpha}{f_i^\alpha} \right) = \omega \wedge \mu_i,
\]
which amounts to

\[(32) \quad \omega \wedge \left( \varpi_i - \mu_i + \sum_{\alpha=1}^k \lambda_\alpha \frac{df_\alpha}{f_\alpha} \right) = 0\]

in each \(U_i\). Thus we get a global closed meromorphic 1-form \(\tilde{\xi}\) with

\[(33) \quad \tilde{\xi}_{|U_i} = \varpi_i - \mu_i + \sum_{\alpha=1}^k \lambda_\alpha \frac{df_\alpha}{f_\alpha}\]

such that

\[(34) \quad \omega \wedge \tilde{\xi} = 0.\]

In this way we can construct \(r + 1 > 1\) global closed meromorphic 1-forms \(\tilde{\xi}_1, \ldots, \tilde{\xi}_{r+1}\) on \(X\) such that

\[(35) \quad \omega \wedge \tilde{\xi}_j = 0, \quad 1 \leq j \leq r + 1.\]

Define \(\alpha_1 = \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_r\) and \(\alpha_2 = \tilde{\xi}_2 \wedge \cdots \wedge \tilde{\xi}_{r+1}\) and remark that \(|\alpha_1|_\infty \neq |\alpha_2|_\infty\), where \(|\cdot|_\infty\) denotes the set of poles. We have the following possibilities:

**Case 1.** \(\alpha_1 \neq 0\) and \(\alpha_2 \neq 0\).

**Case 2.** \(\alpha_1 = 0\) or \(\alpha_2 = 0\).

**Case 1:** Since \(|\alpha_1|_\infty \neq |\alpha_2|_\infty\) we have that \(\alpha_1\) and \(\alpha_2\) are linearly independent over \(\mathbb{C}\). We can prove that there exist \(R_1, R_2 \in \mathcal{M}(X)\) such that \(\omega = R_1 \alpha_i, i = 1, 2\). This gives us that

\[d \left( \frac{R_1}{R_2} \right) \wedge \omega = 0.\]

Moreover, since \(\alpha_1, \alpha_2\) are linearly independent over \(\mathbb{C}\), then \(\frac{R_1}{R_2}\) is not constant. This shows that the meromorphic function \(\frac{R_1}{R_2}\) is a first integral for \(\omega\).

**Case 2:** Suppose without loss of generality that \(\alpha_1 = 0\). Let \(m\) be the largest integer such that \(\tilde{\xi}_1, \ldots, \tilde{\xi}_m\) are linearly independent over \(\mathcal{M}(X)\). Then,

\[(36) \quad \tilde{\xi}_{m+1} = \sum_{i=1}^m R_i \tilde{\xi}_i\]

with \(R_1, \ldots, R_m \in \mathcal{M}(X)\). Since \(\tilde{\xi}_j\) is closed, for \(i = 1, \ldots, m + 1\), we get

\[(37) \quad 0 = \sum_{i=1}^m dR_i \wedge \tilde{\xi}_i.\]

Then, for each \(j = 1, \ldots, m\), multiplying (37) by \(\tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_j \wedge \cdots \wedge \tilde{\xi}_m\) we obtain

\[0 = \sum_{i=1}^m dR_i \wedge \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_j \wedge \cdots \wedge \tilde{\xi}_m = (-1)^{j+1} dR_j \wedge \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_m.\]
there exists that if either is a foliation with infinitely many compact leaves. In general, if foliations which are pull-backs of foliations on surfaces by rational maps. A Darboux-Jouanolou-Ghys type Theorem by providing a characterization of codimension one to the fibers of a rational map.

Here which admits (iii) If \( s \geq \ell + 2 \) then \( \mathcal{F} \) has a rational first integral.

Moreover, from (36) and Lemma 6.2 there exists \( i_0 \in \{1, \ldots, m\} \) such that \( R_{i_0} \) is not constant. That is, \( R_{i_0} \) is a meromorphic first integral for \( \omega \). This proves the theorem.

6.1. Comments. We observe that if \( \mathcal{F} \) is a codimension one distribution, defined by \( \omega \in H^0(X, \Omega^1_X \otimes \mathcal{L}) \) which admits invariant irreducible analytic hypersurfaces, then \( \mathcal{F} \) has a meromorphic first integral, in particular \( \mathcal{F} \) is a foliation with infinitely many compact leaves. In general, if \( \mathcal{F} \) has codimension \( r \), then it follows from the proof of Theorem 6.1 that if either \( \alpha_1 = \xi_1 \wedge \cdots \wedge \xi_r \neq 0 \) or \( \alpha_2 = \xi_2 \wedge \cdots \wedge \xi_{r+1} \neq 0 \), then the Pfaff system \( \mathcal{F} \) is given by the meromorphic decomposable closed form \( \omega/R_i = \alpha_i \), where \( R_i \) is a meromorphic function. This shows us that the Pfaff system \( \mathcal{F} \) is integrable and it is given by a global decomposable meromorphic differential \( r \)-form which is a wedge product of closed logarithmic 1-forms.

In [64] Gomez–Mont provides an extension of a theorem due to R. D. Edwards, K. C. Millett and D. Sullivan concerning foliations with all leaves algebraic, then \( \mathcal{F} \) is tangent to the fibers of a rational map.

In [88] Pereira and Spicer prove an improvement for the Darboux-Jouanolou theorem in the codimension one case. They proved that if \( \mathcal{F} \) is a codimension one foliation on a projective manifold \( X \), \( s \) is the number of hypersurfaces invariant by \( \mathcal{F} \) and

\[
\ell := \dim NS(X) + \dim \mathbb{C} H^0(X, T \mathcal{F}^*) - \dim \mathbb{C} H^0(X, \Omega^1_X),
\]

then the following assertions hold true.

i) If \( s \geq \ell \) then \( \mathcal{F} \) is transversely affine.
ii) If \( s \geq \ell + 1 \) then \( \mathcal{F} \) is defined by a closed logarithmic 1-form.
iii) If \( s \geq \ell + 2 \) then \( \mathcal{F} \) has a rational first integral.

Here \( NS(X) \) denotes the Neron-Severi group of \( X \). In the same work, Pereira and Spicer show a Darboux-Jouanolou-Ghys type Theorem by providing a characterization of codimension one foliations which are pull-backs of foliations on surfaces by rational maps.

We refer to [96] for general results on transversely affine and projective structures for foliations, where Scardua also has considered the Poincaré problem for codimension one foliations, see [96, Theorem 4.2 and 4.3].

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