Some Prospects for the Determination of the Unitarity Triangle before the LHC Era*

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Note to the preprint reader: Due to limitations of the plotting program the axis labels in figs. 2 and 3 are missing the bars on $\rho$ and $\eta$.

Abstract

Anticipating improved determinations of $m_t$, $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $B_K$ and $F_B\sqrt{B_B}$ in the next five years we make an excursion into the future in order to find a possible picture of the unitarity triangle, of quark mixing and of CP violation around the year 2000. We then analyse what impact on this picture will result from the measurements of the four possibly cleanest quantities: $BR(K^+ \to \pi^+\nu\bar{\nu})$, $x_d/x_s$, $\sin(2\alpha)$ and $\sin(2\beta)$. In the course of our investigations we extend the analysis of the unitarity triangle beyond the leading order in the Wolfenstein parameter $\lambda$. We will also shortly present the status of direct CP violation in $K \to \pi\pi$ and $K_L \to \pi^0e^+e^-$. 

1. CKM Matrix and Unitarity Triangle

1.1. Wolfenstein Parametrization Beyond Leading Order

In the Standard Model (SM) with three fermion generations, CP violation arises from a single phase in the unitary $3 \times 3$ Cabibbo-Kobayashi-Maskawa (CKM) matrix. For phenomenological applications it is useful to expand each element of the CKM matrix as a power series in the small parameter $\lambda = |V_{us}| = 0.22$. For the leading order in $\lambda$ the result is

$$V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \frac{\lambda}{2} - \frac{\lambda^2}{2} & A\lambda^3(\varrho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda(1 - \varrho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + O(\lambda^4)$$

(1)

This parametrization being an expansion in $\lambda$ respects unitarity of the CKM matrix only approximately up to terms of order $O(\lambda^5)$. With e.g. LHC expected to test unitarity to a very high precision one has to extend the expansion (1) to higher order terms in $\lambda$. As always with next-to-leading order expansions the definition of higher terms is not unique. A particularly nice form is to relate the parameters ($\lambda$, $A$, $\varrho$, $\eta$) of the approximate Wolfenstein parametrization to the parameters $s_{ij}$ and $\delta$ of the fully unitary standard parametrization $\mathcal{B}$ of the...
2. The UT from Present Day Experiments

2.1. Tree Level B-Decays

Measurements of tree level B-decays can be used to derive the CKM elements $|V_{cb}|$, $|V_{ub}/V_{cb}|$. This then allows to determine

$$R_b = \frac{|V_{ub} V_{cb}^*|}{|V_{cd} V_{cb}^*|} = \sqrt{\bar{\eta}^2 + \bar{\rho}^2} = \left(1 - \frac{\lambda^2}{2}\right) \frac{1}{\lambda} \left|\frac{V_{ub}}{V_{cb}}\right|$$

which represents a circle centered around $(0,0)$ in the complex $(\bar{\rho}, \bar{\eta})$ plane. Thus $R_b$ is simply the length $AB$ in the rescaled UT of fig. 1.

2.2. Indirect CP Violation

The usual box diagram calculation together with the experimental value for $\varepsilon_K$ from $K^0$-$\bar{K}^0$ mixing specifies a hyperbola in the $(\bar{\rho}, \bar{\eta})$ plane with $\bar{\eta} > 0$.

Here $S(x_i)$, $S(x_i, x_j)$, $x_i = n_i^2/M_t^2$ are the Inami-Lim functions, $B_K$ is the renormalization group invariant non-perturbative parameter describing the size of $<K^0|\bar{s}d|\bar{V}_{-A}(\bar{s}d)V_{-A}K^0>$ and $\eta_1 = 1.1$, $\eta_2 = 0.57$.

Thus we can represent (3) as a triangle, the UT, in the complex $(\bar{\rho}, \bar{\eta})$ plane. This is shown in fig. 1.

Figure 1. Unitarity triangle in the complex $(\bar{\rho}, \bar{\eta})$ plane.

We observe that beyond the leading order in $\lambda$ the point ‘A’ does not correspond to $(\bar{\rho}, \bar{\eta})$. Clearly within 3% accuracy $\bar{\rho} = \rho$ and $\bar{\eta} = \eta$. Yet in the distant future the accuracy of experimental results and theoretical calculations may improve considerably so that the more accurate formulation given here will be appropriate. For instance the experiments at LHC should measure $\sin(2\beta)$ to an accuracy of $(2\sim3)$

With fig. 1 it is then a matter of simple trigonometry to calculate $\sin(2\phi_i)$ in terms of $(\bar{\rho}, \bar{\eta})$ and vice versa.

In sects. 2.3 we will now summarize the phenomenological analysis of the UT presented in ref. [8].

2.3. $B^0$-$\bar{B}^0$ Mixing

The experimental knowledge of the $B^0_d - \bar{B}^0_d$ mixing described by the parameter $x = \Delta M/\Gamma_B$ determines $|V_{td}|$. This then specifies via

$$R_t = \frac{|V_{td} V_{cb}^*|}{|V_{cd} V_{cb}^*|} = \sqrt{(1-\bar{\rho})^2 + \bar{\rho}^2} = \frac{1}{\lambda} \left|\frac{V_{td}}{V_{cb}}\right|$$

a circle centered around $(1,0)$ in the complex $(\bar{\rho}, \bar{\eta})$ plane. Here $R_t$ is simply the length $AB$ in the rescaled UT of fig. 1.

All the QCD corrections to $\varepsilon_K$, $B^0$-$\bar{B}^0$ mixing and $BR(K^+ \rightarrow \pi^+ \nu \bar{\nu})$ used here include except for $\eta_3$ the next-to-leading order. Hence, in all formulae of this paper $m_t$ corresponds to the running top quark mass in the $\overline{MS}$ scheme evaluated at $m_t$ i.e. $m_t = \overline{m}_t(m_t)$. The physical top quark mass as the pole of the renormalized propagator is for the range of $m_t$ considered here by $(7 \pm 1)$ GeV higher than $m_t$.

Using eqs. (3)–(8) together with present day and envisioned future ranges of input parameters as of tab. 1 and $\Lambda_{\overline{MS}} = 300$ MeV, $m_c = 1.3$ GeV, one can determine the allowed ranges for the upper corner ‘A’ of the UT and make predictions for various quantities. The result is shown in fig. 2 and tab. 2 respectively.
Looking at tab. one sees that by the year 2000 one can expect predictions for \( \sin(2\beta) \), \( |V_{td}| \), \( BR(K^+ \to \pi^+\nu\bar{\nu}) \) good to \((10-15\%)\) and for \( x_s \) up to \( \pm 20\% \). Thus a measurement of \( BR(K^+ \to \pi^+\nu\bar{\nu}) \) or \( x_s \) at the level of \( \pm 10\% \) could serve as a possible test of the corresponding SM predictions. Huge uncertainties for predicting \( \sin(2\alpha) \) and \( \sin(2\gamma) \) remain however, even with improved input in the future. Turning the argument around, this signals that a measurement of one of these angles would allow to put stringent constraints on some of the input parameters of tab. \ref{tab:input_parameters} e.g. the non-perturbative ones \( B_K \) and \( F_{B_d}/\sqrt{B_{B_d}} \).

### 3. The UT from CP Violating B-Asymmetries

Measuring the CP-asymmetries in neutral B-decays will give the definitive answer whether the CKM description of CP violation is correct. Assuming that this is in fact the case, we want to investigate the impact of the measurements of \( \sin(2\phi_i) \) on the determination of the unitarity triangle. Since in the rescaled triangle of fig. \ref{fig:UT} one side is known, it suffices to measure two angles to determine the triangle completely.

With the CP-asymmetries simply given by

\[
A_{CP}(B^0 \to \psi K_S) = -\sin(2\beta) \frac{x_d}{1 + x_d^2} \quad (9)
\]

\[
A_{CP}(B^0 \to \pi^+\pi^-) = -\sin(2\alpha) \frac{x_d}{1 + x_d^2} \quad (10)
\]

one can determine \( \sin(2\beta) \) without any theoretical uncertainties from measuring the CP-asymmetry in \( B^0 \to \psi K_S \), while for \( \sin(2\alpha) \) the measurement of several other channels is required in order to remove the penguin contributions.

Assuming a measurement of \( \sin(2\beta) \) and \( \sin(2\alpha) \) to give

\[
\sin(2\beta) = \left\{ \begin{array}{ll}
0.60 \pm 0.18 & \text{(a) HERA-B} \ [13] \\
0.60 \pm 0.06 & \text{(b) SLAC} \ [14] 
\end{array} \right. \quad (11)
\]

\[
\sin(2\alpha) = \left\{ \begin{array}{ll}
-0.20 \pm 0.10 & \text{(I)} \\
0.10 \pm 0.10 & \text{(II) SLAC} \ [14] \\
0.70 \pm 0.10 & \text{(III)} 
\end{array} \right. \quad (12)
\]

with the errors expected from different experiments indicated, one can again determine the UT in \((\bar{\rho}, \bar{\eta})\) space. The result is shown in fig. \ref{fig:UT}. Here the solid line labeled ‘superweak’ reflects the implicit relation between \( \bar{\rho} \) and \( \bar{\eta} \) in the superweak scenario where \( \sin(2\beta) = -\sin(2\alpha) \).

Comparing figs. \ref{fig:UT} and \ref{fig:UT} it is obvious that a combined measurement of \( \sin(2\beta) \) and \( \sin(2\alpha) \) at the expected precision will have a large impact on the determination of the UT and CKM parameters (For a discussion of \( \sin(2\beta) \) and \( \sin(2\gamma) \) see ref. \cite{3}). E.g. using \( \sin(2\beta) = 0.6 \pm 0.06 \), \( \sin(2\alpha) = 0.1 \pm 0.1 \) and range (II) of tab. \ref{tab:input_parameters} for \( |V_{cb}| \), \( x_d \) and \( m_t \) one obtains \( \sin(2\gamma) = 0.54 \pm 0.12 \),
Determination of the unitarity triangle in the \( (\bar{\theta}, \bar{\beta}) \) plane by measuring \( \sin(2\beta) \) and \( \sin(2\alpha) \) as of eqs. (11) and (12), respectively. For \( \sin(2\alpha) \) we always find two solutions in \( (\bar{\theta}, \bar{\eta}) \) and for \( \sin(2\beta) \) we only use the solution consistent with \( |V_{ub}/V_{cb}| \leq 0.1 \).

\[
|V_{ud}| = (8.8 \pm 0.4) \times 10^{-3}, \quad x_s = 16.3 \pm 1.3 \quad \text{and} \quad BR(K^+ \rightarrow \pi^+\nu\bar{\nu}) = (1.01 \pm 0.11) \times 10^{-10}.
\]

The ability to make predictions for \( |V_{ud}|, x_s \) and \( BR(K^+ \rightarrow \pi^+\nu\bar{\nu}) \) to an accuracy of \( \pm(5-10)\% \) stems from the absent or small theoretical uncertainties in eqs. (8) and (11), as well as from the expected high precision for the measurement of CP violating B-asymmetries. However, this predictive power can only be achieved through a measurement of both \( \sin(2\beta) \) and \( \sin(2\alpha) \). Finally, we note that the predictions resulting from a measurement of CP violating B-asymmetries are generally more precise than those using \( \varepsilon_K \), \( x_d \), \( |V_{cb}| \) and \( |V_{ub}/V_{cb}| \) as input data.

4. \( \sin(2\beta), \sin(2\alpha) \) from Indirect CP Violation and \( B^0-\bar{B}^0 \) Mixing versus a Direct Measurement

It is useful to combine the results of sects. II and III by making the customary \( \sin(2\beta) \) versus \( \sin(2\alpha) \) plot \( (\bar{\theta},\bar{\eta}) \). This plot demonstrates very clearly the correlation between \( \sin(2\alpha) \) and \( \sin(2\beta) \). The allowed ranges for \( \sin(2\alpha) \) and \( \sin(2\beta) \) corresponding to the choices of parameters in tab. I are shown in fig. 4 together with the results of the independent measurements of \( \sin(2\beta) = 0.60 \pm 0.06 \) and \( \sin(2\alpha) \) given by (12). The latter are represented by dark shaded rectangles. The black rectangles illustrate the accuracy of future LHC measurements (\( \Delta \sin(2\alpha) = \pm 0.04, \Delta \sin(2\beta) = \pm 0.02 \)). We also show the results of an analysis in which the accuracy of various parameters is as in range (II) of tab. I but with the central values modified. Parameter range (IV) is given by

\[
|V_{cb}| = 0.038 \pm 0.002 \quad |V_{ub}/V_{cb}| = 0.08 \pm 0.01 \quad B_K = 0.70 \pm 0.07 \quad \sqrt{B_{B_d}F_{B_d}} = (185 \pm 15) \text{ MeV} \quad x_d = 0.72 \pm 0.04 \quad m_t = (165 \pm 7) \text{ GeV}
\]

In addition we show the prediction of superweak theories which in this plot is represented by a straight line.

There are several interesting features visible on this plot: First, the impact of the direct measurements of \( \sin(2\beta) \) and \( \sin(2\alpha) \) is clearly visible in fig. 4. Next, in cases (III) and (IV) we have examples where the measurements of \( \sin(2\alpha) \) are incompatible with the predictions coming from \( \varepsilon_K \) and \( B^0-\bar{B}^0 \) mixing. This would be a signal for physics beyond the standard model. The measurement of \( \sin(2\alpha) \) is essential for this. Furthermore, case (IV) shows that for a special choice of parameters the predictions for the asymmetries coming...
from $\varepsilon_K$, $B^0 - \bar{B}^0$ mixing, $|V_{cb}|$ and $|V_{ub}/V_{cb}|$ can be quite accurate when these four constraints can only be satisfied simultaneously in a small area of the $(\bar{\rho}, \bar{\eta})$ space. Decreasing $|V_{cb}|$, $|V_{ub}/V_{cb}|$ and $m_t$ and increasing $F_B$ would make the allowed region in the case (IV) even smaller.

Finally, we also observe that the future measurements of B-asymmetries and the improved ranges for the parameters relevant for $\varepsilon_K$ and $B^0 - \bar{B}^0$ mixing will probably allow to rule out the superweak models. This was also already indicated by fig. 3 (III).

5. Direct CP Violation in $K \rightarrow \pi\pi$ and $K_L \rightarrow \pi^0 e^+ e^-$

$Re(\varepsilon'/\varepsilon)$ measures the ratio of direct to indirect CP violation in $K \rightarrow \pi\pi$ decays. The short distance QCD corrections to $\varepsilon'/\varepsilon$ have been calculated at the next-to-leading order level [16, 17]. The result of these analyses can be summarized in an analytic formula for $Re(\varepsilon'/\varepsilon)$ as a function of $m_t$, $\Lambda_{\overline{MS}}$, $m_b$, hadronic matrix element parameters $B_0$, $B_8$ and CKM elements [18]. A simplified version of this formula is given by

$$Re(\varepsilon'/\varepsilon) = 12 \times 10^{-4} \left[ \frac{\eta 5^2 A^2}{1.7 \times 10^{-4}} \right] \frac{150 \text{ MeV}}{m_b(m_c)} \left[ B_6 - Z(x_1) B_8 \right],$$

(14)

where $Z(x_1) = 0.175 \cdot x_1^{0.93}$. Eq. (14) clearly shows that in the SM $\varepsilon'/\varepsilon$ is governed by QCD ($B_6$) and electroweak ($B_8$) penguin contributions. For $m_t = (170 \pm 10) \text{ GeV}$ and using $\varepsilon_K$-analysis to determine $\eta$ one finds $16$ $Re(\varepsilon'/\varepsilon) = (6 \pm 4) \times 10^{-4}$ for $B_8 = B_6 \approx 1$ (lattice, 1/N expansion). For $B_8 \approx 2$, $B_6 \approx 1$ (QCD penguin domination) values as high as $Re(\varepsilon'/\varepsilon) = (15 \pm 5) \times 10^{-4}$ are possible. Thus the remaining theoretical uncertainty stemming from hadronic parameters somehow resembles the still existing experimental discrepancy between E731 $Re(\varepsilon'/\varepsilon) = (7.4 \pm 5.9) \times 10^{-4}$ [19] and NA31 $Re(\varepsilon'/\varepsilon) = (23 \pm 7) \times 10^{-4}$ [20].

For the decay $K_L \rightarrow \pi^0 e^+ e^-$ a recent next-to-leading order analysis [21] of the directly CP violating contribution indicates this part of the amplitude to be the dominant one. One obtains $BR(K_L \rightarrow \pi^0 e^+ e^-)_{\text{air}} = (6 \pm 3) \times 10^{-12}$ [21] and $BR(K_L \rightarrow \pi^0 e^+ e^-)_{\text{indir}} \leq 1.6 \times 10^{-12}$, $BR(K_L \rightarrow \pi^0 e^+ e^-)_{\text{cons}} = (1.0 \pm 0.8) \times 10^{-12}$ for the indirectly CP violating and CP conserving contributions, respectively [22, 23]. The present experimental bound is $BR(K_L \rightarrow \pi^0 e^+ e^-) \leq 4.3 \times 10^{-9}$ [24].

6. Summary and Conclusions

We have shown that in order to compete with the accuracy expected from LHC for the determination of the UT one needs to extend the usual Wolfenstein parametrization of the CKM matrix to the next-to-leading order in the expansion in terms of $\lambda$. To this end we have proposed a form of the next-to-leading order expansion for which the UT at next-to-leading order in $\lambda$ nicely resembles the UT in leading order when coordinates are expressed in $(\bar{\rho}, \bar{\eta})$ instead of the usual ones $(\rho, \eta)$.

Our analysis investigated how well the UT can possibly be determined around the year 2000 from data on $\varepsilon_K$, $B^0 - \bar{B}^0$ mixing, $|V_{cb}|$ and $|V_{ub}/V_{cb}|$. We have found that along this line it will be possible to make predictions for $|V_{td}|$, $\sin(2\beta)$ and $BR(K^+ \rightarrow \pi^+ \nu\bar{\nu})$ up to an error of $(\pm(10-15)\%)$. However, for $x_s$ and $\sin(2\alpha)$, $\sin(2\gamma)$ there will remain sizeable/huge uncertainties, respectively. This results from theoretical uncertainties being present already in the determination of some of the input parameters of this approach.

On the other hand, the future determination of $\sin(2\alpha)$ and $\sin(2\beta)$ from CP violating B-asymmetries at HERA-B, SLAC, KEK being (almost) free of theoretical uncertainties turns out to have an impressive impact on our knowledge of the UT. Along this line it will e.g. be possible to predict $|V_{td}|$, $x_s$ and $BR(K^+ \rightarrow \pi^+ \nu\bar{\nu})$ up to an error of $(\pm(5-10)\%)$. Future LHC B-physics experiments around the year 2005 will refine these studies as evident from fig. 3 and ref. [4].

Any discrepancy found between the indirect determination of $\sin(2\alpha)$, $\sin(2\beta)$ from $\{\varepsilon_K$, $x_d$, $|V_{cb}|$, $|V_{ub}/V_{cd}|\}$ and a direct measurement in CP violating B-asymmetries would signal new physics beyond the SM.

Finally, we shortly summarized the status of direct CP violation in $K \rightarrow \pi\pi$ where for $\varepsilon'/\varepsilon$ both experiment and the non-perturbative part of theory need some improvements. While direct CP violation is known to give only a small contribution to the whole amplitude in $K \rightarrow \pi\pi$, our recent analysis of the direct CP violating part in the decay $K_L \rightarrow \pi^0 e^+ e^-$ indicates that there this contribution seems to be the dominant one.

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