Entanglement thermodynamics

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Abstract

Entanglement entropy is a statistical entropy measuring information loss due to coarse-graining corresponding to a spatial division of a system. In this paper we construct a thermodynamics (entanglement thermodynamics) which includes the entanglement entropy as the entropy variable, for a massless scalar field in Minkowski, Schwarzschild and Reissner-Nordström spacetimes to understand the statistical origin of black-hole thermodynamics. It is shown that the entanglement thermodynamics in Minkowski spacetime differs significantly from black-hole thermodynamics. On the contrary, the entanglement thermodynamics in Schwarzschild and Reissner-Nordström spacetimes has close relevance to black-hole thermodynamics.

1 Introduction

In the classical theory of black-hole there appears a thermodynamical relation among the parameters describing black-holes. We call it black-hole thermodynamics. For example, for a one-parameter family of Schwarzschild black-holes parameterized by their mass $M$, the same relation as the first law of thermodynamics holds:

$$dE_{BH} = T_{BH} dS_{BH},$$

where $E_{BH}$, $S_{BH}$ and $T_{BH}$ are defined by

$$E_{BH} \equiv M,$$
$$S_{BH} \equiv \frac{A_H}{4l_{pl}^2},$$
$$T_{BH} = \frac{1}{4\pi R_H}.$$

Here $R_H = 2Ml_{pl}^2$ and $A_H = 4\pi R_H^2$ are the area radius and the area of the event horizon. It is well-known that in general the area of the event horizon does not decrease in time classically. Hence $S_{BH}$ defined by (3) does not decrease in time as thermodynamical entropy (the second law of black-hole). It can also be shown that sum of $S_{BH}$ and entropy of matter field outside the horizon does not decrease in time semi-classically (the generalized second law). Thus $S_{BH}$ can be regarded as an entropy of a black-hole. Moreover, a black-hole with surface gravity $\kappa$ emits thermal radiation at the so-called Hawking temperature $\kappa/2\pi$. For the Schwarzschild black-hole this Hawking temperature is given by $T_{BH}$. From these facts $S_{BH}$ and $T_{BH}$ are called black-hole entropy and black hole temperature, respectively.

There have been many attempts to understand the statistical origin of the black-hole entropy. Among them a strong candidate is the so-called entanglement entropy. In this paper we attempt to construct a thermodynamics which includes the entanglement entropy as the entropy. We call it thermodynamics of entanglement or entanglement thermodynamics.

2 Basic ingredients of the entanglement thermodynamics

First we review the basic concepts of entanglement thermodynamics. Let $\mathcal{F}$ be a Hilbert space constructed from two Hilbert spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ as

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2,$$
where $\otimes$ denotes a tensor product followed by a suitable completion. From an element $u$ of $\mathcal{F}$ with unit norm we construct an operator $\rho$ (‘density operator’) by

$$\rho v = (u,v)u \quad \forall v \in \mathcal{F},$$

(4)

where $(u,v)$ is the inner product which is antilinear with respect to $u$. From $\rho$ we define so-called reduced density operators $\rho_{1,2}$ by

$$\rho_1 y = \sum_{i,j} e_i (e_i \otimes f_j, \rho (y \otimes f_j)) \quad \forall y \in \mathcal{F}_1,$$

$$\rho_2 z = \sum_{i,j} f_j (e_i \otimes f_j, \rho (e_i \otimes z)) \quad \forall z \in \mathcal{F}_2,$$

(5)

where $\{e_i\}$ and $\{f_j\}$ are orthonormal bases of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The entanglement entropy is defined by $S_{ent} = S[\rho_2]$ or $S_{ent} = S[\rho_1]$ or $S_{ent} = S[\rho_1 \otimes \rho_2]$, where

$$S[\rho_{1,2}] = -\text{Tr}[\rho_{1,2} \ln \rho_{1,2}],$$

$$S[\rho_1 \otimes \rho_2] = S[\rho_1] + S[\rho_2].$$

(6)

Since it can be shown that $S[\rho_1] = S[\rho_2]$ in general, the three options for $S_{ent}$ are identical up to the factor 2. For definition of the entanglement energy we consider the following four options: $E_{ent} = \langle : H_1 : \rangle$ or $E_{ent} = \langle : H_2 : \rangle$ or $E_{ent} = \langle : H_1 + : H_2 : \rangle$ or $E_{ent} = \langle : H_{tot} : \rangle_{\rho_1 \otimes \rho_2}$, where

$$\langle : H_{1,2} : \rangle = \text{Tr}[\rho : H_{1,2}],$$

$$\langle : H_{tot} : \rangle_{\rho_1 \otimes \rho_2} = \text{Tr}[\rho_1 \otimes \rho_2 : H_{tot}].$$

(7)

Here the total hamiltonian $H_{tot}$ is assumed to be decomposed as $H_{tot} = H_{1} + H_{2} + H_{int}$, and $\cdots$ denotes the normal ordering. Finally entanglement temperature $T_{ent}$ is obtained by imposing the first law of entanglement thermodynamics:

$$T_{ent}dS_{ent} = dE_{ent}.$$  

(8)

### 3 Discretized theory of a scalar field

In this paper we consider a massless real scalar field described by the action

$$I = -\frac{1}{2} \int dx^4 \sqrt{-g} \partial \mu \partial \phi \phi ,$$

(9)

where the background geometry is fixed to be a spherically symmetric static spacetime with the metric

$$ds^2 = -N(\rho)^2 dt^2 + d\rho^2 + r(\rho)^2 (d\theta + \sin^2 \theta d\psi^2).$$

(10)

For this system we calculate the entanglement entropy and the entanglement energy to construct entanglement thermodynamics using the methods developed in Ref. 3 and Ref. 8. Those methods are both based on the following form of the Hamiltonian describing a discrete system $\{q^A\} (A = 1, 2, \cdots, n_{tot})$:

$$H_0 = \frac{1}{2a} \delta^{AB} p_A p_B + \frac{1}{2} V_{AB} q^A q^B,$$

(11)

where $p_A$ is a momentum conjugate to $q^A$ and $a$ is a fundamental length of the system. For this discrete system it is easy to divide the whole Hilbert space $\mathcal{F}$ into the form $\mathcal{F} = \otimes \mathcal{F}_1 \otimes \mathcal{F}_2$: $\mathcal{F}_1$ is defined as a Fock space constructed from $\{q^A\} (A = 1, 2, \cdots, n_B)$; $\mathcal{F}_2$ is defined as a Fock space constructed from $\{q^\alpha\} (\alpha = n_B + 1, \cdots, n_{tot})$. In order to apply this scheme to our problem we have to construct a discretized theory of the scalar field whose Hamiltonian is of the form (11).
First we expand the field $\phi$ in terms of the spherical harmonics as

$$
\phi(\rho, \theta, \psi) = \sum_{l,m} \frac{N^{1/2}}{r} \phi_{lm}(\rho) Z_{lm}(\theta, \psi), \tag{12}
$$

where $Z_{lm} = \sqrt{2} \Re Y_{lm}$ for $m > 0$, $\sqrt{2} \Im Y_{lm}$ for $m < 0$, and $Z_{l0} = Y_{l0}$. Then the Hamiltonian corresponding to the Killing energy for the action (9) is decomposed into a direct sum of contributions from each harmonics component $H_{lm}$ as

$$
H = \sum_{lm} H_{lm}. \tag{13}
$$

Here $H_{lm}$ is given by

$$
H_{lm} = \frac{1}{2} \int d\rho \left[ \pi_{lm}^2 + N r^2 \left( \frac{N^{1/2}}{r} \phi_{lm} \right)^2 + l(l+1) \left( \frac{N \phi_{lm}}{r} \right)^2 \right], \tag{14}
$$

where $\pi_{lm}(\rho)$ is a momentum conjugate to $\phi_{lm}(\rho)$.

Note that for any $(l,m)$ (14) has the form

$$
\bar{H} = \frac{1}{2} \int d\rho \left[ \frac{\pi_{lm}^2}{a^2} + \frac{1}{2} \int d\rho d\rho' q(\rho) (\rho, \rho') q(\rho') \right], \tag{15}
$$

where the following algebra of Poisson brackets is understood:

$$
\{ q(\rho), p(\rho') \} = \delta(\rho - \rho') , \quad \{ q(\rho), q(\rho') \} = 0 , \quad \{ p(\rho), p(\rho') \} = 0 . \tag{16}
$$

Each subsystem described by the Hamiltonian (15) can be discretized by the following procedure:

$$
\rho \rightarrow (A - 1/2) a , \quad \delta(\rho - \rho') \rightarrow \delta_{AB}/a , \tag{17}
$$

where $A, B = 1, 2, \ldots$ and $a$ is a cut-off length. The corresponding Hamiltonian of the discretized system is of the form (17) with

$$
q(\rho) \rightarrow q^A , \quad p(\rho) \rightarrow p_A/a , \quad V(\rho, \rho') \rightarrow V_{AB}/a^2 . \tag{18}
$$

In this way we obtain a discretized system with the Hamiltonian (17) with the matrix $V$ given by the direct sum

$$
V = \bigoplus_{l,m} V_{(l,m)} , \tag{19}
$$

where $V_{(l,m)}$ is explicitly expressed as

$$
V_{AB}^{(l,m)} \phi_{lm}^A \phi_{lm}^B = a \sum_{A=1}^{\infty} \left[ N_{A+1/2} \left( \frac{x_{A+1/2}}{a} \right)^2 \left( \frac{N_{A+1}^{1/2}}{x_{A+1}} \phi_{lm}^A - \frac{N_{A}^{1/2}}{x_{A}} \phi_{lm}^A \right)^2 + \frac{l(l+1)}{r_0^2} \left( \frac{N_A \phi_{lm}}{x_A} \right)^2 \right]. \tag{20}
$$

Here

$$
x_A = r(\rho = (A - 1/2)a)/r_0 , \quad x_{A+1/2} = r(\rho = An)/r_0 , \quad N_A = N(\rho = (A - 1/2)a) , 
$$

$$
N_{A+1/2} = N(\rho = Aa) , \quad \phi_{lm}^A = \phi_{lm}(\rho = (A - 1/2)a) . \tag{21}
$$
In the matrix representation $V^{(l,m)}$ is given by the $n_{\text{tot}} \times n_{\text{tot}}$ matrix

$$
(V^{(l,m)})_{AB} = \frac{2\alpha}{r_0} \left( \frac{\Sigma_A}{\Delta_A} \right)^l \left( \frac{\Delta_1}{\Sigma_2} \right)^l \left( \frac{\Delta_2}{\Sigma_2} \right)^l \cdots \left( \frac{\Delta_{A-1}}{\Sigma_A} \right)^l \left( \frac{\Delta_A}{\Sigma_A} \right)^l,
$$

$$
\Sigma_A = \frac{1}{2} (r_0/a)^2 N_A x_A^{-2} \left[ N_{A-1/2} x_{A-1/2} + N_{A+1/2} x_{A+1/2} \right] + \frac{1}{2}(l+1)N_A x_A^{-2},
$$

$$
\Delta_A = -\frac{1}{2} (r_0/a)^2 N_A^{1/2} N_{A+1/2} x_A^{-1} x_{A+1/2} x_{A+1/2}^{-1},
$$

where we have imposed the boundary condition $\phi^{\alpha}_{lm} = 0$. In these expressions $r_0$ is an arbitrary constant, which we set to be the area radius $R_B$ of the boundary in the following arguments.

4 Entanglement thermodynamics in Minkowski spacetime

In the case of Minkowski spacetime $r(\rho) = \rho$ and $N(\rho) = 1$. The discretized system $\{\phi^A_{lm}\}$ ($A = 1, \cdots, n_{\text{tot}}$) is described by Hamiltonian of the form (11) with the potential given by (22).

We split the system $\{\phi^A_{lm}\}$ ($A = 1, \cdots, n_{\text{tot}}$) into the two subsystems, $\{\phi^a_{lm}\}$ ($a = 1, \cdots, n_B$) and $\{\phi^\alpha_{lm}\}$ ($\alpha = n_B + 1, \cdots, n_{\text{tot}}$). For this splitting, the entanglement entropy $S_{\text{ent}}$ and the entanglement energy $E_{\text{ent}}$ for the ground state can be calculated numerically. The results are expressed as

$$
S_{\text{ent}} \propto \frac{R_B^2}{a^2},
$$

$$
E_{\text{ent}} \propto \frac{R_B^2}{a^3},
$$

for all definitions of $S_{\text{ent}}$ and $E_{\text{ent}}$, where $R_B$ is the radius of the boundary defined by $R_B = n_B a$. Hence the entanglement temperature $T_{\text{ent}}$ is proportional to $a^{-1}$:

$$
T_{\text{ent}} \propto \frac{1}{a}.
$$

Thus the entanglement thermodynamics in Minkowski spacetime is quite different from the black-hole thermodynamics since $E_{\text{BH}} \propto R_H / l_{\text{pl}}^2$ and $T_{\text{BH}} \propto 1 / R_H$.

5 Entanglement thermodynamics in Schwarzschild spacetime

In Schwarzschild spacetime the metric is given by

$$
ds^2 = -\left( 1 - \frac{R_H}{r} \right) dt^2 + \left( 1 - \frac{R_H}{r} \right)^{-1} dr^2 + r^2 (d\theta + \sin^2 \theta d\psi^2),
$$

where $R_H$ is the area radius of the horizon. As the radial coordinate $\rho$ we take the proper distance from the horizon:

$$
\rho = \frac{R_H}{2} \sqrt{y^2 - 1 + \ln \left( y + \sqrt{y^2 - 1} \right)},
$$

where the variable $y$ is defined by $y = 2r / R_H - 1$.

As in the case of the Minkowski spacetime we split the system $\{\phi^A_{lm}\}$ ($A = 1, \cdots, n_{\text{tot}}$) into the two subsystems, $\{\phi^a_{lm}\}$ ($a = 1, \cdots, n_B$) and $\{\phi^\alpha_{lm}\}$ ($\alpha = n_B + 1, \cdots, n_{\text{tot}}$), and calculated the entanglement entropy and the entanglement energy $E_{\text{ent}}$ for the ground state numerically. Note that we consider only
degrees of freedom corresponding to positive $A$ (see Ref. [9] for the reason why we do not consider those corresponding to $A \leq 0$).

The results are expressed as

\[ S_{\text{ent}} \propto \frac{R_B^2}{a^2}, \]
\[ E_{\text{ent}} \propto \frac{R_B}{a^2}, \]  

(27)

for all definitions of $S_{\text{ent}}$ and $E_{\text{ent}}$, where $R_B$ is the area radius of the boundary defined by $R_B = r(\rho = n_B a)$ with $n_B = O(1)$ [9]. Here note that the proportionality in (27) holds for a fixed value of $n_B$. Hence the entanglement temperature $T_{\text{ent}}$ is proportional to $R_B^{-1}$:

\[ T_{\text{ent}} \propto \frac{1}{R_B}. \]  

(28)

Thus the entanglement thermodynamics in Schwarzschild spacetime has the same structure as that of the black-hole thermodynamics [9].

6 Entanglement thermodynamics in R-N spacetime

In Reissner-Nordström spacetime the metric is given by

\[ ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q}{r^2}\right)^{-1}dr^2 + r^2(d\theta + \sin^2\theta d\psi)^2, \]  

(29)

where $M$ and $Q$ are the mass and the charge of the black-hole. The area radii of the outer and the inner horizon $R_{H\pm}$ are

\[ R_{H\pm} = M \pm \sqrt{M^2 - Q^2}. \]  

(30)

As the radial coordinate $\rho$ we take the proper distance from the outer horizon

\[ \rho = \sqrt{(M^2 - Q^2)(y^2 - 1)} + M \ln \left(y + \sqrt{y^2 - 1}\right), \]  

(31)

where the variable $y$ is defined by

\[ y = \frac{r - M}{\sqrt{M^2 - Q^2}}. \]  

(32)

Now the numerical results for the entanglement entropy $S_{\text{ent}}$ and the entanglement energy $E_{\text{ent}}$ for the ground state are expressed as

\[ S_{\text{ent}} \propto \frac{R_B^2}{a^2}, \]
\[ E_{\text{ent}} \propto c(q) \frac{R_B}{a^2}, \]  

(33)

for all definitions of $S_{\text{ent}}$ and $E_{\text{ent}}$, where $R_B$ is the area radius of the boundary defined by $R_B = r(\rho = n_B a)$ with $n_B = O(1)$, and the coefficient $c(q)$ is a function of $q = Q/M$ given by

\[ c(q) = \frac{\sqrt{1 - q^2}}{1 + \sqrt{1 - q^2}}. \]  

(34)

The coefficient $c(q)$ approaches to zero in the limit $q \to 1$. Hence the entanglement temperature $T_{\text{ent}}$ is proportional to $R_B^{-1}$:

\[ T_{\text{ent}} \propto \frac{c(q)}{R_B}. \]  

(35)

Note that $T_{\text{ent}}$ is zero in the extremal limit of the background spacetime. Therefore as in the case of the Schwarzschild spacetime, the entanglement thermodynamics in the Reissner-Nordström spacetime has the same structure as that of the black-hole thermodynamics.
7 Conclusion

In this paper we have constructed entanglement thermodynamics for a massless scalar field in Minkowski, Schwarzschild and Reissner-Nordström spacetimes. The entanglement thermodynamics in Minkowski spacetime differs significantly from black-hole thermodynamics. On the contrary, the entanglement thermodynamics in Schwarzschild and Reissner-Nordström spacetimes has the same structure as that of black-hole thermodynamics. In particular, it has been shown that entanglement temperature in the Reissner-Nordström spacetime approaches zero in the extremal limit.

Finally we comment on possible extensions of the entanglement thermodynamics. The first is the inclusion of a charged field as matter. In particular, it will be valuable to analyze the entanglement thermodynamics for a charged field in Reissner-Nordström spacetime. In this case we can define the entanglement charge as an expectation value of the charge of the field for the coarse-grained state. This is now under investigation. The second is a generalization to non-spherically-symmetric spacetimes. For example it is expected that construction of entanglement thermodynamics in Kerr spacetime requires the introduction of a concept of an entanglement angular-momentum.

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