Analogues of gravity-induced instabilities in anisotropic metamaterials

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In the context of field theory in curved spacetimes, it is known that suitable background spacetime geometries can trigger instabilities of fields, leading to exponential growth of their (quantum and classical) fluctuations — a phenomenon called vacuum awakening in the quantum context, which in some classical scenarios seeds spontaneous scalarization/vectorization. Despite its conceptual interest, an actual observation in nature of this effect is uncertain since it depends on the existence of fields with appropriate masses and couplings in strong-gravity regimes. Here, we propose analogues for this gravity-induced instability based on nonlinear optics of metamaterials which could, in principle, be observed in laboratory.

I. INTRODUCTION

The influence of a background material medium on the propagation of mechanic and electromagnetic waves is well known to be formally analogous to that of an effective curved spacetime geometry. This idea was first presented, in the electromagnetic/optical context, by Gordon in 1923 [1] and it has since been developed in a number of different scenarios, particularly after Unruh’s [2] and Visser’s [3] works on acoustic analogues of black holes and their associated Hawking-like radiation. More recent applications of this formal analogy include mimicking in material media quantum lightcone fluctuations [4] and anisotropy in cosmological spacetimes [5]. The most appealing feature of these condensed-matter analogues of gravitational backgrounds is the possibility of observing in laboratory subtle but conceptually interesting effects which can be virtually unobservable in their original contexts — Hawking radiation being certainly the most emblematic among them, with claims of having already been observed in laboratory [6–8].

An interesting effect in the context of (quantum) fields in curved spacetimes is the triggering of field instabilities due to the background spacetime geometry — a phenomenon called vacuum awakening in the quantum context [9–12]. These gravity-induced instabilities exponentially amplify vacuum fluctuations to the point they decohere and seed classical perturbations [13], which, depending on field parameters, eventually evolve to a nonzero classical field configuration (“spontaneous scalarization” in the case of scalar fields [14–17]), stabilizing the whole system. More recently, this mechanism was also predicted to occur for massless spin-1 fields through appropriate nonminimal couplings [18] and, in analogy with the scalar case, the stabilization process was termed “spontaneous vectorization.” To the best of our knowledge, condensed-matter and optical analogues of these gravity-induced instabilities have not been proposed to this date. In this work, we propose and explore possible analogues of gravity-induced instabilities in the context of electromagnetism in polarizable/magnetizable anisotropic (meta)materials.

Electromagnetic instabilities in flat spacetime are expected to occur in some materials. One celebrated example appeared in the context of plasma physics in the late 1950s and became known as Weibel instability [19]. The system, a neutral plasma whose components have anisotropic velocity distribution, possesses growing electromagnetic transverse waves. Related effects have been studied since then, with recent applications to solar plasma instability [20] and solid state devices [21]. Moreover, causal aspects of classical propagation in active materials were discussed in Ref. [22], where properties of the refractive index were established. Nevertheless, besides the fairly recurrence in the literature, usually quantization in such scenarios is not considered [23–25] or it is regarded as inconsistent [26, 27].

It is noteworthy that instability of the electromagnetic field is always accompanied by evolution of the background, ending with the stabilization of the system as a whole. In the case of gravity-induced instability, the gravitational field changes with time, whereas electromagnetic instability in the presence of plasmas involves growing plasmons. In the case of electromagnetic fields in the presence of matter, for whatever form of the interaction with the background, the field’s evolution is ruled by Maxwell’s equations in the presence of polarizable/magnetizable media, and the interaction with the background is encapsulated in the functional dependence of the electric displacement (magnetic) vector field \( \mathbf{D} (\mathbf{H}) \) with the true (microscopic) fields \( \mathbf{E} \) and \( \mathbf{B} \). If the magnitudes involved are small (e.g., in the beginning of the instability action), these functional relations become linear and one may find the form of the coefficients for such systems. For the case of Weibel instability, for instance, if the velocity anisotropy is taken in the \( z \) direction, the instability is modelled by a negative squared refractive index in the direction perpendicular to \( z \).

We apply Gordon’s method to propose a family of optical-based analogue models for electromagnetic fields presenting instabilities in curved spacetimes. We show
how anisotropies of the background enter the effective equations in the form of nonminimal couplings, and in the case of strong anisotropy (just like for the Weibel instability), this coupling results in unstable solutions. We also discuss that for these systems the stabilization process occurs through the nonlinear nature of the background, which may seed spontaneous vectorization in analogy to the Einstein’s field equations in the gravitational scenario.

The paper is organized as follows. In Sec. II we present the covariant formalism of electromagnetism in anisotropic polarizable/magnetizable materials, establishing the formal analogy with nonminimally-coupled electromagnetism in curved spacetimes. In Subsec. II A we consider a particular type of nonminimal coupling inspired by one-loop quantum electrodynamics (QED) corrections to electromagnetism in curved spacetimes. In Sec. III, we apply the formalism presented in the previous section to the scenario of a plane-symmetric anisotropic polarizable/magnetizable materials, establishing the formal analogy with nonminimally-coupled electromagnetism in curved spacetimes. Conditions for triggering instabilities and their types (Subsec. III A), and present a concrete example (homogeneous medium; Subsec. III B) where calculations can be carried over to the end. In Sec. IV we repeat the treatment of the previous section, but now for a more appealing scenario on the gravitational side: spherically-symmetric, stationary anisotropic media. Conditions for triggering instabilities and their types are shown to be very similar to those in the plane-symmetric case (Subsec. IV A). As a concrete application, in Subsec. IV B we show how to mimic QED-inspired nonminimally-coupled electromagnetism in the background spacetime of a Schwarzschild black hole. Then, Sec. V is dedicated to discuss possible stabilization mechanisms which might bear analogy to some curved-spacetime phenomena, such as spontaneous vectorization [18] and particle bursts due to tachyonic instability [22]. Finally, in Sec. V we present some final remarks. We leave for an appendix tedious calculations related to the orthonormalization of modes of Sec. IV.

We adopt the abstract-index notation to represent tensorial quantities (see, e.g., Ref. [29]) and, unless stated otherwise, we use natural units (in which $c = 1$).

II. COVARIANT ELECTROMAGNETISM IN ANISOTROPIC MATERIAL MEDIA

Electromagnetism in material media, in flat spacetime and in the absence of free charges, is described by two antisymmetric (observer-independent) tensors, $F_{ab}$ and $G^{ab}$, satisfying the macroscopic covariant Maxwell’s equations,

$$\partial_a G^{ab} = 0,$$
$$\partial_a F_{bc} = 0,$$

where $\partial_a$ is the derivative operator compatible with the flat metric $\eta_{ab}$ (but in arbitrary coordinates) and the square brackets denote antisymmetrization over the indices enclosed by them. These equations must be supplemented by medium-dependent constitutive relations between $F_{ab}$ and $G^{ab}$ as well as initial and boundary conditions, in order to provide a well-posed problem. These constitutive relations are usually set at the level of (observer-dependent) fields $E_a$, $B^a$, $D^a$, and $H_a$, related to $F_{ab}$ and $G^{ab}$ through

$$E_a = F_{ab} u^b,$$
$$D^a = G^{ab} u_b,$$
$$B^a = -\frac{1}{2} \epsilon^{abcd} F_{bc} u_d,$$
$$H_a = -\frac{1}{2} \epsilon_{abcd} G^{bd} u^c,$$

where $u^a$ is the four-velocity of the observer measuring these fields and $\epsilon_{abcd}$ is the Levi-Civita pseudo-tensor (with $\epsilon_{0123} = \sqrt{-\eta}$, $\eta := \det(\eta_{\mu \nu})$). Moreover, the constitutive relations usually take a simpler form in the reference frame in which the medium is (locally and instantaneously) at rest.

Here, we consider a polarizable and magnetizable medium whose constitutive relations in its instantaneous rest frame take the form

$$D^a = \varepsilon^{ab} E_b,$$
$$H_a = \mu_{ab} D^b,$$

where the tensors $\varepsilon^{ab}$ and $\mu_{ab}$ may depend on spacetime coordinates, and the system is assumed dispersionless. We return to this point later. The fact that Eqs. (7) are valid in the medium’s instantaneous rest frame means that the fields $E_a$, $B^a$, $D^a$, and $H_a$ appearing in them are related to $F_{ab}$ and $G^{ab}$ through Eqs. (3-6) with $u^a = v^a$, the medium’s four-velocity field. We proceed by splitting the “spatial” [30] tensors $\varepsilon^{ab}$ and $\mu_{ab}$ into isotropic and traceless anisotropic parts,

$$\varepsilon^{ab} = \varepsilon h^{ab} + \chi^{ab},$$
$$\mu_{ab} = \mu^{-1} h_{ab} + \chi_{(ab)},$$

where $h^{ab} := \delta^{ab} + v^a v_b$ is the projection operator orthogonal to $v^a$. Inverting Eqs. (3-6) (with $u^a = v^a$),

$$G^{ab} = 2v^a (D^b - \epsilon^{abcd} H_c v_d),$$

and substituting Eqs. (7-10) and (11), we obtain

$$G^{ab} = (g^{ac} g^{bd} + \chi^{abcd}) F_{cd},$$

where $g^{ac} g^{bd}$ is the derivative operator compatible with the flat metric $g^{ac} g^{bd}$ (but in arbitrary coordinates) and the square brackets denote antisymmetrization over the indices enclosed by them. These equations must be supplemented by medium-dependent constitutive relations between $F_{ab}$ and $G^{ab}$ as well as initial and boundary conditions, in order to provide a well-posed problem. These constitutive relations are usually set at the level of (observer-dependent) fields $E_a$, $B^a$, $D^a$, and $H_a$, related to $F_{ab}$ and $G^{ab}$ through

$$E_a = F_{ab} u^b,$$
$$D^a = G^{ab} u_b,$$
$$B^a = -\frac{1}{2} \epsilon^{abcd} F_{bc} u_d,$$
$$H_a = -\frac{1}{2} \epsilon_{abcd} G^{bd} u^c,$$
where we have defined the tensors

\[ g^{ab} := \frac{1}{\sqrt{n}} \left[ \eta^{ab} - (n^2 - 1) \nu^a \nu^b \right], \]

\[ \chi^{abcd} := \frac{n - 1}{\mu} g^{[a[c} g^{d]b} - 2 \chi^{[\alpha(c} \nu^{d) b]} + \frac{1}{2\epsilon} \epsilon_{abcd} \epsilon_{\alpha \beta \gamma \delta} \eta^{\alpha} \nu^b \nu^c \nu^d, \]

and the squared refractive index \( n^2 = \mu \varepsilon \). The idea, then, is to consider the symmetric tensor \( g_{ab} \), defined through \( g_{ab} g^{bc} = \delta^c_a \), as an effective metric of a curved background spacetime perceived by the electromagnetic field \( F_{ab} \). Note that the components of \( g^{ab} \) and \( \eta^{ab} \) satisfy

\[ \det (g^{\alpha \beta}) = \det (\eta^{\alpha \beta}) \]

and, thus, \( \sqrt{\eta} = \sqrt{-g} \), where \( g := \det(g_{ab}) \). One can easily check that \( g_{ab} \) is explicitly given by

\[ g_{ab} = \sqrt{n} \left[ \eta_{ab} + \frac{1}{n^2} \nu^a \nu^b \right]. \]

Therefore, in an arbitrary coordinate system, Eq. (1) reads

\[ 0 = \frac{1}{\sqrt{-\eta}} \partial_a \left( \sqrt{-\eta} G^{\alpha \beta} \right) = \frac{1}{\sqrt{-g}} \partial_a \left( \sqrt{-g} G^{\alpha \beta} \right). \]  

Up to this point, it was understood that the physical background metric \( \eta_{ab} \) and its inverse \( \eta^{ab} \) were responsible for lowering and raising tensorial indices. Now, with the introduction of an effective metric \( g_{ab} \), we should be careful when performing these isomorphisms. In order to minimize chances of confusion, we shall avoid lowering and raising tensorial indices using the effective metric, making explicit most appearances of \( g_{ab} \) and \( g^{ab} \) in the equations below, with few exceptions which will be clearly stated. One obvious exception is the definition of \( g_{ab} \) as the inverse of \( g^{ab} \). Another such exception is the use of \( \nabla_a \) to denote covariant derivative compatible with \( g_{ab} \). With this in mind, from Eqs. (2) and (17), the electromagnetic tensor \( F_{ab} \) satisfies

\[ 0 = \nabla_a (g^{ac} g^{bd} + \chi^{abcd}) F_{cd}, \]

\[ 0 = \nabla_a \left( (g^{ac} g^{bd} + \chi^{abcd}) F_{cd} \right). \]  

Notice that Eqs. (15) and (19) applied to homogeneous \((\nabla_a \varepsilon = \nabla_a \mu = 0)\), isotropic \((\chi^{ab}_{(c)} = 0 = \chi^{(ab)}_{(c)})\) materials, with arbitrary 4-velocity field \( \nu^a \), lead to the same equations which rule minimally-coupled vacuum electromagnetism in a curved spacetime with metric \( \sqrt{\mu/n} g_{ab} \). Optical analogue models in these configurations with \( \mu = 1 \) were studied in [31 32]. Here, we shall focus on electromagnetism in anisotropic materials, more specifically, materials with only "shear-like" anisotropies: \( \chi^{(a)}_{(c)} = 0 = \chi^{(ab)}_{(c)} \). In this case, the tensor \( \chi^{abcd} \) defined in Eq. (14) has the same algebraic symmetries as the Riemann curvature tensor, namely, \( \chi^{abcd} = \chi^{[cdb]} \) and \( \chi^{[abcd]} = 0 \) — in addition to \( \chi^{abcd} = \chi^{[a][b][c][d]} \), which is always true.

The Eqs. (18) and (19) can be seen as analogous to some nonminimally-coupled electromagnetic field equations in curved spacetime. Although in general \( \chi^{abcd} \) is independent of the Riemann tensor associated with the effective metric \( g_{ab} \), one can construct cases where they are related. This is interesting because some one-loop QED corrections to Maxwell’s field equations in curved spacetime [33 34] can be emulated by such nonminimal coupling, as we shall discuss below, in Subsec. II A.

Before considering particular applications of the equations above, let us define a sesquilinear form on the space of complexified solutions, which will be relevant when applying the canonical quantization procedure. As usual, let us solve Eq. (18) by introducing the 4-potential \( A_a \) such that \( F_{ab} = \nabla_a A_b - \nabla_b A_a \). Then, let \( F_{ab} \) and \( F'_{ab} \) be two complex solutions of Eq. (19), associated to \( A_a \) and \( A'_a \), respectively. With overbars representing complex conjugation, we contract \( A_b \) (resp., \( A'_b \)) with Eq. (19) applied to \( F_{cd} \) (resp., \( F'_{cd} \)) and subtract one from the other, arriving at

\[ \nabla_a \left[ (g^{ac} g^{bd} + \chi^{abcd}) (A_b F_{cd} - A'_b F'_{cd}) \right] = 0. \]

This continuity-like equation ensures that the quantity \((A, A') := i \int \Sigma dS \left( g^{ac} g^{bd} + \chi^{abcd} \right) (A_b F_{cd} - A'_b F'_{cd})\) is independent of the space-like hypersurface \( \Sigma \) where the integration is performed — provided we restrict attention to solutions satisfying "appropriate" boundary condition —, where \( dS \) is the physical volume element on \( \Sigma \) and \( N_a = \eta_{ab} N^b \), with \( N^a \) being a unit, future-pointing vector orthogonal to \( \Sigma \) (according to \( \eta_{ab} \)). More specifically, considering that the system of interest is contained in the spacetime region \( M \equiv T \times \Sigma \), where \( T \subset \mathbb{R} \) is a real open interval, then the appropriate boundary condition amounts to imposing that the flux of the (sesquilinear) current appearing in Eq. (20) vanishes through \( T \times \Sigma \) (where \( \mathcal{S} \) denotes the boundary of the space \( \mathcal{S} \)). In particular, in stationary situations which we shall treat here, this condition translates to

\[ \int_\mathcal{S} dS_s \left( g^{ac} g^{bd} + \chi^{abcd} \right) (A_b F_{cd} - A'_b F'_{cd}) = 0, \]

where \( dS \) is the physical area element on \( \Sigma \) and \( s^a \) is the unit vector field normal to \( T \times \Sigma \) (according to \( \eta_{ab} \)). Thus, these conditions being satisfied, Eq. (21) provides a legitimate sesquilinear form on the space \( \mathcal{C} \) of complex-valued solutions of Eqs. (18) and (19). Notice that for pure-gauge solutions — i.e., \( A_a = \nabla_a \psi \), for some scalar function \( \psi \), \( (A, A) = 0 \). (The converse, however, is not true.)

The relevance of this sesquilinear form is that it provides a legitimate inner product on a (non-unique choice of) subspace \( \mathcal{S}^\perp \subseteq \mathcal{C} \) of “positive-norm solutions,” which, together with its complex conjugate \( \mathcal{S}^\perp \subseteq \mathcal{C} \), generates all solutions: \( \mathcal{S}^\perp \oplus \mathcal{S}^\perp = \mathcal{C} \). Loosely speaking, upon completion, \( \mathcal{S}^\perp \) yields a Hilbert space \( \mathcal{H} \) from
A. QED-inspired nonminimal couplings

As mentioned earlier, Eqs. (18) and (19) can be interpreted as ruling electromagnetism in curved spacetimes with some QED-inspired nonminimal coupling $\chi_{abcd}$ with the background geometry. In fact, in the one-loop-QED approximation \[ \chi_{abcd} = \alpha_1 R_{abcd} + \alpha_2 R_{[a[c} g_{d]b]} + \alpha_3 g_{a[c} g_{d]b}, \]
with $\alpha_1 = -\alpha_2/13 = 2\alpha_3 = -\alpha/9(90\pi m_e^2)$, where $\alpha$ is the fine-structure constant, $m_e$ is the electron’s mass, and $R_{abcd}$, $R_{ab}$, and $R$ are, respectively, the Riemann, Ricci and Ricci-scalar curvature tensors associated with the (effective) metric $g_{ab}$. By leaving $\alpha_1, \alpha_2, \alpha_3$ unconstrained, Eq. (23) represents a three-parameter family of couplings of the electromagnetic field with the background effective geometry — see Ref. 33 for some interesting particular cases.

For a generic medium, $\chi_{abcd}$ is not related to the geometry associated with $g_{ab}$. However, we can simulate couplings given by Eq. (23) by conveniently relating $n$ and $u^a$ (which determine $g_{ab}$) with $\mu$ and the anisotropic tensors $\chi_{ab}^{(e)}$ and $\chi_{ab}^{(\mu)}$ (which appear in $\chi_{abcd}$). From Eqs. (14) and (23), and their contractions with $g_{ab}$,
\[ g_{bd} \chi_{abcd} = \frac{3}{2} \left( \frac{n}{\mu} - 1 \right) g^{ac} + \frac{\chi_{ab}^{(e)}}{2n^{1/2}} + \frac{n^{3/2}}{2} g^{ab} g_{cd} \chi_{ab}^{(\mu)}, \]
\[ g_{ad} g_{bd} \chi_{abcd} = 6 \left( \frac{n}{\mu} - 1 \right) (\alpha_1 + 3\alpha_2/2 + 6\alpha_3) R, \]
we can solve for $\mu$ and the anisotropic tensors, obtaining:
\[ \mu = \frac{n}{1 + (\alpha_1/6 + \alpha_2/4 + \alpha_3) R}, \]
\[ n^{3/2} \chi_{ab}^{(e)} = -2\alpha_1 \left( R_{abcd} V_d V + \frac{\alpha_2}{12} H_{ab}^{(e)} \right) + \frac{\alpha_2}{2} \left( R_{ab} - \frac{R}{4} g_{ab} \right), \]
\[ n^{3/2} \chi_{ab}^{(\mu)} = 2\alpha_1 \left( R_{abcd} V_d V + \frac{\alpha_2}{12} H_{ab}^{(e)} \right) + \frac{4\alpha_1 + \alpha_2}{2} \left( R_{ab} - \frac{R}{4} g_{ab} \right), \]
where $V^a = n^{3/4} u^a$ is the 4-velocity of the medium normalized according to the effective metric $g_{ab}$ and $H_{ab} := g_{ab} + V_a V_b$. In Eqs. (27) and (28) indices are lowered and raised by the effective metric and its inverse. Notice that, unless $\alpha_1 = \alpha_2 = 0$ — which implies $\chi_{ab}^{(e)} = 0 = \chi_{ab}^{(\mu)}$ —, as a consequence of $\chi_{ab}^{(e)} v_b = 0 = \chi_{ab}^{(\mu)} v_b$, only geometries associated with $g_{ab}$ which can be put in the form given by Eq. (16) and satisfying
\[ R_{ab} v^b = \frac{R}{4} v^a, \]
for some timelike 4-vector $u^a$, can be emulated by these anisotropic media — with $u^a$ then set as the medium’s 4-velocity. Using Einstein’s equations to map this constraint to the stress-energy-momentum tensor $T_{ab}$ of the corresponding gravitational source, we have that
\[ T_{ab} = \frac{\rho u^a u^b}{4} + \sum_{j=1}^{3} p_j e_j^a e_j^b \]
with $\{ u^a, e_1^a, e_2^a, e_3^a \}$ being a tetrad and $u^a$ timelike —, then
\[ \rho + \frac{1}{3} \sum_{j=1}^{3} p_j = 0 \]
\[ (V_a e_j^a) (\rho + p_j) = 0, \quad j = 1, 2, 3. \]
In particular, if $V^a = u^a$, then Eq. (32) is the only additional constraint to be enforced.

Returning attention to the background effective geometry and recalling that all the geometric tensors are obtained from $g_{ab}$ given in Eq. (16), we see that Eq. (29) actually comprises a system of four differential equations which $n$ and $u^a$ must satisfy. Electromagnetism with nonminimal coupling described by Eq. (23) can only be simulated in these anisotropic media if the background spacetime geometry is associated to solutions of this system [via Eq. (16)]. We shall treat a particular solution to these differential equations later.

III. PLANE-SYMMETRIC ANISOTROPIC MEDIUM AT REST

In this section, we consider the simplest case of an anisotropic medium: a plane-symmetric medium at rest
in the inertial lab frame. The purpose of this section is not yet to establish an analogy with some interesting gravitational system, but to present the analysis in a simple context. In Sec. [4], we apply the analysis to a more appealing scenario.

Let us consider a medium at rest in an inertial laboratory, such that in inertial Cartesian coordinates \( \{(x, y, z)\} = \mathbb{R}^3 \times \mathcal{I} \subseteq \mathbb{R}^4 \) we have \( \omega^\nu = (1, 0, 0, 0), \) \( \mu = \mu(z), \) \( \varepsilon = \varepsilon(z), \)

\[ \lambda_{(c)} = \frac{(\Delta^{(c)})(2 \varepsilon_1^x \varepsilon_2^y - \varepsilon_1^y \varepsilon_2^x - \varepsilon_1^z \varepsilon_2^z),} {3} \]

and

\[ \lambda_{(\mu)} = \frac{(\Delta^{(c)})(2 \delta_1^x \delta_2^y - \varepsilon_1^y \delta_2^x - \varepsilon_1^z \delta_2^z),} {3} \]

with \( \Delta^{(c)} = \Delta^{(c)}(z), \) \( \Delta^{(\mu)} = \Delta^{(\mu)}(z), \) \( z \in \mathcal{I}. \) (\( \mathcal{I} \) is an open real interval.) This simply means that

\[ D^j = \varepsilon_j E_j, \]  \( j = x, y, \)

\[ D^z = \varepsilon_2 E_z, \]

\[ H_j = \mu_1 E_j, \]  \( j = x, y, \)

\[ H_z = \mu_1 E_z, \]

where \( \varepsilon_1 - \varepsilon_1 = \Delta^{(c)}, \) \( (2 \varepsilon_1 + \varepsilon_1)/3 \geq \varepsilon, \mu_1^{-1} - \mu_1^{-1} = \Delta^{(\mu)}, \)

and \( 2 \mu_1^{-1} + \mu_1^{-1} = \varepsilon^{-1} \).

In these coordinates, \( g_{\omega \omega} = \sqrt{n} \text{ diag}(-n^{-2}, 1, 1, 1). \) For convenience, we shall work in the generalized Coulomb gauge \([36]\) in which \( A_\omega = (0, A) \) and \( \partial_{\mu} \cdot (\varepsilon_{\mu} A_\mu) = 0, \) where we have defined \( A_{\mu} := (A_x, A_y, A_z), \) \( \varepsilon_\mu := (\varepsilon_x, \varepsilon_y, \varepsilon_z). \) In this gauge, the t component of Eq. \([19]\) is automatically satisfied, while the spatial components lead to

\[ \left[ -\frac{\varepsilon_1}{\mu_1} \partial_1^2 + \frac{1}{\mu_1} \partial_1^2 + \left( \frac{1}{\mu_1} \partial_1 \right)^2 \right] A_1
\]

\[ = \frac{1}{\mu_1} \left[ \mu_1 \partial_1 (\mu_1^{-1} \varepsilon_1 - \varepsilon^{-1} \partial_1 \varepsilon_1) \right] A_1,
\]

\[ \left[ -\frac{\mu_1}{\varepsilon_1} \partial_1^2 + \frac{1}{\varepsilon_1} \partial_1^2 + \left( \frac{1}{\varepsilon_1} \partial_1 \right)^2 \right] (\varepsilon_{\mu} A_\mu) = 0.\]

First, let us consider solutions \( A \) such that \( A_z = 0, \) which describe electric fields which are perpendicular to the \( z \) direction — transverse electric modes, \( A^{(TE)} \), for short \([37]\). In this case, our gauge condition ensures that there exists a scalar field \( \psi \) such that \( A_x^{(TE)} = \partial_\mu \psi \) and \( A_y^{(TE)} = -\partial_\mu \psi. \) Moreover, making use of the staticity and planar symmetry of the present scenario, we can write \( \psi \epsilon = e^{-i(\omega t - k_z z)} f^{(TE)}(z) \), where \( x_\perp = (x, y, 0), \)

\( k_\perp = (k_x, k_y, 0), \)

\( k_z = |k_z|, \)

and \( f^{(TE)}(z) \) satisfies

\[ \left[ -\frac{d^2}{dz^2} + \left( \frac{k_\perp^2}{\mu_1} - \varepsilon_1 \right) \right] f^{(TE)}(z) = 0.\]

with \( \zeta \) being a spatial coordinate such that \( d\zeta = \mu_1 dz. \) The Eq. \([42]\) must be supplemented by boundary conditions for \( f^{(TE)}(z). \) Imposing Eq. \([42]\) to these modes leads to

\[ \left[ f^{(TE)}(k_\perp, k_z) \frac{d}{d\zeta} f^{(TE)}(k_\perp, k_z) \right] = 0,\]

where \( []^{(I)} \) denotes the flux of the quantity in square brackets through \( \mathcal{I}. \) This condition restricts the possible values of \( \omega^2. \) Let \( \varepsilon^{(TE)}_{k_\perp} \) be the \((k_\perp)-dependent\) set of \( \omega \) values for which Eqs. \([42]\) and \([43]\) are satisfied for \( (k_\perp, k_z) \neq 0. \) For \( \omega, \omega \in \varepsilon^{(TE)}_{k_\perp}, \) we can orthonormalize these modes according to

\[ \left( A^{(TE)}_{k_\perp, k_z}, A^{(TE)}_{k_\perp, k_z'} \right) = \delta_{\omega \omega}', \delta(k_\perp - k_\perp'), \]

\[ \left( A^{(TE)}_{k_\perp, k_z}, A^{(TE)}_{k_\perp', k_z'} \right) = 0 \]

\((\delta_{\omega \omega}', \) being the appropriate Dirac-delta distribution on \( \varepsilon^{(TE)}_{k_\perp}, \) where the sesquilinear form given in Eq. \([21]\), applied to the current scenario, takes the form

\[ (A, A') := i \int_\mathcal{I} d^3 x \left[ \varepsilon_1 (A_1, \partial_1 A_1 - A_1', \partial_1 A_1') + \varepsilon_1 (A_1, A_1 - A_1', \partial_1 A_1) \right] \]

We obtain (up to a global phase)

\[ A^{(TE)}_{k_\perp, k_z} = \frac{k_\perp \times n_\perp}{2 \pi k_z \sqrt{2 \omega}} e^{-i(\omega t - k_z z)} f^{(TE)}(z), \]

with

\[ \int_\mathcal{I} d\zeta \varepsilon_1 f^{(TE)}(z) f^{(TE)}(z) = \delta_{\omega \omega}', \]

which describe magnetic fields which are perpendicular to the \( z \) direction — transverse magnetic modes, \( A^{(TM)} \), for short \([37]\) —, is obtained by conveniently setting \( A^{(TM)} = \varepsilon^{-1/2} \partial_\mu \), where \( \phi \) is an auxiliary function.

Our gauge condition then leads to \( A^{(TM)} = -\varepsilon^{-1/2} \partial_\mu \partial_\mu \phi, \)

\( j = x, y, \) Using, again, staticity and planar symmetry, we find solutions of the form \( \phi = e^{-i(\omega t - k_z z)} f^{(TM)}(z), \)

where \( f^{(TM)}(z) \) satisfies

\[ \left[ \frac{d^2}{d\zeta^2} + \left( \frac{k_\perp^2}{\mu_1} - \varepsilon_1 \right) \right] f^{(TM)}(z) = 0.\]

with \( \xi \) being a spatial coordinate such that \( d\xi = \varepsilon_1 dz. \) The boundary condition imposed by Eq. \([22]\) now leads to

\[ \left[ \omega^2 f^{(TM)}(k_\perp, k_z) - \omega^2 f^{(TM)}(k_\perp, k_z) \frac{d}{d\zeta} f^{(TM)}(k_\perp, k_z) \right] = 0.\]

Let \( \varepsilon^{(TM)}_{k_\perp} \) be the \((k_\perp)-dependent\) set of \( \omega \) values for which Eqs. \([49]\) and \([50]\) are satisfied for \( f^{(TM)}(k_\perp, k_z) \neq 0. \) For
\( \omega, \omega' \in \mathcal{E}^{(\text{TM})}_{k, z} \cap \mathbb{R}^*_+ \), we can normalize these modes according to
\[
\left( A^{(\text{TM})}_{\omega k_i}, A^{(\text{TM})}_{\omega' k_i} \right) = - \left( A^{(\text{TM})}_{\omega k_i}, A^{(\text{TM})}_{\omega' k_i} \right) = \delta_{\omega\omega'} \delta(k_i - k'_i),
\]
(51)
\[
\left( A^{(\text{TM})}_{\omega k_i}, A^{(\text{TM})}_{\omega' k_i} \right) = 0
\]
(52)
(\( \delta_{\omega\omega'} \) now being the appropriate Dirac-delta distribution on \( \mathcal{E}^{(\text{TM})}_{k, z} \), obtaining (up to a global phase)
\[
A^{(\text{TM})}_{\omega k_i} = \frac{e^{-i(\omega t - k \cdot x)}}{2\pi k_i \sqrt{2\omega^3}} \left( \frac{k_i^2}{\varepsilon_1} \mathbf{n}_z + i \frac{k_i}{\varepsilon_1} \frac{d}{dz} \right) f^{(\text{TM})}_{\omega k_i}(z),
\]
(53)
with
\[
\int_{\mathcal{I}} dz \mu_1(z) f^{(\text{TM})}_{\omega k_i}(z) f^{(\text{TM})}_{\omega' k_i}(z) = \delta_{\omega\omega'}.
\]
(54)

Moreover, modes \( A^{(\text{TM})}_{\omega k_i} \) and \( A^{(\text{TM})}_{\omega' k_i} \) are orthogonal to modes \( A^{(\text{TE})}_{\omega k_i} \) and \( A^{(\text{TE})}_{\omega' k_i} \).

The solutions expressed in Eqs. (47) and (53), dubbed positive-frequency normal modes, play a central role in the construction of the Fock (Hilbert) space of the quantized theory, as described at the end of the previous section. With these solutions, the quantum-field operator \( \hat{A} \) is represented by
\[
\hat{A} = \sum_{J \in \{\text{TE,TM}\}} \int_{\mathbb{R}^2} d^2 \mathbf{k}_i \int_{\mathcal{I}} d\omega \left[ \hat{a}^{(J)}_{\omega k_i} A^{(J)}_{\omega k_i} + \text{H.c.} \right],
\]
(55)
where “H.c.” stands for “Hermitian conjugate” of the preceding term and \( \hat{a}^{(J)}_{\omega k_i} \) (respectively, \( \hat{a}^{(J)\dagger}_{\omega k_i} \)) is the annihilation (resp., creation) operator associated with mode \( A^{(J)}_{\omega k_i} \), satisfying the canonical commutation relations:
\[
\left[ \hat{a}^{(J)}_{\omega k_i}, \hat{a}^{(J')\dagger}_{\omega' k_i} \right] = \delta^{JJ'} \delta_{\omega\omega'} \delta(k_i - k'_i),
\]
(56)
\[
\left[ \hat{a}^{(J)}_{\omega k_i}, \hat{a}^{(J')}_{\omega' k_i} \right] = 0.
\]
(57)

As an application of our quantization scheme one can use the above formulas to obtain, for instance, the Carniglia-Mandel quantization in a straightforward way. The system in this case is composed by a dielectric-vacuum interface at \( z = 0 \) and a non-magnetizable (\( \mu_\parallel = 1 \)) homogeneous isotropic non-dispersive dielectric \( (\varepsilon_1 = \varepsilon_\perp = \varepsilon = n^2) \) filling the half-space \( z < 0 \). These data enter Eqs. (42) and (49), thus describing the background in terms of effective potentials of one-dimensional Schrödinger-like problems.

### A. Instability analysis

In the analysis presented above, it was implicitly assumed that all constitutive functions \( \varepsilon_\perp, \varepsilon_\parallel, \mu_\perp, \) and \( \mu_\parallel \) are positive functions of \( z \in \mathcal{I} \). This condition ensures that the field modes presented in Eqs. (47) and (53), together with their complex conjugates, constitute a complete set of (complexified) solutions of Maxwell equations in \( \mathbb{R}^3 \times \mathcal{I} \); in other words, the boundary-value problems defined by Eqs. (42) and Eqs. (49) admit solutions only for (a subset of) \( \omega^2 > 0 \). This is easily seen by interpreting them as null-eigenvalue problems for the linear operators defined in the square brackets of Eqs. (42) and (49). Experience with Schrödinger-like equations teaches us that these equations have solutions provided the associated effective potentials (terms in parentheses) become sufficiently negative in a given region — which implies \( \omega^2 > 0 \) and, typically, the larger the \( k_i^2 \), the larger the \( \omega^2 \).

Here, however, we shall consider a more interesting situation. It has been known for almost two decades that materials can be engineered so that some of their constitutive functions can assume negative values. These exotic materials have been termed metamaterials. In this case, the effective potentials appearing in Eqs. (42) and (49) may become sufficiently negative — granting solutions to these boundary-value problems — without demanding \( \omega^2 > 0 \). For instance, if \( \mu_\parallel < 0 \) (with \( \mu_\perp, \varepsilon_\parallel > 0 \)), then the larger the value of \( k_i \), the more negatively it contributes to the effective potential of Eq. (42), favoring the appearance of solutions with smaller (possibly negative) values of \( \omega^2 \). The same is true for Eq. (49) if \( \varepsilon_\parallel < 0 \) and similar analysis can be done if any other constitutive function becomes negative.

At this point, we must introduce an element of reality concerning the constitutive functions. We have been treating these quantities as given functions of \( z \) alone — neglecting dispersion effects, since we are, here, interested in gravity analogues. However, these material properties generally depend on characteristics of the electromagnetic field itself, particularly on its time variation (i.e., on \( \omega \)), in which case Eqs. (7) and (8) would be valid mode by mode, with the constitutive tensors \( \varepsilon^{ab} \) and \( \mu^{ab} \) possibly being different for different modes. When translated to spacetime-dependent quantities, Eqs. (7) and (8) would be substituted by sums over the set of allowed field modes. Therefore, the precise key assumption about our metamaterial media is that some of their anisotropic constitutive functions \( \varepsilon_\perp, \varepsilon_\parallel, \mu_\perp, \mu_\parallel \) can become negative for some \( \omega \) on the positive imaginary axis, \( \omega^2 < 0 \). Notwithstanding, the less restrictive condition \( \text{Im}(\omega) > 0 \) would suffice for our purposes. However, dealing with the case \( \text{Im}(\omega) \text{Re}(\omega) \neq 0 \) would involve quantization in active media, which we shall treat elsewhere. Moreover, our focus here is to show that the electromagnetic field itself can exhibit interesting behavior without need to exchange energy with the medium (which occurs in dispersive/active media). This justifies our focus on \( \omega^2 < 0 \) in what follows. The possibility of having this type of material will be discussed later.

Let \( \omega^2 = -\Omega^2 \) (with \( \Omega > 0 \)) be such value for which at least one of the constitutive functions is negative for \( z \in \mathcal{I} \). Thus, both the effective potentials of Eqs. (42) and (49)
and (49) take the general form

$$V_{\text{eff}} = C_1 k_1^2 + C_2 \Omega^2,$$

with $C_1$ and $C_2$ being functions of $z$. Two interesting possibilities arise:

- (i) $C_1 < 0$: In this case, the larger the value of $k_1$, the more negative the effective potential gets. Therefore, it is quite reasonable to expect that, for a given size of the interval $\mathcal{I}$, one can always find “large enough” values of $k_1$ — certainly satisfying $k_1^2 > C_2 \Omega^2/C_1$ — such that the Schrödinger-like equation with effective potential $V_{\text{eff}}$ admits null-eigenvalue solutions. We shall refer to this situation as large-$k_1$ instability.

- (ii) $C_1 > 0$ and $C_2 < 0$: Under these conditions, the effective potential $V_{\text{eff}}$, as a function of $k_1$, is bounded from below: $V_{\text{eff}} \geq -|C_2| \Omega^2$. Therefore, a Schrödinger-like equation with effective potential $V_{\text{eff}}$ only admits null-eigenvalue solutions provided $k_1$ is “sufficiently small” — certainly satisfying $k_1^2 < |C_2| \Omega^2/C_1$ — and the size of the interval where $V_{\text{eff}}$ is negative is “sufficiently large.” We shall refer to this situation as minimum-width instability.

Let us call $g_{\Omega k_1}^{(i)}$ the null-eigenvalue solutions mentioned in either case above, with $J \in \{\text{TE, TM}\}$ depending on whether it refers to Eq. (42) or (49) with $\omega^2 = -\Omega^2$ (without loss of generality, $\Omega > 0$). These solutions are associated with unstable electromagnetic modes whose temporal behavior is proportional to $e^{\pm \Omega t}$. Although it might be tempting not to consider these “runaway” solutions, they are essential, if they exist, to expand any arbitrary initial field configuration satisfying the boundary-value problems set by Eqs. (42) and (49); in other words, the stationary modes alone do not constitute a complete set of solutions of Maxwell’s equations with the given boundary conditions. And even if, on the classical level, one might want to restrict attention to initial field configurations which have no contribution coming from these unstable modes — which is certainly unnatural, for causality forbids the system to constrain its initial configuration based on its future behavior —, inevitable quantum fluctuations of these modes would grow, making them dominant some time $e$-foldings $(t \sim N \Omega^{-1}, N \gg 1)$ after the proper material conditions having been engineered. Therefore, these modes are as physical as the oscillatory ones. In fact, artificial inconsistencies have been reported in the literature, regarding field quantization in active media [25, 26], which are completely cured when unstable modes are included in the analysis [44].

It is interesting to note that depending on which constitutive function is negative, Eqs. (42) and (49) may incur in different types of instabilities. For instance, if $\mu_1 < 0$ for a given $\omega^2 = -\Omega^2 < 0$, with all other constitutive functions being positive, then Eq. (42) exhibits case-(i) instability, while Eq. (49) incur in case-(ii) instability. This means that unstable TE modes — with some $k_1 > \sqrt{|\mu_1| \Omega}$ — would certainly be present, while unstable TM modes — with some $k_1 < \sqrt{|\mu_1| \Omega}$ — would only appear if the width of the material (size of the interval $\mathcal{I}$) is larger than some critical value. We shall illustrate these facts in a simple example below. But first, let us analyze some features of these unstable modes. In order not to rely on particular initial field configurations, let us focus on the inevitable quantum fluctuations of these modes.

### 1. Unstable TE modes

Repeating the procedure which led us from Eq. (42) to Eq. (47) for the stable modes, unstable TE modes, $A_{\Omega k_1}^{(\text{TE})}$, properly orthonormalized according to

$$\left( A_{\Omega k_1}^{(\text{TE})}, A_{\Omega k'_1}^{(\text{TE})} \right) = -\left( A_{\Omega k_1}^{(\text{TE})}, A_{\Omega k'_1}^{(\text{TE})} \right) = \delta_{\Omega \Omega'} \delta(k_1, k'_1),$$

and (orthogonal to all other modes) read (up to a time translation)

$$A_{\Omega k_1}^{(\text{TE})} = \frac{k_1 \times \mathbf{n}_z}{4\pi k_1 \sqrt{\Omega \sin \kappa}} e^{ik_1 \cdot \mathbf{x}_z} g_{\Omega k_1}^{(\text{TE})}(z) \left( e^{\Omega t - is^1_xk_1} + e^{-\Omega t + is^1_xk_1} \right),$$

with $0 < \kappa < \pi$, $g_{\Omega k_1}^{(\text{TE})}$ normalized to

$$\int_{\mathcal{I}} dz \epsilon_m(z) g_{\Omega k_1}^{(\text{TE})}(z) g_{\Omega k'_1}^{(\text{TE})}(z) = \delta_{\Omega \Omega'},$$

and $s^1_x$ being the sign of the integral above. Calculating the electric $E_{\Omega k_1}^{(\text{TE})}$ and magnetic $B_{\Omega k_1}^{(\text{TE})}$ fields associated to these modes, we have:

$$E_{\Omega k_1}^{(\text{TE})} = \frac{\sqrt{\Omega}}{4\pi k_1 \sqrt{\sin \kappa}} \left( \mathbf{n}_z \times k_1 \right) e^{ik_1 \cdot \mathbf{x}_z} g_{\Omega k_1}^{(\text{TE})}(z) \left( e^{\Omega t - is^1_xk_1} - e^{-\Omega t + is^1_xk_1} \right),$$

$$B_{\Omega k_1}^{(\text{TE})} = \frac{e^{ik_1 \cdot \mathbf{x}_z}}{4\pi k_1 \sqrt{\Omega \sin \kappa}} \left( -ik_1^2 \mathbf{n}_z + k_1 \frac{d}{dz} \right) g_{\Omega k_1}^{(\text{TE})}(z) \left( e^{\Omega t - is^1_xk_1} + e^{-\Omega t + is^1_xk_1} \right).$$

### 2. Unstable TM modes

Now, turning to the TM modes, we repeat the procedure which led us from Eq. (49) to Eq. (53) for the stable modes. Unstable TM modes, $A_{\Omega k_1}^{(\text{TM})}$, properly orthonormalized according to

$$\left( A_{\Omega k_1}^{(\text{TM})}, A_{\Omega k'_1}^{(\text{TM})} \right) = -\left( A_{\Omega k_1}^{(\text{TM})}, A_{\Omega k'_1}^{(\text{TM})} \right) = \delta_{\Omega \Omega'} \delta(k_1, k'_1),$$

and (orthogonal to all other modes) read (up to a time translation)

$$A_{\Omega k_1}^{(\text{TM})} = \frac{k_1 \times \mathbf{n}_z}{4\pi k_1 \sqrt{\Omega \sin \kappa}} e^{ik_1 \cdot \mathbf{x}_z} g_{\Omega k_1}^{(\text{TM})}(z) \left( e^{\Omega t - is^1_yk_1} + e^{-\Omega t + is^1_yk_1} \right),$$

with $0 < \kappa < \pi$, $g_{\Omega k_1}^{(\text{TM})}$ normalized to

$$\int_{\mathcal{I}} dz \epsilon_m(z) g_{\Omega k_1}^{(\text{TM})}(z) g_{\Omega k'_1}^{(\text{TM})}(z) = \delta_{\Omega \Omega'},$$

and $s^1_y$ being the sign of the integral above. Calculating the electric $E_{\Omega k_1}^{(\text{TM})}$ and magnetic $B_{\Omega k_1}^{(\text{TM})}$ fields associated to these modes, we have:

$$E_{\Omega k_1}^{(\text{TM})} = \frac{\sqrt{\Omega}}{4\pi k_1 \sqrt{\sin \kappa}} \left( \mathbf{n}_z \times k_1 \right) e^{ik_1 \cdot \mathbf{x}_z} g_{\Omega k_1}^{(\text{TM})}(z) \left( e^{\Omega t - is^1_yk_1} + e^{-\Omega t + is^1_yk_1} \right),$$

$$B_{\Omega k_1}^{(\text{TM})} = \frac{e^{ik_1 \cdot \mathbf{x}_z}}{4\pi k_1 \sqrt{\Omega \sin \kappa}} \left( -ik_1^2 \mathbf{n}_z + k_1 \frac{d}{dz} \right) g_{\Omega k_1}^{(\text{TM})}(z) \left( e^{\Omega t - is^1_yk_1} - e^{-\Omega t + is^1_yk_1} \right).$$
The modes given by Eqs. (61) and (67), if present, must be added to the expansion of the field operator $\hat{A}$ given in Eq. (55), along with their complex conjugates— with corresponding annihilation $\hat{a}_{\Omega k_\perp}$ and creation $\hat{a}_{\Omega k_\perp}^{\dagger}$ operators, $J \in \{\text{TE}, \text{TM}\}$. The resulting operator expansion can then be used to calculate electromagnetic-field fluctuations and correlations. In the presence of unstable modes, it is easy to see that the field's vacuum fluctuations are eventually $(t \gg \Omega^{-1})$ dominated by these exponentially-growing modes. Obviously, this instability cannot persist indefinitely as these wild fluctuations will affect the medium's properties, supposedly leading the whole system to a final stable state. In some gravitational contexts, stabilization occurs by decoherence of these growing vacuum fluctuations [13], giving rise to a nonzero classical field configuration — a phenomenon called spontaneous scalarization (for spin-0) [14,17] or vectorization (for spin-1 fields) [18]. It is possible that something similar might occur in the analogous system. We shall discuss this point further in Sec. V.

B. Example

Let us consider a very simple system just to illustrate the results above in a concrete scenario: a slab of width $L$ (in the region $-L/2 < z < L/2$), made of a homogeneous material with, say, $\mu_1 < 0$ for a given $\omega^2 = -\Omega^2$ ($\Omega > 0$) and all other constitutive functions positive. For concreteness sake, here we assume that this value $\omega^2 = -\Omega^2$ is isolated and that it is the most negative value of $\omega^2$ for which $\mu_1 < 0$. This latter assumption is merely a matter of choice, while the former only affects the measure on the set of quantum numbers $k_\parallel$: $\int d^2k_\perp \rightarrow \int d\theta \sum_{k_\parallel} 2\pi k_\parallel / L_\parallel$, $\delta(k_\parallel - k_\parallel') \rightarrow L_\parallel \delta(\theta - \theta')/(2\pi k_\parallel)$, where $L_\parallel$ is the length scale associated with the area of the “infinite” slab ($L_\parallel \gg L$).

According to the discussion presented earlier, in this scenario, TE modes incur in case-(i) (large-$k_\parallel$) instability, while TM modes undergo case-(ii) (minimum-width) instability. The solutions $g_{\Omega k_\parallel}^{(J)}$ of Eqs. (42) and (49) with $\omega^2 = -\Omega^2$ are given by the normalizable — according to Eqs. (62) and (68) — solutions of the null-eigenvalue, Schrödinger-like equation

$$\left(-\frac{d^2}{dz^2} + V_{\text{eff}}\right) g_{\Omega k_\parallel}^{(J)} = 0,$$

with $V_{\text{eff}}$ being the well potential represented in Fig. 1. The depth of the potential is given by

$$V_0 = \frac{|\mu_1| \varepsilon_{\parallel} \Omega^2 + |\mu_1| k_\parallel^2}{\varepsilon_{\parallel} k_\parallel^2}, \quad J = \text{TE}$$

$$V_0 = \frac{|\mu_1| \varepsilon_{\perp} \Omega^2 - |\mu_1| k_\parallel^2}{\varepsilon_{\parallel} k_\parallel^2}, \quad J = \text{TM}.$$

Although here we focus only on unstable modes, associated with $g_{\Omega k_\parallel}^{(J)}$, note that in this example there would also appear stationary bound solutions associated with $\omega_{\parallel k_\parallel}$ — if $\mu_1 < 0$ for some $\omega_0 \in \mathbb{R}$ —, for some $k_\parallel^2 > \max\{\omega_0^2, (n_{\parallel}^{(\text{TE})})^2, (n_{\parallel}^{(\text{TM})})^2\}$, where $n_{\parallel}^{(\text{TE})} = \sqrt{|\mu_1|}$ is the transverse refractive index for the TE modes. For such a hypothetical mode, the slab would act as a waveguide, keeping the mode confined due to total internal reflections at its boundaries. The only peculiar feature here is that $k_\parallel$ would assume arbitrarily large values (in practice, limited only by the inverse length scale below which...
the continuous-medium idealization breaks down) for a given ω₀.

Back to the unstable modes, a straightforward calculation leads to the familiar even and odd solutions to the square-well potential, with \( g^{(1)}_{ΩK_ε} (z) \) exponentially suppressed for \( |z| > L/2 \) and

\[
g^{(1)}_{ΩK_ε} (z) = \begin{cases} \frac{N^{(1)}_m}{\cos a_m} \cos (2a_m z/L), & 0 \leq m \text{ even} \\ \frac{N^{(1)}_m}{\sin a_m} \sin (2a_m z/L), & 1 \leq m \text{ odd} \end{cases}
\]

(72)

\(-L/2 \leq z \leq L/2\), where \( N^{(1)}_m \) are normalization constants and, for the TE modes, \( a_m \geq ΩL/n||/2 \) are solutions of the transcendental equations

\[
\sqrt{|μ_||μ|} \left[ 1 - \frac{Ω^2 L^2}{4a_m^2} |n||^2 \right] - 1 = \begin{cases} -\tan a_m, & m \text{ even} \\ \cot a_m, & m \text{ odd} \end{cases}
\]

(73)

while for the TM modes, \( 0 \leq a_m \leq ΩL/n||/2 \) and

\[
\sqrt{ε_1 ε_2} \left[ \frac{Ω^2 L^2}{4a_m^2} |n||^2 \right] - 1 = \begin{cases} \tan a_m, & m \text{ even} \\ -\cot a_m, & m \text{ odd} \end{cases}
\]

(74)

The transverse momentum \( k_\parallel \) is given in terms of \( a_m \) by

\[
k_\parallel = k^{(m)} \parallel = \begin{cases} \frac{2}{\pi} \sqrt{\frac{μ_||μ||}{|n||}} \left( a_m^2 - |n||^2 \frac{Ω^2 L^2}{4} \right), & \text{TE modes} \\ \frac{2}{\pi} \sqrt{\frac{ε_1}{ε_2}} \left( |n||^2 \frac{Ω^2 L^2}{4} - a_m^2 \right), & \text{TM modes} \end{cases}
\]

(75)

The explicit form of \( N^{(1)}_m \) is not particularly important, so we only present its asymptotic behavior for \( k_\parallel \rightarrow ∞ \) for the TE modes,

\[
N^{(TE)}_m \approx \begin{cases} \sqrt{\frac{2(1+|μ||μ|)}{Lε}}, & m \text{ even}, \ k_\parallel \gg Ω \\ \sqrt{\frac{2(1+|μ||μ|)}{Lε}}, & m \text{ odd}, \ k_\parallel \gg Ω \end{cases}
\]

(76)

and for \( k_\parallel \rightarrow 0 \) for both TE and TM modes,

\[
N^{(TE)}_m \approx \begin{cases} \sqrt{\frac{2(ε_1+|μ||μ|)}{Lε}}, & m \text{ even}, \ k_\parallel \ll Ω \\ \sqrt{\frac{2(ε_1+|μ||μ|)}{Lε}}, & m \text{ odd}, \ k_\parallel \ll Ω \end{cases}
\]

(77)

\[
N^{(TM)}_m \approx \begin{cases} \sqrt{\frac{2(ε_2+|μ||μ|)}{Lε}}, & m \text{ even}, \ k_\parallel \ll Ω \\ \sqrt{\frac{2(ε_2+|μ||μ|)}{Lε}}, & m \text{ odd}, \ k_\parallel \ll Ω \end{cases}
\]

(78)

In Fig. 2 we plot — for different values of ΩL and given values of \( μ_||, μ_±, ε_1, \) and \( ε_± \) — the left-hand side of Eq. (73) (solid black curves in the upper half plane), minus the right-hand side of Eq. (74) (solid black curves in the lower half plane) — substituting, in both, \( a_m \) by the \( m = 1/5 \) \( μ_± = -1 \) \( ε_1 = 2 \) variable \( a \rightarrow \), and the functions \( -\tan a \) and \( \cot a \) (blue dashed lines and red dotted lines, respectively). Crossing of the blue dashed lines (respectively, red dotted lines) with a fixed solid black curve determines values \( a = a_m \) for even (resp., odd) solutions \( g^{(1)}_{ΩK_ε} \) for the corresponding value of ΩL. The figure clearly corroborates our preliminary analysis, showing that unstable TE modes appear with arbitrarily large values of \( a_m \) (and, therefore, of \( k_\parallel \)) and that unstable TM modes only appear if \( L \) is larger than some minimum width \( L_0 \), given by

\[
L_0 = \frac{2Ω^{-1}}{|n||} \tan^{-1} \left( \sqrt{\frac{ε_1}{|μ||}} \right).
\]

(79)

The unstable TE and TM modes inside the slab can then be put in the form

\[
A^{(nTE)}_{ΩK_ε} = \frac{N^{(n)}}{4π k^{(m)}_\parallel / \sqrt{Ω/\sin κ_\parallel}} \cos (2a_m z/L + mπ/2) e^{ik^{(m)}_\parallel \times n} \left( e^{Ωt-ιx_m/2} + e^{-Ωt+ιx_m/2} \right), \ m \geq m^{(TE)}
\]

(80)
\[
A^{(\text{TM})}_{\Omega_k^{(m)}} = \frac{\mathcal{N}_m^{(\text{TM})}}{4\pi \sqrt{\Omega} \sin \kappa_m} \left( e^{i\Omega t - i\kappa_m/2} + e^{-i\Omega t + i\kappa_m/2} \right) \\
\times \left[ k^{(m)}_k n_z \frac{\cos \left( 2a_m z / L + m\pi/2 \right)}{\cos \left( a_m + m\pi/2 \right)} - i \frac{k^{(m)}_k}{\varepsilon_i k^{(m)}_k} \frac{2a_m \sin \left( 2a_m z / L + m\pi/2 \right)}{L \cos \left( a_m + m\pi/2 \right)} \right], \quad 0 \leq m \leq m^{(\text{TM})},
\]
with

\[
m^{(\text{TE})} := \left[ 1 + \left( \frac{L}{L_0} - 1 \right) \frac{2}{\pi} \tan^{-1} \left( \frac{\varepsilon_i}{|\mu_i|} \right) \right], \quad (82)
\]

\[
m^{(\text{TM})} := \left[ \left( \frac{L}{L_0} - 1 \right) \frac{2}{\pi} \tan^{-1} \left( \frac{\varepsilon_i}{|\mu_i|} \right) \right] \quad (83)
\]

\[
B^{(\text{TE})}_{\Omega_k^{(m)}} = \frac{\mathcal{N}_m^{(\text{TE})}}{4\pi \sqrt{\Omega} \sin \kappa_m} \left( e^{i\Omega t - i\kappa_m/2} - e^{-i\Omega t + i\kappa_m/2} \right) \\
\times \left[ -ik^{(m)}_k n_z \frac{\cos \left( 2a_m z / L + m\pi/2 \right)}{\cos \left( a_m + m\pi/2 \right)} + \frac{k^{(m)}_k}{k^{(m)}_k} \frac{2a_m \sin \left( 2a_m z / L + m\pi/2 \right)}{L \cos \left( a_m + m\pi/2 \right)} \right], \quad m \geq m^{(\text{TE})}, \quad (84)
\]

\[
B^{(\text{TM})}_{\Omega_k^{(m)}} = \frac{i\mathcal{N}_m^{(\text{TM})}}{4\pi \sqrt{\Omega} \sin \kappa_m} \left( e^{i\Omega t - i\kappa_m/2} - e^{-i\Omega t + i\kappa_m/2} \right) \\
\times \left[ \frac{1}{L} \frac{2a_m \sin \left( 2a_m z / L + m\pi/2 \right)}{L \cos \left( a_m + m\pi/2 \right)} - i \frac{k^{(m)}_k}{\varepsilon_i k^{(m)}_k} \frac{2a_m \sin \left( 2a_m z / L + m\pi/2 \right)}{L \cos \left( a_m + m\pi/2 \right)} \right], \quad 0 \leq m \leq m^{(\text{TM})}, \quad (85)
\]

Let us recall that these modes give information about fluctuations and correlations of the electromagnetic field; as long as decoherence does not come into play, the expectation values of the field are null, \( \langle A \rangle = \langle E \rangle = \langle B \rangle = 0 \). We shall use these modes later, when discussing possible consequences of these analogue instabilities. But first, let us explore more interesting analogies.

\[\text{IV. SPHERICALLY-SYMMETRIC, STATIONARY ANISOTROPIC MEDIUM}\]

In the previous section, we presented with great amount of detail the canonical quantization scheme for the electromagnetic field in flat spacetime in the presence of arbitrary plane-symmetric anisotropic polarizable/magnetizable media at linear order. The vacuum of such system was then identified with the vacuum of some nonminimally-coupled spin-1 field in a true curved spacetime described by the effective metric \( g_{\alpha\beta} = \sqrt{-\eta} \text{diag}(-n^{-2}, 1, 1, 1) \). The analysis had the advantage of generalizing in a unified language the quantization of various interesting models coming from quantum optics in terms of simple equations (e.g., the Carniglia-Mandel modes \( \text{ES} \)). However, the analogue spacetime for these configurations is of mathematical interest only and does not capture the symmetry of physical spacetimes. In order to study more appealing analogues, in this section we turn to spherically symmetric configurations, presenting them in a more concise way — for the nuances of the quantization were already explained previously. In this context, we may obtain interesting analogues by also assuming that the medium is able to flow. If the refractive index in a flowing material is high enough, such that the velocity of light becomes smaller than the medium’s velocity, then it is clear that a sort of event horizon will form (restricted only to some frequency band which may contain unstable modes). This kind of phenomenon enable us to study analogues of unstable black holes, for instance.
We start working in standard spherical coordinates \((t, r, \theta, \varphi)\), such that \(\eta_{\mu\nu} = \text{diag}(-1, 1, r^2, r^2 \sin^2 \theta)\). Let the medium’s four-velocity field be \(v^\mu = \gamma(1, v, 0, 0)\), where \(v = v(r)\) and \(\gamma = (1 - v^2)^{-1/2}\). The effective-metric components then take the form
\[
\gamma_{\alpha\beta} = \sqrt{n} \begin{pmatrix}
-\gamma^2(n^2 - v^2) & -(1 - n^2)\gamma^2 v & 0 & 0 \\
-(1 - n^2)\gamma^2 v & \gamma^2(1 - n^2v^2) & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix},
\]
where the isotropic parts of the constitutive tensors (in the local, instantaneous rest frame of the medium) are functions of \(r - \varepsilon = \varepsilon(r)\), \(\mu = \mu(r)\) — and, as usual, \(n^2 = n\varepsilon\). As for the traceless anisotropic tensors \(\chi_{(e)}^{\alpha\beta}\) and \(\chi_{ab}^{(\mu)}\), their components read
\[
\chi_{(e)}^{\alpha\beta} = (\Delta^e/3) \left(2\gamma^2 \varepsilon^{\alpha\beta} - \gamma^2 \varepsilon\delta_{\alpha\beta} + 4\gamma^2 v \varepsilon_{\alpha\beta} \right)
+ 2\gamma^2 \delta_{\alpha\beta} \delta_{\nu}^r
\quad -\delta_{\alpha\beta} \gamma^2 \varepsilon_{r} - \delta_{\alpha} \varepsilon_{\nu} \gamma^2 \varepsilon_{r} \sin^2 \theta
\]
and
\[
\chi_{(\mu)}^{\alpha\beta} = (\Delta^\mu/3) \left(2\gamma^2 \varepsilon^{\alpha\beta} - \gamma^2 \varepsilon\delta_{\alpha\beta} + 4\gamma^2 v \varepsilon_{\alpha\beta} \right)
+ 2\gamma^2 \delta_{\alpha\beta} \delta_{\nu}^r
\quad -\delta_{\alpha\beta} \gamma^2 \varepsilon_{r} - \delta_{\alpha} \varepsilon_{\nu} \gamma^2 \varepsilon_{r} \sin^2 \theta.
\]
Similarly to the plane-symmetric case, these anisotropic tensors simply mean that in the instantaneous local rest frame of the medium, its electric permittivity and magnetic permeability in the radial direction \((\varepsilon_1, \mu_1)\) and in the angular directions \((\varepsilon_2, \mu_2)\) satisfy the same relations given below Eqs. \[37,39\] : \(\varepsilon_1 - \varepsilon_2 \equiv \Delta^e\), \(2(\varepsilon_1 + \varepsilon_2)/3 \equiv \varepsilon, \mu_1 - \mu_2 \equiv \Delta^\mu\), and \(2(\mu_1 + \mu_1)/3 \equiv \mu_1^{-1}\).

Not surprisingly, the lab coordinates \((t, r, \theta, \varphi)\) are not the most convenient ones to express Eqs. \[18\] and \[19\] in the case of a moving medium. One might initially think that coordinates \((\tau, r, \theta, \varphi)\) which diagonalize the components of the effective metric, obtained by defining \(\tau = t - p(r)\), with \(p(r)\) satisfying
\[
\frac{dp}{dr} = -\frac{(n^2 - 1)v}{1 - n^2v^2}
\]
would lead to the simplest form of the field equations. In these coordinates, the effective line element \(ds^2_{\text{eff}}\) becomes
\[
ds_{\text{eff}}^2 = \sqrt{n}\left[-n^2F dt^2 + F^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)\right],
\]
where \(F = \gamma^2(1 - n^2v^2)\). It is noteworthy that for \(n = \text{constant} > 0\) (such that the factors of \(n\) in \(ds^2_{\text{eff}}\) can be absorbed via \(r \rightarrow n^{-1}r\) and \(r \rightarrow n^{-1/2}r\)), then the line element above can be made to represent Schwarzschild spacetime by tuning \(v\) so that \(F \equiv (1 - r_s/r)\), where \(r_s\) is some positive constant. This is achieved by a velocity field satisfying \(v^2 = \left[1 + (n^2 - 1)r/r_s\right]^{-1} (n \neq 1)\).

Despite this apparent simplification, the coordinate \(\tau = t - p(r)\) with \(p\) satisfying Eq. \[91\] is not convenient to express Maxwell’s equations in anisotropic media. This is due to the kinematic polarization (resp., magnetization) caused by the magnetic (resp., electric) field. In the case of small velocities and isotropic materials, this effect is modeled by Minkowski’s equations \[45\]. The coordinates \((\tau, r, \theta, \varphi)\) defined using Eq. \[91\] “diagonalizes” only the isotropic part of the theory and do not take into account the anisotropies. It turns out that a much better choice is obtained by setting \(\tau := t - p(r)\) and replacing condition given in Eq. \[91\] by
\[
\frac{dp}{dr} = -\frac{(n^2 - 1)v}{1 - n^2v^2},
\]
where, again, \(n^2_\parallel = \mu_1\varepsilon_1\). This choice fully decouples the electromagnetic field modes in the anisotropic, moving medium, as we shall see below.

Introducing again the 4-potential \(A_\mu\) via \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), in these new coordinates \((\tau, r, \theta, \varphi)\), the convenient (generalized Coulomb) gauge conditions read \(A_\tau = 0\) and
\[
\partial_\tau \left(\varepsilon r^2 A_r\right) + \partial_1 \cdot A_1 = 0,
\]
where \(\varphi\) is merely an auxiliary variable such that \(dr/d\varphi \equiv \gamma^2(1 - n^2v^2)/(\varepsilon_1 r_1)\), \(A_1 = (A_\theta, A_\varphi)\), \(\partial_1\) is the derivative operator on the unit sphere compatible with its metric, and it is understood that \(r\) is a function of the auxiliary variable \(\varphi\). In this gauge, Maxwell’s equations lead to
\[
\begin{align}
-\mu_1 \partial_\tau^2 + \partial_\varphi^2 + \gamma^2(1 - n^2v^2)/\varepsilon_1 r_1^2 \Delta_S^{(0)}(A_1) &= 0, \\
-\varepsilon_1 \partial_\tau^2 + \partial_\varphi^2 + \gamma^2(1 - n^2v^2)/\mu_1 r_1^2 \Delta_S^{(1)}(A_1) &= 0,
\end{align}
\]
where \(\rho\) appearing in Eq. \[96\] is another auxiliary variable defined through \(dr/d\rho \equiv \gamma^2(1 - n^2v^2)/\mu_1\) and \(\Delta_S^{(1)}\) are the Laplacian operators defined on the unit sphere, acting on scalar and vector fields, respectively. In order to solve these equations, we proceed in close analogy to the plane-symmetric case. First, let us find solutions with \(A_r = 0\) — the transverse electric modes, \(A^{(\text{TE})}\). The gauge conditions imply that these solutions can be written as \(A^{(\text{TE})} = (0, \partial_\psi/\sin \theta, -\sin \theta \partial_\psi)\), where \(\psi\) is an auxiliary function to be determined. Making use of the stationarity and spherical symmetry of the present scenario, we can look for field modes of the form \(\psi = e^{-i\omega t}Y_{\ell m}(\theta, \varphi)\) \(f_{\omega\ell}^{(\text{TE})}(r)\), where \(Y_{\ell m}\) are the scalar spherical harmonics. Substituting this into Eq. \[96\], \(f_{\omega\ell}^{(\text{TE})}\) must satisfy
\[
\begin{align}
-\frac{d^2}{dr^2} + \left(\frac{\gamma^2(1 - n^2v^2)\ell(\ell + 1)}{r^2 \mu_1 \mu_1} - \frac{\varepsilon_1 \omega^2}{\mu_1}\right) f_{\omega\ell}^{(\text{TE})} &= 0.
\end{align}
\]
where it is understood that \( r \) is a function of the auxiliary variable \( \rho \). Notice the similarity between this equation and Eq. (12). In fact, the boundary condition given by Eq. (22) assumes the same form here as it does in the plane-symmetric case:

\[
\left[\frac{f_{\omega \ell}^{(TE)}}{d\rho} - \frac{d}{d\rho} f_{\omega \ell}^{(TE)} - \omega^2 f_{\omega \ell}^{(TM)} \frac{d}{d\rho} f_{\omega \ell}^{(TM)}\right]_{\sigma} = 0.
\]  

(98)

This boundary condition ensures that these modes can be orthonormalized according to the sesquilinear form given in Eq. (21), which in this spherically-symmetric scenario assumes the form

\[
(A, A') = \int_{\Sigma} (A_\ell \partial_\ell A'_\ell + \frac{\varepsilon}{\gamma}(1 - n_0^2 v^2) \frac{\gamma^2(n_0^2 - 1)v}{\mu_\perp} \frac{\varepsilon}{\gamma} (A_\perp \partial_\perp A'_\perp) - (\Lambda_\perp \partial_\perp A'_\perp - (\Lambda_\perp \partial_\perp) A'_\perp),
\]  

(99)

with \( \Sigma \) being a spacelike surface \( t = \) constant. After some tedious but straightforward manipulations (presented in the appendix), we obtain the final form of normalized, positive-frequency TE modes:

\[
A_{\omega \ell m}^{(TE)} = \frac{(0, im/\sin \theta, -\sin \theta \partial_\theta)}{\sqrt{2\omega(\ell + 1)}} e^{-i\omega t} Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(TE)}(r),
\]  

(100)

with \( f_{\omega \ell}^{(TE)} \) satisfying Eqs. (97) and (98), and normalized according to

\[
\int_{\mathcal{I}_\rho} d\rho f_{\omega \ell}^{(TE)} f_{\omega \ell}^{(TE)} = \delta_{\omega \omega'}. \]  

(101)

Note that the integration variable is \( \rho \) [instead of \( \rho \) appearing in Eq. (97)] and \( \mathcal{I}_\rho \) stands for the domain of integration in this variable corresponding to \( \mathcal{I} \) in coordinate \( r \).

Now, let us look for solutions with \( A_r \neq 0 \) — the transverse magnetic modes, \( A^{(TM)} \). Let \( \phi \) be such that \( \Delta^{(0)} S \phi = -r^2 \partial_\rho A_r \). Thus, the gauge conditions lead to \( A^{(TM)} = (-r^2 - \varepsilon_\parallel^2 \Delta^{(0)} S, \partial_\rho \partial_\rho, -\partial_\varphi \partial_\varphi) \psi \). Using again stationarity and spherical symmetry, \( \phi = e^{-i\omega t} Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(TM)}(r) \), we obtain that \( f_{\omega \ell}^{(TM)}(r) \) satisfies

\[
\left[ -\frac{d^2}{d\rho^2} + \frac{\gamma^2(1 - n_0^2 v^2)\ell(\ell + 1)}{r^2 \varepsilon_\parallel \varepsilon_\parallel} - \frac{\mu_\perp \omega^2}{\varepsilon_\perp} \right] f_{\omega \ell}^{(TM)} = 0. \]  

(102)

Notice, again, the similarity between this equation and Eq. (49). And, again, the boundary condition imposed by Eq. (22) to these modes take the same form as in the plane-symmetric case:

\[
\left[\omega^2 f_{\omega \ell}^{(TM)} d f_{\omega \ell}^{(TM)} - \omega^2 f_{\omega \ell}^{(TM)} d f_{\omega \ell}^{(TM)}\right]_{\sigma} = 0. \]  

(103)

Properly orthonormalizing these modes using Eq. (A1) — see appendix —, leads to the positive-frequency TM normal modes

\[
A_{\omega \ell m}^{(TM)} = \frac{r^{-2\varepsilon_\parallel^2 - 1} f(\ell + 1, \partial_\rho \partial_\rho, \partial_\varphi \partial_\varphi)}{\sqrt{2\omega^2(\ell + 1)}} e^{-i\omega t} Y_{\ell m}(\theta, \varphi) f_{\omega \ell}^{(TM)}(r),
\]  

(104)

with \( f_{\omega \ell}^{(TM)} \) satisfying Eqs. (102) and (103), and normalized according to

\[
\int_{\mathcal{I}_\rho} d\rho f_{\omega \ell}^{(TM)} f_{\omega \ell}^{(TM)} = \delta_{\omega \omega'}. \]  

(105)

Similarly to the TE case, note that the integration variable is not the same which appears in the differential equation, Eq. (102) (\( \mathcal{I}_\rho \) stands for the domain of integration in the variable \( \rho \) corresponding to \( \mathcal{I} \) in coordinate \( r \)).

The electromagnetic field operator can be represented in terms of the TE and TM modes (and their complex conjugates) as

\[
\hat{A} = \sum_{J(TE,TM)} \sum_{\ell m} \int_{\mathcal{I}_\rho} d\omega \left( a_{\omega \ell m}^{(J)} A_{\omega \ell m}^{(J)} + \text{H.c.} \right),
\]  

(106)

where \( \mathcal{E}^{(J)} := \mathcal{E}^{(J)} \cap \mathbb{R}^+ \), with \( \mathcal{E}^{(J)} \) being the set of \( \omega \) values for which Eqs. (97) and (98), for \( J = \text{TE} \), and Eqs. (102) and (103), for \( J = \text{TM} \), have nontrivial solutions. The orthonormality of TE and TM modes,

\[
\left( A_{\omega \ell m}^{(J)}, A_{\omega \ell m'}^{(J)} \right) = \delta_{\omega \omega'} \delta_{\ell \ell'} \delta_{mm'},
\]

(107)

requires that the canonical commutation relations

\[
\left( a_{\omega \ell m}^{(J)}, a_{\omega \ell m'}^{(J)} \right) = \delta^{J \ell '} \delta_{\omega \omega'} \delta_{\ell \ell'} \delta_{mm'},
\]

(108)

\[
\left( a_{\omega \ell m}^{(J)}, a_{\omega \ell m'}^{(J)} \right) = 0,
\]

(109)

hold.

**A. Instability analysis**

The close similarity between Eqs. (42) and (47) and between Eqs. (49) and (102) make the instability analysis in this spherically-symmetric scenario essentially identical to the one performed in the plane-symmetric case, with \( \ell(\ell + 1) \) playing the role of \( k_1^2 \) did in Eq. (58). So, putting the effective potentials of Eqs. (47) and (102), with \( \omega^2 = -\Omega^2 \), in the form

\[
V_{\text{eff}} = C_1 \ell(\ell + 1) + C_2 \Omega^2,
\]

(111)
we again have two types of instabilities: (i) large-\(l\) instability, when \(C_1 < 0\) somewhere, and (ii) minimum-thickness instability, when \(C_1 > 0\) but \(C_2 < 0\) in a sufficiently thick spherical shell — see discussion below Eq. (58). The only additional feature is that, by allowing the medium to flow, type-(i) (large-\(l\), sufficiently thick spherical shell — see discussion below) instability, when \(\omega^2 = -\Omega^2\) (\(\Omega > 0\), without loss of generality). The normalized, unstable modes are presented below — see appendix for details.

1. Unstable TE modes

Unstable TE modes orthonormalized according to the analogous of Eqs. (107) and (108) read (up to global phase and time translation)

\[
A_{\Omega \ell m}^{(\text{TE})} = \frac{\left( e^{\Omega r - is^*_{\ell} \kappa/2} + e^{-\Omega r + is^*_{\ell} \kappa/2} \right) g_{\Omega \ell}^{(\text{TE})}(r)}{2\sqrt{\Omega(l+1)\sin\kappa}} \times (0, im/\sin \theta, -\sin \theta \partial_\theta) Y_{\ell m}(\theta, \varphi),
\]

(112)

with \(\kappa\) being a constant \((0 < \kappa < \pi)\), \(g_{\Omega \ell}^{(\text{TE})}\) normalized according to

\[
\left| \int_\mathbb{R} d\xi \frac{\varepsilon_+}{\gamma^2(1 - n_\|^2v^2)} g_{\Omega \ell}^{(\text{TE})}(r) g_{\Omega \ell}^{(\text{TE})}(r) \right| = \delta_{\Omega \ell r},
\]

(113)

and \(s^\dagger_{\ell}\) being the sign of the integral above. Calculating the electric \(E_{\Omega \ell m}^{(\text{wTE})}\) and magnetic \(B_{\Omega \ell m}^{(\text{wTE})}\) vector fields associated to these modes in the \textit{lab frame}, we have:

\[
E_{\Omega \ell m}^{(\text{wTE})} = \frac{\sqrt{\Omega}(-im \mathbf{e}_\theta/\sin \theta + \mathbf{e}_\varphi \partial_\varphi)}{2r\sqrt{l(l+1)\sin\kappa}} g_{\Omega \ell}^{(\text{TE})}(r) Y_{\ell m}(\theta, \varphi) \left( e^{\Omega r - is^*_{\ell} \kappa/2} + e^{-\Omega r + is^*_{\ell} \kappa/2} \right),
\]

(114)

\[
B_{\Omega \ell m}^{(\text{wTE})} = \left[ \frac{(l+1) \mathbf{e}_r + (im \mathbf{e}_\varphi/\sin \theta + \mathbf{e}_\theta \partial_\theta)}{2r^2\sqrt{l(l+1)\sin\kappa}} \right] g_{\Omega \ell}^{(\text{TE})}(r) Y_{\ell m}(\theta, \varphi) \left( e^{\Omega r - is^*_{\ell} \kappa/2} + e^{-\Omega r + is^*_{\ell} \kappa/2} \right).
\]

(115)

with, again, \(\kappa\) being a constant \((0 < \kappa < \pi)\), \(g_{\Omega \ell}^{(\text{TM})}\) normalized according to

\[
\left| \int_\mathbb{R} d\xi \frac{\mu_+}{\gamma^2(1 - n_\|^2v^2)} g_{\Omega \ell}^{(\text{TM})}(r) g_{\Omega \ell}^{(\text{TM})}(r) \right| = \delta_{\Omega \ell r},
\]

(117)

and \(s^\dagger_{\ell}\) being the sign of the integral above. Calculating the electric \(E_{\Omega \ell m}^{(\text{wTM})}\) and magnetic \(B_{\Omega \ell m}^{(\text{wTM})}\) vector fields associated to these modes in the \textit{lab frame}, we have:

\[
E_{\Omega \ell m}^{(\text{wTM})} = \frac{(-r^2s^\dagger_{\ell} \mathbf{e}_r/\sin\kappa - (im \mathbf{e}_\varphi/\sin \theta + \mathbf{e}_\theta \partial_\theta) \partial_\varphi)}{2r^2\sqrt{l(l+1)\sin\kappa}} g_{\Omega \ell}^{(\text{TM})}(r) Y_{\ell m}(\theta, \varphi) \left( e^{\Omega r + is^*_{\ell} \kappa/2} + e^{-\Omega r - is^*_{\ell} \kappa/2} \right),
\]

(118)

\[
B_{\Omega \ell m}^{(\text{wTM})} = \frac{\mu_+\sqrt{\Omega}(-im \mathbf{e}_\theta/\sin \theta + \mathbf{e}_\varphi \partial_\varphi)}{2r^2\sqrt{l(l+1)\sin\kappa}} g_{\Omega \ell}^{(\text{TM})}(r) Y_{\ell m}(\theta, \varphi) \left( e^{\Omega r + is^*_{\ell} \kappa/2} + e^{-\Omega r - is^*_{\ell} \kappa/2} \right).
\]

(119)

As argued in the previous case, when instability is triggered and modes \(A_{\Omega \ell m}^{(u)}\) appear, they must be included in the field expansion given by Eq. (106), along with their complex conjugates. Eventually \((t \gg \Omega^{-1})\), these modes
dominate the field fluctuations.

B. Example

Now, let us consider a concrete scenario where electromagnetism in a gravitationally interesting system, nonminimally coupled to the background geometry via $\chi^{abcd}$ given by Eq. (23) (but with arbitrary $\alpha_1, \alpha_2, \alpha_3$), can be mimicked by an anisotropic, stationary moving medium. We have already seen that setting $n = \text{constant}$ and $v^2 = [1 + (n^2 - 1) r/r_s]^{-1}$, leads to an effective line element which describes the vacuum Schwarzschild spacetime. In this case, Eq. (29) is trivially satisfied and Eqs. (26–28) give

$$\mu = n, \quad \Delta((\varepsilon)) = 3\alpha_1 n^{1/2} r_s / r^3, \quad \Delta((\mu)) = 3\alpha_1 r_s / n^{3/2} r^3,$$

which lead to the material properties

$$\varepsilon_\perp = n \left(1 - \frac{3\alpha_1 r_s}{n^{1/2} r^3}\right), \quad \varepsilon_\parallel = n \left(1 + \frac{2\alpha_1 r_s}{n^{1/2} r^3}\right), \quad \mu_\perp = n \left(1 - \frac{3\alpha_1 r_s}{n^{1/2} r^3}\right)^{-1}, \quad \mu_\parallel = n \left(1 + \frac{2\alpha_1 r_s}{n^{1/2} r^3}\right)^{-1}.$$

We promptly see that $n_\parallel := \frac{3\alpha_1 r_s}{n^{1/2} r^3} = n$, which shows that the analogue horizon for these nonminimally-coupled modes, located where $v^2 = n_\parallel^2$, coincides with the analogue Schwarzschild radius $r_s$. [Note, however, that this system is analogous to a physical black hole with Schwarzschild radius $R_s = n^{1/2} r_s$, due to absorption of $\sqrt{n}$ in Eq. (92)] As for the other refractive indices, $n_\perp(\text{TE}) := \frac{3\alpha_1 r_s}{n^{1/2} r^3}$ and $n_\perp(\text{TM}) := \frac{2\alpha_1 r_s}{n^{1/2} r^3} = n^2 / n_\parallel(\text{TE})$, Fig. 3 shows their squared values (in black and red, respectively) for positive (solid lines) and negative (dashed lines) values of $\alpha_1$. Note that, depending on the values of $\alpha_1 / (n^{1/2} r_s^2)$, some kind of metallic material (possibly with some negative squared refractive indices) may be needed in order to mimic this nonminimal coupling of the electromagnetic field with the Riemann curvature tensor in the exterior region of a Schwarzschild black hole. Conversely, regardless how difficult it may be to set up such an experimental configuration in the lab, it is interesting in its own that QED-inspired nonminimally-coupled electromagnetism in the background of a black hole behaves as in such an exotic metamaterial in flat spacetime.

Turning to the question of possible instabilities, in Fig. 4 we show the behavior of the terms $C_1$ and $C_2$ appearing in Eq. (111) for the TE (in blue) and TM (in red) modes — extracted, respectively, from Eqs. (97) and (102):

$$C_1 = \left\{ \begin{array}{ll}
\frac{n^{-2} r^{-9} (r - r_s)}{2} & \left(\frac{r^3 + \alpha_1 r_s}{\sqrt{n}}\right) \left(\frac{r^3 - \alpha_1 r_s}{\sqrt{n}}\right), \\
\frac{r^{-9} (r - r_s)^3}{2} & \left(\frac{r^{-3} - \alpha_1 r_s / \sqrt{n}}{r^{-3} + \alpha_1 r_s / \sqrt{n}}\right),
\end{array} \right. \quad (127)$$

$$C_2 = \left\{ \begin{array}{ll}
\left(1 - \frac{\alpha_1 r_s}{r^3 / \sqrt{n}}\right)^2 & \left(1 - \frac{\alpha_1 r_s}{\sqrt{n} \sqrt{n}}\right)^2,
\end{array} \right. \quad (128)$$

where the first and second lines in the expressions above refer to the TE and TM modes, respectively. The Fig. 4(a) is representative of the behavior of $C_1$ for $-r_s^2 \sqrt{n}/2 < \alpha_1 < r_s^2 \sqrt{n}$, while Fig. 4(b) gives the correct qualitative behavior of $C_1$ for $\alpha_1 < -r_s^2 \sqrt{n}/2$ or $\alpha_1 > r_s^2 \sqrt{n}$. Figs. 4(c) and 4(d) show the behavior of $C_2$ for the same values of $\alpha_1$ used in Figs. 4(a) and 4(b).

It is clear, from the expressions above, that $C_2$ is everywhere non-negative, while $C_1$ assumes negative values in the region with radial coordinate $r$ between $(\alpha_1 r_s / \sqrt{n})^{1/3}$ and $r_s$ (if $\alpha_1 > 0$) or between $\left\lceil (\alpha_1 r_s / (2 \sqrt{n}))^{1/3}\right\rceil$ and $r_s$ (if $\alpha_1 < 0$). Therefore, according to the discussion of Subsec. [IV A] this nonminimally-coupled electromagnetic theory in Schwarzschild spacetime exhibits large-$\ell$ instability. In particular, if $\alpha_1 > r_s^2 \sqrt{n}$ or $\alpha_1 < -2r_s^2 \sqrt{n}$, then the unstable modes influence the exterior region of the back hole.

V. STABILIZATION: SPONTANEOUS VECTORIZATION, PHOTO PRODUCTION, AND LONG-RANGE INDUCED CORRELATIONS

We now turn our attention to discussing what can possibly happen to the analogous system when the vacuum instability is triggered. In the gravitational scenario, it has been shown that in some cases (for instance,
FIG. 4: Plot of the coefficients $C_1$ — (a) and (b) — and $C_2$ — (c) and (d) — appearing in Eq. (111) for electromagnetic modes TE (blue curves) and TM (red curves), nonminimally coupled to the background geometry of a Schwarzschild black hole via Eq. (23). Figs. (a) and (c) illustrate the general behavior of $C_1$ and $C_2$ for $-\frac{r_s^2}{\sqrt{n}n} < \alpha_1 < \frac{r_s^2}{\sqrt{n}n}$, while (b) and (d) are representative of the behavior of $C_1$ and $C_2$ for $\alpha_1 < -\frac{r_s^2}{\sqrt{n}n}$ or $\alpha_1 > \frac{r_s^2}{\sqrt{n}n}$. According to the instability discussion, only large-$\ell$ instability can appear in this case, since $C_2 \geq 0$ everywhere. Moreover, for $\alpha_1 < -\frac{r_s^2}{\sqrt{n}n}$ or $\alpha_1 > \frac{r_s^2}{\sqrt{n}n}$, the unstable modes can be mostly supported outside the analogous event horizon, $r > r_s$.

depending on the field-background coupling), stabilization occurs due to the appearance of a nonzero value for the field (spontaneous scalarization/vectorization) [14–18], seeded by decoherence of the growing initial-vacuum fluctuations [13]. In this process, field particles/waves are produced [14, 28] and carry away the energy excess of the initial vacuum state in comparison to the stabilized configuration.

If we transpose these conclusions, mutatis mutandis, to our analogous systems, then an electromagnetic field should spontaneously appear in the material, bringing the whole system to a new equilibrium configuration — through nonlinear effects brought in by field-dependent constitutive tensors $\varepsilon^{ab}$ and $\mu^{ab}$ [see Eqs. (7,8)] —, with photons being emitted, carrying away the energy excess. Although the detailed dynamics of the stabilization processes in the gravitational and in the analogous systems are quite different — ruled by Einstein equations in the gravitational case and by the macroscopic Maxwell’s equations with field-dependent $\varepsilon^{ab}$ and $\mu^{ab}$ in the analogous systems —, the qualitative features of the whole process, described above, seem quite reasonable to occur in generic field stabilization processes.

It is important to mention that the time scale set by the instability, $\Omega^{-1}$, is typically of the order of the time light takes to travel the typical size of the system, $L$. Therefore, in the analogous lab scenarios, the stabilization process would occur almost instantaneously ($\sim L/(1\text{ cm}) \times 10^{-10}\text{ s}$) once the instability conditions are met — which, for a given system, may depend on external parameters such as temperature, external fields, etc., through their influence on the constitutive functions $\varepsilon_1$, $\varepsilon_3$, $\mu_1$, $\mu_3$. The whole process would most likely be interpreted as a kind of phase transition, where the
and vectorization take over. Notice that once minimum-

in Fig. 2 and four different values of Ω

L

are obtained — in particular, creation \( \hat{a}_{\mathbf{k}}^{(u)\dagger} \) operators. It is easy to see that the field’s vacuum fluctuations and correlations are eventually \( (t, t' \gg \Omega^{-1}) \) dominated by these unstable modes — at least as long as decoherence does not come into play. The dominant contribution to the vacuum correlations in the example of Subsec. III B reads (the reader should refer to Subsec. III B for the definition of all quantities appearing in these expressions):

\[
\begin{align*}
\langle A_j(x) A_l(x') \rangle &= \frac{2\pi}{L_k \Omega} \int_0^{2\pi} d\phi \left( \sum_{m=0}^{(TM)} k_{j}^{(m)} \left[ A_{TMj}^{(u)}(x) \right] \right) \left[ A_{TMl}^{(u)}(x') \right] + \sum_{m=0}^{(TE)} k_{j}^{(m)} \left[ A_{TEj}^{(u)}(x) \right] \left[ A_{TEl}^{(u)}(x') \right] \\
&\sim \frac{e^{\Omega(t+t')}}{4L_k \Omega} \left\{ \sum_{m=0}^{(TM)} \frac{k_{j}^{(m)}}{\sin \kappa_m} D_{TMj}^{(TM)}(d_j) g_{TMj}^{(TM)}(z) g_{TMl}^{(TM)}(z') \right. \right. \\
&\left. \left. + \sum_{m=0}^{(TE)} \frac{k_{j}^{(m)}}{\sin \kappa_m} D_{TEj}^{(TE)}(d_j) g_{TEj}^{(TE)}(z) g_{TEl}^{(TE)}(z') \right\},
\end{align*}
\]

where \( \varphi \) is the angle between \( \mathbf{k}_j \) and \( (\mathbf{x}_j - \mathbf{x}_j') \), \( d_j \equiv \parallel \mathbf{x}_j - \mathbf{x}_j' \parallel \), and the operators \( D_{TMj}^{(j)}(d_j) \) acting on \( g_{TMj}^{(j)}(z) g_{TMl}^{(j)}(z') \) are defined by

\[
\begin{align*}
D_{TMj}^{(j)}(d_j) &:= \frac{1}{\Omega^2 z_{j1}^2} \left[ J_1 \left( k_{j}^{(m)} d_j \right) \delta_{j}^{z} \delta_{j}^{z} + J_1 \left( k_{l}^{(m)} d_l \right) \delta_{j}^{z} \delta_{l}^{z} \right] \frac{d^2}{dz dz'} + \frac{\left( k_{j}^{(m)} \right)^2 J_0 \left( k_{j}^{(m)} d_j \right)}{\Omega^2 z_{j1}^2} \delta_{j}^{z} \delta_{j}^{z} \\
&= \frac{k_{j}^{(m)} J_1 \left( k_{j}^{(m)} d_j \right)}{\Omega^2 z_{j1}^2} \left( \delta_{j}^{z} \delta_{j}^{z} \frac{d}{dz} - \delta_{j}^{z} \delta_{j}^{z} \frac{d}{dz'} \right),
\end{align*}
\]

\[
\begin{align*}
D_{TEj}^{(j)}(d_j) &:= \frac{J_1 \left( k_{j}^{(m)} d_j \right)}{k_{j}^{(m)} d_j} \delta_{j}^{z} \delta_{j}^{z} + J_1 \left( k_{l}^{(m)} d_l \right) \delta_{j}^{z} \delta_{l}^{z},
\end{align*}
\]

with indices \( \ell \) and \( \varphi \) standing for vector components along \( (\mathbf{x}_j - \mathbf{x}_j') \) and \( \mathbf{n}_x \times (\mathbf{x}_j - \mathbf{x}_j') \), respectively; \( J_n \) and \( J'_n \) stand for the Bessel functions of first kind and their first derivatives, respectively. Field correlations \( \langle E_j(x) E_l(x') \rangle \) and \( \langle B_j(x) B_l(x') \rangle \) can be similarly obtained — in particular, \( \langle E_j(x) E_l(x') \rangle \sim \Omega^2 (A_j(x) A_l(x')) \). As an illustration, in Fig. 2 we plot the equal-time \( (t, t' \gg \Omega^{-1}) \), longitudinal correlation function \( \langle A_l(x) A_l(x') \rangle \) for points \( x, x' \) in the plane \( z = 0 \), for the same values of constitutive functions used in Fig. 2 and four different values of \( \Omega L \). The vertical-axis scale is arbitrary — but the same in all plots —, since the correlations grow exponentially in time, from their typical (stable-vacuum) values of order \( h/(cL_d) \sim \left[ 1 \text{ cm}^2/(L_d \text{ cm}) \right] \times 10^{-8} \text{ eV}/(\text{cm}^2 \text{ GHz}^2) \), until decoherence and vectorization take over. Notice that once minimum-

width (TM) instability sets in, macroscopic \( (~ L \) field correlations are enhanced. It is an interesting question whether any such “long-range” correlation would survive or leave an imprint in the final stable configuration. Although not directly relevant for the analogy with gravity-induced instability itself, such correlations might lead to interesting material behavior.

VI. FINAL REMARKS

We have shown that gravity-induced instabilities, related to the vacuum-awakening effect in the quantum context [9,11,28] and spontaneous scalarization/vectorization in the classical one [13,18], can be mimicked by electromagnetism in anisotropic metama-
materials with appropriate constitutive functions. This follows from the formal analogy between electromagnetism in anisotropic media and nonminimally-coupled electromagnetism in curved spacetimes, presented in Sec. [I]. We explored two concrete scenarios: (i) a plane-symmetric, static slab — whose main interest is its simplicity regarding experimental setup (see Sec. [III]) — and (ii) a spherically-symmetric, moving media — whose main feature is its analogy with QED-inspired nonminimally-coupled electromagnetism in Schwarzschild spacetime anisotropic media — whose main inter-

Once instability is triggered in the analogous systems, some stabilization process must take place, leading the system to a new stable configuration. The details of this stabilization process and of the final configuration will most likely depend on specific nonlinear properties of the metamaterial, but it seems reasonable that they might involve the appearance of nonzero electromagnetic fields in the material (analogous to spontaneous vectorization in curved spacetimes) and photo production which carries away the energy excess with respect to the stable configuration. As discussed earlier, the time scale involved in the stabilization process can be very short ($\sim 10^{-10}$ s), which would make it very difficult to even identify the unstable phase. This is similar to what might occur with negative conductivity, which has never been directly measured but which is predicted to lead to zero-dec-resistance states observed in laboratory — although an alternative explanation has been proposed.

Clearly, the feasibility of such analogues is bound to the existence of material configurations with the required constitutive functions. As briefly pointed out in the introduction, this can be achieved at least for anisotropic neutral plasmas, and the recent advances in metamater-

\[ \Omega L = 2 \]

\[ \Omega L = 10 \]

\[ \Omega L = 5 \]

\[ \Omega L = 20 \]

**FIG. 5:** Equal-time ($t = t' \gg \Omega^{-1}$), two-point correlation function $\langle A_\ell(x)A_\ell(x') \rangle$ of the component of the quantum field $\hat{A}$ along the vector-separation $\vec{x}_i - \vec{x}'_i$, for points in the $z = 0$ plane, for different values of $\Omega L$ — with same values of constitutive functions given in Fig. 2. The dotted (blue) lines represent the contribution coming from the TE modes, while the dashed (red) lines depict the contribution coming from the TM modes. The solid (black) lines give the sum of both contributions. Notice that long-range ($|\vec{x}_i - \vec{x}'_i| \geq L$) correlations are mainly due to the TM modes, which undergo minimum-width instability.
rial science offer a plethora of possible candidates, specially the hyperbolic metamaterials \cite{12,13}, that possess precisely the form given in Eqs. (9,10) with the required “negativeness.” In particular, we call attention to the increase in the spontaneous light emission in such configurations, which may be related to the process of stabilization in active scenarios.

It is also important to mention that the QED-inspired analogues (Subsec. II A) are not restricted to the study of vacuum instability. For instance, they can be used to study light ray propagation in the corresponding spacetimes and one possible application is the QED-induced birefringence in the Schwarzschild spacetime \cite{33}. For this particular experiment, one can work far from the effective horizon, where the constitutive coefficients \cite{123,126} are positive.

Our main purpose here was to lay down a novel class of analogue models of curved-spacetime phenomena, with main interest on the gravitational side of the analogy. Notwithstanding, the consequences of the analogue gravity-induced instability to the metamaterial side may be interesting on its own. The electromagnetic field instability may mark, lead or mediate some kind of phase transition in the metamaterial, where the spontaneously created field and/or its amplified “long-range” correlations may play some important role (see discussion in Sec. V). Investigation in these lines are currently in course and will be presented elsewhere.

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Appendix A: Normalization of stable and unstable modes in the spherically symmetric case

Here, we present in detail the calculations involved in normalizing the electromagnetic modes in the spherically symmetric case. Since we are dealing with analogues to which there is a natural physical notion of time — the lab-frame time \( t \) —, it is convenient to use \( t = \text{constant} \) surfaces (\( \Sigma_t \)) to normalize the modes. Obviously, this choice bears no physical consequence on our results.

The sesquilinear form given in Eq. (21), applied to the scenario described in Sec. IV, takes the form — notice that the integrand is a scalar and, as such, can be evaluated in any coordinate system:

\[
(A, A') = i \int_{\Sigma_t} d^2 \epsilon[\hat{A}_l \cdot \partial_\tau A'_l + \frac{\epsilon_\perp \hat{A}_l \cdot \partial_\tau A'_l}{\gamma^2 (1 - n_\parallel^2 u^2)} + \frac{\gamma^2 (n_\parallel^2 - 1) v}{\mu_\perp} [\hat{A}_l \cdot \partial_\tau A'_l - (\hat{A}_l \cdot \partial_\tau) A'_l] - (\hat{A} \leftrightarrow A').
\]  

(A1)

Below, we evaluate this expression for each type of mode.

1. TE modes

a. Stable

Substituting \( A^{(TE)} = (0, \partial_\varphi / \sin \theta, -\sin \theta \partial_\theta \psi) \) into Eq. (A1), with \( \psi = e^{-i \omega \tau} Y_{\ell m}(\theta, \varphi) f^{(TE)}_{\omega \ell}(r) \), one gets:

\[
(A^{(TE)}, A'^{(TE)}) = \int_{\Sigma_2} dS \left[ (\partial_\varphi Y_{\ell m})(\partial_\varphi Y_{\ell' m'}) + \frac{mn'}{\sin^2 \theta} Y_{\ell m} Y_{\ell' m'} \right] 
\]

\[
\times \int_{I} d\rho e^{i(-\omega - \omega') \tau} \left[ (\omega + \omega') \frac{\epsilon_\perp}{\mu_\perp} f^{(TE)}_{\omega \ell} f^{(TE)}_{\omega' \ell'} + \frac{\gamma^2 (n_\parallel^2 - 1) v}{\mu_\perp} \left( f^{(TE)}_{\omega \ell} \frac{d}{d\rho} f^{(TE)}_{\omega' \ell'} - f^{(TE)}_{\omega' \ell'} \frac{d}{d\rho} f^{(TE)}_{\omega \ell} \right) \right],
\]  

(A2)

where \( S_2 \) is the unit sphere, recall that \( dr/d\rho = \gamma^2 (1 - n_\parallel^2 u^2)/\mu_\perp \), and it is understood that this last integral must be evaluated at \( \tau + p(r) = t = \text{constant} \) [recall definition of \( \tau \) right above Eq. (21)]. It is straightforward to show that the first integral evaluates to \( \ell (\ell + 1) \delta_{\ell \ell'} \delta_{m m'} \) provided we normalize \( Y_{\ell m} \) according to \( \int_{S_2} dS Y_{\ell m} Y_{\ell' m'} = \delta_{\ell \ell'} \delta_{m m'} \). As for the second integral, let us first consider the quantity

\[
W_{\omega \omega'}^{(\ell)} := \frac{1}{(\omega - \omega')} \left( f^{(TE)}_{\omega \ell} \frac{d}{d\rho} f^{(TE)}_{\omega' \ell} - f^{(TE)}_{\omega' \ell} \frac{d}{d\rho} f^{(TE)}_{\omega \ell} \right).
\]  

(A3)

Making use of Eq. (97), \( W_{\omega \omega'}^{(\ell)} \) clearly satisfies

\[
\frac{d}{d\rho} W_{\omega \omega'}^{(\ell)} = \frac{\epsilon_\perp}{\mu_\perp} (\omega + \omega') f^{(TE)}_{\omega \ell} f^{(TE)}_{\omega' \ell'}.
\]  

(A4)

Therefore,

\[
(A^{(TE)}, A'^{(TE)}) = \ell (\ell + 1) \delta_{\ell \ell'} \delta_{m m'} 
\]

\[
\times \int_{I} d\rho e^{i(-\omega - \omega') \tau} \left[ \frac{d}{d\rho} W_{\omega \omega'}^{(\ell)} - i(\omega - \omega') \frac{d}{d\rho} W_{\omega \omega'}^{(\ell)} \right] 
\]

\[
\times \int_{I} d\rho e^{i(-\omega - \omega') \tau + p(r)} \frac{d}{d\rho} \left( e^{-i(\omega - \omega') p(r)} W_{\omega \omega'}^{(\ell)} \right) 
\]

\[
= \ell (\ell + 1) \delta_{\ell \ell'} \delta_{m m'} e^{i(-\omega - \omega') \tau} \left[ e^{-i(\omega - \omega') p(r)} W_{\omega \omega'}^{(\ell)} \right]_{I}.
\]  

(A5)

where we made use that \( t = \tau + p(r) \) is kept constant along integration in \( r \) (or \( \rho \)) and \( \left[ \right]_{I} \) indicates that we must calculate the flux of the quantity in square brackets at the boundaries of \( I \). We see that in order to guarantee orthogonality between modes with different \( \omega \), without
worrying about the specific form of \( p(r) \), we must impose boundary conditions at \( z \) such that, in Eq. (A3), 
\[ W_{\omega'}^{(l)} \big|_{z} = 0 \] 
for \( \omega \neq \omega' \). Then, referring back to Eq. (A4) and writing
\[ W_{\omega'}^{(l)}(\rho) = (\omega + \omega') \int_{\rho_{\omega'}}^{\rho} d\rho' \frac{\varepsilon_{z}}{\mu_{z}} f_{\omega'}^{(TE)} f_{\omega'}^{(TE)}, \quad (A6) \]
we finally obtain
\[ (A^{(TE)}, A'^{(TE)}) = 2\omega\ell(\ell + 1)\delta_{\ell\ell'}\delta_{\ell m m'} \int_{z} d\rho f_{\omega'}^{(TE)} f_{\omega'}^{(TE)}, \quad (A7) \]
which justifies the normalization of the TE modes in Sec. IV (Notice that the integration variable is \( \rho \), defined through \( d\rho/d\varphi = \gamma^{2}(1 - n_{\parallel}^{2}v^{2})/c_{\perp} \)).

---

### Unstable TE modes

Generic unstable TE modes are given by
\[ A^{(uTE)} = (0, \partial_{\varphi} \psi / \sin \theta_{e}, -\sin \theta \varphi \psi) \]
with
\[ \psi = (\alpha_{\Omega_{t}} e^{\Omega_{t}} + \beta_{\Omega_{t}} e^{-\Omega_{t}}) Y_{m}(\theta, \varphi) g_{\Omega_{t}}^{(TE)}(r), \quad (A8) \]
where \( \alpha_{\Omega_{t}} \) and \( \beta_{\Omega_{t}} \) are complex constants and \( g_{\Omega_{t}}^{(TE)}(r) \) is a solution of Eq. (97) with \( \omega_{0}^{-2} = -\Omega_{0}^{2} (\Omega > 0) \), without loss of generality and proper boundary conditions (see below). Sesquilinearity of Eq. (A11) makes it easy to calculate \( (A^{(uTE)}, A'^{(uTE)}) \) from Eq. (A5) with the appropriate substitution \( \omega \rightarrow \pm i \Omega \) and \( \omega' \rightarrow \pm \Omega' \):

\[ (A^{(uTE)}, A'^{(uTE)}) = \ell(\ell + 1)\delta_{\ell\ell'}\delta_{\ell m m'} \int_{z} d\rho \left[ g_{\Omega_{t}}^{(TE)} f_{\omega'}^{(TE)} + \frac{\alpha_{\Omega_{t}}}{\beta_{\Omega_{t}}} f_{\omega'}^{(TE)} \right], \quad (A9) \]

---

### TM modes

#### a. Stable

Now, substituting \( A^{(TM)} = (r^{-2} \varepsilon_{z}^{-1} \Delta_{S}^{(0)}, \partial_{\rho} \theta_{e}, \partial_{\rho} \partial_{\theta}) \phi \) into Eq. (B1), \( \phi = e^{i \omega t} Y_{m}(\theta, \varphi) f_{\omega}^{(TM)}(r) \), and evaluating the angular integrals (similarly to the previous TE case), we obtain:

\[ (A^{(TM)}, A'^{(TM)}) = \ell(\ell + 1)\delta_{\ell\ell'}\delta_{\ell m m'} \int_{z} d\rho e^{i(\omega - \omega')} \left\{ (\omega + \omega') \left[ \frac{d}{d\rho} f_{\omega}^{(TM)} \frac{d}{d\rho} f_{\omega'}^{(TM)} + \frac{\gamma^{2}(1 - n_{\parallel}^{2}v^{2})(\ell + 1)}{c_{\perp}^{2}} f_{\omega}^{(TM)} f_{\omega'}^{(TM)} \right] \right\} \]

---

19
where recall that \(dr/d\varrho = \gamma^2(1 - \frac{n_1^2}{n^2})/\varepsilon_1\). The strategy to simplify the expression above is the same applied in the TE case. Define

\[
\mathcal{W}_{\omega'}^{(t)} := \frac{1}{(\omega - \omega')}
\left(\frac{\varepsilon}{v + \varepsilon_1} f_{\omega'}^{(TM)} \frac{d}{d\varrho} f_{\omega'}^{(TM)} - \omega f_{\omega'}^{(TM)} \frac{d}{d\varrho} f_{\omega'}^{(TM)}\right),
\]

(A15)

One can easily check, using Eq. (102), that

\[
\frac{d}{d\varrho} \mathcal{W}_{\omega'}^{(t)} = (\omega + \omega') \left[ \frac{d}{d\varrho} f_{\omega'}^{(TM)} \frac{d}{d\varrho} f_{\omega'}^{(TM)} + \frac{\gamma^2(1 - \frac{n_1^2}{n^2})}{\varepsilon_1} f_{\omega'}^{(TM)} f_{\omega'}^{(TM)} \right],
\]

(A16)

Therefore, we can put Eq. (A14) in the same form as Eq. (A5), with \(W \rightarrow \mathcal{W}\) and \(p \rightarrow \varrho\). Now, orthogonality of the modes demand that \(f_{\omega'}^{(TM)}\) satisfy either Dirichlet or Neumann boundary conditions at \(I\), which leads to

\[
(A_{TM}^{(TM)}, A'_{TM}^{(TM)}) = \ell(\ell + 1) \delta_{\ell\ell'} \delta_{mn}\mathcal{W}_{\omega'}^{(t)}|_I
\]

(A17)

In order to simplify even further the expression above, note that using again Eq. (102) in Eq. (A16) we can write

\[
\frac{d}{d\varrho} \mathcal{W}_{\omega'}^{(t)} = (\omega + \omega') \left[ \frac{d^2}{d\varrho^2} + \frac{\mu_{\perp}}{\varepsilon_1} (\omega^2 + \omega'^2) \right] f_{\omega'}^{(TM)} f_{\omega'}^{(TM)},
\]

(A18)

whose integration on \(I\) gives us \(\mathcal{W}_{\omega'}^{(t)}|_I\), which substituted into Eq. (A17) finally leads to

\[
(A_{TM}^{(TM)}, A'_{TM}^{(TM)}) = 2\omega^3 \ell(\ell + 1) \delta_{\ell\ell'} \delta_{mn},
\]

(A19)

(Notice that the integration variable is \(\rho\).)

b. Unstable

Generic unstable TM modes are given by \(A_{(u)TM} = (r^{-2} v^{-1} A_{S}^{(0)}, \partial_\vartheta \vartheta, \partial_\varphi, \partial_\varrho \varrho)\phi\) with

\[
\phi = (\alpha_{\Omega\ell} e^{\Omega \tau} + \beta_{\Omega\ell} e^{-\Omega \tau}) Y_{\ell m}(\theta, \varphi) g_{\Omega\ell}^{(TM)}(r),
\]

(A20)

with \(0 < \kappa < \pi\).
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