Effect of the symmetry breaking on the stability of persistent currents in a two-component Bose-Einstein condensate

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We study the stability of persistent currents in a two-component Bose-Einstein condensed gas that is confined in a ring potential. The symmetry imposed by the assumption of equal couplings for intra- and interatomic collisions affects the stability of the persistent currents in a dramatic way. We examine the effect of breaking of this symmetry, identifying the major differences in the behavior of the system as compared to the “symmetric” case of equal couplings.

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I. INTRODUCTION

The recent experimental developments in cold atomic gases allow us to realize “ideal” physical systems. For example, within the general class of problems associated with the superfluid properties of Bose-Einstein condensed atoms it has become possible to create annular/toroidal potentials and even to create and observe persistent currents of atoms which circulate around the annulus/torus [1,2]. Clearly our ability to create persistent currents in such trapping geometries is advantageous, since – especially in the limit of a narrow annulus/torus – the complications of two/three-dimensional motion are absent. In addition, it is interesting that the motion in such potentials is topologically non-trivial.

In the recent experiment of Ref. [2] it has also become possible to study the stability of persistent currents in the case of a mixture of two species. As demonstrated in Ref. [2], the addition of a second component has very drastic effects on the stability of persistent currents, essentially because of the extra degrees of freedom which are associated with the second component. Reference [9] has extended the results of Ref. [8] to the case of an annular potential of a finite width, and analyzed the effect of the deviations from one-dimensional motion. More recently Refs. [11] and [12] have paid special attention to the limit of a large population imbalance and have demonstrated that there is a rather rich structure in the behavior of the system in this limit. The numerical study of Ref. [12] has performed numerical simulations that correspond to the experimental conditions of Ref. [2], taking into account the spin-spin interactions. Finally, the study of Ref. [12] has also attempted to explain the results of the experiment of Ref. [2].

References [8], [10], and [11] assumed purely one-dimensional motion of the atoms and, for the problem of persistent currents, equal scattering lengths of atom-atom collisions between the same and different species (which we label as A and B) \(a_{AA}, a_{BB}, \) and \(a_{AB}, \) and also equal masses. According to Ref. [8], in the case of equal scattering lengths the dispersion relation, i.e., the energy of the system as function of the total angular momentum per particle \(\ell, \) is exactly linear for \(0 \leq \ell \leq x_B \) and \(x_A \leq \ell \leq 1, \) where \(x_k = N_k/N, \) with \(N_k (k = A, B) \) being the population of the two species and \(N = N_A + N_B \) the total number of atoms. This linearity of the spectrum is due to the fact that in the range of \(\ell \) mentioned above the total density \(n_A + n_B \) is homogeneous, where \(n_k(\theta) \) is the density of the two species. In Ref. [8] it has thus been shown that metastability of persistent currents for \(0 \leq \ell \leq 1 \) is possible at \(\ell = \max(x_A, x_B). \) In addition, it has been shown that at a certain population imbalance, \(N_A/N_B = 3, \) the critical value of the coupling for stability of persistent currents (with a winding number equal to unity for the larger component A and zero for the smaller component B) diverges. For \(N_A/N_B < 3 \) persistent currents are absent. Finally, in the region with \(\ell > 1 \) stability of persistent currents is not possible at the corresponding values of \(\ell = \nu + x_A = \nu + 1 - x_B (\text{due to Bloch’s theorem} [14]), \) where \(\nu = 0, 1, \ldots, \) when \(\nu > 0, \) in agreement with Ref. [11]. Still, according to Ref. [11], persistent currents may be possible at other values of \(\ell \) for sufficiently small values of \(x_B. \)

The results that are mentioned above rely on the assumption of equal scattering lengths. Although this assumption seems to be innocent and also it is experimentally relevant (as in the case of Rubidium atoms, for example, where these scattering lengths are roughly the same [13]), it imposes a symmetry on the problem, which has very serious consequences. The purpose of the present study is to examine the effect of the breaking of this symmetry on the stability of persistent currents. As we show, even in the limit where this symmetry is broken “weakly” (i.e., \(a_{AA} \approx a_{AB} \approx a_{BB}, \)) the stability of persistent currents is affected dramatically.

According to our results, there are three basic differences as compared to the case of equal scattering lengths: firstly, stability of persistent currents is possible at numerous values of the angular momentum per particle \(\ell \) in the interval \(0 \leq \ell \leq 1, \) depending on the population imbalance between the two components. Secondly, similar results for the stability of the currents also hold for \(\ell > 1. \) Thirdly, stability of the currents is possible for any population imbalance.

In what follows we first present our model in Sec. II.
Then in Sec. III we investigate the stability of persistent currents for \( \ell = x_A, x_B, \) and \( \ell = 1, \) and also at \( \ell = 1 + x_A, 1 + x_B, \) and \( \ell = 2 \) breaking the symmetry via \( \gamma_{AA}. \) Then, we examine the same question when the symmetry is broken via \( \gamma_{AB}. \) In Sec. IV we examine the stability of currents at other values of the angular momentum \( \ell, \) which depend on the population imbalance. In Sec. V we present our numerical results, while in Sec. VI we give a physical interpretation of our findings. Finally in Sec. VII we discuss the relevance of our study with the experimental data and summarize our conclusions.

II. MODEL

Let us assume that we work with atoms which are confined in a narrow toroidal potential of radius \( R \) and cross section \( S, \) with \( R \gg \sqrt{S}. \) The Hamiltonian of this system is

\[
H = H_{AA} + H_{BB} + U_{AB} \sum_{i=1,j=1}^{N_A,N_B} \delta(\theta_i - \theta_j),
\]

with \( k = A, B. \) Here \( M \) is the atom mass and \( U_{jk} = 4\pi\hbar^2 a_{jk}/(MRS) \) are the matrix elements for zero-energy elastic atom-atom collisions (all assumed to be positive). Within the mean-field approximation, the coupled Gross-Pitaevskii equations for the order parameters the two components \( \Psi_A \) and \( \Psi_B \) are

\[
i\hbar \frac{\partial \Psi_k}{\partial t} = -\frac{\hbar^2}{2MR^2} \frac{\partial^2 \Psi_k}{\partial \theta^2} + N(U_{kk}|\Psi_k|^2 + U_{kj}|\Psi_j|^2)\Psi_k.
\]

III. STABILITY OF PERSISTENT CURRENTS

Before we proceed, we need to examine the conditions which guarantee that the two species do not phase separate, in order for the analysis that follows to be valid. From Ref. [3] it turns out that the condition for phase coexistence is that \( \tilde{\gamma}_{AB} - \tilde{\gamma}_{AA} \tilde{\gamma}_{BB} < 1/4 + (\tilde{\gamma}_{AA} + \tilde{\gamma}_{BB})/2, \) where \( \tilde{\gamma}_{jk} = U_{jk} \sqrt{N_j N_k} / (2\pi\epsilon), \) with \( \epsilon = \hbar^2/(2MR^2); \) the parameters \( \tilde{\gamma}_{jk} \) give the ratio between the typical interaction energy and the kinetic energy \( \epsilon. \)

Under the above condition of phase coexistence we may now examine the stability of persistent currents. We first focus on the range \( 0 \leq \ell \leq 1 \) and then examine the same problem for higher values of \( \ell, \) making use of the quasi-periodicity of the energy spectrum that results from Bloch's theorem \([3,4].\) The starting point in our analysis relies on the fact that the lowest-energy state is trivial for \( \ell = 0, x_B, x_A, \) and \( 1. \) If \( \Phi_m(\theta) = e^{im\theta}\sqrt{2\pi} \) are the eigenstates of the ring potential with angular momentum \( m\hbar, \) then for \( \ell = 0: (\Psi_A, \Psi_B) = (\Phi_0, \Phi_0), \) for \( \ell = x_A : (\Psi_A, \Psi_B) = (\Phi_0, \Phi_1), \) for \( \ell = x_A: (\Psi_A, \Psi_B) = (\Phi_1, \Phi_0), \) and finally for \( \ell = 1: (\Psi_A, \Psi_B) = (\Phi_1, \Phi_1). \) Therefore, we have possible candidates for stability of persistent currents at \( \ell = x_A, \ell = x_B, \) and \( \ell = 1. \)

Starting with \( \ell = x_A, \) then

\[
\Psi_A = c_0\Phi_0 + c_1\Phi_1 + c_2\Phi_2,
\]

\[
\Psi_B = d_{-1}\Phi_{-1} + d_0\Phi_0 + d_1\Phi_1.
\]

The energy per particle of the system in the above states is

\[
\frac{E}{N\epsilon} = x_A(c_1^2 + 4c_2^2) + x_B(d_2^2 + d_1^2) + \frac{1}{2}x_A^2\gamma_{AA}(c_0^2 + c_1^2 + c_2^2 + 4c_0c_1c_2) + \frac{1}{2}x_B^2\gamma_{BB}(d_1^2 + d_0^2 + d_1^2)
\]

\[
+ 4d_0d_1c_1c_2 + 4d_1^2d_2 + 4d_0^2d_1 + 4d_1d_2d_1,
\]

\[
+ 2x_Ax_B\gamma_{AB}(c_0 + c_2)^2 + x_B(d_1^2 + d_2^2) + c_1c_2d_1 + c_1c_2d_0 + d_0c_0c_2d_{-1}d_1,
\]

where \( \gamma_{jk} = U_{jk}N/(2\pi\epsilon), \) while the angular momentum is given by \( \ell = x_A(c_1^2 + 2c_2^2) + x_B(d_1^2 - d_2^2). \)

A. Linearisation and Lagrange multipliers

For \( |c_1| \) and \( |d_0| \) being much larger than all the other four coefficients,

\[
\frac{E}{N\epsilon} = x_A(c_1^2 + 4c_2^2) + x_B(d_2^2 + d_1^2) + \frac{1}{2}(x_A^2\gamma_{AA} + x_B^2\gamma_{BB} + 2x_Ax_B\gamma_{AB})
\]

\[
= \ell + 2(x_Ac_1^2 + x_Bd_1^2 + \lambda d_1)
\]

\[
+ \gamma_{AA}2x_A(c_1 + c_2)^2 + \gamma_{BB}2x_B(d_1^2 + d_2^2) + 2x_Ax_B\gamma_{AB}(c_0 + c_2)(d_1 + d_1^2).
\]

Let us consider also the case \( \gamma_{AB} = \gamma_{BB} = \gamma, \) and introduce the Lagrange multiplier \( \lambda \) and the parameter \( g = x_A(c_1^2 - c_0^2) + x_B(d_1^2 - d_1^2) + \epsilon, \) where \( \epsilon = x_A - \ell, \) and minimize \( E/(N\epsilon) + \lambda g. \) The resulting equations are

\[
-\lambda c_0 + \gamma_{AA}x_A(c_0 + c_2) + \gamma_{AB}x_B(d_1 + d_1^2) = 0
\]

\[
(\lambda + 2)c_2 + \gamma_{AA}x_A(c_0 + c_2) + \gamma_{BB}x_B(d_1 + d_1^2) = 0
\]

\[
-\lambda d_1 + \gamma_{AA}c_0 + c_2 + \gamma_{BB}x_B(d_1 + d_1^2) = 0
\]

\[
\lambda d_1 + \gamma_{AA}(c_0 + c_2) + \gamma_{BB}(d_1 + d_1^2) = 0.
\]

Demanding that there are non-zero solutions we get that

\[
4\kappa x_Ax_B\gamma^2 - 2\lambda[x_A(1 + \kappa)(\lambda - 2) + x_B(\lambda + 2)]\gamma
\]

\[
+ \lambda^2(\lambda^2 - 4) = 0,
\]

where \( \kappa \equiv \gamma_{AA}/\gamma - 1. \) For large values of \( \gamma \) and \( \gamma_{AA}, \) the condition for phase coexistence demands that \( \kappa > 0. \)
B. Stability of persistent currents at \( \ell = x_A \) and \( \ell = x_B \) and also at \( \ell = 1 + x_A \) and \( 1 + x_B \)

From the above equation one gets that the critical value of \( \gamma \) for the stability of persistent currents is (setting \( \lambda = 1 \), since the slope of the dispersion relation for \( 0 \leq \ell \leq 1 \) is \( 1 - \lambda \))

\[
4\kappa_A x_B \gamma_{cr}^2 + 2[x_A(1 + \kappa) - 3x_B] \gamma_{cr} - 3 = 0,
\]

(8)

(the second root gives a negative \( \gamma \), assuming that \( \kappa > 0 \), and thus is not acceptable) and therefore

\[
\gamma_{cr} = \frac{3x_B - x_A(1 + \kappa) + \sqrt{[3x_B - x_A(1 + \kappa)]^2 + 12\kappa_A x_B}}{4\kappa_A x_B}.
\]

For \( x_A \to 1 \), it turns out that \( \gamma_{cr} \to 3/[2(1 + \kappa)] \), which is obvious, since this may also be written as \( \gamma_{AA,cr} = 3/2 \), as it is known already \( \square \). For \( x_A \to 0 \),

\[
\gamma_{cr} \to \frac{3}{2\kappa A}.
\]

Since we have not imposed any restriction of the value of \( x_A \), the above formula is valid for the whole range of \( \ell = x_A \) between 0 and 1 and therefore it also gives \( \gamma_{cr} \) for \( \ell = x_B \). The two left curves in Fig. 1 show \( \gamma_{cr}(x_A) \) for \( \kappa = 0.1 \) and 0.01, while the right curve is the result \( \gamma_{cr} = 3/[2(4x_A - 3)] \) of Ref. \( \square \) (for the case of equal couplings, i.e., \( \kappa = 0 \)). From Fig. 1 it is clear that the critical coupling for \( x_A < 1/2 \) is higher than that for \( x_A > 1/2 \).

Turning now to the question of local minima at \( \ell = 1 + x_A \) (and \( \ell = 1 + x_B \)), these may be found by setting \( \lambda = 3 \) as Bloch’s theorem implies \( \square \) (since the slope of the dispersion relation for \( 1 \leq \ell \leq 3 \) is \( 3 - \lambda \)). Then, from Eq. (7),

\[
4\kappa_A x_B \gamma_{cr}^2 - 6[x_A(1 + \kappa) + 5x_B] \gamma_{cr} + 45 = 0.
\]

Therefore,

\[
\gamma_{cr} = \frac{3[x_A(1 + \kappa) + 5x_B] - \sqrt{D}}{4\kappa_A x_B},
\]

(12)

where \( D = 9[x_A(1 + \kappa) + 5x_B]^2 - 180\kappa_A x_B \). For \( x_A \to 1 \), \( \gamma_{cr} \to 15/[2(1 + \kappa)] \), while for \( x_A \to 0 \), \( \gamma_{cr} \to 3/2 \), as expected from the known results of the case with one component.

C. Stability of persistent currents at \( \ell = 1 \) and \( \ell = 2 \)

The case \( \ell = 1 \) may also be examined using the same method as above. In this case we need to consider the states

\[
\Psi_A = c_0 \Phi_0 + c_1 \Phi_1 + c_2 \Phi_2,
\]

\[
\Psi_B = d_0 \Phi_0 + d_1 \Phi_1 + d_2 \Phi_2.
\]

(13)

FIG. 1: The critical value \( \gamma_{cr} \) for which the slope of the dispersion relation vanishes at \( \ell \to \ell_0 \), with \( \ell_0 = x_A \), plotted as a function of \( x_A \), for \( \kappa = 0.1 \) (left), 0.01 (middle), and 0 (right). Here \( \gamma_{AA} = \gamma_{BB} = \gamma \) and \( \gamma_{AB} = \gamma_{AB} = \gamma(\kappa + 1) \).

For \( |c_1| \) and \( |d_1| \) being much larger than all the other coefficients and for \( \gamma_{AB} = \gamma_{BB} = \gamma \), the energy per particle is

\[
\frac{E}{N\epsilon} - \frac{1}{2}(x_A^2 \gamma_{AA} + x_B^2 \gamma + 2x_A x_B \gamma) = \ell + 2(x_A c_2^2 + x_B d_2^2) + \gamma(x_A(c_0 + c_2) + x_B(d_0 + d_2)) + (\gamma_{AB} - \gamma) x_A(c_0 + c_2)^2.
\]

(14)

Following the same method as before, we then introduce the Lagrange multiplier \( \lambda \) and the parameter \( g = x_A(c_2^2 - c_0^2) + x_B(d_2^2 - d_0^2) + \varepsilon \), where \( \varepsilon = 1 - \ell \) and thus minimize \( E/(N\epsilon) + \lambda g \). The resulting equations are

\[
-\lambda c_0 + \gamma_{AA} x_A(c_0 + c_2) + \gamma x_B(d_0 + d_2) = 0
\]

(15)

\[
(\lambda + 2)c_2 + \gamma_{AA} x_A(c_0 + c_2) + \gamma x_B(d_0 + d_2) = 0
\]

\[
-\lambda d_0 + \gamma_{AA} x_A(c_0 + c_2) + \gamma x_B(d_0 + d_2) = 0
\]

\[
(\lambda + 2)d_2 + \gamma x_A(c_0 + c_2) + \gamma x_B(d_0 + d_2) = 0.
\]

For equal couplings, demanding that there are non-zero solutions we get that

\[
\lambda(\lambda + 2)[2\gamma - \lambda(\lambda + 2)] = 0.
\]

(16)

The actual solution is the one for \( \lambda = 0 \), for which the slope is indeed equal to 1. Turning to the case \( \gamma_{AA} \neq \gamma \), then

\[
4\kappa \gamma^2 x_A x_B - 2\lambda(\lambda + 2)\gamma(1 + \kappa x_A) + \lambda^2(\lambda + 2)^2 = 0.
\]

(17)

For the stability at \( \ell = 1 \) we set \( \lambda = 1 \),

\[
4\kappa \gamma^2 x_A x_B - 6\gamma_{cr}(1 + \kappa x_A) + 9 = 0.
\]

(18)

The solution is

\[
\gamma_{cr} = \frac{3(1 + \kappa x_A) - \sqrt{9(1 + \kappa x_A)^2 - 36\kappa x_A x_B}}{4\kappa x_A x_B}.
\]

(19)
The solution is \( \gamma_{cr} = 1 \), \( \lambda = 3 \), \( \gamma_{AB} = \gamma \) and \( \gamma_{AA} = \gamma(\kappa+1) \).

For \( x_A \to 1 \), \( \gamma_{cr} \to 3/[2(1 + \kappa)] \), while for \( x_A \to 0 \), \( \gamma_{cr} \to 3/2 \), again as expected from the known results of the case with one component.

For the stability at \( \ell = 2 \), we set \( \lambda = 3 \),

\[
4\kappa\gamma_{cr}^2x_Ax_B - 30\gamma_{cr}(1 + \kappa x_A) + 225 = 0. \tag{20}
\]

The solution is

\[
\gamma_{cr} = \frac{15(1 + \kappa x_A) - \sqrt{225(1 + \kappa x_A)^2 - 900\kappa x_A x_B}}{4\kappa x_A x_B} \tag{21}
\]

For \( x_A \to 1 \), \( \gamma_{cr} \to 15/[2(1 + \kappa)] \), while for \( x_A \to 0 \), \( \gamma_{cr} \to 15/2 \).

\[\text{D. Breaking the symmetry via } \gamma_{AB}\]

Breaking the symmetry via \( \gamma_{BB} \) gives similar results. What is more interesting is the case \( \gamma_{AA} = \gamma_{BB} = \gamma \) and \( \gamma_{AB} \neq \gamma \). Starting with the stability at \( \ell = x_A \) (and \( \ell = x_B \)), we follow the same method as above and get that

\[
4x_Ax_B\kappa(\kappa + 2)\gamma_{cr}^2 - 2\gamma_{cr}[2(x_A - x_B) - \lambda] + \lambda^2(4 - \lambda^2) = 0. \tag{22}
\]

Here \( \kappa = \gamma_{AB}/\gamma - 1 \) is between \(-1\) and \(0\) for large \( \gamma \) and \( \gamma_{AB} \). For the stability at \( \ell = x_A \) (and \( \ell = x_B \)) we set \( \lambda = 1 \),

\[
4x_Ax_B\kappa(\kappa + 2)\gamma_{cr}^2 - 2\gamma_{cr}[2(x_A - x_B) - 1] + 3 = 0. \tag{23}
\]

Therefore,

\[
\gamma_{cr} = \frac{2(x_A - x_B) - 1 - \sqrt{D}}{4\kappa(\kappa + 2)x_A x_B}. \tag{24}
\]

Here \( D = [2(x_A - x_B) - 1]^2 - 12\kappa(\kappa + 2)x_A x_B \). For \( x_A \to 1 \), then \( \gamma_{cr} \to 3/2 \), in agreement with older results \cite{3}. For \( x_A \to 0 \), then

\[
\gamma_{cr} = \frac{-3}{2\kappa(\kappa + 2)x_A}. \tag{25}
\]

For the stability at \( \ell = 1 + x_A \) (and \( \ell = 1 + x_B \)) we set \( \lambda = 3 \),

\[
4x_Ax_B\kappa(\kappa + 2)\gamma_{cr}^2 - 6\gamma_{cr}[2(x_A - x_B) - 3] - 45 = 0. \tag{26}
\]

Therefore,

\[
\gamma_{cr} = \frac{6(x_A - x_B) - 9 + \sqrt{D}}{4\kappa(\kappa + 2)x_A x_B}, \tag{27}
\]

with \( D = [6(x_A - x_B) - 9]^2 + 180\kappa(\kappa + 2)x_A x_B \). For \( x_A \to 1 \), then \( \gamma_{cr} \to 15/2 \), while for \( x_A \to 0 \), \( \gamma_{cr} \to 3/2 \).

Turning to the case \( \ell = 1 \) and \( \ell = 2 \),

\[
\lambda^2(\lambda + 2)^2 - 2\lambda(\lambda + 2)\gamma - 4\kappa\gamma^2(1 + \gamma_{AB}/\gamma)x_A x_B = 0. \tag{28}
\]

For \( \lambda = 1 \) and \( \gamma_{AB} \approx \gamma \), then

\[
-8\kappa x_A x_B\gamma_{cr}^2 - 6\gamma_{cr} + 9 = 0, \tag{29}
\]

or

\[
\gamma_{cr} = \frac{3 - \sqrt{9 + 72\kappa x_A x_B}}{-8\kappa x_A x_B}. \tag{30}
\]

For \( x_A \to 0 \) and \( x_A \to 1 \), then \( \gamma_{cr} \to 3/2 \). Finally, for \( \lambda = 3 \) and \( \gamma_{AB} \approx \gamma \), then

\[
-8\kappa x_A x_B\gamma_{cr}^2 - 30\gamma_{cr} + 225 = 0, \tag{31}
\]

or

\[
\gamma_{cr} = \frac{15 - \sqrt{225 + 1800\kappa x_A x_B}}{-8\kappa x_A x_B}. \tag{32}
\]

In this case, for \( x_A \to 0 \) and \( x_A \to 1 \), then \( \gamma_{cr} \to 15/2 \).

\[\text{IV. METASTABILITY FOR OTHER VALUES OF THE ANGULAR MOMENTUM AND MORE GENERAL ANALYSIS}\]

Up to now we focused only at the specific values of \( \ell = \nu + x_B, \nu + x_A, \) and \( \nu + 1, \) with \( \nu = 0, 1, \ldots \). For these values of the angular momentum the yrast state is trivial to determine exactly, for any value of the couplings \( \gamma_{ij} \). More specifically, provided that the condition of phase coexistence of the ground state (with \( \ell = 0 \)) discussed in Sec. III is fulfilled, the yrast state is

\[
(\Psi_A, \Psi_B) = (\Phi_{\nu}, \Phi_{\nu+1}) \text{ for } \ell = \nu + x_B, \tag{33}
\]

\[
(\Psi_A, \Psi_B) = (\Phi_{\nu+1}, \Phi_{\nu}) \text{ for } \ell = \nu + x_A, \tag{34}
\]

\[
(\Psi_A, \Psi_B) = (\Phi_{\nu+1}, \Phi_{\nu+1}) \text{ for } \ell = \nu + 1. \tag{35}
\]
This is obvious since the above combination minimizes both the kinetic, as well as the interaction energy of the system and thus it has to be the yrast state.

Still, based on very general arguments which we present below, for sufficiently strong values of the coupling constants one may examine the possibility of additional local minima in the dispersion relation. These minima correspond to the solutions of Eqs. (2) of the form

$$\left(\Psi_A, \Psi_B\right) = (\Phi_m, \Phi_n).$$  \hspace{1cm} (34)

We stress that unless $\Psi_A$ and $\Psi_B$ belong to the specific pairs given by Eq. (34), any other combination does not correspond necessarily to the yrast state, since the kinetic energy is not minimized. On the other hand, since the states of Eq. (34) have a constant density distribution, the interaction energy is minimized, indeed. As a result, for sufficiently strong interaction strengths these pairs of wavefunctions are possible candidates for local minima in the dispersion relation.

Apparently the solutions of Eq. (34) constitute a discrete set of solutions, which correspond also to a discrete set of values of the angular momentum

$$\ell = x_A m + x_B n,$$  \hspace{1cm} (35)

with an energy

$$\frac{E}{N\epsilon} = E_0 + x_A m^2 + x_B n^2,$$  \hspace{1cm} (36)

where $E_0$ is the interaction energy measured in units of $N\epsilon$,

$$E_0 = \frac{1}{2}x_A^2\gamma_{AA} + \frac{1}{2}x_B^2\gamma_{BB} + x_A x_B\gamma_{AB}.$$  \hspace{1cm} (37)

One important feature of these solutions is that they all have the same potential energy and thus the difference in the energy between them is purely kinetic and also independent of the coupling constants $\gamma_{ij}$. On the other hand, the potential energy $E'_0$ of any other solution with an inhomogeneous density distribution will differ from $E_0$ by terms which will depend in general on the coupling constants and it will increase for sufficiently strong values of the $\gamma_{ij}$. Under the two conditions which we mention below, the solution $(\Psi_A, \Psi_B) = (\Phi_m, \Phi_n)$ will thus correspond to an energy minimum in the dispersion relation for sufficiently large values of the coupling constants $\gamma_{ij}$. The two conditions are: (i) There is no direction in the $(n_A, n_B)$ plane along which the potential-energy density $\gamma_{AA}n_A^2/2 + \gamma_{BB}n_B^2/2 + \gamma_{AB}n_An_B$ remains constant in the neighbourhood of the solution, which holds if $\gamma_{AA}\gamma_{BB} - \gamma_{AB}^2 \neq 0$. (ii) In the neighbourhood of the point with angular momentum $\ell = mx_A + nx_B$ there are no other points of similar nature. This happens when the ratio of the two particle fractions is a rational number (clearly, it is always true that the ratio of the particle fractions is rational, but it may be that the irreducible denominator of the ratio is very large, making it behave as if it were irrational).

To understand better the above arguments, let us suppose that the greatest common divisor of $N_A$ and $N_B$ is equal to 1. In this case the values of $\ell$ that support solutions of constant density $(\Psi_A, \Psi_B) = (\Phi_m, \Phi_n)$ are $\ell = mx_A + nx_B$. Since there exist integers $m'$ and $n'$ such that $m'N_A + n'N_B = 1$, it is possible to construct solutions of constant density at all values of $\ell = k/N$, with $k = 0, \pm 1, \pm 2, \ldots$. In fact, there is an infinite number of solutions of constant density at each of these values of $\ell$, since for each pair of the indices $m$ and $n$, one may consider another pair $(m', n') = (m - kx_B, n + kx_A)$, $k = 0, \pm 1, \pm 2, \ldots$, with the same value of $\ell$. Clearly the pair(s) with the lowest kinetic energy will then correspond to the yrast state.

Interestingly enough, even if the difference in the energy between all solutions of constant density is independent of the couplings, still the energy difference between them may vary wildly, even for small differences of $\ell$. If, for example, $N_A = 256$ and $N_B = 243$, then the solutions of constant density differ in their angular momentum by $\delta\ell = 1/499$. Furthermore, when $\ell = x_B = 243/499$, the solution of lowest energy is the one with $(m, n) = (0, 1)$, hence the kinetic energy is $243/499$. If the angular momentum increases to $\ell = 244/499$, then the solution which minimizes the kinetic energy is the one with $(m', n') = (m - 56, n + 59) = (-56, 60)$. This solution has a kinetic energy that is equal to $1677616/499 \approx 3362$, which is roughly $7 \times 10^3$ times the kinetic energy of the neighbouring state. Clearly, the above example suggests a number-theoretically driven wild behaviour in the dispersion relation for strong couplings.

According to the arguments presented above, the combination $(\Psi_A, \Psi_B) = (\Phi_m, \Phi_n)$ are possible yrast states (with $m$ greater than $n + 1$, since if $m = n + 1$ the above pair of states is the yrast state, as explained earlier.) Provided this is so, one may follow the same procedure that was described in Sec. III and examine the condition for stability of persistent currents, considering the states

$$\Psi_A = c_{m-1}\Phi_{n-1} + c_m\Phi_m + c_{m+1}\Phi_{m+1},$$
$$\Psi_B = d_{n-1}\Phi_{n-1} + d_n\Phi_n + d_{n+1}\Phi_{n+1},$$ \hspace{1cm} (38)

where $|c_m|, |d_n| \sim 1$ and $|c_{m\pm 1}|, |d_{n\pm 1}| \ll 1$. It turns out that the condition for stability of the current is given by

$$[4m^2 - 2\gamma_{AA}x_A - 1] [4n^2 - 2\gamma_{BB}x_B - 1] = 4\gamma_{AB}^2x_Ax_B.$$ \hspace{1cm} (39)

Let us examine the validity of the above equation now. First of all, the case $\gamma_{AA}\gamma_{BB} = \gamma_{AB}^2$ is special and it will be analysed in a future publication [16]. In addition, as we mentioned above, Eq. (39) assumes that the pair $(\Psi_A, \Psi_B) = (\Phi_m, \Phi_n)$ is the yrast state. However, this assumption introduces a separate and independent condition on the values of the parameters $\gamma_{ij}$. In other words, one has to find a second phase boundary in the phase diagram that involves the parameters $x_A$ and $\gamma$, where, as before, the three values of $\gamma_{ij}$ are assumed proportional to some $\gamma$. Denoting the corresponding critical
value of $\gamma$ as $\gamma_{cr}^x$, the condition for Eq. (39) to be valid is $\gamma_{cr}^x < \gamma_{cr}$ for some given value of $x_A$.

We thus distinguish the following three cases: (i) $n > 0$ and $m > 2n$, (ii) $n = 0$, and finally (iii) $n > 0$ and $m \leq 2n$.

In the first case, $n > 0$ and $m > 2n$, for $x_A \rightarrow 1$, $\gamma_{cr}$ is determined by $\gamma_{AA,cr} \rightarrow (4m^2 - 1)/2$, which is the known result for the stability of persistent currents in the case of a single component [8], while the value of $\gamma_{cr}^x$ is determined by $\gamma_{AA,cr}^x$, which is $2(m - n)(m - n - 1)$.

Clearly for $x_A \rightarrow 1$ the condition $\gamma_{cr}^x < \gamma_{cr}$ is satisfied. In the opposite limit, $x_A \rightarrow 0$, $\gamma_{cr}$ is determined by $\gamma_{BB,cr} \rightarrow (4n^2 - 1)/2$, while $\gamma_{cr}^x$ is determined by $\gamma_{BB,cr}^x \rightarrow 2(m - n)(m - n - 1)$. In this case the condition $\gamma_{cr}^x < \gamma_{cr}$ is violated. In other words, only for values of $x_A$ which are sufficiently close to unity is Eq. (39) valid.

Turning to the second case, $n = 0$, for $x_A \rightarrow 1$, then again $\gamma_{cr} = \gamma_{AA,cr} \rightarrow (4m^2 - 1)/2$, while $\gamma_{cr}^x$ is determined by $\gamma_{AA,cr}^x$, which is $2m(m - 1)$. In the opposite limit, $x_A \rightarrow 0$, $\gamma_{cr}$ diverges, while $\gamma_{cr}^x$ is determined by $\gamma_{BB,cr}^x \rightarrow 2m(m - 1)$ again. The condition $\gamma_{cr}^x < \gamma_{cr}$ is satisfied in both limits, and therefore Eq. (39) is valid at least for values of $x_A$ sufficiently close to both zero and unity. Figure 4 shows the two phase boundaries for $(\Psi_A, \Psi_B) = (\Phi_3, \Phi_1)$, with $\gamma_{cr}^x$ evaluated numerically, where $\Psi_A$ includes the states $\Phi_0, \Phi_3,$ and $\Phi_4$ and $\Psi_B$ includes the states $\Phi_1, \Phi_4,$ and $\Phi_2$. Here $\gamma = \gamma_{AA} = \gamma_{BB} = (5/4)\gamma_{AB}$.

Turning to the second case, $n = 0$, for $x_A \rightarrow 1$, then again $\gamma_{cr} = \gamma_{AA,cr} \rightarrow (4m^2 - 1)/2$, while $\gamma_{cr}^x$ is determined by $\gamma_{AA,cr}^x$, which is $2m(m - 1)$. In the opposite limit, $x_A \rightarrow 0$, $\gamma_{cr}$ diverges, while $\gamma_{cr}^x$ is determined by $\gamma_{BB,cr}^x \rightarrow 2m(m - 1)$ again. The condition $\gamma_{cr}^x < \gamma_{cr}$ is satisfied in both limits, and therefore Eq. (39) is valid at least for values of $x_A$ sufficiently close to both zero and unity.

A similar picture develops also for $n > 0$ and $m \leq 2n$. For $x_A \rightarrow 1$, then $\gamma_{cr} = \gamma_{AA,cr} \rightarrow (4m^2 - 1)/2$, while $\gamma_{cr}^x = \gamma_{AA,cr}^x \rightarrow 2(m - n)(m - n - 1)$. For $x_A \rightarrow 0$, $\gamma_{cr} = \gamma_{BB,cr} \rightarrow (4n^2 - 1)/2$, while $\gamma_{cr}^x = \gamma_{BB,cr}^x \rightarrow 2(m - n)(m - n - 1)$ and therefore Eq. (39) is valid at least for values of $x_A$ sufficiently close to zero and unity.

V. NUMERICAL RESULTS

The analysis of the previous section suggests that persistent currents which correspond to the solutions of the form of Eq. (34) are possible, provided the couplings are strong enough. We have gotten numerical evidence for this effect via numerical solutions of the two coupled Gross-Pitaevskii like equations of Eq. (2), by employing the method of imaginary-time propagation.

In the data presented in Figs. 5 and 6 we choose $x_B = 0.2$ and $x_A = 0.8$. The couplings are chosen to be $\gamma_{AA} = \gamma_{BB} = 1250/\pi^2$, and $\gamma_{AB} = 750/\pi^2$. Since these dimensionless quantities give roughly the ratio between the interaction energy and the kinetic energy, this choice of parameters corresponds to a rather strong interaction strength.

As seen from Fig. 5 the dispersion relation develops local minima, which may give rise to persistent currents, for values of $\ell$ which are the integer multiples of $x_B$, i.e., $\ell = 0.2, 0.4, 0.6, 0.8,$ and 1.0, as expected. Note, however, that the dispersion relation smooths out when the couplings become equal $\gamma_{AA} = \gamma_{BB} = \gamma_{AB} = 1250/\pi^2$, as seen in Fig. 6.

VI. PHYSICAL INTERPRETATION OF THE RESULTS

To get some insight into the above results it is instructive to examine the problem of equal scattering lengths and more specifically the stability of persistent currents at $l = x_A$. For $\ell \rightarrow x_A^+$ the slope of the dispersion relation ranges in the interval between $1 - 2(x_A - x_B)$ and $1$, depending on the value of the coupling $\gamma$. The lower bound is accomplished for $\gamma \rightarrow \infty$, while the upper one is accomplished for $\gamma = 0$. 

![FIG. 3: The two phase boundaries in the $x_A - \gamma$ plane, for $(\Psi_A, \Psi_B) = (\Phi_3, \Phi_4)$. The solid curve gives $\gamma_{cr}$, while the dashed curve gives $\gamma_{cr}^x$.](image)

![FIG. 4: The two phase boundaries in the $x_A - \gamma$ plane, for $(\Psi_A, \Psi_B) = (\Phi_A, \Phi_0)$. The solid curve gives $\gamma_{cr}$, while the dashed curve gives $\gamma_{cr}^x$.](image)
One remarkable observation in the above result is the saturation of the slope to the value $1 - 2(x_A - x_B)$ for large values of the coupling. It is thus interesting to examine the limiting case $\gamma \to \infty$. For $\ell \to x_A$, the energy (keeping only terms which are linear in $x_A - \ell$) is

$$
\frac{E}{N\epsilon} - \frac{\gamma}{2} = l + 2x_AC_0^2 + 2x_Bd_1^2
+ \gamma[x_A(c_0 + c_2) + x_B(d_{-1} + d_1)]^2.
$$

(40)

However, it turns out that $x_A(c_0 + c_2) + x_B(d_{-1} + d_1)$ vanishes, and as a consequence the term which is proportional to $\gamma$ drops out completely. Therefore,

$$
\frac{E}{N\epsilon} - \frac{\gamma}{2} = l + 2(x_A - x_B)(x_A - \ell),
$$

(41)

and the slope is thus $1 - 2(x_A - x_B)$, as expected.

The reason that underlies this effect is that the density is, in this limit,

$$
n_A(\theta) = \frac{x_A}{2\pi}[1 + 2(c_0 + c_2)\cos \theta]
$$

(42)

which implies that $n_A(\theta) + n_B(\theta)$ is constant. In other words, for $x_A \leq \ell \leq 1$ the total density is exactly homogeneous, and the (extra) energy due to the rotational motion is exactly equal to the associated kinetic energy. On the other hand, for $\ell \to x_A$, to leading order in $x_A - \ell$ the system resembles the case $x_A \leq \ell \leq 1$, keeping its total density homogeneous. However, in order for it to have the required angular momentum, the states $\Phi_{-1}$ and $\Phi_{+1}$ also need to be included in the order parameters $\Psi_A$ and $\Psi_B$, respectively. These extra components make it possible for the density to be homogeneous (sufficiently close to $\ell = x_A$), however they contribute the (extra) term $2(x_A - x_B)(x_A - \ell)$ to the energy, which is purely kinetic energy. In a sense this effect is analogous to the fermionization of bosons for infinite coupling, where the interaction energy is screened completely, and as a result of the screening, there is some finite kinetic energy involved.

The extra degrees of freedom due to the presence of a second component give the system the chance to make its total density as homogeneous as possible, and thus lower its interaction energy. This is what lies behind all the above observations. We are now in a position to understand the importance of the breaking of the symmetry due to the unequal values of the scattering lengths. In this case the “screening” of the interaction energy as the couplings increase is not possible and the above arguments break down. The major consequence of this effect is that there is no saturation of the slope of the dispersion relation, which in turn gives rise to the results of Figs. 2 and 5.

**VII. DISCUSSION AND CONCLUSIONS**

In the experiment of Beattie et al.\cite{7}, although the three scattering lengths which correspond to elastic collisions of the Rubidium atoms which are in the two different spin states are close to each other, they are not equal. Therefore, the results presented in this study are highly relevant. Of course, in this experiment there are also serious differences which have not been considered here and mostly the relatively large width of the annular potential and the fact that the transverse profile is in the Thomas-Fermi limit. Still, the basic conclusions of our study are expected to show up also in this experiment.

The sign of the parameter $\kappa$ is also worth commenting on. In a finite ring this sign may be negative or positive, without having any phase separation (as mentioned above, the condition for phase coexistence is $\tilde{\gamma}_{AB} - \tilde{\gamma}_{AA} \tilde{\gamma}_{BB} < 1/4 + (\tilde{\gamma}_{AA} + \tilde{\gamma}_{BB})/2$). In the limit of a large ring, when the symmetry is broken via $\tilde{\gamma}_{AB}$, then $\kappa > 0$, while in the case that it is broken via $\tilde{\gamma}_{AA}$, then $\kappa < 0$. Interestingly, these are the signs for $\kappa$ that one needs (in both cases) in order for the results to make sense.
The numerical study of Ref. [9] considered an annular potential, assuming motion on a plane due to the action of a very tight potential along the third axis. According to the results of this study, the stability of the currents for $0 \leq \ell \leq 1$ does not change significantly. For higher values of $\ell$ the atomic currents may become stable for sufficiently small values of $x_B$ and sufficiently high values of the coupling. Presumably these differences are due to the deviations from one-dimensional motion, since Ref. [9] considered equal scattering lengths and equal masses, and thus the “symmetric” case. We should also mention that the recent study of Ref. [11] has argued that local minima in the dispersion relation may also appear for $\ell > 1$, even for the case of equal values of the $\gamma_{ij}$. However, the stability of currents does not take place at the values $\ell = \nu + x_A$, where stability is not possible for $\nu > 0$, as shown in Ref. [8], but rather at $\ell = \nu x_A$.

The analysis presented in Secs. IV and V implies that the dispersion relation may develop a very interesting and rich structure that is associated with the ratio between the two populations. The existence of many local minima with a large energy difference between them may give rise to interesting effects. Furthermore, the connection of this analysis with number-theoretic arguments may allow connections of this system with the problem of quantum computation.

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[1] S. Gupta, K. W. Murch, K. L. Moore, T. P. Purdy, and D. M. Stamper-Kurn, Phys. Rev. Lett. 95, 143201 (2005).
[2] Spencer E. Olson, Matthew L. Terraciano, Mark Bashkansky, and Fredrik K. Fatemi, Phys. Rev. A 76, 061404(R) (2007).
[3] C. Ryu, M. F. Andersen, P. Cladé, Vasant Natarajan, K. Helmerson, and W. D. Phillips, Phys. Rev. Lett. 99, 260401 (2007).
[4] B. E. Sherlock, M. Gildemeister, E. Owen, E. Nugent, and C. J. Foot, Phys. Rev. A 83, 043408 (2011).
[5] A. Ramanathan, K. C. Wright, S. R. Muniz, M. Zelan, W. T. Hill, C. J. Lobb, K. Helmerson, W. D. Phillips, and G. K. Campbell, Phys. Rev. Lett. 106, 130401 (2011).
[6] Stuart Moulder, Scott Beattie, Robert P. Smith, Naaman Tammuz, and Zoran Hadzibabic, Phys. Rev. A 86, 013629 (2012).
[7] Scott Beattie, Stuart Moulder, Richard J. Fletcher, and Zoran Hadzibabic, Phys. Rev. Lett. 110, 025301 (2013).
[8] J. Smyrnakis, S. Bargi, G. M. Kavoulakis, M. Magiropoulos, K. Kärkkäinen, and S. M. Reimann, Phys. Rev. Lett. 103, 100404 (2009).
[9] S. Bargi, F. Malet, G. M. Kavoulakis, and S. M. Reimann, Phys. Rev. A 82, 043631 (2010).
[10] K. Anoshkin, Z. Wu, and E. Zaremba, Phys. Rev. A 88, 013609 (2013).
[11] Zhigang Wu, Eugene Zaremba, eprint arXiv:1309.1734.
[12] A. I. Yakimenko, K. O. Isaieva, S. I. Vilchinskii, and M. Weyrauch, eprint arXiv:1309.3978.
[13] M. Abad, A. Sartori, S. Finazzi, and A. Recati, eprint arXiv:1310.0400.
[14] F. Bloch, Phys. Rev. A 7, 2187 (1973).
[15] M. Egorov, B. Opanchuk, P. Drummond, B. V. Hall, P. Hannaford, and A. I. Sidorov, Phys. Rev. A 87, 053614 (2013).
[16] J. Smyrnakis, M. Magiropoulos, N. K. Efremidis, and G. M. Kavoulakis, to be submitted.