Abstract

We review recent work on the study of $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(N)$ from the point of view of the Whitham hierarchy, mainly focusing on three main results: (i) We develop a new recursive method to compute the whole instanton expansion of the low-energy effective prepotential; (ii) We interpret the slow times of the hierarchy as additional couplings and promote them to spurion superfields that softly break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 0$ through deformations associated to higher Casimir operators of the gauge group; (iii) We show that the Seiberg–Witten-Whitham equations provide a set of non-trivial constraints on the form of the strong coupling expansion in the vicinity of the maximal singularities. We use them to check a proposal that we make for the value of the off-diagonal couplings at those points of the moduli space.

I Introduction

The study of non-perturbative phenomena in quantum field theory has experienced drastic advances since, in 1994, Seiberg and Witten gave an ansatz for the exact effective action governing the low-energy excitations of $SU(2) \mathcal{N} = 2$ super Yang-Mills theory [1]. It is given in terms of an auxiliary complex algebraic curve, whose moduli space is identified with the quantum moduli space of the low-energy theory $\mathcal{M}_A$, and a given meromorphic differential, $dS_{SW}$, that induces a special geometry on $\mathcal{M}_A$. Apart form its unquestionable beauty, it proved to contain nontrivial dynamical information about the non perturbative behaviour of the theory. For example, the existence of a confinement mechanism when breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by addition of a mass term. The solution was soon extended to the case of $SU(N)$ [2, 3]. Still, the price to pay was the need for $\mathcal{N} = 2$ supersymmetry. Soft supersymmetry breaking was shown to preserve the analytic properties of the solution in such a way that exact results in the $\mathcal{N} = 0$ theory could be obtained [4, 5, 6].

Interestingly enough, it was soon realized that the Seiberg–Witten solution could be reformulated in terms of certain integrable systems, $dS_{SW}$ being a solution of their averaged (Whitham) dynamics [7]. For example, the periodic Toda lattice is the proper integrable system whose averaged dynamics corresponds to pure $\mathcal{N} = 2$ super Yang-Mills theory for the whole ADE series [8]. The spectral curve $\Gamma_\gamma$ of the particular integrable system is identified with the auxiliary Seiberg–Witten algebraic curve. Its moduli, in spite of being local invariants, evolve with respect to the so-called slow times $T_n$. The
system of non-linear equations that describe this evolution, that amounts to adiabatic deformations of an hyperelliptic curve, was developed by Whitham. Surprisingly, this system turns out to be itself integrable and receives the generic name of Whitham hierarchy (see and references therein).

The Whitham dynamics can be thought of as a generalization of the Renormalization Group (RG) flow (see for a review). The corresponding RG equations were recently derived by Gorsky, Marshakov, Mironov and Morozov: the second derivatives of the prepotential with respect to Whitham slow times \(T_i\) result to be given in terms of Riemann Theta-functions. We would like to show, in this talk, that this framework is very fruitful both to study many features of the low-energy dynamics of \(N=2\) super Yang-Mills theory, as well as to implement natural generalizations of the Seiberg–Witten solution.

We shall start by giving a telegraphical account of the Seiberg–Witten solution to the low-energy dynamics of \(SU(N)\) \(\mathcal{N}=2\) super Yang-Mills theory and of the basic ideas involved in the Whitham hierarchies. We will mainly focus on the fact that they lead naturally to the concept of a prepotential and establish thereby the concrete link between both formalisms. Within this framework, we first develop a new recursive method to compute the whole instanton expansion of the low-energy effective prepotential. Then, we interpret the slow times of the hierarchy as additional couplings and promote them to spurion superfields that softly break \(\mathcal{N} = 2\) supersymmetry down to \(\mathcal{N} = 0\) through deformations associated to higher Casimir operators of the gauge group. We discuss in some detail the case of \(SU(3)\). Finally, we show that the Seiberg–Witten–Whitham equations provide a set of non-trivial constraints on the form of the strong coupling expansion in the vicinity of the maximal singularities. We use them to check a proposal that we make for the value of the off-diagonal couplings at those points of the moduli space. Most of the work presented in this talk (the first two applications) was developed by the authors in collaboration with Marcos Mariño. The analysis of the strong coupling expansion near the maximal singularities was done in. Further generalizations, like extensions to other Lie algebras and/or inclusion of matter, remain interesting problems of research.

II The Seiberg–Witten solution

The classical potential of \(\mathcal{N}=2\) super Yang-Mills theory with a vector multiplet in the adjoint representation of \(SU(N)\) has flat directions. There is a family of inequivalent ground states that constitutes the classical moduli space \(\mathcal{M}_0\), parametrized by a constant \(\text{vev}\) of the scalar field \(\phi\) in the Cartan sub-algebra

\[
\langle \phi \rangle = \sum_{i=1}^{N-1} a^i H_i = \text{diag} (e_1(a^i), \ldots, e_N(a^i)) ,
\]

where \(e_i(a) = \lambda^i \cdot a\), \(\lambda^i\) being the \(i\)-th fundamental weight of the Lie algebra \(A_{N-1}\). At a generic point of \(\mathcal{M}_0\), the unbroken gauge symmetry is \(U(1)^{N-1}\). In fact, for every positive root \(\alpha_+\), a couple of gauge bosons \(W_{\mu \alpha_+}\) gets a mass \(M_{\alpha_+}(a) = \sqrt{2 |\alpha_+ \cdot a|} \) through the Higgs mechanism. They are BPS states with central charge \(Z_{\alpha_+}, Z_a \equiv \alpha \cdot a\) and \(a = a^i \alpha_i\) with \(\alpha_i\) the simple roots. Gauge invariant coordinates for \(\mathcal{M}_0\) can be constructed from the characteristic polynomial

\[
W_{A_{N-1}}(\lambda, \hat{u}_k) = \text{det} (\lambda - \langle \phi \rangle) = \lambda^N - \hat{u}_2(a)\lambda^{N-2} - \hat{u}_3(a)\lambda^{N-3} - \cdots - \hat{u}_N(a) ,
\]

whose coefficients, \(\hat{u}_k(a)\), are nothing but the Casimir operators. Each microscopic theory, characterized by a value of the \(\hat{u}_k\), leads to one effective field theory at low energies. We may therefore think of the quantum moduli space \(\mathcal{M}_A\) of effective field theories as being parametrized by \(u_k(a) \sim \langle \text{Tr} \phi^k \rangle\). It will look like a deformation of \(\mathcal{M}_0\), with the quantum generated scale \(\Lambda\) as the deformation parameter. The microscopic relations, thus, should be recovered for \(\Lambda \to 0\).
The low-energy effective action can be entirely written in terms of a single holomorphic function of the Cartan variables, $F(a^i)$, known as the effective prepotential. By means of symmetry considerations \[1, 2, 3\], the most general expression for $F(a^i)$ is

$$F = \frac{1}{2N} \tau_0 \sum_{\alpha_+} Z_{\alpha_+}^2 + \frac{i}{4\pi} \sum_{\alpha_+} Z_{\alpha_+}^2 \log \frac{Z_{\alpha_+}^2}{\Lambda^2} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} F_k(a) \Lambda^{2Nk},$$

(3)

where $\tau_0$ is the bare coupling constant and $F_k(a)$ are Weyl invariant combinations of $a^i$, whereas $k$ is the instanton number. The full prepotential is homogeneous of degree two in $a^i$ and $\Lambda$. What remains to be computed are the instanton corrections, $F_k(a)$. It is their exact determination the whole point of the Seiberg–Witten solution. In the semiclassical limit, the first few terms can be computed explicitly in the microscopic theory \[16\], and their agreement with the output of the Seiberg–Witten solution provides a non-trivial consistency check of it.

Since the effective prepotential \[3\] is holomorphic, the imaginary part of its second order derivatives with respect to the Cartan variables, $\text{Im}\tau_{ij}$, are harmonic functions. Therefore, they cannot have a global minimum. However, $\text{Im}\tau_{ij}$ enters the low-energy Lagrangian as an effective coupling constant: it must be positive definite. So, $\tau_{ij}$ cannot be globally defined. These conditions are automatically fulfilled if $\tau_{ij}$ is the period matrix of some Riemann surface. This observation led Seiberg and Witten to the following ansatz:

**First:** Over each point on $\mathcal{M}_\Lambda$ labelled by $u_k$, consider a certain hyperelliptic curve. For $SU(N)$ the relevant curve $\Gamma_g$ is \[2\]

$$y^2 = P(\lambda, u_k)^2 - 4\Lambda^{2N}$$

(4)

with $P = W_{A_{N-1}}$. The moduli space of the family of Riemann surfaces of genus $g = N - 1$ written above is to be identified with $\mathcal{M}_\Lambda$. In the classical limit $\Lambda \to 0$, the rational function $y = W_{A_{N-1}}(\lambda, u_k)$ has roots $e_i(a)$. At the quantum level, as $y^2$ factors into the product $y^2 = y_+ y_-$ with $y_\pm = (P(\lambda, u_k) \pm 2\Lambda^N)$, the points $e_i$ split into two sets of roots of $y_\pm$,

$$e_i(\bar{u}_k) \to e^\pm_i(u_k, \Lambda) \equiv e_i(u_{k<N}, u_N \pm 2\Lambda^N),$$

(5)

that become the $2N$ branch points of the Riemann surface.

**Second:** At the point $u_k$ on $\mathcal{M}_\Lambda$, the (quantum) relations between $a^i, a_{Dj}$ and $u_k$ is given by the period integrals

$$a^i(u) = \oint_{A^i} dS_{SW}(u) \quad a_{Dj}(u) = \oint_{B_j} dS_{SW}(u).$$

(6)

with $dS_{SW}$ a meromorphic differential given by

$$dS_{SW} = \frac{\lambda P'(\lambda, u_k)}{\sqrt{P^2(\lambda, u_k) - 4\Lambda^{2N}}} d\lambda,$$

(7)

and $A^i$ and $B_j$ constitute a symplectic basis of homology cycles of the hyperelliptic curve, with the canonical intersections $A^i \cap A^j = B_i \cap B_j = 0$ and $A^i \cap B_j = \delta^i_j$, $i, j = 1, \ldots, N - 1$. The prepotential $F(a)$ is implicitely defined by the equation

$$a_{Di} = \frac{\partial F(a)}{\partial a^i}.$$ 

(8)

The exact determination of $F(a)$ involves, in general, the integration of functions $a_{Di}(a)$ for which there is not a closed form available. Therefore, $F(a)$ will only be calculable in a series expansion.
Third: The BPS spectrum is obtained by integrating the Seiberg–Witten differential $dS_{SW}$ along all nontrivial cycles of the Riemann surface, $\nu(n^e, n^m) = n^e \cdot A + n^m \cdot B$. In fact, this is immediate from the previous point and the fact that the central charge of a state with $n^e_i$ units of electric charge and $n^m_i$ units of magnetic charge with respect to the $i$-th $U(1)$ unbroken subgroup can be written as

$$Z(n^e, n^m) = n^e \cdot a + n^m \cdot a_D.$$  

(9)

Appart from the mass, what remains invariant is the intersection number of two BPS states, given by

$$\nu \cap \nu' = n^e \cdot n'^m - n'^e \cdot n^m \in \mathbb{Z}.$$  

(10)

Notice that this is nothing but the Dirac-Schwinger-Zwanzinger quantization condition [17]. Two dyons are mutually local if they have zero intersection $\nu \cap \nu' = 0$.

Changes in the symplectic basis of homology cycles are performed by means of a symplectic matrix $\Gamma \in Sp(2r, \mathbb{R})$. Accordingly, $a = (a^i, a^D_j)$ transforms as a vector and $\tau_{ij}$ as a modular form.

Since the central charge, $Z_n = n^e \cdot a$ with $n = (n^e_i, n^m_j)$, is an observable, the invariance of the non-perturbative BPS spectrum breaks the continuous duality group $Sp(2r, \mathbb{R})$ down to the discrete subgroup $Sp(2r, \mathbb{Z})$.

Fourth: There are singularities in $M_{\Lambda}$, encoded in the quantum discriminant $\Delta_{\Lambda}$,

$$\Delta_{\Lambda}(u_k, \Lambda) = \prod_{i<j}(e^+_i - e^+_j)^2(e^-_i - e^-_j)^2 = c \Lambda^{2N^2} \Delta_+ \Delta_-,$$  

(11)

at whose zero locus, $\Sigma_{\Lambda}$, two branch points $e^+_i, e^+_j$ collide. What is the same, a certain homology cycle on the Riemann surface shrinks to zero at $\Sigma_{\Lambda}$, signaling the appearance of an extra massless state which is generically a dyon. These singularities lie on curves that intersect at points where many BPS states become simultaneously massless. In particular, at the so-called $N-1$ points, exactly $N-1$ mutually local monopoles become massless. This is the maximal number of mutually local simultaneously massless BPS states. The physics of these points was first investigated in Ref.[18]. They remain the vacua of the $\mathcal{N} = 1$ theory upon perturbation of the $\mathcal{N} = 2$ theory by a mass term. For $SU(N)$, $N > 2$, there are regions in $M_{\Lambda}$ where mutually non-local dyons become simultaneously massless, and the corresponding effective low-energy dynamics seems to be given by a superconformal field theory [19].

III The universal Whitham hierarchy

The name Whitham hierarchy stands for a wide class of integrable systems of differential equations that describe modulations of solutions of soliton equations [9, 20, 21]. Following Krichever [10], we define the moduli space of the Whitham hierarchy by

$$\hat{M}_{g,p} \equiv \{ \Gamma_g, P_a, \xi_a(P), a = 1, \ldots, p \}$$  

(12)

containing the following set of algebraic-geometrical data:

- $\Gamma_g$ denotes a smooth algebraic curve of genus $g$.
- $P_a$ is a set of $p$ points (punctures) on $\Gamma_g$ in generic positions (we will consider, for simplicity, $p = 1$).
- $\xi_a$ are local coordinates in the neighbourhood of the $p$ points, i.e. $\xi_a(P_a) = 0$.

From the general theory of meromorphic differentials over Riemann surfaces we know that there are three basic types of Abelian differentials:
i. **Holomorphic differentials, \( dw_i \).** In any open set \( U \in \Gamma_g \), with complex coordinate \( \xi \), they are of the form \( dw = f(\xi) d\xi \) with \( f \) an holomorphic function. The vector space of holomorphic differentials on a genus \( g \) Riemann surface has complex dimension \( g \). If the curve is hyperelliptic \([4]\), a canonical basis \( \{ dw_j \} \) of this vector space is defined through

\[
\oint_{A_i} dw_j = \delta^i_j \quad \oint_{B_i} dw_j = \tau_{ij} ,
\]

where \( \tau_{ij} \) is the period matrix of the complex curve.

ii. **Meromorphic differentials of the second kind, \( d\Omega_{P,n} \).** They have a single pole of order \( n + 1 \) at point \( P \in \Gamma \), and zero residue. In local coordinates \( \xi \), we shall adopt the normalization

\[
d\Omega_{P,n} = (\xi^{-n-1} + O(1)) d\xi .
\]

This fixes \( d\Omega_{P,n} \) up to an arbitrary combination of holomorphic differentials. There are several ways to fix this normalization. In the context of integrable theories, the standard way to do it is to require that \( d\Omega_{P,n} \) has vanishing \( A_i \)-periods

\[
\oint_{A_i} d\Omega_{P,n} = 0 .
\]

iii. **Meromorphic differentials of the third kind, \( d\Omega_{P,0} \).** They have first order poles at \( P \) and \( P_0 \) (a reference point) with opposite residues taking values \(+1\) and \(-1\) respectively. In local coordinates \( \xi(\xi_0) \) about \( P(P_0) \),

\[
d\Omega_{P,0} = (\xi^{-1} + O(1)) d\xi = -(\xi_0^{-1} + O(1)) d\xi_0 .
\]

The regular part is normalized by demanding that \( d\Omega_{P,0} \) has vanishing \( A_i \)-periods. The appearance of simple poles in the Seiberg–Witten solution is related to the inclusion of matter hypermultiplets in the fundamental representation \([1]\). We will only consider in this talk the case of pure \( SU(N), N = 2 \) super Yang-Mills theory. Thus, we are going to rule out the meromorphic differentials of the third kind from our discussion.

The standard Whitham equations take the following form \([21]\)

\[
\frac{\partial d\Omega_n}{\partial T^m} = \frac{\partial d\Omega_m}{\partial T^n} \tag{17}
\]

where \( d\Omega_n \) is short for \( d\Omega_{P,n} \), and \( T^n \) are a set of slow times the 1-forms may depend upon. According to our previous remark, \( n \) will be considered greater or equal than one, unless the contrary is stated. The Whitham hierarchy can be enhanced to incorporate also holomorphic differentials \( dw_i \), with associated parameters \( \alpha^i \), such that

\[
\frac{\partial dw_i}{\partial \alpha^j} = \frac{\partial dw_j}{\partial \alpha^i} \quad \frac{\partial dw_i}{\partial T^n} = \frac{\partial d\Omega_n}{\partial \alpha^i} \quad \frac{\partial d\Omega_n}{\partial T^m} = \frac{\partial d\Omega_m}{\partial T^n} . \tag{18}
\]

Equations (18) are nothing but the integrability conditions implying the existence of a generating differential \( dS \) satisfying

\[
\frac{\partial dS}{\partial \alpha^i} = dw_i \quad \frac{\partial dS}{\partial T^n} = d\Omega_n . \tag{19}
\]

The Whitham equations hide a certain holomorphic function named prepotential \( F(\alpha^i, T^n) \), that can be defined implicitly through the following set of equations

\[
\frac{\partial F}{\partial \alpha^j} = \oint_{B_j} dS \quad \frac{\partial F}{\partial T^n} = \frac{1}{2\pi i n} \oint_{P} \xi^{-n} dS . \tag{20}
\]
The local behaviour of the generating differential near the puncture $P$ is then

$$dS \sim \left\{ \sum_{n \geq 1} T^n \xi^{n-1} + 2\pi i \sum_{n \geq 1} n \frac{\partial F}{\partial T^n} \xi^{n-1} \right\} d\xi .$$ (21)

An interesting class of solutions, and certainly that which is relevant in connection with $N = 2$ super Yang-Mills theories, is given by those prepotentials that are homogeneous of degree two:

$$\sum_{n \geq 1} \frac{N-1}{n} \alpha_i \frac{\partial F}{\partial \alpha_i} + \sum_{n \geq 1} T^n \frac{\partial F}{\partial T^n} = 2F .$$ (22)

The generating differential $dS$ for homogeneous solutions admits the following form [8, 10]:

$$dS = \sum_{i=1}^{N-1} \alpha_i dw_i + \sum_{n \geq 1} T^n d\Omega_n ,$$ (23)

and, after (13)–(15), the parameters $\alpha_i$ and $T^n$ can be recovered from $dS$ as follows:

$$\alpha_i = \oint_{A_i} dS T^n = \text{res}_P \xi^n dS .$$ (24)

Inserting (19) and (24) into (22), a formal expression for $F$ in terms of $dS$ can be obtained [22],

$$F = \frac{1}{2} \sum_{i=1}^{N-1} \oint_{A_i} dS \oint_{B_i} dS + \frac{1}{4\pi i} \sum_{n \geq 1} \frac{1}{n} \oint_P \xi^n dS \oint_P \xi^{-n} dS .$$ (25)

Following [22], let us consider the decomposition of $dS$ in a different basis of Abelian differentials,

$$dS = \sum_{n \geq 1} T^n d\hat{\Omega}_n ,$$ (26)

where $d\hat{\Omega}_n$ are meromorphic differentials of the second kind (with the same local behaviour than $d\Omega_n$), whose regular part is fixed by the condition:

$$\frac{\partial d\hat{\Omega}_n}{\partial \text{moduli}} = \text{holomorphic} .$$ (27)

Notice that we have not added explicitly holomorphic differentials in $dS$: they are somehow hidden inside the differentials $d\hat{\Omega}_n$. In more concrete terms, the definition of the $\alpha^i$ parameters as given in (24), now forces them to depend on $T^n$ and $u_k$. Conversely, provided we impose that $d\alpha^i/dT^n = 0$, an implicit set of homogeneous functions $u_k(T^n, \alpha^i)$ of degree zero is obtained, and they solve the Whitham equations:

$$\frac{\partial u_k}{\partial T^n} = -\left( \frac{\partial \alpha^i}{\partial u_k} \right)^{-1} \frac{\partial \alpha^i}{\partial T^n} .$$ (28)

Finally, from (24) and (26), it is clear that

$$d\hat{\Omega}_m = d\Omega_m + \sum_{i=1}^{N-1} \frac{\partial \alpha^i}{\partial T^m} dw_i .$$ (29)
The next task is to look for an embedding of the Seiberg–Witten ansatz within the Whitham hierarchy. We will follow very closely, to this end, the approach of Gorsky, Marshakov, Mironov and Morozov [12]. As already noticed in Ref. [7], the curve (3) is the hyperelliptic representation for the spectral curve of the periodic Toda chain of length \( N \). It can be written in terms of a complex parameter \( w \) as follows (we set \( \Lambda = 1 \) for convenience)

\[
P = \left( w + \frac{1}{w} \right) \quad y = \left( w - \frac{1}{w} \right) .
\]  

(30)

This defines a natural coordinate in the vicinity of the two points at infinity \( \infty_+ \sim (\pm y = \infty, \lambda = \infty) \). In fact, from Eq. (30), \( w \) can be written as a meromorphic function

\[
w = \frac{1}{2} (P + y) ,
\]

(31)

that near \( \infty_\pm \) goes as \( w \sim \lambda^\pm N \). Then, \( \xi_\pm = w^{\pm 1/N} \) are local coordinates at the punctures \( \infty_\pm \), which are the points where the relevant meromorphic differential of the Seiberg–Witten solution, \( dS_{SW} \), has its (second order) poles. The times associated to each puncture will be denoted with positive and negative subindices, \( i.e. T_{\infty_\pm,n} = T_{\pm,n} \). Also, it is convenient to slightly change the normalization of our second-kind differentials to be \( d\Omega_{\pm n} \sim \frac{N}{n} dw^{\pm n/N} \).

The Seiberg–Witten differential, \( dS_{SW} \), belongs to the class of solutions of the Whitham hierarchy that fulfill

\[
\frac{\partial dS}{\partial \text{moduli}} = \text{holomorphic} .
\]

(32)

In fact, since its \( A^i \)-periods are \( a^i \), one is tempted to identify \( dS_{SW} \) as the generating form of the Whitham hierarchy at \( \alpha^j = a^j \) and \( T_{\pm 1} = 1, T_{|n| > 1} = 0 \),

\[
dS_{SW} = a^i dw_i + d\Omega_{\infty,1} + d\Omega_{\infty,-1} .
\]

(33)

Varying Eq. (30) for a given curve, \( i.e. \) for fixed \( u_k \) and \( \Lambda \), the Seiberg–Witten differential can be written as

\[
dS_{SW} = \frac{\lambda P'}{y} d\lambda = \lambda \frac{dw}{w} ,
\]

(34)

this making extremely easy to verify the defining property of \( dS_{SW} \),

\[
\frac{\partial dS_{SW}}{\partial u_k} \bigg|_{w=\text{const.}} = \frac{\lambda^{N-k}}{P'} \frac{dw}{w} = \frac{\lambda^{N-k} d\lambda}{y} = d\nu_k , \quad k = 2, 3, ..., N .
\]

(35)

If we hold \( \lambda \) fixed –instead of \( \omega \)–, there is an additional total derivative in the previous expression. Notice that, from the point of view of the Whitham equations, it matters which coordinates are held fixed as long as there are residues to be computed. It will always be understood that derivatives w.r.t. the moduli are taken at constant \( w \).

Lemma [12]: The meromorphic differentials \( d\hat{\Omega}_n \) have the form

\[
d\hat{\Omega}_n = R_n \frac{dw}{w} = P_n^{\pm N} \frac{dw}{w} ,
\]

(36)

where the projection \( (\sum_{k=-\infty}^{\infty} c_k \lambda^k)^+ = \sum_{k=0}^{\infty} c_k \lambda^k \). Notice that, from their defining equation (27), the \( d\hat{\Omega}_n \) have poles at both punctures, \( i.e. \) there are not \( d\hat{\hat{\Omega}}_{\pm n} \) but \( d\hat{\Omega}_n = d\hat{\hat{\Omega}}_n + d\hat{\hat{\Omega}}_{-n} \).
At this point, as shown in Ref. [12], the first and second order derivatives of the prepotential can be computed from Eq. (20), with the following result

\[
\frac{\partial \mathcal{F}}{\partial T_n} = \frac{\beta}{2\pi i} \mathcal{H}_2 \\
\frac{\partial^2 \mathcal{F}}{\partial T_n \partial T_m} = \frac{\beta}{2\pi i} \mathcal{H}_{n+1} ,
\]

(37)

\[
\frac{\partial^2 \mathcal{F}}{\partial \alpha^i \partial \log \Lambda} = \frac{\beta}{2\pi i} \mathcal{H}_2 \\
\frac{\partial^2 \mathcal{F}}{\partial \alpha^i \partial \alpha^j} = \frac{\beta}{2\pi i} \mathcal{H}_{n+1} ,
\]

(38)

where \( \beta = 2N \) is the coefficient of the beta function, whereas \( \mathcal{H}_{m+1,n+1} \) and \( \mathcal{H}_{n+1} \) are homogeneous combinations of the Casimirs constructed as follows:

\[
\mathcal{H}_{m+1,n+1} = \frac{N}{m n} \text{res}_{\infty} \left( P^{m/N}(\lambda) dP^{n/N}(\lambda) \right) = \mathcal{H}_{n+1,m+1} \\
\mathcal{H}_{m+1} \equiv \mathcal{H}_{m+2} .
\]

(39)

The important ingredient is the Riemann’s Theta function \( \Theta_E(z|\tau) \), \( E \) being the following even and half-integer characteristic [13]:

\[
\tilde{\alpha} = (0, \ldots, 0) \quad \tilde{\beta} = (1/2, \ldots, 1/2) .
\]

(40)

As they stand, however, the expressions given in (37)–(38) are not yet suitable for application to the Seiberg–Witten solution. It is still necessary to define the rescaled times \( \hat{T}_n = T_n^{-1} T_n \) and moduli \( \hat{u}_k = T_1 u_k \) (correspondingly, \( \hat{\mathcal{H}}_{m,n+1} = T_1^{m+1} \mathcal{H}_{m+1,n+1} \)) after which, the prepotential of the Seiberg–Witten solution is obtained by identifying \( T_1 \) with \( \Lambda \) in the submanifold \( \hat{T}_{n>1} = 0 \), provided that the moduli space be parametrized by the \( \hat{u}_k \) (notice that \( \hat{a}^i \equiv \alpha^i(u_k,T_1,\hat{T}_{n>1} = 0) = T_1 a^i(u_k,1) = a^i(\hat{u}_k, T_1) \) [14]. The restriction to the submanifold \( \hat{T}_{n>1} = 0 \), yields formulae which are suited for the Seiberg–Witten solution. In particular, from (37) and (38) we obtain

\[
\frac{\partial \mathcal{F}}{\partial \log \Lambda} = \frac{\beta}{2\pi i} \hat{\mathcal{H}}_2 \\
\frac{\partial^2 \mathcal{F}}{\partial \log \Lambda \partial T_n} = \frac{\beta}{2\pi i} \hat{\mathcal{H}}_{n+1} ,
\]

(41)

\[
\frac{\partial^2 \mathcal{F}}{\partial \alpha^i \partial \log \Lambda} = \frac{\beta}{2\pi i} \hat{\mathcal{H}}_2 \\
\frac{\partial^2 \mathcal{F}}{\partial \alpha^i \partial \alpha^j} = \frac{\beta}{2\pi i} \hat{\mathcal{H}}_{n+1} ,
\]

(42)

\[
\frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} = -\frac{\beta^2}{2\pi i} \hat{\mathcal{H}}_2 \hat{\mathcal{H}}_2 \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) ,
\]

(43)

\[
\frac{\partial^2 \mathcal{F}}{\partial \log \Lambda \partial T_n} = -\frac{\beta^2}{2\pi i} \hat{\mathcal{H}}_2 \hat{\mathcal{H}}_{n+1} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) ,
\]

(44)

\[
\frac{\partial^2 \mathcal{F}}{\partial T_n \partial T_m} = \frac{\beta}{2\pi i} \left( \hat{\mathcal{H}}_{m+1,n+1} + \frac{\beta}{mn} \hat{\mathcal{H}}_{m+1} \hat{\mathcal{H}}_{n+1} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) \right) ,
\]

(45)

with \( m, n \geq 2 \).

The first equation in (41) is precisely the RG equation derived in Ref. [28]. Combining the second equation in (41) and (44), it is easy to obtain an interesting relation between Casimir operators [12]:

\[
\frac{\partial \hat{\mathcal{H}}_m}{\partial \log \Lambda} = -\frac{\beta}{2\pi i} \hat{\mathcal{H}}_2 \hat{\mathcal{H}}_m \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) .
\]

(46)

Hereafter, we will always work with the scaled coordinates and hats will be omitted everywhere.
V Instanton Corrections

Instanton calculus provides one of the few non-perturbative links between the Seiberg–Witten solution and the microscopic non-abelian field theory that it is supposed to describe effectively at low energies. From the microscopic theory point of view, instanton contributions to the asymptotic semiclassical expansion of the effective prepotential have been computed, and a remarkable agreement with the Seiberg–Witten solution has been found [16]. We shall see in this section that the connection of $SU(N)$ $\mathcal{N} = 2$ super Yang–Mills theory with Toda–Whitham hierarchies embodies in a natural way a recursive procedure to compute the instanton expansion of the effective prepotential up to arbitrary order.

To begin with, let us fix our conventions. We choose the basis $H_k = E_{k,k} - E_{k+1,k+1}$ for the Cartan subalgebra and $E_{k,j}, k \neq j$ for the raising and lowering operators. Let $\{\alpha_i\}_{i=1,...,N-1}$ stand for the simple roots of $SU(N)$ and $(\alpha, \beta)$ denote the usual inner product constructed with the Cartan-Killing form. The dot product $\alpha \cdot \beta \equiv 2(\alpha, \beta)/(\beta, \beta) = (\alpha, \beta')$. We have that $\alpha_i \cdot \alpha_j = C_{ij}$, with $C_{ij}$ the Cartan matrix, while $\lambda^i \cdot \alpha_j = \delta^i_j$ define the fundamental weights. In particular this means that $\alpha_i = \sum_j C_{ij} \lambda^j$. The simple roots generate the root lattice $\Delta = \{\alpha = n^i \alpha_i | n^i \in \mathbb{Z}\}$.

The instanton expansion of the prepotential was given in eq. (3). We then have, for the LHS of (43),

$$\frac{\partial^2 \mathcal{F}}{\partial (\log \Lambda)^2} = \frac{1}{2\pi i} \sum_{k=1}^{\infty} (2Nk)^2 \mathcal{F}_k(Z) \Lambda^{2Nk}.$$  \hspace{1cm} (47)

The derivative of the quadratic Casimir also has an expansion that can be obtained from the RG equation (first equation in (33)) and the expansion of the prepotential

$$\frac{\partial \mathcal{H}_2}{\partial a^i} = \frac{2\pi i}{\beta} \frac{\partial^2 \mathcal{F}}{\partial a^i \partial \log \Lambda} = C_{ij} a^j + \sum_{k=1}^{\infty} k \mathcal{F}_{k,i} \Lambda^{2Nk} \equiv \sum_{k=0}^{\infty} H_i^{(k)} \Lambda^{2Nk},$$  \hspace{1cm} (48)

where $\mathcal{F}_{k,i} = \partial \mathcal{F}_k/\partial a^i$. The term involving the couplings that appear in the Theta function $\Theta_E$ is

$$i\pi n^i \tau_n n^j = \sum_{\alpha_+} \log \left( \frac{Z_\alpha}{\Lambda} \right)^{-(\alpha \cdot \alpha_+)^2} + \frac{1}{2} \sum_{k=1}^{\infty} (\alpha \cdot \mathcal{F}^{n^i}_{k} \cdot \alpha) \Lambda^{2Nk},$$  \hspace{1cm} (49)

where $\alpha = n^i \alpha_i$ and

$$\alpha \cdot \mathcal{F}^{n^i}_{k} \cdot \alpha \equiv \sum_{i,j} n^i \frac{\partial^2 \mathcal{F}_k}{\partial a^i \partial a^j} n^j = \sum_{\beta, \gamma \in \Delta} (\alpha \cdot \beta) \frac{\partial^2 \mathcal{F}_k}{\partial Z^\beta \partial Z^\gamma} (\gamma \cdot \alpha).$$  \hspace{1cm} (50)

For convenience, we have adjusted the bare coupling to $2\pi i \tau_0 = 3N$. We may shift $\tau_0$ to any value by an appropriate rescaling of $\Lambda$. This will be reflected in the normalization of the $\mathcal{F}_k$.

Inserting (33) in the Theta function, we obtain

$$\Theta_E(0|\tau) = \sum_{\rho=0}^{\infty} \sum_{\alpha_+ \in \Delta_\rho} (-1)^{\rho \cdot \alpha} \prod_{\alpha_+} Z_\alpha^{-(\alpha \cdot \alpha_+)^2} \prod_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{2m!} (\alpha \cdot \mathcal{F}^{n^i}_{k} \cdot \alpha)^m \Lambda^{2Nkm} \Lambda^{2N\rho} \equiv \sum_{p=0}^{\infty} \Theta(p) \Lambda^{2Np},$$  \hspace{1cm} (51)

In the previous expression, $\Delta_\rho \subset \Delta$ is a subset of the root lattice composed of those lattice vectors $\alpha$ that fulfill the constraint $\sum_{\alpha_+} (\alpha \cdot \alpha_+)^2 = 2N\rho$. In particular $\Delta_1$ is the root system, i.e. the simple
roots together with their Weyl reflections. On the other hand $\Delta_r$, for $r > 1$, will be in general a union of Weyl orbits, since Weyl reflections are easily seen to be an automorphisms of $\Delta_r$. Therefore, $\Theta^{(p)}$ is Weyl invariant by construction. In the logarithmic derivative, $\Theta_E$ appears in the denominator, so we need the expansion of the inverse of the Theta function (see Ref. [13] for details):

$$\Theta(0|\tau)^{-1} = \sum_{l=0}^{\infty} \Xi(l)(\Theta) \Lambda^{2Ni}.$$  \hspace{1cm} (52)

Finally, the derivative of the Theta function with respect to the period matrix is given by

$$\frac{1}{i\pi} \partial_{ij} \Theta_E(0, \tau) = \sum_{r=1}^{\infty} \sum_{\alpha \in \Delta_r} (-1)^{\rho\alpha}(\alpha \cdot \lambda^i)(\alpha \cdot \lambda^j) \prod_{\alpha^+} Z_{\alpha^+}^{-((\alpha \cdot \alpha)^2)} \prod_{k=1}^{\infty} \exp \left( \frac{1}{2}(\alpha \cdot F''_k \cdot \alpha) \Lambda^{2Nk} \right) \Lambda^{2Np}$$

$$\equiv \sum_{p=1}^{\infty} \Theta_{ij}^{(p)} \Lambda^{2Np}.$$  \hspace{1cm} (53)

Collecting all the pieces and inserting them back into (43), we find for $F_k(Z)$ the following expression:

$$F_k(Z) = -k^{-2} \sum_{p, q, l=0}^{p+q+l=k-1} \sum_{ij} H_i^{(p)} H_j^{(q)} \Theta_{ij}^{(k-p-q-l)} \Xi(l),$$  \hspace{1cm} (54)

in terms of the previously defined coefficients. If we look at the coefficients in the r.h.s. of Eq. (54), it is easy to see that the expressions they involve depend on $F_1, F_2, \ldots$ up to $F_{k-1}$. In fact, although both $H^{(p)}$ and $\Theta^{(p)}$ depend on $F_1, \ldots, F_p$, the indices within parenthesis reach at most the value $k - 1$ as $\Theta_{ij}^{(0)} = 0$. Moreover $\Theta_{ij}^{(k)}$ depends on $F_1, \ldots, F_{k-1}$ since the vector $\alpha = 0$ is missing from the lattice sum. This fact implies the possibility to build up a recursive procedure to compute all the instanton coefficients by starting just from the perturbative contribution to $F(a)$ in (3). The first few instanton contributions, for example, are simply [13]:

$$F_1 = - \sum_{\alpha \in \Delta_1} (-1)^{\rho \alpha} Z_{\alpha^+}^{2} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha \cdot \alpha)^2},$$  \hspace{1cm} (55)

$$F_2 = - \frac{1}{4} \left( \sum_{\alpha \in \Delta_1} (-1)^{\rho \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha \cdot \alpha)^2} \left[ F_1 + 2(\alpha \cdot F'_1) Z_{\alpha^+} + \frac{1}{2}(\alpha \cdot F''_1 \cdot \alpha) Z_{\alpha^+} \right] \right.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} $\left. + \sum_{\beta \in \Delta_2} Z_{\beta}^{2} \prod_{\alpha^+} Z_{\alpha^+}^{-(\beta \cdot \alpha)^2} \right),  \hspace{1cm} (56)$

$$F_3 = - \frac{1}{9} \left( \sum_{\alpha \in \Delta_1} (-1)^{\rho \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha \cdot \alpha)^2} \left[ 4F_2 + 4(\alpha \cdot F'_2) Z_{\alpha^+} + (\alpha \cdot F''_2 \cdot \alpha) Z_{\alpha^+} + \frac{1}{2}(\alpha \cdot F''_1 \cdot \alpha) (F_1 + 2(\alpha \cdot F'_1) Z_{\alpha^+}) \right. \right.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} $\left. \left. + \frac{1}{8}(\alpha \cdot F''_1 \cdot \alpha) Z_{\alpha^+}^{2} + \frac{1}{2}(\alpha \cdot F''_1 \cdot \alpha) Z_{\alpha^+} \right) + \sum_{\beta \in \Delta_2} (-1)^{\rho \beta} \prod_{\alpha^+} Z_{\alpha^+}^{-(\beta \cdot \alpha)^2} [F_1 + 2(\beta \cdot F'_1) Z_{\beta^+}] \right.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} $\left. + \frac{1}{2}(\beta \cdot F''_1 \cdot \beta) Z_{\beta^+}^{2} + \sum_{\gamma \in \Delta_3} (-1)^{\rho \gamma} \prod_{\alpha^+} Z_{\alpha^+}^{-(\gamma \cdot \alpha)^2} Z_{\gamma^+}^{2} \right),  \hspace{1cm} (57)$

etc. The above expressions make patent the recursive character of the procedure.
VI Soft SUSY Breaking with Higher Casimir Operators

Sofly–broken supersymmetric models offer the best phenomenological candidates to solve the hierarchy problem in grand–unified theories. The spurion formalism provides a tool to generate soft supersymmetry breaking in a neat and controlled manner. To illustrate the method, start from a supersymmetric lagrangian, \( L(\Phi_0, \Phi_1, \ldots) \) with some set of chiral superfields, and single out a particular one, say \( \Phi_0 \). If you let this superfield acquire a constant vev along a given direction in superspace like, for example, \( \langle \Phi_0 \rangle = c_0 + \theta^2 F_0 \), it will induce soft breaking terms, and a vacuum energy of order \( |F_0|^2 \). Turning the argument around, you could promote any parameter in your lagrangian to a chiral superfield, and then freeze it along a supersymmetry breaking direction in superspace giving a vev to its highest component.

From embedding the Seiberg–Witten solution within the Toda–Whitham framework, we have obtained the analytic dependence of the prepotential on some new parameters \( T_n \). In this section, we will interpret these slow times as parameters of a non-supersymmetric family of theories, by promoting them to spurion superfields. In Refs.[4, 5, 25] this program was initiated with the scale parameter \( \Lambda \) and the masses of additional hypermultiplets, \( m_i \), as the only sources for spurions. The slow times, as Eq.(41) shows, are dual to the \( H_{m+1} \), which are homogeneous combinations of the Casimir operators of the group. This means that we will be able to parametrize soft supersymmetry breaking terms induced by all the Casimirs of the group, and not just the quadratic one. In this way, we shall extend to the \( N = 0 \) case the family of \( N = 1 \) supersymmetry breaking terms first considered by Argyres and Douglas [19].

We define the spurion variables \( s_n \) as follows

\[
s_1 = -i \log \Lambda \quad \quad \quad \quad s_n = -i T_n, \quad n = 2, \ldots, r = N - 1.
\]  

Our independent coordinates in the prepotential are \( \alpha^i, s_n \). Using (37) one can find explicit expressions for the dual spurions:

\[
\begin{align*}
s_D^1 &= \frac{\beta}{2\pi} \left[ \mathcal{H}_2 + i \sum_{m \geq 2} m s_m \mathcal{H}_{m+1} - \sum_{m,n \geq 2} m s_m s_n \mathcal{H}_{m+1,n+1} \right], \\
s_D^n &= \frac{\beta}{2\pi n} \left[ \mathcal{H}_{n+1} + i \sum_{m \geq 2} m s_m \mathcal{H}_{m+1,n+1} \right].
\end{align*}
\]  

Notice that, when the spurions \( s_m \) are zero, we recover for the variable \( s_1 \) the results of [3]. Under the symplectic group \( \text{Sp}(2r, \mathbb{Z}) \), the spurions are taken to be scalars, \( s_D^i = s_i \). From the point of view of the Toda–Whitham hierarchy, this is natural in that the slow times parametrize deformations of the curve, and should not be affected by duality transformations (which are transformations among symplectic basis of homology cycles of the curve). From the point of view of physics, this invariance is important because their vev is an external unambiguous input. We see from (59) that the dual times are also invariant under duality transformations.

To break \( N = 2 \) supersymmetry down to \( N = 0 \), as anticipated above, we promote the variables \( s_n \) to \( N = 2 \) vector superfields \( S_n \), and then freeze the scalar and auxiliary components to constant vacuum expectation values. We would like to restrict our framework to non-supersymmetric deformations of the original pure SU(\( N \)) super Yang–Mills theory. Thus, for all \( S_n, n \geq 2 \), we only keep the top components \( F_n \) as a supersymmetry breaking parameter (by SU(\( 2R \)) symmetry, we can always rotate the \( D_n \) components away). In terms of \( N = 1 \) superfields we have,

\[
S \equiv S_1 = s_1 + \theta^2 F_1 \quad \quad S_n = \theta^2 F_n, \quad n \geq 2,
\]  

11
where $s_1$ is related, as seen in (58) to the dynamical scale of the theory. The analysis of the soft breaking induced only by $S_1$ has been done in Refs. 4, 5.

As the prepotential has an analytic dependence on the spurion superfields, the effective Lagrangian up to two derivatives and four fermion terms for the $\mathcal{N} = 0$ theory is given by the exact Seiberg–Witten solution once the spurion superfields are taken into account. This gives the exact effective potential at leading order and the vacuum structure can be determined. That is, all over the quantum moduli space, the effective action will be

$$L_{VM} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial F}{\partial \Phi^I} \bar{\Phi}^I + \frac{1}{2} \int d^2\theta \frac{\partial^2 F}{\partial \Phi^I \partial \Phi^J} W_a W^{a,J} \right], \quad (61)$$

where the capital indices $I, J$ stand both for $i, j = 1, \ldots, N - 1$ labelling abelian chiral $\mathcal{N} = 2$ multiplets $(\Phi^i, V^i)$, and for $m, n = 1, \ldots, N - 1$ that label spurion multiplets $(S^m, V^m)$. If we are near a submanifold of the moduli space of vacua where $n_H$ hypermultiplets become massless, the full Lagrangian also contains the hypermultiplet contribution involving pairs of chiral superfields $H_a, \bar{H}_a, a = 1, \ldots, n_H$,

$$L_{HM} = \sum_a \int d^4\theta \left( H_a^* e^{2n_H V_1} H_a + \bar{H}_a^* e^{-2n_H V_1} \bar{H}_a \right) + \sum_{a,i} \left( \int d^2\sqrt{2} \Phi^i n^a_i H_a \bar{H}_a + \text{h.c.} \right), \quad (62)$$

where the charge of the $a$-th hypermultiplet with respect to the $i$-th $U(1)$ factor has been denoted by $n^a_i$ and, in the previous equation, a particular choice of duality frame, $a^i$, has been made. Namely the vector multiplets $(\Phi^i, V^i)$, are such that near the singular subvariety, the light BPS states in the previous lagrangian are weakly coupled, and perturbation theory is reliable. Of course, this amounts to an appropriate choice of the basis of homology cycles $(A_i, B^i)$. Now, if the BPS states becoming massless are mutually local, we can always fix a basis of cycles such that each $U(1)$ couples to one and only one hypermultiplet. This means that $n^a_i = \delta^a_i$ or vanishes, the later case being possible when $n_H < N - 1$.

The full effective lagrangian will be the sum of (61) and (62). The effective potential can be computed explicitly resulting in

$$V = B^{mn} F_m F_n^{*} + \sqrt{2} (n^a, b^m) \left(F_m \bar{h}_a h_a + \bar{F}_m \bar{h}_a h_a \right) + 2 (n^a, b^b)(h_a \bar{h}_a \bar{h}_b h_b) + \frac{1}{2} (n^a, b^b)(|h_a|^2 - |\bar{h}_a|^2)(|h_b|^2 - |\bar{h}_b|^2) + 2 |n^a, a|^2 (|h_a|^2 + |\bar{h}_a|^2), \quad (63)$$

where $n^a = \sum_i n_i^a a^i$, and $h_a (\bar{h}_a)$ is the scalar component of $H_a (\bar{H}_a)$. Also we have used the quantities

$$(n^a, b^b) = n_i^a b^{-1} i_j b_j^b, \quad (n^a, b^m) = n_i^a b^{-1} i_j b_j^b, \quad B^{mn} = b_{a^m} b^{-1} a^b b^m - b^{mn}, \quad (64)$$

where $b$ is given in terms of the generalized $2(N - 1) \times 2(N - 1)$ matrix of couplings $\tau_{IJ}$,

$$\tau_{ij} = \frac{\partial^2 F}{\partial \alpha^i \partial \alpha^j}, \quad \tau_{ni} = \frac{\partial^2 F}{\partial \alpha^i \partial s_n}, \quad \tau_{mn} = \frac{\partial^2 F}{\partial s_m \partial s_n}, \quad (65)$$

as

$$b_{IJ} = \frac{1}{4\pi} \text{Im} \tau_{IJ}, \quad (66)$$

and it can be computed all over the moduli space from Eqs. (62) - (65). This is precisely the point where the Seiberg–Witten solution enters the calculations.
To obtain the values of the condensates, we first minimize $V$ with respect to $h_a$, $\tilde{h}_a$, resulting in 

$$|h_a| = |\tilde{h}_a|.$$ 

It is convenient to fix the gauge in the $U(1)^{N-1}$ factors in such a way that

$$h_a = \rho_a \quad \quad \tilde{h}_a = \rho_a e^{i\beta_a}.$$ 

(67)

If the charge vectors $n^a$ are linearly independent, the non-trivial condensates satisfy the equation

$$|n^a \cdot a|^2 + \sum_b (n^a, n^b)\rho_b^2 e^{i(\beta_b - \beta_a)} + \frac{1}{\sqrt{2}}(n^a, b^m)F_m e^{-i\beta_a} = 0,$$

(68)

and the effective potential takes the value

$$V = B^m F_m F^*_n - 2\sum_{ab} (n^a, n^b)\rho_a^2 \rho_b^2 \cos(\beta_a - \beta_b).$$ 

(69)

We will consider in what follows, as an explicit example, the case of $SU(3)$.

VII Analysis of $SU(3)$

The $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(3)$ has been analyzed in detail in Refs.[3, 19].

There are two sets of distinguished singularities in the moduli space of this theory:

i. The three $\mathbb{Z}_2$ vacua, known as $\mathcal{N} = 1$ or maximal points [18], located at $u^3 = \frac{27\Lambda^6}{4}$, $v = 0$, that give rise to the $\mathcal{N} = 1$ vacua when the theory is perturbed with a mass term of the form $\text{Tr}\Phi^2$ (we denote $u_2 = u$, $u_3 = v$).

ii. The two $\mathbb{Z}_3$ vacua, known as Argyres–Douglas (AD) points [19], located at $u = 0$ and $v = \pm 2\Lambda^3$, where three mutually nonlocal BPS states become simultaneously massless. The low-energy theory there is an $\mathcal{N} = 2$ superconformal theory.

We will briefly describe the situation near both kind of singularities. We set $\Lambda^6 = 4$ for convenience.

The $\mathbb{Z}_2$ vacua

In this subsection we study the soft breaking of the theory near the $\mathcal{N} = 1$ points where two magnetic monopoles become simultaneously massless. To evaluate the second derivatives of the prepotential, we need the values of the periods of the hyperelliptic curve and the structure of the gauge couplings. We will focus on the $\mathcal{N} = 1$ point $(u = 3, v = 0)$, whereas the values of the quantities at the other two points can be obtained by using the $\mathbb{Z}_3$ unbroken symmetry. The derivatives of the Casimir operators with respect to the dual variables are given by [13]

$$\frac{\partial u}{\partial a_{Dj}} = -2i \sin \frac{\pi j}{N} \quad \quad \frac{\partial v}{\partial a_{Dj}} = -2i \sin \frac{2\pi j}{N}.$$ 

(70)

The gauge couplings near the $\mathcal{N} = 1$ point have the structure

$$\tau_{ij}^D = \frac{1}{2\pi i} \log \left(\frac{a_{Dj}}{2\sqrt{3}}\right) \delta_{ij} + (1 - \delta_{ij})\tau_{ij}^{\text{off}} + O(a_{Di}),$$

(71)

where $\tau_{ij}^{\text{off}}$, $i \neq j$ are the off-diagonal entries of the coupling constant at the $\mathcal{N} = 1$ point, which will be discussed in some detail in the next section. For $SU(3)$, $\tau_{12} = \frac{i}{\pi} \log 2$ [3]. To compute the Theta function, we have to take into account the change of the electric characteristic under the symplectic
transformation to the magnetic variables, \( a_{D_i} \). Indeed, the transformation law for the Theta function is given by

\[
\Theta[\alpha, \Gamma] \cdot \xi^{\Gamma} = e^{i\phi(\det(C\tau + D))^{1/2} \exp[\pi i \xi^t (C\tau + D)^{-1} C\xi]} \Theta[\alpha, \Gamma](\tau|\xi) ,
\]

where \( \phi \) is a \( \xi \)-independent phase and

\[
a^{\Gamma} = D\alpha - C\beta + \frac{1}{2} \text{diag}(CD^t) \quad \beta^{\Gamma} = -B\alpha + A\beta + \frac{1}{2} \text{diag}(AB^t) .
\]

Thus, the magnetic or dual characteristic, \( D \), is

\[
(1/2, 1/2) \quad (0, 0) .
\]

The leading behaviour of the Theta function with dual characteristic, we can compute its derivative at the \( N = 1 \) point of \( SU(3) \),

\[
\frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_D(0|\tau^D) \bigg|_{a_{D_i} = 0} = \frac{1}{4} \delta_{ij} - \frac{1}{12} (1 - \delta_{ij}) .
\]

Using (70) and (75), it is easy to check the relation (46) for the \( \Lambda \) derivatives of the Casimir operators.

At the \( N = 1 \) point, there is a symplectic basis for the hyperelliptic curve, such that the magnetic charge vectors are given by \( n^a_j = \delta^a_j, a_{D_i} = 0 \), and from eq.(68), the condensates are given by

\[
\rho^2_1 = \sqrt{\frac{3}{2\pi^2}} |F_1 + \frac{3}{2} F_2| \quad \rho^2_2 = \sqrt{\frac{3}{2\pi^2}} |F_1 - \frac{3}{2} F_2| .
\]

We see that the soft breaking induced by the quadratic and cubic Casimirs gives rise to monopole condensation in both \( U(1) \) factors, although the condensates are bigger for the soft breaking coming from \( u \) (for equal values of the supersymmetry breaking parameters \( F_1, F_2 \)). In the same way, the vacuum energy associated to these condensates is

\[
V_{\text{eff}} = -b^{mn} F_m F_n^* = -\frac{9}{4\pi^2} \left( |F_1|^2 + \frac{1}{2} |F_2|^2 \right) .
\]

As expected, the soft breaking associated to \( u \) gives lower energy to this vacuum.

The \( Z_3 \) vacua

Next we explore the behaviour near the Argyres–Douglas point at \((u = 0, v = 4)\). It is convenient to use the parameters \( \rho \) and \( \epsilon \) already introduced in Ref.[19],

\[
u = 3\epsilon^2 \rho \quad v - 4 = 2\epsilon^3 .
\]

The three submanifolds \( \rho^3 = 1 \) correspond to three massless BPS states which after an appropriate symplectic transformation can be seen to be charged with respect to only one of the \( U(1) \) factors, with variables denoted by \( a^1, a_{D1} \). They can be seen to be an electron, a dyon, and a monopole. These submanifolds come together at the AD point, where a nontrivial superconformal field theory is argued to exist [14]. To leading order, the hyperelliptic curve splits at the AD point into a small torus (corresponding to two mutually nonlocal periods \( a^1, a_{D1} \) which go to zero) and a big torus with periods \( a^2, a_{D2} \sim \Lambda \). The small torus is given by the elliptic curve

\[
w^2 = z^3 - 3\rho z - 2 ,
\]
and the meromorphic Seiberg–Witten differential degenerates on it to
\[ \lambda_{SW} = \frac{1}{2\sqrt{2\pi}} \epsilon^{5/2} w dz . \] (80)

The matrix of couplings near the AD point, at leading order, reads [19, 27, 28]
\[ \tau_{11} = \tau(\rho) + \mathcal{O}(\epsilon) \quad \tau_{12} = -\frac{i \epsilon^{1/2}}{c \omega_{\rho}} + \mathcal{O}(\epsilon^{3/2}) \quad \tau_{22} = \omega + \mathcal{O}(\epsilon) , \] (81)
where \( \omega_{\rho} \) is the period of the small torus (with \( \text{Im}(\omega_{\rho}/d\rho) > 0 \)), \( c \) is a nonzero constant and \( \omega = e^{\pi i/3} \).

To analyze the Theta function in these variables, we need the appropriate characteristic which, using (73) and the results in [27], is
\[ \vec{\alpha} = \vec{\beta} = (1/2, 1/2) . \] (82)

We can already obtain the behaviour of the Theta function as an expansion in \( \epsilon \):
\[ \Theta(0|\tau) = -\frac{1}{2\pi c} \frac{\epsilon^{1/2}}{\omega_{\rho}} \vartheta_1'(0|\tau(\rho)) \vartheta_1'(0|\omega) + \mathcal{O}(\epsilon^{3/2}) , \] (83)
where \( \vartheta_1(\xi|\tau) \) is the Jacobi theta function with characteristic \([1/2, 1/2]\). Now, using that
\[ \frac{\vartheta''_1(0|\tau)}{\vartheta'_1(0|\tau)} = -\pi^2 E_2(\tau) , \] (84)
we find the leading contribution to the derivative of the Theta function
\[ \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta = \left( \frac{1}{4} E_2(\tau(\rho)) \right) \left( \frac{1}{4} E_2(\omega) \right) . \] (85)
Again, it can be checked that the relation (46) for \( v \) holds (for \( u \), it is necessary to know the explicit values of the constants).

The analysis of the condensates near the AD point is difficult because one has to take into account mutually nonlocal degrees of freedom, and there is not a Lagrangian description of this theory. In fact, one expects that, in the softly broken theory, a cusp singularity will appear in the effective potential near the AD point, as it happens in \( \mathcal{N} = 2 \) QCD with gauge group \( SU(2) \) and one massive flavour [25]. But we can analyze the monopole condensates along the divisors \( \rho^3 = 1 \) and their evolution as we approach the AD point. Near each of the submanifolds \( \rho^3 = 1 \) there is a massless BPS state, and we expect it to condense after breaking supersymmetry down to \( \mathcal{N} = 0 \). These condensates correspond to mutually nonlocal states but we can assume, following the discussion in Refs. [5, 25], that these states do not interact, the condensates being given by the equation
\[ \rho^2 = -\frac{1}{(b^{-1})_{ij}} |a_k|^2 - \frac{e^{-i\beta_k}}{\sqrt{2(b^{-1})_{11}}} \sum_{n=1,2} F_n(b^{-1})_{kj} b_n^j , \] (86)
where \( k = 1, 2, 3 \) and \( a_k \) are the appropriate local coordinates for each of the massless states (\( i.e. \ a_k = a^1, a_{D1}, a^1 - a_{D1} \)). The quantities \( (b^{-1})_{ij}, b^n_j \) should be also computed in the duality frame dictated by the \( a_k \). This approximation should be good far enough from the AD point. These condensates give only a magnetic Higgs mechanism in one of the \( U(1) \) factors, and correspond to the half-Higgsed vacua.
of \cite{19}. Notice that one should perform a careful numerical study of the equations for the condensates and for the effective potential to know if these partial condensates give the true vacua of the $\mathcal{N} = 0$ theory. As we approach the AD point, $\epsilon \to 0$, we see that the parameters for condensation go to zero for both the quadratic and the cubic Casimirs:

$$\frac{\partial u}{\partial a_1^T}, \frac{\partial v}{\partial a_1^T} \sim O(\epsilon^{1/2}),$$

and the mass gap associated to the condensates vanishes at the AD point, like in the $\mathcal{N} = 1$ breaking considered in Ref.\cite{19}.

\section{Strong Coupling Expansion near the Maximal Points}

Let us end by applying the Seiberg–Witten–Whitham equations in the strong coupling regime of $\mathcal{N} = 2$ super Yang–Mills theory near its maximal singularities. The case of $SU(2)$ is special in that, being its Cartan subalgebra one–dimensional, the whole strong coupling expansion of the prepotential can be recursively computed \textit{without} an explicit knowledge of the actual solution $(a(u), a_D(u))$ \cite{14}, much in the same way than the previously derived instanton corrections. For generic $SU(N)$, however, the SWW equations do not give a closed procedure to obtain the strong coupling expansion of the effective prepotential. Aside from some technical difficulties, the main problem is that, in spite of the fact that a grading in $a_D$ and $\Lambda$ still exists, higher terms of the expansion appear in the equations corresponding to lowest powers of the dual variables spoiling recursivity (see Ref.\cite{14} for details). The SWW equations do not seem to be instrumental to study the full strong coupling expansion of $SU(N)$ $\mathcal{N} = 2$ super Yang–Mills theory.

Other methods have been derived in the literature to tackle the problem of computing the higher threshold corrections to the effective prepotential. For example, in Ref.\cite{29} this has been accomplished by parametrizing the neighborhood of the maximal singularities with a family of deformations of the corresponding auxiliary (singular) Riemann manifold. However, this formalism is not sensitive to quadratic terms in the prepotential. Thus, in particular, it does not give an answer for the couplings between different magnetic $U(1)$ factors at the maximal singularities of the moduli space, $\tau^{\text{off}}_{ij}$, introduced in the previous section. The existence and importance of such terms has been first pointed out in Ref.\cite{18} by using a scaling trajectory that smoothly connects the maximal singularities with the semiclassical region. These terms are also important ingredients in the expression of the Donaldson–Witten functional for gauge group $SU(N)$ \cite{27}. To our knowledge, a closed formula for these off-diagonal couplings has not been obtained so far, except for the gauge group $SU(3)$ \cite{3}. Let us then consider the uses of the SWW equations in the solution of this problem.

We have seen before that physical quantities in the neighborhood of any maximal singularity can be translated to a patch in the vicinity of any other by the action of the unbroken discrete subgroup $\mathbb{Z}_N$. We will consider in what follows the point where $u_2$ is real and positive. The strong coupling expansion of the prepotential at such singular point can be written in terms of appropriate $a_D$ variables as

$$\mathcal{F} = \frac{N^2}{2\pi i} \Lambda^2 + \frac{2N\Lambda}{\pi} \sum_{k=1}^{N-1} a_{Dk} \sin \hat{\theta}_k + \frac{1}{4\pi i} \sum_{k=1}^{N-1} a_{Dk}^2 \log \frac{a_{Dk}}{\Lambda_k} + \frac{1}{2} \sum_{k \neq l=1}^{N-1} \tau^{\text{off}}_{kl} a_{Dk} a_{Dl} + \frac{1}{2\pi i} \sum_{s=1}^{\infty} \mathcal{F}_s(a_D) \Lambda^{-s},$$

where $\hat{\theta}_k = \pi k/N$ and the logarithmic term, coming from the one-loop diagram that involves the light monopole, has the appropriate sign and factor making manifest that the theory is non-asymptotically

\footnote{We follow here the conventions of Ref.\cite{18} to fix the first three terms of the expansion.}
free and that there is a monopole hypermultiplet weakly coupled to each dual photon for \( a_D i \to 0 \). The remaining power series expansion comes from the integration of infinitely many massive BPS states: \( \mathcal{F}_s(a_D) \) are polynomials of degree \( s + 2 \) in dual variables and \( \tilde{\Lambda}_k = e^{3/2} \Lambda \sin \hat{\theta}_k \).

We have seen that the SWW formalism allow us to relate the strong coupling expansion of homogeneous combinations of higher Casimir operators \( \mathcal{H}_m \) with that of the prepotential through Eq.(46). Let us first remark that this equation is also valid for the higher Casimirs \( h_n \) themselves [12]: they, as well as their particular combinations encoded in \( \mathcal{H}_n \), are homogeneous functions of \( a_D \) and \( \Lambda \) of degree \( n \).

Thus, at the \( N = 1 \) singularities, the LHS of Eq.(46) is simply

\[
\frac{\partial h_n}{\partial \log \Lambda} = n h_n = \sum_{k=1}^{N} (2 \cos \theta_k)^n ,
\]

where we used the fact that the eigenvalues of \( \phi \) are \( \phi_i = 2 \cos \theta_i \) with \( \theta_i = (i - 1/2) \pi / N \) [18]. The derivative of the Casimir operators with respect to the dual variables can be computed at the same point of the moduli space, by using the explicit representation of the curve in terms of the Chebyshev polynomials [13], resulting in

\[
\frac{\partial h_n}{\partial a_D j} = -2 i \sum_{l=0}^{[n/2]-1} \binom{n-1}{l} \sin(n-2l-1) \hat{\theta}_j ,
\]

whereas, from the expansion of the effective prepotential, the leading couplings at the maximal singularity are given by

\[
\tau_{ij}^D = \frac{1}{2 \pi i} \log \left( \frac{a_{D i}}{\Lambda_i} \right) \delta_{ij} + \tau_{ij}^{\text{off}} .
\]

The derivative of the Theta function \( \Theta_D \) with respect to the period matrix has the following expression when evaluated at the \( N = 1 \) singularity

\[
\frac{1}{i \pi} \partial_{\tau_{ij}} \log \Theta_D(0, \tau_D) = \frac{1}{4} \left( \sum_{\xi^k = \pm 1} \exp \left( i \frac{\pi}{4} \xi^l \tau_{lm} \xi^m \right) \right)^{-1} \sum_{\xi^k = \pm 1} \xi^i \xi^j \exp \left( i \frac{\pi}{4} \xi^l \tau_{lm} \xi^m \right) .
\]

Now, we can insert the results (88)–(91) in the SWW equations (46) obtaining

\[
\frac{1}{2N} \sum_{k=1}^{N} (2 \cos \theta_k)^n = \sum_{l=0}^{[n/2]-1} \binom{n-1}{l} \sin \hat{\theta}_i \sin(n-2l-1) \hat{\theta}_j \left( \sum_{\xi^k = \pm 1} \exp \left( i \frac{\pi}{4} \xi^l \tau_{lm} \xi^m \right) \right)^{-1} \times \sum_{\xi^k = \pm 1} \xi^i \xi^j \exp \left( i \frac{\pi}{4} \xi^l \tau_{lm} \xi^m \right) .
\]

We have \( N - 1 \) equations and \((N - 1)(N - 2)/2\) unknowns (the components of the symmetric matrix \( \tau_{ij}^{\text{off}} \)). Thus, Eq.(92) has predictive power in its own only for \( SU(3) \) and \( SU(4) \). Indeed, we obtain for these two cases the following values:

\[
SU(3) : \quad \tau_{12}^{\text{off}} = i / \pi \log 2 \quad \quad SU(4) : \quad \begin{cases} 
\tau_{12}^{\text{off}} = \tau_{23}^{\text{off}} = -i / \pi \log (\sqrt{2} - 1) \\
\tau_{13}^{\text{off}} = i / \pi \log \sqrt{2}
\end{cases} .
\]

Notice that our result for \( SU(3) \) coincides with that of Ref.[3] while the ones for \( SU(4) \) have not been found previously. For higher \( SU(N) \), further ingredients would be necessary in order to obtain the
off-diagonal couplings at the $\mathcal{N} = 1$ singularity. Instead, we can think of Eq. (92) as a new constraint that $\tau_{mn}^{\text{off}}$ must obey. In fact, inspired by the findings in the last section of Ref. [18], we propose the following ansatz for $\tau_{mn}^{\text{off}}$:

$$
\tau_{mn}^{\text{off}} = \frac{2i}{N^2\pi} \sum_{k=1}^{N-1} \sin k\theta_m \sin k\theta_n \sum_{i,j=1}^{N} \tau_{ij}^{(0)} \cos k\theta_i \cos k\theta_j ,
$$

with $\tau_{ij}^{(0)}$ being given by

$$
\tau_{ij}^{(0)} = \delta_{ij} \sum_{k \neq i} \log(2 \cos \theta_i - 2 \cos \theta_k)^2 - (1 - \delta_{ij}) \log(2 \cos \theta_i - 2 \cos \theta_j)^2 .
$$

There is no equivalent expression available in the literature to compare with. Nevertheless, we can use precisely the SWW equations (92) in order to make a non-trivial check of our ansatz for the off-diagonal couplings (94)–(95). We have done it numerically up to SU(11) with remarkable success [14]. There is a second check that we can do using results that do not rely on Whitham equations at all. Douglas and Shenker showed that the matrix $\tau_{D}^{mn}$ at any point of the scaling trajectory, diagonalizes in the basis \{\sin k\theta_n\} with certain particular eigenvalues (see Eqs.(5.9)–(5.12) of Ref. [18]). The couplings (94)–(95) satisfy this restrictive condition in the limit of the scaling trajectory ending at the maximal singularity. As long as our solution (94)–(95) matches two very stringent and independent conditions, we believe that it provides a faithful answer for $\tau_{mn}^{\text{off}}$ as well as a highly non-trivial test of the Seiberg–Witten–Whitham formalism.

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