Dmitrij Volkov, super-Poincare group and Grassmann variables

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A fundamental role of the Hermann Grassmann anticommuting variables both in physics and mathematics is discussed on the example of supersymmetry. The talk describes how the D. Volkov question about possibility of the existence of Nambu-Goldstone fermions, realized by the Grassmannian variables, resulted in the discovery of the super-Poincare group, its spontaneous breaking and gauging.

1 Introduction

The famous Grassmannian variables discovered by Hermann Günther Grassmann in XIX century created the mathematical ground for the description of fermionic degrees of freedom in quantum field theory, and resulted in many outstanding discoveries in physics and mathematics.

Here I shall remind only one example demonstrating the fundamental role of the Grassmannian variables and their algebra in the discovery of supersymmetry.

2 Spontaneous symmetry breaking and the Nambu-Goldstone bosons

The phenomena of the spontaneous symmetry breaking was studied by Nambu [1], Goldstone [2], Bogolyubov [3], Schwinger [4], Weinberg [5] and others in the 1960s. The massless and spinless Nambu-Goldstone particles associated with arbitrary spontaneously broken group $G$ were identified with the co-ordinates of the coset space $G/H$ [6,7]. The internal symmetry group $G$ describes the symmetry of the Lagrangian of a physical system and its subgroup $H$ is the vacuum state symmetry. The approach of Volkov was based on the works by Elie Cartan on symmetric spaces and exterior differential forms and resulted in the construction of $G$-invariant Phenomenological Lagrangians of the interacting N-G bosons [7]

$$\mathcal{L} = \frac{1}{2} S_p (G^{-1} dG)_k (G^{-1} dG)_k, \quad G = KH,$$

where the differential 1-forms $G^{-1} dG = H^{-1}(K^{-1} dK)H + H^{-1} dH$ represent the vielbeins $(G^{-1} dG)_k$ and the connection $(G^{-1} dG)_h$ associated with the symmetry subgroup $H$ transforming one vacuum state to another. Ferromagnet is a well-known example of the system with the spontaneously broken rotational $SO(3)$ symmetry with respect to spins in the Heisenberg Hamiltonian. Since for ferromagnet the symmetry of vacuum state is a subgroup $O(2)$ of $SO(3)$ the N-G excitations turn out to be spin waves. The phenomenological Lagrangian of spin waves in (anti)ferromagnets, ferrits and in general case of spatially

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disordered media, taking into account new possible phases caused by total spontaneous breaking of the spin rotation group, was constructed in [8]. At that time Volkov [9] put forward the question about the possibility of existence of the N-G fermions with spin 1/2 and its study resulted in the super-Poincare group discovery [10]. The super-Poincare group was also independently discovered by Yu. Gol’fand, E. Lichtman [11], and J. Wess, B. Zumino [12]. The basic motivation of Gol’fand, Lichtman was to construct quantum field theory with the parity violation. Wess, Zumino proposed generalization to four dimensions for the two-dimensional world-sheet supergauge (superconformal) symmetry revealed by P. Ramond [13], A. Neveu, J. Schwartz [14], J. Gervais, B. Sakita [15] in the models of the spinning strings and their world-sheet realization. A crucial step in this way was the use of the anticommuting c-number variables introduced by Hermann Grassmann. I shall talk about it on the basis of the original papers [10].

3 The Lorentz and SL(2C) groups and their matrix realizations

The proper Lorentz group in $D = 4$ Minkowski space is presented by real $4 \times 4$ matrices $\Lambda$ preserving both the scalar product of 4-vectors $xy := x_m y^m$ and the symmetric metric tensor $\eta_{mn} = \text{diag}(-1,1,1,1)$

$$\Lambda \eta \Lambda^T = \eta, \quad \det \Lambda = 1, \quad \eta^T = \eta. \quad (2)$$

The Lorentz group is locally isomorphic to the group $SL(2C)$ of complex $2 \times 2$ matrices $L$ preserving the scalar product $\psi \chi := \psi_\alpha \chi^\alpha$ of the Weyl spinors $\psi_\alpha, \chi^\alpha := \varepsilon^{\alpha\beta} \chi_\beta$ and the antisymmetric metric tensor $\varepsilon_{\alpha\beta} (\varepsilon_{12} = \varepsilon_{21} = -1)$

$$L \varepsilon L^T = \varepsilon, \quad \det L = 1, \quad \varepsilon^T = -\varepsilon. \quad (3)$$

The correspondence $\Lambda \rightarrow \pm L$ gives a two-valued representation of the proper Lorentz group and the $SL(2C)$ group has played essential role in the super-Poincare group discovery. The Pauli matrices $\sigma_i$ together with the identity matrix $\sigma_0$ form a basic set $\sigma_m = (\sigma_0, \sigma_i)$ in the space of $SL(2C)$ matrices. The Lorentz covariant description demands the second set of the Pauli matrices with the upper spinor indices $\tilde{\sigma}_m := (\tilde{\sigma}_0, \tilde{\sigma}_i) := (\sigma_0, -\sigma_i)$ such that

$$\{ \sigma_m, \tilde{\sigma}_n \} = -2\eta_{mn}, \quad \text{Sp} \sigma_m \tilde{\sigma}_n = -2\varepsilon_{mn}, \quad \sigma^m_{\alpha\beta} \tilde{\sigma}^\beta_m = -2\varepsilon^\alpha_\beta \delta^\beta_\alpha. \quad (4)$$

The relativistic Pauli matrices $\sigma_m$ and $\tilde{\sigma}_m$ are Lorentz invariant analogously to the tensors $\eta_{mn}$ and $\varepsilon_{\alpha\beta}$. Using two sets of the Pauli matrices yields the Lorentz covariant realization of the $\Lambda$ matrices in terms of the $L$ matrices

$$\Lambda^m_n = -\frac{1}{2} \text{Sp}(\tilde{\sigma}_m L \sigma^n L^+) \quad (5)$$

showing the mentioned two-valuedness of the $\Lambda \rightarrow L$ mapping. The relation [5] follows from the known correspondence between the real Minkowski vectors $x^m$ and the Hermitian matrices $X = X^\dagger$ of the $SL(2C)$ group

$$X := x^m \sigma_m = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad x^m = -\frac{1}{2} \text{Sp}(\tilde{\sigma}^m X). \quad (6)$$

Since $\det X = -x^m x^m$, the matrices $X$ realize $SL(2C)$ transformations

$$X' = L X L^+ \rightarrow \det X' = -x'^2 = \det X = -x^2, \quad (7)$$

which proves that the 4-vectors $x^m$ and $x^m$ are connected by a Lorentz group transformation. The representation [5] follows from Eq. [7] and the relation

$$X' = x'_m \sigma^m = \Lambda^m_n x_n \sigma^m = L x_m \sigma^m L^+. \quad (8)$$

This information is all necessary to consider the Poincare group realization by $2 \times 2$ matrices.
4 The triangle matrix realization of the Poincare group

The Hermitian matrices $X$ realize the transformations of the proper Poincare group $x'_m = \Lambda_m^nx_n + t_m$ defined as follows

$$X' = LX L^+ + T,$$

(9)

where $T = t_m\sigma_m$. As a result, any Poincare group element may be presented by the couple $(T, L)$ with their composition law given by

$$(T_2, L_2)(T_1, L_1) = (T_3, L_3) := (L_2T_1L_2^+ + T_2, L_2L_1).$$

(10)

It is well known that the group elements $(T, L)$ are presented by complex $4 \times 4$ matrices $G_{\psi}$

$$G_{\psi} = \begin{pmatrix} L & iTL^+L^{-1} \\ 0 & L^{-1} \end{pmatrix} = \begin{pmatrix} 1 & iT \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}. $$

(11)

This fact directly follows from the matrix multiplication

$$ \begin{pmatrix} L_2 & iT_2L_2^{-1} \\ 0 & L_2^{-1} \end{pmatrix} \begin{pmatrix} L_1 & iT_1L_1^{-1} \\ 0 & L_1^{-1} \end{pmatrix} = \begin{pmatrix} L_2L_1 & i(L_2T_1L_2^+ + T_2)(L_2L_1)^{-1} \\ 0 & (L_2L_1)^{-1} \end{pmatrix}. $$

The factorization $G_{\psi} = K_{\psi}H$ (11) shows that the translation matrices $T$ form a homogenous space under the Poincare group transformations

$$G'_{\psi}G_{\psi} = G''_{\psi} = \begin{pmatrix} 1 & iT'' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L'' & 0 \\ 0 & L''^{-1} \end{pmatrix},$$

(12)

where $T' = t'_m\sigma_m$, $T'' = t''_m\sigma_m$, which yield the linear transformation $t'_m = \Lambda_m^nt_n + t'_m$ of the translation parameters $t_m$. It means that the parameters $t_m$ are identified with the space-time coordinates $x_m$, respectively $T = X$ (6), and give the representation of the triangle matrices $G_{\psi}$ (11) in the form

$$G_{\psi} = \begin{pmatrix} L & iXL^+L^{-1} \\ 0 & L^{-1} \end{pmatrix} = \begin{pmatrix} 1 & iX \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} = K_{\psi}H.$$  

(13)

The presentation (13) was one of the key elements Dmitrij Volkov used in the construction of the super-Poincare group.

5 The Grassmannian variables and the super-Poincare group

The Volkov’s idea was to separate the blocks of the $4 \times 4$ matrix $K_{\psi}$ (13) transforming it into the following $5 \times 5$ triangle matrix

$$K_{\psi} = \begin{pmatrix} 1 & iX \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & iX \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

(14)

and then to fill the empty places in the upper triangle block (13) with the anticommuting Grassmannian Weyl spinors $\theta_\alpha$, because they were supposed to play the role of N-G particles with spin $1/2$ subjected to the Fermi statistics $\theta_1\theta_2 = -\theta_2\theta_1$, and consequently $(\theta_1)^2 = (\theta_2)^2 = 0$,

$$K_{\psi} = \begin{pmatrix} 1 & \theta & iZ \\ 0 & 1 & \theta^+ \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \theta^+ = (\bar{\theta}_1, \bar{\theta}_2).$$

(15)
where the complex matrix $Z$, defined by the nilpotent shift of $X$

$$Z = X - \frac{i}{2} \theta \theta^+ \rightarrow z_{\alpha \dot{\beta}} = x_{\alpha \dot{\beta}} - \frac{i}{2} \partial_{\alpha} \tilde{\theta}_{\dot{\beta}},$$  

(16)

was substituted instead of $X$. Such a nontrivial complexification of $X$ has been dictated by the condition to preserve the composition law

$$G'_{S\psi} G_{S\psi} = G''_{S\psi}$$

(17)

for the matrices $G_{S\psi}$

$$G_{S\psi} = \begin{pmatrix}
1 & \theta & iX + \frac{i}{2} \theta \theta^+ \\
0 & 1 & \theta^+ \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
L & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & L^{+1}
\end{pmatrix}$$

(18)

extending the Poincare group matrices $G_{\psi}$ [11]. The substitution of $G_{S\psi}$ together with $G'_{S\psi}$ and $G''_{S\psi}$, presented in the form similar to [13], in Eq. (17) has revealed the transformation law [10]

$$L'' = L' L, \quad \theta'' = L' \theta + \theta', \quad X'' = L' X L^{+} + X' + \frac{i}{2} (L' \theta \theta^+ - \theta^+ \theta^* L^+ + \theta \theta^* L^+)$$

(19)

generalizing the Poincare group law [12]. The corresponding transformations of the complex $Z$-matrices [16] have the form: $Z'' = L' Z L^+ + i \theta^+ L^+$.

The relations (19) are the transformations of the required super-Poincare group. They demonstrate a fundamental role of the Grassmannian variables for the discovery of the super-Poincare group.

6 Supersymmetry and superalgebra

The supersymmetry transformations in the component form

$$\theta'_{\alpha} = \theta_{\alpha} + \xi_{\alpha}, \quad \tilde{\theta}'_{\dot{\alpha}} = \tilde{\theta}_{\dot{\alpha}} + \xi_{\dot{\alpha}}, \quad x'_{\alpha \dot{\alpha}} = x_{\alpha \dot{\alpha}} + \frac{i}{2} (\theta_{\alpha} \tilde{\xi}_{\dot{\alpha}} - \xi_{\alpha} \tilde{\theta}_{\dot{\alpha}})$$

(20)

are extracted from the matrix representation [19] by choosing $L = I_{2 \times 2}, X' = 0$ and $\theta' = \xi$, where $I_{2 \times 2}$ is the $2 \times 2$ identity matrix. The supersymmetry generators $Q^\alpha$ and their complex conjugate $\tilde{Q}^\alpha := -(Q^\alpha)^*$

$$Q^\alpha = \frac{\partial}{\partial \theta_{\alpha}} - \frac{i}{2} \theta_{\alpha} \frac{\partial}{\partial x_{\alpha \dot{\alpha}}}, \quad \tilde{Q}^\dot{\alpha} = \frac{\partial}{\partial \tilde{\theta}_{\dot{\alpha}}} - \frac{i}{2} \tilde{\theta}_{\dot{\alpha}} \frac{\partial}{\partial x_{\alpha \dot{\alpha}}}$$

(21)

together with the translation generator $P^m = i \frac{\partial}{\partial x_m}$ form the superalgebra

$$\{Q^\alpha, \tilde{Q}^\dot{\alpha}\} = -i \frac{\partial}{\partial x_{\alpha \dot{\alpha}}} = \frac{1}{2} \delta^\alpha_{\dot{\alpha}} P^m,$$

(22)

$$\{Q^\alpha, Q^\beta\} = \{\tilde{Q}^\dot{\alpha}, \tilde{Q}^\dot{\beta}\} = [Q^\alpha, P^m] = [\tilde{Q}^\dot{\alpha}, P^m] = 0$$

which is the supersymmetry algebra. The supersymmetry transformations (20) together with their non-zero anticommutator are presented in the equivalent Dirac bispinor form after transition to the Majorana spinors

$$\delta \theta = \xi, \quad \delta \tilde{\theta} = \xi, \quad \delta x_m = -\frac{i}{4} (\xi_{\gamma m} \theta), \quad \{Q_a, Q_b\} = \frac{1}{2} (\gamma_m C^{-1})_{ab} P^m,$$

(23)

where $\tilde{\theta} = \theta^T C$ with the antisymmetric matrix of the charge conjugation $C^{ab} = \left( \begin{array}{cc} \epsilon^{\alpha \beta} & 0 \\ 0 & \epsilon_{\dot{\alpha} \dot{\beta}} \end{array} \right)$ and $Q_a = \frac{\partial}{\partial \theta^a} - \frac{i}{4} (\gamma_m \theta^a) \frac{\partial}{\partial x_m}$. The Majorana spinors and the Dirac $\gamma$-matrices in (23) are defined as follows

$$\theta_a = \left( \begin{array}{c} \theta_{\alpha} \\ \tilde{\theta}_{\dot{\alpha}} \end{array} \right), \quad \xi_a = \left( \begin{array}{c} \xi_{\alpha} \\ \tilde{\xi}_{\dot{\alpha}} \end{array} \right), \quad \gamma_m = \left( \begin{array}{cc} 0 & \sigma_m \\ \sigma_m & 0 \end{array} \right), \quad \{\gamma_m, \gamma_n\} = -2\eta_{mn},$$

(24)
The internal symmetry. The procedure resulted in the factorizable triangle matrix was extended to the enumerated the columns of the matrix relations in general form the Volkov’s idea was to split the blocks of the \(2 \times N\) matrix by the addition of the Grassmannian Weyl spinors \(\theta^{I}_{\alpha}\) which had the index \(I = 1, 2, \ldots, N\) of an internal symmetry group (e.g. \(SU(N)\)). The new index enumerated the columns of the \(2 \times N\) block submatrix. Simultaneously the Lorentz matrix \(H^{\text{ext}}\) was extended to the \((4 + N) \times (4 + N)\) matrix by the addition of the \(N \times N\) block submatrix \(U_{N \times N}\) of the internal symmetry. The procedure resulted in the factorizable triangle matrix

\[
G_{S^{\phi}}^{(\text{ext})} = K_{S^{\phi}}^{(\text{ext})} H^{(\text{ext})} = \begin{pmatrix}
1 & \theta & iX + \frac{i}{2}\theta^{+} \\
0 & I_{N \times N} & \theta^{+} \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
L & 0 & 0 \\
0 & U_{N \times N} & \theta^{+} \\
0 & 0 & L^{+1}
\end{pmatrix}
\]

(25)

associated with the \(N\)-extended super-Poincare group and its transformations are given by the following matrix relations

\[
L'' = L' L, \quad U'' = U' U, \quad \theta'' = L' \theta U^{t-1} + \theta', \\
X'' = L' XL^{+} + X' + \frac{i}{2} (L' \theta U^{t-1} \theta^{+} - \theta' U' \theta^{+} L^{+}),
\]

(26)

where \(U \equiv U_{N \times N}\). The law (26) was derived by the substitution of \(G_{S^{\phi}}^{(\text{ext})}\) \((23)\) for \(G_{S^{\phi}}^{(\text{ext})}\) \((18)\) in the composition law \((17)\). The \(N\)-extended supersymmetry transformations, encoded in the matrix representation \((26)\), generalize the \(N = 1\) supersymmetry transformations \((20)\)

\[
\theta^{I}_{\alpha} = \theta^{I}_{\alpha} + \xi^{I}_{\alpha}, \quad \theta^{I}_{\alpha I} = \theta^{I}_{\alpha I} + \xi^{I}_{\alpha I}, \quad x^{I}_{\alpha \alpha} = x^{I}_{\alpha \alpha} + \frac{i}{2} (\xi^{I}_{\alpha I} \xi^{I}_{\alpha I} - \xi^{I}_{\alpha} \bar{\xi}^{I}_{\alpha})
\]

(27)

for the extended superspace \((x^{m}, \theta^{I}_{\alpha}, \bar{\theta}^{I}_{\alpha I})\). The extension of the Minkowski space to the superspace has revealed the way to bypass the Coleman-Mandula no-go theorem for the unification of the internal and space-time symmetries.

7 Unification of the Poincare group and internal symmetries

In general form the Volkov’s idea was to split the blocks of the \(4 \times 4\) matrix \(K_{\phi}\) in the representation \((13)\) replacing it by the \((4 + N) \times (4 + N)\) triangle matrix \(K_{S^{\phi}}^{(\text{ext})}\) including the Grassmannian Weyl spinors \(\theta^{I}_{\alpha}\) associated with the \(N\)-extended super-Poincare group and its transformations are given by the following matrix relations

\[
L'' = L' L, \quad U'' = U' U, \quad \theta'' = L' \theta U^{t-1} + \theta', \\
X'' = L' XL^{+} + X' + \frac{i}{2} (L' \theta U^{t-1} \theta^{+} - \theta' U' \theta^{+} L^{+}),
\]

(26)

The next important step made by Volkov was the introduction of the supersymmetry invariant differential forms generalizing the famous Cartan \(\omega\)-forms for the case of space with the Grassmannian coordinates. For the above considered coset space \(G_{S^{\phi}}^{(\text{ext})} / H^{(\text{ext})}\) \((25)\), realized by the matrices \(K_{S^{\phi}}^{(\text{ext})}\), the corresponding \(\omega\)-forms appear as the matrix blocks in the product

\[
K_{S^{\phi}}^{(\text{ext})}^{-1} dK_{S^{\phi}}^{(\text{ext})} = \begin{pmatrix}
1 & -\theta & (iZ)^{+} \\
0 & I_{N \times N} & -\theta^{+} \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & d\theta & idZ \\
0 & 0_{N \times N} & d\theta^{+} \\
0 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & d\theta & idX + \frac{i}{2} (d\theta^{+} - \theta d\theta^{+}) \\
0 & 0_{N \times N} & d\theta^{+} \\
0 & 0 & 0
\end{pmatrix}
\]

(28)

and in the component form they are given by the following expressions

\[
\omega^{I}_{\alpha} = d\theta^{I}_{\alpha}, \quad \bar{\omega}_{\alpha I} = d\bar{\theta}_{\alpha I}, \quad \omega_{\alpha \alpha} = dx_{\alpha \alpha} - \frac{i}{2} (d\bar{\theta}_{\alpha I} \theta^{I}_{\alpha} - \theta^{I}_{\alpha} d\bar{\theta}_{\alpha I}).
\]

(29)
In the bispinor representation the fermionic and bosonic one-forms (29) are
\[
\omega = d\theta, \quad \bar{\omega} = d\bar{\theta}, \quad \omega_m = dx_m - \frac{i}{4}(d\bar{\theta} \gamma_m \theta),
\]
(30)
The \(\omega\)-forms are the building blocks for the construction of supersymmetric actions of the interacting N-G particles. To construct the invariant actions Volkov generalized the Cartan method of the exterior differential forms for the case of superspace and found invariant hyper-volume, imbedded in the superspace, and other invariants. Since the invariant action of the N-G fermions has to include the factorized volume element \(d^4x\) it strongly restricts the structure of the admissible combinations of the \(\omega\)-forms. If the combination is given by a product of the \(\omega\)-forms (29) and their differentials, it should have the general number of the differentials equal to four. The condition is satisfied by the well known invariant [10]
\[
d^4V = \frac{1}{4!} \varepsilon_{mnpq} \omega^m \wedge \omega^n \wedge \omega^p \wedge \omega^q,
\]
(31)
where the symbol \(\wedge\) sets the external product, that gives the natural supersymmetric extension of the volume element \(d^4x\) of the Minkowski space. The supersymmetric volume (31), invariant also under the Lorentz and unitary groups, does not contain the spinorial one-forms \(\omega^I\) and \(\bar{\omega}_I\), but they appear, e.g. in the following invariant products
\[
\Omega^{(4)} = \omega^I \wedge \bar{\omega}_I \wedge \bar{\omega}_J, \quad \bar{\Omega}^{(4)} = \varepsilon^{\alpha\beta} \omega^I \wedge \bar{\omega}_I \wedge \bar{\omega}_J \varepsilon_{\alpha\beta},
\]
(32)
where \(d \wedge \omega^m\) denotes the external differential of \(\omega^m\) [10]. The Volkov’s idea for the construction of the N-G fermion action was to use the pullback of the differential form \(d^4V\) (31), and its generalizations similar to (32), on the 4-dimensional Minkowski subspace of the superspace. The pullback is realized by the parametrization of \(\theta\) by the Minkowski coordinates \(x_m\). As a result, the differential forms \(\omega_m\) (30) and \(d^4V\) (31), e.g. for the case \(N = 1\), take the form
\[
\omega_m = (\delta_m^n - \frac{i}{4} \frac{d\bar{\theta}}{dx_n} \gamma_m \theta)dx_n = \omega^n_m dx_n, \quad d^4V = \det W d^4x.
\]
(33)
Due to the spinor \(\theta\) dependence on \(x\), it was identified with the N-G fermionic field \(\psi(x) = a^{-1/2} \theta(x)\) with spin \(1/2\), where \(a\) is the coupling constant \([a] = L^4\). The constant adds the correct dimension \(L^{-3/2}\) to the fermionic field \(\psi(x)\) and its substitution for \(\theta\) in (33) gives the Volkov-Akulov action [10]
\[
S = \frac{1}{a} \int \det W d^4x.
\]
(34)
An explicit form of the action \(S\) (34) with \(W = W(\psi, \partial_m \psi)\) is
\[
S = \int d^4x \left\{ \frac{1}{a} + T_m^m + \frac{a}{2}(T_m^m T^n_n - T^n_m T^m_n) + a^2 T^{(3)} + a^3 T^{(4)} \right\},
\]
(35)
where \(T^{(3)}\) and \(T^{(4)}\) code the interaction terms of the N-G fermions that are cubic and quartic in the particle momentum \(\partial^m \psi\), and \(T_m^m\) is defined by the following relations
\[
W_m^n = \delta_m^n + a T_m^n, \quad T_m^m = -\frac{i}{4} \partial^m \bar{\psi} \gamma_m \psi.
\]
(36)
The first term in (35), unessential for the Minkowski space, is interesting because of its possible connection with the cosmological term in a curved superspace. The second term coincides with the free Dirac action of the massless fermion field \(\psi(x)\) and has the form
\[
S_0 = \int d^4x T_m^m = -\frac{i}{4} \int d^4x \partial^m \bar{\psi} \gamma_m \psi,
\]
(37)
proving that the N-G field $\psi(x)$ actually carries spin $1/2$. The cubic $T^{(3)}$ and quartic $T^{(4)}$ terms in the particle momenta have the following structure [10]

$$T^{(3)} = \frac{1}{3!} \sum_p (-)^p T^m \hat{T}^n \hat{T}^l, \quad T^{(4)} = \frac{1}{4!} \sum_p (-)^p T^m \hat{T}^n \hat{T}^l \hat{T}^k,$$

(38)

where the sum $\sum_p$ corresponds to the sum in all permutations of the subindices in the products of the tensors $T^m$, and these terms describe the vertexes with six and eight N-G fermions respectively. The vertexes (38) were analyzed in [16] and the vanishing of the quartic term $T^{(4)}$ was observed there.

The Volkov’s approach clearly shows how to construct the higher degree terms in the derivatives of the N-G fields to get possible supersymmetric generalizations of the Volkov-Akulov action. In general case the combinations of the $\omega$-forms (29), admissible for the higher order invariant action, are the homogenous functions of the degree four with respect of the differentials $dx$ and $d\psi$. The latter condition guarantees the factorization of the volume element $d^4x$ in the generalized action integral. To restrict the number of such type invariants Volkov proposed to use the minimality condition with respect to the degree of the derivatives $\partial \psi/\partial x$ in the general nonlinear action for the N-G fermions

$$S = \int d^4x L(\psi, \partial \psi/\partial x)$$

(39)

which corresponds to taking into account only the lowest degrees of the momenta of N-G fermions in their scattering matrix. To find the degree of $\partial \psi/\partial x$ in different invariants it was observed that such derivatives appear from the differentials $d\psi$ in the fundamental one-forms (29). In addition, the spinor one-forms create one derivative $\partial \psi/\partial x$, but the vector form terms either do not contain the $\psi$ fields or contain one derivative $\partial \psi/\partial x$ accompanied by $\psi$. As a result, the number of the derivatives $\partial \psi/\partial x$ with respect to the whole number of the fields is lower in the vector differential one-form than in the spinor ones. Also, the invariants including the differential of the $\omega$-forms, like $\tilde{\Omega}^{(4)}$ in (32), have the higher degree in $\partial \psi/\partial x$ in comparison with the $\omega$-forms themselves. Thus, the demand of the minimality of the degree of derivatives in $S$ (39) will be satisfied if the admissible invariants contain only the vector differential one-forms $\omega_m$.

Moreover, Volkov has developed the general method for the supersymmetric inclusion of the N-G particle interactions with other fields. For a given field $\Phi$, carrying the spinor and unitary indices, its differential $d\Phi$ has to be used on the same level as the supersymmetric $\omega$-forms (29) in the application of the above described procedure. The only restriction on the admissible terms including $\Phi$ is the demand of their invariance under the Lorentz and the unitary groups. The invariant interaction of the massive Dirac field with the N-G fermions was considered in [10] as an instructive example of the described procedure.

The interest to the problem of the spontaneous symmetry breaking was recently strengthened in connection with the paper [17] (and Refs. there), where an approach to this fundamental problem, based on new physical observations and superfield effective Lagrangian for the N-G fermions, has been discussed.

One can hope that the Volkov’s geometrical approach will strongly help in the approach development.

9 Gauging the super-Poincare group and supergravity

Taking into account spontaneously broken character of the super-Poincare group, Volkov had immediately understood that its gauging would create the massive Rarita-Schwinger gauge field (gravitino) of spin $3/2$ absorbing the N-G fermion because of the Higgs effect. Realization of this idea led Volkov to the discovery of supergravity - the new physical theory - where graviton gets the gravitino as a superpartner. The corresponding supergravity action was published by D. Volkov and V. Soroka in 1973 [18]. Next papers in supergravity were published in 1976 by S. Deser, B. Zumino [19], S. Ferrara, D. Friedmann, P. Van Niewenhuizen [20] and towards the end of the 1970’s SUSY and SUGRA became the generally

3 Sergei Kuzenko kindly informed me about Ref. [16].
5 Paolo Di Vecchia attracted my attention to Ref. [17].
accepted approaches in the field theory. The exciting story of this discovery was presented by Volkov in his talk in Erice [21]. The maximally extended $N = 8$ supergravity and supersymmetry are discussed by H. Nicolai in this volume [22].

10 From "Ausdehnungslehre" by Hermann Grassmann to superstrings

The Volkov’s method of the construction of the action integrals for the N-G fermions as supersymmetric hypervolumes (and their generalizations) imbedded in superspace has shown the general way of constructing the action integrals for superparticles and for extended supersymmetric objects superstrings, super p-branes [26], [27]. Actually, in this way $d^4V (31)$ may be interpreted as the world-volume of a super 3-brane imbedded in the superspace, if the super coordinates $(x, \theta)$ are parametrized by the world-volume coordinates $(\tau, \sigma^1, \sigma^2, \sigma^3)$ of the super 3-brane. All that refers us to the revolutionary book "Ausdehnungslehre" by Hermann Grassmann [23] with his theory of extensions and the algebra of exterior forms that so inspired William Clifford and Elie Cartan. Due to the clear explanations by Felix Klein [24] about the Grassmann idea to treat the finite pieces of lines, planes and hypersurfaces as pure geometric objects out of coordinates and metric properties, one can see that Grassmann had anticipated the role of superstrings and super p-branes that so naturally unified the Grassmannian variables with his theory of extensions. On behalf of mathematics the Grassmann ideas inspired Felix Berezin and others to create the supermathematics [25]. The limited space will not allow me to remind about many outstanding physicists and mathematicians and their great contributions to the development of supersymmetry, supergravity and supermathematics.

Thus, the brilliant algebraic, geometric and physical ideas of Hermann Günther Grassmann ran way ahead of their time and have created the basic mathematical and physical entities which became the foundation for the construction of space-time theory, the modern quantum field theory, supersymmetry and string theory.

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