A Right Inverse Operator for $\text{curl} + \lambda$ and Applications

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Abstract. A general solution of the equation $\text{curl} \vec{w} + \lambda \vec{w} = \vec{g}$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$ is obtained for an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$ with a Liapunov boundary and $\vec{g} \in W^{p, \text{div}}(\Omega) = \{ \vec{u} \in L^p(\Omega) : \text{div} \vec{u} \in L^p(\Omega) \}$, $1 < p < \infty$. The result is based on the use of classical integral operators of quaternionic analysis. Applications of the main result are considered to a Neumann boundary value problem for the equation $\text{curl} \vec{w} + \lambda \vec{w} = \vec{g}$ as well as to the nonhomogeneous time-harmonic Maxwell system for achiral and chiral media.

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1. Introduction

We study the nonhomogeneous equation

$$\text{curl} \vec{w} + \lambda \vec{w} = \vec{g}, \quad \lambda \in \mathbb{C}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ and propose an integral representation for its general solution or in other words, we construct a right inverse operator for the operator $\text{curl} + \lambda$ in an appropriate functional space. In the situation when the boundary values of $\vec{w}$ are given a representation of $\vec{w}$ is known (e.g., in the context of quaternionic analysis it is obtained directly

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from the Borel-Pompeiu formula [17, pp. 59, 60]). However, such representation not always is convenient. It is clear, e.g., that the Newton potential
\[ L[u](\vec{x}) := -\frac{1}{4\pi} \int_{\Omega} \frac{u(\vec{y})}{|\vec{x} - \vec{y}|} \, d\vec{y} \]
(the right inverse operator for the Laplacian) is often used with no relation to concrete boundary values of a solution to a Poisson equation.

The right inverse operator for the operator \( \text{curl} + \lambda \) is obtained by using a quaternionic approach. Equation (1) is considered as a vector part of a quaternionic equation
\[ (D + \lambda)\vec{w} = g \]
where \( D \) is the Moisil–Teodorescu operator and \( g_0 = \text{Sc} \, g \) is related with \( \vec{g} \) by the equality \( \text{div} \, \vec{g} + \lambda g_0 = 0 \). A right inverse operator for \( D + \lambda \) is well known (see, e.g., [10, 17, 20]). Sometimes it is called the Teodorescu transform and denoted by \( T_\lambda \). However \( T_\lambda \,[g] \) is in general a complete quaternion (whose scalar part is not necessarily zero), and, of course, simply resting a scalar part from \( T_\lambda \,[g] \) does not lead to a solution of (2). Thus, the main problem for constructing a right inverse for \( \text{curl} + \lambda \) reduces to finding a right inverse for the operator \((D + \lambda) V\) where \( V \) is a projection operator \( V \vec{w} = \text{Vec} \,(\vec{w}) = \vec{w} \).

This problem we solve in three steps. First, we introduce certain component operators conforming \( T_\lambda \) and study their properties. Second, we give a complete solution to the problem of constructing so-called metaharmonic conjugate functions, considering the \((D + \lambda)\vec{w} = 0\) (for a full quaternion \( \vec{w} \)) construct \( \vec{w} \) from a given \( w_0 = \text{Sc} \, \vec{w} \) and vice versa, given \( \vec{w} \) find \( w_0 \). Finally, with the aid of these results we find out what term should be rested from \( T_\lambda \,[g] \) in order that the resulting function still be a solution of the equation \((D + \lambda)\vec{w} = g\) at the same time being purely vectorial.

The outline of this paper is as follows. In Sect. 2 is given the notation and some basic results on Eq. (2). In Sect. 3 we introduce a decomposition of the \( \lambda \)-Teodorescu transform and some properties of the component operators. In Sect. 4 the procedure for constructing metaharmonic conjugate functions is presented. It is worth mentioning that in the case \( \lambda \neq 0 \) it resulted to be far more elementary and explicit than in the case \( \lambda = 0 \) (for which we refer to [8, Prop. 2.3] and [9]). In Sect. 5 the main result of this work is presented which consists of a general solution of (1) and an explicit expression for the right inverse operator for the operator \( \text{curl} + \lambda \). In the rest of the paper, we show some applications of this result. In Sect. 6 a Neumann problem for (1) is reduced to a boundary integral equation. In Sect. 7 a general weak solution of the nonhomogeneous time-harmonic Maxwell system is obtained. In Sect. 8 it is applied to a standard boundary value problem for the Maxwell system, well studied in the homogeneous case (e.g., in [5]) but not in the nonhomogeneous situation. Finally, in Sect. 9 a general weak solution of the nonhomogeneous Maxwell system for chiral media is presented.

2. Background for the \( D + \lambda \) System

Together with the equation
\[ \text{curl} \, \vec{w} + \lambda \vec{w} = \vec{g} \]
it is convenient to consider a quaternionic equation whose vectorial part co-
incides with (3). We begin by introducing the necessary notations.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. We are interested in functions
$w = w_0 + \tilde{w} : \Omega \to \mathbb{B}$ with $w_0 = \text{Sc} \ w$, $\tilde{w} = \text{Vec} \ w$, where $\mathbb{B}$ denotes the algebra
of biquaternions [11, 13, 17]. In what follows we often consider biquaternion
valued functions belonging to usual functional spaces in the component-wise
sense, e.g., $w \in C^k(\Omega, \mathbb{B})$ which means that every component of the biquater-
nion valued function $w$ is $k$-times continuously differentiable in the domain
$\Omega$.

From now on $\bar{x} \in \mathbb{R}^3$. The Moisil-Teodorescu differential operator $D$
(also known as the generalized Cauchy–Riemann or occasionally the Dirac
operator but in fact, it was introduced by Hamilton) is defined by
$$D = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3,$$
where $\partial_i = \partial/\partial x_i$, $i = 1, 2, 3$ and $e_i$ stand for basic quaternionic units. We
remind that in terms of the classical differential operators of vector calculus
the action of $D$ can be written as
$$Dw = - \text{div} \ \tilde{w} + \text{grad} \ w_0 + \text{curl} \ \tilde{w},$$
meaning that $\text{Sc} \ (Dw) = - \text{div} \ \tilde{w}$ and $\text{Vec} \ (Dw) = \text{grad} \ w_0 + \text{curl} \ \tilde{w}$.

**Definition 1.** Let $w \in C^1(\Omega, \mathbb{B})$ and $\lambda \in \mathbb{C}$. We will say that $w$
is $\lambda$-monogenic in $\Omega$ if $w$ belongs to the kernel of $D + \lambda$ in $\Omega$.

Or equivalently,
$$D + \lambda)w = 0 \iff \begin{cases} \text{div} \ \tilde{w} = \lambda w_0, \\ \text{curl} \ \tilde{w} + \lambda \tilde{w} = - \text{grad} \ w_0. \end{cases} \quad (4)$$

When $\lambda = 0$ the system (4) represents the Moisil–Teodorescu system
which defines the biquaternionic monogenic functions (see, e.g., [10, 12]).

If $w$ is $\lambda$-monogenic, it necessarily satisfies the **Helmholtz equation**
$$ (\Delta + \lambda^2)w = 0. \quad (5)$$

**Definition 2.** The purely vectorial $\lambda$-monogenic functions (when $\lambda \neq 0$) are
called force-free fields (or sometimes force-free magnetic fields).

They satisfy the equation
$$\text{curl} \ \tilde{u} + \lambda \tilde{u} = 0 \quad (6)$$
which additionally implies that $\text{div} \ \tilde{u} = 0$. Observe that equation (6) implies
that $\text{curl} \ \tilde{u} \times \tilde{u} = 0$. Some references for the force-free fields are [4, 18, 21–
23, 30] and references therein.

We are especially interested in purely vectorial solutions of the nonho-
mogeneous equation
$$ (D + \lambda)\tilde{w} = g \quad (7)$$
where $g = g_0 + \tilde{g} \in L^p(\Omega, \mathbb{B})$, $1 < p < \infty$. A purely vectorial function $\tilde{w}$ is a
solution of (7) iff it solves the system of equations
$$- \text{div} \ \tilde{w} = g_0, \quad (8)$$
$$\text{curl} \ \tilde{w} + \lambda \tilde{w} = \tilde{g}. \quad (9)$$
The second equation of the system coincides with (3) meanwhile the first one is not independent. Indeed, application of div to (9) leads to the compatibility condition \( \text{div} \bar{g} + \lambda g_0 = 0 \). Defining the subspace of functions in \( L^p(\Omega, \mathbb{B}) \), \( 1 < p < \infty \), where the system (8), (9) is well-posed,
\[
\text{Sol}_p^\lambda(\Omega) := \{ g = g_0 + \bar{g} \in L^p(\Omega, \mathbb{B}) : \text{div} \bar{g} + \lambda g_0 = 0 \}
\]
we obtain that (3) is equivalent to (7) for \( g \in \text{Sol}_p^\lambda(\Omega) \).

Remark 3. In the special case \( \lambda = 0 \) the system (8), (9) reduces to the classical div-curl system. In [8] a general solution for this first order partial differential system for star-shaped domains was presented and in [9] for Lipschitz domains in \( \mathbb{R}^3 \) with connected complement.

Using the well-known generalized Green’s formulas [7] a weak characterization of the solutions of (8), (9) is
\[
\int_{\Omega} (\text{grad} v_0 \bar{w}) \, d\bar{y} = \int_{\Omega} (\lambda \bar{w} - g) v_0 \, d\bar{y}, \quad \forall v_0 \in W_0^{1,q}(\Omega), \quad 1/p + 1/q = 1.
\]
In particular, by (10) the elements of \( \text{Sol}_p^\lambda(\Omega) \) satisfy the equality
\[
\int_{\Omega} (\text{grad} v_0) \cdot \bar{g} \, d\bar{y} = \lambda \int_{\Omega} g_0 v_0 \, d\bar{y}, \quad \forall v_0 \in W_0^{1,q}(\Omega),
\]
where \( W_0^{1,q}(\Omega) \) is the subspace consisting of the elements of the Sobolev space \( W^{1,q}(\Omega) = \{ \bar{u} \in L^q(\Omega) : \text{grad} \, u_i \in L^q(\Omega) \} \) whose boundary trace is zero.

### 3. Some Integral Operators

In this section we study the component operators of the \( \lambda \)-Teodorescu transform.

Due to the fact that \( \theta(\bar{x}) = -e^{i\lambda|\bar{x}|}/(4\pi|\bar{x}|) \) is a fundamental solution of the Helmholtz operator \( \Delta + \lambda^2 \), the corresponding fundamental solutions of the operators \( D \pm \lambda \) are given by [20, Th. 3.16], [17]
\[
E_{\pm \lambda}(\bar{x}) = \pm \lambda \theta(\bar{x}) - \text{grad} \, \theta(\bar{x}) \left( \pm \lambda + \frac{\bar{x}}{|\bar{x}|^2} - i\lambda \frac{\bar{x}}{|\bar{x}|} \right), \quad \bar{x} \in \mathbb{R}^3 \setminus \{0\}.
\]

The \( \lambda \)-Teodorescu transform is defined as follows (see, e.g., [17])
\[
T_\lambda[w](\bar{x}) = \int_{\Omega} E_\lambda(\bar{x} - \bar{y}) w(\bar{y}) \, d\bar{y}, \quad \bar{x} \in \mathbb{R}^3. \tag{11}
\]

**Proposition 4.** ([20, Th. 4.14], [10]) Let \( \Omega \) be a bounded domain with a Liapunov boundary, \( w \in L^p(\Omega, \mathbb{B}) \), \( p > 1 \). Then in the generalized sense
\[
(D + \lambda) T_\lambda[w](\bar{x}) = w(\bar{x}), \quad \bar{x} \in \Omega.
\]

**Proposition 5.** Let \( \Omega \) be a bounded domain with a Liapunov boundary, \( w \in L^p(\Omega, \mathbb{B}) \), \( 1 < p < \infty \). Then
\begin{enumerate}
  \item \( \text{Sc} \ T_\lambda[w] \) is a solution of the Helmholtz equation (5) if and only if \( w \in \text{Sol}_p^\lambda(\Omega) \).
\end{enumerate}
(ii) Vec $T_\lambda^+[w]$ is a solution of the Helmholtz equation (5) if and only if curl $\vec{w} + \nabla w_0 = \lambda \vec{w}$.

**Proof.** Due to the factorization of the Helmholtz operator

$$\Delta + \lambda^2 = -(D - \lambda)(D + \lambda),$$

and by Proposition 4 it follows that

$$(\Delta + \lambda^2)T_\lambda[w] = -(D - \lambda)[w] = \text{div} \vec{w} + \lambda w_0 - \text{curl} \vec{w} - \nabla w_0 + \lambda \vec{w},$$

from where, after separating the scalar and the vector parts, the assertion follows. \qed

Following the decomposition of the Teodorescu transform for $\lambda = 0$ [8] and using the relations Sc $E_{\pm\lambda} = \pm \lambda \theta$ and Vec $E_{\pm\lambda} = - \text{grad} \theta$, let us define

$$T_{0,\pm\lambda}[w](\vec{x}) = \int_\Omega [\text{Sc} E_{\pm\lambda}(\vec{x} - \vec{y}) w_0(\vec{y}) - \text{Vec} E_{\pm\lambda}(\vec{x} - \vec{y}) \cdot \vec{w}(\vec{y})] \, d\vec{y},$$

$$= \int_\Omega [\pm \lambda \theta(\vec{x} - \vec{y}) w_0(\vec{y}) + \text{grad}_x \theta(\vec{x} - \vec{y}) \cdot \vec{w}(\vec{y})] \, d\vec{y},$$

$$\vec{T}_{1,\pm\lambda}[w_0](\vec{x}) = \int_\Omega \text{Vec} E_{\pm\lambda}(\vec{x} - \vec{y}) w_0(\vec{y}) \, d\vec{y}$$

$$= - \int_\Omega \text{grad}_x \theta(\vec{x} - \vec{y}) w_0(\vec{y}) \, d\vec{y}$$

and

$$\vec{T}_{2,\pm\lambda}[\vec{w}](\vec{x}) = \int_\Omega [\text{Sc} E_{\pm\lambda}(\vec{x} - \vec{y}) \vec{w}(\vec{y}) + \text{Vec} E_{\pm\lambda}(\vec{x} - \vec{y}) \times \vec{w}(\vec{y})] \, d\vec{y},$$

$$= \int_\Omega [\pm \lambda \theta(\vec{x} - \vec{y}) \vec{w}(\vec{y}) - \text{grad}_x \theta(\vec{x} - \vec{y}) \times \vec{w}(\vec{y})] \, d\vec{y}.$$  

Thus,

$$T_{\pm\lambda}[w] = T_{0,\pm\lambda}[w] + \vec{T}_{1,\pm\lambda}[w_0] + \vec{T}_{2,\pm\lambda}[\vec{w}].$$

Notice that $\vec{T}_{1,\lambda} = \vec{T}_{1,-\lambda}$, meanwhile the difference for the other pair of operators consists in a change of one sign. Under the hypothesis of Proposition 5, we have that $T_{0,\lambda}[w]$ and $\vec{T}_{1,\lambda}[w_0] + \vec{T}_{2,\lambda}[\vec{w}]$ are scalar and vector solutions of the Helmholtz equation (5), respectively.

Let us consider the Newton potential $L_\lambda: L^p(\Omega) \rightarrow W^{2\cdot p}(\Omega)$ defined by

$$L_\lambda[w](\vec{x}) = \int_\Omega \theta(\vec{x} - \vec{y}) w(\vec{y}) \, d\vec{y}$$

(14)

representing a right inverse for the Helmholtz operator $\Delta + \lambda^2$ (see, e.g., [3, p. 155]). Using the fact that $E_\lambda = -(D - \lambda)\theta$, we obtain the following relations.

**Proposition 6.** Let $\Omega$ be a bounded domain with a Liapunov boundary and $w \in L^p(\Omega, \mathbb{B})$, $1 < p < \infty$. Then

$$T_{0,\pm\lambda}[w](\vec{x}) = \text{div} L_\lambda[\vec{w}](\vec{x}) \pm \lambda L_\lambda[w_0](\vec{x}),$$

$$\vec{T}_{1,\pm\lambda}[w_0](\vec{x}) = - \text{grad} L_\lambda[w_0](\vec{x})$$
and
\[
\overrightarrow{T}_{2,\pm\lambda}[\vec{w}](\vec{x}) = -\text{curl } L_\lambda[\vec{w}](\vec{x}) \pm \lambda L_\lambda[\vec{w}](\vec{x}).
\]
Moreover, \(\overrightarrow{T}_{1,\lambda}[w_0]\) is an irrotational vector field and \(-T_0,\lambda[\vec{w}] + \overrightarrow{T}_2,\lambda[\vec{w}] \in \text{Sol}^p_\lambda(\Omega)\).

The proof is straightforward.

The result of Proposition 6 can be summarized in the following expression for the \(\lambda\)-Teodorescu operator \(T_\lambda[w] = -(D - \lambda)L_\lambda[w]\).

4. Construction of Metaharmonic Conjugate Functions

**Definition 7.** We will say that a scalar function \(w_0\) and a vector function \(\vec{w}\) are metaharmonic conjugate functions to each other in \(\Omega\) if the biquaternion valued function \(w := w_0 + \vec{w}\) is \(\lambda\)-monogenic in \(\Omega\), that is \((D + \lambda)(w_0 + \vec{w}) = 0\).

In this section we propose a procedure for constructing \(\vec{w}\) when \(w_0\) is known and vice versa.

**Remark 8.** The construction of the conjugate functions in the case \(\lambda = 0\), in general, is not an elementary operation. For star-shaped domains, it can be performed explicitly based on a radial integral operator [8, Prop. 2.3] (see also [31] for star-shaped domains in \(\mathbb{H}\)). Meanwhile, for a more general domain, the procedure is far less explicit and can be defined in terms of the Cauchy integral operator requiring the inversion of a boundary integral operator (see [9, Appendix] where this procedure was introduced for bounded Lipschitz domains with a connected complement). Both methods were fundamental in the solution of the div-curl system provided in their respective references.

It is worth mentioning that the construction of the metaharmonic conjugate functions (for \(\lambda \neq 0\)) is far more simple and can be performed explicitly in general.

**Theorem 9.** (i) Let \(w_0\) be a solution of \((\Delta + \lambda^2)w_0 = 0\) in \(\Omega\). Then the function \(\tilde{w} := -\frac{1}{\lambda} \nabla w_0\) is its metaharmonic conjugate. It is defined up to an arbitrary purely vectorial solution of \((D + \lambda)\tilde{v} = 0\).

(ii) Let \(\vec{w} \in \text{Ker } (\Delta + \lambda^2)(\Omega)\).

Then a necessary and sufficient condition for the existence of a \(\lambda\)-monogenic function \(w\) in \(\Omega\) such that \(\text{Vec } w = \vec{w}\) is the equality

\[
\text{curl } (\text{curl } + \lambda)\vec{w} = 0,
\]

and if it is satisfied then the unique metaharmonic conjugate to \(\vec{w}\) has the form \(w_0 = \frac{1}{\lambda} \text{div } \vec{w}\).

**Proof.** As a corollary of (12) we have that if \((\Delta + \lambda^2)w_0 = 0\), the function \(w := -\frac{1}{\lambda}(D - \lambda)w_0\) is a solution of (4) and additionally \(\text{Sc } w = w_0\). Thus, (i) is proved.
Let us prove (ii). The necessity of (16) follows directly from (4). For the sufficiency, consider \( w_0 = \frac{1}{\lambda} \text{div} \ \bar{w} \). Hence the scalar equation in (4) is satisfied. Application of the gradient leads then to the equalities

\[
\text{grad } w_0 = \frac{1}{\lambda} \text{grad } \text{div} \ \bar{w} = \frac{1}{\lambda} (\Delta \bar{w} + \text{curl curl} \ \bar{w}).
\]

Now, using (15) and (16) we obtain

\[
\text{grad } w_0 = -\left( \lambda \bar{w} + \text{curl} \ \bar{w} \right)
\]

which is the vector equation in (4).

\[
\square
\]

5. Solution of System (1)

The importance of decomposition (13) of the Teodorescu operator \( T_\lambda \) can be observed in the following result. From now on, the right-hand side \( \vec{g} \) in (3) is assumed to belong to the following functional space [7]

\[
W^{p, \text{div}} (\Omega) := \{ \vec{u} \in L^p (\Omega) : \text{div} \ \vec{u} \in L^p (\Omega), 1 < p < \infty \}.
\]

Theorem 10. Let \( \Omega \) be a bounded domain with a Liapunov boundary, \( \lambda \in \mathbb{C} \), \( \lambda \neq 0 \) and \( \vec{g} \in W^{p, \text{div}} (\Omega), 1 < p < \infty \). Then a weak general solution of the system (3) in \( \Omega \) is given by

\[
\vec{w} = \frac{1}{\lambda} \left( \vec{g} - \text{curl} \ \widetilde{T}_{2, \lambda}[\vec{g}] \right) + \bar{u},
\]

(17)

where \( \bar{u} \) is an arbitrary solution of (6).

Proof. Let \( g_0 := -\frac{1}{\lambda} \text{div} \ \vec{g} \). Then \( g := g_0 + \vec{g} \in W^{p, \text{div}} (\Omega) \). Using Proposition 5 (i), we have that \( T_{0, \lambda}[g] \) is a solution of the Helmholtz equation (5). Thus, by Theorem 9 (i),

\[
(D + \lambda) \left( T_{0, \lambda}[g] - \frac{\text{grad} T_{0, \lambda}[g]}{\lambda} \right) = 0.
\]

Let us consider

\[
\vec{w} := T_\lambda[g] - T_{0, \lambda}[g] + \frac{\text{grad} T_{0, \lambda}[g]}{\lambda} + \bar{u}.
\]

Since \( T_\lambda \) is a right inverse of \( D + \lambda \) (Proposition 4), we obtain

\[
(D + \lambda)\vec{w} = (D + \lambda)T_\lambda[g] - (D + \lambda) \left( T_{0, \lambda}[g] - \frac{\text{grad} T_{0, \lambda}[g]}{\lambda} \right) = g.
\]

Due to the decomposition (13) of the Teodorescu transform, the vector part of the last equality can be written as follows

\[
\text{curl} \left( \widetilde{T}_{1, \lambda}[g_0] + \widetilde{T}_{2, \lambda}[\vec{g}] \right) + \nabla T_{0, \lambda}[g] + \lambda \left( \widetilde{T}_{1, \lambda}[g_0] + \widetilde{T}_{2, \lambda}[\vec{g}] \right) = \vec{g}.
\]

Thus,

\[
\widetilde{T}_{1, \lambda}[g_0] + \widetilde{T}_{2, \lambda}[\vec{g}] + \frac{\text{grad} T_{0, \lambda}[g]}{\lambda} = \frac{1}{\lambda} \left( \vec{g} - \text{curl} \left( \widetilde{T}_{1, \lambda}[g_0] + \widetilde{T}_{2, \lambda}[\vec{g}] \right) \right)
\]

or, equivalently,

\[
T_\lambda[g] - T_{0, \lambda}[g] + \frac{\text{grad} T_{0, \lambda}[g]}{\lambda} = \frac{1}{\lambda} \left( \vec{g} - \text{curl} \left( \widetilde{T}_{1, \lambda}[g_0] + \widetilde{T}_{2, \lambda}[\vec{g}] \right) \right).
\]
Hence (5) takes the form
\[ \vec{w}(\vec{x}) = \frac{1}{\lambda} \left( \vec{g} - \text{curl} \left( \overrightarrow{T_{1,\lambda}}[g_0] + \overrightarrow{T_{2,\lambda}}[\vec{g}] \right) \right) + \vec{u}. \]
By Proposition 6, \( \text{curl} \overrightarrow{T_{1,\lambda}}[g_0] = 0 \), thus (17) is obtained. \( \square \)

**Corollary 11.** Under the hypothesis of Theorem 10 the operator
\[ R_\lambda := \frac{1}{\lambda} \left( I - \text{curl} \overrightarrow{T_{2,\lambda}} \right), \tag{18} \]
is a right inverse of the operator \( \text{curl} + \lambda I \) on \( W^{p,\text{div}}(\Omega) \), \( 1 < p < \infty \).

**Remark 12.** Theorem 10 allows us to solve also slightly more general systems of the form
\[ \text{curl} \vec{v} + \lambda \vec{v} + \nabla \varphi \times \vec{v} = \vec{h} \]
where \( \varphi \) is an arbitrary continuously differentiable scalar function. Indeed, \( \vec{v} \) is a solution of (12) iff \( \vec{w} := e^{\varphi} \vec{v} \) satisfies (3) with \( \vec{g} = e^{\varphi} \vec{h} \).

### 6. A Neumann Boundary Value Problem

Following [23] we consider the Neumann problem for the equation \( \text{curl} \vec{w} + \lambda \vec{w} = \vec{g} \) in a domain with a connected boundary belonging to the class \( C^2 \). The problem consists in finding \( \vec{w} \in C^1(\Omega) \cap C(\Omega) \) such that
\[ \text{curl} \vec{w} + \lambda \vec{w} = \vec{g}, \quad \text{in} \ \Omega, \tag{19} \]
\[ \vec{w}|_{\partial \Omega} \cdot \vec{n} = \varphi_0, \quad \text{on} \ \partial \Omega, \tag{20} \]
where \( \vec{n} \) is the outward pointing normal vector to \( \partial \Omega \), \( \vec{g} \in C^{1,\gamma}(\Omega) \), \( \varphi_0 \in C^{0,\gamma}(\partial \Omega) \) and \( 0 < \gamma < 1 \). A necessary condition for the existence of a solution of (19), (20) is
\[ \int_{\partial \Omega} [\vec{g}|_{\partial \Omega} \cdot \vec{n} - \lambda \varphi_0] \, ds_{\vec{g}} = 0. \tag{21} \]

The next definition was introduced in [23] in order to study the Neumann problem (19), (20) using the relation with the Helmholtz equation.

**Definition 13.** We will say that \( \lambda \) is regular with respect to the Neumann problem (19), (20) if for all solutions \( \vec{w} \in C^1(\Omega) \cap C(\Omega) \), \( a \in C^2(\Omega) \cap C(\Omega) \) of the system of differential equations
\[ \text{curl} \vec{w} + \lambda \vec{w} = \vec{g} + \text{grad} \ a, \]
\[ \Delta a + \lambda^2 a = 0, \quad \text{in} \ \Omega, \]
satisfying the boundary conditions
\[ \vec{w}|_{\partial \Omega} \cdot \vec{n} = \varphi_0, \quad a|_{\partial \Omega} = 0, \quad \text{on} \ \partial \Omega, \]
there follows \( a \equiv 0 \) in \( \Omega \).

Using the operator \( R_\lambda \) (18) we transform the Neumann boundary value problem (19), (20) into a Neumann problem for force-free fields.
Theorem 14. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a $C^2$ connected boundary. Let $\lambda$ be regular with respect to (19), (20), $\vec{g} \in W^{p,\text{div}}(\Omega)$, $\vec{g}|_{\partial\Omega} \cdot \vec{n} \in C^{0,\gamma}(\partial\Omega)$, and $\varphi_0 \in C^{0,\gamma}(\partial\Omega)$. Then a solution of the Neumann problem (19), (20) is given by

$$\vec{w} = R\lambda[\vec{g}] + \vec{u},$$

where

$$\vec{u}(\vec{x}) = -\text{grad} \int_{\partial\Omega} \theta(\vec{x} - \vec{y}) \psi_0(\vec{y}) \, d\vec{g} + (\text{curl } -\lambda) \int_{\partial\Omega} \theta(\vec{x} - \vec{y}) \vec{v}(\vec{y}) \, d\vec{g}, \quad \vec{x} \in \Omega,$$

$$\psi_0 = \varphi_0 - R\lambda[\vec{g}] \bigg|_{\partial\Omega} \cdot \vec{n}, \quad \text{on } \partial\Omega,$$

and $\vec{v} = \vec{u}|_{\partial\Omega} \times \vec{n}$ is a continuous solution of the boundary integral equation

$$\frac{1}{2} \vec{v}(\vec{x}) + \int_{\partial\Omega} \vec{n}(\vec{x}) \times \left( \lambda \theta(\vec{x} - \vec{y}) \vec{v}(\vec{y}) - \text{grad}_z \theta(\vec{x} - \vec{y}) \times \vec{v}(\vec{y}) \right) \, d\vec{g}$$

$$= \vec{n}(\vec{x}) \times \int_{\partial\Omega} \text{grad}_z \theta(\vec{x} - \vec{y}) \psi_0(\vec{y}) \, d\vec{g}, \quad \vec{x} \in \partial\Omega. \quad (23)$$

Proof. By Theorem 10 we have that $\vec{w} = R\lambda[\vec{g}] + \vec{u}$ satisfies (19), where $\vec{u}$ is an arbitrary force-free field. Therefore solution of the Neumann problem (19), (20) reduces to solution of the Neumann problem for a force-free field with a corresponding boundary condition

$$\text{curl } \vec{u} + \lambda \vec{u} = 0, \quad (24)$$

$$\vec{u}|_{\partial\Omega} \cdot \vec{n} = \psi_0 = \varphi_0 - R\lambda[\vec{g}] \bigg|_{\partial\Omega} \cdot \vec{n}. \quad (25)$$

Since (21) is assumed to be fulfilled we have that $\psi_0$ also satisfies (21),

$$\int_{\partial\Omega} \lambda \psi_0 \, d\vec{g} = \int_{\partial\Omega} \text{curl } \overrightarrow{T_{2,\lambda}[\vec{g}]|_{\partial\Omega} \cdot \vec{n}} \, d\vec{g} = 0$$

(due to the Divergence Theorem). The rest of the proof consists in applying the solution given in [23, Th. 3.3] to the Neumann problem for force-free fields (24), (25). □

Notice that the boundary conditions in Theorem 14 were assumed to satisfy the conditions posed in [23]. See also [4,21,22] for Neumann problems for force-free fields in more general domains. For instance, in [4] the existence, uniqueness, and regularity of solutions of boundary value problems (also with $C^2$ boundary) for force-free fields in multiply connected domains as well as in exterior domains was studied.

7. Time-Harmonic Maxwell Equations

Let us consider the time-harmonic Maxwell equations

$$\text{curl } \vec{H} = -i \omega \epsilon \vec{E} + \vec{j}, \quad \text{div } \vec{H} = 0, \quad (26)$$

$$\text{curl } \vec{E} = i \omega \mu \vec{H}, \quad \text{div } \vec{E} = \frac{\rho}{\epsilon}, \quad (27)$$
for a homogeneous isotropic medium. The quantities $\epsilon$ and $\mu$ are complex numbers. The wave number $\lambda = \omega \sqrt{\epsilon \mu}$ is chosen such that $\text{Im} \lambda \geq 0$. The charge density and the current density are related by the equality $\rho = \frac{1}{i\omega} \text{div} \, \vec{j}$. Some references to the theory of time-harmonic Maxwell’s equations are [5, 6, 28].

Following [16] (see also [20] and [17]) the Maxwell system can be diagonalized with the aid of a pair of purely vectorial biquaternionic valued functions $\vec{\varphi} := -i\omega \epsilon \vec{E} + \lambda \vec{H}$ and $\vec{\psi} := i\omega \epsilon \vec{E} + \lambda \vec{H}$, obtaining

$$ (D - \lambda)\vec{\varphi} = \text{div} \, \vec{j} + \lambda \vec{j}, \quad (D + \lambda)\vec{\psi} = -\text{div} \, \vec{j} + \lambda \vec{j}. \tag{28} $$

Notice that the pair of equations (28) is equivalent to the system (26), (27).

Thus, considering $\vec{j} \in W^{p, \text{div}}(\Omega)$ we can apply Theorem 10 to system (26), (27).

**Theorem 15.** Let $\Omega$ be a bounded domain with a Liapunov boundary and $\vec{j} \in W^{p, \text{div}}(\Omega)$, $1 < p < \infty$. Then a general weak solution of the time-harmonic Maxwell system (26), (27) is given by

$$ \vec{E} = \frac{1}{2i\omega} \left(2\vec{j} - \text{curl} \, (\overrightarrow{T}_{2, \lambda} + \overrightarrow{T}_{2, -\lambda}) \, [\vec{j}]\right) + \frac{1}{2i\omega} (\vec{u} - \vec{v}), $$

$$ \vec{H} = -\frac{1}{2\lambda} \text{curl} \, (\overrightarrow{T}_{2, \lambda} - \overrightarrow{T}_{2, -\lambda}) \, [\vec{j}] + \frac{1}{2\lambda} (\vec{u} + \vec{v}), \tag{29} $$

where $\vec{u}$ and $\vec{v}$ are arbitrary force-free fields associated to the wave numbers $\lambda$ and $-\lambda$, respectively. That is, $(D + \lambda)\vec{u} = 0$ and $(D - \lambda)\vec{v} = 0$.

**Proof.** Notice that the right hand sides of (28) satisfy the compatibility conditions $-\text{div} \, \vec{j} + \lambda \vec{j} \in \text{Sol}^p_\lambda(\Omega)$ and $\text{div} \, \vec{j} + \lambda \vec{j} \in \text{Sol}^p_{-\lambda}(\Omega)$, respectively. By Theorem 10, the weak solutions of (28) are given by

$$ \vec{\varphi} = -\vec{j} + \text{curl} \, \overrightarrow{T}_{2, -\lambda}[\vec{j}] + \vec{v}, \quad \vec{\psi} = \vec{j} - \text{curl} \, \overrightarrow{T}_{2, \lambda}[\vec{j}] + \vec{u}, $$

where $\vec{u}$ and $\vec{v}$ are arbitrary force-free fields associated to the wave numbers $\lambda$ and $-\lambda$, respectively.

Consequently, (29) is obtained by the existing relation between the pairs of vector fields $(\vec{E}, \vec{H})$ and $(\vec{\varphi}, \vec{\psi})$: $2i\omega \vec{E} = \vec{\psi} - \vec{\varphi}$ and $2\lambda \vec{H} = \vec{\psi} + \vec{\varphi}$. \hfill \Box

**Remark 16.** Using Proposition 6 to compute $\overrightarrow{T}_{2, \lambda} + \overrightarrow{T}_{2, -\lambda}$ and $\overrightarrow{T}_{2, \lambda} - \overrightarrow{T}_{2, -\lambda}$ we can write (29) in the form

$$ \vec{E} = \frac{1}{i\omega} \left(\vec{j} + \text{curl} \, \text{curl} \, L_\lambda [\vec{j}]\right) + \frac{1}{2i\omega} (\vec{u} - \vec{v}), $$

$$ \vec{H} = -\text{curl} \, L_\lambda [\vec{j}] + \frac{1}{2\lambda} (\vec{u} + \vec{v}), \tag{30} $$

where $L_\lambda$ is the right inverse of the operator $\Delta + \lambda^2$ defined in (14).

Analyzing again the factors $\overrightarrow{T}_{2, \lambda} + \overrightarrow{T}_{2, -\lambda}$ and $\overrightarrow{T}_{2, \lambda} - \overrightarrow{T}_{2, -\lambda}$, by (13) we have that

$$ (\overrightarrow{T}_{2, \lambda} + \overrightarrow{T}_{2, -\lambda}) \, [\vec{j}] = -2 \int_{\Omega} \text{grad}_x \theta(x - \vec{y}) \times \vec{j}(\vec{y}) \, d\vec{y}, $$

$$ (\overrightarrow{T}_{2, \lambda} - \overrightarrow{T}_{2, -\lambda}) \, [\vec{j}] = 2\lambda \int_{\Omega} \theta(x - \vec{y}) \, \vec{j}(\vec{y}) \, d\vec{y}. $$
Therefore the weak solution (29) of the time-harmonic Maxwell system can be rewritten as follows

\[
\vec{E} = \frac{1}{i\omega \epsilon} \left( \tilde{j} + \text{curl} \int_{\Omega} \text{grad} \tilde{z} \theta(\tilde{x} - \tilde{y}) \times \tilde{j}(\tilde{y}) \, d\tilde{y} \right) + \frac{1}{2i\omega \epsilon} (\tilde{u} - \tilde{v}),
\]
\[
\vec{H} = -\text{curl} \left( \int_{\Omega} \theta(\tilde{x} - \tilde{y}) \tilde{j}(\tilde{y}) \, d\tilde{y} \right) + \frac{1}{2\lambda} (\tilde{u} + \tilde{v}).
\]

There exist other integral representations of the time-harmonic Maxwell system using the quaternionic approach in the literature \cite{10,17}. Compare (31) with the solution given in \cite[p. 62]{17} where the boundary values of \( \vec{E} \) and \( \vec{H} \) are assumed to be known.

8. Boundary Value Problems for the Time-Harmonic Maxwell Equations

With the aid of Theorem 15 the method of integral equations developed for boundary value problems for homogeneous Maxwell equations (see, e.g., \cite[Ch. 4]{5}) can be extended onto the nonhomogeneous Maxwell equations. As an example, we study the following boundary value problem. Find a solution of the Maxwell system (26), (27) provided with the boundary condition

\[
\vec{E}|_{\partial \Omega} \times \tilde{n} = \tilde{\varphi}. \quad (32)
\]

The system is considered in a bounded domain \( \Omega \) with a \( C^2 \)-boundary and \( \tilde{\varphi} \in \mathcal{F}_{\text{Div}}(\partial \Omega) := \{ \tilde{\varphi} \in C^{0,\gamma}(\partial \Omega) : \text{Div} \tilde{\varphi} \in C^{0,\gamma}(\partial \Omega) \} \) for \( 0 < \gamma < 1 \). That is, the surface divergence of \( \tilde{\varphi} \) (see \cite[Def. 2.28]{5}) exists and belongs to the Hölder space \( C^{0,\gamma}(\partial \Omega) \).

Analogously to the procedure used in Sect. 6 this boundary value problem is transformed into a boundary value problem for the homogeneous Maxwell system. Denote

\[
\vec{E}^* = i(\tilde{u} - \tilde{v}), \quad \vec{H}^* = \tilde{u} + \tilde{v},
\]

where \( \tilde{u} \) and \( \tilde{v} \) are arbitrary force-free fields from (29). Since \( \text{curl} (\tilde{u} \pm \tilde{v}) = -\lambda (\tilde{u} \mp \tilde{v}) \), it is immediate that \( (\vec{E}^*, \vec{H}^*) \) satisfy the homogeneous time-harmonic Maxwell equations. By Theorem 15 and (30), the boundary value problem (26), (27), (32) is equivalent to finding a pair of vector fields \( (\vec{E}^*, \vec{H}^*) \in C^1(\Omega) \cap C(\overline{\Omega}) \) satisfying

\[
\text{curl} \vec{E}^* + i\lambda \vec{H}^* = 0, \quad \text{curl} \vec{H}^* - i\lambda \vec{E}^* = 0, \quad (33)
\]
\[
\vec{E}^*|_{\partial \Omega} \times \tilde{n} = -2i \left( \tilde{j} + \text{curl} \text{c} \text{l} \text{u} \text{r} \text{l} \text{u} \text{m} \text{l} \text{o} \omega \left[ \tilde{j} \right] \right) \bigg|_{\partial \Omega} \times \tilde{n} - 2\omega e \tilde{\varphi}. \quad (34)
\]

**Theorem 17.** Let \( \Omega \) be a bounded domain with a \( C^2 \)-boundary. Let \( \tilde{\varphi} = \vec{E}|_{\partial \Omega} \times \tilde{n} \in \mathcal{F}_{\text{Div}}(\partial \Omega) \), \( \tilde{j} \in W^{p,\text{div}}(\Omega) \) and \( \tilde{j}|_{\partial \Omega} \times \tilde{n} \in \mathcal{F}_{\text{Div}}(\partial \Omega) \). If \( \text{Im} \lambda > 0 \), then there exists at most one solution of the Maxwell boundary value problem (26), (27), (32).
Proof. Due to the reduction of the problem (26), (27), (32) to the homogeneous problem (33), (34) we just need to verify that \( \vec{E}^*|_{\partial \Omega} \times \vec{n} \) belongs to \( \mathcal{F}_{\text{Div}}(\partial \Omega) \) and then application of [5, Sect. 4.3] gives us the result. Let \( \{S_n\} \) be a sequence of surfaces contained in \( \partial \Omega \) with boundaries \( \partial S_n \) of class \( C^2 \) that converges to the point \( \vec{x} \in \partial \Omega \) in the sense of [5, Def. 2.28] and \( \vec{n}_n \) be the outward unit normal vector to \( \partial S_n \). Then

\[
\text{Div} (\text{curl curl} \ L_\lambda [\vec{j}]) (\vec{x}) = \lim_{S_n \to \vec{x}} \frac{1}{|S_n|} \int_{\partial S_n} \vec{n}_n \cdot \text{curl} \ L_\lambda [\vec{j}] \, ds \vec{y} = \lim_{S_n \to \vec{x}} \frac{1}{|S_n|} \int_{S_n} \text{div} (\text{curl} \ L_\lambda [\vec{j}]) \, d\vec{y} = 0.
\]

Hence by the regularity of \( \vec{\varphi} \) and \( \vec{j} \) we obtain that \( \vec{E}^*|_{\partial \Omega} \times \vec{n} \in \mathcal{F}_{\text{Div}}(\partial \Omega) \). See [5, Th. 4.16] for the uniqueness of the solution. □

For the homogeneous Maxwell system, there exists an extensive literature about the study of boundary value problems. For example, in [25, 27] the method of layer potentials for studying the boundary value problem in Lipschitz domains in \( \mathbb{R}^3 \) was employed.

9. Maxwell’s Equations in Chiral Media

In this section we apply the general solution of the system (8) provided by Theorem 10 to Maxwell’s equations in chiral media. The concept of chirality has played an important role in chemistry, optics, among others fields (see, e.g., [14, 24]). Consider the corresponding Maxwell equations

\[
\begin{align*}
\text{curl} \ E &= i \omega \mu (\vec{H} + \beta \text{curl} \ \vec{H}), \\
\text{curl} \ \vec{H} &= -i \omega \epsilon (\vec{E} + \beta \text{curl} \ \vec{E}) + \vec{j}, \\
\text{div} \ E &= \frac{\rho}{\epsilon}, \quad \text{div} \ \vec{H} = 0,
\end{align*}
\]

where \( \beta \) is the chirality measure of the medium. Following the notation and results of [15, 17], we denote

\[
\vec{E} = -\sqrt{\mu} \vec{E}, \quad \vec{H} = \sqrt{\epsilon} \vec{H} \quad \text{and} \quad \vec{j} = \sqrt{\epsilon} \vec{j}
\]

and consider the following purely vectorial biquaternion valued functions

\[
\vec{\varphi} = \vec{\varphi} = \vec{E} + i \vec{H}, \quad \vec{\psi} = \vec{E} - i \vec{H}.
\]

Then the system (35) can be written in a biquaternionic form as follows

\[
\begin{align*}
\left(D + \frac{\lambda}{1 + \lambda \beta}\right) \vec{\varphi} &= \frac{\rho}{\epsilon \sqrt{\mu}} + \frac{i \vec{j}}{1 + \lambda \beta}, \\
\left(D - \frac{\lambda}{1 - \lambda \beta}\right) \vec{\psi} &= -\frac{\rho}{\epsilon \sqrt{\mu}} - \frac{i \vec{j}}{1 - \lambda \beta},
\end{align*}
\]

where \( \lambda = \omega \sqrt{\epsilon \mu} \). Or equivalently,

\[
(D + \alpha_1) \vec{\varphi} = \frac{i}{\lambda} (-\text{div} \ \vec{j} + \alpha_1 \vec{j}), \quad (D - \alpha_2) \vec{\psi} = -\frac{i}{\lambda} (\text{div} \ \vec{j} + \alpha_2 \vec{j}),
\]

where \( \alpha_1 \) and \( \alpha_2 \) are parameters that depend on the specific problem.
where the new wave numbers \( \alpha_1 = \lambda/(1+\lambda\beta) \) and \( \alpha_2 = \lambda/(1-\lambda\beta) \) physically correspond to the propagation of waves of opposing circular polarizations. This reduction of (35) to a couple of equations (39), sometimes called Beltrami fields, is often used in relation with electromagnetics in chiral media (see, e.g., [1,30]).

**Theorem 18.** Let \( \Omega \) be a bounded domain with a Liapunov boundary and \( \vec{j} \in W^{p,\text{div}}(\Omega), 1 < p < \infty \). Then a general weak solution of the Maxwell system (35) is given by

\[
E = -\frac{i}{\omega \epsilon} \left( 2\vec{j} - \alpha_1 \text{curl } L_{\alpha_1}[\vec{j}] + \alpha_2 \text{curl } L_{\alpha_2}[\vec{j}] + \text{curl} \begin{pmatrix} \text{curl} & \text{curl} \end{pmatrix} \right)
\]

\[
H = -\frac{1}{\lambda} \begin{pmatrix} \alpha_1 \text{curl } L_{\alpha_1}[\vec{j}] + \alpha_2 \text{curl } L_{\alpha_2}[\vec{j}] - \text{curl} \begin{pmatrix} L_{\alpha_1} & L_{\alpha_2} \end{pmatrix} \end{pmatrix}
\]

\[
\vec{u}_{\alpha_1} \quad \text{and} \quad \vec{u}_{-\alpha_2}
\]

are arbitrary force-free fields associated to the wave numbers \( \alpha_1 \) and \( -\alpha_2 \), respectively.

**Proof.** Notice that the \( L^p \) functions on the right hand side of (39) belong to \( \text{Sol}_{\alpha_1}^p(\Omega) \) and \( \text{Sol}_{-\alpha_2}^p(\Omega) \), respectively. By Theorem 10, we have that

\[
\vec{\varphi} = \frac{i}{\lambda} \left( \vec{j} - \text{curl } T_{2,\alpha_1}[\vec{j}] \right) + \vec{u}_{\alpha_1} \quad \text{and} \quad \vec{\psi} = \frac{i}{\lambda} \left( \vec{j} - \text{curl } T_{2,-\alpha_2}[\vec{j}] \right) + \vec{u}_{-\alpha_2},
\]

are weak solutions of (39), where \( \vec{u}_{\alpha_1} \) and \( \vec{u}_{-\alpha_2} \) are arbitrary force-free fields associated to the wave numbers \( \alpha_1 \) and \( -\alpha_2 \), respectively.

Since \( \vec{E} = \frac{1}{2} \left( \vec{\varphi} + \vec{\psi} \right) \) and \( \vec{H} = \frac{1}{2\lambda} \left( \vec{\varphi} - \vec{\psi} \right) \) we have

\[
\vec{E} = \frac{i}{\lambda} \left( 2\vec{j} - \text{curl } T_{2,\alpha_1}[\vec{j}] + \vec{u}_{\alpha_1} + \vec{u}_{-\alpha_2} \right),
\]

\[
\vec{H} = -\frac{1}{\lambda} \text{curl } \begin{pmatrix} T_{2,\alpha_1} - T_{2,-\alpha_2} \end{pmatrix} \quad \text{and} \quad \frac{1}{i} \left( \vec{u}_{\alpha_1} - \vec{u}_{-\alpha_2} \right).
\]

Due to Proposition 6 and (36) we obtain the integral representation (40).

\( \square \)

Thus, for a homogeneous chiral media, the boundary value problems for the non homogeneous Maxwell system reduce to boundary value problems for Beltrami fields. Some references for the treatment of boundary value problems for the homogeneous Maxwell system in chiral media are [2,26,29].

**10. Conclusions**

A right inverse operator for the operator \( \text{curl} + \lambda \) is constructed for any bounded domain with a Liapunov boundary and thus a convenient representation for a general weak solution of the Eq. (1) is presented. Several applications of this result are developed which include the nonhomogeneous time-harmonic Maxwell system for achiral and chiral media. No doubt that the restriction on the smoothness of the boundary can be weakened. Not less
interesting would be obtaining an analogous result for unbounded domains. The main result of the present work admits a natural extension onto the \( n \)-dimensional situation with the aid of analogous Clifford analysis’ tools.

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**References**

[1] Athanasiadis, C., Costakis, G., Stratis, I.G.: Transmission problems in contrasting chiral media. Rep. Math. Phys. **53**, 143–156 (2004)

[2] Athanasiadis, C., Costakis, G., Stratis, I.G.: Electromagnetic scattering by a perfectly conducting obstacle in a homogeneous chiral environment: solvability and low frequency theory. Math. Methods Appl. Sci. **25**, 927–944 (2002)

[3] Babich, V.M., Kapilevich, M.B., Mikhlin, S.G., Natanson, G.I., Riz, P.M., Sobolodetskij, L.N., Smirnov, M.M.: Linear equations of mathematical physics. Nauka, Moscow (1964). (in Russian)

[4] Boulmezaoud, T.-Z., Maday, Y., Amari, T.: On the linear force-free fields in bounded and unbounded three-dimensional domains. ESAIM Math. Model. Numer. Anal. **33**, 359–393 (1999)

[5] Colton, D., Kress, R.: Integral equations methods in scattering theory. Wiley, New York (1983)

[6] Colton, D., Kress, R.: Inverse acoustic and electromagnetic scattering theory. Springer, New York (1992)

[7] Dautray, R., Lions, J.-L.: Mathematical analysis and numerical methods for science and technology, vol. 3. Springer, New York (1985)

[8] Delgado, B.B., Porter, R.M.: General solution of the inhomogeneous div-curl system and consequences. Adv. Appl. Clifford Algebras **27**, 3015–3037 (2017). [https://doi.org/10.1007/s00006-017-0805-z](https://doi.org/10.1007/s00006-017-0805-z)

[9] Delgado, B.B., Porter, R.M.: Hilbert transform for the three-dimensional Vekua equation. Complex Variables and Elliptic Equations (2018). [https://doi.org/10.1080/17476933.2018.1555246](https://doi.org/10.1080/17476933.2018.1555246)

[10] Gürlebeck, K., Sprößig, W.: Quaternionic analysis and elliptic boundary value problems. Birkhäuser Verlag, Berlin (1990)

[11] Gürlebeck, K., Sprößig, W.: Quaternionic and Clifford calculus for physicists and engineers. Wiley, Chichester (1997)

[12] Gürlebeck, K., Habetha, K., Sprößig, W.: Holomorphic functions in the plane and \( n \)-dimensional space. Birkhäuser, Basel (2008)

[13] Gürlebeck, K., Habetha, K., Sprößig, W.: Application of holomorphic functions in two and higher dimensions. Birkhäuser, Basel (2016)

[14] Jaggard, D.L., Mickelson, A.R., Papas, C.H.: On electromagnetic waves in chiral media. Appl. Phys. **18**, 211–216 (1979)

[15] Khmelnytskaya, K .V., Kravchenko, V .V., Oviedo, H.: Quaternionic integral representations for electromagnetic fields in chiral media. Telecommun. Radio Eng. **56**(4 & 5), 53–61 (2001)

[16] Kravchenko, V. V.: On the relation between holomorphic biquaternionic functions and time-harmonic electromagnetic fields. *Deposited in UkrINTEI*, 29.12.1992; # 2073 - Uk - 92; 18pp. (in Russian)
[17] Kravchenko, V.V.: Applied quaternionic analysis. Heldermann Verlag, Lemgo (2003)
[18] Kravchenko, V.V.: On force free magnetic fields. Quaternionic approach. Math. Methods Appl. Sci. 28, 379–386 (2005)
[19] Kravchenko, V.V., Shapiro, M.V.: Helmholtz operator with a quaternionic wave number and associated function theory. Deformations of Mathematical Structures. II. Ed. by J. Lawrynowicz, Kluwer Acad. Publ., 101–128 (1994)
[20] Kravchenko, V.V., Shapiro, M.V.: Integral representations for spatial models of mathematical physics. Addison Wesley Longman Ltd, Harlow (1996)
[21] Kress, R.: A remark on a boundary value problem for force-free fields. Z. Angew. Math. Phys. 28, 715–722 (1977)
[22] Kress, R.: The treatment of a Neumann boundary value problem for force-free fields by an integral equation method. Proc. R. Soc. Edinburgh 82A, 71–86 (1978)
[23] Kress, R.: A boundary integral equation method for a Neumann boundary problem for force-free fields. Z. J. Eng. Math. 15(1), 29–48 (1981)
[24] Lakhtakia, A., Varadan, V.K., Varadan, V.V.: Time-harmonic electromagnetic fields in chiral media. Lecture Notes in Physics 355. Springer, Berlin (1989)
[25] Mitrea, M.: The method of layer potentials in electro-magnetic scattering theory on non-smooth domains. Duke Math. J. 77(1), 111–133 (1995)
[26] Mitrea, M.: The method of layer potentials for electromagnetic waves in chiral media. Forum Mathematicum 13(3), 423–446 (2001)
[27] Mitrea, M.: Boundary value problems for Dirac operators and Maxwell’s equations in nonsmooth domains. Math. Methods Appl. Sci. 25(16–18), 1355–1369 (2002)
[28] Müller, C.: Foundations of the mathematical theory of electromagnetic waves. Springer, Berlin (1969)
[29] Ola, P.: Boundary integral equations for the scattering of electromagnetic waves by a homogeneous chiral obstacle J. Math. Phys., 3969–3980 (1994)
[30] Roach, G.F., Stratis, I.G., Yannacopoulos, A.N.: Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics. Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ (2012)
[31] Sudbery, A.: Quaternionic analysis. Math. Proc. Camb. Phil. Soc. 85, 99–225 (1979)

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