Duality of a Generalized Gauge Invariant Ising Model on Random Surfaces

Z.B. Li, B. Zheng and L. Schülke

Universität-GH Siegen, D-57068 Siegen, Germany

Abstract

A generalized gauge invariant Ising model on random surfaces with non-trivial topology is proposed and investigated with the dual transformation. It is proved that the model is self-dual in case of a self-dual lattice. In special cases the model reduces to the known solvable Ising-type models.

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1 Introduction

In recent years great interest has turned to the non-critical string theory with central charge $C \leq 1$, which was claimed to be non-perturbatively solvable by relating its dual model to certain matrix representations.

The string theory with $C = 1/2$ can be viewed as the Ising model on random surfaces. In the non-perturbative calculation, the continuum surface is tessellated by polygons with $p$ edges. Then through the dual transformation, the Ising model is related to its dual (Ising) model on the $\phi^p$ lattice. The latter has a two-matrix representation which is non-perturbatively solvable around the double scaling point [1, 2, 3].

The dual transformation, however, is not straightforward for non-planar lattices because the high-temperature expansion contains non-boundary one-cycles. As early as 1971, Wegner [4] pointed out that the traditional duality relation is valid only if the completeness condition is fulfilled. For two-dimensional lattices Wegner’s completeness condition means nothing but the planar lattice.

Recently a generalized duality relation valid for both planar and non-planar lattices has been obtained [5, 6]. The remarkable fact is that the dual model of an Ising model is not simply an Ising model, as one implicitly supposed in the two-matrix representation of the string theory with $C = 1/2$. It has also been shown that the model, whose dual model is an Ising model coupled to an external field, turns out to be a gauge invariant Ising model where non-boundary one-cycles do not appear due to the gauge invariance. This was first suggested by Wegner [4] and Balian et al. [7] for a regular planar lattice. For what has recently been done, the exact statement should, therefore, be that it is the gauge invariant Ising model on random surfaces that has the solvable two-matrix representation. Whether the gauge invariant Ising model and the Ising model coupled to an external field belong to the same universality class remains open.

All these indicate that the gauge field in two dimensions is far from trivial, especially in the case of non-planar lattices. It is therefore interesting to investigate Ising-type models on random surfaces with gauge interaction and to study the topological effects of the gauge field.

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3 One may directly discretize the continuum theory by putting the spins on the dual lattice, but in this case one also has to know whether this discretized model is in the same universality class of the original one or not.
On the other hand, as is well known, the self-duality is one of the most attractive points of the Ising model, which help us to understand the phase structure. The self-duality is in general valid only when the lattice is self-dual. For the planar random surfaces one may get the self-duality by summing up all the lattices. This has recently been discussed [8]. On non-planar lattices the situation is more complicated. The self-duality is spoiled by the non-trivial topology of the lattice even if the lattice is self-dual. What is then a self-dual model on non-planar random surfaces?

Furthermore the dual transformation itself is also useful in studying the relation between two Ising-type models [9]. In two dimensions the topological properties of the surfaces are closely related to boundary conditions. Therefore it is important to know how the dual transformation goes for topologically different lattices.

In order to tackle these problems, in this paper the simple gauge invariant Ising model [4, 7] is generalized such that two Ising-spins interact with a gauge field. It will be seen that this model is self-dual even for a non-planar lattice if the lattice is self-dual, because the gauge field kills the non-boundary one-cycles in the high-temperature representation. This is the simplest self-dual Ising-type model on the lattice with non-trivial topology. Under certain limits the model reduces to relatively simpler ones, which are already known to be physically interesting and important. We can demonstrate the calculation of Wilson loops and show how the topology plays its role.

In the next section the dual transformation will be carried out. Several limiting cases are considered in section 3. Finally some discussions follow.

2 A self-dual model

Let us consider a system on a random lattice $L$ with two sets of Ising spins and a $Z_2$ gauge field, whose partition function is

$$Z = \sum_{\{S_i, S'_i, U_{ij}\}} \exp \left\{ - \sum_{<ij> \in L^1} \left( J U_{ij} S_i S_j + J' U_{ij} S'_i S'_j \right) + \sum_{m \in L^2} \beta U_m + \sum_{i \in L^0} \lambda S_i S'_i \right\} \right.$$  \hspace{1cm} (1)

2
where $L^0, L^1$ and $L^2$ are respectively the set of sites, bonds and plaquettes; $S_i = \pm 1$ and $S'_i = \pm 1$ are the spin variables on a site, $U_{ij} = \pm 1$ is the gauge field on a bond and $U_m$ is the plaquette variable defined as the product of the gauge fields along the boundary of the plaquette. This model looks like the ‘spin-Schwinger model’. $\sum_{i \in L^0} \lambda S_i S'_i$ corresponds to the ‘mass’ term.

Here we should mention, as in the case of the simple gauge invariant Ising model [7], in the ‘vacuum sector’ the spins $\{S'_i\}$ can be gauged away by the following transformation

$$\tilde{U}_{ij} = U_{ij} S'_i S'_j, \quad \tilde{S}_i = S_i S'_i$$

However, for the convenience of discussions we will keep it because it preserves explicitly the gauge invariance of the system.

For a given random lattice $L$, one can construct a dual lattice $L^D$, and a model on it with the partition function

$$Z_D = \sum_{\{\tau_m, \tau'_m, V_{mn}\}} \exp \left\{ \sum_{<mn> \in L^D} \left( J_D V_{mn} \tau_m \tau_n + J'_D V_{mn} \tau'_m \tau'_n \right) + \sum_{i \in L^D_0} \beta_D V_i + \sum_{m \in L^D_0} \lambda_D \tau_m \tau'_m \right\}$$

where $L^0_D, L^1_D$ and $L^2_D$ are respectively the set of dual sites, dual bonds and dual plaquettes; $\tau_m, \tau'_m = \pm 1$ are dual spin variables on dual sites, $V_{mn} = \pm 1$ is the dual gauge field on a dual bond and $V_i$ is the dual plaquette variable defined as the product of the dual gauge fields along the boundary of the dual plaquette.

Following the standard procedure for the dual transformation, we will prove that the model with $Z$ is dual to the one with $Z_D$. Rigorously speaking,

$$Z = c Z_D, \quad c = 2^{3(N-P)/2} (\sinh 2J \sinh 2J')^{B/2}(\sinh 2\beta)^{P/2}(\sinh 2\lambda)^{N/2}$$

if

$$\beta_D = \frac{1}{2} \ln (\coth \lambda), \quad \lambda_D = \frac{1}{2} \ln (\coth \beta)$$

and either

$$J_D = \frac{1}{2} \ln (\coth J), \quad J'_D = \frac{1}{2} \ln (\coth J')$$

3
or
\[ J_D = \frac{1}{2} \ln (\coth J), \quad J'_D = \frac{1}{2} \ln (\coth J), \] (7)

where \( B, N \) and \( P \) are respectively the numbers of the bonds, sites and plaquettes on the original lattice.

Note that if \( L \neq L_D \), \( Z \) and \( Z_D \) are respectively partition functions for two different models, which are dual each other. But in case of \( L = L_D \), these two model become the same, i.e. we have a self-dual model. Then the dual relations in (5-7) may give some insight in the phase structure. From the dual relation (5) we know that the critical points appear always by pair in the \((\beta, \lambda)\) plane. It is, however, not possible to locate the critical point even when it is assumed to be unique, as in the case of the simple Ising model. From the dual relations (6-7) one can see that the critical points are grouped in four in the \((J, J')\) plane. If there is only a unique critical point in the \((J, J')\) plane, it is \((J^*, J^*)\) with \( J^* = \frac{1}{2} \ln (\coth J^*) \). To make full use of the dual relations further investigations are needed.

In order to get the duality relations (4-7), we simply write down the high-temperature representation for \( Z \) and the low-temperature representation for \( Z_D \) and compare the two expressions.

### 2.1 The high-temperature representation of \( Z \)

Noting that \( S_i, S'_i \) and \( U_{ij} \) all take values \( \pm 1 \), the partition function \( Z \) in Eq.(1), can be written as

\[
Z = \sum_{\{S_i, S'_i, U_{ij}\} \in L} \prod_{<ij> \in L^1} (\cosh J + U_{ij} S_i S_j \sinh J) \prod_{<ij> \in L^1} (\cosh J' + U_{ij} S'_i S'_j \sinh J') \prod_{i \in L^0} (\cosh \lambda + S_i S'_i \sinh \lambda) \exp \left\{ \sum_{m \in L^2} \beta U_m \right\}
\] (8)

The product \( \prod_{<ij> \in L^1}(\cosh J + U_{ij} S_i S_j \sinh J) \) can be expanded into \( 2^B \) terms. For a given term, a bond \( <ij> \) contributes a factor of either \( \cosh J \) or \( U_{ij} S_i S_j \sinh J \). Let us mark the bond in case of \( U_{ij} S_i S_j \sinh J \). All the marked bonds for this given term assemble as a one-chain \( \gamma \) on the lattice. It is easy to see that this is a one-to-one correspondence between all the terms in the expansion and all the one-chains on the lattice. Similarly
\[ \gamma + \gamma' = \gamma_0 \]

\[ \gamma + \gamma' = \gamma_0 \]

Figure 1: This is a handle of the random surface. The \( \gamma_0 \) on the right side is a boundary one-cycle. The \( \gamma_0 \) on the left is a non-boundary one-cycle

\[ \prod_{<ij> \in L^1} (\cosh J + U_{ij} S_i S_j' \sinh J') \] can be expanded according to another set of all one-chains \( \gamma' \). For \( \prod_{i \in L^0} (\cosh \lambda + S_i S_i' \sinh \lambda) \) we mark the site \( i \) if a term in the expansion picks up \( S_i S_i' \sinh \lambda \). All the marked sites in the term form a zero-chain \( \theta \). This is also a one-to-one correspondence. After performing the summation over \( \{ S_i \} \) and \( \{ S_i' \} \), we get

\[ Z = 2^{2N} \sum_{\{U_{ij}\}} \sum_{\gamma, \gamma' \in C^1} \sum_{\theta \in C^0} \left( \prod_{<ij> \in \gamma} \sinh J \prod_{<ij> \in \gamma'} \cosh J \prod_{<ij> \in \gamma} \sinh J' \prod_{<ij> \not\in \gamma} \cosh J \prod_{<ij> \not\in \gamma'} \sinh J' \right) \prod_{<ij> \not\in \gamma} \cosh J' \prod_{i \in \theta} \sinh \lambda \prod_{i \not\in \theta} \cosh \lambda \prod_{<ij> \in \gamma_0} U_{ij} \exp \left\{ \sum_{m \in L^2} \beta U_m \right\} \right) \] \hspace{1cm} (9)

where \( C^0 \) and \( C^1 \) are the sets of zero-chains and one-chains respectively, \( \gamma_0 = \gamma + \gamma' \), and

\[ \theta + \partial \gamma = 0, \quad \theta + \partial \gamma' = 0 \] \hspace{1cm} (10)

Here \( \partial \gamma \) and \( \partial \gamma' \) denote respectively the boundaries of \( \gamma \) and \( \gamma' \). The equation (10) simply means that \( \gamma \) and \( \gamma' \) are connected by \( \theta \) resulting that \( \gamma_0 \) must be a one-cycle.

Here we should stress that up to now the gauge invariance has not been considered and \( \gamma_0 \) can be either a boundary one-cycle, which circles a certain area of the surface, or a non-boundary one-cycle, which bounds no area. This is shown in Fig 1.

Similarly one can carry out the summation over gauge fields. If \( \gamma_0 \) is a boundary one-cycle,

\[ \sum_{\{U_{ij}\}} \prod_{<ij> \in \gamma_0} U_{ij} \exp \left\{ \sum_{m \in L^2} \beta U_m \right\} = 2^{B} \left\{ \prod_{m \in \delta \gamma_0} \sinh \beta \prod_{m \not\in \delta \gamma_0} \cosh \beta + \prod_{m \in \delta \gamma_0} \cosh \beta \prod_{m \not\in \delta \gamma_0} \sinh \beta \right\} \] \hspace{1cm} (11)
where $\delta \gamma_0$ is the set of all plaquettes with boundary $\gamma_0$. The two terms on the right-hand side represent the two ways to fill up $\gamma_0$, ‘inside’ and ‘outside’. If $\gamma_0$ is a non-boundary one-cycle, the summation over gauge fields gives zero. Therefore

$$Z = 2^{B+2N} \sum_{\gamma \in C^1} \sum_{\gamma_0 \in \Gamma_0} \prod_{<ij> \in \gamma} \sinh J \prod_{<ij> \notin \gamma} \cosh J \prod_{i \in \partial \gamma} \sinh \lambda \prod_{i \notin \partial \gamma} \cosh \lambda$$

(12)

where $\Gamma_0$ is the set of all boundary one-cycles. This is the so-called high-temperature representation of $Z$. The gauge field damps the non-boundary one-cycles induced by the non-trivial topology of the lattice.

2.2 The low-temperature representation

The configurations of the dual gauge fields can graphically be represented by all the one-chains $C^1$ on the original lattice. Let us denote the dual bond of $< mn >$ by $< mn >^*$. For a given configuration, the corresponding one-chain $\gamma$ is obtained as follows: if $V_{mn} = -1$, draw a line on the bond $< mn >^*$ on the original lattice; if $V_{mn} = 1$, do nothing. It is not difficult to see that on the boundary of $\gamma$ the dual plaquette variables are $-1$ and otherwise $+1$. This is shown in Fig 2.
Figure 3: (a) and (b) are the two configurations of dual spins \( \{\tau_m\} \) corresponding to the boundary one-cycle \( \gamma_0 \).

Therefore \( Z_D \) can be written as

\[
Z_D = \sum_{\{\tau_m, \tau'_m\}} \sum_{\gamma \in C^1} \prod_{<mn> \in \gamma} \exp \left\{ -J_D \tau_m \tau_n - J'_D \tau'_m \tau'_n \right\} \prod_{<mn> \notin \gamma} \exp \left\{ J_D \tau_m \tau_n + J'_D \tau'_m \tau'_n \right\} \prod_{i \in \partial \gamma} e^{-\beta D} \prod_{i \notin \partial \gamma} e^{\beta D} \exp \left\{ \sum_{m \in L_D^0} \lambda_D \tau_m \tau'_m \right\}
\]

(13)

For a given dual spin configuration \( \{\tau_m\} \), as shown in Fig. 3 we can draw a boundary one-cycle \( \gamma_0 \) on the original lattice, such that all the dual spins \( \tau_m \) inside \( \gamma_0 \) have the same sign. This is a two-to-one correspondence. In other words, for a given boundary one-cycle, one can assign either positive spins or negative spins inside it. For \( \{\tau'_m\} \) we get similarly a correspondence to \( \gamma'_0 \). Then we have

\[
Z_D = 2 \sum_{\gamma \in C^1} \sum_{\gamma_0, \gamma'_0 \in \Gamma_0} \prod_{<mn> \in \gamma + \gamma_0} e^{-\lambda D} \prod_{<mn> \notin \gamma + \gamma_0} e^{\lambda D} \prod_{i \in \partial \gamma} e^{-\beta D} \prod_{i \notin \partial \gamma} e^{\beta D} \\
\prod_{<mn> \in \gamma + \gamma'_0} e^{-\lambda D} \prod_{<mn> \notin \gamma + \gamma'_0} e^{\lambda D} \prod_{i \in \partial \gamma} e^{-\beta D} \prod_{i \notin \partial \gamma} e^{\beta D} \\
\left( \prod_{m \in \delta(\gamma + \gamma'_0)} e^{-\lambda D} \prod_{m \notin \delta(\gamma + \gamma'_0)} e^{\lambda D} + \prod_{m \in \delta(\gamma + \gamma'_0)} e^{\lambda D} \prod_{m \notin \delta(\gamma + \gamma'_0)} e^{-\lambda D} \right)
\]

(14)

After renaming \( \gamma + \gamma_0 \) as \( \gamma \) and \( \gamma_0 + \gamma'_0 \) as \( \gamma_0 \).
\[
Z_D = 2^P \sum_{\gamma \in C^1} \sum_{\gamma_0 \in \Gamma_0} \prod_{<mn>^* \in \gamma} e^{-J_D} \prod_{<mn>^* \notin \gamma} e^{J_D} \prod_{\gamma \in \gamma_0} e^{-\beta_D} \prod_{\gamma \notin \gamma_0} e^{\beta_D} \left( \prod_{m \in \delta \gamma_0} e^{-\lambda_D} \prod_{m \notin \delta \gamma_0} e^{\lambda_D} + \prod_{m \in \delta \gamma_0} e^{\lambda_D} \prod_{m \notin \delta \gamma_0} e^{-\lambda_D} \right)
\] (15)

This is the low-temperature representation of \(Z_D\). Comparing (12) and (15), we can easily arrive at the dual relations (4-7).

3 Some limiting cases

For some special limiting cases the model reduces to known ones which are solvable. Then the dual relations in (5-7) may be regarded as the dual relations between two different models. Sometimes the reduction of the model is non-trivial and interesting.

3.1 The limit \(\lambda \to 0\)

From eq.(5) we have \(\beta_D \to \infty\). All the dual plaquettes will be frozen to \(V_i = 1\). This is a constraint for the dual gauge fields \(\{V_{mn}\}\). In the language of cohomology theory, \(\{V_i\}\) are two-forms, gauge fields \(\{V_{mn}\}\) are one-forms. Due to the constraint, \(\{V_{mn}\}\) reduce to one-cocycles.

The exact one-cocycles can be written as

\[
V_{mn}^{(exact)} = \eta_m \eta_n
\] (16)

where \(\eta_m, \eta'_m = \pm\) are also spin variables. That is, \(\{V_{mn}^{(exact)}\}\) are pure gauges. It is known that all one-cocycles can be classified into \(\alpha = 2^{2g}\) cohomology classes by the equivalent relation: \(\{V_{mn}\}\) and \(\{V'_{mn}\}\) are cohomologically equivalent if \(\{V_{mn}, V'_{mn}\}\) is an exact one-cocycle; otherwise, they are not cohomologically equivalent.

In our case, the cohomological equivalence is just the same as the local gauge equivalence. Cohomology classes are trajectories of gauge fields under

\[8\]
the local gauge transformation. If one can find an arbitrarily specified one-cocycle for each cohomology class, say \( \{ \tilde{V}^k_{mn} \} \), then the general solution of the constraint is

\[
V_{mn} = \tilde{V}^k_{mn} \eta_m \eta_n, \quad k = 0, 1, \ldots, \alpha - 1
\]  

(17)

The specified one-cocycle of the \( k \)-th cohomology class can be constructed from a one-cycle in the \( k \)-th homology class. Let \( \gamma_k \) be an arbitrarily specified one-cycle of the \( k \)-th homology class on the original lattice, one can define

\[
\tilde{V}^k_{mn} = \tilde{V}_{mn}(\gamma_k) \equiv \begin{cases} 
-1 & < mn >^* \in \gamma_k \\
1 & < mn >^* \notin \gamma_k
\end{cases}
\]  

(18)

Therefore the dual partition function in the limit \( \lambda \to 0 \) is

\[
Z_D = \frac{1}{2} \sum_{k=0}^{\alpha-1} \sum_{\{\tau_m, \tau'_m, \eta_m\}} \exp \left\{ \sum_{<mn> \in L_D^1} \tilde{V}_{mn}(\gamma_k) \eta_m \eta_n (J_D \tau_m \tau_n + J'_D \tau'_m \tau'_n) + \sum_{m \in L_D^0} \lambda_D \tau_m \tau'_m \right\}
\]  

(19)

Here the factor \( 1/2 \) arises due to the fact that \( \{ \eta_m \} \) and \( \{ -\eta_m \} \) correspond to the same gauge field \( V_{mn} \). One can redefine the spin variables \( \tau_m \eta_m \to \tau_m \) and \( \tau'_m \eta'_m \to \tau'_m \), then the dual relation \( Z = c \ Z_D \), (4), reduces to

\[
\sum_{\{S_i, S'_i, U_{ij}\}} \exp \left\{ \sum_{<ij> \in L^1} (J U_{ij} S_i S_j + J' U_{ij} S'_i S'_j) + \sum_{m \in L^2} \beta U_m \right\} = \\
\frac{c}{2} \sum_{k=0}^{\alpha-1} \sum_{\{\tau_m, \tau'_m\}} \exp \left\{ \sum_{<mn> \in L_D^1} \tilde{V}_{mn}(\gamma_k) \left( J_D \tau_m \tau_n + J'_D \tau'_m \tau'_n \right) + \sum_{m \in L_D^0} \lambda_D \tau_m \tau'_m \right\}
\]  

(20)

with \( c = 2^{2N-1-P/2}(\sinh 2J \sinh 2J')^{P/2}(\sinh 2\beta)^{P/2} \). The left-hand side is a ‘massless’ two-spin system with gauge interaction. The right-hand side is its
dual model, whose partition function contains several terms due to the non-trivial topology of the lattice, of which each describes a ‘massive’ two-spin system with an anti-ferromagnetic chain. Both models are very interesting. The techniques of matrix representation may also be applied for solving them.

For the planar lattice we have $\alpha = 1$ ($g = 0$). When $\lambda_D \to \infty$, the dual model reduces to a simple Ising model and the phase transition appears at $J_D + J'_D = J^*$. This is a critical line. If one comes back the original model, the critical points lie on the curve $1/2 \ln \coth J + 1/2 \ln \coth J' = J^*$ when $\beta$ also goes to zero.

3.2 The limit $\lambda \to 0, \beta \to \infty$

Let us further reduce the model in last subsection by taking the limit $\beta \to \infty$, which implies $\lambda = 0$, see eq.(3). Now the model becomes self-dual and the self-dual relation is

$$
\sum_{k=0}^{\alpha-1} \sum_{\{S_i, S'_i\}} \exp \left\{ \sum_{<ij> \in L^1} \tilde{U}_{ij} \left( \gamma_k^D \cdot (J S_i S_j + J' S'_i S'_j) \right) \right\} = c \sum_{k=0}^{\alpha-1} \sum_{\{\tau_m, \tau'_m\}} \exp \left\{ \sum_{<mn> \in L^1_D} \tilde{V}_{mn} \left( \gamma_k \cdot (J_D \tau_m \tau_n + J'_D \tau'_m \tau'_n) \right) \right\}
$$

(21)

where $c = 2^{N-P} (\sinh 2J \sinh 2J')^{B/2}$,

$$
\tilde{U}_{ij}(\gamma_k^D) = \begin{cases} -1 & <ij>* \in \gamma_k^D \\ 1 & <ij>* \notin \gamma_k^D \end{cases}
$$

(22)

and $\gamma_k^D$ is an arbitrarily specified one-cycle in the $k$-th homology class of the dual lattice.

For the planar lattice the model breaks down into two decoupled simple Ising models. The critical points are two lines $J = J^*$ and $J' = J^*$ in the $(J, J')$ plane.

3.3 The limit $J' \to 0$

Here the spin $S'_i$ are decoupled. For the dual model $J'_D \to \infty$ (or $J_D \to \infty$). Therefore $V_{mn} \tau_m \tau'_n \equiv 1$, i.e. $V_{mn} = \tau_m \tau'_n$. Now $\{V_{mn}\}$ are pure gauges and
$\lambda = \infty$, $J + J' = J^*$

Figure 4: This is a phase diagram for a planar lattice. Here the real line corresponds to critical points at $\lambda = 0$, the bold real line corresponds to that at $\lambda = \infty$, and the squares are those for arbitrary $\lambda$. The phase structure inside the box is unknown.

$V_i \equiv 1$. Therefore the dual relation (4) becomes

$$
\sum_{\{S_i, U_{ij}\}} \exp \left\{ \sum_{<ij> \in L^1} J U_{ij} S_i S_j + \sum_{m \in L^2} \beta U_m \right\} = c \sum_{\{\tau_m\}} \exp \left\{ \sum_{<mn> \in L^1_D} J_D \tau_m \tau_n + \sum_{m \in L^2_D} \lambda_D \tau_m \right\}
$$

(23)

with $c = 2^{N+B/2-P/2}(\sinh 2J)^{B/2}(\sinh 2\beta)^{P/2}$ This is the known dual relation discussed by Wegner \[4\] and Balian et al [7].

In this case when $\beta \to \infty$ a phase transition occurs at $J = J^*$. But note that now $\lambda$ is arbitrary. Correspondingly when $\lambda_D = 0$ and $J'_D \to \infty$ there are critical points at $J_D = J^*$ for arbitrary $\beta_D$.

Including two trivial cases, i.e. $J = J' = 0$, $\beta \to \infty$ and $\lambda = 0$, $J \to \infty$, $J' \to \infty$, all the critical points discussed above for a planar lattice are plotted in Fig 4. According to the analysis of the mean field method, inside the box there should be a critical surface. How to locate this critical surface analytically or numerically is at present under consideration.

4 Discussion

We close the paper by adding some remarks on the Wilson loop.

In the limit $\lambda \to 0$ and $\beta \to \infty$, the Wilson loop along an one-cycle $\omega$ (when $\omega$ is a non-boundary one-cycle the Wilson loop is also called Polyakov
Figure 5: This is a regular torus. $\gamma_1$ and $\tilde{\gamma}_1$ divide the torus into two parts, and inside each part the spins have the same sign but the two parts are in opposite signs.

The string) is

$$\langle \prod_{<ij>\in\omega} U_{ij} \rangle = \frac{1}{Z_D} \sum_{k=0}^{\alpha-1} \sum_{\{\tau_m,\tau_m'\}} \exp \sum_{<mn>\in L} J_D V_{mn}(\gamma_k)\tau_m\tau_n + J_D' \tilde{V}_{mn}(\gamma_k + \omega)\tau_m'\tau_n'$$

(24)

The Wilson loop is a topological invariant, i.e. its value only depends on the homological class of $\omega$. This is because when a boundary one-cycle is added to $\omega$, it can be absorbed into $\{\tau_m'\}$ and the result is unchanged. Particularly, when $\omega$ is a boundary one-cycle, the Wilson loop is equal to one.

Let us further take the limit $J \to 0$ ($J_D \to \infty$). Then

$$\langle \prod_{<ij>\in\omega} U_{ij} \rangle = \frac{1}{Z_D} \sum_{\{\tau_m'\}} \exp \sum_{<mn>\in L} J_D' \tilde{V}_{mn}(\omega)\tau_m'\tau_n'$$

(25)

Consider a $L_1 \times L_2$ regular square lattice with periodic boundary conditions. The lattice is topologically equivalent to a torus, i.e. $g = 1$.

In the limit $J_D' \to 0$, everything is frozen and $\langle \prod_{<ij>\in\omega} U_{ij} \rangle$ is trivially equal to one. However, suppose $J_D'$ is very large, for a Polyakov string $\omega = \gamma_1$ the configuration shown in Fig 5 is dominating, where $\tilde{\gamma}_1$ can freely move. Therefore

$$\langle \prod_{<ij>\in\gamma_1} U_{ij} \rangle \sim L_1 e^{-2L_2 J_D'}$$

(26)

This implies the existence of topological excitations.
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