Abstract—A framework of monomial codes is considered, which includes linear codes generated by the evaluation of certain monomials. Polar and Reed-Muller codes are the two best-known representatives of such codes, which can be considered as two extreme cases from a certain point of view.

We introduce a new family of codes, partially symmetric codes. Partially symmetric monomial codes have a smaller group of automorphisms than RM codes and are in some sense "between" Reed-Muller and polar codes. A lower bound on their parameters is introduced along with the explicit construction which achieves it. Structural properties of these codes are demonstrated and it is shown that in some cases partially monomial codes also have a recursive structure.

I. INTRODUCTION

Reed-Muller (RM) codes [1], [2] are very well known in the classical coding theory. It was recently proved that this family of codes achieves the capacity of the binary erasure channel under ML decoding [3]. The equivalent question for general BMS channels is still open and so is low-complexity near-ML decoding of RM codes.

Reed-Muller codes have rich structural properties, such as invariance w.r.t. a large group of permutations that can be expressed as actions on monomials. The best decoding algorithms for RM codes exploit this invariance and in particular the fact that RM code can be represented as Plotkin concatenation of two smaller RM codes in several different ways. The projection-aggregation algorithm [4] approaches ML performance for short and moderate lengths but has a complexity that grows exponentially with the code order. This makes it feasible only for low-rate codes. Recursive list algorithm with permuted factor graphs of Dumer-Shabunov [5] and its variations achieve ML performance with list size that grows exponentially with a code length. The use of multiple factor graph permutations brings the additional benefit of parallelism, which might lead to latency reduction and simplification of hardware implementation [6].

Polar codes [7] achieve the capacity of an arbitrary BMS channel. Contrary to RM codes, they are specifically constructed so that successive cancellation list (SCL) decoding with small list size [8] (which works similarly to Dumer-Shabunov list decoder) is sufficient for near-ML performance. However, this property comes at the cost of the code structure, so both methods designed for RM codes are inefficient for polar codes and polar-like constructions with better finite-length performance, such as CRC-aided polar codes [8] or polar subcodes [9].

One can now ask whether it would be possible to find codes that have a smaller group of symmetries than Reed-Muller codes but also require a smaller decoding complexity for near-ML performance. Previous works include an efficient construction for two factor graph permutations [10]. It performs worse compared to polar codes under list decoding but allows turbo-like decoding with significantly smaller latency. Polar codes tailored to decoding with permuted factor graphs are proposed in [11] with the main focus on performance.

In this paper, we continue this line of research and investigate this question from a code structure point of view. Here we consider codes that can be obtained via evaluations of monomials [12]. Polar and Reed-Muller codes can both be described in this framework. We introduce a family of codes with certain symmetries and show the lower bound on their parameters. A channel-independent construction achieving this bound is proposed and it is shown that in some cases the obtained codes have the recursive structure.

II. MONOMIAL CODES

A. Polar codes

A \((n = 2^m, k)\) polar code with the set of frozen symbols \(F\) is a binary linear block code generated by rows with indices \(i \in [n] \setminus F\) of the matrix \(A_m = A^{\otimes m}\), where \(A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\) and \(\otimes m\) denotes \(m\)-times Kronecker product of a matrix with itself. For a given BMS channel \(W\), the set \(F\) contains indices of the least reliable bit subchannels under successive cancellation decoding.

B. Reed-Muller codes

The Reed-Muller code \(\text{RM}(r, m)\) has length \(2^n\), dimension \(\sum_{i=0}^r \binom{m}{i}\) and minimum distance \(2^{m-r}\). Its generator matrix consists of all rows of \(A_m\) with Hamming weight at least \(2^{m-r}\). As a consequence, any code \(\text{RM}(r, m)\) can be represented as a polar code with the set of frozen symbols

\[F_{r,m} = \{i | \text{wt}(i) < m - r\},\]

where \(\text{wt}(i)\) denotes the number of nonzero bits in binary representation of integer \(i\).

C. Monomial codes

Given some integers \(r, m\) such that \(r \leq m\), consider the generating set \(M_{r,m}\) of monomials

\[M_{r,m} = \left\{ \prod_{i=1}^{s} x_{j_i} | s \leq r, 0 \leq j_i < m \right\} \cup \{1\}, \]

(1)
where all $x_j$ take binary values. It is easy to see that $|M_{r,m}| = \sum_{i=0}^{r} \binom{m}{i}$ and $\max_{g \in M_{r,m}} \deg g = r$.

For some polynomial $f \in \mathbb{F}_2[x_0, \ldots, x_{m-1}]$, let us denote by $\text{ev}(f)$ its evaluation vector, i.e. length-$2^m$ vector obtained by the evaluation of this polynomial in all points of $\mathbb{F}_2^m$. For convenience, we use natural digit ordering, i.e. $b = (b_0, \ldots, b_{m-1}) = \sum_{i=0}^{m-1} b_i 2^i$.

Consider now some $M_{C_m} \subseteq M_{m,m}$ and an enumeration of its elements with natural numbers. It defines $(n = 2^m, k = |M_C|)$ monomial code

$$C = \{\text{ev}(f)|f(x_0, \ldots, x_{m-1}) = \sum_{g_i \in M_C} u_i g_i(x_0, \ldots, x_{m-1}),$$

$$u_i \in \mathbb{F}_2, 0 \leq i < k\}$$

where $u_i$ are information symbols. Minimum distance of the monomial code can be calculated as $2^{m-r^+(C)}$, where

$$r^+(C) = \max_{g \in M_C} \deg g$$

Another way of defining a monomial code is to see the correspondence between monomials and rows of matrix $A_m$. Indeed, $i$-th row of $A_m$ is the evaluation vector of monomial

$$\text{mon}(i) = \prod_{j \in \text{zeros}(i)} x_j,$$

where $\text{zeros}(i)$ is the set of zero bit positions in binary representation of integer $i$.

Examples of monomial codes:

- $(n = 2^m, k)$ polar code with the set of frozen symbols $\mathcal{F} (|\mathcal{F}| = n-k)$ is a monomial code with $M_{\text{polar}} = \{\text{mon}(i), i \in [n] \setminus \mathcal{F}\}$
- Reed-Muller code $\text{RM}(r,m)$ is a monomial code with $M_{\text{RM}} = M_{r,m}$.

III. SYMMETRIES

A. Factor graph permutations

The propagation of LLR values during the decoding of polar code can be represented with $m$-layer factor graph [13]. An example of this graph is presented at Figure 1. Consider a permutation $\pi$ of its layers, or, equivalently, permutation $\tilde{\pi}$ of the input LLR vector performed by applying $\pi$ to the binary representation of elements’ positions. It can be seen that the bit estimation order also becomes permuted according to $\tilde{\pi}$, and thereafter the subchannel reliabilities.

It is easy to see that any factor graph layer permutation can also be expressed as action on monomials $x \rightarrow Ax$ for $m \times m$ permutation matrix $A$.

Polar codes are perfectly fitted for the particular $\tilde{\pi}^*$ but their performance for $\tilde{\pi} \neq \tilde{\pi}^*$ is rather poor since $\mathcal{F}$ no longer contains $n-k$ worst bit positions. On the other hand, the automorphism group of Reed-Muller codes includes permutations $x \rightarrow Ax + b$ for any invertible $A$, so they are invariant under factor graph permutations and hence all $\tilde{\pi}$ have similar performance under SC/SCL decoding.

B. Projections

Consider a linear code $C$ with generating set $M_C$ and the action of the translation group $T_m$: $x \rightarrow x + b$, $b \in \mathbb{F}_2^n$. One can see that it permutes the codewords of $C$ (in particular, swaps the evaluation points $u$ and $u + b$). On the other hand, it performs a change of variables $x_i \rightarrow x_i + b_i$, so that the permuted code has generating set

$$M_{C,b} = \left\{ \prod_{i=1}^{s}(x_j + b_j) \prod_{i=1}^{s} x_j \in M_C \right\} \cup \{1\}.$$

It can be seen that the permutation defined by $b$ belongs to $\text{Aut}(C)$ iff all elements of $M_{C,b}$ can be represented as sum of monomials from $M_C$ (one can also recall the definition of the weakly decreasing codes from [12] and verify that $T_m \in \text{Aut}(C)$ is equivalent to the code being weakly decreasing). Note that in general $M_{C,b}$ may not define a monomial code.

A directional derivative of code $C$ is the linear code with a generating set

$$M_{C\rightarrow b} = \left\{ \prod_{i=1}^{s}(x_j + b_j) + \sum_{i=1}^{s} x_j \prod_{i=1}^{s} x_j \in M_C \right\}$$

It can be seen that this code has the same values at position $u$ and $u + b$, so it has generator matrix of the form $(G^{(b)} G^{(b)}) P'$, where $P'$ is a column permutation matrix and $G^{(b)}$ is the generator matrix of some $(n^b = 2^{m-1}, k^b, d^b)$ code $C^{(b)}$. $C^{(b)}$ will subsequently be denoted as projected code or projection. Let us also introduce the set of trivial projections $\text{Triv}_m = \{2^q|0 \leq q < m\}$. Note that for any $2^q \in \text{Triv}_m$, the corresponding projection is the partial derivative w.r.t. $x_q$:

$$M_{C\rightarrow 2^q} = \{g_i | x_q \cdot g_i \in M_C\}.$$

Since the monomial code is defined by its generating set, we will also use the equivalent notation $M_{M_C \rightarrow 2^q} \equiv M_{C \rightarrow 2^q}$.

Trivial projections can be considered as a first step of the SC algorithm for different factor graph permutations. Polar codes are optimized for one particular $\pi$, so they are invariant under factor graph permutations and hence all $\pi$ have similar performance under SC/SCL decoding.
IV. PARTIALLY SYMMETRIC MONOMIAL CODES

For BMS channel $W$, consider the decoding of some rate-$I(W)$ code $C$ via its projections. Here we focus on the SCL algorithm, but a similar argument holds for other projection-based approaches. If the rate $R_{C^{(b)}}$ of the projection $C^{(b)}$ exceeds the capacity of channel $W^-$, it means that asymptotically one gets a constant fraction of "bad" polarized data subchannels (in particular, $R_{C^{(b)}} - I(W^-)$) and therefore the list size for ML decoding of $C^{(b)}$ is exponential in the code length.

Polar codes are constructed for one particular projection, corresponding to factor graph permutation $\pi$, so that it has rate $\approx I(W^-)$. However, for all $\pi \neq \pi^*$ the projected codes have much higher rates, which makes the decoding with permuted factor graphs inefficient. Reed-Muller codes demonstrate similar performance for any $\pi$ due to their automorphism group, but these symmetries also constrain the projection-based decoding performance. Namely, since any projection of RM$(r, m)$ gives RM$(r - 1, m - 1)$ (with the dimension greater than $I(W^-)$), we can immediately deduce that ML performance cannot be achieved with the fixed list size. It is also possible to show that e.g. if we fix the code rate to $1/2$, the rate of the projected code also converges to $1/2$ when the code length goes to infinity.

It is possible to establish the link between symmetries of Reed-Muller codes and the (in)efficiency of SCL decoding. By construction, all projections of RM codes are identical. As we will show further in this section, it puts a lower bound on the dimension of projections and RM codes, in fact, achieve this bound.

One can now ask the question: if we sacrifice some of the code symmetries, what can we potentially gain? Is it possible to achieve near-ML low-complexity decoding? In this paper, we try to answer this question and investigate the properties of such codes. We only focus on trivial projections due to their connection with SC/SCL decoding with different factor graph permutations.

Definition 1. A partially symmetric code $C_{m,t}$ is a binary linear code such that the dimensions of $t$ of its trivial projections are equal and the dimensions of $m - t$ others are strictly greater.

In other words, there exists some set of target projections $\mathcal{H}_t \subset \text{Triv}_m$, $|\mathcal{H}_t| = t$, such that $\forall h \in \mathcal{H}_t \quad \dim C^{(h)} = \tilde{k}_{t,m}$ and $\forall h \notin \mathcal{H}_t \quad \dim C^{(h)} > \tilde{k}_{t,m}$. Note that Reed-Muller codes are partially monomial with $t = m$. Without loss of generality, we assume $\mathcal{H}_t = \{2^i | i \in [t]\}$. In this paper, only monomial codes $C_{m,t}$ are considered. General case is a topic for future research.

Since we are concerned about the performance under low-complexity decoding via projections, the achievable values of $\tilde{k}_{t,m}$ for a given code dimension $k_{t,m}$ are of interest and in particular, the lower bound

$$\tilde{k}_{t,m} = \min_{\dim C_{m,t} = k_{t,m}} \tilde{k}_{t,m}. \quad (4)$$

A. Lower bounds

The problem of determining the achievable $\tilde{k}_{t,m}^*$, or equivalently the minimum achievable rate $\tilde{r}_{t,m}^*$, for monomial codes is essentially combinatorial. Consider the set $\mathcal{M}_t = \{x_0, \ldots, x_{t-1}\}$. The number of monomials that have $l \leq t$ variables in $\mathcal{M}_t$ is $\binom{t}{l}2^{m-l}$ ($l$ variables out of $t$ can be selected in $\binom{t}{l}$ ways with any combination of the remaining $m - t$). Removing a monomial with $l$ variables in $\mathcal{M}_t$ from $\mathcal{M}_{m,m}$ decreases the dimension of $l$ trivial projections by 1. Consequently, for $l < t$, we need to remove more than one monomial to keep the dimensions of the projected codes equal. This is summarized in Table 1. Since the goal is to calculate the lower bound, we first remove monomials that correspond to the first row of Table 1 then to the second, etc. until the target $k_{t,m}$ is reached. Note that some values of $k_{t,m}$ cannot be achieved due to the granularity constraints. Setting $t = m$ and removing all monomials corresponding to the first several rows of Table 1 give Reed-Muller codes.

Figure 2 demonstrates the lower bound on $\tilde{r}_{t,m}^*$, computed for the case $m = 10$ and $1 \leq t \leq m$. This result can be interpreted as follows. If a curve lies above the reference BEC/BSC lines, the rate of the projected code exceeds the capacity of the underlying channel and SCL decoding needs exponential list size for its near-ML decoding. We can see that for $t > 4$ efficient decoding cannot be performed except for high and low rate regimes. For $t = 3$ something can be potentially done for the BSC, and for $t = 2$ efficient decoding can be performed. Reed-Muller codes are on the $t = m$ curve, and for polar codes the bound $t = 1$ is trivial since they lie exactly on the reference curves. An important observation here is that even though partially symmetric codes do not admit efficient SCL decoding, their rates scale better compared to RM codes and therefore they might perform better with a rather small list size.

The procedure used for obtaining the lower bound also allows to estimate the best possible minimum distance of codes achieving this bound. In order to maximize $d_{\text{min}}$, for every
we sort the monomials with \( l \) variables in \( M_t \) by the total degree and perform the removal procedure from highest to lowest.

**Proposition 1.** Consider some code \( C_{m,t} \) which achieves the combinatorial lower bound. Assume the monomial removal procedure stopped at some \( l = \hat{l} \). Then the upper bound on the minimum distance of \( C_{m,t} \) is \( 2^{\hat{m} - \hat{t} - 1} \), where \( \hat{m} = l + m - t \) if less than \( \binom{l}{t} \) first entries at stage \( l = \hat{l} \) were removed and \( z = l - 1 + m - t \) otherwise.

This result follows directly from the construction. For any \( l \), the maximal total degree of monomials considered in this stage is \( l + m - t \) \( (l \) variables from \( M_t \) and all \( m - t \) from \( M_{t-1} \), and there are \( \binom{l}{t} \) such monomials.

It can be seen that for small values of \( t \) partially symmetric codes achieving the lower bound would have rather small minimum distance (worse than polar codes) and therefore poor ML performance. To overcome this issue, we suggest in practice to first remove all monomials up to a certain degree \( d \), i.e. to construct partially symmetric codes as subcodes of Reed-Muller codes.

**B. Code construction**

In order to construct codes that achieve \( \hat{k}_{l,m} \) for a given \( k_{t,m} \), one can directly follow the procedure used to compute the bound. It is easy to see that the first three steps of the code construction are rather trivial. In particular, for \( d = m \):

1. Take \( M_{t} = M_{m,m} \), \( k = |M_{m,m}| = 2^m \).
2. Set \( \hat{l} = t \). While \( k - 2^{m-t}(\binom{l}{t}) \geq k_{t,m} \), remove from \( M_{c} \), all monomials with \( l \) variables in \( M_{t} \) and decrease \( l \) by one, \( k \) by \( 2^{m-t}(\binom{l}{t}) \).
3. Set \( \hat{d} = m-t+\hat{l} \). While \( k - \binom{l}{t}(\binom{m-t}{d-t}) \geq k_{t,m} \), remove all degree-\( \hat{d} \) monomials with \( \hat{l} \) variables in \( M_{t} \) and decrease \( \hat{d} \) by 1, \( k \) by \( \binom{l}{t}(\binom{m-t}{d-t}) \).

**Example 1.** Consider \( m = 4, t = 3 \) and \( k_{3,4} = 11 \), \( M_3 = \{x_0, x_1, x_2\} \). Table II contains all monomials with at least one variable in \( M_3 \).

**TABLE II: Monomials to remove.**

| \( l \) | Impact on dimension | Monomials |
|---|---|---|
| 3 | Remove 1 monomial | \( x_0x_1x_2x_3, x_0x_1x_3 \) |
| 2 | Remove 3 monomials | \( x_0x_1x_3, x_0x_2x_3, x_1x_2x_3 \) |
| 1 | Remove 3 monomials | \( x_0x_3, x_1x_2, x_2x_3 \) |

Start from \( M_{4,4} \) and \( k = 16 \) and go to step 2. Take \( \hat{l} = 3 \) and remove all such monomials \( \{x_0x_1x_2x_3, x_0x_1x_2\} \), now \( k = 14 \). For \( \hat{l} = 2 \), there are 6 such monomials and \( 14 - 6 < 11 \), so we go to step 3. Take \( \hat{d} = 3 \) and remove all such monomials \( \{x_0x_1x_3, x_0x_2x_3, x_1x_2x_3\} \), now \( k = 11 \) and the construction procedure is terminated.

The constructed \((16,11,4)\) code has generating set \( M_C = \{x_0x_1, x_0x_2, x_0x_3, x_1x_3, x_1x_2x_3, x_1x_3x_2x_3, x_0x_1x_2x_3, x_0x_1x_2x_3, x_0x_1x_3x_2x_3, x_0x_1x_2x_3 \} \). Its target direction al derivatives have generating sets \( M_{C_{\rightarrow 2}} = \{x_0x_1, x_0x_2, x_0x_3, x_0, x_0x_1x_2x_3, x_1x_3x_2x_3 \} \) and it is easy to see that each of them has cardinality 4.

For the case \( d < m \), at step 1 we need to take \( M_{d,m} \) instead of \( M_{m,m} \), at step 2 term \( 2^{m-t}(\binom{l}{t}) \) is replaced with \( \binom{l}{t}(\binom{m-t}{d-t}) \) and at step 3 the initial value of \( \hat{d} \) becomes \( \min(m - t + 1, d) \) (since after step 1 all monomials with degree greater than \( d \) are already removed and therefore out of consideration).

**Objective. (Final step of the procedure)**

Given \( \hat{l} \), \( \hat{d} \) and \( k \), remove \( k - k_{t,m} \) degree-\( \hat{d} \) monomials with \( \hat{l} \) variables in \( M_t \) so that all \( t \) target projections have the same dimensions.

The solution for this step can be found with the following model. Again for simplicity assume that \( d = m \).

Consider bipartite graph \( \bar{G} = (V_L, V_R, E) \) with left vertices \( v_{h_j} \in V_L \) isomorphic to trivial projections \( h_j \in H_t \) and right vertices \( v_{g_i} \in V_R \) isomorphic to all \( \binom{m-1}{\hat{d}-1} \) degree-\( \hat{d} \) monomials \( g_i \) with \( \hat{l} \) variables in \( M_t \). We draw an edge \( e \in E \) between \( v_{h_j} \) and \( v_{g_i} \) if \( g_i \) contains variable \( x_j \) (i.e. \( g_i \) is in the generating set of code \( C^{(h_j)} \)).

Graph \( \bar{G} \) has partitions of size \( t \) and \( \binom{l}{t}(\binom{m-t}{d-t}) \). Vertices in these partitions have degrees \( \binom{l-1}{t-1}(\binom{m-t}{d-t}) \) and \( \hat{l} \), respectively.

We need to keep all \( \dim C^{(h)} \) for \( h \in H_t \) equal, so we want to construct some \((x, \hat{l})\)-regular subgraph \( G' \) of \( \bar{G} \). Observe that for \( t \neq m \) we can split \( \bar{G} \) into partitions that correspond to different monomials from \( M_t \), so it is sufficient to focus only on the case \( t = m \). To simplify the notations, we also assume \( \hat{l} = r \). Figure 3 demonstrates graph \( \bar{G} \) for \( m = 4, r = 2 \) and one of its possible \((1, 2)\)-regular subgraphs \( G' \) (in red).

To keep the graph regular, \( m \cdot y = x \cdot r \) should hold, where \( y \) is the left degree and \( x \) is the number of right vertices, which gives

\[
x = j \cdot \frac{\text{lcm}(m, r)}{r}
\]
Fig. 3: (3, 2)-regular $G$ and its (1, 2)-regular subgraph $G'$. 

\[ y = j \frac{\text{lcm}(m, r)}{m} \]

for some integer $j$. The maximum value of $j$ is $\frac{\text{lcm}(m, r)}{m}$. Let us denote this value as $j^*$ and consider for each $j$ the corresponding graph $G_j$.

**Proposition 2.** Any $G_j$ can be viewed as a valid solution of the certain max-flow problem.

Consider the flow network that consists of source with capacity-$y$ edges to all $v_{h_j} \in V_L$, sink with capacity-$r$ edges from all $v_{g_l} \in V_R$ and edges $e \in E$ of capacity 1. It is easy to see that edges of $G_j$ corresponds to the flow of size $j \cdot \text{lcm}(m, r)$ and this flow is maximal (it matches the total capacity of source edges). Note that the minimum cut of this network corresponds to the set of source edges.

Proposition 2 implies that $G_j$ can be constructed e.g. with Ford-Fulkerson algorithm.

**Theorem 1.** Consider an increasing sequence $j = 0, \ldots, j^*$. Take some value $j$ and the corresponding graph $G_j$, where $G_0$ is an empty graph and $G_{j^*} = G$. Then one can always construct graphs $G_{j-1}, G_{j+1}$ such that $G_{j-1} \subset G_j \subset G_{j+1}$.

**Proof.** Consider flow network from proposition 2 and its max-flow solution $G_j$.

1) Remove from the network edges that are not present in $G_j$ and set source edges capacities equal to $\frac{\text{lcm}(m, r)}{m}$ (as for $j = 1$). Max-flow in this network is $\text{lcm}(m, r)$, and the corresponding solution can be represented with graph $G_{j-1}$. Removing the edges of $G_{j-1}$ from $G_j$ gives the desired graph $G_{j-1}$.

2) Remove from the network edges that are present in $G_j$. As above, change the source capacities and find max-flow solution $G_{j+1}$. Adding edges of $G_{j+1}$ to $G_j$ gives the desired graph $G_{j+1}$.

Note that only partial symmetry is used for the code design and therefore the whole construction is channel-independent.

V. STRUCTURE OF PARTIALLY SYMMETRIC CODES

We define $\hat{C}_{m,t}$ as the code obtained using steps 1-3 of the procedure from Section [LV-B] and $\bar{C}_{m,t,j}$ as the code obtained after its final step for some graph $G_j$. Observe that graph $G_j$ defines a generating set which we denote as $M_G$.

**Proposition 3.** For any $\hat{C}_{m,t}$ holds $\mathcal{T}_m \in \text{Aut}(\hat{C}_{m,t})$.

This property follows directly from the code construction. Consider a monomial $g(x)$ from $M_{\hat{G}_{m,t}}$, such that $\deg g = d$ and assume it has $\tilde{l}$ variables in $\mathcal{M}_l$. Any of its divisors has either less than $\tilde{l}$ variables in $M_t$ or smaller degree, so it could not be removed from $M_{\hat{G}_{m,t}}$ in code construction process. As a consequence, $\hat{C}_{m,t}$ is also monomial and $\mathcal{T}_m \in \text{Aut}(\hat{C}_{m,t})$.

**Theorem 2.** Assume that the code $\hat{C}_{m,t}$ achieves $\hat{k}^*_m$ (i.e. the lower bound is achieved after steps 1-3 of the construction procedure). Then

1) $\hat{C}_{m,t}$ is invariant under the permutation of variables from $\mathcal{M}_l$ (or equivalently for any $h_1, h_2 \in \mathcal{H}_t$, $\hat{C}_{m,t}^{(h_1)} \equiv \hat{C}_{m,t}^{(h_2)}$).

2) For any $h \in \mathcal{H}_t$, code $\hat{C}_{m,t}^{(h)}$ achieves $\hat{k}^*_t$.

**Proof.** Without loss of generality assume $d = m$ and consider the generating set $M_{\hat{C}_{m,t}}$. It contains all monomials of two types:

1) Monomials with $l < \tilde{l}$ variables in $\mathcal{M}_l$.

2) Degree-$d < \tilde{d}$ monomials with $\tilde{l}$ variables in $\mathcal{M}_l$.

If we now look at the generating sets $M_{\hat{C}_{m,t} \rightarrow h}$, $h \in \mathcal{H}_t$ of the directional derivatives, first-type monomials ensure that all monomials of $l < \tilde{l} - 1$ variables in $\mathcal{M}_l$ are in $M_{\hat{C}_{m,t} \rightarrow h}$, and the second-type ensure that all degree-$d < \tilde{d} - 1$ monomials of $\tilde{l} - 1$ variables from $\mathcal{M}_l$ are also in $M_{\hat{C}_{m,t} \rightarrow h}$. Both of these sets are invariant under the permutation of elements of $\mathcal{M}_l$. Since $M_{\hat{C}_{m,t}}$ contains no other monomials, neither does $M_{\hat{C}_{m,t} \rightarrow h}$. This proves the first part of the theorem.

As for the second part, one just needs to take the generating set of code $\hat{C}_{m,t-1}$, and notice that there is a one-to-one correspondence between its elements and the entries of $M_{\hat{C}_{m,t-1} \rightarrow h}$.

Note that permutation-tailored polar codes from [IT] can be considered as a variation of the procedure from Section [LV-B] where a set of degree-$d$ monomials with all possible $l$ variables in $\mathcal{M}_l$ is removed if it contains a monomial that corresponds to "bad" channel in polar notation. It is easy to see that first part of theorem 2 also holds for these codes.

**Proposition 4.** For any code $\hat{C}_{m,t,j}$, there exist codes $\hat{C}_{m,t,j-1}$ and $\hat{C}_{m,t,j+1}$ such that $\hat{C}_{m,t,j-1} \subset \hat{C}_{m,t,j} \subset \hat{C}_{m,t,j+1}$.

Since code $\hat{C}_{m,t,j}$ corresponds to a certain graph $G_j$, one can apply theorem 2 to obtain graphs $G_{j-1}, G_{j+1}$ and use them to construct the corresponding codes.

**Conjecture.** For any code $\hat{C}_{m,t,j}$ and any $h_1, h_2 \in \mathcal{H}_t$ codes $\hat{C}_{m,t,j}^{(h_1)}$ and $\hat{C}_{m,t,j}^{(h_2)}$ are permutation equivalent.
From theorem [1] we get \( G_j = \bigcup_{i \leq j} (G_i \setminus G_{i-1}) \), where \( \setminus \) denotes the edge removal. Each bipartite graph \( G_1 \setminus G_{i-1} \) has the same structure:

- Its biadjacency matrix \( N \) has constant row and column weight (not equal to each other in general);
- Rows of \( N \) are isomorphic to monomials from \( M_{G_i \setminus G_{i-1}} \);
- Rows of \( N \) with nonzero values in column \( q \) are isomorphic to monomials from the directional derivative \( \frac{\partial }{\partial q} M_{G_i \setminus G_{i-1}} \rightarrow 2^n \).

Hence, in order to prove this conjecture one needs to show the following. Consider a binary \( n \times m \) matrix \( A_{\tilde{r},\tilde{m},q} \) with row weight \( \tilde{r}q \) and column weight \( \tilde{m}r \). Define as \( A_{\tilde{r},\tilde{m},q}^{(j)} \) the set of its rows with nonzero entries in column \( j \). For any \( 0 \leq j_1, j_2 < \tilde{m}q \) there exists a column permutation which maps \( A_{\tilde{r},\tilde{m},q}^{(j_1)} \) to \( A_{\tilde{r},\tilde{m},q}^{(j_2)} \) for any \( A_{\tilde{r},\tilde{m},q} \).

VI. NUMERIC RESULTS

We demonstrate the performance of partially symmetric monomial codes in BEC(\( \varepsilon \)), where the polynomial-time ML decoding is available. Figure 4 shows the frame error rate of codes of length 512 and rate 1/2, constructed for different values of \( t \) with \( d = 5 \). The minimum distance of the constructed codes for \( t = 7 \) is equal to 16, for \( t > 7 \) we get the Reed-Muller code with \( d_{min} = 32 \). The polar code given at the figure is constructed for each value of \( \varepsilon \). Its minimum distance depends on the target erasure probability and jumps from 8 to 16 between \( \varepsilon = 0.36 \) and \( \varepsilon = 0.38 \), which can be observed at the figure.

One conclusion that can be made from the picture is that the ML performance does not strictly improve with \( t \). However, partially symmetric codes for some values of \( t \) demonstrate better performance compared to the polar code, although the gap is not large.

As for future work, there are two main research directions. The first one regards the low-complexity decoding of these codes, especially in the AWGN channel. The second is about how to construct the partially symmetric polynomial codes. It is known that non-symmetric polynomial codes can bring significant performance boost under SCL decoding with a small list size [9]. One can wonder whether a similar improvement can be achieved with (partially) symmetric codes.

VII. CONCLUSION

In this paper, we introduce the new family of monomial codes. These codes have a smaller group of automorphisms compared to RM codes but are more adapted for low-complexity decoding. A construction of such codes is proposed and it is shown that the obtained codes in some cases demonstrate the recursive structure.

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