Subword complexes and 2-truncated cubes

M. A. Gorsky

Let $W$ be a finite Coxeter group, and let $S = \{s_1, \ldots, s_n\}$ be a set of simple reflections generating $W$. Consider a word $Q := q_1 \ldots q_m$ in the alphabet of simple reflections ($q_i \in S$ for all $i = 1, \ldots, m$) and an element $\pi$ of the group $W$. The subword complex $\Delta(Q; \pi)$ is a pure simplicial complex on the set of vertices $\{q_1, \ldots, q_m\}$ corresponding to the letters (more precisely, to their positions) in the word $Q$. A set of vertices yields a simplex if the complement in $Q$ to the corresponding subword contains a reduced expression of $\pi$. The maximal simplices correspond to the complements of reduced expressions of $\pi$ in the word $Q$. Subword complexes were introduced by Knutson and Miller in [1]. They showed in [2] that $\Delta(Q; \pi)$ is spherical if and only if the Demazure product of the word $Q$ equals $\pi$; otherwise, $\Delta(Q; \pi)$ is a triangulated ball. For spherical subword complexes, there arise the natural questions of the existence, of a combinatorial description, and of geometric realizations of their polar dual polytopes.

In the group $W$ there exists a unique longest element, denoted by $w_o$. We will consider subword complexes of the form $\Delta(cw_o; w_o)$, where $c$ is a reduced expression of a Coxeter element, and $w_o$ is an arbitrary reduced expression of $w_o$. Such complexes admit a realization by brick polytopes of Pilaud–Stump [3] that we will denote by $B(cw_o; w_o)$. For each Coxeter element $c$ in the group $W$ there are defined the c-cluster complex of type $W$ and its dual polytope, the (generalized) c-associahedron of type $W$. These objects are important in the theory of cluster algebras. Ceballos, Labbé, and Stump [4] proved that the complexes $\Delta(cw_o(c); w_o)$, where $w_o(c)$ is the so-called c-sorting word for $w_o$, are the c-cluster complexes of type $W$. Therefore, the polytopes $B(cw_o(c); w_o)$ realize the c-associahedra of type $W$. The choice of a Coxeter element $c$ is equivalent to the choice of a quiver $Q$, which is an orientation of the Coxeter diagram of the group $W$. Let $c'$ be a reduced expression of an arbitrary (possibly coinciding with $c$) Coxeter element $c'$ in the group $W$, and let $Q'$ be the corresponding quiver. Let $Q_{c,c'}$ be the quiver obtained from $Q$ by erasing all edges oriented differently in $Q$ and $Q'$ (we do not remove any vertices), and let $\tilde{W}_{c,c'}$ be the Coxeter group such that $\tilde{Q}_{c,c'}$ is an orientation of its Coxeter diagram. We can regard $c$ as a Coxeter element in $\tilde{W}_{c,c'}$.

Theorem 1. $\Delta(cw_o(c'); w_o)$ is the c-cluster complex of type $\tilde{W}_{c,c'}$, and $B(cw_o(c'); w_o)$ realizes the corresponding generalized associahedron. In particular, if $c_{rev}$ denotes the word $c$ written in the opposite direction, then $B(cw_o(c_{rev}); w_o)$ is a (combinatorial) cube.

The proof is based on arguments similar to those in the article [4]. We use Lemma 2.3 in [4] and check that all the results in §5 of [4], except for Lemma 5.5 and Proposition 5.6, hold for an arbitrary reduced expression $w_o$, and not only for the c-sorting word.

Consider now an arbitrary reduced expression $w_o$ of the element $w_o$. There exists a bijection $L_{r,c,w_o}$ between the set of letters in the word $cw_o = c_1c_2 \ldots c_nw_1w_2 \ldots w_N$ and the set $\Phi_{\geq-1} = -\Pi(\Phi_+)$ of almost positive (that is, simple negative and all positive) roots.

AMS 2010 Mathematics Subject Classification. Primary 05E45, 52B11, 13F60, 20F55.

DOI 10.1070/RM2014v069n03ABEH004903.

© 2014 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
in the root system \( \Phi \) associated with \( W \):

\[
\text{Lr}(c_i) = -\alpha_{c_i}; \quad \text{Lr}(w_j) = w_1 w_2 \ldots w_{j-1}(\alpha_{w_j}),
\]

where \( \alpha_w \in \Phi \) is the root corresponding to \( w \in W \). The choice of \( w_o \) yields a total order \( \prec_{w_o} \) on \( \Phi^+ \): \( \alpha \prec_{w_o} \beta \) if \( \text{Lr}^{-1}(\alpha) \) goes in \( w_o \) before \( \text{Lr}^{-1}(\beta) \). It is known that such an order satisfies the following condition: for any root subsystem of rank 2 in \( \Phi \) with the canonical generators \( \alpha, \beta \) one has either \( a_1 \alpha + b_1 \beta \prec_{w_o} a_2 \alpha + b_2 \beta \) if \( a_1 < a_2 \), or \( a_1 \alpha + b_1 \beta \prec_{w_o} a_2 \alpha + b_2 \beta \) if \( a_1 > a_2 \). In other words, the positive roots of any rank-2 subsystem are ordered in one of two natural ways. The choice of \( c \) also provides a total order on \( \Phi^+ \): \( \prec_c = \prec_{w_c(c)} \). We will say that a root \( \gamma \in \Phi^+ \) is \( (c,w_o) \)-stable if for any non-commutative rank-2 subsystem \( \langle \alpha, \beta \rangle \) containing \( \gamma \) and such that \( \gamma \neq \alpha, \beta \), we have

\[
\alpha \prec_c \beta \Leftrightarrow \alpha \prec_{w_o} \beta.
\]

This condition depends on \( c \), but not on \( c \). Let \( \text{Stab}(c,w_o) \) be the set of \( (c,w_o) \)-stable roots. The main result of this article is the following theorem.

**Theorem 2.** (i) The vertices of \( \Delta(cw_o;w_o) \) and, equivalently, the facets of \( B(cw_o;w_o) \) are in a one-to-one correspondence with the negative and the \( (c,w_o) \)-stable positive roots of the system \( \Phi \).

(ii) Let the expressions \( w_o, w'_o \) be such that \( \text{Stab}(c,w_o) \subseteq \text{Stab}(c,w'_o) \). Then the complex \( \Delta(cw'_o;w_o) \) can be obtained from the complex \( \Delta(cw_o;w_o) \) by a sequence of edge subdivisions. Similarly, the polytope \( B(cw'_o;w_o) \) can be obtained from the polytope \( B(cw_o;w_o) \) by a sequence of truncations of faces of codimension 2.

The proof is based on results in [5], where it was shown how braid moves in the group \( W \) induce compositions of edge subdivisions and the inverse operations on subword complexes. Then we show the there is a pair of roots \( \alpha, \beta \in \text{Stab}(c,w_o) \) with \( \alpha \prec_{w_o} \beta \) and \( \beta \prec_{w_o} \alpha \) such there are no other roots in \( \text{Stab}(c,w_o) \) between them (with respect to the order \( \prec_{w_o} \)). It remains to show that by a sequence of braid moves one can reorder the letters corresponding to the roots in the interval \( [\alpha, \beta]_{w_o} \) in such a way that \( \beta \) goes before \( \alpha \). One can take as a permutation the \( c \)-sorting word, and then the result follows from the word property of the group \( W \).

The class of 2-truncated cubes, that is, polytopes which can be obtained from the cube of a fixed dimension by a sequence of truncations of faces of codimension 2, has interesting properties and includes important families of polytopes (cf. [6]). Each 2-truncated cube is a flag polytope (that is, any set of pairwise intersecting facets of it has a non-empty intersection). By Theorems 1 and 2, we get the following statement.

**Corollary 1.** Each polytope of the form \( B(cw_o;w_o) \) is a 2-truncated cube and, therefore, is a flag polytope. In particular, any generalized associahedron is a 2-truncated cube.

**Bibliography**

[1] A. Knutson and E. Miller, *Ann. of Math.* (2) 161:3 (2005), 1245–1318.
[2] A. Knutson and E. Miller, *Adv. Math.* 184:1 (2004), 161–176.
[3] V. Pilaud and C. Stump, *Brick polytopes of spherical subword complexes: a new approach to generalized associahedra*, 2011 (v3 – 2014), 52 pp., arXiv:1111.3349.
[4] C. Ceballos, J.-P. Labbé, and C. Stump, *J. Algebraic Combin.* 39:1 (2014), 17–51.
[5] M. Gorsky, *Subword complexes and edge subdivisions*, 2013, 12 c., arXiv:1305.5499; М. Горский, Тр. МИАН (в печати). [M. Gorsky, Tr. Mat. Inst. Steklov. (to appear).]
[6] V. M. Buchstaber and V. D. Volodin, *Associahedra, Tamari lattices, and related structures*, Tamari memorial Festschrift, Progr. Math., vol. 299, Birkhäuser, Basel 2012, pp. 161–186.

Mikhail A. Gorsky
Steklov Mathematical Institute
of Russian Academy of Sciences
E-mail: mike.gorsky@gmail.com