HEREDITARY COMPLETENESS FOR SYSTEMS OF EXPONENTIALS AND REPRODUCING KERNELS

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Abstract. We prove that any complete and minimal system of exponentials \( \{ e^{i\lambda_n t} \} \) in \( L^2(-a, a) \) is hereditarily complete up to a one-dimensional defect. This means that there is at most one (up to a constant factor) function \( f \) which is orthogonal to all the summands in its formal Fourier series \( \sum_n (f, \tilde{e}_n) e^{i\lambda_n t} \), where \( \{ \tilde{e}_n \} \) is the system biorthogonal to \( \{ e^{i\lambda_n t} \} \). An analogous result is obtained for systems of reproducing kernels in some de Branges spaces. On the other hand, we show that in general there exist non-hereditarily complete systems of reproducing kernels in de Branges spaces, thus answering a question posed by N. Nikolski.

1. INTRODUCTION AND MAIN RESULTS

1.1. Hereditary completeness in general setting. A system of vectors \( \{ x_n \}_{n \in \mathbb{N}} \) in a separable Hilbert space \( H \) is said to be exact if it is both complete (i.e., \( \overline{\text{Span}} \{ x_n \} = H \)) and minimal (i.e., \( \overline{\text{Span}} \{ x_n \}_{n \neq n_0} \neq H \) for any \( n_0 \)). Given an exact system we consider its (unique) biorthogonal system \( \{ x'_n \}_{n \in \mathbb{N}} \) which satisfies \( (x_m, x'_n) = \delta_{mn} \). Then to every element \( x \in H \) we associate its formal Fourier series

\[
\sum_{n \in \mathbb{N}} (x, x'_n) x_n.
\]

A natural condition is that this correspondence is one-to-one: no nonzero vector generates zero series, in other words the biorthogonal system \( \{ x'_n \} \) is also complete. Another important property is the possibility of reconstructing the vector \( x \) from its Fourier series

\[
x \in \overline{\text{Span}} \{ (x, x'_n) x_n \}.
\]

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If this holds, we say that the system \( \{x_n\}_{n \in \mathbb{N}} \) is hereditarily complete. We will use an equivalent description: for any partition \( N = N_1 \cup N_2, N_1 \cap N_2 = \emptyset \), the system

\[
\{x_n\}_{n \in N_1} \cup \{x'_n\}_{n \in N_2}
\]

is complete in \( H \). In the opposite situation (i.e., when \( \{x_n\} \) and \( \{x'_n\} \) are complete, but \( \{x_n\} \) is not hereditarily complete) we say that system is non-hereditarily complete.

The importance of this notion is related to the spectral synthesis problem for linear operators. If \( \{x_n\} \) is the sequence of eigenfunctions and root functions of some compact operator, then the hereditary completeness of \( \{x_n\} \) is equivalent to the possibility of the so-called strong spectral synthesis for this operator, i.e., its restriction to any invariant subspace is complete (see [13] or [11, Chapter 4]).

The condition that the biorthogonal system \( \{x'_n\} \) is also complete in \( H \) is by no means automatic and corresponding examples can be easily constructed. It is less trivial to give examples of the situations where both \( \{x_n\} \) and \( \{x'_n\} \) are complete, but the system \( \{x_n\} \) fails to be hereditarily complete. In fact, first examples go back to Hamburger [10] who constructed a compact operator with a complete set of eigenvectors, whose restriction to an invariant subspace is a nonzero Volterra operator (and, hence, is not complete). Further examples of non-hereditarily complete systems were found by Markus [13] and Nikolski [14], while a general approach to constructing non-hereditarily complete systems was developed by Dovbysh, Nikolski and Sudakov [6, 7]. Any non-hereditarily complete system gives an example of an exact system which is not a summation basis. On the other hand, uniform minimality and closeness to an orthonormal system may be combined with nonhereditary completeness [7].

1.2. Hereditary completeness for exponential systems. It is natural to study the problem of hereditary completeness for special systems in functional spaces, e.g., those which appear as families of eigenvectors and root vectors of a certain operator. Exponential systems form an important class in this respect. Let \( \Lambda = \{\lambda_n\} \subset \mathbb{C} \) and let \( e_{\lambda}(t) = \exp(i\lambda t) \). We consider the exponential system \( \{e_{\lambda}\}_{\lambda \in \Lambda} \) in \( L^2(-a, a), a > 0 \). It was shown by Young [16] that, in contrast to the general situation, for any exact system of exponentials its biorthogonal system is complete. Another approach to this problem was suggested in [9], where it is shown that any exact system of exponentials is the system of eigenfunctions of
the differentiation operator $i\frac{d}{dz}$ in $L^2(-a,a)$ with a certain generalized boundary condition.

The problem of hereditary completeness for exponential systems remains open. We will show that the hereditary completeness holds up to a possible one-dimensional defect.

Applying the Fourier transform $\mathcal{F}$ one reduces the problem for exponential systems in $L^2(-\pi,\pi)$ to the same problem for systems of reproducing kernels in the Paley–Wiener space $\mathcal{PW}_\pi = \mathcal{F}L^2(-\pi,\pi)$. Recall that the reproducing kernel of $\mathcal{PW}_\pi$ corresponding to a point $\lambda \in \mathbb{C}$ is of the form

$$K_\lambda(z) = \frac{\sin \pi(z - \lambda)}{\pi(z - \lambda)}, \quad f(\lambda) = (f, K_\lambda)_{\mathcal{PW}_\pi}.$$

Let $\Lambda \subset \mathbb{C}$ be such that the system of reproducing kernels $\{K_\lambda\}_{\lambda \in \Lambda}$ is exact in the Paley–Wiener space $\mathcal{PW}_\pi$. Then the biorthogonal system $\{g_\lambda\}$ is given by

$$g_\lambda(z) = \frac{G(z)}{G'(\lambda)(z - \lambda)},$$

where $G$ is the so-called generating function of the set $\Lambda$. By the above-mentioned result of Young, $\{g_\lambda\}$ is also an exact system. It is well known that $G$ is a function of exponential type $\pi$ and has only simple zeros at the points of $\Lambda$.

**Theorem 1.1.** If $\{K_\lambda\}_{\lambda \in \Lambda}$ is exact in the Paley–Wiener space $\mathcal{PW}_\pi$, then for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the orthogonal complement in $\mathcal{PW}_\pi$ to the system

$$\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$$

(1.1)

is at most one-dimensional.

Moreover, this exceptional one-dimensional complement may happen only in the case when the sequence $\Lambda_1$ has zero density. Given a sequence $\Lambda$ set

$$D_+(\Lambda) = \limsup_{r \to \infty} \frac{n_r(\Lambda)}{2r},$$

where $n_r(\Lambda)$ is the usual counting function of the sequence $\Lambda$, $n_r(\Lambda) = \#\{\lambda \in \Lambda, |\lambda| \leq r\}$.

**Theorem 1.2.** Let $\Lambda \subset \mathbb{R}$, let the system $\{K_\lambda\}_{\lambda \in \Lambda}$ be exact in $\mathcal{PW}_\pi$, and let the partition $\Lambda = \Lambda_1 \cup \Lambda_2$ satisfy $D_+(\Lambda_1) > 0$. Then the system (1.1) is complete in $\mathcal{PW}_\pi$. 
1.3. Reproducing kernels of the de Branges spaces. The above results may be extended to the de Branges spaces. Let $E$ be an entire function in the Hermite–Biehler class, that is $E$ has no zeros in $\mathbb{C}_+$, and

$$|E(z)| > |E^*(z)|, \quad z \in \mathbb{C}_+,$$

where $E^*(z) = \overline{E(\overline{z})}$. With any such function we associate the de Branges space $\mathcal{H}(E)$ which consists of all entire functions $F$ such that $F/E$ and $F^*/E$ restricted to $\mathbb{C}_+$ belong to the Hardy space $H^2 = H^2(\mathbb{C}_+)$. The inner product in $\mathcal{H}(E)$ is given by

$$(F,G)_E = \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2} \, dt.$$

The Hilbert spaces of entire functions $\mathcal{H}(E)$ were introduced by L. de Branges [5] in connection with inverse spectral problems for differential operators. These spaces are also of a great interest from the function theory point of view. The Paley–Wiener space $\mathcal{PW}_a$ is the de Branges space corresponding to $E(z) = \exp(-iaz)$.

An important characteristics of the de Branges space $\mathcal{H}(E)$ is its phase function, that is, an increasing $C^\infty$-function $\varphi$ such that $E(t)\exp i\varphi(t) \in \mathbb{R}$, $t \in \mathbb{R}$ (thus, essentially, $\varphi = -\arg E$ on $\mathbb{R}$). Clearly, for $\mathcal{PW}_a$, $\varphi(t) = at$. If $\varphi' \in L^\infty(\mathbb{R})$ (in which case we say that $\varphi$ has sublinear growth), the space $\mathcal{H}(E)$ shares certain properties with the Paley–Wiener spaces.

A crucial property of the de Branges spaces is the existence of orthogonal bases of reproducing kernels corresponding to real points [5]. For $\alpha \in [0,\pi)$ we consider the set of points $t_n \in \mathbb{R}$ such that

$$(1.2) \quad \varphi(t_n) = \alpha + \pi n, \quad n \in \mathbb{Z}.$$ 

It should be mentioned that the points $t_n$ may exist not for all $n \in \mathbb{Z}$ (e.g., the sequence $\{t_n\}$ may be one-sided, that is, $t_n$ may exist only for $n \geq n_0$). If the points $t_n$ are defined by (1.2), then the system of reproducing kernels $\{K_{t_n}\}$ is an orthogonal basis for $\mathcal{H}(E)$ for each $\alpha \in [0,\pi)$ except, may be, one ($\alpha$ is an exceptional value if and only if $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$).

The completeness of a system biorthogonal to an exact system of reproducing kernels was studied in [2, 8]. In particular, it was shown in [8] that such biorthogonal systems are always complete when $\varphi' \in L^\infty(\mathbb{R})$. The following extension of this result is obtained in [2]: if, for some $N > 0$, $\varphi'(t) = O(|t|^N)$, $|t| \to \infty$, then
either \( e^{i\alpha} E - e^{-i\alpha} E^* \in \mathcal{H}(E) \) for some \( \alpha \in [0, \pi) \), or any system biorthogonal to an exact system of reproducing kernels is complete in \( \mathcal{H}(E) \).

The method of the proof of Theorem 1.1 extends to the case of the de Branges spaces with sublinear growth of the phase.

**Theorem 1.3.** Let \( \mathcal{H}(E) \) be a de Branges space such that \( \varphi' \in L^\infty(\mathbb{R}) \). If the system of reproducing kernels \( \{K_\lambda\}_{\lambda \in \Lambda} \) is exact in \( \mathcal{H}(E) \), then for any partition \( \Lambda = \Lambda_1 \cup \Lambda_2 \), the orthogonal complement in \( \mathcal{H}(E) \) to the system

\[
\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}
\]

is at most one-dimensional.

A crucial step in the proofs of Theorems 1.1 and 1.3 is the use of expansions of functions in \( P W_\pi \) or in \( \mathcal{H}(E) \) with respect to two different orthogonal bases of reproducing kernels. At first glance it may look like an artificial trick; however it should be noted that the existence of two orthogonal bases of reproducing kernels is a property which characterizes de Branges spaces among all Hilbert spaces of entire functions (see [3, 4]). Therefore, we believe this method to be intrinsically connected with the deep and complicated geometry of de Branges spaces.

We do not know if there really exists one-dimensional defect in the conditions of Theorem 1.3 (and even in the Paley–Wiener space \( P W_\pi \)), but we are able to show that in general de Branges spaces nonhereditarily completeness is possible.

**Theorem 1.4.** There exists a de Branges space \( \mathcal{H}(E) \) and an exact system of reproducing kernels \( \{K_\lambda\} \) such that its biorthogonal system is complete, but the original system \( \{K_\lambda\} \) is non-hereditarily complete.

This theorem answers a question about hereditary completeness of systems of reproducing kernels in the model subspaces of the Hardy space (de Branges spaces being a special case) posed by N. Nikolski.

Throughout this paper the notation \( U(z) \lesssim V(z) \) (or equivalently \( V(z) \gtrsim U(z) \)) means that there is a constant \( C \) such that \( U(z) \leq CV(z) \) holds for all \( z \) in the set in question, which may be a Hilbert space, a set of complex numbers, or a suitable index set. We write \( U(z) \asymp V(z) \) if both \( U(z) \lesssim V(z) \) and \( V(z) \lesssim U(z) \).

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2. Preliminaries

Note that if $\Lambda = \Lambda_1 \cup \Lambda_2$, and one of the sets $\Lambda_1$ or $\Lambda_2$ is finite, then the corresponding system (1.1) is complete by a simple Hilbert space argument. Therefore, from now on we exclude the case when one of the sets $\Lambda_1, \Lambda_2$ is finite.

Let $h \in PW_\pi$ be a function orthogonal to the system (1.1). Assume that $\Lambda \cap \mathbb{Z} = \emptyset$ and write the expansion of the vector $h$ with respect to the Shannon–Kotelnikov–Whittaker orthonormal basis $K_n(z) = \frac{\sin \pi(z-n)}{\pi(z-n)}$,

$$h(z) = \sum_n \overline{a_n}K_n(z) = \frac{1}{\pi} \sum_n \overline{a_n}(-1)^n \frac{\sin \pi z}{z-n},$$

where $a_n = \overline{h(n)}$ and $\|h\|^2 = \sum_n |a_n|^2 < \infty$. The fact that $h$ is orthogonal to \{\frac{G(z)}{z-\lambda}\}_{\lambda \in \Lambda_1}$ is equivalent to

$$\left( \frac{G(z)}{z-\lambda}, h \right) = \frac{1}{\pi} \sum_n a_n G(n) \frac{1}{n-\lambda} = 0, \quad \lambda \in \Lambda_1,$$

while $(h, K_\lambda) = 0$, $\lambda \in \Lambda_2$, implies that

$$\sum_n \frac{\overline{a_n}(-1)^n}{\lambda - n} = 0, \quad \lambda \in \Lambda_2.$$

Without loss of generality we may assume that $h$ does not vanish at integers, that is, $a_n \neq 0$, $n \in \mathbb{Z}$. Otherwise we can expand $h$ with respect to some other basis \{\overline{K}_{n+\alpha}\}, $\alpha \in (0, 1)$.

Now let $G_2$ be a canonical product of genus 1 constructed by the zero set $\Lambda_2$, and let $G_1 = G/G_2$. The function $G_2$ is defined uniquely up to an exponential factor $e^{\gamma z}$. Note that the zeros of $G$ satisfy the Blaschke condition in $\mathbb{C}_+$ and in $\mathbb{C}_-$. Therefore, we may choose $\gamma$ such that $G_2^*/G_2 = B_1/B_2$ for some Blaschke products $B_1$ and $B_2$. Hence $G_1^*/G_1$ is the ratio of two Blaschke products as well, since $G^*/G$ is of this form for any generating function $G$ of a complete minimal system of reproducing kernels.

We can rewrite conditions (2.1)–(2.2) as

$$\sum_n a_n G(n) \frac{1}{z-n} = \frac{G_1(z)S_1(z)}{\sin \pi z},$$

where $S_1(z) = \sum_n \frac{1}{z-n}$.
(2.4)  \[ \sum_n \overline{a}_n (-1)^n \frac{z - n}{z - n} = \frac{G_2(z)S_2(z)}{\sin \pi z}, \]

where \( S_1 \) and \( S_2 \) are some entire functions.

The pairs \((S_1, S_2)\) of entire functions satisfying (2.3)–(2.4) parametrize all functions orthogonal to (1.1). We will denote the set of such pairs by \( \Sigma(\Lambda_1, \Lambda_2) \). Note that the function \( S_2 = h/G_2 \) does not depend on the choice of the orthogonal basis \( \{k_{n+}\} \) up to a nonzero scalar factor (we will use this fact repeatedly), while \( S_1 \) will depend on the choice of the basis.

Comparing the residues at \( n \) we get

(2.5)  \[ S_1(n) = (-1)^n a_n G_2(n), \quad G_2(n)S_2(n) = \overline{a}_n. \]

Put \( S = S_1S_2 \). Then

(2.6)  \[ S(n) = S_1(n)S_2(n) = (-1)^n |a_n|^2. \]

**Lemma 2.1.** The function \( G_1S_1 \) is in \( \mathcal{PW}_\pi + z\mathcal{PW}_\pi \).

**Proof.** If \( w \) is a zero of \( G_1S_1 \), then it follows from (2.3) and the inclusion \( G(n)(1 + |n|)^{-1} \in \ell^2 \) that

(2.7)  \[ \frac{G_1(z)S_1(z)}{z - w} = \sin \pi z \sum_n \frac{a_n G(n)}{(n - w)(z - n)} \in \mathcal{PW}_\pi. \quad \square \]

**Lemma 2.2.** Let \( h \in \mathcal{PW}_\pi \) be orthogonal to some system of the form (1.1) and let \( (S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2) \). Then \( S \in \mathcal{PW}_\pi + c\sin \pi z \) for some \( c \in \mathbb{C} \).

**Proof.** Consider the function \( Q \in \mathcal{PW}_\pi \) which solves the interpolation problem \( Q(n) = (-1)^n |a_n|^2, \ n \in \mathbb{Z} \) (where \( a_n \) are the coefficients in the expansion \( h = \sum_n \overline{a}_n K_n \)) and put \( \tilde{S} = S - Q \). Then \( \tilde{S} \) vanishes on \( \mathbb{Z} \) and so \( \tilde{S}(z) = H(z) \sin \pi z \).

It remains to show that \( H \) is a constant. Note that \( G_2S_2 = h \in \mathcal{PW}_\pi \) and, by Lemma 2.1, \( G_1S_1 \in \mathcal{PW}_\pi + z\mathcal{PW}_\pi \). Hence,

(2.8)  \[ G(z)S(z) = \sin^2 \pi z \cdot \sum_n a_n G(n) \frac{z - n}{z - n} \cdot \sum_n \overline{a}_n (-1)^n \frac{z - n}{z - n} \in \mathcal{PW}_{2\pi} + z\mathcal{PW}_{2\pi}, \]

and, since \( G \in \mathcal{PW}_\pi + z\mathcal{PW}_\pi \) and \( GQ \in \mathcal{PW}_{2\pi} + z\mathcal{PW}_{2\pi} \), also

\[ G(z)\tilde{S}(z) = G(z)H(z) \sin \pi z \in \mathcal{PW}_{2\pi} + z\mathcal{PW}_{2\pi}. \]

We may divide by \( \sin \pi z \) in the space \( \mathcal{PW}_\pi \), and so

\[ G(z)H(z) \in \mathcal{PW}_\pi + z\mathcal{PW}_\pi. \]
Since $G$ is an entire function of exponential type $\pi$, we conclude that $H$ is of zero exponential type. Now if $H$ has at least one zero $z_1$, we conclude that $\frac{H(z)G(z)}{z-z_1} \in \mathcal{PW}_\pi$ which contradicts the fact that $\Lambda$ is a uniqueness set for the Paley–Wiener space. Thus, $H$ is a constant. \hfill \Box 

**Lemma 2.3.** Let $h \in \mathcal{PW}_\pi$ be orthogonal to some system of the form (1.1) and let $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Then both functions $S_1/S_1^*$ and $S_2/S_2^*$ are ratios of two Blaschke products.

**Proof.** The zero sets of $S_1$ and $S_2$ satisfy the Blaschke condition in $\mathbb{C}_+$ and in $\mathbb{C}_-$ since $G_1S_1 \in \mathcal{PW}_\pi + z\mathcal{PW}_\pi$ and $h = G_2S_2 \in \mathcal{PW}_\pi$. Thus, it remains to show that $S_1/S_1^*$ and $S_2/S_2^*$ have no exponential factors. By Lemma 2.2 we know that $S$ satisfies this property. Indeed, if $c \neq 0$ this is obvious, whereas if $c = 0$, then the function $S$ coincides with the function $Q \in \mathcal{PW}_\pi$ which is real on $\mathbb{R}$ and has at least one zero in each interval $(n, n + 1)$. So the size of the conjugate indicator diagram of the function $GS$ equals $4\pi$. Hence, the size of the conjugate indicator diagram both for $G_1S_1$ and for $G_2S_2$ equals $2\pi$. Since $G_2S_2 \in \mathcal{PW}_\pi$, we obtain that $G_2S_2/(G_2^*S_2^*)$ is a ratio of two Blaschke products. By the construction of $G_2$, the same is true for $S_2$ and, hence, for $S_1$. \hfill \Box 

**Lemma 2.4.** If $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$, then also $(S_1^*, S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$.

**Proof.** By Lemma 2.3, $S_1^*/S_1$ is of the form $B_1/B_2$ for some Blaschke products $B_1$ and $B_2$. We consider the following representation

$$
(2.9) \quad \frac{G_1(z)S_1(z)}{\sin \pi z} \cdot \frac{S_1^*(z)}{S_1(z)} = \sum_n a_n G(n) \cdot \frac{S_1^*(n)}{S_1(n)} + H(z),
$$

where $H$ is an entire function (which holds since the residues at integers coincide). On the other hand, $G_1S_1^* \in \mathcal{PW}_\pi + z\mathcal{PW}_\pi$, whence $|H(z)| \lesssim 1 + |z|$ and so $H$ is a polynomial of degree at most 1. Finally, (2.3) implies that $e^{-|y|}|G_1(iy)S_1(iy)| \to 0$, $|y| \to \infty$. Since the function $S_1^*/S_1$ is reciprocal to itself at conjugate points, we conclude that $\min(|H(iy)|, |H(-iy)|) \to 0$, $|y| \to \infty$, and so $H \equiv 0$.

Set $b_n = a_n S_1^*(n)/S_1(n)$. We can use an analogous argument to show that

$$
(2.10) \quad \frac{G_2(z)S_2^*(z)}{\sin \pi z} = \sum_n \overline{a_n} (-1)^n \cdot \frac{S_2^*(n)}{S_2(n)}.
$$

Comparing the residues we get

$$
\overline{a_n} (-1)^n S_2^*(n)/S_2(n) = \overline{b_n} (-1)^n.
$$
Thus, the pair \((S_1^*, S_2^*)\) corresponds to the sequence \(\{b_n\}\) in equations \((2.3)\) and \((2.4)\). This means that \((S_1^*, S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)\). \(\Box\)

By Lemma 2.4, if \((S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)\), then \((S_1 + S_1^*, S_2 + S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)\) and \((iS_1 - iS_1^*, -iS_2 + iS_2^*) \in \Sigma(\Lambda_1, \Lambda_2)\). Thus, in what follows we may assume that the functions \(S_1\) and \(S_2\) are real on \(\mathbb{R}\). In this case we have an immediate corollary from \((2.6)\).

**Corollary 2.5.** If \(S_1\) and \(S_2\) are real on \(\mathbb{R}\), then each open interval \((n, n+1), n \in \mathbb{Z}\), contains exactly one zero of \(S\), and \(S\) has no other zeros.

**Proof.** Since \(S\) is real on \(\mathbb{R}\) and changes the sign at \(n \in \mathbb{Z}\), it has at least one zero in every interval \((n, n+1)\). Choosing a zero in each interval we construct the (principal value) canonical product \(S_0\). Then \(S = S_0H\) for some entire function \(H\) of zero exponential type which is real on \(\mathbb{R}\). Clearly, \(|S_0(iy)| \gtrsim |y|^{-1}e^{\pi|y|}, |y| \to \infty\). By Lemma 2.2 we have \(S \in \mathcal{PW}_x + c\sin \pi z\). Hence, \(|H(iy)| \lesssim |y|, |y| \to \infty\), which implies that \(H\) is a polynomial of degree at most 1. Since the signs of \(S(n)\) interchange, \(S\) cannot have two zeros in any of the intervals \((n, n+1)\). Thus, \(H\) is a constant. \(\Box\)

3. Proofs of Theorems 1.1 and 1.2

We are now ready to prove the main results on hereditary completeness for exponential systems.

3.1. Completeness up to a one-dimensional defect. **Proof of Theorem 1.1.** Without loss of generality assume that \(\Lambda \cap \mathbb{Z} = \emptyset\). Let \(f = \sum_{n \in \mathbb{Z}} x_nK_n\) and \(h = \sum_{n \in \mathbb{Z}} b_nK_n\) be two linearly independent vectors orthogonal to \((1.1)\) and let \((S_1, S_2)\) and \((T_1, T_2)\) be the corresponding pairs of entire functions from \(\Sigma(\Lambda_1, \Lambda_2)\). Since, by Lemma 2.4, the pairs \((S_1 + S_1^*, S_2 + S_2^*), (iS_1 - iS_1^*, -iS_2 + iS_2^*), (T_1 + T_1^*, T_2 + T_2^*), (iT_1 - iT_1^*, -iT_2 + iT_2^*)\) also belong to \(\Sigma(\Lambda_1, \Lambda_2)\), we may assume from the very beginning that the pairs \((S_1, S_2)\) and \((T_1, T_2)\) are linearly independent and all functions \(S_1, S_2, T_1, \text{and } T_2\) are real on \(\mathbb{R}\).

Using equations \((2.5)\) for \(S\) and \(T\) we get

\[S_1(n)T_2(n)G_2(n)(S_2 + (S_2)^* - G_2(n)) = (-1)^{n}G_2(n)a_nb_n,\]

and hence,

\[S_1(n)T_2(n) = S_2(n)T_1(n) = \beta_na_nb_n,\]

with \(|\beta_n| = 1|\).
Denote by $Q$ the function in $\mathcal{PW}_\pi$ which solves the interpolation problem $Q(n) = \beta_n a_n b_n$. Then
\[ T_1(z)S_2(z) = Q(z) + a(z) \sin \pi z, \quad S_1(z)T_2(z) = Q(z) + b(z) \sin \pi z, \]
for some entire functions $a$ and $b$. We show now that $a$ and $b$ are constants.

In what follows we denote by $\mathcal{PW}_\pi + \mathbb{C} \sin \pi z$ the class of functions of the form $f + c \sin \pi z$, where $f \in \mathcal{PW}_\pi$, $c \in \mathbb{C}$. Note that the functions $S = S_1S_2$ and $T = T_1T_2$ are in $\mathcal{PW}_\pi + \mathbb{C} \sin \pi z$ by Lemma 2.2. Moreover, note that the pair $(S_1 + T_1, S_2 + T_2)$ corresponds to the vector $f + h$ while the pair $(S_1 + iT_1, S_2 - iT_2)$ corresponds to the vector $f + ih$. Applying again Lemma 2.2 we get $U = (S_1 + T_1)(S_2 + T_2)$ and $V = (S_1 + iT_1)(S_2 - iT_2)$ are in $\mathcal{PW}_\pi + \mathbb{C} \sin \pi z$. Hence the functions
\[ S_1T_2 + S_2T_1 = U - S - T, \quad i(S_2T_1 - S_1T_2) = V - S - T \]
belong to $\mathcal{PW}_\pi + \mathbb{C} \sin \pi z$. Thus, $S_1T_2, S_2T_1 \in \mathcal{PW}_\pi + \mathbb{C} \sin \pi z$, and we conclude that $a$ and $b$ are constants.

Assume that $a \neq 0$. Let us denote by $s_m$ the zero of $S_2$ in the interval $[m - 1/2, m + 1/2]$ for those $m$ for which such a zero exists. Then
\[ Q(s_m) + a(-1)^m \sin \pi (s_m - m) = 0, \]
whence
\[ \sum |s_m - m|^2 \gtrsim \sum \sin^2 \pi (s_m - m) \gtrsim \sum |Q(s_m)|^2 < \infty. \]
On the other hand, the zeros of $S_2$ do not depend on the choice of the basis, they are the zeros of $h/G_2$. Expanding with respect to another basis (say, $\{n + \delta\}$ with small $\delta$) we conclude that $\sum |s_m - m - \delta|^2 < \infty$ for the zeros $s_m$ of $S_2$ with $s_m \in [m + \delta - 1/2, m + \delta + 1/2]$. This is obviously wrong.

Thus, we have proved that $a = b = 0$, and so $S_1T_2 = T_1S_2 = Q$. Since $S_1$ has no common zeros with $S_2$ (we choose the basis so that all $a_n$ are nonzero) we conclude that the zero sets of $S_2$ and $T_2$ coincide, and, thus, $g = ch$ for some constant $c$, a contradiction.

3.2. Proof of Theorem 1.2. The following proposition plays the key role in the proof of Theorem 1.2. In Section 4 we prove a slightly stronger result which applies to general de Branges spaces (see Proposition 4.4). We prefer, however, to include an elementary proof to make the exposition concerning exponential systems more self-contained.
Proposition 3.1. Let $S \in \mathcal{P}W_{\pi} + \mathbb{C}\sin \pi z$ be a real entire function with real zeros interlacing with $\mathbb{Z}$. If $\sum_{n \in \mathbb{Z}} |S(n)| < \infty$, then for any $\delta > 0$ we have

$$L_\delta := \lim_{N \to \infty} \frac{1}{N} \text{card} \{ |k| \leq N : \text{dist} (\mathbb{Z}_S \cap [k, k+1], \mathbb{Z}) > \delta \} = 0.$$  

Proof. Let $S(n) = (-1)^n c_n$. Without loss of generality we may assume that $c_n > 0$ and $\sum_{n \in \mathbb{Z}} c_n = 1$. Then $S(z)/\sin \pi z$ is a Herglotz function in $\mathbb{C}^+$ and

$$S(z) \sin \pi z = b + \sum_{n \in \mathbb{Z}} c_n \frac{z}{z-n}$$

for some $b \in \mathbb{R}$. Set $s(x) = \sum_{n \in \mathbb{Z}} \frac{c_n}{x-n}$.

Case 1. If $b \neq 0$, then

$$\lim_{x \in \mathbb{Z}_S, |x| \to \infty} \text{dist} (x, \mathbb{Z}) = 0.$$  

This follows from the fact that for any $\delta > 0$ we have $s(x) \to 0$ as $|x| \to \infty$ and $\text{dist} (x, \mathbb{Z}) \geq \delta$.

Case 2. Suppose that $b = 0$. Fix two positive numbers $\delta < 1/4$ and $\eta < \delta^3$ and choose $M$ so that $\sum_{|n| \leq M} c_n > 1 - \eta$.

Now let the integer $N$ be so large that $\delta N > M$. Put

$$E_N = \left\{ x \in \mathbb{R} : \left| \sum_{n \in \mathbb{Z}} \frac{c_n}{x-n} \right| \geq \frac{1}{N} \right\}.$$

By Boole’s lemma, $|E_N| = 2N$ (by $|E|$ we denote the Lebesgue measure of the set $E$).

Next, set

$$F_N = \left\{ x \in \mathbb{R} : \left| \sum_{|n| > M} \frac{c_n}{x-n} \right| \geq \frac{\delta}{2N} \right\}.$$

Then

$$|F_N| \leq \frac{4N\eta}{\delta}.$$  

Let $J_N = [-N - \delta N - M, N + \delta N + M]$. Since

$$\left| \sum_{|n| \leq M} \frac{c_n}{x-n} \right| \leq \frac{1}{(1+\delta)N}, \quad x \notin J_N,$$

we have, for $x \in E_N \setminus J_N$,

$$\left| \sum_{|n| > M} \frac{c_n}{x-n} \right| \geq \frac{1}{N} - \frac{1}{(1+\delta)N} = \frac{\delta}{(1+\delta)N},$$

and so $x \in F_N$. We conclude that $E_N \setminus J_N \subset F_N$. 
Consider the family \( I_N \) of the intervals of the form \( I_k = [k, k + 1] \subset J_N \) with \(|k| \geq M + \delta N\) and with the following two properties:

\[
\begin{align*}
(3.1) & \quad (I_k^* \cap E_N) \setminus F_N \neq \emptyset, \quad I_k^* = [k + \delta, k + 1 - \delta]; \\
(3.2) & \quad |I_k \cap F_N| < \delta.
\end{align*}
\]

We will show that, for sufficiently large \( N \), we have

\[
(3.3) \quad \text{card } I_N \geq (1 - A_1 \delta)|J_N|,
\]

where \( A_1 \) is some absolute (numeric) constant. In what follows, symbols \( A_1, A_2, \) etc. will denote different absolute constants.

If \((I_k^* \cap E_N) \setminus F_N = \emptyset\) (i.e., the interval \( I_k^* \) does not satisfy (3.1)), then \( I_k^* \subset (J_N \setminus E_N) \cup F_N \) and

\[
|I_k^*| \leq |J_N| = |J_N \setminus E_N| + |E_N \setminus J_N| + |F_N| 
\]

\[
\leq 2N + 2\delta N + 2M - 2N + \frac{8N\eta}{\delta} \leq A_2 \delta N.
\]

Hence, for the number \( N_1 \) of those intervals \( I_k^* \) which do not satisfy (3.1), we have the estimate

\[
N_1(1 - 2\delta) \leq A_3 \delta N.
\]

On the other hand, for the number \( N_2 \) of those intervals \( I_k \) which do not satisfy (3.2), we get \( N_2 \delta \leq \frac{4N\eta}{\delta} \), and so \( N_2 \leq \frac{4N\eta}{\delta} \leq A_4 \delta N \), since \( \eta < \delta^3 \). Thus, for sufficiently large \( N \),

\[
\text{card } I_N \geq 2N - N_1 - N_2 \geq 2N - A_5 \delta N.
\]

The latter inequality implies (3.3).

Now, if \( I_k \in I_N \), then there exists a point \( y \in (I_k^* \cap E_N) \setminus F_N \) and so we have

\[
\left| \sum_{|n| \leq M} \frac{c_n}{y - n} \right| \geq \frac{1}{N} - \frac{\delta}{2N} > \frac{1}{2N}.
\]

For any \( x \in I_k^* \) using the fact that \(|k| \geq M + \delta N\) we get

\[
(3.4) \quad \left| \sum_{|n| \leq M} \frac{c_n}{x - n} - \sum_{|n| \leq M} \frac{c_n}{y - n} \right| \leq \sum_{|n| \leq M} \frac{|c_n| |x - y|}{|n|(x - n)(y - n)|} \leq \frac{1}{\delta^2 N^2} \leq \frac{1}{4N}.
\]
for sufficiently large $N$, and hence,
\[
\left| \sum_{|n| \leq M} \frac{c_n}{x-n} \right| \geq \frac{1}{4N}, \quad x \in I_k^*.
\]

Suppose that for some $w \in I_k^*$ we have $s(w) = \sum_{n \in \mathbb{Z}} \frac{c_n}{w-n} = 0$. Then
\[
\left| \sum_{|n| > M} \frac{c_n}{w-n} \right| \geq \frac{1}{4N} > \frac{\delta}{N}.
\]
So $w \in F_N$ and, moreover, since the function under the modulus sign is monotone on $I_k$ we obtain that either $[k, w] \subset F_N$ or $[w, k+1] \subset F_N$, which is impossible due to (3.2).

Thus, the zeros of $s$ (and hence of $S$) on $I_k \in \mathcal{I}_N$ are in $I_k \setminus I_k^*$. It follows from (3.3) that
\[
L_\delta = \limsup_{N \to \infty} \frac{1}{N} \text{card} \left\{ |k| \leq N : \text{dist} (Z_\delta \cap [k, k+1], \mathbb{Z}) > \delta \right\} \leq A\delta
\]
for some absolute constant $A$. Since $L_\delta$ is a non-increasing nonnegative function of $\delta$ on $(0, 1/4)$, it follows that $L_\delta \equiv 0$. \qed

**Proof of Theorem 1.2.** Assume that there is a nontrivial function $h$ orthogonal to the system (1.1) such that $D_+(\Lambda_1) > 0$. Denote by $Z_1$ and $Z_2$ the zero sets of $S_1$ and $S_2$, respectively.

Since $G_1 S_1 \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$, we have
\[
D(\Lambda_1 \cup Z_1) = \lim_{r \to \infty} \frac{n_r(\Lambda_1 \cup Z_1)}{2r} \leq \pi,
\]
and so
\[
D_-(Z_1) = \liminf_{r \to \infty} \frac{n_r(Z_1)}{2r} < \pi.
\]
Since $S$ is of exponential type $\pi$, we have $D_+(Z_2) > 0$.

The function $S_2 = h/G_2$ does not depend on the choice of the basis, and replacing if necessary the basis $\{K_n\}$ by the basis $\{K_{n+\alpha}\}$ we may find $\alpha$ such that for a subsequence $\tilde{Z}_2$ of $Z_2$ with positive upper density we have $\text{dist} (\tilde{Z}_2, \mathbb{Z} + \alpha) \geq 1/4$. Without loss of generality assume that this holds for $\alpha = 0$. Construct the function $S_1$ corresponding to this basis by formula (2.3). Then for $S = S_1 S_2$ we have $\sum_{n \in \mathbb{Z}} |S(n)| < \infty$. Note that by Corollary 2.5 the zeros of $S$ interlace with $\mathbb{Z}$. By Proposition 3.1 all zeros of $S$ except the set of zero density are close to $\mathbb{Z}$, and we come to a contradiction. \qed
4. Extensions to the de Branges spaces

4.1. Preliminary remarks. We start with a general construction of functions biorthogonal to a system of reproducing kernels. Let $\mathcal{H}(E)$ be a de Branges space, let $\varphi$ be the corresponding phase function. As usual, we write $E = A - iB$. To avoid inessential difficulties we will always assume that $A \notin \mathcal{H}(E)$. Let $\Lambda \subset \mathbb{C}$ be such that the system of reproducing kernels $\{K_\lambda\}_{\lambda \in \Lambda}$ of the space $\mathcal{H}(E)$ is exact. Then there exists the generating function, that is, an entire function $G \in \mathcal{H}(E) + z\mathcal{H}(E)$, such that $GH \notin \mathcal{H}(E)$ for any nontrivial entire function $H$, and vanishing exactly on the set $\Lambda$. The biorthogonal system to $\{K_\lambda\}_{\lambda \in \Lambda}$ is given by

$$g_\lambda(z) := \frac{G(z)}{G'(\lambda)(z - \lambda)}.$$

We will assume that $\{g_\lambda\}_{\lambda \in \Lambda}$ is also an exact system in $\mathcal{H}(E)$ (recall that this is the case, e.g., when $\varphi' \in L^\infty(\mathbb{R})$ [8] or when $\varphi'$ has at most power growth and $\Theta = E^*/E$ has no finite derivative at $\infty$ [2]).

Let $h \in \mathcal{H}(E)$ be orthogonal to $\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$. Denote by $T = \{t_n\}$ the zero set of $A$ (assume that $T \cap \Lambda = \emptyset$) and recall that the functions

$$\frac{A(z)}{z - t_n} = \pi i \frac{K_{t_n}(z)}{E(t_n)}$$

form an orthogonal basis in $\mathcal{H}(E)$ [5, Theorem 22] and $\|\frac{A(z)}{z - t_n}\|^2 = \pi \varphi'(t_n)$. Expanding

$$h(z) = A(z) \sum_n \frac{a_n \mu_n^{1/2}}{z - t_n}, \quad \{a_n\} \in \ell^2,$$

where $\mu_n = 1/\varphi'(t_n)$, we conclude that

$$\sum_n \frac{a_n \mu_n^{1/2}}{z - t_n} = \frac{G_2(z)S_2(z)}{A(z)}$$

for some entire function $S_2$. As in the Paley–Wiener case we assume that $G_2$ is an entire function which vanishes exactly on $\Lambda_2$ and $G_2/G_2 = B_1/B_2$ for some Blaschke products $B_1$ and $B_2$. On the other hand, since $h \perp g_\lambda$, $\lambda \in \Lambda_1$, we obtain

$$\sum_n \frac{G(t_n)}{E(t_n)} \frac{a_n \mu_n^{1/2}}{z - t_n} = \frac{iG_1(z)S_1(z)}{A(z)}$$
for some entire function $S_1$ (argue as in the Paley–Wiener case). Comparing the residues we get

\begin{equation}
S_1(t_n)G_1(t_n) = -i a_n \mu_n^{1/2} A'(t_n) \frac{G(t_n)}{E(t_n)},
\end{equation}

and

\begin{equation}
S_2(t_n)G_2(t_n) = \overline{a}_n \mu_n^{1/2} A'(t_n).
\end{equation}

Hence, for $S = S_1S_2$, we have

\begin{equation}
S(t_n) = -|a_n|^2 \mu_n (A'(t_n))^2 / E(t_n).
\end{equation}

Since $A'(t_n) = (-1)^n |E(t_n)| \varphi'(t_n)$ (the phase $\varphi$ is chosen so that $\varphi(t_n) = \pi n$), we get

\begin{equation}
S(t_n) = |a_n|^2 A'(t_n).
\end{equation}

From now on we assume that $\varphi$ is of tempered growth, that is,

\begin{equation}
\varphi'(t) = O(|t|^N), \quad |t| \to \infty,
\end{equation}

for some $N$. It follows from (4.6) that, for any $F \in \mathcal{H}(E)$,

\begin{equation}
\frac{|F(x)|}{|E(x)|} \leq \frac{\|K_x\| E\|F\|_E}{|E(x)|} = \left(\frac{\varphi'(x)}{\pi}\right)^{1/2} \|F\|_E \lesssim (|x| + 1)^{N/2}, \quad x \in \mathbb{R}.
\end{equation}

Using the same arguments as in the proof of Lemma 2.1 we get $G_1S_1 \in \mathcal{H}(E) + z\mathcal{H}(E)$. Hence,

\begin{equation}
G(z)S(z) = A^2(z) \sum_n \overline{a}_n \mu_n^{1/2} \frac{1}{z - t_n} \cdot \sum_n G(t_n) a_n \mu_n^{1/2} \frac{1}{E(t_n) (z - t_n)} \in \mathcal{P}_M \cdot \mathcal{H}(E),
\end{equation}

where $\mathcal{P}_M$ is the set of polynomials of degree at most $M$.

Arguing analogously to the proof of Lemma 2.2 we obtain the following growth restriction.

**Lemma 4.1.** Assume that $\varphi$ is of tempered growth. Let $h \in \mathcal{H}(E)$ be orthogonal to some system $\{g_{\lambda}\}_{\lambda \in \Lambda_1} \cup \{K_{\lambda}\}_{\lambda \in \Lambda_2}$ and let $(S_1, S_2)$ be the corresponding pair. Then $S \in \mathcal{P}_M \cdot \mathcal{H}(E)$ for some $M$.

**Proof.** We give a sketch of the proof. Let $M \geq \frac{N}{2} + 1$. By (4.5), we have

\begin{equation}
\frac{|S(t_n)|}{|E(t_n)(\varphi'(t_n))^{1/2}|} = |a_n|^2 (\varphi'(t_n))^{1/2} \lesssim |a_n|^2 |t_n|^{N/2},
\end{equation}

\begin{equation}
(4.7)
G(z)S(z) = A^2(z) \sum_n \overline{a}_n \mu_n^{1/2} \frac{1}{z - t_n} \cdot \sum_n G(t_n) a_n \mu_n^{1/2} \frac{1}{E(t_n) (z - t_n)} \in \mathcal{P}_M \cdot \mathcal{H}(E),
\end{equation}

where $\mathcal{P}_M$ is the set of polynomials of degree at most $M$. 

Arguing analogously to the proof of Lemma 2.2 we obtain the following growth restriction.
and, dividing out sufficiently many zeros $s_1, \ldots, s_M$ of $S$ we obtain that
\[
\sum_n \frac{|\tilde{S}(t_n)|^2}{|E(t_n)|^2 \varphi'(t_n)} < \infty, \quad \tilde{S}(z) = \frac{S(z)}{(z - s_1) \cdots (z - s_M)}.
\]

Now let $Q$ be the (unique) function in $\mathcal{H}(E)$ which solves the interpolation problem $Q(t_n) = \tilde{S}(t_n)$. Using (4.7) and an analogous estimate for $GQ$, we obtain that $G(\tilde{S} - Q) \in \mathcal{P}_M \mathcal{H}(E^2)$. Since $\tilde{S} - Q$ vanishes on $\{t_n\}$, we have $G(\tilde{S} - Q) = GAH \in \mathcal{P}_M \mathcal{H}(E^2)$ for some entire function $H$. We want to show that $H$ is a polynomial of degree at most $M + 1$, whence $\tilde{S} = Q + AH \in \mathcal{P}_{M+1} \mathcal{H}(E)$. Clearly, $H$ is of zero exponential type. If $H$ has at least $M + 2$ zeros, then dividing them out we obtain an entire function $\tilde{H}$ such that $GAH \in \mathcal{H}(E^2)$ and $|G(iy)\tilde{H}(iy)|/|E(iy)| = o(y^{-1})$, $|y| \to \infty$ (we use the fact that $|A(iy)|/|E(iy)| \gtrsim y^{-1}$, $y \to +\infty$). Let $v_n$ be such that $\varphi(v_n) = \pi/2 + \pi n$ (thus, $\{v_n\}$ is the support of another orthogonal family of reproducing kernels). Since $|A(v_n)| = |E(v_n)|$, we conclude that
\[
G(v_n)\tilde{H}(v_n)/E(v_n) \in L^2(\nu), \quad \nu = \sum_n (\varphi'(v_n))^{-1} \delta_{v_n}.
\]

Now it remains to apply [5, Theorem 26] to conclude that $G\tilde{H} \in \mathcal{H}(E)$, a contradiction to the fact that $G$ is the generating function of a complete system of kernels.

By (the proof of) Lemma 4.1, $S = QH_1 + AH_2$ for some polynomials $H_1$, $H_2$, and, hence, $S/S^*$ is a Blaschke product. Suppose that $G_2S_2/(G_2S_2^*)$ is not a Blaschke product. Then for small $\varepsilon$, we have $G_2S_2e^{i\varepsilon z} \in \mathcal{H}(E)$; analogously, $G_1S_1e^{-i\varepsilon z} \in \mathcal{H}(E)$, and, hence, $(S_1e^{i\varepsilon z}, S_2e^{-i\varepsilon z}) \in \Sigma(\Lambda_1, \Lambda_2)$ and $(1 + e^{i\varepsilon z})S_1$, $(1 + e^{-i\varepsilon z})S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Applying (4.5) and Lemma 4.1 to the function $z \mapsto S(z) (1 + e^{i\varepsilon z})(1 + e^{-i\varepsilon z})$, we obtain a contradiction. Thus, $S_2/S_2^*$ and $S_1/S_1^*$ are Blaschke products.

Now by an argument, analogous to Lemma 2.4, the pair $(S_1^*, S_2^*)$ also corresponds to some function orthogonal to $\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$. Thus, we may always find functions $S_1, S_2$ which are real on $\mathbb{R}$. By (4.5) the function $S$ changes its sign at adjacent points $t_n$ (as usual we assume that the basis is chosen so that all coefficients $a_n$ are nonzero), and thus, there is a zero of $S$ in each of the intervals $(t_n, t_{n+1})$. We have an analog of Corollary 2.5.

**Lemma 4.2.** Assume that $\varphi$ is of tempered growth. If a pair $(S_1, S_2)$ corresponds to a function $h \in \mathcal{H}(E)$ orthogonal to some system $\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$ and $S_1$
and $S_2$ are real on $\mathbb{R}$, then $S = S_0H$, where $S_0$ has exactly one zero in any interval $(t_n, t_{n+1})$ and $H$ is a polynomial.

4.2. Proof of Theorem 1.3. Without loss of generality assume that $\Lambda \cap \{t_n\} = \emptyset$, where $\varphi(t_n) = \pi n, n \in \mathbb{Z}$. Let $f$ and $h$ be orthogonal to the system (1.3),

$$f(z) = A(z) \sum_n \frac{\overline{a_n} \mu_n^{1/2}}{z - t_n}, \quad h(z) = A(z) \sum_n \frac{b_n \mu_n^{1/2}}{z - t_n}, \quad \{a_n\}, \{b_n\} \in \ell^2.$$

Let $(S_1, S_2)$ and $(T_1, T_2)$ be the corresponding pairs of entire functions such that $S_1, S_2, T_1$ and $T_2$ are real on $\mathbb{R}$. Using the equations (4.3)–(4.4) in the same way as in the proof of Theorem 1.1, we obtain

$$S_1(t_n)T_2(t_n) = T_1(t_n)S_2(t_n) = a_n b_n |E(t_n)| \varphi'(t_n) \beta_n,$$

where $|\beta_n| = 1$. The hypothesis $\sup_n |\varphi'(t_n)| < \infty$ implies that

$$\sum_n \frac{|S_1(t_n)T_2(t_n)|^2}{|E(t_n)|^2 \varphi'(t_n)} = \sum_n \frac{|S_1(t_n)T_2(t_n)|^2 \varphi'(t_n)}{(A'(t_n))^2} < \infty.$$

Since $\{K_{t_n}\}$ is an orthogonal basis in $\mathcal{H}(E)$ and $\|K_{t_n}\|_E^2 = |E(t_n)|^2 \varphi'(t_n)/\pi$, we conclude that there exists a unique function $Q \in \mathcal{H}(E)$ which solves the interpolation problem $Q(t_n) = a_n b_n |E(t_n)| \varphi'(t_n) \beta_n$. Then

$$T_1(z)S_2(z) = Q(z) + a(z) A(z), \quad S_1(z)T_2(z) = Q(z) + b(z) A(z),$$

for some entire functions $a$ and $b$. We show now that $a$ and $b$ are polynomials.

Note that by Lemma 4.1 the functions $S = S_1S_2$ and $T = T_1T_2$ as well as $(S_1 + T_1)(S_2 + T_2)$ and $(S_1 + i T_1)(S_2 - i T_2)$ are in $\mathcal{P}_M \cdot \mathcal{H}(E)$. Hence, the functions $S_1T_2$ and $S_2T_1$, and, consequently, the functions $S_1T_2 - Q$ and $S_2T_1 - Q$ are in $\mathcal{P}_M \cdot \mathcal{H}(E)$. Now assume that $F = AH$ for some entire function $H$ of zero exponential type and $F \in \mathcal{P}_M \cdot \mathcal{H}(E)$. We claim that $H$ must be a polynomial. Indeed, if $H$ has at least $M$ zeros $z_j$, then dividing $F$ by $\prod_{j=1}^M (z - z_j)$ we obtain a function in $\mathcal{H}(E)$ which vanishes on $\{t_n\}$ and, thus, is identically zero. Applying this argument to $S_1T_2 - Q$ and $S_2T_1 - Q$ we conclude that $a$ and $b$ are polynomials.

Now assume that $a \neq 0$. Let us denote by $s_m$ the zero of $S_2$ such that $|\varphi(s_m) - \varphi(t_m)| \leq \pi/2$ whenever such a zero exists. Then

$$Q(s_m) + a(s_m) A(s_m) = 0.$$

Note that $\{t_m\}$ is separated sequence (i.e., $\inf_{n \neq m} |t_n - t_m| > 0$) and so $s_m$ is at most a union of two separated sequences. By a simple variant of Carleson embedding theorem for the de Branges spaces with $\varphi' \in L^\infty(\mathbb{R})$ (an explicit
statement may be found in [1, Theorem 5.1], though the proof may be recovered already from [15, Theorem 2]) we have

$$\sum_{m} \frac{|Q(s_m)|^2}{|E(s_m)|^2} < \infty$$

for any $Q \in \mathcal{H}(E)$, whence

$$\sum_{m} \frac{|A(s_m)|^2}{|E(s_m)|^2} < \infty.$$  

By the definition of the phase function, $|A(s_m)| = |E(s_m)| \sin(\varphi(s_m) - \varphi(t_m))$.

Thus, we obtain that

$$\sum_{m} \sin^2(\varphi(s_m) - \varphi(t_m)) \leq \sum_{m} (\varphi(s_m) - \varphi(t_m))^2 < \infty.$$  

To complete the proof we apply once again the argument with the shift of the basis. The zeros of $S_2$ do not depend on the choice of the basis. Expanding with respect to another basis, say $\{K_{\tilde{t}_n}\}$, with $\varphi(\tilde{t}_n) = \delta + \pi n$ for some small $\delta$, we get that $\sum_{m}(\varphi(s_m) - \varphi(\tilde{t}_m))^2 < \infty$. However, $|\varphi(t_m) - \varphi(\tilde{t}_m)| = \delta$ and we come to a contradiction.

Thus, we have proved that $a = b = 0$, and so $S_1T_2 = T_1S_2 = Q$. Since $S_1$ has no common zeros with $S_2$ (we choose the basis so that all $a_n$ are nonzero) we conclude that the zero sets of $S_2$ and $T_2$ coincide, and, thus, $f$ is proportional to $h$.

**Remark 4.3.** It is easy to show that if $\varphi$ is of tempered growth, then the orthogonal complement to the system (1.3) is always finite dimensional, with a bound on the dimension depending only on $N$. Indeed, by Lemma 4.2, there exists $M = M(N)$ such that for any pair $(S_1,S_2)$ which corresponds to a function $f$ in the orthogonal complement to (1.3) and is real on $\mathbb{R}$, we have $S = S_0H$, $H \in \mathcal{P}_M$. In particular, any interval $(t_n, t_{n+1})$ contains at most $M + 1$ zeros of $S$.

Now assume that the orthogonal complement to (1.3) contains at least $M + 3$ linearly independent vectors $f_{j,0}$, $j = 1, \ldots M + 3$, such that the corresponding functions $S_{1,j,0}$, $S_{2,j,0}$ are real on $\mathbb{R}$. Considering linear combinations (with real coefficients) $f_{j,1} = f_{j,0} - \alpha_j f_{M+3,0}$, $j = 1, \ldots, M + 2$, we may achieve that the functions $S_{1,j,1}$ corresponding to $f_{j,1}$ have a common zero at $x_1 \in (t_0, t_1)$. Repeating this procedure we obtain a nonzero function $f_{M+2,1}$ in the orthogonal complement to (1.3) such that the corresponding function $S_{1,M+2,1}$ vanishes at $M + 2$ distinct points $x_1, \ldots x_{M+2} \in (t_0, t_1)$ which gives a contradiction.
4.3. **Density results.** Let a pair \((S_1, S_2)\) correspond to a function \(h \in \mathcal{H}(E)\) orthogonal to some system \(\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}\) and let \(S_1\) and \(S_2\) be real on \(\mathbb{R}\). We will show that most of the zeros of \(S\) are in a certain sense close to the set \(\{t_n\}\) (the support of a Clark basis). Thus, the zeros of \(S_2\) which do not depend on the choice of the basis form a small proportion of the zeros of \(S\) (see Corollary 4.5 below).

By Lemma 4.2, \(S = S_0 H\), where \(S_0\) has exactly one zero in each of the intervals \((t_n, t_{n+1})\) and \(H\) is a polynomial. Moreover, by (4.5) we have \(\{S(t_n)/A'(t_n)\} \in \ell^1\), whence \(\{S_0(t_n)/A'(t_n)\} \in \ell^1\). By Lemma 4.1 we have \(S \in \mathcal{P}_M \cdot \mathcal{H}(E)\) for some \(M\), whence \(S_0/A\) grows at most polynomially along \(i\mathbb{R}^+\). Since the zeros of \(A\) and \(S_0\) interlace, the function \(S_0/A\) is a Herglotz function and thus has a representation

\[
\frac{S_0(z)}{A(z)} = az + b + \sum_n \frac{c_n}{z - t_n}, \quad \{c_n\} \in \ell^1.
\]

We will show that in this case the zeros of \(S_0\) (and \(S\)) must be necessarily close (in some sense) to the points \(t_n\). The case when \(a \neq 0\) or \(b \neq 0\) should be treated exactly as in Proposition 3.1. The remaining case follows from the following proposition (apparently, known to experts).

**Proposition 4.4.** Let \(t_n \in \mathbb{R}, n \in \mathbb{Z}, t_n \to \pm \infty, n \to \pm \infty\), and let \(\mu_n > 0, \sum_n \mu_n = M < \infty\). Let \(A\) be an entire function which is real on \(\mathbb{R}\) and has only simple real zeros coinciding with the points \(\{t_n\}\). Define an entire function \(B\) by the Herglotz representation

\[
B(z) = \frac{A(z)}{z - t_n} = \sum_n \frac{\mu_n}{z - t_n}.
\]

If we denote by \(s_n\) the zero of \(B\) in \((t_n, t_{n+1})\), then

\[
\sum_{s_n > 0} \frac{t_{n+1} - s_n}{s_n} < \infty, \quad \sum_{s_n < 0} \frac{s_n - t_n}{|s_n|} < \infty.
\]

**Proof.** The zeros of \(B\) are simple and interlace with the zeros of \(A\). Since \(\text{Im} \frac{B}{A} > 0\) in \(\mathbb{C}^+\), the function \(E = A - iB\) is in the Hermite–Biehler class and so we can define the de Branges space \(\mathcal{H}(E)\). The measure \(\mu = \sum_n \mu_n \delta_{t_n}\) is a corresponding Clark measure for which the embedding operator \(E^{-1} \mathcal{H}(E) \to L^2(\mu)\) is unitary.

Consider the inner function \(\Theta = E^{+}/E\). Since \(2A/E = 1 + \Theta\) and \(2B/E = i(\Theta - 1)\), we have

\[
i \frac{1 - \Theta(z)}{1 + \Theta(z)} = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \sim i \frac{M}{y}, \quad z = iy, y \to \infty.
\]
Hence,
\begin{equation}
\frac{1 + \Theta(iy)}{1 - \Theta(iy)} \sim \frac{y}{M}, \quad y \to \infty.
\end{equation}

It is well known and easy to see that the function $\Theta$ may be reconstructed from the sets \{\(t_n\)\} = \{\(\Theta = 1\)\} and \{\(s_n\)\} = \{\(\Theta = -1\)\} by the formula

$$
\log \frac{\Theta + 1}{\Theta - 1} = c + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{t^2 + 1} \right) f(t) dt,
$$

where

$$
f(t) = \begin{cases} 
-1/2, & t \in (t_n, s_n), \\
1/2, & t \in (s_n, t_{n+1}), 
\end{cases}
$$

and $c \in \mathbb{R}$ (essentially, this is a very special case of the Krein spectral shift formula [12]). Then, by (4.10), we have

$$
\text{Re} \int_{\mathbb{R}} \left( \frac{1}{t-iy} - \frac{t}{t^2 + 1} \right) f(t) dt = \int_{\mathbb{R}} \frac{(1 - y^2)t}{(t^2 + y^2)(t^2 + 1)} f(t) dt = \log y + O(1), \quad y \to \infty.
$$

A direct computation shows, however, that

$$
\int_{\mathbb{R}} \frac{(y^2 - 1)|t|}{(t^2 + y^2)(t^2 + 1)} |f(t)| dt = \log y + O(1), \quad y \to \infty,
$$

whence

$$
\int_{\{t: tf(t) > 0\}} \frac{(y^2 - 1) tf(t)}{(t^2 + y^2)(t^2 + 1)} dt = O(1), \quad y \to \infty,
$$

and therefore

$$
\int_{\{t: tf(t) > 0\}} \frac{|tf(t)|}{t^2 + 1} dt < \infty.
$$

Since $tf(t) > 0$ for $t \in (s_n, t_{n+1})$, $s_n > 0$, or $t \in (t_n, s_n)$, $s_n < 0$, we have

$$
\sum_{s_n > 0} \int_{t_n}^{t_{n+1}} \frac{dt}{t} = \sum_{s_n > 0} \ln \frac{t_{n+1}}{s_n} < \infty, \quad \sum_{s_n < 0} \int_{t_n}^{s_n} \frac{dt}{|t|} = \sum_{s_n < 0} \ln \frac{|t_n|}{|s_n|} < \infty.
$$

The latter convergences are obviously equivalent to (4.9). \hfill \Box

As a corollary we immediately obtain a slightly refined version of Proposition 3.1. Moreover, if $t_n = n$, $n \in \mathbb{Z}$, $A(z) = \sin \pi z$, and $S = S_1S_2$ is the function arising from the possible one-dimensional defect in the Paley–Wiener space, then

$$
\sum_{s \in \mathbb{Z}_2} \frac{1}{|s|} < \infty.
$$
Indeed, the zero set $Z_2$ of the function $S_2$ does not depend on the choice of the basis, therefore applying Proposition 4.4 to $t_n = n$ and $t_n = n + \delta$ (e.g., $\delta = \frac{1}{2}$), $n \in \mathbb{Z}$, we obtain
\[
\sum_{s \in Z_2, s > 0} \frac{[s] + 1 - s}{s} < \infty, \quad \sum_{s \in Z_2, s > 0} \frac{[s - \delta] + 1 + \delta - s}{s} < \infty.
\]

Under natural regularity conditions, Proposition 4.4 implies the following closeness of the sequences $\{t_n\}$ and $\{s_n\}$.

**Corollary 4.5.** Let $A, B, \{t_n\}$ and $\{s_n\}$ be as in Proposition 4.4. Put $I_n = [t_n, t_{n+1}]$. Assume that $|I_k| \asymp |I_n|$, $n \leq k \leq 2n$, with the constants independent on $k, n$, and also that $|t_{2n}| \geq \rho |t_n|$ with some $\rho > 1$. Then for any $\delta > 0$ the set $\mathcal{N}$ of indices $n$ such that $t_n > 0$ and $t_{n+1} - s_n \geq \delta |I_n|$ (respectively, $t_n < 0$ and $s_n - t_n \geq \delta |I_n|$) has zero density.

**Proof.** Note that $|t_k| \asymp |t_n|$, $n \leq k \leq 2n$. If the upper density of $\mathcal{N}$ is positive, then there exists a sequence $M_j \to \infty$ such that
\[
\sum_{n \in [M_j, 2M_j] \cap \mathcal{N}} \frac{t_{n+1} - s_n}{s_n} \asymp \sum_{n \in [M_j, 2M_j] \cap \mathcal{N}} \frac{t_{n+1} - t_n}{t_n} \asymp \sum_{n \in [M_j, 2M_j]} \frac{t_{n+1} - t_n}{t_n} \gtrsim \log \frac{t_{2M_j}}{t_{M_j}} \geq \log \rho,
\]
and the first series in (4.9) diverges, a contradiction. \qed

5. **AN EXAMPLE OF A NONHEREDITARILY COMPLETE SYSTEM OF REPRODUCING KERNELS**

In this section we prove Theorem 1.4, i.e., we construct a de Branges space $\mathcal{H}(E)$ and a complete and minimal system of reproducing kernels $\{K_\lambda\}_{\lambda \in \Lambda}$ such that its biorthogonal system is also complete, but the system $\{K_\lambda\}_{\lambda \in \Lambda}$ is not hereditarily complete.

We have seen in the previous section that the existence of a non-hereditarily complete system of reproducing kernels generated by some function $G$ in the de Branges space $\mathcal{H}(E)$ is equivalent to the solvability of the equations
\[
\sum_n \frac{\sigma_n w_{1/2}^n}{z - t_n} = \frac{G_2(z) S_2(z)}{A(z)},
\]
(5.1) \[
\sum_{n} G(t_n) \frac{a_n \mu_{n}^{1/2}}{E(t_n)} \cdot \frac{i G_1(z) S_1(z)}{A(z)} \cdot \frac{a_n^{1/2}}{n} \cdot z - n = i G_1(z) S_1(z)
\]
for some nonzero \( \{a_n\} \in \ell^2 \) and some entire functions \( S_1 \) and \( S_2 \). If all the above objects are found, then \( h = G_2 S_2 \) is orthogonal to the corresponding system.

The corresponding equations will be constructed as small perturbations of an orthogonal expansion in a de Branges space with respect to a reproducing kernels basis.

We construct the space \( \mathcal{H}(E) \) and the functions \( G_1, G_2, S_1 \) and \( S_2 \) in the reverse order. Namely, we start with the construction of the function \( S \). Let \( a_n \in \mathbb{R} \) be such that \( |a_n| = |n|^{-3/4}, \quad n \neq 0 \) (the signs will be specified later on, the value of \( |a_0| > 0 \) is not important). Put

\[
\frac{S(z)}{\sin \pi z} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n^2}{z - n}.
\]

Then \( S \) has exactly one zero \( z_n \) in each interval \((n, n + 1)\) and, by Proposition 3.1, we have \(|z_n - (n + 1)| \to 0, n \to +\infty, \) with a possible exception of a set of zero density (in fact, using the explicit form of the masses \( a_n^2 \) one can show that the limit over all indices \( n \) is zero). We represent \( S \) as the product

\[
S = S_1 S_2 = T_0 T_1 S_2,
\]

where \( T_0 \) is a canonical product with the zeros \( s_n = z_{4^n} \) in intervals \((4^n, 4^n + 1)\) and \( S_2 \) is a canonical product with the zeros \( z_{4^n + 2^n} \) in \((4^n + 2^n, 4^n + 2^n + 1)\), \( n \geq 2 \). Next we construct \( h \). We will construct it as \( h = \tilde{T}_0 T_1 S_2 \) where \( \tilde{T} \) is a perturbation of the function \( T_0 \) such that

(5.2) \[
\frac{h(z)}{\sin \pi z} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{e_n |a_n|}{z - n},
\]

(5.3) \[
\sum_{n \in \mathbb{Z}} e_n^2 = \infty, \quad \sum_{n \neq 0} \frac{e_n^2}{n^2} < \infty.
\]

Condition (5.2) means that

\[
\frac{S(z)}{\sin \pi z} \cdot \tilde{T}_0(z) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \tilde{T}_0(n) \cdot \frac{a_n^2}{z - n},
\]

\[
\sum_{n \in \mathbb{Z}} \tilde{T}_0(n) \cdot \frac{a_n^2}{z - n}.
\]
and $c_n = |a_n|\tilde{T}_0(n)/T_0(n)$. Let us show that all these conditions may be satisfied. Without loss of generality assume that $|s_n - 4^n| > |s_n - (4^n + 1)|$. Then we shift each zero $s_n$ of $T_0$ from $(4^n, 4^n + 1)$ in the following way:

$$\tilde{s}_n = 4^n + 1 - (4^n)^{3/4} \text{ dist}(s_n, \mathbb{Z}) \rho_n,$$

$n \geq 2$.

We may choose $\rho_n \in (1, 2)$ such that $\text{dist}(\tilde{s}_n, \mathbb{Z} \setminus \{4^n + 1\}) > \frac{1}{10}$ and zero sets of $\tilde{T}_0$ and $T_1S_2$ do not intersect. It is easy to see that with such choice of zeros for $\tilde{T}_0$ for $x \in (4^n - 4^n - 1, 4^n + 4^n - 1)$ we have

$$\left| \frac{\tilde{T}_0(x)}{T_0(x)} \right| \lesssim \left| x - \tilde{s}_n \right|/\left| x - s_n \right|.$$ 

Then we obtain

$$|c_{4^n+1}| \lesssim \left| \frac{\tilde{T}_0(4^n + 1)}{T(4^n + 1)} \right| \cdot |a_{4^n+1}| \approx 1.$$ 

Moreover, it is easy to see that for any $n \in \mathbb{Z}, n \geq 0$, we have

$$(5.4) \quad |n|^{-1} \lesssim \left| \frac{T_0(n)}{T(n)} \right| \lesssim |n|^{3/4}.$$ 

Hence, $|c_n| \lesssim 1$ and, thus, (5.3) is satisfied. Moreover, $|\tilde{T}_0(iy)/T_0(iy)| \approx 1$, and so both terms in (5.2) tend to zero along $i\mathbb{R}$. We conclude that the interpolation formula holds.

Next we introduce a de Branges space $\mathcal{H}(E)$. Put $\mu_n = c_n^2$ and $\mu = \sum_{n \in \mathbb{Z}} \mu_n \delta_n$. By (5.3), $\int (1 + t^2)^{-1} d\mu(t) < \infty$, and we can define a meromorphic inner function $\Theta$ by the formula

$$\frac{1 + \Theta(z)}{1 - \Theta(z)} = \frac{1}{\pi i} \int \frac{1}{t - z} - \frac{t}{t^2 + 1} d\mu(t), \quad z \in \mathbb{C}_+.$$ 

Then $\Theta = E^* / E$ for some entire function $E$ in the Hermite–Biehler class. We may assume that $E$ is of finite exponential type and does not vanish on $\overline{\mathbb{C}_+}$. Moreover, since the zero set of $2A = E + E^*$ coincides with $\mathbb{Z}$, we may choose $E$ so that $A(z) = \sin \pi z$. Now, if we choose the signs of $a_n$ so that $\text{sign } a_n = \text{sign } c_n$, formula (5.2) becomes

$$\frac{h(z)}{\sin \pi z} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} a_n|c_n| \approx \frac{1}{\pi} \sum_{n \in \mathbb{Z}} a_n h_n^{1/2}.$$ 

Hence, $h \in \mathcal{H}(E)$.

We have $h = \tilde{T}_0T_1S_2$. Put $G_2 = \tilde{T}_0T_1$. Then $h = G_2S_2$ and it remains to construct $G_1$ so that $G$ is the generating function of a complete and minimal system and (5.1) is satisfied.
We will construct $G_1$ as a small perturbation of $S_2$ as we did above. We need to satisfy $G \notin \mathcal{H}(E)$, $G \in \mathcal{H}(E) + z\mathcal{H}(E)$ and (5.1) which is rewritten as

$$\frac{S(z)}{\sin \pi z} \cdot \frac{G_1(z)}{S_2(z)} = -i \sum_{n \in \mathbb{Z}} \frac{G_1(n)}{S_2(n)} \cdot \frac{h(n)}{E(n)} \cdot \frac{a_{n+1/2}}{z - n}.$$  

Note that in any de Branges space we have $iA'(t_n) = -E(t_n)\varphi'(t_n)$. Since in our case $A(z) = \sin \pi z$, we have $E(n) = -\pi i(-1)^n(\varphi'(n))^{-1} = -\pi i(-1)^n\mu_n$. Then (5.5) simplifies to

$$\frac{S(z)}{\sin \pi z} \cdot \frac{G_1(z)}{S_2(z)} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{G_1(n)}{S_2(n)} \cdot \frac{h(n)}{(-1)^n|c_n|} \cdot \frac{a_n}{z - n}.$$  

The residues, obviously, coincide.

Applying the above construction to $S_2$ in place of $T_0$ we construct $G_1$ (again we may assume that $G_1$ has no common zeros with $\bar{T}_0T_1$) so that

$$|n|^{-1} \lesssim \left| \frac{G_1(n)}{S_2(n)} \right| \lesssim |n|^{3/4}$$  

and

$$\left| \frac{G_1(4^n + 2^n + 1)}{S_2(4^n + 2^n + 1)} \right| \cdot |a_{4^n+2^n+1}| \asymp 1.$$  

Hence,

$$|G(4^n + 2^n + 1)| = \left| \frac{G_1(4^n + 2^n + 1)}{S_2(4^n + 2^n + 1)} \right| \cdot |a_{4^n+2^n+1}| \cdot |c_{4^n+2^n+1}| \asymp |c_{4^n+2^n+1}|.$$  

Hence, $\sum_{n \in \mathbb{Z}} |G(n)|^2|c_n|^{-2} = \infty$. Thus, $G \notin \mathcal{H}(E)$. However,

$$\sum_{n \neq 0} |G(n)|^2|n|^{-2}|c_n|^{-2} < \infty,$$

whence $\frac{G(z)}{(z-\lambda)E(z)} \in L^2(\mu)$ for the zeros $\lambda$ of $G$. Also $|G_1(iy)/S_2(iy)| \asymp 1$, so

$$\frac{|G(iy)|}{\sin \pi iy} \asymp \frac{|S(iy)|}{\sin \pi iy} \asymp |y|^{-1}, \quad |y| \to \infty,$$  

and by [5, Theorem 26], $G \in \mathcal{H}(E) + z\mathcal{H}(E)$. Estimate (5.6) also implies the interpolation formula (5.5).

It remains to show that $G$ is the generating function of a complete and minimal system of kernels such that its biorthogonal is also complete. The first follows from (5.6) since if $G \mathcal{H}(E)$ for some entire function $H$ of zero exponential type, then $H$ should be at most a polynomial, which contradicts to the fact that $G \notin \mathcal{H}(E)$. Note also that, by (5.4) $|c_n| \gtrsim |n|^{-4}|a_n|$, thus $\mu_n \gtrsim |n|^{-4}$ and also $\sum_{n \in \mathbb{Z}} \mu_n = \infty$. Then, by [2, Theorem 1.2], the system biorthogonal to
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\[ \{ k_\lambda : G(\lambda) = 0 \} \] is automatically complete. This completes the construction of the example (and, thus, the proof of Theorem 1.4).

REFERENCES

[1] A. Baranov, Differentiation in the Branges spaces and embedding theorems, *J. Math. Sci.* 101 (2000), 2881–2913.

[2] A. Baranov, Yu. Belov, Systems of reproducing kernels and their biorthogonal: completeness or incompleteness? *Int. Math. Res. Notices* (2011), 22, 5076–5108.

[3] Yu. Belov, T. Mengestie, K. Seip, Discrete Hilbert transforms on sparse sequences, *Proc. London Math. Soc.* 103 (2011), 3, 73–105.

[4] Yu. Belov, T. Mengestie, K. Seip, Unitary discrete Hilbert transforms, *J. Anal. Math.* 112 (2010), 383–395.

[5] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice–Hall, Englewood Cliffs, 1968.

[6] L. Dovbysh, N. Nikol’ski, Two methods for avoiding hereditary completeness, *Zap. Nauchn. Semin. LOMI* 65 (1976), 183–188; English transl.: *J. Soviet Math.* 16(3) (1981), 1175–1179.

[7] L. Dovbysh, N. Nikol’ski, V. Sudakov, How good can a nonhereditary family be? *Zap. Nauchn. Semin. LOMI* 73 (1977), 52–69; English transl.: *J. Soviet Math.* 34(6) (1986), 2050–2060.

[8] E. Frîcain, Complétude des noyaux reproduisants dans les espaces modèles, *Ann. Inst. Fourier (Grenoble)* 52(2) (2002), 661–686.

[9] G. Gubreev, A. Kovalenko, Tests for completeness of root subspaces of a differentiation operator with abstract boundary conditions, *Math. Notes* 30(4) (1981), 765–770.

[10] H. Hamburger, Über die Zerlegung des Hilbertschen Raumes durch vollstetige lineare Transformaionen, *Math. Nachr.* 4 (1951), 56–69.

[11] S. Khurschev, N. Nikol’ski, A function model and some problems in spectral function theory, *Trudy Math. Inst. Steklov* 176 (1987), 97–210; English transl. in *Proc. Steklov. Inst. Math.* 3 (1988), 101–214.

[12] M. Krein, On perturbation determinants and the trace formula for unitary and selfadjoint operators, *Sov. Math. Dokl.* 3 (1962), 707–710.

[13] A. Markus, The problem of spectral synthesis for operators with point spectrum, *Math. USSR-Izv.* 4(3) (1970), 670–696.

[14] N. Nikol’ski, Complete extensions of Volterra operators, *Math. USSR-Izv.* 3(6) (1969), 1271–1276.

[15] A. Volberg, S. Treil, Embedding theorems for invariant subspaces of the inverse shift operator, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 149 (1986), 38–51; English transl. in *J. Soviet Math.* 42 (1988), 1562–1572.

[16] R. Young, On complete biorthogonal system, *Proc. Amer. Math. Soc.* 83(3) (1981), 537–540.
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