Chaotic motion of time-delay fractional order financial dynamic system of single sliding mode control

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Abstract. In this paper, the equations of time-delay fractional order financial dynamic system are studied. Firstly, the dynamic characteristics of the time-delay fractional order financial system with are analyzed. Secondly, the attractor state change of the time-delay fractional order financial system induced by disturbance is studied. By adding specific uncertainties and disturbances to the specified state variables of the system, the attractor of the system is induced to change. Finally, the classical integral sliding mode control method is used to design a single controller to control the time-delay fractional order financial system from chaos to fixed point, and the feasibility and robustness of the control scheme have been proved theoretically and verified by the simulation.

1. Introduction
Fractional order calculus and integral order calculus have almost the same history of development. It is found that integral order calculus is a special case of fractional order calculus, and integral order chaotic systems are idealized treatments of actual chaotic systems [1-3]. If fractional order differential operators are introduced into chaotic systems, fractional order chaotic systems can produce more complex dynamic behaviors, with strong randomness and unpredictability. At the same time, that also promotes the application research of fractional order chaotic systems and the development of fractional order calculus theory [1-5].

Compared with ordinary differential equations, time-delay differential equations have more complex dynamic characteristics. For example, chaotic solutions of delay differential equations are easier to obtain. In literature [6], it is pointed out that for a single free oscillator under periodic excitation, the time-delay vibration absorber with the application of time-delay displacement feedback, time-delay velocity feedback and time-delay acceleration feedback can achieve the purpose of complete vibration absorption [6]. At present, the research on time-delay dynamic system is mostly still focused on the integral order dynamic system [7-8], and there are few reports about the research on the time-delay fractional order dynamic system. The combination of time-delay factors and fractional order calculus is bound to make the dynamic characteristics of the system more complex. The research on time-delay fractional order dynamic system is in the initial stage [9]. In literature [10], the time delay is introduced into the fractional order Liu system, and it is found that the time delay makes the dynamic characteristics of the fractional order Liu system get qualitative change. The fractional order Liu system can transfer between the fixed point attractor, the periodic attractor and the
chaotic attractor by changing the size of the time delay in a certain range. When the system fixes the
time delay and uses the fractional order as the control parameter, the similar phenomenon can also
come. Some literature studies show that the chaotic phenomenon in financial system can make the
economic operation unpredictable, which will greatly limit the macro-control behaviors of the state
and also threaten the safety of individual investment. In literature [11], the fractional order financial
system is taken as an example, and the stability problem of fractional order financial system is
analyzed through the stability theory of fractional order linear system. On that basis, based on the
financial system’s starting from the practical significance of chaos control, it is of certain theoretical
value to study the control of the chaotic motion state of time-delay fractional order financial system to
the stable fixed point [12].

Therefore, the dynamic characteristics of the time-delay fractional order financial system are firstly
analyzed in this paper. On that basis, the attractor change of the time-delay fractional order financial
system induced by disturbances is studied. Finally, a single sliding mode controller is designed to
realize the control of the time-delay fractional order chaotic system when the target variable and the
sub-time-delay systems are asymptotically stable at the zero point.

2. Dynamic characteristics analysis of time-delay fractional order financial system

On the basis of the traditional fractional order financial system, time delay is added to obtain a new
time-delay fractional order system [11], and that system can be described as follows:

\[
\begin{align*}
D_\alpha^0 x(t) &= z + (y(t - \tau) - a)x, \\
D_\alpha^0 y(t) &= 1 - by(t) - x^2(t - \tau), \\
D_\alpha^0 z(t) &= -x(t - \tau) - cz,
\end{align*}
\]

Among them, x, y and z are state variables, which respectively represent interest rate, input and
price index, a, b and c are system parameters, which respectively are (3, 0.1 and 1), \(\tau\) (\(\tau > 0\)) is the time
delay, \(\alpha_i\) (i = 1, 2, 3) is the fractional order. Research shows that, when \(\tau\) or \(\alpha_i\) takes different values,
the attractor can be induced to change and produce complex dynamic behaviors. In this paper, the
dynamic characteristics of time-delay fractional order nonlinear dynamic system are analyzed by
fractional order differential algorithm and solving the maximum Lyapunov index [13] in the case of
\(\alpha_1=0.93, \alpha_2=0.97, \alpha_3=0.95\) and \(\tau=0.06\), the phase diagram obtained is shown in Fig. 1, and the \(\lambda_{\text{max}}\)
of the maximum Lyapunov index is equal to 0.0714187, which indicates that the system (1) is in
chaotic state under the above parameters.

![Fig. 1. Chaotic attractor of time-delay fractional order financial system](image)

3. Attractor change of time-delay fractional order financial system induced by disturbance

Research found that, under the condition of the above parameters being unchanged, adding specific
uncertainties and disturbances to the state variables of the system (1) could induce the attractor of the
system (1) to change, thus making the system change from chaotic motion to periodic motion. That
process can be described as follows:
\[ D_t^\alpha x(t) = z + (y(t - \tau) - a)x + \Delta f(x, y, z) + d(t), \]
\[ D_t^\alpha y(t) = 1 - by(t) - x^2(t - \tau), \]
\[ D_t^\alpha z(t) = -x(t - \tau) - cz, \]

(2)

Among them, \( \Delta f(x, y, z) = 0.5 \sin(\pi x) \cos(\pi y) \sin(2\pi z) \), \( d(t) = 0.1 \cos(2t) \), and the numerical simulation results of system (2) are shown in Fig. 2-4.

Fig.2. Evolution curve of the state variables over time

Fig.3. Partial enlarged drawing of evolution curve

Fig. 4. Phase diagram of attractor

4. Chaotic motion of time-delay fractional order financial system of single sliding mode control

In this paper, a single sliding mode controller is designed to realize the chaos control of time-delay fractional order chaotic system. The sliding mode surface is constructed and the controller is designed by the target variable, and the system variables are analyzed to find a target variable, which will make the sub-time-delay system asymptotically stable at the zero point when that target variable is zero. The following control methods are provided:

\[ D_t^\alpha x(t) = z + (y(t - \tau) - a)x, \]
\[ D_t^\alpha y(t) = 1 - by(t) - x^2(t - \tau) + u, \]
\[ D_t^\alpha z(t) = -x(t - \tau) - cz, \]

(3)

The constructed sliding mode surface is as follows:

\[ s = D_t^{\alpha - 1} y(t) + \int_0^t by(\tau)d\tau \]

(4)
The derivation of equation (4) is as follows:

\[ s = D_t^{\alpha_2} y(t) + by(t) \]  

(5)

In the sliding mode motion, the sliding mode surface and the rules of sliding mode motion shall meet the following:

\[ s = 0, \dot{s} = 0 \]  

(6)

Combine the equations (3), (5) and (6) to obtain the following:

\[ \dot{s} = D_t^{\alpha_2} y(t) + by(t) = 1 - by(t) - x^2(t - \tau) + by + u_{eq} = 0 \]  

(7)

The equivalent control rate derivated from the above equation is as follows:

\[ u_{eq} = -1 + x^2(t - \tau) \]

The switching rate of sliding mode is designed as follows:

\[ u_r = k|s| \quad (k < 0) \]

By combining the above two parts, a complete sliding mode controller can be obtained as follows:

\[ u = u_{eq} + u_r = -1 + x^2(t - \tau) + k|s| \]  

(8)

In order to illustrate the feasibility of the provided control method, detailed theoretical proof is offered here. The whole proof process is divided into two steps. The first step is to construct Lyapunov function of \( V = \frac{s^2}{2} \) by the provided integral sliding mode control method to prove that the sliding mode motion can reach the sliding mode surface and the target variable can be controlled to zero. The second step: Based on the first step, it is further proved that the sub-time-delay system can evolve over time to be asymptotically stable to the zero point.

The first step:

The derivation of Lyapunov function of \( V = \frac{1}{2} s^2 \) is as follows:

\[ \dot{V} = ss \]

\[ = s(D_t^{\alpha_2} y(t) + by(t)) \]

\[ = s(1 - by(t) - x^2(t - \tau) + u + by(t)) \]

\[ = s(k|s|) \]

\[ = k|s| < 0 \]  

(9)

That theoretical proof shows that the control method in this paper satisfies the sliding mode reaching condition. Combined with equations (5) and (6), the sliding mode dynamic expression can be obtained as follows:

\[ D_t^{\alpha_2} y(t) = -by(t) \]  

(10)

According to the stability theory, the system (10) is asymptotically stable in the case of \( b > 0 \), and when \( y(t) \) converges to zero, the evolution equation of the original dynamic system can be simplified as follows:
\[ D_t^\alpha x(t) = z - ax, \]
\[ D_t^\alpha z(t) = -x(t - \tau) - cz, \]
\[ (11) \]

Thus, it will be proved that the effectiveness of the controller can be further transformed into solving the problem of global asymptotic stability at the zero point in the equation (11).

The second step:
To prove that the sub-time-delay system (11) of time-delay fractional order financial system is asymptotically stable at the zero point, so the stability theory of time-delay fractional order system in Chapter 2 is quoted here to carry out the theoretical proof.

Theorem 3: If \( \alpha_i \in (0, 1), (i = 1 \sim n) \), M is the least common multiple of \( u_i, (u_i, v_i) = 1, u_i, v_i \in z^+ \). If all characteristic values of the system (9) meet \( |\arg(\lambda)| > \pi / 2M \), and there is no pure imaginary root for any \( \tau(t > 0) \) characteristic equation of \( \det(\Delta(s)) = 0 \), the zero solution of the system (11) is globally asymptotically stable for Lyapunov.

The Jacobian matrix of equation (11) at \((0, 0)\) is as follows:
\[ J = \begin{pmatrix} -a & 1 \\ -1 & -c \end{pmatrix} \]

The characteristic equation of Jacobian matrix is as follows:
\[ \lambda^2 + 4\lambda + 4 = 0 \]

The solution is obtained as follows:
\[ \lambda_1 = \lambda_2 = -2 < 0 \]

After laplace transformation on both ends of the equation (11), the following equations can be obtained:
\[ s^{u_i} X(s) = Z(s) - aX(s) \]
\[ s^{v_i} Z(s) = -X(s)e^{-\tau} - cZ(s) \]
\[ (12) \]

Among them, \( X(s) = Lx(t), Z(s) = Lz(t) \), separate coefficient matrix to obtain characteristic value as follows:
\[ \det(\Delta(s)) = \begin{vmatrix} s^{u_i} + a & -1 \\ e^{-\tau} & s^{v_i} + c \end{vmatrix} \]

Let \( \det(\Delta(s)) = 0 \) and obtain the following:
\[ s^{u_i + v_i} + as^{v_i} + cs^{u_i} + e^{-\tau t} = 0 \]
\[ (13) \]

Hereby, assume that \( s = wi = |w| (\cos \frac{\pi}{2} + i \sin (\pm \frac{\pi}{2})) \) is the root of the above characteristic polynomial, and substitute the assumption into equation (13) to obtain the following:
\[ |w|^{\alpha_i + \alpha_3} (\cos \frac{(\alpha_1 + \alpha_3)\pi}{2} + i\sin(\pm \frac{(\alpha_1 + \alpha_3)\pi}{2})) + a|w|^{\alpha_1} (\cos \frac{\alpha_1\pi}{2} + i\sin(\pm \frac{\alpha_1\pi}{2})) + c|w|^{\alpha_3} (\cos \frac{\alpha_3\pi}{2} + i\sin(\pm \frac{\alpha_3\pi}{2})) + \cos wt - i\sin wt = 0 \]
\[ (14) \]

Separate the imaginary part and the real part to obtain the following:
Combine the equation (15) and the equation (16) to obtain the following:

\[
|w|^{\alpha_1 + \alpha_3} \cos \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \right) + a|w|^{\alpha_3} \cos \left( \frac{\alpha_3\pi}{2} \right) + c|w|^{\alpha_5} \cos \left( \frac{\alpha_5\pi}{2} \right) = -\cos wt \tag{15}
\]

\[
|w|^{\alpha_1 + \alpha_3} \sin \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \right) + a|w|^{\alpha_3} \sin \left( \frac{\alpha_3\pi}{2} \right) + c|w|^{\alpha_5} \sin \left( \frac{\alpha_5\pi}{2} \right) = \sin wt \tag{16}
\]

The above equation can be simplified as follows:

\[
\left( |w|^{\alpha_1 + \alpha_3} \cos \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \right) + a|w|^{\alpha_3} \cos \left( \frac{\alpha_3\pi}{2} \right) + c|w|^{\alpha_5} \cos \left( \frac{\alpha_5\pi}{2} \right) \right)^2 +
\]

\[
\left( |w|^{\alpha_1 + \alpha_3} \sin \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \right) + a|w|^{\alpha_3} \sin \left( \frac{\alpha_3\pi}{2} \right) + c|w|^{\alpha_5} \sin \left( \frac{\alpha_5\pi}{2} \right) \right)^2 = 1 \tag{17}
\]

The above equation can be simplified as follows:

\[
|w|^{2(\alpha_1 + \alpha_3)} + k_1|w|^{(2\alpha_1 + \alpha_3)} + k_2|w|^{2\alpha_3} + k_3|w|^{2\alpha_5} + k_4|w|^{\alpha_1 + \alpha_3} + k_5|w|^{\alpha_3} + k_6|w|^{\alpha_5} + k_7|w|^{\alpha_1} = 0 \tag{18}
\]

Among them:

\[
k_1 = 2a(\sin \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \right) \sin \frac{\alpha_3\pi}{2} + \cos \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \cos \frac{\alpha_3\pi}{2} \right))
\]

\[
k_2 = 2c(\cos \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \right) \cos \frac{\alpha_3\pi}{2} + \sin \left( \frac{(\alpha_1 + \alpha_3)\pi}{2} \sin \frac{\alpha_3\pi}{2} \right))
\]

\[
k_3 = a^2
\]

\[
k_4 = 2ac(\sin \left( \frac{\alpha_3\pi}{2} \right) \sin \frac{\alpha_5\pi}{2} + \cos \left( \frac{\alpha_3\pi}{2} \cos \frac{\alpha_5\pi}{2} \right) + \cos \frac{\alpha_5\pi}{2} \cos \frac{\alpha_3\pi}{2})
\]

\[
k_5 = c^2
\]

\[
k_6 = 2a^2c \cos \left( \frac{\alpha_3\pi}{2} \right)
\]

\[
k_7 = 2ac^2 \cos \left( \frac{\alpha_3\pi}{2} \right)
\]

After calculation, when \( \alpha_1 = 0.93 \) and \( \alpha_3 = 0.95 \), \([k_1, k_2, k_3, k_4, k_5, k_6, k_7]\) are equal to \([0.219469, 0.282639, 1, 0.06203, 9, 0.282639, 1.975218]\) respectively. By substituting the calculation results into the equation (18), it can be obtained that there is no real root in the equation (18), which indicates that the assumption is not tenable and means that there is no pure imaginary root in \( \det(\Delta(s)) = 0 \) under the fractional order. The above results of derivation meet the requirements of the theorem, so the theoretical proof of the stability of time-delay fractional order system has been realized, and the whole proof process has proved the feasibility of the provided control method from a theoretical point of view. Hereby, the control method in the paper is further verified by numerical simulation. The step length is \( h = 0.01 \), and the initial values are \([0.1, 4, 0.5]\). The simulation results are shown in Fig. 5 and
Fig. 6 respectively. It can be seen in figures that the state variables of the system gradually converge to zero, and the sliding mode motion can reach and finally stay on the sliding mode surface. The simulation results have verified the effectiveness of the control method in the paper.

In order to further study the effectiveness of the controller designed in the paper when the system is affected by external disturbances, the bounded uncertainties and external disturbances in system (2) are introduced into the system at the same time to test the robustness of the control method. After introducing disturbance, the control system is as follows:

\[
\begin{align*}
D_t^\alpha x(t) &= z + (y(t - \tau) - a)x, \\
D_t^\alpha y(t) &= 1 - by(t) - x^2(t - \tau) + \Delta f(x, y, z) + d(t) + u, \\
D_t^\alpha z(t) &= -x(t - \tau) - cz, 
\end{align*}
\]  

When the system parameters, time delay and fractional order values remain unchanged, research found that the system (19) could still be stabilized to a fixed point by the above control method. The theoretical proof is similar to that of the undisturbed part, so the proof is omitted. The numerical simulation was carried out in the control scheme, and the simulation results are shown in Fig. 7. It can be seen in the figure that the state variables gradually converge to zero, which indicates that the system is controlled to the fixed point of zero.

5. Conclusion

The time delay is ubiquitous and inevitable in the real world. The time-delay dynamic system has complex dynamic behaviors which are often used to describe some engineering phenomena with time delay. It becomes unpredictable that the chaotic phenomenon of financial system reflects the economic operation, which will greatly limit the macro-control behaviors of the state and also threaten the safety of individual investment. Therefore, the dynamic characteristics of time-delay fractional order
financial system are analyzed in this paper. On that basis, the attractor change of time-delay fractional order financial system induced by disturbance is studied. Finally, a single sliding mode controller is designed to realize the chaos control of time-delay fractional order financial system when the target variables of system and the sub-time-delay systems are asymptotically stable at the zero point. Thus, it is of great theoretical significance to study the control of the chaotic motion of time-delay fractional order financial dynamic system.

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