Rationality of Almost Simple Algebraic Groups*

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Abstract

We prove the stable rationality of almost simple algebraic groups,
the connected components of the Dynkin diagram of anisotropic kernel
of which contain at most two vertices. The (stable) rationality of many
isotropic almost simple groups with small anisotropic kernel and some
related results over $p$-adic and arbitrary fields are discussed.

Introduction

Let $G$ be a linear algebraic group defined over a field $k$. The well-known
results of Chevalley and Cartier showed that if $k$ is an algebraically closed
field and $G$ is connected and reductive then $G$ is rational over $k$ as $k$-variety,
i.e., the field $k(G)$ of rational functions defined over $k$ of $G$ is a pure transcendental
extension of $k$. However this is no longer true if $k$ is not algebraically
closed and one of basic geometric problems of algebraic groups over non-
algebraically closed fields is the rationality problem. A milder notion of sta-
bile rationality (and unirationality) is in sequence : An irreducible $k$-variety
$X$ is $k$-stably rational (resp. $k$-unirational) if there is an affine $k$-space $A$
such that $X \times A$ is $k$-birationally equivalent to an affine $k$-space (resp. such
that there is a surjective $k$-morphism $A \to X$). In general, it is difficult to

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verify if a given $k$-group (or $k$-variety) is rational (or irrational). We refer the readers to [Ch], [CT], [MT], [M1-2], [P], [V], [VK] and references thereof for various problems and progress related with the rationality problem.

Up to now there is no general criterion to decide which almost simple groups are stably rational over the field of definition by looking at their Dynkin diagram, except the trivial cases of split and quasi-split groups. Quite recently it became known that for many division algebras of degree 4 (or greater) over $k$, there are many examples of $k$-groups, isotropic or not, related with them which are not stably rational over $k$. The purpose of this note is to show that this result can be used to get such a general criterion. In particular we show that many almost simple groups with relatively big $k$-rank and the degree of the related division algebra is $\leq 3$ are $k$-stably rational. Hence in certain sense, our results are optimal. More precisely the main result of the paper is the following.

**Theorem.** Let $G$ be an almost simple algebraic group over a field $k$. Let $m(G)$ be the maximal number of vertices of connected components of the Dynkin diagram of anisotropic kernel of $G$.

a) If $m(G) \leq 2$, then $G$ is rational or stably rational over $k$.

b) The number 2 in a) is best possible. For any natural number $n$, there exist non stably rational groups $G$ and fields $k$ with $m(G) = 4n − 1$.

In particular over certain "nice" fields, such as local ($p$-adic or real) fields, many isotropic almost simple $k$-group are $k$-(stably) rational. The method of the proof is based on a detailed analyis of the Tits index of the groups under consideration.

**Notation.** For an almost simple group $G$ defined over a field $k$ which has characteristic either 0 or relatively big, e.g. relatively prime with the order of the center $\text{Cent}(G)$ of $G$. Let $S$ be a maximal $k$-split torus of $G$, $T$ a maximal $k$-torus of $G$ containing $S$. If $\dim T = n$, we denote by

$$\Delta = \{\alpha_1, \ldots, \alpha_n\}$$

a basis of simple roots for the root system $\Phi$ of $G$ with respect to $T$. We may consider the relative root system $k\Phi$ of $G$ relatively to $S$ and let $k\Delta$ a basis of $k\Phi$ compatible with $\Delta$. For $1 \leq i \leq n$ we denote by $S_i$ the standard $k$-split torus corresponding to the root $\alpha_i$. We denote by $x_{\alpha}(t)$ the multiplicative one-parameter unipotent subgroup (resp. $h_{\alpha}(t)$ the multiplicative
one-parameter diagonal subgroup) of $G$ corresponding to a root $\alpha \in \Delta$ where we keep the same notation used in [St]. For $\alpha = \alpha_i$ we denote $x_i(t) = x_{\alpha_i}(t)$, $h_i(t) = h_{\alpha_i}(t)$, $1 \leq i \leq n$ and $X_i$ the image of $x_i$ in $G$. In particular, if $G$ is simply connected then $T$ is the direct product of the images of $h_i := h_{\alpha_i}$, $1 \leq i \leq n$. We use intensively the notion and results of Tits’ classification theory of almost simple algebraic groups as presented in [Ti1] and refer also to [BT] for other notions in algebraic groups. We often identify a simple root with the vertex representing it in the Tits index.

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1 Some general useful facts

1.1. Let $G$ be a connected reductive $k$-group, $S$ a maximal $k$-split torus of $G$. The Bruhat decomposition for $G$ (see [BT, Section 4]) implies that

$$G \simeq Z_G(S) \times A$$

as varieties, where $A$ is an affine space defined over $k$. Thus the study of rationality of $G$ is reduced, in certain sense, to that of $Z_G(S)$. Namely $G$ is stably rational over $k$ if and only if $Z_G(S)$ is and if $Z_G(S)$ is $k$-rational, then so is $G$. However in certain cases the group $Z_G(S)$ is hard to handle with and we are forced to find a substitute, which can be studied easier. In many cases it is possible to do so. Namely let $S_0$ be a nontrivial $k$-subtorus of $S$. Another version of Bruhat decomposition says that

$$G \simeq Z_G(S_0) \times A,$$

the direct product of $Z_G(S_0)$ with an affine space $A$ over $k$.

Therefore we are reduced to studying the connected reductive $k$-groups $Z_G(S_0)$. The problem here is to choose a "nice" torus $S_0$ so that we can prove the rationality or stable rationality of $Z_G(S_0)$, which is possible if the $k$-rank is relatively big. First we need the following simple but very useful observation.

1.2. Proposition. [DT] Let $S_0$ be a standard $k$-split torus of $G$ and
$Z_G(S_\theta) = S_\theta T_0 H$ (almost direct product), where $T_0$ is a $k$-torus, $H$ a semisimple $k$-subgroup of $G$. Then the Tits index of $H$ is obtained from that of $G$ by removing all vertices not belonging to the preimage $\tilde{\theta}$ of $\theta \cup \{0\}$ under the restriction map $\Delta \to k\Delta \cup \{0\}$. Moreover $T_0$ is anisotropic and $ST_0 = (Z_T(H))^0$.

**Remark.** The equality in the last statement is not in [DT] but it is clear by comparing the dimension of both side and by making use of the previous part of the proposition. In particular it shows that if $Z_T(H)$ is connected then $ST_0$ contains the center of $H$.

Another interesting remarks are the following observations due to Tits. (See [Ti2] and [Se].) In fact, $S$ can be replaced by a standard $k$-split torus, but we do not need this here.

**1.3. Proposition.** Let $G$ be an almost simple adjoint $k$-group with a maximal $k$-split torus $S$. Then the center of $Z_G(S)$ is connected if $G$ is either adjoint or simply connected.

**1.4. Proposition.** Let $G$ be an almost simple $k$-group, $S_\theta$ a standard $k$-split torus of $G$. Let

$$Z_G(S_\theta) = S_\theta T_0 H.$$  

Then $S_\theta T_0$ is a quasi-split $k$-torus hence also cohomologically trivial.

The following result essentially is due to Ono - Rosenlicht (see [O]).

**1.5. Proposition.** If $T'$ is a central $k$-torus of a connected reductive $k$-group $G$, which is quasi-split torus over $k$, then there is a rational $k$-cross section $G/T' \to G$. In particular the $k$-variety $G$ is birationally equivalent to the product $T' \times (G/T')$.

¿From above we see that it is essential to know the group $Z_G(S)/(ST_0)$ (which we call the *semisimple anisotropic quotient* of $G$) if we want to know the rationality property of $G$. In the next section we examine various computations of this group. The following remark is useful in the sequel.

**1.6. Remark.** If $\pi : G \to G'$ is a central $k$-isogeny and $S' = \pi(S)$ the
image of maximal $k$-split torus of $G$, then $\pi$ induces a central isogeny

$$Z_G(S)/S \to Z_{G'}(S')/S'$$

of semisimple anisotropic quotients.

2 Some computations related with centralizers of split tori

2.1. We keep the above notation and we assume that $G$ is an almost simple $k$-group. Let $P_i$, $1 \leq i \leq s$, be $k$-groups with $F_i$ a central $k$-subgroup of $P_i$. Assume that all $F_i$ are $k$-isomorphic. By a suitable factoring out a central $k$-subgroup of the direct product of $P_i$ we will obtain an almost direct product $P'_1 \ldots P'_s$ with the property that the set-theoretic intersections $P'_i \cap P'_j$ are all equal and $k$-isomorphic to $F_i$. We call such a group the product of $P_i$ with glued central subgroups $F_i$.

2.2. Proposition. Let $G$ be an almost simple $k$-group of type $\Lambda_n$ with $k$-rank $r > 0$.

a) If $G$ is an inner form, the anisotropic semisimple quotient $Z_G(S)/S$ is the product of $k$-conjugate almost simple anisotropic $k$-groups of type $^1\Lambda_{d-1}$ with glued center.

b) If $G$ is an outer form, $Z_G(S)/ST_0$ is the product of anisotropic groups of type $^1\Lambda_{d-1}$ with an anisotropic $k$-group of type $^2\Lambda_{n-2rd}$ with glued central subgroup of order dividing $d$.

Proof. a) First we begin with simply connected isotropic groups of type $^1\Lambda_n$. From [Ti1] we know that such groups have the following Tits index

$$\bullet^1 - - \ldots - - \bullet^{d-1} - - \odot^d - - \bullet - - \ldots - - \bullet^{rd-1} - - \odot^{rd} - - \bullet - - \ldots - - \bullet^n$$

Let $S_i$ be the standard $k$-split torus corresponding to the isotropic vertex
$i, \ i = d, 2d, \ldots, rd$, where $d$ is the index of the division $k$-algebra $D$ related with this type and $r$ denotes the $k$-rank of $G$. Then

$$S = \prod_i S_i$$

is a maximal $k$-split torus of $G$. We have $Z_G(S) = SH$, where $H = \prod H_j$ is a semisimple $k$-group which is an almost direct product of anisotropic $k$-groups $H_j$ of type $1A_{d-1}$ (see Proposition 1.2).

We show that all these groups $H_j$ are in fact $k$-isomorphic to the simply connected almost simple $k$-group $G_0$ of type $1A_{d-1}$ defined by $G_0(k) = SL_1(D)$. Indeed, we may assume that in certain basis, the maximal $k$-split torus $S(k)$ consists of all diagonal matrices from $GL_n(D)$ with coefficients from $k^*$ and of determinant 1:

$$S(k) = \{\text{diag}(t_1, \ldots, t_n) : t_1 \cdots t_n = 1, 1 \leq i \leq n\},$$

thus

$$Z_G(S)(k) = S(k)\{\text{diag}(d_1, \ldots, d_n) : d_i \in SL_1(D), 1 \leq i \leq n\}.$$ 

It follows that all the groups $H_j$ above are $k$-conjugate (i.e. conjugate by elements from $G(k)$). Therefore if $G'$ is a quotient of $G$ by a central $k$-subgroup then we can form the corresponding centralizer of a maximal $k$-split torus $S'$ of $G'$, which is an almost direct product of $S'$ and $k$-conjugate almost simple $k$-groups $H'_j$ of type $1A_{d-1}$; and they are the homomorphic image of the simply connected almost simple $k$-groups $\tilde{H}_j$ of type $1A_{d-1}$ such that $\tilde{H}_j(k) \simeq SL_1(D)$. Some tedious computations shows that $S$ contains the products $z_i z_j$ of two generators $z_i$ and $z_j$ of centers of the groups $H_i$ and $H_j$, respectively. Since this and 1.6, the assertion a) follows.

b) Let $l$ be the separable quadratic extension of $k$ over which $G$ becomes of inner type. Assume first that $G$ is simply connected. Since [Ti1] we know that the Tits index of $G$ is as follows
Denote by $G(\Psi)$ the semisimple regular subgroup of $G$ generated by the root subgroups $X_i, \alpha_i \in \Psi$, where $\Psi$ is a subset of $\Delta$. By Proposition 1.2 in order to compute the intersection $ST_0 \cap H$ we are reduced to computing the intersection of the torus 

$$S_dS_{n-d+1}S_{2d}S_{n-2d+1} \cdots S_{rd}S_{n-rd+1}$$

with the semisimple subgroup 

$$G'_1 \cdots G'_r A,$$

where $G'_i, 1 \leq i \leq r$ is the semisimple $k$-group $G(\Psi_i)$ with the root system generated by the basis 

$$\Psi_i = \{\alpha_{(i-1)d+1}, \ldots, \alpha_{id-1}, \alpha_{n-(i-1)d}, \ldots, \alpha_{n-id+2}\},$$

and $A$ is the group $G(\alpha_{rd+1}, \ldots, \alpha_{n-rd})$. Then as in the part a) the part b) follows. The general case also follows from this as in a) by making use of 1.6. 

The case $d \leq 3$ is of special interest to us due to the following results.

2.3. Proposition. a) Let $G$ be an almost simple $k$-group of type $2A_{n,r}^{(1)}$ (i.e. $d=1$). With above notation, $ST_0$ contains the center of $H$.
b) If $G$ is above and $n$ is even then $G$ is $k$-rational.
c) If $G$ is as above and $n$ is odd, then any almost simple $k$-groups which is isogeneous to $G$ is also $k$-birationally isomorphic (as varieties) to $G$.

Proof. a) We assume first that $G$ is simply connected. The general case follows from this since if $\pi : G \rightarrow G'$ is a central $k$-isogeny, then $S' := \pi(S)$ is a maximal $k$-split torus of $G'$ and $\pi(Z_G(S)) = Z_{G'}(S')$ and we use 1.6. For
simplicity we assume that $r = 1$ and we give a complete computation in this case. From Proposition 1.2 it follows that we have only to check that

$$\text{Cent}(H) \subset S_1 S_d,$$

(1)

where $H = G(\alpha_2, \ldots, \alpha_{n-1})$. From above we see that for an element $t \in T$,

$$t = \prod_{1 \leq i \leq n} h_i(t_i)$$

is in $Z_T(H)$ if and only if $t$ commutes with all one-parameter unipotent subgroups $X_i$ for $2 \leq i \leq n-1$. Hence we have the following system of equations for $t_i$:

$$
\begin{align*}
  t_2^2 &= t_1 t_3, \\
  t_3^3 &= t_2 t_4,
\end{align*}
$$

One checks that

$$t_i = t_{i-1}^{i-1}/t_i^{-2}, 3 \leq i \leq n,$$

while the center of $H$ is generated by

$$h_2(\zeta) \cdots h_{n-1}(\zeta^{n-2}),$$

where $\zeta$ is a primitive $(n - 1)$-root of unity. Hence (1) is verified and a) follows.

b) Note that (with notation as above), the isogeny $\pi$ induces an isogeny (denoted by the same symbol)

$$\pi : Z_G(S)/ST_0 \to Z_G(S'T_0')/S'T_0'$$

between the anisotropic semisimple quotients. Now by a) $ST_0$ contains the center of $H$, the corresponding anisotropic semisimple quotient is an adjoint group, hence the above induced isogeny is in fact a $k$-isomorphism.

Now b) follows from a), results of Section 1 and a result of [VK] that any
adjoint $k$-group of type $A_{2m}$, $m \geq 1$, is $k$-rational. From Section 1 it follows that

$$Z_G(S) \simeq ST_0 \times (Z_G(S)/(ST_0)),$$

and from $ST_0 \simeq ST'_0$ it follows that $G$ and $G'$ are birationally isomorphic over $k$ hence c). □

From 2.2 we derive the following result regarding $d \leq 3$.

**2.4. Proposition.** Let $k$ be a field with a division algebra $D$ of degree $d \leq 3$, $G_0$ an anisotropic semisimple $k$-group which is an almost direct product of $k$-groups of type $A_{d-1}$ isogeneous over $k$ to either $\text{SL}_{1,D}$ or $\text{PGL}_{1,D}$ such that any simply connected factor contains the center of the other. Then $G$ is stably rational over $k$. In particular, if $k$ has a unique up to isomorphism quaternion division algebra (e.g. $k$ is a local field), then any such almost direct product of anisotropic $k$-groups of type $A_1$ is stably rational over $k$.

**Proof.** Since $\text{PGL}_{1,D}$ has no center and is $k$-rational, we may assume that all almost simple factors of $G$ are isomorphic to $\text{SL}_{1,D}$ and they have common center (i.e. product of groups of type $A_{d-1}$ with glued center). Let the number of almost simple components of $G$ be $r$. Then from 2.2 we see that for the group $G_1$ with $G_1(k) = \text{SL}_{r+1}(D)$ and $S$ a maximal $k$-split torus of $G_1$ we have

$$Z_{G_1}(S) = SG_2,$$

and

$$Z_{G_1}(S)/S \simeq G.$$

Since the group $\text{SL}_{1,D}$ is $k$-rational by assumption on $d$, the group $\text{SL}_{n,D}$ is also for any $n$ (see [M1], [V]), and it follows that $G$ is stably rational over $k$. □

**2.5. Remarks.** a) Another close formulation of the proposition is as follows. Let $G$ be an almost simple $k$-group with the following Tits index

$$
\begin{array}{cccccccccc}
\bullet & - & - & \circ & - & - & \bullet & - & - & \circ & \cdots & \bullet & - & - & \circ & - & - & \circ .
\end{array}
$$


i.e. of type $^1\text{A}_{2r+1,r}$, or the following
\[ \cdot - - \cdot - - \circ - - \cdot - - \cdots \cdot - - \circ - - \cdot - - \cdot, \]
i.e. of type $^1\text{A}_{3r+2,r}$. Then $G$ is stably rational over $k$, and it is rational over $k$ if it is adjoint.

Proof. Only the case of adjoint groups needs a proof. But in this case we make use of Proposition 1.3 and notice that adjoint groups of type $\text{A}_s$, $s \leq 2$ are rational.

b) It is not clear if $G$ (above) is always rational, though in general the cancelation of rational varieties does not hold.

c) The well-known examples of non-rational almost simple groups are related with certain division algebras of index 4 or greater. The results of Merkurjev [M1-2] show that if $G$ is simply connected of type $^1\text{A}_n$ such that $G(k) = \text{SL}_n(D)$ where the index of $D$ is divisible by 4 then $G$ is not stably rational over $k$ and also there exist adjoint groups of type $\text{A}_3$ which are not stably rational. It is not known if there are non-rational (or non stably rational) adjoint groups of type $^1\text{A}_{n,r}$ related with division algebra $D$ of degree $d \leq 2$.

3 Rationality of almost simple groups over local fields

3.1. First we assume that $k$ is a $p$-adic field. We have the following result about rationality of almost simple $k$-groups over $k$.

3.2. Theorem. Assume that $G$ is an almost simple group over a $p$-adic field $k$.

a) If $G$ is adjoint then $G$ is $k$-rational.

b) If $G$ is of type different from $^1\text{A}^{(d)}$ ($d \geq 4$), simply connected types $^1\text{D}_{2r+3,r}$
or $2D_{2r+2},r \ (r \ odd)$ then $G$ is rational or stably rational over $k$.

Proof. \( a \) From Section 1 we know that if $S$ is a maximal $k$-split torus of $G$ then the center of $Z_G(S)$ is connected. A theorem of Kneser says that anisotropic semisimple $k$-group is necessary an almost product of groups of type $^1A$. Since the groups $PGL_{1,D}$ are $k$-rational, \( a \) follows.

\( b \) First we assume that $G$ is of classical type. If $G$ is of type $A$ then from [Ti1] and the assumption we deduce that it is of outer type $2A_{2r},r \text{ or } 2A_{2r+1},r$. From Section 1 it follows easily that $G$ is rational. If $G$ is simply connected of type $B$ or $^{1D}(1)$, then the assertion follows from the fact that the Spin group of an isotropic non-degenerate quadratic form is rational (see [P]). If $G(k)$ is given by the special orthogonal or unitary group of some quadratic or skew-hermitian form, $G$ is rational due to Cayley transformation.

Now we assume that $G$ is simply connected of type $D^{(2)}$. From [Ti1] it follows that $G$ is of type $^1D_{2r},r$ with the following Tits index

\[ \bullet^{1} \bullet^{2} \cdots \bullet^{2r-3} \bullet^{2r-2} \circ^{2r-1} \circ^{2r} \]

or $2D_{2r+1},r$ with the Tits index

\[ \bullet^{1} \bullet^{2} \cdots \bullet^{2r-2} \circ^{2r-1} \circ^{2r} \circ^{2r+1} \]

In either case, for a maximal split $k$-torus $S$ of $G$ we have

\[ Z_G(S) = ST_0H, \]

where $H$ is the direct product of simply connected anisotropic groups of type $A_1$. One can check that the anisotropic quotient $Z_G(S)/ST_0$ is an almost direct product of groups isomorphic to $SL_{1,D}$ with common centers, or just direct product of adjoint groups of type $A_1$. The latter groups are known from above (see 2.3) to be stably rational over $k$. Thus $G$ is also. Moreover, for any central $k$-isogeny $\pi: G \to G'$, the anisotropic semisimple quotient of $G'$ is the image of that of $G$, hence is also the product of groups of type $A_1$ with common center hence is stably rational as above. This covers also
the case of almost simple groups of type $D_{2m}$ with center of order 2 which are obtained by factoring the Spin group $\tilde{G} = \text{Spin}(\Phi)$ by the subgroups \{1, $z$\}, \{1, $z'$\}, where the center of $\tilde{G}$ is \{1, $z$, $z'$, $zz'$\} and SO($\Phi$) (or SU($\Phi$)) is $\simeq \text{Spin}(\Phi)/\{1, zz'\}$. (Note that the roles of the two elements $z$ and $z'$ in certain cases are not symmetric.)

Now we consider the case $2D_{2r+2}$, $r$ is even, $r = 2s$. First we assume $G$ simply connected. The Tits index of $G$ is as follows

\[ \begin{array}{cccccccc}
1 & - & \otimes^2 & - & \cdots & - & 2r-1 & - & 2r \otimes \\
\downarrow & & & & & & & & \\
2r+1 & & & & & & & & \\
& 2r+2 & & & & & & & \\
\end{array} \]

Note that

\[ Z_G(S) = SG_1G_3\cdots G_{2r-1}AA', \]

where $G_i$, $1 \leq i \leq 2r-1$ (odd) are anisotropic simply connected $k$-group of type $A_1$, $AA' = R_{l/k}(A'')$ with a quadratic extension $l$ of $k$, over which $G$ is of inner form, and $A''$ is an anisotropic $l$-group of type $A_1$. We may assume that all $G_i$ are identified with the $k$-group $SL_{1,D}$ for a unique (up to isomorphism) quaternion division algebra $D$ over $k$. We notice that the product $H = G_1G_3\cdots G_{2r-1}AA'$ is stably equivalent to the group $R_{l/k}(H')$, where $H' = G_1G_3\cdots G_{r-1}A$. (Here $G_i$ are $k$-groups but considered as $l$-groups.) We can give a similar interpretation for the center of $H, H'$. Since the product $P'$ of $l$-groups $G_i$ ($1 \leq i \leq r - 1$) and $A$ with glued centers are stably rational over $l$, it follows that $P = R_{l/k}(P')$ is stably rational over $k$. But one checks that $P \simeq Z_G(S)/S$ (the same computations as above). Thus $G$ is also stably rational over $k$. The same is also true for the factor groups $\text{Spin}(\Phi)/\{1, z\}$, $\text{Spin}(\Phi)/\{1, z'\}$ (see notation above).

Finally we consider the case of exceptional groups. The non trivial cases are groups of type $^1E_{6,2}$ and $E_{7,4}$ and we assume that they are simply connected. For the groups of type $^1E_{6,2}$, we have the Tits index of $G$ as follows

\[ \begin{array}{cccccccc}
1 & - & 2 & - & \otimes^4 & - & 5 & - & 6 \\
\end{array} \]

and let $S_2$ be the standard split torus corresponding to the root $\alpha_2$. Then
we have

\[ Z_G(S_2) = S_2 A, \]

where \( A \) is a simply connected of type \( ^1\text{A}^{(3)}_{5,1} \). There is a division \( k \)-algebra \( D \) such that \( A(k) = \text{SL}_2(D) \). Since the \( k \)-group \( L \) of type \( A \) defined by \( L(k) = \text{SL}_1(D) \) has rank 2 hence is rational, the group \( A \) is also \( k \)-rational. Now one can check that

\[ S_2 \cap A = \{1\}, \]

thus from Section 1 we know that \( G \) is also \( k \)-rational.

The case of groups of type \( ^7\text{E}_{7,4} \) with the Tits index

\[
\begin{array}{cccccc}
\bullet^2 \\
\circ^7 - - \circ^6 - - \circ^5 - - \circ^4 - - \circ^3 - - \circ^1
\end{array}
\]
is reduced to the case of groups which are product of groups of type \( \text{A}_1 \) with glued center.

3.3. Remark. The exclusion of groups of type \( ^1\text{A}^{(d)} \) with \( d \geq 4 \) is necessary (see the end of Section 2). However it is not clear if it is so regarding the groups of type \( D \) in the above proposition.

3.4. Now we assume that \( k = \mathbb{R} \). Recently Chernousov [Ch] has proved that if \( G \) is an anisotropic semisimple \( \mathbb{R} \)-group with no factors of type \( ^6\text{E}_6, ^7\text{E}_7, ^8\text{E}_8 \) then \( G \) is stably rational over \( \mathbb{R} \). The main idea there (as in [M2]) is to use the group of similarity factors of the forms involved, which goes back to [T1-3], where we considered the problem of weak approximation in a close relation with the problem of rationality. In view of results above, we can generalize the result of Chernousov as follows.

3.5. Proposition. Let \( G \) be a semisimple \( \mathbb{R} \)-group with no anisotropic factors of types \( ^i\text{E}_i, \ i=6,7,8 \). Then \( G \) is stably rational over \( \mathbb{R} \).
4 Rationality over arbitrary field of some isotropic almost simple groups

One may notice that some of arguments used above can be applied to a more general situation. In fact we have the following

4.1. Theorem. Let $G$ be an isotropic almost simple $k$-group of one of the following types $^{1}A^{(d)}$, $d \leq 3$; $B$; $C^{(d)}_{n,r}$, $n - dr \leq 2$, $d \leq 2$; $D^{(d)}_{n,r}$, $d \leq 2$, with $n \leq r + 2$ $(d = 1)$, $n \leq 2r + 1$ $(d = 2)$. Then $G$ is either rational or stably rational over $k$.

Proof. We need only to prove the cases C and D. For the case of type C we make use of Propositions 1.3, 1.5. For the case of type D, we take the centralizer of the standard 1-dimensional split torus $S_0$ of $G$ corresponding to the last circled vertex (in the case $n = 2r$) or vertices (in the case $n = 2r + 1$) (see the Tits indices drawn above for these groups) to get the group

$$Z_G(S_0) = S_0 T_0 A,$$

where $A$ is a $k$-group of type $^{1}A^{(2)}$, which has the factor group $A/(A \cap S_0 T_0)$ of the same type hence stably rational as shown above (see Sec. 2). Hence from Section 1 we know that $G$ is also stably rational. The other cases are treated in a similar way. 

Now we focus our attention to exceptional groups and we have the following result.

4.2. Theorem. Let $G$ be an isotropic almost simple group of exceptional type over a field $k$.

a) If $G$ is of type $^{1}D_4,^{2}F_4$ then $G$ is $k$-rational.

b) Let $G$ be one of the following types : $^{1}E_6$; $^{2}E_6, r \geq 2$ and $G$ is adjoint if $G$ is of type $^{2}E_6$; $E_6^{26}$, $(r \geq 2)$ (simply connected); $E_7, r \geq 2$ (and $G$ is simply connected if $G$ is of type $^{3}E_7$); $E_8, r \geq 3$. Then $G$ is either $k$-rational or stably rational over $k$.

c) If $G$ is of type $^{1}E_7^{18}$, then any $k$-group which is $k$-isogeneous to $G$ is also birationally isomorphic to $G$ over $k$ as $k$-varieties.
Proof. a) The proof follows from results of Section 1.
b) For the group of type $^{1}E_{6}$ we consider only the group $^{1}E_{6,2}^{(28)}$ with Tits index

\[
\begin{array}{c}
\circ 1 \rightarrow \bullet 3 \rightarrow \bullet 4 \rightarrow \bullet 5 \rightarrow \circ 6 \\
\bullet 2
\end{array}
\]

since for the other, the proof is the same as in $p$-adic case above. Let $\tilde{G}$ (resp. $\bar{G}$) be the simply connected covering (resp. adjoint group) of $G$ and $\tilde{S}$ a maximal $k$-split torus of $\tilde{G}$. It is well-known that $Z_{G}(\tilde{S}) = SD$, where $D$ is the Spin group of a quadratic form $f$ which is a norm form of a division Cayley algebra, i.e., $f$ is a Pfister form. By a result of [M2] (Prop. 7) the adjoint group of $SO(f)$ is stably rational over $k$. Hence $\tilde{G}$ is stably rational over $k$. Now we can check that $\tilde{S}$ contains the center of $D$. Thus $\tilde{G}$, being birationally equivalent to $G$, is also stably rational over $k$.

Let $G$ be simply connected of type $E_{6,2}^{16'}$ with the following Tits index

\[
\begin{array}{c}
\circ 2 \rightarrow \bullet 4 \\
\circ 1 \rightarrow \bullet 3 \rightarrow \circ 5 \rightarrow \circ 6
\end{array}
\]

From this we see that the anisotropic kernel of $G$ is of type $^2A_3^{(1)}$, which is also the anisotropic kernel $A$ for the simply connected semisimple group of type $^2A_5$ with the root system spanned by $\Delta \setminus \{\alpha_2\}$. By a result of [M2] (Prop. 8), the adjoint group of $A$ is stably rational and one checks easily that for a maximal $k$-split torus $S$ of $G$, the center of $A$ is contained in $S$. Therefore $G$ and its adjoint group are stably rational.

The case of type $^2E_{6,2}$ is considered similarly as above: If $\tilde{S}$ (resp. $\bar{G}$) has meaning as above, then $\tilde{S}$ contains the center of $Z_{\bar{G}}(\tilde{S})$ hence $\bar{G}$ is rational. Now we assume that $G$ is of type $E_7$. We claim that if $G$ is simply connected of type $E_{7,1}^{66}$ then $G$ is rational. Indeed, for $S$ a maximal $k$-split torus of $G$, we have

\[Z_{G}(S) = SH,\]
where $H$ is a simply connected $k$-group of type $D_6$, $H \simeq \text{Spin}(\Phi)$ for some form $\Phi$. One checks that we have

$$Z_G(S)/S \simeq \text{SU}(\Phi),$$

thus is rational and so is $G$. The only other non-trivial case considered here is the case of type $E_{7.2}$ with the Tits index

```
  2
 7 - - ⊙6 - - 5 - - 4 - - 3 - - ⊙1
```

Assume that $S$ is a maximal $k$-split torus of the simply connected group $G$ of this type. We have

$$Z_G(S_6) = S_6(\tilde{A}_1 \times \tilde{D}_5),$$

where we denote by $\tilde{A}_1$ (resp. $\tilde{D}_5$) the group of type $A_1$ (resp. $D_5$) with the root system spanned on $\alpha_7$ (resp. $\{\alpha_1, \ldots, \alpha_5\}$). We have

$$Z_G(S_6)/S_6 = G_1G_2,$$

where $G_1$ (resp. $G_2$) is simply connected of type $A_1$ (resp. $G_2 \simeq \text{SO}(f)$, $f$ is an isotropic non-degenerate quadratic form in 10 variables over $k$ with Witt index 1). Here we have

$$G_1 \cap G_2 = \{\pm 1\}.$$

To see this one just needs to compute the intersection $S_6 \cap (\tilde{A}_1 \times \tilde{D}_5)$ and the factor

$$F/(S_6 \cap F),$$

where $F = \text{Cent}(\tilde{A}_1 \times \tilde{D}_5)$.

Now we want to show that $H = G_1G_2$ is rational over $k$. Let $S_1$ be a (unique up to conjugacy over $k$) $k$-split torus of $G_2$. As above, we show that $Z_H(S_1)$ is rational over $k$. We have $Z_H(S_1) = S_1G_1G_3$, where $G_3 = \text{SO}(f_0)$, where $f_0$ is the anisotropic part of $f$. We can check that in fact we have a direct product decomposition

$$Z_H(S_1) = S_1 \times G_1 \times G_3,$$

which is clearly rational over $k$.

c) It rests to consider the case $E_{7.1}^{31}$. Let $S$ be a maximal $k$-split torus of $G$, 


where we assume that $G$ is simply connected. One checks that $S$ contains the center of the anisotropic kernel of type $E_6$ of $G$. Thus

$$Z_G(S)/S \simeq Z_{\bar{G}}(\bar{S})/\bar{S}$$

and we are done.

---

4.3. We say that a segment in the Tits index of an almost simple $k$-group $G$ is black (resp. white) if it consists of only black (resp. white, i.e., distinguished) vertices. The length of a segment is the number of vertices it contains. We say that a segment is defined over $k$, if the almost simple subgroup of $G$ with root system spanned on this segment is defined over $k$. In other words, the black segments are the connected components of the Dynkin diagram of anisotropic kernel of $G$. From results proved above we derive the following main result of this paper.

4.4. Theorem. Let $G$ be an almost simple $k$-group and $m(G)$ be the maximal length of the black segments of its Tits index defined over $k$.

a) If $m(G) \leq 2$ then $G$ is either rational or stably rational over $k$.

b) The number 2 in a) is best possible. For any natural number $n$ there exist non stably rational groups $G$ and fields $k$ with $m(G) = 4n - 1$.

Proof. a) It follows from above results and the Tits classification of indices [Ti1].

b) The number 2 in the above theorem is the best possible since [M1] shows that it fails if 2 is replaced by 3. Namely if $k$ is a field such that there exist division algebras $D$ of index $4n$ (e.g. a number field), then for the group $G$ with $G(k) = SL_m(D)$, the subgroup of reduced norm 1 of $M_m(D)$, then $G$ is not stably rational over $k$ and $m(G) = 4n - 1$.

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