Soliton Solutions on Noncommutative Orbifold $T^2/Z_4$

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Abstract

In this paper, we explicitly construct a series of projectors on integral noncommutative orbifold $T^2/Z_4$ by extended GHS construction. They include integration of two arbitrary functions with $Z_4$ symmetry. Our expression possess manifest $Z_4$ symmetry. It is proved that the expression include all projectors with minimal trace and in their standard expansions, the eigen value functions of coefficient operators are continuous with respect to the arguments $k$ and $q$. Based on the integral expression, we alternately show the derivative expression in terms of the similar kernel to the integral one. Since projectors correspond to soliton solutions of the field theory on the noncommutative orbifold, we thus present a series of corresponding solitons.

Keywords: Soliton, Projection operators, Noncommutative orbifold.

1 Introduction

String theory is a very promising candidate for an unified description of the fundamental interactions, including quantum gravity. It may provide a conceptual framework to resolve the clash between two of the greatest achievements of 20th century physics: general relativity and quantum mechanics. Noncommutative geometry is originally an interesting topic in mathematics [1, 2, 3]. In the past few years, it has been shown that some noncommutative gauge theories can be embedded in string theories [4, 5, 6] and noncommutative geometry can also be applied to condensed matter physics. The currents and density of a system of electrons in a strong magnetic field may be described by a noncommutative quantum field theory [7,8,9]. The connection between a finite quantum Hall system and a noncommutative Chern-Simon Matrix model first proposed by [8] was further elaborated in papers [10, 11]. Many papers are concentrated on the research for the related questions about the quantum hall effect [12-18]. Since the noncommutative space resemble a quantum phase space, it exhibits an interesting spacetime uncertainty relation, which cause a $UV/IR$ mixing [19, 20] and a teleological

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behavior. Noncommutative field theories can be regarded as highly constrained deformation of local field theory. Thus it may help us to understand non-locality at short distances in quantum gravity.

Solitons in various noncommutative theories have played a central role in understanding the physics of noncommutative theories and certain situations of string theories. The quantum Hall effect practically provides a good illustration of the combination of the three theories [13, 17, 18, 21]. The existence and form of these classical solutions are fairly independent of the details of the theory, making them useful to probe the string behavior. In fact these solitons are the (lower-dimensional) D-branes of string theory manifested in a field theory limit while still capturing many string features.

Starting from the celebrated paper of Gopakumar, Minwalla and Strominger [22], there are many works to study soliton solutions of noncommutative field theory and integrable systems in the background of noncommutative spaces [23-30]. Although Derrick’s theorem forbids solitons in ordinary 2+1 dimensional scalar field theory [31], solitons in noncommutative scalar field theory on the plane were constructed in terms of projection operators in [22]. It was soon realized that noncommutative solitons represent D-branes in string field theory with a background $B$ field, and many of Sen’s conjectures [32, 33] regarding tachyon condensation in string field theory have been beautifully confirmed using properties of noncommutative solitons. Gopakumar, Minwalla and Strominger made an important finding that in a noncommutative space, a projector may correspond to a soliton in the field theory [22], which proves the significance of the study of projection operators in various noncommutative space. Reiffel [34] constructed the complete set of projection operators on the noncommutative torus $T^2$. On the basis Boca studied the projection operators on noncommutative orbifold [35] obtaining many important results and showed the well-known construction of projector operator for $T^2/Z_4$ in terms of the elliptic function. Soliton solutions in noncommutative gauge theory were introduced by Polychronakos in [23]. Martinec and Moore in their important article deeply studied soliton solutions namely projectors on a wide variety of orbifolds, and the relation between physics and mathematics in this area [27]. Gopakumar, Headrick and Spradlin have shown a rather apparent method to construct the multi-soliton solution on noncommutative integral torus with generic $\tau$[24]. This approach can be generalized to construct the projection operators on the integral noncommutative orbifold $T^2/Z_N$ [30].

In this paper, in the case of integral noncommutative orbifold $T^2/Z_4$ generated by $u_1$ and $u_2$ with

$$u_1u_2 = u_2u_1e^{2\pi i/A}, \quad A = 1, 2, 3, \cdots$$

we generalize the GH S construction, presenting the explicit symmetric form of a series of projectors with manifest $Z_4$ symmetry. It includes all the solutions with minimal trace, and in the standard expansions for the projectors (see equation (9))

$$P = \sum_{s,t} u_1^s u_2^t \Psi_{s,t}(u_1^A, u_2^A)$$

where the eigen value function $\Psi_{s,t}(v_1^A, v_2^A)$ is continuous (where $v_1^A$ and $v_2^A$ are eigenvalues of $u_1^A$ and $u_2^A$). The solutions include two arbitrary complex functions with $Z_4$ symmetry. The kernels of the integrations are closed analytic functions of $u_1$ and $u_2$. In the simplest case, when $A$ is an even number, we reobtain the Boca’s classic result [35] and obtain a new result when $A$ is an odd number. Moreover the above construction is also applicable to the integral $T^2/Z_N$ cases.

This paper is organized as following: In Section 2, we introduce operators on the noncommutative orbifold $T^2/Z_N$. In Section 3, we introduce the $|k, q>$ representation and provide the matrix element relation for the projectors and deduce the relation between the eigen value functions of coefficients and the matrix elements of operators in the $|k, q>$ representation. In Section 4, we study the general projectors with minimal trace when the eigen value functions of coefficients are continuous. In Section 5, we present two kinds of explicit expressions for the projectors with elliptical functions as kernel.
2 Noncommutative Orbifold $T^2/Z_N$

In this section, we introduce operators on the noncommutative orbifold $T^2/Z_N$. First we introduce two hermitian operators $\hat{y}_1$ and $\hat{y}_2$, which satisfy the following commutation relation:

$$[\hat{y}_1, \hat{y}_2] = i. \quad (3)$$

The operators made up of $\hat{y}_1$ and $\hat{y}_2$

$$\hat{O} = \sum_{m,n} C_{mn} \hat{y}_1^m \hat{y}_2^n \quad (4)$$

form the noncommutative plane $R^2$. All operators on $R^2$ which commute with $U_1$ and $U_2$

$$U_1 = e^{-i\hat{y}_2}, \quad U_2 = e^{i(\tau_2\hat{y}_1 - \tau_1\hat{y}_2)}, \quad (5)$$

where $l, \tau_1, \tau_2$ are all real numbers and $l, \tau_2 > 0, \tau = \tau_1 + i\tau_2$, constitute the noncommutative torus $T^2$. We have

$$U_1^{-1}\hat{y}_1U_1 = \hat{y}_1 + l, \quad U_2^{-1}\hat{y}_1U_2 = \hat{y}_1 + l\tau_1,$$

$$U_1^{-1}\hat{y}_2U_1 = \hat{y}_2, \quad U_2^{-1}\hat{y}_2U_2 = \hat{y}_2 + l\tau_2. \quad (6)$$

The operators $U_1$ and $U_2$ are two different wrapping operators around the noncommutative torus and their commutation relation is $U_1U_2 = U_2U_1e^{-2\pi i\tau_2/A}$. When $A = \frac{l^2\tau_2}{2\pi}$ is an integer, we call the noncommutative torus integral. Introduce two operators $u_1$ and $u_2$:

$$u_1 = e^{-i\hat{y}_2/A}, \quad u_2 = e^{-i(\tau_2\hat{y}_1 - \tau_1\hat{y}_2)/A},$$

$$u_1u_2 = u_2u_1e^{2\pi i/A}, \quad u_1^A = U_1, \quad u_2^A = U_2^{-1}. \quad (7)$$

The operators on the noncommutative torus are composed of the Lauarant series of $u_1$ and $u_2$,

$$\hat{O}_{T^2} = \sum_{m,n} C'_{mn}u_1^m u_2^n \quad (8)$$

where $m, n \in Z$ and $C'_{00}$ is called the trace of the operators. Eq.(8) includes all operators on the noncommutative torus $T^2$, satisfying the relation $U_i^{-1}\hat{O}_{T^2}U_i = \hat{O}_{T^2}$. From (7) we can rewrite the equation (8) as

$$\hat{O}_{T^2} = \sum_{s,t=0}^{A-1} u_1^s u_2^t \Psi_{st}(u_1^A, u_2^A) \quad (9)$$

where $\Psi_{st}$ is Laurant series of the operators $u_1^A$ and $u_2^A$. We call this formula the standard expression for the operator on the noncommutative torus $T^2$. The trace for the operator is the constant term’s coefficient of $\Psi_{00}$. Nextly we introduce rotation $R$ in noncommutative space $R^2$

$$R(\theta) = e^{-i\theta \hat{y}_2^2/2 + i\hat{y}_1^2} \quad (10)$$

with

$$R^{-1}\hat{y}_1R = \cos \theta \hat{y}_1 + \sin \theta \hat{y}_2, \quad R^{-1}\hat{y}_2R = \cos \theta \hat{y}_2 - \sin \theta \hat{y}_1. \quad (11)$$

When $\tau = \tau_1 + \tau_2 = e^{2\pi i/N}$, setting $\theta = 2\pi /N (N \in Z)$. The noncommutative torus $T^2$ keep invariant under rotation $R_N \equiv R(2\pi /N)$ [35, 27, 30]. Namely $R_N^{-1}\hat{O}_{T^2}R_N$ is still the operators on the noncommutative Torus $T^2$. Now $U_i' \equiv R_N^{-1}U_iR_N$ can be expressed by monomial of $\{U_i\}$ and their inverses [27]. In this case, we call the operators invariant under rotation $R_N$ on the noncommutative torus as
operators on noncommutative orbifold $T^2/Z_N$. We can also realize these operators in Fock space.

Introduce

$$a = \frac{\hat{y}_2 - i\hat{y}_1}{\sqrt{2}}, \quad a^+ = \frac{\hat{y}_2 + i\hat{y}_1}{\sqrt{2}},$$

then

$$[a, a^+] = 1, \quad R_N = e^{-i\theta a^+ a}.\tag{13}$$

In this paper, we study the projector $P$ on the orbifold $T^2/Z_4$:

$$\tau = i, \quad P^2 = P, \quad U_j^{-1} PU_j = P, \quad j = 1, 2 \tag{15}$$

$$R_4^{-1} PR_4 = P. \tag{16}$$

3 The $|k, q>$ representation, standard form and eigen value function

From the above discussion, we know that the operators $U_1$ and $U_2$ commute with each other on the integral torus $T^2$ when $A$ is an integer. So we can introduce a complete set of their common eigenstates, namely $|k, q>$ representation [36, 37]

$$|k, q> = \sqrt{\frac{l}{2\pi}} e^{-i\pi \hat{y}_2^2/2r_2} \sum_j e^{ijkl}\sqrt{l}q + jl>, \tag{19}$$

where the ket on the right is a $\hat{y}_1$ eigenstate. We have

$$U_1|k, q> = e^{-ilk}|k, q>, \quad U_2|k, q> = e^{ilr_2q}|k, q> = e^{2\pi i q A/l}|k, q>, \tag{20}$$

It also satisfies

$$|k, q> = |k + 2\pi, q> = e^{ilk}|k, q + l>.$$

Consider the equation (9), namely the standard expansion of operators on $T^2$ we have

$$\Psi_{st}(u_1^A, u_2^A)|k, q> = \Psi_{st}(e^{-ilk}, e^{-2\pi i q A/l})|k, q> = \psi_{st}(k, q)|k, q>, \tag{22}$$

where $\psi_{st}$ is a function of the independent variables $k$ and $q$, called the eigen value function of $\Psi_{st}(u_1^A, u_2^A)$. From (22), we see that the function $\psi_{st}$ is invariant when $q \rightarrow q + l/A$,

$$\psi_{st}(k, q + \frac{ln}{A}) = \psi_{st}(k, q). \tag{23}$$

As long as the eigen value function is obtained, the operator on the noncommutative torus can be completely determined. Introducing new basis $|k, q_0; n > = |k, q_0 + \frac{ln}{A} >, k \in [0, \frac{2\pi}{A}), q_0 \in [0, \frac{1}{A})$, we have from (20)

$$\sum_{n=0}^{A-1} \int_0^{2\pi} dk \int_0^{\frac{A}{l}} dq |k, q_0 + \frac{ln}{A} > < k, q_0 + \frac{ln}{A} | = id. \tag{24}$$
From the above equation and (21), we see that when any power of the operators \( u_1 \) and \( u_2 \) act on the \(|k, q_0 + \frac{ln}{A}\rangle\), the result can be expanded in the basis \(|k, q_0 + \frac{ln'}{A}\rangle\) with the same \( k, q_0 \). So the operators on the noncommutative torus have the same property, namely don’t change \( k \) and \( q_0 \). Thus, for every \( k \) and \( q_0 \) we get a \( A \times A \) matrix, called reduced matrix for the operator, as well as the projector:

\[
P_{T^2}|k, q_0 + \frac{ln}{A}\rangle = \sum_{n'} M(k, q_0)_n |k, q_0 + \frac{ln'}{A}\rangle,
\]

It is easy to find that the sufficient and necessary condition for \( P^2 = P \) is [30]

\[
M(k, q_0)^2 = M(k, q_0).
\]

When \( T^2 \) satisfies \( Z_N \) symmetry, since after \( R_N \) rotation \( U_i' \) can be expressed by monomial of \( \{U_i\} \) and their inverses, the state vector \( R_N|k, q_0 + \frac{ln}{A}\rangle \) is still the common eigenstate of the operators \( U_i' \) and \( U_2 \). With the completeness of \( \{|k, q + \frac{ln'}{A}\rangle\} \) and the \( A \)-fold degeneracy eigenvalues of \( U_i \) in the \( kq \) representation, the state can be expanded in the basis \( \{|k', q' + \frac{ln'}{A}\rangle\} \)

\[
R_N|k, q_0 + \frac{ln}{A}\rangle = \sum_{n'} A(k, q_0)_{n' n} |k, q_0 + \frac{ln'}{A}\rangle
\]

where \( k' \in [0, 2\pi/l], q' \in [0, l/A] \) are definite and

\[
R_N^{-1}|k', q_0 + \frac{ln'}{A}\rangle = \sum_{n'} A^{-1}(k, q_0)_{n' n} |k', q_0 + \frac{ln'}{A}\rangle.
\]

We can get the expression for the relation between \( k', q_0' \) and \( k, q_0 \). The mapping \( W : (k, q_0) \rightarrow (k', q_0') \), \( W^N = id \), is essentially a linear relation, and area-preserving. By this fact and since \( R_N \) is unitary, we conclude that the matrix \( A \) is a unitary matrix, that is to say

\[
A^*(k, q_0)_{n' n} = A^{-1}(k, q_0)_{n' n}.
\]

The projector on the noncommutative orbifold \( T^2 / Z_N \) satisfies \( R_N^{-1} P R_N = P \), then from (27)(29)(30) one obtains

\[
R_N^{-1} P R_N |k, q_0 + \frac{ln}{A}\rangle = \sum_{n'} \left[ A^{-1}(k, q_0) M(k', q_0') A(k, q_0) \right]_{n' n} |k, q_0 + \frac{ln'}{A}\rangle,
\]

which should be equal to:

\[
P |k, q_0 + \frac{ln}{A}\rangle = \sum_{n'} M(k, q_0)_{n' n} |k, q_0 + \frac{ln'}{A}\rangle.
\]

So, we have

\[
M(k', q_0') = A(k, q_0) M(k, q_0) A^{-1}(k, q_0)
\]

\(^1\)It is necessary to point out that the matrix \( A \) defined here is the transposed matrix of \( A \) defined in Formula (93) in paper [30].
and the sufficient and necessary condition for the projector on noncommutative orbifold $T^2/Z_N$ to satisfy is:

$$M(k, q_0)^2 = M(k, q_0),$$
$$M(k', q_0) = A(k, q_0)M(k, q_0)A^{-1}(k, q_0).$$

Next we will study the relation between coefficient function $\psi_{st}(k, q)$ and the reduced matrix $M(k, q_0)$. From (23)(25)(26) and (27) we have

$$P|k, q_0 + \frac{ln}{A} > = \sum_{s,t} u_1^s u_2^t \Psi_{st}(u_1^A, u_2^A)|k, q_0 + \frac{ln}{A} >$$
$$= \sum_{s,t} e^{-2\pi i (q_0/l + n/A)t} \psi_{st}(k, q_0)|k, q_0 + \frac{l(n + s)}{A} >$$
$$= \sum_{n'} M(k, q_0)n'|k, q_0 + \frac{ln'}{A} > .$$

So for $n + s < A$ case, we have

$$M(k, q_0)_{n+s,n} = \sum_{t=0}^{A-1} e^{-2\pi i (q_0/l + n/A)t} \psi_{st}(k, q_0)$$

and for $n + s \geq A$ case, we have

$$M(k, q_0)_{n+s-A,n} = \sum_{t=0}^{A-1} e^{-2\pi i (q_0/l + n/A)t} \psi_{st}(k, q_0)e^{-ilk}. $$

Setting

$$M(k, q_0)_{n+s,n} = M(k, q_0)_{n+s-A,n}e^{ilk},$$

We can uniformly write as:

$$M(k, q_0)_{n+s,n} = \sum_{t=0}^{A-1} e^{-2\pi i (q_0/l + n/A)t} \psi_{st}(k, q_0)$$

and have

$$\psi_{st}(k, q_0) = \frac{1}{A} \sum_{n=0}^{A-1} M(k, q_0)_{n+s,n}e^{2\pi i (q_0/l + n/A)t}.$$  

Eq.(41) and (42) is the relation between $\psi_{st}$ and the elements of reduced matrix $M$.

4 Continuous solution for the Projector with Minimal Trace

Now one may ask what property the reduced matrix $M$ possess when the coefficient function $\psi_{st}$ is a continuous function. In this section, we mainly answer this question. First we prove the $A \times A$ matrix satisfying the condition $M^2 = M$ is always diagonalizable. For any vector $\psi$, $M\psi$ is invariant under $M$, namely

$$M(M\psi) = M\psi.$$

Assume there are totally $B$ linear independent invariant vectors under transformation $M$, then

(1) for $A = B$ case, the matrix $M$ is identity of the space expanded by the vectors, namely $A \times A$ unit
matrix. Of course it is diagonal.

(2) for $B < A$ case, considering any vector $a$ and setting $b = Ma - a$, we find $Mb = 0$. Namely any vector $a$ can be expressed as linear combination of invariant vector $c = Ma$ and null vector $b$ under action of $M$. So the whole linear space is composed of certain invariant vectors and null vectors under action of $M$. $M$ can be diagonalized in the representation with these vectors as basis. So we have:

$$M(k, q_0) = S^{-1}(k, q_0)\overline{M}(k, q_0)S(k, q_0),$$ (44)

where

$$\overline{M}(k, q_0) = \text{diag}(1, 1, \cdots, 1, 0, 0, \cdots, 0).$$ (45)

Due to (41), when $\psi_{st}(k, q_0)$ is continuous, $M(k, q_0)$ is also continuous. However $\text{tr } M(k, q_0) = \text{tr } \overline{M}(k, q_0) = 0, 1, 2, \cdots, A$, which is discrete, so when $\psi_{st}$ is continuous, the value of $\text{tr } M(k, q_0) = A\psi_{00}(k, q_0)$ is invariant for all $k$ and $q_0$. The trace of the projector is the zero order term of $\psi_{00}(k, q_0)$ in Laurant expression of $e^{-ik}$ and $e^{-2\pi iqA/l}$, so we have

$$\text{tr } P = \int_0^{2\pi} dk \int_0^{2\pi} dq A \frac{1}{2\pi} \psi_{00}(k, q_0)$$

$$= \int_0^{2\pi} dk \int_0^{2\pi} dq \frac{1}{2\pi} \text{tr } M(k, q_0)$$

$$= \frac{1}{A} \text{tr } M(k, q_0).$$ (46)

The projector is trivial for $\text{tr } M(k, q_0) = 0$, indicating $P = 0$ and identity. The nontrivial $\text{tr } P = \frac{1}{A}, \frac{2}{A}, \cdots, \frac{A - 1}{A}$. In this paper, we only study the nontrivial projector with minimal trace($\text{tr } M(k, q_0) = 1$). Thus

$$M(k, q_0) = s^{-1}(k, q_0) \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots \\ \end{pmatrix} s(k, q_0),$$ (47)

$$M(k, q_0)_{nm} = s^{-1}(k, q_0)_{n0} s(k, q_0)_{0m} \equiv a(k, q_0)_n b(k, q_0)_m.$$ (48)

Explicit calculation about $R_N$ acting on $|k, q; n >$ shows that we can divide the complete area $\Sigma : \{k \in [0, 2\pi/l], q \in [0, l/A]\}$ into $N$ subarea $\sigma_0, \cdots, \sigma_{N-1}$, making $W : \sigma_i \rightarrow \sigma_{i+1}, (i = 0, 1, \cdots, N - 2), \sigma_{N-1} \rightarrow \sigma_0$. If we construct a reduced matrix $M(k, q_0)$ to satisfy (48) in the area $\sigma_0$, then the projector corresponding to continuous $\psi_{st}$ with minimal trace is completely determined. In area $\sigma_0$, set

$$a_n =< k, q_0 + \frac{ln}{A} | \phi_1 >, \quad b_n =< \phi_2 | k, q_0 + \frac{ln}{A} >,$$ (49)

where

$$\sum_n a_n b_n = \text{tr } M(k, q_0) = 1.$$ (50)

In the other areas $\sigma_j$ with $(k, q_0) \rightarrow (k_j, q_{0j})$ by mapping $W^j$, we demand

$$a_n(k_j, q_{0j}) = < k_j, q_{0j} + \frac{ln}{A} | \phi_1 >$$

$$= A^j(k, q_0)_{nn} a_n(k, q_0),$$

$$b_n(k_j, q_{0j}) = < \phi_2 | k_j, q_{0j} + \frac{ln}{A} >$$

$$= b_n(k, q_0) A^{-j}(k, q_0)_{nn}.$$ (52)
We thus have all coefficients of $|\phi_1>, <\phi_2|$ in $\sigma_0, \ldots, \sigma_{N-1}$. Owing to the completeness of $|k, q_0 + \frac{ln}{A}>$ in the area $\Sigma$, $|\phi_1>$ and $<\phi_2|$ can be determined by the coefficient (49) of $|\phi_1>$ and $<\phi_2|$. Meanwhile, in the area $\sigma_j$, we have

$$M(k_j, q_{0j})_{n'n'} = a_n(k_j, q_{0j})b_{n'}(k_j, q_{0j'})$$

$$= [A^j(k, q_0)M(k, q_0)A^{-j}(k, q_0)]_{n'n'}.$$  \hspace{1cm} (53)

The matrix $M(k, q_0)$ really satisfies the equation (34). Consider the state vector

$$|\phi_1> = \int dk dq_0 \sum_n |k, q_0 + \frac{ln}{A}> <k, q_0 + \frac{ln}{A}|\phi_1>$$

$$= \sum_{j=0}^{N-1} \int \sigma_j dk dq_0 \sum_n |k_j, q_{0j} + \frac{ln}{A} > a_n(k_j, q_{0j})$$

$$= \sum_{j=0}^{N-1} \int \sigma_j dk dq_0 \sum_{n'n} |k, q_0 + \frac{ln}{A} > A^j_{nn}(k, q_0) a_n(k, q_0)$$

$$= \sum_{j=0}^{N-1} R^j_N \int \sigma_0 dk dq_0 \sum_n |k, q_0 + \frac{ln}{A} > a_n(k, q_0).$$  \hspace{1cm} (54)

Thus we have

$$R_N|\phi_1> = |\phi_1>.$$  \hspace{1cm} (55)

In the same way, we get

$$<\phi_2|R_N = <\phi_2|.$$  \hspace{1cm} (56)

That is to say that the state vectors $|\phi_1>$ and $<\phi_2|$ are invariant under the rotation $R_N$.

More generally, we can take any state vectors $|\phi_1>$ and $<\phi_2|$ satisfying

$$R_N|\phi_1> = e^{i\alpha_1}|\phi_1>, \hspace{1cm}<\phi_2|R_N^{-1} = e^{-i\alpha_2} <\phi_2|$$  \hspace{1cm} (57)

to construct a projection operator on noncommutative orbifold $T^2/Z_N$. Let $M(k, q_0)$ be given by (48) with

$$a_n(k, q_0) = \frac{<k, q_0 + \frac{ln}{A}|\phi_1>}{\sqrt{\sum_n' <k, q_0 + \frac{ln'}{A}|\phi_1> <\phi_2|k, q_0 + \frac{ln'}{A}>}},$$  \hspace{1cm} (58)

$$b_n(k, q_0) = \frac{<\phi_2|k, q_0 + \frac{ln}{A}>}{\sqrt{\sum_n' <k, q_0 + \frac{ln'}{A}|\phi_1> <\phi_2|k, q_0 + \frac{ln'}{A}>}}.$$  \hspace{1cm} (59)

The projector of minimal trace and with continuous coefficient functions is surely of this form. It can be verified that $M^2 = M$. And it is also covariant under $R_N$. From (30) we have

$$<\phi_2|k', q_0 + \frac{ln}{A}>$$

$$= <\phi_2|R_N \sum_n A^{-1}(k, q_0)n'n'k, q_0 + \frac{n'n}{A}>$$

$$= e^{i\alpha_2} \sum_n A^{-1}(k, q_0)n'n' <\phi_2|k, q_0 + \frac{n'n}{A}>.$$
and similarly

\[ <k', q_0' + n' l / A | \phi_1 > = e^{-i \alpha_1} \sum_{n} <k, q_0 + n'' l / A | \phi_1 > A(k, q_0)_{n' n''}, \]

giving

\[ \sum_{n} <k', q_0' + \ln A | \phi_1 > < \phi_2 | k', q_0' + \ln A > = \sum_{n} <k, q_0 + \ln A | \phi_1 > < \phi_2 | k, q_0 + \ln A > e^{-i (\alpha_1 - \alpha_2)}. \]  

Thus

\[ M(k', q_0')_{mn'} = a_n(k', q_0') b_{n'}(k', q_0') = [AMA^{-1}](k, q_0)_{mn'}, \]

\[ P \] is invariant under rotation \( R_N \) due to (36) and really gives the projection operator on noncommutative orbifold \( T^2/Z_N \). The form of (58) is a generalization of GHS construction.\(^2\). From the above result, we have

\[ M(k, q_0)_{mn'} = \frac{<k, q_0 + \ln A | \phi_1 > < \phi_2 | k, q_0 + \ln A >}{\sum_{n} <k, q_0 + \ln A | \phi_1 > < \phi_2 | k, q_0 + \ln A >}. \]  

Noticing that this equation satisfies (40), we have

\[ \psi_{st}(k, q_0) = \frac{1}{A} \sum_{n=0}^{A-1} M(k, q_0)_{n+s,n} e^{2\pi i (q_0/1 + n/A)t} \]

\[ = \frac{1}{A} \sum_{n=0}^{A-1} <k, q_0 + \frac{l(n+s)}{A} | \phi_1 > < \phi_2 | k, q_0 + \frac{l n}{A} > e^{2\pi i (q_0/1 + n/A)t} \]

\[ \sum_{n} <k, q_0 + \frac{l n}{A} | \phi_1 > < \phi_2 | k, q_0 + \frac{l n}{A} > \]

\[ = \frac{F_{st}(k, q_0)}{AF_{00}(k, q_0)}, \]  

where

\[ F_{st}(k, q_0) = \sum_{n=0}^{A-1} <k, q_0 + \frac{l(n+s)}{A} | \phi_1 > < \phi_2 | k, q_0 + \frac{l n}{A} > e^{2\pi i (q_0/1 + n/A)t}, \]  

with

\[ F_{st}(k, q_0) = F_{st}(k, q_0 + l/A) = F_{st}(k + 2\pi /l, q_0), \]  

\[ F_{st}(k, q_0) = F_{s+A,l}(k, q_0) e^{-ilk} = F_{s,l+A}(k, q_0) e^{-2\pi i q_0 A/l}. \]

So the function \( F_{st} \) is the function of independent variables \( X = e^{-ilk} \) and \( Y = e^{-2\pi i q_0 A/l} \), namely \( F_{st}(k, q_0) = \Phi_{st}(X, Y) \). Similarly

\[ \psi_{st}(k, q_0) = \psi_{st}(X, Y) = \frac{\Phi_{st}(X, Y)}{A\Phi_{00}(X, Y)}. \]  

\[ \] If we change the variable \( X \) and \( Y \) into \( u_1^A \) and \( u_2^A \) respectively, the standard form (9) of the projection operator can be easily obtained. So the key question is to find out \( F_{st}(k, q_0) \).

\[^{2}\text{The condition } P^t = P \text{ isn’t satisfied by } P \text{ like this, which might represent the solitons in a "complex" field.}\]
5 Coherent State Representation

Introduce coherent states

$$|z> = e^{-\frac{1}{2}z\bar{z}}e^{a^+z}|0>,$$

where $z = x + iy, \bar{z} = x - iy$, which satisfies

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d^2z |z><z| = \text{identity},$$

$$R_N|z> = |\omega_N z>.$$  \hspace{1cm} (69)

We can show \[30\]

$$<k,q|z> = \frac{1}{\sqrt{l\pi^{1/4}}} \theta\left(\frac{q + \frac{\omega}{2}k - i\sqrt{2}z}{l}, \frac{\tau}{A}\right)e^{-\frac{\pi}{2}k^2/2 + ikq + \sqrt{2}kz - (z^2 + \bar{z})/2},$$  \hspace{1cm} (71)

where

$$\theta(z, \tau) \equiv \theta\left[\begin{array}{c}
0 \\
0
\end{array}\right](z, \tau)$$

and

$$\theta\left[\begin{array}{c}
a \\
b
\end{array}\right](z, \tau) = \sum_m e^{\pi i (m+a)^2} e^{2\pi i (m+a)(z+b)}.$$  \hspace{1cm} (72)

Thus we can expand the state vectors $|\phi_1>$ and $<\phi_2|$ in terms of coherent state,

$$|\phi_1> = \frac{1}{\pi} \int_{-\infty}^{\infty} dx dy |z><z| \phi_1 >$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dx dy f_1(z)|z>,$$  \hspace{1cm} (73)

$$<\phi_2| = \frac{1}{\pi} \int_{-\infty}^{\infty} dx dy <\phi_2|z><z|$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dx dy f_2(z) <z|.$$  \hspace{1cm} (74)

The condition (57) is satisfied if and only if

$$f_1(\omega_N^{-1}z) = f_1(z)e^{i\alpha_1},$$  \hspace{1cm} (75)

$$f_2(\omega_N^{-1}z) = f_2(z)e^{-i\alpha_2}.$$  \hspace{1cm} (76)

Here $\omega_N = e^{-i\frac{2\pi}{N}}$. We have

$$F_{st}(k, q_0) = \frac{1}{\pi^2} \sum_{n=0}^{A-1} \int <k, q_0 + \frac{l(n+s)}{A}|z_1 > f_1(z_1) dx_1 dy_1$$

$$\times \int <z_2|k, q_0 + \frac{ln}{A} > f_2(z_2) dx_2 dy_2 e^{2\pi i (q_0/2 + n/A)t}$$

$$= \frac{1}{\pi^2} \int dx_1 dy_1 dx_2 dy_2 g_{st}(k, q_0, z_1, z_2) f_1(z_1) f_2(z_2),$$  \hspace{1cm} (77)
where
\[
g_{st}(k, q_0, z_1, z_2) = \sum_{n=0}^{A-1} <k, q_0 + \frac{l(n + s)}{A}|z_1 > < z_2 |k, q_0 + \frac{ln}{A} > e^{2\pi i(q_0/l + n/A)t}.
\]  

We call the kernel \( g \) as generating function in coherent state representation. Next, we study the expression of \( g \) for \( Z_4 \) case. Through \( g \) we can give the integration expression for all the projection operators on the \( T^2/Z_4 \) with minimal trace and continuous eigen value function. Consider the equation
\[
\theta(z, \tau)^* = \theta(z^*, -\tau^*).
\]  

For the \( z_4 \) case, \( \tau = i, A = \frac{l^2}{2\pi} \), from (71)(73) and (74) we get
\[
< k, q + \frac{lq}{A}|z_1 > |z_2 |k, q + \frac{lq'}{A} > = \frac{1}{l/\sqrt{\pi}} \theta(q/l + i\frac{k}{2} + s/A - \frac{i\sqrt{2}z_1}{l}, \frac{i}{A}) \theta(q/l - i\frac{k}{2} + s'/A + \frac{i\sqrt{2}z_2}{l}, \frac{i}{A}) \times e^{-\frac{k^2}{4} + \frac{i}{2}(z_1 + z_2) + \frac{2i}{\pi}(\frac{s + s'}{A})} e^{-\frac{i\sqrt{2}z_1^*}{l}} \equiv K_{ss'}.
\]

Let \( u = \frac{ik}{2\pi}, v = \frac{i}{2}, \mu = -i\frac{\sqrt{A}}{2}z_1, \nu = i\frac{\sqrt{A}}{2}z_2 \), then
\[
K_{ss'} = \frac{C_1}{l/\sqrt{\pi}} \theta(u + A, \frac{s + \mu i}{A}) \theta(v - A, \frac{s' + \nu}{A}) \times e^{\pi i\frac{2}{A^2} + \pi i\left[\frac{\frac{1}{A}|s|^2 + \frac{1}{A}|\nu|^2}{2}\right]}.
\]

where
\[
C_1 = e^{2\pi i\left[\pi i\left(\frac{1}{A}|s|^2 + \frac{1}{A}|\nu|^2\right)^2\right] + \frac{\pi}{4\pi}}\left[\frac{1}{|s|^2 + |\nu|^2}\right].
\]

It can be proved that for integer \( A \):
\[
\begin{align*}
\sum_{r=0}^{A-1} e^{2\pi i\frac{rt}{A}} \theta(x + r/A, \tau/A) \theta(y + r/A, \tau/A) & = A \sum_{d=0,1} \theta(-\frac{\tau}{A}(Ad - t) + x - y, \frac{2\tau}{A}) \theta(\frac{\tau}{A}(-At + A^2d) + A(x + y), 2\tau A) \\
& \times e^{\pi i\frac{1}{A^2}(Ad - t)^2} \times e^{2\pi i(Ad - t)y}
\end{align*}
\]

and
\[
\theta(z, \tau) = \sqrt{i} e^{-\pi iz^2/\tau} \theta(\pm \frac{z}{\tau}, -\frac{1}{\tau}).
\]

Thus we have
\[
G_{at}(u, v) \equiv g_{st}(k, q, z_1, z_2) = e^{2\pi iut} \sum_{r} e^{2\pi iAt} K_{ss't'}(u, v)
\]
\[
= AC_1 \sqrt{\frac{A}{2\pi}} \sum_{d=0,1} \sum_{r} e^{-\frac{\sqrt{A}}{2}(Ad - t)^2 + 2\pi i(Ad - t)\frac{r}{A} + \frac{\pi}{4\pi}((s + \mu - \nu)^2}
\]
\[
\times e^{\frac{\pi}{\sqrt{A}}(s + \mu - \nu)(Ad - t) + 2\pi iAd} \theta(u - \frac{1}{2}(Ad - t) + \frac{i}{2}(s + \mu - \nu), \frac{Ai}{2})
\]
\[
\times \theta(2Av + s + \mu + \nu - t\tau + iAd, 2Ai).
\]
Due to
\[
\sum_{a=0,1} \theta(x + \frac{a}{2}, \tau) = 2\theta(2x, 4\tau), \tag{86}
\]
\[
\sum_{a=0,1} (-1)^a \theta(x + \frac{a}{2}, \tau) = 2e^{2\pi i(x + \frac{a}{2})} \theta(2x + 2\tau, 4\tau), \tag{87}
\]
when \(d\) is equal to 0, 1,
\[
\theta(2x + 2\tau d, 4\tau) = \frac{1}{2} \sum_{a=0,1} (-1)^{ad} e^{2\pi i(x + \frac{a}{2})} \theta(x + \frac{a}{2}, \tau). \tag{88}
\]

The function \(G_{st}(u, v)\) can be rewritten as
\[
G_{st}(u, v) = \frac{A}{4\pi} e^{2\pi i \phi} \sum_{a,d=0,1} (-1)^{ad} \theta(-u + \frac{1}{2} (Ad + t) + i \frac{1}{2} (s + \mu - \nu), \frac{A_i}{2}) \times \theta(-Av + \frac{1}{2} (s + a - \mu - \nu) + i \frac{1}{2} t, \frac{Av}{2}), \tag{89}
\]
where
\[
\phi = \frac{i}{4A} (s^2 + t^2) - \frac{st}{2A} + \frac{i}{2A} s(\mu - \nu) - \frac{t}{2A} (\mu + \nu) + \frac{i}{4A} (|\mu|^2 + |\nu|^2 - 2\mu \nu) \tag{90}
\]
From the above discussion, we know that when \(A\) is an even number, only \(a = 0\) contributes, so we have
\[
G_{st}(u, v) = \frac{A}{4\pi} e^{2\pi i \phi} \theta(-u + \frac{1}{2} t + i \frac{1}{2} (s + \mu - \nu), \frac{A_i}{2}) \theta(-Av + \frac{1}{2} (s - \mu - \nu) + i \frac{1}{2} t, \frac{Av}{2})
\]
and when \(A\) is an odd number,
\[
G_{st}(u, v) = \frac{A}{4\pi} e^{2\pi i \phi} \sum_{a,d=0,1} (-1)^{ad} \theta(-u + \frac{1}{2} (Ad + t) + i \frac{1}{2} (s + \mu - \nu), \frac{A_i}{2}) \times \theta(-Av + \frac{1}{2} (Aa + s - \mu - \nu) + i \frac{1}{2} t, \frac{Av}{2}), \tag{91}
\]
They can be uniformly written as
\[
G_{st}(u, v) = \frac{A}{4\pi} e^{2\pi i \phi} \sum_{a,d=0,1} (-1)^{ad} \theta(-u + \frac{1}{2} (Ad + t) + i \frac{1}{2} (s + \mu - \nu), \frac{A_i}{2}) \times \theta(-Av + \frac{1}{2} (Aa + s - \mu - \nu) + i \frac{1}{2} t, \frac{Av}{2}), \tag{92}
\]
which is \(Z_4\) covariant. So we have from (63)(77) and (85),
\[
\psi_{st}(k, q) = \frac{1}{A} \left\{ \int d^2 \mu d^2 \nu G_{st}(u, v) f_1(\mu) f_2(\nu) \right\} \tag{93}
\]
where functions \(f_1\) should satisfy
\[
f_1(\omega_N \xi) = e^{i\alpha} f_1(\xi). \tag{94}
\]
Let \(\hat{u} = \frac{i}{2\pi} \hat{y}_2\) and \(\hat{A} \hat{v} = \frac{i}{2\pi} \hat{y}_1\), we may replace \(u, v\) by \(\hat{u}\) and \(\hat{v}\) in (93) and get
\[
\Psi_{st}(u_1^A, u_2^A) = \frac{1}{A} \int d^2 \mu d^2 \nu G_{st}(\frac{i}{2\pi} \hat{y}_2, \frac{i}{2\pi} \hat{y}_1) f_1(\mu) f_2(\nu), \tag{95}
\]
The operators $u_1$ and $u_2$ commute with the operators $u_1^A$ and $u_2^A$, and from (7)

\[ u_1^A = e^{-2\pi i \frac{\theta}{\nu}} \]
\[ u_2^A = e^{-2\pi i \tilde{\nu}}. \]

Further taking $u_1^0$ and $u_2^0$ into account, we can insert them to the corresponding operator form of Eq.(89). This leads to the function of $\hat{u}$ and $\hat{v}$

\[ h_{ad} = u_1^0 u_2^0 G_{st}(\hat{u}, \hat{v}) \]
\[ = \frac{A}{4\pi} \sum_{a,d=0} (-1)^{ad} e^{2\pi i \frac{\theta}{\nu}} e^{-2\pi i \frac{\tilde{\nu}}{\mu} \theta}(-\hat{u} + \frac{1}{2}(Ad + \frac{i}{2}(s + \mu - \nu), \frac{A_i}{2}) \]
\[ \times e^{-2\pi i \tilde{\nu} t} \theta(-A\hat{v} + \frac{1}{2}(Aa + s - \mu - \nu) + \frac{i}{2}t, \frac{A_i}{2}) \]
\[ = \frac{A}{4\pi} \sum_{a,d=0} (-1)^{ad} e^{2\pi i \frac{\theta}{\nu}} e^{-2\pi i \frac{\tilde{\nu}}{\mu} \theta}(-\hat{u} + \frac{1}{2}Ad + \frac{i}{2}(\mu - \nu), \frac{A_i}{2}) \]
\[ \times \theta\left[ \frac{i}{2}, \frac{i}{2} \right](-A\hat{v} + \frac{1}{2}Aa - \frac{1}{2}(\mu + \nu), \frac{A_i}{2}). \]

Due to $|\omega_{\nu\mu}| = |\mu|, |\omega_{\nu\nu}| = |\nu|, e^{2\pi i \frac{\theta}{\nu}(|\mu|^2 + |\nu|^2)}$ in the above formula can be attributed to $f_1(\mu)$ and $f_2(\nu)$. Finally, we have

\[ P = \sum_{s,t} u_1^s u_2^t \Psi_{st}(u_1^A, u_2^A) \]
\[ = \sum_{s,t} e^{2\pi i \frac{\theta}{\nu}} \frac{1}{2} \sum_{a,d=0} (-1)^{ad} \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{\nu \theta}{\nu}} e^{-2\pi i \frac{\tilde{\nu}}{\mu} \theta} \]
\[ \times \theta\left[ \frac{i}{2}, \frac{i}{2} \right](-\frac{1}{2}Ad + \frac{i}{2}(\mu - \nu), \frac{A}{2}) \]
\[ \times \theta\left[ \frac{i}{2}, \frac{i}{2} \right](-\frac{1}{2}Ad - \frac{1}{2}(\mu + \nu), \frac{A}{2}) \]
\[ \times \theta(-\frac{1}{2}Ad + \frac{i}{2}(\mu - \nu), A). \]

In the above equation, the two $\theta$ functions can not exchange orders with each other. It holds for any integer number $A$. In the following, we present some discussion.

(1) $A$ is an even number, so $\frac{A_i}{2}$ and $A$ are integers too. Due to (72) we have

\[ P = \sum_{s,t} e^{-3\pi i \frac{\theta}{\nu}} \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{\nu \theta}{\nu}} \]
\[ \times \theta\left[ \frac{i}{2}, \frac{i}{2} \right](-\frac{1}{2}Ad - \frac{1}{2}(\mu + \nu), \frac{A}{2}) \]
\[ \times \theta\left[ \frac{i}{2}, \frac{i}{2} \right](-\frac{1}{2}Ad + \frac{i}{2}(\mu - \nu), \frac{A}{2}) \]
\[ \times \theta\left(-\frac{1}{2}Ad - \frac{1}{2}(\mu + \nu), \frac{A}{2}\right)^{-1}. \]
the above equation is the generalization of the Boca's formula Proposition 3.1(i) [35].

(2) A is an odd number

\[ P = \sum_{s,t=0}^{A-1} e^{-3\pi i \frac{st}{A}} \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{st}{A}} \]

\[ \times \sum_{a,d=0}^{1} (-1)^{ad} e^{-2\pi i (\frac{ad}{2} + \frac{t}{2})} \theta \left[ \frac{s}{A} \right] (-\frac{l_y}{2\pi} + \frac{Ad}{2} + \frac{i}{2}(\mu - \nu), \frac{Ai}{2}) \]

\[ \times \theta \left[ \frac{\frac{1}{2} + \frac{t}{2}}{2A} \right] (-\frac{l_y}{2\pi} + \frac{Aa}{2} - \frac{1}{2}(\mu + \nu), \frac{Ai}{2}) \]

\[ \times \{ A \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{st}{A}} \sum_{a,d=0}^{1} (-1)^{ad} \theta (-\frac{l_y}{2\pi} + \frac{Ad}{2} + \frac{i}{2}(\mu - \nu), \frac{Ai}{2}) \]

Due to

\[ \theta \left[ \frac{\frac{1}{2}}{2A} \right] (x + \frac{Ad}{2}, \tau) \]

\[ \theta \left[ \frac{\frac{1}{2} + \frac{t}{2}}{2A} \right] (2x, 4\tau)(-1)^{ad} e^{2\pi i \frac{st}{A}} + \theta \left[ \frac{s+1}{2A} \right] (2x, 4\tau)(-1)^{(s+1)d} e^{2\pi i \frac{(s+1)d}{2A}} \]

(102)

the numerator of \( P \) can be written as

\[ 2 \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{st}{A}} \sum_{s,t=0}^{A-1} \{ \bar{q}_0 \theta_0 + + \bar{q}_1 \theta_0 (-1)^s + \bar{q}_0 \theta_1 (-1)^t + \bar{q}_1 \theta_1 (-1)^{s+t-1} \} \]

\[ = 2 \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{st}{A}} \sum_{s,t=0}^{2A-1} e^{\pi i \frac{st}{A}} \bar{q}_0 \theta_0 \]

(103)

Where \( \bar{q}_0 = \theta \left[ \frac{\frac{1}{2} + \frac{A}{2}}{2A} \right] (2y, 4\tau), \bar{q}_1 = \theta \left[ \frac{\frac{1}{2} + \frac{A}{2}}{2A} \right] (2x, 4\tau), \delta = 0, 1 \) with \( x = -\frac{l_y}{2\pi} + \frac{y}{2}(\mu - \nu) \) and \( y = -\frac{l_y}{2\pi} - \frac{1}{2}(\mu + \nu) \). The denominator of \( P \) is \( 2A \int d^2 \mu d^2 \nu f_1(\mu) f_2(\nu) e^{\pi i \frac{st}{A}} \sum_{s,t=0}^{1} e^{\pi i \frac{st}{A}} \bar{q}_0 \theta_0 \theta_1 \theta_1 \)

This formula gives another result compared with the Boca’s when \( f_1(z) = f_2(z) = \delta^2(z - 0) \).

Finally, we will give another explicit form of \( P \) in terms of the derivative of elliptic functions. Note that the basis \( \{|n>\} \) of Fock space produce a phase \( \omega^n \) under action of \( R_N \). It is not difficult to find that \( e^{i\alpha_1} \) and \( e^{i\alpha_2} \) in (57) are both integral powers of \( \omega_N \) because of \( (R_N)^N = \text{identity} \). Therefore \( |\phi_1> \) and \( <\phi_2| \) can respectively be expanded in the basis \( \{|n>\} \) and \( \{<n|\} \), where

\[ \{n> = \frac{(a^+)^n}{\sqrt{n!}} |0> \]

(104)

We have the relation between coherent state and particle number eigenstate as following:

\[ <z|n> = e^{\frac{1}{2}z\bar{z}} \frac{z^n}{\sqrt{n!}}. \]

(105)
Subsequently, we substitute (105)(106)(107) into (64) and make use of the formula
\[\langle k | z \rangle = \sum_{m=0}^{\infty} d_m < j + 4m | z >\]  
(107)

We let \( R_N \) act on \(|\phi_1>\) and \(<\phi_2|\) and get
\[R_N |\phi_1> = \omega^j |\phi_1>\]
(108)
\[<\phi_2|R_N^{-1} = <\phi_2|\omega^{-j}.
\] (109)

Subsequently, we substitute (105)(106)(107) into (64) and make use of the formula
\[<k,q|n> = \frac{1}{\sqrt{n!}} \frac{d^n}{dz^n} (e^{\frac{i}{2}z^2} < k,q | z >) |_{z=0}
\] (110)
to obtain
\[F_{st}(k,q_0) = \sum_{m,n}^{\infty} c_m d_n < k,q_0 + \frac{(h+s)}{A} | i + 4m > \times < j + 4n | k,q_0 + \frac{lh}{A} > \times e^{2\pi i (q_0/l + h/A)t}
\] (111)

So, we get the projector in the case of \( z_1 \)
\[P = \sum_{m,n} c_m d_n \frac{1}{\sqrt{(i + 4m)! (j + 4n)!}} d_z^{n+m} (e^{\frac{i}{2}z_1^2 + z_2^2} \times e^{2\pi i \frac{m}{A} + \frac{2\pi}{4} A} \times e^{4\pi^2 i z_1 z_2} \times e^{-2\pi i \frac{m}{A} + \frac{2\pi}{4}})
\] \times \sum_{a,d=0}^{1} (-1)^{ad} \theta \left( \frac{1}{2} \right) \left( \frac{-i \gamma_2}{2\pi} + \frac{1}{2} Ad + \frac{\sqrt{2} A}{2l} (z_1 + z_2) \right) \frac{A}{2}
\] \times \left( \frac{1}{2} \right) \left( \frac{-i \gamma_1}{2\pi} + \frac{1}{2} Aa + \frac{i \sqrt{2} A}{2l} (z_1 - z_2) \right) |_{z_1 = z_2 = 0}
\] multiplying by \( (\theta(\gamma_1 + \frac{1}{2} Aa + \frac{i \sqrt{2} A}{2l} (z_1 - z_2) \left( \frac{A}{2} \right)))^{-1} \)
\] \times \theta(-\frac{i \gamma_1}{2\pi} + \frac{1}{2} Aa + \frac{i \sqrt{2} A}{2l} (z_1 - z_2) \left( \frac{A}{2} \right))^{-1} |_{z_1 = z_2 = 0}.
\] (112)
Thus, we derive two forms of explicit expressions of the projector $P$ in terms of the integration and derivative of the classical theta functions.

6 Discussion

In this paper, $P$ is represented by a form of fraction which make sense only when the denominator has inverse. The formula demands:

$$D = A \int d^2\mu d^2\nu f_1(\mu)f_2(\nu)G_{00}(u,v)$$  \hspace{1cm} (113)

is unequal to zero for any real variables $u$ and $v$. It is easy to prove that when $f_1$ is equal to $f_2^*$, the related denominator

$$D = A \sum_n <k,q + \frac{\ln A}{A}\mid \phi > < k,q + \frac{\ln A}{A}\mid \phi > = A \sum_n < k,q + \frac{\ln A}{A}\mid \phi >^2.$$  \hspace{1cm} (114)

Thus if $D = 0$, then

$$< k,q + \frac{\ln A}{A}\mid \phi > = 0 \quad n = 0,1,\ldots, A - 1.$$  \hspace{1cm} (115)

The zero points of the state vector $\mid \phi >$ in $\mid k,q >$ representation should be points equally spaced along $q$ with interval of $\frac{1}{A}$. The mapping from $k$ and $q$ to $< k,q + \frac{\ln A}{A}\mid \phi > \in C$ is a mapping from plane to plane. In general, $< k,q + \frac{\ln A}{A}\mid \phi > = 0$ are some discrete points, and thus it is casual that $D$ is equal to zero. So in this sense, for most of $f_1 = f_2^*$, this still not happen (in some sense, the measure of $D = 0$ event is zero.) Specially, when the state $\mid \phi_1 > = \mid \phi_2 > = \mid 0 >$, It can be proved [35] that $D$ is not equal to zero everywhere. Thus set

$$\mid \phi_1 > = \mid 0 > + \epsilon \mid \psi_1 >, \quad < \phi_2 > = < 0 > + \epsilon < \psi_2 |$$  \hspace{1cm} (116)

$D$ is also not equal to zero everywhere for small enough $\epsilon$. But we don’t know the situation for general $f_1 \neq f_2^*$.

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