Different approaches to the second-order Klein–Gordon equation

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Abstract
We derive the Klein–Gordon equation for a single scalar field coupled to gravity at second order in perturbation theory and leading order in slow-roll. This is done in two ways: we derive the Klein–Gordon equation first using the Einstein field equation and then directly from the action after integrating out the constraint equations. We also point out an unexpected result regarding the treatment of the field equations.

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1. Introduction

Cosmological perturbation theory is an essential tool which connects theories of the early universe with observation. Until recently, it was sufficient to use linear perturbation theory to make and compare theoretical predictions with cosmic microwave background (CMB) experiments and data from large scale structure surveys. One of the major breakthroughs at first order was the development of the gauge-invariant formalism, initiated by Bardeen and others [1–4]. The next round of CMB observations, which will be carried out by observatories such as \textit{Planck}, together with data from anisotropies in matter fields such as the neutral hydrogen, will require gauge-invariant second-order perturbation theory. A vigorous effort is therefore under way to develop higher-order perturbation theory, and change it from what was once of mere academic interest and a calculational nuisance, to a powerful tool, see for example \cite{5-25}.

Considerable progress has already been made. Within the last five years, reliable predictions have become available for the degree of nonlinearity imprinted by inflation in the CMB temperature anisotropy beyond linear order, e.g. \cite{26-34}. Indeed, there is already some interesting tension between large classes of theoretically well-motivated models and
observation on the basis of the nonlinearity they predict [35–38]. For example, one of the strongest bounds on the parameter space in the curvaton scenario comes from the limit on the non-Gaussianity parameter $f_{NL}$ [39–42]. However, despite the considerable effort expended on the development of the theory, some interesting surprises can be found in unexpected ‘corners’.

Much of this research has been framed in the context of the Lagrangian formalism, where one begins with an action and quantizes it. In doing so, one encounters a variant of the Feynman diagrams which are used to compute scattering amplitudes in particle physics. This method of calculation has the advantage that a good deal of pre-existing knowledge can be imported wholesale from high-energy physics. Its principal drawback arises when making comparisons with the cosmological literature, which has traditionally approached perturbation theory from the standpoint of the Einstein equations.

In this paper we explore the relationship between these two approaches by returning to the Einstein equations and calculating the Klein–Gordon equation at second order in perturbation theory, specializing to the case of a single scalar field and working to first order in the slow-roll approximation. This result has already been published in [21] in the general case, but here we relate it to a different method of derivation, namely variation of the Einstein action coupled to a scalar field. In doing so we highlight a potential pitfall in the reduction of the full Einstein equations, which if unobserved can lead to errors at second order. In particular, this means that techniques which have been applied successfully at first order [3] may need treating with more care once one incorporates the effect of nonlinearities. We obtain consistent results in both cases.

Throughout this paper we use natural units where $8\pi G \equiv M_{P}^{-1/2}$ is set equal to unity, where $M_{P}$ is the so-called reduced Planck mass. Derivatives with respect to conformal time are denoted by a prime. Greek indices, $\mu, \nu, \lambda$, label spacetime coordinates and run from 0 to 3, while lower case Latin indices, $i, j, k$, label purely spatial coordinates and run from 1 to 3. The metric convention is $(-, +, +, +)$ and the background spacetime is taken to be of Friedmann–Robertson–Walker (FRW) form,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 \delta_{ij} dx^i dx^j,$$

(1.1)

where $a = a(t)$ is the scale factor. It is usually more convenient to work in so-called conformal time, defined by $\eta = \int_{t}^\infty dt/a(t)$. Note that throughout this paper we work in a spatially flat background spacetime, and evaluate all perturbed quantities on uniform curvature hypersurfaces [18].

The paper is organized as follows. In the following section we define the metric and the matter content of our spacetime. In section 3 we derive the Klein–Gordon equation using the field equations and highlight possible difficulties involving the spatial part of the second-order field equations. In section 4 we derive the Klein–Gordon equation directly from the action. We conclude with a discussion in section 5.

2. Preliminaries

In this section we define the metric and the matter variables we will use in the following sections:

2.1. Notation

We begin by defining the metric tensor in the uniform curvature gauge up to second order in the perturbations, including scalar perturbations only,
$g_{00} = -a^2(1 + 2\phi_1 + \phi_2), \quad g_{0i} = a^2(B_1 + \frac{1}{2}B_2)i, \quad g_{ij} = a^2\delta_{ij}, \quad (2.1)$

where $\phi_1$ and $\phi_2$ together describe the lapse function, and $B_1$ and $B_2$ comprise the scalar part of the shift function (describing the shear in this gauge). Numerical subscripts denote the order of the perturbation and a comma denotes a partial derivative, that is, $X_{,i} \equiv \partial X/\partial x^i$. The notation used in the present paper coincides with the conventions used, for example, in [21]. In this notation, any tensorial quantity is expanded in a truncated power series, as shown above for the metric tensor (2.1). Similarly the scalar field $\varphi$ can be split into a background part and a perturbation, which we take to be

$$\varphi = \varphi_0 + \delta\varphi_1 + \frac{1}{2}\delta\varphi_2, \quad (2.2)$$

where $\varphi_0$ is the homogeneous background, $\delta\varphi_1$ is chosen to obey Gaussian statistics, and the higher terms $\delta\varphi_n$, $\delta\varphi_n$ for $n \geq 2$ are polynomials in $\delta\varphi_1$ of order $n$. Note that since we work in the flat gauge, the field fluctuation at first and second orders is merely the associated Sasaki–Mukhanov variable [9, 43, 44]. The authors of [28, 29] use a different convention for the perturbations. However, by observing that their field perturbation $\delta\varphi_{SL}$ is related to our perturbations $\delta\varphi_1$ and $\delta\varphi_2$ by $\delta\varphi_{SL} = \delta\varphi_1 + \frac{1}{2}\delta\varphi_2$, we can readily compare our results.

2.2. Action and energy–momentum tensor

The action for a single scalar field minimally coupled to standard Einstein gravity is [45, 46]

$$S = \int d^4 x \sqrt{-g} \left( \frac{1}{2} R + L_\varphi \right) + \int d^3 x \sqrt{h} K, \quad (2.3)$$

where $g_{\mu\nu}$ is the spacetime metric, $g \equiv \det g_{\mu\nu}$ is its determinant, $R$ is the spacetime Ricci scalar, and $\partial$ denotes the boundary of spacetime, if one exists. If a boundary is present, then $h_{ij}$ is its first fundamental form (also referred to as the induced metric, obeying $h_{ij} = g_{ij}$ in the case of the boundary being a hypersurface of constant time), and $K_{ij}$ its second fundamental form (or extrinsic curvature), where $K = \text{tr} K_{ij}$.

The matter Lagrangian is given by

$$L_\varphi = -\frac{1}{2}g^{\mu\nu}\varphi_\mu\varphi_\nu - U(\varphi), \quad (2.4)$$

where $\varphi$ is the scalar field, $U$ is its potential and $\varphi_\mu = \partial \varphi / \partial x^\mu$. The energy–momentum tensor is obtained by varying the action with respect to the metric, giving

$$T_{\mu\nu} = -2\frac{\partial L_\varphi}{\partial g_{\mu\nu}} + g_{\mu\nu}L_\varphi. \quad (2.5)$$

When the field equations are satisfied, this is automatically conserved as a consequence of Noether’s theorem. For a single scalar field, $T_{\mu\nu}$ takes the form

$$T_{\mu\nu} = \left[ \varphi_\mu \varphi_\nu - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\varphi_\rho\varphi_\sigma \right] - g_{\mu\nu}U(\varphi). \quad (2.6)$$

3. Deriving the second-order Klein–Gordon equation using the field equations

We now proceed to derive the governing equation for the scalar field, namely the Klein–Gordon equation, using the field equations. We therefore start by writing Einstein’s field equations

$$G_{\mu\nu} = T_{\mu\nu}, \quad (3.1)$$

which can be found by varying the action (2.3). For this purpose the inclusion of the Gibbons–Hawking term on the boundary is essential. The Bianchi identities then imply the conservation
of energy–momentum,
\[ \nabla^\mu T_{\mu\nu} = 0, \]  
(3.2)
where \( \nabla^\mu \) denotes the covariant derivative.

### 3.1. Field dynamics

The energy conservation equation, equation (3.2), gives an evolution equation for the \( \delta\phi_n \) order-by-order (see e.g. [18] for details). At zeroth order in perturbations, one finds the background equation
\[ \phi''_0 + 2H\phi'_0 + a^2 U_{,\phi} = 0, \]  
(3.3)
where \( H \equiv a'/a \) and is related to the Hubble parameter \( H \) by \( \mathcal{H} = aH \). At first order we find
\[ \delta\phi'^{\prime}_1 + 2H\delta\phi'_1 - \nabla^2\delta\phi_1 + 2a^2 U_{,\phi}\phi_1 - \phi'_0 \nabla^2 B_1 - \phi'_0 \phi'_1 + a^2 U_{,\phi,\phi} \delta\phi_1 = 0, \]  
(3.4)
whereas after some calculation the second-order equation becomes
\[ \delta\phi''_2 + 2H\delta\phi'_2 - \nabla^2\delta\phi_2 + a^2 U_{,\phi,\phi} \delta\phi_2 + a^2 U_{,\phi,\phi,\phi} \delta\phi_1 - 2a^2 U_{,\phi}\phi_2 - \phi'_0 (\nabla^2 B_2 + \phi_2) 
+ 4\phi'_0 B_1 \delta^k \phi_1 + 2(2H\phi'_0 + a^2 U_{,\phi}) \phi_1 \partial_k B_1 + 4\phi_1 (a^2 U_{,\phi,\phi} \delta\phi_1 - \nabla^2 \delta\phi_1) 
+ 4\phi'_0 \phi'_1 \phi_1 - 2\delta\phi_1 (\nabla^2 B_1 + \phi'_1) - 4\partial_k \delta\phi_1 \delta^k B_1 = 0. \]  
(3.5)

In order to arrive at an equation which can reasonably be thought of as a second-order version of the Klein–Gordon equation, we must eliminate the terms \{\( B_1, B_2, \phi_1, \phi_2 \)\} which come from the metric and rewrite them in terms of the field fluctuations \{\( \delta\phi_1, \delta\phi_2 \)\}. This is done by using the constraint part of the Einstein equations. Since we will use the slow-roll approximation to control this part of the calculation, it is first necessary to decide to what order in the approximation our calculations must be carried out.

The slow-roll expansion can be thought of as an expansion in powers of \( 2\epsilon \equiv (\phi'_0/\mathcal{H})^2 \), together with other small quantities obtained by differentiating \( \epsilon \) with respect to cosmic time \( t \). It is easy to see from equation (3.5) that all intrinsically second-order terms (namely, \( B_2 \) and \( \phi_2 \)) are accompanied by a factor of \( \phi'_0 \), whereas some terms formed from the product of first-order quantities appear without any factors of the time derivative of the background field. Therefore we will aim to compute only to \( \phi'_0/\mathcal{H} \sim \sqrt{\epsilon} \) for which purpose it is sufficient to obtain second-order quantities only to zeroth order in slow-roll, but we must carry out the calculation of first-order quantities to the first non-trivial order.

When \( \epsilon \ll 1 \), one says that the field is in the slow-roll regime. Extracting the leading slow-roll part of the background equation, equation (3.3), we obtain
\[ 3\mathcal{H}\phi'_0 + a^2 U_{,\phi} = 0. \]  
(3.6)

### 3.2. Einstein equations

We now turn to the Einstein equations. These break into ten independent equations, of which some are evolution equations and some are constraints. In the present case there is only one degree of freedom, the field fluctuation \{\( \delta\phi_1, \delta\phi_2 \)\}, and therefore there will ultimately be only one non-trivial evolution equation. In this section we aim to use the remaining equations, which are constraints, to obtain the metric quantities \{\( B_1, B_2, \phi_1, \phi_2 \)\} in terms of the Gaussian-field fluctuation, \( \delta\phi_1 \).
This gives an evolution equation for $\mathcal{H}$, namely $\mathcal{H}' = \mathcal{H}^2$. 

### 3.2.2. First order

Now consider the terms which are first order in perturbations. The $(0, 0)$ component of the Einstein equations is

$$2a^2\mathcal{H}_0\phi_1 + \phi_0'\delta\phi_1' + a^2\delta U_1 + 2\mathcal{H}\nabla^2 B_1 = 0,$$

and the $(0, i)$ part gives

$$\partial_i(2\mathcal{H}\phi_1 - \phi_0'\delta\phi_1) = 0,$$

where the gradient can readily be removed, e.g. by working Fourier space. Working to first order in slow roll and using the background equations given above, we find that equation (3.8) can be rewritten as

$$\phi_0'\delta\phi_1' + 2\mathcal{H}^2B_1 = 0.$$ 

This is a constraint which allows us to eliminate $B_1$ from equation (3.5) in favour of the field fluctuation $\delta\phi_1$.

The $(i, j)$ component of the Einstein equation is given by

$$(2\mathcal{H}\phi_1' - \phi_0'\delta\phi_1' + 2a^2U_0\phi_1 + a^2\delta U_1\delta_{ij} + (\nabla^2\delta_{ij} - \partial_i\partial_j)(B_1' + 2\mathcal{H}B_1 + \phi_1) = 0.$$ 

(3.11)

Taking the trace of equation (3.11) gives one scalar equation,

$$3(2\mathcal{H}\phi_1' - \phi_0'\delta\phi_1' + 2a^2U_0\phi_1 + a^2\delta U_1) + 2\nabla^2(B_1' + 2\mathcal{H}B_1 + \phi_1) = 0.$$ 

(3.12)

At linear order we have now two options to 'extract' another scalar equation from equation (3.11):

- It is possible to simply read off the off-diagonal (or $i \neq j$) part of the $(i, j)$ equation, equation (3.11). This method was used by Mukhanov, Feldman and Brandenberger [3] at first order, in conjunction with invariance arguments which allowed them to extend the three independent $i \neq j$ equations to a single scalar equation which constrained the $i \neq j$ term to vanish.

- Another possibility, advocated here, is to apply the operator $\partial^i\partial^j$ to equation (3.11). The benefit in doing so is that one arrives at a scalar equation via manipulations which are manifestly tensorial. Observe that the operator $\nabla^2\delta_{ij} - \partial_i\partial_j$ is transverse to $\partial^i\partial^j$ and therefore vanishes, leaving an evolution equation for $\phi_1$.

Although both methods yield valid results at first order, it is not clear how the former should be extended to second order. On the other hand, the second method requires only textbook manipulations, leading to a correct scalar equation via an essentially mechanical process. Its principal disadvantage is the necessity to introduce inverse Laplacian operators. In either case, one can extract an evolution equation for the shear,

$$\nabla^2(B_1' + 2\mathcal{H}B_1 + \phi_1) = 0,$$

(3.13)

and an evolution equation for the lapse function

$$2\mathcal{H}\phi_1' + a^2\delta U_1 + 2a^2U_0\phi_1 - \phi_0'\delta\phi_1' = 0.$$ 

(3.14)

After dropping terms which are higher order in slow roll this simplifies substantially, yielding

$$2\mathcal{H}\phi_1' - \phi_0'\delta\phi_1' = 0.$$ 

(3.15)

Any combination of equation (3.10) with equations (3.13) or (3.15) allows us to determine $\phi_1$ in terms of $\delta\phi_1$. We therefore have both metric potentials which are necessary to rewrite equation (3.4) as a single equation for $\delta\phi_1$. 

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3.2.3. Second order. As pointed out in section 3.1 we only require the second-order field equations to zeroth order in slow roll. Reducing the calculation to the leading order slow-roll effect considerably simplifies both the calculation and presentation of our results.

The (0, i) Einstein equation is given by

$$\partial_i (\mathcal{H} \phi_2 - \delta \phi'_1 \partial_i \delta \phi_1) = 0.$$  \hspace{1cm} (3.16)

This can be rewritten as a scalar equation by taking its divergence, yielding

$$\mathcal{H} \phi_2 - \nabla^2 (\delta \phi'_1 \nabla^2 \delta \phi_1 + \partial_i \delta \phi'_1 \partial^i \delta \phi_1) = 0,$$  \hspace{1cm} (3.17)

where we introduce the inverse Laplacian, which satisfies the identity $\nabla^{-2}(\nabla^2)X = X$.

The second order $(i, j)$-equation is given by

$$(2\mathcal{H} \phi'_2 + 2a^2 U_i \phi_2) \delta_{ij} + (\nabla^2 \delta_{ij} - \partial_i \partial_j)(B^2 + 2\mathcal{H} B_2 + \phi_2)$$

$$+ (\partial_k \delta \phi_1 \partial^k \delta \phi_1 - \delta \phi_1 \delta \phi_1) \delta_{ij} - 2\partial_i \delta \phi_1 \partial_j \delta \phi_1 = 0.$$  \hspace{1cm} (3.18)

The spatial trace is obtained from equation (3.18) and satisfies

$$3\mathcal{H} \phi'_2 + 3a^2 U_i \phi_2 + \nabla^2 (B^2 + 2\mathcal{H} B_2 + \phi_2) + \frac{1}{2}(-3\delta \phi'_1 \delta \phi_1 + \partial_i \partial_j \delta \phi'_1 \delta \phi_1) = 0.$$  \hspace{1cm} (3.19)

Following the discussion above, we now let $\partial_i \partial_j$ act on equation (3.18) to extract another scalar equation from the $(i, j)$ part, and observe that $\nabla^2(\nabla^2 \delta \phi_1 - \partial_i \partial_j)X = X$. This gives

$$\mathcal{H} \phi'_2 + 3\mathcal{H} \nabla^2 (\delta \phi'_1 \nabla^2 \delta \phi_1 + \partial_i \partial_j \delta \phi'_1 \partial^k \delta \phi_1) - \nabla^{-2}(\nabla^2 \delta \phi_1 \nabla^2 \delta \phi_1)$$

$$+ \delta \phi'_1 \nabla^2 \delta \phi'_1 + \partial_i \partial_j \delta \phi'_1 \partial^k \delta \phi_1 + \partial_i \delta \phi_1 \partial^k \delta \phi'_1 = 0.$$  \hspace{1cm} (3.20)

Note that this usefulness of this procedure is not specific to our choice of gauge or dependent on invoking the slow-roll approximation. We believe it is simplest to employ our method for extracting a scalar equation from the $(i, j)$ part of the Einstein equation, (3.18), whenever the equation cannot be written as a total derivative. This is in general the case at second order and higher, when terms of the form $X_{ij}Y_{ij}$ etc will appear.

How can we be confident that we are proposing the correct method? The first reason is that only this approach leads to a consistent set of Einstein equations. The other is that we get the same result for the Klein–Gordon equation using the action method, as shown in section 4.

Since equation (3.20) is just an evolution equation for the second-order lapse function, another way of deriving it is to take the time derivative of the constraint (3.17). In this case we also need the first-order Klein–Gordon equation in the extreme slow-roll limit, which follows from equation (3.4) as

$$\delta \phi''_1 + 2\mathcal{H} \delta \phi'_1 - \nabla^2 \delta \phi_1 = 0.$$  \hspace{1cm} (3.21)

Substituting equation (3.21) into the time derivative of equation (3.17), we recover equation (3.20), as required.

3.3. The Klein–Gordon equation at second order

We can now substitute the field equations at first and second orders into the Klein–Gordon equation (3.5). Working to leading order in slow roll we get

$$\delta \phi''_2 + 2\mathcal{H} \delta \phi'_2 - \nabla^2 \delta \phi_2 + a^2 U_{,\phi \phi \phi} \delta \phi_1 + \frac{\delta \phi'_2}{\mathcal{H}} \left[ \frac{1}{2} (\delta \phi'_1 \delta \phi_1 + \partial_k \delta \phi_1 \partial^k \delta \phi_1) - 2\delta \phi_1 \nabla^2 \delta \phi_1$$

$$- \nabla^{-2}(\nabla^2 \delta \phi_1 \nabla^2 \delta \phi_1 + \delta \phi_1 \nabla^2 \delta \phi'_1 + \partial_i \delta \phi_1 \nabla^2 \delta \phi_1 + \partial_i \delta \phi_1 \partial^k \delta \phi_1)$$

$$+ 2\partial_i \delta \phi_1 \nabla^2 \delta \phi'_1 \right] = 0.$$  \hspace{1cm} (3.22)

When applying the slow-roll expansion, one ordinarily thinks of the higher derivatives of the potential, such as $U_{,\phi \phi \phi}$, as being negligible in comparison with the leading slow-roll terms.
In equation (3.22) this term has been retained. This is because although $U,\phi/U \ll 1$ is necessary in order for inflation to occur at all, and $U_{\phi\phi\phi}/U \ll 1$ is necessary in order to have sufficient e-foldings, there are no such constraints for higher derivatives. Therefore it may well be that models exist in which $U_{\phi\phi\phi\phi}$ is unusually large. In such a model, $U_{\phi\phi\phi\phi}$ will make a contribution to the non-Gaussianity imprinted in the cosmic microwave background which cannot be ignored [47].

4. Derivation from the third-order action

Let us now return to the action and derive the Klein–Gordon equation from it. This follows a somewhat different procedure than that used with the field equations, owing to the way constraints are implemented in the action formalism.

The action is found to depend only algebraically on the lapse and shift functions and their spatial derivatives; it involves no time derivatives of these quantities. Since one finds the equation of motion for each field, say $W$, by varying the action with respect to $W$ and demanding that the resulting variation $\delta S/\delta W$ is zero, the equations of motion for the lapse $\phi$ and the shift $B$ are also purely algebraic. They can be solved to give expressions in terms of the field fluctuation which hold at all times and do not require integrating any equation of motion. Once $\phi$ and $B$ have been determined, they can be eliminated from the Lagrangian. Varying the resulting action with respect to the field gives the equation of motion directly, which must coincide with the Klein–Gordon equation derived above.

This calculation was initially given (in the flat and comoving slicings) by Maldacena [26] and refined in the flat slicing in [29]. In conformal time, the part of the action quadratic in $\delta \phi^2$ can be written at leading order in slow roll,

$$S_2 = \frac{1}{8} \int d\eta d^3x a^2 \left( (\delta \phi_2')^2 - (\partial \delta \phi_2)^2 \right).$$

(4.1)

On the other hand, there is also an ‘interaction’ term which involves a product of the $\delta \phi_n$. The leading term in this interaction is linear in $\delta \phi_2$ and quadratic in $\delta \phi_1$,

$$S_3 = \int d\eta d^3x a^2 \left[ \frac{1}{3!} a^2 U_{\phi\phi\phi\phi} \delta \phi_2 (\delta \phi_1)^2 + \frac{4 \phi_0}{4\pi} \delta \phi_2 \delta^4 \nabla^{-2} \delta \phi_1 \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial \phi_1} \delta \phi_1 - \frac{1}{2} \delta \phi_2 \left( (\delta \phi_1)^2 + (\partial \delta \phi_1)^2 \right) \right] + \text{permutations},$$

(4.2)

where the permutations are formed by swapping the position of $\delta \phi_2$ among the three possible locations. The higher derivative $U_{\phi\phi\phi\phi}$ has been included to account for the possibility that it is unusually large, as discussed above. The field equation for $\delta \phi_2$ is obtained by demanding that $\delta S/\delta (\delta \phi_2) = 0$, ignoring any boundary terms.

Consider first $S_2$. It is clear that $\delta S_2/\delta (\delta \phi_2)$ can be written as

$$\delta S_2 = \frac{1}{4} \int d\eta d^3x \delta (\delta \phi_2) a^2 (-\delta \phi'' - 2\partial \delta \phi' + \nabla^2 \delta \phi_2) + \frac{1}{4} \int d^3x a^2 \delta (\delta \phi_2) \delta (\delta \phi_2).$$

(4.3)

Now consider $S_3$. From left to right this breaks into four terms, the first of which is the variation of the $U_{\phi\phi\phi\phi}$ term and is trivial. The variation of the second term gives (including the effect of permuting the location of $\delta \phi_2$)

$$\delta S_3 \supset \int d^3x a^2 \frac{\phi_0}{4\pi} \delta (\delta \phi_2) \left[ d^4 \nabla^{-2} \delta \phi_1 \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial \phi_1} \delta \phi_1 - \nabla^2 \left( d^4 \delta \phi_1 \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial \phi_1} \delta \phi_1 + \delta \phi_1 \right) \right]
+ \int d\eta d^3x a^2 \frac{\phi_0}{4\pi} \delta (\delta \phi_2) \left\{ \frac{4}{a^2} \delta^4 \delta \phi_1 \nabla^2 \delta (\delta \phi_1) - \frac{4}{a^2} \nabla^2 \right)$$
that one needs to obtain explicit expressions only to the
precise way these appear in the action, there is a special simplification which implies
eliminate the constraints associated with the lapse function and shift vector. However, owing
present paper. For the action, one must compute the Lagrangian to the
convenient to obtain an evolution equation from the Bianchi identities, as we have done in the
principle this is contained in the Einstein equations themselves, but in practice it is often more
appropriate for the task in hand. In the case of the Einstein equations,
equations are more appropriate for the task in hand. In this paper, we have derived the Klein–Gordon equation at second-order in the perturbations
in slow roll. We have shown explicitly for the first time that the two most
popular approaches in the literature, based on the Einstein field equations and the action
principle, yield equivalent results. This goes some way to demonstrating the consistency of
and, likewise, the fourth term contributes
These terms all combine to give the total variation. Discarding those terms proportional to
δS2/δ(δφ1) together with all surface terms, one finds the overall Klein–Gordon equation
δφ2'' + 2Hδφi′' − V^2 δφ2 + a^2 U_{ϕϕϕ}(δφ1)^2
= \frac{\dot{\phi}''}{\mathcal{H}} \left\{ -\frac{1}{2} \partial^k \delta \phi_1 \partial_k \delta \phi_1 - \frac{1}{2} (\delta \phi_1')^2 + 2 \delta \phi_1 V^2 \delta \phi_1 - 2 \partial^k \delta \phi_1' \partial_k V^{-2} \delta \phi_1' \right\} + V^{-2} \left[ \delta \phi_1^2 \delta \phi_1' + \partial^2 \delta \phi_1 \partial_k \delta \phi_1 + \partial^2 \delta \phi_1' \partial_k \delta \phi_1' + \delta \phi_1' \delta \phi_1' \right].

(4.7)

Equation (4.7) agrees exactly with the Klein–Gordon equation which was found using the
Einstein equations, equation (3.22).

5. Discussion

In this paper, we have derived the Klein–Gordon equation at second-order in the perturbations
and leading order in slow roll. We have shown explicitly for the first time that the two most
popular approaches in the literature, based on the Einstein field equations and the action
principle, yield equivalent results. This goes some way to demonstrating the consistency of
current beyond-leading-order calculations, including those of the nonlinearity parameters f_{NL}
and τ_{NL} [26, 29, 31] which will be important for future CMB experiments.

In many cases one can choose freely whether the action principle or the Einstein field
equations are more appropriate for the task in hand. In the case of the Einstein equations,
one needs the relevant constraint equations at the nth order, and an evolution equation. In
principle this is contained in the Einstein equations themselves, but in practice it is often more
convenient to obtain an evolution equation from the Bianchi identities, as we have done in the
present paper. For the action, one must compute the Lagrangian to the (n + 1)th order, and then
eliminate the constraints associated with the lapse function and shift vector. However, owing
to the precise way these appear in the action, there is a special simplification which implies
that one needs to obtain explicit expressions only to the (n − 1)th order. The result, at least
for low orders of perturbation theory, is that the amount of work required in both approaches seems much the same.

These two methods are traditionally used for different purposes. The field equations are ordinarily the method of choice for obtaining numerical solutions. On the other hand, the action is typically preferred when one aims to compute scattering amplitudes, or more generally the correlation functions of a quantized theory. This is because the action, being a scalar, manifestly exhibits the full group of symmetries associated with the theory. Moreover, it facilitates the use of the so-called interaction picture which is the most common approach for dealing with fields beyond first order in perturbation theory.

In some cases, however, there is no choice and one must work with the equations of motion. For example, this is the case with dissipative systems such as fluids for which it is difficult or impossible to formulate an action principle. In this case one would like to extend the existing second-order framework to allow the computation of quantities traditionally associated with the action, such as the correlation functions of the quantized theory. The analysis given in this paper is a first step in that direction.

We have pointed out that some care is required in reducing the full set of Einstein equations. In general this necessitates contracting spatial tensors into scalar equations by the application of a suitable derivative operator. Together with the spatial trace, this gives two scalar equations which at first order coincide with the well-known standard results [3]. At second order, the analysis requires some more care in order to obtain correct results.

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