ALGEBRAIC TOPOLOGY OF POLISH SPACES: I. FINE SHAPE

SERGEY A. MELIKHOV

ABSTRACT. Fine shape of Polish spaces is a common “correction” of strong shape and compactly generated strong shape (which differ from each other essentially by permuting a direct limit with an inverse limit). For compacta fine shape coincides with strong shape, and in general it involves a stronger form of coherence which amounts to taking into account a natural topology on the indexing sets. Both Steenrod–Sitnikov homology and Čech cohomology are proved to be invariant under fine shape, which cannot be said of any previously known shape theory. In general, fine shape invariance of a (co)homology theory is a strong form of homotopy invariance which implies the map excision axiom.

1. Introduction

The use of \( \lim^1 \) (and in extreme cases also \( \lim^2, \lim^3, \ldots \)) provides a reasonable description of any limiting behavior in homology and cohomology for simplicial complexes (or CW complexes) and, on the other hand, for compact spaces. In contrast, homology and cohomology (even ordinary) of non-triangulable non-compact spaces have been quite poorly understood until recently, due to the lack of any clues on how direct limits (colim) interact with inverse limits (lim). The present paper along with its companion papers [1], [2], [3] make a few first steps in this direction. While actual computations in homology and cohomology take place in [1] and [2] and to some extent [3], the present paper is concerned with “conceptualizing” them by describing a shape-theoretic framework where they naturally belong.

A shape theory is something that is supposed to

- fully agree with homotopy theory on polyhedra (i.e., simplicial complexes with the metric topology) and other ANRs, and
- provide a “geometrically substantial” extension of homotopy theory to more general spaces by means of approximating them by polyhedra or ANRs.

For (metrizable) compacta, there is just one reasonable shape theory, best known as strong shape, which does everything that one expects of a shape theory (see e.g. [25] for a treatment). One definition of the strong shape category for compacta [13], [21], [12], motivated by Chapman’s characterization of shape, is that its objects are complements of compact Z-sets in the Hilbert cube, and morphisms are proper homotopy classes of maps between these. (It is this definition that will be generalized in the present paper.) Strong shape of compacta was originally introduced in a Princeton dissertation by D. E. Christie.

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(supervised by Lefschetz), which was published in 1944 [10] but remained unnoticed for about 30 years, until the theory was rediscovered by Borsuk (in a weak form, which became known as “shape”) [8] and “corrected” by a number of others in the 70s [32], [17], [13], [21], [12], [16; §5].

Shape theories of non-compact spaces is a different matter: while a lot has been written about them in the past 50 years, still very little is actually understood (as discussed below). In fact, given a considerable number of various shape theories in the literature, and with an obvious trend towards degeneration in the entire field, the author does not feel easy about introducing yet another shape theory. But hopefully this can be excused by the expectations that this new shape theory is capable of supersed all previously available ones (in the context of Polish spaces). Some reasons for such expectations are discussed below.

1.1. Fine shape. The definition of fine shape category is given in §2; the basic idea is already present in the first page (§2.1). The name “fine shape” is chosen because it sounds better than e.g. “very strong shape”, while being compatible with the literature. Y. Kodama and his students have used “fine shape” to call what is now better known as strong shape of compacta [21]. As far as I know, they have only applied this terminology in the compact case, so it appears to be free in the non-compact case.

One remarkable feature of fine shape of Polish spaces is that three naturally available approaches to it:

- one based on neighborhoods,
- one based on compacta and
- one based simultaneously on neighborhoods and compacta

yield the same result. A basic form of this somewhat miraculous symmetry appears already in Proposition 2.1. Borsuk [9] has observed that in his original setting (of non-strong shape defined by countable fundamental sequences) the corresponding three approaches all differ from each other on non-compact metrizable spaces, even though they all coincide on compact metrizable spaces. This observation of Borsuk dates from 1972, when, in his own words, “the attempts to extend [the theory of shape of compacta] onto arbitrary metrizable spaces [w]ere only at the beginning” [9]. However, subsequent literature on the subject only aggravated the situation by producing still more of inequivalent definitions (see e.g. [35]). The basic dichotomy was between (strong) shape and compactly generated (strong) shape; but as the two approaches are now reconciled by fine shape, this appears to have been a false dichotomy after all.

Another remarkable feature of fine shape is that both Steenrod–Sitnikov homology\(^1\) and Čech cohomology\(^2\) are proved to be its invariants (see §3). This appears to be a first result of its kind. It is quite trivial that Čech cohomology is an invariant of strong shape, whereas Steenrod–Sitnikov homology is an invariant of compactly generated strong shape. However, it remains unknown whether Steenrod–Sitnikov homology

\(^1\) Also known as Massey homology and as Borel–Moore homology with constant coefficients.
\(^2\) Also known as Alexander–Spanier cohomology and as sheaf cohomology with constant coefficients.
of Polish spaces is an invariant of strong shape. In 1995 Sklyarenko emphasized the question of strong shape invariance of Steenrod–Sitnikov homology in the wider class of paracompact spaces and remarked that its solution “will likely be a test not only for Steenrod–Sitnikov homology, but also for the strong shape theory itself” [37].

1.2. Borsuk versus Fox. There also exist “inverted” theories: strong homology, which is an invariant of strong shape, and strong cohomology, which is an invariant of compactly generated strong shape. However, their usefulness for topology (as opposed to foundations of mathematics) is questionable, as both of them cannot be computed in ZFC for the very simplest examples (see [3; Example 1.1]). Moreover, it turns out that they can be “corrected” by taking into account a natural topology of the indexing set, and upon such correction turn into the usual Steenrod–Sitnikov homology and Čech cohomology [3].

The set-theoretic troubles that strong homology brings to light are of course implicit in the very concept of strong shape for non-compact spaces, and stem directly from the set-theoretic treatment of the uncountable indexing sets of inverse systems. It was Fox who first opened the Pandora’s box, but actually he was rather cautious about it himself, reverting in a later paper from arbitrary inverse systems to inverse sequences: “In [[14]] I gave a somewhat more general definition (inverse system) but I now believe that that more general definition, although perfectly workable, may have been a tactical error on my part. (In topology I know of no convincing reason for the general definition, although I have recently heard that in algebraic geometry there may be.)” [15]

By contrast, Borsuk ignored the new fashion and remained faithful to his (countable) fundamental sequences. But, of course, they are even more obviously inadequate, since a countable sequence is almost never cofinal in the non-compact case (see [2; Theorem 3.1]). The present paper proposes a new take on this notorious issue, which in some sense might be seen as reconciling Borsuk’s and Fox’s approaches.

1.3. Filtrations versus telescopes. In the approach of the present paper, we do without any telescopes of ANRs (which are at least implicit in both Borsuk’s and Fox’s approaches to shape). Instead, we are content with a single ANR $M_X$ containing our Polish space $X$ of interest as a closed $Z$-set, but we also record a filtration $F_X$ on $M_X \setminus X$ that remembers something about the topology of $X$. A fine shape map $X \leadsto Y$ is modeled by a continuous map $M_X \setminus X \to M_Y \setminus Y$ respecting the filtrations. There are at least two good possibilities for what the filtration $F_X$ could be (and also their combination), but as mentioned above they miraculously yield the same result.

When $X$ is a compactum, the filtration has a countable cofinal subset, and it is not hard to see that a filtration-respecting map of the ANRs is equivalent to a usual level map between the telescopes of the countable subfiltrations (see, in particular, [2; proof of Theorem 2.13]).

In the general case, one can similarly show that a filtration-respecting map of the ANRs is equivalent to a level map between modified telescopes of the original uncountable filtrations, where the modification takes into account a natural topology on the indexing
sets; in contrast to the usual telescopes of uncountably-indexed filtrations, the modified 
telescopes are separable and metrizable (see, in particular, [3; proof of Theorem 3.4]).

2. Fine shape

By a space we mean a topological space (rather than a uniform space). By default all spaces are assumed to be metrizable. By ARs and ANRs we mean those for metrizable spaces.

2.1. Filtered and cofiltered spaces. A poset $P$ is called directed if any two elements of $P$ have an upper bound; and codirected if any two elements of $P$ have a lower bound. (This is a special case of filtered and cofiltered categories.)

A space $M$ endowed with a directed (resp. codirected) family of subsets $\kappa$ (resp. $\nu$) will be called filtered (resp. cofiltered) and will be denoted $M_\kappa$ (resp. $M_\nu$).

Let $X$ be a closed subset of a space $M$. If $U$ and $V$ are open neighborhoods of $X$ in $M$, then $U \cap V$ is another one. Thus

$$\nu_M(X) := \{U \setminus X \mid U \text{ is an open neighborhood of } X \text{ in } M\}$$

is a codirected family of subsets of $M \setminus X$. We will call it the neighborhood cofiltration and often abbreviate to $\nu X$. Clearly, each member of $\nu X$ is open in $M \setminus X$.

$\nu_M(X)$ contains the same information as the co-neighborhood filtration

$$\bar{\nu}_M(X) := \{F \subset M \setminus X \mid F \text{ is closed in } M\},$$

which is often more convenient to use, and will often be abbreviated as $\bar{\nu} X$. It is a directed family of subsets of $M \setminus X$ whose members are closed in $M \setminus X$.

On the other hand, if $K$ and $L$ are compact subsets of $M$, then $K \cup L$ is another one. Thus

$$\kappa_M(X) := \{K \setminus X \mid K \text{ is a compact subset of } M\}$$

is a directed family of subsets of $M \setminus X$. We will call it the compact filtration and often abbreviate to $\kappa X$. Each member of $\kappa X$ is locally compact (since it is an open subset of a compact space) and closed in $M \setminus X$.

**Proposition 2.1.** Let $X$ be a closed subset of a metric space $M$ and let $S$ be a closed subset of $M \setminus X$. Then

(a) $S \in \bar{\nu} X$ if and only if $S$ meets every member of $\kappa X$ in a compact set.

(b) $S \in \kappa X$ if and only if $S$ meets every member of $\nu X$ in a compact set.

Thus $\kappa X$ and $\nu X$ in fact determine each other.

**Proof.** First we note that if $U$ is an open neighborhood of $X$ in $M$ and $K$ is a compact subset of $M$, then $(M \setminus U) \cap K$ is a closed subset of $K$, and so must be compact. Also it is disjoint from $X$, and hence coincides with $((M \setminus X) \setminus (U \setminus X)) \cap (K \setminus X)$. This implies the “only if” assertions in (a) and (b).

Suppose that $S$ is not a member of $\bar{\nu} X$. Then $S$ is not closed in $M$. Since $S$ is closed in $M \setminus X$, some sequence of points $x_i \in S$ converges to some $x \in X$. Let $K = \{x_i \mid i \in \mathbb{N}\}$.
Then $K \cup \{x\}$ is compact, so $K \in \kappa X$. But $S \cap K = K$, which is non-compact. Thus $S$ meets the element $K$ of $\kappa X$ in a non-compact set.

Suppose that $S$ is not an element of $\kappa X$. Then the closure $\bar{S}$ of $S$ in $M$ is non-compact. Hence there exists a sequence of points $x_i \in \bar{S}$ which has no limit points in $\bar{S}$. Since $\bar{S}$ is closed in $M$, it also has no limit points in $M$. Each $x_i$ is a limit of a sequence of points $x_{ij} \in S$. We may assume up to a renumbering that $d(x_{ij}, x_i) < 1/j$ for each $j$. Then the sequence $x_{ii}$ also has no limit points. Hence $F := \{x_{ii} \mid i \in \mathbb{N}\}$ is non-compact and closed in $M$. Then, in particular, $M \setminus F$ is an open neighborhood of $X$, and therefore $F$ is a member of $\nu X$. But $S$ meets $F$ in $F$, which is non-compact. □

2.2. Approaching maps. A continuous map $f : M \to N$ will be called a filtered map between filtered spaces $M_\kappa$ and $N_\lambda$ if every $K \in \kappa$ is sent by $f$ into some $L \in \lambda$; and a cofiltered map between cofiltered spaces $M^\nu$ and $N^\mu$ if every $U \in \mu$ contains the $f$-image of some $V \in \nu$.

If we write $L := f_0(K)$ and $V := f^0(U)$, then $f_\kappa : \kappa \to \lambda$ and $f^\mu : \mu \to \nu$ must be (not necessarily order-preserving) maps such that $f(K) \subset f_0(K)$ for all $K \in \kappa$ and $f^0(U) \subset f^{-1}(U)$ for all $U \in \mu$. While these maps $f_\kappa$ and $f^\mu$ are useful, they are not a part of the structure; only their existence is required.

If $X$ is a closed subset of a space $M$ and $Y$ a closed subset of a space $N$, and $F : M \to N$ is a continuous map such that $F^{-1}(Y) = X$, then its restriction $f : M \setminus X \to N \setminus Y$ is a filtered map $(M \setminus X)_\kappa X \to (N \setminus Y)_\kappa Y$ and a cofiltered map $(M \setminus X)^\nu X \to (N \setminus Y)^\nu Y$. Indeed, we may set $f^\nu(U \setminus Y) = f^{-1}(U) \setminus X$ and $f_\kappa(K \setminus X) = f(K) \setminus Y$. Let us note that, in fact, $f^\nu(U \setminus Y) = f^{-1}(U \setminus Y)$ and $f_\kappa(K \setminus X) = f(K \setminus X)$. Due to this, $f^\nu : \nu Y \to \nu X$ and $f_\kappa : \kappa X \to \kappa Y$ are monotone maps of posets.

Moreover, since $F^{-1}(Y) = X$, the restriction of $f$ to each member of $\kappa$ is a proper map. We recall that a map $\varphi$ is called proper if the $\varphi$-inverse of every compact set is compact.

Proposition 2.2. Let $X$ be a closed subset of a space $M$ and $Y$ a closed subset of a space $N$. Let $f : M \setminus X \to N \setminus Y$ be a continuous map. The following are equivalent:

1. $f$ is a cofiltered map $(M \setminus X)^\nu X \to (N \setminus Y)^\nu Y$;
2. $f$ is a filtered map $(M \setminus X)_\kappa X \to (N \setminus Y)_\kappa Y$ and its restriction to each member of $\kappa X$ is a proper map.

Proof. Suppose that $f$ is cofiltered and let $K \in \kappa X$. Given a $U \in \nu Y$, $f(K)$ meets $(N \setminus Y) \setminus U$ in $f(L)$, where $L$ is the intersection of $K$ with $(M \setminus X) \setminus f^{-1}(U)$. Since $f^{-1}(U) \subset \nu X$, by 2.1(b) $L$ is compact. Hence $f(L)$ is compact. So by 2.1(b) $f(K) \subset \kappa Y$.

Next, let $Q$ be a compact subset of $f(K)$. Then $Q$ is closed in $N$, and therefore $f^{-1}(Q)$ is closed in $M$. Since $K \in \kappa X$, we have $K = \bar{K} \cap (M \setminus X)$, where $\bar{K}$ is a compact subset of $M$. Then $f^{-1}(Q) \cap \bar{K}$ is compact. But since $f^{-1}(Q) \subset M \setminus X$, we have $f^{-1}(Q) \cap \bar{K} = f^{-1}(Q) \cap K$. Thus $(f|_K)^{-1}(Q) = f^{-1}(Q) \cap K$ is compact.

Conversely, suppose that $f$ is filtered and its restrictions to the members of $\kappa X$ are proper maps, and let $U \in \nu Y$. Given a $K \in \kappa X$, $(M \setminus X) \setminus f^{-1}(U)$ meets $K$ in $(f|_K)^{-1}(Q)$,
where $Q$ is the intersection of $f(K)$ with $(N \setminus Y) \setminus U$. By 2.1(a) $Q$ is compact. Since $f|_K$ is proper, $(f|_K)^{-1}(Q)$ is also compact. Hence by 2.1(a) $f^{-1}(U) \in \nu X$.

If $X$ is a closed subset of a space $M$ and $Y$ is a closed subset of a space $N$, a continuous map $f : M \setminus X \to N \setminus Y$ will be called an $X$-$Y$-approaching map (or simply an approaching map) if it satisfies any of the equivalent conditions of Proposition 2.2.

If $f : M : M' \to M'' : g : M'_\nu \to M''_\nu$ are filtered maps, then the continuous map $gf : M \to M''$ is clearly a filtered map $M_k \to M''_\nu$, with $(gf)_o = g_o f_\circ$. Dually, if $f : M'_1 : M'_2 \to M'_3$ and $g : M'_2 : M'_3 \to M'_3$ are cofiltered maps, then the continuous map $gf : M'_1 \to M'_3$ is clearly a cofiltered map $M'_1 \to M'_3$, with $(gf)_o = f_\circ g^\circ$.

In particular, given an $X$-$Y$-approaching map $f : L \setminus X \to M \setminus Y$ and a $Y$-$Z$-approaching map $g : M \setminus Y \to N \setminus Z$, their composition is an $X$-$Z$-approaching map.

2.3. Cofinal and coinitial subsets. A subset $S$ of a poset $P$ is called cofinal (resp. coinitial) if every element of $P$ has an upper (resp. lower) bound in $S$. Clearly, a cofinal (resp. coinitial) subset of a directed (resp. codirected) poset is also a directed (resp. codirected) poset.

It is easy to see that if $\kappa'$ is cofinal in $\kappa$ and $\lambda'$ is cofinal in $\lambda$, a continuous map $f : M \to N$ is a filtered map $M_\kappa \to N_\lambda$ if and only if it is a filtered map $M_{\kappa'} \to N_{\lambda'}$. Dually, if $\nu'$ is coinitial in $\nu$ and $\mu'$ is coinitial in $\mu$, then a continuous map $f : M \to N$ is a cofiltered map $M^\nu \to N^\mu$ if and only if it is a cofiltered map $M'^{\nu'} \to N'^{\mu'}$.

**Lemma 2.3.** Let $X$ be a closed subset of a space $M$ and $Y$ a closed subset of a space $N$. Then (a) $\kappa X \times \{I\}$ is cofinal in $\kappa(X \times I)$ and (b) $\nu X \times \{I\}$ is coinitial in $\nu(X \times I)$.

**Proof.** If $K \subset M \times I$ is compact, then so is $\pi(K)$, where $\pi : M \times I \to M$ is the projection. Therefore $K$ is contained in the compact set $\pi(K) \times I$. Thus $\kappa X \times \{I\} = \{(L \setminus X) \times I \mid L \setminus X \in \kappa X\}$ is cofinal in $\kappa(X \times I)$.

If $U$ is a neighborhood of $X \times I$ in $M \times I$, then $U$ contains $V \times I$, where $V = M \setminus \pi(M \times I \setminus U)$. Since $I$ is compact, $\pi$ is a closed map, so $V$ is open. (In more detail, if $x \in V$, then $(x, t) \in U$ for each $t \in I$. Each $(x, t)$ lies in a basic open set $V_x \times W_x$ of $M \times I$ that itself lies in $U$. The open cover $\{W_x \mid x \in I\}$ of $I$ has a finite subcover $\{W_i \mid i \in J\}$. Then $\bigcap_{i \in J} V_i$ is an open neighborhood of $x$ in $V$.) Therefore $U$ contains an open neighborhood $V \setminus X \times I$ of $X \times I$. Thus $\nu X \times \{I\} = \{(V \setminus X) \times I \mid V \setminus X \in \nu X\}$ is coinitial in $\nu(X \times I)$.

2.4. Approaching homotopies. A homotopy $M \times I \to N$ between $f, g : M \to N$ will be called a (co)filtered homotopy between filtered maps $f, g : M_\kappa \to N_\lambda$ (resp. cofiltered maps $f, g : M^\nu \to N^\mu$) if it is a filtered map $(M \times I)_{\kappa \times (I)} \to N_\lambda$ (resp. a cofiltered map $(M \times I)^{\nu \times (I)} \to N^\mu$), where the product (co)filtration $\kappa \times \lambda = \{K \times L \mid K \in \kappa, L \in \lambda\}$ (resp. $\mu \times \nu = \{U \times V \mid U \in \mu, V \in \nu\}$).

**Proposition 2.4.** Let $X$ be a closed subset of a space $M$ and $Y$ a closed subset of a space $N$. Let $H : (M \setminus X) \times I \to N \setminus Y$ be a homotopy. The following are equivalent:

1. $H$ is an $X \times I$-$Y$-approaching map.
(2) H is a cofiltered homotopy \((M \times I \setminus X \times I)_{\kappa X \times \{I\}}^* \to (N \setminus Y)_{\kappa Y}\);
(3) H is a filtered homotopy \((M \times I \setminus X \times I)_{\kappa X \times \{I\}} \to (N \setminus Y)_{\kappa Y}\) and its restriction to each member of \(\kappa X \times \{I\}\) is a proper homotopy.

We recall that a homotopy \(H : A \times I \to B\) is called proper if it is a proper map.

**Proof.** The equivalence of (1) and (2) follows from Lemma 2.3(b). The equivalence of (1) and (3) follows from Lemma 2.3(a) and the fact that the restriction of a proper map to a closed subset of the domain is proper. \(\square\)

If \(X\) is a closed subset of a space \(M\) and \(Y\) is a closed subset of a space \(N\), a homotopy \(H : (M \setminus X) \times I \to N \setminus Y\) will be called an \(X\)-\(Y\)-approaching homotopy (or simply an approaching homotopy) if it satisfies any of the equivalent conditions of Proposition 2.4.

Let us note that \(h_\xi : \kappa X \times \{I\} \to \kappa Y\) and \(H_\circ : \nu Y \to \nu X \times \{I\}\) are in fact monotone maps.

If \(f, g : M \setminus X \to N \setminus Y\) are \(X\)-\(Y\)-approaching maps, \(f \simeq g\) abbreviates the assertion that they are approaching homotopic; and \([f]\) denotes the \(X\)-\(Y\)-approaching homotopy class of \(f\). Let \([X, Y]_{MN}\) denote the set of \(X\)-\(Y\)-approaching homotopy classes of \(X\)-\(Y\)-approaching maps \(M \setminus X \to N \setminus Y\). Clearly, the composition map \([X, Y]_{LM} \times [Y, Z]_{MN} \to [X, Z]_{LN}\) is well-defined. An \(X\)-\(Y\)-approaching map \(f : M \setminus X \to N \setminus X\) is an \(X\)-\(Y\)-approaching homotopy equivalence if there exists an \(Y\)-\(X\)-approaching map \(g : N \setminus Y \to M \setminus X\) such that \(gf \simeq \text{id}_M\) and \(fg \simeq \text{id}_N\).

**2.5. Z-sets in ANRs.** If a subset \(A\) is dense in a subset \(B\) of \(C(X, Y)\), we say that every \(f \in A\) is approximable by a \(g \in B\). This means that for every cover \(C\) of \(Y\) every \(f \in A\) is \(C\)-close to some \(g \in B\); or equivalently that for every metric on \(Y\), for every \(\varepsilon > 0\), every \(f \in A\) is \(\varepsilon\)-close to some \(g \in B\). When either \(X\) or \(Y\) is compact, “for every metric on \(Y\)” can be replaced by “there exists a metric on \(Y\) such that” (see e.g. [40; Lemma 1.1.1]).

We call a closed subset \(X\) of an ANR \(M\) a Z-set if \(M \setminus X\) is homotopy dense in \(M\), that is, there exists a homotopy \(h_t : M \to M\) such that \(h_0 = \text{id}\) and \(h_t(M) \subset M \setminus X\) for each \(t > 0\). It is well-known that a closed subset \(X\) of an ANR \(M\) is a Z-set in \(M\) if and only if for every \(n > 0\), every map \(I^n \to M\) is approximable by a map whose image is disjoint from \(X\) (see [5; Theorem 1.4.4]). Another useful equivalent definition, intermediate between the two just described, is that \(\text{id} : M \to M\) is approximable by maps whose image is disjoint from \(X\).

**Lemma 2.5.** Let \(X\) be a closed Z-set in an ANR \(M\) and \(Y\) a closed Z-set in an ANR \(N\). Let \(\varphi : X \to Y\) be a continuous map. Then

(a) \(\varphi\) extends to a continuous map \(F : M \to N\) such that \(F^{-1}(Y) = X\)

(b) every other extension \(\tilde{F}\) of \(\varphi\) such that \(\tilde{F}^{-1}(Y) = X\) is homotopic to \(F\) by a homotopy through extensions \(H_t\) of \(\varphi\) such that \(H_t^{-1}(Y) = X\) for each \(t \in I\).

**Proof.** (a). Since \(N\) is an ANE, \(\varphi\) extends to a continuous map \(f : M \to N\). Since \(X\) is closed, there exists a continuous map \(g : M \to I\) such that \(X = g^{-1}(0)\). Since
Y is a Z-set in N, there exists a homotopy \( h: N \times I \to N \) such that \( h_0 = \text{id} \) and 
\( h_t(Y) \subseteq N \setminus Y \) for \( t > 0 \). Then \( h^{-1}(Y) = Y \times \{0\} \), and consequently the composition 
\( F: M \xrightarrow{f \times g} N \times I \xrightarrow{h} N \) is as desired.

\( \square \)

\( (b) \). This follows from (a), since \( X \times I \cup M \times \partial I \) is a closed Z-set in \( M \times I \) and \( Y \times I \cup N \times \partial I \)
is a closed Z-set in \( N \times I \).

\( \square \)

2.6. **Fine shape category.** If \( F \) is a map as in Lemma 2.5(a), i.e., such that \( F^{-1}(Y) = X \), then by the above its restriction \( f: M \setminus X \to M \setminus Y \) is an \( X-Y \)-approaching map.

Similarly, if \( H: M \times I \to N \) is a homotopy between \( F \) and \( \tilde{F} \) as in Lemma 2.5(b), i.e., such that \( H^{-1}(Y) = X \times I \), then its restriction \( h: M \times I \setminus X \times I \to N \setminus Y \) is an \( X \times I-Y \)-approaching map, hence an \( X-Y \)-approaching homotopy. We have proved

**Proposition 2.6.** Let \( X \) be a closed Z-set in an ANR \( M \) and \( Y \) be a closed Z-set in an ANR \( N \). Then every continuous map \( \varphi: X \to Y \) gives rise to an \( X-Y \)-approaching map \( M \setminus X \to N \setminus Y \) whose \( X-Y \)-approaching homotopy class \( \varphi_{MN} \in [X,Y]_{MN} \) is well-defined.

Given additionally a continuous map \( \psi: W \to X \), where \( W \) is a closed Z-set in an ANR \( L \), we clearly have \( \varphi_{MN} \psi_{LM} = (\varphi \psi)_{LN} \in [W,Y]_{LN} \).

If \( X \) is identified with closed Z-sets in two ANRs \( M \) and \( M' \), then the approaching homotopy classes \((\text{id}_X)_{MM'}\) and \((\text{id}_X)_{M'M} \) are mutually inverse, in the sense that
\( (\text{id}_X)_{MM'}(\text{id}_X)_{M'M} = [\text{id}_M] \in [X,X]_{MM} \) and \((\text{id}_X)_{M'M}(\text{id}_X)_{MM'} = [\text{id}_M] \in [X,X]_{M'M'} \).

If additionally \( Y \) is identified with closed Z-sets in two ANRs \( N \) and \( N' \), a map 
\( t_{MN}^{M'M'}: [X,Y]_{MN} \to [X,Y]_{M'M'} \), is defined by \( F \mapsto (\text{id}_X)_{NN'}F(\text{id}_X)_{M'M} \). This map is a bijection, with inverse \( t_{M'M}^{MN} \).

Finally we define an equivalence relation on the disjoint union of the approaching homotopy sets \( \bigsqcup_{M,N}[X,Y]_{MN} \) (where \( M \) is an ANR containing \( X \) as a closed Z-set and \( N \) is an ANR containing \( Y \) as a closed Z-set) by declaring \( F \in [X,Y]_{MN} \) and \( \tilde{G} \in [X,Y]_{M'N'} \) equivalent if \( t_{MN}^{M'M'}(F) = G \). It follows from the above that this relation is reflexive, symmetric and transitive.

Thus we may define a **fine shape morphism** \( [f]: X \rightsquigarrow Y \) of metrizable spaces as the equivalence class of the approaching homotopy class \( [f]_{MN} \in [X,Y]_{MN} \) of an approaching map \( f: M \setminus X \to N \setminus Y \).

**Composition of fine shape morphisms** \( [F]: W \rightsquigarrow X \) and \( [G]: X \rightsquigarrow Y \), where \( F \in [W,X]_{LM} \) and \( G \in [X,Y]_{MN} \) is defined to be the equivalence class \( [GF]: W \rightsquigarrow Y \) of the composition of the approaching homotopy classes, and is easily seen to be independent of the choice of representatives.

Corollary 2.6 yields a map from the set \([X,Y]_f\) of homotopy classes of maps \( X \to Y \) into the set \([X,Y]_f \) of fine shape morphisms \( X \rightsquigarrow Y \).

A fine shape morphism \( \varphi: X \rightsquigarrow Y \) is a **fine shape equivalence** if it is represented by an approaching homotopy equivalence; or equivalently if there exists a \( \psi: Y \rightsquigarrow X \) such that 
\( \psi \varphi = [\text{id}_X] \) and \( \varphi \psi = [\text{id}_Y] \). When such a \( \varphi \) exists, \( X \) and \( Y \) are said to be of the same **fine shape**.
3. Invariance of homology and cohomology

3.1. Čech cohomology. Let $X$ be a closed $Z$-set in an ANR $M$. Then $X$ is also a closed $Z$-set in every open neighborhood $U$ of $X$ in $M$, in the sense that maps $f^n : U \to U \setminus X$. Since $U$ is an ANR, $X$ is a $Z$-set in $U$ also in the sense that there exists a homotopy $H_t : U \to U$ such that $H_t(U) \subset X \setminus U$ for $t > 0$. Consequently $U$ is homotopy equivalent to $X$.

Therefore the Čech cohomology of $X$ (possibly extraordinary) can be computed in terms of the neighborhood cofiltration $\nu_M(X)$ on $M \setminus X$:

$$H^n(X) \simeq \operatorname{colim}_{U \in \nu X} H^n(U).$$

We recall that

$$\operatorname{colim}_{U \in \nu X} H^n(U) \simeq \bigoplus_{U \in \nu X} H^n(U) / \langle x - i_*(x) \mid x \in H^n(V), i : U \to V; U, V \in \nu X \rangle.$$ 

Let us consider a cofiltered map $f : (M \setminus X)^{\nu X} \to (N \setminus Y)^{\nu Y}$. It induces homomorphisms $f^*_U : H^n(U) \to H^n(f(U))$ for each $U \in \nu Y$. These in turn determine a homomorphism $f^* : \operatorname{colim}_{U \in \nu Y} H^n(U) \to \operatorname{colim}_{V \in \nu X} H^n(V)$, which is easily seen to be independent of the choice of the map $f^* : \nu Y \to \nu X$. Thus we obtain a well-defined homomorphism $f^* : H^n(Y) \to H^n(X)$ of the Čech cohomology groups.

If cofiltered maps $f, g : (M \setminus X)^{\nu X} \to (N \setminus Y)^{\nu Y}$ are related by a cofiltered homotopy, it is easy to see that $f^* = g^*$. Consequently, an approaching homotopy class $[f]_{MN} \in [X, Y]_{MN}$ induces a well-defined homomorphism $[f]_{MN}^* : H^n(Y) \to H^n(X)$.

Clearly, $(\operatorname{id}_X)_{MN} = \operatorname{id} : H^n(X) \to H^n(X)$, and it follows that a fine shape morphism $\varphi : X \rightsquigarrow Y$ induces a well-defined homomorphism $[\varphi]^* : H^n(Y) \to H^n(X)$ of the Čech cohomology groups.

Since approaching homotopic approaching maps induce the same homomorphism, an approaching homotopy equivalence induces an isomorphism on Čech cohomology. Hence so does a fine shape equivalence. In particular, spaces of the same fine shape have isomorphic Čech cohomology groups.

3.2. Fine $Z$-sets. In order to establish fine shape invariance of Steenrod–Sitnikov homology, it will be convenient to work with “fine $Z$-sets”. This is really only a matter of convenience: one can avoid their use altogether at the cost of replacing subsets with maps (see Remark 4.6).

We say that a subset $X$ of an ANR $M$ is a fine $Z$-set in $M$ if $X$ is a closed $Z$-set in $M$ and each compact $K \subset M$ lies in a compact ANR $N \subset M$ such that $N \cap X = K \cap X$ and $N \cap X$ is a $Z$-set in $N$.

It turns out that:

- Every closed subset of the boundary of a PL manifold is a fine $Z$-set (see 4.1).
- Every Polish space $X$ is a fine $Z$-set in a certain ANR $M$ such that $M \setminus X$ is a polyhedron (see 4.4).
• Every closed Z-set in an $l_2$-manifold or an $l^d_2$-manifold or a $\Sigma$-manifold or an $\mathbb{R}^\omega$-manifold, or a $Q^\omega$-manifold is a fine Z-set (see §4.4). Also, every Polish space is homeomorphic to a closed Z-set in the Hilbert space $l_2$ (see Remark 4.12).

Let $X$ be a fine Z-set in an ANR $M$. Then

$$\tau_M(X) := \{ P \setminus X \mid P \text{ is a compact ANR in } M \text{ such that } P \cap X \text{ is a Z-set in } P \}$$

is a cofinal set in the directed family $\kappa_M(X) = \{ K \setminus X \mid K \text{ is a compact subset of } M \}$ of subsets of $M \setminus X$. In particular, $\tau_M(X)$ is itself a directed family of subsets of $M \setminus X$. We will call it the compact ANR filtration and often abbreviate it as $\tau X$.

Since $\tau X$ is cofinal in $\kappa X$, a continuous map $f: M \setminus X \to N \setminus Y$, where $X$ is a fine Z-set in $M$ and $Y$ is a fine Z-set in $N$, is an $X$-$Y$-approaching map if and only if it is a filtered map $(M \setminus X)_{\tau X} \to (N \setminus Y)_{\tau Y}$ and its restriction to every member of $\tau X$ is a proper map. Let us note, however, that $f_\circlearrowleft: \tau X \to \tau Y$ is not monotone in general.

3.3. Steenrod–Sitnikov homology. Let $K$ be a closed Z-set in a compact ANR $\bar P$. We want to compute $H_n(K)$ in terms of $P := \bar P \setminus K$.\(^3\)

Since $K$ is a Z-set, the inclusion $P \to \bar P$ is a homotopy equivalence. Then it is not hard to see that the inclusion of $\bar P \times 1$ into $R := P \times I \cup \bar P \times 0$ is also a homotopy equivalence. Hence it induces an isomorphism $H_n(P \times 1) \to H_n(R)$ on Steenrod–Sitnikov homology. Therefore $H_n(R, P \times 1) = 0$ for all $n$. Consequently $H_n(K \times 0) \simeq H_n(K \times 0 \cup P \times 1, P \times 1) \simeq H_{n+1}(R, K \times 0 \cup P \times 1)$ for all $n$. The quotient space $R/(K \times 1)$ is homeomorphic to $P \times I \cup P_+ \times 0$, where $P_+ = P \cup \{ \infty \}$ is the one-point compactification of $P$. (If $P$ is compact, $P_+$ is defined to be $P \cup \{ \infty \}$.) Since $K$ is compact, by the map excision axiom $H_{n+1}(R, K \times 0 \cup P \times 1) \simeq H_{n+1}(P \times I \cup P_+ \times 0, \infty \times 0 \cup P \times 1)$. To simplify notation, we will refer to the latter group as $G_n(P)$.

If $L$ is a closed Z-set in a compact ANR $\bar Q$, and $f: P \to Q$ is a proper map, where $Q = \bar Q \setminus L$, then $f$ descends to a continuous map $R_+ \to Q_+$ and hence induces a homomorphism $f_+: G_n(P) \to G_n(Q)$. If $f$ extends to a continuous map $\bar f: \bar P \to \bar Q$ such that $\bar f^{-1}(L) = K$, then the homomorphism $\varphi_+: H_n(K) \to H_n(L)$ induced by the restriction $\varphi: K \to L$ of $\bar f$ is clearly identified with $f_+$. In particular, if $X$ is a fine Z-set in an ANR $M$, then for any inclusion $i: P \to Q$ between members of $\tau_M X$ the homomorphism $i_+: H_n(\bar P \cap X) \to H_n(\bar Q \cap X)$ is identified with $i_+: G_n(P) \to G_n(Q)$.

Therefore the Steenrod–Sitnikov homology (possibly extraordinary) of a fine Z-set $X$ in an ANR $M$ can be computed in terms of the compact ANR cofiltration $\tau_M(X)$ on $M \setminus X$:

$$H_n(X) \simeq \colim_{P \in \tau_X} G_n(P).$$

While we cannot compute $H_n(X)$ in terms of the compact filtration $\kappa_M(X)$, we know that the notion of an approaching map can be defined in terms of either $\kappa X$ of $\tau X$. Then similarly to the case of cohomology, it follows that a fine shape morphism $\varphi: X \rightsquigarrow Y$\(^3\)

\(^3\)For the reader who is familiar with Steenrod–Sitnikov homology of nonclosed pairs (see [29, Proof of Proposition 3.1] for an elementary definition) we note that the reduced homology $\bar H_n(K) \simeq H_{n+1}(P_+, P)$, where $P_+$ is the one-point compactification of $P$.\(^3\)
induces a well-defined homomorphism \( \varphi_* : H_n(X) \to H_n(Y) \) of the Steenrod–Sitnikov homology groups, and if \( \varphi \) is a fine shape equivalence, then \( \varphi_* \) is an isomorphism.

In particular, spaces of the same fine shape have isomorphic Steenrod–Sitnikov homology.

3.4. Relative (co)homology. Given a pair of spaces \((X, X_0)\), we may embed it as a Z-pair or a fine Z-pair in a pair of ANRs \((M, M_0)\) and relativize the preceding constructions. However, there is a shortcut. If we choose \(M_0\) to be an AR, then \(H_n(X, X_0) \simeq H_n(X \cup M_0, M_0) \simeq \tilde{H}_n(X \cup M_0)\) and \(H^n(X, X_0) \simeq H^n(X \cup M_0, M_0) \simeq \tilde{H}^n(X \cup M_0)\). Thus we may simply use a version of the preceding constructions for reduced (but absolute) homology and cohomology. It follows that relative Čech cohomology and relative Steenrod–Sitnikov homology are invariants of (appropriately used) absolute fine shape.

3.5. Steenrod–Sitnikov homotopy. Let \(K\) be a closed Z-set in a compact ANR \(N\) and let \(x \in K\). By Lemma 2.5(a) the inclusion \(\{x\} \to K\) extends to a map \(r : [0, \infty] \to N\) such that \(r^{-1}(K) = \{\infty\}\). Then the Steenrod–Sitnikov homotopy group \(\pi_n(K, x)\) (pointed set for \(n = 0\) is isomorphic to the group \([\{S^n, pt\} \times [0, \infty], (N \setminus K, r)]\) (pointed set for \(n = 0\)) of base ray preserving proper homotopy classes of base ray preserving proper maps \(f : S^n \times [0, \infty) \to N \setminus K\).

Let \(X\) be a fine Z-set \(X\) in an ANR \(M\), and let \(x \in X\). Let us fix a map \(r : [0, \infty] \to N\) such that \(r^{-1}(K) = \{\infty\}\), and let \(\tau X\) be the subset of \(\tau X\) consisting of those \(N \in \tau X\) that contain the image of \(r\). Clearly, \(\tau r X\) is cofinal in \(\tau X\).

Then the Steenrod–Sitnikov homotopy groups of \(X\) can be computed in terms of the filtration \(\tau X\):

\[
\tilde{\pi}_n(X, x) \simeq \colim_{N \in \tau X} [(S^n, pt) \times [0, \infty), (N, r)]_{\infty}.
\]

Similarly to the cases of homology and cohomology, it follows that a fine shape morphism \(\varphi : X \rightsquigarrow Y\) induces a well-defined homomorphism \(\varphi_* : \pi_n(X) \to \pi_n(Y)\) of the Steenrod–Sitnikov homotopy groups (pointed sets if \(n = 0\)), and if \(\varphi\) is a fine shape equivalence, then \(\varphi_*\) is an isomorphism.

In particular, spaces of the same fine shape have isomorphic Steenrod–Sitnikov homotopy groups.

3.6. Map excision. Let us recall that a (co)homology theory on an appropriate category \(\mathcal{C}\) of closed pairs of topological spaces satisfies the map excision axiom if every closed map \(f : (X, A) \to (Y, B)\) in \(\mathcal{C}\) that restricts to a homeomorphism between \(X \setminus A\) and \(Y \setminus B\), induces for each \(n\) an isomorphism \(H_n(X, A) \simeq H_n(Y, B)\) (respectively, \(H^n(X, A) \simeq H^n(Y, B)\)). See [1] for further details and references.

As observed by Mrozik [31], for a (co)homology theory on pairs of compacta, map excision is equivalent to invariance under fine shape. Indeed, the quotient map \(X \to X/A\) factors as \((X, A) \xrightarrow{i} (X \cup CA, CA) \xrightarrow{j} (X/A, pt)\), where \(i\) induces an isomorphism due to usual excision along with homotopy invariance, whereas \(j\) is easily seen to be a fine shape equivalence. Conversely, to see that map excision and homotopy invariance
imply fine shape invariance, in the case of homology (the case of cohomology is similar), we note that a fine shape morphism between compacta $X$, $Y$ is represented by a proper map $f: P_{[0,\infty)} \to Q_{[0,\infty)}$ between their infinite mapping telescopes, which extends to a map between their one-point compactifications and therefore induces a homomorphism $H_n(P_{[0,\infty)}, \infty) \to H_n(Q_{[0,\infty)}, \infty)$. By map excision, this yields a homomorphism $H_n(P_{[0,\infty)}, X) \to H_n(Q_{[0,\infty)}, Y)$, and hence, as long as $P_0 = Q_0 = pt$, also a homomorphism $f_*: H_{n-1}(X) \to H_{n-1}(Y)$. But if $f'$ is proper homotopic to $f$, it follows from homotopy invariance that $f'_* = f_*$.  

The first part of this argument can be generalized:

**Proposition 3.1.** If a (co)homology theory on closed pairs of metrizable spaces is fine shape invariant, then it satisfies map excision.

**Proof.** Let $f: (X, A) \to (Y, B)$ be a closed map that restricts to a homeomorphism between $X \setminus A$ and $Y \setminus B$. Then $Y_0 := f(X)$ is closed in $Y$ and contains $Y \setminus B$. It is well-known that for every closed surjection $f: X \to Y_0$ between metrizable spaces there exists a closed subset $X_0 \subset X$ such that $f_0 := f|_{X_0}: X_0 \to Y_0$ is a perfect surjection (i.e. a closed surjection with compact point-inverses) [30; proof of Corollary 1.2]. It follows that $f_0: X_0 \to Y$ is a perfect map that restricts to a homeomorphism between $X \setminus A$ and $Y \setminus B$. In particular, $X_0$ contains $X \setminus A$. Also, $A_0 := X_0 \cap A$ is closed in $X_0$, and $f_0(A_0) \subset B$, so we may write $f_0: (X_0, A_0) \to (Y, B)$. On the other hand, the inclusion $f_1: (X_0, A_0) \hookrightarrow (X, A)$ is a perfect map that restricts to the identity on $X \setminus A$. Since $ff_1 = f_0$, it suffices to show that both $f_0$ and $f_1$ induce isomorphisms on (co)homology.

Thus we may assume that the original map $f: (X, A) \to (Y, B)$ is perfect. It factors as $(X, A) \xrightarrow{i} (X \cup MC(f|_A), MC(f|_A)) \xrightarrow{j} (Y, B)$, where $i$ induces an isomorphism on (co)homology by usual excision and homotopy invariance, and one can show that $j$ is a fine shape equivalence by considering mapping telescopes, using the assumption that $f$ is perfect, and building on a proof of the fact that if $f$ is in $C_\nu$, then $j$ is a homotopy equivalence. 

\[\blacksquare\]

4. Fine Z-sets in ANRs

**Proposition 4.1.** If $P$ is a locally compact polyhedron and $X$ is a closed subset of $P$, then $X \times \{0\}$ is a fine Z-set in $P \times I$.

**Proof.** Clearly, $X \times \{0\}$ is a Z-set in $P \times I$. Let us fix some metric on $P$ and let $T$ be some triangulation of $P$. Given a compact $K \subset P \times I$, let $L$ be the union of all simplexes of $T$ that meet the image of $K$ under the projection $P \times I \to P$. Since $T$ is locally finite, $L$ is a compact ANR. Let us define $f: L \to I$ by $f(x) = \inf_{(y,t) \in K} t + d(x,y)$. Then $f(x) = 0$ if and only if $x \in K \cap P \times \{0\}$, and for each $(x,t) \in K$ we have $f(x) \leq t$. These two properties persist if we pass from $f(x)$ to $\varphi(x) := \min \{f(x), \frac{1}{2}\}$. Let $N = \{(x,t) \in P \times I \mid x \in L, \ t \geq \varphi(x)\}$. Then $N$ contains $K$ and meets $P \times \{0\}$.
in $K \cap P \times \{0\}$; in particular, it meets $X \times \{0\}$ in $K \cap X \times \{0\}$. Let us now turn everything upside down, i.e. let $r: P \times I \to P \times I$ send $(x,t)$ to $(x,1-t)$. Then $r(N)$ is the shadow of the graph of $\psi := 1 - \varphi$, and so $h: L \times I \to r(N)$, $(x,t) \mapsto (x,\psi(x)t)$, is a homeomorphism which is the identity on $L \times \{0\} \cup r(K \cap P \times \{0\})$. (Here $h$ is injective since $\psi(x) \geq \frac{1}{2}$ for all $x$.) Thus $rhr: L \times I \to N$ is a homeomorphism which is the identity on $L \times \{1\} \cup (K \cap P \times \{0\})$. Since $L \times I$ is a compact ANR containing $K \cap X \times \{0\}$ as a $Z$-set, so is $N$. \hfill $\square$

4.2. The mapping telescope.

**Theorem 4.2.** (Isbell) Every Polish space $X$ is the limit of an inverse sequence of finite-dimensional locally compact polyhedra. Moreover, if $X$ is locally compact, the bonding maps can be assumed to be proper.

**Remark 4.3.** Let us briefly review known proofs of Theorem 4.2.

(1) It is not hard to show, by using a relative version of Tikhonov’s embedding theorem, that $X$ is homeomorphic to a closed subset of the countable product $\mathbb{R}^\infty$ [26; Proposition 8.7]. It follows that $X$ is an inverse limit of subcomplexes $Q_n$ of the standard cubical lattices in $\mathbb{R}^n$ with vertex sets $(2^{-n}\mathbb{Z})^n$ [26; Proposition 18.4]. (This argument was apparently known to Isbell around 1960: see [18; p. 246], [19; p. 301].) The locally compact case is still easier. If $\varphi: X \to [0,\infty)$ be a proper map and $g: X \to I^\infty$ is an embedding, then $\varphi \times g: X \to [0,\infty) \times I^\infty$ is an embedding whose image is closed since it meets each $[i,i+1] \times I^\infty$ in a compact set. It then similarly follows that $X$ is an inverse limit of subcomplexes of the standard cubical lattices in $[0,\infty) \times I^n$ with vertex sets $\mathbb{N} \times (2^{-n}\mathbb{Z})^n$.

(2) One can show [26; Theorem 8.8] (see also [26; Proposition 8.6] and [19; Proof of 3.6] for a more direct approach) that there exists a sequence of open covers $C_0, C_1, \ldots$ of $X$ such that

(a) each $C_{i+1}$ star-refines $C_i$ (that is, for each $x \in X$, the union of all elements of $C_{i+1}$ containing $x$ lies in some element of $C_i$);

(b) any two distinct points $x, y \in X$ are separated by some $C_i$ (that is, no element of $C_i$ contains both $x$ and $y$); and

(c) each nerve $N(C_i)$ is locally compact and finite-dimensional. If $X$ is locally compact, then each element of $C_i$ may be further assumed to contain only finitely many elements of $C_{i+1}$ (see [26; Proposition 8.9]). Since each $C_{i+1}$ refines $C_i$, there is a canonical semilinear (i.e., affine on each simplex) map $\varphi^{C_{i+1}}: N(C_{i+1}) \to N(C_i)$ defined by sending each vertex $U \in C_{i+1}$ of $N(C_{i+1})$ to the barycenter of the simplex of $N(C_i)$ spanned by all $V \in C_i$ satisfying $V \supset U$. With appropriate metric on each $N(C_i)$, their inverse limit is homeomorphic to $X$ if the dimensions of the $N(C_i)$ are bounded (which can be achieved if $X$ is finite-dimensional) [20; V.33] (a minor error is corrected in [27; 8.2]). In the general case, the inverse limit of the maps $\varphi^{C_{ni+1}}$ will work for a certain subsequence $n_i$, using that each $N(C_i)$ is finite-dimensional [20; V.34], [11; Lemma 1.6].
(3) The previous construction (2) can be somewhat improved. The star-refinement condition (a) yields a canonical simplicial map\(^4\) \(\psi_i: N_{i+1}' \to (N_i^\#)'\) from the barycentric subdivision of \(N_{i+1}\) to the barycentric subdivision of a standard cubical subdivision \(N_i^\#\) of the simplicial complex \(N_i\) [27; 6.16]. In fact, \((N_i^\#)'\) is also a subdivision of the barycentric subdivision \(N_i^\#\) [27; 6.18]. With appropriate metric on each \(N(C_i)\), their inverse limit is homeomorphic to \(X\) [27; 8.2]. (This approach works even if the \(N(C_i)\) are not assumed to be locally finite and finite-dimensional.)

Let us also mention some other constructions that come close to proving Theorem 4.2.

(4) By Aharoni’s theorem (see [26; Theorem 8.4]), \(X\) is homeomorphic to a closed subset of the Banach space \(c_0\) of all zero-convergent real sequences with the sup norm. Since the space of finite sequences \(c_{00}\) is dense in \(c_0\), it follows that \(X\) is an intersection of the closures of some subcomplexes \(Q_n\) of the standard cubical lattice in \(c_{00}\) with vertex sets \(\bigcup_{i=1}^\infty (2^{-n}\mathbb{Z})^i\) [26; Lemma 18.1]. Although the \(Q_n\) are neither finite-dimensional nor locally compact, they and their closures are ANRs [26; Theorems 14.23, 14.3].

(5) Sakai [34; 4.10.11] proves that \(X\) is the limit of an inverse sequence of locally compact (but possibly infinite dimensional) polyhedra, with combinatorial (but non-canonical) bonding maps. His proof also works to make the bonding maps proper if \(X\) is locally compact [34; 4.10.12].

**Proposition 4.4.** If \(X\) is the limit of an inverse sequence of locally compact polyhedra \(P_i\), then \(X\) is a fine \(Z\)-set in the extended mapping telescope \(P_{[0,\infty]}\), which is an ANR.

Here \(P_{[0,\infty]}\) is the inverse limit of the obvious retractions between the finite mapping telescopes \(P_{[0,n]}\). The infinite mapping telescope \(P_{[0,\infty]}\) is naturally identified with a subset of \(P_{[0,\infty]}\) so that \(P_{[0,\infty]} \setminus P_{[0,\infty]} = X\). (See [26; §17.2] for further details.)

**Proof.** It is easy to see that \(P_{[0,\infty]}\) deformation retracts onto each \(P_{[0,n]}\) by a deformation retraction that is \(2^{-n}\)-close to the identity at each time instant. Hence \(P_{[0,n]}\), which is certainly an ANR (see e.g. [34; Theorem 6.2.6]), homotopy \(2^{-n}\)-dominates \(P_{[0,\infty]}\). Hence by Hanner’s criterion (see e.g. [34; Theorem 6.6.2]), \(P_{[0,\infty]}\) is also an ANR.

Clearly, \(X\) is a closed \(Z\)-set in \(P_{[0,\infty]}\). If \(K \subset P_{[0,\infty]}\) is compact, then \(K \cap P_{[k-1,\frac{1}{k+1},\infty]}\) has a compact image \(K_{ki}\) in \(P_{k-1+\frac{1}{k+1}} = P_k\) for each \(k\) and \(i\). We may assume that the \(P_i\) are triangulated so that each bonding map \(p_i\) is simplicial as a map from \(P_{i+1}\) to a subdivision of \(P_i\). Let \(N_{ki}\) be the union of all closed simplexes that meet \(K_{ki}\). Since \(P_k\) is locally finite, \(N_{ki}\) is a compact ANR. The bonding maps need not be proper, so \(N_k := N_{k1} \cup N_{k2} \cup \ldots \) need not be compact. Let \(N_{[0,\infty]}\) be the compactified mapping telescope, defined as the union of the sets \(N_k \times (k-1, k]\) and \(K \cap X\) in \(P_{[0,\infty]}\) (note that the image of each \(N_k\) lies in \(N_{k-1}\)). Then \(N_{[0,\infty]}\) is an ANR, but it need not be compact

\(^4\)This appears to be the only known construction of a canonical (that is, not involving any choices) combinatorial bonding map. If \(C\) refines \(D\), there is a non-canonical simplicial map \(\varphi: N(C) \to N(D)\) defined by sending each vertex \(U \in C\) of \(N(C)\) to some vertex \(V \in D\) of \(N(D)\) satisfying \(V \supset U\).
(since each $N_k$ need not be compact). Let $N$ be the compactified mapping telescope of mapping telescopes, defined as the union of the sets $N_{ki} \times [k - 1 + \frac{1}{i+1}, k - 1 + \frac{1}{i}]$ and $K \cap X$ in $P_{[0,\infty]}$. Clearly, $N$ is compact, meets $X$ in $K \cap X$ and contains $K \cap X$ as a closed Z-set. Also, it is easy to see that $N_{[0,\infty]}$ retracts (actually, deformation retracts) onto $N$, and therefore $N$ is an ANR. \qed

4.3. Emulating fine Z-sets.

**Lemma 4.5.** For every closed Z-set $X$ in an ANR $M$ and every compact $K \subset M$, the inclusion $K \to M$ factors into the composition of an embedding $i$ of $K$ into a compact ANR $N$ such that $i(K \cap X)$ is a Z-set in $N$ and a map $f$: $N \to M$ such that $f^{-1}(X) = i(K \cap X)$.

Moreover, every neighborhood of $i(K)$ in $N$ contains a compact ANR $N'$ such that $i(K \cap X)$ is a Z-set in $N'$.

**Proof.** Let us represent $K$ as the limit of an inverse sequence of compact polyhedra $P_i$, and let $P_{[0,\infty]}$ be the compactified mapping telescope. For some $k$ the inclusion $K \subset M$ factors through a map $\varphi: P_{[k,\infty]} \to M$. Then $N := P_{[k,\infty]}$ is a compact ANR which contains $K$ as a Z-set; in particular, it contains $K \cap X$ as a Z-set. Moreover, every neighborhood of $K$ in $N$ contains $N' := P_{[n,\infty]}$ for a suitable $n$. Let $p: P_{[k,\infty]} \to I$ be the composition of the projection $P_{[k,\infty]} \to [k,\infty]$ and a homeomorphism. Since $X$ is a Z-set, there exists a homotopy $H: M \times I \to M$ such that $H|_{X \times \{0\}}$ is the inclusion and $H(M \times (0,1]) \subset M \setminus X$. The composition $f: N \to M$ satisfies $f^{-1}(X) = K \cap X$. \qed

**Remark 4.6.** Using Lemma 4.5 one could avoid the use of fine Z-sets altogether (and use only usual closed Z-sets) by the cost of working with the category of arbitrary maps of compact ANRs to $M$ (instead of the poset of compact ANR subsets of $M$).

4.4. Infinite-dimensional manifolds. Let us say that an ANR $M$ satisfies Property $E$ (respectively, Property $E_f$) if for every compactum (resp. finite-dimensional compactum) $K$, every closed $K_0 \subset K$ and every closed Z-set $X \subset M$, every embedding $K_0 \to M$ extends to an embedding $g$ of some neighborhood $U$ of $K_0$ in $K$ such that $g(U \setminus K_0) \subset M \setminus X$.

**Corollary 4.7.** (a) If $M$ is an ANR satisfying Property $E$, then every closed Z-set in $M$ is a fine Z-set.

(b) If $M$ is an ANR satisfying Property $E_f$ and such that all its compact subsets are finite-dimensional, then every closed Z-set in $M$ is a fine Z-set.

The special case of Properties $E$ and $E_f$ with $X = \emptyset$ was considered by K. Sakai [33; Lemma 1.1], and his argument works to show that if $M$ satisfies Property $E$ (resp. $E_f$), then all $M$-manifolds (i.e., spaces where every point has a neighborhood homeomorphic to an open subset of $M$) also satisfy Property $E$ (resp. $E_f$).
Example 4.8. It is easy to see that the direct limit $Q^\omega$ of inclusions $Q^1 \subset Q^2 \subset \cdots$, where $Q = I^\omega$ is the Hilbert cube, satisfies Property E, using that every compact subset of $Q^\omega$ is contained in some $Q^n$. Hence all $Q^\omega$-manifolds also satisfy Property E.

Example 4.9. It is easy to see that the direct limit $\mathbb{R}^\omega$ of inclusions $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots$ satisfies Property $E_f$, using that every compact subset of $\mathbb{R}^\omega$ is contained in some $\mathbb{R}^n$. Hence all $\mathbb{R}^\omega$-manifolds also satisfy Property $E_f$. Also, every $\mathbb{R}^\omega$-manifold is a countable direct limit of its finite-dimensional compact metric subspaces (see [33]), so all its compact subsets are finite-dimensional.

An ANR $M$ is said to have the \textit{compact Z-set property} if every compact subset of $M$ is a Z-set.

A topologically complete ANR $M$ is said to have the \textit{disjoint cubes property} if every map $f_1 \sqcup f_2: I^n \sqcup I^n \to M$ is approximable by a link map (i.e. by a map sending the two copies of $I^n$ to disjoint sets). This is equivalent to saying that every map of a compactum into $M$ can be approximated by an embedding (see [40; Theorem 7.3.5]).

**Proposition 4.10.** If $M$ is a topologically complete ANR satisfying the compact Z-set property and the disjoint cubes property, then $M$ satisfies Property E.

**Proof.** This follows from another equivalent form of the disjoint cubes property: for every compactum $K$, every closed $K_0 \subset K$ and every closed Z-set $X \subset M$, every map $f: K \to M$ such that $f(K_0)$ is a closed Z-set is approximable by an embedding $g: K \to M$ extending $f|_{K_0}$ and such that $g(K)$ is a closed Z-set and $f(K \setminus K_0) \subset M \setminus X$ [40; Theorem 7.3.5]. \hfill $\square$

Example 4.11. It is easy to see that every compact subset $K$ of the infinite product $\mathbb{R}^\omega$ is a Z-set. Indeed, for each $i \in \mathbb{N}$ let $x_i \in \mathbb{R}$ be some point not in the image of the projection of $K$ onto the $i$th coordinate; then the compositions $h_n: \mathbb{R}^\omega \to \mathbb{R}^\omega$ of the projection $\mathbb{R}^\omega \to \mathbb{R}^n$ and the inclusion $\mathbb{R}^n \times \{(x_{n+1}, x_{n+2}, \ldots)\} \to \mathbb{R}^\omega$ converge to the identity and each $h_n(\mathbb{R}^\omega) \cap K = \emptyset$.

It is also easy to see that $\mathbb{R}^\omega$ has the disjoint cubes property, by composing $f_1$ with an $h_n: \mathbb{R}^\omega \to \mathbb{R}^\omega$ as above such that $h_n(\mathbb{R}^\omega) \cap f_2(I^n) = \emptyset$.

Thus by Proposition 4.10, $\mathbb{R}^\omega$ satisfies Property E. Since $\mathbb{R}^\omega$ is homeomorphic to the Hilbert space $l_2$ by a well-known result of R. D. Anderson (see [7; Theorem VI.2.1]), we conclude that all $l_2$-manifolds satisfy Property E.

**Remark 4.12.** Let us note that every Polish space $X$ is homeomorphic to a closed Z-set in the Hilbert space $l_2$. Indeed, it is well-known that $X$ embeds onto a closed subset of $l_2$ (see e.g. [34; Theorem 6.2.4]). Let $\mathbb{R}^{2\omega}$ be the subspace of $\mathbb{R}^\omega$ consisting of points of the form $(0, x_2, 0, x_4, 0, x_6, \ldots)$. Then $\mathbb{R}^{2\omega}$ is homeomorphic to $\mathbb{R}^\omega$, hence also to $l_2$, and consequently $X$ is homeomorphic to a closed subset $Y$ of $\mathbb{R}^{2\omega}$. On the other hand, $\mathbb{R}^{2\omega}$ is closed in $\mathbb{R}^\omega$ and is a Z-set, since the compositions $h_n: \mathbb{R}^\omega \to \mathbb{R}^\omega$ of the projection $\mathbb{R}^\omega \to \mathbb{R}^{2n+1}$ and the inclusion $\mathbb{R}^{2n+1} \times \{(1, 0, 0, \ldots)\} \to \mathbb{R}^\omega$ converge to the identity and each $h_n(\mathbb{R}^\omega) \cap \mathbb{R}^{2\omega} = \emptyset$. Hence $Y$ is also a closed Z-set in $\mathbb{R}^\omega$. 

Example 4.13. Let us consider the subspace $c_{00}$ of $l_\infty$ consisting of all eventually zero sequences. If $K$ is a compact subset of $c_{00}$, then $r_i(K) := \sup_{(x_1, x_2, \ldots) \in K} |x_i|$ is finite for each $i$. If there exists an $\varepsilon > 0$ such that $r_i(K) > \varepsilon$ for arbitrarily large $i$, then $K$ contains an infinite subset whose points are at distance $> \varepsilon$ from each other, which is a contradiction. Thus $r_1(K), r_2(K), \ldots$ converges to 0. Using this, it is easy to see that $c_{00}$ satisfies the compact Z-set property and the disjoint cubes property (by composing the given map of a cube with the translation by a sufficiently small distance $d$ along the $i$th coordinate for a sufficiently large $i$).

Thus by Proposition 4.10, $c_{00}$ satisfies Property E. Since $c_{00}$ is homeomorphic to the subspace $l_1'$ of $l_2$ consisting of all eventually zero sequences (see [7; Theorem VIII.3.1(b)]), we conclude that all $l_1'$-manifolds satisfy Property E.

Example 4.14. It is easy to see that if $M$ satisfies the compact Z-set property or the disjoint cubes property, then so does $M \times N$. It follows, in particular, that $l_2' \times Q$ satisfies Property E. Since $l_2' \times Q$ is homeomorphic to the pseudo-boundary $\Sigma := [-1, 1]^\infty \setminus (-1, 1)^\infty$ of the Hilbert cube (see [7; Proposition VIII.4.3(b)]), we conclude that all $\Sigma$-manifolds satisfy Property E.

5. Appendix: Descriptive Topology

To put Polish spaces in a more general perspective, let us consider the following classes $\Sigma_n$ and $\Pi_n$ of separable metrizable spaces:

- $\Pi_0$ consists of compact metrizable spaces and $\Sigma_0$ of locally compact polyhedra;
- $X$ is of class $\Sigma_{n+1}$ if it is a countable union of its subspaces of class $\Pi_n$;
- $X$ is of class $\Pi_{n+1}$ if it is the limit of an inverse sequence of spaces of class $\Sigma_n$.

Here $\Pi_1$ consists of all Polish spaces, since Polishness is preserved by sequential inverse limits (the inverse limit is a closed subset of the product) and every Polish space is the limit of an inverse sequence of locally compact polyhedra (see Theorem 4.2).

The classes $\Sigma_n$ and $\Pi_n$ admit a simple description in terms of the usual Borel hierarchy. We recall that for any space $X$, its additive and multiplicative Borel classes $\Sigma_i = \Sigma_i(X)$, $\Pi_i = \Pi_i(X)$ are defined as follows:

- $\Sigma_0$ consists of open subsets of $X$, and $\Pi_0$ consists of closed subsets of $X$;
- $\Sigma_{n+1}$ consists of countable unions of sets in $\Pi_n$;
- $\Pi_{n+1}$ consists of countable intersections of sets in $\Sigma_n$.

Clearly, a subset $S$ of $X$ is in $\Sigma_n$ if and only $X \setminus S$ is in $\Pi_n$. It is not hard to see that each $\Sigma_i \cup \Pi_i \subset \Sigma_{i+1} \cap \Pi_{i+1}$. The classes $\Sigma_1, \Pi_2, \Sigma_3, \ldots$ are also written as $\mathcal{F}_\sigma, \mathcal{F}_{\sigma\delta}, \mathcal{F}_{\sigma\delta\sigma}, \ldots$ and $\Pi_1, \Sigma_2, \Pi_3, \ldots$ as $\mathcal{G}_\delta, \mathcal{G}_{\delta\sigma}$, $\mathcal{G}_{\delta\sigma\delta}, \ldots$.

A separable metrizable space $Y$ is said to be of absolute class $\Sigma_n$ (resp. $\Pi_n$) if every homeomorphic image of $Y$ in every separable metrizable space $X$ is of class $\Sigma_n$ (resp. $\Pi_n$) in $X$. By a theorem of Lavrentiev (see e.g. [7; Theorem VII.1.1]), for $n \geq 2$, every $\Sigma_n$ subset of a Polish space is an absolute $\Sigma_n$; and for $n \geq 1$, every $\Pi_n$ subset of a Polish space is an absolute $\Pi_n$.
By considering an embedding of $Y$ into the Hilbert cube it is easy to see that $Y$ is an absolute $\Pi_0$ if and only if it is compact; and an absolute $\Sigma_1$ (=absolute $F_\sigma$) if and only if it is $\sigma$-compact (i.e. a countable union of its compact subsets). It is well-known that absolute $\Pi_1$ (=absolute $G_\delta$) spaces coincide with Polish spaces (P. S. Alexandrov, 1924; see e.g. [7; Corollary I.3.2]); obviously, absolute $\Sigma_0$ spaces do not exist. By a theorem of Alexandrov, absolute $\Sigma_1$ spaces that are also absolute $\Pi_1$ coincide with those separable metrizable spaces that can be reduced to $\emptyset$ by (necessarily countable) transfinite applications of the operation of removing all points of local compactness [4; Theorem 5.31].

**Proposition 5.1.** $\Pi_n$ coincides with absolute $\Pi_n$ for all $n$, and $\Sigma_n$ coincides with absolute $\Sigma_n$ for all $n \geq 1$.

So the only difference is that absolute $\Sigma_0$ is empty, whereas $\Sigma_0$ consists of all locally compact polyhedra.

**Proof.** We know that both $\Pi_0$ and absolute $\Pi_0$ consist of compacta; that both $\Pi_1$ and absolute $\Pi_1$ consist of Polish spaces; and that coincidence of $\Pi_n$ with absolute $\Pi_n$ implies coincidence of $\Sigma_{n+1}$ with absolute $\Sigma_{n+1}$. Assuming that $\Sigma_n$ coincides with absolute $\Sigma_n$, we also have that $\Pi_{n+1}$ contains absolute $\Pi_{n+1}$, since countable intersection can be replaced by intersection of a decreasing sequence, which is a kind of sequential inverse limit. It remains to prove the reverse inclusion.

Suppose that $X$ is the limit of a (countable) inverse sequence of separable metrizable spaces $X_i$ and continuous maps $f_i: X_{i+1} \to X_i$. Then $X$ is a closed subset of $\prod_i X_i$, so in particular it separable metrizable. By considering the graphs of $f_i$, we can modify any given metrics on the $X_i$ (consecutively) so as to make each $f_i$ uniformly continuous (see [25; §6.7]). Let $\bar{X}_i$ be the completion of $X_i$. Since the $f_i$ are uniformly continuous, they extend to $\bar{f}_i: \bar{X}_{i+1} \to \bar{X}_i$, whose inverse limit is the completion $\bar{X}$ of $X$. Thus the subset $X$ of the complete metric space $\prod_i \bar{X}_i$ is the intersection of the closed subset $\bar{X}$ and the subset $\prod_i X_i$. We have $\prod_i X_i = \bigcap_j \left( X_i \times \prod_{j \neq i} \bar{X}_j \right)$. It follows that if each $X_i$ is of class $\Pi_n$ in $\bar{X}$, then $X$ is of class $\Pi_n$ in $\prod_i \bar{X}_i$. Consequently, the limit of an inverse sequence of absolute $\Pi_n$ spaces is an absolute $\Pi_n$. Since absolute $\Sigma_n$ spaces are absolute $\Pi_{n+1}$, their inverse limit is an absolute $\Pi_{n+1}$. \qed

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Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia

E-mail address: melikhov@mi.ras.ru