The Replicator Coalescent

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Abstract

We consider a stochastic model, called the replicator coalescent, describing a system of blocks of \(k\) different types which undergo pairwise mergers at rates depending on the block types: with rate \(C_{ij}\) blocks of type \(i\) and \(j\) merge, resulting in a single block of type \(i\). The replicator coalescent can be seen as a generalisation of Kingman’s coalescent death chain in a multi-type setting, although without an underpinning exchangeable partition structure. The name is derived from a remarkable connection we uncover between the instantaneous dynamics of this multi-type coalescent when issued from an arbitrarily large number of blocks, and the so-called replicator equations from evolutionary game theory. By dilating time arbitrarily close to zero, we see that initially, on coming down from infinity, the replicator coalescent behaves like the solution to a certain replicator equation. Thereafter, stochastic effects are felt and the process evolves more in the spirit of a multi-type death chain.

Key words: Markov chain, coalescent, coming down from infinity.

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1 Introduction

In this article, we are interested in developing a multi-type analogue of Kingman’s coalescent as a death chain, called a replicator coalescent, with the following interpretation. Blocks take one of \(k\) different types. Mergers within blocks may take place as well as mergers of blocks with two different types. In the latter case, we will need to specify what type the two merging blocks of different type will take. To this end, let us introduce the \(k \times k\) rate matrix \(C = (C_{i,j})\) the merger rate matrix with \(C_{i,j} > 0\) for all \(i, j \in \{1, \cdots, k\}\). This matrix (which is not an intensity matrix) encodes the evolution of a continuous-time Markov chain, say \((n_t, t \geq 0)\), on the state space

\[
\mathbb{N}_0^k = \left\{ \eta \in \mathbb{N}_0^k : \sum_{i=1}^k \eta_i \geq 1 \right\},
\]

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where $N_0 = \{0, 1, 2, \cdots\}$, which is defined in the following way. Given that $n(t) = (n_1, \cdots, n_k) \in \mathbb{N}_k^\times$ such that $\sum_{i=1}^k n_i > 1$:

- For $i \in \{1, \cdots, k\}$ any two specific blocks of type $i$ will merge at rate $C_{i,i}$, and hence a total merger rate of type $i$ blocks equal to $C_{i,i} \frac{n_i}{2}$.

- For $i \neq j$, both selected from $\{1, \cdots, k\}$, any block of type $i$ will merge with any block of type $j$, producing a single block of type $i$, at rate $C_{i,j}$. The total rate of events of this type is thus $C_{i,j} n_i n_j$.

We can interpret $(n(t), t \geq 0)$ as the evolution of the population of $k$ types that exhibit both inter-type and intra-type competition. The rate $C_{i,i}$ is the rate at which two individuals of type $i$ compete for resource resulting in one of them not surviving. Moreover, at rate $C_{i,j}$ individuals of type $i$ and $j$ encounter one another in a competition for resource, resulting in $j$ not surviving. In this respect, our replicator coalescent echos features of the so-called O.K. Corral model describing a famous 19th Century Arizona shoot-out between lawmen and outlaws in [5, 6, 7] as well as the (birth-)death process in [1]. An example of the sample path of the process $n$ is given in Figure 1 in the setting $k = 3$.

Figure 1: A path of a replicator coalescent block numbers with $k = 3$, initiated from $n(0) = (5, 6, 2)$ and reducing to a population of one with $n(\gamma_1) = (1, 0, 0)$. The diagram represents the range of the process and there is no time axis.

The reader will note that the rate at which within-type mergers occur is that of Kingman’s coalescent. When there is only one type, and hence only within-type coalescence occurs, the
replicator coalescent is therefore nothing more than the death chain of a Kingman coalescent. In this sense, the process \((\mathbf{n}, \mathbb{P})\) may be thought of as a multi-type variant of the Kingman death chain.

Unfortunately, the specific structure of the replicator coalescent, does not permit an interpretation in terms of exchangeable partition structures as is the case for Kingman’s coalescent. For an alternative view of multi-type coalescent structures which do embrace a mathematical notion of multi-type exchangeability, see [4].

Although the replicator coalescent lives in the space \(\mathbb{N}_k^*\), we prefer to describe it via a so-called \(L^1\)-polar decomposition in the spirit of e.g. [1]. To this end, define
\[
\sigma(t) = ||\mathbf{n}(t)||_1 = n_1(t) + \cdots + n_k(t) \in \mathbb{N}
\]
and let
\[
r(t) = \text{arg}(\mathbf{n}(t)) := \sigma(t)^{-1}\mathbf{n}(t) \in S^k,
\]
where
\[
S^k := \left\{ (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : \sum_{i=1}^k x_i = 1, x_i \geq 0 \ \forall i \right\}
\]
is the \(k\)-dimensional simplex, with vector entries \(r_i(t) = \sigma(t)^{-1}n_i(t), i = 1, \ldots, k\). We will additionally write and occasionally use \(S^k_+\) to have the same definition as \(S^k\), albeit each of the \(x_i > 0\).

As such, we often refer to the process \(\mathbf{n}\) as \((r, \sigma^{-1})\). In particular, if \(\nabla = (\eta_1, \ldots, \eta_k) \in \mathbb{N}_k^*\), then, we will use \(\mathbb{P}_\nabla\) for the law of the replicator coalescent issued from state \(\mathbf{n}(0) = \nabla\). The usual convention would be to think of the family of probabilities \(\mathbb{P} = (\mathbb{P}_\eta, \eta \in \mathbb{N}_k^*)\), however we interchangeably also think of \(\mathbb{P} = (\mathbb{P}_\nabla, \nabla \in S^k \times \mathbb{N}^{-1})\), where \(S^k \times \mathbb{N}^{-1} := \{(x, 1/n) : x \in S^k \text{ and } n \in \mathbb{N}\}\).

In the setting of the block-counting process for Kingman’s coalescent, there are three fundamental facts which are now taken for granted in mainstream literature. Firstly Kingman’s coalescent block-counting process comes down from infinity almost surely. Secondly, it comes down from infinity in such a way that the number of blocks divided by \(1/t\) converges to a constant as \(t \to 0\). Secondly, and somewhat trivially, the block counting process is a death chain with an absorbing state which is a single block. This inspires us to address the following three main questions of our replicator coalescent:

(1) Does it ‘come down from infinity’ in an appropriately prescribed sense?

(2) What is the distribution on \(\{1, \ldots, k\}\) of the terminal block?

We are interested in characterising the behaviour of the replicator coalescent as we start it from an initial population that ‘tends to infinity’ in a prescribed way, and as such we will give a response to (1). In doing so, we will unravel a remarkable connection with the theory of evolutionary dynamical systems, described by so-called replicator equations, hence our choice of the name replicator coalescent. Our response to (1) is by no means a complete story. For example we do don’t show the existence of entrance laws on Skorokhod space, but rather we focus on the behaviour of the process as we limit its initial state to a boundary state ‘at infinity’, which means an initial condition for \((r, \sigma^{-1}) \in S^k_+ \times \{0\}\).
Definition 1. Henceforth we will say that \( (\eta^N, N \geq 1) \) tends to \((r_0, 0) \in S^k_+ \times \{0\} \)', if \( \eta^N \in \mathbb{N}_k \) such that \( \|\eta^N\| = N \) and \( \arg(\eta^N) \to r_0 \) as \( N \to \infty \).

We are unable to provide any results for (2) and believe this to be an extremely difficult problem; even in light of related results e.g. on the aforementioned OK Coral model in [5, 6, 7]. This short article is but an introduction to replicator coalescence, offering the opportunity for further analysis to take place. Indeed, in future work, we aim to give a more precise statement on the convergence on the Skorokhod space of the process to a unique entrance law which exhibits continuity at time zero (we comment further on this below).

2 Main results

For our first result, we show that the replicator coalescent comes down from infinity in a relatively precise sense. Specifically, we study the time \( \gamma_m = \inf\{t > 0 : \sigma(t) \leq m\}, m \in \mathbb{N} \) that the coalescent first reaches a state with \( m \) blocks in total, which can be bounded in probability for large \( N \) according to the following theorem.

Theorem 1. Suppose \( (\eta^N, N \geq 1) \) tends to \((r_0, 0) \), the replicator coalescent comes down from infinity in the sense that

\[
\lim_{N \to \infty} P_{\eta^N}(0 < \gamma_m < \infty) = 1, \quad \forall m \in \mathbb{N},
\]

and

\[
\lim_{m \to \infty} \lim_{N \to \infty} P_{\eta^N}(\gamma_m < \varepsilon) = 1, \quad \forall \varepsilon > 0.
\]

As it comes down from infinity, the standard Kingman coalescent with merger rate \( c > 0 \) has block count \((\nu(t), t \geq 0)\) that is approximately described by the ordinary differential equation

\[
\dot{\nu}(t) = -c\nu(t)^2/2.
\]

It turns out that the corresponding ODE for our coalescent is already known in the evolutionary game theory literature as the replicator equation.

Replicator equations are used to describe a population of \( k \) types, with the proportion of the total population of type \( i \in \{1, \cdots, k\} \) at time \( t \geq 0 \) denoted \( x_i(t) \in [0,1] \). These values sum to one, so \( \mathbf{x}(t) := (x_1(t), \cdots, x_k(t)) \) lives in \( S^k \), the \( k \)-dimensional simplex. The replicator equations are then written as

\[
\dot{x}_i(t) = x_i(t)(f_i(\mathbf{x}(t)) - \bar{f}(\mathbf{x}(t))), \quad i = 1, \cdots, k, \quad t \geq 0,
\]

where \( f_i : \mathcal{S}^k \mapsto \mathbb{R} \) describes the “fitness” of type \( i \) as a function of the current population density and \( \bar{f}(\mathbf{x}(t)) = \sum_{i=1}^n x_if_i(\mathbf{x}(t)) \) is the average population fitness.

Fitness is often assumed to depend linearly upon the population distribution, with coefficients organised in the “payoff matrix” \( A \). Specifically, let \( A_{i,j} \) denote the payoff for a player of type \( i \) facing an opponent of type \( j \). Then

\[
f_i(\mathbf{x}) = \sum_{j=1}^n A_{i,j}x_j.
\]
This replicator equation, henceforth referred to as the \textit{A-replicator equation} may be written
\begin{equation}
\dot{x}_i(t) = x_i(t) ([A x(t)]_i - x(t)^T A x(t)), \quad i = 1, \cdots, k, \ t \geq 0.
\end{equation}

If the system (2) admits a fixed point in the simplex, i.e. \(x_i(t) = x_i^*, \ i = 1, \cdots, k,\) for some vector \(x^* = (x_1^*, \cdots, x_k^*) \in S^k,\) then we see that, necessarily,
\begin{equation}
[A x^*]_i = (x^*)^T A x^* \quad i = 1, \cdots, k.
\end{equation}

In turn, this implies that there is a constant, \(c > 0,\) such that \(A x^* = c 1,\) so, \(x^* = c A^{-1} 1,\) where \(1\) is the vector in \(\mathbb{R}^k\) with unit entries. Since \(x^* \in S^k,\) it follows that \(1^T x^* = 1,\) and hence \(c = (1^T A^{-1} 1)^{-1},\) thus (2) is solved by
\begin{equation}
x^* = \frac{A^{-1} 1}{1^T A^{-1} 1}.
\end{equation}

If \(x^*\) satisfies the relation
\[(x^*)^T A x > x^T A x, \quad x \in S^k,
\]then it is called an \textit{evolutionary stable state} (ESS). Theorem 7.2.4 of [3] states that if \(x^*\) is a ESS, then
\begin{equation}
\lim_{t \to \infty} x(t) = x^*.
\end{equation}

The following results give us a remarkable connection between the theory of replicator equations and our coalescent model. To this end we define our \(A\) matrix by
\begin{equation}
A_{i,j} = - \left( C_{j,i} 1_{j \neq i} + \frac{1}{2} C_{i,j} \right).
\end{equation}

For the remainder of the paper we will assume the rates \(C\) are such that the matrix \(A\) admits an ESS \(x^*\).

Let us now state the connection between the notion of coming down from infinity for the replicator coalescent and the corresponding replicator equations.

\textbf{Theorem 2.} Suppose that \(A\) has an ESS \(x^*\) and that \((\eta^N, N \geq 1)\) tends to \((r_0, 0)\). Then for all \(T > 0,\)
\[
\lim_{N \to \infty} \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \| R(t) - x(t) \|_2 \right] = 0, \quad i = 1, \cdots, k,
\]
where \(R(t) = r(\tau(t)), \ t \geq 0, \ x(t) = (x_1(t), \cdots, x_k(t))\) solves the \(A\)-replicator equation and
\[
\tau(t) = \inf \{ s > 0 : \int_0^s \sigma(u) du > t \}, \quad t \geq 0.
\]
In particular,
\[
\lim_{t \to \infty} \lim_{N \to \infty} \mathbb{E}_{\eta^N} [\| R(t) - x^* \|_2] = 0, \quad i = 1, \cdots, k.
\]
In words, Theorem 2 says that, by dilating time arbitrarily close to zero, as the replicator coalescent comes down from infinity, its process \( r \) in the simplex behaves deterministically like the solution to an \( A \)-replicator equation. For a special choice of the matrix \( C_{i,j} \), Figure 2 shows simulations of the process \((R(t), t \geq 0)\) which resonate with the statement of Theorem 2. As part of the proof of Theorem 1, we will see that under \( \mathbb{P}_\eta \), the process \( \sigma \) is comparable to a Kingman coalescent with some collision rate, say \( c > 0 \). Noting that

\[
\int_0^{\tau(t)} \sigma(u) du = t, \quad \text{which implies} \quad \dot{\tau}(t) = 1/\sigma(\tau(t)),
\]

if, in heuristic terms, we treat \( \sigma \) as a solution to (1) with \( \sigma(0) = N \), so that \( \sigma(t)^{-1} = N^{-1} + ct/2 \), we have that \( \tau(t) \approx 2(\exp\{ct/2\} - 1)/cN \).

Figure 2: Simulations of a replicator coalescent with \( k = 3 \) initiated from a variety of initial states with an initial number of blocks \( \sigma(0) = 10^{15} \). Each path represents a simulation from a different initial state, presented in barycentric coordinates in the 3-simplex and a logarithmic axis for the total number of blocks. The matrix \( C \) has entries \( C_{i,i} = C_{i,i+1} = 1 \) and other entries zero.

The remainder of this paper is structured as follows. In the next section, we discuss how we can compare the process \((\sigma(t), t \geq 0)\) with Kingman’s coalescent on the same probability space, when it is issued from a finite number of blocks. This comparison is used frequently in several of our proofs. In Section 4 we treat the Markov process \( n \) as a semimartingale and study its decomposition as the sum of a martingale and a bounded variation compensator. This provides the basis for the proofs of Theorem 2, which are given in Section 5. Finally in Section 6 we conclude with some technical remarks and two conjectures concerning further behaviour of the entrance law.

3 Stochastic comparison with Kingman’s coalescent

As alluded to above, there are various points in our reasoning where we will compare the number of blocks in a replicator coalescent with the number of blocks in an appropriately
formulated Kingman coalescent on the same probability space. The first such result is reasoning gives us the proof that the replicator coalescent comes down from infinity.

**Proof of Theorem 1.** The process $\sigma$ decreases by one with each block merger, analogously to the block counting process of a standard Kingman coalescent. It is therefore sufficient to prove that $\sigma$ decreases at least as fast as a Kingman block counting process, this will show that $\gamma_m$ is asymptotically finite. Conversely, by showing that $\sigma$ decreases at most as fast as another Kingman block counting process, then this ensures that $\gamma_m$ is not asymptotically zero.

Formally speaking, from finite starting states, we want to construct such a Kingman coalescent $\nu$ on the same space as $\sigma$ with that $\sigma \leq \nu$ by considering the minimal rate of $\sigma$.

The total rate of mergers in state $n$ is given by

$$
\rho(n) = \sum_{i=1}^k \left( \sum_{j \neq i} C_{ij} n_j n_i + \frac{C_{i,i}}{2} n_i^2 - \frac{C_{i,i}}{2} n_i \right),
$$

which depends not just on the total number of blocks, but also on the distribution of block types. However, since $C_{i,j} > 0$ for all $i, j$, we can choose a constant $0 < C < \overline{C} := \min_{i,j} C_{i,j}$ such that

$$
\rho(n) > C \left( \frac{\|n\|_1}{2} \right) - \sum_{i=1}^k \frac{|C_{i,i} - C|}{2} n_i > C \left( \frac{\|n\|_1}{2} \right) - \frac{(\overline{C} - C)}{2} \|n\|_1,
$$

where $\overline{C} = \max_i C_{i,i}$. For each $\varepsilon > 0$, we can therefore choose $N_\varepsilon > 0$ such that, for all $\|n\|_1 > N_\varepsilon$,

$$
\rho(n) > (1 - \varepsilon)C \left( \frac{\|n\|_1}{2} \right).
$$

Appealing to the skip free property, it follows that we can stochastically couple a Kingman death chain $(\nu^+(t), t \geq 0)$ with collision rate $(1 - \varepsilon)C$ and the process $(n(t), t \geq 0)$ on the same space such that, for any $\eta = (\sigma(0), \arg(n(0)))$ such that $\sigma(0) > m > N_\varepsilon$, when $\nu^+(0) = \sigma(0) = \|\eta\|_1$, $\beta_m^+ \leq \gamma_m$, where $\beta_m^+ = \inf\{t > 0 : \nu^+(t) = m\}$, $m \in \mathbb{N}$.

Now suppose that the sequence $(\eta^N, N \geq 1)$ in $\mathbb{N}_k$ such that $(\|\eta\|_1, \arg(\eta^N)) \rightarrow (\infty, r_0)$ and $r_0 \in S_k^+$. The previous observation allows us to conclude that

$$
\lim_{N \rightarrow \infty} \mathbb{P}_{\eta^N}(\gamma_m < \infty) \geq \lim_{N \rightarrow \infty} \mathbb{P}\left(\beta_m^+ < \infty \mid \nu^+(0) = \|\eta\|_1\right) = 1,
$$

and similarly

$$
\lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{P}_{\eta^N}(\gamma_m < \varepsilon) \geq \lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{P}\left(\beta_m^+ < \varepsilon \mid \nu^+(0) = \|\eta\|_1\right) = 1, \quad \varepsilon > 0.
$$

Conversely, there also exists a constant $C > 0$ such that

$$
\rho(n) = \sum_{i=1}^k \left( \sum_{j \neq i} C_{ij} n_j n_i + \frac{C_{i,i}}{2} n_i^2 - \frac{C_{i,i}}{2} n_i \right) \leq C \left( \frac{\|n\|_1}{2} \right).
$$
In the same spirit, follows that we can stochastically embed another Kingman death chain $(\nu^{-}(t), t \geq 0)$ and the process $(\mathbf{n}(t), t \geq 0)$ on the same space such that, for any $\eta = (\sigma(0), \text{arg}(\mathbf{n}(0)))$ such that

$$\mathbb{P}_\eta[\gamma_m > 0] \geq \mathbb{P}[\beta_m^{-} > 0|\nu(0) = \sigma(0)],$$

where $\beta_m^{-} = \inf\{t > 0 : \nu^{-}(t) = m\}, m \in \mathbb{N}$. This gives us the required behaviour that $\lim_{N \to \infty} \mathbb{P}_\eta^N(\gamma_m > 0) = 1$ for each $m \in \mathbb{N}$, thus completing the proof.

4 Semimartingale representation

We would like to treat the replicator coalescent $(\mathbf{n}(t), t \geq 0)$ as a semimartingale. It turns out to be more convenient to consider instead the vectorial process

$$\mathbf{y}(t) = \begin{pmatrix} r(t \wedge \gamma_1) \\ 1/\sigma(t \wedge \gamma_1) \end{pmatrix}, \quad t \geq 0,$$

where

$$\gamma_1 = \inf\{t > 0 : \sigma(t) = 1\}.$$

Naturally, by expressing the evolution of $(\mathbf{y}(t), t \geq 0)$ as a that of a semimartingale our interest is predominantly in the process $(r(t), t \geq 0)$ so as to make a link with the replicator equations in (2).

Lemma 1. For each $\eta \in \mathbb{N}_*^k$, the process $\mathbf{y}$ under $\mathbb{P}_\eta$ has a semimartingale decomposition

(8) $$\mathbf{y}(t) = \mathbf{y}(0) + \mathbf{m}(t) + \alpha(t), \quad t \geq 0$$

where $(\mathbf{m}(t), t \geq 0)$ is a martingale taking the form

(9) $$\mathbf{m}(t) = \sum_{s \leq t \wedge \gamma_1} \Delta \mathbf{y}(s) - \alpha(t), \quad t \geq 0,$$

such that $\Delta \mathbf{y}(t) = \mathbf{y}(t) - \mathbf{y}(t-)$ and $(\alpha(t), t \geq 0)$ is a compensator taking the form

(10) $$\alpha(t) = \int_0^{t \wedge \gamma_1} \frac{\sigma(s)}{\sigma(s) - 1} \sum_{i=1}^k \left( \frac{\sigma(s)(r(s) - e_i)}{1} \right) r_i(s)[\sigma(s)^{-1}\text{diag}(A)1 - Ar(s)], ds.$$

Proof. A standard computation using the compensation formula tells us that $\mathbf{m}$ is a martingale providing $\sum_{s \leq t} \|\Delta \mathbf{y}(s)\|_1$ has finite expectation for each $t \geq 0$, which is equivalent to the existence of the compensator $\alpha(t)$. The latter is given by the rates that define the replicator
coalescent. More precisely, recalling the definition of the matrix $A$ in (6),

$$
\alpha(t) = \int_0^{t \wedge \gamma_1} \sum_{i=1}^k \left( \frac{n(s) - e_i}{\sigma(s) - 1} - \frac{\lambda_i}{\sigma(s)} \right) \left[ \sum_{j=1, j \neq i}^k n_j(s)C_{ji} + \frac{1}{2}n_i(s)(n_i(s) - 1)C_{ii} \right] ds
$$

as required. Note, if we identify the compensator via the density $(\lambda(t), t \geq 0)$, where

$$
\alpha(t) = \int_0^{t \wedge \gamma_1} \lambda(s) ds
$$

then, in the above representation of $\alpha$, the largest term is of order $\sigma(s)$, which, because the process $n$ is non-increasing, we can easily conclude that, for all $\eta \in \mathbb{N}_+^k$ and any time $t \geq 0$,

$$
\mathbb{E}_{\eta}^N \left[ \sum_{s \leq t \wedge \gamma_2} \|\Delta y(s)\|_2 \right] \leq \mathbb{E}_{\eta}^N \left[ \int_0^{t \wedge \gamma_1} \|\lambda(s)\|_2 ds \right] \leq C\|\eta\|_2 \mathbb{E}_{\eta}^N[t \wedge \gamma_1] \leq C\|\eta\|_2 t,
$$

for an unimportant constant $C > 0$. This ensures that $m$ is a martingale and that $\alpha$ is well defined.

For the proof of Theorems 2, we are interested in the behaviour of the process $r$ under $\mathbb{P}_\eta$ for any $\eta$ such that $\|\eta\|_1 \to \infty$ and $\text{arg}(\eta) \to r$ for some $r \in S^C$. Heuristically speaking, the term $\sigma(s)(r(s) - e_i)$ in the expression for $\alpha$ suggests that $\alpha(t)$ explodes for as $t \to 0$. The undesirable factor $\sigma(s)$ can be removed however by an appropriate time change and in doing so, we begin to see where the relationship with the replicator equations emerges.

**Lemma 2.** Suppose we define the sequence of stopping times $(\tau(t), t \geq 0)$, which are defined by the right inverse,

$$
\tau(t) = \inf\{u > 0 : \int_0^u \sigma(s)ds > t\}, \quad t \geq 0.
$$

Then $y^\tau := y \circ \tau$ has semimartingale decomposition $y^\tau = m^\tau + \alpha^\tau$, where $m^\tau := m \circ \tau$ is a
martingale and, for \( t \geq 0 \),

\[
\alpha^\tau(t) = \int_0^{t^\wedge \gamma_1} \frac{\sigma(\tau(s))}{\sigma(\tau(s)) - 1} \sigma(\tau(s))^{-1} \text{diag}(A) \mathbf{1} - A \tau(s))_i ds \\
\sum_{i=1}^k \left( \frac{1}{\sigma(\tau(s))} \right) r_i(\tau(s)) \sigma(\tau(s))^{-1} \text{diag}(A) \mathbf{1} - A r(\tau(s))_i ds
\]

(15)

Proof. We use basic Steiætes calculus to tell us that

\[
d\alpha^\tau(t) = d\alpha(s)|_{s=\tau(t)} d\tau(t).
\]

Moreover,

\[
\int_0^{\tau(t)} \sigma(s) ds = t \quad \text{and hence} \quad \sigma(\tau(t)) d\tau(t) = dt.
\]

(16)

Combining these observations with the conclusion of Lemma 1 the result follows. In particular, from (12),

\[
\int_0^{\tau(t) \wedge \gamma_1} \lambda(s) ds = \int_0^{t^\wedge \gamma_1} \frac{\lambda(\tau(u))}{\sigma(\tau(u))} du.
\]

(17)

Technically, we need to verify that \( m^\tau \) is a martingale rather than a local martingale, however a computation similar to (13) taking advantage of (15) is easily executed affirming the required martingale status.

Next we look at how controllable the martingale components in the above to semimartingale decompositions are. We consider their second moment behaviour.

**Lemma 3.** For each \( \eta \in \mathbb{N}^k \), the martingale \( m^\tau \) under \( \mathbb{P}_\eta \) satisfies

\[
E_{\eta^N} \left[ \| m^\tau(t) \|^2 \right] \leq C E_{\eta} \left[ \int_0^{\tau(t) \wedge \gamma_1} \frac{\sigma(s)^2}{(\sigma(s) - 1)^2} ds \right], \quad t \geq 0.
\]

Proof. Steiætes calculus, or equivalently general semi-martingale calculus (see for example Theorem 33 of [8]), tells us that, since \( m \) has bounded variation,

\[
\| m(t) \|^2 = 2 \int_0^{\tau(t) \wedge \gamma_1} m(s-) \cdot dm(s) + \sum_{0 < s \leq t^\wedge \gamma_1} \{ \Delta m(s) \|^2 - 2 m(s-) \cdot \Delta m(s) \}
\]

(18)

\[
= 2 \int_0^{t^\wedge \gamma_1} m(s-) \cdot dm(s) + \sum_{0 < s \leq t^\wedge \gamma_1} (\Delta m(s))^2.
\]

As all vector entries are bounded, it is easy to show that \( \int_0^{t^\wedge \gamma_1} m(s-) \cdot dm(s), t \geq 0 \), is a martingale.
Next, we identify the adapted increasing bounded variation process, say $\beta$, which is the compensator of $\sum_{0<s\leq t\wedge \gamma_1} (\Delta m(s))^2$, $t \geq 0$ so that

$$\|m(t)\|_2^2 - \beta(t), t \geq 0$$

is a martingale with mean 0.

To this end, note that $\Delta m(t) = \Delta y(t)$. Hence, we have, on the event that $t$ is a time at which the number of blocks of type $i$ decreases, the mean increment of $(\Delta m(s))^2$ given $(y(u), u < t)$ is given by

$$\chi_i(t) := \left(\begin{array}{c} n(s) - \sigma(s)e_i \\ (\sigma(s) - 1)\sigma(s) \end{array}\right) \cdot \left(\begin{array}{c} n(s) - \sigma(s)e_i \\ (\sigma(s) - 1)\sigma(s) \end{array}\right) = \frac{n(t) \cdot n(t) - 2\sigma(t)n_i(t) + \sigma(t)^2 + 1}{\sigma(t)^2(\sigma(t) - 1)^2}.$$

It is now straightforward to see that, there exists a $C > 0$ such that

$$\beta(t) = \int_0^{t \wedge \gamma_1} \sum_{i=1}^k \chi_i(s) \left[ \sum_{j=1}^k n_j(s)n_i(s)C_{ji} + \frac{1}{2}n_i(s)(n_i(s) - 1)C_{ii} \right] ds$$

$$\leq C \int_0^{t \wedge \gamma_1} \frac{\sigma(s)^2}{(\sigma(s) - 1)^2} ds,$$

for some $C > 0$, where we have used that

$$\sigma(t)^2 = (n_1(t) + \cdots + n_k(t))^2 \geq n(t) \cdot n(t)$$

Taking expectation gives the desired inequality.

As the reader may now expect, our ultimate objective is to show that for any sequence of starting initial configurations $\eta^N \in \mathbb{N}^k$ such that $\|\eta^N\|_1 \to \infty$ as $N \to \infty$ and $\arg(\eta^N) \to r \in S_+^k$, the martingale component $m^r$ disappears. This tells us that the behaviour of $(r(t), t \geq 0)$ behaves increasingly like the compensator term, which is further key to controlling its behaviour. To this end, we conclude this section with two more results that provide us with the desired control of the aforesaid martingale component.

**Lemma 4.** Suppose $(\eta^N, N \geq 1)$ tends to $(r_0, 0)$,

$$\tau^{-1}(t) = \int_0^t \sigma(s) ds \to \infty, \ \tau(t) \to 0, \ and \ |\tau(t) \wedge \gamma_1 - \tau(t)| \to 0,$$

weakly as $N \to \infty$, for any $t \geq 0$. 
Proof. In the spirit of the comparison with an auxiliary Kingman coalescent used in the proof of Theorem 1, using the notation from there, we note from (7) that there exists a constant $C$ such that

$$\rho(n) = \sum_{i=1}^{k} \left( \sum_{j \neq i} C_{ji} n_j n_i + \frac{C_{i,i}}{2} n_i^2 - \frac{C_{i,i}}{2} n_i \right) \leq C \left( \|n\| \right).$$

This shows that there is a death chain $(\nu(t), t \geq 0)$, representing the number of blocks in a Kingman coalescent with merger rate $C$ such that, for any $\eta \in \mathbb{N}^k$, on the same probability space, we can stochastically bound $\sigma(t) \geq \nu(t)$, $t \geq 0$.

We now note that, for any large $M > 0$ there exists a constant $C > 0$ (not necessarily the same as before) such that, for any $m$ sufficiently large,

$$\lim_{N \to \infty} \mathbb{P}_{\eta^N} \left( \int_0^m \sigma(s) ds > M \right) \geq \lim_{N \to \infty} \mathbb{P}_{\eta^N} \left( \int_0^m \nu(s) ds > M \right)$$

$$= \Pr \left( \sum_{n=m+1}^{\infty} \frac{n}{C(n^2)} \mathbf{e}_1^{(n)} > M \right)$$

$$= \Pr \left( \sum_{n=m+1}^{\infty} \frac{1}{n-1} \mathbf{e}_1^{(n)} > \frac{CM}{2} \right),$$

(19)

where $(\mathbf{e}_1^{n}, n \geq 1)$, is a sequence of iid unit-mean exponentially distributed random variables. If we write $(N(t), t \geq 0)$ for a Poisson process with unit rate, then almost surely we have

$$\sum_{n=m+1}^{\infty} \frac{1}{n-1} \mathbf{e}_1^{(n)} = \int_0^\infty \frac{1}{N(s) + m} ds = \int_0^\infty \frac{s}{N(s) + m} ds = \infty$$

where the final equality follows by the strong law of large numbers for Poisson processes. As such the right-hand side of (19) is equal to 1.

Since $M$ and $m$ can be arbitrarily large, this shows the first claim as soon as we note that $\tau^{-1}(t) = \int_0^t \sigma(u) du$, which is an easy consequence of (16). On the other hand, note that since $\int_0^{\tau(t)} \sigma(s) ds = t$, when $\sigma(0) = \eta^N \to \infty$, the above comparison with Kingman’s coalescent shows that almost surely that, since $\int_0^u \sigma(s) ds$ converges weakly to infinity for all $u > 0$, then $\tau(t)$ converges weakly to 0. Indeed, if with positive probability $\tau(t) > \varepsilon$ in the limit as $N \to \infty$, then, on that event, $\int_0^{\tau(t)} \sigma(s) ds \geq \int_0^\varepsilon \sigma(s) ds$, which explodes in distribution. This in turn contradicts the definition of $\tau(t)$. This proves the second and third statements of the corollary. 

In addition, since the second moment of the martingale $m^\tau$ can be controlled by its associated time change, we also get a helpful $L^2$ corollary from Lemma 4.

Corollary 1. From Lemma 3 and Lemma 4, we deduce that, if $(\eta^N, N \geq 1)$ tends to $(r_0, 0)$, then, for each $t > 0$

$$\lim_{N \to \infty} \mathbb{E}_{\eta^N} \left[ \sup_{s \leq t} \|m^\tau(s)\|^2 \right] = 0.$$
Proof. We have from Lemma 3 and a change of variable similar to (17) that

\[
\mathbb{E}_\eta \left[ \left\| m^\tau(t) \right\|_2^2 \right] \leq C \mathbb{E}_\eta \left[ \int_0^{\tau(t) \wedge \tau_1} \frac{\sigma(s)^2}{(\sigma(s) - 1)^2} ds \right]
\]

(20) \[= C \mathbb{E}_\eta \left[ \int_0^{\tau(t) \wedge \tau_1 - 1} \frac{(\sigma(\tau(u)))^2}{(\sigma(\tau(u)) - 1)^2} \frac{1}{\sigma(\tau(u))} du \right] \leq C t \mathbb{E}_\eta \left[ \frac{1}{\sigma(\tau(t))} \right]. \]

We can choose \( N \) sufficiently large such that \( \tau(t) \) is less than \( \delta \) with probability at least \( 1 - \epsilon \) so that,

\[
\mathbb{E}_\eta \left[ \frac{1}{\sigma(\tau(t))^2} \right] \leq \mathbb{E}_\eta \left[ \frac{1}{\sigma(\delta)^2} ; \tau(t) < \delta \right] + \mathbb{P}_\eta(\tau(t) \geq \delta)
\]

\[ \leq \delta \mathbb{E}_\eta \left[ \frac{1}{\delta \nu(\delta)} \right] + \epsilon, \]

where we have again compared with a lower bounding Kingman coalescent \((\nu(t), t \geq 0)\) on the same space, as in Lemma 4. Recall the classical result for Kingman’s coalescent coming down from infinity that, when the collision rate is \( C > 0 \),

\[ \delta \nu(\delta) \to 2/C \]

almost surely as \( \delta \to 0 \); cf [2]. We can now easily conclude with the help of dominated convergence that

\[ \lim_{N \to \infty} \mathbb{E}_\eta \left[ \left\| m^\tau(t) \right\|_2^2 \right] = 0, \]

and this concludes the proof once we invoke Doob’s submartingale inequality. \( \square \)

5 Proof of Theorem 2

Recall from (5) that the existence of an ESS \( \mathbf{x}^* \) implies that the solution to the \( A \)-replicator equation has the limit \( \mathbf{x}(t) \to \mathbf{x}^* \). Reinterpreting (2) in its integral form, this tells us that

(21) \[x_i(t) = x_i(0) + \int_0^t x_i(s)([A\mathbf{x}(s)]_i - \mathbf{x}(s)^T A\mathbf{x}(s))ds, \quad t \geq 0.\]

This representation makes it easier to give the heuristic basis of the proof of Theorem 2.

Following our earlier heuristic reasoning, we can now see that, under \( \mathbb{P}_\eta \) as \( \|\eta\|_1 \to \infty \) the integrand in the expression for \( \alpha^\tau \) appears to have a similar structure to the replicator equations (2). That is, under \( \mathbb{P}_\eta \) as \( \|\eta\|_1 \to \infty \) and as \( t \to 0 \),

\[
\frac{d\alpha^\tau(t)}{dt} \approx \begin{pmatrix} \theta(t) \\ 0 \end{pmatrix}
\]

where

\[\theta_i(t) = r_i(\tau(t))( [Ar(\tau(t))]_i - r(\tau(t))^T Ar(\tau(t)) ).\]
On the other hand, if we can show that \( \mathbf{m}^\tau \to 0 \) as \( \| \eta \|_1 \to \infty \), then, since \( \mathbf{y}^\tau = \mathbf{m}^\tau + \mathbf{\alpha}^\tau \), reading off the first component of \( \mathbf{y}^\tau \), i.e. \( \mathbf{R}(t) := \mathbf{r}(\tau(t)) \), roughly speaking we see that

\[
R_i(t) \approx r_i(0) + \int_0^t R_i(s) \left( [A\mathbf{R}(s)]_i - \mathbf{R}(s)^T A\mathbf{R}(s) \right) ds,
\]

as \( \| \eta \|_1 \to \infty \), given that Corollary 1 shows the martingale component is negligible. In other words, the process \((\mathbf{R}(t), t \geq 0)\), begins to resemble the replicator equation in its integral form \((21)\). It now looks like a reasonable conjecture that \( \mathbf{R}(t) \to \mathbf{x}^* \), just as the solution to the replicator equation does.

Let us thus move to the proof of Theorem 2, which, as alluded to earlier, boils down to the control we have on the martingale \( \mathbf{m}^\tau \) under \( \mathbb{P}_\eta \) as \( \| \eta \|_1 \to \infty \), thanks to Corollary 1.

**Proof of Theorem 2.** Write \( R_i(t) = r(\tau(t)) \) on the event

\[
A_t := \{ t < \tau^{-1}(\gamma_1) \}, \quad t \geq 0,
\]

and note from Lemma 4 and the proof of Theorem 1 that \( \lim_{N \to \infty} \mathbb{P}_{\eta^N}(A_t) = 1 \). We have, for each \( T > 0 \),

\[
\mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \left( \mathbf{R}(t) - \mathbf{r}(0) - \int_0^t \left( \sum_{i=1}^k e_i R_i(s) [A\mathbf{R}(s)]_i - (\mathbf{R}(s)^T A\mathbf{R}(s)) \mathbf{R}(s) \right) ds \right)^2 \right]^{1/2}
\leq \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \left( \mathbf{R}(t) - \mathbf{r}(0) - \int_0^t \frac{\sigma(\tau(s))}{\sigma(\tau(s)) - 1} \sum_{i=1}^k (\mathbf{R}(s) - e_i R_i(s)) [\sigma(\tau(s))^{-1} \text{diag}(\mathbf{A}) \mathbf{1} - A\mathbf{R}(s)]_i ds \right)^2 \right]^{1/2}
+ \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \left( \int_0^t \frac{1}{\sigma(\tau(s)) - 1} \left( \sum_{i=1}^k e_i R_i(s) [A\mathbf{R}(s)]_i - (\mathbf{R}(s)^T A\mathbf{R}(s)) \mathbf{R}(s) \right) ds \right)^2 \right]^{1/2} A_t
\]

\[
+ \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \left( \int_0^t \frac{1}{\sigma(\tau(s)) - 1} \left( (\mathbf{R}(s)^T \text{diag}(\mathbf{A}) \mathbf{1}) \mathbf{R}(t) - \sum_{i=1}^k e_i R_i(s) [\text{diag}(\mathbf{A}) \mathbf{1}]_i \right)^2 \right) \right]^{1/2} A_t
\]

(22)

From Corollary 1, the first term after the inequality tends to zero as \( N \to \infty \). Up to a multiplicative constant, the second and third terms after the inequality can be bounded by

\[
T \mathbb{E}_{\eta^N} \left[ \frac{1}{(\sigma(\tau(T)) - 1) \wedge 1} \right],
\]

where we have used the monotonicity of \( \tau(\cdot) \) and \( \sigma(\cdot) \). As noted in the proof of Corollary 1, the latter tends to zero as \( N \to \infty \).
It now follows that
\[(23)\]
\[
\lim_{N \to \infty} E_{\eta^N}\left[ \sup_{t \leq T} \left( R(t) - r(0) - \int_0^t \sum_{i=1}^k e_i R_i(s)[AR(s)]_i - (R(s)^TAR(s))R(s) \right)^2 1_{A_i} \right]^{\frac{1}{2}} = 0
\]

As all of the vectorial and matrix terms in (23) are bounded, it is also easy to see with the help of Lemma 4 that
\[
\lim_{N \to \infty} E_{\eta^N}\left[ \sup_{t \leq T} \left( R(t) - r(0) - \int_0^t \sum_{i=1}^k e_i R_i(s)[AR(s)]_i - (R(s)^TAR(s))R(s) \right)^2 1_{A_i} \right]^{\frac{1}{2}} \leq \lim_{N \to \infty} CT^p_{\eta^N}(A_T^2) = 0,
\]

for some constant $C > 0$, which gives us
\[(24)\]
\[
\lim_{N \to \infty} E_{\eta^N}\left[ \sup_{t \leq T} \left( R(t) - r(0) - \int_0^t \sum_{i=1}^k e_i R_i(s)[AR(s)]_i - (R(s)^TAR(s))R(s) \right)^2 \right]^{\frac{1}{2}} = 0.
\]

Consider the replicator equation initiated from any $r(0) \in S^k_+$. Similarly to (21), albeit in vectorial form, we can write the solution to (2), when issued from $r(0)$ as
\[(25)\]
\[
x(t) - r(0) - \int_0^t \sum_{i=1}^k e_i x_i(s)[Ax(s)]_i - (x(s)^T Ax(s))x(s) \, ds = 0.
\]

Subtracting (25) off in (24), we see that, for each $T > 0$,
\[
E_{\eta^N}\left[ \sup_{t \leq T} \| R(t) - x(t) \|_2 \right] \\
\leq E_{\eta^N}\left[ \sup_{t \leq T} \| R(t) - r(0) - \int_0^t \sum_{i=1}^k e_i R_i(s)[AR(s)]_i - (R(s)^TAR(s))R(s) \|_2 \right] \\
+ \int_0^T \sum_{i=1}^k e_i E_{\eta^N}\left[ |R_i(s) - x_i(s)| [AR(s)]_i - R(s)^T AR(s) \right] \, ds \\
+ \int_0^T \sum_{i=1}^k e_i x_i(s) E_{\eta^N}\left[ |A(R(s) - x(s))|_i - R(s)^T A(R(s) - x(s)) \right] \, ds \\
+ \int_0^T \sum_{i=1}^k e_i x_i(s) E_{\eta^N}\left[ |(R(s) - x(s))^T Ax(s)| \right] \, ds
\]
Noting that $0 \leq R_i(s), x_i(s) \leq 1$ for all $i = 1 \cdots, k, s \geq 0$, we have,

$$
\mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \| R(t) - x(t) \|_2 \right] \\
\leq \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \| R(t) - r(0) - \int_0^t \sum_{i=1}^k e_i R_i(s) \left( [AR(s)]_i - R(s)^T AR(s) \right) ds \|_2 \right] \\
\quad + C \int_0^T \mathbb{E}_{\eta^N} \left[ \sup_{u \leq s} \| R(u) - x(u) \|_2 \right] ds,
$$

where $C > 0$ is an unimportant constant. Hence, using (24), the monotonicity of norms and dominated convergence,

$$\lim_{N \to \infty} \sup_{N' \geq N} \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \| R(t) - x(t) \|_2 \right] = 0,$$

By taking $t \to \infty$, we easily deduce that

$$\lim_{N \to \infty} \mathbb{E}_{\eta^N} \left[ \sup_{t \leq T} \| R(t) - x(t) \|_2 \right] = 0.$$

This completes the proof of the theorem. 

\section{Concluding remarks}

In further work, one may pursue the issue of Skorokhod continuity with respect to any entrance laws that we can identify. That is to say, whether, for any $(\eta^N, N \geq 1)$, seen as a sequence in $\mathcal{S}_{k} \times N^{-1}$, is tending to $(r_0, 0)$, we have $\mathbb{P}_{\eta^N} \to \mathbb{P}^\infty$, where $\mathbb{P}^\infty$ is an entrance law, in the sense of Skorokhod convergence, where $\mathbb{P}^\infty$ is an entrance law. In the setting that there is an entrance law which is continuous in the aforementioned sense, we have a relatively strong sense in which the replicator coalescent comes down from infinity.

\textbf{Conjecture 1.} Suppose $(\eta^N, N \geq 1)$ tends to $(r_0, 0)$. For $i = 1, \cdots, k$, 

$$\lim_{m \to \infty} \lim_{N \to \infty} \mathbb{E}_{\eta^N} \left[ \sup_{j \geq m} \| r_i(\gamma_j) - x_i^* \|_2 \right] = 0, \quad i = 1, \cdots, k,$$

if and only if $r_0 = x^*$, where we recall $\gamma_j = \inf \{ t > 0 : \sigma(t) \leq j \}$, for $j \geq 1$. 

In contrast to Theorem 2, Conjecture 1 claims that, moving backwards through time, towards the instantaneous event at which the replicator coalescent comes down from infinity at the origin of time, the process \( r \) necessarily approaches the ESS \( x^* \). As such, working backwards in time, the replicator coalescent never gets to see the ‘initial state’ \( (r, 0) \), from which \( \mathbb{P}^\infty \) is constructed in Theorem 1.

Put together Theorems 2 and Conjecture 1 is really claiming that \( x^* \) is a ‘bottleneck’ for the replicator coalescent. Figure 1 simulates an example where \( k = 3 \), in which the bottleneck phenomenon can clearly be seen.

As alluded to above, one of the issues we have with establishing the existence of a law \( \mathbb{P}^\infty \) is that of uniqueness. The next conjecture claims that we can invoke uniqueness among entrance laws synonymously with continuous convergence on the Skorokhod space \( (\mathbb{D}, \mathcal{D}) \).

**Conjecture 2.** Suppose \( (\eta^N, N \geq 1) \) tends to \( (r_0, 0) \), there exists an entrance law, denoted by, \( \mathbb{P}^{(r_0,0)} \). Moreover, \( \lim_{N \to \infty} \mathbb{P}_{\eta^N} \to \mathbb{P}^{(r_0,0)} \) continuously on \( (\mathbb{D}, \mathcal{D}) \) if and only if \( r_0 = x^* \).

Theorem 2 shows that under \( \mathbb{P}_{\eta^N} \) as \( \eta^N \to (r_0, 0) \), in an arbitrarily small amount of time the process \( r \) will jump from \( r_0 \) to \( x^* \). It is for this reason, Conjecture 2 predicts that Skorokhod continuity can only follow if \( r_0 = x^* \).

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