EQUIVALENCE PROBLEM FOR SECOND ORDER PDE AND DOUBLE FIBRATION AS A FLAT MODEL SPACE

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Dedicated to Professor Hajime Sato on his first retirement

Abstract. In this paper, we consider an equivalence problem of second order partially differential equations (PDE) and a duality of the flat differential equation. For the equivalence problem, explicit form of invariants (curvatures) are given. In particular, if all of the curvatures vanish, then PDE are equivalent to the flat equation. We also investigate a duality associated with the flat equation using double fibrations. These double fibrations are described in terms of transformation groups.

1. Introduction

In this paper, we investigate second order PDE for one unknown function of two variables. That is, we consider a problem for when these equations are equivalent to the flat equation, and we also consider a duality for the flat equation. The equivalence problem for differential equations is simply explained as follows. We fix classes of differential equations and a group of coordinate transformations. Then, we consider a problem how differential equations change under coordinate transformations. We can also express this problem in terms of group actions. Let $G$ be a coordinate transformation group and $X$ be a set of certain differential equations. Then the equivalence problem for differential equations in $X$ is interpreted as the problem of determining the orbit decomposition with respect to the action of $G$ on $X$.

The equivalence problem is studied deeply by Sophus Lie and Élie Cartan, and many other authors. We mention a few historical background here. (See [10] for a detailed history of the equivalence problem.) Sophus Lie studied an action of the contact diffeomorphism group $G := \text{Cont}(\mathbb{R}^3)$ on $X := \{y'' = f(x, y, y')\}$, and obtained the fact that this action is transitive. In this case, the orbit decomposition of $X$ for the action of $G$ has just one orbit. After the work of S.Lie, A.Tresse studied the following case. Let $G$ be the subgroup $\text{Diff}(\mathbb{R}^2)^{\text{cont}}$ consisting of lifts of diffeomorphisms on $\mathbb{R}^2$ to the jet space $J^1(\mathbb{R}, \mathbb{R})$, and same set of differential equations $X := \{y'' = f(x, y, y')\}$ . Under this setting up, Tresse considered an orbit decomposition of the action of $G$ on $X$. Contrary to the above problem considered by Lie, Tresse proved that this action is not transitive.

Key words and phrases. second order partially differential equations, equivalence problem, $G$-structure, duality, double fibration.
On the other hand, Élie Cartan also considered the same problem from a different method, which is now called the equivalence method ([3], [12], [18]).

Along this historical background, we consider an equivalence problem for second order PDE for one unknown function of two variables $y = y(x_1, x_2)$:

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = f_{ij}(x_1, x_2, y, z_1, z_2),$$

(1)

where, $f_{ij}$ ($1 \leq i, j \leq 2$) satisfying $f_{ij} = f_{ji}$ are $C^\infty$ functions on $J^1(\mathbb{R}^2, \mathbb{R}) := \{(x_1, x_2, y, z_1, z_2)\}$, and $z_1 = y_{x_1}$, $z_2 = y_{x_2}$. If $f_{ij}$ all vanish, (1) is called the flat equation. We take the group ScaleDiff($\mathbb{R}^3)^{cont}$ of lifts of scale transformations on $\mathbb{R}^3$ as a transformation group $G$.

We will calculate explicitly the curvatures for this equivalence problem by using Cartan’s equivalence method. We obtain the necessary and sufficient condition when the second order PDE satisfying integrability condition is equivalent to the flat equation via a vanishing condition of these curvatures ([18]). Then, our main theorem can be stated as follows.

**Main Theorem 1.** For the above equivalence problem, we determine the fifteen curvatures $M_i, S_j$ explicitly (curvatures $M_i, S_j$ are given in page 11). In particular, we consider the equation (1) for the following functions $f_{ij}$:

$$f_{11} = P(x_1, x_2, y), \quad f_{12} = Q(x_1, x_2, y), \quad f_{22} = R(x_1, x_2, y).$$

Then, this equation is (locally) equivalent to the flat equation under lifts of scale transformations if and only if this equation is integrable.

Compare with equivalence problems of second order ODEs, there is a lot of curvatures in this theorem. The reason is given by the following consideration. In general, orbit decompositions for PDEs are more complicated than orbit decompositions for ODEs. Moreover, $G = \text{ScaleDiff}(\mathbb{R}^3)^{cont}$ is a very strongly restricted group. Therefore, this result is obtained. Conversely, if we take groups larger than ScaleDiff($\mathbb{R}^3)^{cont}$, then we obtain a few of curvatures. For example, we can consider the group $G = \text{Diff}(\mathbb{R}^3)^{cont}$ as a such group.

We also discuss a duality associated with differential equations via double fibration. In particular, we consider a duality between the coordinate space and the solution space of the flat equation. Double fibrations play an important role for a study of this duality. Moreover, these fibrations are usually described via some transformation groups appeared in equivalence problem ([2], [12]). For the group ScaleDiff($\mathbb{R}^3)^{cont}$, we can not obtain a fibration of compact type, because the group ScaleDiff($\mathbb{R}^3)^{cont}$ is too small. Hence, it is natural to consider an existence problem of groups from which double fibration of compact-type is obtained as a flat model space. For this problem, we find a non-trivial
group which gives a fibration of compact-type:

\[ G = \{ g \in SL(4, \mathbb{R}) \mid g[e_3] = [e_3], \quad ^t g^{-1}[e_3] = [e_3] \} \]

\[ = \left\{ \begin{pmatrix} * & * & 0 & * \\ * & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & 0 & * \end{pmatrix} \in SL(4, \mathbb{R}) \right\} \]

For this group, we obtain the following fibration of compact-type.

**Main Theorem 2.** A double fibration constructed by the above group is the following fibration of compact-type.

\[
\begin{aligned}
G/(G \cap H) &\cong \mathbb{F}_{1,3}(1, 2) \\
(G \cap H_4)/(G \cap H) &\cong S^1 \\
G/(G \cap H_4) &\cong RP^2 \\
G/(G \cap H_1) &\cong RP^2
\end{aligned}
\]

The coordinate transformation group \( G \) corresponding to this group \( G \) is constructed by the transformations of the form:

\[ X_1 = X_1(x_1, x_2), \quad X_2 = X_2(x_1, x_2), \quad Y = \frac{y}{A(x_1, x_2)}. \]

Finally, we will consider the dual equations of original equations and calculate these dual equations explicitly ([17]).

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## 2. Equivalence Problem and \( G \)-structure

In this section, we introduce an equivalence problem and explain the \( G \)-structure associated with this problem. For this purpose, we prepare some terminology and notation.

For functions of two variables \( y = y(x_1, x_2) \), we consider the second order PDE (1), and diffeomorphisms \( \phi \) on \( \mathbb{R}^3 \) of the form

\[ \phi(x_1, x_2, y) = (X_1(x_1), X_2(x_2), Y(x_1, x_2, y)). \]

The map \( \phi \) of this form is called a scale transformation. A scale transformation \( \phi \) lifts naturally to a contact diffeomorphism \( \hat{\phi} \) of \( J^1(\mathbb{R}^2, \mathbb{R}) \) defined by:

\[ \hat{\phi}(x_1, x_2, y, z_1, z_2) = (X_1(x_1), X_2(x_2), Y(x_1, x_2, y), Z_1, Z_2), \]
where, \( Z_1 = \frac{Y_{y}x + Y_{y}z_1}{(X_1)_{x_1}} \), \( Z_2 = \frac{Y_{y}x + Y_{y}z_2}{(X_2)_{x_2}} \). We can easily check that the map \( \hat{\phi} \) is a contact diffeomorphism:

\[
\hat{\phi}^*(dy - z_1dx_1 - z_2dx_2) = dY - Z_1dX_1 - Z_2dX_2 \\
= Y_{x_1}dx_1 + Y_{x_2}dx_2 + Y_{y}dy \\
- \frac{Y_{x_1} + Y_{y}z_1}{(X_1)_{x_1}}(X_1)_{x_1}dx_1 - \frac{Y_{x_2} + Y_{y}z_2}{(X_2)_{x_2}}(X_2)_{x_2}dx_2 \\
= Y_{y}(dy - z_1dx_1 - z_2dx_2).
\]

We introduce the following terminology:

\[
\text{ScaleDiff}(\mathbb{R}^3) : = \{ \text{Scale transformation on } \mathbb{R}^3 \}, \\
\text{Diff}(\mathbb{R}^3)_{\text{cont}} : = \{ \text{The lift of Diff}(\mathbb{R}^3) \text{ to } J^1(\mathbb{R}^2, \mathbb{R}) \}, \\
\text{ScaleDiff}(\mathbb{R}^3)_{\text{cont}} : = \{ \text{The lift of ScaleDiff}(\mathbb{R}^3) \text{ to } J^1(\mathbb{R}^2, \mathbb{R}) \}, \\
X : = \{ \text{second order PDE (1)} \}.
\]

The main problem in the present paper is the following.

**Problem 2.1.** Examine the orbit decomposition under the action of ScaleDiff(\( \mathbb{R}^3 \))\(_{\text{cont}} \) on \( X \).

In order to resolve the above problem, we use a \( G \)-structure associated with the equation (1). First, we replace from data of second order PDE (1) to data of differential system ([3], [12], [19]). We choose the following coframe of \( J^1(\mathbb{R}^2, \mathbb{R}) \) corresponding to the equation (1),

\[
\theta_0 : = dy - z_1dx_1 - z_2dx_2, \\
\theta_1 : = dz_1 - f_{11}dx_1 - f_{12}dx_2, \\
\theta_2 : = dz_2 - f_{21}dx_1 - f_{22}dx_2, \\
\omega_1 : = dx_1, \\
\omega_2 : = dx_2.
\]

We consider the Frobenius system

\[
\mathcal{I} : = \{ \theta_0, \theta_1, \theta_2 \}_{\text{diff}} \text{ with } \omega_1 \wedge \omega_2 \neq 0
\]

constructed by this coframe. The correspondence between second order PDE (1) and the Frobenius system \( \mathcal{I} \) is described as follows. Consider vector fields on \( J^1(\mathbb{R}^2, \mathbb{R}) \) which are annialated by \( \theta_i \), while are not annialated by \( \omega_i \). At any point on \( J^1(\mathbb{R}^2, \mathbb{R}) \), such vector fields are generated by two vector fields \( v_1, v_2 \). The integral surfaces which are tangent to the 2-plane \( \text{span}\{v_1, v_2\} \) at any point are the graphs of solutions of the second order PDE (1). Then, the parameters \( (x_1, x_2) \) are regarded as a local coordinate system of this integral surface.
The Frobenius condition (integrability condition) of the Frobenius system $I$ is:

$$d\theta_i \equiv 0 \quad \text{(mod } \theta_0, \theta_1, \theta_2) \quad (i = 0, 1, 2).$$

Then, the above integrability condition is equivalent to $A = B = 0$, where $A$ and $B$ are given by

$$A = (f_{11}x_x - (f_{12})_x + (f_{11})y_z + (f_{11})z_1f_{12} + (f_{11})z_2f_{22}$$
$$- (f_{12})y_z - (f_{12})z_1f_{11} - (f_{12})z_2f_{12};$$

$$B = (f_{12}x_x - (f_{22})_x + (f_{12})y_z + (f_{12})z_1f_{12} + (f_{12})z_2f_{22}$$
$$- (f_{22})y_z - (f_{22})z_1f_{11} - (f_{22})z_2f_{12}. $$

**Remark 2.2.** Hereafter, we discuss only the second order PDE (1) with respect to $f_{ij}$ satisfying $A = B = 0$.

A family of integral surfaces of $I$ gives a 2-dimensional foliation on $J^1(\mathbb{R}^2, \mathbb{R})$. We describe an infinitesimal automorphism group of the foliation, and consider a principal bundle over $J^1(\mathbb{R}^2, \mathbb{R})$ with this group as a structure group.

The contact lift $\hat{\phi}$ of the scale transformation $\phi$ preserving $I$ satisfies the following equations:

$$\hat{\phi}^*\theta_0 = a\theta_0, \quad (a \neq 0),$$
$$\hat{\phi}^*\theta_1 = b\theta_0 + c\theta_1, \quad (c \neq 0),$$
$$\hat{\phi}^*\theta_2 = e\theta_0 + g\theta_2, \quad (g \neq 0),$$
$$\hat{\phi}^*\omega_1 = h\omega_1, \quad (h \neq 0),$$
$$\hat{\phi}^*\omega_2 = k\omega_2 \quad (k \neq 0).$$

The equation (4) can be written in the following form:

$$\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix} =
\begin{bmatrix}
a & 0 & 0 & 0 \\
b & c & 0 & 0 \\
e & 0 & g & 0 \\
0 & 0 & 0 & h \\
0 & 0 & 0 & k
\end{bmatrix}
\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix} \quad \text{(5)}$$

where, $a, b, c, e, g, h, k$ are functions. Thus we have linear transformations of coframes determined by $\hat{\phi}$. Moreover, the lift $\hat{\phi}$ of the scale transformation satisfies:

$$d\theta_0 \equiv -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2 \quad \text{(mod } \theta_0),$$
$$d\theta_1 \equiv 0 \quad \text{(mod } \theta_0, \theta_1, \theta_2),$$
$$d\theta_2 \equiv 0 \quad \text{(mod } \theta_0, \theta_1, \theta_2).$$

(6)
These relations give conditions $a = ch = gk$. From these conditions, we get the linear transformations of coframes of the following form:

$$
\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix} =
\begin{bmatrix}
ch & 0 & 0 & 0 & 0 \\
0 & b & c & 0 & 0 \\
e & 0 & g & 0 & 0 \\
0 & 0 & 0 & h & 0 \\
0 & 0 & 0 & 0 & k
\end{bmatrix}
\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix}.
$$

(7)

Therefore, we obtain the following 5-dimensional Lie group as infinitesimal automorphism group:

$$
G := \left\{ \begin{bmatrix}
ch & 0 & 0 & 0 & 0 \\
b & c & 0 & 0 & 0 \\
e & 0 & g & 0 & 0 \\
0 & 0 & 0 & h & 0 \\
0 & 0 & 0 & 0 & k
\end{bmatrix} \in GL(5, \mathbb{R}) \mid ch = gk \right\}.
$$

(8)

Then, we choose the reduced $G$-bundle $\mathcal{F}_G$ of the coframe bundle $\mathcal{F}_{GL(\mathbb{R}^5)}$ over $J^1(\mathbb{R}^2, \mathbb{R})$. This bundle $\mathcal{F}_G$ is called $G$-structure associated with the second order PDE (1).

### 3. Cartan’s equivalence method

In the previous section, we introduced a $G$-structure $\mathcal{F}_G$ associated with the second order PDE (1). In this section we compute curvatures for the equivalence problem. For this purpose, we adopt Cartan’s equivalence method ([3], [12], [18]).

First, we compute the structure equation on $\mathcal{F}_G$. From (7), we can choose $(\theta_0, \theta_1, \theta_2, \omega_1, \omega_2)$ as $\mathbb{R}^5$-valued tautological 1-form on $\mathcal{F}_G$. To obtain the structure equation, we compute the exterior derivative of the tautological 1-forms $(\theta_0, \theta_1, \theta_2, \omega_1, \omega_2)$.

$$
\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix} =
\begin{bmatrix}
\frac{dc}{c} + \frac{dk}{k} & 0 & 0 & 0 & 0 \\
\frac{db}{ch} - \frac{bcd}{c^2h} & \frac{dc}{c} & 0 & 0 & 0 \\
\frac{dc}{ch} - \frac{edg}{cgh} & 0 & \frac{da}{g} & 0 & 0 \\
0 & 0 & 0 & \frac{dh}{h} & 0 \\
0 & 0 & 0 & 0 & \frac{dk}{k}
\end{bmatrix}
\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix}.
$$

(9)
where,

\[
T_1 = -\frac{b}{ch}, \quad T_2 = -\frac{e}{ch}, \quad T_3 = \frac{b^2}{(ch)^2} - \frac{(f_{11})_y}{h^2} + \frac{b(f_{11})_{z_1}}{ch^2} + \frac{e(f_{11})_{z_2}}{gh^2},
\]

\[
T_4 = \frac{be}{(ch)^2} - \frac{(f_{12})_y}{hk} + \frac{b(f_{12})_{z_1}}{chk} + \frac{e(f_{12})_{z_2}}{ch^2}, \quad T_5 = -\frac{b}{ch} - \frac{(f_{11})_{z_1}}{h},
\]

\[
T_6 = -\frac{(f_{12})_{z_1}}{k}, \quad T_7 = -\frac{e(f_{11})_{z_2}}{gh}, \quad T_8 = -\frac{b}{ch} - \frac{(f_{12})_{z_2}}{h},
\]

\[
T_9 = \frac{be}{(ch)^2} - \frac{g(f_{12})_y}{ch^2} + \frac{bg(f_{12})_{z_1}}{(ch)^2} + \frac{e(f_{12})_{z_2}}{ch^2},
\]

\[
T_{10} = \frac{e^2}{(ch)^2} - \frac{g(f_{22})_y}{chk} + \frac{bg(f_{22})_{z_1}}{c^2hk} + \frac{e(f_{22})_{z_2}}{chk},
\]

\[
T_{11} = -\frac{e}{ch} - \frac{g(f_{12})_{z_1}}{ch}, \quad T_{12} = -\frac{g(f_{22})_{z_1}}{ck}, \quad T_{13} = -\frac{(f_{12})_{z_2}}{h},
\]

\[
T_{14} = -\frac{e}{ch} - \frac{(f_{22})_{z_2}}{k}.
\]

**Remark 3.1.** We put \(\omega := (\theta_0, \theta_1, \theta_2, \omega_1, \omega_2)\) and write the structure equation (9):

\[
d\omega = -\theta \wedge \omega + T\omega \wedge \omega.
\]

In the above, we note that \(\theta\) is a \(g\)-valued 1-form and \(T\omega \wedge \omega\) is a \(\mathbb{R}^5\)-valued 2-form. In fact,

\[
d\omega = d(g\omega) = dg \cdot g^{-1} \wedge \omega + T\omega \wedge \omega,
\]

where \(g \in G\) and \(\omega = (\theta_0, \theta_1, \theta_2, \omega_1, \omega_2)\). The above equation shows that \(\theta\) is the Maurer-Cartan form. In the structure equation (9), each component of \(\theta\) is called the pseudo-connection form and \(T\omega \wedge \omega\) is called the torsion 2-form, and coefficient functions of 2-forms in each component of \(T\omega \wedge \omega\) are called torsions ([6]).

To simplify the structure equation (9), we set:

\[
\alpha := \frac{dc}{c} - \frac{b}{ch}\omega_1 - \frac{e}{ch}\omega_2,
\]

\[
\beta := \frac{db}{ch} - \frac{bdc}{c^2h} - \left\{ \frac{b^2}{(ch)^2} - \frac{(f_{11})_y}{h^2} + \frac{b(f_{11})_{z_1}}{ch^2} + \frac{e(f_{11})_{z_2}}{gh^2} \right\} \omega_1
\]

\[
- \left\{ \frac{be}{(ch)^2} - \frac{(f_{12})_y}{hk} + \frac{b(f_{12})_{z_1}}{chk} + \frac{e(f_{12})_{z_2}}{ch^2} \right\} \omega_2,
\]

\[
\varepsilon := \frac{de}{ch} - \frac{edg}{cgh} - \left\{ \frac{e^2}{(ch)^2} - \frac{g(f_{22})_y}{chk} + \frac{bg(f_{22})_{z_1}}{c^2hk} + \frac{e(f_{22})_{z_2}}{chk} \right\} \omega_1
\]

\[
- \left\{ \frac{e^2}{(ch)^2} - \frac{g(f_{22})_y}{chk} + \frac{bg(f_{22})_{z_1}}{c^2hk} + \frac{e(f_{22})_{z_2}}{chk} \right\} \omega_2,
\]

\[
\delta := \frac{dg}{g} - \frac{b}{ch}\omega_1 - \frac{e}{ch}\omega_2, \quad \gamma := \frac{dh}{h}, \quad \psi := \frac{dk}{k}.
\]

By substituting the above terms into the equation (9), we get the following proposition.
Proposition 3.2. The structure equation on $\mathcal{F}_G$ is written as:

\[
d\begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix} = \begin{bmatrix}
\alpha + \gamma & 0 & 0 & 0 & 0 \\
\beta & \alpha & 0 & 0 & 0 \\
\varepsilon & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & \gamma & 0 \\
0 & 0 & 0 & 0 & \psi
\end{bmatrix} \wedge \begin{bmatrix}
\theta_0 \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
-\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2 \\
L_1 \theta_1 \wedge \omega_1 + L_2 \theta_1 \wedge \omega_2 + L_3 \theta_2 \wedge \omega_1 + L_4 \theta_2 \wedge \omega_2 \\
L_2 \theta_1 \wedge \omega_1 + L_5 \theta_1 \wedge \omega_2 + L_4 \theta_2 \wedge \omega_1 + L_6 \theta_2 \wedge \omega_2 \\
0 \\
0
\end{bmatrix}
\]

(10)

where,
\[
L_1 := -\frac{2b}{ch} - \frac{(f_{11}) z_1}{h}, \quad L_2 := -\frac{e}{ch} - \frac{(f_{12}) z_1}{k}, \quad L_3 := -\frac{c(f_{11}) z_2}{gh},
\]
\[
L_4 := -\frac{b}{ch} - \frac{(f_{12}) z_2}{h}, \quad L_5 := -\frac{g(f_{22}) z_1}{ck}, \quad L_6 := -\frac{2e}{ch} - \frac{(f_{22}) z_2}{k},
\]
\[
\alpha + \gamma = \delta + \psi.
\]

Remark 3.3. In (10), some torsions in the structure equation (9) are absorbed.

To eliminate the ambiguity of the pseudo-connection forms, we need to choose a reduction of $G$-structure $\mathcal{F}_G$. We choose the reduction of $G$-structure by setting $L_2 = L_4 = 0$. We denote this reduced bundle by $\mathcal{F}_{G_1}$, where $G_1$ is the following 3-dimensional Lie group:

\[
G_1 := \left\{ \begin{bmatrix}
ch & 0 & 0 & 0 \\
0 & c & 0 & 0 \\
0 & 0 & g & 0 \\
0 & 0 & 0 & h \\
0 & 0 & 0 & k
\end{bmatrix} \in GL(5, \mathbb{R}) \mid ch = gk \right\}.
\]

We have the tautological 1-form on $\mathcal{F}_{G_1}$ given by:

\[
\begin{bmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\hat{\omega}_1 \\
\hat{\omega}_2
\end{bmatrix} = \begin{bmatrix}
\frac{ch\theta_0}{\delta} \\
-c(f_{12}) z_2 \theta_0 + c\theta_1 \\
-g(f_{12}) z_2 \theta_0 + g\theta_2 \\
\frac{h\omega_1}{\delta} \\
\frac{k\omega_2}{\delta}
\end{bmatrix}.
\]
Then, the structure equation on $\mathcal{F}_{G_1}$ is given by

$$
\begin{bmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\hat{\omega}_1 \\
\hat{\omega}_2
\end{bmatrix} =
\begin{bmatrix}
\alpha + \gamma & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 \\
0 & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & \gamma & 0 \\
0 & 0 & 0 & 0 & \psi
\end{bmatrix}
\begin{bmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\hat{\omega}_1 \\
\hat{\omega}_2
\end{bmatrix}
\wedge
\begin{bmatrix}
M_{12} \hat{\omega}_1 \wedge \hat{\theta}_0 + M_{11} \hat{\omega}_2 \wedge \hat{\theta}_0 - \hat{\theta}_1 \wedge \hat{\omega}_1 - \hat{\theta}_2 \wedge \hat{\omega}_2 \\
M_1 \hat{\theta}_2 \wedge \hat{\omega}_1 + M_2 \hat{\theta}_1 \wedge \hat{\theta}_0 + M_3 \hat{\theta}_2 \wedge \hat{\theta}_0 + M_4 \hat{\theta}_1 \wedge \hat{\theta}_0 + M_5 \hat{\omega}_2 \wedge \hat{\theta}_0 + M_{10} \hat{\omega}_1 \wedge \hat{\theta}_1 + M_{11} \hat{\omega}_2 \wedge \hat{\theta}_1 \\
M_6 \hat{\theta}_1 \wedge \hat{\omega}_2 + M_7 \hat{\theta}_1 \wedge \hat{\theta}_0 + M_8 \hat{\theta}_2 \wedge \hat{\theta}_0 + M_9 \hat{\omega}_1 \wedge \hat{\theta}_0 + M_9 \hat{\omega}_2 \wedge \hat{\theta}_0 + M_{12} \hat{\omega}_1 \wedge \hat{\theta}_2 + M_{13} \hat{\omega}_2 \wedge \hat{\theta}_2
\end{bmatrix}
\begin{bmatrix}
M_1 = -\frac{c (f_{11})_{z_2}}{gh}, & M_2 = -\frac{(f_{12})_{z_2 z_1}}{ck}, & M_3 = -\frac{(f_{12})_{z_2 z_2}}{gh}, \\
M_4 = -\frac{1}{h^2} \{(f_{12})_{y}^2 - (f_{11})_{y} - (f_{12})_{z_2} (f_{11})_{z_1} - (f_{11})_{z_2} (f_{12})_{z_1} \\
+ (f_{12})_{z_2 x_1} + (f_{12})_{z_2 y z_1} + (f_{12})_{z_2 z_1} f_{11} + (f_{12})_{z_2 z_2} f_{21}\}, & M_5 = \frac{1}{hk} \{(f_{12})_{y} + (f_{12})_{z_2} (f_{12})_{z_1} - (f_{12})_{z_2 x_2} - (f_{12})_{z_2 y z_2} - (f_{12})_{z_2 z_1} f_{12} - (f_{12})_{z_2 z_2} f_{22}\}, \\
M_6 = -\frac{g (f_{22})_{z_1}}{ck}, & M_7 = -\frac{(f_{12})_{z_1 z_1}}{ck}, \\
M_8 = \frac{1}{hk} \{(f_{12})_{y} + (f_{12})_{z_1} (f_{12})_{z_2} - (f_{12})_{z_1 x_1} - (f_{12})_{z_1 y z_1} - (f_{12})_{z_1 z_1} f_{11} - (f_{12})_{z_1 z_2} f_{21}\}, & M_9 = -\frac{1}{k^2} \{(f_{12})_{z_1}^2 - (f_{22})_{y} - (f_{12})_{z_2} (f_{22})_{z_1} - (f_{12})_{z_1} (f_{22})_{z_2} \\
+ (f_{12})_{z_1 x_2} + (f_{12})_{z_1 y z_2} + (f_{12})_{z_1 z_1} f_{12} + (f_{12})_{z_1 z_2} f_{22}\}\}, \\
M_{10} = \frac{1}{h} \{(f_{11})_{z_1} - (f_{12})_{z_2}\}, & M_{11} = \frac{(f_{12})_{z_1}}{k}, \\
M_{12} = \frac{(f_{12})_{z_2}}{h}, & M_{13} = \frac{1}{k} \{(f_{22})_{z_2} - (f_{12})_{z_1}\}.
\end{bmatrix}
$$

By absorption of torsions $M_2$, $M_{10}$, $M_{11}$, $M_{12}$, $M_{13}$, we obtain the following:
Proposition 3.4. We have the following structure equation on $\mathcal{F}_{G_1}$:

$$
\begin{bmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\hat{\omega}_1 \\
\hat{\omega}_2
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\alpha} + \hat{\gamma} & 0 & 0 & 0 & 0 \\
0 & \hat{\alpha} & 0 & 0 & 0 \\
0 & 0 & \hat{\delta} & 0 & 0 \\
0 & 0 & 0 & \hat{\gamma} & 0 \\
0 & 0 & 0 & 0 & \hat{\psi}
\end{bmatrix}
\begin{bmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\hat{\omega}_1 \\
\hat{\omega}_2
\end{bmatrix}
$$

(11)

where we set

$$
\hat{\alpha} := \alpha - M_2 \hat{\theta}_0 + M_{10} \hat{\omega}_1 + M_{11} \hat{\omega}_2,
$$

$$
\hat{\gamma} := \gamma + (M_{12} - M_{10}) \hat{\omega}_1,
$$

$$
\hat{\delta} := \delta - M_2 \hat{\theta}_0 + M_{12} \hat{\omega}_1 + M_{13} \hat{\omega}_2,
$$

$$
\hat{\psi} := \psi + (M_{11} - M_{13}) \hat{\omega}_2.
$$

We note that the structure equation (11) defines uniquely the pseudo-connection forms $\hat{\alpha}, \hat{\gamma}, \hat{\delta}, \hat{\psi}$. Hence, we can obtain the invariant 1-forms $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\omega}_1, \hat{\omega}_2, \hat{\alpha}, \hat{\gamma}, \hat{\psi})$ on $\mathcal{F}_{G_1}$. To consider the curvatures for the equivalence problem, we need to use the $\{e\}$-structure by choosing a prolongation of $\mathcal{F}_{G_1}$. Then, we obtain the following structure equation on the $\{e\}$-structure by taking the exterior derivation of tautological 1-forms $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\omega}_1, \hat{\omega}_2, \hat{\alpha}, \hat{\gamma}, \hat{\psi})$:

$$
\begin{bmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\hat{\omega}_1 \\
\hat{\omega}_2 \\
\hat{\alpha} \\
\hat{\gamma} \\
\hat{\psi}
\end{bmatrix}
= 
\begin{bmatrix}
(\hat{\alpha} + \hat{\gamma}) \land \hat{\theta}_0 + \hat{\omega}_1 \land \hat{\theta}_1 + \hat{\omega}_2 \land \hat{\theta}_2 \\
\hat{\alpha} \land \hat{\theta}_0 + M_1 \hat{\theta}_2 \land \hat{\omega}_1 + M_3 \hat{\omega}_1 \land \hat{\theta}_0 + M_2 \hat{\omega}_0 \land \hat{\theta}_0 + M_{10} \hat{\omega}_1 \land \hat{\theta}_0 + M_{11} \hat{\omega}_2 \land \hat{\theta}_0 \\
(\hat{\alpha} + \hat{\gamma} - \hat{\psi}) \land \hat{\omega}_2 + M_6 \hat{\theta}_1 \land \hat{\omega}_3 + M_2 \hat{\theta}_1 \land \hat{\theta}_0 + M_{13} \hat{\omega}_1 \land \hat{\theta}_0 + M_{17} \hat{\omega}_2 \land \hat{\theta}_0 \\
\hat{\gamma} \land \hat{\omega}_1 \\
\hat{\psi} \land \hat{\omega}_2 \\
S_1 \hat{\omega}_1 \land \hat{\theta}_0 + S_2 \hat{\omega}_2 \land \hat{\theta}_0 + S_3 \hat{\theta}_2 \land \hat{\theta}_0 + S_4 \hat{\theta}_2 \land \hat{\omega}_1 + S_5 \hat{\omega}_1 \land \hat{\omega}_2 + S_7 \hat{\theta}_2 \land \hat{\omega}_1 + M_7 \hat{\theta}_1 \land \hat{\omega}_2 \\
S_6 \hat{\omega}_1 \land \hat{\omega}_2 + S_8 \hat{\omega}_1 \land \hat{\theta}_0 + S_9 \hat{\theta}_1 \land \hat{\omega}_1 + S_{10} \hat{\theta}_2 \land \hat{\omega}_1 + S_{11} \hat{\omega}_1 \land \hat{\omega}_2 + S_{12} \hat{\omega}_2 \land \hat{\theta}_0 + S_{13} \hat{\theta}_1 \land \hat{\omega}_2 + S_{14} \hat{\theta}_2 \land \hat{\omega}_2
\end{bmatrix}
$$

Here, the torsions $M_i$ are given by previous page. To write down the torsions explicitly, we use the dual frame of the coframe $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\omega}_1, \hat{\omega}_2)$:

$$
\begin{align*}
\hat{\partial}_\hat{\omega}_1 &= \frac{\partial}{\partial \hat{y}}, \\
\hat{\partial}_\hat{\omega}_2 &= \frac{\partial}{\partial \hat{z}_1}, \\
\hat{\partial}_\hat{\omega}_3 &= \frac{\partial}{\partial \hat{z}_2}, \\
\hat{\partial}_\hat{\theta}_0 &= \frac{\partial}{\partial \hat{x}_1} + z_1 \frac{\partial}{\partial \hat{y}} + f_{11} \frac{\partial}{\partial \hat{z}_1} + f_{12} \frac{\partial}{\partial \hat{z}_2}, \\
\hat{\partial}_\hat{\theta}_1 &= \frac{\partial}{\partial \hat{x}_2} + z_2 \frac{\partial}{\partial \hat{y}} + f_{21} \frac{\partial}{\partial \hat{z}_1} + f_{22} \frac{\partial}{\partial \hat{z}_2}.
\end{align*}
$$
By using the Frobenius condition \(A = B = 0\) (i.e. \((f_{11})_\omega = (f_{12})_\omega, (f_{12})_\omega = (f_{22})_\omega\)), each torsions of the above structure equation can be written as follows.

\[
M_1 = -\frac{c}{gh}(f_{11})_g, \quad M_3 = -\frac{1}{gh}(f_{12})_g, \\
M_4 = -\frac{1}{h^2}\left\{ (f_{11})_\omega + 2(f_{11})_g(f_{12})_g - (f_{11})_\omega(f_{22})_g \right\}, \\
M_5 = \frac{1}{hk}\left\{ (f_{12})_g + (f_{12})_g(f_{12})_g - (f_{12})_\omega \right\}, \\
M_6 = -\frac{g}{ck}(f_{22})_g, \quad M_7 = -\frac{1}{ck}(f_{12})_g, \\
M_8 = \frac{1}{hk}\left\{ (f_{12})_g + (f_{12})_g(f_{12})_g - (f_{12})_\omega \right\}, \\
M_9 = -\frac{1}{k^2}\left\{ -2(f_{12})_g(f_{22})_g + (f_{22})_g(f_{22})_g + (f_{11})_g(f_{22})_g \right\}, \\
S_1 = \frac{1}{ch^2}\left\{ (f_{11})_g_\omega + (f_{11})_g(f_{22})_g + (f_{11})_g(f_{22})_g - (f_{12})_g_\omega - (f_{12})_g_\omega \right\}, \\
S_2 = \frac{1}{ck}\left\{ (f_{12})_g_\omega - (f_{12})_g_\omega - (f_{12})_g_\omega \right\}, \\
S_3 = \frac{(f_{12})_g_\omega_\omega + (f_{12})_g_\omega_\omega + (f_{12})_g_\omega_\omega}{cgh}, \quad S_4 = \frac{(f_{12})_g_\omega_\omega + (f_{12})_g_\omega_\omega}{cgh}, \quad S_5 = \frac{2(f_{12})_g_\omega_\omega - (f_{11})_g_\omega_\omega}{ch}, \\
S_6 = \frac{1}{hk}\left\{ -2(f_{12})_g_\omega - (f_{12})_g_\omega - (f_{12})_g_\omega - (f_{12})_g_\omega \right\}, \\
S_7 = \frac{(f_{11})_g_\omega_\omega + (f_{12})_g_\omega_\omega}{gh}, \quad S_8 = \frac{1}{hk}\left\{ (f_{11})_g_\omega_\omega - 2(f_{12})_g_\omega_\omega \right\}, \\
S_9 = \frac{1}{ck}\left\{ (f_{11})_g_\omega_\omega + (f_{12})_g_\omega_\omega + (f_{12})_g_\omega_\omega \right\}, \\
S_{10} = \frac{2(f_{12})_g_\omega_\omega - (f_{12})_g_\omega_\omega}{gh}, \quad S_{11} = \frac{1}{hk}\left\{ (f_{12})_g_\omega_\omega - (f_{22})_g_\omega_\omega \right\}, \\
S_{12} = \frac{1}{ck}\left\{ -2(f_{12})_g_\omega_\omega - (f_{12})_g_\omega_\omega - (f_{12})_g_\omega_\omega - (f_{12})_g_\omega_\omega \right\}, \\
S_{13} = \frac{2(f_{12})_g_\omega_\omega - (f_{22})_g_\omega_\omega}{ck}, \quad S_{14} = \frac{2(f_{12})_g_\omega_\omega - (f_{22})_g_\omega_\omega}{gh}.
\]

In the above torsions, there are the following relations.
Proposition 3.5. Torsions $M_4$, $M_9$, $S_3$, $S_4$, $S_7$, $S_{10}$, $S_{13}$ are given by:

\[
M_4 = -\frac{1}{h^2}\left\{ -\frac{gh}{c} (M_1)_{\omega_2} + \frac{2gh}{c} M_1 (f_{12})_{\omega_1} - \frac{gh}{c} M_1 (f_{22})_{\omega_2} \right\},
\]
\[
M_9 = -\frac{1}{k^2}\left\{ -\frac{ck}{g} (M_6)_{\omega_1} - \frac{ck}{g} M_6 (f_{11})_{\omega_1} + \frac{2ck}{g} M_6 (f_{12})_{\omega_2} \right\},
\]
\[
S_3 = -\frac{k}{c\hbar} (M_7)_{\omega_1}, \quad S_4 = -\frac{1}{c} (M_3)_{\omega_2}, \quad S_7 = -\frac{1}{c} (M_1)_{\omega_1} + M_3,
\]
\[
S_{10} = -\frac{1}{c} (M_1)_{\omega_1} + 2M_3, \quad S_{13} = -2M_7 + \frac{1}{g} (M_6)_{\omega_2}.
\]

Hence, the vanishing of $M_4$, $M_9$, $S_3$, $S_4$, $S_7$, $S_{10}$, $S_{13}$ is given by vanishing of other curvatures. By the theory of $G$-structure ([12], [18]), a vanishing condition of curvatures $M_i$, $S_j$ ($i = 1, 3, 5, 6, 7, 8$, $j = 1, 2, 5, 6, 8, 9, 11, 12, 14$) gives the following theorem.

Theorem 3.6. Suppose that the second order PDE (1) satisfies the integrability condition $A = B = 0$. Then, the equation (1) is (locally) equivalent to the flat equation under lifts of scale transformations if and only if curvatures $M_i$, $S_j$ vanish.

First, it is easy to check that the functions $f_{ij}$ satisfying $A = B = M_i = S_j = 0$ are written as quadratic polynomials in $z_1$, $z_2$. Hence, if there is a polynomial $z_1$, $z_2$ of degree three among $f_{ij}$, then corresponding equation (1) is not equivalent to the flat equation under lifts of scale transformations.

Next, we give some examples of equation which is equivalent to the flat equation. To show the vanishing condition of the curvatures more explicitly, we consider the functions $f_{ij}$ given by:

\[
f_{11} = P(x_1, x_2, y), \quad f_{12} = Q(x_1, x_2, y), \quad f_{22} = R(x_1, x_2, y).
\]

Then, Theorem 3.6 gives the following Corollary.

Corollary 3.7. Suppose that the functions $f_{ij}$ in (1) are given in the above form. Then the equation (1) is (locally) equivalent to the flat equation under the lifts of scale transformations if and only if $P_y = Q_y = R_y = 0$, $P_{x_2} = Q_{x_1}$, $Q_{x_2} = R_{x_1}$.

Remark 3.8. The conditions $P_y = Q_y = R_y = 0$, $P_{x_2} = Q_{x_1}$, $Q_{x_2} = R_{x_1}$ in Corollary 3.7 are obtained by the integrability condition $A = B = 0$. Namely, a vanishing condition of curvatures (i.e. $M_i = S_j = 0$) is absorbed into the integrability condition. Therefore, it is shown that the second order PDE (1) for the functions $f_{ij}$ given by the above form are equivalent to the flat equations if and only if it is integrable.

4. Duality associated with differential equations

In this section, we discuss a duality between the coordinate space and the solution space associated with the following flat equation:

\[
\frac{\partial^2 y}{\partial x_i \partial x_j} = 0 \quad (1 \leq i, j \leq 2).
\]
For the purpose, we consider the following double fibration.

\[ J^1(\mathbb{R}^2, \mathbb{R}) \]
\[ \pi_1 \]
\[ \mathbb{R}^3 := \{(x_1, x_2, y)\} \]
\[ \pi_2 \]
\[ \mathbb{R}^3 := \{(a, b, c)\} \]

where, projections \( \pi_1, \pi_2 \) are defined by

\[ \pi_1(x_1, x_2, y, z_1, z_2) = (x_1, x_2, y), \]
\[ \pi_2(x_1, x_2, y, z_1, z_2) = (z_1, z_2, y - z_1 x_1 - z_2 x_2). \]

We call the double fibration (13) the model space of the flat equation or flat model space. In this fibration, we regard the left base space as a coordinate space \( \mathbb{R}^3 := \{(x_1, x_2, y)\} \), and a right base space as a solution space \( \mathbb{R}^3 := \{(a, b, c)\} \). Solutions of (12) are written as \( y = ax_1 + bx_2 + c \) for real parameters \( a, b, c \). Graphs of solutions are planes on \( \mathbb{R}^3 \) or \( J^1(\mathbb{R}^2, \mathbb{R}) \), and the 3-parameter family of solutions yields a 2-dimensional foliation on \( J^1(\mathbb{R}^2, \mathbb{R}) \). Then the leaf space of this foliation is interpreted as a solution space of (12). We discuss the compactification of the flat model space. The fibration (13) can be embedded naturally into the following (global) double fibration:

\[ \mathbb{P}(V) \]
\[ \pi_1 \]
\[ \mathbb{F}_V(1, 3) \]
\[ \pi_2 \]
\[ \mathbb{P}(V^*) \cong Gr(3, 4) \]

where, \( V = \mathbb{R}^4 \), and \( Gr(3, 4) \) is a Grassmannian manifold and \( \mathbb{F}_V(1, 3) \) is a flag variety:

\[ Gr(3, 4) = \{ E \mid E \text{ is a hyperplane of } V := \mathbb{R}^4 \} , \]
\[ \mathbb{F}_V(1, 3) = \{(l, E) \mid l \in \mathbb{R}P^3, E \in Gr(3, 4) \cong \mathbb{R}P^3, l \subset E \} . \]

The, projections \( \pi_1, \pi_2 \) are defined by

\[ \pi_1([u], H) := [u], \quad \pi_2([u], H) := [f_H], \]

where, \( f_H \) is a linear functional satisfying \( ker(f_H) = H \) of \( V^* \setminus \{0\} \). (Since \( f_H \) is uniquely defined up to scalar multiplication, \( \pi_2 \) is well-defined.) The double fibration (14) do not depend on coordinate transformation group \( \mathcal{G} \). So, we introduce the flat model space depending on \( \mathcal{G} \).

We fix a coordinate transformation group \( \mathcal{G} \subset Diff(\mathbb{R}^3) \). First, we define the following symmetry group. ([11], [12])
Definition 4.1. Let $G$ be an isotropy subgroup of the flat equation (12) in $\mathcal{G}$. This group $G$ is called symmetry group of the flat equation for $\mathcal{G}$.

In the case of $\mathcal{G} = \text{Diff}(\mathbb{R}^3)$, the symmetry group is $SL(4, \mathbb{R})$ and the action on the coordinate space $\mathbb{R}^3$ is given by:

$$
\begin{pmatrix}
    a_1 & a_2 & a_3 & a_4 \\
    b_1 & b_2 & b_3 & b_4 \\
    c_1 & c_2 & c_3 & c_4 \\
    d_1 & d_2 & d_3 & d_4
\end{pmatrix} \in SL(4, \mathbb{R}),
$$

$$(x_1, x_2, y) \mapsto \left(\frac{a_1 x_1 + a_2 x_2 + a_3 y + a_4}{d_1 x_1 + d_2 x_2 + d_3 y + d_4}, \frac{b_1 x_1 + b_2 x_2 + b_3 y + b_4}{d_1 x_1 + d_2 x_2 + d_3 y + d_4}, \frac{c_1 x_1 + c_2 x_2 + c_3 y + c_4}{d_1 x_1 + d_2 x_2 + d_3 y + d_4}\right).$$

(15)

Next, we introduce subgroups of $SL(4, \mathbb{R})$ as follows:

$$H_i := \{ g \in SL(4, \mathbb{R}) \mid g[e_i] = [e_i] \},$$

$$H_i^\perp := \{ g \in SL(4, \mathbb{R}) \mid t^{-1} g^{-1} [e_i] = [e_i] \},$$

where, $e_i$ ($i=1, \cdots, 4$) are standard basis of $\mathbb{R}^4$, and $[e_i]$ are corresponding elements in $RP^3$.

The subgroups $H_i$ are isotropy subgroups which preserve lines $[e_i]$, and the subgroups $H_i^\perp$ are isotropy subgroups which preserve hyperplanes spanned by $e_j$ ($j \neq i$) respectively.

We used Cartan involution $\tilde{\theta}(g) = t^{-1} g^{-1}$ in the definition of $H_i^\perp$. We consider the following double fibration.

$$
\begin{array}{c}
G/(G \cap H) \\
(G \cap H_4)/(G \cap H) \quad (G \cap H_1)/(G \cap H) \\
G/(G \cap H_4) \quad G/(G \cap H_1)
\end{array}
$$

(16)

where, $H = H_4 \cap H_1$. We call this fibration as a model space of the flat equation (12) with respect to $\mathcal{G}$.

In the case of $\mathcal{G} = \text{Diff}(\mathbb{R}^3)$, we obtain the following well-known fibration using corresponding symmetry group $G = SL(4, \mathbb{R})$.

$$
\begin{array}{c}
SL(4, \mathbb{R})/H \cong FV(1, 3) \\
RP^2 \cong H_4/H \quad \overline{H_1}/H \cong RP^2 \\
RP^3 \cong SL(4, \mathbb{R})/H_4 \quad SL(4, \mathbb{R})/\overline{H_1} \cong RP^3
\end{array}
$$

(17)

This fibration equals the fibration (14).
In the case of $\mathcal{G} = \text{ScaleDiff}(\mathbb{R}^3)$, we calculate the corresponding flat model space. From the action (15) of $SL(4, \mathbb{R})$ on $\mathbb{R}^3$, we have the following symmetry group from restriction of variables associated with the scale transformation.

$$G = \left\{ \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in SL(4, \mathbb{R}) \right\}.$$  

Then, we have the following.

**Proposition 4.2.** We obtain the following double fibration as the flat model space associated with $\mathcal{G} = \text{ScaleDiff}(\mathbb{R}^3)$:

$$\begin{align*}
(G \cap H_4) / (G \cap H) &\cong \mathbb{R}^3 \\
(G \cap \overline{H_1}) / (G \cap H) &\cong \mathbb{R}^2 \\
G / (G \cap H_4) &\cong \mathbb{R}^3 \\
G / (G \cap \overline{H_1}) &\cong \mathbb{R} \\
\end{align*}$$  

(18)

**Proof.** We prove the correspondence $G / (G \cap H_4) \cong \mathbb{R}^3$. We consider the following injective group homomorphism $\Phi : \mathbb{R}^3 \to G / (G \cap H_4)$ defined by:

$$\Phi(a) := \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where, $a = (a_1, a_2, a_3) \in \mathbb{R}^3$. It is clear that $\Phi$ is bijective. Similarly, we have the other correspondences. Thus, we complete the proof. \qed

This fibration is degenerate. Since $\mathcal{G} = \text{ScaleDiff}(\mathbb{R}^3)$ is very strongly restricted from Diff($\mathbb{R}^3$), this degeneration arises. Hence, we consider the following problem.

**Problem 4.3.** Find a symmetry group for proper subgroup $\mathcal{G}$ of Diff($\mathbb{R}^3$), from which has double fibration of compact-type as a flat model space.

To consider this problem, we characterize groups $H_4, \overline{H_1}, H$. To this purpose, we prepare some terminology and notation. We use the Iwasawa decomposition:

$$SL(4, \mathbb{R}) = KAN,$$

where, $K = SO(4)$ and $A = \{ \text{diag}(a_1, a_2, a_3, a_4) \in SL(4, \mathbb{R}) \}$ and $N$ is a group of upper triangle matrices whose diagonal components are all 1. By using the Iwasawa decomposition, we have decompositions of the subgroups $H_4, \overline{H_1}, H$: 

\[ H_4 = K_4 M_4 A \overline{N}, \quad \overline{H}_1 = K_1 M_1 A \overline{N}, \quad H = (K_1 \cap K_4) M_{1,4} A \overline{N}, \]

where, \( K_i \cong SO(3) \) are isotropy subgroups of \( e_i \in S^3 \) and \( \overline{N} = \tilde{\theta}(N) \) and \( M_1, M_4 \) are the following subgroups of \( M \):

\[
M_1 := (1, m_1) \cong \mathbb{Z}_2, \quad M_4 := (1, m_4) \cong \mathbb{Z}_2, \quad M_{1,4} := M_1 \cap M_4,
\]

where, \( m_1 = \text{diag}(-1, -1, 1, 1) \) and \( m_4 = \text{diag}(1, 1, -1, -1) \).

By using these facts, we find the fibration of compact-type. We consider the following subgroups of \( M \):

\[
K_i = 16 T. NODA
\]

\[
\alpha, \gamma \in \mathbb{R}^* \quad \text{and} \quad 1 \leq i \leq 4.
\]

\[
\text{Note that } G \text{ is a subgroup invariant under Cartan involution } \tilde{\theta}. \text{ We show that a double fibration defined by the group } G \text{ is a fibration of compact-type. To prove this assertion, we consider the characterization of groups } G, \ G \cap H_4, \ G \cap \overline{H}_1, \ G \cap H. \text{ In fact, these groups are decomposed as follows.}
\]

\[
\text{LEMMA 4.4. The above Lie groups } G, \ G \cap H_4, \ G \cap \overline{H}_1, \ G \cap H \text{ have the following decompositions:}
\]

\[
G = (K \cap G) A (\overline{N} \cap G) \quad \text{(21)}
\]

\[
G \cap H_4 = (K_4 \cap G) M_4 A (\overline{N} \cap G) \quad \text{(22)}
\]

\[
G \cap \overline{H}_1 = (K_1 \cap G) M_1 A (\overline{N} \cap G) \quad \text{(23)}
\]

\[
G \cap H = (K_1 \cap K_4 \cap G) M_{1,4} A (\overline{N} \cap G) \quad \text{(24)}
\]

\[
\text{PROOF. We prove (21). Since } K \cap G, \ A, \ \overline{N} \cap G \text{ are subgroups of } G, \text{ we have } G \supset (K \cap G) A (\overline{N} \cap G). \text{ Hence we prove } G \subset (K \cap G) A (\overline{N} \cap G). \text{ For } g \in G, \text{ we write } g = \text{kan} \quad (k \in SO(4), \ a \in A, \ n \in \overline{N}). \text{ By definition (19) of } G, \text{ we assume as follows:}
\]

\[
g e_3 = \alpha e_3, \quad ^t g^{-1} e_3 = \gamma e_3,
\]

where, \( \alpha, \gamma \in \mathbb{R}^* \). We write \( n \in \overline{N} \) and \( a \in A \) explicitly as follows:

\[
n = \begin{pmatrix}
1 & 0 & 0 & 0 \\
n_1 & 1 & 0 & 0 \\
n_2 & n_3 & 1 & 0 \\
n_4 & n_5 & n_6 & 1
\end{pmatrix}, \quad a = \begin{pmatrix}
a_1 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & a_3 & 0 \\
0 & 0 & 0 & a_4
\end{pmatrix}.
\]

\[
(25)
\]
Then,
\[ \alpha e_3 = k a e_3 = a_3 k e_3 + a_4 n_6 k e_4, \]
\[ t k e_3 = \alpha^{-1} a_3 e_3 + \alpha^{-1} a_4 n_6 e_4. \] (26)

On the other hand, if we write
\[ t n^{-1} e_3 = l_2 e_1 + l_3 e_2 + e_3, \]
then,
\[ e_3 = l_2 e_1 + l_3 (n_1 e_1 + e_2) + n_2 e_1 + n_3 e_2 + e_3. \]
Hence, we have the equalities \( l_3 = -n_3, \) \( l_2 = n_1 n_3 - n_2 \) by using a linear independence of \( e_1, e_2, e_3, e_4. \) By using these relations,
\[ t^g e_3 = k a^{-1} (n_1 n_3 - n_2) e_1 - k a_2^{-1} n_3 e_2 + k a_3^{-1} e_3, \]
\[ t k e_3 = \gamma^{-1} a_1^{-1} (n_1 n_3 - n_2) e_1 - \gamma^{-1} a_2^{-1} n_3 e_2 + \gamma^{-1} a_3^{-1} e_3. \] (27)

In the equality between right sides of (26) and (27), we have \( n_2 = n_3 = n_6 = 0. \) Hence, we have \( n \in G. \) Since \( g, a, n \in G, \) we have \( k \in G. \) Thus we complete the proof of (21).

By similar method, we can prove the (22), (23), (24).

By using these decomposition formula, we have the following fibration of compact-type:

**Theorem 4.5.** A double fibration constructed by the group (19) is the following fibration of compact-type.

\[
\begin{align*}
G/(G \cap H) &\cong \mathbb{F}_V(1,2) \\
(G \cap H_4)/(G \cap H) &\cong S^1 \\
G/(G \cap H_4) &\cong RP^2 \\
(G \cap \overline{H}_1)/(G \cap H) &\cong S^1 \\
G/(G \cap \overline{H}_1) &\cong RP^2 \\
\end{align*}
\] (28)

**Proof.** By a direct computation, we have
\[ G \cap K = SO(3) \times \mathbb{Z}_2, \quad G \cap K_1 = SO(2) \times \mathbb{Z}_2, \]
\[ G \cap K_4 = SO(2) \times \mathbb{Z}_2, \quad G \cap K_1 \cap K_4 = \mathbb{Z}_2. \]
Thus, we obtain the statement by the following correspondence:
\[ G/(G \cap H_4) \cong RP^2, \quad G/(G \cap \overline{H}_1) \cong RP^2, \quad G/(G \cap H) \cong \mathbb{F}_V(1,2), \]
\[ (G \cap H_4)/(G \cap H) \cong S^1, \quad (G \cap \overline{H}_1)/(G \cap H) \cong S^1. \] □
We note that the coordinate transformation group $G$ corresponding to this symmetry group (19) is constructed by the transformations of the form:

$$X_1 = X_1(x_1, x_2), \quad X_2 = X_2(x_1, x_2), \quad Y = \frac{y}{A(x_1, x_2)}.$$

5. The dual equations

In this section, we compute explicitly the dual equations of the second order PDE (1) ([17]).

First, we assume that solutions of (1) are written by three parameters $X_1, X_2, Y$ as follows:

$$y = h(x_1, x_2, X_1, X_2, Y)$$

(29)

Then, we can choose a local coordinate $(X_1, X_2, Y)$ on a solution space of (1). A family of solutions corresponding to the above solutions is given by moving parameters $X_1, X_2, Y$ in the solution space. Let $Y = Y(X_1, X_2)$ be the surface on the solution space. Then, $Y$ can be written as $Y(X_1, X_2) = g(X_1, X_2, x_1, x_2, y)$. We calculate $Y_X_1, Y_{X_2}$ by using this representation:

By taking a derivation of $y = h(x_1, x_2, X_1, X_2, Y(X_1, X_2))$, we have

$$Y_{X_i} = -\frac{h_{X_i}}{h_Y}.$$  

From this fact, the dual equation of (1):

$$\frac{\partial^2 Y}{\partial X_i \partial X_j} = F_{ij}(X_1, X_2, Y, Z_1, Z_2)$$

(30)

are written:

$$\frac{\partial^2 Y}{\partial X_i \partial X_j} = -\frac{1}{h_Y^2} \left\{ (h_{X_i X_j} + h_{X_i Y} Y_{X_j}) h_Y - h_{X_i} (h_{Y X_j} + h_{Y Y} Y_{X_j}) \right\}$$

$$= \frac{h_{X_i} h_{Y X_j} - h_Y h_{X_i X_j} + Z_j (h_{X_i} h_{Y Y} - h_Y h_{X_i Y})}{h_Y^2}.$$  

In particular, we calculate the dual equation of (12). Solutions of (12) are written as $y = X_1 x_1 + X_2 x_2 + Y$, $z_i \equiv y_{x_i} = X_i$. Hence, $Y_{X_i} = -x_i$ and we obtain the following equations by substituting this term into above dual equations:

$$\frac{\partial^2 Y}{\partial X_i \partial X_j} = 0$$

(31)

This equation is the dual equation of the flat equations (12). Namely, the dual equation of the flat equation is also the flat equation. This fact is supported by the double fibration (13). In the double fibration (13), the solution space of the original flat equation (12) is a right base space $\mathbb{R}^3$, and the solution space of the dual equation (31) is a left base space $\mathbb{R}^3$. 
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