Quantum stochastic equation for a test particle interacting with a dilute Bose gas

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Abstract. We use the stochastic limit method to study long time quantum dynamics of a test particle interacting with a dilute Bose gas. The case of arbitrary form-factors and an arbitrary, not necessarily equilibrium, quasifree low density state of the Bose gas is considered. Starting from microscopic dynamics we derive in the low density limit a quantum white noise equation for the evolution operator. This equation is equivalent to a quantum stochastic equation driven by a quantum Poisson process with intensity $S^{-1}$, where $S$ is the one-particle $S$ matrix. The novelty of our approach is that the equations are derived directly in terms of correlators, without use of a Fock-antiFock (or Gel’fand-Naimark-Segal) representation. Advantages of our approach are the simplicity of derivation of the limiting equation and that the algebra of the master fields and the Ito table do not depend on the initial state of the Bose gas. The notion of a causal state is introduced. We construct master fields (white noise and number operators) describing the dynamics in the low density limit and prove the convergence of chronological (causal) correlators of the field operators to correlators of the master fields in the causal state.

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1. INTRODUCTION

The fundamental equations in quantum theory are the Heisenberg and Schrödinger equations. However, it is a very difficult problem to solve explicitly these equations for realistic physical models and one uses various approximations or limiting procedures such as weak coupling, low density, and hydrodynamical limits. These scaling limits describe the long time behavior of physical systems in different physical regimes.

One of the powerful methods to study the long time behavior in quantum theory is the stochastic limit method developed by Accardi, Lu and Volovich [1]. Many interesting physical models have been investigated by using this method. In particular, it has been applied to study the long time quantum dynamics of a system interacting with a reservoir in the case of a weak interaction between the system and reservoir, i.e. in the weak coupling limit. It was applied to study the spin-boson model [2], polaron model and nonrelativistic quantum electrodynamics [3, 4], quantum Hall effect [5], relations between Hepp-Lieb and Alli-Sewell laser models [6], bifurcation phenomenon in a spin relaxation [7], etc.

An important problem is to study the long time dynamics of a quantum system interacting with a reservoir in the case the interaction is not weak but the density of particles of the reservoir is small, i.e. in the low density limit. To describe a quantum physical model to which the low density limit can be applied let us consider an $N$-level atom (test particle) immersed in a free gas whose molecules can collide with the atom; the gas is supposed to be very dilute. Then the reduced time evolution for the atom will be Markovian, since the characteristic time $t_S$ for appreciable action of the surroundings on the atom (time between collisions) is much larger than the characteristic time $t_R$ for relaxation of correlations in the surroundings. The dynamics of the $N$-level atom interacting with the free gas should converge, in the low density limit, to the solution of a quantum stochastic differential equation driven by quantum Poisson noise. The quantum Poisson process, introduced by Hudson and Parthasarathy [8] (for a description of the quantum Poisson process see also Kumerrer [9]), should arise naturally in the low density limit, as conjectured by Frigerio and Maassen [10] and later by Frigerio and Alicki [11]. For a general survey of quantum stochastic calculus we refer to the review by Attal [12].

The quantum stochastic equation for the low density limit was derived by Accardi and Lu [13–15] using perturbation series for the evolution operator. A nonperturbative white noise approach for the investigation of dynamics in the low density limit is developed in [16, 17], where the mathematical procedure, the so called stochastic golden rule for the low density limit, was formulated. This derivation uses the white noise technique developed for the case of weak coupling limit by Accardi, Lu and Volovich [11]. The approach to derivation of the stochastic equations in [13–17] is based on use of the Fock-antiFock (Gel’fand-Naimark-Segal, or GNS) representation for the canonical commutation relations (CCR) algebra of the Bose gas. The approach of the present paper does not use the Fock-antiFock representation.
We study the low density limit for an $N$-level atom (test particle) interacting with a Bose gas. Starting from microscopic quantum dynamics we derive quantum white noise and quantum stochastic differential equations for the limiting evolution operator. A useful tool is the energy representation introduced in [16, 17] where the case of orthogonal formfactors was considered. In the present paper we consider the case of arbitrary formfactors and an arbitrary, not necessarily equilibrium, quasifree low density state of the reservoir. To each initial low density state of the Bose gas we associate in the low density limit a special ”state” (which is called a causal state) on the limiting master field algebra. We prove the convergence of time-ordered (or causal) correlators of the initial Bose field to the correlators of master fields (which are number operators constructed from some white noise operators) in these causal states. These states are determined by the diagrams which give nontrivial contribution to the limit. The leading diagrams can be interpreted as a new statistics arising in the low density limit (new statistics arising in the weak coupling limit is discussed in [1, 18]).

One of the main results of the paper is that the dynamics in the low density limit is given by the solution of a quantum white noise equation, which is equivalent to the quantum stochastic equation

$$dU_t = dN_t(S - 1)U_t$$

where $U_t$ is the evolution operator at time $t$ describing the limiting dynamics, $S$ is the one-particle $S$ matrix describing scattering of the test particle on one particle of the reservoir, and $N_t(S - 1)$ is the quantum Poisson (number, gauge) process with intensity $S - 1$. The equation describes the evolution of the total system+reservoir and can be applied, in particular, to the important problem of derivation of the linear quantum Boltzmann equation describing the irreversible reduced dynamics of the test particle in the low density limit. Such an equation for the reduced density matrix can be easily obtained from the quantum Langevin equation, which can be derived by using the quantum stochastic differential equation and quantum Itô table (see sect. 7) for stochastic differential $dN_t$ (for a derivation of the quantum Langevin equation see [17]). However, in the present paper we are mainly concentrated on further understanding in what sense the Poisson process is an approximation of the usual quantum field (Theorem 11) and in mechanism through which the quantum stochastic equation arises as a limit of the usual Hamiltonian equation.

In order to describe the objects appearing in (1) let us introduce two Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}$, which are called in this context the system and one-particle reservoir Hilbert spaces, and the Fock space $\Gamma(L^2(\mathbb{R}^+_+; \mathcal{H}))$ over the Hilbert space of square-integrable measurable vector-valued functions from $\mathbb{R}^+_+ = [0, \infty)$ to $\mathcal{H}$. With these notations the solution of the equation is a family of operators $U_t; t \geq 0$ in $\mathcal{H}_S \otimes \Gamma(L^2(\mathbb{R}^+_+; \mathcal{H}))$ (adapted process); $S$ is a unitary operator in $\mathcal{H}_S \otimes \mathcal{H}$, which is explicitly defined in section 6.

Let us introduce the notion of a Poisson process. Let $X$ be a self-adjoint operator in a Hilbert space $\mathcal{K}$ and $\Psi(f)$ the normalized coherent vector in the Fock space $\Gamma(\mathcal{K})$ with test function $f \in \mathcal{K}$. The number operator is the generator of one-parameter unitary
group $\Gamma(e^{i\lambda X})$ characterized by

$$\Gamma(e^{i\lambda X})\Psi(f) = \Psi(e^{i\lambda X} f); \quad \lambda \in \mathbb{R}$$

The number operator is characterized by the property

$$\langle \Psi(f), N(X)\Psi(g) \rangle = \langle f, Xg \rangle \langle \Psi(f), \Psi(g) \rangle$$

The definition of $N(X)$ is extended by complex linearity to any bounded operator $X$ on $\mathcal{K}$. Let us consider $\mathcal{K}$ of the form $L^2(\mathbb{R}^+; H) \cong L^2(\mathbb{R}^+) \otimes H$. For any bounded operators $X_0 \in B(\mathcal{H}_S), X_1 \in B(\mathcal{H})$, and for any $t \geq 0$ we define $N_t(X_0 \otimes X_1) := X_0 \otimes N_{\chi_{[0,t]} \otimes X_1}$, extend this definition by linearity to any bounded operator $K$ in $\mathcal{H}_S \otimes \mathcal{H}$, and call the family $N_t(K); t \geq 0$ of operators in $\mathcal{H}_S \otimes \Gamma(L^2(\mathbb{R}^+; H))$ as quantum Poisson process with intensity $K$. The existence and uniqueness of the solution of the equation in this case follows from the general theory of quantum stochastic differential equations. Moreover, unitarity of $S$ leads to the conclusion that for each $t \geq 0$ $U_t$ is a unitary operator (see Lemma 2).

For the vacuum state of the reservoir (zero density) such an equation was derived by Accardi and Lu [14]. In the present paper we derive this equation for an arbitrary quasifree initial state of the Bose gas. The main feature of the present paper is that the stochastic equations are derived directly in terms of correlators, without use of a Fock-antiFock (or GNS) representation. This simplifies the derivation of the limiting quantum white noise equation and allows us to express the intensity of the quantum Poisson process directly in terms of one-particle $S$-matrix. In our approach the limiting equation, the algebra of the master fields, and the Itô table do not depend on the initial state of the Bose gas.

We obtain that the dynamics of the compound system in the low density limit is described by:

1) the solution of quantum white noise equation [36] or equivalently, quantum stochastic differential equation in forms [11], [44] and

2) the family of causal states $\varphi_L$ on the algebra of master fields.

The reduced dynamics of the system (test particle) in the low density limit for the model under consideration, with completely different methods, based on a quantum Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, has been investigated by Dümcke [19], where it is proved that, under some conditions, the reduced dynamics is given by a quantum Markovian semigroup.

In the approach of the present paper the reduced dynamics can be easily derived from the solution of the limiting quantum stochastic differential equation. Namely, the limiting evolution operator $U_t$ and the limit state $\varphi_L$ determine the reduced dynamics by

$$T_t(X) = \varphi_L(U_t^+(X \otimes 1)U_t),$$

where $X$ is any system observable (bounded operator in $\mathcal{H}_S$), $\varphi_L(\cdot)$ denotes partial expectation, and $T_t$ is the limiting semigroup. This equality shows that $U_t$ is a stochastic dilation of the limiting Markovian semigroup. Using the quantum Itô table
for stochastic differential \( dN_t \) one can derive a quantum Langevin equation for the quantity \( \mathcal{U}_t(X \otimes 1) \mathcal{U}_t \). Then taking partial expectation one gets an equation for \( T_t(X) \); in particular, one can get the generator of the semigroup. This is a general feature of the white noise approach: one at first obtains the equation for the evolution operator of the total system and then gets the reduced dynamics of the test particle. Let us note that although quantum stochastic equations, which are derived in [13, 17], are different from (1) they give the same reduced dynamics.

The low density limit can be applied to the model of a test particle moving through an environment of randomly placed, infinitely heavy scatterers (Lorentz gas) (see the review of Spohn [20]). In the Boltzmann–Grad limit successive collisions become independent and the averaged over the positions of the scatterers the position and velocity distribution of the particle converges to the solution of the linear Boltzmann equation. An advantage of the stochastic limit method is that it allows us to derive equations not only for averaged over reservoir degrees of freedom dynamics of the test particle but for the total system+reservoir. For a rigorous treatment of a classical Lorentz gas we refer to [21–26]. The convergence results and derivation of the linear Boltzmann equation for a quantum Lorentz gas in the low density and weak coupling limits are presented in [27, 28]. The Coulomb gas at low density is considered in [29].

The hydrodynamical limit is described by the Euler equation. In [30] the Euler equation for fermions in the hydrodynamical limit is derived under some assumptions.

Let us describe the plan of the paper. In section 3 we construct the master fields, which are number operators acting in some Hilbert space, and the limit causal states on the master field’s algebra. We prove that the time-ordered (or causal) correlators of the free evolution of the initial field converge in the low density limit to the correlators of the master field in these causal states. In section 4 the stochastic Schrödinger equation which describes the dynamics in the low density limit is derived. In section 5 we bring this equation to the causally normally ordered form. This form is convenient for study the reduced dynamics of the system. In section 6 the expressions for the one-particle S-matrix and T-operator are given. In section 7 quantum stochastic differential equation (1) for the limiting evolution operator is derived.

2. AN ATOM INTERACTING WITH A DILUTE BOSE GAS

Let us explain our notations. We consider a quantum system (test particle) interacting with a boson reservoir (heat bath). Let \( \mathcal{H}_S \) be the Hilbert space of the system. For example, for an \( N \)-level atom \( \mathcal{H}_S = \mathbb{C}^N \). The system Hamiltonian \( H_S \) is a self-adjoint operator in \( \mathcal{H}_S \). The reservoir is described by the boson Fock space \( \Gamma(\mathcal{H}) \) over the one particle Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^d) \) (with scalar product \( \langle \cdot, \cdot \rangle \)), where \( d = 3 \) in the physical case. Moreover, the Hamiltonian of the reservoir is given by \( H_R := d\Gamma(H_1) \) (the second quantization of the one-particle Hamiltonian \( H_1 \)) and the total Hamiltonian \( H_{tot} \) of the compound system is given by a self-adjoint operator on the total Hilbert
space $\mathcal{H}_S \otimes \Gamma(\mathcal{H})$:

$$H_{\text{tot}} := H_{\text{free}} + H_{\text{int}} = H_S \otimes 1 + 1 \otimes H_R + H_{\text{int}}.$$  

Here $H_{\text{int}}$ is the interaction Hamiltonian between the system and reservoir. The one-particle Hamiltonian $H_1$ is the operator of multiplication by some real-valued function $\omega(k)$. The interaction Hamiltonian will be assumed to have the following form:

$$H_{\text{int}} := i(D \otimes A^+(g_0)A(g_1) - D^+ \otimes A^+(g_1)A(g_0))$$

where $D$ is a bounded operator in $\mathcal{H}_S$, $D \in \mathcal{B}(\mathcal{H}_S)$; $A(g_n)$ and $A^+(g_n)$, $n = 0, 1$, are annihilation and creation operators, and $g_0, g_1 \in \mathcal{H}$ are formfactors describing the interaction of the system with the reservoir. This Hamiltonian describes scattering of particles of the Bose gas on the test particle and can be obtained by quantization of the classical interaction potential between particles of two different types with an infinite number of particles of one type (particles of the gas) and finite number of particles of the second type (test particles). This Hamiltonian preserves the particle number of the reservoir, and therefore the particles of the reservoir are only scattered on the test particle and not created or destroyed. Such a Hamiltonian was considered by Davies [31] in the analysis of the weak coupling limit.

The initial state of the compound system is supposed to be factorized:

$$\rho = \rho_S \otimes \varphi_{L,\xi}.$$  

Here $\rho_S$ is an arbitrary density matrix of the system and the initial state of the reservoir $\varphi_{L,\xi}$ is the gauge invariant mean zero Gaussian state, characterized by

$$\varphi_{L,\xi}(A^+(f)A(g)) = \xi \left\langle g, \frac{L}{1 - \xi L} f \right\rangle$$

for each $f, g \in \mathcal{H}$. Here $\xi > 0$ is a small positive number and $L$ is a bounded positive operator in $\mathcal{H}$ commuting with $S_t$ (an operator of multiplication by some function $L(k)$). In the case $L = e^{-\beta H_1}$, so that $L(k) = e^{-\beta \omega(k)}$, where $\beta > 0$ is a positive number, the state $\varphi_{L,\xi}$ is just the Gibbs state, at inverse temperature $\beta$ and fugacity $\xi$, of the free evolution. The fugacity $\xi = e^{\beta \mu}$; $\mu$ is the chemical potential.

The dynamics of the total system is determined by the evolution operator which in interaction representation has the form:

$$U(t) := e^{iH_{\text{free}}t}e^{-itH_{\text{tot}}}.$$  

It satisfies the differential equation

$$\frac{dU(t)}{dt} = -iH_{\text{int}}(t)U(t),$$

where the quantity $H_{\text{int}}(t)$ will be called the evolved interaction and defined as

$$H_{\text{int}}(t) = e^{itH_{\text{free}}}H_{\text{int}}e^{-itH_{\text{free}}}.$$  

The iterated series for the evolution operator is

$$U(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n H_{\text{int}}(t_1) \ldots H_{\text{int}}(t_n)$$  

(3)
With the notations
\[ S_t := e^{iH_t}, \quad D(t) := e^{iH_S}D e^{-iH_S} \]
the evolved interaction can be written in the form
\[ H_{\text{int}}(t) := i(D(t) \otimes A^+(S_t g_0)A(S_t g_1) - D^+(t) \otimes A^+(S_t g_1)A(S_t g_0)). \] (4)
We assume the rotating wave approximation
\[ e^{iH_S}D e^{-iH_S} = D, \]
although generalization to the case of arbitrary \( D \) is not difficult.

We study the dynamics generated by the Hamiltonian (4) in the low density limit: \( n \to 0, \ t \sim 1/n \) (\( n \) is the density of particles of the reservoir). The density of particles with momentum \( k \) in the state \( \varphi_{L,\xi}(\cdot) \) is equal to
\[ \frac{\xi L(k)}{1 - \xi L(k)} \]
and goes to zero as \( \xi \to 0 \). Therefore the limit \( n \to 0, \ t \sim 1/n \) is equivalent to the limit \( \xi \to 0, \ t \sim 1/\xi \).

Let us consider the time rescaling \( t \to t/\xi \) so that \( U(t) \to U(t/\xi) \). With the notation
\[ N_{f,g,\xi}(t) = \frac{1}{\xi} A^+(S_{t/\xi}f)A(S_{t/\xi}g) \] (5)
for any \( f, g \in \mathcal{H} \), the equation for the evolution operator \( U(t/\xi) \) becomes
\[ \frac{dU(t/\xi)}{dt} = (D \otimes N_{g_0,g_1,\xi}(t) - D^+ \otimes N_{g_1,g_0,\xi}(t))U(t/\xi) \] (6)

The reduced dynamics of any test particle’s observable \( X \) in the low density limit is defined by the limit
\[ \lim_{\xi \to 0} \varphi_{L,\xi}(U^+(t/\xi)(X \otimes 1)U(t/\xi)) \]
where \( \varphi_{L,\xi}(\cdot) \) denotes partial expectation. In [19] it was proved that, under some conditions, the limit exists in a small time interval and is equal to \( T_t(X) \), where \( \{T_t; t \geq 0\} \) is a quantum Markovian semigroup. The dynamics of the reduced density matrix \( \rho_S(t) \) is determined through the duality \( \text{Tr}(\rho_S T_t(X)) = \text{Tr}(\rho_S(t)X) \). As was mentioned in the Introduction, in the approach of the present paper the limiting semigroup can be obtained by using the solution \( U_t \) of the quantum stochastic equation as
\[ T_t(X) = \varphi_L(U^+_t(X \otimes 1)U_t) \]
and the generator of the semigroup can be easily derived from quantum Langevin equation. The limiting semigroup can be obtained also from quantum Langevin equation in [17], which is based on a quantum stochastic equation similar to [11] but much more complicated.

The first step to study the low density limit of the model is to find the limit of the field \( N_{f,g,\xi}(t) \). This limit we call master fields or number operators.
3. THE MASTER FIELDS AND THE LIMIT STATES

In this section we construct the algebra of the master fields arising in the low density limit and the limit causal states on this algebra. We prove (Theorem 1) that time-ordered correlators of initial fields converge in the low density limit to correlators of number operators constructed from some white noise operators. Theorem 2 states a useful factorization property of the limiting causal states.

It is convenient to use the "projections"

\[ P_E := \frac{1}{2\pi} \int_{-\infty}^{\infty} dt S_t e^{-itE} = \delta(H_1 - E) \]

with the properties

\[ P_E P_{E'} = \delta(E - E') P_E, \quad P_E^* = P_E, \quad S_t = \int dE P_E e^{itE} \]

For the \( \delta \)-function of a self-adjoint operator cf. Definition (1.2.1) in \[1\].

Let us construct the master space (which is Fock space over some Hilbert space) and master fields. For a given Hilbert space \( \mathcal{H} \) and a self-adjoint operator \( H_1 \) in \( \mathcal{H} \) we define the Hilbert space \( \mathcal{X}_{\mathcal{H},H_1} \) as the completion of the quotient of the set

\[ \left\{ F : \mathbb{R} \to \mathcal{H} \text{ s.t. } ||F||^2 := 2\pi \int dE \langle F(E), P_E F(E) \rangle < \infty \right\} \]

with respect to the zero-norm elements. The inner product in \( \mathcal{X}_{\mathcal{H},H_1} \) is defined as

\[ \langle F, G \rangle = 2\pi \int dE \langle F(E), P_E G(E) \rangle. \]

We denote by \( B^+_g(E,t), B^+_g(E',t') \) time-energy white noise creation and annihilation operators acting in the symmetric Fock space \( \Gamma(L^2(\mathbb{R}_+, \mathcal{X}_{\mathcal{H},H_1})) \) where \( L^2(\mathbb{R}_+, \mathcal{X}_{\mathcal{H},H_1}) \) is the Hilbert space of square integrable functions \( f : \mathbb{R}_+ \to \mathcal{X}_{\mathcal{H},H_1} \). These operators (operator-valued distributions) satisfy the canonical commutation relations

\[ [B_g(E,t), B^+_f(E',t')] = \delta(t' - t)\delta(E' - E) \tilde{\gamma}_{g,f}(E) \]

and causal commutation relations

\[ [B_g(E,t), B^+_f(E',t')] = \delta_+(t' - t)\delta(E' - E) \gamma_{g,f}(E) \]

where \( \delta_+(t' - t) \) is the causal \( \delta \)-function and

\[ \gamma_{g,f}(E) = \int dE' \frac{\langle g, P_{E'} f \rangle}{i(E' - E - \imath 0)} \]

\[ \tilde{\gamma}_{g,f}(E) = 2\pi \langle g, P_E f \rangle \]

In the Appendix we review the definition of the causal \( \delta \)-function; for a detailed discussion of distributions over the simplex and the meaning of two different commutators \[7] and \[9] for the same operators we refer to Sect. 7 in \[1\]. These operators are called time-energy quantum white noise due to the presence of \( \delta(t' - t)\delta(E' - E) \) in \[7\].
For any positive bounded operator $L$ in $\mathcal{H}$ we define the causal gauge-invariant mean-zero Gaussian state $\varphi_L$ by the properties (9)-(12):

$$\varphi_L(B_{i_1}^{e_1} \ldots B_{i_n}^{e_n}) = \sum \varphi_L(B_{i_1}^{e_1} \ldots B_{j_1}^{e_1}) \ldots \varphi_L(B_{i_k}^{e_k} B_{j_k}^{e_k})$$ (9)

where the sum is taken over all permutations of the set $(1, \ldots, 2k)$ such that $i_1 < j_1, \ldots, i_k < j_k, B_{m}^{e_m} := B_{m}^{e_m}(E_m, t_m)$ for $m = 1, \ldots, n$, are time-energy quantum white noise operators with causal commutation relations $\mathcal{K}$, and $\epsilon_m$ means either creation or annihilation operator;

$$\varphi_L(B_{i_1}^{e_1} \ldots B_{i_n}^{e_n}) = 0$$ (10)

$$\varphi_L(B_{f}(E, t)B_{g}(E', t')) = \varphi_L(B_{f}^{+}(E, t)B_{g}^{+}(E', t')) = 0$$ (11)

$$\varphi_L(B_{f}^{+}(E, t)B_{g}(E', t')) = \chi_{[0, t]}(t') \langle g, P_{E}L_f \rangle$$ (12)

Notice that the ”state” $\varphi_L$ does not satisfy the positivity condition. This is a well-known situation for the weak coupling limit (see $\mathcal{I}$) and is due to the fact that we work with time-ordered, or causal correlators. Therefore it is natural to call such ”states” causal states.

**Definition 1** Causal time-energy white noise is a pair $(B_{f}(E, t), \varphi_L)$, where $B_{f}(E, t)$ satisfy the causal commutation relations $\mathcal{K}$ and $\varphi_L$ is a causal gauge-invariant mean-zero Gaussian state characterized by (9)-(12).

Using the operators $B_{f}^{+}(E, t), B_{g}(E, t)$ we define the number operators as

$$N_{f,g}(t) = \int dE B_{f}^{+}(E, t)B_{g}(E, t)$$ (13)

Finally, for a given Hilbert space $\mathcal{H}$ and a self-adjoint operator $H_1$ we have the following objects: for any $\xi > 0$ the family of operators $N_{f,g,\xi}(t)$ defined by $\mathcal{E}$ together with the gauge-invariant quasifree mean-zero Gaussian state $\varphi_{L,\xi}$ and the number operators $N_{f,g}(t)$ together with the causal state $\varphi_L$.

The following theorem describes the relation between these objects and states the master field in the low density limit.

**Theorem 1** There exists causal time-energy white noise $(B_{f}(E, t), \varphi_L)$ such that $\forall n \in \mathbb{N}$

$$\lim_{\xi \to 0} \varphi_{L,\xi}(N_{f_1,g_1,\xi}(t_1) \ldots N_{f_n,g_n,\xi}(t_n)) = \varphi_L(N_{f_1,g_1}(t_1) \ldots N_{f_n,g_n}(t_n))$$

where the equality is understood in the sense of distributions over simplex $t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$. The limit causal state $\varphi_L$ is characterized by $\mathcal{K}$, $\mathcal{L}$ and the number operators are defined by $\mathcal{E}$.

**Remark 1** This convergence is called convergence in the sense of time-ordered correlators. The fact that we use the distributions over simplex is motivated by iterated series $\mathcal{K}$ for the evolution operator.
\textbf{Proof.} Notice that
\[ N_{f,g,\xi}(t) = \int dEN_{f,g,\xi}(E,t) \]
where
\[ N_{f,g,\xi}(E,t) := \frac{e^{iE/t}}{\xi} A^+(P_{E,f})A(S_{t}/\xi g) \]
Therefore
\[ \varphi_{L,\xi}(N_{f_1,g_1,\xi}(t_1) \ldots N_{f_n,g_n,\xi}(t_n)) = \int dE_1 \ldots dE_n \varphi_{L,\xi}(N_{f_1,g_1,\xi}(E_1,t_1) \ldots N_{f_n,g_n,\xi}(E_n,t_n)) \]
Let us denote for shortness of notation for \( l = 1, \ldots, n, \)
\[ A_l^+ := \frac{e^{i\xi l}}{\sqrt{\xi}} A^+(P_{E,f_l}); \quad A_l := \frac{1}{\sqrt{\xi}} A(S_{t_l}/\xi g_l) \]
In this notation
\[ \varphi_{L,\xi}(N_{f_1,g_1,\xi}(E_1,t_1) \ldots N_{f_n,g_n,\xi}(E_n,t_n)) = \varphi_{L,\xi}(A_1^+ A_1 \ldots A_n^+ A_n) \quad (14) \]
The state \( \varphi_{L,\xi} \) is a gauge-invariant mean-zero Gaussian state. Therefore (14) is equal to the sum of terms of the form
\[ \varphi_{L,\xi}(A_{i_1}^+ A_{j_1}) \ldots \varphi_{L,\xi}(A_{i_k}^+ A_{j_k}) \varphi_{L,\xi}(A_{j_{k+1}}^+ A_{i_{k+1}}) \ldots \varphi_{L,\xi}(A_{j_n} A_{i_n}^+) \quad (15) \]
where \( k = 1, \ldots, n, 1 = i_1 < i_2 < \ldots < i_k, j_{k+1} < \ldots < j_n, i_l \leq j_l \) for \( l = 1, \ldots, k \) and \( j_l < i_l \) for \( l = k + 1, \ldots, n \). We say that (15) corresponds to a nonconnected diagram if there exists \( m \in \{1, \ldots, n\} \) such that \( i_l \leq m \Leftrightarrow j_l \leq m \). Otherwise we say that (15) corresponds to a connected diagram.

Let us prove that all the connected diagrams except only one corresponding to the case \( k = 1 \) are equal to zero in the limit. One can write (15) as
\[
\frac{1}{\xi^n} \exp\left\{ i((t_1 - t_{j_1})E_1 + \ldots + (t_{i_n} - t_{j_n})E_{i_n})/\xi\right\} \left(\xi^k F(E) + O(\xi^{k+1})\right)
\]
\[
= \frac{1}{\xi^n} \exp\left\{ i[t_n(E_n - E_{\alpha_n}) + \ldots + t_1(E_1 - E_{\alpha_1})]/\xi\right\} \left(\xi^k F(E) + O(\xi^{k+1})\right)
\]
\[
= \frac{1}{\xi^{n-1}} \exp\left\{ i[(t_n - t_{n-1})\omega_n(E) + \ldots + (t_2 - t_1)\omega_2(E)]/\xi\right\} \left(\xi^{k-1} F(E) + O(\xi^{k})\right)
\]
\[
= \frac{e^{i(t_n - t_{n-1})\omega_n(E)/\xi}}{\xi} \ldots \frac{e^{i(t_2 - t_1)\omega_2(E)/\xi}}{\xi} \left(\xi^{k-1} F(E) + O(\xi^{k})\right) \quad (16)
\]
where \( (\alpha_1, \ldots, \alpha_n) \) is the permutation of the set \( (1, \ldots, n) \), \( \omega_l(E) = E_n + \ldots + E_l - E_{\alpha_n} - \ldots - E_{\alpha_l} \) for \( l = 2, \ldots, n \) and
\[
F(E) = \prod_{l=1}^k \langle g_{j_l}, P_{E_l} L f_{i_l} \rangle \prod_{l=k+1}^n \langle g_{j_l}, P_{E_l} f_{i_l} \rangle
\]
Notice that for a connected diagram all the functions \( \omega_l(E) \) are not identically zero. In fact, suppose that \( \omega_m(E) \equiv 0 \) for some \( m \in \{2, \ldots, n\} \). In this case one has the identity
\[
E_m + \ldots + E_n \equiv E_{\alpha_m} + \ldots + E_{\alpha_n}
\]
(where $E_{\alpha}, E_{\alpha'}$ for $\alpha \neq \alpha'$ are independent variables) which means that $(\alpha_m, \ldots, \alpha_n)$ is a permutation of the set \{m, \ldots, n\} and hence $(\alpha_1, \ldots, \alpha_{m-1})$ is a permutation of the set \{1, \ldots, m-1\}. Let us choose any $l \in \{1, \ldots, n\}$ and consider the term $t_{ji}(E_{ji} - E_{li})$ in the exponent in the second line of (16). If $j_l < m$, then since $i_l \equiv \alpha_{ji}$ and $\alpha_{ji}$ belongs to the set \{1, \ldots, m-1\} one has $i_l \equiv \alpha_{ji} \in \{1, \ldots, m-1\}$, and vice versa if $\alpha_{ji} \equiv i_l \in \{1, \ldots, m-1\}$, then $j_l \leq m-1$. This means that if $\omega_l$ are not identically zero, then \(15\) corresponds to a connected diagram.

Let us consider the case $k > 1$. Then, if \(13\) corresponds to a connected diagram, the functions $\omega_l(E)$ are not identically zero. In this case, since there exists the limit

$$
\lim_{\xi \to 0} \frac{\exp{i(t_1 - t_{n-1})/\xi}}{\xi} \prod_{l=1}^{n-1} \frac{1}{\xi} = \delta_+(t_1 - t_{n-1}) \prod_{l=1}^{n-1} \frac{1}{\xi} \frac{1}{i(\omega_l(E) - i0)}
$$

and the limit of the product of such terms in \(13\), and $k - 1 > 0$, the limit of \(16\) is equal to zero.

Now let us consider the case $k = 1$. In this case \(13\) has the form

$$
\varphi_L(\xi(A_1^+A_n^0\varphi_L(A_1A_2^0) \ldots \varphi_L(A_{n-1}A_n^0 = \\
\frac{1}{\xi^n} \exp{i(t_1 - n)E_1 + (t_2 - t_1)E_2 + \ldots + (t_n - t_{n-1})E_n}/\xi}\left(F(E) + O(\xi^2)\right) = \\
\frac{e^{i(t_1 - n)\omega_1(E)/\xi}}{\xi} \prod_{l=1}^{n-1} \frac{1}{\xi} \frac{1}{i(\omega_l(E) - i0)} \left(F(E) + O(\xi)\right)
$$

where $\omega_l(E) = E_l - E_1$. Using the limit \(19\) one finds that the limit of the right-hand side (RHS) of \(17\) is equal to

$$
\delta_+(t_2 - t_1) \ldots \delta_+(t_n - t_{n-1}) \langle g_n, P_{E_1L}f_1 \rangle \langle g_1, P_{E_2f_2} \rangle \frac{1}{i(E_2 - E_1 - i0)} \ldots \langle g_{n-1}, P_{E_{n-1}f_{n-1}} \rangle \frac{1}{i(E_n - E_1 - i0)}
$$

After integration over $E_1 \ldots E_n$ it becomes equal to

$$
\delta_+(t_2 - t_1) \ldots \delta_+(t_n - t_{n-1}) \int dE \langle g_n, P_{E_L}f_1 \rangle \gamma_{g_1}f_2(E) \ldots \gamma_{g_{n-1}}f_{n-1}(E)
$$

This proves that only one connected diagram survives in the limit.

Now let us consider the quantity

$$
\varphi_L(N_{f_1, g_1}(t_1) \ldots N_{f_n, g_n}(t_n))
$$

With the notation

$$
B^+_l := B^+_l(E_l, t_l); \quad B_l := B_{g_l}(E_l, t_l),
$$

it can be written as

$$
\int dE_1 \ldots dE_n \varphi_L(B^+_1B_1 \ldots B^+_nB_n)
$$

(19)

Notice that on the simplex $t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$ causal $\delta$-functions $\delta_+(t_{l+m} - t_l)$ for $m \geq 2$ are equal to zero. Therefore for $m \geq 2$ one has $\varphi_L(B^+_lB_{l+m}) \propto \delta_+(t_{l+m} - t_l) = 0$ and hence the integrand in \(19\) can be written as

$$
\varphi_L(B^+_1B_1 \ldots B^+_nB_n) = \sum_{k=1}^{n-1} \varphi_L(B^+_1B_k)\varphi_L(B_1B^+_2) \ldots \varphi_L(B_{k-1}B^+_k)\varphi_L(B^+_kB_{k+1} \ldots B^+_nB_n) + \varphi_L(B^+_1B_n)\varphi_L(B_1B^+_2) \ldots \varphi_L(B_{n-1}B^+_n)
$$

(20)
The terms in the sum correspond to nonconnected diagrams. The last term corresponds to a unique nonzero connected diagram. Moreover
\[
\int dE_1 \ldots dE_n \varphi_L(B_1^+B_n)\varphi_L(B_1B_2^+) \ldots \varphi_L(B_{n-1}B_n^+)=\delta_+(t_2-t_1)\ldots\delta_+(t_n-t_{n-1})
\]
\[
\times \int dE < g_n, P_EL f_1 > \gamma_{g_1,f_2}(E) \ldots \gamma_{g_{n-1},f_n}(E),
\]
which is equal to (18).

For \( n = 1 \) the statement of the theorem is clear. In fact,
\[
\lim_{\xi \to 0} \varphi_{L,\xi}(N_{f,g,\xi}(t)) = \lim_{\xi \to 0} \left< g, \frac{L}{1-\xi L}f \right> = \left< g, Lf \right> = \int dE \varphi_L(B_f^+(E,t)B_g(E,t))
\]
Then proof of the theorem follows by induction using the fact that only one connected diagram survives in the limit.

**Remark 2** The fact that in each order of iterated series only one connected diagram survives in the limit can be interpreted as emergence of a new statistics (different from Bose) in the low density limit. For a discussion of new statistic arising in the weak coupling limit we refer to [1] (see also [18]).

The following theorem is important for investigation of the limiting white noise equation for the evolution operator.

**Theorem 2** The limit state \( \varphi_L \) has the following factorization property: \( \forall n \in \mathbb{N} \),
\[
\varphi_L(B_f^+(E,t)N_{f_1,g_1}(t_1) \ldots N_{f_n,g_n}(t_n)B_g(E,t))
\]
\[
= \varphi_L(B_f^+(E,t)B_g(E,t))\varphi_L(N_{f_1,g_1}(t_1) \ldots N_{f_n,g_n}(t_n))
\]
(21)

where the equality is understood in the sense of distributions over simplex \( t \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq 0 \).

**Proof.** From Gaussianity of the causal state \( \varphi_L \) (property [9]) it follows that
\[
\varphi_L(B_f^+(E,t)N_{f_1,g_1}(t_1) \ldots N_{f_n,g_n}(t_n)B_g(E,t))
\]
\[
= \varphi_L(B_f^+(E,t)B_g(E,t))\varphi_L(N_{f_1,g_1}(t_1) \ldots N_{f_n,g_n}(t_n))
\]
\[
+ \int dE_1 \ldots dE_n \sum \varphi_L(B_f^+(E,t)B_g(E,t)) \ldots \varphi_L(B_f^+(E,t)B_g(E,t))
\]
The sum is equal to zero since the last multiplier
\[
\varphi_L(B_f^+(E,t)B_g(E,t)) = \chi_{[0,t]}(t) < g, P_{E_f}Lf_j > \]
is equal to zero almost everywhere on the simplex \( t \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq 0 \) and hence is equal to zero in the sense of distributions on the simplex. This proves the theorem.

Theorem 1 allows us to calculate, in particular, the partial expectation of the evolution operator and Heisenberg evolution of any system observable in the low density limit. In fact, partial expectation of the \( n \)-th term of the iterated series for the evolution operator [3] (or equivalent series for Heisenberg evolution of a system observable) after time rescaling \( t \to t/\xi \) includes the quantity
\[
\int_0^t dt_1 \ldots \int_0^{t_n-1} dt_n \varphi_{L,\xi}(N_{f_1,g_1,\xi}(t_1) \ldots N_{f_n,g_n,\xi}(t_n))
\]
(where $f_\alpha, g_\alpha$ are equal to $g_0$ or $g_1$). The limit as $\xi \to 0$ of this quantity can be calculated using Theorem 1. For example, the contribution of the connected diagram is equal to

$$
\int_0^t dt_1 \int_0^{t_1} dt_2 \delta_+(t_2 - t_1) \int_0^{t_2} dt_3 \delta_+(t_3 - t_2) \ldots \int_0^{t_{n-1}} dt_n \delta_+(t_n - t_{n-1})
$$

$$
\times \int dE \langle g_n, P_ELf_1 \rangle \gamma_{g_1, f_2} (E) \ldots \gamma_{g_{n-1}, f_n} (E)
$$

$$
= t \int dE \langle g_n, P_ELf_1 \rangle \gamma_{g_1, f_2} (E) \ldots \gamma_{g_{n-1}, f_n} (E)
$$

Similarly one can calculate the contribution of nonconnected diagrams (they give terms proportional to higher orders of $t$). Summing over all orders of the iterated series one can find the reduced dynamics of the system. But in the present paper we will get the limiting dynamics in a nonperturbative way, without direct summation of the iterated series. This procedure includes derivation of the white noise equation for the limiting evolution operator and then bringing this equation to the causally normally ordered form. After that one can easily find, for example, the reduced dynamics of the system. For the weak coupling limit such a procedure was developed in [1]. A nontrivial generalization to the low density limit was developed in [16] [17], where the derivation is based on the Fock-antiFock representation for the CCR algebra of the Bose field determined by the state $\varphi_{L,\xi}$. The approach of the present paper does not require a GNS representation and is different from approach of [16] [17].

4. THE WHITE NOISE SCHröDINGER EQUATION

In this section we derive, using the results of previous section, the white noise Schrödinger equation for the limiting evolution operator.

The evolution operator $U(t/\xi)$ satisfies equation (6) which can be written as

$$
\frac{dU(t/\xi)}{dt} = -iH_\xi(t)U(t/\xi),
$$

where

$$
H_\xi(t) = i(D \otimes N_{g_0,g_1,\xi}(t) - D^+ \otimes N_{g_1,g_0,\xi}(t))
$$

The results of the preceding section allow us to write the limit as $\xi \to 0$ of the Hamiltonian $H_\xi(t)$. In the notation (13) the limiting Hamiltonian is the following operator in $H_S \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{X}_{H_1}))$:

$$
H(t) = i(D \otimes N_{g_0,g_1}(t) - D^+ \otimes N_{g_1,g_0}(t))
$$

$$
= i \int dE \left( D \otimes B_{g_0}^+(E,t)B_{g_1}(E,t) - D^+ \otimes B_{g_1}^+(E,t)B_{g_0}(E,t) \right)
$$

(22)

The dynamics of the total system (system+reservoir) in the low density limit $\xi \to 0$ is given by a new evolution operator $U_t$ which is the solution of the white noise Schrödinger equation

$$
\frac{dU_t}{dt} = -iH(t)U_t, \quad U_0 = 1,
$$

(23)
or equivalent integral equation

\[ U_t = 1 + \int_0^t \left( D \otimes N_{g_0,g_1}(t_1) - D^+ \otimes N_{g_1,g_0}(t_1) \right) U_{t_1}. \]  

(24)

5. NORMALLY ORDERED FORM OF THE WHITE NOISE EQUATION

Our next step is to bring the white noise Schrödinger equation to the causally normally ordered form (Theorem 3), i.e., the form in which all annihilation operators are on the right side of the evolution operator and all creation operators are on the left side. Such a form is convenient for study of the limiting dynamics (see remark 3 and text after remark). In particular, it can be used for derivation of (linear) Boltzmann equation.

We assume that for each \( E \in \mathbb{R} \), the inverse operators

\[ T_0(E) := \left( 1 + \gamma_{g_0,g_1}(E)D^+ - \gamma_{g_1,g_0}(E)D + (\gamma_{g_0,g_0}\gamma_{g_1,g_1} - \gamma_{g_1,g_0}\gamma_{g_0,g_1})(E)D^+D \right)^{-1} \]

\[ T_1(E) := \left( 1 + \gamma_{g_0,g_1}(E)D^+ - \gamma_{g_1,g_0}(E)D + (\gamma_{g_0,g_0}\gamma_{g_1,g_1} - \gamma_{g_1,g_0}\gamma_{g_0,g_1})(E)D^+D \right)^{-1} \]

exist.

**Lemma 1** If the evolution operator \( U_t \) satisfies (23) with \( H(t) \) given by (22) then one has

\[ B_{g_0}(E,t)U_t = \gamma_{g_0,g_0}(E)T_0(E)DU_1B_{g_1}(E,t) + T_0(E)(1 - \gamma_{g_1,g_0}(E)D)U_1B_{g_0}(E,t) \]

(25)

\[ B_{g_1}(E,t)U_t = -\gamma_{g_1,g_1}(E)T_1(E)D^+U_1B_{g_0}(E,t) + T_1(E)(1 + \gamma_{g_0,g_1}(E)D^+)U_1B_{g_0}(E,t) \]  

(26)

**Remark 3** Notice that in the RHS of these equalities the annihilation operators \( B_f(E,t) \) are on the right of the evolution operator.

**Proof.** It follows from (8) and (13) that

\[ [B_{f'}(E,t), N_{f,g}(t_1)] = \delta_+(t_1 - t)\gamma_{f',f}(E)B_g(E,t) \]  

(27)

Therefore using the integral equation (24) for the evolution operator one gets

\[ B_{f}(E,t)U_t = [B_{f}(E,t), U_t] + U_tB_f(E,t) \]

\[ = \int_0^t dt_1 \left( D \otimes [B_f(E,t), N_{g_0,g_1}(t_1)] - D^+ \otimes [B_f(E,t), N_{g_1,g_0}(t_1)] \right) U_{t_1} + U_tB_f(E,t) \]

\[ = \left( D\gamma_{f,g_0}(E)B_{g_1}(E,t) - D^+\gamma_{f,g_1}(E)B_{g_0}(E,t) \right) U_t + U_tB_f(E,t) \]  

(28)

The second equality in (28) holds because, due to the time consecutive principle

\[ [B_{f}(E,t), U_{t_1}] = 0 \quad \text{for } t_1 < t. \]

In fact, let us consider the quantity

\[ \int_0^t dt_1 [B_f(E,t), U_{t_1}^{(n-1)}] = (-i)^{n-1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n [B_f(E,t), H(t_2) \ldots H(t_n)] \]  

(29)
where the $n$-th term of the iterated series (3) for $U_t$ has the form
\[ U_t^{(n)} := (-i)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n H(t_1) \ldots H(t_n) \]

The commutator $[B_f(E, t), H(t_k)]$ proportional to $\delta_+(t_k - t)$, hence the commutator $[B_f(E, t), H(t_2) \ldots H(t_n)]$ is equal to zero on the simplex $t \geq t_1 \geq t_2 \ldots \geq t_n \geq 0$ and therefore (29) is equal to zero.

The third equality in (28) holds since from (27) and the definition of causal $\delta$-function one has
\[ \int_0^t dt_1 \delta_+(t_1 - t) B_f(E, t_1) U_{t_1} = B_f(E, t) U_t \]
For a detailed discussion of the time consecutive principle and causal $\delta$-function we refer to [1].

After the substitution $f = g_0$ and $f = g_1$ in (28) one gets
\[ B_{g_0}(E, t) U_t = \left( D\gamma_{g_0,g_0}(E) B_{g_1}(E, t) - D^+ \gamma_{g_0,g_1}(E) B_{g_0}(E, t) \right) U_t + U_t B_{g_0}(E, t) \]
\[ B_{g_1}(E, t) U_t = \left( D\gamma_{g_1,g_0}(E) B_{g_1}(E, t) - D^+ \gamma_{g_1,g_1}(E) B_{g_0}(E, t) \right) U_t + U_t B_{g_1}(E, t) \]
or equivalently
\[ (1 + \gamma_{g_0,g_0}(E) D^+) B_{g_0}(E, t) U_t = \gamma_{g_0,g_0}(E) D B_{g_1}(E, t) U_t + U_t B_{g_0}(E, t) \]  \hspace{1cm} (30)
\[ (1 + \gamma_{g_1,g_0}(E) D) B_{g_1}(E, t) U_t = -\gamma_{g_1,g_1}(E) D^+ B_{g_0}(E, t) U_t + U_t B_{g_1}(E, t) \]  \hspace{1cm} (31)

After left multiplication of both sides of equality (30) by $(1 + \gamma_{g_1,g_0}(E) D)$ and both sides of (31) by $\gamma_{g_0,g_0}(E) D$ one gets
\[ (1 + \gamma_{g_0,g_0}(E) D) (1 + \gamma_{g_0,g_0}(E) D^+) B_{g_0}(E, t) U_t = \gamma_{g_0,g_0}(E) D (1 + \gamma_{g_0,g_0}(E) D) B_{g_1}(E, t) U_t \]
\[ + (1 + \gamma_{g_1,g_0}(E) D) U_t B_{g_0}(E, t) \]  \hspace{1cm} (32)
\[ \gamma_{g_0,g_0}(E) D (1 + \gamma_{g_1,g_0}(E) D) B_{g_1}(E, t) U_t = -\gamma_{g_0,g_0}(E) D^+ \gamma_{g_1,g_1}(E) B_{g_0}(E, t) U_t \]
\[ + \gamma_{g_0,g_0}(E) D U_t B_{g_1}(E, t) \]  \hspace{1cm} (33)

Now after substitution of expression (33) into (32) one has
\[ \left( 1 + \gamma_{g_0,g_0}(E) D^+ - \gamma_{g_1,g_0}(E) D + (\gamma_{g_0,g_0} \gamma_{g_1,g_1} - \gamma_{g_0,g_0} \gamma_{g_0,g_1})(E) D D^+ \right) B_{g_0}(E, t) U_t \]
\[ = \gamma_{g_0,g_0}(E) D U_t B_{g_1}(E, t) + (1 - \gamma_{g_1,g_0}(E) D) U_t B_{g_0}(E, t) \]  \hspace{1cm} (34)

One can show by similar computations that
\[ \left( 1 + \gamma_{g_0,g_0}(E) D^+ - \gamma_{g_1,g_0}(E) D + (\gamma_{g_0,g_0} \gamma_{g_1,g_1} - \gamma_{g_0,g_0} \gamma_{g_0,g_1})(E) D^+ D \right) B_{g_1}(E, t) U_t \]
\[ = -\gamma_{g_1,g_1}(E) D^+ U_t B_{g_0}(E, t) + (1 + \gamma_{g_0,g_1}(E) D^+) U_t B_{g_1}(E, t) \]  \hspace{1cm} (35)

Now since we suppose that the inverse operators $T_0(E)$ and $T_1(E)$ exist, we can solve the above equations (34) and (35) with respect to $B_{g_0}(E, t) U_t$ and $B_{g_1}(E, t) U_t$. The solutions are given by (25) and (26), and that proves the lemma.
Denote
\[ R_{0,0}(E) := \gamma_{g_1,g_1}(E)DT_1(E)D^+ \]
\[ R_{1,1}(E) := \gamma_{g_0,g_0}(E)D^+T_0(E)D \]
\[ R_{0,1}(E) := -DT_1(E)(1 + \gamma_{g_0,g_1}(E)D^+) \]
\[ R_{1,0}(E) := D^+T_0(E)(1 - \gamma_{g_1,g_0}(E)) \]

**Theorem 3** The normally ordered form of equation (23) is
\[
\frac{dU_t}{dt} = -\sum_{n,m=0,1} \int dE R_{m,n}(E)B_{g_m}^+(E,t)U_tB_{g_n}(E,t) \tag{36}
\]

**Proof.** Using (22) white noise Schrödinger equation (23) can be rewritten in a more detailed form
\[
\frac{dU_t}{dt} = \int dE (D \otimes B_{g_0}^+(E,t)B_{g_1}(E,t) - D^+ \otimes B_{g_1}^+(E,t)B_{g_0}(E,t))U_t \tag{37}
\]
It follows from Lemma 1 that
\[
D^+B_{g_0}(E,t)U_t = R_{1,1}(E)U_tB_{g_0}(E,t) + R_{1,0}(E)U_tB_{g_0}(E,t)
\]
\[
DB_{g_1}(E,t)U_t = -R_{0,0}(E)U_tB_{g_0}(E,t) - R_{0,1}(E)U_tB_{g_1}(E,t)
\]
The statement of the theorem is obtained after substitution of these expressions in (37).

**Remark 4** An immediate consequence of Theorem 2 is the following factorization property of the limiting state \( \varphi_L \):
\[
\varphi_L(B_{g_m}^+(E,t)U_tB_{g_n}(E,t)) = \varphi_L(B_{g_m}^+(E,t)B_{g_n}(E,t))\varphi_L(U_t)
\]
This property of the state \( \varphi_L \) similar to the factorization property of the state determined by a coherent vector \( \Psi, \|\Psi\| = 1 \):
\[
(\Psi, B_{g_m}^+(E,t)U_tB_{g_n}(E,t)\Psi) = (\Psi, B_{g_m}^+(E,t)B_{g_n}(E,t)\Psi)(\Psi, U_t\Psi)
\]
which is usually used to define quantum stochastic differential equations (the general notion of adaptedness and adapted domains which are much larger than the coherent ones is given in [12]).

Taking the partial expectation of both sides of equation (36) in the state \( \varphi_L \), using the factorization property and noticing that
\[
\varphi_L(B_{g_m}^+(E,t)B_{g_n}(E,t)) = \langle g_n, P_ELg_m \rangle,
\]
one gets the equation
\[
\frac{d\varphi_L(U_t)}{dt} = -\Gamma \varphi_L(U_t), \tag{38}
\]
where \( \Gamma \) is being called drift and is equal to
\[
\Gamma = \sum_{n,m=0,1} \int dE R_{m,n}(E)\langle g_n, P_ELg_m \rangle
\]
The solution of (38) is
\[ \varphi_L(U_t) = e^{-\Gamma t} \]
In the case of orthogonal test functions, i.e. \( \langle g_0, S t g_1 \rangle = 0 \) this expectation value for the evolution operator was obtained in [16]. Let us note that the expectation value is obtained in a nonperturbative way, without direct summation of the iterated series for the evolution operator, and is a result of the procedure of causal normal ordering.

6. ONE-PARTICLE T OPERATOR AND S MATRIX

In the low density limit the role of multiparticle collisions is negligible and the dynamics of the test particle should be determined by the interaction of the test particle with one particle of the reservoir. In the present section we give the expressions for the one-particle T-operator and S-matrix. In the next section we will rewrite normally ordered white noise equation (36) in a form of the quantum stochastic equation (44) and show (Theorem 5) that the coefficients of this equation can be expressed in terms of the one-particle S-matrix.

Because of number conservation, the closed subspace of \( \mathcal{H}_S \otimes \Gamma(\mathcal{H}) \) generated by vectors of the form \( u \otimes A^+(f) \Phi \) \((u \in \mathcal{H}_S, f \in \mathcal{H} = L^2(\mathbb{R}^d), \Phi \) is the vacuum vector), which is naturally isomorphic to \( \mathcal{H}_S \otimes \mathcal{H} \), is globally invariant under the time evolution operator \( \exp[i(\mathcal{H}_S \otimes 1 + 1 \otimes \mathcal{H}_R + V)t] \). The restriction of the time evolution operator to this subspace corresponds to the evolution operator on \( \mathcal{H}_S \otimes \mathcal{H} \) given by
\[ \exp[i(H_S \otimes 1 + 1 \otimes H_1 + V_1)t] \]
where
\[ V_1 = i(D \otimes |g_0\rangle\langle g_1| - h.c.) \] (39)

The one-particle Møller wave operators are defined as
\[ \Omega_\pm = s - \lim_{t \to \pm \infty} \exp[i(H_S \otimes 1 + 1 \otimes H_1 + V_1)t] \exp[-i(H_S \otimes 1 + 1 \otimes H_1)t] \]
The one-particle T-operator is defined as
\[ T = V_1 \Omega_+ \] (40)
and the one-particle S-matrix as
\[ S = \Omega^*_+ \Omega_+ \] (41)

**Theorem 4** For the interaction (39) the one-particle T-operator and S-matrix have the form
\[ T = -i \sum_{n,m \in \{0,1\}} \int dER_{m,n}(E) \otimes |g_m\rangle\langle P_E g_n| \] (42)
\[ S = 1 - 2\pi \sum_{n,m \in \{0,1\}} \int dER_{m,n}(E) \otimes |P_E g_m\rangle\langle P_E g_n| \] (43)

**Proof.** For the case \( \langle g_0, S t g_1 \rangle = 0 \) equality (42) was proved in [17]. The proof of (42) and (43) for the general case can be done in a similar way.

Expression (43) will be used in the next section for derivation of equation (47).
7. QUANTUM STOCHASTIC EQUATION FOR THE LIMITING EVOLUTION OPERATOR

Normally ordered white noise equation (36) equivalent, through identification

\[ B_m^+(E,t)U_tB_n(E,t)dt = 2\pi dN_t(|P_Eg_m\rangle\langle P_Eg_n|)U_t \]

to the quantum stochastic differential equation

\[ dU_t = -2\pi \sum_{n,m\in\{0,1\}} \int dER_{m,n}(E) dN_t(|P_Eg_m\rangle\langle P_Eg_n|)U_t \] (44)

where \( N_t \) is the quantum Poisson process in \( \Gamma(L^2(\mathbb{R}_+) \otimes \mathcal{H}) \) defined by \( N_t(X) := N(\chi_{[0,t]} \otimes X) \), if \( X \) is an operator in \( \mathcal{H} \). The stochastic differential \( dN_t \) satisfies the usual Ito table

\[ dN_t(X)dN_t(Y) = dN_t(XY) \] (45)

where \( X, Y \) are operators in \( \mathcal{H} \), and the limit state \( \varphi_L \) characterized by the property

\[ \varphi_L(2\pi dN_t(|P_Ef\rangle\langle P_Eg|)) = \langle g, P_ELf \rangle dt \]

The coefficients of quantum stochastic equation (44) can be expressed in terms of one-particle S-matrix describing scattering of the test particle on one particle of the reservoir. To show this we will use Hilbert module notation. For any pair of Hilbert spaces \( \mathcal{X}_0, \mathcal{X}_1 \), if \( N_t \) denotes the Poisson process on the Fock space \( \Gamma(L^2(\mathbb{R}_+) \otimes \mathcal{X}_1) \), then for bounded operators \( X_0 \in B(\mathcal{X}_0), X_1 \in B(\mathcal{X}_1) \), the Hilbert module notation is [10]:

\[ N_t(X_0 \otimes X_1) := X_0 \otimes N_t(X_1) \]

With this notation equation (44) can be written as

\[ dU_t = dN_t\left(-2\pi \sum_{n,m\in\{0,1\}} \int dER_{m,n}(E) \otimes |P_Eg_m\rangle\langle P_Eg_n| \right)U_t \] (46)

An immediate conclusion from (43) and (46) is the following theorem which is one of the main results of the paper.

**Theorem 5** The evolution operator in the low density limit satisfies the quantum stochastic equation driven by the quantum Poisson process with intensity \( S - 1 \):

\[ dU_t = dN_t(S - 1)U_t \] (47)

Equation (47) describes the dynamics of the compound system in the low density limit. Using this equation and the Ito table for stochastic differentials one can obtain a quantum Langevin equation for the Heisenberg evolution of any system observable. Then the corresponding master equation or, equivalently, quantum (linear) Boltzmann equation for reduced density matrix of the system can be obtained simply by taking the partial expectation of this Langevin equation in the causal state \( \varphi_L \).

**Lemma 2** The solution of (47) is unitary.
Proof. Let us show that $d(U_t^+U_t) = 0$. The operator $U_t^+$ satisfies the equation

$$dU_t^+ = U_t^+dN_t(S^+ - 1)$$

One has

$$d(U_t^+U_t) = dU_t^+U_t + U_t^+dU_t + dU_t^+dU_t$$

$$= U_t^+dN_t(S^+ - 1)U_t + U_t^+dN_t(S - 1)U_t + U_t^+dN_t(S^+ - 1)dN_t(S - 1)U_t$$

Using the Ito table [5] one gets

$$dN_t(S^+ - 1)dN_t(S - 1) = dN_t((S^+ - 1)(S - 1))$$

This and unitarity of $S$ leads to

$$d(U_t^+U_t) = U_t^+dN_t(S^+ - 1 + S - 1 + (S^+ - 1)(S - 1))U_t = 0$$

Now it follows from the initial condition $U_{t=0} = 1$ that, for any $t \geq 0$, $U_t^+U_t = 1$. The proof of $U_tU_t^+ = 1$ can be done in a similar way.

8. CONCLUSIONS

In the present paper we consider the dynamics of a test particle ($N$-level atom) interacting with a dilute Bose gas. It is proved that the dynamics of the total system converges in the low density limit to the solution of the quantum stochastic equation driven by a quantum Poisson process with intensity $S - 1$, where $S$ is the one-particle scattering matrix. The limiting equation is derived in a nonperturbative way, without use of iterated series for the evolution operator. The derivation is based on the white noise approach and on the procedure of causal normal ordering developed for the weak coupling limit by Accardi, Lu and Volovich [1]. The novelty of the present derivation is that it does not use the Fock-antiFock (or GNS) representation for the CCR algebra of the Bose gas, determined by the state $\varphi_{L,\xi}$. This simplifies the derivation and allows us to express the intensity of the Poisson process directly in terms of the one-particle $S$-matrix. The notion of causal states is introduced and the convergence of the correlators of the free evolution of the initial number operators to correlators of quantum white noise operators in causal states is proved. The causal states satisfy the factorization property similar to that satisfied by states determined by coherent vectors. This property is crucial for study of the reduced dynamics of the system.

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9. APPENDIX: CAUSAL $\delta$-FUNCTION

Let us recall the construction for distributions on the standard simplex (cf. [1]). Define

$C_0 := \{ \phi : \mathbb{R}_+ \to \mathbb{C} \mid \phi = 0 \text{ a.e.} \}$,

$C_1 := \{ \phi : \mathbb{R}_+ \to \mathbb{C} \mid \phi \text{ is bounded and left-continuous at any } t > 0 \}$,

$C := \text{linear span of } \{ C_0 \cup C_1 \}$.

For any $a > 0$ define $\delta_+(\cdot - a)$ as the unique linear extension of the map:

$\delta_+(\cdot - a) : \phi \in C_1 \to \phi(a)$

$\delta_+(\cdot - a) : \phi \in C_0 \to 0$.

In [1] the following results are proved.

**Lemma 3.** In the sense of distributions one has the limit

$$\lim_{\lambda \to 0} \frac{e^{i(t' - t)E/\lambda^2}}{\lambda^2} = 2\pi \delta(t' - t)\delta(E) \tag{48}$$

**Lemma 4.** In the sense of distributions over the simplex $t \geq t' \geq 0$ one has the limit

$$\lim_{\lambda \to 0} \frac{e^{i(t' - t)E/\lambda^2}}{\lambda^2} = \delta_+(t' - t) \frac{1}{i(E - i0)} \tag{49}$$

The last equality means that for any $f \in C$, $g \in S(\mathbb{R})$, one has the limit

$$\lim_{\lambda \to 0} \int_0^t dt' \int dE \frac{e^{i(t' - t)E/\lambda^2}}{\lambda^2} f(t')g(E) = f(t) \lim_{\varepsilon \to 0^+} \int dE \frac{g(E)}{i(E - i\varepsilon)}.$$
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