On the Measurability of Stochastic Fourier Integral Operators

Michael Oberguggenberger∗ Martin Schwarz†

Abstract

This work deals with the measurability of Fourier integral operators (FIOs) with random phase and amplitude functions. The key ingredient is the proof that FIOs depend continuously on their phase and amplitude functions, taken from suitable classes. The results will be applied to the solution FIO of the transport equation with spatially random transport speed as well as to FIOs describing waves in random media.

1 Introduction

In the theory of hyperbolic partial differential equations (PDEs), Fourier integral operators (FIOs) have become an important tool to examine certain properties of the solution, e.g., the propagation of singularities [1, 6, 12]. When studying waves in random media, the coefficients of the underlying PDEs are random fields. This has become important in seismology [3, 7] and in material science [5, 9]. As the phase and amplitude functions of the FIOs producing a solution or a parametrix are functions of the coefficients of the underlying PDE, one has to ensure that the FIO, respectively its action, stays measurable. The question of measurability of a FIO arises also in its own right, when a deterministic phase or amplitude function is subjected to a stochastic perturbation [8]. This work is dedicated to providing a rigorous proof of various continuity and measurability properties.

Consider a FIO of the form

\[ A_{\Phi, a}[u](x) = \frac{1}{(2\pi)^n} \int \int e^{i\Phi(x, y, \xi)} a(x, y, \xi) u(y) \, dy \, d\xi. \]

The first task will be to show that, for \( \psi \in D(Y) \), respectively \( u \in E'(Y) \), the maps

\[ (\Phi, a) \mapsto A_{\Phi, a}[\psi], \quad (\Phi, a) \mapsto A_{\Phi, a}[u] \]

are continuous with values in \( C^\infty(X) \), respectively \( D'(X) \), on suitable spaces of phase and amplitude functions. (Here \( X \) and \( Y \) are open subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^n \).) Equipping these spaces with their Borel \( \sigma \)-algebra, we consider random functions

\[ \omega \mapsto (\Phi_\omega(x, y, \xi), a_\omega(x, y, \xi)) \]

on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The second task will be to infer that the maps

\[ \omega \mapsto A_{\Phi_\omega, a_\omega}[\psi], \quad \omega \mapsto A_{\Phi_\omega, a_\omega}[u] \]

are measurable as well. The applicability of these results will be demonstrated in three examples: the transport equation and the half wave equation, both with a spatially random propagation speed, and a random perturbation of the solution operator to the wave equation.

The plan of the paper is as follows. We start by recalling required facts from the classical theory of Fourier integral operators. In the subsequent section, we prove the continuity and measurability results. The final section addresses the announced applications. The paper is part of a larger program aiming at studying wave propagation in random media by means of stochastic Fourier integral operators [8, 10]; it provides the probabilistic basis for this program.

∗Unit of Engineering Mathematics, University of Innsbruck, Technikerstraße 13, 6020 Innsbruck, Austria, (michael.oberguggenberger@uibk.ac.at)
†Unit of Engineering Mathematics, University of Innsbruck, Technikerstraße 13, 6020 Innsbruck, Austria, (martin.schwarz@uibk.ac.at)
2 Classical theory of oscillatory integrals and FIOs

2.1 Oscillatory integrals

Let \( n_1, n_2 \in \mathbb{N} \) and let \( Y \subset \mathbb{R}^{n_2} \) be an open set. The subsequent short exposition follows [11]. Let

\[
I_\Phi(au) = \int_{\mathbb{R}^{n_2}} \int_Y e^{i\Phi(y,\xi)} a(y,\xi)u(y) \, dy \, d\xi.
\]

Here \( u \in \mathcal{D}(Y) \) is a smooth function with compact support and \( \Phi : Y \times \mathbb{R}^{n_2} \) is a phase function, which means that \( \Phi|_{Y \times (\mathbb{R}^{n_2} \setminus \{0\})} \) is smooth, real valued and positively homogeneous of degree 1 in \( \xi \). Furthermore, \( \Phi \) does not have any critical points in \( \mathbb{R}^{n_2} \setminus \{0\} \), i.e., for all \( y \in Y \) and \( \xi \in \mathbb{R}^{n_2} \setminus \{0\} \)

\[
[\partial_{y_1} \Phi(y,\xi), \ldots, \partial_{y_{n_2}} \Phi(y,\xi), \partial_{\xi_1} \Phi(y,\xi), \ldots, \partial_{\xi_{n_2}} \Phi(y,\xi)]^T \neq 0.
\]

The function \( a : Y \times \mathbb{R}^{n_2} \) is a Hörmander symbol of class \( S^{0,\delta}_{\rho}(Y \times \mathbb{R}^{n_2}) \), \( \delta \in \mathbb{R}, \, 0 \leq \delta < 1 \) and \( 0 < \rho \leq 1 \). That means, it is smooth and for any given multi-indices \( k \in \mathbb{N}_{\rho}, l \in \mathbb{N}_{\rho} \) and any compact \( K \subset Y \) there exists a constant \( C_{l,k,K} \) such that

\[
\left| \partial^k_{\xi} \partial_{y}^l a(y,\xi) \right| \leq C_{l,k,K} \langle \xi \rangle^{d-\rho|I|+|k|},
\]

where \( y \in K \) and \( \xi \in \mathbb{R}^{n_2} \). As usual, we write \( \langle \xi \rangle = (1 + ||\xi||^2)^{1/2} \).

In general, \( (e^{i\Phi(y,\xi)} a(y,\xi)u(y)) \) is not absolutely integrable; the oscillatory integral has to be regularized. We recall the usual procedure, as presented, e.g., in [11].

Let \( \chi \in \mathcal{D}(\mathbb{R}^{n_2}) \) with \( \chi(\xi) \equiv 1 \) for \( ||\xi|| < 1 \) and \( \chi(\xi) \equiv 0 \) for \( ||\xi|| > 2 \). Furthermore, let

\[
r(\xi, y) = \left( \sum_{i=1}^{n_2} ||\xi||^2 |\partial_{\xi_i} \Phi(y,\xi)|^2 + \sum_{k=1}^{n_2} |\partial_{y_k} \Phi(y,\xi)|^2 \right),
\]

and

\[
\omega_\xi = \frac{e^{i\phi}(y,\xi)}{r}(1 - \chi) ||\xi||^2 (\partial_{\xi_i} \Phi), \quad \beta_\xi = \frac{e^{i\phi}(y,\xi)}{r}(1 - \chi)(\partial_{y_k} \Phi), \quad \gamma = \chi.
\]

Let \( L \) be the differential operator

\[
Lf = -\sum_{i=1}^{n_2} \partial_{\xi_i} (\omega_\xi f) - \sum_{k=1}^{n_2} \partial_{y_k} (\beta_\xi f) + \gamma f.
\]

Then, the formal adjoint operator \( \dagger L \)

\[
\dagger L \left( \sum_{i=1}^{n_2} \omega_\xi \partial_{\xi_i} + \sum_{k=1}^{n_2} \beta_\xi \partial_{y_k} + \gamma \right).
\]

satisfies

\[
\dagger L e^{i\Phi} = e^{i\Phi}.
\]

Furthermore, \( \omega_\xi \in S^{0,1}_{1,0}(Y \times \mathbb{R}^{n_2}), \beta_\xi \in S^{-1,1}_{1,0}(Y \times \mathbb{R}^{n_2}) \) and \( \gamma \in S^{-1,1}_{1,0}(Y \times \mathbb{R}^{n_2}) \).

Finally, one can choose a \( \kappa \in \mathbb{N} \) large enough and iteratively apply \( L \) to \( \omega_\xi \), and get a convergent integral by

\[
I_\Phi(au) = \int_{\mathbb{R}^{n_2}} \int_Y e^{i\Phi(y,\xi)} L^*(a(y,\xi)u(y)) \, dy \, d\xi.
\]

2.2 Classical theory of FIOs

Let \( n_1, n_2, n_3 \in \mathbb{N} \) and \( X \) resp. \( Y \) be an open subset of \( \mathbb{R}^{n_1} \) resp. \( \mathbb{R}^{n_2} \). A Fourier integral operator is an operator of the form

\[
A_{\Phi,a}[u](x) = \int_{\mathbb{R}^{n_2}} \int_Y e^{i\Phi(x,y,\xi)} a(x,y,\xi)u(y) \, dy \, d\xi,
\]

where \( d\xi = (2\pi)^{-n} \, d\xi \). Let \( \Phi : X \times Y \times \mathbb{R}^{n_2} \rightarrow \mathbb{R} \) be a phase function on \( X \times Y \times \mathbb{R}^{n_2} \) and \( a \in S^{0,\delta}_{\rho,\kappa}(X \times Y \times \mathbb{R}^{n_2}) \) an amplitude function. Furthermore, the following conditions for \( \xi \neq 0 \) are assumed:

\[
[\partial_{x_1} \Phi(x, y, \xi), \ldots, \partial_{x_{n_1}} \Phi(x, y, \xi), \partial_{\xi_1} \Phi(x, y, \xi), \ldots, \partial_{\xi_{n_2}} \Phi(x, y, \xi)]^T \neq 0,
\]

\[
[\partial_{y_1} \Phi(x, y, \xi), \ldots, \partial_{y_{n_2}} \Phi(x, y, \xi), \partial_{\xi_1} \Phi(x, y, \xi), \ldots, \partial_{\xi_{n_2}} \Phi(x, y, \xi)]^T \neq 0.
\]
The function $\Phi$ is then called an operator phase function.

By classical theory, if condition (3) is satisfied, operator (5) continuously maps $D(Y)$ into $C^\infty(X)$. Under condition (7), operator (6) can be extended to a continuous map from $E'(Y)$ into $D'(X)$ by

$$\langle A[u], \phi \rangle = \langle u, A[\phi] \rangle,$$

where

$$\chi_{\Phi, m}[\phi](y) = \int_{\mathbb{R}^n} \int_X e^{i\phi(x, y, \xi)} a(x, y, \xi) \phi(x) \, dx \, d\xi.$$

### 3. Stochastic Fourier integral operators

The set of operator phase functions and the space of amplitudes are equipped with a natural metrizable topology, which we now describe. The first task in this section will be to prove the continuity of the maps (1). In order to derive the required inequalities, it will be necessary to replace the conditions on nondegeneracy topology, which we now describe. The first task in this section will be to prove the continuity of the maps (1).

The topology of $M_{hg}(X, Y, \Xi)$ is given by

$$M_{\alpha}(X, Y, \Xi) = \left\{ \Phi \in M_{hg}(X, Y, \Xi) \text{ such that } \forall x \in X, y \in Y, \xi \in \Xi : \right.$$

$$\left\| \left\| \xi \right\|^{-1} \nabla_x \Phi(x, y, \xi) \right\|^2 \geq \alpha \left\| \left\| \xi \right\|^{-1} \nabla_y \Phi(x, y, \xi) \right\|^2 \geq 2 \alpha \right\}.$$

Let $\eta > 0$. The subspace $M_\alpha(X, Y, \Xi)$ is given by

$$M_\alpha(X, Y, \Xi) = \left\{ \Phi \in M_{hg}(X, Y, \Xi) \text{ such that } \forall x \in X, y \in Y, \xi \in \Xi : \right.$$

$$\left\| \left\| \xi \right\|^{-1} \nabla_x \Phi(x, y, \xi) \right\|^2 \geq \alpha \left\| \left\| \xi \right\|^{-1} \nabla_y \Phi(x, y, \xi) \right\|^2 \geq 2 \alpha \right\}.$$
Now let \( \partial B = \{ \xi \in \Xi, \|\xi\| = 1 \} \) be the unit sphere in \( \mathbb{R}^{n_\Xi} \). The space \( C^\infty(X \times Y \times \partial B) \) with its usual topology \( T \) is a separable Fréchet space and thus metrizable and complete [2, Chapter XVII, Section 2]. We have the following result

**Proposition 3.4.** The space \( (M_{h_g}(X,Y,\Xi), (p_m)_{m \in \mathbb{N}}) \) is isomorphic to the space \( (C^\infty(X \times Y \times \partial B), T) \) and hence separable, metrizable, and complete.

**Proof.** The isomorphism is explicitly given by

\[
I : M_{h_g}(X,Y,\Xi) \rightarrow C^\infty(X \times Y \times \partial B) \\
(\mathbf{x}, y, \xi) \mapsto f(\mathbf{x}, y, \xi) \\
\left( (x, y, \xi) \mapsto f(x, y, \frac{\xi}{\|\xi\|}) \right).
\]

The bicontinuity of \( I \) can be most easily seen by employing local spherical coordinates on \( \partial B \).

**Proposition 3.5.** Let \( \alpha > 0 \), then \( M_\alpha(X,Y,\Xi) \) is a closed subset of \( M_{h_g}(X,Y,\Xi) \).

**Proof.** We show that

\[
\mathcal{B}(\phi) := \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^\infty \chi_{[1-j,1]}(\|\mathbf{x}\|) \right\}
\]

continuously maps \( M_{\alpha}(X,Y,\Xi) \) to \( \mathbb{R} \): Fix \( \mathbf{x}, y, \xi \in (X \times Y \times \Xi) \), and choose \( m \geq 1 \) large enough, such that \( x \in K_{X,m} \) and \( y \in K_{Y,m} \). Then, by \( (\mathbf{9}) \)

\[
\left\| \|\xi\|^{-1} \partial_{\xi} \phi(x, y, \xi) \right\| \leq p_m(\phi)
\]

and

\[
|\partial_{\xi} \phi(x, y, \xi)| \leq p_m(\phi).
\]

So in total

\[
\left\| \left( \|\xi\|^{-1} \nabla_x \right) \phi(x, y, \xi) \right\|^2 \leq (n_x + n_\xi)(p_m(\phi))^2.
\]

Since the 2-norm is continuous as well, the limit of a convergent sequence \( (\phi_n)_{n \in \mathbb{N}} \) can be pulled out

\[
\left\| \left( \|\xi\|^{-1} \nabla_x \right) \lim_{n \rightarrow \infty} \phi_n(x, y, \xi) \right\|^2 = \lim_{n \rightarrow \infty} \left\| \left( \|\xi\|^{-1} \nabla_x \right) \phi_n(x, y, \xi) \right\|^2 \geq \alpha,
\]

and the continuity is shown. Since the preimage of a closed set is closed,

\[
\left\{ \phi \in M_{h_g}(X,Y,\Xi) : \left\| \left( \|\xi\|^{-1} \nabla_x \right) \phi(x, y, \xi) \right\|^2 \geq \alpha \right\}
\]

is closed. The proof works analogously for \( \left\| \left( \|\xi\|^{-1} \nabla_y \right) \right\|^2 \), and the intersection of closed sets is closed again.

**Definition 3.6.** The space of amplitude functions on \( X \times Y \times \Xi \) is defined by

\[
S_{\phi,\delta}(X,Y,\Xi) = \left\{ a_{X \times Y \times \Xi} : a \in S^d_{\phi,\delta}(X \times Y \times \mathbb{R}^{n_\Xi}) \right\}.
\]

This space is equipped with the seminorms

\[
q_m(a) = \sup \left\{ \langle \xi \rangle^{-\delta + \delta |l| + \delta |j| + \delta |k|} \left| \partial_{\xi}^l \partial_{\xi}^j \partial_{\xi}^k a(x, y, \xi) \right| : x, y \in K_{X,m}, y \in K_{Y,m}, \xi \in \Xi, |j| + |k| + |l| \leq m \right\}.
\]

Note that by [4, Chapter VII, Section 7.8] the space \( (S^d_{\phi,\delta}(q_m)_{m \in \mathbb{N}}) \) forms a Fréchet space, and thus it is complete and metrizable. Furthermore, one can check that it is separable. Actually, \( S^d_{\phi,\delta}(X,Y,\Xi) \) is isomorphic with \( S^d_{\phi,\delta}(X \times Y \times \mathbb{R}^{n_\Xi}) \).

In any case, both the phase function space and the amplitude function space are closed subsets of separable, metrizable, complete spaces. For any further consideration we will deal with

\[
S^d_{\phi,\delta}(X,Y,\Xi) = M_\alpha(X,Y,\Xi) \times S^d_{\phi,\delta}(X,Y,\Xi),
\]

which is equipped with the product topology of the two spaces, induced by the seminorms \( p_m(\phi, a) := p_m(\phi) + q_m(a) \), \( m \in \mathbb{N} \). We call \( S^d_{\phi,\delta}(X,Y,\Xi) \) the space of FIO operator functions.
Lemma 3.7. Let $\Phi \in \mathcal{M}_\alpha(X, Y, \Xi)$ and let $\alpha$, resp. $\beta$, be the coefficients of the regularizing operator $L$ (see (1)). Then for $m \in \mathbb{N}$ there exists a polynomial $P_m$ such that for any $x \in K_{X,m}, y \in K_{Y,m}$ and $\xi \in \Xi$

$$|\partial_x^i \partial_y^j \partial_{\xi}^l \alpha(x, y, \xi)| \leq P_m(p_m(\Phi)) \langle \xi \rangle^{-|l|}$$

(11)

for $i \in \{1, \ldots, n\}$ and

$$|\partial_x^i \partial_y^j \partial_{\xi}^l \beta(x, y, \xi)| \leq P_m(p_m(\Phi)) \langle \xi \rangle^{-|l|}$$

(12)

for $i \in \{1, \ldots, n\}$ and all $|j| + |k| + |l| + 1 \leq m$.

Proof. For this proof we will show the inequalities only for the zeroth and first derivative of (11). Any higher derivative can be treated the same way.

Recall the notation from Equation (9):

$$r(x, y, \xi) = |||\xi|| \nabla_{\xi} \Phi(x, y, \xi)||^2 + ||\nabla_y \Phi(x, y, \xi)||^2.$$

Since $\Phi \in \mathcal{M}_\alpha(X, Y, \Xi)$ one has that

$$\alpha \langle \xi \rangle^2 \leq r(x, y, \xi),$$

and therefore,

$$|\alpha_i(x, y, \xi)| = \left| \frac{(1 - \chi(\xi)) \langle \xi \rangle^2 \partial_{\xi_i} \Phi(x, y, \xi)}{r(x, y, \xi)} \right|$$

$$\leq \left| \frac{1 - \chi(\xi)}{\alpha} \partial_{\xi_i} \Phi(x, y, \xi) \right|$$

$$\leq C_m^\alpha p_m(\Phi) =: P_m^\alpha(p_m(\Phi))$$

We note that for all multi-indices $j, k$ and $l$ with $|j| + |k| + |l| + 1 \leq m$ there exists a constant $K_m$ such that

$$|\partial_x^i \partial_y^j \partial_{\xi}^l r(x, y, \xi)| \leq K_m \langle \xi \rangle^{2 - |l|} (p_m(\Phi))^2 \leq K_m \langle \xi \rangle^{2 - |l|} (p_m(\Phi))^2,$$

for $\xi \in \Xi, \langle \xi \rangle \geq 1$. Therefore,

$$|\partial_x^i \partial_y^j \partial_{\xi}^l \alpha_i(x, y, \xi)|$$

$$= \left| \partial_y^j \left( \frac{(1 - \chi(\xi)) \langle \xi \rangle^2 \partial_{\xi_i} \Phi(x, y, \xi)}{r(x, y, \xi)} \right) \right|$$

$$\leq \left| (1 - \chi(\xi)) \partial_{y_j} \langle \xi \rangle^2 \partial_{\xi_i} \Phi(x, y, \xi) \right|$$

$$\leq \left| (1 - \chi(\xi)) \langle \xi \rangle^2 \partial_{\xi_i} \Phi(x, y, \xi) \right|$$

$$\leq \left| (1 - \chi(\xi)) \frac{\partial_{\xi_i} \Phi(x, y, \xi) r(x, y, \xi)}{r^2(x, y, \xi)} \right|$$

$$\leq \left| (1 - \chi(\xi)) \frac{\partial_{\xi_i} \Phi(x, y, \xi) r(x, y, \xi)}{\alpha^2 \langle \xi \rangle^2} + \frac{\partial_{y_j} \Phi(x, y, \xi) \partial_{\xi_i} \Phi(x, y, \xi) \partial_{y_j} r(x, y, \xi)}{\alpha^2 \langle \xi \rangle^2} \right|$$

$$\leq C_m^\alpha(p_m(\Phi))^3 =: P_m^\alpha(p_m(\Phi))$$

where the last inequality is due to the fact that $\partial_{\xi_i} \partial_{y_j} \Phi(x, y, \xi)$ and $\partial_{\xi_i} \Phi(x, y, \xi)$ are bounded by a constant times $p_m(\Phi)$ applying (9). Since $(1 - \chi(\xi))$ is nonzero only for $\langle \xi \rangle > 1$, one has no difficulties in the neighborhood of $0$.

Derivation with respect to $\xi_l$ yields

$$|\partial_{\xi_l} \alpha_i(x, y, \xi)| = \left| \partial_{\xi_l} \left( \frac{(1 - \chi(\xi)) \langle \xi \rangle^2 \partial_{\xi_i} \Phi(x, y, \xi)}{r(x, y, \xi)} \right) \right|.$$
which is less or equal to
\[
\leq \left| \partial_{\xi} \chi(\xi) \frac{\|\xi\|^2 \partial_{\xi} \Phi(x, y, \xi)}{r(x, y, \xi)} \right| + \left( 1 - \chi(\xi) \right) \frac{2|\xi| \partial_{\xi} \Phi(x, y, \xi)}{r(x, y, \xi)}
\]
\[
+ \left| (1 - \chi(\xi)) \frac{\|\xi\|^2 \partial_{\xi} \xi \Phi(x, y, \xi)}{\alpha \|\xi\|^2} \right| + \left( 1 - \chi(\xi) \right) \frac{2|\xi| \partial_{\xi} \Phi(x, y, \xi)}{\alpha \|\xi\|^2}
\]
\[
+ \left| (1 - \chi(\xi)) \frac{\|\xi\|^2 \partial_{\xi} \xi \Phi(x, y, \xi)}{\alpha \|\xi\|^2} \right| + \left( 1 - \chi(\xi) \right) \frac{2|\xi| \partial_{\xi} \Phi(x, y, \xi)}{\alpha \|\xi\|^2}
\]
\[
\leq \left| (C_m^2 (\Phi) + C_n^3 (\Phi)^3) \right| \langle \xi \rangle^{-1} =: \mathcal{P}_m^2 (p_m(\Phi)) \langle \xi \rangle^{-1},
\]
where we used that \( \xi / \|\xi\| \leq 1 \) and the same arguments as before.

Having done the estimation for all \( j, k, l \), in the end one can set
\[
\mathcal{P}_m = \sum_i p_m^i,
\]

since all coefficients of \( \mathcal{P}_m^2 \) are nonnegative. To prove (12) one can use the same arguments. \qed

**Definition 3.8.** The seminorms \( (\pi_{X, m})_{m \in \mathbb{N}} \) on \( C^\infty(X) \) are defined by
\[
\pi_{X, m}(v) = \sup \left\{ \left| \partial_x^j v(x) \right| : x \in K_{X, m}, |j| \leq m \right\}, \quad v \in C^\infty(X).
\]

**Proposition 3.9.** Let \( (\Phi_a, \alpha_a)_{a \in \mathbb{N}} \) be a convergent sequence in the product space \( (\mathcal{F}^d_{\alpha_a, p_m(\Phi_a)}(X, \Xi), (p_m(\Phi_a))_{m \in \mathbb{N}}) \) with limit \( (\Phi, a) \). Furthermore, let \( L_n \) be the regularizing operator (cf. Section 2.7) for \( (\Phi_a, \alpha_a) \). Let \( m \in \mathbb{N} \) and \( \psi \in D(Y) \) be fixed. Furthermore, let \( |j| \leq m \). Choose \( \kappa \in \mathbb{N} \) large enough, such that
\[
\left| (e^{a(x, y, \xi)} L_n(a_n(x, y, \xi)) \psi(y)) \right| = O((\xi)^{-\kappa}).
\]

Choose \( m \) large enough such that \( \kappa + m + 1 \leq m \) and \( \text{supp}(\psi) \subset K_{Y, m} \). Then
(a) there exists a constant \( C_{m, p_m(\Phi_a), q_m(\alpha_a), \psi} \), depending on \( m, p_m(\Phi_a), q_m(\alpha_a) \) and \( \psi \), such that
\[
\sup_{x \in K_{X, m}, y \in Y} \left| \partial_x^j \left( e^{a(x, y, \xi)} L_n(a_n(x, y, \xi)) \psi(y) \right) \right| \leq C_{m, p_m(\Phi_a), q_m(\alpha_a), \psi}(\xi)^{-\kappa}.\]

(b) the sequence of FIOs \( A_{\Phi_a, \alpha_a} \) converges in the following sense:
\[
\lim_{n \to \infty} \pi_{X, m}(A_{\Phi, \alpha_a}[\psi] - A_{\Phi, \alpha_a}[\psi]) = 0,
\]
where
\[
A_{\Phi, \alpha_a}[\psi](x) = \int_Y \int_Y e^{a(x, y, \xi)} L_n(a_n(x, y, \xi)) \psi(y) dy \ d\xi,
\]
and
\[
A_{\Phi, \alpha_a}[\psi](x) = \int_Y \int_Y e^{a(x, y, \xi)} L_n(a_n(x, y, \xi)) \psi(y) dy \ d\xi.
\]

**Proof.** (a) We will examine only \( a_n \) and \( L_n(a_n) \). The term \( L_n(a_n) \psi \) can be estimated in the same way, but it is much more tedious.

Since \( \psi \) has compact support, there exists a constant \( C_{\psi, m} \), depending on \( m \) and \( \psi \) such that
\[
\left| \partial_{\xi}^k \psi(y) \right| \leq C_{\psi, m},
\]
for any \( |k| \leq m \). By assumption, \( \text{supp}(\psi) \subset K_{Y, m} \) and therefore
\[
\sup_{x \in K_{X, m}, y \in Y} |a_n(x, y, \xi) \psi(y)| = \sup_{x \in K_{X, m}, y \in K_{Y, m}} |a_n(x, y, \xi) \psi(y)|
\]
\[
\leq C_{\psi, m} q_m(\alpha_n) (\xi)^d
\]
\[
\leq C_{m, p_m(\Phi), q_m(\alpha_n), \psi}(\xi)^d.
\]
Furthermore, 

\[ L_n(a_n(x, y, \xi)\psi(y)) \]

\[ = - \sum_{i=1}^{n_c} \partial_{\xi_i} (\alpha_{aln}(x, y, \xi) a_n(x, y, \xi)\psi(y)) - \sum_{k=1}^{n_c} \partial_{y_k} (\beta_{aln}(x, y, \xi) a_n(x, y, \xi)\psi(y)) \]

\[ + \sum_{n} (x, y, \xi) a_n(x, y, \xi)\psi(y). \]

Using again that \(\text{supp } \psi \subset K_{y,m} \), the first term of (13) can be bounded by

\[ \sup_{x \in K_{X,m}, \ y \in Y} |\partial_{\xi_i} (\alpha_{aln}(x, y, \xi) a_n(x, y, \xi)\psi(y))| \]

\[ \leq \sup_{x \in K_{X,m}, \ y \in K_{Y,m}} |\partial_{\xi_i} (\alpha_{aln}(x, y, \xi) a_n(x, y, \xi)\psi(y))| \]

\[ + \sup_{x \in K_{X,m}, \ y \in K_{Y,m}} |\alpha_{aln}(x, y, \xi) \partial_{\xi_i} a_n(x, y, \xi)\psi(y)| \]

\[ \leq \mathcal{P}_m(p_m(\Phi_n)) (\xi)^{-1} |a_n(x, y, \xi)| C_{\psi,m} \]

\[ + \mathcal{P}_m(p_m(\Phi_n)) (\xi)^{-1} \partial_{y_k} a_n(x, y, \xi) C_{\psi,m} \]

\[ \leq C_{\psi,m} \mathcal{P}_m(p_m(\Phi_n)) q_m(a_n) \left( (\xi)^{d-1} + (\xi)^{d+\bar{\theta}} \right). \]

The second term of (13) can be bounded by

\[ \sup_{x \in K_{X,m}, \ y \in K_{Y,m}} |\partial_{y_k} (\beta_{aln}(x, y, \xi) a_n(x, y, \xi)\psi(y))| \]

\[ \leq \sup_{x \in K_{X,m}, \ y \in K_{Y,m}} |\partial_{y_k} (\beta_{aln}(x, y, \xi) a_n(x, y, \xi)\psi(y))| \]

\[ + \sup_{x \in K_{X,m}, \ y \in K_{Y,m}} |\beta_{aln}(x, y, \xi) \partial_{y_k} a_n(x, y, \xi)\psi(y)| \]

\[ \leq \mathcal{P}_m(p_m(\Phi_n)) (\xi)^{-1} |a_n(x, y, \xi)| C_{\psi,m} \]

\[ + \mathcal{P}_m(p_m(\Phi_n)) (\xi)^{-1} \partial_{y_k} a_n(x, y, \xi) C_{\psi,m} \]

\[ \leq C_{\psi,m} \mathcal{P}_m(p_m(\Phi_n)) q_m(a_n) \left( (\xi)^{d-1} + (\xi)^{d+\bar{\theta}} \right) + (\xi)^{d-1} \].

Since \( \gamma_n(x, y, \xi) \equiv \chi(\xi) \), which is compactly supported, there exists a constant \( \widetilde{C}_m \) such that

\[ \sup_{x \in X, y \in Y, \xi \in \Xi} |\gamma_n(x, y, \xi) a_n(x, y, \xi)\psi(y)| \leq \widetilde{C}_m C_{\psi,m} q_m(a_n) \langle \xi \rangle^{-(n_c+1)}. \]

So for

\[ C_{m,p_m(\Phi_n),q_m(a_n),\psi} = C_{\psi,m} (n_c + n_v + \widetilde{C}_m) \ q_m(a_n) \ \mathcal{P}_m(p_m(\Phi_n)) \land 1 \]

it holds that

\[ \sup_{x \in K_{X,m}} |L_n(a_n(x, y, \xi)\psi(y))| \leq C_{m,p_m(\Phi_n),q_m(a_n),\psi} \langle \xi \rangle^{d-(1-\bar{\theta})\wedge \bar{\omega}}. \]

Iterative application of \( L \) decreases the growth with respect to \( \xi \). For the \( k \)th application of \( L \) we get a constant \( C_{m,p_m(\Phi_n),q_m(a_n),\psi} \) such that

\[ \sup_{x \in K_{X,m}} |L^k_n(a_n(x, y, \xi)\psi(y))| \leq C_{m,p_m(\Phi_n),q_m(a_n),\psi} \langle \xi \rangle^{d-k(1-\bar{\theta})\wedge \bar{\omega}}. \]

If \( k \) is large enough, \( d - k((1 - \bar{\theta}) \wedge \bar{\omega}) \leq -(n_c + 1) \). In the end we set

\[ C_{m,p_m(\Phi_n),q_m(a_n),\psi} = \sum_{i=1}^{\kappa} C_{m,p_m(\Phi_n),q_m(a_n),\psi}. \]
Corollary 3.10. Let \( \psi \in \mathcal{D}(Y) \) fixed. The map
\[
A_{\Phi,a} : \mathcal{F}^d_{\alpha,\varrho,\delta}(X, Y, \Xi) \to C^\infty(X)
\]
\[
(\Phi, a) \mapsto \int_Y \int e^{i\Phi(x,y,\xi)} a(x, y, \xi) \psi(y) \, dy \, d\xi
\]
is continuous with respect to the topologies generated by \((p_m)_{m \in \mathbb{N}}\) and \((\pi_{X, m})_{m \in \mathbb{N}}\).

**Remark 3.11.** Using the same principles of the proof, one can also show that the map
\[
\mathcal{F}^d_{\alpha,\varrho,\delta}(X, Y, \Xi) \times \mathcal{D}(Y) \to C^\infty(X)
\]
\[
(\Phi, a, \psi) \mapsto A_{\Phi,a}[\psi]
\]
is continuous.

**Corollary 3.12.** Let \( u \in \mathcal{E}'(Y) \) be a compactly supported distribution. Then
\[
A_{\Phi,a} : \mathcal{F}^d_{\alpha,\varrho,\delta}(X, Y, \Xi) \to \mathcal{D}'(X)
\]
\[
(\Phi, a) \mapsto A_{\Phi,a}[u]
\]
is measurable again. Furthermore, let
\[ a(x, y, \xi) = 1 \]
and let \( u_0 \in D(X) \) resp. \( u_0 \in E'(X) \). Then, for fixed \( \omega \), the solution to (14) is given by \( A_{\Phi, a}[u_0] \), and the maps
\[
\Omega \to C^\infty(X) \quad \text{resp.} \quad \Omega \to D'(X)
\]
\[
\omega \mapsto A_{\Phi, a}[u_0] \quad \text{resp.} \quad \omega \mapsto A_{\Phi, a}[u_0]
\]
are measurable.

### 4 Applications

In this section we apply Theorem 3.13 in three typical situations.

**Example 4.1.** Let \( X = Y = \mathbb{R}, \Xi = \mathbb{R} \setminus \{0\} \), \( \alpha > 0 \) be a real constant and
\[
C^\infty_\alpha(X) = \{ f \in C^\infty(X) : \forall x \in X : f(x) \geq \alpha \},
\]
and let
\[
c : \Omega \to C^\infty_\alpha(X)
\]
\[
\omega \mapsto (x \mapsto c(\omega, x))
\]
be a measurable map. The transport equation with speed \( c \) is then
\[
\begin{cases}
(\partial_t + c(\omega, x)\partial_x) u(\omega, x, t) = 0 \\
u(\omega, x, 0) = u_0(x).
\end{cases}
\]

**The characteristic curves** satisfy
\[
\begin{cases}
\frac{d}{d\tau} \gamma(\omega, x, t; \tau) = c(\omega, \gamma(\omega, x, t; \tau)) \\
\gamma(\omega, x, t; t) = x.
\end{cases}
\]

Then one can check by classical ODE theory that for fixed \( t \in \mathbb{R}, \omega \in \Omega \) the map
\[
S : C^\infty_\alpha(X) \to C^\infty(X)
\]
\[
(x \mapsto c(\omega, x)) \mapsto (x \mapsto \gamma(\omega, x, x; t))
\]
is continuous in the \( (\pi_X, \bar{\pi}) \) sense. Therefore, for \( t \in \mathbb{R} \) fixed,
\[
\Phi_\omega : \Omega \to M_\alpha(X, Y, \Xi)
\]
\[
\omega \mapsto ((x, y, \xi) \mapsto \xi \cdot (\gamma(\omega, x, t; 0) - y))
\]
is measurable again. Furthermore, let \( a(x, y, \xi) = 1 \) and let \( u_0 \in D(X) \) resp. \( u_0 \in E'(X) \). Then, for fixed \( \omega \), the solution to (14) is given by \( A_{\Phi, a}[u_0] \), and the maps
\[
\Omega \to C^\infty(X) \quad \text{resp.} \quad \Omega \to D'(X)
\]
\[
\omega \mapsto A_{\Phi, a}[u_0] \quad \text{resp.} \quad \omega \mapsto A_{\Phi, a}[u_0]
\]
are measurable.
Example 4.2. Consider as before $X = Y = \mathbb{R}$, $\Xi = \mathbb{R}\setminus \{0\}$. The half wave equation in one dimension is

$$
\begin{align*}
\left(\partial_t + ic(\omega, x)P(D_x)\right) u(\omega, x, t) &= 0 \\
u(\omega, x, 0) &= u_0(x),
\end{align*}
$$

(16)

where $P$ is the pseudodifferential operator with symbol $P(\xi) = |\xi| \left(1 - \chi(\xi)\right)$, where $\chi \in \mathcal{D}(\mathbb{R})$, $\chi(\xi) \equiv 1$ for $|\xi| < 1/4$ and $\chi(\xi) \equiv 0$ for $|\xi| > 1/2$. We assume that $c : \Omega \rightarrow \mathcal{C}_c^\infty(X)$ is measurable and that $c(\omega, x)$ and $|\partial_\xi c(\omega, x)|$ are bounded from above by a constant $R$ for any $x \in \mathbb{R}$ and $\omega \in \Omega$. Further, all seminorms $\pi_{X,m}(c(\omega, \cdot))$ are assumed to be bounded independently of $\omega$.

We fix $\omega \in \Omega$ for now and for notational simplicity we will drop it. A FIO parametrix for (10) can be constructed following [12, Chapter VIII, §3]. The phase function of the parametrix is of the form $\Phi(x, y, \xi) = |\xi| \phi(x, \frac{1}{\omega}(t) - y \cdot \xi)$, where $\phi$ satisfies the eikonal equation,

$$
\begin{align*}
\partial_t \phi(x, t) + c(x)P(\partial_x \phi(x, t)) &= 0 \\
\phi(x, 0) &= x \cdot \xi.
\end{align*}
$$

(17)

This is a nonlinear differential equation of the form $Q(x, t, \phi, \phi_t) = 0$. In [13, Chapter II, Sect 19] one can find an explicit representation of the solution to this problem, which we will write down in the following. Let $F(t; x_1, \xi_1)$ and $G(t; x_1, \xi_1)$ satisfy the following equations

$$
\begin{align*}
\frac{dF}{dt} &= c(F)P'(G) \\
F(0; x_1, \xi_1) &= x_1 \\
\frac{dG}{dt} &= -c'(F)P(G) \\
G(0; x_1, \xi_1) &= \xi_1.
\end{align*}
$$

(18)

There exists a $T \in \mathbb{R}$, such that

$$
P(G(t; x_1, \xi_1)) \equiv |\xi_1| \quad \text{and} \quad P'(G(t; x_1, \xi_1)) \equiv \text{sign } \xi_1
$$

for $0 \leq t \leq T$ and for all $\xi_1, x_1 \in \mathbb{R}$, $|\xi_1| \geq 1$. In that case $F$ does not depend on $G$ and $x = F(t; x_1, \xi_1)$ is just the flow from $(0, x_1)$ to $(t, x)$. The inverse flow is then $F(-t, x, \xi_1)$, which goes from $(t, x)$ to $(0, x_1)$.

Finally, $\phi$ is given by

$$
\phi(x, t, \xi) = x \xi - \int_0^t c(P(G(-s; x; \xi), s, \xi)) \, ds.
$$

(19)

for $t \in [0, T]$. We observe that $F$ depends only on the sign of $\xi_1$ and $\lambda G(t; x_1, \xi_1) = G(t; x_1, \lambda \xi_1)$ for $\lambda \geq 1$, $|\xi_1| > 1$ and all $x \in \mathbb{R}$ and $t \in [0, T]$. Thus,

$$
\phi(x, t, \xi) = |\xi| \phi\left(x, \frac{\xi}{|\xi|}\right)
$$

is compactly supported with respect to $\xi$ for all $x \in \mathbb{R}$ and $t \in [0, T]$. Noting that $\phi(x, 0, \xi) = x \xi$ and by the representation (19) one can also see that for a sufficiently small time interval, there is an $\alpha$ such that $\Phi \in \mathcal{M}_\alpha(X, Y, \Xi)$.

Finally, by classical ODE theory, one can show that $F$ and $G$ continuously depend on $c$ and thus also $\Phi$ (at least for a short time interval).

To obtain the amplitude one has to solve a cascade of transport equations of the form

$$
(\partial_t - c(x)P'(\xi)\partial_x + H(x, t, \xi))a_j(x, t, \xi) = 0, \quad j \leq 0,
$$

where $H$ is a smooth function, depending on derivatives of $c, \phi$ and $a_k$, $k > j$ (for details see again [12, Chapter VIII, §3]). Similarly to the previous example one can show that $a_j$ continuously depends on $c$. In [11] one can find an explicit construction of a function $a$ such that $(a - \sum_{j \leq 0} a_j(x, \xi, t))$ is smoothing, namely

$$
a(x, t, \xi) = \sum_{j \leq 0} (1 - \chi(\xi/n_j))a_j(x, t, \xi),
$$

where $\chi \in \mathcal{D}(\mathbb{R})$, as above, $\chi(\xi) \equiv 1$ for $|\xi| < 1/4$ and $\chi(\xi) \equiv 0$ for $|\xi| > 1/2$, and $n_j \in \mathbb{N}$ approaches $+\infty$ quickly enough as $j$ goes to $-\infty$. Since the $a_j$ depend continuously on $c$ in the $C_\alpha^\infty$-topology, one can chose $n_j$ such that the series converges uniformly together with all derivatives when $c$ is taken from a bounded set in $C_\alpha^\infty(X)$. Thus, $a$ depends continuously on $c$ as well in the corresponding topologies.

Consequently, the phase function $\Phi$ and the amplitude function $a$ of the parametrix $\Lambda_{\Phi, a}$ depend continuously on $c$ in the corresponding topologies.
Recall that $c : \Omega \to C_0^\infty(X)$ is a random function. At fixed $\omega \in \Omega$, one obtains the parametrix $A_{\Phi, a, \omega}$. By Theorem 3.13 if $u_0 \in D(\mathbb{R})$ resp. $u_0 \in \mathcal{E}'(\mathbb{R})$, then the map
\[
\omega \to A_{\Phi, a, \omega}[u_0](x, t)
\]
is measurable.

**Example 4.3.** The standard bottom-up approach to modeling waves in random media would place the randomness in the coefficients of the underlying PDEs. However, the solution depends in a strongly nonlinear way on the coefficients of the equation, even if the PDEs are linear. This makes it hard to track or compute the stochastic features of the solution, such as the expectation, the variance or the autocovariance function. Accordingly, a top-down approach has been proposed in [9, 10]. Starting from the mean field equations as constant coefficient PDEs, the deterministic solution can readily be represented by FIOs. The stochastic properties of the medium are then modeled a posteriori by random perturbations of the phase and amplitude functions.

In applications to material sciences, e.g., damage detection, these random perturbations can be calibrated to measurement data (just as in the bottom-up approach, where the coefficients are calibrated to measurement data). This program has been carried through in [10] in the case of three-dimensional linear elasticity.

The approach leads to stochastic FIOs, whose measurability has to be proven. We present a simplified example using the scalar wave equation.

Let $Y = \mathbb{R}^n$, $X = \mathbb{R}^n \times \mathbb{R}$ and $\Xi = \mathbb{R}^n \setminus \{0\}$. One can solve the deterministic wave equation with constant wave speed $c_0$
\[
\begin{cases}
(\partial_t - c_0 \Delta)u(x,t) = 0 \\
u(x,0) = u_0(x) \\
\partial_t u(x,0) = 0
\end{cases}
\]
using a linear combination of FIOs with phase functions
\[
\Phi(x, t, \xi, y) = (x - y, \xi) \pm c \|\xi\| t
\]
and amplitude function
\[
a(x, y, \xi) \equiv \frac{1}{2}.
\]
Then, instead of using the deterministic $\Phi$ and $a$, randomly perturbed version are introduced. One has to make sure that the randomly perturbed phase function is still an operator phase function. Thus fix $\alpha > 0$ and let
\[
c : \Omega \to C_0^\infty(\mathbb{R}^\nu) \\
\omega \mapsto (x \mapsto c(\omega, x))
\]
be a measurable, smooth random field with $\mathbb{E}(c(x)) = c_0$. Let $\Phi_{\omega}(x, t, y, \xi) = (x - y, \xi) \pm \|\xi\| c(\omega, x)$. Then, $\Phi_{\omega} \in M_{\omega}(X, Y, \Xi)$. Note that this is guaranteed, since we included the time into the image space $X$ and $\|\xi\|^{-1} |\partial_\xi \Phi_{\omega}(x, t, y, \xi)| \geq \alpha$.

Furthermore, let
\[
a : \Omega \to S_{\nu, \delta}(X, Y, \Xi) \\
\omega \mapsto a_{\omega}(x, t, y, \xi)
\]
be random with $\mathbb{E}(a_{\omega}(x, y, \xi)) = \frac{1}{2}$. Then, $(\Phi_{\omega}, a_{\omega}) \in F_{\nu, \delta}(X, Y, \Xi)$ and, as above, $\omega \to A_{\Phi_{\omega}, a_{\omega}, \omega}[u_0]$ is measurable for a given $u_0 \in D(X)$ resp. $u_0 \in \mathcal{E}'(X)$ in the corresponding topology. If $\Phi$ and $a$ are independent, the expectation, the variance and the autocovariance function of $A_{\Phi_{\omega}, a_{\omega}, \omega}[u_0](x)$, resp. $(\Phi_{\omega}, a_{\omega}, \omega)[u_0], \psi)$ can be computed without much difficulty, observing that $\mathbb{E}\left(\exp(i\Phi_{\omega}(x, t, \xi, y))\right)$ is just the characteristic function of the random variable $\Phi_{\omega}(x, t, \xi, y)$, see [5].

**Acknowledgement.** This work was supported by the grant P-27570-N26 of FWF (The Austrian Science Fund).

**References**

[1] P. Boggiatto, E. Buzano, L. Rodino. *Global hypoellipticity and spectral theory.* Akademie Verlag, Berlin, 1996.

[2] J, Dieudonné. *Treatise on Analysis, Volume III.* Academic Press, New York, 1972.
[3] J.-P. Fouque, J. Garnier, G. Papanicolaou, K. Sølna. *Wave propagation and time reversal in randomly layered media*. Springer, New York, 2007.

[4] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer, Berlin Heidelberg, 2003.

[5] L. Lamplmayr, M. Oberguggenberger, M. Schwarz. Stochastic Fourier Integral Operators for Damage Detection. In: M. Voigt, D. Proske, W. Graf, M. Beer, U. Hänfler-Combe, P. Voigt (Eds.), *A Proceedings of the 15th International Probabilistic Workshop & 10th Dresdner Probabilistik Workshop*. TUDpress, Dresden, 2017, 73-84.

[6] M. Mascarello, L. Rodino. *Partial differential equations with multiple characteristics*. Akademie Verlag, Berlin, 1997.

[7] B. Nair, B.S. White. High-Frequency Wave Propagation In Random Media—A Unified Approach. *SIAM J. Appl. Math.*, 51 (1991), no. 2, 374-411.

[8] M. Oberguggenberger, M. Schwarz. Fourier integral operators in stochastic structural analysis. In: F. Werner, M. Huber, T. Lahmer, T. Most, D. Proske (Eds.), *Proceedings of the 12th International Probabilistic Workshop*. Bauhaus-Universitätsverlag, Weimar, 2014, 250-257.

[9] M. Oberguggenberger, M. Schwarz. Stochastic Methods in Damage Detection. In: M. De Angelis (Ed.), *REC2018 - Proceedings of the 8th International Workshop on Reliable Engineering Computing*. University of Liverpool, 2018, 1-11.

[10] M. Schwarz. *Stochastic Fourier Integral Operators and Hyperbolic Differential Equations in Random Media*. PhD thesis, University of Innsbruck, Austria, 2019.

[11] M. A. Shubin. *Pseudodifferential Operator and Spectral Theory*. Springer, Berlin Heidelberg, 1987.

[12] M. E. Taylor. *Pseudodifferential Operators*. Princeton University Press, Princeton, NJ, 1981.

[13] F. Trèves. *Basic Linear Partial Differential Equations*. Academic Press, New York, NY, 1975.