Hölder continuity of the pluricomplex Green function 
and Markov brothers’ inequality

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Abstract. Let $V_E$ be the pluricomplex Green function associated to a compact subset $E$ of $\mathbb{C}^N$. The well known Hölder Continuity Property of $E$ means that there exist constants $B > 0, \gamma \in (0, 1]$ such that $V_E(z) \leq B \text{dist}(z, E)\gamma$. The main result of this paper says that this condition is equivalent to a Vladimir Markov type inequality, i.e. $\|D^\alpha P\|_E \leq M^{\alpha_1}(\deg P)^{\alpha_2}|\alpha_2|^{1-m} \|P\|_E$, where $m, M > 0$ are independent of the polynomial $P$ of $N$ variables. We give some applications of this equivalence and we present its generalization related to a notion of a fit majorant. Moreover, as a consequence of the main result we obtain a criterion for the Hölder Continuity Property in several complex variables of the type of Siciak’s $L$-regularity criterion.

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1. Introduction

Let $E$ be a compact set in $\mathbb{C}^N$. The pluricomplex Green’s function (with pole at infinity) of $E$ can be defined by

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L}_N \text{ and } u \leq 0 \text{ on } E\}, \quad z \in \mathbb{C}^N,$$

where $\mathcal{L}_N$ is the Lelong class of all plurisubharmonic functions in $\mathbb{C}^N$ of logarithmic growth at the infinity, i.e.

$$\mathcal{L}_N := \{u \in PSH(\mathbb{C}^N) : u(z) - \log \|z\|_2 \leq O(1) \text{ as } \|z\|_2 \to \infty\}$$

(for background information, see \cite{14}). Here $\|z\|_2$ stands for the Euclidean norm in $\mathbb{R}^N$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. In the univariate case $V_E$ coincides with the Green’s function

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$g_E$ of the unbounded component of $\hat{\mathbb{C}} \setminus E$ with logarithmic pole at infinity (as usual $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$).

If $V^*_E(z)$ is the standard upper regularization of $V_E$ then it is well known (Siciak’s theorem) that either $V^*_E \in \mathcal{L}_N$ or $V^*_E \equiv +\infty$. It is also equivalent to the fact that $E$ is a non-pluripolar or pluripolar set, respectively. If we define the $L$-capacity of $E$ as $C(E) = \liminf_{||z|| \to \infty} \frac{||z||^2}{\exp V^*_E(z)}$, then $E$ is a pluripolar set if and only if $C(E) = 0$.

A set $E$ is $L$-regular if $\lim_{w \to z} V^*_E(w) = 0$ for every $z \in E$. Siciak has proved that this is equivalent to the continuity of $V_E$ in the whole space $\mathbb{C}^N$. Therefore, $L$-regularity is one of the global properties of $E$ and a crucial role is played here by the continuity of $V_E$ near $E$.

Another global property of the set $E$ that depends only on the behaviour of $V_E$ near $E$ is the H"older continuity property (HCP for short) of the pluricomplex Green’s function $V_E$ (see the result due to B/suppress locki in [24, Prop.3.5] or Prop.2.6 below). By Cauchy’s inequality, one can prove that HCP implies the A.Markov inequality, i.e. there exist constants $m \geq 1$, $M > 0$ such that for every polynomial $P$ of $N$ variables

$$\|\text{grad } P\|_E \leq M (\deg P)^m \|P\|_E. \quad (1)$$

If $E$ admits inequality (1) then it is said to be a Markov set and we write $E \in \text{AMI}(m,M)$. To reveal the importance of this property, we quote the following result due to Plešniak.

**Theorem 1.1** ([20, Th.3.3]). If $E$ is a $C^\infty$ determining compact subset of $\mathbb{R}^N$ then the following statements are equivalent to property (1):

- (Bernstein’s Theorem) If the distance of a function $f : E \to \mathbb{R}$ from the space of polynomials of degree at most $n$ forms a rapidly decreasing sequence (as $n \to \infty$) then $f$ is a restriction to $E$ of a function $\bar{f} \in C^\infty(\mathbb{R}^N)$.

- (ii) The space $(C^\infty(E), \tau_1)$ is complete, where $\tau_1$ is the topology in $C^\infty(E)$ determined by the Jackson’s seminorms.

- (iii) There exists a continuous linear operator $L : (C^\infty(E), \tau_1) \to (C^\infty(\mathbb{R}^N), \tau_0)$ such that $L f|_E = f$ for each $f \in C^\infty(E)$, where $\tau_0$ is the natural topology in $C^\infty(\mathbb{R}^N)$.

- (iv) There exist positive constants $C$ and $\mu$ such that for every polynomial $P$ of degree at most $n$

$$\|P\|_{E_1/\mu} \leq C \|P\|_E,$$

where $E_r = \{z \in \mathbb{C}^N : \text{dist } (z, E) \leq r\}$.

An exciting question is whether there exists a relationship between the A.Markov inequality and the behaviour of the Green’s function near the considered set. It is known (see [7]) that every Markov set $E \subset \mathbb{C}$ is not polar and $E$ is L-regular if $E \subset \mathbb{R}$ ([9]). It seems that A.Markov inequality (1) implies Hölder continuity property but a proof is an open problem mentioned e.g. in [20]. Actually, even the question about L-regularity of Markov sets in the general case remains still open.

We shall make an attempt in the direction of solving this problem by concentrating on a generalization of an inequality proved by A.Markov’s younger brother, V.Markov. He discovered in 1892, after a very detailed investigation, a precise but intricate estimate for the $k$-th derivative of polynomials (see e.g. [22]): for any polynomial $P$ of
degree not greater than \( n \)

\[
\|P^{(k)}\|_{[-1,1]} \leq T_n^{(k)}(1) \|P\|_{[-1,1]} = \frac{n^2[n^2 - 1][n^2 - (k - 1)^2]}{1 \cdot 3 \cdot \ldots \cdot (2k - 1)} \|P\|_{[-1,1]} \quad (2)
\]

where \( T_n(x) = \cos(n \arccos x) \) is the \( n \)-th Chebyshev polynomial (for \( k = 1 \) it was proved by A. Markov in 1889).

Inequality (2) inspired us to consider a new type of Markov inequality (see Def. 2.8 below). It turns out that this inequality is equivalent to Hölder continuity property of the pluricomplex Green’s function. This is the main result of the paper (Th. 2.12).

Although the definition of HCP is simple, its verification for particular sets can be very complicated (see e.g. [1] [21]). The Carleson-Totik criterion (see [12, Th. 1.2, Th. 1.7]) merits mentioning here. It gives an equivalent condition for HCP expressed in terms of capacities in a similar way to Wiener’s criterion for L-regularity. This criterion can be used for proving HCP for a large family of sets. However, the Carleson-Totik criterion holds only in the univariate complex case (or in \( \mathbb{R}^N \)) and the equivalence is valid under certain additional assumption on sets e.g. for sets satisfying an exterior cone condition. In this context, Th. 2.12 of this paper provides a useful tool for showing HCP especially when sets do not satisfy the assumptions of the criterion mentioned above. We give some examples of such an application of Th. 2.12. Moreover, we prove a rather surprising fact that it is sufficient to verify the Hölder continuity property of \( V_E \) only in \( N \) canonical directions (Cor. 2.13). This allows us to show HCP for a large class of sets.

The paper is organized as follows. A statement of the main results is presented in Section 2. The next section contains proofs of these results. In Section 4 we give a generalization of Th. 2.12 related to a notion of a fit majorant. The next section deals with compact subsets of \( \mathbb{R}^N \subset \mathbb{R}^N + i \mathbb{R}^N = \mathbb{C}^N \), especially with convex bodies and also with UPC sets, i.e. uniformly polynomially cuspidal sets. In the last section we show some applications of Th. 2.12 for disconnected compact sets.

2. Notations and statement of the main results

The pluricomplex Green’s function is closely related to polynomials (see [23] or [14, Th. 5.1.7]) in view of the formulas

\[
V_E(z) = \log \Phi_E(z), \quad z \in \mathbb{C}^N;
\]

where \( \Phi_E \) is the Siciak extremal function, i.e.

\[
\Phi_E(z) = \sup \left\{ \|P(z)^{1/n}/\|P\|_E^{1/n} : P \in \mathcal{P}(\mathbb{C}^N), \deg P = n \geq 1, P|_E \not\equiv 0 \right\},
\]

\( \mathcal{P}(\mathbb{K}^N) \) denotes the vector space of polynomials of \( N \) variables with coefficients in \( \mathbb{K} \in \{ \mathbb{C}, \mathbb{R} \} \) and \( \| \cdot \|_E \) is the maximum norm on \( E \).

In order to investigate the behaviour of \( V_E \) near \( E \), we define

\[
V_E^*(z) := \sup \{ V_E(x - w) : x \in E, \|w\|_2 \leq \|z\|_2 \}, \quad z \in \mathbb{C}^N,
\]

that is, a radial modification of \( V_E \). The definition and main properties of \( V_E^* \) were presented by M. Baran, L. Bialas-Ciez, Comparison principles for compact sets in \( \mathbb{C}^N \) with HCP and Markov properties during the Conference on Several Complex Variables
on the occasion of Professor’s Józef Siciak’s 80th birthday, Kraków, 4-8 July 2011. We set out (without proofs) the following examples:

- if $E$ is a unit ball in $\mathbb{C}^N$ (with respect to a fixed complex norm) then $V_E^\star(z) = \log(1 + ||z||_2/C(E))$,
- if $E$ is a convex symmetric body in $\mathbb{R}^N$ then $V_E^\star(z) = \log h(1 + ||z||_2/(2C(E)))$, where $h(t) = t + \sqrt{t^2 - 1}$ for $t \geq 1$.
- if $E$ is a polar set then $V_E^\star(0) = 0$, $V_E^\star|_{\mathbb{C}^N \setminus \{0\}} \equiv +\infty$.

For the non-polar sets we can obtain a very important fact which is derived from Prop.1.4 in [15] (cf. [5, Th.2.1c]):

**Proposition 2.1.** If $E$ is a non-pluripolar compact subset of $\mathbb{C}^N$ and

$$\rho_E(r) := V_E^\star(z) \quad \text{for} \quad ||z||_2 = r,$$

then $t \mapsto \rho_E(e^t)$ is an increasing convex function.

**Remark 2.2.** The function $\rho_E$ has the following basic properties:

a) $\rho_{\lambda E}(r) = \rho_E(\lambda^{-1}r)$, $\lambda > 0$,

b) $\rho_{E \times F}(r) = \max(\rho_E(r), \rho_F(r))$,

c) $\lim_{r \to \infty} (\rho_E(r) - \log r) = -\log C(E)$,

d) $\rho_E$ is increasing, continuous on $(0, +\infty)$ and consequently, $0 = \rho_E(0) \leq \lim_{r \to 0^+} \rho_E(r)$.

Therefore, L-regularity is equivalent to the equality $\lim_{r \to 0^+} \rho_E(r) = 0$.

Indeed, equality a) can be checked by a standard verification. Formula b) is a consequence of the well-known product property of the pluripotential Green function. A behavior of $\rho_E$ for $r$ near the infinity is related to the L-capacity of $E$: like in the proof of Th.2.3 in [5] we can show equality c). Statement d) is deduced directly from Prop.2.1.

On can easily check that $\rho_{[-1,1]}(r) = \log h(1 + r)$. In general, it is rather difficult to calculate the exact values of $\rho_E$. However, for some investigations of the behavior of $\rho_E$ near 0, it will be profitable to find a simple majorant sufficiently close to $\rho_E$.

**Definition 2.3.** A function $\rho: (0, 1] \to (0, +\infty)$ satisfies conditions of a fit majorant if

1) $t \mapsto \rho(e^t)$ is an increasing, $C^1((-\infty, 0])$ strictly convex function,

2) $\rho'(1) \geq 1$ and $\lim_{r \to 0^+} \rho(r) = 0$.

Note that for every function $\rho$ satisfying the above conditions we have $\lim_{t \to -\infty} \psi(t) = 0$ and $\psi^{-1}((0, \rho'(1))] = (-\infty, 0]$ where $\psi(t) := (\rho(e^t))'$.

**Example 2.4.** The following functions satisfy conditions of a fit majorant:

i) $\rho(r) = Ar^\gamma$ for $\gamma \in (0, 1]$ and $A \geq 1/\gamma$,

ii) $\rho(r) = \frac{1}{s} (1/\log(e/r))^s$ for $s > 0$,

iii) $\rho(r) = Ar^\sigma (\log(1/r) + \frac{2}{\sigma})$ for $\sigma \in (0, 1]$ and $A \geq 1$.

We are interested in seeing how the Hölder continuity of the pluricomplex Green’s function $V_E$ is connected with Markov-type inequalities for polynomials on $E$. The question will be made more precise by the next definition:

**Definition 2.5.** Let $\gamma \in (0, 1]$, $B > 0$. A compact set $E \subset \mathbb{C}^N$ admits the Hölder continuity property of the pluricomplex Green’s function $V_E$ ($E \in HCP(\gamma, B)$ in short)
if for every \( z \in \mathbb{C}^N \)
\[
V_E(z) \leq B \text{ dist}(z, E)^\gamma. \tag{4}
\]

It seems appropriate to mention here five equivalents for this property.

**Proposition 2.6.** If \( E \) is a compact subset of \( \mathbb{C}^N \) and \( \gamma \in (0, 1] \) then the following statements are equivalent:

(i) \( \exists B_1 \geq 1 \) \( E \in HCP(\gamma, B_1) \),

(ii) \( \exists B_2 \geq 1 \) \( \rho_E(r) \leq B_2 r^\gamma \) for \( r \geq 0 \),

(iii) \( \exists B_3 \geq 1 \) \( |\rho_E(r) - \rho_E(s)| \leq B_3 |r - s|^\gamma \) for \( r, s \geq 0 \),

(iv) \( \exists B_4 \geq 1 \) \( \Phi_E(z) \leq 1 + B_4 \text{ dist}(z, E)^\gamma \) for \( z \in \mathbb{C}^N \), \( \text{dist}(z, E) \leq 1 \),

(v) \( \exists B_5 \geq 1 \) \( |V_E(z) - V_E(w)| \leq B_5 \|z - w\|^\gamma \) for \( z, w \in \mathbb{C}^N \),

(vi) \( \forall R > 0 \exists B_6 \geq 1 \) \( \Phi_E(z) - \Phi_E(w) \| \leq B_6 \|z - w\|^\gamma \) for \( z, w \in E_R := \{z \in \mathbb{C}^N : \text{dist}(z, E) \leq R\} \).

Moreover, in the equivalences (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (v) we have \( B_1 = B_2 = B_3 = B_5 \).

If \( E \in HCP(\gamma, B) \) then \( E \) is \( L \)-regular and therefore \( C(E) > 0 \). However, a lower bound for \( C(E) \) in terms of the constants \( \gamma, B \) was not known. In this paper we simply solve this problem (see Th.2.12).

A close inspection of the proof of [16, Th.3.5] and use of Stirling’s approximation lead us to

**Proposition 2.7.** If \( E \subset \mathbb{C} \) and there exists \( M_k = M_k(E) \) independent of \( n \) and a polynomial \( P \) of degree at most \( n \) such that
\[
\|P^{(k)}\|_E \leq M_k n^{mk} \|P\|_E \tag{5}
\]
then \( M_k \geq B^k/[(k!)^{m-1}] \) for certain absolute constant \( B > 0 \).

This fact was the motivation for concentrating on the following generalization of the V.Markov inequality.

Let \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

**Definition 2.8.** Fix \( m \geq 1 \), \( M > 0 \). A compact set \( E \subset \mathbb{C}^N \) admits the V.Markov inequality (\( E \in VMI(m, M) \) in short) if for every \( \alpha \in \mathbb{N}_0^N \), \( P \in \mathcal{P}(\mathbb{C}^N) \)
\[
\|D^\alpha P\|_E \leq M^{(\alpha)} \frac{(\deg P)^{m|\alpha|}}{(|\alpha|!)^{m-1}} \|P\|_E \tag{6}
\]
where \( |\alpha| = \alpha_1 + \ldots + \alpha_N \), \( D^\alpha = \frac{\partial^{(|\alpha|)}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \) for \( \alpha = (\alpha_1, \ldots, \alpha_N) \).

In other words, (6) is a version of inequality (5) (and also its analogue in higher dimensional space) with the strongest possible constants \( M_k \) (compare with [6]).

**Example 2.9.** The simplest example of a set admitting V.Markov inequality is the unit disc \( \overline{D} \) in the complex plane. By Bernstein inequality, for every polynomial \( P \) of degree not greater than \( n \) we have \( \|P^{(k)}\|_{\overline{D}} \leq \frac{n!}{(n-k)!} \|P\|_{\overline{D}} \) and thus \( \overline{D} \in VMI(1, 1) \). In the multidimensional space for a polydisc \( D(a, r) = \{z \in \mathbb{C}^N : |z_1 - a_1| \leq r_1, \ldots, |z_N - a_N| \leq r_N\} \) of polyradius \( r = (r_1, \ldots, r_N) \in (0, +\infty)^N \) we get (see [8, Example 2.2])
\[
\|D^\alpha P\|_{D(a, r)} \leq \frac{\nu!}{(\nu - \alpha)!} r^\alpha \|P\|_{D(a, r)}
\]
whenever $P$ is a polynomial of $N$ variables $z_1, \ldots, z_N$ of degree at most $\nu_1$ in $z_1, \ldots, \nu_N$ in $z_N$. As usual, $\nu_1 = \nu_1! \ldots \nu_N!$ and $r^\alpha = r_1^{\alpha_1} \ldots r_N^{\alpha_N}$. Hence

$$\|D^\alpha P\|_{D(a,r)} \leq r^{-\alpha} \nu_1^{\alpha_1} \ldots \nu_N^{\alpha_N} \|P\|_{D(a,r)} \leq r^{-\alpha} (\deg P)^{\alpha} \|P\|_{D(a,r)}$$

for any polynomial $P \in \mathcal{P}(\mathbb{C}^N)$. Therefore $D(a, r) \in VMI(1, \max_1/r_j)$.

**Example 2.10.** Due to the classical inequality proved by V.Markov (see (2)), we have $B, M \gg 0$ whenever $P$ is a polynomial of $N$ variables $z_1, \ldots, z_N$. As usual, $\nu_1 = \nu_1! \ldots \nu_N!$ and $r^\alpha = r_1^{\alpha_1} \ldots r_N^{\alpha_N}$. Hence

$$\|D^\alpha P\|_E \leq \frac{2^{\alpha}}{(b-a)^\alpha} T^{(\alpha)}(1) \ldots T^{(\alpha)}(1) \|P\|_E \leq \frac{2^{\alpha}}{(b-a)^\alpha} \cdot \frac{\nu^{2\alpha}}{\alpha!} \|P\|_E$$

with $a = (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N)$, $\nu = (\nu_1, \ldots, \nu_N)$. Since $N^{|\alpha|} \alpha! \geq |\alpha|!$, we obtain $E \in VMI\left(2, 2N \max_1/(b_j - a_j)\right)$.

It is evident that $VMI(m, M) \Rightarrow AMI(m, M \sqrt{N})$. On the other hand, property (1) easily implies that

$$\|D^\alpha P\|_E \leq M|\alpha| \left(\frac{n!}{(n - |\alpha|)!}\right)^m \|P\|_E \leq M|\alpha| n^{m|\alpha|} \|P\|_E$$

for any $\alpha \in \mathbb{N}_0^N$, $P \in \mathcal{P}(\mathbb{C}^N)$ of degree at most $n$.

**Remark 2.11.** If $E \in AMI(m_1, M_1)$ and if we fix an arbitrary $\delta \in (0, 1)$ then for every polynomial $P$ of degree at most $n$ and for all $|\alpha| \leq n^\delta$, inequality (1) holds with $m = \frac{m_1 - \delta}{1 - \delta}$ and $M = M_1$. Indeed,

$$\|D^\alpha P\|_E \leq (M_1 n^{m_1})^{\alpha} \|P\|_E$$

and

$$n^{m_1|\alpha|} = \left(\frac{n^{m_1}|\alpha|^{m_1 - 1}}{|\alpha|^{m_1 - 1}}\right)^{|\alpha|} \leq \frac{(n^{m_1 + (m_1 - 1)\delta})^{|\alpha|}}{|\alpha|^{(m_1 - 1)|\alpha|}} \leq \frac{n^{m|\alpha|}}{|\alpha|^{(m_1 - 1)|\alpha|}}.$$

By the above, in the particular case of $m_1 = 1$, we get $AMI(1, M_1) \Leftrightarrow VMI(1, M)$.

In the general case, we do not know whether or not the V.Markov inequality is equivalent to that of A.Markov. However, we can show that the Hölder continuity property is equivalent to (6).

**Theorem 2.12 (Main theorem).** If $E$ is a compact subset of $\mathbb{C}^N$, $0 < \gamma \leq 1 \leq m$, $B, M > 0$ then

$$E \in HCP(\gamma, B) \implies E \in VMI(m, M) \text{ with } m = 1/\gamma, \ M = \sqrt{N}(B\gamma e)^{1/\gamma}$$

$$E \in VMI(m, M) \implies E \in HCP(\gamma, B) \text{ with } \gamma = 1/m, \ B = M^{\gamma} N^{\gamma m}.$$  

Moreover, if $E \in VMI(m, M)$, then $C(E) \geq e^{-m - 1}/NM$. Hence, if $E \in HCP(\gamma, B)$, then $C(E) \geq \left(N^{3/2}(B\gamma e^2)^{1/\gamma}\right)^{-1}$.

As a consequence of the above theorem, the well known open problem concerning the conjectured implication $AMI \Rightarrow HCP$ is equivalent to a new question of whether
AMI implies VMI. The first problem regards the properties related to the notions in two different fields: the pluricomplex Green’s function and polynomials, whereas the new question is formulated only in terms of derivatives of polynomials.

Due to the above theorem, we can give new, somewhat unexpected equivalents to the Hölder continuity property of the pluricomplex Green’s function:

**Corollary 2.13.** If $E$ is a compact subset of $\mathbb{C}^N$ and $\gamma \in (0, 1]$ then the following conditions are equivalent:

(i) $E \in HCP(\gamma, B_1)$ with some $B_1 \geq 1$,

(ii) $\exists B_2 > 0 \ \forall z_0 \in E \ \forall j \in \{1, \ldots, N\} \ \forall \zeta \in \mathbb{C}$ such that $|\zeta| \leq 1$ we have

$$V_E(z_0 + \zeta e_j) \leq B_2 |\zeta|^\gamma,$$

(iii) $\exists M_3 > 0 \ \forall j \in \{1, \ldots, N\} \ \forall P \in \mathcal{P}(\mathbb{C}^N) \ \forall k \in \mathbb{N}$ we have

$$\|D^{ke_j}P\|_E \leq M_3^k \frac{(\deg P)^{k/\gamma}}{k!^{1/\gamma - 1}} \|P\|_E,$$

where $e_1, \ldots, e_N$ are the canonical vectors in $\mathbb{C}^N$: $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the value 1 in the $j$th entry.

It seems to be rather surprising that condition (ii) in Cor.2.13 that holds only in $N$ canonical directions, is sufficient to guarantee the Hölder continuity property of $V_E$ in all directions.

We can generalize the main theorem to the case where $\rho_E(r)$ and $k$th derivatives of polynomials have bounds related to fit majorants with some additional properties (Th.4.2). We show that the required properties are satisfied for functions given in Example 2.4 i and ii (Th.4.4, 4.5).

For the compact subsets of $\mathbb{R}^N \subset \mathbb{R}^N + i\mathbb{R}^N = \mathbb{C}^N$ we prove that if inequality (11) holds for $x \in \mathbb{R}^N$ then so is for all $z \in \mathbb{C}^N$ (Cor.5.4). As a consequence, we obtain that $E \in HCP\left(1/2, B/\sqrt{C(E)}\right)$ for any convex body in $\mathbb{R}^N$, where $C(E)$ is the L-capacity of $E$ and $B$ is an absolute constant independent of $E$ and even of $N$ (Example 5.7). Moreover, we prove that every set $E \subset \mathbb{R}^N$ uniformly polynomially cuspidal in direction $v$ with exponent $s$, has the following property: $V_E(x + \zeta v) \leq B|\zeta|^{1/(2s)}$ for $x \in E$, $|\zeta| \leq r_0$ (Th.5.10). Hence we deduce from Cor.2.13 that every UPC compact subset of $\mathbb{R}^N$ admits HCP and thus V.Markov inequality (Cor.5.11). In this way, we obtain a wide class of sets that have such a property. This is the first essential generalization of V.Markov’s result from the end of XIX century.

As another application of the main theorem, we can prove HCP for disconnected sets. Prop.6.1 regards some onion type sets in the complex plane that may not satisfy the assumptions of the Carleson-Totik criterion. These sets are particularly interesting in view of certain properties of compacts admitting so-called local Markov’s inequality (see L.Bialas-Ciez and R.Eggink, *Equivalence of the global and local Markov inequalities in the complex plane*, in preparation). The second example of such an application of Th.2.12 concerns some compact sets consisting of infinitely many pairwise disjoint subsets of $\mathbb{C}^N$ (Prop.6.2).
3. Proofs of the main results

To prove Prop.2.6 we need the following

Lemma 3.1. If \( E \subset \mathbb{C}^N \) is a compact L-regular set then

\[
\begin{align*}
\text{a)} & \quad |V_E(z) - V_E(w)| \leq \rho_E(\|z - w\|_2) \quad \text{for } z, w \in \mathbb{C}^N; \\
\text{b)} & \quad |\rho_E(r) - \rho_E(s)| \leq \rho_E(\|r - s\|) \quad \text{for } r, s \geq 0.
\end{align*}
\]

Proof of Lemma 3.1. Modifying an argument due to Blocki (see [24, Prop. 3.5]), consider the function \( u_\xi(z) := V_E(z + \xi) - \rho_E(\|\xi\|_2) \) for \( z \in \mathbb{C}^N \). We have \( u_\xi \in \mathcal{L}_N \) for any fixed \( \xi \in \mathbb{C}^N \). Moreover,

\[
V_E(z + \xi) \leq \|V_E\|_{\text{dist}(z+\xi, E)} \leq \|V_E\|_{E[\xi]} = \rho_E(\|\xi\|_2)
\]

whenever \( z \in E \). Hence \( u_\xi \leq 0 \) on \( E \) and, by the definition of the pluricomplex Green’s function, \( V_E(z) \geq u_\xi(z) \) for all \( z \in \mathbb{C}^N \) and we can easily obtain statement a).

To prove b), fix \( r \geq s \geq 0 \). We can take \( w \in E_r \) such that \( V_E(w) = \rho_E(r) \). Choose \( z \in E_s \) at distance \( r - s \) from the point \( w \). By inequality \((10)\), we have

\[
0 \leq \rho_E(r) - \rho_E(s) \leq V_E(w) - V_E(z) \leq \rho_E(\|w - z\|_2) = \rho_E(r - s)
\]

and we get property b).

Proof of Proposition 2.6. The implication \((i) \Rightarrow (ii)\) is an easy consequence of the definition of \( \rho_E \).

Lemma 3.1 immediately implies \((iii)\) whenever we assume \((ii)\) and we take into account Remark 2.2.

If we consider \( s = 0 \) and \( r = \text{dist}(z, E) \) for a fixed \( z \in \mathbb{C}^N \), we obtain \((i)\) from \((iii)\).

The equivalence \((i) \iff (iv)\) is an easy consequence of \((3)\) and the elementary inequalities: \(1 + x \leq e^x \) for \( x \geq 0 \) and \( e^b t \leq 1 + (e^b - 1) t \) for \( t \in [0, 1], b > 0 \).

By Blocki’s argument mentioned in the proof of Lemma 3.1, property \((i)\) implies the Hölder continuity of the pluricomplex Green’s function \( V_E \) in the whole space, i.e. condition \((v)\).

To prove \((v) \Rightarrow (vi)\), it is sufficient to apply formulas \((3)\) and the fact that \( V_E \) is lower semicontinuous. Indeed, for \( z, w \in E_R \)

\[
|\Phi_E(z) - \Phi_E(w)| = |\exp(V_E(z)) - \exp(V_E(w))| \\
\leq |V_E(z) - V_E(w)| \exp(\max\{V_E(z), V_E(w)\}) \leq B_5 \|z - w\|^\gamma \exp(\|V_E\|_{E_R}).
\]

The evident implication \((vi) \Rightarrow (iv)\) finishes the proof.

Proof of Theorem 2.12. To show the first implication, consider an arbitrary polynomial \( P \in \mathcal{P}(\mathbb{C}^N) \) of degree at most \( n \) and \( \alpha \in \mathbb{N}_0^N \). By Cauchy’s integral formula and the Bernstein-Walsh-Siciak inequality, for fixed \( z=(z_1, \ldots, z_N) \in E, r \in (0, 1] \) we can obtain

\[
|D^\alpha P(z)| \leq \frac{\alpha!}{(r/\sqrt{N})^\alpha} \|P\|_{D(z, r/\sqrt{N})} \leq \sqrt{N}^{|\alpha|} \frac{\alpha!}{r^{|\alpha|}} \|P\|_E \exp\left(n\|V_E\|_{D(z, r/\sqrt{N})}\right)
\]
where $D(z,r/\sqrt{N}) = \{ w \in \mathbb{C}^N : |w_1 - z_1| \leq r/\sqrt{N}, \ldots, |w_N - z_N| \leq r/\sqrt{N} \}$. From (4) we have

$$|D^\alpha P(z)| \leq \frac{\sqrt{N}^{|\alpha|}\alpha!}{r^{|\alpha|}} \|P\|_E \exp(n Br^\gamma)$$

and for $r = (|\alpha|/(B\gamma n))^m$, $m = 1/\gamma$ we get

$$|D^\alpha P(z)| \leq \frac{\sqrt{N}^{|\alpha|}\alpha!}{|\alpha||\alpha|m} n^{m|\alpha|} \|P\|_E (B\gamma e)^{|\alpha|/\gamma} \leq \left( \sqrt{N}(B\gamma e)^{1/\gamma} \right)^{|\alpha|} \frac{|\alpha|!}{(|\alpha|!)^m n^{m|\alpha|}} \|P\|_E,$$

and (8) is proved.

We now proceed to show implication (9). For this purpose, observe that from (3), it is sufficient to prove

$$|P(z)| \leq \|P\|_E \exp(M^\gamma N^\gamma mn^r \gamma)$$

for any polynomial $P \in \mathcal{P}(\mathbb{C}^N)$ of degree at most $n$ and $z \in \mathbb{C}^N \setminus E$ such that $\text{dist}(z, E) = r$. By Taylor’s formula, we have

$$|P(z)| \leq \sum_{|\alpha| \leq n} \frac{1}{\alpha!} |D^\alpha P(w)| r^{\alpha} \leq \sum_{k=0}^n \sum_{|\alpha| = k} \frac{1}{\alpha!} r^{\alpha} \|D^\alpha P\|_E$$

whenever $w \in E$ and $\text{dist}(z, E) = \|z - w\|_2$. From (6) the above inequality gives

$$|P(z)| \leq \|P\|_E \sum_{k=0}^n M^k N^k \frac{n^{mk}}{(k!)^m} r^k \sum_{|\alpha| = k} \frac{1}{\alpha!}.$$  

Since $\sum_{|\alpha| = k} 1/\alpha! = N^k/k!$, for $\gamma = 1/m$ we have

$$|P(z)| \leq \|P\|_E \sum_{k=0}^n M^k N^k \frac{n^{mk}}{(k!)^m} r^k \leq \|P\|_E \sum_{k=0}^\infty \left[ \frac{(M^\gamma N^\gamma r^\gamma n)^k}{k!} \right]^m = \|P\|_E \mathcal{G}_m(M^\gamma N^\gamma r^\gamma n)$$

where

$$\mathcal{G}_m(x) := \sum_{k=0}^\infty \left( \frac{x^k}{k!} \right)^m = \left\| \left( \frac{x^k}{k!} \right)_{k \in \mathbb{N}_0} \right\|^m_{l_m}$$

and $\| \cdot \|_m$ is the usual norm in the space $l_m$. As $\| \cdot \|_m \leq \| \cdot \|_1$, we have

$$\mathcal{G}_m(x) = \left\| \left( \frac{x^k}{k!} \right)_{k \in \mathbb{N}_0} \right\|^m_{l_m} \leq \left\| \left( \frac{x^k}{k!} \right)_{k \in \mathbb{N}_0} \right\|^m_{l_1} = e^{xm}$$

and

$$|P(z)| \leq \|P\|_E \exp(M^\gamma N^\gamma r^\gamma mn^r \gamma),$$

which gives (11).

If $w \in E$ and $\zeta \in \mathbb{C}^N$, $\|\zeta\|_2 = r$, then in a similar way as above we get

$$|P(w + \zeta)| \leq \|P\|_E \sum_{k=0}^n N^k M^k \frac{n^{mk}}{(k!)^m} r^k.$$
In the case of $NM r \geq 1$ we obtain
\[ |P(w + \zeta)| \leq \|P\|_E (NM r)^n \left( \sum_{k=0}^{n} \frac{n^k}{k!} \right)^m \leq \|P\|_E (NM r)^n e^{nm}. \]

Thus $\rho_E(r) \leq \log(N M e^n) + \log r$ for $r \geq e^{-m} (\frac{1}{NM})$ and consequently (see Remark 2.2),
\[ - \log C(E) = \lim_{r \to \infty} (\rho_E(r) - \log r) \leq \log(N M e^n) \]
and the proof is completed. \( \square \)

**Proof of Corollary 2.13.** First, we prove (ii) \( \Rightarrow \) (iii). Put $p_j(\zeta) = P(z_0 + \zeta e_j)$ for $\zeta \in \mathbb{C}$, $z_0 \in E$ and for a fixed polynomial $P \in \mathcal{P}(\mathbb{C}^N)$ of degree at most $n$. Obviously, $|D^{ke_j}P(z_0)| = |p_j^{(k)}(0)|$ and by Cauchy’s integral formula,
\[ |D^{ke_j}P(z_0)| \leq \left( \frac{k!}{r^k} \right) \max \{|p_j(\zeta)| : |\zeta| = r\} \leq \frac{k!}{r^k} \|P\|_E \max \{|\exp(nV_E(z_0 + \zeta e_j))| : |\zeta| = r\} \]
the last inequality being a consequence of (3). If $r = (k/n)^{1/\gamma}$ then from (ii) we get
\[ \|D^{ke_j}P\|_E \leq k! \left( \frac{n}{k} \right)^{\frac{k}{\gamma}} e^{B_2 k} \|P\|_E \leq M_3 k^{\frac{n^{k/\gamma}}{k^{1/\gamma-1}}} \|P\|_E \]
with some positive constant $M_3$, and (iii) is proved.

In view of Th.2.12, to show (iii) \( \Rightarrow \) (i) it is sufficient to prove that (iii) implies inequality (3). Fix $N^N \ni \alpha = \alpha_1 e_{1} + \ldots + \alpha_N e_{N}$. If $P$ is a polynomial of degree $n_j$ in $z_j$ where $z = (z_1, \ldots, z_N)$, we have
\[ \|D^\alpha P\|_E \leq M_3^{\alpha_1} \frac{n_1^{\alpha_1/\gamma}}{\alpha_1!^{1/\gamma-1}} \|D^{\alpha - \alpha_1 e_{1}} P\|_E \leq \ldots \]
\[ \leq M_3^{[\alpha]} \frac{n_1^{\alpha_1/\gamma} \ldots n_N^{\alpha_N/\gamma}}{\alpha!^{1/\gamma-1}} \|P\|_E \leq M_3^{[\alpha]} N^{[\alpha]} \frac{(\deg P)^{[\alpha]/\gamma}}{|\alpha!|^{1/\gamma-1}} \|P\|_E \]
since $N^{[\alpha]} \alpha! \geq |\alpha|!!$, and (i) follows.

The implication (i) \( \Rightarrow \) (ii) is obvious and the proof is completed. \( \square \)

**Remark 3.2.** It follows from the proof of Cor.2.13 that we can replace condition $V_E(x + \zeta e_j) \leq C_2 |\zeta|$ for $|\zeta| \leq 1$ by the same but only for $|\zeta| \leq r_0 \leq 1$. Indeed, in the proof of implication (ii) \( \Rightarrow \) (iii) it is sufficient to put $r = r_0(k/n)^{1/\gamma}$. We shall use this remark later.

4. Majorants of $V_E$ and a bound of $k$th derivative of polynomials

In this section we shall present a generalization of properties related to HCP. To do this, we need the following basic definition.

**Definition 4.1.** Let $\rho$ satisfy conditions of a fit majorant.

a) We say that $\rho$ is $m$-bounded if for all $c \in (0, 1]$ and $r \in [0, 1]$
\[ \limsup_{n \to \infty} \log \left( 1 + \sum_{k=1}^{n} \exp \left( -k \psi^{-1} (ck/n) + n \rho \left( \exp \psi^{-1} (ck/n) \right) \right) r^k \right)^{1/n} \leq \frac{A}{c} \rho(r), \]
(12)
where $\psi(t) = (\rho(e^t))'$ and $A$ is a constant independent of $c$ and $r$.

b) We say that $\rho$ is \textit{mb-bounded} if for all $s \in \mathbb{N}$ there exist constants $C_s \geq 0, c_s \in (0, 1]$ such that for all $1 \leq k_1 + \cdots + k_s \leq n$

$$- \sum_{j=1}^s k_j \psi^{-1} \left( \frac{k_j}{n} \right) + n \sum_{j=1}^s \rho \left( \exp \psi^{-1} \left( \frac{k_j}{n} \right) \right)$$

$$\leq - \sum_{j=1}^s k_j \psi^{-1} \left( \frac{1}{n} \sum_{j=1}^s k_j \right) + n \rho \left( \exp \psi^{-1} \left( \frac{c_s}{n} \sum_{j=1}^s k_j \right) \right) + C_s \sum_{j=1}^s k_j,$$

where $\sum'$ means that we consider only $k_j$ such that $k_j \neq 0$.

c) We say that $\rho$ is \textit{doubly bounded} if $\lim_{r \to 0^+} \sup_{r} \rho(2r)/\rho(r) < +\infty$.

\textbf{Theorem 4.2.} Let $E$ be a compact subset of $\mathbb{C}^N$ and let $\rho$ satisfy conditions of a fit majorant. Put $\psi(t) = (\rho(e^t))'$.

a) Assume that $\rho_E(r) \leq \rho(r)$ for $r \in (0, 1]$. For all $n \geq 1, \alpha \in \mathbb{N}_0^N, 1 \leq |\alpha| \leq n$ and for an arbitrary $P \in \mathcal{P}(\mathbb{C}^N)$ of degree not greater than $n$ we have

$$||D^\alpha P||_E \leq \alpha!N^{(|\alpha|/2)} \exp \left( -|\alpha|\psi^{-1}(\frac{1}{n}) + n \rho \left( \exp \psi^{-1}(\frac{1}{n}) \right) \right) ||P||_E.$$ 

b) If $\rho$ is $m$-bounded (with a constant $A$) and there exist constants $C > 0, c \in (0, 1]$ such that for all polynomials $P$ of degree not greater than $n$ and $1 \leq |\alpha| \leq n$

$$||D^\alpha P||_E \leq \alpha!C^{(|\alpha|)} \exp \left( -|\alpha|\psi^{-1}(c|\alpha|/n) + n \rho \left( \exp \psi^{-1}(c|\alpha|/n) \right) \right) ||P||_E$$

then we have the inequalities

$$\rho_E(r) \leq \frac{A}{c} \rho(Cr), \quad \text{for} \quad 0 \leq r \leq 1/C;$$

$$\rho_E(r) \leq \frac{A}{c} \rho(1) + \log(Cr), \quad \text{for} \quad r \geq 1/C$$

and $C(E) \geq \exp(-A\rho(1)/c)/C$. Moreover, if $\rho$ is doubly bounded, then there exists a constant $A'$ such that $\rho_E(r) \leq A'\rho(r)$, $r \in [0, 1]$.

c) If $\rho$ is $m$, mb- and doubly bounded and $V_E(x + \zeta e_j) \leq \rho(|\zeta|)$, $x \in E, \zeta \in \overline{B}$, then there exists a constant $B$ such that $V_E(x + \zeta) \leq B\rho(||\zeta||_2)$ for $x \in E, \zeta \in \mathbb{C}^N, ||\zeta||_2 \leq 1$.

\textbf{Proof.} a) By Cauchy's inequality, we have

$$||D^\alpha P(x)||_E/\alpha! \leq \inf_{r>0} \{r^{-|\alpha|} \sup_{||\zeta|| \leq r} \exp(nV_E(x + (\zeta_1, \ldots, \zeta_N)))\}$$

$$\leq \inf_{r>0} \{r^{-|\alpha|} \sup_{||\zeta||_2 \leq \sqrt{Nr}} \exp(nV_E(x + \zeta))\} \leq \inf_{r>0} r^{-|\alpha|} \exp(n\rho_E(\sqrt{N}r))$$

$$= N^{\frac{|\alpha|}{2}} \inf_{r>0} r^{-|\alpha|} \exp(n\rho_E(r)).$$

Hence

$$||D^\alpha P(x)||_E/\alpha! \leq N^{\frac{|\alpha|}{2}} \inf_{r \in (0, 1]} r^{-|\alpha|} e^{n\rho(r)} = N^{\frac{|\alpha|}{2}} \inf_{t \leq 0} \exp \left( -|\alpha| t + n\rho(e^t) \right).$$
Replacing operations gives the inequality deduced from the assumption that \( \rho \leq C r \) if
\[
\inf_{t \leq 0} (-|\alpha| t + n \rho(\epsilon^t)) = -|\alpha| \psi^{-1}(|\alpha|/n) + n \rho(\exp^{-1}(|\alpha|/n)),
\]
and assertion a) follows.
b) By Taylor's theorem applied to a polynomial \( P \) of degree \( n \geq 1 \), \( ||P||_E = 1 \), for \( x \in E, \zeta \in \mathbb{C}^N, ||\zeta||_2 = r \) we can write
\[
|P(x+\zeta)| \leq |P(x)| + \sum_{0<|\alpha| \leq n} \frac{|D^\alpha P(x)|}{\alpha!} ||\zeta||^{|\alpha|}
\]
\[
\leq 1 + \sum_{k=1}^{n} \sum_{|\alpha|=k} C^{|\alpha|} \exp \left( -|\alpha| \psi^{-1} \left( \frac{|\alpha|}{n} \right) + n \rho \left( \exp^{-1} \left( \frac{|\alpha|}{n} \right) \right) \right) r^{|\alpha|}
\]
\[
= 1 + \sum_{k=1}^{n} \binom{N+k-1}{k} C^k \exp \left( -k \psi^{-1} (ck/n) + n \rho \left( \exp^{-1} (ck/n) \right) \right) r^k
\]
\[
\leq 1 + \sum_{k=1}^{n} \frac{(N+k-1)^{N-1}}{(N-1)!} \exp \left( -k \psi^{-1} (ck/n) + n \rho \left( \exp^{-1} (ck/n) \right) \right) (Cr)^k
\]
\[
\leq \frac{(N+n-1)^{N-1}}{(N-1)!} \left( 1 + \sum_{k=1}^{n} e^{(N-1)k} \exp \left( -k \psi^{-1} (ck/n) + n \rho \left( \exp^{-1} (ck/n) \right) \right) (Cr)^k \right).
\]
Replacing \( P \) by \( P^m \) we get
\[
\log |P(x+\zeta)|^{1/n} \leq \log \left( (mn+N-1)^{N-1}/(N-1)! \right)^{1/mn}
\]
\[
+ \log \left( 1 + \sum_{k=1}^{mn} \exp \left( -k \psi^{-1} (ck/mn) + mn \rho \left( \exp^{-1} (ck/mn) \right) \right) (Cr)^k \right)^{1/mn}.
\]
Hence, from (12) we obtain for \( Cr \leq 1 \) the inequality
\[
\log |P(x+\zeta)|^{1/n} \leq \frac{A}{c} \rho(Cr)
\]
and consequently,
\[
\rho_E(r) \leq \frac{A}{c} \rho(Cr), \ r \leq 1/C.
\]
If \( Cr \geq 1 \) then \( (Cr)^k \leq (Cr)^n \) for \( k \leq n \) and a small modification of the above considerations gives the inequality \( \rho_E(r) \leq \frac{A}{c} \rho(1) + \log(Cr) \). The last property in b) can be deduced from the assumption that \( \rho \) is doubly bounded and from the L-regularity of \( E \).
c) If \( V_E(x+\zeta e_j) \leq \rho(||\zeta||) \) then for \( 1 \leq \deg P \leq n \) and \( ||P||_E = 1 \) we can write
\[
|D^{k_\epsilon_j} P(x)| \leq k! \inf_{0 < r \leq 1} r^{-k} \exp(n \sup_{|\zeta|=r} V_E(x+\zeta e_j))
\]
\[
\leq k! \inf_{0 < r \leq 1} r^{-k} \exp(n \rho(r)) = k! \exp((-k \psi^{-1}(k/n)) \exp(n \rho(\psi^{-1}(k/n)))).
\]
Hence
\[ \| D^{k\sigma} P \|_E \leq k! \exp(-k \psi^{-1}(k/n)) \exp(n \rho(\psi^{-1}(k/n))) \]
and thus
\[ \| D^\sigma P \|_E \leq \alpha! \exp \left( - \sum_{j=1}^{N} \alpha_j \psi^{-1} \left( \frac{\alpha_j}{n} \right) + n \sum_{j=1}^{N} \rho \left( \exp \psi^{-1} \left( \frac{\alpha_j}{n} \right) \right) \right) \]
\[ \leq \alpha! \exp \left( - |\alpha| \psi^{-1}(c_N |\alpha|/n) + n \rho \left( \exp \psi^{-1}(c_N |\alpha|/n) \right) + C_N |\alpha| \right). \]

We can see that the assumptions of b) are satisfied and if we make use of it, then we prove assertion c).

As an application of Th.4.2.a we get the following bounds

**Corollary 4.3.** a) If \( \rho(r) = A r^\sigma \), \( A \geq 1/\sigma \) then
\[ \| D^\sigma P \|_E \leq \alpha! \left( A \sigma \sqrt{N} e \right)^{|\alpha|/\sigma} \left( \frac{n}{|\alpha|} \right)^{|\alpha|/\sigma} \| P \|_E \leq \left( A \sigma \sqrt{N} e \right)^{|\alpha|/\sigma} \left( \frac{1}{|\alpha|!} \right)^{1/\sigma - 1} n^{|\alpha|/\sigma} \| P \|_E. \]

b) If \( \rho(r) = (1/\log(e/r))^s \) then
\[ \| D^\sigma P \|_E \leq \alpha!(\sqrt{N}/e)^{|\alpha|} \exp \left( (1 + 1/s)|\alpha|^s n^{1+s} \right) \| P \|_E. \]

c) If \( \rho(r) = r^\sigma (\log(1/r) + 2/\sigma), \sigma \in (0, 1], m = 1/\sigma \) then
\[ \| D^\sigma P \|_E \leq \alpha! N^{m|\alpha|/2} \left( \frac{e - 1}{e} \right)^{2m} \left( \frac{n}{|\alpha|} \right)^{m|\alpha|} \left( 1 + \log(n/|\alpha|) \right)^m \| P \|_E. \]

Th.4.2 generalizes implication \( \text{VMI} \Rightarrow \text{HCP} \) in view of the following

**Theorem 4.4.** If \( \sigma \in (0, 1] \) and \( A \geq 1/\sigma \) then the function \( \rho(r) = Ar^\sigma \) satisfies conditions of a fit majorant and is \( m-, mb- \) and doubly bounded.

**Proof.** It is sufficient to check that \( \rho \) is \( m- \) and \( mb \)-bounded. Fix \( c \in (0, 1] \). Then \( \psi(t) = A \sigma e^{\alpha t} \), which implies \( \psi^{-1}(ck/n) = \log(ck/(A\sigma n))^{1/\sigma} \) and for \( 1 \leq k \leq n \) we obtain
\[ \exp(-k \psi^{-1}(ck/n)) + n \rho \left( \exp \psi^{-1}(ck/n) \right) = \left( A \sigma e^c \right)^{k/\sigma} \left( \frac{n}{k} \right)^{k/\sigma} e^{ck/\sigma} \leq \left( A \sigma e^c \right)^{n_k^{1/\sigma}} \left( \frac{1}{k!} \right)^{1/\sigma}. \]

From this we conclude that
\[ \log \left( 1 + \sum_{k=1}^{n} \exp(-k \psi^{-1}(ck/n) + n \rho \left( \exp \psi^{-1}(ck/n) \right) \right)^{1/n} \]
\[ \leq \frac{1}{n} \log \left( 1 + \sum_{k=1}^{n} \left( A \sigma e^c n^{r^\sigma}/c \right)^{k/k!} \right)^{1/\sigma} \leq \frac{1}{n^\sigma} \log \exp \left( A \sigma e^c n^{r^\sigma}/c \right) = \frac{A}{c} e^{r^\sigma} \leq e \rho(r). \]

It is easy to check that \( \rho \) is \( mb \)-bounded if we take \( c_s = 1 \) and \( C_s = \frac{c}{\sigma} \).

The next theorem concerns another class of fit majorants.
Lemma 4.6. Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. There exists a positive constant $B(p)$ such that for an arbitrary $s > 0$ the following inequality holds

$$\sum_{k=0}^{\infty} \exp(qk^{1/q})e^{-ks} \leq \exp(B(p)s^{-(p-1)}).$$

Proof. If $s \geq 1$ then

$$\sum_{k=0}^{\infty} \exp(qk^{1/q})e^{-ts} \leq 1 + e^{-s} \sum_{k=0}^{\infty} \exp(qk^{1/q})e^{-(k-1)} \leq 1 + B_1(p)s^{-(p-1)} \leq \exp(B_1(p)s^{-(p-1)}).$$

Now we will consider the more difficult case $0 < s < 1$. We replace the series $\sum_{k=0}^{\infty} \exp(qk^{1/q})$ by the Laplace transform of the function $\exp(qt^{1/q})$.

We start with the simple observation that $\exp(qk^{1/q})e^{-ks} \leq e^s \int_k^{k+1} \exp(qt^{1/q})e^{-ts}dt$, which implies

$$\sum_{k=0}^{\infty} \exp(qk^{1/q})e^{-ks} \leq e \int_0^{\infty} \exp(qt^{1/q})e^{-ts}dt \leq \exp(s^{-(p-1)}) \int_0^{\infty} \exp(qt^{1/q})e^{-ts}dt.$$

$$\int_0^{\infty} \exp(qt^{1/q})e^{-ts}dt = q \int_0^{\infty} e^{qt^{1/q}t^{q-1}t}dt = q^{-1} \int_0^{\infty} \exp(qt/s^{1/q})e^{-t^{q-1}}dt.$$

Put $b = s^{-\frac{1}{q-1}}$. We can write $\int_0^{\infty} \exp(qt/s^{1/q})e^{-t^{q-1}}dt = I_1 + \exp(qs^{-(p-1)})I_2$, where

$$I_1 = \int_0^{b} \exp(qt/s^{1/q})e^{-t^{q-1}}dt, \quad I_2 = \int_b^{\infty} \exp(qt/s^{1/q})e^{-(t+b)^q(t+b)^{q-1}}dt.$$

We have

$$I_1 \leq \exp(qs^{-(p-1)}) \int_0^{\infty} e^{-t^{q-1}}dt = q^{-1} \exp(qs^{-(p-1)}) < \exp(qs^{-(p-1)}).$$

Since $(b+t)^q = b^q + qt/s^{1/q} + \frac{1}{2}q(q-1)(b+\theta t)^{q-2}t^2, \; \theta \in (0, 1)$, we obtain

$$I_2 = \exp(-s^{-(p-1)}) \int_0^{\infty} \exp\left(-\frac{1}{2}q(q-1)(b+\theta t)^{q-2}t^2\right)(t+b)^{q-1}dt.$$

If $1 < q \leq 2$ then $(b+\theta t)^{q-2} \geq (b+t)^{q-2}$ and we get

$$\int_0^{\infty} \exp\left(-\frac{1}{2}q(q-1)(b+\theta t)^{q-2}t^2\right)(t+b)^{q-1}dt \leq \int_0^{\infty} \exp\left(-\frac{1}{2}q(q-1)(b+t)^{q-2}t^2\right)(t+b)^{q-1}dt.$$
Observe that for $c u$

If we now put $s$ with $t$ which implies that $S^{(p-1)}(t+1)^{q-1}dt = s^{(p-1)}B_2(p) \leq \exp(B_2(p)s^{(p-1)})$.

If $q \geq 2$ then

$$\int_0^\infty \exp\left(-\frac{1}{2}q(q-1)(b + \theta t)^{-2}t^2\right)(t + b)^{q-1}dt \leq \int_0^\infty \exp\left(-\frac{1}{2}q(q-1)b^{q-2}t^2\right)(t + b)^{q-1}dt$$

$$\leq (s^{(p-1)})^{1/p} \int_0^\infty \exp\left(-\frac{1}{2}q(q-1)t^2\right)(t + 1)^{q-1}dt \leq (s^{(p-1)})^{1/p}B_3(p) \leq \exp(s^{(p-1)}B_3(p)^p/p).$$

The proof is completed by combining all particular cases. $\Box$

**Proof of Theorem 4.5.** We have

$$1 + \sum_{k=1}^n \exp\left(-k \psi^{-1} \left(\frac{ck}{n}\right)\right) + n \rho \left(\exp \psi^{-1} \left(\frac{ck}{n}\right)\right) \leq 1 + \sum_{k=1}^n \exp\left(\left(\frac{1}{q}c^{1/p} + \frac{1}{p}c^{1/q}\right)qk^{1/q}n^{1/p}\right) \left(\frac{p}{e}\right)^k.$$  

Observe that for $c \in (0, 1]$ we have $g(c) := \frac{1}{q}c^{-1/p} + \frac{1}{p}c^{1/q} \geq 1$. Applying Lemma 4.6. with $s = u/(g(c)n^{1/p})$ we obtain

$$1 + \sum_{k=1}^n \exp(qk^{1/q}n^{1/p}) \exp(-ku/(g(c)n^{1/p})) \leq \exp(B(p)g(c)^{p-1}n^{1/q}u^{-(p-1)}).$$

Moreover, we see that

$$\left(1 + \sum_{k=1}^n \exp(g(c)qk^{1/q}n^{1/p})e^{-ku}\right)^{1/g(c)n^{1/p}} \leq 1 + \sum_{k=1}^n \exp(qk^{1/q}n^{1/p}) \exp(-ku/(g(c)n^{1/p}))$$

$$\leq \exp(B(p)g(c)^{p-1}n^{1/q}u^{-(p-1)}).$$

which implies that

$$1 + \sum_{k=1}^n \exp(g(c)qk^{1/q}n^{1/p})e^{-ku} \leq \exp(B(p)g(c)^{p}nu^{-(p-1)}).$$

If we now put $u = \log(e/r)$ for $r \in (0, 1]$ then

$$\log\left(1 + \sum_{k=1}^n \exp(g(c)qk^{1/q}n^{1/p})(r/e)^k\right)^{1/n} \leq B(p)g(c)^p(\log(e/r))^{-(p-1)}$$

$$= (p - 1)B(p)\left(\frac{1}{q}c + \frac{1}{p}c\right)^p \frac{1}{c p - 1} (\log(e/r))^{-(p-1)} \leq \frac{(p - 1)B(p)}{c} \rho(r),$$

which gives (12).
We leave it to the reader to verify that \( \rho \) is \( mb \)-bounded if we take \( C_s = 0 \) and if \( c_s \in (0, 1] \) is chosen so that

\[
q \sup \{ \sum_{j=1}^{s} \lambda_j^{1/q} : \lambda_j \geq 0, \sum_{j=1}^{s} \lambda_j = 1 \} \leq c_s^{-1/p}(1 + \frac{c_s}{p - 1}).
\]

\[\square\]

We end this section with the following example.

**Example 4.7.** If \( E \) is a unit ball in \( \mathbb{C}^N \) then (cf. Section 2) \( \rho_E(r) = \log(1 + r/C(E)) \leq r/C(E) \) that is equivalent to \( V_E(z) \leq \text{dist}(z, E)/C(E) \), \( z \in \mathbb{C}^N \). We can take \( \rho(r) = \max(1, 1/C(E))r \) and thus we get for \( P \in \mathcal{P}(\mathbb{C}^N) \), \( \deg P \leq n, \)

\[
\|D^\alpha P\|_E \leq (\max(1, 1/C(E))(\sqrt{Ne})^{\|\alpha\|n}|\|P\|_E.
\]

**5. HCP of compact subsets of \( \mathbb{R}^N \)**

**Remark 5.1.** For any set \( E \subset \mathbb{R}^N \) it is sufficient to consider only polynomials with real coefficients. Indeed, if \( P \in \mathcal{P}(\mathbb{C}^n) \) then \( P = Q + iR \) where \( P, Q \in \mathcal{P}(\mathbb{R}^N) \), \( \deg P = \max(\deg Q, \deg R) \) and

\[
\|P\|_E = \sup_{|\theta|\leq \pi} \|\cos \theta Q + \sin \theta R\|_E, \quad \|D^\alpha P\|_E = \sup_{|\theta|\leq \pi} \|D^\alpha (\cos \theta Q + \sin \theta R)\|_E.
\]

Hence, if we have the bound \( \|D^\alpha P\|_E \leq C(n, k)||P||_E \) for all \( P \in \mathcal{P}(\mathbb{R}^N), \deg P \leq n, |\alpha| \leq n \) then the same is true for polynomials with complex coefficients.

Observe that the following identity holds for real polynomials \( P \)

\[
\|\text{grad} \, P(x)\|_E^2 = \frac{1}{2}\Delta(P^2(x)) - P(x)\Delta P(x)
\]

and consequently,

\[
E \in AMI(m, M) \iff \exists M' \forall P \in \mathcal{P}(\mathbb{R}^N) \|\Delta P\|_E \leq M'(\deg P)^{2m}\|P\|_E.
\]

Note also that, if \( N = 2 \), then

\[
E \in AMI(m, M) \iff \exists M' \forall P \in \mathcal{P}(\mathbb{R}^N) \|\frac{\partial}{\partial z} P(x, y)\|_E \leq M'(\deg P)^m\|P\|_E
\]

\[
\iff \exists M' \forall P \in \mathcal{P}(\mathbb{R}^N) \|\frac{\partial}{\partial z} P(x, y)\|_E \leq M'(\deg P)^m\|P\|_E
\]

where, as usual,

\[
\frac{\partial}{\partial z} P(x, y) = \frac{1}{2} \left( \frac{\partial P(x, y)}{\partial x} - i \frac{\partial P(x, y)}{\partial y} \right), \quad \frac{\partial}{\partial \overline{z}} P(x, y) = \frac{1}{2} \left( \frac{\partial P(x, y)}{\partial x} + i \frac{\partial P(x, y)}{\partial y} \right).
\]

For compact subsets of \( \mathbb{R}^N \) we can take only real polynomials in the definition of Siciak’s extremal function (cf. [2]):

\[
\Phi_E(z) = \sup \left\{ \frac{|P(z)|^{1/\deg P}}{\deg P} : P \in \mathcal{P}(\mathbb{R}^N), \deg P \geq 1, \|P\|_E \leq 1 \right\}, \quad z \in \mathbb{C}^N.
\]

The following result is a consequence of [4] Th.2.4.
Proposition 5.2. If $E$ is a compact subset of $\mathbb{R}^N$ then
\[ V_E(x + iy) \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} V_E(x + ty) \frac{dt}{1 + t^2} \] (13)
for every $z = x + iy \in \mathbb{C}^N$. Equality holds in (13) if $N = 1$ (for any $z \in \mathbb{C}$).

As a corollary of Prop.5.2 we obtain

Theorem 5.3. Let $E$ be a compact set in $\mathbb{R}^N$. Assume that for every $x \in \mathbb{R}^N$ the inequality holds
\[ V_E(x) \leq B (\text{dist}(x, E))^\gamma \] (14)
where $B > 0$, $\gamma \in (0, 1)$ are constants independent of $x$. Then for all $z \in \mathbb{C}^N$
\[ V_E(z) \leq \tilde{B}(\text{dist}(z, E))^\gamma, \quad \text{with} \quad \tilde{B} = \frac{B}{\pi} \int_{-\infty}^{+\infty} (1 + t^2)^{\gamma/2 - 1} dt = \frac{B}{\Gamma(1/2 - \gamma/2)} \frac{\Gamma(1/2)}{\Gamma(1 - \gamma/2)}. \]

Proof. Evidently, for $z = x + iy \in \mathbb{C}^N$ we have
\[ \text{dist}(z, E) = (\text{dist}(x, E)^2 + \|y\|_2^2)^{1/2}. \]
Inequality (14) is equivalent to
\[ V_E(x) \leq B \|x - x_0\|_2^\gamma \quad \text{for all} \quad x_0 \in E. \]
By Prop.5.2, we get
\[ V_E(z) = V_E(x + iy) \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} V_E(x + ty) \frac{dt}{1 + t^2} \leq \frac{B}{\pi} \int_{-\infty}^{+\infty} \|x - x_0 + ty\|_2^\gamma \frac{dt}{1 + t^2} \]
\[ \leq \frac{\tilde{B}}{\pi} \int_{-\infty}^{+\infty} (\|x - x_0\|_2^2 + \|y\|_2^2)^{\gamma/2} (1 + t^2)^{\gamma/2} \frac{dt}{1 + t^2} \]
\[ = \tilde{B}(\|x - x_0\|_2^2 + \|y\|_2^2)^{\gamma/2}. \]
As $x_0$ is an arbitrary point of $E$, we obtain $V_E(z) \leq \tilde{B}(\text{dist}(z, E))^\gamma$, which completes the proof. $\Box$

It may be worth reminding the reader that if a compact set $E \subset \mathbb{R}^N$ admits the A.Markov inequality then the exponent $m$ in (1) is at least equal to 2 (see e.g. [13]). Therefore, the exponent $\gamma$ in (14) may be at most equal to $\frac{1}{2}$.

Corollary 5.4. If for every $x \in \mathbb{R}^N$ the inequality holds
\[ V_E(x) \leq B (\text{dist}(x, E))^\gamma \] (15)
with $B > 0$, $\gamma \in (0, \frac{1}{2}]$ independent of $x$, then for all $z \in \mathbb{C}^N$
\[ V_E(z) \leq \tilde{B}(\text{dist}(z, E))^\gamma, \quad \tilde{B} = \frac{B}{\sqrt{\pi}} \frac{\Gamma(1/4)}{\Gamma(3/4)}. \]
Corollary 5.5. If $E$ is a compact subset of $\mathbb{R}^N$ and $\gamma \in (0,1]$, then the following conditions are equivalent:

(i) $E \in HCP(\gamma, B_1)$ with some $B_1 \geq 1$,

(ii) inequality \[\text{[14]}\] holds for all $x \in \mathbb{R}^N$ with some $B_2 \geq 1$ independent of $x$,

(iii) $E \in VMI(\frac{\gamma}{\gamma}, B_3)$ with some $B_3 \geq 1$,

(iv) inequality \[\text{[6]}\] holds with some $B \geq 1$ for all polynomials $P$ of real coefficients.

Example 5.6. For $E = [-1, 1]$ we have $V_E(x) = \log h(\max(1, |x|))$ where $h(t) = t + \sqrt{t^2 - 1}$ if $t \geq 1$. If $x_0 \in E$ then

$$V_E(x) \leq \log h(1 + |x - x_0|) = \log \left(1 + |x - x_0|^{1/2}(|x - x_0|^{1/2} + (|x - x_0| + 2)^{1/2})\right).$$

Since $\log(1 + t) \leq \frac{1}{\alpha} t^\alpha$ for $t \geq 0$, $0 < \alpha \leq 1$, we obtain

$$V_E(x) \leq \left\{ \begin{array}{ll}
(1 + \sqrt{3})|x - x_0|^{1/2} & ; |x - x_0| \leq 1,
2(1 + \sqrt{3})^{1/2}|x - x_0|^{1/2} & ; |x - x_0| > 1,
\end{array} \right.$$ 

hence $V_E(x) \leq 2(1 + \sqrt{3})^{1/2}|x - x_0|^{1/2}$ for all $x$, $x_0$ and thus

$$V_E(x) \leq 2(1 + \sqrt{3})^{1/2}(\text{dist}(x, E))^{1/2}.$$ 

Example 5.7. Let $E$ be a convex body in $\mathbb{R}^N$ that is not symmetric with respect to the origin. Fix $\xi \in S^{N-1}$ and put

$$a_\xi(E) = \min_{x \in E} \langle x, \xi \rangle, \quad b_\xi(E) = \max_{x \in E} \langle x, \xi \rangle, \quad \rho_\xi(E) = b_\xi(E) - a_\xi(E).$$

The last value is called the width of $E$ in the direction $\xi$. The minimal width of $E$ is given by $\omega(E) = \inf_{\xi \in S^{N-1}} \rho_\xi(E)$. For $x \in \mathbb{R}^N$ it follows that (see \[\text{[11]}\])

$$V_E(x) = \sup_{\xi \in S^{N-1}} V_{[a_\xi(E), b_\xi(E)]}(\langle x, \xi \rangle) = \sup_{\xi \in S^{N-1}} \left\{ \begin{array}{ll}
V_{[-1, 1]}(2\langle x, \xi \rangle/\rho_\xi(E) - (b_\xi(E) + a_\xi(E))/\rho_\xi(E))
\end{array} \right.$$

Therefore, in the same way as in Example 5.6, we have

$$V_E(x) \leq \sup_{\xi \in S^{N-1}} \log h \left(1 + 2|\langle x - x_0, \xi \rangle/\rho_\xi(E)\right) \leq \log h(1 + 2\|x - x_0\|_2/\omega(E))$$

$$\leq 2(1 + \sqrt{3})^{1/2}(2\|x - x_0\|_2/\omega(E))^{1/2}$$

for any fixed $x_0 \in E$ and in consequence we get

$$V_E(x) \leq (2 + 2\sqrt{3})^{1/2}(4\text{dist}(x, E)/\omega(E))^{1/2} \leq (2 + 2\sqrt{3})^{1/2}(\text{dist}(x, E)/C(E))^{1/2},$$

where $C(E)$ is the L-capacity of $E$ (see \[\text{[5]}\], Example 3.4]). In particular, there exists an absolute constant $B$ such that for all dimensions $N$ and for all convex bodies $E \subset \mathbb{R}^N$ the inequality holds

$$V_E(z) \leq B(\text{dist}(z, E)/C(E))^{1/2}, \quad z \in \mathbb{C}^N.$$
By Th.2.12, we can deduce that these sets belong to \( VMI(2, \sqrt{N}B^2e^2/[4C(E)]) \).

Now we recall a definition of a class of UPC sets introduced by Pawlucki and Pleśniak \cite{17} who have shown its importance in approximation theory. In particular, they have proved a deep result (cf. \cite{17} Cor. 6.5) that every fat compact subanalytic subset of \( \mathbb{R}^N \) belongs to this class (see also \cite{18}).

Let \( s \geq 1, S > 0 \) and \( d \in \{1, 2, \ldots \} \).

**Definition 5.8.** A compact set \( E \subset \mathbb{R}^N \) is called \textit{uniformly polynomially cuspidal} \((E \in UPC(s, S, d) \text{ in short})\) if for every \( x_0 \in E \) we can find a polynomial mapping \( \varphi : \mathbb{R} \rightarrow \mathbb{R}^N \) of degree at most \( d \) such that \( \varphi(1) = x_0 \) and

\[
\text{dist}(\varphi(t), \mathbb{R}^N \setminus E) \geq S(1 - t)^s \quad \text{for} \quad t \in [0, 1].
\]

It is rather difficult to find the optimal constant \( s \) in the last inequality. However, calculations are much simpler for the following modification of the above definition.

**Definition 5.9.** (cf. \cite{3}) Let \( v \) be a fixed unit vector in \( \mathbb{R}^N \). A compact set \( E \subset \mathbb{R}^N \) is called \textit{uniformly polynomially cuspidal in direction} \( v \) \((E \in UPC_v(s, S, d) \text{ in short})\) if for every \( x_0 \in E \) we can find a polynomial mapping \( \varphi : \mathbb{R} \rightarrow \mathbb{R}^N \) of degree at most \( d \) such that \( \varphi(1) = x_0 \) and

\[
\text{dist}_v(\varphi(t), \mathbb{R}^N \setminus E) \geq S(1 - t)^s \quad \text{for} \quad t \in [0, 1].
\]

Here \( \text{dist}_v(x, \mathbb{R}^N \setminus E) := \sup \{r \geq 0 : [x - rv, x + rv] \subset E\} \).

If \( E \in UPC(s, S, d) \) then \( E \in UPC_v(s, S, d) \) for every unit vector \( v \). An open problem is whether conditions \( E \in UPC_v(s_j, S_j, d_j), \ j = 1, \ldots, N, \ v_1, \ldots, v_N \) that are linearly independent imply \( E \in UPC(s, S, d) \) with some \( S, s, d \). It seems that this may not be true for \( N \geq 3 \). However, as an application of the proposition given below, we prove that if \( E \in UPC_v(s_j, S_j, d), \ j = 1, \ldots, N, \) where \((e_j)_j\) is the canonical basis, then we get \( E \in HCP(\frac{1}{2s}) \), where \( s = \max_{1 \leq j \leq N} s_j \). In particular, if \( E \in UPC(s, S, d) \) then \( E \in HCP(\frac{1}{2s}) \) that essentially improves earlier result by Pawlucki and Pleśniak \cite{17} Th.4.1 (see also \cite{19}). As a corollary we get a wide class of non-convex sets that satisfy VMI.

**Theorem 5.10.** If \( E \in UPC_v(s, S, d) \) and \( \varepsilon_0 \in (0, 1) \) then there exists \( C_0 = C_0(\varepsilon_0) > 0 \) such that for every \( |\zeta| \leq r_0 = \sqrt{s}(1 - \varepsilon_0)^s \) the inequality holds

\[
V_E(x + \zeta v) \leq C_0|\zeta|^{1/(2s)}.
\]

**Proof.** Let \( L_0 = \sqrt{2}/S \). Put \( g(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1}) \), \( \hat{g}(\zeta) = \frac{1}{2}(\zeta - \zeta^{-1}) \). For \( \rho > 1 \) write \( a = a(\rho) = ((1 + g(\rho))/2)^{-1}, b = b(\rho) = ((g(\rho) - 1/2)^{-1}, c = c(\rho) = 1/\hat{g}(\rho) \). We have \( b = a(1 - a)^{-1}, c = b(1 - a)^{-1/2}. \) Fix \( \zeta_0 = \alpha_0 + i\beta_0 \in \mathbb{C}^N \) and \( x_0 \in E \) and put \( u_0 = \alpha_0 v, \ v_0 = \beta_0 v \). Define

\[
\psi(\zeta) = \varphi(a(\rho)\frac{1}{2}(g(\zeta) + 1)) + \frac{1}{2}(g(\zeta) - 1)b(\rho)u_0 + i\hat{g}(\zeta)c(\rho)v_0, \quad \zeta \in \mathbb{C}, \quad |\zeta| \geq 1
\]

where \( \varphi \) is a polynomial mapping chosen to \( x_0 \) by the definition of the UPC property.
Since \( \psi(e^{i\theta}) = \varphi(a(\rho)r\tau) + (\tau - 1)b(\rho)u_0 \pm 2\sqrt{\tau(1-\tau)c(\rho)v_0} \) for \( \tau = \frac{\cos \theta + 1}{2} \), \( \theta \in [0, 2\pi] \), we have \( \psi(e^{i\theta}) \in E \) whenever

\[
(1-\tau)b(\rho)|\alpha_0| + 2\sqrt{\tau(1-\tau)c(\rho)|\beta_0|} \leq S(1-ar)^s, \quad \tau \in [0, 1].
\]

The last condition is satisfied if

\[
a \left( \frac{1-\tau}{1-a} |\alpha_0| + \sqrt{\frac{\tau(1-\tau)}{1-a}|\beta_0|} \right) \leq S(1-ar)^s
\]

and consequently, if

\[
a \left( \frac{1-\tau}{1-a}(1-ar)^-(s-1) \frac{|\alpha_0|}{1-a} + \sqrt{\frac{\tau(1-\tau)}{1-a}(1-ar)^{(s-1)/2}} \frac{|\beta_0|}{\sqrt{1-a}} \right) \leq S.
\]

Since \( a, \tau \in [0, 1], 1-\tau \leq 1-ar \) and \( |\alpha_0| + |\beta_0| \leq \sqrt{2} |\xi_0| \), the last condition will hold if \( \sqrt{2} |\xi_0| \leq (1-ar)^s \).

Assuming \( L_0|\xi_0| \leq (1-\varepsilon_0)^s \) and taking \( \rho = h \left( \frac{1+(L_0\delta)^{1/s}}{1-(L_0\delta)^{1/s}} \right), \quad h(t) = t + \sqrt{t^2-1} \) we have \( L_0|\xi_0| = (1-a(\rho))^s \), that is \( \psi(\{ |z| = 1 \}) \subset E \). By the maximum principle for subharmonic functions (applied to the domain \( \{ z \in \mathbb{C} : |z| > 1 \} \)), we get

\[
\log V_E(\psi(\xi)) \leq d \log |\xi|, \quad |\xi| \geq 1.
\]

In particular,

\[
V_E(x_0 + \xi_0 v) = V_E(\psi(\rho)) \leq d \log \rho = d \log h \left( \frac{1+(L_0|\xi_0|)^{1/s}}{1-(L_0|\xi_0|)^{1/s}} \right).
\]

The inequality \( 1 - (L_0|\xi_0|)^{1/s} \geq \varepsilon_0 \) implies that

\[
h \left( \frac{1+(L_0\delta)^{1/s}}{1-(L_0\delta)^{1/s}} \right) \leq h(1 + (2/\varepsilon_0)(L_0\delta)^{1/s}) \leq 1 + A\delta^{1/(2s)}
\]

where

\[
A = (2/\varepsilon_0)^{1/2} L_0^{1/(2s)}(\sqrt{2/\varepsilon_0} + \sqrt{2(1-\varepsilon_0)/\varepsilon_0}) \leq (4/\varepsilon_0)L_0^{1/(2s)} = B.
\]

Since for every \( d \geq 1 \) the function \((1+x)^d - 1)/x \) is increasing for \( x > 0 \), we obtain

\[
\left( 1 + B|\xi_0|^{1/(2s)} \right)^d \leq 1 + C_0|\xi_0|^{1/(2s)}
\]

where \( C_0 = \max_{r \leq r_0} (\left( 1 + B r_0^{1/(2s)} \right)^d - 1) / r_0^{1/(2s)} = (\left( 1 + B r_0^{1/(2s)} \right)^d - 1) / r_0^{1/(2s)} \).

Finally \( V_E(x_0 + \xi_0 v) \leq \log(1 + C_0|\xi_0|^{1/(2s)}) \leq C_0|\xi_0|^{1/(2s)} \).

\( \square \)

Applying Cor.2.13 and Remark 3.2 we get the following result which specifies an earlier result by Pawlucki and Pleśniak (cf. [17, Th.2.1]).

**Corollary 5.11.** If \( E \in UPC_{e_j}(s_j, S_j, d_j), \quad j = 1, \ldots, N \) then there exists a constant \( B \) such that \( E \in HCP(\gamma, B) \) with \( \gamma = 1/(2 \min_j s_j) \). In particular, if \( E \in UPC(s, S, d) \) then \( E \in HCP(1/(2s), B) \).
6. Applications of Theorem 2.12 for disconnected sets.

The first proposition regards certain onion type sets in the complex plane that are very useful in a problem concerning local and global Markov’s properties (see L.Bialas-Ciez and R.Eggink, *Equivalence of the global and local Markov inequalities in the complex plane*, in preparation).

**Proposition 6.1.** Let \((a_j)\) be a strictly decreasing sequence of positive numbers such that \(a_1 = 1\), \(a_j \to 0\) as \(j \to \infty\) and let \(\varphi_j \in (0, \frac{\pi}{2})\) for \(j = 1, 2, \ldots\). Put

\[
C_j := \{a_j e^{it} : t \in [\varphi_j, 2\pi]\} \quad \text{for } j = 1, 2, \ldots,
\]

\[E := \{0\} \cup \bigcup_{j=1}^{\infty} C_j.
\]

If \(|1 - e^{i\varphi_j}| \leq a_{j+1}\) for \(j = 1, 2, \ldots\) then \(E \in HCP(\frac{1}{6}, B)\) for some \(B > 0\).

**Proof.** First, we note that

\[
F := \{e^{it} : t \in [\pi/2, 2\pi]\}
\]

is a connected compact set and so \(F \in HCP(\frac{1}{2}, B_F)\) with some constant \(B_F \geq 1\) (see e.g. [10, Cor.2.2]). From implication (8) we see that \(F \in VMI(2, M_F)\) with \(M_F = (eB_F/2)^2\). We can assume that \(M_F \geq \max\{2e, 1/C(E)\}\).

For any polynomial \(P\) of degree at most \(n\), for \(k \in \{1, \ldots, n\}\) and \(z_0 \in E\), we will prove the inequality

\[
|P^{(k)}(z_0)| \leq M^k n^k \frac{\|P\|_E}{k^{5k}} \quad \text{where} \quad M = 3M_F \exp\left(3M_F(1 + e^{3M_F})\right).
\]

(16)

By Th.2.12 and since \(k^k \geq k!\), condition (16) implies that \(E \in HCP(\frac{1}{6}, B)\) with \(B = 6 M^{1/6}\). Therefore, the proof is completed by showing (16).

Observe first that for any monic polynomial \(P\) of degree \(n\) and for \(k = n\) we have

\[
\|P^{(k)}\|_E = \left(\frac{1}{C(E)}\right)^k n! (C(E))^{n} \leq \left(\frac{1}{C(E)}\right)^k n! \|P\|_E,
\]

because \(C(E)\) is equal to the Chebyshev constant of \(E\). Consequently,

\[
\|P^{(k)}\|_E \leq (M_F)^k \frac{n^6 k^{6k}}{k^{5k}} \|P\|_E
\]

for all polynomials of degree at most \(n\) not necessary monic. Thus condition (16) is fulfilled for \(k = n\).

Consider now \(k < n\). We first examine \(z_0 = 0\) and we will show that

\[
|P^{(k)}(0)| \leq \left(\frac{3M_F e^{3M_F} n^4}{k^3}\right)^k \|P\|_E.
\]

(17)

For this purpose, find \(j \in \mathbb{N}\) such that \(\frac{1}{a_j} \leq \frac{a_j^2}{k^2} < \frac{1}{a_{j+1}}\). By Cauchy’s integral formula,

\[
|P^{(k)}(0)| \leq \frac{1}{a_j} \|P^{(k-1)}\|_{C(0, a_j)},
\]
where \( C(0, a_j) \) is the circle with the radius \( a_j \) about the origin. The norm of \( P \) on \( C(0, a_j) \) is attained at some point \( w_0 \in C(0, a_j) \). Put \( F_j := a_j F \). Obviously, \( F_j \in VMI(2, M_F / a_j) \). If \( w_0 \in F_j \), we have

\[
|P^{(k)}(0)| \leq \frac{1}{a_j} \|P^{(k-1)}\|_{F_j} \leq \left( \frac{1}{a_j} \right)^k \left( \frac{M_F n^2}{k - 1} \right)^{k-1} \|P\|_{F_j} \leq \left( \frac{1}{a_j} \right)^k \left( \frac{3M_F n^2}{k - 1} \right)^{k-1} \|P\|_{F_j}
\]

\[
\leq \left( \frac{n^2}{k^2} \right)^k \left( \frac{3M_F n^2}{k} \right)^k \|P\|_{F_j} \leq \left( \frac{3M_F n^4}{k^3} \right)^k \|P\|_E \quad \text{for } k \geq 2,
\]

because \( t \mapsto \left( \frac{M_F n^2}{t} \right)^t \) is an increasing function whenever \( t \in (0, n+1] \subset \left(0, \frac{M_F n^2}{e} \right) \) and \( F_j \subset C_j \subset E \). The case of \( k = 1 \) is easy to verify and thus inequality (17) is fulfilled for all \( k \in \{ 1, \ldots, n \}, \ w_0 \in F_j \).

If \( w_0 \in C(0, a_j) \setminus F_j \), by Taylor’s formula and VMI for \( F \), we get

\[
|P^{(k)}(0)| \leq \frac{1}{a_j} \|P^{(k-1)}(w_0)\| \leq \frac{1}{a_j} \sum_{l=0}^{n-k+1} \frac{1}{l!} \|P^{(k-1+l)}(a_j)\| |a_j - w_0|^l
\]

\[
\leq \frac{1}{a_j} \sum_{l=0}^{n-k+1} \frac{1}{l!} \left( \frac{1}{a_j} \right)^{k-l+1} \left( \frac{3M_F n^2}{k - 1 + l} \right)^{k-l+1} \|P\|_{F_j} a_j^l |1 - e^{i \varepsilon_j}|^l
\]

\[
\leq \left( \frac{1}{a_j} \right)^k \sum_{l=0}^{n-k+1} \frac{1}{l!} \left( \frac{3M_F n^2}{k + l} \right)^k a_j^l \|P\|_{F_j},
\]

the last inequality being a consequence of the assumption of Prop.6.1. Since \( F_j \subset E \) and \( \frac{1}{a_j} \leq \frac{n^2}{k^2} < \frac{1}{a_j+1} \), we can write

\[
|P^{(k)}(0)| \leq \left( \frac{n^2}{k^2} \right)^k \left( \frac{3M_F n^2}{k} \right)^k \sum_{l=0}^{n-k+1} \frac{1}{l!} \left( \frac{3M_F n^2}{k} \right)^l \left( \frac{k^2}{n^2} \right)^l \|P\|_E \leq \left( \frac{3M_F e^{3M_F} n^4}{k^3} \right)^k \|P\|_E
\]

and this yields inequality (17).

We now turn to the case \( z_0 \neq 0 \). Clearly, \( z_0 \in C_j \) for some \( j \in \{ 1, 2, \ldots, \} \). If \( a_j \leq \frac{k^4}{n^4} \) then by (17) we have

\[
|P^{(k)}(z_0)| \leq \sum_{l=0}^{n-k} \frac{1}{l!} |P^{(k+l)}(0)| |z_0|^l \leq \sum_{l=1}^{n} \frac{1}{l!} \left( \frac{3M_F e^{3M_F} n^4}{(k + l)^3} \right)^l \|P\|_E a_j^l
\]

\[
\leq \left( \frac{3M_F e^{3M_F} n^4}{k^3} \right)^k \sum_{l=1}^{n} \frac{1}{l!} \left( \frac{3M_F e^{3M_F} n^4}{k^3} \right)^l \left( \frac{k^4}{n^4} \right)^l \|P\|_E
\]

\[
\leq \left( \frac{3M_F e^{3M_F} \exp(3M_F e^{3M_F} n^4)}{k^3} \right)^k \|P\|_E
\]

and (16) is proved in this case.

It remains to show estimate (16) if \( a_j > \frac{k^4}{n^4} \). Let \( F_j' \) be a set obtained by a rotation of \( F_j \) about the origin such that \( z_0 \in F_j' \subset C_j \). Since \( F_j' \in VMI(2, M_F / a_j) \), we have

\[
|P^{(k)}(z_0)| \leq \|P^{(k)}\|_{F_j'} \leq \frac{1}{a_j} \left( \frac{3M_F n^2}{k} \right)^k \|P\|_{F_j'} \leq \left( \frac{n^4}{k^4} \right)^k \left( \frac{3M_F n^2}{k} \right)^k \|P\|_E
\]
Proof. Fix with some \( B > 0 \) positive numbers such that

\[
By the above, it follows that \( \alpha \) shall show that for each \( r \)

and inequality (18) is proved.

Then the set \( E \) defined by

\[
E := \{0\} \cup \bigcup_{j=1}^{\infty} E_j, \quad E_j := \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : |z_1-a_j| \leq r_j, |z_2| \leq r_j, \ldots, |z_N| \leq r_j\}.
\]

admits the Hőlder continuity property of the pluricomplex Green function \( HCP(\frac{1}{2r_j^\alpha}, B) \) with some \( B > 0 \).

Proof. Fix \( n \in \{1, 2, \ldots\} \) and a polynomial \( P \) of degree at most \( n \). As a first step we shall show that for each \( \alpha \in \mathbb{N}_0^n, |\alpha| \leq n \)

\[
|D^{(\alpha)} P(0)| \leq \left( \frac{e_N}{b} \right)^{|\alpha|} \left( \frac{n^{1+\mu}}{|\alpha|^\mu} \right)^{|\alpha|} \|P\|_E.
\]

For this purpose, find \( j \geq 2 \) such that \( r_j < \frac{|\alpha|}{n} \leq r_{j-1} \) where \( |\alpha| \leq n \) is fixed. From (7) we have

\[
\|D^\alpha P\|_{E_j} \leq \frac{n^{\alpha}}{r_j^{\alpha}} \|P\|_{E_j}
\]

and thus, by Th.2.12 and Example 2.9, \( E_j \in HCP(1, \frac{N}{r_j}) \). In particular, we get

\[
V_{E_j}(0) \leq \frac{N}{r_j} \text{ dist } (0, E_j) = \frac{N}{r_j} (a_j - r_j) = N r_j < \frac{|\alpha|}{n}.
\]

Formula (3) leads us to

\[
|D^{(\alpha)} P(0)| \leq \left( e^{V_{E_j}(0)} \right)^{n-|\alpha|} \|D^\alpha P\|_{E_j},
\]

By the above, it follows that

\[
|D^{(\alpha)} P(0)| \leq e^{n^{\alpha}} \frac{n^{\alpha}}{r_j^{\alpha}} \|P\|_{E_j} \leq e^{n^{\alpha}} \frac{n^{\alpha+\mu|\alpha|}}{(b^\alpha)^{|\alpha|}} \|P\|_E
\]

and inequality (18) is proved.

Now consider \( z_0 \in E \setminus \{0\} \) and \( |\alpha| \leq n \). If \( z_0 \in E_j \) and \( r_j \geq \left( \frac{|\alpha|}{n} \right)^{1+\mu} \) then

\[
|D^{(\alpha)} P(z_0)| \leq \frac{n^{\alpha}}{r_j^{\alpha}} \|P\|_{E_j} \leq \frac{n^{2|\alpha|+\mu|\alpha|}}{|\alpha|^{(1+\mu)|\alpha|}} \|P\|_E.
\]

In the case of \( r_j \leq \left( \frac{|\alpha|}{n} \right)^{1+\mu} \), Taylor’s formula and inequality (18) yield

\[
|D^{(\alpha)} P(z_0)| \leq \sum_{|\beta| \leq n-|\alpha|} \frac{1}{\beta!} |D^{\alpha+\beta} P(0)| \|z_0\|_2^{\beta}
\]
\[
\sum_{|\beta| \leq n-|\alpha|} \frac{1}{b} \left( \frac{e^N}{b} \right)^{|\alpha|+|\beta|} \left( \frac{n^{1+\mu}}{|\alpha|+|\beta| \mu} \right)^{|\alpha|+|\beta|} ||P||_E \left( r_j \sqrt{N+8} \right)^{|\beta|} 
\]

\[
\leq \left( \frac{e^N}{b} \right)^{|\alpha|} \left( \frac{n^{1+\mu}}{|\alpha| \mu} \right)^{|\alpha|} ||P||_E \sum_{l=0}^{n-|\alpha|} \frac{N^l}{l!} \left( \frac{e^N \sqrt{N+8}}{b} \right)^l \left( \frac{n^{1+\mu}}{|\alpha| \mu} \right)^l \left( \frac{|\alpha|}{n} \right)^{(1+\mu)l} 
\]

\[
\leq \left( \frac{1}{b} e^{N+N\sqrt{N+8} e^N/b} \right)^{|\alpha|} \left( \frac{n^{1+\mu}}{|\alpha| \mu} \right)^{|\alpha|} ||P||_E \leq \left( \frac{1}{b} e^{N+N\sqrt{N+8} e^N/b} \right)^{|\alpha|} \left( \frac{n^{1+\mu}}{|\alpha| \mu} \right)^{|\alpha|} ||P||_E. 
\]

Hence and from inequalities \((18,19)\) we conclude that \(E \in VMI(2+\mu,b e^N+\sqrt{N+8} e^N/b)\), and Th.2.12 leads to \(E \in HCP(1/2+\mu,B)\) with 
\(B = \left( \frac{N}{b} e^{N+N\sqrt{N+8} e^N/b} \right)^{1/2+\mu} (2+\mu)\), which proves the assertion.

\[\blacksquare\]

**Remark 6.3.** We close this paper by offering two questions for further research:

1. Does the continuity of the pluricomplex Green’s function \(V_E\) with respect to each variable separately imply the L-regularity of \(E\)?
2. Has the pluricomplex Green’s function \(V_E\) of a Markov set \(E\) the continuity property with respect to each variable separately?

For the univariate case the answer to the second question is partially known because if \(E \subset \mathbb{R}\) then it is L-regular (see [9]).

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