On the non-equivalence of two standard random walks

O. Bénichou\textsuperscript{a}, K. Lindenberg\textsuperscript{b}, G. Oshanin\textsuperscript{a}

\textsuperscript{a}Laboratoire de Physique Théorique de la Matière Condensée (UMR CNRS 7600), Université Pierre et Marie Curie (Paris 6) - 4 Place Jussieu, 75252 Paris, France
\textsuperscript{b}Department of Chemistry and Biochemistry and BioCircuits Institute, University of California San Diego, La Jolla, CA 92093-0340, USA

Abstract

We focus on two models of nearest-neighbour random walks on \(d\)-dimensional regular hyper-cubic lattices that are usually assumed to be identical - the discrete-time Polya walk, in which the walker steps at each integer moment of time, and the Montroll-Weiss continuous-time random walk in which the time intervals between successive steps are independent, exponentially and identically distributed random variables with mean 1. We show that while for symmetric random walks both models indeed lead to identical behaviour in the long time limit, when there is an external bias they lead to markedly different behaviour.

Keywords: Polya random walk, Montroll-Weiss random walk, external bias

1. Introduction

The random walk is a paradigmatic stochastic process which underlies a vast variety of phenomena in physics, chemistry and biology (see, e.g., \cite{1}). In particular, two models of lattice random walks are found in the literature with equal frequency. The first is the so-called Polya walk (see, e.g., \cite{1}), in which the random walker moves at discrete time moments \(n = 1, 2, \ldots \) on a \(d\)-dimensional lattice so that if it is at site \(X\) it steps on neighbouring site \(X'\) with probability \(p(X)\). The second model is the Montroll-Weiss random walk \cite{2}, in which the same random walk evolves in continuous time, and the times \(t_j\) between successive jumps are independent, identically distributed random variables with a common probability density function \(\psi(t)\). It is often tacitly assumed that in the canonical case when the waiting (or pausing) time density is exponential, i.e., when \(\psi(t_j) = \exp(-t_j)\), the Montroll-Weiss
random walk leads to behaviour \textit{identical} to that of the Polya walk in the limit $t \to \infty$.

The purpose of this short note is to demonstrate that this is indeed the case for \textit{symmetric} random walks, but that in the presence of an external bias the two models exhibit markedly different behaviour. Although this non-equivalence of the two standard models can be explained on intuitive grounds, to the best of our knowledge it has been never spelled out explicitly.

2. Polya random walk vs Montroll-Weiss random walk

Consider a Polya random walk on a $d$-dimensional hyper-cubic lattice with axes $x$ and $y_j$, $j = 1, \ldots, d - 1$, and suppose that there is a constant force $E$ acting on the walker which points in the positive $x$-direction. In this case, the standard transition probabilities $p(X)$ which obey detailed balance are given by

$$p_x = Z^{-1} e^{\beta E/2}, \quad p_{-x} = Z^{-1} e^{-\beta E/2}, \quad p_0 = Z^{-1}, \quad \text{(1)}$$

where $\beta$ is the reciprocal temperature, $p_x$, $p_{-x}$, and $p_0$ are the probabilities to jump in the direction of the applied force, in the direction opposite to the applied force, and in any direction perpendicular to the force, respectively, and $Z$ is the normalisation constant

$$Z = 2 \cosh (\beta E/2) + 2(d - 1). \quad \text{(2)}$$

Consider the site occupation probability $P_n(x, \{y_j\})$, that is, the probability of finding the walker at site $X = (x, \{y_j\})$ at discrete time moment $n$. The characteristic function $\Phi_n(\omega)$ of the site occupation probability is defined as its discrete Fourier transform,

$$\Phi_n(\omega) = \sum_{x,\{y_j\}=-\infty}^{\infty} \exp \left( i\omega_x x + i \sum_{j=1}^{d-1} \omega_j y_j \right) P_n(x, \{y_j\}). \quad \text{(3)}$$

For a Polya walk this function can easily be calculated to give

$$\Phi_n(\omega) = \left( p_x e^{i\omega_x} + p_{-x} e^{-i\omega_x} + Z^{-1} \sum_{j=1}^{d-1} \cos (\omega_j) \right)^n. \quad \text{(4)}$$

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Differentiating the latter equation once and twice over $\omega_x$ leads to the following expressions for the first two moments of the displacement along the $x$-axis:

$$\overline{x}_n = (p_x - p_{-x}) \ n,$$

(5)

and

$$\overline{x^2}_n = (p_x - p_{-x})^2 (n - 1) + (p_x + p_{-x}) \ n,$$

(6)

so that the variance for motion along the $x$-axis is given by

$$\text{Var}_n(x) = (p_x (1 - p_x) + p_{-x} (1 - p_{-x}) + 2p_x p_{-x}) \ n.$$  

(7)

Similarly, differentiating once and twice with respect to any given $\omega_j$ we find that the variance of the site occupation distribution in any direction perpendicular to the field is given by

$$\text{Var}_n(y_j) = Z^{-1} \ n.$$  

(8)

To calculate analogous properties for the continuous-time Montroll-Weiss random walk with an exponential pausing time distribution, we take advantage of the Montroll-Weiss theorem \[2\] which establishes a general relation between $P_n(x, \{y_j\})$ for the discrete-time walk and its continuous-time counterpart $\tilde{P}(x, \{y_j\}; t)$ describing the site-occupation probability for the continuous-time process.

Let $P_\xi(x, \{y_j\})$ denote the generating function of the site occupation probability $P_n(x, \{y_j\})$, i.e.,

$$P_\xi(x, \{y_j\}) = \sum_{n=0}^{\infty} P_n(x, \{y_j\}) \xi^n.$$  

(9)

The Montroll-Weiss theorem states that the Laplace transform of the site-occupation probability $\tilde{P}(x, \{y_j\}; t)$ for the continuous-time walk and $P_\xi(x, \{y_j\})$ are related as follows:

$$\tilde{P}(x, \{y_j\}; u) = \int_0^\infty dt \ exp(-ut) \tilde{P}(x, \{y_j\}; t)$$

$$= \frac{1 - \phi(u)}{u} P_\psi(u)(x, \{y_j\}).$$  

(10)

Here $\phi(u)$ is the Laplace transformed pausing-time density. Multiplying both sides of Eq. (10) by $x$ and $x^2$, and summing over all lattice sites, we readily find the following results for the moments of the displacement of the
continuous-time random walk along the $x$-axis:

$$\bar{x}_t = (p_x - p_{-x}) \, t,$$  \hspace{1cm} (11)

and

$$\text{Var}_t(x) = (p_x + p_{-x}) \, t.$$  \hspace{1cm} (12)

In a similar fashion we find that the variance in any direction perpendicular to the field obeys

$$\text{Var}_t(y_j) = Z^{-1} \, t.$$  \hspace{1cm} (13)

![Figure 1: (Color online) The reduced variance $D$ along the $x$-axis as a function of $\beta E$ for two-dimensional lattices. The red line is the result for the Polya walk, $D = \text{Var}_n(x)/n$, Eq. (7), while the blue line is the result for the continuous-time random walk, $D = \text{Var}_t(x)/t$, Eq. (12).](image)

3. Conclusions

We conclude with the comparison of the results in Eqs. (5) and (11), Eqs. (7) and (12), and Eqs. (8) and (13). One immediately notices that both models predict exactly the same values for the “velocities” $\bar{x}_n/n$ and $\bar{x}_t/t$ (first moments of the site occupation distributions), as well as for the reduced variances $\text{Var}_n(y_j)/n$ and $\text{Var}_t(y_j)/t$ obtained from these distributions in the direction perpendicular to the field. On the contrary, for these two models, often assumed to lead to identical results, the diffusion coefficients (variances) along the $x$-axis exhibit markedly different behaviours as depicted in Fig.1. The variance along the $x$-direction for the Polya walk is a monotonically decreasing function of the field, while for the continuous-time random
walk with exponential waiting time distribution it is (for \( d \geq 2 \)) a monotonically increasing function of the field which saturates at a finite value as \( \beta E \to \infty \). In one-dimension the variance is independent of the value of the applied field. This difference in behaviours stems from the fact that for the discrete-time random walk the number of steps does not fluctuate so that the system tends toward a deterministic ballistic motion as \( \beta E \to \infty \), that is, all realisations in this limit cover the same ground and the variance vanishes. On the other hand, for the continuous-time random walk the number of steps fluctuates from realisation to realisation, which contributes to the variance no matter how strong the field (however, only along the \( x \)-direction). Note that such a striking difference between these two models is completely lost, of course, when one turns to the continuous-space and time description in the diffusion limit. Note as well that for \( \beta E \ll 1 \), both results show the same dependence on the field, which ensures the validity of the Einstein relation between diffusion coefficient and mobility for both models.

References

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[2] Montroll E W and Weiss G H (1965) J. Math. Phys. 6 167.