Precidence thinness in graphs

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\begin{abstract}
Interval and proper interval graphs are very well-known graph classes, for which there is a wide literature. As a consequence, some generalizations of interval graphs have been proposed, in which graphs in general are expressed in terms of \( k \) interval graphs, by splitting the graph in some special way.

As a recent example of such an approach, the classes of \( k \)-thin and proper \( k \)-thin graphs have been introduced generalizing interval and proper interval graphs, respectively. The complexity of the recognition of each of these classes is still open, even for fixed \( k \geq 2 \).

In this work, we introduce a subclass of \( k \)-thin graphs (resp. proper \( k \)-thin graphs), called precedence \( k \)-thin graphs (resp. precedence proper \( k \)-thin graphs). Concerning partitioned precedence \( k \)-thin graphs, we present a polynomial-time recognition algorithm based on \( PQ \) trees. With respect to partitioned precedence proper \( k \)-thin graphs, we prove that the related recognition problem is \textit{NP}-complete for an arbitrary \( k \) and polynomial-time solvable when \( k \) is fixed. Moreover, we present a characterization for these classes based on threshold graphs.

\textit{Keywords:} (proper) \( k \)-thin graphs, precedence (proper) \( k \)-thin graphs, recognition algorithm, characterization, threshold graphs.
\end{abstract}

1. Introduction

The class of \( k \)-thin graphs has recently been introduced by Mannino, Oriolo, Ricci and Chandran in [1] as a generalization of interval graphs. Motivated by this work, Bonomo and de Estrada [2] defined the class of proper \( k \)-thin graphs, which generalizes proper interval graphs. A \textit{k-thin graph} \( G \) is a graph for which

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there is a \( k \)-partition \((V_1, V_2, \ldots, V_k)\), and an ordering \( s \) of \( V(G) \) such that, for any triple \((p, q, r)\) of \( V(G) \) ordered according to \( s \), if \( p \) and \( q \) are in the same part \( V_i \) and \((p, r) \in E(G)\), then \((q, r) \in E(G)\). Such an ordering and partition are said to be consistent. A graph \( G \) is called a proper \( k \)-thin graph if \( V(G) \) admits a \( k \)-partition \((V_1, \ldots, V_k)\), and an ordering \( s \) of \( V(G) \) such that both \( s \) and its reversal are consistent with the partition \((V_1, \ldots, V_k)\). An ordering of this type is said to be a strongly consistent ordering. The interest on the study of both these classes comes from the fact that some \( \text{NP} \)-complete problems can be solved in polynomial time when the input graphs belong to them [1, 2, 3]. Some of those efficient solutions have been exploited to solve real world problems as presented in [1].

On a theoretical perspective, defining general graphs in terms of the concept of interval graphs has been of recurring interest in the literature. Firstly, note that these concepts measure “how far” a given graph \( G \) is from being an interval graph, or yet, how \( G \) can be “divided” into interval graphs, mutually bonded by a vertex order property. Namely, the vertices can be both partitioned and ordered in such a way that, for every part \( V' \) of the partition and every vertex \( v \) of the ordering, the vertex ordering obtained from the original by the removal of all vertices except from \( v \) and those in \( V' \) that precede \( v \), no matter which part \( v \) belongs, is a canonical ordering of an interval graph. Characterizing general graphs in terms of the concept of interval graphs, or proper interval graphs, is not new. A motivation for such an approach is that the class of interval graphs is well-known, having several hundreds of research studies on an array of different problems on the class, and formulating general graphs as a function of interval graphs is a way to extend those studies to general graphs. Given that, both \( k \)-thin and proper \( k \)-thin graphs are new generalizations of this kind. The complexity of recognizing whether a graph is \( k \)-thin, or proper \( k \)-thin, is still an open problem even for a fixed \( k \geq 2 \). For a given vertex ordering, there are polynomial time algorithms that compute a partition into a minimum number of classes for which the ordering is consistent (resp. strongly consistent) [2, 3]. On the other hand, given a vertex partition, the problem of deciding the existence of a vertex ordering which is consistent (resp. strongly consistent) with that partition is \( \text{NP} \)-complete [2].

Other generalizations of interval graphs have been proposed. As examples, we may cite the \( k \)-interval and \( k \)-track interval graphs. A \( k \)-interval is the union of \( k \) disjoint intervals on the real line. A \( k \)-interval graph is the intersection graph of a family of \( k \)-intervals. Therefore, the \( k \)-interval graphs generalize the concept of interval graphs by allowing a vertex to be associated with a set of disjoint intervals. The interval number \( i(G) \) of \( G \) [4] is the smallest number \( k \) for which \( G \) has a \( k \)-interval model. Clearly, interval graphs are the graphs with \( i(G) = 1 \). A \( k \)-track interval is the union of \( k \) disjoint intervals distributed in \( k \) parallel lines, where each interval belongs to a distinct line. Those lines are called tracks. A \( k \)-track interval graph is the intersection graph of \( k \)-track intervals. The multitrack number \( t(G) \) of \( G \) [5, 6] is the minimum \( k \) such that \( G \) is a \( k \)-track interval graph. Interval graphs are equivalent to the 1-interval graphs and 1-track interval graphs. The problems of recognizing \( k \)-interval and
2. Preliminaries

All graphs in this work are finite and have no loops or multiple edges. Let $G$ be a graph. Denote by $V(G)$ its vertex set, by $E(G)$ its edge set. Denote the size of a set $S$ by $|S|$. Unless stated otherwise, $|V(G)| = n$ and $|E(G)| = m$. Let $u,v \in V(G)$, define $u$ and $v$ as adjacent if $(u,v) \in E(G)$.

Let $V' \subseteq V(G)$. The induced subgraph of $G$ by $V'$, denoted by $G[V']$, is the graph $G[V'] = (V', E')$, where $E' = \{(u,v) \in E(G) \mid u,v \in V'\}$. Analogously, for some $E' \subseteq E(G)$, let the induced subgraph of $G$ by $E'$, denoted by $G[E']$, be the graph $G[E'] = (V', E')$, where $V' = \{u,v \in V(G) \mid (u,v) \in E'\}$. A graph $G'$ obtained from $G$ by removing the vertex $v \in V(G)$ is defined as $G' = (V(G) \setminus \{v\}, E')$, where $E' = \{(u,w) \in E \mid u \neq v \text{ and } w \neq v\}$.

Let $v \in V(G)$, denote by $N(v) = \{u \in V(G) \mid v \text{ and } u \text{ are adjacent}\}$ the neighborhood of $v \in V(G)$, and by $N[v] = N(v) \cup \{v\}$ the closed neighborhood of $v$. We define $u,v \in V(G)$ as true twins (resp. false twins) if $N[u] = N[v]$ (resp. $N(u) = N(v)$). A vertex $v$ of $G$ is universal if $N[v] = V(G)$. We define the degree of $v \in V(G)$, denoted by $d(v)$, as the number of neighbors of $v$ in $G$, i.e. $d(v) = |N[v]|$.

A clique or complete set (resp. stable set or independent set) is a set of pairwise adjacent (resp. nonadjacent) vertices. We use maximum to mean
maximum-sized, whereas maximal means inclusion-wise maximal. The use of minimum and minimal is analogous. A vertex \( v \in V(G) \) is said to be simplicial if \( G[N(v)] \) is a clique.

A coloring of a graph is an assignment of colors to its vertices such that any two adjacent vertices are assigned different colors. The smallest number \( t \) such that \( G \) admits a coloring with \( t \) colors (a \( t \)-coloring) is called the chromatic number of \( G \) and is denoted by \( \chi(G) \). A coloring defines a partition of the vertices of the graph into stable sets, called color classes.

A graph \( G(V, E) \) is a comparability graph if there exists an ordering \( v_1, \ldots, v_n \) of \( V \) such that, for each triple \((r, s, t)\) with \( r < s < t \), if \( v_r \sim v_s \) and \( v_s \sim v_t \) are edges of \( G \), then so is \( v_r \sim v_t \). Such an ordering is a comparability ordering. A graph is a co-comparability graph if its complement is a comparability graph.

A tree \( T \) is a connected graph that has no cycles. A rooted tree \( T_v \) is a tree in which a vertex \( v \in V(T) \) is labeled as the root of the tree. All vertices, known as nodes of \( T_v \), have implicit positions in relation to the root. Let \( u, w \in V(T_v) \), \( w \) is a descendant of \( u \) if the path from \( w \) to \( v \) includes \( u \). The node \( w \) is said to be a child of \( u \) if it is a descendant of \( u \) and \((w, u) \in E(T_v)\). The children of \( u \) are defined as the set that containing all child nodes of \( u \). The node \( u \) is said to be a leaf of \( T_v \) if it has no child in \( T_v \). The subtree \( T_u \) of \( T_v \) rooted at the node \( u \) is the rooted tree that consists of \( u \) as the root and its descendants in \( T_v \) as the nodes.

A directed graph, or digraph, is a graph \( D = (V, E) \) such that \( E \) consists of ordered pairs of \( V(G) \). A directed cycle of a digraph \( D \) is a sequence \( v_1, v_2, \ldots, v_i, 1 \leq i \leq n \), of vertices of \( V(D) \) such that \( v_i = v_1 \) and, for all \( 1 \leq j < i \), \((v_j, v_{j+1}) \in E(D)\). A directed acyclic graph (DAG) \( D \) is a digraph with no directed cycles. A topological ordering of a DAG \( D \) is a sequence \( v_1, v_2, \ldots, v_n \) of \( V(D) \) such that there are no \( 1 \leq i < j \leq n \) such that \((v_j, v_i) \in E(D)\). Determining a topological ordering of a DAG can be done in time \( O(n + m) \) [9].

An ordering \( s \) of elements of a set \( C \), denoted by \( V(s) \), consists of a sequence \( e_1, e_2, \ldots, e_n \) of all elements of \( C \). We define \( \bar{s} \) as the reversal of \( s \), that is, \( \bar{s} = e_n, e_{n-1}, \ldots, e_1 \). We say that \( e_i \) precedes \( e_j \) in \( s \), denoted by \( e_i < e_j \), if \( i < j \). An ordered tuple \((a_1, a_2, \ldots, a_k)\) of some elements of \( C \) is ordered according to \( s \) when, for all \( 1 \leq i < k \), if \( a_i = e_j \) and \( a_{i+1} = e_z \), then \( j < z \).

Given orderings \( s_1 \) and \( s_2 \), the ordering obtained by concatenating \( s_1 \) to \( s_2 \) is denoted by \( s_1s_2 \).

2.1. Interval graphs

The intersection graph of a family \( \mathcal{F} \) of sets is the graph \( G \) such that \( V(G) = \mathcal{F} \) and \( S, T \in V(G) \) are adjacent if and only if \( S \cap T \neq \emptyset \). An interval graph \( G \) is the intersection graph of a family \( \mathcal{R} \) of intervals of the real line; such a family is called an interval model, or a model, of the graph. We say that \( \mathcal{R} \) is associated to \( G \) and vice-versa. It is worth mentioning that an interval graph can be associated to several models but an interval model can be associated to a unique graph. Concerning interval graphs, there are some characterizations which are relevant to this work, that we present next.
Figure 1: (a) Canonical ordering and (b) proper canonical ordering.

Theorem 1 ([10]). A graph $G$ is an interval graph if, and only if, there is an ordering $s$ of $V(G)$ such that, for any triple $(p,q,r)$ of $V(G)$ ordered according to $s$, if $(p,r) \in E(G)$, then $(q,r) \in E(G)$.

The ordering described in Theorem 1 is said to be a canonical ordering. Figure 1(a) depicts an interval graph in which the vertices are presented from left to right in one of its canonical orderings.

Let $R$ be an interval model of an interval graph $G$. Note that, if we consider a vertical line that intersects a subset of intervals in $R$, then these intervals consist of a clique in $G$. This is true because they all contain the point in which this line is defined. Moreover, if the given line traverses a maximal set of intervals, then the corresponding clique is maximal. Figure 2 depicts an interval model and the vertical lines that correspond to maximal cliques of the graph in Figure 1(a). The following is a characterization of interval graphs in terms of the maximal cliques of the graph.

Theorem 2 ([11]). A graph $G$ is an interval graph if, and only if, there is an ordering $s_C = C_1, C_2, \ldots, C_k$ of its maximal cliques such that if $v \in C_i \cap C_z$ with $1 \leq i \leq z \leq k$, then $v \in C_j$, for all $i \leq j \leq z$.

The ordering described in Theorem 2 is said to be a canonical clique ordering. The ordering of the maximal cliques depicted in Figure 2, read from left to right, represents a canonical clique ordering.

A proper interval graph is an interval graph that admits an interval model in which no interval properly contains another. There is a characterization
of proper interval graphs which is similar to that presented in Theorem 1, as
described next.

**Theorem 3** \([\text{[1]}]\). A graph \(G\) is a proper interval graph if, and only if, there
is an ordering \(s\) of \(V(G)\) such that, for any triple \((p,q,r)\) of \(V(G)\) ordered
according to \(s\), if \((p,r) \in E(G)\), then \((p,q),(q,r) \in E(G)\).

The ordering specified in the previous theorem is defined as a *proper canonical*
ordering. The ordering of the vertices of the graph depicted in Figure 1(b)
from left to right is proper canonical. Note that a proper canonical ordering is
also a canonical ordering. An important property of proper canonical orderings
is the following:

**Lemma 4** \([\text{[1]}]\). If \(G\) is a connected proper interval graph. Then a proper
canonical ordering of \(G\) is unique up to reversion and permutation of mutual
twin vertices.

As a consequence, if a proper interval graph \(G\) is disconnected, since each
component has a unique proper canonical ordering, then the canonical ordering
of \(G\) consists of a permutation of canonical orderings of each of those components.

Let \(s\) be an ordering of \(V(G)\) and \(s_C = C_1, C_2, \ldots, C_k\) an ordering of the
maximal cliques of \(G\). The sequence \(s\) is said to be *ordered according to \(s_C\)*
if for all \(u, v \in V(G)\) such that \(u < v\) in \(s\), there are no \(1 \leq i < j \leq k\) such
that \(v \in C_i \setminus C_j\) and \(u \in C_j\). As an example, note that the canonical ordering
of the Figure 1(b) is ordered according to the maximal clique ordering \(s_C =
C_1, C_2, \ldots, C_6\) of the Figure 2. The following lemma relates the characterization
of canonical orderings and canonical clique orderings.

**Lemma 5.** Let \(G\) be a graph and \(s\) be an ordering of \(V(G)\). The ordering \(s\) is a
canonical ordering of \(V(G)\) if, and only if, \(s\) is ordered according to a canonical
clique ordering of \(G\).

*Proof.* Consider \(s_C\) a canonical clique ordering of \(G\). We will show that is
possible to build a canonical ordering \(s\) from \(s_C\), respecting its clique ordering.
To achieve this, first start with an empty sequence \(s\). Iteratively, for each
element \(X\) of the sequence \(s_C\), choose all simplicial vertices of \(X\), adding them
to \(s\) in any order and removing them from \(G\). Note that some of the vertices of
\(X\) that were not removed from \(G\), because they were not simplicial in \(G\), now
may be turned simplicial by the removal of vertices. Clearly, at the end of the
process, \(s\) will contain all the vertices of \(G\). Suppose \(s\) is not a canonical ordering,
that is, there are \(p, q, r \in V(G), p < q < r\) in \(s\), such that \((p,r) \in E(G)\) and
\((q,r) \notin E(G)\). Let \(C_p, C_q\) and \(C_r\) be the maximal cliques of \(s_C\) being processed
at moment \(p, q\) and \(r\) were choose, respectively. Note that \(C_p < C_q < C_r\) in \(S_e\)
and, as \(C_p\) is the last maximal clique in \(s_C\) that contains \(p\) and \((p,r) \in E(G),
\(r \in C_p\). Besides, as \((q,r) \notin E(G)\), \(r \notin C_q\). Therefore, \(r \in C_p \cap C_r\) and \(r \notin C_q\).
A contradiction with the fact that \(s_C\) is a canonical clique ordering. Hence, \(s\)
is a canonical ordering.
Consider \( s = v_1, v_2, \ldots, v_n \) be a canonical ordering of \( G \). We prove by induction in \(|V(G)| = n\) that \( s \) is ordered according to a canonical clique ordering. Clearly, the statement is true for \( n = 1 \). Suppose that the statement is true for any \( 1 \leq n' < n \). Let \( s' \) be the sequence obtained from \( s \) by removing \( v_n \). Clearly \( s' \) is also a canonical ordering and, by the induction hypothesis, \( s' \) is ordered according with a canonical clique ordering \( s'_C = C_1, C_2, \ldots, C_k \).

Let \( C_i \), with \( 1 \leq i \leq k \), be the first clique of \( s'_C \) such that \( v_n \in N(C_i) \). Let \( C_i' = \{ C_j \cap N(v_n) \} \cup \{ v_n \} \), \( i \leq j \leq k \). Note that, as \( s \) is a canonical ordering, for all \( C_j \) of \( s'_C \) such that \( i < j \leq k \), \( \{ C_j \cap N(v_n) \} \cup \{ v_n \} \) is a maximal clique of \( G \).

If \( C_i \subseteq N(v_n) \), then \( s'_C = C_1, C_2, \ldots, C'_i, C'_{i+1}, \ldots, C_k \) is a canonical clique ordering that matches \( s \). Otherwise, \( s'_C = C_1, C_2, \ldots, C_i, C'_i, C'_{i+1}, \ldots, C_k \) is a canonical clique ordering that matches \( s \). Therefore, \( s \) is ordered according to a canonical clique ordering of \( G \).

### 2.1.1. PQ trees

A **PQ tree** \([12]\) \( T \) is a data structure consisting of an ordered tree that describes a family \( F \) of permutations of elements from a given set \( C \). In a PQ tree \( T \), the set of leaves is \( C \) and the permutation being represented by \( T \) is the sequence of the leaves from left to right. Regarding internal nodes, they are classified into two types, the \( P \) and the \( Q \) nodes. An equivalent PQ tree \( T' \) to \( T \) is a tree obtained from \( T \) by any sequence of consecutive transformations, each consisting of either permuting the children of a \( P \) node, or reversing the children of a \( Q \) node. The family \( F \) of permutations of elements from \( C \) represented by \( T \) is that of permutations corresponding to all equivalent PQ trees to \( T \).

Graphically, in a PQ tree, leaves and \( P \) nodes are represented by circles and \( Q \) nodes by rectangles. In representations of schematic PQ trees, a node represented by a circle over a rectangle will denote that, in any concrete PQ tree conforming the scheme, such a node is either a \( P \) node, or a \( Q \) node. Figure 3 depicts the described operations. In Figure 3(a), we have a partial representation of a PQ tree. Figure 3(b) depicts an equivalent PQ tree obtained from a permutation of children of a \( P \) node of this tree and Figure 3(c) exemplifies an equivalent PQ tree from the reversion of the children of a \( Q \) node.

One of the applications from the seminal paper introducing PQ trees is that of recognizing interval graphs. In such an application, each leaf of the PQ tree is a maximal clique of an interval graph \( G \) and the family of permutations the tree represents is precisely all the canonical clique orderings of \( G \) \([12]\). A PQ tree can be constructed from an interval graph in time \( O(n + m) \) \([12]\).

We will say that a vertex \( v \) belongs to a node \( X \) of a PQ tree, and naturally denote by \( v \in X \), if it belongs to any leaf that descends from this node. Figure 4 depicts a PQ tree of the interval graph of Figure 1(a) according to the maximal cliques in Figure 2. The permutations implicitly represented by this PQ tree are:
Figure 3: A PQ tree (a) and an example of permutations (b) and reversions (c) of children of its nodes.

- C_1, C_2, C_3, C_4, C_5, C_6
- C_1, C_2, C_3, C_4, C_5
- C_1, C_2, C_4, C_3, C_5, C_6
- C_1, C_2, C_4, C_3, C_6, C_5
- C_6, C_5, C_4, C_3, C_2, C_1
- C_5, C_6, C_4, C_3, C_2, C_1
- C_6, C_5, C_3, C_4, C_2, C_1
- C_5, C_6, C_3, C_4, C_2, C_1

Figure 4: A PQ tree of the interval graph of Figure 1(a) according to the maximal cliques in Figure 2.

2.2. Thinness and proper thinness

A graph G is called a k-thin graph if there is a k-partition (V_1, V_2, ..., V_k) of V(G) and an ordering s of V(G) such that, for any triple (p, q, r) of V(G) ordered according to s, if p and q are in a same part V_i and (p, r) ∈ E(G), then (q, r) ∈ E(G). An ordering and a partition satisfying that property are called consistent. That is, a graph is k-thin if there is an ordering consistent with some k-partition of its vertex set. The thinness of G, denoted by thin(G), is the minimum k for which G is a k-thin graph.

A graph G is called a proper k-thin graph if G admits a k-partition (V_1, ..., V_k) of V(G) and an ordering s of V(G) consistent with the partition and, addition-
ally, for any triple \((p, q, r)\) of \(V(G)\), ordered according to \(s\), if \(q\) and \(r\) are in a same part \(V_i\) and \((p, r) \in E(G)\), then \((p, q) \in E(G)\). Equivalently, an ordering \(s\) of \(V(G)\) such that \(s\) and its reverse are consistent with the partition. Such an ordering and partition are called strongly consistent. The proper thinness of \(G\), or \(pthin(G)\), is the minimum \(k\) for which \(G\) is a proper \(k\)-thin graph.

Figures 5(a) and 5(b) depict two bipartitions of a graph, in which the classes are represented by distinct colors, and two different vertex orderings. The ordering of Figure 5(a) is consistent with the corresponding partition but not strongly consistent, while the ordering of Figure 5(b) is strongly consistent with the corresponding partition.

Note that \(k\)-thin graphs (resp. proper \(k\)-thin graphs) generalize interval graphs (resp. proper interval graphs). The 1-thin graphs (resp. proper 1-thin graphs) are the interval graphs (resp. proper interval graphs). The parameter \(\text{thin}(G)\) (resp. \(\text{pthin}(G)\)) is in a way a measure of how far a graph is from being an interval graph (resp. proper interval graph).

For instance, consider the graph \(C_4\). Since \(C_4\) is not an interval graph, \(\text{pthin}(C_4) \geq \text{thin}(C_4) > 1\). Figure 5 proves that \(\text{thin}(C_4) = \text{pthin}(C_4) = 2\).

A characterization of \(k\)-thin or proper \(k\)-thin graphs by forbidden induced subgraphs is only known for \(k\)-thin graphs within the class of cographs \[2\]. Graphs with arbitrary large thinness were presented in \[1\], while in \[2\] a family of interval graphs with arbitrary large proper thinness was used to show that the gap between thinness and proper thinness can be arbitrarily large. The relation of thinness and other width parameters of graphs like boxicity, pathwidth, cutwidth and linear MIM-width was shown in \[1, 2\].

Let \(G\) be a graph and \(s\) an ordering of its vertices. The graph \(G_s\) has \(V(G)\) as vertex set, and \(E(G_s)\) is such that for \(v < w, (v, w) \in E(G_s)\) if and only if there is a vertex \(z\) in \(G\) such that \(v < w < z, (z, v) \in E(G)\) and \((z, w) \notin E(G)\). Similarly, the graph \(\overline{G}_s\) has \(V(G)\) as vertex set, and \(E(\overline{G}_s)\) is such that for \(v < w, (v, w) \in E(\overline{G}_s)\) if and only if either there is a vertex \(z\) in \(G\) such that \(v < w < z, (z, v) \in E(G)\) and \((z, w) \notin E(G)\) or there is a vertex \(x\) in \(G\) such that \(x < v < w, (x, w) \in E(G)\) and \((x, v) \notin E(G)\).

**Theorem 6.** \[2, 3\] Given a graph \(G\) and an ordering \(s\) of its vertices, a partition of \(V(G)\) is consistent (resp. strongly consistent) with the ordering \(s\) if and only if the partition is a valid coloring of \(G_s\) (resp. \(\overline{G}_s\)), which means that each part corresponds to a color in the coloring under consideration.

Figure 5: (a) A consistent ordering and a (b) strongly consistent ordering of \(V(C_4)\), for the corresponding 2-partitions.
2.3. Precedence thinness and precedence proper thinness

In this work, we consider a variation of the problems described in the last subsection by requiring that, given a vertex partition, the (strongly) consistent orderings hold an additional property.

A graph $G$ is precedence $k$-thin (resp. precedence proper $k$-thin), or $k$-PT (resp. $k$-PPT), if there is a $k$-partition of its vertices and a consistent (resp. strongly consistent) ordering $s$ for which the vertices that belong to a same part are consecutive in $s$. Such an ordering is called a precedence consistent ordering (resp. precedence strongly consistent ordering) for the given partition. We define $\text{pre-thin}(G)$ (resp. $\text{pre-pthin}(G)$) as the minimum value $k$ for which $G$ is $k$-PT (resp. $k$-PPT).

The Figure 6 illustrates a graph that is a 2-PPT graph. The convention assumed is that the strongly consistent ordering being represented consists of the vertices ordered as they appear in the figure from bottom to top and, for vertices arranged in a same horizontal line, from left to right. Therefore, the strongly consistent ordering represented in Figure 6 is $s = a, b, c, a', b', c'$. The graph $C_4$ is not 2-PPT, despite $\text{pthin}(C_4) = 2$. It can be easily verified by brute-force that, for all possible bipartitions of its vertex set and for all possible orderings $s$ in which the vertices of a same part are consecutive in $s$, the ordering and the partition are not strongly consistent. On the other hand, a $k$-PPT graph is a proper $k$-thin graph. Therefore, the class of $k$-PPT graphs is a proper subclass of that of proper $k$-thin graphs.

Figure 6: A 2-PPT graph.

If a vertex order $s$ is given, by Theorem 6, any partition which is precedence (strongly) consistent with $s$ is a valid coloring of $G_s$ (resp. $\tilde{G}_s$) such that, additionally, the vertices on each color class are consecutive according to $s$. A greedy algorithm can be used to find a minimum vertex coloring with this property in polynomial time. Such method is described next, in Theorem 7.

**Theorem 7.** Let $G$ be a graph and $s$ an ordering of $V(G)$. It is possible to obtain a minimum $k$-partition $V$ of $V(G)$, in polynomial time, such that $s$ is a precedence (strongly) consistent ordering concerning $V$.

**Proof.** Consider the following greedy algorithm that obtains an optimum coloring of $G_s$ (resp. $\tilde{G}_s$) in which vertices having a same color are consecutive.
in $s$. That is, $s$ is a precedence consistent (resp. precedence strongly consistent) ordering concerning the partition defined by the coloring. Color $v_1$ with color $1$. For each $v_i$, $i > 1$, let $c$ be the last color used. Then color $v_i$ with color $c$ if there is no $v_j$, $j < i$, colored with $c$ such that $(v_j, v_i) \in E(G_s)$ (resp. $(v_j, v_i) \in E(\tilde{G}_s)$). Otherwise, color $v_i$ with color $c + 1$. We show, by induction on $|s| = n$, that the algorithm finds an optimal coloring in which each vertex has the least possible color.

The case where $n = 1$ is trivial. Suppose that the algorithm obtains an optimum coloring of $G_s$ (resp. $\tilde{G}_s$) and use the given algorithm to color the resulting graph. By the induction hypothesis, the chosen coloring for $v_1, v_2, \ldots, v_{n-1}$ is optimal. Moreover, the colors are non-decreasing and each vertex is colored with the least possible color. Now, add the removed vertex $v_n$ to $s$ and to the graph $G_s$ (resp. $\tilde{G}_s$), with its respective edges, and let the algorithm choose a coloring for it. If the color of $v_n$ is equal to the color of $v_{n-1}$ the algorithm is optimal by the induction hypothesis. Otherwise, $v_n$ is colored with a new color $c'$. Suppose the chosen coloring is not optimal. That is, it is possible to color $v$ with an existing color. This implies that there is at least a neighbor $v_j$ of $v_n$, in $G_s$ (resp. $\tilde{G}_s$), that can be recolored with a smaller color. This is an absurd because the algorithm has already chosen the least possible color for all the vertices of $G_s \setminus \{v_n\}$ (resp. $\tilde{G}_s \setminus \{v_n\}$), relative to $s$. □

In the following sections, we will deal with the case where the vertex partition is given and the problem consists of finding the vertex ordering. From now on, we will then simply call precedence consistent ordering (resp. precedence strongly consistent ordering) to one that is such for the given partition.

### 3. Precedence thinness for a given partition

In this section, we present an efficient algorithm to precedence $k$-thin graph recognition for a given partition. This algorithm uses $PQ$ trees and some related properties to validate precedence consistent orderings in a greedy fashion, iteratively choosing an appropriate ordering of the parts of the given partition that satisfies precedence consistence, if one does exist. Formally, the problem addressed in this chapter is the following.

| Problem: | PARTITIONED $k$-PT (Recognition of $k$-PT graphs for a given partition) |
| Input: | A natural $k$, a graph $G$ and a partition $(V_1, \ldots, V_k)$ of $V(G)$. |
| Question: | Is there a consistent ordering $s$ of $V(G)$ such that the vertices of $V_i$ are consecutive in $s$, for all $1 \leq i \leq k$? |

It should be noted that a precedence consistent ordering $s$ consists of a concatenation of the consistent orderings of $G[V_i]$, for all $1 \leq i \leq k$. That is, $s = s_1 s_2 \ldots s_k$, where $s_1, s_2, \ldots, s_k$ is a permutation of $s'_1, s'_2, \ldots, s'_k$ and $s'_i$ is a canonical ordering of $G[V_i]$, for all $1 \leq i \leq k$. The following property is straightforward from the definition of a precedence consistent ordering.
Property 1. Let \((V_1, V_2, \ldots, V_k)\) be a partition of \(V(G)\), \(s\) a precedence consistent ordering and \(1 \leq i, j \leq k\). If \(V_i\) precedes \(V_j\) in \(s\), then, for all \(u, v \in V_i\) and \(w \in V_j\), if \((u, w) \notin E(G)\) and \((v, w) \in E(G)\), then \(u\) precedes \(v\) in \(s\).

Property 1 shows that, for any given consistent ordering \(s = s_1s_2 \ldots s_k\), the vertices of \(s_j\) impose ordering restrictions on the vertices of \(s_i\), for all \(1 \leq i < j \leq k\). This relation is depicted in Figure 7. This property will be used as a key part of the greedy algorithm to be presented later on.

\[\text{Figure 7: Precedence relations among the vertices in a precedence consistent ordering.}\]

Let \(s_T = C_1, C_2, \ldots, C_q\) be an ordering of the maximal cliques of an interval graph \(G\) obtained from a \(PQ\) tree \(T\). Recall from Section 2 that it is possible to obtain a canonical ordering \(s\) ordered according to \(s_T\). Let \(u, v \in V(G)\).

We define \(T\) as compatible with the ordering restriction \(u < v\) if there exists a canonical ordering \(s\) ordered according to \(s_T\) such that \(u < v\) in \(s\). The following theorem describes compatibility conditions between a \(PQ\) tree and an ordering restriction \(u < v\).

**Theorem 8.** Let \(G\) be an interval graph and \(T\) be a \(PQ\) tree of \(G\). Let \(X\) be a node of \(T\) with children \(X_1, \ldots, X_k\) and \(u, v \in X\). Denote by \(T_X\) the subtree rooted at \(X\). The following statements are true.

(i) if \(v\) belongs to all leaves of \(T_X\), then \(T_X\) is compatible with \(u < v\) (see Figure 8(a)).

(ii) Let \(X_i, X_j \in \{X_1, \ldots, X_k\}\). If \(u \in X_i\), \(v \not\in X_i\), \(v \in X_j\) and \(T_X\) is compatible with \(u < v\), then \(X_i\) precedes \(X_j\) in \(T_X\) (see Figure 8(b)).

**Proof.**

Let \(G_X\) be the graph induced by the union of the leaves of \(T_X\).

(i) Let \(s_x\) be a canonical ordering of \(G_X\) and \(s'_x\) the ordering obtained from \(s_x\) by moving \(v\) to the last position. As \(v\) is an universal vertex from \(G_X\), \(s'_x\) is also a canonical ordering of this graph. Consequently, \(T_X\) is compatible with \(u < v\).
(ii) Suppose $X_j$ precedes $X_i$ in $T_X$ and $T_X$ is compatible with $u < v$. As $v \not\in X_i$, then by Theorem 2 there is no $X_z$ such that $X_i$ precedes $X_z$ and $v \in X_z$. Otherwise, there would exist three maximal cliques $C_i, C_j, C_z$ such that $C_j < C_i < C_z$ in the ordering of cliques represented in $T_X$ and such that $v \not\in C_i, v \in C_j \cap C_z$. Therefore $v < u$ in any canonical ordering $s_X$ of $G$ (Lemma 3), a contradiction because $T_X$ is compatible with $u < v$. Thus, $X_i$ precedes $X_j$ in $T_X$.

Figure 8: Ordering imposed by Theorem 8 item i (a) and item ii (b).

Theorem 8 can be used to determine the existence of a PQ tree compatible with an ordering restriction $u < v$. This task can be achieved by considering as the node $X$ of Theorem 8 each one of the nodes of a given PQ tree $T$. If $T$ violates the conditions imposed by the theorem, $T$ is “annotated” in a way that the set of equivalent PQ trees is restricted, avoiding precisely the violations. This procedure continues until all nodes produce their respective restrictions, in which case any equivalent PQ tree allowed by the “annotated” tree $T$ is compatible with the given ordering restrictions. If there is no equivalent PQ tree to the “annotated” tree $T$, which means there is no way to avoid the violations, then there is no tree which is compatible with such an ordering restriction.

The general idea to “annotate” a PQ tree is to use an auxiliary digraph to represent required precedence relations among the vertices. This digraph is constructed for each part and any topological ordering of it results in a precedence consistent ordering concerning the vertices of this part. Property 1 is applied to determine the ordering restrictions of the vertices and Theorem 8 to ensure that at the end of the algorithm the vertices are ordered according to a canonical clique ordering. Such a procedure is detailed next.

First, the algorithm validates if each part of the partition induces an interval graph. This step can be accomplished, for each part, in linear time [12]. If at least one of these parts does not induce an interval graph, then the answer
is NO. Otherwise, the algorithm tries each part as the first of a precedence consistent ordering. For each candidate part $V_i$, $1 \leq i \leq k$, it builds a digraph $D$ to represent the order conditions that the vertices of $V_i$ must satisfy in the case in which $V_i$ precedes all the other parts of the partition. That is, $V_i$ must be ordered in such a way that it is according to a canonical clique ordering and respects the restrictions imposed by Property $[\text{1}].$ In this strategy, the vertex set of $D$ is $V_i$ and its directed edges represent the precedence relations among its vertices. Namely, $(u, v) \in E(D)$ if, and only if, $u$ must precede $v$ in all precedence consistent orderings that have $V_i$ as its first part. The algorithm uses Property $[\text{1}]$ to find all the ordering restrictions $u < v$ among the vertices of $V_i$ imposed by the others parts, adding the related directed edges to $D$. Then, by building a $PQ$ tree $T$ of $G[V_i]$, Theorem $[\text{3}]$ is used to transform $T$ into $T'$ ensuring that all those ordering restrictions $u < v$ are satisfied. If there is a $PQ$ tree $T'$ of $G[V_i]$ that is compatible with all the ordering restrictions imposed by Property $[\text{1}]$, then the algorithm adds directed edges to $D$ according to the canonical clique ordering represented by $T'$. This step is described below in the Algorithm $[\text{1}]$ and it is similar to the one described in Lemma $[\text{5}]$. At this point, $D$ is finally constructed and the existence of a topological ordering for its vertices determines whether $V_i$ can be chosen as the first part of a precedence consistent ordering for the given partition. If that is the case, $V_i$ is chosen as the first part and the process is repeated in $G \setminus V_i$ to choose the next part. If no part can be chosen at any step, the answer is NO. Otherwise, a feasible ordering of the parts and of the vertices within each part is obtained and the answer is YES. Next, the validation of the compatibility of $T$ is described in more detail.

\begin{algorithm}
\caption{Adding edges from $PQ$-tree to $D$}
\label{addEdgesFromPQTree}
\textbf{Input:} $G$: an interval graph; $D$: a digraph; $T$: a $PQ$-tree;
\begin{algorithmic}
\Procedure{addEdgesFromPQTree}{$G$, $D$, $T$}
\State Let $s_C$ be the canonical clique ordering relative to $T$
\For {each $C_i \in s_C$} 
\State Let $S$ be the set of simplicial vertices of $C_i$
\For {each $v \in S$} 
\For {each $u \in C_{i+1}$} 
\State $E(D) \leftarrow E(D) \cup \{(v, u)\}$
\EndFor
\EndFor
\State $G \leftarrow G \setminus S$
\EndFor
\EndProcedure
\end{algorithmic}
\end{algorithm}

For each imposed ordering $u < v$ (that is, $(u, v) \in E(D)$), $T$ is traversed node by node, applying Theorem $[\text{8}]$. As a consequence of such an application, if the order of the children of some node $X$ of $T$ must be changed to meet some restrictions, directed edges are inserted on $T$ to represent such needed reorderings. Those directed edges appear among nodes that are children of a same node in $T$. At the end, to validate if $T$ is compatible with all needed reorderings of children of nodes, a topological ordering is applied to the children of each node. If there are no cycles among the children of each node, then there
is an equivalent \( PQ \) tree \( T' \) compatible with all the restrictions. In this case, \( T' \) is obtained from \( T \) applying the sequence of permutations of children of \( P \) nodes and reversals of children of \( Q \) nodes which are compliant to the topological orderings. Otherwise, if it is detected a cycle in some topological sorting, then there is no \( PQ \) tree of \( G[V_i] \) which is compatible with all the set of restrictions. In other words, \( V_i \) cannot be chosen as the first part in a precedence consistent ordering for the given partition. Algorithm 2 formalizes the procedure.

To illustrate the execution of Algorithm 2, consider the graph \( G \) as defined in Figure 9 and the 3-partition \( \mathcal{V} = (V_1, V_2, V_3) \) of \( \mathcal{V}(G) \) where \( V_1 = \{a, b, c, d, e, f, g, h, i, j, k, l\} \), \( V_2 = \{a', b', c', d', e', f', g, h', i', j', k', l'\} \), and \( V_3 = \{a'', b'', c'', d'', e'', f'', g'', h'', i'', j'', k'', l''\} \). For the sake of clearness, in Figure 9 the edges with endpoints in distinct parts are depicted in black, the edges with endpoints in a same part are in light gray and the vertices belonging to distinct parts are represented with different colors. Moreover, the vertices of each part, read from the left to right, consist of a canonical ordering of the graph induced by that part. Each part of \( \mathcal{V} \) induces the interval graph \( G' \) depicted in Figure 10(a). Figure 10(b) depicts a model of \( G' \). In this model, all maximal cliques are represented by vertical lines. Figure 10(c) represents a \( PQ \) tree of \( G' \) in which each maximal clique is labeled according to the model in Figure 10(b).

Figure 9: A graph \( G \) and a 3-partition \( \mathcal{V} = (V_1, V_2, V_3) \) of its vertices where \( V_1 = \{a, b, c, d, e, f, g, h, i, j, k, l\} \), \( V_2 = \{a', b', c', d', e', f', g, h', i', j', k', l'\} \), and \( V_3 = \{a'', b'', c'', d'', e'', f'', g'', h'', i'', j'', k'', l''\} \).

Suppose that, at the first step, the algorithm tries to choose \( V_1 \) as the first part of the precedence consistent ordering. As mentioned, \( G[V_1] \cong G' \) and, according to the model in Figure 10(b), \( G[V_1] \) has maximal cliques \( C_1 = \{a, b, c\} \).
Algorithm 2: Partitioned $k$-PT

Input: $G$: a graph; $k$: a natural number; $V$: a $k$-partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$;

function partitioned-$k$-PT($G$, $k$, $V$)
    $s \leftarrow \emptyset$
    for each $V_i \in V$
        if $G[V_i]$ is not an interval graph then
            return (NO, $\emptyset$)
    while $V \neq \emptyset$
        for each $V_i \in V$
            $foundFirstPart \leftarrow$ TRUE
            Create a digraph $D = (V_i, \emptyset)$
            Build a PQ tree $T_i$ of $G[V_i]$
            for each $V_j \in V$ such that $V_j \neq V_i$
                Let $S$ be the set of precedence relations among the vertices of $V_i$ concerning $V_j$ (Property $[1]$)
                for each $(u < v) \in S$
                    $E(D) \leftarrow E(D) \cup \{(u, v)\}$
                for each node $X$ of $T_i$
                    Add the direct edges, deriving from $(u < v)$, among the children of $X$ (Theorem $8$)
            for each node $X$ of $T_i$
                Let $D_X = (V', E')$ be the digraph where $V'$ is the set of the children of $X$ and $E'$ are the directed edges added among them
                if there is a topological ordering $s_X$ of $D_X$ then
                    Arrange the children of $X$ according to $s_X$
                else
                    $foundFirstPart \leftarrow$ FALSE
            if $foundFirstPart$ then
                addEdgesFromPQTree($G[V_i]$, $D$, $T_i$)
                if there is a topological ordering $s_i$ of $D$ then
                    $s \leftarrow ss_i$
                    $V \leftarrow V \setminus V_i$
                else
                    $foundFirstPart \leftarrow$ FALSE
            else
                $foundFirstPart \leftarrow$ FALSE
        if not $foundFirstPart$ then
            return (NO, $\emptyset$)
    return (YES, $s$)
Figure 10: An interval graph $G'$ (a); an interval model (b) and a PQ tree (c) of $G$.

$C_2 = \{c,d\}$, $C_3 = \{d,l\}$, $C_4 = \{e,l\}$, $C_5 = \{f,g,l\}$, $C_6 = \{g,j,l\}$, $C_7 = \{h,i,j,l\}$ and $C_8 = \{i,j,k,l\}$. Concerning the edges between $V_1$ and $V_2$ and according to Property 1, the vertex $a$ of $V_1$ must succeed all the other vertices of this part in any valid canonical ordering. This requirement is translated into the corresponding PQ tree through directed edges as depicted in Figure 11(a). Let $X$ be the current node of $T$. Note that if $X$ is a $Q$ node, then any imposed ordering of a pair of its children implies in ordering all of them. In Figure 11(a), the oriented edges deriving from Property 1 are represented in blue and the edges deriving from the orientation demanded by $Q$ nodes, due to the presence of the blue ones, are represented in orange. Clearly, there is a valid PQ tree that satisfies such orientations. Now the algorithm adds the directed edges to the tree deriving from fact that $V_1$ must also precede $V_3$. Considering the edges
between $V_1$ and $V_3$, and according to Property 1, the vertex $f$ of $V_1$ must succeed all the other vertices of $V_1$ in any valid canonical ordering. This requirement is translated into the $PQ$ tree in Figure 11(a) resulting in the $PQ$ tree in Figure 11(b). Clearly, no $PQ$ tree can satisfy the given orientation due to the directed cycle at the first level of the tree. Then, $V_1$ cannot precede both $V_2$ and $V_3$.

![Diagram](image)

Figure 11: The edges added to the $PQ$ tree of Figure 10(c) through the example of execution of the Algorithm 2.

As $V_1$ cannot be chosen as the first part, the algorithm tries another part as the first of the precedence consistent ordering. Suppose it now chooses $V_2$ as the first part. The interval graph $G[V_2]$ has as maximal cliques $C_1 = \{a', b', c'\}$, $C_2 = \{c', d'\}$, $C_3 = \{d', l'\}$, $C_4 = \{e', l'\}$, $C_5 = \{f', g', l'\}$, $C_6 = \{g', j', l'\}$, $C_7 = \{h', j', j', l'\}$ and $C_8 = \{i', j', k', l'\}$. Note that, as there are no edges between $V_2$ and $V_3$, $V_2$ can precede $V_3$ in any valid consistent ordering. The algorithm must
decide whether $V_2$ can precede $V_1$. According to the Figure 9 and Property 1, the vertices \{a', b', c', d'\} of $V_2$ must succeed all the other vertices of the same part in any valid consistent ordering. This requirement is again translated into directed edges in the PQ tree of $V_2$ as depicted in Figure 11(c). Clearly, there is a PQ tree $T'$ satisfying those directed edges. Then, the algorithm adds the edges deriving from the canonical clique ordering represented by $T'$ to $D$. Figure 12(a) depicts the final state of $D$ once the necessary edges has been added. In this figure, the edges deriving from Property 1 are presented with an orange color and the edges related to $T'$ are presented in a blue color. For readability, in these figures the edges that can be obtained by transitivity are omitted. As there no cycles in $D$, $V_2$ can precede $V_1$ and $V_3$. A topological ordering of $D$ leads to the canonical ordering $s = e', f', g', h', i', j', k', l', d', a', b', c'$ of $G[V_2]$.

After deciding $V_2$ as the first part, the algorithm uses the same process to choose the second part. Suppose it tries $V_1$ as the second part. Figure 11(d) depicts the edges added to the PQ tree of $G[V_1]$. As there are no cycles in the edges added, there is a tree which is compatible with the precedence relations associated with $V_1 < V_3$. Figure 12(b) depicts the final state of the digraph $D$ related to $V_1$. As there are no cycles in $D$, $V_1$ can precede $V_3$, so the algorithm chooses it as the second part. Finally, the algorithm chooses $V_3$ as the last
part and determine that there is a precedence consistent ordering such that \( V_2 < V_1 < V_3 \).

Concerning the complexity of the given strategy, each time one of the \( k \) parts is tried to be the first, we build a new digraph \( D \), a new PQ tree \( T \) and obtain the precedence relations according to Property 1. Enumerating all the precedence relations requires at most \( O(n^3) \) steps, which is the time that it takes to iterate over all triples of vertices of the graph. Moreover, each one of these relations must be mapped to \( D \), which takes time \( O(1) \), and \( T \). First, note that the number of nodes of \( T \) is asymptotically bounded by its number of leaves, that is, by the number of maximal cliques of the part being processed. As the number of maximal cliques is bounded by the number of vertices of the given part, the number of nodes of \( T \) is \( O(n) \). Consequently, it is possible to model a precedence relation of type \( u < v \) into \( T \), using Theorem 8, in time \( O(n^2) \).

To achieve this, first \( T \) is traversed, in order to decide which nodes contain (resp. not contain) \( u \) and \( v \). A traversal of \( T \) can be done in time \( O(n) \), and \( T \) can be constructed in \( O(n + m) \) time. Additional steps will be necessary and generate new traversals in \( T \) following the tree levels, with the purpose to add the necessary directed edges among the vertices that are children of the same node. This step can be done in

\[
\sum_{v \in V(T)} d^2(v) \leq \sum_{v \in V(T)} d(v)|V(T)| = O(|E(T)||V(T)|) = O(|V(T)|^2) = O(n^2)
\]

as \( |E(T)| = O(|V(T)|) \) and \( |V(T)| = O(n) \). Aiming to verify the existence of a compatible tree, the algorithm applies a topological ordering to the children of each node of \( T \), which takes overall time \( O(n(n + m)) \). Then, the ordering in \( T \) is translated to \( D \) through directed edges. By using the Algorithm 1, this step requires no more than \( O(n^2 + m) \) operations. Finally, a topological ordering is applied to \( D \). Thus, the algorithm has

\[
O(k^2(n + m + n^3n^2 + n^2 + nm + n^2 + m + n + m)) = O(k^2n^5)
\]

time complexity.

4. Precedence Proper Thinness for a Given Partition

In this section, we discuss precedence proper thinness for a given partition. First, we prove that this problem is \( \text{NP} \)-complete for an arbitrary number of parts. Then, we propose a polynomial time algorithm for a fixed number of parts based on the one presented in Section 3. Formally, we will prove that the following problem is \( \text{NP} \)-complete.

| Problem: | PARTITIONED \( k \)-PPT (Recognition of \( k \)-PPT graphs for a given partition) |
|----------|--------------------------------------------------------------------------------|
| Input:   | A natural \( k \), a graph \( G \) and a partition \( (V_1, \ldots, V_k) \) of \( V(G) \). |
| Question:| Is there a strongly consistent ordering \( s \) of \( V(G) \) such that the vertices of \( V_i \) are consecutive in \( s \), for all \( 1 \leq i \leq k \)? |
The NP-hardness of the previous problem is accomplished by a reduction from the problem **NOT ALL EQUAL 3-SAT**, which is NP-complete [13]. The details are described in Theorem 9.

**Problem:** NOT ALL EQUAL 3-SAT
**Input:** A formula \( \varphi \) on variables \( x_1, \ldots, x_r \) in conjunctive normal form, with clauses \( C_1, \ldots, C_s \), where each clause has exactly three literals.
**Question:** Is there a truth assignment for \( x_1, \ldots, x_r \) such that each clause \( C_i \), \( i = 1, \ldots, s \), has at least one true literal and at least one false literal?

**Theorem 9.** Recognition of \( k \)-PPT graphs for a given partition is NP-complete, even if the size of each part is at most 2.

**Proof.** A given precedence strongly consistent ordering for the partition of \( V(G) \) can be easily verified in polynomial time. Therefore, this problem is in NP.

Given an instance \( \varphi \) of **NOT ALL EQUAL 3-SAT**, we define a graph \( G \) and a partition of \( V(G) \) in which each part has size at most two. The graph \( G \) is defined in such a way that \( \varphi \) is satisfiable if, and only if, there is a precedence strongly consistent ordering of \( V(G) \) for the partition. The graph \( G \) is constructed as follows.

For each variable \( x_i \) appearing in the clause \( C_j \), create the part

\[
X_{ij} = \{x_{ij}^T, x_{ij}^F\}
\]

For each variable \( x_i \), create the parts

\[
X_i^T = \{x_i^T\} \quad \text{and} \quad X_i^F = \{x_i^F\}
\]

The edges of the graph between these parts are \((x_i^T, x_{ij}^T)\) and \((x_i^F, x_{ij}^F)\) for every \( i, j \) such that variable \( x_i \) appears in clause \( C_j \).

Notice that in any strongly consistent ordering, part \( X_{ij} \) must be between parts \( X_i^T \) and \( X_i^F \). Moreover, if \( x_i^F < x_i^T \), then \( x_{ij}^F < x_{ij}^T \), and conversely. In particular, in any valid vertex order, for each \( i \in \{1, \ldots, r\} \), either \( x_{ij}^F < x_{ij}^T \) for every \( j \in \{1, \ldots, s\} \) or \( x_{ij}^T < x_{ij}^F \) for every \( j \in \{1, \ldots, s\} \).

The **Partitioned** \( k \)-PPT instance will be such that if there is a precedence strongly consistent ordering for the vertices with respect to the given parts, then the assignment \( x_i = (x_i^F < x_i^T) \) (that is, \( x_i \) is true if \( x_i^F \) precedes \( x_i^T \)) in such an ordering and \( x_i \) is false otherwise) satisfies \( \varphi \) in the context of **NOT ALL EQUAL 3-SAT** and, conversely, if there is a truth assignment satisfying \( \varphi \) in that context, then there exists a strongly consistent ordering for the **Partitioned** \( k \)-PPT instance in which \( x_i^F < x_i^T \) if \( x_i \) is true and \( x_i^T < x_i^F \) otherwise.

In what follows, if the \( k \)-th literal \( \ell_{ij} \) of \( C_j \) is the variable \( x_i \) (resp. \( \neg x_i \)), we denote by \( O_{ij} \) the ordered part \( \{x_{ij}^F, x_{ij}^T\} \) (resp. \( \{x_{ij}^T, x_{ij}^F\} \)).

Given a 2-vertex ordered part \( C \), we denote by \( C^1 \) and \( C^2 \) the first and second elements of \( C \). By \( \pm C \), we denote “either \( C \) or \( \neg C \).
For each clause $C_j = \ell_{i_1} \lor \ell_{i_2} \lor \ell_{i_3}$, we add the 2-vertex ordered parts $Y_{i_1j}$, $Y_{i_2j}$, and $Y_{i_3j}$, and the edges $(O_{i_1j}^1, Y_{i_1j}^1)$, $(O_{i_1j}^2, Y_{i_1j}^2)$, $(O_{i_2j}^3, Y_{i_2j}^3)$, $(O_{i_2j}^4, Y_{i_2j}^4)$, $(O_{i_3j}^5, Y_{i_3j}^5)$, $(O_{i_3j}^6, Y_{i_3j}^6)$, $(O_{i_3j}^7, Y_{i_3j}^7)$, $(O_{i_3j}^8, Y_{i_3j}^8)$, $(O_{i_3j}^9, Y_{i_3j}^9)$, $(O_{i_3j}^{10}, Y_{i_3j}^{10})$. These edges ensure the following properties in every strongly consistent ordering of the graph with respect to the defined partition.

1. Since $(O_{i_1j}^2, Y_{i_1j}^1)$ is the only edge between $O_{i_1j}$ and $Y_{i_1j}$, their only possible relative positions are $O_{i_1j} < Y_{i_1j}$ and its reverse $Y_{i_1j} < O_{i_1j}$.
2. Since $(O_{i_1j}^1, Y_{i_1j}^2)$ and $(O_{i_1j}^2, Y_{i_1j}^2)$ are the edges between $O_{i_1j}$ and $Y_{i_1j}$, their possible relative positions are $\pm O_{i_1j} < Y_{i_1j}$ and $Y_{i_1j} < \pm O_{i_1j}$.
3. Since $(O_{i_2j}^3, Y_{i_2j}^3)$ and $(O_{i_2j}^4, Y_{i_2j}^4)$ are the edges between $O_{i_2j}$ and $Y_{i_2j}$, their possible relative positions are $O_{i_2j} < \pm Y_{i_2j}$ and $\pm Y_{i_2j} < O_{i_2j}$.
4. Since $(O_{i_2j}^3, Y_{i_2j}^4)$ and $(O_{i_2j}^4, Y_{i_2j}^4)$ are the edges between $O_{i_2j}$ and $Y_{i_2j}$, their possible relative positions are $O_{i_2j} < \pm Y_{i_2j}$ and $\pm Y_{i_2j} < O_{i_2j}$.
5. Since $(O_{i_3j}^5, Y_{i_3j}^3)$ and $(O_{i_3j}^6, Y_{i_3j}^6)$ are the edges between $O_{i_3j}$ and $Y_{i_3j}$, their possible relative positions are $O_{i_3j} < \pm Y_{i_3j}$ and $\pm Y_{i_3j} < O_{i_3j}$.
6. Since $(O_{i_3j}^5, Y_{i_3j}^5)$ and $(O_{i_3j}^6, Y_{i_3j}^6)$ are the edges between $O_{i_3j}$ and $Y_{i_3j}$, their possible relative positions are $O_{i_3j} < \pm Y_{i_3j}$ and $\pm Y_{i_3j} < O_{i_3j}$.
7. Since $(O_{i_3j}^7, Y_{i_3j}^7)$ and $(O_{i_3j}^8, Y_{i_3j}^8)$ are the edges between $O_{i_3j}$ and $Y_{i_3j}$, their possible relative positions are $O_{i_3j} < \pm Y_{i_3j}$ and $\pm Y_{i_3j} < O_{i_3j}$.
8. Since $(O_{i_3j}^9, Y_{i_3j}^9)$ and $(O_{i_3j}^{10}, Y_{i_3j}^{10})$ are the edges between $O_{i_3j}$ and $Y_{i_3j}$, their possible relative positions are $O_{i_3j} < \pm Y_{i_3j}$ and $\pm Y_{i_3j} < O_{i_3j}$.

Notice that, by items [1] and [6] (resp. [2] and [7]), the vertices of $Y_{i_1j}$ (resp. $Y_{i_2j}$) are forced to lie between those of $O_{i_1j}$ and those of $O_{i_3j}$. More precisely, the possible valid orders are $O_{i_1j} < Y_{i_1j}, Y_{i_2j} < \pm O_{i_3j}$ and their reverses $\pm O_{i_3j} < Y_{i_2j}, Y_{i_1j} < O_{i_1j}$.

By items [3] and [4], the vertices of $O_{i_2j}$ are forced to be between those of $Y_{i_1j}$ and those of $Y_{i_2j}$. More precisely, the possible valid orders are $Y_{i_1j} < O_{i_2j} < \pm Y_{i_2j}$ and their reverses $\pm Y_{i_2j} < O_{i_2j} < Y_{i_1j}$.

By items [5] and [8], the vertices of $Y_{i_3j}$ and $Y_{i_3j}$ are forced to be on the same side with respect to the vertices of $O_{i_3j}$, either $O_{i_3j} < \pm Y_{i_3j}, \pm Y_{i_3j} < O_{i_3j}$.

Hence, taking also into account item [8] the possible valid orders are

- $O_{i_1j} < Y_{i_1j}, Y_{i_3j} < \pm Y_{i_3j} < O_{i_2j}$
- $O_{i_1j} < Y_{i_2j} < \pm O_{i_2j} < Y_{i_1j}, \pm Y_{i_3j}$
- $O_{i_1j} < Y_{i_2j} < O_{i_2j} < Y_{i_1j} < O_{i_3j} < Y_{i_3j}$

and their reverses,

- $O_{i_3j} < Y_{i_2j} < \pm O_{i_2j} < Y_{i_1j}, \pm Y_{i_3j}$
- $O_{i_3j} < Y_{i_1j}, \pm Y_{i_3j} < O_{i_2j} < Y_{i_2j} < O_{i_1j}$
\[ ±Y_{3j} < \bar{O}_{3j} < Y_{1j} < O_{2j} < Y_{2j} < \bar{O}_{1j} \]

and will correspond to truth assignments that make true, respectively,

1. \( \ell_{1j} \land \ell_{2j} \land \neg \ell_{3j} \)
2. \( \ell_{1j} \land \neg \ell_{2j} \land \neg \ell_{3j} \)
3. \( \ell_{1j} \land \neg \ell_{2j} \land \ell_{3j} \)
4. \( \neg \ell_{1j} \land \neg \ell_{2j} \land \ell_{3j} \)
5. \( \neg \ell_{1j} \land \ell_{2j} \land \ell_{3j} \)
6. \( \neg \ell_{1j} \land \ell_{2j} \land \neg \ell_{3j} \)

Suppose first that there is a precedence strongly consistent ordering of \( V(G) \) with respect to its vertex partition. Define a truth assignment for variables \( x_1, \ldots, x_r \) as \( x_i = (x_i^F < x_i^T) \), for \( i \in \{1, \ldots, r\} \).

As observed above, if the value of \( x_i \) is true (resp. false), then for every \( j \in \{1, \ldots, s\} \), the part \( X_{ij} \) is ordered \( x_{ij}^F \ leq x_{ij}^T \) (resp. \( x_{ij}^T < x_{ij}^F \)). So, for each clause \( C_j \), the part corresponding to its \( k \)-th literal will be ordered as \( O_{kj} \) if the literal is assigned true and as \( \bar{O}_{kj} \) if the literal is assigned false. Since for each valid order of the vertices there exist \( k, k' \in \{1, 2, 3\} \) such that the part corresponding to the \( k \)-th literal is ordered \( O_{kj} \) and the part corresponding to the \( k' \)-th literal is ordered \( \bar{O}_{k'j} \), the truth assignment satisfies the instance \( \varphi \) of \( \text{NOT ALL EQUAL 3-SAT} \).

Suppose now that there is a truth assignment for variables \( x_1, \ldots, x_r \) that satisfies the instance \( \varphi \) of \( \text{NOT ALL EQUAL 3-SAT} \). Define the order of the vertices in the following way. The first \( r \) vertices are \( \{x_{1i}^F : x_i \text{ is true}\} \cup \{x_{1i}^T : x_i \text{ is false}\} \), and the last \( r \) vertices are \( \{x_{ri}^F : x_i \text{ is true}\} \cup \{x_{ri}^T : x_i \text{ is false}\} \). Between these first and last \( r \) vertices, place all the parts \( X_{ij}, Y_{1j}, Y_{2j}, \) and \( Y_{3j} \) associated with each clause \( C_j, j = 1, \ldots, s \). In particular, the parts \( X_{ij}, Y_{1j}, Y_{2j}, \) and \( Y_{3j} \) are ordered accordingly to which of the conditions (a)-(f) is satisfied. By the analysis above, this is a precedence strongly consistent ordering of the vertices of \( G \), with respect to the defined parts. As an example, Figure 3 depicts the instance of the \( \text{PARTITIONED } k\text{-PPT} \) problem built from the instance \( \varphi = \{(x_1 \land x_2 \land x_3)\} \) of \( \text{NOT ALL EQUAL 3-SAT} \) problem.

The remaining of this section is dedicated to discuss a polynomial time solution to a variation of the \( \text{PARTITIONED } k\text{-PPT} \) problem. This variation consists in considering a fixed number of parts for \( V(G) \), that is, \( k \) is removed from the input and taken as a constant for the problem. The strategy that will be adopted is the same used for \( \text{PARTITIONED } k\text{-PPT} \) problem. It is not difficult to see that Property 1 is not sufficient to describe the requirements that must be imposed in the ordering of vertices in a precedence strongly consistent ordering. This is so because, unlike what occurs in a precedence consistent ordering, in a precedence strongly consistent ordering the vertices of each part \( V_i \) may impose an ordering to vertices that belong to parts that precede and succeed \( V_i \). Given this fact, we observe the following property to describe such relation.
Figure 13: Instance of the Partitioned $k$-PPT problem built from the instance $\varphi = \{ (x_1 \land x_2 \land x_3) \}$ of NOT ALL EQUAL 3-SAT problem.

**Property 2.** Let $(V_1, V_2, \ldots, V_k)$ be a partition of $V(G)$, $s$ a precedence strongly consistent ordering and $1 \leq i, j \leq k$. If $V_i$ precedes $V_j$ in $s$, then for all $u, v \in V_i$ and $w \in V_j$, if $(u, w) \notin E(G)$ and $(v, w) \in E(G)$, then $u$ precedes $v$ in $s$. Moreover, for all $u \in V_i$ and $w, x \in V_j$, if $(u, w) \notin E(G)$ and $(u, x) \in E(G)$, then $x$ precedes $w$ in $s$.

Figure 14: Precedence relations among the vertices in a precedence strongly consistent ordering.

Notice that the greedy strategy used in Section 3 does not work in the problem being considered. This is so because, according to Property 2 and visually depicted in Figure 14, the ordering of vertices of $V_i$ in a precedence strongly consistent ordering $s$ is influenced by both the parts that precede and succeed $V_i$ in $s$. Despite this, the method described in the Section 3 to validate whether a part can precede a set of parts is also useful to present a solution to this problem.

Let $G$ be a graph, $\mathcal{V} = (V_1, V_2, \ldots, V_k)$ a partition of $V(G)$ and $s$ a precedence strongly consistent ordering of $V(G)$ for the given partition. Clearly, for all $1 \leq i \leq k$, $G[V_i]$ must be a proper interval graph for $s$ to be a precedence...
strongly consistent ordering. Verifying whether $G[V_i]$ is a proper interval graph can be accomplished in linear time. If one of the parts does not induce a proper interval graph, then the answer is NO. Otherwise, each part has a PQ tree associated to it.

For a given sequence $s_V$ of parts of $V$, suppose that $V_j < V_i < V_z$ in $s_V$, for $1 \leq j, i, z \leq k$. Let $T_i$ be a PQ tree of $G[V_i]$. Notice that, considering the Property 2 if we apply Theorem 8 to get the ordering constraints imposed by $V_j$ and $V_z$ to $T_i$, and add the directed edges to $T_i$ in the same way that has been done in Section 3 and $T_i$ can meet the constraints, then $T_i$ is compatible to being at that position. That is, the vertices of $V_i$ can precede the vertices of $V_z$ and succeed the vertices of $V_j$ in any precedence strongly consistent ordering.

We show that for any $s_V$, it is possible to verify whether there is a precedence strongly consistent ordering $s$ in which the ordering of the parts in $s$ is precisely $s_V$. To solve the problem, we will test all $k!$ possible permutations $s_V = V'_1, V'_2, \ldots, V'_k$ among the parts of $V$ and validate, using a digraph and PQ trees, if each part $V'_i$ can precede $V'_j$ and succeed $V'_z$, for all $1 \leq j < i < z \leq k$. This validation is done exactly as described in Section 3 except for using Property 2 instead of Property 1. If there is some $s$ that satisfies this condition, then there is a precedence strongly consistent ordering with respect to $s$ and $G$ is a $k$-PPT graph concerning $V$. Otherwise, $G$ is not a $k$-PPT graph with respect to $V$. Algorithm 3 formalizes the procedure.

Concerning the time complexity of the algorithm, first note that to create in $T_i$ the directed edges derived from Property 2 related to $V'_j$ (resp. $V'_z$) can be done in $O(n^5)$ time. Also, for each $V_i$ we apply this property considering all the other parts, that is, $O(k)$ times, and therefore $O(k^2)$ times overall considering each $V_i$. As this operation must be executed for all $k!$ possible permutations, and considering the analysis of this same method in Section 3, the given strategy yields a worst case time complexity of $O(k!k^2n^5) = O(n^5)$ as $k$ is fixed.

We end this section by mentioning an even more restricted case of the problem. Namely, the recognition of $k$-PPT graphs for a fixed number of parts such that each part induces a connected graph. Note that, as each part induces a connected graph, the proper interval graph induced by each part has an unique proper canonical ordering but reversion or mutual true twins permutation. This fact implies that the PQ tree related to each one of these proper interval graphs is formed by one node of type Q, which is the root, that has all the maximal cliques as its children. That is, there are only two possible configurations for each one of these PQ trees. As the number of possible configurations is a constant, this property leads to a more efficient algorithm. Instead of using Theorem 8 to map restrictions to the PQ tree in order to obtain a compatible tree, the algorithm can check both configurations of the PQ tree independently. As the step which uses Theorem 8 is no longer required, this approach leads to an algorithm that yields a worst case time complexity of $O(k!k^22^k n^3) = O(n^3)$. This strategy is presented in Algorithm 4.

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Algorithm 3: Partitioned \(k\)-PPT

Input: \(G\): a graph; \(V\): a \(k\)-partition \((V_1, V_2, \ldots, V_k)\) of \(V(G)\) for some fixed \(k\)

function Partitioned-\(k\)-PPT\((G, V)\)

for each \(V_i \in V\) do
    if \(G[V_i]\) is not a proper interval graph then
        return (NO, \(\emptyset\))

for each permutation \(s_V\) of \(V\) do
    \(s \leftarrow \emptyset\)
    \(foundValidPermutation \leftarrow \text{TRUE}\)
    for each \(V_i \in s_V\) do
        Create a digraph \(D = (V_i, \emptyset)\)
        Build a PQ tree \(T_i\) of \(G[V_i]\)
        for each \(V_j \in s_V\) such that \(V_j \neq V_i\) do
            Let \(S\) be the set of precedence relations among the vertices of \(V_i\) concerning \(V_j\) (Property [2])
            for each \((u < v) \in S\) do
                \(E(D) \leftarrow E(D) \cup \{(u,v)\}\)
                for each node \(X\) of \(T_i\) do
                    Add the direct edges, deriving from \((u < v)\), among the children of \(X\) (Theorem [8])
            for each node \(X\) of \(T_i\) do
                Let \(D' = (V', E')\) be the digraph where \(V'\) is the set of the children of \(X\) and \(E'\) are the directed edges added among them
            if there is a topological ordering \(s\) of \(D\) then
                Arrange the children of \(X\) according to \(s\)
            else
                \(foundValidPermutation \leftarrow \text{FALSE}\)
            if \(foundValidPermutation\) then
                addEdgesFromPQTree\((G[V_i], D, T_i)\)
            if there is a topological ordering \(s_i\) of \(D\) then
                \(s \leftarrow ss_i\)
            else
                \(foundValidPermutation \leftarrow \text{FALSE}\)
                break
        if \(foundValidPermutation\) then
            return (YES, \(s\))
    return (NO, \(\emptyset\))
Algorithm 4: Partitioned $k$-PPT

**Input:** $G$: a graph; $V$: a $k$-partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$, for some fixed $k$, such that $G[V_i]$ is connected for all $1 \leq i \leq k$

function Partitioned-$k$-PPT($G$, $V$)

for each $V_i \in V$ do
    if $G[V_i]$ is not a proper interval graph then
        return (NO, $\emptyset$)

for each permutation $s_V$ of $V$ do
    $s \leftarrow \emptyset$
    foundValidPermutation $\leftarrow$ TRUE
    for each $V_i \in s_V$ do
        foundValidTree $\leftarrow$ FALSE
        Build a PQ tree $T_i$ of $G[V_i]$
        Let $T'_i$ be the PQ tree obtained from $T_i$ by reversing the order of the children of the root
        for each $T \in \{T_i, T'_i\}$ do
            Create a digraph $D = (V_i, \emptyset)$
            for each $V_j \in s_V$ such that $V_j \neq V_i$ do
                Let $S$ be the set of precedence relations among the vertices of $V_i$ concerning $V_j$ (Property 2)
                for each $(u < v) \in S$ do
                    $E(D) \leftarrow E(D) \cup \{(u, v)\}$
            addEdgesFromPQTree($G[V_i]$, $D$, $T$)
            if there is a topological ordering $s_i$ of $D$ then
                $s \leftarrow ss_i$
                foundValidTree $\leftarrow$ TRUE
                break
            if not foundValidTree then
                foundValidPermutation $\leftarrow$ FALSE
                break
        if foundValidPermutation then
            return (YES, $s$)
    return (NO, $\emptyset$)
5. Characterization of $k$-PT and $k$-PPT Graphs

This section describes a characterization of $k$-PT and $k$-PPT graphs for a given partition. First, we define some further concepts.

A graph $G$ is a split graph if there is a bipartition $(V_1, V_2)$ of $V(G)$ such that $V_1$ is a clique and $V_2$ a stable set of $G$. A graph $G$ is called a threshold graph if $G$ is a split graph and there is an ordering of $V_1$ (resp. $V_2$), named threshold ordering, such that the neighborhood of vertices of $V_1$ (resp. $V_2$) are ordered by inclusion, that is, if $u$ precedes $v$ in the threshold ordering, $N[u] \subseteq N[v]$ (resp. $N(u) \subseteq N(v)$).

For the following characterization, we will define the split graph $S_G(V_1, V_2)$ with respect to a bipartition $(V_1, V_2)$ of a graph $G$. Such a graph is obtained from $G$ by the completion of edges among the vertices of $V_1$ and the removal of all edges among the vertices of $V_2$, hence transforming $V_1$ into a clique and $V_2$ into a stable set. The Figures 15(b) and 15(d) illustrate the corresponding split graphs of the graphs in the Figures 15(a) and 15(c), respectively.

Let $s = s_1s_2$ be an ordering of $V(G)$. We define $(s_1, s_2)$ as in accordance with $G$ if $s_1$ is a threshold ordering of $S_G(V(s_1), V(s_2))$, and if $s_1$ and $s_2$ are canonical orderings of $G[V(s_1)]$ and $G[V(s_2)]$, respectively. Additionally, we define $(s_1, s_2)$ as strongly in accordance with $G$ if $s_1$ and $s_2$ are proper canonical orderings of $G[V(s_1)]$ and $G[V(s_2)]$, respectively, and both $s_1$ and $s_2$ are threshold orderings of $S_G(V(s_1), V(s_2))$.

As an example, let $s_1$ and $s_2$ be the orderings represented in the Figure 15 by reading the vertices of each part, of each graph, from left to right. The related pair $(s_1, s_2)$ of Figure 15(a) is in accordance, but is not strongly in accordance, with the given graph. In the order hand, Figure 15(c) depicts a pair $(s_1, s_2)$ which is strongly in accordance with the associated graph. Finally, Figure 15(e) exemplifies a case where the given orderings are neither in accordance or strongly in accordance with its correlated graph. In fact, since both $V_1$ and $V_2$ in Figure 15(e) induce subgraphs that admit only four canonical orderings each, it can be easily verified that there is no $(s_1, s_2)$ which is in accordance, or strongly in accordance, with the graph.

Lemma 10. Let $G$ be a graph. Then, $G$ is 2-PT if, and only if, there is a consistent ordering $s = s_1s_2$ for which $V = (V(s_1), V(s_2))$ is a partition of $V(G)$ such that $(s_1, s_2)$ is in accordance with $G$.

Proof. Let $\mathcal{V} = \{V_1, V_2\}$ be a partition of $V(G)$ and $s_1$ and $s_2$ be two total orderings of $V_1$ and $V_2$, respectively.

Consider $G$ is 2-PT concerning $\mathcal{V}$. Let $s = s_1s_2$ be a precedence consistent ordering of $V(G)$. Thus, $s_1$ is a canonical ordering of $G[V_1]$. Suppose by absurd that $s_1$ is not a threshold ordering of $S_G(V_1, V_2)$. That is, there are $u, v \in V_1$ and $w \in V_2$, with $u < v$ in $s_1$, such that $w \in N[u]$ and $w \notin N[v]$. As $u < v < w$ em $s$, there is a contradiction with the fact that $s$ is a precedence consistent ordering of $V(G)$. Therefore, $(s_1, s_2)$ is in accordance with $G(V_1, V_2)$.

On the other hand, consider that $(s_1, s_2)$ is in accordance with $G$. Thus, both $s_1$ and $s_2$ are canonical orderings of $G[V_1]$ and $G[V_2]$, respectively, and
Figure 15: Corresponding split graphs ($V_1$ is the set of orange vertices and $V_2$ the black ones).

$s_1$ is a threshold ordering of $S_G(V_1,V_2)$. Now we prove that $s = s_1 s_2$ is a precedence consistent ordering of $V(G)$ concerning $V$. Suppose by absurd that this statement does not hold. That is, there are $u, v \in V_1$ and $w \in V_2$, with $u < v$ in $s$, such that $(u, w) \in E(G)$ and $(v, w) \notin E(G)$. This is a contradiction with the fact that $s_1$ is a threshold ordering ordering of $S_G(V_1,V_2)$. Hence, $s = s_1 s_2$ is a precedence consistent ordering of $V(G)$ concerning $V$.

Lemma 11. Let $G$ be a graph. Then, $G$ is 2-PPT if, and only if, there is a strongly consistent ordering $s = s_1 s_2$ for which $V = (V(s_1), V(s_2))$ is a bipartition of $V(G)$ such that $(s_1, s_2)$ is strongly in accordance with $G$.

Proof. Let $V = \{V_1, V_2\}$ be a bipartition of $V(G)$ and $s_1$ and $s_2$ be two total orderings of $V_1$ and $V_2$, respectively.

Consider $G$ is 2-PPT concerning $V$. Let $s = s_1 s_2$ be a precedence strongly consistent ordering of $V(G)$. Thus, $s_1$ is a proper canonical ordering of $G[V_1]$ and, as $G$ is also a 2-PT graph, $s_1$ is a threshold ordering of $S_G(V_1,V_2)$, according Lemma 10. Suppose by absurd that $s_2$ is not a threshold ordering of $S_G(V_1,V_2)$. That is, there are $u, v \in V_2$ and $w \in V_1$, with $u < v$ in $s_2$, such that $w \in N(v)$ and $w \notin N(u)$. That is, a contradiction with the fact that $s$ is a precedence strongly
consistent ordering, as \( w < u < v \) in \( s \). Hence, \((s_1, s_2)\) is strongly in accordance with \( G(V_1, V_2) \).

Now consider \((s_1, s_2)\) is strongly in accordance with \( G \). That is, both \( s_1 \) and \( s_2 \) are threshold orderings (proper canonical orderings) of \( S_G(V_1, V_2) \) (resp. \( G[V_1] \) and \( G[V_2] \), respectively). By Lemma \( 10 \) \( s = s_1s_2 \) is a precedence consistent ordering of \( V(G) \). Next, we prove that \( s = s_1s_2 \) is also a precedence strongly consistent ordering of \( V(G) \) concerning \( V \). For the sake of contradiction, suppose that the statement does not hold. That is, there are \( u, v \in V_2 \) and \( w \in V_1 \), with \( u < v \) in \( s \), such that \((v, w) \in E(G)\) and \((u, w) \notin E(G)\). This is an absurd, as \( s_2 \) is a threshold ordering ordering of \( S_G(V_1, V_2) \). Consequently, \( s = s_1s_2 \) is a precedence strongly consistent ordering of \( V(G) \) concerning \( V \).

The above lemmas can be generalized to an arbitrary number of parts as follows.

**Theorem 12.** Let \( G \) be a graph. For all \( k > 2 \), \( G \) is \( k \)-PT (resp. \( k \)-PPT) if, and only if, there is a precedence consistent (resp. strongly consistent) ordering \( s = s_1 \ldots s_k \) for which \( V = (V(s_1), V(s_2)), \ldots, V(s_k)) \) is a \( k \)-partition of \( V(G) \) such that for all \( 1 \leq i < j \leq k \), \((s_i, s_j)\) is in accordance (resp. strongly in accordance) with \( G[V(s_i) \cup V(s_j)] \).

**Proof.** Suppose there are a total ordering \( s = s_1 \ldots s_k \) and a partition \( V = (V(s_1), V(s_2)), \ldots, V(s_k)) \) of \( V(G) \). Notice that \( s \) is a precedence consistent (resp. strongly consistent) ordering if, and only if, for all \( 1 \leq i < j \leq k \), \((s_i, s_j)\) is a precedence consistent (resp. strongly consistent) ordering of \( G[V_i \cup V_j] \) concerning the bipartition \((V_i, V_j)\). By Lemma \( 10 \) (resp. Lemma \( 11 \)), it holds if, and only if, \((s_i, s_j)\) is in accordance (resp. strongly in accordance) with \( G[V(s_i) \cup V(s_j)] \).

6. Conclusions and open problems

In this work, we study two classes of graphs: precedence \( k \)-thin and precedence proper \( k \)-thin graphs, subclasses of \( k \)-thin and proper \( k \)-thin graphs, respectively. Concerning precedence \( k \)-thin graphs, we present a polynomial time algorithm that receives as input a graph \( G \) and a \( k \)-partition of \( V(G) \) and decides whether \( G \) is a precedence \( k \)-thin graph with respect to the given partition. This result is presented in Section 3. Regarding precedence proper \( k \)-thin graphs, for the same input, we prove that if \( k \) is a fixed value, then it is possible to decide whether \( G \) is a precedence proper \( k \)-thin graph with respect to the given partition in polynomial time. For variable \( k \), the related recognition problem is \( NP \)-complete. These results are presented in Section 4. Also, using threshold graphs, we characterize both precedence \( k \)-thin and precedence proper \( k \)-thin graphs.

Concerning the classes defined in this paper, some open questions are highlighted:
• Given a graph $G$, what is the complexity of evaluating $pre-thin(G)$ and $pre-pthin(G)$?

• Given a graph $G$ and an integer $k$, what is the complexity of determining if $pre-thin(G)$, or $pre-pthin(G)$, is at most $k$?

• How do $pre-thin(G)$ and $pre-pthin(G)$ relate to $thin(G)$ and $pthin(G)$, respectively?

• Is it possible to extend the results of this paper to consider other types of orderings (partial orders) and restrictions?

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