STRUCTURAL THEOREMS FOR ULTRADISTRIBUTION SEMIGROUPS

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Abstract. We consider exponential ultradistribution semigroups with non-densely defined generators and give structural theorems for ultradistribution semigroups. Also structural theorems for exponential ultradistribution semigroups are given.

1. Introduction and preliminaries

In the previous paper [16] the first two authors have analyzed, on the basis of well developed theory, ultradistribution semigroups through the existence of an ultradistributional fundamental solution to \( u_t - Au = \delta \), where \( A \) is the corresponding closed operator as well as through the sub-exponential estimate of the resolvent \( ||R(\lambda,A)|| \leq Ce^{M(k,|\lambda|)} \) in an appropriate domain defined by the associated function \( M \). In Theorems 8 and 9 of [16] they have given examples based on these characterizations. In this paper we give complete structural characterization of ultradistribution semigroups with the aim of their full characterizations and connections with the corresponding Cauchy problems.

As we mentioned, the literature related to ultradistribution semigroups is pretty reach. It is based on the generalizations of \( C_0 \)-semigroups, especially of various classes of integrated semigroups of W. Arendt [1] and further extensions, and [23] (see also [2], [8], [9], [20], [24]) Especially, we refer to the excellent monograph [3] and the references therein. Ultradistribution semigroups with densely defined generators were considered by J. Chazarain in [6] (see also [5], [7], [9] and references therein) while H. Komatsu [12] considered ultradistribution semigroups with non-densely defined generators as well as Laplace hyperfunction semigroups. We also refer to R. Beals [4]-[5] for the theory of \( \omega \)-ultradistribution semigroups with the densely defined generators, to P. C. Kunstmann [19] and to the monograph of I. Melnikova and A. Filinkov [23] for ultradistribution semigroups with the non-densely defined generators and applications to abstract Cauchy problems (see [21], [22]). In [16] are analyzed ultradistribution semigroups following the approaches of P. C. Kunstmann [18] and S. Wang [27], where distributions semigroups are considered. The most recent theory of ultradistribution semigroups is given in the monograph of M. Kostic [14].

We recall in Section 2 some of definitions and results from [16] related to ultradistribution semigroups. Ultradifferentiable operators are used in order to clarify relations between exponentially bounded and tempered ultradistribution semigroups and convoluted semigroups.

In Section 3 we give a structural characterizations for ultradistribution semigroups. The main results are given in Theorem 2.6. We give five conditions for
ultradistribution semigroups and the corresponding five conditions for exponential ultradistribution semigroups and we give relations between them.

1.1. Notation from ultradistribution theory. Here we use the same notation like in [16] and we follow approach of H. Komatsu [10] in defining ultradistribution spaces. If \((M_p)\) verifies (M.1), (M.2) and (M.3), then the spaces of Beurling, respectively, Roumieu ultradifferentiable functions, are \(\mathcal{D}^{(M_p)}(\mathbb{R})\) and \(\mathcal{D}^{(M_p)}_{*}(\mathbb{R})\) With the notation \(*\) for both cases of brackets, we define \(\mathcal{D}^{*}(\mathbb{R}, E) := L(\mathcal{D}^{*}(\mathbb{R}), E)\) as the space of continuous linear functions from \(\mathcal{D}^{*}(\mathbb{R})\) into \(E\); \(\mathcal{D}^{*}_{0}(\mathbb{R})\) denotes the space of elements in \(\mathcal{D}^{*}(\mathbb{R})\) which are supported by \([0, \infty)\) while \(\mathcal{E}_{0}^{*}\) denotes the space of ultradistributions whose supports are compact subsets of \([0, \infty)\). We also use the traditional notation \(\mathcal{D}_{*}^{*}(\mathbb{R}, E)\) for the space of vector valued ultradistributions supported by \([0, \infty)\). We refer to [11] for the basic material related to vector-valued ultradistribution spaces.

Spaces of tempered ultradistributions of Beurling and Roumieu type are given in [25] (see also [26]) as duals of the test spaces \(\mathcal{S}^{(M_p)}(\mathbb{R})\) and \(\mathcal{S}^{(M_p)}_{*}(\mathbb{R})\), respectively.

Recall ([10]), an entire function of the form \(P(\lambda) = \sum_{p=0}^{\infty} a_{p}\lambda^{p}, \ \lambda \in \mathbb{C}\), is of \((M_p)\)-class, respectively, of \(\{M_p\}\)-class, (i.e., an ultrapolynomial of the respective class) if there exist \(k > 0\) and \(C > 0\), respectively, for every \(k > 0\) there exists a constant \(C > 0\), such that \(|a_{p}| \leq Ck^{p}/M_{p}, \ p \in \mathbb{N}\). The corresponding ultradifferential operator \(P(d/dt) = \sum_{p=0}^{\infty} a_{p}d^{p}/dt^{p}\) is of \((M_p)\)-class, respectively of \(\{M_p\}\)-class. The composition and the sum of ultradifferential operators of the Beurling, resp., the Roumieu class, are ultradifferential operators of the Beurling, resp., the Roumieu class.

The following assertion is well known in the theory of ultradistributions (cf. [10] and [12] Theorem 4.7).

Let \(T \in \mathcal{D}_{*}^{*}(\mathbb{R}, E)\). Then for every \(a > 0\) there exist an ultradifferential operator of \((M_p)\)-class, formally of the form

\[
P_{L}(d/dt) = \sum_{p=1}^{\infty} \left(1 + \frac{L^2}{m^{2}_{p}}\right) a_{p}d^{p}/dt^{p},
\]

where \(L > 0\) is some constant, resp., of \(\{M_p\}\)-class, formally of the form

\[
P_{L_{p}}(d/dt) = \sum_{p=1}^{\infty} \left(1 + \frac{L_{p}^2}{m^{2}_{p}}\right) a_{p}d^{p}/dt^{p},
\]

where \((L_{p})_{p}\) is a sequence decreasing to 0, and a continuous function \(f: (-a, a) \to E\) such that

\[
T = P_{L}(-id/dt)f, \ \text{on} \ \mathcal{D}^{(M_p)}((-a, a)), \ \text{in} \ (M_p) - \text{case, resp.,}
\]

\[
T = P_{L_{p}}(-id/dt)f, \ \text{on} \ \mathcal{D}^{(M_p)}((-a, a)), \ \text{in} \ \{M_p\} - \text{case.}
\]

Due to [25] Theorem 2, we have the following representation theorems for tempered ultradistributions in the case when (M.1), (M.2) and (M.3) are valid.

Let \(T \in \mathcal{S}_{*}^{*}(\mathbb{R}, E)\). Then there exist an ultradifferential operator of \((M_p)\)-class, \(P_{L}(d/dt)\), \(L > 0\), formally of the form \((1.1)\), resp., of \(\{M_p\}\)-class, \(P_{L_{p}}(d/dt)\), \((L_{p})_{p}\) is a sequence tending to zero, formally of the form \((1.2)\), and a continuous function \(f: \mathbb{R} \to E\) with the properties \(\text{supp} f \subset (-a, \infty)\), for some \(a > 0\), \(|f(t)| \leq Ae^{M(k|t|)}\), \(t \in \mathbb{R}\), for some \(k > 0\) and \(A > 0\), resp., for every \(k > 0\) and a
corresponding $A > 0$, and that $T = \rho_L (-id/dt)f$ in $(M_p)$-case on $S^{(M_p)}(\mathbb{R})$, resp., $T = \rho_L (-id/dt)f$ in $\{M_p\}$-case on $S^{(M_p)}(\mathbb{R})$.

2. ULTRADISTRIBUTION SEMIGROUPS

2.1. Some results from ultra distribution theory. We will consider ultradistribution semigroups in the framework of exponential ultradistributions which we define through tempered ultradistributions. We assume here that $(M_p)$ satisfies (M.1), (M.2) and (M.3). The purpose of (M.3) is again the use of [11] Theorem 4.8.

Definition 2.1. Let $a \geq 0$. Then $\mathcal{SE}_a^+(\mathbb{R}) := \{ \phi \in C^\infty(\mathbb{R}) : e^{a\phi} \in \mathcal{S}^+(\mathbb{R}) \}$.

The convergence in this space is given by

$\phi_n \to 0$ in $\mathcal{SE}_a^+(\mathbb{R})$ iff $e^{a\phi_n} \to 0$ in $\mathcal{S}^+(\mathbb{R})$.

We denote by $\mathcal{SE}_a^+(\mathbb{R}, E)$ the space of all continuous linear mappings from $\mathcal{SE}_a^+(\mathbb{R})$ into $E$ equipped with the strong topology.

We have

$F \in \mathcal{SE}_a^+(\mathbb{R}, E)$ iff $e^{-a} F \in \mathcal{S}^+(\mathbb{R}, E)$.

Theorem 2.2. Let $G \in \mathcal{SE}_a^+(\mathbb{R}, E)$. Then there exists an ultrapolynomial $P$ of $\ast$-class and a function $g \in C(\mathbb{R}, E)$ with the property that there exist $k > 0$ and $C > 0$, resp., for every $k > 0$ there exists an appropriate $C_k > 0$ such that

$e^{-ax} \| g(x) \| \leq C_k e^{M(k|x|)}$, $x \in \mathbb{R}$ and $G = P(d/dt)g$.

Proof. Let us prove the assertion in the Beurling case. Since $e^{-a} G \in \mathcal{S}^{(M_p)}(\mathbb{R}, E)$, one can use the same arguments as in [25] in order to see that there exist an ultrapolynomial $P$ of $(M_p)$-class and a function $g_1 \in C(\mathbb{R}, E)$ with the property that there exist $k > 0$ and $C_k > 0$ such that

$\| g_1(x) \| \leq C_k e^{M(k|x|)}$ and that $G = e^{ax} P(d/dt)g_1(x)$.

Put $g(x) = e^{ax} g_1(x)$, $x \in \mathbb{R}$. By Leibnitz formula, we have

$e^{ax} P(d/dt)g_1(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{j+k}{j} (-1)^k a^k a_{k+j} (e^{ax} g_1(x))^{(j)}$

and we will prove the assertion if we show that $b_j \leq C_{M_j}, j \in \mathbb{N}_0$, for some $C$, $L > 0$, where $b_j = \sum_{k=0}^{\infty} \binom{k+j}{j} a^k a_{k+j}, j \in \mathbb{N}_0$.

We will use the following inequality, $\binom{j+k}{j} \leq 2^{k+1} k^j e^j$, $j, k \in \mathbb{N}$.

This follows from

$\binom{j+k}{j} \leq (j+k)^k \leq 2^k j^k + 2^k k^k \leq 2^k (k^j e^j + k^k) = 2^k k^j (e^j + 1)$, $j, k \in \mathbb{N}$,

where we use $j^k \leq k^j e^j$, $j, k \in \mathbb{N}$. This is clear for $k \geq j$. Let us prove this for $k < j$. Put $k = \varepsilon j$ and note, if $\varepsilon \in (0, 1)$, then $\varepsilon \ln \varepsilon \in (-1, 0)$ and

$\varepsilon \ln j \leq \varepsilon \ln j + \varepsilon \ln \varepsilon + j$.

This implies $j^k \leq k^j e^j, k < j$. Now we will estimate $b_j$ using the estimate

$|a_{k+j}| \leq C_{M_k}^{h+j}$ for some $h > 0$, $C > 0$.
and that for every $\varepsilon > 0$ there exists $C_{\varepsilon} > 0$ such that $M_j k^k \leq C_{\varepsilon} \varepsilon^{k+j} M_{k+j}$.

With this we have
\[ M_j |b_j| \leq 2 \sum_{k=0}^{\infty} \frac{h^{k+j} M_j 2^k k! e^{k^2}}{M_{k+j}} \leq 2C(h) \sum_{k=0}^{\infty} \frac{(2a)^k M_j k^k}{M_{k+j}}, \quad j \in \mathbb{N} \]
and choosing $\varepsilon$ enough small, we obtain the convergence of the last series. This implies that there exist $L > 0$ and $C > 0$ such that $|b_j| \leq CL^j/M_j$, $j \in \mathbb{N}$.

We need the following estimations of ultrapolynomials:

**Lemma 2.3.**

(a) Let $P_L$ be of the form \((1.1)\). Then there exist $C, C_1 > 0$, $L_1, L_2 > 0$ such that
\[ e^{2M(L|\zeta|)} \leq |P_L(\zeta)| \leq Ce^{M(L_1|\zeta|)} \text{ if } |\text{Re}\zeta| < \frac{|\text{Im}\zeta|}{2} + \frac{1}{L} \]
and $|a_p| \leq C_1 L_2^p/M_j$, $p \in \mathbb{N}_0$.

(b) Let $(L_p)_p$ be a sequence which strictly decreases to zero and $P_{L_p}$ be defined by \((1.2)\). Then there exists $C > 0$ such that, for every $k > 0$, there exists $C_k > 0$, such that
\[ |P_{L_p}(\zeta)| \leq C_k e^{M(k|\zeta|)} \text{ if } |\text{Im}\zeta| < \frac{|\text{Re}\zeta|}{2} + \frac{1}{L_1}, \]
and (with another $C_k$, for given $k > 0$) $|a_p| \leq C_k k^p/M_j$, $p \in \mathbb{N}_0$. Moreover, there exists a subordinate function $\varepsilon(\rho)$, $\rho \geq 0$, such that
\[ e^{2M(\varepsilon(|\zeta|))} \leq |P_{L_p}(\zeta)| \text{ if } |\text{Im}\zeta| < \frac{|\text{Re}\zeta|}{2} + \frac{1}{L}. \]

**Proof.** We will prove only the part
\[ e^{2M(L|\zeta|)} \leq |P_L(\zeta)| \text{ if } |\text{Im}\zeta| < \frac{|\text{Re}\zeta|}{2} + \frac{1}{L}. \]

Note that for any $c > 0$, the inequality $x^2 - y^2 \geq 0$ ($\zeta = x + iy$) implies $|1 + c\zeta^2| \geq |c|^2$. Also, $|1 + c\zeta^2| \geq |c|^2$, for all sufficiently small $|\zeta|$. Thus, by the simple calculation we have that
\[ |1 + \frac{L^2 \zeta^2}{m_p^2}| \geq \frac{L^2}{m_p^2} |\zeta|^2 \text{ if } |\text{Im}\zeta| < \frac{|\text{Re}\zeta|}{2} + \frac{1}{L}. \]

This implies
\[ |P_L(\zeta)| = \prod_{p=1}^{\infty} \left(1 + \frac{L^2 \zeta^2}{m_p^2}\right) \geq \prod_{p=1}^{\infty} \left(\frac{L^2}{m_p^2} |\zeta|^2\right) \geq e^{2M(|\zeta|)} \text{ if } |\text{Im}\zeta| < \frac{|\text{Re}\zeta|}{2} + \frac{1}{L}. \]

\[ \square \]

**Lemma 2.4.** Let $P_L(d/dt)$ and $P_{L_p}(d/dt)$ be of the form \((1.1)\) and \((1.2)\), respectively. The mappings
\[ P_L(d/dt) : S^{(M_p)}(\mathbb{R}) \to S^{(M_p)}(\mathbb{R}), \quad \phi \mapsto P_L(d/dt)\phi, \]
\[ P_{L_p}(d/dt) : S^{(M_p)}(\mathbb{R}) \to S^{(M_p)}(\mathbb{R}), \quad \phi \mapsto P_{L_p}(d/dt)\phi, \]
are continuous linear bijections.
Proof. We will prove the lemma in the Beurling case. Let \( \phi \in \mathcal{S}^{(M_p)}(\mathbb{R}) \). Then

\[
\mathcal{F}(P_L(\text{id}/dt)\phi)(\xi) = P_L(-\xi)\hat{\phi}(\xi) = P_L(\xi)\hat{\phi}(\xi), \quad \xi \in \mathbb{R}.
\]

One can prove by standard arguments that \( P_L(\xi)\hat{\phi} \in \mathcal{S}^{(M_p)}(\mathbb{R}) \). We have to prove that \( \hat{\phi}/P_L(\xi) \in \mathcal{S}^{(M_p)}(\mathbb{R}) \).

Notice that there exists \( r > 0 \) such that, for every \( \xi \in \mathbb{R} \), the circle \( k_\xi(r) \), with the center \( \xi \) and the radius \( r \), is contained in the domain \( \{\text{Im}\zeta| < 1/C \} \) where the estimates of Lemma \([4,3]\) are satisfied. By Cauchy’s formula, with suitable constants, it follows

\[
|P_{L}^{-1}(\xi)| \leq C_{n}^{1} \sup\{|P_{L}^{-1}(\xi + re^{i\theta})| : \theta \in [0, 2\pi]\} \leq C_{n}^{1} e^{M(|\xi|+r)}, \quad \xi \in \mathbb{R}, \quad n \in \mathbb{N}_{0}.
\]

Now it is easy to prove that for every \( h > 0 \),

\[
\sup\left\{ \frac{h^{n}|(\hat{\phi}/P_{L})(\xi)|e^{M(|\xi|)}}{M_{n}} : \xi \in \mathbb{R}, \quad n \in \mathbb{N}_{0} \right\} < \infty
\]

which is equivalent with \( \hat{\phi}/P_{L} \in \mathcal{S}^{(M_p)}(\mathbb{R}) \). \( \square \)

2.2. Structural theorems. Let \( A \) be a closed operator and \( K \) be a locally integrable function on \([0, \tau], 0 < \tau \leq \infty\), and let \( \Theta(t) := \int_{0}^{t} K(s) ds \), \( 0 \leq t \leq \tau \). Recall (see \([14, 15, 20]\) for example), if there exists a strongly continuous operator family \((S_{K}(t))_{t \in [0, \tau]}\) such that \( S_{K}(t)C = CS_{K}(t) \), \( S_{K}(t)A \subset AS_{K}(t) \), \( \int_{0}^{\tau} S_{K}(s)x ds \in D(A) \), for \( t \in [0, \tau] \), \( x \in E \) and

\[
A \int_{0}^{t} S_{K}(s)x ds = S_{K}(t)x - \Theta(t)C x, \quad x \in E,
\]

then \((S_{K}(t))_{t \in [0, \tau]}\) is called a (local) \( K \)-convoluted \( C \)-semigroup having \( A \) as a subgenerator.

If \( \tau = \infty \), then it is said that \((S_{K}(t))_{t \geq 0}\) is an exponentially bounded \( K \)-convoluted \( C \)-semigroup generated by \( A \) if, additionally, there exist \( M > 0 \) and \( \omega \in \mathbb{R} \) such that \( ||S_{K}(t)|| \leq Me^{\omega t} \), \( t \geq 0 \). \((S_{K}(t))_{t \in [0, \tau]}\) is called non-degenerate, if the assumption \( S_{K}(t)x = 0 \), for all \( t \in [0, \tau] \), implies \( x = 0 \).

We recall from \([10]\) the definitions of \( L \)-ultradistribution semigroups and ultradistribution semigroups (following \([18]\) and \([27]\)) and define exponential ultradistribution semigroups.

**Definition 2.5.** Let \( G \in \mathcal{D}^{*}_{+}(\mathbb{R}, L(E)) \). It is an exponential \( L \)-ultradistribution semigroup of \( s \)-class if the following conditions (U.1)–(U.5) hold:

(U.1) \( G(\phi * \psi) = G(\phi)G(\psi) \), \( \phi, \psi \in \mathcal{D}_{0}^{*}(\mathbb{R}) \);

(U.2) \( \mathcal{N}(G) := \bigcap_{\phi \in \mathcal{D}_{0}^{*}(\mathbb{R})} N(G(\phi)) = \{0\} \);

(U.3) \( \mathcal{R}(G) := \bigcup_{\phi \in \mathcal{D}_{0}^{*}(\mathbb{R})} R(G(\phi)) \) is dense in \( E \);

(U.4) For every \( x \in \mathcal{R}(G) \) there exists a function \( u \in C([0, \infty), E) \) satisfying

\[
u(0) = x \quad \text{and} \quad G(\phi)x = \int_{0}^{\infty} \phi(t)u(t) dt, \quad \phi \in \mathcal{D}^{*}(\mathbb{R}).
\]

(U.5) There exists \( a \geq 0 \) such that \( G \in \mathcal{SE}^{*}_{a}(\mathbb{R}, L(E)) \);
Recall, \( f \ast g(t) := \int_0^t f(t-s)g(s) \, ds \), \( t \in \mathbb{R} \). If \( G \in \mathcal{D}_+^\omega(\mathbb{R}, L(E)) \) satisfies

\[(U.6) \ G(\phi \ast_0 \psi) = G(\phi)G(\psi), \ \text{for} \ \phi, \ \psi \in \mathcal{D}^\omega(\mathbb{R}), \ \text{and} \ (U.5), \ \text{then it is a exponential}
\]

pre-ultradistribution semigroup, in short, pre-(EUDSG) of \(*\)-class.

If \((U.6), (U.5)\) and \((U.2)\) are fulfilled for \( G \), then \( G \) is an exponential ultradistribution semigroup of \(*\)-class, in short, (EUDSG). A pre-(EUDSG) \( G \) it is said that \( G \) is dense ultradistribution semigroup.

If only \((U.6)\) holds then we call \( G \) pre-ultradistribution semigroup or pre-(UDSG).

If \((U.6)\) and \((U.2)\) holds, \( G \) is ultradistribution semigroup, in short (UDSG), and if additionally \((U.3)\) holds then \( G \) is dense ultradistribution semigroup.

\[\text{If} \ G \in \mathcal{D}_+^\omega(\mathbb{R}, L(E)), \ \text{then the condition:} \]

\[(U.2) \ \text{supp}G(x) \subset \{0\}, \ \text{for every} \ x \in E \setminus \{0\}, \ \text{is equivalent to} \ (U.2).\]

Let \( D \) be another Banach space and \( P \in \mathcal{D}_+^\omega(\mathbb{R}, L(D, E)). \ \text{Then, as in the case of} \]

distribution semigroups, \( G \in \mathcal{D}_+^\omega(\mathbb{R}, L(E, D)) \) is an ultradistribution fundamental solution for \( P \) if

\[P \ast G = \delta \otimes I_E \ \text{and} \ G \ast P = \delta \otimes I_D.\]

If additionally \( G \in \mathcal{SE}_\omega^\omega(\mathbb{R}, L(E, [D(A)])) \), holds for some \( a \geq 0 \), then it is said that \( G \) is exponential ultradistribution fundamental solution for \( P \).

As in the case of distributions, an ultradistribution fundamental solution for \( P \in \mathcal{D}_+^\omega(\mathbb{R}, L(D, E)) \) is uniquely determined.

In the sequel, we will use the phrase “\( G \) is an ultradistribution fundamental solution for \( A \)” if \( G \) is an ultradistribution fundamental solution for \( P := \delta^t \otimes I_{D(A)} - \delta \otimes A \in \mathcal{D}_+^\omega(\mathbb{R}, L([D(A)], E)). \)

Following the investigation of H. Komatsu [12], in the framework of Denjoy-Karleman-Komatsu theory of ultradistributions and P. C. Kunstmann [19], in the theory of \( \omega \)-ultradistributions, we define the next regions:

\[\Omega^{(M_p)} := \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq M(k|\lambda|) + C \}, \ \text{for some} \ k > 0, \ C > 0, \ \text{resp.,}\]

\[\Omega^{(M_p)} := \{ \lambda \in \mathbb{C} : \text{Re} \lambda \geq M(k|\lambda|) + C_k \}, \ \text{for every} \ k > 0 \ \text{and a corresponding} \ C_k > 0. \]

\( B_0 \Omega^* \) is denoted either \( \Omega^{(M_p)} \) or \( \Omega^{(M_p)}. \)

In Theorem 2.7 which is to follow, in the case of tempered ultradistribution semigroups (and similarly in the case of exponentially bounded ultradistribution semigroups), we use [14] Theorem 3.5.14, where the inverse Laplace transform is performed on the straight line connecting points \( \widetilde{a} - i \infty \) and \( \widetilde{a} + i \infty \), where \( \widetilde{a} > 0. \)

With a suitable choice of \( L, \ \text{resp.,} \ (L_p)_p, \ \text{we have that this line lies in the domain} \)

\[|\text{Im}(i\zeta)| < \frac{|\text{Re}(i\zeta)|}{2} + \frac{1}{2}, \ \text{resp.,} \ |\text{Im}(i\zeta)| < \frac{|\text{Re}(i\zeta)|}{2} + \frac{1}{2}, \ \text{where we have the quoted estimates for} \ P_L(-i\lambda), \ \text{resp.,} \ P_{L_p}(-i\lambda). \]

Let us explain this in the Beurling case with more details. Choose any \( L \in (0, \frac{1}{2}) \) and put

\[K(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{b+i\infty} \frac{e^{\lambda t}}{P_L(-i\lambda)} \, d\lambda, \ t \geq 0. \ \text{Then} \ K \ \text{is an exponentially bounded, continuous function defined on} \ [0, \infty) \ \text{and we shall simply write}\]

\[K = L^{-1}(\frac{1}{P_L(-i\lambda)}).\]

Now, we will give the structural characterizations for \( \text{(UDSG)}'s \) and exponential \( \text{(UDSG)}'s \). Some of these characterizations are proved in [12, 14, 15, 19], and [23]. We will indicate this in Theorem 2.7.

First, we list the statements:

(a) \( A \) generates a \( \text{(UDSG)} \) of \(*\)-class \( G \).

(a') \( A \) generates a \( \text{(EUDSG)} \) of \(*\)-class \( G \).
(b) A generates a (UDSG) of *-class $G$ such that, for every $a > 0$, $G$ is of the form $G = P_L^a(-id/dt)S_K^a$ on $D(M^a_p)\{(\cdot,\cdot)\}$ in $(M^a_p)$-case, (resp., $G = P_L^a(-id/dt)S_K^a$ on $D(M^a_p)\{(\cdot,\cdot)\}$ in $(M^a_p)$-case), where $S_K^a : (\cdot,\cdot) \rightarrow L(E,[D(A)])$ is continuous, $S_K^a(t) = 0$, $t \leq 0$.

(b’) A generates a (EUdSG) of *-class $G$ so that $G$ is of the form $G = P_L(-id/dt)S_K$ on $S_a^{(M^a_p)}(\mathbb{R})$ in $(M^a_p)$-case, (resp., $G = P_L(-id/dt)S_K$ in $(M^a_p)$-case), where $S_K : \mathbb{R} \rightarrow L(E,[D(A)])$ is continuous, $S_K(t) = 0$, $t \leq 0$ and $e^{-at}\|S_k(t)\| \leq Ae^{M(\lambda|t|)}$, for some $k > 0$ and $A > 0$, resp., for every $k > 0$ and corresponding $A > 0$, $t \in \mathbb{R}$.

(c) For every $a > 0$, $A$ is the generator of a local non-degenerate $K_a$-convoluted semigroup $(S_K^a(t))_{t \in [0,a]}$, where $K_a = \mathcal{L}^{-1}\left(\frac{1}{P_L(-i\lambda)}\right)$ in $(M^a_p)$-case, resp., $K_a = \mathcal{L}^{-1}\left(\frac{1}{P_L(-i\lambda)}\right)$ in $(M^a_p)$-case and $P_L^a$, resp., $P_L^a$, is an ultradifferential operator of *-class such that for $0 < a < b$ the restriction of $P_L^a S_K^b$, resp., $P_L^a S_K^b$, on $D^\ast((\cdot,\cdot))$ is equal to $P_L^a S_K^b$, resp., $P_L^a S_K^b$.

(c’) $A$ is the generator of a global, exponentially bounded non-degenerate $K$-convoluted semigroup $(S_K(t))_{t \geq 0}$, where $K = \mathcal{L}^{-1}\left(\frac{1}{P_L(-i\lambda)}\right)$ in $(M^a_p)$-case, resp., $K = \mathcal{L}^{-1}\left(\frac{1}{P_L(-i\lambda)}\right)$ in $(M^a_p)$-case.

(d) There exists an ultradistribution fundamental solution of *-class for $A$, denoted by $G$, with the property $\mathcal{N}(G) = \{0\}$.

(d’) There exists an exponential ultradistribution fundamental solution of *-class $G$ for $A$, with the property $\mathcal{N}(G) = \{0\}$.

(e) $\rho(A) \supset \Omega^\ast$ and

\[\|R(\lambda : A)\| \leq Ce^{M(\lambda|\lambda|)}, \lambda \in \Omega^\ast(M^a_p),\]

for some $k > 0$ and $C > 0$ in $(M^a_p)$-case, resp.,

\[\|R(\lambda : A)\| \leq C_\ast e^{M(\lambda|\lambda|)}, \lambda \in \Omega^\ast(M^a_p),\]

for every $k > 0$ and a corresponding $C_\ast > 0$ in $(M^a_p)$-case.

(e’) $\rho(A) \supset \{\lambda \in \mathbb{C} : \Re \lambda > a\}$ and

\[\|R(\lambda : A)\| \leq Ce^{M(\lambda|\lambda|)}, \Re \lambda > a,\]

for some $a, k > 0$ and $C > 0$ in $(M^a_p)$-case, resp.,

\[\|R(\lambda : A)\| \leq C_\ast e^{M(\lambda|\lambda|)}, \Re \lambda > a,\]

for every $k > 0$ and a corresponding $a, C_\ast > 0$ in $(M^a_p)$-case.

**Theorem 2.6.** (a) $\Leftrightarrow$ (d); (a’) $\Leftrightarrow$ (d); (c) $\Rightarrow$ (d); (e) $\Rightarrow$ (d); (d) $\Rightarrow$ (e); (e’) $\Rightarrow$ (e); if $(M^a_p)$ additionally satisfies (M.3), then (a’) $\Rightarrow$ (c’).

**Proof.** (a) $\Leftrightarrow$ (d): This equivalence is proved in [15], when $\mathcal{N}(G) \neq \{0\}$. The statement (a) $\Rightarrow$ (d) is direct consequence of [16] Theorem 2 (c). We give here the sketch of the proof of the opposite direction. Let $G \in D^\ast(L, L(E,[D(A)])$ be an ultradistributional fundamental solution of *-class for $A$. By the direct calculation we have that $A$ is closable operator.

Let $A$ generates $G$. If $(x,y)$ belongs to the closure of $A$, then there exists a sequence $(x_n, y_n)\rightarrow (x, y)$, when $n \rightarrow \infty$, in $E \times E$. Let $\phi \in D_0^\ast(\mathbb{R})$ be fixed. For $\varphi \in D_0^\ast(\mathbb{R})$ we have

\[\|G(\varphi)(G(-\phi)x - G(\phi)y)\| = \text{...}\]
for $G$ similar. Let $\omega \in \mathbb{C}$ for $\Re \omega > 0$. Let $G$ be a fundamental ultradistribution solution of $\delta$-class for $A$ we have that $A \subset A$. It implies that $\mathcal{D}^*_{+}(\mathbb{R}, [D(A)])$ is an isomorphic to a subspace of $\mathcal{D}^*_{+}(\mathbb{R}, [D(A)])$. From the first part of the theorem we have that $G$ is a fundamental ultradistribution solution for \( P := \delta' \otimes I_d_{D[A]} - \delta \otimes \mathbb{A} \). So $G^*$ is an isomorphism from $\mathcal{D}^*_{+}(\mathbb{R}, E)$ onto $\mathcal{D}^*_{+}(\mathbb{R}, [D(A)])$ and onto $\mathcal{D}^*_{+}(\mathbb{R}, [D(A)])$ which implies that $\mathcal{D}^*_{+}(\mathbb{R}, [D(A)]) = \mathcal{D}^*_{+}(\mathbb{R}, [D(A)])$, so $[D(A)] = [D(A)]$

The statement (a) $\iff$ (d) can be proved similarly using that $G$ can be extended continuously on $\mathcal{E}^{\mathcal{S}^*}(\mathbb{R})$, [16].

The proof of (d) $\implies$ (e) is given in [23].

(d) $\implies$ (e) [14]: We will give a proof for Beurling case. The Roumeiu case is quite similar. Let $G$ be a exponential fundamental ultradistribution solution of $(M_p)$-class for $A$, i.e., $G$ is a fundamental ultradistribution solution and $G \in \mathcal{E}^{\mathcal{S}^*}_{\mathcal{M}_p}(\mathbb{R}, L(E))$ for $\omega \geq 0$. Let $s > 0$. We define a function $g \in \mathcal{E}^{(M_p)}(\mathbb{R})$ such that $g(t) = 0$ for $t < -s$ and $g(t) = 1$ for $t \geq 0$. The definition of $\tilde{G}(\lambda) := G(g(t)e^{-\lambda t}) := G(e^{-\lambda t}(g(t)e^{(\omega - \lambda)t}))$ have meaning since the function $t \mapsto g(t)e^{(\omega - \lambda)t}$, when $t \in \mathbb{R}$ and for all $\lambda \in \mathbb{C}$ such that $\Re \lambda > \omega$, is in $\mathcal{S}^{(M_p)}(\mathbb{R})$. Because $G$ is a fundamental ultradistribution solution for $A - \omega I$, for $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}, x \in E)$ we have that,

$$(A - \omega I)G(e^{-\omega t}\varphi)x = G(-e^{-\omega t}\varphi)x - \varphi(0)x.$$  

Using that $\mathcal{D}^{(M_p)}(\mathbb{R})$ is dense in $\mathcal{S}^{(M_p)}(\mathbb{R})$, we get that the previous equation holds for all $\mathcal{S}^{(M_p)}(\mathbb{R})$. Let we put $\varphi(t) = g(t)e^{(\omega - \lambda)t} \in \mathcal{S}^{(M_p)}(\mathbb{R})$. Then supp$G \subseteq [0, \infty)$ and we obtain:

$$AG(\lambda)x = AG(e^{-\lambda t}\varphi)x = \lambda \tilde{G}(\lambda)x - \varphi(0)x, \quad \Re \lambda > \omega.$$  

From this equation, $(\lambda I - A)\tilde{G}(\lambda)x = x, \quad x \in E, \quad \Re \lambda > \omega$. $\tilde{G}(\lambda)A \subseteq AG(\lambda)$ holds for $\Re \lambda > \omega$ we have $\tilde{G}(\lambda)(\lambda I - A)x = x$, for $x \in D(A)$ and $\Re \lambda > \omega$. We put $\omega = a$ so we have proved the first part of the statement. From the discussion above, it is clear that $R(\lambda : A)x = \tilde{G}(\lambda)x$, for $x \in E$, $\Re \lambda > a$. Using (M.1) we obtain that

$$\|R(\lambda : A)\| = \|\tilde{G}(\lambda)\| = \|G(e^{-\omega t}(g(t)e^{(\omega - \lambda)t}))\| \leq \leq C'' \sup_{t \in K} \frac{g(t)e^{(\omega - \lambda)t}(p)}{M_p h^p} \leq C'' \sum_{j \in K} C_j g^{(p-j)}(t) \cdot \frac{e^{(\omega - \lambda)t}(j)}{M_p h^p} \leq \leq C'' \sum_{j \in K} C_j \frac{(\omega - \lambda)^j e^{(\omega - \lambda)t}}{M_j h^j} \leq C e^{M(\|\lambda\|)}.$$  

(a) $\implies$ (c):

We will prove this assertion in the Beurling case by the use of already mentioned structural theorem for elements of $\mathcal{E}^{(M_p)}_{\mathcal{S}^*}(L(E))$:

$$G(\phi) = \langle \phi, P_L(-id/dt)S(t) \rangle, \quad \phi \in \mathcal{S}^{(M_p)}(\mathbb{R}),$$

where $P_L$ is the principal value.
where, for an appropriate $k > 0$,

$$e^{-at}||S(t)|| \leq e^{M(k|t|)}, \quad t \in \mathbb{R}.$$  

Fix an $x \in E$. By Theorem [16, Theorem 2 (c)],

$$AG(\phi)x = -\langle \phi', P_L(-id/|t|)S(t)x \rangle - \phi(0)x, \quad \text{for all } \phi \in S^{(M_p)}(\mathbb{R}).$$  

Since $1 = P_L(-id/|t|)L^{-1}(1/P_L(-i \cdot))$ in the sense of ultradistributions, we have, for every $\phi \in S^{(M_p)}(\mathbb{R}),$

$$0 = \langle \phi'(t), (P_L(-id/|t|)A \int_0^t S(s)x ds - P_L(-id/|t|)S(t)x \rangle$$

$$+ \int_0^t \mathcal{L}^{-1}(1/P_L(-i \cdot))(s)x ds \rangle$$

$$= \langle P_L(id/|t|)\phi'(t), (A \int_0^t S(s)x ds - S(t)x) + \int_0^t \mathcal{L}^{-1}(1/P_L(-i \cdot))(s)x ds \rangle.$$  

Assume that $\psi \in \mathcal{D}(\mathbb{R})$ and $\phi \in S^{(M_p)}(\mathbb{R})$ so that $\psi = P_L(id/|t|)\phi$ (cf. Lemma [2,3]). This implies

$$(2.1) \quad A \int_0^t S(s)x ds - S(t)x + \int_0^t \mathcal{L}^{-1}(1/P_L(-i \cdot))(s)x ds = \text{const},$$

in the sense of Beurling ultradistributions on $(0, \infty)$. We obtain that $\text{const} = 0$ by putting $x = 0$ in (2.1). Since the left side of (2.1) is continuous on $\mathbb{R}$, we have

$$A \int_0^t S(s)x ds = S(t)x - \Theta(t)x = 0, \quad \text{where } \Theta(t) = \int_0^t \mathcal{L}^{-1}(1/P_L(-i \cdot))(s) ds,$$

for all $t \geq 0$. This completes the proof of (a)' $\Rightarrow$ (c)'.

Let us show (c) $\Rightarrow$ (d) in the Beurling case. The proof of (c)' $\Rightarrow$ (d)' is similar. Define $G$ on $\mathcal{D}^{(M_p)}((-\infty, a))$, for all $a > 0$, by

$$G := P^\alpha_L(-id/|t|)S^\alpha_{K_a}, \quad \text{where } P^\alpha_L = \sum_{p=0}^{\infty} a_p (d/|t|)^p.$$  

Then $G$ is a continuous linear mapping from $\mathcal{D}^{(M_p)}(\mathbb{R})$ into $L(E)$ which commutes with $A$. Moreover, supp$G \subset [0, \infty)$. Let $\phi \in \mathcal{D}^{(M_p)}((-\infty, a))$ and $x \in E$. We have,

$$G(-\phi)x - AG(\phi)x = -\sum_{p \geq 0} a_p (-i)^p \int_0^a \phi^{(p+1)}(s)S^\alpha_{K_a}(s)x ds$$

$$- \sum_{p \geq 0} a_p (-i)^p \int_0^a \phi^{(p)}(s)AS^\alpha_{K_a}(s)x ds = -\sum_{p \geq 0} a_p (-i)^p \int_0^a \phi^{(p+1)}(s)S^\alpha_{K_a}(s)x ds$$

$$+ \sum_{p \geq 0} a_p (-i)^p \int_0^a \phi^{(p+1)}(s)(S^\alpha_{K_a}(s)x - \Theta_a(s)x) ds =$$
\[
\sum_{p \geq 0} a_p (-i)^p \int_0^a \phi^{(p)}(s) K_\alpha(s)x \, ds = \phi(0)x.
\]

Hence, \( G \in H^{(M_p)}(\mathbb{R}, L(E, [D(A)])) \) is an ultradistribution fundamental solution for \( A \). Clearly, \( \mathcal{N}(G) = \{0\} \). □

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