A convergent finite element approximation for the quasi-static Maxwell–Landau–Lifshitz–Gilbert equations

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Abstract

We propose a \(\theta\)-linear scheme for the numerical solution of the quasi-static Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations. Despite the strong non-linearity of the Landau–Lifshitz–Gilbert equation, the proposed method results in a linear system at each time step. We prove that as the time and space steps tend to zero (with no further conditions when \(\theta \in (\frac{1}{2}, 1]\)), the finite element solutions converge weakly to a weak solution of the MLLG equations. Numerical results are presented to show the applicability of the method.

Key words: Maxwell–Landau–Lifshitz–Gilbert, finite element, ferromagnetism

AMS subject classifications: 65M12, 35K55

1 Introduction

The Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations describe the electromagnetic behavior of a ferromagnetic material. In this paper, for simplicity, we suppose that there is a bounded cavity \(\tilde{D} \subset \mathbb{R}^3\) (with perfectly conducting outer surface \(\partial\tilde{D}\)) into which a ferromagnet \(D\) is embedded. We assume further that \(\tilde{D}\setminus\bar{D}\) is a vacuum.

We will consider the quasi-static case of the MLLG system. Letting \(D_T := (0, T) \times D\) and \(\tilde{D}_T := (0, T) \times \tilde{D}\), the magnetization field \(m : D_T \rightarrow \mathbb{S}^2\), where \(\mathbb{S}^2\) is the unit sphere in \(\mathbb{R}^3\), and the magnetic field \(H : \tilde{D}_T \rightarrow \mathbb{R}^3\) satisfy

\[
\begin{align*}
\dot{m}_t &= \lambda_1 m \times H_{\text{eff}} - \lambda_2 m \times (m \times H_{\text{eff}}) \quad \text{in } D_T, \\
\mu_0 H_t + \sigma \nabla \times (\nabla \times H) &= -\mu_0 \ddot{m}_t \quad \text{in } \tilde{D}_T,
\end{align*}
\]

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in which \( \lambda_1 \neq 0, \lambda_2 > 0, \sigma \geq 0, \) and \( \mu_0 > 0 \) are constants. Here \( \tilde{m} : \tilde{D}_T \to \mathbb{R}^3 \) is the zero extension of \( m \) onto \( \tilde{D}_T \), i.e.,

\[
\tilde{m}(t, x) = \begin{cases} 
m(t, x), & (t, x) \in D_T \\
0, & (t, x) \in \tilde{D}_T \setminus D_T. \end{cases}
\]

For simplicity the effective field \( \mathbf{H}_{\text{eff}} \) is taken to be \( \mathbf{H}_{\text{eff}} = \Delta \mathbf{m} + \mathbf{H} \).

The system \((1.1) - (1.2)\) is supplemented with initial conditions

\[
m(0, .) = m_0 \text{ in } D \quad \text{and} \quad H(0, .) = H_0 \text{ in } \tilde{D}, \tag{1.3}
\]

and boundary conditions

\[
\partial_n m = 0 \text{ on } (0, T) \times \partial D \quad \text{and} \quad (\nabla \times \mathbf{H}) \times n = 0 \text{ on } (0, T) \times \partial \tilde{D}. \tag{1.4}
\]

The equation \((1.1)\) is the first dynamical model for the precessional motion of a magnetization, suggested by Landau and Lifshitz \[12\]. The existence and uniqueness of a local strong solution of \((1.1) - (1.4)\) is shown by Cimrak \[8\]. He also proposes \[7\] a finite element method to approximate this local solution and provides error estimation.

Gilbert introduces a different approach for description of damped precession in \[9\]:

\[
\lambda_1 \mathbf{m}_t + \lambda_2 \mathbf{m} \times \mathbf{m}_t = \mu \mathbf{m} \times \mathbf{H}_{\text{eff}}, \tag{1.5}
\]

in which \( \mu = \lambda_1^2 + \lambda_2^2 \). A proof of the equivalence between \((1.5)\) and \((1.1)\) can be found in \[13\]. It is easier to numerically solve \((1.5)\) than \((1.1)\) because the latter has a double cross term, namely \( \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \).

Instead of solving \((1.1) - (1.4)\), Banas, Bartels and Prohl \[2\] propose an implicit nonlinear scheme to solve problem \((1.2) - (1.5)\), and prove that the finite element solution converges to a weak global solution of the problem. Their method requires a condition on the time step \( k \) and space step \( h \) (namely \( k = O(h^2) \)) for the convergence of the nonlinear system of equations resulting from the discretization.

Following the idea developed by Alouges and Jaison \[1\] for the Landau–Lifshitz–Gilbert (LLG) equation \((1.5)\), we propose a \( \theta \)-linear finite element scheme to find a weak global solution to \((1.2) - (1.5)\). We prove that the numerical solutions converge to a weak solution of the problem with no condition imposed on time step and space step as \( \theta \in (\frac{1}{2}, 1] \). It is required that \( k = o(h^2) \) when \( \theta \in [0, \frac{1}{2}) \), and \( k = o(h) \) when \( \theta = \frac{1}{2} \). The implementation aspect of the algorithm is reported in \[13\] where no convergence analysis is carried out.

The paper is organized as follows. Weak solutions of the MLLG equations are defined in Section \[2\]. We also introduce in this section the \( \theta \)-linear finite element scheme. Some technical lemmas are presented in Section \[3\]. In Section \[4\] we prove that the finite element solutions converge to a weak solution of the MLLG equations. Numerical experiments are presented in the last section.
2 Weak solutions and finite element schemes

Before presenting the definition of a weak solution to the MLLG equations, it is necessary to introduce some function spaces and to assume some conditions on the initial functions \( m_0 \) and \( H_0 \).

The function spaces \( H^1(D, \mathbb{R}^3) \) and \( H(\text{curl}; \tilde{D}) \) are defined as follows:

\[
H^1(D, \mathbb{R}^3) = \left\{ u \in L^2(D, \mathbb{R}^3) : \frac{\partial u}{\partial x_i} \in L^2(D, \mathbb{R}^3) \text{ for } i = 1, 2, 3. \right\},
\]

\[
H(\text{curl}; \tilde{D}) = \left\{ u \in L^2(\tilde{D}, \mathbb{R}^3) : \nabla \times u \in L^2(\tilde{D}, \mathbb{R}^3) \right\}.
\]

Here, for a domain \( \Omega \subset \mathbb{R}^3 \), \( L^2(\Omega, \mathbb{R}^3) \) is the usual space of Lebesgue squared integrable functions defined on \( \Omega \) and taking values in \( \mathbb{R}^3 \). Throughout this paper, we denote

\[
\langle \cdot, \cdot \rangle_{\Omega} := \langle \cdot, \cdot \rangle_{L^2(\Omega, \mathbb{R}^3)} \quad \text{and} \quad \| \cdot \|_{\Omega} := \| \cdot \|_{L^2(\Omega, \mathbb{R}^3)}.
\]

In order to define a weak solution of MLLG equations, we assume that the given functions \( m_0 \) and \( H_0 \) satisfy

\[
m_0 \in H^1(D, \mathbb{R}^3), \quad |m_0| = 1 \text{ a.e. in } D \quad \text{and} \quad H_0 \in H(\text{curl}; \tilde{D}). \tag{2.1}
\]

For physical reasons (see [10]), these initial fields must satisfy

\[
\text{div}(H_0 + \chi_D m_0) = 0 \text{ in } \tilde{D} \quad \text{and} \quad (H_0 + \chi_D m_0) \cdot n = 0 \text{ on } \partial \tilde{D}. \tag{2.2}
\]

Since equations (1.1) and (1.5) are equivalent (a proof of which can be found in [13]), instead of solving (1.1)–(1.4) we solve (1.2)–(1.5). A weak solution of the problem is defined in the following definition.

**Definition 2.1.** Let the initial data \( (m_0, H_0) \) satisfy (2.1) and (2.2). Then \( (m, H) \) is called a weak solution to (1.2)–(1.5) if, for all \( T > 0 \), there hold

1. \( m \in H^1(D_T, \mathbb{R}^3) \) and \( |m| = 1 \text{ a.e. in } D_T; \)
2. \( H, H_t, \nabla \times H \in L^2(\tilde{D}_T, \mathbb{R}^3); \)
3. for all \( \phi \in C^\infty(D_T) \) and \( \zeta \in C^\infty(\tilde{D}_T), \)

\[
\lambda_1 \langle m_t, \phi \rangle_{D_T} + \lambda_2 \langle m \times m_t, \phi \rangle_{D_T} = \mu \langle \nabla m, \nabla (m \times \phi) \rangle_{D_T} + \mu \langle m \times H, \phi \rangle_{D_T} \tag{2.3}
\]

and

\[
\mu_0 \langle H_t, \zeta \rangle_{\tilde{D}_T} + \sigma \langle \nabla \times H, \nabla \times \zeta \rangle_{\tilde{D}_T} = -\mu_0 \langle \tilde{m}_t, \zeta \rangle_{\tilde{D}_T}, \tag{2.4}
\]

where \( \mu = \lambda_1^2 + \lambda_2^2; \)}
4. In the sense of traces there holds

\[ m(0, \cdot) = m_0, \quad (2.5) \]

5. For almost all \( T' \in (0, T) \),

\[ \mathcal{E}(T') + \lambda_2 \mu^{-1} \| m_t \|_{D_{T'}}^2 + \lambda_2 \mu^{-1} \| H_t \|_{D_{T'}}^2 + 2 \mu_0^{-1} \sigma \| \nabla \times H \|_{D_{T'}}^2 \leq \mathcal{E}(0), \quad (2.6) \]

where

\[ \mathcal{E}(T') = \| \nabla m(T') \|_{D}^2 + \| H(T') \|_{D}^2 + \lambda_2 \mu^{-1} \mu_0^{-1} \sigma \| \nabla \times H(T') \|_{D}^2. \]

We next introduce the \( \theta \)-linear finite element scheme which approximates a weak solution \((m, H)\) defined in Definition 2.1.

Let \( T_h \) be a regular tetrahedral partition of the domain \( \bar{D} \) into tetrahedra of maximal mesh-size \( h \), and let \( T_{h}|_D \) be its restriction to \( D \subset \bar{D} \). We denote by \( \mathcal{N}_h := \{ x_1, \ldots, x_N \} \) the set of vertices and by \( \mathcal{M}_h := \{ e_1, \ldots, e_M \} \) the set of edges.

To discretize the LLG equation (2.3), we introduce the finite element space \( \mathbb{V}_h \subset \mathbb{H}^1(D, \mathbb{R}^3) \) which is the space of all continuous piecewise linear functions on \( T_{h}|_D \). A basis for \( \mathbb{V}_h \) can be chosen to be \( \{ \phi_n \}_{1 \leq n \leq N} \), where \( \phi_n(x_m) = \delta_{n,m} \). Here \( \delta_{n,m} \) stands for the Kronecker symbol. The interpolation operator from \( \mathbb{C}^0(D, \mathbb{R}^3) \) onto \( \mathbb{V}_h \) is denoted by \( I_{\mathbb{V}_h} \),

\[ I_{\mathbb{V}_h}(v) = \sum_{n=1}^{N} v(x_n)\phi_n(x) \quad \forall v \in \mathbb{C}^0(D, \mathbb{R}^3). \]

To discretize Maxwell’s equation (2.4), we use the space \( \mathbb{Y}_h \) of lowest order edge elements of Nedelec’s first family [14]. It is known [14] that \( \mathbb{Y}_h \) is a subspace of \( \mathbb{H}(\text{curl}; \bar{D}) \) and that the set \( \{ \psi_1, \ldots, \psi_M \} \) is a basis for \( \mathbb{Y}_h \) if it satisfies

\[ \psi_q \in \{ \psi : \bar{D} \to \mathbb{R}^3 \mid \psi|_K(x) = a_K + b_K \times x, \ a_K, b_K \in \mathbb{R}^3, \forall K \in T_h \}, \]

\[ \int_{e_p} \psi_q \cdot \tau_p \, ds = \delta_{qp}, \]

where \( \tau_p \) is the unit vector in the direction of edge \( e_p \). We also define the following interpolation operator \( I_{\mathbb{Y}_h} \) from \( \mathbb{C}^\infty(\bar{D}) \) onto \( \mathbb{Y}_h \),

\[ I_{\mathbb{Y}_h}(u) = \sum_{q=1}^{M} u_q \psi_q \quad \forall u \in \mathbb{C}^\infty(\bar{D}, \mathbb{R}^3), \]

where \( u_q = \int_{e_q} u \cdot \tau_q \, ds \).

Fixing a positive integer \( J \), we choose the time step \( k \) to be \( k = T/J \) and define \( t_j = jk \), \( j = 0, \ldots, J \). For \( j = 1, 2, \ldots, J \), the functions \( m(t_j, \cdot) \) and \( H(t_j, \cdot) \) are approximated by \( \hat{m}^{(j)}_h \in \mathbb{V}_h \) and \( \hat{H}^{(j)}_h \in \mathbb{Y}_h \), respectively.
We define the space $W_h^{(j)}$ by

$$W_h^{(j)} := \left\{ w \in W_h \mid w(x_n) \cdot m_h^{(j)}(x_n) = 0, \ n = 1, \ldots, N \right\},$$

and denote

$$H_h^{(j+1/2)} := \frac{H_h^{(j+1)} + H_h^{(j)}}{2} \quad \text{and} \quad d_t H_h^{(j+1)} := k^{-1}(H_h^{(j+1)} - H_h^{(j)}).$$

Algorithm 2.1.

Step 1: Set $j = 0$. Choose $m_h^{(0)} = I_{V_h} m_0$ and $H_h^{(0)} = I_{V_h} H_0$.

Step 2: Find $(v_h^{(j+1)}, H_h^{(j+1)}) \in W_h^{(j)} \times Y_h$ satisfying

$$\lambda_2 \left\langle \mathbf{v}_h^{(j+1)}, \mathbf{w}_h^{(j)} \right\rangle_D - \lambda_1 \left\langle \mathbf{m}_h^{(j)} \times \mathbf{v}_h^{(j+1)}, \mathbf{w}_h^{(j)} \right\rangle_D = - \mu \left\langle \nabla (\mathbf{m}_h^{(j)} + k \phi v_h^{(j+1)}), \nabla \mathbf{w}_h^{(j)} \right\rangle_D + \mu \left\langle H_h^{(j+1/2)}, \mathbf{w}_h^{(j)} \right\rangle_D \quad \forall \mathbf{w}_h^{(j)} \in W_h^{(j)},$$

and

$$\mu_0 \left\langle d_t H_h^{(j+1)}, \zeta_h \right\rangle_D + \sigma \left\langle \nabla \times H_h^{(j+1/2)}, \nabla \times \zeta_h \right\rangle_D = - \mu_0 \left\langle \mathbf{v}_h^{(j+1)}, \zeta_h \right\rangle_D \quad \forall \zeta_h \in Y_h.$$  \hspace{1cm} (2.8)

Step 3: Define

$$m_h^{(j+1)}(x) := \sum_{n=1}^{N} \frac{m_h^{(j)}(x_n) + k v_h^{(j+1)}(x_n)}{|m_h^{(j)}(x_n) + k v_h^{(j+1)}(x_n)|^2} \phi_n(x).$$

Step 4: Set $j = j + 1$, and return to Step 2.1 if $j < J$. Stop if $j = J$.

The parameter $\theta$ in (2.8) can be chosen arbitrarily in $[0, 1]$. The method is explicit when $\theta = 0$ and fully implicit when $\theta = 1.$

By the Lax–Milgram Theorem, for each $j > 0$ there exists a unique solution $(v_h^{(j+1)}, H_h^{(j+1)}) \in W_h^{(j)} \times Y_h$ of equations (2.8)–(2.9). Since $\left|m_h^{(j)}(x_n)\right| = 1$ and $v_h^{(j+1)}(x_n) \cdot m_h^{(j)}(x_n) = 0$ for all $n = 1, \ldots, N$, there holds

$$\left|m_h^{(j)}(x_n) + k v_h^{(j+1)}(x_n)\right| \geq 1.$$  \hspace{1cm} (2.10)

Therefore, the algorithm is well defined. There also holds $\left|m_h^{(j+1)}(x_n)\right| = 1$ for $n = 1, \ldots, N.$
3 Some technical lemmas

In this section we present some lemmas which will be used in the rest of the paper. We start by recalling the following lemma proved in [3].

**Lemma 3.1.** If there holds
\[
\int_D \nabla \phi_i \cdot \nabla \phi_j \, dx \leq 0 \quad \text{for all } i, j \in \{1, 2, \ldots, J\} \text{ and } i \neq j, \tag{3.1}
\]
then for all \( u \in \mathbb{V}_h \) satisfying \( |u(x_l)| \geq 1, l = 1, 2, \ldots, J \), there holds
\[
\int_D \left| \nabla I_h \left( \frac{u}{|u|} \right) \right|^2 \, dx \leq \int_D |\nabla u|^2 \, dx. \tag{3.2}
\]
Condition (3.1) holds if all dihedral angles of the tetrahedra in \( T_h \mid D \) are less than or equal to \( \pi/2 \); see [3]. In the sequel we assume that (3.1) holds.

The next lemma defines a discrete \( L^p \)-norm in \( \mathbb{V}_h \) which is equivalent to the usual \( L^p \)-norm.

**Lemma 3.2.** There exist \( h \)-independent positive constants \( C_1 \) and \( C_2 \) such that for all \( u \in \mathbb{V}_h \) there holds
\[
C_1 \|u\|_{L^p(\Omega)}^p \leq h^d \sum_{n=1}^N |u(x_n)|^p \leq C_2 \|u\|_{L^p(\Omega)}^p,
\]
where \( \Omega \subset \mathbb{R}^d, d=1,2,3. \)

A proof of this lemma for \( p = 2 \) and \( d = 2 \) can be found in [11, Lemma 7.3] or [6, Lemma 1.12]. The result for general values of \( p \) and \( d \) can be obtained in the same manner.

The following lemma can be proved by using the technique in [11, Lemma 7.3].

**Lemma 3.3.** There exists an \( h \)-independent positive constant \( C \) such that for each tetrahedron \( K \in T_h \) and \( v \in \mathbb{V}_h \) there holds
\[
||v(x) - v(x_i)|| \leq C h|\nabla v(x)| \quad \text{for all } x \in K,
\]
where \( \{x_i\}_{i=1,2,3} \) are the vertices of \( K \).

Finally the following lemma is elementary; the proof of which is included for completeness.

**Lemma 3.4.** The solutions \( (m_h^{(j)}, v_h^{(j+1)}), j = 0, 1, \ldots, J, \) obtained from Algorithm 2.1 satisfy
\[
\left| m_h^{(j+1)}(x_n) - m_h^{(j)}(x_n) \right| \leq \left| v_h^{(j+1)}(x_n) \right| \quad \forall n = 1, 2, \ldots, N, \quad j = 0, \ldots, J. \tag{3.3}
\]
Proof. By using the definition of \( m_{h}^{(j+1)} \), the property \( m_{h}^{(j)}(x_n) \cdot v_{h}^{(j+1)}(x_n) = 0 \), and the identity

\[
|m_{h}^{(j)}(x_n) + k v_{h}^{(j+1)}(x_n)| = \sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2}
\]

we obtain

\[
\frac{|m_{h}^{(j+1)}(x_n) - m_{h}^{(j)}(x_n)|^2}{k} = \frac{|m_{h}^{(j)}(x_n) + k v_{h}^{(j+1)}(x_n)|}{k} - \frac{|m_{h}^{(j)}(x_n)|}{k} = \frac{m_{h}^{(j)}(x_n)(1 - |m_{h}^{(j)}(x_n) + k v_{h}^{(j+1)}(x_n)|) + k v_{h}^{(j+1)}(x_n)|^2}{k^2 |m_{h}^{(j)}(x_n) + k v_{h}^{(j+1)}(x_n)|^2}
\]

\[
= \frac{2 + 2k^2 |v_{h}^{(j+1)}(x_n)|^2 - 2\sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2}}{k^2 \left(1 + k^2 |v_{h}^{(j+1)}(x_n)|^2\right)}
\]

\[
= \frac{2\sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2} - 1}{k^2 \sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2}}.
\]

Using the fact that

\[
2 \leq \sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2} + 1
\]

we deduce

\[
\frac{|m_{h}^{(j+1)}(x_n) - m_{h}^{(j)}(x_n)|^2}{k} \leq \frac{(\sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2} + 1) (\sqrt{1 + k^2 |v_{h}^{(j+1)}(x_n)|^2} - 1)}{k^2}
\]

\[
= |v_{h}^{(j+1)}(x_n)|^2,
\]

proving the lemma.

In the following section, we show that our numerical solution converges to a weak solution of the problem \((1.2) - (1.5)\).

4 Existence of weak solutions

The next lemma provides a bound in the \(L^2\)-norm for the discrete solutions.
Lemma 4.1. The sequence \( \left\{ (m_h^{(j)}, v_h^{(j+1)}, H_h^{(j)}) \right\}_{j=0,1,...} \) produced by Algorithm 2.1 satisfies

\[
\mathcal{E}_h^{(j)} + C \sum_{i=0}^{j-1} k\|v_h^{(i+1)}\|^2_D + \lambda_2 \mu^{-1} \sum_{i=0}^{j-1} k\|d_t H_h^{(i+1)}\|^2_D \\
+ 2\mu_0^{-1} \sigma \sum_{i=0}^{j-1} k\|\nabla \times H_h^{(i+1/2)}\|^2_D \leq \mathcal{E}_h^0, \tag{4.1}
\]

where

\[
\mathcal{E}_h^{(j)} = \|\nabla m_h^{(j)}\|^2_D + \|H_h^{(j)}\|^2_D + \lambda_2 \mu^{-1}\|\nabla \times H_h^{(j)}\|^2_D,
\]

and

\[
C = \begin{cases} 
\lambda_2 \mu^{-1}, & \theta \in \left[\frac{1}{2}, 1\right] \\
\lambda_2 \mu^{-1} - (1 - 2\theta)C_1 h^{-2}, & \theta \in \left[0, \frac{1}{2}\right),
\end{cases}
\]

in which \( C_1 \) is a positive constant which is independent with \( j, k \) and \( h \).

Proof. Choosing \( w_h^{(j)} = v_h^{(j+1)} \) in (2.8) and \( \zeta_h = H_h^{(j+1/2)} \) in (2.9), we obtain

\[
\lambda_2 \|v_h^{(j+1)}\|^2_D + k\theta \mu \|\nabla v_h^{(j+1)}\|^2_D = -\mu \left\langle \nabla m_h^{(j)}, \nabla v_h^{(j+1)} \right\rangle_D + \mu \left\langle H_h^{(j+1/2)}, v_h^{(j+1)} \right\rangle_D \tag{4.2}
\]

\[
\frac{\mu_0}{2} d_t \|H_h^{(j+1)}\|^2_D + \sigma \|\nabla \times H_h^{(j+1/2)}\|^2_D = -\mu_0 \left\langle v_h^{(j+1)}, H_h^{(j+1/2)} \right\rangle_D. \tag{4.3}
\]

Multiplying \( \mu_0^{-1} \) to both sides of (4.3) and adding the resulting equation to (4.2), we deduce

\[
\lambda_2 \|v_h^{(j+1)}\|^2_D + k\theta \mu \|\nabla v_h^{(j+1)}\|^2_D + \frac{\mu_0}{2} d_t \|H_h^{(j+1)}\|^2_D \tag{4.4}
\]

\[
+ \mu_0^{-1} \sigma \|\nabla \times H_h^{(j+1/2)}\|^2_D = -\mu \left\langle \nabla m_h^{(j)}, \nabla v_h^{(j+1)} \right\rangle_D.
\]

Since \( m_h^{(j)} + k v_h^{(j+1)} \in V_h \) and

\[
m_h^{(j+1)} = I_h \left( \frac{m_h^{(j)} + k v_h^{(j+1)}}{|m_h^{(j)} + k v_h^{(j+1)}|} \right),
\]

it follows from (2.10) and Lemma 3.1 that

\[
\|\nabla m_h^{(j+1)}\|^2_D \leq \|\nabla (m_h^{(j)} + k v_h^{(j+1)})\|^2_D.
\]

Equivalently, we have

\[
\|\nabla m_h^{(j+1)}\|^2_D \leq \|\nabla m_h^{(j)}\|^2_D + k^2 \|\nabla v_h^{(j+1)}\|^2_D + 2k \left\langle \nabla m_h^{(j)}, \nabla v_h^{(j+1)} \right\rangle_D. \tag{4.5}
\]
Equality (4.4) is used to obtain from (4.5) the following inequality
\[
\|\nabla m_h^{(j+1)}\|_D^2 \leq \|\nabla m_h^{(j)}\|_D^2 - k^2(2\theta - 1)\|\nabla v_h^{(j+1)}\|_D^2 - 2k_0\mu_0\|\nabla H_h^{(j+1/2)}\|_D^2
\]
\[
- k \rho_d \|H_h^{(j+1)}\|_D^2 - 2k_0\mu_0\|\nabla H_h^{(j+1)}\|_D^2.
\]
Hence,
\[
\|\nabla m_h^{(j+1)}\|_D^2 + \|H_h^{(j+1)}\|_D^2 + 2k_0\mu_0\|\nabla v_h^{(j+1)}\|_D^2 + 2k_0\mu_0\|\nabla \times H_h^{(j+1)}\|_D^2
\]
\[
+ k^2(2\theta - 1)\|\nabla v_h^{(j+1)}\|_D^2 \leq \|\nabla m_h^{(j)}\|_D^2 + \|H_h^{(j)}\|_D^2.
\] (4.6)

Next choosing \(\zeta_h = d_tH_h^{(j+1)}\) in equation (2.9), we obtain
\[
2k_0\|d_tH_h^{(j+1)}\|_D^2 + \sigma\|\nabla \times H_h^{(j+1)}\|_D^2 = \sigma\|\nabla \times H_h^{(j)}\|_D^2
\]
\[
- 2k_0\left\langle v_h^{(j+1)}, d_tH_h^{(j+1)} \right\rangle_D.
\]
The term \(-2k_0\left\langle v_h^{(j+1)}, d_tH_h^{(j+1)} \right\rangle_D\) can be estimated by
\[
-2k_0\left\langle v_h^{(j+1)}, d_tH_h^{(j+1)} \right\rangle_D \leq k_0\|v_h^{(j+1)}\|_D^2 + k_0\|d_tH_h^{(j+1)}\|_D^2.
\]
Therefore, we deduce
\[
k_0\|d_tH_h^{(j+1)}\|_D^2 + \sigma\|\nabla \times H_h^{(j+1)}\|_D^2 \leq \sigma\|\nabla \times H_h^{(j)}\|_D^2 + k_0\|v_h^{(j+1)}\|_D^2.
\] (4.7)

Multiplying \(\lambda_2\mu_1\mu_0^{-1}\) to both sides of (4.1) and adding the resulting equation to (4.6), we obtain
\[
\|\nabla m_h^{(j+1)}\|_D^2 + \|H_h^{(j+1)}\|_D^2 + \lambda_2\mu_1\mu_0^{-1}\|\nabla \times H_h^{(j+1)}\|_D^2
\]
\[
+ k\lambda_2\mu_1\|v_h^{(j+1)}\|_D^2 + k\lambda_2\mu_1\|d_tH_h^{(j+1)}\|_D^2 + 2k_0\mu_0^{-1}\sigma\|\nabla \times H_h^{(j+1/2)}\|_D^2
\]
\[
+ k^2(2\theta - 1)\|\nabla v_h^{(j+1)}\|_D^2 \leq \|\nabla m_h^{(j)}\|_D^2 + \|H_h^{(j)}\|_D^2 + \lambda_2\mu_1\mu_0^{-1}\|\nabla \times H_h^{(j)}\|_D^2.
\]
Replacing \(j\) by \(i\) in the above inequality and summing over \(i\) from 0 to \(j - 1\) yield
\[
\|\nabla m_h^{(j)}\|_D^2 + \|H_h^{(j)}\|_D^2 + \lambda_2\mu_1\mu_0^{-1}\sigma\|\nabla \times H_h^{(j)}\|_D^2 + \lambda_2\mu_1\mu_0^{-1}\|v_h^{(j+1)}\|_D^2
\]
\[
+ \lambda_2\mu_1\|d_tH_h^{(i+1)}\|_D^2 + 2\mu_0^{-1}\sigma\|\nabla \times H_h^{(i+1/2)}\|_D^2
\]
\[
+ k^2(2\theta - 1)\|\nabla v_h^{(i+1)}\|_D^2 \leq \|\nabla m_h^{0}\|_D^2 + \|H_h^{0}\|_D^2 + \lambda_2\mu_1\mu_0^{-1}\sigma\|\nabla \times H_h^{0}\|_D^2.
\] (4.8)
When $\theta \in \left[\frac{1}{2}, 1\right]$, the term $k^2(2\theta - 1) \sum_{i=0}^{j-1} \|\nabla v_h^{(i+1)}\|_D^2$ is nonnegative. Hence, from inequality (4.8) we obtain (4.1) where $C = \lambda_2 \mu^{-1}$. When $\theta \in [0, \frac{1}{2})$, using the inverse estimate we obtain

$$C_1 k^2 h^{-2}(2\theta - 1) \sum_{i=0}^{j-1} \|v_h^{(i+1)}\|_D^2 \leq k^2(2\theta - 1) \sum_{i=0}^{j-1} \|\nabla v_h^{(i+1)}\|_D^2,$$

(4.9)

where, $C_1$ is a positive constant which is independent with $j$, $k$ and $h$. Hence, from inequality (4.8) we obtain (4.1) where $C = \lambda_2 \mu^{-1} - C_1 k h^{-2}(1 - 2\theta)$. This completes the proof of the lemma. \hfill $\square$

**Remark 4.2.** The constant $C$ in the above lemma is positive when $\theta \in [1/2, 1]$. When $\theta \in [0, 1/2)$ the additional condition $k = o(h^2)$ assures us that $C$ is positive when $h$ and $k$ are sufficiently small. This condition will be required later in the following lemma and theorem.

The discrete solutions $m_h^{(j)}$, $v_h^{(j+1)}$ and $H_h^{(j)}$ constructed via Algorithm 2.1 are interpolated in time in the following definition.

**Definition 4.3.** For each $t \in [0, T]$, let $j \in \{0, \ldots, J\}$ be such that $t \in [t_j, t_{j+1})$. We define for $t \in [0, T]$ and $x \in D$

$$m_{h,k}(t, x) := \frac{t-t_j}{k} m_h^{(j+1)}(x) + \frac{t_{j+1}-t}{k} m_h^{(j)}(x),$$

$$m_{h,k}^-(t, x) := m_h^{(j)}(x),$$

$$v_{h,k}(t, x) := v_h^{(j+1)}(x),$$

$$H_{h,k}(t, x) := \frac{t-t_j}{k} H_h^{(j+1)}(x) + \frac{t_{j+1}-t}{k} H_h^{(j)}(x),$$

$$\tilde{H}_{h,k}(t, x) := \frac{1}{2} \left( H_h^{(j+1)}(x) + H_h^{(j)}(x) \right),$$

and

$$H_{h,k}^-(t, x) := H_h^{(j)}(x).$$

The following lemma shows that $\{m_{h,k}\}$, $\{m_{h,k}^\prime\}$ and $\{v_{h,k}\}$ converge (up to the extraction of subsequences) as $h$ and $k$ tend to 0.

**Lemma 4.4.** Assume that $h$ and $k$ go to 0 with a further condition $k = o(h^2)$ when $\theta \in [0, \frac{1}{2})$ and no condition otherwise. There exist $m \in \mathbb{H}^1(D_T, \mathbb{R}^3)$ and
\( H \in \mathbb{H}^1(0, T, L^2(\tilde{D})) \) such that \( \nabla \times H \) belongs to \( L^2(\tilde{D}_T) \) and

\[
\begin{align*}
\mathbf{m}_{h,k} &\to \mathbf{m} \text{ strongly in } L^2(D_T), & \text{(4.10)} \\
\frac{\partial \mathbf{m}_{h,k}}{\partial t} &\rightharpoonup \mathbf{m}_t \text{ weakly in } L^2(D_T), & \text{(4.11)} \\
\mathbf{v}_{h,k} &\to \mathbf{m}_t \text{ weakly in } L^2(D_T), & \text{(4.12)} \\
\mathbf{m}_h &\to \mathbf{m} \text{ strongly in } L^2(D_T), & \text{(4.13)}
\end{align*}
\]

\[ |\mathbf{m}| = 1 \text{ a.e. in } D_T, \]

\[
\begin{align*}
\mathbf{H}_{h,k} &\rightharpoonup \mathbf{H} \text{ weakly in } \mathbb{H}^1(0, T, L^2(\tilde{D})), & \text{(4.15)} \\
\nabla \times \mathbf{H}_{h,k} &\rightharpoonup \nabla \times \mathbf{H} \text{ weakly in } L^2(\tilde{D}_T), & \text{(4.16)} \\
\text{and } \nabla \times \mathbf{H}_{h,k} &\rightharpoonup \nabla \times \mathbf{H} \text{ weakly in } L^2(\tilde{D}_T). & \text{(4.17)}
\end{align*}
\]

\[ \mathbf{v}_{h,k} \rightharpoonup \mathbf{m}_t \text{ weakly in } L^2(D_T), \]

Proof of (4.10) and (4.11):

Our goal is to prove that \( \{\mathbf{m}_{h,k}\} \) is bounded in \( \mathbb{H}^1(D_T, \mathbb{R}^3) \) and then use the Banach–Alaoglu Theorem. We note from Definition 4.3 that it suffices to prove that

\[ \|\mathbf{m}^{(j)}\|_D \leq c, \]

\[ \|\nabla \mathbf{m}^{(j)}\|_D \leq c, \]

where the generic constant \( c \) is independent of \( j, h, \) and \( k \). Indeed, it follows from Definition 4.3 and the Cauchy–Schwarz inequality that

\[ \|\mathbf{m}_{h,k}\|_{D_T}^2 \leq ck \sum_{j=0}^{J-1} \left( \|\mathbf{m}^{(j+1)}\|_D^2 + \|\mathbf{m}^{(j)}\|_D^2 \right), \]

\[ \|\partial t \mathbf{m}_{h,k}\|_{D_T}^2 \leq c \sum_{j=0}^{J-1} \left( \|\nabla \mathbf{m}^{(j+1)}\|_D^2 + \|\nabla \mathbf{m}^{(j)}\|_D^2 \right), \]

\[ \left\| \frac{\partial \mathbf{m}_{h,k}}{\partial t} \right\|_{D_T}^2 = \sum_{j=0}^{J-1} k \left\| \frac{\mathbf{m}^{(j+1)} - \mathbf{m}^{(j)}}{k} \right\|_{D}^2. \]

In order to prove (4.18) we note that for every \( \mathbf{x} \in D \) there are at most 4 basis functions \( \phi_{n_1}, \phi_{n_2}, \phi_{n_3} \) and \( \phi_{n_4} \) being nonzero at \( \mathbf{x} \). This together with \( |\mathbf{m}^{(j)}_{h}(\mathbf{x}_{n_i})| = 1 \) and \( \sum_{i=1}^{4} \phi_{n_i}(\mathbf{x}) = 1 \) yields

\[ |\mathbf{m}^{(j)}_{h}(\mathbf{x})|^2 = \left\| \sum_{i=1}^{4} \mathbf{m}^{(j)}_{h}(\mathbf{x}_{n_i})\phi_{n_i}(\mathbf{x}) \right\|^2 \leq 1. \]
This implies (4.18) with a constant $c = |D|^{1/2}$ where $|D|$ is the measure of the
domain $D$.

Inequality (4.19) is proved in Lemma 4.1. In order to prove inequality (4.20), we
note that Lemma 3.4 and Lemma 3.2 imply
\[
\|m_{h,k}^{(j+1)} - m_{h}^{(j)}\|_D \leq c \|v_h^{(j)}\|_D.
\]
By using this inequality, Lemma 4.1 and Remark 4.2 we deduce
\[
k \sum_{j=0}^{J-1} \left\| \frac{m_{h,k}^{(j+1)} - m_{h}^{(j)}}{k} \right\|_D^2 \leq c \sum_{j=0}^{J-1} \|v_h^{(j)}\|_D^2 \leq c.
\]

The Banach–Alaoglu Theorem implies the existence of a subsequence of \{\(m_{h,k}\)\}
which converges weakly to a function $m \in H^1(D_T)$ as $k$ and $h$ tend to zero. This
implies (4.10) and (4.11).

Proof of (4.12):

From (4.1) and and Remark 4.2 it is straightforward to show that \{\(v_{h,k}\)\}
is bounded in $L^2(D_T)$. Hence, there exists a subsequence of \{\(v_{h,k}\)\} which converges
weakly to a function $v$ in $L^2(D_T)$. The problem reduces to proving that $m_t$ equals
$v$ in $L^2(D_T)$. In order to show this we choose for each $\psi \in L^2(D_T)$ a sequence
\{\(\psi_i\)\} $\in C_0^\infty(D_T)$ converging to $\psi$ in $L^2(D_T)$ as $i$ tends to infinity. We then have
\[
|\langle m_t - v, \psi \rangle_{D_T}| \leq |\langle m_t - v, \psi_i - \psi \rangle_{D_T}| + |\langle m_t - \partial m_{h,k}/\partial t, \psi_i \rangle_{D_T}|
+ |\langle \partial m_{h,k}/\partial t - v_{h,k}, \psi_i \rangle_{D_T}| + |\langle v_{h,k} - v, \psi_i \rangle_{D_T}|
\leq \| m_t - v \|_{D_T} \| \psi_i - \psi \|_{D_T} + |\langle m_t - \partial m_{h,k}/\partial t, \psi_i \rangle_{D_T}|
+ |\langle \partial m_{h,k}/\partial t - v_{h,k}, \psi_i \rangle_{D_T}||\psi_i||_{L^\infty(D_T)} + |\langle v_{h,k} - v, \psi_i \rangle_{D_T}|
=: T_1 + \cdots + T_4.
\]

By letting $h, k \to 0$ and then $i \to \infty$ we have $T_i \to 0$ for $i = 1, 2$ and 4. It remains to
show that $T_3 \to 0$. It is clear from the definition of $m_{h,k}^{(j+1)}$ in Algorithm 2.1 that
\[
\| m_{h,k}^{(j+1)}(x_n) - m_{h}^{(j)}(x_n) - k v_{h}^{(j+1)}(x_n) \| = \| m_{h}^{(j)}(x_n) + k v_{h}^{(j+1)}(x_n) \| - 1.
\]
It easily follows from \( |m_h^{(j)}(x_n)| = 1 \) and \( v_h^{(j+1)}(x_n) \cdot m_h^{(j)}(x_n) = 0 \) that

\[
|m_h^{(j)}(x_n) + kv_h^{(j+1)}(x_n)| \leq \frac{1}{2}k^2 |v_h^{(j+1)}(x_n)|^2 + 1.
\]

The above inequality and (4.13) yield

\[
\left| \frac{m_h^{(j+1)}(x_n) - m_h^{(j)}(x_n)}{k} - v_h^{(j+1)}(x_n) \right| \leq \frac{1}{2}k |v_h^{(j+1)}(x_n)|^2.
\]

By using Lemma 3.2 we deduce

\[
\left\| \frac{\partial m_{h,k}}{\partial t} - v_{h,k}(t) \right\|_{L^1(D)} \leq ck \|v_{h,k}(t)\|_{D}^2 \quad \text{for } t \in [t_j, t_{j+1}).
\]

Integrating both sides of this inequality with respect to \( t \) over an interval \([t_j, t_{j+1})\) and summing over \( j \) from 0 to \( J - 1 \) yield, noting the boundedness of \( \{\|v_{h,k}\|_{D}\} \),

\[
\left\| \frac{\partial m_{h,k}}{\partial t} - v_{h,k} \right\|_{L^1(D_T)} \leq ck \|v_{h,k}\|_{D_T}^2 \leq ck \to 0 \quad \text{as } h, k \to 0.
\]

Thus \( T_3 \to 0 \) as \( h, k \to 0 \) and \( i \to \infty \). It follows from (4.22) that

\[
\langle m_t - v, \psi \rangle_{D_T} = 0 \quad \forall \psi \in L^2(D_T).
\]

This proves (4.12).

**Proof of (4.13):**

It is clear from the definition of \( m_h^{-} \) and \( m_h \) that for \( t \in [t_j, t_{j+1}) \) there holds

\[
\left\| m_{h,k}(t) - m_h^{-}(t) \right\|_D = \left\| (t - t_j) \frac{m_h^{(j+1)} - m_h^{(j)}}{k} \right\|_D \leq k \left\| \frac{\partial (m_{h,k}(t_x))}{\partial t} \right\|_D.
\]

Integrating both sides of this inequality with respect to \( t \) over an interval \([t_j, t_{j+1})\) and summing over \( j \) from 0 to \( (J - 1) \) yield

\[
\left\| m_{h,k} - m_h^{-} \right\|_{D_T} \leq k \left\| \frac{\partial m_{h,k}}{\partial t} \right\|_{D_T} \leq ck \to 0 \quad \text{as } h, k \to 0.
\]

The above result and (4.10) imply (4.13).

**Proof of (4.14):**

Using Lemma 3.3 and noting that \( |m_h^{(j)}(x_n)| = 1 \) for \( n = 1, \cdots, N \), we deduce

\[
\left| m_h^{(j)}(x) \right| - 1 \leq C h^2 \left| \nabla m_h^{(j)}(x) \right|^2 \quad \text{for all } x \in D.
\]
Integrating both sides of the above inequality on \([t_j, t_{j+1}) \times D\), using Lemma 4.1 and noting Remark 4.2, we obtain
\[
\int_{t_j}^{t_{j+1}} \int_D \left| 1 - |m_h^{(j)}(x)| \right|^2 \, dx \, dt \leq ch^2 \int_{t_j}^{t_{j+1}} \| \nabla m_h^{(j)} \|^2_D \leq ckh^2.
\]
Hence
\[
\int_{D_T} \left| 1 - |m_{h,k}^-| \right|^2 \, dx \, dt \to 0 \text{ as } h, k \to 0.
\]
We infer from (4.13) that
\[|m| = 1 \text{ a.e. in } D_T.\]

Proof of (4.15), (4.16) and (4.17):

By using the same arguments as above, we obtain these results, completing the proof of the lemma.

We are now able to prove the main result of this paper.

**Theorem 4.5.** Assume that \(h\) and \(k\) go to 0 with the following conditions

\[
\begin{cases}
  k = o(h^2) & \text{when } 0 \leq \theta < 1/2, \\
  k = o(h) & \text{when } \theta = 1/2, \\
  \text{no condition} & \text{when } 1/2 < \theta \leq 1.
\end{cases}
\]

Then the limits \((m, H)\) given by Lemma 4.4 is a weak solution of the MLLG equations (2.3)–(2.4).

**Proof.** For any \(\phi \in C^\infty(D_T)\), \(\zeta \in C^\infty(\tilde{D}_T)\), and \(t \in [t_j, t_{j+1})\), we define
\[
w_{h,k}(t, \cdot) := I_{V_h}(m_{h,k}^- \times \phi(t, \cdot)) \quad \text{and} \quad \zeta_h(t, \cdot) := I_{V_h}(\zeta(t, \cdot)).
\]

In equations (2.8) and (2.9), replacing \(w_h^{(j)}\) and \(\zeta_h\) by \(w_{h,k}(t)\) and \(\zeta_h(t)\), respectively, and using Definition 4.3, we rewrite (2.8)–(2.9) as
\[
-\lambda_1 \left\langle m_{h,k}^{-}(t) \times v_{h,k}(t), w_{h,k}(t) \right\rangle_D + \lambda_2 \left\langle v_{h,k}(t), w_{h,k}(t) \right\rangle_D \\
= -\mu \left\langle \nabla (m_{h,k}^- + k\theta v_{h,k}(t)), \nabla w_{h,k}(t) \right\rangle_D + \mu \left\langle \nabla \tilde{H}_{h,k}(t), w_{h,k}(t) \right\rangle_D,
\]
and
\[
\mu_0 \left\langle \frac{\partial \tilde{H}_{h,k}(t)}{\partial t}, \zeta_h(t) \right\rangle_{\tilde{D}} + \sigma \left\langle \nabla \times \tilde{H}_{h,k}(t), \nabla \times \zeta_h(t) \right\rangle_{\tilde{D}} = -\mu_0 \left\langle v_{h,k}(t), \zeta_h(t) \right\rangle_{\tilde{D}}.
\]
Integrating both sides of these equations with respect to $t$ over an interval $[t_j, t_{j+1})$ and summing over $j$ from 0 to $J - 1$ yield

$$- \lambda_1 \langle m_{h,k} \times v_{h,k}, w_{h,k} \rangle_{D_T} + \lambda_2 \langle v_{h,k}, w_{h,k} \rangle_{D_T}$$

$$= -\mu \langle \nabla (m_{h,k} + k \theta v_{h,k}), \nabla w_{h,k} \rangle_{D_T} + \mu \langle \widetilde{H}_{h,k}, w_{h,k} \rangle_{D_T}$$

(4.25)

and

$$\mu_0 \left\langle \frac{\partial H_{h,k}}{\partial t}, \zeta_h \right\rangle_{\bar{D}_T} + \sigma \left\langle \nabla \times \tilde{H}_{h,k}, \nabla \times \zeta_h \right\rangle_{\bar{D}_T} = -\mu_0 \langle v_{h,k}, \zeta_h \rangle_{\bar{D}_T}.$$  

(4.26)

In order to prove that $m$ and $H$ satisfy (2.3) and (2.4), respectively, we prove that as $h$ and $k$ tend to 0 there hold

$$\langle m_{h,k} \times v_{h,k}, w_{h,k} \rangle_{D_T} \rightarrow \langle m \times m_t, m \times \phi \rangle_{D_T},$$

(4.27)

$$\langle v_{h,k}, w_{h,k} \rangle_{D_T} \rightarrow \langle m_t, m \times \phi \rangle_{D_T},$$

(4.28)

$$\langle \nabla m_{h,k}, \nabla w_{h,k} \rangle_{D_T} \rightarrow \langle \nabla m, \nabla (m \times \phi) \rangle_{D_T},$$

(4.29)

$$k \langle \nabla v_{h,k}, \nabla w_{h,k} \rangle_{D_T} \rightarrow 0,$$

(4.30)

$$\langle \widetilde{H}_{h,k}, w_{h,k} \rangle_{D_T} \rightarrow \langle H, m \times \phi \rangle_{D_T},$$

(4.31)

and

$$\left\langle \frac{\partial H_{h,k}}{\partial t}, \zeta_h \right\rangle_{\bar{D}_T} \rightarrow \langle H_t, \zeta \rangle_{\bar{D}_T},$$

(4.32)

$$\left\langle \nabla \times \tilde{H}_{h,k}, \nabla \times \zeta_h \right\rangle_{\bar{D}_T} \rightarrow \langle \nabla \times H, \nabla \times \zeta \rangle_{\bar{D}_T},$$

(4.33)

$$\langle v_{h,k}, \zeta_h \rangle_{\bar{D}_T} \rightarrow \langle m_t, \zeta \rangle_{\bar{D}_T}.$$  

(4.34)

We now prove (4.27) and (4.30); the others can be obtained in the same manner.

Using the triangular inequality and Holder’s inequality, we estimate

$$I_{h,k} := \left| \langle m_{h,k} \times v_{h,k}, w_{h,k} \rangle_{D_T} - \langle m \times m_t, m \times \phi \rangle_{D_T} \right|$$

as follows:

$$I_{h,k} \leq \left| \langle m_{h,k} \times v_{h,k}, w_{h,k} - m_{h,k} \times \phi \rangle_{D_T} \right| + \left| \langle m_{h,k} \times v_{h,k}, (m_{h,k} - m) \times \phi \rangle_{D_T} \right|$$

$$+ \left| \langle (m_{h,k} - m) \times v_{h,k}, m \times \phi \rangle_{D_T} \right| + \left| \langle m \times (v_{h,k} - m_t), m \times \phi \rangle_{D_T} \right|$$

$$\leq \| m_{h,k} \|_{L^\infty(D_T)} \| v_{h,k} \|_{D_T} \| w_{h,k} - m_{h,k} \times \phi \|_{D_T}$$

$$+ \| m_{h,k} \|_{L^\infty(D_T)} \| v_{h,k} \|_{D_T} \| m - m_{h,k} \|_{L^\infty(D_T)} \| \phi \|_{L^\infty(D_T)}$$

$$+ \| m - m_{h,k} \|_{D_T} \| v_{h,k} \|_{D_T} \| \phi \|_{L^\infty(D_T)}$$

$$+ \| v_{h,k} - m_t \|_{D_T} \| \phi \|_{L^\infty(D_T)}$$

$$\leq c \left( \| w_{h,k} - m_{h,k} \times \phi \|_{D_T} + \| m - m_{h,k} \|_{D_T} + \| v_{h,k} - m_t \|_{D_T} \right).$$
where we have used (4.21) and Lemma 4.1 noting Remark 4.2. The interpolation operators $I_{Vh}$ and $I_{Yh}$ have the following properties (see e.g., [5] and [14])

$$\| \mathbf{m}_{h,k} - \mathbf{w}_{h,k} \|_{L^2([0,T],H^1(D))} \leq C h \| \mathbf{m}_{h,k} \|_{H^1(D_T)} \| \mathbf{\phi} \|_{W^{2,\infty}(D_T)},$$

$$\| \zeta(t) - \zeta_h(t) \|_D + \| \nabla \times (\zeta(t) - \zeta_h(t)) \|_D \leq C h \| \nabla^2 \zeta \|_D.$$  

(4.35)

This implies

$$\lim_{k,h \to 0} I_{h,k} = 0,$$

proving (4.27).

In order to prove (4.30) we first note that

$$\| \nabla \mathbf{w}_{h,k} \|_{D_T} \leq \| \nabla (\mathbf{m}_{h,k} \times \mathbf{\phi} - \mathbf{w}_{h,k}) \|_{D_T} + \| \nabla (\mathbf{m}_{h,k} \times \mathbf{\phi}) \|_{D_T} \leq c h \| \mathbf{m}_{h,k} \|_{H^1(D_T)} \| \mathbf{\phi} \|_{W^{2,\infty}(D_T)} + \| \nabla \mathbf{m}_{h,k} \|_{D_T} \| \nabla \mathbf{\phi} \|_{L^\infty(D_T)} \leq c \| \mathbf{\phi} \|_{W^{2,\infty}(D_T)},$$

where we have used (4.35) and the boundedness of $\| \mathbf{m}_{h,k} \|_{H^1(D_T)}$. Now using Holder’s inequality we obtain

$$k \langle \nabla \mathbf{v}_{h,k}, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \leq c k \| \nabla \mathbf{v}_{h,k} \|_{D_T} \leq c k h^{-1} \| \mathbf{v}_{h,k} \|_{D_T} \leq c k h^{-1} \| \mathbf{v}_{h,k} \|_{D_T} \leq c k h^{-1}$$

(4.36)

when $\theta \in [0,1/2]$. Therefore, under the assumption (4.24) there holds

$$k \langle \nabla \mathbf{v}_{h,k}, \nabla \mathbf{w}_{h,k} \rangle_{D_T} \to 0.$$

We now prove (2.5). Since $\mathbf{m}_0^h = I_{Vh}(\mathbf{m}_0)$, the sequence $\{\mathbf{m}_0^h\}$ converges to $\mathbf{m}_0$ in $L^2(D)$ as $h$ tends to 0. Using the weak continuity of the trace operator we obtain that $\mathbf{m}(0, \cdot) = \mathbf{m}_0$ in the sense of traces.

Finally, applying weak lower semicontinuity of norms in inequality (4.1) we obtain the energy inequality (2.6), which completes the proof. 

\[ \square \]
5 Numerical experiments

In order to carry out physically relevant experiments, the initial fields \( m_0, H_0 \) must satisfy condition (2.2). This can be achieved by taking

\[
H_0 = H_0^* - \chi_D m_0,
\]

where \( \text{div} H_0^* = 0 \) in \( \tilde{D} \). In our experiment, for simplicity, we choose \( H_0^* \) to be a constant. We solve an academic example with \( D = \tilde{D} = (0,1)^3 \) and

\[
m_0(x) = \begin{cases} 
(0,0,-1), & |x^*| \geq \frac{1}{2}, \\
(2x^*A, A^2 - |x^*|^2)/(A^2 + |x^*|^2), & |x^*| \leq \frac{1}{2},
\end{cases}
\]

\[
H_0^*(x) = (0,0,H_s), \quad x \in \tilde{D},
\]

where \( x = (x_1, x_2, x_3), x^* = (x_1-0.5, x_2-0.5, 0) \) and \( A = (1-2|x^*|)^4/4. \) The constant \( H_s \) represents the strength of \( H_0 \) in the \( x_3 \)-direction. We compute the experiments for \( H_s = 0, \pm 30, \pm 100 \) and \( \pm 1000 \). We set the values for the other parameters in (1.1) and (1.2) as \( \lambda_1 = \lambda_2 = \mu_0 = \sigma = 1. \)

The domain \( D \) is partitioned into uniform cubes with the mesh size \( h = 1/2^3 \), where each cube consists of six tetrahedra. We choose the time step \( k = 10^{-3} \) and the parameter \( \theta \) in Algorithm 2.1 to be 0.7. The construction of the basis functions for \( W_h^{(j)} \) and \( Y_h \) in this algorithm is discussed in [13]. At each iteration we need to solve a linear system of size \( (2N + M) \times (2N + M) \), recalling that \( N \) is the number of vertices and \( M \) is the number of edges in the triangulation. The code is written in Fortran90.

The evolution of \( \|\nabla m_{h,k}\|_{\tilde{D}}, \|H_{h,k}\|_{\tilde{D}} \) and \( \|\nabla \times H_{h,k}\|_{\tilde{D}} \) are depicted in Figures 1, 2 and 3 respectively. Figure 4 shows that the solution satisfies condition (2.6) in Definition 2.1.

Remark 5.1. By the time this paper was written up, we learnt that Bañás, Page and Praetorius [4] independently solved a similar problem. They also used a linear scheme similar to our scheme, even though their variational formulation was different.

References

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Figure 1: Plot of $t \mapsto \|\nabla m_{h,k}(t)\|_D$.

Figure 2: Plot of $t \mapsto \|H_{h,k}(t)\|_{\tilde{D}}$.
Figure 3: Plot of $t \mapsto \| \nabla \times H_{h,k}(t) \|_{\tilde{D}}$.

Figure 4: Plot of $\log t \mapsto \mathcal{E}(t)$
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