Aliasing free for evolutionary spectrum of non-stationary processes

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Abstract. This paper proposes a non parametric discrete estimate of evolutionary spectral density of non-stationary process with continuous time. To solve a problem related to aliasing effects, we assume that the evolutionary spectral density has a compact support and we use a specific uniform sampling depending on the width of the compact support. An asymptotically unbiased and consistent estimate is given.

1. Introduction
Stationarity is an important assumption in the spectral analysis of time series. However, in many applications the signal is modeled as a non-stationary process. Priestley in [9]),[10]),[11]) developed the basic theory of Evolutionary Spectrum ES for non-stationary oscillatory processes. Other studies have been made as in [7], which were based on Wold-Cramér decomposition of the process. The ES is time-dependant and describes the local power-frequency distribution at each instant of time. The application of the ES covers various scientific fields: signal and image processing [4], [1], [20], seismic [18], oceanography, music [2]. The estimation of the evolutionary spectral density is studied in [12], [7], [4], [16], [19] and [8].

In this work, as in [9], [10],[11], we consider a non-stationary oscillatory process \( X(t) \) defined by:

\[
X(t) = \int_{-\infty}^{+\infty} e^{it\omega} A_t(\omega) dZ(\omega); \quad t \in R
\]

where, \( A_t(\omega) \), as function of \( t \), is the generalized Fourier transform whose module has an absolute maximum at origin. \( Z \) is a process with orthogonal increments and \( \mathbb{E} \left[ |dZ(\omega)|^2 \right] = d\mu(\omega) \), for some positive measure \( \mu \). Processes which admit such representation are called oscillatory processes. The evolutionary spectral measure, with respect to the oscillatory family \( \mathcal{F} = A_t(\omega)e^{i\omega t} \), is defined at each instant \( t \) by \( dH_t(\omega) = |A_t(\omega)|^2 d\mu(\omega) \). The choice of the oscillatory process is motivated by its physical interpretation namely the variance of the process interpreted as a measure of the total power of the process at time \( t \): \( \text{Var}(X(t)) = \int_{-\infty}^{+\infty} dH_t(\omega) \) see [9]). The evolutionary spectral density of the process \( \{X(t)\} \) is given by \( h_t(\omega) \) and defined as follows: \( h_t(\omega) = \frac{dH_t(\omega)}{d\omega} \), \( \omega \in R \). Priestley in [12] proposed an estimator of \( h_t(\omega) \) defined by:

\[
\hat{h}(\omega) = \int_{-\infty}^{\infty} w_T(u) U(t-u, \omega)^2 du \quad \text{where} \quad U(t, \omega) = f_{t-T}^{T} g_u X(t-u)e^{-i\omega(t-u)} du
\]

The filter \( g_u \) and the weight function \( w_T(u) \) are satisfying the usual conditions given in [9]. To calculate this estimator, we must observe the process on a time interval \([t-T, t]\) which is impossible in practice.
The first idea is to use the discrete estimator proposed in [12] for the time-discrete non-stationary processes, but this estimator is not appropriate because our process is time-continuous. Indeed, we show in this paper that this estimator converges to \( \sum_{k \in \mathbb{Z}} h_k(\omega + 2k\pi) \) instead of converging to \( h_t(w) \), this phenomenon is called the ”aliasing effects” see [6]. Several studies have provided a solution for aliasing effects using non-uniform sampling. Except that non-uniform sampling is often perceived as not suitable when we have need to observe the signal at precise instants as it frequently happens in astronomy, medicine or geosciences. It is also not suitable when there are missing observations. These difficulties are illustrated in Lomb [5], Scargle [15] and Tarczynski [17]. Our contribution is to provide a solution for a uniform sampling when the evolutionary spectral density has compact support. Indeed, as in Sabre ([14] and [13]), we assume that the evolutionary spectral density has a compact support i.e. it is vanishing outside an interval \( I = [-\Omega, \Omega] \). We construct a periodogram where the process \( \{X(t), \ t \in R\} \) is sampled at discrete instants chosen such that the interval between successive observations depends on the width of the interval \( I \). Using usual technique, we give a smoothing estimator of \( h_t(w) \) and show that it is asymptotically unbiased and consistent.

The paper is organized as follows: Section 2 shows the existence of the aliasing effects. Section 3 constructs a periodogram and shows that it is an asymptotically unbiased estimator of \( h_t(\omega) \) (lemma 2.1, theorem 2.2). Section 4 presents a smoothing estimator in the neighborhood of time-instant \( t \) using a weight function and shows that it is an asymptotically unbiased and consistent estimator of the (weighted) average of \( h_t(\omega) \) in the neighborhood of the time-instant \( t \) (theorems 3.1 and 3.2). Section 5 contains the concluding remarks, the potential applications and open research problems.

2. The aliasing effects

According to [9], a non-stationary oscillatory centered processes with continuous time can be written as (1), where \( A_t(\omega) = \int_{-\infty}^{\infty} e^{it\omega} dK_\omega(u) \). \( K_\omega \) is a measure such that: \( \int_{-\infty}^{\infty} |dK_\omega(\theta)| = 1 \)

To estimate the evolutionary spectral density \( h_t(w) \) from a discrete sample, we use the estimator proposed by [9] for process with discrete time defined as follows:

\[
I_t(w) = \sum u g_u X(t - u)e^{-iw(t-u)}
\]

where \( g_u \) is a filter respecting the usual conditions defined by:

\[
g_u = O(e^{-|u|})
\]

\[
2\pi \sum_{u=-\infty}^{+\infty} |g_u|^2 = \int |\Gamma(u)|^2 du = 1
\]

\( \Gamma \) is transform Fourier of \( g \): \( \Gamma(\omega) = \sum_{u=-\infty}^{+\infty} g_u e^{-iu\omega} \). The width of the filter is described as \( B_g = \sum_{u=-\infty}^{+\infty} |u||g_u| < \infty \)

For each family \( \mathcal{F} = \{e^{i\theta}A_t(\omega)\} \), the function \( \mathcal{B}_\mathcal{F}(\omega) \) is defined by: \( \mathcal{B}_\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} |\theta||dK_\omega(\theta)| \). This function is a measure of the ”width” of \( |dK_\omega(\theta)| \). Assume that the family \( \mathcal{F} \) is semi stationary i.e. the function \( \mathcal{B}_\mathcal{F}(\omega) \) is bounded for all \( \omega \). the function \( \mathcal{B}_\mathcal{F} = (\sup_{\omega} \mathcal{B}_\mathcal{F}(\omega))^{-1} \) is the characteristic ”width” of the family \( \mathcal{F} \). The filter \( g \) is chosen such that its width \( B_g \) is enough smaller than \( \mathcal{B}_\mathcal{F} \): \( B_g << \mathcal{B}_\mathcal{F} \), and \( \frac{B_g}{\mathcal{B}_\mathcal{F}} < \varepsilon \), \( \varepsilon \) is a small nonnegative number. As in [9],[10], [12], suppose that \( \Gamma \) is pseudo \( \delta \)-function with respect to \( h_t \) in order \( \frac{B_g}{\mathcal{B}_\mathcal{F}} \), which means:

\[
\left| \int \Gamma(s)h_t(s+k)ds - h_t(k) \int \Gamma(s)ds \right| < \frac{B_g}{\mathcal{B}_\mathcal{F}}
\]
Let us show that $|I_t(w)|^2$ is an asymptotically unbiased estimator of $\sum_{k \in \mathbb{Z}} h_t(2k\pi + w)$ precisely $E|I_t(w)|^2 = \sum_{k \in \mathbb{Z}} h_t(2k\pi + w) + O(\varepsilon)$. Indeed, it is easy to show that

$$E|I_t(w)|^2 = \int |\Gamma_{t,v+w}(v)|^2 |A_t(w+v)|^2 d\mu(w+v)$$

where

$$\Gamma_{t,\lambda,\tau}(\theta) = \sum_{n=-\infty}^{+\infty} g_n e^{-in\theta} \frac{A_{t-\tau u}(\lambda)}{A_t(\lambda)}. \tag{4}$$

From the theorem 11.2.1 in Priestley [13], we have $\Gamma_{t,v+w}(v) = \Gamma(v) + r(t, w)$. where

$$|r(t, w)| < \frac{\varepsilon}{|A_t(w+v)|}. \tag{5}$$

Therefore $E|I_t(w)|^2 = \int |\Gamma(v)|^2 h_t(w+v)dv + L_1 + L_2 + L_3$, where

$$L_1 = \int \Gamma(v) r^*(t, w) |A_t(w+v)|^2 d\mu(v+w)$$

$$L_2 = \int \Gamma^*(v) r(t, w) |A_t(w+v)|^2 d\mu(v+w)$$

$$L_3 = \int |r(t, w)|^2 |A_t(w+v)|^2 d\mu(v+w)$$

$r^*$ is the conjugate of $r$. Since the measure $\mu$ is finite and using the inequality (5), we have

$$L_2 \leq \varepsilon \int_A d\mu(v+w) + \varepsilon \int_{-\infty}^{0} |\Gamma(v)|^2 dH_t(v+w)$$

where the set $A$ is defined as follows: $A = \{ v ; |\Gamma(v)| |A_t(v + \omega_0)| \leq 1 \}$. $L_2 \leq \varepsilon \int_A d\mu(v+w) + \varepsilon \sup(h_t) \int_{-\infty}^{+\infty} |\Gamma(v)|^2 dv$

Thus $L_2 = O(\varepsilon)$. Similarly we show that $L_1 = O(\varepsilon)$ and $L_3 = O(\varepsilon)$. Therefore

$$E|I_t(w)|^2 = \sum_{k \in \mathbb{Z}} \int_{-\pi}^{+\pi} |\Gamma(v + 2k\pi)|^2 h_t(v + 2k\pi + w)dv + O(\varepsilon)$$

Since $\Gamma$ is $2\pi$-periodic, we obtain $E|I_t(w)|^2 = \int_{-\pi}^{+\pi} |\Gamma(v)|^2 \sum_{k \in \mathbb{Z}} h_t(v + 2k\pi + w)dv + O(\varepsilon)$ From (3) and (4), we get $E|I_t(w)|^2 = \sum_{k \in \mathbb{Z}} h_t(2k\pi + w) + O(\varepsilon)$.

3. Periodogram and its proprieties

To solve this problem, we assume that the evolutionary spectral density $h_t(\omega)$ is vanishing outside an interval $]-\Omega, \Omega[$, where $\Omega$ is a non negative real and we construct the following periodogram:

$$U_t(\omega_0) = \tau^\frac{1}{2} \sum_{u=-N}^{N} g_u X(t-\tau u) e^{-i\omega_0(t-\tau u)}; \quad \omega_0 \in ]-\Omega, \Omega[$$

where $\tau$ is a real number such as $0 < \tau < \inf \left\{ \frac{2\pi}{|2\pi - \Omega|}, \frac{4\pi}{4\pi - \omega_0} \right\}$ and $g$ is a filter defined above chosen such that $|\Gamma(v)|^2$ is pseudo $\delta$-function with respect to the function $x \rightarrow h_t(x \pi)$ in order $\varepsilon$. 

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Thus, let us, in the following lemma, give the results about the generalized transform of the filter $g_u$.

**Lemma 3.1** For all real numbers $\lambda$ and $\theta$, we have

$$|A_t(\lambda)| \Gamma_{t, \lambda, \tau}(\theta) \leq \tau \varepsilon.$$

where $\Gamma_{t, \lambda, \tau}(\theta)$ is defined in (12).

The proof of this lemma is similar to that used in Priestley [10].

The following theorem shows that $|U_t(\omega_0)|^2$ is an asymptotically unbiased estimator of $h_t(\omega_0)$.

**Theorem 3.2** Let $\omega_0$ be a real number, we have

$$\left[ E |U_t(\omega_0)|^2 \right] \simeq h_t(\omega_0) + O((\tau + 1)\varepsilon).$$

**Proof:** We have $U_t(\omega_0) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \Gamma_{t, \omega, \tau}(\tau \omega - \tau \omega_0) A_t(\omega) e^{i(t(\omega - \omega_0))} d\omega$. Let $\omega = \omega_0 = v$, then

$$U_t(\omega_0) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \Gamma_{t, v+\omega_0, \tau}(v \tau) A_t(\omega_0 + v) e^{i(tv)} d\omega.$$ Since $Z$ is a process with orthogonal increments, we obtain

$$E \left[ |U_t(\omega_0)|^2 \right] = \tau \int_{-\infty}^{\infty} \Gamma_{t, v+\omega_0, \tau}(v \tau)^2 |A_t(\omega_0 + v)|^2 d\mu(\omega_0 + v).$$

From the lemma 2.1, we have $\Gamma_{t, v+\omega_0, \tau}(v \tau) = \Gamma(v \tau) + r(t, \omega_0, \tau)$ where

$$|r(t, \omega_0, \tau)| \leq \frac{\varepsilon}{A_t(\omega_0 + v)}.$$ (6)

Thus,

$$E \left[ |U_t(\omega_0)|^2 \right] = \tau \int_{-\infty}^{\infty} \Gamma(v \tau)^2 h_t(v + \omega_0) dv + I_1 + I_2 + I_3,$$

where

$$I_1 = \tau \int_{-\infty}^{\infty} \Gamma(v \tau) r^*(t, \omega_0, \tau) |A_t(v + \omega_0)|^2 d\mu(v + \omega_0).$$

$$I_2 = \tau \int_{-\infty}^{\infty} \Gamma^*(v \tau) r(t, \omega_0, \tau) |A_t(v + \omega_0)|^2 d\mu(v + \omega_0),$$

$$I_3 = \tau \int_{-\infty}^{\infty} |r(t, \omega_0, \tau)|^2 |A_t(v + \omega_0)|^2 d\mu(v + \omega_0).$$

Using the inequality (6), and the fact that the measure $\mu$ is finite, we get $I_3 \leq \tau \varepsilon^2 \int_{-\infty}^{\infty} d\mu(v) = O(\tau \varepsilon^2)$, and $|I_2| \leq \varepsilon \tau \int_{-\infty}^{\infty} |\Gamma(v \tau)| |A_t(v + \omega_0)| d\mu(v + \omega_0).$ To show that $I_2 = O(\tau \varepsilon)$, it remains to be proven that the right side of the last integral is finite. Indeed, let the set $\Lambda$ be defined as follows:

$$\Lambda = \{ v; |\Gamma(v \tau)| |A_t(v + \omega_0)| \leq 1 \}.$$ 

$\bar{\Lambda}$ being the complementary of $\Lambda$. Therefore,

$$\int_{-\infty}^{\infty} |\Gamma(v \tau)| |A_t(v + \omega_0)| d\mu(v + \omega_0) \leq \int_{\Lambda} d\mu(v + \omega_0) + \int_{-\infty}^{+\infty} |\Gamma(v \tau)|^2 h_t(v + \omega_0) dv$$

$$\leq \frac{1}{\tau} \sup_{v'} h_t(v') \int_{-\infty}^{+\infty} |\Gamma(v')|^2 dv'.$$
Thus, $|I_2| = O ((\tau + 1)\varepsilon)$. In the same way, we can show $|I_1| = O ((\tau + 1)\varepsilon)$. Therefore,

$$E \left[ U_t(\omega_0) \right]^2 = \tau \int_{-\infty}^{+\infty} |\Gamma(v\tau)|^2 h_t(v + \omega_0)dv + O ((\tau + 1)\varepsilon).$$

Let $v\tau = \omega$, we get

$$E \left[ U_t(\omega_0) \right]^2 = \sum_{j \in Z} \int_{2j\pi}^{2(j+1)\pi} |\Gamma(\omega)|^2 h_t(\frac{\omega}{\tau} + \omega_0)d\omega + O ((\tau + 1)\varepsilon).$$

Using the following change of variable $v = \omega - 2j\pi$, and the fact that $\Gamma(v)$ is 2\pi-periodic, we can write

$$E \left[ U_t(\omega_0) \right]^2 = \int_0^{2\pi} |\Gamma(\omega)|^2 \sum_{j \in Z} h_t(\frac{v + 2j\pi}{\tau} + \omega_0)dv + O ((\tau + 1)\varepsilon).$$

Let $j$ be an integer number such that $h_t(\frac{v + 2j\pi}{\tau} + \omega_0) \neq 0$ for all $v \in [0, 2\pi]$. Thus, $\frac{2j\pi}{\tau} + \omega_0 \leq v + 2j\pi + \omega_0 \leq \frac{2(j+1)\pi}{\tau} + \omega_0$, the integer $j$ verifies $\frac{2(j+1)\pi}{\tau} + \omega_0 < \Omega$ and $-\Omega < \frac{2j\pi}{\tau} + \omega_0$. Thus, $\frac{(-\Omega - \omega_0)\pi}{2\pi} < j < \frac{(\Omega - \omega_0)\pi}{2\pi} - 1$. As $\tau < \inf \left\{ \frac{2\pi}{\Omega + \omega_0}, \frac{2\pi}{\Omega - \omega_0} \right\}$, we have $\frac{(\Omega - \omega_0)\pi}{2\pi} - 1 < 1$ and $-1 < \frac{(-\Omega - \omega_0)\pi}{2\pi}$. Consequently, $j = 0$. Then

$$E \left[ U_t(\omega_0) \right]^2 = \int_0^{2\pi} |\Gamma(\omega)|^2 h_t(\frac{v}{\tau} + \omega_0)dv + O ((\tau + 1)\varepsilon).$$

Since $|\Gamma(v)|^2$ is pseudo $\delta$-function with respect to the function $x \rightarrow h_t(\frac{x}{\tau})$ in order $\varepsilon$, we deduce that $E \left[ U_t(\omega_0) \right]^2 = h_t(\omega_0) + O ((\tau + 1)\varepsilon)$.

### 4. Smoothing periodogram

In order to give a consistent estimate of $h_t(\omega_0)$, we propose the smoothing estimator defined as follows:

$$\hat{h}_{t,T'}(\omega_0) = \sum_{v=-m}^{m} w_{T',v} |U_{t-v}(\omega_0)|^2,$$

where $w_{T',v}$ is a positive weight function depending on the parameter $T'$, and satisfying the following conditions:

- a) $w_{T',v} = 0$, for all $v$ such that $|v| > m$,
- b) $w_{T',v} = w_{T',-v}$,
- c) $\sum_{v=-m}^{m} w_{T',v} = 1$,
- d) $\sum_{v=-m}^{m} w_{T',v}^2 < \infty$.
- e) Suppose that there exists a constant $C$ such that $\lim_{T' \rightarrow \infty} \left\{ T' \sum_{u=-m}^{m} |W_{T',u}|^2 \right\} = C$, where $W_{T',u} = \sum_{v=-m}^{m} e^{-ivu}w_{T',v}$. The following theorem shows that $\hat{h}_{t,T'}(\omega_0)$ is an estimator asymptotically unbiased of the average values of $h_t(\omega_0)$ in the neighborhood of the time-instant $t$.

**Theorem 4.1** Let $\omega_0 \in ]-\Omega, +\Omega[$, then

$$E \left[ \hat{h}_{t,T'}(\omega_0) \right] = \bar{h}_t(\omega_0) + O ((\tau + 1)\varepsilon)$$
where $\bar{h}_t(\omega_0) = \frac{1}{\gamma} \sum_{u,v} w_{T_u v} h_{T_{uv}} (\omega_0)$.

It is easy to prove this theorem by using the theorem 2.2. When the process is gaussian, the following theorem shows that the variance of $\hat{h}_{T_{uv}} (\omega_0)$ converges to zero and the estimator $\hat{h}_{T_{uv}} (\omega_0)$ converges to $\bar{h}_t(\omega_0)$ in mean square.

**Theorem 4.2** Suppose that the process $X_t$ is gaussian. Let $\omega \in ]-\Omega, +\Omega[$, then

$$V ar [\hat{h}_{T_{uv}} (\omega)] = O\left(\frac{\tau^2}{T^2}\right).$$

$$E \left[ \hat{h}_{T_{uv}} (\omega) - \bar{h}_t(\omega_0) \right] = O\left(\frac{\tau^2}{T^2}\right) + O \left((\tau + 1)^2 \varepsilon^2\right)$$

**Proof** We have

$$E \left\{ \left| U_t(\omega) \right|^2 \left| U_s(\omega') \right|^2 \right\} = \tau^2 \sum_{u,v} \sum_{r,z} g_u g_v g_r g_z e^{i\omega(t - \tau)} e^{i\omega(s - \tau)} e^{i\omega'(t - \tau)} e^{i\omega'(s - \tau)}$$

Since $X_t$ is Gaussian we get

$$E \left[ X_{t-\tau} X_{t-v} X_{t-r} X_{t-z} \right] = R(t - \tau, t - v) R(s - \tau, s - z) + R(t - \tau, s - z) R(t - v, s - r) + R(t - \tau, s - r) R(t - v, s - z)$$

where

$$R(p, q) = \int e^{i\omega(p - q)} A_p(\omega) A_q^*(\omega) d\omega = E \left[ X_p X_q^* \right]. \quad (7)$$

To simplify, we suppose that $d\mu(\omega) \equiv d\omega$. Using (7) and the integral representation, we obtain

$$E \left\{ \left| U_t(\omega) \right|^2 \left| U_s(\omega') \right|^2 \right\} = E \left[ \left| U_t(\omega) \right|^2 \right] E \left[ \left| U_s(\omega') \right|^2 \right] + S_1 + S_2. \quad (8)$$

where

$$S_1 = \tau^2 \sum_{u,v,r,z} g_u g_v g_r g_z e^{i\omega(t - \tau)} e^{i\omega(s - \tau)} e^{i\omega'(t - \tau)} e^{i\omega'(s - \tau)}$$

$$\times \int \int e^{i(\tau-t)\theta} e^{-i(s-r)\phi} A_{t-\tau}(\theta) A_{s-z}^*(\theta) e^{i(\tau-t)\phi} e^{i(s-r)\phi} A_{s-r}^*(\phi) A_{t-\tau}(\phi) d\theta d\phi$$

$$S_2 = \tau^2 \int \left\{ \sum_u g_u e^{-i\omega(t-\tau+\theta)} A_{t-\tau}(\theta) \right\} \left\{ \sum_r g_r e^{i\omega(s-\tau+\phi)} A_{s-r}^*(\phi) \right\} e^{i\theta(t-s)} e^{i\phi(s-t)} d\theta d\phi.$$

The lemma 2.1 and the definition of $\Gamma$, imply

$$S_1 \simeq \tau^2 \int \int \Gamma(\theta - \omega - \tau) A_t(\theta) \Gamma^*(\phi - \omega - \tau) A_s^*(\phi) d\theta d\phi \times \Gamma^*(\theta - \omega - \tau) A_s^*(\theta) e^{i\theta(t-s)} e^{i\phi(s-t)} d\theta d\phi.$$
Since \(g_u\) and \(X(t)\) are real, we have

\[ A_t(-\theta) = A_t^*(\theta), \quad \Gamma(-\theta + \omega) = \Gamma^*(\theta - \omega). \]

Consider the function \(B_{t,\tau}\) defined by \(B_{t,\tau}(x) = A_t(\frac{x}{\tau})\) and putting \(\phi = -\phi\), the last equality becomes

\[
S_1 \approx \tau^2 \left\{ \int \Gamma(\theta_\tau - \omega_\tau) \Gamma^*(\theta_\tau - \omega'_\tau) B_{t,\tau}(\tau \theta) B_{s,\tau}^*(\tau \theta) e^{i\theta(t-s)} d\theta \right\} + \left\{ \int \Gamma^*(\phi_\tau + \omega'_\tau - \Gamma(\phi_\tau + \omega_\tau) B_{s,\tau}^*(\tau \phi) B_{t,\tau}(\tau \phi) e^{-i\phi(s-t)} d\phi \right\}.
\]

Since \(|\Gamma(\omega)|^2\) is pseudo \(\delta\)-function with respect to \(f_{t,\tau}(\omega) = h_t(\frac{\omega}{\tau}) = |A_t(\frac{\omega}{\tau})|^2\), it is reasonable to suppose that \(\Gamma(\omega)\) is pseudo \(\delta\)-function with respect to \(A_t(\frac{\omega}{\tau})\) which is equal to \(B_{t,\tau}(\omega)\). From this approximation, we have

\[
S_1 \approx \frac{1}{4} \tau^2 \left\{ B_{t,\tau}(\omega_\tau) B_{s,\tau}^*(\omega_\tau) + B_{t,\tau}(\omega_\tau) B_{s,\tau}^*(\omega_\tau) \right\} \left\{ B_{t,\tau}^*(\tau \omega) B_{s,\tau}(\tau \omega) + B_{s,\tau}^*(\tau \omega) B_{t,\tau}(\tau \omega) \right\}
\]

\[
\int \int \Gamma(\theta_\tau - \omega_\tau) \Gamma^*(\theta_\tau - \omega'_\tau) \Gamma(\phi_\tau + \omega_\tau) \Gamma^*(\phi_\tau + \omega'_\tau) e^{-i(\theta + \phi)(s-t)} d\theta d\phi.
\] (9)

In the same way, we give a similar expression for \(S_2\). The covariance can be written as:

\[
\text{Cov} \left[ |U_t(\omega)|^2, |U_s(\omega')|^2 \right] = S_1 + S_2
\]

From (8), we deduce that

\[
\text{Cov} \left[ \hat{f}_t(\omega), \hat{f}_s(\omega') \right] = \sum_{u,v=-m}^{m} w_{T',u} w_{T',v} (S_1 + S_2) = R_1 + R_2
\]

\(R_1\) and \(R_2\) correspond to the respective contributions of \(S_1\) and \(S_2\). From (9), \(R_1\) becomes:

\[
R_1 \approx \frac{1}{4} \tau^2 \sum_{i=1}^{4} \int \int \sum_{u=-m}^{m} \sum_{v=-m}^{m} w_{T',u} w_{T',v} b_{t-u}^{(i)} b_{s-v}^{(i)} e^{-iu \theta + \phi} e^{iv \theta + \phi} \Gamma(\theta_\tau - \omega_\tau)
\]

\[
\Gamma^*(\theta_\tau - \omega'_\tau) \Gamma(\phi_\tau + \omega_\tau) \Gamma^*(\phi_\tau + \omega'_\tau) e^{-i(s-t)(\theta + \phi)} d\theta d\phi
\]

where \(b_{t-u}^{(1)} = |B_{t-u,\tau}(\omega_\tau)|^2\), \(b_{t-u}^{(2)} = B_{t-u,\tau}(\omega_\tau) B_{s-u,\tau}^*(\omega'_\tau)\), \(b_{t-u}^{(3)} = B_{s-u,\tau}(\omega_\tau) B_{t-u,\tau}(\omega'_\tau)\), and \(b_{t-u}^{(4)} = |B_{t-u,\tau}(\omega'_\tau)|^2\)

\[
R_1 \approx \frac{1}{4} \tau^2 \sum_{i=1}^{4} \int \int \Gamma(\theta_\tau - \omega_\tau) \Gamma^*(\theta_\tau - \omega'_\tau) \Gamma(\phi_\tau + \omega_\tau) \Gamma^*(\phi_\tau + \omega'_\tau) K_i^s(\theta + \phi)
\]

\[
K_i^s(\theta + \phi) e^{-i(s-t)(\theta + \phi)} d\theta d\phi.
\]

with \(K_i^s(\theta) = \sum_u w_{T',u} b_{t-u}^{(i)} e^{-iu \theta}\). Then \(w_{T',v} b_{s-v}^{(i)} = \sum_\theta K_i^s(\theta) e^{iv \theta}\). Let \(H_{t,\tau}(\theta) = K_i^s(\frac{\theta}{\tau})\), we have

\[
R_1 \approx \frac{1}{4} \tau^2 \sum_{i=1}^{4} \int \int \Gamma(\theta_\tau - \omega_\tau) \Gamma^*(\theta_\tau - \omega'_\tau) \Gamma(\phi_\tau + \omega_\tau) \Gamma^*(\phi_\tau + \omega'_\tau) H_{t,\tau}^s(\theta + \tau \phi) H_{s,\tau}^s(\theta + \tau \phi)
\]

\[e^{-i(s-t)(\theta + \phi)} d\theta d\phi.\]
where \( H_{t,\tau}^i(\theta) = \sum_u w_{T',u} b_{t-u}^{(i)} e^{-iu \frac{\theta}{T}} \). Therefore \( H_{t,\tau}^i(\theta) \) is the Fourier transform of the product of two functions: \( S_1 : v \rightarrow w_{T',v,\tau} \) and \( S_2 : v \rightarrow b_{t-v,\tau}^{(i)} \). As in Priestley [10], [13], suppose that Fourier transform of \( S_2 \) is pseudo \( \delta \)-function with respect to \( \Gamma(x) \). Choose \( T' \) such that Fourier transform of \( S_1 \) is pseudo \( \delta \)-function with respect to \( \Gamma(x) \). Consequently, \( H_{t,\tau}^i(x)H_{s,\tau}^s(x) \) is pseudo \( \delta \)-function with respect to \( \Gamma(x) \). We obtain
\[
\int K^i_{\tau}(\theta)K^s_{\tau}(\theta)d\theta = \int H_{t,\tau}^i(\tau \theta)H_{s,\tau}^s(\tau \theta)d\theta = 2\pi \sum_{u=-m}^m w_{T',u}^2 b_{t-u}^{(i)} b_{t-u}^{(i)\ast}
\]
Set \( \widetilde{H}^i(\tau \theta) = \frac{H_{t,\tau}^i(\tau \theta)H_{s,\tau}^s(\tau \theta)}{H_{t,\tau}^i(\tau \theta)H_{s,\tau}^s(\tau \theta)d\theta} \). We can write
\[
R_1 \approx \frac{\pi}{2} \tau^2 \left\{ \sum_{u=-m}^m (w_{T',u})^2 \right\} \left\{ \int |\Gamma(\phi \tau + \omega \tau)|^2 |\Gamma(\phi \tau + \omega' \tau)|^2 d\theta \right\} \times \left\{ \sum_{i=1}^4 b_{t}^{(i)} b_{s}^{(i)\ast} \int e^{-i(s-t)(\phi)} H^i(\tau \phi)d\phi \right\}
\]
For \( s = t \) and \( \omega = \omega' \) the expression \( R_1 \) becomes
\[
R_1 \approx \frac{\pi}{2} \tau^2 \left\{ \sum_{u=-m}^m (w_{T',u})^2 \right\} \left\{ \int |\Gamma(\phi \tau + \omega \tau)|^2 d\phi \right\} \left\{ \sum_{i=1}^4 |b_{t}^{(i)}|^2 \right\} 4h_t(\omega)
\]
Using Parseval equality \( \sum (w_{T',u})^2 = |W_{T',v}|^2 \), we obtain
\[
R_1 \approx \frac{2\pi \tau^2}{T'} h_t(\omega)^4 \sum T'(W_{T',v})^2 \int |\Gamma(\phi \tau + \omega \tau)|^4 d\phi.
\]
From the condition e) satisfied by the weight function \( w_{T',v} \), we conclude that \( R_1 = O \left( \frac{\tau^2}{T'} \right) \). In the same way we show that \( R_2 = O \left( \frac{\tau^2}{T'} \right) \). Thus,
\[
\text{Var}[\tilde{h}_{t,T'}(\omega)] = O \left( \frac{\tau^2}{T'} \right).
\]
It is easy to show that \( E \left[ \tilde{h}_{t,T'}(\omega_0) - \overline{h}_t(\omega_0) \right]^2 \) is equal to the sum of the square of the bias and the variance of the estimator.

5. Conclusion
We have proposed in this paper some results about the estimation of the evolutionary spectral density for non-stationary time-continuous processes, which were sampled at discrete instants. The approach is based on the technique used by Sabre ([?]) for stable processes combining estimates of evolutionary spectrum introduced by Priestley ([9]). This work could be applied to several cases when the process is non-stationary with continuous time as in examples of:

- the spatial and temporal movement of some planets in astronomy can be considered as a non-stationary continuous-time processes
• the study of chemicals on agricultural soil which are considered as a non-stationary processes with continuous-space-parameter.
• the study of the telephone networks to increase the flow through non-stationary signals with continuous time.

This work could be supplemented by the study of optimal smoothing parameters using cross validation methods that have been proven in the field.

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