Existence and measure of ergodic leaves in Novikov’s problem on the semiclassical motion of an electron

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Abstract

We show that “ergodic régime” appears for generic dispersion relations in the semiclassical motion of electrons in a metal and we prove that, in the fixed energy picture, the measure of the set of such directions is zero.

The problem of semiclassical motion of an electron in a lattice under a strong magnetic field leads to a very interesting problem of low dimensional “periodic” topology [Nov82], namely the structure of plane sections of 2-dimensional submanifolds of \( T^3 \). A.V. Zorich [Zor84] and I.A. Dynnikov [Dyn93a] proved Novikov’s conjecture, i.e. that open orbits are strongly asymptotics to some straight line, respectively in the almost rational and generic case; from their analysis Novikov [Nov95] extracted the following beautiful picture: once a dispersion relation is given, to every magnetic field direction giving rise to the generic situation (in which open leaves fill a genus-1 component of the Fermi surface) it corresponds one and only one indivisible integer 2-cycle in \( T^3 \) (Miller index). Such triple of integers are locally constant with respect to the direction of magnetic field.

From this construction we get two kinds of pictures on the 2-sphere. One is the so-called global picture, in which we label a point with the relative integer 2-cycle if it gives rise to the generic situation for some value of energy. In this case on the 2-sphere it is defined a countable set of disjoint open sets whose union is dense. It is still unclear whether this set has full measure or not.

The second kind of pictures is the fixed energy one, in which fix a value of the energy and we label a point with the corresponding integer 2-cycle only if it gives rise to generic open orbits at that energy, and we label a point with the null 2-cycle if all orbits are closed at the same energy.

Novikov and Maltsev [NM98] applied these results to the conductivity theory of normal metals describing new universal observable topological phenomena. It is hence very important, also from the physical point of view, to understand more about the appearance of these special directions of the magnetic field.

Below we present a series of statements that represent the stronger properties known at this time about their measure. These theorems are needed for applications in physics. They are based on the techniques developed by Dynnikov and were missing in the previous publications. As I was informed in process of publication of this article, Dynnikov also published some of these results in his new article [Dyn99].

**Lemma 1.** For a generic dispersion relation \( f \in C^\infty(T^3) \), boundaries of stabilities zones are piecewise smooth and stability zones meet in a countable set of points.

**Corollary 1.** For every generic dispersion relation \( f \in C^\infty(T^3) \) there are uncountably many ergodic directions.

**Theorem 1.** For any \( f \in C^\infty(T^3) \), the set of directions generating “ergodic régime” in the fixed energy picture has zero measure for almost all values of \( f \).

**Theorem 2.** For a generic \( f \in C^\infty(T^3) \), the set of directions generating “ergodic régime” in the fixed energy picture has zero measure for all values of \( f \).
The idea that leads to the corollary is that on every generic loop in $S^2$ the set of non generic directions forms a Cantor set, so that the set of all non generic directions forms some kind of fractal on the sphere.

An analytical study of the properties of this fractal appears to be very difficult to perform. It is know from theorems 1 and 2 that for a generic oriented surface the set of non generic direction has measure 0, but it is still an open question even whether this measure remains zero when we consider the set of ergodic direction relative to an interval of energies, that is relevant for [NM98].

It has been conjectured by S.P. Novikov that generically the set of “ergodic” directions at fixed energy has fractal dimension $\alpha \in (0,1)$ on the sphere, and similarly that the set of all of them should have fractal dimension $\beta \in (1,2)$.

Numerical computations have been performed by the author [DL99] for the function

$$f(x, y, z) = \cos(x) + \cos(y) + \cos(z).$$

Because of the symmetry $f(x^a + \pi) = -f(x^a)$, this function has the property to give rise to all stability zones at their biggest size at the same energy, namely at $E = 0$, so that the global picture can be obtained just studying a fixed energy picture.

The numerical simulation works in the following way: the surface $M^2 = f^{-1}(0)$ has genus 3 and the embedding $i : M^2 \to \mathbb{T}^3$ has rank 3, so that we can find a canonical base of cycles $\{e_j, f_j\}$ on $M^2$ s.t. $i_*(e_j) = 0$ and $i_*(f_j)$ are a base in $H_1(\mathbb{T}^3, \mathbb{Z})$. The program scans then all rational values of the magnetic field $H = (m/N, n/N, 1)$, $N \ge n \ge m$, for some fixed $N$, and for any of them finds the critical points and evaluates the homology class of the two critical loops in $\mathbb{T}^3$ and its intersection numbers with the cycles $i_*(f_j)$ in $M^2$. If just one of the critical loops is homologous to zero in $\mathbb{T}^3$, then the intersection number of the other gives automatically the homology class of the indivisible 2-cycle $l \in H_2(\mathbb{T}^3, \mathbb{Z})$ that corresponds to that magnetic field.

The following table shows the results obtained for $N = 400$:

| Hom Class | Area       | Hom Class | Area       |
|-----------|------------|-----------|------------|
| (0,0,1)   | (2.83 ± .02)10^{-1} | (1,5,5)   | (4.1 ± .2)10^{-3} |
| (1,1,1)   | (2.03 ± .01)10^{-1} | (2,5,8)   | (4.1 ± .4)10^{-3} |
| (1,2,2)   | (8.2 ± .2)10^{-2}  | (2,6,7)   | (3.4 ± .4)10^{-3} |
| (0,1,2)   | (5.1 ± .1)10^{-2}  | (4,7,8)   | (3.0 ± .3)10^{-4} |
| (1,3,3)   | (2.1 ± .1)10^{-2}  | (0,3,4)   | (2.9 ± .4)10^{-3} |
| (2,3,4)   | (1.7 ± .1)10^{-2}  | (3,5,7)   | (2.7 ± .3)10^{-3} |
| (1,3,5)   | (9.6 ± .5)10^{-3}  | (1,6,6)   | (2.0 ± .1)10^{-3} |
| (1,4,6)   | (9.6 ± .5)10^{-3}  | (4,5,8)   | (2.0 ± .4)10^{-3} |
| (0,2,3)   | (9.0 ± .6)10^{-3}  | (5,8,10)  | (1.9 ± .4)10^{-3} |
| (2,4,5)   | (8.6 ± .6)10^{-3}  | (4,6,9)   | (1.8 ± .3)10^{-3} |
| (1,4,4)   | (8.3 ± .3)10^{-3}  | (1,6,10)  | (1.7 ± .1)10^{-3} |
| (1,2,4)   | (6.2 ± .5)10^{-3}  | (5,9,11)  | (1.6 ± .2)10^{-3} |
| (3,4,6)   | (4.7 ± .5)10^{-3}  | (4,6,7)   | (1.5 ± .2)10^{-3} |

We used two different standard techniques to evaluate the Minkowski fractal dimension of the set of “ergodic” directions from the data we obtained puttin $N = 400$. The values obtained for the fractal dimension agree very well with each other and give an estimate of $d \simeq 1.8$, consistently with Novikov’s conjecture. New calculations for $N = 1000$ are on the way so that it will be possible to have a better estimate in a short time.
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