High-order inertial phase shifts for time-domain atom interferometers

Kai Bongs, Romain Launay, and Mark A. Kasevich
Physics Department, Yale University, New Haven CT, 06520

(Dated: March 31, 2022)

High-order inertial phase shifts are calculated for time-domain atom interferometers. We obtain closed-form analytic expressions for these shifts in accelerometer, gyroscope, optical clock and photon recoil measurement configurations. Our analysis includes Coriolis, centrifugal, gravitational, and gravity gradient-induced forces. We identify new shifts which arise at levels relevant to current and planned experiments.

PACS numbers: PACS numbers: 39.20.+Q, 32.80.Pj, 03.75.Dg, 04.80.-Y

Atom interferometric measurements have growing applications in basic and applied science. For example, atom interferometric techniques have been recently used to measure rotations [1], gravity gradients [2], $\hbar/m_{Cs}$ [3] and accelerations [3] with unprecedented precision. Future applications range from tests of General Relativity to the development of next generation inertial navigation systems. The accuracy of atom interferometric instruments, however, hinges on the accuracy of the theory used to connect the measured interferometric phase shift to the physically relevant quantities.

In this paper, we develop analytic expressions for the response of commonly used atom interferometer measurement configurations to experimentally relevant combinations of rotation, gravity and gravity gradient-induced forces. We identify new classes of high-order phase shifts which are observable in current experiments, and which seem to be of vital importance for proposed future experiments. In particular, we show below that current acceleration measurements need to be corrected at the part per billion (ppb) level for high-order rotation effects, that measurements of the photon recoil need a 10 ppb correction for gravity gradients, and that next generation optical time standards will need gravity gradient corrections to achieve fractional frequency accuracies of $\delta\nu/\nu < 10^{-17}$. Related theoretical work in this area has treated simple (one-dimensional) models [2] or provided general frameworks [4].

For simplicity we confine our discussion to the case of time-domain light-pulse atom interferometers [4]. However, with minor modification our results can be applied to other de Broglie wave interferometry approaches [5]. For light-pulse atom interferometers, coherent division, redirection and recombination of atomic wavepackets is accomplished via momentum exchange with an external driving laser field [6]. Complete descriptions of experimental realizations based on this principle are given in Ref. [5]. In brief, an optical pulse of area $\pi/2$, coupling two stable internal atomic states $|1\rangle$ and $|2\rangle$, serves to coherently divide an incident atomic wavepacket: an atom initially in state $|1\rangle$ is driven into a coherent superposition of internal states $|1\rangle$ and $|2\rangle$, with the momenta of the wavepackets associated with these states differing by the momentum of the photon used to drive the transition. Pulses of area $\pi$ induce stimulated transitions which exchange the internal states associated with the atomic wavepackets, while changing their momenta by the photon recoil momentum. Atom interferometers are realized from sequences of these pulses. For example, $\pi/2 - \pi - \pi/2$ pulse sequences have been used to build atomic interferometers analogous to optical Mach-Zehnder interferometers, as illustrated in Fig. 1.

The total phase shift between interfering paths, $\Delta \phi_{\text{total}}$, can be broken into three contributions:

$$\Delta \phi_{\text{total}} = \Delta \phi_{\text{prop}} + \Delta \phi_{\text{laser}} + \Delta \phi_{\text{sep}},$$

where $\Delta \phi_{\text{prop}}$ is the phase shift due to wavepacket propagation between the interrogating optical pulses, $\Delta \phi_{\text{laser}}$ is the shift acquired during the laser-atom interactions used to manipulate the atomic wavepackets, and $\Delta \phi_{\text{sep}}$ is the shift due to the (possible) final spatial separation of the interfering wavepackets at the interferometer output port [5]. Our approach neglects terms originating in the spatial extent of the wavepackets [5] and invokes the short-pulse (high Rabi frequency) limit for the optical interactions in order to clarify contributions arising from inertial forces. We briefly summarize expressions for $\Delta \phi_{\text{prop}}, \Delta \phi_{\text{laser}}$ and $\Delta \phi_{\text{sep}}$ below.
The $\Delta \phi_{\text{prop}}$ term is obtained using the Feynman path integral approach \cite{13}, which involves calculation of the difference between the classical actions associated with the interfering wavepacket trajectories. This is illustrated in Fig. 1 for the case of the accelerometer implementation. The actions $S$ are obtained by integrating the Lagrangian $L$ over the classical trajectories $\Gamma$ associated with the mean positions of each wavepacket, e.g. $S = \int \mathcal{L} \left( r(t), v(t) \right) dt$, with the path $\Gamma$ determined by the classical trajectory $\left( r(t), v(t) \right)$. This approach is formally correct when $L$ is at most second order in position $r(t)$ and velocity $v(t)$ (see Ref. \cite{12}). We find the classical trajectories $\Gamma$ through integration of the Euler-Lagrange equations for $L$.

The phase difference resulting from laser interactions $\Delta \phi_{\text{laser}}$ is discussed in detail in Ref. \cite{8}. This shift is obtained from solution of the Schrödinger equation for a resonantly driven two-level quantum system. In the limit of impulse excitations, this solution reduces to the following rules for the evolution of the probability amplitudes of states $|1 \rangle$ and $|2 \rangle$:

$$|1 \rangle \rightarrow i \exp \left[ i \mathbf{k} \cdot \mathbf{r} (t_p) \right] |2 \rangle; \quad |2 \rangle \rightarrow i \exp \left[ -i \mathbf{k} \cdot \mathbf{r} (t_p) \right] |1 \rangle. \quad (1)$$

Here $\mathbf{k}$ is the propagation vector associated with the coupling laser field(s) and $\mathbf{r} (t_p)$ is the mean position of the atomic wavepacket at the interaction time $t_p$. Time-dependent terms associated with the frequency of the driving fields have been suppressed (as they cancel for the time symmetric pulse sequences considered here), and we assume the initial laser phase to be constant during the interferometer sequence. As an example, for the pulse sequence of Fig. 1, the resulting laser-induced phase difference for an ensemble initially prepared in state $|1 \rangle$ and detected in state $|1 \rangle$ is:

$$\Delta \phi_{\text{laser}} = \mathbf{k} \cdot (\mathbf{r}_A - \mathbf{r}_C - \mathbf{r}_D + \mathbf{r}_B). \quad (2)$$

The shift $\Delta \phi_{\text{sep}}$ arises when the classical positions of the two interfering trajectories do not coincide at the exit beamsplitter (as illustrated in Fig. \cite{8}). This shift is adequately approximated by

$$\Delta \phi_{\text{sep}} = \mathbf{p} \cdot \Delta \mathbf{r} / \hbar, \quad (3)$$

where $\mathbf{p}$ is the mean momentum of the wavepackets in a given output port, and $\Delta \mathbf{r}$ is the spatial separation between the centers of each wavepacket at the time of the last optical pulse \cite{13}. In certain special cases – for example, that of uniform gravity – this contribution is zero.

The above phase shifts were evaluated in a geocentric reference frame fixed to the surface of the Earth, as shown in Fig. \cite{8}. In this frame, the Lagrangian can be written as:

$$L (\mathbf{r}, \mathbf{v}) = m \left( \frac{\mathbf{v}^2}{2} + g \cdot \mathbf{r} + \frac{1}{2} r_i T_{ij} r_j + \Omega \cdot (\mathbf{r} + \mathbf{R}) \times \mathbf{v} \right) \right. \nonumber $$

$$\left. + \frac{1}{2} (\Omega \times (\mathbf{r} + \mathbf{R}))^2 \right), \quad (4)$$

where $m$ is the atomic mass, $\mathbf{R}$ is the displacement from the center of the Earth, $\Omega$ the Earth rotation rate, $g$ the acceleration due to gravity and $T_{ij}$ is the gravity gradient tensor.

Analytic solutions were obtained in a perturbative approach \cite{12}, where the action was evaluated for the full Lagrangian $L$ over trajectories determined by the Lagrangian $L$ which neglected gravity gradients, it e.g.:

$$L (\mathbf{r}, \mathbf{v}) = m \left( \frac{\mathbf{v}^2}{2} + g \cdot \mathbf{r} + \Omega \cdot (\mathbf{r} + \mathbf{R}) \times \mathbf{v} \right) \right. \nonumber $$

$$\left. + \frac{1}{2} (\Omega \times (\mathbf{r} + \mathbf{R}))^2 \right). \quad (5)$$

This approximation was validated numerically as discussed below.

We evaluated accelerometer \cite{4, 7}, gyroscope \cite{1}, photon recoil \cite{3}, and optical clock interferometer \cite{9} configurations, as summarized in Tables \cite{8, 9}. Each configuration is discussed briefly below. We tabulate only the most significant terms in Taylor expansions of the total phase shifts. In order to give a quantitative measure for the different phase contributions, the tabulated expressions were evaluated for parameters corresponding to current experiments. In particular, $m = 2.21 \times 10^{-25}$ kg (Cs atomic mass), $\lambda_{\text{eff}} = 426$ nm (effective wavelength for two-photon Raman transitions), latitude 41°, Earth radius $R = 6.72 \times 10^6$ m, gravity $g_z = -9.8$ m/sec$^2$ and gravity gradient $2T_{xx} = 2T_{yy} = -2g_z = 2g_z / R$ with all other gravity gradient terms being zero (for a spherically symmetric Earth). The coordinate system origin is taken as the mean wavepacket position at the time of the first optical pulse, and the phase shift is determined just after the final pulse.

**Gravimeter.** Phase shifts for an atomic fountain gravime-
ter/accelerometer, based on a $\pi/2 - \pi - \pi/2$ sequence of vertically propagating optical pulses, are presented in Table I. For the numerical estimates we used a time $T = 0.4$ s between the pulses (corresponding to a fountain height of 78 cm) and a vertical launch velocity $v_z = -g_z T$. Note that in a satellite microgravity experiment the first two terms would cancel, such that the gravity gradient contribution would dominate. Recent terrestrial experiments have demonstrated the capability to resolve phase shifts at or below the $10^{-9}$ g level [7].

Optical clock. The optical Ramsey pulse sequence, $\pi/2 \uparrow -\pi/2 \uparrow -\pi/2 \downarrow -\pi/2 \downarrow$ (arrows indicate propagation directions of the interrogating pulses) is presented in Table I. Results are shown for pulses propagating along a vertical axis, with $T = 0.4$ s between the first and second and the third and fourth pulse, infinitesimal time between the second and third pulse and initial vertical velocity $v_z = -g_z T$. Atomic clocks based on this sequence offer the prospect of pushing the accuracy of the definition of the second below the $10^{-15}$ level and are a longstanding goal in frequency metrology. For an operational time standard, terms linear in $k_z$ can be suppressed by pulse reversal techniques [7] ($k_z \rightarrow -k_z$, for example), leaving terms quadratic in $k_z$ as possible systematic shifts. The largest of these tabulated is the well-known recoil-shift. The smaller term, which is linear in $T_{zz}$, depends on the location of the measurement, and represents a possible systematic offset at the $\delta\nu/\nu \sim 10^{-17}$ level. This term exists for both horizontal and vertical interrogation geometries.

Photon recoil measurement. Chu and coworkers have shown that a modified form of the optical Ramsey method can be used for precise determination of the quantity $\hbar/m_{Cs}$ [13]. The modification involves insertion of a series of $N - 1$ $\pi$ pulses, of alternating propagation direction, between the second and third $\pi/2$ pulses, and has the effect of enhancing the photon recoil phase shift terms. Following Refs. [13, 14], we calculate the phase difference between the two possible closed interferometer branches (see Table II). The phase terms are evaluated using the following parameters (chosen to correspond to the experiment in Ref. [14]): $T = 0.13$ s for the time between the first pair as well as the second pair of $\pi/2$ pulses, $T_{rec} = (1/3000)$ s between the second $\pi/2$ pulse and the first $\pi$ pulse, $T_{rec} = (1/3000)$ s between each of the subsequent $N - 1$ $\pi$ pulses, and $N - 1 = 30$ $\pi$ pulses. The second correction term due to gravity gradients is at the anticipated level of precision achieved in recent $\hbar/m_{Cs}$ measurements.

Gyroscope. A $\pi/2 - \pi - \pi/2$ sequence with optical propagation vectors nominally perpendicular to the mean atomic velocity can be viewed as a Sagnac-type rotation sensor [7]. In Table IV we estimate the phase shifts for a time-domain interferometer with parameters which correspond to the precision gyroscope of Ref. [1]. In particular, the light pulses are chosen to propagate horizontally in the west-east direction, with the atomic velocity $v_y = 290$ m/s in the north-south direction and $T = 1/290$ s time between pulses (corresponding to a 1 m spatial separation of the laser interaction regions). The largest correction to the well known Sagnac shift (the leading term) is near the resolution limit of current instruments, and is compensated by atomic beam reversal techniques.

It is interesting to compare the above results with those from a perturbative treatment of gravity gradients and

| TABLE I: Gravimeter phase shifts. |
|-----------------------------------|
| Term | Phase (rad) | Relative phase |
| $k_zT^2g_z$ | $-2.32 \cdot 10^{-7}$ | 1.0 |
| $k_zT^2\Omega_z^2R$ | $4.44 \cdot 10^{-4}$ | $1.9 \cdot 10^{-3}$ |
| $k_zT^3\Omega_z\Omega_y$ | $1.08 \cdot 10^{-3}$ | $4.7 \cdot 10^{-7}$ |
| $\frac{7k_zT^2g_zT_{zz}}{T}$ | $-6.32$ | $2.7 \cdot 10^{-7}$ |
| $-3k_zT^3\Omega_z^2$ | $-3.11 \cdot 10^{-2}$ | $1.3 \cdot 10^{-9}$ |
| $-\frac{k_zT^4\Omega_z\Omega_y}{T}$ | $1.81 \cdot 10^{-2}$ | $7.8 \cdot 10^{-10}$ |
| $\frac{7k_zT^2T_{zz}\Omega_z^2R}{T}$ | $1.21 \cdot 10^{-2}$ | $5.2 \cdot 10^{-10}$ |
| $\frac{k_zT^2}{T^2}\Omega_z^2$ | $9.71 \cdot 10^{-3}$ | $4.2 \cdot 10^{-10}$ |
| $\frac{-3k_zT^4\Omega_z^2}{T}$ | $-3.47 \cdot 10^{-5}$ | $1.5 \cdot 10^{-12}$ |
| $\frac{-3k_zT^4\Omega_z\Omega_y}{T}$ | $-2.79 \cdot 10^{-5}$ | $1.2 \cdot 10^{-12}$ |
| $\frac{-3k_zT^4\Omega_z\Omega_y}{T}$ | $-2.62 \cdot 10^{-5}$ | $1.1 \cdot 10^{-12}$ |

| TABLE II: Phase terms for an optical clock with light pulses parallel to gravity. |
|-----------------------------------|
| Term | Phase (rad) | Relative phase |
| $k_zT^2g_z$ | $-2.32 \cdot 10^{-7}$ | 1.0 |
| $k_zT^2\Omega_z^2R$ | $4.44 \cdot 10^{-4}$ | $1.9 \cdot 10^{-3}$ |
| $-\frac{k_z^2T^2}{T^3}\Omega_z$ | $-4.16 \cdot 10^{-4}$ | $1.8 \cdot 10^{-3}$ |
| $k_zT^3\Omega_z\Omega_y$ | $1.08 \cdot 10^{-1}$ | $4.7 \cdot 10^{-7}$ |
| $\frac{-k_zT^4\Omega_z\Omega_y}{T}$ | $-6.32$ | $2.7 \cdot 10^{-7}$ |
| $-3k_zT^3\Omega_z^2$ | $-3.11 \cdot 10^{-2}$ | $1.3 \cdot 10^{-9}$ |
| $\frac{-\Omega_z}{T}\frac{k_zT^4\Omega_z\Omega_y}{T}$ | $1.81 \cdot 10^{-2}$ | $7.8 \cdot 10^{-10}$ |
| $\frac{7k_zT^2T_{zz}\Omega_z^2R}{T}$ | $1.21 \cdot 10^{-2}$ | $5.2 \cdot 10^{-10}$ |
| $\frac{k_zT^2}{T^2}\Omega_z^2$ | $9.71 \cdot 10^{-3}$ | $4.2 \cdot 10^{-10}$ |
| $\frac{-3k_zT^4\Omega_z^2}{T}$ | $-3.47 \cdot 10^{-5}$ | $1.5 \cdot 10^{-12}$ |
| $\frac{-3k_zT^4\Omega_z\Omega_y}{T}$ | $-2.79 \cdot 10^{-5}$ | $1.2 \cdot 10^{-12}$ |
| $\frac{-3k_zT^4\Omega_z\Omega_y}{T}$ | $-2.62 \cdot 10^{-5}$ | $1.1 \cdot 10^{-12}$ |

| TABLE III: Phase terms for a photon recoil measurement. |
|-----------------------------------|
| Term | Phase (rad) | Relative phase |
| $\frac{2\Omega_z}{T}k_z^2T$ | $8.39 \cdot 10^{-8}$ | 1.0 |
| $\frac{2N_{10}k_z^2T^3T_{zz}}{T}$ | $6.89 \cdot 10^{-3}$ | $8.2 \cdot 10^{-9}$ |
| $\frac{N_{29}k_z^2T^3T_{rec}T_{zz}}{T}$ | $8.22 \cdot 10^{-4}$ | $9.8 \cdot 10^{-10}$ |
| $\frac{6k_zT^2T_{rec}T_{zz}}{4\Omega_z\Omega_y}$ | $4.36 \cdot 10^{-5}$ | $5.2 \cdot 10^{-11}$ |

| TABLE IV: Phase terms for a Sagnac rotation sensor. |
|-----------------------------------|
| Term | Phase (rad) | Relative phase |
| $2k_zT^2\Omega_zv_y$ | $4.69$ | 1.0 |
| $-2k_zT^3\Omega_yg_z$ | $6.28 \cdot 10^{-4}$ | $1.3 \cdot 10^{-4}$ |
| $-2k_zT^3\Omega_yR$ | $-1.20 \cdot 10^{-6}$ | $2.6 \cdot 10^{-7}$ |
| $-2k_zT^3\Omega_y\Omega_y^2$ | $-9.09 \cdot 10^{-7}$ | $1.9 \cdot 10^{-7}$ |
| $\frac{k_zT^2}{T^2}\Omega_z^2T_{zz}$ | $3.11 \cdot 10^{-9}$ | $6.6 \cdot 10^{-10}$ |
rotations, \emph{i.e.} using the Lagrangian $\tilde{L} = m \left( \frac{\mathbf{v}^2}{2} + \mathbf{g} \cdot \mathbf{r} \right)$ to determine the classical trajectories, but evaluating the action with respect to the full Lagrangian $L$ above. As an example, Table \ref{tab:v} shows these terms for the gravimeter configuration. Comparison with Table \ref{tab:v} indicates that the phase shift error associated with this commonly used approximation \ref{tab:v} is $\sim 5 \cdot 10^{-3}$ rad or $\sim 2 \cdot 10^{-10}$ g, which might be resolved by current experiments. In contrast, using $\tilde{L}$ to estimate the classical trajectories, we obtain agreement at the $\sim 1$ µrad level between our analytic calculations and numeric calculations which use the full Lagrangian $L$ to determine the classical trajectories. µrad level agreement is also obtained for the optical clock (Table \ref{tab:v}) and photon recoil (Table \ref{tab:v}) configurations, while 200 picorad agreement is obtained for the gyroscope configuration (Table \ref{tab:v}).

In conclusion, we have analyzed accelerometer, gyroscope, photon recoil and optical clock interferometer configurations. We have identified new phase shift terms which have their origin in cross-couplings between rotation, acceleration and gravity gradient perturbations on wavepacket motion.

This work was supported by the Office of Naval Research, Army Research Office, National Science Foundation and NASA. K. Bongs thanks the DFG (Deutsche Forschungsgemeinschaft) for financial support. We thank Neelima Sehgal for technical assistance and acknowledge valuable conversations with C. Bordé and S. Chu.

\begin{table}[h]
\centering
\caption{Phase terms for the accelerometer sequence derived treating rotations and gravity gradients as perturbations.}
\begin{tabular}{|c|c|c|}
\hline
Term & Phase (rad) & Relative phase \\
\hline
$k_z T^2 g_z$ & $-2.32 \cdot 10^3$ & 1.0 \\
$k_z T^2 \Omega^2 y R$ & $4.44 \cdot 10^4$ & $1.9 \cdot 10^{-3}$ \\
k_z T^2 v_z T_{zz} & $1.08 \cdot 10^1$ & $4.7 \cdot 10^{-7}$ \\
$\frac{k_z T^2 v_z \Omega^2 y}{2}$ & $-6.32$ & $2.7 \cdot 10^{-7}$ \\
k_z T^2 v_z \Omega^2 y & $1.04 \cdot 10^{-2}$ & $4.5 \cdot 10^{-10}$ \\
$\frac{1}{2} k_z k_z T^2 \Omega^2 y$ & $9.71 \cdot 10^{-3}$ & $4.2 \cdot 10^{-10}$ \\
$\frac{1}{2} \Omega^2 y$ & $-6.05 \cdot 10^{-3}$ & $2.6 \cdot 10^{-10}$ \\
\hline
\end{tabular}
\end{table}

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