AN APPROXIMATE INVERSE RIESZ-SOBOLEV INEQUALITY

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Abstract. The Riesz-Sobolev inequality relates the convolution of nonnegative functions with domains $\mathbb{R}^d$ to the convolution of their symmetric nonincreasing rearrangements. We show that for dimension $d = 1$, for indicator functions of sets, if the inequality is sufficiently close to an equality then the sets in question must nearly coincide with intervals.

1. Introduction

Let $|S|$ denote the Lebesgue measure of $S \subset \mathbb{R}^d$. Denote by $f^*$ the equimeasurable symmetric nonincreasing rearrangement of a nonnegative measurable function $f$, and if $|S| < \infty$, denote by $S^*$ the ball $B$ centered at $0 \in \mathbb{R}^d$ which satisfies $|B| = |S|$.

Consider any nonnegative measurable functions $f, g, h$ defined on $\mathbb{R}^d$ which tend to zero in the sense that for any $t > 0$, $|\{x : f(x) > t\}|$ is finite, and the same holds for $g, h$. The inequality of Sobolev and Riesz \cite{9,10} states that

$$\langle f^* \ast g, h^* \rangle \leq \langle f^* \ast g^* \ast, h^* \rangle.$$  

In particular, for indicator functions of measurable sets $A, B, C$ with finite Lebesgue measures,

$$\langle 1_A \ast 1_B, 1_C \rangle \leq \langle 1_A^* \ast 1_B^*, 1_C^* \rangle.$$  

This foundational inequality directly implies the formally more general (1.1).

Inverse theorems have been used to characterize those functions which extremize certain specific inequalities. One element of Lieb’s \cite{7} characterization of extremizers of the Hardy-Littlewood-Sobolev inequality was the fact that if $h = h^*$, and if $h^*$ is positive and strictly decreasing, then equality holds in (1.1) only if $f = f^*$ and $g = g^*$ up to translations. See for instance Theorem 3.9 in \cite{8}. Christ \cite{3} relied on a sharper inverse theorem of Burchard \cite{1} to characterize extremizers for an inequality for the Radon transform. The simple one-dimensional case of Burchard’s theorem states that if

$$\max(|A|, |B|, |C|) \leq \min(|A| + |B|, |B| + |C|, |A| + |C|)$$  

then if equality holds in (1.2), then $A, B, C$ must be intervals, up to null sets. Equality also implies that the centers $a, b, c$ of the intervals $A, B, C$ satisfy $a + b = c$.

In this paper we establish an inverse result which describes cases of near equality in (1.2) for $\mathbb{R}^1$. This will be applied in a companion paper \cite{4} to characterize those functions which nearly extremize Young’s convolution inequality for $\mathbb{R}^d$.

Let $S \triangle T$ denote the symmetric difference between sets $S, T$.

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Theorem 1.1. For any $\varepsilon, \varepsilon' > 0$ there exists $\delta > 0$ with the following property. Let $A, B, E, F \subset \mathbb{R}$ be Lebesgue measurable sets with positive, finite measures. Suppose that
$$
(1.4) \quad (1 + \varepsilon')(\max(|A|, |B|) - \min(|A|, |B|)) \leq |E| \leq \frac{1}{2}(1 - \varepsilon')(|A| + |B|)
$$
and that $|F| = 3|E|$. If
$$
(1.5) \quad \langle 1_A \ast 1_B, 1_E \rangle \geq \langle 1_A \ast 1_B, 1_{E'} \rangle - \delta \max(|A|, |B|)^2
$$
and
$$
(1.6) \quad \langle 1_A \ast 1_B, 1_F \rangle \geq \langle 1_A \ast 1_B, 1_{F'} \rangle - \delta \max(|A|, |B|)^2
$$
then there exists an interval $I \subset \mathbb{R}$ such that
$$
(1.7) \quad |A \triangle I| < \varepsilon|A|.
$$

The hypotheses of Theorem 1.1 may benefit from clarifications. Let $S_{t,A,B}$ denote the superlevel set
$$
(1.8) \quad S_{t,A,B} = \{ x : (1_A \ast 1_B)(x) > t \}.
$$
We often write $S_t$ as shorthand for $S_{t,A,B}$. When $A, B$ are intervals, $|S_{t,A,B}| = |A| + |B| - 2t$ for all $t \in [0, \|1_A \ast 1_B\|_\infty)$.

1. The unexpected, and perhaps unsatisfactory, feature of this formulation is that a lower bound for $\langle 1_A \ast 1_B, 1_S \rangle$ is hypothesized for two sets $S$, rather than merely for a single set. Worse yet, the measures of these two sets are required to be coupled.
2. The condition that $|F| = 3|E|$ can be relaxed, for trivial reasons, to $|F| = 3|E| + O(\delta \max(|A|, |B|))$.
3. The hypotheses are vacuous unless $\min(|A|, |B|) > \frac{1}{2} \max(|A|, |B|)$.
4. In a companion paper [4] in which Theorem 1.1 is applied, its hypotheses are satisfied in a much more robust form. Indeed, (1.5) is known in that application to hold for a family of sets $E$ whose measures take on essentially all values in the range $\max(|A|, |B|) - \min(|A|, |B|) < |E| < |A| + |B|$. Thus the requirement that $|F| = 3|E|$ is no encumbrance there. The general form of the analysis in [4] suggests that this robust form of the hypotheses might arise naturally in other applications, as well.
5. Define $\alpha, \beta$ by $|E| = |A| + |B| - 2\alpha$ and $|F| = |A| + |B| - 2\beta$. As will be proved below in Lemmas 2.1 and 2.2 it follows from the hypotheses (1.5), (1.6) that
$$
| |S_\alpha| - |E| | \leq C\delta^{1/2} \max(|A|, |B|),
$$
$$
\langle 1_A \ast 1_B, 1_{S_\alpha} \rangle \geq \langle 1_A \ast 1_B, 1_{S_\beta} \rangle - C\delta^{1/2} \max(|A|, |B|)^2
$$
with corresponding statements for $S_\beta, F$.
6. The hypothesis (1.6) involving $F$ can be replaced by its weaker consequence
$$
(1.9) \quad |S_\beta| \leq |A| + |B| - 2\beta - \delta \max(|A|, |B|)
$$
established in Lemma 2.2 where $\beta = \frac{1}{2}(|A| + |B| - 3|E|)$. Taken at face value, (1.9) is an upper bound on $1_A \ast 1_B$, rather than a lower bound. This seeming paradox hints at the structure of our analysis, which is related to the Brunn-Minkowski inequality $|U + V| \geq |U| + |V|$. A well-known inverse principle is that if equality holds in the Brunn-Minkowski inequality, then $U, V$ are equal to intervals, up to null sets. Here an approximate inverse principle, governing the case in which $|U + V|$ is relatively small, is exploited.
Theorem 1.2. Let \( h \) be a nonnegative function such that \( |\{x : h(x) > t\}| < \infty \) for all \( t > 0 \), and \( |\{x : h(x) > 0\}| > 0 \). Suppose that its symmetric nonincreasing rearrangement \( h^* \) is continuous and strictly decreasing on its support. Let \( K \) be a compact subset of \((0, \|h\|_\infty)\).

For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the following property. Let \( A, B \subseteq \mathbb{R} \) be Lebesgue measurable sets with \(|A|, |B| \in K\). Suppose that \( \max(|A|, |B|) \leq (2 - \rho) \min(|A|, |B|) \). If
\[
(1_A \ast 1_B, h) \geq (1_{A^*} \ast 1_{B^*}, h^*) - \delta \max(|A|, |B|)^2
\]
then there exists an interval \( I \subseteq \mathbb{R} \) such that
\[
|A \triangle I| < \varepsilon |A|.
\]
The constants in this result do depend on \( h, K \).

The structure of the analysis is as follows: (i) The hypotheses of Theorem 1.1 imply a lower bound for \( |S_\alpha| \) and an upper bound for \( |S_\beta| \), with \( \alpha, \beta \) as in the statement. (ii) \( S_\beta \supset S_\alpha - S_\alpha + S_\alpha \). Therefore (i) becomes an upper bound for the measure of a sumset associated to \( S_\alpha \). (iii) An inverse theorem of additive combinatorics, concerning sets whose sumsets are small, adapted to the continuum setting, implies that \( S_\alpha \) nearly coincides with an interval. (iv) A compactness argument establishes the special case of Theorem 1.1 in which the set \( E \) is nearly an interval.

2. On Measures of Superlevel Sets of Convolutions

For \( a \in (0, \infty) \) let \( I_a = \left[-\frac{1}{2}a, \frac{1}{2}a\right] \). Define
\[
(2.1) \quad \Theta(a, b, c) = (1_{I_a} \ast 1_{I_b}, 1_{I_c}).
\]
This function \( \Theta : (0, \infty)^3 \rightarrow (0, \infty) \) is continuous, is strictly positive on \((0, \infty)^3\), and is a symmetric function of its three arguments \( a, b, c \). If \( 0 < b < a \) and \( a - b < c < a + b \), then
\[
(2.2) \quad \Theta(a, b, c) = b(a - b) + \frac{1}{2} \int_{a-b}^{c} (a + b - t) dt.
\]
In this section, we deduce certain bounds on the measures of superlevel sets from the near equality \( (1_A \ast 1_B, 1_E) \geq \Theta(|A|, |B|, |E|) - \delta \max(|A|, |B|)^2 \).

Recall the notation \( S_t = S_{t, A, B} = \{x : (1_A \ast 1_B)(x) > t\} \).
Lemma 2.1. Let $A, B, E$ be Lebesgue measurable sets of finite, positive measures satisfying
\begin{equation}
\max(|A|, |B|) - \min(|A|, |B|) < |E| < |A| + |B| \tag{2.3}
\end{equation}
\begin{equation}
(1_A * 1_B, 1_E) \geq \Theta(|A|, |B|) - |E| \tag{2.4}
\end{equation}
for some $\delta \in (0, 1]$. Define $\alpha$ by $|E| = |A| + |B| - 2\alpha$. Then
\begin{equation}
|S_\alpha \cap E| \geq |A| + |B| - 2\alpha - C\delta^{1/2} \max(|A|, |B|). \tag{2.5}
\end{equation}
In particular, $|S_\alpha| \geq |A| + |B| - 2\alpha - C\delta^{1/2} \max(|A|, |B|)$.

Proof. Set $f = 1_A * 1_B$. Define $E' = E \cap S_\alpha$. A simple calculation using (2.2) demonstrates that
\[ \Theta(|A|, |B|, |E'|) + \alpha(|E| - |E'|) + c(|E| - |E'|)^2 \leq \Theta(|A|, |B|, |E|). \]
Consequently
\[ \langle 1_A * 1_B, 1_E \rangle = \int_E f = \int_{E'} f + \int_{E \setminus E'} f \leq \Theta(|A|, |B|, |E'|) + \alpha|E \setminus E'| \leq \Theta(|A|, |B|, |E|) - c(|E| - |E'|)^2. \]
Since by hypothesis $\int_E f \geq \Theta(|A|, |B|, |E|) - \delta \max(|A|, |B|)^2$, it follows that
\[ |E \setminus E'|^2 = (|E| - |E'|)^2 \leq C\delta \max(|A|, |B|)^2, \]
so
\[ |E \setminus E'| \leq C\delta^{1/2} \max(|A|, |B|). \]
Therefore
\begin{equation}
|S_\alpha \cap E| = |E'| = |E| - |E \setminus E'| = |A| + |B| - 2\alpha - |E \setminus E'| \geq |A| + |B| - 2\alpha - C\delta^{1/2} \max(|A|, |B|). \tag{2.6}
\end{equation}

Lemma 2.2. Let $A, B, E$ be Lebesgue measurable sets of finite, positive measures satisfying
\begin{equation}
\max(|A|, |B|) - \min(|A|, |B|) < |E| < |A| + |B| \tag{2.3}
\end{equation}
\begin{equation}
(1_A * 1_B, 1_E) \geq \Theta(|A|, |B|, |E|) - \delta \max(|A|, |B|)^2 \tag{2.4}
\end{equation}
for some $\delta \in (0, 1]$. Define $\alpha$ by $|E| = |A| + |B| - 2\alpha$. Then
\begin{equation}
|E \triangle S_\alpha| \leq C\delta^{1/2} \max(|A|, |B|) \tag{2.7}
\end{equation}
and consequently
\begin{equation}
|S_\alpha| \leq |A| + |B| - 2\alpha + 2\delta^{1/2} \max(|A|, |B|). \tag{2.8}
\end{equation}

Proof. Consider any measurable set $S$ such that $E \subset S \subset S_\alpha \cup E$ and $|S| \leq |A| + |B|$. Then
\[ \langle 1_A * 1_B, 1_S \rangle \geq \alpha|S \setminus E| + \langle 1_A * 1_B, 1_E \rangle \geq \Theta(|A|, |B|, |E|) - \delta \max(|A|, |B|)^2 + \alpha(|S| - |E|). \]
On the other hand, by the Riesz-Sobolev inequality and the integral formula (2.2) for $\Theta$,
\[ \langle 1_A * 1_B, 1_S \rangle \leq \Theta(|A|, |B|, |S|) = \Theta(|A|, |B|, |E|) + \frac{1}{2} \int_{|E|}^{\infty} (|A| + |B| - t) \, dt. \]
Therefore
\[ \alpha(|S| - |E|) \leq \delta \max(|A|, |B|)^2 + \frac{1}{2} \int_{|E|}^{\infty} (|A| + |B| - t) \, dt, \]
which can be rewritten as
\[ \frac{1}{2} \int_{|E|} |t - (|A| + |B| - 2\alpha)| dt \leq \delta \max(|A|, |B|)^2. \]

Since \(|A| + |B| - 2\alpha = |E|\), the left-hand side is simply \(\frac{1}{4}(|S| - |E|)^2\). Thus
\[ |S| \leq |E| + 2\delta^{1/2} \max(|A|, |B|) = |A| + |B| - 2\alpha + 2\delta^{1/2} \max(|A|, |B|). \]

We conclude that \(|S_\alpha \cup E| \leq |A| + |B| - 2\alpha + 2\delta^{1/2} \max(|A|, |B|)\). Since it was shown in the preceding lemma that \(|S_\alpha \cap E| \geq |A| + |B| - 2\alpha - C\delta^{1/2} \max(|A|, |B|)\), the required bound for \(|S_\alpha \triangle E|\) follows.

**Corollary 2.3.** Under the hypotheses of Lemmas 2.1 and 2.2,
\[ (2.9) \quad | |S_\alpha| - |E| | \leq \Theta(1) \delta^{1/2} \max(|A|, |B|) \]
and
\[ (2.10) \quad \langle 1_A * 1_B, 1_{S_\alpha} \rangle \geq \Theta(|A|, |B|, |S_\alpha|) - \Theta(1) \delta^{1/2} \max(|A|, |B|)^2. \]

**Proof.** The first conclusion follows from our upper bound for \(|S_\alpha \triangle E|\). The final conclusion follows from the inequality
\[ |\langle 1_A * 1_B, 1_{S_\alpha} - 1_E \rangle| \leq \|1_A * 1_B\|_{\infty} |S_\alpha \triangle E| \leq \min(|A|, |B|)C\delta^{1/2} \max(|A|, |B|) \]
and the fact that the function \(r \mapsto \Theta(|A|, |B|, r)\) is Lipschitz continuous with norm equal to \(\max(|A|, |B|)\).

\[ \square \]

**3. Additive structure of superlevel sets of convolutions**

For any sets \(A, B\) define \(A + B = \{a + b : a \in A \text{ and } b \in B\}\). For any positive integers \(\lambda, \mu\) and any set \(S\) define
\[ (3.1) \quad \lambda S - \mu S = \left\{ \sum_{i=1}^{\lambda} x_i - \sum_{j=1}^{\mu} y_j : x_i, y_j \in S \right\}; \]

define \(\lambda S\) and \(-\mu S\) by replacing the appropriate sums by zero.

The following result provides a criterion for a set to be contained in an interval of only slightly larger measure.

**Proposition 3.1.** Let \(A \subseteq \mathbb{R}^1\) be a Lebesgue measurable set with finite, positive measure. If \(|A + A| < 3|A|\) then \(A\) is contained in an interval of length \(\leq |A + A| - |A|\).

The proof is a straightforward reduction to a corresponding result for sums of finite sets due to Freiman [5]. It is deferred to [8].

Proposition 3.1 is the only element of our analysis which does not extend in a straightforward way to higher dimensions. Thus in order to establish the analogue of Theorem 1.1 in all dimensions, it would suffice to establish the analogue of this Proposition.

Let \(U, V \subseteq \mathbb{R}^1\) be Lebesgue measurable sets with finite measures. Then \(|U \triangle V| + 2|U \cap V| = |U| + |V|\), and \(\|1_U - 1_V\|_1 = |U \triangle V|\). Therefore
\[ (3.2) \quad \|1_U - 1_V\|_1 + 2|U \cap V| = |U| + |V|. \]

The triangle inequality for the \(L^1\) norm has the following consequence.
Lemma 3.2. Let $A, B \subset \mathbb{R}$ be measurable sets with finite, positive measures. For $0 < t < \min(|A|, |B|)$, consider the superlevel sets $S_t = \{x \in \mathbb{R} : 1_A * 1_B(x) > t\}$ of the convolution product $1_A * 1_B$. Let $k$ be any positive integer, and let $\alpha_i > 0$ for $1 \leq i \leq 2k + 1$. Define $\beta$ by

(3.3) \[
\left(\beta - \frac{|A| + |B|}{2}\right) = \sum_{i=1}^{2k+1} \left(\alpha_i - \frac{|A| + |B|}{2}\right).
\]

Then

(3.4) \[S_{\alpha_1} - S_{\alpha_2} + S_{\alpha_3} - S_{\alpha_4} + \cdots + S_{\alpha_{2k+1}} \subset S_\beta.
\]

A corollary, by the one-dimensional Brunn-Minkowski inequality $|U + V| \geq |U| + |V|$, is that

(3.5) \[|S_\beta| \geq \sum_{i=1}^{2k+1} |S_{\alpha_i}|.
\]

Proof. To prove the inclusion, set $\tilde{B} = \{z : -z \in B\}$ and $A_x = \{x + y : y \in A\}$. For any $t > 0$,

\[\{x : (1_A * 1_B)(x) > t\} = \{x : |A_x \cap \tilde{B}| > t\}.
\]

Indeed,

\[1_A * 1_B(x) = \int 1_A(x-y)1_B(y) \, dy = \int 1_A(x+y)1_{\tilde{B}}(y) \, dy = |A_x \cap \tilde{B}|.
\]

For $x \in S_t$,

(3.6) \[\|1_{A_x} - 1_{\tilde{B}}\|_1 = |A_x| + |\tilde{B}| - 2|A_x \cap \tilde{B}| = |A| + |B| - 2|A_x \cap \tilde{B}| < |A| + |B| - 2t.
\]

Therefore by the triangle inequality, if $x \in S_{\alpha_1}$ and $x' \in S_{\alpha_2}$ then

(3.7) \[\|1_{A_{x'}} - 1_{A_{x'}}\|_1 < 2|A| + 2|B| - 2\alpha_1 - 2\alpha_2.
\]

Since $\|1_{A_{x'}} - 1_{A_{x'}}\|_1 = \|1_{A_{x'}} - 1_A\|_1$,

(3.8) \[\|1_{A_{x'}} - 1_A\|_1 < 2|A| + 2|B| - 2\alpha_1 - 2\alpha_2 \text{ for any } z \in S_{\alpha_1} - S_{\alpha_2},
\]

In the same way, for any $z \in S_{\alpha_1} - S_{\alpha_2} + S_{\alpha_3} - \cdots + S_{\alpha_{2k+1}}$,

\[\|1_{A_z} - 1_{\tilde{B}}\|_1 < (2k + 1)(|A| + |B|) - 2 \sum_i \alpha_i
\]

and consequently

(3.9) \[|A_z \cap \tilde{B}| = \frac{1}{2}|A| + \frac{1}{2}|B| - \frac{1}{2}\|1_{A_z} - 1_{\tilde{B}}\|_1
\]

\[> \frac{1}{2}|A| + \frac{1}{2}|B| - \frac{2k+1}{2}|A| - \frac{2k+1}{2}|B| + \sum_i \alpha_i = \beta.
\]

A variant of Lemma 3.2 follows from the same reasoning. If $z \in S_{\alpha} - S_{\alpha}$ then $|A_z - A_{\alpha}| < 2(|A| + |B| - 2\alpha)$. For any $z \in kS_{\alpha} - kS_{\alpha}$, $|A_z - A_{\alpha}| < 2k(|A| + |B| - 2\alpha)$. Therefore $|A_z \cap A| > \frac{1}{2}|A| + \frac{1}{2}|A| - \frac{1}{2}(2k|A| + 2k|B| - 4k\alpha) = 2k\alpha - (k - 1)|A| - k|B|$. Thus $kS_{\alpha} - kS_{\alpha} \subset \{x : (1_A * 1_A)(x) > \gamma\}$ where $\gamma = 2k\alpha - (k - 1)|A| - k|B|$.\]
Corollary 3.3. Let $A, B \subset \mathbb{R}$ be Lebesgue measurable sets with finite, positive measures. For $t \geq 0$ define $S_t = \{x : 1_A * 1_B(x) > t\}$. Let $k$ be a positive integer, and suppose that $\varepsilon > 0$ satisfies

$$
(4k + 1)\varepsilon \max(|A|, |B|) \leq |S_\alpha|.
$$

Let $\alpha \geq 0$. Set $\beta = (2k + 1)\alpha - k|A| - k|B|$, and assume that $\beta \geq 0$. If both

$$
|S_\beta| < |A| + |B| - 2\beta + (2k + 1)\varepsilon \max(|A|, |B|)
$$

and

$$
|S_\alpha| > |A| + |B| - 2\alpha - \varepsilon \max(|A|, |B|)
$$

then $S_\alpha$ is contained in some interval $I$ satisfying

$$
|I| < |S_\alpha| + (4k + 2)\varepsilon \max(|A|, |B|).
$$

Moreover,

$$
|S_\alpha| < |A| + |B| - 2\alpha + (4k + 1)\varepsilon \max(|A|, |B|)
$$

and

$$
|S_\beta| > |A| + |B| - 2\beta - (2k + 1)\varepsilon \max(|A|, |B|).
$$

This conclusion is of interest primarily when $\varepsilon \max(|A|, |B|) \ll |S_\alpha|$. It is trivial unless both $\alpha, \beta$ lie in the range $[0, \min(|A|, |B|))$. For $k = 1$, the only case which will be needed below, this range is nonvacuous if and only if

$$
\max(|A|, |B|) < 2 \min(|A|, |B|),
$$

Proof. By the Brunn-Minkowski inequality,

$$
|S_\beta| \geq |S_\alpha + S_\alpha| + (k - 1)S_\alpha - kS_\alpha| \text{ and } |(k - 1)S_\alpha - kS_\alpha| \geq (2k - 1)|S_\alpha|
$$

so

$$
|S_\alpha + S_\alpha| \leq |S_\beta| - (2k - 1)|S_\alpha|
$$

$$
\leq 2|A| + 2|B| - 4\alpha + 4k\varepsilon \max(|A|, |B|)
$$

$$
\leq 2|S_\alpha| + (4k + 1)\varepsilon \max(|A|, |B|).
$$

Since $(4k + 1)\varepsilon \max(|A|, |B|) \leq |S_\alpha|$, it follows from Proposition 3.1 that $S_\alpha$ is contained in some interval $I$ whose length satisfies

$$
|I| < |S_\alpha| + (4k + 1)\varepsilon \max(|A|, |B|).
$$

The inclusion $(k + 1)S_\alpha - kS_\alpha \subset S_\beta$, together with the Brunn-Minkowski inequality, imply that $|S_\beta| \geq |(k + 1)S_\alpha - kS_\alpha| \geq (2k + 1)|S_\alpha|$. The indicated lower bound for $|S_\beta|$ and upper bound for $|S_\alpha|$ follow from this relation together with the hypothesized upper and lower bounds for these same quantities.

The Riesz-Sobolev inequality gives integral bounds for the superlevel set measures $|S_t|$, since the inequality can be reformulated as

$$
\int_0^x (1_A * 1_B)(y) \, dy \leq \int_0^x (1_{A^*} * 1_{B^*})(y) \, dy \text{ for all } x > 0,
$$

and the left-hand side equals $s|S_x| + \int_{t \geq x} |S_t| \, dt$ where $s = (1_{A^*} * 1_{B^*})(x)$. The following example illustrates the nonexistence of useful upper bounds, in general, for the unintegrated quantities $|S_t|$. Let $\lambda$ be a large positive integer. Choose sets $A, B \subset \mathbb{Z}$ of cardinality $\lambda$, which satisfy $|A + B| = |A| + |B| = \lambda^2$. Define $A = A + [-\frac{1}{2}\lambda^{-1}, \frac{1}{2}\lambda^{-1}] \subset \mathbb{R}$ and $B = B + [-\frac{1}{2}\lambda^{-1}, \frac{1}{2}\lambda^{-1}] \subset \mathbb{R}$. Then $|A| = |B| = 1$. Then $|S_t| = |\{x : (1_A * 1_B)(x) > t\}|$
is equal to 0 for $t \geq \lambda^{-1}$, and equals $2\lambda(1 - \lambda t)$ for $0 < t < \lambda^{-1}$. For two intervals $\tilde{A}, \tilde{B}$ of lengths equal to one, the corresponding distribution function satisfies $|\tilde{S}_t| = 2(1 - t)$ for $t \in (0, 1)$. For all $t < (1 + \lambda)^{-1}$, $|S_t| > |\tilde{S}_t|$; moreover, $|S_t|/|\tilde{S}_t| \approx \lambda$ as $t \to 0$.

4. A preliminary inverse Riesz-Sobolev inequality

The special case in which one of the three sets appearing in the expression $\langle 1_A * 1_B, 1_C \rangle$ is an interval is simpler than the general case, but will be an essential step in our analysis. We treat it here.

**Proposition 4.1.** Let $K$ be a compact subset of $(0, \infty)$, and let $\eta > 0$. For each $\varepsilon > 0$ there exists $\delta > 0$, depending also on $\eta, K$, with the following property. Let $A, B \subset \mathbb{R}$ be measurable subsets with finite, positive measures, and let $I \subset \mathbb{R}$ be a bounded interval, such that $|A|, |B|, |I|$ all belong to $K$. Assume further that for any permutation $(a, b, c)$ of $(|A|, |B|, |I|)$, $c \leq (1 - \eta)(a + b)$. Suppose finally that

$$
(4.1) \quad \langle 1_A * 1_B, 1_I \rangle \geq (1 - \delta)\langle 1_a * 1_{b^*}, 1_{b^*} \rangle.
$$

Then there exists an interval $J \subset \mathbb{R}$ such that

$$
(4.2) \quad |A \triangle J| < \varepsilon.
$$

This result is not formulated in a scale-invariant way, but via the action of the affine group it directly implies a scale-invariant generalization. Theorem 4.1 will later be deduced directly from Proposition 4.1 and Corollary 3.3. Observe that in contrast to the setup of Theorem 4.1, $1_A * 1_B, 1_I$ is assumed to be large for only one interval $I$, not for two.

Fix $\eta, K$. Proposition 4.1 is equivalent to the assertion that if $(A_j, B_j, I_j)$ is a sequence of ordered triples satisfying all of these hypotheses, with a sequence of parameters $\delta_j \to 0$, then there exist intervals $J_j$ such that $|A_j \triangle J_j| < \varepsilon_j$ where $\varepsilon_j \to 0$ as $j \to \infty$. We prove this by contradiction. If there were to exist a sequence for which the conclusion failed, then there would necessarily exist a subsequence for which

$$
(4.3) \quad (|A_j|, |B_j|, |I_j|) \to (\alpha, \beta, \gamma)
$$

for some $(\alpha, \beta, \gamma) \in K^3$, so we may restrict attention to such a subsequence.

Without loss of generality, we may assume that each interval $I_j$ is centered at 0, by translating $A_j, B_j, I_j$ by appropriate quantities. Set $I = [-\frac{1}{2}\gamma, \frac{1}{2}\gamma]$. Now

$$
|\langle 1_{A_j} * 1_{B_j}, 1_I - 1_{I'} \rangle| \leq C_K |I \triangle I'|
$$

for any intervals $I, I'$. If $I, I'$ are centered at 0, then $|I \triangle I'| = |I| - |I'|$. Therefore $|I_j \cap I| \to 0$, and consequently

$$
(4.4) \quad \langle 1_{A_j} * 1_{B_j}, 1_I \rangle - \langle 1_{A_j} * 1_{B_j}, 1_{I_j} \rangle \to 0
$$
as $j \to \infty$. Therefore we may replace $I_j$ by $I$ throughout the remainder of the discussion.

**Lemma 4.2.** There exist a function $\Lambda$ such that $\Lambda(r) \to 0$ as $r \to \infty$, a sequence $R_j \to \infty$, and a sequence of real numbers $\tau_j$ such that

$$
(4.5) \quad |A_j \setminus [\tau_j - \rho, \tau_j + \rho]| \leq \Lambda(\rho) \text{ for all } \rho \in [0, R_j].
$$

---

1 The meaning of the symbols $\alpha, \beta$ here is unrelated to their role than in the statement of Theorem 4.1.
Proof. If not, then after replacing the sequence of pairs \((A_j, B_j)\) by an appropriate subsequence, there exists a sequence of bounded intervals \(L_j = [\lambda_j^-, \lambda_j^+]\) such that
\[
|A_j \cap L_j| \to 0,
|L_j| \to \infty,
\lim_{j \to \infty} |A_j^-| = \alpha^- > 0,
\lim_{j \to \infty} |A_j^+| = \alpha^+ > 0,
\]
where \(A_j^- = A_j \cap (-\infty, \lambda_j^-]\) and \(A_j^+ = A_j \cap [\lambda_j^+, \infty)\). Since \(\alpha^- + \alpha^+ = \alpha\), both \(\alpha_-, \alpha_+\) are strictly less than \(\alpha\).

Denote by \(\tilde{S}\) the reflection of a subset \(S \subset \mathbb{R}\) about 0. Decompose \(\tilde{B}_j\) as the disjoint union
\[
\tilde{B}_j = \tilde{B}_j^+ \cup \tilde{B}_j^- \cup (\tilde{B}_j \cap [\lambda_j^- + |I|, \lambda_j^+ - |I|])
\]
where
\[
\tilde{B}_j^- = \tilde{B}_j \cap (-\infty, \lambda_j^- + |I|) \quad \tilde{B}_j^+ = \tilde{B}_j \cap (\lambda_j^+ - |I|, \infty).
\]
Write
\[
\langle 1_{A_j} \ast 1_{B_j}, 1_I \rangle = \langle 1_{A_j} \ast 1_I, 1_{\tilde{B}_j} \rangle
\]
where \(\tilde{B}_j\) is the reflection of \(B_j\) about 0. Then since \(|A_j|, |B_j|\) belong to the fixed compact set \(K\),
\[
\langle 1_{A_j} \ast 1_I, 1_{\tilde{B}_j} \rangle = \langle (1_{A_j^+} + 1_{A_j^-}) \ast 1_I, 1_{\tilde{B}_j} \rangle + O(|A_j \cap L_j|)
= \langle 1_{A_j^+} \ast 1_I, 1_{\tilde{B}_j^+} \rangle + \langle 1_{A_j^-} \ast 1_I, 1_{\tilde{B}_j^-} \rangle + O(|A_j \cap L_j|)
\leq \Theta(|A_j^+|, |B_j^-|, |I|) + \Theta(|A_j^-|, |B_j^+|, |I|) + O(|A_j \cap L_j|)
\to \Theta(\alpha^-, \beta^-, \gamma) + \Theta(\alpha^+, \beta^+, \gamma)
\]
as \(j \to \infty\). The last inequality is justified by the Riesz-Sobolev inequality. On the other hand,
\[
\Theta(|A_j|, |B_j|, |I|) \geq \langle 1_{A_j} \ast 1_B, 1_I \rangle \geq (1 - \delta_j) \Theta(|A_j|, |B_j|, |I|) \to \Theta(\alpha, \beta, \gamma).
\]
The left-hand side converges to \(\Theta(\alpha, \beta, \gamma)\). Therefore
\[
\Theta(\alpha, \beta, \gamma) = \Theta(\alpha^-, \beta^-, \gamma) + \Theta(\alpha^+, \beta^+, \gamma),
\]
with \(\alpha^- + \alpha^+ = \alpha, \beta^- + \beta^+ = \beta, \text{ and } \alpha^\pm \neq 0\).

This is impossible. Indeed, the right-hand side of \((4.7)\) has the following interpretation. Consider intervals \(I^\pm, J^\pm\) of lengths \(\alpha^\pm, \beta^\pm\) respectively, such that distance \((I^-, I^+)\) is sufficiently large, \(J^+\) has the same center as \(I^+\), and \(J^-\) has the same center as \(I^-\). Then
\[
\langle 1_{I^+ \cup I^-} \ast 1_{J^+ \cup J^-}, 1_I \rangle = \langle 1_{I^+} \ast 1_{J^+}, 1_I \rangle + \langle 1_{I^-} \ast 1_{J^-}, 1_I \rangle
= \Theta(\alpha^+, \beta^+, \gamma) + \Theta(\alpha^-, \beta^-, \gamma).
\]
But
\[
\langle 1_{I^+ \cup I^-} \ast 1_{J^+ \cup J^-}, 1_I \rangle < \Theta(|I^+ \cup I^-|, |J^+ \cup J^-|, |I|) = \Theta(\alpha, \beta, \gamma)
\]
by Burchard’s inverse theorem, since \(I^+ \cup I^-\) is not an interval. This contradicts \((4.7)\). \(\square\)
So far, we have shown that the sets $A_j$ satisfy the decay bounds (15). The same reasoning applies to the sets $B_j$. By replacing $A_j$ by $A_j - \tau_j$, we may assume henceforth that $\tau_j = 0$. One cannot simultaneously translate $A_j, B_j$ by independent amounts without disturbing the hypothesis that $I$ is centered at 0. But from that restriction on $I$, it now follows easily from the decay bounds for $B_j$ that the sequence $\tau_j^j$ remains uniformly bounded, hence that $B_j$ satisfies the same bounds with $\tau_j^j \equiv 0$; otherwise necessarily $\langle 1_{A_j} \ast 1_{B_j}, 1_I \rangle \to 0$ as $j \to \infty$ for some subsequence of the indices $j$.

Next pass to a further subsequence, for which weak limits exist in $L^2$:

$$1_{A_j} \to f$$

$$1_{B_j} \to g$$

for certain functions $f, g \in L^2(\mathbb{R})$. By this we mean that for any test function $\varphi \in L^2(\mathbb{R})$, $(1_{A_j}, \varphi) \to \langle f, \varphi \rangle$ as $j \to \infty$. Because $|A_j|, |B_j|$ belong to the compact set $K$, some subsequence must converge in this sense. The uniform decay estimate (4.5), in conjunction with the normalization $\tau_j \equiv 0$, preclude the escape to spatial infinity of any mass, so $\|f\|_1 = \lim_{j \to \infty} |A_j| = \alpha$. Likewise, $\|g\|_1 = \lim_{j \to \infty} |B_j| = \beta$. Moreover, $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$.

**Lemma 4.3.**

(4.9) \[ \langle 1_{A_j} \ast 1_{B_j}, 1_I \rangle \to \langle f \ast g, 1_I \rangle \quad \text{as} \quad j \to \infty. \]

**Proof.** Let $\psi_i^\pm$ be continuous functions with ranges in $[0,1]$ such that $\psi_i^- < 1_I < \psi_i^+$, $\psi_i^\pm$ is supported within distance $i^{-1}$ of $I$, and $\psi_i^\pm \to 1_I$ from above and from below, respectively. As $j \to \infty$, $(1_{A_j} \ast 1_{B_j}, \psi_i^\pm) = (1_{A_j} \ast \psi_i^\pm, 1_{B_j}) \to \langle f \ast g, \psi_i^\pm \rangle$ for every $i$, since weak convergence of the sequence $1_{A_j}$ in $L^2$ implies strong $L^2$ convergence of the sequence $1_{A_j} \ast \psi_i^\pm$ for fixed $i$.

Finally, let $i \to \infty$ and use the comparison

$$\langle 1_{A_j} \ast 1_{B_j}, \psi_i^- \rangle \leq \langle 1_{A_j} \ast 1_{B_j}, 1_I \rangle \leq \langle 1_{A_j} \ast 1_{B_j}, \psi_i^+ \rangle$$

along with the corresponding upper and lower bounds for $\langle f \ast g, 1_I \rangle$. \hfill \Box

Therefore $\langle f \ast g, 1_I \rangle = \Theta(\alpha, \beta, \gamma)$. Recall that $\|f\|_\infty \leq 1$, $\|g\|_\infty \leq 1$, $\|f\|_1 = \alpha$ and $\|g\|_1 = \beta$. The next lemma guarantees that under these circumstances, $f, g$ are indicator functions of sets of measures $\alpha, \beta$ respectively.

**Lemma 4.4.** Let $A, B, I \subset \mathbb{R}$ be intervals centered at 0 of finite, positive lengths $|A|, |B|, |I|$ which satisfy

(4.10) \[ \max(|A|, |I|) - \min(|A|, |I|) < |B| < |A| + |I|. \]

Let $f, g \in L^1(\mathbb{R})$ be nonnegative functions satisfying $\|f\|_\infty, \|g\|_\infty \leq 1$, $\|f\|_1 = |A|$ and $\|g\|_1 = |B|$. Then $\langle f \ast g, 1_I \rangle \leq \langle 1_A \ast 1_B, 1_I \rangle$. Moreover, equality can hold only if $f, g$ are indicator functions of sets of measures $|A|, |B|$ respectively.

**Proof.** By the Riesz-Sobolev inequality, $\langle f \ast g, 1_I \rangle \leq \langle f^\ast \ast g^\ast, 1_I \rangle$. Therefore it suffices to prove the result under the additional assumption that $f = f^\ast$ and $g = g^\ast$, which we assume henceforth.

Both $1_A, f$ are symmetric nonincreasing, and $f(x) \leq 1_A(x)$ for every $x \in \mathbb{R}$, so for any symmetric nonincreasing function $h$, $\int fh \leq \int 1_A h$. Since $g \ast 1_I$ is a symmetric nonincreasing function,

$$\langle f \ast g, 1_I \rangle = \langle f, g \ast 1_I \rangle \leq \int 1_A \cdot (g \ast 1_I) = \langle 1_A \ast g, 1_I \rangle.$$
Repeating the argument with \( f, g \) replaced by \( g, 1_A \) respectively gives \( \langle f \ast g, 1_I \rangle \leq \langle 1_A \ast 1_B, 1_I \rangle \). Therefore
\[
\langle f \ast g, 1_I \rangle \leq \langle 1_A \ast g, 1_I \rangle \leq \langle 1_A \ast 1_B, 1_I \rangle.
\]

If \( \langle f \ast g, 1_I \rangle = \langle 1_A \ast 1_B, 1_I \rangle \), then the preceding inequality forces \( \langle 1_A \ast g, 1_I \rangle = \langle 1_A \ast 1_B, 1_I \rangle \). Write \( \langle 1_A \ast g, 1_I \rangle \) as \( \langle g, h \rangle \) where \( h = 1_A \ast 1_I \) is symmetric nonincreasing, and is strictly decreasing on the set of all \( x \) which satisfy \( \max(|A|, |I|) - \min(|A|, |I|) < 2|x| < |A| + |I| \). Under the assumption (4.10), it is apparent that among all symmetric nonincreasing functions \( g \) which satisfy \( \|g\|_1 = |B| \) and \( \|g\|_\infty \leq 1 \), \( f h \) is maximized when \( g = 1_B \), and in no other cases. Therefore \( g = 1_B \). By symmetry, \( f = 1_A \).

We have shown so far that there are sets \( A, B \) such that \( 1_A_j \to 1_A \) where \( |A_j| \to |A| \), and likewise \( 1_B_j \to 1_B \) and \( |B_j| \to |B| \).

**Lemma 4.5.** Let \( E_j, E \subset \mathbb{R}^d \) be Lebesque measurable sets. Suppose that as \( j \to \infty \), \( |E_j| \to |E| < \infty \) and \( 1_E \to 1_E \). Then \( |E_j \triangle E| \to 0 \) as \( j \to \infty \).

**Proof.** Let \( \varepsilon > 0 \). Let \( K, \mathcal{O} \) respectively be a compact set and an open set such that \( K \subset E \subset \mathcal{O} \) and \( |\mathcal{O} \setminus K| < \varepsilon \). Let \( \varphi : \mathbb{R}^d \to [0, 1] \) be a continuous function which satisfies \( \varphi \equiv 1 \) on \( K \) and \( \varphi \equiv 0 \) outside of \( \mathcal{O} \). Then
\[
|E_j \cap \mathcal{O}| \geq \int \varphi 1_{E_j} \to \int \varphi 1_E \geq \int \varphi 1_K = |K| - |E| - \varepsilon.
\]

Therefore
\[
\liminf_{j \to \infty} |E_j \cap \mathcal{O}| \geq |E| - \varepsilon,
\]
and consequently
\[
\liminf_{j \to \infty} |E_j \cap E| \geq |E| - \varepsilon - |\mathcal{O} \setminus E| \geq |E| - 2\varepsilon.
\]
Since \( |E_j| \to |E| \), this implies that \( \limsup_{j \to \infty} |E_j \triangle E| < 2\varepsilon \).

Thus \( 1_{A_j} \to 1_A \) and \( 1_{B_j} \to 1_B \) in \( L^1 \) norm. Since we already know that \( \langle 1_A \ast 1_B, 1_I \rangle \) converges to \( \langle f \ast g, 1_I \rangle \) and on the other hand \( \langle 1_A \ast 1_B, 1_I \rangle \) by hypothesis, we conclude that \( \langle 1_A \ast 1_B, 1_I \rangle = \Theta(|A|, |B|, |I|) \). These three measures \( |A|, |B|, |I| \) satisfy the hypothesis (4.3) of Burchard’s inverse theorem. Therefore \( A, B \) are intervals, modulo null sets. Since \( |A_j \triangle A| = \|1_{A_j} - 1_A\|_1 \) and the latter has been shown to converge to zero, the proof of Proposition 4.1 is complete.

To extend Proposition 4.1 to higher dimensions, with the interval \( I \) replaced by a compact convex set \( K \) of positive Lebesgue measure, requires only a small modification. Let \( B(z, R) \) denote the ball in the norm associated to \( K \), with center \( z \) and radius \( R \). In place of Lemma 4.2, it suffices to show that there cannot exist a radius \( R \in (0, \infty) \) and center \( z \in \mathbb{R}^d \) such that \( |A \cap B(z, R)| \) and \( |A \cap (\mathbb{R}^d \setminus B(z, 2R))| \) are bounded below while \( |A \cap (B(z, 2R) \setminus B(z, R))| \) is nearly equal to zero. This follows from the proof of Lemma 4.2. Precompactness is obtained by bounding \( K \), inside and outside, by comparable ellipsoids, then exploiting affine symmetries to reduce to the case where the ellipsoids are balls.
5. Conclusion of proof

Proof of Theorem 6.1. Lemmas 2.1 and 2.2 together with Corollary 3.3 demonstrate that $E$ is well approximated by some interval $I$, in the sense that $|E \triangle I| \leq C\delta^{1/2} \max(|A|, |B|)$. Then

$$
(1_A * 1_B, 1_I) \geq (1_A * 1_B, 1_E) - \max(|A|, |B|)|E \triangle I|
$$

$$
\geq \Theta(|A|, |B|, |I|) - \delta \max(|A|, |B|)^2 - \max(|A|, |B|)|E \triangle I|
$$

$$
\geq \Theta(|A|, |B|, |I|) - C\delta^{1/2} \max(|A|, |B|)^2.
$$

By Proposition 4.1 there exists an interval $J$ such that $|J \triangle A| < \varepsilon$, where $\varepsilon \to 0$ as $\delta \to 0$. \qed

6. Proof of Proposition 3.1

Write $\#(S)$ to denote the cardinality of a finite set $S$, and $|S|$ for the Lebesgue measure of a subset $S \subset \mathbb{R}$. The proof of Proposition 3.1 uses the following theorem of Freiman [5]. See Theorem 5.11 of [11] for an exposition, and [6] for an extension to two sets.

The theorem of Freiman states the following: Let $A$ be a finite subset of $\mathbb{Z}$. If $\#(A + A) < 3\#(A) - 3$, then $A$ is contained in a rank one arithmetic progression of cardinality $\leq \#(A) - \#(A) + 1$.

Proof of Proposition 3.1. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with finite, positive measure. Assume that $|A + A| < 3|A| - \rho$ for some $\rho > 0$. $\rho$ will remain fixed throughout the discussion.

Let $\varepsilon, \delta > 0$ be small parameters. In particular, we require that $\delta < \frac{1}{2}$. For $n \in \mathbb{Z}$ consider the interval $I_n = (\varepsilon n - \frac{\varepsilon}{2}, \varepsilon n + \frac{\varepsilon}{2})$. Let $A \subset \mathbb{Z}$ be the set of all $n$ for which $|A \cap I_n| \geq (1 - \delta)|I_n|$, and let $\tilde{A} = \bigcup_{n \in A} I_n$. By the Lebesgue differentiation theorem, the symmetric difference $A \triangle \tilde{A}$ satisfies $|A \triangle \tilde{A}| \to 0$ as $\max(\varepsilon, \delta) \to 0$. Therefore $\varepsilon\#(A) - |A| \to 0$ as $\max(\varepsilon, \delta) \to 0$, as well. In particular,

$$
(1 - \eta)\varepsilon^{-1}|A| \leq \#(A) \leq (1 + \eta)\varepsilon^{-1}|A|
$$

where $\eta = \eta(\varepsilon, \delta)$ tends to zero as $\max(\varepsilon, \delta) \to 0$.

If $S, T \subset (-\frac{1}{2}, \frac{1}{2})$ are measurable sets of measures $> \frac{1}{2}$, then $S \cap T$ has positive Lebesgue measure and therefore $0 \in S - T$. It follows from this fact that if $m, n \in A$, then $\varepsilon n + \varepsilon m \in A + A$. Therefore $k \in A + A \Rightarrow \varepsilon k \in A + A$. Therefore

$$
\#(A + A) \leq \varepsilon^{-1}|A + A|.
$$

Now

$$
\#(A + A) + 3 \leq \varepsilon^{-1}|A + A| + 3
$$

$$
< \varepsilon^{-1}(3 - \rho)|A| + 3
$$

$$
\leq (3 - \rho)(1 - \eta)^{-1}\#(A) + 3
$$

$$
= 3\#(A) + (3 - \rho\#(A))
$$

where $\rho > 0$ may be taken to be independent of $\varepsilon, \delta, \eta$ provided only that these quantities are sufficiently small. Since $\#(A) \to \infty$ as $\max(\varepsilon, \delta) \to 0$, $(3 - \rho\#(A)) < 0$ provided that $\varepsilon\delta$ are chosen to be sufficiently small, and thus $\#(A + A) < 3\#(A) - 3$. 

The theorem of Freiman cited above now implies that there exists an arithmetic progression \( P = P(\varepsilon, \delta) \subset \mathbb{Z} \) such that \( A \subset P \) and
\[
\#(P) \leq \#(A + A) - \#(A) + 1.
\]
The set \( P = P(\varepsilon, \delta) = \bigcup_{n \in P} I_n \) then satisfies
\[
|P| = \varepsilon \#(P) \leq \varepsilon \#(A + A) - \varepsilon \#(A) + \varepsilon \leq |A + A| - (1 - \eta)|A| + \varepsilon = |A + A| - |A| + \eta|A| + \varepsilon.
\]

\( P \) is an arithmetic progression of rank 1 in \( \mathbb{Z} \), of some step \( d \) which without loss of generality can be taken to be positive. We claim that \( d = 1 \). Suppose not. Since \( P \subset k + d\mathbb{Z} \) for some \( k \in \mathbb{Z} \), and \( A \subset P \), for any \( m, m', n, n' \in A \)
\[
|(m + n) - (m' + n')| \geq 2 \text{ unless } m + n = m' + n'.
\]

Represent \( A \) as the set of all \( \varepsilon n + \varepsilon s \), where \( n \in A \) and \( s \in S_n \), where \( S_n \subset (-\frac{1}{2}, \frac{1}{2}) \). We have already arranged that \( |S_n| \geq (1 - \delta) \) for every \( n \in A \).

For any measurable sets \( S, T \subset (-\frac{1}{2}, \frac{1}{2}) \), the associated sumset \( S + T \) is contained in \((-1, 1)\) and satisfies \( |S + T| \geq |S| + |T| \) by the Brunn-Minkowski inequality. Thus for each element \( n \) of \( A + A \), the set \( A + A \) intersected with the interval of length \( 2 \varepsilon \) centered at \( \varepsilon n \) has measure \( \geq (2 - 2\delta)\varepsilon \). As \( n \) varies over \( A + A \), these intersections are pairwise disjoint by \((6.2)\). Since \( \#(A + A) \geq 2\#(A) - 1 \) by the Cauchy-Davenport inequality \([11]\),
\[
|A + A| \geq (2 - 2\delta)\varepsilon \#(A + A) \geq (2 - 2\delta)\varepsilon (2\#(A) - 1).
\]

Therefore
\[
|A + A| \geq (2 - 2\delta)(1 - \eta)2|A| - 2\varepsilon \geq (4 - \varrho)|A| - 2\varepsilon
\]
where \( \varrho \to 0 \) as \( \max(\varepsilon, \delta) \to 0 \). For a sufficiently small choice of the parameters \( \varepsilon, \delta \), this contradicts the hypothesis \( |A + A| < 3|A| \). Therefore \( d = 1 \).

The union of \( P \) with finitely many points \( n \pm \frac{1}{2} \) is an interval. We have thus proved that for any \( \gamma > 0 \), there exists an interval \( I_\gamma \subset \mathbb{R} \) such that \( A \subset I_\gamma \) and \( |I_\gamma| \leq |A + A| - |A| + \gamma \).

Then \( I = \bigcap_{n=1}^{\infty} I_{1/n} \) is an interval; it contains \( A \); it satisfies \( |I| \leq |A + A| - |A| \). \( \square \)

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