Space-Time Spectral Analysis of 2-D Signal on the Globe Using Spherical Harmonics and Wavelet Transform Methods

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Abstract. In this research, a method to analyze wave-number and frequency spectra from time series of global 2-D geophysical data has been developed. Spherical Harmonics (SH) transform method is used to analyze wave structures in both longitude and latitude, while the temporal evolution of frequency content in the data is analyzed using complex wavelet transform. The frequency spectra of longitudinally propagating waves are separated into their westward- and eastward components by wavelet transforms of the corresponding to complex SH coefficients and their complex conjugate. The method was applied to analyze daily OLR (Outgoing Long-wave Radiation) data of 27 years record (1987-2013) as a proxy for large-scale convective pattern. It proved that the method enables us to identify the existence of various wave structures representing the convectively coupled planetary atmospheric waves. As SH and wavelet transform methods are generic tools for signal analysis, our space-time spectral analysis method should also be applicable to other geophysical data whose properties are characterized with embedded planetary scale waves.

1. Introduction
Most of geophysical time series can be represented as combinations of wave signals, which vary in time and space. Space-time spectral analysis (STSA) is a general approach to extract wave composition of geophysical signals in frequency and wave-number domains. Fourier transform is a commonly used method to analyze wave frequency contents in time series data, which can be readily expanded to two or more dimensions in Cartesian coordinate to perform STSA. Fourier transform analysis is practically implemented as fast Fourier-transform (FFT), which can accurately give information about wave spectrum of the signals (e.g., [8]). In this method, signals are assumed to have uniform wave packet throughout the time domain. However, geophysical signals may have frequency content that change with time. Moreover, global geophysical data are defined on spherical surface requiring a special treatment.

Spherical harmonics (SH) transform is a method to analyze wave structures on spherical surface. It uses spherical harmonics function to transform a global data as function of space (longitude and latitude) into complex coefficients as functions of wave numbers [3]. Time series of the complex coefficients can be analyzed for its frequency contents to perform an alternative STSA of global data. However, instead of FFT, we consider to use wavelet transform method to analyze frequency contents of complex time series in order to obtain frequency spectra that are more localized in time (e.g., [4]).
In atmospheric science, it is well known that strong wave activities take place in the tropics and are coupled with convection. However, atmospheric waves are not continuously monochromatic propagating wave, but they can be observed as transient wave-like patterns that irregularly or quasi-regularly emerge and disappear. One of such phenomena is the Madden Julian Oscillation (MJO), which is an intra-seasonal (30-60 day) variability whose pattern is similar to eastward propagating wave from Indian Ocean to Pacific Ocean with a phase speed of about 5 m/s. It may be initiated by Kelvin and Rossby waves, which propagate eastward and westward respectively [11]. The fundamental theory of planetary-scale atmospheric waves in the tropics has been proposed by Matsuno [6].

Convections in the tropics are marked by large-scale convective clouds, which can be identified as anomalies of outgoing longwave radiation (OLR) observed from satellites. Jie et al. [2] developed a method to analyze OLR data by combining cylindrical parabolic and wavelet transform. By using this method, the complex coefficients (wavenumber and time function) resulted from cylindrical parabolic transform for each wavenumber could be analyzed by wavelet transform to see the frequency evolution. In addition, they also show that with wavelet transform it is possible to separate eastward and westward propagating wave components. In this work, we modified their method by combining SH and wavelet transforms to analyze global OLR data and to study frequency evolution of the wave components for eastward and westward propagating wave-like patterns.

2. Data and Methods

In this research, two types of data were used, synthetic and real data. The synthetic data was used to confirm the effectiveness of westward and eastward wave separation analyzed from data on the globe, while the previously mentioned OLR data is used as real data. Detailed descriptions of the datasets are given below.

2.1. Synthetic Data

The nondimensional space-time synthetic data \( H(\lambda, \theta, t) \) is generated from superposition of four sinusoidal waves:

\[
H(\lambda, \theta, t) = \sin \left( \frac{2\pi}{73} \right) \left( 5 \sin \left( \frac{2\pi}{T_1} t - m\lambda \right) + \sin \left( \frac{2\pi}{T_2} t - m\lambda \right) + 3 \sin \left( \frac{2\pi}{T_3} t + m\lambda \right) + \sin \left( \frac{2\pi}{T_4} t + m\lambda \right) \right)
\]

where the first two terms represent eastward propagating waves and the rests represent westward propagating waves. \( \lambda \) and \( \theta \) are longitudinal and latitudinal grids, respectively, and \( t \) is time (day). \( T_1, T_2, T_3, \) and \( T_4 \) are 60, 365, 10, and 3650-day period, respectively, and \( m \) is zonal wavenumber, here taken to be 2. Spatial depiction of \( H(\lambda, \theta, t) \) for \( t = 1-4 \) is presented in Figure 1. Note that in meridional direction, the zero crossing of \( H(\lambda, \theta, t) \) only occurs at the equator.

![Figure 1](image1)

2.2. Real Data

The real data consist of daily OLR data from 1 January 1987 to 31 December 2013 with 2.5° resolution [5]. This data, provided by NOAA/OAR/ESRL PSD, Boulder, Colorado, USA, can be downloaded from http://www.esrl.noaa.gov/psd/. OLR have long been used as a proxy indicator of convective activity. Lower (higher) OLR indicates enhanced (suppressed) convection, especially over the tropic. OLR
patterns undergo space-time variations affected by many phenomena with different temporal and spatial scales. Figure 2 shows time series of spatial original OLR with time increment of 5 days. On 1 December 1996, lower OLR (high convection) is observed over Indian Ocean (black circle). It can also be seen that the low OLR pattern propagates eastward with time, so that on 16 December 1996 it was observed over the eastern part of Indonesia. The propagating pattern of actively convective region belongs to MJO, which is a well known atmospheric phenomenon.

Figure 2. Time evolution of OLR [Wm\(^{-2}\)] on (a) 1\(^{st}\), (b) 6\(^{th}\), (c) 11\(^{th}\), and (d) 16\(^{th}\) December 1996. Black circles show high convection of MJO.

2.3. Spherical Harmonics (SH) Transform

It has been mentioned that we propose to use a combination of SH and wavelet transforms to perform STSA. The SH representation of a global data at time \( t \) can be written as

\[
H(\lambda, \theta, t) = \sum_{n} \sum_{m} H_n^m(t) Y_n^m
\]

(2)

where \( Y_n^m(\lambda, \mu) = P_n^m(\mu) e^{im\lambda} \) is SH function with latitude defined as \( \mu = \sin \theta \) and longitude \( \lambda \). \( P_n^m \) is the Legendre polynomial with order \( m \) and degree \( n \) that represents meridional (latitudinal) variation, while \( e^{im\lambda} \) is sinusoidal function that represents longitudinal variation. Here, \( n \) and \( m \) represent a total and zonal wavenumber, respectively. Additionally, if \( n - m \) is even (odd), then \( P_n^m \) is symmetric (asymmetric) about equator. \( H_n^m \) are complex coefficients corresponding to the data at time \( t \).

An application of SH transform to the global data at any time during the valid data period can be expressed as

\[
H_n^m(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-1}^{1} H(\lambda, \mu, t) Y_n^m* d\mu d\lambda
\]

(3)

SH transform in this study was implemented by using SPHEREPACK, a Fortran program set for geophysical modeling [1]. Examples of SH decompositions \( H_n^m(t)Y_n^m \) of synthetic and real data for arbitrary time are shown in Figure 3 for \( (n=1, m=1), (n=3, m=2) \), and \( (n=3, m=3) \). Note that for synthetic data, only \( (n=3, m=2) \) pattern has large spatial variation corresponding to the amplitude defined in Equation (1).
Figure 3. SH decomposition $H_n^m(t)Y^m_n$ of (a) synthetic at $t = 1$ and (b) real data at $t = 1^{st}$ December 1996. The unit for (a) is nondimensional and for (b) is Wm$^{-2}$. As an example, the decompositions are for $(n=1,m=1)$, $(n=3,m=2)$, and $(n=3,m=3)$.

For further analysis, $m$ and $n$ must be chosen so that the SH coefficients can be associated with certain geophysical phenomena. In this research $n$ and $m$ are determined based on spectrum of equatorial atmospheric waves analyzed from OLR data using from Wheeler and Kiladis [10] (hereafter WK99) method as shown in Figure 4.

This study focused on MJO and Equatorial Rossby (ER) wave as examples of eastward and westward propagating waves, respectively. From Figure 4, it is shown that MJO spectrum (black box) has peak around $m = 1$–$4$ (with the highest peak at $m=1$) with 30–60 day period, while ER (blue box) has peak around $m = 2$–$5$ with 16–64 day period. In this case, only two pairs of $n$ and $m$ i.e. $(n=1,m=1)$ and $(n=3,m=3)$ are chosen for further analysis. Note that the corresponding SH components are symmetric about the equator as shown in Figure 3.
Figure 4. Symmetric component of daily OLR [15°S-15°N] spectrum from 1979 to 2000 overlaid by atmospheric wave dispersion curves for equivalent depth of 12, 25, and 50 m resulted from WK99 method. Black and blue boxes are MJO and ER domain.

Unfiltered time series of complex SH coefficients $H_1^1$ and $H_3^3$ were obtained by applying SH transform to the global data for each consecutive time $t$. Figure 5 shows the time variation of $H_1^1$ and $H_3^3$ analyzed from the OLR anomaly. In this case, the OLR anomaly was obtained by removing the first three harmonics of annual cycle from the real data in order to focus on higher frequency (sub seasonal) disturbances.

Figure 5. Time variation of daily real and imaginary coefficients of (a) $H_1^1$ and (b) $H_3^3$ of the OLR anomaly

2.4. Wavelet Transform

Following the work of Jie et al. [2], wavelet transform was applied to time series of $H_n^m(t)$ and its complex conjugate $H_n^m*(t)$ for certain pair of $n$ and $m$. Here, Morlet wavelet was used as the analyzing complex mother wavelet:

$$\Psi(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-t^2/2}$$

(4)

where $\omega_0$ is non-dimensional frequency. The outputs of the wavelet transform are wavelet power spectra:


\[
W_{tt}(a,b) = \frac{1}{(a)^{1/2}} \int \Psi^* \left( \frac{t-b}{a} \right) H_{\frac{2}{1}} dt
\]  

(5)

and

\[
W_{tt^*}(a,b) = \frac{1}{(a)^{1/2}} \int \Psi^* \left( \frac{t-b}{a} \right) H_{\frac{2}{1}}^* dt
\]  

(6)

where \( \Psi^* \) is a complex conjugate of \( \Psi \) with dilatation (scale) \( a \) and translation (position) \( b \). The wavelet transform was implemented using wavelet tools developed by Torrence and Compo [7]. The output wavelet power spectra of the OLR anomaly are then analyzed and interpreted for the MJO and ER wave activities based on the time localization of wave frequency.

3. Results

3.1 Separation of Eastward and Westward Propagating Waves in Synthetic Case

Wavelet amplitude spectrum of \( H_{\frac{2}{1}} \) and \( H_{\frac{2}{1}}^* \) (\( W_t \) and \( W_{tt^*} \)) is shown in Figure 6(a) and (b) respectively. From Figure 6(a), there are dominant amplitudes at period about 10- and 3650-day. These are periods of westward propagating wave. On the other hand, in Figure 6b, dominant amplitudes appear at period about 60- and 365-day, which are periods of eastward propagating wave. Hence, it is shown here that wavelet transforms of \( H_{\frac{2}{1}} \) (\( H_{\frac{2}{1}}^* \)) indicate the frequency components of westward (eastward) propagating waves.

![Figure 6](image)

**Figure 6.** Wavelet power spectrum (a) eastward and (b) westward components and of Equation (1).

It should be noted that, although the eastward and westward propagating waves in Equation (1) are defined by negative and positive wave number \( m \), the wavelet separation actually operates on negative and positive frequency. This is because the negative \( m \) is not defined in the output of SH transform (Adams and Swartztrauber, 1977).

3.2 Eastward and Westward Propagating Waves Analyzed from OLR Data

Figure 7 shows the wavelet spectrum of \( H_{\frac{1}{1}} \) with marked period of 32-64 days. It can be seen that, by selecting \( m=1 \) and \( n=1 \), the global wavelet spectrum of \( H_{\frac{1}{1}} \) indicates dominant amplitudes at periods corresponding to MJO. Moreover, Figure 7 also shows the time localization of MJO activities and their coincidence with variabilities at other frequencies. At this point, we limit our interpretation to MJO and restrain from discussing other related atmospheric phenomena.
We further demonstrated the application of our STSA method for analyzing westward propagating ER wave by calculating $W_{H_3}$, which is the wavelet spectrum of $H_3$ time series, as shown in Figure 8. Referring to Figure 4, ER has about 16–64 day period with wavenumber around $m = 3$. It can be seen from Figure 8 that strong wave amplitudes appear at period 16-64 day, which is in agreement with ER period from WK99 spectrum (Figure 4).

It is also of interest herein to show that the wavelet spectrum in Figure 7 can be used for monitoring of MJO activities by constructing an index from averaged $W_{H_3}$ at period range of 32-64 day. The resulted index can then be compared with already established and well-known MJO index i.e. Real Time Multivariate MJO (RMM) developed by Wheeler and Hendon [9]. As it can be seen from Figure 9, time series of both MJO indices shows good agreement. Moreover, the MJO index derived from SH coefficients show more contrast during active and non-active MJO periods.
Figure 9. Time series of RMM amplitude (red) and averaged power spectrum of $H_1^1(W_{H_1^1})$ at period of 32-64 days

4. Conclusion
SH and wavelet transform combination is applicable to analyze wave components of global data which characterize planetary scale waves. This method could be applied to see time localization of wave’s period and separate wave components based on its wavenumber and zonal propagating direction, where westward (eastward) components are represented by power spectrum of complex (complex conjugate) coefficient resulted from SH transform. From this study, using OLD data it is shown that MJO characteristic appeared at wavelet power spectrum of $H_1^1$ at 32-64 days period, while ER appeared at wavelet power spectrum of $H_1^1$ at 16-64 days period. This method should be potential to analyze wave components of other kinds of global geophysical data. Furthermore, it is potential to be a method to get wave activity indexes.

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