Default Distances Based on the KMV-CEV Model

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Abstract
This paper presents a new method to assess default risk based on applying non constant volatility to the KMV model, taking the CEV model as an instance. We find the evidence that the classical KMV model could not distinguish ST companies in China stock market. Aiming at improve the accuracy of the KMV model, we assume the firm’s asset value dynamics are given by the CEV process $dV_A = \mu_A dt + \delta V_A^{\beta - 1} dB$ and use fixed effects model and equivalent volatility method to estimate parameters. The estimation results show the $\beta > 1$ for non ST companies while $\beta < 1$ for ST companies and the equivalent volatility method estimate the parameters much more precisely. Compared with the classical KMV model, our CEV-KMV model fits the market better in forecasting the default probability. We also provide an insight that other volatility model can be applied, too.

Keywords: KMV, Default Risk, CEV Process, Equivalent Volatility

1 Introduction

Due to the outbreak of the worldwide financial crisis in 2008, risk management especially credit risk management becomes one of the most centralized topics in finance. Bankrupt or default forecasting models, like the KMV model and Credit Metrics model, now play a key role in assessing credit risk.

The well-known KMV model was developed by Kealhofer, McQuown, Vasicek and their company which is also called KMV Corporation in 1997, based on the seminal study of Merton (1974) [1]. What KMV model contributes is it offers an insight that we can regard the equity as a call option on the firm’s asset. In the past several decades, the power of the KMV model in assessing credit risk has been tested. Kealhofer and Kurbat (2001) [2] focus on the listed companies in North America and find the KMV model captures all the information of the traditional accounting variables in financial statements when forecasting the default risk. Compared with Altman’s (1968) Z-Score and Ohlson’s (1980) O-Score, Hillegeist et al. (2004) also argue that the KMV model provides significantly more information than either of the two accounting-based measures.

In order to improve accuracy of forecasting default probability, various advanced theories and methods are applied by other researchers. Bharath and Tyler (2008) put up a naive predictor and find it performs slightly better in hazard models and in out-of-sample forecasts than both the KMV model and a reduced-form model that uses the same inputs. In order to find the best default point, Zhang and Shi (2016) use particle swarm optimization and fuzzy clustering to modify the KMV model and the results show their FC-PSO-KMV model provides more accurate predictions. Focusing on the default point, too, Song et al. (2020) also use particle swarm optimization to adjust the KMV model since they find the classical KMV model could not do distinguish the ST companies, a fact which will be also seen in this paper. However, very few pay attention to modification on the GBM assumption in the KMV model, this paper will point out disadvantages of the GBM assumption and try to modify it.
Initially, Black and Scholes (1973) assumes the underlying stock returns dynamics are given by a GBM, i.e., constant volatility. However, the implied volatility smile, skew or smirk indicates that constant volatility may be wrong. More and more volatility models are developed in order to deal with volatility smile, like CEV model proposed by Cox (1975), Heston model (1993) and Hagan’s SABR model (2002).

In this paper, we take the CEV model as an instance. The constant elasticity of variance (CEV) model is considered to be an effective model which deal with the volatility smile. After Cox proposed the CEV model, more and more researchers are taking further study and testing the power of the CEV model, like Beckers (1980) and Davydov et al. (2001). Though we can see these volatility models including CEV model now play an important role in finance, yet still few study apply them to the KMV model. In this paper, we will offer an insight for how to apply the non-constant volatility to the KMV model.

The remainder of this paper proceeds as follows. Section 2 introduces some preliminary methodology, including the KMV model, CEV process, fixed effect model and equivalent volatility. In Section 3, we describe how we collect the data and deduce some test approaches. Section 4 tests the classical KMV model and find it hard to distinguish ST and non ST companies. After we analyse the volatility problem, in section 5 we propose a CEV-KMV model and the empirical results show the CEV process do improve the forecasting ability of the KMV model. Finally, section 6 concludes our results and looks into the future.

2 Methodology

2.1 The KMV model

Default forecasting is one of the most enduring themes of credit risk. In the well-known KMV model, the key point is computing default probability. Firstly, we assume that the firm’s asset value ($V_A(t)$) dynamics are given by a Geometric Brownian Motion (GBM):

$$\frac{dV_A(t)}{V_A(t)} = \mu_A dt + \sigma_A dB(t), \quad t \in [0, T],$$

(1)

where $B(t)$ is a standard Brownian motion, $\mu_A$ and $\sigma_A$ represents the firm’s asset value drift rate and volatility respectively. Let $D$ represents the debt of the company at a given maturity $T$, then naturally $V_A(T) < D$ means the company has no ability to afford its debt, which finally leads to default. In the KMV model $D$ is named default point and usually chosen to be a half of the long-term debt (LTD) plus the short-term debt (STD) at $t = 0$. The default probability can be expressed as

$$P (V_A (T) < D) = N \left( -\frac{\ln \frac{V_A(0)}{D} + (r - \frac{\sigma_A^2}{2}) T}{\sigma_A \sqrt{T}} \right) = N (-d_2),$$

(2)

where $N(\cdot)$ is the standard normal cumulative distribution function, $r$ is the risk free interest rate, SHIBOR may be a suitable choice in China market.

Unfortunately, both the firm’s asset value $V_A(0)$ and the volatility $\sigma_A$ may be not easy to obtained, which makes it hard to compute. However, the volatility of the firm’s equity can be estimated by using the historical stock data. In order to make use of this, we regard the firm’s equity ($V_E$) as a call option on $V_A$ with strike price $D$, so $V_E$ can be expressed as, of course, Black-Scholes-Merton formula:

$$V_E = V_A N(d_1) - e^{-r(T-t)} D N(d_2),$$

(3)
where \( d_2 \) is defined in (2) and \( d_1 = d_2 + \sigma_A \sqrt{T} \). Then \( dV_E \) follows from Itô’s lemma:

\[
dV_E = \left( \frac{\partial V_E}{\partial t} + rV_A \frac{\partial V_E}{\partial V_A} + \frac{\sigma_A^2 V_A^2}{2} \frac{\partial^2 V_E}{\partial V_A^2} \right) dt + \sigma_A V_A \frac{\partial V_E}{\partial V_A} dB_t,
\]

implying that \( V_E \) have local volatility

\[
\sigma_E = \sigma_A \frac{V_A \frac{\partial V_E}{\partial V_A}}{V_E} = \sigma_A \frac{V_A}{V_E} N(d_1).
\]

Since \( V_E(0) \) and \( \sigma_E \) can be determined or estimated, \( V_A(0) \) and \( \sigma_A \) can be calculated by combining (3) and (5).

Besides, it is an interesting fact that rather than just pay attention to the theoretical default probability \( N(-d_2) \), the KMV model defines a default distance (or distance to default) \( DD = d_2 \) or

\[
DD' = \frac{V_A - D}{V_A \sigma_A} \approx \frac{\ln \frac{V_A}{D} + \left( r - \frac{\sigma_A^2}{2} \right)}{\sigma_A} = d_2,
\]

and then creates a map from default distance to the historical expected default frequency (EDF). The more details about EDF can be found in Crosbie and Bohn (2003) [15]. However, the rare default data in China do not allow us to do this, but it may bring some advantages in statistical inference for us to focus on the default distances.

### 2.2 The CEV Model

The constant elasticity of variance (CEV) model was proposed by Cox (1975) [10]. Cox assumes that the firm’s asset value dynamics are given by

\[
dV_A(t) = \mu_A dt + \delta V_A^{\beta - 1} dB(t), \quad t \in [0, T],
\]

where \( \delta > 0, \beta > 0 \) are constant. The CEV process means \( V_A \) has local volatility \( \sigma_A = \delta V_A^{\beta - 1} \), implying

\[
\ln \sigma_A = \ln \delta + (\beta - 1) \ln V_A.
\]

From (7) we find if \( \beta = 1 \) then (7) turns back to the GBM. If \( \beta > 1 \), the volatility is increasing with respect to the underlying. If \( \beta < 1 \), the volatility is decreasing with respect to the underlying.

Similarly, if we know \( \delta \) and \( \beta \), it is theoretically available to compute default probability \( P(V_A(T) < D) \). However, since \( V_A(T) \) is not normal distributed, an analytical solution for default probability is hard to get, so the numerical solution may be a better choice.

By the Feynman-Kac Theorem, \( g(t, x) = E[1_{\{V_A(T) < D\}} | \mathcal{F}_t] \) solves the partial differential equation (PDE):

\[
\begin{align*}
\frac{\partial u}{\partial t} + r x \frac{\partial u}{\partial x} + \frac{\delta^2 x^{2\beta}}{2} \frac{\partial^2 u}{\partial x^2} &= 0, \quad t \in [0, T], \\
u(T, x) &= 1_{\{x < D\}},
\end{align*}
\]

it is very convenient for us to use the Finite Difference Method (FDM) to solve (9) and then

\[
P(V_A(T) < D) = u(0, V_A(0)).
\]
2.3 Fixed Effect Model

An efficient and simple method to estimate the parameters in the CEV model is that we could assume every company in a group (ST group or non ST group) has a common $\beta$ but a unique $\delta$ in CEV process. Thus (8) implies a fixed effects model:

$$\ln \sigma_A(i,t) = \ln \delta_i + (\beta - 1) \ln V_A(i,t), \quad i = 1, \ldots, n, \quad t = 1, \ldots, T. \quad (11)$$

Under general conditions, ordinary least squares (OLS) is a consistent estimation procedure for these parameters. The OLS estimators using the previous data are

$$\hat{\beta} = 1 + \frac{\sum_{i=1}^{n} \sum_{t=1}^{T} \left( \ln V_A(i,t) - \ln \sigma_A(i) \right) \left( \ln \sigma_A(i,t) - \ln \sigma_A(i) \right)}{\sum_{i=1}^{n} \sum_{t=1}^{T} \left( \ln V_A(i,t) - \ln \sigma_A(i) \right)^2}, \quad (12)$$

$$\ln \delta_i = \ln \sigma_A(i) - (\hat{\beta} - 1) \ln V_A(i), \quad i = 1, \ldots, n, \quad (13)$$

where $\ln \sigma_A(i) = \frac{1}{T} \sum_{t=1}^{T} \ln \sigma_A(i,t)$ and $\ln V_A(i) = \frac{1}{T} \sum_{t=1}^{T} \ln V_A(i,t)$. Despite great convenience, this method has a “model risk”: the model we use to calculate default distance is CEV process however the data ($V_A, \sigma_A$) for the fixed effects model are computed under GBM. To overcome this drawback, the main results of Hagan and Woodward (1999) \[16\] offer an insight, which we will discuss in next section.

2.4 Equivalent Volatility

Consider an European call option pricing problem, if the underlying asset value dynamics $dS$ (maybe not GBM) and terminal payoff $c(T,S) = (S_T - K)^+$ are given, consequently the option price $c(0,S)$ could be obtained theoretically. Define

$$d_1(\sigma) = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \quad d_2(\sigma) = d_1(\sigma) - \sigma \sqrt{T}. \quad (14)$$

Because of the 1-1 relation between Black-Scholes-Merton formula and implied volatility, option prices are often quoted by stating implied volatility. Thus the equivalent volatility $\sigma_B$ for $dS$ is defined as the unique solution of the following equation

$$c(0,S) = SN(d_1(\sigma)) - e^{-rT} KN(d_2(\sigma)), \quad (15)$$

where $c(0,S)$ is the option price calculated under assumption $dS$, and the “B” in the equivalent volatility $\sigma_B$ means the right hand side of $[16]$ is Black’s model (Black-Scholes-Merton model).

Hagan and Woodward (1999) \[16\] use singular perturbation methods to deduce the equivalent volatility for CEV model:

$$\sigma_B = \frac{\delta}{f^{1-\beta}} \left\{ 1 + \frac{(1-\beta)(2+\beta)(F-K)^2}{24f^2} + \frac{(1-\beta)^2\delta^2T}{24f^{2-2\beta}} + \cdots \right\}, \quad (16)$$

where $F = e^{rT}S_0$ is the future price and $f = \frac{1}{2}(F + K)$.

Obviously, if we assume the asset value dynamics $dV_A$ is given by CEV process, the $\sigma_A$ computed by combining $[3]$ and $[5]$ is nothing but the equivalent volatility. Go on considering every company has
Table 1: Summary statistics of equity, default point and volatility

| Variable                  | ST/Non ST | Mean   | Median | Std.    | Obs.  |
|---------------------------|-----------|--------|--------|--------|-------|
| Equity (billion CNY)      | ST        | 3.819  | 2.530  | 4.942  | 1674  |
|                           | Non ST    | 19.648 | 8.165  | 45.481 | 1674  |
| Default point (billion CNY)| ST        | 4.899  | 1.140  | 16.511 | 1674  |
|                           | Non ST    | 28.404 | 4.433  | 108.625| 1674  |
| Volatility of equity      | ST        | 0.480  | 0.471  | 0.116  | 1674  |
|                           | Non ST    | 0.411  | 0.402  | 0.102  | 1674  |

a common $\beta$ and a unique $\delta$, it is easy for us to estimate these parameters by methods like calibration. Since equivalent volatility method overcome the “model risk”, we guess if we use this method instead of the fixed effects model as an estimation procedure, the performance of the CEV-KMV model will be better.

3 Empirical Design

3.1 Data

In China, the listed companies under special treatment (ST) are always be considered as the ones with higher credit risk, while the non specially treated (non ST) listed companies usually have a lower default probability. In order to test the performance of the KMV model, we will use the data in China stock market to see if the default distances of the ST companies are significantly less than those of non ST companies.

We obtain data from Choice Database established by EastMoney. For ST companies, we only focus on those which at least have one-year continuous transaction records, ensuring there are enough data for us to estimate $\sigma_E$, the estimator of $\sigma_E$ is the annualized sample standard deviation of the latest 250 daily return:

$$\hat{\sigma}_E(i,t) = \sqrt{\frac{250}{249} \sum_{s=t-250}^{t-1} (r_{is} - \bar{r}_t)^2}, \quad \bar{r}_t = \frac{1}{250} \sum_{s=t-250}^{t-1} r_{is}.$$

We calculate the $\sigma_E$ quarterly (at the last day in every quarter), from 2019Q1 to 2021Q1. The equities $V_E$ and default points $D$ could be obtained directly or computed from database or financial statements, too. The missing data have been filled with the nearest neighbor. We find 186 ST companies are available during 2019Q1-2021Q1. Correspondingly, we randomly choose 186 non ST companies.

Table 1 reports the descriptive statistics of our data. The average equity amount of ST companies is 3.819 billion CNY, far less than that of non ST companies, 19.648. Similarly, the mean default point of non ST companies is approximately five times as much as that of ST companies. However, the volatility of non ST companies is a little less than that of ST companies. In both ST and non ST companies, default point is greater than equity. Since we collect the all available ST companies and the non ST companies sampled are chosen randomly, the summary statistics could reveal the general rule: ST companies may be more unstable than non ST companies.
3.2 Test Approaches

It can be seen from Fig. that the approximate distribution of default distances is gamma distribution, so testing whether the default distances of ST companies are less than those of non ST companies turns to be deducing a large sample test under gamma assumption. If the default distances of ST and non ST companies are assumed to be \( X \sim \text{Ga}(\alpha_1, \beta_1) \), \( Y \sim \text{Ga}(\alpha_2, \beta_2) \), respectively, then the Central Limit Theorem applies and the asymptotic sampling distribution of the sample mean \( \bar{X}_m \) and \( \bar{Y}_n \) is normal. Under null hypothesis \( H_0 : \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \), the test statistics

\[
Z_1 = \frac{\bar{X}_m - \bar{Y}_n}{\sqrt{\frac{\hat{\alpha}_1}{m\hat{\beta}_1^2} + \frac{\hat{\alpha}_2}{n\hat{\beta}_2^2}}} \quad (17)
\]

is asymptotically standard normal implied by the Slutsky Theorem, where \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2 \) are the maximum likelihood estimators for these parameters. The \( p \)-value for alternative hypothesis \( H_1 : \frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\beta_2} \) can also be derived.

![Histogram and gamma density of default distances (2021Q1)](image)

Figure 1: Histogram and gamma density of default distances (2021Q1)

In order to check the robustness of results based on parametric tests, the Wilcoxon test may be helpful. By using only the information of the sample ranks, Wilcoxon test proposed by Wilcoxon (1945) is one of the most popular non parametric tests. Define \( W_X \) is the sum ranks of the ST default distance \( X \) in the hybrid sample. Fixed the ratio \( m/n \), when \( n \) is large, under the null hypothesis the statistic

\[
Z_2 = W_X - \frac{m(m+n+1)}{2} \sqrt{\frac{mn(m+n+1)}{12}} \quad (18)
\]

is asymptotically standard normal too. We reject the null hypothesis if \( Z_2 < c \), where \( c \) is a given constant corresponding to the significance level. Since the Pitman’s asymptotic relative efficiency (ARE) of Wilcoxon test on a skewed distribution is far less than 1, \( Z_2 \) may reject more often than \( Z_1 \).
Table 2: Default distances based on the KMV model

| Quarter | ST/Non ST | Mean   | Std.   | Alpha | Beta  |
|---------|-----------|--------|--------|-------|-------|
| 2019Q1  | ST        | 4.498  | 2.524  | 4.183 | 1.075 |
|         | Non ST    | 4.484  | 2.006  | 6.686 | 0.671 |
| 2019Q2  | ST        | 4.264  | 2.617  | 3.771 | 1.131 |
|         | Non ST    | 4.245  | 1.9    | 6.642 | 0.639 |
| 2019Q3  | ST        | 4.215  | 2.686  | 3.616 | 1.166 |
|         | Non ST    | 4.264  | 1.941  | 6.384 | 0.668 |
| 2019Q4  | ST        | 4.578  | 2.874  | 3.531 | 1.297 |
|         | Non ST    | 4.617  | 2.124  | 6.202 | 0.744 |
| 2020Q1  | ST        | 4.366  | 2.774  | 3.447 | 1.267 |
|         | Non ST    | 4.361  | 2.085  | 6.087 | 0.716 |
| 2020Q2  | ST        | 4.349  | 2.778  | 3.277 | 1.327 |
|         | Non ST    | 4.657  | 2.168  | 6.046 | 0.77  |
| 2020Q3  | ST        | 4.263  | 2.659  | 3.38  | 1.261 |
|         | Non ST    | 4.306  | 2.018  | 6.019 | 0.715 |
| 2020Q4  | ST        | 4.109  | 2.412  | 3.667 | 1.121 |
|         | Non ST    | 4.106  | 1.79   | 6.549 | 0.627 |
| 2021Q1  | ST        | 4.026  | 2.266  | 3.845 | 1.047 |
|         | Non ST    | 4.305  | 1.999  | 5.878 | 0.732 |

Note: The default distances are computed by using \( DD = d_2 \). Alpha and beta in the table means the maximum likelihood estimators in gamma distribution.

4 The Performance of the KMV Model

4.1 Empirical Results

By using the KMV model, default distances could be computed by combining (3) and (5). Table 2 summarizes the results. The intuitive evidence against the effectiveness of the classical KMV model is that we could see seldom difference between the average default distances of ST and non ST companies in most of these quarters. Only in two quarters (2020Q2 and 2021Q1) we find the default distances of ST companies is far less than that of non ST companies. However, we find the standard deviations of ST companies default distances are almost in \([2.5, 2.7]\), while those of non ST companies default distances are almost in \([1.9, 2.1]\). Besides, the histograms of default distances have been shown in Fig.1

Both parametric and non parametric tests results are reported in the Table 3. The parametric tests results present evidence against the KMV model. The \( p \)-values for all quarters are greater than 0.1, indicating that the null hypothesis could not be rejected. The results of 2019Q1, 2019Q2, 2020Q1 and 2020Q4 suggest most strongly that there is no evidence for ST default distances are less than non ST default distances. A comparison of the results for \( Z_1 \) versus \( Z_2 \) illustrates the previously discussed fact that the non parametric test tends to reject too often. In a word, though by using \( Z_2 \), the KMV model performs well in distinguishing ST and non ST companies, the parametric tests results can not satisfy anyone. We believe that the classical KMV model do have some disadvantages.
Table 3: Tests results for the KMV model

| Quarter | $Z_1$ | $p$-value | $Z_2$ | $p$-value |
|---------|-------|-----------|-------|-----------|
| 2019Q1  | 0.045 | 0.518     | -1.294| 0.098     |
| 2019Q2  | 0.059 | 0.523     | -1.785| 0.037     |
| 2019Q3  | -0.160| 0.436     | -2.239| 0.013     |
| 2019Q4  | -0.146| 0.442     | -2.188| 0.014     |
| 2020Q1  | 0.020 | 0.508     | -2.071| 0.019     |
| 2020Q2  | -1.213| 0.113     | -3.452| 0.000     |
| 2020Q3  | -0.159| 0.437     | -2.096| 0.018     |
| 2020Q4  | 0.011 | 0.504     | -1.597| 0.055     |

Note: $Z_1$ represents the test statistic in parametric test given by (17), $Z_2$ represents the test statistic in non parametric test given by (18).

4.2 Further Discussion

Recently, some researchers have pointed out disadvantages of the KMV model, like Zhang and Shi (2018) [17] find that KMV model varies with firm size and what they pay attention to is the default point. Actually, we focus on the relation between volatility and firm size: does volatility varies with the firm size?

In theory, the volatility $\sigma_A$ in the classical KMV model is a constant. In practice, however, $\sigma_A$ may be not a constant. Fig.2 illustrates that at least with respect to $V_A$, volatility $\sigma_A$ is not a constant. Instead, $\sigma_A$ is negatively and positively correlated to $V_A$ in the picture (a) and (b), respectively. Besides companies in figures, there is a general linear correlation between $\ln \sigma_A$ and $\ln V_A$ in all companies which matches (8) deduced from the CEV process. What’s more? It seems in the group of ST companies $\beta$ is less than 1 while for non ST companies $\beta$ is great than 1. The tests results and the facts above suggest that KMV model do vary with the firm size, applying non-constant volatility may improve it.

5 The CEV-KMV model

5.1 Default Distance

We propose a CEV-KMV model by applying CEV process rather than GBM in the classical KMV model. We have illustrated that the default probability under CEV process can be calculated by FDM, but in order to in order to match the test approaches used previously and gain advantages in statistical inference, like we used to do in the classical KMV model, though $V_A(T)$ is not normal distributed, we can redefine the default distance as

$$ DD = N^{-1}(P(V_A(T) < D)), \quad (19) $$

where $N^{-1}(\cdot)$ is the inverse function of the standard normal cumulative distribution function. Therefore, to test the performance of CEV process, we need only estimate $\delta$ and $\beta$.

5.2 Empirical Results

The estimation results under fixed effects model and equivalent volatility model differ. The estimator of $\beta$ are 0.9957 and 1.1850 under fixed effects model, for ST companies and non ST companies,
Figure 2: The relation between $\ln V_A$ and $\ln \sigma_A$. The horizontal axis represents $\ln V_A$ and the longitudinal axis represents the $\ln \sigma_A$. 
| Quarter | ST/Non ST | Mean | Std. | Alpha | Beta |
|---------|-----------|------|------|-------|------|
| 2019Q1  | ST        | 4.803| 1.842| 7.274 | 0.660|
|         | Non ST    | 4.910| 1.697| 9.147 | 0.537|
| 2019Q2  | ST        | 4.496| 1.886| 6.171 | 0.714|
|         | Non ST    | 4.795| 1.749| 8.374 | 0.573|
| 2019Q3  | ST        | 4.353| 1.944| 5.595 | 0.778|
|         | Non ST    | 4.714| 1.794| 8.018 | 0.588|
| 2019Q4  | ST        | 4.499| 1.989| 5.932 | 0.795|
|         | Non ST    | 4.682| 1.722| 8.313 | 0.563|
| 2020Q1  | ST        | 4.406| 2.045| 5.120 | 0.861|
|         | Non ST    | 4.575| 1.840| 7.459 | 0.613|
| 2020Q2  | ST        | 4.388| 2.062| 4.922 | 0.892|
|         | Non ST    | 4.654| 1.822| 7.630 | 0.610|
| 2020Q3  | ST        | 4.398| 1.988| 5.260 | 0.836|
|         | Non ST    | 4.612| 1.711| 8.389 | 0.550|
| 2020Q4  | ST        | 4.557| 2.160| 4.906 | 0.929|
|         | Non ST    | 4.753| 1.799| 8.052 | 0.590|
| 2021Q1  | ST        | 4.494| 2.184| 4.776 | 0.941|
|         | Non ST    | 4.939| 1.998| 7.133 | 0.692|

**Panel B: Equivalent Volatility Method**

| Quarter | ST/Non ST | Mean | Std. | Alpha | Beta |
|---------|-----------|------|------|-------|------|
| 2019Q1  | ST        | 4.725| 1.883| 6.686 | 0.707|
|         | Non ST    | 5.712| 2.133| 7.949 | 0.719|
| 2019Q2  | ST        | 4.365| 1.925| 5.757 | 0.758|
|         | Non ST    | 5.600| 2.393| 6.487 | 0.863|
| 2019Q3  | ST        | 4.412| 1.946| 5.581 | 0.791|
|         | Non ST    | 5.489| 2.489| 5.841 | 0.940|
| 2019Q4  | ST        | 4.464| 2.019| 5.273 | 0.847|
|         | Non ST    | 5.424| 2.410| 6.200 | 0.875|
| 2020Q1  | ST        | 4.340| 2.020| 5.027 | 0.863|
|         | Non ST    | 5.445| 2.489| 5.582 | 0.975|
| 2020Q2  | ST        | 4.352| 1.999| 5.071 | 0.858|
|         | Non ST    | 5.371| 2.467| 5.802 | 0.926|
| 2020Q3  | ST        | 4.505| 1.911| 5.825 | 0.773|
|         | Non ST    | 5.470| 2.311| 6.403 | 0.854|
| 2020Q4  | ST        | 4.424| 1.953| 5.366 | 0.825|
|         | Non ST    | 5.591| 2.366| 6.461 | 0.865|
| 2021Q1  | ST        | 4.344| 1.947| 5.280 | 0.823|
|         | Non ST    | 5.750| 2.458| 6.133 | 0.938|

*Note: Alpha and beta in the table means the maximum likelihood estimators in gamma distribution.*
respectively. While under equivalent volatility model, the $\beta$ estimated for ST companies is 0.9841 and for non ST companies is 1.1413. Besides, the estimated $\delta$’s under two models are quite dissimilar, which may lead to the difference between the two groups of default distances. The estimation results imply that for ST companies, local volatility is a decreasing function with respect to $V_A$, while for non ST companies local volatility is an increasing function, illustrating the previously discussed fact.

Table 4 contains some statistics for default distances under the CEV-KMV model. Panel A reports the mean and standard deviation of default distances by using fixed effects model. The mean of the ST default distances is less than that of the non ST default distances in every quarter, which reveals that the CEV-KMV model performs better in distinguishing ST companies than the classical KMV model. But what panel A of Table 5 summaries tells us though in most of these quarters the null hypothesis can be rejected by using $Z_2$, the test results by using $Z_1$ still satisfy nobody. The only good news is, the $p$-values for the parametric tests decrease a little, by contrast to the classical KMV model.

Focusing on the equivalent volatility method, panel B of Table 4 reports the main statistics of the default distances. Recall we have discussed that the performance of the CEV-KMV model will be better if we use the equivalent volatility method instead of the fixed effects model as an estimation procedure. The supportive evidence for that is the average default distance of ST companies is below 4.75 in all quarters while that of non ST companies is larger than 5.35 in all quarters. Besides, panel B of Table 5 reports the tests results. The values of $Z_1$ and the corresponding $p$-values imply the null hypothesis is rejected. The story is the same for the non parametric tests, which ensures the robustness.

Figure 3: $\overline{DD}_{nST} - \overline{DD}_{ST}$ under different models. The black line represents the classical KMV model, the blue line represents the CEV-KMV model using the fixed effects model as estimation procedure, the purple line represents the CEV-KMV model using the equivalent volatility method.

In order to show the improvement of applying CEV process to the KMV model clearly, Fig 3 plots the differences between the mean non ST default distances and mean ST default distances $\overline{DD}_{nST} - \overline{DD}_{ST}$. The greater this difference is, the better the model performs. Obviously from the figure, it brings great advantages in distinguishing ST companies that we assume the firm’s asset value follows CEV process rather than GBM in the KMV model. Besides, the different methods for
Table 5: Tests results for the CEV-KMV model

| Quarter | $Z_1$  | $p$-value | $Z_2$  | $p$-value |
|---------|--------|-----------|--------|-----------|
| Panel A: Fixed Effects Model |
| 2019Q1  | -0.209 | 0.417     | -0.897 | 0.185     |
| 2019Q2  | -0.865 | 0.193     | -2.788 | 0.003     |
| 2019Q3  | -0.862 | 0.194     | -2.676 | 0.004     |
| 2019Q4  | -0.538 | 0.295     | -2.199 | 0.014     |
| 2020Q1  | -0.448 | 0.327     | -1.659 | 0.049     |
| 2020Q2  | -0.703 | 0.241     | -2.270 | 0.012     |
| 2020Q3  | -0.491 | 0.312     | -1.970 | 0.024     |
| 2020Q4  | -0.499 | 0.309     | -1.882 | 0.030     |
| 2021Q1  | -1.349 | 0.089     | -2.870 | 0.002     |
| Panel B: Equivalent Volatility Method |
| 2019Q1  | -2.508 | 0.006     | -4.659 | 0.000     |
| 2019Q2  | -3.893 | 0.000     | -5.657 | 0.000     |
| 2019Q3  | -3.727 | 0.000     | -4.633 | 0.000     |
| 2019Q4  | -3.332 | 0.000     | -4.357 | 0.000     |
| 2020Q1  | -4.244 | 0.000     | -4.791 | 0.000     |
| 2020Q2  | -3.762 | 0.000     | -4.463 | 0.000     |
| 2020Q3  | -3.059 | 0.001     | -4.265 | 0.000     |
| 2020Q4  | -3.916 | 0.000     | -5.089 | 0.000     |
| 2021Q1  | -4.988 | 0.000     | -5.875 | 0.000     |

Note: $Z_1$ represents the test statistic in parametric test given by (17), $Z_2$ represents the test statistic in non parametric test given by (18).

estimation procedure also lead to the different results. Equivalent volatility is a more suitable method in estimation than the fixed effects model since it overcomes the “model risk”.

5.3 Further Discussion

The fact we find that the monotonicity of local volatility vary with whether the company is ST is surprising. In general, for listed companies, since the greater volume and higher transaction frequency of a larger size firm must lead to a greater volatility (see in Weigand (1996) \[18\] and Lee et al. (2000) \[19\]), most people may consider the larger firm size is, the greater the volatility becomes. However, we find ST companies does not follow this rule, the volatility of ST companies decreases when the firm size increases.

We guess one of the main reasons for ST companies disobey the general rule may be the ST companies always have higher risk, the less firm size is, the higher bankrupt or delisted probability will be. A higher risk in stock market often manifests itself as a great volatility. In another aspect concerned about internal control, McMullen et al. (1996) \[20\] found overall 26.5% firms offer their internal control report but among those companies in whose financial statements there are mistakes only 10.5% are willing to report their internal control, which maybe also contributes to a higher credit risk. In a word, this interesting fact we found may be caused by many other reasons and deserves further study.
6 Conclusion

Yet still no consensus exists in assessing default risk even if it has intrigued researchers for nearly 70 years. More and more tools and techniques have been used to assess default risk. This paper presents some useful approaches to apply CEV process to the KMV model. We focus on the default distances and compare them under different methods.

The first thing we find is, the classical KMV model performs not well in distinguishing ST companies in China. A contributing factor we point out is that volatility may be not a constant, and the figures suggest applying CEV process instead of GBM. We also notice that it is not easy to estimate the parameters in CEV model. We present two types of estimation procedure in the paper: fixed effects model and equivalent volatility. The estimation results of both two methods show the local volatility is an increasing function with respect to the non ST firm’s asset, but it is a decreasing function for the ST companies, i.e. $\beta$ is greater than 1 for non ST companies while $\beta$ is less than 1 for ST companies. Besides, after we applied these two methods, the tests results show the equivalent volatility method is much more suitable in estimating CEV parameters. Finally, we find that if the estimated parameters are accurate, the CEV-KMV model can bring great advantages in distinguishing ST companies.

What can be improved is that we can try some advanced parameters estimating procedures like MCMC or other machine learning algorithm. Besides, we offer an insight that other volatility model like Heston and SABR can be used based on the KMV model if suitable because the 1-1 relation between equivalent volatility and option price as well as the effective numerical methods allow us to do it.

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