The gap equation for fermions in a version of thermal QED in three dimensions is studied numerically in the Schwinger-Dyson formalism. The interest in this theory has been recently revived since it has been proposed as a model of high-temperature superconductors. We include wave-function renormalization in our equations and use a non-bare vertex. We then have to solve a system of two integral equations by a relaxation algorithm. Fermion and photon self-energies varying independently with energy and momentum are used, which should produce more accurate results than in the previous literature. The behaviour of the theory with increasing temperature and number of fermion flavours is then carefully analyzed.

I. THE PROBLEM

The study of QED in three dimensions during the past years has revealed very interesting physics in this theory. At finite temperature, a version of QED, i.e. $\tau_3 - QED$, has been proposed as a model of high-temperature superconductivity. The behaviour of the theory with increasing temperature $T$ and number of fermion flavours $N_f$ was recently studied numerically in the Schwinger-Dyson formalism in a bare-vertex approximation. This approximation corresponds to a particular truncation of the Schwinger-Dyson hierarchy, and its effects, along with wave-function renormalization effects, could be important, especially in connection with the behaviour of the theory when the number of fermion flavours is varied. This has led to several interesting studies at zero-temperature trying to include these effects, in order to solve the controversy on whether there is a critical number of fermion flavours beyond which there is no mass generation. At finite temperature, similar studies have so far used several approximations that we will relax in this study.

The first purpose of this work is to provide an accurate value for the ratio $r = 2M(0, 0)/kB T_c$, where $M(0, 0)$ is the fermion mass function at zero momentum and energy and $T_c$ is the critical temperature beyond which there is no mass generation. Apart from being theoretically very interesting, this quantity is of direct physical relevance since it can be compared to corresponding values measured in certain high-temperature superconductors. The second purpose is to provide a reliable phase diagram of the theory with respect to $T$ and $N_f$. We continue to consider momentum- and energy-dependent self-energies, but we include wave-function renormalization effects and a non-bare vertex.

This complicates the study considerably and constitutes a non-trivial step forward, since instead of having only one gap equation, a system of two integral equations has to be solved. We continue to neglect the imaginary parts of the photon polarization functions and of the fermion self-energy for simplicity, as other studies have done so far. Their inclusion, even though in principle necessary, would double the number of coupled equations to be solved, which would render the numerical study too complicated for our algorithm and the computer power at hand.

II. THE EQUATIONS

We will employ the Schwinger-Dyson formalism in order to gain information on the momentum-dependent fermion self-energy and the non-perturbative physics behind it. The Schwinger-Dyson equation for the fermion self-energy is given by

$$S^{-1}(p) = S_0^{-1}(p) - e^2 \int \frac{d^3k}{(2\pi)^3} \gamma^{\mu} S(q) \Delta_{\mu\nu}(k) \Gamma^\nu(p, q)$$

(1)

where $q = p - k$, $e$ is the dimensionful gauge coupling of the theory which we will take to be constant throughout this study, $\Delta_{\mu\nu}$ is the photon propagator with $\mu, \nu = 0, 1, 2$, $\Gamma^\nu$ is the full photon-fermion vertex, $\gamma^{\mu}$ is a four-dimensional representation of the $\gamma$-matrices, $S_0$ is the bare fermion propagator, and the finite-temperature fermion propagator in the real-time formalism is given by

$$S(p) = \left( (1 + A(p)) \hat{p} + \Sigma(p) \right) \times \frac{1}{\left( (1 + A(p))^2 p^2 + \Sigma^2(p) \right) - \frac{2\pi\delta((1 + A(p))^2 p^2 + \Sigma^2(p))}{e^{\beta|p|} + 1}},$$

(2)

where $\beta = 1/k_B T$, $A(p)$ is the wave-function renormalization function, $\delta$ is the usual Dirac function and we have made a rotation to Euclidean space. Note that we avoid the matrix form that the propagator has in this formalism, since the Schwinger-Dyson equation that we have written down involves only a one-loop diagram directly, so complications due to the field-doubling problem do not arise. However, it has to be noted that a more careful treatment involving the imaginary parts of the self-energies would involve the full matrix propagators
The resulting equations would in that case be unfortunately too complicated to be solved even numerically, without serious truncations. Moreover, due to the broken Lorentz invariance at finite temperature, the wave-function renormalization could in principle affect differently the $p_0$ and $|p|$ propagator components, i.e. we should replace $(1 + A(p))\gamma_i$ by $((1 + A(p))^0 + a)p_0 + (1 + B(p))\gamma_i p_i$ with $i = 1, 2$. For simplicity we will restrict ourselves to situations where $a = 0$, which correspond to a zero chemical potential, and we will work in the approximation where $A(p) = B(p)$ also for non-zero temperatures as done in similar studies.

For the vertex $\Gamma^{\mu}(p, q) = (1 + A(q))\gamma^\mu$, where $A(q)$ is the same wave-function renormalization function appearing in the fermion propagator. It is unfortunately not symmetric in the vertex momenta, but its use simplifies the numerical algorithm considerably. Even though this vertex does not satisfy a priori the Ward-Takahashi identities, it is expected to incorporate the basic qualitative features of a non-perturbative vertex at zero temperature when used in a Schwinger-Dyson context. It has also been used in a finite-temperature case, supported by the qualitative agreement of the results of this ansatz with the ones obtained by more elaborate treatments. Furthermore, in the problem of the present study it gives results similar to a symmetrized vertex.

Moreover, the photon propagator in the Landau gauge is given by

$$\Delta_{\mu\nu}(k) = \frac{Q_{\mu\nu}}{k^2 + \Pi_L(k)} + \frac{P_{\mu\nu}}{k^2 + \Pi_T(k)}$$

(3)

where

$$Q_{\mu\nu} = \left( \delta_{\mu0} - k_\mu k_0/k^2 \right) k^2 \left( \delta_{\nu0} - k_\nu k_0/k^2 \right)$$

$$P_{\mu\nu} = \delta_{\mu i} \delta_{\nu j} A_{ij}$$

(4)

with $i, j = 1, 2$, and where we neglect its temperature-dependent delta-function part since it is expected to give a vanishingly small contribution. The longitudinal and transverse photon polarization functions $\Pi_L$ and $\Pi_T$ are given explicitly in and taken from, where they are calculated in a massless-fermion approximation and where wave-function renormalization and vertex effects cancel only for specific vertex choices. One should in principle couple the expressions for $\Pi_L, T$ in our system of integral equations in a self-consistent manner, but unfortunately this would render our numerical algorithm too complicated.

Identifying the parts of this equation with the same spinor structure, we reduce the problem to that of a system of two three-dimensional integral equations involving two functions varying independently with $p_0$ and $|p|$. The equations take the following form:

$$M(p_0, |p|) = \frac{\alpha}{N_f(1 + A(p_0, |p|))} \int \frac{dk_0|k|d|k|d\theta}{(2\pi)^3} \times$$

$$\times \frac{M(q_0, |q|)}{q^2 + M^2(q_0, |q|)} \sum_{p = L, T} \frac{1}{k^2 + \Pi_F(k_0, |k|)}$$

$$- \frac{\alpha}{N_f(1 + A(p_0, |p|))} \int \frac{dk_0|k|d|k|d\theta}{(2\pi)^3} \frac{M(E, |q|)}{2E(e^{\beta E} + 1)} \times$$

$$\sum_{\epsilon = 1, -1} \sum_{p = L, T} \frac{1}{(p_0 - \epsilon E)^2 + k^2 + \Pi_F(p_0 - \epsilon E, |k|)}$$

$$A(p_0, |p|) = \frac{\alpha}{N_f p^2} \int \frac{dk_0|k|d|k|d\theta}{(2\pi)^3} \frac{1}{q^2 + M^2(q_0, |q|)} \times$$

$$\left( \frac{Q(p_0, p, k_0, k)}{k^2 + \Pi_L(k_0, |k|)} + \frac{P(p_0, p, k_0, k)}{k^2 + \Pi_T(k_0, |k|)} \right)$$

$$- \frac{\alpha}{N_f p^2} \int \frac{|k|d|k|d\theta}{(2\pi)^3} \frac{1}{(2\pi)^3} \frac{1}{2E(e^{\beta E} + 1)} \times$$

$$\sum_{\epsilon = 1, -1} \left( \frac{Q(p_0, p, p_0 - \epsilon E, k)}{(p_0 - \epsilon E)^2 + k^2 + \Pi_L(p_0 - \epsilon E, k)} + \frac{P(p_0, p, p_0 - \epsilon E, k)}{(p_0 - \epsilon E)^2 + k^2 + \Pi_T(p_0 - \epsilon E, k)} \right),$$

(5)

where $\alpha = e^2 N_f$, and it is more convenient to work with the mass function $M(p_0, |p|) = \Sigma(p_0, |p|)/(1 + A(p_0, |p|))$. We also sum over the photon polarizations $P = L, T$ and over the two roots of the delta function by introducing $\epsilon = 1, -1$. The quantity $E$ is approximated by the relation $E^2 \approx |q|^2 + M^2(0, 0)$, where use of the delta-function property $\delta(ax) = \delta(x)/|a|$ has been made. Furthermore, the functions $Q$ and $P$ are given by

$$Q(p_0, p, k_0, k) = 2 \left( p_0 \left( \frac{p k_0}{k^2} \right) \right) \left( \frac{k^2}{k^2} \right) \left( \frac{q_0 - (q k) k_0}{k^2} \right)$$

$$P(p_0, p, k_0, k) = 2 \left( p q - \left( \frac{p k (q k)}{k^2} \right) \right) - pq.$$
physical quantities of interest do not vary substantially with our ultra-violet (UV) cut-off, since at the effective UV cut-off $\alpha$ of the theory they are already negligibly small [16]. The solution at $T = 0$ and $N_f = 2$ for the functions $\Sigma(p_0, |p|)$ and $-A(p_0, |p|)$ is given in Figs. 1 and 2 respectively. The general variation of these functions with momentum does not change with increasing temperature or varying of $N_f$, even though the overall scale of $\Sigma$ drops fast with $N_f$. The function $\Sigma(p_0, |p|)$ falls as expected with increasing momentum, and is of the same form as the mass function $M(p_0, |p|)$. The function $A(p_0, |p|)$ is always in the range between -1 and 0 as required [18], it is tending to zero for increasing momenta, and it is of the same form and magnitude as the approximate form used in [12].

For a given number of fermion flavours $N_f$, when the temperature exceeds some critical value there is no solution for the fermion mass function but the trivial one.

![Figure 1](image1.png)

**FIG. 1.** The fermion self-energy $\Sigma(p_0, |p|)$ at zero temperature and for $N_f = 2$, $\Lambda_{UV}/\alpha = 0.1$, as a function of energy and momentum in logarithmic scale. All quantities are scaled by $\alpha$.

![Figure 2](image2.png)

**FIG. 2.** The opposite of the wave-function renormalization $-A(p_0, |p|)$ at zero temperature and for $N_f = 2$, $\Lambda_{UV}/\alpha = 0.1$.

The value of the ratio $r$ is concentrated approximately around $r \approx 10.6$, which is comparable to the values obtained in [3] which neglected $A(p_0, |p|)$. The ratio $r$ found is also comparable to typical values obtained in Ref. [12], which includes wave-function renormalization effects but uses several approximations which we were able to by-pass in this study. We confirm therefore that adding these effects does not influence the behaviour of the theory in a significant way. We have to note moreover that our $r$ values are somewhat larger than the value $r \approx 8$ measured for some high-temperature superconductors [19]. However, we could be overestimating this ratio because of a possibly poor convergence of the algorithm for temperatures close to the critical one.

At zero-temperature, this theory is known to exhibit also an interesting behaviour with the number of fermions $N_f$. In Fig. 3 we plot the zero-momentum and zero-temperature fermion mass function with respect to $N_f$. We fit the data with the exponential curve $e^{-1.48N_f}/13.25$. At $N_f \approx 3.35$, the mass function is still roughly four

![Figure 3](image3.png)

**FIG. 3.** The fermion mass function at zero momentum and zero temperature, scaled by $\alpha$ and on a logarithmic scale, with respect to $N_f$ for a ratio $\Lambda_{UV}/\alpha = 0.1$. We fit our results with the curve $M(0,0)/\alpha = e^{-1.48N_f}/13.25$. Values of $N_f$ larger than 3.35 are not considered, because then the self-energy falls below the IR-cut-off.

![Figure 4](image4.png)

**FIG. 4.** The phase diagram of the theory with respect to the temperature scaled by $\alpha$ and in logarithmic scale and the number of fermion flavours, for a ratio $\Lambda_{UV}/\alpha = 0.1$. We fit the data points with the curve $k_B T/\alpha = e^{-1.48N_f}/70$. This curve should not be extrapolated for $N_f \gtrsim 3.35$.
times larger than the cut-off. When \( N_f \gtrsim 3.35 \), our algorithm does not converge and the mass function tends fast below the IR cut-off. This behaviour could indicate that \( N_f \approx 3.35 \) is some critical point beyond which dynamical mass generation is impossible.

The value of \( N_f \) we find is remarkably close to the one quoted in the numerical study of \cite{3}, and it is also close to our previous result \cite{4}. A similar study \cite{5}, which includes a calculation of the fermion-field anomalous dimension to second-order in \( 1/N_f \), predicts a critical value \( N_f \approx 3.28 \), which is only slightly larger than the theoretical prediction neglecting wave-function renormalization which gives \( N_f = \frac{22}{3} \approx 3.24 \), and quite close to the value we find numerically.

In Fig. 4 we plot the phase diagram of the theory with respect to \( N_f \) and \( k_BT \). It separates two regions of the parameter space which either allow or do not allow dynamical mass generation. The choice of an exponential fitting curve was only made to describe “phenomenologically” the general tendency of the data and to provide a measure for a \( r \)-ratio independently of \( N_f \), and is reminiscent of the results in Ref. \cite{1} but with a somewhat steeper slope. However, there are also studies that predict a non-analytic behaviour of \( \Sigma \) for \( N_f \) near its critical value \cite{6}. Lack of convergence of the algorithm and fall of the mass function below the IR cut-off however do not allow us to test the precise behaviour of the theory near \( N_f \approx 3.35 \), and it is also clear that the fitting curves should not be extrapolated for \( N_f \) larger than this value.

\[ \text{IV. DISCUSSION} \]

We were able to solve a system of two coupled integral equations for the fermion mass function and wave-function renormalization for a finite-temperature version of three-dimensional \textit{QED}, by applying a numerical relaxation technique. One main result is a \( r \)-ratio of about 10.6, which is close to previous numerical studies, confirming that including wave-function renormalization, in conjunction with a particular non-bare fermion-photon vertex, does not affect the theory in a significant way. The other important result is the existence of a possibly critical fermion flavour number of roughly 3.35, which is also consistent with some theoretical expectations and other numerical results.

It is the first time that such a study includes the momentum and energy dependence of the fermion and photon self-energies and of wave-function renormalization in a gap equation with a non-bare vertex, and this allows a more reliable description of the behaviour of the theory. We estimate the numerical uncertainty for the values quoted, which comes mainly from the convergence criteria imposed, at about \( \pm 10\% \). The next step for future studies should be the inclusion of the imaginary parts of the self-energies in the equations and the relaxing of the approximation \( A(p) = B(p) \), which could in principle influence the results of this investigation.

\section*{Acknowledgments}

I thank N. Mavromatos, P. Henning and A. Smilga for useful discussions. This research is supported by an Alexander von Humboldt Fellowship.

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