Fast quantum subroutines for the simplex method

Giacomo Nannicini*

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Abstract

We propose quantum subroutines for the simplex method that avoid classical computation of the basis inverse. We show how to quantize all steps of the simplex algorithm, including checking optimality, unbound edness, and identifying a pivot (i.e., pricing the columns and performing the ratio test) according to Dantzig’s rule or the steepest edge rule. The quantized subroutines obtain a polynomial speedup in the dimension of the problem, but have worse dependence on other numerical parameters. For example, for a problem with $m$ constraints, $n$ variables, at most $d$ nonzero elements per column of the constraint matrix, at most $d$ nonzero elements per column or row of the basis, condition basis number $\kappa$, and optimality tolerance $\epsilon$, pricing can be performed in $O(\kappa n + dm \log m)$ time, and pricing requires $O(\kappa^7 n^1 + m^{2+o(1)} + d m)$ time in the worst case using the fastest known algorithm for sparse matrix multiplication. For well-conditioned sparse problems the quantum subroutines scale better in $m$ and $n$, and may therefore have an advantage for very large problems. The running time of the quantum subroutines can be improved if the constraint matrix admits an efficient algorithmic description, or if quantum RAM is available.

1 Introduction

The simplex method is one of the most impactful algorithms of the past century; to this day, it is widely used in a variety of applications. This paper studies some opportunities for quantum computers to accelerate the simplex method. An extended abstract of this work appeared in the proceedings of IPCO 2021 (Nannicini 2021).

The use of quantum computers for optimization is a central research question that has attracted significant attention in recent years. It is known that a quadratic speedup for unstructured search problems can be obtained using Grover’s algorithm (Grover 1996). Thanks to exponential speedups in the solution of linear systems (Harrow et al. 2009, Childs et al. 2017), it seems natural to try to translate those speedups into faster optimization algorithms, since linear systems appear as a building block in many optimization procedures. However, few results in this direction are known. A possible reason for the paucity of results is the difficulty encountered when applying a quantum algorithm to a problem whose data is classically described, and a classical description of the solution is required. We provide a simple example to illustrate these difficulties.

Suppose we want to solve the system $Ax = b$, where $A$ is an $m \times m$ invertible matrix with at most $d$ nonzero elements per column or row. Using the quantum linear systems algorithm of Childs et al. (2017), the gate complexity of this operation is $O(\kappa n + \max\{T_A, T_b\})$, where $T_A, T_b$ indicate the gate complexity necessary to describe $A, b$ in a certain input model, and $\kappa$ is the condition number of $A$. We remark that here and in the rest of this paper, we measure the running time for quantum subroutines as the number of basic gates (i.e., gate complexity), as is usual in the literature. Notice that $m$ does not appear in the running time, as the dependence is polylogarithmic. However, we need $O(dm)$ gates to implement $T_A$ for sparse $A$ in the gate model, and $O(m)$ gates are necessary to implement $T_b$. This is natural for an accurate representation of the input data, since $A$ has $O(dm)$ nonzero elements and $b$ has $O(m)$ nonzero elements. If we also want to extract the solution $x = A^{-1}b$ with precision $\delta$, using the (optimal) tomography algorithm of van Apeldoorn et al. (2022) we end up with running time $O(\kappa^2d^2m^2)$. This is slower than the time taken to classically compute an LU decomposition of $A$, which is $O(d^7m^{1.9} + m^{2+o(1)})$ (Yuster and Zwick 2003). Thus, naive application of quantum linear system algorithms (QLSAs) does not give any advantage.

Despite the aforementioned difficulties, a few fast quantum optimization algorithms exist — not necessarily based on QLSAs; here we briefly discuss some representative examples, and defer a more detailed comparison.

*IBM Quantum, IBM T.J. Watson research center, Yorktown Heights, NY 10598, nannicini@us.ibm.com
with our work to Sect. 2. Brandao and Svore (2017), van Apeldoorn et al. (2017) give polynomial speedups for the solution of semidefinite programs and therefore also linear programs (LPs). These two papers give a quantum version of (Arora and Kale 2016): while the algorithm is essentially the same as its classical counterpart, the basic subroutines admit faster quantum algorithms. The running time for LPs is $O\left(\sqrt{mn} \left(\frac{\kappa}{\epsilon}\right)^{9}\right)$, where $R, r$ are bounds on the size of the optimal primal/dual solution, and $\epsilon$ an optimality tolerance: this is faster than any known classical algorithm when $mn \gg \frac{\kappa}{\epsilon}$ — although as van Apeldoorn et al. (2017) note, many natural SDP formulations do not satisfy this requirement and in fact, $r$ and $R$ may depend on $n$. To achieve this speedup, Brandao and Svore (2017), van Apeldoorn et al. (2017) assume that there exists an efficient quantum subroutine to describe $A$ (i.e., in polylogarithmic time), and output only a dual solution — the primal solution is encoded in a quantum state. If we insist on classical input and output for the optimization problem, the overall running time increases significantly. Kerenidis and Prakash (2018), Casares and Martin-Delgado (2020) also give polynomial speedups for LPs, using different variants of an interior point algorithm. Specifically, Kerenidis and Prakash (2018) give a running time of $O\left(\frac{\kappa}{\epsilon}^{3}\right)$, where $\delta$ is a feasibility tolerance (with a non-standard definition of feasibility), and Casares and Martin-Delgado (2020) give a running time of $O\left(\frac{\kappa}{\epsilon}K\sqrt{n} \left(\frac{\kappa}{\epsilon}\right)^{3}\right)$, where $\|M\|_F$ is an upper bound on the Frobenius norm of all intermediate matrices appearing during the optimization. Both papers follow the classical algorithm, but accelerate the basic subroutines performed at each iteration. To achieve this speedup, these papers rely on QRAM, a form of quantum storage. QRAM assumes that classical data can be accessed in superposition, and this allows data preparation subroutines that are exponentially faster than their equivalent implementation under the standard gate model, in general. Assuming QRAM, the algorithms of Kerenidis and Prakash (2018), Casares and Martin-Delgado (2020) have classical input and output. Similar considerations apply to the work of van Apeldoorn and Gilyén (2019a), whose quantum algorithm for LPs is based on computing the Nash equilibrium of a two-player zero-sum game.

Summarizing, there are few known examples of faster quantum optimization algorithms, and all of them have strong assumptions on the availability of efficient data preparation or data readout subroutines. In particular, the quantum optimization algorithms of Brandao and Svore (2017), van Apeldoorn et al. (2017), Kerenidis and Prakash (2018), Casares and Martin-Delgado (2020), van Apeldoorn and Gilyén (2019a) have one of these two assumptions: (i) that having quantum input/output is acceptable, ignoring the cost of a classical translation, or (ii) that QRAM, a form of quantum storage whose physical realizability is still the subject of debate, is available. Both assumptions have the merit of leading to interesting algorithmic developments, but it is still an open question to find practical situations in which they are satisfied, particularly in the context of traditional optimization applications. We remark that the assumptions can be dropped and the algorithms can be implemented in the standard gate model, but the running time increases. In this paper we propose quantum subroutines that may yield asymptotic speedups even without these two assumptions. A more detailed analysis of the existing literature, together with a comparison with our results, is given in Sect. 2.

**Our results.** For brevity, from now on we assume that the reader is familiar with standard linear optimization terminology; we refer to Bertsimas and Tsitsiklis (1997) for a comprehensive treatment of LPs. The simplex method aims to solve $\min \ c^\top x$, s.t.: $Ax = b, x \geq 0$, where $A \in \mathbb{R}^{m \times n}$ with at most $d$ nonzero elements per column. It keeps a basis, i.e., a set of linearly independent columns of $A$, and repeatedly moves to a different basis that defines a solution with better objective function value. As is common in the literature, we use the term “basis” to refer to both the set of columns, and the corresponding submatrix of $A$, depending on context. We denote by $B$ the set of basic columns, $N$ the set of nonbasic columns, with corresponding submatrices $A_B, A_N$. The maximum number of nonzero elements in any column or row of the basis submatrix is denoted $d$. The basis change (called a pivot) is performed by determining a new column that should enter the basis, and removing one column from the current basis. The operations performed to determine a pivot also identify if the current basis is optimal, or if the problem is unbounded from below. Assessing which columns can enter the basis is called pricing, and it is asymptotically the most expensive step: it requires computing the basis inverse and looping over all the columns in the worst case, for a total of $O\left(d^2 \cdot n^{1.9} + m^{2+o(1)} + d_1 n\right)$ operations using the matrix multiplication algorithm of Yuster and Zwick (2003). In practical implementations, the basis inverse is computed from scratch every few iterations (this is often a user parameter); in other iterations, inexpensive updates are used instead, see e.g., Chvátal (1983), with a more favorable worst-case running time of $O\left(m^2 + d_1 n\right)$. With the steepest edge pivoting rule, that can achieve better performance on real-world problems by reducing the number of required iterations (Forrest and Goldfarb 1992), the term $d_1 n$ in the two running time expressions above increases to $m^2 n$. (In practical implementations of the simplex method, steepest edge is typically used only for dual simplex algorithms, due to its high computational cost; an approximate variant, with more efficient updates, is used in primal simplex instead.)
In the following, we denote by $T_{LS}(L, R, \epsilon)$ the running time of a QLSA on the linear system $Lx = r$ with precision $\epsilon$, where $r$ is a column of $R$ specified by some index. Note that this accounts for the cost of preparing a quantum state encoding any column of $R$, which is important if we want the ability to solve $Lx = r$ for all columns $r \in R$ in superposition. We denote $c_{\text{max}} := \max_j c_j$ and $q := \max(\|b\|, \max_j(\|A_j\|))$.

We show that we can apply Grover search to choose an entering column, so that the running time scales as $O(\sqrt{n})$ rather than $O(n)$ for looping over all the columns. To apply Grover search we need a quantum oracle that determines if a column is eligible to enter the basis, i.e., if it has negative reduced cost. We propose a construction for this oracle using a QLSA, several gadgets to make state amplitudes interfere in a certain way, and amplitude estimation \cite{Brassard2002}. The construction avoids classical computation of the basis inverse. The overall running time of the oracle is $O(\sqrt[3]{\delta} T_{LS}(A_B, A_N, \frac{\epsilon}{\eta \kappa})$, where $\epsilon$ is the precision for the reduced costs (i.e., the optimality tolerance). Using the QLSA of \cite{Childs2017}, in the circuit model and without taking advantage of the structure of $A$ besides sparsity, this gives a total running time of $O(\sqrt[3]{\delta} d \sqrt{m(dn + dm)})$. If the ratio $n/m$ is large, we can find a better tradeoff between the Grover speedup and the data preparation subroutines, and improve the running time of the quantum pricing algorithm to $O(\sqrt[3]{\delta} d^{3/5} \sqrt{c_n} \sqrt{m})$. We can also apply the steepest edge pivoting rule increasing the running time by a factor $c_{\text{max}}$. We summarize this below.

**Theorem 1** There exist quantum subroutines to identify if a basis is optimal, or determine a column with negative reduced cost, with running time $\tilde{O}(\frac{1}{\sqrt{n}} T_{LS}(A_B, A_N, \frac{\epsilon}{\eta \kappa}))$. In the gate model without QRAM, this is $\tilde{O}(\frac{1}{\sqrt{n}} d \sqrt{m(dn + dm)})$, which can be reduced to $\tilde{O}(\frac{1}{\sqrt{n}} d^{3/5} \sqrt{c_n} \sqrt{m})$ if the ratio $n/m$ is larger than $2\frac{\epsilon}{\eta \kappa}$. If QRAM to store $A$ is available, the running time is $\tilde{O}(\frac{1}{\sqrt{n}} \sqrt{k} \sqrt{m})$. With the steepest edge pivoting rule, the running time of the subroutine to determine a column entering the basis increases by a factor $c_{\text{max}}$.

The regime with large $n/m$ is interesting because it includes many natural LP formulations; e.g., the LP relaxations of cutting stock problems, vehicle routing problems, or any other formulation that is generally solved by column generation \cite{Lubbecke2002}. The optimality tolerance for the quantum pricing subroutine is slightly different from the classical algorithm: we need a tolerance relative to some norm for efficiency. For steepest edge, the increased running time reflects the precision necessary to run an approximate quantum minimum finding algorithm; these details are discussed subsequently in the paper. Note that the running time of the quantum subroutines depends explicitly on the condition number of the basis and the precision of reduced costs $\epsilon$ is fixed, while classically $\kappa$ is not explicit when using Gaussian elimination, but the $\epsilon$ obtained would depend on it (because the basis inverse could be inaccurate). Since the algorithm may fail if the condition number grows too large (classically, because the calculations become too imprecise and we can no longer accurately check optimality or feasibility; quantumly, because the running time increases too much), we quantize a heuristic rule commonly used in classical implementations of the simplex method to reduce “bad pivots”, i.e., basis changes that deteriorate the condition number. If $A$ is structured, the quantum running time can decrease significantly: for example, if $A$ differs from the assignment problem constraint matrix only for a polylogarithmic number of elements, then its description in the sparse oracle access model used in this paper requires time $O(1)$, rather than $O(d,n)$. Our running time analysis assumes that the matrix is sparse but the sparsity pattern is unstructured. An important remark is that all quantum subroutines succeed with some constant probability, that can be boosted to at least $1 - \gamma$ with $O(\log \frac{1}{\delta})$ repetitions; in our informal theorem statements we omit this aspect for simplicity.

If pricing is performed via our quantum subroutine, we obtain the index of a column that has negative reduced cost with arbitrarily high probability. To determine which column should leave the basis, we have to perform the ratio test; scanning the rows during the ratio test also allows us to detect if the problem is unbounded. Using techniques similar to those used for the pricing step, we can identify the column that leaves the basis in time $O(\frac{1}{\sqrt{n}} \kappa \kappa^2 \sqrt{d} \sqrt{m^{1.5}})$, where $\delta$ is the precision parameter of this step and quantifies the maximum infeasibility after the pivot. Classically, the ratio test requires time $O(m^2)$ in the worst case, because the basis inverse would be dense even if the basis is sparse. We summarize this result below.

**Theorem 2** There exists a quantum subroutine to identify if a nonbasic column proves unboundedness of the LP in time $\tilde{O}(\frac{1}{\sqrt{n}} T_{LS}(A_B, A_N, \delta))$. There also exists a quantum subroutine to perform the ratio test in time $\tilde{O}(\frac{1}{\sqrt{n}} T_{LS}(A_B, A_N, \delta))$, where $\delta$ is the maximum infeasibility of the basic solution after pivoting. In the gate model without QRAM, the running times are respectively $\tilde{O}(\frac{1}{\sqrt{n}} \kappa \kappa^2 \sqrt{d} \sqrt{m^{1.5}})$ and $\tilde{O}(\frac{1}{\sqrt{n}} \kappa \kappa^2 \sqrt{d} \sqrt{m^{1.5}})$. If QRAM to store $A$ and $b$ is available, the running times are respectively $\tilde{O}(\frac{1}{\sqrt{n}} \kappa \kappa^2 m)$ and $\tilde{O}(\frac{1}{\sqrt{n}} \kappa \kappa^2 m)$.

The factor $\kappa$ in the expressions above comes from requiring an absolute feasibility tolerance $\delta$; if precision relative to some column norms is sufficient, the factor disappears. The exact statement of these theorems is given subsequently in the paper.

It is known that for most practical LPs the maximum number of nonzeros in a column is essentially constant; for example, on the entire benchmark set MIPLIB2010, less than 1% of the columns have more
than 200 nonzeroes (and less than 5% have more than 50 nonzeroes). Similarly, the number of nonzeroes per row of the basis is small: on MIPLIB2010, looking at the optimal bases of the LP relaxations, less than 0.01% of the rows have more than 50 nonzeroes. As \(m, n\) increase, so typically does the sparsity. For example, the largest problem available in the benchmark set MIPLIB2017 has \(m \approx 7.1 \times 10^6, n \approx 3.9 \times 10^7\), and 99.999% of the columns have less than 30 nonzero elements; for the second largest problem, which has \(m \approx 2.0 \times 10^7, n \approx 2.1 \times 10^7, 99.998\%\) of the columns have this property. Hence, although LPs may have “global” constraints leading to dense columns or rows, we expect many bases arising from real-world LPs to be extremely sparse, and it is interesting to look at the scaling of the running time under the assumption that the sparsity parameters are at most polylogarithmic in \(m\) and \(n\). In this case, the running time of the oracle for the reduced costs in the gate model without QRAM is \(\tilde{O}(\sqrt{mn})\), giving a total running time to choose an entering column of \(\tilde{O}(\sqrt{mn(n+m)})\), and the steepest edge pivoting rule is a factor \(c_{\text{max}}\) slower. Hence, for a well-conditioned basis and under the assumption (often verified in practice) that \(d = \tilde{O}(\log mn)\), we obtain running time \(\tilde{O}(\sqrt{mn(n+m)})\) for the quantum pricing subroutine, which can be reduced to \(\tilde{O}(\sqrt{m})\) if the ratio \(n/m\) is large; and running time \(\tilde{O}(n^{1.5})\) for the quantum ratio test subroutine. With QRAM, the gate complexity decreases further, and the proposed algorithms achieve essentially linear scaling in \(m\) and \(n\): \(\tilde{O}(\sqrt{mn})\) and \(\tilde{O}(\sqrt{m})\), respectively.

Summarizing, the quantum subroutines that we propose can be asymptotically faster than the best known classical version of them, under the assumption that the LPs are extremely sparse — an assumption that is generally verified in practice — and the bases are well-conditioned. In addition to the classical input parameters \((m, n, \text{and the sparsity parameters})\), the gate complexity of the quantum subroutines depends on some numerical parameters: optimality and feasibility tolerances, maximum norm \(\eta\) of a column of \(A\) or \(b\), and the maximum cost coefficient (for steepest edge). In practice the tolerances are typically chosen independently of \(m\) and \(n\), and they can be treated as constants, while the parameter \(\eta\) depends on the sparsity. For well-conditioned problems, the quantum subroutines have better scaling in \(m\) and \(n\), and this could turn into an asymptotic advantage. To achieve this potential advantage, we never explicitly compute the basis inverse, and rely on the quantum computer to indicate which columns should enter and leave the basis at each iteration. Similar to other papers in the quantum optimization literature, we use a classical algorithm (the simplex method) and accelerate the subroutines executed at each iteration of the simplex. However, our asymptotic speedup (when the condition number and some other numerical parameters are small) does not depend on the availability of QRAM or of the data in “quantum form”. The key insight to obtain an asymptotic speedup even with classical input and output is to interpret the simplex method as a collection of subroutines that output only the changes in the basis, avoiding the cost of performing full quantum state tomography (i.e., obtaining a classical description of the quantum state). The main disadvantage of our method is arguably the inverse dependence on the precision parameters \(\epsilon\) and \(\delta\); as discussed in the next section, existing quantum optimization algorithms methods in the literature have worse dependence on the precision parameters (although the precision parameters have a different meaning and are not directly comparable). However, for very large problems, the quantum subroutine may be able to perform low-precision pivots faster than a classical algorithm.

Our algorithms require a fault-tolerant quantum computer, therefore they are not suitable for implementation on the noisy intermediate-scale quantum computers currently available. Some recent studies (Sanders et al. 2020, Babbush et al. 2021) estimate the running time of a few quantum algorithms (including optimization algorithms) and compare it with their quadratically slower classical counterparts. These studies conclude that with the current state of quantum technologies, a small polynomial speedup of the quantum algorithm over classical is entirely offset by the cost of performing error correction. The algorithms discussed in our paper likely suffer from the same weakness, and might be numerically advantageous only with significant progress in fault-tolerant quantum computation.

We remark that even with sophisticated pivoting rules, the number of iterations of the classical simplex method could be exponential. The quantum version proposed in this paper does not circumvent this issue, and its worst-case running time is slower than that of the fastest quantum algorithm for LPs (van Apeldoorn and Gilyén 2019). It is well established that in practice the simplex method performs much better than the worst case, in terms of the number of iterations (Spielman and Teng 2004, Dadush and Huiberts 2018), as well as the complexity of a single iteration, see e.g. Chvátal (1983). The most attractive feature of the simplex method is its excellent practical performance, and we hope that a quantized version, closely mimicking the classical counterpart while accelerating the linear algebra carried out at each iteration, would inherit this trait.

The rest of this paper is organized as follows. We start with a more detailed review of existing literature in Sect. 2 Sect. 3 contains a brief summary of the simplex method to establish terminology. In Sect. 4 we define our notation, describe useful results from the literature, and give an overview of our quantum subroutines. Sect. 5 gives a detailed explanation of each step in the gate model without QRAM, while Sect. 6 explains how
to modify the algorithm to improve the running time if we assume that QRAM is available. Sect. 7 concludes the paper.

2 Comparison with the existing literature

The simplex method has been extensively studied in the operations research and computer science literature. It was known since the 70s that depending on the pivoting rule, the algorithm may take exponential time on certain inputs ([Klee and Minty 1972]), which is in contrast with its excellent practical performance. Attempts at explaining this behavior led to the idea of “smoothed analysis” ([Spielman and Teng 2004]), see the recent work of Dadush and Hübner ([2018]) for an overview. This paper proposes a quantization of the simplex method with Dantzig’s rule, the steepest edge pivoting rule, or with a randomized rule that chooses uniformly at random among the possible pivots. It is known that in expectation, the “random edge” pivot rule may not work at explaining this behavior led to the idea of “smoothed analysis” (Spielman and Teng 2004), see the recent work of Dadush and Hübner (2018) for an overview. This paper proposes a quantization of the simplex method with Dantzig’s rule, the steepest edge pivoting rule, or with a randomized rule that chooses uniformly at random among the possible pivots. It is known that in expectation, the “random edge” pivot rule may not

| Algorithm                  | Iteration cost                       | # Iterations | QRAM size          | Comments                                                                 |
|----------------------------|--------------------------------------|--------------|--------------------|--------------------------------------------------------------------------|
| Multiplicative weights update (van Apeldoorn and Gilyén 2019b) | $O \left( \left( \sqrt{d} + \sqrt{\frac{n \log n}{\epsilon}} \right) d \left( \frac{R}{\epsilon} \right)^2 \right)$ | $O(\frac{R \log n}{\epsilon^2})$ | $O \left( \left( \frac{R}{\epsilon} \right)^2 \right)$ | Outputs dual solution and (quantum) primal solution; $R, r$ could depend on $n, m$ |
| Interior point (Augustino et al. 2021) | $O(\kappa^2 \frac{n^2}{\epsilon})$ | $O(\sqrt{n} \log \frac{1}{\epsilon})$ | $O(d, n + m)$ | $\kappa$ comes from intermediate matrices                                |
| Game-theoretical (van Apeldoorn and Gilyén 2019a) | $\tilde{O} \left( \sqrt{d} \left( \frac{R}{\epsilon} \right)^{1.5} \right)$ | $\tilde{O} \left( \left( \frac{R}{\epsilon} \right)^2 \right)$ | $\tilde{O} \left( \left( \frac{R}{\epsilon} \right)^2 \right)$ | $R, r$ could depend on $n, m$ |
| This paper                 | $O(\tilde{O}(\sqrt{dn} + dm))$, plus $O(\tilde{O}(\log n))$ | N/A (exp)    | No QRAM            | $\kappa$ comes from the current basis; outputs only basis information    |
| This paper                 | $O(\tilde{O}(\kappa^{\sqrt{m}n}))$, plus $O(\tilde{O}(\kappa^{2m}))$ | N/A (exp)    | $O(d, n + m)$     |                                                                            |
| Classical simplex          | $O(d, n^{1.9} + m^{2+o(1)} + d, n)$ | N/A (exp)    | No QRAM            |                                                                            |

On the quantum side, to the best of our knowledge all algorithms for LPs are derived from some classical algorithm. In the introduction we mentioned the papers ([Brandao and Svore 2017], [van Apeldoorn et al. 2017], [Kerendis and Prakash 2018], [Casares and Martin-Delgado 2020], [van Apeldoorn and Gilyén 2019a], [Brandao and Svore 2017], [van Apeldoorn et al. 2017]) are based on the multiplicative weights update method. ([Kerendis and Prakash 2018], [Casares and Martin-Delgado 2020]) are based on the interior point method. ([van Apeldoorn and Gilyén 2019a]) is based on a reduction of LPs to two-player zero-sum games and the classical algorithm of ([Grigoriadis and Khachiyan 1995]). For these methods the number of iterations is polynomial (or better), and is taken directly from the classical algorithm; the computational complexity of each iteration is reduced taking advantage of quantum subroutines. A faster quantum interior point algorithm is described in ([Augustino et al. 2021]), whereas a faster version of the quantum multiplicative weights update method for LPs is given in (van Apeldoorn and Gilyén 2019a).

We summarize key features of several papers in Table 1. For our algorithm, we report the running time of the most general (and therefore slowest) version; multiple opportunities for speedups are discussed throughout the paper. The table highlights the main advantages of our method, in particular the fact that the iteration running time is a polynomial with very low degree even without QRAM: for all other methods, each iteration is significantly more expensive, and even more so if we consider the gate complexity in case QRAM is not available (i.e, the gate complexity increases by a factor equal to the size of the QRAM, also reported in the
Table). For some methods, the steep dependence on $\epsilon$ could be a limiting factor. If QRAM is available, our subroutines have the very attractive feature of essentially linear dependence on $m$ or $n$ (i.e., $O(\sqrt{m}n)$ or in $O(m)$). On the other hand, the methods proposed in this paper suffer from the same weakness as the classical simplex method: giving a sub-exponential upper bound on the number of iterations is difficult (we mark the upper bounds as “$N/A$ (exponential)” in the table). However, this has not prevented the simplex method from being extremely efficient in practice. Furthermore, due to the dependence on $\kappa$, the proposed subroutines can be faster than classical only for well-conditioned problems.

As remarked in the Table 1, the papers using the multiplicative weights update framework as well as (van Apeldoorn and Gilyén 2019a) have a running time that depends on a parameter $\frac{m}{n}$, which may in turn depend on $n$, $m$ — see the discussion in van Apeldoorn et al. (2014), as well as the application to experimental design discussed in (van Apeldoorn and Gilyén 2019a) to understand the necessary tradeoffs to obtain a quantum speedup in $n$ and $m$. Note that $\epsilon$-optimality of the reduced cost, as used in the simplex method, is not a global optimality guarantee, and therefore our $\epsilon$ parameter is not directly comparable to the $\epsilon$ used in the other algorithms discussed in Table 1. We discuss this in more detail in Sect. 5.2. We also remark that in our paper, the running time depends on the condition number $\kappa$ of the basis at each iteration of the simplex method. It is known that $\kappa$ of the intermediate bases could be worse than that of the initial basis — this is true also for the classical simplex method. We discuss some steps to numerically keep the condition number under control in Sect. 5.7, similarly to existing implementations of the classical simplex. An analogous dependence on $\kappa$ can be found in (Kerenidis and Prakash 2018, Augustino et al. 2021), but in those papers it is exacerbated by the fact that $\kappa$ of the Newton linear system for interior point methods grows as we approach optimality, whereas for the simplex method, $\kappa$ comes from submatrices of the initial constraint matrix and does not necessarily grow large. Regarding input and output, the algorithm presented in this paper has fully classical input, and outputs the current basis at each iteration; to obtain the (primal) solution, it is necessary to classically solve a single linear system of size $m \times m$ (this is more efficient than obtaining a solution via a QLSA).

3 Overview of the simplex method

The simplex method solves the following linear optimization problem: $\min c^\top x$ s.t. $Ax = b$, $x \geq 0$, where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. A basis is a subset of $m$ linearly independent columns of $A$. Given a basis $B$, assume that it is an ordered set and let $B(j)$ be the $j$-th element of the set. The set $N := \{1, \ldots, n\} \backslash B$ is called the set of nonbasic variables. We denote by $A_B$ the square invertible submatrix of $A$ corresponding to columns in $B$, and $A_N$ the remaining submatrix. The term “basis” may refer to the set of column indices $B$ or to the submatrix $A_B$, depending on context. The simplex method can be described compactly as follows; see, e.g., Bertsimas and Tsitsiklis (1997) for a more detailed treatment.

- Start with any basic feasible solution. (This is w.l.o.g. because it is always possible to find one.) Let $B$ be the current basis, $N$ the nonbasic variables, $x = A_B^{-1}b$ the current solution.
- Repeat the following steps:
  1. Compute the reduced costs for the nonbasic variables $\bar{c}_N = c_N - c_B A_B^{-1} A_N$. This step is called the pricing. If $\bar{c}_N \geq 0$ the basis is optimal: the algorithm terminates. Otherwise, choose $k : \bar{c}_k < 0$.
  2. Compute $u = A_B^{-1} A_N$. If $u \leq 0$, the optimal cost is unbounded from below; the algorithm terminates.
  3. If some component of $u$ is positive, compute (this step is called the ratio test):

$\mathbf{r}^* := \min_{j = 1, \ldots, m} \frac{x_B(j)}{u_j}$

(1)

4. Let $\ell$ be such that $\mathbf{r}^* = \frac{x_B(\ell)}{u_\ell}$. Row $\ell$ is the pivot row. Form a new basis replacing $B(\ell)$ with $k$.

This step is called the pivot. Update $x = A_B^{-1} b$.

To perform the pricing step, the standard classical approach is to compute an LU factorization of the basis $A_B$; this requires time $O(d_B^2 m^{1.9} + m^{2+\omega(1)})$ using fast sparse matrix multiplication techniques (Yuster and Zwick 2002). (In practice, the traditional $O(m^3)$ Gaussian elimination is used instead, but the factorization is not computed from scratch at every iteration.) Then, we can compute the vector $c_B A_B^{-1}$ and finally perform the $O(n)$ calculations $c_k - c_B A_B^{-1} A_N$ for all $k \in N$; this requires an additional $O(d_B n)$ time, bringing the total time to $O(d_B^2 m^{1.9} + m^{2+\omega(1)} + d_B n)$. To perform the ratio test, we need the vector $u = A_B^{-1} A_N$, which takes time $O(m^2)$ assuming the LU factorization of $A_B$ is available from pricing. As remarked earlier, $A_B^{-1}$ or the LU factors may not be dense if $A_B$ is sparse; furthermore, the vectors $c_B A_B^{-1}$ and $A_B^{-1} b$ can be updated from
previous iterations exploiting the factorization update, so all these steps could take significantly less time in practice. Finally, since the calculations are performed with finite precision, we use an optimality tolerance $\epsilon$ and the optimality criterion becomes $\epsilon_k \geq -\epsilon$.

It is well known that the performance of the simplex method in practice depends on the pivoting rule. One of the simplest rules is Dantzig’s rule, which chooses $k = \arg \min_h \bar{c}_h$. Modern implementations of the simplex method typically rely on more sophisticated pivoting rules. Among these, the steepest edge pivoting rule has been shown to lead to a significant reduction in the number of iterations [Forrest and Goldfarb 1992], but the per-iteration cost increases. With steepest edge pivoting, the choice of the column entering the basis becomes:

$$k = \arg \min_h \frac{\bar{c}_h}{\| A_B^\top A_h \|}$$

This is more expensive to compute than Dantzig’s rule, as it requires knowledge of the norms $\| A_B^{-1} A_h \|$. Forrest and Goldfarb [1992] show how to update these norms in time $O(m^3 \nu)$, which asymptotically is essentially the same as recomputing them from scratch, provided the basis inverse is available (in practice, sparsity of the basis inverse may make the updates easier). We remark that the same paper introduces dual steepest edge pivoting rules, for which the norm updates cost $O(m^3)$ instead, but in this paper we focus our attention on the primal simplex.

4 Quantum implementation: overview

Before giving an overview of our methodology, we introduce some notation and useful results from the literature. The state of a quantum computer with $q$ qubits is a unit vector in $(\mathbb{C}^2)^\otimes q = \mathbb{C}^{2^q}$; we denote the standard basic vectors by $| j \rangle$, where $j \in \{0, 1\}^q$ (e.g., when $q = 2$, $|01\rangle$ denotes the standard basic vector $(0, 1, 0, 0)^\top$). Their conjugate transpose is $(j) = (\langle j \rangle)^\dagger$. A quantum state is therefore of the form $| \psi \rangle = \sum_{j \in \{0, 1\}^q} \alpha_j | j \rangle$, with $\sum_{j \in \{0, 1\}^q} | \alpha_j |^2 = 1$. The final state is obtained by applying a unitary matrix $U \in \mathbb{C}^{2^n \times 2^n}$ to the initial state $|0^n\rangle$, where $0^n$ denotes the $n$-digit all-zero binary string (i.e., the first standard basic vector). We assume that the reader is familiar with quantum computing notation; an introduction for non-specialists is given in [Nannicini 2017], and a comprehensive reference is [Nielsen and Chuang 2002]. Given a vector $v$, $\| v \|$ denotes its $\ell_2$-norm; given a matrix $A$, $\| A \|$ denotes the spectral norm, whereas $\| A \|_F$ is the Frobenius norm. Given two matrices $C, D$ (including vectors or scalars), $(C, D)$ denotes the matrix obtained stacking $C$ on top of $D$, assuming that the dimensions are compatible. Given a classical vector $v \in \mathbb{R}^{2^n}$, we denote $| v \rangle := \sum_{j \in \{0, 1\}^n} \frac{v_j}{\sqrt{\sum_j v_j^2}} | j \rangle$ its amplitude encoding. If we have binary digits or strings $a_1, a_2$, we denote by $a_1 \cdot a_2$ their concatenation. Given a binary string $a = a_1 \cdot a_2 \cdots \cdot a_m$ with $a_j \in \{0, 1\}$, we define $0.a := \sum_{j=1}^m a_j 2^{-j}$. The symbol $\oplus$ denotes bitwise addition modulo 2, i.e., binary XOR. $1_m$ is the all-one vector of size $m$.

4.1 Preliminaries

We define the following symbols:

- $d_c$: maximum number of nonzero entries in any column of $A$.
- $d_r$: maximum number of nonzero entries in any row of $A_B$.
- $d := \max\{d_c, d_r\}$: sparsity of $A_B$.
- $\kappa$: ratio of largest to smallest nonzero singular value of $A_B$. Throughout this paper, we assume that $\kappa$, or an upper bound on it, is known. Note that since $\kappa$ is provided as an input to the QLSA, replacing it with an upper bound increases the running time by a corresponding amount.
- $\eta := \max\{\max_j \| A_j \|, \| b \|\}$: maximum norm of a column of $A$ or $b$.
- $L$: maximum nonzero entry of $A_B$, rounded up to a power of 2.
- $r := (\lceil \log m \rceil + \lceil \log d \rceil)$: number of bits to index entries of $A_B$.

The notation $O$ is used to suppress polylogarithmic factors in the input parameters, i.e., $O(f(x)) = O(f(x) \poly(\log m, \log m, \log \frac{1}{\epsilon}, \log \kappa, \log d, \log L))$. Note that $L$ is exponential in the input size, but we include $\log L$ among the factors suppressed in $O$ notation because it simplifies the analysis: keeping track of the number of bits used to represent entries is cumbersome and largely influenceless. As stated in the introduction, we assess the complexity of quantum algorithms in terms of basic gates. We assume that the cost of a controlled unitary is the same as that of the unitary, because the number of additional gates required by the controlled unitary is generally the same in the $O$ notation.

We use several common building blocks for quantum algorithms: phase estimation, amplitude amplification [Grover 1996], amplitude estimation [Brassard et al. 2002]. We state a version of these building blocks below.
Theorem 3 (Sect. 5.2 in Nielsen and Chuang (2002)) Let $U$ be a unitary and $|\psi\rangle$ an eigenvector of $U$ with eigenvalue $2\pi i \theta$. Let $\varepsilon = 2^{-q}$ and $\gamma > 0$. With probability at least $1 - \gamma$, the phase estimation algorithm determines a $q$-qubit approximation of $\theta$ of, i.e., $|\theta - \hat{\theta}| < 2^{-q} = \varepsilon$, using $O(2^q + \log \frac{1}{\varepsilon}) = O(\frac{1}{\varepsilon} + \log \frac{1}{\varepsilon})$ applications of $U$ and $O(\log \frac{1}{\varepsilon} + \log \frac{1}{\gamma})$ additional gates. In particular, if we let $0.a$ be the output of the procedure and we use $q + \lfloor \log(2 + \frac{1}{2\gamma}) \rfloor$ qubits of precision, the first $q$ bits of $a$ are accurate with probability at least $1 - \gamma$, i.e., $Pr(|\theta - \hat{\theta}| < 2^{-q} | 0.a) \geq 1 - \gamma$.

Theorem 4 (Grover (1996), Brassard et al. (2002)) Let $U$ be a q-qubit unitary such that $U|0^q\rangle = \sqrt{p}|\psi_{\text{good}}\rangle + \sqrt{1-p}|\psi_{\text{bad}}\rangle$, where for some $G \subseteq \{0,1\}^q$, we have $|\psi_{\text{good}}\rangle = \frac{1}{\sqrt{p}} \sum_{j \in G} |\alpha_j\rangle$, $|\psi_{\text{bad}}\rangle = \sum_{j \notin G} |\alpha_j\rangle$ and $p = \sum_{j \in G} |\alpha_j|^2$. Let $F$ be a unitary that maps $|\psi_{\text{good}}\rangle \rightarrow -|\psi_{\text{good}}\rangle$, $|\psi_{\text{bad}}\rangle \rightarrow |\psi_{\text{bad}}\rangle$. The amplitude amplification algorithm outputs $j \in G$ with probability at least $2/3$ using $O\left(\frac{2}{\varepsilon}\right)$ applications of $U$ and $F$, and additional gates. In particular, if we have a quantum oracle implementing a Boolean function $f : \{0,1\}^q \rightarrow \{0,1\}$ and let $G = \{x \in \{0,1\}^q : f(x) = 1\}$, we can determine an element of $G$ with $O\left(\frac{2}{\varepsilon}\sqrt{q}\right)$ applications of $U$, if $|G|$ is unknown; if $|G|$ is known, we can determine an element of $G$ with $O(\sqrt{q})$ applications in expectation.

Theorem 5 (Brassard et al. (2002), Sect. 6.3 in Nielsen and Chuang (2002)) Let $U, p, |\psi_{\text{good}}\rangle, |\psi_{\text{bad}}\rangle, F, G$ be defined as in Theorem 3. Let $\gamma = \sin \theta$, with $0 \leq \theta \leq \frac{\pi}{4}$. Let $R$ be a unitary that maps $|0^q\rangle \rightarrow -|0^q\rangle$ and $|j\rangle \rightarrow |\bar{j}\rangle$ for all $j$. Then the state $U|0^q\rangle$ is an equally weighted superposition of eigenvectors of the operator $Q = U^1 RU_F$, with eigenvalues $e^{2p}\epsilon, e^{-2p}\epsilon$ respectively. In particular, applying phase estimation to the operator $Q$ and the state $U|0^q\rangle$, with $q$ qubits of precision, yields an estimate $\hat{\theta}$ such that $|\sin^2 \hat{\theta} - p| \leq 2\sqrt{\frac{p(1-p)}{3}} + \frac{\pi^2}{24}$ with probability at least $2/3$. If we simply want to determine if $G$ is empty or not, $\lfloor q/2 \rfloor + 3$ qubits of precision suffice, yielding an algorithm that succeeds with probability at least $2/3$ and requires $O(\sqrt{q})$ calls to $U$.

In several parts of this paper we use amplitude amplification on oracles with bounded error: it has been shown that this is possible while achieving the same running time as with deterministic oracles (Høyer et al. 2003). We also use an approximate version of quantum minimum finding; we state its complexity below. Its correctness follows from the result of Durr and Høyer (1998). Detailed proofs of this result, and all other results in this paper, are available in the appendix.

Theorem 6 (Durr and Høyer (1998)) Let $S := \{0,\ldots,2^q - 1\}$ and $f : S \rightarrow \mathbb{R}$. Suppose we have access to a unitary $U_f$ such that given $|j\rangle, |k\rangle$ with $j, k \in S$, the unitary $U_f$ returns $1$ if $f(j) \leq f(k)$ or $\lnot (j = k)$, and $0$ otherwise. There exists a quantum algorithm that finds $y \in S$ such that $f(y) \leq \min_{x \in S} f(x) + \epsilon$ using at most $\sqrt{\frac{2}{\epsilon}}\sqrt{2^q} + \frac{\pi^2}{24} = O(\sqrt{q})$ calls to $U_f$ in expectation.

We remark the (well-known) fact that statements about expected running time, such as in Theorem 6 and 7, imply that the right answer can be obtained with probability at least $2/3$ by executing the algorithm for three times the expected running time (by Markov’s inequality). We frequently use this alternative characterization of the algorithms.

Finally, we discuss the QLSA introduced in Childs et al. (2017), see also Chakraborty et al. (2018), Gilven et al. (2019). For the system $A_B x = b$ with integer entries, the input to the algorithm is encoded by two unitaries, $P_{A_B}$ and $P_b$, which are queried as oracles defined as follows:

- $P_{A_B}$: specified by two maps; the map $|j, \ell\rangle \rightarrow |j, \nu(j, \ell)\rangle$ provides the index $\nu(j, \ell)$ of the $\ell$-th possibly nonzero element of column $j$, the map $|j, k, z\rangle \rightarrow |j, k, z \oplus (A_B)_{jk}\rangle$ provides the value of the $k$-th element of column $j$. (This is the typical sparse encoding for vectors or matrices in the classical world as well.)
- $P_b$: maps $|0|^{\lfloor \log m \rfloor} \rightarrow |b\rangle$.

Theorem 7 (Thm. 5 in Childs et al. (2017)) Let $A_B$ be such that $\|A_B\| \leq 1$. Given $P_{A_B}, P_b,$ and $\varepsilon > 0$, there exists a quantum algorithm that produces the state $|\tilde{x}\rangle$ with $\|A_B^{-1} b - |\tilde{x}\rangle\| \leq \varepsilon$ using $O(\text{dc})$ queries to $P_{A_B}$ and $P_b$, with additional gate complexity $O(\text{dc})$.

Note that (Childs et al. 2017) discusses the case of a symmetric matrix $A_B$, but it is known that this restriction can be relaxed by considering the system

$$
\begin{pmatrix}
0 & A_B \\
A_B^T & 0
\end{pmatrix}
\begin{pmatrix}
|0\rangle \\
|\tilde{x}\rangle
\end{pmatrix} =
\begin{pmatrix}
|b\rangle \\
|0\rangle
\end{pmatrix}.
$$

See Harrow et al. (2003). If the original (non-symmetric) $A_B$ is invertible, so is the symmetrized matrix, and the singular values are the same (but with different multiplicity). In the rest of this paper, we refer to $A_B$ as a shorthand for the above symmetrized matrix; the r.h.s. and the number of rows $m$ are adjusted accordingly. Our running time analysis takes into account this transformation, i.e., we use $d$ rather than $d_s$ where appropriate. The last result concerns estimation of the norm of the solution of a linear system; this is useful because the QLSA outputs a normalized solution, but sometimes we need the norm of the unnormalized vector as well. We state a simplified version, as the details are cumbersome and unimportant in the context of our paper.
Theorem 8 (Corollary 32 in Chakraborty et al. (2018)) In the setting of Thm. 5 we can output \( \hat{c} \in \mathbb{R} \) such that \( (1 - \xi)\|A^{-1}_{B}b\| \leq \hat{c} \leq (1 + \xi)\|A^{-1}_{B}b\| \) using \( O(\frac{\xi}{\epsilon}) \) queries to \( P_{A_{B}} \) and \( P_{c} \).

Thm. 5 immediately yields an estimate of \( A_{B}^{-1}b \) (without the normalization for \( b \)) with the same error provided that \( \|b\| \) is known; since this is always the case in this paper (we can assume that the norm of \( b \) and of the columns of \( A \) computed in a preprocessing step), in the remainder we do not worry about the r.h.s. normalization in the context of Thm. 5.

A quantum RAM (QRAM) is a quantum-accessible form of storage that allows for querying a superposition of addresses. Given a QRAM that stores the classical vector \( v_{j} \in \mathbb{R}^{2^{q}} \), and a quantum state \( \sum_{j=0}^{2^{q}-1} \alpha_{j}|j\rangle \), we can perform the following mapping in time \( \tilde{O}(1) \):

\[
\sum_{j=0}^{2^{q}-1} \alpha_{j}|j\rangle \otimes |0\rangle \rightarrow \sum_{j=0}^{2^{q}-1} (\alpha_{j}|j\rangle \otimes |v_{j}\rangle).
\]

Note that this mapping has gate complexity \( \tilde{O}(2^{q}) \) in the standard gate model without QRAM, because it would have to be constructed as a “lookup table” in the general case. Therefore, if an algorithm has gate complexity \( C \) assuming access to a QRAM of size \( 2^{q} \), there exists a quantum algorithm for the same task that does not require QRAM access and has gate complexity \( \tilde{O}(2^{C}) \). We should note that the physical realizability of QRAM is still an open question. While there have been proposals for its construction, such as (Giovannetti et al. 2008), the community has not reached a consensus regarding the practical feasibility of these proposals. Nonetheless, QRAM is used in virtually all papers on quantum optimization, because the cost of describing classical data is often prohibitive in the gate model without QRAM, negating any quantum advantage.

4.2 High-level description of the quantum subroutines
As stated in the introduction, a naive application of a QLSA (with explicit classical input and output) in the context of the simplex method does not give any advantage over the classical algorithm. To gain an edge, we need a different approach. The idea of this paper is based on the observation that an iteration of the simplex method can be reduced to a sequence of subroutines that have scalar output. Indeed, the simplex method does not require explicit knowledge of the full solution vector \( A_{B}^{-1}b \) associated with a basis, or of the full simplex tableau \( A_{B}^{-1}A_{N} \), provided that we are able to:

- Identify if the current basis is optimal or unbounded;
- Identify a pivot, i.e., the index of a column with negative reduced cost that enters the basis, and the index of a column leaving the basis.

While subroutines to perform these tasks require access to \( A_{B}^{-1}b \) and/or \( A_{B}^{-1}A_{N} \), we show that we can get an asymptotic speedup by never computing a classical description of \( A_{B}^{-1}, A_{B}^{-1}b \) or \( A_{B}^{-1}A_{N} \). This is because extracting from the quantum state the classical description of an amplitude-encoded vector (an operation called “state tomography”) is much more expensive than obtaining scalar outputs, as these can be digitally encoded as basis states and read from a single measurement with some probability.

Throughout this overview of the quantum algorithms, we assume that the LP data is properly normalized. The normalization can be carried out as a classical preprocessing step, whose details are given in Sect. 5 and the running time of this step is negligible compared to the remaining subroutines.

Our first objective is to implement a quantum oracle that determines if a column has negative reduced cost, so that we can apply Grover search to this oracle. To reach this goal we rely on the QLSA of Thm. 4. Using that algorithm, with straightforward data preparation we can construct an oracle that, given a column index \( k \) in a certain register, outputs \( A_{B}^{-1}A_{k} \) in another register. We still need to get around three obstacles:

(i) the output of the QLSA is a renormalization of the solution, rather than the (unscaled) vector \( A_{B}^{-1}A_{k} \); (ii) we want to compute \( c_{B} - c_{B}A_{B}^{-1}A_{k} \), while so far we only have access to \( |A_{B}^{-1}A_{k}\rangle \); (iii) we need the output to be a binary yes/no condition (i.e., not just some information encoded in an amplitude) so that Grover search can be applied to it. We overcome the first two obstacles by: extending and properly scaling the linear system so that \( c_{B} \) is suitably encoded in the QLSA output; and using the inverse of the unitary that maps \( |0\rangle^{\otimes \log m+1} \) to \( |(-c_{B}, 1)\rangle \) to encode \( c_{B} - c_{B}A_{B}^{-1}A_{k} \) in the amplitude of one of the basis states. To determine the sign of such amplitude, we rely on interference to create a basis state with amplitude \( \alpha \) such that \( |\alpha| \geq \frac{1}{2} \) if and only if \( c_{B} - c_{B}A_{B}^{-1}A_{k} \geq 0 \). At this point, we apply amplitude estimation (Thm. 5) for \( O(1/\epsilon) \) iterations to test the condition with precision \( \epsilon \). We therefore obtain a unitary operation that overcomes the three obstacles. A similar scheme can be used to determine if the basis is optimal, i.e., no column with negative reduced cost exists.
Some further complications merit discussion. The first one concerns the optimality tolerance: classically, this is typically \( \bar{c}_N \geq -\epsilon \) for some given \( \epsilon \). Note that an absolute optimality tolerance is not invariant to rescaling of \( c \). In the quantum subroutines, to ensure that the precision of the amplitude estimation part is not too demanding, we use the inequality \( \tilde{c}_k \geq -\epsilon \| (A_k^{-1} A_k, c_k) \| \) as optimality criterion, and show that precision \( O(\epsilon) \) is sufficient for this task. We remark that if \( \| A_k^{-1} A_k \| \) is small, the ratio test \([\text{I}]\) has a small denominator, hence the basic feasible solution \( A_k^{-1} b \) may move by a large amount if column \( k \) enters the basis; it is therefore reasonable to require that \( \tilde{c}_k \) is very close to zero to declare that the basis is optimal. The converse is true if \( \| A_k^{-1} A_k \| \) is large. Thus, our approach can be interpreted as some relative optimality criterion. As is typical in the simplex method, the stopping criterion is based on optimality of the reduced costs; even if this does not give guarantees on the gap with respect to the optimal solution of the problem, this is the method used in practical implementations of the algorithm. Since we never explicitly compute the basis inverse, we do not have classical access to \( \| (A_k^{-1} A_k, c_k) \| \). To alleviate this issue, we show in Sect. \([\text{II}9]\) that there exists a quantum subroutine to compute the root mean square of \( \| A_k^{-1} A_k \| \) over all \( k \); thus, if desired we can also output this number to provide some characterization of the optimality tolerance used in the pricing step (\( \| A_k^{-1} A_k \| \) for a specific \( k \) can be obtained with a direct application of Thm. \([\text{II}6]\)). It is important to remark that while the optimality tolerance is relative to a norm, which in turn depends on the problem data, the evidence provided in Sect. \([\text{II}1]\) suggests that due to sparsity, in practice we expect these norms to grow very slowly with \( m \) and \( n \) (i.e., polylogarithmically). To implement Dantzig’s pivoting rule and determine the smallest reduced cost, as opposed to a random column with negative reduced cost, the running time increases by a factor \( c_{\max} \), because we increase precision to perform pairwise comparisons of reduced costs in quantum minimum finding. To implement the steepest edge pivoting rule the running time is the same as for Dantzig’s rule, but the analysis is a bit more involved, as we need to properly renormalize the encoding of the reduced cost to account for denominators of the form \( \| A_k^{-1} A_k \| \). For this, we use Thm. \([\text{II}1]\) and very simple block-encoding techniques (Gilvéñ et al. \(2019\)). Note that in the classical case, steepest edge is more expensive than Dantzig’s rule, increasing the running time expression by an additive term \( mn^2 \).

The second complication concerns the condition number of the basis: the running time of the quantum routines explicitly depends on it, but this is not the case for the classical algorithms based on an LU decomposition of \( A_B \) (although precision may be affected). We do not take any specific steps to improve the worst-case condition number of the basis (e.g., Clader et al. \(2013\)), but we note that similar issues affect the classical simplex method: even if the running time does not depend on \( \kappa \), when \( \kappa \) grows very large the algorithm may fail because the computation of primal solutions or pivots becomes too imprecise. Many approaches have been proposed to prevent this from happening in practice, e.g., modifications of the pivoting rule to select pivot elements that are not too small, such as the two-pass Harris ratio test. We discuss in Sect. \([5.7]\) how to quantize the Harris procedure; we suspect that other similar procedures can be quantized as well.

With the above construction we have a quantum subroutine that determines the index of a column with negative reduced cost, if one exists. Such a column can enter the basis. To perform a basis update we still need to determine the column leaving the basis: this is our second objective. For this step we need knowledge of \( A_k^{-1} A_k \). Classically, this is straightforward because the basis inverse is available, since it is necessary to compute reduced costs anyway. With the above quantum subroutines the basis inverse is not known, and in fact part of the benefit of the quantum subroutines comes from always working with the original, sparse basis, rather than its possibly dense inverse. Thus, we describe another quantum algorithm that uses a QLSA as a subroutine, and identifies the element of the basis that is an approximate minimizer of the ratio test \([\text{I}]\). Special care must be taken in this step, because attaining the minimum of the ratio test is necessary to ensure that the basic solution after the pivot is feasible (i.e., satisfies the nonnegativity constraints \( x \geq 0 \)).

However, in the quantum setting we are working with continuous amplitudes, and determining if an amplitude is zero is impossible due to finite precision. Our approach to give rigorous guarantees for this step relies on binary search for the optimal value of \( r \) in \([\text{I}]\), i.e., the maximum “step length” when moving along an edge of the polyhedron, while ensuring that the violation of the nonnegativity constraints is at most \( \delta \). With similar techniques we can determine if column \( k \) proves unboundedness of the LP. Note that because of the inexactness of the ratio test, we could pivot to slightly infeasible solutions, i.e., we only guarantee \( x \geq -\delta 1_m \) for a given \( \delta > 0 \); to prevent this from breaking the algorithm, we borrow two strategies from the classical simplex. The first strategy is a strict zero step for basic variables that are negative after a pivot: this is the same as in the classical world, see, e.g., Huangfu \(2013\). The second strategy is a periodic recomputation of the basis to eliminate infeasibilities: we give a quantum subroutine to determine \( \delta \)-feasibility, and if the basis is infeasible, we can switch to Phase 1 of the simplex method, possibly increasing precision, until we recover (approximate) feasibility. (Recall that Phase 1 is equivalent to solving the problem \( \min \sum_{i=1}^{m} s_i, Ax + s = b, x \geq 0, s \geq 0 \), which is always feasible assuming \( b \geq 0 \), and can be used to determine a feasible basis for the original problem if one exists.)
5 Details of the quantum implementation

We now discuss data preparation and give the details of the quantum subroutines. In this section we work with the standard gate model and do not assume that we have access to QRAM. Some of the data preparation routines become significantly faster with QRAM: this is the subject of Sect. 5 where we also indicate the (small) algorithm modifications that are required to take advantage of quantum-accessible storage.

The steps followed at every iteration are listed below: all the subroutines used in the algorithm are fully detailed in subsequent sections. For brevity we do not explicitly uncompute auxiliary registers, but it should always be assumed that auxiliary registers are reset to their original state at the end of a subroutine; this does not affect the running time analysis. Furthermore, we are only concerned with giving constant lower bounds on the probability of success of the subroutines; all probabilities of success can be made arbitrarily large with standard approaches and polylogarithmic cost increase.

Algorithm 1 Run one iteration of the simplex method: \textsc{SimplexIter}(A, B, c, \epsilon, \delta).

1: \textbf{Input:} Matrix \(A\), basis \(B\), cost vector \(c\), precision parameters \(\epsilon, \delta, t\).
2: \textbf{Output:} Flag “optimal”, “unbounded”, or a pair \((k, \ell)\) where \(k\) is a nonbasic variable with negative reduced cost, \(\ell\) is the basic variable that should leave the basis if \(k\) enters.
3: Normalize \(c\) so that \(|c_B| = 1\). Normalize \(A\) so that \(|A_B| \leq 1\).
4: Apply \textsc{IsOptimal}(\(A, B, \epsilon\)) to determine if the current basis is optimal. If so, return “optimal”.
5: Apply \textsc{FindColumn}(\(A, B, \epsilon\)) to determine a column with negative reduced cost. Let \(k\) be the column index returned by the algorithm. \(k\) is the pivot column.
6: Apply \textsc{IsUnbounded}(\(A_B, k, \delta\)) to determine if the problem is unbounded. If so, return “unbounded”.
7: Apply \textsc{FindRow}(\(A_B, k, \delta\)) to determine the index \(\ell\) of the row that minimizes the ratio test (1).

All data normalization is performed within the subroutine \textsc{SimplexIter}(\(A, B, c, \epsilon, \delta\)), on line 3 this prepares the input for all other subroutines. Normalizing \(c\) has cost \(O(n)\), while finding the leading singular value of \(A_B\) has cost \(O(\frac{1}{\epsilon}md \log m)\) using, e.g., the power method, with precision \(\epsilon\) \cite{Kuczynski and Wozniakowski 1992}. The QLSA requires the eigenvalues of \(A_B\) to lie in \([-1, -1/\kappa]\) \cup \([1/\kappa, 1]\); those outside this set are discarded by the algorithm. We can choose \(\kappa\) to be some arbitrary constant, say, \(10^{-4}\), rescale \(A\) by \((1-\epsilon')/\sigma_{max}\); where \(\sigma_{max}\) is the estimate obtained with the power method; and inflate \(\kappa\) by a factor \(1/(1-\epsilon')\). This ensures that the spectrum of the rescaled \(A_B\) satisfies the requirements. The overall running time is not affected, because the time for the remaining subroutines dominates \(O(\frac{1}{\epsilon}md \log m)\), as shown in the rest of this section (e.g., simply loading \(A_B\) and \(A\) for the QLSA requires time \(O(d_n + dm)\), see the proof of Thm. 4).

5.1 Implementation of the oracles \(P_{A_B}\) and \(P_b\)

Recall that the quantum linear system algorithm of Thm. 7 requires access to two quantum oracles describing the data of the linear system. We now discuss their implementation, so as to compute their gate complexity. Given computational basis states \(|k, j\rangle\), in this section we denote \(|k, j\rangle := |k\rangle \otimes |j\rangle\) for brevity. Recall that \(\oplus\) denotes the binary XOR; and, for readers not familiar with quantum computers, recall that the \(X\) gate is equivalent to a bit-flip (i.e., \(X(0) = |1\rangle\), \(X(1) = |0\rangle\)). We start with the two maps necessary for \(P_{A_B}\).

Since the map \(|j, \ell, \nu(j, \ell)\rangle \rightarrow |j, \nu(j, \ell)\rangle\) is in-place, we implement it as the sequence of mappings \(|j, \ell, 0^{\lceil \log m \rceil}\rangle \rightarrow |j, \ell, \nu(j, \ell)\rangle \rightarrow |j, \ell, \nu(j, \ell), 0^{\lceil \log m \rceil}\rangle\).

To implement the first part \(|j, \ell, 0^{\lceil \log m \rceil}\rangle \rightarrow |j, \ell, \nu(j, \ell)\rangle\), for \(j\) and \(\ell\) we use \(O(\log m)\) controlled single-qubit operations. Let \(U\) be the transformation that maps the basis state \(|\ell\rangle\) into \(|\nu(j, \ell)\rangle\): this can be computed classically in \(O(\log m)\) time, since it requires at most \(\lceil \log m \rceil\) bit-flips. Then the desired mapping requires at most \(O(\log m)\) \(r\)-controlled \(X\) gates, which require \(O(r)\) basic gates each plus polylogarithmic extra space. The mapping \(|j, \ell, \nu(j, \ell)\rangle \rightarrow |j, \nu(j, \ell), \ell, \ell, \nu(j, \ell)\rangle\) is then easy to construct, as it requires at most \(\lceil \log m \rceil\) CX gates to compute \(\ell, \ell, \nu(j, \ell) = \nu(j, \ell)\) in the second register (by taking the XOR of the third register with the second). We then undo the computation in the third register. Thus, we obtain a total of \(O(r \log m)\) basic gates for each \(j\) and \(\ell\). There are at most \(d\) nonzero elements per column (recall the transformation to make \(A_B\) symmetric), yielding \(O(d)\) basic gates to construct the first map of \(P_{A_B}\) for a given column \(j\); since there are \(m\) columns, the overall gate complexity of this oracle is \(O(dm)\).

The implementation of \(|j, k, z\rangle \rightarrow |j, k, z \oplus A_{jk}\rangle\) is similar. Since \(j, k \in \{0, \ldots, m - 1\}\) and the largest entry of \(A\) is \(L\), the map requires \(O(r \log L)\) basic gates for each \(j\) and \(k\). Summarizing, the implementation of the two maps for \(P_{A_B}\) requires \(O(d)\) basic gates per column, for a total of \(O(dm)\) gates. This is intuitive as
Algorithm 2 Interference routine: INTERFERENCE(U, V).

1. Input: two controlled state preparation unitaries $U, V$ on $q$ qubits with $U|0\rangle = \sum_{j=0}^{2^q-1} \alpha_j |j\rangle$ and $V|0\rangle = \sum_{j=0}^{2^q-1} \beta_j |j\rangle$.
2. Output: State $|\psi\rangle = \frac{1}{2^q} |0\rangle \otimes \sum_{j=0}^{2^q-1} (\alpha_j + \beta_j)|j\rangle + \frac{1}{2^q} |1\rangle \otimes \sum_{j=0}^{2^q-1} (\beta_j - \alpha_j)|j\rangle$.
3. Introduce an auxiliary qubit in state $|0\rangle$ and apply a Hadamard gate, yielding the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$; assume that this is the first qubit.
4. Apply the controlled unitaries $|0\rangle \langle 0| \otimes I^{\otimes q} + |1\rangle \langle 1| \otimes U$ and $|0\rangle \langle 0| \otimes V + |1\rangle \langle 1| \otimes I^{\otimes q}$ to the state, obtaining $\sum_{j=0}^{2^q-1} |j\rangle \otimes U|0\rangle \otimes V|0\rangle + |1\rangle \otimes U|0\rangle \otimes V|0\rangle$.
5. Apply a Hadamard gate on the first qubit. Return the resulting state $|\psi\rangle$.

Proposition 2 Let $U|0\rangle = \sum_{j=0}^{2^q-1} \alpha_j |j\rangle$ and $V|0\rangle = \sum_{j=0}^{2^q-1} \beta_j |j\rangle$. Then INTERFERENCE(U, V) (Alg. 2) returns the state $|\psi\rangle = \frac{1}{2^q} |0\rangle \otimes \sum_{j=0}^{2^q-1} (\alpha_j + \beta_j)|j\rangle + \frac{1}{2^q} |1\rangle \otimes \sum_{j=0}^{2^q-1} (\beta_j - \alpha_j)|j\rangle$, using one call to controlled-U and controlled-V, and two extra gates.

The proof follows immediately by definition of the Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

To determine the sign of a reduced cost we use the subroutines SIGNEST-NFN, given in Alg. 3 and SIGNEST-NFP, given in Alg. 13. The acronyms "NFN" and "NFP" stand for "no false negatives" and "no false positives", respectively, based on an interpretation of these subroutines as classifiers that need to assign a 0-1 label to the input. We need two such subroutines because the quantum phase estimation, on which they are based, is a continuous transformation. Therefore, a routine that has "no false negatives", i.e., with high probability it returns 1 if the input data’s true class is 1 (in our case, this means that a given amplitude is $\geq -\epsilon$), may have false positives: it may also return 1 with too large probability for some input that belongs to class 0 (i.e., the given amplitude is $< -\epsilon$). The probability of these undesirable events decreases as we get away from the threshold $-\epsilon$. We therefore adjust the precision and thresholds used in the sign estimation routines to construct one version that with high probability returns 1 if the true class is 1 (no false negatives), and one version that with high probability returns 0 if the true class is 0 (no false positives). The subroutines are similar to the traditional Hadamard test, but rather than taking a measurement, we apply amplitude estimation on the Hadamard qubit.

Proposition 3 Let $U|0\rangle = |\psi\rangle = \sum_{j=0}^{2^q-1} \alpha_j |j\rangle$ with real $\alpha_j$, and $k \in \{0, \ldots, 2^q - 1\}$ a basis state index. Then with probability at least $3/4$:

- if $\alpha_k \geq -\epsilon$ Algorithm 3 (SIGNEST-NFN) returns 1, and if Algorithm 3 returns 1 then $\alpha_k \geq -2\epsilon$;
- if $\alpha_k < -2\epsilon$ Algorithm 3 (SIGNEST-NFN) returns 0, and if Algorithm 3 returns 0 then $\alpha_k < -\epsilon$.

The algorithm makes $O(1/\epsilon)$ calls to $U, U^\dagger$ or their controlled version, and requires $O(q + \log^2 \frac{1}{\epsilon})$ additional gates.
Algorithm 3 Sign estimation routine: SIGNESTN FN(U, k, ϵ).

1: Input: state preparation unitary \( U \) on \( q \) qubits (and its controlled version) with \( U|0^q\rangle = \sum_{j=0}^{2^q-1} \alpha_j |j\rangle \) and \( \alpha_j \) real, index \( k \in \{0, \ldots, 2^q - 1\} \), precision \( \epsilon \).
2: Output: if \( \alpha_k \geq -\epsilon \), with probability at least 3/4.
3: Let \( V \) map \( |0^q\rangle \) to \( |k\rangle \). Compute \( |\psi\rangle = \text{INTERFERE}(U, V) \).
4: Apply amplitude estimation to the state \( |0\rangle \otimes |k\rangle \) with \( \lfloor \log_2 \frac{3\epsilon}{2\sqrt{\epsilon}} \rfloor + 2 \) qubits of accuracy; let \( |a\rangle \) be the bitstring produced by the phase estimation portion of amplitude estimation.
5: If \( 0.a < \frac{1}{2} \), return 1 if \( 0.a \geq \frac{1}{6} - \frac{2\epsilon}{\sqrt{3\pi}} \), 0 otherwise; if \( 0.a \geq \frac{1}{2} \), return 1 if \( 1 - 0.a \geq \frac{1}{6} - \frac{2\epsilon}{\sqrt{3\pi}} \), 0 otherwise.

The alternative version of this routine, i.e., the one that with high probability has “no false positives”, can be constructed similarly, by adjusting the thresholds. We give its pseudocode in the appendix.

Proposition 4 Let \( U|0\rangle = |\psi\rangle = \sum_{j=0}^{2^q-1} \alpha_j |j\rangle \) with real \( \alpha_j \), and \( k \in \{0, \ldots, 2^q - 1\} \) a basis state index. Then with probability at least 3/4:
- if \( \alpha_k \leq -\epsilon \) Algorithm 4 (SIGNESTNFP) returns 0, and if Algorithm 4 returns 0 then \( \alpha_k \leq \frac{\epsilon}{\sqrt{3\pi}} \);
- if \( \alpha_k > \frac{\epsilon}{\sqrt{3\pi}} \) Algorithm 4 (SIGNESTNFP) returns 1, and if Algorithm 4 returns 1 then \( \alpha_k > -\epsilon \).

The algorithm makes \( O(1/\epsilon) \) calls to \( U, U^\dagger \) or their controlled version, and requires \( O(q + \log^2 \frac{1}{\epsilon}) \) additional gates.

It is important to discuss the assumption that \( \alpha_j \) is real. In this paper, the subroutines SIGNESTNFP and SIGNESTNFP are applied to the output of a QLSA; by definition of the QLSA problem, the output state is the amplitude encoding of the solution, see [Childs et al. (2017)], thus it has real coefficients for real data. However, the implementation of a quantum algorithm as a circuit is generally assumed to be modulo a global phase. Unfortunately, the two subroutines described in this section (as well as the Hadamard test) are sensitive to the relative phase of controlled-\( U \) with respect to the remaining part of the quantum state, so a global phase in the implementation of \( U \) would affect the result of the sign test. This issue can be avoided by ensuring that the mapping controlled-\( V \) used in the algorithm has the same phase as controlled-\( U \), so that there is no relative phase. From a theoretical point of view, a careful implementation of a QLSA (e.g., through the Chebyshev polynomial [Childs et al. (2017)] or the singular value transformation [Gilyén et al. (2019)] framework) is able to track the phase applied by the algorithm, and therefore we can account for this phase in subsequent steps: the only restriction is that we cannot take the controlled-QLSA unitary to be a “black-box”, and need to know its phase.

5.3 Determining if the basis is optimal, or the variable entering the basis

Algorithm 4 Determining the reduced cost of a column: \( \text{RedCost}(A_B, A_k, c, \epsilon) \).

1: Input: basis \( A_B \) with \( \|A_B\| \leq 1 \), nonbasic column \( A_k \), cost vector \( c \) with \( \|c\| = 1 \), precision \( \epsilon \).
2: Output: A quantum state such that the reduced cost of column \( k \) is encoded in the amplitude of \( |0^{\lfloor \log_2 m \rfloor}\rangle \), and a flag indicating success, with bounded probability.
3: Solve the linear system \( \begin{pmatrix} A_B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_k \\ c_k \end{pmatrix} \) using the quantum linear system algorithm with precision \( \frac{\epsilon}{10\sqrt{2}} \). Let \( |\psi\rangle \) be the quantum state obtained this way.
4: Let \( U_c \) be a unitary such that \( U_c|0^{\lfloor \log_2 m+1 \rfloor}\rangle = \{(-c_B, 1)\} \); apply \( U_c^\dagger \) to \( |\psi\rangle \) to obtain \( |\psi'\rangle \).
5: Return \( |\psi'\rangle \) and the flag indicating success as set by the quantum linear system algorithm.

To find a column that can enter the basis, we use the subroutines \( \text{RedCost} \) and \( \text{CanEnter} \), described in Alg.s 4, 5. They are building blocks of \( \text{FindColumn} \), detailed in Alg. 5.

Proposition 5 With bounded probability, Algorithm 4 (\( \text{RedCost} \)) returns a quantum state such that the amplitude \( \alpha_0 \) of the basis state \( |0^{\lfloor \log_2 m+1 \rfloor}\rangle \) satisfies:

\[
|\alpha_0 - \frac{\tilde{c}_k}{\|A_B^{-1}A_k, c_k\|}| \leq \frac{\epsilon}{10}.
\]

The gate complexity of the algorithm is \( O(7T_{LS}(A_B, A_k, s) + m) \).

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Algorithm 5 Determine if a column is eligible to enter the basis: CANENTER($A_B, A_k, c, \epsilon$).

1: **Input:** Basis $A_B$ with $\|A_B\| \leq 1$, nonbasic column $A_k$, cost vector $c$ with $\|c_B\| = 1$, precision $\epsilon$.  
2: **Output:** 1 if the nonbasic column has reduced cost $<-\epsilon$, 0 otherwise, with bounded probability.  
3: Let $U_r$ be the unitary implementing $\text{RedCost}(A_B, A_k, c, \epsilon)$.  
4: Return 1 if $\text{SignEstNFP}(U_r, 0^{[\log m+1]}, \frac{14\epsilon}{10\sqrt{2}})$ is 0 and the success flag for $\text{RedCost}(A_B, A_k, c, \epsilon)$ is 1; return 0 otherwise.

Proposition 6 With bounded probability, if Algorithm 5 (CANENTER) returns 1 then column $A_k$ has reduced cost $<-\epsilon\|A_B^{-1}A_k, c_k\|$. The gate complexity of the algorithm is $O(T_{LS}(A_B, A_k, \t\kappa) + \log^2 \frac{1}{\epsilon} + m)$.

In Prop. 6 we are concerned with ensuring that only columns with negative reduced cost are returned, but using Prop. 8 it is easy to characterize an implication in the opposite direction as well; in this specific case, it can be easily established that if column $A_k$ has reduced cost $<-2.2\epsilon\|A_B^{-1}A_k, c_k\|$, then CANENTER($A_B, A_k, c, \epsilon$) returns 1 with bounded probability.

Algorithm 6 Determine the next column to enter the basis: FINDCOLUMN($A, B, c, \epsilon$).

1: **Input:** Matrix $A$, basis $B$, cost vector $c$, precision $\epsilon$, with $\|A_B\| \leq 1$ and $\|c_B\| = 1$.  
2: **Output:** index of a column $k$ with $\tilde{c}_k < -\epsilon\|A_B^{-1}A_k, c_k\|$ if one exists, with bounded probability.  
3: Let $U_{ rhs}$ be the unitary that maps $|k\rangle \otimes |0^{[\log m]}\rangle$ to $|k\rangle \otimes |A_k\rangle$ for any $k \in \{1, \ldots, n\} \setminus B$.  
4: Apply the quantum search algorithm (Thm. 3) with search space $\{1, \ldots, n\} \setminus B$ to the function $\text{CANENTER}(A_B, U_{rhs}(k), c, \epsilon)$.  
5: Return $k$ as determined by the quantum search algorithm.

Theorem 9 Algorithm 8 (FINDCOLUMN) returns a column $k \in N$ with reduced cost $\tilde{c}_k < -\epsilon\|A_B^{-1}A_k, \frac{c_k}{\|A_B^{-1}\|}\|$, with expected number of iterations $O(\sqrt{n})$. The total gate complexity of the algorithm is $\tilde{O}(\frac{\sqrt{m}}{\epsilon}(d_n + dm))$.

We remark that Thm. 9 concerns the case in which at least one column eligible to enter the basis (i.e., with negative reduced cost) exists. Following Alg. 9, $\text{SIMPLEXITER}$, the subroutine FINDCOLUMN is executed only if $\text{ISOPTIMAL}$ returns false.

The subroutine ISOPTIMAL can be constructed in almost the same way as FINDCOLUMN, hence we do not give the pseudocode. There are two differences. First, at line 3 of CANENTER we use $\text{SignEstNFP}$: this ensures that if the (rescaled) reduced cost is $\leq -\epsilon$, CANENTER returns 1 by Prop. 8. Second, at line 3 of FINDCOLUMN, rather than calling the quantum search algorithm, we use amplitude estimation (Thm. 9) to determine if there is any index $k$ for which (modified) CANENTER($A_B, U_{rhs}(k), c, \epsilon$) has value 1; if no such index exists, ISOPTIMAL returns 1. If all algorithms are successful, which happens with large probability, and ISOPTIMAL returns 1, we are guaranteed that the current basis $B$ is optimal. The gate complexity of ISOPTIMAL is exactly the same as in Thm. 9; see Thm. 9.

We can also examine the two possible cases of failure. A failure might occur if the basis is optimal, but ISOPTIMAL returns 0. In this case, failure can be detected and recovered from, because by Prop. 8 there can be no index $k$ for which CANENTER returns 1, so we observe 0 in the output register of CANENTER — again, assuming all algorithms are successful. Another failure might occur if the basis is not optimal, but FINDCOLUMN fails to return a column with negative reduced cost. This could happen if all columns have (rescaled) reduced cost close to the threshold $-\epsilon$ (i.e., between $-2.2\epsilon$ and $-\epsilon$). We can recover from this failure using $\text{SignEstNFP}$ at line 3 of CANENTER, ensuring that FINDCOLUMN returns some index with CANENTER equal to 1.

Note that when ISOPTIMAL returns 1, the current basis $B$ is optimal but we do not have a classical description of the solution vector. It is straightforward to write a quantum subroutine to compute the optimal objective function value, using the techniques discussed in this paper. To obtain a full description of the solution vector, the fastest approach is to classically solve the system $A_Bx = b$, which requires time $O(d_n^{0.7}m^{1.9} + m^{2+o(1)})$. We remark that this is only necessary in the last iteration, hence it does not dominate the running time unless the problem instance is trivial.

The column selection subroutine FINDCOLUMN presented above returns a randomly chosen column with negative reduced cost. This is sufficient for convergence. To implement a pivoting rule akin to Dantzig’s rule, i.e., that selects the column with minimum reduced cost, a bit more work is needed. To use the quantum minimum finding algorithm (Thm. 9) we need a subroutine that can approximately compare the reduced costs of two columns, say $j$ and $k$. For the normalized reduced cost discussed so far, i.e., $\tilde{c}_j/\|A_B^{-1}A_j, c_j\|$,
this is straightforward: we use subroutine RedCost to compute the normalized reduced cost of \( A_j \) and \( A_k \), Interfere to encode their difference, and SignEst\(\text{StNFN} \) to determine if the difference is at least \( \epsilon \). This gives the same running time as in Thm. 9. If we want to implement Dantzig’s rule for the original reduced costs we need to remove the normalization before applying Interfere: this increases the cost by a factor \( \max_{j=1,\ldots,n} c_j \). The implementation details are very similar to those for the steepest edge pivoting rule, that we discuss next.

To implement the steepest edge pivoting rule \cite{ForrestGoldfarb1992} we apply quantum minimum finding with the comparison subroutine for the reduced cost described in Alg. 7.

Algorithm 7 Compare the steepest edge price of two columns: \textsc{SteepestEdgeCompare}(\( A_B, A_j, A_k, c, \epsilon \)).

1. Input: Basis \( A_B \) with \( \|A_B\| \leq 1 \), columns \( A_j, A_k \), cost vector \( c \) with \( \|c_B\| = 1 \), precision \( \epsilon \).
2. Output: 1 if \( \frac{c_k}{A_B^{\top} A_k} \leq (1 - \epsilon) \frac{c_j}{A_B^{\top} A_j} - \epsilon \), 0 otherwise.
3. Compute an estimate of the norms \( \|A_B^{-1} A_j\|, \|A_B^{-1} A_k\| \) in separate registers, with relative error \( \frac{\epsilon}{4} \).
   Call them \( \hat{d}_j, \hat{d}_k \) respectively.
4. Compute an estimate of the norms \( \|A_B^{-1} A_j\|, \|A_B^{-1} A_k\| \) in separate registers, with relative error \( \frac{\epsilon}{4} \).
   Call them \( \hat{e}_j, \hat{e}_k \) respectively.
5. Let \( U \) be the unitary that applies \( \text{RedCost}(A_B, A_j, c, \epsilon/4) \), then multiplies the coefficient of \( |0^{\lceil \log m \rceil} \rangle \) by \( \frac{\hat{d}_j}{\max_{\ell \in N} c_\ell c_j} \).
6. Let \( V \) be the unitary that applies \( \text{RedCost}(A_B, A_k, c, \epsilon/4) \), then multiplies the coefficient of \( |0^{\lceil \log m \rceil} \rangle \) by \( \frac{\hat{d}_k}{\max_{\ell \in N} c_\ell c_k} \).
7. Let \( W \) be the unitary that applies \( \text{Interfere}(U, V) \).
8. Return 1 if \( \text{SignEst\(\text{StNFN}(W, 0^{\lceil \log m \rceil} + 1, \epsilon/(8 \max_{\ell \in N} c_\ell)) \) is 0 and the success flag for all algorithms is 1.

Theorem 10 Let \( h := \arg \min_{A \in N} \frac{c_h - c_j A_B^{-1} A_h}{\|A B^{-1} A_h\|} \). Using quantum minimum finding with the comparison subroutine \textsc{SteepestEdgeCompare}(\( A_B, A_j, A_k, c_B, \epsilon \)), optimizing over the set \( N \), we can determine a column index with steepest edge price at most:

\[
(1 + \epsilon) \frac{c_h - c_j A_B^{-1} A_k}{\|A_B^{-1} A_k\|} + \epsilon,
\]

with \( O(\sqrt{n}) \) iterations. The total gate complexity of the algorithm is \( \tilde{O}(\frac{\epsilon c_{\max}}{\epsilon^{\epsilon}} (d_n + dm)) \), where \( c_{\max} := \max_{\ell \in N} c_\ell \).

We remark that the normalization factor \( c_{\max} \), which appears in the running time, is used to ensure that the (normalized) steepest edge prices are at most 1 in absolute value, so that they can be encoded in an amplitude. It is not difficult to change the algorithm to determine if a potentially smaller normalization factor is sufficient, but since in the worst case the normalization is of the order of \( c_{\max} \), we do not pursue this approach.

5.4 Improving the running time when the ratio \( n/m \) is large

The pricing algorithms discussed in Sect. 5.3 highlights a tradeoff between the number of iterations and time necessary to load the data. Indeed, if we apply the quantum search algorithm over all columns we need to perform \( O(\sqrt{n}) \) iterations, but the unitary to prepare the data for the QLSA requires time that scales as \( O(d_n) \). In some cases it may therefore be advantageous to split the set of columns into multiple sets, and apply the search algorithm to each set individually. To formalize this idea, suppose that we split the \( n \) columns into \( h \) sets of equal size. The running time of the algorithm discussed in Thm. 8 then becomes \( O(\frac{dm}{\epsilon}\sqrt{d_n + dm}) \).

We can determine the optimal \( h \) by minimizing the above expression. Ignoring the multiplication factor and setting the derivative with respect to \( h \) equal to zero, the optimal \( h \) must satisfy:

\[
-\frac{1}{2} h^{-3/2} n^{3/2} d_n + \frac{1}{2} h^{-1/2} dm \sqrt{n} = 0,
\]

which yields \( h = \frac{dm}{\sqrt{n}} \). Since \( h \) must be integer, we can use this approach only if \( \frac{n}{m} \geq 2 \frac{d_n}{d_m} \), hence, \( n/m \) may have to be large if the columns can be much sparser than the rows. Substituting the optimal \( h \) in the running time expression computed above yields \( O(\frac{dm^{3/2}}{\epsilon}\sqrt{d_n + dm}) \). Notice that this choice of \( h \) is optimal for Thm. 10 as well, as the corresponding running time is simply multiplied by \( c_{\max} \), yielding a total running time of \( O(\frac{dm \epsilon^{\epsilon}}{c_{\max}} \sqrt{d_n + dm}) \) with steepest edge pivoting.
5.5 Estimating the error tolerance

As detailed in Sect. 5.3, the quantum pricing algorithm with the equivalent of a Dantzig rule uses the optimality tolerance $-\epsilon \| (A_B^{-1} A_k, \frac{\alpha_k}{\|A_B^{-1} A_k\|}) \|$. It may therefore be desirable to compute some estimate of $\|A_B^{-1} A_k\|$, since the optimality tolerance used by the algorithm depends on it. However, $A_B^{-1}$ is not classically available. We give a quantum subroutine to compute the root mean square of $\|A_B^{-1} A_k\|$ over all $k$.

**Proposition 7** Let $\|A_B\| = 1$. Using amplitude estimation, it is possible to estimate $\|A_B^{-1} A_N\|^2_2$ up to relative error $\epsilon$ with gate complexity $O(\frac{\sqrt{\alpha}k}{\epsilon} + \kappa^2d^2m)$.

Given an estimate of $\|A_B^{-1} A_N\|^2_2$ with relative error $\epsilon$, dividing by $|N|$ and taking the square root yields an estimate of the root mean square of $\|A_B^{-1} A_k\|$ for $k \in N$, with relative error $\approx \sqrt{\epsilon}$ (up to first order approximation). Obviously, we could also use Thm. 5 to estimate just the norm of $\|A_B^{-1} A_k\|$ for a specific column $k$, e.g., the column entering the basis.

5.6 Determining the variable leaving the basis

In this section we use subroutines $\text{SignEstNFP}^+(U, k, \epsilon)$ and $\text{SignEstNFP}^+(U, k, \epsilon)$ that check if a real amplitude (up to global phase) has positive sign, i.e., they return 1 if $\alpha_k \geq \epsilon$, 0 otherwise, with the same structure as the subroutines described in Sect. 5.2 (i.e., NFN does not have false negatives, NFP does not have false positives). Such subroutines can easily be constructed in a manner similar to $\text{SignEstNFPN}(U, k, \epsilon)$ and $\text{SignEstNFPN}(U, k, \epsilon)$: rather than estimating the amplitude of the basis state $|0 \rangle \otimes |k \rangle$ in the proof of Prop. 3 and Prop. 4, we estimate the amplitude of $|1 \rangle \otimes |k \rangle$ and adjust the return value of the algorithm (e.g., for $\text{SignEstNFP}^+(U, k, \epsilon)$, if $0.\alpha < \frac{1}{2}$, we return 1 if $0.\alpha \leq \frac{1}{2} - 2 \sqrt{\epsilon}$, 0 otherwise). The gate complexity is the same.

**Algorithm 8** Determine if the problem is unbounded from below: $\text{IsUnbounded}(A_B, A_k, \delta)$.

1. **Input:** Basis $A_B$ with $\|A_B\| \leq 1$, nonbasic column $A_k$, precision $\delta$.
2. **Output:** 1 if $A_B^{-1} A_k \leq \delta 1_m \| A_B^{-1} A_k \|$, 0 otherwise, with bounded probability.
3. Let $U_{LS}$ be the unitary implementing the QLSA for the system $A_B x = A_k$ with precision $\frac{\delta}{2m}$.
4. Define a function $g(\ell) := \text{SignEstNFP}^+(U_{LS}, \ell, \frac{\delta}{2m}) \wedge (\text{the success flag for QLSA is 1})$.
5. Use amplitude estimation (Thm. 5) to determine if there exists $\ell : 1, \ldots, m$ such that $g(\ell) = 1$.
6. If no such $\ell$ is found, return 1; otherwise, return 0.

We first describe an algorithm to determine if column $k$ of the LP proves unboundedness, then describe how to perform the ratio test.

**Proposition 8** With bounded probability, if Algorithm 8 (IsUnbounded) returns 1 then $A_B^{-1} A_k \leq \delta 1_m \| A_B^{-1} A_k \|$, with total gate complexity $O(\frac{\sqrt{\kappa}d^2m^{1.5}}{\epsilon})$. Choosing $\delta = 1/\|A_B^{-1} A_k\|$, we can test if $A_B^{-1} A_k \leq \delta 1_m$ with total gate complexity $O(\frac{\sqrt{\kappa}d^2m^{1.5}}{\epsilon})$.

If $\text{IsUnbounded}(A_B, A_k, \delta)$ returns 1, we have a proof that the LP is unbounded from below, up to the given tolerance. Otherwise, we have to perform the ratio test. This is described in the next subroutine. On line 4 of the subroutine, we omit several details for readability; in particular, we use slightly different unitaries $U_r$ to test whether we are above or below the threshold $-\delta/2$. The details are given in the proof of Thm. 11.

**Algorithm 9** Determine the basic variable (row) leaving the basis: $\text{FindRow}(A_B, A_k, b, \delta, t)$.

1. **Input:** Basis $A_B$ with $\|A_B\| \leq 1$, nonbasic column $A_k$, $\text{r.h.s.} \ b$, precision $\delta$.
2. **Output:** index of the row that should leave the basis according to the ratio test 11, with bounded probability.
3. For $r \geq 0$, define a unitary $U_r$ that performs the following operations: it applies the QLSA to solve the system $A_B x = b - rA_k$, then uses amplitude estimation (Thm. 5) and $\text{SignEstNFPN}$ with precision $O(\frac{\|A_B^{-1} A_k\|}{\epsilon})$ to determine if any component $\ell = 1, \ldots, m$ of the solution vector are $< -\frac{\delta}{2}$. If so it returns 1 and a corresponding index $j$, otherwise it returns 0.
4. Perform binary search on $r \geq 0$ to determine a value $\hat{r}$ such that $U_{\hat{r}}$ returns 1 and $U_{\hat{r} + \frac{r}{2^{\lceil \log_2 \frac{\delta}{2m} \rceil}}} \text{returns 0}$.
5. Return the row index $\ell$ identified by $U_r$. 


Theorem 11 Let \( \ell \) be the row index returned by Algorithm 2 (FindRow), and let \( \hat{B} \) be the basis obtained from \( B \) by replacing column \( k \) with the basic variable corresponding to row \( \ell \). Then, with bounded probability, the basic solution corresponding to \( \hat{B} \) is \( \delta \)-feasible, i.e., \( A_{\hat{B}}^{-1}b \geq -\delta 1_m \). The gate complexity of the algorithm is \( O(\sqrt{d} \kappa^2 m^{1.5}) \).

Notice that the ratio test is performed approximately, i.e., the solution found after pivoting might be infeasible, but the maximum infeasibility after pivoting is bounded by \( \delta \). We discuss how to deal with slightly infeasible solutions in Sect. 5.7.

5.7 Two-pass ratio test: improving condition number, and dealing with infeasibilities

The classical simplex method benefits from decades of experience, yielding multiple modifications that, while not necessary in theory, are fundamental for its practical success. One such modification is the two-pass Harris ratio test, a description of which can be found in [Gill et al. 1989]. This modification to the pivoting step is known to improve the condition number of the bases explored in the course of the algorithm. The two-pass Harris ratio test works as follows (to be consistent with the rest of this paper we describe it for a problem in standard form, but it can easily be extended to problems with lower and upper bounds on the decision variables). Let \( \delta \) be the feasibility tolerance. Instead of using (1), we compute, in the first pass:

\[
\tilde{r} := \min_{j=1, \ldots, m, u_j > 0} \frac{x B(j) + \delta}{u_j} \tag{2}
\]

This is equivalent to relaxing \( x \geq 0 \) to \( x \geq -\delta u \). Then, in the second pass, we determine:

\[
\ell := \arg \max_{j=1, \ldots, m, u_j > 0} \left\{ u_j : \frac{x B(j)}{u_j} \leq \tilde{r} \right\}. \tag{3}
\]

In other words, we first determine the minimum value of a relaxed ratio test, then we choose the pivot row as the one that maximizes the value of the pivot element \( u_j \), while still giving a ratio that does not exceed the relaxed ratio test value \( \tilde{r} \), thereby ensuring that \( \delta \)-feasibility is preserved.

Implementing this strategy as part of our quantum subroutines is straightforward. Algorithm 9 already uses the relaxed feasibility definition \( x \geq -\delta u \), see Thm. 11. Let \( \tilde{r} \) be the value computed on line 9 of Algorithm 9. We can use repeated applications of \( U_\ell \) to return all row indices \( \ell \) that satisfy \( (A_{\hat{B}}^{-1}(b - \tilde{r}A_\ell)) \leq -\frac{1}{\sqrt{\kappa}} \) or a similar tolerance. If there are \( t \) such indices, this increases the running time by a factor \( \sqrt{t} \), see, e.g., [Ambainis 2004]; in practice we expect \( t \) to be small, and we do not have to find all indices if it becomes too time consuming. Let \( L \) be the set of such indices. We can find the maximum pivot element among indices in \( L \) by using approximate quantum minimum finding (Thm. 10) combined with the sign estimation subroutines of Sect. 5.2 in a manner similar to the construction discussed in Thm. 14. The running time of this last step is essentially the same as in Thm. 11.

Similarly to practical implementations of the classical simplex algorithm, the basis obtained after a pivot may be slightly infeasible, as controlled by the feasibility tolerance \( \delta \). It is therefore important to discuss how to recover from this situation. We propose two approaches. The first approach is the implementation of a strict zero step for basic variables that are negative after a pivot; see [Gill et al. 1984, Huangfu 2013]. To do so, we simply estimate the value of the basic variable that is leaving the basis, i.e., \( (A_{\hat{B}}^{-1}b) \), where \( \ell \) is the index returned by Algorithm 9. Such an estimate can be computed with Euclidean norm error \( \delta \) in time \( \tilde{O}(2d^2 \kappa^2 m) \), which is faster than Algorithm 9. If the variable leaving the basis has a negative value, we fix it to its slightly infeasible value until it enters the basis again, or the basis is recomputed (see below). Note that fixing a nonbasic variable to some nonzero value is equivalent to adjusting the right-hand side \( b \).

The second approach is a periodic recomputation of the basis to eliminate any infeasibilities. More specifically, given a basis \( B \), we determine if it is feasible according to a specified tolerance, and if it is not, we switch to Phase 1 of the simplex method, possibly increasing precision, to attain feasibility. Since we can increase the precision arbitrarily, we can assume that Phase 1 is always successful if the original problem admits a feasible basis. Note, however, that if Phase 1 is not able to regain feasibility within a number of iterations smaller than the basis recomputation frequency, the algorithm fails (this can also happen in the classical simplex method [Gill et al. 1984]). With the proposed quantum subroutines we do not have classical, unrestricted access to the basic feasible solution \( A_B^{-1}b \). Thus, to check feasibility of the solution associated with a basis \( B \), we can use Algorithm 10 instead, keeping the same running time guarantees.

Proposition 9 With bounded probability, if \( A_B^{-1}b \geq -\delta 1_m \) then Algorithm 14 (IsFeasible) returns 0. The total gate complexity of the algorithm is \( \tilde{O}(\frac{2}{\delta} \kappa^2 d^2 m T^3) \).
Algorithm 10 Determine if a basic solution is feasible: IsFeasible($A_B, b, \delta$).

1. Input: Basis $A_B$ with $\|A_B\| \leq 1$, right-hand side vector $b$, precision $\delta$.
2. Output: 1 if $A_B^{-1}b \geq -\delta 1_m$, 0 otherwise, with bounded probability.
3. Let $U_{LS}$ be the unitary implementing the QLSA for the system $A_Bx = b$ with precision $\frac{\delta}{\|A_B^{-1}b\|}$.
4. Define a function $g(\ell) := (\neg \text{SignEstNFP}(U_{LS}, \ell, \frac{9\delta}{20\|A_B^{-1}b\|})) \land (\text{the success flag for QLSA is 1})$.
5. Use amplitude estimation (Thm. 5) to determine if there exists $\ell \in \{1, \ldots, m\}$ such that $g(\ell) = 1$.
6. If no such $\ell$ is found, return 1; otherwise, return 0.

If IsFeasible returns 1 and the algorithm is successful, the basic solution is feasible with tolerance $\delta$, so that we do not need to change the basis; otherwise, as discussed, we can switch to Phase 1 aiming to minimize infeasibilities.

6 Faster implementation with QRAM

We now describe how to modify the quantum subroutines when QRAM is available. To solve linear systems, we rely on a QLSA that constructs a black encoding of the matrix from QRAM data structures, see e.g., (Chakraborty et al. 2018, Gilven et al. 2019) for a definition. We require a QRAM of size $\tilde{O}(d_n + m)$ to store $b$ as well as all the nonzero entries of $A$, arranged in a data structure similar to the one described in Prop. 1.

For $p \in [0, 1]$, define $\mu(A) := \min \{\|A\|_F, \sqrt{s_{2p}(A)s_{1-p}(A)^T}\}$ where $s_p(A) = \max \sum_j A_{ij}^p$; the running time of the QLSA depends on $\mu(A)$.

**Proposition 10** If the matrix $A_B$ and the columns of $A_N$ are stored in QRAM, we can implement a QLSA with running time $T_{LS}(A_B, A_N, \epsilon) = \tilde{O}(\mu(A_B)\kappa)$.

An iteration of the simplex method proceeds exactly as described in Alg. 1, replacing each call to the QLSA of Chakraborty et al. (2017) in the subroutines with the QLSA of Chakraborty et al. (2018), as in Prop. 10.

The running time of FindColumn is $\tilde{O}(\sqrt{\kappa}(T_{LS}(A_B, A_k, \frac{\delta}{\kappa}) + m))$, and by Prop. 10 this is $\tilde{O}(\sqrt{\kappa}\mu(A_B)\kappa + m)$). Notice that because $A_B$ is $m \times m$ and $\|A_B\| = 1$, $\mu(A_B) \leq \sqrt{m}$, hence we obtain running time $\tilde{O}(\sqrt{\kappa}\sqrt{m} \kappa)$. The running time calculation to determine if the basis is optimal is similar. With the steepest edge version of pricing, the running time increases by a factor $c_{\max}$ once again. The running time of FindRow is $\tilde{O}(\sqrt{\kappa^2}(T_{LS}(A_B, A_k, \frac{\delta}{\kappa}A_k^{\top}(b - r_A)))$, and by Prop. 10 this is $\tilde{O}(\sqrt{\kappa^2}\mu(A_B))$; using the upper bound $c_{\max}$, we can upper bound it by $\tilde{O}(\sqrt{\kappa^2}\mu(A_B))$. Similar calculations for the subroutines IsUnbounded and IsFeasible yield running time $\tilde{O}(\sqrt{\kappa^2}\mu(A_B))$. Finally, we remark that at every iteration of Alg. 1 on top of the $\tilde{O}(md)$ classical operations to normalize data, we need to update $\tilde{O}(m)$ memory locations in QRAM, as discussed in Prop. 10.

7 Future research

This paper proposes several quantum subroutines for simplex method. For several reasons, it is conceivable that the factors $\sqrt{m}, d_n, \kappa$ in the running time of the pricing step cannot be further decreased when using a similar scheme to what is presented in this paper. This is based on the following observations: first, the dependence of QLSA cannot be improved to $\kappa^{1-\delta}$ for $\delta > 0$ unless BQP = PSPACE (Harrow et al., 2009); second, the factor $O(\sqrt{m})$ is optimal for quantum search, relative to an oracle that identifies the acceptable solutions (Bennett et al., 1997); and third, to be able to determine if a column has negative reduced cost (in superposition), we would expect that the running time cannot be reduced below $O(d_n)$ in general (since this is the number of nonzero elements in the matrix). However, it is possible to exploit structure in the constraint matrix to reduce this factor: this requires specialized data preparation algorithms, but it may be worth the effort for certain structures that appear frequently in LPs. Regarding the condition number, even though improving the theoretical dependency may not be possible, there could exist efficient practical strategies to keep it under control: this is routinely done in classical implementations of the simplex method, and some examples are of this discussed in this paper. A systematic study of such strategies, in theory and in practice, could prove informative. Of course, it may be possible to give altogether better algorithms using a different scheme than the one presented here.
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A Proofs

A.1 Proofs from Section 4.1

Proof. Proof of Thm. 2. Note that, by virtue of how the comparison condition is structured, in each application of the quantum search algorithm there is always at least one “good” element. Thus, the proof and the running time analysis given by Durr and Hoyer (1996) for the case $\epsilon = 0$ (i.e., exact minimum finding) also work for $\epsilon > 0$, using the same argument given by Durr and Hoyer (1996, Sect. 3) for the case in which the values of $f(x)$ are not distinct. \hfill \blacksquare
A.2 Proofs from Section 5.1

Proof. Proof of Prop. 4. We can rely on the construction of Grover and Rudolph (2002), that consists in a sequence of controlled rotations. The construction can be understood on a binary tree with \( m \) leaves, representing the elements of \( b \), and where each inner node has value equal to the sum of the squares of the children nodes. Each inner node requires a controlled rotation with an angle determined by the square root of the ratio between the two children nodes. Assuming that \( m \) is a power of 2 for simplicity, the tree has \( \log m \) levels and the construction when \( b \) is dense (\( d = m \)) requires \( \sum_{j=0}^{\log m} 2^j = O(m) \) controlled rotations. Notice that at each inner node, a controlled rotation is necessary only if both children have nonzero value. If only one child has nonzero value the operation requires at most a controlled rotation. The construction when \( d < m \), we can take at most \( d \) nodes at each level that require a controlled rotation, and in fact the deepest level of the binary tree such that all nodes contain nonzero value is \( \lceil \log d \rceil \).

We need \( O(d) \) controlled rotations up to this level of the tree, and for each of these nodes we may need at most \( O(\log m) \) subsequent operations, yielding the total gate complexity \( \tilde{O}(d) \).

A.3 Proofs from Section 5.2

Proof. Proof of Prop. 3. After line 3 using Prop. 2 we are in the state:

\[
\frac{1}{2}(1 + \alpha_k)|0\rangle \otimes |k\rangle + \frac{1}{2}(1 - \alpha_k)|1\rangle \otimes |k\rangle + \frac{1}{2}(0)|0\rangle \otimes \sum_{j=0}^{2^n-1} \alpha_j |j\rangle.
\]

We now apply amplitude estimation to the state \(|0\rangle \otimes |k\rangle\) to determine the magnitude of \( \frac{1}{2}(1 + \alpha_k) \). The exact phase that must be estimated by the phase estimation portion of the algorithm is the number \( \theta \) such that:

\[
\sin \pi \theta = \frac{1}{2}(1 + \alpha_k).
\]

Suppose \( \alpha_k \geq -\epsilon \). Then \( \sin \pi \theta \geq \frac{1}{2}(1 - \epsilon) \), implying, by monotonicity of \( \sin^{-1} \) over its domain:

\[
\theta > \sin^{-1}\left(\frac{1}{2}(1 - \epsilon)\right) \geq \frac{\pi}{6} + \frac{\epsilon}{\sqrt{3}}\left(\frac{1}{2}(1 - \epsilon) - \frac{1}{2}\right) \geq \frac{\pi}{6} - \frac{\epsilon}{\sqrt{3}} \geq \frac{1}{6} - \frac{\epsilon}{\sqrt{3}},
\]

where for the second inequality we have used the Taylor expansion of \( \sin^{-1}(x) \) at \( x = \frac{1}{2} \):

\[
\sin^{-1}(x) \approx \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - \frac{1}{2}) + \frac{2\sqrt{3}}{9}(x - \frac{1}{2})^2.
\]

Rather than \( \theta \), we obtain an approximation \( \hat{\theta} \) up to a certain precision. By Thm. 3 using \( \left\lceil \log \frac{2\epsilon}{\pi} \right\rceil + 2 \) qubits of precision, then

\[
|\theta - \hat{\theta}| \leq \frac{\epsilon}{\sqrt{3}\pi}, \quad (4)
\]

with probability at least \( 3/4 \). Then we must have \( \hat{\theta} \geq \frac{1}{6} - \frac{\epsilon}{\sqrt{3}\pi} \), with probability at least \( 3/4 \). In this case, the algorithm returns 1 (recall that if the first bit of the expansion is 1, i.e., \( 0.a > \frac{1}{2} \), we must take the complement \( 1 - 0.a \) because we do not know the sign of eigenvalue on which phase estimation is applied, see (Brassard et al. 2002) for details).

Now suppose the algorithm returns 1, implying \( \hat{\theta} \geq \frac{1}{6} - \frac{\epsilon}{\sqrt{3}\pi} \). By (4) we must have \( \theta \geq \hat{\theta} - \frac{\epsilon}{\sqrt{3}\pi} \geq \frac{1}{6} - \frac{3\epsilon}{\sqrt{3}\pi} \).

Thus,

\[
\alpha_k = 2 \sin \pi \theta - 1 \geq 2 \sin\left(\frac{\pi}{6} - \frac{\epsilon}{\sqrt{3}\pi}\right) - 1 \geq 2\left(\frac{1}{2} \cos(-\sqrt{3}\epsilon) + \frac{\sqrt{3}}{2} \sin(-\sqrt{3}\epsilon)\right) - 1
\]

\[
\geq 2\left(\frac{1}{2}(1 - \frac{2\sqrt{3}}{\pi}\epsilon) - \frac{3\epsilon}{\pi}\right) - 1 \geq -\frac{\epsilon}{\pi}(3 + 2\sqrt{3}) \geq -2\epsilon.
\]

The remaining part of the proposition’s statement follows immediately from the first part.

Regarding the gate complexity, amplitude estimation with \( O(\log \frac{1}{\epsilon}) \) digits of precision requires \( O(\frac{1}{\epsilon}) \) calls to \( U \), controlled-\( U \), or controlled-\( U^\dagger \), and the reflection circuits of the Grover iterator (which can be implemented
Algorithm 11: Sign estimation routine: SignEstNFP(U, k, \epsilon).

1: **Input:** state preparation unitary \(U\) on \(q\) qubits (and its controlled version) with \(U|0^q\rangle = \sum_{j=0}^{2^q-1} \alpha_j |j\rangle\) and \(\alpha_j\) real up to a global phase factor, index \(k \in \{0, \ldots, 2^q - 1\}\), precision \(\epsilon\).
2: **Output:** 0 if \(\alpha_k \leq -\epsilon\), with probability at least \(3/4\).

3: Let \(V\) map \(|0^q\rangle\) to \(|k\rangle\). Compute \(|\psi\rangle = \text{INTERFERE}(U, V)\).
4: Apply amplitude estimation to the state \(|0\rangle \otimes |k\rangle\) with \(\lceil \log \frac{2\sqrt{\pi\epsilon}}{\epsilon} \rceil + 2\) qubits of accuracy; let \(|a\rangle\) be the bitstring produced by the phase estimation portion of amplitude estimation.
5: If \(0.a < \frac{1}{2}\), return 1 if \(0.a > \frac{1}{6} - \frac{2\epsilon}{3\sqrt{3}\pi}\), 0 otherwise; if \(0.a \geq \frac{1}{2}\), return 1 if \(1 - 0.a > \frac{1}{6} - \frac{2\epsilon}{3\sqrt{3}\pi}\), 0 otherwise.

**Proof. Proof of Prop. 4.** The proof is similar to Prop. 3 and we use the same symbols and terminology. We apply amplitude estimation to the state \(|0\rangle \otimes |k\rangle\) to determine the magnitude of \(\frac{1}{2}(1 + \alpha_k)\). Suppose \(\alpha_k \leq -\epsilon\). Then \(\sin \pi \theta \leq \frac{1}{2}(1 - \epsilon)\), implying:

\[
\theta \leq \frac{\sin^{-1}\left(\frac{1}{2}(1 - \epsilon)\right)}{\pi} \leq \frac{\frac{2}{\sqrt{3}}(\frac{1}{2}(1 - \epsilon) - \frac{1}{2}) + \frac{8\sqrt{3}}{27}(x - \frac{1}{2})^3}{\pi} \\
\leq \frac{\frac{2}{\sqrt{3}} - \frac{1}{2} + \frac{8\sqrt{3}}{27}}{\pi} \leq \frac{1}{6} - \frac{\sqrt{\epsilon}}{3\sqrt{3}\pi},
\]

where we used \(\epsilon \leq \frac{1}{2}\), and for the second inequality we have used the Taylor expansion of \(\sin^{-1}(x)\) at \(x = \frac{1}{2}\):

\[
\sin^{-1}(x) \approx \frac{\pi}{2} + \frac{2\sqrt{3}}{9}(x - \frac{1}{2}) + \frac{2\sqrt{3}}{9}(x - \frac{1}{2})^2 + \frac{8\sqrt{3}}{27}(x - \frac{1}{2})^3,
\]

coupled with the fact that the third-order term is negative at \(\frac{1}{2}(1 - \epsilon)\). Rather than \(\theta\), we obtain an approximation \(A\theta\) up to a certain precision. By Thm. 4 using \(\lceil \log \frac{2\sqrt{\pi\epsilon}}{\epsilon} \rceil + 2\) qubits of precision, then

\[
|\theta - \hat{\theta}| \leq \frac{\epsilon}{9\sqrt{3}\pi}
\]

(5)

with probability at least \(3/4\). Then we must have \(\hat{\theta} \leq \frac{1}{\pi} - \frac{2\epsilon}{3\sqrt{3}\pi}\) with probability at least \(3/4\). In this case, the algorithm returns 0.

Now suppose the algorithm returns 0, implying \(\hat{\theta} \leq \frac{1}{\pi} - \frac{2\epsilon}{3\sqrt{3}\pi}\). By \(4\) we must have so that \(\theta \leq \hat{\theta} + \frac{\epsilon}{9\sqrt{3}\pi} \leq \frac{1}{\pi} + \frac{\epsilon}{9\sqrt{3}\pi}\). Thus,

\[
\alpha_k = 2\sin \pi \theta - 1 \leq 2\sin \frac{\pi}{6} + \frac{5\epsilon}{9\sqrt{3}} - 1 \leq \left(\frac{1}{2} \cos \frac{5\epsilon}{9\sqrt{3}} + \frac{\sqrt{3}}{2} \sin \frac{5\epsilon}{9\sqrt{3}}\right) - 1
\]

\[
\leq \frac{5}{9\sqrt{3}} \epsilon \leq \frac{\epsilon}{3}.
\]

The remaining part of the proposition’s statement follows immediately from the first part, and the running time analysis is the same as in Prop. 3.

**A.4 Proofs from Section 5.3.**

**Proof. Proof of Prop. 4.** Let us analyze \(\text{REDCost}(A_B, A_k, c, \epsilon)\). The QLSA encodes the solution to:

\[
\begin{pmatrix} A_B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_k \\ c_k \end{pmatrix}
\]

(6)
in a state $|\psi\rangle$ that is guaranteed to be $\frac{1}{\sqrt{2}}$ close to the exact normalized solution $|(A_B^{-1}A_k,c_k)\rangle$. Call this state $|x,y\rangle$, where $x,y$ correspond to the (approximate) solution of $\mathfrak{Q}$.

On line 4 we apply the unitary $U_1$, obtaining $U_1^{|\langle x,y\rangle\rangle}$. We are now interested in tracking the value of the coefficient of the basis state $|0|^{\log m+1}\rangle$, which is the input to the SignEstNFN routine on line 4 of CANENTER. This coefficient is equal to:

$$\langle \langle 0|^{\log m+1} |U_1^{|\langle x,y\rangle\rangle} \rangle \rangle = \langle \langle -c_B,1 | (x,y) \rangle \rangle,$$

because by definition $\langle \langle 0|^{\log m+1} |U_1^{|\langle x,y\rangle\rangle} \rangle \rangle = \langle \langle c_B,1 | \rangle \rangle$. Furthermore,

$$\langle \langle -c_B,1 | (x,y) \rangle \rangle = \left| \frac{1}{||-c_B,1||} \left( y - \sum_{j=1}^{m} c_{B(j)} x_j \right) \right|.$$

Again by definition, $y$ is approximately equal to $c_k/||A_B^{-1}A_k,c_k||$ whereas $\sum_{j=1}^{m} c_{B(j)} x_j$ is approximately equal to $c_B A_B^{-1} A_k/||A_B^{-1}A_k,c_k||$. The total error is $\frac{1}{10 \sqrt{2}}$, hence, recalling that $||-c_B,1|| = \sqrt{2}$, we have:

$$\left| \frac{1}{\sqrt{2}} \left( y - \sum_{j=1}^{m} c_{B(j)} x_j \right) - \frac{1}{\sqrt{2}} \frac{c_k}{||A_B^{-1}A_k,c_k||} \right| \leq \frac{\epsilon}{10 \sqrt{2}}.$$

This concludes the proof. \hfill $\Box$

**Proof. Proof of Prop. 8** CANENTER relies on Algorithms 3 (SignEstNFN) and 4 (RedCost).

On line 4 of CANENTER, we apply SignEstNFN to the unitary that implements RedCost, with $|0|^{\log m+1}\rangle$ as the target basis state and precision $\frac{11 \epsilon}{10 \sqrt{2}}$. Suppose this returns 0. By Prop. 3 and Prop. 5 we have:

$$-\frac{11 \epsilon}{10 \sqrt{2}} \geq \frac{1}{\sqrt{2}} \left( \tilde{c}_k - \sum_{j=1}^{m} c_{B(j)} x_j \right) \geq \frac{1}{\sqrt{2}} \frac{\tilde{c}_k}{||A_B^{-1}A_k,c_k||} - \frac{\epsilon}{10 \sqrt{2}}.$$

This implies:

$$\frac{\tilde{c}_k}{||A_B^{-1}A_k,c_k||} \leq -\epsilon.$$

The above discussion guarantees that if the return value of the CANENTER is 1, then $\tilde{c}_k < -\epsilon(||A_B^{-1}A_k,c_k||)$, with probability at least 3/4, as desired. The gate complexity is easily calculated: SignEst makes $O(\frac{1}{\epsilon})$ calls to RedCost and requires an additional $O(\log m + \log^2 \frac{n}{\epsilon})$ gates. RedCost requires one call to the QLSA and additional $O(m)$ gates for $U_s$. Thus, the total gate complexity is $O(\frac{2}{\epsilon} T_{LS}(A_B,A_k) + m + \log^2 \frac{n}{\epsilon})$. \hfill $\Box$

**Proof. Proof of Thm. 9** FindColumn relies on three subroutines: $U_{rbs}$, CANENTER, and quantum search, as described in Thm. 4. $U_{rbs}$ can be constructed by repeatedly applying the procedure of Prop. 1 controlled on the register containing the column index $k$; the total gate complexity is $O(d_k n)$. By Prop. 8 if the routine CANENTER returns 1 then the reduced cost of column $A_k$ is $< -\epsilon(||A_B^{-1}A_k,c_k||)$ with respect to the rescaled data, with bounded probability. We can boost this probability with repeated applications and a majority vote to make it as close to 1 as desired. Notice that CANENTER is applied to the rescaled data. In terms of the original data, the condition on the reduced cost becomes $\tilde{c}_k < -\epsilon(||A_B^{-1}A_k,c_k||)$, as claimed in the theorem statement (the rescaling of $A_k$ and $A_B^{-1}$ cancel out).

Finally, we apply the quantum search algorithm (Thm. 4) using CANENTER as the target function. The expected number of iterations before success is $O(\sqrt{n})$; the gate complexity is therefore $O(\sqrt{n} (T_{LS}(A_B,A_k) + m))$. By Thm. 4 $T_{LS}(A_B,A_k) = O(kdd_n + kd^2 m)$, because $P_{A_B}$ has gate complexity $O(dm)$ and $P_B$ is the routine $U_{rbs}$, which has gate complexity $O(d_k n)$. Thus, we obtain a total gate complexity of $O(\sqrt{n} T_{LS}(A_B,A_k) + d_k n + dm)$, as claimed. \hfill $\Box$

**Proof. Proof of Thm. 10**

The main ingredient of the proof is to analyze the subroutine SteepestEdgeCompare, Alg. 2 which is used to run the quantum minimum finding algorithm of Thm. 6.

At Steps 3-4 we compute norm estimates using Thm. 8. (In a practical implementation, the estimates at Step 4 could be derived from those at Step 3 with properly adjusted error tolerances.) This requires time $O(\sqrt{d_k n} T_{LS}(A_B,A_k,\epsilon))$.

Recall that $c_{max} := \max_{e \in E} c_e$. At Steps 5-6 we define two unitaries whose first building block is RedCost. By Prop. 5 RedCost encodes an approximation of $\frac{1}{||A_B^{-1}A_j,c_j||}$ with additive error at most $\frac{\epsilon}{10}$.
in the amplitude of the all-zero basis states; let $\alpha_0, \beta_0$ be these amplitudes. We then multiply $\alpha_0$ by $\frac{d_j}{c_{\text{max}}}$ and $\beta_0$ by $\frac{d_j}{c_{\text{max}}}$. Note that these terms are $\leq 1$ thanks to the factor $c_{\text{max}}$ at the denominator, and multiplying the all-zero basis state by such a coefficient is simply an application of a unitary with that coefficient in the top-left corner. It is known that such a unitary can be efficiently constructed with high precision with cost $O(1)$; for an explicit construction, see, e.g., (Gilyén et al. 2019, Lemma 48).

We then apply $\text{INTERFERE}$ and $\text{SignEstNFN}$, to estimate the sign of $\frac{1}{2}(\beta_0 - \alpha_0)$ with error $\frac{\epsilon}{c_{\text{max}}}$. Suppose \( \frac{\bar{c}_k}{\|A_B^{-1}A_k\|} < (1 - \epsilon) \frac{c_{\text{max}}}{\|A_B^{-1}A_j\|} \) - $\epsilon$. We have:

$$\beta_0 \leq \left( \frac{\bar{c}_k}{\|A_B^{-1}A_k\|} + \frac{\epsilon}{10} \right) \frac{\|A_B^{-1}A_k, c_k\|}{c_{\text{max}} \|A_B^{-1}A_k\|} \right) \left( 1 + \frac{\epsilon}{2} \right) \frac{\bar{c}_j}{\|A_B^{-1}A_j\|} - \frac{\epsilon}{c_{\text{max}}} + \frac{\epsilon}{c_{\text{max}}} \leq \alpha_0 - \frac{\epsilon}{c_{\text{max}}}.
$$

By Prop. 5 this implies that $\text{SignEstNFN}$ returns $0$. Thus, applying the quantum minimum finding algorithm of Thm. 6 returns an approximate minimizer in $O(\sqrt{n})$ iterations, i.e., a column $k$ such that:

$$\bar{c}_k \left\| A_B^{-1}A_k \right\| \leq (1 + \epsilon) \bar{c}_j \left\| A_B^{-1}A_j \right\| + \epsilon,$$

where $h$ is the arg min as defined in the theorem statement.

The running time is obtained by multiplying the gate complexity of the STEEPESTEDGECOMPARE subroutine by the number of iterations $O(\sqrt{n})$. In each call to STEEPESTEDGECOMPARE, we perform a constant number of norm estimations using Thm. 6, we apply $\text{REDCost}$ and $\text{SignEstNFN}$ with precision $O(\frac{1}{c_{\text{max}}})$.

By Prop. 5 and Prop. 6 the running time is $O(\frac{d^2}{\epsilon}(d,n + dm))$.

### A.5 Proofs from Section 5.5

**Proof. Proof of Prop. 4.** The algorithm works as follows. We apply the unitary $U_{\text{for}}$ that maps $|0^{\log n} \rangle \otimes |0^{\log m} \rangle \rightarrow \sum_{k \in N} |k \rangle \otimes |A_k \rangle$; we call the first register, that loops over $k \in N$, the “column register”. This unitary can be constructed with $\tilde{O}(d,n)$ gates, using Prop. 4 (The time to classically compute the norms can be ignored, as this only needs to be done once, and its time complexity is less than the total complexity of the algorithm.) We then apply the QLSA of (Childs et al. 2017, Thm. 3), using $U_{\text{for}}$ as the oracle $P_B, P_{\text{SteepestEdge}}$ as the oracle for the constraint matrix, and precision $\frac{\epsilon}{\sqrt{n}}$. As a result, conditioned on the column register being $|k \rangle$, we obtain $|A_B^{-1}A_k \rangle$ in the output register of the QLSA algorithm. Following (Childs et al. 2017), there exists an auxiliary register that has value $|0 \rangle$ with amplitude $\frac{1}{\alpha} \|A_B^{-1}A_k\| = \frac{1}{\alpha} \|A_B^{-1}A_k\|$, where $\alpha$ is a known number with $\alpha = O(\sqrt{n \log(nk) / \epsilon})$, and $A_B^{-1}$ is an operator that is $\frac{\epsilon}{\sqrt{n}}$-close to $A_B^{-1}$ in the operator norm. Since this is true for all $k$, the probability of obtaining $|0 \rangle$ in the auxiliary register is:

$$\sum_{k \in N} \frac{\|A_k\|^2}{\|A_N\|^2} \frac{1}{\alpha^2 \|A_k\|^2} \left\| A_B^{-1}A_k \right\|^2 = \frac{\|A_B^{-1}A_N\|^2}{\alpha^2 \|A_N\|^2}. \tag{7}$$

Using Thm. 6 to estimate this probability, amplitude estimation with precision $\frac{\epsilon}{4\alpha^2}$ yields an estimate $\tilde{\alpha}$ of $\alpha$ with error $\pm \left( \frac{\epsilon}{4\alpha^2} + \frac{\epsilon^2}{16\alpha^4} \right)$, see (Brassard et al. 2002, Thm. 12). Our estimate for $\|A_B^{-1}A_N\|^2$ is $\rho := \tilde{\alpha}^2 \|A_N\|^2$. We have:

$$\rho \leq \frac{\|A_B^{-1}A_N\|^2}{\|A_B^{-1}A_N\|^2} \left( \frac{\|A_B^{-1}A_N\|^2}{\alpha^2 \|A_N\|^2} \right) \left( \frac{\epsilon}{4\alpha^2} + \frac{\epsilon^2}{16\alpha^4} \right) \leq 1 + \frac{\epsilon}{2} + \frac{\epsilon^2}{4\alpha^2} \left( \frac{\epsilon}{4\alpha^2} + \frac{\epsilon^2}{16\alpha^4} \right) \leq 1 + \epsilon,$$

where we have used the fact that $\frac{\|A_B^{-1}A_N\|^2}{\alpha^2 \|A_N\|^2} \leq 1$ because the smallest singular value of of $A_B^{-1}$ is 1, and the fact that we can assume $\epsilon \leq \alpha^2$ so that $\frac{\epsilon^2}{16\alpha^4} \leq \frac{\epsilon}{16\alpha^4}$. A similar calculation yields the lower bound, yielding the desired approximation. Regarding the running time, amplitude estimation with precision $\frac{\epsilon}{4\alpha^2}$ requires $\frac{\alpha^2}{\epsilon} = \hat{O}(\frac{\epsilon}{\sqrt{n}})$ applications of the QLSA. The running time for the QLSA is $\hat{O}(nk,dm)$, because $P_B$ has gate complexity $\hat{O}(dm)$ and $P_{\text{SteepestEdge}}$, which has gate complexity $\hat{O}(d,n)$. Thus, we obtain a total running time of $\hat{O}(\frac{\epsilon}{\sqrt{n}}(kd,n + kd^2m))$. \qed
A.6 Proofs from Section 5.6

Proof. Proof of Prop. [5] Let \( \bar{x} \) be the state produced by the QLSA, where \( \bar{x} \) is an approximate solution to the linear system; by Thm. [7] we have \( \| \bar{x} - \frac{A^\top_j A_k}{\| A^\top_j A_k \|} \| \leq \frac{\delta}{m} \).

Suppose the algorithm returns 1; this implies that the index \( \ell \) is not found. Then if all algorithms are successful, the routine \text{SignEstNFN}\(^*\) (\( U_{LS}, \ell, \frac{\delta}{m} \)) must have returned 0 for all \( \ell \). By Prop. [3] \( \bar{x} \leq \frac{\delta}{m} \), thus:

\[
\left( \frac{A^\top_j A_k}{\| A^\top_j A_k \|} \right) \leq \bar{x}_\ell + \frac{\delta}{10} < \frac{9\delta}{10} + \frac{\delta}{10} = \delta,
\]

which implies \( (A^\top_j A_k)/ \| A^\top_j A_k \| < \delta \| A^\top_j A_k \| \). Thus, if \text{ISUnbounded} returns 1 and all algorithms are successful, it must be that \( (A^\top_j A_k)/ \| A^\top_j A_k \| \) for all \( \ell \), which is the condition in the statement of the proposition; in this case, the LP is indeed unbounded.

We now analyze the running time. To determine with constant probability if the sought index \( \ell \) exists, i.e., if \( g(\ell) = 1 \) for some value of \( \ell \), we apply amplitude estimation with \( O(\sqrt{m}) \) applications of the Grover operator, see Thm. [4]. Each application takes time \( O(\frac{1}{\delta}kd^2m) \), which is the complexity of running the QLSA and the sign estimation. The probability of success for the QLSA is bounded, and we have a way of determining success, therefore we can boost the probability of correctness arbitrarily high with enough repetitions of the algorithm.

\[
\text{Proof of Thm. [11].} \quad \text{Proof.} \quad \text{Proof of Prop. [8].} \quad \text{Let } \|
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\text{Proof of Prop. [8].} \quad \text{Let } \|
A.7 Proofs from Section 5.7

Proof. Proof of Prop. 9. Let $|\tilde{x}\rangle$ be the state produced by the QLSA; by Thm. 7, we have $\|\tilde{x} - \frac{A^{-1}b}{\|A^{-1}b\|}\| \leq \frac{\delta}{10}$.

Suppose $A^{-1}b \not\geq -\delta 1_m$, i.e., the basic solution is infeasible. Then, for some index $\ell$, we must have

$$\left(\frac{A^{-1}b}{\|A^{-1}b\|}\right)_\ell \|A^{-1}b\| < -\delta \|A^{-1}b\|.$$ 

This implies:

$$\tilde{x}_\ell \leq \frac{(A^{-1}b)_\ell}{\|A^{-1}b\|} + \frac{\delta}{10\|A^{-1}b\|} < -\frac{\delta}{10\|A^{-1}b\|} = -\frac{9\delta}{10\|A^{-1}b\|}.$$ 

By Prop. 3 if the routine $\text{SignEstNFN}(U_{LS}, \ell, \frac{9\delta}{20\|A^{-1}b\|})$ is successful it returns zero, so that the function $g(\ell)$ evaluates to 1. This implies that if all subroutines are successful, $\text{IsFeasible}$ returns 0, as desired. The running time analysis is the same as in Prop. 9.$\square$

A.8 Proofs from Section 6

Proof. Proof of Prop. 10. By (Gilyén et al. 2019, Lemma 50), we can construct a $(\mu(A_B), O(\log n), \epsilon/(\kappa^2 \log^3 \frac{3}{\epsilon}))$ block encoding for $A_B$ using suitable data structures, in time $O(1)$. Using the techniques in (Chakraborty et al. 2018, Sect. 4.3), we can then compute a normalized version of $A^{-1}_B |x\rangle$ in time $O(\mu(A_B) \kappa)$.

The QRAM data structures describing $A_B$ and $A_N$ can be prepared in the claimed time following Kerenidis and Prakash (2018) and Prop. 1. After each iteration of the simplex method, we simply need to reindex the structures in memory, i.e., swap one nonbasic column with a basic column, which takes at most $O(m)$ operations since each column has size $m$. $\square$