DUAL CANONICAL BASIS FOR UNIPOTENT GROUP AND BASE AFFINE SPACE

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Abstract. Denote by $N \subset SL_k$ the subgroup of unipotent upper triangular matrices. In this paper, we show that the dual canonical basis of $\mathbb{C}[N]$ can be parameterized by semi-standard Young tableaux. Moreover, we give an explicit formula for every element in the the dual canonical basis. Let $N^- \subset SL_k$ be the subgroup of unipotent lower triangular matrices and let $\mathbb{C}[SL_k]^{N^-}$ be the coordinate ring of the base affine space $SL_k/N^-$. Denote by $\mathbb{C}[\widehat{SL_k}^{N^-}]$ the quotient of $\mathbb{C}[SL_k]^{N^-}$ by identifying the leading principal minors with 1. We also give an explicit description of the dual canonical basis of $\mathbb{C}[\widehat{SL_k}^{N^-}]$ and give a conjectural description of the dual canonical basis of $\mathbb{C}[SL_k]^{N^-}$.

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1. Introduction

Quantum groups (or quantized universal enveloping algebras) was introduced independently by Drinfeld [15] and Jimbo [31] around 1985.

Let $\mathfrak{g}$ be a simple complex Lie algebra of type $A, D, E$. Denote by $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ a triangular decomposition of $\mathfrak{g}$. Let $v$ be an indeterminate and let $U_v(\mathfrak{g}) = U_v(\mathfrak{n}) \otimes U_v(\mathfrak{h}) \otimes U_v(\mathfrak{n}^-)$ be the Drinfeld-Jimbo quantum group over $\mathbb{C}(v)$. Inspired by a seminal work of Ringel [50], Lusztig introduced a canonical basis $\mathbf{B}$ of $U_v(\mathfrak{n})$ with remarkable properties in [43, 44]. In
[33], Kashiwara found an alternative approach to the canonical basis of [43] which made sense in the more general context of Kac-Moody Lie algebras.

The quantum algebra $U_v(n)$ is endowed with a distinguished scalar product. Let $B^*$ be the basis of $U_v(n)$ adjoint to the canonical basis $B$ with respect to this scalar product. The graded dual $A_v(n)$ of $U_v(n)$ can be regarded as the quantum coordinate ring of the unipotent group $N$ with Lie algebra $n$ (see e.g. [27, 30]). The basis $B^*$ can be identified with a basis of $A_v(n)$ called the dual canonical basis. When $v \to 1$, the basis $B^*$ specializes to a basis of the coordinate ring $\mathbb{C}[N]$ and it is called the dual canonical basis of $\mathbb{C}[N]$.

Canonical basis and dual canonical basis (in particular, the dual canonical basis of $\mathbb{C}[N]$) has been studied intensively in the literature using different methods and many important results are obtained, see e.g. [3, 4, 5, 6, 20, 21, 22, 23, 24, 25, 26, 27, 30, 32, 36, 38, 39, 40, 48, 49, 51, 52].

On the other hand, more work is needed to give a full description of the dual canonical basis, see e.g. the paragraph before the last paragraph of Section 2 in [25].

The aim of this paper is to give an explicit description of the dual canonical basis of $\mathbb{C}[N]$ in the case that $N \subset SL_k$ is the subgroup of unipotent upper triangular matrices, and the dual canonical basis of $\mathbb{C}[SL_k] N^-$ which is closely related to $\mathbb{C}[N]$.

Let $N^- \subset G = SL_k$ be the subgroup of unipotent lower triangular matrices. The group $N^-$ acts on $G$ by left multiplication. Denote by $\mathbb{C}[SL_k] N^-$ the ring of $N^-$-invariant regular functions on $SL_k$. Explicit description of the dual canonical basis of $\mathbb{C}[SL_k] N^-$ is still an open problem, see e.g. the end of Section 6.5 in [19].

The ring $\mathbb{C}[N]$ has a cluster algebra structure which can be obtained from a cluster algebra structure on $\mathbb{C}[SL_k] N^-$ by identifying leading principal minors with 1 [19]. Denote by $\mathbb{C}[SL_k] N^-$ the quotient of $\mathbb{C}[SL_k] N^-$ by identifying the leading principal minors with 1. The algebras $\mathbb{C}[N]$ and $\mathbb{C}[SL_k] N^-$ have the same cluster algebra structure (cf. Section 2.2).

Denote by $SSYT(k − 1, [k], ∼)$ a certain quotient of the monoid $SSYT(k − 1, [k])$ of semi-standard tableaux with at most $k − 1$ rows and with entries in $[k]$ (cf. Section 3). Our main result is the following.

**Theorem 1.1** (Theorems 4.8 and 5.3). The set $\{ch_{\mathbb{C}[N]}(T) : T \in SSYT(k − 1, [k], ∼)\}$ (respectively, $\{ch_{\mathbb{C}[SL_k] N^-}(T) : T \in SSYT(k − 1, [k], ∼)\}$) is the dual canonical basis of $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k] N^-$), where

$$ch_{\mathbb{C}[N]}(T) = \sum_{u \in S_m} (-1)^{f(u_{\text{wr}})} p_{uw_0, w_T w_0} (1) \Delta_{u, T'} \in \mathbb{C}[N],$$

$$ch_{\mathbb{C}[SL_k] N^-}(T) = \sum_{u \in S_m} (-1)^{f(u_{\text{wr}})} p_{uw_0, w_T w_0} (1) \Delta_{u, T'} \in \mathbb{C}[SL_k] N^-,$$

$T' \sim T$, the columns of $T'$ are fundamental tableaux, $T'$ has $m$ columns, $w_0 \in S_m$ is the longest permutation, $w_T \in S_m$ is determined by $T$, $\Delta_{u, T'}$ is the product of certain flag minors related to $T'$, and $p_{y, y'}(t)$ is a Kazhdan-Lusztig polynomial [37].

The difference between the formulas for $ch_{\mathbb{C}[N]}(T)$ and $ch_{\mathbb{C}[SL_k] N^-}(T)$ is that the flag minors in the formula for $ch_{\mathbb{C}[N]}(T)$ are flag minors in $\mathbb{C}[N]$ while the flag minors in the formula for
ch\(\sim_{C[SL_k]}(T)\) are flag minors in \(\mathbb{C}[SL_k]^{N^-}\). We write \(ch_{C[N]}(T)\) (respectively, \(ch_{C[SL_k]}(T)\)) as \(ch(T)\) if there is no confusion.

To prove Theorem 1.1, we applied Hernandez-Leclerc’s monoidal categorification of \(C[N]\) [30], a \(q\)-character formula in [14, Theorem 1.3] which is obtained from a result due to Arakawa-Suzuki [1] (see also Section 10.1 in [42], and [2, 28]) and from the quantum affine Schur-Weyl duality [12], and the following theorem.

**Theorem 1.2** (Theorem 4.6). There is an isomorphism \(\mathcal{P}_{k,\Delta}^+ \to SSYT(k - 1, [k], \sim)\) of monoids.

Here \(\mathcal{P}_{k,\Delta}\) is a certain submonoid of the monoid of dominant monomials (cf. Section 2.3).

By Theorem 1.1, the dual canonical basis of \(C[N]\) (respectively, \(C[SL_k]^{N^-}\)) is parametrized by semi-standard tableaux in \(SSYT(k - 1, [k], \sim)\) and every dual canonical basis element is of the form \(ch(T)\) for some \(T \in SSYT(k - 1, [k], \sim)\). In [36, 49], it is shown that cluster monomials in \(C[N]\) (respectively, \(C[SL_k]^{N^-}\)) belong to the dual canonical basis. Therefore every cluster variable in \(C[N]\) (respectively, \(C[SL_k]^{N^-}\)) is also of the form \(ch(T)\).

**Example 1.3.** The cluster variables (not including frozen variables) of \(C[N]\), \(N \subset SL_4\), (respectively, \(C[SL_4]^{N}\)) are \(ch(T)\), where \(T\)'s are the following tableaux:

\[
\begin{array}{cc}
2 & 3 \\
1 & 3 \\
4 & 3 \\
\end{array}
\quad
\begin{array}{cc}
1 & 4 \\
2 & 3 \\
3 & 4 \\
\end{array}
\quad
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
4 & 4 \\
\end{array}
\quad
\begin{array}{cc}
1 & 3 \\
2 & 4 \\
4 & 4 \\
\end{array}
\quad
\begin{array}{cc}
1 & 3 \\
2 & 4 \\
4 & 4 \\
\end{array}
\quad
\begin{array}{cc}
1 & 3 \\
2 & 4 \\
4 & 4 \\
\end{array}
\]

In \(C[SL_4]^{N}\) and \(C[N]\), we have that \(ch(\begin{array}{c}1 \\ 2 \\ 3 \\ 4 \end{array}) = \Delta_3 \Delta_{124} - \Delta_4 \Delta_{123}\). In \(C[N]\), this is also equal to \(x_{13} x_{34} - x_{14} = \Delta_{13,34}\).

Every tableau \(T\) in \(SSYT(k - 1, [k])\) can be written as \(T = T'' \cup T'\) where “\(\cup\)” is the multiplication in the monoid \(SSYT(k - 1, [k])\) (cf. Section 3), \(T''\) is a tableau whose columns are fundamental tableaux and \(T''\) is a fraction of two trivial tableaux (cf. Section 3).

For a tableau \(T\) with columns \(T_1, \ldots, T_r\), we denote by \(\Delta_T = \Delta T_1 \cdots \Delta T_r\) the *standard monomial* of \(T\). For a fraction \(ST^{-1}\) of two tableaux \(S, T\), we denote \(\Delta_{ST^{-1}} = \Delta S \Delta_T^{-1}\) (cf. Section 4.2).

For \(T \in SSYT(k - 1, [k])\), we define \(ch'(T) = \Delta_{T''} ch_{C[SL_k]}(T'')\). We conjecture that \(\{ch'(T) : T \in SSYT(k - 1, [k])\}\) is the dual canonical basis of \(C[SL_k]^{N^-}\), see Conjecture 5.6.

The paper is organized as follows. In Section 2, we give some background on cluster algebras, quantum affine algebras, cluster structure on \(C[N]\) and \(C[SL_k]^{N^-}\), and Hernandez-Leclerc’s monoidal categorification of \(C[N]\). In Section 3, we describe the monoid of semi-standard Young tableaux. In Section 4, we show that a certain submonoid of the monoid of dominant monomials is isomorphic to the monoid of semi-standard tableaux. In Section 5, we give a formula for every element in the dual canonical basis of \(C[N]\) (respectively, \(C[SL_k]^{N^-}\)). In Section 6, we describe the mutation rule in \(C[N]\) (respectively, \(C[SL_k]^{N^-}\)) in terms of tableaux.
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2. Preliminary

2.1. Cluster algebras. Fomin and Zelevinsky introduced cluster algebras [21] in order to understand in a concrete and combinatorial way the theory of total positivity (cf. [45, 46]) and canonical bases in quantum groups (cf. [43, 44, 33]). We recall the definition of cluster algebras.

A quiver $Q$ is an oriented graph given by a set of vertices $Q_0$, a set of arrows $Q_1$, and two maps $s, t : Q_1 \to Q_0$ taking an arrow to its source and target, respectively.

Let $Q$ be a finite quiver without loops or 2-cycles. For a vertex $k \in Q_0$, the mutated quiver $\mu_k(Q)$ is a quiver with the same set of vertices as $Q$, and its set of arrows is obtained by the following procedure:

(i) add a new arrow $i \to j$ for every existing pair of arrows $i \to k$, $k \to j$;
(ii) reverse the orientation of every arrow with target or source equal to $k$;
(iii) erase every pair of opposite arrows possibly created by (i).

Let $m \geq n$ be positive integers and let $\mathcal{F}$ be an ambient field of rational functions in $n$ independent variables over $\mathbb{Q}(x_{n+1}, \ldots, x_m)$. A seed in $\mathcal{F}$ is a pair $(x, Q)$, where $x = (x_1, \ldots, x_m)$ is a free generating set of $\mathcal{F}$, and $Q$ is a quiver (without loops or 2-cycles) with vertices $[m]$ whose vertices $1, \ldots, n$ are called mutable and whose vertices $n+1, \ldots, m$ are called frozen. For a seed $(x, Q)$ in $\mathcal{F}$ and $k \in [n]$, the mutated seed $\mu_k(x, Q)$ in direction $k$ is $(x', \mu_k(Q))$, where $x' = (x'_1, \ldots, x'_m)$ with $x'_j = x_j$ for $j \neq k$ and $x'_k \in \mathcal{F}$ is determined by the exchange relation:

$$x'_k x_k = \prod_{\alpha \in Q_1, s(\alpha) = k} x_{t(\alpha)} + \prod_{\alpha \in Q_1, t(\alpha) = k} x_{s(\alpha)}.$$

The mutation class of a seed $(x, Q)$ is the set of all seeds obtained from $(x, Q)$ by a finite sequence of mutations. For every seed $((x'_1, \ldots, x'_{n}, x_{n+1}, \ldots, x_m), Q')$ in the mutation class, the set $\{x'_1, \ldots, x'_n, x_{n+1}, \ldots, x_m\}$ is called a cluster, $x'_1, \ldots, x'_n$ are called cluster variables, and $x_{n+1}, \ldots, x_m$ are called frozen variables. The cluster algebra $\mathcal{A}(x, Q)$ is the $\mathbb{Z}[x_{n+1}, \ldots, x_m]$-subalgebra of $\mathcal{F}$ generated by all cluster variables. A cluster monomial is a product of non-negative powers of cluster variables belonging to the same cluster.

2.2. Cluster structure on $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$. In this subsection, we recall the cluster structure on $\mathbb{C}[N]$ and $\mathbb{C}[SL_k]^{N^-}$, cf. [5, 6, 20, 22, 25].

Let $V \cong \mathbb{C}^k$ be a $k$-dimensional complex vector space. By choosing a basis in $V$, one can identify $G = SL_k$ with the special linear group $SL(V)$ complex matrices with determinant 1. The subgroup $N^- \subset G$ of unipotent lower triangular matrices acts on $G$ by left multiplication. This action induces the action of $N^-$ on the coordinate ring $\mathbb{C}[G]$. Denote by $\mathbb{C}[G]^{N^-}$ the ring of $N^-$-invariant regular functions on $G$. The ring $\mathbb{C}[SL_k]^{N^-}$ has a cluster algebra structure whose initial cluster is given as follows.

For a $n \times n$ matrix $z$ and $J, J' \subset [n]$ (\(|J'| = |J|\)), denote by $\Delta_{J,J'}(z)$ the determinant of the submatrix of $z$ with rows labeled by $J'$ and columns labeled by $J$. In the case that $J' = \{1, 2, \ldots, |J|\}$, we write $\Delta_J = \Delta_{J,J'}$ and it is called a flag minor.
Let \( I = [k - 1] \) be the set of the vertices of the Dynkin diagram of \( \mathfrak{sl}_k \). Let \( Q_{k,\Delta} \) be a quiver with the vertex set \( V_{k,\Delta} = \{(i, p) : i \in I \cup \{k\}, p \in [i]\} \setminus \{(k, k)\} \) and with edge set:

\[(i, p) \to (i + 1, p + 1), \quad (i, p) \to (i, p - 1), \quad (i, p) \to (i - 1, p),\]

see Figure 1. The vertices \((i, i), i \in I \) and \((k, p), p \in I \) are frozen.

For \( i \in I, p \in [i], \) denote \( \Delta^{(i,p)} = \Delta_J \), where \( J = \{1, 2, \ldots, p - 1, p + k - i\} \). Attach to the vertex \((i, p)\) the flag minor \( \Delta^{(i,p)} \), \( i \in I, p \in [i] \). An initial cluster of \( \mathbb{C}[SL_k]^{N^-} \) consists of the initial quiver \( Q_{k,\Delta} \) and initial cluster variables \( \Delta^{(i,p)}, i \in I, p \in [i] \). Figure 1 is the initial cluster for \( \mathbb{C}[SL_5]^{N^-} (k = 5) \) if we replace \( \Delta_1 \) by \( \Delta_{1,i}, i \in [k - 1] \).

In Figure 1,

\[
\Delta_5, \Delta_4, \Delta_3, \Delta_2, \Delta_1, \Delta_{45}, \Delta_{34}, \Delta_{23}, \Delta_{12}, \Delta_{345}, \Delta_{234}, \Delta_{123}, \Delta_{2345}, \Delta_{1234},
\]
sit at the vertices

\[
(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (4, 4), (5, 4),
\]

respectively.

Denote by \( \mathbb{C}[\widehat{SL}_k]^{N^-} \) the quotient of \( \mathbb{C}[SL_k]^{N^-} \) by identifying the leading principal minors \( \Delta_{1,i} (i \in [k - 1]) \) with 1. The cluster algebra structure on \( \mathbb{C}[SL_k]^{N^-} \) induces a cluster algebra structure on \( \mathbb{C}[\widehat{SL}_k]^{N^-} \).

Denote by \( N \subset SL_k \) the subgroup of unipotent upper triangular matrices. The ring map \( \mathbb{C}[SL_k]^{N^-} \to \mathbb{C}[N] \) defined by restricting \( N^- \)-invariant functions on \( SL_k \) to the subgroup \( N \). This map is onto and transforms the above described cluster structure on \( \mathbb{C}[SL_k]^{N^-} \) into a cluster structure on \( \mathbb{C}[N] \) (cf. [19]). This cluster structure on \( \mathbb{C}[N] \) has an initial cluster consisting of the initial quiver \( Q_{k,\Delta} \) and initial cluster variables \( \Delta^{(i,p)}, i \in I, p \in [i], \) see Figure 1.

2.3. Monoidal categorification of the cluster algebra structure on \( \mathbb{C}[N] \). Hernandez and Leclerc introduced the notion of a monoidal categorification of a cluster algebra in
[29, 34]. For a monoidal category \((C, \otimes)\), a simple object \(S\) of \(C\) is called real if \(S \otimes S\) is simple. A simple object \(S\) is called prime if there exists no non-trivial factorization \(S \cong S_1 \otimes S_2\). The monoidal category \(C\) is called a monoidal categorification of a cluster algebra \(A\) if the Grothendieck ring of \(C\) is isomorphic to \(A\) and if (1) any cluster monomial of \(A\) corresponds to the class of a real simple object of \(C\), and (2) any cluster variable of \(A\) corresponds to the class of a real simple prime object of \(C\).

Let \(Q\) be an orientation of the Dynkin diagram of \(\mathfrak{g}\). Hernandez and Leclerc [30] constructed a tensor category \(C_Q\) and showed that \(C_Q\) is a monoidal categorification of the ring \(\mathbb{C}[N]\) and its dual canonical basis. To our purpose, we use a special case \(C_{k,\triangle}\) of \(C_Q\). We recall the definition of \(C_{k,\triangle}\) in the following.

Let \(\mathfrak{g}\) be a simple Lie algebra and \(I\) the set of the vertices of the Dynkin diagram of \(\mathfrak{g}\). Denote by \(P\) the weight lattice of \(\mathfrak{g}\) and by \(Q \subset P\) the root lattice of \(\mathfrak{g}\). There is a partial order on \(P\) given by \(\lambda \leq \lambda'\) if and only if \(\lambda' - \lambda\) is equal to a non-negative integer linear combination of positive roots.

In this paper, we take \(q\) to be a non-zero complex number which is not a root of unity, \(\mathfrak{g} = \mathfrak{sl}_k\), and \(I = [k - 1]\) be the set of vertices of the Dynkin diagram of \(\mathfrak{g}\). The quantum affine algebra \(U_q(\widehat{\mathfrak{g}})\) is a Hopf algebra that is a \(q\)-deformation of the universal enveloping algebra of \(\mathfrak{g}\) [15, 16, 31].

We fix \(a \in \mathbb{C}^\times\) and denote \(Y_{i,s} = Y_{i,a^s}, i \in I, s \in \mathbb{Z}\). Denote by \(P\) the free abelian group generated by \(Y_{i,s}^{\pm 1}, i \in I, s \in \mathbb{Z}\), denote by \(P^+\) the submonoid of \(P\) generated by \(Y_{i,s}, i \in I, s \in \mathbb{Z}\), and denote by \(P_{k,\triangle}^+\) the submonoid of \(P^+\) generated by \(Y_{i,i-2p}, i \in I, p \in [i]\).

An object \(V\) in \(C_{k,\triangle}\) is a finite dimensional \(U_q(\widehat{\mathfrak{g}_k})\)-module that satisfies the condition: for every composition factor \(S\) of \(V\), the highest \(l\)-weight of \(S\) is a monomial in \(Y_{i,i-2p}, i \in I, p \in [i]\). Simple modules in \(C_{k,\triangle}\) are of the form \(L(M)\) (cf. [11], [29]), where \(M \in P_{k,\triangle}^+\) and \(M\) is called the highest \(l\)-weight of \(L(M)\). The elements in \(P^+\) are called dominant monomials. Denote by \(K(C_{k,\triangle})\) the Grothendieck ring of \(C_{k,\triangle}\).

Let \(\mathbb{Z}P = \mathbb{Z}[Y_{i,s}^{\pm 1}]_{i \in I, s \in \mathbb{Z}}\) be the group ring of \(P\). The \(q\)-character of a \(U_q(\widehat{\mathfrak{g}})\)-module \(V\) is given by (cf. [18])

\[
\chi_q(V) = \sum_{m \in P} \dim(V_m)m \in \mathbb{Z}P,
\]

where \(V_m\) is the \(l\)-weight space with \(l\)-weight \(m\) (\(l\)-weights of \(V\) are identified with monomials in \(P\)). It is shown in [18] that \(q\)-characters characterize simple \(U_q(\widehat{\mathfrak{g}})\)-modules up to isomorphism.

Denote by \(wt : P \to P\) the group homomorphism defined by sending \(Y_{i,a}^{\pm 1} \mapsto \pm \omega_i, i \in I\), where \(\omega_i\)'s are fundamental weights of \(\mathfrak{g}\). For a finite dimensional simple \(U_q(\widehat{\mathfrak{g}})\)-module \(L(M)\), we write \(wt(L(M)) = wt(M)\) and call it the highest weight of \(L(M)\).

Let \(Q^+\) be the monoid generated (in the case that \(\mathfrak{g} = \mathfrak{sl}_k\)) by

\[
A_{i,s} = Y_{i,s+1}Y_{i,s-1} \prod_{j \in I, |j-i|=1} Y_{j,s}^{-1}, \quad i \in I, s \in \mathbb{Z}.
\]

There is a partial order \(\leq\) on \(P\) (cf. [17, 47]) defined by

\[
M \leq M' \text{ if and only if } M'M^{-1} \in Q^+.
\]

For \(i \in I, s \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 1}\), \(L(X_{i,k}^{(s)})\), where \(X_{i,k}^{(s)} = Y_{i,s}Y_{i,s+2} \cdots Y_{i,s+2k-2}\), are called Kirillov-Reshetikhin modules. The modules \(L(X_{i,1}^{(s)}) = L(Y_{i,s})\) are called fundamental modules.
Hernandez and Leclerc [30] proved that the tensor category $C_{k,\triangle}$ is a monoidal categorification of the ring $\mathbb{C}[N]$ and its dual canonical basis. The Grothendieck ring $K(C_{k,\triangle})$ has a cluster algebra structure with an initial seed consisting of the initial quiver $Q_{k,\triangle}$ and initial cluster variables $X^{(i-2p)}_{i,p}$, $i \in I$, $p \in [i]$, where $X^{(i-2p)}_{i,p}$ sits at the position $(i,p)$ of the quiver $Q_{k,\triangle}$, see Figure 2. We put trivial modules $\mathbb{C}$ at the positions $(k,i)$, $i \in [k-1]$, in order to compare with the quiver in Figure 1.

Recall that in Section 2.2, for $i \in I$, $p \in [i]$, we denote $\Delta^{(i,p)} = \Delta_J$, where $J = \{1, 2, \ldots, p-1, p+k-i\}$.

**Theorem 2.1** ([30, Theorems 1.1, 1.2, and 6.1]). The assignments $L(Y_{i,i-2p}) \mapsto \Delta^{(i,p)}$, $i \in I$, $p \in [i]$, induce an algebraic isomorphism $\Phi_{\mathbb{C}[N]} : K(C_{k,\triangle}) \rightarrow \mathbb{C}[N]$.

The assignments $L(Y_{i,i-2p}) \mapsto \Delta^{(i,p)}$, $i \in I$, $p \in [i]$, induce an algebraic isomorphism $\Phi_{\widetilde{C}(SL_k)^{N-}} : K(C_{k,\triangle}) \rightarrow \mathbb{C}[SL_k]^{N-}$.

We usually write $\Phi_{\mathbb{C}[N]}$ (respectively, $\Phi_{\widetilde{C}(SL_k)^{N-}}$) as $\Phi$ if there is no confusion.

### 3. The Monoid of Semi-Standard Young Tableaux

In this section, we show that the set of semi-standard Young tableaux with at most $k$ rows and with entries in a set $[m]$ form a monoid under certain product "∪".

For $k, m \in \mathbb{Z}_{\geq 1}$, denote by $SSYT(k,[m])$ the set of all semi-standard Young tableaux (including the empty tableau denoted by $\emptyset$) with less or equal to $k$ rows and with entries in $[m]$. For a tableau $T \in SSYT(k,[m])$ with $k'$ ($k' \leq k$) rows, when we say the $i$th ($i > k'$) row of $T$, we understand that the $i$th row is empty.

For $T, T' \in SSYT(k,[m])$, we denote by $T \cup T'$ the row-increasing tableau whose $i$th row is the union of the $i$th rows of $T$ and $T'$ (as multisets).

![Figure 2. The initial cluster for $C_{5,\triangle}$.](image-url)
**Example 3.1.** In $\text{SSYT}(5,[6])$, we have that
\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
\end{array}
\cup
\begin{array}{ccc}
2 & 3 & 4 \\
5 & & \\
\end{array}
= \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}.
\]

For $S, T \in \text{SSYT}(k,[m])$, we say that $S$ is a factor of $T$ (denoted by $S \subseteq T$) if for every $i \in [k]$, the $i$th row of $S$ is contained in the $i$th row of $T$ (as multisets). For a factor $S$ of $T$, we define $T_S = S^{-1}T = TS^{-1}$ to be the row-increasing tableau whose elements in the $i$th row are the elements in the multiset-difference of $i$th row of $T$ and the $i$th row of $S$, for every $i \in [k]$. We now prove that for $S, T, T' \in \text{SSYT}(k,[m])$, we have $S \cup T, T' \in \text{SSYT}(k,[m])$. Denote by $S(i)$ the $i$th row of a tableau $S$. We need to prove that for any $i < j$, the 2-row tableau with the first row $T(i) \cup T'(i)$ and the second row $T(j) \cup T'(j)$ is semi-standard. It suffices to prove this in the case that $T'$ has one column. Let $i, j$ rows of $T$ be
\[
a_1 \ a_2 \ \cdots \ a_{r_1} \\
b_1 \ b_2 \ \cdots \ b_{r_2},
\]
for some $r_1 \geq r_2$. We have the following cases.

**Case 1.** $T'$ does not have entry in rows $i$ and $j$. In this case, the result is trivial.

**Case 2.** $T'$ has an entry $a'$ in row $i$ and the row $j$ is empty. There exists $k \in [0, r_1]$ such that $a_1 \leq \cdots \leq a_k \leq a' \leq a_{k+1} \leq \cdots \leq a_{r_1}$. The $i, j$ rows of $T \cup T'$ are
\[
a_1 \ a_2 \ \cdots \ a_k \ a' \ a_{k+1} \ \cdots \ a_{r_1} \\
b_1 \ b_2 \ \cdots \ b_k \ b_{k+1} \ b_{k+2} \ \cdots \ b_{r_2}.
\]
We have that $a' \leq a_{k+1} < b_{k+1}$ and for all $d \in [k+1, r_2 - 1]$, $a_d < b_d \leq b_{d+1}$. Therefore the $i, j$ rows of $T \cup T'$ form a 2-row semi-standard tableau.
Case 3. $T'$ has entries $a'$ and $b'$ in rows $i$ and $j$. There are $k \in [0,r_1]$, $l \in [0,r_2]$ such that $a_1 \leq \cdots \leq a_k \leq a' \leq a_{k+1} \leq \cdots \leq a_r$, and $b_1 \leq \cdots \leq b_l \leq b' \leq b_{k+1} \leq \cdots \leq b_r$.

If $k = l$, then the $i,j$ rows of $T \cup T'$ form a 2-row semi-standard tableau. If $k > l$, then the $i,j$ rows of $T \cup T'$ are

$$a_1 \ a_2 \ \cdots \ a_l \ a_{l+1} \ a_{l+2} \ \cdots \ a_k \ a' \ a_{k+1} \ \cdots \ a_r$$
$$b_1 \ b_2 \ \cdots \ b_l \ b' \ b_{l+1} \ \cdots \ b_k \ b_{k+1} \ \cdots \ b_r.$$  

We have $a' < b' \leq b_k$, $a_{k+1} \leq a' < b'$, and for all $d \in [l+2,k]$, $a_d \leq a' < b' \leq b_{d-1}$. Therefore the $i,j$ rows of $T \cup T'$ form a 2-row semi-standard tableau.

If $k < l$, then the $i,j$ rows of $T \cup T'$ are

$$a_1 \ a_2 \ \cdots \ a_k \ a' \ a_{k+1} \ \cdots \ a_{l-1} \ a_l \ a_{l+1} \ \cdots \ a_r$$
$$b_1 \ b_2 \ \cdots \ b_k \ b_{k+1} \ b_{k+2} \ \cdots \ b_l \ b' \ b_{l+1} \ \cdots \ b_r.$$  

We have $a' \leq a_{k+1} < b_{k+1}$, $a_l < b_l \leq b'$, and for all $d \in [k+1,l-1]$, $a_d < b_d \leq b_{d+1}$. Therefore the $i,j$ rows of $T \cup T'$ form a 2-row semi-standard tableau. $\square$

4. ISOMORPHISMS OF MONOIDS $\mathcal{P}_{k,\Delta}^+$ AND SSYT($k-1,[k],\sim$)

In this section, we show that the monoids $\mathcal{P}_{k,\Delta}^+$ and SSYT($k-1,[k],\sim$) are isomorphic.

4.1. Factorization of a tableau as a product of fundamental tableaux. For $i \in I$, $p \in [r]$, denote by $T^{(i,p)}$ the one-column tableau with entries $\{1,2,\ldots,p-1,p+k-i\}$. We call the tableau $T^{(i,p)}$ a fundamental tableau. We also use $T_{(l,a)}$ to denote a fundamental tableau with $l_a$ rows and whose last entry $a$. We have that $T_{(l,a)} = T_{(l+a-k-a,l_a)}$.

There is a total order on the set of one-column fundamental tableaux in SSYT($k,[m]$): for two one column fundamental tableaux $T = T_{(l_a,a)}$, $T' = T_{(l_{a'},a')}$, $T \leq T'$ if either $l_a > l_{a'}$ or $l_a = l_{a'}$, $a \leq a'$. For example,

\[
\begin{array}{c|c|c|c|c}
1 & 1 & 1 \\
2 & 2 & < & \ 3 & < \\
5 & 6 & < & \ 4 & < \\
\end{array}
\]

If the columns $T_1,\ldots,T_r$ ($T_i$ is the $i$th column of $T$) of a tableau $T \in \text{SSYT}(k,[m])$ are all fundamental tableaux, then $T_1 \leq T_2 \leq \cdots \leq T_r$ in the above described total order.

Lemma 4.1. For $k,m \in \mathbb{Z}$, every $T \in \text{SSYT}(k,[m],\sim)$ can be uniquely factorized as a $\cup$-product of fundamental tableaux and there is a unique $T' \in \text{SSYT}(k,[m],\sim)$ such that $T' \sim T$ and the columns of $T'$ are fundamental tableaux.

Proof. First we prove the existence. It suffices to prove the existence in the case that $T$ is a one-column tableau. Denote by $i_1 < \ldots < i_r$ the entries of $T$. If $i_1 = 1$, then $T \sim T'$, where $T'$ is the union of the fundamental tableaux $T^{(j,i)}$, where the entries of $T^{(j,i)}$ are $\{1,2,\ldots,j-1,i\}$, $j \in [2,r]$. If $i_1 > 1$, then $T \sim T'$, where $T'$ is the union of the fundamental tableaux $T^{(j,i)}$, $j \in [r]$.

Now we prove uniqueness. Suppose that $T \sim T'$, $T \sim T''$, and the columns of $T',T''$ are fundamental tableaux. Then $T' \sim T''$. It follows that there are trivial tableaux $A,B$ such that $A \cup T' = B \cup T''$. Since the columns of $A,B$ are trivial tableaux and the columns of $T',T''$ are fundamental tableaux, we have that $A = B$. It follows that $T' = T''$ since SSYT($k,[m],\sim$) is cancellative by Lemma 3.2. $\square$
Example 4.2. In SSYT(5, [6], ∼), we have that
\[
\begin{array}{cccc}
1 & 2 & \sim & 1 \\
3 & 4 & \cup & 3 \\
5 & 6 & \cup & 5 \\
\end{array}
\quad
\begin{array}{cccc}
2 & 4 & \cup & 1 \\
\cup & 2 & \cup & 2 \\
\cup & 4 & \cup & 6 \\
\end{array}
= \begin{array}{cccc}
1 & 2 & \sim & 1 \\
3 & 5 & \cup & 4 \\
6 & 5 & \cup & 2 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & \sim & 1 \\
1 & 1 & \cup & 1 \\
2 & 3 & \cup & 4 \\
\end{array}
.
\]

4.2. Weights on semi-standard tableaux and on products of flag minors. There is a bijection between the set of one-column semi-standard tableaux in SSYT(k − 1, [k], ∼) and the set of (non-trivial) flag minors of \( \mathbb{C}[N] \) sending the one-column tableau with entries in \( J \subset [k] \) to the flag minor \( \Delta_J \). Denote by \( T_\Delta \) the tableau corresponding to a flag minor \( \Delta \) and \( \Delta_T \) the flag minor corresponding to a one-column tableau \( T \). For a tableau \( T \) with columns \( T_1, \ldots, T_r \), we denote by \( \Delta_T = \Delta_{T_1} \cdots \Delta_{T_r} \) the standard monomial \( T \). For a fraction \( ST^{-1} \) of two tableaux \( S, T \), we denote \( \Delta_{ST^{-1}} = \Delta_S \Delta_T^{-1} \).

Definition 4.3. For a fundamental tableau \( T^{(i,p)} \in \text{SSYT}(k − 1, [k], ∼) \), \( i \in I \), \( p \in [i] \), we define the weight of the tableau as \( \text{wt}(T^{(i,p)}) = \omega_i \in P \), where \( \omega_i \) is a fundamental weight of \( \mathfrak{g} \). We define \( \text{wt}(1) = 0 \).

For a tableau \( T \in \text{SSYT}(k − 1, [k], ∼) \), we define the weight of \( T \) as \( \text{wt}(T) = \sum_j \text{wt}(T^{(j)}) \), where \( T = \bigcup_T T^{(j)} \) is the unique factorization of the tableau \( T \) into fundamental tableaux.

Definition 4.4. For a flag minor \( \Delta \in \mathbb{C}[N] \), we define the weight of \( \Delta \) as \( \text{wt}(T_\Delta) \). For a product \( \prod_j \Delta^{(j)} \) of flag minors, we define \( \text{wt}(\prod_j \Delta^{(j)}) = \sum_j \text{wt}(\Delta^{(j)}) \).

4.3. Isomorphism of monoids. By Theorem 2.1, \( \{\Delta_T : T \in \text{SSYT}(k − 1, [k], ∼)\} \) is an additive basis of \( \mathbb{C}[N] \), \( N \subset SL_k \). Therefore for any module \( [L(M)] \in K(\mathcal{C}_{k,\Delta}) \),

\[
\Phi([L(M)]) = \sum_{T \in \text{SSYT}(k-1,[k],\sim)} c_T \Delta_T \in \mathbb{C}[N],
\]

for some \( c_T \in \mathbb{C}^\times \).

Define \( \text{Top}(\Phi([L(M)])) \) to be the tableau which appears on the right hand side of (4.1) with the highest weight. By the same proof as the proof of Lemma 3.22 in [14] using \( q \)-character theory, we have that \( \text{Top}(\Phi(L(M))) \) exists for every \( L(M) \in K(\mathcal{C}_{k,\Delta}) \). Moreover, \( \text{wt}(L(M)) = \text{wt}(\text{Top}(\Phi([L(M)]))) \).

We define a map

\[
\Phi : \mathcal{P}^+_{k,\Delta} \rightarrow \text{SSYT}(k − 1, [k], ∼), \quad M \mapsto \text{Top}(\Phi(L(M))),
\]

and denote \( T_M = \Phi(M) \).

Recall that for \( i \in I \), \( p \in [i] \), \( T^{(i,p)} \) is the one-column tableau with entries \( \{1, 2, \ldots, p − 1, p + k − i\} \). The following lemma follows from Theorem 2.1 and the definition of \( \Phi \).

Lemma 4.5. For fundamental modules \( L(Y_{i,i-2p}) \in \mathcal{C}_{k,\Delta}, \ i \in I, \ p \in [i], \) we have that \( \Phi(Y_{i,i-2p}) = T^{(i,p)} \) and \( \text{wt}(Y_{i,i-2p}) = \text{wt}(T^{(i,p)}) = \omega_i \).

Recall that \( T_{(l_a,a)} \) is a one-column fundamental tableau with \( l_a \) rows and whose last entry is \( a \), and \( T_{(l_a,a)} = T_{l_a+k-a,l_a} \).

By Lemma 4.1, every \( T \in \text{SSYT}(k − 1, [k], ∼) \) has a unique factorization \( T \sim \bigcup_{a=1}^r T_{(l_a,a)} \).

We define

\[
\Psi : \text{SSYT}(k − 1, [k], ∼) \rightarrow \mathcal{P}^+_{k,\Delta}, \quad T \mapsto \prod_{a=1}^r Y_{l_a+k-a,k-a-l_a},
\]
and denote $M_T = \Psi(T)$. We will show that $\Psi$ is the inverse of $\tilde{\Phi}$.

**Theorem 4.6.** The map $\tilde{\Phi} : \mathcal{P}_{k,\Delta}^+ \to \text{SYT}(k-1, [k], \sim)$ is an isomorphism of monoids and its inverse is $\Psi$.

**Proof.** We first show that $\tilde{\Phi}$ is a homomorphism of monoids. By the theory of $q$-characters, for any $M, M' \in \mathcal{P}_{k,\Delta}^+$, we have that

$$[L(M)][L(M')] = [L(MM')] + \sum_{\tilde{M}, \text{wt}(\tilde{M}) < \text{wt}(MM')} c_{\tilde{M}}[L(\tilde{M})],$$

for some $c_{\tilde{M}} \in \mathbb{Z}_{\geq 0}$. Since $\Phi : \mathbb{K}([C_k]) \to \mathbb{C}[N]$ is an algebra isomorphism, we have that

$$\Phi(L(M))\Phi(L(M')) = \Phi(L(MM')) + \sum_{\tilde{M}, \text{wt}(\tilde{M}) < \text{wt}(MM')} c_{\tilde{M}}\Phi(L(\tilde{M})).$$

It follows that $\text{Top}(\Phi(L(M))\Phi(L(M'))) = \text{Top}(\Phi(L(MM'))).$ Therefore $\tilde{\Phi}(MM') = \tilde{\Phi}(M) \cup \tilde{\Phi}(M').$

We now show that $\Psi$ is a homomorphism of monoids. Since $\Psi(T)$ only depends on the equivalence class of $T$, it suffices to check that $\Psi(T)\Psi(T') = \Psi(T \cup T')$ when $T, T'$ are tableaux whose columns are fundamental tableaux. It is clear that the columns of the product $T \cup T'$ are also fundamental tableaux. By definition, the value of $\Psi$ on a tableau whose columns are fundamental tableaux is product of the values of $\Psi$ on every column of the tableau. It follows that $\Psi(T)\Psi(T') = \Psi(T \cup T').$

We now check that both composites $\Psi \tilde{\Phi}$ and $\tilde{\Phi} \Psi$ are the identity map. It suffices to check this on generators. For any $i \in I$, $p \in [i]$, by Lemma 4.5 and the definition of $\Psi$, we have

$$\Psi \tilde{\Phi}(Y_{i,i-2p}) = \Psi(T^{(i,p)}) = \Psi(T_{(p,k+p-i)}) = Y_{i,i-2p}.$$  

Every fundamental tableau in $\text{SYT}(k-1, [k], \sim)$ is a one-column tableau of the form $T_{(k,a)}$ for some $a \in [2, k]$ and $l_a \in [a-1]$. We have

$$\tilde{\Phi} \Psi(T_{(k,a)}) = \tilde{\Phi}(Y_{k,k-a,a-1}) = T_{(k+k-a,a)} = T_{(k,a)}.$$  

$\square$

In Table 1, the first column consists of all fundamental modules in $C_{5,\Delta}$ and the second column consists of the corresponding fundamental tableaux in $\text{SYT}(4, [5], \sim)$.

**Definition 4.7.** For a tableau $T \in \text{SYT}(k-1, [k], \sim)$, we define an element $\text{ch}_{C[N]}(T) \in \mathbb{C}[N]$ (respectively, $\text{ch}_{C[SL_k]}^N(T) \in \mathbb{C}[SL_k]^N$) to be the $\Phi_{C[N]}([L(M_T)])$ (respectively, $\Phi_{C[SL_k]}^N(T)$).

Usually we write $\text{ch}_{C[N]}(T)$ (respectively, $\text{ch}_{C[SL_k]}^N(T)$) as $\text{ch}(T)$ when we know that we are working on $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]^N$).

By Theorems 1.1, 1.2, and 6.1 in [30] and Theorem 4.6, we have that following.

**Theorem 4.8.** The set $\{\text{ch}_{C[N]}(T) : T \in \text{SYT}(k-1, [k], \sim)\}$ (respectively, $\{\text{ch}_{C[SL_k]}^N(T) : T \in \text{SYT}(k-1, [k], \sim)\}$) is the dual canonical basis of $\mathbb{C}[N]$ (respectively, $\mathbb{C}[SL_k]^N$).
In this section, we give an explicit formula for every element $\text{ch}_{C[N]}(T)$ (respectively, $\text{ch}_{\overline{C}[SL_k]^N}$) in the dual canonical basis of $C[N]$ (respectively, $\overline{C}[SL_k]^N$).

### 5.1. Formula for $\text{ch}(T)$

Let $T \in \text{SSYT}(k-1,[k],\sim)$ be a tableau which is $\sim$-equivalent to a tableaux $T'$ whose columns are fundamental tableaux and which has $m$ columns. We have that the columns of $T'$ are $T_{(a_i,b_i)}, i = 1,\ldots,m$, for some $a_1,\ldots,a_m \in [k-1]$, $b_1,\ldots,b_m \in [k]$. Denote $P_T = \{(a_i,b_i) : i \in [m]\}$ (as a multi-set). We define $i_T = (i_1,\ldots,i_m)$ and $j_T = (j_1,\ldots,j_m)$, where $i_1 \leq \cdots \leq i_m$ are $a_1,\ldots,a_m$ written in weakly increasing order and $j_1 \leq \cdots \leq j_m$ are the elements $b_1,\ldots,b_m$ written in weakly increasing order. For $c = (c_1,\ldots,c_m)$, $d = (d_1,\ldots,d_m) \in \mathbb{Z}^m$, we denote $P_{c,d} = \{(c_i,d_i) : i \in [m]\}$ (as a multi-set).

Let $S_m$ be the symmetric group on $[m]$. Denote by $\ell(w)$ the length of $w \in S_m$ and denote by $w_0 \in S_m$ be the longest permutation. For $i = (i_1,\ldots,i_m) \in S_m$, denote by $S_i$ the subgroup of $S_m$ consisting of elements $\sigma$ such that $i_{\sigma(j)} = i_j, j \in [m]$. It is clear that for $i,j \in S_m$, $P_{w,i,j} = P_{w,i,j}$ if and only if $w' \in S_j w S_i$. By [9, Sections 2.4, 2.5], [35, Proposition 2.3], and [8, Proposition 2.7], there is a unique permutation of maximal length in $S_j w S_i$.

For any $T \in \text{SSYT}(k-1,[k],\sim)$, there exists $w \in S_m$ such that $P_T = P_{w,i_T,j_T}$. Define $w_T \in S_{j_T} w S_{i_T}$ to be the unique permutation with maximal length. Then $P_T = P_{w_T,i_T,j_T}$. It is clear that $w_T$ is also the unique permutation in $S_m$ of maximal length such that $P_T = P_{w_T,i_T,j_T}$.

**Definition 5.1.** Let $T \in \text{SSYT}(k-1,[k],\sim)$ and $T \sim T'$, where $T'$ has $m$ columns and all the columns are fundamental tableaux. For $u \in S_m$, we define $\Delta_{u:T} \in \mathbb{C}[SL_k]^N$ as follows. If $j_a \in [i_u(a),i_u(a)+k]$ for all $a \in [m]$, define the tableau $\alpha(u;T)$ to be the semi-standard tableau whose columns are $T_{(i_u(a),j_a)}, a \in [m]$, and define $\Delta_{u:T} = \Delta_{\alpha(u;T)} \in \mathbb{C}[SL_k]^N$ to be the standard monomial of $\alpha(u;T)$ (cf. Section 4.2). If $j_a \not\in [i_u(a),i_u(a)+k]$ for some $a \in [m]$, then the tableau $\alpha(u;T)$ is undefined and $\Delta_{u:T} = 0$.

| module | tableau |
|--------|---------|
| $L(Y_{1,-1})$ | $\{5\}$ |
| $L(Y_{2,0})$ | $\{4\}$ |
| $L(Y_{2,-2})$ | $\{1,5\}$ |
| $L(Y_{3,1})$ | $\{3\}$ |
| $L(Y_{3,-1})$ | $\{1,4\}$ |
| $L(Y_{3,-3})$ | $\{1,2,5\}$ |
| $L(Y_{4,2})$ | $\{2\}$ |
| $L(Y_{4,0})$ | $\{1,3\}$ |
| $L(Y_{4,-2})$ | $\{1,2,4\}$ |
| $L(Y_{4,-4})$ | $\{1,2,3,5\}$ |

**Table 1.** Correspondence between fundamental monomials and fundamental tableaux in $\text{SSYT}(4,[5],\sim)$. Since all tableaux in the table are one-column tableaux, we represent them by their entries.
Denote by $M_{GL}$ corresponding to a monomial $M$. Formula (5.3) \[ \text{Sections 2.4, 2.5}, \]\[ \text{tisegments and dominant monomials} \]

Example 5.2. Let $T = \begin{array}{ccc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \in \text{SSYT}(5, [6], \sim)$. Then $T \sim T'$, $T' = \begin{array}{ccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 5 & 6 \end{array}$.

We have that $i_T = (1, 2, 2, 3, 3)$, $j_T = (2, 3, 4, 5, 6)$, and $w_T = s_2 s_4$. For $u = s_2 \in S_5$, $\alpha(u; T)$ is the semi-standard tableau whose columns are $T_{(1,2)}$, $T_{(2,3)}$, $T_{(3,4)}$, $T_{(2,5)}$, $T_{(3,6)}$. We have $\Delta_{u, T} = \Delta_2 \Delta_{15} \Delta_{13} \Delta_{126} \Delta_{124}$.

We have the following theorem.

**Theorem 5.3.** Let $T \in S\text{SYT}(k - 1, [k], \sim)$ and $T \sim T'$ for some tableau $T'$ whose columns are fundamental tableaux and which has $m$ columns. Then

\begin{equation}
\chi_{C[N]}(T) = \sum_{u \in S_m} (-1)^{\ell(u_T)} p_{uw_0, w_T w_0}(1) \Delta_u; T' \in \mathbb{C}[N],
\end{equation}

\begin{equation}
\chi_{\overset{\sim}{C[S_L]}}(T) = \sum_{u \in S_m} (-1)^{\ell(u_T)} p_{uw_0, w_T w_0}(1) \Delta_u; T' \in \mathbb{C}[\overset{\sim}{S_L}]^N\sim.
\end{equation}

5.2. **Proof of Theorem 5.3.** Let $F$ be a non-archimedean local field. Complex, smooth representations of $GL_n(F)$ of finite length are parameterized by multisegments $[7, 53]$. A multisegment $m = \sum_{i=1}^m \Delta_i$ of segments. A segment $\Delta$ is identified with an interval $[a, b]$, $a, b \in \mathbb{Z}$, $a \leq b$.

By quantum Schur-Weyl duality [12, Section 7.6], there is a correspondence between multisegments and dominant monomials

\begin{equation}
[a, b] \mapsto Y_{b-a+1, a+b-1}, \quad Y_{i, s} \mapsto \left[ \frac{s - i + 2}{2}, \frac{s + i}{2} \right].
\end{equation}

Denote by $M_m$ the monomial corresponding to a multisegment $m$ and $M_M$ the multisegment corresponding to a monomial $M$.

We interpret $M_{[a, a-1]}$ as the trivial monomial $1 \in \mathcal{P}^+$ and interpret $M_{[a, b]}$ with $b < a - 1$ as 0. For any $m$-tuples $(\mu, \lambda) \in \mathbb{Z}^m \times \mathbb{Z}^m$, we define multi-set:

\[ \text{Fund}_M(\mu, \lambda) = \{ M_{[\mu_i, \lambda_i]} : i \in [m] \}. \]

For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$, denote by $S_\lambda$ the subgroup of $S_m$ consisting of elements $\sigma$ such that $\lambda_{\sigma(i)} = \lambda_i$. For $\mu = (\mu_1, \ldots, \mu_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$, we denote $M_{[\mu, \lambda]} = \sum_{i=1}^m [\mu_i, \lambda_i]$.

For a multisegment $m$ with $m$ terms, there exist unique weakly decreasing tuples $\mu, \lambda \in \mathbb{Z}^m$ and unique permutation of maximal length $w_m \in S_m$ such that $M_m = M_{[w_m, w_m, \ldots, w_m]}$ ([9, Sections 2.4, 2.5], [35, Proposition 2.3], and [8, Proposition 2.7]). Note that for any $w, w' \in S_m$ and any $\mu, \lambda \in \mathbb{Z}^m$, $M_{w', \mu, \lambda} = M_{w, w', \mu, \lambda}$ if and only if $w' \in S_\lambda w S_\mu$. The element $w_m \in S_m$ is also the unique permutation of maximal length in $S_\lambda w_m S_{\mu_m}$. We write $\lambda_m = \lambda_M$, $\mu_m = \mu_M$, $w_m = w_M$ for $M = M_m$.

**Proof of Theorem 5.3.** We will prove the formula (5.1) for $\chi_{C[N]}(T)$. The proof of the formula (5.2) for $\chi_{\overset{\sim}{C[S_L]}}(T)$ is the same.

For every finite dimensional $U_q(\overset{\sim}{S_k})$-module $L(M)$, we have that

\begin{equation}
\chi_q(L(M)) = \sum_{u \in S_m} (-1)^{\ell(u, M)} p_{uw_0, w, w_0}(1) \prod_{M' \in \text{Fund}_M(u, M, \lambda)} \chi_q(L(M')).
\end{equation}
This formula is given in Theorem 5.4 in [14]. It is derived from a result due to Arakawa-Suzuki [1] (see also Section 10.1 in [42], and [2, 28]) and from the quantum affine Schur-Weyl duality [12]. In Theorem (5.4), we interpret \( \chi_{\mu}(L(M_{a,a-1})) = 1 \) and \( \chi_{\mu}(L(M_{a,b})) = 0 \) if \( b < a - 1 \).

By (5.3) and Theorem 4.6, there is a correspondence between multisegments and tableaux induced by the following correspondence between segments and fundamental tableaux:

\[
(\mu, \lambda) \mapsto T_{(1-\mu, k-\lambda)}, \quad T_{(i_a, a)} \mapsto [1 - l_a, k - a],
\]

where \( T_{(1-\mu, k-\lambda)} \) is the one-column tableau with entries \( \{1, 2, \ldots, -\mu, k - \lambda\} \). Denote by \( T_m \) the tableau corresponding to the multisegment \( m \) and denote by \( m_T \) the multisegment corresponding to the tableau \( T \).

Denote \( i_T = (i_1, \ldots, i_m) \), \( j_T = (j_1, \ldots, j_m) \). By (5.5), we have that \( i_a = 1 - \mu_a, j_a = k - \lambda_a \) for \( a \in [k] \). Therefore \( w_T \) defined in Subsection 5.1 and \( w_{m_T} \) defined in this subsection are the same.

Apply the isomorphism \( \Phi_{C[N]} \) in Theorem 2.1 and the isomorphism \( \Phi \) in Theorem 4.6 to the formula (5.4), we obtain the formula (5.1).

\[
\begin{array}{c}
1 \\
3 \\
2 \\
4 \\
5 \\
6
\end{array}
\]

Remark 5.4. The difference between the formulas for \( \chi_{C[N]}(T) \) and \( \chi_{C[SL_k]^N}(T) \) is that the flag minors in (5.1) are flag minors in \( C[N] \) while the flag minors in (5.2) are flag minors in \( C[SL_k]^N \).

For example, in \( C[SL_4]^N \) and \( C[N] \), we have that \( \chi(\begin{array}{ccc}
1 & 3 \\
2 & \\
4 & \\
5 & 6
\end{array}) = \Delta_4 \Delta_{124} - \Delta_4 \Delta_{123} \). On the other hand, in \( C[N] \), this is equal to \( x_{13}x_{34} - x_{14} = \Delta_{13,34} \).

We give an example of a computation of \( \chi(T) \).

Example 5.5. We take \( T = \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
5 & 6
\end{array} \in \text{SSYT}(5, [6], \sim) \) as in Example 5.2. Then \( i_T = (1, 2, 2, 3, 3, 4, 5, 6) \), \( j_T = (2, 3, 4, 5, 6) \), and \( w_T = s_2s_4 \). By Theorem 5.3, we have that

\[
\begin{align*}
\chi(T) &= \Delta_2 \Delta_{14} \Delta_{13} \Delta_{126} \Delta_{125} + \Delta_3 \Delta_{15} \Delta_{12} \Delta_{126} \Delta_{124} + \Delta_2 \Delta_{16} \Delta_{15} \Delta_{124} \Delta_{123} \\
&\phantom{=} + \Delta_5 \Delta_{14} \Delta_{12} \Delta_{126} \Delta_{123} + \Delta_4 \Delta_{16} \Delta_{12} \Delta_{125} \Delta_{123} - \Delta_3 \Delta_{14} \Delta_{12} \Delta_{126} \Delta_{125} \\
&\phantom{=} - \Delta_2 \Delta_{16} \Delta_{14} \Delta_{125} \Delta_{123} - \Delta_2 \Delta_{15} \Delta_{13} \Delta_{126} \Delta_{124} - \Delta_5 \Delta_{16} \Delta_{12} \Delta_{124} \Delta_{123} \\
&\phantom{=} - \Delta_1 \Delta_{15} \Delta_{12} \Delta_{126} \Delta_{123}.
\end{align*}
\]

(5.6)

Recall that in Section 4.2, for a fraction \( ST^{-1} \) of two tableaux \( S, T \), we denote \( \Delta_{ST^{-1}} = \Delta_{S} \Delta_{T}^{-1} \). For \( T \in \text{SSYT}(k - 1, [k]) \), we have that \( T = T'' \cup T' \), where \( T' \) is a tableau whose columns are fundamental tableaux and \( T'' \) is a fraction of two trivial tableaux. Define \( \chi'(T) = \Delta_{T''} \chi_{C[SL_k]^N}(T') \). We have the following conjecture.

**Conjecture 5.6.** For every \( T \in \text{SSYT}(k - 1, [k]) \), \( \chi'(T) \in C[SL_k]^N \). Moreover, \( \{ \chi'(T) : T \in \text{SSYT}(k - 1, [k]) \} \) is the dual canonical basis of \( C[SL_k]^N \).

We give an example to explain Conjecture 5.6.
Example 5.7. We take \( T = \begin{array}{cccc}
1 & 2 \\
3 & 4 \\
5 & 6
\end{array} \in \text{SSYT}(5,[6]) \). Then \( T = T'' \cup T' \), where \( T' = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 3 & 4 \\
5 & 6
\end{array}, \ T'' = \begin{array}{cc}
1 & 1 \\
2 & 2 
\end{array} \). We have that

\[
\text{ch}(T') = \frac{\text{ch}(T'')}{\Delta_1 \Delta_{12} \Delta_{12}} = \Delta_{136} \Delta_{245} - \Delta_{126} \Delta_{345} \in \mathbb{C}[SL_6]^{N^-},
\]

where \( \text{ch}(T') \) is equal to (5.6).

6. Mutation of Tableaux

In this section, we give a mutation rule for the cluster algebra \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)) using tableaux.

A finite dimensional \( U_q(\mathfrak{g}) \)-module is called prime if it is not isomorphic to a tensor product of two nontrivial \( U_q(\mathfrak{g}) \)-modules (cf. [13]). A simple \( U_q(\mathfrak{g}) \)-module \( M \) is real if \( M \otimes M \) is simple (cf. [41]). We say that a tableau \( T \in \text{SSYT}(k-1,[k],\sim) \) is real (respectively, prime) if \( M_T \) is real (respectively, prime).

By Theorem 4.8, every element in the dual canonical basis of \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)) is of the form \( \text{ch}(T), \ T \in \text{SSYT}(k-1,[k],\sim) \). In [36, 49], it is shown that cluster monomials in \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)) belong to the dual canonical basis and they correspond to real modules in \( C_{k,\Delta} \). The cluster variables in \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)) correspond to real prime modules in \( C_{k,\Delta} \). Therefore cluster monomials (respectively, cluster variables) in \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)) are also of the form \( \text{ch}(T) \), where \( T \) is a real (respectively, real prime) tableau in \( \text{SSYT}(k-1,[k],\sim) \).

In [14, Section 4], it is shown that the mutation rule in Grassmannian cluster algebras can be described using semi-standard Young tableaux of rectangular shape. Similarly, we now show that the mutation rule in \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)) can be described using semi-standard Young tableaux.

Starting from the initial seed of \( \mathbb{C}[N] \) (respectively, \( \mathbb{C}[SL_k]^{N^-} \)), each time we perform a mutation at a cluster variable \( \text{ch}(T_r) \), we obtain a new cluster variable \( \text{ch}(T'_r) \) defined recursively by

\[
\text{ch}(T'_r) \text{ch}(T_r) = \prod_{i \to r} \text{ch}(T_i) + \prod_{r \to i} \text{ch}(T_i),
\]

where \( \text{ch}(T_i) \) is the cluster variable at the vertex \( i \). On the other hand, by Theorem 2.1 and the formula (4.4), we have that

\[
(6.1) \quad \text{ch}(T_r) \text{ch}(T'_r) = \text{ch}(T_r \cup T'_r) + \sum_{T''} c_{T''} \text{ch}(T'')
\]

for some \( T'' \in \text{SSYT}(k-1,[k],\sim) \), \( \text{wt}(T'') < \text{wt}(T_r \cup T'_r) \), \( c_{T''} \in \mathbb{Z}_{\geq 0} \). Therefore one of the two tableaux \( \cup_{i \to r} T_i \) or \( \cup_{r \to i} T_i \) has strictly greater weight than the other, and moreover the one
with higher weight is equal to $T_i \cup T'_i$ in $SSYT(k-1, [k], \sim)$. Denote by $\max\{\cup_{i \to r} T_i, \cup_{r \to i} T_i\}$ this higher weight tableau. Then

$$T'_r = T^{-1}_r \max\{\cup_{i \to r} T_i, \cup_{r \to i} T_i\}.$$  

**Remark 6.1.** There is a partial order called *dominance order* in the set of semi-standard Young tableaux (cf. [10, Section 5.5]).

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, with $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0$, $\mu_1 \geq \cdots \geq \mu_\ell \geq 0$, be partitions. Then $\lambda \leq_{\text{dom}} \mu$ in the dominance order if $\sum_{j \leq i} \lambda_j \leq \sum_{j \leq i} \mu_j$ for $i = 1, \ldots, \ell$.

For a semi-standard tableau $T$ in $SSYT(k, [m])$ and $i \in [m]$, denote by $T[i]$ the sub-tableau obtained from $T$ by restriction to the entries in $[i]$. For a tableau $T$, let $\text{sh}(T)$ denote the shape of $T$. For $T, T' \in SSYT(k, [m])$ of the same shape, $T \leq_{\text{dom}} T'$ in the dominance order if for every $i \in [m]$, $\text{sh}(T[i]) \leq_{\text{dom}} \text{sh}(T'[i])$ in the dominance order on partitions.

The *content* of a tableau $T \in SSYT(k, [m])$ is the vector $(\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m$, where $\nu_i$ is the number of $i$-filled boxes in $T$. By a similar proof as the proof of Proposition 3.28 in [14], for $T, T' \in SSYT(k-1, [k])$ with the same content and with the same shape, $T \leq_{\text{dom}} T'$ in the dominance order if and only if $M_T \leq M_{T'} \in \mathbb{P}^+$ in the monomial order in (2.2).

In the mutation described above, if we use tableaux in $SSYT(k-1, [k])$ (not other tableau representatives of equivalence classes in $SSYT(k-1, [k], \sim)$), then in every step, $\cup_{i \to r} T_i$ and $\cup_{r \to i} T_i$ have the same shape and the same content. Therefore in the mutations, one can also use tableaux in $SSYT(k-1, [k])$ and use the dominance order on tableaux to compute $\max\{\cup_{i \to r} T_i, \cup_{r \to i} T_i\}$ in (6.2).

**Example 6.2.** The following are some examples of exchange relations in $\mathbb{C}[N], N \subset SL_6$, (respectively, $\mathbb{C}[\overline{SL_6}]^N$):

$$\text{ch}(\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}) \text{ch}(\begin{array}{ccc}
3 & 4 & 5 \\
1 & 2 & 6
\end{array}) = \text{ch}(\begin{array}{ccc}
3 & 4 & 5 \\
2 & 6 & 1
\end{array}) \text{ch}(\begin{array}{ccc}
2 & 4 & 5 \\
3 & 6 & 1
\end{array}) + \text{ch}(\begin{array}{ccc}
2 & 4 & 5 \\
3 & 6 & 1
\end{array}) \text{ch}(\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}).$$

$$\text{ch}(\begin{array}{ccc}
2 & 3 & 4 \\
5 & 6
\end{array}) \text{ch}(\begin{array}{ccc}
3 & 5 & 6 \\
1 & 2
\end{array}) = \text{ch}(\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5
\end{array}) \text{ch}(\begin{array}{ccc}
1 & 4 & 5 \\
2 & 3
\end{array}) + \text{ch}(\begin{array}{ccc}
1 & 4 & 5 \\
2 & 3
\end{array}) \text{ch}(\begin{array}{ccc}
2 & 3 & 4 \\
1 & 5
\end{array}).$$

**Example 6.3.** The cluster variables (not including frozen variables) of $\mathbb{C}[N], N \subset SL_5$, (respectively, $\mathbb{C}[\overline{SL_5}]^N$) are $\text{ch}(T)$, where $T$'s are the following tableaux:

$$\begin{array}{cccccccccccccccc}
2 & 3 & 1 & 1 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 3 & 4 & 1 & 1 & 1 \\
1 & 3 & 4 & 5 & 3 & 4 & 5 & 5 & 4 & 4 & 5 & 5 & 4 & 5 & 5
\end{array}$$
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