Induced loop quantum cosmology on a brane via holography

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Based on the holographic principle, it is demonstrated that loop quantum Friedmann equations can be induced on a brane, corresponding to a strongly coupled string regime in the bulk, and have braneworld cosmology equations as its low energy limit. Such result can establish a possible connection between loop quantum gravity and string theory.

String theory and loop quantum gravity (LQG) offer descriptions of the spacetime in contexts where its quantum behavior should arises. Particularly, these theories suggests possible descriptions of the near Big-Bang physics. In this way, Loop Quantum Cosmology (LQC) [1], which comes from LQG, and braneworld cosmology [2], arising from string theory, offer descriptions of the universe which take into account effects of quantum gravity. LQC and braneworld cosmology, each on their side, have mapping directions for several areas of cosmology and related research in particle physics and field theory. They have consequences, for example, for the primordial gravitational wave spectrum, for the description of the inflationary period, and for structure formation. Moreover, such approaches have become very popular during the last decades because, among other things, they allow making contact with observational activity [2–4].

LQC and braneworld cosmology however remain at variance until now, leading physics to sit on the fence about the beginning state of our universe. Actually, in despite of the fact that, a possible duality between these two ways to describe the behavior of the universe at its beginning has been pointed, they disagree about the role of the quantum corrections in the Friedmann equations [3], which implies in a controversy about the resolution of the Big Bang singularity. This in turn would have also consequences for other important and broad issues in physics. As an example, the primordial gravitational waves spectrum in LQC differs from the braneworld cosmology case, since in the first exists a contribution from a pre-bounce phase. In this way, in opposition to the braneworld inflationary models where low energy gravitational modes come from a high energy region, in LQC these modes have their origin in a low energy state of the universe at a time before the Big-bang. An important implication of this is that the largest scale structures observed at the present time must have their origin not from the quantum fluctuations shortly after the big-bang, but in a semiclassical pre-Big bang phase [3,7].

On the other standing point, among the possible signals of the existence of a quantum structure for spacetime stays the holographic principle [8–11]. According to this principle, when gravitational phenomena becomes important, the representation of the physics in a volume of space can be considered as encoded on its boundary. In this way, holographic hypothesis provide us with an equivalence, or duality, between a theory containing gravity, that works in a spacetime volume, and a gravity-free theory describing the physics on the boundary surface of that region.

In this sense, an important result by Jacobson has been the derivation of Einstein’s field equations from the holographic principle [11]. The basic assumption behind this result is consider that the Clausius relation, \(dQ = TdS\), which connects entropy, temperature and heat, can be held through each spacetime point, for all the local Rindler causal horizon, by taking \(T\) as the Unruh temperature and \(dQ\) as the energy flux as measured by an observer in an accelerated frame just inside the Rindler horizon. The most relevant lesson we bring from this result is that the spacetime comes to be viewed as a gas of atoms with an associated entropy given by the Bekenstein-Hawking formula [12], and the Einstein’s gravitational field equation must be interpreted as an equation of state for the spacetime gas. After, the results obtained by Padmanabhan who, by the use of the principle of the equipartition of energy, has found out a bridge between the spacetime macroscopic description provided by Einstein’s equations and spacetime microscopic degrees of freedom, have reinforced such interpretation of the spacetime introduced by Jacobson [13].

The validity of the holographic principle in the context of cosmology was firstly addressed by Fischler and Suskind [14]. Following these authors, a lot of efforts have been made to implement the Bekenstein bound in this context [15–22]. More recently, observational evidence has been pointed to the existence of a holographic phase in the earlier times of our universe [23].

The relation between quantum cosmology and holography has been investigated both in the context of braneworld cosmology [21] and LQC [23], by the use of the Jacobson’s formalism. In the case of braneworld cosmology, holographic principle has been satisfied, since corrected Friedmann equations have been found out in agreement with the usual braneworld cosmology. Moreover, the relationship between braneworld cosmology and ADS/CFT duality (an important and broadly applied version of the holographic principle), has been established [2]. On the other hand, the attempt to obtain the

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LQC equations from Bekenstein-Hawking entropy has led initially to entropic corrected Friedmann equations which give a different scenario from LQC ones, when the standard logarithmic quantum corrected Bekenstein-Hawking formula, which comes from LQG [25], is considered. However, as has been pointed recently by Silva, the LQC equations can be met when the entropy-area relation that arises in the context of loop quantum black holes is taken into account [27].

In the present article we shall assume that our universe stays on a Randall-Sundrum II 3-brane and obeys the holographic principle, and show how the LQC equations appear as induced on the brane. Moreover, we shall demonstrate that LQC and braneworld cosmology can be connected, where the last arises from the first when one takes the low energy limit of the theory. In order to obtain the LQC and braneworld cosmology equations, we shall use an adaptation of method developed by Jacobson introduced by Cai et al in order to apply the holographic principle to cosmological contexts [28]. Through out this paper, we shall take \( \hbar = c = k_B = G = 1 \).

**Modified black hole entropy-area relation in a braneworld scenario**

In this section, we shall derive a modified black hole entropy-area relation in a braneworld context by suitable gauge choice of the radial coordinate in a braneworld black hole spacetime.

In order to do this, we have that the most general black hole static solution of a four dimensional black hole metric induced on a Randall-Sundrum II 3-brane is written as

\[
ds^2 = -A(r)^2 dt^2 + B(r)^2 dr^2 + C(r)^2 d\Omega^2
\]

where the choice of the function \( C(r) \) is arbitrary.

In the case of the area gauge, we have that \( C(r) = r \). However, in this work, we shall consider that, for a scenario near the Planck regime, the spacetime geometry is distorted by quantum gravity effects in a way that a more general gauge could be considered.

Let us write the equation (1) as

\[
ds^2 = \frac{C^2}{r^2} ds^2 ,
\]

where

\[
ds^2 = \frac{A(r)^2}{C(r)} dr^2 dt^2 - \frac{B(r)^2}{C(r)} dr^2 - r^2 d\Omega^2
\]

In this way, the metrics (1) and (2) are conformally equivalent.

An important consequence is that the way how the black hole temperature and entropy are related with the black hole mass is the same in the spacetime described by \( ds^2 \) and in the spacetime described by \( ds^2 \), since such relations do not changes by conformal transformations [29].

In this work, we shall not be interested in the specific form of the function \( A(r) \) and \( B(r) \). However, we shall done the minimal assumption that such functions must be fixed in a way that the line element \( ds^2 \) describes a black hole solution that obeys the usual Bekenstein-Hawking entropy-area relation \( S = \frac{A}{4} \). On the other hand, if we consider the line element \( ds^2 \), we shall have that, depending on the choice of the function \( C(r) \), the entropy-area relation for this solution must have a different form since the radial distance in a scenario described by the metric (2) is different from the radial distance described by the metric (3).

In this work, we shall consider the line element (3) as given the physical metric by the choice of the following gauge:

\[
C^2 = \left( r^2 + \frac{L^4}{r^2} \right).
\]

Such conformal factor has been used in [29, 31] in order to solve the Schwarzschild black hole singularity, and for \( L \to 0 \), we recover the area gauge.

By considering the factor above a new radial coordinate will be defined as

\[
R = \left( r^2 + \frac{L^4}{r^2} \right)^{1/2}.
\]

where \( r \) is the radial coordinate in the spacetime described by \( ds^2 \). In particular, the value of \( R \) associated with the event horizon will be

\[
R_{EH} = \left( r_+^2 + \frac{L^4}{r_+^2} \right)^{1/2}.
\]

where \( r_+ \) is the black hole event horizon radius in the \( ds^2 \) spacetime.

Since the event horizon surface area \( A \) changes by the conformal transformation, the black hole entropy-area relation must be modified. In fact, the rescaled event horizon area will be given by

\[
A = 4\pi \left( r_+^2 + \frac{L^4}{r_+^2} \right)
\]

As a consequence, the black hole entropy will be expressed in terms of the area (7) as

\[
S = \frac{\sqrt{A^2 - A_0^2}}{4} + O(\gg A_0^2)
\]
where $A_L = 2\sqrt{2\pi}L^2$.

The expression \[8\] to the black hole entropy also appears in the context of loop quantum black holes [32]. In the reference [27], it has been demonstrated that the LQC semiclassical equations can be obtained from the entropy area relation [8], in the context of loop quantum black holes. In the following sections, we shall see how LQC dynamical equations can be induced on a brane by the use of the holographic principle when one takes into account such modified entropy-area relation.

**Quantum corrected Friedmann equations from modified black hole entropy**

In braneworld scenario, it has been conceived that the holographic principle can be realized in the cosmological context if one considers that our four-dimensional universe stays in a 3-brane with a FRW metric on the boundary of $AdS^5$ space containing a 5-dimensional black hole. In this context, the brane motion through the $AdS^5$ space induces the evolution of our 4-dimensional universe [22, 33].

However, in our work, we shall make our investigation from the perspective of an observer on the brane. In this way, we shall be interested in a scenario where the holographic principle can be satisfied in four dimensions in a way that the entropy of the 4-dimensional universe can be associated with the area of its boundary. This kind of scenario has been conceived by some authors by admitting that near the big bang or big crunch singularities our four dimensional universe could go through a holographic phase due to the formation of a gas of 4-dimensional black holes able to saturated the holographic bound [14–21].

In order to do such investigation, we shall take into account other realization of the holographic principle that has arisen from a result by Jacobson [11] and adapted by Cai et al for cosmological contexts [25, 28]. From this approach, the Friedmann equation can be regarded as the first law of thermodynamics for the universe apparent horizon coincides with the universe apparent horizon radius. In this way, we can identify the entropy associated with such horizon with the universe-size black hole entropy.

In this way, by considering that the black holes induced on the brane are described by the metric [2], with the function $C(r)$ given by [4], we shall take into account the modified entropy-area relation [8] in order to find out modified Friedmann equations in the braneworld scenario. Moreover, in order to fulfill the condition described above, we must to emphasize that the radial distance used in our calculations must be given by the function $C(r)$.

Now, with all these points fixed, let us consider that the energy-momentum tensor $T_{\mu\nu}$ of the matter present in the universe is written as the ones for a perfect fluid:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}.$$  \hspace{1cm} (11)

From the energy conservation law, we obtain the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0.$$  \hspace{1cm} (12)

Now, let us write the expressions for the work density $W$ and the energy-supply vector $\psi$:

$$W = -\frac{1}{2}T^{ab}h_{ab} \quad \text{and} \quad \psi_a = T_a^b\partial_b\tilde{r} + W\partial_a\tilde{r}.$$  \hspace{1cm} (13)

In the present context, we have

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}.$$  \hspace{1cm} (10)

In this point, it is necessary to say that the choice of the holographic boundary in the context of holographic cosmology has been a point of controversy. However, in order to get a thermodynamical description of the evolution of the universe based on the Jacobson formalism, it has been shown that the choice of the cosmological apparent horizon as the holographic boundary [17, 28] is more convenient. It is because, at the apparent horizon, the first law of thermodynamics and the Friedmann equations have been shown to be equivalent [34, 36]. Moreover, it has been shown that the obedience to the generalized second law of thermodynamics is fulfilled in the scenario of an accelerating expanding universe when one identifies the cosmological holographic boundary as the universe apparent horizon [37, 39].

In order to calculate the entropy associated with the universe apparent horizon, we can consider a black hole filling the volume of the universe enclosed inside the apparent horizon in a way that the black hole radius coincides with the universe apparent horizon radius. In this way, we can identify the entropy associated with such horizon with the universe-size black hole entropy.
\[ W = \frac{1}{2}(\rho - p - \frac{1}{6}\frac{k_4^2}{k_4^2}(\rho p)) \]  
(14)

and

\[ \psi_t = -\frac{1}{2}(\rho + p)H\tilde{r} - \frac{1}{12}k_4^2\rho p H\tilde{r}. \]  
(15)

\[ \psi_r = \frac{1}{2}(\rho + p)a + \frac{1}{12}k_4^2\rho p H\tilde{r}. \]  
(16)

By the use of the relations above, we can find out the expression for the amount of energy that goes through the universe apparent horizon during a time \( dt \) as

\[ \delta Q = A(\rho + p)\tilde{r}H\left(1 + \frac{\rho}{\sigma}\right)dt. \]  
(17)

where \( A = 4\pi\tilde{r}_A^2 \) and \( \sigma = 6k_4^2/k_4^2 \) is the brane tension.

As has been pointed in the reference [25], the horizon temperature is defined by the Friedmann metric, independently of the gravity theory in concern. In contrast, the horizon entropy depends on the gravity theory we are taking into account. Consequently, the temperature related to the universe apparent horizon will be given by \( T = \left(2k_B\pi \tilde{r}_A\right)^{-1} \), which was found out in [30], by the use of the tunneling formalism. On the other side, the apparent horizon entropy will be defined by the Eq. [8].

From the results above, by the use of the Clausius relation \( dQ = TdS \), we obtain

\[ \dot{H} - \frac{k}{a^2} = 4\pi\frac{\sqrt{A^2 - A_L^2}}{A}(\rho + p)\left(1 + \frac{\rho}{\sigma}\right), \]  
(18)

where we have used the following relation

\[ \dot{\tilde{r}}_A = -H\tilde{r}_A^2\left(\dot{H} - \frac{k}{a^2}\right). \]  
(19)

In this point, the use of the continuity Eq. [12] give us

\[ \frac{8\pi d\rho}{3dt} \left(1 + \frac{\rho}{\sigma}\right) = \frac{A}{\sqrt{A^2 - A_L^2}} \frac{d(H^2 + k/a^2)}{dt}, \]  
(20)

which provide

\[ \frac{4\pi}{3} \sqrt{1 + \frac{\rho}{\sigma}} = 2\pi \int \frac{dA}{A\sqrt{A^2 - A_L^2}}. \]  
(21)

The integration in the equation (21) give us

\[ H^2 + \frac{k}{a^2} = \frac{4\pi}{A_L} \cos(\Theta). \]  
(22)

In the equation above, we have \( \Theta = \pm \left[\frac{2\pi A L (1 + \frac{\rho}{\sigma})}{\alpha} - \alpha\right] \), and \( \alpha \) is a phase constant.

The Eqs. (18) and (22) are modified versions of the Friedmann equations. The corrections present in these equations provide an effective density term in the form of a harmonic function of the classical density. The most important implications of this first result is that such modified Friedmann equations gives us a scenario where a quantum bounce takes the place of the Big Bang initial singularity when the universe density assumes a critical value. The usual Friedmann equations are obtained when one takes \( A_L \to 0 \).

**Relation between LQC and braneworld cosmology —**

In order to address the relation between LQC and braneworld cosmology, shall put all quantum corrections on the right-hand side of the Eq. (22). In this way, we shall express the geometric contend of this equation in terms of the classical radial coordinate, i.e, the radial coordinate \( \tilde{r}_A \) in the relation \( \tilde{r}_A = (\tilde{r}_A^2 + L^4/\tilde{r}_A^2)^{1/2} \) (See the Eq. [5]).

By the use of the Eq. (10), we get

\[ H^2 + \frac{k}{a^2} = \frac{H^2 + \frac{k}{a^2}}{1 + L^4(H^2 + \frac{k}{a^2})^2} = \frac{4\pi}{A_{mim}} \cos(\Theta), \]

where the prime indicates the dependence of the functions on \( \tilde{r}_A \).

Two identical solutions can be find out by solve the equation above:

\[ H^2 + \frac{k}{a^2} = \frac{1 + \sin\left[(2/3)A L \rho + (1 + \frac{1}{2}\rho) - \alpha\right]}{L^2 \cos\left[(2/3)A L (1 + \rho) - (1 + \frac{1}{2}\rho) - \alpha\right]). \]  
(23)

If we expand the right-hand side of the equation above around \( A_L \), we get

\[ H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \left[A(\rho^2 + B(\rho)) + \Lambda(\alpha)\right], \]  
(24)
where

\[
A(\alpha) = \frac{2A_L}{3\sqrt{8\pi}} \frac{(1-P)^2}{(1+P)^2} \sec(\alpha) \left( \sec(\alpha) - \tan(\alpha) \right)^2 + \frac{\sigma}{\sec(\alpha)(\sec(\alpha) - \tan(\alpha))},
\]

\[
B(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{(1-P)}{(1+P)} \sec(\alpha)(\sec(\alpha) - \tan(\alpha)),
\]

\[
\Lambda(\alpha) = \frac{\sqrt{8\pi}}{A_L} (\sec(\alpha) - \tan(\alpha)).
\]

(25)

Since the quantum corrections present in the LQC and braneworld cosmology Friedmann equations contains thermals till the order of the minimal area $[1, 2, 30]$, from now on, we shall discrider the terms of order $O(\geq A_L^2)$ front of the other ones. Moreover, the term $\Lambda(\alpha)$ arises as the cosmological constant.

In this point, we shall redefine the universe density by the simple transformation $\rho \rightarrow (\beta \rho - \delta)$. In this way, we shall obtain, dropping the primes:

\[
H^2 = \frac{k}{a^2} = \frac{8\pi}{3} \rho \left(1 - \frac{\rho}{\varrho(\alpha, \sigma)}\right),
\]

(26)

with $\varrho = 4\Lambda - B^2/A$, $\beta = \sqrt{B^2 - 4\Lambda A}$, and $\delta = (B^2 - 4\Lambda A \pm B \sqrt{(B^2 - 4\Lambda A)})/2A$. The redefined universe density consists also in a solution of the continuity equation. Moreover, it gives us the usual universe density where one takes the limit of $\alpha \rightarrow \pi/2(A_L \rightarrow 0)$.

On the other hand, the Raychaudhuri equation can be obtained from the time derivative of the equation (26) which, by the use of the continuity equation, expressed in terms of the redefined density and the classical radial coordinate, give us

\[
\dot{H} = -4\pi(\rho + p) \left(1 - \frac{2\rho}{\varrho(\alpha, \sigma)}\right)
\]

(27)

The fig. 1 shows the critical density $\varrho$ as a function of the phase constant $\alpha$ and the brane tension $\sigma$. In this way, for positive values of $\varrho(\alpha, \sigma)$, we shall have that the equations (26) and (27) will give us the LQC semiclassical Friedmann equations [1]. On the other hand, for negative values of $\varrho(\alpha, \lambda)$, we shall obtain from the equations (26) and (27) the braneworld cosmology Friedmann equations [2]. As one can see, the transition between four dimensional LQC and braneworld cosmology occurs as $\alpha \rightarrow \pi/2$ and $\sigma$ becomes bigger.

Conclusions and remarks

In the present work, modified Friedmann equations have been found out by the use of an entropy-area relation which arises from an alternative choice of the radial coordinate on a brane black hole spacetime. From these equations, both the usual semiclassical LQC and braneworld cosmology equations are recovered.

By fixing the constant of phase $\alpha$, we have that the value of the brane tension has a crucial role in the determination of what kind of cosmology is induced on the brane, since the transition between LQC and braneworld Friedmann equation would be driven by the choice of the brane tension value. By the way, if one relates the physics on the brane with the physics in the bulk, it is known that the brane tension is connected with the string couplings. In particular, we have $\sigma \sim 1/g_s$, where $g_s$ corresponds to the closed string coupling. Consequently, braneworld Friedmann equations on the brane will be related with a weakly coupled string regime in the bulk. On the other hand, a more interestingly relationship is established between a strongly coupled regime of strings in the bulk and LQC Friedmann equations on the brane.

Interestingly, based on the results demonstrated recently by Singh and Soni [11], if one considers the Raychaudhuri equation in the form of (27), the double possibility in the signal of $\varrho(\alpha, \sigma)$ implies in two possible situations. In this way, we shall have that a strongly coupling string regime in the bulk, which implies $\varrho(\alpha, \sigma) > 0$ on the brane, as occurs in the context of LQC, will induce a geometry on the brane that can be quantized in a background independent way, in terms of polymer quantization. On the other hand, a weakly coupled string regime in the bulk, which will imply in $\varrho(\alpha, \sigma) < 0$ on the brane, as occurs in the context of braneworld cosmology, will induce a theory that does not underlie polymer quantiza-
tion and the geometry will be classical and fixed. In this way, the results found out in this article could point to a deep connection between LQG and string theory, where a strongly coupled strings are connected with LQC via holography.

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