ON THE CAYLEY DEGREE OF AN ALGEBRAIC GROUP

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Abstract. A connected linear algebraic group $G$ is called a Cayley group if the Lie algebra of $G$ endowed with the adjoint $G$-action and the group variety of $G$ endowed with the conjugation $G$-action are birationally $G$-isomorphic. In particular, the classical Cayley map

$$X \mapsto (I_n - X)(I_n + X)^{-1}$$

between the special orthogonal group $SO_n$ and its Lie algebra $so_n$, shows that $SO_n$ is a Cayley group. In an earlier paper we classified the simple Cayley groups defined over an algebraically closed field of characteristic zero. Here we consider a new numerical invariant of $G$, the Cayley degree, which “measures” how far $G$ is from being Cayley, and prove upper bounds on Cayley degrees of some groups.

1. Introduction

Let $G$ be a connected linear algebraic group and let $\mathfrak{g}$ be its Lie algebra. We say that $G$ is a Cayley group if there is a birational isomorphism

$$\varphi: G \rightarrow \mathfrak{g}$$

which is equivariant with respect to the conjugation action of $G$ on itself and the adjoint action of $G$ on $\mathfrak{g}$; see [LPR, Definition 1.5]. In particular, the classical Cayley map [C]

$$X \mapsto (I_n - X)(I_n + X)^{-1}$$

between the special orthogonal group $SO_n$ and its Lie algebra $so_n$ shows that $SO_n$ is a Cayley group. (The same formula shows that $Sp_{2n}$ is Cayley as well.) In the sequel we will always assume that the base field $k$ is algebraically closed and of characteristic zero. (Problem 1 below is of interest for arbitrary $k$ but the partial answers we would like to discuss here require this assumption.)

In 1975 D. Luna [L1], [L3] asked the second-named author a question that, in the above terminology, can be restated as follows: For what $n$ is the group $SL_n$ Cayley? In [LPR] we showed that $SL_n$ is Cayley if and only if $n \leq 3$ and, more generally, proved the following classification theorem.

**Theorem 1.** ([LPR, Theorem 3.31(a)]) A connected simple algebraic group $G$ is Cayley if and only $G$ is isomorphic to one of the following groups: $SL_2$, $SL_3$, $SO_n$ ($n \neq 2, 4$), $Sp_{2n}$, $PGL_n$ ($n \geq 1$).

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Note that $\text{SO}_n$ is a Cayley group for every $n \geq 1$; we have excluded $\text{SO}_2$ and $\text{SO}_4$ from the above list because these groups are not simple.

A generalized Cayley map of $G$ is a rational $G$-equivariant map $\varphi : G \to \mathfrak{g}$, as in (1), except that instead of requiring it to be a birational isomorphism, we only require it to be dominant, see [LPR, Definition 10.9]. Every generalized Cayley map of $G$ has finite degree,

$$\deg \varphi = [k(G) : \varphi^*(k(\mathfrak{g}))] < \infty,$$

where, as usual, $k(X)$ and $k[X]$ denote respectively the field of rational and the algebra of regular functions on an irreducible algebraic variety $X$). A generalized Cayley map (1) exists for every linear algebraic group $G$; see [LPR, Proposition 10.5]. Hence the following natural number is well defined.

**Definition 1.** The Cayley degree $\text{Cay}(G)$ of $G$ is the minimal value of $\deg \varphi$, as $\varphi$ ranges over all generalized Cayley maps of $G$.

Note that, by definition, $G$ is a Cayley group if and only if $\text{Cay}(G) = 1$. Therefore Theorem 1 may be viewed as a first step toward a solution of the following more general problem.

**Problem 1.** Find the Cayley degrees of connected simple algebraic groups.

We do not have any general methods for proving lower bounds on the Cayley degree, beyond those provided by Theorem 1; in particular, we do not have an example of a linear algebraic group $G$ with $\text{Cay}(G) > 2$. Thus in this note we will primarily concentrate on upper bounds. Our main results are Theorems 2 and 3 below.

**Theorem 2.** If $n \geq 3$, then $\text{Cay}(\text{SL}_n) \leq n - 2$.

Our proof of Theorem 2 is self-contained. For $n = 3$ this argument gives a new proof of the fact that $\text{Cay}(\text{SL}_3) = 1$ (i.e., $\text{SL}_3$ is a Cayley group), which is simpler than either of the two proofs in [LPR]. For $n = 4$, Theorem 2 implies that $\text{Cay}(\text{SL}_4) = 2$; see Example 4.

To motivate our second main result, we note that the exceptional group $G_2$ plays a special role in this theory. While $G_2$ is not a Cayley group, it is close to being one, in the sense that $G_2 \times G_2^m$ is Cayley; see [LPR, Theorem 1.31(b)]. In fact, $G_2$ is the unique simple group $G$ which is stably Cayley but is not Cayley; see [LPR, Theorems 1.29 and 1.31]. (Recall that $G$ is called stably Cayley if $G \times G_2^m$ is Cayley for some $r \geq 1$.) Theorem 3 below shows that $G_2$ is also close to being Cayley in the sense of having a small Cayley degree.

**Theorem 3.** $\text{Cay}(G_2) = 2$.

The rest of this note is structured as follows. In Section 2 we determine the Cayley degrees of Spin groups and some groups of type $A$. In Section 3 we prove a lemma that reduces the computation of the Cayley degree of a reductive group $G$ to a question about finite group actions. This lemma is then used as a starting point for the proofs of Theorems 2 and 3 in Sections 4 and 5 respectively. In Section 6 we give a representation theoretic interpretation of the Cayley degree.

## 2. First examples

**Lemma 1.** (a) Let $\pi : G \to H$ be an isogeny between connected linear algebraic groups and let $d$ be the order of its kernel.
(a1) Then
\[ \text{Cay}(G) \leq d \cdot \text{Cay}(H). \]

(a2) If \( G \) is not Cayley but \( H \) is Cayley, and \( d = 2 \), then \( \text{Cay}(G) = 2 \).

(b) Let \( \varphi_i \) be a generalized Cayley map of a connected linear algebraic group \( G_i \), where \( i = 1, \ldots, n \). Then \( \varphi_1 \times \ldots \times \varphi_n \) is a generalized Cayley map of \( G_1 \times \ldots \times G_n \), and
\[ \deg (\varphi_1 \times \ldots \times \varphi_n) = \deg \varphi_1 \ldots \deg \varphi_n. \]

Proof. (a1) The groups \( G \) and \( H \) have the same Lie algebra \( \mathfrak{g} \). Let \( \varphi: H \to \mathfrak{g} \) be a generalized Cayley map of \( H \). Since \( \ker \pi \) is a finite central subgroup of \( G \) and \( \deg \pi = d \), the composition \( \varphi \circ \pi: G \to \mathfrak{g} \) is a generalized Cayley map of \( G \). Its degree is \( d \cdot \deg \varphi \), and part (a1) follows.

(a2) Since \( G \) is not Cayley, we have \( \text{Cay}(G) \geq 2 \). The opposite inequality follows from part (a1).

Part (b) follows from the interpretation of degree of a rational map as the number of points in a general fiber. □

From (b) and Definition 1 we obtain the following upper bound.

**Corollary 1.** \( \text{Cay}(G_1 \times \ldots \times G_n) \leq \text{Cay}(G_1) \cdots \text{Cay}(G_n) \).

The following example shows that, in general, equality does not hold.

**Example 1.** Since \( \text{Cay}(G_2) \geq 2 \) by Theorem 1, but \( \text{Cay}(G_2 \times G^2_m) = 1 \) (see [LPR, Theorem 1.31]), we see that
\[ \text{Cay}(G_2 \times G^2_m) < \text{Cay}(G_2) \cdot \text{Cay}(G^2_m). \]

(In fact, the right hand side of this inequality is equal to 2, because \( \text{Cay}(G_2) = 2 \) by Theorem 2 and \( \text{Cay}(G^2_m) = 1 \); see [LPR, Example 1.21].)

**Example 2.** (see [LPR, p. 962]) The groups
\[ \text{Spin}_2 \simeq G_m, \quad \text{Spin}_3 \simeq \text{SL}_2, \quad \text{Spin}_4 \simeq \text{SL}_2 \times \text{SL}_2, \quad \text{Spin}_5 \simeq \text{Sp}_4 \]
are easily seen to be Cayley. On the other hand, \( \text{Spin}_n \) is not Cayley if \( n \geq 6 \). Since \( \text{SO}_n \) is Cayley for every \( n \), applying Lemma 1(b) to the natural 2-sheeted isogeny \( \text{Spin}_n \to \text{SO}_n \) (where \( n \geq 6 \)), we obtain
\[ \text{Cay}(\text{Spin}_n) = \begin{cases} 2 & \text{for } n \geq 6, \\ 1 & \text{for } n \leq 5. \end{cases} \tag{3} \]

**Example 3.** Since \( \text{PGL}_n \) is a Cayley group for every \( n \geq 1 \), Lemma 1, applied to the natural isogeny \( \text{SL}_n/\mu_d = G \to H := \text{PGL}_n \) yields
\[ \text{Cay}(\text{SL}_n/\mu_d) \leq n/d. \tag{4} \]

In particular,
\[ \text{Cay}(\text{SL}_{2d}/\mu_d) = \begin{cases} 2 & \text{for } d \geq 3, \\ 1 & \text{for } d \leq 2. \end{cases} \]

Note also that setting \( d = 1 \) in (4) yields \( \text{Cay}(\text{SL}_n) \leq n \). Theorem 2 strengthens this bound.
3. The maximal torus

In this section we reduce the problem of finding Cay(G) for a connected reductive group G, to a question about finite group actions.

**Lemma 2.** Let G be a connected linear algebraic group, let T be its maximal torus, let C and N be the centralizer and normalizer of T in G respectively, and let W := N/C be the Weyl group. Denote the Lie algebras of G, T, and C by g, t, and c, respectively.

(a) Then

\[
\text{Cay}(G) = \min_{\psi} \deg \psi, \tag{5}
\]

where \(\psi\) ranges over all dominant rational \(N\)-equivariant maps \(C \to c\).

(b) Moreover, if \(G\) is reductive, then (5) holds, where \(\psi\) ranges over all W-equivariant dominant rational maps \(T \to t\).

**Proof.** Recall that \(G \cong G_1 \times \cdots \times G_n\) and \(g \cong g_1 \times \cdots \times g_n\), where \(\cong\) stands for a birational isomorphism of G-varieties. Moreover, if \(\psi : G \times N \to G \times N\) is a dominant rational G-map, then \(\psi := \varphi|_C : C \to c\) is a dominant rational \(N\)-map and \(\varphi^{-1}(x) = \psi^{-1}(x)\) for a general point \(x \in c\); see [LPR, Lemma 2.17]. Hence

\[
\deg \varphi = |\varphi^{-1}(x)| = |\psi^{-1}(x)| = \deg \psi. \tag{6}
\]

Thus we have a degree preserving bijection between generalized Cayley maps of \(G\) and dominant rational \(N\)-equivariant maps \(C \to c\). This immediately implies (a). If \(G\) is reductive, then \(C = T\), \(c = t\), and the \(N\)-actions on \(C\) and \(c\) descend to the \(W\)-actions (since \(T\), being commutative, acts trivially). Hence part (b) follows from part (a). \(\square\)

**Corollary 2.** Let \(\varphi\) be a generalized Cayley map of a connected reductive group G. Then

\[
\deg \varphi = [k(G)^G : \varphi^*(k(g)^G)].
\]

**Proof.** We will continue to use the notations of Lemma 2 and set \(\psi := \varphi|_T\). Since \(W\) is a finite group acting on \(T\) and \(t\) faithfully, we have \([k(T) : k(T)^W] = |W|\) and \([k(t) : k(t)^W] = |W|\). From this we deduce that \(\deg \psi := [k(T) : \psi^*(k(t))] = [k(T)^W : \psi^*(k(t)^W)]\). Since we have \([k(T)^W : \psi^*(k(t)^W)] = [k(G)^G : \varphi^*(k(g)^G)]\), see [P, Theorem (1.7.5)], [LPR, (3.4)], the claim now follows from (6). \(\square\)

**Remark 1.** If \(\varphi\) is a morphism, Corollary 2 can be deduced from [L3, Lemme Fondamental]. For certain particular morphisms \(\varphi\), a proof can be found in [KM, Corollary (3.3)].

4. Proof of Theorem 2

By Lemma 2 it suffices to construct a dominant rational \(W = S_n\)-equivariant map between the maximal torus \(T\) in \(\text{SL}_n\) and its Lie algebra \(t\).

To keep the notation clear in the construction to follow, we will work with two copies of the affine space \(A^n\), with the same natural (permutation) action of \(S_n\). We will denote one by \(A^n_x\) and the other by \(A^n_y\) and use the variables \(x_1, \ldots, x_n\) and, respectively, \(y_1, \ldots, y_n\) as standard coordinate functions on \(A^n_x\) and \(A^n_y\). We will now embed \(t\) and, respectively, \(T\) into \(A^n_x\) and \(A^n_y\) as the following \(S_n\)-invariant subvarieties:

\[
t = \{(a_1, \ldots, a_n) \in A^n_x | \ a_1 + \cdots + a_n = 0\},
\]

\[
T = \{(b_1, \ldots, b_n) \in A^n_y | \ b_1 \cdots b_n = 1\}.
\]
Consider the mutually inverse $S_n$-equivariant rational maps $\varphi: \mathbb{A}^n_x \to \mathbb{A}^n_y$ and $\psi: \mathbb{A}^n_y \to \mathbb{A}^n_x$ given by

$$\varphi := \left( \frac{x_1 + 1}{x_1}, \ldots, \frac{x_n + 1}{x_n} \right) \quad \text{and} \quad \psi := \left( \frac{1}{y_1 - 1}, \ldots, \frac{1}{y_n - 1} \right).$$

These maps give rise to a (biregular) isomorphism between the open subsets

$$U_x := \{ (a_1, \ldots, a_n) \in \mathbb{A}^n_x \mid a_1 \ldots a_n \neq 0 \}$$

and

$$U_y := \{ (b_1, \ldots, b_n) \in \mathbb{A}^n_y \mid (b_1 - 1) \ldots (b_n - 1) \neq 0 \}$$

in $\mathbb{A}^n_x$ and $\mathbb{A}^n_y$ respectively. Substituting $y_i = \frac{x_i + 1}{x_i}$ into the equation $y_1 \ldots y_n - 1 = 0$ of $T$, we see that $\psi(T \cap U_y) = X \cap U_x$, where $X$ is the hypersurface in $\mathbb{A}^n_x$ cut out by the equation

$$f(x_1, \ldots, x_n) := (x_1 + 1) \ldots (x_n + 1) - x_1 \ldots x_n = 0.$$

Since $X \cap U_x$ is isomorphic to $T \cap U_y$ (which is irreducible) and $X$ does not contain any of the $n$ components $\{ x_i = 0 \}$ of the complement of $U_x$, we conclude that $X$ is irreducible $S_n$-invariant hypersurface in $\mathbb{A}^n_x$. Hence $f$ is a power of an irreducible polynomial. Since $\deg f(1, \ldots, 1, x_i, 1, \ldots, 1) = 1$ for every $i$, we conclude that in fact $f$ is irreducible. As $\deg f = n - 1$, this implies that $X$ is a hypersurface of degree $n - 1$. By our construction $X$ is birationally isomorphic to $T$ (via $\varphi$), as an $S_n$-variety.

Let $\pi$ be the projection $X \dasharrow t$ from a point $a = (a, \ldots, a) \in \mathbb{A}^n_x$. That is, for any point $b \in X$, $b \neq a$, the point $\pi(b)$ is the intersection point of the line passing through $a$ and $b$ with the hyperplane $t \subset \mathbb{A}^n_x$. Moreover, we choose $a$ so that it lies on $X$. Note that this automatically means that it does not lie in $t$. Indeed, since zero does not satisfy the equation

$$f(a, \ldots, a) = (1 + a)^n - a^n = 0,$$

if $a \in X$, then $a$ cannot lie in $t$. Since our base field $k$ is algebraically closed and of characteristic zero, such an $a$ exists for every $n \geq 2$. Note that $\pi$ is well-defined, unless $X$ is a hyperplane parallel to $t$. Since $\deg X = n - 1$, it is not a hyperplane for every $n \geq 3$. Thus $\pi$ is well-defined for every $n \geq 3$. Note also that since $a$ is fixed by $S_n$, the map $\pi$ is $S_n$-equivariant.

We claim that $\pi: X \dasharrow t$ is dominant. Since $\pi$ is a projection map from a point on a hypersurface $X$, and $\deg X = n - 1$, this claim implies that $\deg \pi = n - 2$. Composing $\pi$ with a birational isomorphism $\psi: T \dasharrow X$, we obtain an $S_n$-equivariant dominant rational map $T \dasharrow t$ of degree $n - 2$, and Theorem 2 is proved.

It remains to show that $\pi$ is dominant. Assume the contrary. Let $X_0$ be the closure of the image of $\pi$ in $t$. Then $X$ is the cone over $X_0$ centered at $a$. Since, as we remarked above, $X$ is not a hyperplane (we are assuming throughout that $n \geq 3$), $X$ has to be singular at $a$. Consequently, $a$ satisfies the system of equations

$$\begin{cases} f(a) = (1 + a)^n - a^n = 0, \\ \frac{\partial f}{\partial x_1}(a) = (1 + a)^{n-1} - a^{n-1} = 0. \end{cases}$$

But this system has no solutions, a contradiction. Theorem 2 is now proved. \qed

**Example 4.** By Theorem 2, $\text{Cay}(\text{SL}_4) \leq 2$. Equivalently, $\text{Cay}(\text{SL}_4) = 2$; indeed, we know that $\text{Cay}(\text{SL}_4) \neq 1$, i.e., $\text{SL}_4$ is not a Cayley group by Theorem 1.
Since $\text{SL}_4/\mu_2 \simeq \text{SO}_4$ is Cayley, the equality $\text{Cay}(\text{SL}_4) = 2$ can also be obtained by applying Lemma 1(b) to the isogeny $\text{SL}_4 \to \text{SL}_4/\mu_2$. Alternatively, since $\text{SL}_4 \simeq \text{Spin}_6$, the equality $\text{Cay}(\text{SL}_4) = 2$ is a special case of (3).

5. Proof of Theorem 3

First recall that $G_2$ is not Cayley (see Theorem 1) and hence $\text{Cay}(G_2) \geq 2$. Thus we only need to prove the opposite inequality. By Lemma 2 it suffices to construct a $W$-equivariant dominant rational map $T \dashrightarrow t$ of degree 2, where $T$ is a maximal torus of $G_2$, $t$ is the Lie algebra of $T$, and $W$ is the Weyl group.

Recall that $W$ is isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$. Once again, we consider two copies of the 3-dimensional affine space, $\mathbb{A}^3_x$ and $\mathbb{A}^3_y$, with the following $W$-actions. The symmetric group $S_3$ acts on both copies in the natural way (by permuting the coordinates). The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^3_x$ by
\[
(a_1, a_2, a_3) \mapsto (-a_1, -a_2, -a_3),
\]
and on $\mathbb{A}^3_y$ by
\[
(b_1, b_2, b_3) \mapsto \left(\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}\right).
\]
We may (and shall) embed $t$ and $T$ into $\mathbb{A}^3_x$ and $\mathbb{A}^3_y$, respectively, as the following $W$-invariant subvarieties:
\[
t = \{(a_1, a_2, a_3) \in \mathbb{A}^3_x \mid a_1 + a_2 + a_3 = 0\},
\]
\[
T = \{(b_1, b_2, b_3) \in \mathbb{A}^3_y \mid b_1 b_2 b_3 = 1\}.
\]

We now consider the mutually inverse $W$-equivariant rational maps $\varphi: \mathbb{A}^3_x \to \mathbb{A}^3_y$ and $\psi: \mathbb{A}^3_y \to \mathbb{A}^3_x$ given by
\[
\varphi := \left(\frac{x_1 - 1}{x_1 + 1}, \frac{x_2 - 1}{x_2 + 1}, \frac{x_3 - 1}{x_3 + 1}\right) \quad \text{and} \quad \psi := \left(-\frac{y_1 + 1}{y_1 - 1}, -\frac{y_2 + 1}{y_2 - 1}, -\frac{y_3 + 1}{y_3 - 1}\right).
\]
These maps give rise to a $W$-equivariant isomorphism between the open subsets
\[
U_x := \left\{(a_1, a_2, a_3) \in \mathbb{A}^3_x \mid (a_1 + 1)(a_2 + 1)(a_3 + 1) \neq 0\right\}
\]
and
\[
U_y := \left\{(b_1, b_2, b_3) \in \mathbb{A}^3_y \mid (b_1 - 1)(b_2 - 1)(b_3 - 1) \neq 0\right\}
\]
in $\mathbb{A}^3_x$ and $\mathbb{A}^3_y$, respectively. Substituting $y_i = \frac{x_i - 1}{x_i + 1}$ into the equation $y_1 y_2 y_3 = 1$ of $T$, we see that $\psi(T \cap U_y) = X \cap U_x$, where $X$ is the $W$-invariant quadric surface in $\mathbb{A}^3_x$ defined by the equation
\[
x_1 x_2 + x_2 x_3 + x_1 x_3 + 1 = 0.
\]
Composing the $W$-equivariant birational isomorphism $\psi: T \dashrightarrow \mathbb{A}^3_x$ with the $W$-invariant linear projection $\alpha: X \to t$ given by
\[
\alpha := \left(x_1 - \frac{x_1 + x_2 + x_3}{3}, x_2 - \frac{x_1 + x_2 + x_3}{3}, x_3 - \frac{x_1 + x_2 + x_3}{3}\right),
\]
we obtain a desired $W$-equivariant rational map $\alpha \circ \psi: T \dashrightarrow t$ of degree 2. \qed
Remark 2. The proofs of Theorems 2 and 3 proceed along similar lines: we begin by defining a birational isomorphism $\psi$ between $T$ and a hypersurface $X$, then project $X$ onto $t$. Note, however, that the projections $\pi$ (in the proof of Theorem 2) and $\alpha$ (in the proof of Theorem 3) are different in the following sense: $\pi$ is a projection from a point on $X$, and $\alpha$ is a linear projection ($\alpha$ may also be viewed as a projection from a point at infinity, which does not lie on $X$). Note that $\alpha$ cannot be replaced by a projection from a point of $X$, since $X$ has no $W$-equivariant points (and also because otherwise $\alpha$ would have degree 1 and our argument would show that $G_2$ is a Cayley group, which we know to be false).

Remark 3. The formula for $\varphi$ is somewhat similar to the formula for the classical Cayley map (2). Note, however, that we cannot replace $\frac{x_1 - 1}{x_1 + 1}, \frac{x_2 - 1}{x_2 + 1}$, etc. by $\frac{1 - x_1}{x_1 + 1}, \frac{1 - x_2}{x_2 + 1}$, etc. in the definition of $\varphi$. If we do this, then, setting $\psi = \varphi^{-1}$, we see that the image of $T$ under $\psi$ becomes the cubic $x_1x_2x_3 + x_1 + x_2 + x_3 = 0$, rather than the quadric $x_1x_2 + x_2x_3 + x_1x_3 + 1 = 0$, and the above argument gives a generalized Cayley map of degree 3, rather than 2.

6. A REPRESENTATION THEORETIC APPROACH

In conclusion we outline a representation theoretic approach to determining the Cayley degree of an algebraic group.

Let $X$ be an irreducible algebraic variety endowed with an action of an algebraic group $H$, and let $V$ be a vector space over $k$ of dimension $\dim X$ endowed with a linear action of $H$. Then rational dominant $H$-maps $X \dashrightarrow V$ are described as follows. Let $M$ be a submodule of the $H$-module $k(X)$ such that

(i) $M$ is isomorphic to the $H$-module $V^*$,

(ii) $k(X)$ is algebraic over the subfield $k(M)$ generated by $M$ over $k$.

By (ii), $k(M)/k$ is a purely transcendental extension of degree $\dim X$. Since $k(V)$ is generated over $k$ by $V^*$, any isomorphism of $H$-modules $V^* \to M$ can be uniquely extended up to an $H$-equivariant embedding $\iota : k(V) \hookrightarrow k(X)$ whose image is $k(M)$. This embedding determines a rational dominant $H$-map $\psi : X \dashrightarrow V$ such that $\psi^* = \iota$. We have

$$\deg \psi = [k(X) : k(M)].$$  \hfill (7)

Any dominant rational $H$-map $X \dashrightarrow V$ is obtained in this way.

Now suppose $G$ is a connected reductive linear algebraic group, $X = T$ is a maximal torus, $V = t$ is the Lie algebra of $T$ and $H = W = N_G(T)/T$ is the Weyl group. In view of Lemma 2(b) the above approach relates generalized Cayley maps of $G$ to the $W$-module structure of $k(T)$. This connection may be used to prove upper bounds on Cay$(G)$.

Example 5. Let $G = G_2$. Use the notation of Section 5. Let $t_i$ be the restriction of $y_i$ to $T$. Then $t_1t_2t_3 = 1$ and $k(T) = k(t_1, t_2)$. Put

$$z_i := t_i - t_i^{-1}.\quad \hfill (8)$$

From the description of the $W$-actions on $T$ and $t$ given in Section 5 it follows that

$$M := \{\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 \mid \alpha_1 + \alpha_2 + \alpha_3 = 0, \ \alpha_i \in k\}$$  \hfill (9)

is a submodule of the $W$-module $k(T)$ that is isomorphic to the $W$-module $t^*$. Let

$s_1 := z_1 - z_2, \ s_2 := z_1 - z_3$  \hfill (10)
Then $s_1, s_2$ is a basis of $M$, so $k(M) = k(s_1, s_2)$. We have $k(t_1, s_1, s_2) = k(T)$ because $t_2 = (t_1^2 - 1)(t_1 s_1 + t_1 s_2 - t_1^2 + t_1 + 1)^{-1}$. It follows from (8), (10) that
\[
\begin{aligned}
-t_2 + t_2^{-1} &= s_1 - t_1 + t_1^{-1}, \\
t_1 t_2 - t_1^{-1} t_2^{-1} &= s_2 - t_1 + t_1^{-1}.
\end{aligned}
\tag{11}
\]
Eliminating $t_2$ and $t_2^{-1}$ from (11), we obtain the following equation:
\[
t_1^6 - (s_1 + s_2)t_1^5 + (s_1 s_2 - 2s_1 - 2s_2 - 1)t_1^4 + (s_1^2 + s_2^2 - 5)t_1^3 \\
+ (s_1 s_2 + 2s_1 + 2s_2 + 1)t_1^2 + (s_1 + s_2 + 1)t_1 + 1 = 0.
\]
Thus for the conjugating and adjoint actions of $H := W$ respectively on $X := T$ and $V := t$, and for $M$ defined by (9), the above conditions (i), (ii) hold and $[k(T) : k(M)] \leq 6$. Hence by (7), (6), and Lemma 2, there exists a generalized Cayley map of $G$ of degree $[k(T) : k(M)]$. In particular, this implies that $\text{Cay}(G_2) \leq 6$ (of course, by Theorem 3, we know that in fact $\text{Cay}(G_2) = 2$).

\section*{References}

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