Carleman estimate and inverse source problem for Biot’s equations describing wave propagation in porous media

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Received 21 April 2013, in final form 9 August 2013
Published 17 September 2013
Online at stacks.iop.org/IP/29/115002

Abstract

According to Biot’s paper in 1956, by using the Lagrangian equations in classical mechanics, we consider a problem of the filtration of a liquid in porous elastic-deformation media whose mechanical behavior is described by the Lamé system coupled with a hyperbolic equation. Assuming the null surface displacement on the whole boundary, we discuss an inverse source problem of determining a body force only by observation of surface traction on a suitable sub-domain along a sufficiently large time interval. Our main result is a H"{o}lder stability estimate for the inverse source problem, which is proved by a new Carleman estimate for Biot’s system.

1. Introduction

In 1956, Biot [9] presented a three-dimensional theory for coupled frame–fluid wave propagation in fluid-saturated porous media, treating the solid frame and the saturating fluid as two separate collocated coupled continua. Two second-order coupled partial differential equations were derived from this theory. More precisely, let us consider an open and bounded domain $\Omega$ of $\mathbb{R}^3$ with $C^\infty$ boundary $\Gamma = \partial \Omega$, and let $v = v(x)$ be the unit outward normal vector to $\partial \Omega$ at $x$. Given $T > 0$, Biot’s equation is written as

\begin{align}
\rho_{11} \frac{\partial^2 u_i}{\partial t^2} + \rho_{12} \frac{\partial^2 u_i}{\partial t} - \Delta u_i(x, t) - \nabla (q \text{ div } u^f) &= F_1, \\
\rho_{12} \frac{\partial^2 u_i}{\partial t^2} + \rho_{22} \frac{\partial^2 u_i}{\partial t} - \nabla (q \text{ div } u^s) - \nabla (r \text{ div } u^f) &= F_2, \\
\end{align}

with the boundary condition

\begin{align}
&u_i(x, t) = 0, \quad u^f(x, t) \cdot v = 0, \quad (x, t) \in \Sigma := \Gamma \times (-T, T) 
\end{align}

where $u_i, u^f, u^s$ are the displacement, fluid and solid velocity, respectively, and $\rho, \mu, \lambda$ are the density and Lamé coefficients, and $q, r$ are the fluid and solid Reynolds coefficients.
and the initial condition
\[
(u^t(x, 0), \partial_t u^t(x, 0)) = (0, 0), \quad (u^f(x, 0), \partial_t u^f(x, 0)) = (0, 0), \quad x \in \Omega,
\]  
(1.3)
where \( F = (F_1, F_2)^T \) is an external force with \( F_\ell = (F^1_\ell, F^2_\ell, F^3_\ell)^T, \ell = 1, 2, \) and \( \Delta_{\mu, \lambda} \) is the elliptic second-order linear differential operator given by
\[
\Delta_{\mu, \lambda} v(x) \equiv \mu \Delta v(x) + (\mu + \lambda)(\nabla \text{div} v(x)) + (\text{div} v(x))\nabla \lambda(x) + (\nabla v + (\nabla v)^T)\nabla \mu(x), \quad x \in \Omega.
\]  
(1.4)
Throughout this paper, \( t \) and \( x = (x_1, x_2, x_3) \) denote the time variable and the spatial variable, respectively, and \( u^t = (u^t_1, u^t_2, u^t_3)^T \) and \( u^f = (u^f_1, u^f_2, u^f_3)^T \) denote the solid frame and fluid phase displacement vectors at the location \( x \) and the time \( t, \) respectively. Here and henceforth \( .^T \) denotes the transpose of matrices under consideration.

We assume that the Lamé coefficients \( \mu \) and \( \lambda \) satisfy
\[
\mu(x) > 0, \quad \lambda(x) + \mu(x) > 0, \quad \forall x \in \overline{\Omega}.
\]
The function \( q(x) > 0, x \in \Omega, \) is the dilatational coupling factor between the fluid phase and the solid frame. The coefficient \( r(x) > 0, x \in \overline{\Omega}, \) is the bulk modulus of the fluid phase and \( q_{11}(x), q_{22}(x) > 0 \) and \( x \in \overline{\Omega} \) are the corrected mass densities for the solid phase and the fluid phase porosity and \( q_{12}(x) \) is the inertial coupling factor, see [16].

We assume that the source terms are given by
\[
F_\ell(t, x) = p_\ell(x) R_\ell(t, x), \quad \ell = 1, 2, \quad (x, t) \in Q,
\]  
(1.5)
where \( p_1, p_2 \in H^2(\Omega) \) are real valued and \( R_\ell = (R^1_\ell, R^2_\ell, R^3_\ell)^T, \ell = 1, 2 \) satisfy
\[
\sum_{\ell=1}^2 \sum_{j=2}^3 \left( \frac{\| \partial_t R_\ell \|^2_{L^2(Q)} + \| \partial_t \nabla R_\ell \|^2_{L^2(Q)}}{C} \right) \leq C.
\]  
(1.6)
In this paper, we assume that \( p_1 \) and \( p_2 \) are unknown, while \( R_1 \) and \( R_2 \) are given.

The main subject of this paper is the inverse problem of determining \( p = (p_1, p_2) \in (H^2(\Omega))^2 \) uniquely from observed data of the displacement vector \( u = (u^t, u^f) \) in a sub-domain \( \omega \subset \Omega. \) It is an important problem, for example, in mechanics to determine the source \( p \) inside a porous body from measurements of the slide frame and fluid phase displacements in \( \omega. \)

### 1.1. Inverse problem

Let \( \omega \subset \Omega \) be an arbitrarily given sub-domain, such that \( \partial \omega \subset \partial \Omega, \) i.e., \( \omega = \Omega \cap \mathcal{F}, \) where \( \mathcal{F} \) is a neighborhood of \( \Gamma \) in \( \mathbb{R}^3 \) and let \( R(x, t) = (R_1(x, t), R_2(x, t)) \) be appropriately given. Then, we want to determine \( p(x) = (p_1(x), p_2(x)), x \in \Omega, \) by measurements \( u_{\omega \times (T,T)}. \)

Our formulation of the inverse problem requires only a finite number of observations. As for inverse problems for non-stationary Lamé system by infinitely many boundary observations (i.e., Dirichlet-to-Neumann map), we refer to [40], for example.

For the formulation with a finite number of observations, Bukhgeim and Klibanov [12] created a method based on a Carleman estimate and established the uniqueness for inverse problems of determining spatially varying coefficients for scalar partial differential equations. See also [2–5, 8, 10, 11, 21–23, 28, 29, 32–34, 36, 37, 43]. In particular, as for inverse problems for the isotropic Lamé system, we can refer to [18, 20, 24–28, 30].

A Carleman estimate is an inequality for a solution to a partial differential equation with weighted \( L^2 \)-norm and effectively yields the unique continuation for a partial differential equation with non-analytic coefficients. As a pioneering work concerning a Carleman estimate,
we refer to Carleman’s paper [13], where what is called a Carleman estimate was proved and it was applied for proving the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since [13], the theory of Carleman estimates has been developed and we refer, for example, to [17, 29] for Carleman estimates for functions having compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). For Carleman estimates for functions without compact supports, we refer to [7, 15, 19, 38, 42]. Moreover, Carleman estimates have been applied for estimating the energy, see, e.g., [27, 31, 35, 36].

Biot’s system is a coupled system. For Carleman estimates and applications to inverse problems for coupled hyperbolic–parabolic systems, we refer to [6, 7], and for coupled parabolic–elliptic systems, we quote [14, 30]. The paper [6] discusses an inverse source problem for the thermoelasticity system and [7] establishes Carleman estimates for various systems such as the thermoelasticity plate equation, the thermoelasticity system with residual stress, while [14] proves stability for an inverse source problem for the linearized Navier–Stokes equations. The paper [30] applies a Carleman estimate to prove stability for a coefficient inverse problem for an elasticity equation with residual stress. Albano and Tataru [1] prove a Carleman estimate for a hyperbolic–parabolic system and establish the observability.

1.2. Notations and statement of main results

In order to formulate our results, we need to introduce some notations. For \( x_0 \in \mathbb{R}^3 \setminus \overline{\Omega} \), we define the following set of scalar coefficients:

\[
\mathcal{C}(m, \theta) = \left\{ c \in C^2(\overline{\Omega}); c(x) > c^* > 0, \ x \in \overline{\Omega}, \|c\|_{C^2(\overline{\Omega})} \leq m, \ \frac{\nabla c \cdot (x - x_0)}{2c} \leq 1 - \theta \right\},
\]

where the constants \( m > 0 \) and \( \theta \in (0, 1) \) are given.

**Assumption A.1.** Throughout this paper, we assume that the coefficients \( (\varrho_{ij})_{1 \leq i, j \leq 2} \), \( \mu \), \( \lambda \), \( q \), \( r \in C^2(\overline{\Omega}) \) satisfy the following conditions:

\[
\varrho(x) = \varrho_{11}(x)\varrho_{22}(x) - \varrho_{12}^2(x) > 0, \ \forall x \in \overline{\Omega},
\]

\[
\lambda(x)r(x) - q^2(x) > 0, \ \forall x \in \overline{\Omega}.
\]

Let \( A(x) = (a_{ij}(x))_{1 \leq i, j \leq 2} \) be the \( 2 \times 2 \)-matrix given by

\[
A(x) = \frac{1}{\varrho} \begin{pmatrix} \varrho_{22} & -\varrho_{12} \\ \varrho_{12} & \varrho_{11} \end{pmatrix} \begin{pmatrix} 2\mu + \lambda & q \\ q & r \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

By (1.8), we can prove that \( (a_{ij}(x))_{1 \leq i, j \leq 2} \) is a positive definite matrix on \( \overline{\Omega} \).

**Assumption A.2.** Let \( A(x) \) have two distinct positive eigenvalues: \( \mu_2(x), \mu_3(x) > 0, \mu_2(x) \neq \mu_3(x) \). Moreover, setting \( \mu_1 := (\varrho^{-1}\varrho_{22})/\mu \), where \( \mu \) is the Lamé coefficient, we assume

\[
\mu_1, \mu_2, \mu_3 \in \mathcal{C}(m, \theta).
\]

**Assumption A.3.** We assume that the solution \( u = (u^t, u^f) \) satisfies the a priori boundedness and regularity:

\[
u \in H^3(Q), \quad \|u\|_{H^3(Q)} \leq M_0,
\]

for some positive constant \( M_0 \).
The main results of this paper can be stated as follows.

Let \( V(\Omega) = (H^1(\Omega))^3 \times H(\text{div}, \Omega) \), where
\[
H(\text{div}, \Omega) = \{ u \in (L^2(\Omega))^3; \text{div} \, u \in L^2(\Omega) \}.
\]

The norm in \( V(\Omega) \) is chosen as follows:
\[
| (v^1, v^2) |^2_{V(\Omega)} = | v^1 |^2_{H^1(\Omega)} + | v^2 |^2_{L^2(\Omega)} + | \text{div} \, v^2 |^2_{L^2(\Omega)}, \quad v = (v^1, v^2) \in V(\Omega).
\]

**Theorem 1.1.** Let \( F \in H^1(-T, T; L^2(\Omega)) \), \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^6 \times (H^1(\Omega))^6 \). Then, there exists a unique solution \( u(x, t) = (u^x(x, t), u^y(x, t)) \) of (1.1)–(1.2) with initial data \((u_0, u_1)\), such that
\[
\| u \|_{C^1([-T, T]; L^2(\Omega))} \leq C(\| F \|_{H^1(-T, T; L^2(\Omega))} + \| u_0 \|_{H^1(\Omega)} + \| u_1 \|_{H^1(\Omega)}).
\]

Moreover, if \( F = 0 \), then the energy of the solution \( u = (u^x, u^y) \) given by
\[
E(t) = \frac{1}{2} \int_{\Omega} (M(x) \partial_t u \cdot \partial_t u + \lambda |\text{div} \, u|^2 + 2\mu |\varepsilon(u')|^2 + r |\text{div} \, u'^2 + 2q (\text{div} \, u') (\text{div} \, u')) \, dx
\]
is conserved, that is,
\[
E(t) = E(0), \quad \forall t \geq 0.
\]
Here, \( M(x) = (\rho_j(x) I_1)_{1 \leq i, j \leq 2} \) and \( \varepsilon(v) = \frac{1}{2} (\nabla v + (\nabla v)^T) \).

The proof is based on the Galerkin method, see [41] for the case \( n = 2 \). For completeness, we will give a proof for dimension 3 in section 4.

**Remark 1.1.** Condition (1.11) is a regularity property for the direct problem, that is, the boundary value–initial value problem, and requires sufficient smoothness and compatibility condition of sufficient order for the initial value \((u_0, u_1)\). Condition (1.11) can be verified similarly to section 4, but we here omit details in order to concentrate on the inverse problem.

In order to formulate our stability estimates for the inverse problem, we introduce some notations.

Let \( \vartheta : \Omega \rightarrow \mathbb{R} \) be the strictly convex function given by
\[
\vartheta(x) = |x - x_0|^2, \quad x \in \overline{\Omega}.
\]

Set
\[
D^2 = \max_{x \in \overline{\Omega}} \vartheta(x), \quad d^2 = \min_{x \in \overline{\Omega}} \vartheta(x), \quad D_0^2 = D^2 - d^2.
\]

By assumption A.2, there exist constants \( c_j^* > 0 \), \( j = 1, 2, 3 \), such that \( \mu_j(x) > c_j^* > 0 \) for all \( x \in \Omega, j = 1, 2, 3 \). Let \( c_k^* = \min\{c_1^*, c_2^*, c_3^*\} \). We choose \( \beta > 0 \), such that
\[
\beta + \frac{mD_0}{\sqrt{c_k^*}} \sqrt{\beta} < \theta c_0^*, \quad c_0^* d^2 - \beta D^2 > 0.
\]

Here, we note that since \( x_0 \notin \overline{\Omega} \), such \( \beta > 0 \) exists.
We set
\[
T_0 = \frac{D_0}{\sqrt{\beta}}.
\]

The main results of this paper can be stated as follows.
**Theorem 1.2. (Stability)** Assume (A.1), (A.2) and (A.3), and \( \partial \omega \supset \partial \Omega \). Let \( T > T_0 \) and \( \mathbf{u} \) be the solution of (1.1)–(1.2) and (1.3). Moreover, let us assume that \( \Phi_j(x) := R_j(x, 0) \) satisfy

\[
\Phi_j(x) \cdot (x - x_0) \neq 0 \quad \text{for all} \quad x \in \overline{\Omega}, \quad j = 1, 2.
\]

Let \( M > 0 \). Then, there exist constants \( C > 0 \) and \( \kappa \in (0, 1) \), such that the following estimate holds:

\[
\| p_1 \|_{H^\ell(\Omega)}^2 + \| p_2 \|_{H^\ell(\Omega)}^2 \leq C \mathcal{E}_\omega(\mathbf{u})^\kappa
\]

for any \( p_\ell \in H^\ell(\Omega), \ell = 1, 2 \), such that \( \| p_\ell \|_{H^\ell} \leq M \) and \( p_\ell = 0 \) on \( \Gamma \). Here,

\[
\mathcal{E}_\omega(\mathbf{u}) = \sum_{j=2}^{3} \| \partial^j \mathbf{u} \|_{H^{l}(\omega \times (-T, T))}^2.
\]

It is another subject to discuss how much we can relax regularity condition (1.11) for proving the conditional stability for our inverse problem (maybe in weaker norms). In this paper, we will not discuss it.

By theorem 1.2, we can readily derive the uniqueness in the inverse problem:

**Corollary 1.1. Under the assumptions in theorem 1.2, we have the uniqueness:** let \( \mathbf{u} = (\mathbf{u}', \mathbf{u}'') \) satisfy Biot’s system (1.1)–(1.3), such that \( \mathbf{u}(x, t) = 0, (x, t) \in \omega \times (-T, T) \). Then, \( p_1(x) = p_2(x) = 0 \) for all \( x \in \Omega \) and \( \mathbf{u}(x, t) = 0 \) in \( \mathcal{Q} \).

In this paper, we discuss the global stability and uniqueness which hold over the whole domain \( \Omega \). Moreover, we assume that the observation sub-domain \( \omega \) is a boundary layer, that is, \( \partial \omega \supset \partial \Omega \). The relaxation of this assumption is important. We have no general theory for Carleman estimates for coupled systems as (1.1). In order to prove a key Carleman estimate (theorem 2.1), we will introduce \( \mathbf{u}' \), \( \mathbf{u}'' \) and \( \mathbf{u}' \), so that we need boundary values for them and the assumption \( \partial \omega \supset \partial \Omega \) is necessary. On the other hand, in [24–26], Carleman estimates are proved for the Lamé system not by assuming that \( \partial \omega \supset \partial \Omega \), the argument there is based on pseudodifferential operators and seems valid for Biot’s system. An application of the method in [24–26] needs substantial pseudodifferential calculus and here we do not argue more.

Our main results are global and so we have to assume boundary condition (1.2) on \( \partial \Omega \). On the other hand, given an observation sub-domain \( \omega \) and not assuming \( \partial \omega \supset \partial \Omega \), we can discuss in which sub-domain we can prove the stability and the uniqueness. This is local stability and uniqueness and it is sufficient to prove a Carleman estimate in some sub-domain for functions with compact supports. We can refer, e.g., to [36] for a basic idea of applications of a Carleman estimate with a cut-off function. Here, we do not discuss local results.

The remainder of the paper is organized as follows. In section 2, we give a Carleman estimate for Biot’s system. In section 3, we prove theorem 1.2. Section 4 is devoted to the proof of theorem 1.1.

### 2. Carleman estimate for Biot’s system

In this section, we will prove a Carleman estimate for Biot’s system, which is interesting in itself. In order to formulate our Carleman estimate, we introduce some notations. Let \( \vartheta : \overline{\Omega} \rightarrow \mathbb{R} \) be the strictly convex function given by (1.14), where \( x_0 \notin \overline{\Omega} \).

We define two functions \( \psi, \varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) of class \( C^\infty \) by

\[
\begin{align*}
\psi(x, t) &= |x - x_0|^2 - \beta |t|^2 \quad \text{for all} \quad x \in \Omega, \quad -T \leq t \leq T, \\
\varphi(x, t) &= e^{\psi(x, t)} - \gamma > 0,
\end{align*}
\]

(2.1)
where \( T > T_0 \). Therefore, by (1.17) and (1.15), we have
\[
\varphi(x, 0) \geq d_0, \quad \varphi(x, \pm T) < d_0, \tag{2.2}
\]
with \( d_0 = \exp(\varphi d^2) \). Thus, for given \( \eta > 0 \), we can choose sufficiently small \( \varepsilon = \varepsilon(\eta) \), such that
\[
\varphi(x, t) \leq d_0 - \eta \equiv d_1 \quad \text{for all } (x, t) \in \{(x, t) \in Q; \ |t| > T - 2\varepsilon\}, \tag{2.3}
\]
\[
\varphi(x, t) \geq d_0 - \frac{\eta}{2} \equiv d_2 \quad \text{for all } (x, t) \in \{(x, t) \in Q; \ |t| < \varepsilon\}.
\]
Let \((u^i, u^f)\) satisfy Biot’s system
\[
\partial_{11} \partial_1^2 u^i(x, t) + \partial_{12} \partial_2^2 u^i(x, t) - \Delta_{u, x, \lambda} u^i(x, t) - \nabla (q \text{div } u^f) = F_1, \\
\partial_{12} \partial_1^2 u^i(x, t) + \partial_{22} \partial_2^2 u^i(x, t) - \nabla (r \text{div } u^f) - \nabla (q \text{div } u^f) = F_2, \quad \text{in } Q. \tag{2.4}
\]
The following theorem is a Carleman estimate for Biot’s system (2.4).

**Theorem 2.1.** There exist \( \tau_\ast > 0 \) and \( C > 0 \), such that the following estimate holds:
\[
\int_Q \tau (|\nabla_{x,t} u^i|^2 + |\nabla_{x,t} (\text{div } u^i)|^2 + |\nabla_{x,t} (\text{div } u^f)|^2) e^{2\tau \varphi} \, dx \, dt \\
+ \int_Q \tau^3 (|u^i|^2 + |\text{div } u^i|^2 + |\text{div } u^f|^2) e^{2\tau \varphi} \, dx \, dt \\
\leq C \int_Q (|F|^2 + |\nabla F|^2) e^{2\tau \varphi} \, dx \, dt \tag{2.5}
\]
for any \( \tau \geq \tau_\ast \) and any solution \((u^i, u^f) \in (H^2(Q))^6 \) to (2.4) which is supported in a fixed compact set \( K \subset \text{int}(Q) \).

In order to prove theorem 2.1, we use a Carleman estimate for a coupling hyperbolic system, which we discuss in the next subsection.

### 2.1. Carleman estimate for a hyperbolic system

First we recall the following Carleman estimate for a scalar hyperbolic equation. As for the proof, we refer to [7, 23], for example.

**Lemma 2.1.** Let \( c \in \mathcal{C}(m, \theta) \). There exist constants \( C > 0 \) and \( \tau_\ast > 0 \), such that the following Carleman estimate holds:
\[
C \int_Q e^{2\tau \varphi} (\tau |\nabla_{x,t} y|^2 + \tau^3 |y|^2) \, dx \, dt \leq \int_Q e^{2\tau \varphi} (|\partial_t^2 y - c \Delta y|^2) \, dx \, dt
\]
whenever \( y \in H^2(Q) \) is supported in a fixed compact set \( K \subset \text{int}(Q) \) and any \( \tau \geq \tau_\ast \).

Let \( v = (v_1, v_2) \in (H^2(\Omega))^2 \) satisfy the following hyperbolic system
\[
\begin{cases}
\partial_t^2 v_1 - b_{11}(x) \Delta v_1 - b_{12}(x) \Delta v_2 = g_1 & \text{in } Q, \\
\partial_t^2 v_2 - b_{21}(x) \Delta v_1 - b_{22}(x) \Delta v_2 = g_2 & \text{in } Q, \tag{2.6}
\end{cases}
\]
for \( g = (g_1, g_2) \in (L^2(Q))^2 \). We assume that the matrix \( B(x) = (b_{ij}(x))_{1 \leq i, j \leq 2} \) is definite positive with two positive eigenvalues \( c_1, c_2 \in \mathcal{C}(m, \theta) \). We note that we may have \( c_1 = c_2 \). Then, by lemma 2.1, we have the following Carleman estimate.

**Lemma 2.2.** There exist constants \( C > 0 \) and \( \tau_\ast > 0 \), such that the following Carleman estimate holds:
\[
C \int_Q e^{2\tau \varphi} (\tau |\nabla_{x,t} v|^2 + \tau^3 |v|^2) \, dx \, dt \leq \int_Q e^{2\tau \varphi} |g|^2 \, dx \, dt
\]
for any \( \tau \geq \tau_\ast \), whenever \( v \in H^2(Q) \) is a solution of (2.6) and supported in a fixed compact set \( K \subset \text{int}(Q) \).
**Proof.** System (2.6) can be written in the equivalent form

\[
\partial_t^2 v - B(x) \Delta v = g \quad \text{in } Q, \tag{2.7}
\]

By the assumption on \(B(x)\), there exists a matrix \(P(x)\), such that

\[
(P^{-1}BP)(x) = \text{Diag}(c_1(x), c_2(x)) = \Lambda(x), \quad x \in \Omega.
\]

Therefore, system (2.7) can be written in an equivalent form

\[
\partial_t^2 \tilde{v} - \Lambda(x) \Delta \tilde{v} = \tilde{g} + \mathcal{B}_1(x, \partial) v,
\]

where

\[
\tilde{v}(x, t) = P^{-1}(x)v(x, t), \quad \tilde{g}(x, t) = P^{-1}(x)g(x, t),
\]

\(\mathcal{B}_1\) is a first-order differential operator.

Since \(c_j \in C(m, \theta)\) for \(j = 1, 2\), we can apply lemma 2.1 for the two components of \(\tilde{v}\) and obtain

\[
C \int_Q e^{2\tau v}(\tau |\nabla x \tilde{v}|^2 + \tau^2 |\tilde{v}|^2) \, dx \, dt \leq \int_Q e^{2\tau v} |\tilde{g}|^2 \, dx \, dt + \int_Q e^{2\tau v}(|v|^2 + |\nabla v|^2) \, dx \, dt
\]

and by (2.8), we easily obtain

\[
|v(x, t)| \leq C |\tilde{v}(x, t)|, \quad |\nabla v(x, t)| \leq C(|\nabla \tilde{v}(x, t)| + |\tilde{v}(x, t)|), \quad |\tilde{g}(x, t)| \leq C |g(x, t)|
\]

for \((x, t) \in Q\). This completes the proof. \(\square\)

### 2.2. Proof of the Carleman estimate for Biot’s system

In this section, we derive a global Carleman estimate for a solution of system (2.4). We consider the \(6 \times 6\)-matrix

\[
M(x) = \begin{pmatrix} q_{11}(x)I_3 & q_{12}(x)I_3 \\ q_{12}(x)I_3 & q_{22}(x)I_3 \end{pmatrix}.
\]

(2.9)

Here, \(I_3\) is the \(3 \times 3\) identity matrix. Then, by assumption A.1, we have

\[
M^{-1}(x) = \frac{1}{q} \begin{pmatrix} q_{22}(x)I_3 & -q_{12}(x)I_3 \\ -q_{12}(x)I_3 & q_{11}(x)I_3 \end{pmatrix}.
\]

Let \(v^\prime = \text{div } u^\prime\), \(v^f = \text{div } u^f\), \(v = (v^\prime, v^f)\) and \(w^\prime = \text{curl } u^\prime\). Put \(G = M^{-1}F\), \(F = (F_1, F_2)^T\) and apply \(M^{-1}\) to system (2.4), we obtain

\[
\begin{align*}
\partial_t^2 u^\prime - \mu_1 \Delta u^\prime - (\mu_1 + \lambda_1) \text{div}(\text{div } u^\prime) &= G_1 + \mathcal{P}_1 u^\prime + \mathcal{R}_1 \text{div } u^\prime, \quad \text{in } Q \\
\partial_t^2 u^f + \mu_2 \Delta u^f - r_1 \text{div}(\text{div } u^f) - q_1 \text{div}(\text{div } u^\prime) &= G_2 + \mathcal{P}_2 u^f + \mathcal{R}_2 \text{div } u^f, \quad \text{in } Q.
\end{align*}
\]

(2.10)

Here, \(\mathcal{P}_j, j = 0, 1\) are differential operators of order \(j\) with coefficients in \(L^\infty(Q)\), and

\[
\begin{align*}
\mu_1 &= \varrho^{-1} \mu q_{22}, & \lambda_1 &= \varrho^{-1} (\lambda q_{22} - q_{12}), & q_1 &= \varrho^{-1} (q_2 q_{22} - r_1 q_{12}) \\
\mu_2 &= \varrho^{-1} \mu q_{12}, & q_2 &= \varrho^{-1} (q_2 q_{12} - (\mu + \lambda) q_{12}), & r_2 &= \varrho^{-1} (r_1 q_{11} - q_2 q_{12}).
\end{align*}
\]

(2.11)

Henceforth \(\mathcal{P}_j, j = 1, \ldots, 4\) denote some first-order operators with \(L^\infty(Q)\)-coefficients.

We apply \(\text{div}\) to the equations in (2.10), and can derive the following equations:

\[
\begin{align*}
\partial_t^2 v^\prime - a_{11} \Delta v^\prime - a_{12} \Delta v^f &= \text{div } G_1 + \mathcal{P}_1 (v^\prime, v^f, u^\prime, w^\prime) \\
\partial_t^2 v^f - a_{21} \Delta v^\prime - a_{22} \Delta v^f &= \text{div } G_2 + \mathcal{P}_2 (v^f, v^\prime, u^\prime, w^\prime),
\end{align*}
\]

(2.12)

where \((a_{ij})_{1 \leq i, j \leq 2}\) is given by (1.9). We apply the curl to first equation (2.10) to obtain

\[
\partial_t^2 w^\prime - \mu_1 \Delta w^\prime = \text{curl } G_1 + \mathcal{P}_3 (v^\prime, v^f, u^\prime, w^f)
\]

(2.13)
and
\[ \partial_t^2 \mathbf{u}' - \mu_1 \Delta \mathbf{u}' = G_1 + \mathcal{P}_d(v', v', \mathbf{u}', \mathbf{w}'). \] (2.14)

Applying lemma 2.2 to system (2.12), we have for \( v = (v', v') \)
\[
C \int_Q e^{2r\tau} \left( |\nabla_x v|^2 + \tau^3 |v|^2 \right) \, dx \, dr \leq \int_Q e^{2r\tau} \left( |F|^2 + |\nabla F|^2 \right) \, dx \, dr
\]
\[
+ \int_Q e^{2r\tau} \left( |\mathbf{u}'|^2 + |\mathbf{w}'|^2 + |\nabla \mathbf{u}'|^2 + |\nabla \mathbf{w}'|^2 \right) \, dx \, dr.
\]

Applying lemma 2.1 to (2.13) and (2.14), we obtain
\[
C \int_Q e^{2r\tau} \left( |\nabla_x (\mathbf{w}')|^2 + \tau^3 |\mathbf{w}'|^2 + \tau |\nabla_x \mathbf{u}'|^2 + \tau^3 |\mathbf{u}'|^2 \right) \, dx \, dr
\]
\[
\leq \int_Q e^{2r\tau} \left( |F|^2 + |\nabla F|^2 \right) \, dx \, dr + \int_Q e^{2r\tau} \left( |v|^2 + |\nabla v|^2 \right) \, dx \, dr.
\]

Therefore, for \( \tau \) sufficiently large, we obtain (2.5). This completes the proof of theorem 2.1.

For the proof, we introduce \( \text{div} \mathbf{u}' \), \( \text{div} \mathbf{u}' \) and \( \text{curl} \mathbf{u}' \), and such an idea is already used for example in \([6, 18, 24–26, 30]\).

3. Proof of theorem 1.2

In this section, we prove the stability (theorem 1.2) for the inverse source problem.

For the proof, we apply the method in [22] which modified the argument in [12] and proved the stability for an inverse coefficient problem for a hyperbolic equation. For it, the Carleman estimate (theorem 2.1) is a key.

3.1. Modified Carleman estimate for Biot’s system

Let \( \sigma_T = \omega \times (-T, T) \). We modify theorem 2.1 for functions which vanish at \( \pm T \) with first \( t \)-derivatives.

**Lemma 3.1.** There exist positive constants \( \tau_*, C > 0 \) and \( C_0 > 0 \), such that the following inequality holds:
\[
\int_Q \tau \left( |\nabla_x v|^2 + |\nabla_x (\text{div} v')|^2 + |\nabla_x (\text{div} v')|^2 \right) e^{2r\tau} \, dx \, dr
\]
\[
+ \int_Q \tau^3 \left( |v|^2 + |\text{div} v'|^2 + |\text{div} v'|^2 \right) e^{2r\tau} \, dx \, dr
\]
\[
\leq C \int_Q \left( |G|^2 + |\nabla G|^2 \right) e^{2r\tau} \, dx \, dr + C e^{C\tau} \|v\|^2_{L^2(\sigma_T)} \] (3.1)

for any \( \tau \geq \tau_* \) and any \( \mathbf{v} = (\mathbf{v}', \mathbf{w}') \in H^2(Q) \) satisfying, for \( G = (G_1, G_2) \)
\[
\partial_{11} \partial_{12}^2 \mathbf{v}' + \partial_{12} \partial_{22}^2 \mathbf{v}' - \Delta_{t, x} \mathbf{v}' - \nabla (q \text{div} v') = G_1,
\]
\[
\partial_{12} \partial_{11}^2 \mathbf{v}' + \partial_{22} \partial_{22}^2 \mathbf{v}' - \nabla (q \text{div} \mathbf{v}') - \nabla (r \text{div} v') = G_2 \quad \text{in} \; Q, \] (3.2)

such that
\[ \partial_t^j \mathbf{v}(x, \pm T) = 0 \quad \text{for all} \quad x \in \Omega, \; j = 0, 1. \] (3.3)
Proof. Let \( \omega^0 \subset \omega \). In order to apply Carleman estimate (2.5), we introduce a cut-off function \( \xi \) satisfying \( 0 \leq \xi \leq 1, \xi \in C^\infty(\mathbb{R}^3), \xi = 1 \) in \( \overline{\Omega \setminus \omega^0} \) and \( \text{Supp} \xi \subset \Omega \). Let \( v \in H^2(\Omega) \) satisfy (3.2) and (3.3). Put
\[
w(x, t) = \xi(x)v(x, t), \quad (x, t) \in Q.
\]
and let \( Q_0 = (\Omega \setminus \overline{\omega}) \times (-T, T) \). Noting that \( w \in H^2(Q) \) is compactly supported in \( Q \) and \( w = v \) in \( Q_0 \) and applying Carleman estimate (2.5) to \( w \), we obtain
\[
\int_{Q_0} \tau (|\nabla_x v|^2 + |\nabla_x (\text{div } v)|^2 + |\nabla_x (\text{div } v^f)|^2) e^{2\tau \rho} \, dx \, dt
\]
\[
+ \int_Q \tau (|v|^2 + |\text{div } v|^2 + |\text{div } v^f|^2) e^{2\tau \rho} \, dx \, dt
\]
\[
\leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau \rho} \, dx \, dt + C \int_Q |v|^2 e^{2\tau \rho} \, dx \, dt
\]
for any \( \tau \geq \tau_* \). Here \( Q_2 \) is a differential operator of order 2 whose coefficients are supported in \( \omega \).

Therefore,
\[
\int_Q \tau (|\nabla_x v|^2 + |\nabla_x (\text{div } v)|^2 + |\nabla_x (\text{div } v^f)|^2) e^{2\tau \rho} \, dx \, dt
\]
\[
+ \int_Q \tau (|v|^2 + |\text{div } v|^2 + |\text{div } v^f|^2) e^{2\tau \rho} \, dx \, dt
\]
\[
\leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau \rho} \, dx \, dt + C e^{\rho \tau} \|v\|^2_{H^2(\omega)}.
\]
This completes the proof of the lemma. \( \square \)

By \( \mathcal{N}_{\tau, \rho}(v) \), we denote
\[
\mathcal{N}_{\tau, \rho}(v) = \int_Q \tau (|\nabla_x v|^2 + |\nabla_x (\text{div } v)|^2 + |\nabla_x (\text{div } v^f)|^2) e^{2\tau \rho} \, dx \, dt
\]
\[
+ \int_Q \tau (|v|^2 + |\text{div } v|^2 + |\text{div } v^f|^2) e^{2\tau \rho} \, dx \, dt,
\]
where \( v = (v, v^f) \).

Now, we recall (2.2) and (2.3) for the definition of \( d_0, \eta \) and \( \varepsilon \) and we introduce a cut-off function \( \xi \) satisfying \( 0 \leq \xi \leq 1, \xi \in C^\infty(\mathbb{R}) \) and
\[
\xi = 1 \text{ in } (-T + 2\varepsilon, T - 2\varepsilon), \quad \text{Supp } \xi \subset (-T + \varepsilon, T - \varepsilon).
\]
Finally, we denote by \( \tilde{v} \) the function
\[
\tilde{v}(x, t) = \xi(t)(v, v^f)(x, t), \quad (x, t) \in Q.
\]

Lemma 3.2. There exist positive constants \( \tau_*, C \) and \( C_0 \), such that the following inequality holds:
\[
C \mathcal{N}_{\tau, \rho}(\tilde{v}) \leq \int_Q (|F|^2 + |\nabla F|^2) e^{2\tau \rho} \, dx \, dt + e^{C_{\text{div}}} \|v\|^2_{H^2(\Omega)} + e^{2d_1} \|v\|^2_{H^1(-T,T;\Omega)}
\]
for any \( \tau \geq \tau_* \) and any \( v = (v, v^f) \in (H^2(\Omega))^6 \) satisfying
\[
\begin{align*}
\partial_1 \tilde{\omega}^2 v^f + \partial_2 \tilde{\omega}^2 v^f - \Delta_{\mu, \lambda} v^f - \nabla (q \text{ div } v^f) &= F_1(x, t) \\
\partial_2 \partial_1 \omega^2 v + \partial_2 \partial_2 \omega^2 v^f - \nabla (q \text{ div } v^f) - \nabla (r \text{ div } v^f) &= F_2(x, t),
\end{align*}
\]
for any \( (x, t) \in Q \).
Proof. We note that $\tilde{\nu} \in (H^2(\Omega))^6$ and
\[
\begin{align*}
\partial_t \partial_x^2 \tilde{\nu} + \partial_t \partial_x^2 \tilde{\nu} - \Delta_{\mu,\lambda} \tilde{\nu} - \nabla (q \text{div} \tilde{\nu}) &= \zeta(t) F_1(x,t) + P_1(\nu, \partial_x \nu), \\
\partial_t \partial_x^2 \tilde{\nu} + \partial_t \partial_x^2 \tilde{\nu} - \nabla (q \text{div} \tilde{\nu}) - \nabla (\tau \text{div} \tilde{\nu}) &= \zeta(t) F_2(x,t) + P_2(\nu, \partial_x \nu),
\end{align*}
\]
where $P_1$ and $P_2$ are zeroth-order operators and supported in $|t| > T - 2\varepsilon$. Therefore, applying lemma 3.1 to $\tilde{\nu}$ and using (2.3), we complete the proof of the lemma. 

3.2. Preliminary estimates

Let $\varphi(x,t)$ be the function defined by (2.1). Then
\[
\varphi(x,t) = e^{\psi(x,t)} = \rho(x) \alpha(t),
\]
where $\rho(x)$ and $\alpha(t)$ are defined by
\[
\rho(x) = e^{\psi(x)} \geq d_0, \quad \forall x \in \Omega \quad \text{and} \quad \alpha(t) = e^{-\beta \gamma r^2} \leq 1, \quad \forall t \in [-T, T].
\]
Next, we present the following Carleman estimate of a first-order partial differential operator:
\[
L(x, D)v = \sum_{i=1}^3 a_i(x) \partial_i v + a_0(x)v, \quad x \in \Omega,
\]
where
\[
a_0 \in C(\overline{\Omega}), \quad a = (a_1, a_2, a_3) \in [C^1(\overline{\Omega})]^3
\]
and
\[
|a(x) \cdot (x - x_0)| \geq c_0 > 0, \quad \text{on} \quad \overline{\Omega}
\]
with a constant $c_0 > 0$. Then

Lemma 3.3. In addition to (3.9) and (3.10), we assume that $\|a_0\|_{C(\overline{\Omega})} \leq M$ and $\|a_i\|_{C^1(\overline{\Omega})} \leq M$, $1 \leq i \leq 3$. Then, there exist constants $\tau_* > 0$ and $C > 0$, such that
\[
\tau \int_{\Omega} |v(x)|^2 e^{2\rho(x)} \, dx \leq C \int_{\Omega} |L(x, D)v(x)|^2 e^{2\rho(x)} \, dx
\]
for all $v \in H^1_0(\Omega)$ and all $\tau > \tau_*$. The proof is direct by integration by parts and see, e.g., [22].

Consider now the following system:
\[
\begin{align*}
\partial_t \partial_x^2 u^t + \partial_t \partial_x^2 u^t - \Delta_{\mu,\lambda} u^t - \nabla (q \text{div} u^t) &= F_1(x,t), \\
\partial_t \partial_x^2 u^t + \partial_t \partial_x^2 u^t - \nabla (q \text{div} u^t) - \nabla (\tau \text{div} u^t) &= F_2(x,t),
\end{align*}
\]
with the boundary condition
\[
u^t(x,t) = 0, \quad u^t(x,t) \cdot v = 0, \quad (x,t) \in \Sigma
\]
and the initial condition
\[
(u^t(x,0), \partial_t u^t(x,0)) = (0,0), \quad (u^t(x,0), \partial_t u^t(x,0)) = (0,0), \quad x \in \Omega,
\]
where the functions $F_1$ and $F_2$ are given by
\[
F_1(x,t) = p_1(x)R_1(x,t), \quad F_2(x,t) = p_2(x)R_2(x,t).
\]
We introduce the following notations:
\[
u = (u^t, v^t), \quad v_j(x,t) = \partial_t u^j(x,t), \quad (x,t) \in Q, \quad j = 0, 1, 2, 3.
\]
The functions \( v_j, j = 1, 2, 3 \) solve the following system:

\[
\begin{align*}
\varrho_{11} \partial_t^2 v_j + \varrho_{12} \partial_t^2 v_j - \Delta_{\mu,\lambda} v_j(x, t) - \nabla (q \text{div } v_j) &= \partial_t^2 F_1(x, t), \\
\varrho_{12} \partial_t^2 v_j + \varrho_{22} \partial_t^2 v_j - \nabla (q \text{div } v_j) - \nabla (r \text{div } v_j) &= \partial_t^2 F_2(x, t), \quad (x, t) \in Q,
\end{align*}
\]

with the boundary condition

\[
\begin{align*}
v_j(x, t) = 0, \\
\partial_n v_j(x, t) - \nu = 0, \quad (x, t) \in \Sigma.
\end{align*}
\]

\( \text{(3.16)} \)

We set

\[ \bar{v}_j = \zeta v_j, \]

where \( \zeta(t) \) is given by (3.5). We apply lemma 3.2 to obtain the following estimate:

\[
CN_{\tau,\varphi}(\bar{v}_j) \leq \int_Q (|\partial_t^2 F|^2 + |\nabla \partial_t^2 F|^2) e^{2\tau} \text{ dx dt} + e^{2\tau} \|v_j\|^2_{H^1(\omega_1)} + e^{2\tau} \|v_j\|^2_{H^1(-T,T;H^1(\Omega))},
\]

\[
j = 0, 1, 2, 3,
\]

\( \text{(3.18)} \)

provided that \( \tau > 0 \) is large enough.

**Lemma 3.4.** There exists a positive constant \( C > 0 \), such that the following estimate

\[
\int_{\Omega} |z(x, 0)|^2 \text{ dx} \leq C \int_Q \left( |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2 \right) \text{ dx dt}
\]

for any \( z \in L^2(Q) \), such that \( \partial_t z \in L^2(Q) \).

**Proof.** Let \( \zeta \) be the cut-off function given by (3.5). By direct computations, we have

\[
\begin{align*}
\int_{\Omega} \zeta^2(0) |z(x, 0)|^2 \text{ dx} &= \int_{-T}^0 \frac{d}{dt} \left( \int_{\Omega} \zeta^2(t) |z(x, t)|^2 \text{ dx} \right) \text{ dt} \\
&= 2 \int_{-T}^0 \int_{\Omega} \zeta^2(t) z(x, t) \partial_t z(x, t) \text{ dx dt} + 2 \int_{-T}^0 \int_{\Omega} \zeta'(t) \zeta(t) |z(x, t)|^2 \text{ dx dt}.
\end{align*}
\]

Then, we have

\[
\int_{\Omega} |z(x, 0)|^2 \text{ dx} \leq C \int_Q \left( |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2 \right) \text{ dx dt}.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 3.5.** Let \( \phi_t(x) = \text{div} (p_t(x) \Phi_t(x)) \). Then, there exists a constant \( C > 0 \), such that

\[
\sum_{\ell = 1}^2 \int_{\Omega} e^{2\tau \rho} (|\phi_t(x)|^2 + |\nabla \phi_t(x)|^2) \text{ dx} \leq C \left( N_{\tau,\varphi}(\bar{v}_2) + N_{\tau,\varphi}(\bar{v}_3) \right) \sum_{\ell = 1}^2 \int_{\Omega} (|p_t|^2 + |\nabla p_t|^2) e^{2\tau \rho} \text{ dx},
\]

provided that \( \tau \) is large.

**Proof.** We set \( v^{(1)} = v_1^j \) and \( v^{(2)} = v_2^j \). Applying lemma 3.4 for \( z_j(x, t) = e^{\tau \varphi(x,t)} \text{div } \bar{v}_2^{(j)}(x, t) \), \( j = 1, 2 \), we obtain the following inequality:

\[
C \tau \int_{\Omega} e^{2\tau \rho} \sum_{j=1}^2 |\text{div } v^{(j)}(x, 0)|^2 \text{ dx dt} \leq \tau \int_{Q} e^{2\tau \rho} \sum_{j=1}^2 |\text{div } \bar{v}^{(j)}(x, t)|^2 \text{ dx dt}
\]

\[
+ \tau \int_Q e^{2\tau \rho} \sum_{j=1}^2 |\partial_t \text{div } \bar{v}^{(j)}(x, t)|^2 \text{ dx dt} \leq N_{\tau,\varphi}(\bar{v}_2).
\]

\( \text{(3.19)} \)
Applying lemma 3.4 again with \( w_j(x, t) = e^{\tau_\text{p}(x,t)} \nabla \text{div} (\vec{\Omega}(x, t)) \), we obtain
\[
C \int_\Omega e^{2\tau_\text{p}} \sum_{j=1}^2 |\nabla \text{div} \, \vec{\Omega}(x, 0)|^2 \, dx \leq \tau \int_Q e^{2\tau_\text{p}} \sum_{j=1}^2 |\nabla \text{div} \, \vec{\Omega}(x, t)|^2 \, dx \, dt
\]
\[
+ \tau^{-1} \int_Q e^{2\tau_\text{p}} \sum_{j=1}^2 (|\nabla \text{div} \, \vec{\Omega}(x, t)|^2 + |\nabla \text{div} \, \vec{\Omega}(x, t)|^2) \, dx \, dt
\]
\[
\leq \mathcal{N}_{\tau, \phi}(\vec{\Omega}_2) + \mathcal{N}_{\tau, \phi}(\vec{\Omega}_3). \tag{3.20}
\]
Adding (3.19) and (3.20), we find
\[
\int_\Omega e^{2\tau_\text{p}} \sum_{j=1}^2 (|\nabla \vec{\Omega}(x, 0)|^2 + |\nabla \text{div} \, \vec{\Omega}(x, 0)|^2) \, dx \leq C(\mathcal{N}_{\tau, \phi}(\vec{\Omega}_2) + \mathcal{N}_{\tau, \phi}(\vec{\Omega}_3)). \tag{3.21}
\]
Since
\[
M(x)(\vec{\Omega}_2(x, 0), \vec{\Omega}_2(x, 0))^T = (p_1(x) \Phi_1(x), p_2(x) \Phi_2(x))^T, \quad x \in \overline{\Omega},
\]
we have
\[
|\phi_\ell(x)|^2 + |\nabla \phi_\ell(x)|^2 \leq C \left( |\vec{\Omega}_2(x, 0)|^2 + |\nabla \vec{\Omega}_2(x, 0)|^2 + |\vec{\Omega}_2(x, 0)|^2 + |\nabla \vec{\Omega}_2(x, 0)|^2 \right)
\]
for \( x \in \overline{\Omega} \). On the other hand, using (3.11), we obtain
\[
|\vec{\Omega}_2(x, 0)|^2 + |\nabla \vec{\Omega}_2(x, 0)|^2 + |\vec{\Omega}_2(x, 0)|^2 + |\nabla \vec{\Omega}_2(x, 0)|^2 \leq C \sum_{\ell=1}^2 (|p_\ell|^2 + |\nabla p_\ell|^2), \quad x \in \overline{\Omega}. \tag{3.23}
\]
Combining (3.23), (3.22) and (3.21), we complete the proof of the lemma. \( \square \)

**Lemma 3.6.** There exists a constant \( C > 0 \), such that
\[
\tau \int_\Omega \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau_\text{p}} \, dx \leq C \int_\Omega \left( |\nabla \phi_\ell(x)|^2 + |\phi_\ell(x)|^2 \right) e^{2\tau_\text{p}(x)} \, dx
\]
for all large \( \tau > 0 \), \( \ell = 1, 2, 3 \).

**Proof.** We have
\[
\text{div}(\partial_t p_\ell(x) \Phi_\ell(x)) = \partial_t \phi_\ell(x) - \text{div}(p_\ell \partial_t \Phi_\ell(x)) \quad \text{for all } k = 1, 2, 3.
\]
Therefore,
\[
\int_\Omega \left( |\text{div}(\partial_t p_\ell(x) \Phi_\ell(x))|^2 + |\text{div}(p_\ell \partial_t \Phi_\ell(x))|^2 \right) e^{2\tau_\text{p}} \, dx \leq \int_\Omega \left( |\nabla \phi_\ell|^2 + |\phi_\ell|^2 \right) e^{2\tau_\text{p}(x)} \, dx
\]
\[
+ C \int_\Omega \left( |p_\ell|^2 + |\nabla p_\ell|^2 \right) e^{2\tau_\text{p}} \, dx. \tag{3.24}
\]
Since \( p_\ell = 0 \) and \( \nabla p_\ell = 0 \) on the boundary \( \Gamma \) and \( \nabla \Phi_\ell \cdot (x - x_0) \neq 0 \), we can apply lemma 3.3 respectively with the choice \( v = p_\ell \) and \( v = \partial_t p_\ell \) to obtain
\[
\tau \int_\Omega \left( |\partial_t p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau_\text{p}} \, dx \leq C \int_\Omega \left( |\text{div}(\partial_t p_\ell(x) \Phi_\ell)|^2 + |\text{div}(p_\ell \Phi_\ell)|^2 \right) e^{2\tau_\text{p}} \, dx \tag{3.25}
\]
for \( \ell = 1, 2 \) and \( k = 1, 2, 3 \). Inserting (3.24) into the left-hand side of (3.25) and choosing \( \tau > 0 \) large, we obtain
\[
\tau \int_\Omega \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau_\text{p}} \, dx \leq C \int_\Omega \left( |\nabla \phi_\ell(x)|^2 + |\phi_\ell(x)|^2 \right) e^{2\tau_\text{p}(x)} \, dx.
\]
The proof is completed. \( \square \)
3.3. Completion of the proof of theorem 1.2

By lemmata 3.5 and 3.6, we obtain
\[
\tau \sum_{\ell=1}^{2} \int_{\Omega} e^{2\tau \rho(x)} (|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2) \, dx \leq C \sum_{\ell=1}^{2} \int_{\Omega} (|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2) e^{2\tau \rho(x)} \, dx + C(N_{\tau,\varphi} (\tilde{V}_2) + \varphi_{\tau,\nu} (\tilde{V}_3)).
\]
Therefore, choosing \( \tau > 0 \) large to absorb the first term on the right-hand side into the left-hand side and applying (3.19), we obtain
\[
\tau \sum_{\ell=1}^{2} \int_{\Omega} e^{2\tau \rho(x)} (|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2) \, dx \leq C \sum_{j=2}^{3} \left( \int_{Q} (|\beta_j^F|^2 + |\beta_j^\nabla|^2) e^{2\tau \rho} \, dt \right.
\]
\[+ C e^{C_0 \tau} \|v_j\|^2_{H^1(\omega_0)} + C e^{C_0 \tau} \|v_j\|^2_{H^1(T,T;H^1(\Omega))} \bigg)
\leq C \sum_{\ell=1}^{2} \int_{Q} (|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2) e^{2\tau \rho} \, dx + C e^{C_0 \tau} E_u + C e^{C_0 \tau} M_0. \tag{3.26}
\]
Then the first term of the right-hand side of (3.26) can be absorbed into the left-hand side, if we take large \( \tau > 0 \).

Since \( \rho(x) \geq d_0 \), we obtain
\[
\sum_{\ell=1}^{2} \int_{\Omega} (|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2) \, dx \leq C e^{2(d_1 - d_0) \tau} + e^{C_0 \tau} E_u \leq C e^{-\epsilon \tau} + e^{C_0 \tau} E_u. \tag{3.27}
\]
At the last inequality, we used \( 0 < d_1 < d_0 \), we can choose \( \epsilon > 0 \), such that \( e^{2(d_1 - d_0) \tau} \leq e^{-\epsilon \tau} \) for sufficiently large \( \tau > 0 \).

4. Proof of theorem 1.1

This section is devoted to the study of the existence, uniqueness and regularity of solutions of the following system:
\[
\begin{align*}
\varrho_{11} \partial_t^2 u^\ell + \varrho_{12} \partial_t^2 u^\ell - \Delta_{\mu, \nu} u^\ell(x, t) - \nabla (q \, \text{div} \, u^\ell) &= F_1(x, t), \\
\varrho_{11} \partial_t^2 u^\ell + \varrho_{12} \partial_t^2 u^\ell - \nabla (r \, \text{div} \, u^\ell) &= F_2(x, t), \quad (x, t) \in Q
\end{align*}
\tag{4.1}
\]
with the boundary condition
\[
u(x, t) \cdot v = 0, \quad (x, t) \in \Sigma = \Gamma \times (-T, T) \tag{4.2}
\]
and the initial condition
\[
\begin{align*}
u^\ell(x, 0), \quad u^\ell_t(x, 0) &= (u_{0}^\ell, \quad u_{1}^\ell), \quad (u^\ell(x, 0), \quad u^\ell_t(x, 0)) = (u_{0}^\ell, \quad u_{1}^\ell), \quad x \in \Omega. \tag{4.3}
\end{align*}
\]
4.1. Function spaces

We denote by \( \mathcal{D}(\Omega) \) the space of compactly supported, infinitely differentiable functions in \( \Omega \) equipped with the inductive limit topology. We denote by \( \mathcal{D}'(\Omega) \) the space dual to \( \mathcal{D}(\Omega) \). In general, we denote by \( X' \) the space dual to the function space \( X \). We denote by \( \langle f, g \rangle \) the inner product in \( L^2(\Omega) \) and by \( \|f\| \) the value of \( f \in X' \) on \( g \in X \). We use usual notations for Sobolev spaces. If \( X \) is a Banach space, then we denote by \( L^p(0, T; X) \) the space of functions \( f : (0, T) \rightarrow X \) which are measurable, take values in \( X \) and satisfy
\[
\left( \int_0^T \|f(t)\|^p_{X} \, dt \right)^{1/p} = \|f\|_{L^p(0, T; X)} < \infty
\]
for $1 \leq p < \infty$, while
\[
\|f\|_{L^p(0,T;X)} = \text{esssup}_{t \in (0,T)} \|f(t)\|_X < \infty
\]
for $p = \infty$. It is known that the space $L^p(0,T;X)$ is complete.

We define the space
\[
H(\text{div} ; \Omega) = \{u \in (L^2(\Omega))^3; \text{div} u \in L^2(\Omega)\},
\]
equipped with the norm
\[
\|u\|_{H(\text{div};\Omega)} = (\|u\|_{L^2(\Omega)}^2 + \|\text{div} u\|_{L^2(\Omega)}^2)^{1/2}.
\]
Let us consider the space
\[
V(\Omega) = (H^1(\Omega)^3) \times H(\text{div} ; \Omega),
\]
equipped with the norm
\[
\|u\|_{V(\Omega)} = (\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\text{div} u\|_{L^2(\Omega)}^2)^{1/2}.
\]

### 4.2. Generalized solution

We introduce the bilinear form on $V(\Omega)$ by
\[
B(u, v) = \frac{1}{2} \int_{\Omega} (\lambda \text{div} (u') \text{div} (v') + 2\mu \varepsilon(u') : \varepsilon(v')) + r \text{div} (u') \text{div} (v') \, dx
\]
\[
+ \frac{1}{2} \int_{\Omega} (\nabla u' + (\nabla u')^T) \Delta v' \, dx
\]
for any $u = (u', u^f) \in V(\Omega)$, $v = (v', v^f) \in V(\Omega)$. We recall that the matrix $M$ is given by
\[
(M)_{ij} = \int_{\Omega} \varepsilon(u^i) : \varepsilon(u^j) \, dx,
\]
and $\varepsilon(u^i) = \frac{1}{2} (\nabla u^i + (\nabla u^i)^T)$.

Here we set $A : B = \sum_{i,j=1}^n a_{ij} b_{ij}$ for $n \times n$ matrices $A = (a_{ij})_{1 \leq i, j \leq n}$ and $B = (b_{ij})_{1 \leq i, j \leq n}$.

**Definition 4.1.** We say that $u = (u', u^f)$ is a generalized solution of problem (4.1)–(4.2), if $u \in L^2(0,T;V(\Omega))$ satisfies initial condition (4.3) and the following identity:
\[
(Mu(t), v(t)) + B(u(t), v(t)) = (F(t), v(t)), \quad \text{almost all } t \in (0,T)
\]
for any $v \in L^2(0,T;V(\Omega))$.

We note that in (4.5), the integration is only in $x$.

**Lemma 4.1.** For $\eta > 0$, we set
\[
B_\eta(u, v) = B(u, v) + \eta (u, v), \quad u, v \in V(\Omega).
\]

Then, there exists sufficiently large constant $\eta$ such that the symmetric bilinear form $B_\eta$ satisfies
\[
(i) \quad |B_\eta(u, v)| \leq C_1 \|u\|_{V(\Omega)} \|v\|_{V(\Omega)}, \text{ for any } u, v \in V(\Omega),
\]
\[
(ii) \quad B_\eta(u, u) \geq C_2 \|u\|_{V(\Omega)}^2, \text{ for any } u \in V(\Omega).
\]

**Proof.** By (4.4), we obtain, for any $u, v \in V(\Omega)$
\[
|B(u, v)| \leq (\|u\|_{H^1(\Omega)}^2 + \|\text{div} u\|_{L^2(\Omega)}^2 + \|\text{div} v\|_{L^2(\Omega)}^2)
\]
\[
\leq C \|u\|_V \|v\|_V.
\]
(4.6)

Then, for any $\eta$, we can derive (i).

Now, we note that for a vector $u^f \in H^1_0(\Omega)$, we have the following Korn’s inequality:
\[
C_1 \|u^f\|_{H^1(\Omega)}^2 \leq \int_{\Omega} \varepsilon(u^f) : \varepsilon(u^f) \, dx.
\]
Then, for $W = (\text{div} \ u^f, \text{div} \ u^f')$, we have
\[
B(u, u) \geq \mu C_1 \|u^f\|^2_{H^1(\Omega)} + \frac{1}{2} \int_{\Omega} M_0 W \cdot W \, dx,
\]
where $M_0$ is the symmetric $2 \times 2$-matrix given by
\[
M_0(x) = \left( \begin{array}{cc} \lambda & q \\ q & r \end{array} \right) \geq \gamma_0 I,
\]
which implies
\[
B(u, u) \geq \mu C_1 \|u^f\|^2_{H^1(\Omega)} + \frac{\gamma_0}{2} (\|\text{div} u^f\|^2_{L^2(\Omega)} + \|\text{div} u^f'\|^2_{L^2(\Omega)}) \\
\geq \mu C_1 \|u^f\|^2_{H^1(\Omega)} + \frac{\gamma_0}{2} \|\text{div} u^f\|^2_{L^2(\Omega)} - \frac{\gamma_0}{2} \|u\|_{L^2(\Omega)}^2 \\
\geq C_2 \|u\|^2_{H^1(\Omega)} - \eta \|u\|_{L^2(\Omega)}^2.
\]
(4.7)

This completes the proof of the lemma.

4.3. Construction of approximate solutions

Let $\mathcal{A} : (L^2(\Omega))^6 \to (L^2(\Omega))^6$ be the self-adjoint operator defined by
\[
\mathcal{A} u = \left( \begin{array}{c} \Delta_{\mu, \lambda} u^f + \nabla(\text{div} u^f) \\ \nabla(\text{div} u^f) + \nabla(\text{div} u^f') \end{array} \right).
\]

Then, system (4.1) can be written as
\[
M \partial_t^2 u - \mathcal{A} u = F, \quad (x, t) \in Q
\]
with initial condition
\[
u(x, 0) = (u^0(x), u^0_t(x)), \quad \partial_t u(x, 0) = (u^1(x), u^1_t(x))
\]
and boundary condition
\[
u(x, t) = 0, \quad u^t \cdot n = 0, \quad (x, t) \in \Sigma.
\]
(4.10)

Let $(w_j)_{j \geq 1}$ be a sequence of solutions in $(H^2(\Omega) \cap H^1_0(\Omega))^6$, such that for all $m \in \mathbb{N}$, $w_1, \ldots, w_m$ are linearly independent and all the finite linear combinations of $(w_j)_{j \geq 1}$ are dense in $(H^2(\Omega))^6$.

We seek approximate solutions of the problem in the form
\[
u_{m}(t) = \sum_{j=1}^{m} g_{jm}(t) w_j.
\]
(4.11)

The functions $g_{jm}(t)$ are defined by the solution of the system of ordinary differential equations
\[
(M \partial_t^2 \nu_{m}, \omega_j) + B(\nu_{m}, \omega_j) = (F(t), \omega_j), \quad 1 \leq j \leq m,
\]
(4.12)

with the initial conditions
\[
u_{m}(0) = \nu_{m0} \to u_0 \quad \text{in} \quad (H^2(\Omega) \cap H^1_0(\Omega))^6, \\
\partial_t \nu_{m}(0) = u_{m0} \to u_1 \quad \text{in} \quad (H^1(\Omega))^6.
\]
(4.13)

System (4.12)–(4.13) depends on $g_{jm}(t)$ and therefore has a solution on some segment $[0, t_m]$; see [39]. From \textit{a priori} estimates below and the theorem on continuation of a solution, we deduce that it is possible to take $t_m = T$. 

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4.4. A priori estimates

Multiplying (4.8) by $g_{jm}(t)$ and summing over $j$ from 1 to $m$, we obtain

$$
(M\partial^2_t u_m, \partial_t u_m) + B(u_m, \partial_t u_m) = (F(t), \partial_t u_m).
$$

(4.14)

Hence,

$$
\frac{1}{2} \frac{d}{dt} \left[ \|M^{1/2} \partial_t u_m(t)\|_{L^2(\Omega)}^2 + B_g(u_m(t), u_m(t)) \right] = (F(t), \partial_t u_m(t)) + \eta \frac{d}{dt} \|u_m(t)\|_{L^2(\Omega)}^2.
$$

(4.15)

Let

$$
\Phi^2(t) = \|M^{1/2} \partial_t u_m(t)\|_{L^2(\Omega)}^2 + B_g(u_m(t), u_m(t)).
$$

From (4.15), we obtain

$$
\frac{1}{2} \frac{d}{dt} \Phi^2(t) \leq C \left[ \|F(t)\|_{L^2(\Omega)}^2 + \|\partial_t u_m(t)\|_{L^2(\Omega)}^2 + \|u_m(t)\|_{L^2(\Omega)}^2 \right].
$$

(4.16)

Integrating with respect to $\tau$ from 0 to $t$, we obtain

$$
\Phi^2(t) \leq C \left[ \|F\|_{L^2(\Omega)}^2 + \Phi^2(0) + \int_0^t \left( \|\partial_t u_m(\tau)\|_{L^2(\Omega)}^2 + \|u_m(\tau)\|_{L^2(\Omega)}^2 \right) d\tau \right].
$$

(4.17)

Since

$$
\Phi^2(t) \geq C \left( \|\partial_t u_m(t)\|_{L^2(\Omega)}^2 + \|u_m(t)\|_{L^2(\Omega)}^2 \right)
$$

(4.18)

and

$$
\Phi^2(0) \leq C + \|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2,
$$

(4.19)

we have from (4.18)

$$
\|u_m(t)\|_{V(\Omega)}^2 + \|\partial_t u_m(t)\|_{L^2(\Omega)}^2 \leq R_0 + \int_0^t \left( \|\partial_t u_m(\tau)\|_{L^2(\Omega)}^2 + \|u_m(\tau)\|_{L^2(\Omega)}^2 \right) d\tau,
$$

(4.20)

where $R_0 = C + \|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2 + \|F\|_{L^2(\Omega)}^2$. By the Gronwall inequality, we conclude that

$$
\|u_m(t)\|_{V(\Omega)}^2 + \|\partial_t u_m(t)\|_{L^2(\Omega)}^2 \leq R_0
$$

(4.21)

for all $t \in (0, T)$ and $m \geq 1$.

In order to obtain the second a priori estimate, we observe that

$$
\|\partial^2_t u_m(0)\|_{H^1(\Omega)} \leq C \left[ \|F(0)\|_{L^2(\Omega)}^2 + \|u_m(0)\|_{H^1(\Omega)}^2 \right] \leq R_1.
$$

(4.22)

Indeed, multiplying (4.8) by $g_{jm}(0)$, summing over $j$ and setting $t = 0$, we obtain

$$
(M\partial^2_t u_m(0), \partial^2_t u_m(0)) + B(u_m(0), \partial^2_t u_m(0)) = (F(0), \partial^2_t u_m(0)).
$$

(4.23)

Consequently,

$$
(M\partial^2_t u_m(0), \partial^2_t u_m(0)) = (F(0), \partial^2_t u_m(0)) + \eta \|u_m\|_{L^2(\Omega)}^2
$$

(4.24)

which implies

$$
\|\partial^2_t u_m(0)\| \leq C \left( \|F(0)\|_{L^2(\Omega)}^2 + \|u_m\|_{H^1(\Omega)}^2 \right) \leq CR_2.
$$

(4.25)

Differentiating (4.14) with respect to $t$, multiplying by $g_{jm}$ and summing over $j$, we obtain the identity:

$$
\frac{1}{2} \frac{d}{dt} \left[ \|M^{1/2} \partial_t u_m(t)\|_{L^2(\Omega)}^2 + B_g(u_m(t), u_m(t)) \right] = (F(t), \partial_t^2 u_m(t)) + \eta \frac{d}{dt} \|\partial_t u_m(t)\|_{L^2(\Omega)}^2.
$$

(4.26)
Then, we conclude that
\[
\|\partial_t^2 u_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t u_m(t)\|_{V}^2 \leq R_2 + \|\partial_t u_m(0)\|_{L^2(\Omega)}^2 + C \int_0^t \left( \|\partial_t^2 u_m(\tau)\|_{L^2(\Omega)}^2 + \|\partial_t u_m(\tau)\|_{V}^2 \right) \, d\tau.
\]
(4.27)
By (4.27) and the Gronwall inequality, we obtain
\[
\|\partial_t^2 u_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t u_m(t)\|_{V}^2 \leq R_1.
\]
(4.28)
Taking into consideration that \(u_m = 0\) in \(\Sigma\), we see
\[
u_m \in L^\infty(0, T; V(\Omega)), \quad \partial_t \nu_m \in L^\infty(0, T; V(\Omega)), \quad \partial_t^2 \nu_m \in L^\infty(0, T; L^2(\Omega)).
\]
(4.29)

4.5. Passage to the limit

By (4.29), we can extract a sub-sequence from \((u_m)_{m \geq 0}\), which we denote again by \((u_m)_m\), such that
\[
u_m \rightarrow \nu \quad \text{in the weak-star topology in} \quad L^\infty(0, T; V(\Omega))
\]
\[
\partial_t \nu_m \rightarrow \partial_t \nu \quad \text{in the weak-star topology in} \quad L^\infty(0, T; V(\Omega))
\]
(4.30)
and
\[
(u_m, \partial_t u_m) \rightarrow (u, \partial_t u) \quad \text{a.e. on} \quad \Sigma.
\]

Multiplying (4.8) by \(\theta \in L^1(0, T)\) and integrating, we have
\[
\int_0^T \left( (M\partial_t^2 u_m(t), w_j) + B(u_m, w_j) \right) \theta(t) \, dt = \int_0^T (F(t), w_j) \theta(t) \, dt.
\]
(4.31)
On the other hand,
\[
\int_0^T B(u_m, w_j) \theta(t) \, dt = - \int_0^T (u_m, \mathscr{A} w_j) \theta(t) \, dt,
\]
so that
\[
\lim_{m \rightarrow \infty} \int_0^T B(u_m, w_j) \theta(t) \, dt = - \int_0^T (u, \mathscr{A} w_j) \theta(t) \, dt = \int_0^T B(u, w_j) \theta(t) \, dt.
\]
(4.33)
Thus, we obtain
\[
\int_0^T \left( (M\partial_t^2 u(t), w_j) + B(u, w_j) \right) \theta(t) \, dt = \int_0^T (F(t), w_j) \theta(t) \, dt.
\]
(4.34)
Taking into account that \(w_j\) are dense in \((H^2(\Omega) \cap H^1_0(\Omega))^6\) and therefore in \(V\), we obtain
\[
(M\partial_t^2 u, v) + B(u(t), v(t)) = (F(t), v), \quad t \in (0, T)
\]
(4.35)
for all \(v \in L^2(0, T; V(\Omega))\).

We have \(B(u(t), v(t)) = -(\mathscr{A}u(t), v(t))\) for any \(v \in \mathcal{D}(\Omega)\), where the application of the differential operator \(A\) to \(u\) is in the distributional sense in \(\mathcal{D}'(\Omega)\). Hence, we obtain
\[
M\partial_t^2 u - \mathscr{A} u = F, \quad \text{in} \ \mathcal{D}'(\Omega), \quad \text{a.e. in} (0, T).
\]
(4.36)
On the other hand, \(\partial_t^2 u, \partial_t u, F \in L^\infty(0, T; L^2(\Omega))\). Hence, (4.36) holds in \(L^\infty(0, T; L^2(\Omega))\).
Boundary condition (4.2) is satisfied by the choice of the space \( V(\Omega) \). We prove that the initial conditions are satisfied. Suppose \( \theta \in C^1(0, T) \) and \( \theta(T) = 0 \). For any \( j \), we have

\[
\int_0^T \left( \frac{d}{dt}(u_m(t) - u(t)), w_j \right) \theta(t) \, dt = -(u_m(0) - u(0), w_j)\theta(0) - \int_0^T (u_m(t) - u(t), w_j) \theta'(t) \, dt. \tag{4.37}
\]

Then, by (4.30), we have

\[
\lim_{m \to \infty} |(u_m(0) - u(0), w_j)| = 0.
\]

Since \( u_{0m}(x) = u_m(0, x) \) and \( u_{0m} \to u_0 \), we obtain \( u(0) = u_0 \) and can argue similarly for \( u_1 \).

Then, we conclude that there exists a solution \( u \) of (4.1), such that

\[
\partial_t u \in L^\infty(0, T; V(\Omega)), \quad \text{and} \quad \partial_t^2 u \in L^\infty(0, T; L^2(\Omega)), \tag{4.38}
\]

which implies

\[
\begin{align*}
\partial_t u' &\in C^1(0, T; H^1_0(\Omega)), & \partial_t^2 u' &\in C(0, T; L^2(\Omega)) \\
\partial_t u' &\in C^1(0, T; H(\text{div}; \Omega)), & \partial_t^2 u' &\in C(0, T; L^2(\Omega)).
\end{align*} \tag{4.39}
\]

On the other hand,

\[
\begin{align*}
\nabla(q \text{ div } u') + \nabla(r \text{ div } u') &\in C(0, T; L^2(\Omega)), \\
\Delta_{\mu, \lambda} u' + \nabla(q \text{ div } u') &\in C(0, T; L^2(\Omega)).
\end{align*} \tag{4.40}
\]

Consequently,

\[
\Delta_{\mu, \lambda} u' \in C(0, T; L^2(\Omega)), \quad \tilde{\lambda} = \lambda - \frac{q^2}{r}.
\]

Then, by the elliptic regularity, \( u' \in H^1_0(\Omega) \) yields

\[
\begin{array}{c}
\partial_t u' \in C(0, T; H^2(\Omega)) \\
\end{array}
\]

By \( \text{div } u' \in C(0, T; H^1(\Omega)) \), we see

\[
\begin{align*}
\partial_t u' &\in C(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T; H^1(\Omega)) \\
\text{div } u' &\in C(0, T; H^1(\Omega) \cap C^1(0, T; L^2(\Omega)). \tag{4.41}
\end{align*}
\]

### 4.6 Uniqueness

Let \( u_1 \) and \( u_2 \) be two solutions to (4.1)–(4.2) with the same initial data, and set \( u = u_1 - u_2 \).

Then for every function \( v \in V(\Omega) \), we have

\[
(M\partial_t^2 u, v) + B(u, v) = 0, \quad \forall t \in (0, T).
\]

Since \( \partial_t u \in V(\Omega) \), we may take \( v = \partial_t u \), and this equation can be reduced to equality

\[
\frac{1}{2} \frac{d}{dt} \left[ \|M^{1/2} \partial_t u \|^2_{L^2(\Omega)} + B(\eta, u, u) \right] = \frac{\eta}{2} \frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)}.
\]

Then,

\[
\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \left( \|\partial_t u(\tau)\|^2_{L^2(\Omega)} + \|u(\tau)\|^2_{L^2(\Omega)} \right) \, d\tau.
\]

This implies that \( \|u(t)\|^2_{L^2(\Omega)} = 0 \) for \( \tau \leq \tau_1 \), and \( u_1 = u_2 \) a.e. in \( Q \).

The proof of theorem 1.1 is completed.
Acknowledgments

Most of this paper was written during the stays of the first author at Graduate School of Mathematical Sciences of the University of Tokyo in 2011 and 2012. The author thanks the school for the hospitality. The authors thank the anonymous referees for valuable comments.

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