ON THE INDEX OF HARMONIC MAPS FROM SURFACES TO COMPLEX PROJECTIVE SPACES

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Abstract. We shall find the dimension of the spaces of holomorphic sections and holomorphic differentials of certain line bundles to give improved lower bounds on the index of complex isotropic harmonic maps from the sphere and torus to complex projective spaces.

1. Introduction

Harmonic maps are smooth maps between Riemannian manifolds which are critical points of the Dirichlet energy functional (see, for example, [6, 15]). The index of a harmonic map is a measure of its stability and is generally difficult to calculate. Examples of stable harmonic maps are constant mappings between Riemannian manifolds and holomorphic maps between Kähler manifolds, which thus both have index 0 [15]. Harmonic maps given by the Gauss transform of a full holomorphic map from a Riemann surface of genus $g$ to complex projective space are unstable and are of particular interest as they form a large class of harmonic maps called complex isotropic, or equivalently of finite uniton number; this class includes all harmonic maps from the 2-sphere (see [7]).

We give new bounds on the index of harmonic maps from the 2-sphere to complex projective space (Theorem 3.7) and complex isotropic harmonic maps from the torus to complex projective space (Theorem 3.15), which improve those in [7]. This is achieved by recalling that holomorphic vector fields along a harmonic map $\phi$ give smooth variations that contribute to the index of $\phi$ [7, p. 258]; we improve the known estimates by calculating the dimension of the space of holomorphic vector fields along a harmonic map $\phi$ by decomposing the tangent bundle using the harmonic sequence of $\phi$.

In [7], J. Eells and J.C. Wood classified all complex isotropic harmonic maps from a Riemann surface $M$ to $\mathbb{CP}^n$. Later F.E. Burstall and J.C. Wood gave an interpretation of this in [2] by defining certain subbundles of the trivial bundle $M \times \mathbb{C}^{n+1}$ and maps between these subbundles, and developing a technique of analysing harmonic maps from a Riemann surface into a complex Grassmannian using “diagrams”. In [2], the harmonic maps of [7] are constructed by a repeated use of a Gauss transform and we will use the interpretation in [2] to calculate bounds on the index of harmonic maps constructed in this way.

2. Preliminaries

We recall the construction of complex isotropic harmonic maps given in [2, 7]; for additional reading related to these constructions see [3, 5, 12, 13, 17] and for a moving frames approach see [4, 16].

2.1. Subbundles of $M \times \mathbb{C}^{n+1}$. Let $M$ be a compact Riemann surface. Let us identify $\mathbb{CP}^n$ with the set of complex lines (i.e. one-dimensional complex subspaces in $\mathbb{C}^{n+1}$) in the usual way, so that each point $V \in \mathbb{CP}^n$ is identified with a complex line in $\mathbb{C}^{n+1}$. The tautological bundle $T$ over $\mathbb{CP}^n$ is the subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$.
whose fibre at \( V \in \mathbb{C}P^n \) is the line \( V \) in \( \mathbb{C}^{n+1} \). By decomposing the complexified tangent bundle \( T^C \mathbb{C}P^n \) using the complex structure in the usual way we have
\[
T^C \mathbb{C}P^n = T^{(1,0)} \mathbb{C}P^n \oplus T^{(0,1)} \mathbb{C}P^n.
\]
There is a well-known connection-preserving isomorphism \( h : T^{(1,0)} \mathbb{C}P^n \to L(T, T^\perp) \) given by
\[
h(Z) \sigma = \pi_{T^\perp} Z(\sigma)
\]
where \( \sigma \) is a local section of \( T \), \( Z \in T^{(1,0)} \mathbb{C}P^n \), \( \pi_{T^\perp} \) denotes the orthogonal projection onto \( T^\perp \) and \( Z(\sigma) \) denotes differentiation with respect to \( Z \). For alternative descriptions of this isomorphism see [2, 3, 7].

Consider a smooth map \( \phi : M \to \mathbb{C}P^n \). We may decompose the \( \mathbb{C} \)-linear extension of its differential \( d\phi \) into components
\[
\partial \phi : T^{(1,0)} M \to T^{(1,0)} \mathbb{C}P^n, \quad \bar{\partial} \phi : T^{(0,1)} M \to T^{(0,1)} \mathbb{C}P^n.
\]
To each map \( \phi : M \to \mathbb{C}P^n \), we may associate the pullback of the tautological bundle \( \bar{\phi} := \phi^{-1} T \); this is the complex line subbundle of the trivial bundle \( M \times \mathbb{C}^{n+1} \) over \( M \) whose fibre at \( z \) is the line \( \phi(z) \). The orthogonal projection \( \pi_\psi \) onto \( \bar{\phi} \) applied to the standard derivation on \( M \times \mathbb{C}^{n+1} \) over \( M \) induces a connection \( \phi \nabla \) on \( \bar{\phi} \) on a (local complex) chart \(( U, z) \) of \( M \) this is given by
\[
\phi \nabla_{\partial/\partial z} v = \pi_\psi \frac{\partial}{\partial z} v, \quad \phi \nabla_{\bar{\partial}/\partial \bar{z}} v = \pi_\psi \frac{\partial}{\partial \bar{z}} v,
\]
for \( v \in \Gamma(\bar{\phi}^{-1} \mathbb{T}) \).

Given mutually orthogonal subbundles \( \phi \) and \( \psi \) as in [2] we define the bundle maps \( A'_{\phi, \psi} : \phi \to \psi \) and \( A''_{\phi, \psi} : \phi \to \psi \) by
\[
A'_{\phi, \psi}(v) = \pi_\psi \frac{\partial}{\partial z} v \quad \text{and} \quad A''_{\phi, \psi}(v) = \pi_\psi \frac{\partial}{\partial \bar{z}} v,
\]
where \( \pi_\psi \) is the orthogonal projection onto \( \psi \). These two maps are adjoint up to sign, more concretely, with \( \langle , \rangle_\phi \) the Hermitian metric on \( \phi \) induced from the flat metric on the trivial bundle \( \mathbb{C}P^n \times \mathbb{C}^{n+1} \) then
\[
-\langle A'_{\phi, \psi} v, w \rangle_\phi = \langle v, A''_{\psi, \phi} w \rangle_\phi \quad \text{for} \quad v \in \phi, \quad w \in \psi.
\]
A very useful special case of the above is the following, we set
\[
A'_{\phi} = A'_{\phi, 1 : \phi \to \phi^\perp} \quad \text{and} \quad A''_{\phi} = A''_{\phi, 1 : \phi \to \phi^\perp}.
\]
Then, using the pullback of (2.1), we have the following isomorphism of bundles over \( M \):
\[
\phi^{-1} T^{(1,0)} \mathbb{C}P^n \cong L(\phi, \phi^\perp),
\]
which can be used to identify the bundle maps \( A'_{\phi} \) and \( A''_{\phi} \) with \( \partial \phi(\partial/\partial z) \) and \( \bar{\partial} \phi(\partial/\partial \bar{z}) \) respectively. Then we have the following for a smooth map \( \phi : M \to \mathbb{C}P^n \).

**Lemma 2.1.** [2]

(i) The map \( \phi \) is holomorphic (respectively antiholomorphic) if and only if \( A''_{\phi} = 0 \) (respectively \( A'_{\phi} = 0 \)).

(ii) The map \( \phi \) is harmonic if and only if \( A'_{\phi} : \phi \to \phi^\perp \) is holomorphic, i.e.,
\[
A'_{\phi} \circ \phi \nabla_{\partial/\partial z} = \phi^\perp \nabla_{\partial/\partial z} \circ A'_{\phi},
\]
or equivalently, \( A''_{\phi} : \phi \to \phi^\perp \) is antiholomorphic.
Let \( \phi : M \to \mathbb{C}P^n \) be a non-antiholomorphic harmonic map. After a process of filling out the zeros of \( A'_0 \) detailed in [2, p. 266], according to Lemma 2.1 the image of \( A'_0 \) is a holomorphic subbundle of \( \phi^\perp \) which we denote by \( \text{Im}A'_0 \). We call this holomorphic subbundle the \( \phi^\perp \)-Gauss bundle and denote it by \( G'(\phi) \); we have \( G'(\phi) := G'(\phi)^{-1}T \) for some smooth map \( G'(\phi) : M \to \mathbb{C}P^n \). Similarly let \( \phi : M \to \mathbb{C}P^n \) be a non-holomorphic harmonic map, then the image of \( A''_0 \) is an antiholomorphic subbundle of \( \phi^\perp \) denoted \( G''(\phi) \) and called the \( \phi^\perp \)-Gauss bundle. As before \( G''(\phi) := G''(\phi)^{-1}T \) for some smooth map \( G''(\phi) : M \to \mathbb{C}P^n \).

**Lemma 2.2.** [2, Proposition 2.3] Let \( \phi : M \to \mathbb{C}P^n \) be a harmonic map. If \( \phi \) is not antiholomorphic then \( G''(\phi) \) is harmonic and \( G''(G'(\phi)) = \phi \). If \( \phi \) is not holomorphic then \( G''(\phi) \) is harmonic and \( G''(G''(\phi)) = \phi \).

We shall now construct a harmonic sequence using the above. A map \( f : M \to \mathbb{C}P^n \) is said to be full if its image does not lie in a proper projective subspace of \( \mathbb{C}P^n \). Let \( f_0 : M \to \mathbb{C}P^n \) be a full holomorphic map, then as above \( G'(f_0) := \text{Im}A'_{f_0} \subset f_0^\perp \) where \( G'(f_0) : M \to \mathbb{C}P^n \) is a harmonic map by Lemma 2.2. Applying the procedure again to \( G'(f_0) : M \to \mathbb{C}P^n \) we have \( G''(f_0) := \text{Im}A''_{G'(f_0)} \subset f_0^\perp \) where \( G''(f_0) : M \to \mathbb{C}P^n \) is again a harmonic map by Lemma 2.2.

For ease of notation write, \( f_j := G'(G'(\ldots G'(f_0)\ldots)) \) where \( G' \) is applied \( j \) times to \( f_0 \) and \( f_{j+1} := \text{Im}A''_{f_j} \subset f_{j+1}^\perp \) where \( f_{j+1} = f_j \cap f_{j+1}^\perp \) its associated subbundle given by the pullback of the tautological bundle. Now the \( j \)th iteration of the procedure above applied to \( f_{j-1} : M \to \mathbb{C}P^n \) gives \( f_j := \text{Im}A''_{f_{j-1}} \subset f_j^\perp \) where \( f_j : M \to \mathbb{C}P^n \) is again a harmonic map.

**Remark 2.3.** One can apply the above to a full antiholomorphic map \( g_0 \) by replacing \( A' \) and \( G' \) with \( A'' \) and \( G'' \) respectively and letting \( g_k := G''(G''(\ldots G''(g_0)\ldots)) \) where \( G'' \) is applied \( k \) times to \( g_0 \).

It was shown in [2] and through a different interpretation in [7] that the \( n \)th iteration of the procedure above gives \( f_n := \text{Im}A'_{f_{n-1}} \subset f_n^\perp \) where \( f_n : M \to \mathbb{C}P^n \) is a full antiholomorphic map. Using Lemma 2.1 we see that \( A''_n = A'_{f_n} = 0 \) since \( f_0 \) and \( f_n \) are holomorphic and antiholomorphic respectively, therefore \( G''(f_0) = G'(f_0) = 0 \) and so do not define maps into \( \mathbb{C}P^n \). Therefore we have the following sequence.

\[
(2.2) \quad \ldots \xleftarrow{A'_{f_{i+2}}} \xrightarrow{f_{i+1}} \text{Im}A'_{f_{i+1}} \xrightarrow{A'_{f_{i+1}}} f_{i+1}^\perp \xleftarrow{\text{Im}A'_{f_{i+1}}} \xrightarrow{f_{i+2}} \text{Im}A'_{f_{i+2}} \xrightarrow{A'_{f_{i+2}}} f_{i+2}^\perp \xleftarrow{\text{Im}A'_{f_{i+2}}} \ldots,
\]

where \( f_0 : M \to \mathbb{C}P^n \) is a full holomorphic map with associated bundle \( f_{i-1}^{-1}T \), \( f_i : M \to \mathbb{C}P^n \) is a full harmonic map with associated bundle \( f_{i+1}^{-1}T \) for each \( i \in \{1, 2, \ldots, n-1\} \) and \( f_n : M \to \mathbb{C}P^n \) is a full antiholomorphic map with associated bundle \( f_{n+1}^{-1}T \).

Let \( (U, z) \) be a chart of \( M \) and let \( z_0 \in U \) be a zero of \( A'_{f_{n-1}} \) where \( p \in \{1, \ldots, n\} \), then we can write \( A'_{f_{n-1}}(z) = (z - z_0)^k \lambda(z) \) where \( \lambda \) is a smooth section of \( L(f_{i-1}^{-1} f_{i+1}^{-1} \ldots) \), non-zero at \( z_0 \) and \( k \in \mathbb{N} \). Then we say that \( f_0 \) is \( p \rho \)th-order) ramified at the point \( z_0 \) with ramification index \( k \). We call the sum of all ramification indices of the points of \( p \rho \)th ramification the \( \rho \)th total ramification index and denote it \( r_{\rho-1} \).

**Remark 2.4.** All harmonic maps \( S^2 \to \mathbb{C}P^n \) are given as above; for higher genera the construction gives all harmonic maps which are complex isotropic [7], or equivalently of finite uniton number cf. [1], alternatively described in [2].
3. Estimates of the Index

In [7] an estimate was given for the index of nonholomorphic harmonic maps \( \phi : M_g \rightarrow \mathbb{C}P^n \) where \( M_g \) is a closed Riemann surface of genus \( g \). Throughout this paper we call a map that is holomorphic or antiholomorphic \( \pm \)-holomorphic.

**Proposition 3.1.** [7] Let \( \phi : M_g \rightarrow \mathbb{C}P^n \) be a non-\( \pm \)-holomorphic harmonic map. Then

\[
\text{index}(\phi) \geq \text{deg}(\phi)(n + 1) + n(1 - g).
\]

In this chapter we shall give improvements to this estimate for genus 0 and for complex isotropic harmonic maps for genus 1.

3.1. The Space of Holomorphic Sections and Holomorphic Differentials. The estimate in Proposition 3.1 was constructed by noting that a holomorphic vector field along \( \phi \) gives a smooth variation of \( \phi \) that contributes to the index of \( \phi \).

**Lemma 3.2.** [7, p. 258] Let \( \phi : M_g \rightarrow \mathbb{C}P^n \) be a non-\( \pm \)-holomorphic harmonic map then

\[
\text{index}(\phi) \geq \dim H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n)
\]

where \( H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) \) is the space of holomorphic sections of \( \phi^{-1}T^{(1,0)}\mathbb{C}P^n \) over \( M_g \).

**Theorem 3.3** (Riemann–Roch [10]). Let \( W \rightarrow M_g \) be a holomorphic vector bundle of rank \( n \) over Riemann surface \( M \) of genus \( g \) then

\[
\dim \mathbb{C} H^0(M_g, W) - \dim \mathbb{C} H^1(M_g, W) = c_1(\wedge^n W) + n(1 - g)
\]

where \( c_1 \) is the first Chern class (evaluated on the canonical generator of \( H_2(M_g, \mathbb{Z}) \)) and \( H^1(M_g, W) \) the space of holomorphic differentials of \( W^* \) over \( M_g \) (i.e. holomorphic \((1,0)\)-forms with values in \( W^* \)).

Let \( \phi : M_g \rightarrow \mathbb{C}P^n \) be a non-\( \pm \)-holomorphic harmonic map. Then using Riemann–Roch for the holomorphic vector bundle \( \phi^{-1}T^{(1,0)}\mathbb{C}P^n \rightarrow M_g \) of rank \( n \) we get

\[
\dim \mathbb{C} H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) - \dim \mathbb{C} H^1(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) = \deg(\phi)(n + 1) + n(1 - g)
\]

and Proposition 3.1 follows directly by disregarding the non-negative number \( \dim \mathbb{C} H^1(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) \). We improve the estimate in Proposition 3.1 by looking at \( \dim \mathbb{C} H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) \) and \( \dim \mathbb{C} H^1(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) \) more closely and either finding their dimension or identifying criteria for them to be trivial.

Considering the harmonic sequence (2.2) above, we say a full harmonic map \( \phi : M \rightarrow \mathbb{C}P^n \) has *directrix* \((f, \rho)\) if \( \phi = G'(G'(\ldots G'(G'(f))\ldots)) \) where \( G' \) is applied \( \rho \) times to a full holomorphic map \( f : M \rightarrow \mathbb{C}P^n \) and \( \rho \in \{0, 1, \ldots, n\} \) [2, 7]. Given a harmonic map \( \phi : M \rightarrow \mathbb{C}P^n \) with *directrix* \((f, \rho)\) then by (2.1) we have

\[
\phi^{-1}T^{(1,0)}\mathbb{C}P^n \cong L(f_{\rho}', f_{\rho}'') = L(f_{\rho}', f_0 \oplus f_1 \oplus \cdots \oplus f_{\rho-1} \oplus f_{\rho+1} \oplus \cdots \oplus f_n)
\]

\[
\cong L(f_{\rho}', f_0) \oplus \cdots \oplus L(f_{\rho}', f_{\rho-1}) \oplus L(f_{\rho}', f_{\rho+1}) \oplus \cdots \oplus L(f_{\rho}', f_n).
\]

So

\[
(3.1) \quad \dim H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{C}P^n) = \sum_{\alpha=0}^{n} \dim H^0(M_g, L(f_{\rho}'), f_{\rho})).
\]

where for each \( \alpha, L(f_{\rho}', f_{\rho}')) is a line bundle, that is, a rank 1 holomorphic bundle. By noting that the degree of \( \phi \) is minus the first Chern class \( c_1 \) of the bundle \( \phi \) (see [7].
Lemma 5.1], we have from [7, p. 246], given a harmonic map \( \phi : M_g \to \mathbb{C}P^n \) with directrix \((f, \rho)\) then

\[
(3.2) \quad c_1(\phi) = \sum_{\alpha=0}^{\rho-1} r_\alpha - \deg(f) + \rho(2 - 2g).
\]

We deduce the following.

**Proposition 3.4.** For each \( i_- < \rho \),

\[
(3.3) \quad \dim H^0(M_g, L(f^0, f^0_{i-})) = 0 \quad \text{when} \quad \sum_{\alpha=i_-}^{\rho-1} r_\alpha > (\rho - i_-)(2g - 2),
\]

\[
(3.4) \quad \dim H^1(M_g, L(f^0, f^0_{i-})) = 0 \quad \text{when} \quad \sum_{\alpha=i_-}^{\rho-1} r_\alpha < (\rho - i_- - 1)(2g - 2),
\]

and for each \( i_+ > \rho \),

\[
(3.5) \quad \dim H^0(M_g, L(f^0, f^0_{i+})) = 0 \quad \text{when} \quad \sum_{\alpha=i_+}^{\rho} r_\alpha < (i_+ - \rho + 1)(2g - 2),
\]

\[
(3.6) \quad \dim H^1(M_g, L(f^0, f^0_{i+})) = 0 \quad \text{when} \quad \sum_{\alpha=i_+}^{\rho} r_\alpha > (i_+ - \rho)(2g - 2).
\]

**Proof.** Let us split the sum (3.1) and consider the two different sums separately:

\[
(3.7) \quad \dim H^0(M_g, \phi^{-1}T^{(1,0)} \mathbb{C}P^n) = \sum_{i_-=0}^{\rho-1} \dim H^0(M_g, L(f^0, f^0_{i-})) + \sum_{i_+=\rho+1}^{\rho} \dim H^0(M_g, L(f^0, f^0_{i+})).
\]

First consider an element of the first sum in (3.7). For a particular \( i_- < \rho \) we have from (3.2),

\[
(3.8) \quad c_1(L(f^0, f^0_{i-})) = c_1(f^0_{i-} \otimes f^0_{i-}) = -c_1(f^0_{i-}) + c_1(f^0_{i-}) = (\rho - i_-)(2g - 2) - \sum_{\alpha=i_-}^{\rho-1} r_\alpha,
\]

and so

\[
(3.9) \quad c_1(L(f^0, f^0_{i-})) < 0 \iff \sum_{\alpha=i_-}^{\rho-1} r_\alpha > (\rho - i_-)(2g - 2).
\]

Noting that for any holomorphic line bundle \( E \to M_g \), \( \dim H^0(M_g, E) = 0 \) if \( c_1(E) < 0 \) [9, p. 154], (3.9) gives (3.3). Considering \( H^1(M_g, L(f^0, f^0_{i-})) \), then using Serre duality we have

\[
H^1(M_g, L(f^0, f^0_{i-})) \cong H^0(M_g, L(f^0, f^0_{i-})^* \otimes T^* M_g).
\]

By a similar argument to (3.8) we have

\[
(3.10) \quad c_1(L(f^0, f^0_{i-})^* \otimes T^* M_g) = -(\rho - i_- - 1)(2g - 2) + \sum_{\alpha=i_-}^{\rho-1} r_\alpha.
\]
and so
\[
c_1(L(f_\rho^*, f_{-i}^*) \otimes T^* M_g) < 0 \iff \sum_{\alpha=i}^{\rho-1} r_\alpha < (\rho - i - 1)(2g - 2),
\]
which gives (3.4). By a similar argument for elements of the second sum in (3.7) we have (3.5) and (3.6). Full details of calculations are given in the author’s thesis [14]. □

3.2. Sphere. We will now restrict to the sphere \((g = 0)\), where we have the following corollary to Proposition 3.4

**Corollary 3.5.** For each \(i_- < \rho\), \(\dim H^0(M_0, L(f_{-i}^*, f_{-i-1}^*)) = 0\) and for each \(i_+ > \rho\), \(\dim H^1(M_0, L(f_{i-1}^*, f_{i+1}^*)) = 0\).

**Remark 3.6.** Note here that for \(g = 0\), (3.4) and (3.5) tell us nothing as \(r_\alpha \geq 0\) for all \(0 \leq \alpha \leq n\).

**Theorem 3.7.** Let \(\phi : S^2 \to \mathbb{CP}^n\) be a non-\(\pm\)-holomorphic harmonic map with directrix \((f, \rho)\) and \(r_\alpha\) the \((\alpha + 1)\)th total ramification index of \(f\), then
\[
\text{index}(\phi) \geq n(n + 1) - \rho(\rho + 1) - (2\rho - 1)(n - \rho) + \sum_{\alpha=\rho}^{n-1} (n - \alpha) r_\alpha
\]
\[
= (n + 1) \deg(\phi) + n + \rho^2 + \sum_{\alpha=0}^{\rho-1} (\alpha + 1) r_\alpha.
\]

**Proof.** For each \(i_+ > \rho\), Theorem 3.3 (Riemann–Roch) gives
\[
\dim H^0(M_0, L(f_{i-1}^*, f_{i+1}^*)) = c_1(L(f_{i-1}^*, f_{i+1}^*)) + 1
\]
and so by (3.1) and using Corollary 3.5
\[
\dim H^0(M_0, \phi^{-1} T^{(1, 0)} \mathbb{CP}^n) = \sum_{i_+ = \rho+1}^{n} \dim H^0(M_0, L(f_{i-1}^*, f_{i+1}^*)) = \sum_{i_+ = \rho+1}^{n} \{ c_1(L(f_{i-1}^*, f_{i+1}^*))+1 \}
\]
\[
= (n - \rho) + \sum_{i_+ = \rho+1}^{n} c_1(L(f_{i-1}^*, f_{i+1}^*)) = (n - \rho) - (n - \rho)c_1(f_{\rho}^*) + \sum_{i_+ = \rho+1}^{n} c_1(f_{i+}^*). \]

By using (3.2), we have
\[
\sum_{i_+ = \rho+1}^{n} c_1(f_{i+}^*) = \sum_{i_+ = \rho+1}^{n} \{ \sum_{\alpha=0}^{i_+ - 1} r_\alpha - \deg(f) + 2i_+ \}
\]
\[
= \sum_{i_+ = \rho+1}^{n} \sum_{\alpha=0}^{i_+ - 1} r_\alpha - (n - \rho) \deg(f) + n(n + 1) - \rho(\rho + 1)
\]
\[
= (n - \rho) \sum_{\alpha=0}^{\rho} r_\alpha + \sum_{\alpha=\rho+1}^{\rho-1} (n - \alpha) r_\alpha - (n - \rho) \deg(f) + n(n + 1) - \rho(\rho + 1).
\]

Putting these together we have
\[
\dim H^0(M_g, \phi^{-1} T^{(1, 0)} \mathbb{CP}^n) = n(n + 1) - \rho(\rho + 1) - (2\rho - 1)(n - \rho) + \sum_{\alpha=\rho}^{n-1} (n - \alpha) r_\alpha.
\]
From [9] p. 271 we have a useful relation involving the \((\alpha + 1)\)th total ramification index

\[ \sum_{\alpha=0}^{n-1} (n - \alpha) r_\alpha = (n + 1) \deg(f) + n(n+1)(g-1). \]  

(3.10)

For \( g = 0 \) we shall split this sum so that we have

\[ \sum_{\alpha=\rho}^{n-1} (n - \alpha) r_\alpha = (n + 1) \deg(f) - n - \sum_{\alpha=0}^{\rho-1} (n - \alpha) r_\alpha. \]

Another useful relation between the degrees of \( \phi \) and \( f \) coming from [7] p. 246 is

\[ \deg(\phi) = \deg(f) - \sum_{\alpha=0}^{\rho-1} r_\alpha - \rho(2g-2). \]  

(3.11)

Using these for \( g = 0 \) we can write

\[ \dim H^0(M_g, \phi^{-1} T^{(1.0)} \mathbb{CP}^n) = (n + 1) \deg(\phi) + n + \rho^2 + \sum_{\alpha=0}^{\rho-1} (\alpha + 1) r_\alpha. \]

By Lemma 3.2 the theorem is proven.

\[ \blacksquare \]

**Corollary 3.8.** Let \( \phi: S^2 \to \mathbb{CP}^2 \) be a non-±-holomorphic harmonic map with directrix \((f,1)\) and \( r_0 \) be the first total ramification index of \( f \) then

\[ \text{index}(\phi) \geq 3 \deg(\phi) + r_0 + 3 = 3 \deg(f) - 2r_0 - 3. \]

**Proof.** Let \( \phi: S^2 \to \mathbb{CP}^2 \) be a non-±-holomorphic harmonic map then as \( \rho = 1 \) and \( n = 2 \) then we have from Theorem 3.7,

\[ \dim H^0(M_0, \phi^{-1} T^{(1.0)} \mathbb{CP}^n) = 3 + r_1. \]

Using (3.10) for \( g = 0, n = 2 \) and (3.11) for \( g = 0, \rho = 1 \) we have that \( r_1 = 3 \deg(f) - 2r_0 - 6 \) and \( \deg(f) = \deg(\phi) + r + 2 \) so

\[ \dim H^0(M_0, \phi^{-1} T^{(1.0)} \mathbb{CP}^n) = 3 \deg(f) - 2r_0 - 3 = 3 \deg(\phi) + r_0 + 3. \]

\[ \blacksquare \]

**Remark 3.9.** For a harmonic map \( \phi: S^2 \to \mathbb{CP}^n \) with directrix \((f, \rho)\), Theorem 3.7 is an improvement on known estimates (Proposition 3.1) by the amount \( \rho^2 + \sum_{\alpha=0}^{\rho-1}(\alpha + 1) r_\alpha \).

In particular, Corollary 3.8 is an improvement for a harmonic map \( \phi: S^2 \to \mathbb{CP}^2 \) with directrix \((f,1)\) by the amount \( 1 + r_0 \).

### 3.3. Torus

Let us restrict to the torus \((g = 1)\), where we have another corollary to Proposition 3.4.

**Corollary 3.10.** for each \( i_- < \rho \),

\[ \dim H^0(M_1, L(\frac{f}{2}, \frac{f}{2-i_-})) = 0 \quad \text{when} \quad \sum_{\alpha=i_-}^{\rho-1} r_\alpha > 0, \]

(3.12)

for each \( i_+ > \rho \),

\[ \dim H^1(M_1, L(\frac{f}{2}, \frac{f}{2+i_+})) = 0 \quad \text{when} \quad \sum_{\alpha=i_+}^{\rho-1} r_\alpha > 0. \]

(3.13)

**Remark 3.11.** Similar to the sphere we see that (3.4) and (3.5) tell us nothing as \( r_\alpha \geq 0 \) for all \( 0 \leq \alpha \leq n \).
Unlike the sphere we cannot conclude that (3.12) or (3.13) is identically zero, as we have special cases when \( \sum_{\alpha=1}^{\rho} r_\alpha = 0 \) and \( \sum_{\alpha=\rho}^{\alpha} r_\alpha = 0 \). Note that, as \( r_\alpha \geq 0 \) for all \( 0 \leq \alpha \leq n \), then \( \sum_{\alpha=1}^{\rho} r_\alpha = 0 \) if and only if \( r_\rho = r_{\rho+1} = \cdots = r_{\rho+1} = 0 \) and similarly \( \sum_{\alpha=\rho}^{\alpha} r_\alpha = 0 \) if and only if \( r_{\rho} = r_{\rho+1} = \cdots = r_{\rho+1} = 0 \). To tackle these special cases we will need the following technical lemma.

**Lemma 3.12.**

(i) Let \( s \) be an antimeromorphic section of line bundle \( E \to M \) equipped with an Hermitian metric and Hermitian connection, then \( s/|s|^2 \) is a meromorphic section of \( E \to M \).

(ii) If \( s \) has a zero (resp. pole) at a point \( z_0 \in M \) of order \( k \) then \( s/|s|^2 \) has a pole (resp. zero) at \( z_0 \in M \) of order \( k \).

**Proof.** Write \( \bar{Z} = \partial/\partial \bar{z} \), then for (i) it suffices to show that, away from the poles of \( s \),

\[
\nabla_{\bar{Z}} \frac{s}{|s|^2} = 0
\]

where \( \nabla \) is the Hermitian connection on \( E \). Now

\[
\nabla_{\bar{Z}} \frac{s}{|s|^2} = \frac{\langle s, s \rangle \nabla_{\bar{Z}} s - s \nabla_{\bar{Z}} \langle s, s \rangle}{\langle s, s \rangle^2} = \frac{1}{\langle s, s \rangle} (\langle s, s \rangle \nabla_{\bar{Z}} s - s \nabla_{\bar{Z}} s, s).
\]

Note that \( \langle s, \nabla_{\bar{Z}} s \rangle = 0 \) as \( s \) is antimeromorphic. Also as \( E \to M \) is a line bundle then

\[
\nabla_{\bar{Z}} s = fs
\]

for some complex-valued function \( f \). So

\[
\nabla_{\bar{Z}} \frac{s}{|s|^2} = \frac{1}{\langle s, s \rangle} (\langle s, s \rangle fs - s(\langle s, s \rangle)) = 0.
\]

To establish that \( s/|s|^2 \) is meromorphic and to show (ii), let \( (U, z) \) be a chart of \( M \) and \( z_0 \in U \) be a zero of \( s \), then we can write

\[
s(z) = (z - z_0)^k \lambda(z)
\]

where \( \lambda \) is a local antiholomorphic section of \( E \to M \), non-zero at \( z_0 \). Then

\[
s = \frac{1}{|z - z_0|^k} \frac{\lambda(z)}{|\lambda(z)|^2}
\]

where \( \lambda(z)/|\lambda(z)|^2 \) is a non-zero local holomorphic section of \( E \to M \) by (i). We argue similarly when \( z_0 \) is a pole. \( \square \)

Now we are ready to examine the special cases when \( \sum_{\alpha=1}^{\rho} r_\alpha = 0 \) for (3.12) and \( \sum_{\alpha=\rho}^{\alpha} r_\alpha = 0 \) for (3.13).

**Lemma 3.13.** For each \( i_+ > \rho \) then

\[
\sum_{\alpha=\rho}^{i+1-1} r_\alpha = 0 \quad \Rightarrow \quad \dim H^1(M_1, L(f_{\rho+i}^\prime, f_{\rho+i}^\prime)) = 1.
\]

**Proof.** Consider the diagram similar to (2.2). Let us set

\[
A''_{i_+ \rho} = A''_{\rho+1} \circ A''_{\rho+2} \circ \cdots \circ A''_{i_+ - 1} \circ A''_{i_+},
\]

so \( A''_{i_+ \rho} \colon f_{\rho+i} \to f_{\rho+i} \) and \( A''_{i_+ \rho} \) is a local antimeromorphic section of \( L(f_{\rho+i}^\prime, f_{\rho+i}^\prime) \). Using Lemma 3.12 we have that \( A''_{i_+ \rho}/|A''_{i_+ \rho}|^2 \) is a local meromorphic section of \( L(f_{\rho+i}^\prime, f_{\rho+i}^\prime) \), that has the same number of poles as \( A''_{i_+ \rho} \) has zeros. By the assumption that \( \sum_{\alpha=\rho}^{i+1-1} r_\alpha = 0 \)
0, $A''_{i+,\rho}$ has no zeros and therefore $A''_{i+,\rho}/|A''_{i+,\rho}|^2$ is a local holomorphic section of $L(f_{i+}, f_{i+})$. In a similar way to [2, p. 264] there is a corresponding global section

$$\frac{A''_{i+,\rho}}{|A''_{i+,\rho}|^2} \, dz \in \Gamma(L(f_{i+}, f_{i+}) \otimes T^*_{(1,0)} M_1).$$

Now by Serre duality we have

$$H^1(M_1, L(f_{i+}, f_{i+}) \otimes T^*_{(1,0)} M_1) \cong H^0(M_1, L(f_{i+}, f_{i+}) \otimes T^*_{(1,0)} M_1)$$

where $T^*_{(1,0)} M_1$ is the holomorphic cotangent bundle that has been canonically identified with $T^* M_1$. Due to $L(f_{i+}, f_{i+}) \otimes T^*_{(1,0)} M_1$ being a line bundle and $A''_{i+,\rho}/|A''_{i+,\rho}|^2$ being a local holomorphic section of $L(f_{i+}, f_{i+})$ then every holomorphic section of $L(f_{i+}, f_{i+}) \otimes T^*_{(1,0)} M_1$ is of the form $\lambda(z)(A''_{i+,\rho}/|A''_{i+,\rho}|^2) \, dz$ for some meromorphic function $\lambda(z)$ such that for each pole $z$ of $A''_{i+,\rho}/|A''_{i+,\rho}|^2$ has no zeros and as meromorphic functions on the torus have the same number of poles and zeros (counted according to multiplicity) [3, p. 153], then $\lambda(z)$ is just a constant and so the claim is proven.

Lemma 3.14. For each $i_- < \rho$ then

$$\sum_{i=0}^{\rho-1} r_\alpha = 0 \implies \dim H^0(M_1, L(f_{i+}, f_{i-})) = 1.$$

Proof. Similar to Lemma 3.13.

Considering Lemmas 3.13 and 3.14 and Theorem 3.3 we have the following breakdown of the special cases: For each $i_+ > \rho$ we have

$$\dim H^0(M_1, L(f_{i+}, f_{i+})) = \begin{cases} c_1(L(f_{i+}, f_{i+})) + 1, & \text{if } \sum_{\alpha=0}^{i_+-1} r_\alpha = 0, \\ c_1(L(f_{i+}, f_{i+})), & \text{otherwise.} \end{cases}$$

Similarly for each $i_- < \rho$ we have

$$\dim H^0(M_1, L(f_{i-}, f_{i-})) = \begin{cases} 1, & \text{if } \sum_{\alpha=0}^{i_-} r_\alpha = 0, \\ 0, & \text{otherwise.} \end{cases}$$

So we have four distinct cases:

**Theorem 3.15.** Let $\phi : M_1 \to \mathbb{CP}^n$ be a non-$\pm$ holomorphic complex isotropic harmonic map with directrix $(f, \rho)$ then

$$\text{index}(\phi) \geq (n+1) \deg(\phi)$$

$$\begin{cases} (i_+ - \rho) + \sum_{\alpha=0}^{\rho-1} (\alpha + 1)r_\alpha, & \text{if } \sum_{\alpha=0}^{i_+-1} r_\alpha = 0 \text{ for some } i_+ > \rho, \\
(i_- - \rho) + \sum_{\alpha=0}^{i_-} (\alpha + 1)r_\alpha, & \text{if } \sum_{\alpha=0}^{i_-} r_\alpha = 0 \text{ for some } i_- < \rho, \\
(i_+ - i_-) + \sum_{\alpha=0}^{i_- - i_+} (\alpha + 1)r_\alpha, & \text{if } \sum_{\alpha=0}^{i_- - i_+} r_\alpha = 0 \text{ for some } i_- < \rho < i_+, \\
\sum_{\alpha=0}^{\rho-1} (\alpha + 1)r_\alpha, & \text{otherwise.} \end{cases}$$
Proof. We will give the proof for the first case; the other cases are proven similarly. Suppose that
\[
\sum_{\alpha=\rho+1}^{i_+-1} r_\alpha = 0 \quad \text{for some } i_+ > \rho \quad \text{and} \quad \sum_{\alpha=1}^{\rho-1} r_\alpha \neq 0 \quad \text{for all } i_- < \rho.
\]
It is evident that if \(\sum_{\alpha=\rho}^{i_+-1} r_\alpha = 0\) for some \(i_+ > \rho\) then \(\sum_{\alpha=\rho}^{k-1} r_\alpha = 0\) for all \(k\) such that \(i_+ \geq k > \rho\). Using (3.14) and (3.15) we have the following: for each \(k \geq \rho\) such that \(i_+ \geq k > \rho\) then \(\dim H^0(M_1, L(f_{+,1}, f_{-,1})) = c_1(L(f_{+,1}, f_{-,1})) + 1\), for each \(j > i_+ > \rho\) then \(\dim H^0(M_1, L(f_{+,1}, f_{-,1})) = c_1(L(f_{+,1}, f_{-,1}))\); for each \(i_- < \rho\) then \(\dim H^0(M_1, L(f_{+,1}, f_{-,1})) = 0\).

So we have
\[
\dim H^0(M_1, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) = \sum_{i_+ = \rho+1}^{n} \dim H^0(M_1, L(f_{+,1}, f_{-,1}))
= \sum_{k=\rho+1}^{i_+} \{c_1(L(f_{+,1}, f_{-,1})) + 1\} + \sum_{j=i_+-1}^{n} \{c_1(L(f_{+,1}, f_{-,1}))\}
= (i_+ - \rho) + \sum_{\alpha=\rho+1}^{n} c_1(L(f_{+,1}, f_{-,1}))
= (i_+ - \rho) - (n - \rho)c_1(L(f_{+,1}, f_{-,1})) + \sum_{\alpha=\rho+1}^{n} c_1(L(f_{+,1}, f_{-,1})),
\]
where, using (3.2), we have
\[
\sum_{\alpha=\rho+1}^{n} c_1(L(f_{+,1}, f_{-,1})) = \sum_{\alpha=\rho+1}^{n} \{\sum_{\beta=0}^{\alpha-1} r_{\beta} - \deg(f)\} = \sum_{\alpha=\rho+1}^{\rho} \sum_{\beta=0}^{\alpha-1} r_{\beta} - (n - \rho) \deg(f)
= (n - \rho) \sum_{\beta=\rho+1}^{\rho} r_{\beta} + \sum_{\beta=\rho+1}^{n-1} (n - \beta)r_{\beta} - (n - \rho) \deg(f).
\]
Together with (3.10) and (3.11) for \(g = 1\) we have
\[
\dim H^0(M_1, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) = (i_+ - \rho) + \sum_{\alpha=i_+}^{n} (n - \alpha) r_{\alpha}
= (n + 1) \deg(\phi) + (i_+ - \rho) + \sum_{\alpha=0}^{\rho-1} (\alpha + 1) r_{\alpha}.
\]

\(\square\)

Corollary 3.16. Let \(\phi : M_1 \to \mathbb{CP}^2\) be a non-\(\pm\)-holomorphic complex isotropic harmonic map then
\[
\text{index}(\phi) \geq \begin{cases} 
1, & \text{if } r_1 = 0, \\
3 \deg(\phi) + 1, & \text{if } r_0 = 0, \\
3 \deg(\phi) + r_0, & \text{if } r_0 > 0, r_1 > 0
\end{cases} = \begin{cases} 
1, & \text{if } r_1 = 0, \\
3 \deg(\phi) + 1, & \text{if } r_0 = 0, \\
3 \deg(\phi) - 2r_0, & \text{if } r_0 > 0, r_1 > 0.
\end{cases}
\]
Proof. From Theorem 3.15 with \( \rho = 1 \) and \( n = 2 \) we have

\[
\dim H^0(M_1, \phi^{-1}T^{(1,0)}\mathbb{CP}^2) = \begin{cases} 
3 \deg(\phi) + r_0 + 1, & \text{if } r_1 = 0, \\
3 \deg(\phi) + 1, & \text{if } r_0 = 0, \\
3 \deg(\phi) + r_0, & \text{if } r_0 > 0, r_1 > 0.
\end{cases}
\]

We know from \( \text{(3.10)} \) and \( \text{(3.11)} \) that \( r_1 = 3 \deg(f) - 2r_0 \) and \( \deg(\phi) = \deg(f) - r_0 \) respectively. Using these together, one can easily see the relations in the corollary. \( \square \)

Remark 3.17. The third case of Theorem 3.15 does not occur in Corollary 3.16 as if \( r_0 + r_1 = 0 \) then both \( r_0 \) and \( r_1 \) are zero and as \( r_1 = 3 \deg(f) - 2r_0 \) for the torus we have \( \deg(f) = 0 \); but this implies that \( f \) is constant.

3.3.1. Example. Corollary 3.16 poses a natural question on the existence of full holomorphic maps \( f : M_1 \to \mathbb{CP}^2 \) where \( r_0 = 0 \) and/or \( r_1 = 0 \).

Let us consider the Weierstrass \( \wp \)-function, \( \wp : \mathbb{C} \to \mathbb{C} \):

\[
\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \frac{1}{(z - \omega)^2} + \frac{1}{\omega^2},
\]

for \( \Lambda = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z} \} \) a period lattice on \( \mathbb{C} \). The map \( \wp \) has a double pole at each lattice point and can also be seen as a map from a torus as \( M_1 = \mathbb{C}/\Lambda \). Therefore \( \wp \) can be seen as a meromorphic function of the torus with a double pole at \( z = 0 \) and two zeros (not necessarily distinct); for more information on the Weierstrass \( \wp \)-function see [11] p. 153 and [8]. Now consider the differential of \( \wp : M_1 \to \mathbb{C} \) given by

\[
\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^3}.
\]

It is known [11] that \( \wp'(z) \) has a triple pole at \( z = 0 \) and simple zeros at \( \omega_1/2, \omega_2/2 \), and \( (\omega_1 + \omega_2)/2 \) where \( \omega_1 \) and \( \omega_2 \) are the periods of the period lattice \( \Lambda \) given above.

Using the Weierstrass \( \wp \)-function above we construct an example of a full holomorphic map \( f : M_1 \to \mathbb{CP}^2 \) with \( r_0 = 0 \), and then using Corollary 3.16 we find a bound for its index.

Example 3.18. Let \( F : M_1 \to \mathbb{C} \setminus \{0\} \), where \( F = (1, \wp, \wp') \). Then let \( f = [F] : M_1 \to \mathbb{CP}^2 \), so \( f = [1, \wp, \wp'] \), and \( f = [0, 0, 1] \) on lattice points. Then, following the construction of [11] (cf. [13]), we have that \( f \) is first ramified at some \( z_0 \) if \( F \wedge F'(z_0) = 0 \) for \( F \wedge F' : M_1 \to \wedge^2 \mathbb{C}^3 \) and \( F' = dF/dz \). Now for \( F = (1, \wp, \wp') \) above we have

\[
F \wedge F' = (\wp \wp'' - (\wp')^2, -\wp'', \wp'),
\]

and we see that \( f \) is ramified at \( z_0 \) if and only if

\[
-\wp''(z_0) = \wp'(z_0) = 0.
\]

But

\[
\wp'(z_0) = 0 \implies z_0 = \frac{\omega_1}{2}, \frac{\omega_2}{2} \quad \text{or} \quad \frac{\omega_1 + \omega_2}{2},
\]

which are simple zeros of \( \wp' \) and so \( -\wp''(z_0) \neq 0 \) when \( \wp'(z_0) = 0 \); hence \( f \) is unramified.

It is shown in [11] that \( f \) is a bijection onto the projective plane curve given in affine form by \( y^2 = 4x^3 - 9gx - g_3 \) with \( g_2 \) and \( g_3 \) constants which are defined by the lattice \( \Lambda \).

Therefore the map \( f \) above can be seen as the diffeomorphism \( \tilde{f} : M_1 \to E \) where

\[
E = \left\{ [1, x, y] \mid y^2 = 4x^3 - 9gx - g_3 \right\} \cup \{[0, 0, 1]\}.
\]

Now consider the projection \( p : E \to \mathbb{CP}^1 ; [1, x, y] \mapsto [1, y] \) and note that this projection makes sense as \( (1, y) \neq (0, 0) \). It can be easily seen that, for any point \( [1, y] \in \mathbb{CP}^1 \) such that \( 4x^3 - y^2 - 9gx - g_3 = 0 \) has distinct roots, then \( [1, y] \) has exactly three pre-images in \( E \). Therefore as the (Brouwer) degree of a map between surfaces is the number of
pre-images of a regular point we can conclude that the projection $p$ has degree 3. Now consider the diagram

$$M_1 \xrightarrow{\tilde{f}} E \xrightarrow{p} \mathbb{CP}^1 \xrightarrow{i} \mathbb{CP}^2$$

where $i: \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ is the inclusion map. The map $f$ is the composition of the diffeomorphism $\tilde{f}$, the degree 3 projection $p$ and the degree 1 inclusion $i$. Therefore $f = [1, \varphi, \varphi']$ has degree 3.

Let $\phi = G'(f): M_1 \to \mathbb{CP}^2$ be the harmonic map with directrix $(f, 1)$ then by Corollary 3.16 we have

$$\text{index}(\phi) \geq \dim H^0(M_1, \phi^{-1}T(1,0)\mathbb{CP}^2) = 3\deg(f) + 1 = 10.$$

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