Unitary Inequivalent Representations in Quantum Physics

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Abstract

First the existence of different unitary inequivalent representations in the Quantum Field Theory (QFT) is discussed. Then it is shown that how they can play a major role for us to understand some phenomena such as Hawking effect.

1 Introduction

Different inequivalent representations are among main and natural properties of Quantum Field Theory. However, in conventional Quantum Field Theory, physicists do not pay proper attention to them. In other words in conventional Quantum Field Theory we take into consideration just one class of these representations and ignore the others. In this article first we make a review of them and then show that although it seems that they play no important role in Quantum Field theory in flat (Minkowski) space-time, they are inevitably part of Quantum Field Theory in curved space-times, without which it is impossible to formalize a consistent QFT.

2 Unitary Inequivalent Representations

2.1 Stone-von Neumann Uniqueness theorem

First we begin with Stone-von Neumann uniqueness theorem [1] which states that for \( \{ \tilde{U}(a) | a \in \mathbb{R} \} \), \( \{ \tilde{V}(b) | b \in \mathbb{R} \} \) be finite sets of weakly continuous unitary operators acting irreducibly on a separable Hilbert space \( H \) such that

\[
\tilde{U}(a) \tilde{V}(b) = \exp \left( -\frac{ab}{\hbar} \right) \tilde{V}(b) \tilde{U}(a), \quad \tilde{U}(a) \tilde{U}(b) = \tilde{U}(a + b) \quad \text{and} \quad \tilde{V}(a) \tilde{V}(b) = \tilde{V}(a + b),
\]

then there is a Hilbert space isomorphism \( W : H \to L^2(\mathbb{R}) \) such that \( W \tilde{U}(a) W^{-1} = U(a) \) and \( W \tilde{V}(a) W^{-1} = V(a) \). This theorem concludes

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that for every system with \( N \) degrees of freedom, where \( N \) is finite, we have unitary equivalent representations in Hilbert space. As an example of this equivalent representations one can consider the wave function of a non-relativistic system described as \( \psi(x) \). In non-relativistic quantum mechanics \( \psi(x) \) is called space representation of wave function. Besides every system we have another representation called momentum representation, denoted by \( \phi(p) \). It is easy to show that both representations are unitary equivalent, since they can transform to each other by a fourier transformation:

\[
\psi(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int \phi(p) \exp\left(\frac{ipx}{\hbar}\right) dp
\]

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\]

Since these transformations preserve the norm of the wave function they are unitary. As mentioned above the Von Neumann uniqueness is valid for all systems with finite degrees of freedom \([2]\). But the situation changes when \( N \to \infty \). In this case instead of a unitary equivalent representation, we have the equivalent classes of representations. Each two representations belonging to one of these classes are unitary equivalent but the ones from different classes do not need to be equivalent. By definition the field is a system with infinite number of degrees of freedom, so naturally we have to deal with unitary inequivalent representations. Then one may ask which class of representations is physical? In conventional Quantum Field Theory the answer to this question is given by the condition:

\[
H|0\rangle = 0
\]

where the \( |0\rangle \) is the vacuum state of quantum field.(after this selection we ignore the existence of other representations.) So after choosing this representation, it was thought that we do not need other class of representations, and this one is sufficient for Quantum Field Theory. But later it became clear that there are some phenomena in nature that can not be explained by regarding only one of these classes. One of them is Hawking effect which we will discuss it in next section. So the correct treatment is that we have to take into account all classes of representations and try to formalize our theory by regarding all of them. In other words all of these classes are equally important.

2.2 Why do these inequivalent representations exist?

Although some people try to ignore the existence of these classes of representations, the existence of them can be proved in the context of conventional Quantum Field Theory as in Algebraic one. In conventional approach to Quantum Field Theory, since we are dealing with a field we have to make
a cut for our system. Remembering that this situation does not take place in systems with finite degrees of freedom. Because in this situation we are able to close the system and specify it. But in fields this can not be done. The reason is that we can not specify the infinity. So by an idealization we make a cut and try to construct a complete set of observables locally. As an example we suppose that our infinity is located in a very far place, say Andromeda galaxy, and \( \{ A_i \} \) is our complete set of observables. So it is clear that we can formalize our theory with \( \{ A_i \} \), but this can be done just locally. Since if someone makes a change in a place beyond Andromeda galaxy, globally our representations will change. But as we have made a cut in Andromeda, this change will have no effect in our local observations. From one side it shows that in local observations and interactions we can neglect these different representations, but on the other side in non-local effects all of them become important. Here we have to emphasize that if we want to construct a complete and self consistent theory which can relate local and global phenomena, dealing with all these representations is necessary and that theory will be the ultimate theory of Quantum Gravity. The existence of this different representations can be easily shown in the context of Algebraic Quantum Field Theory too. Where one can associate a \( C^* \) – algebra to a quantum field. This follows by GNS construction \([3, 4]\) which states that for every element on \( C^* \) – algebra like \( \omega \), there is a representation \( \pi \) of the algebra by linear operators on a dense subspace \( D \subseteq H \) such that

\[
\omega(A) = (\Omega, \pi(A)\Omega)
\]  

where \( \Omega \) is the unit vector in \( D \). So we can conclude that in each representation which we construct with GNS construction, the specified state \( \omega \) in \( C^* \)-algebra is related to unit vector \( \Omega \) in Hilbert space. Thus if one chooses another state say \( \nu \) and constructs another representation, then there is no need to these two representations be unitary equivalent.

### 3 An example: Hawking effect

In this section we want to discuss the Hawking effect and show that by regarding these unitary inequivalent representations it is possible, not in a complete mathematical way, to describe this phenomena.

#### 3.1 A brief explanation of Hawking effect

Stephen Hawking \([5]\) showed that by regarding quantum effects, it is possible to attribute a thermal radiation to black holes. This radiation, called Hawking radiation, has a temperature which is

\[
T_H = \frac{\hbar c^3}{8\pi GMk_B}
\]  

3
This formula contains four fundamental constants in nature, \( h, G, k \) and \( c \). In other words Hawking radiation showed that there is a connection between Thermodynamics, General Relativity and Quantum Field Theory. Although this was a great achievement, at first glance Hawking’s original calculations suffer from transplanckian problem. Because of this, some physicists by considering the fact that in our world there is no transplanckian energy concluded that Hawking effect is not physical. The situation is changed when Bekenstein \[6\] introduced the generalized second law of thermodynamics which states that in a system with a black hole the total amount of entropy is given by

\[ \Delta S_{\text{outside}} + \Delta S_{\text{B.H}} \geq 0 \]  

where for a black hole the entropy is equal to

\[ S_{\text{B.H}} = \frac{A}{4hG} \]  

where \( A \) denotes the area of the black hole. Since the generalized second law indicates that when an object falls into a black hole, increases the entropy of the black hole and all the information of the object will be lost, it is possible to relate a thermal radiation to the black hole. In other words one can conclude that in the presence of gravitational collapse the vacuum state of quantum field, \(|0\rangle\), becomes unstable and finally is changed to a thermal state, \(|\beta\rangle\). The main problem arises when we try to discuss this effect in the context of Quantum field theory in curved space-time. Consider a vacuum state\(|0\rangle\). In order to calculate the Hawking effect we have to use the semiclassical General Relativity,

\[ G_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle \]  

where \( \langle T_{\mu\nu} \rangle \) is the expectation value of the quantum field. Now it seems that for a vacuum state we have

\[ \langle 0 | T_{\mu\nu} | 0 \rangle = 0 \]  

. By comparing to the left hand side of the equation, it is evident that it is compatible with the black hole metric. But the situation changes when we arrive to final state of quantum field, the thermal state \(|\beta\rangle\). In this case we have

\[ \langle \beta | T_{\mu\nu} | \beta \rangle \neq 0 \]  

But this seem contradictory in two cases. First, we have

\[ G_{\mu\nu} \neq \kappa \langle \beta | T_{\mu\nu} | \beta \rangle \]  

which indicates that the given background for black hole is not compatible with the new stress tensor. The second contradiction takes place when we
look at the vacuum state $|0\rangle$ in the context of conventional Quantum Field Theory. As mentioned above the conventional QFT is formalized in just one of these representations where

$$H |0\rangle = 0 \quad (12)$$

Since all representations in this class are equivalent to each other up to a unitary transformation, it is impossible to find a unitary transformation like $U$ which will be able to transform $|0\rangle$ into $|\beta\rangle$ such that

$$U |0\rangle = |\beta\rangle \quad (13)$$

The strange results introduced above divide theoretical physicists into two groups. The first are those who try to resolve these contradictions using different mathematical and physical methods and concepts. Others, those who conclude that these contradictions tell us that Hawking effect is unphysical, even QFT in curved space-time does not exist.

But these two contradictions may be easily solved if we forget the one class of representations and regard all of them simultaneously. In this case we can say that during the Hawking effect, the vacuum state $|0\rangle$ belonging to some class of representation changes into another class of representations and becomes $|\beta\rangle$. Meanwhile the given background denoted by $G_{\mu\nu}$ changes into another one. So it becomes clear that considering all unitary inequivalent representations in QFT will give us a powerful tool such that we will be able to show the consistency of some phenomena like Hawking effect.

## 4 Conclusion

As mentioned above, the existence of these inequivalent unitary representations is the inevitable part of the QFT. But the main problem is that when we try to construct a physical theory, by considering the Poincare symmetry, we select just one of these classes and simply forget about the existence of others. This causes some problems such as Haag’s no-go theorem [2]. On the other hand, the formulation of S-Matrix is such that one can find the final state $|f\rangle$ at $t = +\infty$ by operating S-matrix on the initial state $|i\rangle$ at $t = -\infty$ without taking into account the moment of interaction, regarding it as a black box. But it is the moment of interaction that all of these classes may become equally important. But their existence is something which is related to the global structure and that is why we will not be able to see the effect of them in our local observations. Another important feature is that Quantum Gravity will enable us to relate the yet unknown transplanckian world and our one to each other. This correlation shows itself in Hawking effect which again can be explain in this manner. Here we wanted to show there may be a new look to the yet unknown quantum theory of gravity and the gravity may have the role of relating one class of representations to another, although it seems a very difficult physical and mathematical task.
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