ON ROOT CATEGORIES OF FINITE-DIMENSIONAL ALGEBRAS

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Abstract. For any finite-dimensional algebra $A$ over a field $k$ with finite global dimension, we investigate the root category $\mathcal{R}_A$ as the triangulated hull of the 2-periodic orbit category of $A$ via the construction of B. Keller in “On triangulated orbit categories”. This is motivated by Ringel-Hall Lie algebras associated to 2-periodic triangulated categories. As an application, we study the Ringel-Hall Lie algebras for a class of finite-dimensional $k$-algebras with global dimension 2, which turn out to give an alternative answer for a question of GIM Lie algebras by Slodowy in “Beyond Kac-Moody algebra, and inside”.

1. Introduction

Root category was first introduced by D. Happel [8] for finite-dimensional hereditary algebra, which was used to characterize a bijection between the indecomposable objects of the root category for the path algebra of Dynkin type and the root system of corresponding complex simple Lie algebra.

Let $A$ be a finite-dimensional hereditary algebra over a field $k$. Let $\mathcal{D}^b(\text{mod } A)$ be the derived category of finitely generated right $A$-modules. Then the root category $\mathcal{R}_A$ of $A$ is defined to be the 2-periodic orbit category $\mathcal{D}^b(\text{mod } A)/\Sigma^2$, where $\Sigma$ is the suspension functor. It was proved by Peng-Xiao [17], the root category $\mathcal{R}_A$ is triangulated via the homotopy category of 2-periodic complexes category of $A$-modules. With this triangle structure, Peng and Xiao [18] constructed a so called Ringel-Hall Lie algebra associated to each root category and realized all the symmetrizable Kac-Moody Lie algebras. In fact, Peng-Xiao’s construction is valid for any $\text{Hom}$-finite 2-periodic triangulated category. In [15], Lin-Peng realized the elliptic Lie algebras of type $D^{(1,1)}_4$, $E^{(1,1)}_6$, $E^{(1,1)}_7$, $E^{(1,1)}_8$ via the 2-periodic orbit categories (which are triangulated) of corresponding tubular algebras. However, in general, for arbitrary finite-dimensional $k$-algebra $A$, the 2-periodic orbit category $\mathcal{D}^b(\text{mod } A)/\Sigma^2$ is not triangulated with the inherited triangle structure from $\mathcal{D}^b(\text{mod } A)$ (cf. section 3.3 or [11]). Up to now, there are no suitable $\text{Hom}$-finite 2-periodic triangulated categories to realize the other elliptic Lie algebras via the Ringel-Hall Lie algebras approach.

Let $A$ be a finite-dimensional $k$-algebra with finite global dimension. In [29], the authors propose to study the homotopy category $\mathcal{K}_2(\mathcal{P})$ of 2-periodic complexes category of finitely generated projective $A$-modules and give a geometric construction of a Lie algebra over $\mathbb{C}$ directly instead of over finite fields in [18]. In this paper, we propose to study another 2-periodic triangulated category $\mathcal{R}_A$ called the root category of $A$ via Keller’s construction [11]. Then Peng-Xiao’s construction [18] gives a Lie algebra $\mathcal{H}(\mathcal{R}_A)$ for arbitrary finite-dimensional $k$-algebra $A$ with finite global dimension. We remark that the root category $\mathcal{R}_A$ is invariant up to derived equivalence. Note that by the 2-universal property of root category, we have an embedding $\mathcal{R}_A \hookrightarrow \mathcal{K}_2(\mathcal{P})$. When the algebra $A$ is hereditary, $\mathcal{R}_A \cong \mathcal{K}_2(\mathcal{P}) \cong \mathcal{D}^b(\text{mod } A)/\Sigma^2$ and coincides with the original definition of Happel. We also remark that by using $\mathcal{R}_A$, one can easily construct 2-periodic triangulated categories.

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such that the Grothendieck groups of these categories characterize the root lattices for any elliptic Lie algebras (cf. section 3.2). This is one of the motivations to introduce the root category in this paper. The relation between the Ringel-Hall Lie algebras of these categories and the corresponding elliptic Lie algebras would be interesting to study in future.

This paper is organized as follows: in section 2, for any finite-dimensional k-algebra A of finite global dimension, we introduce the root category $R_A$ and study its basic properties. It is $\text{Hom}$-finite 2-periodic triangulated category and admits AR-triangles. We also give an explicitly characterization of its Grothendieck group. In section 3, we study some motivating examples. In particular, we give a minimal example such that the 2-periodic orbit category is not triangulated with the inherited triangle structure. In section 4, we consider the root categories of representation-finite hereditary algebras, we show that such root categories characterize the algebras up to derived equivalence. In the last section, we study the Ringel-Hall Lie algebras of a class of finite-dimensional k-algebras with global dimension 2, which turn out to give a negative answer for a question on GIM Lie algebra by Slodowy. Let us mention that different counterexamples have been discovered in [1] by using different approach. In the appendix, we discuss the universal property of root category and study recollement associated to root categories, which can be use to construct various examples inductively such that the 2-periodic orbit category is not triangulated with the inherited triangle structure from the bounded derived category.

Throughout this paper, we fix a field $k$. All algebras are finite-dimensional k-algebras with finite global dimension. All modules are right modules. Let $\mathcal{C}$ be a k-category. For any $X, Y \in \mathcal{C}$, we write $\mathcal{C}(X, Y)$ for $\text{Hom}_\mathcal{C}(X, Y)$. For a subcategory $\mathcal{M}$ in a triangulated category $\mathcal{T}$, we denote by $\text{tria}(\mathcal{M})$ the thick subcategory of $\mathcal{T}$ contains $\mathcal{M}$.

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2. ROOT CATEGORIES FOR FINITE-DIMENSIONAL ALGEBRAS

2.1. 2-periodic orbit categories. Let $A$ be a finite-dimensional k-algebra of finite global dimension. Let $\mathcal{D}^b(\text{mod } A)$ be the bounded derived category of finitely generated $A$-modules and $\Sigma$ the suspension functor. Consider the left total derived functor of $A \otimes_k A^{op}$-module $\Sigma^2 A$

$$\Sigma^2 = ? \otimes_A \Sigma^2 A : \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } A),$$

which is an equivalence. For all $L, M$ in $\mathcal{D}^b(\text{mod } A)$, the group

$$\mathcal{D}^b(\text{mod } A)(L, \Sigma^{2n} M)$$

vanishes for all but finitely many $n \in \mathbb{Z}$. The 2-periodic orbit category

$$\mathcal{D}^b(\text{mod } A) / \Sigma^2$$

of $A$ is defined as follows:

- the objects are the same as those of $\mathcal{D}^b(\text{mod } A)$;
- if $L$ and $M$ are in $\mathcal{D}^b(\text{mod } A)$ the space of morphisms is isomorphic to the space

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{D}^b(\text{mod } A)(L, \Sigma^{2n} M).$$

The composition of morphisms is obviously. Suppose that $A$ is hereditary. Then the orbit category is called root category of $A$ which was first introduced by D. Happel in [8].
A Hom-finite \( k \)-additive triangulated category \( D \) is called 2-periodic triangulated if:

- \( \Sigma^2 \cong 1 \), where \( \Sigma \) is the suspension functor of \( D \);
- the endomorphism ring \( \text{End}(X) \) for any indecomposable object \( X \) is a finite-dimensional local \( k \)-algebra.

In particular, the 2-periodic orbit category of a hereditary algebra \( A \) is 2-periodic triangulated with canonical triangle structure proved by Peng-Xiao [17]. However, this is not true in general. The first example is due to A. Neeman who considers the algebra \( A \) of dual numbers \( k[X]/(X^2) \). Then the 2-periodic orbit category of \( A \) is not triangulated (cf. section 3 of [11]). No example of algebra with finite global dimension seems to be known.

2.2. Root category via Keller’s construction. As shown by Keller in [11], if \( D^b(\text{mod } A) \) is triangulated equivalent to the bounded derived category of a hereditary category, then the orbit category \( D^b(\text{mod } A)/\Sigma^2 \) is triangulated. In general, the orbit category is not triangulated. But a triangulated hull was defined in [11] as the algebraic triangulated category \( R_A \) with the following universal properties:

- There exists an algebraic triangulated functor \( \pi : D^b(\text{mod } A) \to R_A \);
- Let \( B \) be a dg category and \( X \) an object of \( D(A^{op} \otimes B) \). If there exists an isomorphism in \( D(A^{op} \otimes B) \) between \( \Sigma^2 A \otimes_A X \) and \( X \), then the triangulated algebraic functor \( \Sigma^2 A \otimes_A X : D^b(\text{mod } A) \to D(B) \) factorizes through \( \pi \).

Consider \( A \) as a dg algebra concentrated in degree 0. Let \( S \) be the dg algebra with underlying complex \( A \oplus \Sigma A \), where the multiplication is that of the trivial extension:

\[
(a, b)(a', b') = (aa', ab' + ba').
\]

Let \( D(S) \) be the derived category of \( S \) and \( D^b(S) \) the bounded derived category, i.e. the full subcategory of \( D(S) \) formed by the dg modules whose homology has finite total dimension over \( k \). Let \( \per(S) \) be the perfect derived category of \( S \), i.e. the smallest subcategory of \( D(S) \) contains \( S \) and stable under shift, extensions and passage to direct factors. Clearly, the perfect derived category \( \per(S) \) is contained in \( D^b(S) \). Denote by \( p : S \to A \) the canonical projection. It induces a triangle functor \( p_* : D^b(\text{mod } A) \to D^b(S) \). By composition we obtain a functor

\[
\pi_A : D^b(\text{mod } A) \to D^b(S) \to D^b(S)/\per(S).
\]

Let \( \text{tria}(p_* A) \) be the thick subcategory of \( D^b(S) \) generated by the image of \( p_* A \). By Theorem 2 of [11], the triangulated hull of the orbit category \( D^b(\text{mod } A)/\Sigma^2 \) is the category

\[
R_A = \text{tria}(p_* A)/\per(S).
\]

Moreover, there is an embedding \( i : D^b(\text{mod } A)/\Sigma^2 \hookrightarrow A >_S /\per(S) \). If \( i \) is dense, then we say that the 2-periodic orbit category \( D^b(\text{mod } A)/\Sigma^2 \) is triangulated with inherited triangle structure of \( D^b(\text{mod } A) \). If \( A \) is an hereditary algebra, the embedding \( i \) is essentially an equivalence of triangulated categories

\[
D^b(\text{mod } A)/\Sigma^2 \cong \text{tria}(p_* A)/\per(S).
\]

Since \( S \) is a negative dg algebra. It is well-known that there is a canonical \( t \)-structure \((D^\leq, D^\geq)\) induced by homology over \( D(S) \). In particular, \( D^\leq \) is the full subcategory of \( D(S) \) whose objects are the dg modules \( X \) such that the homology groups \( H^p(X) \) vanishes for all \( p > 0 \). Obviously, the \( t \)-structure restricts to the subcategory \( D^b(S) \) of \( D(S) \). It is not hard to see that \( D^b(S) = \text{tria}(p_* A) \). Then the root category \( R_A = D^b(S)/\per S \) in this case. In the following, we call the triangulated hull \( R_A \) the root category of \( A \) and \( \pi_A : D^b(\text{mod } A) \to D^b(S)/\per(S) = R_A \) the canonical functor.
Remark 2.1. One can also consider the construction for the orbit category $D^b(\text{mod } A)/\Sigma^{-2}$ which in fact the same as $D^b(\text{mod } A)/\Sigma^{2}$. Then one replaces the dg algebra $S$ by $S' = A \oplus \Sigma^{-3}A$. The root category defined as $\mathcal{R}_A = \text{tria}(p_A)/\text{per } S'$.

2.3. Alternative description of $\mathcal{R}_A$. There is another description of $\mathcal{R}_A$ in $[11]$. Let $\mathcal{A}$ be the dg category of bounded complexes of finitely generated projective $A$-modules. Naturally, the tensor product of $\Sigma^2A$ define a dg functor from $\mathcal{A}$ to $\mathcal{A}$. Then one can form the $\text{dg orbit category } \mathcal{B}$ as the dg category with the same objects of $\mathcal{A}$ and such that for any $X, Y \in \mathcal{B}$, we have

$$\mathcal{B}(X, Y) \cong \bigoplus_{n \in \mathbb{Z}} A(X, \Sigma^{2n}Y).$$

Now we have an equivalence of categories

$$D^b(\text{mod } A)/\Sigma^{2} \cong \mathcal{H}^0(\mathcal{B}).$$

Let $\mathcal{D}(\mathcal{B})$ be the derived category of the dg category $\mathcal{B}$. Let the ambient triangulated category $\mathcal{M}$ be the triangulated subcategory of $\mathcal{D}(\mathcal{B})$ generated by the representable functors. Then theorem 2 of $[11]$ implies that $\mathcal{R}_A \cong \mathcal{M}$.

Proposition 2.2. Let $A$ be a finite-dimensional $k$-algebra of finite global dimension. Then the root category $\mathcal{R}_A$ is a Hom-finite 2-periodic triangulated category.

Proof. The Hom-finiteness follows from the description of $\mathcal{M}$, since the homomorphisms between representable functors of $\mathcal{B}$ are finite-dimensional over $k$. Consider the $\mathcal{B} \otimes \mathcal{B}^{op}$-module $X : X(A, B) = B(A, B)$ for any $A, B \in \mathcal{B}$, it induces the identity functor

$$1 : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}).$$

One can also consider the $\mathcal{B} \otimes \mathcal{B}^{op}$-module $Y(A, B) = \Sigma^2B(A, B)$ for any $A, B \in \mathcal{B}$. Clearly, the module $Y$ induces the triangle functor

$$\Sigma^2 : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}).$$

By the definition of the dg orbit category $\mathcal{B}$, we know that $X$ is isomorphic to $Y$ as $\mathcal{B} \otimes \mathcal{B}^{op}$-modules, which will induce an invertible morphism $\eta : 1 \to \Sigma^2$ by Lemma 6.1 of $[10]$. Thus, to show that $\mathcal{R}_A$ is 2-periodic triangulated category, it suffices to show that $\mathcal{R}_A$ is Krull-Schmidt category. It suffices to prove that each idempotent morphism of $\mathcal{R}_A$ is split, i.e. $\mathcal{R}_A$ is idempotent completed. Recall that $\mathcal{R}_A = \mathcal{M} \subset \mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{B})$ is idempotent complete since $\mathcal{D}(\mathcal{B})$ admits arbitrary direct sums. Moreover, $\mathcal{M}$ is closed under direct summands in $\mathcal{D}(\mathcal{B})$. Now the result follows from the well-known fact that if an additive category $\mathcal{C}$ is idempotent completed, then a full subcategory $\mathcal{D}$ of $\mathcal{C}$ is idempotent completed if and only if $\mathcal{D}$ is closed under direct summands. \hfill \square

2.4. Serre functor over $\mathcal{R}_A$. Keep the notations above. Let $D = \text{Hom}_k(?, k)$ be the usual duality over $k$. The $S \otimes_k \mathcal{S}^{op}$-module $DS$ induces a triangle functor

$$? \otimes_S DS : \mathcal{D}(S) \to \mathcal{D}(S).$$

We have the following well-known fact (see e.g. lemma 1.2.1 of $[2]$).

Lemma 2.3. There is a non-degenerate bilinear form

$$\alpha_{X,Y} : \mathcal{D}(S)(X,Y) \times \mathcal{D}(S)(Y,X) \to k$$

which is bifunctorial for $X \in \text{per}(S)$ and $Y \in D^b(S)$. 
**Proposition 2.4.** The functor $\overset{L}{\otimes_S} DS$ restricts to auto-equivalences

$$\overset{L}{\otimes_S} DS : \mathcal{D}^b(S) \to \mathcal{D}^b(S), \overset{L}{\otimes_S} DS : \text{per}(S) \to \text{per}(S).$$

**Proof.** Since $A$ has finite global dimension, we know that $DA \in \text{per} A$. One can easily deduce that $DS \in \text{per} S$. Similarly, we have $S \in \text{tria}(DS) \subseteq \mathcal{D}(S)$. This particular implies that $DS$ is a small generator of $\mathcal{D}(S)$. It is not hard to show that

$$\mathcal{D}(S)(S, \Sigma^n S) \cong \mathcal{D}(S)(DS, \Sigma^n DS), n \in \mathbb{Z}.$$ 

Thus by Lemma 4.2 of [10], we infer that $\overset{L}{\otimes_S} DS$ is an equivalence over $\mathcal{D}(S)$. Now the functor $\overset{L}{\otimes_S} DS$ restricts to $\text{per}(S)$ follows from $\text{tria}(DS) = \text{per}(S)$. Similarly, recall that we have $\text{tria}(p_* A) = \mathcal{D}^b(S)$ and $A \overset{L}{\otimes_S} DS \cong \Sigma^{-1} DA \in \mathcal{D}^b(S)$. Now again by the finite dimension of $A$, we have $A \in \text{tria}(p_*(DA)) \subseteq \mathcal{D}^b(S)$. In particular, we have $\text{tria}(p_*(DA)) = \mathcal{D}^b(S)$. Thus, $\overset{L}{\otimes_S} DS$ restricts to an equivalence $\overset{L}{\otimes_S} DS : \mathcal{D}^b(S) \to \mathcal{D}^b(S)$. \qed

Before going to state the next result, we recall Amiot’s construction [3] of bilinear form for quotient category. Let $\mathcal{T}$ be a triangulated category and $\mathcal{N} \subset \mathcal{T}$ a thick subcategory of $\mathcal{T}$. Assume $\nu$ is an auto-equivalence of $\mathcal{T}$ such that $\nu(\mathcal{N}) \subset \mathcal{N}$. Moreover we assume that there is a non degenerate bilinear form:

$$\beta_{N,X} : \mathcal{T}(N, X) \times \mathcal{T}(X, \nu N) \to k$$

which is bifunctorial in $N \in \mathcal{N}$ and $X \in \mathcal{T}$. Let $X, Y \in \mathcal{T}$. A morphism $p : N \to X$ is called a local $\mathcal{N}$-cover of $X$ relative to $Y$ if $N$ is in $\mathcal{N}$ and it induces an exact sequence:

$$0 \to \mathcal{T}(X, Y) \overset{p^*}{\to} \mathcal{T}(N, Y).$$

The following theorem is due to Amiot [3].

**Theorem 2.5.**

1) The bilinear form $\beta$ naturally induces a bilinear form:

$$\beta'_{X,Y} : \mathcal{T}/\mathcal{N}(X, Y) \times \mathcal{T}/\mathcal{N}(Y, \nu \Sigma^{-1} X) \to k$$

which is also bifunctorial for $X, Y \in \mathcal{T}/\mathcal{N}$;

2) Assume further $\mathcal{T}$ is Hom-finite. If there exists a local $\mathcal{N}$-cover of $X$ relative to $Y$ and a local $\mathcal{N}$-cover of $\nu Y$ relative to $X$, then the bilinear form $\beta'_{X,Y}$ is non-degenerate.

Recall that $\mathcal{R}_A = \mathcal{D}^b(S)/\text{per}(S)$. Now we have the following

**Proposition 2.6.**

1) The bilinear form $\alpha$ induces a bifunctorial bilinear form $\alpha'$:

$$\alpha'_{X,Y} : \mathcal{R}_A(X, Y) \times \mathcal{R}_A(Y, \Sigma^{-1} X \overset{L}{\otimes_S} DS) \to k;$$

2) The bilinear form $\alpha'$ is non-degenerate over $\mathcal{R}_A$.

**Proof.** The first statement follows from lemma [2,3], proposition [2,4] and theorem [2,5] directly.

Let $P_A = \text{Tot}(\cdots \to \Sigma^n S \to \Sigma^{n-1} S \to \cdots \to \Sigma S \to S \to 0 \to \cdots)$, i.e. $P_A$ is the projective resolution of $S$-module $A$. Then one can easily see that $\mathcal{D}^b(S)(A, \Sigma^m A)$ is finite dimension over $k$ for any $m \in \mathbb{Z}$. In particular, we have

$$\mathcal{D}^b(S)(A, \Sigma^{2m} A) \cong A \text{ and } \mathcal{D}^b(S)(A, \Sigma^{2m+1} A) = 0,$$

for $m \geq 0$ and $\mathcal{D}^b(S)(A, \Sigma^m A) = 0$ for $m < 0$. Since $p_* A$ generates the category $\mathcal{D}^b(S)$, which implies that $\mathcal{D}^b(S)$ is Hom-finite, i.e. for any $X, Y \in \mathcal{D}^b(S)$, we have
\[ \dim_k \mathcal{D}^b(S)(X, Y) < \infty. \] Since the non-degeneracy is extension closed, it suffices to show that \( \alpha'_{\Sigma^2 A, \Sigma^n A} \) is non-degenerate. Equivalently, it suffices to show that \( \alpha'_{\Sigma^2 A, \Sigma^n A} \) is non-degenerate for any \( n \in \mathbb{Z} \). By Theorem 2.4, it suffices to show that there exists a local \( \text{per} \ S \)-cover of \( A \) relative to \( \Sigma^n A \) and a local \( \text{per} \ S \)-cover of \( \Sigma^n A \) relative to \( A \). For \( n < 0 \), since \( \mathcal{D}^b(S)(A, \Sigma^n A) = 0 \), one can take \( p : S \to A \) be the local \( \text{per} \ S \) of \( A \) relative to \( \Sigma^n A \).

Suppose that \( n \geq 0 \). Let
\[
P_{A, \Sigma^n A} := \operatorname{Tot}(\cdots \to 0 \to \Sigma^n S \to \Sigma^{n-1} S \to \cdots \to \Sigma S \to S \to 0 \to \cdots).
\]
Clearly \( P_{A, \Sigma^n A} \in \text{per} \ S \). One can easily to see that \( p : P_{A, \Sigma^n A} \to A \) is a local \( \text{per} \ S \)-cover of \( A \) relative to \( \Sigma^n A \). Note that \( A \otimes_S DS \cong \Sigma^{-1} DA \). A local \( \text{per} \ S \)-cover of \( \Sigma^n A \) relative to \( \Sigma^{-1} DA \) is equivalent to a local \( \text{per} \ S \)-cover of \( A \) relative to \( \Sigma^{-n-1} DA \). If \( n \geq 0 \), one can easily show that \( \mathcal{D}^b(S)(A, \Sigma^{-n-1} DA) = 0 \). Suppose that \( n < 0 \). One can show that
\[
P_{A, \Sigma^{-n-1} DA} := \operatorname{Tot}(\cdots \to 0 \to \Sigma^{-n-1} S \to \Sigma^{-n-2} S \to \cdots \to \Sigma S \to S \to 0 \to \cdots) \to A
\]
is a local \( \text{per} \ S \)-cover of \( A \) relative to \( \Sigma^{-n-1} DA \).

**Theorem 2.7.** The root category \( \mathcal{R}_A \) admits Auslander-Reiten triangles.

**Proof.** By proposition 2.6, we know that \( \Sigma^{-1} \otimes_S DS \) is the Serre functor of \( \mathcal{R}_A = \mathcal{D}^b(S)/\text{per} \ S \). Now the result follows form Reiten-Van den Bergh’s result in [20]. \( \square \)

2.5. **The Grothendieck group of \( \mathcal{R}_A \).** Suppose that the algebra \( A \) has \( n \) non-isomorphic simple modules, say \( S_1, \ldots, S_n \) and \( P_1, \ldots, P_n \) the corresponding projective covers. Since \( \mathcal{S} \) is the trivial extension of \( A \) with non-standard gradation, \( S_1, \ldots, S_n \) are also non-isomorphic simple modules for \( \mathcal{S} \). By the existence of \( t \)-structure over \( \mathcal{D}^b(S) \), it is not hard to see that the Grothendieck group \( G_0(\mathcal{D}^b(S)) \) of \( \mathcal{D}^b(S) \) is isomorphic to \( \mathbb{Z}[S_1] + \cdots + \mathbb{Z}[S_n] \).

Indeed, consider the inclusion algebra homomorphism \( i : A \to \mathcal{S} \), which induces a triangle functor \( i_* : \mathcal{D}^b(S) \to \mathcal{D}^b(\text{mod} A) \). Compose with the functor \( p_* : \mathcal{D}^b(\text{mod} A) \to \mathcal{D}^b(S) \), the result follows from that \( G_0(\mathcal{D}^b(\text{mod} A)) \cong \mathbb{Z}^n \).

We have the following exact sequence of triangulated categories
\[
\text{per} \ S \to \mathcal{D}^b(S) \to \mathcal{R}_A,
\]
which induces an exact sequence of Grothendieck groups
\[
G_0(\text{per} \ S) \overset{i^*}{\to} G_0(\mathcal{D}^b(S)) \overset{\phi}{\to} G_0(\mathcal{R}_A) \to 0.
\]
In particular, we have \( G_0(\mathcal{R}_A) \cong G_0(\mathcal{D}^b(S))/\im i_* \). Let \( \tilde{P}_i = P_i \oplus \Sigma P_i \). It is not hard to see that \( \tilde{P}_i \) are all the indecomposable projective objects in \( \mathcal{D}(\mathcal{S}) \) up to shifts. Note that \( \mathcal{S} \) is negative, as remark in [10], each compact object is an extension of direct sum of \( \Sigma^n \tilde{P}_i, n \in \mathbb{Z} \). In particular, in the Grothendieck group \( G_0(\text{per} \ S) \), for any \( X \in \text{per} \ S \), \([X]\) is a finite sum of \([P_i]\), \( i = 1, \ldots, n \). It is easy to see that \( i^*([P_i]) = 0 \in G_0(\mathcal{D}^b(S)) \). Thus, the image of \( i^* \) is zero and the induced linear map \( \phi : G_0(\mathcal{D}^b(S)) \to G_0(\mathcal{R}_A) \) is an isomorphism.

Compose \( \phi \) with \( p^* : G_0(\mathcal{D}^b(\text{mod} A)) \to G_0(\mathcal{D}^b(S)) \) induced by \( p_* : \mathcal{D}^b(\text{mod} A) \to \mathcal{D}^b(S) \), which is exactly the induced map \( \pi_A^* : G_0(\mathcal{D}^b(\text{mod} A)) \to G_0(\mathcal{R}_A) \) by the canonical functor \( \pi_A : \mathcal{D}^b(\text{mod} A) \to \mathcal{R}_A \). In particular, \( \pi_A^* \) is an linear isomorphism.

Define the Euler bilinear form \( \chi_{\mathcal{R}_A}(\cdot, \cdot) \) on \( G_0(\mathcal{R}_A) \) by
\[
\chi_{\mathcal{R}_A}([X], [Y]) = \dim_k \mathcal{R}_A(X, Y) - \dim_k \mathcal{R}_A(X, \Sigma Y).
\]
for any $X, Y \in \mathcal{R}_A$. Clearly, it is well-defined due to the 2-periodic property of $\mathcal{R}_A$. Let $\chi_A(\cdot, \cdot)$ be the Euler bilinear form over $\mathcal{D}^b(\text{mod } A)$, i.e.
\[
\chi_A([X], [Y]) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathcal{D}^b(\text{mod } A)(X, \Sigma^i Y)
\]
for any $X, Y \in \mathcal{D}^b(\text{mod } A)$. Since $G_0(\mathcal{R}_A)$ is generated by $[\pi_A S_i], i = 1, \ldots, n$, where $S_i$ are simple $A$-modules, we have $\chi_{\mathcal{R}_A}([\pi_A S_i], [\pi_A S_j]) = \chi_A([S_i], [S_j])$ for any $1 \leq i, j \leq n$. Thus the symmetric bilinear form $(-,-)$ over $G_0(\mathcal{R}_A)$ given by
\[
([X],[Y]) = \chi([X],[Y]) + \chi([Y],[X])
\]
is the same over $G_0(\mathcal{D}^b(\text{mod } A))$ via the isomorphism $\pi_A$.

In particular, we have proved the following

**Proposition 2.8.** The canonical functor $\pi_A : \mathcal{D}^b(\text{mod } A) \to \mathcal{R}_A$ induces an isometry $\pi_A^* : G_0(\mathcal{D}^b(\text{mod } A)) \to G_0(\mathcal{R}_A)$, i.e. a linear map such that $\chi_A(x, y) = \chi_{\mathcal{R}_A}(\pi_A^* x, \pi_A^* y)$ for any $x, y \in G_0(\mathcal{D}^b(\text{mod } A))$.

**Remark 2.9.** If $A$ is finite-dimensional hereditary $k$-algebra. We have $\mathcal{R}_A \cong \mathcal{D}^b(\text{mod } A)/\Sigma^2$, this shows that $G_0(\mathcal{D}^b(\text{mod } A)/\Sigma^2) \cong G_0(\mathcal{D}^b(\text{mod } A))$. Let $\pi : \mathcal{D}^b(\text{mod } A) \to \mathcal{R}_A$ which is dense in this case. Let $G_0(\mathcal{R}_A)$ be the Grothendieck group of $\mathcal{R}_A$ induced by the triangles of image $\pi$. We have $G_0(\mathcal{R}_A) \cong G_0(\mathcal{R}_A)$.

Let $c_{\mathcal{R}_A}$ be the automorphism of $G_0(\mathcal{R}_A)$ induced by the Auslander-Reiten translation of $\mathcal{R}_A$. Let $c_A$ be the automorphism of $G_0(\mathcal{D}^b(\text{mod } A))$ induced by the AR translation of $\mathcal{D}^b(\text{mod } A)$.

**Proposition 2.10.** $c_A$ identifies with $c_{\mathcal{R}_A}$ via the isomorphism $\pi_A^* : G_0(\mathcal{D}^b(\text{mod } A)) \to G_0(\mathcal{R}_A)$.

**Proof.** Let $P_i, i = 1, \ldots, n$ be the non-isomorphic indecomposable projective modules of $A$. Since $A$ has finite global dimension, we know that $[P_i], i = 1, \ldots, n$ also form a basis of $G_0(\mathcal{D}^b(\text{mod } A))$. Via the isomorphism $\pi_A^* : [\pi_A P_i]$ is also a basis of $G_0(\mathcal{R}_A)$. Thus, it suffices to show that $c_{\mathcal{R}_A}(\pi_A P_i)$ coincides with $\pi_A^* c_A(P_i), i = 1, \ldots, n$. By the definition of $c_{\mathcal{R}_A}$ and $c_A$, we have
\[
c_{\mathcal{R}_A}([P_i]) = [\Sigma^{-1} P_i \otimes_A D A], \quad c_A([\pi_A P_i]) = [\Sigma^{-2} P_i \otimes_S D S],
\]
One can easily check that $[\Sigma^{-1} P_i \otimes_A D A] = [\Sigma^{-2} P_i \otimes_S D S]$ in $G_0(\mathcal{R}_A)$.

3. Motivating Examples

3.1. 14 exceptional unimodular singularities. Inspired by the theory that the universal deformation and simultaneous resolution of a simple singularity are described by the corresponding simple Lie algebras [5] K. Saito associated in [21], a generalization of root system to any regular weight systems [22], and asks to construct a suitable Lie theory in order to reconstruct the primitive forms for the singularities. This is well-done for simple singularities and simple elliptic singularities. But, in general, it is not clear how to construct a suitable Lie theory even for 14 exceptional unimodular singularities.

Based on the duality theory of the weight systems [23] and the homological mirror symmetry, Kajiura-Saito-Takahashi [12] (Takahashi [27]) propose to study the triangulated category $HMFM^A(f_W)$ of matrix factorizations of the homogenous polynomial $f_W$ associated to a simple singularity $W$, then the root system appears as the set of the isomorphism classes of the exceptional objects via the Grothendieck group of the triangulated category.
This approach has been generalized to the case of regular weight systems with smallest exponent $\epsilon = -1$ in [13] which includes the 14 exceptional unimodular singularities. In [12], the authors show that the category $\text{HMF}_A^W(f_W)$ is triangulated equivalent to the bounded derived category of finitely generated modules over the path algebra of the corresponding $ADE$-type. In [13], they show that the category $\text{HMF}_A^W(f_W)$ is triangulated equivalent to the bounded derived category of finitely generated modules of certain finite-dimensional algebras $A_W$. Moreover, the Grothendieck groups of these triangulated categories characterize the strange duality for the 14 exceptional unimodular singularities. In his survey article [24], K. Saito proposes three methods to construct Lie algebras for each exceptional singularity and asks which Lie algebra satisfies some extra requirements, for more details see [24]:

1) the Lie algebra defined by the Chevalley generators and generalized Serre relations for the Cartan matrix associated to algebra $A_W$;
2) the Lie subalgebra comes from vertex operator algebra for the Grothendieck group $K_0(\mathcal{D}^b(\text{mod } A_W))$;
3) the algebra constructed by Ringel-Hall construction for the derived category of $\mathcal{D}^b(\text{mod } A_W)$.

We remark that for the simple singularities which are self-dual, if one consider the Lie algebra iii) in the sense of Peng-Xiao [18] for the root category, then these three Lie algebras are isomorphic to each other. For the case of $\epsilon = -1$, we remark that the algebra $A_W$ has global dimension 2 and it is not derived equivalent to any hereditary category. It is not clear that whether the 2-periodic orbit category $\mathcal{D}^b(\text{mod } A_W)/\Sigma^2$ is triangulated or not. Now the root category $\mathcal{R}_{A_W}$ seems to be a suitable consideration for the Lie algebra iii). Then Peng-Xiao’s Theorem [18] (see also [30]) implies that there is a Lie algebra $\mathcal{H}(\mathcal{R}_{A_W})$ associated with $\mathcal{R}_{A_W}$. We remark that we do not know whether the Grothendieck group of $\mathcal{R}_A$ is proper or not. In this case, the automorphism $c_{\mathcal{R}_{A_W}}$ has finite order $h_W$, where $h_W$ is the order of the Milnor monodromy of the corresponding singularity $W$.

### 3.2. An algebra of global dimension 2

Let $Q$ be the following quiver

$$
\begin{align*}
\begin{array}{ccc}
3 & \alpha_1 & 2 \\
\beta_1 & \beta_2
\end{array}
\end{align*}
$$

Let $I$ be the ideal generated by the relations $\beta_i \circ \alpha_i = 0$. Let $A = k/Q/I$ be the quotient algebra. The global dimension of $A$ is 2. Let $S_i, i = 1, 2, 3,$ be the non-isomorphic simple modules of $A$. Consider the Euler symmetric bilinear form of $G_0(\mathcal{R}_A)$ given by

$$
([X][Y]) = \chi(X,Y) + \chi(Y,X),
$$

for any $X, Y \in \mathcal{R}_A$. One can check that $([S_1][S_2]) = -2, ([S_1][S_3]) = 2, ([S_2][S_3]) = -2$. The bilinear form $(-,-)_{G_0(\mathcal{R}_A)}$ is degenerate over $G_0(\mathcal{R}_A)$, one extends $G_0(\mathcal{R}_A)$ to $\mathcal{L}$ such that the bilinear form $(-,-)_{\mathcal{L}}$ over $\mathcal{L}$ is non-degenerate, i.e. the restriction $(-,-)_{\mathcal{L}|_{G_0(\mathcal{R}_A)}}$ coincides with $(-,-)_{G_0(\mathcal{R}_A)}$.

Let $V_L$ be the lattice vertex operator algebra associated to $\mathcal{L}$. If we consider the Lie algebra $\mathfrak{g}_A$ generated by the vertex operators $e^{\pm[S_i]}$ in the Lie algebra $V_L/DC_L$, where $D$ is the derivative operator, then $\mathfrak{g}_A$ is isomorphic to the elliptic algebra [31] of type $A_1^{(1,1)}$ and also isomorphic to the toroidal algebra [19] of $\mathfrak{sl}_2$.

Now consider the root category $\mathcal{R}_A$ of $A$, there is a Lie algebra $\mathcal{H}(\mathcal{R}_A)(\text{Ringel-Hall Lie algebra})$ associated with $\mathcal{R}_A$. We would like to know what is the relation between $\mathcal{H}(\mathcal{R}_A)$ and the elliptic algebra $A_1^{(1,1)}$. At this moment, we can only show that $u_{[S_1]}, u_{[S_2]}, u_{[S_3]}$ satisfy the GIM Lie algebra relations [26].
We also remark that up to now, there is not any triangulated category to realize the elliptic algebras of type $A$ and $D$ (except for $D_{4}^{(1,1)}$) via the approach of Ringel-Hall Lie algebras. Similar to the above example, by triangular extension of algebras, for any elliptic Lie algebra, one can construct a 2-periodic triangulated category (possibly not unique) such that the Grothendieck group with the symmetric Euler bilinear form characterizes the root lattice for the corresponding elliptic Lie algebra.

3.3. A minimal example. Let $Q$ be the following quiver with relation

\[
\begin{array}{c}
\alpha \\
2 \\
\beta \\
1
\end{array}
\]

where $\beta \circ \alpha = 0$. Let $A$ be the quotient algebra of path algebra $kQ$ by the ideal generated by $\beta \circ \alpha$. Then $A$ is representation-finite and has global dimension 2. Let $D^b(\text{mod} A)$ be the bounded derived category of finitely generated right $A$-modules. Let $\mathcal{A}$ be the dg enhance of bounded complexes of finite generated projective $A$-modules. Let $\Sigma^2 : A \to A$ be the dg enhance of the square of suspension functor of $D^b(\text{mod} A)$. Let $B$ be the dg orbit category of $A$ respect to $\Sigma^2$ (cf. section 2.3). The canonical dg functor $\pi : A \to B$ yields a $B \otimes_k A^{\text{op}}$-module

\[(B, A) \to B(B, \pi A),\]

which induce the standard functors

\[
\mathcal{D}(A) \xrightarrow{\pi_*} \mathcal{D}(B).
\]

Note that $\mathcal{R}_A$ is the triangulated subcategory of $\mathcal{D}(B)$ generated by the representable functors. We also have a triangle equivalence $F : \mathcal{D}(\text{Mod} A) \to \mathcal{D}(A)$. Now the composition

\[
D^b(\text{mod} A) \hookrightarrow D(\text{Mod} A) \xrightarrow{F} D(A) \xrightarrow{\pi_*} D(B)
\]

gives the canonical functor $\pi_* : D^b(\text{mod} A) \to \mathcal{R}_A$.

**Proposition 3.1.** The canonical function $\pi_* : D^b(\text{mod} A) \to \mathcal{R}_A$ is not dense.

**Proof.** We will construct an object in $\mathcal{R}_A$ which is not in the image of $\pi_*$. Let $S_i$, $i = 1, 2$ be the simple $A$-modules associated to the vertices $i$ and $P_i$, $i = 1, 2$ be the corresponding indecomposable projective modules. Let $l : P_2 \to P_1$ be the embedding and $\gamma : P_1 \to S_1 \hookrightarrow P_2$. Let $X$ be the complex $\cdots \to 0 \to P_2 \xrightarrow{(l,0)} P_1 \oplus P_2 \xrightarrow{(0,1)^t} P_1 \to 0 \cdots$, where $P_1 \oplus P_2$ is in the 0-th component. Let $Y$ be the complex $\cdots \to 0 \to 0 \to P_2 \xrightarrow{0} P_2 \to 0 \cdots$, where the left $P_2$ is in the 0-th component. Let $f \in \text{Hom}_{D^b(\text{mod} A)}(X, Y)$ and $g \in \text{Hom}_{D^b(\text{mod} A)}(X, \Sigma^2 Y)$ be the followings

\[
\begin{array}{cccccccc}
0 & \xrightarrow{0} & P_2 & \xrightarrow{(l,0)} & P_1 \oplus P_2 & \xrightarrow{(0,1)^t} & P_1 & \xrightarrow{0} 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{0} & 0 & \xrightarrow{(\gamma,1)^t} & P_2 & \xrightarrow{0} & P_2 & \xrightarrow{0} 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{0} & P_2 & \xrightarrow{(l,0)} & P_1 \oplus P_2 & \xrightarrow{(0,1)^t} & P_1 & \xrightarrow{0} 0 \\
\end{array}
\]
Consider the mapping cone of $\pi_*(f+g)$ in $R_A$, we claim that the mapping cone of $\pi_*(f+g)$ is not in the image of $\pi_*$. Consider the triangle

$$\pi_*(X) \xrightarrow{\pi_*(f+g)} \pi_*(Y) \to Z \to \Sigma \pi_*(X).$$

Applying the functor $\pi_\rho$, we get a triangle in $D(\text{Mod } A)$

$$\pi_\rho \pi_*(X) \xrightarrow{\pi_\rho \pi_*(f+g)} \pi_\rho \pi_*(Y) \to \pi_\rho Z \to \Sigma \pi_\rho \pi_*(X).$$

Note that for any $X \in D^b(\text{mod } A)$, we have $\pi_\rho \pi_*(X) \cong \oplus_{i \in \mathbb{Z}} \Sigma^i X$. Thus, $\pi_\rho Z$ is isomorphic to the mapping cone of the following chain map of complexes

$$\cdots \to P_1 \oplus P_2 \to P_1 \oplus P_2 \to P_1 \oplus P_2 \to P_1 \oplus P_2 \to \cdots$$

In particular, the mapping cone is

$$\cdots \to P_2 \oplus P_1 \oplus P_2 \to P_2 \oplus P_1 \oplus P_2 \to P_2 \oplus P_1 \oplus P_2 \to \cdots$$

Let $h : P_1 \to P_1$ consider the complex $P : \cdots \to P_1 \xrightarrow{h} P_1 \xrightarrow{h} P_1 \to \cdots$, one can check that

$$\cdots \to P_2 \oplus P_1 \oplus P_2 \to P_2 \oplus P_1 \oplus P_2 \to P_2 \oplus P_1 \oplus P_2 \to \cdots$$

is a quasi-isomorphism. In particular, $\pi_\rho Z$ is isomorphic to $P$ in $D(\text{Mod } A)$. If there exists $U \in D^b(\text{mod } A)$ such that $\pi_\rho(U) = Z$, then $\pi_\rho Z \cong \oplus_{i \in \mathbb{Z}} \Sigma^i U$. But one can easily show that $P$ is indecomposable in $D(\text{Mod } A)$. This completes the proof.

The above example implies that in general the orbit category $D^b(\text{mod } A)/\Sigma^2$ is not triangulated even with small global dimension. In the appendix, we propose a way to construct various examples from a known one by using recollement associates to root categories. It would be interesting to know that whether one can give an example without oriented cycles such that the orbit category $D^b(\text{mod } A)/\Sigma^2$ is not triangulated with inherited triangle structure.

4. The ADE root categories

In this section, we focus on the root categories of finite-dimensional hereditary algebras of Dynkin type. We will show that such root categories characterize the algebras up to derived equivalence.
4.1. Separation of AR-components. Let $A$ be a finite dimensional $k$-algebra with finite global dimension. Let $\pi_A : \mathcal{D}^b(\text{mod } A) \to \mathcal{R}_A$ be the canonical triangle functor. By theorem, we know that $\mathcal{R}_A$ has Serre functor, equivalently, Auslander-Reiten triangles (AR-triangles). When $\pi_A$ is dense, it is quite easy to show that $\pi_A$ preserves the AR-triangles, i.e. each AR-triangle of $\mathcal{R}_A$ comes from an AR-triangle of $\mathcal{D}^b(\text{mod } A)$ via the canonical functor $\pi_A$. In particular, the AR-quiver of $\mathcal{D}^b(\text{mod } A)$ determines the AR-quiver of $\mathcal{R}_A$. In general, we have the following

**Theorem 4.1.** Let $A$ be a finite-dimensional $k$-algebra with finite global dimension and $\pi_A : \mathcal{D}^b(\text{mod } A) \to \mathcal{R}_A$ the canonical functor. Then the functor $\pi_A$ maps AR-triangles of $\mathcal{D}^b(\text{mod } A)$ to AR-triangles of $\mathcal{R}_A$. As a consequence, there is no irreducible morphism between $\text{im } \pi_A$ and $\mathcal{R}_A \setminus \text{im } \pi_A$.

**Proof.** Recall that for arbitrary objects $X, Y \in \mathcal{D}^b(\text{mod } A)$, we have canonical isomorphism

$$\mathcal{R}_A(\pi_A(X), \pi_A(Y)) \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{D}^b(\text{mod } A)(\Sigma^2iX, Y),$$

and $\mathcal{D}^b(\text{mod } A)(\Sigma^2iX, Y)$ vanishes for all but finitely many $i$. Let $S$ and $\tilde{S}$ be the Serre functors of $\mathcal{D}^b(\text{mod } A)$ and $\mathcal{R}_A$ respectively. Firstly, we show that $\pi_A(S)(X) \cong \tilde{S}\pi_A(X)$ for any indecomposable object $X \in \mathcal{D}^b(\text{mod } A)$. Consider the functor $DR_A(?, \pi_A(S)(X))$ over $\mathcal{R}_A$, where $D = \text{Hom}_k(?, k)$ is the usual duality of $k$. We have the following canonical isomorphism

$$DR_A(\pi_A(X), \pi_A(S)(X)) \cong D(\bigoplus_{i \in \mathbb{Z}} \mathcal{D}^b(\text{mod } A)(\Sigma^2iX, S(X)))$$

$$\cong \bigoplus_{i \in \mathbb{Z}} DD^b(\text{mod } A)(\Sigma^2iX, S(X))$$

$$\cong \bigoplus_{i \in \mathbb{Z}} D^b(\text{mod } A)(X, \Sigma^2iX)$$

$$\cong \mathcal{R}_A(\pi_A(X), \pi_A(S)(X))$$

The indecomposable property implies that $\mathcal{R}_A(\pi_A(X), \pi_A(S)(X))$ is a local $k$-algebra. Let $\eta \in DR_A(\pi_A(X), \pi_A(S)(X))$ be the image of $1_{\pi_A(X)} \in \mathcal{R}_A(\pi_A(X), \pi_A(X))$ via the canonical isomorphism. Let $\eta^* : \mathcal{R}_A(\pi_A(X), ?) \to DR_A(?, \pi_A(S)(X))$ be the natural transformation corresponding to $\eta$. It is clear that $\eta^*|_{\text{im } \pi_A}$ is an isomorphism. Since $\mathcal{R}_A$ is the triangulated hull of $\text{im } \pi_A$, one deduces that $\eta^*$ is an isomorphism over $\mathcal{R}_A$. In particular, $DR_A(?, \pi_A(S)(X))$ is representable. On the other hand, the Serre functor $\tilde{S}$ implies $DR_A(?, S\pi_AX)$ is also represented by $\mathcal{R}_A(\pi_A(X), ?)$. Thus, we have $\pi_A(S)(X) \cong \tilde{S}\pi_A(X)$.

Let $\Sigma^{-1}SX \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} S(X)$ be an AR-triangle of $\mathcal{D}^b(\text{mod } A)$. Let $\pi_A(\Sigma^{-1}SX) \xrightarrow{u} W \xrightarrow{v} \pi_A(X) \xrightarrow{g} \pi_A(S)(X)$ be the AR-triangle in $\mathcal{R}_A$. Clearly, $\pi_A(f)$ is not a split monomorphism. Thus, by the definition of AR-triangle, there is a morphism $t : W \to \pi_AY$ such that $\pi_A(f) = t \circ u$. Namely, we have the following commutative diagram of triangles

$$\pi_A(\Sigma^{-1}SX) \xrightarrow{u} W \xrightarrow{v} \pi_A(X) \xrightarrow{g} \pi_A(S)(X)$$

$$\pi_A(\Sigma^{-1}SX) \xrightarrow{\pi_A(f)} \pi_A(Y) \xrightarrow{\pi_A(g)} \pi_A(X) \xrightarrow{\pi_A(h)} \pi_A(S)(X)$$

We claim that $s$ is an isomorphism. Suppose not, then $s$ is nilpotent by the indecomposable of $X$. Then $\pi_A(h) \circ s = 0$ follows form that $\Sigma^{-1}SX \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} S(X)$ is an AR-triangle,
which implies \( v = 0 \), contradiction. Thus, \( t \) is isomorphism. In particular, the image of \( \Sigma^{-1}SX \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} S(X) \) is indeed an AR-triangle of \( \mathcal{R}_A \).

Now one can easily deduce that there is no irreducible morphism between \( \text{im} \pi_A \) and \( \mathcal{R}_A \setminus \text{im} \pi_A \), which completes the proof. \( \square \)

**Remark 4.2.** Theorem 4.1 has been proved for the generalized cluster category in [4] by using different approach. We remark that one can adapt a variant proof to deduce the result for generalized cluster category. Indeed, by the 2-Calabi-Yau property of generalized cluster category, one can deduce that the Serre functor of the derived category coincides with the Serre functor of the generalized cluster category on the objects. Then one shows that the functor \( \pi_A \) preserves AR-triangles. By the universal property of root category, the Serre functor \( S : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{D}^b(\text{mod} A) \) will induce a functor \( S \) : \( \mathcal{R}_A \to \mathcal{R}_A \). It would be interesting to compare it with the Serre functor \( \tilde{S} \).

### 4.2. The ADE root categories

Let \( A \) and \( B \) be finite-dimensional \( k \)-algebras with finite global dimension. If \( A \) and \( B \) are derived equivalent, it is clear that \( \mathcal{R}_A \cong \mathcal{R}_B \). It would be interesting to characterize all the algebras which have the same root category up to triangle equivalence. In general, this question seems to be very hard. In the following we will characterize the algebras share the root category with a path algebra of Dynkin quiver. Since the derived category of Dynkin quiver is not dependent on the choice of orientation, we assume \( Q \) be the following quiver for simplicity.

\[
\begin{align*}
A_n : & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n \\
D_n : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow n \\
E_6 : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \\
E_7 : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \\
E_8 : & \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8
\end{align*}
\]

**Theorem 4.3.** Let \( A \) be a finite-dimensional \( k \)-algebra with finite global dimension. If the root category \( \mathcal{R}_A \cong \mathcal{R}_{kQ} \) for some Dynkin quiver \( Q \), then \( A \) is derived equivalent to \( kQ \).

**Proof.** Since \( Q \) is finite Dynkin quiver, the AR-quiver of \( \mathcal{D}^b(\text{mod} \ kQ) \) is connected. The canonical functor \( \pi_{kQ} : \mathcal{D}^b(\text{mod} \ kQ) \to \mathcal{R}_{kQ} \) is dense, which implies that the AR-quiver of \( \mathcal{R}_{kQ} \) is connected. By theorem 4.1 we inform that the functor \( \pi_A : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{R}_A \) is dense. In particular, any \( X \in \mathcal{R}_A \) has preimage in \( \mathcal{D}^b(\text{mod} \ A) \). Let \( P_i, i = 1, \cdots, n \) be the indecomposable projective \( kQ \)-modules. It is clear that

\[
\dim_k \mathcal{R}_{kQ}(\pi_{kQ}P_i, \pi_{kQ}P_j) \leq 1 \text{ for } i \leq j \text{ and } \mathcal{R}_{kQ}(\pi_{kQ}P_i, \pi_{kQ}P_j) = 0 \text{ for } i > j.
\]
Let $F : \mathcal{R}_{kQ} \to \mathcal{R}_A$ be the triangle equivalent functor. We claim that there is an object $M = M_1 \oplus \cdots \oplus M_n$ in $\mathcal{D}^b(\text{mod } A)$ such that
\[
\pi_A(M) = F(\pi_{kQ}(kQ)) \text{ and } \mathcal{D}^b(\text{mod } A)(M, \Sigma^t M) = 0 \text{ for } t \neq 0.
\]

Let $\{\Sigma^{2r} X_i | r \in \mathbb{Z}\}$ be the preimages of $F(\pi_{kQ}(P_i))$ in $\mathcal{D}^b(\text{mod } A)$. Let us prove this claim for case $Q = A_n$, the other cases are similar. Note that $n$ is a sink vertex, we can choose $M_n = X_n$. Since $\mathcal{R}_A(F(\pi_{kQ}(P_{n-1})), F(\pi_{kQ}(P_n))) \cong k$, there is a unique $r_{n-1} \in \mathbb{Z}$ such that $\mathcal{D}^b(\text{mod } A)(\Sigma^{2r_{n-1}} X_{n-1}, M_n) \cong k$ and $\mathcal{D}^b(\text{mod } A)(\Sigma^{2r} X_{n-1}, X_n) = 0$ for $t \neq r_{n-1}$. We can take $M_{n-1} = \Sigma^{2r_{n-1}} X_{n-1}$. Replace $M_n$ by $M_{n-1}$, one can construct $M_i$ for any $i = 1, \cdots, n$. Clearly, we have $\pi_A(M) \cong F(\pi_{kQ}kQ)$ and $\mathcal{D}^b(\text{mod } A)(M, \Sigma^{2r} M) = 0$ for $r \neq 0$. $\mathcal{D}^b(\text{mod } A)(M, \Sigma^{2r+1} M) = 0, r \in \mathbb{Z}$ follows from $\mathcal{R}_kQ(\pi_{kQ}kQ, \Sigma \pi_{kQ}kQ) = 0$. In particular, $M$ is a (partial) tilting complex of $\mathcal{D}^b(\text{mod } A)$. We have $\mathcal{D}^b(\text{mod } kQ) \cong \mathcal{D}^b(\text{mod } \text{End}_{\mathcal{D}^b(\text{mod } A)}(M)) \cong \text{tria}(M)$, where $\text{tria}(M)$ is the thick subcategory of $\mathcal{D}^b(\text{mod } A)$ contains $M$. If we can show that $\text{tria}(M) = \mathcal{D}^b(\text{mod } A)$, then we are done. Let $i : \mathcal{D}^b(\text{mod } kQ) \to \text{tria}(M) \to \mathcal{D}^b(\text{mod } A)$ be the composition. By the universal property of root category, we have the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{D}^b(\text{mod } kQ) & \xrightarrow{i} & \mathcal{D}^b(\text{mod } A) \\
\downarrow{\pi_{kQ}} & & \downarrow{\pi_A} \\
\mathcal{R}_{kQ} & \xrightarrow{\tilde{i}} & \mathcal{R}_A
\end{array}
\]
where $\tilde{i}$ is induced by the full embedding $i$. It is clear that $\tilde{i}$ is also full and faithful, thus an equivalence, which implies $i$ is dense and an equivalence. $\square$

### 4.3. Tame quiver of type $\widehat{D}$ and $\widehat{E}$
Assume $Q$ be the following quiver

\[
\widehat{D}_n : \begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & \cdots & \rightarrow & n-2 & \rightarrow & n & \rightarrow & n+1 \\
\downarrow & & & & & & & & & & & \\
3 & \rightarrow & 4 & \rightarrow & & & & & & & & & \\
\end{array}
\]

\[
\widehat{E}_6 : \begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 \\
\downarrow & & & & & & & & & & \\
4 & \rightarrow & & & & & & & & & & \\
\end{array}
\]

\[
\widehat{E}_7 : \begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8 \\
\downarrow & & & & & & & & & & & & \\
3 & \rightarrow & & & & & & & & & & & & \\
\end{array}
\]

\[
\widehat{E}_8 : \begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 8 & \rightarrow & 9 \\
\downarrow & & & & & & & & & & & & & & \\
3 & \rightarrow & & & & & & & & & & & & & & \\
\end{array}
\]

The theorem 4.3 also holds for tame quiver of type $\widehat{D}$ and $\widehat{E}$. One can adapt a variant proof of theorem 4.3.

**Proposition 4.4.** Let $A$ be a finite dimensional $k$-algebra with finite global dimension. If the root category $\mathcal{R}_A \cong \mathcal{R}_{kQ}$ for some tame quiver $Q$ of type $\widehat{DE}$, then $A$ is derived equivalent to $kQ$. 

Proof. It suffices to prove this proposition for $Q$ be the above quiver. Clearly the canonical functor $\pi_{kQ}: D^b(\text{mod }kQ) \to \mathcal{R}_{kQ}$ is dense. In this case, $D^b(\text{mod }kQ)$ is the union of preprojective component, preinjective component and tubes up to shift. If the image $\text{im }\pi_A$ intersects with preprojective (resp. preinjective) component nonempty, by theorem 4.1, every object in this component belongs to $\text{im }\pi_A$. Then one can adapt the proof of theorem 4.3. Now suppose that $\text{im }\pi_A$ intersects with both preprojective and preinjective component empty. Let $T$ be the union of $kQ$-modules in the tubes. It is clear that $T$ is a hereditary abelian subcategory of $kQ$-modules. By theorem 9.1 of [11], we know that $D^b(T)/\Sigma^2$ is triangulated and we have the following commutative diagram

$$
\begin{array}{ccc}
D^b(T) & \longrightarrow & D^b(\text{mod }kQ) \\
\downarrow \pi & & \downarrow \pi_{kQ} \\
D^b(T)/\Sigma^2 & \longrightarrow & \mathcal{R}_{kQ}
\end{array}
$$

where $\tilde{i}$ is induced by $i$. In particular, we know that $\tilde{i}$ is a full embedding. Now $\text{im }\pi_A \subset T \cup \Sigma T$ implies that $\text{im }\pi_A \subset D^b(T)/\Sigma^2$, which contradicts to $\text{tria}(\text{im }\pi_A) = \mathcal{R}_A$. □

5. Ringel-Hall Lie algebras and GIM Lie algebras

Throughout this section, let $k$ be a field with $|k| = q$. We study the Ringel-Hall Lie algebras of a class of finite-dimensional $k$-algebras with global dimension 2. Building on the representation theory of these algebras, we will give a negative answer for a question on GIM-Lie algebras by Slodowy in [26]. We remark that different counterexamples of this question have been discovered by Alpen [1] by considering fixed point subalgebras of certain Lie algebras.

5.1. Generalized intersection matrix Lie algebras. We recall the generalized intersection matrix Lie algebra (GIM-Lie algebra for short) following Slodowy [26]. A matrix $A \in M_l(\mathbb{Z})$ is called a generalized intersection matrix, or GIM for short, if the followings are satisfied

$$
\begin{align*}
A_{ii} &= 2 \\
A_{ij} < 0 &\iff A_{ji} < 0 \\
A_{ij} > 0 &\iff A_{ji} > 0
\end{align*}
$$

If moreover $A$ is symmetric, then $A$ is called an intersection matrix. Given a GIM $A \in M_l(\mathbb{Z})$, a root basis associated to $A$ is a triplet $(H, \nabla, \Delta)$ consisting of

- a finite dimensional $\mathbb{Q}$-vector space $H$;
- a family $\nabla = \{\alpha^\vee_1, \ldots, \alpha^\vee_l\}$, where $\alpha^\vee_i \in H$;
- a family $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i \in H^* = \text{Hom}_\mathbb{Q}(H, \mathbb{Q})$

satisfy the following

1) both sets $\Delta$ and $\nabla$ are linearly independent;
2) $\alpha_j(\alpha_i^\vee) = A_{ij}$ for all $1 \leq i, j \leq l$;
3) $\dim_\mathbb{Q} H = 2l - \text{rank }A$. 
The GIM-Lie algebra \( \mathfrak{g} = \text{GIM}(A) \) attached to the root basis \((H, \triangle, \Delta)\) is given by the generators \( h = H \otimes \mathbb{C} \) and \( e_{\pm \alpha}, \alpha \in \Delta \) satisfying the following relations:

1. \([h, h'] = 0, h, h' \in \mathfrak{h}\)
2. \([h, e_\alpha] = \alpha(h)e_\alpha, h \in \mathfrak{h}, \alpha \in \pm \Delta\)
3. \([e_\alpha, e_{-\alpha}] = \alpha^\vee, \alpha \in \Delta\)
4. \(ad(e_\alpha)^{\max(1,1-\beta(\alpha^\vee))}e_\beta = 0, \alpha \in \Delta, \beta \in \pm \Delta\)
5. \(ad(e_{-\alpha})^{\max(1,1-\beta(-\alpha^\vee))}e_\beta = 0, \alpha \in \Delta, \beta \in \pm \Delta\).

If \( A \) is a symmetric Cartan matrix, then the \( \text{GIM}(A) \) is essentially the Kac-Moody algebras associated to \((H, \triangle, \Delta)\).

Let \( ad : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) be the adjoint representation of \( \mathfrak{g} \). Consider the restriction of \( ad \) to \( \mathfrak{h} \), the Lie algebra \( \mathfrak{g} \) decomposes into a direct sum

\[
\mathfrak{g} = \bigoplus_{\gamma \in \mathfrak{h}^*} \mathfrak{g}_\gamma
\]

of eigenspaces

\[
\mathfrak{g} = \{ x \in \mathfrak{g} | [h, x] = \gamma(h)x \text{ for all } h \in \mathfrak{h}\}.
\]

Clearly, we have \( \mathfrak{h} \subseteq \mathfrak{g}_0 \). The following question has been addressed in \([26]\) by Slodowy: Does equality hold?

If we consider the derived subalgebra \([\mathfrak{g}, \mathfrak{g}]\) of \( \mathfrak{g} \), the above question is equivalent to the following: Do we have \( \dim_{\mathbb{C}}[\mathfrak{g}, \mathfrak{g}]_0 = l^2 \)? We remark that the derived subalgebra \([\mathfrak{g}, \mathfrak{g}]\) can be presented by generators \( \alpha_i^\vee, 1 \leq i \leq l \) and \( e_\alpha, \alpha \in \pm \Delta \) with the same relations in \( \mathfrak{g} \). In \([1]\), Alperen has given a negative answer for this question by using Lie theory. In the following, we will give a negative answer for this question via representation-theoretic approach.

5.2. The Ringel–Hall Lie algebra. We recall the definition of the Ringel–Hall Lie algebra of a 2-periodic triangulated category following \([13]\). Let \( \mathcal{R} \) be a Hom-finite \( k \)-linear triangulated category with suspension functor \( \Sigma \). By \( \text{Ind} \mathcal{R} \) we denote a set of representatives of the isoclasses of all indecomposable objects in \( \mathcal{R} \).

Given any objects \( X, Y, L \) in \( \mathcal{R} \), we define

\[
W(X, Y; L) = \{(f, g, h) \in \text{Hom}_\mathcal{R}(X, L) \times \text{Hom}_\mathcal{R}(L, Y) \times \text{Hom}_\mathcal{R}(Y, \Sigma X) \mid
X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} \Sigma X \text{ is a triangle}\}.
\]

The action of \( \text{Aut}(X) \times \text{Aut}(Y) \) on \( W(X, Y; L) \) induces the orbit space

\[
V(X, Y; L) = \{(f, g, h)^\wedge | (f, g, h) \in W(X, Y; L)\}
\]

where

\[
(f, g, h)^\wedge = \{(af, gc^{-1}, ch(\Sigma a)^{-1}) | (a, c) \in \text{Aut}(X) \times \text{Aut}(Y)\}.
\]

Let \( \text{Hom}_\mathcal{R}(X, L)^Y \) be the subset of \( \text{Hom}_\mathcal{R}(X, L) \) consisting of morphisms \( l : X \to L \) whose mapping cone \( \text{Cone}(l) \) is isomorphic to \( Y \). Consider the action of the group \( \text{Aut}(X) \) on \( \text{Hom}_\mathcal{R}(X, L)^Y \) by \( d : l = dl \), the orbit is denoted by \( l^* \) and the orbit space is denoted by \( \text{Hom}_\mathcal{R}(X, L)^Y \). Dually one can also consider the subset \( \text{Hom}_\mathcal{R}(L, Y)^X \) of \( \text{Hom}_\mathcal{R}(L, Y) \) with the group action \( \text{Aut}(Y) \) and the orbit space \( \text{Hom}_\mathcal{R}(L, Y)^X \). The following proposition is an observation due to \([25]\).

Lemma 5.1. \(|V(X, Y; L)| = |\text{Hom}_\mathcal{R}(X, L)^Y| = |\text{Hom}_\mathcal{R}(L, Y)^X|\).
We assume further that the \( R \) is 2-periodic, i.e., \( R \) is Krull–Schmidt and \( \Sigma^2 \cong 1 \).

Let \( G_0(\mathcal{R}) \) be the Grothendieck group of \( \mathcal{R} \) and \( I_R(-, -) \) the symmetric Euler form of \( \mathcal{R} \). For an object \( M \) of \( \mathcal{R} \), we denote by \([M]\) the isoclass of \( M \) and by \( h_M = \dim \mathcal{R} \) the canonical image of \([M]\) in \( G_0(\mathcal{R}) \). Let \( \mathfrak{h} \) be the subgroup of \( G_0(\mathcal{R}) \otimes \mathbb{Z} \mathbb{Q} \) generated by \( h_M \), \( M \in \text{ind} \mathcal{R} \), where \( d(M) = \dim_k(\text{End}(X)/\text{rad End}(X)) \). One can naturally extend the symmetric Euler form to \( \mathfrak{h} \times \mathfrak{h} \). Let \( n \) be the free abelian group with basis \( \{u_X | X \in \text{ind} \mathcal{R}\} \).

\[
\mathfrak{g}(\mathcal{R}) = \mathfrak{h} \oplus n,
\]
a direct sum of \( \mathbb{Z} \)-modules. Consider the quotient group

\[
\mathfrak{g}(\mathcal{R})_{(q-1)} = \mathfrak{g}(\mathcal{R})/(q-1)\mathfrak{g}(\mathcal{R}).
\]

Let \( F^i_{XY} = |V(X, Y; L)| \). Then by Peng and Xiao [18] we know that \( \mathfrak{g}(\mathcal{R})_{(q-1)} \) is a Lie algebra over \( \mathbb{Z}/(q-1)\mathbb{Z} \), called the Ringel–Hall Lie algebra of \( \mathcal{R} \). The Lie operation is defined as follows.

1. for any indecomposable objects \( X, Y \in \mathcal{R} \),
\[
[u_X, u_Y] = \sum_{L \in \text{ind} \mathcal{R}} (F^i_{YX} - F^j_{XV})u_L - \delta_{XY} \frac{h_X}{d(X)},
\]
where \( \delta_{XY} = 1 \) for \( X \cong \Sigma Y \) and 0 else.

2. \( [h_X, h_Y] = 0 \).

3. for any objects \( X, Y \in \mathcal{R} \) with \( Y \) indecomposable,
\[
[h_X, u_Y] = I_R(h_X, h_Y)u_Y, \quad [u_Y, h_X] = -[h_X, u_Y].
\]

A triangulated category \( \mathcal{T} \) is called proper, if for any nonzero indecomposable object \( X \in \mathcal{T} \), \( \dim X \neq 0 \) in the Grothendieck group \( G_0(\mathcal{T}) \). If the 2-periodic triangulated category \( \mathcal{R} \) is proper, then \( [u_X, u_{\Sigma X}] = -\frac{h_X}{d(X)} \), which coincides the origin definition in [18]. However, the proof in [18] is still valid for non-proper 2-periodic triangulated category for the Lie bracket defined above (cf. [29]).

5.3. A class of finite-dimensional \( k \)-algebras. Let \( Q \) be the following quiver

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \alpha & \beta & \gamma & \circ & n + m \\
\end{array}
\]

We assume \( m \geq 1, n \geq 2 \). Let \( A \) be the quotient algebra of path algebra \( kQ \) by the ideal generated by \( \beta \circ \alpha, \gamma \circ \alpha \). It has global dimension 2.

Let \( E \) be a field extension of \( k \) and set \( V^E = V \otimes_k E \) for any \( k \)-space \( V \). Then \( A^E \) is an \( E \)-algebra and, for \( M \in \text{mod} A \), \( M^E \) has a canonical \( A^E \)-module structure. Clearly, \( A^E \) still has global dimension 2. Let \( \mathcal{R}_{A^E} \) be the root category of \( A^E \). Thus, one has the Ringel-Hall Lie algebra \( \mathfrak{g}(\mathcal{R}_{A^E}|_{(E|-1)}) \), which is a Lie algebra over \( \mathbb{Z}/(|E| - 1)\mathbb{Z} \).

Let \( \overline{k} \) be the algebraic closure of \( k \) and set
\[
\Omega = \{E|k \subseteq E \subseteq \overline{k} \text{ is a finite field extension}\}.
\]

We consider the direct product \( \prod_{E \in \Omega} \mathfrak{g}(\mathcal{R}_{A^E}|_{(E|-1)}) \) of Lie algebras and let \( \mathcal{L}\mathfrak{g}(\mathcal{R}_A) \) be the Lie subalgebra of \( \prod_{E \in \Omega} \mathfrak{g}(\mathcal{R}_{A^E}|_{(E|-1)}) \) generated by \( u_{S_i} = (u_{S_i})_{E \in \Omega} \) and \( u_{\Sigma S_i} = (u_{\Sigma S_i})_{E \in \Omega} \) for all simple \( A \)-modules \( S_i, 0 \leq i \leq m + n \). Clearly, \( h_i = (h_{S_i^E})_{E \in \Omega}, 0 \leq i \leq n + m \) belong to \( \mathcal{L}\mathfrak{g}(\mathcal{R}_A) \). We call \( \mathcal{L}\mathfrak{g}(\mathcal{R}_A) \) the integral Ringel-Hall Lie algebra of
A. Clearly, the algebra $\mathcal{L}(\mathcal{R}_A)$ has a grading by the Grothendieck group $G_0(\mathcal{R}_A)$ of $\mathcal{R}_A$, namely,

$$\mathcal{L}(\mathcal{R}_A) = \bigoplus_{\alpha \in G_0(\mathcal{R}_A)} \mathcal{L}(\mathcal{R}_A)_{\alpha}$$

such that $\deg u_{S_i} = \dim S_i, \deg(u_{S_i}) = \dim \Sigma S_i$. In particular, $h_i \in \mathcal{L}(\mathcal{R}_A)_0$.

Let $(-, -)$ be the symmetric Euler form of $\mathcal{R}_A$ (cf. section 2.3). Then image of simple $A$-modules $[S_i], 0 \leq i \leq n + m$ form a $\mathbb{Z}$-basis of $G_0(\mathcal{R}_A)$. Define the matrix $C = (c_{ij})$, $c_{ij} = ([S_i], [S_j])$ for $0 \leq i, j \leq n + m$. One can easily show that $C$ is an intersection matrix. Let $(H, \nabla, \Delta)$ be a root basis of $C$. Thus one can form the GIM Lie algebra $\mathfrak{g}(C) = \text{GIM}(C)$ associated to $C$. We are interested in its derived subalgebra $\mathfrak{g}(C)' = [\mathfrak{g}(C), \mathfrak{g}(C)]$.

**Theorem 5.2.** There is a surjective Lie algebra homomorphism $\phi : \mathfrak{g}(C)' \to \mathcal{L}(\mathcal{R}_A) \otimes \mathbb{C}$ defined by

$$\alpha_i \mapsto h_i,$$

$$e_{\alpha_i} \mapsto u_{S_i},$$

$$e_{-\alpha_i} \mapsto -u_{S_i}, 0 \leq i \leq n + m.$$ 

Moreover, $\phi$ keeps the gradations and $\dim \mathcal{C}(\mathcal{L}(\mathcal{R}_A) \otimes \mathbb{C})_0 \geq m + n + 2$. As a consequence, we infer that $\dim \mathcal{C}(\mathcal{L}(\mathcal{R}_A) \otimes \mathbb{C})_0 \geq m + n + 2$.

The proof of this theorem carries throughout the rest of this section.

**Lemma 5.3.** Let $M$ be the unique indecomposable $A$-module with composition series $S_0, S_1, S_2, S_{n+1}$. Then $u_M = (u_{ME})_{E \in \Omega} \in \mathcal{L}(\mathcal{R}_A)$ and $0 \neq [u_M, u_{M}] \in \mathfrak{h} \otimes \mathbb{C}$, where $\mathfrak{h}$ is the subspace of $\mathcal{L}(\mathcal{R}_A)$ spanned by $h_i, 0 \leq i \leq n + m$.

**Proof.** One can easily check that $u_{ME} = [[[u_{S_0}, u_{S_1}], u_{S_2}], u_{S_{n+1}}]$ by using lemma 5.1 for any $E \in \Omega$. Thus, both $u_M, u_{M}$ belong to $\mathcal{L}(\mathcal{R}_A)$. Let $P_i$ be the indecomposable projective $A$-modules associated to each vertex $i$. Let $\to P_0 \to P_1 \to P_2 \oplus P_{n+1} \to M \to 0$ be the projective cover of $M$. We infer that $\text{Hom}_{\mathcal{R}_A}(M, M) = \text{Hom}_{\mathcal{D}^b(\text{mod} A)}(M, M) \oplus \text{Hom}_{\mathcal{D}^b(\text{mod} A)}(M, \Sigma^2 M)$. Moreover, $\dim_k \text{Hom}_{\mathcal{R}_A}(M, M) = 2$ and $\dim_k \text{rad} \text{Hom}_{\mathcal{R}_A}(M, M) = 1$.

Now consider triangles in $\mathcal{R}_A$

$$M \to L \to \Sigma M \xrightarrow{f} \Sigma M \to \Sigma^2 M,$$

we can write $f = f_0 + f_1$, where $f_0 \in \text{Hom}_{\mathcal{D}^b(\text{mod} A)}(\Sigma M, \Sigma M)$ and $f_1 \in \text{Hom}_{\mathcal{D}^b(\text{mod} A)}(\Sigma M, \Sigma^2 M)$. If $f_0 \neq 0$, then $f$ is an isomorphism and $L \cong 0$. Thus, it suffices to consider for $f_0 = 0$, i.e. $0 \neq f \in \text{rad} \text{Hom}_{\mathcal{R}_A}(M, M)$, and then the triangle $M \to L \to \Sigma M \xrightarrow{f} \Sigma M$ is induced by a triangle in $\mathcal{D}^b(\text{mod} A)$. By computing the mapping cone of $f$ in $\mathcal{D}^b(\text{mod} A)$, we infer that $L$ isomorphic to the complex $\cdots \to 0 \to P_0 \xrightarrow{(f_0, f_1)} M \oplus P_1 \to P_2 \oplus P_3 \to 0 \cdots$, where $P_2 \oplus P_3$ is the $-1$-th component. We claim that $L$ is indecomposable in $\mathcal{D}^b(\text{mod} A)$. Indeed, suppose $L \cong X \oplus Y$ in $\mathcal{D}^b(\text{mod} A)$. Then $H^*(L) \cong H^*(X) \oplus H^*(Y)$, where $H^*(-)$ be the homology groups of corresponding complex. Now the only nonzero homology groups of $L$ are $H^{-1}(L) \cong H^{-2}(L) \cong M$, which are indecomposable $A$-modules. Thus, we may assume $X \cong \Sigma M$ and $Y \cong \Sigma M$. Now in the root category $\mathcal{R}_A$, we have $\Sigma^2 M \cong M$.

In particular, we get a triangle $M \to M \oplus \Sigma M \to \Sigma M \xrightarrow{f} \Sigma M$. By a well-known fact, the triangle is split and $f = 0$, contradiction. Since $\dim_k \text{rad} \text{Hom}_{\mathcal{R}_A}(M, M) = 1$, for any $f, h \in \text{rad} \text{Hom}_{\mathcal{R}_A}(M, M)$, the mapping of $f$ and $h$ are isomorphic to each other.

Similarly, one can discuss for $g$ and show that $N$ is indecomposable if and only if $0 \neq g \in \text{Hom}_{\mathcal{D}^b(\text{mod} A)}(M, \Sigma^2 M)$. In this case, we have $N \cong \Sigma^{-1} L$. Now by the definition of
the Lie bracket, we have

\[ [u_M, u_{\Sigma M}] = -h_M + \sum_{L \in \text{ind} \mathcal{R}_A} (F^L_{\Sigma M, M} - F^L_{M, \Sigma M}) u_L \]

\[ = -h_M + F^L_{\Sigma M, M} u_L - F^{\Sigma L - 1}_M u_{\Sigma L}. \]

One can show that \( \dim_k \text{Hom}_{\mathcal{R}_A}(M, L) = 1 \). Therefore, by lemma 5.1 we have \( F^L_{\Sigma M, M} = F^{\Sigma L - 1}_M u_L = 1 \). In particular, we have \([u_M, u_{\Sigma M}] = -h_M + u_L - u_{\Sigma L} \) in \( \mathfrak{g}(\mathcal{R}_A)(q-1) \). We remark that the proof above is valid for any finite field extension of \( k \). Thus, in the integral Ringel-Hall Lie algebra \( \mathcal{L}\mathfrak{g}(\mathcal{R}_A) \), we also have \([u_M, u_{\Sigma M}] = -h_M + u_L - u_{\Sigma L} \), which implies the desired result. \( \square \)

Now we are in a position to prove the theorem 5.2

**Proof.** The relations (1)(2)(3) follows from the definition of Lie bracket of Ringel-Hall Lie algebra. It suffices to show \( u_{S_i}, u_{\Sigma S_j}, 0 \leq i, j \leq m + n \) satisfy the relations (4) and (5). We discuss for \( i, j \) in 4 cases.

Case 1: \( i, j \in \{0, 1\} \). We consider the quotient algebra \( B = A/A(e_2 + e_3 + \cdots + e_{n+m}) A \), where \( e_i \) is the idempotent associated to the vertex \( i \). Note that \( B \) is projective as a right \( A \)-module. Then the derived functor \( F = -\otimes_B A \mathcal{B} : \mathcal{D}^b(\text{mod} B) \to \mathcal{D}^b(\text{mod} A) \) is an embedding by theorem 3.1 in [7]. Now, lemma 5.1 implies the induced functor \( \overline{F} : \mathcal{R}_B \to \mathcal{R}_A \) is also fully faithful. In particular, we have a injective algebra homomorphism \( \mathcal{L}\mathfrak{g}(\mathcal{R}_B) \to \mathcal{L}\mathfrak{g}(\mathcal{R}_A) \). Moreover, we can identify the simple \( B \)-modules with simple \( A \)-modules via the functor \( F \). Thus, to check the a relations for \( \mathcal{L}\mathfrak{g}(\mathcal{R}_A) \) involve 0 \( \leq i, j \leq 1 \), it suffices to check it in \( \mathcal{L}\mathfrak{g}(\mathcal{R}_B) \). Note that the algebra \( B \) is hereditary of type \( A_1 \), we infer that \( \mathcal{L}\mathfrak{g}(\mathcal{R}_B) \otimes \mathbb{C} \) is isomorphic to the affine Kac-Moody algebra of type \( \tilde{A}_1 \) by the main theorem of [18], which implies relations (4) and (5) hold.

Case 2: \( i, j \in \{1, 2, \cdots, n+m\} \). Let \( B = A/Ae_0 A \). It is easy to see that \( \text{Ext}_A(B_A, B_A) = 0 \).

Again by theorem 3.1 [7], we have \( F = -\otimes_B A \mathcal{B} : \mathcal{D}^b(\text{mod} B) \to \mathcal{D}^b(\text{mod} A) \) is an embedding. Note that in this case \( B \) is hereditary of Dynkin type \( A_{m+n} \). Thus, the integral Ringel-Hall algebra \( \mathcal{L}\mathfrak{g}(\mathcal{R}_B) \otimes \mathbb{C} \) is isomorphic to simple Lie algebra of type \( A_{m+n} \). Now the result follows from the proof of case 1.

Case 3: \( i = 0, j \neq 1, 2, n + 1 \). In particular, by the definition of Lie bracket we only need to show that \([u_{S_i}, u_{S_j}] = 0 \) and \([u_{S_j}, u_{\Sigma S_j}] = 0 \). This follows from the fact that \( S_j \) has projective dimension 2 and the projective resolution of \( S_j \) does not involve \( P_0 \).

Case 4: \( i, j \in \{0, 2, n+1\} \). For the case \( i = 0, j = 2 \), we consider the quotient algebra \( B = A/A(e_3 + \cdots + e_{m+n}) A \), which turns out to be a tilted algebra of tame hereditary algebra of type \( \tilde{A}_2 \). Thus the integral Ringel-Hall algebra \( \mathcal{L}\mathfrak{g}(\mathcal{R}_B) \otimes \mathbb{C} \) is isomorphic to the Kac-Moody algebra of type \( \tilde{A}_2 \). Now the result follows from the proof of case 1, since we still have full embeddings \( F : \mathcal{D}^b(\text{mod} B) \to \mathcal{D}^b(\text{mod} A) \) and \( \overline{F} : \mathcal{R}_B \to \mathcal{R}_A \). For the case \( i = 0, j = n + 1 \), one considers the quotient algebra \( B = A/A(e_2 + \cdots + e_n + e_{n+2} + \cdots + e_{n+m}) A \).

Thus \( \phi \) is indeed a algebra homomorphism. It is obviously surjective and keeps the gradation. Clearly, \( h_i = (h_{S_i})_{E \mathcal{S}_0} \) is linearly independent in \( (\mathcal{L}\mathfrak{g}(\mathcal{R}_A) \otimes \mathbb{C})_0 \). By lemma 5.3, we infer that \( u_M - u_{\Sigma M} \in (\mathcal{L}\mathfrak{g}(\mathcal{R}_A) \otimes \mathbb{C})_0 \), which is linearly independent to \( h_0, h_1, \cdots, h_{m+n} \). Thus, \( \dim_\mathbb{C} (\mathcal{L}\mathfrak{g}(\mathcal{R}_A) \otimes \mathbb{C})_0 \geq m + n + 2 \). This completes the proof. \( \square \)

**Remark 5.4.** Firstly, theorem 5.2 essentially give a negative answer to Slodowy’s question. If the equality holds for \( \mathfrak{g} = GIM(C) \), i.e. \( \dim_\mathbb{C} \mathfrak{g}_0 = m + n + 2 \), then for the derived
subalgebra $g' = [g, g]$, we must have $\dim_{\mathbb{C}} g'_0 = m + n + 1$. In fact, following the proof of lemma 5.3, one can even show that $\dim_{\mathbb{C}} (\mathcal{L}g(\mathcal{R}_A) \otimes \mathbb{Z})_0 \geq (m + 1)n + 1$. Secondly, by lemma 5.3, we know that $(\dim M, \dim M) = 4$ and $u_M \in \mathcal{L}g(\mathcal{R}_A)$. In particular, this also shows that the GIM Lie algebra $g$ has root with length greater than 2. Thirdly, one can easily see that the root basis of GIM Lie algebra is never isomorphic to an affine Kac-Moody algebra of type $\tilde{A}_{m+n}$. By theorem 5.2, we know that GIM $(C)$ is never isomorphic to an affine Kac-Moody algebra of type $\tilde{A}_{m+n}$, this also show that the GIM Lie algebras are not invariant under braid equivalent in general.

**Appendix A. Recollement lives in root categories**

In the appendix, we show that a recollement of bounded derived categories lives in the corresponding root categories under suitable assumption. This can be use to construct various algebras inductively such that the 2-periodic orbit category is not triangulated with the inherited triangle structure from the bounded derived category.

**A.1. Derived category of dg category.** Let $\mathcal{A}$ be a small differential graded (dg) $k$-category. We identify a dg $k$-algebra with a dg category with one object. Let $\text{Diff}\mathcal{A}$ be the dg category of right dg $\mathcal{A}$-modules. A dg $\mathcal{A}$-module $P$ is called $K$-projective if $\text{Diff}\mathcal{A}(P, ?)$ preserves acyclicity. For any dg category $\mathcal{B}$, let $\mathcal{Z}_0(\mathcal{B})$ be the category with the same objects of $\mathcal{A}$ whose Hom-space is given by

$$\mathcal{Z}_0(\mathcal{B})(X, Y) = Z^0(B(X, Y)),$$

i.e. the 0th cocycle of dg $k$-module $B(X, Y)$. Let $\mathcal{H}^0(\mathcal{B})$ be the category with the same objects of $\mathcal{B}$ whose Hom-space is given by

$$\mathcal{H}^0(\mathcal{B})(X, Y) = H^0(B(X, Y)),$$

i.e. the 0th homology of dg $k$-module $B(X, Y)$. For the dg category $\text{Diff}\mathcal{A}$, we define $\mathcal{C}(\mathcal{A}) := \mathcal{Z}_0(\text{Diff}\mathcal{A})$ and $\mathcal{H}(\mathcal{A}) := \mathcal{H}^0(\text{Diff}\mathcal{A})$. A morphism $L \to N$ in $\mathcal{C}(\mathcal{A})$ is called quasi-isomorphism if it induces an isomorphism in homology. Let $\mathcal{D}(\mathcal{A})$ be the derived category of $\mathcal{A}$, i.e. the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphism. A dg $\mathcal{A}$-module $L$ is called compact if $\mathcal{D}(\mathcal{A})(L, ?)$ commutes with arbitrary direct sums. For instance, the projective $\mathcal{A}$-module $\mathcal{A}(?, A), A \in \mathcal{A}$ is both $K$-projective and compact. Let $\text{per}(\mathcal{A})$ be the perfect derived category of $\mathcal{A}$, i.e. the smallest subcategory of $\mathcal{D}(\mathcal{A})$ contains $\mathcal{A}$ and stable under shift,extensions and passage to direct factors. For any subcategory $\mathcal{M} \subseteq \mathcal{D}(\mathcal{A})$, let $\text{tri}(\mathcal{M})$ be the thick subcategory of $\mathcal{D}(\mathcal{A})$ contains $\mathcal{M}$.

Let $X$ be a dg $B \otimes_k \mathcal{A}^{op}$-module. It gives rise to a pair of adjoint dg functors

$$\text{Diff} \mathcal{A} \xrightarrow{T_X} \text{Diff} B.$$

Assume $X$ is $K$-projective as $B \otimes_k \mathcal{A}^{op}$-module, then $(T_X, H_X)$ induces an adjoint pair triangle functors $(LT_X, RH_X)$ over the derived categories, where $LT_X$ is the left derived functor of $T_X$. If both $\mathcal{A}$ and $B$ are dg $k$-algebras, we also write $? \otimes \mathcal{A} X_B$ for $LT_X$.

**A.2. The universal property of root category.** Let $A$ and $B$ be finite-dimensional $k$-algebras with finite global dimension. Let $L : \mathcal{D}^b(\text{mod} A) \to \mathcal{D}^b(\text{mod} B)$ be a standard functor, i.e. $F \cong ? \otimes \mathcal{A} X_B$ for some complex of $A^{op} \otimes_k B$-module. Since for any triangle functor $L : \mathcal{D}^b(\text{mod} A) \to \mathcal{D}^b(\text{mod} B)$, we have $L \circ \Sigma^2_A \cong \Sigma^2_B \circ L$. By the universal
property of dg orbit category (cf. section 9.4 in [II]), $F$ naturally induces a triangle functor $\overline{F} : \mathcal{R}_A \to \mathcal{R}_B$ and we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}^b(\text{mod } A) & \xrightarrow{F} & \mathcal{D}^b(\text{mod } B) \\
\pi_A \downarrow & & \downarrow \pi_B \\
\mathcal{R}_A & \xrightarrow{\tau} & \mathcal{R}_B
\end{array}
$$

where $\pi_A, \pi_B$ are the canonical functors. In the following, we will study the induced functor $\overline{F}$ explicitly.

We may assume $A X_B$ is $K$-projective as $A^{op} \otimes_k B$-module. Clearly, $X$ has finite total homology. Moreover, $A X_B$ is compact as left $A$-module and right $B$-module respectively due to the fact $A$ and $B$ have finite global dimension. Then we have the canonical isomorphism $R\text{Hom}_B(A X_B, ?) \cong L \otimes_B R\text{Hom}_B(A X_B, B)_A$. Let $B Y_A \to B R\text{Hom}_B(A X_B, B)_A$ be a $K$-projective resolution of $B R\text{Hom}_B(A X_B, B)_A$ as $B^{op} \otimes_k A$-module. Thus, the right adjoint $G$ of $F$ naturally isomorphic to $\tilde{\otimes}_B Y_A$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the dg category of bounded complexes of finitely generated projective $A$-modules and $B$-modules respectively. The tensor product by $X$ and $Y$ define dg functors $\otimes_A X : \mathcal{A} \to B$ and $\otimes_B Y : B \to \mathcal{A}$. By abuse of notation, we denote these dg functors by $F$ and $G$ as well. Similarly, one can lift the square of the shift functors $\Sigma^2_A : \mathcal{D}^b(\text{mod } A) \to \mathcal{D}^b(\text{mod } A)$ and $\Sigma^2_B : \mathcal{D}^b(\text{mod } B) \to \mathcal{D}^b(\text{mod } B)$ to dg functors $\Sigma^2_A : \mathcal{A} \to \mathcal{A}$ and $\Sigma^2_B : \mathcal{B} \to \mathcal{B}$.

Let $\mathcal{R}_A$ be the dg orbit category (cf. section 5 of [II]) of $\mathcal{A}$ respects to $\Sigma^2_A$. Let $\mathcal{R}_B$ be the dg orbit category of $\mathcal{B}$ respects to $\Sigma^2_B$. We have canonical dg functors $\pi_A : \mathcal{A} \to \mathcal{R}_A$ and $\pi_B : \mathcal{B} \to \mathcal{R}_B$. We have natural isomorphisms $\Sigma^2_B \circ F \cong F \circ \Sigma^2_A$ and $\Sigma^2_A \circ G \cong G \circ \Sigma^2_B$ of dg functors. Thus, by the universal property of dg orbit categories, $F$ and $G$ induce dg functors $\overline{F} : \mathcal{R}_A \to \mathcal{R}_B$ and $\overline{G} : \mathcal{R}_B \to \mathcal{R}_A$. Clearly, $\overline{F}$ yields a $\mathcal{R}_B \otimes_k \mathcal{R}_A^{op}$-bimodule $X_{\overline{F}}$

$$X_{\overline{F}}(B, A) \mapsto \mathcal{R}_B(B, \overline{F}(A)).$$

Similarly, $\overline{G}$ induces an $\mathcal{R}_A \otimes_k \mathcal{R}_B^{op}$-bimodule $Y_{\overline{G}}$

$$Y_{\overline{G}}(A, B) \mapsto \mathcal{R}_A(A, \overline{G}(B)).$$

Let $L T_{X_{\overline{F}}} : \mathcal{D}(\mathcal{R}_A) \to \mathcal{D}(\mathcal{R}_B)$ be the derived tensor functor of $X_{\overline{F}}$. Let $L T_{Y_{\overline{G}}} : \mathcal{D}(\mathcal{R}_B) \to \mathcal{D}(\mathcal{R}_A)$ be the derived tensor functor of $Y_{\overline{G}}$. In the following, we identify the objects of $\mathcal{A}$ with $\mathcal{R}_A$ and the objects of $\mathcal{B}$ with $\mathcal{R}_B$ respectively.

**Lemma A.1.** $L T_{X_{\overline{F}}}$ is left adjoint to $L T_{Y_{\overline{G}}}$.  

**Proof.** Clearly, $X_{\overline{F}}$ is $K$-projective for any $A \in \mathcal{A}$ and $L T_{X_{\overline{F}}}$ is left adjoint to $R H_{X_{\overline{F}}}$. It suffices to show that $L T_{Y_{\overline{G}}} \cong R H_{X_{\overline{F}}}$. For any $\tilde{A} \in \mathcal{A}$, $X_{\overline{F}}(\tilde{A}, \tilde{A}) \cong \mathcal{R}_B(\tilde{A}, \overline{F}(\tilde{A}))$ which is compact in $\mathcal{D}(\mathcal{R}_B)$. By Lemma 6.2 (a) in [II], we have $L T_{X_{\overline{F}}} \cong R H_{X_{\overline{F}}}$, where $X_{\overline{F}}$ is defined by

$$X_{\overline{F}}(\tilde{A}, \tilde{B}) = \text{Dif } \mathcal{R}_B(X_{\overline{F}}(\tilde{A}, \tilde{A}), \tilde{B}).$$
Thus, it suffices to show that we have a quasi-isomorphism $Y_{\mathfrak{C}} \rightarrow X^T_{\mathfrak{F}}$ as $\mathcal{R}_A \otimes_k \mathcal{R}_B^{op}$-bimodule. For any $\tilde{A} \in A$ and $\tilde{B} \in B$, we have

$$X^T_{\mathfrak{F}}(\tilde{A}, \tilde{B}) = \text{Def} \mathcal{R}_B(X_{\mathfrak{F}^c}(?, \tilde{A}), \tilde{B}^\wedge)$$
$$= \text{Def} \mathcal{R}_B(F(\tilde{A})^\wedge, \tilde{B}^\wedge)$$
$$\cong \mathcal{R}_B(F(\tilde{A}), \tilde{B})$$
$$\cong \bigoplus_{n \in \mathbb{Z}} B(F(\tilde{A}), \Sigma_B^{2n} \tilde{B})$$
$$\cong \bigoplus_{n \in \mathbb{Z}} R\text{Hom}_B(\tilde{A} \otimes_A X_B, \Sigma_B^{2n} \tilde{B})$$
$$\cong \bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, R\text{Hom}_B(X, \Sigma_B^{2n} \tilde{B}))$$

Recall that we have quasi-isomorphism $\Sigma_B^{2n} \tilde{B} \otimes_B R\text{Hom}_B(X, B) \rightarrow R\text{Hom}_B(A_X B, \tilde{B})$ and $\tilde{A}$ is $K$-projective as right $A$-module. In particular, we have a quasi-isomorphism

$$\bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, \Sigma_B^{2n} \tilde{B} \otimes_B R\text{Hom}_B(X, B)) \xrightarrow{q.is} \bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, R\text{Hom}_B(A_X B, \tilde{B})).$$

Again, we also have quasi-isomorphism $\Sigma_B^{2n} \tilde{B} \otimes_B Y \rightarrow \Sigma_B^{2n} \tilde{B} \otimes_B R\text{Hom}_B(X, B)$, which implies

$$\bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, \Sigma_B^{2n} \tilde{B} \otimes_B Y) \xrightarrow{q.is} \bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, \Sigma_B^{2n} \tilde{B} \otimes_B R\text{Hom}_B(X, B)).$$

The first term

$$\bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, \Sigma_B^{2n} \tilde{B} \otimes_B Y) \cong \bigoplus_{n \in \mathbb{Z}} R\text{Hom}_A(\tilde{A}, \Sigma_A^{2n} (\tilde{B} \otimes_B Y))$$
$$\cong \bigoplus_{n \in \mathbb{Z}} A(\tilde{A}, \Sigma_A^{2n} G(\tilde{B}))$$
$$\cong \mathcal{R}_A(\tilde{A}, \overline{G}(\tilde{B}))$$
$$= Y_{\mathfrak{C}}(\tilde{A}, \tilde{B})$$

Thus, we have obtained a quasi-isomorphism $Y_{\mathfrak{C}}(\tilde{A}, \tilde{B}) \rightarrow X^T_{\mathfrak{F}}(\tilde{A}, \tilde{B})$, which is natural in both $\tilde{A}$ and $\tilde{B}$. This completes the proof. \qed

The following lemma is quite obviously.

**Lemma A.2.** If $F : \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } B)$ is fully faithful, then $LT_{\mathfrak{C}} : \mathcal{D}(\mathcal{R}_A) \rightarrow \mathcal{D}(\mathcal{R}_B)$ is fully faithful.

**Proof.** It follows from the Lemma 4.2 (a) and (b) of [10] directly. \qed

Let $\mathcal{R}_A$ be the perfect derived category of $\mathcal{R}_A$. Let $\mathcal{R}_B$ be the perfect derived category of $\mathcal{R}_B$. In other word, $\mathcal{R}_A$ and $\mathcal{R}_B$ are the root categories of $A$ and $B$ respectively. Clearly, the triangle functors $LT_{\mathfrak{C}}$ and $LT_{\mathfrak{C}}$ restrict to an adjoint pair of triangle functors

$$\mathcal{R}_A \xrightarrow{LT_{\mathfrak{C}}} \mathcal{R}_B.$$
For simplicity, westill denote $\overline{F} := \operatorname{L} T_{X_{\psi}} : \mathcal{R}_A \to \mathcal{R}_B$ and $\overline{G} := \operatorname{L} T_{Y_{\psi}} : \mathcal{R}_B \to \mathcal{R}_A$.

A.3. Recollement lives in root categories. Suppose we are given triangulated categories $\mathcal{D}', \mathcal{D}, \mathcal{D}''$ with triangle functors

\[
\begin{array}{ccc}
\mathcal{D}' & \xrightarrow{i^*} & \mathcal{D} \\
\mathcal{D} & \xrightarrow{j^* = j_!} & \mathcal{D}'' \\
i' & \xrightarrow{j_*} & j
\end{array}
\]

such that

- $(i^*, i_*, j_!)$ and $(j!, j^*, j_*)$ are adjoint triples;
- $i_*, j_!, j_*$ are fully faithful;
- $j^* \circ i_* = 0$;
- any $X$ in $\mathcal{D}$, there are distinguished triangles

\[
i_! i^! X \to X \to j_* j^! X \to \Sigma i_* i^! X \to X, j_! j^! X \to X \to i_* i^! X \to \Sigma j_* j^! X
\]

where the morphisms $i_! i^! X \to X \to j_* j^* X$, etc. are adjunction morphisms.

Then we say that $\mathcal{D}$ admits recollement relative to $\mathcal{D}'$ and $\mathcal{D}''$. This notation was first introduced by Beilinson-Bersstein-Deligne \cite{1} in geometric setting with the idea that $\mathcal{D}$ can be viewed as being glued together from $\mathcal{D}'$ and $\mathcal{D}''$. It is not hard to show that if both $\mathcal{D}'$ and $\mathcal{D}''$ are Krull-Schmidt categories, so is $\mathcal{D}$. Recollement in algebraic setting was studied extensively due to the close relation with tilting theory \cite{9} \cite{11}, etc.

Let $A, B, C$ be finite-dimensional $k$-algebras with finite global dimension. Suppose that the bounded derived category $\mathcal{D}^b(\mod B)$ admits a recollement relative to $\mathcal{D}^b(\mod A)$ and $\mathcal{D}^b(\mod C)$. In particular, we have the following diagram of triangulated categories and triangle functors

\[
\begin{array}{ccc}
\mathcal{D}^b(\mod A) & \xrightarrow{i_* = i_!} & \mathcal{D}^b(\mod B) \\
\mathcal{D}^b(\mod B) & \xrightarrow{j^* = j_!} & \mathcal{D}^b(\mod C) \\
i' & \xrightarrow{j_*} & j
\end{array}
\]

Assume further that both the functors $i^*$ and $j_!$ are standard. Then we have the following

**Theorem A.3.** Keep the notations above. Let $A, B$ and $C$ be finite-dimensional $k$-algebras with finite global dimension such that the derived category $\mathcal{D}^b(\mod B)$ admits a recollement relative to $\mathcal{D}^b(\mod A)$ and $\mathcal{D}^b(\mod C)$. Assume that the functor $i^*$ and $j_!$ are standard. The root category $\mathcal{R}_B$ admits a recollement relative to $\mathcal{R}_A$ and $\mathcal{R}_C$. Moreover, we have the following commutative diagram of recollements

\[
\begin{array}{ccc}
\mathcal{D}^b(\mod A) & \xrightarrow{i_* = i_!} & \mathcal{D}^b(\mod B) \\
\mathcal{D}^b(\mod B) & \xrightarrow{j^* = j_!} & \mathcal{D}^b(\mod C) \\
i' & \xrightarrow{j_*} & j
\end{array}
\]

- $\pi_A : \mathcal{R}_A \to \mathcal{R}_B$
- $\pi_B : \mathcal{R}_B \to \mathcal{R}_C$
- $\pi_C : \mathcal{R}_C \to \mathcal{R}_A$

**Proof.** Since $i^*$ and $j_!$ are standard, then all the functors $i_*, i^!, j^*, j_*$ are standard due to fact $A, B, C$ have finite global dimension. Thus, we have the corresponding induced functors $i^*, i_*, i^!, j_!, j^!, j_*$. The commutativity of the above diagram follows from the universal
We prove the existence of the first triangle, the second one is similar. By Lemma A.1, one infers that \( \tilde{\iota}_* \) and \( \tilde{\iota}_* \) are adjoint triples follows Lemma A.1. By Lemma A.2, one infers that \( \tilde{\iota}_* \) and \( \tilde{\iota}_* \) are fully faithful. Since \( \mathcal{R}_A \) is generated by \( \pi_A(A) \), to show that \( \tilde{j}^* \circ \tilde{\iota}_* = 0 \), it suffices to show \( \tilde{j}^* \circ \tilde{\iota}_*(\pi_A(A)) = 0 \). By the commutativity of the above diagram, this result follows from \( \tilde{j}^* \circ \tilde{\iota}_* = 0 \). It remains to show that for any \( X \in \mathcal{R}_B \) there are triangles

\[
\begin{array}{c}
\eta_i^* X \to X \to \tilde{j}_* \tilde{j}^* X \to \Sigma \eta_i^* X, \quad \tilde{\iota}_* \tilde{\iota}^* X \to X \to \iota_* \iota^* X \to \Sigma \tilde{\iota}_* \tilde{\iota}^* X.
\end{array}
\]

We prove the existence of the first triangle, the second one is similar.

If \( X \in \text{im} \pi_B \), there is \( Y \in \mathcal{D}^b(\text{mod } B) \) such that \( X = \pi_B(Y) \). By the recollement of \( \mathcal{D}_B^b(\text{mod } B) \) relative to \( \mathcal{D}_B(\text{mod } A) \) and \( \mathcal{D}_B^b(\text{mod } C) \), we have

\[
\eta^i_i Y \to Y \to j_* j^* Y \to \Sigma \eta^i_i Y.
\]

Applying the triangle functor \( \pi_B \), we get a triangle in \( \mathcal{R}_B \)

\[
\pi_B(\eta^i_i Y) \to \pi_B(Y) \to \pi_B(j_* j^* Y) \to \Sigma \pi_B(\eta^i_i Y).
\]

By the commutativity of the functors, we have

\[
\iota^i_i \pi_B(Y) \to \pi_B(Y) \to j_* j^* \pi_B(Y) \to \Sigma \iota^i_i \pi_B(Y).
\]

Clearly, this triangle is isomorphic to

\[
\eta^i_i \pi_B(Y) \xrightarrow{\eta^i_i} \pi_B(Y) \xrightarrow{\epsilon^i_i} j_* j^* \pi_B(Y) \xrightarrow{\epsilon^i_i} \Sigma \iota^i_i \pi_B(Y)
\]

where \( \eta^i_i, \epsilon^i_i \) are adjunction morphisms, which implies the later one is a distinguished triangle.

Consider the triangle \( X \xrightarrow{f} Y \to Z \to \Sigma X \), where \( X, Y \in \text{im} \pi_B \). Consider the following commutative square

\[
\begin{array}{ccc}
\eta_i^* X & \xrightarrow{\eta^-} & X \\
\downarrow{j_* j^* f} & & \downarrow{f} \\
\tilde{j}_* \tilde{j}^* Y & \xrightarrow{\eta^+} & Y \\
\end{array}
\]

By nine lemma, one can embed the square to the following commutative diagram of triangles

\[
\begin{array}{cccc}
\eta_i^* X & \xrightarrow{\eta^i} & X & \xrightarrow{\epsilon^i} j_* \tilde{j}^* X \\
\downarrow{j_* j^* f} & & \downarrow{f} & \downarrow{j_* j^* f} \\
\tilde{j}_* \tilde{j}^* Y & \xrightarrow{\eta^+} & Y & \xrightarrow{\epsilon^i} j_* \tilde{j}^* Y \\
\end{array}
\]

\[
\begin{array}{cccc}
\tilde{\iota}_* \tilde{\iota}^* U_Z & \xrightarrow{u} & Z & \xrightarrow{v} j_* \tilde{j}^* Z \\
\downarrow{\eta^i} & & \downarrow{\epsilon^i} & \downarrow{\tilde{j}^* g_2} \\
\tilde{\iota}_* \tilde{\iota}^* Z & \xrightarrow{u} & Z & \xrightarrow{v} j_* \tilde{j}^* Z \\
\end{array}
\]

Let \( \phi(u) : U_Z \to \tilde{\iota}^* Z \) be the morphism corresponds to \( u \) under the natural isomorphism. Let \( \phi(v) : \tilde{j}^* Z \to V_Z \) be the morphism corresponds to \( v \). It is clear that \( \phi(u) \) and \( \phi(v) \) are isomorphisms. Thus, one gets the following commutative diagram

\[
\begin{array}{ccc}
\tilde{\iota}_* U_Z & \xrightarrow{u} & Z & \xrightarrow{v} j_* \tilde{j}^* Z & \xrightarrow{w} \Sigma \tilde{\iota}_* U_Z \\
\downarrow{\eta^i} & & \downarrow{\epsilon^i} & \downarrow{\tilde{j}^* \phi^{-1}} & \downarrow{\Sigma \tilde{\iota}_* \phi(u)} \\
\tilde{\iota}_* Z & \xrightarrow{\eta^i} & Z & \xrightarrow{\epsilon^i} j_* \tilde{j}^* Z & \xrightarrow{\delta} \Sigma \tilde{\iota}_* Z
\end{array}
\]
where $\delta = J_{\ast} \phi (v) \circ w \circ \sum_i \phi (u)$. Thus, one informs that
\[ i_{\ast} i^1 Y \overset{\pi_B}{\rightarrow} Z \overset{\xi_Z}{\rightarrow} j_{\ast} j^1 Y \rightarrow \sum_i i_{\ast} i^1 Z \]
is a distinguished triangle. Now this holds for any $Z \in R_B$ by 'devissage'.

Corollary A.4. Keep the assumptions in theorem A.3. If the canonical functor $\pi_B$ is dense, then both $\pi_A$ and $\pi_C$ are dense.

Proof. For any $X \in R_A$, consider $i_{\ast} X \in R_B$. By the dense of $\pi_B$, there is a $Y \in D^b (\text{mod } B)$ such that $\pi_B (Y) \cong i_{\ast} X$. For $Y$, one have the canonical triangle $i_{\ast} i^1 Y \rightarrow Y \rightarrow j_{\ast} j^1 Y \rightarrow$.

Applying the functor $\pi_B$, we have
\[ \pi_B (i_{\ast} i^1 Y) \rightarrow \pi_B (Y) \rightarrow \pi_B (j_{\ast} j^1 Y) \rightarrow \]
which have to isomorphic to the canonical triangle
\[ i_{\ast} i^1 (i_{\ast} X) \rightarrow X \rightarrow 0 \rightarrow .\]

One gets $X \cong \pi_A (i^1 Y)$. In particular, $\pi_A$ is dense. Similar proof implies that $\pi_C$ is also dense. \hfill \Box

Remark A.5. If only one of $i^\ast$ and $j_!$ is standard, say $i^\ast$ is standard. Then lemma A.1 A.2 and a result of [16] imply that there is a recollement
\[ R_A \overset{i_{\ast} i^1 \simeq}{\rightarrow} R_B \overset{\bar{j}_{\ast} j^1 \simeq}{\leftarrow} R_B / i_{\ast} R_A. \]

The corollary A.4 also holds in this case (one should replace the functor $\pi_C$).

The following is now quite obviously.

Corollary A.6. Let $A$ and $B$ be finite-dimensional $k$-algebras with finite global dimension. Assume the root category $R_A$ is not triangulated with the inherited triangle structure. For any finite dimensional $A \otimes_k B^{op}$-module $M$, the root category of the triangular extension of $A$ and $B$ by $M$ is not triangulated with the inherited triangle structure.

References

[1] Dagmar Alpen, *Zur Struktur von GIM-Liealgebren*, Hamburger Beiträge Zur Mathematik aus dem Mathematischen Seminar, Heft 28 (1984).
[2] Claire Amiot, *Sur les petites catégories triangulées*, Ph. D thesis (2008), http://www.institut.math.jussieu.fr/~amiot/these.pdf
[3] Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Annales de l’Institut Fourier, vol. 59 (2009), no. 6, 2525-2590.
[4] Claire Amiot and Steffen Oppermann, *The image of the derived category in the cluster category*, preprint, arXiv: 1010.1129v1.
[5] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, Actes du Congrès International des Mathématiciens (Nice 1970), Tome 2, 279-284, Gauthier-Villars, Paris, 1971.
[6] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque 100 (1982), 5-171.
[7] E. Cline, B. Parshall and L. Scott, *Algebraic stratification in representation categories*, J. algebra 117 (1988), 504-521.
[8] Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. 62 (1987), no. 3, 339-389.
[9] Peter Jørgensen, *Recollement for differential graded algebras*, J. Algebra 299 (2006), 589-601.
[10] Bernhard Keller, *Derived dg categories*, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63-102.
[11] Bernhard Keller, *On triangulated orbit categories*, Doc. Math. 10 (2005), 551-581.
[12] Hiroshige Kajiura, Kyoji Saito and Atsushi Takahashi, *Matrix factorizations and representations of quivers II: type ADE case*, Adv. in Math. 211 (2007), 327-362.
[13] Hiroshige Kajiura, Kyoji Saito and Atsushi Takahashi, *Triangulated categories of matrix factorizations for regular systems of weights with $\epsilon = -1$*, preprint, arXiv:0708.0210.
[14] Steffen Koenig, *Tilting complexes, perpendicular categories and recollements of derived categories of rings*, J. Pure Appl. Algebra 73 (1991), 211-232.
[15] Yanan Lin and Liangang Peng, *Elliptic Lie algebras and tubular algebras*, Adv. in Math. 196 (2005), 487-530.
[16] B. Parshall and L. Scott, *Derived categories, quasi-hereditary algebras, and algebraic groups*, Proc. of the Ottawa-Moosone Workshop in Algebra 1987, Math. Lecture Note Series, Carleton University and Université d’ Ottawa (1988).
[17] Liangang Peng and Jie Xiao, *Root categories and simple Lie algebras*, J. Algebra 198 (1997), no. 1, 19-56.
[18] Liangang Peng and Jie Xiao, *Triangulated categories and Kac-Moody algebras*, Inventiones Math. 140 (2000), 563-603.
[19] S. E. Rao and R. V. Moody, *Vertex representations for $n$-toroidal Lie algebras and a generalization of the Virasoro algebra*, Commun. Math. Phys. 159 (1994), 239-264.
[20] I. Reiten, M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. 15 (2002), 295-366.
[21] Kyoji Saito, *Extended affine root systems I*, Publ. Res. Inst. Math. Sci. 21 (1985), no. 1, 75-179.
[22] Kyoji Saito, *Regular systems of weights and associated singularities*, Complex analytic singularities, 479-526, Adv. Stud. Pure Math., 8, North-Holland, Amsterdam, 1987.
[23] Kyoji Saito, *Duality for regular systems of weights*, Asian. J. Math. 2 (1998), 983-1048.
[24] Kyoji Saito, *Towards a categorial construction of Lie algebras*, Advanced Studies in Pure Mathematics 50 (2008), 101-175.
[25] Kyoji Saito and D. Yoshii, *Extended affine root systems IV (Simply-laced Elliptic Lie algebras)*, Publ. Res. Inst. Math. Sci. 36 (2000), 385-408.
[26] Peter Slodowy, *Beyond Kac-Moody algebras, and inside*, Canadian Mathematical Society Conference Proceedings, Volume 5 (1986), 361-371.
[27] A. Takahashi, *Matrix factorizations and representations of quivers I*, preprint, arXiv: math.AG/0506347.
[28] Jie Xiao and Fan Xu, *Hall algebras associated to triangulated categories*, Duke Math. J. 143 (2008), no. 2, 357-373.
[29] Jie Xiao, Fan Xu and Guanglian Zhang, *Derived categories and Lie algebras*, arXiv:math/0604564v2.
[30] Fan Xu, *On triangulated categories and enveloping algebras*, preprint, arXiv: math.RT/0710.5588v2.
[31] D. Yoshii, *Elliptic Lie algebras (inhomogeneous cases)*, preprint.

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