A NOTE ON REPRESENTATIONS OF SOME AFFINE VERTEX ALGEBRAS OF TYPE $D$

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Abstract. In this note we construct a series of singular vectors in universal affine vertex operator algebras associated to $D^{(1)}_\ell$ of levels $n - \ell + 1$, for $n \in \mathbb{Z}_{>0}$. For $n = 1$, we study the representation theory of the quotient vertex operator algebra modulo the ideal generated by that singular vector. In the case $\ell = 4$, we show that the adjoint module is the unique irreducible ordinary module for simple vertex operator algebra $L_{D_4}(-2,0)$. We also show that the maximal ideal in associated universal affine vertex algebra is generated by three singular vectors.

1. Introduction

The classification of irreducible modules for simple vertex operator algebra $L_{\mathfrak{g}}(k,0)$ associated to affine Lie algebra $\mathfrak{g}$ of level $k$ is still an open problem for general $k \in \mathbb{C}$ ($k \neq -h^\vee$). This problem is connected with the description of the maximal ideal in the universal affine vertex algebra $N_{\mathfrak{g}}(k,0)$. One approach to this classification problem is through construction of singular vectors in $N_{\mathfrak{g}}(k,0)$.

The known (non-generic) cases include positive integer levels (cf. [FZ], [L], [MP]) and some special cases of rational admissible levels, in the sense of Kac and Wakimoto [KW] (cf. [A1], [A3], [AM], [AL], [DLM], [P1], [P2]). It turns out that negative integer levels also have some interesting properties. They appeared in bosonic realizations in [FF], and also recently in the context of conformal embeddings (cf. [AP]).

In this note we study a vertex operator algebra associated to affine Lie algebra of type $D^{(1)}_\ell$ and negative integer level $-\ell + 2$. This level appeared in [AP] in the context of conformal embedding of $L_{B_{\ell-1}}(-\ell + 2,0)$ into $L_{D_\ell}(-\ell + 2,0)$. This conformal embedding is in some sense similar to the conformal embedding of $L_{D_\ell}(-\ell + \frac{3}{2},0)$ into $L_{B_\ell}(-\ell + \frac{3}{2},0)$.

We will show that there are also similarities in singular vectors in universal affine vertex algebras $N_{B_\ell}(-\ell + \frac{3}{2},0)$ (studied in [P1]) and $N_{D_\ell}(-\ell + 2,0)$. More generally, we construct a series of singular vectors

$$v_n = \left( \sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1)e_{\epsilon_1 + \epsilon_i}(-1) \right)^n 1$$

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in $N_D(n-\ell+1,0)$, for any $n \in \mathbb{Z}_{>0}$. For $n = 1$, we study the representation theory of the quotient $N_D(\ell+2,0)$ modulo the ideal generated by $v_1$. Using the methods from [A1], [A2], [AM], [MP], we obtain the classification of irreducible weak modules in the category $\mathcal{O}$ for that vertex algebra. It turns out that there are infinitely many of these modules.

In the special case $\ell = 4$, we obtain the classification of irreducible weak modules from the category $\mathcal{O}$ for simple vertex operator algebra $L_{D_4}(-2,0)$. This vertex algebra also appeared in [AP] in the context of conformal embedding of $L_{G_2}(-2,0)$ into $L_{D_4}(-2,0)$. It follows that there are finitely many irreducible weak $L_{D_4}(-2,0)$–modules from the category $\mathcal{O}$, that the adjoint module is the unique irreducible ordinary $L_{D_4}(-2,0)$–module, and that every ordinary $L_{D_4}(-2,0)$–module is completely reducible. We also show that the maximal ideal in $N_{D_4}(-2,0)$ is generated by three singular vectors.

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2. Preliminaries

We assume that the reader is familiar with the notion of vertex operator algebra (cf. [Bor], [FHL], [FLM], [FB], [FZ], [K2], [L], [LL]) and Kac-Moody algebra (cf. [K1]).

Let $V$ be a vertex operator algebra. Denote by $A(V)$ the associative algebra introduced in [Z], called the Zhu’s algebra of $V$. As a vector space, $A(V)$ is a quotient of $V$, and we denote by $[a]$ the image of $a \in V$ under the projection of $V$ onto $A(V)$. We recall the following fundamental result from [Z]:

**Proposition 2.1.** The equivalence classes of the irreducible $A(V)$–modules and the equivalence classes of the irreducible $\mathbb{Z}_+$–graded weak $V$–modules are in one-to-one correspondence.

Let $\mathfrak{g}$ be a simple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $\hat{\mathfrak{g}}$ the (untwisted) affine Lie algebra associated to $\mathfrak{g}$. Denote by $V(\mu)$ the irreducible highest weight $\mathfrak{g}$–module with highest weight $\mu$, and by $L(k,\mu)$ the irreducible highest weight $\hat{\mathfrak{g}}$–module with highest weight $k\Lambda_0+\mu$.

Furthermore, denote by $N(k,0)$ (or $N_\mathfrak{g}(k,0)$) the universal affine vertex algebra associated to $\hat{\mathfrak{g}}$ of level $k \in \mathbb{C}$. For $k \neq -h^\vee$, $N(k,0)$ is a vertex operator algebra with Segal-Sugawara conformal vector, and $L(k,0)$ is a simple vertex operator algebra. The Zhu’s algebra of $N(k,0)$ was determined in [FZ]:

**Proposition 2.2.** The associative algebra $A(N(k,0))$ is canonically isomorphic to $U(\mathfrak{g})$. The isomorphism is given by $F : A(N(k,0)) \rightarrow U(\mathfrak{g})$

$$F([x_1(-n_1-1)\cdots x_m(-n_m-1)1]) = (-1)^{n_1+\cdots+n_m}x_m\cdots x_1,$$

for any $x_1,\ldots,x_m \in \mathfrak{g}$ and any $n_1,\ldots,n_m \in \mathbb{Z}_+$. 
We have:

**Proposition 2.3.** Assume that a $\hat{\mathfrak{g}}$–submodule $J$ of $N(k, 0)$ is generated by $m$ singular vectors ($m \in \mathbb{Z}_{>0}$), i.e. $J = U(\hat{\mathfrak{g}})v^{(1)} + \ldots + U(\hat{\mathfrak{g}})v^{(m)}$. Then

$$A(N(k, 0)/J) \cong U(\mathfrak{g})/I,$$

where $I$ is the two-sided ideal of $U(\mathfrak{g})$ generated by $u^{(1)} = F([v^{(1)}]), \ldots, u^{(m)} = F([v^{(m)}])$.

Let $J = U(\hat{\mathfrak{g}})v^{(1)} + \ldots + U(\hat{\mathfrak{g}})v^{(m)}$ be a $\hat{\mathfrak{g}}$–submodule of $N(k, 0)$ generated by singular vectors $v^{(1)}, \ldots, v^{(m)}$. Now we recall the method from [A1], [A2], [AM], [MP] for the classification of irreducible $A(N(k, 0)/J)$–modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations.

Denote by $L$ the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f = [X, f]$ for $X \in \mathfrak{g}$ and $f \in U(\mathfrak{g})$. Let $R^{(j)}$ be a $U(\mathfrak{g})$–submodule of $U(\mathfrak{g})$ generated by the vector $u^{(j)} = F([v^{(j)}])$ under the adjoint action, for $j = 1, \ldots, m$. Clearly, $R^{(j)}$ is an irreducible highest weight $U(\mathfrak{g})$–module. Let $R^{(j)}_0$ be the zero-weight subspace of $R^{(j)}$.

The next proposition follows from [A1], [AM], [MP]:

**Proposition 2.4.** Let $V(\mu)$ be an irreducible highest weight $U(\mathfrak{g})$–module with the highest weight vector $v_\mu$, for $\mu \in \mathfrak{h}^*$. The following statements are equivalent:

1. $V(\mu)$ is an $A(N(k, 0)/J)$–module,
2. $R^{(j)}_0 V(\mu) = 0$, for every $j = 1, \ldots, m$,
3. $R^{(j)}_0 v_\mu = 0$, for every $j = 1, \ldots, m$.

Let $r \in R^{(j)}_0$. Clearly there exists the unique polynomial $p_r \in \mathcal{S}(\mathfrak{h})$ such that

$$rv_\mu = p_r(\mu)v_\mu.$$

Set $\mathcal{P}^{(j)} = \{ p_r \mid r \in R^{(j)}_0 \}$, for $j = 1, \ldots, m$. We have:

**Corollary 2.5.** There is one-to-one correspondence between

1. irreducible $A(N(k, 0)/J)$–modules from the category $\mathcal{O}$,
2. weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathcal{P}^{(j)}$, for every $j = 1, \ldots, m$.

In the case $m = 1$, we use the notation $R$, $R_0$ and $\mathcal{P}$ for $R^{(1)}$, $R^{(1)}_0$ and $\mathcal{P}^{(1)}$, respectively.

3. **Vertex operator algebra associated to $D^{(1)}_\ell$ of level $-\ell + 2$**

In this section we study the representation theory of the quotient of universal affine vertex operator algebra associated to $D^{(1)}_\ell$ of level $-\ell + 2$, modulo the ideal generated by a singular vector of conformal weight two.

Denote by $\mathfrak{g}$ the simple Lie algebra of type $D_\ell$. We fix the root vectors for $\mathfrak{g}$ as in [Bo1], [FF]. We have:
Theorem 3.1. Vector
\[ v_n = \left( \sum_{i=2}^{\ell} e_{\epsilon_i - \epsilon_i} (-1) e_{\epsilon_1 + \epsilon_i} (-1) \right)^n 1 \]
is a singular vector in \( N_{D_n}(n - \ell + 1, 0) \), for any \( n \in \mathbb{Z}_{>0} \).

Proof. Direct verification of relations \( e_{\epsilon_k - \epsilon_{k+1}}(0)v_n = 0 \), for \( k = 1, \ldots, \ell - 1 \),
\( e_{\epsilon_{\ell-1} + \epsilon_\ell}(0)v_n = 0 \) and \( f_{\epsilon_1 + \epsilon_2}(1)v_n = 0 \).

In the case \( n = 1 \), we obtain the singular vector
\[ v = \sum_{i=2}^{\ell} e_{\epsilon_i - \epsilon_i} (-1) e_{\epsilon_1 + \epsilon_i} (-1) 1 \]
in \( N_{D_1}(-\ell + 2, 0) \).

Remark 3.2. Vector \( v \) from relation (3.1) has a similar formula as singular vector
\[ \frac{1}{4} e_{\epsilon_1} (-1)^2 1 + \sum_{i=2}^{\ell} e_{\epsilon_i - \epsilon_i} (-1) e_{\epsilon_1 + \epsilon_i} (-1) 1 \]
for \( B_{\ell}^{(1)} \) in \( N_{B_\ell}(-\ell + \frac{3}{2}, 0) \). The representation theory of the quotient of \( N_{B_\ell}(-\ell + \frac{3}{2}, 0) \) modulo the ideal generated by that vector was studied in [P1].

We will consider representations of the vertex operator algebra
\[ V_{D_\ell}(-\ell + 2, 0) = \frac{N_{D_\ell}(-\ell + 2, 0)}{U(\mathfrak{g})v}. \]

Proposition 2.3 gives:

Proposition 3.3. The associative algebra \( A(V_{D_\ell}(-\ell + 2, 0)) \) is isomorphic to the algebra \( U(\mathfrak{g})/I \), where \( I \) is the two-sided ideal of \( U(\mathfrak{g}) \) generated by
\[ u = \sum_{i=2}^{\ell} e_{\epsilon_i - \epsilon_i} e_{\epsilon_1 + \epsilon_i}. \]

We have the following classification:

Theorem 3.4. For any subset \( S = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, \ell - 2\}, \)
\( i_1 < \ldots < i_k \), and \( t \in \mathbb{C} \), we define weights
\[ \mu_{S,t} = \sum_{j=1}^{k} \left( i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1}(t + \ell - 1) \right) \omega_{i_j} + t \omega_{\ell-1}, \]
\[ \mu'_{S,t} = \sum_{j=1}^{k} \left( i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1}(t + \ell - 1) \right) \omega_{i_j} + t \omega_{\ell}, \]
where \( \omega_1, \ldots, \omega_\ell \) are fundamental weights for \( \mathfrak{g} \). Then the set
\[ \{ L_{D_\ell}(-\ell + 2, \mu_{S,t}), L_{D_\ell}(-\ell + 2, \mu'_{S,t}) \mid S \subseteq \{1, 2, \ldots, \ell - 2\}, t \in \mathbb{C} \} \]
provides the complete list of irreducible weak $V_{D_{s}}(-\ell + 2, 0)$–modules from the category $\mathcal{O}$.

**Proof.** We use the method for classification of irreducible $A(V_{D_{s}}(-\ell + 2, 0))$–modules in the category $\mathcal{O}$ from Corollary \[2.5\] In this case $R \cong V_{D_{s}}(2\omega_1)$, and similarly as in [P1, Lemma 28] one obtains that

$$\dim R_0 = \ell - 1.$$  

Furthermore, one obtains by direct calculation that

$$p_i(h) = h_i(h_{\ell} + \ell - i - 1), \quad \text{for } i = 1, \ldots, \ell - 1$$

are linearly independent polynomials in $P(0)$. Here $h_i (i = 1, \ldots, \ell)$ denote the simple coroots for $g$ and

$$h_{\ell} = h_{\ell} + 2h_{\ell+1} + \ldots + 2h_{\ell-2} + h_{\ell-1} + h_{\ell}, \quad \text{for } i < \ell - 1.$$  

Corollary \[2.5\] now implies that the highest weights of irreducible $A(V_{D_{s}}(-\ell + 2, 0))$–modules from the category $\mathcal{O}$ are given as solutions of polynomial equations

$$p_i(h) = 0, \quad i = 1, \ldots, \ell - 1.$$  

First we note that for $i = \ell - 1$, we obtain the equation

$$h_{\ell-1}h_{\ell} = 0.$$  

Thus, either $h_{\ell-1} = 0$ or $h_{\ell} = 0$. Assume first that $h_{\ell-1} = 0$, and let $S = \{i_1, \ldots, i_k\}, i_1 < \ldots < i_k$ be the subset of $\{1, 2, \ldots, \ell - 2\}$ such that $h_i = 0$ for $i \not\in S$ and $h_i \neq 0$ for $i \in S$. Then we have the system

$$h_{i_1} + 2h_{i_2} + \ldots + 2h_{i_k} + h_{\ell} + \ell - i_1 - 1 = 0,$$

$$h_{i_2} + 2h_{i_3} + \ldots + 2h_{i_k} + h_{\ell} + \ell - i_2 - 1 = 0,$$

$$\vdots$$

$$h_{i_{k-1}} + 2h_{i_k} + h_{\ell} + \ell - i_{k-1} - 1 = 0,$$

$$h_{i_k} + h_{\ell} + \ell - i_k - 1 = 0.$$  

The solution of this system is given by

$$h_{ij} = i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j}s + (-1)^{k-j+1}(t + \ell - 1), \quad \text{for } j = 1, \ldots, k;$$

$$h_{\ell} = t \quad (t \in \mathbb{C}).$$  

It follows that $V_{D_{s}}(\mu_{S,\ell}^{(t)})$ is an irreducible $A(V_{D_{s}}(-\ell + 2, 0))$–module. Similarly, the case $h_{\ell} = 0$ gives that $V_{D_{s}}(\mu_{S,\ell})$ is irreducible $A(V_{D_{s}}(-\ell + 2, 0))$–module. We conclude that the set

$$\{V_{D_{s}}(\mu_{S,\ell}), V_{D_{s}}(\mu_{S,\ell}^{(t)}) \mid S \subseteq \{1, 2, \ldots, \ell - 2\}, t \in \mathbb{C}\}$$
provides the complete list of irreducible $A(\mathcal{V}_{D_4}(-\ell+2,0))$-modules from the category $\mathcal{O}$. The claim of theorem now follows from Zhu’s theory. □

**Example 3.5.** For $\ell = 4$, we have subsets $S = \emptyset, \{1\}, \{2\}, \{1, 2\}$ of the set $\{1, 2\}$, so we obtain that the set
\[
\{L_{D_4}(-\ell + 2, t\omega_2), L_{D_4}(-\ell + 2, t\omega_3), L_{D_4}(-\ell + 2, (-2 - t)\omega_1 + t\omega_3),
L_{D_4}(-\ell + 2, (-2 - t)\omega_1 + t\omega_4), L_{D_4}(-\ell + 2, (-1 - t)\omega_2 + t\omega_3),
L_{D_4}(-\ell + 2, (-1 - t)\omega_2 + t\omega_4), L_{D_4}(-\ell + 2, t\omega_1 + (-1 - t)\omega_2 + t\omega_3),
\}
\]
provides the complete list of irreducible weak $\mathcal{V}_{D_4}(-2,0)$-modules from the category $\mathcal{O}$.

Recall that a module for vertex operator algebra is called ordinary if $L(0)$ acts semisimply with finite-dimensional weight spaces. We have:

**Corollary 3.6.** The set
\[
\{L_{D_4}(-\ell + 2, t\omega_{\ell-1}), L_{D_4}(-\ell + 2, t\omega_\ell) \mid t \in \mathbb{Z}_{\geq 0}\}
\]
provides the complete list of irreducible ordinary $\mathcal{V}_{D_4}(-\ell + 2,0)$-modules.

**Proof.** If $L_{D_4}(-\ell + 2, \mu)$ is an ordinary $\mathcal{V}_{D_4}(-\ell + 2,0)$-module, then $\mu$ is a dominant integral weight. Then $\mu(h_{\ell i + \ell-1}) \in \mathbb{Z}_{\geq 0}$, for $i = 1, \ldots, \ell - 1$. Relations (3.2) and (3.3) then give that
\[
\mu(h_i) = 0, \quad \text{for } i = 1, \ldots, \ell - 2,
\]
and $\mu(h_{\ell-1}) = 0$ or $\mu(h_\ell) = 0$. Thus, $\mu = t\omega_{\ell-1}$ or $\mu = t\omega_\ell$, and $t \in \mathbb{Z}_{\geq 0}$ since $\mu$ is a dominant integral weight. □

It follows that:

**Corollary 3.7.** The set of irreducible ordinary $L_{D_4}(-\ell + 2,0)$-modules is a subset of the set
\[
\{L_{D_4}(-\ell + 2, t\omega_{\ell-1}), L_{D_4}(-\ell + 2, t\omega_\ell) \mid t \in \mathbb{Z}_{\geq 0}\}.
\]

4. **Case $\ell = 4$**

In this section we study the case $\ell = 4$. We determine the classification of irreducible weak $L_{D_4}(-2,0)$-modules from the category $\mathcal{O}$. It turns out that there are finitely many of these modules and that the adjoint module is the unique irreducible ordinary $L_{D_4}(-2,0)$-module. We also show that the maximal ideal in $\mathfrak{N}_{D_4}(-2,0)$ is generated by three singular vectors.

Denote by $\theta$ the automorphism of $\mathfrak{N}_{D_4}(-2,0)$ induced by the automorphism of the Dynkin diagram of $D_4$ of order three such that
\[
\theta(e_1 - e_2) = e_3 - e_4, \quad \theta(e_2 - e_3) = e_2 - e_3, \quad \theta(e_3 - e_4) = e_3 + e_4, \quad \theta(e_3 + e_4) = e_1 - e_2.
\]
Relation (3.1) implies that
\[
v = (e_{e_1 - e_2}(-1)e_{e_1 + e_2}(-1) + e_{e_1 - e_3}(-1)e_{e_1 + e_3}(-1) + e_{e_1 - e_4}(-1)e_{e_1 + e_4}(-1))1
\]
is a singular vector in $N_{D_4}(-2,0)$. Furthermore,

$$\theta(v) = (e_{\epsilon_3-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1)-e_{\epsilon_2-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1-\epsilon_4}(-1))1,$$

and

$$\theta^2(v) = (e_{\epsilon_3+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1)-e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))1$$

are also singular vectors in $N_{D_4}(-2,0)$. We consider the vertex operator algebra

$$\tilde{L}_{D_4}(-2,0) = \frac{N_{D_4}(-2,0)}{J},$$

where $J$ is the ideal in $N_{D_4}(-2,0)$ generated by vectors $v$, $\theta(v)$ and $\theta^2(v)$.

Proposition 2.3 gives that the associative algebra $A(\tilde{L}_{D_4}(-2,0))$ is isomorphic to the algebra $U(\mathfrak{g})/I$, where $I$ is the two-sided ideal of $U(\mathfrak{g})$ generated by $u$, $\theta(u)$ and $\theta^2(u)$, and

$$u = e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + e_{\epsilon_1-\epsilon_3}e_{\epsilon_1+\epsilon_3} + e_{\epsilon_1-\epsilon_4}e_{\epsilon_1+\epsilon_4}.$$ 

**Proposition 4.1.** We have:

(i) The set

$$\{L_{D_4}(-2,0), L_{D_4}(-2,-2\omega_1), L_{D_4}(-2,-2\omega_3), L_{D_4}(-2,-2\omega_4), L_{D_4}(-2,-\omega_2)\}.$$ 

provides a complete list of irreducible weak $\tilde{L}_{D_4}(-2,0)$–modules from the category $\mathcal{O}$.

(ii) $L_{D_4}(-2,0)$ is the unique irreducible ordinary module for $\tilde{L}_{D_4}(-2,0)$.

**Proof.** (i) We use the method for classification from Corollary 2.5. In this case $R^{(1)} \cong V_{D_4}(-2\omega_1)$, $R^{(2)} \cong V_{D_4}(-2\omega_3)$, $R^{(3)} \cong V_{D_4}(-2\omega_4)$ and

$$\dim R^{(1)}_0 = \dim R^{(2)}_0 = \dim R^{(3)}_0 = 3.$$

Using polynomials from relation (3.2) and automorphisms $\theta$ and $\theta^2$, one obtains that the highest weights $\mu$ of $A(\tilde{L}_{D_4}(-2,0))$–modules $V_{D_4}(\mu)$ are obtained as solutions of these 9 polynomial equations:

$$h_{\epsilon_1-\epsilon_2}(h_{\epsilon_1+\epsilon_2}+2) = 0$$

$$h_{\epsilon_2-\epsilon_3}(h_{\epsilon_2+\epsilon_3}+1) = 0$$

$$h_{\epsilon_3-\epsilon_4}h_{\epsilon_3+\epsilon_4} = 0$$

$$h_{\epsilon_3-\epsilon_4}(h_{\epsilon_1+\epsilon_2}+2) = 0$$

$$h_{\epsilon_2-\epsilon_3}(h_{\epsilon_1+\epsilon_4}+1) = 0$$

$$h_{\epsilon_3+\epsilon_4}h_{\epsilon_1-\epsilon_2} = 0$$

$$h_{\epsilon_3+\epsilon_4}(h_{\epsilon_1+\epsilon_2}+2) = 0$$

$$h_{\epsilon_2-\epsilon_3}(h_{\epsilon_1-\epsilon_4}+1) = 0$$

$$h_{\epsilon_1-\epsilon_2}h_{\epsilon_3-\epsilon_4} = 0.$$ 

This easily gives that $\mu = 0$, $-2\omega_1$, $-2\omega_3$, $-2\omega_4$ or $-\omega_2$, and the claim follows from Zhu’s theory.
Claim (ii) follows from the fact that \( \mu = 0 \) is the only dominant integral weight such that \( L_{D_4}(-2, \mu) \) is in the set given in the claim (i).

We have:

**Theorem 4.2.** Vertex operator algebra \( \widetilde{L}_{D_4}(-2, 0) \) is simple, i.e.

\[
L_{D_4}(-2, 0) = \frac{N_{D_4}(-2, 0)}{U(\hat{\mathfrak{g}}).v + U(\hat{\mathfrak{g}}).\theta(v) + U(\hat{\mathfrak{g}}).\theta^2(v)}.
\]

**Proof.** Let \( w \) be a singular vector for \( \hat{\mathfrak{g}} \) in \( \widetilde{L}_{D_4}(-2, 0) \). The classification result from Proposition 4.1 (ii) implies that \( U(\hat{\mathfrak{g}}).w \) is a highest weight \( \hat{\mathfrak{g}} \)–module with highest weight \(-2\Lambda_0\) and that \( w \) is proportional to \( 1 \). The claim follows. \( \Box \)

We conclude:

**Theorem 4.3.**

(i) The set

\[ \{L_{D_4}(-2, 0), L_{D_4}(-2, -2\omega_1), L_{D_4}(-2, -2\omega_2), L_{D_4}(-2, -2\omega_3), L_{D_4}(-2, -\omega_2)\} \]

provides a complete list of irreducible weak \( L_{D_4}(-2, 0) \)–modules from the category \( \mathcal{O} \).

(ii) \( L_{D_4}(-2, 0) \) is the unique irreducible ordinary module for \( L_{D_4}(-2, 0) \).

(iii) Every ordinary \( L_{D_4}(-2, 0) \)–module is completely reducible.

**Proof.** Proposition 4.1 and Theorem 4.2 imply claims (i) and (ii).

(iii) Let \( M \) be an ordinary \( L_{D_4}(-2, 0) \)–module, and let \( w \) be a singular vector for \( \hat{\mathfrak{g}} \) in \( M \). The classification result from (ii) implies that \( U(\hat{\mathfrak{g}}).w \) is a highest weight \( \hat{\mathfrak{g}} \)–module with highest weight \(-2\Lambda_0\). Claim (ii) also implies that any singular vector in \( U(\hat{\mathfrak{g}}).w \) has highest weight \(-2\Lambda_0\) and it is proportional to \( w \). Thus, \( U(\hat{\mathfrak{g}}).w \) is an irreducible \( \hat{\mathfrak{g}} \)–module and the claim follows. \( \Box \)

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