

Noise rectification and fluctuations of an asymmetric inelastic piston

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Abstract – We consider a massive inelastic piston, whose opposite faces have different coefficients of restitution, moving under the action of an infinitely dilute gas of hard disks maintained at a fixed temperature. The dynamics of the piston is Markovian and obeys a continuous Master Equation: however, the asymmetry of restitution coefficients induces a violation of detailed balance and a net drift of the piston, as in a Brownian ratchet. Numerical investigations of such non-equilibrium stationary state show that the velocity fluctuations of the piston are symmetric around the mean value only in the limit of large piston mass, while they are strongly asymmetric in the opposite limit. Only taking into account such an asymmetry, i.e. including a third parameter in addition to the mean and the variance of the velocity distribution, it is possible to obtain a satisfactory analytical prediction for the ratchet drift velocity.

Introduction. – Granular materials have been the subject of intense research in the last 20 years in physics [1]. Most of the non-trivial phenomena that can be observed in a shaken box of sand are due to the inelasticity of collisions among grains [2]. Kinetic energy is dissipated into heat, introducing an intrinsic time irreversibility in the “microscopic” dynamics which can have consequences at a more macroscopic level: for instance species segregation [3], breakdown of energy equipartition [4], apparent Maxwell-demon–like properties such as heat currents against a temperature gradient [5] or mass current against a density gradient [6], ratchet–like net drift of a asymmetrically shaped tracers [7], and so on. It is well known, moreover, that the space asymmetry of an adiabatic piston results in a stationary macroscopic motion [8]. Here we consider, instead, a model of asymmetric granular piston with different coefficients of restitution which, with respect to a previously presented model of granular Brownian ratchet [7], has the advantages of being simpler to be studied analytically as well as realized in the laboratory, and at the same time displays unexpected peculiar properties: in particular we will show how its stationary state is characterized by asymmetric velocity fluctuations, and that this asymmetry becomes crucial when the piston mass is smaller than the mass of surrounding disks. Such a light piston limit, which could be regarded as purely academic in the case of a sect separating two molecular gases, is instead realistic in the case of granular gases, where the role of “molecules” is played by large grains.

Model. – The 2D model consists of a piston of mass $M$ and height $L$ surrounded by a dilute gas of $N$ hard disks of mass $m$ and density $p$. The faces of the piston have two different values of inelasticity, characterized by the coefficients of restitution $\alpha_1$, on the left, and $\alpha_2$, on the right. The piston can only slide, without rotating, along the direction $x$, perpendicular to its faces. The thickness of the piston can be neglected for the purpose of our simulations and calculations. The particles-piston binary collisions are described by the rule:

\begin{align}
V &= V' - (1 + \alpha_i) \frac{e^2}{1 + \epsilon^2} (V' - v'_x), \\
v_x &= v'_x + (1 + \alpha_i) \frac{1}{1 + \epsilon^2} (V' - v'_x), \\
v_y &= v'_y,
\end{align}

\[(1)\]
where $v$ and $v'$ are the post-collisional and pre-collisional disk velocities, respectively, while $V$ and $V'$ are the corresponding velocities of the piston, while $\epsilon^2 = m/M$. Because of the constraint on the piston, its vertical velocity is always 0. The energy of this granular system is not conserved and an external driving mechanism is needed to attain a stationary state. In our model, the surrounding gas is coupled to a thermal bath which keeps its temperature constant: the exact nature of the thermostat is not discussed here (many models have been introduced in the literature, see for example [9,10]), since we focus on the dynamics of the piston which is assumed not to couple directly with the thermostat, but only with the gas particles. For the sake of simplicity we assume that the surrounding gas is homogeneous in space and time, with a Maxwell-Boltzmann velocity probability density function (pdf) $\phi(v_x, v_y)$ with zero average and given variance $\langle v_x^2 \rangle = \langle v_y^2 \rangle$ (see below). Such an assumption can be considered realistic as long as the gas is dilute and the characteristic time of coupling with the external heat bath is much shorter than that with the piston. In this case also Molecular Chaos for piston-disk collisions can safely be assumed, allowing for the use of the Direct Simulation Monte-Carlo (DSMC) algorithm to simulate the piston dynamics [11]. Moreover, without loss of generality, we can choose one side of the ratchet to be elastic. Based on the rules (1), in fact, a system with mass $M$ and coefficients of restitution $0 \leq \epsilon_2 < \epsilon_1 \leq 1$ is equivalent to one with $\epsilon_1 = 1$ and effective parameters $M'$ and $0 \leq \epsilon_2' \leq 1$ given by the following relations:

$$M' = \frac{m(1 - \epsilon_1) + 2M}{1 + \epsilon_1},$$

$$\epsilon_2' = 1 - 2\frac{\epsilon_1 - \epsilon_2}{1 + \epsilon_1}. \tag{2}$$

In the following, therefore, we choose $\epsilon_1 = 1$ and we study the behavior of the piston for different values of the mass ratio $\epsilon^2$ and coefficient of restitution $\epsilon_2$.

**Results and discussion.** — Our choice of the initial conditions for the piston are $V(0) = 0$ and $X(0) = 0$. After an initial transient whose duration depends on all control parameters such as collision frequency, coefficients of restitution, masses, etc., the piston reaches a stationary regime. In the following we shall indicate as $T_g = m\langle v_x^2 \rangle$ the value of the gas temperature and $T_r = M\langle (V' - V)^2 \rangle$ the ratchet temperature. The DSMC simulations have been performed using $T_g = 1$, $m = 1$. We focus here on the behavior of the piston, whose position and velocity at time $t$ are denoted as $X(t)$ and $V(t)$, respectively. Trajectories of the piston in its non-equilibrium stationary state, i.e. discarding transients, for particular choices of the parameters $M$ and $\epsilon_2$, are displayed in fig. 1a. When the system is totally elastic or $\epsilon_1 = \epsilon_2$ no average motion occurs. On the contrary, if the inelasticity of the left and right side are different, the piston shows a mechanism of rectification of disorder-induced fluctuations driven by the collisions and it drifts with average velocity $\langle V \rangle \neq 0$. In particular this drift appears to be always oriented towards the side of the piston with smaller restitution coefficient, the positive direction in our case, so that $\langle V \rangle > 0$. This phenomenon can be understood by considering the average momentum transferred by the gas to the piston, $M\langle V' - V \rangle$. Assuming that the piston is slower than the thermal velocity of the gas, it is easy to show that it will experience a viscous drag force $-\gamma V$, where $\gamma$ is given by $(2 + \alpha_1 + \alpha_2)\rho \sqrt{M/(M + m)} \sqrt{2mT_g/\pi}$ plus a net force

$$F = \frac{\rho T_g}{2} \frac{M}{M + m} (\epsilon_1 - \epsilon_2). \tag{3}$$

In other words, the momentum transferred by the gas is smaller on the more inelastic side, originating the drift in that direction. From the simulation one also observes...
that the velocity of the drift is bigger if the elasticity is smaller, while it seems not much influenced by the piston mass. Mass, on the contrary, influences fluctuations in the trajectories, which are larger the lighter the ratchet. In order to characterize the behavior of these fluctuations, we show, in fig. 1b and fig. 1c, the pdf $P(V)$ of the rescaled velocity of the piston. When the piston is massive $\epsilon^2 < 1$, the distribution $P(V)$ does not display any appreciable difference with respect to a Gaussian with finite mean. On the contrary, when the mass $M$ of the piston is equal or smaller than that of surrounding disks, $P(V)$ results asymmetric with a larger tail for positive velocities. Such an effect occurs at smaller elasticity and it is stronger if $\epsilon^2 \gg 1$. Note also that in all our simulations we always have $\langle V \rangle > 0$, while in some cases the maximum of the velocity pdf occurs at negative values.

In the dilute gas limit, it is possible to study the piston dynamics by means of a Master Equation (ME) for $P(V, t)$ which can be written as [7]

$$\frac{\partial P(V, t)}{\partial t} = \int dV' W(V|V') P(V', t) - W(V'|V) P(V, t),$$

where the transition rate is

$$W(V|V') = \frac{\rho L}{\eta} \sum_{k=1}^{2} (-1)^k \int d\Theta (-1)^k (V' - v_x)$$

$$= \sum_{k=1}^{2} \langle V' - v_x \rangle \delta(V' - V + \mu_k(V' - v_x))$$

(5)

with $\mu_k = (1 + \alpha_k) \epsilon^2/(1 + \epsilon^2)$ (see eq. (1)) and $\Theta$ is the Heaviside step function.

In order to obtain quantitative information about the velocity and granular temperature of the ratchet, we can write a system of equations describing the evolution of the first moments of the distribution, starting from (4) and (5), and invoke some approximation to obtain a closed set. By direct experience we have found that, due to the asymmetry of $P(V)$, it is essential to consider a further parameter $\xi = (\langle V - \langle V \rangle^2 \rangle)^3$ in addition to the average velocity and granular temperature [12]. To this aim we assume that the pdf of the piston can be written as

$$P(V) = \sqrt{\frac{M}{2\pi T_r}} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{\pi^n}} \exp \left[ -\frac{M}{2T_r} (V - \langle V \rangle)^2 \right].$$

(6)

Imposing the normalization and that $\langle (V - \langle V \rangle)^2 \rangle = T_r/M$, the series (6), with a truncation at $n = 3$, becomes

$$P(V) = \sqrt{\frac{M}{2\pi T_r}} \left( 1 - \frac{\xi}{6 \sqrt{3}} \right) \exp \left[ -\frac{M}{2T_r} (V - \langle V \rangle)^2 \right].$$

(7)

In the above expression $\xi$ represents the first term that is a measure of the asymmetry of $P(V)$ about the average value. Under these assumptions the equations for $\langle V \rangle$, $T_r$, and $\xi$ can be obtained multiplying both sides of eq. (4) by $V$, $M(V - \langle V \rangle)^2$ and $(V - \langle V \rangle)^3$, respectively and performing the integrations. After some calculations, expressions in analytical form for these equations can be derived if it is assumed that $\langle V \rangle \ll V_{th}$, where $V_{th} = \sqrt{T_r/M}$. By retaining only the terms of first order in $\langle V \rangle$, one obtains the following differential equations:

$$\frac{\partial \rho}{\partial t} = -\frac{\rho T_r}{2M} a_1(\eta) - \rho \sqrt{\frac{2T_r}{\pi M}} a_2(\eta) \langle V \rangle$$

$$-\frac{\rho}{3} \sqrt{\frac{M}{2\pi T_r}} a_3(\eta) \xi,$$

(8)

$$\frac{\partial T_r}{\partial t} = \frac{\rho}{2} \sqrt{2\pi T_r^3} b_1(\eta) + \frac{\rho}{2} T_r b_2(\eta) \langle V \rangle + \frac{M \rho}{2} b_3 \xi$$

$$+ \frac{\rho}{6} \sqrt{\frac{M^3}{2\pi T_r}} b_4(\eta) \langle V \rangle,$$

(9)

$$\frac{\partial \xi}{\partial t} = -\frac{3T_r^2}{2M^2} c_1(\eta) - \frac{\rho}{2} \sqrt{\frac{2T_r^3}{\pi M}} c_2(\eta) \langle V \rangle$$

$$+ \frac{\rho}{2} \sqrt{\frac{T_r^3}{2\pi M^2}} c_3(\eta) \xi + \frac{\rho}{2} c_4(\eta),$$

(10)

where $\eta = T_r/T_g$. The explicit expressions for coefficients $a_1$, $a_2$, $a_3$, $b_1$, $b_2$, $b_3$, $c_1$, $c_2$, $c_3$ and $c_4$ in terms of $\eta$ can be found in [13].

We can solve eqs. (8)–(10) in the stationary state (all time derivatives are put to zero), obtaining

$$\langle V \rangle = -\frac{3}{2} \sqrt{\frac{\pi}{V_{th}}} \frac{a_1(\eta) g_{34}(\eta) + 2a_2(\eta) g_{14}(\eta)}{a_2(\eta) g_{34}(\eta) + a_3(\eta) g_{24}(\eta)}$$

(11)

$$\xi = -\frac{3V_{th}^2}{2a_3(\eta)} \left[ \sqrt{2\pi V_{th}} a_1(\eta) + 4a_2(\eta) \langle V \rangle \right],$$

(12)

$$3\sqrt{2\pi V_{th}} [2b_2(\eta) V_{th}^2 \langle V \rangle + 2b_3 \xi] + 2b_4(\eta) \langle V \rangle \xi + 24 V_{th}^4 b_1(\eta) = 0,$$

(13)

where $g_{14}(\eta) = b_4(\eta) c_1(\eta) + 4b_3(\eta) c_4(\eta)$, $g_{24}(\eta) = b_2(\eta) c_4(\eta) + 3\pi b_2(\eta) c_4$ and $g_{34}(\eta) = b_4(\eta) c_3(\eta) - 6\pi b_3 c_4$ [13].

A numerical solution of system (11)–(13) gives the stationary values of $\langle V \rangle$, $\eta$ and $\xi$. In the symmetric case $a_1 = a_2 = a$, it turns out that $a_1(\eta) = b_2(\eta) = b_3 = c_1(\eta) = c_4 = 0$ the system has the solutions $\langle V \rangle = \xi = 0$ and from (13) we get $\eta = (1 + \alpha)/[\epsilon^2 (1 + \epsilon)^2 - 1]$, that is the same solution already obtained by Martin et al. [14].

When $\alpha_1 = \alpha_2 = 1$ the temperature of the piston is equal to that of the gas. The results in the asymmetric case are shown in figs. 2–4 together with the simulation data.

Figure 2 displays a good agreement between theory and DSMC for the rescaled observable $\langle V \rangle/V_{th}$ for all values of $\alpha_2$ and $\epsilon^2$ studied. It is remarkable that our results are good even if the above assumption does not hold true, i.e. when $\langle V \rangle \approx V_{th}$. In the inset of fig. 2 we show the behavior of $\langle V \rangle/V_{th}$ as a function of the coefficient of restitution $\alpha_2$ for different $\epsilon^2$. The theoretical results for $(M \leq m)$, derived assuming a Gaussian $P(V)$ (i.e. $\xi = 0$) are also displayed. They appear to disagree with the numerical
Fig. 2: The velocity of the piston, rescaled with its thermal velocity $V_{th} = \sqrt{T_r/M}$, as function of the coefficient of the restitution $\alpha_2$ for different values of the parameter $\epsilon^2$: 10 (circles), 1 (squares), 0.1 (diamonds) and 0.01 (triangles). The symbols correspond to the simulation data while the lines correspond to the solutions obtained from eqs. (11)–(13). Inset: The velocity of the piston, rescaled with the thermal velocity of the gas $v_{gth} = \sqrt{T_g/m}$, as a function of the coefficient of the restitution $\alpha_2$ for the same values of $\epsilon^2$. The dashed and dot-dashed lines correspond to the theoretical calculation for the cases $\epsilon^2 = 10$ and $\epsilon^2 = 1$, respectively, assuming $P(V)$ exactly Gaussian.

Fig. 3: The temperature ratio $\eta = T_r/T_g$ as a function of the coefficient of the restitution $\alpha_2$ for the same cases of fig. 2. The symbols correspond to the simulation data while the lines correspond to the solutions obtained from eqs. (11)–(13).

Fig. 4: The third moment $\xi = \langle (V - \langle V \rangle)^3 \rangle$, rescaled with the quantity $v_{gth}^3$ as a function of the parameter $\epsilon^2$ for $\alpha_2 = 0.3$. The symbols correspond to the simulation data while the line corresponds to the solution obtained from eqs. (11)–(13).

Fig. 5: The diffusion coefficient as a function of $\epsilon^2$ calculated in two different ways (see text) for $\alpha_2 = 0.3$ and $\alpha_2 = 0.9$. Inset: a) the self-diffusion $d_2(t) = \langle (\langle d(t) \rangle - \langle d(t) \rangle)^2 \rangle$ of the piston vs. time; b) the self-correlation function of the piston-velocity vs. time. The dashed line corresponds to the theoretical prediction of $D_r$ (see text) for $\epsilon^2 = 0.01, \alpha_2 = 0.9$ and $\tau_c = 2.5$.

results, suggesting that in these cases a Gaussian pdf for the piston velocity fluctuations is a very poor assumption. The third moment $\xi$, that is a marker of the asymmetry of the pdf, is not small (see fig. 4) and expansion (6) is crucial. The small differences between theory and DSMC for $\epsilon^2 = 10$ (more visible in the inset of fig. 2) are probably due to the fact that we have considered only the first three terms of the expansion. In fact the numerical results show for large values of $\epsilon^2$ the fourth moment becomes relevant so that one should take into account the evolution of such a quantity in the coupled set of equations in order to give more accurate results.

We would like to add a comment concerning the high-velocity tails of the distribution function $P(V)$. In spite of the fact that the body of such a distribution is not Gaussian and skewed, we have been able to prove that its tails are of Gaussian nature, but with different effective temperatures in each tail, reflecting the asymmetry of the piston. In particular, the temperature of each tail is given by a formula similar to the one presented by Martin and Piasecki for the inelastic intruder:

$$T_i = T \frac{1 + \alpha_i}{(1 - \alpha_i) m/M + 2}.$$  \hspace{1cm} (14)

Finally, we have studied, in fig. 5, the diffusion coefficient $D$ of the piston for different values of $\epsilon^2$. In order
to better characterize the situation of a light ratchet, we have evaluated $D$ in two different ways. First we have measured it by its definition $D = \lim_{t \to \infty} \frac{(\langle d(t) - \langle d \rangle \rangle^2)}{2t}$, where $d(t) = X(t) - X(0)$ is the displacement of the tracer with respect to its initial position at a time $t = 0$, taken when the whole system has become stationary. Second we have used the self-correlation function of the piston velocity defined by $C(t) = \langle (V(t) - \langle V \rangle)(V(0) - \langle V \rangle) \rangle$: its time integral $\int_0^\infty C(t') dt'$ gives an exact measure of the diffusion coefficient. This time integral can also be estimated by assuming an exponential decay $C(t) = T_e \exp(-t/\tau)$: in this case $D_e = T_e \tau / M$. The asymptotic decay, in fact, is always exponential, since the process is Markovian by construction. This is checked in the insets of fig. 5. From eqs. (11)–(13) it can be deduced that, in the large $M/m$ limit the dynamics of the piston is analogous to an Ornstein-Uhlenbeck process with an effective friction $1/\tau$ and noise intensity $2T_e/\tau$, with $\tau = \tau_e/\sqrt{[\epsilon^2(3 + \alpha_2)]}$. $\tau_e$ is the average collision time. From such an expression it is seen that the diffusion coefficient does not depend on the mass and very little on the inelasticity. For smaller values of $M/m$, on the contrary, the diffusion coefficient increases, as displayed in fig. 5. A growing discrepancy between $D_e$ and $D$ is observed in this limit, due to the fact that the first part of the correlation decay is not exactly exponential.

**Conclusion.** – We have discussed a simple example of noise rectification when two identically driven granular gases are separated by a moving piston. The difference of piston inelasticities between the two faces, induces a stationary drift and the appearance of non-Gaussian fluctuations, which become highly asymmetric when inertia is reduced. A description of the dynamics in terms of the first three moments of the velocity distribution is sufficient to predict most of the physics of this system. It is quite rare to reach such a detailed knowledge of a system with asymmetric non-Gaussian fluctuations: this is achieved, here, thanks to the empirical observation that the third cumulant of the distribution is dominant over cumulants of higher order, a fact that is not guaranteed in general [15]. We are also confident that the assumption used, mainly that the two gases on the sides of the piston are very dilute and driven at high frequency, can be reproduced in the laboratory and could be at the base of ratcheting mechanisms observed in real granular materials.

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