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Abstract. Let $X$ be a real reflexive Banach space and $X^*$ be its dual space. Let $G_1$ and $G_2$ be open subsets of $X$ such that $G_2 \subseteq G_1$, $0 \in G_2$, and $G_1$ is bounded. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, $A : X \supset D(A) \to 2^{X^*}$ be a maximal monotone and positively homogeneous operator of degree $\gamma > 0$, $C : X \supset D(C) \to X^*$ be a bounded demicontinuous operator of type $(S_{\gamma})$ with respect to $D(L)$, and $T : G_1 \to 2^{X^*}$ be a compact and upper-semicontinuous operator whose values are closed and convex sets in $X^*$. We first take $L = 0$ and establish the existence of nonzero solutions of $Ax + Cx + Tx \ni 0$ in the set $G_1 \setminus G_2$. Secondly, we assume that $A$ is bounded and establish the existence of nonzero solutions of $Lx + Ax + Cx \ni 0$ in $G_1 \setminus G_2$. We remove the restrictions $\gamma \in (0, 1]$ for $Ax + Cx + Tx \ni 0$ and $\gamma = 1$ for $Lx + Ax + Cx \ni 0$ from such existing results in the literature. We also present applications to elliptic and parabolic partial differential equations in general divergence form satisfying Dirichlet boundary conditions.

1. Introduction and preliminaries

Let $X$ be a real reflexive Banach space and $X^*$ be its topological dual space. The symbol $2^{X^*}$ denotes the collection of all subsets of $X^*$. The norm on $X$ is denoted by $\| \cdot \|_X$. When there is no risk of misunderstanding, the norms on $X$ and $X^*$ are both denoted by $\| \cdot \|$. The pairing $\langle x^*, x \rangle$ denotes the value of the functional $x^* \in X^*$ at $x \in X$. The symbols $\partial Z, \text{Int} Z, \overline{Z}$ and $\text{co} Z$ denote the boundary, interior, closure, and convex hull of the set $Z \subseteq X$, respectively. The symbol $B_X(0, R)$ denotes the open ball of radius $R > 0$ with center at 0 in $X$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$, respectively. For a sequence $\{x_n\}$ in $X$ and $x_0 \in X$, we denote by $x_n \to x_0$ and $x_n \rightharpoonup x_0$ the strong convergence and weak convergence, respectively. Given another real Banach $Y$, an operator $T : X \supset D(T) \to Y$ is said to be bounded if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of $Y$. The operator $T$ is said to be compact if it maps bounded subsets of $D(T)$ onto relatively compact subsets in $Y$. The operator $T$ is said to be demicontinuous if it is strong-to-weak continuous on $D(T)$.
A multivalued operator $A$ from $X$ to $X^*$ is written as $A : X \supset D(A) \to 2^{X^*}$, where $D(A) = \{x \in X : Ax \neq \emptyset\}$ is the effective domain of $A$. Here, $Ax$ means $A(x)$, and these notations are used interchangeably in the sequel. We denote the graph of $A$ by $\text{Gr}(A)$, i.e., $\text{Gr}(A) = \{(x, y) : x \in D(A), y \in Ax\}$.

**Definition 1.1.** An operator $A : X \supset D(A) \to 2^{X^*}$ is said to be positively homogeneous of degree $\gamma > 0$ if $(x, y) \in \text{Gr}(A)$ implies $sx \in D(A)$ for all $s \geq 0$ and $(sx, s^\gamma y) \in \text{Gr}(A)$.

**Remark 1.2.** An equivalent condition for an operator $A : X \supset D(A) \to 2^{X^*}$ to be positively homogeneous of degree $\gamma > 0$ is that $x \in D(A)$ implies $sx \in D(A)$ for all $s \geq 0$ and $s^\gamma Ax \subset A(sx)$. It follows that a positively homogeneous operator $A$ of degree $\gamma > 0$ satisfies $0 \in A(0)$. When $A$ is positively homogeneous of degree $\gamma > 0$, it can be verified that $x \in D(A)$ implies $sx \in D(A)$ for all $s > 0$ and $s^\gamma Ax = A(sx)$. However, in general, the property $s^\gamma Ax = A(sx)$ may not be true for $s = 0$. For example, let $A : \mathbb{R} \supset [0, \infty) \to 2^\mathbb{R}$ be given by

$$Ax = \begin{cases} (-\infty, 0] & \text{for } x = 0 \\ x^{-\gamma} & \text{for } x > 0. \end{cases}$$

Clearly, $A(0) = (-\infty, 0] \neq \{0\}$.

A gauge function is a strictly increasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$. The duality mapping of $X$ corresponding to a gauge function $\varphi$ is the mapping $J_\varphi : X \supset D(J_\varphi) \to 2^{X^*}$ defined by

$$J_\varphi x = \{x^* \in X^* : \langle x^*, x \rangle = \varphi(\|x\|)\|x\|, \|x^*\| = \varphi(\|x\|)\}, \quad x \in X.$$ 

The Hahn-Banach theorem ensures that $D(J_\varphi) = X$, and therefore $J_\varphi : X \to 2^{X^*}$ is, in general, a multivalued mapping. The duality mapping corresponding to the gauge function $\varphi(r) = r^\gamma$ is called the normalized duality mapping and denoted by $J$. It is well-known that the duality mapping $J_\varphi$ satisfies

$$J_\varphi x = \frac{\varphi(\|x\|)}{\|x\|} Jx, \quad x \in X \setminus \{0\}.$$ 

Since $J$ is homogeneous of degree 1, we have

$$J_\varphi (sx) = \frac{\varphi(s\|x\|)}{\|x\|} Jx, \quad (s, x) \in \mathbb{R}_+ \times (X \setminus \{0\}).$$

In particular, when $\varphi(r) = r^{p-1}$, $1 < p < \infty$, we obtain $J_\varphi x = \|x\|^{p-2} Jx$, $x \in X \setminus \{0\}$, which implies

$$J_\varphi (sx) = s^{p-1} J_\varphi x, \quad (s, x) \in \mathbb{R}_+ \times X,$$

i.e., $J_\varphi$ is positively homogeneous of degree $p - 1$.

When $X$ is reflexive and both $X$ and $X^*$ are strictly convex, the inverse $J^{-1}_\varphi$ of $J_\varphi$ is the duality mapping of $X^*$ with the gauge function $\varphi^{-1}(r) = r^{q-1}$, where $q$ is given by $1/p + 1/q = 1$. It is easy to verify that

$$J^{-1}_\varphi (sx^*) = s^{q-1} J^{-1}_\varphi x^*, \quad (s, x^*) \in \mathbb{R}_+ \times X^*.$$ 

(1.1)

It is clear that $J_\varphi$ is positively homogeneous of degree $\gamma > 0$ if and only if $\varphi$ is positively homogeneous of degree $\gamma > 0$. Additional properties of duality mappings in connection with Banach space geometry can be found in Alber and Ryazantseva [17] and Cioranescu [19].
Definition 1.3. An operator \( A : X \ni u \mapsto Ax + Cu \in 2^X \) is said to be monotone if for all \( (x,u), (y,v) \in \text{Gr}(A) \) we have \( \langle u - v, x - y \rangle \geq 0 \). A monotone operator \( A : X \ni u \mapsto Ax + Cu \in 2^X \) is said to be maximal monotone if \( \text{Gr}(A) \) is maximal in \( X \times X^* \), when \( X \times X^* \) is partially ordered by set inclusion.

In what follows, we assume that \( X \) is reflexive and both \( X \) and \( X^* \) are strictly convex. It is well-known that the duality mapping \( J_\varphi \) is maximal monotone. A monotone operator \( A \) is maximal if and only if \( \text{Gr}(A) = X^* \) for all \( \lambda \in (0, \infty) \) and all gauge functions \( \varphi \). For a proof of this result for \( \varphi(v) = r^{p-1}, 1 < p < \infty \), the reader is referred, for example, to Barbu [10] Theorem 2.3.

Definition 1.4. Let \( L : X \ni u \mapsto Lx \in X^* \) be a densely defined linear maximal monotone operator. An operator \( C : X \ni u \mapsto Cx \in X^* \) is said to be of type \( (S_+) \) with respect to \( D(L) \) if for every sequence \( \{x_n\} \subset D(L) \cap D(C) \) with \( x_n \to x_0 \) in \( X \), \( Lx_n \to Lx_0 \) in \( X^* \) and
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0,
\]
we have \( x_n \to x_0 \) in \( X \). In this case, if \( L = 0 \), then \( C \) is said to be of type \( (S_+) \).

Definition 1.5. A family of operators \( C(s) : X \ni u \mapsto C(s)x \in X^* \), \( s \in [0, 1] \), is said to be a homotopy of type \( (S_+) \) with respect to \( D(L) \) if for every sequence \( \{x_n\} \subset D(L) \cap G \) with \( x_n \to x_0 \) in \( X \) and \( Lx_n \to Lx_0 \) in \( X^* \), \( \{s_n\} \subset [0, 1] \) with \( s_n \to s_0 \) and
\[
\limsup_{n \to \infty} \langle C(s_n)x_n, x_n - x_0 \rangle \leq 0,
\]
we have \( x_n \to x_0 \) in \( X \), \( x_0 \in G \) and \( C(s_0)x_n \to C(s_0)x_0 \) in \( X^* \). In this case, if \( L = 0 \), then \( C(s) \) is said to be a homotopy of type \( (S_+) \). A homotopy \( C(s) \) of type \( (S_+) \) with respect to \( D(L) \) is bounded if the set \( \{C(s)x : s \in [0, 1], x \in G\} \) is bounded.

Definition 1.6. An operator \( T : X \ni u \mapsto Tx \in 2^X \) is said to be of class \( (P) \) if
\begin{itemize}
  \item[(i)] it maps bounded sets to relatively compact sets;
  \item[(ii)] for every \( x \in D(T) \), \( Tx \) is a closed and convex subset of \( X^* \); and
  \item[(iii)] \( T(\cdot) \) is upper-semi-continuous, i.e., for every closed set \( F \subset X^* \), the set \( T^-(F) = \{x \in D(T) : Tx \cap F \neq \emptyset\} \) is closed in \( X \).
\end{itemize}

Hu and Papageorgiou introduced the operators of class \( (P) \) in [21]. We recall a compact-set valued upper-semi-continuous operator \( T \) is closed. Furthermore, given an operator \( T \) of class \( P \) and a sequence \( \{(x_n, y_n)\} \subset \text{Gr}(T) \) such that \( x_n \to x \in D(T) \), the sequence \( \{y_n\} \) has a cluster point in \( Tx \).

This paper is organized as follows. In Section 2 we study variants of the standard Yosida approximants introduced in Brézis, Crandall, and Pazy [14] and their fundamental properties. Since the topological degree theory for \( (S_+) \)-operators is employed to establish the main existence results in the later sections, we provide several results involving variants of Yosida approximants related to the Browder degree theory [16].

In Section 3 we first prove the existence of nonzero solutions of \( Ax + Cx + Tx \ni 0 \) by utilizing the topological degree theories developed by Browder [18] and Skrypnik [32]. In this case, \( A \) is maximal monotone with \( A(0) = \{0\} \) and positively homogeneous of degree \( \gamma > 0 \), \( C \) is bounded demicontinuous of type \( (S_+) \), and \( T \) is of class \( (P) \). This result extends an analogous result for \( \gamma \in (0, 1] \) established in [2] to an arbitrary degree of homogeneity \( \gamma > 0 \). Another main result established in
this section is the existence of nonzero solutions of \( Lx + Ax + Cx \ni 0 \), where \( L, C \) are as above, and \( A \) is a bounded maximal monotone and positively homogeneous of degree \( \gamma > 0 \). This result extends an analogous result for \( \gamma = 1 \) established in [2] to an arbitrary degree of homogeneity \( \gamma > 0 \).

In Section 4, we present some applications of the theories developed in Section 3 to elliptic and parabolic partial differential equations, in general, divergence form that include \( p \)-Laplacian with \( 1 < p < \infty \) and satisfy Dirichlet boundary conditions.

For additional facts and various topological degree theories related to the subject of this paper, the reader is referred to Adhikari and Kartsatos [4, 5], Kartsatos and Skrypnik [24, 26], and Zeidler [34].

2. Variants of Yosida approximants and related properties

Let \( X \) be a strictly convex and reflexive Banach space with strictly convex \( X^* \). By using the duality mapping \( J_\varphi \) corresponding to an arbitrary gauge function \( \varphi \), we study variants of the Yosida approximants in Brézis et al. [14] and resolvents of a maximal monotone operator \( A : X \supset D(A) \to 2^{X^*} \). For each \( \lambda > 0 \) and each \( x \in X \), the inclusion

\[
0 \in J_\varphi(x_\lambda - x) + \lambda Ax_\lambda \tag{2.1}
\]

has a unique solution \( x_\lambda \in D(A) \) (see Proposition 2.1 (i)). We define \( J_\varphi^\lambda : X \to D(A) \subset X \) and \( A_\lambda^\varphi : X \to X^* \) by

\[
J_\varphi^\lambda x := x_\lambda \quad \text{and} \quad A_\lambda^\varphi x := \frac{1}{\lambda} J_\varphi(x - J_\varphi^\lambda x), \quad x \in X. \tag{2.2}
\]

The operators \( A_\lambda^\varphi \) and \( J_\varphi^\lambda \) are variants of the standard Yosida approximant \( A_\lambda \) and resolvent \( J_\lambda \) of \( A \). For each \( x \in X \), we have

\[
A_\lambda^\varphi x \in A(J_\varphi^\lambda x) \quad \text{and} \quad x = J_\varphi^\lambda x + J_\varphi^{-1}(\lambda A_\lambda^\varphi x).
\]

When \( \varphi(r) = r^{p-1} \), a splitting of \( x \) in terms of \( A_\lambda^\varphi \) and \( J_\varphi^\lambda \) is

\[
x = J_\varphi^\lambda x + \lambda^{p-1} J_\varphi^{-1}(A_\lambda^\varphi x), \tag{2.3}
\]

and therefore

\[
A_\lambda^\varphi x = (A^{-1} + \lambda^{p-1} J_\varphi^{-1})^{-1} x, \quad x \in X. \tag{2.4}
\]

It is easy to verify that \( A = A_\lambda^\varphi \) if and only if \( A = 0 \). In fact, if \( A = 0 \), then \( J_\varphi^\lambda = I \), the identity operator on \( X \). Moreover, if \( 0 \in D(A) \) and \( 0 \in A(0) \), then \( A_\lambda^\varphi 0 \equiv 0 \).

The choice of an appropriate gauge function is essential for the main existence results in this paper. The following proposition summarizes some important properties of \( A_\lambda^\varphi \) and \( J_\varphi^\lambda \) along the lines of analogous properties of \( A_\lambda \) and \( J_\lambda \). A complete proof is provided here for the reader’s convenience.

**Proposition 2.1.** Let \( X \) be a strictly convex and reflexive Banach space with strictly convex dual \( X^* \) and \( A : X \supset D(A) \to 2^{X^*} \) be a maximal monotone operator. Then the following statements hold.

(i) The operator \( A_\lambda^\varphi \) is single-valued, monotone, bounded on bounded subsets of \( X \), and demicontinuous from \( X \) to \( X^* \).

(ii) For every \( x \in D(A) \) and \( \lambda > 0 \), we have

\[
\|A_\lambda^\varphi x\| \leq |Ax| := \inf\{\|x^*\| : x^* \in Ax\}.
\]
(iii) The operator $J_\varphi^\varphi$ is bounded on bounded subsets of $X$, demicontinuous from $X$ to $D(A)$, and

$$
\lim_{\lambda \to 0} J_\varphi^\varphi x = x \quad \text{for all } x \in \co D(A).
$$

(iv) If $\lambda_n \to 0$, $x_n \rightharpoonup x$ in $X$, $A_\lambda^\varphi x_n \rightharpoonup y$ and

$$
\limsup_{n,m \to \infty} (A_\lambda^\varphi x_n - A_\lambda^\varphi x_m, x_n - x_m) \leq 0,
$$

then $(x, y) \in \Gr(A)$ and

$$
\lim_{n,m \to \infty} (A_\lambda^\varphi x_n - A_\lambda^\varphi x_m, x_n - x_m) = 0.
$$

(v) For every sequence $\{\lambda_n\}$ with $\lambda_n \to 0$, $A_\lambda^\varphi x \to A^{(0)} x$ for all $x \in D(A)$. In addition, if $X^*$ is uniformly convex, then $A_\lambda^\varphi x \to A^{(0)} x$ for all $x \in D(A)$.

(vi) If $\lambda_n \to 0$ and $x \notin D(A)$, then

$$
\lim_{n \to \infty} \|A_\lambda^\varphi x\| = \infty.
$$

Proof. (i) We first show that $J_\varphi^\varphi$ is single-valued. Given $x \in X$ and $\lambda > 0$, let $x_\lambda$ and $\tilde{x}_\lambda$ be solutions of (2.1). Take $y \in Ax_\lambda$ and $\tilde{y} \in A\tilde{x}_\lambda$ such that

$$
J_\varphi(x_\lambda - x) + \lambda y = 0 \quad \text{and} \quad J_\varphi(\tilde{x}_\lambda - x) + \lambda \tilde{y} = 0.
$$

This along with the monotonicity of $A$ and $J_\varphi$ implies

$$
\langle J_\varphi(x_\lambda - x) - J_\varphi(\tilde{x}_\lambda - x), (x_\lambda - x) - (\tilde{x}_\lambda - x) \rangle = 0. \quad \text{(2.5)}
$$

Since $X$ is strictly convex, it follows that $J_\varphi$ is strictly monotone, i.e., for $u_1, u_2 \in X$, we have

$$
\langle J_\varphi u_1 - J_\varphi u_2, u_1 - u_2 \rangle > 0 \text{ if and only if } u_1 \neq u_2.
$$

It follows from (2.5) that $x_\lambda = \tilde{x}_\lambda$. Thus, $J_\varphi^\varphi$ is single-valued, and therefore $A_\lambda^\varphi$ is also single-valued. It is easy to verify the monotonicity of $A_\lambda^\varphi$.

To show $A_\lambda^\varphi$ is bounded, let $B \subset X$ be bounded. For each $x \in B$, let $x_\lambda = J_\varphi^\varphi x$. Let $(u, v) \in \Gr(A)$. Using (2.1), it follows that

$$
\langle J_\varphi(x_\lambda - x), x_\lambda - u \rangle = 0,
$$

where $y_\lambda \in Ax_\lambda$. This implies

$$
\langle J_\varphi(x_\lambda - x), x_\lambda - u \rangle = -\lambda \langle y_\lambda, x_\lambda - u \rangle \leq \lambda \langle v, u - x_\lambda \rangle.
$$

The last inequality follows from the monotonicity of $A$. It then follows that

$$
\langle J_\varphi(x_\lambda - x), x_\lambda - x \rangle = \langle J_\varphi(x_\lambda - x), x_\lambda - u \rangle + \langle J_\varphi(x_\lambda - x), u - x \rangle

\leq \lambda \langle v, u - x_\lambda \rangle + \langle J_\varphi(x_\lambda - x), u - x \rangle

= \lambda \langle v, u - x \rangle + \lambda \langle v, x - x_\lambda \rangle + \langle J_\varphi(x_\lambda - x), u - x \rangle. \quad \text{(2.6)}
$$

This implies

$$
\varphi(\|x_\lambda - x\|) \|x_\lambda - x\| \leq \lambda \|v\| (\|u - x\| + \|x_\lambda - x\|) + \varphi(\|x_\lambda - x\|) \|u - x\|. \quad \text{(2.7)}
$$

If $\{x_\lambda : x \in B\}$ is unbounded, the inequality (2.7) yields a contradiction. Thus, $J_\varphi^\varphi$ is bounded on $B$. Since $\varphi$ is bounded on $B$, it follows from (2.2) that $A_\lambda^\varphi$ is also bounded on $B$.

Let $\{x_n\} \subset X$ be such that $x_n \rightharpoonup x_0 \in X$ as $n \to \infty$. Denote $u_n = J_\lambda^\varphi x_n$ and $v_n = A_\lambda^\varphi x_n$, so that

$$
J_\varphi(u_n - x_n) + \lambda v_n = 0. \quad \text{(2.8)}
$$
Since \( J^\varphi \) and \( A^\varphi \) are bounded on bounded sets, both \( \{u_n\} \) and \( \{v_n\} \) are bounded. Since \( J^\varphi \) and \( A \) are monotone, it follows from

\[
\langle J^\varphi(u_n - x_n) - J^\varphi(u_m - x_m), (u_n - x_n) - (u_m - x_m) \rangle
\]

\[
= -\lambda(v_n - v_m, (u_n - x_n) - (u_m - x_m))
\]

that

\[
\lim_{n, m \to \infty} \langle v_n - v_m, u_n - u_m \rangle = 0,
\]

\[
\lim_{n, m \to \infty} \langle J^\varphi(u_n - x_n) - J^\varphi(u_m - x_m), (u_n - x_n) - (u_m - x_m) \rangle = 0.
\]

Passing to subsequences, we may assume that \( u_n \to u_0 \) in \( X \), \( v_n \to v_0 \) in \( X^* \), and \( J^\varphi(u_n - x_n) \to w_0 \) in \( X^* \) for some \( u_0 \in X \) and some \( v_0, w_0 \in X^* \). By [10, Lemma 2.3], it follows that \( (u_0, v_0) \in \text{Gr}(A) \) and \( (u_n - x_n, w_0) \in \text{Gr}(J^\varphi) \). Using all these in (2.8), we obtain \( J^\varphi(u_0 - u_0) + \lambda v_0 = 0 \), which implies \( u_0 = J^\varphi x_0 \) and \( v_0 = A^\varphi x_0 \), i.e., \( J^\varphi x_n \to J^\varphi x_0 \) and \( A^\varphi x_n \to A^\varphi x_0 \) as \( n \to \infty \). This proves the demicontinuity of \( J^\varphi \) and \( A^\varphi \).

(ii) Let \( x \in D(A) \) and \( \lambda > 0 \). Let \( y \in Ax \) and \( x_\lambda = J^\varphi x \). Then

\[
0 \leq \langle y - Ax, x - x_\lambda \rangle
\]

\[
= \langle y, x - x_\lambda \rangle - \frac{1}{\lambda} \varphi(||x - x_\lambda||)||x - x_\lambda||
\]

\[
\leq ||y|| ||x - x_\lambda|| - \frac{1}{\lambda} \varphi(||x - x_\lambda||)||x - x_\lambda||,
\]

which implies \( \varphi(||x - x_\lambda||) \leq \lambda ||y|| \), and therefore

\[
||A^\varphi x|| = \frac{1}{\lambda} ||J^\varphi(x - x_\lambda)|| \leq ||y||.
\]

Consequently, \( ||A^\varphi x|| \leq ||Ax|| := \inf\{||y|| : y \in Ax\} \).

(iii) The boundedness of \( J^\varphi \) on bounded subsets of \( X \) and its demicontinuity are already proved in (i). Let \( x \in \text{co}D(A) \) and \( (u, v) \in \text{Gr}(A) \). Following the arguments that lead to (2.7), we find that \( \{x_\lambda - x : \lambda > 0\} \) is bounded, and therefore \( \{J^\varphi(x_\lambda - x) : \lambda > 0\} \) is bounded. Let \( \{\lambda_n\} \subset (0, \infty) \) be such that \( \lambda_n \to 0 \). Let \( y \in X^* \) be such that \( J^\varphi(x_\lambda_n - x) \to y \) in \( X^* \). Then (2.6) yields

\[
\lim_{n \to \infty} \sup_{\lambda > 0} \varphi(||x_\lambda_n - x||)||x_\lambda_n - x|| \leq \langle y, u - x \rangle.
\]

It is clear that this argument applies to all \( u \in \text{co}D(A) \). Taking \( u = x \), we obtain

\[
\lim_{n \to \infty} \varphi(||x_\lambda_n - x||)||x_\lambda_n - x|| = 0.
\]

By the homeomorphic property of the gauge function \( \varphi \), it follows that we must have \( x_\lambda_n \to x \) as \( n \to \infty \). This completes the proof of (iii).

(iv) Let \( u_n = J^\varphi x_n \) for all \( n \). Since \( \{A^\varphi x_n\} \) is bounded, it follows that

\[
\varphi(||x_n - u_n||) = \varphi(||x_n - J^\varphi x_n||) = ||J^\varphi(x_n - J^\varphi x_n)|| = \lambda_n ||A^\varphi x_n|| \to 0
\]

as \( n \to \infty \). This implies \( ||x_n - u_n|| \to 0 \) as \( n \to \infty \). Since

\[
\langle A^\varphi x_n - A^\varphi x_m, x_n - x_m \rangle
\]

\[
= \langle A^\varphi x_n - A^\varphi x_m, u_n - u_m \rangle + \langle A^\varphi x_n - A^\varphi x_m, (x_n - u_n) - (x_m - u_m) \rangle
\]
and $A$ is monotone, it follows as in Brézis et al. \cite{bouk} that
\[
\lim_{n,m \to \infty} \langle A_{\lambda_n}^n x_n - A_{\lambda_m}^m x_m, x_n - x_m \rangle = 0 \quad \text{and} \quad \lim_{n,m \to \infty} \langle A_{\lambda_n}^n x_n - A_{\lambda_m}^m x_m, u_n - u_m \rangle = 0.
\]

The conclusion of (iv) now follows from \cite{bouk} Lemma 2.3.

(v) Let $x \in D(A)$. Since $X^*$ is reflexive and strictly convex and $Ax$ is a closed and convex subset of $X^*$, it follows that there exists a unique element of $Ax$, denoted by $A(0)x$, such that $\|A(0)x\| = \inf \{\|x^*\| : x^* \in Ax\}$. Let $\{\lambda_n\} \subset (0, \infty)$ be such that $\lambda_n \to 0$ and $A_{\lambda_n}^n x \to y$ in $X^*$ as $n \to \infty$. As in the proof of (iv), with $x_n = x$, we have $y \in Ax$. In view of part (ii), it follows that
\[
\|y\| \leq \liminf_{n \to \infty} \|A_{\lambda_n}^n x\| \leq \limsup_{n \to \infty} \|A_{\lambda_n}^n x\| \leq \|A(0)x\|,
\]
and therefore we must have $y = A(0)x$ and $A_{\lambda_n}^n x \to A(0)x$ in $X^*$. Moreover, if $X^*$ is uniformly convex, then, by \cite{bouk} Lemma 1.1, we obtain $A_{\lambda_n}^n x \to A(0)x$ in $X^*$.

(vi) Suppose, on the contrary, that there is a sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ and an element $x \notin \overline{D(A)}$ such that $\{\|A_{\lambda_n}^n x\|\}$ is bounded. Let $R > 0$ be such that $\|A_{\lambda_n}^n x\| \leq R$ for all $n$. Then, by (2.2), we have
\[
\varphi(\|x - J_{\lambda_n}^n x\|) = \|J_{\varphi}(x - J_{\lambda_n}^n x)\| \leq R\lambda_n.
\]
Since $\varphi^{-1}$ is also a gauge function, we obtain $J_{\lambda_n}^n x \to x$ as $n \to \infty$. This implies $x \in \overline{D(A)}$, a contradiction. \hfill \Box

A proof of the following lemma for $\varphi(r) = r$ can be found in Boubakari and Kartsatos \cite{bouk}. Since we are dealing here with an arbitrary gauge function $\varphi$, we provide a complete proof.

**Lemma 2.2.** Let $A : X \supset D(A) \to 2^{X^*}$ be maximal monotone and $G \subset X$ be bounded. Let $0 < \lambda_1 < \lambda_2$. Then there exists a constant $K$, independent of $\lambda$, such that
\[
\|A_{\lambda}^\varphi x\| \leq K
\]
for all $x \in G$ and $\lambda \in [\lambda_1, \lambda_2]$.

**Proof.** For every $x \in X$, we have
\[
A_{\lambda}^\varphi x = \frac{1}{\lambda} J_{\varphi}(x - x_{\lambda}),
\]
where $x_{\lambda} = J_{\lambda}^\varphi x$. Let $(u, v) \in Gr(A)$. In view of (2.7) in the proof of (i) in Proposition 2.1, we have
\[
\varphi(\|x_{\lambda} - x\|) \geq \lambda \varphi(\|u - x\| + \|x_{\lambda} - x\|) \geq \lambda \varphi(\|u - x\| + \|x_{\lambda} - x\|) + \varphi(\|x_{\lambda} - x\|) \geq \lambda \varphi(\|u - x\| + \|x_{\lambda} - x\|) + \varphi(\|x_{\lambda} - x\|) \|u - x\|.
\]
By the properties of the gauge function $\varphi$, it follows that $\varphi(\|x_{\lambda} - x\|)$ must be bounded, i.e., there exists a constant $K_0 > 0$ such that
\[
\varphi(\|x_{\lambda} - x\|) \leq K_0
\]
for all $x \in G$ and all $\lambda \in [\lambda_1, \lambda_2]$. Consequently, we have
\[
\|A_{\lambda}^\varphi x\| = \frac{1}{\lambda} \varphi(\|x_{\lambda} - x\|) \leq \frac{1}{\lambda_1} K_0 =: K
\]
for all $x \in G$ and all $\lambda \in [\lambda_1, \lambda_2]$. \hfill \Box
By a well-known renorming theorem due to Troyanski [33], a reflexive Banach space \( X \) can be renormed with an equivalent norm with respect to which both \( X \) and \( X^* \) become locally uniformly convex (therefore strictly convex). With such a renorming, the duality mapping \( J_x \) is a homeomorphism from \( X \) onto \( X^* \). Henceforth, we assume that both \( X \) and \( X^* \) are reflexive and locally uniformly convex.

The following lemma involving \( A \) is essentially due to Brézis et al. [14].

**Lemma 2.3.** Let \( A : X \supset D(A) \to 2^{X^*} \) and \( S : X \supset D(S) \to 2^{X^*} \) be maximal monotone operators such that \( 0 \in D(A) \cap D(S) \) and \( 0 \in S(0) \cap A(0) \). Assume that \( A + S \) is maximal monotone and that there is a sequence \( \{\lambda_n\} \subset (0, \infty) \) such that \( \lambda_n \to 0 \), and a sequence \( \{x_n\} \subset D(S) \) such that \( x_n \to x_0 \in X \) and \( A_{\lambda_n}x_n + w_n \rightharpoonup y_0^* \in X^* \), where \( w_n \in Sx_n \). Then the following statements are true.

(i) The inequality

\[
\lim_{n \to \infty} \langle A_{\lambda_n} x_n + w_n, x_n - x_0 \rangle < 0
\]

is impossible.

(ii) If

\[
\lim_{n \to \infty} \langle A_{\lambda_n} x_n + w_n, x_n - x_0 \rangle = 0,
\]

then \( x_0 \in D(A + S) \) and \( y_0^* \in (A + S)x_0 \).

**Definition 2.4.** An operator \( A : X \supset D(A) \to 2^{X^*} \) is said to be **strongly quasibounded** if for every \( S > 0 \) there exists \( K(S) > 0 \) such that \( \|x\| \leq S \) and \( \langle x^*, x \rangle \leq S \) for some \( x^* \in Ax \) imply \( \|x^*\| \leq K(S) \).

It is obvious that a bounded operator is strongly quasibounded. With regard to possibly unbounded operators, Browder and Hess [18] and Pascali and Sburlan [28] have shown that a monotone operator \( A \) with \( 0 \in \text{Int}D(A) \) is strongly quasibounded. The following lemma with the particular case \( \varphi(r) = r \) addressed in Kartsatos and Quarcoo [23] Lemma D] is needed in the sequel.

**Lemma 2.5.** Let \( A : X \supset D(A) \to 2^{X^*} \) be a strongly quasibounded maximal monotone operator such that \( 0 \in A(0) \). Let \( \{\lambda_n\} \subset (0, \infty) \) and \( \{x_n\} \subset X \) be such that

\[
\|x_n\| \leq S \quad \text{and} \quad \langle A_{\lambda_n} x_n, x_n \rangle \leq S_1 \quad \text{for all} \quad n,
\]

where \( S, S_1 \) are positive constants. Then there exists a number \( K > 0 \) such that

\[
\|A_{\lambda_n} x_n\| \leq K \quad \text{for all} \quad n.
\]

**Proof.** Denote \( w_n = A_{\lambda_n} x_n \) and \( u_n = J_{\varphi} x_n \) for all \( n \). Then we have

\[
w_n \in Ax_n \quad \text{and} \quad x_n = u_n + J_{\varphi}^{-1}(\lambda_n w_n).
\]

In view of \( 0 \in A(0) \), we obtain

\[
0 \leq \langle w_n, u_n \rangle = \langle w_n, x_n - J_{\varphi}^{-1}(\lambda_n w_n) \rangle = \langle w_n, x_n \rangle - \langle w_n, J_{\varphi}^{-1}(\lambda_n w_n) \rangle = \langle w_n, x_n \rangle - \varphi^{-1}(\lambda_n \|w_n\|)\|w_n\|
\]
This yields $\langle w_n, u_n \rangle \leq S_1$ and $\varphi^{-1}(\lambda_n \|w_n\|)\|w_n\| \leq S_1$ for all $n$. Suppose \{w_n\} is unbounded. Then there exists a subsequence, denoted again by \{w_n\}, such that $\|w_n\| \to \infty$ and $1 \leq \|w_n\|$ for all $n$. Consequently, $\varphi^{-1}(\lambda_n \|w_n\|) \leq S_1$ for all $n$, and since $x_n = u_n + J_{\varphi^{-1}}(\lambda_n w_n)$, it follows that

$$\lambda_n \|w_n\| = \|J_{\varphi}(x_n - u_n)\| = \varphi(\|x_n - u_n\|).$$

This implies $\|x_n - u_n\| = \varphi^{-1}(\lambda_n \|w_n\|) \leq S_1$ for all $n$. Since \{x_n\} is bounded, we obtain the boundedness of \{u_n\} and \{(w_n, u_n)\}, which contradicts the strong quasiboundedness of $A$. Consequently, \{w_n\} is bounded.

For the rest of this paper, we take the gauge function $\varphi(r) = r^{p-1}$, $p > 1$. For the special case $\varphi(r) = r$, the reader can find proofs of Lemma 2.6 in Kartsatos and Skrypnik [25] when $0 \in A(0)$ and in Asfaw and Kartsatos [8], without the condition $0 \in A(0)$. We note that Zhang and Chen in [35, Lemma 2.7] proved the continuity of $x \mapsto A_\lambda x$ on $D(A)$ for each $\lambda > 0$, also without the condition $0 \in A(0)$. In [8, Lemma 6], however, the continuity of $x \mapsto A_\lambda x$ on $X$ is used with no mention of its validity. We next provide a detailed proof of the continuity of the mapping $(\lambda, x) \mapsto A_\lambda^\varphi x$ on $(0, \infty) \times X$.

**Lemma 2.6.** Let $A : X \supset D(A) \to 2^{X^*}$ be a maximal monotone operator. Then the mapping $(\lambda, x) \mapsto A_\lambda^\varphi x$ is continuous on $(0, \infty) \times X$.

**Proof.** We first prove the continuity of $x \mapsto A_\lambda^\varphi x$ on $X$ for each fixed $\lambda > 0$. To this end, let \{x_n\} $\subset$ $X$ be such that $x_n \to x_0 \in X$. By Lemma 2.2 we have the boundedness of \{A_\lambda^\varphi x_n\}, and therefore

$$\lim_{n \to \infty} \langle A_\lambda^\varphi x_n - A_\lambda^\varphi x_0, x_n - x_0 \rangle = 0. \quad (2.11)$$

We know that

$$x_n = J_{\lambda_0} A_\lambda^\varphi x_n + \lambda_0^{-1} J_{\varphi}^{-1}(A_\lambda^\varphi x_n) \quad \text{and} \quad x_0 = J_{\lambda_0} A_\lambda^\varphi x_0 + \lambda_0^{-1} J_{\varphi}^{-1}(A_\lambda^\varphi x_0). \quad (2.12)$$

Since $A_\lambda^\varphi x_n \in A(J_{\lambda_0} A_\lambda^\varphi x_n)$ and $A_\lambda^\varphi x_0 \in A(J_{\lambda_0} A_\lambda^\varphi x_0)$, the monotonicity of $A$ together with (2.11) and (2.12) yields

$$\lim_{n \to \infty} \langle A_\lambda^\varphi x_n - A_\lambda^\varphi x_0, J_{\varphi}^{-1}(A_\lambda^\varphi x_n) - J_{\varphi}^{-1}(A_\lambda^\varphi x_0) \rangle = 0. \quad (2.13)$$

Since $J_{\varphi}^{-1}$ is a duality mapping from $X^*$ to $X$, it follows, in view of [19, Proposition 2.17], that

$$A_\lambda^\varphi x_n \to A_\lambda^\varphi x_0 \quad \text{as} \quad n \to \infty.$$

This proves the continuity of $A_\lambda^\varphi$ on $X$.

We now proceed to prove the continuity of $(\lambda, x) \mapsto A_\lambda^\varphi x$ on $(0, \infty) \times X$. Let \{\lambda_n\} $\subset$ $(0, \infty)$ and \{x_n\} $\subset$ $X$ be such that $\lambda_n \to \lambda_0 \in (0, \infty)$ and $x_n \to x_0 \in X$ as $n \to \infty$. Let $G \subset X$ be a bounded set that contains $x_n$ for all $n$. Rename $\lambda_1, \lambda_2 > 0$ such that $\lambda_n \in [\lambda_1, \lambda_2]$ for all $n$. Since

$$J_{\lambda_n}^\varphi x_n \in A^{-1}(A_\lambda^\varphi x_n) \quad \text{and} \quad x_n = J_{\lambda_n}^\varphi x_n + \lambda_n^{-1} J_{\varphi}^{-1}(A_\lambda^\varphi x_n),$$

it follows that

$$J_{\lambda_n}^\varphi x_n + \lambda_n^{-1} J_{\varphi}^{-1}(A_\lambda^\varphi x_n) \in A^{-1}(A_\lambda^\varphi x_n) + \lambda_n^{-1} J_{\varphi}^{-1}(A_\lambda^\varphi x_n) \quad = \quad (A^{-1} + \lambda_n^{-1} J_{\varphi}^{-1})(A_\lambda^\varphi x_n).$$
This implies
\[
A^\infty_{\lambda_n} x_n = \left( A^{-1} + \lambda_0^{-1} J_{\varphi}^{-1} \right)^{-1} \left( J_{\lambda_n} x_n + \lambda_0^{-1} J_{\varphi}^{-1} (A^\infty_{\lambda_n} x_n) \right)
\]
\[
= A^\infty_{\lambda_0} \left( J_{\lambda_n} x_n + \lambda_0^{-1} J_{\varphi}^{-1} (A^\infty_{\lambda_0} x_n) \right)
\]
\[
= A^\infty_{\lambda_0} \left( x_n + (\lambda_0^{-1} - \lambda_n^{-1}) J_{\varphi}^{-1} (A^\infty_{\lambda_0} x_n) \right).
\]

By Lemma 2.2, \( \{ A^\infty_{\lambda_n} x_n \} \) is bounded, and so is \( \{ J_{\varphi}^{-1} (A^\infty_{\lambda_0} x_n) \} \). Since \( \lambda_n \to \lambda_0 \), we have \( (\lambda_0^{-1} - \lambda_n^{-1}) J_{\varphi}^{-1} (A^\infty_{\lambda_0} x_n) \to 0 \) as \( n \to \infty \). The continuity of \( A^\infty_{\lambda_0} \) implies \( A^\infty_{\lambda_0} x_n \to A^\infty_{\lambda_0} x_0 \) as \( n \to \infty \). This completes the proof. \( \square \)

**Remark 2.7.** We anticipate that Lemma 2.6 holds for any gauge function \( \varphi \). Since the formula (2.4) may not hold for \( A^\infty_\varphi \) with a general \( \varphi \), the above proof does not go through and this subject may be of independent research interest.

Let \( G \) be an open and bounded subset of \( X \). Let \( L : X \supseteq D(L) \to X^* \) be densely defined linear maximal monotone, \( A : X \supseteq D(A) \to 2^{X^*} \) maximal monotone, and \( C(s) : X \supseteq G \to X^* \), \( s \in [0,1] \), a bounded homotopy of type \( (S_+) \) with respect to \( D(L) \). Since \( \text{Gr}(L) \) is closed in \( X \times X^* \), the space \( Y = D(L) \) associated with the graph norm \( \| x \|_Y = \| x \|_X + \| Lx \|_{X^*} \), \( x \in Y \), becomes a real reflexive Banach space. We may assume that \( Y \) and its dual \( Y^* \) are locally uniformly convex.

Let \( j : Y \to X \) be the natural embedding and \( j^* : X^* \to Y^* \) its adjoint. Since \( j : Y \to X \) is continuous, we have \( D(j^*) = X^* \). This implies that \( j^* \) is also continuous. Since \( j^{-1} \) is not necessarily bounded, we have, in general, \( j^*(X^*) \neq Y^* \). Moreover, \( j^{-1}(G) = G \cap D(L) \) is closed and \( j^{-1}(G) = G \cap D(L) \) is open, \( j^{-1}(G) \subset j^{-1}(\overline{G}) \), and \( \partial(j^{-1}(G)) \subset j^{-1}(\partial G) \).

We define \( M : Y \to Y^* \) by \( \langle Mx, y \rangle := \langle Ly, j^{-1}(Lx) \rangle \), \( x, y \in Y \), where the duality pairing \( (\cdot, \cdot) \) is in \( Y^* \times Y \), and \( J^{-1} \) is the inverse of the duality map \( J : X \to X^* \) and is identified with the duality map from \( X^* \) to \( X^{**} = X \). Also, for every \( x \in Y \) such that \( Mx \in j^*(X^*) \), we have \( J^{-1}(Lx) \in D(L^*) \), \( Mx = j^* \circ L^* \circ J^{-1}(Lx) \), and
\[
\langle Mx - My, y - x \rangle = \langle Lx - Ly, J^{-1}(Lx) - J^{-1}(Ly) \rangle \geq 0
\]
for all \( y \in Y \) such that \( My \in j^*(X^*) \). Moreover, it is easy to see that \( M \) is continuous on \( Y \), and therefore \( M \) is maximal monotone.

We now define \( \tilde{L} : Y \to Y^* \) and \( \tilde{C}(s) : j^{-1}(\overline{G}) \to Y^* \) by \( \tilde{L} = j^* \circ L \circ j \) and \( \tilde{C}(s) = j^* \circ C(s) \circ j \), respectively, and for each \( t > 0 \), we also define \( \tilde{A}_{\tilde{T}}^\infty : Y \to Y^* \) by \( \tilde{A}_{\tilde{T}}^\infty = j^* \circ \tilde{A}_{\tilde{T}}^\infty \circ j \), where \( \tilde{A}_{\tilde{T}}^\infty \) is the Yosida approximant of \( A \) corresponding to the gauge function \( \varphi \).

The next lemma employs Lemma 2.5 and follows as in [5, Lemma 5], and therefore its proof is omitted.

**Lemma 2.8.** Let \( G \subset X \) be open and bounded. Assume the following:

(i) \( L : X \supseteq D(L) \to X^* \) is linear, maximal monotone with \( D(L) = X \);

(ii) \( A : X \supseteq D(A) \to 2^{X^*} \) is strongly quasibounded, maximal monotone with \( 0 \in A(0) \); and

(iii) \( C(t) : X \supseteq \overline{G} \to X^* \) is a bounded homotopy of type \( (S_+) \) with respect to \( D(L) \).

Then, for a continuous curve \( f(s), 0 \leq s \leq 1 \), in \( X^* \), the set
\[
F = \{ x \in j^{-1}(\overline{G}) : \tilde{L} + \tilde{A}_{\tilde{T}}^\infty + \tilde{C}(s) + tMx = j^* f(s) \text{ for some } t > 0, s \in [0,1] \}
\]
is bounded in $Y$.

The next two propositions are essential for the existence results in Section 2 and Section 3.

**Proposition 2.9.** Let $A : X \supset D(A) \to 2^{X^*}$ be maximal monotone and $C : X \supset D(C) \to X^*$ be bounded, demicontinuous and of type $(S_+)$. Suppose that $G \subset X$ is open and bounded such that $0 \in A(0)$, $p \in X^*$, and
\[
p \not\in (A + C)x
\]
for all $x \in \partial G \cap D(A) \cap D(C)$. Then the following statements hold.

(i) There exists $t_0 > 0$ such that
\[
A_t^x x + Cx \neq p
\]
for all $x \in \partial G \cap D(C)$ and $t < t_0$.

(ii) For fixed $t_1, t_2 > 0$, define $q(t) := tt_1 + (1 - t)t_2$, $t \in [0, 1]$. Then the operator
\[
H(t, x) = A_{q(t)}^x x + Cx, \quad (t, x) \in [0, 1] \times \overline{G}
\]
is a homotopy of type $(S_+)$.\[\]
(iii) For every sequence $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$, $\lim_{n \to \infty} d_{S_+}(A_{t_n}^x + C, G, p)$ exists and does not depend on the choice of $\{t_n\}$.

**Proof.** (i) Without loss of generality, we assume that $p = 0$. In fact, if $p \neq 0$, then we replace $C$ with $C - p$. Suppose that (iii) is false. Then there exist $\{t_n\} \subset (0, \infty)$ and $\{x_n\} \subset \partial G$ such that $t_n \to 0$ and
\[
A_{t_n}^x x_n + Cx_n = 0 \tag{2.14}
\]
for all $n$. Since $C$ is bounded, $\{Cx_n\}$ is bounded. This implies that $\{A_{t_n}^x x_n\}$ is also bounded. We may assume that there exist $x_0 \in X$ and $w_0 \in X^*$ such that $x_n \to x_0$ in $X$ and $A_{t_n}^x x_n \to w_0$ in $X^*$. If
\[
\limsup_{n \to \infty} (Cx_n, x_n - x_0) > 0,
\]
we find a subsequence of $\{x_n\}$, denoted again by itself, such that
\[
\lim_{n \to \infty} (Cx_n, x_n - x_0) > 0.
\]
In view of (2.14), we obtain
\[
\lim_{n \to \infty} (A_{t_n}^x x_n, x_n - x_0) < 0;
\]
however, this is impossible by (i) of Lemma 2.3. We then must have
\[
\lim_{n \to \infty} (Cx_n, x_n - x_0) \leq 0.
\]
By the $(S_+)$-property of $C$, we have $x_n \to x_0$, and consequently
\[
\lim_{n \to \infty} (A_{t_n}^x x_n, x_n - x_0) = 0.
\]
By (ii) of Lemma 2.3, we obtain $x_0 \in D(A)$ and $w_0 \in Ax_0$. Since $C$ is demicontinuous, $Cx_n \to Cx_0$ in $X^*$. This implies $w_0 = -Cx_0$, i.e., $0 \in (A + C)(\partial G)$, contradicting $0 \notin (A + C)(\partial G)$.\[\]
(ii) Let $t_1, t_2 \in (0, t_0]$ be such that $t_1 < t_2$. Consider the following one-parameter family of operators:

$$H(t, x) := A^\varphi_{q(t)}x + Cx, \quad (t, x) \in [0, 1] \times \overline{C}.$$ 

We prove that $H(t, \cdot)$ is a bounded homotopy of type $(S_+)$.

\[ \text{Remark 2.10.} \quad \text{We prove that } H(t, \cdot) \text{ is a bounded homotopy of type } (S_+). \]

The boundedness of $H(t, \cdot)$ follows from Lemma 2.2 and the boundedness of $C$. Let $\{t_n\} \subset [0, 1]$ and $\{x_n\} \subset \overline{C}$ satisfy $t_n \to t_0$ and $x_n \to x_0$ in $X$, and

$$\lim_{n \to \infty} \sup_{t \in [0, 1]} (A^\varphi_{q(t_n)}x_n + Cx_n, x_n - x_0) \leq 0. \quad (2.15)$$

Using the monotonicity of $A^\varphi_{q(t)}$ in (2.15), we obtain

$$\lim_{n \to \infty} \sup_{t \in [0, 1]} (A^\varphi_{q(t_n)}x_0 + Cx_n, x_n - x_0) \leq 0. \quad (2.16)$$

By Lemma 2.6, we have $A^\varphi_{q(t_n)}x_0 \to A^\varphi_{q(t_0)}x_0$, and therefore (2.16) yields

$$\lim_{n \to \infty} \sup_{t \in [0, 1]} (C x_n, x_n - x_0) \leq 0.$$

Since $C$ is demicontinuous and of type $S_+$, it follows that $x_n \to x_0$ in $X$ and $C x_n \to Cx_0$ in $X^*$. Consequently, we have

$$A^\varphi_{q(t_n)}x_n + Cx_n \to A^\varphi_{q(t_0)}x_0 + Cx_0$$

as $n \to \infty$. This proves that $H(t, \cdot)$, $t \in [0, 1]$, is a homotopy of type $(S_+)$. (iii) By the invariance of the degree, $d_{S_+}$, for $(S_+)$-mappings under the homotopies of type $(S_+)$, we have

$$d_{S_+}(A^\varphi_{1_1}, G, 0) = d_{S_+}(H(0, \cdot), G, 0) = d_{S_+}(H(1, \cdot), G, 0) = d_{S_+}(A^\varphi_{1_2}, G, 0).$$

It follows that $d_{S_+}(A^\varphi_{1_1}, G, 0)$ exists and is independent of $t \in (0, t_0]$. \hfill $\square$

**Remark 2.10.** Let $A$, $C$, $G$, and $p$ be the same as in Proposition 2.9. When we define a degree mapping of $A + C$, denoted by $D(A + C, G, p)$, by

$$D(A + C, G, p) = \lim_{t \to 0^+} d_{S_+}(A^\varphi_{t}, G, p),$$

we can verify that the degree mapping $D$ has the same four basic properties as the Browder degree in [10]. By the uniqueness of the Browder degree established by Berkovits and Miettunen [12], the degree $D$ coincides with the Browder degree for $A + C$.

By replacing $\hat{T}$ everywhere in [5, Lemma 5, Lemma 6, and Lemma 8] with $\hat{A}^\varphi$ with the gauge function $\varphi(r) = r^{p-1}$ and by following the methodology used in [5] in conjunction with Lemmas 2.3, 2.5, 2.6, and 2.8 we obtain Proposition 2.11 below. Its proof is omitted here because the method of proof is similar to that in [5] and Proposition 2.9 except for having to deal with $\hat{A}^\varphi$. For further properties of $L + A + C$ in relation to the following proposition for $\varphi(r) = r$, the reader is referred to Addou and Mermri [11] and Adhikari and Kartsatos [5].

**Proposition 2.11.** Let $G \subset X$ be open and bounded. Assume that $L : X \supset D(L) \to X^*$ is linear, maximal monotone with $D(L) = X$; $A : X \supset D(A) \to 2^{X^*}$ is strongly quasibounded, maximal monotone with $0 \in A(0)$; and $C(t) : X \supset \overline{C} \to X^*$, $t \in [0, 1]$, is a bounded homotopy of type $(S_+)$ with respect to $D(L)$. Suppose that

$$0 \not\in (L + A + C(t))x$$

for all $x \in \partial G \cap D(L) \cap D(A)$. Then the following statements hold.
(i) There exists $t_0 > 0$ such that
\[ \hat{L}x + \hat{A}_q^\varphi x + \hat{C}(t)x + tMx \neq 0 \]
for all $(t,x) \in [0,1] \times (\partial G \cap D(L))$ and $t < t_0$.
(ii) For fixed numbers $t_1, t_2 > 0$, define $q(t) := tt_1 + (1 - t)t_2$, $t \in [0,1]$. Then the operator
\[ \hat{H}(t,x) = \hat{L}x + \hat{A}_q^\varphi x + \hat{C}(t)x + s(t)Mx, \]
with $(t,x) \in [0,1] \times (\partial G \cap D(L))$, is a homotopy of type $(S_\gamma)$.
(iii) For every sequence $\{t_n\} \subset (0,\infty)$ such that $t_n \to 0$,
\[ \lim_{n \to \infty} d_{S_\gamma}(\hat{L} + \hat{A}_{t_n}^\varphi + \hat{C}(t) + t_n M, G, 0) \]
exists and does not depend on the choice of $\{t_n\}$.

3. Existence of nontrivial solutions

Hu and Papageorgiou generalized in [21] the Browder degree theory [16] to the mappings of the form $A + C + T$, where $A : X \supset D(A) \to 2^{X^*}$ is maximal monotone with $0 \in A(0)$, $C : X \supset D(C) \to X^*$ is bounded demicontinuous of type $(S_\gamma)$, and $T$ is of class $(P)$. With an application of the $(S_\gamma)$-degree developed by Browder [10] and Skrypnik [32], we prove in Theorem 3.3 the existence of nonzero solutions of $Ax + Cx + Tx \ni 0$ when $A + C + T$ satisfies certain boundary conditions, and the operator $A$, in addition, is positively homogeneous of degree $\gamma > 0$. This result extends the existence result for $\gamma \in (0,1]$ in [2] to $\gamma > 0$ (see also [6] Theorem 6) for $\gamma = 1$.

The following lemma, which is crucial to the existence results in this section, shows that positively homogeneous maximal monotone operators transmit the homogeneity into their Yosida approximants corresponding to $J_\varphi$ with $\varphi(r) = r^{p-1}$, $p > 1$, and a suitable value of $p$.

**Lemma 3.1.** Let $A : X \supset D(A) \to 2^{X^*}$ be maximal monotone and positively homogeneous of degree $\gamma > 0$. Then, for each $t > 0$, the Yosida approximant $A_t^\varphi$ corresponding to the gauge function $\varphi(r) = r^{p-1}$, $p > 1$, satisfies
\[ A_t^\varphi(sx) = \begin{cases} s^\gamma A_t^{\varphi+1-\gamma}x & \text{for } (s,x) \in (\mathbb{R}_+ \setminus \{0\}) \times X \\ 0 & \text{for } (s,x) \in \{0\} \times X. \end{cases} \]  

Consequently, if $p = \gamma + 1$, then $A_t^\varphi$ is positively homogeneous of degree $\gamma$, i.e.,
\[ A_t^\varphi(sx) = s^\gamma A_t^\varphi x \text{ for all } (s,x) \in \mathbb{R}_+ \times X. \]

**Proof.** Let $t > 0$ be fixed. The case $s = 0$ is trivial. Assume $s > 0$, and let
\[ y = A_t^\varphi(sx) = (A^{-1} + t^{q-1}J_\varphi^{-1})^{-1}(sx), \quad x \in X, \]
where $q$ satisfies $1/p + 1/q = 1$. Then
\[ y \in A(-t^{q-1}J_\varphi^{-1}y + sx) = A \left( s \left(-t^{q-1}s^{-1}J_\varphi^{-1}y + x \right) \right). \]

This means
\[ (s \left(-t^{q-1}s^{-1}J_\varphi^{-1}y + x \right), y) \in \text{Gr}(A). \]
Since $A$ is positively homogeneous of degree $\gamma > 0$, we obtain
\[ (-t^{q-1}s^{-1}J_\varphi^{-1}y + x, s^{-\gamma}y) \in \text{Gr}(A), \]
Theorem 3.3. Clearly, $A$ is positively homogeneous of degree $1$ if $p = \gamma + 1$.

Remark 3.2. In the settings of Lemma 3.1 with $p = \gamma + 1$, it follows from (2.3) that the resolvent $J_p^\gamma$ is positively homogeneous of degree 1 in the following sense: for each $t > 0$, we have $J_p^\gamma(sx) = s\gamma J_p^\gamma x$ for all $x \in X$ and all $s \geq 0$.

Theorem 3.3. Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $A : X \supset D(A) \rightarrow 2^{X^*}$ be maximal monotone and positively homogeneous of degree $\gamma > 0$ with $A(0) = \{0\}$; $C : \overline{G_1} \rightarrow X^*$ bounded, demicontinuous and of type $(S_\gamma)$; and $T : \overline{G_1} \rightarrow 2^{X^*}$ of class $(P)$. Assume, further, that

(H1) there exists $u_0^* \in X^* \setminus \{0\}$ such that $Ax + Cx + Tx \nsubseteq \lambda u_0^*$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(A) \cap \partial G_1)$, and

(H2) $Ax + Cx + Tx + \lambda Jx \nsubseteq 0$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(A) \cap \partial G_2)$.

Then the inclusion $Ax + Cx + Tx \ni 0$ has a nonzero solution $x \in D(A) \cap (G_1 \setminus G_2)$.

Proof. To study the solvability of the inclusion

$$Ax + Cx + Tx \ni 0, \quad x \in \overline{G_1},$$

we consider the associated approximate equation

$$A_p^\gamma x + Cx + q_\epsilon x = 0, \quad t > 0, \quad x \in \overline{G_1}, \quad \epsilon > 0. \quad (3.2)$$

Here, the gauge function is taken to be $\varphi(r) = r^{p-1}$, $1 < p < \infty$ so that $\gamma = p - 1$, and $q_\epsilon : \overline{G_1} \rightarrow X^*$ is an approximate continuous Cellina-selection as in [3] Lemma 6 and [21] satisfying $q_\epsilon x \in T(B_\epsilon(x) \cap \overline{G_1}) + B_\epsilon(0)$ for all $x \in \overline{G_1}$ and $q_\epsilon(\overline{G_1}) \subset \co T(\overline{G_1})$.

We show that the equation (3.2) has a solution $x_{t, \epsilon}$ in $G_1 \setminus G_2$ for all sufficiently small $t$ and $\epsilon$. To this end, we first show that there exist $\tau_0 > 0, t_0 > 0$ and $\epsilon_0 > 0$ such that the equation

$$A_p^\gamma x + Cx + q_\epsilon x = \tau u_0^* \quad (3.3)$$

has no solution in $G_1$ for every $\tau \geq \tau_0, \ t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$.

Assuming the contrary, let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, \infty)$, $\{\epsilon_n\} \subset (0, \infty)$ and $\{x_n\} \subset G_1$ be such that $\tau_n \rightarrow \infty$, $t_n \rightarrow 0$, $\epsilon_n \rightarrow 0$ and

$$A_p^\gamma x_n + Cx_n + q_\epsilon x_n = \tau_n u_0^*. \quad (3.4)$$
We can assume that \( q_n x_n \to g^* \in X^* \) in view of the properties of \( T \). Then \( \|A_{t_n}^\infty x_n\| \to \infty \) as \( \tau_n v_0^* \to \infty \) and \( \{C x_n\} \) is bounded. Thus, from (3.4), we obtain

\[
\frac{A_{t_n}^\infty x_n}{\|A_{t_n}^\infty x_n\|} + \frac{C x_n}{\|A_{t_n}^\infty x_n\|} + \frac{q_n x_n}{\|A_{t_n}^\infty x_n\|} = \frac{\tau_n}{\|A_{t_n}^\infty x_n\|} v_0^*.
\]

This implies

\[
\frac{\tau_n}{\|A_{t_n}^\infty x_n\|} \to 1 \quad \text{so that} \quad \frac{\tau_n}{\|A_{t_n}^\infty x_n\|} \to \frac{1}{\|v_0^*\|} \quad \text{as} \quad n \to \infty.
\]

Since \( p - 1 = \gamma \), by Lemma [3.1] \( A_{t_n}^\infty \) is also homogeneous of degree \( \gamma = p - 1 \), and therefore we obtain

\[
\frac{A_{t_n}^\infty x_n}{\|A_{t_n}^\infty x_n\|} = A_{t_n}^\infty \left( \frac{x_n}{\|A_{t_n}^\infty x_n\|^{1/\gamma}} \right).
\]

Let \( u_n = x_n/\|A_{t_n}^\infty x_n\|^{1/\gamma} \). It is clear that \( u_n \to 0 \). In view of (3.5), (3.6), and (3.7), we obtain \( A_{t_n}^\infty u_n \to h \) with \( h = v_0^*/\|v_0^*\| \). This implies

\[
\lim_{n \to \infty} \langle A_{t_n}^\infty u_n, u_n \rangle = \langle h, 0 \rangle = 0.
\]

Since \( t_n \to 0 \), by (ii) of Lemma [2.3] with \( S = 0 \) we obtain \( 0 \in D(A) \) and \( h \in A(0) = \{0\} \), a contradiction to \( ||h|| = 1 \).

We now consider the homotopy mapping

\[
H_1(s, x, t, \epsilon) = A_{t_n}^\infty x + C x + q_n x - s \tau_0 v_0^*, \quad s \in [0, 1], \quad x \in \overline{G_1},
\]

where \( t \in (0, t_0] \) and \( \epsilon \in (0, \epsilon_0] \) are fixed. By following the arguments as in [2] Theorem 3.1, we can show that, for every \( s \in [0, 1] \) the operator \( x \mapsto C x - s \tau_0 v_0^* \) is bounded, demicontinuous and of type \((S_+)^m\) on \( \overline{G_1} \), and that the equation \( H_1(s, x, t, \epsilon) = 0 \) has no solution in \( \partial G_1 \) for all sufficiently small \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \) and all \( s \in [0, 1] \). In doing this, we need to use Lemma 2.3. The details are omitted.

It follows from Proposition 2.9 that the mapping \( H_1(s, x, t, \epsilon) \) is an admissible homotopy for the degree, \( \mathcal{d}_{S_+}(H_1(s, \cdot, t, \epsilon), G_1, 0) \) is well-defined and is a constant for all \( s \in [0, 1] \) and for all \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \).

Assume that

\[
\mathcal{d}_{S_+}(H_1(1, \cdot, t_1, \epsilon_1), G_1, 0) \neq 0,
\]

for some sufficiently small \( t_1 \in (0, t_0] \) and \( \epsilon_1 \in (0, \epsilon_0] \). Then the equation

\[
A_{t_n}^\infty x + C x + q_n x = \tau_0 v_0^*
\]

has a solution in \( G_1 \). However, this contradicts our choice of the number \( \tau_0 \) in (3.3).

Consequently,

\[
\mathcal{d}_{S_+}(A_{t_n}^\infty + C + q_n, G_1, 0) = \mathcal{d}_{S_+}(H_1(0, \cdot, t, \epsilon), G_1, 0) = 0, \quad t \in (0, t_0], \epsilon \in (0, \epsilon_0].
\]

We next consider the homotopy mapping

\[
H_2(s, x, t, \epsilon) = s(A_{t_n}^\infty x + C x + q_n x) + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \overline{G_2}.
\]

We claim that there exist \( t_1 \in (0, t_0] \) and \( \epsilon_1 \in (0, \epsilon_0] \) such that \( H_2(s, x, t, \epsilon) = 0 \) has no solution on \( \partial G_2 \) for any \( s \in [0, 1] \), any \( t \in (0, t_1] \) and any \( \epsilon \in (0, \epsilon_1] \). To prove the claim, we assume the contrary and then follow the argument used in [2] Theorem 3.1 along with the properties of \( A_{t_n}^\infty \) established in Lemma 2.3 to arrive at a contradiction to (H2). For the sake of convenience, we assume that \( t_0 \) and \( \epsilon_0 \) are sufficiently small so that we may take \( t_1 = t_0 \) and \( \epsilon_1 = \epsilon_0 \).
It follows from Proposition 2.9 that \( H_2(s, x, t, \epsilon) \) is an admissible homotopy for the degree of \((S_+)\)-mappings and \( d_{S_+}(H_2(s, \cdot, t, \epsilon), G_2, 0) \) is well-defined and constant for all \( s \in [0, 1] \), all \( t \in (0, t_0] \) and all \( \epsilon \in (0, \epsilon_0] \). By the invariance of the \((S_+)\)-degree, for all \( t \in (0, t_0] \) and \( \epsilon \in (0, \epsilon_0] \), we have
\[
d_{S_+}(H_2(1, \cdot, t, \epsilon), G_2, 0) = d_{S_+}(A_t^G + C + q_\epsilon, G_2, 0) = d_{S_+}(H_2(0, \cdot, t, \epsilon), G_2, 0) = d_{S_+}(J, G_2, 0) = 1.
\]
Thus, for all \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \), we have
\[
d_{S_+}(A_t^G + C + q_\epsilon, G_1, 0) \neq d_{S_+}(A_t^G + C + q_\epsilon, G_2, 0).
\]

Using the excision property of the \((S_+)\)-degree, which is an easy consequence of its finite-dimensional approximations, for every \( t \in (0, t_0] \) and every \( \epsilon \in (0, \epsilon_0] \), there exists a solution \( x_{t,\epsilon} \in G_1 \setminus G_2 \) of \( A_t^G x + Cx + q_\epsilon x = 0 \). Let \( t_n \to 0 \), \( \epsilon_n \to 0 \) and let \( x_n \in G_1 \setminus G_2 \) be the corresponding solutions of \( A_t^G x + Cx + q_\epsilon x = 0 \), i.e.,
\[
A_t^G x_n + Cx_n + q_\epsilon x_n = 0.
\]
We may assume that \( x_n \to x_0 \) in \( X \) and \( q_\epsilon x_n \to g^* \in X^* \). We observe that
\[
\langle A_t^G x_n, x_n - x_0 \rangle = -\langle Cx_n + q_\epsilon x_n, x_n - x_0 \rangle.
\]
If
\[
\limsup_{n \to \infty} \langle Cx_n + q_\epsilon x_n, x_n - x_0 \rangle > 0,
\]
then we obtain a contradiction from (i) of Lemma 2.3 with \( S = 0 \) there. Consequently,
\[
\limsup_{n \to \infty} \langle Cx_n + q_\epsilon x_n, x_n - x_0 \rangle \leq 0,
\]
and hence
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0.
\]

By the \((S_+)\)-property of \( C \), we obtain \( x_n \to x_0 \in G_1 \setminus G_2 \). Then \( Cx_n \to Cx_0 \) and \( A_t^G x_n \to -Cx_0 - g^* \). Using this in (ii) of Lemma 2.3 with \( S = 0 \) there, we obtain \( x_0 \in D(A) \) and \( -Cx_0 - g^* \in Ax_0 \). By a property of the selection \( q_\epsilon x_n \) as in Hu and Papageorgiou [21], we have \( g^* \in Tx_0 \), and therefore \( Ax_0 + Cx_0 + Tx_0 \ni 0 \). We also have
\[
x_0 \in G_1 \setminus G_2 = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.
\]

By (H1) and (H2), we have \( x_0 \notin \partial G_1 \cup \partial G_2 \), and hence \( x_0 \in D(A) \cap (G_1 \setminus G_2) \). 

**Remark 3.4.** We point out that the condition \( A(0) = \{0\} \) on the homogeneous maximal monotone operator \( A \) used in Theorem 3.3 is rather mild in view of Rockafellar’s result [29] which says that a monotone map is locally bounded at every point in the interior of its domain.

The existence of nonzero solutions of \( Lx + Ax + Cx \ni 0 \), where the maximal monotone operator \( A \) is strongly quasibounded and positively homogeneous of degree \( \gamma = 1 \), is established in [2]. In the following theorem, we extend this result to an arbitrary degree \( \gamma > 0 \) for the same combination of operators in the spirit of the Berkovits-Mustonen theory in [11] and the theories developed in [6].
recall that the maximal monotone operator $A$ investigated in \cite{[6]} is strongly quasi-bounded. However, by a result of Hess \cite{[20]}, a strongly quasibounded and positively homogeneous operator of degree $\gamma > 0$ is necessarily bounded. Therefore, in the following theorem, we assume that the maximal monotone operator $A$ is bounded.

**Theorem 3.5.** Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $G_2 \subset G_1$. Let $L : X \supset D(L) \to X^*$ be linear maximal monotone with $\overline{D(L)} = X$, and $A : X \supset D(A) \to 2^{X^*}$ bounded, maximal monotone and positively homogeneous of degree $\gamma > 0$. Also, let $C : \overline{G_1} \to X^*$ be bounded, demicontinuous and of type $(S_+)$ with respect to $D(L)$. Moreover, assume that

(H3) there exists $v^* \in X^* \setminus \{0\}$ such that $Lx + Ax + Cx \not\in \lambda v^*$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(A) \cap \partial G_1)$, and

(H4) $Lx + Ax + Cx + \lambda Jx \not\in 0$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(A) \cap \partial G_2)$.

Then the inclusion $Lx + Ax + Cx \ni 0$ has a solution $x \in D(L) \cap D(A) \cap (G_1 \setminus G_2)$.

**Proof.** We begin by observing that a positively homogeneous and bounded maximal monotone operator $A$ of degree $\gamma > 0$ satisfies $0 \in D(A)$ and $A(0) = \{0\}$. To solve the inclusion

$$Lx + Ax + Cx \ni 0, \quad x \in \overline{G_1},$$

(3.10)

let us consider the associated equation

$$\hat{L}x + \hat{A}_t^\gamma x + \hat{C}x + tMx = 0, \quad t \in (0, \infty), \quad x \in j^{-1}(\overline{G_1}).$$

(3.11)

Here, the gauge function is $\varphi(r) = r^{p-1}, 1 < p < \infty$, and $\gamma = p-1$. We can show as in \cite{[5]} Lemma 5] that there exists $R > 0$ such that the open ball $B_Y(0, R)$ contains all the solutions of (3.11). We recall that $Y = D(L)$.

We shall prove that (3.11) has a solution $x_t \in j^{-1}(G_1 \setminus G_2)$ for all sufficiently small $t > 0$. We first claim that there exist $\tau_0 > 0$, $t_0 > 0$ such that

$$\hat{L}x + \hat{A}_t^\gamma x + \hat{C}x + tMx = \tau j^*v^*$$

(3.12)

has no solution in $G_1^1(Y) := j^{-1}(G_1) \cap \overline{B_Y(0, R)}$ for all $t \in (0, t_0]$ and all $\tau \in (\tau_0, \infty)$. Assume the contrary and let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, 1)$ and $\{x_n\} \subset G_1^1(Y)$ such that $\tau_n \to \infty$, $t_n \to 0$ and

$$\hat{L}x_n + \hat{A}_{t_n}^\gamma x_n + \hat{C}x_n + t_nMx_n = \tau_n j^*v^*.$$  

(3.13)

We note that $j^*$ is one-to-one because $j(Y) = Y$, which is dense in $X$. This implies that $j^*v^*$ is nonzero, and therefore $\tau_n j^*v^* \in X^* \to +\infty$. Also, the sequence $\{x_n\}$ is bounded in $Y$ and so we may assume that $x_n \to x_0$ in $X$ and $Lx_n \to Lx_0$ in $X^*$. In particular, $\{Lx_n\}$ is bounded in $X^*$. Since $Mx_n \in j^*(X^*)$, we have $J^{-1}(Lu) \in D(L^*)$ and $Mx_n = j^*L^*J^{-1}(Lu)$. Since $j^*$, $L^*$, $J^{-1}$ are bounded, we have the boundedness of $\{Mx_n\}$. It is clear that $\hat{L}x_n$ and $\hat{C}x_n$ are bounded in $Y^*$, and therefore \cite{[5]} implies that $\|\hat{A}_{t_n}^\gamma x_n\|_{Y^*} \to \infty$. Since $A$ is positively homogeneous of degree $\gamma$, applying Lemma 3.1 for $\gamma = p-1$ shows that each $A_{t_n}^\gamma$ is also positively homogeneous of $\gamma = p - 1$. Consequently,

$$\frac{\hat{A}_{t_n}^\gamma x_n}{\|\hat{A}_{t_n}^\gamma x_n\|_{Y^*}} = \hat{A}_{t_n}^\gamma \left(\frac{x_n}{\|\hat{A}_{t_n}^\gamma x_n\|_{Y^*}^{1/\gamma}}\right)$$

(3.14)
for all \( n \). Define \( \beta_n := 1/\|\hat{A}^\tau_n x_n\|_{\gamma^*} \) and \( \delta_n := \beta_n^{1/\tau} \). Since \( \|\hat{A}^\tau_n x_n\|_{\gamma^*} \to \infty \), it follows that \( \beta_n x_n \to 0 \) and \( \delta_n x_n \to 0 \) in \( X \) as \( n \to \infty \). From (3.13) and (3.14), we find
\[
\hat{L}(\beta_n x_n) + \hat{A}^\tau_n (\delta_n x_n) + \beta_n \hat{C} x_n + t_n \beta_n M x_n = \tau_n \beta_n j^* v^*.
\] (3.15)

Because \( \|\hat{A}^\tau_n (\delta_n x_n)\|_{\gamma^*} = 1 \) and the remaining terms on the left in (3.15) converge to 0 in \( X^* \) as \( n \to \infty \), we obtain \( \tau_n \beta_n \to 1/\|j^* v^*\|_{\gamma^*} \), and therefore \( \hat{A}^\tau_n (\delta_n x_n) \to y_0 \), where \( y_0 = j^* v^*/\|j^* v^*\|_{\gamma^*} \). Since \( u_n := \delta_n x_n \) is fixed, it can be easily seen that \( \|y_0\|_{\gamma^*} = 1 \). Also, \( \phi := \hat{C} x_n \) is an admissible homotopy for the \( (S_+)\)-degree, \( d_{S_+} \), and therefore \( d_{S_+}(H(s, \cdot), G^1_R(Y), 0) \) is well-defined and remains constant for all \( s \in [0, 1] \). We now follow the arguments as in [2, Theorem 2.2] in conjunction with Lemma 2.3 to arrive at
\[
\langle \hat{L} x_n + \hat{A}^\tau_n x_n + \hat{C} x_n + t_n M x_n = s_0 j^* \gamma^* v^*.
\] (3.17)

Here, the boundedness of \( \{\hat{A}^\tau_n x_n\} \) follows as in Step I of [3, Proposition 1], except that we now use \( \hat{A}^\tau_n \) in place of the operators \( T_{s_0} \) used in [3]. Since \( x_n \to x_0 \) in \( Y \), we have \( x_n \to X_0 \) in \( X \) and \( J \hat{L} x_n = L x_0 \) in \( X^* \). Also, since \( x_n \in B_Y(0, R) \) and
\[
\partial (J^{-1}(G^1_1 \cap B_Y(0, R)) \cap \partial B_Y(0, R) \subset \partial (G^1_1 \cap B_Y(0, R)) \subset \partial (\gamma^* v^*),
\]
we have \( x_n \in J^{-1}(\gamma^* v^*) \) and therefore \( \gamma^* v^* \) is well-defined and \( \gamma^* v^* \) is constant for all \( s \in [0, 1] \). Also, by Proposition 2.11, the limit
\[
\lim_{t \to 0^+} d_{S_+}(H(1, \cdot), G^1_R(Y), 0) = 0
\]
exists. By shrinking \( t_0 \) further, if necessary, we find that \( d_{S_+}(H(1, \cdot), G^1_R(Y), 0) \) is a constant for all \( t \in (0, t_0) \). Suppose, if possible, that
\[
\gamma^* v^* \neq 0
\]
for some \( t_1 \in (0, t_0) \). Then there exists \( x_0 \in G^1_R(Y) \) such that
\[
\hat{L} x + \hat{A}^\tau_n x + \hat{C} x + t_1 M x = \tau_0 j^* v^*.
\]
This contradicts the choice of $\tau_0$ as stated in (3.12). Since
\[ d_{S_+}(H(0, \cdot), G^1_R(Y), 0) = d_{S_+}(H(1, \cdot), G^1_R(Y), 0), \]
we have
\[ d_{S_+}(\hat{L} + \hat{A}_t^x + \hat{C} + tM, G^1_R(Y), 0) = d_{S_+}(H(0, \cdot), G^1_R(Y), 0) = 0 \quad (3.18) \]
for all $t \in (0, t_0]$.

Next, we consider the homotopy $\tilde{H} : [0, 1] \times Y \to Y^*$ defined by
\[ \tilde{H}(s, x) = s(\hat{L}x + \hat{A}_t^x x + \hat{C}x) + tMx + (1 - s)\hat{J}x, \quad s \in [0, 1], \quad x \in j^{-1}(G^2_R). \]

As in [3] Step III, p. 29, it can be shown that there exists $t_0 > 0$ (shrink it to a smaller number if necessary) such that all the solutions of
\[ \tilde{H}(s, x) = 0, \quad t \in (0, t_0], \quad s \in [0, 1] \]
are bounded in $Y$. We enlarge the previous number $R > 0$, if necessary, so that all solutions of $\tilde{H}(s, x) = 0$ as described above are contained in $B_Y(0, R)$ in $Y$.

Again, by following arguments similar to that in [2] Theorem 2.2, we can show the existence of $t_1 \in (0, t_0]$ such that the equation $\tilde{H}(s, x) = 0$ has no solutions on $\partial G^2_R(Y)$ for any $t \in (0, t_1]$ and any $s \in [0, 1]$. Here, $G^2_R(Y) := j^{-1}(G^2_R) \cap B_Y(0, R)$. In fact, if we assume the contrary, we can arrive at a situation that contradicts (H4). At this point, we replace the number $t_0$ chosen previously with $t_1$ and call it $t_0$ again. Let us fix $t \in (0, t_0]$ and consider the homotopy equation
\[ \tilde{H}(s, x) = s(\hat{L}x + \hat{A}_t^x x + \hat{C}x) + tMx + (1 - s)\hat{J}x = 0, \quad s \in [0, 1], \quad x \in \overline{G^2_R(Y)}. \quad (3.19) \]

It is already discussed that (3.19) has no solution on $\partial G^2_R(Y)$. We note that $\tilde{H}$ is an affine homotopy of bounded demicontinuous operators of type $(S_+)$ on $\overline{G^2_R(Y)}$; namely, $\hat{L} + \hat{A}_t^x + \hat{C} + tM$ and $tM + \hat{J}$. We also note here that $tM + \hat{J}$ is strictly monotone. In view of Proposition 2.11, it follows that $\tilde{H}(s, x)$ is an admissible homotopy for the $(S_+)$-degree, $d_{S_+}$, which satisfies
\[ d_{S_+}(\tilde{H}(1, \cdot), G^2_R(Y), 0) = d_{S_+}(\tilde{H}(0, \cdot), G^2_R(Y), 0). \quad (3.20) \]

This implies
\[ d_{S_+}(\hat{L} + \hat{A}_t^x + \hat{C} + tM, G^2_R(Y), 0) = d_{S_+}(tM + \hat{J}, G^2_R(Y), 0) = 1 \quad (3.21) \]
for all $t \in (0, t_0]$. The last equality follows from [15] Theorem 3, (iv). From (3.18) and (3.21), we obtain
\[ d_{S_+}(\hat{L} + \hat{A}_t^x + \hat{C} + tM, G^1_R(Y), 0) \neq d_{S_+}(\hat{L} + \hat{A}_t^x + \hat{C} + tM, G^2_R(Y), 0) \]
for all $t \in (0, t_0]$. By the excision property of the $(S_+)$-degree, for each $t \in (0, t_0]$, there exists a solution $x_t \in G^1_R(Y) \setminus G^2_R(Y)$ of the equation
\[ \hat{L}x + \hat{A}_t^x x + \hat{C}x + tMx = 0. \]

We now pick a sequence $\{t_n\} \subset (0, t_0]$ such that $t_n \to 0$ and denote the corresponding solution $x_t$ by $x_n$, i.e.,
\[ \hat{L}x_n + \hat{A}^x_{t_n} x_n + \hat{C}x_n + t_n Mx_n = 0. \]

Since $Y$ is reflexive, we have $x_n \to x_0 \in Y$ by passing to a subsequence. This implies $x_n \to x_0$ in $X$ and $Lx_n \to Lx_0$ in $X^*$. By the boundedness (therefore strong quasiboundedness) of $A$, we may assume, in view of Lemma 2.3, that $A^x_{t_n} x_n \to w^* \in X^*$. By a standard argument in conjunction with Lemma 2.3 and the $(S_+)$-property
of C with respect to D(L), we obtain \( x_n \to x_0 \in G^1_R(Y) \setminus G^2_R(Y) \). By Lemma 2.3 and the demicontinuity of C, we have \( x_0 \in D(A) \), \( w^* \in Ax_0 \), and \( Cx_n \to Cx_0 \) in \( X^* \). Thus, \( Lx_0 + Ax_0 + Cx_0 = 0 \).

Finally, to show \( x_0 \in G_1 \setminus G_2 \), we note that
\[
G^1_R(Y) \setminus G^2_R(Y) = (G_1 \setminus G_2) \cap Y \cap B_Y(0, R) \subset G_1 \setminus G_2.
\]
Consequently, \( x_n \in G_1 \setminus G_2 \) for all \( n \), and therefore \( x_0 \in \overline{G_1 \setminus G_2} \subset (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2 \).

By (H3) and (H4), \( x_0 \notin \partial G_1 \cup \partial G_2 \) and hence \( x_0 \in D(L) \cap D(T) \cap (G_1 \setminus G_2) \).

3.1. Open Problem. Does Theorem 3.5 hold true if the boundedness of A is dropped? Since a positively homogeneous operator that is strongly quasibounded is necessarily bounded, it is desirable to determine whether Theorem 3.5 holds if \( A \) is assumed to be “quasibounded”. An operator \( A : X \supseteq D(A) \to X^* \) is said to be quasibounded if for every \( S > 0 \) there exists \( K(S) > 0 \) such that \( \|x\| \leq S \) and \( \langle x^*, x \rangle \leq S\|x\| \) for some \( x^* \in Ax \) imply \( \|x^*\| \leq K(S) \).

4. Applications

In this section, we apply Theorems 3.3 and 3.5 to elliptic and parabolic boundary value problems in general divergence form which are obtained by modifying relevant examples from Berkovits and Mustonen [11], Kittilä [27], and Adhikari [2].

**Application 4.1.** We consider the space \( X = W^{m,p}_0(\Omega) \) with the integer \( m \geq 1 \), the number \( p \in (1, \infty) \), and the domain \( \Omega \subset \mathbb{R}^N \) with smooth boundary. We let \( N_0 \) denote the number of all multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_N) \) such that \( |\alpha| = \alpha_1 + \cdots + \alpha_N \leq m \). For \( \xi = (\xi_\alpha)_{|\alpha| \leq m} \in \mathbb{R}^{N_0} \), we have a representation \( \xi = (\eta, \zeta) \), where \( \eta = (\eta_\alpha)_{|\alpha| \leq m-1} \in \mathbb{R}^{N_1} \), \( \zeta = (\zeta_\alpha)_{|\alpha| = m} \in \mathbb{R}^{N_2} \) and \( N_0 = N_1 + N_2 \). We let
\[
\xi(u) = (D^\alpha u)_{|\alpha| \leq m}, \quad \eta(u) = (D^\alpha u)_{|\alpha| \leq m-1}, \quad \text{and} \quad \zeta(u) = (D^\alpha u)_{|\alpha| = m},
\]
where \( D^\alpha = \prod_{i=1}^N \left( \frac{\partial}{\partial x_i} \right)^{\alpha_i} \). We write \( \nabla u := (D^\alpha u)_{|\alpha| = 1} \), and when \( |\alpha| = k \in \{1, 2, \ldots, m\} \), we simply write \( D^k u := (D^\alpha u)_{|\alpha| = k} \). Also, define \( q := p/(p-1) \).

We now consider the partial differential expression in divergence form
\[
\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u)), \quad x \in \Omega.
\]
The functions \( A_\alpha : \Omega \times \mathbb{R}^{N_0} \to \mathbb{R} \) are assumed to be Carathéodory, i.e., each \( A_\alpha(x, \xi) \) is measurable in \( x \) for fixed \( \xi \in \mathbb{R}^{N_0} \) and continuous in \( \xi \) for almost all \( x \in \Omega \). We assume the following conditions on \( A_\alpha \):

(H5) There exist \( p \in (1, \infty) \), \( c_1 > 0 \), and \( \kappa_1 \in L^q(\Omega) \) such that
\[
|A_\alpha(x, \xi)| \leq c_1 \xi^{|p-1} + \kappa_1(x), \quad x \in \Omega, \quad \xi \in \mathbb{R}^{N_0}, \quad |\alpha| \leq m.
\]

(H6) The Leray-Lions condition
\[
\sum_{|\alpha| = m} [A_\alpha(x, \eta_1 \zeta_1) - A_\alpha(x, \eta_1 \zeta_2)](\zeta_1 - \zeta_2) > 0
\]
is satisfied for every \( x \in \Omega \), \( \eta_1, \zeta_1, \zeta_2 \in \mathbb{R}^{N_2} \) with \( \zeta_1 \neq \zeta_2 \).
If an operator \( A \) implies that 
\( A \) the operator we have
\[
|\alpha| \leq m
\]

is satisfied for every \( x \in \Omega \) and \( \xi_1, \xi_2 \in \mathbb{R}^N_0 \).

(H8) There exist \( c_2 > 0, \kappa_2 \in L^1(\Omega) \) such that
\[
\sum_{|\alpha| \leq m} A_\alpha(x, \xi) \xi_\alpha \geq c_2 |\xi|_p^p - \kappa_2(x), \quad x \in \Omega, \ \xi \in \mathbb{R}^N_0.
\]

(H9) Each \( A_\alpha(x, \xi) \) is homogeneous of degree \( \gamma > 0 \) with respect to \( \xi \).

If an operator \( A : W^{m,p}_0(\Omega) \to W^{-m,q}(\Omega) \) is given by
\[
\langle Au, v \rangle = \int_{\Omega} \sum_{|\alpha| \leq m} A_\alpha(x, \xi) D^\alpha v, \quad u, v \in W^{m,p}_0(\Omega), \quad (4.1)
\]

then the conditions (H5), (H7) imply that \( A \) is bounded, continuous, and monotone
as discussed in Kittilä [27, pp. 25-26] and Pascali and Sburlan [28, pp. 274-275].
Since \( A \) is continuous, it is maximal monotone. Moreover, the condition (H9)
implies that \( A \) is positively homogeneous of degree \( \gamma > 0 \). For example, for \( m = 1 \),
we have \( |\alpha| \leq 1 \), and when
\[
A_\alpha(x, \eta, \zeta) = \begin{cases} 
|\zeta|^{p-2} \zeta & \text{for } |\alpha| = 1 \\
0 & \text{for } |\alpha| = 0,
\end{cases}
\]

the operator \( A \) in (4.1) is given by \( A := -\Delta_p \), where \( \Delta_p \) is the \( p \)-Laplacian from
\( W^{1,p}_0(\Omega) \) to \( W^{-1,q}(\Omega) \) defined as
\[
\Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad u \in W^{1,p}_0(\Omega).
\]
It is clear that \( \Delta_p \) is positively homogeneous of degree \( p - 1 \).

Similarly, the condition (H5), with \( A_\alpha \) replaced by \( C_\alpha \), implies that the operator
\[
\langle C u, v \rangle = \int_{\Omega} \sum_{|\alpha| \leq m} C_\alpha(x, \xi(u)) D^\alpha v, \quad u, v \in W^{m,p}_0(\Omega), \quad (4.2)
\]
is a bounded continuous mapping. We also know that conditions (H5), (H6), and (H8), with \( C_\alpha \) in place of \( A_\alpha \) everywhere, imply that the operator \( C \) is of type \((S_+)\)
(see Kittilä [27, p. 27]).

We also consider a multifunction \( H : \Omega \times \mathbb{R}^{N_1} \to 2^\mathbb{R} \) such that

(H10) \( H(x, r) = [\varphi(x, r), \psi(x, r)] \) is measurable in \( x \) and upper semicontinuous in \( r \),
where \( \varphi, \psi : \Omega \times \mathbb{R}^{N_1} \to \mathbb{R} \) are measurable functions; and

(H11) \( |H(x, r)| = \max \{|\varphi(x, r)|, |\psi(x, r)|\} \leq a(x) + c_2 |r| \) a.e. on \( \Omega \times \mathbb{R}^{N_1} \),
where \( a(\cdot) \in L^q(\Omega), \ c_2 > 0 \).

Define \( T : W^{m,p}_0(\Omega) \to 2^{W^{-m,q}(\Omega)} \) by
\[
Tu = \left\{ h \in W^{-m,q}(\Omega) : \exists w \in L^q(\Omega) \text{ such that } w(x) \in H(x, u(x)) \right\}
\]
and \( \langle h, v \rangle = \int_{\Omega} w(x) v(x) \) for all \( v \in W^{m,p}_0(\Omega) \).

It is well-known that \( T \) is upper-semicontinuous and compact with closed and convex values
(see [21, p. 254]), and therefore \( T \) is of class \((P)\).

We now state the following theorem as an application of Theorem 3.3.
Theorem 4.2. Assume that the operators $A$, $C$, and $T$ are defined as above. Assume, further, that the rest of the conditions of Theorem 3.3 are satisfied for two balls $G_1 = B_{\delta_1}(0)$ and $G_2 = B_{\delta_2}(0)$, where $0 < \delta_2 < \delta_1$. Then the Dirichlet boundary value problem

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \left( A_\alpha(x,\xi(u)) + C_\alpha(x,\xi(u)) \right) + H(x,u) \geq 0, \quad x \in \Omega,$$

has a “weak” nonzero solution $u \in B_{\delta_1}(0) \setminus B_{\delta_2}(0) \subset W^{m,p}_0(\Omega)$, which satisfies the inclusion $Au + Cu + Tu \geq 0$.

Application 4.3. Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ with smooth boundary, $m \geq 1$ an integer, and $a > 0$. Set $Q = \Omega \times [0,a]$. Consider differential operators of the form

$$\frac{\partial u}{\partial t}(x,t) + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \left( A_\alpha(x,t,\xi(u(x,t))) + C_\alpha(x,t,\xi(u(x,t))) \right) \quad (4.3)$$

in $Q$. The functions $A_\alpha = A_\alpha(x,t,\xi)$ and $C_\alpha = C_\alpha(x,t,\xi)$ are defined for $(x,t) \in Q$, $\xi = (\xi_\alpha)_{|\alpha| \leq m} = (\eta,\zeta) \in \mathbb{R}^{N_0}$ with $\eta = (\eta_\gamma)_{|\alpha| \leq m-1} \in \mathbb{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha| = m} \in \mathbb{R}^{N_2}$, and $N_1 + N_2 = N_0$. We assume that each $A_\alpha(x,t,\xi)$ satisfies the usual Carathéodory condition. We consider the following conditions.

(H12) (Continuity) For some $p > 1$, $c_1 > 0$, $g \in L^q(Q)$ with $q = p/(p-1)$, we have

$$|A_\alpha(x,t,\eta,\zeta)| \leq c_1(|\xi|^{p-1} + |\eta|^{p-1} + g(x,t)),$$

for $(x,t) \in Q$, $\xi = (\eta,\zeta) \in \mathbb{R}^{N_0}$ and $|\alpha| \leq m$.

(H13) (Monotonicity)

$$\sum_{|\alpha| \leq m} (A_\alpha(x,t,\xi_1) - A_\alpha(x,t,\xi_2)) (\xi_{1\alpha} - \xi_{2\alpha}) \geq 0 \text{ for } (x,t) \in Q \text{ and } \xi_1,\xi_2 \in \mathbb{R}^{N_0}.$$

(H14) (Leray-Lions)

$$\sum_{|\alpha| = m} (A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*)) (\zeta_\alpha - \zeta^*_\alpha) > 0,$$

for $(x,t) \in Q$, $\eta \in \mathbb{R}^{N_1}$ and $\zeta,\zeta^* \in \mathbb{R}^{N_2}$.

(H15) (Coercivity) There exist $c_0 > 0$ and $h \in L^1(Q)$ such that

$$\sum_{|\alpha| \leq m} A_\alpha(x,t,\xi) \geq c_0 |\xi|^p - h(x,t), \quad (x,t) \in Q \text{ and } \xi \in \mathbb{R}^{N_0}.$$

(H16) Each $A_\alpha(x,t,\xi)$ is homogeneous of degree $\gamma > 0$ with respect to $\xi$.

Under condition (H12), the second term of (4.3) with $C_\alpha = 0$ generates a continuous bounded operator $A : X \to X^*$ defined by

$$\langle Au, v \rangle = \sum_{|\alpha| \leq m} \int_Q A_\alpha(x,t,\xi(u(x,t))) D^\alpha v, \quad u, v \in X,$$

where $X = L^p(0,a;V)$, $X^* = L^q(0,a;V^*)$, and $V = W^{m,p}_0(\Omega)$. With the additional conditions (H13) and (H16), the operator $A$ is maximal monotone and positively homogeneous of degree $\gamma$. Under (H12), (H14), and (H15) with $A_\alpha$ replaced by
$C_{\alpha}$ and other obvious changes, the second term in (4.3) with $A_{\alpha} = 0$ generates a continuous, bounded operator $C$ defined as

$$\langle Cu, v \rangle = \sum_{|\alpha| \leq m} \int_{Q} C_{\alpha}(x,t,\xi(u(x,t)))D^{\alpha}v, \quad u, v \in X,$$

which satisfies the condition $(S_{+})$ with respect to $D(L)$, where the operator $L$ is defined as follows. The operator $\partial/\partial t$ generates an operator $L : X \supset D(L) \rightarrow X^*$, where

$$D(L) = \{ v \in X : v' \in X^*, \; v(0) = 0 \},$$

via the relation

$$\langle Lu, v \rangle = \int_{0}^{a} \langle u'(t), v(t) \rangle_v \, dt, \quad u \in D(L), \; v \in X,$$

where $\langle \cdot, \cdot \rangle_v$ is the duality pairing in $V^* \times V$. The symbol $u'(t)$ is the generalized derivative of $u(t)$, i.e.,

$$\int_{0}^{a} u'(t) \varphi(t) \, dt = - \int_{0}^{a} \varphi'(t) u(t) \, dt, \quad \varphi \in C_{0}^{\infty}(0,a).$$

We can verify, as in Zeidler [34], that $L$ is densely defined, linear and maximal monotone.

Given $h \in L^{q}(Q)$, define $h^* \in X^*$ by

$$\langle h^*, v \rangle = \int_{Q} hv, \quad v \in X.$$

As an application of Theorem 3.5 we obtain the following theorem.

**Theorem 4.4.** Assume that the operators $L, A,$ and $C$ are as above, with $A_{\alpha}$ satisfying (H12), (H13), and (H16), and $C_{\alpha}$ in place of $A_{\alpha}$ satisfying (H12), (H14), and (H15). Assume, for a given $h \in L^{q}(Q)$, that the rest of the conditions of Theorem 3.5 are satisfied when $C$ is replaced with $C - h^*$ for two balls $G_{1} = B_{\delta_{1}}(0)$ and $G_{2} = B_{\delta_{2}}(0)$ in $X = L^{p}(0,a;V)$, where $0 < \delta_{2} < \delta_{1}$ and $V = W^{m}_{0}(\Omega)$. Then the initial-boundary value problem

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} \left( A_{\alpha}(x,t,\xi(u)) + C_{\alpha}(x,t,\xi(u)) \right) = h(x,t),$$

$$D^{\alpha}u(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0,a], \quad |\alpha| \leq m - 1,$$

$$u(x,0) = 0, \quad x \in \Omega,$$

has a “weak” nonzero solution $u \in B_{\delta_{1}}(0) \setminus B_{\delta_{2}}(0) \subset L^{p}(0,a;V)$ satisfying

$$Lu + Au + Cu = h^*.$$

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References

[1] A. Addou, B. Mermri; Topological degree and application to a parabolic variational inequality problem, \textit{Int. J. Math. \& Sci.}, \textbf{25} (2001), no. 4, 273–287.
[2] D. R. Adhikari; Nontrivial solutions of inclusions involving perturbed maximal monotone operators, \textit{Electron. J. Differential Equations}, \textbf{2017} (2017), no. 151, 1–21.
[3] D. R. Adhikari, A. G. Kartsatos; Invariance of domain and eigenvalues for perturbations of densely defined linear maximal monotone operators, \textit{Appl. Anal.}, \textbf{95} (2016), no. 1, 24–43.
[4] D. R. Adhikari, A. G. Kartsatos; A new topological degree theory for perturbations of the sum of two maximal monotone operators, \textit{Nonlinear Anal.}, \textbf{74} (2011), no. 14, 4622–4641.
[5] D. R. Adhikari, A. G. Kartsatos; Strongly quasibounded maximal monotone perturbations for the Berkovits-Mustonen topological degree theory, \textit{J. Math. Anal. Appl.}, \textbf{348} (2008), no. 1, 12–136.
[6] D. R. Adhikari, A. G. Kartsatos; Topological degree theories and nonlinear operator equations in Banach spaces, \textit{Nonlinear Anal.}, \textbf{69} (2008), no. 4, 1235–1255.
[7] Y. Alber, I. Ryazantseva; Nonlinear ill-posed problems of monotone type, \textit{Springer, Dordrecht}, 2006.
[8] T. M. Asfaw, A. G. Kartsatos; A Browder topological degree theory for multi-valued pseudomonotone perturbations of maximal monotone operators, \textit{Adv. Math. Sci. Appl.}, \textbf{22} (2012), no. 1, 91–148.
[9] J. P. Aubin, A. Cellina; Differential inclusions, \textit{Springer-Verlag}, 1984.
[10] V. Barbu; Nonlinear semigroups and differential equations in Banach spaces, \textit{Noordhoff Int. Publ.}, Leyden (The Netherlands), 1975.
[11] J. Berkovits, V. Mustonen; Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems, \textit{Rend. Mat. Appl.}, \textbf{12} (1992), no. 3, 597–621.
[12] J. Berkovits, M. Miettunen; On the uniqueness of the Browder degree, \textit{Proc. Amer. Math. Soc.}, \textbf{136} (2008), no. 10, 3467–3476.
[13] I. Boubakari, A. G. Kartsatos; The Leray-Schauder approach to the degree theory for \((S_+)-perturbations\) of maximal monotone operators in separable reflexive Banach spaces, \textit{Nonlinear Anal.}, \textbf{70} (2009), no. 12, 4350–4368.
[14] H. Brézis, M. G. Crandall, A. Pazy; Perturbations of nonlinear maximal monotone sets in Banach spaces, \textit{Comm. Pure Appl. Math.}, \textbf{23} (1970), 123–144.
[15] F. E. Browder; The degree of mapping and its generalizations, \textit{Contemp. Math.}, \textbf{21} (1983), 15–40.
[16] F. E. Browder; Fixed point theory and nonlinear problems, \textit{Bull. Amer. Math. Soc.}, \textbf{9} (1983), no. 1, 1–39.
[17] F. E. Browder; Nonlinear operators and nonlinear equations of evolution in Banach spaces, nonlinear functional analysis, \textit{Proc. Sympos. Pure Appl. Math.}, \textbf{18} (1976), 1–308.
[18] F. E. Browder, P. Hess; Nonlinear mappings of monotone type in Banach spaces, \textit{J. Functional Analysis}, \textbf{11} (1972), 251–294.
[19] I. Cioranescu; Geometry of Banach spaces, duality mappings and nonlinear problems, \textit{Kluwer Acad. Publ.}, Dordrecht, 1990.
[20] P. Hess; On nonlinear mappings of monotone type homotopic to odd operators, \textit{J. Functional Analysis}, \textbf{11} (1972), 138–167.
[21] S. Hu, N. S. Papageorgiou; Generalizations of Browder’s degree, \textit{Trans. Amer. Math. Soc.}, \textbf{347} (1995), no.1, 233–259.
[22] A. G. Kartatsos, J. Lin; Homotopy invariance of parameter-dependent domains and perturbation theory for maximal monotone and m-accretive operators in Banach spaces, \textit{Adv. Differential Equations}, \textbf{8} (2003), no.2, 129–160.
[23] A. G. Kartatsos, J. Quarcoo; A new topological degree theory for densely defined \((S_+)-perturbations\) of multivalued maximal monotone operators in reflexive separable Banach spaces, \textit{Nonlinear Anal.}, \textbf{69} (2008), no. 8, 2339–2354.
[24] A. G. Kartatsos, I. V. Skrypnik; Degree theories and invariance of domain for perturbed maximal monotone operators in Banach spaces, \textit{Adv. Differential Equations}, \textbf{12} (2007), no. 11, 1275–1320.
[25] A. G. Kartsatos, I. V. Skrypnik; A new topological degree theory for densely defined quasi-bounded ($\tilde{S}_+$)-perturbations of multivalued maximal monotone operators in reflexive Banach spaces, *Abstr. Appl. Anal.*, (2005), no. 2, 121–158.

[26] A. G. Kartsatos, I. V. Skrypnik; On the eigenvalue problem for perturbed nonlinear maximal monotone operators in reflexive Banach spaces, *Trans. Amer. Math. Soc.*, 358 (2006), no. 9, 3851–3881.

[27] A. Kittilä; On the topological degree for a class of mappings of monotone type and applications to strongly nonlinear elliptic problems, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertations*, 91 (1994), 48pp.

[28] D. Pascali, S. Sburlan; Nonlinear mappings of monotone type, *Sijthoff and Noordhoof*, Bucharest, 1978.

[29] R. Rockafellar, Local boundedness of nonlinear monotone operators, *Michigan Math. J.*, 16 (1969) 397–407.

[30] S. Simons; Minimax and monotonicity, vol. 1693, *Springer-Verlag*, Berlin, 1998.

[31] I. V. Skrypnik; Nonlinear elliptic boundary value problems, *BG Teubner*, 1986.

[32] I. V. Skrypnik; Methods for analysis of nonlinear elliptic boundary value problems, vol. 139, *American Mathematical Society*, 1994.

[33] S. L. Trojanski; On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, *Studia Math.*, 37 (1971), 173–180.

[34] E. Zeidler; Nonlinear functional analysis and its applications, II/B, *Springer-Verlag*, New York, 1990.

[35] S.-S. Zhang, Y.-Q. Chen; Degree theory for multivalued ($S$)-type mappings and fixed point theorems, *Appl. Math. Mech.*, 11 (1990), no. 5, 441–454.

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