The Hopf-Lax formula for multiobjective costs with non-constant discount via set optimization

D. Visetti*

May 6, 2021

Abstract

The minimization of a multiobjective Lagrangian with non-constant discount is studied. The problem is embedded into a set-valued framework and a corresponding definition of the value function is given. Bellman’s optimality principle and Hopf-Lax formula are derived. The value function is shown to be a solution of a set-valued Hamilton-Jacobi equation.

Keywords: multicriteria calculus of variations, value function, discount factor, Hopf-Lax formula, Bellman’s principle, Hamilton-Jacobi-Bellman equation, set relations.

1 Introduction

In this paper the following optimization problem is considered

$$\inf_{y \in A(t,x)} \left( \int_t^T d_t(s) L(\dot{y}(s))ds + g(y(T)) \right)$$

where $L$ is a vector-valued Lagrangian or utility function, $d_t(s)$ is a discount factor and $A(t,x)$ is a set of admissible arcs that take the value $x$ at time $t$. Often it is the case that there are more than one function to minimize (for example production cost and holding inventory cost). Sometimes these functions can contradict each other, in the sense that trying to minimize one of them leads to the increase of another one. It can be sensible to discount the expenditures: for example we can think of a discount at a continuous rate $r$ $d_t(s) = e^{-r(s-t)}$ or $d_t(s) = e^{-\int_t^s \rho(s)ds}$, where $\rho$ is in $L^\infty([0,T])$. In the Preface of [8] the author writes that “Forward-looking individuals recognize that decisions made today affect those to be made in the future, at least in part, by expanding or contracting the set of admissible choices, that is, by lowering or raising the cost of a future choice. Such intertemporal linkages reside at the core of all dynamic processes in economics. Consequently, mathematical methods that account for such intertemporal linkages are fundamental, in principle, to all economic decisions”. In the

*Faculty of Economics and Management, Free University of Bozen-Bolzano, Italy.
same book an economic interpretation of control problems and of Hamilton-Jacobi-Bellman equations are provided.

In [12], the problem \((P)\) has been studied in the case without discount: in order to have the Hopf-Lax formula the Lagrangian depends only on the derivative \(\dot{y}\). In particular, it does not depend on time. The idea in this paper and in [12] is to embed the vector-valued problem into a set-valued one. More precisely, it is fundamental to work in a complete lattice, so that infima and suprema are well defined. For this approach see [10].

In [22], the author proposed problem \((P)\) for a real-valued utility function, with a non-constant discount, provided a Hop-Lax formula and deduced a dynamic programming equation.

The present paper generalizes both [12] and [22]. With respect to the first one, there is the non-constant discount. With respect to the second one, there is the multi-objective Lagrangian. To the knowledge of the author, this kind of generalization has never been addressed.

When the problem \((P)\) is embedded into a set-valued framework, a complete lattice structure is obtained and so the value function has a straightforward definition. It is important to notice that, because of the discount factor, it can be called a current value optimal value function, for its value is discounted back to time \(t\).

The Hopf-Lax formula was found in the 50’s (see [17] and [15], the first one in dimension one and the second one in general dimension). This result came from the fact that straight lines are optimal trajectories. This happens when the Hamiltonian function only depends on the derivative of the arc \((H = H(p))\). In [6] it is proved that this happens also when the Hamiltonian depends also on the arc \(H = H(y, p)\), it is nondecreasing in \(y\) and convex and positively homogeneous of degree 1 in \(p\).

More recent developments can be found in [23], [1], [4], [14].

In [22] as well as in this paper, the optimal trajectories are not linear, but all the same it is possible to write a generalization of the Hopf-Lax formula, in the sense that it is still possible to shift the infimum from an infinite dimensional space to a finite dimensional one.

As regards the importance of considering a non-constant discount factor, this choice arises in many economic models. The utility function \(L\), that measures the satisfaction of an agent, changes during the time, in the sense that an earlier attainment of the utility gives the agent a higher satisfaction. This is why the discount factor is also called impatience rate (see for example [5], [16], [18]).

The problem of [22] has been generalized to the field of deterministic differential games with a stochastic terminal time in [20]. See also [19].

## 2 Preliminaries

First, some basic concepts and definitions of set optimization and of complete lattice approach are recalled. For more on the subject, see [10].

Let \(C\) be a closed and convex cone in \(\mathbb{R}^d\) with nonempty interior. Its dual cone is defined as

\[
C^+ = \{\zeta \in \mathbb{R}^d \mid \forall z \in C \quad \zeta \cdot z \geq 0\},
\]
where $\cdot$ denotes the scalar product in $\mathbb{R}^d$.

If $\mathcal{P}(\mathbb{R}^d)$ is the power set of $\mathbb{R}^d$, it can be endowed with the Minkowski sum, i.e., for $A, B \in \mathcal{P}(\mathbb{R}^d)$ one sets $A + B = \{ a + b \mid a \in A, \ b \in B \}$. For the empty set, the sum is defined as $A + \emptyset = \emptyset + A = \emptyset$. We shall consider also the closure with respect to the usual topology of the sum:

$$A \oplus B = \text{cl} (A + B).$$

We denote by $\mathcal{F}(\mathbb{R}^d, C)$ the subset of $\mathcal{P}(\mathbb{R}^d)$ of those sets which are invariant with respect to the sum of the cone and closed:

$$\mathcal{F}(\mathbb{R}^d, C) = \{ A \subseteq \mathbb{R}^d \mid A = A \oplus C \}.$$

The pair $(\mathcal{F}(\mathbb{R}^d, C), \supseteq)$ is a complete lattice, where the infimum and the supremum over a collection $A \subseteq \mathcal{F}(\mathbb{R}^d, C)$ can be found as

$$\inf A = \text{cl} \bigcup_{A \in A} A, \quad \sup A = \bigcap_{A \in A} A.$$

For any $\zeta \in C^+ \setminus \{0\}$, we define the half-space

$$H^+(\zeta) = \{ z \in \mathbb{R}^d \mid \zeta \cdot z \geq 0 \}.$$

We can also consider the $\zeta$-difference of two sets $A, B \in \mathcal{F}(\mathbb{R}^d, C)$:

$$A - \zeta B = \{ z \in \mathbb{R}^d \mid z + B \subseteq A \oplus H^+(\zeta) \}.$$

It is possible to see that the difference can also be written as

$$A - \zeta B = \{ z \in \mathbb{R}^d \mid \zeta \cdot z + \inf_{b \in B} \zeta \cdot b \geq \inf_{a \in A} \zeta \cdot a \}$$

and that it is always a closed half-space or the empty set or the whole space.

We give here a definition of limit. Let $\{A_m\}_{m \in \mathbb{N}}$ be a sequence of sets in $\mathcal{F}(\mathbb{R}^d, C)$. The notation $\lim_{m \to \infty} A_m$ identifies the set

$$\lim_{m \to \infty} A_m = \{ z \in \mathbb{R}^d \mid \forall m \in \mathbb{N}, \ \exists z_m \in A_m \text{ and } \lim_{m \to \infty} z_m = z \},$$

which is still in $\mathcal{F}(\mathbb{R}^d, C)$. This definition coincides with Painlevé-Kuratowski upper limit (see [2]). Generalizing, if $\{A_s\}_{s \in S}$ with $S \subseteq \mathbb{R}$ is a family of sets in $\mathcal{F}(\mathbb{R}^d, C)$ and $s_* \in \mathbb{R}$, we denote by $\lim_{s \to s_*} A_s$ the set which satisfies for any sequence $\{s_m\}_{m \in \mathbb{N}} \subseteq S$, with $\lim_{m \to \infty} s_m = s_*$,

$$\lim_{s \to s_*} A_s = \lim_{m \to \infty} A_{s_m}.$$

Let $X$ be a vector space and $f : X \to \mathcal{F}(\mathbb{R}^d, C)$ be a set-valued function. The graph of $f$ is the set

$$\text{graph } f = \{(x, z) \in X \times \mathbb{R}^d \mid z \in f(x) \} \subseteq X \times \mathbb{R}^d$$

and its (effective) domain is the set

$$\text{dom } f = \{ x \in X \mid f(x) \neq \emptyset \} \subseteq X.$$
Solutions of set optimization problems split into sets generating the infimum (infimizers) and points with a minimal function value with respect to the order in the corresponding lattice. The solution concept is due to [13] (see also [10] for more comments and references). For $M \subseteq X$, we set

$$f[M] = \{ f(x) \mid x \in M \}.$$ 

The set $M \subseteq X$ is called an infimizer of $f$ if

$$\inf f[M] = \inf f[X]. \tag{1}$$

We want now to define a set-valued function, which generalizes a linear function. Let $\eta \in \mathbb{R}^n$ and $\zeta \in C^+$ be given. We consider $S_{(\eta,\zeta)} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^d, C)$:

$$S_{(\eta,\zeta)}(x) = \{ z \in \mathbb{R}^d \mid \zeta \cdot z \geq \eta \cdot x \}. \tag{2}$$

In fact this function is half-space valued and is additive and positively homogeneous in $x$.

The Fenchel conjugate of the function $f : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^d, C)$ is defined as the function

$$f^* : \mathbb{R}^n \times C^+ \setminus \{0\} \to \mathcal{F}(\mathbb{R}^d, C) \quad (\eta, \zeta) \mapsto \sup_{x \in \mathbb{R}^n} S_{(\eta,\zeta)}(x) - 

\quad \star \quad f(x). \tag{3}$$

3 Variational problem with discount

Let $L : \mathbb{R}^n \to \mathbb{R}^d$, $g : \mathbb{R}^n \to \mathbb{R}^d$ be two functions mapping into $\mathbb{R}^d$. Fix $T > 0$ and consider a variable discount factor $d_t(s)$ for $t$ in $[0, T]$ and $s$ in $[t, T]$, taking positive values.

Let $\mathcal{L}_t : [0, T] \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^d, C)$ denote the set-valued function

$$\mathcal{L}_t(s, w) = d_t(s)L(w) + C.$$

We define the functionals $I_t : W^{1,1}([t, T], \mathbb{R}^n) \to \mathbb{R}^d$, $J_t : W^{1,1}([t, T], \mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^d, C)$ by

$$I_t(y) = \int_t^T d_t(s)L(\dot{y}(s)) \, ds + d_t(T)g(y(T)),$$

$$J_t(y) = \int_t^T \mathcal{L}_t(s, \dot{y}(s)) \, ds + d_t(T)g(y(T)),$$

where the second integral is understood in the Aumann sense (see [3]) and where for every $F : [0, T] \to \mathcal{F}(\mathbb{R}^d, C)$, $t \in [0, T]$, we define

$$\int_t^t F(s) \, ds = C.$$

For any $x \in \mathbb{R}^n$ we shall consider the problem:

$$\text{minimize } J_t(y) \quad \text{over the set } A(t, x) = \{ y \in W^{1,1}([t, T], \mathbb{R}^n) \mid y(t) = x \}. \tag{4}$$

4
Since the problem has been now embedded into a set-valued problem, we are now working on the complete lattice $\mathcal{F}(\mathbb{R}^d, C)$. This means that the infimum and the supremum are well defined and the value function is (see also [12]):

$$
U(t,x) = \inf_{y \in A(t,x)} J_t(y). \quad (5)
$$

For simplicity for any $\zeta \in C^+ \setminus \{0\}$, we denote

$$
L_\zeta : \mathbb{R}^n \rightarrow \mathbb{R}
$$

$$
w \mapsto L(w) \cdot \zeta
$$

$$
g_\zeta : \mathbb{R}^n \rightarrow \mathbb{R}
$$

$$
w \mapsto g(w) \cdot \zeta
$$

$$
I_{t,\zeta} : W^{1,1}([t,T], \mathbb{R}^n) \rightarrow \mathbb{R}
$$

$$
y \mapsto I_t(y) \cdot \zeta
$$

Let $B^+$ be a base of $C^+$, i.e., for each element $\zeta \in C^+ \setminus \{0\}$ there exist a unique $\xi \in B^+$ and a unique $\lambda > 0$ such that $\zeta = \lambda \xi$. For example $B^+$ can be formed by the unitary vectors in $C^+$. Another possibility is, if there is an element $c_0 \in C$ such that $\zeta \cdot c_0 > 0$ for all $\zeta \in C^+ \setminus \{0\}$, then the set $B_{c_0}^+ = \{ \zeta \in C^+ \mid \zeta \cdot c_0 = 1 \}$ is a base of $C^+$.

We will consider the following hypotheses:

(h1) All the scalarizations $L_\zeta$, $\zeta \in B^+$ of $L$ are $C^2$, strictly convex and coercive

$$
\lim_{|w|_n \rightarrow \infty} \frac{L_\zeta(w)}{|w|_n} = +\infty,
$$

where $| \cdot |_n$ is the standard norm in $\mathbb{R}^n$.

(h2) All the scalarizations $g_\zeta$, $\zeta \in B^+$ of $g$ are globally Lipschitz in $\mathbb{R}^n$.

(h3) For any $t \in [0,T]$, $d_t : [t,T] \rightarrow (a,1]$, with $a > 0$, is continuous and $d_t(t) = 1$ for each $t \in [0,T]$.

Throughout the paper, $\nabla$ denotes the gradient of a real function or the Jacobian matrix of a vector function and $\nabla^2$ the Hessian matrix of a real function.

Remark 3.1. By Lemma 3.1 in [22] (see also [9]) and hypotheses (h1) and (h3), for any $\zeta \in C^+ \setminus \{0\}$ and fixed $s \in (t,T]$,

(i) $\nabla L_\zeta$ is a homeomorphism of $\mathbb{R}^n$;

(ii) the mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$
p \mapsto (\nabla L_\zeta)^{-1}\left(\frac{p}{d_t(s)}\right)
$$

$$
p \mapsto \int_t^s (\nabla L_\zeta)^{-1}\left(\frac{p}{d_t(r)} \right) dr
$$

are of class $C^1$ and surjective.
For any $p \in \mathbb{R}^n$, $\zeta \in C^+ \setminus \{0\}$, we consider the arc
\[
Y_{t,x,p,\zeta}(s) = x + \int_t^s \left( \nabla L_{s,\zeta} - \frac{p}{d_t(r)} \right) dr.
\]
(6)
It is an element of $A(t,x)$.

In the following lemma we study some property of concavity of the Hamiltonian function, recalling the definition of the function $S$ in (2).

**Lemma 3.2.** Let $\zeta \in C^+ \setminus \{0\}$, $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\mathcal{H}_{t,\zeta} : [t,T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R}^d, C^+_+)$ be defined by
\[
\mathcal{H}_{t,\zeta}(s,w,p) = S_{(p,\zeta)}(w) - \zeta \mathcal{L}(s,w).
\]
Then
\[
\mathcal{L}^*(s,p,\zeta) = \sup_{w \in \mathbb{R}^n} \mathcal{H}_{t,\zeta}(s,w,p) = \mathcal{H}_{t,\zeta} \left( s, \dot{Y}_{t,x,p,\zeta}(s), p \right).
\]
**Proof.** It is immediate to see that
\[
z \in \mathcal{H}_{t,\zeta}(s,w,p)
\]
if and only if
\[
z \cdot \zeta + d_t(s)L_{\zeta}(w) \geq p \cdot w.
\]
Since the real-valued function $p \cdot w - d_t(s)L_{\zeta}(w)$ is concave in $w$ and has a maximizer in $\dot{Y}_{t,x,p,\zeta}(s)$, we have
\[
p \cdot \dot{Y}_{t,x,p,\zeta}(s) - d_t(s)L_{\zeta}(\dot{Y}_{t,x,p,\zeta}(s)) \geq p \cdot w - d_t(s)L_{\zeta}(w)
\]
for any $w \in \mathbb{R}^n$. Now, if $z \in \mathcal{H}_{t,\zeta} \left( s, \dot{Y}_{t,x,p,\zeta}(s), p \right)$,
\[
z \cdot \zeta \geq p \cdot \dot{Y}_{t,x,p,\zeta}(s) - d_t(s)L_{\zeta} \left( \dot{Y}_{t,x,p,\zeta}(s) \right) \geq p \cdot w - d_t(s)L_{\zeta}(w)
\]
for any $w \in \mathbb{R}^n$ and
\[
z \in \bigcap_{w \in \mathbb{R}^n} \mathcal{H}_{t,\zeta}(s,w,p).
\]

As a consequence of the previous lemma, the following property holds.

**Lemma 3.3.** For any $(t,x) \in [0,T] \times \mathbb{R}^n$, $w,p \in \mathbb{R}^n$, $\zeta \in C^+ \setminus \{0\}$ and $s \in [t,T]$, there holds
\[
p \cdot \left( \dot{Y}_{t,x,p,\zeta}(s) - w \right) - \frac{\zeta}{|\zeta|_d} - d_t(s) \left( L \left( \dot{Y}_{t,x,p,\zeta}(s) \right) - L(w) \right) \in H^+(\zeta),
\]
where $| \cdot |_d$ is the norm in $\mathbb{R}^d$. 

6
Proof. By Lemma 3.2

\[ S_{(p, \zeta)}(\dot{Y}_{t,x,p,\zeta}(s)) - \zeta L_t(s, \dot{Y}_{t,x,p,\zeta}(s)) \subseteq S_{(p, \zeta)}(w) - \zeta L_t(s, w). \]  

(7)

These sets are half-spaces because they are \( \zeta \)-differences and can be written in the following way:

\[ S_{(p, \zeta)}(w) - \zeta L_t(s, w) = p \cdot w \frac{\zeta}{|\zeta|_d} - d_t(s)L(w) + H^+(\zeta). \]

Then the inclusion (7) can be written

\[ p \cdot (\dot{Y}_{t,x,p,\zeta}(s) - w) \frac{\zeta}{|\zeta|_d} - d_t(s) \left( L(\dot{Y}_{t,x,p,\zeta}(s)) - L(w) \right) \in H^+(\zeta). \]

\( \square \)

4 Bellman’s optimality principle

Usually, in Bellman’s optimality principle an inequality and an equation are involved that link the value function evaluated at two different times \( t < \tau \). This is true for example when the Lagrangian is real-valued (see Lemma 4.1 and Corollary 4.1 in [22]). Instead, if the Lagrangian is vector-valued, the situation is more complex and it is not possible to obtain the value function at time \( \tau \). In fact the infimum is taken over the sum of two parts, as one can see in the following theorem.

**Theorem 4.1.** For every initial condition \((t, x) \in [0, T] \times \mathbb{R}^n\), admissible arc \( y \in A(t, x) \) and \( \tau \in [t, T] \), we have

\[ U(t, x) \supseteq \int_t^T L_t(s, \dot{y}(s)) ds \oplus \inf_{\eta \in A(\tau, y(\tau))} [W(t, \tau, \eta) + J_\tau(\eta)], \]  

(8)

where \( W : [0, T] \times [t, T] \times W^{1,1}([\tau, T], \mathbb{R}^n) \rightarrow \mathbb{R}^d \),

\[ W(t, \tau, \eta) = \int_\tau^T (d_t(s) - d_\tau(s))L(\dot{y}(s)) ds + (d_\tau(T) - d_\tau(T))g(\eta(T)). \]

Moreover, the set \( M \subset A(t, x) \) is an infimizer for problem (4) (see definition (1)) if and only if for all \( \tau \in [t, T] \)

\[ U(t, x) = \inf_{y \in M} \left[ \int_t^T L_t(s, \dot{y}(s)) ds \oplus \inf_{\eta \in A(\tau, y(\tau))} [W(t, \tau, \eta) + J_\tau(\eta)] \right]. \]  

(9)

**Proof.** Consider \( y \in A(t, x) \). If \( \tau = T, A(T, y(T)) = \{ y(T) \} \) and

\[ \inf_{\eta \in A(T, y(T))} [W(t, T, \eta) + J_T(\eta)] = (d_t(T) - 1)g(y(T)) + g(y(T)) + C = d_t(T)g(y(T)) + C \]  

(10)

and

\[ \int_t^T L_t(s, \dot{y}(s)) ds \oplus d_t(T)g(y(T)) + C \subseteq U(t, x). \]
Also for \( \tau = t \) the inclusion is trivially true. Let us now consider \( t < \tau < T \). For any \( \eta \in A(\tau, y(\tau)) \), we can consider the arc
\[
y_\eta(s) = \begin{cases} 
  y(s) & \text{if } t \leq s \leq \tau \\
  \eta(s) & \text{if } \tau < s \leq T
\end{cases}
\]
and \( y_\eta \in A(t, x) \). It is possible to write \( J_t(y_\eta) \) as
\[
J_t(y_\eta) = \int_t^\tau L_t(s, \dot{y}_\eta(s))ds + d_t(T)g(y_\eta(T)) \\
= \int_t^\tau L_t(s, \dot{y}(s))ds + \int_\tau^T L_t(s, \dot{\eta}(s))ds + d_t(T)g(\eta(T)) \\
= \int_t^\tau L_t(s, \dot{y}(s))ds + [W(t, \tau, \eta) + J_\tau(\eta)] .
\]
(11)

Since
\[
U(t, x) \supseteq J_t(y_\eta)
\]
for every \( \eta \), then, using (11), one gets
\[
U(t, x) \supseteq \inf_{\eta,A(\tau,y(\tau))} \left( \int_t^\tau L_t(s, \dot{y}(s))ds \oplus [W(t, \tau, \eta) + J_\tau(\eta)] \right) \\
= \int_t^\tau L_t(s, \dot{y}(s))ds + \inf_{\eta,A(\tau,y(\tau))} [W(t, \tau, \eta) + J_\tau(\eta)]
\]
and (8) is proved.

If for all \( \tau \in [t, T] \) (9) holds, in particular for \( \tau = T \), using (10),
\[
U(t, x) = \inf_{\eta \in M} \left( \int_t^T L_t(s, \dot{y}(s))ds \oplus d_t(T)g(y(T)) + C \right) = \inf_{\eta \in M} J_t(y)
\]
and \( M \) is an infimizer.

Finally, we want to prove that, if \( M \) is an infimizer, then for any \( \tau \in [t, T] \)
\[
U(t, x) \subseteq \inf_{\eta \in M} \left[ \int_t^\tau L_t(s, \dot{y}(s))ds \oplus \inf_{\eta \in A(\tau,y(\tau))} [W(t, \tau, \eta) + J_\tau(\eta)] \right] .
\]
It is sufficient to prove that for any \( y \in M \)
\[
J_t(y) \subseteq \inf_{\eta \in M} \left[ \int_t^\tau L_t(s, \dot{y}(s))ds \oplus \inf_{\eta \in A(\tau,y(\tau))} [W(t, \tau, \eta) + J_\tau(\eta)] \right] .
\]
We can write \( J_t(y) \) in a similar way to (11), using \( \eta(s) = y(s) \), and this concludes the proof. 
\[ \square \]
5 Hopf-Lax formula

Before stating the Hopf-Lax formula, we need the following lemmas. The first one is a coercivity result.

**Lemma 5.1.** Given \( \zeta \in C^+ \setminus \{0\} \) and \((t, x) \in [0, T] \times \mathbb{R}^n\), the following limits hold:

(i) \( \lim_{|w|_n \to +\infty} |\nabla L_\zeta (w)|_n = +\infty \),

(ii) \( \lim_{|p|_n \to +\infty} |(\nabla L_\zeta)^{-1}(p)|_n = +\infty \),

(iii) for any \( s \in [t, T] \), \( \lim_{|p|_n \to +\infty} |Y_{t,x,p,\zeta}(s)|_n = +\infty \),

(iv) \( \lim_{|p|_n \to +\infty} I_{t,\zeta}(Y_{t,x,p,\zeta}) = +\infty \).

**Proof.** We want to prove that the functions \( |\nabla L_\zeta (w)|_n \) and \( |(\nabla L_\zeta)^{-1}(p)|_n \) are coercive. Since \( L_\zeta \) is convex, there holds

\[
L_\zeta(0) - L_\zeta(w) \geq \nabla L_\zeta(w) \cdot (-w)
\]

for any \( w \in \mathbb{R}^n \) and this gives that

\[
|\nabla L_\zeta(w)|_n |w|_n \geq \nabla L_\zeta(w) \cdot w \geq L_\zeta(w) - L_\zeta(0).
\]

(Hypothesis (h1)) implies that for \( w \in \mathbb{R}^n \), with \( |w|_n \) sufficiently big, there exist \( K_\zeta, \epsilon_\zeta > 0 \) such that

\[
L_\zeta(w) \geq K_\zeta |w|_n^{1+\epsilon_\zeta}.
\]

From (12) and (13), dividing by \( |w|_n \), the following inequality is obtained

\[
|\nabla L_\zeta(w)|_n \geq K_\zeta |w|_n^{\epsilon_\zeta} - \frac{L_\zeta(0)}{|w|_n}
\]

and, taking the limit, one obtains (i).

For the inverse function let us assume that there exists a sequence \( \{p_m\}_{m \in \mathbb{N}} \subseteq \mathbb{R}^n \) such that

\[
\lim_{m \to +\infty} |p_m|_n = +\infty
\]

and

\[
\lim_{m \to +\infty} |(\nabla L_\zeta)^{-1}(p_m)|_n = \alpha.
\]

Then the sequence \( \{w_m\}_{m \in \mathbb{N}} \subseteq \mathbb{R}^n \), defined by

\[
w_m = (\nabla L_\zeta)^{-1}(p_m)
\]

is bounded and there exists a converging subsequence (that we still denote \( \{w_m\}_{m \in \mathbb{N}} \)) \( w_m \to \hat{w} \). Then we obtain that

\[
\lim_{m \to +\infty} |p_m|_n = \lim_{m \to +\infty} |\nabla L_\zeta(w_m)|_n = |\nabla L_\zeta(\hat{w})|_n.
\]
This contradicts (14) and so (ii) holds. This implies in particular (iii).

By hypothesis (h2) there exists a Lipschitz constant $G_\zeta > 0$ for the function $g_\zeta$:

$$g_\zeta(0) - g_\zeta(Y_{t,x,p,\zeta}(T)) \leq |g_\zeta(Y_{t,x,p,\zeta}(T)) - g_\zeta(0)| \leq G_\zeta|Y_{t,x,p,\zeta}(T)|.$$ 

In order to find the coercivity with respect to $p$ of the following function, we calculate for $|p|_n$ sufficiently big

$$I_{t,\zeta}(Y_{t,x,p,\zeta}) = \int_t^T d_t(s) L_\zeta \left( \dot{Y}_{t,x,p,\zeta}(s) \right) ds + d_t(T)g_\zeta(Y_{t,x,p,\zeta}(T))$$

$$\geq \int_t^T d_t(s) K_\zeta \left| \dot{Y}_{t,x,p,\zeta}(s) \right|^{1+\epsilon_\zeta}_n ds + d_t(T)g_\zeta(0)$$

$$\geq \int_t^T d_t(s) K_\zeta \left| (\nabla L_\zeta)^{-1} \left( \frac{p}{d_t(s)} \right) \right|^{1+\epsilon_\zeta}_n ds + d_t(T)g_\zeta(0)$$

$$- d_t(T)G_\zeta \left( |x| + \sqrt{n} \int_t^T \left| (\nabla L_\zeta)^{-1} \left( \frac{p}{d_t(s)} \right) \right| ds \right).$$

Since, if $F \in L^1([a,b]; \mathbb{R}^n)$, then

$$\left| \int_a^b F(s) ds \right|_n \leq \sqrt{n} \int_a^b |F(s)|_n ds,$$

we have that

$$I_{t,\zeta}(Y_{t,x,p,\zeta}) \geq \int_t^T d_t(s) K_\zeta \left| (\nabla L_\zeta)^{-1} \left( \frac{p}{d_t(s)} \right) \right|^{1+\epsilon_\zeta}_n ds + d_t(T)g_\zeta(0)$$

$$- d_t(T)G_\zeta \left( |x| + \sqrt{n} \int_t^T \left| (\nabla L_\zeta)^{-1} \left( \frac{p}{d_t(s)} \right) \right| ds \right)$$

$$= \int_t^T \left( d_t(s) K_\zeta \left| (\nabla L_\zeta)^{-1} \left( \frac{p}{d_t(s)} \right) \right|^{\epsilon_\zeta}_n - d_t(T)\sqrt{nG_\zeta} \right) \left| (\nabla L_\zeta)^{-1} \left( \frac{p}{d_t(s)} \right) \right|_n ds$$

$$+ d_t(T)g_\zeta(0) - d_t(T)G_\zeta |x|_n$$

and this proves (iv). \qed

In the following lemma, an arc is given, that minimizes the functional with respect to every direction of the dual cone.

**Lemma 5.2.** Given $\zeta \in C^+ \setminus \{0\}$ and $(t, x) \in [0, T] \times \mathbb{R}^n$, there exists $p(t, x, \zeta) \in \mathbb{R}^n$ such that

$$\inf_{p \in \mathbb{R}^n} I_{t,\zeta}(Y_{t,x,p,\zeta}) = I_{t,\zeta}(Y_{t,x,p(t,x,\zeta),\zeta}),$$

$$\inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) = J_t(Y_{t,x,p(t,x,\zeta),\zeta}) + H^+(\zeta). \quad (15)$$

We observe that

$$\inf_{p \in \mathbb{R}^n} [J_t(Y_{t,x,p,\zeta})] + H^+(\zeta) = \inf_{p \in \mathbb{R}^n} [J_t(Y_{t,x,p,\zeta}) + H^+(\zeta)].$$
Proof. By (iv) of Lemma 5.1 the function of $p$ $I_{t,\zeta}(Y_{t,x,p,\zeta})$ attains its minimum at $p(t,x,\zeta) = p_0 \in \mathbb{R}^n$.

It is obvious that

$$\inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) \supseteq J_t(Y_{t,x,p_0,\zeta}) + H^+(\zeta).$$

In order to prove that

$$\inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) \subseteq J_t(Y_{t,x,p_0,\zeta}) + H^+(\zeta),$$

we consider $z \in J_t(Y_{t,x,p,\zeta}) + H^+(\zeta)$ for some $p \in \mathbb{R}^n$, then we can write

$$z = I_{t,\zeta}(Y_{t,x,p,\zeta}) \frac{\zeta}{|\zeta|^2} + h$$

with $h \in H^+(\zeta)$. Since we have

$$z = I_{t,\zeta}(Y_{t,x,p_0,\zeta}) \frac{\zeta}{|\zeta|^2} + (I_{t,\zeta}(Y_{t,x,p_0,\zeta}) - I_{t,\zeta}(Y_{t,x,p_0,\zeta})) \frac{\zeta}{|\zeta|^2} + h'$$

with $h', h'' \in H^+(\zeta)$, so $z \in J_t(Y_{t,x,p_0,\zeta}) + H^+(\zeta)$ and this completes the proof. \hfill \Box

To simplify the notation we define now

$$Y_{t,x,\zeta}(s) = Y_{t,x,p(t,x,\zeta),\zeta}(s), \quad (16)$$

where $p(t,x,\zeta)$ is defined in the previous lemma.

Now the Hopf-Lax formula can be stated.

**Theorem 5.3.** Let $g$ have convex components. If $x \in \mathbb{R}^n$ and $0 \leq t < T$, the value function (5) of problem (4) can be written as

$$U(t,x) = \sup_{\zeta \in C^+} \left( \inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) \right). \quad (17)$$

**Proof.** Let us define

$$V(t,x) = \sup_{\zeta \in C^+} \left( \inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) \right) = \sup_{\zeta \in C^+} \left( J_t(Y_{t,x,\zeta}) + H^+(\zeta) \right), \quad (18)$$

where Lemma 5.2 has been used. We consider an arc $y \in A(t,x)$. By Remark 3.1 (ii), for any $\zeta \in C^+ \setminus \{0\}$, there exists $\overline{p}$ such that $Y_{t,x,\overline{p},\zeta}(T) = y(T)$. By Lemma 3.3 there holds

$$\overline{p} \cdot \left( \dot{Y}_{t,x,\overline{p},\zeta}(s) - \dot{y}(s) \right) \frac{\zeta}{|\zeta|^2} - d_t(s) \left( L \left( \dot{Y}_{t,x,\overline{p},\zeta}(s) \right) - L(\dot{y}(s)) \right) \in H^+(\zeta).$$
This implies that
\[
\mathcal{L}_t(s, \dot{y}(s)) \subseteq \mathcal{L}_t \left( s, \dot{Y}_{t,x,p,z}(s) \right) - \overline{p} \cdot \left( \dot{Y}_{t,x,p,z}(s) - \dot{y}(s) \right) \frac{\zeta}{|\zeta|^2} + H^+(\zeta).
\]
Integrating the previous inclusion from \( t \) to \( T \), one obtains
\[
\int_t^T \mathcal{L}_t(s, \dot{y}(s)) \, ds \subseteq \int_t^T \mathcal{L}_t \left( s, \dot{Y}_{t,x,p,z}(s) \right) \, ds - \left( \int_t^T \overline{p} \cdot \left( \dot{Y}_{t,x,p,z}(s) - \dot{y}(s) \right) \, ds \right) \frac{\zeta}{|\zeta|^2} + H^+(\zeta)
\]
\[
= \int_t^T \mathcal{L}_t \left( s, \dot{Y}_{t,x,p,z}(s) \right) \, ds + H^+(\zeta).
\]
Adding \( d_t(T)g(y(T)) = d_t(T)g(Y_{t,x,p,z}(T)) \) to both sets, we obtain that for any arc \( y \) there exists \( \overline{p} \) such that
\[
\int_t^T \mathcal{L}_t(s, \dot{y}(s)) \, ds + d_t(T)g(y(T)) \subseteq \int_t^T \mathcal{L}_t \left( s, \dot{Y}_{t,x,p,z}(s) \right) \, ds + d_t(T)g(Y_{t,x,p,z}(T)) + H^+(\zeta)
\]
and this proves that \( U(t,x) \subseteq \inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,z}) + H^+(\zeta) \) for every \( \zeta \in C^+ \setminus \{0\} \) and consequently
\[
U(t,x) \subseteq \sup_{\zeta \in C^+} \left( \inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,z}) + H^+(\zeta) \right) = V(t,x).
\]

Let us suppose that there exists \( z_0 \in V(t,x) \setminus U(t,x) \). For every \( \zeta \in C^+ \setminus \{0\} \),
\[
z_0 \in J_t(Y_{t,x,\zeta}) + H^+(\zeta) \tag{19}
\]
and \( z_0 = I_t(Y_{t,x,\zeta}) + h_\zeta \). If we fix \( c_0 \in \text{int} \, C \) and consider the half-line \( z_0 + r c_0 \) with \( r > 0 \), for \( r \) sufficiently large \( h_\zeta + r c_0 \in C \) and consequently \( z_0 + r c_0 \in U(t,x) \). Then there exists \( r_0 > 0 \) such that \( z_1 = z_0 + r_0 c_0 \in U(t,x) \) and it is on the boundary of \( U(t,x) \). Since \( U(t,x) \) is convex, by the supporting hyperplane theorem (see for example [7]) there exists \( \xi \in \mathbb{R}^d \setminus \{0\} \) such that for any \( z \in U(t,x) \)
\[
\xi \cdot z \geq \xi \cdot z_1.
\]
From the fact that \( z_1 + c \in U(t,x) \) for any \( c \in C \), we have that \( \xi \cdot (z_1 + c) \geq \xi \cdot z_1 \) and \( \xi \in C^+ \setminus \{0\} \). From (19) with \( \zeta = \xi \), we obtain that
\[
\xi \cdot z_0 \geq I_t,\xi(Y_{t,x,\xi}) \tag{20}
\]
Since \( I_t(Y_{t,x,\xi}) \subseteq U(t,x) \), we have
\[
I_t,\xi(Y_{t,x,\xi}) \geq \xi \cdot z_1 = \xi \cdot (z_0 + r_0 c_0).
\]
This inequality and inequality (20) give
\[
r_0 \xi \cdot c_0 \leq 0,
\]
but this implies that \( \xi \cdot c_0 = 0 \) and this is not possible because \( c_0 \in \text{int} \, C \).
6 The Hamilton-Jacobi-Bellman equation

In this section we assume that:

(h4) the discount factor is of class $C^1$ in $t$

(h5) all the scalarizations $g_\zeta$, with $\zeta$ in a base $B^+$ of $C^+$ are $C^2$ and convex.

In the following lemma the differentiability of the arcs $Y_{t,x,p,\zeta}(s)$ and of their derivatives $\dot{Y}_{t,x,p,\zeta}(s)$ with respect to the parameters is studied.

**Lemma 6.1.** Given $\zeta \in C^+ \setminus \{0\}$, $(t, x) \in [0, T] \times \mathbb{R}^n$ and $p \in \mathbb{R}^n$ the arcs $Y_{t,x,p,\zeta}(s)$ and the derivatives $\dot{Y}_{t,x,p,\zeta}(s)$ admit the partial derivatives with respect to $t$, $x$ and $p$:

$$\frac{\partial Y_{t,x,p,\zeta}(s)}{\partial t} = -\nabla L_\zeta^{-1}(p) - \int_t^s \frac{\partial d_t(r)}{\partial t} \nabla(\nabla L_\zeta)^{-1}\left(\frac{p}{d_t(r)}\right)\left(\frac{p}{(d_t(r))^2}\right) dr,$$

$$\frac{\partial \dot{Y}_{t,x,p,\zeta}(s)}{\partial t} = -\frac{\partial d_t(s)}{\partial t} \nabla(\nabla L_\zeta)^{-1}\left(\frac{p}{d_t(s)}\right)\left(\frac{p}{(d_t(s))^2}\right),$$

$$\nabla_x Y_{t,x,p,\zeta}(s) = I,$$

$$\nabla_x \dot{Y}_{t,x,p,\zeta}(s) = 0,$$

$$\nabla_p Y_{t,x,p,\zeta}(s) = \int_t^s \frac{1}{d_t(r)} \nabla(\nabla L_\zeta)^{-1}\left(\frac{p}{d_t(r)}\right) dr,$$

$$\nabla_p \dot{Y}_{t,x,p,\zeta}(s) = \frac{1}{d_t(s)} \nabla(\nabla L_\zeta)^{-1}\left(\frac{p}{d_t(s)}\right),$$

where $I$ is the identity in $\mathbb{R}^n$ and $0$ is the null matrix.

**Proof.** By Remark 3.1 the arcs $Y_{t,x,p,\zeta}(s)$ are of class $C^1$ in $p$. We recall that $L_\zeta$ is of class $C^2$ and it is easy to see that

$$\nabla(\nabla L_\zeta)^{-1}(x) = \left[\nabla^2 L_\zeta((\nabla L_\zeta)^{-1}(x))\right]^{-1},$$

so it exists and it is continuous. \qed

In the following proposition the function $p(t, x, \zeta)$ introduced in Lemma 5.2 is studied.

**Proposition 6.2.** Given $\zeta \in C^+ \setminus \{0\}$ and $(t, x) \in [0, T] \times \mathbb{R}^n$

(i) $p(t, x, \zeta)$ in Lemma 5.2 is a solution of

$$F(t, x, p, \zeta) = 0 \quad \text{(21)}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times C^+ \setminus \{0\} \rightarrow \mathbb{R}^n$ is defined by

$$F(t, x, p, \zeta) = p + d_t(T)\nabla g_\zeta(Y_{t,x,p,\zeta}(T))$$

13
(ii) The Jacobian matrix
\[ \nabla_p F(t, x, p, \zeta) = I + A(t, x, p, \zeta), \]
where \( I \) is the identity in \( \mathbb{R}^n \) and
\[ A(t, x, p, \zeta) = \frac{d_t(T) \nabla^2 g_\zeta(Y_{t,x,p,\zeta}(T)) \nabla_p Y_{t,x,p,\zeta}(T)}{\partial t}, \]
is nonsingular.

(iii) The function \( p(t, x, \zeta) \) is well defined in a neighborhood of \( (t, x) \) and admits the partial derivatives
\[ \frac{\partial p}{\partial t}(t, x, \zeta) = -(I + A(t, x, p, \zeta))^{-1} \left[ \frac{\partial dt(T)}{\partial t} \nabla g_\zeta(Y_{t,x,p,\zeta}(T)) + d_t(T) \nabla^2 g_\zeta(Y_{t,x,p,\zeta}(T)) \frac{\partial Y_{t,x,p,\zeta}(T)}{\partial t} \right], \]
\[ \frac{\partial p}{\partial x}(t, x, \zeta) = -(I + A(t, x, p, \zeta))^{-1} \left[ d_t(T) \nabla^2 g_\zeta(Y_{t,x,p,\zeta}(T)) \nabla_x Y_{t,x,p,\zeta}(T) \right]. \]

Proof. From the definition in the first equation of (15), \( p(t, x, \zeta) \) must solve the equation
\[ \nabla_p I_{t,\zeta}(Y_{t,x,p,\zeta}) = 0. \]
Calculating the previous derivative, one obtains
\[ [p + d_t(T) \nabla g_\zeta(Y_{t,x,p,\zeta}(T))] \cdot \int_t^T \frac{1}{d_t(s)} \nabla(L_{\zeta})^{-1} \left( \frac{p}{d_t(s)} \right) ds = 0. \]
The solutions of equation (21) are obviously also solutions of the previous equation.

The matrix \( A(t, x, p, \zeta) \) is the product of two matrices. The first one is the Hessian matrix of a \( C^2 \) function, so it is symmetric. The second one is the integral of the matrix
\[ \frac{1}{d_t(r)} \nabla(L_{\zeta})^{-1} \left( \frac{p}{d_t(r)} \right) = \frac{1}{d_t(r)} \left[ \nabla^2 L_{\zeta} \left( \nabla(L_{\zeta})^{-1} \left( \frac{p}{d_t(r)} \right) \right) \right]^{-1}. \]
Since the Hessian matrix of \( L_{\zeta} \) is symmetric, so it is its inverse and its integral. Moreover, the first matrix \( \nabla^2 g_\zeta \) is positively semidefinite, while the second one \( \nabla_p Y_{t,x,p,\zeta}(T) \) is positively definite. Then their product \( A(t, x, p, \zeta) \) is also positively semidefinite (because they can be simultaneously diagonalized). If \( \nabla_p F(t, x, p, \zeta) \) were singular, \( A(t, x, p, \zeta) \) should have an eigenvector of \(-1\) and this is in contradiction with the fact that it is positively semidefinite.

For \( (t, x) \in [0, T] \times \mathbb{R}^n, q \in \mathbb{R}^n \) and \( \zeta \in C^+ \setminus \{0\} \), we consider the partial derivatives:
\[ U_{t,\zeta}(t, x) = \lim_{s \to 0^+} \frac{1}{s} [U(t + s, x) - \zeta U(t, x)], \]
\[ U_{q,\zeta}(t, x) = \lim_{s \to 0^+} \frac{1}{s} [U(t, x + sq) - \zeta U(t, x)]. \]
Similar definitions are used in [11] and [21]. These derivatives, if they exist, are closed half-spaces with normal \( \zeta \) or in the extreme cases they are the empty set or \( \mathbb{R}^d \).
Proposition 6.3. Given $\zeta \in C^+ \setminus \{0\}$ and $(t, x) \in [0, T] \times \mathbb{R}^n$, the partial derivatives with respect to $t$ and with respect to $x$ in the direction $q$ exist and are the following ones:

$$U_{t,\zeta}(t, x) = S(u_{t,\zeta}(t, x), \zeta)$$  \hspace{1cm} (23)

$$U_{q,\zeta}(t, x) = S(\nabla u_{\zeta}(t, x), \zeta)(q)$$  \hspace{1cm} (24)

where

$$u_{t,\zeta}(t, x) = -L_{\zeta}(Y_{t,x,\zeta}(t)) + \int_t^T \frac{\partial d_t(s)}{\partial t} L_{\zeta}(Y_{t,x,\zeta}(s)) ds + \frac{\partial d_t(T)}{\partial t} g_{\zeta}(Y_{t,x,\zeta}(T))$$

$$-d_t(T) \nabla g_{\zeta}(Y_{t,x,\zeta}(T)) \cdot (\nabla L_{\zeta})^{-1}(p(t, x, \zeta)),$$

$$\nabla u_{\zeta}(t, x) = d_t(T) \nabla g_{\zeta}(Y_{t,x,\zeta}(T)).$$

Proof. First of all, we calculate, using Hopf-Lax formula and (15),

$$\inf \{ \zeta \cdot z \mid z \in U(t, x) \} = \inf \left\{ \zeta \cdot z \mid z \in \sup_{\zeta \in C^+} \left( \inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) \right) \right\}$$

$$\geq \inf \left\{ \zeta \cdot z \mid z \in \inf_{p \in \mathbb{R}^n} J_t(Y_{t,x,p,\zeta}) + H^+(\zeta) \right\}$$

$$= \inf \{ I_{t,\zeta}(Y_{t,x,p,\zeta}) \mid p \in \mathbb{R}^n \} = I_{t,\zeta}(Y_{t,x,\zeta}).$$

Since $I_t(Y_{t,x,\zeta}) \in U(t, x)$, we can conclude that

$$\inf \{ \zeta \cdot z \mid z \in U(t, x) \} = I_{t,\zeta}(Y_{t,x,\zeta}).$$

Now it is possible to write for $h > 0$:

$$\frac{1}{h}[U(t+h,x)-\zeta U(t, x)] = \left\{ z \in \mathbb{R}^d \mid \zeta \cdot z \geq \frac{1}{h}[I_{t+h,\zeta}(Y_{t+h,x,\zeta}) - I_{t,\zeta}(Y_{t,x,\zeta})] \right\}.$$
and this concludes the proof.

Remark 6.4. It is not difficult to see that

\[ U(t+h, x+hq) - \zeta U(t, x) = (U(t+h, x+hq) - \zeta U(t+h, x)) + (U(t+h, x) - \zeta U(t, x)) \]

and that

\[ \lim_{h \to 0} \frac{1}{h} (U(t+h, x+hq) - \zeta U(t+h, x)) = U_{q, \zeta}(t, x). \]
This means that

$$\lim_{h \to 0} \frac{1}{h} (U(t + h, x + hq) - \zeta U(t, x)) = U_{t,\zeta}(t, x) + U_{q,\zeta}(t, x).$$

Recalling the Fenchel conjugate (3), the following theorem can be stated.

**Theorem 6.5.** Given $\zeta \in C^+ \setminus \{0\}$ and $(t, x) \in [0, T] \times \mathbb{R}^n$, considering $\mathcal{L}_t(t, \cdot) : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^d, C)$, the value function $U(t, x)$ is a solution of the equation

$$U_{t,\zeta}(t, x) = \mathcal{L}^*_t(t, -\nabla u_\zeta(t, x), \zeta) + w(t, x, \zeta),$$

where

$$w(t, x, \zeta) = \int_t^T \frac{\partial d_t(s)}{\partial t} L_t(\hat{Y}_{t, x, \zeta}(s)) \, ds + \frac{\partial d_t(T)}{\partial t} g(\hat{Y}_{t, x, \zeta}(T)).$$

**Proof.** Given $q \in \mathbb{R}^n$, we consider $y(s) = x + (s - t)q$, $y \in A(t, x)$. Using Bellman’s inequality (8), we have for $h > 0$ sufficiently small

$$U(t, x) \supseteq \int_t^{t+h} \mathcal{L}_t(s, q) ds \oplus \inf_{\eta \in A(t+h, x+hq)} [W(t, t+h, \eta) + J_{t+h}(\eta)]$$

$$\supseteq \int_t^{t+h} \mathcal{L}_t(s, q) ds \oplus W(t, t+h, Y_{t+h, x+hq, \zeta}) + J_{t+h}(Y_{t+h, x+hq, \zeta}).$$

Using this inclusion of sets, we obtain

$$U(t + h, x + hq) - \zeta U(t, x) \subseteq$$

$$U(t + h, x + hq) - \zeta \left[ \int_t^{t+h} \mathcal{L}_t(s, q) ds \oplus W(t, t+h, Y_{t+h, x+hq, \zeta}) + J_{t+h}(Y_{t+h, x+hq, \zeta}) \right].$$

We can calculate that

$$\inf_{z \in U(t+h, x+hq)} \zeta \cdot z = I_{t+h,\zeta}(Y_{t+h, x+hq, \zeta})$$

and

$$\inf_{z \in \int_t^{t+h} \mathcal{L}_t(s, q) ds} \int_t^{t+h} d_t(s) L_q(s) ds$$

$$\inf_{z \in W(t+h, Y_{t+h, x+hq, \zeta}) + J_{t+h}(Y_{t+h, x+hq, \zeta})} \zeta \cdot z = \zeta \cdot W(t, t+h, Y_{t+h, x+hq, \zeta}) + I_{t+h,\zeta}(Y_{t+h, x+hq, \zeta}).$$

Then we have the following inclusion:

$$U(t + h, x + hq) - \zeta U(t, x) \subseteq - \int_t^{t+h} d_t(s) L_q(s) ds - W(t, t+h, Y_{t+h, x+hq, \zeta}) + H^+(\zeta)$$

and, taking the limit, the partial derivatives fulfill the inclusion

$$U_{t,\zeta}(t, x) + U_{q,\zeta}(t, x) \subseteq - L(q) + w(t, x, \zeta) + H^+(\zeta).$$
As a consequence, we can write that

$$U_{t,\zeta}(t, x, \zeta) \subseteq (S_{(\nabla u_{\zeta}(t, x), \zeta)} (q) - \zeta L(q)) + w(t, x, \zeta)$$

for every $q \in \mathbb{R}^n$. It is then possible to take the supremum with respect to $q$

$$U_{t,\zeta}(t, x, \zeta) \subseteq \sup_{q \in \mathbb{R}^n} (S_{(\nabla u_{\zeta}(t, x), \zeta)} (q) - \zeta L(q)) + w(t, x, \zeta)$$

$$= L^*_t(t, -\nabla u_{\zeta}(t, x)) + w(t, x, \zeta).$$

(26)

It is easy to check that for $q = \nabla L^{-1}(p(t, x, \zeta))$

$$U_{t,\zeta}(t, x, \zeta) + U_{\nabla L^{-1}(p), \zeta}(t, x) = -L \left( \nabla L^{-1}(p(t, x, \zeta)) \right) + w(t, x, \zeta) + H^+(\zeta)$$

and so

$$U_{t,\zeta}(t, x) = \left[ S_{(-\nabla u_{\zeta}(t, x), \zeta)} \left( \nabla L^{-1}(p(t, x, \zeta)) \right) - \zeta L \left( \nabla L^{-1}(p(t, x, \zeta)) \right) \right] + w(t, x, \zeta).$$

From (26) and the previous equation, one concludes that

$$U_{t,\zeta}(t, x) \subseteq L^*_t(t, -\nabla u_{\zeta}(t, x))$$

$$\subseteq \left[ S_{(-\nabla u_{\zeta}(t, x), \zeta)} \left( \nabla L^{-1}(p(t, x, \zeta)) \right) - \zeta L \left( \nabla L^{-1}(p(t, x, \zeta)) \right) \right] + w(t, x, \zeta) = U_{t,\zeta}(t, x).$$

This proves (25). $\square$

In the following corollary, the Hamilton-Jacobi-Bellman equation is written independently of the directions in the dual cone. It is easy to see that equation (25) can also be written

$$U_{t,\zeta}(t, x) - \zeta L^*_t(t, -\nabla u_{\zeta}(t, x), \zeta) - w(t, x, \zeta) = H^+(\zeta).$$

However, the neutral element with respect to $\oplus$ in $\mathcal{F}(\mathbb{R}^d, C)$ is $C$ and not $H^+(\zeta)$. So, in order to have a “complete” equation and not one that describes only one direction, an intersection of the corresponding sets must be taken.

**Corollary 6.6.** Given $(t, x) \in [0, T] \times \mathbb{R}^n$, the value function $U(t, x)$ is a solution of the equation

$$\sup_{\zeta \in C^+ \setminus \{0\}} \left[ U_{t,\zeta}(t, x) - \zeta L^*_t(t, -\nabla u_{\zeta}(t, x), \zeta) - w(t, x, \zeta) \right] = C.$$

7 Conclusions

Two features are often present in economic problems. One is that the optimization required involves more than one function, which can be in conflict. It is obvious that a linearization is a very strong simplification of the problem. A cone can help to model the preferences of an agent (or the probabilities of using one optimization function or the other one). An approach to the multicriteria case was proposed in [12] in the classical framework. The second feature
is a discount factor that permits to determine the current value of the Lagrangian. This was done in [22] for the real-valued case.

This paper aims at both targets: it gives a mathematical model that includes the discount factor and the multiobjective nature of the problem.

There are still many open problems. One of them is the question how to use these techniques to handle control problems. Another line of research could be to generalize results like [6] to the multicriteria case.

Acknowledgement

The author is indebted to Andreas Hamel for fruitful suggestions and discussions.

This work was supported within the project Optimal Control Problems with Set-valued Objective Function by Free University of Bozen-Bolzano (Grant OptiConSOF).

References

[1] J.-P. Aubin, Lax-Hopf formula and Max-Plus properties of solutions to Hamilton-Jacobi equations, Nonlinear Differential Equations and Applications NoDEA 20 (2013), 187–211.

[2] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston-Basel-Berlin 1990.

[3] R.J. Aumann, Integrals of set-valued functions, J. Math. Anal. Appl. 12 (1965), 1–12.

[4] A. Avantaggiati, P. Loreti, Lax type formulas with lower semicontinuous initial data and hypercontractivity results, Nonlinear Differential Equations and Applications NoDEA 20 (2013), 385–411.

[5] R. Barro, Ramsey meets Laibson in the neoclassical growth model, The Quarterly Journal of Economics 114 (1999), 1125–1152.

[6] E.N. Barron, R. Jensen, W. Liu, Hopf–Lax-Type Formula for $u_t + H(u, Du) = 0$, Journal of Differential Equations 126 (1996), 48–61.

[7] S.P. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press (2004), 50–51.

[8] M. Caputo, Foundations of Dynamic Economic Analysis: Optimal Control Theory and Applications, Cambridge University Press (2005).

[9] G. De Marcio, G. Gorni, G. Zampieri, Global inversion of functions: an introduction, Nonlinear Differential Equations and Applications 1 (1994), 229–248.

[10] A.H. Hamel, F. Heyde, A. Löhne, B. Rudloff, C. Schrage, Set optimization – A rather short introduction. In: A.H. Hamel, F. Heyde, A. Löhne, B. Rudloff, C. Schrage (Eds.), Set Optimization and Applications – The State of the Art, Springer 2015, 65–141.
[11] A.H. Hamel, C. Schrage, Directional derivatives, subdifferentials and optimality conditions for set-valued convex functions, Pacific Journal of Optimization 10 4 (2014), 667–689.

[12] A.H. Hamel, D. Visetti, The value functions approach and Hopf-Lax formula for multiobjective costs via set optimization, Journal of Mathematical Analysis and Applications 483 1 (2020), 123605.

[13] F. Heyde, A. Löhne, Solution concepts in vector optimization: a fresh look at an old story, Optimization 60 12 (2011), 1421–1440.

[14] N. Hoang, Hopf-Lax formula and generalized characteristics, Applicable Analysis 96 2 (2013).

[15] E. Hopf, Generalized solutions of non-linear equations of first order, Journal of Mathematics and Mechanics 14 (1965), 951–973.

[16] L. Karp, Non-constant discounting in continuous time, Journal of Economic Theory 132 (2007), 557–568.

[17] P.D. Lax, Hyperbolic systems of conservation laws II, Communications on Pure and Applied Mathematics 10 (1957), 537–566.

[18] J. Marín-Solano, J. Navas, Non-constant discounting in finite horizon: The free terminal time case, Journal of Economic Dynamics and Control 33 (2009), 666–675.

[19] J. Marín-Solano, C. Patxot, Heterogeneous discounting in economic problems, Optimal Control Applications and Methods 33 1 (2012), 32–50.

[20] J. Marín-Solano, E.V. Shevkoplyas, Non-constant discounting and differential games with random time horizon, Automatica 47 (2011), 2626–2638.

[21] M. Pilecka, Set-valued optimization and its application to bilevel optimization, PhD-thesis 2016, Technische Universität Bergakademie Freiberg.

[22] J.P. Rincón-Zapatero, Hopf-Lax formula for variational problems with non-constant discount, Journal of Geometric Mechanics 1 3 (2009), 357–367.

[23] T. Strömberg, The Hopf-Lax formula gives the unique viscosity solution, Differential Integral Equations 15 1 (2002), 47–52.