Stochastic Finite State Control of POMDPs with LTL Specifications

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Abstract—Partially observable Markov decision processes (POMDPs) provide a modeling framework for autonomous decision making under uncertainty and imperfect sensing, e.g. robot manipulation and self-driving cars. However, optimal control of POMDPs is notoriously intractable. This paper considers the quantitative problem of synthesizing sub-optimal stochastic finite state controllers (sFSCs) for POMDPs such that the probability of satisfying a set of high-level specifications in terms of linear temporal logic (LTL) formulae is maximized. We begin by casting the latter problem into an optimization and use relaxations based on the Poisson equation and McCormick envelopes. Then, we propose an stochastic bounded policy iteration algorithm, leading to a controlled growth in sFSC size and an any time algorithm, where the performance of the controller improves with successive iterations, but can be stopped by the user based on time or memory considerations. We illustrate the proposed method by a robot navigation case study.

I. INTRODUCTION

Robots and autonomous systems must interact with uncertain and dynamically changing environments under complex rules that specify desired system behavior. This inherent uncertainty presents the challenge of synthesizing and verifying the control and sensing algorithms against safety and abstract rules. For example, autonomous robot manipulation tasks are characterized by (i) imperfect actuation; (ii) the inability to accurately localize the robot, its end effector and the obstacles in the workspace; and (iii) noisy and error-prone sensing. In fact, imperfect observation makes the decoupling of planning and execution difficult, if not impossible.

Such operations can often be abstracted to a discrete system representation at a sensible level of abstraction, yielding a partially observable Markov decision processes (POMDP) [1]. This paper considers the task of designing finite state control systems for POMDPs with linear temporal logic (LTL) specifications. Since both sensing and actuation are imperfect and partially observable, it is only possible to probabilistically guarantee adherence to the given LTL specifications. The procedure developed in this paper maximizes the probability of satisfaction within a given class of stochastic finite state controllers (sFSCs).

A. Literature Review

During the past twenty years, formal methods have become increasing popular in robotics and controls [2]–[7], where simultaneous motion and task planning is a challenging problem.

LTL is a useful choice for robot goal and safety specification as it has an intuitive correlation to natural language [8]. Notably, LTL formulas can represent goals over infinite executions. This is useful for representing persistent surveillance and perpetually online applications. In order to capture environmental disturbances, it is often useful to model the dynamics in a probabilistic fashion. Markov decision processes [9] are a popular choice for the discrete abstraction of noisy systems. In the case of (fully observable) Markov decision process (MDP), synthesis of controllers with probabilistic satisfaction guarantees of LTL specification is well understood [10]. In fact, for fully observable MDPs under LTL specifications, robust [11] and receding horizon controllers [12] have been formulated.

For POMDPs, the design of optimal controllers or policies to meet LTL specifications is largely an open problem. In general such policies are stochastic (randomized) and require infinite memory. For unbounded memory strategies EXPTIME-completeness of a broad set of objectives (parity objectives) is proven in [13]. Also, in [14] the existence and construction of finite memory strategy for (strictly) positive probability of satisfaction is shown to be an EXPTIME-complete problem. To surmount this difficulty, many approximate, point-based, and Monte Carlo based methods have been proposed [15]. However, these techniques do not provide guarantees for LTL satisfaction. Approaches based on incremental satisfiability modulo theory solvers [16] and simulations over belief spaces [17] are also fettered by scalability issues. In [18], a recurrent neural network based method is proposed to synthesize stochastic but memoryless policies and [19] synthesizes sub-optimal FSCs for POMDPs using parametric synthesis for Markov chains and a convex-concave relaxation [20]. However, these cannot be used to handle LTL specifications and require assumptions on the structure of the FSC.

B. Contributions

We propose a methodology to design sFSCs for POMDPs with LTL specifications. Our method presents an any time algorithm, which can optimally add states to the finite state controller to improve the probability of satisfaction. The main contributions of this paper are as follows:

- We represent the LTL specifications as a deterministic Rabin automaton (DRA) and construct a product-POMDP.
- We then show that closing the loop with an sFSC leads to a Markov chain with a set of free parameters. Then we cast the problem of maximizing the probability of LTL specifications over the free parameters into an optimization problem;
- We use the Poisson equation and the Reformulation-Linearization technique to convexify a set of the con-
strains of this optimization problem;
- We propose a bounded policy iteration (BPI) method to design the sFSCs with efficient policy improvement steps;
- We mitigate the conservatism of the proposed methodology by formulating algorithms for finding initial feasible controllers and modifying the number of states in the sFSC to improve the probability of LTL satisfaction.

C. Outline

We briefly review some notions and results used throughout the paper in the next section. In Section III, we describe how POMDP traces produced by POMDP executions can be verified against an LTL formula. In Section IV we formulate an optimization problem to maximize the probability of LTL satisfaction. In Section VI we propose a method based on bounded policy iteration to design sFSCs. In Section VII we elucidate the proposed methodology with a robot navigation example. Finally, Section VIII concludes the paper.

II. PRELIMINARIES

A. Linear Temporal Logic

Temporal logic enables representation and reasoning about temporal aspects of a system [10], [21], [22]. It has been utilized to formally specify and verify behavior in many applications [23]. This paper considers the Linear Temporal Logic (LTL) subset of temporal logic. LTL is built upon a set of atomic propositions \( AP \), and is closed under the logic connectives, \( (\neg, \lor, \land, \rightarrow) \), and the temporal operators “next” \( (\rightarrow) \), “always” \( (\Box) \), “eventually” \( (\Diamond) \), and “until” \( (\mathcal{U}) \). An LTL formula can be constructed as \( \varphi = \text{true} \lor \text{false} \lor \varphi \land \varphi \lor \varphi \lor \varphi \rightarrow \varphi \rightarrow \Diamond \varphi \lor \Diamond \varphi \lor \varphi \rightarrow \varphi \mathcal{U} \varphi \).

LTL semantics are given by interpretations over infinite executions of a finite transition system with state space \( S \). For an infinite execution \( \sigma = s_0 s_1 \ldots s_i \in S \), LTL formula \( \varphi \) holds at position \( i \geq 0 \) of \( \sigma \), denoted \( s_i \models \varphi \), iff \( \varphi \) holds for the remainder of the execution \( \sigma \), starting at position \( i \).

For any LTL formula \( \varphi \) over atomic propositions, \( AP \), one can construct a Deterministic Rabin Automaton (DRA), with the input alphabet \( 2^{AP} \omega \), that accepts all and only those infinite words, \( \sigma \in (2^{AP})^\omega \), where \( A^\omega \) denotes infinite words composed of elements of \( A \), that satisfy \( \varphi \) [24], [25]. Algorithms for converting an LTL formula \( \varphi \) to an equivalent DRA can be found in [26] and a popular tool is described in [27]. While the worst case complexity of this conversion is doubly exponential, sufficiently expressive subsets of LTL can be translated to a DRA in polynomial time [28].

Definition 1 (DRA): A Deterministic Rabin Automaton (DRA) is a five-tuple \( \mathcal{D} \mathcal{R} \mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega) \), where
- \( Q \) is the set of states,
- \( \Sigma \) is the input alphabet. For our purposes, \( \Sigma = 2^{AP} \).
- \( \delta : Q \times \Sigma \to Q \) is the deterministic transition function,
- \( q_0 \in Q \) is the initial state,
- \( \Omega = \{(\text{Avoid}_r, \text{Repeat}_r)|r \in \{1, \ldots, N_\Omega\}, \text{Avoid}_r, \text{Repeat}_r \subseteq S\} \) is the Rabin acceptance condition.

Definition 2 (Rabin Acceptance): A run \( \pi = q_0 q_1 \ldots \) of a \( \mathcal{D} \mathcal{R} \mathcal{A} \) with acceptance condition \( \Omega = \{(\text{Avoid}_1, \text{Repeat}_1), \ldots, (\text{Avoid}_{N_\Omega}, \text{Repeat}_{N_\Omega})\} \) is accepting if there exists an \( r \in \{1, \ldots, N_\Omega\} \), such that \( \text{Inf}(\pi) \cap \text{Avoid}_r = \emptyset \) and \( \text{Inf}(\pi) \cap \text{Repeat}_r \neq \emptyset \), where \( \text{Inf}(\pi) \) is the set of states that occur infinitely often in \( \pi \).

The Rabin acceptance conditions implies that for some pair \( (\text{Avoid}_r, \text{Repeat}_r) \in \Omega \), no state in \( \text{Avoid}_r \) is visited infinitely often, while some state in \( \text{Repeat}_r \) is visited infinitely often.

To use a DRA to verify an LTL formula \( \varphi \), one assumes that a system’s interesting properties are given by a set of atomic propositions \( AP \) over system variables \( V \). An execution \( \sigma = v_0 v_1 \ldots \) of the system leads to a unique (infinite) trace over the truth evaluations of \( AP \), given by \( h(\sigma) = h(v_0)h(v_1) \ldots \). Here \( h(v_t) \in 2^{AP} \) denotes the truth value of all atomic propositions in \( AP \) at time step \( t \) using the state \( v_t \). At the start of the system’s execution, the DRA corresponding to \( \varphi \) is initialized to its initial state \( q_0 \). As the system execution progresses, the evaluations \( h(v_t) \) for \( t = 0, 1, \ldots \) dictate how the DRA evolves via the transition function \( \delta \). The execution \( \sigma \) satisfies \( \varphi \) iff the DRA accepts \( h(\sigma) \).

B. Markov Chains

A Markov chain \( \mathcal{M} \) with state space \( S \), transition probability defined as the conditional distribution \( T(.|s) : S \to [0,1] \) such that \( \sum_{s \in S} T(s'|s) = 1, \forall s \in S \), and the initial distribution \( \nu_{\text{init}} \) such that \( \sum_{s \in S} \nu_{\text{init}}(s) = 1 \). An infinite path, denoted by the superscript \( \omega \), of the Markov chain \( \mathcal{M} \) is a sequence of states \( \pi = s_0 s_1 s_2 \ldots \in S^\omega \) such that \( T(s_{t+1}|s_t) > 0 \) for all \( t \) and \( \nu_{\text{init}}(s_0) > 0 \). The probability space over such paths is the defined as follows. The sample space \( \Xi \) is the set of infinite paths with initial state \( s \in S \), i.e., \( \Xi = \text{Paths}(s) \). \( \sum_{s \in S} \text{Paths}(s) \) is the least \( \sigma \)-algebra on \( \text{Paths}(s) \) containing \( Cyl(\omega) \), where \( Cyl(\omega) = \{\omega' \in \text{Paths}(s)| \omega \text{ is a prefix of } \omega'\} \) is the cylinder set. To specify the probability measure over all sets of events in \( \Sigma_{\text{Paths}(s)} \), we provide the probability of each cylinder set as follows

\[
\Pr_{\mathcal{M}}[Cyl(s_0 \ldots s_n)] = \nu_{\text{init}}(s_0) \prod_{0 \leq t < n} T(s_{t+1}| s_t). \tag{1}
\]

Once the probability measure is defined over the cylinder sets, the expectation operator \( E_{\mathcal{M}} \) is also uniquely defined. In the sequel, we remove the subscript \( \mathcal{M} \) whenever the Markov chain is clear from the context.

The transition probabilities \( T \) form a linear operator which can be represented as a matrix, hereafter denoted by \( T \).

\[
T := \begin{bmatrix}
T_{11} & T_{12} & \ldots & T_{1|S|} \\
T_{21} & T_{22} & \ldots & T_{2|S|} \\
\vdots & \vdots & \ddots & \vdots \\
T_{|S|1} & T_{|S|2} & \ldots & T_{|S||S|}
\end{bmatrix} : M_S \to M_S,
\]

where \( T_{ij} = T(s_j|s_i) \). Let \( \tilde{b}_t \) denote a distribution, or belief, over states of the Markov chain at some time \( t \):

\[
\tilde{b}_t = \{b(s_1) b(s_2) \ldots b(s_{|S|})\}.
\]

The operator \( T \) maps a belief at time \( t \), \( b_t \), to a belief \( b_{t+1} \) at \( t+1 \):

\[
b_{t+1} = \tilde{b}_t T.
\]

Definition 3: Let \( \pi = s_0 s_1 \ldots \) be a path in the global Markov chain. The occupation time of set \( A \subseteq S \) is

\[
f_A := \sum_{t=1}^{\infty} 1(s_t \in A), \tag{2}
\]
where $\mathbb{I}(\phi) = \begin{cases} 1 & \text{the statement } \phi \text{ holds.} \\ 0 & \text{otherwise.} \end{cases}$ is the indicator function. Thus $f_A$ counts the number of times the set $A$ is visited after $t = 0$. The first return time, $\tau_A$, denotes the first time after $t = 0$ that set $A$ is visited $\tau_A := \min\{t \mid t \geq 1 | s_t \in A\}$.

The return probability describes the probability of set $A$ being visited in finite time when the start state is $s$, $L(s, A) := \Pr(\tau_A < \infty | s_0 = s)$.

If $A$ is a singleton set, i.e., $A = \{s'\}$ for some $s' \in S$, then $f_{s'}, \tau_{s'}$ and $L(s, s')$ will respectively denote the occupation time, first return time and return probability.

**Definition 4 (Communicating Classes):** The state $s \in S$ leads to state $s' \in S$, denoted $s \rightarrow s'$, if $L(s, s') > 0$. Distinct states $s, s'$ are said to communicate, denoted $s \leftrightarrow s'$ when $L(s, s') > 0$ and $L(s', s) > 0$. Moreover, the relation “$\leftrightarrow$” is an equivalence relation, and equivalence classes $C(s) = \{s' : s \leftrightarrow s'\}$ cover $S$, with $s \in C(s)$ [29].

**Definition 5 (Irreducibility and Absorbing Sets):** If $C(s) = S$ for some $s \in S$, the Markov chain, $\mathcal{M}$, is irreducible— all states communicate. In addition, $C(s)$ is absorbing if $\sum_{s' \in C(s)} T(s'|s') = 1$, $\forall s' \in C(s)$.

**Definition 6 (Restriction of $\mathcal{M}$ to an Absorbing Set):** Let $C \subseteq S$ be an absorbing set. By Definition 5 if initial state $s_0$ lies in $C$, then for any path $\pi = s_0 s_1 \ldots$, the state $s_t$ lies in $C$ for all $t \geq 0$. Hence, the Markov chain can be studied exclusively in the smaller set $C$. The restriction of $\mathcal{M}$ to $C$ is denoted by $\mathcal{M}_{|C}$. An absorbing set is minimal if it does not contain a proper absorbing subset.

**Definition 7 (Recurrence and Transience):** The state $s \in S$ is called recurrent if $\mathbb{E}[f_s | s_0 = s] = \infty$ and transient if $\mathbb{E}[f_s | s_0 = s] < \infty$, with $f_s$ given by [2].

Recurrence and transience are class properties. Recurrent classes are also minimally absorbing classes. Furthermore, let $m_s = \mathbb{E}[\tau_s]$. State $s \in S$ is positive recurrent if $m_s < \infty$, and null recurrent if $m_s = \infty$. All states in a recurrent class are either positive recurrent or all null recurrent. For a finite state discrete-time Markov chain, all recurrent classes are positive recurrent [29].

**Definition 8 (Invariant and Ergodic Probability Measures):** Let $\nu \in \mathcal{M}_S$ be a probability measure (p.m.) on $S$. $\nu$ is an invariant p.m. if $\nu T = \nu$.

**Definition 9 (Occupation Measures):** Define the $t$-step expected occupation measure with initial state $s_0$ as

$$T^{(t)}(A|s_0) := \sum_{s \in A} \frac{1}{t} \sum_{k=0}^{t-1} T^k(s_0|s), \quad A \subseteq S, \quad t = 1, 2, \ldots$$

where $T^k$ denotes the composition of $T$ with itself $k-1$ times.

A pathwise occupation measure is defined as follows

$$\pi^{(t)}(A) = \frac{1}{t} \sum_{k=1}^t \mathbb{I}(s_k \in A), \quad A \subseteq S, \quad t = 1, 2, \ldots$$

**Proposition 1 ([29]):** The expected value of the path-wise occupation measure is the $t$-step occupation measure

$$\mathbb{E}\left[\pi^{(t)}(A)|s_0\right] = T^{(t)}(A|s_0), \quad \forall t \geq 1.$$
scheme will be sufficient for synthesizing controllers that satisfy LTL formulas over POMDPs. If the world state transitions from $s_i^{mod}$ to $s_j^{mod}$, then reward $r(s_j^{mod})$ is issued. The world’s initial state, $s^{mod}(t = 0)$, gathers reward $r(s^{mod}(t = 0))$.

Finally, the world model is assumed to be time invariant: $S^{mod}, \mathcal{O}, \mathcal{A}, \mathcal{P}, T, O, h$, and $r$ do not vary with time. At this point, we are ready to define a path in a POMDP.

**Definition 12 (Path in a POMDP):** An infinite path in a (labeled) POMDP, $\mathcal{P}\mathcal{M}$, with states $s \in S$ is an infinite sequence $\pi = s_0 o_0 \alpha_1 s_1 o_1 \alpha_2 \cdots \in (S \times \mathcal{O} \times \mathcal{A})^\omega$, such that for all $t \geq 0$ we have $T(s_{t+1}|s_t, \alpha_{t+1}) > 0$, $O(s_t|s_t) > 0$, and $\epsilon_{init}(s_0) > 0$. Any finite prefix of $\pi$ that ends in either a state or an observation is a finite path fragment.

Given a POMDP, we can define beliefs or distributions over states at each time step to keep track of sufficient statistics with finite description \[32\]. The beliefs for all $s \in S$ can be computed using Bayes’ law as follows:

$$b_0(s) = \frac{\epsilon_{init}(s) O(s_0|s)}{\sum_{s' \in S} \epsilon_{init}(s') O(s'|s)},$$

$$b_t(s) = \frac{O(s_t|s_{t-1}, \alpha_t) \sum_{s' \in S} T(s|s', \alpha_t)b_{t-1}(s')}{\sum_{s' \in S} O(s_t|s_{t-1}, \alpha_t) \sum_{s'' \in S} T(s|s', \alpha_t)b_{t-1}(s'')},$$

for all $t \geq 1$. It is also worth mentioning that (4) is referred to as the belief update equation.

**D. Stochastic Finite State Control of POMDPs**

It is well established that designing optimal policies for POMDPs based on the (continuous) belief states require uncountably infinite memory or internal states \[13, 14\]. This paper focuses on a particular class of POMDP controllers, namely, stochastic finite state controllers. These controllers lead to a finite state space Markov chain for the closed loop controlled system, allowing tractable analysis of the system’s infinite executions in the context of satisfying LTL formulae. For a finite set $A$, let $M_A$ denote the set of all probability distributions over $A$.

**Definition 13 (Stochastic Finite State Controller (sFSC)):** Let $\mathcal{P}\mathcal{M}$ be a POMDP with observations $\mathcal{O}$, actions $\mathcal{A}$, and initial distribution $\epsilon_{init}$. A stochastic finite state controller (sFSC) for $\mathcal{P}\mathcal{M}$ is given by the tuple $\mathcal{G} = (G, \omega, \kappa)$ where

- $G = \{g_1, g_2, \ldots, g_{|G|}\}$ is a finite set of internal states (I-states).
- $\omega : G \times \mathcal{O} \rightarrow M_{G \times \mathcal{A}}$ is a function of internal sFSC states $g_k$ and observation $o$, such that $\omega(g_k, o)$ is a probability distribution over $G \times \mathcal{A}$. The next internal state and action pair $(g_t, \alpha_t)$ is chosen by independent sampling of $\omega(g_t, \alpha_t)$. By abuse of notation, $\omega(g_t, \alpha_t|g_k, o)$ will denote the probability of transitioning to internal sFSC state $g_t$ and taking action $\alpha_t$, when the current internal state is $g_k$ and observation $o$ is received.
- $\kappa : M_{\mathcal{S}} \rightarrow M_{G}$ chooses the starting internal FSC state $g_0$, by independent sampling of $\kappa(\epsilon_{init})$, given initial distribution $\epsilon_{init}$ of $\mathcal{P}\mathcal{M}$. $\kappa(\epsilon_{init})$ will denote the probability of starting the FSC in internal state $g$ when the initial POMDP distribution is $\epsilon_{init}$.

A deterministic FSC can be written as a special case of the sFSC just defined.
denoted \( A^\varphi \) with the Rabin acceptance condition given in Definition \([2]\). Then, the product-POMDP, \( PM^\varphi \), has state space \( S = S^{mod} \times Q \), the same action set \( Act \), and observations \( O \). Furthermore,

- The transition probabilities of \( PM^\varphi \) are given by
  \[
  T^\varphi ((s_{mod}^t, q)| (s_{mod}^t, q_k), \alpha) = \begin{cases} T(s_{mod}^{mod} | s_{mod}^t, \alpha) & \text{if } \delta(q_k, h(s_{mod}^{mod})) = q_k, \\ 0 & \text{otherwise.} \end{cases}
  \]
- The initial state probability distribution is given by
  \[
  \iota^\varphi_{init}(s_{mod}^t, q) = \begin{cases} \iota_{init}(s_{mod}^{mod}) & \text{if } \delta(q_0, h(s_{mod}^{mod})) = 0, \\ 0 & \text{otherwise.} \end{cases}
  \]
- The observation probabilities are \( O^\varphi(o | s_{mod}^{mod}) = O(o | s_{mod}^{mod}) \).
- If rewards \( r(s_{mod}^{mod}) \) are defined over the POMDP \( PM \), new rewards over the product states are defined as \( r^\varphi(s_{mod}^{mod}, q) = r(s_{mod}^{mod}) \).

From the Rabin acceptance pairs \( \Omega \) of \( A^\varphi \), define the accepting pairs \( \Omega^PM^\varphi = \{(\text{Repeat}^PM^\varphi, \text{Avoid}^PM^\varphi)\} \), \( 0 \leq i \leq |\Omega| \} \) for the product-POMDP as follows. A product state \( s = (s_{mod}, q) \) of \( PM^\varphi \) is in Repeat\(^PM^\varphi \) if \( q \in \text{Repeat}_t \) and \( s \) is in Avoid\(^PM^\varphi \) if \( q \in \text{Avoid}_t \). Note that \( |\Omega^PM^\varphi| = |\Omega| \).

### A. Inducing an sFSC for PM from that of \( PM^\varphi \)

To control the POMDP, \( PM \), it is necessary to derive a policy for \( PM \) from a policy computed for \( PM^\varphi \).

**Definition 16 (Induced sFSC):** Let sFSC \( G = (G, k, \omega) \) control product-POMDP \( PM^\varphi \). The sFSC \( G_{mod} = (G^{mod}, k^{mod}, \omega^{mod}) \) that controls \( PM \) is induced as follows.

- I-states of the induced sFSC are given by \( G^{mod} \).
- The initial state of the induced sFSC is given by \( k^{mod}(g_k | \iota^\varphi_{init}) = k(g_k | \iota^\varphi_{init}) \).
- The probability of transitioning between I-states and issuing an action \( \alpha \) is given by \( \omega^{mod}(g_t, \alpha | g_k, \omega) = \omega(g_t, \alpha | g_k, \alpha) \).

### B. Verifying LTL Satisfaction via the Product-POMDP

Now, we consider the criterion for an (infinite) execution of \( PM \) to satisfy \( \varphi \). Let \( \sigma^\varphi = s_0 s_1 \ldots \) \( s_t = (s_{mod}^t, q_1) \) be an execution of the product-POMDP under some sFSC \( G \).

**Definition 17 (Accepting execution):** We say that \( \sigma^\varphi \) is an accepting execution if, for some \( (\text{Repeat}^PM^\varphi, \text{Avoid}^PM^\varphi) \in \Omega^PM^\varphi \), \( \sigma^\varphi \) interacts with \( \text{Repeat}^PM^\varphi \) infinitely often, while it intersects \( \text{Avoid}^PM^\varphi \) only a finite number of times.

The notion of verifying LTL properties using product transition systems is well known in the literature \([3, 10]\) and the following lemma can be derived for the product-POMDP.

**Lemma 1:** Let \( \sigma^\varphi = s_0 s_1 \ldots \) \( s_t = (s_{mod}^t, q_1) \) be an execution of \( PM^\varphi \) and the corresponding execution of \( PM \) be given by \( s = s_{mod}^0 s_{mod}^1 \ldots \). Then, \( \sigma^\varphi \) satisfies \( \varphi \), i.e., \( \sigma^\varphi \models \varphi \), if and only if \( \pi \models q_0 \pi \models q_1 \ldots \) is an accepting run on \( A^\varphi \).

**Proof:** The proof follows from the construction of the product-POMDP. The run \( \sigma^\varphi \) can be projected onto its POMDP and DRA components as runs \( \sigma \) and \( \pi \). Next, the trace generated by \( \sigma \), given by \( h(\sigma) = h(s_{mod}^0 h(s_{mod}^1 \ldots \) leads to the same unique path \( \pi \) in the DRA \( A^\varphi \). Thus, if \( \pi \) is an accepting run in the DRA, then \( \sigma^\varphi \models \varphi \).

### IV. An Optimization Problem for LTL Satisfaction

In this section, we formulate the problem of synthesizing sFSCs for POMDPs with LTL specification into an optimization problem.

#### A. Measuring the Probability of LTL Satisfaction

This section culminates in Proposition \([4]\), which presents the principal problem that must be solved to find the sFSC maximizing the probability of LTL specifications on \( PM \).

Section \([3, 4]\) described how the accepting executions of the product-POMDP, \( PM^\varphi \), under a given sFSC controller, have a one-to-one correspondence to the executions of the original POMDP, \( PM \), that satisfy \( \varphi \).

Recall from Section \([3, 4]\) that a product-POMDP, \( PM^\varphi \), controlled by an sFSC, \( G \), induces a Markov chain, denoted as \( \hat{M}_{S \times G}^{PM^\varphi, G} \), evolving on the finite state space \( S \times G = (S^{mod} \times Q) \times G \). Using the probability measure defined over the paths of the global Markov chain (Section \([3, 4]\)), the probability of satisfaction of \( \varphi \) over the controlled system is defined as:

**Definition 18 (Probability of satisfaction of \( \varphi \)):** For product-POMDP \( PM^\varphi \) controlled by sFSC \( G \), the probability of satisfaction of \( \varphi \), defined over Paths\( (M_{S \times G}^{PM^\varphi, G}) \), is:

\[
Pr(PM^\varphi \models \varphi | G) = Pr(M_{S \times G}^{PM^\varphi, G} | [\sigma^g \in \text{Paths}(M_{S \times G}^{PM^\varphi, G}) ],
\]

where \( \downarrow_{S} (.) \) projects paths \( \sigma^g \) of the induced Markov chain

\[
\sigma^g = [s_0, q_0] [s_1, q_1] \ldots = [x_0^{mod}, q_0] [x_1^{mod}, q_1, g_1] \ldots
\]

to the associated product-POMDP execution

\[
\downarrow_S (\sigma^g) = [x_0^{mod}, q_0] [x_1^{mod}, q_1] \ldots
\]

Since the global Markov chain is unique given \( PM \), \( \varphi \), and \( G \), thereafter the subscript on the probability operator in the r.h.s. of \( (5) \) will be dropped, and expectation will be defined using the probability measure over the global Markov chain.

Finally, define \( \Pr(PM^\varphi \models \varphi | G) = \Pr(PM^\varphi \models \varphi | G) \) as the probability that the original uncontrolled model satisfies \( \varphi \).

Recall that the global Markov chain \( M_{S \times G}^{PM^\varphi, G} \) induced by the sFSC \( G \) controlling the product-POMDP, \( PM^\varphi \), evolves over the global state space \( S \times G \), where the product-POMDP state space is given by \( S = (S^{mod} \times Q) \). Since the state space is finite, every state is either positive recurrent or transient.

Consider a product state \( s \in S \). If there exists \( g \in G \) such that the global state \( [s, g] \) is recurrent in \( M_{S \times G}^{PM^\varphi, G} \), \( s \) is said to be recurrent under \( G \). For a set \( A = \{[s_0, g_0], \ldots \} \in (S \times G) \) the projection \( \downarrow_S \) is \( \downarrow_S (A) = \{s_0, \ldots \} \) (taken uniquely).

Let \( R^\varphi \) denote the set of all recurrent states of \( M_{S \times G}^{PM^\varphi, G} \).

Partition the recurrent states into disjoint recurrent classes

\[
\text{RecsRec}^S = \{R_1, R_2, \ldots R_N\}
\]

such that

\[
R_1 \cup R_2 \cup \cdots \cup R_N = R, \quad R_i \cap R_j = \emptyset, i \neq j.
\]

The partitioning is required to be maximal. Formally, this means that for each \( R_k, R_l \in \text{RecsRec}^S \), \( s_i \leftrightarrow s_j \), \( \forall s_i, s_j \in R_k \), and \( s_i \leftrightarrow s_j \), \( s_i \in R_k, s_j \in R_l, k \neq l \). The first
equation states that within each recurrent class, $R_k$, all states are reachable from one another. The second equation states that no two distinct recurrent classes can be combined to make a larger recurrent class, thus making the partitions maximal.

**Definition 19 ($\varphi$-feasible Recurrent Set):** For sFSC $G$, a (maximal) recurrent set class $R_k$ is a $\varphi$-feasible recurrent set if $\exists (\text{Repeat}^{P,M^\varphi}, \text{Avoid}^{P,M^\varphi})$ such that,

$$\downarrow_S (R_k) \cap \text{Repeat}^{P,M^\varphi} \neq \emptyset,$$

$$\downarrow_S (R_k) \cap \text{Avoid}^{P,M^\varphi} = \emptyset.$$  \hfill (7)

Let $\varphi \text{-RecSets}^G \supseteq \bigcup R_k$, such that $R_k$ is $\varphi$-feasible.

The problem of maximizing the probability of satisfaction can be solved as follows.

**Proposition 4:** The satisfaction probability of an LTL formula can be maximized by optimizing the following objective

$$\max_{G} \sum_{R \in \varphi \text{-RecSets}^G} \Pr[\pi \rightarrow R],$$  \hfill (8)

where $\pi \rightarrow R$ implies the path entering the recurrent set.

**Proof:** Recall that recurrence implies absorption, i.e., if the Markov chain path enters a state in a recurrent set, the path is forever confined to that set. This implies the following long term behavior of path probabilities:

$$\Pr[\pi \rightarrow (R_k \cup R_l)] = \Pr[\pi \rightarrow R_k] + \Pr[\pi \rightarrow R_l], \quad k \neq l,$$

wherein we used (9). Over infinite executions, the path must end up in some recurrent set,

$$\sum_{R_k \in \varphi \text{-RecSets}^G} \Pr[\pi \rightarrow R_k] = 1.$$  \hfill (9)

Furthermore, conditions (7) imply the existence of a recurrent state in $\text{Repeat}^{P,M^\varphi}$, while simultaneously avoiding those states from $\text{Avoid}^{P,M^\varphi}$ that are recurrent under sFSC $G$. Therefore, if

$$\sum_{R \in \varphi \text{-RecSets}^G} \Pr[\pi \rightarrow R] = 1,$$

then the LTL specifications are satisfied. Hence, maximizing the $\sum_{R \in \varphi \text{-RecSets}^G} \Pr[\pi \rightarrow R]$ term implies maximizing the probability of satisfying the LTL specifications.

To further understand the solution to (8), note that there are two main components in the choice of an sFSC, $G$;

1) **Structure:** The sFSC has two structural components:

   a) The number of I-states, $|G|$, which impacts the size of the global Markov chain state space.

   b) The set of parameters in $\omega$ and $\kappa$ with non-zero values. This set determines the global state space connectivity graph, whose nodes represent states of the global Markov chain, and whose directed edges indicate that a one-step transition can be made from $[s,g]$ to $[s',g']$. The underlying graph completely and unambiguously determines the global Markov chain recurrent and transient classes.

2) **Quality:** The values of non-zero parameters of $\omega$ and $\kappa$ determine the probability with which the global Markov chain paths reach some $R \in \text{RecSets}^G$.

![Figure 2. Assigning rewards for frequent visits to Repeat$^{P,M^\varphi}$. The diagram depicts a product-POMDP state space. The edges depend on an action, $\alpha \in Act$, chosen arbitrarily here. There is only one Rabin pair (Repeat$^{P,M^\varphi}$, Avoid$^{P,M^\varphi}$). In order to incentivize visiting Repeat$^{P,M^\varphi}_t$, the state $s_5 \in S$ is assigned a reward of 1, while all other states are assigned a reward of 0.](image)

**B. Reward Design for LTL Satisfaction**

This section introduces an any time algorithm to optimize over both the sFSC quality and structure. This algorithm is based on the fact that finite state Markov chains evolve in two distinct phases: a transient phase, and a steady state phase in which the execution has been absorbed into a recurrent set. Therefore, rewards are designed with the following goals:

- During the transient phase, the global state is absorbed into a $\varphi$-feasible recurrent set quickly.
- During the steady state phase, the system visits the states in $\text{Repeat}_t^{P,M^\varphi}$ frequently.

1) **Incentivizing Frequent Visits to Repeat$^{P,M^\varphi}_t$:** In classical POMDP planning, an agent collects rewards as it visits different states. To quickly accumulate useful goals rewards collected at later times are discounted. While there exist temporal logics that allow explicit verification/design for known finite time horizon [10], it may be hard to predict the horizon for a given POMDP and LTL formula a-priori. In such scenarios, a discounted reward scheme, which does not affect feasibility, thus offers a viable solution.

Consider, a particular product-POMDP with Rabin acceptance pair (Repeat$^{P,M^\varphi}_t$, Avoid$^{P,M^\varphi}_t$). The aim is to visit states in $\text{Repeat}_t^{P,M^\varphi}$ often. To achieve this, we assign the following “repeat” reward scheme (see Figure 2):

$$r^\beta_\tau(s) = \begin{cases} 1 & \text{if } s \in \text{Repeat}_t^{P,M^\varphi}, \\ 0 & \text{otherwise}. \end{cases}$$  \hfill (10)

The discounted reward problem takes the form:

$$\eta_\beta(\tau) = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t r^\beta_\tau(s_t) \left| \nu^{ex}_t \right|,$$

where $\beta$ is the discount factor. Note that in (11), while the objective incentives early visits to states in $\text{Repeat}_t^{P,M^\varphi}$ in order to accrue maximum rewards, it has two drawbacks:
1) The objective becomes exponentially less dependent, with decay rate $\beta$, on visits to $\text{Repeat}_{t}^{P,M^e}$ at later time steps. Thus, frequent visits are incentivized mainly during the initial time steps.

2) Due to partial observability, the transition from a transient to a recurrent phase cannot be reliably detected. Hence, visits to $\text{Avoid}_{t}^{P,M^e}$ cannot be precluded in steady state.

To tackle the first problem, if a stationary policy that is independent of the initial product-POMDP distribution can be found, then the expected visiting frequency will remain the same for later time steps, including during steady state, when the global Markov chain evolves in a recurrent set. A suboptimal solution for the second problem is discussed next.

2) The Steady State Probability of Visiting $\text{Avoid}_{t}^{P,M^e}$. This section develops a method to compute the probability of visiting a state in $\text{Avoid}_{t}^{P,M^e}$. If this quantity can be computed, then a discounted reward criterion can be optimized under the constraint that this probability is zero, or extremely low. In order to compute the probability of visiting $\text{Avoid}_{t}^{P,M^e}$ regardless of the global Markov chain execution phase (transient or steady state), we first define a transition rule that makes every state in $\text{Avoid}_{t}^{P,M^e}$ a sink. To this end, consider the following modified product-POMDP. For $\forall \alpha \in \mathcal{A}$, let

$$T^\varphi_{mod}(s_k|s_j,\alpha) = \begin{cases} 0 & \text{if } s_j \neq s_k \text{ and } s_j \in \text{Avoid}_{t}^{P,M^e} \\ T^\varphi(s_k|s_j,\alpha) & \text{otherwise}. \end{cases}$$

Then, assign a different, “avoid” reward scheme

$$r^\varphi_{av}(s) = \begin{cases} 1 & \text{if } s \in \text{Avoid}_{t}^{P,M^e} \\ 0 & \text{otherwise}. \end{cases} \quad (12)$$

For sFSC $\mathcal{G}$, consider the expected long term average reward

$$\eta_{av}(t) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{mod} \left[ \sum_{t=0}^{T} r^\varphi_{av}(s_t) \right] \left( \pi_{init}^{P,M^e} \right), \quad (14)$$

where the expectation is taken over the global Markov chain arising from the transition distribution $T^\varphi_{mod}$ of $\text{Avoid}_{t}^{P,M^e}$.

**Lemma 2:** Let $\pi \in \text{Paths}(M^{P,M^e,\varphi})$ be a global Markov chain path arising from the execution of the original unmodified product-POMDP. Then

$$\Pr \left[ \pi \to (\text{Avoid}_{t}^{P,M^e} \times G) \mid \pi_{init}^{\varphi} \right] = \eta_{av}(t). \quad (15)$$

**Proof:** See Appendix A

Lemma 2 provides a tractable way to compute the probability of visiting $\text{Avoid}_{t}^{P,M^e}$. Note the conditional dependence on $\pi_{init}$ in (14). Recall that to satisfy an LTL formula, it is only required to guarantee that the probability of visiting an avoid state is zero in steady state. Requiring this probability to be zero during the transient period may render a solution infeasible.

Unfortunately, using the formulation presented so far it is not possible to know if a particular POMDP path has entered steady state behavior. At most, it is possible to know the probability of being in steady state by taking the sum of all beliefs over recurrent states that form the steady state behavior.

Next, assume that the controller has access to an oracle that can declare the end of the transient period during which visits to $\text{Avoid}_{t}^{P,M^e}$ may be allowed. The oracle can also indicate when the system enters a sub-Markov chain where $\text{Avoid}_{t}^{P,M^e}$ is never visited. Of course, no such oracle exits, but we show below that a product-POMDP and reward assignment can be designed such that the controller implicitly incorporates an oracle function.

3) Partitioned sFSC and Steady State Detecting Global Markov Chain: Suppose that the sFSC I-states, $G$, are divided a-priori into two sets—transient states $G^{tr}$ and steady states $G^{ss}$—such that $G = G^{tr} \cup G^{ss}$, and $G^{tr} \cap G^{ss} = \emptyset$.

As explained below, this state partitioning can indicate if the execution of the global Markov chain has zero probability of future visits to $\text{Avoid}_{t}^{P,M^e}$.

Let the global state at time $t$ be given by $[s_t, g_t]$. We seek to create a global Markov chain whose underlying product-POMDP has the following property

$$\Pr \left[ [s_{t'}, g_{t'}] \in \text{Avoid}_{t}^{P,M^e} \times G \mid \exists t \leq t', \text{ s.t.} \right.$$

$$[s_t, g_t] \in \text{Repeat}_{t}^{P,M^e} \times G^{ss} \right] = 0. \quad (16)$$

In other words, let the product-POMDP visit states in $\text{Repeat}_{t}^{P,M^e}$ while the sFSC executes in steady state, i.e., $g_t \in G^{ss}$. Then, it must be ensured that the probability for the product-POMDP to visit $\text{Avoid}_{t}^{P,M^e}$ at anytime in the future is zero. This requirement can be achieved in three steps.

First, constrain the sFSC to prevent a transition from an I-state in $G^{ss}$ to any other I-state in $G^{tr}$. Formally, $\forall \alpha \in \mathcal{A}, o \in \mathcal{O}$,

$$\omega(g', \alpha|g, o) = 0, \quad g \in G^{ss}, g' \in G^{tr}. \quad (17)$$

This constraint ensures that the controller transitions to steady state only once during an execution, mimicking the fact that for each infinite path in the Markov chain, the transition to a recurrent set occurs once.

Second, the method of evaluating the global Markov chain transition distribution is based on the following definition.

**Definition 20 (Steady State Detecting Global Markov Chain):** The steady state detecting global (ssd-global) Markov chain is defined by its transition distribution function

$$T_{ssd}^{P,M^e,\varphi}([s', g']) = \begin{cases} \sum_{\alpha,o} \omega(o|s) \omega(g', \alpha|g, o) T^\varphi(s'|s, \alpha) & \text{if } g \in G^{tr}, g' \in G^{ss}, \\ \sum_{\alpha,o} \omega(o|s) \omega(g', \alpha|g, o) T^\varphi(s'|s, \alpha) & \text{if } g, g' \in G^{ss} \text{, due to (17)}. \end{cases} \quad (18)$$

Note the use of modified transition function from (12) in Definition 20. This modification transforms all states in $\text{Avoid}_{t}^{P,M^e}$ to sinks. This construction prevents visits to $\text{Avoid}_{t}^{P,M^e}$ during steady state, while allowing the execution to visit these states in the transient phase.
Third, in addition to the reward schemes of (10) and (13), assign the following rewards to the I-states:

\[ r^G_c(g) = \begin{cases} 1 & \text{if } g \in G^{ss}, \\ 0 & \text{if } g \in G^{rr}. \end{cases} \tag{19} \]

C. Casting into an Optimization Problem

Let us define \( t^s_{\text{init}} \) as a distribution over the ssd-global Markov chain states as follows.

\[ t^s_{\text{init}}([s,g]) = \begin{cases} 1 & \text{if } s \in \text{Repeat}_t^{P,M^G}, g \in G^{ss}, \\ 0 & \text{otherwise}. \end{cases} \tag{20} \]

Using the rewards (10), (13) and (19), for an sFSC of fixed size and partitioning \( G = \{G^{tr}, G^{ss}\} \), the following conservative optimization criterion is derived:

**Conservative Optimization Criterion**

\[
\begin{align*}
\max_{\omega, \kappa} & \quad \eta_{\beta}(\tau) \\
\text{subject to} & \quad \eta_{\text{av}}^{\text{ssd}}(\tau) = 0 \\
& \quad (g'|g, \alpha, o) = 0 \\
& \quad g' \in G^{tr}, g \in G^{ss} \\
& \quad \sum_{(g'|g, \alpha, o) \in \text{Act}} \omega(g'|g, \alpha, o) = 1 \\
& \quad \forall g, o \in \mathcal{O}, \alpha \in \text{Act} \\
& \quad \sum_{g \in \mathcal{G}} \kappa(g) = 1.
\end{align*} \tag{21} \]

In (21), frequent visits to Repeat\(_t^{P,M^G}\) are incentivized by maximizing

\[ \eta_{\beta}(\tau) = \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=0}^{T} \beta^t r^G_c(s_t) t^G_c(s_t) \big| \tau^s_{\text{init}} \right], \quad 0 < \beta < 1, \]

while steady state visits to Avoid\(_t^{P,M^G}\) are forbidden via the first constraint in (21)

\[ \eta_{\text{av}}^{\text{ssd}}(\tau) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{\text{ssd}} \left[ \sum_{t=0}^{T} r(s_t) t^G_c(s_t) \big| \tau^s_{\text{init}} \right] = 0, \]

The constraints relate to the I-state partitioning introduced above and the probabilities of admissible sFSC parameters. Note that \( \eta_{\text{av}}^{\text{ssd}} \) is computed using the ssd-global Markov chain transition distribution from (13), but the expression \( \eta_{\beta} \) uses the unmodified Markov chain transition distribution. The product terms \( r^G_cG \) and \( r^G_c\eta \) ensure that only those visits to Repeat\(_t^{P,M^G}\) are rewarded when the controller I-state lies in \( G^{ss} \), implying that it can now guarantee no more visits to Avoid\(_t^{P,M^G}\).

Note that the choice of initial condition (20) implies that in steady state,

\[ \forall [s,g] \in (\text{Repeat}_t^{P,M^G} \times G^{ss}), [s,g] \rightarrow (\text{Avoid}_t^{P,M^G} \times G). \tag{22} \]

Compare (22) to (16), which can be re-written as

\[ \Pr \left[ \pi \rightarrow [s,g] \in (\text{Repeat}_t^{P,M^G} \times G) \right] > 0 \quad \implies \quad [s,g] \rightarrow (\text{Avoid}_t^{P,M^G} \times G). \]

The condition of the latter statement is only required to hold for those states in \( (\text{Repeat}_t^{P,M^G} \times G) \) under the current \( \omega \) and \( \kappa \). If some repeat state is not visited by the controller during steady state, then the proposed choice of \( t^s_{\text{init}} \) adds additional feasibility constraints, which may severely reduce the obtainable reward, \( \eta_{\beta} \), and possibly render the problem infeasible. This is why (21) is called a *Conservative Optimization Criterion*. While sub-optimal, this criterion has some significant advantages. In the sequel, we show that the Conservative Optimization Criterion can be framed as a policy iteration algorithm, with efficient policy improvement steps. Moreover, the improvement steps can also add I-states to the sFSC, which help to escape the local maxima encountered during optimization of the total reward \( \eta_{\beta}(\tau) \). The added sFSC I-states allow the generation and differentiation of many new observation and action sequences. This implies that many new paths in the global Markov chain can be explored for the purpose of improving the optimization objective. We begin by leveraging the Poisson Equation method to convert the Conservative Optimization Criterion into a bilinear program.

V. THE POISSON EQUATION FOR THE GLOBAL MARKOV CHAIN

The discussion in this section is restricted to time homogeneous, discrete time, finite state space Markov chains [35]. The main focus is the ssd-global Markov chain of Definition 20 which can differentiate whether states in Avoid\(_t^{P,M^G}\) can be visited. Recall that the ssd-global Markov chain is generated by partitioning the sFSC I-states into transient and steady state sets, \( G^{tr} \) and \( G^{ss} \). The transition probabilities \( T^P_{\text{ssd}} \) were then computed using (18). In addition, recall the average reward function \( r^{av}([s, g]) = r^G_c(s) r^G_c(g) \). A vectorized representation is needed for ordering the global state space \( S \times G \) denoted as \( r^{av} \).

**Definition 21 (Poisson Equation (PE) for \( T^P_{\text{ssd}} \))**

(a) \( \widetilde{g} = T^P_{\text{ssd}} \overset{\beta}{\rightarrow} \widetilde{g} \) and (b) \( \widetilde{g} + \beta - T^P_{\text{ssd}} \overset{\beta}{\rightarrow} \beta = r^{av} \). \tag{23}

where the matrix form (18) of \( T^P_{\text{ssd}} \) has been used. If (23) holds, the pair \( (\widetilde{g}, \beta) \) is called a solution to the PE with charge \( r^{av} \).

More generally, the reward \( r^{av} \) can be replaced with any measurable function, \( f : S \times G \to \mathbb{R} \). The PE is developed in (29), (35), and the conditions for existence and uniqueness of its solutions can be found in (36).

When a Markov chain has a single recurrent class and possibly some transient states, the PE solves the long term average cost criterion for a given initial state \( s_0 \).

\[ \eta_{av} = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T} f(s) \big| s_0 \right] \]

for the reward \( r^{av}(t) \). In fact, the value for the scalar \( \eta_{av} \) is the solution to the following slightly different version of the PE (23):

\[ \eta_{av} + \beta - T^P_{\text{ssd}} \overset{\beta}{\rightarrow} \beta = r^{av} \]. \tag{24}

Note that (24) is obtained from (23)(b) by replacing the vector \( \widetilde{g} \) by the scalar \( \eta_{av} \). For a finite Markov chain with a single recurrent class, this has a unique solution for \( \eta_{av} \).
The multi-chain PE as introduced in (23) is used when the average reward accounts for the probability of absorption into the different $R_i$ in the computation of the average cost given the initial distribution $i_{\text{init}}(b)$. Further discussion of the PE in the context of dynamic programming is provided in Section VI-A2.

For the finite state closed loop global Markov chain under study in this work, a solution for the PE always exists.

**Lemma 3 (29):** (a) For a finite state space Markov chain with transition matrix $T_{ssd}^{\text{PMte},G}$ and charge $r_{ssd}^\beta$, a solution pair $(\vec{g}, \vec{h})$ to the PE always exists. (b) Moreover, $\vec{g}$ is unique and is given by

$$\vec{g} = \Pi_{ssd} r_{ssd}^\beta,$$

(25)

where $\Pi_{ssd}$ is the limiting matrix introduced in Definition 10.

(c) The solution $\vec{h}$ in (25), when paired with $\vec{h} = H^\text{PMte}$ solves the PE, where $H$ is called the deviation matrix given by

$$H = (I - T_{ssd}^{\text{PMte},G} + \Pi_{ssd})^{-1} (I - \Pi_{ssd}).$$

(d) $\vec{h}$ is not unique. If $(\vec{g}, \vec{h})$ is a solution then any $(\vec{g}, \vec{h} + \Pi_{ssd})$ is also a solution.

The PE yields the quantity $g$, which can be used to compute the probability of visiting $\text{Avoid}_{\text{PMte}}^{G} \times G_{ss}$ in the following theorem. This probability can be used to enforce the constraint $\eta_{ssd}^\beta = 0$ in the optimization problem (21).

**Theorem 1:** The probability that the ssd-global Markov chain visits $(\text{Avoid}_{\text{PMte}}^{G} \times G_{ss})$ for an initial distribution $i_{\text{init}} \in M_{S \times G}$ is given by

$$\Pr \left[ \pi \rightarrow (\text{Avoid}_{\text{PMte}}^{G} \times G_{ss}) \mid i_{\text{init}}^T \right] = i_{\text{init}}^T \vec{g}.$$

**Proof:** Note that under $T_{ssd}^{\text{PMte},G},$ each state in $(\text{Avoid}_{\text{PMte}}^{G} \times G_{ss})$ is a sink by construction and therefore recurrent. Applying Lemma 2 gives

$$\Pr \left[ \pi \rightarrow (\text{Avoid}_{\text{PMte}}^{G} \times G_{ss}) \mid i_{\text{init}}^T \right] = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{T} \left[ \sum_{t=0}^{T} r_{ssd}^\beta (s_t, g_t) \right] i_{\text{init}}^T$$

$$= i_{\text{init}}^T \Pi_{ssd} \mathbb{E}_{T} \left[ (\text{Avoid}_{\text{PMte}}^{G} \times G_{ss}) \right]$$

$$= i_{\text{init}}^T \Pi_{ssd} r_{ssd}^\beta$$

$$= i_{\text{init}}^T \vec{g},$$

where line 1 implies line 2 due to (15) in Lemma 2 and (14), and line 3 follows from the fact that $r_{ssd}^\beta$ can be re-written as an indicator vector $r_{ssd}^\beta = \mathbb{E}_{T} \left[ (\text{Avoid}_{\text{PMte}}^{G} \times G_{ss}) \right].$

**Theorem 1** will be used in the sequel to enforce the constraint, $\eta_{ssd}^\beta(\tau) = 0$ in optimization (21).

VI. BOUNDED POLICY ITERATION FOR LTL REWARD MAXIMIZATION

We employ an stochastic dynamic programming approach to solve the Conservative Optimization Criterion. In the general setting of an arbitrary reward function and infinite state space, the existence of an optimal solution for the average case is not guaranteed. However, for the set of problems of interest in this paper, the global Markov chain is a discrete time system that evolves over finite state space, in which case the average reward does have an optimum. Additionally, as will be seen in Section VI-D the optimal solution for the average case is not required for the algorithm proposed herein. Only the evaluation of the average reward value function under a given sFSC is required to guarantee LTL satisfaction. Therefore, the Bellman equation for the average reward case is sufficient for this work. Next, the relevant dynamic programming equations for both discounted and average rewards are summarized for the specific case of POMDPs controlled by sFSCs.

A. Dynamic Programming Variants for POMDPs with sFSCs

For POMDPs controlled by sFSCs, the dynamic program is developed in the global state space $S \times G.$ The value function is defined over this global state space, and policy iteration techniques must also be carried out in the global state space.

1) Value Function for Discounted Reward Criterion: For a given sFSC, $\vec{g},$ and the unmodified product-POMDP, the value function $V^\beta$ is the expected discounted sum of rewards under $\vec{g},$ and can be computed by solving a set of linear equations:

$$V^\beta([(s_i, g_k)]) = r^\beta([(s_i, g_i)]) + \beta \sum_{o \in O, a \in \text{Act}} \omega(g_i, a|g_k, o) T^\omega(s_j|s_i, a)V^\beta([(s_j, g_j)]).$$

For the global Markov chain, the above can be written in vector notation as follows

$$\vec{V}^\beta = \vec{r}^\beta + \beta T_{ssd}^{\text{PMte},G} \vec{V}^\beta.$$  

(26)

Remark that (26) is the Bellman Equation for the discounted reward criterion. The value function of the POMDP states, for a given I-state $g,$ and the unmodified product-POMDP, the value function $V^\beta$ is the expected discounted sum of rewards under $\vec{g},$ and can be computed by solving a set of linear equations:

$$V_g^\beta(b) = \vec{r}_g^\beta + \beta T_{ssd}^{G} \vec{V}_g^\beta.$$  

(27)

If $i_{\text{init}}^T$ is the initial distribution of the product-POMDP then, the best sFSC I-state can be selected as

$$\kappa(g|i_{\text{init}}^T) = \begin{cases} 1 & \text{if } g = \arg \max_{g'} V_{g'}^\beta(i_{\text{init}}^T) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the sFSC is started in the I-state with maximum expected value for the belief.

**Definition 22 (Value Function):** The value function gives the value at any belief $b$ using the following

$$V_g^\beta(b) = \max_{g'} V_{g'}^\beta(b).$$  

(28)

Clearly, (27) shows that the value of a particular I-state is a linear function of the belief state. The value function itself is piece-wise linear by taking the pointwise maximum of all the I-state values at each belief state.
Computational Complexity and Efficient Approximation: Solving a system of linear equations by direct methods is $O(n^3)$ where $n$ is the number of equations. (26) represents $|S||G|$ equations. However, a basic Richardson iteration can be applied. One starts with an arbitrary value of $V^\beta$, typically 0, and repeatedly computes $V^\beta_i(t+1) = r^\beta_i + \beta T P M^\beta V^\beta_i(t)$, until $\|V^\beta_i(t+1) - V^\beta_i(t)\|_\infty < \varepsilon_\beta$, where $\varepsilon_\beta > 0$. During each iteration, the maximum number of operations required are $O(|S|^2|G|^2)$, however if the ssd-global Markov chain can be represented as a sparse matrix, then the complexity is linear.

2) Value Function for Average Reward Criterion: For a given sFSC $G$, the value function $V^\beta$ is the expected discounted sum of rewards under $G$, and can be computed by solving a set of linear equations:

$$V^{av}([s_i, g_k]) = -\eta^{av}([s_i, g_k]) + r^{av}([s_i, g_k]) + \sum_{o \in O, \alpha \in Act \atop g_s \in G, s_i \in S} O(s_i, o, \alpha|g_k, o)T(s_j|s_i, \alpha)V^{av}([s_j, g_k]).$$

Writing the above in vector notation for the ssd-global Markov chain gives $\vec{V}_{av} = (-P^{av} + r^{av}) + T_{ssd}^{av} \vec{V}_{av}$. The latter system of equations constitutes the Bellman equation for the average reward criterion. Note that this is the same as the second part of the PE (23) by substituting $\vec{g} = \vec{\eta}^{av}$.

Computational complexity: Since the value function of the average reward criterion is identical to the Poisson Equation, the following considers the complexity of solving the Poisson Equation. Again, the exact methods of solving the linear system of equations is cubic in number of equations, which is $2|S||G|$ in (23) with as many variables, which comprise of both $\vec{V}_{av}$ and $\vec{g}$. However, as will be shown later in this section, direct computation of the full $\vec{g}$ and $\vec{V}_{av}$ vectors will not be required frequently in the algorithm proposed in this section. The PE will be directly inserted into the optimization software as a set of constraints in order to compute the values for the unknown vectors $\vec{g}$ and $\vec{V}_{av}$.

B. Bellman Optimality / DP Backup - Discounted Case

When the discounted case does not have constraints other than probability constraints on $\omega$ and $\kappa$, then at optimality the discounted value function satisfies the Bellman Optimality Equation, which is also known as the DP Backup Equation:

$$V^\beta(b) = \max_{o \in Act} \left(r^\beta(b) + \beta \sum_{o \in O} Pr(o|b)V^\beta(b'_o)\right)$$

where $Pr(o|b) = \sum_{s \in S} O(s, o|b, s'_o) = \sum_{s} T(s'|s, o)O(s, o|s'_o)$, and $V^\beta(b'_o)$ is computed using (28) and (27). The r.h.s. of the DP Backup Equation can be applied to any value function. The effect is an improvement (if possible) at every belief state. However, DP backup is difficult to use directly as it must be computed at each belief state in the belief space, which is uncountably infinite.

C. Bounded Policy Iteration for sFSCs

Policy iteration incrementally improves a controller by alternating between two steps: Policy Evaluation and Policy Improvement, until convergence to an optimal policy. For the discounted reward criterion, policy evaluation amounts to solving (26). During policy improvement, a dynamic programming update using the DP Backup Equation is used. This results in the addition, merging, and pruning of I-states of the sFSC.

In (59) a methodology called the Bounded Policy Iteration is proposed, in which the sFSC is allowed to be stochastic. Next, we briefly outline this methodology before showing how it can be adapted for solving the Conservative Optimization Criterion given by (21).

We are concerned with maximizing the expected long term discounted reward criterion over a general POMDP. The state transition probabilities are given by $T(s'|s, \alpha)$, and observation probabilities by $O(o|s)$. Most of this section follows from (59) and (40), where the authors showed that (1)Allowing stochastic I-state transitions and action selection (i.e., sFSC I-state transitions and actions sampled from distributions) enables improvement of the policy without having to add more I-states. (2) If the policy cannot be improved, then the algorithm has reached a local maximum. Specifically, there are some belief states at which no choice of $\omega$ for the current size of the sFSC allows the value function to be improved. In such a case, a small number of I-states can be added that improve the policy at precisely those belief states, thus escaping the local maximum.

Definition 23 (Tangent Belief State): A belief state $b$ is called a tangent belief state, if $V^\beta(b)$ touches the DP Backup of $V^\beta(b)$ from below. Since $V^\beta(b)$ must equal $V^\beta_g$ for some $g$, we also say that the I-state $g$ is tangent to the backed up value function $V^\beta$ at $b$.

Equipped with this definition, the two steps involved in policy improvement can be carried out as follows.

Improving I-States by Solving a Linear Program: An I-state $g$ is said to be improved if the tunable parameters associated with that state can be adjusted so that $\hat{V}^\beta_g$ is increased. The improvement is posed as a linear program (LP) as follows:

I-state Improvement LP: For the I-state $g$, the following LP is constructed over the unknowns $\epsilon, \omega(g', \alpha|g, o), \forall g', \alpha, o$

$$\max_{\epsilon, \omega(g', \alpha|g, o)} \epsilon \quad \text{subject to}$$

Improvement Constraint:

$$V^\beta([s, g]) + \epsilon \leq V^\beta([s, g])$$

$$\sum_{s': g', \alpha, o} O(s'|g, o)T(s'|s, \alpha)V^\beta([s', g']), \forall s,$$

Probability Constraints:

$$\sum_{(g', \alpha) \in G \times Act} \omega(g', \alpha|g, o) = 1, \quad \forall o \in O,$$

$$\omega(g', \alpha|g, o) \geq 0, \quad \forall g' \in G, \alpha \in Act, o \in O. \quad (29)$$

The above LP searches for $\omega$ values that improve the I-state value vector $\hat{V}^\beta_g$ by maximizing the parameters $\epsilon$. If an improvement is found, i.e., $\epsilon > 0$, the parameters of the I-state are updated by the corresponding maximizing $\omega$. 

Algorithm 1 Bounded PI: Adding I-States to Escape Local Maxima

**Input:** Set $B$ of tangent beliefs from policy improvement LPs for each I-state, $N_{\text{new}}$ the maximum number of I-states to add.

1. $N_{\text{added}} \leftarrow 0$.
2. repeat
3. Pick $b \in B$, $B = B \setminus \{b\}$.
4. $F_{\text{wd}} = \emptyset$.
5. for all $(\alpha, o) \in (\mathcal{A} \times \mathcal{O})$ do
6. if $Pr(o|b) = \sum_{s \in S} b(s)\mathcal{O}(o|s) > 0$ then
7. Look ahead one step to compute forward beliefs $b_{o,\alpha}(s') = \sum_{s} T(s'|s, \alpha) \sum_{o|s} \mathcal{O}(o|s)b(o|s)$.
8. $F_{\text{wd}} \leftarrow F_{\text{wd}} \cup \{b_{o,\alpha}\}$.
9. for all $b_{\text{fwd}} \in F_{\text{wd}}$ do
10. Apply the r.h.s. of DP Backup to $b_{\text{fwd}}$, $V_\beta^{\text{backward}}(b_{\text{fwd}}) = \max_{\alpha \in \mathcal{A}} \max_{o \in \mathcal{O}} \left\{ r_\beta(b_{\text{fwd}}) + \beta \sum_{o \in \mathcal{O}} Pr(o|b_{\text{fwd}}) \max_{\alpha \in \mathcal{A}} b_{o,\alpha}^{\beta} + \max_{\alpha \in \mathcal{A}} \mathcal{O}(o|s)b(o|s) \right\}$.
11. Note the maximizing action $\alpha^*$ and I-state $g^*$.
12. if $V_\beta^{\text{backward}}(b_{\text{fwd}}) > V_\beta(b_{\text{fwd}})$ then
13. Add new deterministic I-state $g_{\text{new}}$ such that $\omega(g_{\text{new}}|g^*, \alpha^*, o) = 1 \forall o \in \mathcal{O}$.
14. $N_{\text{added}} \leftarrow N_{\text{added}} + 1$.
15. if $N_{\text{added}} \geq N_{\text{new}}$ then
16. return
17. until $B = \emptyset$.

**Escaping Local Maxima by Adding I-States:** Eventually no I-state can be improved with further iterations, i.e., $\forall g \in G$, the corresponding LP yields an optimal value of $\epsilon = 0$.

**Theorem 2** ([39]): Policy Iteration has reached a local maximum if and only if $V_\gamma$ is tangent to the backed up value function for all $g \in G$.

In order to escape local maxima, the controller can add more I-states to its structure. Here the tangency criterion becomes useful. First, note that the dual variables corresponding to the Improvement Constraints in the LP provides the tangent belief state(s) when $\epsilon = 0$. At a local maximum, each of the $|G|$ linear programs yield some tangent belief states. Most implementations of LP solvers solve the dual variables simultaneously and so these tangent beliefs are readily available as a by-product of the optimization process introduced above.

Algorithm [1][39] uses the tangent beliefs to escape the local maximum.

**D. Bounded Policy Iteration for LTL Rewards**

This section shows how the bounded policy iteration methodology described in the previous section can be modified to solve the Conservative Optimization Criterion [21].

Algorithm [2] outlines the main steps in the bounded policy iteration for the Conservative Optimization Criterion. Again, there are two distinct parts of the policy iteration. First, policy evaluation in which $V_\beta$ is computed whenever some parameters of the controller changes (Steps 2, 10 and 18). The actual optimization algorithm to accomplish this step is found in Section VI-D1. Second, after evaluating the current value function, an improvement is carried out either by changing the parameters of existing nodes, or if no new parameters can improve any node, then a fixed number of nodes are added to escape the local maxima (Steps 14-17). This is described in Section VI-D3.

The two parts of policy improvement, namely the optimization to improve a given node, and addition of new nodes to escape local maxima are explained in detail in the subsequent sections.

1) **Node Improvement:** The first observation is that the search over $\kappa$ can be dropped. This simplification occurs because the initial node is chosen by computing the best valued node for the initial belief, i.e., $\kappa(g_{\text{init}}) = 1$, where $g_{\text{init}} = \text{argmax}_{g} (r_\gamma^{g_{\text{init}}})^{T} V_\beta^{g_{\text{init}}}$.

Once this initial node has been selected, the above objective differs from the typical discounted reward maximization problem due the presence of the new constraint $\eta_{av}^{g_{\text{init}}}(t) = 0$, which must be incorporated into the optimization algorithm.
Using Theorem 1, the above constraint can be rewritten as
\( \eta_{av}(x) = 0 \iff (\tilde{g}_{init})^T \tilde{g} = 0 \), where \( \tilde{g} \) uniquely solves the PE. This allows the node improvement to be written as a bilinear program. Again, one node \( g \) is improved at a time while holding all other nodes constant as follows.

**I-state Improvement Bilinear Program:**

\[
\begin{align*}
\max \quad & \epsilon, \omega(g', \alpha|g, o, \tilde{g}) \tilde{V}_{av} \\
\text{subject to} \quad & \text{Improvement Constraints:} \\
& V^{g}(\{s, g\}) + \epsilon \leq r^{g}(s) + \beta \sum (O(o)s) \\
& \times \omega(g', \alpha|g, o) T^{PM}_{\omega} (s'|s, \alpha) V^{g'}(\{s', g'\}), \forall s \\
\text{Poisson Equation (if } g \in G^{ss}:) \\
& \tilde{V}_{av} + \tilde{g} = \tilde{r}^{av} + T^{PM}_{\omega} \tilde{V}_{av} \\
& \tilde{g} = T^{PM}_{\omega} \tilde{g} \\
\text{Feasibility Constraints (if } g \in G^{ss}:) \\
& (\tilde{r}^{av}_{init})^T \tilde{g} = 0 \\
\text{FSC Structure Constraints (if } g \in G^{ss}:) \\
& \omega(g', \alpha|g, o) = 0 \iff g' \in G^{tr} \\
\text{Probability Constraints:} \\
& \sum_{g'} \omega(g', \alpha|g, o) = 1, \forall o, \\
& \omega(g', \alpha|g, o) \geq 0, \forall g', \alpha, o.
\end{align*}
\]

(30)

Note that a node in \( G^{tr} \) does not have to guarantee that product-POMDP states are not allowed to visit \( \text{Avoid}_{PM}^{\omega} \) and hence the extra Poisson and Feasibility Constraints that appear above need only be applied to I-state \( g \in G^{ss} \). Furthermore, the sFSC structure constraints ensure that once the execution has transitioned to steady state, the I-states in \( G^{tr} \) can no longer be visited.

The Poisson Constraints introduce bilinearity in the optimization. This is because the term \( T^{PM}_{\omega} \omega(\cdot|\cdot, \cdot) \), which is linear in \( \omega(g', \alpha|g, o) \), is multiplied by the unknowns \( \tilde{V}_{av} \) and \( \tilde{g} \) in the two sets of constraints that form the PE.

2) **Convex Relaxation of Bilinear Terms:** Bilinear problems are in general hard to solve unless they are equivalent to positive semidefinite or second order cone programs, which make the problem convex. Neither of these convexity assumptions hold for the bilinear constraints in (30). However, several convex relaxation schemes exist for bilinear problems. In this paper, we utilize a linear relaxation resulting from the Reformulation-Linearization Technique (RLT), which is summarized below, to obtain a possibly sub-optimal solution at each improvement step.

While RLT can be applied to a wide range of problems including discrete combinatorial problems, it is introduced here for the case of Quadratically Constrained Quadratic Problems (QCQPs) over unknowns \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \). The notation follows from [43]. A QCQP can be written as

\[
\begin{align*}
\max \quad & x^T Q_x x + a^T_x x + b^T y \\
\text{subject to} \quad & x^T Q_k x + a^T_k x + b^T y \leq c_k \quad \text{for } k = 1, 2, \ldots, p, \\
& l_x \leq x \leq u_x \quad \text{for } i = 1, 2, \ldots, n, \\
& l_y \leq y \leq u_y \quad \text{for } j = 1, 2, \ldots, m.
\end{align*}
\]

RLT is carried out as follows, for each \( x_i, x_j \) such that the product term \( x_i x_j \) is non zero in either the objective or the constraints, a new variable \( X_{ij} \) is introduced, which replaces the product \( x_i x_j \) in the problem. In addition, the bounds \( l_x, l_x, u_x, u_x, u_x, u_x \) are utilized to produce four new linear constraints

\[
\begin{align*}
X_{ij} - l_x - x_i - x_j & \geq -l_x \tilde{x}_x, \\
X_{ij} - u_x - x_i - x_j & \geq -u_x \tilde{x}_x, \\
X_{ij} - x_i - x_j - l_x & \leq -u_x \tilde{x}_x, \\
X_{ij} - x_i - x_j - u_x & \leq -l_x \tilde{x}_x.
\end{align*}
\]

The above constraints are the McCormick convex envelopes [44]. For bilinear programming with bounded variables, the McCormick convex envelopes are successively used in algorithms such as branch and bound [45] to successively obtain tighter relaxations to obtain globally optimal solutions. An efficient solver that incorporates this methodology is [46].

The bilinearity arises because the rows \( T^{PM}_{\omega} \omega(\cdot|\cdot, \cdot) \), which are linear terms of the unknowns \( \omega(\cdot|\cdot, \cdot) \), are multiplied by \( \tilde{V}_{av} \) and \( \tilde{g} \). The other rows of \( T^{PM}_{\omega} \) are not functions of the unknowns and their values are used from the values in the previous policy evaluation step. The total number of bilinear terms in both sets of equations is given by \( 2 \times |S| |O| |G| |Act| \). Moreover, applying the convex relaxation requires that all terms appearing in bilinear products must have finite bounds. For the unknowns \( \omega(\cdot|\cdot, \cdot) \), \( \tilde{g} \) and \( \tilde{V}_{av} \) these bounds are given by

\[
\begin{align*}
\tilde{0} \leq \omega(\cdot|\cdot, \cdot) \leq \tilde{1}, \\
\tilde{0} \leq \tilde{g} \leq \tilde{1}, \\
-M_1 \leq \tilde{V}_{av} \leq M_2,
\end{align*}
\]

where \( M_1, M_2 \) are large positive constants that are manually selected. This is because the feasibility set for \( \tilde{V}_{av} \) is dependent on the eigenvalues of \( I - T^{PM}_{mod, g, \omega} \), which is difficult to represent in terms of the optimization variables. During numerical implementation, this issue was not found to adversely affect the solution quality. This may be due to the fact that \( \tilde{V}_{av} \) does not appear in either the objective or in the feasibility constraints of (30). In fact, for the choice of \( \omega \) only constrains the value \( \tilde{g} \), whereas given \( \omega \) and \( \tilde{g} \) a feasible value of \( \tilde{V}_{av} \) can always be found.

3) **Addition of I-States to Escape Local Maxima:** When no I-state Improvement LP yields \( \epsilon > 0 \), a local maxima for the bounded policy iteration has been reached. The dual variables corresponding to the Improvement Constraints in (30) again give those belief states that are tangent to the backed up value function. The process for adding I-states involves forwarding the tangent beliefs one step and then checking if the value of those forward beliefs can be improved. However, an additional check for recurrence constraints has to be made.
Algorithm 3 Adding I-states to Escape Local Maxima of Conservative Optimization Criterion

Input: (a) Set \( G \) of tangent beliefs for each I-state. (b) A function \( node : B \to G \) identifying the I-state which yields each tangent belief. (c) \( N_{\text{new}} \) the maximum number of I-states to add.

1: \( N_{\text{added}} \leftarrow 0 \).
2: repeat
3: Pick \( b \in B \), \( B \leftarrow B \setminus \{b\} \), \( g \leftarrow node(b) \).
4: Compute the set of forward beliefs, \( F_{\text{wd}} \), as in Steps 4-10 of Algorithm 1.
5: for all \( b_{\text{fwd}} \in F_{\text{wd}} \) do
6: if \( g \in G^{tr} \) then
7: \( \text{candidates} \leftarrow G \times \text{Act} \).
8: else
9: \( \text{candidates} \leftarrow G^{ss} \times \text{Act} \).
10: \( \text{candidates} \leftarrow \text{PruneCandidates(candidates, } b_{\text{fwd}}, \hat{V}^{\text{av}}, \hat{g}) \) using Algorithm 4.
11: if \( \text{candidates} \leftarrow \emptyset \) then
12: Go to step 5.
13: Apply the r.h.s. of DP Backup to \( b_{\text{fwd}} \)
14: \( V_\beta^{\text{fwd}}(b_{\text{fwd}}) = \max_{\alpha \in \text{Act}} \{ r_\beta(b_{\text{fwd}}) + \beta \sum_{o \in \Omega} \Pr(o|b_{\text{fwd}})(b_{\text{fwd},o}(s)V_\beta^{\text{fwd}}(s)) \} \), where, \( b_{\text{fwd}} \) is computed for each product state \( s' \in S \) as follows
15: \( b_{\text{fwd},o}(s') = \sum_{s} T(s'|s, o) \sum_{o' \in \Omega} G(o'|b_{\text{fwd}}(s)) \).
16: Note the maximizing action \( \alpha^* \) and I-state \( g^* \).
17: if \( V_\beta^{\text{fwd}}(b_{\text{fwd}}) > V_\beta^{\text{fwd}}(b_{\text{fwd}}) \) then
18: Add new deterministic I-state \( g_{\text{new}} \) such that \( \omega(g_{\text{new}}) = g^*, \alpha^*, o \)\) = 1 \( \forall o \in \Omega \).
19: Assign \( g_{\text{new}} \) to correct sFSC partition as follows:
20: \( g_{\text{new}} \in G^{tr} \) if \( g \in G^{tr} \)
21: \( g_{\text{new}} \in G^{ss} \) otherwise.
22: \( N_{\text{added}} \leftarrow N_{\text{added}} + 1 \).
23: return
24: until \( B = \emptyset \).

if the involved I-state belongs to the \( G^{ss} \) states of the sFSC controller. In addition, if an I-state is added to the sFSC, it must also be assigned to either \( G^{tr} \) or \( G^{ss} \), because the next policy evaluation iteration depends on the I-state partitioning in the computation of \( T^{PM}_{\text{mod}} \). The procedure for adding I-states is provided in Algorithm 3. 

Algorithm 3 can be understood as follows. Assume that a tangent belief \( b \) exists for some I-state \( g \). Similar to Algorithm 1, instead of directly improving the value of the tangent belief, the algorithm tries to improve the value of forwarded beliefs reachable in one step from the tangent beliefs. This is given in Step 4 of Algorithm 3. Recall from Section VI-C that when a new I-state is added, its successor states are chosen from the existing I-states. A similar approach is used in Algorithm 3. However, a new node may be added to either \( G^{tr} \) or \( G^{ss} \) depending on the I-state that generated the original tangent belief. Recall that I-states in \( G^{ss} \) have two additional constraints. First, no state in \( G^{ss} \) can transition to any state in \( G^{tr} \). This is enforced by limiting the successor state candidates in Steps 6-9. Secondly, for improving a node in \( G^{ss} \), the allowed actions and transitions must satisfy the Poisson Constraints of (30). This further reduces or prunes the possible successor candidates in Step 10, which is elaborated as a separate procedure in Algorithm 4. The rest of the procedure is identical to Algorithm 1 except for Step 20, in which any newly added I-state is placed in the correct partition of \( G^{tr} \) or \( G^{ss} \).

Algorithm 4 Pruning candidate successor I-states and actions to satisfy recurrence constraints.

Input: Set of candidate successor states and actions \( \text{candidates} \subseteq G^{ss} \times \text{Act} \).

1: \text{for all } \( (g, \alpha) \in \text{candidates} \) do
2: Add new state \( g_{\text{phantom}} \) to \( G^{ss} \) to create a larger sFSC where \( \omega(g, \alpha|g_{\text{phantom}}, \alpha) = 1, \forall \alpha \in \Omega \).
3: Compute \( T^{PM}_{\text{mod}} \) and \( T^{PM}_{\text{init}} \) for the new larger global ssd Markov chain.
4: Solve PE for the new larger global Markov chain to obtain solutions \( \bar{g}, \bar{V}^{\text{av}} \).
5: if Any Feasibility Constraints in (30) are violated under the larger sFSC then
6: \( \text{candidate} \leftarrow \text{candidates} \setminus \{(g, \alpha)\} \).
7: return candidates

4) Finding an Initial Feasible Controller: So far, it has not been shown how an initial feasible controller may be found to begin the policy iteration. A feasible sFSC is one which produces at least one \( \omega \)-feasible recurrent set (Definition 19). This problem can be posed as a bilinear program, as well. Assume a size \( |G| \) and partitioning \( G = \{G^{tr}, G^{ss}\} \) of the sFSC has been chosen such that \( |G^{tr}| > 0 \) and \( |G^{ss}| > 0 \). Next, consider the PE for the ssd-global Markov chain, in which the states in \( Avoid^{PM}_{\text{mod}} \times G^{ss} \) are sinks. However, instead of the charge of the PE being \( r^{\text{avg}} \), consider the charge \( r^{\beta} \) in which the states in \( Repeat^{PM}_{\text{mod}} \times G^{ss} \) are rewarded. This is given by \( \bar{g}_{\text{feas}} = T^{PM}_{\text{mod}} \bar{g}_{\text{feas}} \) and \( \bar{V}^{\text{av}} + \bar{g}_{\text{feas}} = r^{\beta} + T^{PM}_{\text{mod}} \bar{V}^{\text{av}} \). Then, it can be shown that some state in \( Repeat^{PM}_{\text{mod}} \times G^{ss} \) is recurrent and can be reached from the initial distribution with positive probability if and only if \( \exists g \in G^{ss} \) such that \( (T^{PM}_{\text{mod}} \bar{g}_{\text{feas}, g} > 0 \). However, the constraint of never visiting the avoid states still applies. These procedures and constraints can be collected
together in the following bilinear maximization problem.

\[
\max_{\omega, V^{av}, V^{feas}, \bar{g}} \left( \epsilon^{PM^{av}}_{\text{init}} \right)^T \bar{g}_{\text{feas}, g}
\]
subject to

Poisson Equation 1:

\[
\bar{V}^{av} + \bar{g} = \pi^{av} + T^{PM^{av}} \bar{V}^{av} \]

Poisson Equation 2:

\[
\bar{V}^{feas} + \bar{g}_{\text{beta}} = \pi^{feas} + T^{PM^{feas}} \bar{V}^{feas} \]

Feasibility constraints (\( g \in G^{ss} \))

\[
(\epsilon^{PM^{av}}_{\text{init}})^T \bar{g} = 0
\]

FSC Structure Constraints:

\[
\omega(g', \alpha|g, o) = 0 \text{ if } g \in G^{tr} \text{ and } g \in G^{ss}
\]

Probability constraints:

\[
\sum_{g', \alpha} \omega(g', \alpha|g, o) = 1 \quad \forall o
\]

\[
\omega(g', \alpha|g, o) \geq 0 \quad \forall g', \alpha, o
\]

Any positive value of the objective \( (\epsilon^{PM^{av}}_{\text{init}})^T \bar{g}_{\text{feas}, g} \) gives a feasible controller, and therefore the optimization need not be carried out to optimality. If the problem is infeasible, then states in \( G^{ss} \) can be successively added to search for a positive objective.

VII. CASE STUDIES: ROBOT NAVIGATION

In this section, case studies for the bounded policy iteration algorithm described in Section VI-D are shown. The first example demonstrates the effectiveness of the algorithm to optimize the transient phase of the controlled system, while the second example illustrates the effectiveness in improving the steady state behavior of the controlled system. The case studies use a grid world system model, whose graphical representation is given in Figure 3.

A. Robot Navigation POMDP Set-Up

The world model is represented by an \( M \times N \) grid, with \( M = 7 \) fixed and varying \( N \geq 1 \). A robot can move from cell to cell. Thus, the state space is given by \( S = \{s_i|i = x + My, x \in \{0, \ldots, M - 1\}, y \in \{0, \ldots, N - 1\}\} \). The action set available to the robot is \( A = \{\text{Right}, \text{Left}, \text{Up}, \text{Down}, \text{Stop}\} \). The actions Right, Left, Up and Down, move the robot from its current cell to a neighboring cell, with some uncertainty. The state transition probabilities for various cell types (near a wall, or interior) are shown for action Right in Figure 3. Actions Left, Up, and Down have analogous definitions. For the deterministic action Stop, the robot stays in its current cell. Partial observability arises because the robot cannot precisely determine its cell location from measurements directly. The observation space is \( O = \{o_i|i = x + My, x \in \{0, \ldots, M - 1\}, y \in \{0, \ldots, N - 1\}\} \). In the robot’s actual cell position (dark blue), the sensed location has a distribution over the actual position and nearby cells (light blue). The robot starts in Cell 1(yellow): \( \epsilon_{\text{init}}(s_1) = 1 \). While the robot’s initial state is known exactly in this example, it is not required. Finally, there are three atomic propositions of interest in this grid world, giving \( AP = \{a, b, c\} \). In cell 0, only \( a \) is true, while, respectively, only \( b \) and \( c \) are true in cells 6 and 3.

B. Case Study I - Stability with Safety

LTL Specification: The LTL specification is given by
\[
\varphi_2 = \diamond \neg a \land \square \neg c,
\]
where \( b \) and \( c \), shown in Figure 3, are requirements for the robot to navigate to cell 6, and stay there, while avoiding cell 3, respectively.

Results: The difficulty in this specification is that the robot must localize itself to the top edge of the corridor before moving rightward to cell 6. Note that a random walk performed by the robot is feasible: there is a finite probability that actions chosen randomly will lead the robot to cell 6 without visiting cell 3. The sFSC used to seed the bounded policy iteration algorithm was chosen to have uniform distribution for I-state transitions and actions. Figure 4 shows the result of the bounded policy iteration in detail. It can be seen that the value of the initial belief increases monotonically with successive policy improvement steps, which includes both the optimization of \( \epsilon^{PM^{av}}_{\text{init}} \) and the addition of I-states to escape local maxima, as discussed in Section VI-D3.

C. Case Study II - Repeated Reachability with Safety

This case study illustrates how the Bounded Policy Iteration, especially the addition of I-states to the sFSC, improves the steady state behavior of the controlled system.

System Model and LTL specification: Let \( N = 3 \) and the LTL specification be given by
\[
\varphi_1 = \square \neg a \land \square \neg b \land \square \neg c.
\]

Results: For this example, the controller was seeded with a feasible sFSC of size \( |G| = 3 \), with \( |G^{ss}| = 2 \), using the method described in Section VI-D4. After the first few policy improvement steps, the initial I-state was found to be in \( G^{ss} \). By construction, once the sFSC transitions to an I-state in \( G^{ss} \) it can no longer visit states in \( G^{tr} \), when local maxima was encountered. Subsequently, all new I-states were assigned to \( G^{ss} \). The improvement in steady state behavior with the addition of each I-state is shown in Figure 5 where it can be seen that the expected frequency of visiting \( \text{Repeat}_{0}^{PM^{av}} \) steadily increases with the addition of I-states.

VIII. CONCLUSIONS

We proposed a methodology to synthesize sFSCs for POMDPs with LTL specification. We used the Poisson Equa-
tion and convex relaxations involving McCormick envelopes to relax a nonlinear optimization problem for designing sFSCs. The stochastic bounded policy iteration algorithm was adapted to the case in which certain states were required to be never visited. The key benefit of using this variant of dynamic programming was that it allowed for a controlled growth in the size of the sFSC, and could be treated as an anytime algorithm, where the performance of the controller improves with successive iterations, but can be stopped by the user based on time or memory considerations.

Future research will explore the extension of the proposed method to multi-agent POMDPs [47] and partially observable stochastic games [48, 49].

APPENDIX

Proof of Lemma 2 Consider a finite path fragment \( \pi = s_0 s_1 \ldots \) in each of the two Markov chains given by \( T^\varphi \) and \( T^\varphi_{mod} \) respectively. Consider the event of visiting a state in Avoid\( \varphi \) for the first time at the \( k \)-th time step. A path that satisfies this can be written as \( \pi_k = s_0 s_1 \ldots s_k \ldots \) such that \( s_0 \ldots s_{k-1} \notin \text{Avoid}\( \varphi \) and \( s_k \in \text{Avoid}\( \varphi \). Then, from the definition of the probability measure of cylinder sets in (1), the probability measures of the cylinder sets under the two Markov chains are identical: \( \Pr(M) Cyl_M(\pi_k) | s_k \varphi \) \( \Pr(M) Cyl_M(\pi_k) | s_k \varphi \). Next, note that the probability of paths visiting Avoid\( \varphi \) in the l.h.s. of the lemma is given by

\[
\Pr \left[ \pi \rightarrow (\text{Avoid}^\varphi_M) \times G \right] | s_k \varphi \] = \sum_{k=0}^{\infty} \Pr(M) Cyl_M(\pi_k) | s_k \varphi \] = \sum_{k=0}^{\infty} \Pr(M) Cyl_M(\pi_k) | s_k \varphi \] = \sum_{k=0}^{\infty} \Pr(M) Cyl_M(\pi_k) | s_k \varphi \].

In addition, since each state in Avoid\( \varphi \) is absorbing under \( T^\varphi_{mod} \) and has a reward 1 under the scheme of (13), for a given infinite path \( \pi \) of \( M_{mod} \), the long term average sum of rewards can be seen to be \( \text{Rew}(\pi) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} r^\varphi_t(s_t) \) for \( \pi \rightarrow \text{Avoid}^\varphi \) 1, otherwise.

This happens because if a path visits any state Avoid\( \varphi \) it forever remains in that state accumulating a reward of 1 at each time step. In the limit as time steps grow to infinity, the average reward per step converges to 1.

Finally, taking the expectation of the function \( \text{Rew}(\pi) \) gives

\[
\eta_{av}(r) = \mathbb{E}_{mod} \left[ \text{Rew}(\pi) \right] = 1 \cdot \Pr(M) \left[ \pi \rightarrow \text{Avoid}^\varphi_M \right] | s_k \varphi \] + 0 \cdot \Pr(M) \left[ \pi \rightarrow \text{Avoid}^\varphi_M \right] | s_k \varphi \] = \sum_{k=0}^{\infty} \Pr(M) Cyl_M(\pi_k) | s_k \varphi \].
