GENERIC INITIAL IDEALS AND FIBRE PRODUCTS

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ABSTRACT. We study the behavior of generic initial ideals with respect to fibre products. In our main result we determine the generic initial ideal of the fibre product with respect to the reverse lexicographic order. As an application we compute the symmetric algebraic shifted complex of two disjoint simplicial complexes as was conjectured by Kalai. This result is the symmetric analogue of a theorem of Nevo who determined the exterior algebraic shifted complex of two disjoint simplicial complexes as predicted by Kalai.

1. INTRODUCTION

Let $K$ be a field of characteristic 0, $S = K[x_1, \ldots, x_n]$ and $I$ be a graded ideal of $S$. Unless otherwise said, initial terms are taken with respect to the degree reverse lexicographic order, $\text{rlex}$ for short. The generic initial ideal $\text{gin}(I)$ of $I$ is an important object. It reflects many homological, geometrical and combinatorial properties of the associated rings and varieties. The ideal $\text{gin}(I)$ is defined as the ideal of initial terms of $\varphi(I)$ where $\varphi$ is a generic element of $\text{GL}_n(K)$. Here $\text{GL}_n(K)$ acts on $S$ as the group of graded $K$-algebra isomorphisms. See, e.g., Eisenbud [6] for the construction, in particular for the precise definition of “generic” element, and [7, 8] for results related to the generic initial ideal.

In practice, to compute $\text{gin}(I)$, one takes a random element $\varphi \in \text{GL}_n(K)$ and computes, with CoCoA [1], Macaulay 2 [12] or Singular [14], the ideal of leading terms of $\varphi(I)$. The resulting ideal is, with very high probability, $\text{gin}(I)$. From the theoretical point of view it is in general difficult to describe $\text{gin}(I)$ explicitly. Just to give an example, important conjectures, such as the Fröberg conjecture on Hilbert functions, would be solved if we could only say what is the $\text{gin}$ of a generic complete intersection.

Generic initial ideals can also be defined in the exterior algebra. In the exterior algebra context Kalai made in [11] conjectures about the behavior of generic initial ideals of squarefree monomial ideals with respect standard operations on simplicial complexes. Two of these operations have purely algebraic counterparts: they corresponds to the tensor product and the fibre product over $K$ of standard graded $K$-algebras. If $A = \bigoplus A_i$ and $B = \bigoplus B_i$ are standard graded $K$-algebras the tensor product $A \otimes_K B = \bigoplus (A_i \otimes_K B_j)$ is a (bi)graded $K$-algebra and the fibre product $A \circ B$ is, by definition, $A \otimes_K B / (A_1 \otimes B_1)$. Note that we have the short exact sequence

$$0 \to A \circ B \to A \oplus B \to K \to 0. \tag{1}$$

If we have presentations $A = K[x_1, \ldots, x_n]/I$ and $B = K[x_{n+1}, \ldots, x_{n+m}]/J$ then $A \otimes_K B$ has presentation $K[x_1, \ldots, x_{n+m}]/I + J$ and $A \circ B$ has presentation $K[x_1, \ldots, x_{n+m}]/F(I, J)$ where

$$F(I, J) = I + J + Q$$

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and \( Q = (x_i : 1 \leq i \leq n)(x_j : n + 1 \leq j \leq m) \).

Denote by \( \text{gin}_n(I) \) the gin of \( I \) in \( K[x_1, \ldots, x_n] \) with respect to rlex induced by \( x_1 > \cdots > x_n \). Similarly denote by \( \text{gin}_m(J) \) the gin of \( J \) in \( K[x_{n+1}, \ldots, x_{n+m}] \) with respect to rlex induced by \( x_{n+1} > \cdots > x_{n+m} \). Kalai’s conjectures \([11]\) correspond, in the symmetric algebra, to the following statements:

\[ \text{(2)} \]
\[ \text{gin}(I + J) = \text{gin}(\text{gin}_n(I) + \text{gin}_m(J)), \]

\[ \text{(3)} \]
\[ \text{gin}(F(I, J)) = \text{gin}(F(\text{gin}_n(I), \text{gin}_m(J))). \]

As observed by Nevo \([13]\), the exterior algebra counterpart of Equality (2) does not hold. Hence it is not surprising to discover that (2) does not hold in the symmetric algebra as well. Here is a simple example:

**Example 1.1.** Set \( n = 2, m = 3 \) and \( I = (x_1, x_2)^2 \) and \( J = (x_3^2, x_3x_4, x_4x_5) \). Then \( \text{gin}_2(I) = I \) since \( I \) is strongly stable, \( \text{gin}_3(J) = (x_3, x_4)^2 \) since \( J \) defines a Cohen-Macaulay ring of codimension 2. But \( \text{gin}(I + J) \) and \( \text{gin}(\text{gin}_n(I) + \text{gin}_m(J)) \) differ already in degree 2 since \( x_1x_4 \) belongs to the second, but not to the first ideal.

On the other hand, Nevo \([13]\) proved the exterior algebra analog of (3) for squarefree monomial ideals. Our main result asserts that (3) holds:

**Theorem 1.2.** Let \( I \subset K[x_1, \ldots, x_n] \) and \( J \subset K[x_{n+1}, \ldots, x_{n+m}] \) be graded ideals. Then we have \( \text{gin}(F(I, J)) = \text{gin}(F(\text{gin}_n(I), \text{gin}_m(J))) \).

Using the results in Section 3 it is also possible to compute explicitly \( \text{gin}(F(I, J)) \) in some cases. As an example we show that

\[ \text{gin}(Q) = (x_ix_j : i + j \leq n + m, i \leq j \text{ and } i \leq \min\{n, m\}) \]

and further that \( \text{gin}(Q^k) = \text{gin}(Q)^k \) for all \( k \in \mathbb{N} \). In Section 4 we present an application related to simplicial complexes. Let \( \Gamma \) be a simplicial complex on the vertex set \( \{x_1, \ldots, x_n\} \). We denote the symmetric algebraic shifted complex of \( \Gamma \) by \( \Delta^s(\Gamma) \). In general a shifting operation \( \Delta \) in the sense of Kalai \([11]\) replaces a complex \( \Gamma \) with a combinatorially simpler complex \( \Delta(\Gamma) \) that still detects some combinatorial properties of \( \Gamma \). Symmetric algebraic shifting \( \Delta^s \) is an example of such an operation and can be realized by using a kind of polarization of the generic initial ideal with respect to the reverse lexicographic order of the Stanley-Reisner ideal of \( \Gamma \) in the polynomial ring \( K[x_1, \ldots, x_n] \). See Section 4 for precise definitions and more details. For two simplicial complexes \( \Gamma_1 \) on the vertex set \( \{x_1, \ldots, x_n\} \) and \( \Gamma_2 \) on the vertex set \( \{x_{n+1}, \ldots, x_{n+m}\} \) we denote by \( \Gamma_1 \uplus \Gamma_2 \) the disjoint union of \( \Gamma_1 \) and \( \Gamma_2 \) on the vertex set \( \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\} \). In Section 4 we prove:

**Theorem 1.3.** We have \( \Delta^s(\Gamma_1 \uplus \Gamma_2) = \Delta^s(\Delta^s(\Gamma_1) \uplus \Delta^s(\Gamma_2)) \).

This result was conjectured by Kalai in \([11]\) and it was proved for exterior shifting by Nevo \([13]\) who noted also that his approach could be used to prove the statement for symmetric shifting. Indeed, our methods and strategy of proofs as presented in Section 2 and Section 3 follow the ideas of Nevo and can be seen as a symmetric version of his results in the exterior algebra.
2. Linear Algebra and Generic Initial Ideals

In this section we present some tools from linear algebra used in this paper. We follow ideas of Nevo [13] who treated the exterior case of the results which will be presented in this section.

Let $K$ be a field with char $K = 0$ and $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring. Given a monomial $x^a \in S_d$ we define a $K$-linear map $\tau_a : S \rightarrow S$ by

$$
\tau_a(x^b) = \begin{cases} 
  x^{b-a} & \text{if } a \preceq b, \\
  0 & \text{else}.
\end{cases}
$$

For a polynomial $g = \sum_{a \in \mathbb{N}^n} \lambda_a x^a \in S$ we extend this definition and consider the $K$-linear map $\tau_g = \sum_{a \in \mathbb{N}^n} \lambda_a \tau_a$. In the following we will use the rule

$$
\tau_{gh}(l) = \tau_g(\tau_h(l)) \text{ for homogeneous } g, h, l \in S
$$

which is easily verified.

Given a graded ideal $I \subset S$ and the reverse lexicographic order $<_{\text{rlex}}$ induced by $x_1 > \cdots > x_n$ on $S$, the algebra $S/I$ has a $K$-basis $\mathcal{B}(S/I)$ consisting of those monomial $x^a$ such that $x^a \not\in \text{in}(I)$ where $\text{in}(I)$ always denotes the initial ideal of $I$ with respect to rlex.

Induced by the inclusion of $\mathcal{B}(S/I)$ into $\mathcal{B}(S)$, we may consider $S/I$ as a graded $K$-vector subspace of $S$. (So we are not using the quotient structure of $S/I$ as an $S$-module in the following if not otherwise stated.) Given a homogeneous element $g \in S_d$ the map $\tau_g$ can be restricted to $S/I$. The main object we are studying are the various kernels

$$
\text{Ker}_{S/I}(\tau_g) = \text{Ker}(\tau_g : (S/I)_e \rightarrow (S/I)_{e-d}).
$$

We fix now a second generic $K$-basis $f_j = \sum_{i=1}^n a_{ij} x_i$ of $S_1$ where $A = (a_{ij}) \in K^{n \times n}$ is invertible and study the map $\tau_{f^a}$ where $f^a$ is a monomial in the new $K$-basis $f_1, \ldots, f_n$. Generic means here that with the automorphism $\varphi : S \rightarrow S, x_i \mapsto f_i$ we can compute the generic initial ideal $\text{gin}(I)$ with respect to rlex induced by $x_1 > \cdots > x_n$ of a given graded ideal $I \subset S$ via $\text{gin}(I) = \text{in}(\varphi^{-1}(I))$.

We need the following alternative way to decide whether a monomial belongs to a generic initial ideal or not. This criterion is well-known to specialists but we include it for the sake of completeness. For a (homogeneous) element $g \in S$ we denote by $\overline{g}$ the residue class in $S/I$ (induced by the quotient structure of $S/I$).

**Lemma 2.1.** Let $I \subset S$ be a graded ideal and $x^a \in S$ be a monomial. Then

$$
x^a \in \text{gin}(I) \iff \overline{f^a} \in \text{span}_K \{ \overline{f^b} : x^b <_{\text{rlex}} x^a, |b| = |a| \} \subset S/I.
$$

**Proof.** We have that

$$
x^a \in \text{gin}(I) = \text{in}(\varphi^{-1}(I))
\iff x^a - \sum_{x^b <_{\text{rlex}} x^a, |b| = |a|} \lambda_b x^b \in \varphi^{-1}(I) \text{ for some } \lambda_b \in K
\iff \overline{x^a} \in \text{span}_K \{ \overline{x^b} : x^b <_{\text{rlex}} x^a, |b| = |a| \} \subset S/\varphi^{-1}(I)
\iff \overline{f^a} \in \text{span}_K \{ \overline{f^b} : x^b <_{\text{rlex}} x^a, |b| = |a| \} \subset S/I.
$$

This concludes the proof. \qed
Let $\mathcal{B}(S/I) = \{ x^c : x^c \not\in \text{in}(I) \}$ be the $K$-basis of $S/I$ as discussed above. Similarly $\mathcal{B}(S/\text{gin}(I))$ is the natural $K$-basis of $S/\text{gin}(I)$, i.e. those monomials $x^c$ with $x^c \not\in \text{gin}(I)$.

We consider these monomials either as monomials in $S$ or in the corresponding residue class ring of $S$. The generic initial ideal $\text{gin}(I)$ can also be characterized via the kernels $\text{Ker}_{S/I}(\tau f^a)$. More precisely, we have:

**Proposition 2.2.** Let $d > 0$ be a positive integer, $x^a \in S_d$ be a monomial and $I \subset S$ be a graded ideal. Then

$$\dim_K \bigcap_{x^b <_{\text{rlex}} x^a, \ |b| = d} \text{Ker}_{S/I}(\tau f^b)_d = |\{ x^c \in \mathcal{B}(S/\text{gin}(I)) : |c| = d, \ x^a \leq_{\text{rlex}} x^c \}|.$$

In particular, $x^a \in \mathcal{B}(S/\text{gin}(I))$ if and only if

$$\dim_K \bigcap_{x^b <_{\text{rlex}} x^a, \ |b| = d} \text{Ker}_{S/I}(\tau f^b)_d > \dim_K \bigcap_{x^b \leq_{\text{rlex}} x^a, \ |b| = d} \text{Ker}_{S/I}(\tau f^b)_d.$$

**Proof.** Let $g \in (S/I)_d$ be a homogenous element expressed as a linear combination of the elements of $\mathcal{B}(S/I)$, i.e.

$$g = \sum_{x^c \in \mathcal{B}(S/I)} \lambda_c x^c \text{ for some } \lambda_c \in K.$$

Write $f^b$ for $b \in \mathbb{N}^n$ with $|b| = d$ as

$$f^b = \sum_{a' \in \mathbb{N}^n, \ |a'| = d} A_{ba'} x^{a'},$$

where $A_{ba'}$ is a polynomial in the $a_{ij}$ of the base change matrix $A = (a_{ij})$. We get that

$$\tau f^b(g) = \sum_{x^c \in \mathcal{B}(S/I)} A_{ba} \lambda_c.$$  

Hence

$$\dim_K \bigcap_{x^b <_{\text{rlex}} x^a, \ |b| = d} \text{Ker}_{S/I}(\tau f^b)_d$$

equals the dimension of the solution space $\mathcal{S}$ of the linear systems of equations $M_a z = 0$ with the matrix $M_a = (A_{bc})$ where $x^b <_{\text{rlex}} x^a$, $x^c \in \mathcal{B}(S/I)$ and $|b| = |c| = d$. It follows from Lemma [2.1] that

$$\text{rank } M_a = |\{ x^c \in \mathcal{B}(S/\text{gin}(I)) : |c| = d, \ x^c <_{\text{rlex}} x^a \}|.$$

We compute

$$\dim_K \bigcap_{x^b <_{\text{rlex}} x^a, \ |b| = d} \text{Ker}_{S/I}(\tau f^b)_d = \dim_K \text{Ker}(M_a : (S/I)_d \to (S/I)_d)$$

$$= \dim_K (S/I)_d - \text{rank } M_a$$

$$= \dim_K (S/\text{gin}(I))_d - \text{rank } M_a$$

$$= \{ x^c \in \mathcal{B}(S/\text{gin}(I)) : |c| = d, \ x^a \leq_{\text{rlex}} x^c \}$$

and this is what we wanted. □
For later applications we give a second characterization of the membership in the generic initial ideals which is similar to Proposition 2.2. At first we need the following Lemma:

**Lemma 2.3.** Let \( a \in \mathbb{N}^n \) and \( d = |a| < e \). Then

\[
\ker_{S/I}(\tau_{f^n})_e = \bigcap_{j=1}^n \ker_{S/I}(\tau_{f^{a+e_j}})_e
\]

**Proof.** Let \( g \in \ker_{S/I}(\tau_{f^n})_e \). Then for \( i = 1, \ldots, n \) we have that \( \tau_{f^{a+e_j}}(g) = \tau_{f^i}(\tau_{f^n}(g)) = 0 \). Thus \( g \in \bigcap_{j=1}^n \ker_{S/I}(\tau_{f^{a+e_j}})_e \).

On the other hand let \( g \in (S/I)_e \setminus \ker_{S/I}(\tau_{f^n})_e \). Then \( 0 \neq \tau_{f^n}(g) \in (S/I)_{e-d} \) is a nonzero element of degree \( > 0 \). Since the set \( \{ f^c : c \in \mathbb{N}^n, |c| = e - d \} \) is a \( K \)-basis for \( S_{e-d} \), there must exist an \( f^c \) such that \( \tau_{f^c}(\tau_{f^n}(g)) \neq 0 \). Choose an \( i \in \mathbb{N} \) with \( c_i > 0 \). Then

\[
0 \neq \tau_{f^c}(\tau_{f^n}(g)) = \tau_{f^{a+e_i}}(g) = \tau_{f^{a+e_i}}(\tau_{f^{a+e_j}}(g)) \neq 0
\]

and hence \( \tau_{f^{a+e_i}}(g) \neq 0 \). Thus \( g \notin \bigcap_{j=1}^n \ker_{S/I}(\tau_{f^{a+e_j}})_e \). This concludes the proof. \( \square \)

For a monomial \( 1 \neq x^b \in S \) with \( b \in \mathbb{N}^n \) we set \( \min(x^b) = \min(b) = \min\{ i : b_i > 0 \} \). Furthermore we define \( \min(1) = 1 \). Given a monomial \( x^d \in S_d \) we denote by

\[\text{Sh}(a) = \{ x^b \in S_{d+1} : x^b/x_{\min(b)} = x^a \}\]

the shadow of \( x^a \) by multiplying with \( x_i \) such that \( i \leq \min(a) \). Let

\[\max\text{Sh}(a) = \max \text{Sh}(a) \text{ and } \min\text{Sh}(a) = \min \text{Sh}(a)\]

be the maximal and minimal element of \( \text{Sh}(a) \) with respect to \( <_{\text{rlex}} \) on \( S \). Now we partition the generic initial ideal using the set \( \text{Sh}(a) \).

**Proposition 2.4.** Let \( d > 0 \), \( x^d \in S_d \) be a monomial and \( I \subset S \) be a graded ideal. Then

\[
|\text{Sh}(a) \cap \mathcal{B}(S/\text{gin}(I))| = \dim_K \bigcap_{x^b <_{\text{rlex}} x^a, |b| = d} \ker_{S/I}(\tau_{f^b})_{d+1} - \dim_K \bigcap_{x^b <_{\text{rlex}} x^a, |b| = d} \ker_{S/I}(\tau_{f^b})_{d+1}
\]

**Proof.** It follows from Lemma 2.3 that

\[
\dim_K \bigcap_{x^b <_{\text{rlex}} x^a, |b| = d} \ker_{S/I}(\tau_{f^b})_{d+1} = \dim_K \bigcap_{x^b <_{\text{rlex}} x^a, |b| = d} \bigcap_{j=1}^n \ker_{S/I}(\tau_{f^{a+e_j}})_{d+1}.
\]

Since \( \{ x^{b'} : x^{b'} <_{\text{rlex}} \min \text{Sh}(a) \} = \{ x^b x_i : x^b <_{\text{rlex}} x^a, 1 \leq i \leq n \} \), the latter dimension equals

\[
\dim_K \bigcap_{x^{b'} <_{\text{rlex}} \min \text{Sh}(a), |b'| = d+1} \ker_{S/I}(\tau_{f^{b'}})_{d+1}
\]

which is by Proposition 2.2 equal to

\[
|\{ x^c \in \mathcal{B}(S/\text{gin}(I)) : |c| = d + 1, \min \text{Sh}(a) <_{\text{rlex}} x^c \}|.
\]
Analogously we compute
\[
\dim_K \bigcap_{x^b \leq \text{rlex} x^a, |b| = d} \ker_{S/I}(\tau_f^b)_{d+1}
\]
\[= |\{ x^c \in \mathcal{B}(S/\gin(I)) : |c| = d + 1, \max \sh(a) < \text{rlex} x^c \}|. \]
Taking the difference in question gives rise to the desired formula. \hfill \Box

Given a graded ideal \( I \subset S \) and a monomial \( x^a \in S \) we define
\[
d_I(a) = |\text{Sh}(x^a/x_{\min(a)}) \cap \mathcal{B}(S/\gin(I))|.
\]
Using \( d_I(a) \) we can decide whether \( x^a \) belongs to \( \gin(I) \) or not. Recall that a monomial ideal \( I \subset S \) is called \textit{strongly stable}, if \( x_j x^a/x_i \in I \) for every \( x^a \in I \) such that \( x_j|x^a \) and \( j \leq i \). Since \( \text{char} K = 0 \) it is well-known that \( \gin(I) \) is strongly stable (see, e.g., Eisenbud [6]).

\textbf{Lemma 2.5.} Let \( I \subset S \) be a graded ideal and \( x^a \in S_d \) be a monomial with \( d > 1 \). Then
\[
x^a \notin \gin(I) \iff \min(x^a/x_{\min(a)}) - \min(x^a) + 1 \leq d_I(a).
\]

\textbf{Proof.} Let \( x^{a'} = x^a/x_{\min(a)} \). Then clearly \( \min(a') \geq \min(a) \) and \( x^a \in \text{Sh}(x^{a'}) \). If \( x^a \notin \gin(I) \), then \( x^b \notin \gin(I) \) for \( x^b = x_j x^{a'} \) and \( \min(a) \leq j \leq \min(a') \) because \( \gin(I) \) is strongly stable. Hence \( \min(x^a/x_{\min(a)}) - \min(x^{a'}) + 1 \leq d_I(a) \) because the considered monomials \( x^b \) are elements of \( \text{Sh}(x^{a'}) \). Similarly we see that \( \min(x^a/x_{\min(a)}) - \min(x^a) + 1 \leq d_I(a) \) implies that \( x^a \notin \gin(I) \). \hfill \Box

Since \( \gin(\gin(I)) = \gin(I) \) (see, e.g., [2] for a proof) it follows right from the definition that:

\textbf{Lemma 2.6.} Let \( I \subset S \) be a graded ideal and \( x^a \in S_d \) be a monomial with \( d > 1 \). Then
\[
d_I(a) = d_{\gin(I)}(a).
\]

3. \textsc{Fibre Products}

We recall some notation. Let \( \gin_n(I) \) be the generic initial ideal of a graded ideal \( I \subset K[x_1, \ldots, x_n] \) with respect to \( \text{rlex} \) induced by \( x_1 > \cdots > x_n \), let \( \gin_m(J) \) be the generic initial ideal of a graded ideal \( J \subset K[x_{n+1}, \ldots, x_{n+m}] \) with respect to \( \text{rlex} \) induced by \( x_{n+1} > \cdots > x_{n+m} \), and let \( \gin(L) \) be the generic initial ideal of a graded ideal \( L \subset S = K[x_1, x_2, \ldots, x_{n+m}] \) with respect to \( \text{rlex} \) induced by \( x_1 > \cdots > x_n > x_{n+1} > \cdots > x_{n+m} \). Furthermore we define the ideals \( m_n = (x_1, \ldots, x_n) \subset K[x_1, x_2, \ldots, x_n], m_m = (x_{n+1}, \ldots, x_{n+m}) \subset K[x_{n+1}, \ldots, x_{n+m}] \) and \( Q = m_n m_m \subset S \). We prove first a special version of our main result:

\textbf{Proposition 3.1.} Let \( I \subset K[x_1, \ldots, x_n] \) and \( J \subset K[x_{n+1}, \ldots, x_{n+m}] \) be graded ideals. Then we have the following formulas for ideals in \( S \):

(i) \( \gin(I + m_m) = \gin(\gin_n(I) + m_m) \).

(ii) \( \gin(J + m_n) = \gin(\gin_m(J) + m_n) \).
Proof. (i): Let \( f_1, \ldots, f_{n+m} \subset S_1 \) be a \( K \)-basis such that for the \( S \)-automorphism \( \varphi : S \to S, x_i \mapsto f_i \) we can compute \( \gin(I + m) \) and \( \gin(\gin_n(I) + m) \) by using \( \varphi^{-1} \). We denote the residue class of \( f_i \) in \( K[x_1, \ldots, x_n] = S/m \) by \( g_i \).

Let \( \psi : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n] \) be an \( K[x_1, \ldots, x_n] \)-automorphism such that the ideals \( \gin(I) \) and \( \gin(\gin_n(I)) \) can be computed by using \( \psi^{-1} \).

For a monomial \( x^a \in S \) we have that
\[
x^a \in \gin(I + m) \\
\iff \overline{f^a} \in \text{span}_K \{ \overline{f}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset S/(I + m)
\]
\[
\iff \overline{g^a} \in \text{span}_K \{ \overline{g}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset K[x_1, \ldots, x_n]/I
\]
\[
\iff \overline{\psi^{-1}(g^a)} \in \text{span}_K \{ \overline{\psi^{-1}(g)}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset K[x_1, \ldots, x_n]/\gin(I)
\]
\[
\iff \overline{\psi^{-1}(g^a)} \in \text{span}_K \{ \overline{\psi^{-1}(g)}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset K[x_1, \ldots, x_n]/\gin_n(I)
\]

Here the first equality follows from Lemma 2.1. The second one is trivial. The third one follows since \( S/\psi^{-1}(I) \) and \( K[x_1, \ldots, x_n]/\gin(I) = K[x_1, \ldots, x_n]/\in_{\text{rlex}}(\psi^{-1}(I)) \) share a common \( K \)-vector space basis. Similarly we compute
\[
x^a \in \gin(\gin_n(I) + m)
\]
\[
\iff \overline{f^a} \in \text{span}_K \{ \overline{f}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset S/(\gin_n(I) + m)
\]
\[
\iff \overline{g^a} \in \text{span}_K \{ \overline{g}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset K[x_1, \ldots, x_n]/\gin_n(I)
\]
\[
\iff \overline{\psi^{-1}(g^a)} \in \text{span}_K \{ \overline{\psi^{-1}(g)}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset K[x_1, \ldots, x_n]/\gin_n(I)
\]
\[
\iff \overline{\psi^{-1}(g^a)} \in \text{span}_K \{ \overline{\psi^{-1}(g)}^b : x^b \prec_{\text{rlex}} x^a, |b| = |a| \} \subset K[x_1, \ldots, x_n]/\gin_n(I)
\]

The last equality is new and it follows from the fact that \( \gin_n(\gin_n(I)) = \gin_n(I) \). Thus we see that
\[
\gin(I + m) = \gin(\gin_n(I) + m)
\]
and this shows (i). Analogously one proves (ii). \( \square \)

We need the following result which relates the numbers \( d_I(a) \) introduced in Section 2 of various ideals.

Proposition 3.2. Let \( I \subset K[x_1, \ldots, x_n] \) and \( J \subset K[x_{n+1}, \ldots, x_{n+m}] \) be graded ideals and \( x^a \in S_d \) be a monomial with \( d > 1 \). We have that
\[
d_{I+J+Q}(a) = d_{I+m_n}(a) + d_{J+m}(a).
\]

Proof. As a \( K \)-vector space we may see \( S/(I + m) \) and \( S/(J + m) \) as subspaces of \( S/(I + J + Q) \) using the standard \( K \)-bases \( \mathcal{B}(S/(I + m)), \mathcal{B}(S/(J + m)) \) and \( \mathcal{B}(S/(I + J + Q)) \) respectively. Using the equality \( (I + m) \cap (J + m) = I + J + Q \) it follows from a direct computation that the homomorphism of \( K \)-vector spaces
\[
\rho_d : S/(I + m)_d \oplus S/(J + m)_d \to S/(I + J + Q)_d, (g, h) \mapsto g + h,
\]
is surjective with trivial kernel for \( d > 0 \). Thus it is an isomorphism for \( d > 0 \).
Let $x'^d = x^d / x_{\min(a)}$. Then $|a'| = d - 1 > 0$. The following diagram

$$
\begin{array}{ccc}
S/(I + m_n)_d \oplus S/(J + m_n)_d & \xrightarrow{P_d} & S/(I + J + Q)_d \\
\oplus \tau_{f_b} & \downarrow & \oplus \tau_{f_b} \\
\oplus \tau_{f_b} & \downarrow & \oplus \tau_{f_b} \\
\oplus \tau_{f_b} & \downarrow & \oplus \tau_{f_b} \\
\end{array}
$$

is commutative where the direct sums are take over monomials of degree $d - 1$. It follows that

$$
dim_K \bigcap_{x^b \leq \text{rlx}^d, |b|=d-1} \ker_{S/(I+J+Q)}(\tau_{f_b})_d = \dim_K \bigcap_{x^b \leq \text{rlx}^d, |b|=d-1} \ker_{S/(I+m_n)}(\tau_{f_b})_d + \dim_K \bigcap_{x^b \leq \text{rlx}^d, |b|=d-1} \ker_{S/(J+m_n)}(\tau_{f_b})_d.
$$

Now Proposition 2.4 implies that

$$
d_{I+J+Q}(a) = \left| \text{Sh}(a') \cap \mathcal{B}(S/\text{gin}(I+J+Q)) \right| = \left| \text{Sh}(a') \cap \mathcal{B}(S/\text{gin}(I+m_n)) \right| + \left| \text{Sh}(a') \cap \mathcal{B}(S/\text{gin}(J+m_n)) \right| = d_{I+m_n}(a) + d_{J+m_n}(a).
$$

This concludes the proof.

We are ready to prove Theorem 1.2.

**Proof of 1.2** Let $x^a \in S_d$ be a monomial for an integer $d > 0$. If $d = 1$ then $\text{gin}(I+J+Q)_1 = \text{gin}(\text{gin}_n(I)+\text{gin}_m(J)+Q)_1$ for trivial reasons. Assume that $d > 1$. Applying Lemma 2.6, Proposition 5.1, and Proposition 5.2 we see that

$$
d_{I+J+Q}(a) = d_{I+m_n}(a) + d_{J+m_n}(a) = d_{\text{gin}(I+m_n)}(a) + d_{\text{gin}(J+m_n)}(a)
= d_{\text{gin}(I)+\text{gin}_m(J)+Q}(a).
$$

Thus it follows from Lemma 2.5 that

$$
x^a \not\in \text{gin}(I+J+Q) \Leftrightarrow x^a \not\in \text{gin}(\text{gin}_n(I)+\text{gin}_m(J)+Q).
$$

**Remark 3.3.** It is clear that rlx plays an important role in the arguments we have used to prove 1.2. For other term orders the statement of 1.2 fails to be true. For instance, for the lex order, $m = n = 3$ the ideals $I = (x^2_1,x_1x_2)$ and $J = (x^2_3,x_3x_6)$ do not fulfill 1.2.
In view of Theorem 1.2, the ideal \( \operatorname{gin}(F(I, J)) \) can be computed by first computing \( \operatorname{gin}_n(I), \operatorname{gin}_n(J) \) and then \( \operatorname{gin}(F(\operatorname{gin}_n(I), \operatorname{gin}_n(J))) \). One would like to describe the generators of the ideal \( \operatorname{gin}(F(I, J)) \) in terms of those of \( \operatorname{gin}_n(I) \) and \( \operatorname{gin}_n(J) \). The first step in this direction is the following:

**Proposition 3.4.** Let \( Q = (x_i : 1 \leq i \leq n)(x_j : n+1 \leq j \leq m) \subset S \). Then:

(i) \( \operatorname{gin}(Q) = (x_i x_j : i + j \leq n + m, \ i \leq j \) and \( i \leq \min\{n, m\}) \).

(ii) \( \operatorname{gin}(Q^k) = \operatorname{gin}(Q)^k \) for all \( k \).

**Proof.** Every ideal which is a product of ideals of linear forms has a linear resolution, see [4]. The \( \text{gin}\)-rlex of an ideal with linear resolution has linear resolution [6]. Hence we know that \( \operatorname{gin}(Q^k) \) is generated in degree \( 2k \) for all \( k \). Therefore it is enough to prove equality (i) in degree 2 and to prove equality (ii) in degree 2k. To check that (i) holds in degree 2, it is enough to prove the inclusion \( \supseteq \) since the two vector spaces involved have vector space dimension equal to \( mn \). It follows from Lemma 2.5 that for a monomial \( x_i x_j \) with \( i \leq j \) we have

\[
x_i x_j \in \operatorname{gin}(Q) \iff j - i + 1 > d_Q(\mathbf{e}_i + \mathbf{e}_j)
\]

where for \( k \in \{1, \ldots, n+m\} \) we denote by \( \mathbf{e}_k \) the \( k \)-th standard basis vector of \( \mathbb{Z}^{n+m} \). By Lemma 2.6 and Proposition 3.2 we know

\[
(4) \ d_Q(\mathbf{e}_i + \mathbf{e}_j) = d_m(\mathbf{e}_i + \mathbf{e}_j) + d_m(\mathbf{e}_i + \mathbf{e}_j) = d_{\operatorname{gin}(m)}(\mathbf{e}_i + \mathbf{e}_j) + d_{\operatorname{gin}(m)}(\mathbf{e}_i + \mathbf{e}_j)
\]

\[
(5) \ d_{(x_1, \ldots, x_m)}(\mathbf{e}_i + \mathbf{e}_j) + d_{(x_1, \ldots, x_n)}(\mathbf{e}_i + \mathbf{e}_j)
\]

where we used the obvious facts \( \operatorname{gin}(m) = (x_1, \ldots, x_m) \) and \( \operatorname{gin}(m_n) = (x_1, \ldots, x_n) \).

Thus we have to compute \( d_{(x_1, \ldots, x_m)}(\mathbf{e}_i + \mathbf{e}_j) \) and \( d_{(x_1, \ldots, x_n)}(\mathbf{e}_i + \mathbf{e}_j) \) respectively. We treat the case \( n \leq m \); if \( n > m \) one argues analogously. At first we consider

\[
1 \leq i, j \leq n + m \text{ with } i + j \leq n + m, \ i \leq j \text{ and } i \leq n.
\]

It follows from the definition that

\[
d_{(x_1, \ldots, x_m)}(\mathbf{e}_i + \mathbf{e}_j) = |\{x_i x_j : m + 1 \leq l, j \leq m + n, \ l \leq j\}|.
\]

This number is only nonzero if \( j \geq m + 1 \) and in this case we have \( d_{(x_1, \ldots, x_m)}(\mathbf{e}_i + \mathbf{e}_j) = j - m \). Similarly, we have that \( d_{(x_1, \ldots, x_n)}(\mathbf{e}_i + \mathbf{e}_j) \) is only nonzero if \( j \geq n + 1 \) and in this case we have \( d_{(x_1, \ldots, x_n)}(\mathbf{e}_i + \mathbf{e}_j) = j - n \). In any case it follow from Equation (4) that

\[
d_Q(\mathbf{e}_i + \mathbf{e}_j) \leq (j - n) + (j - m) \leq (m - i) + (j - m) < j - i + 1
\]

where we used that \( i + j \leq m + n \). Hence we get that \( x_i x_j \in \operatorname{gin}(Q) \) as desired. As explained above, this proves (i).

As observed above, to prove (ii) it is enough to prove the asserted equality in degree \( 2k \). Consider the \( K \)-algebra \( K[\varphi(Q_2)] \) generated by \( \varphi(Q_2) \) for a generic change of coordinates \( \varphi \). The assertion we have to prove is equivalent to the assertion that the initial algebra of \( K[\varphi(Q_2)] \) is generated as a \( K \)-algebra by the elements \( x_i x_j \) described in (i) which generate \( \operatorname{gin}(Q) \) as an ideal. We claim that the algebra relations among those \( x_i x_j \) are of degree 2. This can be easily seen by eliminating from the second Veronese ring of \( S \) using the know Gröbner bases [3] or [15]. Hence, by the Buchberger-like criterion for being a Sagbi basis [5], it suffices to prove that \( \operatorname{gin}(Q^2) = \operatorname{gin}(Q)^2 \) in degree 4. This can be done by
checking that the two vector spaces involved have the same dimension. The dimension of \( \text{gin}(Q^2)_4 \) is equal to that of \( Q^2_4 \) which is \((n+1)\binom{m+1}{2}\). It remains to compute the dimension of \( \text{gin}(Q)_2^2 \). Observe that a basis of \( \text{gin}(Q)_2^2 \) is given by the monomials \( x_1 x_j x_k x_l \) such that the following conditions hold:

\[
\begin{cases}
1 \leq i \leq j \leq h \leq k \leq m + n, \\
j \leq \min(n, m), \\
i + k \leq n + m, \\
j + h \leq n + m.
\end{cases}
\]

(6)

It would be nice to have a bijection showing that the number, call it \( W(n, m) \), of \((i, j, h, k)\) satisfying (6) is equal to \((n+1)\binom{m+1}{2}\). We content ourself with a non-bijective proof. Assuming that \( n \leq m \), a simple evaluation of the formula

\[
\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{h=j}^{m-n-j} \sum_{k=h}^{m-n-i} 1
\]

allows us to write \( W(n, m) \) as

(7)

\[
\sum_{i=1}^{n} \binom{n + m - 2i + 3}{3}
\]

plus

(8)

\[-\binom{m+2}{4} - \binom{n+2}{4} + \binom{m-n+2}{4}.
\]

The expression (7) can be evaluated, depending on whether \( m + n \) is odd or even, using the formulas:

(9)

\[
\sum_{i=0}^{p} \binom{2i + u}{3} = \begin{cases} 
\frac{p(p + 1)(2p^2 + 2p - 1)}{6} & \text{if } u = 1, \\
\frac{(p^2 - 1)p^2}{3} & \text{if } u = 0.
\end{cases}
\]

It follows that the expression of (7) is indeed equal to

(10)

\[
\frac{1}{6}n^2m + 1/6nm^3 + 1/6n^3 + 1/2nm^2 + 1/6nm - 1/6n
\]

That summing (10) with (8) one gets \((m+1)/2\binom{n+1}{2}\) is a straightforward computation. \(\square\)

**Remark 3.5.** (a) Refining the arguments [3,4], one can show that statement (i) and (ii) hold for every term order satisfying \( x_1 > \cdots > x_{m+n} \).

(b) The equality \( \text{gin}(Q^k) = \text{gin}(Q)^k \) is quite remarkable. We do not know other families of ideals satisfying it. It is difficult to guess a generalization of it. For instance that equality does not hold for a product of 3 or more ideals of linear forms, e.g. \((x_1, x_2)(x_3, x_4)(x_5, x_6)\). And it does not hold for the ideal defining the fibre product of 3 of more polynomial rings, e.g. \((x_1, x_2)(x_3, x_4) + (x_1, x_2)(x_5, x_6) + (x_3, x_4)(x_5, x_6)\).

(c) The statements (i) and (ii) of 3.4 can be rephrased as follows: The \((n,m)\)-th Segre product in generic coordinates has a toric degeneration to a ladder of the second Veronese ring of the polynomial ring in \( m + n - 1 \) variables and this holds independently of the term order.
4. Symmetric shifted simplicial complexes

Recall that $\Gamma$ is called a \textit{simplicial complex} on the vertex set $\{x_1, \ldots, x_n\}$ if $\Gamma$ is a subset of the power set of $\{x_1, \ldots, x_n\}$ which is closed under inclusion. The elements $F$ of $\Gamma$ are called \textit{faces}, and the maximal elements under inclusion are called \textit{facets}. If $F$ contains $d+1$ elements of $[n]$, then $F$ is called a \textit{$d$-dimensional face}, and we write $\dim F = d$. The empty set is a face of dimension $-1$. If $\Gamma \neq \emptyset$, then the \textit{dimension} $\dim \Gamma$ is the maximum of the dimensions of the faces of $\Gamma$. Let $f_i$ be the total number of $i$-dimensional faces of $\Gamma$ for $i = -1, \ldots, \dim \Gamma$. The vector $f(\Gamma) = (f_{-1}, \ldots, f_{\dim \Gamma})$ is called the \textit{$f$-vector} of $\Gamma$.

Let $\mathcal{C}(x_1, \ldots, x_n)$ be the set of simplicial complexes on $\{x_1, \ldots, x_n\}$. Following constructions of Kalai we define axiomatically the concept of \textit{algebraic shifting}. See [8] and [11] for surveys on this subject. We call a map $\Delta: \mathcal{C}(x_1, \ldots, x_n) \to \mathcal{C}(x_1, \ldots, x_n)$ an \textit{algebraic shifting operation} if the following is satisfied:

(S1) If $\Gamma \in \mathcal{C}(x_1, \ldots, x_n)$, then $\Delta(\Gamma)$ is a shifted complex, i.e. for all $F \in \Delta(\Gamma)$ and $i < j$ with $x_i \in F$ we have that $F \setminus \{x_j\} \cup \{x_i\} \in \Delta(\Gamma)$.

(S2) If $\Gamma \in \mathcal{C}(x_1, \ldots, x_n)$ is shifted, then $\Delta(\Gamma) = \Gamma$.

(S3) If $\Gamma \in \mathcal{C}(x_1, \ldots, x_n)$, then $f(\Gamma) = f(\Delta(\Gamma))$.

(S4) If $\Gamma', \Gamma \in \mathcal{C}(x_1, \ldots, x_n)$ and $\Gamma' \subseteq \Gamma$ is a subcomplex, then $\Delta(\Gamma') \subseteq \Delta(\Gamma)$ is a subcomplex.

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ indeterminates. We define the \textit{Stanley-Reisner ideal} $I_\Gamma = \left( \prod_{x_i \in F} x_i : F \subseteq \{x_1, \ldots, x_n\}, F \not\subseteq \Gamma \right)$ and the \textit{Stanley-Reisner ring} $K[\Gamma] = S/I_\Gamma$. Denote by $x_F$ the squarefree monomial $\prod_{x_i \in F} x_i$.

Note that a simplicial complex $\Gamma$ is shifted if and only if the Stanley-Reisner ideal $I_\Gamma \subseteq S = K[x_1, \ldots, x_n]$ is a \textit{squarefree strongly stable ideal} with respect to $x_1 > \cdots > x_n$, i.e. for all $F \subseteq \{x_1, \ldots, x_n\}$ with $x_F \in I_\Gamma$ and all $x_i \in F$, $j < i$, $x_j \not\in F$ we have that $(x_jx_F)/x_i \in I_\Gamma$.

Here we are interested in the following shifting operation introduced by Kalai. We follow the algebraic approach of Aramova, Herzog and Hibi. Assume that $K$ is a field of characteristic 0. We consider the following operation on monomial ideals of $S$. For a monomial $m = x_{i_1} \cdots x_{i_t}$ with $i_1 \leq i_2 \leq \cdots \leq i_t$ of $S$ we set $m^\sigma = x_{i_1}x_{i_2+1} \cdots x_{i_t+1}$. For a monomial ideal $I$ with unique minimal system of generators $G(I) = \{m_1, \ldots, m_s\}$ we set $I^\sigma = \langle m_1^\sigma, \ldots, m_s^\sigma \rangle$ in a suitable polynomial ring with sufficiently many variables.

Let $\Gamma$ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$ with Stanley-Reisner ideal $I_\Gamma$. The \textit{symmetric algebraic shifted complex} of $\Gamma$ is the unique simplicial complex $\Delta'(\Gamma)$ on the vertex set $\{x_1, \ldots, x_n\}$ such that $I_{\Delta'(\Gamma)} = \text{gin}(I_\Gamma)^\sigma \subseteq S$ where $\text{gin}(\cdot)$ is the generic initial ideal with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n$. It is not obvious that $\Delta'(\cdot)$ is indeed a shifting operation. The first difficulty is already that $I_{\Delta'(\Gamma)}$ is an ideal of $S$. This and the proofs of the other properties can be found in [8].

We want to give a combinatorial application of the results of Section 3. For two simplicial complexes $\Gamma_1$ on the vertex set $\{x_1, \ldots, x_n\}$ and $\Gamma_2$ on the vertex set $\{x_{n+1}, \ldots, x_{n+m}\}$ we denote by

$$\Gamma_1 \uplus \Gamma_2$$

the \textit{disjoint union} of $\Gamma_1$ and $\Gamma_2$ on the vertex set $\{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\}$. The following theorem was conjectured by Kalai in [11]. Observe that it was proved for exterior shifting by Nevo [13].
**Theorem 4.1.** Let \( \Gamma_1 \) be a simplicial complex on the vertex set \( \{x_1, \ldots, x_n\} \) and \( \Gamma_2 \) be a simplicial complex on the vertex set \( \{x_{n+1}, \ldots, x_{n+m}\} \). Then we have that
\[
\Delta^s(\Gamma_1 \cup \Gamma_2) = \Delta^s(\Delta^s(\Gamma_1) \cup \Delta^s(\Gamma_2)).
\]

**Proof.** The Stanley-Reisner ideal of \( \Gamma_1 \cup \Gamma_2 \) is \( I_{\Gamma_1 \cup \Gamma_2} = I_{\Gamma_1} + I_{\Gamma_2} + Q \). To determine the simplicial complex \( \Delta^s(\Gamma_1 \cup \Gamma_2) \) we compute
\[
I_{\Delta^s(\Gamma_1 \cup \Gamma_2)} = \text{gin}(I_{\Gamma_1} + I_{\Gamma_2} + Q)^\sigma
= \text{gin}(\text{gin}_n(I_{\Gamma_1}) + \text{gin}_m(I_{\Gamma_2}) + Q)^\sigma
= \text{gin}(\text{gin}_n(I_{\Gamma_1})^\sigma + \text{gin}_m(I_{\Gamma_2})^\sigma + Q)^\sigma
= \text{gin}(\text{gin}_n(I_{\Gamma_1})^\sigma + \text{gin}_m(I_{\Gamma_2})^\sigma + Q)^\sigma = I_{\Delta^s(\Gamma_1 \cup \Gamma_2)}.
\]
The first and fifth equality are the definitions of the corresponding ideals. The second and forth equality follow from [12]. The third equality follows from the proof in [8, Theorem 8.19]. \( \square \)

5. **Algebraic properties of fibre products**

It is easy to see how homological properties transfer from \( A \) and \( B \) to \( A \circ B \). The main point is that the short exact sequence (17) allows to compare how to control local cohomology. In this section we show that the componentwise linearity behaves well with respect to fibre products. Recall the following definition of Herzog-Hibi [9]. A graded ideal \( L \) in some polynomial ring over the field \( K \) is called componentwise linear if for all \( k \in \mathbb{N} \) the ideal \( L_{(k)} \) has a \( k \)-linear resolution (i.e. \( \text{Tor}_i(L_{(k)}, K) = 0 \) for \( j \neq k \)) where \( L_{(k)} \) is the ideal generated by all elements of degree \( k \) of \( L \).

In the proof we will use the following criterion of Herzog-Reiner-Welker [10, Theorem 3.2] to check whether an ideal is componentwise linear: Let \( L \) be a graded ideal in a polynomial ring \( T \). Then \( L \) is componentwise linear if and only if for all \( k \in \mathbb{Z} \) we have \( \text{reg} T/L_{\leq k} \leq k-1 \) where \( L_{\leq k} \) is the ideal generated by all elements of \( L \) of degree smaller or equal to \( k \) and \( \text{reg} M \) denotes the Castelnuovo-Mumford regularity of a f.g. graded \( T \)-module \( M \). Note that in [10] this result is only stated for monomial ideals, but the proof works also, more generally, for graded ideals. Recall that the Castelnuovo-Mumford regularity can be computed via local cohomology as \( \text{reg} M = \max \{ j \in \mathbb{Z} : H^i_m(M) \neq 0 \} \) for some \( i \) where \( m \) denotes also the graded maximal ideal of \( T \).

**Theorem 5.1.** Let \( I \subseteq K[x_1, \ldots, x_n] \) and \( J \subseteq K[x_{n+1}, \ldots, x_{n+m}] \) be graded ideals such that \( I \subseteq m_1^n \) and \( J \subseteq m_2^m \). Then \( I \) and \( J \) are componentwise linear if and only if \( F(I,J) \) is componentwise linear.

**Proof.** At first we note that \( F(I,J)_{\leq k} = 0 \) for \( k \leq 1 \), and for \( k \geq 2 \) we have
\[
F(I,J)_{\leq k} = (I + J + Q)_{\leq k} = I_{\leq k} + J_{\leq k} + Q = N(I_{\leq k}, J_{\leq k}).
\]
The short exact sequence (11) with the current notation is:
\[
0 \to S/F(I,J)_{\leq k} \to S/(I_{\leq k} + m_m) \oplus S/(J_{\leq k} + m_n) \to S/m \to 0.
\]
It induces the long exact sequence of local cohomology modules in degree \( j-i \)
\[
\cdots \to H^{i-1}_m(S/m)_{j-1-(i-1)} \to H^i_m(S/F(I,J)_{\leq k})_{j-i} \to H^{i+1}_m(S/m)_{j-(i+1)} \to \cdots
\]
→ \( H^i_m(S/(I_{≤k} + m^n)) j-i \oplus H^i_m(S/(J_{≤k} + m^n)) j-i \rightarrow H^i_m(S/m) j-i \rightarrow \ldots \)

Observe that for \( j ≥ 2 \) we have that \( H^i_m(S/m) j-1-/(i-1) = H^i_m(S/m) j-i = 0 \) because \( S/m \)
has regularity 0. The ring change isomorphism for local cohomology modules yields

\[
H^i_m(S/(I_{≤k} + m^n)) \cong H^i_{m_m}(K[x_1, \ldots, x_n]/I_{≤k})
\]

and

\[
H^i_m(S/(J_{≤k} + m^n)) \cong H^i_{m_m}(K[x_{n+1}, \ldots, x_{n+m}]/J_{≤k})
\]

as graded \( K \)-vector spaces. Thus we get for \( j ≥ 2 \) the isomorphisms

\[
H^i_m(S/F(I,J)_{≤k}) j-i \cong H^i_{m_m}(K[x_1, \ldots, x_n]/I_{≤k}) j-i \oplus H^i_{m_m}(K[x_{n+1}, \ldots, x_{n+m}]/J_{≤k}) j-i
\]

as \( K \)-vector spaces. Since the regularities of all modules in questions are greater or
equal to 2, we obtain that \( \text{reg} S/F(I,J)_{≤k} ≤ k \) if and only if \( \text{reg} K[x_1, \ldots, x_n]/I_{≤k} ≤ k \) and
\( \text{reg} K[x_{n+1}, \ldots, x_{n+m}]/J_{≤k} ≤ k \). As noted above, this concludes the proof.

\[\square\]

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