Geometry and measurement: specialized knowledge of future teachers within a pedagogical laboratory

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Abstract. Based on the need to train teachers for the teaching of school mathematics, it is proposed to carry out a pedagogical laboratory, which allows monitoring successful practices of teachers within the university in order to promote teaching and research processes within the courses. This document aims to analyze the teaching and learning practices of prospective teachers in a didactic course of geometric and metric thinking. In this way, under a qualitative research perspective, the development of the laboratory is analyzed as a process developed in five stages: identification of the population, study of documents, identification of the mathematical object, planning, implementation and reflection and systematization of the experience. This article contributes mainly to the representation of phenomena and proposal of models of physics and natural sciences. One of the main conclusions proposes training for the teaching of school mathematics with real experiences given by natural contexts where pedagogical practices are carried out. From these practices, the reflections made by the students reveal the contributions of adequate planning of the classes for the success in the development of this, in addition to the security that these practices generate when taking classes from non-own courses.

1. Introduction

In the training of prospective teachers of the Bachelor of mathematics at the Universidad del Norte, didactic and disciplinary aspects of mathematics are addressed, the basis for professional practice; when approaching aspects of planning, several questions are identified about how to put in one place everything learned in didactics of the mathematical objects studied [1]. One of the first aspects found in different works [2,3] correspond to the imbalance in studies at the undergraduate level compared to didactics, pedagogy and mathematics. In some cases, priority is given to the domain of mathematical formalism, and not in training for the teaching of school mathematics. That is, many graduates get a degree with a notable theoretical mastery of mathematics (university), but the impact of this on school mathematics is not seen. In this sense, the mathematics learned does not respond to the learning needs, as demonstrated in national standardized tests, where the satisfactory or advanced level is achieved only by 51% in third grade, 27% in fifth grade, and 24% in ninth grade [4].

National quality guidelines for the teaching of mathematics [5,6] suggest curricular paths that can be developed in basic and secondary education. However, it is not evident how all these elements converge in the reality of the regular classroom to guarantee that this knowledge produces reasoning and questioning that transcends the procedural aspects of the mathematical object, spending more time learning mathematical objects such as algebra and numerical operations, rather than the one dedicated to developing spatial or random thinking [7]. In this sense, this communication aims to analyze the teaching and learning practices of prospective teachers in a didactic course of geometric and metric
thinking, pondering that a new generation of professors from the Universidad del Norte will be closer to their own teaching experiences of school mathematics. For this, we will initially present the budgets of the mathematics teachers' specialized knowledge (MTSK) as a theoretical reference. Later, the stages in which this qualitative research has been developed will be detailed. We will place a strong emphasis on the recognition of lesson study as a professional development methodology in prospective teachers.

This paper aims to analyze the teaching and learning practices of prospective teachers in a didactic course of geometric and metric thinking, to this effect, it aims to respond to the hypothesis: mathematics undergraduates favor mathematical content over pedagogical and didactic content, so planning is not an easy process to develop during joint teaching practices. This proposal shows the need to analyze the pedagogical practices of mathematics students when they participate in a course on didactics of geometric and metric thinking under the structure of a pedagogical laboratory; based on this, a class episode is presented in which characteristics of specialized knowledge of prospective teachers are analyzed. Finally, the conclusions that allow the adoption of specific recommendations on the training of future generations of teachers are presented.

2. Theoretical framework
This research considers knowledge from the theoretical perspective of MTSK [8]. In addition, it constitutes a conceptualization that assumes the specialization of knowledge of the teacher who teaches mathematics as a central element of teaching knowledge, thus considering all knowledge of the teacher as specialized. Within this model, six subdomains are proposed, embedded in two large domains, namely, the pedagogical content knowledge (PCK) domain and the mathematical knowledge (MK) domain. The location of the domains aims to develop indicators that show how pedagogical, didactic and mathematical knowledge are put into play in professional or prospective teacher practices. In this way, the MK contains the subdomains: knowledge of topics (KoT), knowledge of the structure of mathematics (KSM) and the knowledges of practices in mathematics (KPM) [9]. For its part, the PCK domain contains the following subdomains: knowledge of mathematics teaching (KMT), knowledge of feature of learning mathematics (KFLM), and knowledge of mathematics learning standards (KMLS) [10]. At the center of the model are the beliefs and conceptions that permeate the sub-domains according to the contexts of teaching practices.

When developing investigations using this theoretical framework, they are frequently framed in the analysis of domains of this model. For this article, we will focus our eyes on the KMT subdomain, understanding as the main characteristics in the development of this domain the specialized knowledge that the prospective teacher has to make mathematical sense of the instructions for teaching, but also the way in which he/she does (mathematical) sense of students' responses. That is, this subdomain of common content knowledge (CCK) [11] is differentiated within pedagogical knowledge, not specialized in mathematics. For this case, students use reasoning for solving tasks. In this sense, teachers are familiar with the examples, strategies and representations used for each situation proposed to the students; From the above, teachers identify possible reasoning errors or unfinished students' reasoning. In this way, it is possible to define the space of examples, strategies and representations for solving problems that allows them to make sense not only for solutions similar to theirs but also to make sense of the responses, reasoning and strategies of the students [12].

3. Method
Based on the theoretical framework of the MTSK, the development of the course "didactics of geometric and metric thinking" is proposed for four students who are in the seventh semester of the Bachelor of mathematics or mathematics.

This course was framed under the methodological structure of lesson study, which is a methodology of Japanese origin that favors the joint learning of those who participate. Initially, it was proposed to improve math and science practices under the slogan of "teachers learning together" [13]. However, the teaching of teachers in our region and the profession of mathematics teachers differs from the conditions proposed for education professionals in Japan. In this sense, this study is projected as a “Glocal” type
proposal [14], in which it respects the global conditions initially proposed by Japanese researchers but adapted to the local vision of teacher training in Colombia, particularly of the Caribbean region. In this way, the proposal corresponds to a lesson study “Glocal” already proposed in previous studies in Brazil [15]. The model in Figure 1 allows us to plan, reflect, and analyze the teaching needs through the following execution phases:

(i) Identification of the prospective teacher population in the didactic course of geometric and metric thinking.
(ii) Study of the documents, in which quality reference documents in the teaching of mathematics in Colombia were presented, focused on the teaching of geometry and measurement at different levels of training.
(iii) Identification of the mathematical object, where the prospective teachers chose a topic from school mathematics to prepare for the development of a class.
(iv) Lesson planning, where a model instrument proposed by Acevedo 2018 is implemented.
(v) Implementation of the class, which was initially conceived in person with students in the classrooms and which, finally, was carried out virtually with the presentation of a video-paper in which the students led the class and made teaching suggestions during the video.
(vi) Reflection and systematization of the experience.

![Diagram](image)

**Figure 1.** Methodological structure of the pedagogical laboratory in teaching geometry with prospective teachers.

The data produced correspond to transcripts of the recordings, photographic records, the researcher's diaries, the 4 students plans, 2 online questionnaires (one initial and one ending the course), 1 video-paper of the presentation of the class or recording of the applied class without an external observer, and a virtual semi-structured interview. Finally, for this article, the analysis of the data produced in the use of systematization and the planning proposed by the four students will be presented, where they reveal the knowledge for the teaching of mathematics, based on the proposal of the expanded format of planning [16]. This case study seeks to understand the complexity of a case, as constituted by a mathematics undergraduate student facing planning practices in a geometry teaching laboratory [17]. For this paper, the results of the planning corresponding to the second stage of the pedagogical laboratory, for a student of the geometry didactics course, will be presented.
4. Results
Planning is the most time-consuming process during the lesson study stage because at this time concepts of didactics and pedagogy explained in the course converge, in addition to the mathematical concepts developed throughout the undergraduate training. For planning the topics were chosen voluntarily. In this way, the topics developed by the students were focused on the topics listed in Table 1.

Table 1. Groups into the course geometry and measurement didactics.

| Prospective teacher number | Planning title                                                                 | Goals-setting for student learning                                                                 | Mathematics Standard                                                                 |
|-----------------------------|--------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
|                             | Teaching of the area and perimeter of polygons by means of covering and Pick's Theorem. | Develop skills and knowledge about the area and perimeter of a plane figure using coatings and Pick's theorem in the Geoboard as an unconventional unit of measurement. | I select units, both conventional and standardized, appropriate for different measurements. |
|                             | The teaching of three-dimensional representations. | Develop the representation of three-dimensional objects from different points of view and positions using various types of representations. | I represent three-dimensional objects from different positions and views. |
|                             | The teaching of the surface coating by means of tessellations in 6th grade. | Use tessellations as a tool for teaching geometric figures, which allow the student to acquire knowledge that they can use in the contexts of daily life. | I make constructions and designs using 3D geometric figures and bodies and 2D geometric figures or drawings. |
|                             | The teaching of the Pythagorean Theorem. | Recognize the concepts used in the Pythagorean Theorem and apply it to situations in the mathematical, everyday and/or interdisciplinary context. | I recognize and contrast geometric properties and relationships used in the proof of basic theorems (Pythagoras). |

The lesson planning is considered as a collaborative plan, which involves all prospective teachers to participate from the initial investigation to the final process of reflection and systematization. The planning that they carried out in this course was classified as "complete", since they stated that, in other undergraduate courses, dedicated to didactics or pedagogy, little thought was given to the planning process. The lesson planning offered to students to prepare the mathematical object to be taught was recorded in a table proposed by the author in 2017, in which it begins with the presentation of the title of the proposal, the topic, the mathematical thinking to which it is directed, the minimum preliminary concepts that the student must have for the success of the planning, proposed time, general objective of the session, the curricular standards and Basic Learning Rights (BLR) to be developed during the session. Later, the activities are specified during 4 moments of the class: (i) action; (ii) formulation; (iii) validation; and (iv) institutionalization. For each one of these moments, the students had to think of one or more activities proportional to the time proposed for the entire activity (120 minutes). In Table 2, the objective was: make 2D representations using what was seen in the previous moment; each activity included a specific objective that contributed to the general objective of the session, the specific time of the activity, in addition to a general description of the activity, the description of the teacher's intervention through questions or guidelines, also, a space dedicated to the expected responses of the student and, finally, a space dedicated to the
responses to the actions of the teacher, or to the continuity with the following actions of the teacher, as described in the example in Table 2.

In this particular planning, student 2 was given the opportunity to think that not only does the teacher’s perspective exist, but that he/she thinks about the student and proposes some other questions that can be asked by the students during their intervention, that is, student response is proposed to each question. For the proposed activity of representation of concrete three-dimensional shapes, such as the soma cube, there are representations of multiple views (left side, right side and front) in which the shape can be deduced from the position in which the figure of colors to the left of the representation. According to the deduction of the parts in which this figure is composed, from the views, students can identify the composition of the figure 19 cubes (2 blue of 4 cubes, 1 green of 4 cubes, 1 yellow of 4 cubes and 1 red of 3 cubes). By viewing the student's responses, possible doubts are identified, but above all, the difficulties that a student would have in class to relate the minimum number of views that are necessary to describe a 3D figure. Besides this, the last question reveals something important in the construction of the views and it is the conservation of the dimensions of the two-dimensional representations. Surely, if the prospective teacher considered the use of color in the representations, it could make it easier to understand the positions that they are identifying and consider it in future planning’s since students often do not face the recognition of the representation of shapes.

Table 2. Lesson planning example from prospective teacher number 2 in formulation stage.

| Description of the activity | Teacher intervention (questions / guidance) | Student (responses / actions / reactions) | Teacher responses |
|-----------------------------|-------------------------------------------|------------------------------------------|------------------|
| Representation of solids: an example will be given for the entire course. | The multiple views of a figure on the board will be explained and, as with these, it is easy to see all the characteristics of the figure in question. For example: | Why is it three views? | Because with three views it is enough to see all the characteristics of the object. No, there are other ways to see them. Yes, it is necessary to maintain the proportions and measurements. |

The planning process went through the joint and collaborative discussion of the prospective teachers of the course during the course of ten classes, however, the version of the planning presented in this article corresponds to the sixth, in which the students were already in moments prior to the implementation of their plans. The first four planning sessions were limited to recognizing the planning format and restructuring the introduction to the topic to be taught, allowing them to make considerable
research on the topic. In particular, this contributed to show the KMT in the prospective teachers, because as they progressed in the investigations on the subject, they also inquired about the way in which this knowledge should be taught, analyzing between different proposals, some suggested by the teacher of the course (author), and others consulted in other media. In this way, the selection of teaching material and the proposal of questions to teach through explanation and participation of students by questions allowed us to identify that prospective teachers reveal manifestations of knowledge of the teaching of school mathematics.

5. Conclusions
Reality leads them to a chain of "trials and errors" that we can avoid from teacher training, making them project themselves on their working lives while experiencing mathematics in an environment that can be modified with experiences that can be carried later to the classroom. The call is, then, for training in the teaching of mathematics based on learning experiences to be a teacher in real settings.

In addition, becoming a professor as soon as they leave university, even during professional practice, can become a daunting task for those who start in the teaching profession. Particularly as a mathematics teacher, which shows the need to develop other skills during the undergraduate training that allows identifying a clear balance between the mathematical, didactic and pedagogical training of school mathematics. Consequently, to identify and convince themselves of having chosen the appropriate profession to their interests. This indicates that there is a latent need to strengthen the teacher's domain of knowledge of school mathematics, but without ignoring the pedagogical and didactic. That is, it is necessary to propose a constant initial training, so that prospective mathematics teachers can identify an adequate school mathematics, and necessary and sufficient knowledge to develop mathematical thinking in their students.

The requirements experienced must be transformed from the participation in the courses of the degree in mathematics. Therefore, it can be transferred to the training of future physics teachers, through the application of strategies that can improve teaching and learning practices, from class planning. In this way, Lesson Study is a model of teacher training that contemplates continuous improvement because developed in the long term, it has demonstrated a positive appreciation in the learning results of mathematics, as revealed by the Japanese proposals, now “glocalized” worldwide. Although conducting these studies is not frequent in Colombia, this training scheme could propose the betterment of teaching practices. Thus, Lesson Study constitutes an opportunity for future teachers as they learn to listen to others and, in turn, to collaborate towards the improvement of practices together.

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