A NEW PROOF OF THE GROMOV’S
THEOREM ON ALMOST FLAT MANIFOLDS

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(Dedicated to Jeff Cheeger’s 75th Birthday)

Abstract. We will give a new proof for the Gromov’s theorem on almost flat manifolds, which is an inductive proof on dimension.

The purpose of this paper is to give a new proof (by induction on dimension) for the well-known Gromov’s theorem on almost flat manifolds.

Theorem 1. ([Gr], [Ru]) Given $n \geq 2$, there are constants, $\epsilon(n), w(n) > 0$, such that if the sectional curvature and diameter of a compact $n$-manifold $M$ satisfies

$$|\sec_M| \cdot \text{diam}(M)^2 < \epsilon(n),$$

then $M$ is diffeomorphic to an infra-nilmanifold, $N/\Gamma$, where $N$ is a simply connected nilpotent Lie group, $\Gamma$ is a discrete subgroup of $N \rtimes \text{Aut}(N)$ (the group of automorphisms) such that the index $[\Gamma : \Gamma \cap N] \leq w(n)$.

A strong converse of Theorem 1 holds that any compact infra-nilmanifold admits one-parameter family of left invariant metrics (constructed via an inhomogeneous rescaling on a left invariant metric, determined by the infra-nil structure), $g_\epsilon$, such that $|\sec_{g_\epsilon}| \cdot \text{diam}(g_\epsilon)^2 \to 0$ as $\epsilon \to 0$ ([Gr], [FH]).

Theorem 1 has been a cornerstone in the collapsing theory of Cheeger-Fukaya-Gromov on collapsed manifolds with bounded sectional curvature ([CFG], [CG1,2], [Fu1-3]), which has important applications ([Fu4], [Ro1] and references within).

In [Gr], Gromov proved that a bounded normal covering of $M$ is diffeomorphic to a nilmanifold, and that $M$ is diffeomorphic to an infra-nilmanifold was due to [Ru] via constructing a flat connection with a parallel torsion on $M$.

Roughly speaking, Gromov’s proof in [Gr] imitates the proof of Bieberbach theorem on a compact flat $n$-manifold $M$, demonstrating that any deck transformation on $\mathbb{R}^n$ satisfies that its rotation component is either trivial or not small i.e., the minimal rotation angle is bounded below by a positive constant depending only on $n$. The core piece in Gromov’s proof is a complicated distortion estimate on iterated commutators of deck transformations of short geodesic loops on the Riemannian universal covering $\tilde{M}$ that can have small non-trivial holonomy. The key discovery

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of Gromov is that when normalizing the length of a short geodesic loop to one, its holonomy is actually much smaller.

Gromov’s proof involves several different fields and some unconventional arguments, which partially motivated the authors in [BK].

Our new proof of Theorem 1 is by induction on \( n \): we show that to conclude that an almost flat manifold has a bounded normal covering space that is diffeomorphic to a nilmanifold, it is enough to explore the central part of a nilmanifold (see Theorem 3), provided with a recent criterion of a nilmanifold (see Theorem 5). In particular, the induction enables us to bypasses the distortion estimate for iterated commutators of short geodesic loops ([Gr]); indeed our proof reduces the above distortion estimate to a single special commutator; one of the short geodesic loops has an almost minimal length i.e., the length is proportional to the injectivity radius of \( M \) (see (4.1)).

In our proof of Theorem 1, for simplicity we will always assume that an almost flat metric satisfies the following extra regularity (Theorem 2, also (4.2) below); an proof without the extra regularity can be carried with a little more work (cf. [Ro3]).

**Theorem 2.** (Smoothing) Let \((M, g)\) be a complete \( n \)-manifold with \(|\text{sec}_M| \leq 1\). For any \( \epsilon > 0 \), there is an \( \epsilon \) \( C^1 \)-close metric \( g_\epsilon \) of \((\epsilon, l)\)-regularity i.e.,

\[
|g - g_\epsilon|_{C^1(M)} < \epsilon, \quad |\text{sec}_{g_\epsilon}| < 1, \quad |\nabla_\epsilon^k \text{Rm}(g_\epsilon)| \leq c(n, k, \epsilon), \quad 0 \leq k \leq l,
\]

where \( \text{Rm} \) denotes the curvature tensor, and \( c(n, \epsilon, l) \) is a constant depending only on \( n, \epsilon \) and \( l \).

A proof of Theorem 2 via a Ricci flows can be found in [Shi], or via a local embedding method was given in [Ab].

The following theorem is the key in the proof of Theorem 1 via induction on \( n \).

**Theorem 3.** (Central geometry/topology of almost flat manifolds) There exists a constant \( \epsilon(n) > 0 \) such that if \( M \) is an almost flat \( n \)-manifold i.e., \(|\text{sec}_M| \cdot (\text{diam}(M))^2 < \epsilon \leq \epsilon(n)\), then the following properties hold:

(3.1) A finite normal covering space of \( M \) with order \( \leq a(n) \) admits a principal \( T^k \)-bundle, \( T^k \rightarrow \hat{M} \rightarrow \hat{M}/T^k \), and a \( C^1 \)-close \( T^k \)-invariant metric such that the quotient metric on \( \hat{M}/T^k \) is \( \epsilon' \)-almost flat, \( 0 < \epsilon' \leq \epsilon(n) \).

(3.2) Normalizing the metric to that \(|\text{sec}_{M}| \leq 1 \) (thus \( \text{diam}(M))^2 < \epsilon(n) \)), the injectivity radius of the Riemannian universal cover, \( \text{injrad}(\hat{M}) \geq \delta(n) > 0 \). If in addition, \( \pi_1(M) \) is nilpotent, then a short basis has almost constant displacement everywhere in a fundamental domain.

Remark 4. (4.1) The proof of (3.1) requires a distortion estimate only for a special commutator of two short geodesic loops; one of which has length proportional to \( \text{injrad}(M) \) by a constant (see (8.1)).

(4.2) The \( T^k \)-action on \( \hat{M} \) almost preserves the almost flat metric; averaging which under the \( T^k \)-action yields a desired invariant metric in (3.1) (here a \((\epsilon, l)\)-regularity in Theorem 2 is used, say \( l = 2 \)). We point it out that without Theorem 2, an inductive proof can still be carried out, with additional work ([Ro3]).

Theorem 3 enables us to prove Theorem 1 by induction on \( n \), with a help from the following topological criterion for a compact nilmanifold.
Theorem 5. (Nilmanifolds: Iterated principal circle bundles, [Na], [Be]) A compact n-manifold \( M \) is diffeomorphic to a nilmanifold if and only if \( M \) admits an iterated principal circle bundles,

\[
S^1 \to M \to M_1, \quad S^1 \to M_1 \to M_2, \quad \cdots, \quad S^1 \to M_n \to \text{pt}.
\]

By induction on \( n \), it is easy to show that a nilmanifold admits an iterated principal circle bundles, and conversely, that the fundamental group is nilpotent and the universal cover is diffeomorphic to \( \mathbb{R}^n \) (hence \( M \) is homeomorphic to a nilmanifold, [FH]). A diffeomorphism in Theorem 5 was mentioned in early literature without a proof (cf. [Be]). The author observed that the main result in [Na] implies Theorem 5, and an elementary proof can be found in [Be].

Let's first give a proof of Theorem 1 by assuming Theorem 3.

**Proof of Theorem 1.**

The proof is divided into two steps: Step 1. Prove that a bounded normal covering space of \( M \) is diffeomorphic to a nilmanifold. Step 2. Prove that \( M \) is diffeomorphic to an infra-nilmanifold.

Step 1. We proceed by induction on \( n \), starting with the trivial case: \( n = 2 \).

For \( n > 2 \), by Theorem 3 we conclude that \( M \) has a bounded normal covering space \( \hat{M} \) which admits a principal \( T^k \)-bundle \( (k \geq 1) \), \( T^k \to \hat{M} \xrightarrow{\pi} M/T^k = B \), and a \( C^1 \)-close \( T^k \)-invariant metric, \( g_{T^k} \), such that the quotient metric on \( B \) is almost flat. Without loss of generality, we may assume \( k < n \). Applying induction on \( B \), we may assume that \( B \) has a bounded normal covering space, \( \hat{B} : \hat{B} \to B \), and \( \hat{B} \) is diffeomorphic to a nilmanifold. Let \( T^k \to \hat{\pi}^*(\hat{M}) \to \hat{B} \) denote the \( \hat{\pi} \)-pullback principal \( T^k \)-bundle. Then \( M' := \hat{\pi}^*(\hat{M}) \) is a bounded normal covering space of \( M \).

By Theorem 5, \( \hat{B} \) admits iterated principal circle bundles, thus \( M' \) admits an iterated principal circle bundles. Again by Theorem 5, \( M' \) is diffeomorphic to a nilmanifold, \( N/\Gamma' \), where \( N \) is a simply connected nilpotent manifold which contains \( \Gamma' = \pi_1(M') \) as a co-compact discrete subgroup of \( N \).

Step 2. Based on the above Step 1, we assume a \( \Gamma' \)-conjugate diffeomorphism, \( \phi : (\hat{M}, \Gamma') \to (N, \Gamma) \). Because \( \Gamma \) is torsion free, by Malcev rigidity the homomorphism induced by conjugation, \( \Gamma \to \text{Aut}(\Gamma') \), embeds \( \Gamma \) into \( N \times \text{Aut}(\Gamma') \), hence \( N/\Gamma \) is an infra-nilmanifold. We shall extend \( \phi \), using the center of mass method, to a \( \Gamma \)-conjugate diffeomorphism, \( (\hat{M}, \Gamma) \to (N, \Gamma) \), thus \( M \) is diffeomorphic to \( N/\Gamma \).

By the above we may assume the following: identifying \( M' \) with \( N/\Gamma' \), \( M' \) admits two finite group \( \Gamma/\Gamma' \)-actions: one isometric free \( \Gamma/\Gamma' \)-action that almost preserves the nilpotent structure i.e., the iterated principal \( S^1 \)-bundles, while the other preserves the nilpotent structure. To apply the center of mass, it is enough to construct a left invariant metric on \( N/\Gamma' \) such that \( \Gamma/\Gamma' \) acts on \( N/\Gamma' \) isometrically and the two metrics are \( C^1 \)-close. A standard method is averaging the almost flat metric (of higher regularity) on \( \hat{M} \) by the nilpotent Lie group action to obtain a left invariant metric.

Based on (3.2) one may give a direct construction of a left invariant metric. Observe that the Gromov’s short basis at \( p' \in M' \), \( \{\gamma_j\} \), determines a basis on the Lie algebra \( h = T_{\pi'(p')}\hat{M}, \{v_k\} \), \( v_k = \gamma_k'(0) \) with \( |\gamma_k'(0)| = \text{length}(\gamma_k) \). We define a left-invariant metric on \( N \) by

\[
\bar{g}_N(v_k, v_l) = \bar{g}(v_k, v_l).
\]
By abusing notation, we use \( g_N \) to denote the averaging \( g_N \) by \( \Gamma/\Gamma' \). By (3.2), it is clear that restricting to any \( R \)-unit ball on \( N \) with respect to \( \tilde{g}_N \) or \( \tilde{g} \), the two metrics are GH-close to a flat metric, thus the two metrics are GH-close on \( B_R(\tilde{p}) \).

Using the two \( C^1 \)-close metrics on \( M' \), \( g' \) and \( g_N \), and identifying \( M \) with \( N \) by \( \phi \), we may assume two free isometric \( \Gamma \)-actions on \( M \), denoted by \( \mu_1 \) and \( \mu_2 \) such that \( \mu_1 \equiv \mu_2 \) on \( \Gamma' \), with respect to \( \tilde{g} \) and \( \tilde{g}_N \) respectively. To apply the method of the center of mass, we normalize \( \tilde{g} \) (and \( \tilde{g}_N \)) to that \( |\sec_{\tilde{g}}| \leq 1 \), thus \( \text{diam}(M')^2 < a(n)^2 \epsilon(n) < \frac{\delta(n)}{2} \), the convexity radius of \( \tilde{g} \) (see (3.2)).

Let \( \Gamma = \bigcup_{k=1}^s \alpha_k \Gamma' \), \( s \leq a(n) \). For fixed \( \alpha = \alpha_k, h \in N \), and any \( \gamma \in \Gamma' \),

\[
d(h, h) = d(\mu_2(\alpha)(\mu_1((\alpha \gamma)^{-1}, h), h) = d(\mu_2(\alpha)\mu_2(\gamma)(\mu_1(\alpha^{-1})h), h) \]

\[
= d(\mu_2(\alpha)\mu_1(\alpha^{-1})h, h).
\]

Then \( A(h) = \{ \mu_2(\alpha_k \gamma)^{-1}\mu_1(\alpha_k \gamma)(h), \gamma \in \Gamma' : 1 \leq k \leq s \} \) is a finite set of size \( |A(h)| \leq a(n) \). We now specify a representative \( \alpha_i \) as a projection of the identity \( e \) to \( \alpha_i \Gamma' \) (with respect to \( \tilde{g} \)). Because \( \text{diam}(M') < c(n) \epsilon(n) \) is small,

\[
\delta(n) > 10a(n) \max\{d(\mu_2(\alpha_k)\mu_1(\alpha_k^{-1})(h, h), 1 \leq k \leq s \},
\]

thus we can define a map, \( f : N \rightarrow N \), \( f(h) = c_h \), where \( c_h \) denotes the center of mass of \( A(h) \subset B_s(h) \) with respect to \( \tilde{g} \). Clearly, \( f \) is \( \Gamma \)-invariant, and it is straightforward to check that for any \( 0 \leq t \leq 1 \), the map, \( F(t, h) = c_h(t) : [0, 1] \times \tilde{N} \rightarrow N \), defines an isotopic between \( F(0, \cdot) = \text{id}_N \) and \( F(1, \cdot) = f \), where \( c_h(t) \) is the unique minimal geodesic from \( h \) to \( f(h) \) (cf. [GK]).

In the rest of the paper, our main effort is to prove Theorem 3. Because \( |\sec_M| \cdot \text{diam}(M)^2 \) is a scaling invariant, by a standard Gromov’s compactness argument Theorem 3 is equivalent to the following:

**Theorem 3’**. (Central geometry/topology of almost flat manifolds) Let a sequence of compact \( n \)-manifolds, \( M_i \rightarrow\text{GH} \) pt, such that \( |\sec_{M_i}| \leq 1 \). Passing to a subsequence if necessary, the following properties hold:

(3.1)’ \( M_i \) has a bounded normal covering space \( \tilde{M}_i \) which admits a principal \( T^k \)-bundle \( (k \geq 1) \) and a \( C^1 \)-close invariant metric so that the base manifolds, equipped with the quotient metrics, satisfy that \( \tilde{M}_i/T^k \rightarrow\text{GH} \) pt and \( |\sec_{\tilde{M}_i/T^k}| \leq c(n) \), a constant depends on \( n \).

(3.2)’ The injectivity radius of the Riemannian universal covering space of \( M_i \), \( \text{injrad}(\tilde{M}_i) \geq \delta(n) > 0 \). If in addition, \( \pi_1(M_i) \) is nilpotent, then a short basis at any \( p_i \in M_i \), the ratio of displacement, \( \frac{d(\gamma_i(\tilde{p}_i), c_i)}{d(\gamma_i(x_i), x_i)} \rightarrow 1 \), for all \( \tilde{x}_i \) in a fundamental domain at \( \tilde{p}_i \) in \( \tilde{M}_i \).

In our proof of (3.1)’, we blow-up twice on the sequence, \( M_i \), by \( \ell_i^{-1} \) and \( \rho_i^{-1} \), where \( \ell_i = \text{diam}(M_i) \) (the minimal collapsing rate) and \( \rho_i = \text{injrad}(M_i) \) (the maximal collapsing rate) respectively, and we investigate structures on the blow-up sequences by standard tools in metric Riemannian geometry (e.g., the equivariant GH-convergence, the Cheeger-Gromov convergence theorem, gluing of \( C^1 \)-close local fiber bundles via the method of the center of mass, etc). Structural results from the blow-up by \( \ell_i^{-1} \) will be used in exploring on a finite normal cover of \( \rho_i^{-1} M_i \), a local product structure of a \( T^k \)-bundle (see Example 10).
Because the proof is fairly involved, for the convenience of readers we first present an outline: let’s start with the following $\ell_i^{-1}$-blow up commutative equivariant GH-convergence,
\[
\begin{align*}
& (B_{\ell_i^{-1}}(0_i), 0_i, \Gamma_i) \xrightarrow{\text{eqGH}} (\mathbb{R}^n, 0, G) \\
& \exp_{p_i} \downarrow \quad \text{proj} \\
& (\ell_i^{-1} M_i, p_i) \xrightarrow{\text{GH}} X = \mathbb{R}^n / G,
\end{align*}
\]
where via path lifting $\Gamma_i = \pi_1(M_i, p_i)$ acts pseudolocally on $B_{\ell_i^{-1}}(0_i) \subset T_{p_i} M_i$. Because $\text{diam}(X) = 1$ (compact), using the generalized Bieberbach theorem (see Theorem 6 below), by the standard equivariant convergence it follows that a bounded normal covering space of $M_i$, $\ell_i^{-1} M_i \xrightarrow{\text{GH}} T^m$, a flat torus. Consequently, one constructs a fiber bundle, $N_i \to \ell_i^{-1} M_i \xrightarrow{f_i} T^m$, such that $f_i$ is an $\epsilon_i$-Gromov-Hausdorff approximation (briefly, GHA) and $\epsilon_i$-Riemannian submersion, and the second fundamental form $|\Pi(N_i)| \leq c(n)$ (with respect to the original metric), see Theorem 7 (a new proof in Appendix) and Lemma 8 below (cf. [Fu1], [CFG]).

Because an $f_i$-fiber is almost flat, we establish, by induction on $n$, the following properties (i large): i) $\tilde{M}_i$ is diffeomorphic to $\mathbb{R}^n$, and (3.2) holds. ii) any $x_i \in \tilde{M}_i$, $c(n)^{-1} \leq \text{injrad}(\tilde{M}_i)/\text{injrad}(x_i) \leq c(n)$. iii) Based on a holonomy distortion estimate for short geodesic loops whose lengths are proportional to $\rho_i$ (see Lemma 9), we can choose $\tilde{M}_i$ so that the pointed GH-limit, $(\rho_i^{-1} \tilde{M}_i, x_i) \xrightarrow{\text{GH}} (T^k \times \mathbb{R}^{n-k}, e \times 0)$, a product of flat $n$-manifolds, such that $k$ is independent of $x_i$, $\text{diam}(T^k) \leq d$ and $\text{injrad}(T^k) \geq d^{-1}$ ($d$ is a constant independent of $x_i$). Applying the Cheeger-Gromov’s convergence theorem ([Ch], [GLP]), we construct a locally finite open cover for $M_i$, $\{U_i, \alpha\}$, such that $\rho_i^{-1} U_{i, \alpha}$ is $C^3 \epsilon_i$-close to a product of flat manifolds, $T^k \times B_{10^{-k}}^n(0_\alpha)$.

Observe that a choice of a base point $e$ on the $T^k$-factor of $T^k \times B_{10^{-k}}^n(0_\alpha)$ and a canonical basis for $\pi_1(T^k, e)$ uniquely determines a group structure on $T^k$-fibers (via parallel translations), hence $T^k \times B_{10^{-k}}^n(0_\alpha)$ is a principal $T^k$-bundle. Moreover, a different choice of $e$ and a canonical basis for $\pi_1(T^k, e)$ yields an automorphism of $T^k$, thus an isomorphism of the resulting principal $T^k$-bundles. If $U_{i, \alpha} \cap U_{i, \beta} \neq \emptyset$, it is clear that the two trivial $T^k$-bundles are $C^1$-close, thus we are able to glue $\{U_{i, \alpha}\}$ to form a $T^k$-bundle, $T^k \to \tilde{M}_i \to B_i$, with an affine structural group (note that our gluing operation can be viewed as a simplest case of the construction of a $T$-structure on collapsed manifolds with bounded sectional curvature, [CG2]). Because $\pi_1(T^k)$ injects into the center of $\pi_1(M_i)$, this is a principal $T^k$-bundle.

We begin to fill in details in the above outline of proof of Theorem 3’.

**Theorem 6.** (Generalized Bieberbach Theorem, [FY]) Let $G$ be a closed subgroup of $\text{Isom}(\mathbb{R}^n)$ such that $\mathbb{R}^n / G \neq \{\text{pt}\}$ is compact. Then

(6.1) The identity component $G_0$ of $G$ satisfies that $\mathbb{R}^n / G_0$ with the quotient metric is isometric to $\mathbb{R}^m$, $m = n - \dim(G_0)$.

(6.2) $G$ contains a normal subgroup $\hat{G}$ of finite index such that $\mathbb{R}^n / \hat{G}$ is isometric to a flat torus $T^m$.

We present a short proof here, based on basic properties of a complete non-compact length space with non-negative curvature in the sense of Toponogov triangle comparison ([BGP]), which are analogous to the classical Soul theorem and
Splitting theorem of Cheeger-Gromoll in Riemannian geometry ([ChG1,2]).

Proof of Theorem 6.

(6.1) Arguing by contradiction, assume that $\mathbb{R}^n/G_0$ is not isometric to $\mathbb{R}^m$ $(1 \leq m < n)$. Then $\mathbb{R}^n/G_0$, as a complete non-compact Alexandrov space of non-negative curvature, splits, $\mathbb{R}^n/G_0 = \mathbb{R}^k \times X$ $(0 \leq k < m)$, and $X$ contains no line ([Mi]). Via horizontal lifting of lines in $\mathbb{R}^k$, $\mathbb{R}^n$ splits as, $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, such that each $\mathbb{R}^{n-k}$-factor is $G_0$-invariant. We claim that if $X = \mathbb{R}^{n-k}/G_0$ is not a point, then $X$ is not compact. Without loss of generality, we may assume that the $G_0$-orbits do not form a fiber bundle i.e., there is a non-principal $G_0$-orbit. If $p$ is a point with a local maximal isotropy group, then $G_0(p)$ is a subplane $\mathbb{R}^\ell$, $1 \leq \ell < n - k$. Without loss of the generality we may assume $p = 0$. Let $H = \{A \in SO(n), (A, b) \in G_0\}$, and $\bar{H}$ denote the closure of $H$ in $SO(n)$. Let $G_0(0) = \mathbb{R}^k$ $(k \geq 1)$, which includes the case that $\mathbb{R}^k \times SO(k) < G_0$, and let $\mathbb{R}^{n-k}$ denote the orthogonal complement of $\mathbb{R}^k$. Then for any $z \in S^{n-1}, G_0(z) \subset \mathbb{R}^k \times S^{n-k-1}$. Consequently for $G_0(z_i) \subset \mathbb{R}^k \times S^{n-k-1}_R, d(G_0(z_1), G_0(z_2)) \geq |R_1 - R_2|$, thus the $G_0$-orbit space is not compact.

Because $X$ is not compact and contains no line, $X$ has a soul $S$. Let $\gamma$ be a ray in $X$ from $p \in S$. Because for any $\alpha \in G/G_0$, $\alpha(p) \in P$ and $\alpha(\gamma)$ is a ray in $X$, $\mathbb{R}^n/G = \mathbb{R}^m/(G/G_0)$ contains a ray, a contradiction to that $\mathbb{R}^m/(G/G_0) = \mathbb{R}^n/G$ is compact.

(6.2) Because $\mathbb{R}^n/G$ is compact, $G/G_0$ is finitely generated. By (6.1) and 6.13 in [Ra], $G/G_0$ contains a torsion-free subgroup $\Gamma$ of finite index. Without loss of the generality, $\Gamma$ can be taken to be normal in $G/G_0$, thus $\mathbb{R}^m/\Gamma$ is a compact flat manifold. Now we apply the Bieberbach theorem to obtain a normal subgroup $\mathbb{Z}^m$ of $\Gamma$, with the desired normal subgroup of $G$ as the inverse image of the normalizer of $\mathbb{Z}^m$ in $G/G_0$. □

Theorem 7. (Fiber bundles, [Fu1], [CFG]) Given $n, d, \rho > 0$, there exist constants, $\epsilon(n, d, \rho), c(n) > 0$, such that if a compact $n$-manifold $M$ and a compact $m$-manifold $N$ satisfy the following conditions:

$$|\sec_{M}| \leq 1, \quad |\sec_{N}| \leq 1, \quad \text{diam}(M) \leq d, \quad \text{injrad}(N) \geq 2\rho,$$

and $d_{GH}(M, N) < \epsilon \leq \epsilon(n, d, \rho)$, then there is smooth fiber bundle map, $F \to M \overset{f}{\to} N$, satisfying the following properties:

(7.1) $f$ is an $\epsilon$-GHA.

(7.2) $f$ is an $\epsilon'$-Riemannian submersion i.e., for any vector $\xi$ orthogonal to $F$, $e^{-\epsilon'} \leq \frac{|df(\xi)|}{|\xi|} \leq e^{\epsilon'}$, where $\epsilon' = \Psi(\epsilon|n, d, \rho) \to 0$ as $\epsilon \to 0$.

(7.3) The second fundamental form of any fiber, $|\Pi(F)| \leq c(n)$.

Note that by (7.3), the intrinsic metric on $F$ is almost flat, and the structural group can be reduced to an affine group ([Fu1], [CFG]).

There have been various extension of Theorem 7 under weak curvature conditions; see [Ya] for $\sec_{M} \geq -1$, and see [Hu] and [Ro3] for $\text{Ric}_{M} \geq -(n - 1)$ and local covering space non-collapsed. In Appendix, we will present a self-contained elementary proof.

In the next two lemmas, we will establish properties that will be used in the proof of Theorem 3'.
Lemma 8. Let the assumption be as in Theorem 3'. Passing to a subsequence if necessary, the following properties hold:

(8.1) There is a bounded normal covering space of $M_i$, $\ell_i^{-1}\hat{M}_i \xrightarrow{GH} T^m$, a flat torus.

(8.2) For $i$ large, there is a smooth fiber bundle, $N_i \to \hat{M}_i \xrightarrow{f_i} \ell_i T^m$, satisfying (7.1)-(7.3) (with respect to the original metric).

(8.3) The universal cover of $\hat{M}_i$ is diffeomorphic to $\mathbb{R}^n$.

(8.4) The injectivity radius, $\text{injrad}(\hat{M}_i) \geq \delta(n) > 0$.

(8.5) For any $x_i \in \hat{M}_i$, $1 \leq \frac{\text{injrad}(x_i, \hat{M}_i)}{\text{injrad}(\hat{M}_i)} \leq b(n)$.

(8.6) For any $x_i \in \hat{M}_i$, $(\rho_i^{-1}\hat{M}_i, x_i) \xrightarrow{GH} (X, x)$. Then $X$ has a soul $F$ of dimension $k \geq 1$, $\text{diam}(F) \leq d$ and injrad($F$) $\geq d^{-1}$, where $k$ and $d$ are constants independent of $x_i$ (which may depend on the sequence $\{\hat{M}_i\}$).

(8.7) $\hat{\Gamma}_i = \pi_1(\hat{M}_i)$ contains a subgroup $\mathbb{Z}^k$, generated by short geodesic loops of length uniformly proportional to $\rho_i$, with normalizer of index, $[\hat{\Gamma}_i : N_{\hat{\Gamma}_i}(\mathbb{Z}^k)] \leq c(n)$.

Proof. (8.1) By the equivariant Gromov’s pre-compactness, passing to a subsequence we may assume the following commutative diagram (see the outline of the proof of Theorem 3.2'):

$$
\begin{array}{ccc}
(B_{\hat{\Gamma}_i}(0_i), 0_i, \hat{\Gamma}_i) & \xrightarrow{\text{eqGH}} & (\mathbb{R}^n, 0_i, G) \\
\exp_{\rho_i} \downarrow & & \downarrow \text{proj} \\
(\ell_i^{-1}\hat{M}_i, p_i) & \xrightarrow{GH} & X = \mathbb{R}^n/G.
\end{array}
$$

By Theorem 3.5, we may assume a normal subgroup of $G$ of finite index, $\hat{G}$, such that $\mathbb{R}^n/G = T^m$ and $G/\hat{G}$ is finite. Then $\Gamma_i$ contains a normal subgroup, $\hat{\Gamma}_i$, such that $\hat{\Gamma}_i \to \hat{G}$, $\hat{\Gamma}_i/\hat{\Gamma}_i \cong G/\hat{G}$, thus $\hat{M}_i = B_{\hat{\Gamma}_i}(0_i)/\hat{\Gamma}_i$ is a normal covering space of $M_i$ of order $|G/\hat{G}|$, and $\ell_i^{-1}\hat{M}_i \xrightarrow{GH} T^m$ (let $\{\alpha_j\}$ be a finite set of generators for $G/\hat{G}_0$, and let $\alpha_{i,j} \in \Gamma_i$ such that $\alpha_{i,j} \to \alpha_j$. Then $\hat{\Gamma}_i = \langle \Gamma_i(\epsilon), \alpha_{i,j} \rangle$, cf. [FY]).

(8.2) can be obtained from (8.1) and Theorem 7; note that $d_{GH}(\ell_i^{-1}\hat{M}_i, T^m) < \epsilon_i$ implies that $d_{GH}(\hat{M}_i, \ell_i T^m) = \ell_i \epsilon_i$, where $\sec_{\ell_i T^m} \equiv 0$ and $\text{injrad}(\ell_i T^m) = \ell_i \text{injrad}(T^m)$. Because

$$
\frac{d_{GH}(\hat{M}_i, \ell_i T^m)}{\text{injrad}(\ell_i T^m)} < \frac{\epsilon_i}{\ell_i} \frac{\ell_i}{\text{injrad}(T^m)} = \frac{\epsilon_i}{\text{injrad}(T^m)} \to 0,
$$

Theorem 7 applies to the pair, $(\hat{M}_i, \ell_i T^m)$, with fixed $i$ large, to obtain a bundle map, $f_i : \hat{M}_i \to \ell_i T^m$, which is the same obtain by rescaling by $\rho_i^{-1}$. To see that $|\Pi(N_i)| \leq c(n)$ with respect to the original metric, equivalently $|\Pi(N_i)| \leq c(n)\ell_i \to 0$ with respect to $\ell_i^{-1}\hat{M}_i$, we will assume the higher regularity of the original metric (Theorem 3) that guarantees that $b_{i,\beta,j}$ is $C^2,\alpha$-close to $b_j$ (see Remark A6).

In the proof of (8.3)-(8.7), we will proceed induction on $n$, using the fiber bundle structure obtained in (8.2).

(8.3) We proceed by induction on $n$, starting with the trivial case that $n = 2$. For $n > 2$, by (8.2) we obtain a smooth fiber bundle map, $N_i \to \hat{M}_i \xrightarrow{f_i} \ell_i T^m$ and $f_i$ satisfying (7.1) and (7.2). Let $\pi : \mathbb{R}^m \to \ell_i T^m$ denote the Riemannian universal
cover, and let $N_i \rightarrow \pi^*(\bar{M}_i) \overset{\bar{f}_i}{\rightarrow} \mathbb{R}^m$ denote the pullback bundle by $\pi$. Then $\pi^*(\bar{M}_i)$ is diffeomorphic to $N_i \times \mathbb{R}^m$, and $\pi^*(\bar{M}_i)$ is a covering space of $\bar{M}_i$. Hence $\bar{M}_i$ is diffeomorphic to $\tilde{N}_i \times \mathbb{R}^m$, where $\tilde{N}_i$ is a universal covering space of $N_i$. Because the intrinsic metric on $N_i$ satisfies that $N_i \overset{GH}{\rightarrow} \text{pt}$ such that $|\sec_{N_i}| \leq c(n)$, applying the induction on $N_i$ we conclude that $\tilde{N}_i$ is diffeomorphic to $\mathbb{R}^{n-m}$, thus $\bar{M}_i$ is diffeomorphic to $\mathbb{R}^n$.

(8.4) As seen from the proof of (8.3), $\bar{M}_i$ has a fiber bundle, $\tilde{N}_i \rightarrow \bar{M}_i \overset{\bar{f}_i}{\rightarrow} \mathbb{R}^m$, such that $\bar{f}_i$ also satisfies (7.2) and (7.3).

We proceed by induction on $n$, starting with $n = 2$, thus $m = 1$ or 2. For $m = 1$, the above implies that $\bar{M}_i$ is a $\mathbb{R}^1$-bundle over $\mathbb{R}^1$. If $\bar{M}_i$ has a short geodesic loop $\tilde{\gamma}_i$ at $\tilde{x}_i$, then $\tilde{\gamma}_i$ can be shortly homotopic to a trivial loop at $\tilde{x}_i$, a contradiction. If $m = 2$, then $\ell_i^{-1}\bar{M}_i$ is diffeomorphic and $C^{1,\alpha}$-close to a flat torus $T^2$, thus $g_i$ is bi-Lipschitz to the flat metric on $T^2$. By loop shortening operation on the projection of $\tilde{\gamma}_i$ in $\bar{M}_i$ with respect to the flat metric, one gets a non-trivial geodesic loop with respect to the flat metric, a contradiction because any geodesic loop on a flat $T^2$ is not homotopically trivial.

For $n > 2$, by applying induction on $N_i$, we may assume that the intrinsic metric on $\tilde{N}_i$ satisfies $\text{injrad}(\tilde{N}_i) \geq \delta_1(n)$. If $\bar{M}_i$ has a short geodesic loop $\tilde{\gamma}_i$ at $\tilde{x}_i$, by (7.2) and (7.3) one sees that via the radial contraction in a 1-tube around $\tilde{N}_i \ni \tilde{x}_i$, followed by a loop shortening process in $\tilde{N}_i$, one shortly deforms $\tilde{\gamma}_i$ to an intrinsic short geodesic loop in $\tilde{N}_i$, thus whose lifting in $\tilde{N}_i \subset \bar{M}_i$ is not a loop, a contradiction.

(8.5) Observe that (8.5) holds if it holds on a bounded normal covering space of $M_i$ (i.e., $M_i$ in (8.1)).

As a preparation, we first show that for any $x_i \in \bar{M}_i$, $1 \leq \frac{\text{injrad}(x_i, N_i)}{\text{injrad}(x_i, M_i)} \leq 2$.

By definition, $\text{injrad}(x_i, N_i) \geq \text{injrad}(x_i, \bar{M}_i)$. If $\gamma_i$ is a geodesic loop at $x_i$ with $|\gamma_i| = 2\text{injrad}(x_i, \bar{M}_i)$, then $\gamma_i$ is contained in a $r$-tubular neighborhood of $N_i$, $r < \text{diam}(M_i)$. By the radial deformation to $N_i$, followed by a path shortening one deforms $\gamma_i$ shortly to a $N_i$-geodesic loop at $x_i$ with almost the same length, because $|\Pi(N_i)| \leq c(n)$ and $r << 1$ (the length distortion is at most exponentially in terms of $c(n)r$).

We now proceed by induction, starting with a trivial case $n = 2$. As seen in (8.3) and (8.4), we consider the fiber bundle, $N_i \rightarrow \bar{M}_i \overset{\bar{f}_i}{\rightarrow} \ell_i T^m$. Applying induction on $N_i$, we assume that $N_i$ has a bounded normal covering space $\tilde{N}_i$ such that $1 \leq \frac{\text{injrad}(x_i, N_i)}{\text{injrad}(N_i)} \leq b_1(n)$. Consequently, $1 \leq \frac{\text{injrad}(x_i, N_i)}{\text{injrad}(N_i)} \leq b_2(n)$.

Observe that for any $p_i, x_i \in \bar{M}_i$, let $\alpha_i$ be a minimal geodesic from $f_i(p_i)$ to $f_i(x_i)$. Let $\tilde{\alpha}_i$ denote the horizontal lifting of $\alpha_i$ at $p_i$, $q_i = \alpha_i(1)$ and $x_i$ are in the same $N_i$-fiber. Because $|\Pi(N_i)| \leq c(n)$ and $d(f_i(p_i), f_i(x_i)) < \text{diam}(M_i) << 1$, similarly we have that $2^{-1} \leq \frac{\text{injrad}(p_i, \bar{M}_i)}{\text{injrad}(\tilde{\alpha}_i(1), M_i)} \leq 2$.

Assume that $\rho_i = \text{injrad}(p_i, \bar{M}_i)$. For any $x_i \in \bar{M}_i$, let $q_i = \tilde{\alpha}_i(1)$. Then

$$1 \leq \frac{\text{injrad}(x_i, \bar{M}_i)}{\text{injrad}(p_i, \bar{M}_i)} = \frac{\text{injrad}(x_i, \bar{M}_i)}{\text{injrad}(q_i, N_i)} \cdot \frac{\text{injrad}(q_i, N_i)}{\text{injrad}(p_i, \bar{M}_i)} \cdot \frac{\text{injrad}(p_i, N_i)}{\text{injrad}(q_i, N_i)} \leq 2^4 \frac{\text{injrad}(x_i, N_i)}{\text{injrad}(q_i, N_i)} \leq 2^4 b_2(n).$$
(8.6) Similar to (8.5), (8.6) holds if it holds on a sequence of bounded normal covering space of $M_i, \bar{M}_i$. Assume that $\rho_i = \text{injrad}(\rho_i, \bar{M}_i)$.

We proceed by induction on $n$, starting with the obvious case that $n = 2$. For $n > 2$, consider $N_i \to \bar{M}_i \xrightarrow{\ell_i} \ell_iT^m$ with (7.1) and (7.2). Without loss of generality, we assume that $\frac{\rho_i}{\ell_i} \to 0$ (otherwise $M_i$ is diffeomorphic to a flat manifold); so $(\rho_i^{-1}\bar{M}_i, x_i) \xrightarrow{GH} (X, x)$ with $X$ a non-compact flat $n$-manifold. Because $|\Pi(N_i)|\rho_i^{-1}\bar{M}_i \leq c(n)\rho_i \to 0, X = Y \times \mathbb{R}^m$ (Splitting theorem of Cheeger-Gromoll), and $(\rho_i^{-1}N_i, x_i) \xrightarrow{GH} (Y, x)$. By (8.5), it is clear that $Y$ has a non-trivial geodesic loop at $x$, thus a soul of $Y, F$, satisfies that $d(x, F) < \infty$ and dim$(F) \geq 1$. Applying induction on $N_i$, we obtain that diam$(F) \leq d$ and injrad$(F) \geq d^{-1}$.

(8.7) In the proof of (8.6) we obtain that $N_i \to \bar{M}_i \to \ell_iT^m$, $(\rho_i^{-1}\bar{M}_i, x_i) \xrightarrow{GH} (Y \times \mathbb{R}^m, x)$ and $(\rho_i^{-1}N_i, x_i) \xrightarrow{GH} (Y, x)$, $Y$ has a soul $F$, a compact flat submanifold. By Cheeger-Gromov $C^{1,\alpha}$-convergence, we may assume that $F_i \subset N_i$ and $B_{d,F,x}(F_i)$ is $C^{1,\alpha}$-close to $B_{d,F,x}(F)$ (in particular, $F_i$ is $C^{1,\alpha}$-close to $F$). By Bieberbach theorem, $\pi_1(F_i) \cong \pi_1(F)$ contains a normal subgroup, $H_i \cong \mathbb{Z}^k$, with $[\pi_1(F_i) : H_i] = a \leq a(k)$. We claim that $\pi_1(F_i)$ is normal in $\hat{\Gamma}_i$, assuming which we first prove (8.6.7).

Let $A_i$ denote the conjugate class of $H_i$ in $\hat{\Gamma}_i$, and let $S(A_i)$ be the group of permutations on $A_i$. Let $\phi: \hat{\Gamma}_i \to S(A_i)$ denote the homomorphism induced by conjugation. Then $N_i\hat{\Gamma}_i(H_i) \supset \ker \phi$, thus $[\hat{\Gamma}_i : N_i\hat{\Gamma}_i(H_i)] \leq [\hat{\Gamma}_i : \ker \phi] = |\text{im}(\phi)| \leq |S(A_i)| \leq |A_i|!$.

We now estimate $|A_i|$. Because $\pi_1(F_i)$ is normal in $\hat{\Gamma}_i$, $A_i$ is a subset of all $\mathbb{Z}^k$-subgroups of $\pi_1(F_i)$. Let $\psi: \pi_1(F_i) \to \pi_1(F_i)/H_i$. Observe the following properties: (i) For any two subgroups of $H_1, H_2 < \pi_1(F_i)$, $H_1 = H_2$ if and only if $H_1 \cap H_i = H_2 \cap H_i$ and $\psi(H_1) = \psi(H_2)$. (ii) $[H_i : H_1 \cap H_i] \leq [\pi_1(F_i) : H_1]$. The two properties imply that $\pi_1(F_i)$ contains at most $c(a)$ many possible $\mathbb{Z}^k$-subgroups of index $a$, thus $|A_i| \leq c(a) \leq c(a(k))$, where $c(a)$ is a constant depending on $a$.

To see the claim, observe that the Gromov’s short generators of $\hat{\Gamma}_i = \pi_1(M_i)$ at $x_i$ are divided into two groups, one that their lengths are uniformly proportional to $\rho_i$ i.e., the short generators, $\alpha_i \in \pi_1(F_i)$, and $\gamma_i \in \hat{\Gamma}_i - \pi_1(F_i)$ such that if $\hat{\ell}_i = \min\{\gamma_i\}, \gamma_i \in \hat{\Gamma}_i - \pi_1(F_i)$, then $\frac{\rho_i}{\ell_i} \to 0$ as $i \to \infty$. Because $|\Pi(N_i)| \leq c(n)$, each $\gamma_i$-conjugation on $\alpha_i$ has a bounded distortion, the $\gamma_i$-conjugation preserves $\pi_1(F_i)$. \hfill \Box

**Lemma 9.** (Local product $T^k$-bundles) Let the assumptions be as in Lemma 8.

(9.1) Let $\mathbb{Z}^k$ be as in (8.7). Then the centerizer of $\mathbb{Z}^k, [\Gamma_i : C_{\Gamma_i}(\mathbb{Z}^k)] \leq c(n)$.

(9.2) For any $\delta > 0$, there is $N(\delta) > 0$, such that for $i \geq N(\delta), x_i \in \bar{M}_i = M_i/C_{\bar{M}_i}(\mathbb{Z}^k)$,

$$d_{GH}(B_{30}(x_i, (d\rho_i)^{-1}\bar{M}_i), B_{30}(T^k \times \mathbb{R}^{n-k}, e \times 0)) < \delta,$$

and $k, \text{diam}(T^k) \leq 1$ and injrad$(T^k) \geq d^{-2}$ hold for all $x_i$.

Note that (9.2) is required to construct a principal $T^k$-bundle on $\bar{M}_i/C_{\bar{M}_i}(\mathbb{Z}^k)$. Note that in general a complete flat manifold $X$ with a soul flat $T^k$ may not isometric to the metric product, $T^k \times \mathbb{R}^{n-k}$ (see Example 10).

**Proof of Lemma 9.**
(9.1) It suffices to prove (9.1) on $\hat{M}_i$; recall that by (8.7) we obtain that $N_{\hat{\Gamma}_i}(\mathbb{Z}^k)$ has a bounded index in $\hat{\Gamma}_i$. Fixing a canonical basis for $\mathbb{Z}^k$, $\gamma_i, 1, ..., \gamma_i, k$ (consisting of closed geodesics in $T^k$ at $x_i$), consider the holonomy representation induced by conjugation, $\psi_i : N_{\hat{\Gamma}_i}(\mathbb{Z}^k) \to \text{Aut}(\mathbb{Z}^k)$. We claim that the $\psi_i$-image is bounded (for all large $i$), thus $\ker(\psi_i) = C_{\Gamma_i}(\mathbb{Z}^k)$ has index in $\hat{\Gamma}_i$, $|\text{Im}(\psi_i)|$.

As a preparation, we first show that for any $\eta_i \in \hat{\Gamma}_i$ with $|\eta_i| \leq 1$,

$$e^{-c(n)} \leq \frac{d(\eta_i^{-1}\gamma_i, \eta_i(x_i), \tilde{x}_i)}{d(\gamma_i, \tilde{x}_i)} \leq e^{c(n)}.$$  

We proceed by induction on $n$, starting with the trivial case $n = 2$. For $n > 2$, we have $N_i \to \hat{M}_i \xrightarrow{f_i} \ell_i T^m$. Applying induction on $N_i$, we assume that for $\alpha_i \in \Lambda_i$ with $|\alpha_i| \leq 1$, the above distortion estimate holds. Let $\alpha_i \beta_i$ denote a horizontal lifting of element from $\pi_1(T^m)$ with $|\alpha_i \beta_i| \leq 1$. Then

$$e^{-c(n)} \leq \frac{d((\alpha_i \beta_i)^{-1}\gamma_i, (\alpha_i \beta_i)(x_i), \tilde{x}_i)}{d(\gamma_i, \tilde{x}_i)} = \frac{d((\alpha_i^{-1}\gamma_i, (\alpha_i \beta_i)(x_i), \beta_i(\tilde{x}_i)))}{d(\gamma_i, \beta_i(\tilde{x}_i))} \cdot \frac{d(\gamma_i, \beta_i(\tilde{x}_i))}{d(\gamma_i, \tilde{x}_i)} \leq e^{c(n)},$$

because the distortion of $\alpha_i^{-1}\gamma_i \alpha_i$ (resp. $\gamma_i$) applies to every point in $\hat{N}_i$ (see (8.5)), which contains $\tilde{x}_i$ and $\beta_i(\tilde{x}_i)$.

Let $\{\alpha_{i,j}\}_{j=1}^{a(n)}$ denote a Gromov’s short generators for $N_{\hat{\Gamma}_i}(\mathbb{Z}^k)$ at $\tilde{x}_i$. Then $S_i = \{\psi_i(\alpha_{i,j})\}_{j=1}^{a(n)}$ is a symmetric set of generators for $\psi_i(N_{\hat{\Gamma}_i}(\mathbb{Z}^k))$, equipped with the word length in terms of $S_i$. Because $|\alpha_{i,j}| \to 0$ as $i \to \infty$, words with length $\leq 1$ looks like more and more, while the above estimate shows that the total number is bounded above by $L \approx \frac{\text{vol}(B_i^{2\gamma_i,1}(0))}{\text{vol}(B_i^{1}(0))}$ (rescaling $|\gamma_i, 1| = 1$, $|\gamma_i, k| \leq d^2$). In particular, $|(S_i)| \leq L$, where $(S_i)$ denotes the set of elements of word length $\leq L$. Consequently, $|\text{Im}(\psi_i)| \leq L$ for all large $i$ ([KPT]). Then

$$[\Gamma_i : C_{\Gamma_i}(\mathbb{Z}^k)] \leq [\Gamma_i, \hat{\Gamma}_i] \cdot [\hat{\Gamma}_i : C_{\hat{\Gamma}_i}(\mathbb{Z}^k)] \leq (c(n)).$$

(9.2) Arguing by contradiction, assuming $\epsilon > 0$ and a sequence, $\hat{M}_i = \hat{M}_i/C_{\hat{\Gamma}_i}(\mathbb{Z}^k)$, $x_i \in \hat{M}_i$, such that $B_{3\delta}(x_i(d\rho_i)^{-1}\hat{M}_i)$ is at least $\epsilon$-away from any $B_{3\delta}(T^k \times \mathbb{R}^{n-k})$, with $\text{diam}(T^k) \leq 1$ and $\text{injrad}(T^k) \geq d^{-2}$. Passing to a subsequence, we may assume that

$$(\rho_i^{-1}\hat{M}_i, x_i) \xrightarrow{\text{GH}} (X, x).$$

We shall show that $X = T^k \times \mathbb{R}^{n-k}$, a contradiction (see (8.6)). We proceed by induction on $n$, starting with $n = 2$. In this case, $\hat{M}_i = T^2$ (see Example 10). For $n > 2$, as seen in the proof of (8.6), bases on $N_i \to \hat{M}_i \to \ell_i T^m$ we have that $$(\rho_i^{-1}\hat{M}_i, x_i) \xrightarrow{\text{GH}} (Y \times \mathbb{R}^m, x)$$ and $$(\rho_i^{-1}\hat{M}_i, x_i) \xrightarrow{\text{GH}} (Y, x).$$ Applying induction on $N_i (\Lambda_i = C_{\Lambda_i}(\mathbb{Z}^k))$ we conclude that $Y = T^k \times \mathbb{R}^{n-m}$, thus $X = T^k \times \mathbb{R}^{n-k}$. \hfill \Box

With the above preparation, we are ready for

Proof of Theorem 3'.
Without loss of generality, we may assume that all $M_i$ satisfies the additional regularities in Theorem 2, say $\ell = 2$. Fixing $i$ large, let $\tilde{M} := \tilde{M}_i = M_i/C_{\Gamma_1}(Z^k)$.

(3.1) We divide the construction of a principal $T^k$-bundle in two steps: Step 1. Using (9.2), we construct a locally finite open cover for $\tilde{M}_i$, $\{U_\alpha\}$, $U_\alpha$ admits an almost isometric $T^k$-action that is determined by (collapsed) metric, such that on $U_\alpha \cap U_\beta \neq \emptyset$, the two $T^k$-actions are at least $C^1$-close. Step 2. Glue $\{(U_\alpha, T^k)\}$ to a $T^k$-bundle with an affine structural group. Because $\pi_1(\tilde{M}_i)$ is the centralizer of $\pi_1(T^k)$, the structural group is trivial i.e., the $T^k$-bundle is a principal.

Step 1. Let $\{p_i\}$ denote a 10-net on $(d\rho)^{-1}\tilde{M}$. Fixing $\epsilon > 0$, by Lemma 9 (i large) we may assume a diffeomorphism, $f_\alpha : B_{30}(p_\alpha, (d\rho)^{-1}\tilde{M}) \to B_{30}(0_\alpha, T^k \times \mathbb{R}^{n-k})$, such that the pullback flat metric is $C^3$-close. Let $U_\alpha := f_\alpha^{-1}(T^k \times B_{10}^{n-k}(0_\alpha))$, $V_\alpha := f_\alpha^{-1}(T^k \times B_{12}^{n-k}(0_\alpha))$, where $\text{diam}(T^k) \leq 1$ and $\text{injrad}(T^k) \geq d^{-2}$. As discussed in the outline of the proof of Theorem 3’, a choice of a base point $e \in T^k$ and a canonical basis for $\pi_1(T^k, e)$ uniquely determine a group structure on $T^k$, hence $U_\alpha$ (resp. $V_\alpha$) has a trivial principal $T^k$-bundle structure, $P_\alpha = \text{proj} \circ f_\alpha : U_\alpha \to B_{10}^{n-k}(0_\alpha)$.

For $U_\alpha \cap U_\beta \neq \emptyset$, let $W_\alpha = P_\alpha(V_\alpha \cap (U_\beta - U_\alpha))$, $W_\beta = B_{10}^{n-k}(0) - P_\beta(U_\alpha)$, and define a map, $\phi_{\alpha\beta} : W_\alpha \to B_{10}^{n-k}(0_\beta)$, $\phi_{\alpha\beta}(z) = \text{cm}(z)$, the center of mass of $P_\beta(P_\alpha^{-1}(z))$, $z \in W_\alpha$. It is clear that $\phi_{\alpha\beta}$ is an embedding, by which we glue together $B_{10}^{n-k}(0_\alpha)$ and $B_{10}^{n-k}(0_\beta)$: $B_{10}^{n-k}(0_\alpha) \cup W_\alpha \sim \phi_{\alpha\beta}(W_\alpha) \cup P_\beta^{-1}(\phi_{\alpha\beta}(W_\alpha))$. Fixing a common base point $x_{\alpha\beta} \in U_\alpha \cap U_\beta$ and a canonical basis, up to an automorphism of $T^k$ we may view the two $T^k$-actions are from (a single) $T^k$, and the two $T^k$-actions are at least $C^0$-close.

Using the fact that the pullback metrics on $U_\alpha$ and $U_\beta$ are $C^3$-close by which the two isometric $T^k$-actions are isometric, by [GK] the two $T^k$-actions are at least $C^1$-close.

Step 2. We will patch $\{(U_\alpha, T^k)\}$ together to obtain a $T^k$-bundle with affine structural group. Observe that the desired result follows from the gluing construction (Lemmas 1.4 and 1.5 of [CG2]), based on the stability of compact group actions ([GK]). For the sake of a self-contained proof, here we present an elementary gluing construction based on the center of mass technique in [GK].

For each $z \in \phi_{\alpha\beta}(W_\alpha)$, we define a diffeomorphism, $\psi_z : P_\alpha^{-1}(z) \to P_\beta^{-1}(z)$, the nearest point projection, which is smooth in $z$. Then replace $\psi_z(x)$ by the center of mass of $\{t^{-1}\phi_{\alpha\beta}(tx), t \in T^k\}$, still denoted by $\psi_z(x)$, thus $\psi_z : P_\alpha^{-1}(z) \to P_\beta^{-1}(z)$ is also $T^k$-conjugate. Let $\gamma_{x, \psi_z(x)}$ denote the unique normal minimal geodesic from $x$ to $\psi_z(x)$ (with respect to a pullback flat metric on $B_{30}(x_{i, \alpha})$).

To patch $U_\alpha$ and $U_\beta$ together, we fix a transition function (which may be thought as the normalized distance function on $A_{11}^{n-k}(0_\alpha) = B_{11}^{n-k}(0_\alpha) - B_{10}^{n-k}(0_\alpha)$), on $B_{10}^{n-k}(0_\alpha) \cup W_\alpha \sim \phi_{\alpha\beta}(W_\alpha) \cup W_\beta$, as follows:

$$\chi(v) = \begin{cases} 0 & v \in B_{10}^{n-k}(0_\alpha) \\ 1 & v \in W_\beta - \phi_{\alpha\beta}(W_\alpha) \end{cases}.$$  

We define $\Phi : [0, 1] \times (U_\alpha \cup U_\beta) \to U_\alpha \cup U_\beta$:

$$\Phi(t, x) = \begin{cases} x & x \in U_\alpha \cup P_\beta^{-1}(W_\beta - \phi_{\alpha\beta}(W_\alpha)) \\ \gamma_{x, \psi_{P_\beta(x)}(x)}(\chi(P_\beta(x))t) & \text{otherwise} \end{cases}$$
Clearly, \(\Psi(t, x)\) is an isometry between \(id_{U_i \cup U_\beta} = \Phi(0, x)\) and \(\Phi(1, x)\), and the new \(T^k\)-action on \(U_\alpha \cup V_\alpha \cap U_\beta\) by \(\Phi(1, T^k)\), will agree with the \(T^k\)-action on the boundary of \(U_\alpha \cup (V_\alpha - U_\alpha) \cap U_\beta\), thus the two principal \(T^k\)-bundles are glued into a \(T^k\)-principal \(T^k\)-bundle (a tool for a gluing in a general situation is the isometry extension theorem).

Because \(\{U_\alpha\}\) is a finite set, we apply induction on the number of the gluing operations. By the above, we glue \(U_1\) and \(U_2\) to form a \(T^k\)-bundle (with an affine structural group). Assume that we obtain a \(T^k\)-bundle on \(U = U_1 \cup \cdots \cup U_m\) (with an affine structural group). To glue \(U_{m+1}\) to \(U\), it suffices to check that the gluing region in \(W \cap U_{s+1}\) satisfies the \(C^3\) \(\epsilon\)-closeness to \(B_{20}(T^k \times \mathbb{R}^{n-k})\). Because each gluing may produce an error on the \(C^3\) \(\epsilon\)-closeness, say \(C^3 c(n)\epsilon\)-close, a cumulation of errors could add up to exceed \(C\epsilon\) for any constant \(C\). This bad possibility is ruled out by the fact that in a region where multiple gluing operations occur, the number of gluing operation is bounded above by \(\leq a(n)\). Hence, taking \(\epsilon\) suitably small we can apply the above model situation and glue \(W\) and \(U_{s+1}\) to obtain a \(T^k\)-bundle with an affine structural group on \(W \cup U_{m+1}\).

Finally, we explain that a desired \(C^1\)-close \(T^k\)-invariant metric is obtained by averaging the metric under the almost isometric \(T^k\)-action: based on the additional regularities of the metric on \(M\), the second fundamental form and the \(A\)-tensor the principal \(T^k\)-bundle has bounded covariant derivative, thus the \(T^k\)-invariant metric satisfies the same regularity at least one order less than the original metric. By the O’Neil’s Riemannian submersion formula, it is clear that the quotient metric on \(\hat{M}_i/T^k\) has bounded sectional curvature.

(3.2) ‘The first part has been proved in (8.4). Consider \(\ell^{-1}_i M_i \xrightarrow{GH} \mathbb{R}^n/G\) in the proof of (8.1). Because \(\Gamma_i\) is nilpotent, \(T^\ell = \mathbb{R}^n/G\) (e.g., if the isometric \(G\)-action is not free, then \(\mathbb{R}^n/G\) is not compact). Applying Theorem 7, there is a fiber bundle map, \(f_i : M_i \to T^\ell\), such that \(|\Pi(f_i)| \to 0\) with respect to \(\ell^{-1}_i M_i\) (comparing with the proof of (8.2)). Because for all \(\tilde{x}_i\) in the fundamental domain at \(\tilde{p}_i\), \(\frac{d(\gamma_i(\tilde{p}_i), \tilde{p}_i)}{d(\gamma_i(\tilde{x}_i), x)}\) is scaling invariant, this ratio is almost one is clear.

We conclude the paper by supplying the following example mentioned several times through the above proofs.

**Example 10.** We construct one-parameter family of flat metrics, \(g_\epsilon\), on a Klein bottle, such that \((K^2, g_\epsilon) \xrightarrow{GH} \text{pt}\), and the blow-up sequence, \((K^2, \ell^{-1}_\epsilon g_\epsilon) \xrightarrow{GH} [0, 1]\), not a manifold. Moreover, a pointed GH-limit, \((K^2, \rho^{-1}_\epsilon g_\epsilon, x) \xrightarrow{GH} (X, x)\) is a complete flat surface which is either isomorphic to \(S^1 \times \mathbb{R}_1\) or a twisted \(\mathbb{R}^1\)-bundle over \(S^1\) (infinite Möbius band), depending on \(x\) (see (8.8) and Lemma 9).  

First, \((K^2, g_0)\) admits an isometric \(T^3\)-action with two exceptional \(T^1\)-orbits, where \(g_0\) is a flat metric. Hence, \(g_0\) at each point can be written \(g_0 = ds^2 + (ds^2)\perp\). For \(\epsilon \in (0, 1]\), define \(g_\epsilon' = \epsilon^2 ds^2 + (ds^2)\perp\) and \(g_\epsilon = \sqrt{\epsilon} g_\epsilon'\). Then \((K^2, g_\epsilon) \xrightarrow{GH} \text{pt}\). It is clear that \(\ell_\epsilon \approx \text{diam}(K^2, g_\epsilon)\), and

\[
(K^2, \ell^{-1}_\epsilon g_\epsilon) \xrightarrow{GH} [0, 1], \quad (T^2, \ell^{-1}_\epsilon g_\epsilon) \xrightarrow{GH} T^1.
\]

Moreover, for any \(x \in K^2\), \(\text{injrad}(x) \approx \epsilon^2\), and

\[
(K^2, \rho^{-1}_\epsilon g_\epsilon, x) \xrightarrow{GH} \left\{ \begin{array}{ll}
(S^1 \times \mathbb{R}_1, x_{\infty}) & T^1(x) \text{ is regular} \\
(M_\infty, x_{\infty}) & T^1(x) \text{ is exceptional}
\end{array} \right.
\]
where $M_\infty$ is a twisted $\mathbb{R}^1$-bundle over $S^1$ (an infinite Möbius band).

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**Appendix. Proof of Theorem 7**

A fiber bundle structure as in Theorem 7 has been generalized under various weak curvature conditions; however one may not have that the induced metric on a fiber is almost flat. Let’s first briefly describe methods of constructions.

(A1) Smoothing method with $|\text{sec}_M| \leq 1$ ([CFG], [Fu1]). Given $M$ a metric $g_\epsilon$ with higher regularity that is $C^1$ $\epsilon$-close to the original metric (Theorem 2), and a continuous $\epsilon$-GHA $h : M \to N$, using the method of center of mass one smooths $h$ with respect to $g_\epsilon$ to be a desired bundle map, $h_\epsilon$ (called $\epsilon$-regular), satisfying (7.1)-(7.3) with respect to the original metric as well. Hence, $h_\epsilon$ satisfies (7.1)-(7.3) with respect to the original metric as well.

(A2) Imbedding method with $\text{sec}_M \geq -1$ ([Fu4], [Ya], [Ro1,2]). Given a $\rho$-net, $\{\bar{x}_\alpha\}_{\alpha=1}^n \subset N$ and a cut-off function $h$, supp$(h) \subset [0,\rho)$, define a smooth map, $\Phi : N \to \mathbb{R}^s$, $\Phi(x) = (h \circ d_{\bar{x}_\alpha}(x))$, and a $C^1$-map, $\Psi : M \to \mathbb{R}^s$, $\Psi(x) = (h \circ \bar{d}_{\bar{x}_\alpha}(x))$, where $\bar{d}$ is the average of $d_{\bar{x}_\alpha}$ over $B_{\bar{\rho}}(x_\alpha)$, and $x_\alpha$ is a lift of $\bar{x}_\alpha$. Then $\Phi : N \to \mathbb{R}^s$ is an embedding, and $f = \Phi^{-1} \circ \text{Proj} \circ \Psi : M \to N$ is a $C^1$-fiber bundle map satisfying (7.1) and a weak version of (7.2), where Proj denotes the projection to a nearest point in $\Phi(N)$ (a fiber may not be an infra-nilmanifold; e.g., a sphere).

(A3) Gluing $(m,\delta)$-splitting maps with $\text{Ric}_M \geq -(n-1)$ and the universal cover of any $\rho$-ball on $M$ is non-collapsed ([Hu]). Given an $\rho$-net, $\{\bar{x}_\alpha\} \subset N$, and let $x_\alpha$ be a lift of $\bar{x}_\alpha$. By [CC], one gets that a smooth $(m,\delta)$-splitting map, $H_\alpha : B_\rho(x_\alpha) \to B_\rho(0) \subset T_{\bar{x}}N$. Let $\{f_\alpha\}$ be a partition of unity associate to the locally finite open cover, $\{B_\rho(x_\alpha)\}$, and the center of mass method, one can glue $\{\exp \circ H_\alpha\}$ together to a smooth $H : M \to N$. Using the non-degeneracy of an $(n,\delta)$-splitting map ([CJN]), Huang showed that each $(m,\delta)$-splitting map $H_\alpha$ is non-degenerate and satisfies (7.1), and $H$ is also non-degenerated (the weak regularity fails to identify a fiber as an infra-nilmanifold).

(A4) Successively blowing up and gluing method ([Ro3]) under the same conditions of (A3); a baby version is similar to the construction of the principal $T^k$-bundle in the proof of Theorem 3'. The result is a fiber bundle map, $f : M \to N$, satisfying (7.1), such that a $f$-fiber diffeomorphic to an infra-nilmanifold, and the structural group.

In view of the above, our proof below does not rely on a smoothing method, nor involving in an tedious calculus estimate. Using that $|\text{sec}_M| \leq 1$, we will get the desired (local) regularities (for ‘free’) based on the following lemma (see Step 2), via directly verification of boundary conditions.

**Lemma A5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $g_i$ be two Riemannian metrics on $\Omega$, and let $u_i$ be solution of the Dirichlet problems:

\[
\begin{cases}
\Delta_i u_i = 0 & \Omega \\
u_i = f_i & \partial \Omega, \quad i = 1, 2
\end{cases}
\]

where $u_1 \neq u_2$. Then $\|u_1 - u_2\|_{C^1} \leq C(g_1, g_2, \Omega, f_1, f_2, \partial \Omega)$. 

*Proof.* Let $\Omega'$ be a transversal slice for $\partial \Omega$ with $\Omega' \cap \Omega = \{x\}$, and $f_i(x) = 0$. Define $v_i = u_i - f_i$, and $u_i = v_i + f_i$. Then the problem for $v_i$ is:

\[
\begin{cases}
\Delta_i v_i = 0 & \Omega \\
v_i = 0 & \partial \Omega.
\end{cases}
\]

Using the mean value theorem and the fact that $|\Delta_i u_i| \leq C(g_i, \Omega)$, one gets $\|v_i\|_{C^1} \leq \|v_i\|_{C^0} \leq C(g_i, \Omega, \partial \Omega)$. Then by the maximum principle and the maximum principle for $u_i$, we have $\|u_1 - u_2\|_{C^1} \leq C(g_1, g_2, \Omega, f_1, f_2, \partial \Omega)$.

*End of proof.*
We define a smooth map, \( H \), such that \( |\partial H M f| \leq \Psi(\epsilon) \).

Observe that \( v = u_1 - u_2 \) satisfies \( \Delta_1 v = (\Delta_2 - \Delta_1) u_2 \) on \( \Omega \) and \( v|_{\partial \Omega} = f_1 - f_2 \). Hence Lemma A5 follows from a regularity estimate of an elliptic equation.

**Proof of Theorem 7.**

By a standard compactness argument, it suffices to consider a sequence of compact \( n \)-manifolds, \( M_i \xrightarrow{GH} N \), such that \( |\sec M_i| \leq 1 \), \( |\sec N| \leq 1 \) and \( \text{injrad}(N) \geq 2\rho \), and prove that for \( i \) large there is a smooth map, \( f_i : M_i \to N \), satisfying (7.1)-(7.3) (note that (7.2) and (7.3) are local properties, so our strategy is to verify (7.2) and (7.3) to the lifting of \( H \)). We will glue the family of local (fiber bundle) maps, \( \{ \hat{\pi}_i \} \), i.e. \( \hat{\pi}_i \to \pi_i \), denote by \( \text{cm}(x_i) \) (called the center of mass). We will glue the family of local (fiber bundle) maps, \( \{ \{ U_{i,\beta, H_{i,\beta}} \} \} \), by the method of center of mass: take a partition of unity, \( \{ \phi \} \), associate to the open cover \( \{ B_{\delta \rho}(x_i) \} \) (see (2.5). For any \( x_i \in M_i \) close to \( x_\beta \in N \), the energy functional,

\[
E(x_i, y) = \frac{1}{2} \sum_\beta f_\beta(H_{i,\beta}(x_i)) d_N^2(H_{i,\beta}(x_i), y) : B_\rho(x_\beta) \to \mathbb{R}.
\]

is a finite sum of strictly convex functions in \( B_\rho(x_\beta) \), \( E(x_i, y) \) is strictly convex, thus \( E(x_i, y) \) achieves a unique minimum in \( B_\rho(x_\beta) \), denote by \( \text{cm}(x_i) \) (called the center of mass). We define \( f_i : M_i \to N \), \( f_i(x_i) = \text{cm}(x_i) \).

Step 2. We will verify that \( H_{i,\beta} \) satisfies (7.1) and (7.3) with \( \delta = \frac{1}{4} \), and \( \delta = \delta(\epsilon) \) suitably small and fixed, \( \delta^{-1} M_i \to \delta^{-1} N \), \( H_{i,\beta} \) satisfies (7.2) which is scaling invariant. Note that (7.2) and (7.3) are local properties, so our strategy is to verify (7.2) and (7.3) to the lifting of \( H_{i,\beta} \) on a `covering space'.

Let \( B_\pi(0_i, \beta) \subset T_{x_i, \beta} M_i \) equipped with the pullback metric by \( \exp_{x_i, \beta}, g^*_{x_i, \beta}, \) let \( i_\beta \), \( \hat{\pi}_i, \beta \), \( \hat{t}_i, \beta \), \( \hat{x}_i \), \( \hat{\beta}_i, \beta \), etc., defined on \( V_i, \beta = \bigcup_{\gamma \in \Lambda_i, \beta} \gamma(D_{i,\beta} \cap B_{\delta \rho}(0_i, \beta)) \), \( B_{\delta \rho}(0_i, \beta) \subset V_i, \beta \subset B_{2\delta \rho}(0_i, \beta) \), where \( D_{i,\beta} \subset B_{\frac{\rho}{2}}(0_i, \beta) \) is a ‘fundamental domain’ of \( \exp_{x_i, \beta} \) at \( 0_i, \beta \) (i.e., \( \exp_{x_i, \beta} : D_{i,\beta} \to B_{\frac{\rho}{2}}(0_i, \beta) \) is bijection), \( \Lambda_i \subset \pi_1(B_\epsilon(x_i, \beta)) \) (\( \epsilon \ll \delta \rho \)) is a finite subset (roughly, one may imagine

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2this observation was pointed out to the author by Zhengchao Han.
$V_{i,\beta}$ as a finite solid cylinder with ‘boundary’, $\partial V_{i,\beta} = V_{i,\beta} \cap \exp^{-1}(\partial B_{\delta\rho}(x_{i,\beta}))$, i.e., without the top/bottom caps). Then $\hat{u}_{i,\beta,j}$ satisfies the following:

$$\begin{cases} \Delta \hat{u}_{i,\beta,j} = 0 & V_{i,\beta} \\ \hat{u}_{i,\beta,j} = \hat{b}_{i,\beta,j} & \partial V_{i,\beta} \end{cases}$$

and $H_{i,\beta}$ satisfies (7.2) if and only if $\hat{H}_{i,\beta}(x) = (\hat{u}_{i,1,\beta}(x), \ldots, \hat{u}_{i,m,\beta}(x))$ does, and $\hat{H}_{i,\beta}$ is a ‘periodic’ function with a ‘period’, $\hat{H}_{i,\beta}|_{D_{i,\beta}\cap B_{\delta\rho}(0_{i,\beta})}$.

Restricting to $D_{i,\beta}$ (with $\delta = \frac{1}{\ell}$), we may assume that $\hat{b}_{i,\beta,j} = d(\rho e_j, \cdot) - \rho$. By standard Schauder interior estimate, (7.3) follows from that $|\hat{u}_{i,\beta,j}|_{C^0(V_{i,\beta})} \leq c(n)$. Note that $\hat{u}_{i,\beta,j} + \rho$ is positive harmonic on $V_{i,\beta}$, by the Cheng-Yau gradient estimate $|\nabla \hat{u}_{i,\beta,j}|_{C^1(V_{i,\beta}\cap B_{\frac{1}{\ell}\delta\rho}(x_{i,\beta}))} \leq c(n)$, if $|\hat{u}_{i,\beta,j} + \rho|_{C^0(D_{i,\beta}\cap B_{\frac{1}{\ell}\delta\rho}(0_{i,\beta}))} \leq c(n)$. By the maximal principle $|\hat{u}_{i,\beta,j} + \rho|$ achieves the maximum on $\partial V_{i,\beta}$; thus the desired bound is obvious, $|\hat{b}_{i,\beta,j} + \rho|_{C^0(\partial V_{i,\beta})} \leq c(n, \rho)$.

In proving that $H_{i,\beta}$ satisfies (7.2), our approach is to construct a model solution with respect to the Euclidean metric on $\mathbb{R}^n = T_{x_{i,\beta}} M_i$, and applying Lemma A5.

Let $b_{j}(x) = d(\mathbb{R}(\rho e_j)_{\perp}, x) - \rho$ the distance function to the hyperplane at $\rho e_j$, where $\mathbb{R}(\rho e_j)$ is the hyperplane at $\rho e_j$ with the normal vector $e_j$. Then $b_j$ satisfies

$$\begin{cases} \Delta b_j = 0 & V_{i,\beta} \\ b_j = b_j & \partial V_{i,\beta}, \end{cases}$$

and $H_{\beta}(x) = (b_1(x), \ldots, b_m(x)) : V_{i,\beta} (\subset \mathbb{R}^n) \rightarrow \mathbb{R}^m$, is the projection; so (7.2) holds for any $\epsilon > 0$. We claim that $\delta = \delta(\epsilon)$ is chosen small, $H_{\beta}$ is $C^{1,\alpha}$-close to $\hat{H}_{i,\beta}$ with respect to $\delta^{-1} N$ and $\delta^{-1} M_i$, thus $\hat{H}_{i,\beta}$ satisfies (7.2). By Lemma A5, the claim follows from that restricting to $\partial V_{i,\beta}$, $\hat{b}_{i,\beta,j}$ is $C^{1,\alpha}$-close to $b_j$.

For $\delta$ suitably small, the pullback metric $g^{\ast}_i$ on $B_{\delta\rho}(0_{i,\beta})$ is $C^{1,\alpha}$-close to the Euclidean metric (see the proof of (8.7)), thus it suffices to check that $\hat{b}_{i,\beta,j}(x) = d_{A_{i,\beta}}(x) - \rho$ is $C^0$ $\epsilon$-close to $b_j$, which equals to, restricting to $V_{i,\beta}$, the Busemann function along the $x_j$-axis; clearly, $\hat{b}_{i,\beta,j}$ approximates the Busemann function along the $x_j$-axis, for $\delta(\epsilon) << 1$ and $i$ large.

Step 3. By the construction of $H_i : M_i \rightarrow N$ and Step 2, it is clear that $H_i$ satisfies (7.1) and (7.3). It remains to check that $H_i$ satisfies (7.2).

Because (7.2) is local property, and because $H_{i,\beta}$ and $b_j$ are at least $C^{1,\alpha}$-close, it reduces to check (7.2) for the gluing of $\{H_{\beta}\}$ via $\{f_{\beta}\}$, which is easily seen via a differentiation of implicit functions.

Remark A6. Inspecting the above proof, it is clear that if one assumes that $g_i$ satisfies a little higher regularity, $|\nabla \text{Rm}(g_i)| \leq C$, then (for $\delta$ suitably small; e.g. replacing $M_i$ with $\sqrt{t_i} M_i \overset{\text{GH}}{\rightarrow} \text{pt}$) $\hat{u}_{i,\beta,j}$ is $C^{2,\alpha}$ close to $b_j$.

References

[Ab] U. Abresch, Über das glatten Riemann’scher metriken, Habilitationsschrift Rheinischen Friedrich-Wilhelms-Universität Bonn (1998).

[Be] I. Belegradek, Iterated circle bundles and infra-nilmanifolds, arXiv:1805.06585.

[BGP] Y. Burago, M. Gromov, and G. Perel’man, A.D. Alexandrov spaces with curvature bounded below, Uspekhi Mat. Nank 47(2) (1992), 3-51.
P. Buser; H. Karcher, *Gromov’s almost flat manifolds*, Astérisque. 81 (1981).

J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. 92 (1970), 61-75.

J. Cheeger; T. Colding, *Lower Bounds on Ricci Curvature and the Almost Rigidity of Warped Products*, Ann. of Math. 144 (Jul., 1996), no. 1, 189-237.

J. Cheeger, K. Fukaya; M. Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, J. Amer. Math. Soc. 5 (1992), 327-372.

J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. 6 (1971), 119-128.

J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. 96 (1972), 413-443.

J. Cheeger; M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded I*, J. Diff. Geom. 23 (1986), 309-346.

J. Cheeger; M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded II*, J. Diff. Geom. 32 (1990), 269-298.

J. Cheeger, W. Jiang; A. Naber, *Rectifiability of singular sets in noncollapsed spaces with Ricci curvature bounded below*, Ann. of Math 193 (2021), 407-538.

F. Farrell; W. Hsiang, *Topological characterization of flat and almost flat Riemannian manifolds $M^n$ ($n \neq 3, 4$)*, Amer. J. Math. 105 (1983), 641-672.

K. Fukaya, *Collapsing of Riemannian manifolds to ones of lower dimensions*, J. Diff. Geom. 25 (1988), 139-156.

K. Fukaya, *A boundary of the set of Riemannian manifolds with bounded curvature and diameter*, J. Diff. Geom. 28 (1988), 1-21.

K. Fukaya, *Collapsing Riemannian manifolds to ones of lower dimensions. II*, Jpn. Math. 41 (1989), 333-356.

K. Fukaya, *Hausdorff convergence of Riemannian manifolds and its applications*, Recent Topics in Differential and Analytic Geometry (T. Ochiai, ed.), Kinokuniya, Tokyo, (1990).

K. Fukaya; T. Yamaguchi, *The fundamental groups of almost non-negatively curved manifolds*, Ann. of Math 136 (1992), 253-333.

M. Gromov, *Almost flat manifolds*, J. Diff. Geom. 13 (1978), 231-241.

M. Gromov, J. Lafontaine; P. Pansu, *Structures metriques pour les varietes riemanniennes*, CedicFernand Paris, (1981).

K. Grove; H. Karcher, *How to conjugate $C^1$-close actions*, Math. Z 132 (1973), 11-20.

H. Huang, *Fibrations, and stability for compact Lie group actions on manifolds with bounded local Ricci covering geometry*, Front. Math. China 15 (2020), 69-89.

V. Kapovitch; A. Petrunin; W. Tuschmann, *Nilpotency, almost nonnegative curvature bound.*, Ann. of Math. 171 (2010), 343-373 2010.

A. D. Milka, *Metric structure of a certain class of spaces that contain straight lines*, Ukrain. Geomtr. Sb. Vyp. 4 (1967), 43-48.

M. Nakayama, *On the $S^1$-fibred nilBott tower*, Osaka J. Math 51 (2014), 67-87.

M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag Berline Heidelberg, New York (1972).

X. Rong, *Collapsed manifolds with bounded curvature and applications*, Surveys in J. Diff. Geom. XI (2007), 1-23.

X. Rong, *Convergence and collapsing theorems in Riemannian geometry*, Handbook of Geometric Analysis, Higher Education Press and International Press, Beijing-Boston (2010), no. II ALM 13, 193-298.

X. Rong, *Collapsed manifolds with a local Ricci bounded covering geometry*, Preprint.

E. Ruh, *Almost flat manifolds*, J. Diff. Geom. 17 (1982), 1-14.

W. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Diff. Geom. 30 (1989), 225-301.

T. Yamaguchi, *Collapsing and pinching under a lower curvature bound*, Ann. of Math. 133 (1991), 317-357.

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