On the Biclique Cover of the Complete Graph

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Abstract Let \( K \) be a set of \( k \) positive integers. A biclique cover of type \( K \) of a graph \( G \) is a collection of complete bipartite subgraphs of \( G \) such that for every edge \( e \) of \( G \), the number of bicliques need to cover \( e \) is a member of \( K \). If \( K = \{1, 2, \ldots, k\} \) then the maximum number of vertices of a complete graph that admits a biclique cover of type \( K \) with \( d \) bicliques, \( n(k, d) \), is the maximum possible cardinality of a \( k \)-neighborly family of standard boxes in \( \mathbb{R}^d \). In this paper, we obtain an upper bound for \( n(k, d) \). Also, we show that the upper bound can be improved in some special cases. Moreover, we show that the existence of biclique cover of type \( K \) of the complete bipartite graph with a perfect matching removed is equivalent to the existence of a cross \( K \)-intersection family.

Keywords Biclique cover · Multilinear polynomial · Cross \( K \)-intersection families

Mathematics Subject Classification 05C70

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1 Introduction

Throughout the paper, we consider only the simple graph. Let \( V(G) \) and \( E(G) \) denote the vertex set and edge set of the graph \( G \). As usual we will use the symbol \([n]\) to denote the set \( \{1, 2, \ldots, n\} \). By a biclique we mean a complete bipartite graph. We will denote by \( K_{m,n}^- \) the complete bipartite graph with a perfect matching removed. A biclique cover of a graph \( G \) is a collection of bicliques of \( G \) such that each edge of \( G \) belongs to at least one of the bicliques. The number of bicliques in a biclique cover is the size of the cover. In the literature, there are several ways to define a biclique cover problem for different purposes, see \([2–4, 7, 10, 11, 13]\). The Graham and Pollak Theorem state that the minimum number of complete bipartite subgraphs needed to partition the edge set of complete graph \( K_n \), is \( n - 1 \). Motivated by problems in geometry, Alon \([2]\) studied an extension of the previous result and defined the following definition.

Definition 1 Let \( K \) be a set of \( k \) positive integers. We say that a biclique cover of the graph \( G \) is of type \( K \) if for every edge \( e \) of the graph \( G \), the number of bicliques that cover \( e \) is an element of the set \( K \).

We write \( n(K, d) \) for the maximum number of vertices of a complete graph that admits a biclique cover of type \( K \) and size \( d \). In the aforementioned definition if we take the set \( K = \{1, 2, \ldots, k\} \) then this biclique cover is called of order \( k \) and the maximum number of vertices of a complete graph that admits a biclique cover of order \( k \) and size \( d \) is denoted by \( n(k, d) \). Let \( C \) be a finite family of \( d \)-dimensional convex polytopes. The family \( C \) is called \( k \)-neighbourly if for every two distinct members \( C \) and \( C' \) of the family \( d-k \leq \text{dime}(C \cap C') \leq d-1 \). A \( k \)-neighbourly family of \( d \)-dimensional boxes with edge parallel to the coordinate axes, is called a \( k \)-neighbourly family of standard boxes in \( \mathbb{R}^d \). The maximum possible cardinality of a 1-neighbourly family of standard boxes in \( \mathbb{R}^d \) is precisely \( d + 1 \). Zaks \([18]\) has proved this result, using a relation between it and the theorem of Graham and Pollak \([9]\) about bipartite decompositions of complete graphs. Alon has shown that the maximum possible cardinality of a \( k \)-neighbourly family of standard boxes in \( \mathbb{R}^d \) is precisely the maximum number of vertices of a complete graph that admits a biclique cover of order \( k \) and size \( d \), (see \([2]\) for more details.) Alon gave upper and lower bounds for this parameter as follows.

Theorem A \([2]\) Let \( d \) be a positive integer and \( 1 \leq k \leq d \) then

1. \( d + 1 = n(1, d) \leq n(2, d) \leq \cdots \leq n(d-1, d) \leq n(d, d) = 2^d \).
2. \( \left( \begin{array}{c} d \end{array} \right)^k \leq \prod_{i=0}^{k-1} \left( \left\lfloor \frac{d+i}{k} \right\rfloor + 1 \right) \leq n(k, d) \leq \sum_{i=0}^{k} 2^i \left( \begin{array}{c} d \end{array} \right) < 2 \left( \frac{2^d}{k} \right)^k \).

In this paper, we obtain a slightly improved bound for the extension of this parameter and prove the following theorem.

Theorem 1 For \( 1 \leq k \leq d \), assume that \( K \) is a set of \( k \) positive integers. Then

\[
n(K, d) \leq 2^k \left\lfloor \begin{array}{c} d \end{array} \right\rfloor + \sum_{i=1}^{k-1} 2^i \left( \begin{array}{c} d-1 \end{array} \right) \left( \begin{array}{c} i-1 \end{array} \right) \).
\]

[Springer]
Example 1 Let $K = \{2, 4, \ldots, 4k\}$. Then there exists a spanning biclique cover $H_1, H_2, \ldots, H_d$ of type $K$ for the complete graph on $(1 + (d/2) + (d/4) + \cdots + (d/2^k))$ vertices. To see this, assume that the vertices of the complete graph are labelled by all subsets of cardinality 0, 2, 4, $\ldots$, $2k$ of $[d]$. Construct the biclique cover $H_1, H_2, \ldots, H_d$ which $H_i$ has $(X_i, Y_i)$ as vertex set such that $X_i$ is the set of all subsets that contain $i$ and $Y_i$ is the set of all subsets that do not contain $i$. It is not difficult to check that for every edge $e$ of the complete graph, the number of bicliques that cover $e$ is an element of the set $K$. Therefore, for $K = \{2, 4, \ldots, 4k\}$, we have $\sum_{i=0}^{k} (d/2^i) \leq n_s(K, d)$.

Example 2 Let $K = \{1, 2, \ldots, \lceil \log n \rceil \}$. Encode the vertices of the complete graph by binary vectors of length $m = \lceil \log n \rceil$. For each $i = 1, \ldots, m$, let $H_i$ be a biclique containing all edges that the codes of whose endpoints differ in the $i$th coordinate. Easily, each $H_i$ is a spanning biclique and for every edge $e$ of the complete graph, the number of bicliques that cover $e$ is at least 1 and at most $\lceil \log n \rceil$. So, we have a spanning biclique cover of type $K$ and size $m$ for a complete graph on $2^m$ vertices. Hence for this $K$ we have $2^m \leq n_s(K, m)$.

We prove the following upper bound for $n_s(K, d)$.

**Theorem 2** If $K$ is a set of $k$ positive integers, then

$$n_s(K, d) \leq 2^{k-1} \binom{d}{k} + 2^{k-1} \binom{d-1}{k} + 2^k - 1.$$  

Using Theorem 2 and Example 2, for $K = \{1, 2, \ldots, m\}$ where $m = \lceil \log n \rceil$, we have

$$2^m \leq n_s(K, m) \leq \frac{3}{2} 2^m - 1. \quad (1)$$

It is interesting to consider the biclique cover of type $K$ of other graphs besides the complete graph. Assume that $K$ is a set of $k$ positive integers and $X$ is an arbitrary
set of \(d\) points. Suppose that \(A = \{A_1, A_2, \ldots, A_m\}\) and \(B = \{B_1, B_2, \ldots, B_m\}\) are two collections of subsets of \(X\) such that \(|A_i \cap B_j| \in K\) for \(i \neq j\) and \(|A_i \cap B_i| = 0\) for every \(i\). The pair \((A, B)\) is called a cross \(K\)-intersection family. The following theorem shows that a cross \(K\)-intersection family is related to biclique cover of type \(K\) of the graph \(K_{m,m}\).

**Theorem 3** Let \(K\) be a finite set. There exists a cross \(K\)-intersection family with \(m\) sets on a set of size \(d\) if and only if there exists a biclique cover of type \(K\) and size \(d\) of \(K_{m,m}\).

In [16] Snevily made the following conjecture.

**Conjecture 1** [16] Let \(A = \{A_1, A_2, \ldots, A_m\}\) and \(B = \{B_1, B_2, \ldots, B_m\}\) be two collections of subsets of an \(d\)-element set. Let \(K = \{l_1, l_2, \ldots, l_k\}\) be a collection of \(k\) positive integers. Assume that for \(i \neq j\) we have \(|A_i \cap B_j| \in K\) and that \(|A_i \cap B_i| = 0\), then

\[
m \leq \binom{d}{k}.
\]

By Theorem 3, we can state the above conjecture in terms of the biclique cover as follows. Let \(K = \{l_1, l_2, \ldots, l_k\}\) be a collection of \(k\) positive integers. The maximum number of the vertices in each part of a complete bipartite graph with a perfect matching removed that admits a biclique cover of type \(K\) and size \(d\) is at most \(\binom{d}{k}\). Note that this bound is sharp by taking all \(k\)-element subsets of \([d]\) as \(A\) and all \((d - k)\)-element subsets of \([d]\) as \(B\). In [17] William Y.C. Chen and Jiuqiang Liu proved the following theorem.

**Theorem B.** [17] Let \(p\) be a prime number and \(K = \{l_1, l_2, \ldots, l_k\}\) \(\subseteq \{1, 2, \ldots, p - 1\}\). Assume that \(A = \{A_1, A_2, \ldots, A_m\}\) and \(B = \{B_1, B_2, \ldots, B_m\}\) are two collections of subsets of \(X\) such that \(|A_i \cap B_j| \pmod{p} \in K\) for \(i \neq j\) and \(|A_i \cap B_i| = 0\) for every \(i\). If \(\max l_j < \min\{|A_i| \pmod{p}| 1 \leq i \leq m\}\), then

\[
m \leq \left(\frac{d-1}{k}\right) + \left(\frac{d-1}{k-1}\right) + \cdots + \left(\frac{d-1}{k-2r+1}\right),
\]

where \(r\) is the number of different set sizes in \(A\).

Theorem B. has the following immediate corollary by choosing prime number \(p\) greater than \(d\) and \(r = 1\).

**Corollary 1** [17] Let \(K = \{l_1, l_2, \ldots, l_k\}\) be a set of \(k\) positive integers and \(\max l_j < s\). Suppose that \(A = \{A_1, A_2, \ldots, A_m\}\) and \(B = \{B_1, B_2, \ldots, B_m\}\) are two collections of subsets of \([d]\) such that \(|A_i \cap B_j| \in K\) for \(i \neq j\) and \(|A_i \cap B_i| = 0\) for every \(i\). If either \(A\) is \(s\)-uniform or \(B\) is \(s\)-uniform, then

\[
m \leq \binom{d}{k}.
\]
Let $K = \{l_1, l_2, \ldots, l_k\}$ be a set of $k$ positive integers. Corollary 1 states that the maximum number of the vertices in each part of a complete bipartite graph with a perfect matching removed that admits a biclique cover of type $K$ and size $d$, with this extra condition that every vertex of this graph lies in exactly $s$ bicliques and $\max l_j < s$, is at most $\binom{d}{k}$. The structure of the rest of this paper is to prove Theorem 1, 2, and 3.

2 Proof of Theorem 1

A polynomial in $n$ variable is called multilinear if every variable has degree 0 or 1. Observe that when each variable in a polynomial attains values 0 or 1, if each variable $x^p_i$ ($p > 1$) is replaced by $x_i$, we can consider this polynomial as a multilinear polynomial. For a subset $A_i$ of $[n]$, the characteristic vector of $A_i$ is the vector $v_{A_i} = (v_1, \ldots, v_n)$, where $v_j = 1$ if $j \in A_i$ and $v_j = 0$ otherwise. Let $\{H_1, H_2, \ldots, H_d\}$ be a biclique cover of type $K$ for the graph $K_n$ such that $H_i$ has $X_i$ and $Y_i$ as its vertex classes. For every $1 \leq i \leq n$ define

$$A_i := \{j \mid i \in X_j\} \quad \text{and} \quad B_i := \{j \mid i \in Y_j\}.$$ 

Now, with each pair $(A_i, B_i)$ we associate a polynomial $P_i(x, y)$ defined by:

$$P_i(x, y) = \prod_{l_j \in K} (v_{A_i}.x + v_{B_i}.y - l_j).$$

Where $v_{A_i}$ (resp. $v_{B_i}$) is the characteristic vector of the set $A_i$ (resp. $B_i$), $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$. The key property of these polynomials is

$$P_i(v_{B_j}, v_{A_j}) \neq 0 \quad \text{and} \quad P_i(v_{B_j}, v_{A_j}) = 0 \quad \text{for all} \quad i \neq j,$$

which follows immediately from this fact that $\{H_1, \ldots, H_d\}$ is a biclique cover of type $K$. Let

$$\mathcal{A} = \{(M, N) \mid M, N \subseteq [d], \ M \cap N = \emptyset, \ d \in N, \ |M \cup N| \leq k\}.$$ 

It is easy to see that the cardinality of $\mathcal{A}$ is equal to $\sum_{i=0}^{k-1} 2^i \binom{d-1}{i}$. For every $(M, N) \in \mathcal{A}$, the polynomial $Q_{(M,N)}(x, y)$ is defined by

$$Q_{(M,N)}(x, y) = \prod_{i \in M} x_i \prod_{i \in N} y_i.$$ 

Throughout the paper we set $\prod_{i \in A} x_i = \prod_{i \in A} y_i = 1$ when $A$ is an empty set. We will now show that the polynomials in the set

$$\mathcal{P} = \{P_i(x, y) \mid 1 \leq i \leq n\} \cup \{Q_{(M,N)}(x, y) \mid (M, N) \in \mathcal{A}\},$$
as the polynomials from \( \{0, 1\}^{2d} \) to \( \mathbb{R} \), are linearly independent. For this purpose, we set

\[
\sum_{i=1}^{n} \alpha_i P_i(x, y) + \sum_{(M,N) \in \mathcal{A}} \beta_{(M,N)} Q_{(M,N)}(x, y) = 0.
\]

We can rewrite the above equality as follows:

\[
\sum_{d \in A_i} \alpha_i P_i(x, y) + \sum_{d \not\in A_i} \alpha_i P_i(x, y) + \sum_{(M,N) \in \mathcal{A}} \beta_{(M,N)} Q_{(M,N)}(x, y) = 0. \tag{3}
\]

The proof will be divided into 3 steps.

**Step 1.** We begin by proving that for every \( i \) which \( d \not\in A_i \), it holds that \( \alpha_i = 0 \). In the contrary assume that \( i_0 \) is a subscript such that \( \alpha_{i_0} \neq 0 \) and \( d \not\in A_{i_0} \). Substituting \((v_{B_{i_0}}, v_{A_{i_0}})\) in the Eq. 3, according to the relation 2 and since \( d \in N \), all terms in the relation 3 but \( \alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) \) vanish. In this way, \( \alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) = 0 \). Finally, as \( P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) \neq 0 \), we have \( \alpha_{i_0} = 0 \) which is a contradiction.

**Step 2.** We will show that for every \( i \) which \( d \in A_i \), it holds that \( \alpha_i = 0 \). According to the step 1 we have

\[
\sum_{d \in A_i} \alpha_i P_i(x, y) + \sum_{(M,N) \in \mathcal{A}} \beta_{(M,N)} Q_{(M,N)}(x, y) = 0. \tag{4}
\]

Assume that \( i_0 \) is a subscript such that \( \alpha_{i_0} \neq 0 \). Let \( v'_{A_{i_0}} = v_{A_{i_0}} - (0, \ldots, 0, 1) \) and evaluate the Eq. 4 in \((v_{B_{i_0}}, v'_{A_{i_0}})\). As \( A_i \cap B_i = \emptyset \) we have that \( d \not\in B_i \), so for every \( i \) in the equation 4 we have \( P_i(v_{B_{i_0}}, v'_{A_{i_0}}) = P_i(v_{B_{i_0}}, v_{A_{i_0}}) \). From this and since \( d \in N \) we conclude that \( \alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) = 0 \), hence \( \alpha_{i_0} = 0 \).

**Step 3.** Note that \( Q_{(M',N')}(v_{M'}, v_{N'}) = 1 \) and \( Q_{(M,N)}(v_{M'}, v_{N'}) = 0 \) for any \((M, N) \in \mathcal{A}\) with \( M \cup N \neq M' \cup N' \) and \( |M \cup N| \geq |M' \cup N'| \). So all the polynomials of the set \( \{ Q_{(M,N)}(x, y) \mid (M, N) \in \mathcal{A} \} \) are linearly independent. Therefore, for every \((M, N) \in \mathcal{A}\) we have \( \beta_{(M,N)} = 0 \).

We have thus proved the polynomials of the set \( \mathcal{P} \) as the polynomials with domain \( \{0, 1\}^{2d} \) are linearly independent. So we can consider these polynomials as multilinear polynomials. On the other hand every polynomial in the set \( \mathcal{P} \) can be written as a linear combination of the multilinear monomials of degree at most \( k \). Furthermore, they do not contain any monomials that contain both \( x_i \) and \( y_i \) for the same \( i \). The number of such monomials are \( \sum_{i=0}^{k} 2i \binom{d}{i} \) and hence,

\[
n + \sum_{i=0}^{k-1} 2i \binom{d-1}{i} \leq \sum_{i=0}^{k} 2i \binom{d}{i}.
\]

Now, by a straightforward calculation the formula of Theorem 1 will be achieved.
3 Proof of Theorem 2

Before embarking on the proof of Theorem 2, we will establish the following lemma.

**Lemma 1** For every $1 \leq i \leq d - 1$, let the set $B_i$ define as follows:

$$B_i = \{(I, J)| I, J \subseteq [d-i+1], \ I \cap J = \emptyset, \ d-i+1 \in I \cup J, \ x | I \cup J | \neq d-i+1, \ |I \cup J| \leq k-1\}.$$ 

Let for every pair $(I, J) \in B_i$, $R_{(I,J)}^i(x, y)$ denote the following polynomial

$$R_{(I,J)}^i(x, y) = \prod_{j \in I} x_j \prod_{j \in J} y_j \left( \sum_{j \notin J, j \leq d-i} x_j + \sum_{j \notin I, j \leq d-i} y_j - (d-i) \right).$$

Then

$$B = \left\{R_{(I,J)}^i(x, y) \mid 1 \leq i \leq d-1, \ (I, J) \in B_i \right\} \tag{5}$$

is a set of linearly independent polynomials.

**Proof** To prove the assertion, assume this is false and let

$$\sum_{(I,J) \in B_1} \gamma_{(I,J)}^1 R_{(I,J)}^1(x, y) + \cdots + \sum_{(I,J) \in B_{d-1}} \gamma_{(I,J)}^{d-1} R_{(I,J)}^{d-1}(x, y) = 0 \tag{6}$$

be a nontrivial linear relation. Suppose that $i_0$ is the greatest superscript and $(I_0, J_0)$ is the subscript such that has minimum cardinality in the set $B_{i_0}$ and $\gamma_{(I_0,J_0)}^{i_0} \neq 0$. Substitute $(v_{i_0}, v_{j_0})$ for $(x, y)$ in the linear relation 6. In view of the definition of $B_i$ all terms in the linear relation 6 but $\gamma_{(I_0,J_0)}^{i_0} R_{(I_0,J_0)}^{i_0}(v_{i_0}, v_{j_0})$ vanish. Since $R_{(I_0,J_0)}^{i_0}(v_{i_0}, v_{j_0}) \neq 0$, we have $\gamma_{(I_0,J_0)}^{i_0} = 0$. This is a contradiction which completes the proof.

Obviously, for every $1 \leq i < d$ we have

$$|B_i| = \begin{cases} \sum_{j=0}^{k-2} 2^{j+1} \frac{d-i}{j} & i \leq d-k+1 \\ \sum_{j=0}^{d-i-1} 2^{j+1} \frac{d-i}{j} & i \geq d-k+2 \end{cases}.$$ 

It is a well-known fact that

$$\sum_{i=0}^{n} \binom{m+i}{m} = \binom{m+n+1}{m+1}.$$ 

By this fact clearly, $|B| = \sum_{j=1}^{k-1} 2^j (d-j) - 2^k + 2$. Let $A_k = \{(M, N) \mid (M, N) \in A, \ |M \cup N| = k\}$ and $B$ is defined as Lemma 1. We claim that $\{P_i(x, y) \mid 1 \leq i \leq
n \} \cup \{ Q_{(M,N)} \mid (M, N) \in \mathcal{A}_k \} \) with all the polynomials \( R^i_{(I,J)}(x, y) \in \mathcal{B} \) remain linearly independent. Before prove the claim, we shall note that all polynomials in the set \( \mathcal{B} \) have this property that vanish in the point \((v_{B_i}, v_{A_j})\) for every \(1 \leq i \leq n\). Now, assume the claim is false and let

\[
\sum_{i=1}^{n} \alpha_i \rho_i(x, y) + \sum_{(M,N) \in \mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x, y) + \sum_{i=1}^{d-1} \sum_{(I,J) \in \mathcal{B}_i} \gamma^i_{(I,J)} R^i_{(I,J)}(x, y) = 0 \tag{7}
\]

be a nontrivial linear relation.

**Step 1.** Let \(i_0\) be a subscript such that \(d \notin A_{i_0}\) and \(\alpha_{i_0} \neq 0\). Substitute \((v_{B_i}, v_{A_{i_0}})\) for \((x, y)\) in the linear relation 7. We know that for every \(i\) and every \((I, J) \in \mathcal{B}_i\), \(R^i_{(I,J)}(v_{B_i}, v_{A_{i_0}}) = 0\). Also, \(d \in N\) so \(Q_{(M,N)}(v_{B_i}, v_{A_{i_0}}) = 0\). Using these and by 2 all terms in the linear relation 7 but \(\alpha_{i_0} \rho_i(v_{B_i}, v_{A_{i_0}})\) vanish. Since \(\rho_i(v_{B_i}, v_{A_{i_0}}) \neq 0\), therefore \(\alpha_{i_0} = 0\).

**Step 2.** According to the step 1 we have

\[
\sum_{d \in A_i}^{n} \alpha_i \rho_i(x, y) + \sum_{(M,N) \in \mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x, y) + \sum_{i=1}^{d-1} \sum_{(I,J) \in \mathcal{B}_i} \gamma^i_{(I,J)} R^i_{(I,J)}(x, y) = 0 \tag{8}
\]

Let \(i_0\) be a subscript such that \(\alpha_{i_0} \neq 0\). We define \(v'_{A_{i_0}}\) to be \(v_{A_{i_0}} - (0, \ldots, 0, 1)\). Substitute \((v_{B_i}, v'_{A_{i_0}})\) for \((x, y)\) in the linear relation 8. For \(1 \leq i \leq d - 1\) and every pair \((I, J) \in \mathcal{B}_j\), by the definition of \(R^i_{(I,J)}(x, y)\), it holds that \(R^i_{(I,J)}(v_{B_j}, v_{A_j}) = R^i_{(I,J)}(v_{B_j}, v'_{A_{i_0}})\) for every \(1 \leq j \leq n\). Now, similar in the step 1 all terms in the relation 8 but \(\alpha_{i_0} \rho_i(v_{B_i}, v'_{A_{i_0}})\) vanish. Since \(\rho_i(v_{B_i}, v'_{A_{i_0}}) \neq 0\), therefore \(\alpha_{i_0} = 0\). So, we have

\[
\sum_{(M,N) \in \mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x, y) + \sum_{i=1}^{d-1} \sum_{(I,J) \in \mathcal{B}_i} \gamma^i_{(I,J)} R^i_{(I,J)}(x, y) = 0 \tag{9}
\]

**Step 3.** Since \(d \in N\) and \(d \notin J\) for all \((I, J) \in \mathcal{B}_i\), \(i = 2, \ldots, d - 1\), if we evaluate equality 9 in \((v_I, v_J)\) then we conclude that \(\gamma_{(I,J)} = 0\). Hence we have

\[
\sum_{(M,N) \in \mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x, y) + \sum_{(I,J) \in \mathcal{B}_i} \gamma^1_{(I,J)} R^1_{(I,J)}(x, y) = 0 \tag{10}
\]

Assume that \((I_0, J_0)\) is a subscript such that has minimum cardinality in the set \(\mathcal{A}_1\) such that \(\gamma_{(I_0,J_0)} \neq 0\). Substituting \((v_{I_0}, v_{J_0})\) in the Eq. 10 then since \(|I_0 \cup J_0| \leq k - 1\) and \(|M \cup N| = k\) all terms in 10 but \(\gamma_{(I_0,J_0)} R^1_{(I_0,J_0)}(v_{I_0}, v_{J_0})\) vanish. So we have \(\gamma_{(I_0,J_0)} = 0\)

**Step 4.** By independence of the polynomials in the set \(\mathcal{P}\), each \(\beta_{(M,N)} = 0\), therefore the claim is true.
So, we have \( n + 2^{k-1} \binom{d-1}{k-1} + \sum_{i=1}^{k-1} 2^i \binom{d}{i} - 2^k + 2 \) linearly independent polynomials which, as the proof of Theorem 1, are in the space generated by \( \sum_{i=0}^{k} 2^i \binom{d}{i} \) monomials. Hence,

\[
n(K, d) \leq 2^k \binom{d}{k} - 2^{k-1} \binom{d-1}{k-1} + 2^k - 1.
\]

That complete the proof of Theorem 2.

### 4 Proof of Theorem 3

Let \( K = \{l_1, l_2, \ldots, l_k\} \), and \((A, B)\) be a cross \( K \)-intersection families with \( m \) blocks on a set of \( d \) points. For every \( 1 \leq j \leq d \), let

\[
X_j \overset{\text{def}}{=} \{ i \mid 1 \leq i \leq m, \; j \in A_i \},
\]

\[
Y_j \overset{\text{def}}{=} \{ i \mid 1 \leq i \leq m, \; j \in B_i \}.
\]

Now, for \( j = 1, 2, \ldots, d \), we construct the complete bipartite graph \( G_j \) with vertex set \((X_j, Y_j)\), where \( X_j \) and \( Y_j \) were defined as above. Let \( ij \) be an arbitrary edge of \( K_{m,m}^- \), consider sets \( A_i \) and \( B_j \). Without loss of generality assume that \( A_i \cap B_j = \{v_1, v_2, \ldots, v_l\} \), which \( 1 \leq l \leq k \). It is not difficult to see that the edge \( ij \) was covered by the graphs \( G_{v_1}, G_{v_2}, \ldots, G_{v_l} \). So we have a biclique cover of type \( K \) and size \( d \) of \( K_{m,m}^- \). Conversely let \( \{G_1, G_2, \ldots, G_d\} \) be a biclique cover of type \( K \) and size \( d \) of the graph \( K_{m,m}^- \). Assume \( G_i \) has \((X_i, Y_i)\) as the vertex set. For every \( 1 \leq j \leq m \) define

\[
A_j \overset{\text{def}}{=} \{ i \mid 1 \leq i \leq d, \; j \in X_i \},
\]

\[
B_j \overset{\text{def}}{=} \{ i \mid 1 \leq i \leq d, \; j \in Y_i \}.
\]

Let \( A = \{A_1, \ldots, A_m\} \) and \( B = \{B_1, \ldots, B_m\} \). Since \( X_i \cap Y_i = \emptyset \) for every \( 1 \leq i \leq d \) then \( A_j \cap B_j = \emptyset \) for every \( 1 \leq j \leq m \). Also, if \( \{G_{v_1}, G_{v_2}, \ldots, G_{v_l}\} \) is the set of graphs that cover the edge \( ij \) then \( A_i \cap B_j = \{v_1, v_2, \ldots, v_l\} \) where \( |A_i \cap B_j| \in K \). Hence \((A, B)\) is a cross \( K \)-intersection family.

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