Nonzero Classical Discord

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Quantum discord is the quantitative difference between two alternative expressions for bipartite mutual information, given respectively in terms of two distinct definitions for the conditional entropy. Whereas nonzero discord is touted as a form of quantum correlation, we show, by constructing a stochastic classical model of shared states, that discord indeed quantifies the presence of some stochasticity in the measurement process. We then establish an operational meaning of classical discord in the context of state merging with noisy measurement and thereby show the quantum-classical separation is not discord being nonzero but rather negative conditional entropy.

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Entanglement exemplifies the mystery of quantum mechanics, for example the conundrum of Schrödinger’s cat [1,3], and embodies the quintessential resource for quantum information processing, such as the consumable ebits for quantum teleportation [4]. Recently significant effort is expended on extending the notion of entanglement to generalized bipartite “quantum correlations”[5] with “quantum discord” the most ubiquitous of these measures and furthermore universal in the sense that only a negligibly few states have zero discord [6].

The huge effort into studying discord is driven by the optimism that, “Algorithms could instead tap into a quantum resource called discord, which would be far cheaper and easier to maintain in the lab” [3]. Just as entanglement is operationalized [6] by teleportation, discord is operationalized by state merging [3,12]. Therefore, discord can be understood not just as a mathematical characterization of a state but also as a quantity relevant for performing a certain information task. The question we address is whether this resource, operationalized by state merging, is quintessentially “quantum”.

To assess the quantumness of discord, we need to transcend the dichotomy of quantum vs. classical mechanics, in which something that is not classical, in the sense of deterministically evolving objects with arbitrarily precise properties, is ipso facto quantum. Development of a quantum “toy theory” [13] is an example of breaking this dichotomy by finding a self-consistent theory that is broader than classical theory yet not as powerful as quantum theory, and the advantage of studying such middle-ground theories is to ascertain how meaningful a resource is insofar as being quintessentially quantum.

A famous middle-ground approach arises by replacing classical mechanics by stochastic mechanics, or stochastic field theory [14], in which case either or both the precision of specifying states and the dynamics are sacrificed. Such theories are weaker than full quantum mechanics, and a quantum resource should then be identified with ruling out any self-consistent theory that is weaker than quantum mechanics. In quantum information science, the full power of quantum computing and the full security of quantum cryptography are expected to require the full power of quantum mechanics and not these lesser beasts. Thus, we now have the context for asking whether discord really is an indicator of this full quantum power or a manifestation of a lesser theory.

We begin by reviewing the essence of discord. Consider two parties sharing some state of information. A precisely known state of information can be represented by an n-bit string i ∈ {0,1}n, which we now generalize to a state in stochastic information theory.

Definition 1. A stochastic-information state is a distribution p(i) ≥ 0 such that ∑ p(i) = 1.

The Shannon entropy of this stochastic-information state is H = −∑ p(i) log p(i) with the base 2 logarithm notation suppressed. In the quantum case the state is a trace-class bounded positive operator ρ on the Hilbert space represented by C2⊗n with basis |i⟩, and the state is pure if ρ2 = ρ. The state’s von Neumann entropy is S = −tr(ρ log ρ).

Mutual information relates to shared information between two parties often called “Alice” (A) and “Bob” (B). In the stochastic classical case the joint state is the probability distribution

\[ P_{AB} = \sum_{i,j} P_A(i) P_B(j) \chi(i,j) \]  

for \( \chi(i,j) \) the \( 2^n \times 2^n \) matrix with zeros everywhere except at the location \( (i,j) \), which is the classical version of the quantum basis tensor product

\[ \chi(i,j) := |i⟩⟨i| ⊗ |j⟩⟨j|. \]

The strings \( i \) and \( j \) need not be equal length, but we make the restriction that Alice and Bob each hold \( n \)-bit strings for convenience of notation. In the quantum case the joint state \( ρ_{AB} \) acts on \( C_2^\otimes n \otimes C_2^\otimes n \).

Let \( H(A) \) signifying the entropy of \( A \)’s marginal distribution and similar for \( B \) and for \( A \) and \( B \). The mutual information \( I(A;B) = H(A) + H(B) - H(A, B) \) between
Alice and Bob is mathematically equivalent to $J(A; B) \equiv H(A) - H(A|B)$ for

$$H(A|B) = \sum_i p_A(i) \sum_j p_B(j|i) \log p_B(j|i).$$  \hspace{1cm} (3)

Operationally, this conditional entropy is obtained by Bob measuring and announcing his results to Alice.

If Alice’s state has zero entropy conditioned on Bob’s results, and vice versa for Bob’s state conditioned on Alice’s results, we refer to this state as pure.

Definition 2. The stochastic-information joint state $p_{AB}(i, j)$ is pure iff $H(A|B) = 0$.

Remark 1. The state is trivially pure if the Shannon entropy of the stochastic-information state is pure, but this extension of the notion of purity allows noisy preparation provided that Alice and Bob can still recover pure states by one-way communication. The state would not be pure if either Alice or Bob’s announcements are based on results from noisy measurements.

In the quantum case, the same expression for $I$ above holds but with Shannon entropy $H$ replaced by von Neumann entropy $S$. In that situation the expression for the conditional entropy $S(A|B) := S(A, B) - S(B)$, can be negative, signalling the presence of entanglement, thereby making its operational interpretation less straightforward.

To address the operational challenge of quantum conditional entropy, an alternative version explicitly dependent on one party’s choice of measurements has been introduced \[16\]. For $\Pi := \{\pi_k\}$ representing Bob’s choice of projective measurement basis and

$$\rho_{A|i} := \frac{\text{tr}_B[(\mathbb{I} \otimes \pi_k)\rho_{AB}]}{\text{tr}[(\mathbb{I} \otimes \pi_k)\rho_{AB}]}$$  \hspace{1cm} (4)

being the resultant states conditioned on Alice’s side, this measurement-dependent conditional quantum entropy is

$$S^\Pi(A|B) := \sum_k p_k S(\rho_{A|i}) \geq 0.$$  \hspace{1cm} (5)

Operationally this non-negative quantity represents how much information, on average, Alice can extract from her system given Bob’s measurement results.

Quantum discord, defined as \[16\]

$$D_{A\to B}(\rho_{AB}) := \min_{\Pi} S^\Pi(A|B) - S(A|B)$$  \hspace{1cm} (6)

is equivalent to

$$D_{A\to B}(\rho_{AB}) = I(A; B) - \max_{\Pi} J^\Pi(A; B),$$  \hspace{1cm} (7)

with

$$J^\Pi(A; B) := S(A) - S^\Pi(A|B),$$  \hspace{1cm} (8)

being the version of mutual information that employs measurement-dependent quantum conditional entropy. The term $J^\Pi(A; B)$ is the “classical” part of the correlations; hence, nonzero quantum discord is attributed to genuine quantum correlations independent of measurements regardless of the measurement basis. In this perspective, zero discord is proclaimed as being classical due to $I \equiv J$. An interpretation of quantum discord as given by Eq. (6) is that of a measure of how much the bipartite quantum system is affected by local measurements. We argue in the following that such interpretation is directly applicable to the classical case, where we define the classical discord and show that its properties closely resembles the ones of quantum discord.

Our aim is to show that nonzero discord can hold classically by treating conditional entropy operationally and seeing that nonzero discord arises from noisy (or imperfect) measurement \[17\]. In other words we now establish that $I \neq J$ if measurement is at all noisy. This noisy measurement arises naturally within the stochastic reconciliation protocol used to construct the conditional entropy.

In the classical case, we treat Bob’s measurement as imperfect, which is equivalent to Bob using a noisy channel $M$, represented by the stochastic transition matrix $M$, followed by perfect measurements of his bit string. Unlike the quantum case, with uncountable basis choices $\{\Pi\}$, the classic noiseless measurement can be performed only in one basis $\{i\}$. As a stochastic matrix, $\sum_i M(i, j) = 1 \forall j$. The noiseless and maximally noisy cases correspond to $M = I$ and to $M(i, j) = 2^{-n} \forall i, j$, respectively.

Now we define classical discord analogous to the quantum discord \[17\] but with quantum measurement-based mutual information $J^\Pi$ replaced by a noisy classical measurement-based counterpart defined below. The noisy measurement apparatus is represented by a channel $M$, which can be represented by a stochastic matrix $M$.

Definition 3. The stochastic-information state with added stochasticity due to Bob’s measurement apparatus (quantified by stochastic matrix $M$) is

$$p_{AB'} := p_{AB} M^T,$$  \hspace{1cm} (9)

for $T$ denoting the matrix transpose, and the subscript $B'$ denotes that the noise is on Bob’s side.

Definition 4. The mutual information of noisy state $p_{AB'}$ is

$$J^M(\rho_{AB}) := I(p_{AB'}) = I(p_{AB} M^T).$$  \hspace{1cm} (10)

Definition 5. The classical discord of state $p_{AB}$ subjected to $B$’s measurement with noise $M$, represented by stochastic matrix $M$,

is

$$D^M_{A\to B}(p_{AB}) := I(A; B) - J^M(p_{AB}).$$  \hspace{1cm} (11)

Proposition 1. Classical discord is non-negative.

Proof. $D^M_{A\to B}(p_{AB}) \geq 0$, $\forall p_{AB}$ follows immediately from the data processing inequality \[18\].
As any classical [1] can be “embedded” into a quantum bipartite state $\rho_{AB}$ by replacing $\chi(i, j)$ by the quantum projector \[2\], namely $\rho_{AB} = \sum_{i, j} \rho_{AB}(i, j)\chi(i, j)$, the necessary and sufficient conditions for zero quantum discord [16] must also be necessary and sufficient for classical discord. Equivalently, $\rho_{AB}$ must be invariant under the measurement $M$ on $B$ side, which we formalize in the following proposition.

**Proposition 2.** The classical discord of the state $\rho_{AB}$ with noise $M$ on $B$ is zero, namely $D_{A \rightarrow B}^M(\rho_{AB}) = 0$, iff

$$p_{AB} = p_{AB}'.$$  

(12)

We now investigate condition $\[12\]$ for zero classical discord in two cases: i) for a fixed noise $M$, we find all possible states that have zero classical discord, and ii) for a given state $p_{AB}$ we show how to find all noisy channels that induce zero classical discord.

**Case i** We employ reshaping to find all possible states $\rho_{AB}$ that satisfy $\[12\]$. A square matrix $M$ can be reshaped into a column vector $C$ by employing a column-major order; i.e., each column is stacked on top of its next adjacent column. Arbitrary $A, B, C, D$ square matrices satisfy the identity

$$ACB^T = D \iff (A \otimes B)C = D$$  

(13)

so Prop.\[2\] is equivalent to

$$D_{A \rightarrow B}^M(\rho_{AB}) = 0 \iff (1 \otimes M)p_{AB} = p_{AB}.$$  

(14)

In other words, $p_{AB}$ must be a fixed point of $1 \otimes M$ and must also be a valid probability vector. Fixed points with such properties are called stationary vectors.

Stochasticity of $M$ implies stochasticity of $1 \otimes M$, and the Perron-Frobenius theorem states that any stochastic matrix has at least one stationary vector. Therefore, a solution to the right-hand side $\[14\]$ always exists. To find the class of all $R$ possible states of zero discord, let $\{m_1, m_2, \ldots, m_R\}$ be the largest set of linearly independent stationary vectors of $M (R = 1$ if all entries of $M$ are strictly positive).

Thus, a stationary vector of $1 \otimes M$ must have the form

$$\sum_{j=0}^{2^n} \sum_{k=0}^{R} q_{jk} m_k = \sum_{j=0}^{2^n} \sum_{k=0}^{R} q_{jk} = 1, \quad q_{jk} \geq 0 \forall j, k,$$  

(15)

and $m_j$ denotes the $j$th eigenvector of $M$, i.e., the vector comprising all zeros except for a unique 1 at the $j$th position. Therefore, a class of zero-discord states $S_0 = \{p_{AB}\}$ exists with each $p_{AB}$ of the form $\[15\].$

**Case ii** For a given fixed state $\rho_{AB}$, we now show how to find all possible noisy channels that induce zero classical discord. The right-hand side of $\[14\]$ can be rewritten as $(1 \otimes M - I)p_{AB}^T = 0$, which holds iff

$$(1 \otimes p_{AB})M = 1.$$  

(16)

Thus, given $p_{AB}$, the class of channels for which the classical discord is zero can be found from solving $\[16\]$ with the additional restriction that $M$ is a valid representation of a classical channel (i.e., its reshaping $M$ is stochastic). This last restriction is a linear constraint, and, therefore, the solutions to $\[16\]$ can be found using linear programming [19].

Just as zero-discord quantum states have zero measure [20], zero-discord classical states have zero measure except in the singular case of perfect measurement, as formalized below.

**Proposition 3.** The set of classical zero-discord states has measure zero in the set of all classical bipartite states except for noiseless measurements.

**Proof.** A zero-discord state can be represented by the distribution $\{q_{jk}\} \[15\]$, which has a $2^nR$-dimensional domain. Unless $R = 2^n$, which pertains to $M = I$, the domain of the distribution is strictly lower in dimension so the set of zero-discord states has measure zero unless the channel is noiseless.

We now have a definition of classical discord, demonstrated its natural correspondence with quantum discord, showed that nonzero classical discord is due to measurement noise on Bob’s side, and proved that states with zero classical discord occupy zero measure. Now we establish operational meaning via state merging, analogous to operationalized quantum discord.

**Definition 6.** State merging is a two-party task whereby the bipartite state $\rho_{AB}$ comprising two shares of size $n$ and $m$ bits is merged into a unipartite state of size $n + m$ bits held by one party such that the merged state has the same distribution as the original bipartite state.

**Remark 2.** The term “state merging” arose in quantum information [13] but quantizes the notion of compressing correlated classical data streams [21, 22]. We use the term state merging for the classical case because our objective is to connect quantum discord with classical information theory so transferring quantum terminology to the classical domain is appropriate here.

We consider two-bit state merging (Alice holds one bit and Bob holds the other bit) as this simple case suffices to establish operational meaning for classical discord. The following argument is readily extended to multiple bits held by each party. In the quantum case, the operational interpretation of quantum discord arises through a Gel’fand-Naimark purification of the bipartite state $\rho_{AB}$ with the resultant tripartite state $\rho_{ABC}$ being pure [23]. The third party, Charlie (C), assists Alice and Bob with merging their states.

We purify $\rho_{AB}$ to a tripartite state according to Def.\[2\] except that in this tripartite case purity means that $H(A|BC) = 0$ and also for $B|CA$ and $C|AB$. Specifically the purified state is

$$p_{ABC} = (1 - q)\chi_{abc} + q\chi_{\bar{a}\bar{b}\bar{c}}, \quad a, b, c \in \{0, 1\}, \quad 0 \leq q \leq 1$$  

(17)

with $q$ an arbitrary mixing parameter associated with noisy preparation and $\bar{a}$ denoting the logical negation of $a$; i.e.,
In the quantum case, the conditional entropy for a single bit with flip probability \( q \) is given by

\[
H(q) = -q \log(q) - (1-q) \log((1-q)).
\]

The last line of (18) follows from the previous line because a simple calculation yields \( H(q) = -q \log(q) - (1-q) \log((1-q)) \) and the fact that the noise on \( B \)’s side is assumed to be the same as the noise on \( C \)’s side.

**Remark 3.** In the quantum case, \( D_{A-C}^M(p_{AC}) \) is modified by including entanglement of formation between \( A \) and \( B \), and entanglement of formation must be zero classically as entanglement is forbidden. Hence, in the quantum case, conditional entropy can be negative as shown in quantum state merging, but not in the classical case due to entanglement of formation being zero.

We have addressed the question of whether discord is an inherently quantum correlation or is rather a manifestation of noise, modelled by stochastic information theory. Our findings demonstrate that discord is indeed a quantifier of stochasticity. In the non-quantum case stochasticity is introduced by imperfect or noisy measurements, while in the quantum case it is introduced by the arbitrariness of the local measurement basis. In both situations the discord measures how much the joint state is affected by a local measurement. In that sense discord is not necessarily a quantifier of a quantum phenomenon or resource.

Our method for showing the classicality of non-zero discord has been to construct a model for stochastic information, which incorporates a noisy measurement represented by a stochastic channel. A direct consequence of measurement being noisy is the inequivalence of defining classical mutual information in terms of joint vs conditional entropy. In an operational treatment of conditional measurement, conditional entropy involves measuring a string and announcing these results to the other party. Nonzero noise in the measurement causes mutual information based on conditional entropy to differ from mutual information based on joint entropy.

A fascinating consequence of our classical discord investigation is that the role of genuine quantumness, manifested as entanglement, is precisely the negativity of conditional information. This negativity only arises if entanglement of formation is nonzero. Thus, discord can be understood classically as a stochastic information figure of merit except when entanglement of formation is non-zero. Operationally nonzero entanglement of formation is precisely what distinguishes our classical state merging from genuine quantum state merging [9,10].

In addition to establishing the classical nature of nonzero discord and the importance of entanglement in making discord quantum, our method of stochastic information theory with noisy preparation (especially the classical analogue of a pure state) and noisy measurement (which breaks the equivalence of the two methods of defining mutual information in terms of conditional vs joint entropy), provides a strong test of the quantumness of a plethora of new quantum correlations measures that have emerged recently [24].

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