TAKENS THEOREM FOR RANDOM DYNAMICAL SYSTEMS

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Dedicated to Björn Schmalfuß on the occasion of his 60th birthday

Abstract. In this paper, we study random dynamical systems with partial
hyperbolic fixed points and prove the smooth conjugacy theorems of Takens
type based on their Lyapunov exponents.

1. Introduction. This paper is a continuation of our previous one [21] on smooth
linearization for random dynamical systems. In [21] we proved the smooth con-
jugacy theorems of Sternberg type for random dynamical systems based on their
Lyapunov exponents, where we assumed that there is no zero Lyapunov exponent.

In this paper, we study the case that has zero Lyapunov exponents.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\theta^n)_{n \in \mathbb{Z}}$ be a measurable
$\mathbb{P}$-measure preserving dynamical system on $\Omega$. A random dynamical systems (or a cocycle) on
the space $\mathbb{C}^d$ over the dynamical system $\theta^n$ is a measurable map

$$\phi : \mathbb{Z} \times \Omega \times \mathbb{C}^d \to \mathbb{C}^d, \; (n, \omega, x) \mapsto \phi(n, \omega, x),$$

such the map $\phi(n, \omega) := \phi(n, \omega, \cdot) : \mathbb{C}^d \to \mathbb{C}^d$ forms a cocycle over $\theta^n$:

$$\phi(0, \omega) = Id, \; \text{for all} \; \omega \in \Omega,$$

$$\phi(n + m, \omega) = \phi(n, \theta^m \omega) \circ \phi(m, \omega), \; \text{for all} \; m, n \in \mathbb{Z}, \; \omega \in \Omega.$$

in A typical example is the time-map of the solution operator for a stochastic
differential equation. When $\phi(n, \omega, \cdot)$ is differentiable for each $n$ and $\omega$, $\phi$ is called
a differentiable random dynamical system. We write the time-one map $\phi(1, \omega, x)$ as $\phi(\omega, x) := \phi(1, \omega, x)$. Then $\phi(\omega, \cdot)$ is the so-called random diffeomorphism that generates the random dynamical system $\phi(n, \omega, x)$

$$\phi(n, \omega, \cdot) = \begin{cases} \phi(\theta^{n-1} \omega, \cdot) \circ \cdots \circ \phi(\omega, \cdot), & n > 0 \\ I, & n = 0 \\ \left(\phi(\theta^n \omega, \cdot)\right)^{-1} \circ \cdots \circ \left(\phi(\theta^{-1} \omega, \cdot)\right)^{-1}, & n < 0. \end{cases}$$

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We consider $C^N$, $1 < N \leq \infty$, random diffeomorphisms $\phi(\omega, x)$ in $\mathbb{R}^d$ with a fixed point $^1 x = 0$, i.e., $\phi(\omega, 0) = 0$ for all $\omega \in \Omega$. We assume that $\phi(\omega, x)$ is a locally tempered $C^N$ random diffeomorphism, that is, there is a tempered ball $B_p(\omega)(0) = \{x \mid |x| < p(\omega)\}$, where $p(\omega)$ is a random variable tempered from below (i.e., $\lim_{n \to \pm \infty} \frac{1}{n} \log^+ \rho(p^\omega(\omega)) = 0$, $\mathbb{P}$ - a.s.), such that each
\[
\sup_{x \in B_p(\omega)} \|D\phi(\omega, x)\| = C_i(\omega), \quad 0 \leq i \leq N
\]
is tempered from above (i.e., $\lim_{n \to \pm \infty} \frac{1}{n} \log^+ C_i(\theta^n(\omega)) = 0$, $\mathbb{P}$ - a.s.). The size of a tempered ball may decrease as $\omega$ varies, but these changes along each orbit $\theta^n\omega$ are at a subexponential rate. The upper bound of $C_i(\omega)$ may grow to infinity as $\omega$ varies. But along each orbit $\theta^n\omega$, it may increase only at a subexponential rate. This nonuniform behavior is one of the intrinsic features of random dynamical systems.

Two local tempered random diffeomorphisms $\phi$ and $\psi$ with fixed point $x = 0$ are $C^k$ locally conjugate for $1 \leq k \leq \infty$ if there exists a $C^k$ random diffeomorphism $h(\omega, x)$ defined on a tempered ball $V(\omega)$ with $h(\omega, 0) = 0$ such that
\[
h(\theta\omega, \phi(\omega, x)) = \psi(\omega, h(\omega, x)) \quad \text{for} \quad x \in V(\omega), \quad \text{a. s.} \quad \omega \in \Omega
\]
This conjugacy relationship implies that $h$ carries orbits of $\phi$ to orbits of $\psi$, when the orbits stay in the corresponding domains.

We write $\phi(\omega, x)$ as
\[
\phi(\omega, x) = A(\omega)x + F(\omega, x),
\]
where $A(\omega) = D\phi(\omega, 0) \in GL(d, \mathbb{R})$, $F(\omega, 0) = 0$ and $DF(\omega, 0) = 0$. We assume that the conditions of the Multiplicative Ergodic Theorem hold, that is,
\[
\log^+ \|A(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad \log^+ \|A^{-1}(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}).
\]
Then, by the celebrated Oseledets’ Multiplicative Ergodic Theorem (see Theorem 2.1), there exists a $\theta$-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$, $\Phi(n, \omega)$ (the linear random dynamical system generated by $A(\omega)$) has $p(\omega)$ Lyapunov exponents:
\[
\lambda_1(\omega) > \cdots > \lambda_{p(\omega)}(\omega),
\]
and the phase space $\mathbb{R}^d$ has an invariant splitting:
\[
\mathbb{R}^d = E_c(\omega) \oplus E_u(\omega) \oplus E_s(\omega),
\]
where $E_c(\omega)$, $E_u(\omega)$, and $E_s(\omega)$ are the Oseledets’s spaces corresponding to the zero Lyapunov exponents, the positive Lyapunov exponents, and the negative Lyapunov exponents, respectively. Furthermore, $p(\omega)$, $\lambda_i(\omega)$ are $\theta$-invariant functions, i.e., $p(\theta\omega) = p(\omega)$ and $\lambda_i(\theta\omega) = \lambda_i(\omega)$. When $\theta^n$ is ergodic, $p(\omega)$ and $\lambda_i(\omega)$ are independent of $\omega$.

For the remainder of the paper, we assume that $\theta^n$ is ergodic. Otherwise, we restrict our study to each ergodic component.

In the next section, we will see that there is a linear random transformation which changes the linear part $A(\omega)$ of $\phi$ to a block diagonal:
\[
\begin{bmatrix}
A^c(\omega) & 0 & 0 \\
0 & A^u(\omega) & 0 \\
0 & 0 & A^s(\omega)
\end{bmatrix},
\]
and preserve the Lyapunov spectrum.

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1 A large class of random dynamical systems may be converted into this case. See [1], page 310
Our main result can be summarized as

**Theorem.** (Takens Theorem for Random Diffeomorphisms) For each integer \( k > 0 \), there exists a positive integer \( N_0 = N_0(k) \) such that

(i) If \( \phi \) and \( \psi \) are \( C^{N_0} \) locally tempered random diffeomorphisms and have a common center manifold \( W^c(\omega) \) with \( (\phi - \psi) = O(|x|^{N_0}) \) for \( x \in W^c(\omega) \), then \( \phi \) and \( \psi \) are \( C^k \) locally conjugate.

(ii) If \( \phi \) is \( C^{N_0} \) tempered and the nonzero Lyapunov exponents satisfy

\[
(\lambda^h, \tau) \neq 0, \quad 2 \leq |\tau| \leq N_0,
\]

where \( (\lambda^h, \tau) := \sum_{i \neq 0} \lambda_i \tau_i \), then \( \phi \) is locally \( C^k \) conjugate to a RDS whose time one map has the form

\[
\psi(x, \omega) = \begin{pmatrix}
A_c^e(\omega)x_c + F_c(\omega, x_c) \\
A_u^e(\omega)x_u + F_u(\omega, x) \\
A_s(\omega)x_u + F_s(\omega, x)
\end{pmatrix},
\]

where \( x = (x_c, x_u, x_s) \in \mathbb{R}^d = E^c \oplus E^u \oplus E^s \).

(iii) If, in addition to the condition (2), the nonzero Lyapunov exponents also satisfy

\[
(\lambda^h, \tau) \neq \lambda_i, \quad 2 \leq |\tau| \leq N_0, \lambda_i \neq 0,
\]

then \( \phi \) is locally \( C^k \) conjugate to a RDS whose time one map has the form

\[
\psi(x, \omega) = \begin{pmatrix}
A_c^e(\omega)x_c + F_c(\omega, x_c) \\
B_u^e(\omega, x_c)x_u \\
B_s(\omega, x_c)x_s
\end{pmatrix}.
\]

**Remark.** (i) When \( \phi \) is a deterministic diffeomorphism, this result was proved by Takens [32]. (ii) When \( \theta^n \) is not ergodic, the quantities such as the dimensions of the stable and unstable Oseledets subspaces may vary as \( \omega \) changes. \( \Omega \) may be decomposed into a union of countably many disjoint \( \theta \)-invariant measurable sets \( \tilde{\Omega}_i \) such that on each \( \tilde{\Omega}_i \), these quantities are constant. The construction of conjugacy is carried out independently over each invariant set \( \tilde{\Omega}_i \). Patching them together gives the conjugacy over the whole \( \Omega \).

The study of smooth conjugacy of deterministic dynamical systems to their normal forms has a long and rich history. Analytic linearization was first studied by Poincaré and Birkhoff, later by Siegel [40], Arnold [3], Moser [28], Zehnder [44], Brjuno [8], and others. There is an extensive literature on \( C^k \) smooth linearization that was initiated by Sternberg [41, 42]. Some classical results may be found in Nagumo and Isé [30], Chen [10], Hartman [16, 17], and Nelson [29]. For more delicate conditions such that the nonlinear system admits a \( C^k \)-smooth linearization we refer to Sell [37, 38], Beleskii [6, 7], ElBialy [13], Zhang and Zhang [45]. A result which preserves the geometric structure of the original equation was recently obtained by Banyaga, de la Llave, and Wayne [5]. A \( C^0 \) linearization (A Hartman-Grobman Theorem) associated with the local structural stability may be found in Grobman [14], Hartman [16], Pugh [31], and Kirchgraber and Palmer [33].

For random dynamical systems, Wanner [43] proved a Hartman-Grobman theorem, also see [1] and [12] for topological conjugacy. A structural stability theorem for deterministic dynamical systems under random perturbations was obtained by P-D. Liu [24]. In [2], also see [1], Arnold and Xu gave a theorem on formal linearization of random diffeomorphisms. In [21], we established theorems of Poincaré type...
for random dynamical systems. In [22], we established theorems of Sternberg type for random dynamical systems. A theorem of Siegel type for random dynamical systems was presented in [23].

In Section 2, we introduce basic concepts on random dynamical systems, the Multiplicative Ergodic Theorem, and some basic lemmas; In Section 3, we prove our main results.

2. Random dynamical systems and center manifolds. In this section, we first review some of the basic concepts and results on random dynamical systems including the Multiplicative Ergodic Theorem, which are taken from [1]. We also introduce basic notations and state the assumptions on the systems. Then, we review the theorems on center-unstable manifolds, center-stable manifolds, and center manifolds for random dynamical systems, which we borrowed from [15].

2.1. Random dynamical systems. In this paper, we consider time-discrete \( C^N \), \( N \geq 2 \) random dynamical systems. We assume that 0 is a fixed point of \( \phi(n, \omega, \cdot) \), i.e., \( \phi(n, \omega, 0) = 0 \) for any \( n \in \mathbb{Z} \) and \( \omega \in \Omega \). Then, we rewrite \( \phi(n, \omega, x) \) as

\[
\phi(n, \omega, x) = \Phi(n, \omega) x + f(n, \omega, x),
\]

where \( \Phi(n, \omega) := D_x \phi(n, \omega, 0) \in \text{Gl}(d; \mathbb{R}) \) and the nonlinear term \( f(n, \omega, x) \) and its derivative in \( x \) are vanish at \( x = 0 \). By the cocycle property of \( \phi \), \( \Phi(n, \omega) \) is a linear cocycle with two-sided time over \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta^n)_{n \in \mathbb{Z}}) \).

Let \( A(\omega) := \Phi(1, \omega) \) and \( F(\omega, x) = f(1, \omega, x) \). Then, we write the time-one map \( \phi(1, \omega, x) \) as

\[
\phi(\omega, x) := \phi(1, \omega, x) = A(\omega)x + F(\omega, x).
\]

Note that \( F(\omega, 0) = 0 \) and \( D_x F(\omega, 0) = 0 \). In addition, \( \Phi(n, \omega) \) is generated by \( A(\omega) \):

\[
\Phi(n, \omega) = \begin{cases} 
A(\theta^{-1}\omega) \cdots A(\omega), & n > 0, \\
I, & n = 0, \\
A^{-1}(\theta^n\omega) \cdots A^{-1}(\theta^{-1}\omega), & n < 0.
\end{cases}
\]

Let \( \{x_n\}_{n \in \mathbb{Z}} \) be an orbit of \( \phi(n, \omega, x) \) with initial value \( x_0 \), i.e., \( x_n := \phi(n, \omega, x) \). Then, \( \{x_n\}_{n \in \mathbb{Z}} \) satisfies the equation

\[
x_{n+1} = A(\theta^n\omega)x_n + F(\theta^n\omega, x_n), \ n \in \mathbb{Z}.
\]

Conversely, a solution of the above equation is also an orbit of \( \phi(n, \omega, x_0) \).

The next concept is of fundamental importance in the study of random dynamical systems.

**Definition 2.1.**  
(1): A random variable \( R : (\Omega, \mathcal{F}) \to (\mathbb{R}^+ \setminus \{0\}, \mathcal{B}(\mathbb{R}^+ \setminus \{0\})) \) is called tempered with respect to a metric dynamical system \( \theta^n \) if

\[
\lim_{n \to \pm\infty} \frac{1}{n} \log R(\theta^n \omega) = 0 \ \mathbb{P}\text{-a.s.}
\]

(2): A random variable \( R : (\Omega, \mathcal{F}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)) \) is called tempered from above if

\[
\lim_{n \to \pm\infty} \frac{1}{n} \log^+ R(\theta^n \omega) = 0 \ \mathbb{P}\text{-a.s.}
\]

(3): A random variable \( R : (\Omega, \mathcal{F}) \to ((0, +\infty], \mathcal{B}(0, \infty]) \) is called tempered from below if \( 1/R \) is tempered from above.
Moreover, we recall that a multifunction \( W = \{ W(\omega) \}_{\omega \in \Omega} \) of nonempty closed sets \( W(\omega), \omega \in \Omega \), contained in \( \mathbb{R}^d \) is called a random set if
\[
\omega \mapsto \inf_{y \in W(\omega)} |x - y|
\]
is a random variable for every \( x \in \mathbb{R}^d \).

**Definition 2.2.** A random set \( W(\omega) \) is called an invariant set for a random dynamical system \( \phi(n, \omega, x) \) if
\[
\phi(n, \omega, W(\omega)) \subset W(\theta^n \omega) \quad \forall n \in \mathbb{Z}^+.
\]

2.2. **Multiplicative Ergodic Theorem and nonuniform partial hyperbolicity.** The following theorem is the Multiplicative Ergodic Theorem [1, pp. 134, 153].

**Theorem 2.1.** (Multiplicative Ergodic Theorem) Let \( \Phi \) be a linear random dynamical system over the metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta^n)_{n \in \mathbb{Z}})\). Assume that
\[
\log^+ \|A(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \quad \log^+ \|A^{-1}(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}).
\]
Then there exists an invariant subset \( \tilde{\Omega} \subset \Omega \) of full measure such that for each \( \omega \in \tilde{\Omega} \) the following hold:

1. The \( \lim_{n \to +\infty} (\Phi(n, \omega)^* \Phi(n, \omega))^{1/2n} =: \Psi(\omega) > 0 \) exists.
2. Let \( e^{\lambda_1(\omega)} < \cdots < e^{\lambda_1(\omega)} \) be the different eigenvalues of \( \Psi(\omega) \), and let \( U_{p(\omega)}(\omega), \ldots, U_1(\omega) \) be the corresponding eigenspaces with multiplicities \( d_i(\omega) \):
\[
\mathbb{R}^d = U_1(\omega) \oplus \cdots \oplus U_{p(\omega)}(\omega).
\]
3. If \((\Omega, \mathcal{F}, \mathbb{P}, (\theta^n)_{n \in \mathbb{Z}})\) is ergodic, the functions \( p(\omega), \lambda_i(\omega) \) and \( d_i(\omega) \) are constant on \( \tilde{\Omega} \).
4. For each \( \omega \in \tilde{\Omega} \), there exists a splitting
\[
\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_{p(\omega)}(\omega)
\]
of \( \mathbb{R}^d \) into random subspaces \( E_i(\omega) \) with dimension \( d_i(\omega) \). Moreover, if \( P^i(\omega) : \mathbb{R} \to E_i(\omega) \) denotes the corresponding projection onto \( E_i(\omega) \), then
\[
A(\omega)P^i(\omega) = P^i(\theta(\omega))A(\omega),
\]
equivalently,
\[
A(\omega)E_i(\omega) = E_i(\theta(\omega)).
\]
5. We have
\[
\lim_{n \to +\infty} \frac{1}{n} \log |\Phi(n, \omega)x| = \lambda_i(\omega) \iff x \in E_i(\omega) \setminus \{0\}.
\]
6. The functions \( \omega \mapsto p(\omega) \in \{1, \ldots, d\}, \omega \mapsto \lambda_i(\omega) \in \mathbb{R}, \omega \mapsto d_i(\omega) \in \{1, \ldots, d\}, \omega \mapsto E_i(\omega), \) and \( \omega \mapsto P^i(\omega) \) are measurable.

Here \( \lambda_i(\omega) \) and \( E_i(\omega) \) are so-called Lyapunov exponents and Oseledets spaces, respectively. In the remainder of this paper, we denote \( \tilde{\Omega} \) by \( \Omega \) and assume that all statements are true for \( \omega \in \Omega \). For \( \phi(n, \omega, x) \), we assume
Hypothesis 1. $\phi(n, \omega, x)$ is a $C^N$ random dynamical system for $2 \leq N \leq \infty$ and $\Phi(n, \omega) = D\phi(n, \omega, 0)$ satisfies the conditions of the Multiplicative Ergodic Theorem.

We divide the Lyapunov exponents into three groups based their signs. Let

$$\sigma_u(\omega) := \{\lambda_i(\omega) > 0\}, \quad \sigma_s(\omega) := \{\lambda_i(\omega) < 0\}, \quad \sigma_c(\omega) := \{\lambda_i(\omega) = 0\}$$

and denote

$$E^u(\omega) := \bigoplus_{\lambda_i(\omega) \in \sigma_u(\omega)} E_i(\omega),$$

$$E^s(\omega) := \bigoplus_{\lambda_i(\omega) \in \sigma_s(\omega)} E_i(\omega),$$

$$E^c(\omega) := \bigoplus_{\lambda_i(\omega) \in \sigma_c(\omega)} E_i(\omega)$$

with corresponding projections

$$P^u(\omega) : \mathbb{R}^d \mapsto E^u(\omega), \quad P^s(\omega) : \mathbb{R}^d \mapsto E^s(\omega), \quad P^c(\omega) : \mathbb{R}^d \mapsto E^c(\omega).$$

Then

$$\mathbb{R}^d = E^u(\omega) \oplus E^s(\omega) \oplus E^c(\omega).$$

We call $E^u(\omega)$ unstable Oseledets subspace, $E^s(\omega)$ stable Oseledets subspace and $E^c(\omega)$ center Oseledets subspace. Let $d_u(\omega)$, $d_s(\omega)$ and $d_c(\omega)$ denote the dimensions of $E^u(\omega)$, $E^s(\omega)$ and $E^c(\omega)$, respectively. From Theorem 2.1, $d_u(\omega)$, $d_s(\omega)$ and $d_c(\omega)$ are measurable functions from $\Omega$ to $\{1, \ldots, d\}$ and $P^u(\omega), P^s(\omega)$ and $P^c(\omega)$ are measurable projections. When $\theta$ is not ergodic, $\Omega$ can be decomposed into a union of $l$ disjoint $\theta$-invariant measurable sets

$$\Omega = \bigcup_{i=1}^l \Omega_i,$$

where on each $\Omega_i$, $d_u(\omega)$, $d_s(\omega)$ and $d_c(\omega)$ are constant. One can build the center-unstable manifold, the center-stable manifold and the center manifold over $\Omega_i$, then patch them together to get dimension-varying invariant manifolds on the whole $\Omega$.

Next, we assume that the fixed point is partially hyperbolic.

Hypothesis 2. $\sigma_c(\omega) \neq \emptyset$, i.e., $\Phi(n, \omega)$ has zero Lyapunov exponents.

The following lemma is on the nonuniform partial hyperbolicity of the system, which is a consequence of the Multiplicative Ergodic Theorem 2.1.

Lemma 2.1. There exists $\theta$-invariant random variable $\beta(\omega) > 0$ such that for each $\theta$-invariant random variable $\alpha(\omega)$, $\beta(\omega) > \alpha(\omega) > 0$, there is a tempered random variable $K(\omega) : \Omega \to [1, +\infty)$ such that

$$\|\Phi(n, \omega)P^u(\omega)\| \leq K(\omega)e^{\beta(\omega)n} \forall n \leq 0,$$

$$\|\Phi(n, \omega)P^s(\omega)\| \leq K(\omega)e^{-\beta(\omega)n} \forall n \geq 0,$$

$$\|\Phi(n, \omega)P^c(\omega)\| \leq K(\omega)e^{\alpha(\omega)|n|} \forall n \in \mathbb{Z}.$$

Here $\beta(\omega)$ is chosen to be smaller than the absolute values of all non-zero Lyapunov exponents. For example, one may choose

$$\beta(\omega) = \frac{1}{2} \min\{|\lambda_i(\omega)|, \lambda_i(\omega) \neq 0, i = 1, \cdots, p(\omega)|.$$
As \( \omega \) varies, \( \beta(\omega) \) may be arbitrarily small and \( K(\omega) \) may be arbitrarily large. However, along each orbit \( \theta^n \omega, \beta(\omega) \) is a constant and \( K(\omega) \) can increase only at a subexponential rate. When \( \theta \) is ergodic, both \( \alpha \) and \( \beta \) are constants.

Note the invariant splitting of \( \mathbb{R}^d = E_1(\omega) \oplus \ldots \oplus E_{p(\omega)}(\omega) \) is dependent of \( \omega \). But, by corollary 4.3.12 in [1], we can choose a new coordinate system so that the corresponding Oseledets spaces are deterministic. We restate the fact as follows.

**Lemma 2.2.** There exists a measurable map 
\[ P : \Omega \to GL(d, \mathbb{R}) \]
such that
\[ P(n, \omega) \text{ is conjugate to a block diagonal random dynamical system, i.e.} \]
\[ P(\theta^n \omega)P(n, \omega)P^{-1}(\omega) = \Psi(n, \omega) = \text{diag} (\Psi_1(n, \omega), \ldots, \Psi_p(\omega)(n, \omega)), \]
where \( \Psi_i(n, \omega) \) are cocycles of size \( d_i(\omega) \).

(2) The transformation \( P \) preserves the Lyapunov spectrum
\[ \{(\lambda_i(\omega), d_i(\omega)) | 1 \leq i \leq p(\omega)\} \]
and the corresponding Oseledets spaces
\[ \tilde{E}_i(\omega) = \{0\} \times \cdots \times \{0\} \times \mathbb{R}^{d_i(\omega)} \times \{0\} \times \cdots \times \{0\} \subset \mathbb{R}^d. \]

(3) \( ||P(\omega)|| \) and \( ||P^{-1}(\omega)|| \) are tempered.

Thus, the unstable, stable and center Oseledets subspaces for \( \Psi \) are
\[ \tilde{E}^u(\omega) := \bigoplus_{\lambda_i(\omega) \in \sigma_+(\omega)} \tilde{E}_i(\omega), \]
\[ \tilde{E}^s(\omega) := \bigoplus_{\lambda_i(\omega) \in \sigma_-(\omega)} \tilde{E}_i(\omega), \]
\[ \tilde{E}^c(\omega) := \bigoplus_{\lambda_i(\omega) \in \sigma_c(\omega)} \tilde{E}_i(\omega). \]

Then \( \mathbb{R}^d \) has an orthogonal decomposition:
\[ \mathbb{R}^d = \tilde{E}^u(\omega) \oplus \tilde{E}^s(\omega) \oplus \tilde{E}^c(\omega). \]

We still use \( P^u(\omega), P^s(\omega), \) and \( P^c(\omega) \) to denote the corresponding projections. The following lemma is a consequence of Lemmas 2.1 and 2.2.

**Lemma 2.3.** Assume that Hypotheses 1 and 2 hold. There exists a \( \theta \)-invariant random variables \( \beta : \Omega \to (0, \infty) \) such that for each \( \theta \)-invariant random variables \( \alpha(\omega) > 0 \) satisfying \( \alpha(\omega) < \beta(\omega)/2N \) there is a tempered random variable \( K(\omega) : \Omega \to [1, \infty) \) such that
\[ ||\Psi(n, \omega)P^u(\omega)|| \leq K(\omega)e^{\beta(\omega)n} \forall n \leq 0, \]
\[ ||\Psi(n, \omega)P^s(\omega)|| \leq K(\omega)e^{-\beta(\omega)n} \forall n \geq 0, \]
\[ ||\Psi(n, \omega)P^c(\omega)|| \leq K(\omega)e^{\alpha(\omega)|n|} \forall n \in \mathbb{Z}. \]

Set
\[ P^{cu}(\omega) = P^c(\omega) + P^u(\omega), \quad P^{cs}(\omega) = P^c(\omega) + P^s(\omega), \]
\[ E^{cu}(\omega) = E^c(\omega) \oplus E^u(\omega), \quad E^{cs}(\omega) = E^c(\omega) \oplus E^s(\omega). \]

We call \( E^{cu}(\omega) \) center-unstable Oseledets subspace and \( E^{cs}(\omega) \) center-stable Oseledets subspace. On \( \Omega_i \), by Lemma 2.2, \( \mathbb{R}^d \) has an invariant splitting \( \mathbb{R}^d = E^{cu} \oplus E^s \).
$(\mathbb{R}^d = E^{cu} \oplus E^n)$ independent of $\omega$. Surely, the projection operators $P^{cu}, P^s$ $(P^{cs}, P^n)$ are also independent of $\omega$ on $\Omega$ since $\theta$ is ergodic. For each $x \in \mathbb{R}^d$, we write it as

$$x = x^{cu} + x^s$$

for some $x^{cu} \in E^{cu}$ and $x^s \in E^s$, 

$$x = x^{cs} + x^n$$

for some $x^{cs} \in E^{cs}$ and $x^n \in E^n$.

By Lemma 2.3, we have

**Corollary 2.1.** Assume that Hypotheses 1 and 2 hold. There exist $\theta$-invariant random variables $\alpha, \beta : \Omega \to (0, \infty)$ satisfying $0 < \alpha(\omega) < \beta(\omega)/2N$ and a tempered random variable $K(\omega) : \Omega \to [1, \infty)$ such that

$$\|\Psi(n, \omega)P^{cu}(\omega)\| \leq K(\omega)e^{-\alpha(n)} \forall n \leq 0,$$

$$\|\Psi(n, \omega)P^{s}(\omega)\| \leq K(\omega)e^{-\beta(n)} \forall n \geq 0,$$

and

$$\|\Psi(n, \omega)P^{n}(\omega)\| \leq K(\omega)e^{\beta(n)} \forall n \leq 0,$$

$$\|\Psi(n, \omega)P^{cs}(\omega)\| \leq K(\omega)e^{\alpha(n)} \forall n \geq 0.$$

The random linear transformation $P(\omega)$ transforms the random dynamical system $\phi(n, \omega, x)$ to

$$\dot{\phi}(n, \omega, x) = P(\theta^n \omega)\phi(n, \omega, P^{-1}(\omega)x) = \Psi(n, \omega)x + \tilde{f}(n, \omega, x)$$

where $\tilde{f}(n, \omega, x) = P(\theta^n \omega)f(n, \omega, P^{-1}(\omega)x)$. Without loss of generality, under Hypotheses 1 and 2 we always assume that the linear part $\Phi(n, \omega)$ of random dynamical system $\phi(n, \omega, x)$ satisfies Lemmas 2.3 and 2.1.

**2.3. Random center manifolds.** In this subsection, we state the random center-unstable, center-stable, and center manifolds, which we take from [15].

By Lemma 2.2, it suffices to consider a random dynamical system in the block diagonal form

$$\phi(n, \omega, x) = \Phi(n, \omega)x + f(n, \omega, x),$$

where $\Phi(n, \omega)$ is a linear random dynamical system in block diagonal form which can be written as

$$\Phi = \begin{bmatrix} \Phi_u & 0 & 0 \\ 0 & \Phi_s & 0 \\ 0 & 0 & \Phi_c \end{bmatrix}$$

and satisfies Lemma 2.3, $f(n, \omega, 0) = 0$, and $Df(n, \omega, 0) = 0$.

For the nonlinear term $f(1, \omega, x)$ we assume that

**Hypothesis 3.** There exists a ball $U(\omega) = B(0, \rho_0(\omega)) = \{x \in \mathbb{R}^d \mid |x| < \rho_0(\omega)\}$, where $\rho_0 : \Omega \to (0, +\infty)$ is tempered from below such that

$$\sup_{x \in U(\omega)} \|D^k_f(1, \omega, x)\|_{L^k(\mathbb{R}^d, \mathbb{R}^d)} \leq \tilde{B}_k(\omega)$$

for all $0 \leq k \leq N < +\infty, \omega \in \Omega$, where the $\tilde{B}_k$ are tempered from above and the $L^k(\mathbb{R}^d, \mathbb{R}^d)$ are the Banach space of all $k$-linear maps from $\mathbb{R}^d$ to $\mathbb{R}^d$ with the norm $\|\cdot\|_{L^k(\mathbb{R}^d, \mathbb{R}^d)}$.

Then, we introduce the cut-off function to modify the nonlinear term $f(1, \omega, x)$. Let $\sigma(s)$ be a $C^\infty$ function from $(-\infty, +\infty)$ to $[0, 1]$ with

$$\sigma(s) = 1 \text{ for } |s| \leq 1, \quad \sigma(s) = 0 \text{ for } |s| \geq 2, \quad \sup_{s \in \mathbb{R}} |\sigma'(s)| \leq 2.$$
Assuming that the random variable $\rho : \Omega \to (0, +\infty)$ be tempered and satisfy $2\rho(\omega) \leq \rho_0(\omega)$. We take a modification of $f(1, \omega, x)$ as follows.

$$F_\rho(\omega, x) = \begin{cases} \sigma \left( \frac{|x|}{\rho(\omega)} \right) f(1, \omega, x), & |x| \leq 2\rho(\omega), \\ 0, & |x| > 2\rho(\omega). \end{cases}$$

Then $f(1, \omega, \cdot)$ is extended to the outside of $U(\omega)$. By simple calculation, we have the following lemma.

**Lemma 2.4.**

1. $F_\rho(\omega, x) = f(1, \omega, x) \forall |x| \leq \rho(\omega)$;
2. $\|D_x F_\rho(\omega, x)\|_{L(R^d, R^d)} \leq 10\tilde{B}_2(\omega)\rho(\omega) \forall \omega \in \Omega$ and $x \in \mathbb{R}^d$;
3. $\sup_{x \in \mathbb{R}^d} \|D_x^k F_\rho(\omega, x)\|_{L^k(R^d, R^d)} \leq B_k(\omega)$ for all $2 \leq k \leq N$ and $\omega \in \Omega$.

By choosing sufficiently small tempered radius $\rho(\omega)$, one has that $\psi(1, \omega, x) := \Phi(1, \omega)x + F_\rho(\omega, x)$ is a $C^N$ diffeomorphism on $\mathbb{R}^d$. Thus $\psi(1, \omega, x)$ generates a $C^N$ modified random dynamical system.

{\{x_n\}_{n \in \mathbb{Z}} is an orbit of $\phi(n, \omega, x_0)$ if and only if

$$x_{n+1} = \Phi(1, \theta^n \omega)x_n + f(1, \theta^n \omega, x_n) \forall n \in \mathbb{Z}. \quad (7)$$

From now on, we consider modified equation (7). To simplify the notation, the modified random dynamical system is still denoted by $\phi(n, \omega, x)$.

**Theorem 2.2.** (Center-Unstable Manifold Theorem) Assume that Hypotheses 1-3 hold. Then for a sufficiently small tempered radius $\rho(\omega)$, $\phi(n, \omega, x)$ has a $C^N$ center-unstable manifold which is given by

$$W^{cu}(\omega) = \{\xi + h^{cu}(\xi, \omega) \mid \xi \in E^{cu}\},$$

where $h^{cu} : E^{cu} \times \Omega \to E^s$ satisfies the following:

1. $h^{cu}(\xi, \omega)$ is $B(E^{cu}) \otimes \mathcal{F}$-measurable and $h^{cu}(\xi, \omega)$ is $C^N$ in $\xi$ with

$$\text{Lip} h^{cu}(\cdot, \omega) < 1, \quad h^{cu}(0, \omega) = 0, \quad D_{\xi} h^{cu}(0, \omega) = 0;$$

2. $\|D_i h^{cu}(\xi, \omega)\| \leq \tilde{K}_i(\omega)$ for all $0 \leq i \leq N < +\infty$, where each $\tilde{K}_i(\omega)$ is random variables tempered from above;

3. $W^{cu}_{\text{loc}}(\omega) = \{x \in W^{cu}(\omega) \mid x \in B(0, \rho(\omega))\}$ is a local unstable-center manifold for the original system (4).

**Theorem 2.3.** (Center-Stable Manifold Theorem) Assume that Hypotheses 1-3 hold. Then for a sufficiently small tempered radius $\rho(\omega)$, $\phi(n, \omega, x)$ has a $C^N$ center-stable manifold which is given by

$$W^{cs}(\omega) = \{\xi + h^{cs}(\xi, \omega) \mid \xi \in E^{cs}\},$$

where $h^{cs} : E^{cs} \times \Omega \to E^s$ satisfies the following:

1. $h^{cs}(\xi, \omega)$ is $B(E^{cs}) \otimes \mathcal{F}$-measurable and $h^{cs}(\xi, \omega)$ is $C^N$ in $\xi$ with

$$\text{Lip} h^{cs}(\cdot, \omega) < 1, \quad h^{cs}(0, \omega) = 0, \quad D_{\xi} h^{cs}(0, \omega) = 0;$$

2. for all $0 \leq i \leq N < +\infty$, we have that $\|D_i h^{cs}(\xi, \omega)\| \leq \tilde{K}_i(\omega)$, where each $\tilde{K}_i(\omega)$ is random variables tempered from above;
Then for a sufficiently small tempered radius $\rho$ (Center Manifold Theorem) Theorem 2.4 gives the center manifold as follows.

$$W^{c}(\omega) = \{ \zeta + h^{c}(\zeta, \omega) \mid \zeta \in E^{c} \},$$

where $h^{c} : E^{c} \times \Omega \to E^{u} \oplus E^{s}$ satisfies the following:

1. $h^{c}(\xi, \omega)$ is $\mathcal{B}(E^{c}) \otimes \mathcal{F}$-measurable and $h^{c}(\xi, \omega)$ is $C^{N}$ in $\xi$ with

$$\text{Lip } h^{c}(\cdot, \omega) < 1, \quad h^{c}(0, \omega) = 0, \quad D_{\xi}h^{c}(0, \omega) = 0;$$

2. for all $0 \leq i \leq N < +\infty$, we have that $\|D_{\xi}h^{c}(\xi, \omega)\| \leq K_{i}^{c}(\omega)$, where each $K_{i}^{c}(\omega)$ is random variables tempered from above;

3. $W^{c}_{loc}(\omega) = \{ x \in W^{c}(\omega) \mid x \in B(0, \rho(\omega)) \}$ is a local center-stable manifold for the original system (4).

3. Smooth conjugacy. In this section, we prove our main results. First, we introduce the following concept.

**Definition 3.1.** Two random dynamical systems $\phi(n, \omega, x)$ and $\psi(n, \omega, x)$ with a fixed point $x = 0$ are said to be $C^{k}$ conjugate locally if there exists a $C^{k}$ random diffeomorphism $h(\omega, x)$ from a tempered ball $U(\omega)$ to a tempered ball $V(\omega)$ with $h(\omega, 0) = 0$ such that

$$h(\theta, \omega, \phi(1, \omega)x) = \psi(1, \omega)h(\omega, x) \quad \text{for} \quad x \in U(\omega), \quad a.s. \ \omega \in \Omega. \quad (8)$$

The conjugacy relationship (8) implies that $h(\theta^{n}, \omega, \phi(n, \omega)x) = \psi(n, \omega)h(\omega, x)$ when the orbits stay in the corresponding domains.

The next theorem gives a smooth conjugacy in a jet class.

**Theorem 3.2.** Let $\phi$ be a random dynamical system satisfying Hypothesis 1-3. Then for any integer $k > 0$, there exists a $\theta$-invariant measurable integer function $N_{0} = N_{0}(\alpha, \beta, k)$ such that if $\phi$ and $\psi$ are $C^{N_{0}}$ ($N_{0} \geq N_{0}$) locally tempered and have a common center manifold $W^{c}(\omega)$ with $\phi - \psi = O(|x|^{N})$ for $\omega \in \Omega$, $x \in W^{c}(\omega)$, then $\phi$ and $\psi$ are $C^{k}$ locally conjugate.

The proof of this theorem is based on the following lemmas and proposition. Some of them are directly taken from our previous work [22]. We first applying the center-unstable manifold theorem (Theorem 2.2) and the center-stable manifold theorem (Theorem 2.3) to $\phi$. Then, $\phi$ has a $C^{N}$ local center-unstable manifold

$$W^{uc}_{loc}(\omega) = \{ \eta + h^{uc}(\eta, \omega) \mid \eta = (x^{u}, x^{c}) \in E_{u} \oplus E_{c}, \ |\eta| \leq \rho(\omega) \}$$

and a $C^{N}$ local center stable manifold

$$W^{sc}_{loc}(\omega) = \{ \xi + h^{sc}(\xi, \omega) \mid \xi = (x^{s}, x^{c}) \in E_{s} \oplus E_{c}, \ |\xi| \leq \rho(\omega) \}$$

We may identify $E_{u}$, $E_{c}$ and $E_{s}$ with $\mathbb{R}^{d_{u}}$, $\mathbb{R}^{d_{c}}$ and $\mathbb{R}^{d_{s}}$, respectively. Next, we use the center-stable, center-unstable and the common center manifolds as new axes to
It also follows from Theorem 2.2 and Theorem 2.3 that this transformation is a $C^N$ random diffeomorphism. By this transformation, the random diffeomorphism $\phi(1,\omega)$ is locally conjugate to a $C^N$ random diffeomorphism
\[
\hat{\phi}(1,\omega) : x = (y,w,z)^T \mapsto A(\omega)x + F_1(\omega,x)
\]
where $A(\omega) = D\phi(1,\omega)$ is written as $\text{diag}(A^s(\omega),A^c(\omega),A^u(\omega))$, $F_1(\omega,x)$ satisfies Hypothesis 3, and $F_1(\omega,0) = 0$, $DF_1(\omega,0) = 0$. Furthermore,
\[
F_1 = \begin{pmatrix}
F_{1,s}(\omega,y,w,z) \\
F_{1,c}(\omega,y,w,z) \\
F_{1,u}(\omega,y,w,z)
\end{pmatrix}, \quad F_{1,s}(\omega,0,w,z) = 0, \quad F_{1,u}(\omega,y,w,0) = 0,
\]
which implies that $y = 0$ is the center-stable manifold, $z = 0$ is the center-unstable manifold and $y = 0, z = 0$ is the common center manifold for the new random dynamical system.

Similarly, $\psi(1,\omega)$ is $C^N$ conjugate to
\[
\hat{\psi}(1,\omega) : x = (y,w,z)^T \mapsto A(\omega)x + F_2(\omega,x)
\]
where $F_2(\omega,x)$ satisfies Hypothesis B, and $F_2(\omega,0) = 0$, $DF_2(\omega,0) = 0$. Furthermore,
\[
F_2 = \begin{pmatrix}
F_{2,s}(\omega,y,w,z) \\
F_{2,c}(\omega,y,w,z) \\
F_{2,u}(\omega,y,w,z)
\end{pmatrix}, \quad F_{2,s}(\omega,0,w,z) = 0, \quad F_{2,u}(\omega,y,w,0) = 0.
\]
Let
\[
R = \begin{pmatrix}
R_s(\omega,y,w,z) \\
R_c(\omega,y,w,z) \\
R_u(\omega,y,w,z)
\end{pmatrix} = F_1 - F_2 = \begin{pmatrix}
F_{1,s}(\omega,y,w,z) - F_{2,s}(\omega,y,w,z) \\
F_{1,c}(\omega,y,w,z) - F_{2,c}(\omega,y,w,z) \\
F_{1,u}(\omega,y,w,z) - F_{2,u}(\omega,y,w,z)
\end{pmatrix}.
\]
Then $R_s(\omega,0,w,z) = 0, R_u(\omega,y,w,0) = 0$. When $\text{jet}^N_y(\phi - \psi) = 0$ for $x \in W^c(\omega)$, the order of the difference between the corresponding center stable manifolds and the order of the difference between center unstable manifolds are $N$, thus $\text{jet}^N_{y=0,z=0}\hat{\phi} - \hat{\psi} = 0$.

In the following, we will decompose $R$ into two parts, one part is dominated by a power of $y$ while another is dominated by a power of $z$.

**Lemma 3.3.** The function $R$ can be written as $R = R_1 + R_2$, where $R_i(\omega,x)$ are measurable and $R_i(\omega,\cdot)$ are $C^{[N/2]}$ tempered functions satisfying
\[
|R_1| \leq C_0(\omega)|y|^{[N/2]}, \quad |R_2| \leq C_0(\omega)|z|^{[N/2]}, \quad x \in U(\omega)
\]
where $C_0(\omega)$ is tempered from above.

**Proof.** Consider the Taylor expansion of $R$ of order $[N/2]$ with respect to the variable $y$:
\[
R(\omega,y,z) = \sum_{i=0}^{[N/2]} a_i(\omega,y,w,z) + R_{[N/2]}(\omega,y,w,z),
\]
where $R_{[N/2]} = O(|y|^{N/2})$ and $a_i$ is a homogeneous polynomial of variable $y$ of degree $i$ whose coefficients are the measurable functions of $(\omega, w, z)$ and $C^{(N/2)}$ with respect to the variable $(w, z)$. Since $\phi_{\rho_i}^{N}_{y=0, z=0}R = 0$, we have $\|A_i\| = O(\|z\|^{N/2})$.

Let $R_1 = R_{[N/2]}$ and $R_2 = \sum_{i=0}^{N/2} a_i(\omega, y, w, z)$. Then (10) holds. Other properties of $R_1$ and $R_2$ follows from the properties of $F_1$ and $F_2$. This completes the proof of the lemma.

Next, we use the cut-off procedure to modify the functions, $F_1$, $F_2$, $R_1$, and $R_2$. We set for $i = 1, 2$

$$\tilde{F}_i(\omega, x) = \sigma_{\rho_i(\omega)}(|x|)F_i(\omega, x),$$
$$\tilde{R}_i(\omega, x) = \sigma_{\rho_i(\omega)}(|x|)R_i(\omega, x).$$

We choose a random variable $\rho_1(\omega)$ tempered from below such that $\rho_1(\omega) \leq \rho(\omega)$. Then, $\tilde{F}_1$ and $\tilde{F}_2$ are $C^N$ tempered function on $\mathbb{R}^d$ and $\tilde{R}_1$ and $\tilde{R}_2$ are $C^{(N/2)}$ tempered function on $\mathbb{R}^d$. Moreover, the following hold:

$$|\tilde{R}_1| \leq C_0(\omega)|y|^{N/2}\sigma_{\rho_1(\omega)}(|y|), \quad |\tilde{R}_2| \leq C_0(\omega)|z|^{N/2}\sigma_{\rho_1(\omega)}(|z|), \quad \text{for } x \in \mathbb{R}^d,$$

(11)

Let $\tilde{\phi} = A(\omega)x + \tilde{F}_1$ and $\tilde{\psi} = A(\omega)x + \tilde{F}_2$. Then

$$\tilde{\phi} = \tilde{\phi}, \quad \tilde{\psi} = \tilde{\psi}, \quad \text{for } |x| < \rho_1(\omega).$$

Let $\phi_\tau(m, \omega, x)$ and $\psi_\tau(m, \omega, x), \tau \in [0, 1]$, be the families of random dynamical systems whose time-one maps have the forms

$$\phi_\tau(1, \omega, x) = \tilde{\phi}(\omega, x) + \tau \tilde{R}_1$$

and

$$\psi_\tau(1, \omega, x) = \tilde{\phi}(\omega, x) + \tilde{R}_1 + \tau \tilde{R}_2$$

respectively.

Lemma 3.4. [22] For any given integer $k > 0$, there exist a constant $M_k(\omega) = M_k(\alpha, \beta)$ and a random variable $d_k(\omega)$ tempered from above such that

$$\max_{1 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^d} \|D_{(\omega, x)}^\alpha \phi_\tau^{n+1}(n, \omega, x)\| \leq d_k(\omega)M_k(\omega)^{|n|}, \quad n \in \mathbb{Z},$$

(12)

$$\max_{1 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^d} \|D_{(\omega, x)}^\alpha \psi_\tau^{n+1}(n, \omega, x)\| \leq d_k(\omega)M_k(\omega)^{|n|}, \quad n \in \mathbb{Z}.$$

(13)

The next lemma states that one may reduce the problem of $C^k$ conjugacy (8) to a problem of solving a linear recurrent functional equation. This idea is based on the so-called homotopy method which was used in [18] and [19] for finitely smooth normal forms of deterministic dynamical systems. We extend it in [22] further so that it can be applied to random dynamical systems where the partial hyperbolicity is nonuniform.

Lemma 3.5. [22] Let $\phi_\tau(n, \omega, x), \tau \in [0, 1], n \in \mathbb{Z}$, be a family of $C^1$ random dynamical systems in $\mathbb{R}^d$ with fixed point 0 whose time-one mapping has the form

$$\phi_\tau(1, \omega, x) = \phi_\tau(\omega, x) = \phi(\omega, x) + \tau R(\omega, x).$$

If there exists a measurable mapping $r : \Omega \times \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ such that

(i) $r(\omega, \cdot, \cdot)$ is $C^k$ and satisfies

$$|r(\omega, x, \tau)| \leq a(\omega)|x|^2, \quad \text{for } |x| \leq 1,$$

(14)

where $a(\omega)$ is a tempered function from above;
then $\phi$ and $\phi_1$ are $C^k$ locally conjugate.

Next, we construct formal solutions of linear functional equation (15) in terms of an infinite series. We will prove later that they are convergent for $R$ being $R_1$ and $R_2$.

For simplicity, we set $(f_*R)(x) := (Df)\circ f^{-1}(x)$ for a diffeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ and a vector field $R$ in $\mathbb{R}^d$. Notice that $f_*(g_*R) = (f \circ g)_*R$. Then, we have two formal solutions to equation (15).

**Lemma 3.6.** [22] The functions

$$ r_1(\omega, x, \tau) = -\sum_{m=1}^{\infty} \left( (\phi_1^{-1}(m, \omega))_* R(\theta^{m-1}\omega, \phi_1^{-1}(\theta^{m-1}\omega, \cdot)) \right)(x) $$

and

$$ r_2(\omega, x, \tau) = \sum_{m=0}^{\infty} \left( (\bar{\phi}_1^{-1}(-m, \omega))_* R(\bar{\theta}^{-m-1}\omega, \bar{\phi}_1^{-1}(\bar{\theta}^{-m-1}\omega, \cdot)) \right)(x) $$

are formal solutions of (15).

We are now ready to show the convergence of the formal series solutions. Let

$$ R_1(\omega, x, \tau) = R_1(\omega, \phi_1^{-1}(1, \omega, x)), \quad R_2(\omega, x, \tau) = R_2(\omega, \psi_1^{-1}(1, \omega, x)), $$

then $R_1$ and $R_2$ are $C^{(N/2)}$ (resp. $C^\infty$) tempered and satisfy

$$ |R_1(\omega, x, s)| \leq C_0(\omega)|y|^{N/2}\sigma_{\rho_1(\omega)}(|y|), $$

$$ |R_2(\omega, x, s)| \leq C_0(\omega)|z|^{N/2}\sigma_{\rho_1(\omega)}(|z|), $$

where functions $C_0$ is tempered from above.

The next lemma gives two smooth solutions of (15) with $R = R_1$ and $R_2$.

**Lemma 3.7.** [22] For any given integer $k > 0$, there exists a positive integer $N_0 = N_0(\alpha, \beta, k)$ such that if $N \geq N_0$ and $\text{jet}^N_{x=0}R = 0$, then the functions $r_1(\cdot, \cdot, \cdot)$ and $r_2(\cdot, \cdot, \cdot)$ defined by

$$ r_1(\omega, x, s) = -\sum_{n=1}^{\infty} \left( (\phi_1^{-1}(n, \omega))_* R_1(\theta^{n-1}\omega, \cdot, \tau) \right)(x) $$

and

$$ r_2(\omega, x, \tau) = \sum_{n=0}^{\infty} \left( (\psi_1^{-1}(-n, \omega))_* R_2(\theta^{-n-1}\omega, \cdot, \tau) \right)(x) $$

are measurable from $\Omega \times \mathbb{R}^d \times [0, 1]$ to $\mathbb{R}^d$ and are $C^k$ smooth with respect to $(x, \tau)$ for each fixed $\omega \in \Omega_1$ and satisfy (14).

Summarizing the above discussions gives

**Proposition 3.8.** For each $k \in \mathbb{N}$, there exists an integer $N_0 = N_0(\alpha, \beta, k)$ such that if $\tilde{\phi}$ and $\tilde{\psi}$ are $C^{N_0}$ and (11) holds with $N = N_0$ for $\omega \in \Omega$, then $\tilde{\phi}$ and $(\tilde{\phi} + R_1)$, $(\tilde{\phi} + R_1 + R_2)$ are $C^k$ locally conjugate.
Lemma 3.9. Let $\phi$ and $\psi$ be two linear cocycles in $\mathbb{R}^r$ and $\mathbb{R}^s$ over the metric DS $(\Omega, F, \mathbb{P}, \theta(t)_{t \in \mathbb{Z}})$ respectively. Let $\phi$ and $\psi$ be generated by $A : \Omega \rightarrow GL(r, \mathbb{R})$ and $B : \Omega \rightarrow GL(s, \mathbb{R})$ respectively. We assume that
\[
\log^+ \|A(\cdot)\|, \log^+ \|A^{-1}(\cdot)\|, \log^+ \|B(\cdot)\|, \log^+ \|B^{-1}(\cdot)\| \in L^1(\Omega, F, \mathbb{P})
\]
and let the Lyapunov spectrum of the cocycle $\phi$ and $\psi$ be $Sp(A) = (\lambda_i)$ and $Sp(B) = (\mu_j)$ respectively. Let
\[
C(A, B)_n : H_{n, r, s} \rightarrow H_{n, r, s}
\]
be the linear cocycle over $\theta$ generated by
\[
f(x) \mapsto B(\omega) f(A^{-1}(\omega)x),
\]
then $C(A, B)_n$ has the spectrum
\[
Sp(C(A, B)_n) = \{\mu_j - (\lambda, \tau), |\tau| = n, j = 1, 2, \ldots, s\}.
\]

Proof. Let $N(A^{-1}(\omega))_n$ denote the linear cocycle generated by
\[
H_{n, r, 1} \rightarrow H_{n, r, 1} : f(x) \mapsto f(A^{-1}(\omega)x),
\]
then $C(A, B)_n = N(A^{-1})_n \otimes B(\omega)$. Now the conclusion follows from (iii) of Proposition 8.2.6 and (ii) of Theorem 5.4.2 in [1].

Lemma 3.10. Let $\theta$ be a metric DS $(\Omega, F, \mathbb{P}, \theta(t)_{t \in \mathbb{Z}})$ and
\[
U(\omega) := B_{\rho(\omega)} = \{x \in \mathbb{R}^r : \|x\| \leq \rho(\omega)\}
\]
be a tempered ball. Let $b(\omega, \cdot) : U(\omega) \rightarrow \mathbb{R}^r$, $M(\omega, \cdot) : U(\omega) \rightarrow GL(s, \mathbb{R})$ and $w(\omega, \cdot) : U(\omega) \rightarrow \mathbb{R}^r$ be $C^N$ tempered functions with $w(\omega, 0) = 0$. Assume that the linear cocycle $\phi_1$ generated by $Du(\omega, 0)$ and the linear cocycle $\phi_2$ generated by $M(\omega, 0)$ satisfy the IC of the MET. If $\phi_2$ is hyperbolic and all Lyapunov exponents of $\phi_1$ are zero, then the equation
\[
h_{\theta_\omega} \circ w - Mh_\omega + b = 0 \quad (22)
\]
has a $C^N$ tempered solution $h_\omega(x) = h(\omega, x) : \Omega \times B_{\rho(\omega)} \rightarrow \mathbb{R}^r$ in some tempered ball $B_{\rho(\omega)}$.

Proof. Consider a RDS in $\mathbb{R}^r \times \mathbb{R}^s$ over $\theta$ whose time one map is
\[
\psi(\omega)(u, v) = (w(\omega, u), M(\omega, u)v - b(\omega, u)).
\]
Let $a(\omega)$ be the solution of the cocycle generated by the affine difference equation
\[
a_{n+1} = M(\theta^n \omega, 0)a_n - b(\theta^n \omega, 0).
\]
Then, by Theorem 5.6.5 in [1], $a(\omega)$ is tempered. Let $u = x, v = y - a(\omega)$. Then, $\psi$ in the new coordinate has the form
\[
\psi(\omega)(u, v) = (w(\omega, u), M(\omega, u)v + M(\omega, u)a(\omega) - b(\omega, u) - a(\theta \omega)).
\]
The origin $(0, 0)$ is a fixed point of $\psi$ and the linearization matrix of $\psi$ at $(0,0)$ is
\[
B(\omega) = \begin{pmatrix}
Dw(\omega, 0) & 0 \\
* & M(\omega, 0)
\end{pmatrix}.
\]
By Theorem 5.2.2 in [1], the linear cocycle generated by \(B(\omega)\) has the spectrum \(\text{Sp}(\theta, B) = \text{Sp}(\theta, Dw(\omega, 0)) \cup \text{Sp}(\theta, M(\omega, 0))\). By center manifold theorem 2.4, there exist a local center manifold \(W^c(\omega)\) which is a graph of \(C^1\) tempered map

\[ v = h^c(\omega, u), \quad u \in B_{\tilde{\rho}(\omega)}, \]

where \(B_{\tilde{\rho}(\omega)}\) is a tempered ball. Since \(W^c(\omega)\) is invariant under the action of \(\psi\), we have

\[ h^c_{\theta, \omega} \circ w = M(\omega, u)h^c + M(\omega, u)a(\omega) - b(\omega, u) - a(\theta\omega), \]

which implies that the function \(h = h^c + a(\omega)\) satisfies the equation (22).

\[ \square \]

**Lemma 3.11.** Let \(\phi\) be a \(C^1\) tempered RDS in \(\mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}\) over a metric DS \(\theta\) whose time one map has the form

\[ \phi(u, v) = (w(\omega, u) + f(\omega, u, v) + O(\|v\|^{n+1}), A(\omega, u)v + O(\|v\|^2)) \]

where \(w(\omega, 0) = 0\) and \(f\) is a homogeneous polynomial of variable \(v\) of degree \(n\) with values in \(\mathbb{R}^{d_1}\) whose coefficients are locally \(C^1\) tempered functions of variable \(u\). If the Lyapunov spectrum \(\text{Sp}(\theta, Dw(\omega, 0)) = \{0\}\) and the Lyapunov spectrum \(\Lambda = \text{Sp}(\theta, A(\omega, 0))\) satisfies

\[ (\lambda, \tau) \neq 0, \quad \text{for} \quad |\tau| = n, \quad (23) \]

then \(\phi\) is locally \(C^{n-1}\) conjugate to a \(C^{n-1}\) tempered RDS whose time one map has the form

\[ \psi(u, v) = (w(\omega, u) + O(\|v\|^{n+1}), A(\omega, u)v + O(\|v\|^2)). \]

**Proof.** Let \(H_{\omega}(u, v) = (u + h(\omega, u, v), v)\), where \(h\) is a homogeneous polynomial of variable \(v\) of degree \(n\) with values in \(\mathbb{R}^{d_1}\) whose coefficients are locally \(C^1\) tempered functions of variable \(u\). Then

\[ H_{\omega} \circ \phi \circ H_{\omega}^{-1}(u, v) = (w(\omega, u) + R(\omega, u, v) + O(\|v\|^{n+1}), A(\omega, u)v + O(\|v\|^2)), \]

where \(R = -Dw(\omega, u)h(\omega, u, v) + f(\omega, u, v) + b(\omega, w(\omega, u), A(\omega, u)v).\) The equation \(R = 0\) is equivalent to

\[ h(\theta\omega, w(\omega, u), v) - Dw(\omega, u)h(\omega, u, A^{-1}(\omega, u)v) + f(\omega, u, A^{-1}(\omega, u)v) = 0. \quad (24) \]

Let \(M(\omega, u)\) be the linear operator

\[ H_{n, d_1, d_1} \to H_{n, d_1, d_1} : f(v) \mapsto Dw(\omega, u)f(A^{-1}(\omega, u)v). \]

If we denote by \(h_{\omega}(u)\) and \(b(\omega, u)\) the coefficient vectors of the polynomials \(h(\omega, u, v)\) and \(b(\omega, u, v)\) respectively, then the equation (24) can be written in the form

\[ h_{\theta\omega} \circ w - M(\omega, u)h_{\omega} + b = 0. \quad (25) \]

By Lemma 3.9 and Lemma 3.10, equation (25) has locally a \(C^{n-1}\) tempered solution \(h_{\omega}\).

\[ \square \]

**Lemma 3.12.** Let \(\phi\) be a \(C^1\) tempered RDS in \(\mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}\) over a metric DS \(\theta\) whose time one map has the form

\[ \phi(u, v) = (w(\omega, u), A(\omega, u)v + f(\omega, u, v) + O(\|v\|^{n+1})) \]

where \(w(\omega, 0) = 0\) and \(f\) is a homogeneous polynomial of variable \(v\) of degree \(n\) with values in \(\mathbb{R}^{d_2}\) whose coefficients are locally \(C^1\) tempered functions of variable \(u\). If the Lyapunov spectrum \(\text{Sp}(\theta, Dw(\omega, 0)) = \{0\}\) and the Lyapunov spectrum \(\Lambda = \text{Sp}(\theta, A(\omega, 0))\) satisfies

\[ (\lambda, \tau) \neq \lambda_i, \quad \text{for} \quad |\tau| = n, \quad (26) \]
then $\phi$ is locally $C^{N-1}$ conjugate to a $C^{N-1}$ tempered RDS whose time one map has the form

$$\psi(u, v) = (w(\omega, u), A(\omega, u)v + O(||v||^{n+1})).$$

**Proof.** Let $H_\omega(u, v) = (u, v + h(\omega, u, v))$, where $h$ is a homogeneous polynomial of variable $v$ of degree $n$ with values in $\mathbb{R}^d$ whose coefficients are locally $C^N$ tempered functions of variable $u$. Then

$$h_\omega \circ \phi \circ H_\omega^{-1}(u, v) = (w(\omega, u), A(\omega, u)v + R(\omega, u, v) + O(||v||^{n+1})),$$

where $R = -A(\omega, u)h(\omega, u, v) + f(\omega, u, v) + h(\omega, w(\omega, u), A(\omega, u)v)$. The equation $R = 0$ is equivalent to

$$h(\theta\omega, w(\omega, u), v) - A(\omega, u)h(\omega, u, A^{-1}(\omega, u)v) + f(\omega, u, A^{-1}(\omega, u)v) = 0. \quad (27)$$

Let $M(\omega, u)$ be the linear operator

$$H_{n, d_1, d_2} \rightarrow H_{n, d_1, d_2} : f(v) \mapsto A(\omega, u)f(A^{-1}(\omega, u)v).$$

If we denote by $h_\omega(u)$ and $b(\omega, u)$ the coefficient vectors of the polynomials $b(\omega, u, v)$ and $b(\omega, u, v)$ respectively, then the equation $(27)$ can be written in the form

$$h_\omega \circ w - M(\omega, u)h_\omega + b = 0. \quad (28)$$

By Lemma 3.9 and Lemma 3.10, equation $(28)$ has locally a $C^{N-1}$ tempered solution $h_\omega$.

**Proof of the main theorem.** The conclusion (i) follows from Lemma 3.11 and Theorem 3.2. The conclusion (ii) follows from the conclusion (i), Lemma 3.12 and Theorem 3.2.

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