Exploring Complex Graphs by Random Walks ‡

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Abstract

We present an algorithm to grow a graph with scale-free structure of in- and out-links and variable wiring diagram in the class of the world-wide Web. We then explore the graph by intentional random walks using local next-near-neighbor search algorithm to navigate through the graph. The topological properties such as betweenness are determined by an ensemble of independent walkers and efficiency of the search is compared on three different graph topologies. In addition we simulate interacting random walks which are created by given rate and navigated in parallel, representing transport with queueing of information packets on the graph.

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I. INTRODUCTION

We are living in the world of networks, majority of which are not static but evolve in time. The evolution of network is governed by microscopic rules that in course of time lead to the emergence of complex structures through the mechanism of self-organization and dynamical constraints. The emergent scale-free structure of links in technological networks the Internet and the Web and in molecular and gene regulatory networks is of particular importance for their functional stability. The networks are adequately represented by graphs with a given structure of links. The obvious question is then how the networks function, making the necessity of modeling dynamic processes on these graphs. The dynamics of random walks on graphs represents a powerful tool, in which we can explore connection between the graphs topology and functional properties of the network.

To explore numerically graphs representing complex evolving networks by random walks, two types of algorithms are necessary: First, an algorithm to grow the graph with a given structure of links; Second, an algorithm that governs a random walk on that graph. Complexity of link structure in scale-free graphs such as the Web graph suggests various possibilities for local navigation. For instance, apart from a naive move strategy, the random walker may learn from nodes linking preferences to get quickly to a well connected node. For intentional random walks directed to a given destination on the graph, we implement local search algorithm that monitors near- and next-near neighbors in search for the destination node.

II. GRAPH GROWTH ALGORITHMS: THE WEB GRAPH

We present an algorithm to grow a graph with scale-free structure and flexible wiring diagram in the class of the world-wide Web. Objectives are to grow a graph that has statistically the same properties as measured in the real Web: scale-free degree distributions both for in- and out-links (exponents $\tau_{in} \approx 2.2$ and $\tau_{out} \approx 2.6$); occurrence of a giant component and the distribution of clusters with the exponent $\tau_s \approx 2.5$. As demonstrated in Ref. a minimal set of microscopic rules necessary to reproduce such graphs include growth, attachment, and rewiring. Time is measured by addition of a node, which attempts to link with probability $\tilde{\alpha}$ to a node $k$. Else, with probability $1 - \tilde{\alpha}$ a preexisting node $n$
rewires or adds a new out-link directed to \( k \). Nodes \( k \) and \( n \) are selected with probabilities
\[
p_{in} \equiv p_{in}(k, t), \quad p_{out} \equiv p_{out}(n, t)
\]
\[
p_{in} = (M\alpha + q_{in}(k, t))/(1 + \alpha)Mt ; \quad p_{out} = (M\alpha + q_{out}(n, t))/(1 + \alpha)Mt ,
\]
which depend on current number of respective links \( q_{in}(k, t) \) and \( q_{out}(n, t) \). \( M \) is average number of links per time step (see \[1, 4\] for more details). The graph flexibility, which is measured by the degree of rewiring \((1 - \tilde{\alpha})/\tilde{\alpha}\), is essential both for the appearance of the scale-free structure of out-links and for occurrence of closed cycles, which affect the dynamic processes on the graph.

By solving the corresponding rate equations we find that the local connectivities \(< q_{in}(s, t) > \) and \(< q_{out}(s, t) > \) at a node added at time \( s \) increase with time \( t \) as
\[
q_{\kappa}(s, t) = A_{\kappa}[(t/s)^{\gamma_{\kappa}} - B_{\kappa}] .
\]
with \( \kappa = in \) and \( out \), and \( \gamma_{in} = 1/(1 + \alpha) \) and \( \gamma_{out} = (1 - \tilde{\alpha})/(1 + \alpha) \). We use the original one-parameter model introduced in \[1\] with \( \tilde{\alpha} = \alpha = 0.25 \) and \( M = 1 \). When \( \tilde{\alpha} = 1 \) the emergent structure is tree like with one out-link per node.

In Fig. 1a we show simulated local connectivities for \( t = N = 10^4 \) nodes in agreement with Eq. \[2\]. This implies the power-law behavior of emergent degree distributions with exponent \( \tau_{\kappa} = 1/\gamma_{\kappa} + 1 \), in agreement with simulations in \[1, 4\].

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**FIG. 1:** Left panel: Average connectivity of node \( s \) for (top to bottom) out-links, in-links and number of visits \(< u > \) for sequential walks in the Web graph. Right panel: Probability distribution \( P(u) \) of visits \( u \) to a given node vs \( u \) in the Web graph, tree graph with same in-link distribution and for randomly grown graph. Curves shifted vertically. All data log-binned.
We now explore the graph with an ensemble of intentional random walks. To stress importance of the graph topology, we compare results obtained on the Web graph, the tree graph with same in-link structure and on randomly grown graph. We first consider dynamics of an ensemble of independent random walkers, that can be implemented sequentially. We then introduce some details of the dynamics of interacting random walks which are navigated in parallel while moving towards their selected destinations, leading to queueing processes on the Web graph.

Sequential Walks and Graph Topology

For each random walk we select a start and destination node at random (see [5, 6] for electrical networks). Then the walk is navigated by next-near-neighbor (nnn-) algorithm to search for its destination. In Fig. 1a also is shown the average number of visits \( < u > \) at node \( s \) for the walker using both in- and out-links equally. As expected, the number of visits to a node are determined by its local connectivities. This is further illustrated in Fig. 1b where the distribution of number of visits \( P(u) \) is shown for three different graph topologies. The distribution falls of faster in the case of Web (slope \( \tau_u = 2.275 \pm 0.015 \)) compared to the tree graph (\( \tau_u = 1.94 \pm 0.018 \)), suggesting that the presence of closed cycles on the Web graph increases its searcability.

FIG. 2: Left panel: Probability distribution of search times \( P(T_s) \) vs \( T_s \) for three graph topologies as in Fig. 1b. All data are logarithmically binned. Right panel: Spectrum of the density, number of active nodes, and number of packets in the traffic (bottom to top).
The higher efficiency of the nnn-search on the Web compared to tree graph is further expressed in the distributions of duration of search on Fig. 2a, where the respective slopes are \( \tau_{ts} = 1.541 \pm 0.008 \) and \( \tau_{ts} = 0.98 \pm 0.016 \). Intuitively, this can be understood as the nnn-search algorithm often directs walks through the hub node, to which a majority of in-links point. In the Web graph, the same algorithm can use two types of well connected nodes—hub and authority nodes. It is interesting to note that a broad distribution of search time is also found on randomly grown graph (the exponent \( \tau_{ts} = 0.425 \pm 0.008 \)), whereas the corresponding distribution of visits is a stretch-exponential function (cf. Fig. 1b).

**Parallel Walks: Transport on Graphs**

To model simultaneous random walks, as for instance information packet transport on the Internet, in addition to graph topology and search algorithm as above, we need additional parameters to specify packet creation rate \( R \) and queueing discipline, as LIFO [7]. In addition, we assume finite maximal queue length \( H \), which then modifies search when a local queue is at its maximum [7]. Technically, the object-oriented programming is necessary. As a consequence of simultaneous intentional walks, we have new phenomena related to dynamically interacting queues on the graph. At low creation rate \( R \to 0 \) the statistics of independent walks as above applies. For \( R > 0 \), the time that a walker spends on the graph, which is interesting for costs planning, consists of search time and time that it spends trapped in queues along the way. For large \( R \) we have dense traffic, in which searchability is limited, and eventually a transition to jammed state occurs. The dynamics of queueing depends on the graph topology and the parameters \( R \) and \( H \). Few details are shown in Fig. 2b for our scale-free graph with cycles. For large enough creation rate the density of packets arriving at a hub node in a single time step exhibits long-range temporal correlations. Similar correlations are found in the number of simultaneously moving packets (activity) and in the number of present packets (moving and queueing) at each time step. An analysis of transit time statistics and queue properties for a scale-free tree graph can be found in [7].

Presence of closed cycles enhances searchability of flexible scale-free graphs. The efficiency of nnn-search algorithm on these graphs is related to occurrence of hub and authority nodes. Sequential walks discover directly the graph topology, which together with parameters \( R, H \)
affects transport and queuing on the graph.

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