On Generalizations of the Newton-Raphson-Simpson Method

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Abstract

We present generalizations of the Newton-Raphson-Simpson method. Specifically, for a positive integer \(m\) and the sequence of coefficients of a Taylor series of a function \(f(z)\), we define an algorithm we denote by \(\text{NRS}(m)\) which is a way to evaluate, in our terminology, a sum of \(m\) formal zeros of \(f(z)\). We prove that \(\text{NRS}(1)\) yields the familiar iterations of the Newton-Raphson-Simpson method. We also prove that \(\text{NRS}(m)\) is a way to evaluate certain \(A\)-hypergeometric series defined by Sturmfels [3]. In order to define these algorithms, we make use of combinatorial objects which we call trees with negative vertex degree.

1 Introduction

The main purpose of this paper is to define a sequence of algorithms \(\text{NRS}(m)\) for positive integer \(m\) which generalize the Newton-Raphson-Simpson method.

We review the Newton-Raphson-Simpson method. Let \(f(z) : \C \to \C\) be a differentiable function and \(c_0 \in \C\). Recall that the Newton-Raphson-Simpson method constructs a sequence \(c_N, N \geq 0\) defined by

\[
 c_{N+1} = c_N - \frac{f(c_N)}{f'(c_N)}.
\]

Then the limit \(c = \lim_{N \to \infty} c_N\), if it exists, is a zero of \(f(z)\). Depending on \(f(z)\) and \(c_0\), the limit may or may not exist. See Kollerstrom [1] for information about the Newton-Raphson-Simpson method.

Given an integer \(d \geq m\), the algorithm \(\text{NRS}(m)\) constructs a sequence of rational functions \(J_m(n), n \geq 0\) in the variables \(a_0, a_1, \ldots, a_d\). We think of the \(a_k\) as being the coefficients of a polynomial

\[
 f(z) = \sum_{k=0}^{d} a_k z^k.
\]

We prove that

\[
 \sum_{n=0}^{N-1} J_1(n)
\]
is equal to the $N$-th iteration $c_N$ of the Newton-Raphson-Simpson method applied to $f(z)$ with $c_0 = 0$. For larger $m$, we claim that

$$\sum_{n=0}^{\infty} J_m(n),$$

if convergent, is equal to a sum of $m$ zeros of $f(z)$. In Section 3 we apply the NRS($m$) for certain polynomials and give tables of values for the partial sums of (1). These tables indicate the series for these examples converges to the sum of the $m$ zeros of $f(z)$ that are closest to 0. In Section 4 we talk more about the claimed sufficient conditions on $f(z)$ that yield these results, namely that the zeros of $f(z)$ be positive.

We obtain the rational functions $J_m(n)$ by considering certain infinite sums $A_m$ in the $a_i$; choosing a certain order of summation for $A_m$ yields the series (1). We define the $A_m$ in Section 3 in terms of combinatorial objects which we call trees with negative vertex degree. Next we just present a high-level description of the $A_m$ and why they are relevant, including their appearance in [3].

The sums $A_m - A_{m-1}$ (where $A_0$ denotes 0) are examples of what we call formal zeros for $f(z)$. For a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we view the $a_k$ as indeterminates in some suitable $R_1$ and we view $f(z)$ as a function $f(z) : R \rightarrow R$. Then a formal zero for $f(z)$ is an element $Z \in R$ such that

$$f(Z) = 0 \in R.$$

There are some different ways to approach the $A_m$.

One way, for example, is to view

$$Z_m = Z_m(a_0, a_1, \ldots, a_{m-2}, a_{m+1}, a_{m+2}, \ldots)$$

as a function of the independent variables $a_k$ for $k \neq m - 1, m$ and to view $a_{m-1}$ and $a_m$ as constants. Then we set

$$Z_m = -\frac{a_{m-1}}{a_m} + \sum_{\vec{n}} c(\vec{n}) a^{\vec{n}}$$

where

$$\vec{n} = (n_0, n_1, \ldots, n_{m-2}, n_{m+1}, n_{m+2}, \ldots)$$

is a sequence of non-negative integers $n_i$, almost all zero; where

$$a^{\vec{n}} = \prod_{i=0, i \neq m-1, m}^{\infty} a_i^{n_i};$$

and where $c(\vec{n})$ are some coefficients. We can solve for the $c(\vec{n})$ by using the set of equations

$$\frac{\partial^{\vec{n}} f}{(\partial a)^{\vec{n}}} (Z_m)|_{a_i=0, i \neq m, m-1} = 0.$$
for all \(\vec{n}\). This method yields a sum for \(Z_m\) that is equal to \(A_m - A_{m-1}\). More generally, for \(m_1 < m_2\), we may also view \(Z_{m_1, m_2}\) as a function of the independent variables \(a_i\) for \(i \neq m_1, m_2\) and view \(a_{m_1}\) and \(a_{m_2}\) as constants. Then we set

\[
Z_{m_1, m_2} = \left( -\frac{a_{m_1}}{a_{m_2}} \right)^{m_2-m_1} + \sum_{\vec{n}} c(\vec{n})a^{\vec{n}}
\]

and solve for \(c(\vec{n})\) as before.

Another method to obtain the \(A_m\) is to consider the limits of functions in a suitable ring. For example, if we let

\[
g_m(z) = z - \frac{f(z)}{a_m z^{m-1}},
\]

then the limit

\[
\lim_{n \to \infty} g^n_m (\frac{a_{m-1}}{a_m})
\]

is equal to \(A_m - A_{m-1}\). Again we view \(a_{m-1}\) and \(a_m\) as constants, and we interpret expressions with denominators as geometric series.

In [3], Sturmfels considers differential equations satisfied by the roots of a polynomial and expresses their solutions using certain \(A\)-hypergeometric series. He gives formulas for the coefficients \(c(\vec{n})\) and denotes some of these solutions by

\[
-\left[ \frac{a_{m-1}}{a_m} \right] + \left[ \frac{a_{m-2}}{a_{m-1}} \right].
\]

In Section 3 we prove that \(A_m = -\left[ \frac{a_{m-1}}{a_m} \right]\).

We now describe the outline of this paper. In Section 2 we prove that NRS(1) is equivalent to Newton-Raphson-Simpson method. In Section 3 we define NRS(\(m\)) using the trees with negative vertex degree and some functions built from \(f(z)\) that we call auxiliary functions. In Section 4 we show how to explicitly compute the auxiliary functions. In Section 5 we apply NRS(\(m\)) to actual polynomials and present numerical tables of the associated quantities. In Section 6 we discuss further work.

## 2 NRS(1) and the type number of a tree

For each plane tree, we define what we call its type number. We then show how the Newton-Raphson-Simpson method is actually summing trees ordered by this type number (Theorem 2). The NRS(\(m\)) will also sum trees by type number, but the trees will have negative vertex degree.

We recall the definition of rooted trees and plane trees. See Chapter 5 of [2]. We will use the convention that each vertex has degree 0 or degree \(\geq 2\).

**Definition 1.** A rooted tree \(R\) is a finite acyclic graph with one vertex distinguished as the root which we denote by \(\text{root}(R)\). Let \(v\) be a vertex of \(R\). The subtrees of \(v\) are the
components of $R \setminus v$ that do not contain the root of $R$. A plane tree $T$ is a rooted tree such that the subtrees of each vertex are linearly ordered. The vertex degree $\deg(v)$, or just degree, of a vertex $v$ in $T$ is the number of subtrees of $v$. We also require that $\deg(v) \neq 1$ for any vertex $v$ of $T$.

**Remark 1.** A plane tree $T$ is equivalent to an ordered sequence $\{T_1, T_2, \ldots, T_k\}$ of other plane trees, for $k \geq 0$. We call $T_i$ the $i$-th root subtree of $T$. We denote the plane tree that consists of a single vertex by $T_0$; this tree corresponds to $k = 0$ and the empty sequence. We let $d_i(T)$ be the number of vertices of $T$ that have degree $i$. We call $\{d_i(T)\}_{i=1}^\infty$ the degree sequence of $T$. We let $\text{Luk}_1$ denote the set of all plane trees (after the Łukasiewicz words reviewed in Section 3).

### 2.1 The ring $R_1$

We define a ring $R_1$ and a map from the set of plane trees into $R_1$.

For integer $k \geq 0$, let $R_1(k)$ be the $\mathbb{Q}$-vector space spanned by monomials of the form

$$\prod_{i=0}^\infty (-a_i/a_1)^{n_i}$$

where $n_i$ are non-negative integers, almost all 0 and with $n_1 = 0$, satisfying

$$\sum_{i=0}^\infty n_i = k.$$ 

Thus an element $r \in R_1(k)$ is a finite sum of the monomials of the form (2). For $r_{k_1} \in R_1(k_1)$ and $r_{k_2} \in R_1(k_2)$, clearly

$$r_{k_1} r_{k_2} \in R_1(k_1 + k_2).$$

We let $R_1$ be the ring consisting of all elements $R_1$ of the form

$$r = \sum_{k=0}^\infty r(k)$$

where $r(k) \in R_1(k)$, and where addition and multiplication in $R_1$ are the usual operations on graded infinite sums. Note that in the sum (3) for $R_1$ we allow infinitely many of the $r(k)$ to be non-zero; in this case we say that $r$ is an infinite sum. Otherwise we say that $r$ is a finite sum. We let $f(z)$ denote a general power series of the form:

$$f(z) = \sum_{k=0}^\infty a_k z^k.$$ 

We view the coefficients $a_k$ as indeterminates. We define expressions $R_1(T)$ using the coefficients $a_k$ and a plane tree $T$.

**Definition 2.** Let $T$ be a rooted planar tree. Define $R_1(T) \in R_1$ to be

$$R_1(T) = \prod_{k=0}^\infty (-a_k/a_1)^{d_k(T)}.$$ 

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2.2 The element $A_1$

We wish to find a way to compute the sum

$$A_1 = \sum_{T \in \text{Luk}_1} R_1(T).$$

Note that this sum $A_1$ is a well-defined element of $R_1$ because, for each $k \geq 0$, there are only finitely many $T$ with

$$k = \sum_{i=0}^{\infty} d_i(T).$$

Thus we write

$$A_1 = \sum_{k=0}^{\infty} A_1(k)$$

with $A_1(k) \in R_1(k)$. For a given power series $f(z)$ whose coefficients $a_k$ are actual complex numbers, we would like to evaluate $A_1$ as a complex number. By direct substitution, the $A_1(k)$ easily yield complex numbers (because the $A_1(k)$ are polynomials in the $-\frac{a_k}{a_1}$). Then an obvious way to evaluate $A_1$ is to take the limit of partial sums

$$\lim_{N \to \infty} \sum_{k=0}^{N} A_1(k).$$

However, we find that for general $f(z)$ this series does not have desirable convergence properties. We thus specify a different way of ordering the sum. We prove that this ordering is equivalent to the Newton-Raphson-Simpson method. If the partial sums of this ordering converge, then $A_1$ corresponds to a zero of $f(z)$.

To specify this different ordering, we define what we call the type number of a plane tree.

2.3 The type number of a plane tree

**Definition 3.** Let $T$ be a plane tree. We define a non-negative integer $\text{type}(T)$, which we call the type number of $T$, and we say that $T$ is of type $n$ if $\text{type}(T) = n$. If $T = T_0$ consists of a single vertex, then define $\text{type}(T)$ to be 0. Otherwise, define $\text{type}(T)$ to be $n+1$ if either of the following two conditions holds:

1. Exactly one of $T$’s root subtrees is of type $n+1$ and the rest are of type at most $n$.
2. Two or more of $T$’s root subtrees are of type $n$ and the rest are of type less than $n$.

If $T$ satisfies the second condition, we say that $T$ is final.

**Definition 4.** Let

\[ \text{Luk}_1(n) = \{ T \in \text{Luk}_1 \text{ and } \text{type}(T) = n \} \]

Let

\[ J_1(n) = \sum_{T \in \text{Luk}_1(n)} R_1(T) \]
and

\[ S_1(0) = 0, \quad S_1(n) = \sum_{k=1}^{n} J(k). \]

Note that the elements \( J_1(n) \) and \( S_1(n) \) are well-defined elements of \( R_1 \).

The ordering that we specify is

\[ A_1 = \sum_{n=0}^{\infty} J_1(n). \]

Now \( J_1(n) \) as an element of \( R_1 \) is itself an infinite sum; instead of trying to find an ordering to evaluate that sum, we establish an equation (Theorem 1) in \( R_1 \) that is linear in \( J_1(n) \) with coefficients in terms of \( J(k) \) for \( k < n \). Then we solve those equations for \( J_1(n) \).

This allows us to express \( J_1(n) \) as a ratio of elements in \( R_1 \).

We show how to establish the equations for \( J_1(n) \). We use the auxiliary function \( f_0(x) \):

**Definition 5.** Define the auxiliary function \( f_0(x) \) by

\[ f_0(x) = \sum_{n=2}^{\infty} -\frac{a_n}{a_1} \left( \frac{a_0}{a_1} + x \right)^n = -\frac{1}{a_1}(f(-\frac{a_0}{a_1} + x) - a_1(-\frac{a_0}{a_1} + x) - a_0). \]

The two necessary properties of \( f_0(x) \) are given below in Property 1.

**Definition 6.** Let \( X \) be a subset of \( \text{Luk}_1 \). Define the set \( \text{Subtrees}_m(X) \subset \text{Luk}_1 \) to be the set of trees \( T \) such that if \( T' \) is a root subtree of \( T \) with \( \text{deg}(\text{root}(T')) \geq 2 \), then \( T' \in X \).

Let \( T_1 \in \text{Luk}_1 \) such that \( T_1 \notin X \). Define the set \( \text{Subtrees}_1(X,T_1) \subset \text{Luk}_1 \) to be the set of trees \( T \) such that \( T \) has exactly one root subtree that is equal to \( T_1 \); and if \( T' \neq T_1 \) is a root subtree of \( T \) with \( \text{deg}(\text{root}(T')) \geq 2 \), then \( T' \in X \).

**Property 1.** For \( X \subset \text{Luk}_m \), let

\[ S(X) = \sum_{T \in X \cap \text{Luk}_1} R_1(T). \]

Then

\[ f_0(S(X)) = \sum_{T \in \text{Subtrees}(X) \cap \text{Luk}_1} R_m(T). \]

Let \( T_1 \in \text{Luk}_1 \) and \( \notin X \). Then

\[ R_1(T_1)f'_0(S(X)) = \sum_{T \in \text{Subtrees}(X,T_1) \cap \text{Luk}_1} R_1(T). \]

The next theorem establishes equations that determine \( J_1(n) \).

**Theorem 1.**

\[ J(1) = f_0(0) + J(1)f'_0(0) \]
\[ J(n+1) = f_0(S_1(n)) - f_0(S_1(n-1)) - J_1(n)f'_0(S_1(n-1)) + J(n+1)f'_0(S_1(n)) \]
Proof. We first establish the equation for \( J(1) \). Let \( T \) be of type 1. If \( T \) is final, then all of its root subtrees of type 0; that is, all the root subtrees are single vertices. Thus

\[
\sum_{\text{type}(T) = 1, \: T \text{ is final}} R_1(T) = f_0(0).
\]

If \( T \) is not final, then it has exactly one root subtree of type 1 and the rest are single vertices.

\[
\sum_{\text{type}(T) = 1, \: T \text{ is not final}} R_1(T) = J(1) f'_0(0).
\]

Therefore

\[
\sum_{\text{type}(T) = 1} R_1(T) = J(1) = f_0(0) + J(1) f'_0(0).
\]

We now determine \( J_1(n) \). For any positive integer \( k \), we have that

\[
f_0(S(k))
\]

represents trees whose root subtrees are of type at most \( k \); and

\[
J(k) f'_0(S(k-1))
\]

represents trees with exactly one root subtree of type \( k \) and the rest of type at most \( k-1 \). Therefore

\[
\sum_{\text{type}(T) = n+1, \: T \text{ is final}} R_1(T) = f_0(S_1(n)) - f_0(S_1(n-1)) - J_1(n) f'_0(S_1(n-1)).
\]

\[
\sum_{\text{type}(T) = n+1, \: T \text{ is not final}} R_1(T) = J(n+1) f'_0(S_1(n)).
\]

\[
\sum_{\text{type}(T) = n+1} R_1(T) = J(n+1) = f_0(S_1(n)) - f_0(S_1(n-1)) - J_1(n) f'_0(S_1(n-1)) + J(n+1) f'_0(S_1(n))
\]

Lemma 1. For \( n \geq 1 \), we have

\[
J_1(n) = \frac{f(-\frac{a_0}{a_1} + S_1(n-1))}{f'(-\frac{a_0}{a_1} + S_1(n-1))}.
\]

Proof. We prove this by induction. When \( n = 1 \), we have by Lemma 1

\[
J(1) = \frac{f_0(0)}{1 - f'_0(0)}.
\]

Now

\[
f_0(0) = \frac{1}{a_1} f(-\frac{a_0}{a_1}), \quad 1 - f'_0(0) = \frac{1}{a_1} f'(-\frac{a_0}{a_1})
\]
so

\[ J(1) = -\frac{f(-\frac{a_0}{a_1})}{f'(-\frac{a_0}{a_1})}. \]

Now assume the statement of the lemma is true for some \( n \geq 1 \). By Lemma 1

\[ J(n + 1) = \frac{f_0(S_1(n)) - f_0(S_1(n - 1)) - J_1(n)f'_0(S_1(n - 1))}{1 - f'_0(S_1(n))}. \]  

(5)

Applying the definition of \( f_0(x) \) and simplifying, we have

\[ f_0(S_1(n)) = -\frac{1}{a_1}f(-\frac{a_0}{a_1} + S_1(n)) + S_1(n), \]

\[ -f_0(S_1(n - 1)) = \frac{1}{a_1}f(-\frac{a_0}{a_1} + S_1(n - 1)) - S_1(n - 1), \]

and

\[ -J_1(n)f'_0(S_1(n - 1)) = \frac{1}{a_1}J_1(n)f'(-\frac{a_0}{a_1} + S_1(n - 1)) - J_1(n). \]

The sum of the above three expressions is

\[ -\frac{1}{a_1}f(-\frac{a_0}{a_1} + S_1(n)) \]

where we use the fact that

\[ S_1(n) = S_1(n - 1) + J_1(n) \]

and the induction hypothesis. Furthermore

\[ 1 - f'_0(S_1(n)) = \frac{1}{a_1}f'(-\frac{a_0}{a_1} + S_1(n)). \]

Substituting these results into (5) proves the induction step and the lemma. \( \square \)

**Theorem 2.** Let \( c_0 = 0 \) and \( c_N \) be defined by the Newton-Raphson-Simpson method applied to \( f(z) \). Let \( J_1(n) \) be as defined above. For \( N \geq 1 \), then

\[ c_N = \sum_{n=0}^{N-1} J_1(n). \]

**Proof.** If \( N = 1 \), then

\[ J(0) = -\frac{a_0}{a_1} \]

and the statement of the theorem is true. We re-express the statement as

\[ c_N = -\frac{a_0}{a_1} + S_1(n - 1) \]

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and assume it is true for some \( N \geq 1 \). Now we apply Lemma \( \PageIndex{1} \) to obtain

\[
-\frac{a_0}{a_1} + S_1(n) = -\frac{a_0}{a_1} + S_1(n - 1) + J_1(n)
\]

\[
= c_N - \frac{f\left( -\frac{a_0}{a_1} + S_1(n - 1) \right)}{f'\left( -\frac{a_0}{a_1} + S_1(n - 1) \right)}
\]

\[
= c_N - \frac{f(c_N)}{f'(c_N)}
\]

\[
= c_{N+1}.
\]

This proves the theorem. \( \Box \)

3 NRS(\( m \)) and trees with negative vertex degree

To define the algorithms, we define generalized Lukasiewicz words (Definition \( \PageIndex{8} \)) and trees with negative vertex degree (Construction \( \PageIndex{1} \)). The plane trees discussed above have possible vertex degrees of either 0 or an integer that is at least 2. We call these classical plane trees. The trees with negative vertex degree have the same structure as classical plane trees but their vertices may have degree that is any integer except 1.

3.1 Generalized Lukasiewicz words and trees with negative vertex degree

Recall that a plane tree is uniquely determined by the preorder (depth-first order) sequence of its vertex degrees. We will call this sequence the prodder sequence. See Chapter 5 of \cite{2} for the definition of preorder. For classical plane trees, this preorder sequence of non-negative integers is called the Lukasiewicz word for the tree. We recall the defining properties of Lukasiewicz words.

**Definition 7.** A Lukasiewicz word \( \ell \) may be defined as a sequence \( \{l_i\}_{i=1}^N \) of integers such that

\[
l_i \geq 0, \quad \sum_{i=1}^n (l_i - 1) \geq 0, \quad \text{and} \quad \sum_{i=1}^N (l_i - 1) = -1
\]

for each \( n < N \). (Note that according to our convention each \( l_i \neq 1 \) as well.)

**Definition 8.** Define a generalized Lukasiewicz word \( \ell \) to be a sequence \( \{l_i\}_{i=1}^N \) of integers such that

\[
l_i \neq 1, \quad \sum_{i=1}^n (l_i - 1) \geq 0, \quad \text{and} \quad \sum_{i=1}^N (l_i - 1) = -1
\]

for each \( n < N \). Define \( \text{minDegree}(\ell) \) to be the smallest (most negative) integer \( l_i \) that occurs in \( \ell \). For \( m \geq 1 \), define \( \text{Luk}_m \) to be the set of all generalized Lukasiewicz words \( \ell \) such that \( \text{minDegree}(\ell) \geq -m + 1 \).
Construction 1. Given a generalized Lukasiewicz word \( l = \{l_i\}_{i=1}^N \), we construct a tree \( T \) with negative vertex degree in the following way. We construct a new word \( U(l) \) from \( l \) by taking each \( l_i \) in \( l \) with \( l_i < 0 \) and replacing it with a sequence of 0’s of length \( |l_i| + 1 \). Thus the generalized Lukasiewicz word

\[
l = \{2, 4, 3, 0, -4, 4, 0, 0, -1\}
\]

yields

\[
U(l) = \{2, 4, 3, 0, 0, 0, 0, 0, 4, 0, 0, 0\}.
\]

By construction \( U(l) \) is a non-generalized Lukasiewicz word and thus is the preorder sequence for some classical plane tree which we call \( U(T) \). Now from \( U(T) \) we construct the tree \( T \): we give \( T \) the same structure as \( U(T) \), but for each set of \( |l_i| + 1 \) vertices of degree 0 in \( U(T) \) that came from an \( l_i < 0 \) in \( l \), we say that the rightmost vertex \( v \) of these vertices in the preorder has degree \( l_i \); that the \( |l_i| \) vertices of degree 0 immediately to the left of \( v \) in the preorder are “canceled” by \( v \); and that these canceled vertices do not contribute to the number of vertices of degree 0 in \( T \). We say that a canceled vertex does not have any degree but we do consider it a subtree of its parent vertex. We say that the classical plane tree \( U(T) \) is the underlying tree of \( T \). We say that \( T \) has the preorder sequence \( l \). See Figure 1.

For \( m \geq 1 \), we identify the set of all plane trees whose vertex degrees are at least \(-m + 1\) with \( \text{Luk}_m \).

Definition 9. Let \( T \) be a tree with negative vertex degree with preorder sequence \( l = \{l_i\}_{i=1}^N \). We define the type number \( \text{type}(T) \) of \( T \) to be equal to \( \text{type}(U(T)) \), where \( U(T) \) is the underlying tree of \( T \). We say that \( T \) is final if \( U(T) \) is final. We define \( \text{terminal}(T) \) to be the number of consecutive 0’s at the right end of \( l \).

Remark 2. We can construct any tree \( T \) with negative vertex degree by specifying a sequence of trees \( \{T_1, T_2, ..., T_k\} \), where each \( T_i \) is a tree of negative vertex degree, and then appropriately assigning negative degrees to those trees \( T_i \) that consist of a single
vertex. That is, suppose $T_i$ is a single vertex and we assign it to have degree $-h < 0$. Then there must be a subsequence of the form

$$\{T_{i-k+1}, T_{i-k+2}, \ldots, T_{i-1}, T_i\}$$

(6)

where $T_j$ consists of a single vertex for $i - k + 2 \leq j < i$, and terminal($T_{i-k+1}$) $\geq h - (k - 2)$. This motivates the following definition.

**Definition 10.** For integers $k$ and $h$ with $m - 1 \geq h \geq k - 1 \geq 1$, define a $(k, h)_m$-block to be a sequence

$$B = \{T_1, T_0, T_0, \ldots, T_0\}$$

of trees in $L_k$, where there are $k - 1$ trees $T_0$ after $T_1$, and terminal($T_1$) $\geq h - (k - 2)$. Define a $1_m$-block to be a sequence consisting of a single tree

$$B = \{T_1\}$$

where $T_1$ is any tree in $L_k$. We refer to both $(k, h)_m$-blocks and $1_m$ blocks as blocks.

**Remark 3.** We identify $L_k$ with the set of sequences

$$\{B_1, B_2, \ldots, B_N\}$$

where $N \geq 0$ and $B_i$ is either a $(k, h)_m$-block or a $1_m$-block. The tree $T_0$ corresponds to the empty sequence (when $N = 0$). We compare this identification to that of Remark 1.

### 3.2 The number of generalized Lukasiewicz words with a given degree sequence

Let

$$\{d_0, d_1, d_2, \ldots\}$$

be a sequence of non-negative integers such that $d_1 = 0$; only finitely many of the $d_k$ are non-zero; and

$$\sum_{k=0}^{\infty} (k-1)d_k = -1.$$ 

Then the number of Lukasiewicz words

$$l = \{l_1, l_2, \ldots, l_N\}$$

such that the integer $k$ appears $d_k$ times in $l$ is equal to

$$\frac{\left(\sum_{k=0}^{\infty} d_k\right)!}{\left(\sum_{k=0}^{\infty} d_k\right)} \prod_{k=0}^{\infty} (d_k)!.$$ 

Theorem 5.3.10 of [2] proves this statement. We present a corresponding result about generalized Lukasiewicz words. The proof in [2] directly carries over and we present it here in that generality.
Theorem 3. Let
\[ d = \{ \cdots, d_{-2}, d_{-1}, d_0, d_1, d_2, \cdots \} \]
be a sequence of non-negative integers such that \( d_1 = 0 \); only finitely many of the \( d_i \)
are non-zero; and
\[ \sum_{i=-\infty}^{\infty} (k-1)d_k = -1. \]
The number of generalized Lukasiewicz words
\[ l = \{ l_1, l_2, \ldots, l_N \} \]
with degree sequence \( d \) is
\[ \frac{\left( \sum_{k=-\infty}^{\infty} d_k \right)!}{\left( \sum_{k=-\infty}^{\infty} d_k \right) \prod_{k=-\infty}^{\infty} (d_k)!}. \]

Proof. Let
\[ \sum_{k=-\infty}^{\infty} d_k = N. \]
Consider the set \( \mathcal{A}_d \) of all sequences
\[ l = \{ l_1, l_2, \ldots, l_N \} \]
such that \( d_k \) of the \( l_i \) equal \( k \) and
\[ \sum_{i=-\infty}^{\infty} (i-1)d_i = -1. \]
The order of \( \mathcal{A}_d \) is thus
\[ |\mathcal{A}_d| = \frac{\left( \sum_{k=-\infty}^{\infty} d_k \right)!}{\prod_{k=-\infty}^{\infty} (d_k)!}. \]
Let \( l \in \mathcal{A}_d \) and let \( C(i, l) \) denote the \( i \)-th conjugate of \( l \):
\[ C(i; l) = \{ l_{i+1}, l_{i+2}, \ldots, l_{N-1}, l_N, l_1, l_2, \ldots, l_{i-1} \} \]
We claim that these \( N \) conjugates are distinct. If \( C(i; l) = C(j; l) \) for \( j > i \), then that
means
\[ l_k = l_{k'} \]
whenever $k \equiv k' \mod (j - i)$. This implies that $j - i$ divides $N$ and that each $d_k$ is a multiple of $\frac{N}{j - i}$. By assumption

$$\sum_{k=-\infty}^{\infty} (k - 1)d_k = -1,$$

so $\frac{N}{j - i}$ divides 1. But that means $j - i = N$, which is impossible since $1 \leq i, j \leq N$. Therefore the $N$ conjugates of $l$ are distinct.

We claim that exactly one of these conjugates is a generalized Lukasiewicz word. First we show that at least one conjugate is a generalized Lukasiewicz word. Suppose that the negative integer $M$ is an attained lower bound for the partial sums:

$$\sum_{i=1}^{k} (l_i - 1) \geq M$$

for all $1 \leq k \leq N$ and that

$$\sum_{i=1}^{k_1} (l_i - 1) = M$$

with $k_1$ minimal (we may assume that $k_1 \neq N$, or else $M = -1$ and we are done). Then we claim that the conjugate $w$

$$w = \{l_{k_1+1}, l_{k_1+2}, ..., l_N, l_1, l_2, ..., l_{k_1}\}$$

is a generalized Lukasiewicz word. We have

$$\sum_{i=k_1+1}^{k} (l_i - 1) \geq 0$$

for all $k_1 \leq k \leq N$, or else $M$ would not be a lower bound.

Now suppose

$$\sum_{i=k_1+1}^{N} (l_i - 1) + \sum_{i=1}^{k} (l_i - 1) < 0$$

for some $1 \leq k < k_1$. Since

$$\sum_{i=k_1+1}^{N} (l_i - 1) = -M - 1,$$

that implies

$$\sum_{i=1}^{k} (l_i - 1) < M + 1,$$

contradicting the minimality of $k_1$. Therefore $w$ is a generalized Lukasiewicz word.

Now suppose

$$w = \{w_1, w_2, ..., w_N\}$$
is a generalized Lukasiewicz word. If some conjugate $w'$

$$w' = \{w_j, w_{j+1}, \ldots, w_N, w_1, w_2, \ldots, w_{j-1}\}$$

for $j \neq 1$ is also a generalized Lukasiewicz word, then

$$\sum_{i=j}^{N}(w_i - 1) \geq 0$$

and

$$\sum_{i=j}^{N}(w_i - 1) + \sum_{i=1}^{j-1}(w_i - 1) = -1.$$ 

Therefore

$$\sum_{i=1}^{j-1}(w_i - 1) < 0.$$ 

But this contradicts the assumption that $w$ is a generalized Lukasiewicz word. Therefore the only conjugate of $w$ that is a generalized Lukasiewicz word is $w$ itself.

Let $L_d$ denote the set of generalized Lukasiewicz words with degree sequence $d$. Now $L_d \subset A_d$, and we have partitioned $A_d$ into subsets that each have order $N$ such that each subset contains exactly one generalized Lukasiewicz word. Thus

$$|L_d| = \frac{|A_d|}{N}.$$

This proves the theorem.

3.3 The ring $R_m$

We define the ring $R_m$. For $k \geq 0$, let $R_m(k)$ be the $\mathbb{Q}$-vector space spanned by monomials of the form

$$\prod_{i=-m+1}^{\infty} \left(-\frac{a_{m-1+i}}{a_m}\right)^{n_i}$$

where $n_i$ are non-negative integers, almost all zero with $n_1 = 0$, satisfying

$$\sum_{i=-m+1}^{\infty} n_i = k.$$ 

Thus an element $r \in R_m(k)$ is a finite sum of the monomials of the form (7). For $r_{k_1} \in R_m(k_1)$ and $r_{k_2} \in R_m(k_2)$, then

$$r_{k_1}r_{k_2} \in R_m(k_1 + k_2).$$

We let $R_m$ be the ring consisting of all elements $r$ of the form

$$r = \sum_{k=0}^{\infty} r(k)$$

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where $r(k) \in R_m(k)$; and where addition and multiplication in $R_m$ are the usual operations on infinite sums. Note that in the sum we allow infinitely many of the $r(k)$ to be non-zero.

**Definition 11.** Let $T \in \text{Luk}_m$. Define

$$R_m(T) = \prod_{k=-m+1}^{\infty} \left(-\frac{a_{m+k-1}}{a_m}\right)^{d_k(T)}$$

We call $R_m(T)$ the $R_m$-expression of $T$.

### 3.4 The element $A_m$

**Definition 12.**

$$A_m = \sum_{T \in \text{Luk}_m} R_m(T).$$

As for $A_1$ in section 2, the elements $A_m$ are well-defined elements of $R_m$ because for any $k \geq 0$, there are only finitely many trees $T \in \text{Luk}_m$ with

$$\sum_{i=\infty}^{\infty} d_i(T) = k.$$ 

We let

$$\begin{bmatrix} a_{j-1} \\ a_j \end{bmatrix}$$

denote the $t$-hypergeometric series of $[3]$.

**Theorem 4.** The $t$-hypergeometric series $\begin{bmatrix} a_{m-1} \\ a_m \end{bmatrix}$ may be viewed as an element of $R_m$. As elements of $R_m$,

$$A_m = -\begin{bmatrix} a_{m-1} \\ a_m \end{bmatrix}.$$

**Proof.** To agree with the notation of $[3]$, we let $j = m$. In equation 4.2 of $[3]$, Sturmfels defines $\begin{bmatrix} a_{j-1} \\ a_j \end{bmatrix}$ to be the infinite sum

$$\begin{bmatrix} a_{j-1} \\ a_j \end{bmatrix} = \sum_i \frac{(-1)^i_j}{i_j-1+1} \left( \frac{i_j}{i_0, i_1, \ldots, i_j-1, i_j+1, \ldots, i_n} \right) \left( \frac{a_{j-1}}{a_{i_j+1}} \right) \prod_{k=0, k \neq j}^{n} a_k^{i_k} \tag{9}$$

where the sum is over all sequences $i$ of non-negative integers $\{i_0, i_1, \ldots, i_n\}$ such that

$$\sum_{k=0, k \neq j}^{n} i_k = i_j \tag{10}$$

and

$$\sum_{k=0, k \neq j}^{n} ki_k = ji_j. \tag{11}$$
Using equation (10), equation (11) may be rewritten as

\[-(i_{j-1} + 1) + \sum_{k=0, k \neq j, j-1}^{n} (k - j) i_k = -1.\]

And

\[\frac{1}{i_{j-1} + 1} \left( \begin{array}{c}
i_j \\
i_0, i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n \end{array} \right) = \frac{1}{i_j + 1} \left( \begin{array}{c}
i_j + 1 \\
i_0, i_1, \ldots, i_{j-2}, i_{j-1} + 1, i_{j+1}, \ldots, i_n \end{array} \right).\]  

(12)

Thus we can interpret each \(i_k, k \neq j, j-1\) as the number of vertices in a tree \(T\) with negative vertex degree that have degree \(1 + k - j\); \(i_{j-1} + 1\) as the number of vertices that have degree 0; and \(i_j\) as the number of vertices that have vertex degree (that is, are not canceled). By Theorem 3, expression (12) counts the number of all such \(T\).

The monomial factor in (9) is then \(-R_j(T)\).

3.5 Strategy to determine \(J_m(n)\)

We perform this sum by ordering the trees \(T\) according to their type number: letting

\[J_m(n) = \sum_{T \in \text{Luk}_m, \text{type}(T) = n} R_m(T),\]

(13)

we write

\[A_m = \sum_{n=0}^{\infty} J_m(n).\]

(14)

As in Section 2 for \(m = 1\), the quantity \(J_m(n)\) is an infinite sum. Instead of specifying an ordering for this sum, we establish equations in \(R_m\) that allow us to solve for \(J_m(n)\). Specifically, we introduce the quantities \(J_{i,m}(n)\) for \(0 \leq i < m - 1\) and establish a system of \(m\) equations that are linear in the \(J_{i,m}(n)\). We define these \(J_{i,m}(n)\) now.

First, recall the construction of trees in \(\text{Luk}_m\) discussed in Remark 2. Given integers \(h\) and \(k\), the number \(\text{terminal}(T)\) determines whether \(T\) is a valid choice for the first tree in the \((k, h)_m\)-block. Therefore we partition \(\text{Luk}_m\) into \(m + 1\) disjoint subsets based on \(\text{terminal}(T)\):

**Definition 13.** Let \(T_0\) denote the tree consisting of a single vertex and let \(m \geq 1\). If \(m > 1\), define

\[\text{Luk}_{1,m} = \{T \in \text{Luk}_m : \text{terminal}(T) = 1 \text{ and } T \neq T_0\}.\]

For \(i \neq 1\) and \(0 \leq i < m - 1\), define

\[\text{Luk}_{i,m} = \{T \in \text{Luk}_m : \text{terminal}(T) = i\}\]

and

\[\text{Luk}_{m-1,m} = \{T \in \text{Luk}_m : \text{terminal}(T) \geq m - 1 \text{ and } T \neq T_0\}.\]
Thus
\[ \text{Luk}_m = \{ T_0 \} \bigcup_{i=0}^{m-1} \text{Luk}_{i,m} \]

Refining this partition by the type number yields the following terms.

**Definition 14.** For \( 0 \leq i \leq m-1 \), let
\[ \text{Luk}_{i,m}(n) = \{ T \in \text{Luk}_{i,m} : \text{type}(T) = n \}. \]

\[ J_{i,m}(n) = \sum_{T \in \text{Luk}_{i,m}(n)} R_m(T) \tag{15} \]

Thus for \( n \geq 1 \)
\[ J_m(n) = \sum_{i=0}^{m-1} J_{i,m}(n). \]

We next explain how to establish the system of linear equations to determine the \( J_{i,m}(n) \). When \( m = 1 \), the quantity
\[ J_{0,1}(n) = J_1(n) \]
and we used the single auxiliary function \( f_0(x) \) to establish an equation for \( J_{0,1}(n) \). For general \( m \), we will use \( m \) auxiliary functions \( f_{i,m}(\vec{x}) \) in \( m \) variables \( \vec{x} = (x_0, x_1, \ldots, x_{m-1}) \).

The two properties that these auxiliary have which generalize the those of \( f_0(x) \) are listed in Property 2 below.

**Definition 15.** Let \( X \) be a subset of \( \text{Luk}_m \). Define the set \( \text{Subtrees}_m(X) \subset \text{Luk}_m \) to be the set of trees \( T \) such that if \( T' \) is a subtree of the root of \( T \) with \( \deg(\text{root}(T')) \geq 2 \), then \( T' \in X \).

Let \( T_1 \in \text{Luk}_m \) such that \( T_1 \notin X \). Define the set \( \text{Subtrees}_m(X,T_1) \subset \text{Luk}_m \) to be the set of trees \( T \) such that the root of \( T \) has exactly one subtree that is equal to \( T_1 \); and if \( T' \neq T_1 \) is a subtree of the root of \( T \) with \( \deg(\text{root}(T')) \geq 2 \), then \( T' \in X \).

**Property 2.** For \( X \subset \text{Luk}_m \), let
\[ S_i(X) = \sum_{T \in X \cap \text{Luk}_{i,m}} R_m(T) \]

and
\[ \bar{S}(X) = (S_0(X), S_1(X), \ldots, S_{m-1}(X)). \]

Then
\[ f_{i,m}(\bar{S}(X)) = \sum_{T \in \text{Subtrees}_m(X \cap \text{Luk}_{i,m})} R_m(T). \]

Let \( T_1 \in \text{Luk}_{i,m} \) and \( \notin X \). Then
\[ R_m(T_1) \frac{\partial f_{i,m}}{\partial x_j}(\bar{S}(X)) = \sum_{T \in \text{Subtrees}_m(X,T_1 \cap \text{Luk}_{i,m})} R_m(T). \]

To construct the auxiliary functions \( f_{i,m}(\vec{x}) \) that satisfy these properties, we define the map \( P \), *partial blocks*, and *partial trees* next.

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3.6 The forgetful map $P$

Recall Remark 3 in which we identify a tree with negative vertex degree with a sequence of blocks. Let $T_1$ denote the first tree in a block. We define a forgetful map $P$ on the set of blocks such that $P$ forgets everything about the tree $T_1$ except the integer terminal($T_1$) and whether $T_1 = T_0$. Specifically, let $B$ be a $(k, h)_m$-block. If $T_1 \neq T_0$, define $P(B)$ to be the triple

$$P(B) = (\min(m - 1, \text{terminal}(T_1)), k, h),$$

and if $T_1 = T_0$, define $P(B)$ to be the triple

$$(T_0, k, h).$$

On the set of $1_m$-blocks, define

$$P(\{T_0\}) = (T_0)$$

and, if $T_1 \neq T_0$,

$$P(\{T_1\}) = (\min(m - 1, \text{terminal}(T_1))).$$

We call the images of $P$ partial blocks which for concreteness are defined next along with their $R_m$-expressions.

**Definition 16.** Let $T_0$ denote the tree consisting of a single vertex. In the following four cases we define a partial block $b$, its length which we denote length($b$), and its $R_m$-expression $R_m(b)$. The expression $R_m(b)$ will be a monomial in $x_i$ and $(-a_m)^{\pm 1}$.

We define $b$ to be:

1. A triple of integers
   $$b = (i, k, h)$$
   where $1 \leq h \leq m - 1$; $2 \leq k \leq h + 1$; and $h - (k - 2) \leq i \leq m - 1$. Define
   $$\text{length}(b) = k$$
   and
   $$R_m(b) = x_i((-a_m^{-1})^{k-2-h}(-a_m^{-1-h})).$$

2. The triple
   $$b = (T_0, k, k - 1)$$
   where $k \geq 2$. Define
   $$\text{length}(b) = k$$
   and
   $$R_m(b) = -\frac{a_m^{-1-(k-1)}}{a_m}.$$ 

3. For $0 \leq i \leq m - 1$, the 1-tuple
   $$b = (i).$$
Define
\[ \text{length}(b) = 1 \]
and
\[ R_m(b) = x_i. \]

4. The 1-tuple
\[ b = (T_0). \]

Define
\[ \text{length}(b) = 1 \]
and
\[ R_m(b) = -\frac{a_{m-1}}{a_m}. \]

The map \( P \) extends naturally to \( \text{Luk}_m \): writing \( T \) as a sequence of blocks
\[ T = \{ B_1, B_2, ..., B_N \}, \]
we set
\[ P(T) = \{ P(B_1), P(B_2), ..., P(B_N) \}. \]

We call these images partial trees which we define next. The part about the \( s \) empty subtrees will be necessary to express a recurrence relation among the \( f_{i,m}(\bar{x}) \) in Section 4.

**Definition 17.** For integer \( s \geq 0 \), define a partial tree with \( s \) empty subtrees to be a sequence:
\[ \mathcal{T} = \{ b_1, b_2, ..., b_N, \emptyset, ..., \emptyset \} \]
where \( N \geq 0 \); each \( b_i \) is either a \((k,h)\)\( _m \)-partial block or a \( 1_m \)-partial block; and there are \( s \) \( \emptyset \)'s representing empty subtrees. Let \( \text{Luk}_m; s \) denote the set of all such partial trees. We say that
\[ \deg(\text{root}(\mathcal{T})) = s + \sum_{j=1}^{N} \text{length}(b_j). \]

Thus
\[ P(\text{Luk}_m) = \text{Luk}_m; 0. \]

For \( \mathcal{T} \in \text{Luk}_m; 0 \), then by construction \( P^{-1}(\mathcal{T}) \subset \text{Luk}_{i,m} \) for some \( i \). Define terminal(\( \mathcal{T} \)) to be this \( i \).

Define
\[ \text{Luk}_{i,m}; s = \{ \mathcal{T}: \mathcal{T} \in \text{Luk}_{m; s} \text{and terminal}(\mathcal{T}) = i \}. \]

Define the \( R_m \)-expression \( R_m(\mathcal{T}) \) of the partial tree \( \mathcal{T} \) to be
\[ 1(s + \sum_{r=1}^{N} \text{length}(b_r) \geq 2)(-\frac{a_{m-1}+s+\sum\text{length}(b_r)}{a_m}) \prod_{r=1}^{N} R_m(b_r). \]

For \( \mathcal{T} \in \text{Luk}_{m; 0} \), we view \( R_m(\mathcal{T}) = R_m(\mathcal{T})(x_0, x_1, ..., x_{m-1}) \) as a function of the variables \( x_i \). Next we show that \( R_m(\mathcal{T}) \) satisfies properties similar to those listed in Property 2.
Lemma 2. Let $\mathcal{T} \in \mathbb{L}_{m,0}$ and $X \subset \mathbb{L}_{m}$. Suppose $T_1 \notin X$ and $T_1 \in \mathbb{L}_{i,m}$.

1. \[ R_m(\mathcal{T})(\mathcal{S}(X)) = \sum_{T \in \mathbb{L}(\mathcal{T}) \cap \text{Subtrees}_{m}(X)} R_m(T) \]

2. \[ R_m(T_1) \frac{\partial R_m(\mathcal{T})}{\partial x_i}(\mathcal{S}(X)) = \sum_{T \in \mathbb{L}(\mathcal{T}) \cap \text{Subtrees}_{m}(X, T_1)} R_m(T) \]

Proof. 1. This follows immediately from the definition of a partial tree. The expression $R_m(\mathcal{T})(\mathcal{S}(X))$ is obtained by substituting each $x_i$ in $R_m(\mathcal{T})$ with

\[ S_i(X) = \sum_{T \in \mathbb{L}(\mathcal{T}) \cap \text{Subtrees}_{m}(X)} R_m(T) \]

Expanding out, we obtain a sum of terms; each term is an $R_m$ expression of a tree $T \in \mathbb{L}(\mathcal{T})$ whose root subtrees are in $X$ if they have root degree $\geq 2$. Every such $T$ corresponds to some term.

2. By the product rule, the expression $R_m(T_1) \frac{\partial R_m(\mathcal{T})}{\partial x_i}(\mathcal{S}(X))$ is obtained by choosing each factor of $x_i$ that appears in $R_m(\mathcal{T})$ and replacing it with $R_m(T_1)$, and then adding all such expressions. This corresponds to making $T_1$ a root subtree of $T$ in the spot where the $x_i$ was removed. Substituting it the $\mathcal{S}(X)$ as in part 1 now yields the sum of terms that correspond to elements in $\mathbb{L}(\mathcal{T}) \cap \text{Subtrees}_{m}(X, T_1)$.

3.7 The system of linear equations for $J_{i,m}(n)$

Now we can define the auxiliary functions $f_{i,m}(\mathcal{x})$. The auxiliary functions will satisfy the properties in Property 2 because the $R_m(\mathcal{T})$ satisfy similar properties. We can then establish the system of equations for $J_{i,m}(n)$ in Theorem 5.

Definition 18.

\[ f_{i,m}(\mathcal{x}, s) = \sum_{\mathcal{T} \in \mathbb{L}_{i,m,s}} R_m(\mathcal{T}) \]

\[ f_{i,m}(\mathcal{x}) = f_{i,m}(\mathcal{x}, 0). \]

Corollary 1. The auxiliary functions $f_{i,m}(\mathcal{x})$ satisfy the two properties in Property 2.

Proof. This is immediate from the definition of $f_{i,m}(\mathcal{x})$, Lemma 2 and the disjoint union

\[ \bigcup_{\mathcal{T} \in \mathbb{L}_{i,m,0}} P^{-1}(\mathcal{T}) = \mathbb{L}_{i,m}. \]
Recall the definitions

\[ J_{i,m}(n) = \sum_{T \in \text{Luk}_{i,m}(n)} R_m(T) \]

and

\[ S_{i,m}(n) = \sum_{k=1}^{n} J_{i,m}(k), \]

\[ \bar{S}_m(n) = (S_{0,m}(n), S_{1,m}(n), ..., S_{m-1,m}(n)). \]

**Theorem 5.** Let \( F(x_0, x_1, ..., x_{m-1}) \) be a function. For \( n \geq 1 \), define \( L_m(F, n) \) to be:

\[ L_m(F, n) = F(\bar{S}_m(n-1)) + \sum_{i=0}^{m-1} J_{i,m}(n) \frac{\partial F}{\partial x_i}(\bar{S}_m(n-1)). \]

The following is a system of linear equations in the unknowns \( J_{i,m}(1) \):

\[ \{ J_{i,m}(1) = L_m(f_{i,m}, 1) : 0 \leq i \leq m-1 \}. \]

Assuming that \( J_{i,m}(k) \) has been evaluated for \( 1 \leq k \leq n \), the following is a system of linear equations in the unknowns \( J_{i,m}(n+1) \):

\[ \{ J_{i,m}(n+1) = L_m(f_{i,m}, n+1) - L_m(f_{i,m}, n) : 0 \leq i \leq m-1 \}. \]

**Proof.** Let \( n = 1 \). Using the properties of the auxiliary functions in Property 2, we obtain

\[ \sum_{T \in \text{Luk}_{i,m}(1), T \text{ is not final}} R_m(T) = \sum_{k=0}^{m-1} J_{k,m}(1) \frac{\partial f_i}{\partial x_k}(\bar{S}_m(0)) \]

and

\[ \sum_{T \in \text{Luk}_{i,m}(1), T \text{ is final}} R_m(T) = f_i(\bar{S}_m(0)). \]

Adding these two equations yields

\[ J_{i,m}(1) = L(f_{i,m}, 1). \]

Considering all \( i, 0 \leq i \leq m-1 \), gives a system of \( m \) equations in the \( m \) unknowns \( J_{i,m}(1) \).

Now let \( n > 1 \). Again using the properties of the auxiliary functions in Property 2, we obtain

\[ \sum_{T \in \text{Luk}_{i,m}(n+1), T \text{ is not final}} R_m(T) = \sum_{k=0}^{m-1} J_{k,m}(n+1) \frac{\partial f_i}{\partial x_k}(\bar{S}_m(n)) \]

and

\[ \sum_{T \in \text{Luk}_{i,m}(n+1), T \text{ is final}} R_m(T) = f_i(\bar{S}_m(n)) - f_i(\bar{S}_m(n-1)) - \sum_{k=0}^{m-1} J_{k,m}(n) \frac{\partial f_i}{\partial x_k}(\bar{S}_m(n-1)). \]
Adding these two equations yields

\[ J_{i,m}(n + 1) = L(f_{i,m}, n + 1) - L(f_{i,m}, n). \]

Considering all \( i, 0 \leq i \leq m - 1 \), gives a system of \( m \) equations in the \( m \) unknowns \( J_{i,m}(n + 1) \).\[\square\]

Solving this system allows us to express \( J_{i,m}(n) \) as a ratio of elements in \( R_m \), and via

\[ J_m(n) = \sum_{i=0}^{m-1} J_{i,m}(n) \]

we can also can be express \( J_m(n) \) as a ratio of elements in \( R_m \). Then \( A_m \) is the sum

\[ A_m = \frac{-a_{m-1}}{a_m} + \sum_{n=1}^{\infty} J_m(n). \]

We will explicitly construct the auxiliary functions and find solutions to these systems in Section 5.

4 Explicit construction of the auxiliary functions

We next show how to compute \( f_{i,m}(\vec{x}) \). We use a recurrence relation (equation 4.3) that can be implemented by a computer. We use the function \( \text{PartialTrees}_m(\vec{x}, s) \) defined next.

**Definition 19.**

\[ \text{PartialTrees}_m(\vec{x}, s) = \sum_{T \in \text{Luk}_m} R_m(T). \]

To compute \( \text{PartialTrees}_m(\vec{x}, s) \), we use the following functions.

The function \( \text{PartialBlock}_m(\vec{x}; k, h) \) for \( h \geq k - 1 \geq 1 \) is the sum of \( R_m \)-expressions of all \((k, h)\) partial blocks:

\[ \text{PartialBlock}_m(\vec{x}; k, h) = (-\frac{a_{m-1}}{a_m})(-\frac{a_{m-1}}{a_m})^{k-2-h} \sum_{i=h-(k-2)}^{m-1} (x_i + (-\frac{a_{m-1}}{a_m}) \mathbf{1}(i = 1))). \]

Let \( \vec{n} \) denote

\[ \vec{n} = (n_1, n_2, ..., n_m), \quad n_i \in \mathbb{Z}, \quad n_i \geq 0. \]

For \( s \geq 0 \), the function \( \text{PartialTrees}_m(\vec{x}; s, \vec{n}) \) is the sum of \( R_m \)-expressions of all partial trees \( \Sigma \) with \( s \) empty subtrees such that the sequence of partial blocks for \( \Sigma \) contains \( n_k \) partial blocks of length \( k \):

\[ \text{PartialTrees}_m(\vec{x}; s, \vec{n}) = \text{multinomial}(\vec{n}) \mathbf{1}(s + \Sigma m_i \geq 2)(-\frac{a_s+m-1+\Sigma n_i}{a_m}) \]

\[ \times \left( \frac{-a_{m-1}}{a_m} + \sum_{i=0}^{m-1} x_i \right)^{n_1} \prod_{k=2}^{m} (\sum_{h=k-1}^{m-1} \text{PartialBlock}_m(\vec{x}; k, h))^{n_k} \]
Then $\text{PartialTrees}_m(\bar{x}; s)$ is the sum of the functions $\text{PartialTrees}_m(\bar{x}; s, \bar{n})$ over all $\bar{n}$:

$$\text{PartialTrees}_m(\bar{x}; s) = \sum_{\bar{n}} \text{PartialTrees}_m(\bar{x}; s, \bar{n}).$$

We note that if $f(z)$ is a polynomial, then the above sum over $\bar{n}$ is a finite sum, and then $\text{PartialTrees}_m(\bar{x}; s)$ is a polynomial in $x_0, x_1, ..., x_{m-1}$ whose coefficients are polynomials in the $-\frac{a_k}{a_m}$.

For an $N \geq 0$ and $s \geq 0$, let $\mathcal{T}$ be a partial tree with $s$ empty subtrees:

$$\mathcal{T} = \{b_1, b_2, ..., b_N, \emptyset, \emptyset, ..., \emptyset\}.$$

Recall that $f_{i,m}(\bar{x}, s)$ is the sum of $R_m$-expressions of all partial trees $\mathcal{T}$ with $s$ empty subtrees and $\text{terminal}(\mathcal{T}) = i$. We consider the three cases of whether $i$ is equal to 0; greater than 0 and less than $m - 1$; and equal to $m - 1$.

1. $i = 0$.

We have the following three subcases.

**subcase(1) $N > 0$ and $b_N = (0)$**

The sum of the $R_m$-expressions for such partial trees is

$$x_0 \text{PartialTrees}_m(\bar{x}; s + 1)$$

because we take any partial tree with $s + 1$ empty subtrees and replace the leftmost empty to be the partial block $b = (0)$. This partial block $b$ has $R_m$-expression $x_0$.

**subcase(2) $N > 0$ and $\text{length}(b_N) = k \geq 2$**

Such partial trees are obtained from taking a partial tree with $s + k$ empty subtrees, and replacing the first $k$ empty subtrees with a partial block $b$ of length $k$. The sum of $R_m$-expressions of all such partial trees is

$$\sum_{k=2}^{m} \left( \sum_{h=k-1}^{m-1} \text{PartialBlock}_m(\bar{x}; k, h) \text{PartialTrees}_m(\bar{x}; s + k) \right)$$

**subcase(3) $N = 0$**

The $R_m$-expression of this partial tree $\mathcal{T}$ is

$$\left( \frac{a_{m-1+s}}{a_m} \right) 1(s \geq 2).$$

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Adding the $R_m$-expressions for these three cases yields

$$ f_{0,m}(\bar{x}, s) = x_0 \text{PartialTrees}_m(\bar{x}; s+1) + \sum_{k=2}^{m} \left( \sum_{h=k-1}^{m-1} \text{PartialBlock}_m(\bar{x}; k, h) \text{PartialTrees}_m(\bar{x}; s+k) \right) $$

$$ + \left( \frac{-a_{m-1+s}}{a_m} \right) 1(s \geq 2). $$

2. $1 \leq i < m - 1$

We have the following two subcases.

**subcase(1) $N > 0$ and $b_N = (i)$**

Such partial trees are obtained by taking a partial tree with $s + 1$ empty subtrees and replacing the leftmost empty subtree with the partial block $b = (i)$. The sum of the $R_m$-expressions of all such $\mathcal{T}$ is

$$ x_i \text{PartialTrees}_m(\bar{x}; s+1) $$

**subcase(2) $N > 0$ and $b_N = (T_0)$**

Such partial trees are obtained by taking a partial tree $\mathcal{T}'$ with $s + 1$ empty subtrees and $\text{terminal}(\mathcal{T}') = i - 1$ and replacing the leftmost empty subtree with the partial block $b = (T_0)$. The sum of the $R_m$-expressions of all such $\mathcal{T}$ created this way is

$$ \left( \frac{-a_{m-1}}{a_m} \right) f_{i-1,m}(\bar{x}, s + 1). $$

Adding the $R_m$-expressions for these two subcases yields

$$ f_{i,m}(\bar{x}, s) = x_i \text{PartialTrees}_m(\bar{x}; s+1) + \left( \frac{-a_{m-1}}{a_m} \right) f_{i-1,m}(\bar{x}, s + 1). \quad (16) $$

3. $i = m - 1$

Then we have the following two sub cases.

**subcase(1)**

There is an integer $k$ where $0 \leq k \leq m - 2$ such that there is a subsequence of $\mathcal{T}$

$$ \{b_{N-k}, b_{N-k+1}, \ldots, b_N\} $$

where

$$ b_{N-k} = (i) $$

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for some \( i \geq m - 1 - k \), and

\[ b_{N-k+r} = (T_0) \]

for \( 1 \leq r \leq k \). Such a \( T \) is obtained from a \( T' \) with \( k + 1 \) empty subtrees and replacing the leftmost empty subtree with the partial block \((i)\), and then replacing each of the next \( k \) empty subtrees with the partial block \((T_0)\). Summing over all \( k \), the sum of \( R_m \)-expressions of all such trees \( T \) is

\[
\sum_{k=0}^{m-2} (-a_{m-1}/a_m)^k \left( \sum_{i=m-k-1}^{m-1} x_i \right) \text{PartialTrees}_m(x; k+1)
\]

**subcase (2)**

We have the subsequence

\[
\{ b_{N-(m-1)-1}, b_{N-(m-1)-2}, \ldots, b_N \}
\]

where \( b_r = (T_0) \) for \( N - (m-1) - 1 \leq r \leq N \). Such a \( T \) is obtained from a \( T' \) with \( m - 1 \) empty subtrees and replacing each of the empty subtrees with the partial block \((T_0)\). The sum of the \( R_m \)-expressions of such trees created this way is

\[
(-a_{m-1}/a_m)^{m-1} \text{PartialTrees}_m(x; m-1).
\]

Therefore

\[
f_{m-1,m}(x) = (-a_{m-1}/a_m)^{m-1} \text{PartialTrees}_m(x; m-1) + \sum_{k=0}^{m-2} (-a_{m-1}/a_m)^k \left( \sum_{i=m-k-1}^{m-1} x_i \right) \text{PartialTrees}_m(x; k+1).
\]

\( \square \)

We list the auxiliary functions for a quintic polynomial

\[
f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5.
\]

**m = 1:**

\[
f_{0,1}(x_0) = \frac{-a_2}{a_1} (-\frac{a_0}{a_1} + x_0)^2 + \frac{-a_3}{a_1} (-\frac{a_0}{a_1} + x_0)^3 + \frac{-a_4}{a_1} (-\frac{a_0}{a_1} + x_0)^4 + \frac{-a_5}{a_1} (-\frac{a_0}{a_1} + x_0)^5
\]

**m = 2:**

\[
f_{0,2}(x_0, x_1) = \frac{-a_0}{a_2} (-\frac{a_1}{a_2} + x_1) (-\frac{a_3}{a_2} - \frac{a_0 a_5}{a_1 a_2} (-\frac{a_1}{a_2} + x_1) - \frac{a_4}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1) - \frac{a_5}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1)^2
\]

\[
+ x_0 (-\frac{a_0 a_4}{a_1 a_2} (-\frac{a_1}{a_2} + x_1) - \frac{a_3}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1) - 2 \frac{a_4 a_5}{a_1 a_2} (-\frac{a_1}{a_2} + x_1) (-\frac{a_1}{a_2} + x_0 + x_1)
\]

\[
- \frac{a_4}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1)^2 - \frac{a_5}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1)^3
\]

\[
f_{1,2}(x_0, x_1) = (\frac{-a_1}{a_2} + x_1) (-\frac{a_0 a_4}{a_1 a_2} (-\frac{a_1}{a_2} + x_1) - \frac{a_3}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1) - \frac{a_4 a_5}{a_1 a_2} (-\frac{a_1}{a_2} + x_1) (-\frac{a_1}{a_2} + x_0 + x_1)
\]

\[
- \frac{a_4}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1)^2 - \frac{a_5}{a_2} (-\frac{a_1}{a_2} + x_0 + x_1)^3)
\]

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\(m = 3:\)

\[f_{0,3}(x_0, x_1, x_2) = -\frac{a_5 a_2}{a_2 a_3}(\frac{-a_2}{a_3} + x_1 + x_2)\]
\[+ (\frac{a_0 a_3}{a_2^2} x_2 + \frac{a_2}{a_3}(\frac{-a_2}{a_3} + x_1 + x_2))(-\frac{a_4}{a_3} x_2 + \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2))\]
\[+ x_0(\frac{a_4}{a_3}(-\frac{a_2}{a_3} + x_0 + x_1 + x_2) - \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2)^2\]
\[- \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_1 + x_2))\]

\[f_{1,3}(x_0, x_1, x_2) = x_1(\frac{a_4}{a_3}(-\frac{a_2}{a_3} + x_0 + x_1 + x_2) - \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2)^2\]
\[- \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2))\]

\[f_{2,3}(x_0, x_1, x_2) = -\frac{a_2}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2)(\frac{a_4}{a_3} - \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2))\]
\[+ x_2(\frac{a_4}{a_3} - \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2)^2\]
\[- \frac{a_5}{a_3}(\frac{-a_2}{a_3} + x_0 + x_1 + x_2))\]

\[m = 4:\]

\[f_{0,4}(x_0, x_1, x_2, x_3) = x_0(\frac{-a_5}{a_4})(\frac{-a_3}{a_4} + x_0 + x_1 + x_2 + x_3)\]
\[- \frac{a_5}{a_4}(\frac{-a_0 a_4}{a_3^2} x_3 + \frac{a_1 a_4}{a_3}(x_2 + x_3) + \frac{a_2}{a_4}(\frac{-a_3}{a_4} + x_0 + x_1 + x_2 + x_3))\]

\[f_{1,4}(x_0, x_1, x_2, x_3) = \frac{a_0 a_5}{a_4} x_4 - x_1(\frac{-a_5}{a_4})(\frac{-a_3}{a_4} + x_0 + x_1 + x_2 + x_3)\]

\[f_{2,4}(x_0, x_1, x_2, x_3) = -\frac{a_4}{a_4}(\frac{a_3 a_5^2}{a_4^2} - \frac{a_5}{a_4} x_1) - x_2(\frac{-a_5}{a_4})(\frac{-a_3}{a_4} + x_0 + x_1 + x_2 + x_3)\]

\[f_{3,4}(x_0, x_1, x_2, x_3) = -\frac{a_4^2}{a_3^2}(x_2 + x_3) - x_3(\frac{-a_5}{a_4})(\frac{-a_3}{a_4} + x_0 + x_1 + x_2 + x_3)\]

\[m = 5:\]

\[f_{i,5}(x_0, x_1, ..., x_{m-1}) = 0\]
5 Numerical examples

5.1 A quintic polynomial with rational zeros

Now we specialize $f(z)$ to be the following polynomial with real coefficients and map the various expressions to real numbers.

$$f(z) = (1 - z)(1 - \frac{z}{2})(1 - \frac{z}{4})(1 - \frac{z}{8})(1 - \frac{z}{16})$$

$$= 1 - \frac{31}{16}z + \frac{155}{128}z^2 - \frac{155}{512}z^3 + \frac{31}{1024}z^4 - \frac{1}{1024}z^5$$

| n  | $J_1(n)$ | $-\frac{a_n}{a_1} + S_1(n)$ |
|-----|----------|-----------------------------|
| 0   | 0        | 5.161290323 × 10^{-1}      |
| 1   | 3.099986240 × 10^{-1} | 8.261276563 × 10^{-1} |
| 2   | 1.403785124 × 10^{-1} | 9.665061687 × 10^{-1} |
| 3   | 3.188553499 × 10^{-2} | 9.983917037 × 10^{-1} |
| 4   | 1.604320113 × 10^{-3} | 9.999960238 × 10^{-1} |
| 5   | 3.976182710 × 10^{-6} | 1.000000000 |
| 6   | 2.439269432 × 10^{-11} | 1.000000000 |
| 7   | 9.180054555 × 10^{-22} | 1.000000000 |

Table 1: $m = 1$

| n  | $J_{0,2}(n)$ | $J_{1,2}(n)$ | $J_2(n)$ | $-\frac{a_n}{a_2} + S_2(n)$ |
|-----|--------------|--------------|----------|-----------------------------|
| 0   | 0            | 0            | 0        | 1.600000000 |
| 1   | -4.659688684 × 10^{-1} | 1.285700049 | 8.197311805 × 10^{-1} | 2.419731181 |
| 2   | -2.893844613 × 10^{-1} | 7.161536865 × 10^{-1} | 4.26769251 | 2.846500406 |
| 3   | -1.070644295 × 10^{-1} | 2.455302622 × 10^{-1} | 1.384658328 × 10^{-1} | 2.984966238 |
| 4   | -1.24371433 × 10^{-2} | 2.730562457 × 10^{-2} | 1.486821024 × 10^{-2} | 2.999834449 |
| 5   | -1.448081738 × 10^{-4} | 3.103391679 × 10^{-4} | 1.655309941 × 10^{-4} | 2.999999980 |
| 6   | -1.827406122 × 10^{-8} | 3.86197182 × 10^{-8} | 2.034291060 × 10^{-8} | 3.000000000 |
| 7   | -2.798594637 × 10^{-16} | 5.864244005 × 10^{-16} | 3.06549367 × 10^{-16} | 3.000000000 |
| 8   | -6.410472972 × 10^{-32} | 1.336321214 × 10^{-32} | 6.952739166 × 10^{-32} | 3.000000000 |

Table 2: $m = 2$
Table 3: $m = 3$

| $n$ | $J_{0,3}(n)$ | $J_{1,3}(n)$ | $J_{2,3}(n)$ | $J_{3,3}(n)$ |
|-----|---------------|---------------|---------------|---------------|
| 0   | 0             | 0             | 0             | 0             |
| 1   | -1.469709756  | -5.895234873 $\times 10^{-1}$ | 3.87351306 | 1.814118062 |
| 2   | -8.795058030 $\times 10^{-1}$ | -6.56193355 $\times 10^{-1}$ | 2.427500757 | 8.918016189 $\times 10^{-1}$ |
| 3   | -3.065450285 $\times 10^{-1}$ | -2.976185716 $\times 10^{-1}$ | 8.727716252 $\times 10^{-1}$ | 2.686080251 $\times 10^{-1}$ |
| 4   | -3.145514266 $\times 10^{-2}$ | -3.311045690 $\times 10^{-2}$ | 9.116888144 $\times 10^{-2}$ | 2.170142720 $\times 10^{-4}$ |
| 5   | -2.842482660 $\times 10^{-4}$ | -3.311045690 $\times 10^{-4}$ | 9.116888144 $\times 10^{-4}$ | 1.587356346 $\times 10^{-8}$ |
| 6   | -2.143169948 $\times 10^{-8}$ | -2.585019447 $\times 10^{-8}$ | 6.315545741 $\times 10^{-8}$ | 8.458625235 $\times 10^{-17}$ |
| 7   | -1.63725119 $\times 10^{-16}$ | -1.433465253 $\times 10^{-16}$ | 3.443052895 $\times 10^{-16}$ | 2.395783540 $\times 10^{-33}$ |
| 8   | -3.33597374 $\times 10^{-33}$ | -4.162565330 $\times 10^{-33}$ | 9.893837244 $\times 10^{-33}$ | 2.395783540 $\times 10^{-33}$ |

Table 4: $m = 3$ continued

| $n$ | $\frac{\partial}{\partial a} + S_3(n)$ |
|-----|--------------------------------------|
| 0   | 4.000000000                           |
| 1   | 5.814118062                           |
| 2   | 6.705919681                           |
| 3   | 6.974527706                           |
| 4   | 6.999782970                           |
| 5   | 6.999999984                           |
| 6   | 7.000000000                           |
| 7   | 7.000000000                           |
| 8   | 7.000000000                           |

Table 5: $m = 4$

| $n$ | $J_{0,4}(n)$ | $J_{1,4}(n)$ | $J_{2,4}(n)$ | $J_{3,4}(n)$ |
|-----|---------------|---------------|---------------|---------------|
| 0   | 0             | 0             | 0             | 0             |
| 1   | -2.901096310  | -1.381474433, | 4.104059794 | 3.730963449 |
| 2   | -1.242997894  | -1.092366178  | 5.026520639 $\times 10^{-1}$ | 3.078851277 |
| 3   | -2.246822248 $\times 10^{-1}$ | -2.526877753 $\times 10^{-1}$ | -6.14652087 $\times 10^{-2}$ | 7.35225534 $\times 10^{-1}$ |
| 4   | -6.21936047 $\times 10^{-3}$ | -7.840976033 $\times 10^{-3}$ | -4.228779292 $\times 10^{-3}$ | 2.330509308 $\times 10^{-2}$ |
| 5   | -4.20354555 $\times 10^{-6}$ | -5.637329986 $\times 10^{-6}$ | -3.932363674 $\times 10^{-6}$ | 1.700311697 $\times 10^{-5}$ |
| 6   | -1.780482352 $\times 10^{-12}$ | -2.477579062 $\times 10^{-12}$ | -1.965670765 $\times 10^{-12}$ | 7.55076540 $\times 10^{-12}$ |
| 7   | -3.047097262 $\times 10^{-25}$ | -4.33903585 $\times 10^{-25}$ | -3.710732333 $\times 10^{-25}$ | 1.332463930 $\times 10^{-24}$ |
\[
\frac{\alpha_1}{a_4} + S_4(n)
\]

| \(n\) | \(J_4(n)\) | \(\frac{\alpha_1}{a_4} + S_4(n)\) |
|-------|-----------|-----------------|
| 0     | 0         | 10.00000000     |
| 1     | 3.552452499 | 13.552452499   |
| 2     | 1.246139269 | 14.798591768   |
| 3     | 1.963890324 \times 10^{-1} | 14.994980800   |
| 4     | 5.015969708 \times 10^{-3} | 14.99996770    |
| 5     | 3.229868758 \times 10^{-6} | 15.000000000   |
| 6     | 1.327032861 \times 10^{-12} | 15.000000000   |
| 7     | 2.226906125 \times 10^{-25} | 15.000000000   |

Table 6: \(m = 4\) continued

### 5.2 A Jensen polynomial for \(\xi(\frac{1}{2} + i\sqrt{t})\)

We apply NRS(\(m\)) algorithms to the third degree Jensen polynomial for \(\xi(\frac{1}{2} + i\sqrt{t})\).

We recall the definition of Jensen polynomials for a power series

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k.
\]

The \(N\)-th degree Jensen polynomial is

\[
\sum_{k=0}^{N} \binom{N}{k} a_k z^k.
\]

It is a theorem that a power series series \(f(z)\) has all real zeros if and only all its Jensen polynomials have all real zeros.

Let

\[
\xi(s) = -s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)
\]

Let \(a_i\) denote the power series coefficients of \(\xi(\frac{1}{2} + i\sqrt{t})\):

\[
\xi\left(\frac{1}{2} + i\sqrt{t}\right) = \sum_{k=0}^{\infty} a_k t^k.
\]

To compute \(a_k\), we use the formula

\[
\xi(s) = 1 - 2\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} \frac{b_{k+1}}{2^{k+1}} \left[ \frac{n}{k} \right] \left( \prod_{j=0}^{k} (-s + 1 - 2j) \right) \left( \prod_{j=0}^{k} (s - 2j) \right)
\]

where

\[
b_k = \sum_{n=1}^{\infty} \frac{e^{-\pi n^2}}{(\pi n^2)^k}
\]

(17)
and \[
(n) \\
(k)
\]
is the unsigned Stirling number of the first kind. In another paper we prove that the formula (17) holds by proving
\[
\int_1^\infty e^{-up}u^s \, du = \frac{e^{-p}}{p} (1 + \sum_{n=1}^\infty g_n(s, p))
\]
where
\[
g_n(s, p) = \frac{1}{(n+1)!} \sum_{k=0}^{n-1} \left( \frac{p^{-k-1}}{k+1} \prod_{j=0}^{k}(s - j) \right)
\]
and \(p > 1\). There we prove that the coefficients of the powers of \(s\) in \(g_n(s, p)\) are positive if \(p > 1\) and the series is absolutely convergent for any \(s \in \mathbb{C}\). We use formula (17) summing up to \(n = 100\) to compute
\[
\begin{align*}
a_0 &= 9.9424 \times 10^{-1} \\
a_1 &= -2.2982 \times 10^{-2} \\
a_2 &= 2.4488 \times 10^{-4} \\
a_3 &= -1.5251 \times 10^{-6}.
\end{align*}
\]
The third degree Jensen polynomial is
\[
f(z) = a_0 + 3a_1z + 3a_2z^2 + a_3z^3.
\]

| \(n\) | \(J_1(n)\) | \(-\frac{a_0}{a_1} + S_1(n)\) |
|---|---|---|
| 0 | 0 | 14.421 \times 10^{-1}\n| 1 | 3.0425 | 17.463\n| 2 | 1.37564 \times 10^{-1} | 17.601\n| 3 | 2.7830 \times 10^{-4} | 17.601\n| 4 | 1.1384 \times 10^{-9} | 17.601\n| 5 | 1.9048 \times 10^{-20} | 17.601\n
| \(n\) | \(J_1(n)\) | \(-\frac{a_0}{a_1} + S_1(n)\) |
|---|---|---|
| 0 | 0 | 14.421 \times 10^{-1}\n| 1 | 3.0425 | 17.463\n| 2 | 1.37564 \times 10^{-1} | 17.601\n| 3 | 2.7830 \times 10^{-4} | 17.601\n| 4 | 1.1384 \times 10^{-9} | 17.601\n| 5 | 1.9048 \times 10^{-20} | 17.601\n
Table 7: \(m = 1\)
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$n$ & $J_{0,2}(n)$ & $J_{1,2}(n)$ & $J_{2}(n)$ & $-\frac{a_1}{a_2} + S_2(n)$ \\
\hline
0 & 0 & 0 & 0 & 93.850 \\
1 & -4.5493 & 28.506 & 23.957 & 117.81 \\
2 & -4.3017 \times 10^{-1} & 2.6095 & 2.1794 & 119.99 \\
3 & -3.7235 \times 10^{-2} & 2.2057 \times 10^{-2} & 1.8333 \times 10^{-2} & 120.00 \\
4 & -2.6905 \times 10^{-7} & 1.5663 \times 10^{-6} & 1.2972 \times 10^{-6} & 120.00 \\
5 & -1.3678 \times 10^{-15} & 7.8611 \times 10^{-15} & 6.4933 \times 10^{-15} & 120.00 \\
6 & -3.4665 \times 10^{-32} & 1.9734 \times 10^{-31} & 1.6268 \times 10^{-31} & 120.00 \\
\hline
\end{tabular}
\caption{$m = 2$}
\end{table}

6 Further work

Suppose
\[ f(z) = \sum_{k=0}^{N} a_k z^k \]
is a polynomial of degree $N$ with $a_i \in \mathbb{C}$ and with all positive zeros $z_k$ such that $z_k < z_{k+1}$.

- The NRS($m$) algorithm applied to the coefficients $a_k$ is convergent and outputs the sum $\sum_{k=1}^{m} z_k$.
- For each $0 \leq i \leq m - 1$, the series $\sum_{n=1}^{\infty} J_{i,m}(n)$ converges quadratically when the zeros $z_i$ are distinct.
- The convergence of NRS($m$) implies that those outputs yield zeros of $f(z)$.

Now let
\[ f(z) = \sum_{k=0}^{N} a_k z^k = a_N \prod_{i=1}^{N} (z - z_i) \]
where $a_k$ and $z_k$ are indeterminates. We say a polynomial is $z_i$-positive if it is a polynomial in the $z_1, ..., z_N$ that has positive coefficients. A rational function is $z_i$-positive if it is a ratio of $z_i$-positive polynomials.

- $J_{m}(n)$ is $z_i$ positive.
- The difference
\[ \left( \sum_{k=0}^{m} z_k \right) - \left( -\frac{a_{m-1}}{a_m} + S_m(n) \right) \]
is $z_i$ positive.
- The expressions arising from formal contour integrals are $z_i$-positive.
- The $J_{i,m}(n)$ may also have $z_i$-positive properties.
• Combinatorial proof that $A_m - A_{m-1}$ is a formal zero. We proof for $m = 1, 2$.
• Interpret the other hypergeometric series in [3] using trees and the Newton-Raphson-Simpson method.
• Use more general rings to express formal zeros. For example, let $I \subseteq \mathbb{N}_0$. Let

$$g(z) = \sum_{i \in I} a_i z^i,$$

$$f(z) = g(z) + \sum_{i \notin I} a_i z^i,$$

and $z_0$ a zero of $g(z)$. Set

$$Z = z_0 + \sum_{\vec{n}} c(\vec{n}) a^{\vec{n}}$$

with $a^{\vec{n}}$ a multinomial in the $a_i, i \notin I$.
• Use other orderings to evaluate $A_m$. For a suitable degree 2 polynomial, the sum $A_1$ can be summed by an ordering that yields the Taylor series of the square root in the quadratic formula. We would find corresponding orderings for formal zeros that generalize this ordering to higher degree, for example by expressing expressing formal zeros in terms of Taylor series, some of evaluate to radical expressions. See how these hypothetical Taylor series are related to Turán inequalities.
• Relationship between Turán inequalities and expressions for $J_m(n)$. See if Turán inequalities or other set of conditions imply convergence of NRS($m$).
• Householder methods expressed in terms of trees and generalized as was NRS($m$). Maybe altering $L_i(F, n)$ to include higher-order terms or making the type number of a tree to include multiple parameters.
• For arbitrary analytic functions $f(z)$, express the convergence of $f_{i,m}(\vec{x})$ in terms of the convergence of $f(z)$.
• Apply NRS($m$) to coefficients of $\xi(\frac{1}{2} + \sqrt{t})$ using (17). Perhaps try $q$-analogues of terms in (17) to try to prove positivity of NRS($m$) quantities. See if $q$-analogues of coefficients of (17) satisfy Turán inequalities. For example, the $q$-analogue of

$$(s + j)$$

may be

$$(1 - q^{s+j}).$$

The unsigned Stirling numbers have $q$-analogues, and the $b_k$ may be expressed via elliptic integrals in terms of a rational number sequence we denote by \{\kappa(n)\}$_{n=0}^{\infty} = 1, -3, 26, -378, 8136, -244728, \ldots$. Each number $\kappa(n)$ is defined as a sum over a certain finite subset $\mathcal{E}(2n + 1)$ of classical plane trees by:

$$wt(T) = (-1)^{|V(T)|} \prod_{n=1}^{\infty} \left( \prod_{j=1}^{n} (4j - 3) \right)^{2d_{2n}(T)} \left( \prod_{j=1}^{n} (4j - 1) \right)^{2d_{2n+1}(T)}$$
\[ \kappa(n) = \frac{1}{2^n} \sum_{T \in \mathcal{E}(2n+1)} wt(T). \]

This allows us to express \( b_k \) as a series of rational numbers. A \( q \)-analogue of \( \kappa(n) \) could lead to \( q \)-analogues of \( b_k \).

- Galois theory applied to formal zeros.
- NRS(\( m \)) applied to the function \( f(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \) or its \( q \)-analogues.

References

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[3] Sturmfels, Bernd. “Solving algebraic equations in terms of \( \mathcal{A} \)-hypergeometric series”. Discrete Mathematics 210, (2000), pp. 171-181.