GALERKIN-TYPE METHODS FOR STRICTLY PARABOLIC
EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. We prove existence of weak solutions to the Cauchy problem corresponding to various strictly parabolic equations on a compact Riemannian manifold \((M, g)\). This also includes strictly parabolic equations with stochastic forcing with linear diffusion. Existence is proved through a variant of the Galerkin method and can be used to construct a convergent finite element method.

1. Introduction

The main subject of this article is the Cauchy problem for a strictly parabolic equation of the form

\[
\partial_t u + \text{div} f_x(u) = \text{div}(\text{div}(A_x(u))), \quad x \in M,
\]

\[
u|_{t=0} = u_0(x) \in L^2(M),
\]

on a smooth (Hausdorff), orientable, compact \(d\)-dimensional Riemannian manifold \((M, g)\). Here, for a fixed \(\lambda \), \(x \mapsto f_x(\lambda) \in \mathcal{X}(M)\) is a vector field on \(M\) while \(x \mapsto A_x(\lambda) \in T^1_1(M)\) is a symmetric \((1, 1)\) tensor field on \(M\).

We suppose that the map \((x, \lambda) \mapsto f_x(\lambda) \equiv f(x, \lambda), f : M \times \mathbb{R} \rightarrow \mathcal{X}(M)\) is \(C^1\) and that, for every \(\lambda \in \mathbb{R}, x \mapsto f_x(\lambda) \in \mathcal{X}(M)\). Also, \((x, \lambda) \mapsto A_x(\lambda) : M \times \mathbb{R} \rightarrow T^1_1(M)\) is supposed to be \(C^2\) and to satisfy \(x \mapsto A_x(\lambda) \in T^1_1(M)\) for each \(\lambda \in \mathbb{R}\) and we assume that the \(\lambda\)-derivative \(\partial_\lambda A_x\), which we denote in the sequel by \(A'_x\), is symmetric positive definite, i.e. that the equation (1) is strictly parabolic. In particular, this also implies that \(\lambda \mapsto \langle A'_x(\lambda)\xi, \xi \rangle\) is strictly increasing for any \(\xi \in T_xM\). In other words, we suppose that for a constant \(c > 0\) independent of \(\lambda\):

\[
\langle A'_x(\lambda)\xi, \xi \rangle \geq c\|\xi\|^2_{g}, \quad \xi \in T_x(M).
\]

Before we proceed, we note that the standard form of the diffusion operator in Euclidean space is

\[
\text{div} a(x, u) \nabla u
\]

with a positive definite matrix-valued function \(a\) (or very often only non-negative definite; see e.g. [6] and references therein). For \(u = u(t, x)\) we can rewrite the latter expression as \(\text{div} \ \text{div} (A(x, u(t, x))) - \text{div} (\text{div} A(x, \lambda)|_{\lambda=u(t,x)})\) where \(A(x, \lambda) = \int_0^\lambda a(x, z)dz\), which reduces the standard form of the diffusion operator (1) to the form from (1). We also note that that (1) locally reduces to a standard parabolic equation (see the proof of Theorem 11 below) and that the form given in (1) is chart independent. For further discussions see Remark 1 below.

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In the next section, we shall precisely define the notions and notations that we have been using above.

The literature on parabolic equations is vast so we will just mention the classical books [19, 26]. They model transport processes governed by convection/advection (first order term) and diffusion (second order term) and therefore are of wide interest in the field of applied mathematics (see e.g., [8, 15, 21, 22] and references therein). Moreover, they play an important role in geometry since the Ricci flow (via the so called DeTurck’s trick) is also an example of a parabolic equation [3].

Interestingly, investigations of general quasilinear parabolic equations on manifolds are fairly rare and recent. In [16, 18, 23], one can find results concerning local existence of smooth solutions to parabolic equations on manifolds. As for weak solutions, a two-dimensional situation was considered in [24], and in a special situation (when the manifold is a torus), one can find an approach similar to ours in [26, p. 319]).

In the current contribution, we introduce a Galerkin scheme for parabolic PDEs, i.e. we consider a finite dimensional approximation to the corresponding Cauchy problem and prove convergence of the resulting sequence of approximate solutions. Once we have developed the method for equation (1), we can easily extend it to numerous similar equations. For instance, we can add the Laplace-Beltrami operator on the right-hand side of (1) and obtain existence of solution for such an equation (see Remark 12), and we can extend the result from [24] to manifolds of arbitrary dimension. Finally, we are able to apply our method to parabolic PDEs with linear diffusion term. Parabolic equations (even degenerate parabolic equations) have already been investigated in the stochastic setting in e.g. [13] (see also references therein) in the Euclidean space and in spatial situations in [9] on manifolds (in the form of the vanishing viscosity approximation for stochastic scalar conservation laws). This is the first such result for parabolic equations on manifolds. To be more precise, we shall investigate convergence of the Galerkin method for stochastic PDEs of the form

\[ \partial_t u + \text{div} f_x(u) = \text{div}(\text{div}(A_x(u))) + \Phi(x, u)dW_t, \quad x \in M, \quad (5) \]

on a smooth, compact, orientable, d-dimensional (Hausdorff) Riemannian manifold \((M, g)\) where for almost every \(x \in M\) the matrix valued mapping \(\lambda \mapsto A_x(\lambda)\) is linear. The object \(W_t\) is the one-dimensional Wiener process defined on the stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) with the sample space \(\Omega\), the \(\sigma\)-algebra \(\mathcal{F}\), the natural filtration \(\mathcal{F}_t\) generated by the Wiener process \(W_t\), and the probability measure \(P\). This means that the unknown function \(u\) depends on \((t, x, \omega) \in \mathbb{R}_+ \times M \times \Omega\).

The paper is organized as follows. In Section 2 we introduce basic notions from Riemannian geometry and Sobolev spaces on manifolds. We note that although the theory of Sobolev spaces on manifolds is well investigated, we find that this part of the paper represents a very useful collection of (not easy to find) information, especially from the viewpoint of the rapidly developing field of PDEs on manifolds. In Section 3 we introduce the Galerkin scheme that we are going to use and prove the convergence of this scheme for bounded flux \(f_x\) and diffusion \(A_x\) under appropriate assumptions on these bounds (see (12)). Section 4 is devoted to proving existence of solutions under so called geometry-compatibility conditions (14) (which are actually incompressibility conditions when it comes to fluid dynamics as explained in [11, Introduction]), but without assuming (12). In the next two remarks, we explain how to deal with the non-strictly parabolic equations on manifolds regularized with the Laplace-Beltrami operator [11, Section 4], and how to obtain solutions and their regularity properties for parabolic equations of higher
order (see also [24]). In the final Section 3 we prove existence of solutions to (5) with initial data (2). In the process, we recall some basic notions from stochastic calculus.

2. Preliminaries from Riemannian geometry and functional analysis

We denote by $T_0^1(M)$ the set of vector fields over $M$, by $\Omega^1(M) \equiv T_0^1(M)$ the set of one forms over $M$, and by $T_1^1(M)$ the set of $(1, 1)$-tensors over $M$.

If $X \in T_0^1 = \mathcal{X}(M)$ is a $C^1$ vector field on $M$ with local representation $X = X^i \frac{\partial}{\partial x^i}$, then its divergence $\text{div} \ X \in C(M)$ is locally given by

$$\text{div} \ X = \frac{\partial X^k}{\partial x^k} + \Gamma^i_{kj} X^k$$  (6)

In this expression, the $\Gamma^i_{kj}$ denote the Christoffel symbols corresponding to $g$, $g^{im}$ are the coefficients of the inverse metric, and throughout this paper we always adhere to the Einstein summation convention. The same expression holds for $\Omega^1$ in abstract index notation, or

$$\text{div} \omega = g^{ij} \partial_i \omega_j - \Gamma^k_{ij} g^{jk} \omega_k,$$  (7)

in local coordinates. If $T = T^a_b \in T_1^1(M)$, then $\text{div} \ T = \nabla_a T^a_b \in \Omega^1(M)$. Locally, $T = T_i^k \frac{\partial}{\partial x^i} \otimes dx^k$. Then $\text{div} \ T = (\text{div} \ T)_i \frac{\partial}{\partial x^i}$, where

$$\text{div} (T)_i = \partial_i T^k_i + \Gamma^j_{ij} T^k_i - \Gamma^k_{ij} T^j_i.$$  (8)

For $T \in T_1^1(M)$ in $C^2$, $\text{div} (\text{div} (T)) \in C(M)$. The explicit form of $\text{div} (\text{div} (T))$ in terms of local coordinates can be found in [14] (it will not be needed here).

Remark 1. Using the notations introduced above, we can add to the observations concerning the relationship between (4) and more standard forms of parabolic operators initiated after (3), now in the full setting of Riemannian manifolds. First, denote by $\delta \in T_1^1(M)$ the Kronecker-delta tensor, $\delta(\omega, X) = \omega(X)$ for $\omega \in \Omega^1(M)$ and $X \in \mathcal{X}(M)$ (cf. [11]), and let $A_\lambda(\lambda) := \lambda \cdot \delta(\lambda)$. In any local chart, $\delta = \partial_i \otimes dx^i$, and using (3) we see that $\text{div}(A_\lambda(u)) = \omega_i dx^i$, where

$$\omega_i = \delta^j_i \partial_j u + (\Gamma^j_{ij} \delta^i_j - \Gamma^i_{j} \delta^j_i) u = \delta^i_i \partial_i u = \partial_i u$$

Since $(\nabla u)^k = g^{ij} \partial_j u$, we get $\omega_i = g_{ik}(\nabla u)^k$. Hence in this special case we have

$$\text{div}(A_\lambda(u)) = g_\nabla u = d u$$

and thereby

$$\text{div}(\text{div}(A_\lambda(u))) = \text{div}(d u) = \nabla a \nabla u = \Delta u,$$  (9)

with $\Delta$ the Laplace-Beltrami operator on $M$. Clearly, $\text{div}(A_\lambda(u))$ in general will not necessarily be of such a simple form. However, our analysis of (1) readily carries over to right hand sides of the form $\text{div}(C(x, u) \nabla u)$ ($C(\cdot, \lambda) \in T_1^1(M)$) as well, cf. also Remark 17 below.
The metric $g$ induces scalar products on any tensor product $(T_X M)^s$, given in abstract index notation by

$$\langle S, T \rangle = S^{a_1 \ldots a_r} b_1 \ldots b_s T_{a_1 \ldots a_r} b_1 \ldots b_s.$$  

We will denote the corresponding norms by $\| \cdot \|_g$ irrespective of $r$ and $s$. Also, for any tensor field $T \in T^s(M)$ we set

$$\| T \|_{L^\infty(M)} := \sup_{x \in M} \| T(x) \|_g.$$

and

$$\| T \|_{L^2(M)} := \left( \int_M \| T(x) \|^2_2 dV(x) \right)^{\frac{1}{2}}.$$

Here, $dV$ is the oriented Riemannian volume measure (given, in any chart of the oriented atlas, by $dV = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$).

In the Cauchy problem (1), (2), ($\lambda$)-tensor field $f \in C^2(M)$, $M$ compact and oriented without boundary. Then

$$\int_M f \text{div}(\text{div} A) dV = - \int_M (\text{div} A)(\nabla f) dV. \tag{10}$$

**Proof:** By Stokes' theorem, $\int_M \text{div} X dV = 0$ for any $C^1$ vector field $X$. If $\omega = X_a$ is the one-form metrically equivalent to $X$, $\omega = X^a$, then $\text{div} \omega = \nabla^a X_a = \nabla_a X^a$, so we also have $\int_M \text{div} \omega dV = 0$ for any $C^1$ one-form $\omega$. Now

$$\text{div}(f \text{div} A) = \nabla^b (f \nabla_a A^a b) = \nabla^b f \nabla_a A^a b + f \nabla^b \nabla_a A^a b = (\text{div} A)(\nabla f) + f \text{div}(\text{div} A),$$

so

$$0 = \int_M (f \text{div} A) dV = \int_M (\text{div} A)(\nabla f) dV + \int_M f \text{div}(\text{div} A) dV.$$

Furthermore, under the same assumptions as in Lemma 2 we have:

**Lemma 3.**

$$\int_M f \text{div}(\text{div} A) dV = \int_M \text{tr}(\tilde{H}^f \circ A^T) dV = \int_M \text{tr}(A \circ \tilde{H}^f) dV. \tag{11}$$

Here, $\text{tr}$ denotes the trace, $(A^T)^a_b = A^a_b$ is the transpose of $A$, and $(\tilde{H}^f)^a_b = \nabla^a \nabla_b f$ is the $(1,1)$-tensor field metrically equivalent to the Hessian $\nabla_a \nabla_b f$ of $f$.

**Proof:** We have

$$\text{div}(\text{div} A)(\nabla f) = \nabla_a A^a b \nabla_b f = \nabla_a (A^a_b \nabla_b f) - A^a_b (\nabla_a \nabla_b f)$$

$$= \text{div}(A(\nabla f)) - (A^T)^a_b (\nabla_a \nabla_b f)$$

$$= \text{div}(A(\nabla f)) - (A^T)^b_a (\nabla^a \nabla_b f)$$

$$= \text{div}(A(\nabla f)) - \text{tr}(\tilde{H}^f \circ A^T).$$

Integrating this identity over $M$, and applying Stokes' theorem and Lemma 2 gives the first equality. The second one follows since $\tilde{H}^f$ is symmetric and $\text{tr} B = \text{tr} B^T$ for any $(1,1)$-tensor $B$.

**Remark 4.**

(i) Since $\text{div} \text{div} A = \text{div} A^T$ and $A^T = A$, on the right hand side of (10) we may replace $A$ by $A^T$, and conversely for (11).
(ii) By density, the conclusion of Lemma 7 remains valid for $A \in H^2(M)$ and $f \in H^1(M)$, and that of Lemma 8 for $A \in H^2(M)$ and $f \in H^2(M)$.

We need the following consequence of the parabolicity of the operator $\text{div div } A_\lambda$. However, we first need to assume that the vector field $f$ and the tensor $A_\lambda$ are bounded in the sense that there exists $C > 0$ such that (recalling that we set $A_\lambda' := \partial_\lambda A_\lambda$) for all $\lambda$:

$$
\|f(\lambda)\|_{L^\infty(M)} + \|A_\lambda(\lambda)\|_{L^\infty(M)} + \|\lambda A_\lambda'\|_{L^\infty(M)} + \|\text{div } A_\lambda(\lambda)\|_{L^\infty(M)} \leq \bar{C}(1 + |\lambda|)
$$

As we shall see in Section 4, the assumption (12) can be avoided in some important situations.

**Lemma 5.** (i) For any $u \in H^2(M)$,

$$
\int_M \text{div div } A_\lambda(u) u(x) dV(x) \leq -c \int_M \|\nabla u(x)\|^2_g dV(x) + C \max\{1, \int_M |u(x)|^2 dV(x)\}.
$$

(ii) If $\lambda \mapsto A_\lambda(\lambda)$ is linear then

$$
\int_M \text{div div } A_\lambda(u) u(x) dV(x) \leq -c \int_M \|\nabla u(x)\|^2_g dV(x) + C \int_M |u(x)|^2 dV(x).
$$

(iii) If $\lambda \mapsto A_\lambda(\lambda)$ is linear and $u \in H^3(M)$, then (with $\Delta u = \text{div } \text{grad } u$)

$$
\int_M \text{div div } A_\lambda(u) \Delta u(x) dV(x) \geq k \int_M |\Delta u(x)|^2 dV(x)
$$

for some constants $k, K, c, C > 0$ independent of $u$.

**Proof:** By density, it suffices to assume $u$ to be smooth.

(i) By (10), we have

$$
\int_M \text{div div } A_\lambda(u) u(x) dV(x) = -\int_M \text{div } A_\lambda(u) \nabla u(x) dV(x).
$$

Here,

$$
\text{div}(A_\lambda(u)) \nabla u = \nabla_b(A_\lambda(u))^{-1}_a \nabla^a u = (\nabla_b A_\lambda^{-1}_a)(x, u(x)) \nabla^a u + (A_\lambda')^b_a \nabla_b u \nabla^a u
$$

$$
= (\text{div } A_\lambda(\lambda)\big|_{\lambda=\mu}) (\nabla u) + (A_\lambda'\big|_{\lambda=\mu}) (\nabla u, \nabla u).
$$

By (12) we have $|\big( \text{div } A_\lambda(\lambda)\big|_{\lambda=\mu}) (\nabla u)| \leq \bar{C}(1 + |\mu|) \|\nabla u\|_g$. Combining this with (3), we obtain

$$
-\text{div}(A_\lambda(u)) \nabla u \leq -c \|\nabla u\|^2_g + \bar{C}(1 + |\mu|) \|\nabla u\|_g
$$

$$
\leq -c \|\nabla u\|^2_g + C^2 N + \frac{1}{N} \|\nabla u\|^2_g + \bar{C} N |u|^2 + \frac{C}{N} \|\nabla u\|^2_g,
$$

where we applied the Peter-Paul inequality twice in the last step. By choosing $N > 0$ large enough, we conclude the proof.

(iii) We get the conclusion by noticing that linearity of $A_\lambda$ implies due to (12):

$$
\|\text{div } A_\lambda(\lambda)\|_{L^\infty(M)} \leq C \lambda
$$
with which we omit the term \( C^2 N \) in (17).

(ii) In this case we have \( A_x(u) = A_x \cdot u \), and by Stokes’ theorem

\[
\int_M \text{div} \text{div}(A_x u) \Delta u \, dV(x) = -\int_M \text{div}(A_x u)(\nabla \Delta u) \, dV(x).
\]

We have

\[
\text{div}(A_x u)(\nabla \Delta u) = u \nabla_a A^a_b \nabla^b \nabla^c \nabla_c u + A^a_b \nabla_a \nabla^b \nabla^c \nabla_c u,
\]

where (with \( R \) the Riemann tensor, cf., e.g., [29, (3.2.3)])

\[
A^a_b \nabla_a \nabla^c \nabla_c u = A^a_b \nabla_a u \nabla^c \nabla^c \nabla^c u + A^a_b \nabla_a u R_{c d} \nabla^c \nabla^c u.
\]

Again by Stokes’ theorem,

\[
\int_M A^a_b \nabla_a u \nabla^c \nabla^c \nabla^c u \, dV = -\int \nabla^c (A^a_b \nabla_a u) \nabla^c \nabla^c u \, dV
\]

\[
= -\int (\nabla^c A^a_b) \nabla_a u \nabla^c \nabla^c u \, dV - \int A^a_b \nabla^c \nabla_a u \nabla^c \nabla^c u \, dV.
\]

The integrand in the last term can be rewritten as

\[
\nabla^c \nabla^c u A^a_b \nabla^b \nabla^c u = tr(\tilde{H}^u \cdot A \cdot \tilde{H}^u) = tr((\tilde{H}^u)^2 \cdot A)
\]

Collecting terms, we arrive at

\[
\int_M \text{div} \text{div}(A_x u) \Delta u \, dV(x) = \int_M \text{tr}((\tilde{H}^u)^2 \cdot A) \, dV - \int_M u \text{div}(A)(\nabla \Delta u) \, dV
\]

\[
+ \int_M (\tilde{H}^u)^b_c \nabla^c A^a_b \nabla_a u \, dV - \int_M A^a_b \nabla a u R_{c d} \nabla^c \nabla^c u \, dV.
\]

Let us consider each term in the previous expression individually. In general, if \( R \) is a symmetric matrix and \( S \) is symmetric and positive definite, then (with \( \| . \|_F \) the Frobenius norm) we have

\[
\text{tr}(R^2 \cdot S) \geq \lambda \text{tr}(R^2) = \lambda \| R \|_F^2,
\]

where \( \lambda \) is the smallest eigenvalue of \( S \). Thus there exists a constant \( k > 0 \) such that

\[
\int_M \text{tr}((\tilde{H}^u)^2 \cdot A) \, dV \geq k \int_M \| \tilde{H}^u \|_F^2 \, dV.
\]

For the second term, we have (with \( \text{dim}(M) = d \))

\[
| \int_M u \text{div}(A)(\nabla \Delta u) \, dV | \leq | u \text{div}(A) \|_H^1 \| \nabla \Delta u \|_{H^{-1}(M)}
\]

\[
\leq K_0 \| u \text{div}(A) \|_H \| \Delta u \|_{L^2(M)}
\]

\[
\leq K_1 \| u \text{div}(A) \|_{H^1}^2 + \frac{k}{4d^2} \int_M |\Delta u|^2 \, dV \leq K_1 \| u \|_{H^1}^2 + \frac{k}{4d^2} \int_M |\Delta u|^2 \, dV,
\]

for an appropriate constant \( K_1 \) (and \( k \) from (19)). We estimate the third term in a similar manner:

\[
| \int_M (\tilde{H}^u)^b_c \nabla^c A^a_b \nabla_a u \, dV |
\]

\[
\leq \frac{k}{4d} \int_M \| \tilde{H}^u \|_g^2 \, dV + K_2 \| \nabla A \|_{L^2}^2 \int_M \| \nabla u \|_{L^2}^2 \, dV
\]

\[
\leq \frac{k}{4} \int_M \| \tilde{H}^u \|_F^2 \, dV + K_2 \| \nabla A \|_{L^2}^2 \int_M \| \nabla u \|_{L^2}^2 \, dV.
\]
Moreover, we have \( \Lambda \) is a linear isomorphism with inverse \( \Lambda^{-1} \). The eigenvalue of an orthonormal basis, so by the spectral theorem for compact operators there exists a countable \( H \) and so the corresponding orthonormal system is \( \{e_n\} \). This implies \( \lambda^1 e_n = \lambda_n e_n \) and, since \( \Lambda^1 \) is elliptic, we have \( e_n \in C^s(M) \). Also, \( \lambda_n \nearrow +\infty \) as \( n \to \infty \).

Set \( \Lambda^s := (I - \Delta)^{s/2} \), then \( (\Lambda^s)^{-1} = \Lambda^{-s} \) and by [5, Th. 8.5], for every \( s \in \mathbb{R} \):

\[
\Lambda^s : H^s(M) \to H^{-s}(M)
\]

is a linear isomorphism with inverse \( \Lambda^{-s} \).

Now, \( \Lambda^{-1} : L^2(M) \to H^1(M) \) is a compact operator (by Rellich's theorem), so by the spectral theorem for compact operators there exists a countable orthonormal basis \( \{e_n\} \) of \( L^2(M) \) consisting of eigenfunctions of \( \Lambda^{-1} \), and we denote the eigenvalue of \( e_n \) by \( \lambda_n^{-1} \). This implies \( \Lambda^1 e_n = \lambda_n e_n \) and, since \( \Lambda^1 \) is elliptic, we have \( e_n \in C^\infty(M) \). Also, \( \lambda_n \nearrow +\infty \) as \( n \to \infty \).

Since \( \Lambda^{-s} : L^2(M) \to H^s(M) \) is an isomorphism, we may introduce on \( H^s(M) \) an equivalent scalar product by

\[
\langle u, v \rangle_s := \langle \Lambda^s u, \Lambda^s v \rangle_{L^2}.
\]

With respect to \( \langle \cdot, \cdot \rangle_s \), \( \{e_m\}_{m \in \mathbb{N}} \) is an orthogonal system:

\[
\langle e_m, e_n \rangle_s = \langle \Lambda^s e_m, \Lambda^s e_n \rangle_{L^2} = \langle \Lambda^{2s} e_m, e_n \rangle_{L^2} = \langle \lambda_m^{2s} e_m, e_n \rangle_{L^2} = \lambda_m^{2s} \delta_{mn},
\]

and so the corresponding orthonormal system is

\[
\{e_m^{(s)}\}_{m \in \mathbb{N}} := (\lambda_m^{-s} e_m)_{m \in \mathbb{N}}.
\]

It is complete in \( H^s(M) \) since

\[
\langle u, e_m \rangle_s = 0 \quad \forall m \in \mathbb{N} \quad \Rightarrow \quad \lambda_m^{2s} \langle \Lambda^{2s} u, e_m \rangle_{L^2} = 0 \quad \forall m \quad \Rightarrow \quad \Lambda^{2s} u = 0 \quad \Rightarrow \quad u = 0.
\]

Moreover, we have

\[
u \in H^s(M) \quad \Leftrightarrow \quad \Lambda^s u \in L^2 \quad \Leftrightarrow \quad \sum_{m \in \mathbb{N}} \|\Lambda^s u, e_m\|_{L^2}^2 = \sum_{m \in \mathbb{N}} \|u, e_m\|_{L^2}^2 \lambda_m^{2s} < \infty.
\]
An important consequence of the above, which we shall make use of repeatedly below, is that projections onto span(e₁, . . . , eₙ) do not depend on the Sobolev-index:
\[
\sum_{m=1}^{n} \langle u, e_m \rangle_{L^2} e_m = \sum_{m=1}^{n} \langle u, e_m^{(s)} \rangle_{L^2} e_m^{(s)}. \tag{26}
\]

Let us show that the standard scalar product in H¹(M) and the one defined in (23) coincide. By [14, Def. 2.1],
\[
\langle u, v \rangle_{H^1} = \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}
\]
and by Green’s second identity
\[
\langle \nabla u, \nabla v \rangle_{L^2} = \int_M \langle \nabla u, \nabla v \rangle dV = -\frac{1}{2} \int_M v \Delta u + u \Delta v dV
\]
Consequently,
\[
2 \langle u, v \rangle_{H^1} = 2 \langle u, v \rangle_{L^2} - \langle \Delta u, v \rangle_{L^2} - \langle u, \Delta v \rangle_{L^2} = \langle (I - \Delta) u, v \rangle_{L^2} + \langle u, (I - \Delta) v \rangle_{L^2}
\]
As we shall make use of certain variants of vector-valued Sobolev spaces, we conclude this section by recalling some basic definitions and properties, referring to [17] for details and references. For any s ∈ ℝ, T > 0, we define L²((0, T), Hˢ(M)) to be the space of measurable functions u : (0, T) → Hˢ(M) such that
\[
\|u\|_{L²((0, T), H^{s}(M))} := \left( \int_0^T \|u(t, \cdot)\|_{H^s(M)}^2 dt \right)^{1/2} < \infty.
\]
By H¹((0, T), Hˢ(M)) we denote the space of all u ∈ L²((0, T), Hˢ(M)) that are weakly differentiable (with respect to t) and such that
\[
\|u\|_{H^1((0, T), H^{s}(M))} := \|u\|_{L²((0, T), H^{s}(M))} + \|\partial_t u\|_{L²((0, T), H^{s}(M))} < \infty.
\]
Finally, we shall require the following fundamental result (cf. [4, Th. II.5.16]):

**Theorem 6.** (Aubin-Lions-Simon) Let E₁ ⊆ E ⊆ E₀ be Banach spaces such that E₁ is compactly embedded in E and E is continuously embedded in E₀. For 1 ≤ p, q ≤ ∞, let
\[
W = \{ u \in L^p((0, T); E₁) : \partial_t u \in L^q((0, T); E₀) \}.
\]
Then,
(i) If p < ∞, then the embedding of W into Lᵖ((0, T); E) is compact.
(ii) If p = ∞ and q > 1, then the embedding of W into C([0, T]; E) is compact.

3. Galerkin approximation

We fix an orthonormal basis \{eₖ\}ₖ∈ℕ in L²(M) consisting of eigenfunctions of the Laplace-Beltrami operator, as described in Section 2 and look for approximate solutions to (1), (2) in the form
\[
u_n(t, x) = \sum_{k=1}^{n} \alpha_k^n(t) e_k(x). \tag{27}
\]
Next, we insert uₙ from (27) into (1) and look for αₖ, k = 1, . . . , n, so that (1) is satisfied in the space span(e₁, . . . , eₙ). In other words, we multiply the expression
\[
\partial_t u_n + \text{div} f_n(u_n) = \text{div}(\text{div}(A_k(u_n))) \tag{28}
\]
by $e_j$ for every $j = 1, \ldots, n$, and integrate over $M$. We get after taking into account orthonormality of the basis $(e_k)_{k \in \mathbb{N}}$:

$$
\hat{\alpha}_j^n(t) = \int_M \left( f_k \left( \sum_{k=1}^n \alpha_k^n(t) e_k(x) \right), \nabla e_j(x) \right) dV(x)
+ \int_M \text{tr} \left( A_x \left( \sum_{k=1}^n \alpha_k^n(t) e_k(x) \right) H^{e_j}(t, x) \right) dV(x).
\tag{29}
$$

Using Lemma 2 and conditions (12), one can prove existence of solutions to the latter system supplemented with appropriate initial data. Indeed, the standard Cauchy theorem provides existence of a local solution and conditions (12) enable us to extend the solution for an arbitrary time interval as shown in e.g. [28, Theorem 5.2.1] even in the stochastic case, which we shall consider later. More precisely, we have the following lemma.

**Lemma 7.** Under the assumption (12), for any fixed $n \in \mathbb{N}$, the system of ODEs (29) with initial data $\alpha_j(0) = \alpha_{j0} \in \mathbb{R}$ has a globally defined solution.

**Proof:** According to the Cauchy theorem, system (29) with the given initial data has a solution defined on $[0, T)$ for some $T > 0$. Then applying the bounds (12) to (29), we obtain an estimate of the form

$$
|\alpha(t)| \leq |\alpha(0)| + C \int_0^t (1 + |\alpha(s)|) ds,
$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\|$ is any norm on $\mathbb{R}^n$. Consequently, $|\alpha(t)| \leq (|\alpha(0)| + C t)e^{Ct}$ by Gronwall’s inequality. Hence the functions $\alpha^n_j, j = 1, \ldots, n$, cannot blow up as $t \to T$, implying that they can be extended to all of $\mathbb{R}$.

If we rewrite (2) as

$$
u_0(x) = \sum_{k=0}^\infty \alpha_k e_k(x)
$$

and take $\alpha^n_j(0) = \alpha_{j0}$ in (29), we know that there exists a sequence of approximate solutions to (1), (2) in the sense that (29) is satisfied for any $n \in \mathbb{N}$.

**Lemma 8.** The sequence $(u_n)$ is bounded in $L^2((0, T) \times M)$.

**Proof:** We first multiply (28) by $u_n$ and integrate over $(0, t) \times M$ for any fixed $t \in (0, T)$. We have after integration by parts (cf. Lemma 2):

$$
\frac{1}{2} \int_M |u_n(t, x)|^2 dV(x) - \frac{1}{2} \int_M |u_0(x)|^2 dV(x)
+ \int_0^t \int_M \nabla (A \nabla u_n) \cdot \nabla u_n dV(x) d\tau
\tag{30}
= \int_0^t \int_M f_n(x) \cdot \nabla u_n dV(x) d\tau.
$$

Abbreviating $u_n(\tau, x)$ by $u_n$ henceforth, Lemma 5 (i) shows

$$
\int_0^t \int_M \nabla (A \nabla u_n) \cdot \nabla u_n dV(x) d\tau 
\geq c_t \int_0^t \int_M |\nabla u_n|^2 dV(x) d\tau
\geq C T \text{vol}(M) - C \int_0^t \int_M |u_n|^2 dV(x) d\tau.
$$
Also, the Peter-Paul inequality gives
\[
\int_0^t \int_M f_k(u_n) \cdot \nabla u_n \, dV(x) \, d\tau \leq \frac{c}{2} \int_0^t \int_M \|\nabla u_n\|^2 \, dV(x) \, d\tau + \frac{2}{c} \int_0^t \int_M \|f_k(u_n)\|^2 \, dV(x) \, d\tau.
\]
Inserting this into (30) and using (12) we see that for certain constants \(C_1, C_2 > 0\)
\[
\int_0^T \int_M |u_n(t,x)|^2 \, dV(x) + \frac{1}{2} \int_0^t \int_M \|\nabla u_n\|^2 \, dV(x)
\leq CT \text{vol}(M) + \frac{1}{2} \int_0^t \int_M |u_0(x)|^2 \, dV(x) + \frac{2}{c} \int_0^t \int_M \|f_k(u_n)\|^2 \, dV(x)
+ \int_0^t \int_M |u_n|^2 \, dV(x) \, d\tau \leq C_1 + C_2 \int_0^t \int_M |u_n|^2 \, dV(x) \, d\tau.
\] (31)
Therefore Gronwall’s inequality implies that \(\int_M |u_n(t,x)|^2 \, dV(x) \leq C_T\) for some constant \(C_T\) depending only on \(T\). Re-inserting into (31), we arrive at the desired conclusion:
\[
\int_0^T \int_M (|u_n|^2 + \|\nabla u_n\|^2) \, dV(x) \, dt \leq \tilde{C}_T.
\] (32)

Next, we need to estimate the \(t\)-derivative of \((u_n)\).

**Lemma 9.** For any \(T > 0\) there exists a constant \(C > 0\) such that
\[
\|u_n\|_{H^1((0,T);H^{-1}(M))} \leq C.
\]

**Proof:** Recall that
\[
\|u_n\|^2_{H^1((0,T);H^{-1}(M))} = \int_0^T \|u_n(t,x)\|^2_{H^{-1}(M)} \, dt + \int_0^T \|\partial_t u_n(t,x)\|^2_{H^{-1}(M)} \, dt.
\] (33)
Since, according to Lemma 8, we have
\[
\|u_n\|^2_{L^2((0,T);H^{-1}(M))} \leq \|u_n\|^2_{L^2((0,T);H^1(M))} \leq C < \infty,
\]
it is enough to bound the second term on the right-hand side of (33).

Here, using (28) and the continuity of the differential operator \(\text{div} : L^2(M,TM) \rightarrow H^{-1}(M)\),
\[
\int_0^T \|\partial_t u_n(t,x)\|^2_{H^{-1}(M)} \, dt = \int_0^T \| - \text{div} f_k(u_n) + \text{div}(\text{div} A_k(u_n))\|^2_{H^{-1}(M)} \, dt
\leq 2 \int_0^T \left( \|\text{div} f_k(u_n)\|^2_{H^{-1}(M)} + \|\text{div}(\text{div} A_k(u_n))\|^2_{H^{-1}(M)} \right) \, dt
\leq \tilde{C} \int_0^T \left( \|f_k(u_n)\|^2_{L^2(M)} + \|\text{div} A_k(u_n)\|^2_{L^2(M)} \right) \, dt.
\]
for some \(\tilde{C} > 0\). Finally, (12), together with Lemma 8 imply the boundedness of this latter expression.

Now, we can prove the existence theorem for (1), (2).

**Theorem 10.** The Cauchy problem (1), (2) admits a weak solution belonging to \(L^2((0,T);H^1(M))\) if \(f \in C^1(M \times \mathbb{R})\) and \(A \in C^2(M \times \mathbb{R})\) satisfy (12).
Proof:  By Lemmas 5 and 9 we have that \((u_n)\) is bounded in \(L^2((0,T), H^1(M))\) and that \((\partial_t u_n)\) is bounded in \(L^2((0,T), H^{-1}(M))\). We may therefore apply Theorem 6(i) to the triple \(H^1(M) \subset L^2(M) \subset H^{-1}(M)\) with \(p = q = 2\) to conclude that \((u_n)\) possesses a subsequence (again denoted by \((u_n)\)) that converges strongly in \(L^2((0,T), L^2(M))\) to some \(u \in L^2((0,T), L^2(M))\).

Let us show that \(u\) will represent a weak solution to (1), (2). Take an arbitrary \(\varphi \in C_c^\infty((0,T) \times M)\) and denote by \(\varphi_n\) its projection on \(\text{span}\{e_k \mid 1 \leq k \leq n\}\), i.e. (cf. (20)),

\[
\varphi_n(t, x) := \sum_{k=1}^n \langle \varphi(t, \cdot), e_k^{(2)} \rangle_2 e_k^{(2)}(x) = \sum_{k=1}^n \langle \varphi(t, \cdot), e_k \rangle_{L^2} e_k(x).
\]

We then have, taking into account that \(u_n\) satisfies (1) in the space \(\text{span}\{e_k \mid 1 \leq k \leq n\}\) and using Lemma 3,

\[
0 = -\int_0^T \int_M (\partial_t u_n + \text{div}(f(x)(u_n)) - \text{div}(\text{div}(A(x)(u_n)))) \varphi_n \, dV(x) \, dt
= \int_0^T \int_M (u_n \partial_t \varphi_n + f(x)(u_n) \cdot \nabla \varphi_n + \text{tr}(A(x)(u_n) \circ \tilde{H}^{\varphi_n})) \, dV(x) \, dt
+ \int_0^T \int_M u_0(x) \varphi_n(0, x) \, dV(x).
\]

Consequently, for any \(n \in \mathbb{N}\) we obtain

\[
\int_0^T \int_M (u \partial_t \varphi + f(x)(u) \cdot \nabla \varphi + \text{tr}(A(x)(u) \circ \tilde{H}^{\varphi})) \, dV(x) \, dt
+ \int_0^T \int_M u_0(x) \varphi(0, x) \, dV(x)
= \int_0^T \int_M ((u - u_n) \partial_t \varphi + (f(x)(u) - f(x)(u_n)) \nabla \varphi) \, dV(x) \, dt
+ \int_0^T \int_M \text{tr}((A(x)(u) - A(x)(u_n)) \circ \tilde{H}^{\varphi}) \, dV(x) \, dt + \int_0^T \int_M u_0(x) (\varphi(0, x) - \varphi_n(0, x)) \, dV(x)
+ \int_0^T \int_M (u_n \partial_t (\varphi - \varphi_n) + f(x)(u_n) \nabla (\varphi - \varphi_n)) \, dV(x) \, dt
+ \int_0^T \int_M \text{tr}(A(x)(u_n) \circ \tilde{H}^{(\varphi - \varphi_n)}) \, dV(x) \, dt.
\]

Letting \(n \to \infty\) and using \(u_n \to u\) in \(L^2((0,T) \times M)\) and \(\varphi_n \to \varphi\) in \(H^2((0,T) \times M)\), we conclude that \(u\) is indeed a weak solution to (1), (2).

If we additionally assume that the initial value is \(C^1\), as well as stronger boundedness assumptions than (12) on flux and diffusion tensor, we can prove that the solution also increases its regularity:

**Theorem 11.** Suppose that

\[
\|A(x)(\lambda)\|_{C^1(\mathbb{R} \times M)} + \|\lambda A'(x)(\lambda)\|_{L^\infty(\mathbb{R} \times M)} + \|A''(x)(\lambda)\|_{L^\infty(\mathbb{R} \times M)}
+ \|f(x)(\lambda)\|_{L^\infty(\mathbb{R} \times M)} \leq C
\]

(34)

for some \(C > 0\). Moreover, let \(u_0 \in C^1(M)\). Then the weak solution to the initial value problem (1), (2) constructed in Theorem 7 belongs to \(H^{1,2}((0,T) \times M)\) (i.e., Sobolev order 1 in \(t\) and order 2 in \(x\)).

Proof:  Since \(M\) is compact, it suffices to show that for any \(\varphi \in C_0^2(M)\) which is compactly supported in the domain of a chart \((U, w)\) of \(M\) we have \(\varphi u \in \mathbb{H}^1(U)\).
For the next term on the right hand side of (35), we have (using (7))
\[ \partial_t(\varphi u) = \varphi \partial_t u = \nabla^a(A'_\lambda(u)^b_a \nabla_b(\varphi u)) - \nabla^a(A'_\lambda(u)^b_a \nabla_b \varphi \cdot u - \varphi \nabla_b A_\lambda(\lambda)^b_a|_{\lambda=u}) \]
\[ - \nabla^a \varphi \nabla_b (A_\lambda(u)^b_a) - \nabla_a(\varphi \xi(u)) + \nabla_a \varphi f^a_\lambda(u). \]

Due to our assumption (3), this means that \( \varphi u \) satisfies
\[ \nabla^a (A'_\lambda(u)^b_a \nabla_b(\varphi u)) = \text{div}(A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) dx^r) \]
\[ = g^{ij} \partial_i (A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u)) - \Gamma^k_{ij} g^{il} A'_\lambda(u)^r_l g^s_a \partial_k(\varphi u) \]
\[ = \partial_i (g^{ij} A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u)) - \partial_i g^{ij} A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) \]
\[ - \Gamma^k_{ij} g^{il} A'_\lambda(u)^r_l g^s_a \partial_k(\varphi u) \]
\[ =: \partial_i (a^{ik} \partial_s(\varphi u)) + b^s \partial_s(\varphi u), \]

with
\[ a^{ik} := g^{ij} A'_\lambda(u)^r_j g^s_a \]
\[ b^s := -(\partial_i g^{ij} + \Gamma^k_{ij} g^{il}) A'_\lambda(u)^r_l g^s_a. \]

For the next term on the right hand side of (35), we have (using (7))
\[ \nabla^a (A'_\lambda(u)^b_a \nabla_b(\varphi u)) = \text{div}(A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) dx^r) \]
\[ = g^{ij} \partial_i (A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) - (\varphi \text{ div } A)_k|_{\lambda=u} dx^k) \]
\[ = g^{ij} \partial_i \left(u A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) - (\varphi \text{ div } A)_j|_{\lambda=u}\right) \]
\[ - \Gamma^k_{ij} g^{il} (u A'_\lambda(u)^r_l g^s_a \partial_k(\varphi u) - (\varphi \text{ div } A)_k|_{\lambda=u}) \]

Then setting
\[ c := -(\Gamma^k_{ij} g^{il} + \partial_i g^{ik}) (u A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) - (\varphi \text{ div } A)_k|_{\lambda=u}) \]
\[ c^i := g^{ij} (u A'_\lambda(u)^r_j g^s_a \partial_s(\varphi u) - (\varphi \text{ div } A)_j|_{\lambda=u}) \]
we conclude
\[ \nabla^a (A'_\lambda(u)^b_a \nabla_b\varphi \cdot u - \varphi \nabla_b A_\lambda(\lambda)^b_a|_{\lambda=u}) = \partial_i c^i + c. \]

Similarly, using (35):
\[ \nabla^a \varphi \nabla_b (A_\lambda(u)^b_a) + \nabla_a(\varphi f^a_\lambda(u)) \]
\[ = g^{ks} \partial_s \varphi (\partial_k A_\lambda(u)^r_k) + \Gamma^r_{ks}(u)^s_j \nabla_t A_\lambda(u)^r_t + \partial_k (\varphi f^s_\lambda(u)^k) + \varphi \Gamma^r_{js} f^j_\lambda(u)^k \]
\[ = \partial_i (g^{ks} \partial_s \varphi A_\lambda(u)^r_k) - A_\lambda(u)^r_k \partial_s (g^{ks} \partial_k \varphi) + g^{ks} \partial_s (\Gamma^r_{js} A_\lambda(u)^r_j - \Gamma^r_{js} A_\lambda(u)^r_j) \]
\[ + \partial_k (\varphi f^s_\lambda(u)^k) + \varphi \Gamma^r_{js} f^j_\lambda(u)^k \]
and from here, defining
\[ d^k := g^{ks} \partial_k \varphi A_\lambda(u)^r_k + \varphi f^j_\lambda(u)^k \]
\[ d := -A_\lambda(u)^r_k \partial_r (g^{ks} \partial_s \varphi) + g^{ks} \partial_s (\Gamma^r_{js} A_\lambda(u)^r_j - \Gamma^r_{js} A_\lambda(u)^r_j) + \varphi \Gamma^r_{js} f^j_\lambda(u)^k, \]
we conclude
\[ \nabla^a \varphi \nabla_b (A_\lambda(u)^b_a) + \nabla_a(\varphi f^a_\lambda(u)) = \partial_i d^i + d. \]
Finally, if we set
\[ f = -c - d + <\text{grad} \varphi, f_x(u)>_g \quad \text{and} \quad f^i = -c^i - d^i, \]
we see that in terms of local coordinates, Lemma \[\text{35}\] satisfies (see \[\text{36}, \text{37}, \text{38}\]).
Due to \[\text{34}\] we have
\[ f, f^i, a^i, b^i \in L^\infty([0, T] \times K). \]
Using the analog of \[\text{30}\] for \(u\), it follows that \((x, t) \mapsto (\varphi u)(t, x) \in V_2^{1,0}([0, T] \times U)\) (in the notation of \[\text{19}\]). We may therefore apply \[\text{19, Ch. III, Th. 7.1 and Cor. 7.1}\] which, due to Theorem \[\text{10}\] and the fact that we have zero boundary conditions, leads to
\[ ||\varphi u||_{L^\infty([0, T] \times U)} \leq C < \infty \]  \[\text{(39)}\]
From here and \[\text{19, Ch. III, Th. 10.1}\], we obtain that for some \(\alpha > 0\)
\[ \varphi u \in C^\alpha_0((0, T) \times U'), \quad U' \subseteq U. \]  \[\text{(40)}\]
Thus, by taking \(U'\) on which \(\varphi \equiv 1\), we see that \(u\) satisfies the following boundary value problem for a quasi-linear equation on \([0, T] \times U'\) (we denote \(S_T = (0, T) \times \partial U'\)):
\[ \partial_t u = \partial_i(a^{ik}\partial_k u) + b^i\partial_i u + \partial_if^i + f \]
\[ u|_{S_T} = \gamma_{S_T}(u) \in C^\alpha(S_T), \]
\[ u|_{t=0} = u_0 \]
where \(\gamma_{S_T}\) is the trace operator.
According to \[\text{19, Ch. V, Th. 6.4}\], our regularity assumptions on the initial data, and again invoking \[\text{33}\], we conclude that \(u \in H^{1,2}((0, T) \times K')\) for any \(K' \subset K\). Since we have chosen an arbitrary chart and the manifold \(M\) is compact, we conclude that \(u \in H^{1,2}((0, T) \times M)\).
Uniqueness of such weak solutions is not automatic, unless we assume additional properties of the coefficients as we shall see in the next section.

Remark 12. We remark that the same method can be used if we add the Laplace-Beltrami operator \(\Delta u\) or even the bi-harmonic operator \(\Delta^2 u\) on the right hand side of \[\text{(1)}\], in which case we merely need to require semi-definiteness of the tensor \(A_k\).
In other words, we may consider the equation
\[ \partial_t u + \text{div} f_x(u) = \text{div}(\text{div}(A_k(u))) + \epsilon \Delta u, \quad x \in M \]  \[\text{(41)}\]
for a positive parameter \(\epsilon > 0\) and a \((1, 1)\)-tensor \(x \mapsto A_k(\lambda) \in T_1^1(M)\) that satisfies:
\[ \langle A_k(\lambda)\xi, \xi \rangle \geq 0, \]  \[\text{(42)}\]
and supplement it with the initial conditions \[\text{(2)}\]. We can then apply the same method as for problem \[\text{(1), (2)}\]. Indeed, in the case of equation \[\text{(11)}\], the system of ODEs obtained after inserting the approximation \[\text{(22)}\] has the form
\[ \partial_j^n = \int_M (f_x(u_n), \nabla e_j(x)) dV(x) + \int_M \text{tr}(A_k(u_n) \circ \tilde{H}^{e_j})(x)dV(x) + \epsilon \lambda_j \int_M e_j^2(x)dx. \]  \[\text{(43)}\]

Then, Lemma \[\text{7}\] remains valid and the rest of the proof is the same since \[\text{(11)}\] is a strictly parabolic PDE. We note that the estimates given in Lemma \[\text{8}\] and Lemma \[\text{9}\] are based on the strict parabolicity of the equation, and that the existence proof in turn is based on those Lemmas. Alternatively, we may also reduce directly
to the situation studied previously by defining a new \((1,1)\)-tensor field \(\tilde{A}_\lambda := A_\lambda + \varepsilon M\), where \(I\) is the identity tensor, \(I = \delta_{ab}\). Indeed, then \(\text{div}(\text{div}(A_\lambda(u))) = \text{div}(\text{div}(\tilde{A}_\lambda(u))) + \varepsilon \Delta u\).

Moreover, we note that the method of proof of Theorem 10 applies on question of \(L^2((0,T);H^1(M))\) convergence of any sequence of approximate solutions to (1) satisfying bounds from Lemma 8 and Lemma 9.

4. The case of unbounded flux and diffusion

In the previous section we proved existence of a global solution to (1), (2) under the assumption that the flux \(f_\lambda\) and the diffusion \(A_\lambda\) are bounded in the sense of (12). Here, we shall show that the initial value problem (1), (2) also has a global solution also in the absence of (12), when imposing additional assumptions that induce a suitable maximum principle.

More precisely, we suppose

- The initial condition satisfies \(0 \leq u_0 \leq 1\);
- The geometry compatibility condition [2] \((44)\) is satisfied:

\[
\text{div}\ f_\lambda(\lambda) = \text{div}(\text{div}(A_\lambda(\lambda))) \quad \text{for every } \lambda \in \mathbb{R}.
\]

Condition \((44)\) means that the divergence of the (diffusive) flux \(f_\lambda(\lambda) - \text{div}(A_\lambda(\lambda))\) is zero. If we model the dynamics of a fluid in the Euclidean case, the latter means that the fluid is incompressible. Indeed, if we denote by \(\rho\) the density of the fluid, then its change in a control volume is given by:

\[
\frac{D\rho}{Dt} = \text{div}(a_\lambda(\rho) \cdot \nabla \rho), \quad a_\lambda(\lambda) = \partial_\lambda A(\mathbf{x}, \lambda)
\]

i.e. the change of the density occurs only due to the diffusion effects. In (45) we use the standard convention in the frame of which \(\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k} \cdot \nabla \rho\) is the material derivative for the flow velocity \(\frac{d\mathbf{x}}{dt} = (\frac{\partial x}{\partial t}, \ldots, \frac{\partial x}{\partial t})\). Under the assumption that we are in the Euclidean situation, equation (11) can be rewritten as

\[
\frac{\partial \rho}{\partial t} + \partial_\lambda (f_\lambda(\lambda) - \text{div}A_\lambda(\lambda)) \bigg|_{\lambda=\rho} \cdot \nabla \rho + \text{Div}(f_\lambda(\lambda) - \text{div}A_\lambda(\lambda)) \bigg|_{\lambda=\rho} = \text{div}(a_\lambda(\rho) \cdot \nabla \rho)
\]

Then, since the velocity of the fluid point is \(\frac{d\mathbf{x}}{dt} = \partial_\lambda (f_\lambda(\xi) - \text{div}A_\lambda(\xi)) \bigg|_{\xi=\rho}\) we get by subtracting (46) from (45):

\[
(\text{div}(f_\lambda(\lambda) - \text{div}(A_\lambda(\lambda)))) \bigg|_{\lambda=\rho} = 0,
\]

which is the geometry compatibility condition (since the latter must hold for any possible density \(\rho\)).

The numerical values in the condition \(0 \leq u_0 \leq 1\) are not essential, but often are a natural choice as the unknown function may describe, for instance, the concentration of fluids in porous media. We refer to [11], where this assumption is imposed and where additional context is provided. In particular, we will use the following result (a modification of [11] Th. 1):

**Theorem 13.** Assume that the geometry compatibility condition \((44)\) holds and that \(u : \mathbb{R}_+ \times M \rightarrow \mathbb{R}\) is a \(H^1(\cdot)([0,T] \times M)\), \(T > 0\), solution to (11). Then for any
convex function $S \in C^2(\mathbb{R})$ such that $S(0) = 0$ we have

$$\partial_t S(u) + \text{div} \int_{u(t,x)} f_x(\xi) S'(\xi) \, d\xi = \text{div} \left( \int_{u(t,x)} A'_x(\xi) S'(\xi) \, d\xi \right) + \epsilon \Delta S(u) - \varepsilon S''(u)|\nabla u|^2 - S''(u)\langle A'_x(u)\nabla u, \nabla u \rangle,$$

(47)

where $f = \partial_t f$ and $A' = \partial_x A$.

**Proof:** By multiplying (41) by $f_x(\xi) S'(\xi) \, d\xi - \text{div} \left( \int_{u(t,x)} A'_x(\xi) S'(\xi) \, d\xi \right)$

(48)

Combining (47) and (48), the claim follows.

As for the Laplace-Beltrami operator, we have

$$S'(u)\Delta u = \Delta S(u) - S''(u)|\nabla u|^2.$$

(49)

Combining (45) and (49), the claim follows.

**Remark 14.** In [11], this result is proved supposing that $u$ be bounded and non-negative, but an inspection of the proof shows that these assumptions are not required.

**Theorem 15.** Let $f \in C^2(M \times \mathbb{R})$ and $A \in C^2(M \times \mathbb{R})$. Then under the assumptions $u_0 \in C^1(M)$, $0 \leq u_0 \leq 1$ and [14], the Cauchy problem [1, 2] admits a unique bounded weak solution belonging to $H^{(1,2)}((0, T) \times M)$.

**Proof:** To begin with, we consider an alternative problem to (1), where $u$ are replaced by suitably truncated versions $\tilde{u}$ and $\tilde{A}$, respectively, satisfying the global bounds (43). Concretely, set

$$\tilde{f}_x(\chi) := f_x(\chi(\lambda))$$

(50)

and

$$\tilde{A}_x(\lambda) = A_x(\chi(\lambda)),$$

(51)

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\chi(\lambda) = \lambda$ for $\lambda \in [0, 1]$, $\chi$ is constant on $(-\infty, -1]$ and on $[2, \infty)$ and such that $\chi' \geq 0$ everywhere.

We consider the following equation

$$\partial_t \tilde{u}_\varepsilon + \text{div} \tilde{f}_x(\tilde{u}_\varepsilon) = \text{div}(\text{div}(\tilde{A}_x(\tilde{u}_\varepsilon))) + \epsilon \Delta \tilde{u}_\varepsilon.$$

(52)

Then (52) is satisfied for $\tilde{f} \in C^2$ and $\tilde{A} \in C^2$, and the geometry compatibility condition (43) holds. By Theorem 11 and Remark 12, the initial value problem corresponding to (52) with $0 \leq u_0 \leq 1$ has a solution $\tilde{u}_\varepsilon \in H^{(1,2)}((0, T) \times M))$. We may now apply Theorem 13 to $\tilde{u}_\varepsilon$. More precisely, we insert for $S$ as limiting cases for the regularity (cf. [12 Sec. 2]) the semi-entropies

$$\eta_+(\tilde{u}_\varepsilon) = \frac{|\tilde{u}_\varepsilon - 1|_+}{0, \quad \text{otherwise}} \quad \eta_-(\tilde{u}_\varepsilon) = \frac{|\tilde{u}_\varepsilon|}{0, \quad \text{otherwise}}$$

Then, integrating (47) over $[0, T) \times M$, we get, taking into account the positive definiteness of $(\tilde{A}'_x(\lambda) + \epsilon I)$,

$$\int_M |\tilde{u}_\varepsilon(T, x) - 1|_+ \, dV(x) \leq \int_M |u_0(x) - 1|_+ \, dV(x) = 0, \quad \text{and}$$

$$\int_M |\tilde{u}_\varepsilon(T, x) - 0|_- \, dV(x) \leq \int_M |u_0(x) - 0|_- \, dV(x) = 0,$$

(53)
implying that $0 \leq \tilde{u}_\varepsilon \leq 1$. Thus $\tilde{u}_\varepsilon$ never leaves the $\lambda$-region in which, according to (50), (22), and the choice of $\chi$, $\tilde{f}_x \equiv f_x$ and $\tilde{A}_x \equiv A_x$.

Thus, in the range of $u_\varepsilon$, the equation (22) is strictly parabolic with the parabolicity constant $\epsilon$ from (3) independent of $\varepsilon$. Therefore, repeating the proof of Theorem 10 we conclude that the sequence $(\tilde{u}_\varepsilon)$ converges in $L^2((0,T);H^1(M))$ toward a function $\tilde{u}$ solving the problem

$$
\partial_t \tilde{u} + \text{div} f_x(\tilde{u}) = \text{div}(\text{div}(A_x(\tilde{u})))
$$

$$
\tilde{u}|_{t=0} = u_0(x),
$$

Hence $\tilde{u}$ is a solution to the original initial value problem (1), (2). From here and since the conditions of Theorem 11 are satisfied on the range of $\tilde{u}$, we conclude that the solution constructed above is in $H^{1,\frac{1}{2}}((0,T) \times M)$ and bounded. Consequently, it is an entropy admissible solution in the sense of [11, Def. 3] (due to [11, Th. 1 and Th. 2]). It is therefore unique by [11, Cor. 14].

**Remark 16.** The assumption $f \in C^2(M \times \mathbb{R})$ can be relaxed by involving mollification with respect to $\lambda \in \mathbb{R}$ into the construction. The mollification does not affect the geometry compatibility conditions and thus it does not affect the proof of the previous theorem. However, in this case we would have several additional technical steps which blur the ideas of the proof. Let us just briefly explain the main steps.

First, we mollify both $f_x(\lambda)$ and $A_x(\lambda)$ with respect to $\lambda \in \mathbb{R}$ via the standard convolution kernel, say $\rho_{\varepsilon}$. This does not affect the geometry compatibility condition. For this new flux $\tilde{f}_x(\lambda)$ and diffusion $A_x^\varepsilon(\lambda)$ we solve (1), (2), obtaining a family of bounded functions $(u_\varepsilon)$ satisfying

$$
\partial_t u_\varepsilon + \text{div} f_x^\varepsilon(u_\varepsilon) = \text{div}(\text{div}(A_x^\varepsilon(u_\varepsilon))), \quad x \in M,
$$

$$
u_\varepsilon|_{t=0} = u_0(x) \in L^\infty(M).
$$

This family $(u_\varepsilon)$ satisfies the estimates from Lemma 8 and Lemma 9 and therefore is strongly precompact in $L^2([0,T] \times M)$. A limiting function $u$ (along a subsequence of $(u_\varepsilon)$) then is the solution of (1), (2).

**Remark 17.** We note that we can also consider a general parabolic equation of the form (see [18])

$$
\partial_t u = Q[u] \quad \text{on} \quad (0,T) \times M,
$$

again with the initial data (2) on a $d$-dimensional compact orientable Riemannian manifold. The operator $Q$ has the following local expression:

$$
Q[u](t,x) = \partial_{j_1 \ldots j_p} \left( A^{1 \ldots p j_1 \ldots j_p}(t,x,\nabla u, \ldots, \nabla^{p-1} u) \partial_{i_1 \ldots i_p} u(t,x) \right)
$$

$$
+ b(t,x,u,\nabla u, \ldots, \nabla^{p-1} u),
$$

where the tensor $A = (A^{1 \ldots p j_1 \ldots j_p})$ is a locally elliptic smooth $(2p,0)$-tensor satisfying uniformly with respect to all the variables

$$
-(-1)^p \sum_{i_1, \ldots, i_p, j_1, \ldots, j_p} A^{1 \ldots p j_1 \ldots j_p} \xi_{i_1 \ldots i_p} \xi_{j_1 \ldots j_p} \geq c \sum_{i_1, \ldots, i_p} |\xi_{i_1 \ldots i_p}|^2
$$

for some $c > 0$ and every $\xi = (\xi_{i_1 \ldots i_p})_{i_1 \in \{1, \ldots, d\}}$.

The Galerkin approximation from Section 3 will induce a system of equations (corresponding to (22)):

$$
\hat{\alpha}_j^n = \int_M Q[u_n](t,x) e_j(x) dV(x) = F(t,\alpha_n), \quad j = 1, \ldots, n,
$$

where $\alpha_n = (\alpha_1^n, \ldots, \alpha_n^n)$. Moreover, under appropriate growth assumptions on $A$ and $b$, repeating the methods from Lemma 8 and Lemma 9 parabolicity of the
We consider the following spaces (not distinguishing between functions and the
L-corresponding equivalence classes):

\[ \text{dW} \]

The stochastic forcing \( F \) filtration

We see that the latter integral is actually a random variable with respect to the
\[ \{28, \text{Definition 3.4.1}\] as the following limit in probability:

\[ t \text{ is continuous in } t, \quad \omega \]

Finally, \( W_t \) is continuous in \( t \).

The stochastic forcing \( dW_t \) is the Itô integral defined for appropriate functions \( g \)
[28, Definition 3.4.1] as the following limit in probability:

\[ \int_0^T g(t, \omega) dW_t = \lim_{n \to \infty} \sum_{k=0}^n g(t_k, \omega)(W(t_{k+1}, \omega) - W(t_k, \omega)). \]

We see that the latter integral is actually a random variable with respect to the
filtration \( \mathcal{F}_t \).

We consider the following spaces (not distinguishing between functions and the
corresponding equivalence classes):

\[ L^p(\Omega; L^2((0, T); H^1(M))) \]

\[ = \{ u : (0, T) \times M \times \Omega \to \mathbb{R} : \int_{\Omega} \int_0^T \| u(t, \cdot, \omega) \|^2_{H^1(M)} dt d\mathbb{P}(\omega) < \infty \} \]

\[ L^p(\Omega; C^{1/2}((0, T); H^{-1}(M))) := \{ u : (0, T) \times M \times \Omega \to \mathbb{R} : \int_{\Omega} \sup_{\Delta t > 0} \frac{\| u(t + \Delta t, \cdot, \omega) - u(t, \cdot, \omega) \|^2_{H^{-1}(M)}}{\Delta t} d\mathbb{P}(\omega) < \infty \} \]

\[ L^p(\Omega; L^2(M))) = \{ u : M \times \Omega \to \mathbb{R} : \int_{\Omega} \int_M |u(x, \omega)|^2 dV(x) d\mathbb{P}(\omega) < \infty \}. \]

We assume the following for the coefficients of \([5]\):

(i) \( f \in C^1(M \times \mathbb{R}) \) is such that \( \sup_{x \in M} \| f(\cdot, \lambda) \|_{L^2(M)} < C \) and \( \| f \|_{C^1(M \times \mathbb{R})} < C \),

and, for simplicity, \( f(x, 0) \equiv 0 \);

(ii) For every \( x \in M \), the map \( \lambda \mapsto A_x(\lambda) \) is linear;

(iii) \( \Phi \in C^1_0(M \times \mathbb{R}) \).

We now supplement equation \([5]\) with the initial condition

\[ u|_{t=0} = u_0(x, \omega) = \sum_{k=1}^\infty \alpha_k(\omega)v_k(x), \quad (x, \omega) \in M \times \Omega, \quad (57) \]

where \( u_0 \in L^\infty(\Omega; L^2(M))) \) in the sequel if not stated otherwise. Note that this implies

\[ \sum_{k=1}^\infty |\alpha_k(\omega)|^2 \leq C < \infty \quad \text{almost surely.} \quad (58) \]

We shall prove the following theorem.

5. STOCHASTIC PARABOLIC DIFFERENTIAL EQUATION

Let us first introduce the notions that we need. By \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) we denote the
stochastic basis with the sample space \( \Omega \), the \( \sigma \)-algebra \( \mathcal{F} \), the natural filtration \( \mathcal{F}_t \)
generated by the Wiener process \( W_t \), and the probability measure \( \mathbb{P} \).

The Wiener process \( W_t \) is a stochastic process which has independent Gaussian
increments in the sense that \( W_{t+u} - W_t \) is independent of the past values \( W_s, s < t \), and \( W_{t+u} - W_t \sim \mathcal{N}(0, u) \). Finally, \( W_t \) has continuous paths i.e. with probability 1, \( W_t \) is continuous in \( t \).

We shall prove the following theorem.
Theorem 18. If the mapping $\lambda \mapsto A_\lambda(x)$ is linear for every $x \in M$, then the Galerkin approximation:

$$u_n(t, x, \omega) = \sum_{k=1}^{n} \alpha_k^n(t, \omega)e_k(x), \quad t \in [0, T), \quad x \in M, \quad \omega \in \Omega,$$

converges in $L^p_\mathcal{P}(\Omega; L^2((0, T); H^1(M)))$ towards $u \in L^p_\mathcal{P}(\Omega; L^2((0, T); H^1(M))) \cap L^p_\mathcal{P}(\Omega; C^{1/2}((0, T); L^2(M)))$ representing a solution to \([5], [77]\) in the sense that for every $\varphi \in H^2(M)$ and every $\Delta t > 0$ it holds

$$\int_M (u(t + \Delta t, x, \omega) - u(t, x, \omega)) \varphi(x) dV(x) = \int_t^{t+\Delta t} \int_M f(x, u)\nabla\varphi(x) dV(x) dt - \int_t^{t+\Delta t} \int_M \text{div}(A_x \cdot u)\nabla\varphi dV(x) dt + \int_t^{t+\Delta t} \int_M \Phi(x, u)\varphi dV(x) dW_t. \quad (60)$$

In order to prove Theorem 18 we shall need two fundamental results from the Itô calculus, the first of which is (see [28, Th. 4.1.2], [7, Th. 5.9]):

Lemma 19. (Itô lemma) Let $X_t$ be an Itô process given by

$$dX_t = \mu_1 dt + \sigma_1 dW_t. \quad (61)$$

For each $f \in C^{1,2}([a, b] \times \mathbb{R})$, $f = f(t, z)$, also $f(t, X_t)$ is an Itô process, and

$$df(t, X_t) = \left( \frac{\partial f}{\partial t} + \mu_1 \frac{\partial f}{\partial z} + \frac{\sigma_1^2}{2} \frac{\partial^2 f}{\partial z^2} \right) dt + \sigma_1 \frac{\partial f}{\partial z} dW_t \quad (62)$$

holds.

We remark here that (61) is actually an informal way of expressing the integral equality

$$X_{t_0+s} - X_{t_0} = \int_{t_0}^{t_0+s} \mu_1 dt + \int_{t_0}^{t_0+s} \sigma_1 dW_t, \quad \forall t_0, s > 0. \quad (63)$$

To formulate the second prerequisite, as in [28, Def. 3.1.4], by $\mathcal{V}(S, T)$ we denote the set of all $f : [0, \infty) \times \Omega \to \mathbb{R}$ that are measurable, $\mathcal{F}_t$-adapted and satisfy $E \left[ \int_T^S f(t, \omega)^2 dt \right] < \infty$. Then by [28, Cor. 3.1.7]:

Lemma 20. (Itô isometry) For any $f \in \mathcal{V}(S, T)$

$$E \left[ \left( \int_S^T f(t, \omega) dW_t \right)^2 \right] = E \left[ \int_S^T f(s, \omega)^2 ds \right].$$

5.1. Proof of Theorem 18. We seek functions $\alpha_k$, $k = 1, \ldots, n$ such that (6) is satisfied in the following sense (cf. [27]): Let

$$u_n(t, x, \omega) = \sum_{k=1}^{n} \alpha_k^n(t, \omega)e_k(x). \quad (64)$$

Then we require that almost surely for any $\varphi \in \text{span}\{e_k\}_{k=1,\ldots,n}$ we have:

$$\int_M du_n \varphi dV(x) = \int_M f(x, u_n)\nabla\varphi dV(x) dt - \int_M \text{div}(A_x \cdot u_n)(\nabla\varphi) dV(x) dt + \int_M \Phi(x, u_n)\varphi dV(x) dW_t. \quad (65)$$
If we put here \( \varphi = e_j, j = 1, \ldots, n \), using orthogonality of \( \{e_k\}_{k \in \mathbb{N}} \), we get the following system of stochastic ODEs:
\[
d\alpha_j^n = \int_M f(x, u_n) \nabla e_j(x) \, dV(x) \, dt
- \int_M \text{div}(A_x \cdot u_n)(\nabla e_j) \, dV(x) \, dt + \int_M \Phi(x, u_n)e_j(x) \, dV(x) \, dW_t.
\]
(66)

According to (12) and since \( \Phi \in C^1_0(M \times \mathbb{R}) \), from [28, Theorem 5.2.1] we infer that (66) satisfying finite initial data \( \alpha_j(\omega) \) (in the sense that \( \alpha_j(\omega) \leq C < \infty \) almost surely) has a unique globally defined \( t \)-continuous solution. In particular, if \( u_0(x, \omega) = \sum_{j \in \mathbb{N}} \alpha_j(\omega)e_j(x) \), and we choose
\[
\alpha_j(0, \omega) = \alpha_j(\omega),
\]
(67)
we obtain a sequence \( \{u_n\} \) satisfying for every \( n \in \mathbb{N} \) the relation (66) (see (58)).

**H\(^s\)-estimates.** Let us now derive \( H^s \)-estimates, \( s = 1, 2 \), for \( \{u_n\} \). To this end, from (66) and the Itô lemma we infer:
\[
\frac{1}{2} d|\alpha_j^n|^2 = \int_M f(x, u_n)(\nabla e_j) \alpha_j^n \, dV(x) \, dt
- \int_M \text{div}(A_x \cdot u_n)((\nabla e_j) \alpha_j^n) \, dV(x) \, dt + \frac{1}{2} \left[ \int_M \Phi(x, u_n)e_j(x) \, dV(x) \right]^2 \, dt
+ \int_M \alpha_j^n \Phi(x, u_n)e_j(x) \, dV(x) \, dW_t.
\]
(68)

By summing the latter relation for \( j = 1, \ldots, n \), we conclude from the fact that the \( e_j \) form a complete orthonormal system in \( L^2(M) \):
\[
\frac{1}{2} d\|u_n(t, \cdot, \omega)\|^2_{L^2(M)} \leq \int_M f(x, u_n) \nabla u_n(t, x, \omega) \, dV(x) \, dt
- \int_M \text{div}(A_x \cdot u_n)(\nabla u_n(t, x, \omega)) \, dV(x) \, dt + \frac{1}{2} \|\Phi(x, u_n)\|^2_{L^2(M)} \, dt
+ \int_M \Phi(x, u_n)u_n(t, x, \omega) \, dV(x) \, dW_t.
\]

From here, integrating the relation over \( (0, T) \) and using Lemma 5(i), we have the following estimate
\[
\int_M \frac{|u_n(T, x, \omega)|^2}{2} \, dV(x) + c \int_0^T \int_M \|\nabla u_n\|^2_0 \, dV(x) \, dt
\leq \int_M \frac{|u_0(x, \omega)|^2}{2} \, dV(x) + \int_0^T (\|\Phi(x, u_n)\|^2_{L^2(M)} + \|\Phi(x, u_n)\|^2_{L^2(M)}) \, dt
+ C \int_0^T \int_M |u_n|^2 \, dV(x) \, dt + \int_0^T \int_M \Phi(x, u_n)u_n \, dV(x) \, dW_t.
\]
(69)

This implies
\[
E\left( \int_M \frac{|u_n(T, x, \cdot)|^2}{2} \, dV(x) \right)
\leq E\left( \int_M \frac{|u_0(x, \cdot)|^2}{2} \, dV(x) \right) + T(\| \sup_{\lambda} |\Phi(\cdot, \lambda)|\|^2_{L^2(M)} + \| \sup_{\lambda} |\Phi(x, \lambda)|\|^2_{L^2(M)})
+ CE\left( \int_0^T \int_M |u_n|^2 \, dV(x) \, dt \right).
\]
Using the Peter-Paul inequality and again (24), we have for appropriate constants $\lambda$

$$E \left( \int_M \frac{|u_n(T, x, \omega)|^2}{2} dV(x) \right) \leq e^{CT} \left( E \left( \int_M \frac{|u_0(x, \omega)|^2}{2} dV(x) \right)
+ T(\|\sup_{\lambda} |f(\cdot, \lambda)|\|_{L^2(M)}^2 + \|\sup_{\lambda} |\Phi(\cdot, \lambda)|\|_{L^2(M)}^2) \right).$$

Combining this with (68), we finally have

$$E \left( \int_M \frac{|u_n(T, x, \omega)|^2}{2} dV(x) + c \int_0^T \int_M \|\nabla u_n\|_2^2 dV(x) \right)
\leq E \left( \int_M \frac{|u_0(x, \omega)|^2}{2} dV(x) + T(\|\sup_{\lambda} |f(\cdot, \lambda)|\|_{L^2(M)}^2 + \|\sup_{\lambda} |\Phi(\cdot, \lambda)|\|_{L^2(M)}^2) \right)
+ CTe^{CT} \left( E \left( \int_M \frac{|u_0(x, \omega)|^2}{2} dV(x) + T(\|\sup_{\lambda} |f(\cdot, \lambda)|\|_{L^2(M)}^2 + \|\sup_{\lambda} |\Phi(\cdot, \lambda)|\|_{L^2(M)}^2) \right) \right).$$

To prove higher order estimates, we multiply (68) by $\lambda^2$ and then sum the obtained expressions. Using (24), we get:

$$\frac{1}{2} d\|u_n(t, \cdot, \omega)\|^2_{H^1(M)} = - \int_M \text{div}(f(x, u_n)) \left( \sum_{j=1}^n \lambda^2 \alpha_j(t, \omega)e_j(x) \right) dV(x) dt
- \int_M \text{div}(A_{\chi} \cdot u_n) \left( \nabla \left( \sum_{j=1}^n \lambda^2 \alpha_j(t, \omega)e_j(x) \right) \right) dV(x) dt
+ \frac{1}{2} \sum_{j=1}^n \lambda^2 \int_M \Phi(x, u_n)e_j(x)dV(x)^2 dt
+ \int_M \Phi(x, u_n) \sum_{j=1}^n \lambda^2 \alpha_j(t, \omega)e_j(x)dV(x) dW_t.$$

Next, since $(I - \Delta)e_j = \lambda^2 e_j$ we can rewrite the latter in the form

$$\frac{1}{2} d\|u_n(t, \cdot, \omega)\|^2_{H^1(M)} = - \int_M \text{div}(f(x, u_n)) \left( \sum_{j=1}^n \lambda^2 \alpha_j(t, \omega)e_j(x) \right) dV(x) dt
- \int_M \text{div}(A_{\chi} \cdot u_n)(\nabla (u_n - \Delta u_n)) dV(x) dt
+ \frac{1}{2} \sum_{j=1}^n \lambda^2 \int_M \Phi(x, u_n)e_j(x)dV(x)^2 dt
+ \int_M \Phi(x, u_n) \sum_{j=1}^n \lambda^2 \alpha_j(t, \omega)e_j(x)dV(x) dW_t.$$

Using the Peter-Paul inequality and again (24), we have for appropriate constants $\tilde{K}$ and $k$:

$$\frac{1}{2} d\|u_n(t, \cdot, \omega)\|^2_{H^1(M)} \leq \tilde{K}\|f(\cdot, u_n)\|_{H^1(M)}^2 dt + \frac{k}{4} \|u_n(t, \cdot, \omega)\|_{H^1(M)}^2 dt
- \int_M \text{div}(A_{\chi} \cdot u_n)(\nabla u_n) dV(x) dt
- \int_M \text{div}(A_{\chi} \cdot u_n) \Delta u_n dV(x) dt
+ \frac{1}{2} \sum_{j=1}^n \lambda^2 \int_M \Phi(x, u_n)e_j(x)dV(x)^2 dt.
Now, we take the expectation of the latter estimate. Then the last term vanishes and using Lemma 5 (ii) we conclude
\[
\frac{1}{2}E\left(\|u_n(t, \cdot, \cdot)\|_H^2(M)\right) + \bar{k} E\left(\int_0^t \|u_n(t', \cdot, \cdot)\|_H^2(M) dt'\right)
\leq \frac{1}{2}E\left(\|u_{n0}(\cdot, \cdot)\|_H^2(M)\right) + KE\left(\int_0^t \|\tilde{\Phi}(\cdot, u_n)\|^2_H(M) dt'\right)
+ \bar{K} E\left(\int_0^t \max\{1, \|u_n(t', \cdot, \cdot)\|_H^2(M) dt'\}\right) + E\left(\int_0^t \|\tilde{\Phi}(\cdot, u_n)\|^2_H(M) dt'\right),
\]
where \(\bar{k}, \bar{K}\) are positive constants depending on \(k, K\) from Lemma 5. According to (70), we see that the right-hand side of the previous expression is bounded and thus
\[
E\left(\|u_n(t, \cdot, \cdot)\|_H^2(M)\right) + E\left(\int_0^t \|u_n(t, \cdot, \cdot)\|_H^2(M) dt'\right) \leq C \quad \forall t \geq 0. \tag{71}
\]

**Convergence.** Using the splitting into \(\text{span}\{e_1, \ldots, e_n\}\) and \(\text{span}\{e_k\}_{k>n}\) and (71)
\[
E\left(\|u_n - u_m\|^2_{L^2([0,T] \times M)}\right) = E\left(\|u_m\|^2_{L^2([0,T] \times M)}\right)
= E\left(\left[\int_0^T \sum_{j=n+1}^m |\alpha_j^m(t, \cdot, \cdot)|^2 dt\right]^{1/2}\right)
\leq \frac{1}{\lambda_N} E\left(\|u_m\|^2_{L^2([0,T]; H^1(M))}\right) \to 0
\]
as \(N \to \infty\), and analogously
\[
E\left(\|u_n - u_m\|^2_{L^2([0,T]; H^2(M))}\right) = E\left(\|u_m\|^2_{L^2([0,T]; H^2(M))}\right)
\leq \frac{1}{\lambda_N} E\left(\|u_m\|^2_{L^2([0,T]; H^2(M))}\right) \to 0
\tag{73}
\]
as \(N \to \infty\).

Using the above and the linearity of the parabolic term, unlike the situation from the previous section, we can establish directly that \((u_n)\) is a Cauchy sequence in \(L^2_p(\Omega; L^2([0,T]; H^1(M)))\). Then, we shall prove continuity of the limit with respect to \(t \in [0, T]\). To this end, we subtract equations (65) for \(m \geq n \geq N \in \mathbb{N}\) to get for any \(j \leq n\)
\[
d(\alpha_j^n - \alpha_j^m) = \int_M ((f(x, u_n) - f(x, u_m)) \nabla e_j(x)) dV(x) dt
- \int_M \text{div} (A_x \cdot (u_n - u_m)) (\nabla e_j(x)) dV(x) dt
+ \int_M (\Phi(x, u_n) - \Phi(x, u_m)) e_j(x) dV(x) dt.
\tag{74}
\]
From here, according to the Itô lemma, we have
\[
\frac{d|\alpha_j^n - \alpha_j^m|^2}{2} = \int_M ((f(x, u_n) - f(x, u_m)) \nabla ((\alpha_j^n - \alpha_j^m) e_j(x)) dV(x) dt
- \int_M \text{div}(A_x \cdot (u_n - u_m)) \nabla ((\alpha_j^n - \alpha_j^m) e_j(x)) dV(x) dt.
\]
\[
+ \left[ \int_M \left( \Phi(x, u_n) - \Phi(x, u_m) \right) e_j(x) dV(x) \right]^2 dt
+ \int_M \left( \Phi(x, u_n) - \Phi(x, u_m) \right) \left( \alpha^m_j - \alpha^n_j \right) e_j(x) dV(x) dW_t.
\]

Keeping in mind (74), we get after summing the latter expressions for \( j = 1, \ldots, n \):

\[
\frac{1}{2} \int_M \left| u_n - u_m \right|^2 dV(x) = \int_M \left( f(x, u_n) - f(x, u_m) \right) \cdot \nabla(u_n - u_m) dV(x) dt
- \int_M \text{div}(A_{x}(u_n - u_m)) \cdot \nabla(u_n - u_m) dV(x) dt
+ \int_M \left( \Phi(x, u_n) - \Phi(x, u_m) \right)^2 dV(x) dt
+ \int_M \left( \Phi(x, u_n) - \Phi(x, u_m) \right) (u_n - u_m) dV(x) dW_t
- \int_M \left( f(x, u_n) - f(x, u_m) \right) \nabla \varphi^+_n dV(x) dt
+ \int_M \text{div}(A_{x}(u_n - u_m)) \cdot \nabla \varphi^+_n dV(x) dt
- \int_M \left[ \left( \Phi(x, u_n) - \Phi(x, u_m) \right) \varphi^+_n \right]^2 dV(x) dt
- \int_M \left( \Phi(x, u_n) - \Phi(x, u_m) \right) \varphi^+_n dV(x) dW_t,
\]

where \( \varphi^+_n = (u_n - u_m) \frac{1}{\partial_n} = - (u_m) \frac{1}{\partial_n} \).

Let us first consider the terms containing \( dt \) in (75). We have:

\[
| \int_M \left( f(x, u_n) - f(x, u_m) \right) \cdot \nabla(u_n - u_m) dV(x) dt |
\leq K_1 \| f \|_{\infty}^2 \| u_n(t, \cdot, \omega) - u_m(t, \cdot, \omega) \|_{L^2(M)}^2 + \frac{C}{2} \| \nabla(u_n(t, \cdot, \omega) - u_m(t, \cdot, \omega)) \|_{L^2(M)}^2.
\]

Next, from Lemma 5 (i'), we have

\[
- \int_M \text{div}(A_{x}(u_n - u_m)) (\nabla(u_n - u_m)) dV(x) dt
= \int_M \text{div} \text{div}(A_{x}(u_n - u_m)) (u_n - u_m) dV(x) dt
\leq -c \| \nabla(u_n(t, \cdot, \omega) - u_m(t, \cdot, \omega)) \|_{L^2(M)}^2 + C \| u_n(t, \cdot, \omega) - u_m(t, \cdot, \omega) \|_{L^2(M)}^2.
\]

Furthermore,

\[
| \int_M \left( \Phi(x, u_n) - \Phi(x, u_m) \right)^2 dV(x) | \leq \| \Phi' \|_{\infty} \| u_n(t, \cdot, \omega) - u_m(t, \cdot, \omega) \|_{L^2(M)}^2.
\]

The rest of the terms containing \( dt \) tend to zero as \( N \to \infty \) (and thus \( n, m \to \infty \)) according to (72) and (73).

From the above, we get after integrating (75) over \([0, T]\) and applying the expectation operator (which makes the terms containing \( dW_t \) vanish):

\[
E \left( \int_M \frac{|u_{n0} - u_{m0}(x, \omega)|^2}{2} dV(x) \right) + \frac{C}{2} E \left( \int_0^T \int_M \| \nabla(u_n - u_m) \|_{L^2(M)}^2 dV(x) dt \right)
\leq E \left( \int_M \frac{|u_{n0} - u_{m0}(x, \omega)|^2}{2} dV(x) \right) + \frac{C}{2} E \left( \int_0^T \int_M |u_n - u_m|^2 dV(x) dt \right) + c_n.
\]
where $c_n \to 0$ as $N \to \infty$.

From here, we see that the Gronwal lemma implies:

$$E\left(\int_M |(u_n - u_m)(T, x, \omega)|^2 dx + \int_0^T \int_M |\nabla(u_n - u_m)(t, x)|^2 dx\right) \leq C(T)c_n, \quad (77)$$

for a locally finite function $C(T)$ depending on $T$ and again $c_n \to 0$.

This proves that the sequence $(u_n)$ is Cauchy in $L^2_p(\Omega; L^2([0, T]; H^1(M)))$, i.e. it converges towards some function $u \in L^2_p(\Omega; L^2([0, T]; H^1(M)))$, which is a solution to (60). Moreover, according to (71), we have $u \in L^2_p(\Omega; L^2([0, T]; H^2(M)))$ as well. It remains to prove that $u \in L^2_p(\Omega; C^{1/2}((0, T); L^2(M)))$.

Consider the integral formulation of the stochastic differential equation (60). We have for every $\varphi \in H^2(M)$:

$$\int_M (u(t + \Delta t, x, \omega) - u(t, x, \omega)) \varphi(x) dV(x) = \int_t^{t + \Delta t} \int_M f(x, u)\nabla \varphi(x) dV(x) dt$$

$$+ \int_t^{t + \Delta t} \int_M \text{div}(A_k \cdot u)(\nabla \varphi) dV(x) dt + \int_t^{t + \Delta t} \int_M \Phi(x, u)\varphi(x) dV(x) dW_t.$$

To proceed from here, write $u = \sum_{i \in \mathbb{N}} a_i e_i$, choose $\varphi(x) = e_k(x)$ and use integration by parts to get:

$$a_k(t + \Delta t, \omega) - a_k(t, \omega) = -\int_t^{t + \Delta t} \int_M \text{div}(f(x, u)e_k(x)) dV(x) dt'$$

$$+ \int_t^{t + \Delta t} \int_M \text{div}(A_k(u)) e_k(x) dV(x) dt' + \int_t^{t + \Delta t} \int_M \Phi(x, u)e_k(x) dV(x) dW_t.$$}

We square the latter expression, find the expectation, and use the Jensen inequality to infer:

$$E\left[ (a_k(t + \Delta t, \omega) - a_k(t, \omega))^2 \right]$$

$$\leq C\left( \Delta t E\left[ \left( \int_t^{t + \Delta t} \text{div}(f(x, u)e_k(x)) dV(x) \right)^2 dt' \right] + \Delta t E\left[ \left( \int_t^{t + \Delta t} \text{div}(A_k(u)) e_k(x) dV(x) \right)^2 dt' \right] + E\left[ \left( \int_t^{t + \Delta t} \Phi(x, u)e_k(x) dV(x) dW_t \right)^2 \right] \right).$$

We divide the expression by $\Delta t$, use here the Ito isometry (Lemma 20), and sum the expression over $k \in \mathbb{N}$. We have:

$$E\left[ \frac{\|u(t + \Delta t, \cdot) - u(t, \cdot, \cdot)\|_2^2}{\Delta t} \right]$$

$$\leq C\left( E\left[ \int_t^{t + \Delta t} \text{div}(f(\cdot, u(t', \cdot)) dV(x)) dt' \right] + E\left[ \int_t^{t + \Delta t} \text{div}(A_k(u)) dV(x) dt' \right] + E\left[ \frac{1}{\Delta t} \int_t^{t + \Delta t} \Phi(\cdot, u(t', \cdot)) dV(x) dt' \right] \right)$$

$$\leq C\left( E\left[ \|u\|_{L^2((0, T); H^2(M))} \right] + \sup_{t \in (0, T)} E\left[ \|\Phi(\cdot, u(t, \cdot, \cdot))\|_2^2 \right] \right) \leq C,$$
since according to (70) and (71)
\[
E\left[ \int_t^{t+\Delta t} \left\| \div (\cdot) \right\|_{L^2(M)}^2 \mathrm{d}t' \right] \leq CE \left[ \left\| u \right\|_{L^2((0,T);H^1(M))}^2 \right] \leq C < \infty
\]
\[
E\left[ \int_t^{t+\Delta t} \left\| \div (A_{\Phi}(u)) \right\|_{L^2(M)}^2 \mathrm{d}t' \right] \leq CE \left[ \left\| u \right\|_{L^2((0,T);H^1(M))}^2 \right] \leq C < \infty
\]
\[
E\left[ \frac{1}{\Delta t} \int_t^{t+\Delta t} \left\| \Phi(\cdot, u(t',\cdot)) \right\|_{L^2(M)}^2 \mathrm{d}t' \right] \leq \sup_{t \in (0,T)} E \left[ \left\| \Phi(\cdot, u(t,\cdot)) \right\|_{L^2(M)}^2 \right] < \infty.
\]
\[
E\left[ \left\| u \right\|_{C^{1/2}(0,T);L^2(M)}^2 \right] \leq C < \infty \quad \Rightarrow \quad u \in L^2_P(\Omega; C^{1/2}(0,T); L^2(M)).
\]
This concludes the proof.

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