Central limit theorems for Markov chains based on their convergence rates in Wasserstein distance

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Abstract: Many tools are available to bound the convergence rate of Markov chains in total variation (TV) distance. Such results can be used to establish central limit theorems (CLT) that enable error evaluations of Monte Carlo estimates in practice. However, convergence analysis based on TV distance is often non-scalable to high-dimensional Markov chains (Qin and Hobert (2018); Rajaratnam and Sparks (2015)). Alternatively, robust bounds in Wasserstein distance are often easier to obtain, thanks to a coupling argument. Our work is concerned with the implication of such convergence results, in particular, do they lead to CLTs of the corresponding Markov chains? One indirect and typically non-trivial way is to first convert Wasserstein bounds into total variation bounds. Alternatively, we provide two CLTs that directly depend on (sub-)geometric convergence rates in Wasserstein distance. Our CLTs hold for Lipschitz functions under certain moment conditions. Finally, we apply these CLTs to three sets of Markov chain examples including an independence sampler, an unadjusted Langevin algorithm (ULA), and a special autoregressive model that generates reducible chains.

Keywords: Geometric ergodicity, Independence sampler, Markov chain Monte Carlo, Martingale approximation, Unadjusted Langevin algorithm.

1. Introduction

Total variation (TV) distance plays a central role in the convergence analysis of Markov chains. Various rates of convergence of Markov chains to their invariant distributions, such as geometric ergodicity (GE) and polynomial ergodicity, are most commonly defined in terms of TV distance, and many tools have been developed to achieve qualitative and quantitative bounds for such rates. See, e.g., Rosenthal (1995); Jarner and Roberts (2002).

Convergence rates in TV distance are closely related to different types of mixing in Markov chains, allowing classical CLTs on mixing processes to be invoked. See Jones (2004) for a survey. Here is an example of proving CLT under this approach: if a chain has GE, then it is exponentially fast strongly mixing (Chan and Geyer, 1994), hence classical CLTs apply (Ibragimov and Linnik, 1971), and asymptotic normality holds for functions with certain moments. CLTs for Markov chains are practically important because they allow the use of standard error to evaluate estimators based on Markov chains,
say, when approximating Bayesian posterior features using Markov chain Monte Carlo (MCMC). Despite available tools and important implications of establishing convergence rates of Markov chains in TV distance, when the dimension of the state space increases, robust results are rarely attained. For example, Rajaratnam and Sparks (2015) investigated a Bayesian linear regression problem empirically, for which the convergence rates in TV to the posterior were observed to deteriorate quickly as the number of regressors, $p$, grew. Technically, a most widely used drift and minorization (d&m) tool to develop upper bounds of convergence rates was shown to be potentially very conservative by Qin and Hobert (2018). They pointed out that the “small set” used in the minorization condition is indeed substantial, in the sense that its probability under the invariant distribution is at least $\frac{1}{2}$. And to compensate for the very large “small set” in the coupling argument of the d&m method, the probability of coupling is confined, with greater severity in state spaces of higher dimension. This yields conservative upper bounds for convergence rates in TV distance.

**Remark 1** An exception is Qin and Hobert (2019), which analyzed an MCMC called the data augmentation (DA) algorithm for a Bayesian probit regression model, and showed that its convergence rate in TV, as a function of data size, $n$, and the number of regressors, $p$, indeed approaches a number strictly smaller than 1 if either $n$ or $p$ increases. Nevertheless, their proof depends on the fact that, when either $n$ or $p$ grows, there exists a Markov chain with constant dimension that has the same convergence rate as that of the DA. It turns out such tricks do not often work for general Markov chains with an increasing dimension. For example, the strict bound on rates for the algorithms studied by Qin and Hobert (2019) can no longer be established if both $n$ and $p$ grow.

Wasserstein distance is a broad class of distances that measure the discrepancy between probability measures. Coupling arguments have been developed to bound the convergence of Markov chains in $L_1$-Wasserstein distance, and they are generally more robust to high-dimensional state spaces than those for the TV distance (Madras and Sezer, 2010; Durmus and Moulines, 2015). Successful examples of bounding Wasserstein distance in high-dimensional Markov chains can be found in Qin and Hobert (2018, 2019) and Durmus and Moulines (2015). Briefly, these papers use coupling arguments based on drift and contraction conditions. In contrast to the minorization condition in the aforementioned d&m tool for bounding the TV distance, the contraction condition only requires the two working chains to move closer, instead of to match exactly, when they visit a small set. We mention that the TV distance can be seen as a special case of the $L_1$-Wasserstein distance when the metric is the discrete metric. However, the contraction condition only works with respect to metrics that are equivalent to the metric of the Polish state space, which excludes the discrete metric unless the state space is countable.

Despite the promising convergence results in Wasserstein distance, there has been limited work to further their practical relevance, say, to establish CLTs for the corresponding Markov chains. One indirect way to establish CLTs is to first convert Wasserstein bounds into TV bounds, before appealing to classical Markov chain CLTs (Gibbs, 2004). A sufficient condition for such a conversion can be found in Madras and Sezer (2010), but the condition is rather difficult to check in practice.
The contribution of this paper is to provide CLTs of Markov chains that are directly based on their convergence in Wasserstein distance, among other regularity conditions. To be more specific, our main results are two CLTs, Theorems 9 and 10, such that if the metric of the \( L_1 \)-Wasserstein distance satisfies certain regular moment conditions, under which the Markov chain of interest converges at fast enough rates, then there exists a martingale approximation (MA, defined in Section 2.2) to the additive functional of the Markov chain for any square-integrable Lipschitz function. As a result, a martingale CLT can be applied to establish CLT for the additive functional. Note that these CLTs are even applicable to certain reducible Markov chains (see e.g. Section 4.3), which can not possibly be handled with classical CLTs that depend on convergence in TV distance. Other existing works on Wasserstein distance based CLTs include Komorowski and Walczuk (2012), which are more general in scope as they concern Markov processes. When applied to Markov chains, their CLT requires somewhat different conditions than ours, most importantly the convergence of Wasserstein distance at the geometric rate, which is stronger than that of our Theorems (A2). In addition, a minor difference is they assume a finite \( 2 + \delta \) moment condition, which is slightly stronger than the second moment condition (A1) needed in our CLTs.

The rest of the paper is organized as follows. In Section 2, we review sufficient conditions for the existence of MA to additive functionals of Markov chains and martingale CLTs. In Section 3, we establish CLTs for Markov chains based on their rates of Wasserstein convergence. Our main results are Theorems 9 and 10, which are CLTs with practically checkable conditions. In Section 4 we demonstrate the application of our CLTs to three examples of Markov chains.

2. Preliminaries for Markov chains and their CLTs

Definitions and results in this section are primarily based on Maxwell and Woodroofe (2000), Douc et al. (2018) and Tierney (1994).

2.1. On general state space Markov chains

Suppose \( \mathcal{X} \) is a Polish space and \( \mathcal{B} \) its Borel \( \sigma \)-field. Denote by \( \Phi = \{X_0, X_1, X_2, \ldots \} \) a Markov chain with state space \( (\mathcal{X}, \mathcal{B}) \) and transition function \( Q \). Let \( \pi \) be an invariant measure for \( Q \), that is,

\[
\pi(C) = \int_{\mathcal{X}} Q(x, C) \pi(dx) \quad \text{for all } C \in \mathcal{B}.
\]

Let

\[
\mathcal{L}^2(\pi) = \left\{ g : \mathcal{X} \to \mathbb{R} \text{ such that } \int_{\mathcal{X}} g^2(x) \pi(dx) < \infty \right\} \quad \text{and} \quad \mathcal{L}^2_0(\pi) = \left\{ g \in \mathcal{L}^2(\pi) : \int_{\mathcal{X}} g(x) \pi(dx) = 0 \right\}.
\]
Let $\| \cdot \|$ denote the $L^2$-norm, that is, $\| g \| = \left( \int_X g^2(x) \pi(dx) \right)^{\frac{1}{2}}$ for any $g \in \mathcal{L}^2(\pi)$. The transition function $Q$ defines an operator on $\mathcal{L}^2(\pi)$, which we denote using the same symbol, and that, \[
 Qg(x) = \int_X g(y)Q(x; dy) \quad \text{for any } g \in \mathcal{L}^2(\pi).
\]

It can be shown that $Q$ is a contraction, in the sense that $\|Qg\| \leq \|g\|$ for any $g \in \mathcal{L}^2(\pi)$.

A Markov chain is said to be Harris ergodic if it is positive Harris recurrent and aperiodic (Nummelin, 1984; Tierney, 1994). The Theorem below says that Harris ergodicity of a Markov chain can be characterized using its convergence in TV distance. Here, the TV distance between any two probability measures $\mu$ and $\nu$ on $(X, \mathcal{B})$ is defined as $d_{TV}(\mu, \nu) = \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|$.

**Theorem 1** (Theorem 5.2.6 of Douc et al. (2018)) If $Q$ admits a unique invariant probability measure $\pi$, then the corresponding Markov chain $\Phi$ is ergodic.

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**Theorem 2** (Nummelin (1984)) A Markov chain $\Phi$ is Harris ergodic if and only if there exists a probability measure $\pi$ on $(X, \mathcal{B})$ such that 
\[
\lim_{n \to \infty} d_{TV}(Q^n\delta_x, \pi) = 0 \quad \text{for all } x \in X.
\]

**Remark 2** Harris ergodicity implies ergodicity, because the former implies the existence of a unique invariant measure, which implies the latter by Theorem 1.

### 2.2. On martingale approximation and martingale CLTs

For a Markov chain $\Phi$, $g \in \mathcal{L}^2_0(\pi)$ and $n = 1, 2, \ldots$, let 
\[
S_n = S_n(g) := g(X_0) + g(X_1) + \cdots + g(X_n),
\]
and 
\[
S^*_n(g) = \frac{S_n(g)}{\sqrt{n}}.
\]

There are at least three approaches to study the asymptotic behavior of $S^*_n(g)$ in the literature of Markov chains. First, if the chain satisfies certain mixing conditions, classical results in mixing processes can be used to derive asymptotic normality for $S^*_n(g)$. See e.g. Jones (2004) for a review. Alternatively, if $\Phi$ can be generated using a method called regenerative simulation, then establishing the asymptotic normality for $S^*_n(g)$ may be reduced to that for the sum of independent components Mykland et al. (1995); Tan et al. (2015). In this paper, we will focus on a third method that is based on an MA (Holzmann, 2004) to $S_n(g)$. 

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Definition 1  There exists an MA to $S_n(g)$ if there are two sequences of random variables $M = \{M_n\}_{n \geq 1}$ and $R = \{R_n\}_{n \geq 1}$ such that

1. $S_n(g) = M_n + R_n$, for $n \geq 1$;
2. $M$ is a martingale adapted to the filtration $\{\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n), n \geq 1\}$, and $E(M_1|X_0) = 0$;
3. $E(R_n^2) = o(n)$ as $n \to \infty$.

Note that if an MA to $S_n(g)$ exists, then $S^*_n(g)$ and $\frac{M_n}{\sqrt{n}}$ are asymptotically equivalent in the sense that

$$S^*_n(g) = \frac{M_n}{\sqrt{n}} + \frac{R_n}{\sqrt{n}} = M_n + o_p(1).$$  \hspace{1cm} (2.1)

Next, we briefly review sufficient conditions that imply the existence of MA. The first condition was developed by Gordin and Lifšic (1978). If there exists a solution $h \in L^2(\pi)$ to Poisson’s equation

$$h - Qh = g,$$  \hspace{1cm} (2.2)

then $S_n(g)$ can be represented as

$$S_n(g) = \sum_{k=1}^{n} (h(X_k) - Qh(X_{k-1})) + Qh(X_0) - Qh(X_n).$$

Let $M_n = \sum_{k=1}^{n} (h(X_k) - Qh(X_{k-1}))$ and $R_n = Qh(X_0) - Qh(X_n)$, then all three statements in Definition 1 of MA hold. Butzer and Westphal (1971) provided the following sufficient condition for Poisson’s equation to be solvable:

$$\sum_{n=0}^{\infty} \|Q^n g\| < \infty.$$  \hspace{1cm} (2.3)

This condition will be needed in establishing one of our main CLTs, Theorem 10.

Alternatively, an important relaxation to needing a solvable Poisson’s equation (2.2) was introduced by Kipnis and Varadhan (1986), which requires instead, for some $\epsilon > 0$, the solution to

$$(1 + \epsilon)h_\epsilon - Qh_\epsilon = g.$$  \hspace{1cm} (2.4)

Denote the solution to (2.4) by $h_\epsilon$, which always exists due to the convergence of the power series in (2.6) below. Then there is a particular way to represent $S_n(g)$ using $h_\epsilon$ and $Qh_\epsilon$, that eventually yields an MA. For details, see Kipnis and Varadhan (1986, Thm 1.3). This result does require reversibility of the Markov chains to guarantee finite asymptotic variances.

Along this line, Maxwell and Woodroofe (2000) developed an even weaker sufficient condition for the existence of an MA. This key result and its proof are summarized below, which is needed in the other one of our main CLTs, Theorem 9.

For $n \geq 1$, let

$$V_ng = \sum_{k=0}^{n-1} Q^k g.$$
Theorem 3 (Maxwell and Woodroofe (2000)) Given a function \( g \in \mathcal{L}_0^2(\pi) \), if
\[
\sum_{n=1}^{\infty} n^{-3/2} \|V_n g\| < \infty,
\]  
then there is an MA to \( S_n(g) \).

Proof. Since \( Q \) is a contraction, its resolvent for any \( \epsilon > 0 \) is given by
\[
R_\epsilon = ((1 + \epsilon)I - Q)^{-1} = \sum_{k=1}^{\infty} \frac{Q^{k-1}}{(1 + \epsilon)^k}.
\]
Therefore the solution to equation (2.4) exists, and is given by
\[
h_\epsilon = \sum_{k=1}^{\infty} \frac{Q^{k-1}g}{(1 + \epsilon)^k} = \epsilon \sum_{n=1}^{\infty} \frac{V_n g}{(1 + \epsilon)^{n+1}},
\]  
and \( h_\epsilon \in \mathcal{L}_0^2(\pi) \) if \( g \in \mathcal{L}_0^2(\pi) \). On \( \mathcal{X} \times \mathcal{X} \), define the function
\[
H_\epsilon(X_{k-1}, X_k) = h_\epsilon(X_k) - Q h_\epsilon(X_{k-1}).
\]
Further, let \( R_{n,\epsilon} = Q h_\epsilon(X_0) - Q h_\epsilon(X_n) \), and \( M_{n,\epsilon} = \sum_{k=1}^{n} H_\epsilon(X_{k-1}, X_k) \). It is straightforward to show that
\[
S_n(g) = M_{n,\epsilon} + \epsilon S_n(h_\epsilon) + R_{n,\epsilon}.
\]
For each fixed \( \epsilon \), \( M_{n,\epsilon} \) is a martingale because
\[
E(M_{n,\epsilon}|\mathcal{F}_{n-1}) = E(M_{n-1,\epsilon} + H_\epsilon(X_{n-1}, X_n)|\mathcal{F}_{n-1}) = M_{n-1,\epsilon} + E(h_\epsilon(X_n) - Q h_\epsilon(X_{n-1})|\mathcal{F}_{n-1}) = M_{n-1,\epsilon}.
\]
Under (2.5), Maxwell and Woodroofe (2000) proved that

(a) The limit \( H = \lim_{\epsilon \downarrow 0} H_\epsilon \) exists and \( H \in \mathcal{L}^2(\pi_1) \), where \( \pi_1(dx_0, dx_1) = Q(x_0, dx_1)\pi(dx_0) \).

Further, define \( M_n = \sum_{k=1}^{n} H(X_{k-1}, X_k) \). Then, \( \lim_{\epsilon \downarrow 0} M_{n,\epsilon} = M_n \) in \( \mathcal{L}^2(P) \) for each fixed \( n \) and \( (M_n)_{n \geq 1} \) is a martingale adapted to \( (\mathcal{F}_n)_{n \geq 1} \).

(b) \( \lim_{\epsilon \downarrow 0} \epsilon S_n(h_\epsilon) = 0 \) in \( \mathcal{L}^2(P) \).

(c) Let \( R_n = S_n(g) - M_n = M_{n,\epsilon} + \epsilon S_n(h_\epsilon) + R_{n,\epsilon} - M_n \). Then \( E(R_n^2) = o(n) \).

In summary, (2.5) implies \( S_n(g) = M_n + R_n \), which is an MA to \( S_n \).\( \square \)

Define the martingale differences by \( m_n = M_n - M_{n-1} \) for \( n \geq 2 \) and \( m_1 = M_1 \). We next review two different martingale CLTs. First, we present a martingale CLT by Brown (1971). This result was used by Derriennic and Lin (2003) to establish Theorem 8 later in this paper, on which our Theorem 9 is based.
Theorem 4 Let \((M_n)_{n \geq 1}\) be a martingale adapted to \((\mathcal{F}_n)_{n \geq 1}\) with \(E(m_n^2) < \infty\) for every \(n\). Define \(U^2_n = \sum_{k=1}^{n} E(m_k^2 | \mathcal{F}_{k-1})\) and \(\sigma_n^2 = E(U_n^2)\). Suppose that

\[
\lim_{n \to \infty} \frac{U_n^2}{\sigma_n^2} = 1 \text{ a.s.,}
\]

and

\[
\lim_{n \to \infty} \frac{\sigma_n^2}{n} \sum_{k=1}^{n} E(m_k^2 1_{|m_k| > \epsilon \sqrt{n}}) = 0, \text{ for every } \epsilon > 0.
\]

Then,

\[
\frac{M_n}{\sqrt{n}} \Rightarrow N(0, \sigma^2), \text{ as } n \to \infty.
\]

Next, we present an alternative martingale CLT based on Hall et al. (1980, Corollary 3.1), which can also be found in Douc et al. (2018, Corollary E.4.2). This CLT will be used in constructing our Theorem 10.

Theorem 5 Let \((M_n)_{n \geq 1}\) be a martingale adapted to \((\mathcal{F}_n)_{n \geq 1}\) with \(E(m_n^2) < \infty\) for every \(n\). Suppose that

\[
\frac{1}{n} \sum_{k=1}^{n} E(m_k^2 | \mathcal{F}_{k-1}) \xrightarrow{P} \sigma^2, \text{ as } n \to \infty,
\]

and that,

\[
\frac{1}{n} \sum_{k=1}^{n} E(m_k^2 1_{|m_k| > \epsilon \sqrt{n}}) \xrightarrow{P} 0, \text{ as } n \to \infty.
\]

Then,

\[
\frac{M_n}{\sqrt{n}} \Rightarrow N(0, \sigma^2), \text{ as } n \to \infty,
\]

where \(\Rightarrow\) denotes weak convergence of probability measures.

3. Main results

In Section 3.1, we introduce practically checkable conditions on Markov chain convergence rates in terms of Wasserstein distance that lead to (2.3) and (2.5), respectively. Recall these are key conditions in establishing Markov chain CLTs based on MA and martingale CLTs. The resulting two versions of Markov chain CLTs are then stated in Section 3.2.

3.1. A martingale approximation based on convergence rates in Wasserstein distance

Let \((\mathcal{X}, \psi)\) be a Polish metric space. Further, let \(\mathcal{P}(\mathcal{X})\) be the set of probability measures on \((\mathcal{X}, \mathcal{B})\) and \(\delta_x\) the point mass at \(x\). For \(\mu, \nu \in \mathcal{P}(\mathcal{X})\), let

\[
\mathcal{C}(\mu, \nu) = \{v \in \mathcal{P}(\mathcal{X} \times \mathcal{X}) : v(A_1 \times \mathcal{X}) = \mu(A_1), v(\mathcal{X} \times A_2) = \nu(A_2) \text{ for } A_1, A_2 \in \mathcal{B}\}.
\]
Then $C(\mu, \nu)$ is called the set of all couplings of $\mu$ and $\nu$, which contains all the probability measures on $(X \times X, B \times B)$ with marginals $\mu$ and $\nu$. The $L_1$-Wasserstein distance between $\mu$ and $\nu$ is defined to be

$$W_\psi(\mu, \nu) = \inf_{\gamma \in C(\mu, \nu)} \int_{X \times X} \psi(x, y) \gamma(dx, dy). \quad (3.1)$$

Fernique (1981) derived the following dual formulation of the $L_1$-Wasserstein distance,

$$W_\psi(\mu, \nu) = \sup_{g \in \mathcal{G}_\psi} \left| \int_X g(x) \mu(dx) - \int_X g(x) \nu(dx) \right|,$$  \quad (3.2)

where $\mathcal{G}_\psi = \{ g : |g(x) - g(y)| \leq \psi(x, y) \text{ for } x, y \in X \}$. For $p \in \mathbb{Z}_+$, define

$$\mathcal{P}^p_\psi = \left\{ \mu \in \mathcal{P}(X) : \int_X \psi(x_0, x)^p \mu(dx) < \infty \text{ for some } x_0 \in X \right\}. \quad (3.3)$$

Since $(X, \psi)$ is a Polish metric space, $(\mathcal{P}^1_\psi, W_\psi)$ is again a Polish metric space (see, e.g., Villani (2008, Definition 6.1 and Theorem 6.18)), and we call $W_\psi$ the $L_1$-Wasserstein metric on $\mathcal{P}^1_\psi$.

In the rest of Section 3.1, we present practically checkable conditions on Markov chain convergence rates in terms of Wasserstein distance that lead to (2.3) and (2.5). We first define the following assumptions. Note that $A'_2$ is stronger than $A_2$.

$A_1$. $\pi \in \mathcal{P}^2_\psi$.

$A_2$. There exists a rate function $r(n)$ such that

$$W_\psi(Q^n \delta_x, Q^n \delta_y) \leq C r(n) \psi(x, y), \quad \text{for some } C > 0, \text{ any } x, y \in X, \text{ and any } n \geq 0,$$  \quad (3.4)

and

$$\sum_{k=0}^{n-1} r(k) = o(n^{\frac{1}{2}}). \quad (3.5)$$

$A'_2$. There exists a rate function $r(n)$ such that (3.4) holds, and

$$\sum_{k=0}^{\infty} r(k) = O(1). \quad (3.6)$$

**Lemma 1** Assumption $A_1$ implies

$$I = \left[ \int_X \left( \int_X \psi(x, y) \pi(dy) \right)^2 \pi(dx) \right]^{\frac{1}{2}} < \infty.$$ 

**Proof.** By assumption $A_1$, there exists $x_0 \in X$ such that $\int_X \psi(x_0, x)^2 \pi(dx) < \infty$. Then the triangle inequality implies

$$I = \left[ \int_X \left( \int_X \psi(x, y) \pi(dy) \right)^2 \pi(dx) \right]^{\frac{1}{2}} \leq \left[ \int_X \left( \int_X \psi(x, x_0) + \psi(x_0, y) \right)^2 \pi(dx) \right]^{\frac{1}{2}} \psi(x_0, x)^2 \pi(dx) < \infty.$$
Lemma 2 For $g \in G_\psi$, condition (3.4) implies that,
\[ |Q^k g(x) - Q^k g(y)| \leq Cr(k)\psi(x, y), \quad k \geq 0. \]

Proof. For any $k \geq 0$,
\[
|Q^k g(x) - Q^k g(y)| \leq \sup_{h \in G_\psi} |Q^k h(x) - Q^k h(y)|
\]
\[
= \sup_{h \in G_\psi} \left| \int_{X} h(z)Q^k(x, dz) - \int_{X} h(z)Q^k(y, dz) \right|
\]
\[
= W_\psi(Q^k \delta_x, Q^k \delta_y) \leq Cr(k)\psi(x, y).
\]

We now state and prove two key results that eventually lead to our main CLTs, Theorem 9 and 10, respectively.

Theorem 6 If $A_1$ and $A_2$ hold, then (2.5) holds for $g \in G_\psi \cap L^2_0(\pi)$.

Proof. The key term in (2.5) is
\[ ||V_n g|| = \left( \int_X (V_n g(x))^2 \pi(dx) \right)^{\frac{1}{2}}. \]

Note that
\[ E_\pi V_n g(X) = E_\pi \left( \sum_{k=0}^{n-1} Q^k g(X) \right) = \sum_{k=0}^{n-1} E_\pi Q^k g(X). \]

The above expectation is indeed 0 due to the following. For $k = 0$, $E_\pi Q^0 g(X) = 0$ since $g \in L^2_0(\pi)$. For $k = 1$, since $Q$ is $\pi$-invariant,
\[ E_\pi Qg(X) = \int_X Qg(x)\pi(dx) = \int_X \int_X g(y)Q(x, dy)\pi(dx) = \int_X g(y)\pi(dy) = 0. \]
Then by induction, $E_\pi Q^k g(X) = 0$ for all $k \geq 0$. Hence

$$
\|V_n g\| = \left[ \int_X (V_n g(x) - E_\pi V_n g(X))^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
= \left[ \int_X \left( V_n g(x) - \int_X V_n g(y) \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
\leq \left[ \int_X \left( \int_X |V_n g(x) - V_n g(y)| \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
= \left[ \int_X \left( \int_X \sum_{k=0}^{n-1} Q^k g(x) - \sum_{k=0}^{n-1} Q^k g(y) \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
\leq \left[ \int_X \left( \int_X \sum_{k=0}^{n-1} |Q^k g(x) - Q^k g(y)| \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
\leq \left[ \int_X \left( \int_X \sum_{k=0}^{n-1} Cr(k) \psi(x,y) \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2} = CI \sum_{k=0}^{n-1} r(k),
$$

where the last inequality follows from condition (3.4) of $A_2$ and Lemma 2, and $I = \left[ \int_X (\int_X \psi(x,y) \pi(dy))^2 \pi(dx) \right]^\frac{1}{2} < \infty$ due to assumption $A_1$ and Lemma 1. Hence, under (3.5) of assumption $A_2$, (2.5) holds.

**Theorem 7** If $A_1$ and $A'_2$ hold, then (2.3) holds for $g \in \mathcal{G}_\psi \cap \mathcal{L}^2_0(\pi)$.

**Proof.** In the proof of Theorem 6, we already established that $E_\pi Q^n g(X) = 0$ for $n \geq 0$. Hence,

$$
\sum_{n=0}^{\infty} \|Q^n g\| = \sum_{n=0}^{\infty} \left[ \int_X (Q^n g(x))^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
= \sum_{n=0}^{\infty} \left[ \int_X \left( Q^n g(x) - \int_X Q^n g(y) \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
\leq \sum_{n=0}^{\infty} \left[ \int_X \left( \int_X |Q^n g(x) - Q^n g(y)| \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2}
$$

$$
\leq \sum_{n=0}^{\infty} \left[ \int_X \left( \int_X C r(n) \psi(x,y) \pi(dy) \right)^2 \pi(dx) \right]^\frac{1}{2} = CI \sum_{n=0}^{\infty} r(n),
$$
where the last inequality holds due to condition (3.4) of \( A'_2 \) and Lemma 2, and \( I = \left[ \int_X \left( \int_{\mathcal{Y}} \psi(x, y) \pi(dy) \right)^2 \pi(dx) \right]^{\frac{7}{2}} < \infty \) due to assumption \( A_1 \) and Lemma 1.

Finally, we introduce two rate functions that satisfy (3.5) of assumption \( A_2 \), to make (3.5) easier to check in practice.

**Proposition 1** For \( 0 < \rho < 1, \gamma \geq 0.5, \) and \( r(n) = \rho^{\gamma} \), condition (3.5) holds.

**Proof.** It suffices to show \( \sum_{k=1}^{n-1} r(k) = o(n^{\frac{\gamma}{2}}) \) for \( \gamma = 0.5 \), because \( \rho^k \leq \rho^{0.5} \) for any \( \rho \in (0, 1) \) and \( k \geq 0 \). Note that \( r(k) \) is decreasing in \( k \), hence

\[
\int_1^n r(k) dk \leq \sum_{k=1}^{n-1} r(k) = \rho + \sum_{k=2}^{n-1} r(k) \leq \int_1^{n-1} r(k) dk.
\]

In the case of \( \gamma = 0.5 \), we have

\[
\int r(k) dk = \int \rho^{0.5} \frac{dk}{\rho} = \int \rho^{0.5} \frac{du}{0.5} (\text{set } u = k^{0.5})
\]

\[
= 2 \left( \frac{\rho u}{\ln \rho} - \frac{\rho}{\ln^2 \rho} \right) + C = 2 \left( \frac{\rho^{0.5} \frac{k^{0.5}}{0.5} \ln \rho - \rho^{0.5} \frac{k^{0.5}}{0.5} \ln^2 \rho} {\ln \rho} \right) + C,
\]

thus,

\[
\int_1^{n-1} r(k) dk = 2 \left( \frac{\rho^{(n-1)^{0.5}} (n-1)^{0.5}} {\ln \rho} - \frac{\rho^{(n-1)^{0.5}} \ln^{2} \rho} {\ln \rho} \right) - 2 \left( \frac{\rho \ln \rho - \rho \ln^2 \rho} {\ln \rho} \right)
\]

\[
= O(n^{0.5}) \cdot \rho^{(n-1)^{0.5}} = o(n^{0.5}).
\]

Hence \( \sum_{k=1}^{n-1} \rho^k = o(n^{0.5}) \) for any \( \gamma \geq 0.5 \). \( \square \)

**Proposition 2** For any \( \beta > 0.5 \), set \( r(0) = 1 \) and \( r(n) = \frac{1}{n^\beta} \) for \( n \geq 1 \). Then, condition (3.5) holds.

**Proof.** It suffices to show that \( \sum_{k=1}^{n-1} \frac{1}{k^\beta} = o(n^{\frac{1}{2}}) \) for \( 0.5 < \beta < 1 \). Let \( f(k) = \frac{1}{k^\beta} \), which is decreasing in \( k \). We have

\[
\int_1^{n+1} \frac{1}{k^\beta} dk \leq \sum_{k=1}^{n} \frac{1}{k^\beta} \leq 1 + \int_1^{n} \frac{1}{k^\beta} dk,
\]

where

\[
\int_1^{n} \frac{1}{k^\beta} dk = \frac{1}{1-\beta} (n^{1-\beta} - 1).
\]

Hence \( \sum_{k=1}^{n-1} \frac{1}{k^\beta} = o(n^{\frac{1}{2}}) \) for \( \beta > 0.5 \). \( \square \)
3.2. CLTs for ergodic Markov chains

As mentioned earlier, the existence of MA in (2.1), combined with the martingale CLT lead to CLTs for Markov chains. An existing result of such is stated below:

**Theorem 8** (Derriennic and Lin (2003)) Let $\Phi$ be an ergodic Markov chain. For $\pi-$almost every point $x \in X$ and $X_0 = x$, if (2.5) holds, and $g \in \mathcal{L}^2(\pi)$, then $\sigma^2(g) = \lim_{n \to \infty} \frac{1}{n} E_\pi(S_n^2(g))$ exists and is finite and

$$\frac{S_n(g)}{\sqrt{n}} \Rightarrow N(0, \sigma^2(g)), \text{ as } n \to \infty.$$ 

**Remark 3** Suppose further that $\Phi$ is Harris ergodic. If the conclusion of Theorem 8 holds for one initial distribution (including degenerate distributions), then it holds for every initial distribution (Meyn and Tweedie, 1993, Proposition 17.1.6).

Note that condition (2.5) in Theorem 8 is not easy to check directly in practice. It can however be replaced by its sufficient conditions derived in our Theorem 6, as follows.

**Theorem 9** Suppose $\Phi$ is an ergodic Markov chain, for which $A_1$ and $A_2$ hold. For $\pi-$almost every point $x \in X$, and $X_0 = x$, and any $g \in \mathcal{G}_\psi \cap \mathcal{L}^2_0(\pi)$, we have that $\sigma^2(g) = \lim_{n \to \infty} \frac{1}{n} E_\pi(S_n^2(g))$ exists and is finite and

$$\frac{S_n(g)}{\sqrt{n}} \Rightarrow N(0, \sigma^2(g)), \text{ as } n \to \infty.$$ 

We next develop a CLT for Markov chains with a compact state space $X$. Compared to Theorem 9, this CLT applies to Markov chains that start at any point $x \in X$, at the price of imposing slightly more stringent conditions on their convergence rates in Wasserstein distance. Let $\mathcal{C}$ be the class of all continuous functions on $X$. A Markov chain is said to be weak Feller if $Q_c \in \mathcal{C}$ for any $c \in \mathcal{C}$.

**Lemma 3** (Breiman (1960)) Suppose $Q$ is a weak Feller Markov transition function on a compact state space $X$ that allows a unique invariant distribution $\pi$. Then for the corresponding Markov chain $\Phi$ with any starting point $X_0 = x \in X$ and any $c \in \mathcal{C}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} c(X_k) = E_\pi c(X), \text{ almost surely.}$$ 

Based on Lemma 3, Theorem 5 and 7, we establish the following CLT.

**Theorem 10** Suppose $Q$ is a Markov transition function on a compact state space $X$ that allows a unique invariant distribution $\pi$. Also assume $Q$ is weak Feller, and that $A_1$ and $A_2'$ hold. Then, for the corresponding Markov chain $\Phi$ from any starting point $X_0 = x \in X$, and $g \in \mathcal{G}_\psi \cap \mathcal{L}^0_0(\pi)$, we have that $\sigma^2(g) = \lim_{n \to \infty} \frac{1}{n} E_\pi(S_n^2(g))$ exists and is finite, and

$$\frac{S_n(g)}{\sqrt{n}} \Rightarrow N(0, \sigma^2(g)), \text{ as } n \to \infty.$$
Proof. For any $g \in G \cap \mathcal{L}_1^2(\pi)$, assumptions $A_1$ and $A'_2$ imply (2.3) according to Theorem 7, which further implies the existence of $h \in \mathcal{L}^2(\pi)$ that solves Poisson’s equation (2.2) (Butzer and Westphal, 1971). Hence there exists a martingale approximation to $S_n(g)$ (Gordin and Lifšic, 1978) in the sense that

$$S_n(g) = M_n + R_n,$$

where $M_n = \sum_{k=1}^{n} m_k$ with $m_k = h(X_k) - Qh(X_{k-1}) = g(X_{k-1}), k \geq 1$ being the martingale differences and $R_n = Qh(X_0) - Qh(X_n)$.

Next, we show the asymptotic normality of $\frac{S_n(g)}{\sqrt{n}}$ through Theorem 5, the conditions of which are checked below. First, since $X$ is compact and $g \in G \psi$, the value of $g$ is bounded on $X$. Hence, $E(m_n^2) = E(g^2(X_{n-1})) < \infty$. Also, for any $\epsilon > 0$, there exists $n$ such that $m_k = g(X_{k-1}) \leq \epsilon \sqrt{n}$ for all $k$, which implies (2.8). Next, $g \in C$ leads to $c = Qg^2 \in C$, which then implies (2.7) by Lemma 3. Hence Theorem 5 applies. Combining all the above, we have

$$\frac{S_n(g)}{\sqrt{n}} = \frac{M_n}{\sqrt{n}} + \frac{R_n}{\sqrt{n}} = \frac{M_n}{\sqrt{n}} + o_p(1) \Rightarrow N(0, \sigma^2), \text{ as } n \to \infty,$$

where $\sigma^2 = E_\pi c(X) = E_\pi Qg^2(X) = E_\pi g^2$.

4. Applications

4.1. CLT for an independence sampler

In this section, we apply our Theorem 9 to establish CLT for Markov chains generated by a class of independence samplers, which belong to the family of Metropolis-Hastings algorithms that is popular for MCMC simulations. The goal of the independence sampler is to generate a Markov chain with invariant distribution $\pi$. Given $X_n, n \geq 0$, the independence sampler proposes $Y_{n+1}$ from a probability distribution with density $f$, and is accepted as the next state $X_{n+1}$ with probability

$$\alpha(X_n, Y_{n+1}) = \min \left( 1, \frac{\pi(Y_{n+1}) f(X_n)}{\pi(X_n) f(Y_{n+1})} \right).$$

Otherwise, $X_{n+1}$ is set to the same value as $X_n$.

We consider the following example of independence sampler from Jarner and Roberts (2002):

[IS] Consider the independence sampler where the state space $X = [0, 1]$, the invariant distribution $\pi$ is the Uniform(0, 1) distribution, and the proposal density is $f(x) = (r + 1)x^r$ for some $r > 0$.

We will show that the Markov chain corresponding to [IS] does not have GE under TV distance, hence classical results such as Chan and Geyer (1994, Thm 2) can not be used.
to establish CLT. However, our Theorem 9 based on convergence rates in Wasserstein distance can be used to show that CLT holds for all bounded functions.

Define \( w(x) = \frac{\pi(x)}{f(x)} \). An independence sampler has GE in TV distance if and only if

\[
\operatorname{esssup}_{x \in \mathcal{X}} w(x) < \infty \quad \text{(Tierney, 1994; Roberts and Rosenthal, 2011; Mengersen and Tweedie, 1996).}
\]

For the independence sampler in [IS], \( w(x) = \frac{1}{(r+1)^r} \) and \( \operatorname{esssup}_{x \in [0,1]} w(x) = \infty \), hence the sampler does not have GE.

We next establish CLT for the independence sampler in [IS] using Theorem 9, which requires verifying \( A_1 \), \( A_2 \) and ergodicity of the Markov chain. Let \( \psi \) be the discrete metric in this section. Then \( G_\psi \), the set of all Lipschitz functions, is the same as the set of all bounded functions, and the Wasserstein distance \( W_\psi(Q^n(x,\cdot),\pi) \) reduces to the total variation distance, \( d_{TV}(Q^n(x,\cdot),\pi) \). First, \( A_1 \) holds because for any \( x_0 \in \mathcal{X} \),

\[
\int_{\mathcal{X}} \psi(x_0,x)^2 \pi(dx) \leq \int_{\mathcal{X}} \pi(dx) = 1.
\]

Next, to establish \( A_2 \), we use results from Jarner and Roberts (2002). For a function \( V \geq 1 \), define the distance \( d_V(\mu,\nu) \) between two probability measures on \( (\mathcal{X},\mathcal{B}) \) by

\[
d_V(\mu,\nu) = \sup_{|g| \leq V} \left| \int_{\mathcal{X}} g(x) \mu(dx) - \int_{\mathcal{X}} g(x) \nu(dx) \right|.
\]

When \( V = 1 \), the distance \( d_V \) becomes the total variation distance. Set \( r < s < r+1 \), \( 1 \leq \beta \leq \frac{s}{r} \) and \( V_\beta(x) = x^{-\beta\frac{s-1}{s}} \). By Theorem 3.6 and Proposition 5.2 from Jarner and Roberts (2002), for any \( 1 \leq \beta \leq \frac{s}{r} \), we have

\[
(n+1)^{-\beta-1} d_{V_\beta}(Q^n(x,\cdot),\pi) \to 0, \quad \text{as} \quad n \to \infty.
\]

It follows that

\[
d_{TV}(Q^n(x,\cdot),\pi) \leq d_{V_\beta}(Q^n(x,\cdot),\pi) \leq C \frac{1}{n^{\beta-1}},
\]

where \( C \) is a positive constant. Thus, by the triangular inequality,

\[
d_{TV}(Q^n(Q^n(x,\cdot),\pi),Q^n(y,\cdot)) \leq 2C \frac{1}{n^{\beta-1}}.
\]

By Proposition 2, \( A_2 \) holds only if \( \beta > 1.5 \). Finally, \( A_2 \) implies that \( \pi \) is the unique limiting distribution, and hence the unique invariant distribution for \( Q \). So the chain is ergodic by Theorem 1. After all, Theorem 9 applies, and CLT holds for all bounded functions for the independence sampler defined in [IS].

### 4.2. CLT for an unadjusted Langevin algorithm

In this section, we study how to use Wasserstein distance based Theorem 9 to establish CLT for unadjusted Langevin algorithms (ULA). Following some general results, a
specific example is carried out concerning the computing of Bayesian logistic regression
models.

Let \( \pi(x) = c e^{-U(x)} \) denote the density of a probability distribution of interest on \( \mathbb{R}^p \), where \( U \) is a known function and \( c \) is a possibly unknown normalizing constant. A
general ULA generates a Markov chain according to

\[
X_{n+1} = X_n - h \nabla U(X_n) + \sqrt{2h} Z_{n+1},
\]

where \( \{Z_n\}_{n \geq 0} \) is an i.i.d. sequence of \( p \)-dimensional standard Gaussian random variables and \( h > 0 \) is a fixed step size. This Markov chain admits a stationary distribution \( \pi_h \),
which is usually different from, but close to \( \pi \) when \( h \) is small. In Durmus and Moulines (2019), convergence behavior of ULA with both constant and decreasing step sizes was
studied under TV distance and \( L_2 \)-Wasserstein distance. Also, explicit dependency of
the convergence bounds on the dimension of the state space was investigated in depth.
Further, Roberts and Tweedie (1996) established standard drift and minorization conditions
that imply GE of the chain in TV distance. When these conditions hold, classical
Markov chain CLTs apply to functions that have finite \((2 + \epsilon)\)-moments.

We will take an alternative route to establish CLT for ULA, based on its convergence
in \( L_1 \)-Wasserstein distance instead of the classical TV distance. Specifically, we resort
to Theorem 9. Note that \( A_2 \) is a key condition of the Theorem, and we first introduce
a Lemma that helps verify \( A_2 \) for Markov chains generated by a general class of ULA.
Consider Markov chain \( \Phi \) with state space \( \mathcal{X} = \mathbb{R}^p \). Let \( \psi(x,y) = \|x - y\| \) be the \( L_2 \)
norm, i.e., the euclidean distance, for \( x, y \in \mathbb{R}^p \).

**Lemma 4** Suppose there exists \( L > 0 \) such that for every \( x, y \in \mathcal{X}, \|\nabla U(x) - \nabla U(y)\| \leq L\|x - y\| \). Further, suppose there exists \( M > 0 \) such that for every \( x, y \in \mathcal{X}, (\nabla U(x) - \nabla U(y))^T (x - y) \geq M\|x - y\|^2 \). Then, for \( 0 < h < \frac{2M}{L^2} \), there exists \( 0 < \gamma < 1 \) such that for any \( x, y \in \mathcal{X} \),
\[
W_\psi(Q^n \delta_x, Q^n \delta_y) \leq \gamma^n \psi(x,y).
\]

**Proof.** Let \( \tilde{\Phi} = \{(X_n, Y_n)\}_{n=0}^\infty \) be a coupled version of the ULA chain \( \Phi \) where \( \{X_n\}_{n=0}^\infty \)
follows (4.1), and

\[
Y_{n+1} = Y_n - h \nabla U(Y_n) + \sqrt{2h} Z_{n+1}, \quad n \geq 0,
\]

with starting value \( (X_0, Y_0) = (x, y) \). Let \( \tilde{K} \) denote the Markov transition kernel of \( \tilde{\Phi} \).
It suffices to show there exists \( 0 < \gamma < 1 \) such that for every \( x, y \in \mathcal{X} \),
\[
\tilde{K} \psi(x,y) \leq \gamma \psi(x,y),
\]

where \( \tilde{K} \psi(x,y) = \int_{\mathcal{X} \times \mathcal{X}} \psi(x', y') \tilde{K} \left( (x,y), (dx', dy') \right) \).
Because (4.2) implies
\[
W_\psi(Q^n \delta_x, Q^n \delta_y) \leq \int_{\mathcal{X} \times \mathcal{X}} \psi(x', y') \tilde{K}^n \left( (x,y), (dx', dy') \right) \leq \gamma^n \psi(x,y).
\]
By the definition of Wasserstein distance, we have
\[
W_\psi(Q \delta_x, Q \delta_y) \leq \tilde{K} \psi(x,y) = E \left( \|X_1 - Y_1\| | X_0 = x, Y_0 = y \right) = \|x-y-h(\nabla U(x) - \nabla U(y))\|.
\]
The right hand side of the above equation is such that
\[
\|x - y - h(\nabla U(x) - \nabla U(y))\|^2 = \|x - y\|^2 - 2h(\nabla U(x) - \nabla U(y))^T(x - y) + h^2\|\nabla U(x) - \nabla U(y)\|^2 \\
\leq (1 + h^2L^2)\|x - y\|^2 - 2h(\nabla U(x) - \nabla U(y))^T(x - y) \\
\leq (1 + h^2L^2 - 2hM)\|x - y\|^2.
\]

Hence, for \(0 < h < \frac{2M}{L^2}\), set \(\gamma = (1 + h^2L^2 - 2hM)^{\frac{1}{2}} < 1\), then the desired inequality (4.2) holds.

**Remark 4** Conditions of Lemma 4 require \(\nabla U\) to be Lipschitz and strongly-convex. They also imply ergodicity of the ULA chain. Specifically, under these conditions, the coupled chain \(\Phi\) can be viewed as a sequence of pointwise contractive iterated random functions. Indeed, for \(0 < h < \frac{2M}{L^2}\), and \(\gamma = (1 + h^2L^2 - 2hM)^{\frac{1}{2}} < 1\), we have
\[
\|X_n - Y_n\| \leq \gamma\|X_{n-1} - Y_{n-1}\|.
\]

Thus, by Diakonis and Freedman (1999, Theorem 1.1), the chain \(\Phi\) has a unique invariant distribution \(\pi\), which implies ergodicity by Theorem 1.

Now, we study a specific example of ULA that is useful for computing Bayesian logistic regression models. For subject \(i\), \(i = 1, \ldots, k\), observed are the binary response \(y_i\) and a \(p\)-dim vector of covariates \(x_i\). Let \(\beta \in \mathbb{R}^p\) denote the regression coefficient. Under the Normal(0, \(G^{-1}\)) prior for \(\beta\), for some user-specified positive definite precision matrix \(G\), the density of the posterior distribution of interest is given by
\[
\pi(\beta \mid \{x_i, y_i\}_{i=1}^k) \propto \exp \left\{ \sum_{i=1}^k \left( y_i \beta^T x_i - \log(1 + e^{\beta^T x_i}) \right) - \frac{\beta^T G \beta}{2} \right\}.
\]

The corresponding ULA Markov chain \(\Phi = \{\beta_n\}_{n \geq 0}\) is generated by
\[
\beta_{n+1} = \beta_n - h\nabla U(\beta_n) + \sqrt{2h}Z_{n+1}, \quad \text{(4.3)}
\]
where
\[
\nabla U(\beta) = G\beta + \sum_{i=1}^n \left[ \frac{1}{1 + \exp(-x_i^T \beta)} - y_i \right] x_i.
\]

For the rest of this section, we establish CLT for the set of all Lipschitz functions of the ULA in (4.3), \(\Phi = \{\beta_n\}_{n \geq 0}\), solely based on convergence rates in Wasserstein distance \(W_\psi\). Specifically, we show that Theorem 9 is applicable by verifying assumptions \(A_1, A_2\) and ergodicity of \(\Phi\).

First, \(A_1\) holds. This is because
\[
\pi(\beta \mid \{x_i, y_i\}_{i=1}^k) = C \prod_{i=1}^k \left[ \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} \right]^{y_i} \left[ \frac{1}{1 + e^{\beta^T x_i}} \right]^{1-y_i} \exp \left( -\frac{\beta^T G \beta}{2} \right) \\
\leq C \exp \left( -\frac{\beta^T G \beta}{2} \right),
\]
for some constant $C$, hence, for any $\beta_0 \in \mathbb{R}^p$,
\[
\int_{\mathbb{R}^p} \|\beta - \beta_0\|^2 \pi(\beta) \{ (x_i, y_i) | i = 1 \} \ d\beta \leq C \int_{\mathbb{R}^p} \|\beta - \beta_0\|^2 \exp \left( -\frac{\beta^T G \beta}{2} \right) \ d\beta < \infty.
\]

Next, we establish $A_2$ using Lemma 4. It suffices to verify the two conditions of this Lemma. We first show that $\nabla U(\beta)$ is Lipschitz. Note that
\[
\nabla^2 U(\beta) = \frac{G}{2} + \sum_{i=1}^{n} \left[ \frac{d}{d\beta} \left( \frac{1}{1 + \exp(-x_i^T \beta)} \right) \right] x_i
\]
\[
= \frac{G}{2} + \sum_{i=1}^{n} \left[ x_i \frac{\exp(-x_i^T \beta)}{[1 + \exp(-x_i^T \beta)]^2} x_i^T \right]
\]
\[
= \frac{G}{2} + X^T A X,
\]
where $X^T = [x_1, \ldots, x_n]$ and $A$ is a diagonal matrix with $A_{ii} = \frac{\exp(-x_i^T \beta)}{[1 + \exp(-x_i^T \beta)]^2}$. Further,
\[
\frac{\exp(-x_i^T \beta)}{[1 + \exp(-x_i^T \beta)]^2} = \frac{1}{1 + \exp(-x_i^T \beta)} \left( 1 - \frac{\exp(-x_i^T \beta)}{1 + \exp(-x_i^T \beta)} \right) \leq \frac{1}{4}.
\]
Hence, by the mean value inequality for vector-valued functions, we have
\[
\|\nabla U(\beta) - \nabla U(\alpha)\| \leq \sup_{\beta^*} \|\nabla^2 U(\beta^*)\| \leq \left( \frac{\lambda_{\max}(G)}{2} + \frac{\lambda_{\max}(X^T X)}{4} \right) \|\beta - \alpha\|,
\]
where $\|\nabla^2 U(\beta^*)\|$ is the norm of the $p \times p$ matrix $\nabla^2 U(\beta^*)$, and $\lambda_{\max}(G)$ is the largest eigenvalue of the positive definite matrix $G$. Next, we verify the second condition of Lemma 4 that concerns the strong convexity of $\nabla U(\beta)$. Note that
\[
(\nabla U(\beta) - \nabla U(\alpha))^T (\beta - \alpha)
\]
\[
= \frac{(\beta - \alpha)^T G (\beta - \alpha)}{2} + \sum_{i=1}^{k} \left[ \frac{\exp(-x_i^T \alpha) - \exp(-x_i^T \beta)}{[1 + \exp(-x_i^T \beta)][1 + \exp(-x_i^T \alpha)]} \right] x_i^T (\beta - \alpha)
\]
\[
\geq \frac{(\beta - \alpha)^T G (\beta - \alpha)}{2} \geq \frac{\lambda_{\min}(G) \|\beta - \alpha\|^2}{2},
\]
where $\lambda_{\min}(G)$ denotes the smallest eigenvalue of $G$. Hence, we’ve shown that the two conditions in Lemma 4 hold with $L = \frac{\lambda_{\max}(G)}{2} + \frac{\lambda_{\max}(X^T X)}{4} > 0$, and $M = \frac{\lambda_{\min}(G)}{2} > 0$, respectively. This further implies $A_2$ and ergodicity of the chain by Remark 4. After all, CLT holds for the ULA in (4.3), $\Phi = \{\beta_n\}_{n \geq 0}$, for any Lipschitz function.
4.3. CLT for a reducible Markov chain

In this section, we use Theorem 10 to derive CLTs for a class of reducible Markov chains induced by a family of AR(1) models. Consider the following Markov chain $\Phi = \{X_n\}_{n \geq 0}$ on the state space $\mathcal{X} = [0, 1]$ with an arbitrary starting point $X_0 = x, x \in \mathcal{X}$. Given $X_n$ for $n \geq 0$, $X_{n+1}$ is generated by

$$X_{n+1} = aX_n + (1 - a)\theta_{n+1},$$

where $0 < a < 1$ and $\theta_n \sim \text{Bernoulli}(\frac{1}{2})$. Several interesting models are special cases of (4.4). For $a = \frac{1}{2}$, (4.4) is called the Bernoulli shift model, and the unique invariant distribution of $\Phi$ is Uniform($0, 1$). For $a = \frac{1}{3}$, the unique invariant distribution of $\Phi$ is the Cantor distribution.

In general, for any $0 < a < 1$ in (4.4), $\Phi$ has a unique invariant distribution (Solomyak, 1995; Jessen and Wintner, 1935), which we denote by $\pi$. It’s also known that $\Phi$ is neither $\pi$-irreducible nor strongly mixing (Andrews, 1984; Wu and Shao, 2004). Since $\pi$-irreducibility is a necessary condition for convergence in TV distance (Nummelin, 1984), CLTs that require any kind of convergence of $\Phi$ in TV distance are inapplicable. Instead, the convergence behavior of $\Phi$ can still be described under a Wasserstein distance. We derive below a CLT for $\Phi$ using Theorem 10.

Applying Theorem 10 requires us to verify assumptions $A_1$, $A_2'$, the ergodicity and the weak Feller property of $\Phi = \{X_n\}_{n \geq 0}$, which we do next.

First, $\Phi$ is weak Feller because for any continuous function $c(\cdot)$ on $\mathcal{X}$, $Qc(x) = E(c(X_1)|X_0 = x) = \frac{1}{2} [c(ax) + c(ax + (1 - a))]$ is also continuous on $\mathcal{X}$. Next, we verify the ergodicity of $\Phi$. Let $\psi(x, y) = |x - y|$ denote the euclidean distance for $x, y \in \mathbb{R}$. Define $\tilde{\Phi} = \{(X_n, Y_n)\}_{n=0}^{\infty}$ to be a coupled version of $\Phi$ such that $X_n$ follows (4.4), and

$$Y_{n+1} = aY_n + (1 - a)\theta_{n+1},$$

where $X_0 = x$ and $Y_0 = y$, for $x, y \in \mathcal{X}$. Let $K$ denote the kernel of $\tilde{\Phi}$. Note that the coupled chain $\tilde{\Phi}$ is a sequence of pointwise contractive iterated random functions in the sense that

$$|X_{n+1} - Y_{n+1}| = a|X_n - Y_n|.$$  

Thus, by Theorem 1.1 from Diaconis and Freedman (1999), $\Phi$ has a unique stationary distribution $\pi$, which implies its ergodicity by Theorem 1. Next, assumption $A_1$ follows from the compactness of $\mathcal{X}$:

$$\int_{\mathcal{X}} |x - x_0|^2 \pi(dx) \leq \int_{\mathcal{X}} \pi(dx) < \infty.$$  

Finally, to check assumption $A_2'$, note that

$$W_\psi(Q^m_\delta_x, Q^m_\delta_y) \leq K\psi(x, y) = E\left(|X_1 - Y_1| \big| X_0 = x, Y_0 = y\right) = a|x - y|,$$

which implies that

$$W_\psi(Q^n_\delta_x, Q^n_\delta_y) \leq a^n|x - y|.$$
Thus, $A'_2$ holds as $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. After all, for any Markov chain $\Phi$ defined by (4.4) with an arbitrary starting point, Theorem 10 applies, and asymptotic normality holds for all Lipschitz functions for $\Phi$.

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