FEYNMAN DIAGRAMS AND THE KDV HIERARCHY

DOMENICO FIORENZA

Abstract. The generating series of the intersection numbers of the stable cohomology classes on moduli spaces of curves satisfies the string equation and a KdV hierarchy. Kontsevich’s original proof of this result uses a matrix model and the matrix Airy equation. Witten then recasted Kontsevich’s results in terms of Virasoro algebras, by means of an ingenious mixture of Feynman diagrams techniques and integrations by parts with some “rather formidable choice” of the integrands. In this note we show how Witten’s formidable choices can be bartered for standard Feynman diagram manipulations, as soon as one suitably enlarges the class of Feynman diagrams occurring in the proof.

Introduction

A classical result of Kontsevich-Witten states that the generating series of the intersection numbers of the stable cohomology classes on moduli spaces of curves satisfies the string equation and a KdV hierarchy (see [Arb02] for a very comprehensive introduction to the subject). Kontsevich’s original proof [Kon92] uses a matrix model, now called the ’t Hooft-Kontsevich model, and the matrix Airy equation. Witten [Wit92] recasted Kontsevich’s results in terms of Virasoro algebras. His proof is an ingenious mixture of Feynman diagrams techniques and integrations by parts with some “rather formidable choice” of the integrands. The aim of this note is to show how the techniques developed in [FM03] allow to straightforwardly rewrite Witten’s proof entirely in terms of Feynman diagrams, trading the formidable choices of the integrands for standard Feynman diagram manipulations. The trick used to accomplish this is to use a wider class of Feynman diagrams than the one considered by Witten in [Wit92]. Namely, the class of Feynman diagrams with $n$-valent vertices labelled by polynomials in $n$ variables, as in [FM03].

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\[ 1 \text{ For instance, to prove the last equation in his paper, Witten uses the familiar fact that} \]
\[ \int dY \frac{\partial}{\partial Y_{ij}} \left( M_{ij} \exp \frac{1}{2} tr \Theta Y^2 - \frac{1}{6} tr Y^3 \right) = 0, \text{ where } Y \text{ is an } N \times N \text{ anti-Hermitian matrix, } \Theta \text{ is a negative definite symmetric } N \times N \text{ real matrix and } M \text{ is any polynomial in } Y \text{ and } \Theta. \text{ Then he makes the rather formidable choice} \]
\[ M = \Theta^4 Y - \Theta^3 Y \Theta + \frac{1}{2} \Theta^2 Y \Theta^2 + \Theta^3 Y^2 - \Theta^2 Y \Theta Y + \frac{3}{2} \Theta Y \Theta^2 Y - \frac{1}{4} \Theta Y \Theta Y \Theta - \Theta^2 Y^2 \Theta \]
\[ + \Theta^3 Y^2 - \Theta Y \Theta Y^2 - \frac{3}{2} \Theta Y^2 \Theta Y + \frac{9}{8} \Theta Y \Theta Y Y + \frac{1}{2} \Theta Y^2 Y^2 + \frac{3}{2} 2 \Theta Y^3 - \frac{11}{8} Y \Theta Y^3 \]
\[ - \frac{1}{16} Y^2 \Theta Y^2 + \frac{1}{32} Y^3 - \frac{N}{8} Y^2 + \frac{5N}{2} Y \Theta + \frac{3N}{2} \Theta^2 - 2 \Theta Y^2 - \frac{1}{16} tr Y^2 - \frac{3}{8} tr \Theta Y \]
\[ + \frac{1}{4} Y \Theta Y^2 - \frac{1}{2} Y \Theta Y + \frac{1}{2} \Theta Y \Theta. \]
The paper is organized as follows

1. Intersection numbers on the moduli space of curves
2. The KdV hierarchy and Virasoro operators
3. The idea of the proof
4. The ‘t Hooft-Kontsevich matrix model
5. Witten’s formula for derivatives
6. Proof of equation (I)
7. Proof of equation (II)
Appendix: Formal differential operators

1. Intersection numbers on the moduli space of curves

For fixed integers \( g \geq 0 \) and \( n \geq 1 \) with \( 2 - 2g - n < 0 \), let \( M_{g,n} \) be the moduli space of smooth complete curves of genus \( g \) with \( n \) marked points and \( \overline{M}_{g,n} \) be its Deligne-Mumford compactification [DM69] (see [ACG11] for an overview). Denote by \( \psi_i \in H^2(M_{g,n}, \mathbb{C}) \) the Miller-Witten cohomology classes [Mil86, Wit91], with \( i = 1, \ldots, n \), and by \( \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} \) the intersection number [Wit91]

\[
\langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} := \int_{M_{g,n}} \psi_{\nu_1} \cdots \psi_{\nu_n}.
\]

The generating series of intersection numbers ("free energy functional" in physics literature) is the formal series in the variables \( t_0, t_1, \ldots \)

\[
F(t_*) = \sum_{g,n} F_{g,n}(t_*) := \sum_{g,n} \left( \frac{1}{n!} \sum_{\nu_1, \ldots, \nu_n} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} t_{\nu_1} \cdots t_{\nu_n} \right).
\]

The exponential of the generating series is called the partition function of the intersection numbers and it is denoted by the symbol \( Z(t_*) \).

It is well-known that the moduli spaces of stable curve have an ideal orbi-cellularization whose cells are indexed by isomorphism classes of closed connected ribbon graphs with numbered holes, see, e.g., [Har88, Kon92, Mon04, Mon09]. As a consequence of this fact, any integral on a moduli space of stable curves can be written as a sum over isomorphism classes of numbered ribbon graphs. In particular, Kontsevich finds the following remarkable identity (Kontsevich’s Main Identity):

\[
\sum_{\nu_1, \ldots, \nu_n} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{(2\nu_i - 1)!!}{\lambda_i^{2\nu_i + 1}} =
\]

\[
= \sum_{(\Gamma, h)} \frac{1}{|\text{Aut}(\Gamma, h)|} \left( \frac{1}{2} \right)^{|\text{Vertices}(\Gamma)|} \prod_{l \in \text{Edges}(\Gamma)} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}}
\]

where: the \( \lambda_i \) are positive real variables; \((\Gamma, h)\) ranges over the set of isomorphism classes of closed connected numbered ribbon graphs of genus \( g \) with \( n \) holes; for any edge \( l \) of \( \Gamma \), \( l^+, l^- \) denote the (not necessarily distinct) holes \( l \) belongs to.

2. The KdV hierarchy and Virasoro operators

Let \( \mathbb{C}[u, \partial_u/\partial_0, \ldots] \) be the differential algebra generated by the variable \( u \). The Gel'fand-Dikii polynomials are the differential polynomials \( R_i(u) \) defined by the...
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Recursion

\[
\frac{\partial R_{n+1}(u)}{\partial t_0} = \frac{1}{2n+1} \left( \frac{\partial u}{\partial t_0} + 2u \frac{\partial}{\partial t_0} + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} \right) R_n(u),
\]

with initial datum \( R_0(u) = 1 \) and boundary condition \( R_n(0) = 0 \) for \( n > 0 \).

The Korteweg-de Vries (KdV for short) hierarchy is the following hierarchy of differential equations for an element \( U \) of \( \mathbb{C}[[t_*]] \):

\[
\frac{\partial U}{\partial t_i} = \frac{\partial}{\partial t_0} R_{i+1}(U).
\]

The first equation of the KdV hierarchy is

\[
(2.1) \quad \frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}.
\]

If we set \( \varphi(x,t) := U \frac{x}{\sqrt{2}}, 3\sqrt{2} t, 0, 0, \ldots \), equation (2.1) becomes

\[
\varphi_t = 6 \varphi \varphi_x + \varphi_{xxx},
\]

which is the classical KdV equation (see [Arb02] or [Arn97]).

We are now ready to state the result we are going to prove in this paper.

**Theorem** (Kontsevich-Witten). Let \( F(t_*) \) be the generating series of intersection numbers on moduli spaces of curves. Then

1. the series \( F \) satisfies the string equation

\[
\frac{\partial F}{\partial t_0} = \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i} + \frac{t_0}{2} t_{i+1}^2;
\]

2. the series \( U = \partial^2 F/\partial t_0^2 \) satisfies the KdV hierarchy.

In [Wit92] Witten showed that this theorem can be recasted by means of Virasoro operators as follows. For any \( \rho \in \mathbb{Z} + 1/2 \), define a differential operator \( \alpha_\rho \), by

\[
\alpha_\rho = \begin{cases} 
\frac{(2\rho!!)}{\sqrt{2}} \frac{\partial}{\partial t_{\rho+\frac{1}{2}}} & \text{if } \rho > 0 \\
\frac{1}{(-2\rho - 2)!!\sqrt{2}} \left( t_{\rho+\frac{1}{2}} - \delta_{\rho+\frac{1}{2},0} \right) & \text{if } \rho < 0
\end{cases}
\]

The operators \( \alpha_\rho \) satisfy the commutation relation

\[
[\alpha_{\rho_1}, \alpha_{\rho_2}] = \rho_1 \delta_{\rho_1+\rho_2,0}.
\]

This implies that the formal differential operators

\[
L_n = \begin{cases} 
\frac{1}{2} \sum_{\rho} \alpha_\rho \alpha_{n-\rho}, & n \neq 0 \\
\sum_{\rho>0} \alpha_{-\rho} \alpha_\rho + \frac{1}{16}, & n = 0
\end{cases}
\]

realize a Virasoro algebra with central charge \((c^3 - c)/12\):\n
\[
(2.2) \quad [L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12}.
\]

Recasted by means of the \( L_n \)'s, the Kontsevich-Witten theorem becomes
Theorem. The partition function \( Z(t_*) \) is a zero vector for the Virasoro operators \( L_n, n \geq -1 \).

By the commutation relations (2.2), to prove the Kontsevich-Witten’s theorem one only needs to check \( L_n Z = 0 \) for \( n = -1, 2 \). Written out explicitly, these two equations are:

\[(KW1) \quad \frac{\partial Z}{\partial t_0} = \sum_{i=0}^{\infty} t_{i+1} \frac{\partial Z}{\partial t_i} + \frac{t_0^2}{2} Z \]

\[(KW2) \quad \frac{\partial Z}{\partial t_3} = \frac{1}{7!!} \left( \sum_{i=0}^{\infty} (2i+5)(2i+3)(2i+1) t_i \frac{\partial Z}{\partial t_{i+2}} + 3 \frac{\partial^2 Z}{\partial t_0 \partial t_1} \right). \]

3. The idea of the proof

As a consequence of its Main Identity among the intersection numbers \( \langle \tau_{\nu_1} \cdots \tau_{\nu_n} \rangle \), Kontsevich proves that the partition function \( Z(t_*) \) is related to the asymptotic expansion of the Hermitian matrix integral

\[(3.1) \quad \int_{H(N)} \exp \left\{ \sqrt{-1} \frac{1}{6} \text{tr} X^3 \right\} d\mu_\Lambda(X), \]

where \( \Lambda = \text{diag}\{\Lambda_1, \ldots, \Lambda_N\} \) is a positive definite Hermitian matrix and \( d\mu_\Lambda(X) \) is the Gaussian measure defined by the inner product

\[(X|Y)_\Lambda := \frac{1}{2} (\text{tr}(XAY) + \text{tr}(YAX)) \]

on the space \( H(N) \) of \( N \times N \) Hermitian matrices. More precisely, let, for any \( k \geq 0 \),

\[t_k(\Lambda) = -(2k+1)!! \text{tr} \Lambda^{-(2k+1)}. \]

Then

\[(3.2) \quad Z(t_*) \Big|_{t_*(\Lambda)} \approx \int_{H(N)} \exp \left\{ \sqrt{-1} \frac{1}{6} \text{tr} X^3 \right\} d\mu_\Lambda(X) \]

as \( |\Lambda| \to \infty \). When \( N \to \infty \), the \( \{t_k(\Lambda)\} \) become independent coordinates (Miwa coordinates) on the space \( H(N)/U(N) \), so, the Kontsevich-Witten’s theorem is equivalent to

\[L_n Z|_{t_*(\Lambda)} = 0, \quad n = -1, 2, \]

i.e., it is reduced to a statement concerning the Hermitian matrix integral (3.1). One of the big advantages of this translation of the original problem in a problem concerning Hermitian matrices is that the differential operators

\[\sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} \quad \text{and} \quad \sum_{i=0}^{\infty} (2i+5)(2i+3)(2i+1) t_i \frac{\partial}{\partial t_{i+2}} \]

appearing in equations (KW1-KW2) are very natural in terms of the Miwa coordinates. Indeed they correspond to the action of the operators \( \text{tr} \Lambda^{-1} \partial / \partial \Lambda \) and
trΛ⁵∂/∂Λ on a function of the t∗(Λ). More precisely, we have

\begin{align}
(KW1') \quad \text{tr} \Lambda^{-1} \frac{\partial}{\partial \Lambda} Z(t_*(\Lambda)) &= - \left( \sum_{i=0}^{\infty} t_{i+1} \frac{\partial Z}{\partial t_i} \right)_{t_*(\Lambda)} \\
&= \left( - \sum_{i=0}^{\infty} (2i+5)(2i+3)(2i+1)t_i \frac{\partial Z}{\partial t_{i+2}} + 3 \text{tr} \Lambda \frac{\partial Z}{\partial t_1} + \text{tr} \Lambda^3 \left( \frac{\partial Z}{\partial t_0} \right) \right)_{t_*(\Lambda)}
\end{align}

\begin{align}
(KW2') \quad \text{tr} \Lambda^5 \frac{\partial}{\partial \Lambda} Z(t_*(\Lambda)) &= \left( - \sum_{i=0}^{\infty} (2i+5)(2i+3)(2i+1)t_i \frac{\partial Z}{\partial t_{i+2}} + 3 \text{tr} \Lambda \frac{\partial Z}{\partial t_1} + \text{tr} \Lambda^3 \left( \frac{\partial Z}{\partial t_0} \right) \right)_{t_*(\Lambda)}
\end{align}

Put together equations (KW1,KW2) and equations (KW1',KW2') to obtain that the Kontsevich-Witten’s theorem is equivalent to the following two equations:

\begin{align}
(\text{I}) \quad \frac{\partial Z}{\partial t_0} \bigg|_{t_*(\Lambda)} &= - \text{tr} \Lambda^{-1} \frac{\partial}{\partial \Lambda} Z(t_*(\Lambda)) + \frac{(\text{tr} \Lambda^{-1})^2}{2} Z(t_*(\Lambda)) \\
(\text{II}) \quad \left( -105 \frac{\partial Z}{\partial t_3} + 3 \text{tr} \Lambda \frac{\partial Z}{\partial t_1} + \text{tr} \Lambda^3 \left( \frac{\partial Z}{\partial t_0} \right) \right)_{t_*(\Lambda)} &= \text{tr} \Lambda^5 \frac{\partial}{\partial \Lambda} Z(t_*(\Lambda))
\end{align}

Having written \( Z(t_*(\Lambda)) \) as an Hermitian matrix integral it is immediate to compute that also (\( \text{tr} \Lambda^{-1} \partial/\partial \Lambda \) \( Z(t_*(\Lambda)) \)) and (\( \text{tr} \Lambda^5 \partial/\partial \Lambda \) \( Z(t_*(\Lambda)) \)) are Hermitian matrix integrals. Moreover, one can show that also the derivatives of the partition function \( Z(t_*) \) with respect to the \( t_* \) variables (evaluated at \( t_*(\Lambda) \)) can be expressed as Hermitian matrix integrals. The proof of equations (I) is therefore reduced to checking two identities between Hermitian matrix integrals.

Such a check is performed by Witten [Wit92] by a clever mixture of Feynman diagrams techniques and tricky integration by parts. We are going to show how introducing suitable Feynman diagrams with vertices decorated by polynomials as in [PM03] reduces everything to checking a few graphical identities.

The rest of this paper is organized as follows: we first recall some general notation and result concerning the relation between Feynman diagrams and Gaussian integrals. Next we introduce the Feynman rules for the \'t Hooft-Kontsevich model and rewrite the right-hand sides of equations (I) as expectation values of suitable Feynman diagrams. We will then discuss Witten’s formula for the derivatives of the partition function \( Z(t_*) \), which allows to write also the left-hand sides of equations (I) as expectation values of suitable Feynman diagrams (with vertices decorated by polynomials). Having done this the proof of the theorem is reduced

\[ \frac{\partial Z}{\partial t_0} \bigg|_{t_*(\Lambda)} = - \text{tr} \Lambda^{-1} \frac{\partial}{\partial \Lambda} Z(t_*(\Lambda)) + \frac{(\text{tr} \Lambda^{-1})^2}{2} Z(t_*(\Lambda)) \]

\[ \left( -105 \frac{\partial Z}{\partial t_3} + 3 \text{tr} \Lambda \frac{\partial Z}{\partial t_1} + \text{tr} \Lambda^3 \left( \frac{\partial Z}{\partial t_0} \right) \right)_{t_*(\Lambda)} = \text{tr} \Lambda^5 \frac{\partial}{\partial \Lambda} Z(t_*(\Lambda)) \]

\[ \text{tr} \Lambda^{2k+1} \frac{\partial}{\partial \Lambda} t_*^{(\Lambda)} = - \frac{(2i+1)!!}{(2i-2k-1)!!} t_{*-k}^{(\Lambda)}, \]

for any \( k \geq 1 \) and any \( i \geq k \).

In [Wit92] Witten shows that for any \( D \) in \{\( \partial/\partial t_0, \partial/\partial t_1, \partial/\partial t_2, \partial/\partial t_3, \partial^2/\partial t_0^2, \partial^2/\partial t_0 \partial t_1 \}\), there exist a polynomial \( P_D \) in the odd traces of \( X \) such that

\[ DZ(t_*) \bigg|_{t_*(\Lambda)} = \int_{\mu_A(N)} P_D(X) \cdot \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr} X^3 \right\} d\mu_A(X), \]

and conjectures that this should be true for any differential operator in the variables \( t_* \). The Witten conjecture has been proved by Di Francesco, Itzykson and Zuber in [DFIZ93] and is henceforth known as the DFIZ theorem. We proved a generalization of the DFIZ theorem in [AM03]. See [AC96, Bin02] for the use of the DFIZ theorem in the investigation of the geometry of moduli spaces of pointed curves.
to checking two identities among expectation values of Feynman diagrams: this is a straightforward (but quite long and tedious) computation.

4. THE ’’T HOOFT-KONTSEVICH MATRIX MODEL

Here we will briefly recall the Feynman rules\footnote{Details on these Feynman rules can be found in \cite{FM03}; we refer the reader to \cite{FM02} and references therein for an introduction to the general theory of Feynman diagrams and their relations with the asymptotic expansion of Gaussian integrals.} for the Gaussian integral

\begin{equation}
\int_{\mathcal{H}(N)} \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr} X^3 \right\} d\mu_\Lambda(X)
\end{equation}

where $d\mu_\Lambda(X)$ is the Gaussian measure on $\mathcal{H}(N)$ induced by the inner product $(X|Y)_\Lambda = \frac{1}{2} (\text{tr} X \Lambda Y + \text{tr} Y \Lambda X)$, i.e.,

$$d\mu_\Lambda(X) := \frac{dX}{\int_{\mathcal{H}(N)} \exp \left\{ -\frac{1}{2} \text{tr} \Lambda X^2 \right\} dX}.$$

The space of fields is the complexification of the space $\mathcal{H}(N)$ of $N \times N$ Hermitian matrices, so it is naturally isomorphic to the space $M_N(\mathbb{C})$ of $N \times N$ complex matrices. We will denote by $\{E_{ij}\}$ the canonical basis of $M_N(\mathbb{C})$ and will write the indices $ij$ near to a leg in a Feynman diagram to denote evaluation or coevaluation at the basis element $E_{ij}$.

The graphical element corresponding to the field $M \in M_N(\mathbb{C})$ will be denoted by

\[
\begin{array}{c}
\text{\large $M$}\\
\text{\large $i$}\\
\text{\large $j$}\\
\end{array}
\end{equation}

In particular, we have

\[
\begin{array}{c}
\text{\large $\Lambda^i$}\\
\text{\large $i$}\\
\end{array}
\end{equation}

The \textit{propagator} corresponds to the copairing dual to the inner product $(-|-)_\Lambda$, so it is given by

$$\begin{array}{c}
\text{\large $i$}\\
\text{\large $j$}\\
\end{array}
\end{equation}

$$
\Rightarrow \frac{2}{\Lambda_i + \Lambda_j}$$

The \textit{interaction} corresponds to the cyclically invariant tensor

$$X \otimes Y \otimes Z \Rightarrow \frac{\sqrt{-1}}{2} \text{tr} XYZ,$$

so it is represented by a trivalent vertex

\[
\begin{array}{c}
\text{\large $j$}\\
\text{\large $i$}\\
\text{\large $k$}\\
\end{array}
\end{equation}

\Rightarrow \frac{\sqrt{-1}}{2}$$
We well call this an ordinary vertex, in order to distinguish it from the special vertices that we will introduce below. Since the interaction enjoys a cyclic invariance, the trivalent vertex representing it is equipped with a cyclic order on the legs; therefore the Feynman diagrams related to the Gaussian integral (4.1) are ribbon graphs. In the displayed diagrams of this paper, the cyclic order on the vertices will be the one induced by the standard orientation of the plane.

We conclude the list of the Feynman rules of the ’t Hooft-Kontsevich model by introducing the special vertices; they correspond to the cyclic interactions

\[ X_1 \otimes \cdots X_n \mapsto \text{tr}(X_1 \cdots X_n) \]

and are graphically represented as

\[ \text{closed trivalent ribbon graph} \]

Closed trivalent ribbon graph corresponding to top-dimensional cells in the ideal orbicellularization of the moduli spaces of curves can be considered as Feynman diagrams in the ’t Hooft-Kontsevich model with only ordinary vertices and no legs. By the Feynman rules described above, is immediate to compute that the amplitude of such a ribbon graph \( \Gamma \) is given by

\[
Z^\Lambda(\Gamma) = (-1)^{|\Gamma^{(2)}|} \left( \frac{1}{2} \right)^{|\Gamma^{(0)}|} \sum_c \prod_{l \in \Gamma^{(1)}} \frac{2}{\Lambda_{c(l^+)} + \Lambda_{c(l^-)}},
\]

where \( \Gamma^{(2)} \) denotes the set of the holes of the ribbon graph \( \Gamma \), \( \Gamma^{(0)} \) is set of the vertices of \( \Gamma \), \( c \) ranges in the set of all maps \( \Gamma^{(2)} \rightarrow \{1, \ldots, N\} \), and \( l^\pm \) are the two (not necessarily distinct) holes \( l \) belongs to. So the graphical elements we are considering are actually ribbon graphs whose holes are coloured by the colours \( \{1, \ldots, N\} \).

The right-hand side of equation (4.2) is very similar to the right-hand side of the Kontsevich’ Main Identity (1.1). Indeed, a bit of combinatorics shows that

\[
Z(t^\ast) \bigg|_{t^\ast(\Lambda)} = \sum_{\Gamma} Z^\Lambda(\Gamma) \frac{1}{|\text{Aut } \Gamma|},
\]

where \( \Gamma \) ranges into the set of closed trivalent ribbon graphs. See [Kon92] or [FM03] for details. The right-hand-side of equation (4.3) is the expectation value of the vacuum in the ’t Hooft-Kontsevich model. To see this, recall that the expectation value of the ribbon graph \( \Psi \) is defined as the sum

\[
\langle \langle \Psi \rangle \rangle^\Lambda := \sum_{\Gamma \in \mathcal{R}_\Psi(0)} \frac{Z^\Lambda(\Gamma)}{|\text{Aut } \Gamma|},
\]

where \( \mathcal{R}_\Psi(0) \) denotes the set of (isomorphism classes of) closed trivalent ribbon graphs containing \( \Psi \) as a distinguished sub-graph and having no special vertex outside \( \Psi \). By saying that the sub-graph \( \Psi \) is distinguished, we require that any automorphism of an object \( \Gamma \in \mathcal{R}_\Psi(0) \) maps \( \Psi \) onto itself. It follows from the definition that \( \mathcal{R}_\Psi(0) \) is the set of isomorphism classes of all closed trivalent ribbon graphs.
If $\Psi$ is a ribbon graph (possibly with special vertices) with $n$ legs, then

$$X \mapsto Z(\Psi)(X^{\otimes n})$$

is a polynomial function (of degree $n$) on $\mathcal{H}(N)$ so it is integrable with respect to the Gaussian measure $d\mu\Lambda$ and, by the general theory of Feynman diagrams, we have:

$$\langle \langle \Psi \rangle \rangle_{\Lambda} = \int_{\mathcal{H}(N)} Z(\Psi)(X^{\otimes n}) \frac{\lambda}{|\text{Aut } \Psi|} \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr } X^3 \right\} d\mu\Lambda(X)$$

In particular, we have:

$$\langle \langle \emptyset \rangle \rangle_{\Lambda} = \int_{\mathcal{H}(N)} \frac{\lambda}{|\text{Aut } \Psi|} \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr } X^3 \right\} d\mu\Lambda(X)$$

so that equation (4.3) can be rewritten as

$$Z(t_*\big|_{t\in(\Lambda)} = \int_{\mathcal{H}(N)} \frac{\lambda}{|\text{Aut } \Psi|} \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr } X^3 \right\} d\mu\Lambda(X) \cdot \int_{\mathcal{H}(N)} \text{tr } X^2 d\mu\Lambda(X).$$

We also introduce the subset $\Gamma$ of $\mathcal{R}_{\Gamma}(0)$ consisting of those Feynman diagrams having no vertices except those in $\Gamma$, i.e., the set of Feynman diagrams that can be obtained by joining the legs of $\Gamma$ by means of edges. We call elements in $\Gamma$ the closures of $\Gamma$. Clearly, if $\Gamma$ has an odd number of legs, then $\Gamma$ is empty. By definition, the expectation value without interactions of the ribbon graph $\Psi$ is the sum

$$\langle \langle \Psi \rangle \rangle_{\Lambda,0} := \sum_{\Gamma \in \Psi} \frac{Z(\Gamma)}{|\text{Aut } \Gamma|},$$

and we have

$$\langle \langle \Psi \rangle \rangle_{\Lambda,0} = \int_{\mathcal{H}(N)} Z(\Psi)(X^{\otimes n}) d\mu\Lambda(X)$$

Note that, if $\Psi$ is a closed ribbon graph, then $\Gamma = \{ \emptyset \}$ so

$$\langle \langle \Psi \rangle \rangle_{\Lambda,0} = \frac{Z(\Psi)}{|\text{Aut } \Psi|},$$

and

$$\langle \langle \Psi \rangle \rangle_{\Lambda} = \langle \langle \Psi \rangle \rangle_{\Lambda,0} \cdot \langle \langle \emptyset \rangle \rangle_{\Lambda}.$$

We now come back to the two equations in the Kontsevich-Witten theorem. It is easy to compute, for any $l \in \mathbb{Z}$,

$$\left( \text{tr } \Lambda^l \frac{\partial}{\partial \Lambda} \right) \int_{\mathcal{H}(N)} \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr } X^3 \right\} d\mu\Lambda(X) =$$

$$= -\frac{1}{2} \int_{\mathcal{H}(N)} \text{tr } \Lambda^l X^2 \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr } X^3 \right\} d\mu\Lambda(X) +$$

$$+ \frac{1}{2} \int_{\mathcal{H}(N)} \exp \left\{ \frac{\sqrt{-1}}{6} \text{tr } X^3 \right\} d\mu\Lambda(X) \cdot \int_{\mathcal{H}(N)} \text{tr } \Lambda^l X^2 d\mu\Lambda(X)$$

The bilinear map

$$X \otimes Y \mapsto \text{tr } X\Lambda Y$$
can be seen as the amplitude of the diagram

\[
\begin{array}{c}
\Lambda^i \\
\end{array}
\]

so that equation (4.4) can be rewritten as

\[
\left( \text{tr} \Lambda^i \frac{\partial}{\partial \Lambda} \right) Z(t_\star(\Lambda)) = \frac{1}{2} \left\langle \left\langle \Lambda \right. \right. \left. \Lambda^i \right\rangle_{\Lambda} + \frac{1}{2} \left\langle \left\langle 0 \right. \right. \left. \right\rangle_{\Lambda} \cdot Z_{\Lambda}\left( \Lambda^i \right) = \frac{1}{2} \left\langle \left\langle \Lambda \right. \right. \left. \Lambda^i \right\rangle_{\Lambda} + \frac{1}{2} \left\langle \left\langle 0 \right. \right. \left. \right\rangle_{\Lambda} \cdot Z_{\Lambda}\left( \Lambda^i \right)
\]

It immediate to compute

\[
Z_{\Lambda}\left( \begin{array}{c}
\Lambda^i \\
\end{array} \right) = \sum_{i,j} \frac{2\Lambda^i}{\Lambda_i + \Lambda_j} = \sum_{i,j} \frac{\Lambda^i + \Lambda^j}{\Lambda_i + \Lambda_j}
\]

So, in particular

\[
Z_{\Lambda}\left( \begin{array}{c}
\Lambda^{-1} \\
\end{array} \right) = (\text{tr} \Lambda^{-1})^2
\]

and

\[
Z_{\Lambda}\left( \begin{array}{c}
\Lambda^2 \\
\end{array} \right) = 2 \text{tr} \Lambda^4 - 2 \text{tr} \Lambda^3 \text{tr} \Lambda + (\text{tr} \Lambda^2)^2
\]
Therefore, equations (I-II) can be rewritten as

(I) \[ 2 \frac{\partial Z}{\partial t_0} \bigg|_{t_0, (\Lambda)} = \Lambda^{-1} \]

(II) \[ \left( 210 \frac{\partial Z}{\partial t_3} - 6 \frac{\partial^2 Z}{\partial t_0 \partial t_1} + 6 \text{tr} \Lambda \frac{\partial Z}{\partial t_1} + 2 \text{tr} \Lambda^2 \frac{\partial Z}{\partial t_0} \right) \bigg|_{t_0, (\Lambda)} = \]

\[ = \left\langle \left\langle \Lambda^{-1} \right\rangle \right\rangle \Lambda - (2 \text{tr} \Lambda^4 - 2 \text{tr} \Lambda^3 \text{tr} \Lambda + (\text{tr} \Lambda^2)^2) \cdot \left\langle \left\langle 0 \right\rangle \right\rangle \Lambda \]

5. Witten’s formula for derivatives

We have seen in Section 4 that the Kontsevich’ Main Identity relates the partition function of the intersection numbers \( Z(t_\ast) \) to the expectation value of the vacuum in the \( N \)-dimensional ‘t Hooft-Kontsevich model. It has been remarked by Witten [Wit92] that the Kontsevich’ Main Identity also relates the first order derivatives of the partition function \( Z(t_\ast) \) to expectation values in the \((N + 1)\)-dimensional ‘t Hooft-Kontsevich model of ribbon graph with one “distinguished” hole, and more generally, the \( k \)th order derivatives of \( Z(t_\ast) \) to expectation values in the \((N + k)\)-dimensional ‘t Hooft-Kontsevich model of ribbon graph with \( k \) “distinguished” holes.

Indeed, by the definition of the free energy \( F(t_\ast) \) it immediately follows that

\[ \frac{\partial F(t_\ast)}{\partial t_k} = \sum_{g,n,\nu_\ast} \frac{1}{n!} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} t_{k} \rangle_{g,n} t_{\nu_1} \cdots t_{\nu_n}, \]

so that taking the derivative with respect to the variable \( t_k \) corresponds to “adding a \( \tau_k \) at the end of the intersection indices”. If we write the Kontsevich’ Main Identity for ribbon graphs with \((n + 1)\) holes, we find

\[ \sum_k \left( \sum_{\nu_1,\ldots,\nu_n} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} t_{k} \rangle_{g,n} \frac{(2\nu_i - 1)!!}{\lambda_i^{2\nu_i + 1}} \right) \frac{(2k - 1)!!}{\lambda_{n+1}^{2k+1}} = \]

\[ = \sum_{(\Gamma, h)} \frac{1}{|\text{Aut}(\Gamma, h)|} \left( \frac{1}{2} \right)^{|\Gamma(0)|} \prod_{l \in \Gamma(i)} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}}, \]

where \((\Gamma, h)\) ranges in the set of isomorphism classes of closed connected trivalent genus \(g\) ribbon graphs, with \(n + 1\) holes, numbered from 1 to \(n + 1\). Therefore,

\[ \sum_{m_\ast, \nu_\ast} \langle \tau_{\nu_1} \cdots \tau_{\nu_n} t_{k} \rangle_{g,n+1} \prod_{i=1}^{n} \frac{(2\nu_i - 1)!!}{\lambda_i^{2\nu_i + 1}} = \]

\[ = \frac{1}{(2k - 1)!!} \text{Coeff}_{\lambda_{n+1}}^{-1} \left( \sum_{(\Gamma, h)} \left( \frac{1}{2} \right)^{|\Gamma(0)|} \prod_{l \in \Gamma(i)} \frac{2}{\lambda_{h(l^+)} + \lambda_{h(l^-)}} \right). \]
Therefore, the derivatives of the generating series \( F(t_*) \) can be seen as Laurent coefficients of the Main Identity with one more variable \( \lambda_{n+1} \). Translating this fact into the language of Hermitian matrix integrals lead us to consider the \((N + 1)\)-dimensional \( 't \) Hooft-Kontsevich model \((N + 1)\)-dimensional \( 't \) Hooft-Kontsevich model \( Z_{z \oplus \Lambda} \) defined by the diagonal matrix
\[
z \oplus \Lambda = \begin{pmatrix} z & 0 \\ 0 & \Lambda \end{pmatrix},
\]
where \( z \) is a positive real variable. The propagators in the \((N + 1)\)-dimensional \( 't \) Hooft-Kontsevich model are
\[
\begin{align*}
i \quad \quad j & \quad \quad \quad \rightarrow \quad \frac{2}{\Lambda_i + \Lambda_j} \\
i \quad \quad z & \quad \quad \rightarrow \quad \frac{2}{z + \Lambda_i}
\end{align*}
\]
so the graphical elements we are considering are actually ribbon graphs with holes coloured by the colours \( \{z, 1, 2, \ldots, N\} \). We will put a \( z \) in the middle of an hole to mean that the hole is given the “special” colour \( z \); colours on a blank hole will be allowed to vary in the set \( \{1, 2, \ldots, N\} \). A little combinatorics (see details in [FM03]) then shows that, for any \( k \geq 0 \), the following identity holds:
\[
\frac{\partial}{\partial t_k} Z_{t_*} \bigg|_{t_*(\Lambda)} = -\frac{1}{(2k - 1)!} \text{Coeff}_{z}^{-((2k+1))} \left( \sum_{\Gamma \in \mathcal{R}_0^{[1]}(0)} \frac{Z_{z \oplus \Lambda}(\Gamma)}{|\text{Aut } \Gamma|} \right),
\]
where \( \mathcal{R}_0^{[1]}(0) \) denotes the set of isomorphism classes of trivalent closed ribbon graphs with exactly one \( z \)-decorated hole (and all the other holes blank).

A completely similar argument works for higher order derivatives. For instance, second order derivatives are related to amplitudes in the \((N + 2)\)-dimensional \( 't \) Hooft-Kontsevich model defined by the diagonal matrix
\[
w \oplus z \oplus \Lambda = \begin{pmatrix} w & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \Lambda \end{pmatrix},
\]
by the formula
\[
\frac{\partial^2}{\partial t_{k_1} \partial t_{k_2}} Z_{t_*} \bigg|_{t_*(\Lambda)} = \text{Coeff}_{w}^{-((2k_1+1))} \text{Coeff}_{z}^{-((2k_2+1))} \left( \sum_{\Gamma \in \mathcal{R}_0^{[1]}(0)} \frac{Z_{w \oplus z \oplus \Lambda}(\Gamma)}{|\text{Aut } \Gamma|} \right),
\]
where \( \mathcal{R}_0^{[1]}(0) \) denotes the set of isomorphism classes of trivalent closed ribbon graphs with only ordinary vertices and exactly one \( z \)-decorated hole and one \( w \)-decorated one (and all the other holes blank).

Note that we are extracting the Laurent coefficient of \( z^{-((2k_2+1))} \) first, and the coefficient of \( w^{-((2k_1+1))} \) later; this means that we are considering the Laurent expansion in the region \( |z| > |w| \). It follows from the Kontsevich’ Main Identity that one would get the same result by considering the Laurent expansion in the region \( |w| > |z| \).
Now, let us give a closer look to equation (5.3). If $\Gamma$ is an element of $R[1]_n$, then its $z$-decorated hole can be regarded as a distinguished sub-diagram of $\Gamma$. So, if we call ($z$-decorated) hole-type a minimal element of $R[1]_n$, i.e., a ribbon graph $\Gamma \in R[1]_n$ such that no proper subgraph of $\Gamma$ lies in $R[1]_n$, then equation (5.3) can be rewritten as:

$$
\frac{\partial Z(t_s)}{\partial t_k} \bigg|_{t_s(\Lambda)} = -\frac{1}{(2k-1)!!} \sum_{\Gamma \in S_z} \text{Coeff}_z^{-(2k+1)} \left( \sum_{\Phi \in R_{[1]}(0)} \frac{Z_{z\oplus\Lambda}(\Phi)}{|\text{Aut } \Phi|} \right),
$$

where $S_z$ denotes the set of isomorphism classes of hole types, and $R_{[1]}(0)$ denotes the set of isomorphism classes of closed ribbon graphs containing the hole type $\Gamma$ as a distinguished subgraph and having no $z$-decorated hole apart from the hole of $\Gamma$. By introducing the shorthand notation

$$(\langle \Gamma \rangle_{z\oplus\Lambda}^{[1]} := \sum_{\Phi \in R_{[1]}(0)} \frac{Z_{z\oplus\Lambda}(\Phi)}{|\text{Aut } \Phi|},$$

where $\Gamma$ is an hole type, Witten’s formula for derivatives finally becomes:

$$
(5.5) \quad \frac{\partial^2 Z(s_1^*; t_s)}{\partial t_{k_1} \partial t_{k_2}} \bigg|_{t_s(\Lambda)} = -\frac{1}{(2k_1-1)!!(2k_2-1)!!} \sum_{\Gamma \in S_{z,w}} \text{Coeff}_w^{-(2k_1+1)} \text{Coeff}_z^{-(2k_2+1)} \langle \langle \Gamma \rangle^{[1]}_{z\oplus\Lambda} \rangle_{z\oplus\Lambda}^{[1,1]} \langle \langle \Gamma \rangle_{z\oplus\Lambda}^{[1]} \rangle_{z\oplus\Lambda}$$

The sum on the right-hand side of equation (5.5) is actually a finite sum. Indeed, according to the Feynman rules (5.1-5.2), each edge of the $z$-decorated hole brings a factor of order $O(z^{-1})$ as $z \to \infty$. Then, the coefficient of $z^{-(2k+1)}$ in the Laurent expansion of the amplitude $Z_{z\oplus\Lambda}(\Gamma)$ will be zero as soon as the hole-type $\Gamma$ has more than $2k + 1$ edges. Since there are finitely many hole types such that the $z$-decorated hole is bounded by at most $2k + 1$ edges, we are actually dealing with a finite sum.

Similarly, we call ($\{z, w\}$-decorated) two-holes-types the minimal elements of $R_{[1,1]}$, so that we can rewrite equation (5.4) as

$$
(5.6) \quad \frac{\partial Z(s_1^*; t_s)}{\partial t_{k_1} \partial t_{k_2}} \bigg|_{t_s(\Lambda)} = \frac{1}{(2k_1-1)!!(2k_2-1)!!} \sum_{\Gamma \in S_{z,w}} \text{Coeff}_w^{-(2k_1+1)} \text{Coeff}_z^{-(2k_2+1)} \langle \langle \Gamma \rangle_{z\oplus\Lambda}^{[1]} \rangle_{z\oplus\Lambda}$$

where the symbol $S_{z,w}$ denotes the set of isomorphism classes of two-holes types. An argument completely similar to the one used above shows that the also sum on the right-hand side of (5.6) is actually a finite sum. More precisely, only the 2-hole-types such that the total number of edges bounding the special holes is at most $2k_1 + 2k_2 + 2$ have to be considered.

6. PROOF OF EQUATION (1)

By equation (5.5) we can rewrite (1) as

$$
-2 \sum_{\Gamma \in S_{z}^{[1,1]}} \text{Coeff}_z^{-1} \langle \langle \Gamma \rangle_{z\oplus\Lambda}^{[1]} = \langle \langle \begin{array}{c} A^{-1} \\ \Lambda \end{array} \rangle_{z\oplus\Lambda} \rangle_{z\oplus\Lambda}.
$$
where $S_{\leq 1}^z$ denotes the set of hole types such that the $z$-decorated hole is bounded by at most one edge. There is only one such hole type. We compute

$$-2 \text{Coeff}_{z}^{-1} \left( \begin{array}{c}
\cdot \n
\end{array} \right) = -2 \text{Coeff}_{z}^{-1} \left( \frac{\sqrt{-1}}{z + \Lambda} \right)$$

$$= -2\sqrt{-1} = -2\sqrt{-1} \begin{array}{c}
i \n
\end{array},$$

so that

$$(6.1) \quad -2 \sum_{\Gamma \in S_{\leq 1}^z} \text{Coeff}_{z}^{-1} \langle \langle \Gamma \rangle \rangle = -2\sqrt{-1} \langle \langle \begin{array}{c}
\circ \n
\end{array} \rangle \rangle_{\Lambda}.$$

Since the leg outgoing from the univalent vertex on the right-hand side of equation (6.1) must end in a trivalent vertex, we find

$$-2 \sum_{\Gamma \in S_{\leq 1}^z} \text{Coeff}_{z}^{-1} \langle \langle \Gamma \rangle \rangle = -2\sqrt{-1} \langle \langle \begin{array}{c}
\circ \n
\end{array} \rangle \rangle_{\Lambda} = \langle \langle \begin{array}{c}
\circ \n
\end{array} \rangle \rangle_{\Lambda}$$

On the other hand,

$$\Lambda^{-1} \begin{array}{c}
\cdot \n
\end{array} = \Lambda_i^{-1} = \begin{array}{c}
\cdot \n
\end{array}$$

so

$$\langle \langle \begin{array}{c}
\circ \n
\end{array} \rangle \rangle_{\Lambda} = \langle \langle \begin{array}{c}
\circ \n
\end{array} \rangle \rangle_{\Lambda}$$

and equation (I) follows.

7. Proof of equation (II)

The proof of equation (II) goes along the same lines of the proof of equation, (I), but it is a bit more involved. We begin by rewriting (II) by means of the Witten's
formulas (5.5-5.6) for the derivatives of the partition function:

\[
-14 \sum_{\Gamma \in S^7} \text{Coeff}_z^{-7} \langle \langle \Gamma \rangle \rangle_{z \oplus \Lambda} - 6 \sum_{\Gamma \in S^4_{z,w}} \text{Coeff}_{w}^{-1} \text{Coeff}_z^{-3} \langle \langle \Gamma \rangle \rangle_{w \oplus z \oplus \Lambda} - \\
-6 \text{tr} \Lambda \sum_{\Gamma \in S^3} \text{Coeff}_z^{-3} \langle \langle \Gamma \rangle \rangle_{z \oplus \Lambda} - 2 \text{tr} \Lambda^3 \sum_{\Gamma \in S^4} \text{Coeff}_z^{-1} \langle \langle \Gamma \rangle \rangle_{z \oplus \Lambda} - \\
(7.1) \\
-\left\langle \begin{array}{c}
\Lambda \\
\end{array} \right\rangle + (2 \text{tr} \Lambda^4 - 2 \text{tr} \Lambda^3 \text{tr} \Lambda + (\text{tr} \Lambda^2)^2) \cdot \langle \langle \emptyset \rangle \rangle_{\Lambda} = 0,
\]

where \( S_{z}^{\leq l} \) denotes the set of isomorphism classes of hole types such that the \( z \)-decorated hole is bounded by at most \( l \) edges, and \( S_{z,w}^{\leq 4} \) denotes the set of isomorphism classes of 2-holes types such that the total number of edges bounding the \( z \)- and \( w \)-decorated holes is at most four.

By the Feynman rules in the ’t Hooft-Kontsevich model, the Laurent coefficients of the amplitude of a hole type are polynomials in the variables \( \Lambda_i \)'s. Indeed:

1. each ordinary vertex brings a factor \( \sqrt{-1/2} \);
2. each internal edge bordering the \( z \)-decorated hole on both sides contributes a factor \( 1/z \);
3. the other internal edges contribute factors of the form \( 2/(z + \Lambda_i) \) for \( i = 1, \ldots, N \).

Similarly, also the Laurent coefficients of the amplitudes of a two-holes type are polynomials in the variables \( \Lambda_i \)'s.

A convenient way to represent graphically these Laurent coefficients is to enlarge the class of special vertices by adding special vertices decorated by polynomials. This is formally done as follows.

Let \( \varphi \) be a polynomial in \( \mathbb{C}[\theta_1, \theta_2, \ldots, \theta_n] \); we say that the polynomial \( \varphi \) is cyclically invariant if it is invariant with respect to the natural action of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) on the coordinates. We will consider an \( n \)-valent special vertex decorated by the polynomial \( \varphi \):

This Feynman rule is well defined due to the cyclical invariance of \( \varphi \).

We also consider special vertices decorated by non-cyclically invariant polynomials:

The rôle of the “★” mark is precisely to break the cyclical symmetry of the graphical element, so that the “★” tells which indeterminate —among those corresponding to indices decorating holes around the vertex— comes first. Note that, when dealing
with ★-marked special vertices, there is actually no need that the valence $n$ of the vertex equals the the number $\nu$ of variables of the polynomial decorating it. Indeed, if $\nu < n$ then the above Feynman rule still makes sense; if $\nu > n$ then wrap around the vertex as many times as needed.

Using these notations, the Laurent coefficients we are interested in are easily written in the form

$$\sum_\Xi p_\Xi (\text{tr } \Lambda^*) Z_\Lambda(\Xi)$$

where the $\Xi$'s are disjoint unions of special vertices decorated by polynomials, and the $p_\Xi$'s are polynomials in the positive traces of $\Lambda$; see $\text{[FM03]}$ for several computation examples; we call $\Xi$ a cluster of special vertices.

Also the term

\[ \langle \langle \begin{array}{c} \Lambda^5 \\ j \end{array} \rangle \rangle \Lambda \]

appearing in (7.1) can be written as the expectation value of a special vertex decorated by a polynomial. Indeed,

\[ \Lambda^5 = \theta_1^{\star} \]

so that

\[ \langle \langle \begin{array}{c} \Lambda^5 \\ j \end{array} \rangle \rangle = \langle \langle \theta_1^{\star} \rangle \rangle \]

Therefore, equation (7.1) is reduced to showing that

(7.2)

$$\sum_\Xi q_\Xi (\text{tr } \Lambda^*) \langle \langle \Xi \rangle \rangle_\Lambda = 0$$

where the $\Xi$ are suitable clusters of special vertices and the $q_\Xi$ are polynomials in the positive traces of $\Lambda$. We remark that all the clusters $\Xi$ and the polynomials $q_\Xi$ are explicitly computable: one just has to write down all the hole types and the two-holes types involved in equation (7.1).

To prove equation (7.2) we will use the Feynman diagrammatic manipulations introduced in $\text{[FM03]}$. These manipulations will reduce the proof to a completely straightforward, but long and tedious, computation. Therefore, after describing the general theory, we will explicitly compute only a few terms of (7.2), leaving the remaining computations as an exercise to the reader.

We define the total degree of a term of the form

$$(\text{tr } \Lambda^0)^{d_0} (\text{tr } \Lambda)^{d_1} \cdots (\text{tr } \Lambda^\nu)^{d_\nu} \langle \langle \Xi \rangle \rangle_\Lambda$$

as the integer $\deg \Xi + \sum i_i d_i$, where $\deg \Xi$ is the sum of the degrees of the polynomials decorating the vertices of $\Xi$. 

The key idea (due to Witten [Wit92]) is that one can lower the total degrees of the terms appearing in equation (7.2) by the following remark. For any \( \psi \in \mathbb{C}[\theta_1, \ldots, \theta_n] \), let \( u_\psi \in \mathbb{C}[\theta_1, \ldots, \theta_n] \) be the polynomial

\[
u_\psi(\theta_1, \theta_2, \ldots, \theta_n) := (\theta_n + \theta_1) \cdot \psi(\theta_1, \theta_2, \ldots, \theta_n),
\]

Then

\[
u_\psi /c56 /c63/c57 /c62/c58 /c61/c59 /c60 \quad \mapsto \quad (\Lambda_{i_1} + \Lambda_{i_n}) \psi(\Lambda_1, \ldots, \Lambda_{i_n})
\]

and then we can use the factor \((\Lambda_{i_1} + \Lambda_{i_n})\) to cancel the factor \(2 / (\Lambda_{i_1} + \Lambda_{i_n})\) coming from the edge stemming from the vertex just before the ciliation (in the cyclic order of the vertex). Now, assume we are dealing with the expectation value

\[
\langle \langle \nu_\psi /c56 /c63/c57 /c62/c58 /c61/c59 /c60 \rangle \rangle \Lambda
\]

By definition, this expectation value is a sum over ribbon graphs (with a distinguished sub-diagram); for any \( \Gamma \) in this sum, the edge stemming from the vertex just before the ciliation must \( \text{either} \) end at another —distinct— vertex \( \text{or} \) make a loop. Therefore, using the definition of expectation value again,

\[
(7.3) \quad \langle \langle \nu_\psi \rangle \rangle \Lambda = \sum_{j=1}^{n-3} \langle \langle \nu_\psi \rangle \rangle \Lambda + \sum_{0 < j < n-2} \langle \langle \nu_\psi \rangle \rangle \Lambda + \sum_{j=1}^{n-3} \langle \langle \delta_j \rangle \rangle \Lambda + \langle \langle \nu_\psi \rangle \rangle \Lambda
\]

Now, each of the terms at right-hand side of (7.3) above, can be rewritten as the expectation value of a linear combination (over \( \mathbb{C}[\text{tr } \Lambda^*] \)) of clusters; indeed, one can directly compute:

\[
(C1) \quad \langle \langle \nu_\psi \rangle \rangle \Lambda = \sqrt{-1} \langle \langle \psi \rangle \rangle \Lambda ;
\]

\[
(C2) \quad \langle \langle \nu_\psi \rangle \rangle \Lambda = 2 \sum_{h=0}^{k} \text{tr } \Lambda^h \langle \langle \psi \rangle \rangle \Lambda ,
\]

\( (n+1) \)-valent

\( (n-2) \)-valent
where the polynomials $\psi_h$ are defined by the equation

$$\psi(\theta_1, \ldots, \theta_{n-1}, \theta_2) = \sum_{h=0}^{k} \theta_1^h \psi_h(\theta_2, \ldots, \theta_{n-1});$$

where the polynomials $\phi'_h$, $\phi''_h$ are defined by

$$\psi(\theta_1, \ldots, \theta_j, \theta_{j+2}, \ldots, \theta_{n-1}, \theta_{j+2}) = \sum_{h=0}^{k} \phi'_h(\theta_1, \ldots, \theta_j) \cdot \phi''_h(\theta_{j+2}, \ldots, \theta_{n-1});$$

where the polynomials $\eta_h$ are defined by:

$$\psi(\theta_1, \ldots, \theta_{n-2}, \theta_1) = \sum_{h=0}^{k} \theta_n^h \eta_h(\theta_1, \ldots, \theta_{n-2}).$$

The same argument also works in the more general case of a cluster of special vertices such that one of the polynomials decorating the vertices has the form $u_\psi$. In this case, a new graphical element may appear, which is not listed in equations (C1)–(C4) above; namely, that the ciliated edge connects the chosen vertex to another one in the same cluster. Direct computation again gives:

$$\psi(\theta_1, \ldots, \theta_{n-j-2}, \theta_1, \theta_n) = \sum_{h=0}^{k} \theta_n^h \eta_h(\theta_1, \ldots, \theta_{n-j-2}).$$

where the polynomial $\psi \ast \zeta$ is defined by

$$(\psi \ast \zeta)(\theta_1, \ldots, \theta_n) = \psi(\theta_1, \ldots, \theta_n) \cdot \zeta(\theta_n, \theta_{n+1}, \ldots, \theta_{n+j}, \theta_1).$$

By this edge-contracting procedure we can lower the total degree of expressions of the form $p_{\Xi}(\zeta)_\Lambda$ as soon as one of the vertices of $\Xi$ is decorated by a polynomial of the form $u_\psi$. In order to apply it to equation (7.2) we have to change the polynomials decorating the vertices in (7.2) with polynomials of the form $u_\psi$. This can be done by using
the following trick: if \( \varphi_1(\theta_1, \ldots, \theta_n) \) and \( \varphi_2(\theta_1, \ldots, \theta_n) \) are two polynomials such that

\[
\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \varphi_1(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}) = \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \varphi_2(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})
\]

then

\[
\langle \langle \varphi_1 \rangle \rangle_{\Lambda} = \langle \langle \varphi_2 \rangle \rangle_{\Lambda}
\]

Indeed, if we set

\[
\varphi(\theta_1, \ldots, \theta_n) := \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \varphi_1(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})
\]

then

\[
\langle \langle \varphi_1 \rangle \rangle_{\Lambda} = \langle \langle \varphi_2 \rangle \rangle_{\Lambda}
\]

and the same equality holds with \( \varphi_2 \) in place of \( \varphi_1 \). Therefore, a way of changing a polynomial \( \varphi \) decorating a vertex into a polynomial of the form \( u \psi \) is the following:

1. make a cyclically invariant polynomial out of \( \varphi \) by setting
   \[
   \overline{\varphi}(\theta_1, \ldots, \theta_n) := \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \varphi(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})
   \]

2. find a cyclic decomposition

\[
\overline{\varphi}(\theta_1, \ldots, \theta_n) = \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} u_{\psi}(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}).
\]

Clearly, not every cyclically invariant polynomial admits a cyclic decomposition of the form \( (7.4) \); for instance the polynomial \( \overline{\varphi}(\theta_1, \theta_2) = \theta_1^2 + \theta_2^2 \) has no such decomposition. Therefore, in general one could be not able to change the polynomial \( \varphi \) into a polynomial of the form \( u_{\psi} \). However, explicit computations show that all the polynomials occurring in the proof of equation \( (7.1) \) can actually be changed in the form \( u_{\psi} \) by using the trick described above. We address the reader interested in dealing with cyclic polynomials which do not admit a cyclic decomposition of the form \( (7.4) \) to [FM03].

We now work out explicitly the first steps in the proof of equation \( (7.1) \) to show how the above general theory applies.

All terms in equation \( (7.2) \) have at most total degree equal to 6. Moreover, there is only one term of total degree 6: it is the term coming from the unique hole type
such that the $z$-decorated hole is bounded by exactly one edge. We have

$$-14 \text{Coeff}_z^{-7} \begin{pmatrix} i & i \\ z \\ i & i \end{pmatrix} = -14 \text{Coeff}_z^{-7} \frac{\sqrt{-1}}{z + \Lambda_i} =$$

$$= -14 \sqrt{-1} \Lambda_i^6 = -14 \text{Coth} \cdot \theta_1^6$$

So, the degree 6 term in (7.2) is

$$-14 \sqrt{-1} \langle \langle \theta_1^6 \rangle \rangle \Lambda$$

The polynomial $\theta_1^6$ is trivially cyclically decomposable:

$$\theta_1^6 = (\theta_1 + \theta_1) \cdot \theta_1^5 \cdot \frac{1}{2}.$$

Therefore, by the edge-contraction procedure described above, we find

$$-14 \sqrt{-1} \langle \langle \theta_1^6 \rangle \rangle \Lambda = 7 \langle \langle \theta_1^6 \rangle \rangle \Lambda$$

This way we have reduced the degree 6 term to a degree 5 term. In equation (7.2) there are other two degree 5 terms. One is the term coming from the unique hole type such that the $z$-decorated hole is bounded by exactly two edges. We have

$$-14 \text{Coeff}_z^{-7} \begin{pmatrix} i & i \\ z & i \\ i & i \end{pmatrix} = 14 \text{Coeff}_z^{-7} \frac{1}{(z + \Lambda_{i_1})(z + \Lambda_{i_2})}$$

$$= -14 \sum_{n_1 + n_2 = 5} \Lambda_{i_1}^{n_1} \Lambda_{i_2}^{n_2} = \frac{i_2}{i_1} \frac{\eta_2}{\Lambda_i}$$

where

$$\eta_2(\theta_1, \theta_2) = -14 \left( \theta_1^5 + \theta_1^4 \theta_2 + \theta_1^3 \theta_2^2 + \theta_1^2 \theta_2^3 + \theta_1 \theta_2^4 + \theta_2^5 \right).$$

The other degree 5 term is the one coming from

$$\langle \langle \rangle \rangle \Lambda$$
Indeed, we have seen at the beginning of this section that this expectation value equals

\[
\left\langle \sigma_1^* \right\rangle_{\Lambda}
\]

Summing up, we obtain that the contribution from terms of total degree at least 5 in equation (7.2) is reduced to:

\[
\left\langle \sigma_2 \right\rangle_{\Lambda}
\]

where

\[
\psi_2(\theta_1, \theta_2) = -8\theta_1^5 - 14\theta_1^4\theta_2 - 14\theta_1^3\theta_2^2 - 14\theta_1^2\theta_2^3 - 14\theta_1\theta_2^4 - 8\theta_2^5.
\]

The polynomial \(\psi_2\) is cyclically decomposable: we have

\[
\psi_2(\theta_1, \theta_2) = \sum_{\sigma \in \mathbb{Z}/2\mathbb{Z}} u \psi_2(\theta_{\sigma(1)}, \theta_{\sigma(2)}),
\]

where

\[
\psi_2(\theta_1, \theta_2) = -4\theta_1^4 - 3\theta_1^3\theta_2 - 4\theta_1^2\theta_2^2 - 3\theta_1\theta_2^3 - 4\theta_2^4.
\]

Applying the edge-contraction procedure again, we get

\[
\left\langle \sigma_2 \right\rangle_{\Lambda} = \sqrt{-1} \left\langle \sigma_2^* \right\rangle_{\Lambda} + \left( -16 \text{tr } \Lambda^4 - 12 \text{tr } \Lambda^3 \text{tr } \Lambda - 8(\text{tr } \Lambda^2)^2 \right) \left\langle \emptyset \right\rangle_{\Lambda}
\]

\[
= \sqrt{-1} \left\langle \psi_2 \right\rangle_{\Lambda} - \left( 16 \text{tr } \Lambda^4 + 12 \text{tr } \Lambda^3 \text{tr } \Lambda + 8(\text{tr } \Lambda^2)^2 \right) \left\langle \emptyset \right\rangle_{\Lambda}
\]

where

\[
\psi_2(\theta_1, \theta_2, \theta_3) = \psi_2(\theta_1, \theta_2) + \psi_2(\theta_2, \theta_3) + \psi_2(\theta_3, \theta_1)
\]

\[
= -8\theta_1^4 - 3\theta_1^3\theta_2 - 4\theta_1^2\theta_2^2 - 3\theta_1\theta_2^3 - 8\theta_2^4 - 3\theta_2^3\theta_3 - 4\theta_2^2\theta_3^2 - 3\theta_2\theta_3^3 - 8\theta_3^4 - 4\theta_3^3\theta_1 - 4\theta_3^2\theta_1^2 - 3\theta_3\theta_1^3.
\]

This way we have reduced the total contribution coming from terms of total degree at least 5 to a sum of terms of total degree 4. There are other terms of total degree 4 in equation (7.2). One is the term

\[
(2 \text{tr } \Lambda^4 - 2 \text{tr } \Lambda^3 \text{tr } \Lambda + (\text{tr } \Lambda^2)^2) \cdot \left\langle \emptyset \right\rangle_{\Lambda}
\]

appearing in equation (7.1). The others are the terms coming from the hole types whose \(z\)-decorated hole is bounded by exactly 3 edges. There are three such hole

\[\text{[Footnote: The computer program Maple V has been used for this and the following computations.]}\]
types. The first of them gives

$$-14 \text{Coeff}_{z}^{-7} \left( \begin{array}{c} i_2 \\ i_1 \\ i_3 \\ z \\ i_2 \\ i_1 \\ i_3 \\ z \\ i_2 \\ i_1 \\ i_3 \\ z \end{array} \right) = 14 \text{Coeff}_{z}^{-7} \sqrt{-1} \frac{1}{(z + \Lambda_{i_1})(z + \Lambda_{i_2})(z + \Lambda_{i_3})}$$

$$= 14 \sqrt{-1} \sum_{n_1 + n_2 + n_3 = 4} \Lambda_{i_1}^{n_1} \Lambda_{i_2}^{n_2} \Lambda_{i_3}^{n_3}$$

where \( \eta_3(\theta_1, \theta_2, \theta_3) = 14 \sqrt{-1} \sum_{n_1 + n_2 + n_3 = 4} \theta_1^{n_1} \theta_2^{n_2} \theta_3^{n_3} \).

The second hole type with three edges bounding the \( z \)-decorated hole gives no contribution to total degree 4 terms. Indeed,

$$-14 \text{Coeff}_{z}^{-7} \left( \begin{array}{c} z \\ i_1 \\ i_2 \\ i_3 \end{array} \right) = \frac{7}{2} \text{Coeff}_{z}^{-7} \frac{1}{z^3} = 0$$

Finally, the third hole type gives

$$-14 \text{Coeff}_{z}^{-7} \left( \begin{array}{c} z \\ i_1 \\ i_2 \\ i_3 \end{array} \right)$$

$$-14 \sum_{i_1, i_2} \text{Coeff}_{z}^{-7} \left( \begin{array}{c} z \\ i_1 \\ i_2 \\ i_1 \end{array} \right)$$

$$= 14 \text{Coeff}_{z}^{-7} \sum_{i_1, i_2} \frac{1}{z(z + \Lambda_{i_1})(z + \Lambda_{i_2})}$$

$$= 2 \left( 14 \text{tr} \Lambda^0 \text{tr} \Lambda^4 + 14 \text{tr} \Lambda \text{tr} \Lambda^3 + 7(\text{tr} \Lambda^2)^2 \right) \langle \emptyset \rangle_{\Lambda}$$

Summing up (taking into account the cardinalities of the automorphism groups involved) the total contribution from terms of total degree at least 4 is just
where
\[
\varphi_4(\theta_1, \theta_2, \theta_3) = \sqrt{-1}(6\theta_1^4 + 11\theta_1^3\theta_2 + 10\theta_1^2\theta_2^2 + 11\theta_1\theta_2^3 + 6\theta_2^4 + \\
+ 11\theta_2^3\theta_3 + 10\theta_2^2\theta_3^2 + 11\theta_2\theta_3^3 + 6\theta_3^4 + 11\theta_3^3\theta_1 + \\
+ 10\theta_3^2\theta_1^2 + 11\theta_3\theta_1^3 + 14\theta_1\theta_2\theta_3^2 + 14\theta_2\theta_3\theta_1^2 + \\
+ 14\theta_3\theta_1\theta_2^2).
\]

One computes \(\varphi_3(\theta_1, \theta_2, \theta_3) = \sum_{\sigma \in \mathbb{Z}/3\mathbb{Z}} u\psi_3(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \theta_{\sigma(3)})\), where
\[
\psi_3(\theta_1, \theta_2, \theta_3) = \frac{\sqrt{-1}}{4}(65\theta_1^3 + 19\theta_1^2\theta_2 + 11\theta_1\theta_2^2 + 9\theta_1^3).
\]

Moreover,
\[
\psi_3(\theta_1, \theta_2) = \frac{\sqrt{-1}}{4}(65\theta_1^3 + 19\theta_1^2\theta_2 + 11\theta_1\theta_2^2 + 9\theta_1^3)
\]

So, applying the edge-contraction procedure again, we find
\[
\begin{align*}
\langle \varphi_4 \rangle = \sqrt{-1} \langle \psi_3 \rangle + 12\sqrt{-1} \text{tr} \Lambda^3 \langle \theta_1 \rangle + \\
+ 28\sqrt{-1} \text{tr} \Lambda^2 \langle \theta_2 \rangle + 16\sqrt{-1} \text{tr} \Lambda \langle \theta_3 \rangle + \\
+ 48\sqrt{-1} \text{tr} \Lambda^0 \langle \theta_4 \rangle
\end{align*}
\]

where
\[
\varphi_3(\theta_1, \theta_2, \theta_3, \theta_4) = \psi_3(\theta_1, \theta_2, \theta_3) + \psi_3(\theta_2, \theta_3, \theta_4) + \psi_3(\theta_3, \theta_4, \theta_1) + \\
+ \psi_3(\theta_4, \theta_1, \theta_2) = \\
= \sqrt{-1}\left(\frac{13}{2}\theta_2\theta_1^2 - \theta_2\theta_3\theta_4 - \theta_1\theta_2\theta_3 + 7\theta_3^3 + 7\theta_1^3 + 7\theta_2^3 + \\
+ 7\theta_4^3 + 4\theta_1\theta_3^2 + 4\theta_3\theta_1^2 - \theta_3\theta_4\theta_1 + \frac{19}{2}\theta_1\theta_2^2 + \\
+ \frac{19}{2}\theta_2\theta_3^2 + \frac{13}{2}\theta_3\theta_2^2 + \frac{13}{2}\theta_1\theta_4^2 + \frac{19}{2}\theta_4\theta_1^2 - \\
- \theta_4\theta_1\theta_2 + 4\theta_4\theta_2^2 + \frac{19}{2}\theta_3\theta_4^2 + \frac{13}{2}\theta_4\theta_3^2 + 4\theta_2\theta_4^2\right)
\]

and we have written the total contribution coming from terms of total degree at least 4 as a sum of terms of total degree 3.

The proof goes on along these lines: after a few more (completely straightforward, but long and tedious) computations one ends up with an explicit expression whose terms are all of total degree zero. These terms cancel out and equation (II) is
Moreover the combinatorial partition function

\begin{equation}
A \text{ formal triangular differential operator in the variables } C
\end{equation}

Similar computations give

\begin{equation}
\text{form a (non commutative) algebra}
\end{equation}

As an immediate corollary, one gets

\begin{equation}
\text{proven. The interested reader can found this computation carried out in full detail in [Fie02].}
\end{equation}

**APPENDIX: Formal differential operators**

In Section 5 we showed how the second order derivative \( \partial^2 Z(t_*)/\partial t_0 \partial t_1 \) appearing in equation (\( \text{[II]} \)) can be computed by considering Feynman diagrams with two distinguished holes.

In this appendix we show how it can be computed by using the main result from FM03. Let \( Z(s_*; t_*) \) be the partition function of the combinatorial intersection numbers (see [Kon92, AC96]). Then \( Z(t_*) = Z(s^0_*; t_*) \), where \( s^0_* = (0, 1, 0, 0, \ldots) \). Also the combinatorial partition function \( Z(s_*; t_*) \) is related to the asymptotic expansion of a matrix integral, namely

\begin{equation}
Z(s_*; t_*) \bigg|_{t_*(\Lambda)} = \int_{\mathcal{H}(N)} \exp \left\{ -\sqrt{-1} \sum_{j=0}^{\infty} (-1/2)^j s_j \frac{\text{tr} X^{2j+1}}{2j+1} \right\} d\mu_{\Lambda}(X),
\end{equation}

Moreover

\begin{equation}
\frac{\partial^{n_1} Z(s_*; t_*)}{\partial s_0^{n_0} \cdots \partial s_l^{n_l}} \bigg|_{s^*_*; t_*(\Lambda)} = \frac{n_0! \cdots n_l!}{\sqrt{-1}^{n_1} (-2)^{\sum_j n_j}} \langle \langle \psi_1 \Pi^{n_0} \cdots \Pi^{n_l} \rangle \rangle_{\Lambda},
\end{equation}

where \( \psi_k \) denotes a \( k \)-valent special vertex and \( \Pi \) denotes the disjoint union.

For any polyindex \( m_* = (m_0, m_1, \ldots, m_l, 0, 0, \ldots) \) set:

\begin{equation}
\| m_* \|_- := \sum_{i=1}^{\infty} (2i-1) m_i, \quad \| m_* \|_+ := \sum_{i=0}^{\infty} (2i+1) m_i.
\end{equation}

A formal triangular differential operator in the variables \( s_* \) is a formal series

\begin{equation}
D(s_*, \partial/\partial s_*) = \sum_{\| n_* \|_+ \leq \| m_* \|_-} a_{m_*, n_*} s_*^{m_*} \frac{\partial^{\| n_* \|}}{\partial s_*^{n_*}}, \quad a_{m_*, n_*} \in \mathbb{C},
\end{equation}

of bounded degree in \( s_* \). Formal triangular differential operators in the variables \( s_* \) form a (non commutative) algebra \( \mathbb{C}[\langle s_*, \partial/\partial s_* \rangle] \) which naturally acts on \( \mathbb{C}[\langle t_*; s_* \rangle] \).

The main result from FM03 can stated as follows: there exists formal triangular differential operators \( D_k(s_*, s_*, \partial/\partial s_*) \) such that, for any \( k_1, \ldots, k_n \) in \( \mathbb{N} \),

\begin{equation}
\frac{\partial^n Z(s_*; t_*)}{\partial t_{k_1} \cdots \partial t_{k_1}} = D_{k_1}(s_*, \partial/\partial s_*) \cdots D_{k_n}(s_*, \partial/\partial s_*) Z(s_*; t_*)
\end{equation}

As an immediate corollary, one gets

\begin{equation}
\frac{\partial^2 Z(t_*)}{\partial t_1 \partial t_0} = \left( D_1(s_*^0, \partial/\partial s_*) D_0(s_*, \partial/\partial s_*) Z(s_*; t_*) \right)_{s_*^0}
\end{equation}

It is computed in FM03 that

\begin{equation}
D_0(s_*, \partial/\partial s_*) = \frac{s_*^0}{2} + \sum_{m=0}^{\infty} (2m+1)s_{m+1} \frac{\partial}{\partial s_m}
\end{equation}

Similar computations give

\begin{equation}
D_1(s_*^0, \partial/\partial s_*) = \frac{1}{2} \frac{\partial}{\partial s_1} + \frac{1}{24}
\end{equation}
We therefore find
\[ \frac{\partial^2 Z(t_*)}{\partial t_1 \partial t_0} = \left( \frac{1}{2} \frac{\partial}{\partial s_1} + \frac{1}{24} \right) \left( \frac{s_0^2}{2} + \sum_{m=0}^{\infty} (2m+1)s_{m+1} \frac{\partial}{\partial s_m} \right) \bigg|_{s_*} \]
and we get the asymptotic expansion
\[ \frac{\partial^2 Z(t_*)}{\partial t_1 \partial t_0} \bigg|_{t_*(\Lambda)} = \frac{1}{4} \left\langle \begin{array}{c} \text{Diagram} \\ \text{of graphs} \end{array} \right\rangle_{\Lambda} - \frac{13}{24} \sqrt{-1} \left\langle \begin{array}{c} \text{Diagram} \\ \text{of graphs} \end{array} \right\rangle_{\Lambda} \]

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Dipartimento di Matematica “Guido Castelnuovo” — Università degli Studi di Roma “La Sapienza” — P.le Aldo Moro, 2 – 00185 – Roma, Italy

E-mail address: fiorenza@mat.uniroma1.it