GENERIC $T$-ADIC EXPONENTIAL SUMS IN ONE VARIABLE

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Abstract. The $T$-adic exponential sum associated to a Laurent polynomial in one variable is studied. An explicit arithmetic polygon is proved to be the generic Newton polygon of the $C$-function of the $T$-adic exponential sum. It gives the generic Newton polygon of $L$-functions of $p$-power order exponential sums.

1. Introduction

Let $W$ be the ring scheme of Witt vectors, $\mathbb{F}_q$ the field of characteristic $p$ with $q$ elements, $\mathbb{Z}_q = W(\mathbb{F}_q)$, and $\mathbb{Q}_q = \mathbb{Z}_q[\frac{1}{p}]$.

Let $\Delta \supseteq \{0\}$ be an integral convex polytope in $\mathbb{R}^n$, and $I$ the set of vertices of $\Delta$ different from the origin. Let

$$f(x) = \sum_{u \in \Delta} (a_u x^u, 0, 0, \ldots) \in W(\mathbb{F}_q[x_1^\pm 1, \ldots, x_n^\pm 1])$$

with $\prod_{u \in I} a_u \neq 0$,

where $x^u = x_1^{u_1} \ldots x_n^{u_n}$ if $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$.

Let $T$ be a variable.

Definition 1.1. The sum

$$S_f(k, T) = \sum_{x \in (\mathbb{F}_q^\times)^n} (1 + T)^{Tr_{\mathbb{Z}_q / \mathbb{Z}_p}(f(x))} \in \mathbb{Z}_p[[T]]$$

is called a $T$-adic exponential sum. And the function

$$L_f(s, T) = \exp\left(\sum_{k=1}^{\infty} S_f(k, T) \frac{s^k}{k!}\right) \in 1 + s\mathbb{Z}_p[[T]][[s]]$$

is called an $L$-function of $T$-adic exponential sums.

We view $L_f(s, T)$ as a power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_p((T))$. Let $\zeta_{p^m}$ be a primitive $p^m$-th root of unity, and $\pi_m = \zeta_{p^m} - 1$. Then $L_f(s, \pi_m)$ is the $L$-function of the $p$-power order exponential sums $S_f(k, \pi_m)$ studied by Liu-Wei [LW].

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Definition 1.2. The function
\[ C_f(s, T) = \exp\left(\sum_{k=1}^{\infty} -\left(q^k - 1\right)^{-n} S_f(k, T) \frac{s^k}{k} \right) \]
is called a $C$-function of $T$-adic exponential sums.

We have
\[ L_f(s, T) = \prod_{i=0}^{n} C_f(q^i s, T)^{(-1)^{n-i+1}(i)}, \]
and
\[ C_f(s, T) = \prod_{j=0}^{\infty} L_f(q^j s, T)^{(-1)^{n-1}(n+j-1)}. \]

So the $C$-function $C_f(s, T)$ and the $L$-function $L_f(s, T)$ determine each other. From the last identity, one sees that
\[ C_f(s, T) \in 1 + s\mathbb{Z}_p[[T]][[s]]. \]

We also view $C_f(s, T)$ as a power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_p((T))$. The $C$-function $C_f(s, T)$ was shown to be $T$-adic entire in $s$ by Liu-Wan [LWn].

Let $C(\Delta)$ be the cone generated by $\Delta$, and $M(\Delta) = M(\Delta) \cap \mathbb{Z}^n$. There is a degree function $\deg$ on $C(\Delta)$ which is $\mathbb{R}_{\geq 0}$-linear and takes the values 1 on each co-dimension 1 face not containing 0. For $a \not\in C(\Delta)$, we define $\deg(a) = +\infty$.

Definition 1.3. A convex function on $[0, +\infty]$ which is linear between consecutive integers with initial value 0 is called the infinite Hodge polygon of $\Delta$ if its slopes between consecutive integers are the numbers $\deg(a)$, $a \in M(\Delta)$. We denote this polygon by $H_\infty^\Delta$.

Liu-Wan [LWn] also proved the following.

Lemma 1.4. We have
\[ T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)(p - 1)H_\infty^\Delta, \]
where NP stands for Newton polygon.

Definition 1.5. The $T^a$-adic Newton polygon of $C_f(s, T; \mathbb{F}_p^a)$ is called the absolute Newton polygon of $C_f(s, T; \mathbb{F}_p^a)$.

Conjecture 1.6. If $p$ is sufficiently large, then the absolute $T$-adic Newton polygon of $C_f(s, T)$ is constant for a generic $f$. We call it the generic Newton polygon of $C_f(s, T)$.

Definition 1.7. The $\pi_m^a$-adic Newton polygon of $C_f(s, \pi_m; \mathbb{F}_p^a)$ is called the absolute Newton polygon of $C_f(s, \pi_m; \mathbb{F}_p^a)$. 
Combine results of Gelfand-Kapranov-Zelevinsky [GKZ], Adolphson-Sperber [AS], Liu-Wei [LW] with Grothendieck specialization lemma [Ka], the absolute Newton polygon of $C_f(s, \pi_m)$ is constant for a generic $f$. We call it the generic Newton polygon of $C_f(s, \pi_m)$.

**Conjecture 1.8.** If $p$ is sufficiently large, then the generic Newton polygon of $C_f(s, \pi_m)$ is independent of $m$, and coincides with the generic Newton polygon of $C_f(s, T)$.

The generic Newton polygon of $C_f(s, \pi_m)$ for $m = 1$ was studied by Wan [Wa1, Wa2].

In the rest of this section we assume that $\triangle \subset \mathbb{Z}$.

**Definition 1.9.** Let $0 \neq a \in M(\triangle)$. We define
\[
\delta \in (a) = \begin{cases} 
1, & \{\deg(a)\} = \{\deg(pi)\} \text{ for some } i \text{ with } ia > 0, \deg(i) < \{\deg(a)\}, \\
0, & \text{otherwise},
\end{cases}
\]
where $\{\cdot\}$ is the fractional part of a real number. We also define $\delta \in (0) = 0$.

**Definition 1.10.** A convex function on $[0, +\infty]$ which is linear between consecutive integers with initial value 0 is called the arithmetic polygon of $\triangle$ if its slopes between consecutive integers are the numbers
\[
\varpi_\triangle(a) = \lceil(p - 1) \deg(a) \rceil - \delta \in (a), a \in M(\triangle),
\]
where $\lceil \cdot \rceil$ is the least integer equal or greater than a real number. We denote this polygon by $p_\triangle$.

We can prove the following.

**Theorem 1.11.** We have
\[
p_\triangle \geq (p - 1) H^\infty_\triangle.
\]
Moreover, they coincide at the point $\text{Vol}(\triangle)$.

Let $D$ be the least common multiple of the nonzero endpoint(s) of $\triangle$.

The main results of this paper are the following theorems.

**Theorem 1.12.** If $p > 3D$, then
\[
T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)p_\triangle.
\]

**Theorem 1.13.** Let $f(x) = \sum_{u \in \triangle} (a_u x^u, 0, 0, \cdots)$, and $p > 3D$. Then there is a non-zero polynomial $H(y) \in \mathbb{F}_q[y_u \mid u \in \triangle]$ such that
\[
T - \text{adic NP of } C_f(s, T) = \text{ord}_p(q)p_\triangle
\]
if and only if $H((a_u)_{u \in \triangle}) \neq 0$.

The above theorem implies Conjecture 1.6 for $\triangle \subset \mathbb{Z}$. 
Theorem 1.14. Let \( f(x) = \sum_{u \in \Delta} (a_u x^{u}, 0, 0, \cdots), \) \( p > 3D, \) and \( m \geq 1. \) Then \( \pi_m - \text{adic NP of } C_f(s, \pi_m) = \text{ord}_p(q) \Delta \) if and only if \( H((a_u)_{u \in \Delta}) \neq 0. \)

The above theorem implies Conjecture 1.8 for \( \Delta \subset \mathbb{Z}. \)

Note that, if \( p \nmid D, \) then \( L(s, \pi_m) \) is a polynomial of degree \( p^m - 1 \) \( \text{Vol}(\Delta) \) for all \( m \geq 1 \) by a result of Adolphson-Sperber [AS] and a result of Liu-Wei [LW]. We shall prove the following.

Theorem 1.15. Let \( f(x) = \sum_{u \in \Delta} (a_u x^{u}, 0, 0, \cdots), \) and \( p > 3D. \) Then \( \pi_m - \text{adic NP of } L_f(s, \pi_m) \geq \text{ord}_p(q) \Delta \) on \([0, p^m - 1 \text{Vol}(\Delta)]\)
with equality holding if and only if \( H((a_u)_{u \in \Delta}) \neq 0. \)

Zhu [Zh1, Zh2] and Blache-Férand [BF] studied the Newton polygon of the \( L \)-function \( L_f(s, \pi_m) \) for \( m = 1. \)

2. Arithmetic estimate

In this section \( \Delta \subset \mathbb{Z}, \) \( A \) is a finite subset of \( M(\Delta) \times \mathbb{Z}/(b), \) and \( \tau \) is a permutation of \( A. \) We shall estimate
\[
\sum_{a \in A} \lceil \text{deg}(pa - \tau(a)) \rceil,
\]
where \( \text{deg}(i, u) = \text{deg}(i). \)

However, except in the ending paragraph, we assume that \( \Delta = [0, d], \) \( A \) is a finite subset of \( \{1, 2, \cdots \} \times \mathbb{Z}/(b). \)

Write
\[
\{x\}' = 1 + x - \lfloor x \rfloor = \begin{cases} \{x\}, & \text{if } \{x\} \neq 0, \\ 1, & \text{if } \{x\} = 0. \end{cases}
\]

Lemma 2.1. We have
\[
\sum_{a=0}^{m} (\delta_{<} - \delta_{\leq})(a) = \#\{1 \leq a \leq d\{\frac{m}{d}\}' \mid \{\frac{m}{d}\}' < \{\frac{pa}{d}\}'\},
\]
where
\[
\delta_{<}(a) = \begin{cases} 1, & \{\frac{a}{d}\}' < \{\frac{pa}{d}\}' \\ 0, & \text{otherwise.} \end{cases}
\]

Proof. Note that both \( \delta_{\leq} \) and \( \delta_{<} \) have a period \( d \) and have initial value 0. So we may assume that \( m < d. \) We have
\[
\sum_{a=0}^{m} \delta_{\leq}(a) = \sum_{a=1}^{m} \sum_{i=1}^{a-1} 1
\]
\[
\sum_{i=1}^{m-1} \sum_{a_i + 1}^{m} 1 = \#\{1 \leq i < m \mid i < d\frac{p_i}{d} \leq m\}.
\]

And, by definition,
\[
\sum_{a=0}^{m} \delta_\epsilon(a) = \#\{1 \leq a \leq m \mid a < d\frac{pa}{d}\}.
\]

The lemma now follows. □

**Lemma 2.2.** Let \( A, B, C \) be sets with \( A \) finite, and \( \tau \) a permutation of \( A \). Then
\[
\#\{a \in A \mid \tau(a) \in B, a \in C\} \\
\geq \#\{a \in A \mid a \in B, a \in C\} - \#\{a \in A \mid a \not\in B, a \not\in C\}.
\]

**Proof.** We have
\[
\#\{a \in A \mid a \in B, \tau(a) \in B, a \in C\} \\
\geq \#\{a \in A \mid a \in B, \tau(a) \not\in B\} - \#\{a \in A \mid a \in B, a \not\in C\},
\]
and
\[
\#\{a \in A \mid a \not\in B, \tau(a) \in B, a \in C\} \\
\geq \#\{a \in A \mid a \not\in B, \tau(a) \in B\} - \#\{a \in A \mid a \not\in B, a \not\in C\}.
\]

Note that
\[
\#\{a \in A \cap B \mid \tau(a) \not\in B\} = \#\{a \in A \setminus B \mid \tau(a) \in B\}.
\]

So
\[
\#\{a \in A \mid \tau(a) \in B, a \in C\} \\
\geq \#\{a \in A \mid a \in B, a \in C\} - \#\{a \in A \mid a \not\in B, a \not\in C\}.
\]

The lemma is proved. □

For \( a \in A \), we define
\[
\delta_\epsilon^\tau(a) = \begin{cases} 1, & \text{if } \deg(\tau(a)) < \text{deg}(a), \\
0, & \text{otherwise}. \end{cases}
\]

**Theorem 2.3.** If \( p > 3d \), and
\[
A = \{(1, \cdots, m-1) \times \mathbb{Z}/(b)\} \cup \{(m, i_0), \cdots, (m, i_{l-1})\},
\]
then
\[
\sum_{a \in A} \delta_\epsilon^\tau(a) \geq b \sum_{a=0}^{m-1} (\delta_\epsilon - \delta_\epsilon')(a) + l(\delta_\epsilon - \delta_\epsilon)(m).
\]
Proof. First we assume that \( l = 0 \). We have
\[
\sum_{a \in A} \delta_<(a) \geq \# \{ a \in A, \{ \text{deg}(\tau(a)) \}' \leq \{ \frac{m-1}{d} \}' < \{ p \text{deg}(a) \}' \}.
\]
Applying the last lemma with
\[
B = \{ a \in A, \{ \text{deg}(a) \}' \leq \{ \frac{m-1}{d} \}' \},
\]
and
\[
C = \{ a \in A, \{ \frac{m-1}{d} \}' < \{ p \text{deg}(a) \}' \},
\]
we get
\[
\sum_{a \in A} \delta_<(a) \geq \# \{ a \in A, \{ \text{deg}(a) \}' \leq \{ \frac{m-1}{d} \}' < \{ p \text{deg}(a) \}' \}
- \# \{ a \in A, \{ \text{deg}(a) \}' > \{ \frac{m-1}{d} \}' \geq \{ p \text{deg}(a) \}' \}.
\]
We have
\[
\# \{ a \in A, \{ \text{deg}(a) \}' \leq \{ \frac{m-1}{d} \}' < \{ p \text{deg}(a) \}' \} = b(\frac{m}{d} + 1)\# \{ 1 \leq a \leq d \{ \frac{m-1}{d} \}' | \{ \frac{m}{d} \}' < \{ \frac{pa}{d} \}' \}.
\]
We also have
\[
\# \{ a \in A, \{ \text{deg}(a) \}' > \{ \frac{m-1}{d} \}' \geq \{ p \text{deg}(a) \}' \}
= \frac{b}{d} m \# \{ d \geq a \geq d \{ \frac{m-1}{d} \}' | \{ \frac{m}{d} \}' \geq \{ \frac{pa}{d} \}' \}.
\]
\[
= \frac{b}{d} m \# \{ 1 \leq a \leq d \{ \frac{m-1}{d} \}' | \{ \frac{m}{d} \}' < \{ \frac{pa}{d} \}' \}.
\]
It follows that
\[
\sum_{a \in A} \delta_<(a) \geq b \# \{ 1 \leq a \leq d \{ \frac{m-1}{d} \}' | \{ \frac{m}{d} \}' < \{ \frac{pa}{d} \}' \}
\geq \sum_{a=0}^{m-1} (\delta_<(a) - \delta_<(m)).
\]
Secondly we assume that \( \delta_<(m) = 0 \). Extend the action of \( \tau \) trivially from \( A \) to \( \{1, \cdots, m\} \times \mathbb{Z}/(b) \). By what we just proved,
\[
\sum_{a \in \{1, \cdots, m\} \times \mathbb{Z}/(b)} \delta_<(a) \geq b \sum_{a=0}^{m} (\delta_<(a) - \delta_<(m)).
\]
It follows that
\[
\sum_{a \in A} \delta_<(a) \geq b \sum_{a=0}^{m} (\delta_<(a) - (b-l)\delta_<(m)).
\]
\[ \sum_{a \in A} \delta_<(a) \geq b \sum_{a=0}^{m-1} (\delta_<(\delta_<(a) + l(\delta_<(a)))(m). \]

Finally we assume that \( \delta_a(m) = 1 \). We have
\[ \sum_{a \in A} \delta_<(a) \geq \#\{a \in A, \{\deg(\tau(a))\}' \leq \{\frac{m-1}{d}' \} < \{p \deg(a)\}' \}. \]

Applying the last lemma with
\[ B = \{a \in A, \{\deg(a)\}' \leq \{\frac{m-1}{d}' \} \}, \]
and
\[ C = \{a \in A, \{\frac{m-1}{d}' \} < \{p \deg(a)\}' \}, \]
we get
\[ \sum_{a \in A} \delta_<(a) \geq \#\{a \in A, \{\deg(a)\}' \leq \{\frac{m-1}{d}' \} < \{p \deg(a)\}' \} \]
\[ -\#\{a \in A, \{\deg(a)\}' > \{\frac{m-1}{d}' \} \geq \{p \deg(a)\}' \}. \]

We have
\[ \#\{a \in A, \{\deg(a)\}' \leq \{\frac{m-1}{d}' \} < \{p \deg(a)\}' \} \]
\[ = b(\lfloor \frac{m}{d} \rfloor + 1) \#\{1 \leq a \leq d\{\frac{m-1}{d}' \} \leq \{\frac{m-1}{d}' \} < \{p \frac{a}{d}' \} \}. \]

We also have
\[ \#\{a \in A, \{\deg(a)\}' > \{\frac{m-1}{d}' \} \geq \{p \deg(a)\}' \} \]
\[ = b(\lfloor \frac{m}{d} \rfloor) \#\{d \geq a > d\{\frac{m-1}{d}' \} \geq \{\frac{m-1}{d}' \} \geq \{p \frac{a}{d}' \} \} + l1\{a \geq \frac{m}{d}' \} \geq \{p \frac{a}{d}' \} \}. \]

It follows that
\[ \sum_{a \in A} \delta_<(a) \geq b \#\{1 \leq a \leq d\{\frac{m-1}{d}' \} \leq \{\frac{m-1}{d}' \} < \{p \frac{a}{d}' \} \} \]
\[ -l1\{\frac{m-1}{d}' \geq \{p \frac{a}{d}' \} \} \geq b \sum_{a=0}^{m-1} (\delta_<(\delta_<(a) + l(\delta_<(a)))(m). \]

The proof of the theorem is completed. \( \square \)

**Theorem 2.4.** If \( p > 3d \), \( A \) is of cardinality \( bm + l \) with \( 0 \leq l < b \), then
\[ \sum_{a \in A} (\lfloor p \deg(a) \rfloor - \lfloor \deg(a) \rfloor + \delta_<(a)) \geq bp_\Delta(m) + l\varpi(m). \]

Moreover, the strict inequality holds if \( A \) is not of the form
\[ (\{1, \cdots, m-1\} \times \mathbb{Z}/(b)) \cup \{(m, i_0), \cdots, (m, i_{l-1})\}. \]
Proof. We may assume that $A$ is not of the form
\[
\{(1, \cdots, m-1) \times \mathbb{Z}/(b)) \cup \{(m, i_0), \cdots, (m, i_{l-1})\}.
\]
There is an element $a_0 \in A$ with $\deg(a_0) > \deg(m)$. Set $A' = A \setminus \{a_0\}$ and
\[
\tau'(a) = \begin{cases} 
\tau(a), & a \neq \tau^{-1}(a_0), \\
\tau(a_0), & a = \tau^{-1}(a_0).
\end{cases}
\]
Then
\[
\sum_{a \in A} \left([p \deg(a)] - \deg(a) + \delta_{\tau'}(a)\right)
> \sum_{a \in A'} \left([p \deg(a)] - \deg(a) + \delta_{\tau'}(a)\right) + \varpi(m).
\]
The theorem now follows by induction. \qed

We now assume that $\triangle \subset \mathbb{Z}$, and $A$ is a finite subset of $M(\triangle) \times \mathbb{Z}/(b)$.
For an integer $m \geq 1$, we define
\[
A_m = \{a \in M(\triangle) \mid \varpi(a) \leq p\triangle(m) - p\triangle(m-1)\}.
\]

**Theorem 2.5.** If $p > 3D$, then
\[
\sum_{a \in A} [\deg(pa - \tau(a))] \geq bp\triangle(m).
\]
Moreover, the strict inequality holds if $m$ is a turning point of $p\triangle$, and $A \neq A_m \times \mathbb{Z}/(b)$.

*Proof.* We define $\text{sgn}((i, u)) = \text{sgn}(i)$. Write
\[
A_i = \{a \in A \mid \text{sgn}(a) = (-1)^i\},
\]
and
\[
A_{ij} = \{a \in A \mid \text{sgn}(a) = (-1)^i, \text{sgn}(\tau(a)) = (-1)^j\}.
\]
Define a new permutation $\tau_0$ as follows:
\begin{itemize}
  \item $\tau_0$ is identity on $A_0$.
  \item $\tau_0 = \tau$ on $A_{11}$ and $A_{22}$.
  \item $\tau_0$ maps $A_1 \setminus A_{11}$ to $A_1 \setminus \tau(A_1)$.
  \item $\tau_0$ maps $A_2 \setminus A_{22}$ to $A_2 \setminus \tau(A_2)$.
\end{itemize}
We have
\[
\sum_{a \in A} [\deg(pa - \tau(a))] \geq \sum_{a \in A} \left([p \deg(a)] - \deg(\tau_0(a)) + 1_{\{\deg(\tau_0(a))' < \{p \deg(a)\}'\}}\right).
\]
The theorem now follows the last one. \qed
3. The \(T\)-adic Dwork Theory

In this section we review the \(T\)-adic analogue of Dwork theory on exponential sums.

Let
\[
E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^i}{p^i}\right) = \sum_{i=0}^{+\infty} \lambda_i t^i \in 1 + t\mathbb{Z}_p[[t]]
\]
be the \(p\)-adic Artin-Hasse exponential series. Define a new \(T\)-adic uniformizer \(\pi\) of \(\mathbb{Q}_p((T))\) by the formula
\[
E(\pi) = 1 + \pi^1.
\]
Let \(\pi_1/D\) be a fixed \(D\)-th root of \(\pi\). Let
\[
L = \{ \sum_{i \in M(\triangle)} c_i \pi^{\deg(i)} x^i : c_i \in \mathbb{Z}_q[[\pi^{1/D}]] \}.
\]

Let \(a \mapsto \hat{a}\) be the Teichmüller lifting. One can show that the series
\[
E_f(x) := \prod_{a_i \neq 0} E(\pi \hat{a}_i x^i) \in L.
\]

Note that the Galois group of \(\mathbb{Q}_q\) over \(\mathbb{Q}_p\) can act on \(L\) but keeping \(\pi^{1/D}\) as well as the variable \(x\) fixed. Let \(\sigma\) be the Frobenius element in the Galois group such that \(\sigma(\zeta) = \zeta^p\) if \(\zeta\) is a \((q-1)\)-th root of unity. Let \(\Psi_p\) be the operator on \(L\) defined by the formula
\[
\Psi_p(\sum_{i \in M(\triangle)} c_i x^i) = \sum_{i \in M(\triangle)} c_i \pi^{\deg(i)} x^i.
\]

Then \(\Psi := \sigma^{-1} \circ \Psi_p \circ E_f\) acts on the \(T\)-adic Banach module
\[
B = \{ \sum_{i \in M(\triangle)} c_i \pi^{\deg(i)} x^i \in L, \ \text{ord}_T(c_i) \to +\infty \text{ if } \deg(i) \to +\infty \}.
\]

We call it Dwork’s \(T\)-adic semi-linear operator because it is semi-linear over \(\mathbb{Z}_q[[\pi^{1/D}]]\).

Let \(b = \log_p q\). Then the \(b\)-iterate \(\Psi^b\) is linear over \(\mathbb{Z}_q[[\pi^{1/D}]]\), since
\[
\Psi^b = \Psi_p^b \circ \prod_{i=0}^{b-1} E_f(x^{p^i}).
\]

One can show that \(\Psi\) is completely continuous in the sense of Serre [Se]. So
\[
\text{det}(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{1/D}]]) \text{ and } \text{det}(1 - \Psi s | B/\mathbb{Z}_p[[\pi^{1/D}]])
\]
are well-defined.

We now state the \(T\)-adic Dwork trace formula [LWn].

**Theorem 3.1.** We have
\[
C_f(s, T) = \text{det}(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{1/D}]]).
\]
4. The Dwork semi-linear operator

Write

\[ E_f(x) = \sum_{i \in M(\Delta)} \gamma_i x^i, \]

and

\[ \det(1 - \Psi s | B/\mathbb{Z}_p[[\pi^{1/2}]]) = \sum_{i=0}^{+\infty} (-1)^i c_i s^i. \]

Let \( O(\pi^\alpha) \) denotes any element of \( \pi \)-adic order \( \geq \alpha \). In this section we prove the following.

**Theorem 4.1.** Let \( p > 3D \). Then

\[ \text{ord}_s(c_{bm}) \geq bp_\Delta(m). \]

Moreover, if \( m < \text{Vol}(\Delta) \) is a turning point of \( p_\Delta \), then

\[ c_{bm} = \pm \text{Norm}(\det(\gamma_{pi-j} i,j \in A_m)) + O(\pi^{bp_\Delta(m)+1/D}), \]

where \( \text{Norm} \) is the norm map from \( \mathbb{Q}_q(\pi^{1/D}) \) to \( \mathbb{Q}_p(\pi^{1/D}) \).

**Proof.** Fix a normal basis \( \bar{\xi}_u, u \in \mathbb{Z}/(b) \) of \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Let \( \xi_u \) be their Tera-
chmüller lift of \( \bar{\xi}_u \). Then \( \xi_u, u \in \mathbb{Z}/(b) \) is a normal basis of \( \mathbb{Q}_q \) over \( \mathbb{Q}_p \), and \( \sigma \) acts on the basis \( \xi_u, u \in \mathbb{Z}/(b) \) as a permutation. Let \( (\gamma_{i,u}(j,\omega))_{i,j \in M(\Delta), 1 \leq u, \omega \leq b} \) be the matrix of \( \Psi \) on \( B \otimes \mathbb{Z}_p \mathbb{Q}_p(\pi^{1/D}) \) with respect to the basis \( \{\xi_u x^i\}_{i \in M(\Delta), 1 \leq u \leq b} \).

Then

\[ c_{bm} = \sum_A \det((\gamma_{i,u}(j,\omega))_{(i,u),(j,\omega) \in A}), \]

where \( A \) runs over all subsets of \( M(\Delta) \times \mathbb{Z}/(b) \) with cardinality \( bm \). One can show that

\[ \det(\gamma_{i,j})_{i,j \in A_m \times \mathbb{Z}/(b)} = \pm \text{Norm}(\det((\gamma_{pi-j} i,j \in A_m))). \]

Therefore the theorem follows from the following.

**Theorem 4.2.** Let \( A \subset M(\Delta) \times \mathbb{Z}/(b) \) be a subset of cardinality \( bm \). If \( p > 3D \), then

\[ \text{ord}_T(\det(\gamma_{i,u}(j,\omega))_{(i,u),(j,\omega) \in A}) \geq bp_\Delta(m). \]

Moreover, if \( m < \text{Vol}(\Delta) \) is a turning point of \( p_\Delta \), and \( A \neq A_m \times \mathbb{Z}/(b) \), then the strict inequality holds.

**Proof.** one can show that \( \gamma_i = O(\pi^{[\deg(i)]}) \). And, from the equality

\[ (\xi_u \gamma_{pr-l})^{\sigma-1} = \sum_{\omega=1}^{b} \gamma(r,w),(l,u)\xi_\omega, \]

we infer that

\[ \gamma_{i,j} = O(\pi^{[\deg(pi-j)]}). \]
So we have
\[ \sum_{a \in A} \text{ord}_\pi(\gamma_{a,\tau(a)}) \geq \sum_{a \in A} \lceil \deg(pa - \tau(a)) \rceil \geq bp_\Delta(m). \]
Moreover, if \( m \) is a turning point of \( p_\Delta \), and \( A \neq A_m \times \mathbb{Z}/(b) \), then the strict inequality holds. \( \square \)

5. The Hasse Polynomial

In this section we study \( \det(\gamma_{pi-j})_{i,j \in A_m} \).

**Definition 5.1.** For each positive integer \( m \), we define \( S^0_m \) to be the set of permutations \( \tau \) of \( A_m \) satisfying \( \tau(0) = 0 \), and
\[ \frac{\tau(a)}{d(\text{sgn}(a))} \geq \deg(pa) - \lceil \deg(pa) - \deg(n) \rceil, \quad a \neq 0, \]
where \( n \) is the element of maximal degree in \( A_m \cap \text{sgn}(a)\mathbb{N} \).

**Lemma 5.2.** Let \( p > 3D \), \( m < \text{Vol}(\Delta) \) a turning point of \( p_\Delta \), and \( \tau \) a permutation of \( A_m \). Then
\[ \sum_{a \in A_m} \lceil \deg(pa - \tau(a)) \rceil \geq p_\Delta(m), \]
with equality holding if and only if \( \tau \in S^0_m \).

**Proof.** We assume that \( M(\Delta) = \mathbb{N} \). The other cases can be proved similarly.
In this case, \( \text{sgn}(a) = +1 \) if \( a \neq 0 \), and the element \( n \) of maximal degree in \( A_m \cap \text{sgn}(a)\mathbb{N} \) is \( m - 1 \). Let \( d \) be the nonzero endpoint of \( \Delta \). For \( a = 0 \), we have
\[ \lceil \deg(pa - \tau(a)) \rceil \geq 0 \]
with equality holding if and only if \( \tau(a) = 0 \). For \( a \neq 0 \), we have
\[ \left\lfloor \frac{pa - \tau(a)}{d} \right\rfloor \geq \left\lfloor \frac{pa - n}{d} \right\rfloor \]
with equality holding if and only if
\[ \frac{\tau(a)}{d} \geq \frac{pa}{d} - \left\lfloor \frac{pa - n}{d} \right\rfloor. \]
It follows that
\[ \sum_{a \in A_m} \lceil \deg(pa - \tau(a)) \rceil \geq \sum_{a=1}^{n} \lceil \deg(pa - n) \rceil = p_\Delta(m) \]
with equality holding if and only if \( \tau \in S^0_m \). The theorem is proved. \( \square \)

**Lemma 5.3.** We have
\[ \gamma_i = \pi^{\lceil \deg(i) \rceil} \sum_{\sum_{j \in \Delta} \lambda_j a_j^n_j} \prod_{j \in \Delta} \lambda_j a_j^n_j + O(\pi^{\lceil \deg(i) \rceil + 1}). \]
Proof. We have
\[ \gamma_i = \sum_{\sum_{j \in \Delta} n_j = i, n_j \geq 0} \prod_{j \in \Delta} P_j^{n_j} \prod_{j \in \Delta} \lambda_{n_j} a_j^{n_j}. \]
We also have that
\[ \sum_{j \in \Delta} n_j \geq \lceil \deg(i) \rceil \text{ if } \sum_{j \in \Delta} jn_j = i. \]
The lemma now follows.

Definition 5.4. For each positive integer \( m \), we define
\[ H_m(y) = \sum_{\tau \in S_m} sgn(\tau) \prod_{i \in A_m} \sum_{\sum_{j \in \Delta} n_j = pi - \tau(i)} \prod_{j \in \Delta} \lambda_{n_j} y_j^{n_j} \in \mathbb{Z}_p[y_j \mid j \in \Delta]. \]

Theorem 5.5. Let \( p > 3D \), and \( m < \text{Vol}(\Delta) \) a turning point of \( p_\Delta \). Then
\[ \det(\gamma_{pi-j})_{i,j \in A_m} = H_m((\bar{a}_j)_{j \in \Delta}) p^{\Delta(m)} + O(p^{\Delta(m)+1/D}). \]

Proof. Let \( S_m \) be the set of permutations of \( A_m \). We have
\[ \det(\gamma_{pi-j})_{i,j \in A_m} = \sum_{\tau \in S_m} \prod_{i \in A_m} \gamma_{pi-\tau(i)}. \]
The theorem now follows from the last two lemmas.

Definition 5.6. The reduction of \( H_m \) modulo \( p \) is denoted as \( \overline{H}_m \), and is called the Hasse polynomial of \( \Delta \) at \( m \).

Theorem 5.7. If \( p > 3D \), and \( m < \text{Vol}(\Delta) \) is a turning point of \( p_\Delta \), then \( \overline{H}_m \) is non-zero.

Proof. Define \( \deg(y_j) = |j| \). Then
\[ \prod_{i \in A_m} \sum_{\sum_{j \in \Delta} n_j = pi - \tau(i)} \prod_{j \in \Delta} \lambda_{n_j} y_j^{n_j} \]
has degree
\[ \sum_{i \in A_m} |pi - \tau(i)| = \sum_{i \in A_m} sgn(i)(pi - \tau(i)) \]
\[ = p \sum_{i \in A_m} sgn(i)i - \sum_{i \in A_m} sgn(i)\tau(i) \]
\[ \geq p \sum_{i \in A_m} sgn(i)i - \sum_{i \in A_m} sgn(\tau(i))\tau(i) \]
\[ \geq (p - 1) \sum_{i \in A_m} sgn(i)i, \]
with equality holding if and only if $\tau$ preserves the sign. Therefore it suffices to show that the reduction of
\[
\sum_{\tau \in S^1_m} \text{sgn}(\tau) \prod_{i \in A_m} \sum_{j \in \Delta} \prod_{j \in \Delta} \lambda_{n_j} y_j^{n_j},
\]
where $S^1_m$ consists of the sign-preserving permutations of $S^0_m$, is nonzero. One can prove this by the maximal-monomial-locating technique of Zhu [Zh1], as was used by Blache-Férard [BF].

**Definition 5.8.** We define $H = \prod_m \mathcal{M}_m$, where the product is over all turning points $m < \text{Vol}(\Delta)$ of $p_\Delta$.

**Theorem 5.9.** If $p > 3D$, then $H$ is non-zero.

**Proof.** This follows from the last theorem. \qed

6. Proof of the main theorem

In this section we prove the main theorems of this paper.

**Lemma 6.1.** The Newton polygon of $\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_q[[\pi^{1/3}]])$ coincides with that of $\det(1 - \Psi s \mid B/\mathbb{Z}_p[[\pi^{1/3}]])$.

**Proof.** Note that
\[
\det(1 - \Psi s \mid B/\mathbb{Z}_p[[\pi^{1/3}]]) = \text{Norm}(\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_q[[\pi^{1/3}]])),
\]
where Norm is the norm map from $\mathbb{Z}_q[[\pi^{1/3}]]$ to $\mathbb{Z}_p[[\pi^{1/3}]]$. The lemma now follows from the equality
\[
\prod_{\zeta^{i} = 1} \det(1 - \Psi \zeta s \mid B/\mathbb{Z}_p[[\pi^{1/3}]]) = \det(1 - \Psi^{b} s \mid B/\mathbb{Z}_p[[\pi^{1/3}]]).
\]
\qed

**Theorem 6.2.** The $T$-adic Newton polygon of $\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_q[[\pi^{1/3}]])$ is the lower convex closure of the points
\[
(m, \text{ord}_T(c_{bm})), \quad m = 0, 1, \ldots .
\]

**Proof.** By the last lemma, the $T$-adic Newton polygon of the power series $\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_q[[\pi^{1/3}]])$ is the lower convex closure of the points
\[
(i, \text{ord}_T(c_{i})), \quad i = 0, 1, \ldots .
\]
It is clear that $(i, \text{ord}_T(c_{i}))$ is not a vertex of that polygon if $b \nmid i$. So that Newton polygon is the lower convex closure of the points
\[
(bm, \text{ord}_T(c_{bm})), \quad m = 0, 1, \ldots .
\]
It follows that the $T$-adic Newton polygon of $\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_q[[\pi^{1/3}]])$ is the lower convex closure of the points
\[
(m, \text{ord}_T(c_{bm})), \quad m = 0, 1, \ldots .
\]
We now prove Theorem 1.12 which says that
\[ T \text{-adic NP of } C_f(s, T) \geq \operatorname{ord}_p(q)p_\Delta \text{ if } p > 3D. \]

**Proof of Theorem 1.12.** Combine the last theorem with the \( T \)-adic Dwork’s trace formula, we see that the \( T \)-adic Newton polygon of \( C_f(s, T) \) is the lower convex closure of the points
\[(m, \operatorname{ord}_T(c_m)), \ m = 0, 1, \ldots \]
The theorem now follows from the estimate
\[ \operatorname{ord}_T(c_m) \geq bp_\Delta(m). \]

We now prove Theorem 1.11 which says that
\[ p_\Delta \geq (p-1)H^\infty_\Delta \]
with equality holding at the point \( \operatorname{Vol}(\triangle) \).

**Proof of Theorem 1.11.** We assume that \( \Delta = [0, d] \). The other cases can be proved similarly. It suffices to show that
\[ p_\Delta \geq (p-1)H^\infty_\Delta \text{ on } [0, d] \]
with equality holding at the point \( \Delta \). Let \( 0 < m \leq d \). We have
\[
p_\Delta(m) = \sum_{0 \leq a < m} \left( \left\lceil \frac{(p-1)a}{d} \right\rceil - \delta \varepsilon(a) \right)
= \sum_{0 \leq a < m} \left( \left\lfloor \frac{pa}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \delta \varepsilon(a) - \delta \varepsilon(a) \right)
= \sum_{1 \leq a < m} \left( \left\lfloor \frac{pa}{d} \right\rfloor - 1 \right) + \sum_{1 \leq a < m} (\delta \varepsilon - \delta \varepsilon)(a)
= \sum_{1 \leq a < m} \left\lfloor \frac{pa}{d} \right\rfloor + \sum_{1 \leq a < m} (\delta \varepsilon - \delta \varepsilon)(a)
= p \sum_{1 \leq a < m} \frac{a}{d} - \sum_{1 \leq a < m} \left\{ \frac{pa}{d} \right\} + \sum_{1 \leq a < m} \frac{1}{d\left\{ \frac{pa}{d} \right\} \geq m} \cdot 1
= (p-1)H^\infty_\Delta(m) + \sum_{1 \leq a < m} \frac{a}{d} - \sum_{1 \leq a < m} \left\{ \frac{pa}{d} \right\} + \# \{ 1 \leq a < m \mid d\left\{ \frac{pa}{d} \right\} \geq m \}.
\]
In particular, we have
\[ p_\Delta(d) = (p-1)H^\infty_\Delta(d). \]

Note that
\[
\sum_{1 \leq a < m} \frac{a}{d} - \sum_{1 \leq a < m} \left\{ \frac{pa}{d} \right\} + \# \{ 1 \leq a < m \mid d\left\{ \frac{pa}{d} \right\} \geq m \}
\]
\[
\sum_{1 \leq a < m} \frac{a}{d} - \sum_{\substack{1 \leq a < m \\ d \left( \frac{a}{p} \right) < m}} \{\frac{pa}{d}\} \geq 0.
\]

It follows that
\[
p_{\Delta}(m) \geq (p - 1)H_{\infty}^m(m).
\]

**Theorem 6.3.** Let \(A(s,T)\) be a \(T\)-adic entrie series in \(s\) with unitary constant term. If \(0 \neq |t|_p < 1\), then
\[
t - \text{adic NP of } A(s,t) \geq T - \text{adic NP of } A(s,T),
\]
where \(NP\) is the short for Newton polygon. Moreover, the equality holds for one \(t\) if and only if it holds for all \(t\).

**Proof.** Write
\[
A(s,T) = \sum_{i=0}^{\infty} a_i(T)s^i.
\]

The inequality follows from the fact that \(a_i(T) \in \mathbb{Z}_q[[T]]\). Moreover,
\[
t - \text{adic NP of } A(s,t) = T - \text{adic NP of } A(s,T)
\]
if and only if
\[
a_i(T) \in T^e\mathbb{Z}_q[[T]]^\times
\]
for every turning point \((i,e)\) of the \(T\)-adic Newton polygon of \(A(s,T)\). It follows that the equality holds for one \(t\) iff it holds for all \(t\). \(\square\)

**Theorem 6.4.** Let \(f(x) = \sum_{u \in \Delta} (a_u, x^u, 0, 0, \cdots)\), and \(p > 3D\). If the equality
\[
\pi_m - \text{adic NP of } C_f(s,\pi_m) = \text{ord}_p(q)p_{\Delta}
\]
for one \(m \geq 1\), then it holds for all \(m \geq 1\), and we have
\[
T - \text{adic NP of } C_f(s,T) = \text{ord}_p(q)p_{\Delta}.
\]

**Proof.** This follows from Theorems 4.1 and 6.3 \(\square\)

**Theorem 6.5.** Let \(f(x) = \sum_{u \in \Delta} (a_u, x^u, 0, 0, \cdots)\). Then
\[
\pi_m - \text{adic NP of } C_f(s,\pi_m) = \text{ord}_p(q)p_{\Delta}
\]
if and only if
\[
\pi_m - \text{adic NP of } L_f(s,\pi_m) = \text{ord}_p(q)p_{\Delta} \text{ on } [0, p^{m-1}\text{Vol}(\Delta)].
\]

**Proof.** Assume that \(L_f(s,\pi_m) = \prod_{i=1}^{p^{m-1}d} (1 - \beta_is)\). Then
\[
C_f(s,\pi_m) = \prod_{j=0}^{\infty} L_f(q^js,\pi_m) = \prod_{j=0}^{\infty} \prod_{i=1}^{p^{m-1}d} (1 - \beta_iq^is).
\]
Therefore the slopes of the $q$-adic Newton polygon of $C_f(s, \pi_m)$ are the numbers

$$j + \text{ord}_q(\beta_i), \ 1 \leq i \leq p^{m-1}\text{Vol}(\triangle), \ j = 0, 1, \ldots.$$ 

One can show that the slopes of $p_\triangle$ are the numbers

$$j(p - 1) + p_\triangle(i) - p_\triangle(i - 1), \ 1 \leq i \leq p^{m-1}\text{Vol}(\triangle), \ j = 0, 1, \ldots.$$ 

It follows that

$$\pi_m - \text{adic NP of } C_f(s, \pi_m) = \text{ord}_p(q)p_\triangle$$

if and only if

$$\pi_m - \text{adic NP of } L_f(s, \pi_m) = \text{ord}_p(q)p_\triangle \text{ on } [0, p^{m-1}\text{Vol}(\triangle)].$$

□

We now prove Theorems 1.13, 1.14 and 1.15. By the above theorems, it suffices to prove the following.

**Theorem 6.6.** Let $f(x) = \sum_{u \in \triangle} (a_u x^u, 0, 0, \cdots)$, and $p > 3D$. Then

$$\pi_1 - \text{adic NP of } L_f(s, \pi_1) = \text{ord}_p(q)p_\triangle \text{ on } [0, \text{Vol}(\triangle)]$$

if and only if $H((a_u)_{u \in \triangle}) \neq 0$.

**Proof.** It is known that the $q$-adic Newton polygon of $L_f(s, \pi_1)$ coincides with $H_\infty^\infty$ at the point $\text{Vol}(\triangle)$. By Theorem 1.11, $p_\triangle$ coincide with $(p - 1)H_\infty^\infty$ at the point $\text{Vol}(\triangle)$. It follows that the $\pi_1$-adic Newton polygon of $L_f(s, \pi_1)$ coincides with $\text{ord}_p(q)p_\triangle$ at the point $\text{Vol}(\triangle)$. Therefore it suffices to show that

$$\pi_1 - \text{adic NP of } L_f(s, \pi_1) = \text{ord}_p(q)p_\triangle \text{ on } [0, \text{Vol}(\triangle) - 1]$$

if and only if $H((a_u)_{u \in \triangle}) \neq 0$.

From the identity

$$C_f(s, \pi_1) = \prod_{j=0}^\infty L_f(q^j s, \pi_1),$$

and the fact the $q$-adic orders of the reciprocal roots of $L_f(s, \pi_1)$ are no greater than 1, we infer that

$$\pi_1 - \text{adic NP of } L_f(s, \pi_1) = \pi_1 - \text{adic NP of } C_f(s, \pi_1) \text{ on } [0, \text{Vol}(\triangle) - 1].$$

Therefore it suffices to show that

$$\pi_1 - \text{adic NP of } C_f(s, \pi_1) = \text{ord}_p(q)p_\triangle \text{ on } [0, \text{Vol}(\triangle) - 1]$$

if and only if $H((a_u)_{u \in \triangle}) \neq 0$. The theorem now follows from the $T$-adic Dwork trace formula and Theorems 4.1 and 5.5. □
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