String Junctions and BPS States in Seiberg-Witten Theory

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Abstract

We argue that certain BPS states in the D3-brane probe realization of N=2 SU(2) Super-Yang-Mills theory correspond to multi-pronged strings connecting the D3-brane to the background 7-branes. This provides a physical realization of the decay of these states on the curve of marginal stability, and explains their absence in the strong coupling regime.

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1 Introduction

String theory provides new ways of approaching problems in field theory which are hard and often intractable using traditional field theory techniques. A case in point is the BPS spectrum of four dimensional $N = 4$ Super-Yang-Mills theory, with gauge group $SU(n)$. The conjectured $SL(2, \mathbb{Z})$ symmetry of this theory predicts the existence of a host of 1/2 BPS dyon states, i.e. states preserving 1/2 of the total supersymmetry. The existence of these states, however, has only been rigorously demonstrated in the one and two monopole cases [1], although there are indications that they exist as well in the multi-monopole case [2, 3]. In string theory one can study $N = 4$ SYM with gauge group $SU(n)$ as the low-energy world-volume theory of $n$ parallel D3-branes [4, 5]. The above BPS states correspond simply to type IIB strings between pairs of D3-branes. Since type IIB strings can carry arbitrary $(p, q)$ charges, where $p$ and $q$ are relatively prime [6], it follows that the 1/2 BPS states in the field theory carry such charges as well, and this gives the complete spectrum of 1/2 BPS states.

For $n > 2$, the $N = 4$ superalgebra [7] also allows states preserving only 1/4 of the supersymmetry, i.e. 1/4 BPS states [8]. One characteristic of these states is that they carry mutually non-local charges under the different $U(1)$ subgroups of $SU(n)$. It would therefore be difficult to establish their existence from the field theory point of view. On the other hand type IIB string theory contains objects carrying mutually non-local $(p, q)$ charges; these are the so-called multi-pronged strings [9, 10]. In [11] it was shown that three-pronged strings ending on three parallel D3-branes indeed correspond to 1/4 BPS states in the world-volume $N = 4$ $SU(3)$ SYM theory. This result can be generalized to $SU(n)$ and multi-pronged strings, with up to $n$ prongs [12].

In this paper we address the BPS spectrum of Seiberg-Witten theory [13], namely $N = 2$ SYM with gauge group $SU(2)$, in the same approach. In field theory the spectrum can be computed in the weak coupling regime, but becomes inaccessible when the coupling is strong. In fact it has been argued that the strong coupling BPS spectrum is different from the weak coupling one, and that the spectrum “jumps” as a curve of marginal stability is crossed [13, 14, 15]. The classical moduli space of the theory is parameterized by a complex number $z$, and has a singularity at the origin corresponding to a point of enhanced gauge symmetry. Quantum mechanically, however, this singularity is split into two points, located at $z = \pm \Lambda^2$, where hypermultiplets carrying charges $(0, 1)$ and $(2, 1)$ become massless. The exact low-energy effective action is expressed in terms of an analytic function $a(z)$ and its magnetic dual $a_D(z)$, which can be expressed as integrals of a meromorphic differential $\lambda$ over cycles of an auxiliary Riemann surface. In particular the SYM coupling constant is given by $\tau = da_D/da$. These functions also determine the mass of a BPS state with electric charge $p$ and magnetic charge $q$, as

$$m_{(p,q)} = |pa(z) + qa_D(z)|. \quad (1.1)$$
The spectrum in the weakly coupled regime consists of electrically charged W-boson vector multiplets, and hypermultiplets with magnetic charge $\pm 1$ and electric charge $2n$ with $n \in \mathbb{Z}$. The moduli space is divided into two regions by a curve of marginal stability $\mathcal{C}$, given by $\text{Im}(a_D/a) = 0$, which is diffeomorphic to a circle. It follows from (1.1) and the triangle inequality that the kinematic threshold for the decay of BPS states into lighter BPS states is saturated on this curve. It is also apparent that since $pa + qa_D = 0$ at the location of a $(p,q)$ singularity, this curve must pass through both hypermultiplet points. On this curve $a_D/a$ takes values between $-2$ and $2$. Specifically, on the upper half plane part of $\mathcal{C}$, $a_D/a \in [-2,0]$, and on the lower half plane part, $a_D/a \in [0,2]$. Let us denote the region outside $\mathcal{C}$, i.e. the weak coupling region, by $\mathcal{M}_+$, and the region inside $\mathcal{C}$, i.e. the strong coupling region, by $\mathcal{M}_-$. The BPS spectrum is expected to be different in the strong coupling regime [13]. In fact it was conjectured in [14], and shown in [15], that the spectrum in $\mathcal{M}_-$ consists solely of the $(0,1)$ and $(2,1)$ hypermultiplets, which become massless at $z = \pm \Lambda^2$. As a consequence, all other BPS states which exist in $\mathcal{M}_+$ must decay on the curve of marginal stability.

In string theory, $N = 2 \text{SU}(2)$ SYM can be understood as the low-energy theory on a D3-brane probe in the background of an orientifold 7-plane ($\Omega 7$) [16]. Sen has shown that this classical background is modified quantum mechanically, in accordance with F-theory, to a background of two mutually non-local 7-branes [19]. In the world-volume theory of the D3-brane this corresponds precisely to the singularity splitting of Seiberg-Witten theory. The two hypermultiplets which become massless at the two singularities correspond to open strings beginning on the D3-brane and ending on the respective 7-brane [20]. Since there are now three branes, one could in principle construct states from multi-pronged strings connecting them. We shall demonstrate that, other than the above two hypermultiplets, all the BPS states, including the W-bosons, correspond to multi-pronged strings connecting the D3-brane with the two 7-branes. Furthermore, this approach provides a physical realization of the decay of these BPS states on the curve of marginal stability.

The paper is organized as follows. In section 2 we review the D3-brane probe approach and the picture of BPS states as open strings. In section 3 we discuss multi-pronged strings in 7-brane backgrounds, and derive the supersymmetry conditions. In section 4 we show that the W-boson corresponds to a certain four-pronged string, and in section 5 we similarly construct multi-pronged string representations for the massive hypermultiplet dyons. Section 6 contains our summary and conclusions.

Note added: We were informed that results similar to ours will be reported in [21].

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1 We will work in the convention in which the charge of the W-boson is 2.

2 For an alternative approach to Seiberg-Witten theory and its BPS spectrum using six-dimensional self-dual strings see [17], and for an approach using M-theory see [18].
2 Review of 3-brane probe

Consider a D3-brane probe in the background of an Ω7-plane in type IIB string theory. Eight of the 32 supersymmetries are preserved, and the theory on the world-volume of the D3-brane is $N = 2 \ SU(2)$ SYM. The Yang-Mills coupling and theta angle are related to the string coupling and RR scalar as

$$\tau_{YM} = \tau_s,$$

where $\tau_{YM} \equiv \theta_{YM}/2\pi + i4\pi/g_Y^2$, and $\tau_s \equiv A + i/g_s$. The classical moduli space is given by the position of the D3-brane relative to the Ω7-plane, which we parameterize by the complex coordinate $z$. At a generic point the gauge group is broken to $U(1)$, and when the brane coincides with the plane ($z = 0$) it is enhanced to $SU(2)$. The photon corresponds to an open string beginning and ending on the D3-brane, and the W-bosons correspond to a string, of either orientation, which begins on the D3-brane, reflects off the Ω7-plane, and ends on the D3-brane with the same orientation. One can also introduce D7-branes into the background. These will give rise to hypermultiplets (quarks) in the D3-brane world-volume, corresponding to open strings beginning on the D3-brane and ending on the D7-branes (see fig. 1).

Figure 1: D3-brane (black circle) in the background of an Ω7-plane (cross) and a D7-brane (white circle). The BPS states are (a) photon, (b) W-boson, and (c) quark.

In such a background the string coupling acquires a $z$ dependence, which near the Ω7-plane or a D7-brane is given by

$$\tau(z) \sim \frac{\alpha}{2\pi i} \ln(z - z_i),$$

where $\alpha = 1$ for a D7-brane and $\alpha = -4$ for the Ω7-plane. If four D7-branes coincide with the Ω7-plane, the RR charge is canceled locally, and $\tau$ is everywhere constant. However, as pointed out by Sen [19], if one tries to move the D7-branes away from the orientifold plane, then sufficiently near the Ω7-plane $\text{Im}(\tau)$ becomes negative. The classical picture must therefore be modified non-perturbatively.
The appropriate modification is specified by F-theory. Consider a compactification of F-theory on a K3, which is given by a torus fibered over $P^1$. This is equivalent to a compactification of type IIB string theory on $P^1$ with 24 7-branes at positions $z_i$, where the fiber degenerates. In the $T^4/\mathbb{Z}_2$ orbifold limit of K3 the 7-branes divide into four groups of six coinciding 7-branes. This limit is equivalent to a type IIB orientifold compactification on $T^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action is given by $(-1)^F \Omega R$. In this language the four fixed points are $\Omega 7$-planes, which carry $-4$ units of RR charge. Consistency requires the addition of four D7-branes at each fixed point, in order to cancel the RR charge. We end up with precisely (four times) the classical background considered above. It is then clear from the F-theory point of view how to deform this configuration; one simply moves away from the orbifold limit of the K3 into a smooth K3. The equation governing such deformations close to one of the fixed points is given by

$$y^2 = x^3 + f(z)x + g(z), \quad (2.3)$$

where $f$ and $g$ are polynomials of degree 2 and 3 respectively. The number of deformations is given by the number of coefficients in the polynomials $f$ and $g$, modulo shifting $z$, and rescaling $z$, $x$, and $y$. This gives five complex parameters, one of which is the asymptotic value of $\tau$. The remaining four parameters specify the positions of the six 7-branes. Thus we have fewer parameters than 7-branes.

It was noted in [19] that eq. (2.3) is identical to the equation for the Seiberg-Witten curve for $SU(2)$ with $N_f = 4$. Furthermore one can identify the four parameters with the masses of the four quarks. In analogy with the Seiberg-Witten solution, one concludes therefore that the classical gauge enhancement singularity, given in this case by the $\Omega 7$-plane, splits into two mutually non-local hypermultiplet points, given in this case by two 7-branes with different $(p, q)$ charges. These can be taken to be $(0, 1)$ and $(2, 1)$, in the convention that a D7-brane has charges $(1, 0)$. The appearance of the Seiberg-Witten curve is not a coincidence, and follows from the fact that the theory on the D3-brane probe in this background is precisely Seiberg-Witten theory [16].

A picture of BPS states in the probe theory was developed in [20]. There it was argued that a $(p, q)$ hypermultiplet dyon state in the probe theory corresponds to a $(p, q)$ string stretched between the D3-brane probe and a 7-brane. Note that $(p, q)$ need not coincide with the charges of the 7-brane, as the string can undergo a monodromy when encircling a 7-brane. A $(p, q)$ state exists as long as there exists a path along which the total monodromy transforms the $(p, q)$ charges of the string to the charges of the 7-brane. For this state to be BPS saturated the path should correspond to a $(p, q)$ “geodesic”. In other words it should minimize the mass of the $(p, q)$ string, given by

$$m_{(p, q)} = \int_T T_{(p, q)} ds, \quad (2.4)$$
Figure 2: Quantum-mechanical resolution of the Ω7-plane into a (0, 1) 7-brane (square) at $z = 8\Lambda^2$, and a (2, 1) 7-brane (triangle) at $z = -8\Lambda^2$. The curve of marginal stability $C$ separates the weak coupling region $\mathcal{M}_+$ from the strong coupling region $\mathcal{M}_-$. 

where $T_{(p,q)}$ is the tension of a $(p, q)$ string

$$T_{(p,q)} = \frac{1}{\sqrt{\text{Im}(\tau)}} |p + q\tau| .$$

(2.5)

It is understood that the $(p, q)$ charges of the string may change along the path $P$. The metric $ds^2$ is given by \[22\]

$$ds^2 = \text{Im}(\tau) \left| \frac{\eta^2(\tau)}{\sqrt{2\eta^2(\tau_0)}} \prod_{i=1}^{6} (z - z_i)^{-1/12} dz \right|^2,$$

(2.6)

where $z_i$ are the locations of the six 7-branes, and $\tau_0$ is the asymptotic value of $\tau$. We are primarily interested in the case $N_f = 0$, which corresponds to removing all four D7-branes to infinity. As shown in \[23\], the metric in this case becomes

$$ds^2 = \text{Im}(\tau) \left| \frac{\eta^2(\tau)}{\sqrt{2\Lambda^2}} \prod_{i=1}^{2} (z - z_i)^{-1/12} dz \right|^2,$$

(2.7)

where

$$z_1 = 8\Lambda^2 \quad , \quad z_2 = -8\Lambda^2$$

(2.8)

are the positions of the $(0, 1)$ and $(2, 1)$ 7-branes respectively (see fig. 2). The above metric is related to the Seiberg-Witten solution as

$$T_{(p,q)} ds = |pda + qda_D| ,$$

(2.9)
which implies that the geodesic equation for a \((p, q)\) string is given by
\[
p \frac{da}{dt} + q \frac{d a_D}{dt} = (p + q \tau) \frac{da}{dt} = c, \tag{2.10}
\]
where \(c\) is a constant. For topologically trivial paths, \textit{i.e.} paths which do not go around 7-branes, the solution is simply given by
\[
 p a(z(t)) + q a_D(z(t)) = c(t - 1), \tag{2.11}
\]
where \(z(0)\) is the position of the D3-brane, and \(z(1)\) is the position of the \((p, q)\) 7-brane. The geodesic is then gotten by solving for \(z(t)\). The mass can now easily be calculated, and is given by
\[
m_{(p, q)} = |c| = |p a(z(0)) + q a_D(z(0))|, \tag{2.12}
\]
in agreement with the mass formula for BPS states in Seiberg-Witten theory \([1,1]\). The only states which correspond to topologically trivial geodesics are the \((0, 1), (2, 1)\) hypermultiplets (assuming \(N_f = 0\)), as these are the charges of the 7-branes (fig. 3a).

The hypermultiplet states with charges \((2n, 1)\) \((n > 1)\) would have to correspond to strings along topologically non-trivial paths, in order to pick up the appropriate monodromies (fig. 3b,c). One could solve eq. (2.10) for fixed values of \(p\) and \(q\), if we allow \(a\) and \(a_D\) to be multivalued functions, by continuing them along the path. We shall denote these continuations as \(\tilde{a}\) and \(\tilde{a}_D\). The solution is given by
\[
 p \tilde{a}(z(t)) + q \tilde{a}_D(z(t)) = c(t - 1). \tag{2.13}
\]
Since \(\tilde{a}\) and \(\tilde{a}_D\) are not single valued however, it is no longer obvious that there exists a solution for \(z(t)\), \textit{i.e.} a geodesic.

The W-boson, which carries the charges \((2, 0)\), would have to correspond to a \((1, 0)\) string that begins and ends on the D3-brane. The orientations of the two ends must be the same, and therefore the string must undergo a monodromy which reverses its orientation. This is possible if the string winds once around the \((0, 1)\) and \((2, 1)\) 7-branes (fig. 3d), for which the monodromy is given by
\[
 M = \begin{pmatrix}
 -1 & 4 \\
 0 & -1
 \end{pmatrix}. \tag{2.14}
\]
The solution satisfying this requirement is given by
\[
 2 \tilde{a}(z(t)) = c(t - 1/2), \tag{2.15}
\]
and it is again not obvious that one can solve for \(z(t)\).

In \([23]\) it was shown that there are no topologically non-trivial geodesics with base points in the strong coupling region \(\mathcal{M}_-\). This is consistent with the conjecture that the BPS spectrum in \(\mathcal{M}_-\) consists of only the \((0, 1)\) and \((2, 1)\) states, which correspond to topologically
trivial geodesics. On the other hand, it is not at all clear that non-trivial geodesics corresponding to the additional BPS states exist even in the weak coupling region $\mathcal{M}_+$. In fact one can argue that a geodesic corresponding to the W-boson does not exist. The equation for such a geodesic (2.15) implies that $\tilde{a}$ vanishes somewhere along the geodesic (at $t = 1/2$). For that to be the case, the geodesic must pass through a 7-brane, whose $(p, q)$ charges agree with those of the string. However, since the only 7-branes present carry charges $(0, 1)$ and $(2, 1)$, and the charges of the string vary from $(1, 0)$ to $(1, 1)$ and back to $(1, 0)$ along the path, this is impossible. In section 4 we shall give another argument against the existence of such a geodesic.

In the following we shall offer an alternative picture for the additional BPS states. We shall argue that these states are given by multi-pronged strings connecting the D3-brane to the two 7-branes. This picture has the advantage that it gives a clear physical explanation for the decay process. When the prongs ending on the D3-brane degenerate, the two other sets of prongs are free to separate along the D3-brane. As we will show, this happens precisely when the D3-brane is on the curve of marginal stability.

3 Multi-pronged strings in 7-brane backgrounds

Multi-pronged strings are generalizations of ordinary $(p, q)$ strings in type IIB string theory, which possess more than two ends. The simplest such object is the three-pronged string, consisting of three strings with charges $(p_i, q_i)$ coming together at a junction point, such that $\sum p_i = \sum q_i = 0$ [9, 10]. Other multi-pronged strings, also known as string networks or string webs, are formed by connecting a number of three-pronged strings.

Supersymmetry restricts the geometry of string networks [24, 25]. In the absence of 7-branes the strings must be straight, the network must be planar, and the relative orientations
of the strings are fixed according to their \((p, q)\) charges. If we fix the coordinates on the plane so that \((1, 0)\) strings are oriented in the real direction, the condition for supersymmetry given in \([24, 25]\) is that \((p, q)\) strings are oriented in the \((p + q\tau)\) direction. In fact an equally supersymmetric network is obtained if \((p, q)\) strings are oriented instead in the \((p + q\tau)\) direction. This network is related to the previous one by reflection about the real axis. More generally, if we denote by \(\phi_i\) the direction in the complex plane of a \((p_i, q_i)\) string in a string network, the condition for supersymmetry is either
\[
e^{i\phi_i} = e^{i\phi} \frac{p_i + q_i\tau}{|p_i + q_i\tau|} \quad \text{for all } i ,
\] (3.1)
or
\[
e^{i\phi_i} = e^{i\phi} \frac{p_i + q_i\tau}{|p_i + q_i\tau|} \quad \text{for all } i ,
\] (3.2)
where \(\phi\) is the direction of the \((1, 0)\) string. In particular, this means that when a number of strings, with \(\sum p_i = \sum q_i = 0\), meet at a junction the relative angles are fixed by supersymmetry in such a way as to give a vanishing force \([10, 24, 27]\).

The amount of supersymmetry preserved by string networks is one fourth of the original amount in type IIB string theory, i.e. 8 supersymmetries. If additional D-branes are present the amount of unbroken supersymmetry depends on the type of D-branes and their orientation relative to the string network. It was shown in \([11]\) that if the D-branes are transverse to the string network, supersymmetry is only preserved for D3-branes and D7-branes. In the former case one has 4 supersymmetries, and in the latter case 8 supersymmetries. The latter case can be generalized to any combination of parallel \((p, q)\) 7-branes, which alone would preserve the same 16 supersymmetries as a D7-brane. This includes the background discussed in the previous section. Furthermore, including D3-branes in such backgrounds will reduce the unbroken supersymmetries to 4.

In a generic background of 7-branes strings are not straight\(^3\), but rather follow geodesics (2.10) which are particular to the \((p, q)\) charges of the string. The orientation of a \((p_i, q_i)\) string along its geodesic is given by
\[
e^{i\phi_i(z)} = \frac{dz/dt_i}{|dz/dt_i|} = \frac{|p_i + q_i\tau(z)|}{(p_i + q_i\tau(z))} \frac{|da/dz|}{|c_i|} .
\] (3.3)
For a network of strings we must apply the supersymmetry condition (3.1) or (3.2) locally. This gives either
\[
e^{i\phi_i(z)} = e^{i\phi(z)} \frac{p_i + q_i\tau(z)}{|p_i + q_i\tau(z)|} \quad \text{for all } i ,
\] (3.4)
\(^3\)There are special 7-brane backgrounds for which \(\tau\) is constant, and strings are therefore straight \([27, 28]\).
\( e^{i\phi_i(z)} = e^{i\phi(z)} \frac{p_i + q_i \tau(z)}{|p_i + q_i \tau(z)|} \) for all \( i \), \hspace{1cm} (3.5)\)

where \( \phi(z) \) is the local orientation of the \((1,0)\) string. From (3.5) we see that this is given by

\[ e^{i\phi(z)} = \left| \frac{da/dz}{da/dz} \right| e^{i\phi}, \hspace{1cm} (3.6) \]

where \( \phi \) is an arbitrary constant angle, which corresponds to the (constant) orientation of the \((1,0)\) string in flat space. Consequently we deduce that only (3.5) is consistent with (3.3)\(^4\), and that

\[ c_i / |c_i| = e^{i\phi} \hspace{1cm} \text{for all } i, \hspace{1cm} (3.7) \]

in other words that the phases of all the \( c_i \) are equal.

## 4 W-bosons

Let us now return to the particular 7-brane background of section 2. In the classical picture the W-boson corresponds to a pair of fundamental strings beginning on the D3-brane and “ending” on the \(\Omega\)-plane. Quantum corrections deform this background into two separated 7-branes, with charges \((0,1)\) and \((2,1)\). At the same time the W-boson state must somehow be deformed as well. One possibility is that it becomes an open string which winds around the two 7-branes (fig. 4a). Another possibility is that it is deformed into a four-pronged string, with external prongs \((1,0),(1,0),(0,1)\) and \((2,1)\), and an internal \((1,1)\) prong \(^5\), such that the first two external prongs end on the D3-brane, and the last two end on the corresponding 7-branes (fig. 4b).

These two distinct configurations are in fact connected by the process of string creation \(\)\(^2\). In the simplest case this phenomenon occurs when two mutually transverse D-branes cross. In particular when a D-string, \( i.e. \) a \((0,1)\) string, crosses a D7-brane, \( i.e. \) a \((1,0)\) 7-brane, a fundamental \((1,0)\) string is created between them. Using \( SL(2,\mathbb{Z}) \) this can be generalized to a \((p,q)\) string crossing a \((p',q')\) 7-brane with \( |pq' - p'q| = 1 \), in which case a \((p',q')\) string is created. The phenomenon can further be generalized using charge conservation to arbitrary values of \( |pq' - p'q| \), in which case \( |pq' - p'q| \) \((p',q')\) strings are created.

Let us apply the supersymmetry conditions to the above configurations. In both cases two ends of a \((1,0)\) string end on the D3-brane with the same orientation. Supersymmetry

\(^4\)It is not too surprising that only one of (3.4) or (3.5) is consistent with supersymmetry, since generic 7-brane backgrounds involve non-trivial monodromies which break the reflection symmetry.

\(^5\)We will drop overall minus signs in \((p,q)\), remembering that the orientations have to be such as to satisfy charge conservation at each junction.
Figure 4: Possible quantum deformations of W-boson: (a) A single string on a topologically non-trivial path. (b) A four pronged string connecting the D3-brane to the two 7-branes. (c) Four-pronged string with degenerate intermediate prong. The last one is the only supersymmetric configuration.

requires that the two (1, 0) strings be parallel at the location of the D3-brane. Since the geodesics for these strings are uniquely determined by their endpoint and tangent at the endpoint, the two (1, 0) strings must follow the same geodesic. This condition cannot be satisfied for a single string following a topologically non-trivial path (fig. 4a), so such a path cannot correspond to a geodesic. On the other hand the four-pronged string configuration (fig. 4b) will satisfy this condition if the internal (1, 1) string shrinks to zero length, and the two three-string junctions coincide (fig. 4c).

It should be noted that a similar conclusion which rules out topologically nontrivial closed geodesics was reached in [28] without the use of supersymmetry.

Let us denote the locations of the (0, 1) 7-brane, (2, 1) 7-brane, and D3-brane by \( z_1, z_2, \) and \( z_3, \) respectively, and the location of the two coincident junction points by \( z_0. \) Parameterize the geodesics for the (0, 1), (2, 1), and (1, 0) strings by \( t_1, t_2 \) and \( t_3, \) respectively, where \( t_i = 0 \) corresponds to the position of the junction, and \( t_i = 1 \) to the position of the brane. The three geodesics are then given by

\[
\begin{align*}
P_1: \quad a_D(z(t_1)) & = c_1 t_1 + a_D(z_0) \\
P_2: \quad (-2a - a_D)(z(t_2)) & = c_2 t_2 - 2a(z_0) - a_D(z_0) \\
P_3: \quad a(z(t_3)) & = c_3 t_3 + a(z_0)
\end{align*}
\]

(4.1)

where

\[
\begin{align*}
c_1 & = a_D(z_1) - a_D(z_0) = -a_D(z_0) \\
c_2 & = 2a(z_0) + a_D(z_0) - 2a(z_2) - a_D(z_2) = 2a(z_0) + a_D(z_0) \\
c_3 & = a(z_3) - a(z_0)
\end{align*}
\]

(4.2)
The second equality in the first two equations follows from the fact that \( pa + qa_D = 0 \) at the location of a \((p, q)\) 7-brane. The supersymmetry condition (3.7) is

\[
c_1/|c_1| = c_2/|c_2| = c_3/|c_3|.
\]

(4.3)

Using (4.2), the first equality implies that

\[
\text{Im} \left( \frac{a_D(z_0)}{a(z_0)} \right) = 0, \quad \frac{a_D(z_0)}{a(z_0)} > -2,
\]

(4.4)

and the second equality implies that

\[
\text{Im} \left( \frac{a(z_3)}{a(z_0)} \right) = 0, \quad \frac{a(z_3)}{a(z_0)} > 1.
\]

(4.5)

The first condition requires \( z_0 \) to be on the curve of marginal stability. The second condition can be viewed as a condition on \( z_3 \), i.e. the position of the D3-brane; namely that \( z_3 \) and \( z_0 \) be located on a curve of constant phase for \( a(z) \), and that \( |a(z_3)| > |a(z_0)| \). This is satisfied only if \( z_3 \in \mathcal{M}_+ \), i.e. outside the curve of marginal stability (see Appendix B for proof).

It follows from the behavior of \( a(z) \) near \( z = \pm 1 \) (Appendix A), that its phase varies on the upper half plane part of \( C \) from 0 at \( z = 1 \) to \( \pi/2 \) at \( z = -1 \). It also follows from the asymptotic behavior for large \( |z| \) that the phase varies on the upper half plane from 0 at \( z \to +\infty \) to \( \pi/2 \) at \( z \to -\infty \). This implies that the constant phase curves which cross the upper half plane part of \( C \) foliate the upper half plane. A similar argument holds for the lower half plane.

We conclude that the state corresponding to the four-pronged string is BPS saturated if, and only if, the D3-brane is in the region \( \mathcal{M}_+ \). When the D3-brane is exactly on the curve of marginal stability, i.e. when \( z_3 = z_0 \), the \((1, 0)\) prongs degenerate, and the remaining \((0, 1)\) and \((2, 1)\) strings can separate along the D3-brane (see fig. 5).

The total mass is given by the sum of the four individual prong masses

\[
M = |c_1| + |c_2| + 2|c_3|.
\]

(4.6)

The supersymmetry condition (4.3) then implies that

\[
M = |c_1 + c_2 + 2c_3| = 2|a(z_3)|,
\]

(4.7)

which is precisely the (BPS) mass of the W-boson.

## 5 Hypermultiplets

In addition to the W-bosons and the two hypermultiplets carrying charges \((0, 1)\) and \((2, 1)\), the BPS spectrum in the region \( \mathcal{M}_+ \) includes hypermultiplets carrying charges \((2n, 1)\), with
Figure 5: Decay of W-boson on curve of marginal stability. When the D3-brane is on this curve, the two $(1,0)$ prongs degenerate, leaving only the $(0,1)$ and $(2,1)$ strings between the D3-brane and the respective 7-brane, which can now separate along the D3-brane.

$n > 1$. These are known to exist as solitons in the weakly coupled field theory, and can in fact be obtained from the $(0,1)$ and $(2,1)$ states by performing monodromy transformations using (2.14).

A single string configuration for these states is shown in figs. 6a,b, and the corresponding topologically trivial multi-pronged string configuration is shown in fig. 6c. As before, the two configurations are connected via string creation. Unlike the W-boson case however, it is not clear whether supersymmetry forbids the single string geodesic. On the other hand, we can still impose the supersymmetry conditions on the multi-pronged string. The geodesics for the three sets of prongs are characterized by the complex numbers $c_i$, which are now given by

\begin{align*}
c_1 &= (1 - n)a_D(z_0) \\
c_2 &= n(2a(z_0) + a_D(z_0)) \\
c_3 &= 2na(z_3) + a_D(z_3) - 2na(z_0) - a_D(z_0) .
\end{align*}

(5.1)

Equating the phases of $c_1$ and $c_2$ gives (4.4) again, which restricts $z_0$ to lie on the curve of marginal stability. Equating the phases of $c_2$ and $c_3$ then gives

\begin{align*}
\text{Im} \frac{2na(z_3) + a_D(z_3)}{2na(z_0) + a_D(z_0)} &= 0 , \\
\frac{2na(z_3) + a_D(z_3)}{2na(z_0) + a_D(z_0)} &> 1 ,
\end{align*}

(5.2)

which means that $z_3$ and $z_0$ must lie on the same curve of constant phase for the function $f(z) = 2na(z) + a_D(z)$, and that $|f(z_3)| > |f(z_0)|$. As before, this condition is satisfied only if $z_3 \in \mathcal{M}_+$ (see Appendix B).

As in the W-boson case, it follows that the multi-pronged string in fig. 6c is BPS only if the D3-brane is in $\mathcal{M}_+$, and decays to $n$ $(2,1)$ strings and $(n-1)$ $(0,1)$ strings on the curve of marginal stability. The mass of this state is given by

\begin{align*}
M &= |c_1| + |c_2| + |c_3| ,
\end{align*}

(5.3)
Figure 6: Possible configurations for the massive hypermultiplets carrying charges $(2n, 1)$. These states could correspond to a single string on a topologically non-trivial path, (a) in the case $n = 2k$, and (b) in the case $n = 2k + 1$, or to a multi-pronged string (c).

which, due the equality of the phases of $c_1$, $c_2$ and $c_3$, becomes

$$M = |c_1 + c_2 + c_3| = |na(z_3) + a_D(z_3)|,$$

(5.4)
in precise agreement with the Seiberg-Witten result (1.1).

There is one subtlety we should point out. Unlike the W-boson case, the constant phase curves of the function $f(z) = 2na(z) + a_D(z)$ which cross the upper half plane part of $C$ do not foliate the upper half plane. This can be seen from the behavior of the phase of $f(z)$ on $C$ and at infinity, which follows from the asymptotic behaviors of $a$ and $a_D$ (Appendix A). Specifically, the phase of $f(z)$ varies on the upper half plane part of $C$ from 0 at $z = 1$ to $\pi/2$ at $z = -1$. On the other hand, at infinity it varies on the upper half plane from $\pi/2$ at $z \to +\infty$ to $\pi$ at $z \to -\infty$, and on the lower half plane from 0 at $z \to -\infty$ to $\pi/2$ at $z \to +\infty$. Consequently, the multi-pronged string state cannot be BPS everywhere in $M_+$, and we expect to see a transition to a single geodesic string state, of the kind studied in [28]. This issue is currently under study [29].

Let us consider more generally a state carrying charges $(2n, m)$. For $|m| > 1$ there does not exist a single string representation of this state. One could however construct a suitable multi-pronged string, consisting of a $(2n, m)$ prong $\mathbb{I}$, $n (2, 1)$ prongs, and $(n - m) (0, 1)$ prongs. The geodesic constants are given by

$$c_1 = (m - n)a_D(z_0)$$
$$c_2 = n(2a(z_0) + a_D(z_0))$$
$$c_3 = 2na(z_3) + ma_D(z_3) - 2na(z_0) - ma_D(z_0),$$

(5.5)

$^6$If $2n$ and $m$ are not relatively prime this would actually be $k$ prongs, where $k$ is the largest common divisor.

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for the $(0, 1)$, $(2, 1)$, and $(2n, m)$ prongs, respectively. The supersymmetry conditions are now

$$\text{Im} \frac{a_D(z_0)}{a(z_0)} = 0,$$

and

$$\text{Im} \frac{2na(z_3) + ma_D(z_3)}{2na(z_0) + ma_D(z_0)} = 0, \quad \frac{2na(z_3) + ma_D(z_3)}{2na(z_0) + ma_D(z_0)} > 1.$$ (5.6) (5.7)

The first condition means that $z_0$ is again on the curve of marginal stability. However, since $|a_D(z_0)/a(z_0)| < 2$ on this curve, a solution exists only if $|m| < |n|$. This is consistent with the fact that from the field theory point of view, states with $|m| > |n|$ would have to become massless somewhere on the curve of marginal stability, even though there are no singularities on this curve, other than the two corresponding to $(0, 1)$ and $(2, 1)$ [15]. The second condition is satisfied, as before, only if $z_3 \in \mathcal{M}_\pm$ (Appendix B).

It thus appears that BPS multi-pronged strings exist for all states with charges $(2n, m)$, under the condition that $|m| < |n|$ when $n > 1$. This includes in particular states with $|m| > 1$, which do not exist in the field theory. As argued in [15], these states are related by the monodromy at infinity (2.14) to states with $|m| > |n|$. This argument is somewhat unsatisfactory from the string theory point of view, as one would like an argument based solely on the consistency of a given string configuration. We do not have such an argument at this time.

6 Conclusions

We have demonstrated that the entire BPS spectrum of Seiberg-Witten theory, at strong coupling as well as at weak coupling, can be accounted for in the D3-brane probe picture, either as open strings on topologically trivial geodesics between the D3-brane and a single 7-brane, or as multi-pronged strings connecting the D3-brane to both 7-branes. The former correspond to the hypermultiplets carrying charges $(0, 1)$ and $(2, 1)$, and exist everywhere in the moduli space. The latter correspond to the W-bosons and hypermultiplets carrying charges $(2n, 1)$ ($n > 1$), and exist only in the region outside the curve of marginal stability. Their decay on this curve is described by a simple string theory process in which strings between the D3-brane and the 7-branes separate along the D3-brane.

For the case of the W-boson we were able to show that the multi-pronged string is the unique BPS representative. There are no topologically non-trivial geodesics beginning and ending on the D3-brane (see also [28]). An analogous argument for the massive hypermultiplets is needed, since one clearly expects the BPS states to be unique.

An equally pressing issue is the apparent existence of BPS multi-pronged strings corresponding to states which do not exist in the field theory, namely hypermultiplets with
charges $(2n, m)$, such that $|n| > |m| > 1$. One would like to rule out these states at the string theory level by some consistency argument, like the “s-rule” for example \[30\].

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**Appendix A**

In this appendix we summarize the asymptotic behaviors of $a(z)$ and $a_D(z)$ near the singularities $z = \pm 1, \infty$, as given in \[13\]. The functions $a(z)$ and $a_D(z)$ can be expressed in terms of hypergeometric functions as

\[
a(z) = \left(\frac{z + 1}{2}\right)^{1/2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2}{z + 1}\right)
\]

\[
a_D(z) = i\frac{z - 1}{2} F\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{1 - z}{2}\right).
\]  

(A.1)

Near the point at infinity, the asymptotic behaviors are given by

\[
a(z) \sim \sqrt{z/2}
\]

\[
a_D(z) \sim \frac{i}{\pi} \sqrt{2z} \left[\ln z + 3 \ln 2 - 2\right].
\]  

(A.2)

Near the branch point at $z = -1$:

\[
a(z) \sim \frac{i}{2\pi} \left[\epsilon \frac{z + 1}{2} \ln \frac{z + 1}{2} + \frac{z + 1}{2} \left(-i\pi - \epsilon(1 + 4 \ln 2)\right) + 4\epsilon\right]
\]

\[
a_D(z) \sim \frac{i}{\pi} \left[-\frac{z + 1}{2} \ln \frac{z + 1}{2} + \frac{z + 1}{2} (1 + 4 \ln 2) - 4\right],
\]  

(A.3)

and near the branch point at $z = 1$:

\[
a(z) \sim \frac{2}{\pi} - \frac{1}{2\pi} \left[\ln \frac{z - 1}{2} + 1 - 4 \ln 2\right]
\]

\[
a_D(z) \sim \frac{i}{2} \frac{z - 1}{2}.
\]  

(A.4)
Appendix B

In this appendix we would like to prove the following theorem, relevant to sections 4 and 5:

**Theorem 1** Let \( f(z) = pa(z) + qa_D(z) \), and let \( z_0 \) be a point on the curve of marginal stability \( C \), defined by \( \text{Im}(a_D/a) = 0 \). Then for any point \( z \in M_+ \), i.e. outside \( C \), such that \( \text{Im}(f(z)/f(z_0)) = 0 \), the magnitude of \( f \) satisfies \(|f(z)| > |f(z_0)|\).

The condition \( \text{Im}(f(z)/f(z_0)) = 0 \) means that the points \( z \) and \( z_0 \) lie on the same curve of constant phase for the function \( f(z) \). We shall prove the above theorem in two steps. First we will show that \(|f(z)|\) is monotonic along curves of constant phase. Then we will show that \(|f(z)|\) is asymptotically increasing as we move away from \( C \) along curves of constant phase. The result \(|f(z)| > |f(z_0)|\) then follows.

**Step 1:** Curves of constant phase are given by

\[
\frac{d}{dt} \text{Im} \left( \ln f(z) \right) = \frac{1}{2} |f|^{-2} \left( \bar{f} \dot{f} - f \ddot{f} \right) = 0 .
\]  
(B.1)

This in turn implies that

\[
\frac{d}{dt} |f|^2 = \bar{f} \dot{f} + f \ddot{f} = 2 \bar{f} \dot{f} ,
\]  
(B.2)

So

\[
\frac{d}{dt} |f|^2 = 0 \iff f = 0 \text{ or } \dot{f} = 0 .
\]  
(B.3)

However, \( f \) can only vanish for \((p, q) = (0, 1)\) or \((2, 1)\), and this happens at the locations of the 7-branes. As we are interested in other values of \((p, q)\), we conclude that \(|f|\) can have an extremum only if \( \dot{f} \) vanishes somewhere. This would imply

\[ p \dot{a} + q \dot{a}_D = 0 , \]
(B.4)

and therefore that

\[ \tau \equiv \frac{d a_D}{d a} = -\frac{p}{q} \in \mathbb{R} . \]
(B.5)

On the other hand, positivity of the metric on moduli space requires \( \text{Im}(\tau) > 0 \), so \( \dot{f} \) cannot vanish anywhere. \(|f(z)|\) is therefore monotonic along curves of constant phase.

**Step 2:** From the asymptotic behaviors of \( a \) and \( a_D \) as \(|z| \to \infty \) (A.2) we see that

\[ f(z) \sim iq \sqrt{r} \ln r e^{i\theta/2} \quad \text{as} \quad r \to \infty , \]
(B.6)

where \( z = re^{i\theta} \). It follows that in the limit \( r \to \infty \) the constant phase curves are asymptotic to radial lines, and that

\[ \frac{d|f|}{dr} > 0 . \]
(B.7)

\(|f(z)|\) is therefore asymptotically increasing along the curves of constant phase.

\[ \text{QED} \]
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