Thermodynamic criticism to the Ginzburg–Landau approach to superconductivity

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Abstract

The Ginzburg–Landau approach postulates an energy density, together with an interpretation for the supercurrent, and invokes Ohm’s law. We consider quasi-one-dimensional nonuniform superconducting loops, either smooth or piecewise uniform, and evaluate the averages of the current and of the power released per unit length, above the critical temperature, due to thermal fluctuations. We consider three averages: canonical ensemble average, time-average using a time-dependent model, and canonical ensemble in the reciprocal space. All the evaluations imply that heat is absorbed in part of the loop and released in other part, despite the assumption that the loop is at uniform temperature.

Keywords: thermal fluctuations, TDGL, second law of thermodynamics, paraconductivity
I. INTRODUCTION

The second law of thermodynamics asserts that within a system at uniform temperature there is no heat flux on average, no matter what its Hamiltonian is.

Here we examine the case of a non-uniform 1D superconducting loop that encloses a non-integer number of magnetic flux quanta, slightly above its critical temperature. The motivation for the present study was Ref. [1].

The questions which we intend to answer will be spelled out in the following two Sections; in Sections IV–VI we develop and apply three different techniques for the evaluation of the quantities of interest, and in Section VII we summarize our results.

II. FORMULATION OF THE PROBLEM

A. Statistical average

If $q_1,...q_f$ are the coordinates of the phase space that describe the microstate of a system, $F(q_1,...q_f)$ is the energy of this system, and the system is in equilibrium with a heat bath at temperature $T$, then the average value of any quantity $Q(q_1,...q_f)$ is predicted to be

$$\langle Q \rangle = \frac{\int Q(q_1,...q_f)e^{-F/k_B T} dq_1,...dq_f}{\int e^{-F/k_B T} dq_1,...dq_f},$$

where $k_B$ is Boltzmann’s constant and the integrals are over the entire phase space.

B. Energy of the system

We consider a superconducting loop of perimeter $L$. In the static Ginzburg–Landau (GL) formalism the microstate of a superconductor is described by means of a complex field $\psi$, known as the “order parameter”, and the magnetic potential $A$. We consider the case in which the linear dimensions of the cross section are much shorter than $L$ and than the typical distances over which $\psi$ and $A$ vary; in this case the system becomes one dimensional and the position is described by the arc length $x$, $0 \leq x \leq L$.

We consider a situation above the critical temperature. The order parameter does not vanish because of thermal fluctuations, but terms of order $O(|\psi|^4)$ can be neglected and the GL energy
of the system can be written as

\[ F = \int_0^L \left[ \alpha |\psi|^2 + \frac{\hbar^2}{2m} \left( i \frac{d}{dx} - \frac{2\pi A}{\Phi_0} \right) \psi \right]^2 w(x) dx, \quad (2) \]

where \( \alpha > 0 \) depends on the superconducting material and on the temperature, \( m \) is the mass of a Cooper pair, \( A \) is the tangential component of \( A \), \( \Phi_0 = \pi \hbar c / e \) is the quantum of magnetic flux, \( e \) is the absolute value of the electron charge, and \( w(x) \) is the cross section of the superconducting wire.

The magnetic potential can be eliminated from (2) by defining

\[ \tilde{\psi}(x) = \exp \left[ \frac{2\pi i}{\Phi_0} \int_{x_1}^x A(x') dx' \right] \psi(x), \quad (3) \]

where \( x_1 \) is arbitrary and \( x \) can be extended beyond the range \( 0 \leq x \leq L \) by identifying \( x + L \) with \( x \). Using (3), (2) becomes

\[ F = \oint \left[ \alpha |\tilde{\psi}|^2 + \left( \frac{\hbar^2}{2m} \right) |d\tilde{\psi}/dx|^2 \right] w(x) dx. \quad (4) \]

It follows from (3) that \( \tilde{\psi}(x + L) = i \tilde{\psi}(x) \exp(2\pi i \Phi/\Phi_0) \), with \( \Phi = \oint A(x) dx \), so that \( \tilde{\psi} \) is multi-valued unless \( \Phi/\Phi_0 \) is integer.

Although the microstates of the system are described by two fields, \( \psi \) and \( A \), the accepted averaging procedure \[3–7\] includes only \( \psi \) in the phase space.

Let us now fix some of the parameters in the system to be considered. The length \( L \) of the loop, its electrical conductivity \( \sigma \), its resistance \( R \) (regarding the loop as an open circuit) and the temperature \( T \) will be fixed. We fix \( \alpha = \hbar^2 / 2mL^2 \). The contribution of the current around the loop to the magnetic flux will be negligible and we fix the applied flux as \( \Phi = \Phi_0 / 4 \), so that \( \tilde{\psi}(x + L) = i \tilde{\psi}(x) \). These values define a typical cross section \( w_0 \) and a dimensionless quantity \( \gamma \):

\[ w_0 = L/\sigma R, \quad \gamma = \hbar / e^2 R. \quad (5) \]

Several possibilities for the cross section \( w(x) \) will be considered. A smooth profile will be

\[ w(x) = \left( 2w_0 / \sqrt{3} \right) [1 - \cos(2\pi x / L) / 2]; \quad (6) \]

a family of discrete profiles will be described in the following section.
C. Discrete profiles

We divide the loop into $N$ segments of length $L/N$, centered at $x = x_k := (\lambda + k - 1)L/N$ with $0 \leq \lambda < 1$ and $k = 1, \ldots, N$. The cross section of segment $k$ will be

$$w_k = \left(\frac{w_0}{N}\right)[1 - \cos(2\pi x_k)/2] \sum_{j=1}^{N} \left[1 - \cos(2\pi x_j)/2\right]^{-1}. \quad (7)$$

In the limit of large $N$, this profile becomes the smooth profile (6).

Let us now build a model for the energy of this system, motivated by the GL energy (4). Instead of the field $\tilde{\psi}(x)$, we assign the value $\tilde{\psi}_k$ to the entire segment $k$; the derivative $d\tilde{\psi}/dx$ is replaced by the finite difference $N(\tilde{\psi}_k + 1 - \tilde{\psi}_k)/L$ and we define

$$F_{N,\lambda} := \sum_{k=1}^{N} \left\{ \frac{\alpha L w_k}{N} |\tilde{\psi}_k|^2 + \frac{\hbar^2 N w_{k+1}}{2mL} |\tilde{\psi}_{k+1} - \tilde{\psi}_k|^2 \right\}, \quad (8)$$

where $w_{k+}$ is some average between $w_k$ and $w_{k+1}$ and $\tilde{\psi}_{N+1} = \exp(2\pi i\Phi/\Phi_0)\tilde{\psi}_1$. We have studied two possibilities for $w_{k+}$: $w_{k+} = (w_k + w_{k+1})/2$ and $w_{k+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$.

We still require discretizations for $\psi$ and $A$. We assign the magnetic potential $A_+^k$ (respectively $A_-^k$) to the half-segment $x_k < x < x_k + L/2N$ (respectively $x_k > x > x_k - L/2N$) and define

$$\tilde{\psi}_1 := \psi_1; \quad \tilde{\psi}_k := \exp\left[\frac{\pi i L}{N\Phi_0} \left(A_+^k + A_-^k + \sum_{j=2}^{k-1} (A_+^j + A_-^j)\right)\right]\psi_k \quad 2 \leq k \leq N. \quad (9)$$

In terms of these variables, the energy is

$$F_{N,\lambda} = \sum_{k=1}^{N} \left\{ \frac{\alpha L w_k}{N} |\tilde{\psi}_k|^2 + \frac{\hbar^2 N w_{k+1}}{2mL} \left(|\tilde{\psi}_k|^2 + |\tilde{\psi}_{k+1}|^2 - 2Re \left[\tilde{\psi}_{k+1}\tilde{\psi}_k^*e^{(\pi i L/N\Phi_0)(A_+^k + A_-^{k+1})}\right]\right) \right\}. \quad (10)$$

Although the systems described here approach the smooth loop only in the limit of large $N$, any of them, for any $N$, $\lambda$ and $\{w_{k+}\}$, can be regarded as a self-consistent model that should obey the laws of thermodynamics.

III. QUANTITIES OF INTEREST

In the case of the smooth loop, the supercurrent density is given by [2]:

$$J_S(x) = -\frac{2e\hbar}{m} \text{Im} \left[\psi^* \left(\frac{d}{dx} + \frac{2\pi i}{\Phi_0} A\right) \psi\right] = -\frac{2e\hbar}{m} \text{Im} \left[\tilde{\psi}^* \frac{d\tilde{\psi}}{dx}\right]. \quad (11)$$

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Although the expression for $J_S$ looks the same as the expected current density of a charged particle described by the Schrödinger equation, we should note that in the Schrödinger case this expression follows from charge conservation, whereas in the GL case it follows from variation of the energy functional with respect to $A(x)$. Similarly, in the discrete case, in the half-segment where the magnetic potential is $A_k^{\pm}$, the supercurrent density is given by $J_{S_k}^{\pm} = -(2N_c/Lw_k)\partial F_{N,\lambda}/\partial A_k^{\pm}$, where the meaning of partial derivative is that the magnetic potential in the other regions of the loop, and also $\{\psi_j\}$, are kept constant. Using (10) and (9) we obtain

$$J_{S_k}^{\pm} = -(2Ne\hbar w_k/mLw_k)\text{Im}[\tilde{\psi}_k^{\pm}\tilde{\psi}_{k+1}]; \quad J_{S_k}^{-} = -(2Ne\hbar w_{(k-1)+}/mLw_k)\text{Im}[\tilde{\psi}_{k-1}^{\pm}\tilde{\psi}_k].$$  (12)

We denote by $I$ the total current around the loop. By Ohm’s law, the electric field is $E(x) = [I/w(x) - J_S(x)]/\sigma$ in the smooth case and

$$E_k^{\pm} = (I/w_k - J_{S_k}^{\pm})/\sigma$$  (13)
in the discrete case. Since the magnetic flux through the loop remains constant, the circulation of the electric field vanishes and the current is

$$I = \oint J_S dx / \oint w^{-1} dx$$  (14)
in the smooth case and, noting that $\sum_{k=1}^N w_k^{-1} = N/w_0$,

$$I = \frac{w_0}{2N} \sum_{k=1}^N (J_{S_k}^- + J_{S_k}^+)$$  (15)
in the discrete case.

The power per unit length delivered by the electric field to the flowing charges is

$$P(x) = IE(x).$$  (16)

If this power is not taken somewhere else by the supercurrent, it has to be passed as heat into the position $x$.

IV. EVALUATION OF AVERAGES FOR DISCRETE SYSTEMS

The sets $\{\psi_k\}$ and $\{\tilde{\psi}_k\}$ cover the same phase space, so that it makes no difference over which of them we average. For further simplification, we diagonalize (8) numerically, taking the form

$$F_{N,\lambda} = \sum_{k=1}^N f_k |\varphi_k|^2.$$  (17)
The passage from \( \{ \psi_k \} \) to \( \{ \varphi_k \} \) is a rotation in the phase space, and any of them can be used for averaging.

The electric field in \((13)\) and the current in \((15)\) are quadratic expressions. \( \langle \varphi_i^* \varphi_j \rangle \) vanishes for \( j \neq i \), and \( \langle \varphi_k^* \varphi_k \rangle = k_BT/f_k \). We found \( \langle E_k^\pm \rangle = 0 \) in all the cases considered and the values of the average currents obtained in this way are shown in the first two rows of Table I.

Table I: Average current in units of \( e k_BT/\hbar \). The ‘method’-column directs to the section where the evaluation method is described. ‘a’ stands for the arithmetic mean \( w_{k+} = (w_k + w_{k+1})/2 \) and ‘h’ for the harmonic mean \( w_{k+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1} \). The pairs of numbers in the headings row stand for \( N \) and \( \lambda \).

| method | \( w_{k+} \) | 3.0 | 3.0.5 | 4.0 | 4.0.25 | 5.0 | 10.0 | 15.0 |
|--------|---------------|-----|-------|-----|--------|-----|------|------|
| IVa    | -1.291       | -1.286 | -1.271 | -1.270 | -1.259 | -1.238 | -1.234 |
| IVh    | -1.215       | -1.252 | -1.227 | -1.230 | -1.230 | -1.230 | -1.230 |
| Vh     | -1.214       | -1.251 | -1.224 | -1.228 | -1.229 | -1.225 | -1.228 |

The average power per unit length, \( \langle P_k^\pm \rangle \), is obtained from Eq. \((16)\). \( IE_k^\pm \) is a long linear combination of components of the form \( \varphi_i^* \varphi_j \varphi_i^* \varphi_m \). Most of these components give no contribution to the average, and the only contributions are those of the forms \( \langle (\varphi_i^* \varphi_i)^2 \rangle = 2(k_BT/f_i)^2 \) and \( \langle \varphi_i^* \varphi_i \varphi_j^* \varphi_j \rangle = (k_BT)^2/f_i f_j \) for \( i \neq j \). Representative results obtained for small values of \( N \) are shown in Fig. 1. From the second law of thermodynamics and the fact that the temperature is uniform, we would expect no local energy flow from the system to the heat bath on average, i.e. \( \langle P_k^\pm \rangle \) should vanish identically (as would be the case if \( I \) and \( E_k^\pm \) were uncorrelated). Instead, we see in Fig. 1 that, with the exception of the case \( (N,\lambda) = (3, 0) \) with arithmetic mean, the system absorbs power at its broadest part and releases it at the thinnest part.

We would like to know whether this disagreement with thermodynamics is a feature of discretization or a property of the GL approach. With this purpose, we investigate the behavior of \( \langle P_k^\pm \rangle \) as \( N \) increases and the smooth limit is approached. Judging by the \( N \)-dependence of \( \langle I \rangle \), we expect a faster convergence when \( \{ w_{k+} \} \) are taken as harmonic means. (The values of \( \langle I \rangle \) for \( N = 100 \) and \( N = 120 \) coincide within seven significant figures.) The results are shown in Fig. 2. Surprisingly, the effect reverses at intermediate values of \( N \) and, as the smooth limit is approached, the system absorbs power at its thinnest part and releases it at the broadest part.
Figure 1: Average power released per unit length [Eq. (10)] as a function of position, evaluated as described in Sec. IV for small values of $N$. The icons at the head of the figure depict the cross section of the system considered, with the position where $x = 0$ and $x = L$ join at the bottom of the icon. Solid lines stand for $w_{k+} = (w_k + w_{k+1})/2$ and dashed lines for $w_{k+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$. Red: $N = 3$, $\lambda = 0$; black: $N = 3$, $\lambda = 0.5$; blue: $N = 4$, $\lambda = 0.25$. $\gamma$ is defined in Eq. (5).

Figure 2: Average power released per unit length at the thinnest (blue) and at the broadest (orange) segment of the loop, as functions of $N$. In all cases $N$ is an integer multiple of 4, $w_{k+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$ and $\lambda = 0$. When passing from $N = 12$ to $N = 16$ the signs of $\langle P(0) \rangle$ and $\langle P(L/2) \rangle$ reverse.
Figure 3: The colored segments show the average power per unit length as a function of position, obtained as described in Section IV for $N = 30$. The purple lines were evaluated taking $w_{k+}$ as an arithmetic mean and the other lines taking $w_{k+}$ as a harmonic mean. Purple: $\lambda = 0$; red: $\lambda = 0$; blue: $\lambda = 0.25$; green: $\lambda = 0.5$. The pink segments at $x = 0$, $x = L/2$ and $x = L$ mark the limit $N \to \infty$, obtained from extrapolation in Fig. 2. The black curve and the pink circles were obtained as explained in Section VI with $M = 15$ for the curve and $M = 50$ for the circles.

Figure 3 enables us to appraise the influence of discretization on the power released, for a moderately large value of $N$. All the lines were obtained for $N = 30$, but using different averages for $w_{k+}$ or different orientations $\lambda$. The fact that all the results nearly coalesce indicates that discretization has no qualitative influence, and suggests that, if $\langle P(x) \rangle$ does not vanish for moderately large $N$, it will also not vanish for the smooth loop.

The evidence of this section leads us to conclude that the GL approach does not comply with thermodynamics, neither for a discrete nor for a smooth nonuniform loop. We might think of three reservations to this conclusion: (i) Johnson noise has to be added to Eq. (13); (ii) the released power should not be interpreted as energy transferred to the heat bath, but rather as potential energy that is converted to kinetic energy of the moving charges and then transported as an energy supercurrent; and (iii) the magnetic potential should be included in the phase space. In order to address these reservations we require a model for time evolution.
V. TIME DEPENDENT MODEL

In this section we discretize time into steps \( \Delta t \), sufficiently small to justify regarding the changes of \( \psi \) and \( A \) during a step as infinitesimal.

A. Johnson noise and evolution equations

During a step \( \Delta t \), the Johnson noise adds to the electric field in Eq. (13) an additional term \( \eta_{k,\pm} \sqrt{4k_BTN/\sigma w_k L \Delta t} \), where \( \eta_{k,+} \) and \( \eta_{k,-} \) stand for random variables with zero average, variance 1 and normal distribution. Choosing a gauge such that \( cE^\pm_k = -\Delta A^\pm_k / \Delta t \), where \( \Delta A^\pm_k(t) = A^\pm_k(t + \Delta t) - A^\pm_k(t) \) and using (5), Eq. (13) becomes an evolution equation for \( A^\pm_k(t) \):

\[
\Delta A^\pm_k = -(cw_0 \hbar / \gamma e^2 L)(I/w_k - J^\pm_{sk}) \Delta t - (2c/eL) \eta_{A,k,\pm} \sqrt{k_BTNw_0 \Delta t / \gamma w_k}. \tag{18}
\]

Equation (18) is implied by the time-dependent Ginzburg–Landau model (TDGL), which has Ohm’s law built into it. TDGL is the simplest dynamical model that converges to GL when equilibrium is attained. Ignoring fluctuations, for a system in which there are no supercurrents in equilibrium (provided that fluctuations are ignored), and taking a gauge with no electrical potential, TDGL states that \( d\psi/dt \propto -\delta F/\delta \psi^* \) and \( dA/dt \propto -\delta F/\delta A \) [8, 9], or, in discretized form, \( \Delta \psi_k / \Delta t \propto -w_k^{-1} \partial F_{N,\lambda} / \partial \psi_k^* \) and \( \Delta A^\pm_k / \Delta t \propto -w_k^{-1} \partial F_{N,\lambda} / \partial A^\pm_k \propto J^\pm_{sk} \). From the equation \( \Delta A^\pm_k / \Delta t \propto J^\pm_{sk} \), subjected to the constriction \( \sum_{k=1}^N (A^-_k + A^+_k) = \text{constant} \), and adjusting the constant of proportionality to the conductivity of the loop, we recover Eq. (18) without the stochastic term. The last term is a Langevin term, the variance of which is determined by the fluctuation-dissipation theorem [10].

Similarly, from the equation \( \Delta \psi_k / \Delta t \propto -w_k^{-1} \partial F_{N,\lambda} / \partial \psi_k^* \),

\[
\frac{\Delta \psi_k}{\Delta t} = -\frac{\Gamma \alpha}{\hbar} \psi_k - \frac{N^2 \hbar \Gamma}{2mL^2 w_k} \left[ w_{k+}(\psi_k - \psi_{k+1}e^{(\pi i L/N \Phi_0)}(A^+_k + A^+_{k+1})) \right. \\
+ \left. w_{(k-1)+}(\psi_k - \psi_{k-1}e^{-(\pi i L/N \Phi_0)}(A^-_{k-1} + A^-_k))] + \eta^\psi_k \sqrt{NTk_BT / \hbar L w_k \Delta t}, \tag{19}
\]

where \( \Gamma \) is a dimensionless material constant and both the real and the imaginary part of \( \eta^\psi_k \) are random variables with zero average, variance 1, and normal distribution. Using (9) and assuming
that $\Delta t$, $\Delta A_j^\pm$ and $\Delta \psi_k$ are sufficiently small, (19) gives the evolution of $\tilde{\psi}_k$:

$$\tilde{\psi}_k(t + \Delta t) = \left\{ (1 - \frac{\Gamma \alpha \Delta t}{\hbar}) \tilde{\psi}_k(t) + \frac{N^2 \hbar \Gamma \Delta t}{2mL^2 w_k} [w_{(k-1)+} \tilde{\psi}_{k-1}(t) - (w_{(k-1)+} + w_{k+}) \tilde{\psi}_k(t) + w_{k+} \tilde{\psi}_{k+1}(t)] + \eta_k \sqrt{\frac{N \Gamma k_B T \Delta t}{\hbar L w_k}} \right\} \text{prod}_k,$$

(20)

with

$$\text{prod}_1 = 1; \quad \text{prod}_{j+1} = \text{prod}_j \exp\left[iLe(\Delta A_j^+ + \Delta A_{j+1})/Nc\hbar\right].$$

(21)

TDGL is valid for gapless superconductivity. This is the situation that we are considering, since $\alpha > 0$. At any rate, in this study we are less concerned with the question of whether there is a system in nature that follows this evolution than with the question of compatibility of this model with thermodynamics.

We note that $\gamma$ and $\Delta t$ enter (18) only through their ratio; likewise, $\Gamma$ and $\Delta t$ enter (20) only through their product. Therefore, changing $\gamma$ and $\Gamma$ while keeping their product unchanged leads to the same evolution, although at a different rate. Anyway, in this study we are interested only in equilibrium values, which should not be affected by the choices of $\gamma$ and $\Gamma$.

Unlike the procedure in Sec. [IV], which did not include the magnetic potential in the phase space, in TDGL the fields $A$ and $\psi$ stand on the same footing.

**B. Energy current**

We may decompose $F_{N,\lambda}$ in (8) into a sum of terms $F_k$ and $F_{k+}$, where $F_k$ is the term that contains $|\tilde{\psi}_k|^2$ and $F_{k+}$ contains $|\tilde{\psi}_{k+1} - \tilde{\psi}_k|^2$. We may associate $F_k$ as located in segment $k$ and $F_{k+}$ as located at the boundary between $k$ and $k + 1$.

A change $\Delta \tilde{\psi}_k$ in $\tilde{\psi}_k$ leads to changes $\Delta F_k$, $\Delta F_{k+}$ and $\Delta F_{(k-1)+}$ in $F_k$, $F_{k+}$ and $F_{(k-1)+}$. The time average of $\text{Re}[\tilde{\psi}_k^* \Delta \tilde{\psi}_k]$ vanishes, and accordingly we take only cross terms into account. $\Delta F_{k+} + \Delta F_{(k-1)+}$ may be interpreted as a temporary change in the local energy, whereas $I_{Ek} \Delta t = (\Delta F_{k+} - \Delta F_{(k-1)+})/2$ may be interpreted as the energy transported by the supercurrent from the negative to the positive boundary of segment $k$. From (8) we obtain

$$I_{Ek} \Delta t = (\hbar^2 N/2 m L) \text{Re}[\Delta \tilde{\psi}_k (w_{(k-1)+} \tilde{\psi}_{k-1} - w_{k+} \tilde{\psi}_{k+1})^*].$$

(22)

Equation (22) can be identified as a discretized version of Eq. (D) in Ref. [8], adapted to the case of a nonuniform wire.
C. Results

The averages in this Section are not obtained from Eq. (11), but are rather time-averages. We took \( \gamma = \Gamma = 1 \) and, for evaluation of the average current, we took \( \Delta t = 2 \times 10^{-4} h / k_B T \) and followed the evolution equations (18) and (20) during \( 2 \times 10^{10} \) steps for \( N < 10 \) (\( 10^{10} \) steps for \( N \geq 10 \)). The initial value of \( \tilde{\psi} \) was zero, and the \( 10^7 \) initial steps were not included in the average. Our results for the current are shown in the last row in Table I.

Comparison of the last two rows in Table I shows that these two apparently independent procedures lead to the same results, but the time-averages have a random uncertainty, slightly larger than \( 10^{-3} k_B T / h \), that can be attributed to the fact that the averaging spanned time was not infinite. There is also a systematic discrepancy of a similar size, that can be attributed to the fact that \( \Delta t \) was not infinitesimal.

Evaluation of \( \langle I_E \rangle \) and \( \langle P \rangle \) required longer periods of time and was limited to small values of \( N \). For \( N = 5 \), the evolution of \( \tilde{\psi} \) and \( A \) was followed during \( 1.6 \times 10^{11} \) steps of \( \Delta t = 2 \times 10^{-3} h / k_B T \) and, for \( N = 8 \), during \( 10^{11} \) steps of \( \Delta t = 10^{-3} h / k_B T \). \( \langle I_E \rangle \) vanished within its statistical uncertainty, which was of the order of \( 10^{-4} (k_B T)^2 / \gamma h \) and therefore had no significant influence on the released powers.

Figure 4 compares the local released powers found in this Section with those obtained using Eq. (11). Within the expected statistical uncertainty, the results obtained by both procedures agree with each other in all the cases, and in most cases they are convincingly different from zero.

Figure 4 shows that Johnson noise does not clear away the local power release. Moreover, if we use (13) instead of (18), we obtain different results for \( \langle I \rangle \), \( \langle I_E \rangle \) and \( \langle P \rangle \), indicating that this noise can actually be regarded as a source of the power release distribution.

VI. AVERAGING IN THE RECIPROCAL SPACE

In the case of a smooth profile we can write \( \tilde{\psi} \) as a modified Fourier series,

\[
\tilde{\psi}(x) = \sum b_n e^{2\pi i (n+1/4)x/L} .
\]  

This sum extends over all integers, but we will take \( |n| \leq M \), where \( M \) will be increased until acceptable convergence is obtained.
Figure 4: Powers released per unit length as functions of position. The horizontal lines were evaluated using Eq. (1) and the vertical lines are time-averages, obtained as explained in Section V. The vertical lines are centered at the time-averages and their half-lengths equal the standard deviation divided by the square root of the number of steps. Black: $N = 8$, $\lambda = 0.5$; red: $N = 5$, $\lambda = 0.1$. $w_{k+}$ was taken as the harmonic mean of $w_k$ and $w_{k+1}$.

From (11) and (23),

$$J_S(x) = -\frac{4\pi e \hbar}{mL} \Re \left[ \sum_{-M \leq \ell, n \leq M} b_\ell^* b_n \left( n + \frac{1}{4} \right) e^{2\pi i (n-\ell)x/L} \right];$$ (24)

from (14) and (24), using (5),

$$I = -\frac{4\pi e \hbar w_0}{mL} \sum_{-M \leq n \leq M} \left( n + \frac{1}{4} \right) b_n^* b_n;$$ (25)

and from (4), (6) and (23)

$$F = \frac{2w_0 L}{\sqrt{3}} \left\{ \sum_{n=-M}^{M} \left[ \alpha + \frac{(2\pi \hbar)^2}{2mL^2} \left( n + \frac{1}{4} \right)^2 \right] b_n^* b_n \\
- \frac{1}{4} \sum_{n=-M}^{M-1} \left[ \alpha + \frac{(2\pi \hbar)^2}{2mL^2} \left( n + \frac{1}{4} \right) \left( n + \frac{5}{4} \right) \right] (b_{n+1}^* b_n + b_{n+1}^* b_n^* + b_{n+1}^* b_n^*) \right\}. \quad (26)$$

We can now proceed as in Section IV. We diagonalize (26), so that it takes the form $F = \sum_{n=-M}^{M} f_n |c_n|^2$. It follows that the only quadratic terms with nonzero average are $\langle c_j^* c_j \rangle = k_B T / f_j$, and the only quartic terms that contribute are those of the forms $\langle (c_j^* c_j)^2 \rangle = 2(k_B T / f_j)^2$.
and \( \langle c_i^* c_i c_j^* c_j \rangle = (k_B T)^2 / f_i f_j \) for \( i \neq j \). Finally, the average of the power per length in (16) is evaluated as

\[
\langle P(x) \rangle = \frac{\hbar w_0 \langle I[I - J_S(x)w(x)] \rangle}{\gamma e^2 L w(x)}.
\]

The values of the average current obtained in this way converge slowly for increasing \( M \). For \( M = 5 \) (respectively 10, 15, 20) we obtain \( \hbar \langle I \rangle / ek_B T = -1.289 \) (respectively -1.261, -1.251, -1.246).

The values of \( \langle P(x) \rangle \) obtained from (27) for \( M = 15 \) are shown by the black curve in Fig. 3. This curve is not expected to coincide with the results obtained in Section IV because the limits \( M, N \to \infty \) have not been reached. Nevertheless, this figure strongly suggests that all the results would coincide in this limit and rebuts the possibility that \( \langle P(x) \rangle \equiv 0 \).

\[ \text{VII. SUMMARY} \]

Based on the Ginzburg–Landau model for superconductivity, we have evaluated the average local power absorbed or released by the circulating current in a family of systems that represent a loop with nonuniform cross section threaded by a magnetic flux. The evaluation was performed following three apparently independent procedures.

Our results are independent of the followed procedure, and violate the requirement that for uniform temperature there should be no heat flow.

\[ \text{Acknowledgements} \]

The author has benefited from correspondence with Armen Allahverdyan. Part of the computations have used resources of the Technion – Israel Institute of Technology.

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